Virtual permutations and polymorphisms

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There is a natural map from a symmetric group $S_n$ to a smaller symmetric group $S_{n-1}$, we write a decomposition of a permutation into a product of disjoint cycles and remove the element $n$ from this expression. For this reason there exists the inverse limit $\mathcal{S}$ of sets $S_n$. We equip $S_n$ with the uniform distribution (or more generally with an Ewens distribution) and get a structure of a measure space on $\mathcal{S}$ (it is called 'virtual permutations' or 'Chinese restaurant process'), a double $S_\infty \times S_\infty$ of an infinite symmetric group acts on $\mathcal{S}$ by left and right 'multiplications'. We discuss the closure of $S_\infty \times S_\infty$ in the semigroup of polymorphisms (spreading maps with spreaded Radon–Nikodym derivatives) of $\mathcal{S}$. We get formulas for some polymorphisms, in particular for the center of the closure. Expressions are sums of multiple convolutions of Dirichlet distributions, summation sets are certain collections of dessins d’enfant.

1. Introduction. Virtual permutations, chips, and polymorphisms

The topic of the paper is formulas for the action of the Olshanski semigroups of chips on the virtual permutations by polymorphisms, in this section we explain these notions and formulate the problem.

1.1. Some notation. Denote by $\mathbb{Z}_+$ (resp., $\mathbb{R}_+$) the set of nonnegative integers (resp. reals), by $\mathbb{R}^\times$ (resp. $\mathbb{R}_{>0}$) the multiplicative group (resp. the set) of positive reals. For a set $A$ we denote by $\# A$ the number of its elements. By $A \sqcup B$, $\sqcup_j A_j$ we denote disjoint unions of sets.

We denote a list $(v_1, v_2, \ldots)$ by $\{v_i\}$ or by $\{v_i\}_i$ (we apply this notation for both ordered or nonordered lists). For finite nonordered lists $\{k_j\}$ of positive integers we use a notation

$$t_m[\{k_j\}] := \text{number of entries of } m \text{ to a list } \{k_j\}.$$ (1.1)

By $\delta_X[[a]]$ we denote the unit atomic measure (delta-function) at a point $a$ of a space $X$. We denote product-measures by $\mu_1 \times \mu_2$ (to not be confused with the symbol $\times$ of a multiplication).

Denote by $S_n$ the symmetric group, which is considered as the group of permutations of the set

$$I_n := \{1, 2, \ldots, n\},$$

by $S_\infty$ the group of all permutations of $\mathbb{N}$ with finite supports,

$$S_\infty = \lim_{\rightarrow} S_n = \cup_{n=1}^\infty S_n.$$ 

This group is countable and is equipped with the discrete topology.

By $\mathfrak{S}_\infty$ we denote the group of all permutations of $\mathbb{N}$. This group is continual and is equipped with the unique separable topology compatible with the structure of the group. A sequence $\sigma_j \in \mathfrak{S}_\infty$ converges to $\sigma$ if for any $p \in \mathbb{N}$ we have $\sigma_j(p) = \sigma(p)$ for sufficiently large $j$.

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We can regard elements of symmetric groups as diagrams of the form

\[
\begin{array}{c}
1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\]

(1.2)

We draw a collection of 'upper circles' enumerated by natural numbers and a collection of 'lower circles' also enumerated by \(\mathbb{N}\). If \(g\) sends \(i\) to \(j\), then we connect by an arc \(i\)-th upper circle with \(j\)-th lower circle.

A product of permutations corresponds to a gluing of diagrams

\[
\begin{array}{c}
1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\]

\[\times \quad \big\vert \quad \times \]

\[\begin{array}{c}
1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\]

\[\times \quad \big\vert \quad \times \]

A. Virtual permutations and the bisymmetric group

1.2. The map \(\Upsilon\). Define a canonical map

\[
\Upsilon_{n-1}^n : S_n \rightarrow S_{n-1}
\]

in the following way. Let \(\sigma \in S_n\).

1) If \(\sigma(n) = n\), we set \(\Upsilon_{n-1}^n \sigma(j) = \sigma(j)\) for all \(j < n\).

2) Let \(\sigma(n) = \alpha \neq n\). Then \(\sigma^{-1}(n) = \beta\) also \(\neq n\). Then we assume \(\Upsilon_{n-1}^n \sigma(\beta) = \alpha\). For \(j \neq \beta\) we set \(\Upsilon_{n-1}^n \sigma(j) = \sigma(j)\).

EXAMPLE. For instance,

\[
\Upsilon_4^5 \left( \begin{array}{c}
1 & 2 & 3 & 4 & 5 \\
3 & 2 & 4 & 1 & 5
\end{array} \right) = \left( \begin{array}{c}
1 & 2 & 3 & 4 \\
3 & 2 & 4 & 1
\end{array} \right);
\]

\[
\Upsilon_4^5 \left( \begin{array}{c}
1 & 2 & 3 & 4 & 5 \\
3 & 2 & 5 & 1 & 4
\end{array} \right) = \left( \begin{array}{c}
1 & 2 & 3 & 4 \\
3 & 2 & 4 & 1
\end{array} \right).
\]

On the language of diagrams we have:

\[
\Upsilon_4^5 :
\begin{array}{c}
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5
\end{array}
\rightarrow
\begin{array}{c}
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5
\end{array} = \begin{array}{c}
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4
\end{array}
\]

Namely, we add an arc connecting the top '5' and the bottom '5'. We get a compound arc \(3^{\text{top-bottom}}_5 5^{\text{top-bottom}}_4\) and consider it as an arc \(3^{\text{top-bottom}}_5 4\) (here and below we consider arcs with fixed ends up to isotopies fixing ends). Similarly,

\[
\Upsilon_4^5 :
\begin{array}{c}
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5
\end{array}
\rightarrow
\begin{array}{c}
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5
\end{array} = \begin{array}{c}
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4
\end{array}
\]

We add an arc \(5^{\text{top-bottom}}\), get a cycle, and remove it.

\[
\Upsilon_{n-1}^n\text{ is } S_{n-1} \times S_{n-1}\text{-equivariant in the following sense:}
\]

(1.3) \[
\Upsilon_{n-1}^n(h_1^{-1}gh_2) = h_1^{-1}\Upsilon_{n-1}^n(g)h_2, \quad \text{where } g \in S_n, \ h_1, \ h_2 \in S_{n-1}.
\]
Another description of the map $\Upsilon_{n-1}^n$. Decomposing $g \in S_n$ as a product of disjoint cycles,

\begin{equation}
g = (k_1^1 k_2^1 \ldots) (k_1^2 k_2^2 \ldots) \ldots
\end{equation}

and removing $n$ from this expression, we get an element $\Upsilon_{n-1}^n(g)$ defined as product of disjoint cycles.

More generally, for $n \geq m$ we define a canonical map $\Upsilon_m^n : S_n \to S_m$,

\begin{equation}
\Upsilon_m^n(g) := \Upsilon_{m-1}^n \Upsilon_{n-2}^n \Upsilon_{n-1}^n(g).
\end{equation}

Equivalently, we remove $m + 1$, $m + 2$, \ldots, $n$ from expression (1.4) and get an element of $S_m$.

1.3. The inverse limit of the spaces $S_n$. Thus we have the following chain of surjective maps

\begin{equation}
\cdots \xleftarrow{\Upsilon_{n+2}^n} S_n \xleftarrow{\Upsilon_{n+1}^n} S_{n+1} \xleftarrow{\Upsilon_{n+2}^n} S_{n+2} \xleftarrow{\Upsilon_{n+3}^n} \cdots
\end{equation}

So we can consider the inverse limit of sets $S_n$

\[ \mathcal{S} = \lim_{\leftarrow} S_n. \]

By the definition, $\mathcal{S}$ consists of sequences

\[ \tilde{\sigma} = (\sigma_1, \sigma_2, \sigma_3, \ldots), \quad \text{where} \ \sigma_k \in S_k \ \text{and} \ \Upsilon_{k-1}^k \sigma_k = \sigma_{k-1}. \]

In particular, we have a canonical map $\Upsilon_{n}^\infty : \mathcal{S} \to S_n$ defined by $\Upsilon_n^\infty \tilde{\sigma} = \sigma_n$.

A point of the space $\mathcal{S}$ can be described in the following way. We fix a partition of $\mathbb{N}$ into a disjoint union of subsets (pre-tables)

\begin{equation}
\mathbb{N} = M_1 \sqcup M_2 \sqcup \ldots
\end{equation}

and fix a cyclic ordering in each subset. A map $\Upsilon_{n}^\infty$ is the removing all elements $> n$. Then we get the set $I_n = \{1, 2, \ldots, n\}$ splitted into a disjoint union of cyclically ordered subsets, i.e., an element of $S_n$.

The space $\mathcal{S}$ is not an inverse limit in the category of groups, since generally

\[ \Upsilon_{n-1}^n(g_1 g_2) \neq \Upsilon_{n-1}^n(g_1) \Upsilon_{n-1}^n(g_2). \]

In particular, $\mathcal{S}$ does not have a group structure.

However, for $n > m$ by (1.3),

\[ \Upsilon_m^n(h_1^{-1} g h_2) = h_1^{-1} \Upsilon_m^n(g) h_2, \quad \text{where} \ g \in S_n, \ h_1, h_2 \in S_m. \]

Therefore $S_m \times S_m$ acts on $\mathcal{S}$. Since $m$ is arbitrary, we get an action of the double $S_\infty \times S_\infty$ on $\mathcal{S}$.

It is more natural to consider such inverse limit as a measure space.

1.4. Virtual permutations. For $g \in S_n$ denote by $[g]$ the number of its disjoint cycles. Fix $z \geq 0$. The Ewens distribution is a probabilistic measure $\mu_n^z$ on $S_n$ defined by

\[ \mu_n^z(g) = \frac{z^{|g|}}{z(z + 1) \ldots (z + n - 1)}, \]

for $z = 1$ we get a uniform distribution on $S_n$.

It easy to show that the pushforward of the Ewens distribution $\mu_n^z$ is the Ewens distribution $\mu_{n-1}^z$ (i. e., for any subset $K \subset S_{n-1}$ the measure of its preimage
A more convenient description of $\mathcal{S}^z$. It is done in two steps.

**Step 1.** We present the limit distribution of lengths of cycles, see Kingman [10]. Consider the measure $\nu$ on the half-line $x > 0$ defined by

$$
\text{d}\nu^z(x) = z x^{-1} e^{-x} \text{d}x.
$$

Consider the Poisson measure $\pi^z$ on the set of (non-ordered) countable subsets (see, e.g., [16], [24], Sect. X.4) $\{s_1, s_2, \ldots\} \subset \mathbb{R}$ defined by the measure $\mathbb{E} \nu^z$. It can be easily shown that the series $\sum s_j$ converges a.s. We define a collection $\{\ell_\omega\}$ as $\ell_\omega = s_\omega / \sum s_j$ and get a measure (the Poisson–Dirichlet distribution) on the set of unordered collections $\{\ell_\omega\}$ such that $\sum \ell_\omega = 1$.

For a given $\{\ell_\omega\}$ consider a collection $U_1, U_2, \ldots$ of oriented circles of lengths $\ell_1, \ell_2, \ldots$. We imagine them as circles on plane oriented clockwise and call them by *tables*, sets $\sqcup U_\omega$ we call by *restaurants* (‘Chinese restaurants’). Denote the space of all restaurants equipped with the Poisson–Dirichlet distribution by $\mathbb{T}^z$.

We can regard points of $\mathbb{T}^z$ as non-ordered collections of tables $\{U_\omega\}$ or as non-ordered collections of lengths $\{\ell_\omega\}$.

**Remark.** Clearly, the numbers $\ell_\omega$ are pairwise distinct a.s.

**Step 2.** We chose a random sequence in $\sqcup U_\omega$ such that each element is distributed uniformly with respect to the Lebesgue measure on $\sqcup U_\omega$. We denote points of a sequence (we call them *guests*) by $\overline{1}, \overline{2}, \overline{3}, \ldots$. Configurations of guests on tables are defined up to rotations of tables. A *virtual permutation* or an *occupied restaurant* is a collection $\{\ell_\omega\}$ distributed according the Poisson–Dirichlet law, the corresponding restaurant $\sqcup U_\omega$, and a random sequence $\overline{1}, \overline{2}, \overline{3}, \ldots \in \sqcup U_\omega$. We denote the measure space of all virtual permutations by $\mathcal{S}^z$, denote by $\mu^z$ the measure on $\mathcal{S}^z$.

We denote *occupied tables* by $\overline{U}_\omega$.

**Remark.** Clearly, for almost all points of $\mathcal{S}^z$ a sequence of guests $\overline{1}, \overline{2}, \ldots$ is dense in $\sqcup U_\omega$ and its elements are pairwise distinct.

---

2For a measurable subset $A \subset \mathbb{R}_{>0}$ denote by $\Omega(A, k)$ the set (‘event’) of all configurations whose intersections with $A$ have precisely $k$ point. We assume that a probability of $\Omega(A, k)$ is $\exp(-\nu^z(A)) \nu^z(A)^k / k!$ and for pairwise disjoint sets $A_j$ and arbitrary $k_j$ the events $\Omega(A_j, k_j)$ are independent.
Remark. The set of all tables of a given restaurant can be identified with \( \mathbb{N} \), we can order sets of tables according a length \( \ell_{\omega} \), or according a minimal number of a guest sitting at the table. We prefer to think that a set of tables of a given restaurant is an abstract countable set and do not identify such sets for different restaurants.

There is a canonical map \( \Upsilon_{\infty}^n : \mathcal{S}^\infty \rightarrow S_n \). Namely, let us ’forget’ all guest \( \overline{n + 1}, \overline{n + 2}, \ldots \) \( \in \sqcup U_{\omega} \). After this, each table \( U_{\omega} \) contains a finite collection of guests \( \overline{\omega}, \overline{\omega}_1, \overline{\omega}_2, \ldots \) seating at the table. So we get a partition of \( \{1, \ldots, n\} \) into a disjoint union of cyclically ordered sets. Thus we come to a permutation decomposed into a product of disjoint cycles.

The pushforward of the measure \( \mu^z \) to \( S_n \) is the Ewens measure \( \mu^z_n \). We also have \( \Upsilon_n^{-1} \circ \Upsilon_{\infty}^n = \Upsilon_{n-1}^\infty \).

It remains to explain how pre-tables defined in Subsect. 1.4 generate tables. Let \( \tilde{\sigma} \) be a point of the inverse limit \( \prod_{n=1}^{\infty} \mathcal{S}_n \). Consider two points \( \tilde{i}, \tilde{j} \) on one pre-table. For each \( N \geq i, j \) consider the cycle of \( \Upsilon_N \tilde{\sigma} \) containing \( i, j \), say \( (\ldots i m_1 m_2 \ldots m_n j \ldots) \). Denote by \( p_{ij}(N) \) the number of elements between \( i \) and \( j \). Then (see [48], [14]) for almost all points of \( \mathcal{S}^\infty \) for all pairs \( \tilde{i}, \tilde{j} \) lying on one pre-table we define

\[
(1.8) \quad \text{length of arc } [\tilde{i}, \tilde{j}] := \lim_{N \to \infty} \frac{p_{ij}(N)}{N}.
\]

The limit exists a.s. for all \( i, j \). Then we get an occupied resturant.

1.6. The action of \( S_\infty \times S_\infty \) on \( \mathcal{S} \). The space \( \mathcal{S}^\infty \) is not a group, but the group \( S_\infty \times S_\infty \) acts on \( \mathcal{S} \) by ’left and right multiplications’. Let \( u \in \mathcal{S}^\infty \), let \( \{ \overline{U} \}_{\alpha} \) be its occupied tables. Set \( u = \Upsilon_n^\infty u \). For each \( i \in I_n \) we denote by \( \overline{U}[i] \) the table containing the guest \( \tilde{i} \), by \( \overline{U}_+[i] \) the arc \( [\tilde{i}, u(i)] \) of \( \overline{U}[i] \), by \( \overline{U}_-[i] \) the arc \( [u^{-1}(i), \tilde{i}] \).

Fix \( g \in S_n \). Cutting tables at points (guests) \( \tilde{1}, \ldots, \tilde{n} \), we get a finite collection of segments \( \{ \overline{U}_+[i] \}_{i \in I_n} \) and a countable number of non-cutted tables. The collection \( \{ \overline{U}_-[j] \}_{j \in I_n} \) coincides with \( \{ \overline{U}_+[i] \}_{i \in I_n} \) up to a reordering.

To obtain \( ug \), for each \( i \in I_n \) we glue the segments \( \overline{U}_-[j] \) and \( \overline{U}_+[g(j)] \) identifying points \( \tilde{j} \in \overline{U}_-[j] \) and \( g(\tilde{j}) \in \overline{U}_+[g(j)] \) and putting the guest \( \tilde{j} \) to the point of gluing. In this way, we get a family of new tables enumerated by disjoint cycles of \( ug \in S_n \), and add the collection of non-cutted tables.

To obtain \( g \overline{u} \) we repeat the same steps, but put the guest \( \overline{g(\tilde{j})} \) to the point of gluing. Notice that \( g \overline{u} = g(\overline{u})g^{-1} \), so the restorants of \( u \) and \( g \overline{u} \) have the same tables and the same positions of all guests with numbers \( n \), the guests \( \tilde{1}, \ldots, \tilde{n} \) differ by a permutation \( g \).

Remark. These rules are simply a rephrasing of a description of multiplication of a permutation defined as map and a permutation defined as a product of cycles.

1.7. Radon–Nykodim derivatives. Thus, we get an action of the group \( S_\infty \times S_\infty \) on \( \mathcal{S}^\infty \) given by \( u \mapsto h_1^{-1}uh_2 \).

Clearly, on the level of \( S_n \) we have

\[
\mu_n^z(h_1^{-1}uh_2) = z^{[h_1^{-1}gh_2]^{-1}[g]} \mu_n^z(u).
\]
For \( u := \mathcal{S}^2 \) the operation of cutting and gluing described above involves only a finite number of tables, so the number

\[
\gamma(h_1, h_2; u) := \{ \text{number of tables of } h_1^{-1}uh_2 \} - \{ \text{number of tables of } u \} := [h^{-1}Y_N^\infty(g)h] - [Y_N^\infty(g)] \quad \text{for large } N
\]

is well-defined. The Radon-Nikodym derivative of the transformation \( u \mapsto h_1^{-1}uh_2 \) is \( z \gamma(h_1, h_2; u) \).

1.8. The inversion of virtual permutations. Clearly,

\[
\Upsilon_n^{-1}(g^{-1}) = g^{-1}.
\]

So the inversion \( g \mapsto g^{-1} \) defines a measure preserving map \( \mathcal{S}^2 \to \mathcal{S}^2 \), we denote it by \( u \mapsto u^{-1} \). This is simply a changing of orientations of all tables from clockwise to counterclockwise. Clearly

\[
(hug)^{-1} = g^{-1}u^{-1}h^{-1}.
\]

1.9. The bisymmetric group. The bisymmetric group \( \mathcal{S} \) is the subgroup in \( S^\infty \times S^\infty \) consisting of pairs \((g_1, g_2)\) such that \( g_1g_2^{-1} \in S^\infty \). Denote by \( K \subset \mathcal{S} \) the diagonal subgroup, i.e., the subgroup consisting of pairs \((g, g)\), where \( g \) ranges in \( S^\infty \). So, \( K \simeq S^\infty \). We define topology on \( \mathcal{S} \) assuming that \( K \) is an open subgroup equipped with the natural topology. The homogeneous space \( \mathcal{S}/K \) is a countable space with the discrete topology.

The bisymmetric group \( \mathcal{S} \) acts on \( \mathcal{S}^2 \). Namely, the subgroup \( K \) acts by permutations of guests, such permutations are measure-preserving maps. The subgroup \( S^\infty \times S^\infty \) acts as above.

Remark. 1) The group \( \mathcal{S} \) was introduced by Olshanski \[38\]. It is a type I group, a classification of its irreducible unitary representations is known, see Olshanski \[38\], Okounkov \[37\], see also \[21\]. The most important family of irreducible representations are \( K \)-spherical representations, i.e., representations having a fixed vector with respect to \( K \) (it is automatically unique). In fact such representations were classified by Thoma \[47\], 1964, see also \[50\] and \[38\].

2) Decompositions of quasiregular representations of \( \mathcal{S} \) in \( L^2(\mathcal{S}^2) \) are known, see Kerov, Olshanski, Vershik \[13\], Borodin, Olshanski \[3\], \[4\]. One of informal aims of the present work is a search of additional possibilities for the analysis on the space \( \mathcal{S}^2 \).

B. The bisymmetric group and its train.

1.10. Multiplication of double cosets. See Olshanski \[38\], see also \[31\]. For \( \alpha \in \mathbb{Z}_+ \) we denote by \( K_\alpha \) the subgroup in \( K \) consisting of \((g, g)\) such that \( g \in S^\infty \) fixes \( 1, \ldots, \alpha \in \mathbb{N} \). We set \( K_0 := K \). Denote by

\[
K_\alpha := K_\alpha \cap (S^\infty \times S^\infty)
\]

the corresponding subgroup of finitely supported permutations.

\[3\]See a formal definition below in Subsect. 1.16, see also, e.g., \[2\], 9.12.
For any $\alpha, \beta$ we consider the double coset space

$$S(\alpha, \beta) = K_\alpha \backslash \mathcal{S} / K_\beta.$$ 

It is easy to see that each double coset $K_\alpha \cdot g \cdot K_\beta$ has a finitary representative $g' \in S_\infty \times S_\infty$. Moreover,

$$(1.9) \quad K_\alpha \backslash \mathcal{S} / K_\beta \simeq K_\alpha \backslash (S_\infty \times S_\infty) / K_\beta,$$

all such quotient spaces are countable and the quotient topologies are discrete.

For any $\alpha, \beta, \gamma \in \mathbb{Z}_+$, there exists a natural multiplication

$$S(\alpha, \beta) \times S(\beta, \gamma) \rightarrow S(\alpha, \gamma)$$

defined in the following way. For each $\beta \in \mathbb{Z}_+$ we consider the following sequence

$$\theta^\beta[j] \in \mathbb{K}_\beta$$

defined by

$$\theta^\beta[j](k) = \begin{cases} 
\alpha & \text{if } k \leq \beta; \\
\alpha + j & \text{if } \beta < k \leq \beta + j; \\
\alpha - j & \text{if } \beta + j < k \leq \beta + 2j; \\
\alpha & \text{if } k > \beta + 2j.
\end{cases}$$

see Fig. 1.

Now we take double cosets $g = K_\alpha g K_\beta, h = K_\beta h K_\gamma$, without loss of generality we can think that $g, h \in S_\infty \times S_\infty$. Consider the sequence

$$K_\alpha \cdot g \theta^\beta[j] h \cdot K_\gamma \in K_\alpha \backslash \mathcal{S} / K_\beta.$$ 

It is easy to show that this sequence is eventually constant. We define a product $g \cdot h \in K_\alpha \backslash \mathcal{S} / K_\gamma$ as the limit (i.e., the stable value) of this sequence. It can be shown that the result does not depend on a choice of representatives $g, h \in S_\infty \times S_\infty$ and this operation is associative. So we get a category $\mathcal{S}$ (the train of $\mathcal{S}$) whose objects are nonnegative integers and morphisms $\beta \rightarrow \alpha$ are double cosets,

$$\text{Ob}(\mathcal{S}) := \mathbb{Z}_+, \quad \text{Mor}_\mathcal{S}(\beta, \alpha) := S(\alpha, \beta) = K_\alpha \backslash \mathcal{S} / K_\beta.$$ 

We also define an involution $g \mapsto g^*$ on the category $\mathcal{S}$. Namely, a map $g \mapsto g^{-1}$ determines bijections $K_\alpha \backslash \mathcal{S} / K_\beta \rightarrow K_\beta \backslash \mathcal{S} / K_\alpha$. Clearly,

$$(g \circ h)^* = h^* \circ g^*.$$ 

1.11. Chips. Recall Olshanski’s description [38] of the product of double cosets. We represent an element of the group $S_\infty \times S_\infty$ as a diagram of the type drawn on Fig. 2a. On Fig. 2b, 2c we define combinatorial data corresponding to a double coset. We get a diagram (a chip) of the form shown on the Fig. 2c.

We use the term proposed by Kerov [11] for elements of the Brauer semigroups. Recall that the Schur–Weyl duality between $GL(N)$ and $S_n$ has a counterpart [5], 1936, for orthogonal groups $O(N)$ and symplectic groups $Sp(2N)$, dual objects are certain semigroups of chips (with horizontal arcs but without crosses). Brauer’s approach also gives a similar statement for tensors over $GL(N)$.
a) An element of $S$. The symbols $l$ and $r$ in subscripts are abbreviations of 'left' and 'right'. Symbols $+$ and $-$ in superscripts correspond to the top and the bottom of the diagram. The left and the right parts of the diagram are symmetric one to another except a finite number of arcs.

b) On Figure $\beta = 3$, $\alpha = 4$. We add 'horizontal' arcs with crosses.

c) We consider compound arcs on the previous diagram up to isotopies with fixed ends. Also, we get a countable family of circles with two crosses and forget them.

**Figure 2.** The construction of a chip from an element of $S$.

It is convenient to think that each arc has a 'length', which is defined as the half of the number of crosses.

For figures in our printed text it is more convenient to draw crosses.

We get diagrams of the following type. On the top we have black circles labeled by $\beta_l^+$, $\ldots$, $1_l^+$ to the left of the dashed line, and $1_r^+$, $\ldots$, $\beta_r^+$ to the right of the

with covariant and contravariant components, in this case diagrams have a separating dashed line (it separates 'co-' and 'contra-') and horizontal arcs corresponding to convolutions. On the other hand, according Wasserman [57] and Olshanski [58], representations of $S$ can be realized in certain tensors of infinite order, their decomposition is controlled by certain 'dual' compact groups.
dashed line. On the bottom we have black circles labeled by $\alpha_l^-, \ldots, 1_l^-$ on the left and $1_r^-, \ldots, \alpha_r^-$ on the right. Each circle is an end of an arc. There are 3 following types of arcs:

1) arcs $i^+ \downarrow i^- [\varphi]$ (resp., $i^+ \downarrow i^- [\varphi]$) of integer length $\varphi \geq 0$ from the top to the bottom located in the right (resp., left) hand side of the diagram;

2) arcs $i^+ \circ j^+[\psi]$ (resp., $i^- \circ j^- [\psi]$) of length $\psi \in (1/2 + \mathbb{Z}_+)$ from left to right on the top (resp., bottom) of the diagram;

3) circles $\odot [k]$ of integer length $k > 0$.

We remove all cycles of length 1 (any diagram obtained by our procedure contains an infinite number of cycles $\odot [1]$, this collection contains no information). So in all cases we get finite objects.

It is easy to show that such diagrams are in one-to-one correspondence with elements of $\mathbb{K}_\alpha \backslash S/\mathbb{K}_\beta$. Multiplication of diagrams is a gluing, see Fig. 3. The involution corresponds to the replacement of the top and of the bottom.

**Remark.** An element of the semigroup $S(0, 0)$ is a collection of cycles $\odot [k_j]$. The product is a union of such collections. In particular, this semigroup is Abelian.

**1.12. The multiplicativity theorem.** Let $\rho$ be a unitary representation of the group $S$ in a Hilbert space $H$. Denote by $H_\alpha$ the subspace of all $\mathbb{K}_\alpha$-fixed vectors, by $P_\alpha$ the operator of orthogonal projection to $H_\alpha$. For $g \in S$ we define the operator

$$\tilde{\rho}_{\alpha, \beta}(g) : H_\beta \to H_\alpha$$

by

$$\tilde{\rho}_{\alpha, \beta}(g) := P_\alpha \rho(g) \big|_{H_\beta}.$$

It is easy to see that $\tilde{\rho}_{\alpha, \beta}(g)$ depends only on the double coset $g = \mathbb{K}_\alpha g \mathbb{K}_\beta$. The following multiplicativity theorem holds (see [38], [31]).

**Theorem 1.1.** For any $\alpha, \beta, \gamma \in \mathbb{Z}_+$ and $g \in \mathbb{K}_\alpha \backslash S/\mathbb{K}_\beta$, $h \in \mathbb{K}_\beta \backslash S/\mathbb{K}_\gamma$ we have

$$(1.10) \quad \tilde{\rho}_{\alpha, \beta}(g) \tilde{\rho}_{\beta, \gamma}(h) = \tilde{\rho}_{\alpha, \gamma}(g \circ h).$$

We assume that a Hilbert space is separable by definition.
So, for any unitary representation of the group $\Sigma$ we get a functor from the category $\Sigma$ to the category of Hilbert spaces and bounded operators. This reduces investigation of representations of the group $\Sigma$ to representations of the category $\Sigma$.

In particular, this theorem can be applied to quasiregular representations of the bisymmetric group $\Sigma$ in $L^2(\Sigma^2)$. There arises a question: \textit{is it possible to describe operators corresponding to chips explicitly?} Below we reformulate this question in terms of polymorphisms.

\section*{1.13. Remark. Infinite chips and weak closures of the group $\Sigma$ in unitary representations.} Denote by $\mathcal{C}(H)$ the semigroup of all operators in a Hilbert space $H$ with norm $\leq 1$ (we call such operators \textit{contractive}). It easy to see that $\mathcal{C}(H)$ is compact and metrizable with respect to the weak operator topology, the multiplication in $\mathcal{C}(H)$ is separately continuous (see, e.g., \cite{24}, Sect. I.1).

Let $G$ be a group, $\rho$ be its unitary representation in a Hilbert space $H$. Consider the subset $\rho(G)$ in the space of all operators in $H$ and consider its closure $\overline{\rho(G)}$ in the semigroup of all contractive operators in the weak operator topology. Clearly, we get a compact separately continuous semigroup, see \cite{40}, \cite{24}, Sect. I.1.

A description of such semigroup can be a nontrivial problem even for $\mathbb{Z}$, i.e., for closure of the set of powers of a given unitary operator $U$. For a comeagre set of unitary operators this semigroup is the unit ball in $L^\infty(S^1,\mu)$, where $\mu$ is the spectral measure of $U$, see \cite{20}.

For reasonable representations of Lie groups (over real and over local fields) the question usually leads to one-point compactifications, see \cite{9}. Other types of locally compact groups and discrete non-Abelian groups are not well-understood. For infinite-dimensional groups this question leads to handable and unexpected algebraic structures as it was observed by Olshanski \cite{40}. Let us describe such compactification for the group $\Sigma$.

The sequence $\theta_{\beta}[j]$ defined above converges to the projector $P_{\beta}$ in the weak operator topology, see, e.g., \cite{24}, Theorem VIII.1.4. Therefore the following operators
\[
\hat{\rho}_{\alpha,\beta}(g) = P_{\alpha} \rho(g) P_{\beta} : H \to H
\]
are contained in $\overline{\rho(\Sigma)}$. It is easy to see that these operators depend only on double coset $g$ containing $g$ and have the following block structure
\[
\hat{\rho}(g) = \begin{pmatrix}
\tilde{\rho}(g) & 0 \\
0 & 0
\end{pmatrix} : \quad H_{\beta} \oplus H_{\beta}^\perp \to H_{\alpha} \oplus H_{\alpha}^\perp.
\]

Clearly, we have
\[
\hat{\rho}_{\alpha,\beta}(g) \hat{\rho}_{\beta,\gamma}(h) = \hat{\rho}_{\alpha,\gamma}(g \circ h),
\]
this identity is a rephrasings of (1.10). So chips act in the space $H$ itself.

More generally, consider diagrams as on Fig. 2.c, where circles on the top and the bottom (from the left and from the right) are enumerated by $\mathbb{N}$. We require the following conditions of finiteness of such diagrams:

\begin{itemize}
\item[a)] the number of cycles $\bigcirc[k]$ is finite (recall that $k \geq 2$);
\item[b)] the number of ’vertical’ arcs $i^+_l \searrow j^-_m \uparrow p^+_r \downarrow q^-_m$ of lengths $m > 0$ is finite;
\item[c)] the number of horizontal arcs $i^+_l \sim j^+_m[1/2 + k], i^-_l \sim j^-_m[1/2 + k]$ of lengths $1/2 + k > 1/2$ is finite.
\end{itemize}
d) a diagram is symmetric with respect to the dashed line up to a finite family of arcs.

Denote the semigroup of such chips by $S(\infty, \infty)$. The group of its invertible elements is $S$.

**Remark.** The embedding $S(n, n) \to S(\infty, \infty)$ is given by adding arcs $i^+_l \sim i^+_r \sim [1/2]$, $i^-_l \sim i^-_r \sim [1/2]$ for all $i > n$.

**Remark.** The semigroup $S(\infty, \infty)$ has a center. It consists of chips whose arcs have the form $j_l \downarrow j_r$, $j_l \downarrow j_r$ for all $j \in \mathbb{N}$ and $\bigcup [k_j]$, where $k_j \geq 2$. The center is isomorphic to $S(0, 0)$.

For a unitary representation $\rho$ of $S$ denote by $\Pi_+$ the operator of orthogonal projection to the subspace of vectors fixed by the whole group $S$, by $\Pi_-$ the operator of projection to the subspace of vectors $v$ satisfying

$$\rho(h_1, h_2)v = \text{sgn}(h_1h_2^{-1})v, \quad \text{where } (h_1, h_2) \in S.$$ 

**Proposition 1.2.** a) Any unitary representation $\rho$ of $S$ extends to a representation of the semigroup $S(\infty, \infty)$.

b) The closure $\overline{\rho(S)}$ consists of the image of $S(\infty, \infty)$ and operators $\Pi_+ + \Pi_-$ and $\Pi_+ - \Pi_-$. 

**Remarks.** a) Representations of the semigroup $S(\infty, \infty)$ can be constructed in the following way. We split $\mathbb{N}$ into a disjoint union of two countable sets $\mathbb{N} = L \sqcup M$. Denote by $K(\infty)$ the subgroup of $K$ consisting of elements fixing all elements of $L$. Then $S(\infty, \infty) \simeq K(\infty) \setminus S/K(\infty)$ and the multiplicativity theorem remains valid with the same proof. The group of invertible elements of $S(\infty, \infty)$ is isomorphic to $S(\infty, \infty)$. It can be shown that the representation of $S$ in the subspace $K(\infty)$-fixed vectors is equivalent to the initial representation of $S$.

b) The second statement of the theorem is a relatively easy corollary of results of Olshanski [38] and Okounkov [37], we omit its proof.

c) We can define sets $S(\infty, m)$ and $S(n, \infty)$ in the same way, and so get a completed category $\overline{S}$,

$$\text{Ob}(\overline{S}) := \mathbb{Z}_+ \sqcup \infty.$$ 

Any unitary representation of $S$ admits a unique extension to a representation of this category.

**C. Polymorphisms**

It is natural to reformulate the question about operators in $S^2$ corresponding to chips in terms of polymorphisms of the space $S^2$.

**1.14. Bistochastic kernels or measure preserving polymorphisms.** Recall that a Lebesgue space is a measure space, which is equivalent to a union of an interval of a line and a finite, countable or empty set of (atomic) points with nonzero measures (see, [44], [2], §9.4). A measure $\mu$ is probabilistic if the measure of the whole space is 1. A measure is continuous if it has no atomic points. Clearly, the spaces $S^2$ of virtual permutations are Lebesgue spaces with continuous measures.
Let \( (M, \mu) \) be a Lebesgue space with a continuous measure \( \mu \). Denote by \( \text{Ams}(M) \) the group of bijective a.s. measure preserving transformations of \( M \). The group \( \text{Ams}(M) \) acts in \( L^2(M, \mu) \) by unitary operators
\[
T(g)f(m) = f(mg).
\]
(1.12)

Let \( (M, \mu), (N, \nu) \) be probabilistic Lebesgue measure spaces. A bistochastic kernel or a measure preserving polymorphism \(^6\) \( q : M \to N \) is a measure \( q \) on \( M \times N \) such that the pushforward \(^6\) of \( q \) under the projection \( M \times N \to M \) coincides with \( \mu \) and the pushforward under the projection \( M \times N \to N \) coincides with \( \nu \).

\(^6\)The notion rises to E. Hopf \(^8\), see also \(^\ref{35, 17, 49}\), such objects are widely used in ergodic theory, see e.g., \(^\ref{15, 10}\). We use the term 'polymorphism' proposed in \(^\ref{49}\).

Example. Let \( g \in \text{Ams}(M) \). Consider the map \( M \to M \times M \) given by \( m \mapsto (m, mg) \). Then the pushforward of \( M \) under this map is a measure preserving polymorphism. In particular, for the identical transformation \( g = 1 \) we get an identical polymorphism.\( \Box \)

A measure preserving polymorphism \( q : M \to N \) can be regarded as a 'spreading map' sending points \( m \in M \) to measures. Namely, consider a partition
\[
M \times N = \coprod_{m \in M} m \times N
\]
and conditional measures \(^8\) \( \kappa_m(n) \) on \( N \) defined for almost all \( m \) from the condition
\[
q(S) = \int_M \kappa_m((m \times N) \cap S) \, d\mu_m(n) \quad \text{for any measurable } S \subset M \times N.
\]
Then we can regard \( q \) as a 'spreading map' sending points \( m \in M \) to measures \( \kappa_m(n) \) on \( N \).

A product of measure preserving polymorphisms corresponds to a double spreading. Namely, let \( q \) be a polymorphism \( M \to N \) and \( p \) be a polymorphism \( N \to K \). Let \( \kappa_m(n) \) be the system of conditional measures corresponding to \( q \), and \( \pi_n(k) \) be the system of conditional measures corresponding to \( p \). Then the system \( \rho_m(k) \) of conditional measures corresponding to the product \( \tau = p \circ q \) is defined by
\[
\rho_m(k) = \int_N \pi_n(k) \, d\kappa_m(n).
\]

The map \( g \mapsto T(g) \), see (1.12), can be extended from \( \text{Ams}(M) \) to polymorphisms \( q : M \to N \). Namely, the operator \( T(q) : L^2(N) \to L^2(M) \) is given by
\[
T(q)f(m) = \int_N f(n) \, d\mu_m(n).
\]
Then for \( q : M \to N, p : N \to K \) we have
\[
T(p \circ q) = T(p) \, T(q).
\]

We say that a sequence \( q^{(j)} : M \to N \) converges to \( q \) if for any measurable \( A \subset M, B \subset N \) the sequence \( q^{(j)}(A \times B) \) converges to \( q(A \times B) \). This convergence is equivalent to the weak operator convergence \( T(q^{(j)}) \to T(q) \).

\(^7\)Let \( (M, \mu) \) be a measure space, let \( S \) be a set, \( \psi : M \to S \) a map. We say that a subset \( C \subset S \) is measurable if \( \psi^{-1}(C) \) is measurable, the pushforward \( \sigma \) of the measure \( \mu \) is defined by \( \sigma(C) = \mu(\psi^{-1}(C)) \).

\(^8\)See \(^\ref{44, 2}\), Chapter 10.
The product of polymorphisms is separately continuous; the group $A_{ms}(M)$ is dense in the semigroup of polymorphisms $M \to M$.

We also have the involution $q \mapsto q^*$ on polymorphisms, it sends a measure $q$ on $M \times N$ to the same measure on $N \times M$. We have $T(q^*) = T(q)^*$.

1.15. Quotient spaces and the corresponding measure preserving polymorphisms. Let $M$ be a Lebesgue measure space, let $M = \bigsqcup_{h \in H} S_h$ be a measurable partition, let $\sigma_h(m)$ be the conditional measures on sets $S_h$ (see \[44\]). Consider the quotient space $H$ with the induced measure $\eta(h)$. Then we have the following polymorphisms (see \[30\], Subsect. 3.10):

a) We have the projection $\pi : M \to H$ sending $m \in M$ to the element $S_h$ containing $m$. We consider the map $M \to M \times H$ given by $m \mapsto (m, \pi(m))$, we denote by $x$ the image of $\mu$ under this map. Then $x$ is a measure-preserving polymorphism $M \to H$. The operator $T(x)f(m) = f(\pi(m))$ is a canonical isometric embedding $L^2(H) \to L^2(M)$. The image consists of $L^2$-functions that are constant on sets $S_h$.

b) We also have a polymorphism $x^*$, the corresponding operator $T(x^*) : L^2(M) \to L^2(H)$ is the operator of conditional expectation $T(x^*)f(h) = \mathcal{E}f(h) := \int_{S_h} f(m) d\sigma_h(m)$.

c) Notice that $x \otimes x^*$ is the identical polymorphism $H \to H$.

d) For the polymorphism $x^* \otimes x : L^2(M) \to L^2(M)$ the corresponding operator is the orthogonal projector to the subspace $L^2(H)$.

1.16. General polymorphisms. Let $M$ be a Lebesgue space with a continuous measure. Recall that a map $g : M \to M$ leaves a measure $\mu$ quasiinvariant if it is bijective almost sure and there is a function $g' : M$ (the Radon–Nikodym derivative, see, e.g., \[2\], 9.12) such that for any measurable $A \subset M$ we have

$$\mu(Ag) = \int_A g'(m) \, d\mu(m),$$

the $g'(m)$ is a natural extension of Jacobians in classical analysis. Denote by $G_{ms}(M)$ the group of transformations of $M$ leaving the measure $\mu$ quasiinvariant.

Denote by $\Pi \subset \mathbb{C}$ the strip

$$0 \leq \text{Im} \lambda \leq 1.$$

For $r + is \in \Pi$ we consider the following transformations of the space of functions on $M$:

$$T_{r+is}(g) f(m) = f(mg) g'(m)^r e^{i s},$$

Then $T_{r+is}(g)$ is an isometric operator $L^{1/r}(M) \to L^{1/r}(M)$.

Remark. The main topic of our interest is the case $r = 1/2$, when we get a unitary operators $L^2 \to L^2$. However, operators in spaces $L^p$ are used in our argumentation in Theorem \[2\]

9This is also equivalent to the condition: $g$ is bijective a.s. and both $g$ and $g^{-1}$ sent sets of zero measure to sets of zero measure.
Denote by $\mathbb{R}^\circ$ the multiplicative group of positive numbers, denote by $t$ the coordinate on it. Let $(M, \mu)$, $(N, \nu)$ be probabilistic Lebesgue spaces. We say that a polymorphism $p : M \rightarrow N$ is a measure on $M \times N \times \mathbb{R}^\circ$ such that

1. The pushforward of $p$ under the projection $M \times N \times \mathbb{R}^\circ \rightarrow M$ is the measure $\mu$.
2. The pushforward of the measure $t \cdot p$ to $N$ coincides with the measure $\nu$.

Denote by $\text{Pol}(M, N)$ the space of all polymorphisms $M \rightarrow N$. See [23], [30].

**Example.** Measure preserving polymorphisms from Subsect. [1.14] are polymorphisms, they are supported by the set $M \times N \times \{1\} \subset M \times N \times \mathbb{R}^\circ$.

**Example.** Let $q \in \text{Gms}(M, \mu)$. Consider the map $M \rightarrow M \times M \times \mathbb{R}^\circ$ defined by

$$m \mapsto (m, \varrho(m), \varrho'(m)).$$

Then the pushforward of the measure $\mu$ under this map is a polymorphism $M \rightarrow M$. ☐

**Example.** Denote by $\mathcal{M}^\circ(\mathbb{R}^\circ)$ the set of all positive finite Borel measures $\sigma$ on $\mathbb{R}^\circ$ such that $t \cdot \sigma$ also is finite. Clearly, any measurable function $(m, \eta) \mapsto s_{m,n}$ from $M \times N$ to $\mathcal{M}(\sigma)$ determines a certain measure $\varrho$ on $M \times N \times \mathbb{R}^\circ$. Namely, for measurable sets $A \subset M$, $B \subset N$, $C \subset \mathbb{R}^\circ$ we set

$$\varrho(A \times B \times C) = \int_{A \times B} s_{m,n}(C) d\mu(m) d\nu(n).$$

We also must assume that

$$\forall m : \int_N s_{m,n}(\mathbb{R}^\circ) d\nu(n) = 1, \quad \forall n : \int_M \int_{\mathbb{R}^\circ} t \cdot s_{m,n}(t) d\mu(m) = 1,$$

under these conditions $\varrho$ is contained in $\text{Pol}(M, N)$. We say that a polymorphism $\varrho$ obtained in such way is continuous. Notice that polymorphisms from the previous example are not continuous. ☐

1.17. **Convergence of polymorphisms.** Let $p_j \in \text{Pol}(M, N)$. Consider measurable subsets $A \subset M$, $B \subset N$. Restrict $p$ to $A \times B \times \mathbb{R}^\circ$ and take its pushforward under the map $A \times B \times \mathbb{R}^\circ \rightarrow \mathbb{R}^\circ$. Denote the resulting measure on $\mathbb{R}^\circ$ by $p[A \times B]$. We say that a sequence $p_j \in \text{Pol}(M, N)$ converges to $p$, if for any measurable subsets $A \subset M$, $B \subset N$ we have weak convergences of measures

$$p_j[A \times B] \rightarrow p[A \times B], \quad t \cdot p_j[A \times B] \rightarrow t \cdot p[A \times B].$$

It is easy to show (see [30], Theorem 5.3) that the group $\text{Gms}(M)$ is dense in $\text{Pol}(M, M)$. Continuous polymorphisms $M \rightarrow N$ are dense in $\text{Pol}(M, N)$.

1.18. **Products of polymorphisms.** There is an obvious way to multiply continuous polymorphisms. Let $u \in \text{Pol}(M, N)$, $v \in \text{Pol}(N, K)$ be determined by functions $(m, n) \mapsto u_{m,n}$, $(n, k) \mapsto v_{n,k}$. Then their product

$$w = v \circ u \in \text{Pol}(M, K)$$

is determined by the function $M \times K \rightarrow \mathcal{M}(\sigma)$ given by

$$w_{m,k} = \int_N u_{m,n} \ast v_{n,k} d\nu(n),$$

where $\ast$ denotes the convolution on $\mathbb{R}^\circ$.

The multiplication $\circ$ extends to a separately continuous map

$$\text{Pol}(M, N) \times \text{Pol}(N, K) \rightarrow \text{Pol}(M, K),$$
and moreover this operation is associative (see [30], Theorem 5.5, Theorem 5.9). So we get a category Pol whose objects are Lebesgue probabilistic measure spaces and morphisms are polymorphisms.

For different ways to define the multiplication of polymorphisms, see [30], however it seems that definitions by continuity (by continuous extension from continuous polymorphisms or from the group Gms(\(M\))) are more convenient.

1.19. Polymorphisms as spreading maps. Informally, polymorphisms are 'maps' \(M \to N\), which spread points of \(M\) along \(N\) and the Radon–Nikodym derivative also is spread. Namely, let \(q \in \text{Pol}(M, N)\). Consider a partition \(M \times N = \bigsqcup_{m \in M} m \times N\) and conditional measures \(\kappa_m(\cdot, t)\) on fibers of the partition. We say that measures \(\kappa_m(\cdot, t)\) are spreaded images of points \(m\). Below in Section 3 we describe polymorphisms of \(S_z\) in terms of such conditional measures.

1.20. Involution. Define an involution in the category Pol. Consider the map \((m, n, t) \mapsto (n, m, t^{-1})\). For \(p \in \text{Pol}(M, N)\) we consider its pushforward \(p'\) under this map and define \(p^* \in \text{Pol}(N, M)\) as the measure \(t \cdot p'\). Then

\[(q \circ p)^* = p^* \circ q^*\]

1.21. Mellin–Markov transforms of polymorphisms. For each \(r + is \in \Pi\) the map \(g \mapsto T_{r+is}(g)\) extends to the category of polymorphisms.

Let \(p \in \text{Pol}(M, N)\). The Mellin–Markov transform of \(q : M \to N\) is a function defined in the strip \(r + is \in \Pi\) taking values in operators \(T_{r+is}(q) : L^{1/r}(N) \to L^{1/r}(M)\) and determined from the equality

\[
\int_M \varphi(m) T_{r+is}(q) \psi(m) d\mu(m) = \iint_{M \times N \times \mathbb{R}^o} \varphi(m) \psi(n) t^{r+is} d\nu(m, n, t)
\]

for any \(\psi \in L^{1/r}(N), \varphi \in L^{1/(1-r)}(M)\).

The operators \(T_{r+is}(q) : L^{1/r}(N) \to L^{1/r}(M)\) are bounded and for any \(p \in \text{Pol}(M, N), q \in \text{Pol}(M, K)\) we have (see [30], Theorem 6.14)

\[T_{r+is}(q) T_{r+is}(p) = T_{r+is}(q \circ p)\]

It is more transparent to determine \(q\) in terms of conditional measures \(\kappa_m(n, t)\) as in Subsect. 1.19. Then

\[T_{r+is}(q)f(m) = \int_{N \times \mathbb{R}^o} f(n) t^{r+is} d\kappa_m(n, t)\]

1.22. Closures of actions of groups on measure spaces. For a group \(G\) acting on a measure space \(M\) we get a question about the closure of \(G\) in the semigroup \(\text{Pol}(M, M)\) of polymorphisms. This question for \(G = \mathbb{Z}\) acting by measure preserving transformations was a subject of numerous works, see e.g., [15], [10], [46], [18], a related topic is intertwiners (joinings) of such actions in class of measure preserving polymorphisms.

For infinite-dimensional groups the first problem of this kind was solved by Nelson [22], who examined the action of the infinite-dimensional orthogonal group on the infinite-dimensional space with a Gaussian measure (the resulting semigroup is
the semigroup of all contractive operators). For the natural group of symmetries of Poisson measure and for the action of the restricted group GL on a space with a Gaussian measure the problems were discussed in [25], [28]. They are unexpectedly non-trivial and lead to formulas, which are at least unusual. One simple case (measures on the space of infinite Hermitian matrices invariant with respect to unitary groups) was examined in [33].

In this paper we consider the action of the bisymmetric group $\mathfrak{S}$ on spaces $\mathfrak{S}^z$ of virtual permutations.

1.23. The purpose of the paper. First, we show that the categories $\mathfrak{S}$ and $\mathfrak{S}^z$ of chips act on the space of virtual permutations by polymorphisms. The category of chips is a representative of a wide zoo of train constructions, in Section 2 we prove a general statement in the following spirit: certain actions of infinite-dimensional group on measure spaces generate actions of their trains by polymorphisms.

In Section 3 we write a formula for the action of the semigroup $\mathfrak{S}(0,0)$ on the space of $\mathfrak{S}^z$ of restaurants. In fact, for a given collection $\{\bigcirc[k_j]\}$ of cycles we write spreaded images of points as sums over a certain sets of dessins d’enfant (or equivalently of checker triangulated surfaces), summands are multiple convolutions of Dirichlet distributions (Theorem 3.5).

In Section 4 we get some further statements in this spirit. In Theorem 4.1 we get formulas for the action of the center $\mathfrak{S}(0,0)$ of the semigroup $\mathfrak{S}(\infty, \infty)$ on $\mathfrak{S}^z$.

2. Trains of $(G,K)$-pairs and polymorphisms.

Train constructions (multiplication of double cosets) and the multiplicativity theorem 1.1 are relatively usual phenomena for infinite-dimensional groups, including infinite symmetric groups [38], [29], [31], classical groups over reals [?] [24], [26] (and also over finite fields [40], [35] and $p$-adic fields [34], [32]) groups of transformations of measure spaces [23], [24] and some exotic cases.

For infinite-dimensional group a lot of quasiinvariant actions (i.e., embeddings to the group $\text{Gms}(\cdot)$) is known. In this section, we wish to show that under some conditions such actions automatically generate actions of corresponding train categories by polymorphisms. We prove Theorem 2.1 which apparently is sufficient to justify this claim for known zoo of train constructions. Also, the proof is simple, short, and can be easily repeated independently in any explicit case.

We do not pretend to any wider purpose.

2.1. Operators corresponding to double cosets. Let $G$ be a separable topological group, let $K$ be a closed subgroup. Let $\rho$ be a unitary representation of $G$ in a separable Hilbert space $H$. Denote by $H(K)$ the subspace consisting of vectors fixed by elements of $K$, by $P(K)$ the orthogonal projection to $H(K)$. For two subgroups $K, L$ and $g \in G$ consider the operator

$$\tilde{\rho}_{(K,L)}(g) := P(K)\rho(g)|_{H(L)} : H(L) \to H(K).$$

It is easy to verify that the operator-valued function $\tilde{\rho}_{(K,L)}(g)$ is constant on double cosets $KgL$, so it is a function on $K\backslash G/L$. We define a more coarse equivalence
relation on $G$,
\[(2.1) \quad g \sim g' \quad \text{if} \quad \tilde{\rho}(K,L)(g) = \tilde{\rho}(K,L)(g') \quad \text{for all} \ \rho.\]

Denote an equivalence class containing $g$ by $[KgL]$, denote the space of equivalence classes by $[K/G/L]$.

2.2. $(G, K)$-pairs. Now let $G$ be a separable topological group, let $K$ be a subgroup. Let $\{K_\alpha\}_{\alpha \in \mathcal{A}}$ be a family of subgroups in $K$. Let the set $\mathcal{A}$ be partially ordered, and for any $\alpha, \alpha' \in \mathcal{A}$ there is $\nu \in \mathcal{A}$ such that $\nu \succ \alpha, \nu \succ \alpha'$. Assume also that $\alpha \succ \alpha'$ implies $K_{\alpha'} \supset K_{\alpha}$.

A $(G, K)$-pair is a group and a family of subgroups $\{K_\alpha\}_{\alpha \in \mathcal{A}}$ as above satisfying the following conditions:

A. For each $\alpha, \beta, \gamma \in \mathcal{A}$, for each $g \in [K_\alpha \backslash G/K_\beta], h \in [K_\beta \backslash G/K_\gamma]$ there is $p \in [K_\alpha \backslash G/K_\gamma]$ such that for each unitary representation $\rho$ of $G$ in a Hilbert space $H$ we have
\[(2.2) \quad \tilde{\rho}(K_\alpha, K_\beta)(g) \tilde{\rho}(K_\beta, K_\gamma)(h) = \tilde{\rho}(K_\alpha, K_\gamma)(p).\]

B. For each unitary representation, the subspace $\bigcup_{\alpha \in \mathcal{A}} H(K_\alpha)$ is dense in $H$.

So we get an operation
\[[K_\alpha \backslash G/K_\beta] \times [K_\beta \backslash G/K_\gamma] \to [K_\alpha \backslash G/K_\gamma].\]

Denote it by $p = gh$ or $p = g \circ h$. Since the product of linear operators is associative, this operation is also associative. So we get a category - the train $\mathcal{I} = \mathcal{I}(G, K)$ of $(G, K)$. Its objects are enumerated by the set $\mathcal{A}$, morphisms $\beta \to \alpha$ are reduced double cosets $[K_\alpha \backslash G/K_\beta]$.

This category is equipped with the *involution* $g \mapsto g^*$, which is the map
\[[K_\alpha \backslash G/K_\beta] \to [K_\beta \backslash G/K_\alpha]\]

induced by the inversion in $G, g \mapsto g^{-1}$.

To simplify notations, denote
\[H[\alpha] := H(K_\alpha), \quad P[\alpha] := P(K_\alpha), \quad \tilde{\rho}_{\alpha, \beta} := \tilde{\rho}(K_\alpha, K_\beta).\]

2.3. Actions of $(G, K)$-pairs by polymorphisms. Consider a $(G, K)$-pair. Let $G$ act on a Lebesgue probabilistic measure space $(M, \mu)$ by transformations leaving the measure quasiinvariant, let $K$ act by measure preserving transformations. In such a case we say that we have an *action of a $(G, K)$-pair on a measure space*.

For each $K_\alpha$ consider the $\sigma$-algebra $\Sigma[\alpha]$ of all $K_\alpha$-invariant sets. Namely, a set $B \subset M$ is contained in $\Sigma[\alpha]$ if and for any $g \in G$ the symmetric sum $B \triangle Bg$ has measure 0; equivalently, the indicator function of $B$ is a $K_\alpha$-fixed element of $L^2(M, \mu)$. This sigma-algebra $\Sigma[\alpha]$ determines a measurable partition.\(^\text{[11]}\)

---

\(^{11}\) Recall that a partition $M = \sqcup_{h \in H} S_h$ of $M$ is measurable if there is a countable family of measurable subsets $A_j$, which are unions of elements of partitions, such that for each distinct elements $S_h, S_{h'}$ there is $A_j$ such that $S_h \subset A, S_{h'} \subset M \setminus A$ or vise versa. Consider the set $\mathcal{I}[\alpha] \subset L^2(M)$ of all indicator functions $1_C$ of elements of $\Sigma[\alpha]$. Choose a dense countable subset \{1_{C_j}\} in $\mathcal{I}[\alpha]$. The sigma-algebra generated by $C_j$ and sets of zero measure coincides with $\Sigma[\alpha]$. We say that $m, m' \in M$ are equivalent if for each $C_j$ we have $m, m' \in C_j$ or $m, m' \notin C_j$. By the definition, we get a measurable partition. By the Rohlin theorem the quotient is a Lebesgue measure space.
of $M$, the quotient space is a Lebesgue measure space, denote them $(M[\alpha], \mu[\alpha])$. We have an identification of $L^2(M[\alpha])$ with the space of $K_{\alpha}$-fixed vectors in $L^2(M)$. By Subset. 1.15 we get a canonical measure preserving polymorphism $\varphi[\alpha]: M \to M[\alpha]$. 

Operators $T_{r+is}(\varphi[\alpha]) : L^{1/r}(M[\alpha]) \to L^{1/r}(M)$ are operators 

$$\mathcal{E}'_{1/r}[\alpha] \varphi(m) = \varphi(m)$$ 

where $m$ is element of the partition containing $m$, these operators do not depend on $s$ and in fact do not depend on $r$.

An operator $T_{r+is}(\varphi[\alpha])$ is the operator of conditional expectation $\mathcal{E}_{1/r}[\alpha] : L^{1/r}(M) \to L^{1/r}(M[\alpha])$. For $r = 1/2$, $T_{1/2+is}(\varphi[\alpha]^*)$ is the orthogonal projection to $L^2(M[\alpha])$.

**Theorem 2.1.** Let a $(G, K)$-pair act on a measure space $(M, \mu)$. Assign for each $\alpha \in A$ the measure space $(M[\alpha], \mu[\alpha])$. For each $\alpha, \beta \in A$ and $g \in G$ define the polymorphism

$$(2.3) \quad \Xi_{\alpha, \beta}(g) := \varphi[\alpha]^* g \varphi[\beta] \in \text{Pol}(M[\alpha], M[\beta]).$$

Then the maps $g \mapsto \Xi_{\alpha, \beta}(g)$ determine a functor from the train $T(G, K)$ to the category of polymorphisms.

**Proof.** We must verify the identity

$$(2.4) \quad \Xi_{\alpha, \beta}([K_{\alpha}gK_{\beta}]) \otimes \Xi_{\beta, \gamma}([K_{\beta}hK_{\gamma}]) = \Xi_{\alpha, \gamma}([K_{\alpha}gK_{\beta} \circ [K_{\beta}hK_{\gamma}])$$

By (2.3), Theorem 6.12, two polymorphisms are equal if and only if their Mellin–Markov transforms are equal. A Mellin–Markov transform is a holomorphic operator-valued function in the strip $\Pi$, see (3), Lemma 6.10, so it is uniquely determined by its values on the line $1/2+is$. Applying functors $T_{1/2+is}$ to both sides of conjectural equality (2.4) we come to the equivalent conjectural equality

$$\left(\mathcal{E}[\alpha]T_{1/2+is}(g)\mathcal{E}'[\beta]\right) \left(\mathcal{E}[\beta]T_{1/2+is}(h)\mathcal{E}'[\gamma]\right) = \mathcal{E}[\alpha]T_{1/2+is}(g \circ h)\mathcal{E}'[\gamma],$$

these operators act $L^2(M[\gamma]) \to L^2(M[\alpha])$. Since $\mathcal{E}'[\beta] \mathcal{E}[\beta] = P[\beta]$, we can write the left hand side as

$$P[\alpha] T_{1/2+is}(g) P[\beta] T_{1/2+is}(h)\Big|_{L^2(M[\gamma])}.$$ 

By the definition of trains, this equals to

$$P[\alpha] T_{1/2+is}(g \circ h)\Big|_{L^2(M[\gamma])},$$

i.e., to the right-hand side. \( \square \)

**2.4. Application to the bysymmetric group.** 1) We take $G = \mathbb{S}$, $K[\alpha] = \mathbb{K}_\alpha$, $M = \mathbb{G}^\times$, and get an action of the category $\mathbb{S}$.

2) In our case $[K_{\alpha} \mathbb{S}/K_{\beta}] = \mathbb{K}_\alpha \mathbb{S}/\mathbb{K}_\beta$, this is clear from explicit constructions of representations of $\mathbb{S}$, see (3). However, in considerations above sets $[K_{\alpha} \mathbb{G}/K_{\beta}]$ are used only for the establishing of the associativity, in our case the associativity is obvious. So we can simply repeat the proof of Theorem 2.4 without a reference to (3).

3) If we want to consider the extended category $\overline{\mathbb{S}}$, see Subsect. 1.13 then we take a countable subset $\Omega \subset \mathbb{N}$ such that $\mathbb{N} \setminus \Omega$ also is countable. Elements of $A$ are subsets $C \subset \mathbb{N}$ such that $\Omega \setminus C$ are finite. A group $K(C)$ is the subgroup in $\mathbb{K} \simeq \overline{\mathbb{S}}_\infty$ fixing all elements of $C$. 

3. Action of the semigroup $S(0,0)$ on the space $\mathcal{T}$ of restaurants

In the construction of the previous section we set:\n\[ M = \mathcal{S}, \quad G = \mathbb{S}, \quad K = \mathbb{K}, \quad K_n = \mathbb{K}_n \]

3.1. Spaces of half-empty restaurants. Denote by $\mathcal{S}_n^z$ the measure space, whose points are restaurants $\{ U_\omega \} \in \mathcal{T}$ equipped with guests $1, \ldots, n$ chosen uniformly, denote such points by $\{ U_\omega \}, \{ j \} j \in I_n$. Denote the measure on $\mathcal{S}_n^z$ by $\mu^z_n$. In particular $\mathcal{S}_0^z = \mathcal{T}$.

We have an obvious forgetting map $\mathcal{S}_n^z \to \mathcal{S}_m^z$, the pushforward of the measure $\mu^z$ is $\mu^z_m$.

**Proposition 3.1.**
\[ \mathcal{S}[n] = \mathcal{S}_n^z. \]

**Proof.** Since $\mathcal{S}_n^z$ is the quotient space of $\mathcal{S}^z$, the space $L^2(\mathcal{S}_n^z)$ is a canonically defined subspace in $L^2(\mathcal{S}^z)$ and elements of this subspace are $\mathbb{K}$-fixed. We must show that the space of $\mathbb{K}_n$-fixed vectors in $L^2(\mathcal{S}^z)$ is precisely the $L^2(\mathcal{S}_n^z)$. Notice that $L^2(\mathcal{S}^z)$ is a direct integral of Hilbert spaces, the base of the integral is the measure space $\mathcal{S}_n^z$ and a fiber over a point $\{ U_\omega \}, \{ j \} j \in I_n$ is $L^2$ on a product of countable number of copies of $\sqcup U_\omega$ enumerated by $n + 1, n + 2, \ldots$. The group $\mathbb{K}_n$ acts trivially on the base $\mathcal{S}_n^z$. In each fiber it acts by permutation of factors $\sqcup U_\omega$. So a $\mathbb{K}_n$-fixed vector in $L^2(\mathcal{S}^z)$ is an integral of $\mathbb{K}_n$-fixed vectors in fibers $L^2(\sqcup U_\omega^{\infty})$. By the zero-one law such a vector is a constant function. \hfill $\Box$

**Remark.** So the polymorphism $\xi[n]^*: \mathcal{S}^z \to \mathcal{S}_n^z$ is the removing of guests $\overline{n + 1}, \overline{n + 2}, \ldots$. The adjoint polymorphism $\xi[n]: \mathcal{S}_n^z \to \mathcal{S}^z$ is a random arrangement of new guests $n + 1, n + 2, \ldots$ (these polymorphisms are measure-preserving).

3.2. Embeddings and projections. Let $m \geq n$. Define the following canonical morphisms $\lambda^m_n \in \mathcal{S}(n, m)$, see Fig. 4a. We take arcs

1) $1^+_1 \downarrow 1^-_1 [0], \ldots, n^+_n \downarrow n^-_n [0]$ in the right hand side;
2) $1^-_1 \downarrow 1^+_1 [0], \ldots, n^-_n \downarrow n^+_n [0]$ in the left hand side;
3) $(n + 1)^+ \sim (n + 1)^- [1/2], \ldots, m^+_m \sim m^-_m [1/2].$

The following statement immediately follows from the definition:

**Proposition 3.2.** a) The polymorphism $\Xi_{m,n}(\lambda^m_n): \mathcal{S}_n^z \to \mathcal{S}_m^z$ is a forgetting of guests $m + 1, \ldots, \overline{m}$.

b) The polymorphism $\Xi(\lambda^m_n): \mathcal{S}_n^z \to \mathcal{S}_m^z$ is a uniform random arrangement of guests $m + 1, \ldots, \overline{m}$ on a given restaurant $\sqcup U_\omega$. 
More formally, let $(\sqcup U_\omega, \{i\}_{i \leq n}) \in \mathcal{S}_n^z$. Then its spreaded image under $\gamma(\lambda^m_n)$ is the measure on $\mathbb{R}^2 \times \mathbb{S}_m^n$ supported by the set consisting of points

$$1 \times (\sqcup U_\omega, \{i\}_{i \leq n}, \{j\}_{n<j \leq m}),$$

where the guests $\{i\}_{i \leq n}$ are the same, and guests $n+1, \ldots, m$ are uniformly distributed on $\sqcup U_\omega$.

**Remark.** Proposition 3.2 also provides us a description of polymorphisms corresponding to diagrams drawn on Fig. 4. We forget part of guests and arrange new guests randomly.

Our next purpose is to describe the action of the semigroup $\mathcal{S}(0,0)$ on the space $\mathcal{S}^z = \mathcal{S}_0^z$. We need to define some additional objects, namely checker surfaces (they also arise in representation theory of infinite symmetric group for other reasons, see [31], [29]).

### 3.3. Checker surfaces

See [29], [31]. Consider an oriented compact two-dimensional closed surface (generally speaking it is disconnected). Consider a graph on this surface separating it into triangles. Let triangles be colored black and white in the checker order, i.e., neighbors of a black triangle are white and neighbors of a white triangles are black. Let edges of the graph be colored 3 colors, denote them by $a$, $b$, $c$. Let these colors be arranged clockwise on a perimeter of each white triangles, let they be arranged anticlockwise on perimeters of black triangles. We call such graphs on surfaces by checker surfaces. Two surfaces are equivalent if they are isotopic. So such a surface is a pure combinatorial object.

Denote by $\mathcal{Q}$ the set of all checker surfaces, by $\mathcal{Q}^\bullet_n \subset \mathcal{Q}$ we denote the set of surfaces with $2n$ triangles ($n$ black and $n$ white).

**Labeled checker surfaces.** Consider an element of $\mathcal{Q}_n$. Assign pairwise different labels 1, $\ldots$, $n$ to black black triangles and pairwise different labels 1, $\ldots$, $n$ to white triangles. We call an object obtained in this way by a labeled checker surface. Denote by $\mathcal{Q}_n^\bullet$ the set of all such surfaces.

Fix an element $\Sigma \in \mathcal{Q}_n^\bullet$. For each color $a$, $b$, $c$ we define an element $g_a$, $g_b$, $g_c \in S_n$ in the following way. For each $k = 1, 2, \ldots, n$ we find a label $k$ on a white triangles. This triangle has a unique edge of color $a$, this edge is contained in a black triangle with some label, say $l$. Then $g_a$ sends $k$ to $l$. In the same way we define permutations $g_b$ and $g_c$ (they are reflections through $b$-edges and $c$ edges respectively). This determines a one-to-one correspondence between the set $\mathcal{Q}_n^\bullet$ and the group $S_n \times S_n \times S_n$.

**The inverse construction.** Fix elements $g_a$, $g_b$, $g_c \in S_n$. Consider a collection of $n$ white triangles with sides colored $a$, $b$, $c$ clockwise, see Fig. 6. Assign to these triangles labels 1, $\ldots$, $n$. Similarly, take a collection of $n$ labeled black
triangles, whose sides are colored $a$, $b$, $c$ counterclockwise. For each $k \leq n$ we glue (according the orientations) the side of type $a$ of $k$-th white triangle with side of the type $a$ of $g_a(k)$-th a black triangle. Repeating this operation for $g_b$ and $g_c$, we get an element of $\mathcal{Q}_n^*$. Denote a checker surface obtained in this way by $\Sigma(g_a, g_b, g_c)$.

**Types of vertices.** Notice that we have 3 types of vertices. We say that a vertex has a type $A$ if it is adjacent to edges of colors $b$, $c$ (equivalently, it is opposite to an edge of the type $a$); a type $B$ if it is adjacent to edges of colors $a$, $c$; a type $C$ if colors of adjacent edges are $a$ and $b$.

Colors of edges containing a vertex interchange, so valences of vertices of checker surfaces are even. We say that the order $\text{ord}(v)$ of a vertex $v$ is the half of its valence. For $\Sigma(g_a, g_b, g_c) \in \mathcal{Q}_n^*$, cycles of $g_a^{-1}g_b$ are in one-to-one correspondence with vertices of the type $C$, etc. Below it is convenient to use a double terminology: vertices of the type $C$ = vertices of the type $(g_a, g_b)$, etc.

3.4. **Remark. Dessins d’enfant.** The set $\mathcal{Q}$ is in one-to-one correspondence with ‘dessins d’enfant’ in the sense of Grothendieck. Recall that a dessin d’enfant is a bipartite graph on an oriented compact two-dimensional surface such that the complement of the graph is a union of disks. According the Belyi theorem, dessins d’enfants are in one-to-one correspondence with pairs $(R, \Phi)$, where $R$ is a non-singular complex curve defined over the algebraic closure $\overline{\mathbb{Q}}$ of rational numbers $\mathbb{Q}$, and $\Phi$ is a holomorphic function from $R$ to the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \infty$, whose critical values are 0, 1, $\infty$. For detailed discussions, applications, and references, see [19], [45].

Consider a checker surface. Removing edges of colors $a$, $b$ we get a dessin. Vertices of the type $C$ are in one to-one correspondence with complementary disks.

3.5. **Dirichlet measures on simplices.** Fix $\ell > 0$. Consider the $(p-1)$-dimensional simplex $\Delta_p(\ell) \subset \mathbb{R}^p$ defined by

$$x_1 + \cdots + x_p = \ell, \quad \text{where } x_j \geq 0.$$ 

Let $k_1, \ldots, k_p$ be positive reals. A *Dirichlet distribution* $\Theta_p[k_1, \ldots, k_p; \ell]$ is a (probabilistic) measure on $\mathbb{R}^p$ supported by $\Delta_p(\ell)$ and given by

$$\frac{\Gamma(k_1 + \cdots + k_p) \ell^{-p+1}}{\prod_{i=1}^p \Gamma(k_i)} \prod_{i=1}^p x_i^{k_i-1} dx_1 \cdots dx_{p-1},$$

see, e.g., [52], 7.7. This is a general definition, actually, we need only integer $k_j$.
Proposition 3.4. Let $u$ where $\Re \Theta$ present proofs of both formulas.

If $p = 1$, we define $\Theta_1[k; \ell]$ as the delta-measure on $\mathbb{R}^1$ supported by $x_1 = \ell$.

More generally, let $m_1, \ldots, m_q \geq 0$. Let $I$ be set of all $j$ such that $m_j > 0$, say $I = \{m_{\alpha_1}, \ldots, m_{\alpha_p}\}$. Then $\mathbb{R}^q$ splits as a product of the space $\mathbb{R}^{q-I}$ with coordinates $x_{\beta}$, where $\beta \in I$, and the space $\mathbb{R}^{q-\#I}$ with coordinates $x_{\gamma}$, where $\gamma \notin I$. We define $\Theta_q[m_1, \ldots, m_q; \ell]$ as a product of the measure $\Theta_{q-I}[m_{\alpha_1}, \ldots, m_{\alpha_p}; \ell]$ on $\mathbb{R}^{q-I}$ and the atomic unit measure on $\mathbb{R}^{q-\#I}$ supported by 0.

We need Dirichlet distributions for the following reason. Let $\Theta$ be a measure distributed uniformly on the circle of length $\beta$. We need Dirichlet distributions for the following reason. Let $\Theta$ be the distribution $\Theta_1[1, \ldots, 1; \ell]$, i.e., by

\[
\Theta_1[1, \ldots, 1; \ell] := \sum_{\ell} \delta_{\ell}.
\]

Remark. In this formula $dx_p$ is absent, but the coordinate $x_p$ is not distinguished, all measures $dx_1 \ldots dx_{j-1} dx_{j+1} \ldots dx_p$ coincide on $\Delta_p(\ell)$.

3.6. Laplace transforms of the Dirichlet distributions. Let $\Psi$ be a measure supported by $\mathbb{R}^p_+$. Its Laplace transform is

\[
\mathcal{L}\Psi(u_1, \ldots, u_p) := \int_{\mathbb{R}^p_+} e^{-\sum u_i} d\Psi(x_1, \ldots, x_p),
\]

where $\Re u_j \geq 0$.

Proposition 3.4. Let $k_j > 0$. The Laplace transform of the Dirichlet distribution $\Theta_p[k_1, \ldots, k_p; a]$ is given by

\[
\mathcal{L}\Theta_p[k_1, \ldots, k_p; a](u_1, \ldots, u_p) = \frac{\Gamma(\sum k_j) a^{-p+1}}{2\pi i} \int_{-\infty}^{\infty} \prod_{j=1}^p e^{az} (z + u_j)^{-k_j} dz.
\]

If $k_j \in \mathbb{N}$, then this expression equals

\[
\frac{(\sum k_j - 1)! a^{-p+1}}{\prod_j (k_j - 1)!} \prod_{j=1}^p \left( -\frac{\partial}{\partial u_j} \right)^{k_j-1} \prod_m e^{-u_m} \prod_{j \neq m} (u_j - u_m).
\]

Below we need (3.4). The statement (3.3) was obtained in Phillips [41]; we present proofs of both formulas.

Proof. Consider functions $f_1(x_1), \ldots, f_p(x_p)$ on $\mathbb{R}_+$ and an integral

\[
\varphi_a := \int_{\Delta_p(a)} f_1(x_1) \ldots f_p(x_p) dx_1 \ldots dx_{p-1}.
\]
Consider its Laplace transform ($\text{Re } z \geq 0$)

$$
\int_0^\infty e^{-za} \varphi_a \, da = \int_{\mathbb{R}_+^p} e^{-z \sum x_j} \prod_{j=1}^p f_j(x_j) \, dx_1 \ldots \, dx_p = \prod_{j=1}^p \int_0^\infty e^{-zx_j} f_j(x_j) \, dx_j.
$$

Applying the inversion formula for the Laplace transform, we get (cf. (3.3.4.1))

$$
(3.5) \quad \varphi_a = \frac{1}{2\pi i} \int_{-\infty}^{i\infty} e^{za} \left( \prod_{j=1}^p \int_0^\infty e^{-zx_j} f_j(x_j) \, dx_j \right) \, dz.
$$

If $f_j(x_j) = x_j^{k_j-1} e^{-u_j x_j}$, then $\varphi_a$ is (upto a constant factor) the Laplace transform of $\Theta_p[k_1, \ldots, k_p]$. We have

$$
\int_0^\infty e^{-zx} f_j(x) \, dx = \int_0^\infty x_j^{k_j-1} e^{-(z+u_j)x} \, dx = \frac{\Gamma(k_j)}{(z + u_j)^{k_j}}.
$$

Applying (3.5), we come to the first statement (3.3) of the proposition. For general $k_j$ the integral in the right-hand side of (3.3) is a kind of multivariate confluent hypergeometric function. For integer $k_j$ this integral can be evaluated by residues.

Now set $k_j = 1$ in (3.3). Then

$$
\prod_{j=1}^p (z + u_j) = \sum_m \frac{1}{\prod_{j: j \neq m} (u_j - u_m)} \cdot \frac{1}{z + u_m}.
$$

Keeping in mind the integral

$$
\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{az}}{z + u_m} \, dz = e^{-a u_m},
$$

we come to the identity

$$
\mathcal{L} \Theta_p[1, \ldots, 1; a] (u_1, \ldots, u_p) =
$$

$$
= \int_{\Delta_p(a)} \prod_{j=1}^p e^{-u_j x_j} \, dx_1 \ldots \, dx_{p-1} = \sum_m \frac{e^{-a u_m}}{\prod_{j: j \neq m} (u_j - u_m)}.
$$

A multiplication of a distribution by $x_j$ implies a differentiation of its Laplace transform by $u_j$. This leads to (3.4).

3.7. The action of $\mathfrak{g}[0,0]$ on $\mathfrak{g}_0$. Consider an element

$$
\{ \circ [k_j] \}_j \in \mathfrak{g}[0,0],
$$

i.e., a union of cycles of lengths $k_1, \ldots, k_p \geq 2$. To write a formula for the corresponding polymorphism, we need some notation.

Fix a collection of tables $\{U_\nu\} \in \mathfrak{g}_0^\infty$, let $\ell_\omega$ be their lengths, recall that they are pairwise different almost sure.

For a given $\{U_\nu\}_\nu \in \mathfrak{g}_0^\infty$ we intend to write the spreaded image of $\{U_\nu\}_\nu$, i.e., the corresponding conditional measure on the space $\mathbb{R}^\infty \times \mathfrak{g}_0^\infty$. This measure is supported by points of the type $z^\omega \times \{V_\nu\}_\nu$, where a set $\{V_\nu\}_\nu$ coincides with $\{U_\nu\}_\nu$ up to a finite number of tables. It is more convenient to speak about collections $\{\ell_\nu\}$ of lengths, which are point configurations on the segment $[0, 1]$. We introduce the operation

$$
\text{Replace} \left[ \{\ell_\omega\}_j \rightarrow \Lambda \right] \{\ell_\nu\}
$$
on such configurations, which replaces a finite subcollection \( \{ \ell_\omega \}_i \) of the collection \( \{ \ell_\nu \} \) by a random collection \( \subset \mathbb{R}_n^\ell \) with a given distribution \( \Lambda \). More generally, we will use such notation for similar transformations of lists of other types.

**Notation.** a) For \( \Sigma \in \mathfrak{F} \) denote by \( A_\omega \) its \( A \)-vertices, by \( B_\beta \) its \( B \)-vertices, by \( C_\gamma \) its \( C \)-vertices. Denote by \( \# \{ B_\beta \} \) (resp. \( \# \{ C_\gamma \} \)) the number of \( B \)-vertices (resp. \( C \)-vertices).

b) For \( \Sigma \in \mathfrak{F} \) and \( \{ U_\nu \}_{\nu \in \Omega} \subset \mathfrak{S}_0^\beta \) we define a framing of \( \Sigma \in \mathfrak{F} \) as an injective map \( \omega \) from the set of \( B \)-vertices of \( \Sigma \) to the set of tables. In particular, for any \( B \)-vertex \( B_\beta \) we assign a number \( \ell_\omega(B_\beta) \). Denote by \( \mathrm{Fr}(\Sigma, \{ \ell_\nu \}) \) the set of all framings of \( \Sigma \).

c) For \( \Sigma \in \mathfrak{F} \) we denote by \( \text{Aut}_B(\Sigma) \) the group of all automorphisms of \( \Sigma \) fixing all \( B \)-vertices.

d) Let \( \{ k_j \} \) be as above. By \( \Gamma[\{ k_j \}] \) denote the set of all \( \Sigma \in \mathfrak{F}_n \) whose \( A \)-vertices have orders \( \{ k_j \} \).

e) Denote by \( t_m[\{ k_j \}] \) the number of entries of \( m \) to a list \( \{ k_j \} \).

**Theorem 3.5.** Fix an element \( \{ \bigcirc [k_j] \}_j \in \mathcal{S}[0,0] \) and a restaurant \( \sqcup U_\nu \) with tables of lengths \( \{ \ell_\nu \} \). For \( \Sigma \in \Gamma[\{ k_j \}] \) denote by \( m_{\beta \gamma} \) the number of edges connecting \( B_\beta \) and \( C_\gamma \). Then the polymorphism \( \Xi_{0,0}(\{ \bigcirc [k_j] \}) \) sends the point \( \{ \ell_\nu \} \) to the following measure on \( \mathbb{R}^d \times \mathfrak{S}_0^\beta \):

\[
\prod_j k_j \prod_{m \in \mathbb{N}} \ell_m[\{ k_j \}]! \sum_{\Sigma \in \Gamma[\{ k_j \}], \omega \in \mathrm{Fr}(\Sigma, \{ \ell_\nu \})} \frac{1}{\# \text{Aut}_B(\Sigma)} \prod_{B_\beta} \frac{\ell_{\text{ord}(B_\beta)}(\omega(B_\beta))}{(\text{ord}(B_\beta) - 1)!} \times \\
\times \left( \delta_{\mathbb{N}}[\| \{ B_\beta \} \| \gamma - \# \{ B_\beta \}] \right) \times \\
\times \text{Replace} \left[ \{ \ell_\omega(B_\beta) \}_\beta \rightarrow \bigotimes_{\beta} \Theta_{\# \{ C_\gamma \}} \left[ \{ m_{\beta \gamma} \}_\gamma ; \ell_\omega(B_\beta) \right] \{ \ell_\nu \} \right],
\]

where the symbol \( \otimes \) denotes a convolution of Dirichlet distributions on \( \mathbb{R}^\# \{ C_\gamma \} \).

Notice that a convolution in a summand of the formula is a measure on the simplex \( \Delta_{\# \{ C_\gamma \}}(\sum \ell_\omega(B_\beta)) \), densities of such measures are piecewise polynomial. In any case, by Proposition 3.3 Laplace transforms of such measures are explicit elementary functions.

Theorem is proved in Subsect. 3.8.3.12

**3.8. A product of two permutations, which are determined in terms of disjoint cycles.** Recall that a ribbon graph is a graph with fixed cyclic orders of edges at each vertex, see Fig. 7. Such a graph can be regarded as an oriented two-dimensional surface consisting of bands along edges. Boundary of this surface is a union of circles, gluing a disk to each component of the boundary we get a compact oriented surface.

**A construction of a bipartite ribbon graph by two permutations.** Let \( g, h \in S_n \) be represented as products of disjoint cycles

\[
g = (\sigma_1)(\sigma_2) \ldots, \quad h = (\tau_1)(\tau_2) \ldots
\]

For each cycle \( \sigma_\alpha \) we draw a ‘chamomile’ \( A_\alpha \) as on Fig. 8 and enumerate petals

\[13\]This group is poor, for a connected surface it is cyclic (and usually trivial), for a disconnected surface it is a product of cyclic groups.
of $A_\alpha$ by elements of the cycle $(\sigma_\alpha)$ clockwise. For each cycle $(\tau_\beta)$ of $h$ we draw a similar chamomile $B_\beta$. Gluing petals of $g$-chamomiles and petals of $h$-chamomiles according to the enumeration of petals and orientations (see Fig. 9), we get a bipartite ribbon graph, say $Gr(g,h)$, whose list of vertices is $\{A_\alpha\}, \{B_\beta\}$ and edges are enumerated by $1, 2, \ldots, n$.

Each component of the boundary of the ribbon graph $Gr(g,h)$ can be regarded as a polygonal path with an even number of sides. Passing it counterclockwise we observe edges of two types, with origins in $A$-vertices and with origins in $B$-vertices. Labels on edges of the first type form cycles of the permutation $hg$, labels on edges of the second type form cycles of $gh$. See Fig. 10a. This correspondence arises at least to Goulden, Jackson [7].

**Insertions of elements to cycles and ribbon graphs.** Such pictures are well-compatible with insertions of additional elements to cycles of $h$. Namely, take a larger group $S_{n+\mathbb{N}}$. Let $\tilde{g} \in S_{n+\mathbb{N}}$ be the trivial extension of $g$. Choose an
element $h^\circ N \in S_{n+N}$ such that
\begin{equation}
(3.7) \quad \Upsilon^N_{\circ}(h^\circ N) = h.
\end{equation}
Then the ribbon graph $\text{Gr}(\tilde{g}, h^\circ N)$ can be easily obtained from the ribbon graph $\text{Gr}(g, h)$. Namely, an insertion of elements $q_1, \ldots, q_r > n$ to a cycle $(\tau_\beta)$ between $j$ and $k$,

$(\ldots j k \ldots) \mapsto (\ldots j q_1 \ldots q_r k \ldots),$

means that we draw at the vertex $\beta$ edges labeled by $q_1, \ldots, q_r$ between edges $j$ and $k$. Opposite vertices of these edges have valence 1 (since $\tilde{g}(q_\mu) = q_\mu$). See, Fig. 10.b.

3.9. Checker surfaces and products of permutations. Reformulate this construction in the terms of checker surfaces. For $g, h \in S_n$ we take the surface $\Sigma(g, 1, h^{-1}) \in \mathcal{P}_{n \cdot \bullet}$. Vertices of this surface have types $(1, g)$, $(h^{-1}, 1)$, $(g, h^{-1})$, they correspond to cycles of permutations $g, h, (hg)^{-1}$ respectively.

For each vertex of $(g, h^{-1})$-type, labels in adjacent white triangles passing counterclockwise give a cycle of $hg$.

Removing vertices of the type $(g, h^{-1})$ and adjacent edges we get a ribbon graph described above, remaining edges have type $(1)$, labels on both sides of such an
Figure 11. To a pass from $\Sigma(g, 1, h^{-1})$ to $\Sigma(\tilde{g}, 1, (h^\circ N)^{-1})$. Additional drawing on former black triangles.

Figure 12. Transformations of a piece of a table with guests.

edge coincide and so these labels can be attributed to edges. We get the ribbon graph $Gr(g, h^{-1})$.

**Insertion of elements to cycles and checker surfaces.** Operation of insertion of an element with number $> n$ to a cycle of $h$ is shown on the Fig. 10.

So the surface $\Sigma(\tilde{g}, 1, (h^\circ N)^{-1})$ is obtained from the surface $\Sigma(g, 1, h^{-1})$ by the following operation. We preserve all white triangles with labels, and black triangles are transformed as it is shown on Fig. 11.

### 3.10. The right action of $S_n$ on the space of virtual permutations $\mathcal{S}^\natural$.

Let $g \in S_n$, $u \in \mathcal{S}^\natural$. Denote $u = \Upsilon_n u$. Consider the surface $\Sigma(g, 1, u^{-1}) \in \mathcal{Q}_n^\natural$. Take a $B$-vertex, let $T$ be an adjacent black triangle. Let the clockwise white neighbor of $T$ has label $j$ and so the counterclockwise white neighbor has label $u(j)$. We assign to $T$ the interval $(j, u(j))$ of the corresponding occupied table, denote this interval by $O(T)$, see Fig. 11.

For each $C$-vertex consider a counterclockwise cyclic chain

$$j_1, O(T_{i_1}), j_2, O(T_{i_2}), \ldots$$

consisting of labels on adjacent white triangles and ordered sets $O(\cdot)$ on adjacent black triangles. Uniting them, we get a table of $ug$.

### 3.11. The action of $S_n$ on $\mathcal{S}_n^\natural$.

For an element $g \in S_n$ consider the polymorphism

$$\Xi_{n,n}(g) = \xi[n] \ast g\xi[n]$$

of $\mathcal{S}_n^\natural$. Consider a point $u$ of $\mathcal{S}_n^\natural$. It determines a permutation $u \in S_n$ and a collection of lengths $\{l_j\}$ between points $\overrightarrow{j}$ and $\overrightarrow{u(j)}$.

Consider the labeled checker surface $\Sigma = \Sigma(g, 1, u^{-1}) \in \mathcal{Q}_n^\natural$. For a black triangle of $\Sigma$ with label $j$ we assign the length of the arc $\overrightarrow{j, u(j)}$. 
Lemma 3.6. The spreaded image of a point \( u \in \mathcal{G}_n^z \) under the polymorphism \( g \in S_n \) is a \( \delta \)-measure on \( \mathbb{R}^\circ \times \mathcal{G}_n^z \) supported by a point \( z^\circ \times v \), where

\[
s = #\{(g^{-1}, u)\text{-vertices}\} - #\{(u^{-1}, 1)\text{-vertices}\}
\]

and \( v \in \mathcal{G}_n^z \) is defined in the following way. For each \( (g^{-1}, u^{-1}) \)-vertex \( C_\gamma \) of \( \Sigma \) we draw a table taking labels \( \overline{\pi}_1, \overline{\pi}_2, \ldots \) from white triangles passing counterclockwise and length of arc \( [\overline{\pi}_{i-1}, \overline{\pi}_i] \) from black triangles. This gives us a collection of tables containing all labels \( \overline{\pi} \). Adding tables of \( u \) that do not contain labels we get \( v \).

In particular, in this case we get a deterministic map \( \mathcal{G}_n^z \to \mathcal{G}_n^z \), so the group \( S_n \) acts on the space \( \mathcal{G}_n^z \).

The statement follows from the the construction of the previous subsection and formula (1.5).

3.12. Proof of Theorem 3.5. Denote \( n := \sum k_j \). Choose an element \( g \in S_n \) whose cycles have lengths \( k_1, \ldots, k_p \). Denote by \( (\sigma_\alpha) \) disjoint cycles of \( g \). This choice is noncanonical and further considerations depend on the last paragraph of the proof. We extend \( g \) to an element \( \tilde{g} \) of \( S_\infty \) in a trivial way. Then the double coset \( \mathcal{Y} \cdot (1, \tilde{g}) \cdot \mathcal{K} \) is \( \{\circ[k_j]\} \). We decompose

\[
\{\circ[k_j]\} = (\lambda^n_0)^* g \lambda^n_0.
\]

Recall that the polymorphism \( \Xi(\lambda^n_0) : \mathcal{G}_n^z \to \mathcal{G}_n^z \) defined in Subsect. 3.2 is a random uniform arrangement of guests \( \overline{T}, \ldots, \overline{\pi} \) in an empty restaurant. This operation can be described in the following (more complicated) way.

Fix a restaurant, let \( \{\ell_\mu\} \) be lengths of tables (recall that they are pairwise distinct a.s).

— First, fix a collection \( \{s_\mu\} \) of nonnegative integers such that \( \sum s_\mu = n \). Assume that a probability of such a choice is

\[
n! \prod_\mu \ell_\mu^{s_\mu} \prod s_\mu!
\]

(products actually are finite, the sum equals 1). Denote \( \mathcal{B} = \mathcal{B}(\{s_\mu\}) \) the set of all \( \mu \), for which \( s_\mu > 0 \).

— Second, for each \( s_\mu > 0 \) we take a cyclically ordered set \( V_\mu \) with \( s_\mu \) elements. Choose a bijection \( \{1, 2, \ldots, n\} \to \cup_{\mu \in \mathcal{B}} V_\mu \) and assume that all variants have the same probability \( 1/n! \). So we get a collection \( \{V_\mu\}_{\mu \in \mathcal{B}} \) of cyclically ordered sets consisting of positive integers \( \leq n \). We consider each set up to cyclic ordering, so there are \( \prod_{\mu \in \mathcal{B}} s_\mu \) ways to obtain the same collection. Thus, we get a permutation \( u \in S_n \) defined as a product of cycles.

— Third, for each cycle \( V_\mu \) of \( u \) we insert arcs between elements of the cycle (guests), lengths of arcs are chosen according to the uniform probabilistic distribution on the simplex \( \Delta_{s_\mu}(\ell_\mu) \) (i.e., the Dirichlet distribution \( \Theta_{s_\mu}[1, \ldots, 1; \ell_\mu] \)).

So we get an element \( u \) of \( \mathcal{G}_n^z \) depending on \( \{s_\mu\}, u \in S_n \), and a point of product of simplices \( \prod_{\mu \in \mathcal{B}} \Delta_{s_\mu}(\ell_\mu) \).

Keeping in mind Lemma 3.6 we can apply \( g \in S_n \) to a random element \( u \in \mathcal{G}_n^z \).

For each \( u \) consider the surface \( \Sigma(g, 1, u^{-1}) \). By definition, labels on both sides of any edge of the color (1) coincide. Vertices of types \( A, B, C \) in formulation of Theorem 3.5 correspond to types \( (1, g), (u^{-1}, 1), (g, u^{-1}) \). Clockwise cycles of labels
on white triangles about a vertex $A_n$ are cycles $(\sigma_n)$ of $g$. Each surface is equipped with a framing $\omega$, namely $B$-vertices correspond to tables $\{U_\mu\}_{\mu \in B}$ of the initial restaurant $\{U_\mu\}$ (comparatively to the set $\Gamma[[k_j]]$ in the formulation of Theorem 3.5 we add labels on triangles). Additionally, we assign a positive number $l(j)$ to each black triangle $T(j)$ with label $j$. Such numbers around a given $B$-vertex are distributed according the measure $\Theta_{\text{ord}(B_\beta)}[1, \ldots, 1; \ell_{\omega(B_\beta)}]$.

New tables of a random restaurant $\Sigma_{0,n}(\lambda^n_0)^* \Sigma_{n,0}(g) \Xi_{n,0}(\lambda^n_0)$ correspond to vertices $C_\gamma$, a length $L(\gamma)$ of such a table is sum of real numbers $l(j)$ attributed to black triangles containing $C_\gamma$.

$$L(\gamma) = \sum_{j: C_\gamma \in T(j)} l(j).$$

We write this sum in the form

$$L(\gamma) = \sum_{B_\beta} \sum_{j: B_\beta, C_\gamma \in T(j)} l(j).$$

By Lemma 3.3 for a fixed $\beta$ a distribution of vectors

$$\left\{ \sum_{j: B_\beta, C_\gamma \in T(j)} l(j) \right\}_\gamma$$

is the Dirichlet distribution $\Theta_{\#(C_\gamma)}[\{m_{\beta_\gamma}\}_\gamma; \ell_{\omega(C_\beta)}]$. Therefore the sums $\{L(\gamma)\}_\gamma$ are distributed as a convolution $\star_{\beta}$ of such Dirichlet distributions.

Hence the desired conditional measure is

$$\sum_{\Sigma(g, 1, u^{-1}) \text{ with } g} \sum_{\omega \in \Pr(\Sigma(g, 1, u^{-1}); \ell_{\omega})} \prod_{B_\beta} \frac{(\ell_{\omega(B_\beta)})^{\text{ord}(B_\beta)}}{\text{ord}(B_\beta)! - 1} \times$$

$$\times \left( \delta_{\mathbb{R}^+}[\#(C_\gamma) - \#(B_\beta)] \right) \times$$

$$\times \text{Replace} \left[ (\ell_{\omega(B_\beta)}) \rightarrow (\star_{\beta} \Theta_{\#(C_\gamma)}[\{m_{\beta_\gamma}\}_\gamma; \ell_{\omega(B_\beta)}]) \{\ell_{\omega}\} \right].$$

In fact, the first summation is taken over the subset in $\mathfrak{g}_n^*$ consisting of labeled surfaces such that:

1) for each $c$-edge labels on two-sides of the edge coincide;

2) clockwise cycles of labels of white triangles about $A$-vertices coincide with the disjoint cycles $(\sigma_n)$ of the permutation $g$.

Forgetting all labels we get an element of the set $\Gamma[[k_j]]$ defined in Subsect. 3.7. Fix a surface $\Lambda \in \Gamma[[k_j]]$ with fixed framing (so we can think that $B$-vertices of $\Lambda$ are enumerated) and evaluate the number of different surfaces of the type $\Sigma(g, 1, u^{-1})$ over $\Sigma$.

1) We must choose a one-to-one correspondence between the set of $A$-vertices and the set of cycles $(\sigma_n)$ of $g$ (an order of a vertex must coincide with an order of the corresponding cycle). This can be done by $\prod_{m \geq 2} \ell_m \{k_j\}!$ ways.

2) After this, the collection of labels on white triangles containing a vertex $A_n$ consists of elements of the corresponding cycle $(\sigma_n)$ in the same cyclic order. The only freedom is rotation of the cycle, this gives us the factor $\prod \text{ord}(A_n) = \prod k_\beta$.

3) Labels on black triangles are uniquely determined by labels on white triangles.
4) Two surfaces $\tilde{\Lambda}_1$, $\tilde{\Lambda}_2$ obtained by such arrangements (including framings) can be equivalent. In this case we have an automorphism of the surface preserving the framing and identifying two labelings. So we must divide the result by the order of the group of such automorphisms.

We came to formula (3.6).

□

4. Further Statements

4.1. A more general case. Next, let us consider a space $S_n^+$ and describe its polymorphism determined by a chip, whose left-hand half is trivial, see Fig. 13.

Data defining a chip. Fix a permutation $\sigma \in S_n$, a collection $\varphi_1, \ldots, \varphi_n \in \mathbb{Z}_+$ and a collection of integers $k_1, \ldots, k_p \geq 2$. Define a chip $b = b[\sigma, \{\varphi_i\}, \{k_j\}] \in S(n, n)$ consisting of the following arcs:

1) $j^+ \downarrow j^-[0]$, where $j = 1, \ldots, n$;
2) $j^+ \downarrow \sigma(j)^-\varphi_j]$, where $j = 1, \ldots, n$;
3) $\circ[k_i]$, where $i = 1, \ldots, p$.

Data determining a half-empty table. Next, fix an element $\tau \in S_n$ represented as a product of disjoint cycles, a collection of positive reals $l_1, \ldots, l_n$, and a countable collection of positive reals $\ell_\nu$ such that $\sum_i l_i + \sum \ell_\nu = 1$. These data define a point $t = t[\tau, \{l_i\}, \{\ell_\nu\}] \in T^+_n$.

Namely, for $i$-th cycle of $\tau$ we create a table $U_i$ containing points of the cycle as guests (in the same cyclic order), and claim that distances between $j$ and $\tau(j)$ are $l_j$. Also, we add label-free tables of lengths $\ell_\nu$.

Now we intend to write the spreaded image of the half-empty restaurant $t$ under the polymorphism $b$. The result is similar to Theorem 3.5 but it must include additional combinatorial data.

The set of summation. Define the following set $\Gamma[b; t]$ consisting of checker surfaces equipped with following additional structures:

1) An injective map $F$ (labeling)

$F: \{1, 2, \ldots, n\} \rightarrow \{\text{set of white triangles}\}$.

2) A bijection between

\[
\{\text{set of $A$-vertices of $\Sigma$}\} \leftrightarrow \{\text{set of cycles of $\sigma$}\} \cup \{\text{set of cycles $\circ[k_i]$ of the chip $b$}\}.
\]
This map must be compatible with the labeling $F$. Namely, if a vertex $A_\alpha$ corresponds to a cycle $[k_i]$ of the chip, then $\text{ord} A_\alpha = k_i$ and white triangles containing $A_\alpha$ are label-less. If a vertex $A_\alpha$ corresponds to a cycle $(\nu_1 \nu_2 \ldots \nu_\rho)$ of $\sigma$, then labeled white triangles containing $A$ are precisely $F(\nu_1), \ldots, F(\nu_\rho)$ in this cyclic order, and for each $\nu_i$ there are precisely $\varphi_i$ label-less white triangles between $F(\nu_i)$ and $F(\nu_{i+1})$, see Fig. 14.

3) An injective map $\omega$ (framing)

$$\omega : \{\text{set of } B\text{-vertices of } \Sigma\} \rightarrow \{\text{set of tables of } t\}.$$  

Moreover, labels on white triangles containing $B_\beta$ must coincide with guests on the corresponding table in reversed cyclic order, see Fig. 15.

**Figure 14.** a) A cycle of $\sigma$ (on the figure we have a cycle (273)) and corresponding arcs of a chip. b) The $A$-vertex corresponding to this cycle $\sigma$.

**Figure 15.** A table, the corresponding $B$-vertex, and sets $T(i)$.

**Some notation.** 1) Vertices $B_\beta$ split into two types. Denote by $\Phi$ the set of all $B_\beta$ such that all white triangles containing $B_\beta$ are label-less. For $B_\beta \in \Phi$ denote by $T(B_\beta)$ the set of black triangles containing $B_\beta$.

For $B_\beta \notin \Phi$ consider the counterclockwise cycle of adjacent white triangles, let $\zeta_1, \ldots, \zeta_s$ be labels on labeled white triangles. For each adjacent pair $(\zeta_i, \zeta_{i+1})$ denote by $T(\zeta_i)$ the set of black triangles between them. Notice that a $B$-vertex is determined by the number $\zeta_i$ and we can omit $\beta$ from the notation, see Fig. 15.

2) Similarly, denote by $\Psi$ the set of $C$-vertices, for which all adjacent triangles are label-less. For $C_\gamma \in \Psi$ denote by $T(C_\gamma)$ the set of black triangles containing $C$. For $C \notin \Psi$ we write counterclockwise labels on adjacent white triangles $(\mu_1, \mu_2, \ldots)$. Denote by $R(\mu_j)$ the set of black triangles between $\mu_j$ and $\mu_{j+1}$.  

Denote by $\mathbf{1}_i$ the vector of dimension $n$ with exactly $1$ in the $i$-th position and $0$ elsewhere.
3) For each vertex of \( C \) we consider the counterclockwise cycle of adjacent labeled triangles. Uniting such cycles we get an element \( \rho \) of \( S_n \).

4) Denote by \( \text{Aut}_B(\Sigma, F) \) the group of all automorphisms of \( \Sigma \) fixing all \( B \)-vertices and all labeled triangles. Such an automorphism automatically is trivial on each connected component of \( \Sigma \) containing a labeled triangle.

**Remark.** In Theorem 3.5 parameters of Dirichlet measures \( \Theta[\ldots] \) are expressed in the terms of the matrix \( m_{\beta, \gamma} \) whose elements are number of black triangles with vertices \( B_\beta \) and \( C_\gamma \). Rows of the matrix are enumerated by \( B \)-vertices and columns by \( C \)-vertices. In our case, rows of a similar matrix are enumerated by chains \( T(i) \) between two labeled triangles at \( B \)-vertices \( \notin \Phi \) and by vertices \( B_\beta \in \Phi \). Columns are enumerated by chains \( R(j) \) between labeled triangles at \( C \)-vertices \( \notin \Psi \) and by vertices \( C_\gamma \in \Psi \). So analog of the matrix \( m_{\beta, \gamma} \) contains elements of types

\[
\begin{align*}
\#(T(i) \cap R(j)) &\quad \#(T(i) \cap R(C_\gamma)) \\
\#(T(B_\beta) \cap R(j)) &\quad \#(T(B_\beta) \cap R(C_\gamma))
\end{align*}
\]

For this reason an expression for convolutions now is longer, but in fact the structure of the formula is the same.

**Theorem 4.1.** The spreaded image of a point \( t \) under a polymorphism \( b \) is the measure on \( \mathbb{R}^\gamma \times S_n \) given by the formula

\[
\begin{align*}
(4.1) \quad &\prod k_i \prod_{\ell_m([k_j])} \sum_{(\Sigma, \omega, F) \in \Gamma([b, t])} \frac{1}{\#G(\Sigma, F)} \prod_{\nu=1}^{n} \ell_{r}^{\#T(\nu)} \prod_{B_\beta \in \Phi} \left( \frac{\ell_{\omega(B_\beta)}}{(\operatorname{ord}(B_\beta) - 1)!} \times \\
&\times \left( \delta_{\mathbb{R}^\gamma} \left[ [z \#(C_\gamma) - \#(B_\beta)] \right] \right) \right) \times \\
&\times \text{Replace} \left( \tau; \{l_i\}; \{\ell_{\omega(B_\beta)}\}_{B_\beta \in \Phi} \right) \rightarrow \\
&\left( B_\beta \in \Phi \right) \left[ \Theta_{n+\#\Psi}\left[ \#(T(i) \cap R(j)) \right]_{j \in I_\omega}; \#(T(i) \cap R(C_\gamma))_{C_\gamma \in \Psi} ; [C_j] \right] \\
&\left( \times \left( B_\beta \in \Phi \right) \left[ \Theta_{n+\#\Psi}\left[ \#(T(B_\beta) \cap R(j)) \right]_{j \in I_\omega}; \#(T(B_\beta) \cap R(C_\gamma))_{C_\gamma \in \Psi} ; [\ell_{\omega(B_\beta)}] \right] \right) \\
&\left( \tau; \{l_i\}; \{\ell_{\omega(B_\beta)}\} \right) \rightarrow \\
&\left( B_\beta \in \Phi \right) \left( \times \left( B_\beta \in \Phi \right) \left( \Theta_{n+\#\Psi}\left[ \#(T(B_\beta) \cap R(j)) \right]_{j \in I_\omega}; \#(T(B_\beta) \cap R(C_\gamma))_{C_\gamma \in \Psi} ; [\ell_{\omega(B_\beta)}] \right) \right) \\
&\left( \tau; \{l_i\}; \{\ell_{\omega(B_\beta)}\} \right) \rightarrow \\
&\left( B_\beta \in \Phi \right) \left( \times \left( B_\beta \in \Phi \right) \left( \Theta_{n+\#\Psi}\left[ \#(T(B_\beta) \cap R(j)) \right]_{j \in I_\omega}; \#(T(B_\beta) \cap R(C_\gamma))_{C_\gamma \in \Psi} ; [\ell_{\omega(B_\beta)}] \right) \right) \\
&\left( \tau; \{l_i\}; \{\ell_{\omega(B_\beta)}\} \right) \rightarrow \\
&\left( B_\beta \in \Phi \right) \left( \times \left( B_\beta \in \Phi \right) \left( \Theta_{n+\#\Psi}\left[ \#(T(B_\beta) \cap R(j)) \right]_{j \in I_\omega}; \#(T(B_\beta) \cap R(C_\gamma))_{C_\gamma \in \Psi} ; [\ell_{\omega(B_\beta)}] \right) \right) \\
&\left( \tau; \{l_i\}; \{\ell_{\omega(B_\beta)}\} \right) \rightarrow \\
&\left( B_\beta \in \Phi \right) \left( \times \left( B_\beta \in \Phi \right) \left( \Theta_{n+\#\Psi}\left[ \#(T(B_\beta) \cap R(j)) \right]_{j \in I_\omega}; \#(T(B_\beta) \cap R(C_\gamma))_{C_\gamma \in \Psi} ; [\ell_{\omega(B_\beta)}] \right) \right) \\
&\left( \tau; \{l_i\}; \{\ell_{\omega(B_\beta)}\} \right) \rightarrow \\
&\left( B_\beta \in \Phi \right) \left( \times \left( B_\beta \in \Phi \right) \left( \Theta_{n+\#\Psi}\left[ \#(T(B_\beta) \cap R(j)) \right]_{j \in I_\omega}; \#(T(B_\beta) \cap R(C_\gamma))_{C_\gamma \in \Psi} ; [\ell_{\omega(B_\beta)}] \right) \right) \\
&\left( \tau; \{l_i\}; \{\ell_{\omega(B_\beta)}\} \right) \rightarrow
\end{align*}
\]
the cyclic order. For each $\beta$ we equip the set of all $\eta_\beta$ with the uniform measure $d\eta_\beta$, assuming that the total measure is $\ell_{\omega(B)}^\text{ord}$ $(\#B_\beta - 1)!$. Let $Q$ be a black triangle containing $B_\beta$. Let $T$ be its white predecessor according to the cyclic order, $T'$ be the white follower. We assign to $Q$ the arc $\mathcal{B}(Q) := [\eta_\beta(T), \eta_\beta(T')]$ of the table $U_{\omega(B)}$ together with guests. Notice that the ends of such arcs are not guests a.s. Define a new collection $\{\zeta_\gamma\} = \{\zeta_\gamma(\{\eta_\beta\})\}$ of occupied tables in the following way: take a vertex $C_\gamma$, write all black triangles $Q_{s_1}, Q_{s_2}, \ldots$ containing $C_\gamma$ in the counterclockwise order. Then $\zeta_\gamma$ is a cyclically ordered set

$$\zeta_\beta := \mathcal{B}(Q_{s_1}) \sqcup \mathcal{B}(Q_{s_2}) \sqcup \ldots$$

equipped with a natural length. We also have a pushforward $\zeta_\gamma(d\eta)$ of the measure $d\eta = d\eta_1d\eta_2 \ldots$, this is a new measure on a collection of occupied tables.

**Theorem 4.2.** The image of a point $u \in \mathbb{S}^n$ under the polymorphism $1\{\{k_j\}\}$ is

$$\prod_j k_j! \prod_{m \geq 2} t_m(\{k_j\}) \sum_{(\Sigma, \omega) \in \Gamma^\nu(\{k_j\})} \left(\delta_{\mathbf{G}}(\mathcal{U}_\mathcal{B}^{(\mathcal{B}(\mathcal{B}(Q_i)) - \mathcal{B}(Q_i))}) \times \text{Replace}\left(\mathcal{U}_\mathcal{B}^{(\mathcal{B}(Q_i))} \rightarrow \{\zeta_\gamma(d\eta)\}\right)\right)$$

To evaluate this image we apply Remark after Proposition 4.2. We take the second copy $\mathbb{N}$ of $\mathbb{N}$ consisting of points $1, 2, \ldots$. Let regard $\mathbb{S}_\infty$ as the group $\mathbb{S}(\mathbb{N}, \mathbb{N})$ of all permutations of $\mathbb{N} \sqcup \mathbb{N}$ and $\mathbb{S}$ as a subgroup in $\mathbb{S}(\mathbb{N} \sqcup \mathbb{N}) \times \mathbb{S}(\mathbb{N} \sqcup \mathbb{N})$. Now we can realize $\mathbb{S}(\mathbb{N}, \mathbb{N})$ as the semigroup of double cosets $\mathbb{S}(\mathbb{N}) \setminus \mathbb{S}/\mathbb{S}(\mathbb{N})$.

Let $\sum k_j = n$. Take $g \in \mathbb{S}$ that is trivial on the left copy of $\mathbb{N} \sqcup \mathbb{N}$ and on the right copy of $\mathbb{N}$ and on the set $\{n + 1, n + 2, \ldots\}$. Let also $g$ induces a permutation of the set $\{1, 2, \ldots\}$ with cycles of lengths $k_j$.

Further considerations as the same as above. We take an analog of $\lambda_0^n$, it corresponds to the chip $1^2 \{1/2\} \sim 1^2 \{1/2\}$, $2^2 \{1/2\} \sim 3^2 \{1/2\}$ and other arcs are strictly vertical lines of weight 0. It corresponds to putting additional guests $1, 2, \ldots$, $n$ to the occupied restaurant $u$.

We fall to the situation of Subsect. 3.10 where the set $\mathbb{N}$ is replaced by $\mathbb{N} \sqcup \{1, 2, \ldots, n\}$ and $g \in \mathbb{S}_n$ acts nontrivially on additional random guests $1, 2, \ldots, n$. These 'numbers' also are labels $1, 2, \ldots, n$ on white triangles. We apply the construction of Subsect. 3.10 and forget random guests.

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