Closed subspaces which are attractors for representations of the Cuntz algebras

Palle E. T. Jorgensen

Abstract. We analyze the structure of co-invariant subspaces for representations of the Cuntz algebras $O_N$ for $N = 2, 3, \ldots, N < \infty$, with special attention to the representations which are associated to orthonormal and tight-frame wavelets in $L^2(\mathbb{R})$ corresponding to scale number $N$.

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1. Introduction: Wavelets

A particular construction of wavelets on the real line $\mathbb{R}$ is based on what is called subband filters. The idea is that a wavelet decomposition of $L^2(\mathbb{R})$ can be organized in frequency bands with adjustment to a system of subspaces (identifying a cascade of resolutions) in $L^2(\mathbb{R})$, and each frequency band having its scaling resolution. The subband filters may be realized as functions on the torus.

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\( \mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \} \). If the scaling is \( N > 1 \) then the \( L^2(\mathbb{R}) \) functions needed in the wavelet decomposition may be obtained (under favorable conditions) as solutions to a system of cocycle conditions, see (1.1) below, involving Fourier transform \( \psi \mapsto \hat{\psi} \) on \( \mathbb{R} \), or rather \( L^2(\mathbb{R}) \). The \( L^2(\mathbb{R}) \)-system consists of a scaling function \( \varphi \) and the wavelet generators \( \psi_1, \ldots, \psi_{N-1} \). The cocycle conditions (see [BrJo02] for details) are

\[
\sqrt{N} \hat{\varphi}(N\xi) = m_0(e^{i\xi}) \hat{\varphi}(\xi), \quad \xi \in \mathbb{R}, \text{ and }
\sqrt{N} \hat{\psi}_j(N\xi) = m_j(e^{i\xi}) \hat{\varphi}(\xi), \quad j = 1, \ldots, N - 1.
\]

For more details on this, we refer the reader to [Dau92], [Mal99], [Jor99], [BrJo99], [Jor01], and [BrJo02]. After the problem is discretized, and a Fourier series is introduced, we then arrive at a certain system of operators on \( L^2(\mathbb{T}) \), where the one-torus \( \mathbb{T} \) is equipped with the usual normalized Haar measure. The operators are defined from a fixed system of functions \( m_0, \ldots, m_{N-1} \) on \( \mathbb{T} \) as follows:

\[
(1.2) \quad (S_j f)(z) = m_j(z) f(z^N), \quad f \in L^2(\mathbb{T}), \quad z \in \mathbb{T}, \quad j = 0, 1, \ldots, N - 1.
\]

The orthogonality conditions which are usually imposed are known to imply the following relations on the operators \((S_j)_{j=0}^{N-1}\) in (1.2):

\[
(1.3) \quad S_j^* S_i = \delta_{j,i} I_{L^2(\mathbb{T})} \quad \text{and} \quad \sum_{j=0}^{N-1} S_j S_j^* = I_{L^2(\mathbb{T})}.
\]

The reader is referred to [Jor01], [Jor00], and [BrJo02] for additional details on this point. Specifically, the properties which must be imposed on the functions \( m_0, \ldots, m_{N-1} \) from (1.2) are known in signal processing as the subband quadrature (if \( N = 2 \)) conditions. They can be checked to be equivalent to the operator relations (1.3), also known as the Cuntz relations. They are satisfied if and only if the \( N \times N \) matrix \((i,j = 0, \ldots, N - 1)\)

\[
(1.4) \quad A_{i,j}(z) := \frac{1}{N} \sum_{w \in \mathbb{T}} m_i(w) w^{-j}, \quad z \in \mathbb{T},
\]

is unitary for all (or almost all, with respect to Haar measure on \( \mathbb{T} \)) \( z \in \mathbb{T} \). The relations (1.3) are called the Cuntz relations, and they are special cases of relations which are defined axiomatically in the theory of representation of \( C^* \)-algebras. But they have an independent life in the science of signal processing; see [BrJo02], [Mal99], and the references given there.

2. The Cuntz relations

It is known [Cun77] that there is a simple \( C^* \)-algebra \( \mathcal{O}_N \) such that the representations \( \rho \) of \( \mathcal{O}_N \) are in a 1–1 correspondence with systems of operators on Hilbert
space satisfying the Cuntz relations. If the generators of $\mathcal{O}_N$ are denoted $s_i$, then
\begin{equation}
 s_i^* s_j = \delta_{i,j} 1 \quad \text{and} \quad \sum_{j=1}^N s_j s_j^* = 1,
\end{equation}
where 1 denotes the unit-element in the $C^*$-algebra $\mathcal{O}_N$. The system of operators
$S_j := \rho(s_j)$ will then satisfy the Cuntz relations on the Hilbert space $\mathcal{H}$ which carries the representation $\rho$. Conversely, every system of operators $S_j$ on a Hilbert space $\mathcal{H}$ which satisfies the Cuntz relations comes from a representation $\rho$ of $\mathcal{O}_N$ via the formula $\rho(s_j) = S_j$, $j = 1, \ldots, N$. It is known that not every representation of $\mathcal{O}_N$ is of the form (1.2). $\mathcal{O}_N$ has many type III representations, and (1.2) implies type I (see [BJO99]). But it was shown in [Jor01] that the analysis of the wavelet representations (1.2) predicts a number of global and geometric properties of the variety of all wavelets subject to a fixed scaling.

In addition the representations of $\mathcal{O}_N$ and their functional models are used in multivariable scattering theory: see, for example, [DKS01], [BaVi1], [BaVi2], and [Kribs].

While our present results apply to general representations of $\mathcal{O}_N$, they are motivated by (1.2). In particular, we will be interested in smaller subspaces of $L^2(T)$ which determine the representation. Further, when a wavelet representation is given, a result in Section 10 makes precise a sense in which these subspaces are attractors, i.e., they arise as limits of a dynamical iteration. It is known from [Jor01] that, if the wavelet system consists of compactly supported functions $\varphi, \psi_1, \ldots, \psi_{N-1}$ on $\mathbb{R}$, then there is a finite-dimensional subspace $\mathcal{L}$ which determines the representation, and therefore the wavelet analysis. The construction of wavelets from multiresolutions (i.e., scales of closed subspaces in $L^2(\mathbb{R})$) is explained in several books on wavelets, starting with [Dau92]. A geometric approach which is close to the present one, based on wandering subspaces, was first outlined in [DaLa98]; see also [BaMe99]. In this paper, we study such determining minimal closed subspaces in the context of the most general representations of the Cuntz algebras $\mathcal{O}_N$.

3. Subspaces of the Hilbert space

The Hilbert space which carries a representation of $\mathcal{O}_N$ must be infinite-dimensional, but we show that it contains “small” distinguished closed subspaces.

First some definitions: there will be eleven in all.

Definitions 3.1. (i) If $\{S_i\}_{i=1}^N$ is a representation of $\mathcal{O}_N$ on a Hilbert space, set $S_I = S_{i_1} \cdots S_{i_k}$ for all multi-indices $I = (i_1, \ldots, i_k)$, and define the multi-index length as $|I| := k$. The set of all such multi-indices $I$ with $i_\nu \in \{1, \ldots, N\}$ is denoted $\mathcal{I}(N)$. The set of multi-indices of length $k$ is denoted $\mathcal{I}_k(N)$.

(ii) If $\mathcal{F}$ is a set of vectors in a Hilbert space $\mathcal{H}$, the notation $\bigvee \mathcal{F}$ stands for the closed linear span of $\mathcal{F}$.
(iii) If $\mathcal{M}$ is a family of subspaces in $\mathcal{H}$, then $\bigvee \mathcal{M}$ stands for the closed linear span of these subspaces.

(iv) If $\{S_i\}$ are the isometries which define a representation of $O_N$, acting on a fixed Hilbert space $\mathcal{H}$, and if $\mathcal{L}$ is a closed subspace in $\mathcal{H}$, then we set

$$S\mathcal{L} := \bigvee_{i=1}^N S_i \mathcal{L}, \quad \text{and} \quad S^* \mathcal{L} := \bigvee_{i=1}^N S_i^* \mathcal{L}.$$  

(Note that, since the operators $S_i$ are isometries, it follows that each space $S_i \mathcal{L}$ is closed. From the identity

$$\sum_{i=1}^N S_i S_i^* = I_{\mathcal{H}},$$

it follows that there are natural conditions which imply that each of the spaces $S_i^* \mathcal{L}$ is also closed. We spell out this point in Lemma 3.2 below.)

(v) A closed subspace $\mathcal{L}$ is said to be co-invariant for a fixed representation $(S_i)$ of $O_N$ acting in a Hilbert space $\mathcal{H}$ if $S^* \mathcal{L} \subset \mathcal{L}$.

(vi) A closed subspace $\mathcal{L}$ is said to be saturated if

$$\bigvee_{k=1}^{\infty} S^k \mathcal{L} = \mathcal{H},$$

where, for every $k$,

$$S^k \mathcal{L} := \bigvee_{(i_1, \ldots, i_k) \in \mathcal{I}_k(N)} S_{i_1} S_{i_2} \cdots S_{i_k} \mathcal{L}.$$  

(vii) A co-invariant closed subspace $\mathcal{L}$ is said to be minimal if the corresponding complementing space $\mathcal{W} := S\mathcal{L} \ominus \mathcal{L}$ generates a maximal subspace, i.e.,

the family

$$\mathcal{F}(\mathcal{W}) := \{ \mathcal{W}, S_I \mathcal{W} : I = (i_1, \ldots, i_k), \ k \geq 1, \ I \in \mathcal{I}(N) \}$$

is such that

$$\sum \oplus \{ \mathcal{K} : \mathcal{K} \in \mathcal{F}(\mathcal{W}) \}$$

is maximal in $\mathcal{H}$.

(viii) We say that $\mathcal{L} \subset \mathcal{H}$ is a core for the representation if it is co-invariant, saturated, and minimal.

(ix) A closed subspace $\mathcal{W} \subset \mathcal{H}$ is called wandering if all the spaces in the family $\mathcal{F}(\mathcal{W})$ are mutually orthogonal.

(x) A subspace (not necessarily closed) $\mathcal{V}$ in $\mathcal{H}$ is said to reduce to a closed subspace $\mathcal{L}$ in a representation $(S_i)$ if, for every $v \in \mathcal{V}$, there is a $k_0 \in \mathbb{N}$ such that $S_i^* S_i v \in \mathcal{L}$ whenever $k \geq k_0$.

(xi) A subspace $\mathcal{L}$ is said to be stable with respect to a representation $(S_i)$ of $O_N$ if it is invariant under each of the projections $E_i := S_i S_i^*$, $i = 1, \ldots, N$. 

Our main result is Theorem \[10.1\]. It states that the standard wavelet representation \[1.2\] corresponding to filter functions \(m_j\) which are Lipschitz always has a natural finite-dimensional co-invariant subspace \(L_{\text{fin}}\). The results leading up to Section \[10\] throw light on the properties of the subspace \(L_{\text{fin}}\), and on co-invariant subspaces more generally.

**Lemma 3.2.** Let a representation \((S_i)\) of \(O_N\) on a Hilbert space \(H\) be given. Let \(L \subset H\) be a closed subspace which is stable. Then each of the linear spaces \(S_i^*L = \{S_i^*x : x \in L\}\) is closed in \(H\).

**Proof.** Set \(i = 1\) for specificity. Let \(x_n\) be a sequence in \(L\) such that \(S_1^*x_n \xrightarrow{n\to\infty} y\). We introduce the projection \(E_1 = S_1S_1^*\). Since \(S_j^*E_1 = \delta_{j,1}S_1^*\), we have \(S_1^*E_1x_n = S_1^*x_n\), \(S_j^*E_1x_n = 0\) if \(j \neq 1\), and \(\|S_i^*E_1(x_n - x_m)\| = \|E_1(x_n - x_m)\| \xrightarrow{n,m\to\infty} 0\).
The sequence \((E_1x_n)\) is convergent, and its limit is in \(L\) since \(L\) is closed and stable. If \(E_1x_n \xrightarrow{n\to\infty} z\), then \(S_1^*z = y\), and we conclude that \(S_1^*L\) is closed. \(\square\)

### 4. Co-invariant closed subspaces

A representation of \(O_N\) on a Hilbert space \(H\) is specified by operators \(S_1, \ldots, S_N\) on \(H\) subject to the Cuntz relations

\[
S_i^*S_j = \delta_{i,j}I_H, \quad \sum_{j=1}^NS_jS_j^* = I_H,
\]

where \(I_H\) denotes the identity operator in the Hilbert space \(H\). Intrinsic to this is the set of \(N\) commuting projections \(E_j := S_jS_j^*\). They enter into the statement of the next lemmas.

In the next lemma we record some general properties about co-invariant subspaces. They are stated in terms of projections. Recall there is a 1–1 correspondence between closed subspaces \(L \subset H\) and projections \(P\) in \(H\), i.e., \(P = P^* = P^2\). If \(L\) is given, there is a unique \(P\) such that \(L = PH = \{x \in H : Px = x\}\), and conversely.

**Lemma 4.1.** Let \(\{S_i\}_{i=1}^N\) be a representation of \(O_N\) on a Hilbert space \(H\), and let \(L\) be a closed subspace in \(H\). Set \(\alpha (A) = \sum_{i=1}^NS_iAS_i^*\), \(A \in \mathcal{B}(H)\).

(a) Then \(\alpha : \mathcal{B}(H) \to \mathcal{B}(H)\) is an endomorphism satisfying \(\alpha (I_H) = I_H\).

(b) If \(P\) denotes the projection onto \(L\), then \(L\) is co-invariant if and only if \(P \leq \alpha (P)\); and \(\alpha (P)\) is then the projection onto \(SL\).

(c) If \(L\) is co-invariant, then \(Q = \alpha (P) - P\) is a projection. Its range is the wandering subspace \(W := (SL) \cap L\).

**Proof.** The details are left to the reader. They are based on standard geometric facts about projections in Hilbert space. \(\square\)
Lemma 4.2. Let \( \{S_i\}_{i=1}^N \) be a representation of \( \mathcal{O}_N \), and let \( \mathcal{L} \) be a closed co-invariant subspace. Set \( W = (S\mathcal{L}) \oplus \mathcal{L} \), and \( \mathcal{F}(W) = W \oplus SW \oplus S^2W \oplus \cdots \). Let \( P_\mathcal{L} \) be the projection onto \( \mathcal{L} \), and let \( \alpha \) be the endomorphism of Lemma 4.1. Then the limit \( P_\alpha = \lim_{n \to \infty} \alpha^n(P_\mathcal{L}) \) exists, and the projection \( P_\mathcal{F} \) onto \( \mathcal{F}(W) \) is given by the formula

\[
(4.2) \quad P_\mathcal{F} = P_\infty - P_\mathcal{L}.
\]

Moreover, the operators \( T_i = P_\mathcal{F}S_iP_\mathcal{F} \), \( i = 1, \ldots, N \), satisfy the following (Cuntz–Toeplitz–Fock) relations:

(a) \( T_i^*T_j = \delta_{i,j}P_\mathcal{F} \), \( i, j = 1, \ldots, N \),

(b) \( \sum_{i=1}^N T_i^*T_i \leq P_\mathcal{F} \),

(c) \( T_i^*w = 0 \) for all \( i = 1, \ldots, N \), and all \( w \in W \).

Proof. The details amount to direct verifications and are left for the reader.

Lemma 4.3. Let \( \{S_i\}_{i=1}^N \) be a representation of \( \mathcal{O}_N \) on a Hilbert space \( \mathcal{H} \). Let \( W \subset \mathcal{H} \) be a closed subspace such that \( \langle x \mid S_Jy \rangle = 0 \) for all \( x, y \in W \) and all multi-indices \( J \in \mathcal{I}(N) \). Then the operators \( T_i := P_{\mathcal{F}(W)}S_iP_{\mathcal{F}(W)} \) satisfy the conditions (4.1)–(4.3) in Lemma 4.2, i.e., the subspace \( \mathcal{F}(W) = W \oplus SW \oplus S^2W \oplus \cdots \) induces a Fock-space representation of the Cuntz–Toeplitz relations.

Proof. It suffices to show that each operator \( S_i^* \) maps \( W \) into \( \mathcal{L} = \mathcal{H} \ominus \mathcal{F}(W) \). Suppose \( x \in W \) and \( y \in \mathcal{F}(W) \): then we claim that \( \langle S_i^*x \mid y \rangle = 0 \). The assertion follows from this. To establish the claim, suppose first that \( y \in W \). Then \( \langle S_i^*x \mid y \rangle = \langle x, S_iy \rangle = 0 \) holds on account of the definition of \( W \). Similarly \( \langle S_i^*x \mid S_Jy \rangle = \langle x \mid S_iS_Jy \rangle = 0 \) for all \( x, y \in W \), and all multi-indices \( J \). It follows, in particular, that the operators \( T_i = P_{\mathcal{F}(W)}S_iP_{\mathcal{F}(W)} \) satisfy \( T_i^*x = 0 \) for all \( x \in W \), in addition to the Cuntz–Toeplitz relations (a)–(c).

Our next result is a partial converse to Lemma 4.2.

Proposition 4.4. Let \( \{S_i\}_{i=1}^N \) be a representation of \( \mathcal{O}_N \) on a Hilbert space \( \mathcal{H} \), and let \( W_m \) be chosen as in Lemma 4.7 below to be maximal (in the sense of Zorn’s lemma) with respect to the property

\[
(4.3) \quad \langle x \mid S_Jy \rangle = 0, \quad x, y \in W_m, \quad J \in \mathcal{I}(N).
\]

Then every co-invariant closed subspace \( \mathcal{L} \) such that \( S^*W_m \subset \mathcal{L} \subset W_m^\perp \) satisfies

\[
(4.4) \quad S\mathcal{L} \ominus \mathcal{L} = W_m.
\]

Conversely, if equality holds in (4.4) for some wandering subspace \( W \) and all \( \mathcal{L} \) with \( S^*W \subset \mathcal{L} \subset W^\perp \), then \( W \) is maximal.

Proof. First note that, if \( W \) is any wandering subspace, then the inclusions

\[
S^*W \subset \mathcal{F}(W)^\perp \subset W^\perp
\]
are automatic; and \( \mathcal{L}_W := \mathcal{F}(\mathcal{W})^\perp \) is co-invariant. Also \( \mathcal{L}_W \) is maximal among the co-invariant subspaces \( \mathcal{L} \) satisfying
\[
S^*\mathcal{W} \subset \mathcal{L} \subset \mathcal{W}^\perp.
\]

Recall that, if \( P_\mathcal{L} \) is the projection onto \( \mathcal{L} \), then by Lemma 4.3, \( \alpha(P_\mathcal{L}) = \sum_{i=1}^{N} S_i P_\mathcal{L} S_i^* \) is the projection onto \( S\mathcal{L} \). If we show that \( \alpha(P_\mathcal{L}) x = x \) for all \( x \in \mathcal{W}_m \), it follows that \( \mathcal{W}_m \subseteq (S\mathcal{L}) \cap \mathcal{L} \). But the vectors in \( (S\mathcal{L}) \cap \mathcal{L} \) satisfy the orthogonality relations \( 4.3 \), and we may then conclude that \( 4.3 \) holds by virtue of the maximality of \( \mathcal{W}_m \); see Lemma 4.7 below. The proof is now completed since \( S_i^* x \in \mathcal{L} \) for all \( x \in \mathcal{W}_m \), and we get \( \alpha(P_\mathcal{L}) x = \sum_{i=1}^{N} S_i P_\mathcal{L} S_i^* x = \sum_{i=1}^{N} S_i S_i^* x = x \).

We leave the verification of the converse implication to the reader, i.e., that equality in \( 4.4 \) for some \( \mathcal{W} \) and all \( \mathcal{L} \) as specified implies that \( \mathcal{W} \) is maximal with respect to \( 4.4 \).

**Remark 4.5.** The argument shows in particular that if \( \mathcal{W} \) is any closed wandering subspace, then \( S^*\mathcal{W} \subset \mathcal{W}^\perp \), and further that there will always be co-invariant subspaces \( \mathcal{L} \) satisfying
\[
(4.5) \quad S^*\mathcal{W} \subset \mathcal{L} \subset \mathcal{W}^\perp.
\]

In fact, if \( \mathcal{W} \) is given, then \( \mathcal{L} = \mathcal{F}(\mathcal{W})^\perp \) is such an intermediate co-invariant subspace. If \( \mathcal{W} \) is also assumed maximal in the sense of Proposition 4.4, then it follows that \( \mathcal{L} = \mathcal{F}(\mathcal{W})^\perp \) is the only closed and co-invariant saturated subspace which is intermediate as stated in \( 4.6 \).

**Remark 4.6.** While, on the face of it, the relations \( 4.1 \) appear to depend on a choice of coordinates, this is not so: the Cuntz C*-algebra is in fact a functor from Hilbert space to C*-algebras, as noted in \[\text{Cun77}\]. Specifically, consider the mapping
\[
(4.6) \quad z = (z_1, \ldots, z_N) \mapsto s(z) = \sum_{j=1}^{N} z_j s_j
\]
from the Hilbert space \( \mathbb{C}^N \) into the C*-algebra \( \mathcal{O}_N \). If 1 denotes the unit element in \( \mathcal{O}_N \), we get \( s(z)^* s(w) = \langle z \mid w \rangle 1 \) where \( \langle z \mid w \rangle = \sum_{j=1}^{N} z_j w_j \) is the usual inner product of \( \mathbb{C}^N \). If \( \| \cdot \| \) denotes the C*-norm on \( \mathcal{O}_N \), i.e., satisfying \( 1 \leq 1 \) and \( \| A^* A \| = \| A \|^2 \), \( A \in \mathcal{O}_N \), then \( \| s(z) \| = \left( \sum_{j=1}^{N} |z_j|^2 \right)^{1/2} = \| z \|_{\mathbb{C}^N} \).

*States* on \( \mathcal{O}_N \) are linear functionals \( \omega: \mathcal{O}_N \to \mathbb{C} \) such that \( \omega(1) = 1 \), \( \omega(A^* A) \geq 0, A \in \mathcal{O}_N \). A special state \( \omega \) on \( \mathcal{O}_N \) is determined by
\[
(4.7) \quad \omega(s_1 s_2 \cdots s_k s_1^* \cdots s_2^* s_1^*) = \prod_{s} \delta_{i_s,j_s} \cdot \delta_{k,l_s} \cdot N\cdot \nu^{-k}.
\]

As is well known, states induce cyclic representations. A representation \( \rho \) of \( \mathcal{O}_N \) in a Hilbert space \( \mathcal{H} \) is *cyclic* if there is a vector \( \Omega \) in \( \mathcal{H} \) such that \( \{ \rho(A) \Omega : A \in \mathcal{O}_N \} \) is dense in \( \mathcal{H} \). If the state is \( \omega \), the corresponding representation \( \rho \) is determined by \( \omega(A) = \langle \Omega \mid \rho(A) \Omega \rangle \). When \( \rho \) is given, we set \( S_i := \rho(s_i) \),
\( i = 1, \ldots, N \). We shall often identify a representation \( \rho \) of \( O_N \) with the corresponding operators \( S_i = \rho(s_i) \).

The representation \( \{S_i\}_{i=1}^N \) of the state (4.7) does not have a closed coinvariant and saturated subspace which is minimal with respect to the usual ordering of closed subspaces. Nonetheless the following general result holds.

**Lemma 4.7.** Let \( \{S_i\}_{i=1}^N \) be a representation of \( O_N \) on a Hilbert space \( \mathcal{H} \).

(i) Then \( \mathcal{H} \) contains a wandering subspace \( \mathcal{W}_m \) which is maximal with respect to inclusion. Specifically, \( \mathcal{W}_m \) satisfies the implication: Whenever \( \mathcal{W} \) is a wandering subspace such that \( \mathcal{W}_m \subseteq \mathcal{W} \), then \( \mathcal{W}_m = \mathcal{W} \).

(ii) Maximal wandering subspaces are non-unique.

**Proof.** Recall that a closed subspace \( \mathcal{W} \) is wandering if and only if the family of spaces \( \{W, S_I W\} \) is orthogonal as \( I \) varies over all the multi-indices, i.e., \( I \in \mathcal{I}(N) \). Equivalently,

\[
P_W S_I P_W = 0 \quad \text{for all } I \in \mathcal{I}(N)
\]

where \( P_W \) denotes the projection onto \( \mathcal{W} \). If \( \{\mathcal{W}_\alpha\} \) is a linearly ordered family of wandering subspaces (i.e., for all \( \alpha, \beta \), one of the inclusions \( \mathcal{W}_\alpha \subseteq \mathcal{W}_\beta \) or \( \mathcal{W}_\beta \subseteq \mathcal{W}_\alpha \) holds), then it can be checked that \( \mathcal{W} = \bigvee_{\alpha} \mathcal{W}_\alpha \) satisfies the condition (4.8). The existence of \( \mathcal{W}_m \) now follows from Zorn’s lemma. The example of (4.7) shows that \( \mathcal{W}_m \) will typically be non-unique.

**Lemma 4.8.** Let \( (S_i)_{i=1}^N \) be a representation of \( O_N \) on a Hilbert space \( \mathcal{H} \), and let \( \mathcal{L} \) be a closed subspace of \( \mathcal{H} \). Then conditions (i) and (ii) are equivalent:

(i) \( S^* \mathcal{L} \subseteq \mathcal{L} \);

(ii) \( \mathcal{L} \subseteq S \mathcal{L} \).

**Proof.** (i) \( \Rightarrow \) (ii). Assume (i). Let \( x \in \mathcal{L} \). Then \( x = \sum_i S_i S_i^* x \in S \mathcal{L} \) since \( S_i^* x \in \mathcal{L} \) for \( i = 1, \ldots, N \).

(ii) \( \Rightarrow \) (i). Assume (ii). If \( x \in \mathcal{L} \), then there are \( y_i \in \mathcal{L} \) such that \( x = \sum_i S_i y_i \).

We get \( S_i x = \sum_i S_i^* S_i y_i = \sum_i \delta_{ij} y_i = y_j \), which proves (i).

**Remark 4.9.** Example 5.3 below shows that there are representations \( (S_i)_{i=1}^N \) of \( O_N \) for every \( N = 2, 3, \ldots \), such that the second inclusion in the lemma is sharp, but not the first; i.e., it may happen that \( S^* \mathcal{L} = \mathcal{L} \) while \( \mathcal{L} \nsubseteq S \mathcal{L} \).

**Example 4.10.** The cyclic representation \( (S_i)_{i=1}^N \) defined from the state \( \omega \) in (4.7) has a co-invariant infinite-dimensional subspace \( \mathcal{L} \) spanned by the vectors \( S_i^* \omega \) where \( I = (i_1, \ldots, i_k) \) runs over all the multi-indices \( k \geq 1 \), and the relative orthocomplement \( S \mathcal{L} \cap \mathcal{L} \) is spanned by the following \( N^2 \) orthogonal vectors:

\[
\{S_i S_j^* \omega : i, j = 1, \ldots, N\}.
\]

**Proof.** The details are left to the reader.
Lemma 4.11. Let \((S_i)_{i=1}^N\) be a representation of \(O_N\) on a Hilbert space \(H\), and let \(\mathcal{L}\) be a closed subspace in \(H\) which is co-invariant. Then the subspace

\[
\mathcal{M} := \bigvee \{ E_i \mathcal{L} : i = 1, \ldots, N \}
\]

is co-invariant and stable. We use the notation \(E_i = S_i S_i^*\) for the projection onto \(S_i H\).

Proof. Since \(E_i^2 = E_i\), it is clear that the space \(\mathcal{M}\) is invariant under each \(E_i\). Since \(S_i^* E_i = \delta_{j,i} S_j^*\), it is clear that \(\mathcal{M}\) is also invariant under each \(S_i^*\). Hence \(\mathcal{M}\) is co-invariant. For every \(x \in \mathcal{L}\), we have \(x = \sum_i E_i x\) by (4.1), and it follows then from (4.9) that \(\mathcal{L} \subset \mathcal{M}\), which was used in the argument above.

5. Resolution subspaces of \(\mathcal{H}\)

We begin this section with an explicit isomorphism between the family Co-inv of all closed co-invariant subspaces, and the family Wan of all closed wandering subspaces. The definitions refer to a specified representation \(\{S_i\}_{i=1}^N\) of \(O_N\) on a Hilbert space \(H\), and we shall work with the corresponding endomorphism \(\alpha(A) = \sum_{i=1}^N S_i A S_i^*, A \in B(H)\). Note that if \(P\) is a projection of \(H\) onto a closed subspace \(K \subset H\), then \(\alpha(P)\) is the projection onto \(S K\).

Lemma 5.1. For \(L \in \text{Co-inv}\) define \(\mu(L) = (SL) \ominus L\). Then \(\mu(L) \in \text{Wan}\). For \(W \in \text{Wan}\), set \(\lambda(W) = H \ominus F(W)\); then \(\lambda(W) \in \text{Co-inv}\). Moreover

\[
\lambda(W) = \mu(\lambda(W)) .
\]

Proof. If \(K \subset H\) is a closed subspace, then we denote by \(P_K\) the projection onto \(K\), i.e., \(K = P_K H\), and \(P_K = P_K^* = P_K^2\). Thus, identifying closed subspaces with projections, we arrive at the formulas

\[
\mu(P_L) = \alpha(P_L) - P_L
\]

and

\[
\lambda(P_W) = I_H - P_{F(W)} = I_H - \sum_{n=0}^{\infty} \alpha^n(P_W) .
\]

If \(W\) is wandering, then the projections \(\alpha^n(P_W)\) in the sum are mutually orthogonal, and it follows that \(\lim_{n \to \infty} \alpha^n(P_W) = 0\). Hence it follows that the two terms in the difference below are convergent, and that

\[
\mu(\lambda(P_W)) = \sum_{n=0}^{\infty} \alpha^n(P_W) - \sum_{n=1}^{\infty} \alpha^n(P_W) = P_W .
\]

Lemma 5.2. Let a representation \((S_i)\) of \(O_N\) be given, and let \(\mathcal{L}\) be a closed subspace which is co-invariant. Let \(\mathcal{V}\) denote the linear span of the spaces \(S^k \mathcal{L}\). Then \(\mathcal{V}\) reduces to \(\mathcal{L}\) in the representation.
Proof. We have $S^* \mathcal{L} \subset \mathcal{L}$. Using $\sum_i S_i S_i^* = I_{\mathcal{H}}$ we conclude that $\mathcal{L} \subset S \mathcal{L}$, and by induction $S^k \mathcal{L} \subset S^{k+1} \mathcal{L}$ for $k \in \mathbb{N}$. Since, clearly, $S_i^* (S^{k+1} \mathcal{L}) \subset S^k \mathcal{L}$, we conclude that $\mathcal{V}$ reduces to $\mathcal{L}$ in the sense of Definition 5.1.5.

The next two examples are the representation of $\mathcal{O}_2$ on the Hilbert space $L^2(\mathbb{T})$ which are defined from the two best known examples of wavelets in $L^2(\mathbb{R})$.

The first is the Haar wavelet. It has $\varphi = \chi_I$, where $I$ is the unit interval $I = [0, 1)$, and

$$\psi = \chi_{[0,1]} - \chi_{[\frac{1}{2},1)}$$

The corresponding functions $m_j$, $j = 0, 1$, which define the wavelet filters are $m_0(z) = \frac{1}{\sqrt{2}} (1 + z)$ and $m_1(z) = \frac{1}{\sqrt{2}} (1 - z)$; see (1.1) and (1.2). The other wavelet is not localized in $x$-space, but rather in the dual Fourier variable $\xi$ of the Fourier transform $\hat{\psi}(\xi) = \int_{-\infty}^{\infty} e^{-i2\pi \xi x} \psi(x) \, dx$. These wavelets are called frequency localized; see [BrJo99]. It is known that there is a wavelet (named after Shannon) for which the wavelet generator $\psi_S$ is characterized by

$$\hat{\psi}_S(\xi) = \chi_{[-\frac{1}{4}, -\frac{1}{2}] \cup [\frac{1}{2}, \frac{3}{4}]}(\xi), \quad \xi \in \mathbb{R}.$$ 

We now compare the $\mathcal{O}_2$-representation of $\psi_S$ with that of the Haar wavelet from (5.3).

More generally, we note that the examples of representations of $\mathcal{O}_N$ which primarily motivate our results are those which arise from discretizing wavelet problems, in the sense of [BrJo02] and [Jor99]. They may be realized on the Hilbert space $\mathcal{H} = L^2(I)$ where $I$ is a compact interval with Lebesgue measure. Using the Fourier series on functions on $I$, it will be convenient to view $\mathcal{H}$ alternately as $\ell^2(\mathbb{Z})$, or as $L^2(\mathbb{T})$, where $\mathbb{T}$ is the one-torus, equipped with the usual normalized Haar measure. When working with $L^2(\mathbb{T})$, it will be convenient to have the orthonormal basis $e_n(z) = z^n$, $n \in \mathbb{Z}$, for use in computations.

The following two examples of representations of $\mathcal{O}_N$ on $\mathcal{H}$ illustrate the concepts in Section 1. We will give them just for $N = 2$, but the reader can easily write out the general case.

**Example 5.3.** Let $S_0 f(z) = f(z^2)$, and $S_1 f(z) = zf(z^2)$, $f \in \mathcal{H} = L^2(\mathbb{T})$.

In terms of the Fourier basis, this representation may be identified as follows:

$$S_0 e_n = e_{2n}, \quad S_1 e_n = e_{2n+1}, \quad n \in \mathbb{Z},$$

and with the adjoint operators

$$S_0^* e_n = \begin{cases} e_{n/2} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \quad \text{and} \quad S_1^* e_{2n+1} = e_n, \quad S_1^* e_m = 0 \text{ if } m \text{ is even.}$$

It is immediate that the system of operators $(S_i)_{i=0}^1$ defines a representation of $\mathcal{O}_2$, and that $\mathcal{L} = \text{span}\{e_{-1}, e_0\}$ is a core for the representation, in the sense of
Attractors for representations

Definition 3.1(viii). Since

\[ S^*_0 e_{-1} = 0, \quad S^*_1 e_{-1} = e_{-1}, \]
\[ S^*_0 e_0 = e_0, \quad S^*_1 e_0 = 0, \]

it is clear that \( L \) is \textit{stable} in the sense of Definition 3.1(xi).

Remark 5.4. For Haar's representation \((S_i)_{i=0}^1\) in Example 5.3 we have two natural finite-dimensional co-invariant subspaces,

\[ L_{\text{min}} = \bigvee \{ e_{-1}, e_0 \} \quad \text{and} \quad L = \bigvee \{ e_{-2}, e_{-1}, e_0, e_1 \}; \]

and \( L_{\text{min}} \) is minimal in the sense of Definition 3.1(viii), while \( L \) is not. This follows from the observations

\[ (S L_{\text{min}}) \ominus L_{\text{min}} = \bigvee \{ e_{-2}, e_1 \}, \]
\[ S L \ominus L = \bigvee \{ e_{-4}, e_{-3}, e_2, e_3 \}, \]

and

\[ S^* L = L_{\text{min}}. \]

Example 5.5. Working with \( I = \left[ -\frac{1}{2}, \frac{1}{2} \right] \), we may view \( L^2(I) \) as a space of \( \mathbb{Z} \)-periodic functions on \( \mathbb{R} \), and we will work with the operations \( x \mapsto 2x, \ x \mapsto \frac{x}{2}, \ x \mapsto \frac{x + 1}{2} \mod \mathbb{Z} \). The last two are branches of inverses of the doubling \( x \mapsto 2x \mod 1 \), or equivalently of \( z \mapsto z^2 \) on \( \mathbb{T} \). In the second incarnation the two branches of the inverse may be thought of as \( \pm \sqrt{z} \) restricted to \( \mathbb{T} \subset \mathbb{C} \), i.e., the two branches of the complex square root. Consider the subsets \( A \) and \( B \) of \( I \):

\[ A = \{ x \in I : |x| \leq \frac{1}{4} \}, \quad \text{and} \quad B = \{ x : -\frac{1}{2} \leq x < -\frac{1}{4} \} \cup \{ x : \frac{1}{4} < x \leq \frac{1}{2} \}, \]

forming a dyadic partition of \( I \). Then set

\[ S_0 f(x) := \sqrt{2} \chi_A(x) f(2x \mod \mathbb{Z}), \quad \text{and} \quad S_1 f(x) := \sqrt{2} \chi_B(x) f(2x \mod \mathbb{Z}). \]

It will be convenient for us to omit the mod \( \mathbb{Z} \) notation when it is otherwise implicit in the formulas. Then it is immediate that the isometries \((S_i)_{i=0}^1\) define a representation of \( O_2 \), and that \( L := \mathbb{C} \mathbb{1} \) is a \textit{core} for the representation, acting on \( \mathcal{H} = L^2(I) \), and viewing \( I \) as \( \mathbb{R}/\mathbb{Z} \). Also we use the notation \( \mathbb{1} \) for the function which is constant \( \equiv 1 \) on \( I \). The formulas

\[ S^*_0 \mathbb{1} = \frac{1}{\sqrt{2}} \mathbb{1}, \quad S^*_1 \mathbb{1} = \frac{1}{\sqrt{2}} \mathbb{1}, \]
\[ S_0 \mathbb{1} = \sqrt{2} \chi_A, \quad S_1 \mathbb{1} = \sqrt{2} \chi_B \]

now make it clear that this one-dimensional core subspace \( L \) is not \textit{stable} in the sense of Definition 3.1(xvii). Specifically, we have:

\[ E_0 \mathbb{1} = \chi_A, \quad \text{and} \quad E_1 \mathbb{1} = \chi_B. \]
As in the discussion below Lemma 5.2, it follows that
\begin{align}
S_0^*\chi_A &= \frac{1}{\sqrt{2}} \mathbb{1}, & S_0^*\chi_B &= 0, \\
S_1^*\chi_A &= 0, & S_1^*\chi_B &= \frac{1}{\sqrt{2}} \mathbb{1}.
\end{align}
Therefore the two-dimensional space \( M := \text{span} \{\chi_A, \chi_B\} \) is co-invariant, and it is \textit{saturated} in the sense of Definition 3.1(vi). The space \( M \) is also stable.

6. Nontrivial co-invariant subspaces
Let \((S_i)_{i=1}^N\) be a representation of \( O_N \) acting on a Hilbert space \( H \). Suppose a closed subspace \( L \) in \( H \) is co-invariant. We then get the resolution
\[ L \subset S L \subset S^2 L \subset \cdots \subset S^n L \subset \cdots \subset H. \]
If \( L \) is saturated, then \( \bigvee_n S^n L = H \).
There are two trivial cases where both conditions are satisfied for a given \( L \). First, if \( L = H \), there isn’t much to say. Secondly, if \( L = S L \), then the given representation on \( H \) simply restricts to a representation on \( L \). Recall, if \( L \) is co-invariant, and if \( L = S L \), then \( L \) is an invariant subspace for all the \( 2N \) operators \( S_1, \ldots, S_N \), \( S_1^*, \ldots, S_N^* \).

We say that \( L \) is a reducing subspace for the given representation. We then get two orthogonal representations of \( O_N \) by restriction to each of the two spaces, \( L \) and its orthocomplement \( H \oplus L \). We say that \( L \) is a reducing subspace for the given representation. We then get two orthogonal representations of \( O_N \) by restriction to each of the two spaces, \( L \) and its orthocomplement \( H \oplus L \).

**Theorem 6.1.** There is a two-way correspondence between the nontrivial, closed, co-invariant, and saturated subspaces \( L \) for a representation \((S_i)_{i=1}^N\) of \( O_N \) on \( H \), and closed subspaces \( W \neq 0 \) in \( H \oplus L \) which provide an orthogonal decomposition
\[ H = L \oplus W \oplus S W \oplus S^2 W \oplus \cdots \]
Note the subspace \( W \) in (6.2) is called wandering, and the conditions imply pairwise orthogonality of all the closed subspaces:
\[ L, W, S_i W, S_1 S_2 W, S_1 S_2 S_3 W, \ldots, S_1 \cdots S_k W, \ldots, \]
where all multi-indices \((i_1, i_2, \ldots, i_k)\) are considered for \( k = 1, 2, \ldots \).

**Proof.** Suppose first that \( 0 \neq W \) satisfies the conditions from (6.2). Then set \( L := H \oplus \bigoplus_{k=0}^\infty S^k W \), with the understanding that \( S^0 W := W \) and
\[ S^k W = \bigvee_{(i_1, i_2, \ldots, i_k)} S_{i_1} \cdots S_{i_k} W. \]
Since $\sum_{k=0}^{\infty} S^k W$ is invariant under all the isometries $S_i$, $i = 1, \ldots, N$, it is clear that $L$ is invariant under the adjoints $S_i^*$, and so $L$ is co-invariant. Since $W \neq 0$, clearly $L \neq \mathcal{H}$. It remains to verify that the strict inclusion $L \subset S L$ holds. Suppose, indirectly, that $L = S L$. Then $L$ is reducing, and so is $\sum_{k=0}^{\infty} S^k W$. To see that this is impossible, let $I(k)$ denote the set of all multi-indices $(i_1, \ldots, i_k)$, where $i_1, i_2, \cdots \in \{1, 2, \ldots, N\}$. If $\sum_{k=0}^{\infty} S^k W$ were $S_i^*$-invariant for all $i$, then

$$
\lim_{k \to \infty} \sum_{I \in I(k)} \|S_I x\|^2 = 0 \quad \text{for all } x \in W,
$$

where $S_I = S_{i_k}^* \cdots S_{i_1}^*$ for $I = (i_1, \ldots, i_k)$. But this is impossible by the Cuntz relations. Recall that

$$
\sum_{I \in I(k)} \|S_I x\|^2 = \|x\|^2 \quad \text{holds for all } x.
$$

This contradicts (6.5) and the condition $W \neq 0$. 

Suppose now that $L$ is a given nontrivial, closed, co-invariant and saturated subspace for the given representation in $\mathcal{H}$. As noted, then $L \subset S L$; so the relative orthocomplement $W := (S L) \ominus L$ is nonzero. We first prove that all the multi-indexed subspaces listed in (6.3) are mutually orthogonal. The argument is by induction, starting with $W \perp S W$. We must check the inner products

$$
\left\langle \sum_i S_i x_i \mid \sum_{j_1,j_2} S_{j_1} S_{j_2} y_{j_1,j_2} \right\rangle
$$

where $x_i, y_{j_1,j_2} \in L$, under the conditions

$$
\sum_i S_i x_i \perp W \quad \text{and} \quad \sum_{j_2} S_{j_2} y_{j_1,j_2} \perp W \quad \text{for all } j_1 = 1, \ldots, N.
$$

But the term in (6.7) simplifies to

$$
\sum_{i,j} \langle x_i \mid y_{i,j} \rangle = 0,
$$

where the vanishing results from the conditions (6.8). We leave the remaining recursive argument to the reader.

It remains to check that the sequence of closed subspaces

$$
L, W, S W, S^2 W, \ldots
$$

is total in $\mathcal{H}$, i.e., that (6.2) holds.

Since $L$ is given to be saturated, the conclusion will follow from

$$
S^{k+1} L \subset L \oplus W \oplus S W \oplus \cdots \oplus S^k W.
$$
If \( k = 0 \), this holds from the ansatz which defines \( W \) in terms of the given closed subspace \( L \), i.e., \( W := (SL) \oplus L \). Suppose (6.9) has been verified up to \( k - 1 \). Then

\[
S^{k+1}L \subset S \left( L \oplus W \oplus \cdots \oplus S^{k-1}W \right) \\
\subset SL \oplus SW \oplus \cdots \oplus S^kW \\
\subset L \oplus W \oplus SW \oplus \cdots \oplus S^kW,
\]

where we used the induction hypothesis and the ansatz.

\[\square\]

7. Existence

While the co-invariant subspaces for the wavelet representations are relatively well understood for the special representations of \( O_N \) which derive from wavelet analysis, the situation is somewhat mysterious in the case of the most general \( O_N \)-representations. Even the existence of nontrivial co-invariant subspaces which form cores for a given \( O_N \)-representation appears to be open in general. However:

**Corollary 7.1.** A representation \((S_i)_{i=1}^N\) of \( O_N \) on a Hilbert space \( \mathcal{H} \) has a nontrivial core subspace \( L \), i.e., a closed co-invariant subspace \( L \subset \mathcal{H} \) which is nontrivial and saturated, if and only if it has a nonzero wandering vector. Further, \( L \) may be chosen to be minimal with respect to being co-invariant and saturated.

**Proof.** We showed in Theorem 6.1 that the general case may be reduced to the consideration of nontrivial co-invariant subspaces. In Theorem 6.1 we also showed that the nontrivial co-invariant subspaces \( L \) may be understood from the corresponding wandering subspaces \( W \). If \( L \) is a nontrivial co-invariant subspace, then \( W := (SL) \oplus L \) is wandering for the representation, and conversely every wandering subspace \( W \) comes from a co-invariant subspace \( L \). The idea is to prove the existence claim in the corollary by establishing the existence of a maximal wandering subspace \( W_{\text{max}} \). We will do this by Zorn’s lemma. We say that a wandering subspace \( W_{\text{max}} \) is maximal if for every wandering subspace \( W \) such that \( W \supseteq W_{\text{max}} \), we may conclude that \( W = W_{\text{max}} \). The existence of \( W_{\text{max}} \) follows by an application of Zorn’s lemma to the family of all wandering subspaces \( W \neq 0 \) ordered by inclusion. To be able to do this, we need to know that every representation of \( O_N \) has at least one wandering subspace \( W \neq 0 \). Some details are given below for the benefit of the reader.

To start the transfinite induction we must assume that there are solutions

\[
x \in \mathcal{H}, \quad \|x\| = 1,
\]

to the system

\[
(x \mid S_I x) = 0, \quad \text{for all } I = (i_1, \ldots, i_k), \ k \geq 1.
\]

This is the stated condition in the formulation of Corollary 7.1 and it is needed for the start of the transfinite induction. It is conceivable that such a nonzero vector
Remark 7.2. It follows from Section 4 above that \( \|1 - s_I\| \geq \sqrt{2} \) for all \( I \) where \( s_I \in \mathcal{O}_N \) satisfies \( \rho(s_I) = S_I = S_{i_1} \cdots S_{i_k} \). Hence it follows that there is a state \( \omega \) on \( \mathcal{O}_N \) such that \( \omega(S_I) = 0 \) for all \( I \in \mathcal{I}(N) \setminus \{\emptyset\} \), but it is not known if such a normal state \( \omega \) may be found. If so we would have

\[
\text{trace}(|x\rangle \langle x| S_I) = \langle x | S_I x \rangle = 0,
\]

where \(|x\rangle \langle x|\) is the rank-1 projection of \( \mathcal{H} \) onto \( \mathbb{C}x \), and where we used Dirac’s bra-ket notation.

Remark 7.3. The existence problem for wandering subspaces \( W \) may be reformulated as a fixed-point problem. This is made clear by the equivalent identities (5.1) and (5.2) of Lemma 5.1. Introducing the transformation \( W \mapsto \mu(\lambda(W)) \) of the right-hand side in (5.1), or \( P_W \mapsto \mu(\lambda(P_W)) \) in (5.2), it is clear that the existence of a wandering subspace \( W \) is equivalent to the existence of a fixed point \( \neq 0 \) in the identity (5.1).

8. Pure co-invariant subspaces

Let \( (S_i)_{i=1}^N \) be a representation of \( \mathcal{O}_N \) on a Hilbert space \( \mathcal{H} \), and let \( \mathcal{L} \) be a co-invariant subspace. For \( x \in \mathcal{H} \), set

\[
S^n x := \bigvee \{ S_{i_1} \cdots S_{i_n} x : I = (i_1, \ldots, i_k) \in \mathcal{I}_n(N) \},
\]

\[
S^0 x := x,
\]

and

\[
S^\infty x := \bigvee_{n \geq 0} S^n x.
\]

The special case \( N = 1 \) covers a single isometry.

Definition 8.1. We say that a co-invariant subspace \( \mathcal{L} \) is pure if the following implication holds:

\[
(8.1) \quad x \in \mathcal{L}, \; S^\infty x \subset \mathcal{L} \implies x = 0.
\]

To motivate the next result, we recall first the case \( N = 1 \), i.e., a single isometry \( S \) in a Hilbert space \( \mathcal{H} \). The Wold decomposition states that \( \mathcal{H} \) decomposes as \( \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_\infty \), \( \mathcal{H}_\infty = \bigcap_{n \geq 1} S^n \mathcal{H} \), \( \mathcal{H}_0 = \mathcal{H} \ominus \mathcal{H}_\infty \), both subspaces invariant, with \( S|_{\mathcal{H}_\infty} \) unitary, and \( S_0 := S|_{\mathcal{H}_0} \) a shift. To say that \( S_0 \) is a shift means \( S_0^* S_0 \to 0 \). A shift is determined up to unitary equivalence by its multiplicity, i.e., \( \dim \{ x \in \mathcal{H}_0 : S_0 x = 0 \} \). The space \( \mathcal{W}_0 := \ker(S_0^*) = \mathcal{H}_0 \ominus S_0 \mathcal{H}_0 \) is wandering, and \( (\mathcal{H}_0, S_0) \) is unitarily equivalent to the obvious shift on vectors, i.e., to \( \bigoplus_{n \geq 0} \mathcal{W}_0 := \mathcal{W}_0 \oplus \mathcal{W}_0 \oplus \cdots \) and the operator given by \( (x_0, x_1, \ldots) \mapsto (0, x_0, x_1, \ldots) \).
So for $N = 1$, our understanding of the invariant and the co-invariant subspaces amounts to the corresponding issues for the shift. Since a subspace $\mathcal{L}$ is co-invariant if and only if its orthocomplement is invariant, the problem is solved by Beurling’s theorem. Identify $\sum_{n \geq 0} W_0$ by the Hardy space $H_+(W_0)$ of analytic $W_0$-valued functions, i.e.,

$$f: \mathbb{T} \rightarrow W_0, \quad f(z) = \sum_{n=0}^{\infty} \xi_n z^n, \quad \xi_n \in W_0, \quad \|f\|^2 = \sum_{n=0}^{\infty} \|\xi_n\|^2.$$  

(8.2)  

An inner function $u$ is a function on $\mathbb{T}$, taking values in the unitary operators on $W_0$, i.e., $u(z)^* u(z) = I_{W_0}$, a.e. $z \in \mathbb{T}$, with $u$ having an analytic operator-valued continuation to $\{z \in \mathbb{C} : |z| < 1\}$. Since, by (6.2), the shift is represented by $M_z f(z) = z f(z), \quad f \in H_+(W_0),$ it follows that the space  

$$H_+ (u) := \{uf : f \in H_+(W_0)\}$$  

is invariant when $u$ is an inner function. Beurling’s theorem [Hel64] states that every invariant subspace for the shift has this form.

**Proposition 8.2.** If an inner function $u$ is given, then the co-invariant subspace  

$$\mathcal{L}(u) := H_+ (u)^\perp = H_+(W_0) \ominus H_+ (u)$$  

is pure.

**Proof.** If $u$ is constant, $u = I_{W_0}$, then $\mathcal{L}(u) = 0$ satisfies the condition, so we assume $u$ to be non-constant. Suppose $f \in \mathcal{L}(u)$ and $S^\infty f \subset \mathcal{L}(u)$. Then  

(8.4)  

$$f(z), \quad zf(z), \quad z^2 f(z), \quad \ldots$$  

are all in $\mathcal{L}(u)$.

Let $f = \sum_{n=0}^{\infty} \xi_n z^n, \quad \xi_n \in W_0,$ and $u(z) = \sum_{n=0}^{\infty} A_n z^n$ where $A_n: W_0 \rightarrow W_0$ is a system of operators in $W_0$ such that

(8.5)  

$$\sum_{n=0}^{\infty} A_n A_n^* = I_{W_0}\quad \text{and} \quad \sum_{n=0}^{\infty} A_n A_{n+k}^* = 0 \quad \text{for } k = 1, 2, \ldots.$$  

(8.6)  

Using (8.3) and (8.4), we get the system:

$$A_n^* \xi_0 + A_{n+1}^* \xi_1 + \ldots = 0,$$

$$A_n^* \xi_1 + A_{n+1}^* \xi_2 + \ldots = 0,$$

$$A_n^* \xi_2 + A_{n+1}^* \xi_3 + \ldots = 0,$$

$$\vdots$$
Multiply each of these equations by $A_n$ and sum over $n = 0, 1, 2, \ldots$. We then get

$$\sum_{n=0}^{\infty} A_n A_n^* \xi_k + \sum_{n=0}^{\infty} A_n A_{n+1}^* \xi_{k+1} + \cdots = 0$$

for $k = 0, 1, \ldots$. An application of (8.8)–(8.6) now yields $\xi_k = 0$ for all $k = 0, 1, \ldots$. This proves the conclusion. \qed

If $N > 1$, we are considering representations of $O_N$, and the examples in Section 5.3 show that not all co-invariant subspaces are pure. But we do have the following:

**Theorem 8.3.** Let $(S_i)_{i=1}^N$, $N > 1$, be a representation of $O_N$ on a Hilbert space $H$, and let $L$ be a co-invariant subspace. Then $L$ contains a pure co-invariant subspace.

**Proof.** If $L = 0$, we are done. If not, consider vectors (if any) $x \in L$, $\|x\| = 1$, such that $S^\infty x \subset L$. If there are no such vectors, $L$ is pure. Otherwise pick a family $F = \{x\}$ such that $S^\infty x \subset L$ and $S^\infty x \perp S^\infty y$ for all $x, y \in F$, $x \neq y$. By Zorn’s lemma, we may pick $F$ to be maximal with respect to these properties. Now set $L_p := L \cup \{S^\infty x : x \in F\}$. This space is clearly co-invariant. If some vector $x_p \in L_p$ satisfies $S^\infty x_p \subset L_p$, then $F \cup \{x_p\}$ satisfies the orthogonality property. Since $F$ was chosen maximal, we conclude that $x_p = 0$. This proves that $L_p$ is pure. \qed

**Examples 8.4.** (a) Consider the representation $(S_i)_{i=0}^1$ of $O_2$ acting on $\ell^2(\mathbb{Z}) \simeq L^2(\mathbb{T})$ outlined in Example 5.3. From (5.3) we note that the closed subspace spanned by $\{e_0, e_1, e_2, \ldots\}$ is invariant for the representation, i.e., invariant under all four of the operators $S_i, S_j^*, i, j = 0, 1$; and moreover this restricted representation, $\rho_+$ say, is irreducible on this space. The space is $H_+ \simeq \ell^2(\{0, 1, 2, \ldots\}) \simeq$ the Hardy space. Recall the Hardy space $H_+ \subset L^2(\mathbb{T})$ has the representation

$$H_+ = \left\{ f \in L^2(\mathbb{T}) : f(z) = \sum_{n=0}^{\infty} \xi_n z^n, \quad z \in \mathbb{T}, \quad \sum_{n=0}^{\infty} |\xi_n|^2 = \|f\|^2 < \infty \right\}. \quad (8.7)$$

Since, in general, *every finite-dimensional* co-invariant subspace is pure, we note that each one of the following subspaces $L_1, L_2, L_3$ in $H_+$ is pure: $L_1 = [e_0], L_2 = [e_0, e_1], L_3 = [e_0, e_1, e_2]$. The corresponding three wandering subspaces, i.e.,

$$W_i := (S L_i) \ominus L_i, \quad i = 1, 2, 3,$$

are:

$$W_1 = [e_1], \quad W_2 = [e_2, e_3], \quad W_3 = [e_3, e_4, e_5]. \quad (8.9)$$

An example of a proper (i.e., nontrivial) co-invariant subspace $(\subset H_+)$ which is not pure is

$$L := [e_0, e_1, e_3, e_6, e_7, e_{12}, e_{13}, e_{14}, e_{15}, e_{24}, e_{25}, \ldots]. \quad (8.10)$$

The stated properties for this last subspace $L$ follow from its representation as

$$L = L_2 \oplus S^\infty e_3. \quad (8.11)$$
(b) Based on (a), one might think that a pure co-invariant subspace cannot be infinite-dimensional. This is not so, as we now illustrate with the representation $\rho$ of Remark 4.6. This is the representation $\rho$ from Remark 4.6. This is the representation $\rho$ of $O_N$ with cyclic vector $\Omega$ which acts on the Hilbert space $\mathcal{H}$, i.e., $\mathcal{H}$ is the Hilbert space spanned by the following vectors: $S_I S_J^* \Omega$, where $I$ and $J$ run over the set of all multi-indices $\mathcal{I}(N)$, i.e., $(i_1, i_2, \ldots, i_n)$, $n = 1, 2, \ldots, i_\nu \in \{1, 2, \ldots, N\}$, with the convention $n = 0$ corresponding to $I = \emptyset$, and $S_\emptyset = I$. The two indices $I$ and $J$ might have different length. It is immediate from (4.7) that the following subspace,

$$L := \bigvee \{ S_I^* \Omega : I \in \mathcal{I}(N) \setminus \{ \emptyset \} \},$$

is a nontrivial co-invariant subspace for the representation $\rho$, and that the corresponding wandering subspace $W$ is $N^2$-dimensional; in fact, $W = \bigvee_{i,j} [S_i S_j^* \Omega]$; see Example 4.10.

**Observation 8.5.** The subspace $L$ in (8.12) is a pure co-invariant subspace for the representation $\rho$ of Remark 4.6.

**Proof.** Let $P = P_L$ denote the (orthogonal) projection onto $L$. We show that

$$\| P S_i P \| \leq N^{-\frac{1}{2}} \quad \text{for } i = 1, \ldots, N. \tag{8.13}$$

Since $N > 1$, the result follows; in fact, if $x$ is any vector $x \in L$ for which $S_i x \in L$ for some $i$, then $x = 0$. Indeed, $\| x \| = \| S_i x \| = \| P S_i P x \| \leq N^{-\frac{1}{2}} \| x \|$. To prove (8.13), we note that the following normalized vectors,

$$\left\{ N^{\frac{k}{2}} S_i^* \Omega : |I| = k' \right\},$$

are mutually orthogonal when $I$ varies over $\mathcal{I}_{k'}(N)$, and also when the respective length of indices $I$, $I'$ are different, i.e., $|I| = k \neq k' = |I'|$. Hence, by (8.12), we have an orthonormal basis for $L$. Now let $P_k$ denote the projection onto the closed subspace spanned by the vectors in (8.14) for all values $k'$ such that $k' \leq k$. Then we get

$$\lim_{k \to \infty} P_k = P_L \ (= P). \tag{8.15}$$

If $x \in L$, then

$$\| P_k S_i x \| = N^k \sum_{|I| = k} |\langle S_i^* \Omega | S_i x \rangle|^2$$

$$\leq N^{1 - k} \sum_{|I| = k} |\langle S_i^* S_i \Omega | x \rangle|^2$$

$$\leq N^{-1} \| P_{k+1} x \|^2. \tag{8.16}$$

Letting $k \to \infty$, and using (8.15), we now arrive at the conclusions: $x = P_L x$ ($= \lim_{k \to \infty} P_k x$), $P_L S_i x = \lim_{k \to \infty} P_k S_i x$, and $\| P_L S_i x \| \leq N^{-\frac{1}{2}} \| x \|$, the last estimate being equivalent to the desired one (8.13). \qed
9. Tight frames of wavelets

We now turn to the representations (1.2)–(1.4) which define tight frames of multiresolution wavelets, and we give a representation of pure co-invariant subspaces in the Hilbert space $L^2(\mathbb{T})$. It is shown in [BrJo02] that to get solutions $\varphi, \psi_i$ as in (1.1) which are in $L^2(\mathbb{R})$, the following condition must be satisfied by the matrix function $T \ni z \mapsto A(z) \in U_N(C)$ from formula (1.4). Specifically, if $\rho = \rho_N = e^{i2\pi/N}$, $i = \sqrt{-1}$, and if

\begin{equation}
A_{j,k}(1) = \frac{1}{\sqrt{N}} \rho^{j-k} = \frac{1}{\sqrt{N}} e^{ik2\pi/N},
\end{equation}

then the solutions $\hat{\varphi}, \hat{\psi}_i$ in (1.4) are in $L^2(\mathbb{R})$, and their inverse Fourier transforms $\varphi, \psi_i$ are in $L^2(\mathbb{R})$ as well. Introducing the triple-indexed functions

\begin{equation}
\psi_{i,j,k}(x) = N^{\frac{j}{2}} \psi_i(N^j x - k), \quad i = 1, \ldots, N - 1, \quad j, k \in \mathbb{Z},
\end{equation}

we say that $\{\psi_{i,j,k}\}$ is a tight frame if and only if, for all $F \in L^2(\mathbb{R})$, the following (Bessel) identity holds:

\begin{equation}
\int_{\mathbb{R}} |F(x)|^2 \, dx = \sum_{i,j,k} \left| \langle \psi_{i,j,k} | F \rangle_{L^2(\mathbb{R})} \right|^2
\end{equation}

The next result is also proved in [BrJo02], but is included here for the convenience of the reader:

**Theorem 9.1.** If $T \ni z \mapsto A(z) \in U_N(C)$ is a unitary matrix function satisfying (9.1) then the wavelet functions $\psi_{i,j,k}$ defined in (1.1) and (9.2) form a tight frame in $L^2(\mathbb{R})$.

**Remark 9.2.** (The stretched Haar wavelet.) The following example for $N = 2$ shows that this system (9.2) might not in fact be an orthonormal basis: Take $A(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & z \\ 1 & -z \end{pmatrix}$. Then $A(1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$,

\begin{equation}
\varphi(x) = \frac{1}{3} \chi_{[0,3]}(x), \quad \text{and}
\end{equation}

\begin{equation}
\psi(x) = \frac{1}{3} \left( \chi_{[0,\frac{3}{2}]}(x) - \chi_{[\frac{3}{2},3]}(x) \right).
\end{equation}

It follows from a direct verification, or from the theorem, that this function $\psi$ in (9.4) makes

\begin{equation}
\psi_{j,k}(x) := 2^j \psi(2^j x - k), \quad j, k \in \mathbb{Z},
\end{equation}

into a tight frame. But since $\|\psi\| = \frac{1}{\sqrt{3}}$, and since the different functions in (9.6) are not orthogonal, we see that this is a wavelet tight frame which is not an orthonormal basis in $L^2(\mathbb{R})$. 
Not all representations as in (1.2) and (1.4) satisfy condition (9.1); for example, the representation 
\( (T_i)_{i=0}^{N-1} \) defined from the constant matrix function
\( \mathbb{T} \ni z \mapsto I_N = \begin{pmatrix} 1 & \cdots & 0 \\ 0 & \ddots & 0 \\ \vdots & \ddots & 1 \end{pmatrix} \in U_N(\mathbb{C}) \) clearly does not satisfy (9.1). Yet as we show, we may use this simple representation as a base-point for a comparison with all other representations. Specifically, we have the following lemma.

**Lemma 9.3.** If \( S_i = S_i^{(A)} \) is any representation defined from some matrix function
\( \mathbb{T} \ni z \mapsto A(z) \in U_N(\mathbb{C}) \), then
\[
(T_j^* S_i^{(A)}) f(z) = A_{i,j}(z) f(z), \quad i,j = 0, \ldots, N-1, \quad f \in L^2(\mathbb{T});
\]
i.e., when \( i,j \) are given, then the operator \( T_j^* S_i^{(A)} \) is a multiplication operator on \( L^2(\mathbb{T}) \), in fact multiplication by the matrix entry \( A_{i,j}(z) \) of the unitary matrix \( A(z) \).

**Proof.** Apply formula (1.4) and the fact that the operators \( T_j \) and \( T_j^* \) are given on \( L^2(\mathbb{T}) \) as
\[
T_j f(z) = z^j f(z^N),
\]
and
\[
(T_j^* f)(z) = \frac{1}{N} \sum_{w \in \mathbb{T}} w^{-j} f(w), \quad f \in L^2(\mathbb{T}).
\]

As a corollary we get the following formula for the adjoint \( S_i^{(A)*} \) in general:
\[
S_i^{(A)*} f(z) = \sum_{j=0}^{N-1} A_{i,j}(z) T_j^* f(z).
\]

When \( A \) is given we introduce the subspace
\[
\mathcal{L} = \mathcal{L}^{(A)} = \bigvee_{I \in \mathcal{I}(N)} [S_I^{(A)*} \mathbb{1}],
\]
where \( \mathbb{1} = e_0 \) is the constant function on \( \mathbb{T} \), consistent with the terminology \( e_n(z) = z^n, \ n \in \mathbb{Z} \). If the matrix entries in \( S_I^{(A)} \) are Fourier polynomials, it is clear that \( \mathcal{L}^{(A)} \) is a finite-dimensional co-invariant subspace, and therefore also pure. In the next result, we give a necessary and sufficient condition for \( \mathcal{L}^{(A)} \) to be a pure co-invariant subspace for the wavelet representation \( (S_j^{(A)})_{j=0}^{N-1} \) on \( L^2(\mathbb{T}) \).

**Lemma 9.4.** Let \( \mathbb{T} \ni z \mapsto A(z) \in U_N(\mathbb{C}) \) be a measurable unitary matrix-valued function, and let \( (S_j) = \left( S_j^{(A)} \right) \) be the corresponding representation of \( O_N \) on the Hilbert space \( L^2(\mathbb{T}) \). Then the following three conditions are equivalent. (We consider \( 0 \leq j < N \) and functions in \( L^2(\mathbb{T}) \).)
Attractors for representations

(9.12) \( S_j f = g \).

(ii) \( S_j^* g = f \) and \( \| g \| = \| f \| \).

(iii) \( T_i^* g = f \cdot A_{j,i} \) for all \( i \).

Here each of the identities in (i)–(iii) is taken in the pointwise sense, i.e., identity for the functions on \( T \) pointwise a.e. with respect to Haar measure on \( T \). The product on the right-hand side in (9.13) is \( f(z) A_{j,i} (z) \) a.e. \( z \in T \).

Proof. (i) \( \Rightarrow \) (ii): This is clear since \( S_j \) is an isometry. Hence \( \| f \| = \| g \| \), and \( S_j^* g = S_j^* S_j f = f \), which is the combined assertion in (ii).

(ii) \( \Rightarrow \) (iii): Assuming (ii), and using (9.10), we get

(9.14) \[ \sum_{i=0}^{N-1} A_{j,i}(z) T_i^* g(z) = f(z), \quad z \in T. \]

Using unitarity of the matrix function \( A \), and the Schwarz inequality for the Hilbert space \( \mathbb{C}^N \), we get the pointwise estimate

(9.15) \[ |f(z)|^2 \leq \sum_{i=0}^{N-1} |T_i^* g(z)|^2, \quad \text{a.e. } z \in T. \]

Integration of this over \( T \) with respect to Haar measure yields

(9.16) \[ \| f \|^2 \leq \sum_{i=0}^{N-1} \| T_i^* g \|^2 = \| g \|^2. \]

But the second condition in (ii) then states that we have equality in Schwarz’s inequality. First we have it in the vector form (9.16). But this means that

\[ \int_T \left( \sum_{i=0}^{N-1} |T_i^* g(z)|^2 - |f(z)|^2 \right) d\mu(z) = 0. \]

In view of (9.15), this means that in fact, (9.15) is an a.e. identity, i.e., that

\[ |f|^2 = \sum_{i=0}^{N-1} |T_i^* g|^2 \quad \text{a.e. on } T, \]

and that therefore

\[ \left| \sum_{i=0}^{N-1} A_{j,i}(z) T_i^* g(z) \right|^2 = \sum_{i=0}^{N-1} |T_i^* g(z)|^2 \quad \text{for a.e. } z \in T. \]

Hence there is a function \( h_j \) on \( T \) such that \( T_i^* g = h_j A_{j,i} \). But an application of (ii) and \( \sum_{i=0}^{N-1} T_i T_i^* = I_{L^2(T)} \) shows that \( h_j = f \), which is the desired conclusion (iii), i.e., the formula (9.13).
\( \Rightarrow (i): \) If \( (iii) \) holds, we get
\[
S_j f(z) = \sum_{i=0}^{N-1} A_{j,i} (z^N) T_i f(z) = \sum_{i=0}^{N-1} z^i T^*_i g(z^N)
\]
\[
= \sum_{i=0}^{N-1} T_i T^*_i g(z) = g(z).
\]

We note two consequences deriving from the condition \( (9.1) \). It is a condition on the given measurable matrix function \( T \ni z \mapsto A(z) \in \mathcal{U}_N(\mathbb{C}) \), and therefore on the corresponding representation \( (S_j(A))_{j=0}^{N-1} \) of \( \mathcal{O}_N \). This representation acts on the Hilbert space \( L^2(T) \). But the wavelet system \( \varphi, \psi_j \) from \( (1.1) \) relates to the line \( \mathbb{R} \), and not directly to \( T \). Indeed, condition \( (9.1) \) ensures that the wavelet functions of the system \( \varphi, \psi_j \), derived from \( A \), are in \( L^2(\mathbb{R}) \), and \( (9.1) \) is called the frequency-subband condition. The functions
\[
m_j^{(A)}(z) = \sum_{k=0}^{N-1} A_{j,k} (z^N) z^k
\]
are called subband filters: \( m_0^{(A)} \) is the low-pass filter, and the others \( m_j^{(A)}, j \geq 1 \), are the higher-pass filter bands.

**Notations 9.5.** Condition \( (9.1) \) gives the distribution of the \( N \) cases with probabilities \( \frac{1}{N} \left|m_j^{(A)}(\cdot)\right|^2 \) on the frequencies \( \frac{j}{N}, j = 0, 1, \ldots, N-1 \), which represent the bands. Recall if \( \rho = e^{i2\pi/N} \), then \( \{\rho^j : 0 \leq j < N\} \) are the \( N \)’th roots of unity, i.e., \( (\rho^j)^N = 1 \). The frequency passes for the bands \( 0, \frac{1}{N}, \ldots, \frac{N-1}{N} \), referring to the low-pass filter, are \( (1, 0, 0, \ldots, 0) \), and similarly
\[
\frac{1}{N} \left|m_j^{(A)}(\rho^k)\right|^2 = \delta_{j,k}.
\]

In fact, these conditions \( (9.18) \) are equivalent to the single matrix condition \( (9.1) \) for \( A \).

We also note that \( (9.1) \) implies that each one of the \( N \) isometries \( S_j^{(A)} \) on \( L^2(T) \) is a shift, i.e., that
\[
\lim_{n \to \infty} S_j^{(A)n} = 0.
\]

This conclusion, while nontrivial, is contained in the result Theorem 3.1 in [BrJo97]. Note that each of the \( N \) shift operators has infinite multiplicity in the sense of Proposition 8.2 above. Recall if \( S : \mathcal{H} \to \mathcal{H} \) is a shift in a Hilbert space \( \mathcal{H} \), then
the multiplicity space is $W_S := (SH)^\perp = \ker (S^*)$: specifically, $H \simeq \sum_{n=0}^{\infty} W_S$ with $S$ represented as

$$
(9.20) \quad (x_0, x_1, \ldots) \mapsto \hat{S}(x_0, x_1, \ldots)
$$

where $x_0, x_1, \cdots \in W_S$. For the particular application to $S_j^{(A)}$, we have

$$
(9.21) \quad W_{S_j^{(A)}} = \sum_{k \neq j}^{\oplus} W_{S_k^{(A)}} (L^2(T)).
$$

The next result is a corollary of Proposition 8.2 and the discussion above.

**Corollary 9.6.** (A Dichotomy.) Let the matrix function $T \ni z \mapsto A(z) \in U_N(\mathbb{C})$, $N > 1$, satisfy (9.1), and let $(S_j^{(A)})_{j=0}^{N-1}$ be the corresponding representation of $O_N$ on $L^2(T)$. If $L \subset L^2(T)$ is a co-invariant subspace, then the following two conditions are equivalent.

1. $L$ is pure.
2. $L = L^2(T)$.

**Proof.** Clearly (1) $\Rightarrow$ (2), and it is immediate from Definition 8.1 that every finite-dimensional co-invariant subspace $L$ is not pure. That follows since, if $f \in L$ satisfies $S^\infty f \subset L$, then for each $k \in \mathbb{Z}_+$, the family $\{S_I f : I \in I(N), \ |I| = k \} \subset S^\infty f \subset L$ consists of orthogonal vectors. If $f \neq 0$, then there are $N^k$ such vectors. Hence the result holds whenever the entries in $A$ are Fourier polynomials; see also [JoKr02]. We now turn to (2) $\Rightarrow$ (1) in the general case: Let $A$ satisfy (9.1), and let $L$ be a co-invariant subspace, referring to the representation of $O_N$ given by $(S_j^{(A)})_{j=0}^{N-1}$. Then $L^\perp = L^2(T) \oplus L$ is invariant for each of the $N$ shift operators $S_j^{(A)}$, $j = 0, \ldots, N - 1$. Now suppose $L \neq L^2(T)$, or equivalently that $L^\perp \neq 0$. Setting $j = 0$, we see that there is a unitary operator-valued function $T \ni z \mapsto u_0(z) \in U(W_{S_0^{(A)}})$ such that

$$
(9.22) \quad L^\perp = \left\{u_0 h_+ : h_+ \in H_+ \left(W_{S_0^{(A)}}\right)\right\}
$$

where $W_{S_0^{(A)}}$ is given by (9.21).

If $f \in L$ satisfies $S^{(A)}f \subset L$, then $f, S_0^{(A)} f, S_0^{(A)} 2 f, \ldots, S_0^{(A)} n f, \cdots \in L$. Relative to the representation (9.20), applied to $S_0^{(A)}$, we get

$$
\hat{f}, z \hat{f}, z^2 \hat{f}, \ldots, z^n \hat{f}, \cdots \in L.
$$

But then we get $f = 0$ by an application of Proposition 8.2 to $S_0^{(A)}$. Hence, $L$ is pure; see Definition 8.1. Note that [BrJo97, Theorem 3.1] was used as well. This result implies that $S_0^{(A)}$ is a shift, and so it has the representation (9.20). $\blacksquare$
We now turn to the space of Lipschitz functions on $\mathbb{T}$. Via the coordinate $z = e^{-i\xi}$, $\xi \in \mathbb{R}$, we identify functions on $\mathbb{T}$ with $2\pi$-periodic functions on $\mathbb{R}$ and we define the Lipschitz space $\text{Lip}_1$ by
\begin{equation}
\| f \|_{\text{Lip}_1} := |f(0)| + \sup_{-\pi \leq \xi < \eta < \pi} \frac{|f(\xi) - f(\eta)|}{|\xi - \eta|} < \infty.
\end{equation}
A matrix function is said to be Lipschitz if its matrix entries are in $\text{Lip}_1$.

**Theorem 9.7.** A matrix function, $\mathbb{T} \ni z \mapsto A(z) \in U_N(\mathbb{C})$ is given. We assume it is in the Lipschitz class, and that it satisfies (9.1). Consider the co-invariant subspace $\mathcal{L}^{(A)} \subset L^2(\mathbb{T})$ defined from the corresponding representation $\left(S_{j}^{(A)}\right)_{j=0}^{N-1}$ of $\mathcal{O}_N$ as follows:
\begin{equation}
\mathcal{L}^{(A)} := \bigvee \left[ S_{j}^{(A)} e_0 : I \in \mathcal{I}(N) \right]
\end{equation}
where $e_0 = 1$ is the constant function 1 on $\mathbb{T}$. Then $\mathcal{L}^{(A)}$ is pure.

**Proof.** In view of Corollary it is enough to show that $\mathcal{L}^{(A)} \neq L^2(\mathbb{T})$, or equivalently that $(\mathcal{L}^{(A)})^\perp \neq 0$. The argument is indirect. If $\mathcal{L}^{(A)} = L^2(\mathbb{T})$, we get a contradiction as follows (we have suppressed the superscript $A$ in the notation):
The set of vectors $\{ S_j^* e_0 : I \in \mathcal{I}(N) \}$ is relatively compact in $C(\mathbb{T})$ by Arzelà-Ascoli. To see this, we use the Lipschitz property, and the formula
\begin{equation}
S_j^* e_0 (z) = \frac{1}{N^k} \sum_{w^{N^k} = z} m_{I}^{(k)} (w)
\end{equation}
where $m_{I}^{(k)} (z) = m_{i_1} (z) m_{i_2} (z^N) \cdots m_{i_k} \left( z^{N^{k-1}} \right)$, $I = (i_1, \ldots, i_k)$. Further note that $\sqrt{N} S_j^* \left( \sqrt{N} f \right) (1) = f(1)$ for all Lipschitz functions $f$, and that the analogous conditions hold for $S_j^* \cdots S_{N-1}^*$. Since $m_{i_j} \left( \rho^k \right) = \delta_{j,k} \sqrt{N}$, there is a Lipschitz solution $f$ to the following system of equations:
\begin{equation}
f(1) = 1, \quad S_0^* f = \frac{1}{\sqrt{N}} f, \quad S_j^* f = 0, \ j \geq 1.
\end{equation}
Using $f = \sum_j S_j S_j^* f = \frac{1}{\sqrt{N}} S_0 f$, it follows that $S_0$ has $\sqrt{N}$ as eigenvalue, contradicting that $S_0$ is isometric. The contradiction proves that $\mathcal{L}^{(A)} \perp \neq 0$.

In the next section, we give additional details on the existence question for the eigenvalue problems related to the operators $S_j^{(A)}$ in the case when the matrix function $A$ is assumed to be Lipschitz.

**10. Finite dimensions**

In this section we offer a construction of a finite-dimensional co-invariant (nonzero, and nontrivial) subspace for the representation of $\mathcal{O}_N$ on $L^2(\mathbb{T})$ which is associated with a multiresolution wavelet of scale $N$. It is both natural and optimal with
respects to the conditions of Sections 6–9. The setting is as in the previous section: Recall, a Lipschitz mapping \( T \ni z \mapsto A(z) \in U_N(\mathbb{C}) \) is given, and it is assumed that the subbands are ordered according to (9.1), i.e., that \( A(1) \) is the Hadamard matrix, or equivalently that

\[
m_j \left( \rho_N^k \right) = \delta_{j,k} \sqrt{N}, \quad j, k = 0, \ldots, N - 1,
\]

where \( \rho_N := \exp \left( i \frac{2\pi}{N} \right) \), and

\[
m_j (z) = \sum_{k=0}^{N-1} A_{j,k} (z^N) z^k.
\]

Then the operators \( S_j = S_j^{(4)} \) on \( L^2(\mathbb{T}) \) are

\[
(S_j f)(z) = m_j (z) f \left( z^N \right), \quad j = 0, \ldots, N - 1, \ f \in L^2(\mathbb{T}), \ z \in \mathbb{T}.
\]

**Theorem 10.1.** Let \((S_j)_{j=0}^{N-1}\) be a representation as in (10.3) determined by a Lipschitz system and subject to conditions (10.1) and (10.2). Then the following two conditions are equivalent.

(i) There is a finite-dimensional co-invariant subspace \( \mathcal{L} \subset L^2(\mathbb{T}) \) which contains the solutions \( f \) to the following affine conditions:

\[
f \in \text{Lip}_1, \quad f(1) = 1, \quad \sqrt{N} S_0^* f = f.
\]

(ii) There is a finite constant \( K \) such that, for all \( f \in \text{Lip}_1 \),

\[
\sup_{J \in \mathcal{I}(N)} \left| N^{\frac{1}{2}} (S_J f)(1) \right| \leq K \|f\|.
\]

The affine dimension of the convex set (10.4) of Lipschitz functions is at least one.

**Proof.** We begin with the conditions (10.4). We will show that there is a well defined linear operator \( T \) on \( \text{Lip}_1 \) with finite-dimensional range (dimension at least one) such that

\[
(T f)(1) = f(1) \quad \text{for all } f \in \text{Lip}_1
\]

and

\[
(T f)(z) = \lim_{k \to \infty} \left( \sqrt{N} S_0^* \right)^k f(z), \quad f \in \text{Lip}_1,
\]

where the limit in (10.6) is uniform for \( z \in \mathbb{T} \).

**Lemma 10.2.** Let the operators \( S_j, j = 0, \ldots, N - 1 \), be as specified above in (10.3), i.e., the Lipschitz property is assumed, as is (10.1). For bounded functions \( f \) on \( \mathbb{T} \), set \( \|f\| := \sup_{z \in \mathbb{T}} |f(z)| \); and if \( f \) is differentiable, set

\[
\tilde{f}(x) := f \left( e^{-i2\pi x} \right),
\]

and \( f' := \tilde{f}' \). Finally, let

\[
M_1 := N^{-\frac{1}{2}} \max_{0 \leq j < N} \|m_j'\|.
\]
Then we have the following estimate:

\[
N^\frac{k}{2} \left\| (S^*_{j_k} \cdots S^*_{j_2} S^*_{j_1} f)' \right\| \leq N^{-\frac{k}{2}} \| f' \| + M_1 \| f \|
\]

for all \( k \in \mathbb{Z}^+ \), all \( f \in \text{Lip}_1 \),

and all multi-indices \( J = (j_1, j_2, \ldots, j_k) \in I_k (N) \).

Proof. Using (10.7), we shall pass freely between any of the four equivalent formulations, functions on \( T \), functions on \([0, 1)\), functions on \( \mathbb{R}/\mathbb{Z} \), or one-periodic functions on \( \mathbb{R} \), omitting the distinction between \( f \) and \( \tilde{f} \) in (10.7). With the multi-index notation in (10.9), we set

\[
m_J(z) := m_{j_1} (z) m_{j_2} (z^N) \cdots m_{j_k} (z^{N^k-1}), \quad z \in T.
\]

Setting \( S_J := S_{j_1} \cdots S_{j_k} \), and \( S^*_J := S^*_{j_1} \cdots S^*_{j_2} S^*_{j_1} \), we find

\[
(S_J f)(z) = m_J(z) f\left(z^{N^k}\right),
\]

and

\[
(S^*_J f)(z) = \frac{1}{N^k} \sum_{w \in T \atop w^N = z} \bar{m}_J(w) f(w).
\]

The sum in (10.12) contains \( N^k \) terms, which is evident from the rewrite in the form below, using instead one-periodic functions on \( \mathbb{R} \), \( x \in \mathbb{R} \):

\[
(S^*_J f)(x) = \frac{1}{N^k} \sum_{y \in \mathbb{R} \atop N^k y \equiv x \mod 1} \bar{m}_J(y) f(y).
\]

In this form \( m_J(y) = m_{j_1} (y) m_{j_2} (Ny) \cdots m_{j_k} (N^{k-1}y) \) (compare with (10.10)), and the points \( y \) may be represented as

\[
y = \frac{x + l_0 + l_1 N + l_2 N^2 + \cdots + l_{k-1} N^{k-1}}{N^k},
\]

with the integers \( l_0, l_1, \ldots, l_{k-1} \) taking values over the mod \( N \) residue classes 0, 1, \ldots, \( N-1 \). Given this, it is clear how the general form of (10.9) follows from the case \( k = 1 \). We will do \( k = 1 \), and leave the induction and the multi-index gymnastics to the reader. Using (10.13)–(10.14) we have

\[
N^\frac{1}{2} (S^*_J f)' (x) = \frac{1}{N^\sqrt{N}} \sum_{l=0}^{N-1} \bar{m}_j \left( \frac{x + l}{N} \right) f' \left( \frac{x + l}{N} \right) + \frac{1}{N^\sqrt{N}} \sum_{l=0}^{N-1} \bar{m}_j' \left( \frac{x + l}{N} \right) f \left( \frac{x + l}{N} \right).
\]
For the individual terms on the right-hand side, we use Schwarz’s inequality, as follows: First,
\[
\frac{1}{N} \left| \sum_{l=0}^{N-1} \bar{m}_j \left( \frac{x+l}{N} \right) f' \left( \frac{x+l}{N} \right) \right|
\leq \left( \frac{1}{N} \sum_{l=0}^{N-1} \left| \bar{m}_j \left( \frac{x+l}{N} \right) \right|^2 \right)^{\frac{1}{2}} \left( \frac{1}{N} \sum_{l=0}^{N-1} \left| f' \left( \frac{x+l}{N} \right) \right|^2 \right)^{\frac{1}{2}}
\leq \|f'\| \;
\]
and second,
\[
\frac{1}{N} \left| \sum_{l=0}^{N-1} \bar{m}_j \left( \frac{x+l}{N} \right) f \left( \frac{x+l}{N} \right) \right|
\leq \left( \frac{1}{N} \sum_{l=0}^{N-1} \left| \bar{m}_j' \left( \frac{x+l}{N} \right) \right|^2 \right)^{\frac{1}{2}} \left( \frac{1}{N} \sum_{l=0}^{N-1} \left| f \left( \frac{x+l}{N} \right) \right|^2 \right)^{\frac{1}{2}}
\leq \sqrt{N} M_1 \|f\|
\]
where the terms on the right are given in (10.8) and the discussion in that paragraph.

Introducing the last two estimates back into (10.15), we get
\[
N^{\frac{1}{2}} \left| (S^*_f)'(x) \right| \leq N^{-\frac{1}{2}} \|f'\| + M_1 \|f\|,
\]
which is the desired estimate.

Note that if the functions \(f\) in (10.9) are restricted by \(\|f\| \leq 1\), then all the functions \(N^{\frac{1}{2}} S^*_f f\) are contained in a compact subset in the Banach space \(C(T)\). This follows from [DuSc, vol. I, p. 245, Theorem IV.3.5]; see also [IoMa50, Section 4], or [Bal00]. The conclusion is that there is a fixed finite-dimensional subspace \(L\) which contains all the functions obtained as limits of the terms \(N^{\frac{1}{2}} S^*_f f\) as the multi-indices \(J = (j_1, \ldots, j_k)\) vary. Introducing \(I_{\infty}(N) = \{0, 1, \ldots, N - 1\}^{\mathbb{Z}^2}\), for each \(\xi = (j_1, j_2, \ldots) \in I_{\infty}(N)\) we may pass to infinity via an ultrafilter \(u(\xi)\), i.e.,
\[
(10.16) \quad T_{\xi} f := \lim_{u(\xi)} N^{\frac{1}{2}} S^*_f f,
\]
and we note that all the operators \(T_{\xi}, \xi \in I_{\infty}(N)\), have their range contained in \(L\). If \(j \in \{0, 1, \ldots, N - 1\}\), then the extension \((j_\xi)\) is a point in \(I_{\infty}(N)\) for every \(\xi \in I_{\infty}(N)\), and
\[
(10.17) \quad N^{\frac{1}{2}} S^*_f T_{\xi} f = T_{(j_\xi)} f \in L.
\]
This proves that \(L\) is co-invariant.
For a reference to ultrafilters and compactifications, we suggest [Ency] and [Bou89].

The limit (10.6) is covered by this discussion, since the point \( \xi = (0, 0, \ldots) \in \mathcal{I}_\infty(N) \), so the corresponding operator \( T \) in (10.6) is just \( T_{(0,0,\ldots)} \). Hence (10.6) is a special case of (10.16). Finally, (10.5) follows from (10.16).

\[
\sqrt{N} (S_0^* f)(0) = \frac{1}{N} \sum_{l=0}^{N-1} m_0 \left( \frac{l}{N} \right) f \left( \frac{l}{N} \right) = f(0),
\]

which is based on (10.1). In the additive form, (10.21) implies \( m_0(0) = \sqrt{N} \), and \( m_0 \left( \frac{k}{N} \right) = \cdots = m_0 \left( \frac{k}{N} \right) = 0 \). Note \( z = \exp(-i2\pi(x = 0)) = 1 \) is used in (10.15).

Let \( f \in \text{Lip}_1 \), and set \( f_J = N^{-\frac{|J|}{2}} S_J f \). We noted that each \( f_J \) is in Lip_1, and we gave a uniform estimate on the derivatives, i.e., on \( f_J'(x) \). But the sequence is also bounded in \( C(T) \), relative to the usual norm \( \| \cdot \| \) on \( C(T) \). Moreover,

\[
\|f_J\| \leq N^{-\frac{|J|}{2}} \|f'\| + (M_1 + K) \|f\|.
\]

To see this, first note the estimate from (iii), i.e.,

\[
\sup_{J \in \mathcal{I}(N)} |f_J(1)| \leq K \|f\|.
\]

The details are understood best in the additive formulation, i.e., with \( T \sim [0, 1) \) and the identification \( z \cong x \) via \( z = \exp(-i2\pi x) \). The first iteration step, starting with \( f \), yields \((x = 0)\)

\[
N^{-\frac{|J|}{2}} (S_0^* f)(0) = N^{-\frac{|J|}{2}} \sum_{l=0}^{N-1} \bar{m}_0 \left( \frac{l}{N} \right) f \left( \frac{l}{N} \right) = f(0),
\]

and subsequent steps yield

\[
N^{-\frac{|J|}{2}} (S_J f)(0) = N^{-\frac{|J|}{2}} \sum_{l=0}^{N-1} \bar{m}_J \left( \frac{l}{N} \right) f \left( \frac{l}{N} \right) = f \left( \frac{j}{N} \right),
\]

where conditions (10.11) are used. So if \( |J| = 1 \), (10.20) holds, and \( \leq \) is equality. Note that in general, if \( |J| = k > 1 \), then

\[
F_J(0) = N^{-\frac{k}{2}} \sum_{N^k x \equiv 0 \mod 1} m_J(x) f(x),
\]

where \( x \) is an \( N \)-adic fraction \( 0 \leq x < 1 \). Note that \( x \) varies over the solutions \( N^k x \equiv 0 \mod 1 \), and \( J \) is fixed. For real values of \( x, 0 \leq x < 1 \), we have

\[
|f_J(x) - f_J(0)| \leq \int_0^x |f_J'(y)| \, dy \leq x \left( N^{-\frac{k}{2}} \|f'\| + M_1 \|f\| \right);
\]

and, using (10.20), we get

\[
|f_J(x)| \leq N^{-\frac{k}{2}} \|f'\| + (M_1 + K) \|f\|,
\]

which is the desired estimate (10.19).
If $\xi \in \mathcal{I}_\infty (N)$, the operator $T_\xi$ is defined on $f$ as in (10.16) by a limit over an ultrafilter,

$$T_\xi f = \lim_{u(\xi)} f_j,$$

as noted in (10.16). Using (10.19), and passing to the limit, we get $\|T_\xi f\| \leq (M_1 + K) \|f\|$. Similarly, an application of (10.20) yields $|(T_\xi f)(1)| \leq K \|f\|$. The fact that the range of $T_\xi$ is finite-dimensional results as noted from Arzelà-Ascoli in view of the uniform estimate on the derivatives $f'_j$.

**Remark 10.3.** Even for the case of the Haar wavelet, both the standard one where the polyphase matrix $A(z)$ is $A(z) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, or the stretched Haar wavelet where $A(z) = \begin{pmatrix} 1 & z \\ 1 & -z \end{pmatrix}$, the space $\{ f \in \text{Lip}_1 : \sqrt{2}S_0^* f = f \}$ is of dimension more than 1. In the first case, the dimension can be checked to be 2, and in the second case, it is 3. The respective subspaces $\mathcal{L}_1$ and $\mathcal{L}_3$ are $\mathcal{L}_1 = [e_0, e_{-1}]$ and $\mathcal{L}_3 = [e_0, e_{-3}, e_{-1} + e_{-2}]$, where $e_n(z) = z^n$, $n \in \mathbb{Z}$, $z \in \mathbb{T}$. Two independent functions $f$ in $\mathcal{L}_1$ satisfying $f(1) = 1$ are $e_0$ and $\frac{1}{2} (e_0 + e_3)$, and $\frac{1}{3} (e_0 + e_{-1} + e_{-2})$.

Despite the fact that $\{ f \in \text{Lip}_1 : S_0^* f = \left( \frac{1}{\sqrt{N}} \right) f \}$ is automatically finite-dimensional for the representations of Theorem 10.1, we show that for the same class of representations, and for every $\lambda \in \mathbb{C}$, $|\lambda| < 1$, the space

$$\{ f \in L^2(\mathbb{T}) : S_0^* f = \bar{\lambda} f \}$$

is infinite-dimensional. If $N = 2$, then in fact it is isomorphic to $W = S_1L^2(\mathbb{T})$. Introducing the projection $P_\mathcal{W} := S_1S_1^*$, it can easily be checked that the operator $(I - \bar{\lambda}S_0)^{-1} P_\mathcal{W}$ is well defined and maps into ker $(\bar{\lambda}I - S_0^*)$. To see that inner products are preserved, note that

$$(I - \bar{\lambda}S_0)^{-1} w_1 \bigg| (I - \bar{\lambda}S_0)^{-1} w_2 \bigg) = \frac{1}{1 - |\lambda|^2} (w_1 | w_2)$$

holds for all $w_1, w_2 \in \mathcal{W}$. The easiest way to see this is via the following known theorem.

**Theorem 10.4.** For every representation $(S_j)_{j=0}^1$ of $\mathcal{O}_2$ on a Hilbert space $\mathcal{H}$ there is a unique unitary isomorphism $T : H_+ (\mathcal{W}) \to \mathcal{H}$ such that $T w = w$ for $w \in \mathcal{W}$, and

$$(10.26) \quad S_0 T = T M_z$$

if and only if $S_0^* x \underset{n \to \infty}{\to} 0$, $x \in \mathcal{H}$. 

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Proof. This result is essentially contained in [BrJo97, Theorem 9.1]. We will just give the formula for \( T \) and its adjoint \( T^* \), and then leave the remaining verifications to the reader. \( T : H_+ (W) \to \mathcal{H} \) is
\[
T \left( \sum_{n=0}^{\infty} z^n w_n \right) = \sum_{n=0}^{\infty} S_0^n w_n
\]
where \( w_n \in W \) and
\[
\left\| \sum_{n=0}^{\infty} z^n w_n \right\|_{H_+ (W)}^2 = \sum_{n=0}^{\infty} \| w_n \|_W^2,
\]
and if \( x \in \mathcal{H} \),
\[
T^* x = \sum_{n=0}^{\infty} z^n P_{W} S_0^* n x \in H_+ (W).
\]
The fact that \( \| T^* x \| = \| x \| \) is based on the assumption that \( S_0^* n x \to 0 \), i.e., that \( S_0^* n \) is a shift.

Introducing the Szegö kernel
\[
C_\lambda (z) := \frac{1}{1 - \lambda z}
\]
and the inner function
\[
u_\lambda (z) = \frac{\lambda - z}{1 - \lambda z},
\]
we arrive at
\[
T (C_\lambda \otimes w) = (I - \lambda S_0)^{-1} w,
\]
which makes (10.25) immediate. For the orthocomplement \( \{ x \in \mathcal{H} : S_0^* x = \bar{\lambda} x \} \perp \), the \( H_+ (W) \) representation is
\[
\{ u_\lambda (z) F_+ (z) : F_+ \in H_+ (W) \} = \{ F \in H_+ (W) : F (\lambda) = 0 \}.
\]

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Department of Mathematics, The University of Iowa, 14 MacLean Hall, Iowa City, IA 52242-1419, U.S.A.

E-mail address: jorgen@math.uiowa.edu