Real-time modeling of three-dimensional granular intrusion

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Granular intrusion is common in a range of processes including ballistic impact and penetration problems, and locomotion of humans, animals, and vehicles in natural terrains. The computational cost of full-scale numerical modeling is high, whereas a capability to model such scenarios in real-time is critical in applications such as path planning and efficient maneuvering of vehicles in sandy terrains and extraterrestrial environments. Existing reduced-order methods that model intrusion have limited capabilities due to their shape- and media-specific forms. This work formulates a reduced-order modeling technique, 3-dimensional Resistive Force Theory (3D-RFT), capable of accurately and quickly predicting the resistive force on arbitrary-shaped bodies moving in grains. Aided by a continuum mechanical description of the granular bed, a comprehensive set of symmetry constraints, and a large amount of reference data, we develop a self-consistent and accurate form for 3D-RFT. We verify the model capabilities in a wide range of cases and show it can be quickly recalibrated to different media and intruder surface types.

1. Introduction

Generic RFT form. The Resistive Force Theory methodology was originally introduced by Gray and Hancock (12) for modeling self-propelling undulatory biological systems in viscous fluids. In this model, a simple approximate formula for the resistive force on a segment of a thin body is derived from the Stokes equations as a function of the segment’s velocity components, orientation, and a few variables characterizing the fluid-segment interaction. Importantly, the theory assumes decoupling of the forces over the various segments of the body (13). The success of fluid RFT motivated multiple studies (14, 16, 17) to explore the existence of a similar theory in granular media.

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Significance Statement

This work proposes a general theory for modeling diverse granular intrusion problems such as animal and human locomotion in sands and other natural terrains. Respecting numerous physical constraints, the theory allows for modeling arbitrary motion of three-dimensional generally-shaped objects in granular media in near real-time. This theory is a crucial step in developing robust, reliable, and fast methods for modeling granular interactions. The work also provides a blueprint for developing physics-informed reduced-order models for other media.

In this work, we introduce a method for predicting the granular resistive forces on arbitrarily shaped three-dimensional bodies moving in arbitrary ways in a granular media. This work extends the work of 2-dimensional granular resistive force theory (2D-RFT) proposed by Li et al.(14) to 3-dimensions. Attempts to extend RFT to 3D intruders have only recently been explored (5, 15). However, the previous approaches have limited robustness as the forms chosen lack a direct connection to granular mechanics, they do not always satisfy the needed self-consistency constraints from symmetry, and they were tested against non-exhaustive input and output data sets which masked certain incorrect force regimes (see Sec 1 of Supplemental Information). We overcome these limitations to propose a 3D-RFT framework that is easy to use and quick to recalibrate for a variety of granular media and surface roughnesses. The analysis is informed by a large data-set generated using a validated granular mechanics continuum model. We then examine predictions of the resulting 3D-RFT against a variety of granular intrusion tests, consisting of the arbitrary motion of many symmetric and asymmetric shapes in beds of granular media, and find an excellent agreement between the reference results and 3D-RFT predictions both globally (total intrusion force and moment) and locally (surface stress distribution).

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low-speeds, the rate-independent nature of granular media (characterized by low values of the non-dimensional (micro-) inertial number $I$ (18–20) and macro-inertial number $I_{mac}$, see Materials and Methods) makes the dependence of intrusion force independent of the velocity magnitude. Assuming material strength increases with pressure and that pressure is primarily due to gravity, Li’s 2D-RFT model has the following form:

$$F_{total} = \int_{surf} \left(|\alpha_{z}^{mat}(\beta, \gamma)| \right) \left| z \right| ds.$$  \hspace{1cm} (1)

Here, $F_{total}$ represents the total force on an intruding surface, which is divided into smaller planar sub-surface elements of area $ds$ and depth $|z|$ from the free surface. The tilt angle $\beta$ and angle of attack $\gamma$ characterize the orientation and motion of each surface element of the intruding body (Fig. 1). The vector-valued function of angles $\alpha_{1}^{mat} = \left(\alpha_{z}^{mat}(\beta, \gamma), \alpha_{z}^{mat}(\beta, \gamma)\right)$ represents the force per area unit per unit depth; this function must be obtained a priori through experiments or simulations of plate drag and depends on the material properties of the granular media, the intruder surface interaction, and the value of gravity. Of note, Eq 1 assumes no correlation between the forces on different sub-surfaces; only details local to a surface element determine the force on that element (16). A comprehensive comparison of various existing reduced order methods for modeling granular intrusions, including 2D-RFT and a terramechanical model, can be referred to from Agarwal et al. (2).

In recent years, it has been shown that plasticity-based PDE models can also obtain the form of 2D granular RFT (21). More recently, the performance of the continuum approach in modeling a variety of granular intrusions has been demonstrated for wheeled locomotion, impact and penetration, and multi-body intrusion (2, 22–25). Additionally, the approach also provides insight into the somewhat surprising observation that granular RFT is often more accurate than its viscous fluid counterpart (12, 13). Thus, while experimental observations primarily drove the original RFT discoveries, the availability of faster computational methods, the success of 2D-RFT, and a need for better real-time 3D granular intrusion methods have driven the exploration of 3D-RFT. Our work combines the capabilities of the continuum approach with a few symmetry requirements and DEM data to accurately and efficiently model the physics of 3-dimensional granular intrusion to develop a 3D-RFT. We briefly discuss the details of the continuum approach next.

2. Continuum modeling

We use continuum modeling as the primary theoretical motivator as well as a reference data generation tool in this work. The constitutive model we use (22, 23) has a dense response characterized by a rate-insensitive, non-dilatant frictional flow rule, but also models the separated state which allows material to become stress-free when below a critical density. The model has been validated in a number of previous studies of granular intrusion and locomotion (21, 25–28).

The constitutive flow equations representing the material’s separation behavior, shear yield condition, and tensorial co-directionality, respectively, are shown below:

$$(\rho - \rho_c)P = 0 \quad \text{and} \quad P \geq 0 \quad \text{and} \quad \rho \leq \rho_c,$$

$$\dot{\gamma}(\tau - \mu_{int}P) = 0 \quad \text{and} \quad \dot{\gamma} \geq 0 \quad \text{and} \quad \tau \leq \mu_{int}P,$$

$$D_{ij}/\dot{\gamma} = \sigma_{ij}^{\prime}/2\tau \quad \text{if} \quad \dot{\gamma} > 0 \quad \text{and} \quad P > 0$$  \hspace{1cm} (2)

where, $i, j = 1, 2, 3$. In these equations, $\sigma$ represents the Cauchy stress tensor and $\sigma_{ij}^{\prime} = (\sigma_{ij} + P\delta_{ij})$ represents the deviatoric part of $\sigma$ where $P = -\rho \tau / 3$ represents the hydrostatic pressure, $\tau = \sqrt{\sigma_{ij}^{\prime} \sigma_{ij}^{\prime}/2}$ represents the equivalent shear stress and $\mu_{int}$ and $\rho_c$ represent the bulk friction coefficient and critical close-packed density of the granular volume. $D_{ij} = (\partial v_i / \partial x_j + \partial v_j / \partial x_i)$ represents the (plastic) flow rate tensor, and $\dot{\gamma} = \sqrt{2D_{ij} D_{ij}}$ represents the equivalent shear rate. We assume the surface friction coefficient $\mu_{int}$ describes the interaction of the granular continuum with intruder surfaces.

We use a 3D Material Point Method (MPM) solver from Baumgarten and Kamrin (29) to implement the continuum modeling in this study, which has been successfully used for modeling complex problems in the past (29, 30). We have also validated the accuracy of the continuum model against experiments for a specific set of 3D plate intrusions, which justify the use of the continuum solver for generating 3D-RFT reference data. The details of the validation studies are provided in the Methods and Materials section.

3. 3D-RFT premises

We begin by systematically summarizing the physical assumptions and constraints we use to arrive at a generic 3D-RFT (the next section will provide further details on how these premises are used). These premises act as the guiding principles for establishing the form of 3D-RFT and are also useful for understanding the limits of 3D-RFT.

1. **Premise-1: Localization and decoupling of forces**
   This hypothesis assumes that the force on any sub-surface of an intruding body is independent of the motion or configuration of any other sub-surface of the body. This is the primary order-reduction hypothesis inherent to all RFT’s, e.g. it is used in Eq 1.

2. **Premise-2: Leading-edge hypothesis**
   This hypothesis assumes that only the leading edges of a body (surfaces moving ‘into’ and not ‘away from’ a granular volume) experience non-negligible resistive force. Mathematically, such surfaces have a positive dot product between their surface normal and velocity direction ($\mathbf{v} \cdot \mathbf{n} \geq 0$). The hypothesis applies only to non-cohesive media. Section S5 in the Supplementary Information provides evidence in support of this hypothesis in three dimensions.
3. **Premise-2: Validity of the continuum model** We suppose that the medium is well-represented by the continuum model shown in Eq 2, which is used to guide the theory for 3D-RFT. This is akin to other advances in RFT stemming from this same basic continuum model (21, 25). Consequently, the resistive force on a plate depends on the same limited set of material parameters that govern the continuum model: \( \rho_c \) (the critical density), \( \mu_{\text{int}} \) (the internal friction), and \( \mu_{\text{surf}} \) (the media-surface friction).

4. **Premise-4: Force dependence on material internal friction** We use extensive data analysis to explore the dependence of resistive force on \( \mu_{\text{int}} \). From this, we assert that the normal force is relatively uninfluenced by \( \mu_{\text{surf}} \) for a large range of internal friction (\( \mu_{\text{int}} = 0.3 - 0.9 \)) (see Fig S2). Between different \( \mu_{\text{int}} \), the normal forces appear to only vary by a multiplicative scalar factor \( \xi_n \) as discussed in Sec 4.

5. **Premise-5: Force dependence on media-intruder surface friction** We use a similar analysis as done in the previous premise to explore the dependence of surface tractions on \( \mu_{\text{surf}} \). From this, we assert that the tangential forces strongly depend on \( \mu_{\text{surf}} \). Specifically, we observe that plate-tangential resistive forces generated at a higher \( \mu_{\text{surf}} \) can be used to generate the tangential force for a lower \( \mu_{\text{surf}} \); the dependences at the lower surface friction can be obtained by limiting the magnitude of the tangential force based on the Coulomb friction limit at the lower \( \mu_{\text{surf}} \). Detailed material response graphs in this regard can be found in Sec S3 of the Supplementary Information.

6. **Premise-6: Isotropy of the drag force relation** Any function providing the intrusion force on an intruder must obey a symmetry relationship whereby if the entire problem is rotated by some amount — that is the free-surface, gravity, intruder orientation/position, and intruder velocity are all rotated the same amount — then the resistive force must also rotate by this common global rotation. As we will show, this constraint, which implies the drag force relations are isotropic functions of their inputs, imposes a rather strong restriction on the three-dimensional form that 3D-RFT can take.

7. **Premise-7: Consistency with lower-dimensional RFT** We desire a 3D-RFT model that collapses back to the previously defined 2D-RFT description in the appropriate limits. Thus, in line with the angle-based characterization of 2D-RFT by Li et al. (14) (Fig 1), we desire to ultimately express 3D-RFT in terms of similar characteristic angles \( \beta \) and \( \gamma \) and a new twist angle \( \psi \) representing the angle between the planes of plate normal and velocity direction with the vertical.

In addition to the above premises, we will utilize a few operational constraints. We limit ourselves to quasi-static intruder motions, with negligible inertial effects in the granular media. This was also assumed in the original 2D-RFT formulation and lets the force on a sub-surface be deemed independent of the surface’s speed. More recently, an inertia-sensitive 2D-RFT has also been proposed and validated (25). We limit our attention to quasi-static cases in this work (See Materials and Methods section for more details). We also require that intruders are only submerged to a shallow depth. This comes from limits on the linearity of granular material’s resistance with depth \( |z| \) in a gravity-loaded system and limits us to considering depths only up to a \( O(1) \) factor of the size of the object being intruded (24, 31). Lastly, the RFT form assumes grains to be small relative to the size-scale of the intruder. RFT is expected to have reduced accuracy along intruder surfaces that sharply vary; direct grain-size effects may be important to determining the resistive force on these subsurfaces.

4. **Proposed form of 3D-RFT**

In light of Premise 1, we propose a 3D-RFT taking the following form:

\[
F_{\text{total}} = \int_{\text{surf}} \alpha^\text{mat}(\hat{n}, \hat{v}, \hat{g})|z| \, ds. \tag{3}
\]

Here, the \( \alpha^\text{mat} \) function is expressed in terms of three vectors, the outward surface normal \( \hat{n} \), velocity direction \( \hat{v} \), and gravity \( \hat{g} \). Note that 2D-RFT could be expressed equally well in this form.

In particular, referring to Premise 3, we write

\[
\alpha^\text{mat}(\hat{n}, \hat{v}, \hat{g}) = \alpha(\hat{n}, \hat{v}, \hat{g}; \rho_c, \mu_{\text{int}}, \mu_{\text{surf}}). \tag{4}
\]

Further, we assume the force per area per depth can be decomposed into normal and tangential parts with the following property dependences, which align with Premises 4 and 5:

\[
\alpha = \alpha_n(\hat{n}, \hat{v}, \hat{g}; \rho_c, \mu_{\text{int}}) + \alpha_t(\hat{n}, \hat{v}, \hat{g}; \rho_c, \mu_{\text{int}}, \mu_{\text{surf}}), \tag{5}
\]

where \( \alpha_n \) and \( \alpha_t \) represent normal and tangential contributions to \( \alpha \). We propose the following simplifying functional forms consistent with Premises 4 and 5

\[
\alpha_n = \rho_c \hat{g} f(\mu_{\text{int}}) \left[ \alpha^\text{gen}_n(\hat{n}, \hat{v}, \hat{g}) \right] (-\hat{n}) \tag{6}
\]

\[
\alpha_t = \rho_c \hat{g} f(\mu_{\text{int}}) \min \left( \frac{\mu_{\text{surf}}}{\mu_{\text{surf}}} \left| \alpha^\text{gen}_n(\hat{n}, \hat{v}, \hat{g}) \right| \left| \alpha^\text{gen}_t(\hat{n}, \hat{v}, \hat{g}) \right|, 1 \right) \alpha^\text{gen}_t(\hat{n}, \hat{v}, \hat{g}). \tag{7}
\]

The prefactor \( \rho_c \hat{g} f(\mu_{\text{int}}) \), which we collectively refer to as \( \xi_n \), is a media dependent scaling coefficient reflecting the overall intrusive strength of the system. The presence of the multiplicative \( \hat{g} \) factor is required on dimensional grounds (21). Additionally, we find that \( f \) follows a roughly cubic dependence on \( \mu_{\text{int}} \) (see Fig S3 in Supplementary Information for more details). This dependence is in accord with the observations of many researchers in the past in simpler vertical intrusions of flat plates in granular volumes (a sub-case of 3D-RFT)(32). Section S2 of Supplementary Information provides the detailed material response graphs in this regard.

The generic RFT functions shown with superscript ‘gen’ are labeled as such because we model them to be approximately universal across all granular/intruder systems. The min function in the formula for \( \alpha_t \) acts as a cut-off that ensures the ratio of tangential and normal force on a plate element does not exceed the media-surface friction coefficient \( \mu_{\text{surf}} \), as per Premise 5.

The largest meaningful value of \( \mu_{\text{surf}} \) is \( \mu_{\text{surf}} = \mu_{\text{int}} \). Which is the fully-rough limit wherein the min function is always 1. In this case, we have \( \alpha = \xi_n \alpha^\text{gen}_n + \alpha^\text{gen}_t \) where

\[
\alpha^\text{gen} \equiv \alpha^\text{gen}_n + \alpha^\text{gen}_t. \tag{8}
\]
Symmetry constraints on 3D-RFT formulation. We use symmetry constraints inherent to the drag problem (Premise 6) to constrain the functional form of $\alpha^{\text{gen}}$. Our strategy is to constrain the function space to satisfy symmetry constraints by design rather than leaving it to chance based on the choice of fit functions. Moreover, by enforcing the symmetry constraints directly, we reduce the space of admissible functions, thereby reducing the amount of fitting that must be done.

Consider a small plate intruder characterized with $\hat{n}$, $\hat{v}$, $ds$, $|z|$, and $g$. For $\mu_{\text{surf}} = \mu_{\text{int}}$, the force on the plate according to RFT is $df = \xi_v \alpha^{\text{gen}}(\hat{n}, \hat{v}, g)|z|ds$. If the entire system is rotated — including the intruder, the granular bed, and gravity — the resistive force on the intruder must rotate by the same amount. This is because rotating the entire system should be consistent with a fixed system and a rotation of the observer. Figure 3A visualizes this action. Thus, for any rotation $R$, we expect that $Rdf = \xi_v \alpha^{\text{gen}}(R\hat{n}, R\hat{v}, Rg)|z|ds$, and thus

$$\alpha^{\text{gen}}(R\hat{n}, R\hat{v}, Rg) = R\alpha^{\text{gen}}(\hat{n}, \hat{v}, \hat{g}).$$

This ‘global rotation constraint’ implies $\alpha^{\text{gen}}$ is an isotropic function of its inputs. Thus, in accord with Isotropic Representation Theory (IRT)(33) the function must have the following specific form:

$$\alpha^{\text{gen}}(\hat{n}, \hat{v}, \hat{g}) = f_1\hat{n} + f_2\hat{v} + f_3\hat{g},$$

where $f_1$, $f_2$, and $f_3$ are three mutually-independent arbitrary scalar-valued functions of coordinate-invariant dot-products between the three direction vectors, that is $f_i = f_i(\hat{g} \cdot \hat{v}, \hat{g} \cdot \hat{n}, \hat{n} \cdot \hat{v})$. Equation 10 has reduced the problem of fitting $\alpha^{\text{gen}}$ from determining a vector-valued function of six independent variables (three vectors with unital constraints) to determining a vector-valued function of three independent variables (three dot products). Note that the form given in Eqs 6-7 for general $\mu_{\text{surf}}$ continues to satisfy the IRT requirement Eq 10. A detailed proof in this regard is provided in section S3 of the Supplementary Information.

We next introduce the methodology for parametrizing sub-surfaces in terms of three angles to arrive at our ultimate description of $\alpha^{\text{gen}}$.

3D-RFT sub-surface characterization.

Use of a local coordinate frame. Equations 6-10 define the stress-per-depth on a sub-surface using $\hat{n}$, $\hat{v}$, and $\hat{g}$ directions (beside material properties). We define a local cylindrical coordinate system at each sub-surface based on its local velocity ($\hat{v}$) and gravity direction ($\hat{g}$). We choose the direction opposite to the gravity (upward in general) as the positive $z$-direction and use the horizontal component of $\hat{v}$ as the positive $r$ direction. The remaining $\theta$ direction is chosen as the cross product between $\hat{r}$ and $\hat{z}$. The free-surface is taken as the reference ($z = 0$) for the $z$-direction.
Surface twist angle ($\psi$): We define $\psi$ as the azimuthal angle between the r-axis and the projection of the surface normal onto the r$\theta$-plane, denoted by $\mathbf{n}_a$.

Surface tilt angle ($\beta$): We define $\beta$ as the polar angle between the r-axis and the r$\theta$-plane. To be clear, $\beta$ measures the angle between the r$\theta$-plane and one of $\mathbf{n}$ or $-\mathbf{n}$, whichever gives a result in the $[-\pi/2,\pi/2]$ range. This choice is not problematic because at any time, only one side of a plate element experiences forces, and this can be identified using the leading edge condition ($\mathbf{v} \cdot \mathbf{n} \geq 0$) of Premise 2 which guarantees $\beta \in [-\pi/2,\pi/2]$.

Next we describe the representation of the velocity direction.

Angle of attack ($\gamma$): The local coordinate frame definitions (Eq 11) keep the velocity vector completely within the rz-plane. Thus, once $\{\mathbf{r}, \hat{\mathbf{r}}, \hat{\mathbf{z}}\}$ are determined, only one angle is needed to represent the velocity direction. We call this angle $\gamma$; see Fig 2B. We define $\gamma$ as the angle between the velocity direction vector and the local positive r-axis.

Based on the above definitions, the variations of each of the system characteristic angles $\{\beta, \gamma, \psi\}$ is restricted to $[-\pi/2,\pi/2]$ for any leading-edge surface. We use these limits in the generation of reference 3D-RFT data. Mathematical formulas for the angles in terms of vector components in a fixed cartesian frame are provided in the Material and Methods section.

We express the final form of $\alpha_{x}^{gen}$ in the local coordinate frame $\{\mathbf{r}, \hat{\mathbf{r}}, \hat{\mathbf{z}}\}$ by expressing $\{\mathbf{n}, \hat{\mathbf{v}}, \hat{\mathbf{g}}\}$ as:

\[ \hat{\mathbf{g}} = -\hat{\mathbf{z}}, \quad \hat{\mathbf{v}} = \cos \gamma \hat{\mathbf{r}} - \sin \gamma \hat{\mathbf{z}} \]
\[ \mathbf{n} = \sin \beta \cos \psi \hat{\mathbf{r}} + \sin \beta \sin \psi \hat{\mathbf{r}} - \cos \beta \hat{\mathbf{z}} \]

Substitution of definitions in Eq 10 gives the expressions for the r, $\theta$, and z components of $\alpha_{x}^{gen}$ as follows:

\[ \alpha_{x}^{gen} = \alpha_{r}^{gen} \hat{\mathbf{r}} + \alpha_{\theta}^{gen} \hat{\mathbf{r}} + \alpha_{z}^{gen} \hat{\mathbf{z}} \]
\[ \alpha_{r}^{gen} (\beta, \gamma, \psi) = f_1 \sin \beta \cos \psi + f_2 \cos \gamma \]
\[ \alpha_{\theta}^{gen} (\beta, \gamma, \psi) = f_1 \sin \beta \sin \psi \]
\[ \alpha_{z}^{gen} (\beta, \gamma, \psi) = -f_1 \cos \beta - f_2 \sin \gamma - f_3 \]

where, $f_1 = f_1(x_1, x_2, x_3)$, $f_2 = f_2(x_1, x_2, x_3)$, and $f_3 = f_3(x_1, x_2, x_3)$ are three functions of $\{x_1, x_2, x_3\}$ defined as:

\[ x_1 = \mathbf{g} \cdot \hat{\mathbf{v}} = \sin \gamma, \quad x_2 = \mathbf{g} \cdot \mathbf{n} = \cos \beta, \]
\[ x_3 = \mathbf{n} \cdot \hat{\mathbf{v}} = \cos \psi \cos \gamma \sin \beta + \sin \gamma \cos \beta \]

The 3D-RFT model we introduce is closed upon fitting the three functions $f_i(x_1, x_2, x_3)$. Note that by building the functional relationships for $\alpha_{r}^{gen}$, $\alpha_{\theta}^{gen}$, and $\alpha_{z}^{gen}$ from IRT (Eq 14), the model automatically satisfies many easy-to-observe local constraints regardless of the choice of the $f_i$’s. These constraints include (i) ‘plate twist symmetry’ (Fig 3B), which requires that the sub-surface forces in the r- and z-direction should be even functions of plate twist ($\psi$), and that force in the z-direction should be an odd function of $\psi$; (ii) ‘plate tilt symmetry’ (Fig 3C) which requires that when the plate faces upwards or downwards ($\beta = 0$), the sub-surface force in the z-direction should vanish, the force magnitude should depend only on $\gamma$, and the twist angle $\psi$ should have no influence on the force; and (iii) ‘vertical motion symmetry’ (figure 3D), which requires that for any tilt $\beta$, as $\gamma \to \pm \pi/2$ (approaching an upward or downward motion) then any azimuthal rotation (changing $\psi$ at constant $\beta$) of a sub-surface should rotate the resultant force on the sub-surface by the same angle.

Fig. 3. 3D-RFT symmetry constraints: (A) Global rotational constraint requiring the drag force to be an isotropic function of the plate normal, motion direction, and gravity direction. Some consequences of this constraint are plate twist symmetry, plate tilt symmetry, and vertical motion symmetry. (B) A special case of plate twist symmetry: $F_0(\beta, \gamma, \psi = 0) = 0$. (C) A special case of plate tilt symmetry: $F_0(\beta = 0, \gamma, \psi) = 0$, and (D) vertical motion symmetry: $F_0(\beta, \gamma = \pm \pi/2, \psi = 0) \to (\beta, \gamma = \pm \pi/2, \psi = \delta)$ causes $(P_1, P_2 = 0, P_3) \to (P_1 \cos \delta, P_2 \sin \delta, P_3)$. Violet, red, and blue arrows show force, velocity, and surface-normal direction, respectively.
5. Reference data

We use a large number of combinations (~3000) of material properties ($\rho$, $\mu_{\text{int}}$, $\mu_{\text{surf}}$) and 3D-RFT angles ($\beta$, $\gamma$, $\psi$) to generate continuum modeling-based reference data for evaluating the 3D-RFT form. The details of the combinations are provided in the Materials and Methods section. Polynomial fits for $f_1$, $f_2$, and $f_3$ are provided in the Supplementary Information (Table S3 and S4). Figure 4 shows the simulation setup used for the data collection. While both the $\beta$ and the $\gamma$ angles are varied over the interval $[-\pi/2, \pi/2]$, $\psi$ was varied only in $[0, \pi/2]$ taking advantage of ‘plate twist symmetry’ discussed earlier.

Figure 5 shows an example of 3D-RFT fittings against reference data. Odd columns in the figure show the data obtained using continuum simulations as a function of $\beta$ and $\gamma$ at four $\psi$ values. The material properties were $\mu_{\text{int}} = 0.4$, $\rho_0 = 3000 \text{ kg/m}^3$, and $\mu_{\text{surf}} = 0.15$. Corresponding 3D-RFT fittings are plotted on the even columns. We find the value of the scaling coefficient $\xi_0$ to be $0.92 \times 10^9 \text{ N/m}^3$ for this material. While Eq 14 represents the most generic form of 3D-RFT, the choice of the $f_i(x_1, x_2, x_3)$ determines the final 3D-RFT form. All the results presented in this work use 3rd degree polynomial fits for the $f_i$ in $\{x_1, x_2, x_3\}$ (Table S3). The performance of 3D-RFT does not change significantly between 3rd and 4th degree polynomial fits. The latter form fits the trends of $|\alpha_i|/|\alpha_n|$ better but has inconsequential effects on 3D-RFT predictions for the test cases used in this study.

The 3D-RFT model we propose is summarized by Eqs 3-8, with $\hat{f}$ fit as shown in Fig S3, and with $\alpha^{\text{gen}}$ expressed using Eqs 12-14 in terms of third degree polynomial fits for the $f_i$, and using directions $\{\hat{r}, \hat{\theta}, \hat{z}\}$ and angles $\{\beta, \gamma, \psi\}$ as shown in Fig 4. To numerically implement the model, we discretize the intruder surface into small plate elements and discretize the objects into elements that are on the leading edge of the intruder. A step-by-step implementation strategy for 3D-RFT is given in the Materials and Methods section.

6. Validation studies

We first test the accuracy of the implied localization rule (Premise 1) of the proposed form of 3D-RFT by comparing its predictions for ten arbitrary intruding objects to full continuum model solutions of the same intrusions. We use the material properties $\mu_{\text{int}} = 0.4$, $\rho_0 = 3000 \text{ kg/m}^3$, and $\mu_{\text{surf}} = 0.4$ for these cases. A representation of the objects and their dimensions are provided in Fig 6 and its caption. The object length scales are kept to be 7 cm in all the cases, and the objects are submerged to an initial depth of 27 cm (vertical distance between the free surface and the geometric center of the shape). The objects are moved at a speed of 0.1 m/s in different directions in the $xz$-plane. These directions are characterized using $\theta$, which represents the angle between the velocity direction ($\theta_0$) and the positive $x$-axis in a clockwise direction (same as $\gamma$ definition for a plate element). Negative $\theta$ represents upward motion, positive $\theta$ represents downward motion, and $\theta = 0$ represents horizontal motion along the $x$-direction. The variations of net-force ($F_x$, $F_y$, and $F_z$) with $\theta$ are plotted in Fig 6. 3D-RFT agrees with the continuum solutions well in modeling all the intrusion test scenarios considered in Fig 6. Objects with sharp corners generally show somewhat weaker fits than those with smoother shapes; this
We use a scaling coefficient (with a grain density of with simultaneous rotation and translation velocities. We use Agarwal et al. figure also shows the variations of force and moment on the dimensions and setup schematic are provided in Fig 7. The particles in a length= along the intruder solid cylindrical intruder in a granular volume. The setup consists of approximately kg/m³ and the granular volumes have an effective bulk density of kg/m³ (0.53) in both the DEM studies. We determine the internal coefficient of friction $\mu_{\text{int}}$ as 0.21 using simple shear simulations. Section S6 of the Supplementary Information provides more details of the simple shear test setup and detailed material properties. We use a scaling coefficient ($\xi_0$) value of $0.12 \times 10^6$ N/m³ based on the $f$ relationship between $\xi_0/\rho g$ and $\mu_{\text{int}}$ shown in Fig S3. See Table S1 and section S1 for more details.

Cylinder Drill: In this test, we model simultaneous rotation and translation (drilling) of a solid cylindrical intruder along the $z$-axis in a granular volume (diameter= 0.05 m, length=0.14 m). The setup consists of approximately $6 \times 10^6$ particles in a $100d \times 100d \times 70d$ sized granular bed. The setup dimensions and setup schematic are provided in Fig 7. The figure also shows the variations of force and moment on the intruder over time from the DEM studies versus 3D-RFT. In addition, the figure shows the variation of stress over the intruder surface from DEM and 3D-RFT. All reported components (net force and moments, as well as stress distributions) show a strong match between the two approaches.

Bunny Drill: In this test, we model the drilling motion ($\omega = 2\pi$ rad/s, $v = 0.1$ m/s) of a Stanford Bunny (34) shaped rigid intruder in a granular volume. The shape is chosen because it is an example of a complex, asymmetric 3D object. The granular bed consists of approximately $2.1 \times 10^6$ particles spread over a $150d \times 150d \times 88d$ sized domain. The bunny shape was slightly modified from the standard shape — the shape was proportionally scaled in such a way that the bunny height measures 0.1 m, and the bunny base was flattened to make the base a plane surface without an inward extrusion. Figure 8 shows the simulation setup where the grains are colored with velocity magnitudes. Figure 8 also shows the variation of stresses over the intruder surface from DEM and 3D-RFT. All the reported components (net force and moments, as well as stress distributions) show a strong match between the two approaches.

7. Conclusion

3D-RFT is an important step towards developing a generic real-time modeling technique capable of modeling granular intrusion of arbitrarily shaped objects over a large range of low and high-speed scenarios in diverse materials and environments. Previously, granular RFT’s usage has focused on the modeling of arbitrary 2D objects moving in-plane. We
have proposed an extension of RFT to three dimensions in a fashion consistent with granular continuum mechanics and necessary symmetry constraints. The accuracy of the proposed 3D-RFT was demonstrated against a variety of full-field
intrusion simulations, both continuum and DEM. Notably, we provide a scheme that determines 3D-RFT in different intrusion systems quickly and directly in terms of basic properties of the granular media ($\rho_s$ and $\mu_{int}$) and the intruder surface ($\mu_{surf}$). The most immediate opportunity to expand 3D-RFT would be to combine 3D-RFT with Dynamic RFT (25) to build a high-speed three-dimensional RFT (3D-DRT). The current form of 3D-RFT does not include a “shadowing effect” i.e. the fact that forces are reduced on leading edge surfaces that lie in the immediate wake behind another part of the intruder (35). Such effects are more pronounced in intruders with complex shapes or fine geometric features such as the Bunny shape we consider in this study. Characterizing this effect would be an important addition to RFT. Effects of multi-body intrusions (24, 36), density variations (37), inertial and non-inertial velocity effects (3, 25, 38), cohesion (39, 40), and inclined domains (41) on the resistive forces experienced by intruding bodies are among other aspects for further exploration toward the ultimate goal of a generic and fast granular intrusion model applicable to terradynamical motions (8), granular impact systems (42, 43), locomotors (44), and many other similar applications. This work also presents a systematic approach towards developing reduced-order models in other similar systems with a combination of mathematical analysis and experiments/simulation-based data collection.

**Materials and Methods**

**Evaluation of quasi-static conditions in a system.** We use the following definitions of the micro-inertial number $I$ and the macro-inertial number $I_{mac}$ for evaluating the applicability of 3D-RFT in modeling the granular resistive forces in a granular intrusion system;

$$I = \dot{\gamma}/\sqrt{\frac{P}{\rho_g}d^2}, \quad I_{mac} = v/\sqrt{\frac{P}{\rho_g}},$$

where $\dot{\gamma}$ represents the material shear rate, $P$ represents the hydrostatic pressure, $\rho_g$ represents the material grain density, $d$ represents the mean grain diameter, and $v$ represents the speed. The $I_{mac}$ formulation is equivalent to the inverse square root of the *Euler number* which measures the ratio of the dynamic pressure $\rho v^2$ to the total pressure $P$.

The macro- and micro-inertial numbers are defined pointwise within a granular media so, to determine if an intrusion is quasi-
static, it is convenient to determine characteristic values for these numbers. For this, we use characteristic values of $\gamma$, $P$, and $v$. We assume that the intruder has an angular velocity $\omega$, a translational velocity $v_{\text{intruder}}$, and a characteristic length $L$. We also assume that the media has a critical density $\rho_c$ and that the system is acted upon by gravity $g$. We characterize $v$ as $v_{\text{max}}(v_{\text{intruder}}, L/\omega)$ and $\gamma$ as $\gamma = v/L$. We consider intrusive loading of the system at characteristic depth $L$ to give a characteristic $P$ as $\xi_n L$. Upon substitution, we get:

\[
I \sim (v/L)/\sqrt{\xi_n L}/\sqrt{\rho d^2} = \frac{v^2 \rho d^2}{\xi_n L^3} = \frac{vd}{L} \frac{\rho d}{\xi_n L},
\]

\[
I_{\text{mac}} \sim v/\sqrt{\xi_n L}/\rho d = \frac{\rho d}{\xi_n L}.
\]

From the above equations, we observe that the characteristic value of $I_{\text{mac}}$ reduces to a multiple of the Froude number ($Fr$) in gravity loaded systems. To this end, Sunday et al. (45) explored the existence of macro-inertial effects during high-speed granular intrusions and observed insignificant contributions of macro-inertial effects in the material force response for $Fr < 1.5$, which gives $I_{\text{mac}} \lesssim 0.48$. Similarly, Agarwal et al. (25), observed insignificant macro-inertial effects (macro-inertial forces < 10% of static resistive forces i.e. $\rho Av^2/K|z| < 10\%$) in granular plate intrusions at $I_{\text{mac}} < 0.16$. Thus, we impose an upper limit of 0.15 on $I_{\text{mac}}$ to be quasi-static. The amount $I$ affects the flow can be quantified by how much it changes the apparent internal friction ($\mu_{\text{int}}$). To keep these changes bounded by 10% we set an upper bound on the characteristic value of $I$ to be 0.010 so as to ensure quasi-static conditions. For all the test cases used in this study, we choose system parameters in such a way that $I$ and $I_{\text{mac}}$ are always below above mentioned limits keeping their motions in quasi-static limits. The test cases 1-10 are continuum simulation that use a rate-independent constitutive law and have $I_{\text{mac}} \sim 0.02$ ($L \approx 0.07 \text{ m}$, $v = 0.1 \text{ m/s}$, $\xi_n = 0.92 \times 10^4$, $\rho_c = 3000 \text{ kg/m}^3$). In the DEM based cylinder drill test cases, we find $I < 0.002$ and $I_{\text{mac}} < 0.07$ ($L \approx 0.10 \text{ m}$, $\omega < \pi \text{ rad/s}$, $v_{\text{intruder}} = 0.1 \text{ m/s}$, $\xi_n = 0.12 \times 10^6$, $\rho_c = 2470 \text{ kg/m}^3$). Similarly, in the bunny drill test case, we find $I \approx 0.004$ and $I_{\text{mac}} \sim 0.13$ ($L \approx 0.10 \text{ m}$, $\omega = 2\pi \text{ rad/s}$, $v_{\text{intruder}} = 0.1 \text{ m/s}$, $\xi_n = 0.12 \times 10^6$, $\rho_c = 2470 \text{ kg/m}^3$). Thus, 3D-RFT is a valid approach for modeling all the test cases considered in this study, based on the insignificance of micro- and macro-inertial force contributions.

Continuous approach accuracy validation. Several studies in the past have verified the accuracy of this constitutive formulation in plane-strain problems. We use the 3D numerical implementation of MPM developed by Baumgarten and Kamrin (29) for this study which has been successfully used for modeling complex problems in the past (29, 30). For the continuum model to be useful to determine input data for a 3D-RFT, it must be shown to reliably match experiments for 3D plate intrusions. We test this in two scenarios.

In the first test case, we check if the 3D-continuum simulations can regenerate the experimental variation of force/depth/area on flat plates in submerged granular beds from Li et al.(14). This experimental data was also used by Li et al.(14) in the generation of 2D-RFT form. We use an effective materials density of $\rho_c = 1450 \text{ kg/m}^3$ (loose glass beads, $\rho_g = 2500 \text{ kg/m}^3$, $\phi_c = 0.58$) inline with Li et al.(14) experiments and an approximate internal friction value for glass beads as $\mu_{\text{int}} = 0.4$. The media-plate surface friction was taken as $\mu_{\text{rail}} = 0.4$. The relative values of the forces from continuum results remarkably match the experimental observations. The absolute values from continuum results, however, are higher than experiments by a constant multiplicative factor of $\sim 1.1$. A smaller value of $\mu_{\text{int}}$ for glass beads could have provided a closer match to the experiments as the graphs are not expected to change.

Fig. 8. DEM based 3D-RFT verification: Bunny drill: (A) A snapshot of the Stanford-Bunny drill setup where a 10 cm high standford-bunny was simultaneously rotated ($\omega = 2\pi \text{ rad/s}$, clockwise) and translated ($v = 0.1 \text{ m/s}$, downwards) along the z-axis. The grains are colored with velocity magnitudes. The simulation domain consisted of $\sim 2.1 \times 10^9$ particles (50)/50 split of 3 mm and 3.4 mm diameter (4) grains spread over a 150d x 150d x 80d physical space. (B) Variation of net force ($F$, left) and moment ($M$, right) components ($x$: yellow, $y$: orange, and $z$: blue) from DEM (solid lines) and 3D-RFT (dotted lines). (C) Components of the surface stress distribution from DEM (Top) and 3D-RFT (Bottom) at a 5 cm bunny-center-depth below the free surface. The DEM material properties are provided in Table S2 of the Supplementary Information.
their shape with changing internal friction values (14). But we do not attempt the exact calibration as the purpose of the test was to verify the accuracy of the continuum formulation and implementation. These results establish sufficient efficacy of the continuum model for plate motions in which the velocity, plate normal, and gravity or co-planar.

In the second test case, we assess the quantitative accuracy of the continuum approach in modeling in-plane as well as out-of-plane forces. We consider a study Maladen et al. (48) which measured the normal and tangential forces on submerged plates moving horizontally in granular media as a function of plate twist (see Fig. 10 (top) for angles definition). The material properties are provided in the figure caption. The continuum results match observations from Maladen et al. (48) well.

The combination of the above two studies establishes the overall accuracy of the continuum model and its implementation for both in-plane and out-of-plane inputs and outputs in plate intrusion problems.

**3D-RFT reference data generation.** We use a large combination of material properties (ρ, μinit, μsurf) and 3D-RFT reference variables (β, γ, ψ) to generate continuum modeling-based reference data for evaluating the 3D-RFT form. In regards to the material properties we use five material internal friction values (μinit = [0.3, 0.4, 0.5, 0.7, 0.9]) with two values of surface friction (μsurf = 0.4) in each case. For μinit = 0.4 we use 3 instead of 2 μsurf values. For each of the 11 combinations of μinit and μsurf, we conduct plate intrusions at 7 combinations of plate tilt angle (β = π/2 : π/6 : π/2 rad), 7 combinations of velocity direction angle (γ = −π/2 : π/6 : π/2 rad), and 4 combinations of plate twist angle (ψ = 0 : π/6 : π/2 rad). For the {μinit = 0.4, μsurf = 0.15}, we use 13 combinations of β (π/2 : π/6 : π/2 rad), 13 combinations of γ (−π/2 : π/6 : π/2 rad), and 4 combinations of ψ (0 : π/6 : π/2 rad). Additionally, we conduct plate intrusion simulations at {μinit = 0.2, μsurf = 0.2} and {μinit = 0.1, μsurf = 0.1} at ψ = 0 rad to evaluate ξn at μinit = 0.1 and 0.2. We do not explore ψ in [−π/2 : 0] rad range for the reference data as αn, αs, and ξn are known to be even, even, and odd (resp.) in ψ from ‘plate twist symmetry’. The polynomial fits for f1, f2, and f3 are provided in the Supplementary Information (Table S3 and table S4).

**3D-RFT Implementation.** We use an explicit iterative scheme to implement 3D-RFT in this study. The strategy primarily consists of three parts — (1) discretizing the intruder surface into small sub-surfaces, (2) finding the sub-surface forces using sub-surface orientation angles (β and γ), velocity angle (γ), area (ds), and depth from the free surface (|z|), and (3) summing over all sub-surfaces to find the net resistive force and moment response. A step-by-step implementation of the strategy is provided below:

**Step 1:** Discretize the intruder surface into small plane sub-surface elements. We use the open-source software, ‘Blender’ (version 2.91) for modeling and discretizing various intruder geometries in our study (using .wrl format).

**Step 2:** Calculate the velocity direction vector ħ, surface normal ĥ, and depth from the free surface |z| for each sub-surface. Repeat Steps 3-11 for each sub-surface.

**Step 3:** Check if ĥ · ħ ≥ 0 (sub-surface is a ‘leading edge’) and z < 0 (sub-surface is submerged in the media). If both the
conditions are met, follow Steps 4-11. If they are not, set the sub-surface resistive force to zero and consider the next sub-surface.

**Step 4:** Find local coordinate frame \( \{ \hat{r}, \hat{\theta}, \hat{z} \} \) using eq 11.

**Step 5:** Find RFT characteristic angles \( \{ \beta, \gamma, \psi \} \) using \( \hat{\theta} \), \( \hat{n} \), and local coordinate frame \( \{ \hat{r}, \hat{\theta}, \hat{z} \} \) as follows:

Find the surface characteristic angle \( \beta \) as:

\[
\beta = -\cos^{-1}(\hat{n} \cdot \hat{z}) \quad \text{if} \quad \hat{n} \cdot \hat{r} \geq 0 \quad \& \quad \hat{n} \cdot \hat{z} \geq 0
\]

\[
+ \pi - \cos^{-1}(\hat{n} \cdot \hat{z}) \quad \text{if} \quad \hat{n} \cdot \hat{r} \geq 0 \quad \& \quad \hat{n} \cdot \hat{z} < 0
\]

\[
- \pi + \cos^{-1}(\hat{n} \cdot \hat{z}) \quad \text{if} \quad \hat{n} \cdot \hat{r} < 0 \quad \& \quad \hat{n} \cdot \hat{z} < 0 \quad [10]
\]

Remember that this \( \beta \) corresponds to only ‘leading edges’ as non ‘leading-edge’ sub-surface never reach this step.

**Step 6:** Calculate the values of \( \alpha \) from vertical plate intrusion experiments (intrusion of a thin flat plate of area \( ds \) at \( \beta = 0, \psi = 0 \), and \( \gamma = \pi/2 \)) using the following formula:

\[
\xi_n = \frac{F_{\text{vertical}}}{\alpha^\text{gen}(\beta = 0, \gamma = \pi/2, \psi = 0)} \times ds \times |z|
\]

**Step 7:** Estimate the media specific scaling factor \( \xi_n \) using expected functional form of \( \xi_n \) from section S1 and Fig S3 of the Supplementary Information if the media effective density \( \rho_e \), gravity magnitude \( g \), and media internal friction coefficient \( \mu \) are known. Alternatively, obtain \( \xi_n \) from other intruder shape discretization of the intruder.

**Step 8:** Calculate the values of \( \{ \alpha_1^\text{gen}, \alpha_2^\text{gen}, \alpha_3^\text{gen} \} \) using Eq 14 and table S3.

**Step 9:** Calculate the system specific \( \alpha \) and \( \alpha \) in the local coordinate frame using Eq 6-7 and add them up (Eq 5) to get \( \alpha \).

**Step 10:** Calculate \( \{ \alpha_x, \alpha_y, \alpha_z \} \) using Eq 13 by substituting triad \( \{ \hat{r}, \hat{\theta}, \hat{z} \} \) from Eq 11.

**Step 11:** Calculate the net resistive force on the sub-surface by multiplying the triad \( \{ \alpha_x, \alpha_y, \alpha_z \} \) with sub-surface depth \( |z_i| \), and area \( ds_i \).

**Step 12:** Sum over all the sub-surfaces to find the final net force and moment on the intruder.

Once the net resistive force on the intruder is known, one can use momentum balance equations to further model the intruder motion. Convergence studies are also done to determine the discretization of the intruder shape.

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