Deformation Quantization of 
Principal Fibre Bundles 
and 
Classical Gauge Theories

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To my family
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Introduction

As typical for a thesis in mathematical physics the aspiration of this work is twofold.

From the physical point of view it is motivated by the aim to find a global and geometric formulation of classical gauge theories on arbitrary noncommutative space-times where the non-commutative structure is given by a star product. In principle, such a formulation is done by an appropriate adaption of all occurring geometric structures to the noncommutative case. As a first step towards this ambitious goal, this work presents and investigates a notion of a deformation quantization of principal fibre bundles and related geometric structures that play a role in physical applications.

The investigation of the underlying mathematical problem, namely the deformation of right modules with respect to a given deformed algebra is of independent mathematical interest. From this point of view the work provides a general algebraic approach to such deformation problems. For certain geometric examples the existence and the uniqueness up to equivalence of the investigated structures is shown by an explicit computation of Hochschild cohomologies.

Motivation

Quantum mechanics and general relativity are indisputably two of the most important theories in physics and build the basis of the present understanding of the fundamental laws of nature. Both concepts are well-established and impressively confirmed by experiments. However, it still is an unsolved problem to unify the quantum mechanical description of nature at microscopic scales and the sophisticated model of space-time geometry and its interaction with matter provided by general relativity. The attempt to combine the two theories leads to conceptual problems and reveals the necessity for new models of the space-time. The promising concept of so-called noncommutative space-times first arose in the different context of quantum electrodynamics where it was supposed to be a way to handle the occurring UV-divergences. In a letter to Peierls from 1930, Heisenberg already suggested to introduce uncertainties in the space coordinates and regretted not to be able to provide a reasonable mathematical framework [67]. In 1947, Snyder was the first who substantiated the idea and formulated a first notion of a quantized space-time by substituting the space-time coordinates by hermitian operators [113,114]. However, with the great achievements of other regularization methods and the concept of renormalization the noncommutative space-times passed out of mind for some years.

Meanwhile, the mathematical field of noncommutative geometry was established in the works of Connes [33], Madore [89], Gracia-Bondía et al. [58], and many others. The developed notions in this new area in particular clarified the mathematical definition of noncommutative spaces and provided the appropriate mathematical framework for all later investigations. Based on the observation that many structures in differential geometry are determined by their algebraic properties, noncommutative geometry extends the correspondences between geometric data and commutative algebras to the noncommutative case. In particular, any smooth manifold is already determined by the commutative algebra of smooth functions on it. Taking the algebraic structure as the funda-
mental one and dropping the notion of a manifold as a set of points, ‘any’ algebra can be interpreted as the function algebra on a corresponding noncommutative manifold.

That this new concept of noncommutative manifolds is physically meaningful became clear with the works \[39, 40\] of Doplicher, Fredenhagen and Roberts. Using the uncertainty relations of quantum mechanics and the Einstein equations of general relativity they pointed out in a gedanken-experiment that it is not possible to determine the space-time coordinates of a point-like particle with arbitrary accuracy. Instead, they deduced uncertainty relations for the space-time coordinates where the Planck length $l_P = 1.6 \times 10^{-35} \text{m}$ has been perceived as the scale where the new effects of noncommutativity should occur. Consequently, the notion of a space-time consisting of single points has no operational physical meaning. In the proposed new framework the space-time coordinates have been replaced by noncommuting operators. In \[40\] this also led to a first approach towards quantum field theories on noncommutative space-times.

Starting with a noncommutative algebra describing a noncommutative manifold it is one of the main purposes of noncommutative geometry to find the algebraically motivated counterparts of any geometric structure that is defined on an ordinary manifold. In particular, vector bundles and principal bundles are of special interest since they provide the appropriate framework to study classical gauge theories. Vector bundles over a noncommutative manifold are simply defined as finitely generated and projective modules over the considered algebra. This definition based on the work of Swan \[115\] is the natural generalization of the fact that vector bundles are determined by their sections which are such modules with respect to the functions on the base manifold.

The situation for principal fibre bundles is not that easy since the geometric structures have no obvious algebraic properties determining them. For trivial bundles and matrix Lie groups as structure groups Dubois-Violette, Kerner and Madore \[42\] presented first approaches towards new geometric models of gauge theory. More general but abstract notions of so-called quantum principal bundles were presented by Brzeziński and Majid \[22\] and further investigated by Hajac \[64, 65\]. Basically, the structure group there is replaced by a Hopf algebra \[91\] co-acting on the algebra which is related to the total space. This definition of quantum principal bundles has later been realized to be nothing but so-called Hopf-Galois extensions. A related definition where the total space of the bundle is also replaced by an algebra was given by Đurđević in \[44, 46\] who also studied notions of corresponding gauge theories \[47\]. Structures in the context of associated vector bundles were also investigated from this point of view, for instance in \[21, 34\]. The underlying geometrical and algebraic background of all these approaches is summarized in the expository articles \[6\] and \[93\]. The influence of noncommutative geometry gave rise to a huge amount of new approaches in various realms of theoretical physics as it is outlined in the articles \[11\] of Douglas and Nekrasov and \[116, 117\] of Szabo.

However, due to the very abstract formulation of these concepts it is often not clear how the occurring structures and features have to be interpreted and how one can construct physically relevant models. For the investigation of noncommutative space-times and their impact on corresponding classical gauge field theories in the aspired geometric setting, the methods of deformation quantization seem to be appropriate. The basic concepts of deformation quantization initially investigated by Bayen and coworkers in \[8\] in order to study the quantization of phase spaces and the relationship between classical and quantum mechanics can be applied in the following way. In the most general case of the pursued approach the noncommutative space-time is assumed to be given by an associative deformation of the pointwise product of functions on the considered classical space-time manifold, for instance by a star product with respect to a Poisson structure. In general, the new multiplication is just defined for the formal power series of functions with respect to some formal parameter. In the spirit of Gerstenhaber’s deformation theory \[51, 52\] which is the basic framework for deformation quantization one should then try to deform all relevant algebraic structures of a principal fibre bundle that are related to the initially deformed multiplication of
functions on the base manifold such that the algebraic properties are preserved. Although the convergence of the formal series is a nontrivial problem this approach, that already gave rise to the notion of deformation quantization of vector bundles, has some important advantages. Since the classical objects are mostly just replaced by formal power series their physical interpretation is clear. If the basic deformed structures are established the found situation may indeed lead to new perceptions that result in further generalizations. Moreover, the framework of deformation quantization allows an understanding of the classical limit. In the new context the formal parameter encodes the influence of the noncommutative aspects of space-time and can be interpreted as the Planck area if the formal series converge. For macroscopic phenomena where this scale is negligibly small, the parameter can be set to zero and one obtains the ordinary gauge theory. These two aspects, the reformulation of gauge theories in stages and the existing notion of a classical limit, are the main reasons for this approach.

Although the notion of a star product is also clear for complicated geometries as typically occurring in general relativity, most of the attempts to formulate physical theories known so far use the controversial assumption of a flat Minkowski space equipped with the Weyl-Moyal product or other star products respecting the relevant commutation relations of the coordinates. Within this setting the work [112] of Seiberg and Witten on string theory initiated the investigations of Yang-Mills and other gauge theories on noncommutative flat space-times which arise as low-energy limits of string theories. Neglecting their geometric background all physically relevant structures can be traced back to functions on the space-time. In the approaches that make use of this simplification all ordinary products are replaced by the considered star product. In the works [77, 79, 90, 123, 129] Wess and coworkers pointed out that the Lie algebra defining the gauge fields and the infinitesimal gauge transformations of such theories has to be extended to its universal enveloping algebra. The initial considerations in [112] further motivated the so-called Seiberg-Witten maps which express the noncommutative gauge and matter fields together with their infinitesimal gauge transformations by formal series of their commutative counterparts in the deformation parameter. For quantum field theoretic models on a flat space-time the works of Grosse and Wohlgenannt [59, 60] contributed results concerning the renormalizability of quantum field theories on noncommutative spaces. An overview over this aspect was given by Rivasseau [110]. Based on these results there already exist concrete formulations of the standard model on noncommutative space-times and proposals how to measure the occurring parameters that encode the noncommutativity in experiments [2, 3, 105].

Although the mentioned models have been investigated to an enormous extent, there are various issues that remain unclear and thus require further investigations and improvements. First of all, the physically adequate notion of a noncommutative space-time is still not well-understood. The use of star products immediately implies that the space-time carries a Poisson structure and there do not exist any physical arguments that distinguish a reasonable choice for it or even guarantee its existence. There is no experimental evidence for a globally defined noncommutative structure, in particular not for a constant Poisson structure as used so far. In contrast to this it is rather natural to assume that the noncommutative features of space-time are a local effect. Waldmann and coworkers have discussed these aspects in [5, 68, 69, 124] and presented an approach to locally noncommutative space-times.

Second, and this is the issue basically motivating the present work, the transparent and clear geometric formulation of classical gauge field theories in terms of principal fibre bundles and associated vector bundles has not been sufficiently clarified and investigated in the context of deformation quantization. The need for a geometrical generalization is supported by the fact that the physical effects at the Planck scale which initially gave rise to the consideration of noncommutative space-times can not be described in the flat Minkowski space and require the full geometric framework of general relativity. Moreover, many of the simplifications used so far are too strong and not sustainable. The naive understanding of fields as functions on the space-time is just a local description of
the geometric objects like vector-valued differential forms or sections of vector bundles and depends on the used local charts. This is especially important for nontrivial bundles which should be taken into account out of the following obvious reasons. The relevant space-times have non-vanishing curvature and due to possible singularities and black holes they may carry nontrivial topologies admitting nontrivial principal and vector bundles over them. Moreover, one usually demands that the fields have a certain behaviour to decrease at infinity in order to obtain a finite amount of energy. Using a one-point compactification the fields that vanish at infinity can be defined as fields on a sphere which also allows nontrivial bundles.

From this point of view it becomes clear that the procedures used in the most models, namely to replace all products of functions by star products has to be improved. The fields are no functions and thus it is simply not possible to multiply them with a star product. If this is done for the functions obtained in the local expressions of the fields the new products heavily depend on the choice of the local charts and have no global meaning. In the geometric formulation fields are multiplied using fibre metrics which in the noncommutative case have to be adapted properly. Besides establishing the deformation quantization of vector bundles, Bursztyn and Waldmann provided a notion of deformed hermitian fibre metrics and discussed some aspects of noncommutative field theories in this setting \[23–25, 120–122, 125\].

This well-understood framework for vector bundles only describes matter fields. Gauge potentials and gauge transformations are not taken into account. In order to do this one has to find the deformed versions of principal fibre bundles. The appropriate notion of a deformation quantization of principal fibre bundles was first given in the authors diploma thesis \[126\]. Using an adapted Fedosov construction \[48, 49\] it was shown that under the assumption of a symplectic base manifold the functions on the total space can be deformed into an invariant right module with respect to a Fedosov star product on the base.

### Results of the work

As a continuation of the above developments it is the main concern of this work to investigate the notion of deformation quantization of principal fibre bundles with arbitrary Poisson manifolds as basis. From the algebraic point of view the considered definition encodes a deformation problem of right modules which also can be discussed in a more general geometric framework. Due to this observation a notion of deformation quantization of surjective submersions is also taken into account.

It is the main result of this work to show that the functions on the total space of any surjective submersion can be deformed into a right module with respect to an arbitrary differential star product on the base manifold. These deformations defined as formal series of bidifferential operators with the pointwise module structure in the lowest order of the formal parameter, are further shown to be unique up to equivalence. The same result concerning the existence and the classification of deformation quantizations is further shown for the special case of principal fibre bundles where the deformations and their equivalence transformations are additionally required to be invariant under the action of the structure Lie group of the bundle.

The attempt to construct deformations of algebraic structures and correspondingly defined equivalence transformations between them order by order in the formal parameter is typically opposed to obstructions which are encoded in certain Hochschild cohomology groups. For star products which are associative deformations of the algebra of smooth functions the theorem of Hochschild, Kostant, and Rosenberg \[73\] states that the relevant Hochschild cohomology of the considered algebra is isomorphic to the Gerstenhaber algebra of antisymmetric multivector fields. Due to this well-known result further discussed in \[29, 99, 108\] the proof for the existence and the classification of star products given by Kontsevich \[83\, 85\] has to make use of a more sophisticated
approach. Due to these facts and other typical examples of algebraic structures where the mentioned obstructions in general are nontrivial, the following facts are rather surprising.

The general investigation of module deformations in this work in particular substantiates the obstructions against the desired orderwise constructions proving the above stated result concerning the classification and the existence of deformation quantizations of surjective submersions and principal fibre bundles. In both cases they are given by the first and the second cohomology group of the Hochschild complex of the algebra of functions on the base manifold with values in the corresponding bimodule of differential operators of the functions on the total space which are required to be invariant operators in the case of principal fibre bundles. An explicit and nontrivial computation points out that the first and all higher cohomology groups are trivial.

The computation of the cohomology based on explicit local homotopies not only provides an elegant proof of the above stated main assertion but, in principle, also provides an explicit prescription how to construct the relevant structures.

The mathematical framework which is necessary to describe and discuss the arising deformation problems is provided in full generality. The work presents a new definition for the deformation of right modules with respect to a deformed algebra. This discussion in the sense of Gerstenhaber’s deformation theory \[52\] extends the purely algebraic notions in the works of Donald and Flanigan \[38\] and Yau \[130\]. In the course of the considerations it turns out that the notion of deformed module structures consisting of bidifferential operators is a subtle point. The aspired local formulas are obtained if the deformed module structure is a formal series of one-cochains of the mentioned Hochschild complex. The developed tools and techniques to perform the crucial computation of the cohomology are also formulated in the most general way and may possibly serve for further investigations of similar problems. With respect to the commutant, this means the algebra of endomorphisms of the functions on the total space respecting their deformed module structure, the vanishing first cohomologies yield the following result.

Depending on the geometric choices of a principal connection and an always existing invariant and torsion-free covariant derivative on the total space of the principal fibre bundle which respects the vertical bundle and which is related to a torsion-free covariant derivative on the base manifold, the commutant of a deformation quantization of a principal fibre bundle within the differential operators is isomorphic to the formal power series of vertical differential operators. In consequence, this induces an invariant associative deformation of the algebra of vertical differential operators and an invariant deformation of the classical left module structure of the functions on the total space with respect to the commutant. For a fixed star product on the base manifold, all deformations in the resulting bimodule structure are unique up to invariant equivalence. Moreover, the two occurring algebras turn out to be mutual commutants.

For surjective submersions one obtains an analogous assertion where the structures are not invariant. It is a basic feature of any principal fibre bundle that a representation of its structure group on a finite dimensional vector space induces an associated vector bundle. Concerning this point the above result establishes a connection to the works \[25\,120\,122\,125\] of Bursztyn and Waldmann. The well-known isomorphism of invariant vector-valued functions on the principal fibre bundle and the sections of the associated vector bundle lead to the following observation.

Every deformation quantization of a principal fibre bundle induces a deformation quantization of any associated vector bundle. Moreover, one always obtains a surjective algebra morphism from the deformed algebra of vertical differential operators to the commutant of the deformed associated vector bundle.

Motivated by the aspired applications, the work presents a generalization of the above statements. The notions concerning the deformation with respect to the functions on the total space can also be defined for the sections of arbitrary vector bundles over the total space which, in the case of principal fibre bundles, are assumed to be equivariant vector bundles. The above results
then induce the following generalization.

The existence and uniqueness up to equivalence can be proved in the same way for the right module of sections of any (equivariant) vector bundle over the total space. The assertion in particular holds for horizontal differential forms on a principal fibre bundle. Depending on the above geometric choices the invariant module deformation then induces a corresponding deformation of the module structure of the forms on the base manifold with values in any associated vector bundle. Although these results can be traced back to the above fundamental notion of deformation quantization it is convenient in the main text to investigate the sections and regard the functions as a special case having further properties. Some of the mathematical results of this work, basically those for the module of functions, have already been published in the article [19] which will appear in the Journal für die reine und angewandte Mathematik (Crelle’s Journal).

Applications and outlook

Due to the fact that surjective submersions and principal fibre bundles are basic geometric structures that are omnipresent in differential geometry, the above results have various applications in mathematical physics.

- For the aspired geometric formulation of gauge theories on noncommutative space-times the above results provide a first step towards a better understanding of the algebraic structures which are crucial for the theory. All the occurring module structures with respect to the function algebra on the space-time allow an adapted deformation for any given star product. The matter fields of a field theory, given as the module of sections on an associated vector bundle can now be deformed in the gauge theoretic setting. The Lie algebra of infinitesimal gauge transformations which is required to respect the module structure then has to be extended to the commutant within the differential operators. This is the geometric and global analogue of the local observations made by Jurčo, Schraml, Schupp and Wess in [78]. It turns out in the geometric formulation that the gauge potentials, given by the principal connections are an affine vector space and have no direct module structure like the matter fields. This is only the case for the underlying vector space whose module structure can be deformed. The physical interpretation of these observations is still unclear. The role of gauge potentials and force fields in the noncommutative setting has to be investigated in more detail. In the symplectic case the adapted Fedosov construction of the deformed module structure for the matter fields reveals a functorial dependence of the necessary choice of a gauge potential, confer [126]. This dependence shows a nice behaviour under local gauge transformations and gives rise to the conjecture that the Seiberg Witten maps for the gauge potentials in the global geometric framework could possibly be seen as the map assigning a deformed module structure to any gauge potential. In order to investigate this in full generality it would be desirable to find a more explicit construction of the module structures in the slightly more realistic case of space-times with a Poisson structure. For the formulation of a complete gauge theory one further has to clarify the notion of deformed fibre metrics as done for vector bundles in [25]. Then the newly defined actions should be invariant under the noncommutative analogues of gauge transformations.

- The structure of a principal fibre bundle also occurs in the context of phase space reduction where the found results provide a deeper understanding of the quantization of the Marsden-Weinstein reduction [1 Sect. 4.3]. If a Lie group acts on a phase space by a Hamiltonian group action with an equivariant momentum map it also acts on the surface determined by the momentum level zero. In the case of a free and proper action this surface is the total space
of a principal bundle over the quotient space given by the orbits. This base manifold inherits a Poisson structure of the initial phase space and is called the reduced phase space. In the framework of deformation quantization one is then usually interested in star products of the initial phase space inducing star products on the reduced one. This problem has already been discussed using the BRST formalism [12,18,70] or more direct constructions of the reduced star product, confer [15,16,56]. Another approach is provided by the fact that the functions on the momentum level surface are a bimodule with respect to the two algebras of functions on the two phase spaces. The left and right module structures are induced by the pullbacks of the embedding map of the surface and the bundle projection. If it is possible to deform the left module structure with respect to a given star product such that the corresponding commutant is isomorphic to the formal series of functions on the reduced phase space, the latter automatically inherits a reduced star product. In [13,14] this approach is presented together with first results in the symplectic case. For general Poisson manifolds there have been found sufficient conditions for a successful reduction [28,30,31]. The results of this work now yield a contribution to this problem from the opposite direction. For every star product on a reduced phase space it is possible to construct a right module structure for the functions on momentum level surface. Together with the commutant one even obtains a deformed bimodule structure which is unique up to equivalence.

- As discussed in the article [19] the results also motivate further investigations with respect to Morita theory.

**Outline of the work**

The present work is organized as follows.

- **Chapter 1** provides a short introduction to the concepts of deformation quantization and it is explained in which sense a star product represents and describes the noncommutative aspects of space-time at the Planck scale. A discussion of the algebraic structures occurring in the geometric formulation of a classical gauge theory will then lead to the basic definitions.

- A general and detailed investigation of deformations of right module structures with respect to deformed algebras is the content of Chapter 2. The new algebraic notion is presented after a detailed discussion of the Hochschild complexes of algebras and modules. The cohomology groups of these complexes are then identified to encode the obstructions for the orderwise construction of such deformations and the corresponding equivalence transformations. These results can moreover be obtained for structures of particular types having further specific properties that should be respected by corresponding deformations. It is further shown that trivial cohomology groups not only imply the existence and uniqueness up to equivalence of deformed module structures but also allow a computation of the corresponding commutants which give rise to deformed bimodules. For modules which are invariant under group actions a separate discussion reveals further general results. Using an explicit homotopy it is finally shown that the obstructions for projective modules always vanish.

- **Chapter 3** starts with a summary of the well-known facts concerning the algebraic notion of multidifferential operators. This is necessary to define the notion of differential algebra and module structures which then are seen to be examples of particular types.

- In Chapter 4 it is pointed out how the general concepts of presheaves and sheaves encoding the relations between global and local data, can be used to compute (invariant) differential Hochschild cohomologies.
• It is well-known from homological algebra that Hochschild cohomologies are certain values of derived functors and that this fact can be used to compute the cohomologies by doing it for certain other complexes. Motivated by this purely algebraic situation it is shown in Chapter 5 that such a result can also be obtained for a more specific situation. The differential Hochschild cohomologies of the algebra of smooth functions on a convex subset of $\mathbb{R}^n$ with values in certain bimodules are isomorphic to cohomologies involving the topological versions of the bar and Koszul resolutions. This result does not follow from abstract arguments. All the isomorphisms have to be established by explicitly given maps whose construction is a nontrivial issue.

• The central result of this work is proved in Chapter 6. The Hochschild cohomologies which are crucial for the deformation quantization of surjective submersions and principal fibre bundles are computed explicitly and shown to be trivial. Based on the results of the previous chapters the computation makes use of a locally given homotopy. Besides proving the existence and the uniqueness up to equivalence the obtained result also presents a way to construct the considered structures explicitly.

• The investigation of the commutant within the differential operators is the basic concern of Chapter 7. The obtained results are based on adapted versions of the symbol calculus of differential operators and the existence of particular covariant derivatives. This is moreover used to show that there always exist deformations which respect a further algebraic property.

• In Chapter 8 it is shown that each deformation quantization of a principal fibre bundle induces a deformation quantization of any associated vector bundle.

• The two Appendices A and B provide the necessary basics of differential geometry and homological algebra which are used in the main text. Appendix C contains a technical but simple proof which has been omitted in Chapter 5.
Chapter 1

Deformation quantization and classical gauge theories

As mentioned in the introduction, it is the main motivation of this work to investigate how the methods of deformation quantization can be used for a geometrically meaningful discussion and formulation of classical gauge theories on noncommutative space-times. First of all, it has to be clarified in which sense a manifold equipped with a star product can be interpreted as a noncommutative space-time. This will become clear after a short review of the basic notions and results of deformation quantization. Based on the obtained interpretation of a star product algebra, the geometric formulation of classical gauge theories in terms of principal fibre bundles will then be investigated. For the implementation of the new noncommutative structure into the theory it turns out that the crucial right module structures have to be adapted. This will finally lead to the definition of a deformation quantization of principal fibre bundles.

1.1 Deformation quantization and noncommutative space-times

In order to formulate the definition of a star product and all other relevant structures occurring in the framework of deformation quantization, one needs the concept of formal power series, confer [87, Chap. IV, §9].

Definition 1.1.1 (Formal power series)
Let \( X \) be a module over a ring \( R \) and \( \lambda \) a letter. Then the set \( X[[\lambda]] \) of formal power series in the formal parameter \( \lambda \) with coefficients in \( X \) is given by the cartesian product \( X[[\lambda]] = \prod_{r=0}^{\infty} X_r \) with \( X_r = X \) for all \( r \in \mathbb{N}_0 \). In this context, all sequences \( (x_r)_{r \in \mathbb{N}_0} = (x_0, x_1, \ldots) \) in \( X \) are denoted as formal power series \( x = \sum_{r=0}^{\infty} \lambda^r x_r \).

Following this purely algebraic prescription one can extend rings, algebras and other algebraic structures to the formal power series. Analogously, one defines the application of formal power series of maps itself and one finds the isomorphism \( \text{Hom}_R(X,Y)[[\lambda]] \cong \text{Hom}_R[[\lambda]](X[[\lambda]],Y[[\lambda]]) \) for the \( R \)-linear maps between two modules \( X,Y \). For associative and unital algebras \( A \) with unit 1 it
is an important fact that every series \( a = \sum_{r=0}^{\infty} \lambda^r a_r \in \mathcal{A}[[\lambda]] \) starting with an invertible element \( a_0 \) is invertible. Writing \( a = a_0 + \lambda b \) with \( b \in \mathcal{A}[[\lambda]] \) the element 
\[
a^{-1} = a_0^{-1} \sum_{s=0}^{\infty} (\lambda ba_0^{-1})^s \in \mathcal{A}[[\lambda]]
\] (1.2)
satisfies \( aa^{-1} = 1 = a^{-1}a \), confer [102, Lemma 2.2.2]. Further details and discussions of formal power series can for example be found in [103, App. A] and [123, Bem. 4.2.36, Sect. 6.2.1]. Within this framework the definition of a star product can be formulated as follows, confer [8].

**Definition 1.1.2 (Formal star product)**

Let \( M \) be a smooth manifold and \( C^\infty(M) \) denote the smooth complex-valued functions. Then a formal star product is an associative \( \mathcal{C}[[\lambda]] \)-bilinear product \( \star \) for \( C^\infty(M)[[\lambda]] \) of the form 
\[
a \star b = \sum_{r=0}^{\infty} \lambda^r C_r(a, b)
\] (1.3)
for all \( a, b \in C^\infty(M)[[\lambda]] \) with \( \mathcal{C} \)-bilinear maps \( C_r : C^\infty(M) \times C^\infty(M) \to C^\infty(M) \) such that

i.) \( C_0(a, b) = a \cdot b \) is the pointwise product of functions and

ii.) \( 1 \star a = a = a \star 1 \) which means that \( C_r(1, a) = 0 = C_r(a, 1) \) for all \( r \geq 1 \).

Two star products \( \star \) and \( \tilde{\star} \) on \( M \) are said to be equivalent if there exists a formal series \( S = id_{C^\infty(M)} + \sum_{r=1}^{\infty} \lambda^r S_r \) of linear maps \( S_r : C^\infty(M) \to C^\infty(M) \) such that 
\[
S(a \star b) = Sa \tilde{\star} Sb \quad \text{and} \quad S(1) = 1.
\] (1.4)

A star product \( \star \) is called differential if the \( C_r \) are bidifferential operators. Then, the equivalence transformations \( S \) have to consist of differential operators.

The associativity of a star product shows that the manifold \( M \) is necessarily a Poisson manifold with a Poisson bracket \( \{\cdot, \cdot\} \) defined by 
\[
C_1(a, b) - C_1(b, a) = i\{a, b\},
\] (1.5)
where \( i \in \mathbb{C} \) is the imaginary unit.

The existence of star products on symplectic manifolds was independently proved by DeWilde and Lecomte [37], and Fedosov [48]. A further contribution was given by Omori, Maeda and Yoshioka [106]. With his famous formality theorem Kontsevich showed that there exist star product for any Poisson manifold such that (1.5) holds for the given Poisson bracket. Moreover, this approach clarified the classification of star products with respect to the given notion of equivalence. Previously, this was done for the symplectic case by Bertelson, Cahen and Gutt [10], Gutt and Rawnsley [63], as well as by Nest and Tsygan [101]. The topic was further pursued by Deligne [36] and Neumaier [103,104].

The notion of star products first arose in the context of deformation quantization as established by Bayen, Flato, Frønsdal, Lichnerowicz, and Sternheimer [8]. It is a remarkable physical phenomenon that the fundamental theory of quantum mechanics has a classical limit. Due to this fact we only have a physical intuition for the observables in classical mechanics, given by the real-valued functions in the Poisson algebra \( (C^\infty(M), \{\cdot, \cdot\}) \) of functions on a Poisson manifold \( M \). In contrast to this classical situation the observables in quantum mechanics are defined as the self-adjoint elements in the noncommutative algebra of operators on a Hilbert space. The reason
1.2 The geometric formulation of a classical gauge theory

for this generally accepted axiomatic point of view can be seen in Heisenbergs uncertainty relations

\[ \Delta Q^i \Delta P^j \geq \frac{\hbar}{2} \delta^{ij} \]

for position and momentum which are satisfied if the corresponding observables \( Q^i \) and \( P^j \) in quantum mechanics satisfy the commutation relation \( [Q^i, P^j] = i\hbar \delta^{ij} \). The resulting quantization problem, namely to construct the quantum theory for any arbitrarily given classical system and to establish a relation between the observables is not trivial. In general, there exists no canonical way of quantization that in particular provides the aspired relation between the Poisson bracket \( \{\cdot, \cdot\} \) for the classical observables and the commutator \( \frac{\hbar}{i}[\cdot, \cdot] \) for the assigned operators. Deformation quantization in its original intention is a geometrically motivated approach to this problem. A star product deforming a Poisson bracket is seen as the noncommutative algebraic structure of the observables in quantum mechanics expressed in terms of the classical functions. Due to the fact that

\[ [a, b] = a \star b - b \star a = i\lambda \{a, b\} + O(\lambda^2) \] (1.6)

it is clear that the deformation parameter can be interpreted as Planck constant \( \hbar = 6.6 \times 10^{-34} Js \). According to the considerations and results of Doplicher, Fredenhagen, and Roberts in [39, 40] the space time coordinates \( x^\mu, \mu = 0, \ldots, 3 \), of an event in the usual Minkowski space-time \( M = \mathbb{R}^4 \) are subjected to the uncertainty relations

\[ \Delta x^0 \sum_{i=1}^3 \Delta x^i \geq l_P^2 \quad \text{and} \quad \sum_{j<k=1}^3 \Delta x^j \Delta x^k \geq l_P^2. \] (1.7)

The occurring Planck length \( l_P = \sqrt{\gamma \hbar c^3} = 1.6 \times 10^{-35} m \) is determined by the gravitational constant \( \gamma \), the Planck constant \( \hbar \) and the velocity \( c \) of light and can be seen as the scale where the predictions of classical mechanics and general relativity lead to contradictions and where a new and more fundamental model of space-time is necessary. As shown in the mentioned works the uncertainty relations can be satisfied in a specific framework where the coordinate functions \( x^\mu \) are replaced by certain generators \( q^\mu \) of an abstract algebra satisfying commutation relations of the form

\[ [q^\mu, q^\nu] = i\hbar \Theta^{\mu\nu} \] (1.8)

with a constant Poisson tensor \( \Theta^{\mu\nu} \). Again, the problem is to construct the fundamental theory of quantum space-time out of its well-known and obviously existing classical limit. Completely analogous to the above quantization problem one has to replace the commutative algebra of functions on the space-time manifold \( M \) by a noncommutative one. With this analogy it is obvious that the general framework of deformation quantization can also be used to describe the physical effects of such a noncommutative space-time in terms of the well-known one. The only additional problem is that there is no physical evidence for a globally defined Poisson structure, confer the discussions in [5, 69, 124]. Nevertheless, in order to investigate the impact of a noncommutative space-time structure on field theories it is still adequate to work with noncommutative space-times that are given by a smooth manifold \( M \) together with a differential star product. The basic algebraic structure for further investigations is thus the star product algebra \( (C^\infty(M)[[\lambda]], \star) \).

1.2 The geometric formulation of a classical gauge theory

It has become evident in the last century that classical gauge theories have a simple and clear geometrical formulation which shall be outlined in the following. Details, concrete applications
and further discussions of the well-known topic can be found in [11, 35, 97, 100, 111, 126]. It is assumed that the reader is familiar with the necessary concepts of differential geometry which can also be found in the above references or in the most books on differential geometry, for instance [80, 81, 88, 94]. Appendix A contains a summary of the most important facts together with an explanation of the used notation.

For the description of the kinematics of a classical gauge theory one needs the following crucial data.

i.) A principal fibre bundle \( p : P \rightarrow M \) with structure Lie group \( G \) and principal right action \( r \).

ii.) A representation \( \pi : G \rightarrow \text{Aut}(V) \) of \( G \) on a finite dimensional vector space \( V \) from the left.

These fundamental structures have the following interpretation. The base manifold \( M \) plays the role of the space-time. Thus it is equipped with a lorentzian metric \( h \), which in particular is a non-degenerate symmetric tensor field \( h \in \Gamma^\infty(M, S^2 TM) \). The total space \( P \) is sometimes referred to as the space of generalized phase factors. Considering the fibres one can say that at any point \( p \in M \) there is an additional degree of freedom which is strongly related with the group \( G \). The name comes from classical electrodynamics where the elements of the relevant group \( G = U(1) = \{ e^{i\Theta} \mid \Theta \in (0, 2\pi] \} \) are determined by the phase \( \Theta \). The structure group \( G \) is the internal symmetry group of the theory and controls the gauge fields by the representations on the vector space \( V \) and the Lie algebra \( \mathfrak{g} \) of \( G \). The meaning of this will become clear with the following notions.

In the global geometric context the \( G \)-invariant vector-valued functions

\[
f \in C^\infty(P, V)^G
\]  

satisfying \( f \circ r_g = \pi_{g^{-1}} \circ f \) for all \( g \in G \) are interpreted as the matter or particle fields of the theory. The principal connections in the form of their connection one-forms

\[
\omega \in \mathfrak{c} \subseteq (\Gamma^\infty(P, T^* P) \otimes \mathfrak{g})^G,
\]  

which are \( \mathfrak{g} \)-valued one-forms that are \( G \)-invariant with respect to the adjoint representation, \( r^*_g \omega = \text{Ad}_{g^{-1}} \omega \), and reproduce the generators of the fundamental vector fields, confer Definition A.3.3, are seen as the gauge potentials. The induced covariant exterior derivative \( d_\omega \) of such a connection then encodes the description of the changing rate of any field. Using this derivative one obtains the corresponding curvature forms

\[
\Omega = d_\omega \omega \in (\Gamma^\infty_{\text{hor}}(P, \bigwedge^2 T^* P) \otimes \mathfrak{g})^G,
\]  

which are horizontal and invariant \( \mathfrak{g} \)-valued two-forms. They are seen as the force fields or field strength tensors. An immediate consequence of the above definition are the structure equation and the Bianchi identity,

\[
\begin{align*}
\Omega &= \quad d_\omega \omega = d \omega + \frac{1}{2} [\omega, \omega]_\wedge, \\
d_\omega \Omega &= \quad 0,
\end{align*}
\]  

where \([ , , ]_\wedge\) is the bracket for \( \mathfrak{g} \)-valued forms defined in (A.27). The covariant exterior derivative further satisfies

\[
\begin{align*}
d_\omega f &= \quad d f + \pi'(\omega) f, \\
d_\omega \alpha &= \quad d \alpha + [\omega, \alpha]_\wedge
\end{align*}
\]
for all $f \in C^\infty(P,V)^G$ and $\alpha \in (\Gamma^\infty(P,\mathfrak{N}T^*P) \otimes \mathfrak{g})^G$. The gauge group or group of gauge transformations is given by the group of $G$-invariant $G$-valued functions

$$C^\infty(P, G)^G$$

with respect to the action of the Lie group $G$ on itself via conjugation. An element $H \in C^\infty(P, G)^G$ thus satisfies $H \circ r_g = \text{Conj}_{g^{-1}} \circ H$. The group structure is the naturally induced via $(H_1 H_2)(u) = H_1(u) H_2(u)$ for all $u \in P$. Correspondingly, the gauge algebra or (Lie) algebra of infinitesimal gauge transformations is given by the algebra of $G$-invariant $\mathfrak{g}$-valued functions

$$C^\infty(P, \mathfrak{g})^G$$

with respect to the adjoint action. Thus one has $\Xi \circ r_g = \text{Ad}_{g^{-1}} \circ \Xi$ for such $\Xi \in C^\infty(P, \mathfrak{g})^G$. Again, the Lie bracket is the one induced by the Lie bracket on $\mathfrak{g}$, $[\Xi_1, \Xi_2](u) = [\Xi_1(u), \Xi_2(u)]$.

Remark 1.2.1 (Gauge group and gauge algebra)

i.) There is a group isomorphism

$$\Phi : C^\infty(P, G)^G \rightarrow \text{Gau}(P) = \{\tau : P \rightarrow P \mid \tau \circ r_g = r_g \circ \tau \text{ and } p \circ \tau = p\}$$

between the gauge group and the gauge transformations $\text{Gau}(P)$ of the principal fibre bundle which is denoted and defined by

$$\Phi_H(u) = (\Phi(H))(u) = r_{H(u)} u = u. H(u)$$

for all $H \in C^\infty(P, G)^G$ and $u \in P$.

ii.) There is a vector space isomorphism

$$\phi : C^\infty(P, \mathfrak{g})^G \rightarrow \mathfrak{gau}(P) = \Gamma^\infty(P, V P)^G = \{V \in \Gamma^\infty(P, TP) \mid r_g^* V = V \text{ and } Tp \circ V = 0\}$$

between the gauge algebra and the infinitesimal gauge transformations of the principal fibre bundle which is given by

$$\phi(\Xi) = \Xi_p, \quad \Xi_p(u) = \frac{d}{dt} \bigg|_{t=0} u. \exp(t \Xi(u))$$

for all $\Xi \in C^\infty(P, \mathfrak{g})^G$. In addition, the isomorphism is an anti-homomorphism of Lie algebras,

$$[\Xi_1, \Xi_2]_P = -[(\Xi_1)_P, (\Xi_2)_P].$$

iii.) The notion infinitesimal comes from the fact that a vector field $V \in \Gamma^\infty(P, TP)$ is an infinitesimal gauge transformation in $\mathfrak{gau}(P)$ if and only if its flow $\text{Fl}^V_t \in \text{Gau}(P)$ is a gauge transformation for all possible $t \in I$ with some $I \subseteq \mathbb{R}$. Under the above isomorphisms this relation is encoded in the generalized exponential map $\exp : C^\infty(P, \mathfrak{g})^G \rightarrow C^\infty(P, G)^G$ from Appendix A.3.2 since

$$\text{Fl}^\Xi_t = \Phi_{\exp(t \Xi)}$$

and $\frac{d}{dt}|_{t=0} \exp(t \Xi(u)) = \Xi(u)$ for the one-parameter group $\exp(t \Xi)$. 

The (local) gauge transformations of the fields are given by the natural action of the gauge group \( \text{Gau}(P) \) via pullback. Then all structures yield transformed ones of the same type. For \( H \in C^\infty(P,G)^G \) the tangent map of \( \Phi_H \in \text{Gau}(P) \) is given by

\[
T_u \Phi_H Z_u = T_u l_{H(u)} Z_u + (T_{H(u)} l_{H^{-1}(u)} T_u H Z_u)_p (\Phi_H(u))
\]

for all \( Z_u \in T_u P \) and with the left multiplication \( l_g h = gh \) in \( G \). Thus one finds the following gauge transformations

\[
\begin{align*}
f & \mapsto f' = \Phi_{H^{-1}} f = \pi(H) f \\
H : \omega & \mapsto \omega' = \Phi_{H^{-1}} \omega = \Lambda_H \omega + \delta H^{-1} \\
\Omega & \mapsto \Omega' = \Phi_{H^{-1}} \Omega = \Lambda_H \omega
\end{align*}
\]

for all matter fields \( f \in C^\infty(P,V)^G \), gauge potentials \( \omega \in \mathcal{C} \) and force fields \( \Omega \in (\Gamma_{\text{hor}}^\infty(P,T^*P))^G \). The expression \( \delta H^{-1} : TP \to \mathfrak{g} \) is the left logarithmic derivative of \( H^{-1} \) defined by \( \delta H^{-1}(Z_u) = T_{H^{-1}(u)} l_{H(u)} T_u H^{-1} Z_u \). Further, it is \( (\pi(H) \alpha)_u = \pi(H(u)) \circ \alpha_u \) for each vector-valued form \( \alpha \in \Gamma^\infty(P,\mathfrak{g} T^*P) \otimes V \) with a corresponding representation. The pullback of a connection one-form \( \omega \in \mathcal{C} \) indeed yields \( \omega' \in \mathcal{C} \) with curvature \( \Omega' = d\omega \omega' \). The gauge covariance of the covariant derivative \( d_\omega \) is encoded in the fact that

\[
(d_\omega \alpha)' = d_\omega' \alpha'.
\]

The infinitesimal (local) gauge transformations with respect to a function \( \Xi \in C^\infty(P,\mathfrak{g})^G \) are obtained by considering the gauge transformation with respect to the one-parameter group \( H_t = \exp(t\Xi) \) and to take the derivative \( \frac{d}{dt}\big|_{t=0} \). The infinitesimal version of (1.25) then is

\[
\begin{align*}
f & \mapsto \delta_\Xi f = - \mathcal{L}_\Xi f = \pi'(\Xi) f \\
\Xi : \omega & \mapsto \delta_\Xi \omega = \text{ad}\omega - d\omega = [\Xi,\omega] - d\omega \\
\Omega & \mapsto \delta_\Xi \Omega = - \mathcal{L}_\Xi \Omega = \text{ad}(\Xi)\Omega.
\end{align*}
\]

There, \( \mathcal{L}_\Xi = \frac{d}{dt}\big|_{t=0} (F_t|_{\mathfrak{g}})^* \) denotes the Lie derivative in direction \( \Xi \in \mathfrak{g} \) and \( \pi' : \mathfrak{g} \to \text{End}(V) \) is the induced Lie algebra representation. For all vector-valued forms one again sets \( (\pi'(\Xi)\alpha)_u = \pi'(\Xi(u))\alpha_u \). For the horizontal forms \( \alpha \in \Gamma_{\text{hor}}^\infty(P,\mathfrak{g} T^*P) \otimes V \) one additionally has

\[
[\delta_{\Xi_1}, \delta_{\Xi_2}]\alpha = \delta_{[\Xi_1,\Xi_2]}\alpha.
\]

The infinitesimal gauge transformations \( \delta_\Xi \) only yield elements \( \delta_\Xi \alpha \) of the same type if the corresponding fields give rise to a vector space. This is not true for the gauge potentials. The principal connections are only an affine space over the invariant and horizontal \( \mathfrak{g} \)-valued one forms. Thus one has

\[
\mathcal{C} \ni \omega \mapsto \delta_\Xi \omega \in T_\omega \mathcal{C} = (\Gamma_{\text{hor}}^\infty(P,T^*P) \otimes \mathfrak{g})^G.
\]

In order to reproduce the formulas physicists are familiar with, one makes use of a local gauge which is nothing but a local section

\[
\sigma \in \Gamma^\infty(U,\mathcal{P}|_U)
\]

of the principal fibre bundle over an open set \( U \subseteq M \). Then one defines

\[
\begin{align*}
\phi &= \sigma^* f \in C^\infty(U,V) \\
A &= \pi' \circ \sigma^* \omega \in \Gamma^\infty(U,\mathfrak{g} T^*U) \otimes \text{End}(V) \\
F &= \pi' \circ \sigma^* \Omega = \pi' \circ (d\sigma^* \omega + \frac{1}{2}[\sigma^* \omega, \sigma^* \omega]) \in \Gamma^\infty(U,\mathfrak{g}^2 T^*U) \otimes \text{End}(V) \\
\mathcal{U} &= \pi \circ \sigma^* H \in C^\infty(U,\text{Aut}(V)) \\
\mathcal{U} &= \pi' \circ \sigma^* \Xi \in C^\infty(U,\text{End}(V))
\end{align*}
\]
for all \( f \in C^\infty(P,V)^G \), \( \omega \in \mathfrak{e} \), \( H \in C^\infty(P,G)^G \), and \( \Xi \in C^\infty(P,\mathfrak{g})^G \). Then the structure equation is \( F = dA + A \wedge A \) and the Bianchi identity reads \( dF = F \wedge A - A \wedge F \) where, for instance, 
\[(A \wedge A)(X,Y) = A(X) \circ A(Y) - A(Y) \circ A(X)\]
for all \( X,Y \in \mathcal{X}(U,TU) \). The local and infinitesimal local gauge transformations \( \text{(1.25)} \) and \( \text{(1.27)} \) now have the form
\[
\phi \mapsto \Phi \phi \quad \text{and} \quad \phi \mapsto \Phi \phi \quad \text{for} \quad A \mapsto \Phi A \quad \text{and} \quad \Phi : \quad F \mapsto \Phi F \quad \text{and} \quad F \mapsto [\Phi,F].
\]
Equation \( \text{(1.14)} \) yields \( \sigma^* d\omega f = (d + A)\sigma^* f = (d + A)\phi \) where \( (A\phi)_p(X_p) = A_p(X_p)(\phi(p)) \). Usually one chooses a basis \( \{e_a\}_{a=1,\ldots,\dim G} \) of the Lie algebra \( \mathfrak{g} \) such that \( [e_a,e_b] = C^c_{ab}e_c \) with the structure constants \( C^c_{ab} \) and sets \( T_a = \pi(e_a) \). Here and in the following we use Einstein’s summation convention. Then one obviously has \( \omega = \omega^a e_a \) and \( \Omega = \Omega^a e_a \) with simple differential forms \( \omega^a \in \Gamma^\infty(P,T^*P) \) and \( \Omega^a \in \Gamma^\infty(P,\mathfrak{N}T^*P) \) and one defines \( A^a = \sigma^* \omega^a \in \Gamma^\infty(U,T^*U) \) if \( U \subseteq M \) is the domain of a chart \( x : U \to x(U) \subseteq \mathbb{R}^n \) the forms have a \( C^\infty(U) \)-module basis generated by the \( d x^\mu \) for \( \mu = 1,\ldots,n = \dim M \). This yields
\[
A = (\sigma^* \omega^a)\pi'(e_a) = A^a T_a = A^a_\mu d x^\mu T_a \tag{1.37}
\]
with \( A^a_\mu \in C^\infty(U) \). Analogously,
\[
F = \frac{1}{2} F^a_{\mu \nu} d x^\mu \wedge d x^\nu T_a \tag{1.38}
\]
and \( \text{(1.12)} \) yields the well-known formula
\[
F^a_{\mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + C^a_{bc} A^b_\mu A^c_\nu. \tag{1.39}
\]
The fields \( A_\mu = A^a_\mu T_a \) and \( F_{\mu \nu} = F^a_{\mu \nu} T_a \in C^\infty(U,\text{End}(V)) \) which have the same transformation behaviour as \( A \) and \( F \) then are the gauge and force fields physicists work with. Moreover, with the \( \text{def} \)inition \( \sigma^* d\omega f = D_\mu(\sigma^* f) \) \( d x^\mu \) and \( \text{(1.14)} \) one finds the usual covariant derivative
\[
D_\mu = \partial_\mu + A^a_\mu T_a = \partial_\mu + A_\mu \tag{1.40}
\]
for the matter fields \( \phi \). Analogously, for \( \mathfrak{g} \)-valued forms \( \alpha \in (\Gamma^\infty(P,\mathfrak{N}T^*P) \otimes \mathfrak{g})^G \) with \( \pi' \circ \sigma^* \alpha = \alpha^a_{\mu_1,\ldots,\mu_k} d x^{\mu_1} \wedge \cdots \wedge d x^{\mu_k} T_a \) one sets \( \pi' \circ \sigma^*(d\omega \alpha) = d_{\pi'} \alpha_{\mu_1,\ldots,\mu_k}^a d x^{\mu_1} \wedge d x^{\mu_1} \wedge \cdots \wedge d x^{\mu_k} T_a \) and obtains
\[
D^a_{\mu_1,\ldots,\mu_k} = \partial_\mu \delta^a_\mu - C^a_{bc} A^c_\mu. \tag{1.41}
\]
With \( \text{(1.15)} \) the Bianchi identity \( \text{(1.13)} \) implies the homogeneous field equations
\[
\sum_{zyk(\mu \nu)} D^a_{\mu,\nu} = 0. \tag{1.42}
\]
for all \( a = 1,\ldots,\dim G \).

**Remark 1.2.2 (Introduction of charges)**

In physical applications one has to take care of the dimensions. By convention one thus introduces the coupling constants which have the interpretation of charges. In electrodynamics one sets \( D_\mu = \partial_\mu + ie A_\mu \) where \( e \) is the electric charge.

**Remark 1.2.3 (Lagrangians, field equations and interactions)**

1) In order to find the dynamics of such a gauge theory one needs some more geometrical data. All this can be found in \( \text{(1.1)} \) and shall only be outlined briefly. A \( G \)-invariant Lagrangian \( L \) yields a function \( L_0 : C^\infty(P,V)^G \to C^\infty(M) \) of the form \( L_0(f) = f(L(f,d f) \)
with \( L(\pi_\theta f, \pi_\theta \mathrm{d} f) = L(f, \mathrm{d} f) \) which in physics literature is referred to as **invariance under global gauge transformations**. The demand for invariance under local gauge transformations is satisfied by

\[
\mathcal{L}(f, \omega) = L(f, \mathrm{d}_\omega f)
\]

(1.43) involving the gauge potential \( \omega \). The metric \( h \) of the space-time \( M \) and a \( G \)-invariant metric \( k \) on the Lie algebra \( \mathfrak{g} \), \( k(\mathrm{Ad}_g \xi, \mathrm{Ad}_g \eta) = k(\xi, \eta) \), induce a metric \( hk \) on the horizontal \( \mathfrak{g} \)-valued forms on \( P \). This gives rise to the **self-action density**

\[
S(\omega) = -\frac{1}{2}hk(d_\omega \omega, d_\omega \omega) = -\frac{1}{2}\|\Omega\|^2.
\]

(1.44)

Taking a volume density \( \mu \) with respect to \( h \) one can define the **action** \( \int_U (\mathcal{L} + S)(f, \omega)\mu \) over any open subset \( U \subseteq M \) with compact closure. The **principle of least action** which claims that \( f \) and \( \omega \) are stationary in the sense that

\[
\frac{d}{dt} \bigg|_{t=0} \int_U (\mathcal{L} + S)(f + tf', \omega + t\alpha)\mu = 0
\]

(1.45) for all \( f' \in C^\infty(P,V)^G \) and \( \alpha \in (\Gamma_{\text{hor}}^\infty(P, T^*P) \otimes \mathfrak{g})^G \), then is equivalent to the **Lagrange equation** for the matter field \( f \) and the **inhomogeneous field equation** for the gauge field \( \omega \).

ii.) In order to find physically relevant examples of matter fields and corresponding Lagrangians one needs the theory of **spin structures, spinors and Dirac operators**, confer [27, 32, 50] for the mathematical background and [88] for some physical considerations. On the Minkowski space, the Lagrangian typically is of the form

\[
\mathcal{L}_0(\phi) = \langle \partial_\mu \phi, \partial^\mu \phi \rangle - m^2 \langle \phi, \phi \rangle - V(\langle \phi, \phi \rangle)
\]

(1.46) where \( \langle \cdot, \cdot \rangle \) is some metric on the considered vector space \( V \). The first summand is seen as kinetic energy, the constant \( m \) is interpreted as mass and \( V \) is some interaction potential. The structure group of the considered principal bundle then is an appropriate subgroup \( G \) of the orthogonal group of \( \langle \cdot, \cdot \rangle \). For matrix Lie groups with the defining representation the self-interaction then is given by the trace,

\[
S(A) = -\frac{1}{2} \text{tr}(F_{\mu\nu}F^{\mu\nu}).
\]

(1.47)

In the physical interpretation the demand of local gauge invariance and the introduction of the covariant derivative in (1.43), usually referred to as **minimal coupling**, determine the interaction of the gauge potentials with the matter fields.

**Remark 1.2.4 (Associated vector bundles)**

It is well-known that the representation \( \pi : G \to \text{Aut}(V) \) of the structure group leads to an associated vector bundle \( q : E = P \times_G V \to M \), confer Appendix A.4.3. Then all structures and operations from above have a corresponding description in terms of the associated vector bundle. Some of the well-known facts are summarized in Proposition A.4.1. The matter fields can be seen as sections of this associated bundle due to the isomorphism \( s : C^\infty(P,V)^G \to \Gamma^\infty(M, E) \) and the gauge potentials and force fields give rise to covariant derivatives \( \nabla^E \) on \( E \) with curvatures \( \mathcal{R}^E \). Further, an element \( H \in C^\infty(P,G)^G \) of the gauge group induces a vector bundle automorphism \( \hat{H} \in \text{Aut}(E) \), this means a fibrewise linear diffeomorphism \( \hat{H} : E \to E \) with \( q \circ \hat{H} = q \), by \( \hat{H}[u, v] = [u, \pi(H(u))v] = [\Phi_H(u), v] \) for all equivalence classes \( [u, v] \in P \times_G V \). Analogously, an element \( \Xi \in C^\infty(P, \mathfrak{g})^G \) of the gauge algebra induces a section \( \hat{\Xi} \in \Gamma^\infty(M, \text{End}(E)) \) in the endomorphism bundle of \( E \) by \( \hat{\Xi}[u, v] = [u, \pi'(\Xi(u))v] \). With the isomorphism \( s \) one finds that \( \hat{H} \circ s(f) = s(\pi(H)f) \).
1.3 Module structures in classical gauge theories

In order to investigate the impact of a noncommutative space-time structure on the above formulation in the aspired framework of deformation quantization, it is necessary to identify the crucial algebraic structures of a gauge theory which are in contact with the algebra of functions on the space-time. Considering a star product algebra \((C^\infty(M)[[\lambda]], \star)\) as the new noncommutative version of the space-time, the idea then is to adapt the related structures by an appropriate deformation in order to maintain the algebraic structures which are assumed to be the fundamental content of the theory. From the purely algebraic point of view it is clear that the simplest structures related to an algebra are corresponding modules. Indeed, the above formulation points out that certain right modules have an important physical interpretation.

The matter fields, given by invariant vector valued functions \(C^\infty(P, V)\) on the principal fibre bundle are obviously a module with respect to the functions \(C^\infty(M)\) on the base with the pointwise multiplication making use of the pullback with the bundle projection. For \(f \in C^\infty(P, V)\) and \(a \in C^\infty(M)\) the new matter field \(fa\) is defined by

\[
(fa)(u) = (p^*a \cdot f)(u) = a(p(u))f(u)
\]  

for all \(u \in P\) where the multiplication on the right side is the scalar one of the vector space \(V\). The required \(G\)-invariance is clear by the property \(p \circ r_g = p\) and the defining property of the representation \(\pi\). Due to the commutativity of the pointwise product of functions this is as well a left as a right module structure but as indicated by the notation it will be considered as a right module structure.

As explained in Remark 1.2.4 and Proposition A.4.1 the matter fields can also be seen as the sections \(\Gamma^\infty(M, P \times_G V)\) of an associated vector bundle and the considered isomorphism is a module isomorphism. The natural right module structure of the sections is the crucial algebraic property and any such finitely generated and projective module can be seen as the module of sections on a vector bundle. This well-known statement, confer the more detailed discussion in Section 8.1, points out the importance of the above right module structure which is obviously induced by the corresponding \(G\)-invariant right module structure of the functions \(C^\infty(P)\) itself. For \(f \in C^\infty(P)\) and \(a \in C^\infty(M)\) one defines \(fa = f \cdot p^*a \in C^\infty(P)\) and finds the property

\[
\delta r_g(fa) = (\delta r_g f)a.
\]  

Without regard to the \(G\)-invariance this basic module structure of functions also occurs in a much more general framework and only makes use of the projection \(p : P \rightarrow M\) which is a surjective submersion.

The local gauge transformations \(\Phi \in \text{Gau}(P)\) and the infinitesimal counterparts \(\Xi_P \in \Gamma^\infty(VP)^G\) which are applied to matter fields are obviously seen to respect the considered module structure and thus are elements of the so-called commutant. For the functions \(f \in C^\infty(P)\) one easily finds that

\[
\Phi^*(fa) = (\Phi^*f)a
\]  

and

\[
\delta\Xi_P(fa) = -\mathcal{L}\Xi_P(fa) = (-\mathcal{L}\Xi_P f)a = (\delta\Xi_P f)a
\]

for all \(a \in C^\infty(M)\).

The set \(\mathcal{C}\) of gauge potentials \(\omega\) has no such right module structure. It is an affine vector space over the horizontal and \(G\)-invariant \(\mathfrak{g}\)-valued one-forms \(\alpha \in (\Gamma^\infty_{\text{hor}}(P, T^*P) \otimes \mathfrak{g})^G\). This vector space and all higher differential forms of the same type are again modules with respect to \(C^\infty(M)\).
1.4 Deformation quantization and classical gauge theories

The aim to adapt the crucial $C^\infty(M)$-module structure of $C^\infty(P)$ to a given star product $\bullet$ on the manifold $M$ gives rise to the following definitions which were already given in the publication \[19\].

**Definition 1.4.1 (Deformation quantization of surjective submersions)**

Let $p : P \to M$ be a surjective submersion and $\star$ be a differential star product on $M$.

i.) A deformation quantization of the surjective submersion is a right $(C^\infty(M)[[\lambda]], \star)$-module structure $\bullet$ of $C^\infty(P)[[\lambda]]$ meaning that

$$f \bullet (a \star b) = (f \bullet a) \star b$$

for all $f \in C^\infty(P)[[\lambda]]$ and $a, b \in C^\infty(M)[[\lambda]]$ such that

$$f \bullet a = f : \mathcal{P}^* a + \sum_{r=1}^\infty \lambda^r \rho_r(f, a)$$

with $\mathbb{C}$-bilinear maps $\rho_r : C^\infty(P) \times C^\infty(M) \to C^\infty(M)$ which are bidifferential operators.

ii.) Two such deformations $\bullet$ and $\cdot$ are said to be equivalent if and only if there exists a formal series $T = \text{id}_{C^\infty(P)} + \sum_{r=1}^\infty \lambda^r T_r$ of differential operators $T_r \in \text{DiffOp}(C^\infty(P))$ such that for all $f \in C^\infty(P)[[\lambda]]$ and $a \in C^\infty(M)[[\lambda]]$

$$T(f \bullet a) = T(f) \cdot a.$$ (1.54)

The notion of bidifferential operators used for the definition means that the local expression of these operators shall always exist and that with respect to local charts they are bidifferential operators in the common sense of calculus. The global description of such maps is a subtle point and will be clarified in the course of the further investigations of these structures. Taking the $G$-invariance into account the following definition is natural.

**Definition 1.4.2 (Deformation quantization of principal fibre bundles)**

Let $p : P \to M$ be a principal fibre bundle with structure group $G$ and principal right action $r : P \times G \to P, r(u, g) = r_g u$, and $\star$ a differential star product on $M$.

i.) A deformation quantization of the principal fibre bundle is a $G$-invariant deformation quantization of the surjective submersion $p : P \to M$ with respect to $\star$, this means a right module structure $\bullet$ as in (1.52) and (1.53) with the additional property

$$r_g^*(f \bullet a) = (r_g^* f) \bullet a$$

for all $f \in C^\infty(P)[[\lambda]], a \in C^\infty(M)[[\lambda]],$ and $g \in G$.

ii.) Two such deformations $\bullet$ and $\cdot$ are said to be equivalent if they are equivalent in the sense of Definition 1.4.1 with $G$-invariant operators $T_r$ which means that in addition to (1.55) for all $g \in G$ one has

$$r_g^* T_r = T_r \circ r_g^*.$$ (1.56)

In the diploma thesis \[126\] the existence of deformation quantizations of principal fibre bundles was already shown for the special case of a symplectic base manifold $M$. There the star product is assumed to be a Fedosov star product \[18\] depending on a symplectic and torsion-free covariant derivative $\nabla^M$ on $M$ and a formal series $\Omega^M = \sum_{r=1}^\infty \lambda^r \Omega_r^M \in \lambda \Gamma^\infty(M, T^* M)[[\lambda]]$ of closed
two-forms, \(d\Omega^M = 0\). With an adapted version of the Fedosov construction for bimodules it is shown that there always exist corresponding deformation quantizations of any principal fibre bundle over \(M\). The construction and thus the right module structure itself further depend on the choice of a principal connection one-form \(\omega\) and an always existing \(G\)-invariant, torsion-free covariant derivative \(\nabla^P\) on \(P\) which respects the vertical bundle and is related to \(\nabla^M\). Although it was shown that deformation quantizations for different choices of \(\nabla^P\) are equivalent with explicitly given equivalence transformations \([126, \text{Satz } 3.1.16]\) it was not possible to clarify the classification in this framework. This and the fact that there is no evidence for a symplectic structure on a space-time gives rise to investigate the above notion of deformation quantization for arbitrary star products on Poisson manifolds. In the present work we prove the following two fundamental theorems.

**Theorem 1.4.3**

Every surjective submersion \(p : P \rightarrow M\) with a star product \(\ast\) on \(M\) admits a deformation quantization which is unique up to equivalence.

**Theorem 1.4.4**

Every principal fibre bundle \(p : P \rightarrow M\) with a star product \(\ast\) on \(M\) admits a deformation quantization which is unique up to equivalence.

Although the Fedosov construction in \([126]\) only works in the symplectic setting, it has further properties that might be useful for a physical interpretation. In particular, it yields a functorial dependence of the module structure \(\ast\) on the connection \(\omega\) and \(\nabla^P\). Under a gauge transformation \(\Phi \in \text{Gau}(P)\) the crucial dependence on \(\omega\), denoted by \(\ast_\omega\), has the behaviour

\[\Phi^* (f \ast_\omega a) = \Phi^* f \ast_{\Phi^* \omega} a.\] (1.57)

Although \(\ast_\omega\) and \(\ast_{\Phi^* \omega}\) are equivalent the pullback \(\Phi^*\) is no equivalence transformation.

Equation (1.57) in particular shows, that the local gauge transformations are no longer in the commutant. If this algebraic property is seen to be fundamental and shall be maintained, the actions of the gauge transformations have to be replaced by the natural action of the endomorphisms in the commutants. For the infinitesimal versions one obviously makes the same observation. Since they are classically given by Lie derivatives, this means by differential operators, it is an interesting task to compute the commutant of the deformation quantization within all differential operators of the algebra \(C^\infty(P)\) which will be done in this work.

The situation for gauge potentials and gauge fields has not been studied so far and is a difficult problem since it is not clear which structures should be adapted. Moreover, the gauge potentials already appear as a necessary choice for the Fedosov construction which gives evidence that the matter and gauge fields already are strongly related in this setting. Nevertheless, it is a first step of understanding also to investigate the deformation problem of the vector space of horizontal and \(G\)-invariant forms together with their commutants. All this will now be done in a very general way which yields a further contribution to the final aim of adapting the whole geometry of principal fibre bundles to a given star product.
Chapter 2

Deformation theory of algebras and modules

The basic ideas of deformation quantization as introduced in [7,8] have their origin in the early works [51–55] of Gerstenhaber on algebraic deformation theory. Concretely, they provide the algebraic setting for the deformation theory of algebras. The main point there is that the obstruction for an order by order construction of an associative deformation of an algebra structure is encoded in a certain Hochschild cohomology of the algebra. In the same way the obstruction for a construction of an equivalence transformation between two associative deformations is given by another Hochschild cohomology. This fundamental ideas can be reformulated for the deformation theory of other kinds of algebraic structures. Donald and Flanigan [38] have first considered the deformation theory of modules. Later this topic has been discussed in [130] by Yau who has also studied various other structures.

In most cases, however, it is not possible to compute the crucial cohomologies explicitly or they are not vanishing and thus this approach does not help for the decision whether it is possible to execute the constructions of the relevant structures or not. However, the deformation theory of Gerstenhaber is one of the fundamental guidelines of this work. With the further assumption that the Hochschild cochains are differential operators it will be possible to compute the cohomologies for the right modules occurring in the framework of surjective submersions and principal fibre bundles.

This chapter presents the general framework for all this. Besides giving a motivation by the well-known concepts and definitions there are presented the modifications and adaptions of the ideas and approaches for the investigation of deformed modules. The first section is dedicated to the well-known notion of Hochschild complexes. It is introduced together with a short summary of the naturally occurring operations like the insertions and the cup product. After this very general statements we concentrate on complexes which are induced by algebras and right modules. For the first case the algebraic structure of the Hochschild cohomology is well-known and given by a so-called Gerstenhaber algebra. Leaving the purely algebraic description it is explained in Section 2.2 what shall be understood by algebra and module structures of particular types. This new definition specifies in an axiomatic way which further properties of the algebraic structures can be implemented in the purely algebraic setting. In comparison to [19] the crucial conditions are formulated in a slightly more general way in the present work. The Sections 2.3 and 2.4 then show that the obstruction theory for deformations of algebras and modules can be reformulated within this refined context. Moreover, Section 2.5 shows that the considered Hochschild cohomologies even play a crucial role for the investigation of the commutant of a deformed right module structure. With respect to the applications Section 2.6 contains a discussion of algebraic structures which are invariant under group actions. Finally this chapter is closed with a first simple example given by
Chapter 2. Deformation theory of algebras and modules

the projective modules. There it is possible to compute the purely algebraic cohomology by an explicit homotopy. In the following presentation it is assumed that the reader is familiar with the basic concepts of homological algebra. The crucial definitions can be found in Appendix B or in the respective literature, for instance [74, 75, 87].

From now on let \( K \) be a field of characteristic 0 like \( \mathbb{Q}, \mathbb{R}, \mathbb{C} \). This will be sufficient for our purposes. However, all considerations of this chapter are still valid for a commutative ring \( K \) with at least \( 1 \neq 0 \) and \( 0 \neq 2, \frac{1}{2} \in K \).

2.1 The Hochschild complex

In order to build up an obstruction theory for the deformations of algebras and modules one needs the notion of Hochschild complexes and Hochschild cohomologies which shall be introduced in the following. All presented structures of this section are well-known and already appear in the early works [71, 72] of Hochschild and [51] of Gerstenhaber. In our presentation, the very general considerations and definitions in the literature are already adapted to the algebraic framework of our later examples. This is convenient in order to introduce the more specific framework which will be used later on, and to emphasize the crucial additional aspects.

2.1.1 The Hochschild complex with values in a bimodule

Let \( A \) and \( M \) be \( K \)-vector spaces. For all integers \( k \in \mathbb{Z} \) consider the derived \( K \)-vector spaces

\[
HC^k(A, M) = \begin{cases}
\{0\} & \text{if } k < 0 \\
M & \text{if } k = 0 \\
\text{Hom}_K(A \times \cdots \times A, M) & \text{if } k \geq 1,
\end{cases}
\]

(2.1)

this means the \( K \)-multilinear maps of \( A \) with values in \( M \) for \( k \geq 1 \). The direct sum \( HC^*(A, M) = \bigoplus_{k \in \mathbb{Z}} HC^k(A, M) \) yields a \( \mathbb{Z} \)-graded space and the corresponding degree of homogeneous elements is often referred to as dimension or tensor degree, since \( HC^k(A, M) \) for \( k \geq 1 \) can also be seen as the linear maps \( \text{Hom}_K(A^\otimes k, M) \) of the \( k \)-fold tensor product \( A^\otimes k \) of \( A \) over \( K \). Besides this, it will be convenient for further definitions and computations to shift the grading by one which means to consider

\[
HC[1]^*(A, M) = \bigoplus_{k \in \mathbb{Z}} HC^{k+1}(A, M)
\]

(2.2)

where the degree of homogeneous elements is defined by

\[
\text{deg} \phi = k \iff \phi \in HC^{k+1}(A, M).
\]

(2.3)

So, an element of \( HC^k(A, M) \) is said to have dimension or tensor degree \( k \) but degree \( k - 1 \). Within this framework one can now define the following basic operations on \( HC^*(A, M) \).

Definition 2.1.1 (The insertion maps and the cup product)

Let \( A, M \) be \( K \)-vector spaces.

i.) For homogeneous elements \( \phi \in HC^{k+1}(A, M) \) and \( \psi \in HC^{m+1}(A, A) \) with \( k, m \in \mathbb{N}_0 \) and for \( i = 0, \ldots, k = \text{deg} \phi \) the insertion \( \phi \circ_i \psi \in HC^{k+m+1}(A, M) \) of \( \psi \) in \( \phi \) after the \( i \)-th position is defined by

\[
(\phi \circ_i \psi)(a_1, \ldots, a_{k+m+1}) = \phi(a_1, \ldots, a_i, \psi(a_{i+1}, \ldots, a_{i+m+1}), a_{i+m+2}, \ldots, a_{k+m+1})
\]

(2.4)

for all \( a_1, \ldots, a_{k+m+1} \in A \).
2.1. The Hochschild complex

ii.) Linear combination of all possible insertions yields a new multiplication

\[ \phi \circ \psi = \sum_{i=0}^{\deg \phi} (-1)^i \deg \psi \phi \circ_i \psi \]  

(2.5)

iii.) Let \( M \) be equipped with the additional structure of an associative \( \mathbb{K} \)-algebra. Then for all \( k, m \in \mathbb{N}_0 \) the cup product

\[ \cup : \text{HC}^k(A, M) \times \text{HC}^m(A, M) \to \text{HC}^{k+m}(A, M) \]  

(2.6)

is defined by

\[ (\phi \cup \psi)(a_1, \ldots, a_{k+m}) = \phi(a_1, \ldots, a_k)\psi(a_{k+1}, \ldots, a_{k+m}) \]  

(2.7)

for all \( \phi \in \text{HC}^k(A, M), \psi \in \text{HC}^m(A, M) \) and \( a_1, \ldots, a_{k+m} \in A \).

All maps allow bilinear extensions defined on all elements in \( \text{HC}^* (A, M) \) and \( \text{HC}^* (A, A) \). For the insertion one therefore simply defines \( \phi \circ_i \psi = 0 \) if \( i > \deg \phi \). If one of the involved tensor degrees is less than zero, all maps are defined to be zero, too. By (2.6), the cup product is homogeneous with respect to the tensor degree. Instead, the maps \( \circ_i, \circ_j : \text{HC}^{k+1}(A, M) \times \text{HC}^{m+1}(A, M) \to \text{HC}^{k+m+1}(A, M) \) are homogeneous with respect to the shifted degree,

\[ \deg (\phi \circ \psi) = \deg \phi + \deg \psi. \]  

(2.8)

**Remark 2.1.2 (Associativity of the cup product)**

The cup product \( \cup \) is obviously associative. Thus, \( (\text{HC}^* (A, M), \cup) \) is a graded, associative algebra with respect to the tensor degree.

In the case \( M = A \) the insertions can be composed. But as stated in the following proposition this does not lead to an associative multiplication.

**Proposition 2.1.3**

Let \( A, M \) be \( \mathbb{K} \)-vector spaces and let \( \phi \in \text{HC}^* (A, M), \psi, \chi \in \text{HC}^* (A, A) \) be homogeneous elements.

i.) With respect to their composition the corresponding insertions satisfy

\[ (\phi \circ_i \chi) \circ_j \psi = \begin{cases} (\phi \circ_j \chi) \circ_{i+\deg \chi} \psi & \text{if } j < i \\ \phi \circ_i (\psi \circ_{j-i} \chi) & \text{for } i \leq j \leq i + \deg \psi \\ (\phi \circ_{j-\deg \psi} \psi) \circ_i \chi & \text{if } j > i + \deg \psi. \end{cases} \]  

(2.9)

ii.) In consequence, the map \( \circ \) satisfies

\[ (\phi \circ \psi) \circ \chi - \phi \circ (\psi \circ \chi) = \sum_{i=1}^{\deg \phi} \sum_{j=0}^{i-1} (-1)^i \deg \psi + j \deg \chi (\phi \circ_i \psi) \circ_j \chi \]

\[ + \sum_{i=0}^{\deg \phi-1} \sum_{j=i+\deg \psi+1}^{\deg \phi+\deg \psi} (-1)^i \deg \psi + j \deg \chi (\phi \circ_i \psi) \circ_j \chi \]

\[ = (-1)^{\deg \psi \deg \chi} ((\phi \circ \chi) \circ \psi - \phi \circ (\chi \circ \psi)). \]  

(2.10)

A complete and detailed version of the combinatorial proof can be found in [123]. There, the case \( M = A \) is treated, but the considerations are exactly the same in the slightly more general case of the present situation.
**Proposition 2.1.4**

In the case \( M = A \) the supercommutator

\[
[\phi, \psi] = \phi \circ \psi - (-1)^{\deg \phi \deg \psi} \psi \circ \phi
\]

is superantisymmetric and satisfies the super Jacobi identity

\[
[\phi, [\psi, \chi]] = [[\phi, \psi], \chi] + (-1)^{\deg \phi \deg \psi} [\psi, [\phi, \chi]]
\]

with respect to the degree \( \deg \). Thus, \((HC^\bullet(A, A), \deg, [\cdot, \cdot])\) becomes a super Lie algebra.

Again, a detailed proof can be found in [123].

**Remark 2.1.5**

As explained in [51], Equation (2.9) shows that \( HC[1]^\bullet(A, A) \) defines a so-called right pre-Lie system and that \( HC[1]^\bullet(A, M) \) is a right module over \( HC[1]^\bullet(A, A) \). Then, for \( M = A \), Equation (2.10) is a general consequence of the fact that \( HC[1]^\bullet(A, A) \) becomes a graded right pre-Lie ring. Further, this right pre-Lie ring gives rise to the graded Lie ring \( HC[1]^{\bullet}(A, [\cdot, \cdot]) \) satisfying (2.12). For more details and the explicit definitions of the mentioned notions, confer [51].

**Definition 2.1.6 (Gerstenhaber bracket)**

Let \( A \) be a \( \mathbb{K} \)-vector space. The nonassociative product \( \circ \) of \( HC^\bullet(A, A) \) is called the Gerstenhaber product and the super Lie bracket \([\cdot, \cdot]\) is called the Gerstenhaber bracket.

The Gerstenhaber bracket yields an important characterization of associative multiplications.

**Lemma 2.1.7 (Associative multiplications)**

Let \( A \) be a \( \mathbb{K} \)-vector space. Then, a bilinear map \( \mu : A \times A \rightarrow A \), seen as an element \( \mu \in HC^2(A, A) \), is an associative multiplication, if and only if

\[
[\mu, \mu] = 0.
\]

**Proof:** By definition, \( \deg \mu = 1 \) and thus \( [\mu, \mu] = \mu \circ \mu + \mu \circ \mu = 2\mu \circ \mu \). Then,

\[
(\mu \circ \mu)(a, b, c) = \sum_{i=0}^{1} (-1)^i (\mu \circ_i \mu)(a, b, c) = \mu(\mu(a, b), c) - \mu(a, \mu(b, c))
\]

and the Lemma is clear since \( \frac{1}{2} \in \mathbb{K} \). 

After this general considerations we now consider the case where \( A \) is an associative \( \mathbb{K} \)-algebra and \( M \) is an \((A, A)\)-bimodule. Here and in the following \( A \) will always be a unital algebra with unit 1. For convenience’s sake the multiplication in the algebra and the module multiplication are simply denoted by \( xy \) with corresponding elements \( x, y \) if the meaning is clear by the context.

Then, consider the \( \mathbb{K} \)-linear maps

\[
\delta^k : HC^k(A, M) \rightarrow HC^{k+1}(A, M)
\]

which for \( k \in \mathbb{N} \) are defined by

\[
(\delta^k \phi)(a_1, \ldots, a_{k+1}) = a_1 \phi(a_2, \ldots, a_{k+1}) + \sum_{i=1}^{k} (-1)^i \phi(a_1, \ldots, a_i a_{i+1}, \ldots, a_{k+1}) + (-1)^{k+1} \phi(a_1, \ldots, a_k) a_{k+1}.
\]
For \( k = 0 \) one has \((\delta^0 \phi)(a) = a\phi - \phi a \) for \( \phi \in \mathcal{M} \) and \( a \in A \). The second term of the sum in (2.15) is given by \( \sum_{i=1}^{k} (-1)^i \phi(a_1, \ldots, a_i a_{i+1}, \ldots, a_{k+1}) = -(\phi \circ \mu)(a_1, \ldots, a_{k+1}) \). It is a straightforward but lengthy computation using the bimodule properties of \( \mathcal{M} \) which shows that

\[ \delta^{k+1} \circ \delta^k = 0. \] (2.16)

By setting the maps \( \delta^k \) to be trivial for \( k < 0 \) all this yields a \( \mathbb{K} \)-linear map \( \delta : \text{HC}^*(A, \mathcal{M}) \rightarrow \text{HC}^{*+1}(A, \mathcal{M}) \) which increases the degree by one and has square zero, \( \delta^2 = 0 \). Thus, these structures define a cochain complex, confer Appendix B, which leads to the following definition.

**Definition 2.1.8 (The Hochschild complex of an algebra with values in a bimodule)**

Let \( A \) be an associative \( \mathbb{K} \)-algebra and \( \mathcal{M} \) be a \( \mathbb{K} \)-vector space with an \((A, A)\)-bimodule structure. Then, the cochain complex \((\text{HC}^*(A, \mathcal{M}), \delta)\) is called the Hochschild complex of \( A \) with values or coefficients in \( \mathcal{M} \) and the Hochschild differential \( \delta \). For all \( k \in \mathbb{N}_0 \) the corresponding \( k \)-th cohomology group

\[ \text{HH}^k(A, \mathcal{M}) = \frac{\ker \delta^k}{\text{im} \delta^{k-1}} \] (2.17)

is called the \( k \)-th Hochschild cohomology group and one sets

\[ \text{HH}^*(A, \mathcal{M}) = \bigoplus_{k \in \mathbb{N}_0} \text{HH}^k(A, \mathcal{M}). \] (2.18)

### 2.1.2 The Hochschild complex of an associative algebra

It is clear from the Definition 2.1.8 that an associative algebra, seen as a bimodule over itself, yields a Hochschild complex where the associativity is crucial for (2.16).

**Definition 2.1.9 (The Hochschild complex of an associative algebra)**

Let \( (A, \mu) \) be an associative algebra. Then \((\text{HC}^*(A, A), \delta)\) is called the Hochschild complex of the algebra \( A \).

Clearly, the algebra multiplication is then seen as a cochain

\[ \mu \in \text{HC}^2(A, A). \] (2.19)

In this section the well-known additional features of Hochschild complexes of associative algebras are presented. As already seen in Proposition 2.1.4 and Lemma 2.1.7 the case \( \mathcal{M} = A \) yields further structures and different ways to describe the algebra and its Hochschild complex. Moreover, the Hochschild cohomology \( \text{HH}^*(A, A) \) is one of the most important structures encoding crucial properties of the associative algebra \( A \). For completeness’ sake all this will be recalled in the following such that finally one can state the well-known algebraic structure of \( \text{HH}^*(A, A) \).

**Lemma 2.1.10 (The Hochschild complex of an algebra)**

Let \((A, \mu)\) be an associative algebra.

i.) The associativity of the multiplication \( \mu \) yields that \( \mu \in \text{HC}^2(A, A) \) is a cocycle, \( \delta \mu = 0 \). It is even a coboundary \( \mu = \delta i \) with respect to the identity cochain \( i \in \text{HC}^1(A, A) \), \( i(a) = a \).

ii.) The Hochschild differential can be expressed by

\[ \delta \phi = (-1)^{\deg \phi} [\mu, \phi] = -[\phi, \mu] \] (2.20)

with the Gerstenhaber bracket \([\cdot, \cdot]\).
iii.) The cup product can be expressed by

$$\phi \cup \psi = (\mu \circ_0 \phi) \circ_k \psi$$

(2.21)

for homogeneous $\phi \in \text{HC}^k(A, A)$ and arbitrary $\psi \in \text{HC}^\bullet(A, A)$.

**Proof:** All assertions are clear by the definitions and are shown in [51].

The expression (2.20) for the Hochschild differential $\delta$, the super Jacobi identity (2.12) and Lemma 2.1.7 yield a different proof for $\delta^2 = 0$ since

$$\delta^2 \phi = [[\phi, \mu], \mu] - [\phi, [\mu, \mu]] = -\delta^2 \phi.$$ 

Further, Equation (2.20) makes it easy to investigate the behaviour of the Hochschild differential with respect to the Gerstenhaber product $\circ$, the Gerstenhaber bracket $[\cdot, \cdot]$ and the cup product $\cup$.

**Proposition 2.1.11 (Compatibilities of the Hochschild differential)**

Let $(A, \mu)$ be an associative $K$-algebra. Then, the Hochschild differential $\delta$ of the corresponding Hochschild complex has the following properties.

i.) For $\phi, \psi \in \text{HC}^\bullet(A, A)$ with homogeneous $\psi$ it is

$$\delta[\phi, \psi] = [\phi, \delta \psi] + (-1)^{\deg \psi} [\delta \phi, \psi].$$

(2.22)

ii.) For $\phi \in \text{HC}^k(A, A)$ and $\psi \in \text{HC}^\bullet(A, A)$ it is

$$\delta(\phi \cup \psi) = \delta \phi \cup \psi + (-1)^k \phi \cup \delta \psi.$$  

(2.23)

iii.) For $\phi \in \text{HC}^k(A, A)$ and $\psi \in \text{HC}^m(A, A)$ it is

$$\phi \circ \delta \psi - \delta(\phi \circ \psi) + (-1)^{m-1} \delta \phi \circ \psi = (-1)^{m-1}(\psi \cup \phi - (-1)^{km} \phi \cup \psi).$$

(2.24)

**Proof:** The first part is a direct consequence of Lemma 2.1.10 and (2.12). Explicitly, one computes $\delta[\phi, \psi] = -[[\phi, \psi], \mu] = -[\phi, [\psi, \mu]] - (-1)^{\deg \psi} [[\phi, \mu], \psi] = [\phi, \delta \psi] + (-1)^{\deg \psi} [\delta \phi, \psi]$. The proofs of the other statements are basic computations using the definitions, Proposition 2.1.3 and Lemma 2.1.10. All this can be found in [123].

**Remark 2.1.12**

i.) The second statement (2.24) still holds in the general framework of Hochschild complexes of algebras $A$ with coefficients in bimodules $M$ which are associative algebras themselves if the additional multiplication of $M$ is compatible with the module structure, this means if $a(m_1m_2) = (am_1)m_2$, $(m_1m_2)a = m_1(m_2a)$ and $(m_1a)m_2 = m_1(m_2a)$ for all $m_1, m_2 \in M$ and $a \in A$.

ii.) Equation (2.24) means that $\delta$ is a super derivation of $(\text{HC}[1]^\bullet(A, A), [\cdot, \cdot])$ from the right of degree one. In the same way, the Leibniz rule (2.23) shows that $\delta$ is a super derivation of $(\text{HC}^\bullet(A, M), \cup)$ from the left.

The Equations (2.22) and (2.23) assure on one hand that the products of cocycles are again cocycles and on the other hand that the coboundaries build an ideal within the cocycles. Since Equation (2.21) further shows that for cocycles $\phi, \psi$ the right hand side is a coboundary the following corollary is obvious.
Corollary 2.1.13
With Proposition 2.1.11 it is clear that the Gerstenhaber bracket \([\cdot, \cdot]\) and the cup product \(\cup\) can be well-defined on the Hochschild cohomology groups \(\text{HH}^*(A, A)\) where the bracket yields a super Lie algebra \((\text{HH}[1]^*(A, A), [\cdot, \cdot])\) and the cup product an associative and super commutative algebra \((\text{HH}^*(A, A), \cup)\).

The assertions of [51, Thm. 5, Cor. 2] which are the content of the next proposition contribute the last part for the characterization of the algebraic structure of the Hochschild cohomology.

Proposition 2.1.14
Let \(A\) be an associative algebra and \(\xi \in \text{HH}^k(A, A), \eta \in \text{HH}^m(A, A), \zeta \in \text{HH}^*(A, A)\). Then, the Leibniz rule
\[
[\eta \cup \zeta, \xi] = [\eta, \xi] \cup \zeta + (-1)^{(k-1)m}\eta \cup [\zeta, \xi]
\]
holds. This means that \([\cdot, \cdot]\) is a graded derivation of \(\cup\) of degree \(\deg \xi\).

Corollary 2.1.15 (The Gerstenhaber algebra \(\text{HH}^*(A, A)\))
Let \(A\) be an associative algebra. Then, the corresponding Hochschild cohomology has the structure of a so-called right Gerstenhaber algebra \((\text{HH}^*(A, A), \cup, [\cdot, \cdot]),\) this means of a \(\mathbb{Z}\)-graded, associative and super commutative algebra with multiplication \(\cup\) and a super Lie algebra structure \([\cdot, \cdot]\) with respect to the shifted grading \(\text{HH}[1]^*(A, A)\) satisfying (2.25). Moreover, if \(A\) is unital with unit \(1 \in A\) the class \([1] \in \text{HH}^0(A, A)\) is a unit with respect to the cup product.

As stated in the beginning of the chapter the Hochschild cohomology of an algebra describes crucial properties. The zeroth Hochschild cohomology \(\text{HH}^0(A, A) = \ker \delta^0\) for instance is nothing but the center of the algebra since for \(a, b \in A = \text{HC}^0(A, A)\) one has \((\delta a)(b) = ba - ab = - \text{ad}(a)b\). This further shows that the coboundaries \(\delta a \in \text{HC}^1(A, A) = \text{End}_K(A, A)\) are inner derivations. For an arbitrary \(D \in \text{HC}^1(A, A)\) one finds \((\delta D)(a, b) = aD(b) - D(ab) + D(a)b\) which shows that the cocycles in \(\ker \delta^1\) are the derivations of \(A\). Thus the first cohomology is the space of outer derivations. As it will be explained in Section 2.3.3 the second and third Hochschild cohomology are crucial for the deformation theory of the algebra.

2.1.3 The Hochschild complex of a right module

Similar to associative algebras, every module over an associative algebra gives rise to a particular Hochschild complex. Without loss of generality the situation is investigated for right modules.

So let \((A, \mu)\) be an associative \(K\)-algebra and \(E\) be a vector space over \(K\) with a right \(A\)-module structure \(\rho : E \times A \rightarrow E\). This can be seen as a \(K\)-linear map
\[
\rho : A \rightarrow \text{End}_K(E, E)
\]
with values in the \(K\)-linear endomorphisms of \(E\) and thus as an element \(\rho \in \text{Hom}_K(A, \text{End}_K(E, E))\). Then, of course, one has the identification \(\rho(e, a) = \rho(a)(e)\) for all \(e \in E\) and \(a \in A\) and the condition for \(\rho\) to be a right module structure with respect to \(\mu\) reads
\[
\rho(b) \circ \rho(a) = \rho(\mu(a, b))
\]
for all \(a, b \in A\). The \(K\)-vector space \(\text{End}_K(E, E)\) can be equipped with the canonical \((A, A)\)-bimodule structure
\[
a \cdot D \cdot b = \rho(b) \circ D \circ \rho(a)
\]
for \(a, b \in A\) and \(D \in \text{End}_K(E, E)\) where \(\circ\) is the usual composition of maps. This gives rise to a particular Hochschild complex.
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Definition 2.1.16 (The Hochschild complex of a right module)
Let $E$ be a right module over an associative algebra $A$. Then $(\text{HC}^\bullet(A, \text{End}_K(E, E)), \delta)$ is called the Hochschild complex of the right module $E$.

Remark 2.1.17 (The Hochschild complex of a right module)
Since the endomorphisms $\text{End}_K(E, E)$ build an algebra the Hochschild complex $\text{HC}^\bullet(A, \text{End}_K(E, E))$ is equipped with a cup product given by

$$\phi \cup \psi(a_1, \ldots, a_{k+m}) = \phi(a_1, \ldots, a_k) \circ \psi(a_{k+1}, \ldots, a_{k+m}).$$

Since the right module structure $\rho$ can be seen as a cochain $\rho \in \text{HC}^1(A, \text{End}_K(E, E))$ (2.30) the Hochschild differential has the form

$$(\delta \phi)(a_1, \ldots, a_{k+1}) = (\phi \cup \rho)(a_2, \ldots, a_{k+1}, a_1) + \sum_{i=1}^{k} (-1)^i (\phi \circ_{i-1} \mu)(a_1, \ldots, a_{k+1}) + (-1)^{k+1}(\rho \cup \phi)(a_{k+1}, a_1, \ldots, a_k).$$

Remark 2.1.18 (Right modules over commutative algebras)
If $(A, \mu)$ is a commutative associative algebra equation (2.27) can be written in the form

$$\rho \circ \mu = \rho \cup \rho.$$

Moreover, there exists an additional choice of the bimodule structure of $\text{End}_K(E, E)$ since (2.28) then can be changed into

$$a \cdot D \cdot b = \rho(a) \circ D \circ \rho(b).$$

This yields a slightly different Hochschild differential with the simple form

$$\delta \phi = \rho \cup \phi - \phi \circ \mu + (-1)^{k+1} \rho \cup \phi.$$

In this chapter the general situation will be discussed. In the later applications, however, it is convenient to use (2.33) which is possible without any restrictions.

Again the Hochschild cohomology encodes crucial information of the right module structure. The zeroth Hochschild cohomology $\text{HH}^0(A, \text{End}_K(E, E)) = \ker \delta^0$ for instance is simply the set of module endomorphisms

$$\text{End}_A(E, E) = \{D \in \text{End}_K(E, E) \mid D \circ \rho(a) = \rho(a) \circ D \text{ for all } a \in A\},$$

which is called the commutant of the given right module structure. The interpretation of the first and second Hochschild cohomology will arise in the framework of the deformation theory discussed in Section 2.4.

2.2 Algebras and modules of a particular type

With the purely algebraic framework built up so far one already could formulate the well-known definitions and results concerning deformations of associative algebras and their generalizations to right modules. There, of course, the corresponding Hochschild cohomologies play the crucial
role. However, since the considered algebra multiplication $\mu$ and the right module structure $\rho$ typically have additional properties which are also required for the higher orders of corresponding deformations one has to introduce a slightly more specific framework containing these aspects. In order to provide the implementation of the additional properties into the considerations it will be crucial that these properties can be expressed in an adequate way and satisfy certain compatibility conditions. With respect to the further applications and since all considerations are based on the framework of Hochschild complexes it is necessary to regard the given structures as corresponding cochains $\mu \in \text{HC}^2(A, A)$ and $\rho \in \text{HC}^1(A, \text{End}_K(E, E))$. Then, the particular properties are assumed to be expressed by particular Hochschild cochains. This reads as

$$\mu \in \text{HC}^2_{\text{type}}(A, A) \subseteq \text{HC}^2(A, A)$$ (2.36)

and

$$\rho \in \text{HC}^1_{\text{type}}(A, D) \subseteq \text{HC}^1(A, \text{End}_K(E, E))$$ (2.37)

where, in principle, ‘type’ describes the properties concerning the arguments of $A$ and $D \subseteq \text{End}_K(E, E)$ is the subset of maps with the correct behaviour with respect to arguments of $E$. This separation is of course not strict. In general, the characterization ‘type’ and $D$ describe the behaviour of the cochains in a whole.

**Remark 2.2.1**

i.) It should be noted that in general the properties for the multiplication $\mu$ and the right module structure $\rho$ could be different. Since we have to ensure that these two are compatible in the sense explained later and with respect to the examples in the following chapters we do not distinguish the different kinds of ‘type’ in the notation.

ii.) The characterization of the right module structure $\rho$, for example the demands to be continuous with respect to some topology or to be a differential operator, really depends on whether it is seen as a map $\rho : E \times A \rightarrow E$ or as a map $\rho : A \rightarrow \text{End}_K(E, E)$ as in (2.37), although these two perspectives are equivalent in the purely algebraic setting. In Remark 6.1.10 this difference is pointed out for the notion of differential operators. For $\mu$, this complication does not appear with (2.36). Of course there could occur further subtleties if one would use the universal property of the tensor product and define the cochains as linear maps of the tensor product instead of multilinear maps. But nevertheless, if the types are fixed in the explained way one can afterwards always identify the multilinear maps with the corresponding linear maps of the tensor product if this view is more convenient.

The above mentioned implementation of the additional properties will be practicable if they are given in the form (2.36) and (2.37) and if these specifications can be extended to the whole Hochschild complexes yielding subcomplexes which are closed under the existing operations. In the following, the structures of such particular types will be the ones of interest. The necessary requirements are summarized in the following two definitions.

**Definition 2.2.2 (Associative algebras of particular types)**

An associative $K$-algebra $(A, \mu)$ is said to be of a particular type if for all $k \in \mathbb{N}$ there are vector subspaces $\text{HC}^k_{\text{type}}(A, A) \subseteq \text{HC}^k(A, A)$ of cochains such that with $\text{HC}^0_{\text{type}}(A, A) = A$ the following assertions hold.

i.) $\mu \in \text{HC}^2_{\text{type}}(A, A)$.

ii.) $\text{HC}^\bullet_{\text{type}}(A, A)$ is closed under the insertions $\circ_i$ after the $i$-th position and the natural action of the symmetric group on the arguments in $A$. 

2.2. Algebras and modules of a particular type
Regarding the structures defined in Section 2.1 it is clear that the above assumptions assure that $HC^\bullet_{\text{type}}(A, A)$ is a subcomplex of $HC^\bullet(A, A)$ with the Hochschild differential $\delta$ which only depends on $\mu$. Further, the related Hochschild cohomology $HH^\bullet_{\text{type}}(A, A)$ is a Gerstenhaber algebra with the same multiplication and bracket.

**Definition 2.2.3 (Module structures of particular types)**

Let $(\mathcal{A}, \mu)$ be an associative $K$-algebra of a particular type. An $\mathcal{A}$-module structure $\rho$ on a $K$-vector space $\mathcal{E}$ is said to be of a particular type which is compatible with $HC^\bullet_{\text{type}}(A, \mathcal{A})$ if for all $k \in \mathbb{N}_0$ there are vector subspaces $HC^k_{\text{type}}(A, \mathcal{D}) \subseteq HC^k(\mathcal{A}, \text{End}_K(\mathcal{E}, \mathcal{E}))$ of cochains with values in a subalgebra $\mathcal{D} \subseteq \text{End}_K(\mathcal{E}, \mathcal{E})$ such that the following assertions hold.

i.) $\rho \in HC^1_{\text{type}}(A, \mathcal{D})$.

ii.) $id_{\mathcal{E}} \in HC^0_{\text{type}}(A, \mathcal{D}) \subseteq \mathcal{D}$.

iii.) $HC^\bullet_{\text{type}}(A, \mathcal{D})$ is closed under the insertions $\partial_i$ after the $i$-th position with respect to the complex $HC^\bullet_{\text{type}}(A, \mathcal{A})$, the cup product $\cup$ in (2.24), and the natural action of the symmetric group on the arguments in $\mathcal{A}$.

The third condition for the cup product implies that $HC^0_{\text{type}}(A, \mathcal{D}) \subseteq \mathcal{D}$ is a subalgebra. With (2.25) it is obvious how the notion of deformations has to be reformulated in this framework. Since all relevant structures, in particular the deformations itself and the equivalence transformations between them, can be expressed in terms of the Hochschild complex the restriction to the subcomplex of the given type yields the desired refined notion. The used notation in the following presentation closely follows [123]. Using the notion of formal power series in a formal parameter $\lambda$ as explained in Section 1.1 the crucial definition now is the following.

**Definition 2.3.1 (Formal associative deformation)**

Let $(\mathcal{A}, \mu_0)$ be an associative $K$-algebra of a particular type as in Definition 2.2.2

i.) A formal associative deformation of $(\mathcal{A}, \mu_0)$ is a $K[[\lambda]]$-bilinear associative multiplication $\mu$ for $\mathcal{A}[[\lambda]]$ of the form

$$
\mu = \sum_{r=0}^{\infty} \lambda^r \mu_r : \mathcal{A}[[\lambda]] \times \mathcal{A}[[\lambda]] \to \mathcal{A}[[\lambda]]
$$

(2.38)

with $K$-bilinear maps $\mu_r \in HC^2_{\text{type}}(\mathcal{A}, \mathcal{A})$.

ii.) Two such deformations $\mu$ and $\tilde{\mu}$ are said to be equivalent if there exists a $K[[\lambda]]$-linear algebra isomorphism $S = \text{id} + \sum_{r=1}^{\infty} \lambda^r S_r : (\mathcal{A}[[\lambda]], \mu) \to (\mathcal{A}[[\lambda]], \tilde{\mu})$ with $S_r \in HC^1_{\text{type}}(\mathcal{A}, \mathcal{A})$. 

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iii.) Associative deformations and their equivalence up to an order \( r \in \mathbb{N} \) are defined in the corresponding way.

With the same argumentation as in Lemma 2.1.7 a formal power series \( \mu = \sum_{r=0}^{\infty} \lambda^r \mu_r \) of \( \mathbb{K} \)-linear maps \( \mu_r : \mathbb{A} \times \mathbb{A} \to \mathbb{A} \) defines an associative multiplication if the correspondingly defined Gerstenhaber bracket \( [\mu, \mu] = 0 \) vanishes. This condition has to be fulfilled in all orders of \( \lambda \). In order \( r \in \mathbb{N}_0 \) it reads

\[
\sum_{s=0}^{r} [\mu_s, \mu_{r-s}] = 0.
\] (2.39)

In lowest order \( \lambda^0 \) this is simply the associativity condition \( [\mu_0, \mu_0] = 0 \) for the undeformed product.

By Lemma 2.1.10 the Hochschild differential can be expressed in terms of the Gerstenhaber bracket with \( \mu_0 \). Since \( [\mu_0, \mu_r] = [\mu_r, \mu_0] = -\delta \mu_r \), condition (2.39) can be read as

\[
\delta \mu_1 = 0
\] (2.40)

for the first order and for all higher orders \( r \geq 2 \) one gets

\[
\delta \mu_r = \frac{1}{2} \sum_{s=1}^{r-1} [\mu_s, \mu_{r-s}].
\] (2.41)

The last equation points out a possibility to construct such deformations order by order and leads to the following well-known result, confer [52, Chap. 1, Prop. 3].

**Proposition 2.3.2**

Let \((\mathbb{A}, \mu_0)\) be an associative \( \mathbb{K} \)-algebra of a particular type. Further, let \( \mu^{(r)} = \mu_0 + \cdots + \lambda^r \mu_r \) be an associative deformation up to order \( r \). Then the condition for \( \mu_{r+1} \in \text{HC}^2_{\text{type}}(\mathbb{A}, \mathbb{A}) \) to define an associative deformation \( \mu^{(r+1)} = \mu^{(r)} + \lambda^{r+1} \mu_{r+1} \) up to order \( r+1 \) is

\[
\delta \mu_{r+1} = R_r
\] (2.42)

with \( R_r \in \text{HC}^3_{\text{type}}(\mathbb{A}, \mathbb{A}) \) explicitly given by \( R_0 = 0 \) and

\[
R_r = \frac{1}{2} \sum_{s=1}^{r} [\mu_s, \mu_{r+1-s}] \quad \text{for } r \geq 1.
\] (2.43)

Moreover, \( \delta R_r = 0 \) whence the obstruction in order \( r+1 \) is the class \([R_r] \in \text{HH}^3_{\text{type}}(\mathbb{A}, \mathbb{A})\).

Concerning equivalence one finds the following well-known statement, confer [52, Chap. 1, Prop. 1].

**Proposition 2.3.3**

Let \((\mathbb{A}, \mu_0)\) be an associative \( \mathbb{K} \)-algebra of a particular type.

i.) An associative deformation of \( \mu \) is always equivalent to a deformation of the form \( \tilde{\mu} = \mu_0 + \sum_{s=1}^{\infty} \lambda^s \tilde{\mu}_s \) where the first non-vanishing cochain \( \tilde{\mu}_r \) is a cocycle, \( \delta \tilde{\mu}_r = 0 \) but no coboundary.

ii.) If \( \text{HH}^2_{\text{type}}(\mathbb{A}, \mathbb{A}) = 0 \) is trivial, all associative deformations are equivalent.

The two propositions show that the second and the third Hochschild cohomology groups of an algebra contain crucial information about its deformation theory. \( \text{HH}^3_{\text{type}}(\mathbb{A}, \mathbb{A}) \) encodes the obstruction for a continuation of a given deformation up to an arbitrary order \( r \) to a deformation up to order \( r+1 \). So, if this cohomology group is trivial there exists a simple way to construct
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associative deformations order by order. The elements in $\text{HH}^2_{\text{type}}(A, A)$ are exactly the equivalence classes of deformations up to order one. This is the case since $\mu = \sum_{s=0}^{\infty} \lambda^s \mu_s$ is an associative deformation up to order one if and only if $\delta \mu_1 = 0$ and two such deformations $\mu$ and $\tilde{\mu}$ are equivalent up to order one if and only if $\mu_1 - \tilde{\mu}_1 = \delta \phi$ is a coboundary. The general classification of associative deformations, however, is more difficult. But nevertheless, if the second Hochschild cohomology group vanishes all deformations are equivalent.

2.4 Deformations of right modules

Motivated by Gerstenhaber’s deformation theory of associative algebras and algebras with other properties, confer [52, Chap. 1], it is now a straightforward generalization which yields the notion of a deformation theory of modules over algebras in the slightly modified framework. A first purely algebraic definition for finite dimensional modules was given in the work of Donald and Flanigan [38]. There and in many other works on the topic, confer [130] and the references therein, the module is seen as a ring homomorphism as in (2.26) which then is the structure to be deformed. In the case which is relevant in this work, though, it is not sufficient only to perform this approach but one has to look for a more general notion where the deformation is a right module with respect to a given deformation of the underlying algebra. Without loss of generality this new concept is only considered for right modules. For left modules all results can be derived similarly.

Definition 2.4.1 (Deformation of a right module structure)

Let $(A, \mu_0)$ be an associative $K$-algebra and $(E, \rho_0)$ be a right $A$-module, both of particular types as in the Definitions 2.2.2 and 2.2.3. Further let $\mu = \sum_{r=0}^{\infty} \lambda^r \mu_r : A[[\lambda]] \times A[[\lambda]] \rightarrow A[[\lambda]]$ be a formal associative deformation of $\mu_0$ with $\mu_r \in \text{HC}^r_{\text{type}}(A, A)$ as in Definition 2.3.1.

i.) A deformation of the right module structure $\rho_0$ with respect to $\mu$ of the given type is a $K[[\lambda]]$-bilinear right $(A[[\lambda]], \mu)$-module structure $\rho$ of $E[[\lambda]]$ of the form

$$\rho = \sum_{r=0}^{\infty} \lambda^r \rho_r : E[[\lambda]] \times A[[\lambda]] \rightarrow E[[\lambda]]$$

with $K[[\lambda]]$-bilinear maps $\rho_r \in \text{HC}^r_{\text{type}}(A, \mathcal{D})$.

ii.) Two such deformations $\rho$ and $\tilde{\rho}$ are said to be equivalent if there exists a formal series

$$T = \text{id}_E + \sum_{r=1}^{\infty} \lambda^r T_r \in \text{HC}^0_{\text{type}}(A, \mathcal{D})[[\lambda]]$$

such that $T(\rho(e, a)) = \tilde{\rho}(Te, a)$, or equivalently

$$T \circ \rho(a) = \tilde{\rho}(a) \circ T,$$

for all $a \in A[[\lambda]]$ and $e \in E[[\lambda]]$.

iii.) The deformations up to an order $r \in \mathbb{N}$ as well as their equivalences are defined in a corresponding way.

Remark 2.4.2

Note that the condition (2.46) for equivalence can be formulated in terms of the cup product (2.29) and then reads

$$T \cup \rho = \tilde{\rho} \cup T.$$

(2.47)
Due to (12) it is clear that a formal series $T$ of operators as in (2.45) is invertible with $T^{-1} \in \HC^{0}_{\text{type}}(A, D)[[\lambda]]$. Thus, such a map $T$ and a deformation $\rho$ of the considered type define a new one via

$$\tilde{\rho} = T \cup \rho \cup T^{-1}.$$  \hfill (2.48)

Note that analogue assertions already hold for deformed algebra structures.

A comparison of the Hochschild complex of an associative algebra $A$ with the one obtained in Section 2.1.3 when regarding the algebra as a right module over itself already yields an indication which Hochschild cohomologies are crucial for the deformations of right modules. It is evident that for all $k \geq 1$ there is a natural way to identify the cochains in $\HC_{k}(A, A)$ with the ones in $\HC^{k-1}_{A}(A, \text{Hom}_{K}(A, A))$. This means that the transition to the module point of view induces a simple shift in the grading of the considered Hochschild complexes. Since the obstructions discussed in Section 2.3 lie in the second and third cohomology one could thus already guess that the obstruction theory for the deformation of general right modules is closely related to the first and second Hochschild cohomology $\HH^{1}_{\text{type}}(A, D)$. That this is really the case is the content of the following two propositions, confer [19, Lemma 2.1 and 2.2].

**Proposition 2.4.3**

In the setting of Definition 2.4.1, let $\rho^{(r)} = \rho_{0} + \cdots + \lambda^{r}\rho_{r}$ be a right module structure with respect to $\mu$ up to order $r$ with $\rho_{s} \in \HC^{1}_{\text{type}}(A, D)$ for all $s = 0, \ldots, r$. Then the condition for $\rho_{r+1} \in \HC^{1}_{\text{type}}(A, D)$ to define a right module structure $\rho^{(r+1)} = \rho^{(r)} + \lambda^{r+1}\rho_{r+1}$ up to order $r + 1$ is

$$\delta \rho_{r+1} = R_{r}$$

with $R_{r} \in \HC^{2}_{\text{type}}(A, D)$ explicitly given by $R_{0} = \rho_{0} \circ \mu_{1}$ and

$$R_{r}(a, b) = \sum_{s=0}^{r} \rho_{s}(\mu_{r+1-s}(a, b)) - \sum_{s=1}^{r} \rho_{s}(b) \circ \rho_{r+1-s}(a) \quad \text{for } r \geq 1.$$  \hfill (2.50)

Moreover, $\delta R_{r} = 0$ whence the recursive obstruction in order $r + 1$ is the class $[R_{r}] \in \HH^{2}_{\text{type}}(A, D)$.

**Proof:** The guideline of the proof is of course given by the well-known considerations in the basic references [38, 52, 130]. Nevertheless, it is carried out completely in order to show that all the slight modifications match together in more specific framework.

The main goal in general is of course to find $\rho_{r} \in \HC^{1}_{\text{type}}(A, D)$ for all $r \in \mathbb{N}$ such that $\rho = \sum_{r=0}^{\infty} \lambda^{r}\rho_{r}$ is a right module structure with respect to $\mu$. The failure of an arbitrary $\rho$ to be such a right module structure is encoded in the $K[[\lambda]]$-bilinear map $C : A[[\lambda]] \times A[[\lambda]] \rightarrow \text{End}_{K[[\lambda]]}(E[[\lambda]], E[[\lambda]])$ with $C(a, b) = \rho(\mu(a, b)) - \rho(b) \circ \rho(a)$. With $\text{End}_{K[[\lambda]]}(E[[\lambda]], E[[\lambda]]) = \text{End}_{K}(E, E)[[\lambda]]$ and the conditions in Definition 2.2.3 it is clear that $C = \sum_{r=0}^{\infty} \lambda^{r}C_{r}$ with elements $C_{r} \in \HC^{0}_{\text{type}}(A, D)$ given by $C_{r}(a, b) = \sum_{s=0}^{r} \rho_{s}(\mu_{r-s}(a, b)) - \rho_{s}(b) \circ \rho_{r-s}(a)$. Due to the associativity of $\mu$ the map $C$ satisfies the equation

$$C(\mu(a, b), c) - C(a, \mu(b, c)) = \rho(\mu(\mu(a, b), c)) - \rho(c) \circ \rho(\mu(a, b)) - \rho(\mu(a, \mu(b, c))) + \rho(\mu(b, c)) \circ \rho(a)$$

$$\rho(\mu(b, c)) \circ \rho(a) - \rho(c) \circ \rho(b) \circ \rho(a) + \rho(a) \circ \rho(b) \circ \rho(a) - \rho(c) \circ \rho(\mu(a, b))$$

$$= C(b, c) \circ \rho(a) - \rho(c) \circ C(a, b).$$

(2.51)

After these preliminary considerations the actual proof is the following. By assumption, there are given $\rho_{0}, \ldots, \rho_{r}$ such that $C_{0} = \cdots = C_{r} = 0$. For an arbitrary $\rho_{r+1} \in \HC^{1}_{\text{type}}(A, D)$ one gets

$$(\delta \rho_{r+1})(a, b) = a \cdot \rho_{r+1}(b) - \rho_{r+1}(\mu_{0}(a, b)) + \rho_{r+1}(a) \cdot b$$

$$\rho_{r+1}(b) \circ \rho_{0}(a) - \rho_{r+1}(\mu_{0}(a, b)) + \rho_{0}(b) \circ \rho_{r+1}(a)$$

(2.52)
and thus the failure of the so defined \( \rho^{(r+1)} \) to be a right module structure up to order \( r + 1 \) is given by

\[
C_{r+1}(a, b) = \rho_{r+1}(\mu_0(a, b)) + \sum_{s=0}^{r} \rho_s(\mu_{r+1-s}(a, b)) - \rho_{r+1}(b) \circ \rho_0(a) - \rho_0(b) \circ \rho_{r+1}(a) - \sum_{s=1}^{r} \rho_s(b) \circ \rho_{r+1-s}(a)
\]

\[
= - (\delta \rho_{r+1})(a, b) + R_r(a, b)
\]

with \( R_r \in HC^2_{type}(A, D) \) as in (2.51). Clearly, \( R_0 = \rho_0 \circ \mu_1 \). Then, condition (2.49) follows immediately. Since \( C_0 = \cdots = C_r = 0 \) Equation (2.51) yields in order \( r + 1 \)

\[
0 = C_{r+1}(b, c) \circ \rho_0(a) - C_{r+1}(\mu_0(a, b), c) + C_{r+1}(a, \mu_0(b, c)) - \rho_0(c) \circ C_{r+1}(a, b)
\]

\[
= (\delta C_{r+1})(a, b, c)
\]

\[
= (\delta R_r)(a, b, c)
\]

where one makes use of \( \delta^2 = 0 \). So \( \delta R_r = 0 \) and the remaining assertions follow.

In particular, the proposition states that if the obstruction vanishes an order by order construction can be done which yields a deformation \( \rho \) of the desired type. So in this case the existence of a deformation is given.

**Corollary 2.4.4 (Existence of module deformations)**

If the second Hochschild cohomology of a right module structure \( \rho_0 \) of a particular type is trivial, \( HH^2_{type}(A, D) = \{0\} \), all deformations up to a finite order can be extended to deformations.

In a similar way one gets the following statement concerning the equivalence of two given deformations.

**Proposition 2.4.5**

In the setting of Definition 2.4.4, let \( \rho \) and \( \tilde{\rho} \) be two deformations of \( \rho_0 \) of a particular type and let \( T^{(r)} = \text{id} + \cdots + \lambda^r T_r \) be an equivalence transformation between them up to order \( r \) with \( T_s \in HC^0_{type}(A, D) \) for \( s = 0, \ldots, r \). Then the condition for \( T_{r+1} \in HC^0_{type}(A, D) \) to define an equivalence transformation \( T^{(r+1)} = T^{(r)} + \lambda^{r+1} T_{r+1} \) up to order \( r + 1 \) is

\[
\delta T_{r+1} = E_r
\]

(2.52)

with \( E_r \in HC^1_{type}(A, D) \) explicitly given by

\[
E_r(a) = \sum_{s=0}^{r} (\tilde{\rho}_{r+1-s}(a) \circ T_s - T_s \circ \rho_{r+1-s}(a)).
\]

(2.53)

Moreover, \( \delta E_r = 0 \) whence the recursive obstruction in order \( r + 1 \) is the class \([E_r] \in HH^1_{type}(A, D)\).

**Proof:** Again, the proof starts with some well-known preliminary considerations. The failure of arbitrary \( T_r \in HC^0_{type}(A, D), r \in \mathbb{N} \), to yield an equivalence transformation \( T = \text{id} + \sum_{r=1}^{\infty} \lambda^r T_r \) is now encoded in the \( K[[\lambda]] \)-linear map \( C : A[[\lambda]] \rightarrow \text{End}_{K[[\lambda]]}(E[[\lambda]], \beta[[\lambda]]) \) with \( C(a) = T \circ \rho(a) - \tilde{\rho}(a) \circ T \). Again, \( C = \sum_{r=0}^{\infty} \lambda^r C_r \) with \( C_r \in HC^1_{type}(A, D) \) given by \( C_r(a) = \sum_{s=0}^{\infty} [T_s \circ \rho_{r-s}(a) - \rho_{r-s}(a) \circ T_{r-s}] \). Due to the right module property of \( \rho \) and \( \tilde{\rho} \) the map \( C \) satisfies the equation

\[
C(\mu(a, b)) = T \circ \rho(\mu(a, b)) - \tilde{\rho}(\mu(a, b)) \circ T
\]

\[
= T \circ \rho(b) \circ \rho(a) - \tilde{\rho}(b) \circ \tilde{\rho}(a) \circ T
\]

(2.54)

\[
= C(b) \circ \rho(a) + \tilde{\rho}(b) \circ C(a).
\]
Now, by assumption there are given $T_0 = \text{id}, \ldots, T_r$ such that $C_0 = \cdots = C_r = 0$. Since $\rho$ and $\tilde{\rho}$ coincide in zeroth order the failure of the extension $T^{(r+1)}$ for an arbitrary $T_{r+1} \in \text{HC}^0_{\text{type}}(A, \mathcal{D})$ to be an equivalence transformation up to order $r + 1$ reads

$$C_{r+1}(a) = T_{r+1} \circ \rho_0(a) - \rho_0(a) \circ T_{r+1} + \sum_{s=0}^{r} (T_s \circ \rho_{r+1-s}(a) - \tilde{\rho}_{r+1-s}(a) \circ T_s)$$

with $E_r \in \text{HC}^1_{\text{type}}(A, \mathcal{D})$ as in (2.54). Then, condition (2.52) follows immediately. Finally, Equation (2.54) yields in order $r + 1$

$$0 = C_{r+1}(b) \circ \rho_0(a) - C_{r+1}(\mu_0(a, b)) + \rho_0(b) \circ C_{r+1}(a)$$

$$= (\delta C_{r+1})(a, b)$$

$$= - (\delta E_r)(a, b),$$

and thus $\delta E_r = 0$.

Again, the proposition assures that if the obstruction vanishes the orderwise construction of equivalence transformations can be carried out.

**Corollary 2.4.6 (Equivalence of module deformations)**

If the first Hochschild cohomology of a right module structure $\rho_0$ of a particular type is trivial, $\text{HH}^1_{\text{type}}(A, \mathcal{D}) = \{0\}$, the module structure is rigid. This means that all deformations of $\rho$ with respect to the same deformation of the underlying algebra, are equivalent.

Note that even if there occur obstructions in higher orders, this means $[E_r] \neq 0$ in $\text{HH}^1_{\text{type}}(A, \mathcal{D})$, it could still be possible to construct equivalence transformations order by order in a more complicated way than in the presented one. For example it could still be possible to change the already found $T_1, \ldots, T_r$ in order to find an adequate $T_{r+1}$.

**Remark 2.4.7**

If the algebra $(A, \mu_0)$ is commutative, it is obvious that all the results concerning module deformations do not depend on the choice between the two possible bimodule structures (2.28) and (2.33) which yield different Hochschild complexes. In both cases the statements are exactly the same and which complex finally will be chosen is just a matter of convenience.

Before we come to a more concrete example in the subsequent subsection we make the following interesting general observation.

**Proposition 2.4.8 (Rigidity of the unit)**

Let $(A, \mu_0)$ be a unital algebra with unit 1 together with a deformation $\mu = \sum_{r=0}^{\infty} \lambda^r \mu_r$ such that $\mu(a, 1) = a = \mu(1, a)$ for all $a \in A[[\lambda]]$. Further, let $(E, \rho_0)$ be a right $A$-module with $\rho_0(1) = \text{id}_E$.

Then, every deformation $\rho = \sum_{r=0}^{\infty} \lambda^r \rho_r$ of $\rho_0$ with respect to $\mu$ satisfies

$$\rho(1) = \text{id}_E[[\lambda]]$$

**Proof:** Due to the assumptions, $\rho(1) = \text{id}_E + \sum_{r=0}^{\infty} \lambda^r \rho_r(1)$ is an invertible map. Then (2.55) follows from $\rho(1) \circ \rho(1) = \rho(1)$ which is a consequence of the given right module property.

Note that the assertion in Proposition 2.4.8 does not depend on a particular type and is thus valid in general.
2.5 The commutant of a deformed module structure

The set of module endomorphisms was already introduced in the end of Section 2.1.3 and can be considered for every module structure. Now we come back to this topic and investigate the relation between the commutants of a given right module structure and its deformations. Whenever one is given a right module structure \( \rho_0 : \mathcal{E} \times \mathcal{A} \to \mathcal{E} \) of a particular type as in Definition 2.4.1 one is interested in the module endomorphisms which are operators with the correct behaviour as well. In general, the commutant of interest is the one within the subalgebra interested in the module endomorphisms which are operators with the correct behaviour as well.

The set of module endomorphisms was already introduced in the end of Section 2.1.3 and can be given a right module structure \( \rho \) between the commutants of a given right module structure and its deformations. Whenever one is considered for every module structure. Now we come back to this topic and investigate the relation for all \( a \in \mathcal{A} \).

\[
\mathcal{B}_0 = \left\{ A \in \mathcal{D} : A \circ \rho_0(a) = \rho_0(a) \circ A \text{ for all } a \in \mathcal{A} \right\}. \tag{2.56}
\]

It is clear that \( \mathcal{B}_0 \) is a subalgebra of \( \mathcal{D} \) with the usual composition \( \circ \) of maps as multiplication. It will be referred to as classical or undeformed commutant.

In analogy to this the commutant of a deformation \( \rho \) of \( \rho_0 \) with respect to a deformation \( \mu \) of \( \mu_0 \) as in Definition 2.4.1 is defined by

\[
\mathcal{K}_0 = \left\{ A \in \mathcal{D}[[\lambda]] : A \circ \rho(a) = \rho(a) \circ A \text{ for all } a \in \mathcal{A}[[\lambda]] \right\}. \tag{2.57}
\]

If a formal series \( A = \sum_{r=0}^{\infty} \lambda^r A_r \in \mathcal{D}[[\lambda]] \) is an element of \( \mathcal{K}_0 \) the condition in the lowest order of \( \lambda \) shows that \( A_0 \in \mathcal{B}_0 \) is an element of the classical commutant. This is of course a motivation to investigate the relation between the commutant \( \mathcal{K}_0 \) and the formal power series \( \mathcal{B}_0[[\lambda]] \).

This investigation is possible with homological techniques for the commutants within the subalgebra \( \text{HC}^0_{\text{type}}(\mathcal{A}, \mathcal{D}) \subseteq \mathcal{D} \). For the undeformed situation this means to consider the zeroth Hochschild cohomology

\[
\mathcal{B} = \text{HH}^0_{\text{type}}(\mathcal{A}, \mathcal{D}) = \ker \delta^0 \subseteq \text{HC}^0_{\text{type}}(\mathcal{A}, \mathcal{D}) \tag{2.58}
\]

which is again a subalgebra \( \mathcal{B} \subseteq \text{HC}^0_{\text{type}}(\mathcal{A}, \mathcal{D}) \) and a subalgebra \( \mathcal{B} \subseteq \mathcal{B}_0 \). The new commutant for the deformed structure is now given by

\[
\mathcal{K} = \left\{ A \in \text{HC}^0_{\text{type}}(\mathcal{A}, \mathcal{D})[[\lambda]] : A \circ \rho(a) = \rho(a) \circ A \text{ for all } a \in \mathcal{A}[[\lambda]] \right\}. \tag{2.59}
\]

Note that one really has

\[
\mathcal{B} = \mathcal{B}_0 \quad \text{and} \quad \mathcal{K} = \mathcal{K}_0 \tag{2.60}
\]

if the right module structure is of a particular type with the additional property that

\[
\text{HC}^0_{\text{type}}(\mathcal{A}, \mathcal{D}) = \mathcal{D}. \tag{2.61}
\]

A comparison of (2.59) with the notion of an equivalence transformation between deformed right module structures as in Definition 2.4.1 shows that the elements in \( \mathcal{K} \) have the same property as the self equivalence transformations of \( \rho \) but do not necessarily start with the identity in the lowest order. Considering the proof of Proposition 2.4.5 this yields a guideline how to construct an element of \( \mathcal{K} \) from one of \( \mathcal{B} \). For the following proposition we will use the fact that it is always possible to decompose \( \text{HC}^0_{\text{type}}(\mathcal{A}, \mathcal{D}) \) into a direct sum

\[
\text{HC}^0_{\text{type}}(\mathcal{A}, \mathcal{D}) = \mathcal{B} \oplus \overline{\mathcal{B}} = \ker \delta^0 \oplus \overline{\mathcal{B}} \tag{2.62}
\]

of \( \mathcal{B} \) and a complementary linear subspace \( \overline{\mathcal{B}} \subseteq \text{HC}^0_{\text{type}}(\mathcal{A}, \mathcal{D}) \) since we work over a field \( \mathbb{K} \).

**Proposition 2.5.1 (The commutant of a deformed right module structure)**

Let \( (\mathcal{E}, \rho_0)_{(\mathcal{A}, \mu_0)} \) be a right module structure of a particular type and let \( \rho \) be a corresponding deformation of \( \rho_0 \) with respect to a deformation \( \mu \) as in Definition 2.4.1. Further, let the first Hochschild cohomology be trivial,

\[
\text{HH}^1_{\text{type}}(\mathcal{A}, \mathcal{D}) = \{0\}. \tag{2.63}
\]
Then, every choice of a complementary subspace \( \overline{\mathcal{B}} \) induces a unique \( \mathbb{K}[[\lambda]] \)-linear bijection
\[
\rho' : \mathcal{B}[[\lambda]] \to \mathcal{K} \subseteq \mathcal{H}C^{0}_{\text{type}}(\mathcal{A}, \mathcal{D})[[\lambda]]
\]
of the form \( \rho' = \text{id} + \sum_{r=1}^{\infty} \lambda^r \rho'_r \) with \( \mathbb{K} \)-linear maps
\[
\rho'_r : \mathcal{B} \to \overline{\mathcal{B}} \subseteq \mathcal{H}C^{0}_{\text{type}}(\mathcal{A}, \mathcal{D})
\]
for \( r \geq 1 \).

**Proof:** As stated above, an element \( A \in \mathcal{B} \) can be seen as the zeroth order of an element \( \rho'(A) \in \mathcal{K} \) which shall be constructed recursively order by order in a unique way so that it is the value of \( A \) under a map \( \rho' = \text{id} + \sum_{r=1}^{\infty} \lambda^r \rho'_r \). To this end the results of Proposition 2.4.5 can be extended to the present situation. Assume to have already found \( \rho_r(A) \in \mathcal{K} \) satisfying the crucial condition up to order \( r \), this means that \( \rho_r(A) = \rho(r) \circ \rho(a) - \rho(a) \circ \rho(r)(A) = \lambda^{r+1}C_{r+1} + \cdots \) for all \( a \in \mathcal{A} \). The condition for extending this to an element \( \rho_{r+1}(A) = \rho_r(A) + \lambda^{r+1}\rho'_{r+1}(A) \) satisfying the crucial condition up to order \( r+1 \) is
\[
(\delta(\rho'_{r+1}(A)))(a) = E_r(a) = \sum_{s=0}^{r} (\rho_{r+1-s}(a) \circ \rho'_{s}(A) - \rho'_{s}(A) \circ \rho_{r+1-s}(a))
\]
with \( \delta E_r = 0 \) in analogy to Proposition 2.4.5. Since the first Hochschild cohomology is trivial such an element \( \rho_{r+1}(A) \in \mathcal{H}C^{0}_{\text{type}}(\mathcal{A}, \mathcal{D}) \) always exists and it is even unique with the further condition
\[
\rho'_{r+1}(A) \in \overline{\mathcal{B}}
\]
for a choice \( \mathcal{H}C^{0}_{\text{type}}(\mathcal{A}, \mathcal{D}) = \ker \delta^0 \oplus \overline{\mathcal{B}} \). This procedure in fact yields an injective map
\[
\rho' = \text{id} + \sum_{r=0}^{\infty} \lambda^r \rho'_r : \mathcal{B} \to (\mathcal{B} + \lambda \overline{\mathcal{B}}[[\lambda]]) \cap \mathcal{K}.
\]
The maps \( \rho'_r : \mathcal{B} \to \overline{\mathcal{B}} \) are \( \mathbb{K} \)-linear which is seen by an easy induction over \( r \) using the linearity of the defining conditions (2.66) and (2.67). The \( \mathbb{K}[[\lambda]] \)-linear extension of \( \rho' \) has again values in \( \mathcal{K} \) and is not only injective but also surjective which is seen as follows. Let \( A = \sum_{r=0}^{\infty} \lambda^r A_r \in \mathcal{K} \). Then \( A_0 \in \mathcal{B} \) and \( A - \rho'(A_0) = \sum_{r=1}^{\infty} \lambda^r B_r \in \mathcal{K} \) starts in order one with \( B_1 \in \mathcal{B} \). This way, induction finally yields a preimage \( A_0 + \lambda B_1 + \cdots \) of \( A \).

By definition, the commutant \( \mathcal{B} \) yields a \( (\mathcal{B}, A) \)-bimodule structure of \( \mathcal{E} \), denoted by
\[
(\mathcal{B}, \circ)(\text{id}_\mathcal{B}, E, \rho_0)(A, \mu_0).
\]

This notation is reasonable, since \( \text{id}_\mathcal{B} : \mathcal{B} \to \text{End}_\mathcal{K}(\mathcal{E}, \mathcal{E}) \) can be seen as a left module structure in the same way as \( \rho_0 \) is a right module structure. The above proposition implies that the bimodule (2.66) has a corresponding counterpart.

**Corollary 2.5.2 (Induced deformations of the classical commutant)**

In the situation of Proposition 2.5.1 the choice of a complementary subspace \( \overline{\mathcal{B}} \) immediately induces the following additional structures.

i.) The isomorphism \( \rho' \) yields an associative deformation
\[
\mu' = \circ + \sum_{r=1}^{\infty} \lambda^r \mu'_r : \mathcal{B}[[\lambda]] \times \mathcal{B}[[\lambda]] \to \mathcal{B}[[\lambda]]
\]
of the classical commutant by
\[
\mu'(A, B) = \rho^{-1}(\rho'(A) \circ \rho'(B)).
\]
ii.) \( \rho' \) further induces a deformed left \((B[[\lambda]], \mu')\)-module structure on \( E[[\lambda]] \) of the usual action of \( B \subseteq \text{End}_K(E, E) \) on \( E \), such that \( E[[\lambda]] \) becomes a bimodule with respect to the two deformed algebras \((B[[\lambda]], \mu')\) and \((A[[\lambda]], \mu)\).

**Proof:** The map \( \mu' \) defined in (2.71) starts with \( \circ \) since the isomorphism \( \rho' \) and its inverse start with the identity in the lowest order in \( \lambda \). The associativity is clear with the one of the composition \( \circ \) of maps. The left module structure in the second part is obvious by (2.71). The bimodule structure is a direct consequence of the fact that \( \rho' \) only takes values in the commutant.

The next definition presents a natural notion of deformations of bimodule structures.

**Definition 2.5.3 (Deformation of bimodules)**

Let \((A, \mu_0)\) and \((B, \mu'_0)\) be associative \( K \)-algebras of particular types. Further, let \( E \) be a \( K \)-vector space endowed with a right module structure \( \rho_0 \in \text{HC}^1_{\text{type}}(A, D) \) compatible with \( \text{HC}^1_{\text{type}}(A, A) \) and a left module structure \( \rho'_0 \in \text{HC}^1_{\text{type}}'(B, D') \) compatible with \( \text{HC}^1_{\text{type}}'(B, B) \) such that \( E \) is a \((B, A)\)-bimodule, denoted by

\[
(B, \mu'_0)(\rho'_0, E, \rho_0)(A, \mu_0).
\] (2.72)

i.) A deformation of this bimodule (2.72) is a bimodule

\[
(B[[\lambda]], \mu')((\rho', E[[\lambda]]), \rho)(A[[\lambda]], \mu)
\] (2.73)

where the left and right module structures \( \rho' \) and \( \rho \) are deformations with respect to corresponding algebra deformations \( \mu' \) and \( \mu \) as in Definition 2.4.1.

ii.) Two such deformations \((\mu', \rho', \rho, \mu)\) and \((\tilde{\mu}', \tilde{\rho}', \tilde{\rho}, \tilde{\mu})\) are said to be equivalent if there exist formal series

\[
S' = \text{id}_B + \sum_{r=1}^{\infty} \lambda^r S'_r, \quad T' = \text{id}_E + \sum_{r=1}^{\infty} \lambda^r T'_r
\] (2.74)

with \( S'_r \in \text{HC}^1_{\text{type}}(B, B), T'_r \in \text{HC}^0_{\text{type}}'(B, D') \) and

\[
S = \text{id}_A + \sum_{r=1}^{\infty} \lambda^r S_r, \quad T = \text{id}_E + \sum_{r=1}^{\infty} \lambda^r T_r
\] (2.75)

with \( S_r \in \text{HC}^1_{\text{type}}(A, A), T_r \in \text{HC}^0_{\text{type}}(A, D) \) such that

\[
S' \circ \mu' = \tilde{\mu}' \circ (S' \otimes S'), \quad S \circ \mu = \tilde{\mu} \circ (S \otimes S)
\] (2.76)

and

\[
T' \circ \rho'(A) = \tilde{\rho}'(S'A) \circ T', \quad T \circ \rho(a) = \tilde{\rho}(Sa) \circ T
\] (2.77)

for all \( A \in B[[\lambda]] \) and \( a \in A[[\lambda]] \).

In the special situation where

\[
\text{HC}^0_{\text{type}}(A, D) = \text{HC}^0_{\text{type}}'(B, D')
\] (2.78)

the two deformations are said to be equivalent if the equivalence transformations on \( E[[\lambda]] \) coincide, this means if

\[
T = T'.
\] (2.79)

iii.) The deformations up to order \( r \in \mathbb{N} \) as well as their equivalence are defined in a corresponding way.
Remark 2.5.4

i.) The equations in (2.76) state nothing but the usual equivalence of deformed algebras where the algebra multiplications are identified with the corresponding linear maps of the tensor product as explained in Remark 2.2.1.

ii.) The equivalence of deformed bimodules in general is just the separate equivalence of the left and the right module structures, now with respect to equivalent algebras and not to fixed ones as in Definition 2.4.1.

iii.) Note that the necessary condition for (2.78) is that either

\[
\text{i.) If the first Hochschild cohomology } \HH^1(A, \mathcal{D}) \text{ is trivial, this means if } \HH^1_{\text{type}}(A, \mathcal{D}) = \{0\} \text{ and } \HH^2_{\text{type}}(A, \mathcal{D}) = \{0\}, \text{ there is a map}
\]

\[
\text{Def}_{\text{type}}(A) \longrightarrow \text{Def}(\HH^0_{\text{type}}(A, \mathcal{D})),
\]

where Def_{\text{type}} denotes the set of equivalence classes of associative deformations of a particular type.

Proposition 2.5.5 (The uniqueness of the induced deformations)

Let \((\mathcal{E}, \rho_0)_{(A, \mu_0)}\) be a right module structure of a particular type.

i.) If the first Hochschild cohomology \(\HH^1_{\text{type}}(A, \mathcal{D}) = \{0\}\) is trivial and \(\mu\) is a given deformation of \(\mu_0\) different choices of a corresponding deformation \(\rho\) of \(\rho_0\) and a complementary subspace \(\mathcal{D}\) in Proposition 2.5.4 yield equivalent deformations \(\mu'\) of the commutant \((\mathcal{B}, \circ)\).

ii.) Furthermore, if the first Hochschild cohomology \(\HH^1_{\text{type}}(A, \mathcal{D}) = \{0\}\) is trivial, equivalent deformations \(\mu_0\) and corresponding choices as above yield equivalent deformations of the bimodule structure (2.69) in the sense of Definition 2.5.3.

iii.) If the first two Hochschild cohomologies are trivial, this means if \(\HH^1_{\text{type}}(A, \mathcal{D}) = \{0\}\) and \(\HH^2_{\text{type}}(A, \mathcal{D}) = \{0\}\), there is a map

\[
\text{Def}_{\text{type}}(A) \longrightarrow \text{Def}(\HH^0_{\text{type}}(A, \mathcal{D})),
\]

where Def_{\text{type}} denotes the set of equivalence classes of associative deformations of a particular type.

Proof: For the first part let \(\rho\) and \(\tilde{\rho}\) be two deformations of \(\rho_0\). Due to the assumption \(\HH^1_{\text{type}}(A, \mathcal{D}) = \{0\}\) and Corollary 2.4.6 these are equivalent in the sense of Definition 2.4.1. So there exists an equivalence transformation \(T = \text{id}_E + \sum_{r=1}^\infty \lambda^r T_r\) with \(T \circ \rho(a) = \tilde{\rho}(a) \circ T\) for all \(a \in A[[\lambda]]\). Obviously, the invertible map \(T\) gives rise to an isomorphism

\[
\text{Conj}_T : K_\rho \longrightarrow K_{\tilde{\rho}}, \quad \text{Conj}_T A = T \circ A \circ T^{-1} \quad \text{for } A \in K_\rho,
\]

where \(K_\rho\) and \(K_{\tilde{\rho}}\) denote the corresponding commutants. For each of these deformations we now consider a choice of a complementary subspace \(\mathcal{D}\) and get the corresponding maps \(\rho'\) and \(\tilde{\rho}'\) as in Proposition 2.5.1. They are invertible and so it is possible to define the map

\[
S' = \tilde{\rho}'^{-1} \circ \text{Conj}_T \circ \rho' = \text{id}_B + \sum_{r=1}^\infty \lambda^r S'_r : B[[\lambda]] \longrightarrow B[[\lambda]]
\]

(2.83)
which obviously is of the stated form with $S'_0 \in \text{HC}^1(\mathcal{B}, \mathcal{B})$. Then this $S'$ is an equivalence transformation between the two deformations $\mu'$ and $\tilde{\mu}'$ induced by $\rho'$ and $\tilde{\rho}'$ since by definition

\[
(S' \circ \mu')(A, B) = (\tilde{\rho}'^{-1} \circ \text{Conj}_T \circ \rho' \circ \rho'^{-1})(\rho'(A) \circ \rho'(B)) = \tilde{\rho}'^{-1}(T \circ \rho'(A) \circ T^{-1} \circ T \circ \rho'(B) \circ T^{-1}) = \tilde{\rho}'^{-1}(\tilde{\rho}'(S'A) \circ \tilde{\rho}'(S'B)) = (\tilde{\mu}' \circ (S' \otimes S'))(A, B). \tag{2.84}
\]

The statement of the second part can be easily traced back to the first part. So let $S$ be an equivalence transformation between the two deformations $\mu$ and $\tilde{\mu}$ as in Definition 2.5.3, this means $S \circ \mu = \tilde{\mu} \circ (S \otimes S)$. For two corresponding deformations $\rho$ and $\tilde{\rho}$ of $\rho_0$ one now defines

\[
\tilde{\rho} = \rho \circ S^{-1} \tag{2.85}
\]

which is a deformation of $\rho_0$ with respect to $\tilde{\mu}$ because of $\tilde{\rho}(\tilde{\mu}(a, b)) = (\rho \circ S^{-1})(\tilde{\mu}(a, b)) = \rho(S^{-1}a, S^{-1}b) = \rho(S^{-1}b) \circ \rho(S^{-1}a) = \tilde{\rho}(b) \circ \tilde{\rho}(a)$ for all $a, b \in A[[\lambda]]$. Thus $\tilde{\rho}$ is equivalent to $\tilde{\rho}$ via some $T$ as before, this means $T \circ \tilde{\rho}(a) = \tilde{\rho}(a) \circ T$. Then, the maps $S, T$ and the there-with constructed map $S'$ as in (2.83) are the equivalence transformations between $(\mu', \rho', \rho, \mu)$ and $(\tilde{\mu}', \tilde{\rho}', \tilde{\rho}, \tilde{\mu})$. The computation (2.84) is exactly the same and the new $T$ immediately leads to $T \circ \rho(a) = T \circ \tilde{\rho}(Sa) = \tilde{\rho}(Sa) \circ T$ and $T \circ \rho'(A) = T \circ \rho'(A) \circ T^{-1} \circ T = (\text{Conj}_T \circ \rho')(A) \circ T = \tilde{\rho}'(S'(A)) \circ T$.

The third part is a direct consequence of the second one. Since in addition the second Hochschild cohomology is trivial the deformations $\rho$ always exist and all the above structures can be constructed which finally yields the stated map due to the given uniqueness up to equivalence.

\section*{2.6 Invariant algebra and module structures}

In this section we investigate the particular situation where the algebra and module structures of a particular type are additionally invariant under the action of a group $G$. Although it is not used in this work it is remarkable that the discussion of $G$-invariant deformations of algebra structures, especially $G$-invariant star products, is still an area of vivid research. The basic notion was given in \cite{9}. Moreover, if $G$ is a Lie group it is also interesting to discuss the infinitesimal version of $G$-invariance, this means the invariance under the induced representation of the Lie algebra $\mathfrak{g}$ of $G$. For more details and the physical applications of invariant star products, in particular in the framework of phase space reduction and quantum momentum mappings, the reader is referred to the references \cite{18, 95, 96}.

For our purpose we concentrate on $G$-invariant module structures and if necessary the $G$-invariant deformation of the underlying algebra is assumed to be given. This general case is discussed but later we will be in the easier situation where the action on the algebra is trivial. In the following, it will be pointed out in which cases the $G$-invariant algebra and module structures define a new particular type. As it will be seen, the crucial condition for that is the invariance of the involved vector spaces. Besides this we can make interesting additional observations concerning the commutant of invariant deformations. This will play an important role in the applications.

First of all we introduce the usual notation and recall some well-known definitions. Given an arbitrary action of a group $G$ on a set $V$, this means a group homomorphism $G \rightarrow \text{Aut}(V)$ into the automorphisms of $V$, we simply write

\[
g \triangleright v \tag{2.86}
\]

for the action of a group element $g \in G$ on an element $v \in V$. If $V$ is a vector space over some field $\mathbb{K}$ the action is called a representation if all group elements act by $\mathbb{K}$-linear maps. The space of all
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$G$-invariant elements is denoted by

$$V^G = \{ v \in V \mid g \triangleright v = v \text{ for all } g \in G \}. \quad (2.87)$$

The orbit of an element $v \in V$ is defined as the set

$$G \triangleright v = \{ g \triangleright v \mid g \in G \} \quad (2.88)$$

Analogously, one defines the sets $g \triangleright W = \{ g \triangleright w \mid w \in W \}$ for linear subspaces $W \subseteq V$. Such a $W \subseteq V$ is called invariant if $G \triangleright W = \{ g \triangleright w \mid g \in G, w \in W \} \subseteq W$. Having representations on $K$-vector spaces $V_1, \ldots, V_k$, $k \in \mathbb{N}$, and $W$ one always has a corresponding representation on the $K$-vector space $\text{Hom}_K(V_1 \times \cdots \times V_k, W)$ of multilinear maps. Explicitly, this representation is defined by

$$(g \triangleright \phi)(v_1, \ldots, v_k) = g \triangleright (\phi(g^{-1} \triangleright v_1, \ldots, g^{-1} \triangleright v_k)) \quad (2.89)$$

for all $g \in G$ with inverse $g^{-1}$, $v_i \in V_i$ and $\phi \in \text{Hom}_K(V_1 \times \cdots \times V_k, W)$. In the literature the $G$-invariant elements $g \triangleright \phi = \phi$ of this representation are often called $G$-equivariant maps since they mediate the actions,

$$g \triangleright \phi(v_1, \ldots, v_k) = \phi(g \triangleright v_1, \ldots, g \triangleright v_k). \quad (2.90)$$

By the above definition the composition of maps is always $G$-invariant, this means

$$g \triangleright (\phi \circ \psi) = (g \triangleright \phi) \circ (g \triangleright \psi) \quad (2.91)$$

for all accordingly given maps $\phi$ and $\psi$ between sets with a $G$-action. The following definition is obvious.

**Definition 2.6.1 (G-invariant algebra and module structures)**

An associative $K$-algebra $(A, \mu)$ is said to be $G$-invariant with respect to a representation of a group $G$ on the vector space $A$ if the group acts by algebra automorphisms, this means if

$$g \triangleright \mu(a, b) = \mu(g \triangleright a, g \triangleright b) \quad \text{for all } a, b \in A, g \in G. \quad (2.92)$$

If this is the case a right $(A, \mu)$-module structure $\rho$ of a $K$-vector space $E$ is said to be $G$-invariant with respect to a further representation of $G$ on $E$ if

$$g \triangleright \rho(e, a) = \rho(g \triangleright e, g \triangleright a) \quad \text{for all } e \in E, a \in A, g \in G. \quad (2.93)$$

The Hochschild complexes of algebras $A$ and modules $E$ as defined in the Sections 2.1.2 and 2.1.3 basically consist of maps between those vector spaces. Thus it is clear that representations of a group $G$ on $A$ and $E$ induce representations on the vector spaces of these complexes. Then, the above definition contains nothing but the $G$-invariance $g \triangleright \mu = \mu$ and $g \triangleright \rho = \rho$ of the corresponding cochains.

**Definition 2.6.2 (Algebra and module structures of a particular $G$-invariant type)**

If $\mu$ and $\rho$ as in Definition 2.6.1 are $G$-invariant structures of a particular type as in Section 2.2 they are said to be of a corresponding $G$-invariant type if all vector spaces occurring in the complexes $\text{HC}_{\text{type}}^{\bullet}(A, A)$ and $\text{HC}_{\text{type}}^{\bullet}(A, D)$ are $G$-invariant, this means if

$$G \triangleright \text{HC}_{\text{type}}^k(A, A) \subseteq \text{HC}_{\text{type}}^k(A, A),$$

$$G \triangleright \text{HC}_{\text{type}}^k(A, D) \subseteq \text{HC}_{\text{type}}^k(A, D),$$

and

$$G \triangleright D \subseteq D. \quad (2.94)$$
Since all operations on the considered complexes are defined by the composition of maps and the given $G$-invariant algebra and module structures, they are $G$-invariant as well. This means that the insertions $\sigma_i$ after the $i$-th position, the cup product $\cup$ and the Hochschild differential $\delta$ have the properties

$$g \triangleright (\phi \circ_i \psi) = (g \triangleright \phi) \circ_i (g \triangleright \psi), \quad (2.95)$$
$$g \triangleright (\phi \cup \psi) = (g \triangleright \phi) \cup (g \triangleright \psi), \quad (2.96)$$
$$g \triangleright (\delta \phi) = \delta(g \triangleright \phi), \quad (2.97)$$

for all possible cochains $\phi, \psi \in HC^\bullet_\text{type}(A, A)$ or $HC^\bullet_\text{type}(A, D)$ and $g \in G$. Further it is clear that $D$ is a $G$-invariant $(A, A)$-bimodule, this means

$$g \triangleright (a \cdot D \cdot b) = (g \triangleright a) \cdot (g \triangleright D) \cdot (g \triangleright b) \quad (2.98)$$

for all $g \in G$, $a, b \in A$, $D \in D$.

These consequences justify Definition 2.6.2 since structures of $G$-invariant types in fact define new particular types in the sense of the Definitions 2.2.2 and 2.2.3. One simply has to consider the complexes $HC^\bullet_\text{type}(A, A)^G$ and $HC^\bullet_\text{type}(A, D)^G$ of $G$-invariant cochains. For degree zero one of course has $HC^0(A, A)^G = A$. Note, that by the definition 2.80 all $G$-invariant $k$-cochains $\phi \in HC^k_\text{type}(A, D)$ indeed obey the equation

$$g \triangleright (\phi(a_1, \ldots, a_k)(e)) = \phi(g \triangleright a_1, \ldots, g \triangleright a_k)(g \triangleright e). \quad (2.99)$$

**Remark 2.6.3 (Representations and cohomology)**

For $G$-invariant types Equation (2.37) implies that the cocycles and coboundaries of the initial Hochschild complex $HC^\bullet_\text{type}(A, D)$ are invariant vector subspaces. Thus one has a representation on the cohomologies as well. But note that the cohomology of the $G$-invariant type in general is not the same as the $G$-invariant cohomology of the initial type. In formulas this means that for all $k \geq 1$

$$H^k(HC^\bullet_\text{type}(A, D)^G) \neq HH_k^\bullet(A, D)^G. \quad (2.100)$$

The only exception is given by the zeroth cohomology. There one has

$$H^0(HC^\bullet_\text{type}(A, D)^G) = HH_0^\bullet(A, D)^G \quad (2.101)$$

since both sides are given by the elements $\phi \in HC^0_\text{type}(A, D)$ with $g \triangleright \phi = \phi$ and $\delta \phi = 0$.

**Remark 2.6.4 (Modules with a trivial representation on the algebra)**

If the representation of $G$ on the algebra $A$ is trivial, meaning that all group elements act by the identity map, the Hochschild complex of the $G$-invariant right module structure $\rho$ is given by

$$HC^\bullet_\text{type}(A, D)^G = HC^\bullet_\text{type}(A, D)^G. \quad (2.102)$$

With the above statements all results from the Sections 2.3, 2.4 and 2.5 can be applied in order to investigate deformations of right modules that inherit a given $G$-invariance. The crucial cohomology then is of course $H^\bullet(HC^\bullet_\text{type}(A, D)^G)$. If the first one is trivial, $H^1(HC^\bullet_\text{type}(A, D)^G) = \{0\}$, Proposition 2.5.1 states that with the notation $\mathcal{B} = HH_0^\bullet(A, D)$ any decomposition

$$HC^1_\text{type}(A, D)^G = \mathcal{B}^G \oplus \mathcal{K}^G \quad (2.103)$$

leads to a bijection $\rho' : \mathcal{B}^G[[\lambda]] \rightarrow \mathcal{K}^G$ between the commutants within $HC^0_\text{type}(A, D)^G[[\lambda]]$. Thus one has the property

$$g \triangleright \rho'(A)(e) = \rho'(A)(g \triangleright e). \quad (2.104)$$
In most applications, however, one is rather interested in the commutants within \( HC_0^{\text{type}}(A, \mathcal{D})[[\lambda]] \), or even \( \mathcal{D}[[\lambda]] \), and wants to find a bijection \( \rho' : \mathcal{B}[[\lambda]] \rightarrow \mathcal{K} \) such that the induced deformed bimodule is \( G \)-invariant like the undeformed one in \( (2.69) \). This is the case if the complement of \( \mathcal{B} \) is an invariant vector space.

**Proposition 2.6.5 (The commutant of a \( G \)-invariant module structure)**

Let \( \rho \) be a \( G \)-invariant deformation of a right module structure \( \rho_0 \) of a particular \( G \)-invariant type and let \( HH_1^{\text{type}}(A, \mathcal{D}) = \{0\} \) as in Proposition 2.5.1. If the complementary subspace \( \overline{\mathcal{B}} \) is \( G \)-invariant, this means

\[
G \triangleright \overline{\mathcal{B}} \subseteq \overline{\mathcal{B}},
\tag{2.105}
\]

the induced bijection \( \rho' : \mathcal{B}[[\lambda]] \cong \mathcal{K} \) and the deformed algebra structure \( (\mathcal{B}[[\lambda]], \mu') \) as in Corollary 2.5.2 are \( G \)-invariant. Explicitly, this means that

\[
g \triangleright (\rho'(A)(e)) = \rho'(g \triangleright A)(g \triangleright e),
\tag{2.106}
\]

\[
g \triangleright (\mu'(A, B)) = \mu'(g \triangleright A, g \triangleright B)
\tag{2.107}
\]

for all \( A, B \in \mathcal{B}[[\lambda]], e \in \mathcal{E}[[\lambda]] \) and \( g \in G \).

Moreover, if additionally the first \( G \)-invariant cohomology is trivial,

\[
H^1(HC_1^{\text{type}}(A, \mathcal{D})) = \{0\},
\tag{2.108}
\]

the so derived \( G \)-invariant deformations \( \rho' \) and \( \mu' \) and the induced \( G \)-invariant bimodule are unique up to \( G \)-invariant equivalence in analogy to Proposition 2.5.7.

**Proof:** First one notes that \( \mathcal{B} = HH_1^{\text{type}}(A, \mathcal{D}) \) always is an invariant subspace. Then the assumption \( (2.105) \) implies the \( G \)-invariance of the decomposition \( HC_1^{\text{type}}(A, \mathcal{D}) = \mathcal{B} \oplus \overline{\mathcal{B}} \). The recursive construction of \( \rho' = \text{id} + \sum_{r=1}^{\infty} \lambda^r \rho_r' \) from the proof of Proposition 2.5.1 shows that each \( \rho_r' \) and thus \( \rho' \) is \( G \)-invariant. With the given assumptions the results of this section imply that for all \( A \in \mathcal{B} \) and \( r \geq 0 \) the elements \( g \triangleright \rho_{r+1}'(A) \) satisfy the same defining Equation \( (2.66) \) as \( \rho_{r+1}'(g \triangleright A) \). The \( G \)-invariance of the complementary subspace \( \overline{\mathcal{B}} \) and the condition \( (2.67) \) finally imply that these elements are equal by uniqueness. Thus one has the stated \( G \)-invariance of \( \rho' \). By definition, this implies \( (2.106) \) and \( (2.107) \).

The assertion concerning \( G \)-invariant equivalence is clear with Proposition 2.5.5 since \( (2.108) \) assures the \( G \)-invariance of all maps occurring in the proof, in particular of the relevant equivalence transformations.

\[
\Box
\]

### 2.7 Projective modules

As a first example we consider projective right modules. The well-known definition of a projective module can be found in Appendix B.2. It turns out that they are rigid and that deformations thereof always exist since the crucial Hochschild cohomologies vanish. In fact it is possible to compute all cohomologies using an explicit homotopy.

So let \( A \) be an associative \( \mathbb{K} \)-algebra and \( \mathcal{E} \) be a projective right \( A \)-module where the module multiplication is written as \( e \cdot a \) for \( e \in \mathcal{E} \) and \( a \in A \). By one of the equivalent characterizations of projective modules stated in Remark B.2.2 we can choose a dual basis, this means families

\[
\{e_i\}_{i \in I} \subseteq \mathcal{E} \quad \text{and} \quad \{e^i\}_{i \in I} \subseteq \text{Hom}_A(\mathcal{E}, A)
\tag{2.109}
\]

for some index set \( I \) where \( \text{Hom}_A(\mathcal{E}, A) \) denotes the set of all right \( A \)-linear homomorphisms. Then, by definition, all \( e \in \mathcal{E} \) can be written as

\[
e = \sum_{i \in I} e_i \cdot e^i(e)
\tag{2.110}
\]
where, respectively, only finitely many $\epsilon^i(e)$ are different from zero even if there should be infinitely many $e_i$ and $\epsilon^i$. Now consider the maps

$$h^k : HC^k(\mathcal{A}, \text{End}_K(\mathcal{E}, \mathcal{E})) \rightarrow HC^{k-1}(\mathcal{A}, \text{End}_K(\mathcal{E}, \mathcal{E}))$$ \hspace{1cm} (2.111)

defined by $h^0 = 0$ and

$$(h^k \phi)(a_1, \ldots, a_{k-1})e = \sum_{i \in I} \phi(\epsilon^i(e), a_1, \ldots, a_{k-1})e_i \hspace{1cm} (2.112)$$

for $k \geq 1$. Using the summation convention for $\sum_{i \in I}$ and with (2.28) we compute for $k \geq 1$

$$(\delta^{k-1} h^k \phi)(a_1, \ldots, a_k)e = (h^k \phi)(a_2, \ldots, a_k)(e \cdot a_1) + \sum_{s=1}^{k-1} (-1)^s (h^k \phi)(a_1, \ldots, a_s a_{s+1}, \ldots, a_k)e + (-1)^k (h^k \phi)(a_1, \ldots, a_{k-1})e \cdot a_k = \phi(\epsilon^1(e), a_1, \ldots, a_k)e_i + \sum_{s=1}^{k-1} (-1)^s \phi(\epsilon^1(e), a_1, \ldots, a_s a_{s+1}, \ldots, a_k)e_i \cdot a_k$$

and analogously

$$(h^{k+1} \delta^k \phi)(a_1, \ldots, a_k)e = \left( (\delta^k \phi)(\epsilon^i(e), a_1, \ldots, a_k) \right) e_i = \phi(a_1, \ldots, a_k)(e_i \cdot \epsilon^i(e)) - \phi(\epsilon^i(e)a_1, a_2, \ldots, a_k)e_i - \sum_{s=1}^{k-1} (-1)^s \phi(\epsilon^i(e), a_1, \ldots, a_s a_{s+1}, \ldots, a_k)e_i \cdot a_k.$$

With the properties of the dual basis this yields $(\delta^{k-1} h^k \phi + h^{k+1} \delta^k \phi) (a_1, \ldots, a_k)e = \phi(a_1, \ldots, a_k)e$. Thus one has

$$\delta^{k-1} \circ h^k + h^{k+1} \circ \delta^k = \text{id}_{HC^k}, \hspace{1cm} (2.113)$$

which shows that $\text{id}_{HC^k}$ is homotopic to zero for all $k \geq 1$, confer Definition [B.1.2]. Consequently, the Hochschild cohomology is trivial. For $h^1$ one further computes

$$(h^1 \delta^0 D)e = (\delta^0 D)(\epsilon^1(e))e_i = D(e_i \cdot \epsilon^1(e)) - D(e_i) \cdot \epsilon^1(e) = D(e) - D(e_i) \cdot \epsilon^1(e).$$

This shows that $\text{id} - h^1 \circ \delta^0$ is a projection onto the commutant $\ker \delta^0$ of the right module since $\epsilon^i$ is right $\mathcal{A}$-linear. Altogether one gets the following statement.

**Proposition 2.7.1 (The cohomology of a projective module)**

*Let $\mathcal{E}$ be a projective right module over an associative algebra $\mathcal{A}$. Then*

$$HH^k(\mathcal{A}, \text{End}_K(\mathcal{E}, \mathcal{E})) = \begin{cases} \text{End}_\mathcal{A}(\mathcal{E}, \mathcal{E}) & \text{for } k = 0 \\ \{0\} & \text{for } k \geq 1. \end{cases} \hspace{1cm} (2.114)$$

*In addition, the choice of a dual basis yields an explicit homotopy and a projection onto the commutant.*
The assertion of course includes the cases where $\mathcal{E}$ is a free module or where $\mathcal{E} = \mathcal{A}$ is the associative algebra itself with unit 1. In the last case the maps $h^k$ can be simply defined by $h^k(\phi)(a_1, \ldots, a_{k-1})b = \phi(b, a_1, \ldots, a_{k-1})1$ as stated in [19, Remark 2.3].

**Remark 2.7.2 (Projective modules of particular types and deformed vector bundles)**

i.) In general it is not true that the map $h^k$ respects a given particular type of cochains. Nevertheless, the simple algebraic definition gives strong evidence that this is indeed the case in many typical situations. The type of differential module structures which will be introduced in the next chapter is such an example, confer Proposition 3.2.13.

ii.) Proposition 2.7.1 shows that projective modules are rigid and that deformations can be constructed order by order. As it will be explained in Section 8.1 an important application of this fact lies in the deformation theory of vector bundles over Poisson manifolds as investigated in [25][120][121].
Chapter 3

Differential algebra and module structures

In this chapter we specify what shall be understood by algebra and module structures acting by differential operators. To this end we use the notion of multidifferential operators between modules with respect to some associative and commutative algebra. The basic definition and some well-known facts of these operators are presented in the first section. After these preliminaries the second section contains a discussion of the general framework which is necessary for the definition of differential algebra and module structures. As it will be shown these definitions determine a particular type of algebraic structures as introduced in Section 2.2. It will further be shown that one is always able to speak of \( G \)-invariant differential algebra and module structures with respect to given representations of a group \( G \). Thus it is possible in the aspired applications to investigate the deformations of such structures using the general results established in Chapter 2. Besides presenting the basic framework we already make some general observations concerning the Hochschild complexes induced by these examples of particular types. Finally, it is shown that the induced cohomology of projective modules of the differential type always vanishes since the homotopy from Section 2.7 still holds in the presented differential framework.

3.1 Algebraically defined multidifferential operators

The motivation for a general notion of differential operators naturally arises from the simple and fundamental definition used within the framework of ordinary analysis. There, a differential operator \( D \) of degree \( l \in \mathbb{N}_0 \) on the functions \( f \in C^\infty(\mathbb{R}) \) is a map \( D : f \mapsto \sum_{i=0}^{l} c_i \frac{\partial^i f}{\partial x^i} \) with functions \( c_i \in C^\infty(\mathbb{R}) \). Considering \( C^\infty(\mathbb{R}) \) as a module over itself the algebraic properties of such a map \( D \) are the guideline for the general algebraic definition. The most characteristic property is derived from the Leibniz rule implying that \( f \mapsto cD(f) - D(cf) \) for any \( c \in C^\infty(\mathbb{R}) \) is a differential operator of degree \( l - 1 \).

In the following summary of the well-known definitions we adapt the notation used in [123, App. A] where one can find further details. However, the basic definition of differential operators was first given in the works of Grothendieck, confer [61, Def. 16.8.1]. The necessary structures for the definition of multidifferential operators used in this work are an associative and commutative \( \mathbb{K} \)-algebra \( \mathcal{C} \) and \( \mathbb{K} \)-vector spaces \( \mathcal{E}_1, \ldots, \mathcal{E}_N, N \in \mathbb{N} \), and \( \mathcal{F} \) with a compatible \( \mathcal{C} \)-module structure.

For these structures one considers the \( \mathbb{K} \)-multilinear maps \( D : \mathcal{E}_1 \times \cdots \times \mathcal{E}_N \to \mathcal{F} \) which, as always, are identified with the \( \mathbb{K} \)-linear maps \( \text{Hom}_\mathbb{K}(\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_N, \mathcal{F}) \) due to the universal property of the tensor product \( \otimes = \otimes_\mathbb{K} \). Any of the given \( \mathcal{C} \)-module structures induces corresponding ones.
on the multilinear maps. One defines them as a left \( \mathcal{C} \)-module structure
\[
(c \cdot D)(e_1, \ldots, e_N) = c(D(e_1, \ldots, e_N))
\] (3.1)
and right \( \mathcal{C} \)-module structures
\[
(D \cdot (i) c)(e_1, \ldots, e_N) = D(e_1, \ldots, c e_i, \ldots, e_N)
\] (3.2)
for \( i = 1, \ldots, N \) which commute with each other and \( \mathcal{C} \). Note that in general \( D \cdot (i) c \neq D \cdot (j) c \) for \( i \neq j \). Denoting by \( L_c \) and \( L_c^{(i)} \) the left multiplication with an element \( c \in \mathcal{C} \) in \( \mathcal{F} \) and \( \mathcal{E}_i \), respectively, these commuting module structures are given by
\[
c \cdot D = L_c \circ D \quad \text{and} \quad D \cdot (i) c = D \circ L_c^{(i)}
\] (3.3)
where in the last case \( L_c^{(i)} \) is the obvious extension to the tensor product \( \mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_N \). Then it is possible to define the operations
\[
D \mapsto \text{ad}^{(i)}_{(1)}(D) = c \cdot D - D \cdot (i) c
\] (3.4)
for all \( c \in \mathcal{C} \) and \( i = 1, \ldots, N \). Note, that all \( \text{ad}^{(i)}_{(1)} \) commute. According to the above stated motivation a multilinear operator \( D \) is called a differential operator if it vanishes after applying a finite number of operations as in (3.3).

For the description of the multorders of differentiation one makes use of multiindices \( L = (l_1, \ldots, l_N), K = (k_1, \ldots, k_N) \in \mathbb{Z}^N \). Their addition and subtraction is defined componentwise. Further one sets \( L \leq K \) if \( l_i \leq k_i \) for all \( i = 1, \ldots, N \). For \( L \geq 0 \), the absolute value of \( L \) is given by \( |L| = l_1 + \cdots + l_N \). Particular multiindices of absolute value one are given by the canonical basis vectors \( e_i \in \mathbb{Z}^N \) with 1 in the \( i \)-th position and 0 at the others. Now a multilinear map \( D \) is called a differential operator of order \( L = (l_1, \ldots, l_N) \) if
\[
\text{ad}^{(1)}_{c l_1} \circ \cdots \circ \text{ad}^{(1)}_{c l_1} \circ \cdots \circ \text{ad}^{(N)}_{c l_N} \circ \cdots \circ \text{ad}^{(N)}_{c l_N}(D) = 0
\] (3.5)
for all \( c l_1, \ldots, c l_N \in \mathcal{C} \). For later applications it is convenient to reformulate this definition as a recursive one. This allows to prove all later assertions by induction over the absolute value \( |L| \).

**Definition 3.1.1 (Multidifferential operator [123, Def. A.4.1])**

Let \( \mathcal{C} \) be an associative and commutative \( \mathbb{K} \)-algebra and \( \mathcal{E}_1, \ldots, \mathcal{E}_N \) and \( \mathcal{F} \) be \( \mathbb{K} \)-vector spaces with a compatible \( \mathcal{C} \)-module structure. The multidifferential operators \( \text{DiffOp}^L_\mathcal{C}(\mathcal{E}_1, \ldots, \mathcal{E}_N; \mathcal{F}) \) with arguments in \( \mathcal{E}_1, \ldots, \mathcal{E}_N \) and values in \( \mathcal{F} \) of multorder \( L = (l_1, \ldots, l_N) \in \mathbb{Z}^N \) are then recursively defined by
\[
\text{DiffOp}^L_\mathcal{C}(\mathcal{E}_1, \ldots, \mathcal{E}_N; \mathcal{F}) = \{0\} \quad \text{if there is some} \ l_i < 0
\] (3.6)
and
\[
\text{DiffOp}^L_\mathcal{C}(\mathcal{E}_1, \ldots, \mathcal{E}_N; \mathcal{F}) = \{ D \in \text{Hom}_\mathbb{K}(\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_N; \mathcal{F}) \mid \\
L_c \circ D - D \circ L_c^{(i)} \in \text{DiffOp}^{L-e_i}_\mathcal{C}(\mathcal{E}_1, \ldots, \mathcal{E}_N; \mathcal{F}) \quad \text{for all} \ c \in \mathcal{C}, i = 1, \ldots, N \}
\] (3.7)
for \( L \geq 0 \).

A direct consequence of Definition 3.1.1 is the following proposition.
3.1. Algebraically defined multidifferential operators

Proposition 3.1.2 (Filtration and module structures [123, Prop. A.4.3])
Let $\mathcal{E}_1, \ldots, \mathcal{E}_N$ and $\mathcal{F}$ be $\mathcal{C}$-modules as in Definition 3.1.1. Then,

$$\text{DiffOp}_c^L(\mathcal{E}_1, \ldots, \mathcal{E}_N; \mathcal{F}) \subseteq \text{DiffOp}_c^K(\mathcal{E}_1, \ldots, \mathcal{E}_N; \mathcal{F}),$$

if $L \leq K$. This implies that

$$\text{DiffOp}_c^*(\mathcal{E}_1, \ldots, \mathcal{E}_N; \mathcal{F}) = \bigcup_{L \geq 0} \text{DiffOp}_c^L(\mathcal{E}_1, \ldots, \mathcal{E}_N; \mathcal{F}) \subseteq \text{Hom}_K(\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_N, \mathcal{F})$$

is a filtered subspace. Moreover, $\text{DiffOp}_c^L(\mathcal{E}_1, \ldots, \mathcal{E}_N; \mathcal{F})$ inherits the $\mathcal{C}$-module structures (3.3).

As it is typical for algebraically defined multidifferential operators, the proofs of the assertions are easy inductions over the absolute value $|L|$. In [123] one finds all the arguments in great detail. Note that it is crucial that $\mathcal{C}$ is commutative. It is an important feature of multidifferential operators that they can be composed.

Proposition 3.1.3 (Composition of differential operators [123, Prop. A.4.4])
Let $\mathcal{E}_1^{(1)}, \ldots, \mathcal{E}_N^{(1)}$, $\mathcal{E}_1^{(M)}$, $\ldots$, $\mathcal{E}_N^{(M)}$ as well as $\mathcal{F}_1, \ldots, \mathcal{F}_M$ and $\mathcal{G}$ be $\mathcal{C}$-modules. Then, for

$$D_i \in \text{DiffOp}_c^{K_i}(\mathcal{E}_i^{(i)}, \ldots, \mathcal{E}_{N_i}^{(i)}; \mathcal{F}_i), \ i = 1, \ldots, M \quad \text{and} \quad D \in \text{DiffOp}_c^L(\mathcal{F}_1, \ldots, \mathcal{F}_M; \mathcal{G})$$

one has

$$D \circ (D_1 \otimes \cdots \otimes D_M) \in \text{DiffOp}_c^{(K_1+\overline{l}_1, \ldots, K_M+\overline{l}_M)}(\mathcal{E}_1^{(1)}, \ldots, \mathcal{E}_N^{(M)}; \mathcal{G}),$$

where $\overline{l}_i = (l_1, \ldots, l_i) \in \mathbb{Z}^{N_i}$ for all $i = 1, \ldots, M$.

The proof can again be found in [123] and is an induction over $r = |K_1| + \cdots + |K_M| + |L|$.

The algebraic definition of multidifferential operators yields invariant subspaces if the relevant structures are compatible with the representations of a group $G$. In detail, one finds the following well-known assertion.

Lemma 3.1.4 (Representations on multidifferential operators)
Let $\mathcal{C}$ a $G$-invariant, associative, and commutative algebra and let $\mathcal{E}_1, \ldots, \mathcal{E}_k, \mathcal{F}$ be $G$-invariant $\mathcal{C}$-modules with respect corresponding representations of a group $G$. Then, the differential operators are invariant subspaces of $\text{Hom}_K(\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_N, \mathcal{F})$ with respect to the induced representation $[2,89]$. For all $L \in \mathbb{N}_0^k$ one has

$$G \triangleright \text{DiffOp}_c^L(\mathcal{E}_1, \ldots, \mathcal{E}_N; \mathcal{F}) \subseteq \text{DiffOp}_c^L(\mathcal{E}_1, \ldots, \mathcal{E}_N; \mathcal{F}).$$

Proof: With the given definition of multidifferential operators the proof is an easy induction over $|L|$ using the compatibility of the structures. In the usual notation the crucial equation is

$$L_{g \triangleright c} \circ (g \triangleright D) - (g \triangleright D) \circ L_{g \triangleright c} = g \triangleright (L_c \circ D - D \circ L_c).$$

In the applications one sometimes uses the fact that the notion of multidifferential operators is functorial.

Lemma 3.1.5 (Functoriality of DiffOp)
Let $N \in \mathbb{N}_0$, $\gamma : \mathcal{C} \rightarrow \mathcal{C}'$ be an isomorphism of associative and commutative $\mathbb{K}$-algebras and $\epsilon_i : \mathcal{E}_i \rightarrow \mathcal{E}'_i$, $i = 1, \ldots, N$, $\zeta : \mathcal{F} \rightarrow \mathcal{F}'$ be isomorphisms along $\gamma$ of $\mathcal{C}$- and $\mathcal{C}'$-modules. Then, for all $L \in \mathbb{N}_0^N$ there is an isomorphism

$$\Omega : \text{DiffOp}_c^L(\mathcal{E}_1, \ldots, \mathcal{E}_N; \mathcal{F}) \rightarrow \text{DiffOp}_{c'}^L(\mathcal{E}_1', \ldots, \mathcal{E}_N'; \mathcal{F}')$$
of $\mathcal{E}$- and $\mathcal{E}'$-modules along $\gamma$ defined by
\[(\Omega D)(e'_1, \ldots, e'_N) = \zeta(D(e^{-1}_1(e'_1), \ldots, e^{-1}_N(e'_N)))\] (3.14)
for all $D \in \text{DiffOp}_{\mathcal{E}}^l(\mathcal{E}_1, \ldots, \mathcal{E}_N; \mathcal{F})$ and $e'_i \in \mathcal{E}'_i$, $i = 1, \ldots, N$.

Thus, $\text{DiffOp}_{\mathcal{E}}^l$, this means the assignment of the space $\text{DiffOp}_{\mathcal{E}}^l(\mathcal{E}_1, \ldots, \mathcal{E}_N; \mathcal{F})$ of differential operators to a tuple $(\mathcal{E}, \mathcal{E}_1, \ldots, \mathcal{E}_N, \mathcal{F})$ consisting of an algebra and modules, can be seen as a functor between correspondingly defined categories.

The functoriality is of course also given in the setting of given representations of a group $G$. If the initial structures and maps are $G$-invariant the same is true for the isomorphism $\Omega$.

**Proof:** That $\Omega$ takes values in the stated multidifferential operators is a simple induction over $r = |L|$ making use of the equation $L_{c'} \circ (\Omega D) - (\Omega D) \circ L_{c'}^{(i)} = \Omega(L_{\gamma^{-1}(c')} \circ D - D \circ L_{\gamma^{-1}(c')}^{(i)})$. Then all further assertions are obvious.

If one considers differential operators $\text{DiffOp}_{\mathcal{E}}^l(\mathcal{E}, \mathcal{E})$ of a free $\mathcal{E}$-module $\mathcal{E}$ it is easy to see that they can be identified with matrices of differential operators in $\text{DiffOp}_{\mathcal{E}}^l(\mathcal{E}, \mathcal{E})$.

**Lemma 3.1.6 (Differential operators of free modules)**

Let $\mathcal{E}$ be a commutative, associative $K$-algebra and $\mathcal{E}$ be a finitely generated free left $\mathcal{E}$-module with a module basis $\mathfrak{B} = \{e_1, \ldots, e_N\} \subseteq \mathcal{E}$. Then, for every differential operator $D \in \text{DiffOp}_{\mathcal{E}}^l(\mathcal{E}, \mathcal{E})$ with $l \in N_0$ there exist $N^2$ uniquely defined differential operators $D_j^i \in \text{DiffOp}_{\mathcal{E}}^l(\mathcal{E}, \mathcal{E})$, $i, j = 1, \ldots, N$, with
\[D(c^i e_j) = D_j^i(c^i)e_i\] (3.15)
for all $c^i \in \mathcal{E}$. The map
\[\Phi_{\mathfrak{B}} : D \mapsto (D_j^i)_{i,j=1,...,N}\] (3.16)
is an isomorphism
\[\Phi_{\mathfrak{B}} : \text{DiffOp}_{\mathcal{E}}^l(\mathcal{E}, \mathcal{E}) \cong \text{Mat}_{N \times N}(\text{DiffOp}_{\mathcal{E}}^l(\mathcal{E}, \mathcal{E}))\] (3.17)
of $\mathcal{E}$-bimodules which depends on the choice of the module basis $\mathfrak{B}$ for $\mathcal{E}$. The module structures $c \cdot D = L_c \circ D$ and $D \cdot c = D \circ L_c$ for the $N \times N$-matrices on the right hand side of (3.17) are defined componentwise and obviously induced by the multiplication in $\mathcal{E}$.

If $\mathcal{E}$ and $\mathcal{E}$ are $G$-invariant structures and if the module basis $\mathfrak{B} = \{e_1, \ldots, e_N\}$ consists of $G$-invariant elements
\[g \triangleright e_i = e_i \quad \text{for all } \quad i \in \{1, \ldots, N\},\] (3.18)
the map $\Phi_{\mathfrak{B}}$ is $G$-invariant with the componentwise representation of $G$ on the matrices.

**Proof:** For arbitrary $j \in \{1, \ldots, r\}$ and $c \in \mathcal{E}$ one surely has $D(c e_j) = D_j^i(c)e_i$ with uniquely defined elements $D_j^i(c) \in \mathcal{E}$. The also uniquely defined maps $D_j^i : \mathcal{E} \rightarrow \mathcal{E}$ are $K$-linear. Thus they satisfy (3.14). The assertion $D_j^i \in \text{DiffOp}_{\mathcal{E}}^l(\mathcal{E}, \mathcal{E})$ can be shown by induction over $l$. For $l = 0$ one computes $D(c e_j) = c D(e_j) = c d_j^i e_i$ with $d_j^i \in \mathcal{E}$. So $D_j^i$ is a multiplication in $\mathcal{E}$ and thus $D_j^i = L_{d_j^i}^c \in \text{DiffOp}_{\mathcal{E}}^l(\mathcal{E}, \mathcal{E})$. For $l > 0$ evaluation of $(L_{c'} \circ D - D \circ L_{c'}) (c e_j)$ with $c, c' \in \mathcal{E}$ leads to $L_{c'}^c \circ D_j^i - D_j^i \circ L_{c'}^c = (L_{c'} \circ D - D \circ L_{c'})^i_j$ for all $i, j \in \{1, \ldots, N\}$ and the induction easily follows. The statement concerning $G$-invariance is clear.
3.2 Differential algebra and module structures

In this section we investigate algebra and module structures which can be seen as differential operators in the sense of the previous section. The main goal is to present the general conditions which are necessary to define a notion of differential such that the properties of the Definitions 2.2.2 and 2.2.3 are achieved. For associative algebras this definition is rather trivial.

Definition 3.2.1 (Associative algebras of differential type)
An associative algebra \((A, \mu)\) is said to be of differential type or simply differential with respect to a commutative, associative algebra \(C\) if \(A\) is a (left) \(C\)-module and if there exist \(m, n \in \mathbb{N}_0\) such that the algebra multiplication is a bidifferential operator
\[
\mu \in \text{DiffOp}^{(m,n)}_C(A;A,A).
\]  
(3.19)

Due to the general properties of algebraically defined multi differential operators this definition yields a particular type as in Definition 2.2.2.

Corollary 3.2.2 (Associative algebras of differential type)
A differential associative algebra \((A, \mu)\) as in Definition 3.2.1 satisfies the conditions of Definition 2.2.2 with the differential Hochschild cochains \(HC^\bullet_{\text{diff}}(A,A)\) given by the multidifferential operators
\[
HC_k^\bullet_{\text{diff}}(A,A) = \bigcup_{L \in \mathbb{N}^L_0} \text{DiffOp}^L_C(A,\ldots,A;A) \quad \text{for } k \in \mathbb{N}
\]  
(3.20)

and \(HC^0_{\text{diff}}(A,A) = A\).

Proof: The only nontrivial point is the closedness under the insertions which is a well-known fact for differential operators, confer Proposition 3.1.3. The closedness under the cup product and all other operations discussed in Section 2.1.2 is clear since they can be expressed in terms of insertions. ■

Remark 3.2.3
If \(A\) is commutative and the \(C\)-module structure is compatible with the algebra multiplication, the latter is always differential with \(m = n = 0\). This is in particular the case for \(A = C\).

For right module structures \(\rho: E \times A \to E\) and the derived Hochschild complexes the situation is more difficult. There, the differential type has to be defined in a more subtle way in order to guarantee the properties of Definition 2.2.3 in particular the closedness under the cup product.

Definition 3.2.4 (Right module structures of differential type)
Let \((A, \mu)\) be a differential associative algebra with respect to a commutative associative algebra \(C\) as in Definition 3.2.1. A right \(A\)-module structure \(\rho\) of a \(\mathbb{K}\)-vector space \(E\) is said to be of differential type or simply differential with respect to \(HC^\bullet_{\text{diff}}(A,A)\) if the following conditions are satisfied.

i.) \(E\) has a left module structure \(l: B \times E \to E\) with respect to a commutative associative algebra \(B\), also seen as \(l: B \to \text{End}_\mathbb{K}(E,E)\).

ii.) There exists an algebra homomorphism \(\gamma: C \to B\).

iii.) The right module structure is a differential operator
\[
\rho \in \text{DiffOp}^{L_\rho}_{\mathcal{C}}(A;\text{DiffOp}^{L_\rho}_{\mathcal{B}}(E,E))
\]  
(3.21)

with \(L_\rho, L_\rho \in \mathbb{N}_0\) and where the spaces \(D^l = \text{DiffOp}^{L_\rho}_{\mathcal{B}}(E,E)\) for all \(l \in \mathbb{N}_0\) are equipped with the natural left \(\mathcal{C}\)-module structure
\[
cD = L_\rho D = l(\gamma(c)) \circ D \quad \text{for all } c \in \mathcal{C}, D \in D^l.
\]  
(3.22)
The left multiplication \( L_c \) with \( c \in \mathcal{C} \) in (3.22) in fact defines a left \( \mathcal{C} \)-module structure since \( l(\gamma(c)) \in \text{DiffOp}_B(\mathcal{E}, \mathcal{E}) \) and \( L_c(L_cD) = l(\gamma(c)) \circ l(\gamma(c')) \circ D = l(\gamma(cc')) \circ D = L_{cc}D \). Using this module structure to define corresponding multidifferential operators of \( A \) with values in \( \text{DiffOp}_B(\mathcal{E}, \mathcal{E}) \) leads to the following proposition.

**Proposition 3.2.5**

A differential right module structure \( \rho \) as in Definition 3.2.4 satisfies the conditions of Definition 2.2.3 with the differential Hochschild cochains \( \text{HC}^*_{\text{diff}}(A, \mathcal{D}) \) defined by

\[
\text{HC}^d_{\text{diff}}(A, \mathcal{D}) = \mathcal{D} = \bigcup_{l \in \mathbb{N}_0} \text{DiffOp}_B^l(\mathcal{E}, \mathcal{E})
\]

and

\[
\text{HC}^k_{\text{diff}}(A, \mathcal{D}) = \bigcup_{L \in \mathbb{N}_0} \bigcup_{l \in \mathbb{N}_0} \text{DiffOp}_B^l(A, \ldots, A; \text{DiffOp}_B^l(\mathcal{E}, \mathcal{E})) \quad \text{for } k \in \mathbb{N}
\]

which are filtered spaces with \( \mathcal{D}^l \subseteq \mathcal{D}^{l'} \) if \( l \leq l' \) and \( \text{HC}^{k,L,l}_{} \subseteq \text{HC}^{k,L',l}_{} \) if \( L \leq L' \) and \( l \leq l' \). By setting \( \text{HC}^{0,L,l}_{} = \mathcal{D}^l \) the proof of the statement (3.25) is an induction over \( k \in \mathbb{N}_0 \).

**Proof:** It is sufficient to prove (3.25) which guarantees the closedness under the cup product. The remaining conditions in Definition 2.2.3 then are obvious or follow from Proposition 3.1.3. The stated filtration is clear by the filtrations of \( \text{DiffOp}^*(\mathcal{E}, \mathcal{E}) \) and \( \text{DiffOp}^*(A, \ldots, A; \text{DiffOp}^l(\mathcal{E}, \mathcal{E})) \) for all \( l \in \mathbb{N}_0 \).

The left multiplication \( l(b) \in \mathcal{D}^0 \) in \( \mathcal{E} \) with an element \( b \in \mathcal{B} \) obviously induces a left multiplication \( L_b \) and a right multiplication \( R_b \) in \( \mathcal{D}^l \) for all \( l \in \mathbb{N}_0 \) which are given by \( L_bD = l(b) \circ D \) and \( R_bD = D \circ l(b) \). Then one first notes that for \( \phi \in \text{HC}^{k,L,l}_{} \) and \( b \in \mathcal{B} \) one has \( L_b \circ \phi \in \text{HC}^{k,L,l}_{} \) which is seen by an easy induction over \( |L| \) because \( L_c \circ (L_b \circ \phi) = (L_b \circ \phi) \circ L_c = L_b \circ (L_c \circ \phi - \phi \circ L_c) \) with the notation introduced in the previous section. Analogously one sees that \( R_b \circ \phi \in \text{HC}^{k,L,l}_{} \) and by the very definition of differential operators one further gets \( L_b \circ \phi - R_b \circ \phi \in \text{HC}^{k,L,l-1}_{} \).

For fixed \( k_1, k_2 \) the proof of the statement (3.25) is an induction over \( r = |L_1| + |L_2| + l_1 + l_2 \). Let \( \phi_1 \in \text{HC}^{k_1,l_1,l_1}_{} \) and \( \phi_2 \in \text{HC}^{k_2,l_2,l_2}_{} \). With the definition

\[
(\phi_1 \circ \phi_2)(a_1, \ldots, a_{k_1+k_2}) = \phi_1(a_1, \ldots, a_{k_1}) \circ \phi_2(a_{k_1+1}, \ldots, a_{k_1+k_2})
\]

it is always clear that \( \phi_1 \circ \phi_2 \) has values in \( \mathcal{D}^{l_1+l_2} \) since the usual composition of maps satisfies \( \mathcal{D}^{l_1} \circ \mathcal{D}^{l_2} \subseteq \mathcal{D}^{l_1+l_2} \). For \( r = 0 \) the \( \mathcal{C} \)-linearity of \( \phi_1 \circ \phi_2 \) is clear by the fact that \( |L_1| = |L_2| = l_1 = 0 \). Under the assumption that the statement holds for \( r \) consider elements \( \phi_s \in \text{HC}^{k_s,l_s,l_s}_{} \) for \( s = 1, 2 \) with \( |L_1| + |L_2| + l_1 + l_2 = r + 1 \). With \( c \in \mathcal{C} \) one then computes for \( i = 1, \ldots, k_1 \)

\[
L_c \circ (\phi_1 \circ \phi_2) = (L_c \circ \phi_1) \circ (L_c \circ \phi_2) \circ L_c^i = (L_c \circ \phi_1 \circ L_c^i)(\phi_2) = \phi_2.
\]

Since \( L_c \circ \phi_1 - \phi_1 \circ L_c^i \in \text{HC}^{k_1,l_1-1, \mathcal{D}}_{} \) it follows by assumption that

\[
L_c \circ (\phi_1 \circ \phi_2) - (\phi_1 \circ \phi_2) \circ L_c^i \in \text{HC}^{k_1+k_2,(L_1+L_2+\mathcal{D})-e_i,1+l_1}_{}.
\]

Since \( \chi = L_c \circ \phi_2 - \phi_2 \circ L_c^i \in \text{HC}^{k_1,l_2-1, \mathcal{D}}_{} \) one obtains for \( i = k_1 + j \) with \( j = 1, \ldots, k_2 \)

\[
L_c \circ (\phi_1 \circ \phi_2) - (\phi_1 \circ \phi_2) \circ L_c^i = (L_c \circ \phi_1) \circ \phi_2 - \phi_1 \circ (\phi_2 \circ L_c^i) = (\phi_1 \circ L_c^i - R_c(c) \circ \phi_1 \circ L_c^i + \phi_2 + \phi_1 \circ \chi).
\]
With $L_{\gamma(c)} \circ \phi_1 - R_{\gamma(c)} \circ \phi_1 \in \text{HC}^{k_1,L_1,l_1-1}$ and the filtration, (3.26) is true for all $i = 1, \ldots, k_1 + k_2$. By the definition of multidifferential operators this yields $\phi_1 \cup \phi_2 \in \text{HC}^{k_1+k_2,(L_1,L_2+\bar{l}_1),l_1+l_2}$.

**Remark 3.2.6**

i.) In the case $A = \mathcal{E}$, what in particular means that $A$ is commutative, $\mathcal{E}$ inherits two different $A$-module structures

\begin{align}
    a \cdot_1 e &= l(\gamma(a))e \quad \text{and} \\
    a \cdot_2 e &= \rho(a)e.
\end{align}

For the definition of $\text{HC}^{k,L,l}$ it is important to use the left $A$-module structure of $\text{DiffOp}^l(\mathcal{E}, \mathcal{E})$ which is induced by (3.28). Otherwise, even in the case $\rho \in \text{HC}^{1,L,0}$, the closedness under the cup product is not valid, confer the crucial point (3.27) in the proof.

If, in addition, $\mathcal{E} = \mathcal{B}$ is a commutative and associative algebra and if all structures have to be traced back to the right module structure $\rho$, the assertions of Proposition 3.2.5 are only true, if $\rho(a) = l(\gamma(a))$ is a left multiplication in $\mathcal{E}$. This is a necessary condition for $\rho$ implying $L_\rho = l_\rho = 0$ for the degrees of differentiation. For associative and commutative algebras $\mathcal{E}$ and $A$ the differential setting is thus determined by an algebra homomorphism $\gamma : A \rightarrow \mathcal{E}$.

ii.) Note that (3.27) becomes false if one neglects the increase $\bar{L}_1$ in the multiderivative. Due to this fact which states that the order $(L_1, L_2 + \bar{L}_1)$ of differentiation of a corresponding cup product $\phi_1 \cup \phi_2$ depends on the order $\bar{L}_1$ of the values of $\phi_1$, one sees that in general the naive definition $\text{HC}^{k}_{\text{diff}}(A, \mathcal{D}) = \bigcup_{l \in \mathbb{N}_0} \text{DiffOp}^l(A, \ldots, A; \mathcal{D})$ would not lead to a well-defined cup product since particular products $\phi_1 \cup \phi_2$ possibly would not lead to differential operators of a certain degree of differentiation. Therefore it is essential to demand that the maps $\phi \in \text{HC}^{k}_{\text{diff}}(A, \mathcal{D})$ have values in the differential operators $\mathcal{D}^l$ with some fixed order $l$ of differentiation which does not depend on the arguments of $\phi$.

**Remark 3.2.7 (G-invariant differential algebra and module structures)**

Due to Lemma 3.1.4 it is clear that the conditions (2.94) in Definition 2.6.2 are satisfied for $G$-invariant and differential algebras $A$ and $A$-modules $\mathcal{E}$ if the additionally introduced structures in the Definitions 3.2.1 and 3.2.4 are $G$-invariant. This means that the left $\mathcal{E}$-module structure of $A$, the left $\mathcal{B}$-module structure of $\mathcal{E}$ and the algebra morphism $\gamma : \mathcal{E} \rightarrow \mathcal{B}$ have to be $G$-invariant with respect to corresponding representations. In this situations we then can consider the particular type of $G$-invariant differential structures.

**Corollary 3.2.8**

With $\mu$ as in (3.19), $\rho$ as in (3.21) and the resulting $(A,A)$-bimodule structure of $\mathcal{D}$ as in (2.28) the differential $\delta$ of the corresponding differential Hochschild complex $(\text{HC}^\bullet(A, \mathcal{D}), \delta)$ has the restrictions

\begin{align}
    \delta : \text{HC}^{k,L,l} \rightarrow \text{HC}^{k+1,\bar{L},l+l_\rho}
\end{align}

with $\bar{L} = (\bar{l}_1, \ldots, \bar{l}_{k+1})$ where for $k \geq 2$

\begin{align}
    \bar{l}_1 &= \max\{l_1 + m, L_\rho + l_1 + l_\rho\} \\
    \bar{l}_i &= \max\{l_i + m, l_{i-1} + n, l_i + l_\rho\} \quad \text{for} \quad i = 2, \ldots, k \\
    \bar{l}_{k+1} &= \max\{l_k + n, l_k, L_\rho\}.
\end{align}

If for commutative $A$ the structure (2.34) is used, Equation (3.31) has to be replaced by

\begin{align}
    \bar{l}_1 &= \max\{L_\rho, l_1 + m\} \\
    \bar{l}_i &= \max\{l_{i-1} + l_\rho, l_i + m, l_{i-1} + n\} \quad \text{for} \quad i = 2, \ldots, k \\
    \bar{l}_{k+1} &= \max\{l_k + l_\rho, l_k + n, L_\rho + l\}.
\end{align}
In both cases every $\text{HC}^k_{\text{diff}}$ is a left $\mathcal{B}$-module via $b \cdot \phi = L_b \circ \phi$ with $L_b$ as in the proof of Proposition 3.2.5. For $l_\rho = 0$, this means if $\mathcal{E}$ is a $(\mathcal{B}, \mathcal{A})$-bimodule, one has

$$\delta(b \cdot \phi) = b \cdot (\delta\phi).$$

\[\text{(3.33)}\]

**Proof:** For $\phi \in \text{HC}^k_{\text{diff}}(\mathcal{B}, \mathcal{M})$ and $a_1, \ldots, a_{k+1} \in \mathcal{A}$ consider the expression (2.31). Due to (2.28), the occurring cochains are $\phi \cup \rho \in \text{HC}^{k+1}((L_L + L_L + L_0)^{l_\rho})$ and $\rho \cup \phi \in \text{HC}^{k+1}((L_L + L_L + L_0)^{l_\rho})$. Further, $\phi \circ_{l_\rho} \mu \in \text{HC}^{k+1}((L_L + L_L + L_0)^{l_\rho})$ by the general property of multidifferential operators and the precondition concerning $\mu$. This shows (3.30) and a simple counting of orders of differentiation leads to (3.31). With the bimodule structure (2.33) the considerations are exactly the same using (2.34). Equation (3.33) is obvious with the definition of $\delta$. \hfill \blacksquare

**Remark 3.2.9**

The grading in (3.30) for the cases $k = 0$ and $k = 1$ can be treated in the same way.

**Remark 3.2.10**

Note that $\mathcal{D}^\rho$ with $l \in \mathbb{N}_0$ is an $(\mathcal{A}, \mathcal{A})$-bimodule via (2.28) or (2.33) only if $\rho \in \text{HC}^{1,L_0}$. As shown, the considered differential structures yield corresponding differential Hochschild complexes. In general this can be defined as follows.

**Definition 3.2.11 (Differential Hochschild complex)**

Let $\mathcal{E}$ be an associative and commutative $\mathbb{K}$-algebra. Further let $\mathcal{A}$ be an associative $\mathbb{K}$-algebra and $\mathcal{M}$ be an $(\mathcal{A}, \mathcal{A})$-bimodule as in Definition 2.1.8 both endowed with a left module structure with respect to $\mathcal{E}$. If the algebraically defined multidifferential operators

$$\text{HC}_{\text{diff}}^k(\mathcal{A}, \mathcal{M}) = \bigcup_{L \in \mathbb{N}_0} \text{DiffOp}_{\mathcal{E}}^L(\mathcal{A}, \ldots; \mathcal{A}; \mathcal{M}) \subseteq \text{HC}^k(\mathcal{A}, \mathcal{M}), \quad k \in \mathbb{N},$$

\[\text{(3.34)}\]

and $\text{HC}_{\text{diff}}^0(\mathcal{A}, \mathcal{M}) = \mathcal{M}$ build a subcomplex $(\text{HC}_{\text{diff}}^k(\mathcal{A}, \mathcal{M}), \delta)$, this is called the (algebraic) differential Hochschild complex of $\mathcal{A}$ with values in $\mathcal{M}$ over $\mathcal{E}$.

In general, (3.31) does not define a subcomplex since the considered subspaces could not be closed under the Hochschild differential $\delta$. In order to guarantee this one has to demand adequate properties of the involved algebraic structures.

**Lemma 3.2.12**

Let $\mathcal{A}$ and $\mathcal{M}$ as above satisfy the following conditions:

i.) The algebra multiplication in $\mathcal{A}$ is a differential operator $\mu \in \text{DiffOp}_{\mathcal{E}}^{(m,n)}(\mathcal{A}, \mathcal{A}; \mathcal{A})$ with $m, n \in \mathbb{N}_0$.

ii.) The left and right $\mathcal{A}$-module structures of $\mathcal{M}$ have the property that for all $k \in \mathbb{N}_0$, $L = (l_1, \ldots, l_k) \in \mathbb{N}^k$ and $\phi \in \text{DiffOp}_{\mathcal{E}}^L(\mathcal{A}, \mathcal{M})$ there exist $s, t, u_\phi, v_\phi \in \mathbb{N}_0$ such that the maps

$$((a_1, \ldots, a_{k+1} \mapsto a_1 \phi(a_2, \ldots, a_{k+1})) \in \text{DiffOp}_{\mathcal{E}}^{L_1}(\mathcal{A}, \ldots; \mathcal{A}; \mathcal{M}), \quad (3.35)$$

$$((a_1, \ldots, a_{k+1} \mapsto \phi(a_1, \ldots, a_k) a_{k+1}) \in \text{DiffOp}_{\mathcal{E}}^{L_2}(\mathcal{A}, \ldots; \mathcal{A}; \mathcal{M}) \quad (3.36)$$

are differential operators of the multiorders $L_1 = (u_\phi, l_1 + s, \ldots, l_k + s) \in \mathbb{N}^{k+1}$ and $L_2 = (l_1 + t, \ldots, l_k + t, v_\phi) \in \mathbb{N}^{k+1}$.
Then (3.34) defines a differential Hochschild complex since for \( L = (l_1, \ldots, l_k) \in \mathbb{N}^k \)

\[
\delta : \text{DiffOp}^L_{\mathcal{C}}(A, \ldots, A; M) \rightarrow \text{DiffOp}^{\tilde{L}}_{\mathcal{C}}(A, \ldots, A; M),
\]

where \( \tilde{L} = (\tilde{l}_1, \ldots, \tilde{l}_{k+1}) \in \mathbb{N}_0^{k+1} \) with

\[
\begin{align*}
\tilde{l}_1 & = \max\{u_\phi, l_1 + t, l_1 + m\} \\
\tilde{l}_i & = \max\{l_{i-1} + s, l_i + t, l_i + m, l_{i-1} + n\} \quad \text{for } i = 2, \ldots, k \\
\tilde{l}_{k+1} & = \max\{l_k + s, v_\phi, l_k + n\}.
\end{align*}
\]

The proof is obvious and a direct consequence of the definition of \( \delta \) and the properties of differential operators. In the subcomplexes derived from differential algebra and module structures we have basically shown that Lemma 3.2.12 is satisfied. In the case of algebras we have \( s = 0 = t \) and \( u_\phi = 0 = v_\phi \) for all \( \phi \in \text{HC}^k_{\text{diff}}(A, A) \). Depending on the possibly given choice of the \((A, A)\)-bimodule structure of \( \mathcal{D} \) we have for right modules either \( s = 0, t = l_\rho \) and \( v_\phi = L_\rho, u_\phi = l \) for \( \phi \in \text{HC}^k_{\text{diff}}(A, A) \) in the case (2.28) or \( s = l_\rho, t = 0 \) and \( u_\phi = L_\rho, v_\phi = l \) for \( \phi \in \text{HC}^k_{\text{diff}}(A, A) \) in the case (2.38).

As a simple but important example one can again consider the case of projective modules. The results of Section 2.7 can be reformulated in the setting of differential module structures.

**Proposition 3.2.13 (Projective modules of the differential type)**

Let \( A \) be a commutative, associative algebra and let \( \mathcal{E} \) be a differential right module as in Definition 3.2.7 with \( A = \mathcal{C} = \mathcal{B} \) and \( \gamma = \text{id}_A \). Further, let \( \mathcal{E} \) be a projective \( A \)-module.

Then, the homotopy map \( h^k \) from Section 2.7 satisfies

\[
h^k : \text{DiffOp}^{(l_1, \ldots, l_k)}(A, \ldots, A; \text{DiffOp}^l(\mathcal{E}, \mathcal{E})) \rightarrow \text{DiffOp}^{(l_2, \ldots, l_k)}(A, \ldots, A; \text{DiffOp}^l(\mathcal{E}, \mathcal{E}))
\]

for all \( l_1, \ldots, l_k, l \in \mathbb{N}_0 \) and \( k \in \mathbb{N} \). This shows that the corresponding cohomology is trivial,

\[
\text{HH}^k_{\text{diff}}(A, \mathcal{D}) = \begin{cases} 
\text{DiffOp}^0_{\mathcal{E}}(\mathcal{E}, \mathcal{E}) & \text{for } k = 0 \\
\{0\} & \text{for } k \geq 1.
\end{cases}
\]

**Proof:** The proof is a twofold induction. Using the defining Equation (2.112), a simple computation shows that for all \( \phi \in \text{HC}^k_{\text{diff}}(A, \mathcal{D}) \) and \( a, a_1, \ldots, a_{k-1} \in A \) one has

\[
L_a \circ h^k \phi(a_1, \ldots, a_{k-1}) - h^k \phi(a_1, \ldots, a_{k-1}) \circ L_a = h^k(L_a \circ \phi - \phi \circ L_a^{(i)})(a_1, \ldots, a_{k-1}).
\]

Due to this equation the first induction over \( l_1 \) shows that \( h^k \phi \) takes values in \( \text{DiffOp}^{(l_2, \ldots, l_k)}(A, \ldots, A; \text{DiffOp}^l(\mathcal{E}, \mathcal{E})) \) if \( \phi \) is a differential operator of multiorier \((l_1, \ldots, l_k)\) for arbitrary \( l_2, \ldots, l_k \in \mathbb{N}_0 \). A second induction over \( r = l_2 + \cdots + l_k \) making use of the fact that

\[
L_a \circ h^k \phi - h^k \phi \circ L_a^{(i)} = h^k(L_a \circ \phi - \phi \circ L_a^{(i+1)})(a_1, \ldots, a_{k-1})
\]

for all \( i = 1, \ldots, k_1 \) then shows the remaining assertion contained in (3.39). The statement concerning the differential cohomology then follows in the same way as the computation of the purely algebraic one in Section 2.7.

The above proposition yields an easy proof for the well-known facts concerning deformed vector bundles, confer Section 8.1.

We close this section with two easy observations concerning differential Hochschild complexes which will be very useful for the later computation of the corresponding cohomologies.
Remark 3.2.14 (Functoriality of differential Hochschild complexes)
It is well-known that Hochschild complexes have a functorial behaviour with respect to their initial data. With Lemma 3.1.5 it is a simple exercise to see that this is still true in the differential or $G$-invariant differential setting. Thus, different choices of structures as $A$, $E$ and $\gamma : C \to B$ in Definition 3.2.1 and 3.2.4 which are related by structure preserving isomorphisms yield isomorphic differential Hochschild complexes with exactly the same properties.

The results of Lemma 3.1.6 induce the following simple observation.

Lemma 3.2.15 (Differential Hochschild complexes and free modules)
Let there be given the structures in Definition 3.2.4 for the following specific situation: Let $A = C$ be commutative and let the right $A$-module structure of $E$ be given by
\[ \rho(a) = l_\gamma(a), \] (3.43)
so that $\rho \in \text{HC}^{1,0,0}$. Then, for all $l \in \mathbb{N}_0$ let the space $D^l = \text{DiffOp}^l_{B}(E, E)$ be equipped with the $(A,A)$-bimodule structure (2.33), this means $a \cdot D \cdot a' = \rho(a) \circ D \circ \rho(a')$ for all $a,a' \in A$ and $D \in D^l$, so that the left module structure of $D^l$ with respect to $C = A$ is unique. In addition, let $E$ be a free $B$-module with module basis $B = \{e_1, \ldots, e_N\}$.

Then, the isomorphisms $\Phi_B : \text{DiffOp}^l_{B}(E, E) \cong \text{Mat}_{N \times N}(\text{DiffOp}^l_{B}(B, B))$ as in Lemma 3.1.6 are isomorphisms of $(A,A)$-bimodules when using the obvious $A$-module structure of $B$ induced by $\gamma$. Further, this gives rise to an isomorphism
\[ (\text{HC}_{\text{diff}}^*(A, \text{DiffOp}^l_{B}(E, E)), \delta) \cong (\text{HC}_{\text{diff}}^*(A, \text{Mat}_{N \times N}(\text{DiffOp}^l_{B}(B, B))), \delta) \] (3.44)
\[ \cong (\text{Mat}_{N \times N}(\text{HC}_{\text{diff}}^*(A, \text{DiffOp}^l_{B}(B, B))), \delta) \] (3.45)
of complexes where the differential in (3.43) is defined componentwise, $\delta(\phi^i_j) := (\delta \phi^i_j)$. In particular, one has
\[ \text{DiffOp}^l_{A}(A, \ldots, A; \text{DiffOp}^l_{B}(E, E)) \cong \text{DiffOp}^l_{A}(A, \ldots, A; \text{Mat}_{N \times N}(\text{DiffOp}^l_{B}(B, B))). \] (3.46)

In the $G$-invariant setting the $G$-invariant module basis guarantees that the isomorphisms (3.44), (3.45), and (3.46) are $G$-invariant.

Proof: That $\Phi_B$ is an isomorphism of $(A,A)$-bimodules is obvious with the given module structures and the fact that $\Phi_B$ is already an isomorphism of $(B,B)$-bimodules. The isomorphisms (3.46) and (3.44) then follow by Lemma 3.1.5 and Remark 3.2.14. The isomorphism (3.45) is an isomorphism of complexes since the Hochschild differential $\delta$ is defined componentwise. 

Note that (3.43) is necessary in order to guarantee that $B$ has a right $A$-module structure which is related to the one of $E$.

Corollary 3.2.16 (The cohomology of free modules)
Due to (3.45) the knowledge of the cohomology $\text{HH}_{\text{diff}}^*(A, \text{DiffOp}^l_{B}(E, E))$ is equivalent to the knowledge of $\text{HH}_{\text{diff}}^*(A, \text{DiffOp}^l_{B}(B, B))$. 

Chapter 4

Sheaf theory and Hochschild cohomologies

It is a basic concept in differential geometry to investigate global structures locally. The definition of a smooth manifold itself makes use of local charts and many other fundamental notions are defined by using local expressions of the involved data as well. In order to compute Hochschild cohomologies that arise in a global geometric context it is thus natural to ask if the problem can be solved by investigating the local situation. This approach is of course only possible if the global objects are related to corresponding local expressions and if the global and local data determine each other in a sufficient way. These aspired relations are the content of the notions of presheaves and sheaves over topological spaces. As it will become clear in this chapter this is the adequate framework to treat the above problem. The basic definitions and general results of sheaf theory that will be used in the subsequent considerations are introduced in the first section. After that we concentrate on sheaves over smooth manifolds and present a different point of view of the sheaves of sheaf homomorphisms and particular subsheaves thereof. This leads to the important result that all differential operators of a finite order of differentiation between particular sheaves carry a sheaf structure themselves. With this conclusion and the observations concerning sheaves over different manifolds made in Section 4.3 it is possible to apply these concepts to the discussion of differential Hochschild complexes of modules as introduced in Section 3.2. It will be pointed out that in typical situations these complexes give rise to corresponding presheaves. Although one has to abandon the desirable properties of a sheaf it is shown that under certain assumptions one is still able to compute the global cohomologies by computing the local ones. Together with the investigation of the $G$-invariant case which enforces a slightly different setting the chapter culminates in the two important Propositions 4.4.7 and 4.4.12.

4.1 Sheaves and sheaf homomorphisms

The origin of sheaf theory can in principle be seen in the attempt to find an axiomatic description of the convenient properties of functions on a topological space with respect to the process of restricting them to open subsets. It is obvious that functions have the property to be determined by their restrictions and thus they yield a generic example of a structure where the knowledge of the global information is equivalent to the knowledge of the local one. The notions of presheaves and sheaves developed from this idea have important applications in many realms of mathematics, for instance in algebraic geometry, complex analysis and many others. The formulation of the essential properties can be performed in different equivalent ways. An important reference for the basics of sheaf theory is the early work of Godement [57]. Slightly different approaches and applications can
moreover be found in [66, Chap. II, Sect. 1], [119, Chap. 2], and [127, Chap. II]. In the following presentation we basically follow the latter two references.

**Definition 4.1.1 (Presheaves in terms of categories and functors)**

Let $M$ be a topological space and $M$-set be the category of its open, nonempty subsets and inclusions. Further, let $\mathcal{C}$ be an arbitrary category.

i. A contravariant functor

$$\mathcal{F} : M\text{-set} \to \mathcal{C} \quad (4.1)$$

is called a **presheaf** $\mathcal{F}$ over or on $M$ with values in $\mathcal{C}$.

ii. A **homomorphism** between two presheaves $\mathcal{F}, \mathcal{G} : M\text{-set} \to \mathcal{C}$ over $M$ is a natural transformation between the two functors. Correspondingly, an **isomorphism** is a natural equivalence.

This precise but abstract definition obviously allows a more concrete formulation which is also used to introduce a notion of sheaves. Although we mainly work with the following description it is often very useful to have the more basic definition in mind. With respect to our further purpose we adjust the notation and restrict ourselves to certain types of categories.

**Definition 4.1.2 (Presheaves and sheaves of sets)**

Let $M$ be a topological space and let $M$ denote the set of its open, nonempty subsets. Further, let $\mathcal{C}$ be a category with certain sets as objects and certain maps as morphisms.

i. A **presheaf** $\mathcal{F}$ of objects of $\mathcal{C}$ over $M$ is given by assigning to each $U \in M$ a set $\mathcal{F}(U)$ in the class of objects of $\mathcal{C}$,

$$U \mapsto \mathcal{F}(U), \quad (4.2)$$

and to each inclusion $V \subseteq U$ of open sets a **restriction map** or **restriction homomorphism**

$$r^U_V : \mathcal{F}(U) \to \mathcal{F}(V), \quad (4.3)$$

such that

(a) for all $U \in M$ the map $r^U_U = id_{\mathcal{F}(U)}$ is the identity morphism of $\mathcal{F}(U)$.

(b) for all $W \subseteq V \subseteq U \subseteq M$ one has $r^W_V \circ r^U_W = r^U_V$.

ii. A presheaf $\mathcal{F}$ is called a **sheaf** if for every open covering $U = \bigcup_{i \in I} U_i$ with open $U \subseteq M$ the following further conditions are satisfied.

(c) If $s, t \in \mathcal{F}(U)$ and $r^V_U(s) = r^V_U(t)$ for all $i \in I$, then $s = t$.

(d) If $s_i \in \mathcal{F}(U_i)$ are given for all $i \in I$ with the property that $r^V_{U_i \cap U_j}(s_i) = r^V_{U_i \cap U_j}(s_j)$ for all $U_i \cap U_j \neq \emptyset$, then there exists an $s \in \mathcal{F}(U)$ such that $r^V_{U_i}(s) = s_i$ for all $i \in I$.

**Definition 4.1.3 ((Pre)sheaf homomorphisms)**

A **(homo)morphism** between two (pre)sheaves $\mathcal{F}, \mathcal{G}$ (in the sense of Definition 4.1.2)

$$h : \mathcal{F} \to \mathcal{G} \quad (4.4)$$

is a collection $h = \{h_U\}_{U \in M}$ of maps

$$h_U : \mathcal{F}(U) \to \mathcal{G}(U) \quad (4.5)$$
which is compatible with the restriction maps. This means that the diagram

\[
\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{h_{UV}} & \mathcal{G}(U) \\
\downarrow r_U & & \downarrow r_V \\
\mathcal{F}(V) & \xrightarrow{h_{UV}} & \mathcal{G}(V)
\end{array}
\]

commutes for all \( V \subseteq U \subseteq M \). The set of all (pre)sheaf morphisms of \( \mathcal{F} \) into \( \mathcal{G} \) is denoted by \( \text{Hom}(\mathcal{F}, \mathcal{G}) \).

If all the maps \( h_U \) are inclusions \( \mathcal{F} \) is called a sub(pre)sheaf of \( \mathcal{G} \). In the case that all \( h_U \) are isomorphisms, \( h \) is called a (pre)sheaf isomorphism.

**Remark 4.1.4 (Notation, (pre)sheaves with additional structures, restricted sheaves)**

i.) Usually, one denotes the value of a restriction map as \( r_U^V(s) = s|_V \).

ii.) In the most cases the considered categories \( \mathcal{C} \) consist of sets with an algebraic structure and structure preserving morphisms. Due to Definition 4.1.1 it is automatically clear that the restriction homomorphisms preserve these structures which therefore is always required in Definition 4.1.2. Then, the same is demanded for corresponding homomorphisms. This way one is able to define sheaves of groups, vector spaces and all other kinds of algebraic structures, as well as the morphisms between them. Further, this notion of a (pre)sheaf morphism can be extended to a notion of a (pre)sheaf morphism where the (pre)sheaves \( \mathcal{F} \) and \( \mathcal{G} \) have different algebraic structures. Definition 4.1.5 gives an example, confer also Remark 4.1.6.

iii.) It is clear that every (pre)sheaf \( \mathcal{F} \) over \( M \) yields a (pre)sheaf \( \mathcal{F}|_U \) over \( U \) for all open subsets \( U \in \mathcal{M} \) by restricting Definition 4.1.2 to open subsets of \( U \).

**Definition 4.1.5 ((Pre)sheaves of modules)**

Let \( A \) be a presheaf of rings and \( \mathcal{E} \) be a (pre)sheaf of abelian groups over \( M \). Then, \( \mathcal{E} \) is called a (pre)sheaf of \( A \)-modules if for every \( U \in \mathcal{M} \) the group \( \mathcal{E}(U) \) is an \( A(U) \)-module and for \( V \subseteq U \) one has

\[
(a \cdot e)|_V = a|_V \cdot e|_V
\]

for all \( a \in A(U) \) and \( e \in \mathcal{E}(U) \).

**Remark 4.1.6**

It is clear that the above notion of a sheaf of modules yields a sheaf homomorphism \( A \times \mathcal{E} \longrightarrow \mathcal{E} \) of (pre)sheaves of sets. But the converse is not true since the module structure is a nontrivial condition.

**Proposition 4.1.7 (The sheaf of local morphisms)**

Let \( \mathcal{F} \) be a presheaf and \( \mathcal{G} \) be a sheaf of sets over \( M \) as in Definition 4.1.2. Then, the presheaf of sets

\[
U \longmapsto \text{Hom}(\mathcal{F}, \mathcal{G})(U) = \text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U) \quad \text{for all } U \in \mathcal{M}
\]

is a sheaf of sets which is called the sheaf of local morphisms of \( \mathcal{F} \) into \( \mathcal{G} \) and denoted by \( \text{Hom}(\mathcal{F}, \mathcal{G}) \).

**Proof:** By definition, the elements in \( \text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U) \) are collections \( \{h_W\}_{W \subseteq U} \) of maps. The restriction maps \( r_U^V: \text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U) \longrightarrow \text{Hom}(\mathcal{F}|_V, \mathcal{G}|_V) \) for \( V \subseteq U \) are defined by

\[
r_U^V(\{h_W\}_{W \subseteq U}) = \{h_W\}_{W \subseteq V}
\]
which obviously satisfy the conditions (a) and (b). Now let \( U = \bigcup_{i \in I} U_i \) as in Definition 4.1.2. For (c) we have to consider collections \( \{ h_W \}_{W \subseteq U} \) and \( \{ k_W \}_{W \subseteq U} \) with \( \{ h_W \}_{W \subseteq U} = \{ k_W \}_{W \subseteq U} \) for all \( i \in I \). Then, for \( W \subseteq U \) and \( f \in \mathcal{F}(W) \) one has for all \( U_i \) with \( U_i \cap W \neq \emptyset \)

\[
  h_W(f)|_{W \cap U_i} = h_{W \cap U_i}(f|_{W \cap U_i}) = k_{W \cap U_i}(f|_{W \cap U_i}) = k_W(f)|_{W \cap U_i}.
\]

Since \( \mathcal{G} \) is a sheaf and \( f \) was arbitrary this yields \( h_W = k_W \). Thus, \( \{ h_W \}_{W \subseteq U} = \{ k_W \}_{W \subseteq U} \) and (c) is shown. For (d) we have to consider collections \( \{ h_W \}_{W \subseteq U} \subseteq \text{Hom}(\mathcal{F}_{U_i}, \mathcal{G}_{U_i}) \) for all \( i \in I \) with \( \{ h_W \}_{W \subseteq U \cap U_j} = \{ h_W \}_{W \subseteq U \cap U_j} \). Then, for \( W \subseteq U \) one can define a map \( h_W : \mathcal{F}(W) \to \mathcal{G}(W) \) by

\[
  h_W(f)|_{W \cap U_i} = h_{W \cap U_i}(f|_{W \cap U_i}) \quad (4.9)
\]

for all \( f \in \mathcal{F}(W) \) and \( W \cap U_i \neq \emptyset \) which is possible since \( \mathcal{G} \) is a sheaf and \( h_{W \cap U_i \cap U_j}(f|_{W \cap U_i \cap U_j}) = h_{W \cap U_i \cap U_j}(f|_{W \cap U_i \cap U_j}) \). Altogether, this leads to a collection of maps \( \{ h_W \}_{W \subseteq U} \) such that by definition \( h_W = h_W \) for all \( W \subseteq U_i \) and \( i \in I \). Further, for \( Z \subseteq W \subseteq U \) it follows

\[
  h_W(f)|_Z = h_Z(f|_Z),
\]

for \( f \in \mathcal{F}(W) \) and \( i \in I \). Again, the sheaf property of \( G \) leads to \( h_W(f)|_Z = h_Z(f|_Z) \), so \( \{ h_W \}_{W \subseteq U} \) is a sheaf morphism which restricts to the given local ones.

\[\Box\]

Remark 4.1.8 (Subsheaves of \( \text{Hom}(\mathcal{F}, \mathcal{G}) \))

If \( \mathcal{F} \) and \( \mathcal{G} \) have further algebraic properties that should be preserved by the maps of corresponding sheaf homomorphisms, this leads to a subsheaf of \( \text{Hom}(\mathcal{F}, \mathcal{G}) \), if the morphism defined in (4.9) has the same properties as the locally given ones.

Example 4.1.9 (Subsheaves of \( \text{Hom}(\mathcal{F}, \mathcal{G}) \))

Let \( \mathcal{F}, \mathcal{G} \) be as in Proposition 4.1.7

1. If \( \mathcal{F} \) is a presheaf and \( \mathcal{G} \) is a sheaf of abelian groups the sheaf homomorphisms consisting of group homomorphisms build a subsheaf of \( \text{Hom}(\mathcal{F}, \mathcal{G}) \), since \( h_W \) in (4.9) is a group homomorphism.

2. Let \( \mathcal{E}_1, \ldots, \mathcal{E}_k \), \( k \in \mathbb{N} \), be (pre)sheaves of abelian groups and let \( \mathcal{F} = \mathcal{E}_1 \times \ldots \times \mathcal{E}_k \) be the (pre)sheaf of sets of the Cartesian product. Completely analogously, the sheaf homomorphisms consisting of maps which are group morphisms in every entry build a subsheaf of \( \text{Hom}(\mathcal{F}, \mathcal{G}) \) which is denoted by \( \text{Hom}_{ab}(\mathcal{E}_1 \times \ldots \times \mathcal{E}_k, \mathcal{G}) \).

4.2 The sheaves of local and differential operators

As seen above the notion of sheaves and the homomorphisms between them provides the appropriate framework to understand the behaviour and the relations between local and global information. In particular, one is able to study an element \( f \in \mathcal{F}(M) \) by considering the restrictions \( f|_U \) to open subsets of a covering of \( M \). In many typical situations, though, one is dealing with sheaves \( \mathcal{F} \) and \( \mathcal{G} \) over \( M \) and is given a ‘global’ map \( h_M : \mathcal{F}(M) \to \mathcal{G}(M) \) of a particular type which one would like to restrict as well and to realize as a sheaf homomorphism. Thus, it is necessary to find criteria for the possibility of reconstructing all the maps of a particular sheaf morphism \( \{ h_U \}_{U \subseteq M} \) of a particular type from \( h_M \). As we will see, this new point of view provides a very useful characterization of the sheaf of local morphisms allowing a convenient investigation of specific subsheaves. Basically, the observations and proofs of this section are generalized versions of the considerations in [123, App. A].
4.2. The sheaves of local and differential operators

In the following we always consider a smooth manifold $M$ as a topological space and denote the sheaf of smooth functions with values in $\mathbb{K}$ on it by $C^\infty_M$ which is a sheaf of associative and commutative algebras.

**Lemma 4.2.1 (Open coverings)**

Let $M$ be a smooth manifold and $V \subseteq M$ be an open subset. Then there exists an open covering $V = \bigcup_{i \in I} O_i$ with the property that for all $i \in I$

i.) $O_i \subset O_i^{cl} \subset V$,

where $O_i^{cl}$ denotes the closure of $O_i$. Additionally, one can achieve that

ii.) there exist open sets $R_i \subset V$ with $O_i^{cl} \subset R_i \subset R_i^{cl} \subset V$.

**Proof:** Using centered charts $x_p : V \supset V_p \longrightarrow x_p(V_p) \subset \mathbb{R}^n$ of $V$ for every point $p \in V$, this means $x_p(p) = 0$, the proof is obvious since there exist balls around $0 \in \mathbb{R}^n$ such that the images under the homeomorphism $x_p^{-1}$ have the required properties. ■

**Lemma 4.2.2 (Sheaves of left $C^\infty_M$-modules)**

Let $\mathcal{E}$ be a sheaf of left $C^\infty_M$-modules over $M$ and $V \subseteq U \subseteq M$ be open subsets.

i.) Every smooth function $\chi \in C^\infty(U)$ with $\text{supp}(\chi) \subseteq V$ defines a map

$$\chi : \mathcal{E}(V) \longrightarrow \mathcal{E}(U),$$

which for all $e \in \mathcal{E}(V)$ is given by

$$\begin{align*}
(\chi e)|_V &= \chi|_V \cdot e, \\
(\chi e)|_{U \setminus \text{supp}(\chi)} &= 0.
\end{align*}$$

In particular, for $e \in \mathcal{E}(U)$ we have

$$\chi(e|_V) = \chi \cdot e. \quad (4.12)$$

ii.) Let $U = \bigcup_{i \in I} U_i$ be an open covering, $\{\chi_i \in C^\infty(M)\}_{i \in I}$ be a subordinate partition of unity and $\{e_i \in \mathcal{E}(U_i)\}_{i \in I}$ be a family of elements. Then there exists a unique element $e \in \mathcal{E}(U)$ with

$$e|_W = \sum_{i \in I, \chi_i|_W \neq 0} (\chi_i e_i)|_W \quad (4.13)$$

for all open subsets $W \subseteq U$ of the open covering

$$\{W \subseteq U \text{ open} | \chi_i|_W = 0 \text{ for all except at most finitely many } i \in I\}. \quad (4.14)$$

This element is denoted by $e = \sum_{i \in I} \chi_i e_i$, since this shows the relevant dependencies. In particular, for $e \in \mathcal{E}(U)$ this yields

$$e = \sum_{i \in I} \chi_i(e|_{U_i}). \quad (4.15)$$

Now, let $O \subset R \subset V$ be open subsets as the ones in the covering of Lemma 4.2.1.
iii.) There exists a smooth function $\chi \in C^\infty(U)$ with

$$\text{supp}(\chi) \subseteq V \quad \text{and} \quad \chi|_{O^c} = 1.$$  \hspace{1cm} (4.16)

Then, for $e \in \mathcal{E}(V)$ one has

$$(\chi e)|_O = e|_O.$$  \hspace{1cm} (4.17)

iv.) There exists a smooth function $\chi \in C^\infty(U)$ with

$$\chi|_{O^c} = 0 \quad \text{and} \quad \chi|_{U \setminus R} = 1.$$  \hspace{1cm} (4.18)

Then, for $e \in \mathcal{E}(U)$ with $e|_V = 0$ one has

$$\chi \cdot e = e.$$  \hspace{1cm} (4.19)

**Proof:** The first assertion holds due to the sheaf properties of $\mathcal{E}$. The last ones are a consequence of the Urysohn lemma, confer [88, Prop. 2.26] or [123, Cor. A.1.5] and [109, Chap. 7], stating that two disjoint closed subsets $A_1, A_2 \subseteq M$ can be separated by a smooth function $\chi \in C^\infty(M)$ with values in $[0,1]$ and the property that $\chi|_{A_1} = 1$ and $\chi|_{A_2} = 0$. The right hand side of (4.13) is a finite sum and thus an element $e_W \in \mathcal{E}(W)$. Now consider $W \cap W' \neq \emptyset$. For $\chi_i$ with $\chi_i|_W \neq 0$ but $\chi_i|_{W'} = 0$ one easily sees that $\chi_i e_i|_{W'} = 0$ by checking it on $W' \cap U_i$ and $W' \cap (U \setminus \text{supp} \chi_i)$. With this it follows

$$e_{W'}|_{W \cap W'} = \sum_{i \in I} \frac{(\chi_i e_i)|_{W \cap W'}}{\chi_i|_{W'} \neq 0} \quad (\chi_i e_i)|_{W \cap W'} = e_{W'}|_{W \cap W'}.$$  \hspace{1cm} (4.22)

The sheaf property of $\mathcal{E}$ then assures the existence of a unique element $e \in \mathcal{E}(U)$ with $e|_W = e_W$. Equation (4.15) is obvious. In the third part one considers the disjoint and closed sets $O^c$ and $U \setminus V$ in the topology of the subspace $U$ and verifies that the functions $\chi \in C^\infty(U)$ with $\chi|_{O^c} = 1$ and $\chi|_{U \setminus V} = 0$ have the stated properties. Equation (4.17) is obvious. The existence of a function with (4.18) follows because $O^c$ and $U \setminus R$ are closed and disjoint sets. For (4.19) one considers the open cover of $U$ consisting of $V$ and $U \setminus R^c$ to check that the corresponding restrictions of $\chi \cdot e$ and $e$ coincide.

**Definition 4.2.3 (Local map)**

Let $\mathcal{E}_1, \ldots, \mathcal{E}_k, \mathcal{G}$ be presheaves of abelian groups over $M$ and $U \in \underline{M}$. Then a map $D : \mathcal{E}_1(U) \times \ldots \times \mathcal{E}_k(U) \rightarrow \mathcal{G}(U)$ which is a group morphism in every argument is said to be local if for all $e_1 \in \mathcal{E}_1(U), \ldots, e_k \in \mathcal{E}_k(U)$ and $V \subseteq U$ one has that

$$D(e_1, \ldots, e_k)|_V = 0 \quad \text{if} \quad e_i|_V = 0 \quad \text{for one} \ i \in \{1, \ldots, k\}.$$  \hspace{1cm} (4.20)

The set of all local maps on $U \in \underline{M}$ is denoted by

$$\text{Loc}(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{G})(U).$$  \hspace{1cm} (4.21)

**Remark 4.2.4 (Local map on functions)**

The presented notion of local maps is obviously consistent with the well-known one for operators $D : C^\infty(M) \rightarrow C^\infty(M)$ on the functions of a manifold. There, $D$ is local if

$$\text{supp}(D \chi) \subseteq \text{supp}(\chi)$$  \hspace{1cm} (4.22)

for all $\chi \in C^\infty(M)$. 

Corollary 4.2.5
Local maps $D$ as above have the useful property that for elements $e_i, e'_i \in \mathcal{E}_i(U)$ with $e_i|_V = e'_i|_V$ for all $i \in \{1, \ldots, k\}$ and $V \subseteq U$

$$D(e_1, \ldots, e_k)|_V = D(e'_1, \ldots, e'_k)|_V. \quad (4.23)$$

Lemma 4.2.6 (Presheaf homomorphisms consist of local maps)
Let $\mathcal{E}_1, \ldots, \mathcal{E}_k$, $k \in \mathbb{N}$, and $\mathcal{G}$ be presheaves of abelian groups over $M$. Then, the maps $h\mathcal{U} \in \{h\mathcal{U}\}_{\mathcal{U} \subseteq M} \in \text{Hom}_{ab}(\mathcal{E}_1 \times \ldots \times \mathcal{E}_k, \mathcal{G})(M)$ are local.

Proof: The assertion is clear with $h\mathcal{U}(e_1, \ldots, e_k)|_V = h\mathcal{V}(e_1|_V, \ldots, e_k|_V) = 0$.

The following proposition shows that local maps and sheaf homomorphisms are really the same for sheaves of $C^\infty$-modules.

Proposition 4.2.7 (Sheaf homomorphisms and local maps)
Let $\mathcal{E}_1, \ldots, \mathcal{E}_k$, $k \in \mathbb{N}$, be sheaves of $C^\infty$-modules and $\mathcal{G}$ be a sheaf of abelian groups over $M$. Then, every sheaf homomorphism $\{h\mathcal{U}\}_{\mathcal{U} \subseteq M} \in \text{Hom}_{ab}(\mathcal{E}_1 \times \ldots \times \mathcal{E}_k, \mathcal{G})(M)$ is already uniquely determined by the map $h_M : \mathcal{F}(M) \rightarrow \mathcal{G}(M)$ and

$$\{h\mathcal{U}\}_{\mathcal{U} \subseteq M} \mapsto h_M \quad (4.24)$$

yields an isomorphism

$$\text{Hom}_{ab}(\mathcal{E}_1 \times \ldots \times \mathcal{E}_k, \mathcal{G})(M) \cong \text{Loc}(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{G})(M) \quad (4.25)$$

of abelian groups.

Proof: Let $\{h\mathcal{U}\}_{\mathcal{U} \subseteq M}$ and $\{h'\mathcal{U}\}_{\mathcal{U} \subseteq M}$ be two sheaf homomorphisms in $\text{Hom}_{ab}(\mathcal{E}_1 \times \ldots \times \mathcal{E}_k, \mathcal{G})(M)$ with $h_M = h'_M$. For an open subset $V \subseteq M$ consider open subsets $O \subseteq V$ as the ones in the covering $\bigcup_{i \in I} O_i = V$ in Lemma 4.2.1, this means with $O^c \subseteq V$, and corresponding functions $\chi \in C^\infty(M)$ as in the third part of Lemma 4.2.2, this means with supp($\chi$) $\subseteq V$ and $\chi|_{O^c} = 1$. Then,

$$h\mathcal{V}(e_1, \ldots, e_k)|_O = h_O((\chi e_1)|_O, \ldots, (\chi e_k)|_O) = h_M(\chi e_1, \ldots, \chi e_k)|_O = h'_\mathcal{V}(e_1, \ldots, e_k)|_O$$

for all $e_1 \in \mathcal{E}_1(V), \ldots, e_k \in \mathcal{E}_k(V)$. Since $\mathcal{G}$ is a sheaf, the two considered sheaf homomorphisms are equal. This shows that if a local map $D \in \text{Loc}(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{G})(M)$ determines a sheaf homomorphism $\{D\mathcal{U}\}_{\mathcal{U} \subseteq M} \in \text{Hom}_{ab}(\mathcal{E}_1 \times \ldots \times \mathcal{E}_k, \mathcal{G})(M)$ with $D_M = D$, this is unique. Indeed, this is the case. For every $V \subseteq M$ a map $D\mathcal{V}$ can be defined by

$$D\mathcal{V}(e_1, \ldots, e_k)|_O = D(\chi e_1, \ldots, \chi e_k)|_O \quad (4.26)$$

for all elements $e_i \in \mathcal{E}_i(V)$ and all open subsets $O$ with corresponding functions $\chi$ as above. The map $D\mathcal{V} : \mathcal{E}_1(V) \times \ldots \times \mathcal{E}_k(V) \rightarrow \mathcal{G}(V)$ is well-defined by (4.26) since $\mathcal{G}$ is a sheaf and since the right hand side does not depend on the choice $(O, \chi)$. If $(O', \chi')$ is another one it follows with $(\chi e_i)|_{O' \cap O'} = (\chi' e_i)|_{O' \cap O'}$ that $D(\chi e_1, \ldots, \chi e_k)|_{O' \cap O'} = D(\chi' e_1, \ldots, \chi' e_k)|_{O' \cap O'}$. In order to show that the so defined maps $D\mathcal{V}$ which are obviously group homomorphisms in every argument, define a sheaf homomorphism $\{D\mathcal{U}\}_{\mathcal{U} \subseteq M} \in \text{Hom}(\mathcal{F}, \mathcal{G})$, let $V \subseteq U \subseteq M$ be fixed open subsets and $O \subseteq V$ as above. Further, let $\chi \in C^\infty(M)$ with supp($\chi$) $\subseteq V$ and $\chi|_O = 1$. Then, the statement follows from

$$D\mathcal{U}(e_1, \ldots, e_k)|_O = D(\chi e_1, \ldots, \chi e_k)|_O = D(\chi(e_1|_V), \ldots, \chi(e_k|_V))|_O = D\mathcal{V}(e_1|_V, \ldots, e_k|_V)|_O.$$
Due to the fact that $D_M = D$ and the proved uniqueness it is clear that this way every sheaf homomorphism $\{h_U\}_{U \subseteq M}$ can be recovered from $h_M$. With the proved results and Lemma 4.2.6 the map (4.24) apparently is an isomorphism.

\[\text{Corollary 4.2.8 (The sheaf of local maps)}\]

With the data of Proposition 4.2.7 the assignment

\[ U \mapsto \text{Loc}(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{G})(U) \quad \text{for all} \quad U \in M \]  \hfill (4.27)

can be extended to a sheaf of abelian groups over $M$ with unique restriction maps

\[ r^U_V : \text{Loc}(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{G})(U) \to \text{Loc}(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{G})(V) \]  \hfill (4.28)

satisfying

\[ D(e_1, \ldots, e_k)|_V = (r^U_V D)(e_1|_V, \ldots, e_k|_V) \]  \hfill (4.29)

for all $D \in \text{Loc}(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{G})(U)$, $e_1 \in \mathcal{E}_1(U), \ldots, e_k \in \mathcal{E}_k(U)$ and $V \subseteq U \subseteq M$.

**Proof:** The statement is clear using the isomorphism (4.25) and Proposition 4.1.7 in order to define $\text{Loc}(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{G})$ as an isomorphic sheaf of $\operatorname{Hom}(\mathcal{E}_1 \times \ldots \times \mathcal{E}_k; \mathcal{G})$. The induced restriction maps are directly seen to be the analogues of $D \mapsto D_V$ in (4.20). Explicitly, for $V \subseteq U \subseteq M$ choose a covering of $V$ of open sets $O$ satisfying $O^1 \subseteq V$ and corresponding functions $\chi \in C^\infty(U)$ with supp($\chi$) $\subseteq V$ and $\chi|_O = 1$. Then $r^U_V$ is defined by

\[ (r^U_V D)(e_1, \ldots, e_k)|_O = D(\chi e_1, \ldots, \chi e_k)|_O \]  \hfill (4.30)

for all $D \in \text{Loc}(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{G})(U)$ and arbitrary elements $e_i \in \mathcal{E}_i(V)$.

**Remark 4.2.9**

The first part in the proof of Proposition 4.2.7 shows that the considered local operators are uniquely defined by their values on restricted elements.

Corollary 4.2.8 is of course trivial and at first sight it seems redundant to introduce the new notion of local maps. But as we will see, this slightly different point of view is very helpful in order to investigate sheaf homomorphisms consisting of maps with further particular properties. This becomes clear with the following important discussion of multidifferential maps and all later applications thereof. First of all we need a generalized version of the statement in [123, Lemma A.3.1].

**Lemma 4.2.10**

Let $\mathcal{E}$ be a presheaf of associative and commutative algebras over a field $\mathbb{K}$ and $\mathcal{F}$ be a sheaf of $\mathcal{E}$-modules over $M$. Further let $V \subseteq M$ be an open subset, $\mathcal{E}_1, \ldots, \mathcal{E}_k, k \in \mathbb{N}$, be left $\mathcal{E}(V)$-modules and let $D \in \operatorname{Hom}_N(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{F}(V))$ be a $\mathbb{K}$-multilinear map. If there exist an open covering $V = \bigcup_{i \in I} O_i$ and differential operators $D_i \in \text{DiffOp}_{\mathcal{E}(V)}^L(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{F}(V))$ of multorder $L \in \mathbb{N}_0^k$ such that

\[ D(e_1, \ldots, e_k)|_{O_i} = D_i(e_1, \ldots, e_k)|_{O_i} \quad \text{for all} \quad i \in I, e_1 \in \mathcal{E}_1, \ldots, e_k \in \mathcal{E}_k, \]  \hfill (4.31)

then $D \in \text{DiffOp}_{\mathcal{E}(V)}^L(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{F}(V))$.

**Proof:** The proof is an easy induction over $|L|$ using the formula $(D \circ L^i_c - L_c \circ D)(e_1, \ldots, e_k)|_{O_i} = (D_i \circ L^i_c - L_c \circ D_i)(e_1, \ldots, e_k)|_{O_i}$ for all $c \in \mathcal{E}(V)$. \hfill \(\blacksquare\)
Proposition 4.2.11 (The sheaves of differential operators)
Let $\mathcal{E}_1, \ldots, \mathcal{E}_k$, $k \in \mathbb{N}$, and $\mathcal{T}$ be sheaves of $C^\infty_M$-modules over $M$. For all open subsets $U \subseteq M$ and $L \in \mathbb{N}_0^k$ let
\[
\text{DiffOp}^L(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{T})(U) = \text{DiffOp}^L_{C^\infty(U)}(\mathcal{E}_1(U), \ldots, \mathcal{E}_k(U); \mathcal{T}(U))
\]
be the sets of algebraically defined multidifferential operators.

i.) For all $U \subseteq M$ and $L \in \mathbb{N}_0^k$
\[
\text{DiffOp}^L(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{T})(U) \subseteq \text{Loc}(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{T})(U). \quad (4.33)
\]

ii.) For all $V \subseteq U \subseteq M$ and $L \in \mathbb{N}_0^k$
\[
r_V^U : \text{DiffOp}^L(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{T})(U) \longrightarrow \text{DiffOp}^L(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{T})(V). \quad (4.34)
\]

iii.) For every $L \in \mathbb{N}_0^k$ the assignment $M \ni U \longmapsto \text{DiffOp}^L(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{T})(U)$ together with the restriction maps $r_V^U$ in (4.33), yield a sheaf of $(C^\infty_M, C^\infty_M)$-bimodules, the so-called sheaf $\text{DiffOp}^L(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{T})$ of multidifferential operators from $\mathcal{E}_1, \ldots, \mathcal{E}_k$ to $\mathcal{T}$ of multiorder $L$.

PROOF: As usual for algebraically defined multidifferential operators, the assertions are in the most cases proved by an induction over the absolute value $|L| \in \mathbb{N}_0$ of the multiindex $L$.

i.) Let $D \in \text{DiffOp}^L(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{T})(U)$, $V \subseteq U$ and $e_i \in \mathcal{E}_i(U)$, $i = 1, \ldots, k$, with $e_i|_V = 0$ for at least one certain $l \in \{1, \ldots, k\}$. Then we choose a covering of $V$ with open sets $O$ as in Lemma 4.2.1 and functions $\chi$ as in the fourth part of Lemma 4.2.2. With the fact that $\chi \cdot e_l = e_l$ we have
\[
D(e_1, \ldots, e_l, \ldots, e_k)|_O = D(e_1, \ldots, \chi \cdot e_l, \ldots, e_k)|_O = \chi|_O \cdot D(e_1, \ldots, e_l, \ldots, e_k)|_O + \left((D \circ L^0_\chi - L \circ D)(e_1, \ldots, e_k)\right)|_O.
\]
Due to the sheaf properties of $\mathcal{T}$ this leads to $D(e_1, \ldots, e_k)|_V = 0$ by an induction over $|L|$ as mentioned above, since $(D \circ L^0_\chi - L \circ D) \in \text{DiffOp}^{L-e_i}(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{T})(U)$ and $\chi|_O = 0$.

ii.) Let $V \subseteq U \subseteq M$, $D \in \text{DiffOp}^L(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{T})(U)$, $e_i \in \mathcal{E}_i(V)$ for $i = 1, \ldots, k$ and $a \in C^\infty(V)$. For the functions $\chi$ with respect to $O \subset V$ as in the definition (4.30) of the restriction map $r_V^U$ one has for $l \in \{1, \ldots, k\}$
\[
\chi^2(a \cdot e_l) = (\chi a) \cdot (\chi e_l), \quad (4.35)
\]
since $(\chi^2(a \cdot e_l))|_V = \chi^2|_V \cdot a \cdot e_l = (\chi a)|_V \cdot (\chi e_l)|_V = ((\chi a) \cdot (\chi e_l))|_V$ and the restrictions to $U \setminus \text{supp} \chi$ vanish. With the fact that $D$ is local and $(\chi(a \cdot e_l))|_O = (\chi^2(a \cdot e_l))|_O$ this leads to
\[
\left((r_V^U D) \circ L^0_\chi - L_a \circ (r_V^U D)\right)(e_1, \ldots, e_k)|_O = (D(\chi e_1, \ldots, \chi^2(a \cdot e_l), \ldots, \chi e_k) - (\chi a) \cdot D(\chi e_1, \ldots, \chi e_k))|_O = (D(\chi e_1, \ldots, (\chi a) \cdot (\chi e_l), \ldots, \chi e_k) - (\chi a) \cdot D(\chi e_1, \ldots, \chi e_k))|_O = \left((D \circ L^0_\chi - L \circ D)\right)(e_1, \ldots, \chi e_l)|_O = \left(D \circ L^0_{\chi a} - L_{\chi a} \circ D\right)(e_1, \ldots, e_l)|_O.
\]
After checking the case $|L| = 0$ the assertion $r_V^U D \in \text{DiffOp}^L(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{T})(V)$ follows again by induction. If $r_V^U \left(D \circ L^0_\chi - L \circ D\right) \in \text{DiffOp}^{L-e_i}(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{T})(V)$, Equation (4.36) and Lemma 4.2.10 show that $(r_V^U D) \circ L^0_\chi - L_a \circ (r_V^U D) \in \text{DiffOp}^{L-e_i}(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{T})(V)$.
Remark 4.2.12

iii.) The sheaf axioms (a)-(c) follow from the fact that Loc(\(E_1, \ldots, E_k; \mathcal{F}\)) is a sheaf and by the already proved statements. The only nontrivial consequence is the last sheaf axiom (d). Consider an open covering \(U = \bigcup_{i \in I} U_i\) as in Definition 4.1.2 and \(D_i \in \text{DiffOp}^L(E_1, \ldots, E_k; \mathcal{F})(U_i)\) with \(D_i|_{U_i \cap U_j} = D_j|_{U_i \cap U_j}\). One has to show that the local map \(D \in \text{Loc}(E_1, \ldots, E_k; \mathcal{F})(U)\), defined by

\[
D(e_1, \ldots, e_k)|_{U_i} = D_i(e_1|_{U_i}, \ldots, e_k|_{U_i})
\]

(4.37)

according to Equation (4.9), Proposition 4.2.7 and Corollary 4.2.8 is a differential operator \(D \in \text{DiffOp}^L(E_1, \ldots, E_k; \mathcal{F})(U)\). Using the module structures (3.3), the property (4.39) implies that the sheaves of differential operators \(\text{DiffOp}^L(E_1, \ldots, E_k; \mathcal{F})(U)\) satisfy

\[
V \cap U \rightarrow \mathcal{F}(U)\text{ is a multidifferential operator by the assumption of the induction. Finally, it is easy to verify that the left and right module structures } a \cdot D = L_a \circ D \text{ and } D \cdot (i) a = D \circ L^{(i)} a \text{ for } i = 1, \ldots, N \text{ as in (4.33) with } a \in C^\infty_M(U) \text{ and } D \in \text{DiffOp}^L(E_1, \ldots, E_k; \mathcal{F})(U) \text{ for } U \subseteq M \text{ satisfy}
\]

\[
(a \cdot D)|_V = a|_V \cdot D|_V \quad \text{and} \quad (D \cdot (i) a)|_V = D|_V \cdot (i) a|_V
\]

(4.39)

for all open subsets \(V \subseteq U \subseteq M\).

\[\square\]

Remark 4.2.12

i.) Proposition 4.2.11 shows that the differential operators \(\text{DiffOp}^L(E_1, \ldots, E_k; \mathcal{F})\) of a finite order \(L \in \mathbb{N}_0^k\) of differentiation are a subsheaf of Loc\((E_1, \ldots, E_k; \mathcal{F})\). This way it is also clear that we have a new example of a subsheaf of \(\text{Hom}(E_1 \times \ldots \times E_k, \mathcal{F})\) where all the maps of a sheaf homomorphism are corresponding differential operators.

ii.) Using the module structures (3.3), the property (4.39) implies that the sheaves of differential operators are sheaves of \(C^\infty_M\)-bimodules.

iii.) Note that in general \(\text{DiffOp}^L(E_1, \ldots, E_k; \mathcal{F})\) is only a presheaf since axiom (d) does not have to be satisfied. As a counterexample consider \(\text{DiffOp}(C^\infty(\mathbb{R}), C^\infty(\mathbb{R}))\). Let \(\chi_0 \in C^\infty(\mathbb{R})\) be a function with \(\text{supp}(\chi_0) \subseteq (-1,1)\) and define \(\chi_i \in C^\infty(\mathbb{R})\) by \(\chi_i(x) = \chi_0(x + 2i) + \chi_0(x - 2i)\) for all \(i \in \mathbb{N}\). Then let \(U_i = (-2i - 1, 2i + 1)\) and consider the differential operators \(D_i \in \text{DiffOp}(C^\infty(U_i), C^\infty(U_i))\) given by \(D_i = \sum_{l=0}^i \chi_l|_{U_i} \frac{d^l}{dx^l}\). Then there exists no \(D \in \text{DiffOp}(C^\infty(\mathbb{R}), C^\infty(\mathbb{R}))\) with \(D|_{U_i} = D_i\) since the order of \(D\) would be infinite.

In later applications we will make use of the following lemma concerning sheaf endomorphisms consisting of differential maps.

Lemma 4.2.13

For a sheaf \(E\) of left \(C^\infty_M\)-modules and \(l \in \mathbb{N}_0\) let \(\mathcal{D}^l = \text{DiffOp}^l(E; E)\) be the corresponding sheaf of differential operators. Then, for all \(l, k \in \mathbb{N}_0\) the usual composition of maps \(\circ : \mathcal{D}^l(U) \times \mathcal{D}^k(U) \rightarrow \mathcal{D}^{l+k}(U)\) for all \(U \subseteq M\) defines a sheaf homomorphism

\[
\circ : \mathcal{D}^l \times \mathcal{D}^k \rightarrow \mathcal{D}^{l+k}.
\]

(4.40)

This means that for \(V \subseteq U \subseteq M\) and \(D \in \mathcal{D}^l(U), D' \in \mathcal{D}^k(U)\) one has

\[
(D \circ D')|_V = D|_V \circ D'|_V.
\]

(4.41)
4.3 Sheaves over different manifolds

Now we investigate the situation where the (pre)sheaves do not have the same base manifold as underlying topological space. For this purpose we introduce two simple definitions whose general versions can be found in [57, Chap. II] and [66, Chap. II]. If \( p : P \to M \) is an open map between two smooth manifolds we always use the notation \( U = p(\tilde{U}) \) to denote open subsets \( \tilde{U} \subseteq P \) and the corresponding open images \( U \subseteq M \).

Lemma 4.3.1 (Pullback of sheaves)

Let \( p : P \to M \) be an open map between two smooth manifolds, this means \( p(\tilde{U}) \in M \) for all \( \tilde{U} \in P \), and let \( \mathcal{G} \) be a (pre)sheaf over \( M \). Then the assignment

\[
P \ni \tilde{U} \mapsto (p^* \mathcal{G})(\tilde{U}) := \mathcal{G}(p(\tilde{U}))
\]

(4.42)

together with the restriction maps

\[
r_{\tilde{V}}^{\tilde{U}} := r_{p(\tilde{U})}^{p(\tilde{V})} : (p^* \mathcal{G})(\tilde{U}) \to (p^* \mathcal{G})(\tilde{V})
\]

(4.43)

yields a presheaf which we call pullback \( p^* \mathcal{G} \) of \( \mathcal{G} \) with \( p \).

The proof of this lemma is obvious.

Remark 4.3.2 (Pullback and inverse image)

i.) Note that, in general, the pullback of a sheaf as defined above is only a presheaf. An open covering \( \tilde{U} = \bigcup_{i \in I} \tilde{U}_i \) induces an open covering \( p(\tilde{U}) = \bigcup_{i \in I} p(\tilde{U}_i) \). Then it is easy to verify that axiom (c) is still fulfilled. This is not always the case for axiom (d) since \( p(\tilde{U} \cap \tilde{V}) \subseteq p(\tilde{U}) \cap p(\tilde{V}) \) can be a proper subset, in which cases it is not possible to trace back property (d) of \( p^* \mathcal{G} \) to the property (d) of \( \mathcal{G} \).

ii.) If \( p \) is continuous instead of open there is a way to define a sheaf \( p^{-1} \mathcal{G} \) over \( P \) out of \( \mathcal{G} \) which is called the inverse image of \( \mathcal{G} \), confer [57, Chap. II, Sect. 1.12] and [66, Chap. II, Sect. 1]. This definition, though, requires some more technical preliminaries and since we do not need the sheaf property for our purposes we have only introduced the above notion of the pullback.

Lemma 4.3.3 (Direct image of sheaves)

Let \( p : P \to M \) be a continuous map between two smooth manifolds and let \( \mathcal{G} \) be a (pre)sheaf over \( P \). Then the assignment

\[
M \ni U \mapsto (p_* \mathcal{G})(U) := \mathcal{G}(p^{-1}(U))
\]

(4.44)

together with the restriction maps

\[
r_{\tilde{V}}^{U} := r_{p^{-1}(U)}^{p^{-1}(V)} : (p_* \mathcal{G})(U) \to (p_* \mathcal{G})(V)
\]

(4.45)

yields a (pre)sheaf, the so-called direct image \( p_* \mathcal{G} \) of \( \mathcal{G} \) under \( p \).

Proof: Clearly, an open covering \( U = \bigcup_{i \in I} U_i \) induces an open covering \( p^{-1}(U) = \bigcup_{i \in I} p^{-1}(U_i) \). Then, the proof of (a), (b) and (c) is again obvious. For (d) one needs \( p^{-1}(U \cap V) = p^{-1}(U) \cap p^{-1}(V) \).
Remark 4.3.4
If \( p \) is open and continuous it follows that \( p_*(\mathcal{F}) = \mathcal{G} \) since \( p(p^{-1}(U)) = U \). But since \( p^{-1}(p(\tilde{U})) \supseteq \tilde{U} \) one has \( p^*(p_*\mathcal{G}) \neq \mathcal{G} \) in general.

Lemma 4.3.5
Let \( p : P \to M \) be an open and continuous map as above. Further let \( \mathcal{F} \) be a (pre)sheaf of sets over \( M \) and \( \mathcal{G} \) be a sheaf of sets over \( P \). Then there is a canonical isomorphism
\[
\text{Hom}(\mathcal{F}, p_*\mathcal{G}) \cong \text{Hom}(p^*\mathcal{F}, \mathcal{G})
\]
(4.46)
between the corresponding presheaf homomorphisms of presheaves over \( M \) and \( P \).

Proof: For \( \{h_U\}_{U \in \mathcal{M}} \in \text{Hom}(\mathcal{F}, p_*\mathcal{G}) \) we define \( \{h_{\tilde{U}}\}_{\tilde{U} \in \mathcal{P}} \) by \( h_{\tilde{U}} := \left(p^{-1}(p(\tilde{U})) \circ h_p(\tilde{U})\right) \). By assumption and the presheaf property of \( \mathcal{G} \) this defines an element in \( \text{Hom}(p^*\mathcal{F}, \mathcal{G}) \). Conversely, such a presheaf homomorphism defines an element in \( \text{Hom}(\mathcal{F}, p_*\mathcal{G}) \) with the definition \( h_U := h_p^{-1}(U) \). It is obvious that both constructions are inverse to each other.

Remark 4.3.6
Since \( \mathcal{G} \) and \( p_*\mathcal{G} \) are sheaves one is able to consider the sheaves of local morphisms \( \text{Hom}(p^*\mathcal{F}, \mathcal{G}) \) over \( P \) or \( \text{Hom}(\mathcal{F}, p_*\mathcal{G}) \) over \( M \).

Example 4.3.7
With \( p : P \to M \) as above there is a presheaf homomorphism \( h = \{h_{\tilde{U}}\}_{\tilde{U} \in \mathcal{P}} : p^*\mathcal{C}_M^\infty \to \mathcal{C}_P^\infty \) with
\[
h_{\tilde{U}} := (p|_{\tilde{U}}^*) : (p^*\mathcal{C}_M^\infty)(\tilde{U}) = \mathcal{C}_\infty^\infty(\tilde{U}) \to \mathcal{C}_\infty^\infty(\tilde{U}).
\]
(4.47)
Since all \( h_{\tilde{U}} \) are inclusions and with Lemma 4.3.3 \( \mathcal{C}_M^\infty \) is a subsheaf of \( p_*\mathcal{C}_P^\infty \).

Remark 4.3.8 (Sheaves over different manifolds)
The assertions of Proposition 4.2.7 and all derived results, now stated for sheaves over \( P \), can be reformulated and extended to the situation where some \( \mathcal{F} \in \{\mathcal{E}_1, \ldots, \mathcal{E}_k\} \) are presheaves \( \mathcal{F} = p^*\mathcal{E} \) of \( p^*\mathcal{C}_M^\infty \)-modules over \( P \) coming from sheaves \( \mathcal{E} \) of \( \mathcal{C}_M^\infty \)-modules over \( M \).

In order to find adequate coverings of \( V \subseteq \tilde{U} \subseteq P \) with subsets \( \tilde{O} \) and functions \( \chi \in \mathcal{C}_\infty^\infty(U) \) as in the crucial Equations (4.26), (4.30) and (4.35) one considers the corresponding open coverings with subsets \( O \) of \( V = p(V) \subseteq M \) and functions \( \chi \) and uses the induced covering of \( p(V) \) with the subsets
\[
\tilde{O} = p^{-1}(O) \cap \tilde{V}.
\]
(4.48)
Then, \( p(\tilde{O}) = O \) and the proofs and definitions can be made in a completely analogous way.

Proposition 4.2.11 has an analogous reformulation. In this case of multidifferential operators of multider \( L \in \mathbb{N}_0^k, k \in \mathbb{N}, \) let \( \mathcal{E}_1, \ldots, \mathcal{E}_k \) be sheaves of \( \mathcal{C}_M^\infty \)-modules over \( M \) and \( \mathcal{F} \) be a sheaf of \( \mathcal{C}_P^\infty \)-modules over \( P \). Using the presheaf homomorphism of Example 4.3.7 \( \mathcal{F} \) inherits a \( p^*\mathcal{C}_M^\infty \)-module structure. Then the sheaf
\[
\text{DiffOp}_{p^*\mathcal{C}_M^\infty}^L(p^*\mathcal{E}_1, \ldots, p^*\mathcal{E}_k; \mathcal{F})
\]
(4.49)
of \( (\mathcal{C}_P^\infty, p^*\mathcal{C}_M^\infty) \)-bimodules over \( P \) is well-defined in the obvious way. By use of Lemma 4.3.5 the given structures also yield a sheaf
\[
\text{DiffOp}_{\mathcal{C}_M^\infty}^L(\mathcal{E}_1, \ldots, \mathcal{E}_k; p_*\mathcal{F})
\]
(4.50)
of \( (p_*\mathcal{C}_P^\infty, \mathcal{C}_M^\infty) \)-bimodules over \( M \) which is the direct image of the afore mentioned sheaf.
4.4 Sheaves and invariant differential Hochschild complexes

After the previous general considerations we now come to the applications with respect to the deformation theory of algebras and modules. In the present section we investigate under which circumstances the concepts of sheaf theory can be used for the task of computing Hochschild cohomologies in the \((G\text{-invariant})\) differential setting introduced in the Sections 2.6 and 3.2. Besides discussing and presenting the necessary framework the main goal is to formulate the important Propositions 4.4.7 and 4.4.12.

The basic setting is the following. Let \(A\) be a sheaf of \(\mathbb{K}\)-algebras and \(E\) be a sheaf of \(\mathbb{K}\)-vector spaces and right \(A\)-modules over a topological space \(M\). This assignment of right modules to all open subsets \(U \subseteq M\) then leads to the induced assignment

\[
U \mapsto \text{HC}^\bullet(A(U), \text{End}_\mathbb{K}(E(U), E(U)))
\]  

of the corresponding Hochschild complexes. According to the basic intention of sheaf theory the natural question then is if these Hochschild complexes are related with each other and if the knowledge of the global cohomology is equivalent to that of the local ones. If this is the case the assertion clearly reveals a natural approach of computing the global cohomology by considering the possibly simpler local problem.

Having this aim in mind a first step is to figure out when the assignment (4.51) can be extended to a presheaf of vector spaces such that the family of Hochschild differentials is a presheaf homomorphism. For the \((G\text{-invariant})\) differential Hochschild complexes this will turn out to be the appropriate guideline to find the necessary general frameworks where all aspired issues are given.

4.4.1 Sheaves and differential Hochschild complexes

First of all we investigate Hochschild complexes \(\text{HC}^\bullet\text{diff}(A, \mathcal{D})\) of the differential type as in Corollary 3.2.8. In order to define the vector spaces of this complex we need a set \(\{A, E, \mathcal{C}, \mathcal{B}, \gamma\}\) of purely algebraic structures as occurring in Definition 3.2.4. Now assume that \(A, E, \mathcal{C}, \text{ and } \mathcal{B}\) are sheaves of corresponding structures over a topological space \(P\) and that \(\gamma: \mathcal{C} \to \mathcal{B}\) is a sheaf homomorphisms consisting of algebra homomorphisms. Due to the results of Section 4.2, especially Proposition 4.2.11, the induced assignments of differential operators to open subsets are presheaves if the corresponding presheaves of commutative algebras, in our case \(\mathcal{C}\) and \(\mathcal{B}\), are presheaves of functions over some smooth manifold.

Remark 4.4.1 (A convenient framework for differential Hochschild complexes)

With respect to the later applications where we have to work with sheaves over different topological spaces we assume from now on to be given the following structures.

i.) A surjective submersion \(p: P \to M\) between smooth manifolds \(P\) and \(M\) inducing

(a) the sheaf \(\mathcal{C} = C^\bullet_M\) of smooth functions on \(M\) which is a sheaf of associative and commutative algebras over \(M\).

(b) the sheaf \(\mathcal{B} = C^\bullet_P\) of smooth functions on \(P\) which is a sheaf of associative and commutative algebras over \(P\).

(c) the presheaf morphism \(\gamma: p^*\mathcal{C} \to \mathcal{B}\) which is given by the pullback of functions, this means \(\gamma_U = (p|_U)^*\) for all open \(U \subseteq P\), confer Example 4.3.7.

ii.) A sheaf \(A\) of \(\mathbb{K}\)-algebras with multiplication \(\mu\) and left \(\mathcal{C}\)-modules over \(M\).

iii.) A sheaf \(E\) of \(\mathbb{K}\)-vector spaces with a left \(\mathcal{B}\)-module structure \(l\) and a right \(p^*A\)-module structure \(\rho\) over \(P\).
The structures $\mathcal{E}$ and $\mathcal{B}$ immediately induce the sheaves $\mathcal{D}^l = \text{DiffOp}^l_{\mathcal{B}}(\mathcal{E}, \mathcal{E})$ of $K$-vector spaces and $\mathcal{B}$-modules over $P$ for all $l \in \mathbb{N}_0$. Since the map $p : P \rightarrow M$ is open it is possible to consider the pullback $p^*\mathcal{E}$. Using the sheaf morphism $\gamma$ in the sense of (3.22), $\mathcal{D}^l = \text{DiffOp}^l_{\mathcal{B}}(\mathcal{E}, \mathcal{E})$ is a sheaf of $p^*\mathcal{E}$-modules over $P$. The pullbacks $p^*\mathcal{E}$ and $p^*\mathcal{A}$ only are presheaves over $P$ but as stated in Remark 4.3.8 this still guarantees that one has the sheaves

$$\text{HC}^{k,L,l} = \text{DiffOp}^L_{p^*\mathcal{E}}(p^*\mathcal{A}, \ldots, p^*\mathcal{A}; \text{DiffOp}^l_{\mathcal{B}}(\mathcal{E}, \mathcal{E}))$$

(4.52)
of $\mathcal{B}$-modules over $P$ for all $L \in \mathbb{N}_0^k$, $k \in \mathbb{N}$, and $l \in \mathbb{N}_0$. Like $U \longmapsto \mathcal{D}(U) = \bigcup_{l \in \mathbb{N}_0} \mathcal{D}^l(U)$, the assignment

$$\tilde{U} \longmapsto \text{HC}^{k,L,l}_{\text{diff}}(p^*\mathcal{A}, \mathcal{D})(\tilde{U}) = \bigcup_{L \in \mathbb{N}_0^k} \bigcup_{l \in \mathbb{N}_0} \text{HC}^{k,L,l}(\tilde{U})$$

(4.53)
of differential operators to open subsets $\tilde{U} \subseteq P$ only induce presheaves of $K$-vector spaces and $\mathcal{B}$-modules. The restriction maps with respect to open subsets $\tilde{V} \subseteq \tilde{U} \subseteq P$ satisfy

$$[\phi(a_1, \ldots, a_k)(e)]|_{\tilde{V}} = \phi|_{\tilde{V}}(a_1|_{\tilde{V}}, \ldots, a_k|_{\tilde{V}})(e|_{\tilde{V}})$$

(4.54)
for all $\phi \in \text{HC}^*(p^*\mathcal{A}, \mathcal{D})(\tilde{U})$, $a_1, \ldots, a_k \in \mathcal{A}(p(\tilde{U}))$, and $e \in \mathcal{E}(\tilde{U})$.

In the same way one finds the presheaf $\text{HC}^*_{\text{diff}}(A, A)$ of $\mathcal{E}$-modules over $M$ which is derived from the sheaf $A$ of algebras over $M$. For this presheaf it is clear that the local cup products, insertions after the $i$-th position and Hochschild differentials are well-defined on the differential subcomplexes only if the family $\mu = \{\mu_U\}_{U \subseteq M}$ of algebra structures consists of differential structures.

Due to Proposition 3.2.3 the cup products as in (2.29) are always well-defined for (4.53). But with the defining Equation (2.31) one further sees that in this case the local Hochschild differentials are well-defined endomorphisms of the differential operators only if both the family $\mu$ and the family $\rho = \{\rho_U\}_{U \subseteq P}$ of right module structures consist of differential elements. This can of course be checked globally since the families $\mu$ and $\rho$ are sheaf homomorphisms $\mu : A \times A \rightarrow A$ and $\rho : \mathcal{E} \times p^*\mathcal{A} \rightarrow \mathcal{E}$.

**Lemma 4.4.2 (Sheaves of differential algebras and modules)**

Let there be given structures as in Remark 4.4.1 such that the algebra multiplication $\mu_M$ of $A(M)$ and the right $(p^*\mathcal{A})(P) = A(M)$-module structure $\rho_P$ of $\mathcal{E}(P)$ are of the differential type as in the Definitions 3.2.7 and 3.3.4.

Then, all algebra structures of the family $\mu = \{\mu_U\}_{U \subseteq M}$ and all right module structures of the family $\rho = \{\rho_U\}_{U \subseteq P}$ are differential with the same degrees of differentiation as the global ones.

**Proof:** As a consequence of Proposition 4.2.7 Corollary 4.2.8 and Remark 4.3.8 one finds $\mu_U = \mu_M |_{U}$ and $\rho_U = \rho_P |_{U}$. Then, the presheaf property of the considered differential cochains yields the statement.

**Remark 4.4.3**

The proof in particular includes the following general observation. If the Hochschild complexes of a particular type induce presheaves of vector spaces and if the global structures are of this type, the same is true for the local ones. The converse assertion of Lemma 4.4.2 is also true since cochains of a fixed order of differentiation are sheaves.

The general property (1.29) implying (4.54) now shows that (4.53) really defines presheaves of Hochschild complexes if $\mu_M$ and $\rho_P$ are differential. The concrete meaning thereof becomes clear in the following lemma.
Lemma 4.4.4 (Presheaf of complexes)
Let there be given structures as in Remark 4.4.1 such that \( \mu_M \) and \( \rho_P \) are of the differential type. Then, the presheaf \( HC^\bullet_{\text{diff}}(p^*A, \mathcal{D}) \) is a presheaf of Hochschild complexes over \( P \) which means that the family \( \{\delta_U\}_{U \subseteq P} \) of Hochschild differentials is a presheaf endomorphism. Thus one has
\[
\delta_V \circ r^V_U = r^V_U \circ \delta_U \quad (4.55)
\]
for all open subsets \( \tilde{V} \subseteq \tilde{U} \subseteq P \), where \( r^V_U \) denotes the restriction map. In particular this induces the sheaf homomorphisms \( \delta : HC^{k,L,l}_C \rightarrow HC^{k,L,l}_C \).

Moreover, one has corresponding (pre)sheaf homomorphisms given by the cup product \( \cup \) and and the insertions \( \circ_i \) after the \( i \)-th position with respect to the presheaf \( p^*HC^\bullet_{\text{diff}}(A, A) \) having analogue morphisms.

PROOF: Using the properties (4.29) and (4.31) of the restriction maps, all assertions can be proved by restricting the defining equations of all structures since the considered differential cochains are uniquely defined by the values on restrictions.

Note the different terminology used in some references, confer [127, Chap. 2]. What here is called a sheaf of complexes is there referred to as differential sheaf. We avoid the latter since it would lead to some confusion with the notion of differential operators.

Corollary 4.4.5 (The sheaves of cocycles)
In the situation of Lemma 4.4.4 equation (4.55) immediately implies that the Hochschild cocycles build subsheaves \( Z^{k,L,l}_C \subseteq HC^{k,L,l}_C \).

The fact that the presheaves \( HC^{k,L,l}_C \) really are sheaves of \( \mathcal{B} = C^\infty_P \)-modules has an important consequence for the later applications since it is possible to obtain global cochains out of local ones as explained in the two first parts of Lemma 4.2.2. So, for all open subsets \( \tilde{V} \subseteq \tilde{U} \subseteq P \) every smooth function \( \tilde{\chi} \in C^\infty(\tilde{U}) \) with \( \text{supp}\tilde{\chi} \subseteq \tilde{V} \) induces a map \( \tilde{\chi} : HC^{k,L,l}(\tilde{V}) \rightarrow HC^{k,L,l}(\tilde{U}) \). By its defining Equation (4.11) and the properties (4.35) and (4.39) of the Hochschild differentials it is easy to see that
\[
\delta_V \circ \tilde{\chi} = \tilde{\chi} \circ \delta_V \quad \text{if} \quad \rho_p = 0, \quad (4.56)
\]
this means if \( \mathcal{E} \) is a sheaf of \( (\mathcal{B}, p^*A) \)-bimodules. Equation (4.55) is clearly verified by checking it on the subsets \( \tilde{V} \) and \( \tilde{U} \setminus \text{supp} \tilde{\chi} \). In the case of \( \rho_p = 0 \) the cocycles \( Z^{k,L,l}_C \) are even subsheaves of \( C^\infty_P \)-modules. Further, this condition has an important consequence for the coboundaries.

Lemma 4.4.6 (The sheaves of coboundaries)
Let \( \mu_M \) and \( \rho_P \) be differential with \( \rho_p = 0 \). Then, the presheaves of Hochschild coboundaries \( B^{k,L,l}_C \subseteq HC^{k,L,l}_C \) are subsheaves of \( C^\infty_P \)-modules if there exist \( \tilde{L} \in \mathbb{N}_0 \) such that \( B^{k,L,l} = \delta HC^{k-L,\tilde{L}} \).

PROOF: The only nontrivial point to check is axiom (d) in Definition 4.1.2. So let \( \bigcup_{i \in I} \tilde{U}_i = \tilde{U} \subseteq P \) be an open covering and let \( \phi_i = \delta_{\tilde{U}_i} \Theta_i \in B^{k,L,l}(\tilde{U}_i) \) be coboundaries with \( \phi_i|_{\tilde{U}_i \cap \tilde{U}_j} = \phi_j|_{\tilde{U}_i \cap \tilde{U}_j} \) and where \( \Theta_i \in HC^{k-1,L,\tilde{L}}(\tilde{U}_i) \). Since \( HC^{k,L,l}_C \) is a sheaf there exists a global cochain \( \phi \in HC^{k,L,l}(\tilde{U}) \) with \( \phi|_{\tilde{U}_i} = \phi_i \) and it remains to show that \( \phi \) is a coboundary. Now, choose a subordinate partition of unity \( \{\tilde{\chi}_i\}_{i \in I} \) and consider the global object \( \Theta = \sum_{i \in I} \tilde{\chi}_i \Theta_i \in HC^{k-1,L,\tilde{L}}(\tilde{U}) \) as explained in Lemma 4.2.2. Then the assertion follows from
\[
\delta_{\tilde{U}} \Theta = \delta_{\tilde{U}} \sum_{i \in I} \tilde{\chi}_i \Theta_i = \sum_{i \in I} \tilde{\chi}_i (\delta_{\tilde{U}_i} \Theta_i) = \sum_{i \in I} \tilde{\chi}_i (\phi|_{\tilde{U}_i}) = \phi,
\]
where the last equation is nothing but (4.15) and
\[ \delta_{\tilde{U}} \sum_{i \in I} \tilde{\chi}_i \Theta_i = \sum_{i \in I} \tilde{\chi}_i (\delta_{\tilde{U}_i} \Theta_i) \]  
follows from the defining Equation (4.13) and (4.56).

Lemma 4.4.6 in particular states that in any case one has the sheaves \( \delta \mathcal{H}^{k,L,\tilde{l}} \). The given proof provides an important tool for the computation of Hochschild cohomologies. If the coboundaries \( B^{k,L,\tilde{l}} \) are a sheaf two global cocycles are in the same cohomology class if and only if the same is true for the restrictions to any open subset. This in particular means that the global cohomology can be determined if the knowledge of the local ones is given. With regard to the applications we formulate these result in the following proposition.

**Proposition 4.4.7 (Global and local differential Hochschild cohomology)**

Let there be given structures as in Remark 4.4.1 such that \( \mu_M \) and \( \rho_P \) are differential with \( l_\rho = 0 \) which means that \( E \) is a \( (B,p^*A) \)-bimodule. Further, let \( \phi, \psi \in \mathcal{H}^{k,L,\tilde{l}}(P) \) be global cocycles, \( \delta_P \phi = 0 = \delta_P \psi \), and let \( \bigcup_{i \in I} \tilde{U}_i = P \) be an open covering. Then the following two assertions are equivalent:

i.) \( \phi \) and \( \psi \) are in the same cohomology class, \( \phi - \psi \in B^{k,L,\tilde{l}}(P) \).

ii.) All local cocycles \( \phi|_{\tilde{U}_i} \) and \( \psi|_{\tilde{U}_i} \) are equivalent, \( (\phi - \psi)|_{\tilde{U}_i} \in B^{k,L,\tilde{l}}(\tilde{U}_i) \), and there exist \( \tilde{l} \in \mathbb{N}_0^{k-1}, \tilde{l} \in \mathbb{N}_0 \) such that \( (\phi - \psi)|_{\tilde{U}_i} = \delta_{\tilde{U}_i} \Theta_i \) with cochains \( \Theta_i \in \mathcal{H}^{k-1,L,\tilde{l}}(\tilde{U}_i) \) for all \( i \in I \).

In particular, if the second statement is true every subordinate partition of unity \( \{\tilde{\chi}_i\}_{i \in I} \) with \( \tilde{\chi} \in C^\infty(P) \) induces an element
\[ \Theta = \sum_{i \in I} \tilde{\chi}_i \Theta_i \in \mathcal{H}^{k-1,L,\tilde{l}}(P) \quad \text{with} \quad \phi - \psi = \delta_P \Theta. \]  

**Proof:** The proof is obvious and a special case of the proof of Lemma 4.4.6 for the family \( \{(\phi - \psi)|_{\tilde{U}_i}\}_{i \in I} \).

**Remark 4.4.8 (Quotients of sheaves)**

If (4.57) holds the above equivalence of global and local statements shows that the equivalence relation defining the cohomology is an equivalence relation on the sheaves \( Z^{k,L,\tilde{l}} \) in the sense of [57, Chap. II, Sect. 1.9] and [66, Chap. II, Sect. 1]. The equivalence classes \( Z^{k,L,\tilde{l}}/\delta(\mathcal{H}^{k-1,L,\tilde{l}}) \) with appropriate orders of differentiation and the full Hochschild cohomology \( \mathcal{H}^{k}_{\text{diff}}(A, D) \) can always be equipped with a canonical presheaf structure. But in general, these are no sheaf structures. Axiom (c) in Definition 4.1.2 still holds but Axiom (d) does not have to be satisfied.

**Remark 4.4.9**

Note that one has analogous statements for complexes and cohomologies induced by algebra structures.

### 4.4.2 Sheaves and \( G \)-invariant differential Hochschild complexes

Now we consider (sub)sheaves \( \mathcal{F} \) of algebraic structures with a compatible action of a Lie group \( G \) over some manifold \( M \). By this we clearly mean that for any open subset \( U \subseteq M \) the group \( G \) acts on \( \mathcal{F}(U) \) by structure preserving maps and that this action is compatible with the restriction maps. Thus one could define the action of \( G \) on a sheaf \( \mathcal{F} \) as a group homomorphism \( G \to \text{Aut}(\mathcal{F}) \) into
4.4. Sheaves and invariant differential Hochschild complexes

the structure preserving sheaf automorphisms. Note that without loss of generality one is able to consider left actions.

The situation for functions on a manifold $P$ shows how typical examples look like and how the previously discussed situation has to be adapted to the $G$-invariant case. Usually, the left action on the functions comes from a right action $r : P \times G \to P$ on the underlying space $P$ and is given by pullback. For all subsets $\hat{U} \subseteq P$ consisting of orbits one has a well-defined action on the functions $f \in C^\infty(\hat{U})$ by

$$g \triangleright f = f \circ r_g$$

for all $g \in G$ where $r_g(u) = r(u, g)$ for all $u \in P$. In the aspired case of a principal fibre bundle $p : P \to M$ with structure group $G$ these invariant subsets are nothing but the preimages $p^{-1}(U)$ of the projection with $U \subseteq M$. This shows that the direct image

$$p_* C^\infty_P$$

is a sheaf of associative and commutative algebras over the base manifold $M$ with a compatible action of $G$. The $G$-invariant functions are nothing but the pullbacks of functions on the base space $M$. This leads to the following important lemma.

**Lemma 4.4.10 (Principal fibre bundles and sheaves with $G$-actions)**

Let $p : P \to M$ be a principal fibre bundle with structure group $G$. Further let $\mathcal{E}_1, \ldots, \mathcal{E}_k, \mathcal{F}$ be sheaves of $C^\infty_P$-modules over $P$ such that the sheaves $p_* \mathcal{E}_1, \ldots, p_* \mathcal{E}_k, p_* \mathcal{F}$ of $p_* C^\infty_P$-modules over $M$ are equipped with a compatible $G$-action.

Then, for all $L \in \mathbb{N}_0^k$ the sheaf

$$p_* D^L = p_* \text{DiffOp}^L(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{F})$$

has a compatible $G$-action given by (4.59) and the $G$-invariant multidifferential operators build a subsheaf $(p_* \text{DiffOp}^L(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{F}))^G$.

**Proof:** One has to show the compatibility of the restriction maps $r^U$ of $p_* D^L$ this means of the restrictions $r_{p^{-1}(U)}^V$ of $D^L$ with the induced $G$-action, where $V \subseteq U \subseteq M$ are open subsets. To this end, choose an open cover of $V$ with sets $O \subseteq V$ as in Lemma 4.2.1 and corresponding functions $\chi$ as in the third part of Lemma 4.2.2. Then it is clear that $p^{-1}(V) \subseteq p^{-1}(U)$ is covered by the sets $p^{-1}(O) \subseteq p^{-1}(V)$ and that the functions $p^* \chi \in C^\infty(p^{-1}(U)) = p_* C^\infty_P(U)$ are $G$-invariant and satisfy $\text{supp} p^* \chi \subseteq p^{-1}(\text{supp} \chi) \subseteq p^{-1}(V)$ and $(p^* \chi)|_{p^{-1}(O)} = 1$. For these functions, the induced maps $p^* \chi : \mathcal{E}_i(p^{-1}(V)) \to \mathcal{E}_i(p^{-1}(U)), i = 1, \ldots, k$, are $G$-invariant since by assumption $(g \triangleright ((p^* \chi)e_i))|_{p^{-1}(V)} = g \triangleright ((p^* \chi)|_{p^{-1}(V)} \cdot e_i) = (g \triangleright p^* \chi)|_{p^{-1}(V)} \cdot (g \triangleright e) = ((p^* \chi)(g \triangleright e))|_{p^{-1}(V)}$, and since the restriction to $p^{-1}(U) \setminus \text{supp} p^* \chi$ vanishes. With this property the defining Equation (4.30) for the restriction maps of $D^L$ leads to

$$(g \triangleright D|_{p^{-1}(V)}(e_1, \ldots, e_k)|_{p^{-1}(O)}) = (g \triangleright (D|_{p^{-1}(V)}(g^{-1} \triangleright e_1, \ldots, g^{-1} \triangleright e_k)))|_{p^{-1}(O)} = (g \triangleright (D((p^* \chi)(g^{-1} \triangleright e_1), \ldots, (p^* \chi)(g^{-1} \triangleright e_k))))|_{p^{-1}(O)}$$

for all $D \in \text{DiffOp}^L(\mathcal{E}_1, \ldots, \mathcal{E}_k; \mathcal{F})$ from which $g \triangleright D|_{p^{-1}(V)} = (g \triangleright D)|_{p^{-1}(V)}$ follows. This compatibility also shows that the invariant differential operators build a subsheaf. The last sheaf axiom is satisfied since the global element $D$ with $D|_{p^{-1}(V)} = D_p^{-1}(V)$ for the locally given and $G$-invariant $D_p^{-1}(V)$ is again $G$-invariant due to $(g \triangleright D)|_{p^{-1}(V)} = g \triangleright D_p^{-1}(V) = D|_{p^{-1}(V)}$.

With the above results it is obvious how the assumptions of the previous subsection have to be specified in the $G$-invariant case.
Further, let $\phi, \psi$ be given structures as in Remark 4.4.11 such that

Let there be given structures as in Remark 4.4.11 such that

Proposition 4.4.12 (Global and local differential Hochschild complexes)

We now assume to be given the following structures.

\begin{enumerate}[i.]
\item A principal fibre bundle $p : P \to M$ with structure group $G$ inducing

\begin{enumerate}[a.)]
\item the sheaf $\mathcal{E} = C^\infty_M$ of smooth functions over $M$ with trivial $G$-action.
\item the compatible action of $G$ on the sheaf $p_* \mathcal{B} = p_* C^\infty_P$ of associative, commutative algebras over $M$ given by the pullback of functions with respect to the principal action.
\item the sheaf morphism $\gamma : \mathcal{E} \to (p_* \mathcal{B})^G$ onto the sheaf of $G$-invariant functions given by the pullback of functions, this means $\gamma_U = (p|_{p^{-1}(U)})^*$ for all open subsets $U \subseteq M$.
\end{enumerate}

\item A sheaf $(A, \mu)$ of $\mathbb{K}$-algebras over $M$ which is a sheaf of $\mathbb{K}$-vector spaces and $\mathcal{E}$-modules with a compatible $G$-action.

Since the $G$-action shall be taken into account we only work with (pre)sheaves over $M$ this time, if necessary by considering the direct images with respect to the projection $p$. With $A = p_*(p^* A)$ we can always use Lemma 4.4.10. Thus one gets that $HC^\bullet_{\text{diff}}(A, A)$ and the direct image $p_* HC^\bullet_{\text{diff}}(p^* A, \mathcal{D}) = HC^\bullet_{\text{diff}}(A, p_* \mathcal{D})$ are presheaves of Hochschild complexes and $C^\infty_M$-modules over $M$ with compatible $G$-actions. In particular, one has the sheaves of $G$-invariant multidifferential operators

$$\text{DiffOp}^L_{\mathcal{E}}(A, \ldots, A; A)^G (4.62)$$

and

$$\left(p_* HC^{k,L}\right)^G = \text{DiffOp}^L_{\mathcal{E}}(A, \ldots, A; p_* \text{DiffOp}^L_{\mathcal{B}}(\mathcal{E}, \mathcal{E}))^G. (4.63)$$

If the action on $A$ is trivial one gets presheaves $HC^\bullet_{\text{diff}}(A, (p_* \mathcal{D})^G)$ consisting of sheaves

$$\left(p_* HC^{k,L}\right)^G = \text{DiffOp}^L_{\mathcal{E}}(A, \ldots, A; (p_* \text{DiffOp}^L_{\mathcal{B}}(\mathcal{E}, \mathcal{E})))^G. (4.64)$$

Note that the $G$-invariance of the families $\mu$ and $\rho$ is given by assumption. If $\mu_M$ and $\rho_P$ are differential one thus has analogue results to those of Lemma 4.4.14 and Corollary 4.4.15 for the $G$-invariant cochains and cocycles. The $G$-invariant coboundaries of the above complexes are given by the presheaves

$$\left(p_* B^k\right)^G = \delta \left(HC^{k-1}_{\text{diff}}(A, p_* \mathcal{D})^G \right). (4.65)$$

The adapted version of Lemma 4.4.16 then states that all $\delta((p_* HC^{k,L}_{\text{diff}}(A, p_* \mathcal{D})$ are subsheaves. In the corresponding proof one considers open coverings $\bigcup_{i \in I} U_i = M$ and the structures assigned to $p^{-1}(U_i)$. Choosing a subordinate partition of unity $\{\chi_i\}_{i \in I}$ with functions $\chi_i \in C^\infty(M)$ the global object $\Theta = \sum_{i \in I}(p^{-1}\chi_i)\Theta_i$ then is $G$-invariant if the local $\Theta_i$ are $G$-invariant since $p^{-1}\chi_i \in C^\infty(P)^G$. These considerations finally yield the following refined version of Proposition 4.4.7.

\begin{enumerate}[i.)]
\item $\phi$ and $\psi$ are in the same $G$-invariant cohomology class, $\phi - \psi \in (B^{k,L}(P))^G$.
\end{enumerate}
ii.) All local cocycles $\phi|_{p^{-1}(U_i)}$ and $\psi|_{p^{-1}(U_i)}$ are equivalent, $(\phi - \psi)|_{p^{-1}(U_i)} \in (B^{k,L,I}(p^{-1}(U_i)))^G$, and there exist $L \in \mathbb{N}_0^{k-1}$, $\tilde{l} \in \mathbb{N}_0$ such that $(\phi - \psi)|_{p^{-1}(U_i)} = \delta_{p^{-1}(U_i)}\Theta_i$ with cochains $\Theta_i \in (HC^{k-1,L,I}(p^{-1}(U_i)))^G$ for all $i \in I$.

In particular, if the second statement is true every subordinate partition of unity $\{\chi_i\}_{i \in I}$ with $\chi_i \in C^\infty(M)$ induces an element

$$\Theta = \sum_{i \in I} (p^{-1}\chi_i)\Theta_i \in (HC^{k-1,L,I}(P))^G \quad \text{with} \quad \phi - \psi = \delta_P\Theta.$$  \quad (4.66)
Chapter 5

Cohomology and projective resolutions

After the rather simple preparations of the last chapter the aim of the present one is to develop
the crucial techniques of cohomology computation which are motivated by general results of ho-
mological algebra, in particular the theory of projective resolutions of modules $A$ over rings. The
fundamental definitions and basic results from homological algebra can be found in [75, Chap. 6]
and are summarized in Appendix B. The basic idea which is the motivation for all subsequent
considerations of this chapter is the following observation concerning derived functors.

Consider a ring $R$, an $R$-module $A$ and an additive functor $F$ from the category $R$-mod of
$R$-modules into the category $\text{Ab}$ of abelian groups. Without loss of generality we consider a
contravariant functor. Then, the application of the $k$-th (right) derived functor $R^kF$ to $A$ by
definition is a cohomology group $H^k(FC)$ with an arbitrary projective resolution $(C, \epsilon)$ of $A$, confer
Definition B.2.3. The functor is well-defined since this definition does not depend on the choice
of the resolution. This has an immediate consequence for the task of computing the cohomology
$H^*$ of a given complex. If it is possible to find a ring $R$, a functor $F$ and a resolution $(C, \epsilon)$ of a
module $A$ as above such that the cohomology groups of interest are isomorphic to the images of $A$
under the derived functors, $H^k \cong R^kF(A) = H^k(F(C))$, the freedom in the choice of the resolution
can be used to find a different complex $F(C')$, clearly with the same cohomology, but where the
problem is easier to handle.

This simple observation yields the guideline for the whole chapter. The first section describes
the well-known fact that the algebraic Hochschild complexes as in Definition 2.1.8 are of the above
form which is seen with the algebraic bar resolution. This algebraic situation and the general
ideas explained above will then serve as a guideline for the more specific situation where $A = C^\infty(V)$ is an algebra of functions and where the Hochschild complexes are of a particular type,
namely the continuous and differential versions. For them we have to introduce the topological
bar resolution in order to see that the new cohomologies are given in a similar way as the purely
algebraic ones. After these preliminary steps we present the corresponding topological version
of the Koszul resolution and prove that this new resolution really provides a different way to
compute the considered differential Hochschild cohomologies. To this end, the necessary chain
homomorphisms and homotopy maps will be given explicitly in order to ensure that the additional
property of the cochains to be differential is respected. The obtained results are already stated
in the publication [19] and have their origin in [17]. In the present work all this is presented in a
self-contained way containing the explicit and refined proofs.
5.1 The Hochschild cohomology as an Ext group

The purely algebraic Hochschild cohomology defined in Definition 2.1.8 is a well-known example of a cohomology, which is coming from a derived functor as explained above. This will now be explained in some detail in order to point out the subtleties of the later modifications. Basically, we follow the considerations in [75, Sect. 6.11].

As in Definition 2.1.8, let \( K \) be a field of characteristic zero and let \((A, \mu)\) be a unital \( K \)-algebra with unit 1. Considering the opposite algebra \((A^{\text{opp}}, \mu^{\text{opp}})\) where \( A = A^{\text{opp}} \) as a vector space and where the multiplication is given by \( \mu^{\text{opp}}(a, b) = \mu(b, a) \) for all \( a, b \in A \) one can consider the extended algebra

\[
\mathcal{A}^e = A \otimes A^{\text{opp}}.
\]

Here and in the following \( \otimes = \otimes_K \) is the tensor product of \( K \)-vector spaces. Then, every \( K \)-vector space \( M \) which is an \((A, A)\)-bimodule can be seen as an \( \mathcal{A}^e \)-module via \((a \otimes b)m = amb \) and vice versa via \( am = (a \otimes 1)m \) and \( mb = (1 \otimes b)m \). With the usual left and right multiplication the same is true for the algebra \( A \) itself. With the ring structure of \( \mathcal{A}^e \) one now comes to the following well-known result, confer [75, Thm. 6.17].

**Proposition 5.1.1**

Let \( A \) be an algebra and \( M \) be an \((A, A)\)-bimodule as in Definition 2.1.8. Then, the Hochschild cohomology is given by a derived functor. In detail, for all \( k \in \mathbb{N}_0 \) one has

\[
\HH^k(A, M) = \Ext^k_A(A, M) = R^k\Hom(A, C_\mathcal{A}(C, M)),
\]

where in the last expression \((C, \epsilon)\) is an arbitrary projective resolution of \( A \) as \( \mathcal{A}^e \)-module.

**Proof:** The proof makes use of a particular projective resolution, namely the so-called bar resolution, showing that \( H^k(\Hom_{\mathcal{A}^e}(C, M)) \) is isomorphic to the Hochschild cohomology defined in Definition 2.1.8. For \( k \geq 0 \) consider the vector spaces

\[
X_k = A \otimes A \otimes \cdots \otimes A \otimes A,
\]

which are \((A, A)\)-bimodules via

\[
a(x_0 \otimes \cdots \otimes x_{k+1})b = ax_0 \otimes x_1 \otimes \cdots x_k \otimes x_{k+1}b.
\]

Moreover, there are isomorphisms

\[
\begin{align*}
X_0 &= A \otimes A \cong \mathcal{A}^e, \\
X_1 &= A \otimes A \otimes A \cong \mathcal{A}^e \otimes A, \\
X_k &= \mathcal{A}^e \otimes X_{k-2},
\end{align*}
\]

between \( \mathcal{A}^e \)-modules. Using (5.5) it is clear that the \( X_k \) are \( \mathcal{A}^e \)-free since they are tensor products of \( \mathcal{A}^e \) and \( K \)-vector spaces.

The \( K \)-linear maps \( d_k : X_k \rightarrow X_{k-1}, \ k \geq 1 \), which are defined by

\[
d_k(x_0 \otimes \cdots \otimes x_{k+1}) = \sum_{i=0}^{k} (-1)^i x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{k+1},
\]

are also \( \mathcal{A}^e \)-linear and satisfy \( d_{k-1} \circ d_k = 0 \). Together with the \( \mathcal{A}^e \)-linear augmentation \( \epsilon : X_0 \rightarrow A \), given by

\[
\epsilon(a \otimes b) = ab,
\]
and satisfying $\epsilon \circ d_1 = 0$, one gets an $\mathcal{A}^e$-free complex $(X, \epsilon)$ over $\mathcal{A}$. The $\mathcal{K}$-homomorphisms $h_1 : \mathcal{A} \to X_0$ and $h_k : X_k \to X_{k+1}$, given by

$$h_k(x_0 \otimes \cdots \otimes x_{k+1}) = 1 \otimes x_0 \otimes \cdots \otimes x_{k+1},$$

(5.8) satisfy

$$\epsilon \circ h_{-1} = \text{id}_{X_0}, \quad d_1 \circ h_0 + h_{-1} \circ \epsilon = \text{id}_{X_0}, \quad \text{and} \quad d_{k+1} \circ h_k + h_{k-1} \circ d_k = \text{id}_{X_k}, \quad \text{for} \quad k \geq 1.$$ (5.9)

Thus, $(C, \epsilon)$ is indeed a projective, even free resolution of $\mathcal{A}$ as an $\mathcal{A}^e$-module. Now, for all $k \in \mathbb{N}_0$ there is an isomorphism

$$\Xi^k : \text{Hom}_{\mathcal{A}^e}(X_k, M) \to \text{HC}^k(\mathcal{A}, M),$$

(5.10)

onto the modules of the Hochschild complex $(\text{HC}^\bullet(\mathcal{A}, M), \delta)$ as in Definition 2.1.8. These are chain homomorphisms

$$\Xi^{k+1} \circ d_{k+1}^e = \delta^k \circ \Xi^k$$

(5.11)

and so the cohomology groups defined by the derived functor are the Hochschild cohomologies from Definition 2.1.8.

Remark 5.1.2

In principle, the Hochschild cohomology can be defined by (5.2) in an even more general algebraic framework, confer [75, Sect. 6.11].

5.2 Continuous Hochschild cohomologies for $C^\infty(V)$

In this section we give a concrete example for Hochschild complexes with continuous cochains and reformulate Proposition 5.1.1 in the topological setting.

From now on let $\mathcal{K}$ be $\mathbb{R}$ or $\mathbb{C}$. Then we consider the $\mathcal{K}$-algebra $\mathcal{A} = C^\infty(V)$ of smooth functions on an arbitrary open subset $V \subseteq \mathbb{R}^n$. Equipped with the usual Fréchet topology of smooth functions, $\mathcal{A}$ is a topological $\mathcal{K}$-algebra. Further, let $\mathcal{M}$ be a locally convex $\mathcal{K}$-vector space and a topological $(\mathcal{A}, \mathcal{A})$-bimodule which means that the trilinear map $(a, m, b) \mapsto a \cdot m \cdot b$ is continuous with respect to the induced product topology. Then, one can consider the Hochschild complex $(\text{HC}^\bullet(\mathcal{A}, \mathcal{M}), \delta)$ of $\mathcal{A}$ with values in $\mathcal{M}$ as in Definition 2.1.8 and therein the cochains which are continuous maps with respect to the induced topology on the $k$-fold products $\mathcal{A} \times \cdots \times \mathcal{A}$ for $k \in \mathbb{N}$. By the well-known properties of locally convex spaces, confer [76, 118], and the defining Equation (2.15), it is clear that this yields a subcomplex

$$(\text{HC}^\bullet_{\text{cont}}(\mathcal{A}, \mathcal{M}), \delta),$$

(5.12)

the so-called continuous Hochschild complex of $\mathcal{A}$ with values in $\mathcal{M}$. The only thing to show is that the Hochschild differential maps continuous cochains into continuous ones. To do this, one basically uses the definition and the fact that a linear map $f : E \to F$ between two locally convex spaces is continuous, if to every continuous seminorm $p_F$ on $F$ there exists a continuous seminorm $p_E$ on $E$ such that for all $e \in E$ one has $p_F(f(e)) \leq p_E(e)$, [118, Prop. 7.7].

For the algebra $\mathcal{A}$ we now investigate the topological versions of the bar complex, confer [33, Sect. III.2.,α] and [17, 108]. In principle, one just considers the completions of the spaces and continuous continuations of the continuous maps occurring in the proof of Proposition 5.1.1 with
respect to the given locally convex topologies. The extended algebra \( \mathcal{A} \otimes \mathcal{A}^{\text{opp}} \) in (5.1) is replaced by its topological counterpart
\[
\mathcal{A}^e = C^\infty(V \times V),
\]
where the tensor product in (5.1) has been completed with respect to the canonical Fréchet topology of smooth functions. Since these are nuclear there are no ambiguities in the definition of the tensor product and its completion, confer \([76, 118]\). In the very same way all the other structures of the topological bar complex are now derived from the purely algebraic ones. So one redefines the spaces
\[
X_0 = \mathcal{A}^e = C^\infty(V \times V) \quad \text{and} \quad X_k = C^\infty(V \times V^k \times V)
\]
for \( k \in \mathbb{N} \) with the \( \mathcal{A}^e \)-module structure
\[
(\hat{a}\chi)(v, q_1, \ldots, q_k, w) = \hat{a}(v, w)\chi(v, q_1, \ldots, q_k, w)
\]
for \( \hat{a} \in \mathcal{A}^e \), \( \chi \in X_k \) and \( v, w, q_1, \ldots, q_k \in V \). This is the adapted version of (5.1). In the same way the boundary operators \( d_k^X : X_k \rightarrow X_{k-1} \) and the augmentation \( \epsilon : X_0 \rightarrow \mathcal{A} \) are defined by
\[
(d_k^X \chi)(v, q_1, \ldots, q_{k-1}, w) = \chi(v, q_1, \ldots, q_{k-1}, w)
\]
\[\begin{split}
&+ \sum_{i=1}^{k-1} (-1)^i \chi(v, q_1, \ldots, q_i, q_{i+1}, \ldots, q_{k-1}, w) + (-1)^k \chi(v, q_1, \ldots, q_{k-1}, w, w)
\end{split}
\]
and
\[
(\epsilon \hat{a})(v) = \hat{a}(v, v).
\]
It is obvious that \( d_k^X \) and \( \epsilon \) are homomorphisms of \( \mathcal{A}^e \)-modules and an easy computation in analogy to the algebraic case yields \( d_{k-1}^X \circ d_k^X = 0 \) for all \( k \geq 2 \) and \( \epsilon \circ d_1^X = 0 \). The homotopies \( h_{-1}^X : \mathcal{A} \rightarrow X_0 \) and \( h_k^X : X_k \rightarrow X_{k+1} \) now are given by
\[
(h_{-1}^X a)(v, w) = a(w) \quad \text{and} \quad (h_k^X \chi)(v, q_1, \ldots, q_{k+1}, w) = \chi(q_1, \ldots, q_{k+1}, w) \quad \text{for } k \geq 0,
\]
and again satisfy
\[
\begin{align*}
\epsilon \circ h_{-1}^X &= \text{id}_\mathcal{A}, \\
(h_{-1}^X + \epsilon) + d_1^X \circ h_0^X &= \text{id}_{X_0} \quad \text{and} \\
h_{k-1}^X \circ d_k^X + d_{k+1}^X \circ h_k^X &= \text{id}_{X_k} \quad \text{for all } k \geq 1.
\end{align*}
\]
Hence the sequence
\[
0 \leftarrow \mathcal{A} \xleftarrow{\epsilon} X_0 \xrightarrow{d_1^X} X_1 \xrightarrow{d_2^X} \cdots \xrightarrow{d_k^X} X_k \xrightarrow{d_{k+1}^X} \cdots,
\]
is exact and the bar complex \((X, d^X)\) defines a resolution \(((X, d^X), \epsilon)\) of \( \mathcal{A} \). Note that the modules \( X_k \) are topologically free as \( \mathcal{A}^e \)-modules, confer \([33]\) for a more general version of this.

With these structures we now reformulate Proposition \( \ref{prop:cohomology} \) for the considered specific situation.

**Proposition 5.2.1**

For \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \) and \( V \subseteq \mathbb{R}^n \) let \( \mathcal{A} = C^\infty(V) \) be the Fréchet algebra of smooth functions on \( V \). Further, let \( \mathcal{M} \) be a locally convex \( \mathbb{K} \)-vector space with a topological \((\mathcal{A}, \mathcal{A})\)-bimodule structure. Then, \( \mathcal{M} \) has a unique \( \mathcal{A}^e = C^\infty(V \times V) \)-module structure with
\[
(a \otimes b) \cdot m = a \cdot m \cdot b \quad \text{for all } a, b \in \mathcal{A}, m \in \mathcal{M}.
\]
For all \( k \in \mathbb{N} \) there is an isomorphism
\[
\Xi^k : \text{Hom}_{\mathcal{A}}^{\text{cont}}(X_k, M) \rightarrow \text{HC}_{\text{cont}}^k(\mathcal{A}, M) \tag{5.22}
\]
between the continuous and \( \mathcal{A}^e \)-linear maps from the topological bar modules \( X_k \) to \( M \) and the continuous Hochschild modules, given by
\[
\left( \Xi^k \psi \right)(a_1, \ldots, a_k) = \psi(1 \otimes a_1 \otimes \cdots \otimes a_k \otimes 1). \tag{5.23}
\]
Further, \( \Xi \) is a chain map, this means
\[
\Xi^{k+1} \circ (d^X_{k+1})^* = \delta^k \circ \Xi^k \quad \text{for all} \ k \in \mathbb{N}_0, \tag{5.24}
\]
so \( \Xi \) is even an isomorphism of complexes and thus the cohomologies are the same,
\[
\text{HH}^\bullet_{\text{cont}}(\mathcal{A}, M) = H^\bullet(\text{Hom}_{\mathcal{A}}^{\text{cont}}(X, M)). \tag{5.25}
\]

**Proof:** By the continuity conditions, the \( \mathcal{A}^e \)-module structure of \( M \) is the unique extension of \( (5.21) \). Further, it is clear that the \( \Xi^k \) defined by \( (5.23) \) take values in the continuous Hochschild cochains. This is seen by using the seminorms which induce the topologies, and the given continuity properties. The injectivity of \( \Xi^k \) follows from the fact that a continuous and \( \mathcal{A}^e \)-linear map in \( \text{Hom}_{\mathcal{A}}^{\text{cont}}(X_k, M) \) is already determined by the values on elements of the form \( 1 \otimes a_1 \otimes \cdots \otimes a_k \otimes 1 \) since the products of those elements with factorising elements \( a \otimes b \in \mathcal{A}^e \) build a dense subset in \( X_k \). The surjectivity is given by of the same reason because every map defined by the right hand side of \( (5.23) \) has a unique \( \mathcal{A}^e \)-linear and continuous continuation on \( X_k \). The boundary operators \( d^X_k \) are continuous maps. This is easily seen for \( (5.6) \) and thus it is true for its continuation \( d^X_k \).

Equation \( (5.24) \) is a direct computation using \( (5.6) \) and the \( \mathcal{A}^e \)-module structure for elements \( a \otimes b \) induced by \( (5.4) \). Then, the rest of the proposition is clear. \( \blacksquare \)

### 5.3 Differential Hochschild cohomologies for \( C^\infty(V) \)

The topological setting described above turns out to be the appropriate starting point in order to specify the aspired characterization of Hochschild complexes and cohomologies in the framework of resolutions for the differential case. As already seen in Section 3.2 it is rather difficult to define a general setting of Hochschild complexes with algebraic multidifferential operators as cochains. However, for smooth functions \( \mathcal{A} = C^\infty(V) \) as above there exists a slightly different but very natural way to define differential operators. As we will see now this can be used to define corresponding differential subcomplexes of the ones in \( (5.22) \) which are isomorphic again. This and the subsequent considerations ending up in Theorem 5.7.5 then will provide a strong tool to compute differential Hochschild cohomologies since the different notions of differential operators coincide in the crucial examples.

In the following it will be necessary to demand that the module \( M \) in Proposition 5.2.1 has more specific properties. So from now on, let \( M \) be a locally convex, complete, topological Hausdorff space which is a topological left \( \mathcal{A} \)-module with respect to the Fréchet topology of \( \mathcal{A} = C^\infty(V) \). This means that the bilinear multiplication
\[
(\mathcal{A} \times M) \ni (a, m) \mapsto a \cdot m \in M \tag{5.26}
\]
is continuous. Furthermore, we demand that \( M \) has a right \( \mathcal{A} \)-module structure with the property that there exists an \( l \in \mathbb{N} \) such that the right module multiplication can be expressed in terms of the left module multiplication by
\[
m \cdot b = \sum_{|\beta| \leq l} \frac{\partial^{|\beta|} b}{\partial v^\beta} \cdot m^{\beta} \quad \text{for all} \ b \in \mathcal{A}, m \in M \tag{5.27}
\]
with elements $m^\beta \in M$ depending continuously on $m$. Such a module $\mathcal{M}$ which shall be referred to
as $(A,A)$-bimodule of order $l$ clearly satisfies the conditions of Proposition 5.2.1.

**Lemma 5.3.1**

Let $\mathcal{M}$ be an $(A,A)$-bimodule of order $l$. Then $\mathcal{M}$ is a topological $(A,A)$-bimodule and the induced
left $A^e$-module structure is explicitly given by

$$\hat{a} \cdot m = \sum_{|\beta| \leq l} \left( \Delta^*_0 \frac{\partial |\beta| \hat{a}}{\partial w^\beta} \right) \cdot m^\beta \quad \text{for all } \hat{a} \in A^e,$$

(5.28)

where $\Delta^*_k$ denotes the pullback with the total diagonal map $\Delta_k : V \to V^{k+2}$ for $k \in \mathbb{N}$ which is
defined by $\Delta_k(v) = (v, \ldots, v)$ and where the differentiation is the one with respect to the second
argument of $\hat{a}$.

**Proof:** Since $\mathcal{A}$ is commutative the right and left module structures combine to a bimodule
structure. Using the common seminorms $p_{K,r}$ for $C^\infty(V)$ defined by

$$p_{K,r}(a) = \max_{w \in K, |\beta| \leq r} \partial |\beta| a \partial v^\beta$$

for all compact subsets $K \subseteq V$ and $r \in \mathbb{N}_0$, the fact that $l$ in (5.27) is uniform for $\mathcal{M}$ implies
the continuity of the trilinear map $(a,m,b) \mapsto a \cdot m \cdot b$. Thus $\mathcal{M}$ is a topological bimodule as
in Proposition 5.2.1. That the induced right $A^e$-module has the form (5.28) follows by continuity
from

$$(a \otimes b) \cdot m = a \cdot m \cdot b = a \cdot \sum_{|\beta| \leq l} \partial |\beta| b \partial v^\beta \cdot m^\beta = \sum_{|\beta| \leq l} \left( \Delta^*_0 \frac{\partial |\beta| (a \otimes b)}{\partial w^\beta} \right) \cdot m^\beta.$$

For the rest of this and the following section we use the definition of differential operators which
is natural for the algebra of smooth functions and, at first sight, slightly different to the purely
algebraic definition of Section 3.1.

**Definition 5.3.2 (Multidifferential maps of $A = C^\infty(V)$)**

Let $k \in \mathbb{N}$, $\mathcal{A} = C^\infty(V)$ be as above and let $\mathcal{M}$ be a left $\mathcal{A}$-module. An $\mathbb{R}$-multilinear map
$\phi : \mathcal{A} \times \ldots \times \mathcal{A} \to \mathcal{M}$ with $k$ arguments is said to be differential of multior
der $L = (l_1, \ldots, l_k) \in \mathbb{N}_0^k$, if it has the form

$$\phi(a_1, \ldots, a_k) = \sum_{|\alpha_1| \leq l_1, \ldots, |\alpha_k| \leq l_k} \left( \frac{\partial |\alpha_1| a_1}{\partial v^{\alpha_1}} \cdots \frac{\partial |\alpha_k| a_k}{\partial v^{\alpha_k}} \right) \cdot \phi^{\alpha_1 \ldots \alpha_k}$$

(5.30)

with multiindices $\alpha_1, \ldots, \alpha_k \in \mathbb{N}_0^k$ and $\phi^{\alpha_1 \ldots \alpha_k} \in \mathcal{M}$. These $L$-differential maps are denoted by
$\text{DiffOp}^L(\mathcal{A}, \ldots, \mathcal{A}; \mathcal{M})$.

**Remark 5.3.3**

i.) If $\mathcal{M}$ is a topological $\mathcal{A}$-module any differential map in the sense of Definition 5.3.2 is contin-
uous.

ii.) As already stated in the beginning, Definition 5.3.2 is consistent with the purely algebraic
definition of multidifferential operators in Section 3.1 for the crucial examples treated in this
work as we will see later in Chapter 6.
iii.) In this sense, \((5.27)\) means that the right module structure is differential with respect to the left one, so \(\phi_m : b \mapsto m \cdot b\) is a differential operator depending on \(m\) of order \(l\).

Using Definition \(5.3.2\) for a module \(M\) as in Lemma \(5.3.1\) we can consider the Hochschild cochains in \(\mathrm{HC}_\text{cont}(A, M)\) consisting of multidifferential operators. Due to the additional structures this yields a subcomplex.

**Proposition 5.3.4**

Let \(A\) and \(M\) be as in Proposition \(5.2.7\) where \(M\) has the additional property \(5.2.7\) with the fixed \(l \in \mathbb{N}\). Then the definition

\[
\mathrm{HC}_\text{diff}^k(A, M) = \bigcup_{L \in \mathbb{N}_0^k} \mathrm{DiffOp}^L(A, \ldots, A; M) \quad \text{for all } k \in \mathbb{N}
\]

and \(\mathrm{HC}_\text{diff}^0(A, M) = M\) yields a subcomplex \((\mathrm{HC}_\text{cont}^*(A, M), \delta)\) of \((\mathrm{HC}_\text{cont}^*(A, M), \delta)\), the so-called differential Hochschild complex of \(A\) with values in \(M\). In particular, the Hochschild differential \(\delta\) restricts to maps

\[
\delta : \mathrm{DiffOp}^L(A, \ldots, A; M) \to \mathrm{DiffOp}^L(A, \ldots, A; M)
\]

for all \(k \in \mathbb{N}_0\) and \(L = (l_1, \ldots, l_k) \in \mathbb{N}_0^k\) where \(\tilde{L} = (\tilde{l}_1, \ldots, \tilde{l}_{k+1}) \in \mathbb{N}_{k+1}\) is given by

\[
\tilde{l}_1 = l_1, \\
\tilde{l}_i = \max\{l_{i-1}, l_i\} \quad \text{for } i = 2, \ldots, k, \\
\tilde{l}_{k+1} = \max\{l_k, l\}.
\]

**Proof:** The continuity properties concerning \(M\) and Definition \(5.3.2\) assure that \(\mathrm{HC}_\text{diff}^k(A, M) \subseteq \mathrm{HC}_\text{cont}^k(A, M)\) is a subset for all \(k \in \mathbb{N}_0\). Then one only has to show \(5.32\) and \(5.33\). For \(L\) as in \(5.32\) and \(\phi \in \mathrm{DiffOp}^L(A, \ldots, A; M)\) as in \(5.30\) one computes with \(5.27\)

\[
(\delta \phi)(a_1, \ldots, a_{k+1})
\]

\[
= \sum_{|\alpha| \leq 1, \ldots, |\alpha| \leq l_k} a_1 \cdot \left( \frac{\partial^{\alpha_1} a_1}{\partial v^{\alpha_1}} \cdots \frac{\partial^{\alpha_k} a_{k+1}}{\partial v^{\alpha_k}} \right) \cdot \phi^{\alpha_1 \ldots \alpha_k}
\]

\[
+ \sum_{i=1}^{k} (-1)^{i} \sum_{|\alpha_i| \leq l_i, \ldots, |\alpha_k| \leq l_k} \left( \frac{\partial^{\alpha_1} a_1}{\partial v^{\alpha_1}} \cdots \frac{\partial^{\alpha_i} (a_{i+1})}{\partial v^{\alpha_i}} \cdots \frac{\partial^{\alpha_k} a_{k+1}}{\partial v^{\alpha_k}} \right) \cdot \phi^{\alpha_1 \ldots \alpha_k} a_{k+1}
\]

\[
+ (-1)^{k+1} \sum_{|\alpha| \leq l_1, \ldots, |\alpha| \leq l_k} \left( a_1 \frac{\partial^{\alpha_1} a_1}{\partial v^{\alpha_1}} \cdots \frac{\partial^{\alpha_k} a_{k+1}}{\partial v^{\alpha_k}} \right) \cdot \phi^{\alpha_1 \ldots \alpha_k}
\]

where the well-known Leibniz rule was used, confer [4 Chap. VII, Sect. 5, Ex. 21]. Given two multiindices \(\alpha, \beta\) of length \(n\) their binomial coefficient is defined by \(\binom{\alpha}{\beta} = \prod_{s=1}^{n} \frac{\alpha_s - \beta_s + 1}{\beta_s}!\). This
shows that \( \delta \phi \) is again of the form (5.30) and a simple counting of orders of differentiation yields (5.35).

\[\text{Lemma 5.3.6}\]

An element \( \psi \in \text{Hom}_{A^e}^{\text{diff}, L}(X_k, M) \) has the form

\[
\psi(\chi) = \sum_{|\alpha_1| \leq l_1, \ldots, |\alpha_k| \leq l_k} \left( \Delta_k^{\alpha_1 \ldots \alpha_k} \frac{\partial^{|\alpha_1|+\ldots+|\alpha_k|+|\beta|} \chi}{\partial q_1^{\alpha_1} \cdots \partial q_k^{\alpha_k} \partial w^\beta} \right) \cdot \psi_{\alpha_1 \cdots \alpha_k \beta} \tag{5.36}
\]

with multiindices \( \alpha_1, \ldots, \alpha_k, \beta \in \mathbb{N}_0^n \) and \( \psi_{\alpha_1 \cdots \alpha_k \beta} \in M \).

\[\text{Proof:}\] The stated form (5.36) follows again by continuity arguments. Let \( \psi \in \text{Hom}_{A^e}^{\text{diff}, L}(X_k, M) \) be such that \( \Xi \psi \in \text{DiffOp}^L(A, \ldots, A; M) \) as in (5.30). Then one computes

\[
\begin{align*}
\psi(a \otimes a_1 \otimes \cdots \otimes a_k \otimes b) &= \psi((a \otimes b)(1 \otimes a_1 \otimes \cdots \otimes a_k \otimes 1)) \\
&= a(\Xi \psi)(a_1, \ldots, a_k \otimes b) \\
&= \sum_{|\alpha_1| \leq l_1, \ldots, |\alpha_k| \leq l_k} a \frac{\partial^{|\alpha_1|} \chi}{\partial a_1} \cdots \frac{\partial^{|\alpha_k|} \chi}{\partial a_k} \frac{\partial^{|\beta|} \chi}{\partial b} \cdot \psi_{\alpha_1 \cdots \alpha_k \beta} \\
&= \sum_{|\alpha_1| \leq l_1, \ldots, |\alpha_k| \leq l_k} \left( \Delta_k^{\alpha_1 \ldots \alpha_k} \frac{\partial^{|\alpha_1|+\ldots+|\alpha_k|+|\beta|} (a \otimes a_1 \otimes \cdots \otimes a_k \otimes b)}{\partial q_1^{\alpha_1} \cdots \partial q_k^{\alpha_k} \partial w^\beta} \right) \cdot \psi_{\alpha_1 \cdots \alpha_k \beta}
\end{align*}
\]

and (5.36) follows.

\[\text{Remark 5.3.5}\]

In comparison with the definitions made in Section 3.2 the new definition (5.31) contains a slight abuse of notation. In the later examples, however, these notions coincide.

By use of the isomorphism \( \Xi \) the differential elements in \( \text{Hom}_{A^e}(X_k, M) \) of multiorder \( L = (l_1, \ldots, l_k) \in \mathbb{N}_0^n \) for all \( k \in \mathbb{N} \) can be defined as

\[
\text{Hom}_{A^e}^{\text{diff}, L}(X_k, M) = (\Xi^k)^{-1}(\text{DiffOp}^L(A, \ldots, A; M)). \tag{5.34}
\]

Equation (5.24) shows that \( \text{Hom}_{A^e}^{\text{diff}}(X, M) \) is a subcomplex of \( \text{Hom}_{A^e}^{\text{cont}}(X, M) \). By the very construction, (5.23) restricts to an isomorphism of complexes

\[
\Xi : \left( \text{Hom}_{A^e}^{\text{diff}}(X, M), (\partial^X)^* \right) \to (\text{HC}_{\text{diff}}^\bullet(A, M), \delta). \tag{5.35}
\]

The concrete form of the elements in (5.34) is clarified in the following lemma.

\[\text{5.4 The topological Koszul resolution of } C^\infty(V)\]

Following the ideas of homological algebra explained in the introduction of this chapter, we now want to find another resolution of \( A = C^\infty(V) \) such that the cohomology derived by the functor \( \text{Hom}_{A^e}(\cdot, M) \) is the same as the continuous and, in particular, the differential Hochschild cohomology defined above. In the following it will be shown that the topological version of the Koszul resolution
(K, ε) of A achieves this aim. It will turn out that the values \( \text{Hom}_{\mathcal{A}^e}(K, \mathcal{M}) \) of the functor \( \text{Hom}_{\mathcal{A}^e}(-, \mathcal{M}) \) will not have to be restricted to homomorphisms of a particular form. This and the fact that the Koszul resolution is finite once more point out the importance of the presented approach since the framework of the aspired computation of the cohomologies becomes easier in two ways. On one hand there only remain finitely many cohomology groups to compute and on the other hand one does not have to care about the rather technical properties of continuous or differential cochains. In addition, this last statement in particular shows that the continuous and the differential cohomologies considered above coincide.

After this very motivating consideration we now recall the topological Koszul complex over \( C^\infty(V) \). In order to guarantee that all occurring maps are well-defined we need the further assumption that

\[
V \subseteq \mathbb{R}^n \quad \text{is a convex open subset.} \tag{5.37}
\]

Then one considers the finitely many (topologically free) \( \mathcal{A}^e \)-modules

\[
K_0 = \mathcal{A}^e \quad \text{and} \quad K_k = \mathcal{A}^e \otimes \mathbb{R}^k(\mathbb{R}^n)^* \cong C^\infty(V \times V, \mathcal{A}^e(\mathbb{R}^n)^*). \tag{5.38}
\]

There \( \mathcal{A}^k(\mathbb{R}^n)^* \) denotes the antisymmetric tensor product of the dual space \( (\mathbb{R}^n)^* \) of \( \mathbb{R}^n \). The boundary operators \( d_k^K : K_k \to K_{k-1} \) for \( k = 1, \ldots, n \) are defined by

\[
((d_k^K \omega)(v, w))(x_1, \ldots, x_{k-1}) = (\omega(v, w))(v - w, x_1, \ldots, x_{k-1}) = (i_v(v - w)\omega(v, w))(x_1, \ldots, x_{k-1}) \tag{5.39}
\]

for \( v, w \in V \) and \( x_1, \ldots, x_{k-1} \in \mathbb{R}^n \) where \( i_v \) denotes the usual insertion maps \( i_v(x) : \mathcal{A}^k(\mathbb{R}^n)^* \to \mathcal{A}^{k-1}(\mathbb{R}^n)^* \) of vectors \( x \in \mathbb{R}^n \) into forms. With a basis \( \{e_i\}_{i=1,\ldots,n} \) of \( \mathbb{R}^n \) and the corresponding coordinate functions \( \{x^j\}_{j=1,\ldots,n} \) defined by \( x^j(v^i e_i) = v^j \) the boundary operators are simply given by

\[
d_k^K = \sum_{j=1}^n \xi_j^i i_v(e_j) \tag{5.40}
\]

with the functions

\[
\xi^j = x^j \otimes 1 - 1 \otimes x^j \in \mathcal{A}^e, \tag{5.41}
\]

this means \( \xi^j(v, w) = (v^j - w^j) \) for all \( v, w \in V \) with \( v = v^i e_i \) and \( w = w^i e_i \).

Again, the maps \( d_k^K \) are \( \mathcal{A}^e \)-module homomorphisms with \( d_k^{k-1} \circ d_k^K = 0 \) for all \( k \geq 2 \) since \( i_v(v - w) i_v(v - w) = 0 \) and \( \epsilon \circ d_1^K = 0 \). Then consider the maps \( h_k^{k-1} = h_{k-1}^k \) and \( h_k^K : K_k \to K_{k+1} \) defined by

\[
(h_k^K \omega)(v, w) = \sum_{j=1}^n \xi^j \wedge \int_0^1 dt \ t^k \frac{\partial \omega}{\partial v^j}(tv + (1 - t)w, w) \tag{5.42}
\]

for \( k \geq 0 \) where \( \frac{\partial \omega}{\partial v^j} \) denotes the element in \( K_k \) obtained from \( \omega \) by taking the partial derivative of its function part in \( \mathcal{A}^e \) with respect to the \( j \)-th component of the first argument. With the basis \( \{e_i\}_{i=1,\ldots,n} \) of \( \mathbb{R}^n \) and the corresponding dual basis \( \{e^i\}_{i=1,\ldots,n} \) of \( (\mathbb{R}^n)^* \) the elements \( \omega \in K_k \) can be written as

\[
\omega = \frac{1}{k!} \sum_{i_1, \ldots, i_k=1}^n \omega_{i_1 \ldots i_k} e^{i_1} \wedge \ldots \wedge e^{i_k} \tag{5.43}
\]

with \( \omega_{i_1 \ldots i_k} \in \mathcal{A}^e \). Then, \( 5.42 \) reads

\[
(h_k^K \omega)(v, w) = \sum_{i_1, \ldots, i_k, j=1}^n \int_0^1 dt \ t^k \frac{\partial \omega_{i_1 \ldots i_k}}{\partial v^j}(tv + (1 - t)w, w)e^j \wedge e^{i_1} \wedge \ldots \wedge e^{i_k}. \tag{5.44}
\]
Lemma 5.4.1
With the above maps and \( \epsilon : K_0 \to A \) as in (5.17) the following homotopy identities hold.

\[
\begin{align*}
\epsilon \circ h_{-1}^K &= \text{id}_A, \\
h_{-1}^K \circ \epsilon + d_1^K \circ h_0^K &= \text{id}_{K_0} \quad \text{and} \\
h_{k-1}^K \circ d_k^K + d_{k+1}^K \circ h_k^K &= \text{id}_{K_k} \quad \text{for all } k \geq 1.
\end{align*}
\]

(5.45)

Proof: For the following computation it is convenient to use the abbreviation \( \omega = \omega_{I_k} e^{I_k} \in K_k \) instead of (5.43) with the obvious meaning. With the summation convention and \( i_s(v - w) = (v^i - w^i) i_s(e_i) \) we compute for \( k \geq 1 \)

\[
(d_{k+1}^K h_k^K \omega)(v, w) = \int_{0}^{1} dt \int_{0}^{1} dt k \frac{\partial \omega_{I_k}}{\partial v^j}(tv + (1 - t)w, w) (v^i - w^i) i_s(e_i)(e^j \wedge e^{I_k})
\]

(5.46)

and

\[
(h_{k-1}^K (d_k^K \omega))(v, w) = \int_{0}^{1} dt \int_{0}^{1} dt k - 1 \frac{\partial \omega_{I_k}}{\partial v^j}(\omega_{I_k}(v, w)(v^i - w^i)) \mid_{(tv + (1 - t)w, w)} e^j \wedge i_s(e_i) e^{I_k}
\]

\[
= \int_{0}^{1} dt \int_{0}^{1} dt k \frac{\partial \omega_{I_k}}{\partial v^j}(tv + (1 - t)w, w) (v^i - w^i) e^j \wedge i_s(e_i) e^{I_k}
\]

(5.47)

\[
+ \int_{0}^{1} dt \int_{0}^{1} dt k \omega_{I_k}(tv + (1 - t)w, w) e^i \wedge i_s(e_i) e^{I_k}.
\]

Since \( e^i \wedge i_s(e_i) e^{I_k} = ke^{I_k} \), the last summand can be written as

\[
\int_{0}^{1} dt \int_{0}^{1} dt k - 1 \omega_{I_k}(tv + (1 - t)w, w) e^i \wedge i_s(e_i) e^{I_k} = \int_{0}^{1} dt \frac{d}{dt} \left( t^k \omega_{I_k}(tv + (1 - t)w, w) \right) e^{I_k}
\]

\[
- \int_{0}^{1} dt \int_{0}^{1} dt k \frac{\partial \omega_{I_k}}{\partial v^j}(tv + (1 - t)w, w) (v^j - w^j) e^{I_k}.
\]

This and the derivation property of \( i_s(e_i) \) show that the sum of (5.46) and (5.47) is

\[
((d_{k+1}^K \circ h_{k}^K + h_{k-1}^K \circ d_k^K) \omega)(v, w) = \omega_{I_k}(v, w) e^{I_k} = \omega(v, w).
\]

This shows the last equation in (5.45). The second follows completely analogously and the first is the same as for the bar resolution. \( \square \)

Remark 5.4.2
i.) Note that in (5.42) the convexity of \( V \) is crucial to get a well-defined map. There are of course different ways to define the homotopies in (5.18) and (5.42). In [19], for example, the slightly more complicated formulas \( (h_{X,1} a)(v, w) = a(v), (h_{X,1} \chi)(v, q_1, \ldots, q_{k+1}, w) = (-1)^{k+1} \chi(v, q_1, \ldots, q_{k+1}) \) and \( (h_{X,1} \omega)(v, w) = - \sum_{j=1}^{n} e^j \wedge \int_{0}^{1} dt \int_{0}^{1} dt k \frac{\partial \omega}{\partial v^j}(v, tw + (1 - t)v) \) have been used.
ii.) The explicit formulas (5.42) and (5.44) of the maps $h^k$ show that it is necessary to use the topological version $\mathcal{A}^e = C^\infty(V \times V)$ of the extended algebra. If $\omega \in K_k$ as in (5.43) has factorising coefficient functions $\omega_{i_1,\ldots,i_k} \in C^\infty(V) \otimes C^\infty(V)$ this does not have to be the case for $h^k(\omega)$. For instance, take $\omega_{i_1,\ldots,i_k}(v,w) = v \sin(w)$. The same behaviour will occur for the explicit chain maps in Section 5.5. This is the main reason why one has to work with the topological completions.

The Equations (5.45) show that $((K,d^K),\epsilon)$ is indeed a topologically free and finite resolution of $\mathcal{A}$,

$$
0 \leftarrow \mathcal{A} \xleftarrow{\epsilon} K_0 \xleftarrow{d^K} K_1 \xleftarrow{d^K} \cdots \xleftarrow{d^K} K_k \xleftarrow{d^K} K_{k+1} \cdots \xleftarrow{d^K} K_n \leftarrow 0.
$$

In an adapted notion, confer [75, Sect. 6.12] for the original one, this shows that $\mathcal{A}$ has finite homological dimension.

### 5.5 Explicit chain maps between the bar and Koszul resolution

The general results concerning projective resolutions, in particular Theorem [13,2.4] give an indication that there exist chain maps between the topological bar and Koszul resolution and that they are always homotopic. For our purpose, however, the pure existence is not enough. Since we are interested in differential Hochschild cochains which obviously is an additional property besides being continuous we do not try to reformulate the general theory in the topological setting but consider explicit chain maps and construct corresponding homotopy maps whose images under the considered functor will turn out to respect differential cochains as well. The following explicit formulas and constructions already appear to some extent in [17]. Here we provide a self-contained presentation including all the relevant proofs.

For all $k \geq 0$ we consider the maps $F_k : K_k \rightarrow X_k$ from [33, Sect. III.2a] defined by

$$
(F_k \omega)(v,q_1,\ldots,q_k,w) = (\omega(v,w))(q_1 - v,\ldots,q_k - v),
$$

and the maps $G_k : X_k \rightarrow K_k$ from [17] defined by

$$
(G_k \chi)(v,w) = \sum_{i_1,\ldots,i_k=1}^n e^{i_1} \wedge \cdots \wedge e^{i_k} \int_0^1 \int_0^{t_1} \cdots \int_0^{t_k} \partial^k \chi (v,t_1 v + (1-t_1)w,\ldots,t_k v + (1-t_k)w,w).
$$

In particular, $F_0 = \text{id}_{\mathcal{A}^e} = G_0$. Note that $G_k$ is well-defined since we assume $V$ to be convex. Clearly, these maps are $\mathcal{A}^e$-module homomorphisms and it is a straightforward computation which shows that $F_k$ and $G_k$ are chain maps, confer Appendix C for the details. This means that for all $k \geq 0$

$$
d_{k+1}^X \circ F_{k+1} = F_k \circ d_{k+1}^K \quad \text{and} \quad d_{k+1}^K \circ G_{k+1} = G_k \circ d_{k+1}^X.
$$

Thus we have the commutative diagram of $\mathcal{A}^e$-module morphisms

$$
\begin{array}{c}
0 \leftarrow \mathcal{A} \xleftarrow{\epsilon} X_0 \xleftarrow{d^X} X_1 \xleftarrow{d^X} \cdots \xleftarrow{d^X} X_k \xleftarrow{d^K} X_{k+1} \xleftarrow{d^X} X_{k+2} \cdots \\
0 \leftarrow \mathcal{A} \xleftarrow{\epsilon} K_0 \xleftarrow{d^K} K_1 \xleftarrow{d^K} \cdots \xleftarrow{d^K} K_k \xleftarrow{d^K} K_{k+1} \xleftarrow{d^K} K_{k+2} \cdots
\end{array}
$$
In Appendix \( \Box \) it is further shown that
\[
G_k \circ F_k = \text{id}_{K_k} \quad \text{for all } k \geq 0,
\]
and thus it is clear that
\[
\Theta_k = F_k \circ G_k : X_k \rightarrow X_k
\]
for all \( k \geq 0 \) is a projection \( \Theta_k \circ \Theta_k = \Theta_k \) on the bar complex with \( \Theta_k \circ d_{k+1}^X = d_{k+1}^X \circ \Theta_{k+1} \). Clearly, \( \Theta_0 = \text{id}_{X_0} \). The explicit form of \( \Theta_k \) is
\[
(\Theta_k \chi)(v, q_1, \ldots, q_{k}, w) = \sum_{i_1, \ldots, i_k = 1}^{n} \sum_{\sigma \in S_k} (-1)^{\sigma} (q_1 - v)^{i_{\sigma(1)}} \cdots (q_k - v)^{i_{\sigma(k)}} \int \frac{d^{k} \chi}{d^{q_1} \cdots d^{q_k}}(v, t_1 v + (1 - t_1) w, \ldots, t_k v + (1 - t_k) w, w),
\]
where \( S_k \) denotes the set of all permutations \( \sigma \) of \( 1, \ldots, k \) and \( (-1)^{\sigma} \) is the signum of \( \sigma \). This is obvious due to the definitions and \( (e^{i_{\sigma}} \wedge \cdots \wedge e^{i_{k}})(q_1, \ldots, q_k) = \sum_{\sigma \in S_k} (-1)^{\sigma} q^{i_{\sigma(1)}} \cdots q^{i_{\sigma(k)}}. \)

If it is possible to show that the images of the chain maps \( F_k \) and \( G_k \) under the functor \( \text{Hom}_A(\cdot, M) \), this means their pullbacks, induce isomorphisms between the corresponding cohomologies, the topological Koszul resolution is identified to have the same cohomologies as the Hochschild complexes of interest. The fact that \( G_k \circ F_k = \text{id} \) and the contravariance of the functor already show that the induced maps \( F_k^* \) are surjective and the \( G_k^* \) are injective. Thus the maps on cohomology level are indeed isomorphisms if the chain maps \( \Theta_k = F_k \circ G_k \) are homotopic to the identity \( \text{id}_{X_k} \) by use of homotopy maps \( s_k : X_k \rightarrow X_{k+1} \). These considerations contain two crucial points which have to be taken into account.

**Remark 5.5.1 (Requirements of the homotopy)**

i.) In contrast to the homotopy maps \( h_k^X \) and \( h_k^K \) used to show the exactness of the topological bar and Koszul resolution, the aspired homotopy maps \( s_k : X_k \rightarrow X_{k+1} \) have to be \( A^e \)-linear since the homotopy equation on the bar complex has to be carried over to the complex of interest by the functor \( \text{Hom}_A(\cdot, M) \).

ii.) In the considered framework of differential Hochschild complexes one has to guarantee that all involved maps with target \( \text{Hom}_A(X_k, M) \), this means \( (s_k)^*, (G_k)^*, \) and \( (\Theta_k)^* \), have values in the subspaces \( \text{Hom}^\text{diff}_A(X_k, M) \) of differential cochains. In contrast to the first point this does not follow from abstract arguments and really makes it necessary to consider the explicit maps.

**Remark 5.5.2**

Regarding the resolutions as complexes of \( \mathbb{K} \)-modules instead of \( A^e \)-modules it is very easy to find an explicit homotopy between \( \Theta_k \) and \( \text{id}_{X_k} \). Application of \( \Theta_k \) to the last equation of (5.19) leads to
\[
\Theta_k \circ h_{k-1}^X \circ d_k^X + d_{k+1}^X \circ \Theta_{k+1} \circ h_k^X = \Theta_k.
\]
Subtraction of (5.50) from (5.19) then yields the homotopy
\[
\text{id}_{X_k} - \Theta_k = t_{k-1} \circ d_k^X + d_{k+1}^X \circ t_k
\]
with the homotopy maps \( t_k = (\text{id}_{X_{k+1}} - \Theta_{k+1}) \circ h_k^X \) which are not \( A^e \)-linear.
5.6 Predifferential maps

After Remark 5.5.1 has made clear the requirements of the desired homotopy, we now have a closer look at a sufficiently large class of $\mathbb{K}$-endomorphisms of the topological bar complex $X_*$ which respect the basic demand mentioned in the second point of Remark 5.5.1. Within this class which contains all examples considered so far, we then present an appropriate procedure to modify arbitrary maps in order to get $A^e$-linear ones. This crucial step will then lead to an explicit construction of the desired homotopy maps. Before defining the mentioned class of maps, we first introduce and investigate a certain kind of matrices which will play the role of coordinate transformations of the convex set $V \subseteq \mathbb{R}^n$ and its cartesian products. So, consider the set $\mathcal{B}_{sr}$ of matrices enjoying the following convexity property

$$\mathcal{B}_{sr} = \left\{ B \in \text{Mat}_{(s+2) \times (r+2)}(\mathbb{R}) \mid \forall i \text{ and } v_0, \ldots, v_{r+1} \in V: \sum_{j=0}^{r+1} B_{ij} = 1 \text{ and } \sum_{j=0}^{r+1} B_{ij} v_j \in V \right\}. \quad (5.58)$$

Each such matrix $B \in \mathcal{B}_{sr}$, written as $B = (B_{ij})$ with $i = 0, \ldots, s+1$ and $j = 0, \ldots, r+1$ acts as a map $B : V^{r+2} \longrightarrow V^{s+2}$ by

$$B(v_0, \ldots, v_{r+1}) = \left( \sum_{j=0}^{r+1} B_{0j} v_j, \ldots, \sum_{j=0}^{r+1} B_{(s+1)j} v_j \right), \quad (5.59)$$

which is the restriction of $B \otimes \text{id}_{\mathbb{R}^n} : \mathbb{R}^{r+2} \otimes \mathbb{R}^n \longrightarrow \mathbb{R}^{s+2} \otimes \mathbb{R}^n$ to $V^{r+2}$. The convexity property of $B$ obviously ensures that $B(v_0, \ldots, v_{r+1})$ is indeed in $V^{s+2}$.

**Remark 5.6.1 (Properties and examples of $\mathcal{B}_{sr}$)**

i.) Due to the defining properties the matrices $\mathcal{B}_{sr} \subseteq \text{Mat}_{(s+2) \times (r+2)}(\mathbb{R})$ build a convex subset. In addition, these matrices are closed under the usual matrix multiplication, this means $\mathcal{B}_{sr} \cdot \mathcal{B}_{rk} \subseteq \mathcal{B}_{sk}$.

ii.) Typical nontrivial examples of elements $B \in \mathcal{B}_{rs}$ are for instance given by all matrices $B$ with rows of the form $(0, \ldots, \lambda, \ldots, (1-\lambda), \ldots, 0)$ for $\lambda \in [0, 1]$. These examples also contain a sort of permutation matrices which lead to maps of the form $B : (v_0, \ldots, v_{r+1}) \longmapsto (v_{\sigma(0)}, \ldots, v_{\sigma(s+1)})$ with $\sigma \in S_{r+1}$. In particular, it is clear that the units $\mathbb{1} \in \text{Mat}_{(s+2) \times (s+2)}(\mathbb{R})$ are elements in $\mathcal{B}_{ss}$.

Using this kind of matrices we now define the above mentioned class of maps.

**Definition 5.6.2 (Predifferential maps on the topological bar complex)**

An endomorphism $A : X_s \longrightarrow X_r$ of the topological bar complex as above is said to be predifferential of multorder $M = (m_0, \ldots, m_{s+1}) \in \mathbb{N}_0^{s+2}$ if it has the form

$$(A\chi)(v, q_1, \ldots, q_r, w) = \sum_{|\alpha_0| \leq m_0, \ldots, |\alpha_{s+1}| \leq m_{s+1}} f^{\alpha_0, \ldots, \alpha_{s+1}}(v, q_1, \ldots, q_r, w) \int_0^1 \int_0^1 t_1 \cdots \int_0^1 t_k \frac{\partial |\alpha_0| + \cdots + |\alpha_{s+1}| \chi}{\partial q_0^{\alpha_0} \cdots \partial q_r^{\alpha_1} \cdots \partial q_w^{\alpha_{s+1}}} (B(t_1, \ldots, t_k)(v, q_1, \ldots, q_r, w)) \quad (5.60)$$

with $f^{\alpha_0, \ldots, \alpha_{s+1}} \in \mathbb{K}_r$, $\alpha : [0, 1]^k \longrightarrow \mathbb{K}$ integrable, and a continuous map $B : [0, 1]^k \longrightarrow \mathcal{B}_{sr}$. The set of all these maps is denoted by $\mathcal{S}_{rs}^{k,M}$. The $\mathbb{R}$-vector space spanned by these elements then is denoted by $\mathcal{S}_{rs}^{k,M}$.
Remark 5.6.3

i.) Note that $S_{rs}^{k,M}$ is not a vector space itself since the sum of two maps has not to be of the form $S_{rs}^{k,M}$. The vector space $S_{rs}^{k,M}$ in fact consists of all finite linear combinations of elements in $S_{rs}^{k,M}$.

ii.) Obviously, $S_{rs} = \bigcup_{k=0}^{\infty} \bigcup_{M \in N_0} S_{rs}^{k,M} \subseteq \text{Hom}_R(X_s, X_r)$ is a filtered subspace with

$$S_{rs}^{k,M} \subseteq S_{rs}^{k',M'} \text{ for } k \leq k' \text{ and } M \leq M'.$$

(5.61)

Remark 5.6.4 (Notation)

In the following considerations and proofs it will be convenient to use a shorter notation for the explicit form (5.60) of an element $A \in S_{rs}^{k,M}$. With $x = (x_0, \ldots, x_{s+1}) = (v, q_1, \ldots, q_s, w)$, $t = (t_1, \ldots, t_k)$ and $\int_0^1 dt_1 \cdots \int_0^1 dt_k \zeta(t_1, \ldots, t_k) = \int d\mu(t)$ one simply writes

$$(A\chi)(v, q, w) = \sum_{|a_0| \leq m_0, \ldots, |a_{s+1}| \leq m_{s+1}} f^{a_0 \cdots a_{s+1}}(x) \int_0^1 d\mu(t) \frac{\partial |a_0| \cdots |a_{s+1}| \chi(B(t)(x))}{\partial x^{a_0} \cdots \partial x^{a_{s+1}}}.$$ (5.62)

When investigating $A^c$-linearity, it will be useful to set $q = (q_1, \ldots, q_r)$ and to highlight the relevant arguments by $\tilde{x} = (v, q, w)$.

Denoting the set of matrix-valued maps used in Definition 5.6.2 by $B_{sr}^k : \{B : [0, 1]^k \rightarrow B_{sr} \mid B \text{ is continuous}\}$, the matrix multiplication mentioned in Remark 5.6.1 naturally induces a multiplication $B_{sr}^{k_1} \times B_{rt}^{k_2} \rightarrow B_{st}^{k_1+k_2}$ for all $k_1, k_2 \in N_0$ and $s, r, t \in N$ by

$$(B_1 B_2)(t_1, \ldots, t_{k_1+k_2}) = B_1(t_1, \ldots, t_{k_1}) B_2(t_{k_1+1}, \ldots, t_{k_1+k_2}) \text{ for all } B_1 \in B_{sr}^{k_1}, B_2 \in B_{rt}^{k_2}.$$ (5.63)

Using this we find the following important property of the maps $S_{rs}^{k,M}$.

Lemma 5.6.5 (Composition of predifferential maps)

The usual composition $\circ$ of homomorphisms restricts to a map

$$\circ : S_{tr}^{k_2,N} \times S_{rs}^{k_1,M} \rightarrow S_{ts}^{k_1+k_2,P},$$ (5.64)

with $P = (p_0, \ldots, p_{s+1})$ and $p_i = m_i + |N|$. In particular, for $A_1 \in S_{rs}^{k_1,M}$ with the matrix-valued function $B_1 \in B_{sr}$ and $A_2 \in S_{tr}^{k_2,N}$ with $B_2 \in B_{rt}$ one has $A_2 \circ A_1 \in S_{ts}^{k_1+k_2,P}$ with the matrix-valued function $B = B_1 B_2 \in B_{st}$ and where $P = (p_0, \ldots, p_{s+1})$ is given by

$$p_i = m_i + \sum_{j=0}^{r+1} T(B_1(t))_{ij} n_j \leq m_i + |N|.$$ (5.65)

There $T : \text{Mat}_{(s+1) \times (r+1)}(R) \rightarrow \text{Mat}_{(s+1) \times (r+1)}(\{0, 1\})$ is given by

$$T(B)_{ij} = \begin{cases} 1 & \text{if } B_{ij} \neq 0 \\ 0 & \text{if } B_{ij} = 0 \end{cases}.$$ (5.66)
For a coefficient functions $f$ via pullback respecting the differential cochains with $A$ Application of Proposition 5.6.6 (Predifferential maps of the bar complex)

Proof: The lemma is proved if one verifies the statement for the elements $A_1$ and $A_2$ as mentioned. With the obvious indexes one simply computes

$$(A_2 \circ A_1)(\chi)(\underline{x})$$

$$= \sum_{|\beta_0| \leq n_0,\ldots,|\beta_r| \leq n_r+1} f^{\beta_0|\cdots|\beta_r+1}_2(x) \int \mu(t_2) d\mu(t_1) \int \mu(t_1) \frac{\partial^{|\beta_0|+\cdots+|\beta_r+1|}(A_1\chi)}{\partial y^{\beta_0} \cdots \partial y^{\beta_r+1}}(B_2(t_2)(\underline{x}))$$

$$= \sum_{|\alpha_0| \leq m_0,\ldots,|\alpha_r+1| \leq m_r+1} f^{\alpha_0|\cdots|\alpha_r+1}_2(x) \int \mu(t_1) \int \mu(t_1) \frac{\partial^{|\alpha_0|+\cdots+|\alpha_r+1|}(A_1\chi)}{\partial x^{\alpha_0} \cdots \partial x^{\alpha_r+1}}(B_1(t_1)(\underline{y}))$$

$$(\beta_0|\cdots|\beta_r+1)(\underline{x}) \int \mu(t_1) \int \mu(t_1) \frac{\partial^{|\gamma_0|+\cdots+|\gamma_r+1|}(A_1\chi)}{\partial x^{\gamma_0} \cdots \partial x^{\gamma_r+1}}(B_1(t_1,\ldots,t_{k_1+k_2})(\underline{x})),$$

where a redefinition of the $t$-variables and the Leibniz rule yield the last form. There, the new coefficient functions $f^{\gamma_0|\cdots|\gamma_r+1}_2$ are given by the product of functions $f^{\beta_0|\cdots|\beta_r+1}_2$ and derivatives of $f^{\alpha_0|\cdots|\alpha_r+1}_2$. A counting of differentiations proves (5.65). Then, the remaining assertions are obvious. ■

The important and eponymous property of the predifferential maps $A \in S_{rs}^{k,M}$ is that their image $A^*$ under the functor $\text{Hom}_{\mathcal{A}^e}(\cdot, \mathcal{M})$ respects the differential cochains.

Proposition 5.6.6 (Predifferential maps of the bar complex)

Let $\mathcal{N}$ be an $(\mathcal{A}, \mathcal{A})$-bimodule of order $l$. Then, every $A \in S_{rs}^{k,M}$ as in Definition 5.6.2 which is $\mathcal{A}^r$-linear defines a map

$$A^* : \text{Hom}_{\mathcal{A}^e}^{\text{diff,L}}(X_\mathcal{N}, \mathcal{M}) \rightarrow \text{Hom}_{\mathcal{A}^e}^{\text{diff,L}}(X_\mathcal{N}, \mathcal{M})$$

via pullback respecting the differential cochains with $\tilde{L} = (\tilde{t}_1, \ldots, \tilde{t}_s)$ and

$$\tilde{t}_i = m_i + |L| + l.$$ (5.68)

For $A \in S_{rs}^{k,M}$ with matrix $B \in B_{sr}$ one has

$$\tilde{t}_i = m_i + \sum_{j=1}^r T(B(t_j))_{ij} l_j + T(B(t_j))_{i(r+1)} l.$$ (5.69)

Proof: By definition, it is sufficient to show that $(A^*\psi)(1 \otimes a_1 \otimes \cdots \otimes a_s \otimes 1) = \psi(A(1 \otimes a_1 \otimes \cdots \otimes a_s \otimes 1))$ of the form (5.30). First, one notes that

$$A(1 \otimes a_1 \otimes \cdots \otimes a_s \otimes 1)(v, q, w)$$

$$= \sum_{|\alpha| \leq m_0, \ldots, |\alpha| \leq m_s} f^{0\alpha_1|\cdots|0\alpha_s}(v, q, w) \int d\mu(t_1) \frac{\partial^{|\alpha_1|+\cdots+|\alpha_s|}(1 \otimes a_1 \otimes \cdots \otimes a_s \otimes 1)}{\partial q_1^{\alpha_1} \cdots \partial q_s^{\alpha_s}}(B(t_1)(v, q, w)).$$

Application of $\psi$ in the form (5.30) basically means to build the partial derivatives and to consider the pullback with respect to the diagonal $\Delta_r : V \rightarrow V^{r+2}$ in order to get functions in $\mathcal{A}$. The derivatives of the functions $f^{\alpha_1|\cdots|\alpha_s}$ thus simply yield values of functions in $\mathcal{A}$ which in the aspired form (5.30) contribute to the elements in $\mathcal{M}$. Because of the particular form of the elements
(1 \otimes a_1 \otimes \cdots \otimes a_s \otimes 1) and the fact that \( B(\xi) \circ \Delta_r = \Delta_s \), the procedure for the terms in the integrals yield functions of the form

\[
\frac{\partial^{\gamma_1,\ldots,\gamma_s}(1 \otimes a_1 \otimes \cdots \otimes a_s \otimes 1)}{\partial q_1^{\gamma_1} \cdots \partial q_s^{\gamma_s}} \circ \Delta_s = \frac{\partial^{\gamma_1}a_1}{\partial q_1^{\gamma_1}} \cdots \frac{\partial^{\gamma_s}a_s}{\partial q_s^{\gamma_s}},
\]

which do not depend on the integration parameter anymore. Together with the inner derivatives, given by components of \( B(\xi) \), the integrations can be carried out. The resulting constants again contribute to the elements in \( M \) and one has found the desired form \((5.30)\). A careful counting of orders of differentiation finally proves the proposition. \( \blacksquare \)

The \( \mathcal{A}^c \)-linearity of the map \( A \) is absolutely necessary in order to guarantee that the considered functor can be applied to it. Out of that reason we first investigate a general procedure to obtain \( \mathcal{A}^c \)-linear maps out of arbitrary ones. For this purpose consider the extended spaces \( \tilde{X}_k = C^\infty(V \times V \times V^k \times V \times V) \simeq X_{k+2}, \ k \in \mathbb{N}_0 \), and the maps

\[
p^*_k : X_k \rightarrow \tilde{X}_k \quad \text{with} \quad (p^*_k \chi)(v', v, q, w, w') = \chi(v', v, q, w).
\]

and

\[
i^*_k : \tilde{X}_k \rightarrow X_k \quad \text{with} \quad (i^*_k \chi)(v, q, w) = \chi(v, v, q, w, w).
\]

Obviously, the maps \( i^*_k \) and \( p^*_k \) are \( \mathcal{A}^c \)-linear and satisfy \( i^*_k \circ p^*_k = \text{id}_{\tilde{X}_k} \). For all \( v', w' \in V \) and \( \chi \in \tilde{X}_s \) there exists an element \( \chi_{v',w'} \in X_s \) defined by

\[
\chi_{v',w'}(v', v, q, w, w') = \chi(v', v, q, w),
\]

Then for all maps \( A : X_s \rightarrow X_r \) there exists a corresponding map \( \tilde{A} : \tilde{X}_s \rightarrow \tilde{X}_r \) defined by

\[
(\tilde{A} \chi)_{v',w'} = A \chi_{v',w'}.
\]

Using this, one further defines the map \( \overline{A} : X_s \rightarrow X_r \) by

\[
\overline{A} = i^*_s \circ \tilde{A} \circ p^*_s.
\]

Then one has the following result.

**Lemma 5.6.7**

*Let \( A : X_s \rightarrow X_r \) be \( \mathbb{R} \)-linear. Then \( \overline{A} \) is \( \mathcal{A}^c \)-linear. Further, one has \( \overline{A+B} = \overline{A} + \overline{B}, \overline{\text{id}_X} = \text{id}_X \) and \( \overline{\lambda A} = \lambda \overline{A} \) for \( \lambda \in \mathbb{R} \).*

**Proof:** The properties follow directly from the \( \mathcal{A}^c \)-linearity of \( \tilde{A} \) and the fact that \( A : X_s \rightarrow X_r \) is an algebra morphism. \( \blacksquare \)

In particular, one additionally notes that \( \overline{A \circ B} = \overline{A} \circ \overline{B} \). This is not true for \( \overline{A \circ B} \) but instead one has the following obvious results.

**Lemma 5.6.8**

*Let \( A : X_r \rightarrow X_t \) and \( B : X_s \rightarrow X_r \) be \( \mathbb{R} \)-linear.

i.) If \( A \circ i^*_s = i^*_t \circ \tilde{A} \), then \( \overline{A \circ B} = A \circ \overline{B} \).

ii.) If \( p^*_s \circ B = \overline{B} \circ p^*_s \), then \( \overline{A \circ B} = \overline{A} \circ B \).

After this general considerations we now come back to the predifferential maps. From the general form \((5.60)\) one directly finds a necessary and sufficient condition for the \( \mathcal{A}^c \)-linearity, namely that there are no partial derivatives with respect to the first and last argument and that those are reproduced under the matrix \( B(\xi) \).
Lemma 5.6.9 (The subspace of $A^e$-linear maps)
An element $A \in \mathcal{S}_{kr}^{k,M}$ as in (5.60) with matrix-valued function $B \in \mathcal{B}_{kr}$ is $A^e$-linear if and only if the two following conditions are satisfied:

i.) The multiindex $M$ has the form $M = (0, m_1, \ldots, m_s, 0)$.

ii.) $B$ takes values in the matrices with first row $(1 0 \ldots 0)$ and last one $(0 \ldots 0 1)$. This means that $B(t)(v, q, w) = (v, B(t)_{0,s+1}(v, q, w), w)$ where $B(t)_{0,s+1}(v, q, w)$ denotes the matrix obtained from $B(t)$ by deleting the first and the last row.

Using this observation one can immediately define a projection onto the subspace of $A^e$-linear predifferential maps which turns out to be the restriction of the general prescription $A \mapsto \mathcal{A}$.

Proposition 5.6.10 (Projection onto $A^e$-linear maps)
Let $A \in \mathcal{S}_{kr}^{k,M}$ be a predifferential map as in (5.62). Then the derived $A^e$-linear map is again an element $\mathcal{A} \in \mathcal{S}_{kr}^{k,M}$ and given by

$$(\mathcal{A} \chi)(v, q, w) = \sum_{|\alpha_1| \leq m_1, \ldots, |\alpha_s| \leq m_s} f^{\alpha_1 \ldots \alpha_s} (v, q, w) \int d\mu(t) \frac{\partial^{\alpha_1} \cdots \partial^{\alpha_s} \chi}{\partial q^{\alpha_1} \cdots \partial q^{\alpha_s}}(v, B(t)_{0,s+1}(v, q, w), w).$$

Further, the map $A \mapsto \mathcal{A}$ on all predifferential maps $\mathcal{S}_{sr}$ is a projection onto the $A^e$-linear maps, this means

$$\mathcal{A} = 0.$$ (5.76)

Further, for $A_1 \in \mathcal{S}_{sr}$ and $A_2 \in \mathcal{S}_{tr}$ one has

$$\mathcal{A}_1 \circ \mathcal{A}_2 = \mathcal{A}.$$ (5.77)

Proof: With $A$ as in (5.62) one gets

$$(\mathcal{A} \chi)(v', q', w') = \sum_{|\alpha_0| \leq m_0, \ldots, |\alpha_{s+1}| \leq m_{s+1}} f^{\alpha_0 \ldots \alpha_{s+1}} (v', q, w) \int d\mu(t) \frac{\partial^{\alpha_0} \cdots \partial^{\alpha_{s+1}} \chi}{\partial x^{\alpha_0} \cdots \partial x^{\alpha_{s+1}}} (v', B(t)(v, q, w), w').$$ (5.78)

With the definitions of the maps $i^*_s$ and $p^*_s$ this yields (5.76) since

$$\frac{\partial^{\alpha_0} \cdots \partial^{\alpha_{s+1}} (p^*_s \chi)}{\partial x^{\alpha_0} \cdots \partial x^{\alpha_{s+1}}} (v', B(t)(v, q, w), w') = \frac{\partial^{\alpha_0} \cdots \partial^{\alpha_{s+1}} \chi}{\partial q^{\alpha_0} \cdots \partial q^{\alpha_s}} (v', B(t)_{0,s+1}(v, q, w), w').$$

The projection property (5.76) is obvious with the explicit form (5.75).

Further, the explicit forms of $\mathcal{A}$ and $\mathcal{A}$ show that $\mathcal{A}$ satisfies $\mathcal{A} \circ i^*_s = i^*_s \circ \mathcal{A}$ since

$$\frac{\partial^{\alpha_1} \cdots \partial^{\alpha_s} (i^*_s \chi)}{\partial q^{\alpha_1} \cdots \partial q^{\alpha_s}} (v, B(t)_{0,s+1}(v, q, w), w) = \frac{\partial^{\alpha_1} \cdots \partial^{\alpha_s} \chi}{\partial q^{\alpha_1} \cdots \partial q^{\alpha_s}} (v, B(t)_{0,s+1}(v, q, w), w).$$

Then (5.77) follows with the first part of Lemma 5.6.8.

Lemma 5.6.11
Let $A \in \mathcal{S}_{kr}^{k,M}$ be as in (5.62) with coefficient functions satisfying $f^{\alpha_0 \ldots \alpha_{s+1}} (v, q, w) = g^{\alpha_0 \ldots \alpha_{s+1}} (v, q)$ and matrix-valued function $B \in \mathcal{B}_{kr}$ satisfying $B(t)(v, q, w) = (v, C(t)(q), w)$ with $C \in \mathcal{B}_{kr}$. Then, $\mathcal{A}$ satisfies $p^*_s \circ \mathcal{A} = \mathcal{A} \circ p^*_s$ and thus $B \circ \mathcal{A} = \mathcal{A} \circ B$.

Proof: The assertions are verified with analogous arguments to those used in the proof of Proposition 5.6.10.
Remark 5.6.12 (Examples)

Note that all previously defined maps are predifferential in the sense of Definition 5.6.2. Indeed, for all \( k \) one has the following.

i.) \( d_k^X \in S_{k-1,k}^0,0^{(0,\ldots,0)} \) and \( d_k^X = d_k^X \).

ii.) \( \Theta_k \in S_{kk}^{k,(0,\ldots,1,0)} \) and \( \Theta^k = \Theta_k \).

iii.) \( h_k^X \in S_{k+1,k}^{0,(0,\ldots,0)} \). However, \( h_k^X \) is not \( A^e \)-linear and hence \( h_k^X \neq h_k^X \).

5.7 Homotopy and isomorphic differential cohomologies

With the above preparations we can now find the desired homotopy.

Remark 5.7.1 (First attempt)

Having the projection onto the \( A^e \)-linear maps, one could conjecture that the projections \( t_k \) of the homotopy map \( t \) found in Remark 5.5.2 yield the desired \( A^e \)-linear maps. This is not the case. With the properties of the involved functions and \( 5.57 \) for \( d_{k+1}^X \) application of the projection to \( 5.57 \) yields

\[
\text{id}_{X_k} - \Theta_k = t_{k-1} \circ d_k^X + d_{k+1}^X \circ t_k,
\]

which is not the desired equation since \( t_{k-1} \circ d_k^X \neq t_{k-1} \circ d_k^X \).

The desired \( A^e \)-linear homotopy map can be defined recursively.

Proposition 5.7.2 (The \( A^e \)-linear homotopy)

Let the \( A^e \)-linear maps \( s_k : X_k \rightarrow X_{k+1} \) for all \( k \in \mathbb{N}_0 \) be defined recursively by

\[
s_k = h_k^X \circ (\text{id}_{X_k} - \Theta_k - s_{k-1} \circ d_k^X) \quad \text{and} \quad s_0 = 0. \tag{5.79}
\]

This yields maps \( s_k \in S_{k+1,k}^{k,M(k)} \) with

\[
M(k) = (0, m_1(k), \ldots, m_k(k), 0) \quad \text{and} \quad m_i(k) = \binom{k}{i-1} \quad \text{for} \quad i = 1, \ldots, k, \tag{5.80}
\]

which yield a homotopy for \( \text{id}_{X_k} \) and \( \Theta_k \), this means

\[
\text{id}_{X_k} - \Theta_k = s_{k-1} \circ d_k^X + d_{k+1}^X \circ s_k \quad \text{for all} \quad k \geq 1. \tag{5.81}
\]

Proof: The assertion (5.80) is a consequence of Lemma 5.6.5 and follows by induction. For the orders of differentiation one directly has by definition that \( m_0(k) = 0 = m_{k+1}(k) \) for all \( k \in \mathbb{N} \). Further, \( M(0) = (0,0) \) and \( M(1) = (0,1,0) = M(\Theta_1) \). Then, it is clear that for \( k \geq 2 \) the orders are controlled by the term \( s_{k-1} \circ d_k^X \) since \( M(\Theta_k) = (0,1,\ldots,1,0) \). In order to apply (5.80) one can combine the occurring matrices in \( d_k^X \) to an effective one which is given by the entries \( B_{ij} = \delta_{ij} + \delta_{i,j+1} \) for \( i = 0, \ldots, k + 1 \) and \( j = 0, \ldots, k \). Then, (5.80) and induction show that for \( i = 1, \ldots, k \) one gets \( m_i(k) = m_i(k-1) + m_{i-1}(k-1) = \binom{k-1}{i-1} \). In the last step one treats \( i = 1 \) and \( i = k \) separately. The other cases follow with \( \binom{k-2}{i-1} + \binom{k-2}{i-1} = \binom{k-1}{i-1} \).

The homotopy (5.51) is shown by induction with use of (5.19). For \( k = 1 \) one has

\[
d_2^X \circ s_1 = d_2^X \circ h_1^X \circ (\text{id}_{X_1} - \Theta_1) = (\text{id}_{X_1} - h_0^X \circ d_1^X) \circ (\text{id}_{X_1} - \Theta_1)
\]  
= id_{X_1} - \Theta_1 - h_0^X \circ d_1^X + h_0^X \circ \Theta_0 \circ d_1^X = id_{X_1} - \Theta_1.

Then, induction yields analogously
\[
d^{X}_{k+1} \circ s_{k} = (\text{id}_{X_{k}} - h^{X}_{k-1} \circ d^{X}_{k}) \circ (\text{id}_{X_{k}} - \Theta_{k} - s_{k-1} \circ d^{X}_{k}) \\
= \text{id}_{X_{k}} - \Theta_{k} - s_{k-1} \circ d^{X}_{k} - h^{X}_{k-1} \circ (d^{X}_{k} - d^{X}_{k} \circ \Theta_{k} - (\text{id}_{X_{k-1}} - \Theta_{k-1} - s_{k-2} \circ d^{X}_{k-1}) \circ d^{X}_{k}) \\
= \text{id}_{X_{k}} - \Theta_{k} - s_{k-1} \circ d^{X}_{k}.
\]

With the above structures and results we can now formulate the following important proposition.

**Proposition 5.7.3**

Let \( k \in \mathbb{N}_{0} \). The pullbacks
\[
G^{\ast}_{k} : \text{Hom}_{A^{\ast}}(K_{k}, \mathcal{M}) \to \text{Hom}^{\text{diff}, L}_{A^{\ast}}(X_{k}, \mathcal{M})
\]
only take values in the differential cochains of multiorder \( L = (l + 1, \ldots, l + 1) \in \mathbb{N}^{k} \). With the same multiindex \( L \) one has
\[
\Theta^{\ast}_{k} : \text{Hom}^{\text{diff}}_{A^{\ast}}(X_{k}, \mathcal{M}) \to \text{Hom}^{\text{diff}, L}_{A^{\ast}}(X_{k}, \mathcal{M}).
\]

Finally, for all \( L \in \mathbb{N}^{k}_{0} \) one has
\[
s^{\ast}_{k} : \text{Hom}^{\text{diff}, L}_{A^{\ast}}(X_{k+1}, \mathcal{M}) \to \text{Hom}^{\text{diff}, L}_{A^{\ast}}(X_{k}, \mathcal{M}),
\]
where \( \tilde{L} = (\tilde{l}_{1}, \ldots, \tilde{l}_{k}) \) is given by \( \tilde{l}_{i} = (k - 1)! + |L| \).

**Proof:** The statement for \( G^{\ast}_{k} \) is a straightforward counting of degrees. For \( X \in \text{Hom}_{A^{\ast}}(K_{k}, \mathcal{M}) \) one simply computes \((G^{\ast}_{k}X)(1 \otimes a_{1} \otimes \cdots \otimes a_{k} \otimes 1)\). This gives an expression of the form \( \hat{a}_{i_{1}, \ldots, i_{k}} X(e^{i_{1}} \wedge \cdots \wedge e^{i_{k}}) \) with elements \( \hat{a}_{i_{1}, \ldots, i_{k}} \in A^{\ast} \) given by
\[
\hat{a}_{i_{1}, \ldots, i_{k}}(v, w) = \int_{0}^{t_{1}} \cdots \int_{0}^{t_{k-1}} \int_{0}^{t_{k}} d t_{1} \cdots d t_{k-1} d t_{k} \frac{\partial a_{1}}{\partial v^{i_{1}}}(t_{1}v + (1 - t_{1})w) \cdots \frac{\partial a_{k}}{\partial v^{i_{k}}}(t_{k}v + (1 - t_{k})w).
\]

Using (5.23), the product can be written as \( \sum_{|\beta| \leq l} \left( \Delta_{0}^{\ast} \frac{\partial^{\beta} a_{i_{1}, \ldots, i_{k}}}{\partial a^{\alpha_{1} \cdots \alpha_{k}}} \right) \cdot X^{\beta}(e^{i_{1}} \wedge \cdots \wedge e^{i_{k}}) \) which then has the form \( \sum_{|\alpha| \leq l+1} \left( \frac{\partial a_{1}}{\partial v^{i_{1}}} \cdots \frac{\partial a_{k}}{\partial v^{i_{k}}} \right) \cdot \phi^{\alpha_{1} \cdots \alpha_{k}} \).

For \( \Theta^{\ast}_{k} \) the assertion follows either from the properties of \( G^{\ast}_{k} \) or from Proposition 5.6.6. The statement concerning \( s^{\ast}_{k} \) follows with the same argumentation where one uses the fact that all the matrices \( B \) occurring in \( s^{\ast}_{k} \) satisfy \( B_{i,k+1} = 0 \) for all \( i = 1, \ldots, k \).

This has an immediate consequence.

**Theorem 5.7.4 (The differential Hochschild cohomologies)**

With the corresponding pullbacks \( F^{\ast}_{k} \) and \( G^{\ast}_{k} \) the topological bar and Koszul complexes induce the commutative diagram
\[
\begin{array}{ccccccc}
\cdots & (d^{X}_{k})^{\ast} & \to & \text{Hom}^{\text{diff}}_{A^{\ast}}(X_{k}, \mathcal{M}) & (d^{X}_{k+1})^{\ast} & \to & \text{Hom}^{\text{diff}}_{A^{\ast}}(X_{k+1}, \mathcal{M}) & (d^{X}_{k+2})^{\ast} & \cdots \\
F^{\ast}_{k} & \downarrow G^{\ast}_{k} & & \downarrow G^{\ast}_{k+1} & & \downarrow G^{\ast}_{k+1} & & \\
\cdots & (d_{k+1}^{X})^{\ast} & \to & \text{Hom}_{A^{\ast}}(K_{k}, \mathcal{M}) & (d_{k+1}^{X})^{\ast} & \to & \text{Hom}_{A^{\ast}}(K_{k+1}, \mathcal{M}) & d_{k+2}^{X} & \cdots
\end{array}
\]
Further, one has a homotopy

$$\text{id}_{\text{Hom}^\text{diff}_{\mathcal{A}}(X_k, \mathcal{M})} - \Theta_k^* = (d_k^X)^* \circ s_{k-1}^* + (s_k)^* \circ (d_{k+1}^X)^* \quad \text{for all } k \geq 1. \quad (5.86)$$

Together with the isomorphism \([5.35]\) this yields the isomorphisms

$$\text{HH}^\bullet\text{diff}(\mathcal{A}, \mathcal{M}) \cong H(\text{Hom}^\text{diff}_{\mathcal{A}}(X_\bullet, \mathcal{M})) \cong H(\text{Hom}_{\mathcal{A}^e}(K_\bullet, \mathcal{M})) \quad (5.87)$$

for the cohomology of the differential Hochschild complex.

Note that every isomorphism in \([5.87]\) is induced by explicitly given maps on the level of cochains. With respect to the application of these results in the next chapter we need the following obvious generalization.

**Theorem 5.7.5**

Let \(\mathcal{M}^\bullet = \bigcup_{l=0}^{\infty} \mathcal{M}^l\) be a filtered \(\mathcal{A}\)-module, this means \(\mathcal{M}^l \subset \mathcal{M}^{l+1}\) and \(\mathcal{A} \cdot \mathcal{M}^l \subset \mathcal{M}^l\) for all \(l \in \mathbb{N}\), such that every \(\mathcal{M}^l\) is an \((\mathcal{A}, \mathcal{A})\)-bimodule of order \(l\). Moreover, the topologies have to respect the filtration which means that for all \(l \in \mathbb{N}\) the topology of \(\mathcal{M}^l\) is given by the induced one from \(\mathcal{M}^{l+1}\). Then we have:

i.) The unions

$$\left( \bigcup_{l=0}^{\infty} \text{HC}^\bullet_{\text{diff}}(\mathcal{A}, \mathcal{M}^l), \delta \right), \left( \bigcup_{l=0}^{\infty} \text{Hom}^\text{diff}_{\mathcal{A}^e}(X_\bullet, \mathcal{M}^l), (d^X)^* \right), \text{ and } \left( \bigcup_{l=0}^{\infty} \text{Hom}_{\mathcal{A}^e}(K_\bullet, \mathcal{M}^l), (d^K)^* \right)$$

are subcomplexes of \(\text{HC}^\bullet(\mathcal{A}, \mathcal{M}^\bullet), \text{Hom}^\text{cont}_{\mathcal{A}^e}(X_\bullet, \mathcal{M}^\bullet), \text{ and } \text{Hom}_{\mathcal{A}^e}(K_\bullet, \mathcal{M}^\bullet)\), respectively. \(\quad (5.88)\)

ii.) The isomorphisms \([5.35]\) for each \(l\) induce an isomorphism of complexes

$$\Xi : \left( \bigcup_{l=0}^{\infty} \text{Hom}^\text{diff}_{\mathcal{A}^e}(X_\bullet, \mathcal{M}^l), \delta_X \right) \longrightarrow \left( \bigcup_{l=0}^{\infty} \text{HC}^\bullet_{\text{diff}}(\mathcal{A}, \mathcal{M}^l), \delta \right). \quad (5.89)$$

iii.) The pullbacks \(G_k^*, F_k^*, \Theta_k^*, \text{ and } s_k^*\) naturally extend to the complexes \([5.88]\). Thus, we have induced isomorphisms for the corresponding cohomologies.
Chapter 6

Deformation theory of right modules on principal fibre bundles

After the general considerations concerning deformation theory of algebras and modules as well as the detailed discussions on Hochschild cohomologies we now come back to our initial issues. As we have seen in the introductory Chapter 1 there is a massive physical interest in the question how the geometry of principal fibre bundles can be reformulated when starting with a star product on the base manifold. From the algebraic point of view such a new formulation is nothing but the adaption or the replacement of all algebraic structures related to the algebra of functions on the base such that the initial algebraic properties are preserved. The natural algebraic joint to the star product algebra was recognized to be given by all corresponding right modules. Within the framework of deformation quantization built up so far this question can now be investigated.

As already seen in Section 1.3 the crucial right modules playing an important role in classical gauge theories are the horizontal differential forms on the considered principal fibre bundles. These forms are of course nothing but sections of the cotangent bundle and tensor products thereof. With respect to the aspired generality we thus investigate the right module structures of the sections of vector bundles over the principal bundle and, even more general, over surjective submersions.

In the first sections of this chapter we substantiate the deformation problem of the mentioned right modules and show that it can be treated in the (G-invariant) differential setting developed in the Chapters 2, 3, and 4. It will become evident that this framework really describes the deformations one is interested in. After the observation of the additional specific properties of the relevant Hochschild complexes we attack the central problem and compute the corresponding cohomology groups. With the results and techniques provided in the previous chapters we will see that it is always possible to reduce the computation to a simpler local problem which is solvable. In the end it turns out that all relevant cohomologies and thus all the obstructions they encode vanish. Due to the general results of Chapter 2 this has the immediate consequence that the aspired deformations, especially the ones playing a role in classical gauge theories, always exist and are uniquely defined up to equivalence.

6.1 Right modules on surjective submersions and principal fibre bundles

In order to define the notion of star products on the algebra of functions on a manifold $M$, this is necessarily equipped with a Poisson structure as explained in Section 1.1. In the following let there be given such a Poisson manifold $M$ together with a corresponding star product $*$ for the formal power series $C^\infty(M)[[\lambda]]$ of functions. In the works $[82,83]$ of Kontsevich it is not only shown that
a star product always exists but also that it can be chosen to be differential. This shall be the case in our situation. The property means that the given deformation \( \mu = \sum_{r=0}^{\infty} \lambda^r \mu_r \) of the pointwise multiplication \( \mu_0 \) consists of bidifferential operators \( \mu_r \in \text{DiffOp}^{\bullet}_{C^\infty(M)}(C^\infty(M), C^\infty(M); C^\infty(M)) \). The notion of differential operators is the one introduced in Section 3.1. Thus one is in the general situation of Definition 3.2.1 with \( \ell = \mathcal{A} = C^\infty(M) \) where \( \mu_0 \) is of the differential type of order \((0, 0)\).

With respect to this algebra structure of the differential type we now investigate the corresponding multiplication \( C^\infty(P,E) \) for all \( r \). Let \( q : E \to P \) be a vector bundle over the total space \( P \) with typical fibre \( W \) of dimension \( r \).

Then, the space of smooth sections \( \Gamma^\infty(P,E) \) of the vector bundle \( q : E \to P \) is a right \( C^\infty(M) \)-module with the pointwise module structure

\[
\rho_0(s,a) = p^* a \cdot s
\]

for all \( s \in \Gamma^\infty(P,E) \) and \( a \in C^\infty(M) \) where one makes use of the pointwise \( C^\infty(P) \)-module structure and the pullback \( p^* : C^\infty(M) \to C^\infty(P) \) which is an algebra morphism. Thus it is obvious that one is in the situation of Definition 3.2.1 and that \( \rho_0 \) is of the differential type with orders \( L_\rho = l_\rho = 0 \). In particular, \( \Gamma^\infty(P,E) \) is a \( (C^\infty(P), C^\infty(M)) \)-bimodule. This and the commutativity of the pointwise multiplication make it possible to equip \( \mathcal{D} = \text{DiffOp}^{\bullet}_{C^\infty(P)}(\Gamma^\infty(P,E); \Gamma^\infty(P,E)) \) with the \( (C^\infty(M), C^\infty(M)) \)-bimodule structure (2.33), this means

\[
(a \cdot D \cdot b)(s) = p^* a \cdot D(p^* b \cdot s)
\]

for all \( a, b \in C^\infty(M), s \in \Gamma^\infty(P,E), D \in \mathcal{D} \). As already stated in Section 2.1.3 this convention does not affect any considerations but is convenient and simplifies many considerations, confer Lemma 3.2.15.

Using these structures we are of course interested in the deformations \( \rho = \sum_{r=0}^{\infty} \lambda^r \rho_r \) of \( \rho_0 \) of the induced differential type. This means that the \( \rho_r \) are 1-cochains

\[
\rho_r \in \bigcup_{l \in \mathbb{N}_0} \text{DiffOp}^l_{C^\infty(M)}(C^\infty(M); \text{DiffOp}^l_{C^\infty(P)}(\Gamma^\infty(P,E); \Gamma^\infty(P,E)))
\]

in the crucial complex

\[
(\text{HC}^\bullet_{\text{diff}}(C^\infty(M), \mathcal{D}), \delta),
\]

where \( \delta \) is induced by (6.1). Due to the observation made in Corollary 3.2.8 the Hochschild differential respects the left \( C^\infty(P) \)-module structure of all vector spaces \( \text{HC}^k_{\text{diff}}(C^\infty(M), \mathcal{D}) \), this means

\[
\delta(f \cdot \phi) = f \cdot \delta \phi
\]

for all \( f \in C^\infty(P), \phi \in \text{HC}^\bullet_{\text{diff}}(C^\infty(M), \mathcal{D}) \). Due to the results of Chapter 2 especially of the Sections 2.4 and 2.5 the first and the second cohomology groups of this differential Hochschild complex represent the obstruction for an order by order construction of the desired deformations and the equivalence transformations between them.

The sections \( \Gamma^\infty(P,E) \) can of course be restricted to open subsets \( \tilde{U} \subseteq P \) and it is clear by considering the restriction \( p|_{\tilde{U}} \) that the sections \( \Gamma^\infty(U, q^{-1}(U)) \) have analogous module structures.
6.1. Right modules on surjective submersions and principal fibre bundles

with respect to $C^\infty(\tilde{U})$ and $C^\infty(p(\tilde{U}))$. The common restrictions of maps are obviously compatible with the considered pointwise structures. Thus, the surjective submersion together with the sheaves

$$\mathcal{E} = \Gamma^\infty_{P\mathcal{E}} \quad \text{and} \quad \mathcal{A} = C^\infty_M$$

of sections of $E$ over $P$ and functions over $M$ are a special example of structures as in Remark 4.4.1 and one is further able to apply the results of Proposition 4.4.7 which will be very important for the computation of the cohomology.

Remark 6.1.1

Note that for the above geometric situation of a surjective submersion $p : P \to M$ one could already consider $C^\infty(P)$ as a right $C^\infty(M)$-module with

$$\rho_0(f,a) = p^*a \cdot f$$

for all $f \in C^\infty(P)$ and $a \in C^\infty(M)$. This is of course a special case of (6.1) for the trivial vector bundle $E = P \times \mathbb{C}$ since $\Gamma^\infty(P, P \times \mathbb{C}) \cong C^\infty(P)$. The corresponding complex then is $H^\bullet_{\text{diff}}(C^\infty(M), \mathcal{D})$ with $\mathcal{D} = \text{DiffOp}^*_{C^\infty(P)}(C^\infty(P), C^\infty(P))$.

Remark 6.1.2 (The zeroth cohomology)

As seen in Section 2.5, the zeroth Hochschild cohomology is given by the commutant of the right module structure within the differential operators. In the considered geometrical context it is clear that the elements of the commutant are nothing but the vertical differential operators $D \in \text{DiffOp}_{\text{ver}}(\Gamma^\infty(P, E), \Gamma^\infty(P, E))$ which are defined by the condition

$$D(p^*a \cdot s) = p^*a \cdot D(s)$$

for all $a \in C^\infty(M)$ and $s \in \Gamma^\infty(P, E)$.

6.1.2 Deformation theory on principal fibre bundles

Since every principal fibre bundle $p : P \to M$ is also a surjective submersion the above considerations still hold in this case. However, being given the additional principal right action $r : G \to \text{Aut}(P)$ of the structure group $G$ it is natural to consider so-called equivariant vector bundles over $P$. This means that $G$ acts on the total space $E$ of the vector bundle $q : E \to P$ by vector bundle automorphisms over the principal right action or the corresponding left action. First, we consider the case of a right action

$$R : G \to \text{Aut}(\mathcal{E})$$

with the property that $R_g : E \to E$ is fibrewise linear and

$$q \circ R_g = r_g \circ q$$

for all $g \in G$. Equation (6.10) can be seen as the $G$-invariance of $q$.

A right action $R$ on a vector bundle $E$ as above naturally induces a corresponding left action

$$L : G \to \text{Aut}(E^*)$$

on the dual vector bundle $q : E^* \to P$ whose fibres $E_u^*$ over $u \in P$ are the dual spaces of the fibres $E_u$ of $E$. For an element $g \in G$ the action is simply given by the transposed maps
L_g = R_g^* : E_g \to E_{g_0}^* defined by (L_g v^*)(x) = v^*(R_g v). It is obvious that the so defined maps
L_g : E^* \to E^* again are fibrewise linear and now satisfy
\[ q \circ L_g = r_{g^{-1}} \circ q. \]  (6.12)

Of course, such a left action induces a right action in the analogous way. In the following it will
not play any role which kind of action on E, left or right, we consider. Together with the principal
right action r both of them induce left representations of G on the space of sections of p : E \to P.
Explicitly, one has for all s \in \Gamma^\infty(P, E)
\[ g \triangleright s = R_{g^{-1}} \circ s \circ r_g \quad \text{or} \quad \quad g \triangleright s = L_g \circ s \circ r_g. \]  (6.13) (6.14)

It is remarkable that the properties of the principal right action r, this means to be free and
proper, confer \[88\ Chap. 9\] or \[43\], are always inherited by the action of an equivariant vector
bundle.

**Lemma 6.1.3 (Equivariant vector bundles over principal fibre bundles)**

*If q : E \to P is an equivariant vector bundle over a principal fibre bundle p : P \to M the action
of the structure Lie group G on E is smooth, free and proper.*

**Proof:** Without loss of generality one can assume to be given a right action R as in (6.9).
Smoothness is clear by definition. That R is free, this means that g \triangleright e \neq e for all e \in E and
all group elements g \in G except the neutral element, is a consequence of the property (6.10)
and the freeness of r. Properness has several equivalent definitions. For our purpose we use the
characterization of \[88\ Prop. 9.13\]. According to the results there, the action R is proper if the
following condition is satisfied. Given a convergent sequence \{e_n\} in E and a sequence \{g_n\} in
G such that the sequence \{R_{g_n} e_n\} converges, then there exists a convergent subsequence of \{g_n\}. That this really is the case can be seen very easily. Due to the continuity of the projection q and
the property (6.10), the convergent sequences \{e_n\} and \{R_{g_n} e_n\} in E induce convergent sequences
\{q(e_n)\} and \{q(R_{g_n} e_n) = r_{g_n} e_n\} in P. But then, the properness of the right action r yields the
assertion. \[\blacksquare\]

Since R_g and L_g are fibrewise linear these definitions yield representations and the module
structures are all G-invariant. One has g \triangleright (f \circ s) = (g \triangleright f)(g \triangleright s) where the action on f \in C^\infty(P)
is given by pullback, g \triangleright f = r_g^* f. Regarding the right module structure (6.11) as cochain and using
the induced action on maps, this implies
\[ g \triangleright \rho_0 = \rho_0. \]  (6.15)

Of course we are interested in deformations of the same type. Since the action on C^\infty(M) is trivial
we are in the situation of Remark 2.2.4 and have to compute the cohomology of the complex
\[ (\text{HC}^\bullet_{\text{diff}}(C^\infty(M), \mathbb{D}^G), \delta) . \]  (6.16)

Of course, all occurring structures are G-invariant. In particular, the zeroth cohomology is given
by the G-invariant vertical differential operators DiffOp_{ver}(\Gamma^\infty(P, E); \Gamma^\infty(P, E))^G.

The properties (6.10) and (6.12) imply that the orbits of R or L are subsets of the preimages
under q of orbits of r. This shows that the right actions are compatible with the restrictions of maps
to \( \hat{U} = p^{-1}(U) \subseteq P \) with open subsets \( U \subseteq M \). Altogether, the sheaves \( \hat{E} = \Gamma_{\hat{P}, E} \) and \( A = C^\infty_M \)
now satisfy the conditions of Remark 4.4.11 and we can use the results of Proposition 4.4.12.
Remark 6.1.4 (Submodules of smooth sections)
In some relevant examples we will not be interested in deformations of the right \( C^\infty(M) \)-module \( \Gamma^\infty(P, E) \) of all sections but rather in the deformations of a particular submodule

\[
\hat{\Gamma}^\infty(P, E) \subseteq \Gamma^\infty(P, E)
\]

consisting of sections with a further particular property. If this submodule gives rise to a subsheaf \( \hat{\Gamma}_{PE}^\infty \) of substructures with all properties of \( \Gamma_{PE}^\infty \) it is evident that all considerations are absolutely the same. In the following we will thus omit the extra notation.

6.1.3 Notation and specific properties of the relevant complexes
Before we come to the computation of the cohomologies of the complexes (6.4) and (6.16) we investigate some further properties of their cochains arising in this specific context.

Remark 6.1.5 (Notation)
For the structures of the previous section the meaning will then always be clear from the context.

Remark 6.1.6 (Notation)
From now on it will be convenient to use the following abbreviations: For any manifold \( N \) and any \( C^\infty(N) \)-(bi)module \( M \) we set

\[
\text{DiffOp}^\bullet(M) = \text{DiffOp}_{C^\infty(N)}^\bullet(M; M)
\]

and in particular

\[
\text{DiffOp}^\bullet(N) = \text{DiffOp}_{C^\infty(N)}^\bullet(C^\infty(N), C^\infty(N)).
\]

Further, we set

\[
\text{DiffOp}^L(N, M) = \text{DiffOp}_{C^\infty(N)}^L(C^\infty(N), \ldots, C^\infty(N); M)
\]

for \( L = (l_1, \ldots, l_k) \in \mathbb{N}_0^k \), \( k \in \mathbb{N} \), and

\[
\text{HC}^\bullet(N, M) = \text{HC}^\bullet(C^\infty(N), M).
\]

For the structures of the previous section the meaning will then always be clear from the context.

The considered cochains are given by differential operators and due to the results of Section 4.4 these operators define presheaves. Thus, the cochains can be restricted to open subsets \( \tilde{U} \subseteq P \) using the natural restriction maps introduced in the Sections 4.2 and 4.3. In the \( G \)-invariant setting these restrictions are only allowed to open subsets \( p^{-1}(U) \) with \( U \subseteq M \). In any case, if \( U = p(U) \) is the domain of a local chart for \( M \) the differential operators have the expected local expressions. We prove the following general statement for the presheaf of differential operators with values in differential operators.

Lemma 6.1.6
Let \( \tilde{U} \subseteq P \) be an open subset such that for \( U = p(\tilde{U}) \) there is a local chart \((U, x)\) of \( M \). Further, let there be given \( \phi \in \text{DiffOp}^L(M, \text{DiffOp}^*_{C^\infty(P, E)}) \) with multiorder \( L = (l_1, \ldots, l_k) \in \mathbb{N}_0^k \) of differentiation. Then there exist uniquely defined differential operators \( \phi_{\tilde{U}}^{a_1, \ldots, a_k} \in \text{DiffOp}^*_{C^\infty(\tilde{U}, E|_{\tilde{U}})} \) with multiindices \( a_i \in \mathbb{N}_0^n \), \( i = 1, \ldots, k \), such that

\[
\phi_{|\tilde{U}}(a_1, \ldots, a_k) = \sum_{|a_1| \leq l_1, \ldots, |a_k| \leq l_k} \left( \frac{\partial^{a_1} \phi_{\tilde{U}}}{\partial x^{a_1}} \cdots \frac{\partial^{a_k} \phi_{\tilde{U}}}{\partial x^{a_k}} \right) \cdot \phi_{\tilde{U}}^{a_1, \ldots, a_k}
\]

for all \( a_1, \ldots, a_k \in C^\infty(U) \).

For \( \phi \in \text{DiffOp}^L(M, \text{DiffOp}^*_{C^\infty(P, E)})^G \) and \( \tilde{U} = p^{-1}(U) \) the same assertion is true with \( G \)-invariant operators \( \phi_{\tilde{U}}^{a_1, \ldots, a_k} \in \text{DiffOp}^*_{C^\infty(\tilde{U}, E|_{\tilde{U}})}^G \).
Theorem 6.1.7

Proof: The proof is a straightforward generalization of the well-known considerations for ordinary differential operators. The assertion is shown by an induction over $|L|$. For $|L| = 0$ one obviously has $\phi|_U(a_1, \ldots, a_k) = (a_1 \ldots a_k) \cdot \phi|_U(1, \ldots, 1)$. Due to the results of Chapter 4 in particular the ones of Section 4.4 the uniqueness of the local expression $\phi|_U$ is already clear. Thus we can assume that $x(U) \subseteq \mathbb{R}^n$ is convex. Now, for $|L| \geq 0$ consider an arbitrary point $u_0 \in \bar{U}$ and $p(u_0) = p_0 \in U$. Further consider an arbitrary section $s \in \Gamma(U)_{|E|}^s$ and functions $a_1, \ldots, a_k \in C^\infty(U)$. The assumed convexity assures that for all points $p \in U$ and $t \in [0,1]$ the convex combination $tx(p) + (1-t)x(p_0)$ is again in the range of the chart $x$ and we can use Hadamard’s trick. For $l = 1, \ldots, k$ one has

$$a_l(p) = (a_l \circ x^{-1})(x(p_0)) + \int_0^1 dt \frac{d}{dt}((a_l \circ x^{-1})(tx(p) + (1-t)x(p_0)))$$

$$= a_l(p_0) + \int_0^1 dt \frac{\partial (a_l \circ x^{-1})}{\partial x^i}(tx(p) + (1-t)x(p_0))(x^i(p) - x^i(p_0))$$

Thus one has $a_l = a_l(p_0) + b^i_l(x^i - x^i(p_0))$. The functions $b^i_l \in C^\infty(U)$ given by

$$b^i_l(p) = \frac{1}{2} \int_0^1 dt \left( \frac{\partial (a_l \circ x^{-1})}{\partial x^i} - \frac{\partial (a_l \circ x^{-1})}{\partial x^i} \right)(tx(p) + (1-t)x(p_0))$$

satisfy

$$\frac{\partial^{|\alpha|+1} b^i_l}{\partial x^i}(p_0) = \int_0^1 dt \frac{\partial^{\alpha} (a_l \circ x^{-1})}{\partial x^\alpha}(tx(p) + (1-t)x(p_0))$$

for all multiindices $\alpha \in \mathbb{N}_0^n$. Using the linearity properties one then has

$$((\phi|_U(a_1, \ldots, a_k))(s))(u_0)$$

$$= \phi|_U(a_1(p_0) + b^i_1 \cdot (x^i - x^i(p_0)), a_2, \ldots, a_k)(s)(u_0)$$

$$= a_1(p_0) \phi|_U(1, a_2, \ldots, a_k)(s)(u_0) + \left( \phi|_U \circ L^{(1)}_{x^i - x^i(p_0)} - L_{x^i - x^i(p_0)} \circ \phi|_U \right)(b^i_1, a_2, \ldots, a_k)(s)(u_0),$$

where the term $L_{x^i - x^i(p_0)} \circ \phi|_U$ could be added since it vanishes in the considered expression due to the module structure, $((a \cdot D)(s))(u_0) = a(p_0)D(s)(u_0)$ with $D \in \text{DiffOp}^s(\Gamma(U, E|_U))$ and $a \in C^\infty(U)$. After setting

$$S^i_l = \phi|_U \circ L^{(i)}_{x^i - x^i(p_0)} - L_{x^i - x^i(p_0)} \circ \phi|_U \in \text{DiffOp}^L_{-s_l}(C^\infty(U), \text{DiffOp}^s(C^\infty(\bar{U})))$$

an iteration yields

$$((\phi|_U(a_1, \ldots, a_k))(s))(u_0) = a_1(p_0) \ldots a_k(p_0) \phi|_U(1, \ldots, 1)(s)(u_0)$$

$$+ \sum_{l=1}^k a_1(p_0) \ldots a_{l-1}(p_0)S^i_l(1, \ldots, 1, b^i_l, a_{l+1}, \ldots, a_k)(s)(u_0).$$

Using the induction hypothesis for the $S^i_l$ and the property (6.23), the form (6.22) follows since $s$ and $u_0$ were arbitrary. The assertion concerning the $G$-invariance is obvious.

Remark 6.1.7

It is easy to see that Lemma 6.1.6 even holds for Hochschild cochains, this means for $\phi \in \text{DiffOp}^L(M, \text{DiffOp}^1(\Gamma(\mathbb{R}, E)))$ with a fixed $l \in \mathbb{N}_0$. Then, the assertion is true with $\phi|_U^{\alpha_1, \ldots, \alpha_k} \in \text{DiffOp}(\Gamma(\mathbb{R}, E|_U))$. Of course the same is true for $G$-invariant cochains.
The local expressions in Lemma 6.1.8 have the interesting consequence that the considered differential operators $\phi$ already have values in the differential operators of some fixed order and thus are Hochschild cochains. In order to prove this we need the following simple lemma.

Lemma 6.1.8
Let $M$ be a manifold and let $D \in \text{DiffOp}^l(M)$ where $l \in \mathbb{N}_0$ is the smallest possible order of differentiation to choose, this means $D \not\in \text{DiffOp}^{l-1}(M)$. Farther let $\{U_i\}_{i \in I}$ be an open cover of $M$. Then the corresponding restrictions of the operator $D$ are differential operators $D|_{U_i} \in \text{DiffOp}^l(U_i)$ with $l_i \leq l$ for all $i \in I$ and there exists at least one $i_0 \in I$ with $D|_{U_{i_0}} \in \text{DiffOp}^l(U_{i_0})$ where $l$ is the smallest possible order.

Proof: The assertion is clear with the sheaf property of differential operators of some fixed order of differentiation. If there existed an $\tilde{l} \in \mathbb{N}_0$ with $l_i \leq \tilde{l} < l$ for all $i \in I$ this would imply that $D \not\in \text{DiffOp}^l(M)$ which is a contradiction to the choice of $l$.

Lemma 6.1.9
For all $k \in \mathbb{N}$ and $L \in \mathbb{N}_0^k$ one has

$$\text{DiffOp}^L(M, \text{DiffOp}^* (\Gamma^\infty(P, E))) = \bigcup_{l \in \mathbb{N}_0} \text{DiffOp}^{l+1}(M, \text{DiffOp}^l(\Gamma^\infty(P, E))).$$

(6.24)
The same is true for operators with values in $G$-invariant maps.

Proof: The nontrivial inclusion to show is

$$\text{DiffOp}^L(M, \text{DiffOp}^* (\Gamma^\infty(P, E))) \subseteq \bigcup_{l \in \mathbb{N}_0} \text{DiffOp}^{l+1}(M, \text{DiffOp}^l(\Gamma^\infty(P, E))).$$

Let $\phi \in \text{DiffOp}^L(M, \text{DiffOp}^* (\Gamma^\infty(P, E)))$ with $L \in \mathbb{N}_0^k$ and $k \in \mathbb{N}_0$. Using the abbreviation $\mathcal{D}^l(P) = \text{DiffOp}^l(\Gamma^\infty(P, E))$ one has to show the following assertion:

There exists an $l \in \mathbb{N}_0$ such that for all $a_1, \ldots, a_k \in C^\infty(M)$ the smallest possible $l(a_1, \ldots, a_k) \in \mathbb{N}_0$ with $\phi(a_1, \ldots, a_k) \in \mathcal{D}^l(\Gamma^\infty(P, E))$ is bounded by $l$, i.e., this means $l(a_1, \ldots, a_k) < l$.

Now assume that this is not the case. Then there exists a strictly monotonically increasing sequence $\{l_n\}_{n \in \mathbb{N}}$ with $l_n \in \mathbb{N}_0$ such that for all $n \in \mathbb{N}$ there exist $a_1^{(n)}, \ldots, a_k^{(n)} \in C^\infty(M)$ with $\phi(a_1^{(n)}, \ldots, a_k^{(n)}) \in \mathcal{D}^{l_n}(P) \setminus \mathcal{D}^{l_{n-1}}(P)$. Due to Lemma 6.1.8 there exist charts $(U_n, x_n)$ of $M$ with $\phi(a_1^{(n)}, \ldots, a_k^{(n)})|_{p^{-1}(U_n)} \in \mathcal{D}^{l_n}(p^{-1}(U_n)) \setminus \mathcal{D}^{l_{n-1}}(p^{-1}(U_n))$. Since $\text{DiffOp}^*(\Gamma^\infty(P, E)) \subseteq \text{Loc}(\Gamma^\infty(P, E))$ is a presheaf one has $\phi(a_1, \ldots, a_k)|_{p^{-1}(U_n)} = \phi|_{p^{-1}(U_n)}(a_1|_{U_n}, \ldots, a_k|_{U_n})$ and (6.22) then shows that the smallest possible degree of $\phi(a_1, \ldots, a_k)|_{p^{-1}(U_n)}$ for all $a_1, \ldots, a_k$ can not be greater than $l_n$. Thus one can assume that the open subsets for different minimal orders are disjoint, $U_n \cap U_m = \emptyset$ for $n \neq m$. Once again with Lemma 6.1.8 we choose for all $n \in \mathbb{N}$ open subsets $V_n \subseteq U_n$ with $V_n^{\text{cl}} \subseteq U_n$ and $\phi(a_1^{(n)}, \ldots, a_k^{(n)})|_{p^{-1}(V_n)} \in \mathcal{D}^{l_n}(p^{-1}(V_n))$ where $l_n$ is minimal. For all $n \in \mathbb{N}$ we now choose $\chi_n \in C^\infty(M)$ with $\chi_n|_{V_n} = 1$ and $\supp(\chi_n) \subseteq U_n$ and consider the well-defined functions $b_s = \sum_{n=1}^\infty \chi_n a_s^{(n)} \in C^\infty(M)$ with $b_s|_{V_n} = a_s^{(n)}|_{V_n}$ for all $s = 1, \ldots, k$. We then get

$$\phi(b_1, \ldots, b_k)|_{p^{-1}(V_n)} = \phi|_{p^{-1}(V_n)}(a_1^{(n)}|_{V_n}, \ldots, a_k^{(n)}|_{V_n}) = \phi(a_1^{(n)}, \ldots, a_k^{(n)})|_{p^{-1}(V_n)} \in \mathcal{D}^{l_n}(p^{-1}(V_n)),$$

where $l_n$ is the smallest possible order. The smallest possible order of $\phi(b_1, \ldots, b_k)$ is at least $l_n$. Since $l_n \to \infty$ for $n \to \infty$, it is clear that $\phi(b_1, \ldots, b_k) \in \text{Loc}(\Gamma^\infty(P, E))$ has no well-defined degree of differentiation, $\phi(b_1, \ldots, b_k) \not\in \text{DiffOp}^*(\Gamma^\infty(P, E))$. This is a contradiction to the assumption on $\phi$. The assertion for $G$-invariant maps is a direct consequence of (6.24) itself.

This result shows that in the present situation the characterization of the differential type is easy and one has not to take care of the subtleties occurring in the general situation.
Remark 6.1.10 (The natural notion of differential right module structures)
The local formulas obtained in the considered situation show that our definitions really discuss
differential right module structures in the natural and desired sense. If \(\tilde{U} \subseteq P\) in Lemma 6.1.6
is the domain of a local chart \((\tilde{U}, z)\) on \(P\) such that one in addition has a local \(C^\infty(\tilde{U})\)-module
basis \(\{s_i\}_{i=1, \ldots, r}\) for \(\Gamma^\infty(\tilde{U}, E|_{\tilde{U}})\) one can further consider the local expression for the operators
\(\text{DiffOp}^i(\Gamma^\infty(\tilde{U}, E|_{\tilde{U}})),\) confer [123, Thm. A.5.2]. Regardless of a possibly existing \(G\)-invariance all
1-cochains \(\rho \in \text{DiffOp}^L(M, \text{DiffOp}^i(\Gamma^\infty(P, E)))\) then have the following local form for all \(a \in C^\infty(p(\tilde{U}))\) with \(f^i \in C^\infty(\tilde{U}),\)
\[
\rho_{\tilde{U}}(s, a) = \rho|_{\tilde{U}}(a)(s) = \sum_{|\alpha| \leq L} \left( \frac{\partial |\alpha| a}{\partial x^\alpha} \cdot \phi_{\tilde{U}}^\alpha \right)(s) = \sum_{|\alpha| \leq L, |\beta| \leq l} \left( p^s \frac{\partial |\alpha| a}{\partial x^\alpha} \right) \cdot \frac{\partial |\beta| f^i}{\partial z^\beta} \cdot s_i^{\alpha \beta}.
\] (6.25)

There, \(s_i^{\alpha \beta} \in \Gamma^\infty(\tilde{U}, E|_{\tilde{U}})\) are uniquely defined sections. Note the different kinds of multiindices
\(\alpha \in \mathbb{N}_0^n\) and \(\beta \in \mathbb{N}_0^{n+k}.

Naively, one would have defined the differential right module structures to be operators in
\[
\text{DiffOp}^i_{\Gamma^\infty(M)}(\Gamma^\infty(P, E), C^\infty(M); \Gamma^\infty(P, E)).
\] (6.26)

However, this does not lead to the desired local expression [6.25]. In the principal bundle case the map
\[
D_g : (s, a) \mapsto p^* a \cdot (g \triangleright s)
\] (6.27)
which is induced by the right action [6.13] or [6.14] defines such a differential operator \(D_g \in \text{DiffOp}^i_{\Gamma^\infty(M)}(\Gamma^\infty(P, E), C^\infty(M); \Gamma^\infty(P, E))\) for all \(g \in G\). But of course such right module structures
containing a nonlocal shift by \(g\) should not be called differential. These considerations justify
and affirm the chosen approach to deformation theory of right modules of a particular type as
introduced in Section 2.2.

6.2 The basic steps of the computation of the cohomology

The strategy to compute the differential Hochschild cohomologies of the complexes [6.4] and [6.16]
basicallly consists of four steps. Although the basic ideas are the same for both complexes we have
to discuss the two cases separately. The differences between surjective submersions and principal
fibre bundles caused by the right action make it necessary to pursue slightly different paths.

6.2.1 The basic steps for surjective submersions

For surjective submersion the four steps are the following.

i.) First, one makes use of the fact that the Hochschild complexes can be restricted to open
subsets of an appropriate covering of \(P\) and that one is able to consider the local situation.

ii.) It is a general fact that the sections of a vector bundle become a free module over the algebra
of smooth functions on the base manifold if one considers the local situation with respect to
a vector bundle chart. Using Lemma 3.2.15 we thus can consider the deformation problem
for the right module of functions described in Remark 6.1.1.

iii.) Due to the fact that a surjective submersion allows a specific choice of charts, we can consider
a surjective submersion \(pr_1 : V \times G \longrightarrow V\) with a manifold \(G\) and an open convex subset
\(V \subseteq \mathbb{R}^n\). In this case, it will be possible to compute the Hochschild cohomologies using the
techniques developed in Chapter 5.
iv.) In the last step we observe that the coboundaries which are used to compute the local cohomologies are images of cochains with globally bounded orders of differentiation. Thus, it will be possible to apply Proposition 4.4.7 and to compute the initial Hochschild cohomology.

The reduction of the problem to the local situation described in the third point makes use of the particular geometric situation. By the constant rank theorem, confer [20, Satz 5.4], for any point \( u \in P \) there exist open subsets \( \tilde{U} \subseteq P \) with \( u \in \tilde{U} \) and \( U \subseteq M \) with \( p = p(u) \in U \) together with diffeomorphisms \( x : U \to V \subseteq \mathbb{R}^n \) and \( \tilde{x} : \tilde{U} \to V \times G \subseteq \mathbb{R}^{n+k} \), such that \( p(\tilde{U}) = U \) and \( x \circ p \circ \tilde{x}^{-1} = \text{pr}_1 \). Clearly, in this case \( G \subseteq \mathbb{R}^k \) is an open subset. It is possible to choose the open subsets in a way that \( V \subseteq \mathbb{R}^n \) is convex. Furthermore, one can achieve that in the considered situation the vector bundle \( q : E \to P \) has a local trivialization over \( \tilde{U} \). This means that there exists a vector bundle chart \( \psi : E|_{\tilde{U}} \to \tilde{U} \times W \) with \( \text{pr}_1 \circ \psi = q \). Such adapted local charts \( (\tilde{U}, \tilde{x}) \) of \( P \) and \( (U, x) \) of \( M \) build adapted atlases and in particular induce corresponding open coverings of \( P \). These adapted structures which will be used throughout the chapter induce the commutative diagram

\[
\begin{array}{cccccc}
E & \xrightarrow{\iota} & E|_{\tilde{U}} & \xrightarrow{\psi} & \tilde{U} \times W \\
q & & q & \text{pr}_1 & \\
P & \downarrow{\iota} & \tilde{U} & \xrightarrow{\tilde{x}} & V \times G \subseteq \mathbb{R}^{n+k} \\
p & & \text{pr}_1 & \\
M & \downarrow{\iota} & U & \xrightarrow{x} & V \subseteq \mathbb{R}^n,
\end{array}
\]

where \( \iota \) in every case denotes the embedding map. All spaces in the first row again are vector bundles over the corresponding spaces in the second row. Considering only the second and third row every column of the diagram is the diagram of a surjective submersion with a canonical right module structure for the corresponding algebras of smooth functions as in (6.7).

Using these adapted charts the reduction of the initial deformation problem to the local one is realized as follows.

**Lemma 6.2.1**

The restriction maps of sections and functions, the local vector bundle chart \( \psi \) and the local charts \( \tilde{x} \) and \( x \) give rise to chain maps and chain isomorphisms

\[
\begin{align*}
\text{HC}_{\text{diff}}^\bullet (M, \text{DiffOp}(\Gamma^\infty (P, E))) & \\
\xrightarrow{\text{restriction}} & \\
\text{HC}_{\text{diff}}^\bullet (U, \text{DiffOp}(\Gamma^\infty (\tilde{U}, E|_{\tilde{U}}))) & \xrightarrow{\cong} \text{Mat}_{r \times r} \left( \text{HC}_{\text{diff}}^\bullet (U, \text{DiffOp}(\tilde{U})) \right) \\
\xrightarrow{\cong} & \\
\text{HC}_{\text{diff}}^\bullet (V, \text{DiffOp}(V \times G)) & \\
\end{align*}
\]

**Proof:** As already mentioned one is in the situation of Section 4.3.1 This in particular means that \( \text{HC}_{\text{diff}}^\bullet (M, \text{DiffOp}(\Gamma^\infty (P, E))) \) can be seen as a presheaf of Hochschild complexes over \( P \). Thus the restriction map is a morphism of complexes.

Choosing a basis \( \{w_1, \ldots, w_r\} \) of the typical fibre \( W \) the homeomorphism \( \psi : E|_{\tilde{U}} \to \tilde{U} \times W \) gives rise to a corresponding \( C^\infty (\tilde{U}) \)-module basis \( \{s_1, \ldots, s_r\} \) of \( \Gamma^\infty (U, E|_{\tilde{U}}) \), thus becoming a finitely generated free module. With (6.1) and (6.2) one thus can apply Lemma 3.2.15 and thus has justified the isomorphism with the complex of matrices. In Lemma 3.2.15 it is further explained...
Chapter 6. Deformation theory of right modules on principal fibre bundles

that the complex of matrices in fact is a matrix of one single complex. Thus the restriction to one of those copies is a surjective chain morphism.

Finally, one considers the pullbacks of the inverse chart maps. It is simple to verify that \((\tilde{x}^{-1})^* : C^\infty(U) \to C^\infty(V \times G)\) is an algebra isomorphism and, in addition, a module isomorphism along the algebra isomorphism \((x^{-1})^* : C^\infty(U) \to C^\infty(V)\). Then, the last isomorphism of complexes is a consequence of the functorial behaviour of differential Hochschild complexes as already stated in Remark \[\text{3.2.14}\].

6.2.2 The basic steps for principal fibre bundles

Considering equivariant vector bundles over principal fibre bundles we have to modify the steps used for surjective submersions taking care of the occurring actions of the structure group \(G\).

\(i.)\) In the \(G\)-invariant case one works with presheaves over the base manifold \(M\). Thus the Hochschild complex \(6.16\) can only be restricted to open subsets \(U \subseteq M\) of an appropriate covering of \(M\) and one considers the local situation with respect to such subsets \(U\).

\(ii.)\) For the second step one has to work with \(G\)-invariant local module bases of the sections. Then, the additional assertion of Lemma \[\text{3.2.15}\] assures that the initial problem for the sections can be traced back to that of the functions even in the \(G\)-invariant setting.

\(iii.)\) Since every principal fibre bundle has corresponding local bundle charts one can consider the trivial bundle \(pr_1 : V \times G \to V\) with the structure Lie group \(G\) and an open convex subset \(V \subseteq \mathbb{R}^n\). In this case the explicit computation of the cohomology will turn out to respect the additional \(G\)-invariance automatically.

\(iv.)\) Finally, it will be possible to apply Proposition \[\text{4.4.12}\] and to compute the initial \(G\)-invariant cohomology.

In detail, the reduction of the problem now makes use of the following geometric situation. Given an arbitrary principal fibre bundle \(p : P \to M\) it is clear by the very definition that for any \(p \in M\) there exists an open subset \(U \subseteq M\) with \(p \in U\) such that the bundle over \(U\) is trivial, confer Definition \[\text{A.2.1}\]. This means that there exists a principal bundle chart \(\varphi : P \supseteq p^{-1}(U) \to U \times G\) with the defining properties \(pr_1 \circ \varphi = p\) and \(\varphi \circ r_g = (\text{id}_U \times r_g) \circ \varphi\). The map \(r_g : G \to G\) is the right multiplication with \(g \in G\), this means \(r_g(h) = hg\). Of course it is possible to achieve that \(U\) is the domain of a chart \(x : U \to V \subseteq \mathbb{R}^n\) with the additional properties that \(U\) is contractible and \(V \subseteq \mathbb{R}^n\) is convex. In the following we will always work with open coverings of \(M\) consisting of such open subsets \(U \subseteq M\). Denoting the unit element of the group by \(e \in G\) one has that \(U \times \{e\}\) and \(\varphi^{-1}(U \times \{e\})\) are contractible. Then it is clear that the bundle \(E\) is trivial over \(\varphi^{-1}(U \times \{e\})\) such that the sections have a module basis \(\{\bar{s}_i\}_{i=1,...,r}\), confer \[\text{[92, Thm. 3.4.35]}\]. This gives rise to a \(G\)-invariant module basis \(\{s_i\}_{i=1,...,r}\) of the sections \(\Gamma^\infty(p^{-1}(U), E|_{p^{-1}(U)})\) over \(p^{-1}(U) = \varphi^{-1}(U \times G)\) by setting

\[s_i(\varphi^{-1}(p, h)) = R_h \bar{s}_i(\varphi^{-1}(p, e)) \quad \text{or} \quad s_i(\varphi^{-1}(p, h)) = L_{h^{-1}} \bar{s}_i(\varphi^{-1}(p, e)). \quad (6.30)\]

The invariance

\[g \triangleright s_i = s_i \quad (6.31)\]

for all \(g \in G\) is a simple computation. Altogether, the above structures and properties are summarized in the following commutative diagram.
6.3. The local Hochschild cohomology $\HH_{\text{diff}}^\bullet(V, \text{DiffOp}(V \times G))$

According to the considerations of Section 6.2.1 we now have to compute the Hochschild cohomologies $\HH_{\text{diff}}^\bullet(V, \text{DiffOp}(V \times G))$ for the trivial surjective submersion $\text{pr}_1 : V \times G \to V$ with a convex subset $V \subseteq \mathbb{R}^n$ and a smooth manifold $G$. It should be emphasized already at the beginning that in the following we do not use the fact that in the situation of Section 6.2.1 $G$ is a subset of $\mathbb{R}^k$. Instead, $G$ can be an arbitrary manifold. Thus it will be very easy to see that the situation of Section 6.2.2, where $G$ is a Lie group and where one has to compute the $G$-invariant Hochschild cohomology $\HH_{\text{diff}}^\bullet(V, \text{DiffOp}(V \times G))^G$ for the trivial principal fibre bundle $\text{pr}_1 : V \times G \to V$, can be treated in the very same way. It will be possible to implement the additional $G$-invariance without any restrictions.

**Lemma 6.2.2**

The restriction maps of sections and functions, the $G$-invariant module basis, the principal bundle chart $\varphi$, and the local chart $x$ give rise to chain maps and chain isomorphisms

$$\HH_{\text{diff}}^\bullet(M, \text{DiffOp}(\Gamma^\infty(P, E))^G)$$

$$\HH_{\text{diff}}^\bullet(U, \text{DiffOp}(\Gamma^\infty(p^{-1}(U), E|_{p^{-1}(U)})^G) \xrightarrow{\cong} \text{Mat}_{r \times r} \left(\HH_{\text{diff}}^\bullet(U, \text{DiffOp}(p^{-1}(U))^G)\right)$$

$$\HH_{\text{diff}}^\bullet(U, \text{DiffOp}(p^{-1}(U))^G) \xrightarrow{\cong} \HH_{\text{diff}}^\bullet(V, \text{DiffOp}(V \times G)^G).$$

**Proof:** For the chain morphism given by the restriction map we now use the results of Section 4.4.2. With the $G$-invariant module basis and Lemma 3.2.15 one gets the first isomorphism and the second chain morphism as in Lemma 6.2.1. For the last isomorphism one needs the $G$-invariant algebra morphism $(\varphi^{-1} \circ (x^{-1} \times \text{id}_G))^* : C^\infty(p^{-1}(U)) \to C^\infty(V \times G)$ which is a module isomorphism along $(x^{-1})^* : C^\infty(U) \to C^\infty(V)$ and Remark 3.2.14.

**6.3 The local Hochschild cohomology $\HH_{\text{diff}}^\bullet(V, \text{DiffOp}(V \times G))$**
We will often take advantage of the fact that an atlas of local charts \((\bar{U},y)\) of \(G\) induces an atlas of adapted local charts 
\[
(\bar{U} = V \times \bar{U}, \bar{x} = \text{id}_V \times y)
\] 
of \(V \times G\).

The computation of the cohomology shall be performed using the arguments and techniques of Chapter 5 in particular Section 5.7. In order to do this, one first has to show the consistency of the two notions of differential cochains. Lemma 6.1.6 already ensures that the local expressions of cochains have the form as in Definition 5.3.2. In the considered case this is sufficient for the global situation.

**Lemma 6.3.1**

Let \(\phi \in \text{DiffOp}^l(V,\text{DiffOp}^l(V \times G))\) with \(L \in \mathbb{N}_0^k\), \(k,l \in \mathbb{N}_0\). Then there exist unique \(\phi^{a_1...a_k} \in \text{DiffOp}^l(V \times G)\) such that

\[
\phi(a_1, \ldots, a_k) = \sum_{|a_1|\leq l_1, \ldots, |a_k|\leq l_k} \left( \frac{\partial |a_1| a_1}{\partial x^{a_1}} \cdots \frac{\partial |a_k| a_k}{\partial x^{a_k}} \right) \phi^{a_1...a_k}.
\]  

(6.35)

for all \(a_1, \ldots, a_k \in C^\infty(V)\).

**Proof:** Using the adapted charts \((6.34)\) for the local expressions in \((6.22)\) the stated uniqueness of the occurring \(\phi^{a_1...a_k} \in \text{DiffOp}(U)\) shows that there exist unique \(\phi^{a_1...a_k} \in \text{DiffOp}^l(V \times G)\) with \(\phi^{a_1...a_k}|_U = \phi^{a_1...a_k}_U\) such that \((6.35)\) holds.

Besides this more or less obvious consistency of definitions one has to prove that \(\text{DiffOp}^\bullet(V \times G)\) satisfies all requirements of the topological bimodule \(M\) in Theorem 5.7.3.

For this purpose we have to consider an appropriate topology of the differential operators \(\text{DiffOp}^l(P)\) on a smooth manifold \(P\) of dimension \(d\). The Fréchet topology of the differential operators of order \(l \in \mathbb{N}_0\) which is induced from the one of smooth functions on \(P\) by using the local expressions will be sufficient for this purpose. Consider all compact subsets \(U \subseteq U\) of a chart \((U,z)\) of \(P\). Then, every differential operator \(D \in \text{DiffOp}^l(P)\) has the local form \(D|_U = \sum_{|\alpha| \leq l} D^\alpha_U \frac{\partial |\alpha|}{\partial z^\alpha}\) with coefficient functions \(D^\alpha_U \in C^\infty(U)\) and multiindices \(\alpha \in \mathbb{N}_0^n\). The common seminorms \(p_{U,r}\) with \(r \in \mathbb{N}_0\) inducing the Fréchet topology of \(C^\infty(P)\), confer \((5.29)\), are then used to define corresponding seminorms \(p^{(l)}_{U,K,r}\) on the spaces \(\text{DiffOp}(P)^l\) via

\[
p^{(l)}_{U,K,r}(D) = \max_{|\alpha| \leq l} \left( \max_{p \in K} \left| \frac{\partial |\beta|}{\partial z^\beta} D^\alpha_U (p) \right| \right)
\]  

(6.36)

It is easy to see that the maps \(p^{(l)}_{U,K,r}\) really are seminorms. Thus, they induce a locally convex topology on \(\mathcal{D}^l\) with the following well-known properties.

**Lemma 6.3.2**

Let \(P\) be a manifold and for each \(l \in \mathbb{N}_0\) let \(\text{DiffOp}^l(P)\) be equipped with the topology induced by the seminorms of \((6.36)\). Then the following statements hold:

i.) The locally convex spaces \(\text{DiffOp}^l(P)\) are Fréchet spaces, this means they are Hausdorff, metrizable and complete.

ii.) The embedding map \(\text{DiffOp}(P)^l \hookrightarrow \text{DiffOp}(P)^k\) for all \(l \leq k\) is a continuous embedding with closed image. In particular, the topology of \(\text{DiffOp}(P)^l\) is the one induced from \(\text{DiffOp}(P)^k\).
iii.) The composition of differential operators is a continuous map \( \text{DiffOp}(P)^l \times \text{DiffOp}(P)^{l'} \rightarrow \text{DiffOp}(P)^{l+l'} \) for all \( l, l' \in \mathbb{N}_0 \).

iv.) With respect to the natural Fréchet topology of \( \mathcal{C}^\infty(P) \) the spaces \( \text{DiffOp}(P)^l \) are topological \( (\mathcal{C}^\infty(P), \mathcal{C}^\infty(P)) \)-bimodules with the left module structure \( a \cdot D = L_a \circ D \) and the right module structure \( D \cdot a = D \circ L_a \) for \( D \in \text{DiffOp}(P) \) and \( a \in \mathcal{C}^\infty(P) \).

v.) If \( P \) is a principal fibre bundle the \( G \)-invariant differential operators are closed subspaces \( \text{DiffOp}^l(P)^G \subseteq \text{DiffOp}(P) \) for all \( l \in \mathbb{N}_0 \).

PROOF: It is obvious that the locally convex spaces \( \text{DiffOp}^l(P) \) are topological vector spaces. For the Hausdorff property one only has to show that for all \( 0 \neq D \in \text{DiffOp}^l(P) \) there exist seminorms with \( p_{U,K,r}^{(l)}(D) \neq 0 \). This is clear with \( K = \{ p \} \) for appropriate \( p \in U \subseteq P \) and \( r = 0 \).

By definition, a manifold is in particular a topological space which is second countable, this means the topology has a countable basis. Thus one can choose a countable covering \( \{ U_n \}_{n \in \mathbb{N}} \) of \( P \) with chart domains and, in addition, a countable covering of each \( U_n \) with compact subsets \( K \subseteq U_n \subseteq P \), confer [123, App. A.1]. Since the locally convex topology is already induced by the corresponding countable set of seminorms \( \{ p_n \}_{n \in \mathbb{N}} \) the topology is metrizable, for instance with the metric \( d(D, D') = \max_{n} \| \frac{1}{2} p_{U,K,r}^{(l)}(D) - \| \frac{1}{2} p_{U,K,r}^{(l)}(D') \| \) for all \( D, D' \in \text{DiffOp}^l(P) \). The countable set of seminorms inducing the topology further implies that the topology is first countable, this means that for all \( D \in \text{DiffOp}^l(P) \) there exists a countable set of neighbourhoods such that any neighbourhood contains one of \( U_n \). For example, the open balls of rational radius with respect to the seminorms \( p_n \) build such a set. Thus, \( \text{DiffOp}^l(P) \) is complete if and only if any Cauchy sequence converges. Due to (6.36) a Cauchy sequence \( \{ D_n \} \) of operators induces Cauchy sequences \( \{ p_{U,K,r}^{(l)}(D_n) \} \) of coefficient functions with respect to the Fréchet topology of \( \mathcal{C}^\infty(U) \) for the relevant chart \((U, z)\). Then, these sequences converge to smooth functions \( D_n^\alpha \in \mathcal{C}^\infty(U) \). Considering different charts \((U, z)\) and \((U', z')\) with \( U \cap U' \neq \emptyset \) the local coefficient functions are related by \( D_{U,n}^\alpha |_{U \cap U'} = \sum_{|\beta| \leq 1} D_{U',n}^\beta |_{U \cap U'} \cdot f_{\beta}^\alpha \) where \( f_{\beta}^\alpha \in \mathcal{C}^\infty(U \cap U') \) contains the Jacobi matrices of the coordinate change. Since the functions are a topological algebra this behaviour is the same for the limits \( D_{U,n}^\alpha \) and \( D_{U',n}^\beta \). Thus, they determine a well-defined differential operator \( D \in \text{DiffOp}^l(P) \) given by the local expressions. By construction, the sequence \( \{ D_n \} \) converges to \( D \). Thus, \( \text{DiffOp}^l(P) \) is a Fréchet space.

The next three assertions are easy consequences of (6.36). If \( D \in \text{DiffOp}^l(P) \) and \( l \leq k \) one has \( p_{U,K,r}^{(k)}(D) = p_{U,K,r}^{(l)}(D) \). This and the first point imply the second assertion. The third and fourth part are a simple consequence of the Leibniz rule. With the obvious notation the third part follows with \( p_{U,K,r}^{(l+l')}((D \circ D') \leq p_{U,K,r}^{(l)}(D) p_{U,K,r}^{(l')}((D') \) and for the fourth part one has \( p_{U,K,r}^{(l)}(f \cdot D) \leq p_{U,K,r}^{(l)}(f) p_{U,K,r}^{(l)}(D) \) and \( p_{U,K,r}^{(l)}((D \cdot f) = p_{U,K,r}^{(l)}(D) p_{U,K,r}^{(l)}(f) \).

For the last statement let \( D_n \in \text{DiffOp}^l(P)^G \) be a sequence of differential operators converging to \( D \in \text{DiffOp}^l(P) \) with respect to the considered topology. The fact that for all compact subsets \( K \subseteq U \) of a chart domain \( U \subseteq P \) one has \( p_{K,r}(D_n(f)) \leq p_{U,K,r}^{(l)}(D_n) p_{K,r}((f) \) implies that for all \( f \in \mathcal{C}^\infty(P) \) the sequence \( D_n(f) \) converges to \( D(f) \) with respect to the usual Fréchet topology of \( \mathcal{C}^\infty(P) \). Since we only work with smooth group actions the right action \( r_g : P \rightarrow P \) is a diffeomorphism for all \( g \). Thus, \( r_g^* : \mathcal{C}^\infty(P) \rightarrow \mathcal{C}^\infty(P) \) is a continuous map and the equation \( r_g^* (D_n(f)) = D_n(r_g(f)) \) implies \( r_g^* (D(f)) = D(r_g(f)) \) for all \( f \in \mathcal{C}^\infty(P) \). This shows \( g \cdot D = D \) which proves the assertion.

The next lemma makes use of the canonical global coordinates \( x^i : \mathbb{R}^n \supseteq V \rightarrow \mathbb{R}, i = 1, \ldots, n \), which extend to maps \( x^i \in \mathcal{C}^\infty(V \times G) \).
Lemma 6.3.3
Every \( D \in \text{DiffOp}^l(V \times G) \) with \( l \in \mathbb{N}_0 \) is of the form
\[
D = \sum_{r=0}^{l} \frac{1}{r!} D^{i_1 \ldots i_r} \frac{\partial^r}{\partial x^{i_1} \ldots \partial x^{i_r}}
\] (6.37)
with uniquely defined vertical operators \( D^{i_1 \ldots i_r} \in \text{DiffOp}^l_{\text{ver}}(V \times G) \) which are symmetric in the indices \( i_1, \ldots, i_r \in \{1, \ldots, n\} \).

Proof: With the adapted local charts (6.34) the considered \( D \in \text{DiffOp}^l(V \times G) \) has the local expression
\[
D|_{\tilde{U}} = \sum_{s=0}^{l} \sum_{r=0}^{l} \frac{1}{r!(s-r)!} D^{i_1 \ldots i_r j_1 \ldots j_{s-r}} \frac{\partial^r}{\partial x^{i_1} \ldots \partial x^{i_r}} \frac{\partial^{s-r}}{\partial y^{j_1} \ldots \partial y^{j_{s-r}}}
\]
(6.38)
with unique functions \( D^{i_1 \ldots i_r j_1 \ldots j_{s-r}} \in C^\infty(V \times \tilde{U}) \) which are symmetric in each group of indices \( i_1, \ldots, i_r \in \{1, \ldots, n\} \) and \( j_1, \ldots, j_{s-r} \in \{1, \ldots, k\} \). The behaviour under a change of the charts \((\tilde{U}, y)\) of \( G \) shows that the locally given vertical differential operators
\[
D^{i_1 \ldots i_r}: \sum_{s=0}^{l} \sum_{r=0}^{l} \frac{1}{(s-r)!} D^{i_1 \ldots i_r j_1 \ldots j_{s-r}} \frac{\partial^{s-r}}{\partial y^{j_1} \ldots \partial y^{j_{s-r}}}
\]
(6.39)
uniquely define global ones \( D^{i_1 \ldots i_r} \in \text{DiffOp}^l_{\text{ver}}(V \times G) \) with \( D^{i_1 \ldots i_r}|_{V \times \tilde{U}} = D^{i_1 \ldots i_r}_{\tilde{U}} \), thus yielding
\(\blacksquare\).

The two lemmas above now imply the applicability of Theorem 5.7.5.

Lemma 6.3.4
The filtered space \( \text{DiffOp}^\bullet(V \times G) \) satisfies the conditions of Theorem 5.7.5. Hence one has
\[
\text{HC}^\bullet_{\text{diff}}(V, \text{DiffOp}(V \times G)) \cong \bigcup_{l=0}^{\infty} \text{Hom}^\bullet_{\text{diff}}(X^\bullet, \text{DiffOp}^l(V \times G))
\] (6.40)
and
\[
\text{HH}^\bullet_{\text{diff}}(V, \text{DiffOp}(V \times G)) \cong H \left( \bigcup_{l=0}^{\infty} \text{Hom}_{A^e}(K^\bullet, \text{DiffOp}^l(V \times G)) \right)
\] (6.41)

Proof: Since \( \text{pr}_1^* : C^\infty(V) \rightarrow C^\infty(V \times G) \) is a continuous map with respect to the corresponding Fréchet topologies, Lemma 6.3.2 immediately implies for \( P = V \times G \) that the spaces \( \text{DiffOp}^l(V \times G) \) are topological \((C^\infty(V), C^\infty(V^\bullet))\)-bimodules with the structures corresponding to (6.2). Lemma 6.3.3 and the Leibniz rule show that the right \( C^\infty(V^\bullet) \)-module structures satisfy condition (5.27), so all \( \text{DiffOp}^l(V \times G) \) are \((C^\infty(V), C^\infty(V^\bullet))\)-bimodules of order \( l \in \mathbb{N}_0 \). Then, the stated isomorphisms follow.

So far, the problem of computing the Hochschild cohomology has only been paraphrased and reformulated for other complexes. But now, it is possible to compute the cohomology of \( \bigcup_{l=0}^{\infty} \text{Hom}_{A^e}(K^\bullet, \text{DiffOp}^l(V \times G)) \) explicitly.

It is a crucial point to recognize that the differential operators on \( C^\infty(V \times G) \) and thus the relevant complex \( \bigcup_{l=0}^{\infty} \text{Hom}_{A^e}(K^\bullet, \text{DiffOp}^l(V \times G)) \) have a particular grading. With Lemma 6.3.3...
it follows immediately that any differential operator \( D \in \text{DiffOp}^l(V \times G) \) can be written as a direct sum \( D = \sum_{r=0}^l D_r \) where the \( D_r \) are given by

\[
D_r = \frac{1}{r!} \sum_{\alpha \in \mathbb{N}_0^n} D^\alpha_{r} \frac{\partial^r}{\partial x_1 \ldots \partial x_r} = \sum_{\alpha \in \mathbb{N}_0^n} D^\alpha_{r} \frac{\partial^r}{\partial (x_1)^{\alpha_1} \ldots (x_n)^{\alpha_n}}
\]

(6.42)

with vertical operators \( D^r \) and \( D^\alpha_r \in \text{DiffOp}^r_{\text{ver}}(V \times G) \) satisfying \( D^\alpha_r(pr_1 a \cdot f) = pr_1^* a \cdot D^\alpha_r(f) \) for all \( a \in C^\infty(V) \) and \( f \in C^\infty(V \times G) \). The \( D_r \in \text{DiffOp}^l(V \times G) \) are obviously specified by the property that the order of differentiation with respect to the horizontal direction \( V \) is exactly given by \( r \). The subset of all such differential operators in \( \text{DiffOp}^l(V \times G) \) of the form (6.42) shall be denoted by \( \text{DiffOp}^l(V \times G) \). Then for all \( l \in \mathbb{N}_0 \) one has the decomposition

\[
\text{DiffOp}^l(V \times G) = \bigoplus_{r=0}^l \text{DiffOp}^l_r(V \times G),
\]

(6.43)

showing that \( \text{DiffOp}^l(V \times G) \) is a graded space. The decomposition (6.43) naturally transfers to \( \text{Hom}_{A^e}(K_\bullet, \text{DiffOp}^l(V \times G)) \) for all \( l \in \mathbb{N} \).

**Definition 6.3.5 (The degree of horizontal differentiation)**

By linear extension of

\[
\deg \psi = r\psi \quad \text{for all } \psi \in \text{Hom}_{A^e}(K_\bullet, \text{DiffOp}^l_r(V \times G))
\]

(6.45)

one obtains a map

\[
\deg : \bigcup_{l=0}^\infty \text{Hom}_{A^e}(K_\bullet, \text{DiffOp}^l(V \times G)) \rightarrow \bigcup_{l=0}^\infty \text{Hom}_{A^e}(K_\bullet, \text{DiffOp}^l(V \times G)),
\]

(6.46)

which, by the obvious reasons, is called the degree of horizontal differentiation.

With \( \text{Hom}_{A^e}(K_0, \text{DiffOp}^l_r(V \times G)) \cong \text{DiffOp}^l_r(V \times G) \) one obviously has a corresponding map \( \deg : \text{DiffOp}^\bullet(V \times G) \rightarrow \text{DiffOp}^\bullet(V \times G) \) defined by

\[
\deg D_r = r D_r \quad \text{for all } D_r \in \text{DiffOp}^l_r(V \times G),
\]

(6.47)

such that \( (\deg \psi)(\omega) = \deg(\psi(\omega)) \) for all \( \psi \in \text{Hom}_{A^e}(K_k, \text{DiffOp}^l(V \times G)) \) and \( \omega \in K_k \) with \( l, k \in \mathbb{N}_0 \).

The \((C^\infty(V), C^\infty(V))\)-bimodule structure of \( \text{DiffOp}^\bullet(V \times G) \) as in (6.2) induces a corresponding \( C^\infty(V \times V)\)-module structure according to Proposition 5.2.1. For the latter structure we can formulate the following simple but useful lemma.

**Lemma 6.3.6**

*Let \( x^i : V \rightarrow \mathbb{R} \) be the coordinate functions with respect to the canonical basis \( \{e_i\}_{i=1,...,n} \) of \( \mathbb{R}^n \) and let \( \xi^i = x^i \otimes 1 - 1 \otimes x^i \in C^\infty(V \times V) \) as in (5.41). Then, for all \( D \in \text{DiffOp}^\bullet(V \times G) \) one has*

\[
(\xi^i \cdot D) \circ \frac{\partial}{\partial x^j} - \xi^i \cdot (D \circ \frac{\partial}{\partial x^j}) = \delta^i_j D, \quad \text{for all } i, j = 1, \ldots, n
\]

(6.48)
by the linear extension of the maps

\[ \sum_{j=1}^{n} \left( ((-\xi^j) \cdot D) \circ \frac{\partial}{\partial x^j} \right) = \deg D. \]  

(6.49)

Moreover, one finds that for all \( r \leq l \in \mathbb{N}_0 \) and \( i = 1, \ldots, n \) the left multiplication \( l_{\xi^i} \) with \( \xi^i \) is a map

\[ l_{\xi^i} : \text{DiffOp}_{l}^{i}(V \times G) \longrightarrow \text{DiffOp}_{l-1}^{i}(V \times G). \]  

(6.50)

**Proof:** With the bimodule structure \([6.2]\) it is obvious that \( x^i \cdot (D \circ \frac{\partial}{\partial x^j}) = (x^i \cdot D) \circ \frac{\partial}{\partial x^j} \) and a simple computation shows \( (D \circ \frac{\partial}{\partial x^j}) \cdot x^i = (D \cdot x^i) \circ \frac{\partial}{\partial x^j} + \delta^j_{\xi^i} D \). Then, \([6.48]\) follows by \([5.21]\). Due to the linearity of both sides it is sufficient to show \([6.49]\) for \( D \in \text{DiffOp}_{l}^{i}(V \times G) \) as in \([6.42]\). Then, one first computes for all \( f \in C^\infty(V \times G) \)

\[
\left( \sum_{j=1}^{n} (D \cdot x^j) \circ \frac{\partial}{\partial x^j} \right)(f) = \sum_{|\alpha|=r} D^\alpha \left( \sum_{j=1}^{n} \prod_{l=1}^{n} \frac{\partial^{\alpha_l} \pr^l_x \cdot x^j \cdot \pr^l_x \cdot f}{\partial (x^j)^{\alpha_l}} \right) = \sum_{|\alpha|=r} D^\alpha \left( \sum_{j=1}^{n} \prod_{l=1}^{n} \frac{\partial^{\alpha_l} \pr^l_x \cdot x^j \cdot \pr^l_x \cdot f}{\partial (x^j)^{\alpha_l}} \right) + \sum_{|\alpha|=r} \prod_{j=1}^{n} \frac{\partial^{\alpha_j} f}{\partial (x^j)^{\alpha_j} \partial x^j} + \sum_{|\alpha|=r} \prod_{j=1}^{n} \frac{\partial^{\alpha_j} f}{\partial (x^j)^{\alpha_j}} \]  

(6.51)

This yields

\[
\sum_{j=1}^{n} \left( (D \cdot x^j) \circ \frac{\partial}{\partial x^j} - (x^j \cdot D) \circ \frac{\partial}{\partial x^j} \right) = \deg D, \]  

(6.52)

and \([6.49]\) follows again with \([5.21]\). The property \([6.50]\) is a simple consequence of the Leibniz rule and \([5.21]\). \( \blacksquare \)

After the above preparing considerations we now define the crucial maps which will lead to a homotopy equation.

**Definition 6.3.7**

We define the map

\[
\delta^{-1}_K : \bigcup_{l=0}^{\infty} \text{Hom}_{A^c}(K_*, \text{DiffOp}^{l}(C^\infty(V \times G))) \longrightarrow \bigcup_{l=0}^{\infty} \text{Hom}_{A^c}(K_{*-1}, \text{DiffOp}^{l}(C^\infty(V \times G))) \]  

(6.53)

by the linear extension of the maps

\[
(\delta^{-1}_K)^k \psi = \begin{cases} 
\frac{1}{k+r}(\delta^{-1}_K)^k \psi & \text{for } k \geq 1 \\
0 & \text{for } k = 0
\end{cases} \]  

(6.54)
for $\psi \in \text{Hom}_{\mathcal{A}^e}(K_k, \text{DiffOp}^l_{\mathcal{A}^e}(C^\infty(V \times G)))$ where the maps

$$(\delta_K^k)^{\sharp} : \bigcup_{l=0}^{\infty} \text{Hom}_{\mathcal{A}^e}(K_k, \text{DiffOp}^l_{\mathcal{A}^e}(C^\infty(V \times G))) \longrightarrow \bigcup_{l=0}^{\infty} \text{Hom}_{\mathcal{A}^e}(K_{k-1}, \text{DiffOp}^l_{\mathcal{A}^e}(C^\infty(V \times G)))$$

(6.55)

are defined by

$$(\delta_K^k)^{\sharp}\psi(e^{i_1} \land \cdots \land e^{i_k}) = -\sum_{j=1}^{n} \xi^j \cdot \psi(i_a(e_j)(e^{i_1} \land \cdots \land e^{i_{k-1}})) \circ \frac{\partial}{\partial x^j}$$

(6.56)

for all $i_s \in \{1, \ldots, n\}$ with the dual basis $\{e^i\}_{i=1,n}$ of the canonical basis $\{e_i\}_{i=1,\ldots,n}$ of $\mathbb{R}^n$.

With (6.40) for the case of the canonical basis of $\mathbb{R}^n$ the coboundary operators

$$\delta_K^k = (d_{k+1}^K)^{\sharp}$$

(6.57)

of the considered complex are given by

$$(\delta_K^k)^{\sharp}\psi(e^{i_1} \land \cdots \land e^{i_{k+1}}) = \sum_{j=1}^{n} \xi^j \cdot \psi(i_a(e_j)(e^{i_1} \land \cdots \land e^{i_{k+1}}))$$

(6.58)

for $\psi \in \bigcup_{l=0}^{\infty} \text{Hom}_{\mathcal{A}^e}(K_k, \text{DiffOp}^l_{\mathcal{A}^e}(C^\infty(V \times G)))$ and $i_s \in \{1, \ldots, n\}$ with $\xi^j$ as in (5.41).

Remark 6.3.8

Note that by definition for all $r \leq l \in \mathbb{N}_0$ and $k \geq 1$ one has

$$\delta_{K-1}^k : \text{Hom}_{\mathcal{A}^e}(K_k, \text{DiffOp}^l_{\mathcal{A}^e}(V \times G)) \longrightarrow \text{Hom}_{\mathcal{A}^e}(K_{k-1}, \text{DiffOp}^{l+1}_{\mathcal{A}^e}(V \times G)).$$

(6.59)

and, even for $k \geq 0$,

$$\delta_K^k : \text{Hom}_{\mathcal{A}^e}(K_k, \text{DiffOp}^l_{\mathcal{A}^e}(V \times G)) \longrightarrow \text{Hom}_{\mathcal{A}^e}(K_{k+1}, \text{DiffOp}^{l-1}_{\mathcal{A}^e}(V \times G))$$

(6.60)

due to (6.50). It is further remarkable that in contrast to $\delta_K$ the maps $\delta_K^k$ and $\delta_{K-1}^k$ are not $A^e$-linear.

Proposition 6.3.9 (The crucial homotopy)

The $\mathbb{K}$-linear map $\delta_{K-1}^k$ yields an explicit homotopy for the identity map of the considered complex $(\bigcup_{l=0}^{\infty} \text{Hom}_{\mathcal{A}^e}(K_\bullet, \text{DiffOp}^l_{\mathcal{A}^e}(C^\infty(V \times G))), \delta_K)$. This means that for $k \geq 1$

$$\delta_{K-1}^{k-1} \circ (\delta_K^k)^{\sharp} + (\delta_K^k)^{\sharp} \circ \delta_{K-1}^{k+1} = \text{id}.$$ 

(6.61)

Consequently, the corresponding cohomologies are trivial, this means

$$\text{H}^k\left(\bigcup_{l=0}^{\infty} \text{Hom}_{\mathcal{A}^e}(K, \text{DiffOp}^l_{\mathcal{A}^e}(V \times G))\right) = \{0\}$$

for $k \geq 1$.

(6.62)

Proof: With the formulas of Lemma 6.3.6 one computes for $k \geq 1$

$$\left(\delta_{K-1}^{k-1} \circ (\delta_K^k)^{\sharp} + (\delta_K^k)^{\sharp} \circ \delta_{K-1}^{k+1}\right)\psi(e^{i_1} \land \cdots \land e^{i_k})$$

$$= -\xi^1 \cdot \left(\psi(e^{i_1} \land \hat{i}_a(e_i)(e^{i_1} \land \cdots \land e^{i_k})) \circ \frac{\partial}{\partial x^j} \right) - (\xi^1 \cdot \psi(\hat{i}_a(e_i)(e^{i_1} \land \cdots \land e^{i_k})) \circ \frac{\partial}{\partial x^j})$$

$$= -\xi^1 \cdot \left(\psi(e^{i_1} \land \hat{i}_a(e_i)(e^{i_1} \land \cdots \land e^{i_k})) \circ \frac{\partial}{\partial x^j} \right) + (\xi^1 \cdot \psi(e^{i_1} \land \cdots \land e^{i_k})) \circ \frac{\partial}{\partial x^j}$$

$$+ (\xi^1 \cdot \psi(e^{i_1} \land \hat{i}_a(e_i)(e^{i_1} \land \cdots \land e^{i_k})) \circ \frac{\partial}{\partial x^j})$$

$$= \deg \psi(e^{i_1} \land \cdots \land e^{i_k}) + \psi(e^{i_1} \land \hat{i}_a(e_i)(e^{i_1} \land \cdots \land e^{i_k}))$$

$$= ((\deg + k\text{id})\psi)(e^{i_1} \land \cdots \land e^{i_k}).$$
This shows that
\[ \delta_K^{k-1} \circ (\delta_K^* + (\delta_K^*)^{k+1} \circ \delta_K^* = \deg + k \id \] (6.63)
for all \( k \in \mathbb{N} \). Due to Remark 6.3.8 and the definition of \( \delta^{-1} \) this yields (6.61). The last statement 6.62 is an immediate consequence.

Proposition 6.3.9 now already states that the local Hochschild cohomologies are trivial. According to the explanations of Proposition 4.4.7 this result extends to the global situation if all cocycles in a determined Hochschild module are images of cochains with globally bounded orders of differentiation. In the present case this is true since there exists an explicit homotopy map \( \delta^{-1} \) yielding the desired property.

**Theorem 6.3.10 (The local Hochschild cohomology \( \text{HH}^1_{\text{diff}}(V, \text{DiffOp}(V \times G)) \))**

Let \( V \subset \mathbb{R}^n \) be an open and convex subset and let \( G \) be a manifold. Then one has for all \( k \in \mathbb{N} \):

1. The \( k \)-th differential Hochschild cohomology with values in \( \text{DiffOp}(V \times G) \) is trivial
   \[ \text{HH}^k_{\text{diff}}(V, \text{DiffOp}(V \times G)) = \{0\} . \] (6.64)
2. There exists an explicit homotopy
   \[ \delta^{-1} \circ (\delta^{-1})^k + (\delta^{-1})^{k+1} \circ \delta^k = \id , \] (6.65)
   where the maps \((\delta^{-1})^k : \text{HC}^k_{\text{diff}}(V, \text{DiffOp}(V \times G)) \to \text{HC}^{k-1}_{\text{diff}}(V, \text{DiffOp}(V \times G))\) are given by
   \[ (\delta^{-1})^k = \Xi^{-1} \circ \left( (G_{k-1})^* \circ (\delta_R)^k \circ (F_k)^* + (s_{k-1})^* \right) \circ (\Xi^k)^{-1} . \] (6.66)
3. In particular, for any multiindex \( L = (l_1, \ldots, l_k) \in \mathbb{N}_0^k \) one has
   \[ (\delta^{-1})^k : \text{DiffOp}^L(V, \text{DiffOp}^L(V \times G)) \to \text{DiffOp}^{\tilde{L}}(V, \text{DiffOp}^{l+1}(V \times G)), \] (6.67)
   where the new multiindex \( \tilde{L} = (\tilde{l}_1, \ldots, \tilde{l}_{k-1}) \in \mathbb{N}_0^{k-1} \) is given by
   \[ \tilde{l}_i = \max\{(k-2)! + |L|, l + 2\} \leq (k-2)! + |L| + l + 2 \quad \text{for all } i = 1, \ldots, k-1. \] (6.68)

**Proof:** The proof of Equation (6.65) is a simple computation which makes use of (6.61), (5.81) and the properties of the involved functions. Then, Equation (6.64) is trivial. The third assertion 6.67 follows with Proposition 5.7.3 and Remark 6.3.8 by counting the orders of differentiation.

As stated in the beginning of the section all previous considerations can be reformulated in the \( G \)-invariant setting.

**Remark 6.3.11 (The \( G \)-invariant local cohomology \( \text{HH}^1_{\text{diff}}(V, \text{DiffOp}(V \times G)^G) \))**

If \( G \) is a Lie group all statements of this section are still true when replacing \( \text{DiffOp}(V \times G) \) by \( \text{DiffOp}(V \times G)^G \). In detail, this is the case because of the following reasons.

1. The assertions of the Lemmas 6.3.1 and 6.3.3 still hold for the \( G \)-invariant operators. This is a direct consequence of the Equations (6.39) and (6.37) due to the stated uniqueness.

2. As seen in the last part of Lemma 6.3.2 \( \text{DiffOp}^L(V \times G)^G \) is a closed subspace of \( \text{DiffOp}^L(V \times G) \) for all \( l \in \mathbb{N}_0 \). With \( \text{pr}^*: C^\infty(V) \to C^\infty(V \times G)^G \) this implies that \( \text{DiffOp}^*(V \times G)^G \) satisfies the conditions of Theorem 5.7.3. Thus, Lemma 6.3.4 can be reformulated.
iii.) The differential operators \( \frac{\partial}{\partial x^j} \in \text{DiffOp}^1(V \times G)^G \) for \( j = 1, \ldots, n \) which occur in the defining Equation (6.50) for the map \( \delta^{K^i}_K \) are \( G \)-invariant. Further, the decomposition (6.43) and the map deg are \( G \)-invariant. Thus, the derived map \( \delta^{K^i}_K \) satisfies

\[
\delta^{K^i}_K^{-1} : \bigcup_{l=0}^\infty \text{Hom}_{A^l}(K^i_\bullet, \text{DiffOp}^l(V \times G)^G) \rightarrow \bigcup_{l=0}^\infty \text{Hom}_{A^l}(K^i_{l-1}, \text{DiffOp}^l(V \times G)^G). \tag{6.69}
\]

Moreover, this implies that the maps \((\delta^{-1})^k \) in Theorem 6.3.10 have restrictions

\[
(\delta^{-1})^k : \text{DiffOp}^L(V, \text{DiffOp}^l(V \times G)^G) \rightarrow \text{DiffOp}^L(V, \text{DiffOp}^{l+1}(V \times G)^G). \tag{6.70}
\]

Thus, Theorem 6.3.10 is also true in the \( G \)-invariant case and one has

\[
\text{HH}^k_{\text{diff}}(V, \text{DiffOp}(V \times G)^G) = \{0\} \quad \text{for all } k \geq 1. \tag{6.71}
\]

6.4 Existence and uniqueness of the relevant deformations

In the last step of our computation we use the obtained local results and apply the Propositions 4.4.7 and 4.4.12. This is only possible due to the found explicit homotopy map \( \delta^{-1} \) which guarantees that all coboundaries are images of cochains with uniformly bounded degrees of differentiation. Altogether, we can reverse the steps of the Lemmas 6.2.1 and 6.2.2 and find the following central theorems.

**Theorem 6.4.1 (Hochschild cohomology for surjective submersions)**

Let \( q : E \rightarrow P \) be a vector bundle over a surjective submersion \( p : P \rightarrow M \). Then,

\[
\text{HH}^k_{\text{diff}}(M, \text{DiffOp}^\bullet(\Gamma^\infty(P, E))) = \begin{cases} \text{DiffOp}^\bullet_{\text{ver}}(\Gamma^\infty(P, E)) & \text{for } k = 0 \\ \{0\} & \text{for } k \geq 1. \end{cases} \tag{6.72}
\]

More specifically, every \( \phi \in \text{DiffOp}^L(M, \text{DiffOp}^l(\Gamma^\infty(P, E))) \) with \( L = (l_1, \ldots, l_k) \in \mathbb{N}_0^k \), \( k \geq 1 \), and \( \delta \phi = 0 \) is of the form

\[
\phi = \delta \Theta \tag{6.73}
\]

with \( \Theta \in \text{DiffOp}^{\tilde{L}}(M, \text{DiffOp}^{l+1}(\Gamma^\infty(P, E))) \) and \( \tilde{L} \) as in (6.68).

**Proof:** The proof is now an easy consequence of Proposition 4.4.7 and Theorem 6.3.10 using a partition of unity which is subordinate to the atlas of adapted charts \( \{U_i\}_{i \in I} \) as used in (6.28). ■

Analogously, one gets the following result for principal fibre bundles.

**Theorem 6.4.2 (Hochschild cohomology for principal fibre bundles)**

Let \( p : P \rightarrow M \) be a principal fibre bundle with structure Lie group \( G \) and let \( q : E \rightarrow P \) be an equivariant vector bundle. Then one has

\[
\text{HH}^k_{\text{diff}}(M, \text{DiffOp}^\bullet(\Gamma^\infty(P, E))^G) = \begin{cases} \text{DiffOp}^\bullet_{\text{ver}}(\Gamma^\infty(P, E))^G & \text{for } k = 0 \\ \{0\} & \text{for } k \geq 1. \end{cases} \tag{6.74}
\]

More specifically, every \( \phi \in \text{DiffOp}^L(M, \text{DiffOp}^l(\Gamma^\infty(P, E))^G) \) with \( L = (l_1, \ldots, l_k) \in \mathbb{N}_0^k \), \( k \geq 1 \), and \( \delta \phi = 0 \) is of the form

\[
\phi = \delta \Theta \tag{6.75}
\]

with \( \Theta \in \text{DiffOp}^{\tilde{L}}(M, \text{DiffOp}^{l+1}(\Gamma^\infty(P, E))^G) \) and \( \tilde{L} \) as in (6.68).
Due to these results and those of the Sections 2.4 and 2.5 we can now find the existence and uniqueness up to equivalence of the aspired deformations of right module structures. Of course we always consider differential and $G$-invariant differential deformations in the sense of the Definitions 2.3.1, 2.4.1, 3.2.1, 3.2.4 and 2.6.2.

Theorem 6.4.3 (Deformations on surjective submersions)
Let $q : E \rightarrow P$ be a vector bundle over a surjective submersion $p : P \rightarrow M$ with a Poisson manifold $M$ as basis. Further, let $\star$ be a star product on $M$. Then there always exists a differential deformation of the right module structure \((6.1)\) of the sections $\Gamma^\infty(P,E)$ which is unique up to equivalence.

Theorem 6.4.4 (Deformations on principal fibre bundles)
Let $q : E \rightarrow P$ be an equivariant vector bundle over a principal fibre bundle $p : P \rightarrow M$ with a Poisson manifold $M$ as basis. Further, let $\star$ be a star product on $M$. Then there always exists a $G$-invariant differential deformation of the right module structure \((6.1)\) of the sections $\Gamma^\infty(P,E)$ which is unique up to equivalence.

Remark 6.4.5 (Notation)
In analogy to $a \star b = \mu(a,b) = \sum_{r=0}^{\infty} \lambda^r \mu_r(a,b)$ we simply write
\[
s \star a = \rho(s,a) = \sum_{r=0}^{\infty} \lambda^r \rho_r(s,a) \tag{6.76}
\]
for the deformed right module structure. All structures of the new right module structure are simply denoted by
\[
(\Gamma^\infty(P,E)[[\lambda]], \star, (C^\infty(M)[[\lambda]], \star)) \tag{6.77}
\]

From the considerations of this chapter, in particular Section 6.2 it becomes clear that the special case of functions $C^\infty(P)$ on the total space $P$ is crucial for all deformations as above. This is the reason why these deformations are called deformation quantizations of the bundles themselves. Besides having clarified the occurring notions we have found the proofs for the Theorems 1.4.3 and 1.4.4.

Corollary 6.4.6 (Deformation quantization of surjective submersions)
Every surjective submersion over a Poisson manifold with a given differential star product admits a deformation quantization which is unique up to equivalence.

Corollary 6.4.7 (Deformation quantization of principal fibre bundles)
Every principal fibre bundle over a Poisson manifold with a given differential star product admits a deformation quantization which is unique up to equivalence.

Example 6.4.8 (Tangent and cotangent bundles)
For a surjective submersion $p : P \rightarrow M$ one can consider the tangent and cotangent bundles $TP \rightarrow P$ and $T^*P \rightarrow P$ as well as arbitrary tensor products thereof. Then the corresponding tensor fields have a deformed right module structure with respect to a star product $\star$ on $M$.

If $P$ is a principal fibre bundle one has a right action $R$ of the structure group $G$ on the tangent bundle $TP$ given by the tangent map
\[
R_g = Tr_g : TuP \rightarrow TuP \tag{6.78}
\]
and the induced left action $L_g = (Tr_g)^*$ on the cotangent bundle. Then it is obvious that the induced actions \((6.13)\) and \((6.14)\) on the sections are nothing but the usual pullbacks of vector fields and differential forms. Thus, the actions are algebra automorphisms with respect to the tensor product.
With respect to the applications in classical gauge theories we find the following important theorem where all results are summarized.

**Theorem 6.4.9 (Deformation quantization of horizontal forms)**

Let \( p : P \to M \) be a principal fibre bundle with structure Lie group \( G \) and right action \( r \). Further, let the base be a Poisson manifold \( M \) with a corresponding differential star product \( \star \).

Then there exists a right \((C^\infty(M)[[\lambda]], \star)\)-module structure \( \bullet \) of the horizontal forms

\[
\Gamma^\infty_{\text{hor}}(P, \wedge T^*P)[[\lambda]]
\]

with the following properties:

i.) The module structure \( \bullet \) is a differential deformation of the pointwise module, this means

\[
\alpha \bullet a = p^* a \cdot \alpha + \sum_{r=1}^{\infty} \lambda^r \rho_r(\alpha, a)
\]

for \( \alpha \in \Gamma^\infty_{\text{hor}}(P, \wedge T^*P)[[\lambda]] \) and \( a \in C^\infty(M)[[\lambda]] \) with \( \rho_r(\alpha, a) = \rho_r(a)(\alpha) \) for differential operators \( \rho_r \in \bigcup_{L,l \in \mathbb{N}_0} \text{DiffOp}^L(M, \text{DiffOp}^l(\Gamma^\infty_{\text{hor}}(P, \wedge T^*P))). \)

ii.) The module structure is \( G \)-invariant, this means that for all \( g \in G \)

\[
r^g_\star(\alpha \bullet a) = (r^g_\star \alpha) \bullet a
\]

iii.) The deformation \( \bullet \) is unique up to equivalence.

**Proof:** One has to show that the horizontal forms \( \Gamma^\infty_{\text{hor}}(P, \wedge T^*P) \) for all \( t \in \mathbb{N}_0 \) give rise to a subsheaf of substructures as in Remark 6.1.4. The defining property of such forms \( \alpha \) to vanish if one argument is vertical, this means \( i_a(V)\alpha = 0 \) if \( V \in \Gamma^\infty(VP) \), is not affected by the right module structure \( \bullet \). The sheaf structure is clear and the pullback \( r^g_\star \) of forms is in fact an action on the horizontal forms since \( T_p \circ T_r g = T_p \) implies that \( T_r g : VP \to VP \). Note that in this case for a local chart \((U, x)\) of \( M \) the tensor products of pullbacks \( p^* d x^i \) for \( i = 1, \ldots, n = \dim M \) build a \( C^\infty(p^{-1}(U)) \)-module basis of \( \Gamma^\infty_{\text{hor}}(p^{-1}(U), \wedge T^*p^{-1}(U)) \) which is obviously \( G \)-invariant. Application of Theorem 6.4.2 shows that

\[
\text{HH}^k_{\text{diff}}(M, \text{DiffOp}(\Gamma^\infty_{\text{hor}}(P, \wedge T^*P))^G) = \{0\}
\]

for all \( k \geq 1 \) and the assertion is again a consequence of the general results of Chapter 2.

**Remark 6.4.10**

With the given definitions it is obvious that in the present situations Proposition 2.4.8 can be applied. Thus, for all deformed right module structures with respect to a star product \( \star \) the unit function \( 1 \in C^\infty(M) \) acts by the identity. This means that

\[
s \bullet 1 = s \quad \text{and} \quad f \bullet 1 = f
\]

for all \( s \in \Gamma^\infty(P, E)[[\lambda]] \) and \( f \in C^\infty(P)[[\lambda]] \) as above.
Chapter 7
The commutants of the deformations

For the further investigations of the deformations discussed in Chapter 6, in particular the computation of the commutants within the differential operators, we will make use of the well-known symbol calculus for differential operators. The basic idea of symbol calculus comes from the general transition from a filtered vector space \( V = \bigcup_{l=0}^{\infty} V_l \) to the corresponding graded vector space \( W = \bigoplus_{l=0}^{\infty} W_l \) by setting \( V_0 = W_0 \) and \( W_l = V_l / V_{l-1} \) for \( l \in \mathbb{N} \). For the filtered spaces of differential operators this procedure yields the spaces of the so-called symbols. In many important geometric examples these symbols are isomorphic to other geometric structures, mostly tensor fields. Then, many crucial properties of a differential operator in fact can be expressed by according properties of the corresponding geometric structures. For many concrete problems and investigations this is a very useful approach.

In the first section of this short chapter we repeat the basic concept for differential operators \( \text{DiffOp}^\bullet(M) \) on the smooth functions on a manifold \( M \). The presentation and discussion of the relevant structures, in particular the symmetrized covariant derivative, result in the observation that every differential operator can be identified with a unique series of symmetric multivector fields. Basically, these considerations are found in [123, Sect. 5.4.1] or [102, Anhang B]. Further information on symbol calculus can be found in [107]. For our purposes we formulate the generalized assertions for the differential operators \( \text{DiffOp}^\bullet(M, C^\infty(P)) \) of the functions on the base manifold \( M \) and the total space \( P \) of a surjective submersion \( p : P \to M \) and for the differential operators \( \text{DiffOp}^\bullet(\Gamma^\infty(M, E)) \) of the sections of a vector bundle \( q : E \to M \). Section 7.2 is dedicated to the investigation of the crucial symmetrized covariant derivatives. It is shown that in the situations of interest there exist adapted covariant derivatives such that the isomorphisms established by symbol calculus preserve further information. As a first application of these observations it is shown in Section 7.3 that there always exist deformation quantizations that respect a simple algebraic property of the pullback \( p^* : C^\infty(M) \to C^\infty(P) \). In combination with the results of Section 2.3 the developed symbol calculus finally allows a detailed investigation of the commutants of the deformed structures which will be performed in Section 7.4.

7.1 Symbol calculus for differential operators

The starting point of the considerations is the already mentioned fact that with respect to a chart \( (U, x) \) of a manifold \( M \) any differential operator \( D \in \text{DiffOp}^l(M) \) of degree \( l \in \mathbb{N}_0 \) on the functions of \( M \) has the local form

\[
D|_U = \sum_{r=0}^{l} \frac{1}{r!} D_{ij}^{r} \cdots \partial_{x^{i_1}} \cdots \partial_{x^{i_r}} \partial^r,
\]

(7.1)
with uniquely defined functions $D_{r}^{i_{1} \ldots i_{r}} \in C^\infty(U)$ for $r = 0, \ldots, l$ which are symmetric in the indices $i_{1}, \ldots, i_{r}$. Conversely, if a $\mathbb{K}$-linear map $D : C^\infty(M) \to C^\infty(M)$ has the local form \ref{eq:symmetry}, for an atlas of $M$ it is a differential operator $D \in \text{DiffOp}^{l}_{r}(M)$. Due to the transformation behaviour under a change of charts it follows that these functions for $r = l$ define a global symmetric tensor field $\sigma_{l}(D) \in \Gamma^\infty(M, S^{l}TM)$, the so-called principal symbol of $D$. This is defined by its restrictions

$$\sigma_{l}(D)|_{U} = \frac{1}{l!} D_{U}^{i_{1} \ldots i_{l}} \frac{\partial}{\partial x^{i_{1}}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_{l}}}$$

\hfill (7.2)

to the chart domains of a corresponding atlas of $M$ where $\sigma$ is the symmetric tensor product which for every $\mathbb{K}$-vector space $V$ is defined by $v_1 \wedge \cdots \wedge v_l = \sum_{\tau \in S_{l}} v_{\tau(1)} \otimes \cdots \otimes v_{\tau(l)}$. Of course, it is $\sigma_{l}(D) = 0$ if and only if $D \in \text{DiffOp}^{l-1}(M)$.

For the further considerations one has to choose a covariant derivative $\nabla^{M} = \nabla^{TM}$ on $TM$, this means a $\mathbb{K}$-linear map $\nabla^{M} : \Gamma^\infty(M, TM) \times \Gamma^\infty(M, TM) \to \Gamma^\infty(M, TM)$ with the properties $\nabla_{a}X = a \nabla_{X}Y$ and $\nabla_{X}(aY) = a \nabla_{X}Y + X(a)Y$ for all $a \in C^\infty(M)$ and $X, Y \in \Gamma^\infty(M, TM)$. Given any symmetric $l$-form $\gamma \in \Gamma^\infty(M, S^{l}TM)$ the symmetrized covariant derivative $D_{M} \gamma \in \Gamma^\infty(M, S^{l+1}T^{*}M)$ induced by $\nabla^{M}$ is defined by

$$(D_{M} \gamma)(X_{1}, \ldots, X_{l+1}) = \sum_{s=1}^{l+1} (\nabla_{X_{s}}^{M} \gamma)(X_{1}, \ldots, \hat{X}_{s}, \ldots, X_{l+1}),$$

\hfill (7.3)

where the dual covariant derivative and its extension to the symmetric tensor product are again denoted by $\nabla^{M}$. The notation $\hat{X}_{s}$ indicates that the argument $X_{s}$ is omitted. For $l = 0$, this means for $a \in C^\infty(M)$, one has $D_{M}a = da$. Due to the defining properties of a covariant derivative it is clear that $\nabla^{M}$ is a differential operator $\nabla^{M} \in \text{DiffOp}^{(l,1)}(\Gamma^\infty(M, TM), \Gamma^\infty(M, TM); \Gamma^\infty(M, TM))$. Likewise, it is easy to see that the symmetrized covariant derivative is a differential operator $D_{M} \in \text{DiffOp}^{1}(\Gamma^\infty(M, S^{l}TM); \Gamma^\infty(M, S^{l+1}T^{*}M))$ for all $l \in \mathbb{N}_{0}$ and thus can be restricted to open subsets $U \subseteq M$. The local expression in a chart $(U, x)$ of $M$ turns out to be

$$D_{M}|_{U} = dx^{i} \wedge \nabla_{\frac{\partial}{\partial x^{i}}}.$$ 

\hfill (7.4)

Thus, $D_{M}$ is a derivation of degree one of the symmetric algebra $(\bigoplus_{l=0}^{\infty} \Gamma^\infty(M, S^{l}T^{*}M), \wedge)$. Note that the local expression \ref{eq:symmetry} may also serve as a definition for $D_{M}$.

Defining $D_{M}^{(l)} = \frac{1}{l!} D_{M}^{l}$ for $l \in \mathbb{N}_{0}$ a simple induction shows that for $a \in C^\infty(M)$

$$\left(D_{M}^{(l)}a\right)|_{U} = \frac{1}{l!} \left( \frac{\partial a}{\partial x_{1}^{i_{1}} \cdots \partial x_{l}^{i_{l}}} + \Gamma_{1 \ldots l}(a|_{U}) \right) dx^{i_{1}} \wedge \cdots \wedge dx^{i_{l}},$$

\hfill (7.5)

where $\Gamma_{1 \ldots l}(a|_{U})$ depends linearly on $a|_{U}$ and its partial derivatives at most up to order $l - 1$. Using the derivation property of $D_{M}$ another simple induction yields

$$D_{M}^{(l)}(a \cdot b) = \sum_{r=0}^{l} (D_{M}^{(r)}a) \wedge (D_{M}^{(l-r)}b)$$

\hfill (7.6)

for all $a, b \in C^\infty(M)$. Since a tensor field $T_{i} \in \Gamma^\infty(M, S^{l}TM)$ can locally be written as

$$T_{i}|_{U} = \frac{1}{l!} T^{i_{1} \ldots i_{l}} \frac{\partial}{\partial x^{i_{1}}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_{l}}}$$

\hfill (7.7)

with unique functions $T^{i_{1} \ldots i_{l}} \in C^\infty(U)$ one can thus define a differential operator $DT_{i} \in \text{DiffOp}(M)$ by

$$DT_{i}a = \frac{1}{l!} \left( T_{i} \cdot D_{M}^{(l)}a \right),$$

\hfill (7.8)
where \( \langle \cdot, \cdot \rangle \) denotes the natural pairing of symmetric multivector fields with symmetric differential forms defined by the local expressions, \( \langle T_i, \gamma \rangle |_U = T_i|_U (d x^{i_1}, \ldots, d x^{i_l}) \gamma|_U \left( \frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_l}} \right) \). Equation (7.8) in fact defines a differential operator since one locally has

\[
(D|_U a)|_U = \frac{1}{l!} \left\{ \frac{1}{l!} T^{i_1 \ldots i_l} \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_l}} \cdot \frac{1}{l!} \left( \frac{\partial^l a|_U}{\partial x^{i_1} \cdots \partial x^{i_l}} + \Gamma_{i_1 \ldots i_l}(a|_U) \right) d x^{i_1} \cdots d x^{i_l} \right\},
\]

The corresponding symbol map \( \sigma \), defined by

\[
\sigma(D) = T_0 + \cdots + T_l \in \bigoplus_{s=0}^l \Gamma^\infty(M, S^sTM)
\]

then is a vector space isomorphism

\[
\sigma : \text{DiffOp}^\bullet(M) \rightarrow \Gamma^\infty(M, S^\bullet TM) = \bigoplus_{l=0}^\infty \Gamma^\infty(M, S^lTM).
\]

If \( \mathfrak{p} : P \rightarrow M \) is a surjective submersion one can not only consider the above explained symbol calculus on the base and on the total space separately but also a combined version. The following proposition shows that a covariant derivative \( \nabla^M \) on \( TM \) and an additional choice of a connection \( TP = VP \oplus HP \) as in Appendix A give rise to a symbol calculus for the differential operators \( \text{DiffOp}^\bullet(M, C^\infty(P)) \) where \( C^\infty(P) \) is equipped with the obvious \( C^\infty(M) \)-module structure which is induced by the pullback \( \mathfrak{p}^* \).

**Proposition 7.1.2 (Symbol calculus for \( \text{DiffOp}^\bullet(M, C^\infty(P)) \))**

Let \( \nabla^M \) be a covariant derivative on a manifold \( M \). Then, every differential operator \( D \in \text{DiffOp}^\bullet(M, C^\infty(P)) \) of order \( l \) can be identified with a unique series \( T_0, \ldots, T_l \) of symmetric multivector fields \( T_j \in \Gamma^\infty(P, S^jHP) \), \( j = 0, \ldots, l \), yielding

\[
D = \sum_{j=0}^l D_{T_j}, \quad (7.9)
\]

where \( T_0 = D(1) \) and \( T_l = \sigma_l(D) \) are independent of the choice of \( \nabla^M \). The corresponding symbol map \( \sigma \), defined by

\[
\sigma(D) = T_0 + \cdots + T_l \in \bigoplus_{s=0}^l \Gamma^\infty(P, S^sTM)
\]

then is a vector space isomorphism

\[
\sigma : \text{DiffOp}^\bullet(M, C^\infty(P)) \rightarrow \Gamma^\infty(M, S^\bullet TM) = \bigoplus_{l=0}^\infty \Gamma^\infty(M, S^lTM).
\]

**Proof:** The proof is a straightforward generalization of the considerations above. As a special case of the results of Lemma 6.1.6 and Remark 6.1.7 one finds that every differential operator \( D \in \text{DiffOp}^\bullet(M, C^\infty(P)) \) has the local form

\[
(D|_{\mathfrak{p}^{-1}(U)}(a) = \sum_{r=0}^l D^{i_1 \ldots i_r}_{\mathfrak{p}^{-1}(U)} \cdot \mathfrak{p}^* \left( \frac{\partial^r a}{\partial x^{i_1} \cdots \partial x^{i_r}} \right), \quad (7.13)
\]
for all \( a \in C^\infty(U) \) where \((U, x)\) is a local chart of \( M \) and where \( D^{i_1 \ldots i_r}_{p^{-1}(U)} \in C^\infty(p^{-1}(U)) \). For \( r = l \) these functions define a global horizontal tensor field \( \sigma_l(D) \in \Gamma^\infty(P, S^lT^*M) \) via

\[
\sigma_l(D)|_{p^{-1}(U)} = \frac{1}{l!} T^{i_1 \ldots i_l}_{p^{-1}(U)} \left( \frac{\partial}{\partial x^{i_1}} \right)^h \cdots \left( \frac{\partial}{\partial x^{i_l}} \right)^h.
\]

(7.14)

Locally, the horizontal lifts \( \{ \left( \frac{\partial}{\partial x^{i_j}} \right)^h \}_{j=1, \ldots, n} \) are a module basis of \( \Gamma^\infty(P, \mathcal{H}P) \) and the dual basis is given by \( \{ p^*d x^i \}_{i=1, \ldots, n} \). This and (7.3) imply that every tensor field \( T_j \in \Gamma^\infty(P, S^jT^*HP) \) gives rise to a differential operator \( D_{T_j} \in \text{DiffOp}^j(M, C^\infty(P)) \) by the natural pairing (7.12). Then, \( \sigma_j(D_{T_j}) = T_j \) leads to \( D - D_{\sigma_l(D)} \in \text{DiffOp}^{l-1}(M, C^\infty(P)) \) and a simple induction yields the assertion.

Besides the generalization made in Proposition 7.1.2 we will need another one for the differential operators \( \text{DiffOp}^*(\Gamma^\infty(M, E)) \) of the sections of a vector bundle \( p : E \to M \). The adoptions are the following. One now always considers charts \((U, x)\) of \( M \) such that \( E|_U \) is trivial and \( \Gamma^\infty(U, E|_U) \) has a local module basis \( \{ e^i \}_{i=1, \ldots, k} \subseteq \Gamma^\infty(U, E|_U) \). Then, every differential operator \( D \in \text{DiffOp}^i(\Gamma^\infty(M, E)) \) has the local form

\[
D_U(s) = \sum_{r=0}^l \frac{1}{r!} \frac{\partial^r s}{\partial x^{i_1} \cdots \partial x^{i_r}} D^{i_1 \ldots i_r}_{U|_i}.
\]

(7.15)

for all local sections \( s = s^i e_i \in \Gamma^\infty(U, E|_U) \) and where \( D^{i_1 \ldots i_r}_{U|_i} \in \Gamma^\infty(U, E|_U) \). This induces a principal symbol \( \sigma_l(D) \in \Gamma^\infty(M, S^lTM \otimes E^* \otimes E) \) by

\[
\sigma_l(D)|_{U} = \frac{1}{l!} \left( \frac{\partial}{\partial x^{i_1}} \right)^h \cdots \left( \frac{\partial}{\partial x^{i_l}} \right)^h .
\]

(7.16)

Note that \( E^* \otimes E \) is isomorphic to the endomorphism bundle \( \text{End}(E) \). With \( \nabla^M \) as before and an additional covariant derivative \( \nabla^E : \Gamma^\infty(M, TM) \times \Gamma^\infty(M, E) \to \Gamma^\infty(M, E) \) on \( E \) one gets a new one on the tensor product \( TM \otimes E \), again denoted by \( \nabla^E \). The symmetrized version \( D_E \) thereof is defined in analogy to (7.3) and (7.4). But now it is a \( (\Gamma^\infty(M, S^*T^*M), \vee) \)-module derivation of \( \Gamma^\infty(M, S^*T^*M \otimes E) \) satisfying

\[
D_E(\alpha \vee \beta \otimes s) = (D_M \alpha) \vee (\beta \otimes s) + \alpha \vee D_E(\beta \otimes s) = (D_M \alpha) \vee (\beta \otimes s) + \alpha \vee (D_M \beta) \otimes s + \alpha \vee \beta \otimes D_E s
\]

(7.17)

for all \( \alpha, \beta \in \Gamma^\infty(M, S^*T^*M) \) and \( s \in \Gamma^\infty(M, E) \). The analogue of (7.3) now reads

\[
(D^{[0]} s)|_{U} = \frac{1}{l!} \left( \frac{\partial^l s}{\partial x^{i_1} \cdots \partial x^{i_l}} + \Gamma^{i_1 \ldots i_l}_{i_1 \ldots i_l}(s|_{U}) \right) d x^{i_1} \cdots d x^{i_l} \otimes e_i.
\]

(7.18)

With the local form

\[
T_l|_{U} = \frac{1}{l!} T^{j_1 \ldots j_l}_{i_1 \ldots i_l} \left( \frac{\partial}{\partial x^{i_1}} \right)^{s} \cdots \left( \frac{\partial}{\partial x^{i_l}} \right)^{s} e_i
\]

(7.19)

of any tensor field \( T_l \in \Gamma^\infty(M, S^lTM \otimes E^* \otimes E) \) and analogous considerations to the previous cases one finds the following extended version of symbol calculus.

**Proposition 7.1.3 (Symbol calculus for \( \text{DiffOp}^*(\Gamma^\infty(M, E)) \))**

Let \( p : E \to M \) be a vector bundle and let \( \nabla^M \) and \( \nabla^E \) be covariant derivatives on \( M \) and \( E \), respectively. Then, every differential operator \( D \in \text{DiffOp}^j(\Gamma^\infty(M, E)) \) of order \( l \) can be identified with a unique series \( T_0, \ldots, T_l \) of tensor fields \( T_j \in \Gamma^\infty(M, S^jTM \otimes E^* \otimes E) \cong \Gamma^\infty(M, S^jTM \otimes \text{End}(E)), j = 0, \ldots, l \), such that \( D = \sum_{j=0}^l T_j \), and where \( T_l = \sigma_l(D) \) is independent of the choices of \( \nabla^M \) and \( \nabla^E \). Thus one has a vector space isomorphism

\[
\sigma : \text{DiffOp}^*(\Gamma^\infty(M, E)) \to \Gamma^\infty(M, S^*T^*M \otimes \text{End}(E)).
\]

(7.20)
7.2 Covariant derivatives on surjective submersions and principal fibre bundles

As seen above, the choice of covariant derivatives is crucial for the symbol calculus of differential operators. In the following we prove the existence of covariant derivatives with particular properties and discuss some resulting consequences which will be used in the later applications. The basic idea of the following lemma comes from the related assertions in [13, Prop. 4.3] and [126, Satz 2.3.20].

**Lemma 7.2.1 (Adapted covariant derivative)**

Let \( p : P \to M \) be a surjective submersion and \( TP = VP \oplus HP \) be a connection. Further, let \( \nabla^M \) be a torsion-free covariant derivative on \( TM \). Then there always exists a covariant derivative \( \nabla^P \) on \( TP \) with the following properties.

i.) \( \nabla^P \) is torsion-free, this means

\[
\nabla^P_Z W - \nabla^P_W Z = [Z, W] \tag{7.21}
\]

for all \( Z, W \in \Gamma^\infty(P, TP) \).

ii.) \( \nabla^P \) respects the vertical bundle, this means

\[
\nabla^P_Z V \in \Gamma^\infty(P, VP) \quad \text{for all } V \in \Gamma^\infty(P, VP), Z \in \Gamma^\infty(P, TP). \tag{7.22}
\]

iii.) \( \nabla^P \) and \( \nabla^M \) satisfy

\[
T_p \circ \nabla^P_X h Y = (\nabla^M_X Y) \circ p \tag{7.23}
\]

for all \( X, Y \in \Gamma^\infty(M, TM) \).

If \( P \) is a principal fibre bundle with structure group \( G \) and \( TP = VP \oplus HP \) is a principal connection, this means \( Tr_u H_u P = H_u g P \) for all \( u \in P \) and \( g \in G \), it is possible to achieve the following additional property.

iv.) \( \nabla^P \) is \( G \)-invariant, this means

\[
t_g^* \nabla^P_Z W = \nabla^P_{t_g^* Z} t_g^* W \tag{7.24}
\]

for all \( Z, W \in \Gamma^\infty(P, TP) \) and \( g \in G \).

**Proof:** The vertical bundle \( VP \) is integrable, this means \( [V, W] \in \Gamma^\infty(P, VP) \) for all \( V, W \in \Gamma^\infty(P, VP) \). Due to this fact it is possible to choose a torsion-free covariant derivative \( \nabla^{VP} : \Gamma^\infty(P, VP) \times \Gamma^\infty(P, VP) \to \Gamma^\infty(P, VP) \). This can now be extended to a covariant derivative \( \nabla^P \) on \( TP \). Due to the Leibniz rule it is possible to define \( \nabla^P \) by the values on vertical and horizontal vector fields. Of course one defines

\[
\nabla^P_V W = \nabla^{VP}_V W \tag{7.25}
\]

for \( V, W \in \Gamma^\infty(P, VP) \). Using the connection in the form of the corresponding projection \( \mathcal{P} : TP \to VP \) onto the vertical space one further sets

\[
\nabla^P_V H = (\text{id} - \mathcal{P})[V, H] \quad \text{and} \quad \nabla^P_H V = \mathcal{P}[H, V] \tag{7.26}
\]

where \( H \in \Gamma^\infty(P, HP) \) and \( V \in \Gamma^\infty(P, VP) \). In addition, this definition can be made locally by using horizontal lifts instead of arbitrary horizontal vectors since there exist local module bases of \( \Gamma^\infty(P, HP) \) consisting of horizontal lifts. The well-known facts concerning related vector fields yielding \( [A, B] \) then show that

\[
\nabla^P_V W = \nabla^{VP}_V W, \quad \nabla^P_V X^h = 0, \quad \text{and} \quad \nabla^P_X h V = [X^h, V]. \tag{7.27}
\]
Note that these equations also yield consistent definitions since \((aX)^h = (p^*a)X^h\) and \(V(p^*a) = 0\) for all \(a \in C^\infty(M)\). For the last case of horizontal lifts we now use \(\nabla^M\) and set
\[
\nabla^P_{X^h}Y^h = \left(\nabla^M_{X^h}Y^h\right) + \frac{1}{2}\left(\left[X^h, Y^h\right] - [X, Y]^h\right). \tag{7.28}
\]
for all \(X, Y \in \Gamma^\infty(M, TM)\). A simple calculation shows that the so defined \(\nabla^P\) is torsion-free and it is obvious that \(\nabla^P\) respects the vertical bundle. With \((A.17)\) one finally verifies equation \((7.28)\). For principal fibre bundles one simply chooses a covariant derivative \(\nabla^{VP}\) that additionally is \(G\)-invariant. Since \(r^*_g\Gamma\infty(P, VP) = \Gamma\infty(P, VP)\) and with \(r^*_gX^h = X^h\) the above defined covariant derivative then is easily verified to be \(G\)-invariant. Confer \cite{126} Satz 2.3.20 for an explicit construction of all this.

**Remark 7.2.2**

i.) Since \(\nabla^P\) as in Lemma 7.2.1 is torsion-free and respects the vertical bundle the covariant derivative of a horizontal lift in vertical direction is vertical, this means
\[
\nabla^P_{V^h}X^h \in \Gamma^\infty(P, VP) \quad \text{for all } V \in \Gamma^\infty(P, VP), X \in \Gamma^\infty(M, TM).
\tag{7.29}
\]
This is obvious with \(\nabla^P_{V^h}X^h = \nabla^P_{X^h}V + [V, X^h]\) and \((A.17)\).

ii.) Note that a \(G\)-invariant covariant derivative \(\nabla^P\) on \(TP\) already defines a covariant derivative \(\nabla^M\) by \((7.23)\). If \(\nabla^P\) is additionally torsion-free the same is true for \(\nabla^M\).

**Lemma 7.2.3**

Let \(\nabla^M\) and \(\nabla^P\) be torsion-free covariant derivatives as in Lemma 7.2.1. Then, for all \(l \in \mathbb{N}\) the corresponding symmetrized covariant derivatives \(D_P^{(l)}\) and \(D_M^{(l)}\) are related by
\[
D_P^{(l)} \circ p^* = p^* \circ D_M^{(l)}.
\tag{7.30}
\]
**Proof:** One has to show that \(D_P(p^*\gamma) = p^*(D_M\gamma)\) for all forms \(\gamma \in \Gamma^\infty(M, S^lT^*M)\). This can be done by evaluation on arbitrary points \(u \in P\) and vectors \(Z_1(u), \ldots, Z_{l+1}(u) \in T_uP\) which of course can be seen as values of vector fields \(Z_1, \ldots, Z_{l+1} \in \Gamma^\infty(P, TP)\). Due to the defining equation \((7.3)\), one has to show the equality of
\[
(D_P(p^*\gamma))_u(Z_1(u), \ldots, Z_{l+1}(u)) = \sum_{s=1}^{l+1} \left( Z_s(u) \left( (p^*\gamma)(Z_1, \ldots, \hat{s} , \ldots, Z_{l+1}) \right) - \sum_{i \neq s} (p^*\gamma)(Z_1, \ldots, \nabla^P_{Z_s}Z_i, \ldots, \hat{s} , \ldots, Z_{l+1})(u) \right)
\tag{7.31}
\]
and
\[
(p^*(D_M\gamma))_u(Z_1(u), \ldots, Z_{l+1}(u)) = (D_M\gamma)_{p(u)}(T_uZ_1(u), \ldots, T_uZ_{l+1}(u)). \tag{7.32}
\]
This can be done by a case differentiation since it is clear with the connection \(TP = VP \oplus HP\) that the relevant vectors can be seen as the direct sum \(Z_i(u) = V_i(u) + X_i^h(u)\) of values of vertical vector fields \(V_i \in \Gamma^\infty(P, VP)\) and horizontal lifts \(X_i^h \in \Gamma^\infty(P, HP)\) of vector fields \(X_i \in \Gamma^\infty(M, TM)\). Since \(\nabla^P\) respects the vertical bundle and with \((7.29)\) it follows that both \((7.31)\) and \((7.32)\) are zero if at least one vector field is vertical and all others are horizontal lifts. Finally, let all vector fields be horizontal lifts. Then, both expressions turn out to be
\[
\sum_{s=1}^{l+1} \left( X_s(\gamma(X_1, \ldots, \hat{s} , \ldots, X_{l+1})) - \sum_{i \neq s} \gamma(X_1, \ldots, \nabla^M_{X_s}X_i, \ldots, \hat{s} , \ldots, X_{l+1}) \right)(p(u)),
\]
since \( X_a^p(p^*a) = p^*(X_a(a)) \) for all \( a \in C^\infty(M) \).

For a principal fibre bundle the \( G \)-invariance of \( \nabla^P \) induces the \( G \)-invariance of the corresponding symmetrized covariant derivative.

**Lemma 7.2.4**

Let \( p : P \to M \) be a principal fibre bundle and let \( \nabla^P \) be a \( G \)-invariant covariant derivative on \( TP \). Then, for all \( l \in \mathbb{N}_0 \) the corresponding symmetrized covariant derivative satisfies

\[
D_p^{(l)} \circ r_g = r_g \circ D_p^{(l)}
\]

for all \( g \in G \).

**Proof:** With the compatibility of the pullbacks \( r_g^* \) and the \( G \)-invariance of \( \nabla^P \) one computes with (7.33) for all \( \gamma \in \Gamma^\infty(P, S^l T^*P) \) and vector fields \( Z_1, \ldots, Z_{l+1} \in \Gamma^\infty(P, TP) \)

\[
(D_p(r_g^*(\gamma)))(r_g^*Z_1, \ldots, r_g^*Z_{l+1})
\]

\[
= \sum_{s=1}^{l+1} (r_g^*Z_s)(r_g^*(r_g^*Z_1, \ldots, r_g^*Z_{s-1}, Z_s, r_g^*Z_{s+1}, \ldots, r_g^*Z_{l+1})) - \sum_{i \neq s} r_g^*\gamma(r_g^*Z_1, \ldots, \nabla_{r_g^*Z_s}r_g^*Z_{s+1}, \ldots, r_g^*Z_{l+1})
\]

\[
= \sum_{s=1}^{l+1} (r_g^*Z_s)(\gamma(Z_1, \ldots, r_g^*Z_{s-1}, \ldots, Z_{l+1})) - \sum_{i \neq s} \gamma(Z_1, \ldots, \nabla_{r_g^*Z_s}r_g^*Z_{s+1}, \ldots, Z_{l+1})
\]

\[
= (r_g^*(D_p\gamma))(r_g^*Z_1, \ldots, r_g^*Z_{l+1}).
\]

This shows (7.33).

The above result concerning \( G \)-invariance has a useful generalization with respect to the covariant derivatives occurring in Proposition 7.1.3. For this purpose we need the following lemma.

**Lemma 7.2.5** *(G-invariant covariant derivatives for equivariant vector bundles)*

Every equivariant vector bundle \( q : E \to P \) over a principal fibre bundle \( p : P \to M \) with structure group \( G \) admits a \( G \)-invariant covariant derivative \( \nabla^E : \Gamma^\infty(P, TP) \times \Gamma^\infty(P, E) \to \Gamma^\infty(P, E) \), this means with

\[
g \triangleright \nabla^E_Z s = \nabla^E_{r_g^*Z} (g \triangleright s)
\]

for all \( Z \in \Gamma^\infty(P, TP) \), \( s \in \Gamma^\infty(P, E) \), and \( g \in G \).

**Proof:** The assertion is nothing but a special case of [62, Cor. B.38] stating that every equivariant vector bundle with a proper group action on the base manifold admits an invariant connection. In our case the principal right action clearly satisfies the required condition. 

**Lemma 7.2.6**

Let \( q : E \to P \) be an equivariant vector bundle over the principal fibre bundle \( p : P \to M \) as in Section 6.1.2. Further, let \( \nabla^E \) and \( \nabla^P \) be \( G \)-invariant covariant derivatives on \( E \) and \( TP \), respectively. Then, the induced covariant derivative \( \nabla^E \) on \( TP \otimes E \) is \( G \)-invariant with respect to the representation on the sections \( \alpha \otimes s \in \Gamma^\infty(P, S^l T^*P \otimes E) \) given by

\[
g \triangleright (\alpha \otimes s) = (r_g^*\alpha) \otimes (g \triangleright s).
\]

Moreover, the same is true for the corresponding symmetrized covariant derivative, this means one has

\[
D^{(l)}_E(g \triangleright s) = g \triangleright (D^{(l)}_E s)
\]

for all \( l \in \mathbb{N}_0 \), \( s \in \Gamma^\infty(E) \), and \( g \in G \).
Proof: The proof is an analogous computation to the one in the proof of Lemma 7.2.4 making use of the involved definitions. In particular, one needs the fact that for \( \gamma \in \Gamma^\infty(P, S^j T^* P \otimes E) \) and \( Z, Z_1, \ldots, Z_l \in \Gamma^\infty(P, TP) \) one has

\[
(\nabla^E_Z \gamma)(Z_1, \ldots, Z_l) = \nabla^E_Z(\gamma(Z_1, \ldots, Z_l)) - \sum_{i=1}^l \gamma(Z_1, \ldots, \nabla^E_Z Z_i, \ldots, Z_l) \tag{7.37}
\]

and

\[
g \triangleright (\gamma(Z_1, \ldots, Z_l)) = (g \triangleright \gamma)(r_g^* Z_1, \ldots, r_g^* Z_l). \tag{7.38}
\]

The above results have the following important consequence.

Proposition 7.2.7

Let \( p : P \rightarrow M \) be a surjective submersion and let \( \nabla^M \) and \( \nabla^P \) be torsion-free covariant derivatives on \( TM \), and \( TP \) respectively, as in Lemma 7.2.4. Further, let \( \nabla^E \) be an arbitrary covariant derivative on the vector bundle \( q : E \rightarrow P \). Then, the corresponding symbol map

\[
\sigma : \text{DiffOp}^\bullet(\Gamma^\infty(P, E)) \rightarrow \Gamma^\infty(P, S^* TP \otimes \text{End}(E)) \tag{7.39}
\]

restricts to a vector space isomorphism

\[
\sigma : \text{DiffOp}^\bullet_{\text{vert}}(\Gamma^\infty(P, E)) \rightarrow \Gamma^\infty(P, S^* VP \otimes \text{End}(E)) \tag{7.40}
\]

from the vertical differential operators to the vertical symmetric multivector fields with values in the endomorphism bundle \( \text{End}(E) \).

Moreover, if \( q : E \rightarrow P \) is an equivariant vector bundle over the principal fibre bundle \( p : P \rightarrow M \) and if the covariant derivatives \( \nabla^P \) and \( \nabla^E \) are \( G \)-invariant, the symbol map \( \sigma \) is also \( G \)-invariant. This means that

\[
g \triangleright \sigma(D) = \sigma(g \triangleright D) \tag{7.41}
\]

with the naturally induced representations of \( G \) on the differential operators \( D \in \text{DiffOp}^\bullet(\Gamma^\infty(P, E)) \) and the sections \( \Gamma^\infty(P, S^* TP \otimes \text{End}(E)) \).

Proof: In order to prove the isomorphism (7.40) consider \( D = \sum_{j=0}^l D_{T_j} \in \text{DiffOp}^\bullet(\Gamma^\infty(P, E)) \) with \( T_j \in \Gamma^\infty(P, S^j TP \otimes \text{End}(E)) \). First assume that \( T_j \in \Gamma^\infty(S^j VP \otimes \text{End}(E)) \) are vertical for all \( j = 1, \ldots, l \). By definition this yields for all \( s \in \Gamma^\infty(P, E) \) and \( a \in C^\infty(M) \) that \( j! D_{T_j} (p^* a \cdot s) = \langle T_j, D_E^{(j)}(p^* a \cdot s) \rangle = \langle T_j, \sum_{r=0}^j p^* D_M^{(r)} p^* a \cdot D_E^{(j-r)} s \rangle = p^* a \cdot \langle T_j, D_E^{(j)} s \rangle \). This shows that \( D \) is vertical. The other implication is slightly more technical. If \( D(p^* a \cdot s) = p^* a \cdot D(s) \) this implies

\[
0 = \sum_{j=1}^l \frac{1}{j!} \langle T_j, \sum_{r=0}^{j-1} p^* D_M^{(j-r)} a \cdot D_E^{(r)} s \rangle = \sum_{j=1}^l \sum_{r=0}^{j-1} \frac{1}{j!} \binom{j}{r} \langle T_j, p^* D_M^{(j-r)} a \cdot D_E^{(r)} s \rangle = \sum_{r=0}^{l-1} \frac{1}{(j-r)!} T_j, p^* D_M^{(j-r)} a \cdot D_E^{(r)} s \rangle
\]

for all \( s \in \Gamma^\infty(P, E) \) and \( a \in C^\infty(M) \). Due to the symbol calculus this shows that

\[
0 = \sum_{j=r+1}^l \frac{1}{(j-r)!} \langle T_j, p^* D_M^{(j-r)} a \rangle \quad \text{for all} \quad r = 0, \ldots, l - 1.
\]
For $r = l - 1$ one gets $\langle T_l, p^* D_M a \rangle$ which locally for a chart $(U, x)$ of $M$ implies that $i_*(p^* \, dx^i) T_l = 0$. Then, the successive treatment of the cases $r = l - 2$ to $r = 0$ yields that $T_0 + \cdots + T_l \in \bigoplus_{r=0}^l \Gamma^\infty(P, S^r V P \otimes \text{End}(E))$. This proves the isomorphism (7.40).

With the additional assumptions, Lemma 7.2.6, and the properties of the naturally involved representations of $G$ a simple computation for $T_j \in \Gamma^\infty(P, S^j TP \otimes \text{End}(E))$ gives $(g \rhd D_T)(s) = g \cdot (D_{(j)} E)(g^{-1} \circ s)$ for all $j$. Here one has used the induced representation of $G$ on the sections $\phi \in \Gamma^\infty(P, \text{End}(E))$. If the action on $E$ is a right action $R_g$ one has $(g \circ \phi) = R_{g^{-1}} \circ \phi \circ R_g$ with the obvious meaning. In any case, the resulting equation

$$g \rhd D_T = D_{g \rhd T}$$

for all $j \in \mathbb{N}_0$ is nothing but the stated $G$-invariance of the symbol map $\sigma$. ■

### 7.3 Deformations that preserve the fibration

In Section 6.4 the question concerning existence and uniqueness up to equivalence of differential and $G$-invariant deformations has been answered completely. In this section we concentrate on a further property of deformation quantizations of surjective submersions and principal fibre bundles, this means the deformed right module structures $\bullet$ of the functions $C^\infty(P)[[\lambda]]$ with respect to a star product algebra $(C^\infty(M)[[\lambda]], \star)$.

**Definition 7.3.1 (Fibration preserving deformation)**

Let $\bullet$ be a deformation quantization of a surjective submersion or a principal fibre bundle $p : P \rightarrow M$. The structure $\bullet$ is said to preserve the fibration if

$$(p^* a) \bullet b = p^*(a \star b)$$ (7.43)

for all $a, b \in C^\infty(M)[[\lambda]]$.

Such deformations are relevant to consider since they respect a classically given algebraic property. In other words, the property states that the pullback $p^* : C^\infty(M)[[\lambda]] \rightarrow C^\infty(P)[[\lambda]]$ in the deformed situation still is a module morphism along the identity when regarding the algebra $(C^\infty(M)[[\lambda]], \star)$ as a module over itself.

In the subsequent considerations we show that there always exist deformations that preserve the fibration. Before doing this we note that there exists an equivalent characterization of the property (7.43).

**Lemma 7.3.2**

A deformation quantization $\bullet$ preserves the fibration if and only if it satisfies the equation

$$1 \bullet a = p^* a$$ (7.44)

for all $a \in C^\infty(M)$.

**Proof:** The assertion is obvious due to the given right module property and the property $1 \star a = a$ of any star product. ■

Given a deformation $\bullet$, the aim will be to find an appropriate equivalence transformation such that the transformed deformation preserves the fibration. In order to find such a series of differential operators we make use of the symbol calculus as explained in the Propositions 7.1.1 and 7.1.2.
Lemma 7.3.3
Let \( D \in \text{DiffOp}^L(M, \text{DiffOp}^l(P)) \) be a differential operator with \( L \in \mathbb{N}_0 \), \( k \in \mathbb{N} \), \( l \in \mathbb{N}_0 \). Then, every function \( f \in C^\infty(P) \) gives rise to a differential operator \( D_f \in \text{DiffOp}^L(M, C^\infty(P)) \) which is defined by
\[
D_f(a_1, \ldots, a_k) = D(a_1, \ldots, a_k)(f)
\]
for all \( a_1, \ldots, a_k \in C^\infty(M) \).

**Proof:** With the present algebraic definition of differential operators the proof is a straightforward induction over \( r = |L| \).

With the preceding lemmas it is now possible to show that for every deformation quantization it is possible to explicitly construct a particular equivalence transformation which relates the both sides of Equation (7.44).

Lemma 7.3.4
Let \( \bullet \) be a deformation quantization of a surjective submersion \( p : P \rightarrow M \). Then there exists a formal series \( T = \text{id} + \sum_{r=1}^\infty \lambda^r T_r \) of differential operators \( T_r \in \text{DiffOp}(P) \) such that
\[
T(p^*a) = 1 \cdot a
\]
for all \( a \in C^\infty(M) \).

If \( \bullet \) is a deformation quantization of a principal fibre bundle it can be achieved that \( T \) is a series of \( G \)-invariant differential operators.

**Proof:** The deformed module structure \( \bullet \) is given by a formal series \( \rho = \sum_{r=0}^\infty \lambda^r \rho_r \) of differential operators \( \rho_r \in \text{DiffOp}^l_r(M, \text{DiffOp}^{m_r}(P)) \) with \( l_r, m_r \in \mathbb{N}_0 \). Lemma 7.3.3 then shows that \( D_r(a) = \rho_r(a)(1) \) defines a differential operator \( D_r \in \text{DiffOp}^l_r(M, C^\infty(P)) \) for all \( r \in \mathbb{N}_0 \). After the choice of a connection \( HP = VP \oplus HP \) and a torsion-free covariant derivative \( \nabla^M \) on \( TM \) Proposition 7.1.2 shows that
\[
D_r(a) = \sum_{s=0}^l \frac{1}{s!} \left< T^r_s, p^*D^s_M a \right> \quad \text{with unique} \quad T^r_s \in \Gamma^\infty(P, S^s HP).
\]

Using a torsion-free covariant derivative \( \nabla^P \) on \( TP \) with the properties of Lemma 7.2.1 we define \( T_r \in \text{DiffOp}^{l_r}(P) \) by
\[
T_r(f) = \sum_{s=0}^{l_r} \frac{1}{s!} \left< T^r_s, D^s_P f \right>.
\]

By the definition of a deformation quantization one has \( D_0 = p^* \in \text{DiffOp}^0(M, C^\infty(P)) \) and with \( D^{(0)}_P = \text{id} \) this implies that \( T_0 = \text{id} \). Due to property (7.30) one finds \( T_r(p^*a) = D_r(a) \) which shows that the so constructed series \( T = \sum_{r=0}^\infty \lambda^r T_r \) has the required property.

In the principal fibre bundle case the proof is slightly different. The given \( G \)-invariance of the \( \rho_r \) implies that \( r^*_g(D_r(a)) = D_r(a) \). Thus, one has \( D_r = p^* \circ D^M_r \) with differential operators \( D^M_r \in \text{DiffOp}^{l_r}(M) \) for which we apply the ordinary symbol calculus with respect to \( \nabla^M \) as above. Choosing a principal connection and using the canonical extension of the horizontal lift to multivector fields one gets
\[
D_r(a) = p^* \sum_{s=0}^{l_r} \frac{1}{s!} \left< T^r_s, D^s_M a \right> = \sum_{s=0}^{l_r} \frac{1}{s!} \left< (T^r_s)^h, D^s_P a \right> \quad \text{with unique} \quad T^r_s \in \Gamma^\infty(M, S^s TM).
\]

Then one defines the \( T_r \in \text{DiffOp}^{l_r}(P) \) in analogy to (7.48). If the used \( \nabla^P \) is \( G \)-invariant the compatibility of the natural pairing with the pullback \( r^*_g \), the \( G \)-invariance of the horizontal lifts, and equation (7.33) show that the \( T_r \) are \( G \)-invariant.
7.4. The commutants of the deformed right modules

With this preceding lemma we finally come to the aspired result.

**Proposition 7.3.5 (Fibration preserving deformations)**

Every surjective submersion and every principal fibre bundle admits a deformation quantization which preserves the fibration.

**Proof:** Let \( \bullet \) be an arbitrary deformation quantization with respect to a given star product \( \ast \). Since the map \( T \) in Lemma 7.3.4 has all properties of an equivalence transformation, \( f \bullet a = T^{-1}(Tf \ast a) \) defines a new deformation quantization \( \tilde{\bullet} \). With (7.46) we then get \( (p^\ast a) \circ b = p^\ast(a \ast b) \) for all \( a, b \in C^\infty(M) \).

The additional \( G \)-invariance of the map \( T \) for principal fibre bundles guarantees the \( G \)-invariance of \( \tilde{\bullet} \) if \( \bullet \) has this property. \( \square \)

**Remark 7.3.6**

Note that the above proof and Lemma 7.3.4 not only yield the existence of deformations which preserve the fibration. Moreover, the above considerations in fact provide an explicit procedure to find such a deformation when starting with an arbitrary one by constructing the corresponding equivalence transformation.

7.4 The commutants of the deformed right modules

As seen in Chapter 2, the fact that the crucial Hochschild cohomologies are trivial not only implies that deformations exist and are unique up to equivalence, but also that one has a lot of information concerning the corresponding commutants. In the subsequent considerations we apply the general results of Section 2.5 to the particular situation of vector bundles \( q : E \rightarrow P \) over surjective submersions and principal fibre bundles.

First of all we concentrate on surjective submersions \( p : P \rightarrow M \) and deformations \( \bullet \) as in Theorem 6.4.3. Due to the vanishing first cohomology group

\[
\text{HH}_1^\text{diff}(M, \text{DiffOp}(\Gamma_\infty(P,E))) = \{0\},
\]

the results of Proposition 2.5.1 ensure that every choice of a deformation quantization \( \bullet \) and a decomposition

\[
\text{DiffOp}(\Gamma_\infty(P,E)) = \text{DiffOp}_{\text{ver}}(\Gamma_\infty(P,E)) \oplus \text{DiffOp}_{\text{ver}}(\Gamma_\infty(P,E))
\]

of all differential operators into the classical commutant and a complementary subspace lead to an isomorphism

\[
\rho' : \text{DiffOp}_{\text{ver}}(\Gamma_\infty(P,E))[[\lambda]] \rightarrow \{ D \in \text{DiffOp}(\Gamma_\infty(P,E))[[\lambda]] \mid D(s \bullet a) = D(s) \bullet a \}
\]

onto the commutant of the deformed module structure with \( \rho' = \text{id} + \sum_{r=1}^\infty \lambda^r \rho_r' \). According to Corollary 2.5.2 this further induces an associative deformation \( (\text{DiffOp}_{\text{ver}}(\Gamma_\infty(P,E))[[\lambda]], \mu') \) of the algebra \( (\text{DiffOp}_{\text{ver}}(\Gamma_\infty(P,E)), \circ) \) of vertical differential operators with the usual composition of maps. Using the definitions

\[
A \bullet' s = \rho'(A)s \quad \text{and} \quad A \ast' B = \mu'(A, B)
\]

for all \( A, B \in \text{DiffOp}_{\text{ver}}(\Gamma_\infty(P,E))[[\lambda]] \) and \( s \in \Gamma_\infty(P,E)[[\lambda]] \), the sections \( C^\infty(P)[[\lambda]] \) inherit a \( (\bullet', \ast') \)-bimodule structure which is shortly denoted by

\[
(\text{DiffOp}_{\text{ver}}(\Gamma_\infty(P,E))[[\lambda]], \bullet')((\bullet, \Gamma_\infty(P,E)[[\lambda]], \ast'), (C^\infty(M)[[\lambda]], \ast)).
\]

In the special case of functions \( C^\infty(P) \), this means when considering deformation quantizations of surjective submersion one gets the following result, see [19, Prop. 4.18]
Proposition 7.4.1 (Mutual commutants for the module $C^\infty(P)$ of functions)
The commutant within the differential operators of $\text{DiffOp}_{\text{ver}}(P)[[\lambda]]$ acting via $\bullet'$ on $C^\infty(P)[[\lambda]]$ is given by $C^\infty(M)[[\lambda]]$ acting via $\bullet$. Thus the two algebras in (7.54) are mutual commutants.

Proof: Let $A = \sum_{r=0}^{\infty} \lambda^r A_r \in \text{DiffOp}^{\bullet}(P)[[\lambda]]$ satisfy $A(D \bullet' f) = D \bullet' (Af)$ for all $D \in \text{DiffOp}_{\text{ver}}(P)[[\lambda]]$ and $f \in C^\infty(P)[[\lambda]]$. It follows that in zeroth order $A_0$ commutes with all undeformed left multiplications with functions $D \in C^\infty(P)$. Since $C^\infty(P)$ is a unital algebra this is only possible if $A_0 \in C^\infty(P)$. Further we know that $A_0$ commutes with all $D \in \text{DiffOp}_{\text{ver}}(P)$ in zeroth order, so $A_0$ is constant in fibre directions. This means nothing but $A_0 = p^*a_0$ for some $a_0 \in C^\infty(M)$. Since $A(D \bullet' f) - (D \bullet' f) \bullet a_0 = D \bullet' (Af - f \bullet a_0)$ and since $Af - f \bullet a_0$ has vanishing zeroth order, a simple induction shows that $Af = f \bullet a$ for a series $a = \sum_{r=0}^{\infty} \lambda^r a_r \in C^\infty(M)[[\lambda]]$.

For $G$-invariant deformations we can apply the results of Section 2.6. In order to achieve a $G$-invariant bimodule structure (7.54) it is necessary to find a $G$-invariant decomposition (7.51). As we will see now, Proposition 7.2.7 guarantees that this crucial condition can be satisfied. Taking a $G$-invariant covariant derivative $\nabla^E$ on the vector bundle $q : E \to P$ one obtains a $G$-invariant symbol map $\sigma$ as in (7.39). The choice of a principal connection, this means a $G$-invariant decomposition $TP = VP \oplus HP$ as in Appendix A naturally induces a $G$-invariant decomposition

$$
\Gamma^\infty(P, S^*TP \otimes \text{End}(E)) = \Gamma^\infty(P, S^*VP \otimes \text{End}(E)) \oplus \Gamma^\infty(P, S^*VP \otimes \text{End}(E)).
$$

(7.55)
The complementary subspace $\overline{\Gamma^\infty(P, S^*VP \otimes \text{End}(E))}$ consists of all tensor fields having a nontrivial horizontal component. The invariance of the composition is clear with the properties $Tr_g HP = HP$ and $Tr_g VP = VP$ and the given action on the vector fields via pullback with the principal right action. With the symbol map $\sigma$ from Proposition 7.2.7 one can define

$$
\text{DiffOp}_{\text{ver}}(\Gamma^\infty(P, E)) = \sigma^{-1}(\Gamma^\infty(P, S^*VP \otimes \text{End}(E))).
$$

(7.56)
Due to the isomorphism (7.40) and the $G$-invariance of $\sigma$ this yields a $G$-invariant decomposition (7.61), this means

$$
G \triangleright \text{DiffOp}_{\text{ver}}(\Gamma^\infty(P, E)) \subseteq \text{DiffOp}_{\text{ver}}(\Gamma^\infty(P, E)).
$$

(7.57)
Note that the classical commutant $\text{DiffOp}_{\text{ver}}(\Gamma^\infty(P, E))$ is $G$-invariant by the compatibility of the differential type with any $G$-action, confer the observations of the Remarks 2.6.3 and 3.2.4. Since the cohomology groups $HH^1_\text{diff}(M, \text{DiffOp}(\Gamma^\infty(P, E))) = \{0\}$ and $HH^1_\text{diff}(M, \text{DiffOp}(\Gamma^\infty(P, E))^G) = \{0\}$ are trivial one can apply Proposition 2.6.5, confer [19] Thm. 5.8.

Theorem 7.4.2
Let $q : E \to P$ be an equivariant vector bundle over the principal fibre bundle $p : P \to M$ and let $\bullet$ be a $G$-invariant deformation of the module structure of the sections $\Gamma^\infty(P, E)$ with respect to a star product $\ast$ as in Theorem 6.4.4. Then there exists a bimodule structure

$$
(\text{DiffOp}_{\text{ver}}(\Gamma^\infty(P, E))[[\lambda]], \ast')(\Gamma^\infty(P, E)[[\lambda]], \bullet)(C^\infty(M)[[\lambda]], \ast)
$$

(7.58)
as in (7.54) with the further property that $\ast'$ and $\bullet'$ are $G$-invariant, this means

$$
g \triangleright (A \ast' B) = (g \triangleright A) \ast' (g \triangleright B)
$$

(7.59)
and

$$
g \triangleright (A \bullet' s) = (g \triangleright A) \bullet' (g \triangleright s)
$$

(7.60)
for all $A, B \in \text{DiffOp}_{\text{ver}}(\Gamma^\infty(P, E))[[\lambda]]$, $s \in \Gamma^\infty(P, E)[[\lambda]]$ and $g \in G$. Moreover, the deformed bimodule structure (7.58) is unique up to $G$-invariant equivalence.
Remark 7.4.3 (Infinitesimal gauge transformations)
Interpreting the deformed module structures in the context of classical gauge theories the consideration of the commutant within the differential operators gives rise to an extension of the algebra of infinitesimal gauge transformations. For two such $V_1, V_2 \in \mathfrak{gau}(P) = \Gamma^\infty(P, V P)^G \subseteq \text{DiffOp}^1_{\text{ver}}(P)$ the deformed action on $f \in C^\infty(P)[[\lambda]]$ gives

$$V_1 \cdot' (V_2 \cdot' f) - V_2 \cdot' (V_1 \cdot' f) = (V_1 \star V_2 - V_2 \star V_1) \cdot' f = \mathcal{L}_{[V_1, V_2]} f + O(\lambda).$$ \hspace{1cm} (7.61)

In general, the higher order contributions are nontrivial. If the former algebraic relation (1.28) shall still be valid in the deformed situation the Lie algebra of infinitesimal gauge transformations $\mathfrak{gau}(P)$ has to be extended to the deformed algebra $(\text{DiffOp}_{\text{ver}}(P)[[\lambda]], \star')$ of vertical differential operators of which $\mathfrak{gau}(P)$ is no longer a Lie subalgebra.
Chapter 8

Deformation quantization of associated vector bundles

The importance of principal fibre bundles in differential geometry comes not least from the fact that every vector bundle can be seen as an associated vector bundle with respect to some principal fibre bundle. In this chapter we investigate how the deformations discussed in Chapter 6 behave under this process of association and how they induce corresponding deformations on the level of vector bundles. As a main result, it will turn out that every deformation quantization of a principal fibre bundle naturally induces a corresponding deformation quantization of any associated vector bundle in the well-known sense that is recalled and discussed in the first section. This observation makes use of the isomorphism between the invariant vector-valued functions on the principal fibre bundle and the sections of the associated bundle. Even more general, we can show that any invariant deformed module structure of the horizontal forms of the principal fibre bundle induces a corresponding deformation of the forms on the base manifold with values in the associated bundle.

8.1 Deformation quantization of vector bundles

It is a well-known fact that the sections $\Gamma^{\infty}(M, E)$ of a vector bundle $q : E \rightarrow M$ have the structure of a finitely generated projective module over the smooth functions $C^{\infty}(M)$ of the base manifold $M$. Moreover, the vector bundle is already determined by the sections, \[115\], and the theorem of Serre and Swan \[58, Thm. 2.10\] states that the category of vector bundles over a manifold $M$ is equivalent to the category of finitely generated projective modules over $C^{\infty}(M)$. This one-to-one correspondence naturally implies what shall be understood by a deformation quantization of a vector bundle $q : E \rightarrow M$ with respect to some star product $\star$ on the Poisson manifold $M$. It is nothing but a deformed module structure

$$ (\Gamma^{\infty}(M, E)[[\lambda]], \bullet)(C^{\infty}(M)[[\lambda]], \star) $$ (8.1)

in the sense of Definition \[24.4\]. The explicit definition and the corresponding well-known results can be found in \[25\] and \[122\]. In our case where $\star$ is a differential star product one of course demands the same for the deformation $\bullet$. In the form $s \bullet a = s \cdot a + \sum_{r=1}^{\infty} \rho_r(s, a)$, this means that the $\rho_r$ are bidifferential operators

$$ \rho_r \in \text{DiffOp}^{(L_r, l_r)}(C^{\infty}(M), \Gamma^{\infty}(M, E); \Gamma^{\infty}(M, E)) \cong \text{DiffOp}^{L_r}(M, \text{DiffOp}^{l_r}(\Gamma^{\infty}(M, E))) $$ (8.2)

for some $L_r, l_r \in \mathbb{N}_0$ and all $r \geq 1$. The stated isomorphism in (8.2) is verified by a simple induction and shows that in the considered case the meaning of a differential deformation is always clear,
confer Remark [6.1.10]. The classical commutant of the considered right module structure \( \Gamma^\infty(M, E) \) is given by the sections of the endomorphism bundle \( \Gamma^\infty(M, \text{End}(E)) \cong \text{DiffOp}^0(\Gamma^\infty(M, E)) \). Thus, one has the bimodule structure

\[
(\Gamma^\infty(M, \text{End}(E)), \circ) \Gamma^\infty(M, E)_{(C^\infty(M), \cdot)}. \tag{8.3}
\]

With the results of Section 2.7 and Proposition 8.2.13 we can apply all considerations of Chapter 2 and find a simple proof for the main assertion of the following theorem, see [25, [122, Thm. 1].

**Theorem 8.1.1 (Deformation quantization of vector bundles)**

Every vector bundle \( q : E \to M \) over a Poisson manifold with a star product \( \star \) has a corresponding deformation quantization which is unique up to equivalence. Moreover, there exists a deformed bimodule structure

\[
(\Gamma^\infty(M, \text{End}(E))[[\lambda]], \cdot')_E (\Gamma^\infty(M, E)[[\lambda]], \cdot)((C^\infty(M)[[\lambda]], \cdot), \tag{8.4}
\]

and for a fixed star product \( \star \) the deformed bimodule structures \( \cdot'_E, \cdot \) and the algebra structure \( \cdot'_E \) are unique up to equivalence.

In addition, the two deformed algebras \( (\Gamma^\infty(M, \text{End}(E))[[\lambda]], \cdot'_E) \) and \( (C^\infty(M)[[\lambda]], \cdot) \) are mutual commutants. In other words, the commutant \( \text{End}_{C^\infty(M)[[\lambda]]}(\Gamma^\infty(M, E)[[\lambda]], \cdot) \) of the right module structure is isomorphic to \( (\Gamma^\infty(M, \text{End}(E))[[\lambda]] \otimes \mathbb{C}[[\lambda]]) \cdot \) as \( \mathbb{C}[[\lambda]] \)-module and as algebra via \( \cdot'_E \).

Note that only the last assertion does not follow from the general considerations of Chapter 2.

### 8.2 Deformation quantization of associated vector bundles

Now we investigate the above situation for associated vector bundles \( E = P \times_G V \) as explained in Appendix A. The following considerations basically follow [19] but the more conceptual proceeding shows that some results can be obtained in a slightly simpler way.

In addition to the results of Proposition A.4.1 we make the following simple observation.

**Lemma 8.2.1 (The endomorphism bundle of an associated vector bundle)**

Let \( p : P \to M \) be a principal fibre bundle, \( \pi : G \to \text{Aut}(V) \) be a representation of the structure group \( G \) on a finite dimensional vector space \( V \) and let \( P \times_G V \) denote the associated vector bundle. Then the induced representation of \( G \) on \( \text{End}(V) \) yields a vector bundle isomorphism

\[
P \times_G \text{End}(V) \cong \text{End}(P \times_G V) \tag{8.5}
\]

over the identity \( \text{id}_M \) for the endomorphism bundle of the associated bundle, given by

\[
[u, L]([u, v]) = [u, L(v)] \tag{8.6}
\]

for equivalence classes \([u, L] \in P \times_G \text{End}(V)\) and \([u, v] \in P \times_G V\). Thus, one has the isomorphism

\[
(C^\infty(P) \otimes \text{End}(V))^G \cong \Gamma^\infty(M, \text{End}(P \times_G V)). \tag{8.7}
\]

With the isomorphism \((C^\infty(P) \otimes V)^G \cong \Gamma^\infty(M, P \times_G V)\) the application of a section of the endomorphism bundle to a section of the bundle reads

\[
(f \otimes L)(h \otimes v) = fh \otimes L(v) \tag{8.8}
\]

for all \( f \otimes L \in (C^\infty(P) \otimes \text{End}(V))^G \) and \( h \otimes v \in (C^\infty(P) \otimes V)^G\).
8.2. Deformation quantization of associated vector bundles

The isomorphism (8.5) is obvious with the given identification (8.6). Then, the remaining statements follow easily using the corresponding isomorphisms of invariant vector-valued functions on $\mathcal{P}$ and sections on the associated bundle, confer [A.34].

In the following it will be shown that a deformation quantization of the principal fibre bundle $P$ as considered in Chapter [B] leads to the bimodule structure (8.4) for any associated vector bundle $P \times_G V$, confer [19, Lemma 6.1].

**Lemma 8.2.2 (Deformed associated bimodules)**

The $G$-invariant bimodule structure

$$(\text{DiffOp}_\text{ver}(P)[[\lambda]], \ast') \left( C^\infty(P)[[\lambda]], \bullet \right) (C^\infty(M)[[\lambda]], \ast)$$

(8.9)

as in Theorem 7.4.2 yields a bimodule structure

$$((\text{DiffOp}_\text{ver}(P) \otimes \text{End}(V))^G[[\lambda]], \ast') \left( (C^\infty(P) \otimes V)^G[[\lambda]], \bullet \right) (C^\infty(M)[[\lambda]], \ast).$$

(8.10)

**Proof:** First of all it is clear that the structures $\ast', \ast$, and $\bullet$ naturally induce corresponding operations with respect to the tensor products $(\text{DiffOp}_\text{ver}(P) \otimes \text{End}(V))[[\lambda]]$ and $(C^\infty(P) \otimes V)[[\lambda]]$, yielding a bimodule structure

$$((\text{DiffOp}_\text{ver}(P) \otimes \text{End}(V))^G[[\lambda]], \ast') \left( (C^\infty(P) \otimes V)^G[[\lambda]], \bullet \right) (C^\infty(M)[[\lambda]], \ast).$$

(8.11)

For factorising elements $A \otimes L, B \otimes K \in \text{DiffOp}_\text{ver}(P) \otimes \text{End}(V)$, $f \otimes v \in C^\infty(P) \otimes V$ and $a \in C^\infty(M)$ the definitions are $(A \otimes L) \ast' (B \otimes K) = (A \ast' B) \otimes (L \circ K)$, $(A \otimes L) \ast (f \otimes v) = (A \ast f) \otimes L(v)$, and $(f \otimes v) \bullet a = (f \ast a) \otimes v$ with some abuse of notation.

Considering only the right module structure, the isomorphism $(C^\infty(P) \otimes V)^G \cong \Gamma^\infty(M, P \times_G V)$ immediately implies the following theorem, see [19, Thm. 6.2].

**Theorem 8.2.3 (Deformation quantization of associated vector bundles)**

By (8.10) every deformation quantization $\bullet$ of a principal fibre bundle induces a corresponding deformation quantization of any associated vector bundle.

In order to find the relation between the left module structures in (8.10) and (8.4) we observe that every $A \otimes L \in (\text{DiffOp}_\text{ver}(P) \otimes \text{End}(V))^G$ can be seen as an element $A \otimes L \in \Gamma^\infty(M, \text{End}(P \times_G V))$ by setting

$$(A \otimes L)(h \otimes v) = A(h) \otimes L(v)$$

(8.12)

for all $h \otimes v \in (C^\infty(P) \otimes V)^G$. With (8.7) it follows that this identification yields a surjective map

$$(\text{DiffOp}_\text{ver}(P) \otimes \text{End}(V))^G \twoheadrightarrow \Gamma^\infty(M, \text{End}(P \times_G V))$$

(8.13)

and that the following diagram commutes,

$$\begin{array}{ccc}
(\text{DiffOp}_\text{ver}(P) \otimes \text{End}(V))^G & \twoheadrightarrow & \Gamma^\infty(M, \text{End}(P \times_G V)) \\
\downarrow \cong & & \downarrow \cong \\
(C^\infty(P) \otimes \text{End}(V))^G & \cong & \Gamma^\infty(M, \text{End}(P \times_G V)).
\end{array}$$

That every element $A \otimes L \in (\text{DiffOp}_\text{ver}(P) \otimes \text{End}(V))^G$ by (8.12) acts in the same way as an element $f \otimes L' \in C^\infty(P) \otimes \text{End}(V)$ by (8.8) is plausible due to the following consideration. The arguments $h \otimes v \in (C^\infty(P) \otimes V)^G$ satisfy $h(u,g)v = h(u)\pi_g^{-1}v$ for all $u \in P$ and $g \in G$. Thus, every vertical differentiation of $h$ is nothing but a linear transformation of $v$. Explicitly, one has
\( \frac{d}{dt} |_{t=0} h(u, \exp(-t \xi)) v = h(u) \pi'_t v \) for each \( \xi \in \mathfrak{g} \) in the Lie algebra and the induced Lie algebra representation \( \pi' : \mathfrak{g} \to \text{End}(V) \).

The bimodule (8.10) shows that for every \( D \in (\text{DiffOp}_{\text{ver}}(P) \otimes \text{End}(V))^G[[\lambda]] \) there exists a unique element \( \phi(D) \in \text{End}_{C^\infty(M)}([\lambda]) (\Gamma^\infty(M, P \times_G V)[[\lambda]], \bullet) \) in the commutant of the right module structure with \( D \bullet s = \phi(D)s \) for all \( s \in \Gamma^\infty(M, P \times_G V)[[\lambda]] \). The map \( \phi : (\text{DiffOp}_{\text{ver}}(P) \otimes \text{End}(V))^G[[\lambda]] \to \text{End}_{C^\infty(M)}([\lambda]) (\Gamma^\infty(M, P \times_G V)[[\lambda]], \bullet) \) is surjective since this is already true for the restriction to \((C^\infty(P) \otimes \text{End}(V))^G[[\lambda]]\). This assertion is a slightly stronger version of [19, Lemma 6.3].

**Lemma 8.2.4**

The map

\[ \phi : (C^\infty(P) \otimes \text{End}(V))^G[[\lambda]] \to \text{End}_{C^\infty(M)}([\lambda]) (\Gamma^\infty(M, P \times_G V)[[\lambda]], \bullet) \tag{8.15} \]

is surjective.

**Proof:** Let \( K = \sum_{r=0}^\infty \lambda^r K_r \in \text{End}_{C^\infty(M)}([\lambda]) (\Gamma^\infty(M, P \times_G V)[[\lambda]], \bullet) \) be an element in the commutant. By definition it is clear that \( K_0 \in \Gamma^\infty(M, \text{End}(P \times_G V)) \subseteq (\text{DiffOp}_{\text{ver}}(P) \otimes \text{End}(V))^G \). With \( \phi(K_0) = \sum_{r=0}^\infty \lambda^r \phi(K_0)_r \), it follows by the definition of \( \phi \) that \( K_0 = \phi(K_0)_0 \). Thus, the element \( K - \phi(K_0) = \sum_{r=1}^\infty \lambda^r K^{(1)}_r \) of the commutant begins in order \( \lambda \). Due to the \( \mathbb{C}[[\lambda]] \)-linearity of \( \phi \), iteration proves the lemma.

As cited in Theorem 8.1.1 there is an isomorphism \( \psi : \text{End}_{C^\infty(M)}([\lambda]) (\Gamma^\infty(M, P \times_G V)[[\lambda]], \bullet) \to \Gamma^\infty(M, \text{End}(P \times_G V)[[\lambda]]) \) such that for each \( D \in (C^\infty(P) \otimes \text{End}(V))^G[[\lambda]] \) one finally has

\[ D \bullet s = \phi(D)s = \psi(\phi(D)) \bullet_E s \tag{8.16} \]

for all \( s \in \Gamma^\infty(M, P \times_G V)[[\lambda]] \). The left module properties in (8.4) and (8.11) then imply the following result for the map \( \gamma = \psi \circ \phi \), confer [19, Thm. 6.4].

**Theorem 8.2.5 (The associated deformed commutant)**

Let \( \bullet \) denote a deformation quantization of a principal fibre bundle as well as the induced deformation quantization of an associated vector bundle \( E = P \times_G V \). Then, for all structures \( \bullet_E \) and \( \bullet_E' \) as in (8.4) there exists a surjective algebra homomorphism

\[ \gamma : (\text{DiffOp}_{\text{ver}}(P) \otimes \text{End}(V))^G[[\lambda]], \bullet' \to (\Gamma^\infty(M, \text{End}(E))[[\lambda]], \bullet_E') \tag{8.17} \]

such that

\[ D \bullet' s = \phi(D) \bullet_E s \tag{8.18} \]

for all \( D \in (\text{DiffOp}_{\text{ver}}(P) \otimes \text{End}(V))^G[[\lambda]] \) and \( s \in \Gamma^\infty(M, E)[[\lambda]] \).

**Remark 8.2.6**

A further investigation that can be found in [19] shows that the extension to the vertical differential operators is necessary. In general, \((C^\infty(P) \otimes \text{End}(V))^G[[\lambda]], \bullet')\) is not deformed into a subalgebra of \((\text{DiffOp}_{\text{ver}}(P) \otimes \text{End}(V))^G[[\lambda]], \bullet'\).

In analogy to the above discussions for functions one gets the following statement for differential forms when using the isomorphism (A.31).

**Proposition 8.2.7 (Deformation quantization of associated forms)**

Every \( G \)-invariant deformed right module structure

\[ (\Gamma^\infty_{\text{hor}}(P, \mathcal{L}T^s P)[[\lambda]], \bullet)_{(C^\infty(M))[[\lambda]], \bullet} \tag{8.19} \]
of the horizontal forms of a principal fibre bundle induces a deformation

\[(\Gamma^\infty(M, \wedge^k \mathcal{C}^\vee T^* M \otimes (P \times_G V)), \bullet)_{(\mathcal{C}^\infty(M)[[\lambda]], \bullet)} \]  

(8.20)

for the differential forms on the base manifold with values in any associated vector bundle.

**Proof:** The proof is a straightforward generalization of Lemma 8.2.2 where one makes use of the isomorphism \( (\Gamma^\infty_{\text{hor}}(P, \wedge^k T^* P) \otimes V)^G \cong \Gamma^\infty(M, \wedge^k T^* M \otimes (P \times_G V)) \).
Appendix A

Geometry of principal fibre bundles

This appendix gives a short overview over some concepts of the differential geometry of Lie groups and principal fibre bundles used in the main text. In order to concentrate on the relevant facts it is assumed that the reader is already familiar with the fundamental concepts in differential geometry. The presented notions can be found in the references [43, 80, 94, 123]. With respect to the aspired physical applications in gauge theories all necessary aspects of differential geometry can also be found in [11, 97, 98].

A.1 Lie groups and group actions

In this first section we collect some basic facts on Lie groups and their actions and representations. All the details omitted in this overview can be found in [43] or in any other book on differential geometry treating Lie groups. By definition, a Lie group \( G \) is a manifold such that the group multiplication and taking the inverse are smooth maps. The Lie algebra of the group is given by \( \mathfrak{g} = T_e G \) at the neutral element \( e \in G \) equipped with the following Lie bracket. Let \( l_g h = gh \) for \( g, h \in G \) denote the left multiplication. Since for any \( \xi \in \mathfrak{g} \) there exists a unique left-invariant vector field \( X^\xi = l_g^* X^\xi \) with \( X^\xi(e) = \xi \), explicitly given by \( X^\xi(g) = T_e l_g(\xi) \), the Lie bracket of vector fields gives rise to the definition \( [\xi, \eta] = [X^\xi, X^\eta](e) \in \mathfrak{g} \) for \( \xi, \eta \in \mathfrak{g} \). Any \( X^\xi \) has a complete flow \( \text{Fl}_s^\xi \) with \( \text{Fl}_1^\xi \circ l_g = l_g \circ \text{Fl}_1^\xi \). The exponential map \( \exp : \mathfrak{g} \to G \) is defined by \( \exp(\xi) = \text{Fl}_1^\xi(e) \). Then the map \( \mathbb{R} \to G \) with \( t \mapsto \exp(t\xi) = \text{Fl}_t^\xi(e) = \text{Fl}^\xi_t(e) \) is a one-parameter group, thus having the important properties \( \exp((t+s)\xi) = \exp(t\xi)\exp(s\xi) \) and \( \exp(0) = e \). Further, one has \( \frac{d}{dt}|_{t=0} \exp(t\xi) = \xi \).

If \( M \) is a manifold a left action of \( G \) on \( M \) is given by a smooth map \( \phi : G \times M \to M \), often denoted by \( \phi(g, p) = \phi_g(p) = \phi^p(g) \) with smooth maps \( \phi_g : M \to M \) and \( \phi^p : G \to M \), such that \( \phi_e = \text{id}_M \) and \( \phi_{gh} = \phi_g \circ \phi_h \). For a right action one clearly has \( \phi_g \circ \phi_h = \phi_{gh} \). An action is called free if \( \phi_g(p) = p \) implies that \( g = e \). Further, an action is called proper if the map \( G \times M \to M \times M \) with \( (g, p) \mapsto (\phi_g(p), p) \) is proper, this means if any compact subset has a compact preimage. It should be noted that properness has many different but equivalent formulations, confer [88, Chap. 9]. In order to describe the infinitesimal version of an action one considers the fundamental vector fields \( \xi_M \in \Gamma^\infty(\mathcal{M}, TM) \) with respect to elements \( \xi \in \mathfrak{g} \) which are defined by

\[
\xi_M(p) = \frac{d}{dt}\bigg|_{t=0} \phi_{\exp(t\xi)}p = T_e \phi^p \xi \tag{A.1}
\]

for all \( p \in M \). The map \( \phi' : \mathfrak{g} \to \Gamma^\infty(\mathcal{M}, TM) \) with \( \xi \mapsto \xi_M \) then satisfies \( [\xi_M, \eta_M] = -[\xi, \eta]_M \) for left actions and \( [\xi_M, \eta_M] = [\xi, \eta]_M \) for right actions.
A representation of a Lie group $G$ on a finite dimensional vector space $V$ is a left action $\pi : G \times V \to V$ such that each $\pi_g : V \to V$ is a linear map. Obviously, a representation can be seen as a group homomorphism

$$\pi : G \to \text{Aut}(V)$$  \hspace{1cm} (A.2)

from $G$ to the group of automorphisms $\text{Aut}(V)$ of $V$ with the composition of maps as group structure. With the identification $T_e V = V$ the fundamental vector fields can be seen as linear maps $\xi_V : V \to V$. Then, the infinitesimal version $\pi' : \xi \mapsto \xi_V$ of the representation is a Lie algebra representation, this means a Lie algebra homomorphism

$$\pi' : \mathfrak{g} \to \text{End}(V),$$  \hspace{1cm} (A.3)

where the Lie bracket $[\cdot, \cdot]_{\text{End}(V)}$ for the endomorphisms is the usual commutator. The crucial point is that $\pi'([\xi, \eta]) = [\xi, \eta]_{\text{TV}} = -[\xi_V, \eta_V]_{\Gamma(V,TV)} = [\xi_V, \eta_V]_{\text{End}(V)}$. With $\text{End}(V) = T_{id} \text{Aut}(V)$ the induced infinitesimal representation of (A.2) can be seen as tangent map $\pi' = T_e \pi$.

Important examples for Lie group actions and representations are the following. Any Lie group $G$ acts on itself from the left by the conjugation

$$\text{Conj} : G \times G \to G, \quad \text{Conj}_g h = ghg^{-1}. \hspace{1cm} (A.4)$$

This induces the adjoint representation $\text{Ad}$ of $G$ on its Lie algebra $\mathfrak{g}$

$$\text{Ad} : G \to \text{End}(\mathfrak{g}), \quad \text{Ad}_g = T_e \text{Conj}_g, \hspace{1cm} (A.5)$$

and the adjoint representation $\text{ad}$ of the Lie algebra $\mathfrak{g}$ on itself as infinitesimal version which turns out to be

$$\text{ad} = \text{Ad}' : \mathfrak{g} \to \text{End}(\mathfrak{g}), \quad \text{ad}_\xi \eta = [\xi, \eta]. \hspace{1cm} (A.6)$$

The adjoint representation $\text{Ad}$ acts by Lie algebra automorphisms which explicitly means that $\text{Ad}_g[\xi, \eta] = [\text{Ad}_g \xi, \text{Ad}_g \eta]$.

### A.2 Principal fibre bundles and surjective submersions

The fundamental geometric structure discussed in this work is the one of a principal fibre bundle. This is nothing but a particular $G$-bundle, confer [94], for a Lie group $G$. The explicit definition is the following.

**Definition A.2.1 (Principal fibre bundle)**

A principal fibre bundle $(p, P, M, G)$ consists of a smooth mapping $p : P \to M$ called projection between two manifolds where $P$ is called total space and $M$ is called base space, together with a Lie group $G$ referred to as structure group such that the following assertions hold.

1. For each $p \in M$ there exists an open neighborhood $U \subseteq M$ and a diffeomorphism $\varphi : P|_U = p^{-1}(U) \to U \times G$ such that the diagram

$$\begin{array}{ccc}
P|_U & \xrightarrow{\varphi} & U \times G \\
p \downarrow & & \downarrow \text{pr}_1 \\
U & \text{pr}_1 & \\
\end{array}$$



commutes where $\text{pr}_1$ is the projection onto the first component.

Such a pair $(U, \varphi)$ is referred to as a principal bundle chart.
There exists a family \( \{ U_i, \varphi_i \}_{i \in I} \) of principal bundle charts with some index set \( I \) such that \( \{ U_i \}_{i \in I} \) is an open cover of \( M \) and for any point \( p \in M \) with \( p \in U_{ij} = U_i \cap U_j \neq \emptyset \) for some \( i, j \in I \) and \( g \in G \) one has
\[
(\varphi_i \circ \varphi^{-1}_j)(p, g) = (p, \varphi_{ij}(p)g),
\]
where the transition function \( \varphi_{ij} : U_{ij} \to G \) is a smooth map and \( \varphi_{ij}(p)g \) denotes the group multiplication in \( G \). Such a family \( \{ U_i, \varphi_i \}_{i \in I} \) is referred to as principal bundle atlas.

If the structures are clear one simply refers to the total space \( P \) as corresponding principal fibre bundle.

The definition has the following immediate consequences.

**Proposition A.2.2 (Cocycle conditions and right action)**

i.) The transition functions of a principal fibre bundle satisfy the so-called cocycle conditions
\[
\varphi_{ij}(p).\varphi_{jk}(p) = \varphi_{ik}(p) \quad \text{and} \quad \varphi_{ii}(p) = e.
\]

ii.) Every principal fibre bundle is equipped with a smooth right action \( r : P \times G \to P \) of \( G \) on the fibres of \( P \) which is free and proper. This right action is well-defined by the local definition in a principal bundle chart \((U, \varphi)\) and reads
\[
r(\varphi^{-1}(p, h), g) = \varphi^{-1}(p, hg)
\]
for all \( p \in U \) and \( h, g \in G \). Sometimes, we will use the abbreviated notation \( r(u, g) = u.g \).

The principal right action is the fundamental structure of any principal fibre bundle. With the following well-known notion of a submersion one can in fact give an alternative definition of a principal bundle based on the right action.

**Definition A.2.3 (Submersion)**

A map \( p : P \to M \) is said to be a submersion at \( u \in P \) if the tangent map \( T_u p : T_u P \to T_{p(u)} M \) at the point \( u \) is surjective. The map \( p \) is called a submersion if it is a submersion at each point \( u \in P \).

Given a free and proper right action \( r : P \times G \to P \) of a Lie group on a manifold \( P \) the quotient \( M = P/G \) is again a smooth manifold and the corresponding projection \( p : P \to M \) is a surjective submersion, confer \[43\ Thm. 1.11.4\]. According to \[94\ Lemma 18.3\], \( P \) is a principal fibre bundle with structure Lie group \( G \) and right action \( r \). Thus one can identify principal fibre bundles with free and proper actions on smooth manifolds. Since it is the case for any surjective submersion any principal bundle admits local sections, this means smooth maps \( \sigma : U \to P \) with \( p \circ \sigma = \text{id}_U \) for sufficiently small and open subsets \( U \subseteq M \).

### A.3 Connections

Before we present the notion of a connection on a fibre bundle together with some of its relevant issues which all can be found in \[94\ Chap. IV\], we clarify the general notion of vector-valued differential forms. For a vector bundle \( q : E \to M \) the \( E \)-valued \( k \)-forms, \( k = 0, \ldots, \text{dim } E \), are given by
\[
\Gamma^\infty(M, \bigwedge^k T^* M \otimes E),
\]
where one makes use of the usual tensor product of vector bundles. For a vector space \( V \), the \( V \)-valued \( k \)-forms on \( M \) are defined by
\[
\Gamma^\infty(M, \Lambda^k T^* M \otimes (M \times V)) = \Gamma^\infty(M, \Lambda^k T^* M) \otimes V, \tag{A.12}
\]
where \( M \times V \) is the trivial vector bundle.

In the most general case a \textit{connection} on a manifold \( P \) is given by a vector-valued one-form \( \mathcal{P} \in \Gamma^\infty(P, T^* P \otimes TP) \cong \Gamma^\infty(P, \text{End}(TP)) \) which is a fibre projection \( \mathcal{P} \circ \mathcal{P} = \mathcal{P} \) when seen as a map \( \mathcal{P} : TP \to TP \). The image \( \text{im} \mathcal{P} \subseteq TP \) is called \textit{vertical space} and \( \ker \mathcal{P} \subseteq TP \) is referred to as \textit{horizontal space}.

### A.3.1 Connections on surjective submersions

The tangent bundle of the total space \( P \) of a surjective submersion \( p : P \to M \) is canonically equipped with a subbundle \( VP \), the so-called \textit{vertical bundle} consisting of all \textit{vertical vectors} \( V \in TP \) with \( TpV = 0 \). In other words,
\[
VP = \ker Tp \subseteq TP
\]
(A.13)
is the kernel of the tangent map \( Tp \) of the projection \( p \).

In this case, a \textit{connection} on \( P \) is a projection \( \mathcal{P} \) onto the vertical bundle \( \text{im} \mathcal{P} = VP \). The kernel defines the so-called \textit{horizontal bundle} \( HP = \ker \mathcal{P} \) which is a complementary subbundle \( HP \) to \( VP \) such that
\[
TP = VP \oplus HP. \tag{A.14}
\]
It is remarkable that the connection is already determined by the choice of a horizontal bundle \( HP \). The \textit{curvature} \( \mathcal{R} \) of the connection \( \mathcal{P} \) is the vector-valued two-form \( \mathcal{R} \in \Gamma^\infty(P, \Lambda^2 T^* P \otimes VP) \) defined by
\[
\mathcal{R}(Z, W) = \mathcal{P}[(id_{TP} - \mathcal{P})Z, (id_{TP} - \mathcal{P})W]. \tag{A.15}
\]

An important structure in the context of connections of surjective submersions is the horizontal lift of vector fields.

**Definition A.3.1 (Horizontal lift)**

Let \( p : P \to M \) be a surjective submersion with a connection given by a horizontal bundle \( HP \). Then the \textit{horizontal lift} of a vector field \( X \in \Gamma^\infty(M, TM) \) is the uniquely defined horizontal vector field \( X^h \in \Gamma^\infty(P, HP) \) which is \( p \)-related to \( X \), this means
\[
Tp \circ X^h = X \circ p. \tag{A.16}
\]

It is helpful for many applications in the main text to summarize some of the obvious properties of horizontal lifts and vertical vector fields.

**Lemma A.3.2 (Horizontal lifts and vertical vector fields)**

Let \( p : P \to M \) be a surjective submersion with a connection \( TP = VP \oplus HP \). Then for all \( X, Y \in \Gamma^\infty(M, TM) \), \( V, W \in \Gamma^\infty(P, VP) \), \( a \in C^\infty(M) \) and \( \alpha \in \Gamma^\infty(M, T^* M) \) the following equations hold.
\[
Tp \circ [X^h, Y^h] = [X, Y] \circ p, \quad Tp \circ [V, X^h] = 0, \quad Tp \circ [V, W] = 0 \tag{A.17}
\]
\[
(ax)^h = (p^* a)X^h, \tag{A.18}
\]
\[
(p^* a)(X^h) = p^*(a(X)), \quad (p^* \alpha)(V) = 0, \tag{A.19}
\]
\[
X^h(p^* a) = p^*(X(a)), \quad V(p^* a) = 0. \tag{A.20}
\]
The first equation of (A.17) yields
\[ [X^h, Y^h] - [X, Y]^h = \mathcal{R}(X^h, Y^h), \] (A.21)
which shows that the curvature of the connection is the obstruction against the horizontal lift \( h : \Gamma^\infty(M, TM) \to \Gamma^\infty(P, HP) \) being a Lie algebra homomorphism. However, Equation (A.18) points out that it is always a \( C^\infty(M) \)-module homomorphism.

Let \( \Gamma^\infty_{\text{hor}}(P, \bigwedge^* T^* P) \) denote the horizontal forms which by definition vanish under the insertion of any vertical vector. Then, for any connection \( HP \) with horizontal projection \( \Gamma^\infty(P, TP) \ni Z \to (\text{id}_{TP} - \mathcal{P})Z = Z^H \in \Gamma^\infty(P, HP) \) one can define the covariant exterior derivative
\[ d_{HP} : \Gamma^\infty(P, \bigwedge^* T^* P) \to \Gamma^\infty_{\text{hor}}(P, \bigwedge^{*+1} T^* P) \] (A.22)
by \( (d_{HP} \alpha)(Z_1, \ldots, Z_{k+1}) = (d \alpha)(Z^H_1, \ldots, Z^H_{k+1}) \) for all \( \alpha \in \Gamma^\infty(P, \bigwedge^* T^* P) \), \( k = 1, \ldots, \dim P \), and \( Z_1, \ldots, Z_{k+1} \in \Gamma^\infty(P, TP) \).

### A.3.2 Principal connections

If \( p : P \to M \) is a principal fibre bundle with structure Lie group \( G \) a connection is said to be a principal connection if it is \( G \)-invariant. For the projection \( \mathcal{P} \) this means \( T_{r_g} \circ \mathcal{P} = \mathcal{P} \circ T_{r_g} \) and the horizontal spaces then have the property \( T_{r_g} H_u P = H_{u, g} \) for all \( g \in G \) and \( u \in P \). In this case it easily follows that the horizontal lifts are \( G \)-invariant, this means
\[ r_g^*X^h = X^h \] (A.23)
for all \( g \in G \).

The exponential map \( \exp : g \to G \) has a natural generalization \( \exp : C^\infty(P, g)^G \to C^\infty(P, G)^G \) from the \( G \)-invariant \( g \)-valued functions \( \Xi \in C^\infty(P, g)^G \) with \( \Xi \circ r_g = \text{Ad}_{g^{-1}} \circ \Xi \) to the \( G \)-valued functions \( F \in C^\infty(P, G)^G \) with \( F \circ r_g = \text{Ad}_{g^{-1}} \circ F \) defined by \( \exp(\Xi)(u) = \exp(\Xi(u)) \) for all \( u \in P \). In analogy to the fundamental vector fields \( \xi_P \in \Gamma^\infty(P, VP) \) for the principal right action which clearly are vertical, one defines vector fields \( \Xi_P \in \Gamma^\infty(P, VP) \) for functions \( \Xi \in C^\infty(P, g) \) by \( \Xi_P(u) = (\Xi(u))(u) \) which have the flow \( t^\xi_P \exp = t^\exp \Xi \). The behaviour under the principal action is given by \( r_g^* \Xi_P = (\text{Ad}_g \circ \Xi \circ r_g)_P \) and \( r_g^* \xi_P = (\text{Ad}_g \xi)_P \).

The fundamental vector fields \( \xi_P \) of the principal right action give rise to a vector bundle isomorphism \( P \times g \cong VP \) by \( (u, \xi) \to \xi_P = T_{r^u} \xi \) showing that the vertical bundle is always trivial. As a consequence, a principal connection \( \mathcal{P} \) induces a \( g \)-valued one-form \( \omega \in \Gamma^\infty(P, T^* P) \otimes g \) by \( \omega(Z) = (T_{r^u})^{-1} \mathcal{P}Z \) with \( Z \in TP \). Such a form \( \omega \) has particular properties and already determines the connection by \( HP = \ker \omega \) and \( \mathcal{P}(Z) = (\omega(Z))_P \) for \( Z \in \Gamma^\infty(P, TP) \). This gives rise to the following definition.

**Definition A.3.3 (Connection one-form)**

A \( g \)-valued one-form \( \omega \in \Gamma^\infty(P, T^* P) \otimes g \) is called connection one-form if it has the following properties.

i.) \( \omega \) reproduces the generators of the fundamental vector fields,
\[ \omega(\xi_P) = \xi \quad \text{for all } \xi \in g. \] (A.24)

ii.) \( \omega : TP \to g \) is \( G \)-invariant, this means
\[ \omega \circ T_{r_g} = \text{Ad}_{g^{-1}} \circ \omega. \] (A.25)
The space of all connection one-forms is denoted by $\mathcal{C}$.

It can be easily verified that the space $\mathcal{C}$ of connection one-forms is an affine vector space over $\Gamma^\infty_{\text{hor}}(P, T^*P) \otimes \mathfrak{g}$. The covariant exterior derivative with respect to $\omega \in \mathcal{C}$ is denoted by $d_\omega$. If $\pi : V \rightarrow \text{Aut}(V)$ is a representation of the structure Lie group $G$ on a finite dimensional vector space $V$ it yields a map

$$d_\omega : \left( \Gamma^\infty(P, \wedge^k T^*P) \otimes V \right)^G \rightarrow \left( \Gamma^\infty_{\text{hor}}(P, \wedge^{k+1} T^*P) \otimes V \right)^G,$$

(A.26)

where $(\Gamma^\infty(P, \wedge^k T^*P) \otimes V)^G$ denotes the $G$-invariant $V$-valued forms $\alpha$ with respect to the induced left action $r^*_g \otimes \pi_g$, this means $r^*_g \alpha = \pi_g^{-1} \circ \alpha$. Note that the connection one-form $\omega$ itself is a particular invariant $\mathfrak{g}$-valued form in the above sense. For $\mathfrak{g}$-valued forms one can define a Lie bracket $[\cdot, \cdot]_\wedge$ by bilinear extension of

$$[\alpha \otimes \xi, \beta \otimes \eta]_\wedge = \alpha \wedge \beta \otimes [\xi, \eta]$$

(A.27)

for all $\alpha, \beta \in \Gamma^\infty(P, \wedge^k T^*P)$ and $\xi, \eta \in \mathfrak{g}$ where $\wedge$ is the anti-symmetric tensor product of differential forms.

In analogy to the connection one-form, the curvature $\mathcal{R}$ gives rise to a $\mathfrak{g}$-valued curvature two-form $\Omega \in \Omega^2(P, \mathfrak{g})^G_{\text{hor}}$ which is defined by $\Omega(Z, W) = -(T_r^* r^*)^{-1} \mathcal{R}(Z, W)$ and satisfies $\mathcal{R}(Z, W) = -([\Omega(Z, W)]_\wedge)$. It turns out that the curvature two-form for a connection one-form $\omega$ is given by the structure equation

$$\Omega = d_\omega \omega = d \omega + \frac{1}{2} [\omega, \omega]_\wedge.$$  

(A.28)

The definition of the covariant exterior derivative and $d \circ d = 0$ immediately imply the Bianchi identity

$$d_\omega \Omega = d \Omega + [\omega, \Omega]_\wedge = 0.$$  

(A.29)

### A.4 Associated vector bundles

Principal fibre bundles have the important property that any representation of the structure group on a vector space $V$ gives rise to a corresponding vector bundle with typical fibre $V$. The following proposition contains some of the most important facts used in this work. The proofs of the well-known assertions can all be found in [94].

**Proposition A.4.1 (Associated vector bundles)**

Let $p : P \rightarrow M$ be a principal fibre bundle with right action $r$ and $\pi : G \rightarrow \text{Aut}(V)$ be a left representation of the structure group $G$ on a finite dimensional vector space $V$. Moreover, let $R : (P \times V) \times G \rightarrow P \times V$ be the right action with $R((u, v), g) = (r_g u, \pi_g^{-1} v)$. Then the following assertions hold.

i.) The quotient $P \times_G V = (P \times V) / G$ given by the set of all orbits of $R$ has a unique structure of a smooth manifold such that the projection $P \times V \rightarrow (P \times V) / G$ is a surjective submersion. The equivalence class of an element $(u, v)$ is denoted by $[u, v]$.

ii.) The map $q : P \times_G V \rightarrow M$ which is well-defined by the prescription $q([u, v]) = p(u)$ defines a vector bundle with typical fibre $V$ which is called the associated vector bundle.

Altogether, one has the following commutative diagram

$$
\begin{array}{ccc}
P \times V & \xrightarrow{p} & M \\
p \downarrow & & \downarrow q \\
P & \xrightarrow{q} & P \times_G V
\end{array}
$$

(A.30)
iii.) With the choice of a principal connection $\omega$ there is a vector space isomorphism

$$s : (\Gamma_\text{hor}^\infty(P, \bigwedge^k T^* P) \otimes V)^G \to \Gamma^\infty(M, \bigwedge^k T^* M \otimes (P \times G V)), \quad (A.31)$$

which is well-defined by the prescription

$$(s(\alpha))(X_1, \ldots, X_k)(p) = [u, \alpha_u(X^h_1(u)), \ldots, X^h_k(u))] \quad (A.32)$$

for $\alpha \in (\Gamma_\text{hor}^\infty(P, \bigwedge^k T^* P) \otimes V)^G$, $X_1, \ldots, X_k \in \Gamma^\infty(M, TM)$, $p \in M$, and some $u \in P$ with $p(u) = p$. As a special case this yields the isomorphism

$$s : (\mathcal{C}^\infty(P) \otimes V)^G \to \Gamma^\infty(M, P \times G V) \quad (A.33)$$

of $\mathcal{C}^\infty(M)$-modules between the $G$-invariant $V$-valued functions on $P$ and the smooth sections of the associated bundle. Explicitly, the formulas are

$$s_f(p) = (s(f))(p) = [u, f(u)] \quad (A.34)$$

and

$$a \cdot s_f = s_{p^* a \cdot f} \quad (A.35)$$

for $f \in (\mathcal{C}^\infty(P) \otimes V)^G$ and $a \in \mathcal{C}^\infty(M)$.

iv.) With the isomorphism [A.33] for the sections of an associated bundle $E = P \times G V$ any principal connection $\omega$ induces a covariant derivative

$$\nabla^E : \Gamma^\infty(M, TM) \times \Gamma^\infty(M, E) \to \Gamma^\infty(M, E) \quad (A.36)$$

by

$$\nabla^E_X s_f = s_{X^h f} \quad (A.37)$$

for all $X \in \Gamma^\infty(M, TM)$ and $f \in \mathcal{C}^\infty(P, V)^G$. The curvature $R \in \Gamma^\infty(P, \bigwedge^2 T^* P \otimes V P)^G$ of the connection induces the curvature $R^E \in \Gamma^\infty(M, \bigwedge^2 T^* M \otimes \text{End}(E))$ of the covariant derivative and one has

$$R^E(X, Y)s_f = s(R(X^h, Y^h)f) = s((\omega([X^h, Y^h]))_P f) = \left(\nabla^E_X \nabla^E_Y - \nabla^E_Y \nabla^E_X - \nabla^E_{[X,Y]}\right)s_f. \quad (A.38)$$
Appendix B

Basic concepts of homological algebra

In the following there are presented some basic notions of homological algebra which are used in the main text. For further details, in particular for the notion of categories and functors, confer [75, Chap. 1, 3, and 6].

B.1 Complexes, cohomology and homotopy

Definition B.1.1 (Complex)
Let \( R \) be a ring. An \( R \)-complex \((C,d)\) is given by two \( \mathbb{Z} \)-indexed sets \( C = \{C_i\}_{i \in \mathbb{Z}} \) of \( R \)-modules and \( d = \{d_i\}_{i \in \mathbb{Z}} \) of \( R \)-homomorphisms \( d_i : C_i \rightarrow C_{i-1} \) such that

\[
d_{i-1} \circ d_i = 0 \tag{B.1}
\]

for all \( i \in \mathbb{Z} \).

If \((C,d)\) and \((C',d')\) are complexes a (chain) homomorphism of \( C \) into \( C' \) is an indexed set \( \alpha = \{\alpha_i\}_{i \in \mathbb{Z}} \) of homomorphisms \( \alpha_i : C_i \rightarrow C'_i \) such that the diagram

\[
\begin{array}{ccc}
C_{i-1} & \xrightarrow{d_i} & C_i \\
\alpha_{i-1} \downarrow & & \downarrow \alpha_i \\
C'_{i-1} & \xrightarrow{d'_i} & C'_i
\end{array} \tag{B.2}
\]

commutes for all \( i \in \mathbb{Z} \). Briefly, this is denoted as

\[
\alpha \circ d = d' \circ \alpha. \tag{B.3}
\]

This definition naturally induces the category \( \mathbf{R\text{-comp}} \) of \( R \)-complexes which is an abelian category, confer [75] Def. 6.1 and 6.2, since the homomorphisms between two complexes have the structure of an abelian group.

Given a complex \((C,d)\) the elements of the submodules \( Z_i = \ker d_i \) given by the kernel are called \( i \)-cycles and the elements in the image \( B_i = \text{im} d_{i-1} \) which is a submodule of \( Z_i \) because of \((B.1)\) are called \( i \)-boundaries. In addition, \( d \) is often called boundary operator.

The resulting \( R \)-module \( H_i = H_i(C) = Z_i/B_i \) is called \( i \)-th homology module or \( i \)-th homology group of the complex \((C,d)\). It follows that a map \( \alpha_i : C_i \rightarrow C'_i \) of a chain homomorphism \( \alpha \) satisfies \( \alpha_i(Z_i) \subseteq Z'_i \) and \( \alpha_i(B_i) \subseteq B'_i \) so there is an induced map \( H_i(\alpha) : H_i(C) \rightarrow H_i(C') \). Altogether one gets additive functors, the so-called \( i \)-th homology functors \( H_i \) from the category \( \mathbf{R\text{-comp}} \) of \( R \)-chain complexes to the category \( \mathbf{R\text{-mod}} \) of \( R \)-modules.
A complex with $C_i = 0$ for $i < 0$ is called positive or chain complex and the elements in a non-vanishing $C_i$ are referred to as $i$-chains. If $C_i = 0$ for $i > 0$ the complex is called negative or cochain complex. In this case one denotes $C_{-i}$ as $C^i$ whose elements are $i$-cochains and $d_{-i}$ as $d^i$, the coboundary operators or differentials. Analogously, one defines the $i$-cocycles and $i$-coboundaries as the elements of $Z^i = \ker d^i$ and $B^i = \text{im} d^{i-1}$, respectively, and finally the $i$-th cohomology group $H^i = Z^i/B^i$.

**Definition B.1.2 (Homotopy)**
Let $\alpha$ and $\beta$ be chain homomorphisms of a complex $(C, d)$ into a complex $(C', d')$. Then $\alpha$ is said to be homotopic to $\beta$ if there exists an indexed set $s = \{s_i\}_{i \in \mathbb{Z}}$ of module morphisms $s_i : C_i \to C'_i$ such that

$$\alpha_i - \beta_i = d'_i + s_i + s_{i-1} \circ d_i. \quad (B.4)$$

With some abuse of notation the set of maps $s$ is often referred to as (chain) homotopy map. It is easy to verify that homotopy, indicated by $\alpha \sim \beta$, is an equivalence relation and that there even exists a multiplication. If $\alpha, \beta : (C, d) \to (C', d')$ are homotopic with homotopy map $s$ and $\gamma, \delta : (C', d') \to (C'', d'')$ are homotopic via $t$, then one easily proves the homotopy $\gamma \circ \alpha \sim \delta \circ \beta$ with homotopy maps $u_i = \gamma_i \circ s_i + t_i \circ \beta_i$. The relevance of homotopy lies in the fact that homotopic chain morphisms $\alpha \sim \beta$ yield the same maps on the homology groups, this means $H_i(\alpha) = H_i(\beta)$.

### B.2 Projective resolutions and derived functors

Projective resolutions are a crucial structure for the investigation of (co)homologies in the purely algebraic framework. Before we come to the definition it is necessary to recall the notion of a projective module, confer [86, Chap. 1, §2A].

**Definition B.2.1 (Projective module)**
An $R$-module $P$ is called projective if for every homomorphism $f : P \to N$ and every epimorphism $p : M \to N$ there exists a homomorphism $h : P \to N$ such that $f = p \circ h$.

The defining property can be visualized in the diagram

\[
\begin{array}{ccc}
P & \xrightarrow{h} & N \\
\downarrow{f} & & \downarrow{p} \\
M & \xrightarrow{p} & N & \to 0.
\end{array}
\]

**Remark B.2.2 (Equivalent definitions)**
There exist the following further equivalent characterizations of a projective (right) $R$-module $P$, see [86, Chap. 1, §2A].

i.) There exists another $R$-module $Q$ such that the direct sum $P \oplus Q$ is a free $R$-module, this means $P \oplus Q \cong R^I = \bigoplus_{i \in I} R_i$ with $R_i = R$ for each $i$ in some index set $I$.

ii.) There exists a dual basis, this means there exist families of elements $\{e_i\}_{i \in I} \subseteq P$ and of $R$-linear functionals $\{e^i\}_{i \in I} \subseteq \text{Hom}_R(P, R)$ for some index set $I$ such that for any $p \in P$ one has $e^i(p) = 0$ for all except finitely many $i \in I$ and

\[
p = \sum_{i \in I} e_i e^i(p). \quad (B.6)
\]

Now it is possible to define the important notion of a projective resolution.
**Definition B.2.3 (Complex over and resolution of a module)**

Let $M$ be an $R$-module. A complex over $M$ is a positive complex $C = (C, d)$ together with a homomorphism $\epsilon : C_0 \to M$, called an augmentation, such that $\epsilon \circ d_1 = 0$. The complex $(C, \epsilon)$ over $M$ is called a resolution of $M$ if the sequence

$$0 \leftarrow M \leftarrow^\epsilon C_0 \leftarrow^d_1 C_1 \leftarrow^\cdots \leftarrow^d_{n-1} C_{n-1} \leftarrow^d_n C_n \leftarrow^\cdots$$  \hspace{1cm} (B.7)

is exact, this means if $\text{im } d_n = \text{ker } d_{n-1}$ for all $n \in \mathbb{N}$, $\text{im } d_1 = \text{ker } \epsilon$ and $\epsilon$ is surjective. Further, the complex $(C, \epsilon)$ over $M$ is called projective, if every $C_i$ is projective.

Projective resolutions and even free resolutions with free $C_i$ always exist, confer [75, Sect. 6.5]. For a resolution the exactness of (B.7) implies that the homologies $H_n(C) = 0$ are trivial for $i > 0$ and that $H_0(C) = C_0/\text{ker } \epsilon \cong M$. Concerning the above structures one now has the following important result.

**Theorem B.2.4**

Let $(C, \epsilon)$ be a projective complex over the module $M$ and let $(C', \epsilon')$ be a resolution of the module $M'$. Further let $\mu : M \to M'$ be a homomorphism. Then there exists a chain homomorphism $\alpha$ of the complex $C$ into $C'$ such that $\mu \circ \epsilon = \epsilon' \circ \alpha_0$. Moreover, any two such homomorphisms $\alpha$ and $\beta$ are homotopic.

The proof consists of a multiple application of the above stated property of projective modules. The assertion of Theorem [B.2.4] belongs to the diagram

$$0 \leftarrow M \leftarrow^\epsilon C_0 \leftarrow^\cdots \leftarrow^\alpha_{n-1} C_{n-1} \leftarrow^\alpha_n C_n \leftarrow^\cdots$$  \hspace{1cm} (B.8)

An important application of the above structures is the definition of derived functors of an additive functor $F$ from the category $R\text{-mod}$ to the category $\text{Ab}$ of abelian groups. Note that a functor between categories where the sets of morphisms all are abelian groups, is called additive if the corresponding map between the morphisms is additive, confer [75, Sect. 3.1].

For an $R$-module $M$ let

$$0 \leftarrow M \leftarrow^\epsilon C_0 \leftarrow^d_1 C_1 \leftarrow^d_2 \cdots$$  \hspace{1cm} (B.9)

be a projective resolution of $M$. Application of a covariant additive functor $F$ then yields a sequence of homomorphisms of abelian groups

$$0 \leftarrow F(M) \leftarrow^\epsilon F(C_0) \leftarrow^d_1 F(C_1) \leftarrow^d_2 \cdots$$  \hspace{1cm} (B.10)

since $F(d_i) \circ F(d_{i-1}) = F(d_i \circ d_{i-1}) = 0$. This sequence is not necessarily exact, so there can occur nontrivial homology groups $H_n(F(C))$. Now consider a projective resolution $(C', \epsilon')$ of a different module $M'$ with a corresponding homomorphism $\mu : M \to M'$. Due to Theorem [B.2.4] there exists a chain homomorphism $\alpha : C \to C'$ with $\mu \circ \epsilon = \epsilon' \circ \alpha_0$ and two such homomorphism $\alpha, \beta$ are always homotopic. The functor $F$ then yields chain homomorphisms $F(\alpha)$ with $F(\mu) \circ F(\epsilon) = F(\epsilon') \circ F(\alpha_0)$ and since it is additive the homotopy $\alpha \sim \beta$ translates to $F(\alpha) \sim F(\beta)$. In particular, if $M = M'$ and $\mu = \text{id}$ it follows that $H_n(F(\alpha)) : H_n(F(C)) \to H_n(F(C'))$ is an isomorphism.
Altogether it follows that for the given covariant functor $F$ and every $n \in \mathbb{N}_0$ one has found another additive functor $L_nF$, the so called $n$-th left derived functor from $\mathcal{R}\text{-mod}$ to $\mathbb{A}b$, given by

$$L_nF(M) = H_n(F(C)),$$
$$L_nF(\mu) = H_n(F(\alpha))$$

for every $\mathcal{R}$-module $M$ and every morphism $\mu : M \rightarrow M'$. In particular, one has $L_0F(M) = F(C_0)/\text{im } F(d_1)$. Due to the above results the right hand sides of (B.11) and (B.12) do not depend on the resolution of $M$ and the chain homomorphism $\alpha$ over $\mu$. If the functor $F$ is contravariant its application to (B.9) yields a cochain complex

$$0 \longrightarrow F(M) \xrightarrow{F(\epsilon)} F(C_0) \xrightarrow{F(d_1)} F(C_1) \xrightarrow{F(d_2)} \cdots$$

With the same arguments as above and using the cohomology functors $H^n$ one gets the $n$-th right derived functor $R^nF$. There one has in particular $R^0F(M) = \ker F(d_1)$.

The following example of derived functors is essential for the discussion of Hochschild cohomologies as explained in Section 5.1. Let $\mathcal{R}$ be a ring and $N$ be an $\mathcal{R}$-module. Then consider the contravariant additive functor

$$\text{hom}(\cdot, N) : \mathcal{R}\text{-mod} \rightarrow \mathbb{A}b,$$

given by

$$\text{hom}(\cdot, N)(M) = \text{Hom}_\mathcal{R}(M, N)$$

for all $\mathcal{R}$-modules $M$ and

$$\text{hom}(\cdot, N)(\mu : M \rightarrow M') = \left( \mu^* : \text{Hom}_\mathcal{R}(M', N) \xrightarrow{\nu \mapsto \mu^* \nu} \text{Hom}_\mathcal{R}(M, N) \right)$$

for all homomorphisms $\mu : M \rightarrow M'$. The $n$-th right derived functor of $\text{hom}(\cdot, N)$ is denoted by

$$\text{Ext}_\mathcal{R}^n(\cdot, N) = R^n\text{hom}(\cdot, N),$$

and its value for a module $M$ with an arbitrary projective resolution $(C, \epsilon)$ is thus given by

$$\text{Ext}_\mathcal{R}^n(M, N) = H^n(\text{Hom}_\mathcal{R}(C, N)).$$

If the context is clear, this means for fixed modules $M$ and $N$, we refer to the latter as the $n$-th Ext group. Since $\text{hom}(\cdot, N)$ is left exact the exactness of $C_1 \rightarrow C_0 \xrightarrow{\epsilon} M \rightarrow 0$ implies that of $0 \rightarrow \text{Hom}_\mathcal{R}(M, N) \xrightarrow{\epsilon^*} \text{Hom}_\mathcal{R}(C_0, N) \rightarrow \text{Hom}_\mathcal{R}(C_1, N)$ and it follows that

$$\text{Ext}_\mathcal{R}^0(M, N) \cong \text{Hom}_\mathcal{R}(M, N).$$
Appendix C

Explicit chain maps of the bar and Koszul complex

This appendix provides the missing computations concerning the maps $F_k : K_k \to X_k$ and $G_k : X_k \to K_k$ defined by the Equations (5.49) and (5.50) in Section 5.4. First, it will be shown that they are indeed chain maps between the bar and Koszul complex. With the definitions (5.16) and (5.39) for the differentials $d^X_k$ and $d^K_k$ and the notation as in Section 5.4 one computes for $k \geq 1$

$$(d^K_k F_k \omega)(v, q_1, \ldots, q_{k-1}, w)$$

$$= (F_k \omega)(v, q_1, \ldots, q_{k-1}, w) + \sum_{i=1}^{k-1} (-1)^i (F_k \omega)(v, q_1, \ldots, q_i, q_i, \ldots, q_{k-1}, w)$$

$$+ (-1)^k (F_k \omega)(v, q_1, \ldots, q_{k-1}, w, w)$$

$$= \omega(v, w)(0, q_1 - v, \ldots, q_{k-1} - v) + \sum_{i=1}^{k-1} (-1)^i \omega(v, w)(q_1 - v, \ldots, q_i - v, q_i - v, \ldots, q_{k-1} - v)$$

$$+ (-1)^k \omega(v, w)(q_1 - v, \ldots, q_{k-1} - v, w - v)$$

$$= (-1)^k (-1)^{-k-1} \omega(v, w)(v - w, q_1 - v, \ldots, q_{k-1} - v)$$

$$= (d^K_k \omega)(v, w)(q_1 - v, \ldots, q_{k-1} - v)$$

$$= (F_{k-1} d^K_k \omega)(v, w)(v, q_1, \ldots, q_{k-1}, w).$$

This shows the desired equation $d^K_k \circ F_k = F_{k-1} \circ d^K_k$. For $G_k$ the computation is longer. In a first step one has

$$(d^K_k G_k \chi)(v, w)$$

$$= i_k(v - w) (e^{i1} \wedge \cdots \wedge e^{ik}) \int_0^1 dt_1 \cdots \int_0^{t_{k-1}} dt_k \frac{\partial^k \chi}{\partial q_1^{i1} \cdots \partial q_k^{i_k}} (v, t_1 v + (1 - t_1)w, \ldots, t_k v + (1 - t_k)w, w)$$

$$= \sum_{j=1}^k (-1)^{j+1} e^{i1} \wedge \cdots \wedge \hat{\wedge} \cdots \wedge e^{ik} (v^{j_1} - w^{j_1}) \int_0^1 dt_1 \cdots \int_0^{t_{k-1}} dt_k \frac{\partial^k \chi}{\partial q_1^{i1} \cdots \partial q_k^{i_k}} (v, q_1(t_1), \ldots, q_k(t_k), w)$$

$$= \sum_{j=1}^k (-1)^{j+1} e^{i1} \wedge \cdots \wedge \hat{\wedge} \cdots \wedge e^{ik} \int_0^1 dt_1 \cdots \int_0^{t_{k-1}} dt_k \frac{\partial}{\partial t_j} \left( \frac{\partial^k \chi}{\partial q_1^{i1} \cdots \hat{j} \cdots \partial q_k^{i_k}} (v, q_1(t_1), \ldots, q_k(t_k), w) \right)$$

where $\hat{j}$ means to leave out the $j$-th term of an expression. Here and in the following we use the abbreviation $q_r(t_i) = t_i v + (1 - t_i) w$ for $r, i = 1, \ldots, k$ where the first index $r$ denotes the
position of the argument. For \( j = 2, \ldots, k - 1 \) we use the fact that for \( f \in C^\infty(\mathbb{R}^2) \) one has
\[
\int_0^{t_j} \frac{d}{dt_j} f(t_j, t_{j+1}) dt_j = \frac{d}{dt_j} \left( \int_0^{t_j} f(t_j, t_{j+1}) dt_j \right) - f(t_j, t_j) \text{ and that subsequent integration yields}
\[
\int_0^{t_{j-1}} \int_0^{t_j} \frac{d}{dt_j} f(t_j, t_{j+1}) dt_j dt_{j+1} = \int_0^{t_{j-1}} f(t_{j-1}, t_{j+1}) dt_{j-1} - \int_0^{t_{j-1}} f(t_j, t_j). \tag{C.1}
\]
For \( j = 1 \) the same argument is applied in the form
\[
\int_0^{1} \int_0^{t_1} \frac{d}{dt_1} f(t_1, t_2) dt_1 dt_2 = \int_0^{1} f(1, t_2) dt_2 - \int_0^{1} f(t_1, t_1) \tag{C.2}
\]
noting that \( q_1(1) = v \). In the case \( j = k \) the integration can be carried out directly. After setting \( t_0 = 1 \) the second summands in the above expressions \( \text{(C.1)} \) and \( \text{(C.2)} \) have the same structure whereby we can combine them into one sum.

\[
(d_k G_k \chi)(v, w) = e^{i1} \wedge \ldots \wedge e^{i_k} \int_0^{1} \int_0^{t_2} \ldots \int_0^{t_{k-1}} \frac{\partial^{k-1} \chi}{\partial q_1^{i_1} \ldots \partial q_k^{i_k}}(v, q_1(t_1), \ldots, q_{k-1}(t_{j-1}), q_j(t_j), q_{j+1}(t_{j+1}) \ldots, q_k(t_k), w)
\]

\[
+ \sum_{j=2}^{k-1} (-1)^{j+1} e^{i_1} \wedge \ldots \wedge e^{i_j} \ldots \int_0^{1} \int_0^{t_2} \ldots \int_0^{t_{j-1}} \frac{\partial^{k-1} \chi}{\partial q_1^{i_1} \ldots \partial q_k^{i_k}}(v, q_1(t_1), \ldots, q_{j-1}(t_{j-1}), q_j(t_j), q_{j+1}(t_{j+1}) \ldots, q_k(t_k), w)
\]

\[
- \sum_{j=1}^{k-1} (-1)^{j+1} e^{i_1} \wedge \ldots \wedge e^{i_j} \ldots \int_0^{1} \int_0^{t_2} \ldots \int_0^{t_{j-1}} \frac{\partial^{k-1} \chi}{\partial q_1^{i_1} \ldots \partial q_k^{i_k}}(v, q_1(t_1), \ldots, q_{j-1}(t_{j-1}), q_j(t_j), q_{j+1}(t_{j+1}) \ldots, q_k(t_k), w)
\]

\[
+ (-1)^{k+1} e^{i_1} \wedge \ldots \wedge e^{i_{k-1}} \int_0^{1} \int_0^{t_{k-2}} \frac{\partial^{k-1} \chi}{\partial q_1^{i_1} \ldots \partial q_{k-1}^{i_{k-1}}}(v, q_1(t_1), \ldots, q_{k-1}(t_{k-1}), q_k(t_{k-1}), w)
\]

\[
- \frac{\partial^{k-1} \chi}{\partial q_1^{i_1} \ldots \partial q_{k-1}^{i_{k-1}}}(v, q_1(t_1), \ldots, q_{k-1}(t_{k-1}), q_k(t_{k-1}), w, w).
\]

Now we rename the indices \((i_1, \ldots, i_k)\) and the variables \((t_1, \ldots, t_k)\) in the first three terms such that the expressions \(e^{i_1} \wedge \ldots \wedge e^{i_{k-1}}\) and the integrals \(\int_0^{1} \int_0^{t_{k-2}}\) are the same for all summands and do not depend on the summation parameter \( j \). As a consequence, the derivatives as well as the arguments of \( \chi \) are changed. After a change \( j' = j - 1 \) of the summation index in the second term
the penultimate expression can be seen as a summand and raises the upper bound of the resulting sum. This gives
\[
(d_k^k G_k \chi)(v, w) = e^{i_1} \wedge \ldots \wedge e^{i_k-1} \int_0^{t_{k-2}} dt_1 \ldots \int_0^{t_{k-1}} dt_k \frac{\partial^{k-1} \chi}{\partial q_1^{i_1} \ldots \partial q_k^{i_{k-1}}} (v, q_1(t_1), \ldots, q_k(t_{k-1}), w)
\]
\[
+ \sum_{j=1}^{k-1} (-1)^j \frac{\partial^{k-1} \chi}{\partial q_1^{i_1} \ldots \partial q_j^{i_j} \partial q_{j+1}^{i_{j+1}} \ldots \partial q_k^{i_{k-1}}} (v, q_1(t_1), \ldots, q_j(t_j), q_{j+1}(t_{j+1}), \ldots, q_k(t_{k-1}), w)
\]
\[
+ (-1)^k \frac{\partial^{k-1} \chi}{\partial q_1^{i_1} \ldots \partial q_k^{i_k}} (v, q_1(t_1), \ldots, q_k(t_{k-1}), w, w).
\]
\[
= e^{i_1} \wedge \ldots \wedge e^{i_k-1} \int_0^{t_{k-2}} dt_1 \ldots \int_0^{t_{k-1}} dt_k \frac{\partial^{k-1} d_k^X \chi}{\partial q_1^{i_1} \ldots \partial q_k^{i_k}} (v, q_1(t_1), \ldots, q_k(t_{k-1}), w)
\]
\[
= (G_{k-1} d_k^X \chi)(v, w).
\]
The penultimate equation becomes clear with the definition (5.16) of the boundary operator $d_k^X$ of the bar complex and so we have shown $d_k^X \circ G_k = G_{k-1} \circ d_k^X$.

The verification of $G_k \circ F_k = \text{id}_{K_k}$ for $k \geq 0$ is a simple computation. With (5.49) and (5.43) one has
\[
(F_k \omega)(v, q_1, \ldots, q_k, w) = \frac{1}{k!} \omega_{j_1 \ldots j_k}(v, w) e^{j_1} \wedge \ldots \wedge e^{j_k} (q_1 - v, \ldots, q_k - v)
\]
\[
= \omega_{j_1 \ldots j_k}(v, w) (q_1^{i_1} - v^{i_1}) \ldots (q_k^{i_k} - v^{i_k}).
\]
With the same notation as above, this and $\int_0^{t_{k-1}} dt_k = \frac{1}{k!}$ gives
\[
(G_k F_k \omega)(v, w) = e^{i_1} \wedge \ldots \wedge e^{i_k} \int_0^{t_{k-1}} dt_k \frac{\partial^{k} (F_k \omega)}{\partial q_1^{i_1} \ldots \partial q_k^{i_k}} (v, q_1(t_1), \ldots, q_k(t_k), w)
\]
\[
= e^{i_1} \wedge \ldots \wedge e^{i_k} \int_0^{t_{k-1}} dt_k \omega_{i_1 \ldots i_k} (v, w)
\]
\[
= \frac{1}{k!} \omega_{i_1 \ldots i_k}(v, w) e^{i_1} \wedge \ldots \wedge e^{i_k} = \omega(v, w)
\]
which proves the statement.
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