Modeling on Vibration of Sandwich Plate Possessing a Compressible Core and Interacting with Viscous Fluid

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Abstract. The paper proposed a mathematical model for analyzing the dynamic response of a three-layered channel wall with a compressible core. The narrow channel formed by two parallel walls is considered. The bottom channel wall was a simply supported three-layered plate. The upper channel wall was considered to be absolutely rigid. The channel was filled with a viscous incompressible liquid with pulsating pressure due to a given harmonic law of pressure pulsation at the channel edges. Longitudinal and transverse vibrations of the three-layered channel wall were studied. The dynamic response of the three-layered channel wall was determined on the basis of hydroelastic oscillations problem solution. The frequency-dependent distribution functions of longitudinal and transverse elastic displacements amplitudes of the three-layered channel wall were constructed. These functions make it possible to obtain amplitude-frequency characteristics of the three-layered channel wall in fixed channel cross-sections. The study was supported by Russian Foundation for Basic Research (projects № 18-01-00127-a and № 19-01-00014-a).

1. Introduction
The development of civil engineering requires analyzing the dynamic response of three-layered structural components interacting with fluid in various objects. For example, such objects may include lubrication channels, cooling or damping systems and so on. Therefore, it is important to work out mathematical models of fluid interaction with a channel walls. For instance, in [1, 2], oscillations models of a circular plate contacting on one side it with an unlimited volume of an ideal incompressible fluid were proposed. The investigation of resonance vibrations of a membrane resting on an elastic foundation and being a part of a tank bottom filled with an ideal liquid with a free surface was carried out in [3]. The natural oscillations of a plate floating on the free surface of an ideal incompressible fluid possessing a finite depth were studied in [4]. In reference [5], natural vibrations of a circular plate on the free surface of an ideal incompressible fluid located in a rigid cylinder are investigated. An analogous model of a plate immersed under the free surface was considered in [6]. The model of acoustic wave propagation in an ideal incompressible liquid caused by forced oscillations of a plate contacting with the liquid was considered in [7]. Reference [8] was devoted to the study of bending oscillations and stability of a plate, being a part of an absolutely rigid channel.
wall with a stream of ideal compressible liquid. Simulation of natural oscillations and stability of a rectangular plate forming a wall of channel filled with an ideal compressible liquid was considered in [9].

The liquid viscosity is a factor of importance in various devices and technological processes, since it determines the damping properties in an oscillatory liquid-elastic body system. For instance, in [10], the model proposed in [1] was generalized to the case of a viscous liquid. The mathematical model of a plate stability and dynamics, being a part of the border, and dividing the volumes filled with two different viscous liquids was formulated in [11]. The response of a finite dimension channel walls interacting with a viscous liquid layer under the channel foundation vibration was investigated in [12]. The hydroelastic vibrations problem of a cantilevered beam immersed in a viscous liquid was solved in [13]. An analogous problem for a piezoelectric beam in a viscous fluid flow was studied in [14]. Transverse vibration of disks interacting with a viscous liquid layer between them was considered in [15]. It is necessary to note, that three-layered structural elements in the form of beams and plates are widely used in various industries. Modeling the behavior of these elements is elaborated in contemporary studies. For instance, a mathematical model of tremoelastic bending of a three-layered beam with compressible core was proposed in [16], and a forced oscillations simulation of a similar beam under local normal loads was carried out in [17]. Mathematical models of hydroelastic vibrations of multilayer elastic elements are not sufficiently presented in the contemporary research and the following references on this subject can be mentioned. The natural hydroelastic vibrations of multilayer plates interacting with an ideal liquid were investigated in [18]. In particular, oscillations plates in the air and immersed in water were considered. The forced vibrations model of a three-layered plate in contact with a viscous liquid layer was considered in [19], reference [20] taking into account the Winkler foundation on which the channel was installed. In these papers, a plate with incompressible core was considered. However, the oscillations of a three-layered channel wall with compressible core, while taking into account the normal and shear stresses of the viscous liquid have not been considered in the above-mention studies.

2. Initial mathematical formulation of the problem

Let us consider a narrow channel with a three-layered wall (Fig. 1). The wall is simply supported at the edges. The thickness of its face sheets are \( h_1, h_2 \) and the core thickness is \( 2c \). Let the Cartesian coordinate system center be located in the center of the median plane of the wall core. The opposite channel wall is absolutely rigid. The channel walls sizes in plan view are \( 2\ell \times b \), let us consider case \( b \gg 2\ell \) and study the plane problem. The distance \( h_0 \) between the channel walls is significantly less than their length, i.e. \( 2\ell \gg h_0 \). The channel is filled with a viscous incompressible liquid and pulsating pressure at the channel edges is \( p^* = p_0 + p_m\sin(\omega t) \). Here \( p_0 \) is a constant pressure, \( p_m \) is the pressure pulsation amplitude, \( \omega \) is the frequency, \( t \) is time. The elastic displacements of the three-layer wall are considerably smaller than \( h_0 \). In the course of the study, we omit the influence of the initial conditions and consider the steady-state forced wall oscillations, since liquid viscosity causes rapid decay of transients [21].

![Figure 1](image.png)

**Figure 1.** A channel with a three-layered wall possessing a compressible core.

1,2 - wall face sheets, 3 - wall core.

The viscous fluid dynamics in the narrow channel is described by the Navier-Stokes and continuity equations for creeping motion [22].
\[ \frac{\partial P}{\partial \zeta} = \nu^2 \frac{\partial^2 U_{\zeta}}{\partial \zeta^2} + \frac{\partial^2 U_{\zeta}}{\partial \zeta^2}, \quad \frac{\partial P}{\partial \zeta} = \nu^2 \left[ \frac{\partial^2 U_{\zeta}}{\partial \zeta^2} + \frac{\partial^2 U_{\zeta}}{\partial \zeta^2} \right], \quad \frac{\partial U_{\zeta}}{\partial \zeta} + \frac{\partial U_{\zeta}}{\partial \zeta} = 0, \]  

(1)  

with no-slip boundary conditions  

\[ U_{\zeta} = 0, U_{\zeta} = 0 \quad \text{at} \quad \zeta = 1, \quad U_{\zeta} = \psi \frac{u_{\text{mol}}}{w_{\text{mol}}} \frac{\partial U_1}{\partial \tau}, \quad U_{\zeta} = \frac{\partial W_\kappa}{\partial \tau} \quad \text{at} \quad \zeta = \lambda W_1, \]  

(2)  

and pressure conditions at the channel edges are  

\[ P = 0 \quad \text{at} \quad \zeta = \pm 1. \]  

(3)  

Here the following notation is used  

\[ \psi = \frac{h_0}{h}, \quad \lambda = \frac{w_{\text{mol}}}{h_0}, \quad \tau = \rho t, \quad \zeta = \frac{x}{l}, \quad \zeta = \frac{z-c-h_1}{h_0}, \quad u_x = w_{\text{mol}} \frac{\rho}{h_0} U_{\zeta}, \quad u_z = w_{\text{mol}} \frac{\rho}{h_0} U_{\zeta}, \quad p = p_0 + p^*(\tau) + \frac{\rho w_{\text{mol}} \dot{\omega}}{h_0 \gamma^2} P, \]  

(4)  

where \( u_x, \ u_z \) are the projections of the fluid velocity vector on the coordinate axes, \( p \) is pressure, \( w_{\text{mol}} \) is the deflection amplitude of the upper face sheet of the three-layered channel wall, \( \rho \) is the liquid density, \( \nu \) is the kinematic viscosity of the liquid, \( \psi \) and \( \lambda \) are small parameters. The elastic displacements of the upper face sheet of the three-layered wall are presented in the form  

\[ u_x = u_{\text{mol}} U_1(\xi, \tau), w_1 = w_{\text{mol}} W_1(\xi, \tau). \]  

The dynamics equations of the three-layered wall with compressible core were obtained in [17], the equations having the form of  

\[ F_1 = a_1 u_1 - a_2 u_2 + a_3 u_3 + a_4 u_4 + a_5 u_5 - a_6 u_6 + a_7 u_7 + a_8 u_8 + a_9 u_9 + a_{10} u_{10} + a_{11} u_{11} + a_{12} u_{12} + a_{13} u_{13} + a_{14} u_{14} + a_{15} u_{15} + a_{16} u_{16} + a_{17} u_{17} + a_{18} u_{18} =aq, \]  

(5)  

where \( a_1, \ldots, a_{18} \) are coefficients presented in [17], \( \rho_k \) is the material density of the \( k \)-th layer, \( q_{zz}, q_{zx} \) are the normal and shear liquid stresses acting on the channel wall. These stresses are written as
\[ q_{zz} = -\frac{\rho'\nu w_{\omega}}{h_j \psi} \left( \nu^2 \frac{\partial U_{zz}}{\partial \xi} + \frac{\partial U_{zz}}{\partial \zeta} \right) \] at \( \zeta = \lambda \frac{w_{\omega}}{\varepsilon_{\omega}} W_1 \), \( q_{zz} = -p_0 - p^* - \frac{\rho'\nu w_{\omega}}{h_j \psi} \left( p - 2\psi^2 \frac{\partial U_{zz}}{\partial \zeta} \right) \) at \( \zeta = \lambda \frac{w_{\omega}}{\varepsilon_{\omega}} W_1 \).

The boundary conditions of Eqs. (5) are written as
\[ w_j = \frac{\partial U_j}{\partial x}, \frac{\partial^2 w_j}{\partial x^2} = 0 \text{ at } x = \pm \ell, \quad (k=1, 2). \tag{6} \]

3. Determining of sandwich plate response
In the case under consideration, \( \psi << 1 \) and \( \lambda << 1 \), therefore, in the zero approximation with respect to small parameters \( \psi \) and \( \lambda \), the Eqs. (1)-(3) are simplified [23] and take the form of
\[ \frac{\partial P}{\partial \xi} = \frac{\partial U_j}{\partial \xi} = 0, \quad \frac{\partial U_j}{\partial \zeta} + \frac{\partial U_j}{\partial \zeta} = 0, \tag{7} \]
\[ U_j = 0, \quad U_j = 0 \text{ at } \zeta = 1, \quad U_j = 0, \quad U_j = \frac{\partial W_1}{\partial \tau} \text{ at } \zeta = 0, \quad P = 0 \text{ at } \zeta = \pm 1. \tag{8} \]

and liquid stresses \( q_{zz}, q_{zz} \) are written as
\[ q_{zz} = -\rho'w_{\omega} \alpha (h_j \psi)^{-1} \frac{\partial U_j}{\partial \xi} | \xi = 0, \quad q_{zz} = -p_0 - p^* - \rho'w_{\omega} \alpha (h_j \psi)^{-1} P | \xi = 0. \tag{9} \]

Solving the fluid dynamics problem (7), (8) we obtain
\[ P_{W_{1,0}} = 12 \frac{\xi}{\sqrt{\zeta + 6(\xi - 1)}} \frac{\partial W_1}{\partial \tau} d\xi + 6 \frac{\xi}{\sqrt{\zeta - 1}} d\xi = \frac{\xi}{\sqrt{\zeta - 1}} \frac{\partial W_1}{\partial \tau} d\xi - 3 \frac{\xi}{\sqrt{\zeta - 1}} d\xi = 6 h_j \frac{\partial W_1}{\partial \zeta}. \tag{10} \]

Considering (6), the solution of Eqs. (5) will be represented as
\[ u_j = \sum_{n=0}^{\infty} R_n^j \cos \frac{2n+1}{2} \pi \frac{x}{\ell}, \quad w_j = \sum_{n=0}^{\infty} P_n^j \cos \frac{2n+1}{2} \pi \frac{x}{\ell}, \quad k = 1, 2. \tag{11} \]

Substituting (10), (11) into (9) and expanding the term \( p^*(t) \) in a series on \( \cos \frac{2n+1}{2} \pi \xi \) we obtain
\[ q_{zz} = \sum_{n=0}^{\infty} \left[ 4 \frac{-1}{(2n+1)\pi} dR_n^j \sin \frac{2n+1}{2} \pi \frac{x}{\ell}, \frac{1}{h_j \psi} \frac{\partial U_j}{\partial \xi} \right] \frac{\partial U_j}{\partial \zeta} \right |_{\xi = 0} \frac{2}{(2n+1)\pi} \right] \frac{dR_n^j}{dt} \cos \frac{2n+1}{2} \pi \frac{x}{\ell}, \tag{12} \]
\[ q_{zz} = -\sum_{n=0}^{\infty} \frac{6\rho'w_{\omega}}{h_j \psi} \left[ \frac{2}{(2n+1)\pi} \right] \frac{dR_n^j}{dt} \sin \frac{2n+1}{2} \pi \frac{x}{\ell}, \frac{1}{h_j \psi} \frac{\partial U_j}{\partial \xi} \right |_{\xi = 0} \frac{1}{h_j \psi} \left[ \frac{1}{2} \frac{\partial U_j}{\partial \xi} \right ] \frac{1}{(2n+1)\pi} \frac{dR_n^j}{dt} \cos \frac{2n+1}{2} \pi \frac{x}{\ell}. \tag{12} \]

According to (12) it follows that \( \frac{1}{h_j \psi} \frac{\partial u_j}{\partial \xi} |_{\xi = 0} = \alpha (h_j \psi)^{-1} \alpha (h_j \psi)^{-1} \) hence the term \( \frac{1}{h_j \psi} \frac{\partial U_j}{\partial \xi} |_{\xi = 0} \) in (5) can be neglected compared to the term \( q_{zz} \). Substituting (11) and (12) into (5) we obtain a system of ordinary differential equations, including two homogeneous algebraic equations. Using these two equations we express \( T_2^* \), \( R_2^* \) in terms of \( T_1^* \) and \( R_1^* \), and taking into account that \( \frac{d^2 R_n^j}{dt^2} = -\omega^2 R_n^j \) we obtain
\[ b_{21}^* T_1^* + b_{22}^* R_1^* = -2K_n \frac{d R_n^j}{dt} \frac{4(-1)^{n+1}}{(2n+1)\pi} p^*(t), \tag{13} \]
\[ T_2^* = (T_1^* (b_{21}^* T_1^* + b_{22}^* R_1^*) + R_n^j (b_{21}^* T_1^* + b_{22}^* R_1^*)) \left( b_{21}^* T_1^* + b_{22}^* R_1^* \right) \), \]
\[ R_2^* = (R_1^* (b_{21}^* T_1^* + b_{22}^* R_1^*) + R_n^j (b_{21}^* T_1^* + b_{22}^* R_1^*)) \left( b_{21}^* T_1^* + b_{22}^* R_1^* \right) \],

and considering the mode of steady harmonic oscillations we find
$$R_n = p_n \frac{4(-1)^{n+1}}{(2n+1)\pi} \left[ \frac{d_1}{d_1^2 + d_2^2\omega^2} \right]^{1/2} p_n^m e^{i\omega t},$$

$$T^n = p_n \frac{4(-1)^{n}}{(2n+1)\pi} \left[ \frac{b_{1n}^3}{b_{1n} d_1} \right]^{1/2} + \frac{4(-1)^{n-1}}{(2n+1)\pi} \sqrt{b_{1n}^2 + (2K_n\omega_2)^2} \left[ \frac{d_1^2 + d_2^2\omega^2}{d_2^2} \right]^{1/2} p_n^m e^{i\omega t},$$

where
$$d_1 = b_{13} - b_{11}b_1^3/b_{11}, \quad d_2 = 2K_3 - b_{13}2K_2/b_{11}, \quad \gamma = -d_2\omega/d_1, \quad \gamma\theta = 2K_3/b_1^3, \quad \Delta = b_2b_{44} - b_2b_{42},$$

$$b_{11} = b_1 + b_2(b_2b_{44} - b_2b_{42})/\Delta + b_3b_4(b_2b_{44} - b_2b_{42})/\Delta, \quad b_{13} = b_3(b_2b_{42} - b_2b_{44})/\Delta + b_4(b_2b_{42} - b_2b_{44})/\Delta,$$

$$b_{14} = \frac{2n+1}{2\ell} \left[ -a_3 + a_1 \left( \frac{2n+1}{2\ell} \right)^2 \right] - m_4\omega^2, \quad b_{21} = -a_2 + a_0 \left( \frac{2n+1}{2\ell} \right)^2 - m_2\omega^2, \quad b_{22} = a_1 + a_0 \left( \frac{2n+1}{2\ell} \right)^2 - m_2\omega^2, \quad b_{23} = a_2 - a_0 \left( \frac{2n+1}{2\ell} \right)^2 + m_4\omega^2, \quad b_{24} = a_0 - a_0 \left( \frac{2n+1}{2\ell} \right)^2 + m_0\omega^2, \quad b_{31} = a_3 - a_1 \left( \frac{2n+1}{2\ell} \right)^2 + a_3 \left( \frac{2n+1}{2\ell} \right)^4 - m_4\omega^2, \quad b_{32} = -a_2 + a_4 \left( \frac{2n+1}{2\ell} \right)^2 - m_0\omega^2, \quad b_{33} = -a_0 + a_2 \left( \frac{2n+1}{2\ell} \right)^2 - a_0 \left( \frac{2n+1}{2\ell} \right)^3 - m_0\omega^2, \quad b_{34} = a_4 - a_4 \left( \frac{2n+1}{2\ell} \right)^2 + a_4 \left( \frac{2n+1}{2\ell} \right)^4 - m_2\omega^2.$$
upper face sheet ones, respectively. It can be noted that, taking into account (13), we also determined the deflection and longitudinal displacement of the lower face sheet of the channel wall.

4. Calculation results
We carried out calculations of functions \( \Pi_{w_i}(\omega,0) = A_{w_i}(\omega) \), \( \Pi_{u_i}(\omega, \ell) = A_{u_i}(\omega) \) using the elaborated mathematical model. These functions are the amplitude-frequency responses of the deflection and longitudinal displacement of the upper face sheet at cross-sections for case \( x=0 \) and \( x=\ell \), correspondingly. Modeling was performed for the following data: \( K_1 = K_2 = 8 \cdot 10^{10} \text{Pa} \), \( K_3 = 4.7 \cdot 10^9 \text{Pa} \), \( G_1 = G_2 = 2.67 \cdot 10^{10} \text{Pa} \), \( G_3 = 9 \cdot 10^7 \text{Pa} \), \( \rho = 1840 \text{kg/m}^3 \), \( \rho_1 = \rho_2 = 2700 \text{kg/m}^3 \), \( \rho_3 = 2150 \text{kg/m}^3 \), \( v = 25 \cdot 10^{-5} \text{m}^2/\text{s} \), \( h_0/\ell = 1/15 \), \( h_1/\ell = 2/100 \), \( h_2/\ell = 3/100 \), \( c/\ell = 5/100 \), \( \ell = 0.1 \text{m} \). We considered only the basic mode of oscillations, i.e. believed \( n = 0 \) in (16). The calculations results are presented in Fig. 2.

\[
A_{w_1}(\omega)
\]

\[
A_{u_1}(\omega)
\]

Figure 2. Charts of the upper face sheet amplitude-frequency responses \( \Pi_{w_i}(\omega,0) = A_{w_i}(\omega) \) and \( \Pi_{u_i}(\omega, \ell) = A_{u_i}(\omega) \) for the case \( n=0 \).

5. Conclusion
As a result of solving the hydroelasticity problem of the three-layered channel wall with compressible core, expressions of elastic displacements of channel wall face sheets are found. Based on the obtained solution, the frequency-dependent distribution functions of the deflection and longitudinal displacement amplitudes of the channel wall face sheets are constructed. These functions at a fixed value of the longitudinal coordinate are transformed into amplitude-frequency characteristics of the considered channel cross-section. Therefore, the mathematical model for studying the three-layered wall of a channel filled with a viscous pulsating fluid is proposed. For example, the model can be used to determine the resonant vibration frequencies of the channel wall. In addition, the model can be used for developing non-destructive testing of the three-layered structural elements interacting with liquid in various objects and technological processes.

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