Dirichlet-type energy of mappings between two concentric annuli

Jiaolong Chen · David Kalaj

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Abstract
Let $\mathbb{A}$ and $\mathbb{A}_s$ be two non-degenerate spherical annuli in $\mathbb{R}^n$ equipped with the Euclidean metric and the weighted metric $|y|^{1-n}$, respectively. Let $\mathcal{F}(\mathbb{A}, \mathbb{A}_s)$ denote the class of homeomorphisms $h$ from $\mathbb{A}$ onto $\mathbb{A}_s$ in the Sobolev space $W^{1,n-1}(\mathbb{A}, \mathbb{A}_s)$. For $n = 3$, the second author (Kalaj in Adv Calc Var, 10.1515/acv-2018-0074, 2019) proved that the minimizers of the Dirichlet-type energy $E[h] = \int_{\mathbb{A}} \|Dh(x)\|^{n-1}_{|h(x)|^{n-1}}dx$ are certain generalized radial diffeomorphisms, where $h \in \mathcal{F}(\mathbb{A}, \mathbb{A}_s)$. For the case $n \geq 4$, he conjectured that the minimizers are also certain generalized radial diffeomorphisms between $\mathbb{A}$ and $\mathbb{A}_s$. The main aim of this paper is to consider this conjecture. First, we investigate the minimality of the following combined energy integral:

$$E[a, b][h] = \int_{\mathbb{A}} a^2 |h|^{n-1} \|DS(x)\|^{n-1} + b^2 |\nabla |h|^{n-1}|dx,$$

where $h = |h|S \in \mathcal{F}(\mathbb{A}, \mathbb{A}_s)$, $|h| = |h|$ and $a, b > 0$. The obtained result is a generalization of [Kalaj (Adv Calc Var, 10.1515/acv-2018-0074, 2019), Theorem 1.1]. As an application, we show that the above conjecture is almost true for the case $n \geq 4$, i.e., the minimizer of the energy integral $E[h]$ does not exist but there exists a minimizing sequence which belongs to the generalized radial mappings.

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✉ Jiaolong Chen
jiaolongchen@sina.com

David Kalaj
davidkalaj@gmail.com

1 Key Laboratory of High Performance Computing and Stochastic Information Processing (HPC SIP) (Ministry of Education of China), School of Mathematics and Statistics, Hunan Normal University, Changsha 410081, Hunan, People’s Republic of China

2 Faculty of Natural Sciences and Mathematics, University of Montenegro, Cetinjski put b.b., 81000 Podgorica, Montenegro
1 Introduction and statement of the main results

For $n \geq 2$, $0 < r < R$ and $0 < r_* < R_*$, let $\mathbb{A} = \{x \in \mathbb{R}^n : r < |x| < R\}$ and $\mathbb{A}_* = \{x \in \mathbb{R}^n : r_* < |x| < R_*\}$ be two spherical annuli in $\mathbb{R}^n$. We write $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.

For natural number $n$, let $A = (a_{i,j})_{n \times n} \in \mathbb{R}^{n \times n}$. We use $A^T$ to denote the transpose of $A$. The Hilbert-Schmidt norm, also called the Frobenius norm, of $A$ is denoted by $\|A\|$, where

$$\|A\|^2 = \sum_{1 \leq i, j \leq n} |a_{i,j}|^2 = \text{tr}(A^T A).$$

For $p \geq 1$, we say that a mapping $h$ belongs to the class $\mathcal{W}^{1,p}(\mathbb{A}, \mathbb{A}_*)$, if $h$ belongs to the Sobolev space $\mathcal{W}^{1,p}(\mathbb{A})$ and maps $\mathbb{A}$ into $\mathbb{A}_*$. Let $h = (h^1, \ldots, h^n)$ belong to $\mathcal{W}^{1,p}(\mathbb{A}, \mathbb{A}_*)$. We denote the Jacobian matrix of $h$ at the point $x = (x_1, \ldots, x_n)$ by $Dh(x)$, where $Dh(x) = \left(\frac{\partial h^i(x)}{\partial x_j}\right)_{n \times n} \in \mathbb{R}^{n \times n}$. Then

$$\|Dh\|^2 = \sum_{1 \leq i, j \leq n} \left|\frac{\partial h^i(x)}{\partial x_j}\right|^2.$$  

Here $\frac{\partial h^i}{\partial x_j}$ denotes the weak partial derivatives of $h^i$ with respect to $x_j$. If $h$ is continuous and belongs to $\mathcal{W}^{1,p}(\mathbb{A}, \mathbb{A}_*)$ with $p \geq 1$, then the weak and ordinary partial derivatives coincide a.e. in $\mathbb{A}$ (cf. [14, Proposition 1.2]). Let $h = \varrho S$, where $S = \frac{h}{|h|}$ and $\varrho = |h|$. By [10, Equality (3.2)], we obtain that

$$Dh(x) = \nabla \varrho(x) \otimes S(x) + \varrho \cdot DS(x)$$

and

$$\|Dh(x)\|^2 = |\nabla \varrho(x)|^2 + \varrho^2 \|DS(x)\|^2,$$

where $\nabla \varrho$ denotes the gradient of $\varrho$.

We say that $h : \mathbb{A} \to \mathbb{A}_*$ is a generalized radial mapping or a quasiradial mapping, if there exists a conformal mapping $T$ of $\mathbb{S}^{n-1}$ onto $\mathbb{S}^{n-1}$, so that $h(x) = \varrho(|x|)T\left(\frac{x}{|x|}\right)$. If $T$ is the identity, then we say that $h$ is a radial mapping.

We use $\mathcal{F}(\mathbb{A}, \mathbb{A}_*)$ to denote the class of homeomorphisms $h$ from $\mathbb{A}$ onto $\mathbb{A}_*$ in $\mathcal{W}^{1,n-1}(\mathbb{A}, \mathbb{A}_*)$. We use $\mathcal{R}(\mathbb{A}, \mathbb{A}_*)$ and $\mathcal{P}(\mathbb{A}, \mathbb{A}_*)$ to denote the class of radial homeomorphisms in $\mathcal{F}(\mathbb{A}, \mathbb{A}_*)$ and the class of generalized radial homeomorphisms in $\mathcal{F}(\mathbb{A}, \mathbb{A}_*)$, respectively.

The general law of hyperelasticity tells us that there exists an energy integral $E[h] = \int_{\mathbb{A}} E(x, h, Dh) dx$ characterizing the elastic properties of a material, where $E : \mathbb{X} \times \mathbb{Y} \times \mathbb{R}^{n \times n} \to \mathbb{R}$ is a given stored-energy function. Here $\mathbb{X}$ and $\mathbb{Y}$ are nonempty bounded domains in $\mathbb{R}^n$, $n \geq 2$. The mathematical models of nonlinear elasticity have been firstly studied by Antman [1], Ball [2] and Ciarlet [3]. One of interesting and important problems in nonlinear elasticity is whether the radially symmetric minimizers are indeed global minimizers of the given physically reasonable energy. For example, Iwaniec and Onninen [7] discussed the minimizers of the conformal energy (or $n$-harmonic energy)

$$E_n[h] = \int_{\mathbb{A}} \|Dh(x)\|^n dx$$

and $(\rho, n)$-energy integral

$$E_{\rho, n}[h] = \int_{\mathbb{A}} \|Dh(x)\|^n \rho(h(x)) dx \quad \text{with} \quad \rho(h(x)) = |h(x)|^{-n}.$$
among all homeomorphisms in \( W^{1,n}(\mathbb{A}, \mathbb{A}_*) \), respectively. Under some assumptions, they proved that the radial minimizers are always the global minimizers. As a generalization of [7], the second author [9] solved the \((\rho, n)\)-energy integral minimization problem for a radial regular metric \( \rho \) among the homeomorphisms \( h \in W^{1,n}(\mathbb{A}, \mathbb{A}_*) \). Koski and Onninen [12] investigated the minimizers of the \( p \)-harmonic energy

\[
E_p[h] = \int_X \|Dh(x)\|^p \, dx
\]

among all homeomorphisms in \( W^{1,p}(X, X_*) \), where \( X \) and \( X_* \) are planar annuli and \( p \in [1, 2) \). They found that the infimum energy cannot be achieved within the radial mappings when \( p = 1 \) and \( X = X_* = \{ x \in \mathbb{R}^2 : 0 < |x| < 1 \} \). Recently, the second author [10] studied the \((\rho, n-1)\)-energy integral (or Dirichlet-type energy)

\[
E_{p,n-1}[h] = \int_A \|Dh(x)\|^{n-1} \rho(h(x)) \, dx \quad \text{with} \quad \rho(h(x)) = |h(x)|^{1-n}
\]

among the mappings in \( \mathcal{F}(\mathbb{A}, \mathbb{A}_*) \) and proved that the minimizers of \( E_{p,n-1}[h] \) are certain generalized radial diffeomorphisms for \( n = 3 \) (cf. [10, Theorem 1.1]). Moreover, he gave the conjecture that:

**Conjecture 1.1** ([10, Conjecture A.2]) For \( n \geq 4 \), the energy \( E_{p,n-1}[h] \) defined in \((1.2)\) among the mappings in \( \mathcal{F}(\mathbb{A}, \mathbb{A}_*) \) achieves its minimum for certain generalized radial diffeomorphisms between \( \mathbb{A} \) and \( \mathbb{A}_* \).

The main purpose of this paper is to investigate this conjecture. For simplicity, we denote \( E_{p,n-1}[h] \) defined in \((1.2)\) by \( \mathcal{E}[h] \).

Recall that if \( h(z) = \varrho(z)e^{i\Theta(z)} \) is a mapping from planar ring \( \mathbb{A} \) onto planar ring \( \mathbb{A}_* \), then

\[
\|Dh\|^2 = |\nabla \varrho|^2 + \varrho^2|\nabla \Theta|^2,
\]

where \( \varrho = |h| \). In [11], the second author studied the minimality of the combined distortion integral

\[
\mathcal{K}[a, b][h] = \int_{\mathbb{A}} a^2 \varrho^2(z)|\nabla \Theta(z)|^2 + b^2|\nabla \varrho(z)|^2 \, dz,
\]

where \( a \) and \( b \) are two positive constants. Inspired by this paper, first we consider the minimality of the combined energy integral

\[
\mathcal{E}[a, b][h] = \int_{\mathbb{A}} a^2 \varrho^{n-1}(x)||DS(x)||^{n-1} + b^2|\nabla \varrho(x)|^{n-1} \, dx
\]

among \( h = \varrho S \in \mathcal{F}(\mathbb{A}, \mathbb{A}_*) \), where \( a, b > 0 \) and \( \varrho = |h| \). Obviously, when \( n = 3 \) and \( a = b = 1 \), it follows from \((1.1)\) and \((1.2)\) that \( \mathcal{E}[a, b][h] \) coincides with \( \mathcal{E}[h] \).

Let

\[
\alpha(t) = \frac{R^{\frac{n-1}{2}} \left( t^{\frac{1}{n-2}} - r^{\frac{1}{n-2}} \right)}{t^{\frac{1}{n-2}} \left( R^{\frac{1}{n-2}} - r^{\frac{1}{n-2}} \right)},
\]

where \( t \in [r, R] \). Then we have the following result on \( \mathcal{E}[a, b][h] \).
Theorem 1.1 For $n \geq 3$ and $h \in \mathcal{F}(\mathbb{H}, \mathbb{A}_\approx)$, 
\[
\mathbb{E}[a, b][h] \geq \omega_{n-1} \left( a^2(n - 1) \frac{n+1}{2} (R - r) + \frac{b^2}{(n - 2)^{n-2}} \frac{Rr}{(R \frac{1}{n-2} - r \frac{1}{n-2})^{n-2}} \log^{n-1} \frac{R_*}{r_*} \right), \tag{1.5}
\]
where $\omega_{n-1}$ is the $(n - 1)$-dimensional Hausdorff measure of $\mathbb{S}^{n-1}$. The equality holds for the following two generalized radial diffeomorphisms with arbitrary conformal $T$ from $\mathbb{S}^{n-1}$ onto $\mathbb{S}^{n-1}$
\[
h_1(x) = r_* \left( \frac{R_*}{r_*} \right)^{\frac{\alpha(|x|)}{n-1}} T \left( \frac{x}{|x|} \right) \quad \text{and} \quad h_2(x) = R_* \left( \frac{r_*}{R_*} \right)^{\frac{\alpha(|x|)}{n-1}} T \left( \frac{x}{|x|} \right),
\]
and the minimizers are unique up to conformal automorphisms of $\mathbb{S}^{n-1}$.

As an application of Theorem 1.1, we find that Conjecture 1.1 is almost true, i.e., the minimizer of the energy integral $\mathcal{E}[h]$ in $\mathcal{F}(\mathbb{H}, \mathbb{A}_\approx)$ does not exist but there exists a minimizing sequence which belongs to $\mathcal{P}(\mathbb{H}, \mathbb{A}_\approx)$. Our result reads as follows.

Theorem 1.2 For $n \geq 4$, the minimizer of the energy integral $\mathcal{E}[h]$ in $\mathcal{F}(\mathbb{H}, \mathbb{A}_\approx)$ does not exist. Further, we have that
\[
\lim_{\lambda \to 0^+} \mathcal{E}[h_1^\lambda] = \lim_{\lambda \to 0^+} \mathcal{E}[h_2^\lambda] = \inf_{h \in \mathcal{P}(\mathbb{H}, \mathbb{A}_\approx)} \mathcal{E}[h] = \inf_{h \in \mathcal{F}(\mathbb{H}, \mathbb{A}_\approx)} \mathcal{E}[h] = \omega_{n-1} \left( (n - 1) \frac{n+1}{2} (R - r) + \frac{1}{(n - 2)^{n-2}} \frac{Rr}{(R \frac{1}{n-2} - r \frac{1}{n-2})^{n-2}} \log^{n-1} \frac{R_*}{r_*} \right),
\]
where $\lambda > 0$, $h_1^\lambda$, $h_2^\lambda \in \mathcal{P}(\mathbb{H}, \mathbb{A}_\approx)$ are defined by
\[
h_1^\lambda(x) = r_* \left( \frac{R_*}{r_*} \right)^{\frac{\alpha(|x|)}{n-1}} \Phi^\lambda \left( \frac{x}{|x|} \right), \quad h_2^\lambda(x) = R_* \left( \frac{r_*}{R_*} \right)^{\frac{\alpha(|x|)}{n-1}} \Phi^\lambda \left( \frac{x}{|x|} \right) \tag{1.6}
\]
and $\Phi^\lambda$ is the mapping from (2.7) (see below).

Further, we establish the following relationships between the Dirichlet-type energy of mappings in $\mathcal{R}(\mathbb{H}, \mathbb{A}_\approx)$ and mappings in $\mathcal{P}(\mathbb{H}, \mathbb{A}_\approx)$. Our results are as follows.

Theorem 1.3 For any $n \geq 4$ and $h \in \mathcal{R}(\mathbb{H}, \mathbb{A}_\approx)$, there exists a mapping $h^\lambda \in \mathcal{P}(\mathbb{H}, \mathbb{A}_\approx)$ such that $\mathcal{E}[h] > \mathcal{E}[h^\lambda]$, where $\lambda \in (0, 1) \cup (1, +\infty)$, $|h^\lambda| = |h|$ and
\[
h^\lambda(x) = H(|x|) \Phi^\lambda \left( \frac{x}{|x|} \right). \tag{1.7}
\]

Theorem 1.4 For $n \geq 4$, there exists a mapping $h_0 \in \mathcal{R}(\mathbb{H}, \mathbb{A}_\approx)$ such that
\[
\mathcal{E}[h_0] = \inf_{h \in \mathcal{R}(\mathbb{H}, \mathbb{A}_\approx)} \mathcal{E}[h] > \inf_{h \in \mathcal{P}(\mathbb{H}, \mathbb{A}_\approx)} \mathcal{E}[h].
\]

The rest of this paper is organized as follows. In Sect. 2, some necessary terminology and preliminary results are given. In Sect. 3, Theorem 1.1 is proved, and Theorem 1.2 is shown in Sect. 4. Section 5 is devoted to the proofs of Theorems 1.3 and 1.4.
2 Preliminary results

For any fixed $x \in \mathbb{A}$, we assume that $\{U_1, \ldots, U_{n-1}, N\}$ is a system of mutually orthogonal vectors of the unit norm, where $N = \frac{x}{|x|}$ and the vectors $\{U_1, \ldots, U_{n-1}\}$ are arbitrarily chosen. For $h \in \mathcal{W}^{1,p}(\mathbb{A}, \mathbb{A}_+)$ with $p \geq 1$, define $h_N(x) = Dh(x)N$ and set $h_{U_i}(x) = Dh(x)U_i$ for $i \in \{1, \ldots, n-1\}$. Since the Hilbert-Schmidt norm of $Dh$ is independent of basis (cf. [4, Page 8]), we have

$$\|Dh\|^2 = |h_N|^2 + |h_{U_1}|^2 + |h_{U_2}|^2 + \cdots + |h_{U_{n-1}}|^2. \tag{2.1}$$

Let $S = \frac{h}{|h|}$ and define the Gram determinant of $S$ at $x \in \mathbb{A}$ by

$$DS(x) = \left| S_{U_1}(x) \times S_{U_2}(x) \times \cdots \times S_{U_{n-1}}(x) \right| \tag{2.2}$$

(cf. [10, Section 3]). Then

$$\|DS(x)\|^{n-1} = \left( |S_{U_1}(x)|^2 + |S_{U_2}(x)|^2 + \cdots + |S_{U_{n-1}}(x)|^2 + |S_N(x)|^2 \right)^{\frac{n-1}{2}} \geq \left( |S_{U_1}(x)|^2 + |S_{U_2}(x)|^2 + \cdots + |S_{U_{n-1}}(x)|^2 \right)^{\frac{n-1}{2}} \geq (n-1)^{\frac{n-1}{2}} |S_{U_1}(x)| \cdot |S_{U_2}(x)| \cdots |S_{U_{n-1}}(x)| \geq (n-1)^{\frac{n-1}{2}} DS(x). \tag{2.3}$$

Further, we have the following result.

**Theorem A** ([10, Lemma 3.1]) Let $h \in \mathcal{F}(\mathbb{A}, \mathbb{A}_+)$ and $S = \frac{h}{|h|}$. Then

$$\int_{\mathbb{A}} DS(x)dx \geq (R - r) \omega_{n-1}.$$  

Let $h(x) = \varrho(t)S(\xi)$ be a generalized radial mapping in $\mathbb{A}$, where $x = t\xi$, $t \in (r, R)$ and $\xi \in S^{n-1}$. It follows from [4, Page 10] that

$$h_N(x) = \dot{\varrho}(t)S(\xi) \quad \text{and} \quad h_{U_i}(x) = \frac{\varrho(t)}{t} \dot{S}_{U_i}(\xi),$$

where $\dot{\varrho}(t) = \frac{d\varrho(t)}{dt}$ and $S_{U_i}(\xi) = DS(\xi)U_i$. Hence, we obtain from (2.1) that

$$\|Dh(x)\|^2 = \dot{\varrho}^2(t) + \frac{\varrho^2(t)}{t^2} (\|DS(\xi)\|^2 - |S_N(\xi)|^2). \tag{2.4}$$

Note that $S$ is a conformal mapping from $S^{n-1}$ onto $S^{n-1}$, which means that

$$|S_N(\xi)| = 0, \quad |S_{U_1}(\xi)| = |S_{U_2}(\xi)| = \cdots = |S_{U_{n-1}}(\xi)|, \tag{2.5}$$

and $S_{U_1}(\xi), S_{U_2}(\xi), \ldots, S_{U_{n-1}}(\xi)$ are mutually orthogonal vectors. By (2.2) and (2.3), we see that

$$DS(\xi) = |S_{U_1}(\xi) \times S_{U_2}(\xi) \times \cdots \times S_{U_{n-1}}(\xi)| = |S_{U_1}(\xi)| \cdot |S_{U_2}(\xi)| \cdots |S_{U_{n-1}}(\xi)|.$$
and

\[ \|DS(\xi)\|^{n-1} = (|SU_1(\xi)|^2 + |SU_2(\xi)|^2 + \cdots + |SU_{n-1}(\xi)|^2)^{\frac{1}{n-1}} = (n-1)^{\frac{n-1}{2}} DS(\xi). \]  

(2.6)

Let \( \Pi : S^{n-1} \rightarrow \hat{\mathbb{R}}^{n-1} \) denote the stereographic projection of \( S^{n-1} \) through the south pole onto \( \hat{\mathbb{R}}^{n-1} \), where \( \hat{\mathbb{R}}^{n-1} = \mathbb{R}^{n-1} \cup \{ \infty \} \). For any \( \lambda > 0 \) and \( x \in \hat{\mathbb{R}}^{n-1} \), let \( g_\lambda(x) = \lambda x \). Then it follows from [9, Proof of Theorem 3.1] that

\[ \Phi^\lambda = \Pi^{-1} \circ g_\lambda \circ \Pi \]  

(2.7)

is a conformal mapping from \( S^{n-1} \) onto \( S^{n-1} \). Obviously, if \( \lambda = 1 \), then \( \Phi^\lambda \) coincides with the identity mapping. Let \( \xi = (\cos \theta, s \sin \theta) \in S^{n-1} \) be a point of longitude \( s \in S^{n-2} \) and meridian \( \theta \in [0, \pi] \). The south pole corresponds to \( \theta = \pi \). Then \( \Phi^\lambda(\xi) = (\cos \varphi(\theta), s \sin \varphi(\theta)) \) (cf. [7, Section 14.3]), where

\[ \varphi(\theta) = 2 \arctan \left( \frac{\lambda \tan \frac{\theta}{2}}{2} \right). \]

Furthermore, by (2.6) and [7, Equalities (14.38)–(14.40)], we get that

\[ \|D\Phi^\lambda(\xi)\|^2 = (n-1) \left( D_{\Phi^\lambda}(\xi) \right)^{\frac{2}{n-1}} = (n-1) \frac{\sin^2 \varphi}{\sin^2 \theta} \]  

(2.8)

and

\[ \int_{S^{n-1}} \|D\Phi^\lambda(\xi)\|^{n-1} d\sigma(\xi) = (n-1)^{\frac{n-1}{2}} \omega_{n-1}, \]  

(2.9)

where \( \sigma \) denotes the \((n-1)\)-dimensional Hausdorff measure so that \( \sigma(S^{n-1}) = \omega_{n-1} \).

3 Proof of Theorem 1.1

The aim of this section is to prove Theorem 1.1. For convenience, in the rest of this paper, we set \( E = (r, R) \) and \( F = (r_*, R_*) \).

3.1 Proof of Theorem 1.1

Let \( h = \varrho S \in \mathcal{F}(\mathcal{A}_*, \mathcal{A}_*) \), where \( \varrho = |h| \) and \( S = \frac{h}{|h|} \). Before we go to the detailed proof, let us make one shortcut. For every constant \( a, b, c > 0 \), we have

\[ \mathbb{E}[a, b] \left[ \frac{ch}{|h|^2} \right] = \mathbb{E}[a, b][h] = \int_{\mathcal{A}_*} \left( a^2 \|DS(x)\|^{n-1} + b^2 \frac{\|\nabla \varrho(x)\|^{n-1}}{\varrho^{n-1}(x)} \right) dx. \]  

(3.1)

In order to prove this statement, we let \( f = \frac{ch}{|h|^2} \). Then

\[ |f(x)| = \frac{c}{\varrho(x)} \quad \text{and} \quad \frac{f(x)}{|f(x)|} = \frac{h(x)}{|h(x)|} = S(x). \]
By calculations, we get that $\nabla |f(x)| = -c \frac{\nabla \varphi(x)}{\varphi^2(x)}$, and so,

\[
\frac{|\nabla f(x)|}{|f(x)|} = \frac{|\nabla \varphi(x)|}{\varphi(x)}.
\]

Then the above equalities and (1.3) imply that (3.1) is true. Thus, in the following, we can assume that

\[
\lim_{|x| \to r} |h(x)| = r_* \quad \text{and} \quad \lim_{|x| \to R} |h(x)| = R_*.
\]

(3.2)

In order to find the minimizer of $\mathbb{E}[h]$, first, we estimate the integral $\int_{\mathcal{A}} \|DS(x)\|^n dx$. It follows from (2.3) and Theorem A that

\[
\int_{\mathcal{A}} \|DS(x)\|^n dx \geq (n - 1) \frac{n-1}{2} \int_{\mathcal{A}} |D_S(x) dx \geq \omega_{n-1}(n - 1) \frac{n-1}{2} (R - r).
\]

(3.3)

Second, we estimate the integral $\int_{\mathcal{A}} |\nabla \varphi(x)|^{n-1} \frac{\varphi}{\varphi(x)^n} dx$. Let $x = t \xi$, where $t = |x|$ and $\xi \in S^{n-1}$. Then

\[
|\varphi_N(x)| = |\langle \nabla \varphi(x), \xi \rangle| \leq |\nabla \varphi(x)|,
\]

(3.4)

where $\varphi_N$ denotes the differentiation of $\varphi$ in the direction $N$ and

\[
\varphi_N(x) = \langle \nabla \varphi(x), N \rangle = \frac{\partial \varphi(x)}{\partial t}.
\]

By Hölder’s inequality and (3.4), we see that

\[
\int_{\mathcal{A}} \frac{|\nabla \varphi(x)|^{n-1}}{\varphi^{n-1}(x)} dx \geq \left( \int_{\mathcal{A}} |\varphi_N(x)| dx \right)^{n-1} \left( \int_{\mathcal{A}} \frac{1}{|x|^{(n-1)/2}} dx \right)^{2-n}.
\]

(3.5)

The equality holds if and only if

\[
\frac{|\nabla \varphi(x)|}{\varphi(x)} = \frac{|\varphi_N(x)|}{|x|^{n-1}} = C_{1,1}|x|^{-\frac{n-1}{2}}
\]

a.e. in $\mathcal{A}$ for some constant $C_{1,1} > 0$ (cf. [13, Page 6]). Since $h = \varphi S$ is a homeomorphism in $\mathcal{F}(\mathcal{A}, \mathbb{A}_s)$, we obtain that (cf. [11, Equality (2.6)])

\[
\int_{\mathcal{A}} \frac{|\varphi_N(x)|}{\varphi(x)|x|^{n-1}} dx \geq \int_{\mathcal{A}} \frac{\varphi_N(x)}{\varphi(x)|x|^{n-1}} dx = \int_{S^{n-1}} \int_{r_*}^R \frac{d\varphi(t\xi)}{\varphi(t\xi)} dt d\sigma(\xi)
\]

\[
= \omega_{n-1} \int_{r_*}^R \frac{d\varphi}{\varphi} = \omega_{n-1} \log \frac{R_*}{r_*}.
\]

(3.6)

Combing (3.1), (3.3), (3.5) and (3.6), we see that (1.5) is true.

To prove the equality statement, assume that the equalities are attained in all inequalities. If the equality is attained in (3.4), then

\[
|\varphi_N(x)| = \left| \langle \nabla \varphi(x), \frac{x}{|x|} \rangle \right| = |\nabla \varphi(x)|,
\]

which implies that the directional derivative of $\varphi_{U_i}$ are all vanished since

\[
|\varphi_{U_i}(x)| = |\langle \nabla \varphi(x), U_i \rangle| = 0
\]
for \( i = 1, \ldots, n - 1 \). Hence, \( \varrho(x) \) depends only on the radial part of \( x = t \xi \), i.e., \( \varrho \) can be expressed as \( \varrho(x) = H(t) \), where \( t = |x| \) (cf. [6, Page 976]). If the equalities are attained in (3.3), then we see from (2.3) that

\[
|D_N S(x)| = 0, \quad |D_{U_1} S(x)| = |D_{U_2} S(x)| = \cdots = |D_{U_{n-1}} S(x)|
\]

and \( D_{U_1} S(x), D_{U_2} S(x), \ldots, D_{U_{n-1}} S(x) \) are mutually orthogonal vectors. Thus there exists a conformal mapping \( T \) from \( S^{n-1} \) onto \( S^{n-1} \) such that \( S(t \xi) = T(\xi) \), where \( \xi \in S^{n-1} \). Therefore, when the equalities in (3.3) and (3.4) hold, \( h \) must be of the form

\[
h(t \xi) = H(t) T(\xi),
\]

where \( t \in E \) and \( \xi \in S^{n-1} \). This means that \( H \) is a homeomorphism from \( E \) onto \( F \) in \( \mathcal{W}^{1,n-1}(E) \). Recall that (3.5) and (3.6) are equalities if and only if

\[
\frac{\dot{H}(t)}{H(t)} = C_{1,1} t^{-\frac{n-1}{n-2}}
\]

a.e. in \( E \) for some \( C_{1,1} > 0 \), where \( \dot{H}(t) = \frac{dH(t)}{dt} \). Let

\[
G(t) = \log H(t) + (n - 2) C_{1,1} t^{-\frac{1}{n-2}}
\]

be a continuous mapping in \( E \). Then \( G \in \mathcal{W}^{1,n-1}(E) \) and it follows from (3.7) that \( \dot{G} = 0 \) a.e. in \( E \). Thus there exists a constant \( C_{1,2} \in \mathbb{R} \) such that \( G = C_{1,2} \) a.e. in \( E \). Since \( G \) is continuous in \( E \), we obtain that \( G \equiv C_{1,2} \) in \( E \), i.e.,

\[
H(t) = C_{1,3} \exp \left\{ C_{1,4} t^{\frac{1}{n-1}} \right\}
\]

for some constants \( C_{1,3} > 0 \) and \( C_{1,4} < 0 \). This, together with (3.2), implies that

\[
C_{1,3} = r_* \left( \frac{R_*}{r_*} \right) \frac{\frac{1}{R^{\frac{n-1}{2}}} - \frac{1}{r^{\frac{n-1}{2}}}}{\frac{1}{R^{\frac{n-1}{2}}} - \frac{1}{r^{\frac{n-1}{2}}}} \quad \text{and} \quad C_{1,4} = \frac{(Rr)^{\frac{1}{n-2}}}{R^{\frac{1}{n-2}} - r^{\frac{1}{n-2}}} \log \frac{r_*}{R_*}.
\]

Thus the proof of the theorem is complete. \( \square \)

### 4 Proof of Theorem 1.2

We shall prove Theorem 1.2 in this section. The proof will be based on three lemmas. The first lemma reads as follows.

**Lemma 4.1** For any \( a \geq b \geq 0 \) and \( s \geq 1 \), we have that

\[
a^s - b^s \leq s(a - b)(a^{s-1} + b^{s-1}).
\]

**Proof** If \( b = 0 \), obviously, we have \( a^s - b^s \leq s(a - b)(a^{s-1} + b^{s-1}) \). In the following, we assume that \( b > 0 \). Note that

\[
s(a - b)(a^{s-1} + b^{s-1}) = s(a^s + ab^{s-1} - ba^{s-1} - b^s).
\]

Hence, it suffices to prove that

\[
\left( \frac{a}{b} \right)^s - 1 \leq s \left( \left( \frac{a}{b} \right)^s + \frac{a}{b} - \left( \frac{a}{b} \right)^{s-1} - 1 \right).
\]

\( \square \) Springer
Let
\[ g(x) = s \left( x^s + x - x^{s-1} - 1 \right) - x^s + 1, \]
where \( x \geq 1 \). Obviously,
\[ g'(x) = s(s - 1)x^{s-1} - s(s - 1)x^{s-2} + s \]
and
\[ g''(x) = s(s - 1)x^{s-3} \left( (s - 1)x - (s - 2) \right). \]
Since \( g''(x) \geq 0 \) in \([1, +\infty)\) and \( g'(1) = s > 0 \), we see that \( g \) is increasing in \([1, +\infty)\). It follows from \( g(1) = 0 \) that \( g(x) \geq 0 \) in \([1, +\infty)\). Therefore, (4.1) is true, and the proof of the lemma is complete.

Next, we establish a general integral representation formula for \( \mathcal{E}[h^\lambda] \), where \( h^\lambda \) is the mapping from (1.7).

**Lemma 4.2** For \( n \geq 4 \) and \( \lambda > 0 \),
\[
\mathcal{E}[h^\lambda] = 2^{n-1} \omega_{n-2} \int_0^1 \int_0^{\infty} \left( \frac{t^2 \dot{H}^2(t)}{H^2(t)} + \frac{\lambda^2 (1 + y^2)^2}{(1 + \lambda^2 y^2)^2} \right)^{\frac{n}{2}} \frac{y^{n-2}}{(1 + y^2)^{n-1}} dy \, dt.
\]

In particular,
\[
\mathcal{E}[h^1] = \omega_{n-1} \int_R \left( n - 1 + \frac{t^2 \dot{H}^2(t)}{H^2(t)} \right)^{\frac{n}{2}} dt.
\]

**Proof** For \( \lambda > 0 \), by (1.2), (1.7), (2.4) and (2.5), we obtain that
\[
\mathcal{E}[h^\lambda] = \int_R \int_{\partial B^1} \left( \frac{t^2 \dot{H}^2(t)}{H^2(t)} + \| D\Phi^\lambda(\xi) \|^2 \right)^{\frac{n}{2}} \frac{1}{\sin^2 \theta} \, d\sigma(\xi) \, d\theta dt.
\]
By (2.8) and (4.4), we get that
\[
\mathcal{E}[h^\lambda] = \omega_{n-2} \int_0^1 \int_0^{\pi} \left( \frac{t^2 \dot{H}^2(t)}{H^2(t)} + (n - 1) \frac{\sin^2 \left( 2 \arctan(\lambda \tan \frac{\theta}{2}) \right)}{\sin^2 \theta} \right)^{\frac{n}{2}} \sin^{n-2} \theta \, d\theta \, dt.
\]
Let \( y = \tan \frac{\theta}{2} \) and \( \theta = \arctan(\lambda y) \). Then \( \sin \theta = \frac{2y}{1+y^2} \), \( d\theta = \frac{2dy}{1+y^2} \) and
\[
\sin \left( 2 \arctan \left( \lambda \tan \frac{\theta}{2} \right) \right) = \sin \left( 2 \arctan(\lambda y) \right) = \frac{2\lambda y}{1 + \lambda^2 y^2}.
\]
Combining the above equalities, we see that (4.2) is true. Further,
\[
\int_0^{+\infty} \frac{y^{n-2}}{(1 + y^2)^{n-1}} \, dy = \frac{1}{2} \int_0^{+\infty} \frac{y^{\frac{n-3}{2}}}{(1 + y)^{n-1}} \, dy = \frac{1}{2} \int_1^{+\infty} \frac{(s - 1)^{\frac{n-3}{2}}}{s^{n-1}} \, ds = \frac{\Gamma^2 \left( \frac{n-1}{2} \right)}{2^n \Gamma \left( \frac{n}{2} \right)}.
\]
and
\[
\frac{\omega_{n-2}}{\omega_{n-1}} = \frac{1}{\int_0^\pi \sin^{n-2} \theta d\theta} = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi} \Gamma(\frac{n-1}{2})}.
\] (4.6)

Then (4.2), (4.5) and (4.6) yield that (4.3) is true. The proof of the lemma is complete. \(\square\)

Based on Lemmas 4.1 and 4.2, we have the following result.

**Lemma 4.3** For \(n \geq 4\), we have that
\[
\lim_{\lambda \to 0^+} \mathcal{E}[h^\lambda] = \inf_{\lambda \in (0, +\infty)} \mathcal{E}[h^\lambda] < \mathcal{E}[h^1]
\]
and
\[
\inf_{\lambda \in (0, +\infty)} \mathcal{E}[h^\lambda] \geq \omega_{n-1} \left( (n - 1)^{\frac{n-1}{2}} (R - r) + \frac{\log^{n-1} R}{(n - 2)^{n-2}} \frac{Rr}{(R - r)^{n-2}} \right)
\]
\[
= \lim_{\lambda \to 0^+} \mathcal{E}[h^1] = \lim_{\lambda \to 0^+} \mathcal{E}[h^2].
\]
where \(h^1\) and \(h^2\) are the mappings from (1.6).

**Proof** In order to prove the lemma, first, we show the following claim.

**Claim 4.1** For \(n \geq 4\),
\[
\inf_{\lambda \in (0, +\infty)} \mathcal{E}[h^\lambda] = \lim_{\lambda \to 0^+} \mathcal{E}[h^\lambda] = \omega_{n-1} \int_r^R \left( (n - 1)^{\frac{n-1}{2}} + \frac{t^{n-1} \frac{\log t}{H^{n-1}(t)}}{H^{n-1}(t)} \right) dt.
\]

For any \(\lambda > 0\), by (4.5) and (4.6), we get that
\[
\int_r^R \int_0^\infty \frac{\lambda^{n-1} (1 + y^2)^{n-1}}{(1 + \lambda^2 y^2)^{n-1}} y^{n-2} dy dt = \int_r^R \int_0^\infty \frac{s^{n-2}}{(1 + s^2)^{n-1}} ds dt = \frac{(R - r) \omega_{n-1}}{2n-1} \omega_{n-2}.
\]

This, together with (1.7), (4.2), (4.5) and (4.6), implies that
\[
\mathcal{E}[h^\lambda] \geq 2^{n-1} \omega_{n-2} \int_r^R \int_0^\infty \left( \frac{t^{n-1} \frac{\log t}{H^{n-1}(t)}}{H^{n-1}(t)} + (n - 1)^{\frac{n-1}{2}} \frac{\lambda^{n-1} (1 + y^2)^{n-1}}{(1 + \lambda^2 y^2)^{n-1}} \right) \times \frac{y^{n-2}}{(1 + y^2)^{n-1}} dy dt
\]
\[
= \omega_{n-1} \int_r^R \left( (n - 1)^{\frac{n-1}{2}} (R - r) + \frac{t^{n-1} \frac{\log t}{H^{n-1}(t)}}{H^{n-1}(t)} \right) dt.
\] (4.7)

In the following, we prove that
\[
\lim_{\lambda \to 0^+} \mathcal{E}[h^\lambda] = \omega_{n-1} \int_r^R \left( (n - 1)^{\frac{n-1}{2}} + \frac{t^{n-1} \frac{\log t}{H^{n-1}(t)}}{H^{n-1}(t)} \right) dt.
\] (4.8)

By (4.2) and (4.7), we see that (4.8) is equivalent to
\[
\lim_{\lambda \to 0^+} \int_r^R \int_0^\infty \left( \left( \frac{t^2}{H^2(t)} + (n - 1)^{\frac{n-1}{2}} \frac{\lambda^2 (1 + y^2)^2}{(1 + \lambda^2 y^2)^2} \right) \frac{y^{n-2}}{(1 + y^2)^{n-1}} \right) dt.
\]
\[
- (n - 1)^{\frac{n-1}{2}} \frac{\lambda^{n-1} (1 + y^2)^{n-1}}{(1 + \lambda^2 y^2)^{n-1}} \frac{y^{n-2}}{(1 + y^2)^{n-1}} dy dt = 0.
\] (4.9)
For fixed $a \geq 0$, obviously, the mapping
\[
x \mapsto (a^2 + x^2)^{n^{-1}} - a^{n-1} - x^{n-1}
\]
is increasing in $[0, +\infty)$ and
\[
\frac{\lambda^2}{(1 + \lambda^2 y^2)^2} \leq \frac{1}{4y^2}
\]
for any $y > 0$ and $\lambda \geq 0$. Then for any $t \in E$, $y > 0$ and $\lambda \geq 0$, it follows from Lemma 4.1 that
\[
(t^2 \dot{H}(t) + (n-1) \frac{\lambda^2(1+y^2)^2}{(1 + \lambda^2 y^2)^2})(\frac{n^{-1}}{\dot{H}^n(t)}) - (n-1) \frac{\lambda^{n-1}(1+y^2)^{n-1}}{(1 + \lambda^2 y^{2n-1})}
\]
\[
\leq \left( t^2 \dot{H}(t) + (n-1) \frac{(1+y^2)^2}{4y^2} \right)^{\frac{n-1}{2}} - (n-1) \frac{\lambda^{n-1}(1+y^2)^{n-1}}{(2y)^{n-1}} + \frac{t^{n-1} \dot{H}^n(t)}{H(t)}
\]
\[
\leq \frac{n-1}{2} t^2 \dot{H}(t) \left( (t^2 \dot{H}(t) + (n-1) \frac{(1+y^2)^2}{4y^2})^{\frac{n-1}{2}} + (n-1) \frac{\lambda^{n-1}(1+y^2)^{n-1}}{(2y)^{n-1}} \right)
\]
\[
+ \frac{t^{n-1} \dot{H}^n(t)}{H(t)}
\]
\[
\leq \left( 1 + (n-1)2^{n-3} \right) \frac{t^{n-1} \dot{H}^n(t)}{H(t)} + (n-1) \frac{(1+y^2)^{n-3}}{2^{n-2}} \frac{t^2 \dot{H}(t)}{y^{n-3}} \frac{t^2 \dot{H}(t)}{H(t)}.
\]
Since the assumption $\hat{h}^\lambda \in \mathcal{P}(\lambda, A_\lambda)$ implies that $H \in \mathcal{W}^{1,n-1}(E, F)$, we obtain
\[
\int_r^R \int_0^{+\infty} \left( \frac{n-1}{2} \frac{t^2 \dot{H}(t)}{H(t)} + (n-1) \frac{t^{n-1} \dot{H}(t)}{H(t)} \frac{t^2 \dot{H}(t)}{y^{n-3}} \frac{t^2 \dot{H}(t)}{H(t)} \right) y^{n-2} dy dt < \infty.
\]
Then it follows from the Lebesgue’s dominated convergence theorem that
\[
\lim_{\lambda \to 0^+} \int_r^R \int_0^{+\infty} \left( \frac{t^2 \dot{H}(t)}{H(t)} + (n-1) \frac{\lambda^2(1+y^2)^2}{(1 + \lambda^2 y^2)^2} \right)^{\frac{n-1}{2}} - \frac{t^{n-1} \dot{H}^n(t)}{H(t)}
\]
\[
- (n-1) \frac{\lambda^{n-1}(1+y^2)^{n-1}}{(1 + \lambda^2 y^{2n-1})} \right) \frac{y^{n-2}}{(1+y^2)^{n-1}} dy dt
\]
\[
= \int_r^R \int_0^{+\infty} \lim_{\lambda \to 0^+} \left( \frac{t^2 \dot{H}(t)}{H(t)} + (n-1) \frac{\lambda^2(1+y^2)^2}{(1 + \lambda^2 y^2)^2} \right)^{\frac{n-1}{2}} - \frac{t^{n-1} \dot{H}^n(t)}{H(t)}
\]
\[
- (n-1) \frac{\lambda^{n-1}(1+y^2)^{n-1}}{(1 + \lambda^2 y^{2n-1})} \right) \frac{y^{n-2}}{(1+y^2)^{n-1}} dy dt = 0.
\]
This implies that (4.9) is true, and so, (4.8) follows. Combining (4.7) and (4.8), we see that Claim 4.1 is true.

Further, it follows from Hölder’s inequality that
\[
\int_r^R \frac{t^{n-1} \dot{H}^n(t)}{H(t)} dt \geq \left( \int_r^R \frac{1}{t^{\frac{1}{n-1}}} dt \right)^{n-1} \left( \int_r^R \frac{1}{t^{\frac{1}{n-1}}} dx \right)^{2-n}
\]
\[
= \frac{\log^{n-1} \frac{R_a}{r_a}}{(n-2)^{n-2} (\frac{1}{(R^{n-2} - r^{n-2})^{n-2}}).
\]
The equality holds if and only if $\hat{H}(t) = C_2, t^{-\frac{n-1}{n-2}}$ a.e. in $E$ for some constant $C_{2, 1} > 0$. Since $H \in W^{1, n-1}(E)$ and $H$ is a homeomorphism from $E$ onto $F$. The same reasoning as in the discussions of Theorem 1.1 shows that

$$H(t) = r_* \left( \frac{R_s}{r_*} \alpha(t) \right) \quad \text{or} \quad H(t) = R_s \left( \frac{r_*}{R_s} \alpha(t) \right), \quad (4.11)$$

where $\alpha(t)$ is the mapping from (1.4).

Now, we are going to finish the proof of Theorem 1.2. By Claim 4.1, (4.3), (4.8), (4.10) and (4.11), we see that

$$\mathcal{E}[h^1] \geq \omega_{n-1} \int_{A_r} \left( (n-1)^{-1} + \frac{t^{n-1} \hat{H}^{n-1}(t)}{H^{n-1}(t)} \right) dt \geq \omega_{n-1} \left( (n-1)^{-1} (R - r) + \frac{\log^{n-1} R_s}{(n-2)^{n-2}} \frac{Rr}{(R^{-\frac{1}{n-2}} - r^{-\frac{1}{n-2}})^{n-2}} \right) = \lim_{\lambda \to 0^+} \mathcal{E}[h_1^\lambda] = \lim_{\lambda \to 0^+} \mathcal{E}[h_2^\lambda],$$

where $h_1^\lambda$ and $h_2^\lambda$ are the mappings from (1.6). The mapping $h_1^\lambda$ preserves the orientation and $h_2^\lambda$ changes the orientation. The proof of the lemma is complete. \hfill $\square$

Now, we are going to finish the proof of Theorem 1.2.

### 4.1 Proof of Theorem 1.2

Let $h = \varrho S \in \mathcal{F}(\mathcal{A}, \mathcal{A}_s)$, where $\varrho = |h|$ and $S = \frac{h}{|h|}$. It follows from (1.1)–(1.3), Theorem 1.1 and Lemma 4.3 that

$$\mathcal{E}[h] = \int_{\mathbb{A}} \left( \|DS(x)\|^2 + \frac{\|\nabla \varrho(x)\|^2}{\varrho^2(x)} \right)^{\frac{n-1}{2}} \varrho^{n-1}(x) dx \geq \int_{\mathbb{A}} \left( \|DS(x)\|^2 + \frac{\|\nabla \varrho(x)\|^2}{\varrho^{n-1}(x)} \right) dx = \mathbb{E}[1, 1][h] \quad (4.12) \geq \omega_{n-1} \left( (n-1)^{-1} (R - r) + \frac{1}{(n-2)^{n-2}} \frac{Rr}{(R^{-\frac{1}{n-2}} - r^{-\frac{1}{n-2}})^{n-2}} \log^{n-1} R_s \right) = \lim_{\lambda \to 0^+} \mathcal{E}[h_1^\lambda] = \lim_{\lambda \to 0^+} \mathcal{E}[h_2^\lambda],$$

where $h_1^\lambda$ and $h_2^\lambda$ are the mappings from (1.6). Thus,

$$\lim_{\lambda \to 0^+} \mathcal{E}[h_1^\lambda] = \lim_{\lambda \to 0^+} \mathcal{E}[h_2^\lambda] = \inf_{h \in \mathcal{F}(\mathcal{A}, \mathcal{A}_s)} \mathcal{E}[h] = \inf_{h \in \mathcal{F}(\mathcal{A}, \mathcal{A}_s)} \mathcal{E}[h] \quad (4.13)$$

Now, we prove that the minimizer of the energy integral $\mathcal{E}[h]$ in $\mathcal{F}(\mathcal{A}, \mathcal{A}_s)$ does not exist. Suppose that there is $\tilde{h} \in \mathcal{F}(\mathcal{A}, \mathcal{A}_s)$ such that $\mathcal{E}[\tilde{h}] = \inf_{h \in \mathcal{F}(\mathcal{A}, \mathcal{A}_s)} \mathcal{E}[h]$, where $\tilde{h} = \tilde{\varrho} S$, ...
\( \tilde{\rho} = |\tilde{h}| \) and \( \tilde{S} = \frac{\tilde{h}}{|\tilde{h}|} \). Then we obtain from (4.12) and (4.13) that

\[
\mathcal{E}[\tilde{h}] = \int_A \left( \| D \tilde{S}(x) \|^2 + \frac{|\nabla \tilde{\rho}(x)|^2}{\tilde{\rho}^2(x)} \right)^{\frac{n-1}{2}}\ dx
\]

\[
= \int_A \left( \| D \tilde{S}(x) \|^2 + \frac{|\nabla \tilde{\rho}(x)|^{n-1}}{\tilde{\rho}^{n-1}(x)} \right)\ dx
\]

\[
= \mathbb{E}[1,1][\tilde{h}]
\]

\[
= \omega_{n-1} \left( (n-1) \frac{n-1}{2} (R-r) + \frac{1}{(n-2)^{n-2}} \frac{R}{(R-r)^{n-2}} \log^{n-2} \frac{R}{r} \right).
\displaystyle \tag{4.14}
\]

Further, by the proof of Theorem 1.1 and (4.14), we see that \( \tilde{S}(x) = T(\frac{x}{|x|}) \) is a conformal mapping from \( \mathbb{S}^{n-1} \) onto \( \mathbb{S}^{n-1} \) and \( \tilde{\rho}(x) = \frac{|\tilde{h}_1^1(|x|)|}{|x|^\frac{n-1}{2}} \) or \( \tilde{\rho}(x) = \frac{|\tilde{h}_2^2(|x|)|}{|x|^\frac{n-1}{2}} \) with

\[
\frac{|\nabla \tilde{\rho}(x)|}{\tilde{\rho}(x)} = C_{1,1} |x|^{-\frac{n-1}{n-2}} > 0
\]
a.e. in \( \mathbb{A} \) for some constant \( C_{1,1} > 0 \). For any \( a \geq b \geq 0 \) and \( s \geq 1 \), since

\[
a^s - b^s \geq (a-b)(a^{s-1} + b^{s-1}),
\]

we obtain that

\[
\int_A \left( \left( \| D \tilde{S}(x) \|^2 + \frac{|\nabla \tilde{\rho}(x)|^2}{\tilde{\rho}^2(x)} \right)^{\frac{n-1}{2}} - \| D \tilde{S}(x) \|^{n-1} - \frac{|\nabla \tilde{\rho}(x)|^{n-1}}{\tilde{\rho}^{n-1}(x)} \right) dx
\]

\[
\geq \int_A \left( \frac{|\nabla \tilde{\rho}(x)|^2}{\tilde{\rho}^2(x)} \left( \| D \tilde{S}(x) \|^2 + \frac{|\nabla \tilde{\rho}(x)|^2}{\tilde{\rho}^2(x)} \right)^{\frac{n-3}{2}} + \| D \tilde{S}(x) \|^{n-3} \right) - \frac{|\nabla \tilde{\rho}(x)|^{n-1}}{\tilde{\rho}^{n-1}(x)} dx
\]

\[
> 0
\]

for \( n \geq 4 \). It’s a contradiction with (4.14).

\[
\square
\]

5 Proofs of Theorems 1.3 and 1.4

The aim of this section is to prove Theorems 1.3 and 1.4.

5.1 Proof of Theorem 1.3

For \( \lambda > 0 \), let \( h^\lambda(x) = H(t) \Phi^\lambda(\xi) \in \mathcal{P}(\mathbb{A}, \mathbb{A}_+^+) \) and \( h(x) = H(t)\xi \in \mathcal{R}(\mathbb{A}, \mathbb{A}_+^+) \), where \( x = t\xi, t = |x| \) and \( \xi \in \mathbb{S}^{n-1} \). If \( \lambda \neq 1 \), we see from (2.8) that \( \| D \Phi^\lambda(\xi) \|^2 \) depends on \( \xi \). Since \( \frac{n-1}{2} > 1 \), we obtain from (2.8), (4.4) and the Minkowski inequality (cf. [13, Page 9]) that

\[
\mathcal{E}[h^\lambda] = \int_r^R \int_{\mathbb{S}^{n-1}} \left( \frac{t^2 \hat{H}^2(t)}{H^2(t)} + \| D \Phi^\lambda(\xi) \|^2 \right)^{\frac{n-1}{2}} d\sigma(\xi) dt
\]

\[
< \int_r^R \left( \left( \int_{\mathbb{S}^{n-1}} \frac{t^{n-1} \hat{H}^{n-1}(t)}{H^{n-1}(t)} d\sigma(\xi) \right)^{\frac{2}{n-1}} + \left( \int_{\mathbb{S}^{n-1}} \| D \Phi^\lambda(\xi) \|^{n-1} d\sigma(\xi) \right)^{\frac{2}{n-1}} \right) dt.
\]
Further, it follows from (2.9) and (4.3) that

\[ E[h^3] < \omega_{n-1} \int_R \left( n - 1 + \frac{t^2 \dot{H}^2(t)}{H^2(t)} \right)^{\frac{n-1}{2}} dt = E[h^1] = E[h]. \]  \tag{5.1}

By the arbitrariness of \( h \), we see that the proof of the theorem is complete. \( \Box \)

In order to prove Theorem 1.4, we shall make some preparation.

Let \( x = t \xi \in \mathbb{A} \) and \( h(x) = H(t)\xi \), where \( t = |x|, \xi \in \mathbb{S}^{n-1}, H \in C^2(E) \cap \mathcal{W}^{1,n-1}(E) \) and \( H \) is an increasing mapping from \( E \) onto \( F \). Then (5.4) admits a unique solution \( H \) in \( E \).

Further, it follows from (2.9) and (4.3) that

\[ \int_R \Lambda(t, H, \dot{H}) dt =: \mathcal{H}[H]. \]  \tag{5.2}

where \( \Lambda(t, H, \dot{H}) = \left( n - 1 + \frac{t^2 \dot{H}^2(t)}{H^2(t)} \right)^{\frac{n-1}{2}} \). Then the Euler-Lagrange equation (or equilibrium equation) for the energy integral \( E[h] \) is

\[ \frac{d}{dt} \left( \frac{\partial}{\partial H} \Lambda \right) = \frac{\partial}{\partial \dot{H}} \Lambda \]  \tag{5.3}

(cf. [8, Section 1.2]). By calculations, (5.3) reduces to

\[ 0 = (n - 3) \frac{t^2 \dot{H}^2}{H^2} \left( \frac{t^2 \dot{H}^2}{H^2} + \frac{t^3 \dot{H}^3}{H^2} - \frac{t^3 \dot{H}^3}{H^3} \right) \]
\[ + \left( \frac{t^2 \dot{H}^2}{H^2} + n - 1 \right) \left( 2 \frac{t^2 \dot{H}^2}{H^2} + \frac{t^3 \dot{H}^3}{H^2} - \frac{t^3 \dot{H}^3}{H^3} \right). \]  \tag{5.4}

**Lemma 5.1** Suppose that \( n \geq 4 \), \( H \in C^2(E) \cap \mathcal{W}^{1,n-1}(E) \) and \( H \) is an increasing mapping from \( E \) onto \( F \). Then (5.4) admits a unique solution \( H_* \) satisfying \( H_* \in C^\infty(E) \) and \( \dot{H}_* > 0 \) in \( E \). Further, we have

\[ \mathcal{H}[H_*] = R\omega_{n-1} \left( w_*^2(r) + n - 1 \right)^{\frac{n-1}{2}} - R\omega_{n-1} \left( w_*^2(r) + n - 1 \right)^{\frac{n-1}{2}} \]
\[ - \tau(n - 1)\omega_{n-1} \left( w_*^2(R) - w_*^2(r) \right), \]  \tag{5.5}

where \( \tau \) is the constant from (5.15) (see below) and \( w_*^2(t) = \frac{\dot{H}_*^2(t)}{H_*^2(t)} \) in \( E \).

**Proof** For \( t \in E \) and \( H \in C^2(E) \), let \( w(t) = t^\dot{H}_*^2(t) \) and \( \dot{w}(t) = \frac{dw(t)}{dt} \). Then

\[ \frac{t^3 \dot{H}^3}{H^2} = w^3 - w^2 + tw\dot{w}, \]

and hence, (5.4) reduces to

\[ 0 = (n - 1)w + w^3 + t\dot{w}((n - 1) + (n - 2)w^2). \]  \tag{5.6}

Solving the differential equation (5.6), we get that

\[ w(n - 1 + w^2)^{\frac{n-3}{2}} = \frac{\tau}{t}, \]  \tag{5.7}

where \( \tau \) is an arbitrary constant. Since \( w = \frac{t^\dot{H}^2}{H^2}, H(r) = r_* \) and \( H(R) = R_* \), we know that \( \tau \neq 0 \). Then the fact that \( H \) is an increasing mapping in \( E \) implies that \( \tau > 0, w > 0 \) and \( \dot{H}_* > 0 \) in \( E \).
For \( w \in \mathbb{R}^+ \), let
\[
\phi(w) = w(n - 1 + w^2)^{\frac{n-3}{2}}.
\]

For any \( \tau_0 > 0 \) and \( t_0 \in E \), since \( \phi \) is a strictly increasing mapping in \( \mathbb{R}^+ \), then there exists a unique \( \tau \) such that \( \phi(t_0) = \tau \). Let
\[
\varphi_0(t, \tau, w) = w(n - 1 + w^2)^{\frac{n-3}{2}} - \tau.
\]

Obviously, \( \varphi_0 \in C^1(E \times \mathbb{R}^+ \times \mathbb{R}^+) \) and \( \frac{\partial}{\partial w} \varphi_0(t, \tau, w) > 0 \). The implicit function theorem yields that there exists a unique function \( \psi(t, \tau) \in C^1(E \times \mathbb{R}^+, \mathbb{R}^+) \) such that \( w = \varphi(t, \tau) \).

Further, for each fixed \( \tau > 0 \), it follows from (5.7) that
\[
\phi'(w)w'(t) = -\frac{\tau}{t^2}.
\]

Since \( \phi(w) \in C^\infty(\mathbb{R}^+) \) and \( w(t) \in C^1(E) \), we get \( (\phi' \circ w)(t) \in C^1(E) \). This, together with the fact that \( \phi'(w) = (n - 1 + w^2)^{\frac{n-3}{2}}(n - 1 + (n - 2)w^2) > 0 \), implies that \( w'(t) \in C^1(E) \). By differentiating with respect to \( t \) on both side of (5.8), we get
\[
\phi''(w)(w'(t))^2 + \phi'(w)w''(t) = -\frac{2\tau}{t^3}.
\]

Similarly, we see that \( w''(t) \in C^1(E) \). Continue this process. Finally, we get that \( w(t) \in C^\infty(E) \), and so, \( H(t) \in C^\infty(E) \).

On the other hand, (5.6) is equivalent to
\[
-\frac{\dot{H}}{H} = \frac{n - 1 + (n - 2)w^2}{n - 1 + w^2} \dot{w}.
\]

Solving this differential equation, we obtain the general solution
\[
H = \kappa \exp \left( -(n - 2)w + (n - 3)\sqrt{n - 1} \arctan \frac{w}{\sqrt{n - 1}} \right),
\]
where \( \kappa > 0 \) is an arbitrary constant and \( w = w(t, \tau) \) is the mapping from (5.7). Hence,
\[
H = \kappa \exp \left( -(n - 2)w(t, \tau) + (n - 3)\sqrt{n - 1} \arctan \frac{w(t, \tau)}{\sqrt{n - 1}} \right). \tag{5.9}
\]

Since \( H(r) = r_* \) and \( H(R) = R_* \), we obtain from (5.9) that
\[
\log \frac{R_*}{r_*} = \log \frac{H(R)}{H(r)} = \psi(r, R, \tau), \tag{5.10}
\]
where
\[
\psi(r, R, \tau) = (n - 2)(w(r, \tau) - w(R, \tau)) + (n - 3)\sqrt{n - 1} \left( \arctan \frac{w(R, \tau)}{\sqrt{n - 1}} - \arctan \frac{w(r, \tau)}{\sqrt{n - 1}} \right). \tag{5.11}
\]

In the following, we prove that for any \( 0 < r < R < +\infty \) and \( 0 < r_* < R_* < +\infty \), there exists a unique \( \tau_* = \tau_*(r, R, r_*, R_*) > 0 \) such that (5.10) holds, where \( \tau_* = \tau_*(r, R, r_*, R_*) \) means that the constant \( \tau_* \) depends only on the quantities \( r, R, r_* \) and \( R_* \).

**Claim 5.1** For any \( 0 < r < R < +\infty \) and \( \tau > 0 \), we have \( \frac{\partial}{\partial \tau} \psi(r, R, \tau) > 0 \).
It follows from the fact \( w(t, \tau) \in C^1(E \times \mathbb{R}^+) \) that \( \psi(r, R, \tau) \in C^1(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+) \). Further, we have that
\[
\frac{\partial}{\partial \tau} \psi(r, R, \tau) = \frac{\partial w(r, \tau)}{\partial \tau} \left( (n-2) - \frac{(n-3)(n-1)}{n-1 + w^2(r, \tau)} \right) - \frac{\partial w(R, \tau)}{\partial \tau} \left( (n-2) - \frac{(n-3)(n-1)}{n-1 + w^2(R, \tau)} \right).
\] (5.12)

By differentiating with respect to \( \tau \) on both side of (5.7), we get that
\[
\frac{\partial w(t, \tau)}{\partial \tau} = \frac{(n-1 + w^2(t-1 + w^2))}{(n-1 + (n-2)w^2)} \frac{w(t, \tau)}{\tau}.
\]

Combining this with (5.7), we obtain that
\[
\frac{\partial w(t, \tau)}{\partial \tau} \left( (n-2) - \frac{(n-3)(n-1)}{n-1 + w^2(r, \tau)} \right) = \frac{w(t, \tau)}{\tau}.
\]

This and (5.12) yield that
\[
\frac{\partial}{\partial \tau} \psi(r, R, \tau) = \frac{w(r, \tau)}{\tau} - \frac{w(R, \tau)}{\tau}.
\]

Since \( w > 0 \), we infer from (5.6) that \( \frac{\partial}{\partial \tau} w(t, \tau) < 0 \). Then \( w(r, \tau) > w(R, \tau) \) and \( \frac{\partial}{\partial \tau} \psi(r, R, \tau) > 0 \). The claim is true.

Obviously, (5.7) implies that
\[
w(t, \tau) = \frac{\tau}{t(n-1)^{\frac{n-3}{2}}} + o(\tau) \to 0^+
\]
as \( \tau \to 0^+ \). Recall that
\[
\arctan x_1 - \arctan x_2 = \arctan \frac{x_1 - x_2}{1 + x_1x_2}
\]
and
\[
\arctan x = x + o(x)
\]
as \( x \to 0 \), where \( x_1, x_2 \in \mathbb{R}^+ \). Hence, when \( \tau \to 0^+ \), we infer from (5.11) that
\[
\psi(r, R, \tau) = \frac{\tau}{r(n-1)^{\frac{n-3}{2}}} - \frac{\tau}{R(n-1)^{\frac{n-3}{2}}} + o(\tau) \to 0^+.
\] (5.13)

When \( \tau \to +\infty \), (5.7) implies that
\[
w(t, \tau) = \left( \frac{\tau}{t} \right)^{\frac{1}{n-2}} + o(\tau^{\frac{1}{n-2}}) \to +\infty,
\]
and hence, we obtain from (5.11) that
\[
\psi(r, R, \tau) = (n-2) \left( \left( \frac{\tau}{r} \right)^{\frac{1}{n-2}} - \left( \frac{\tau}{R} \right)^{\frac{1}{n-2}} \right) + o(\tau^{\frac{1}{n-2}}) \to +\infty.
\] (5.14)
Therefore, for any $0 < r < R < +\infty$ and $0 < r_* < R_* < +\infty$, we infer from Claim 5.1, (5.13) and (5.14) that there exists a unique $\tau_\kappa = \tau_\kappa(r, R, r_*, R_*) > 0$ such that
\[
\psi(r, R, \tau_\kappa) = \log \frac{R_\kappa}{r_\kappa}.
\] (5.15)

Further, by letting $t = r$, $H = r_*$ and $\tau = \tau_\kappa$ in (5.9), we can obtain the constant $\kappa_* = \kappa_*(r, R, r_*, R_*) > 0$. Then (5.5) follows from the above equality and (5.15). The proof of the lemma is complete.

(5.2), we know that there exists a mapping $B$...Page 17 of 20 205

By (5.2) and the fact that $w = \frac{tH}{H}$ and $w \in C^\infty(E)$, we obtain
\[
\mathcal{H}[H] \left[ \omega_{n-1} \right] = \int_R^R (n - 1 + w^2(t)) \frac{n-1}{2} dt
= t(n - 1 + w^2(t)) \frac{n-1}{2} - (n - 1) \int_R^R t \cdot w(t) w'(t) (n - 1 + w^2(t)) \frac{n-3}{2} dt.
\]
Combining this with (5.7), we see that
\[
\mathcal{H}[H] = \omega_{n-1} \left( t(n - 1 + w^2(t)) \frac{n-1}{2} - \tau(n - 1) w(t) \right) \bigg|_{t=R}^{t=R}.
\]
Then (5.5) follows from the above equality and (5.15). The proof of the lemma is complete.

Now, we are going to prove Theorem 1.4.

5.2 Proof of Theorem 1.4

Let $B$ denote the family of increasing homeomorphisms from $E$ onto $F$ in $W^{1,n-1}(E)$. For any $h \in \mathcal{R}(\lambda, \lambda_*), \text{ set } h(x) = H(t)\xi$, where $x = t\xi$ and $t = |x|$. Then $H \in B$. By (5.1) and (5.2), we know that there exists a mapping $h^\lambda(t\xi) = H(t)\Phi^\lambda(\xi) \in \mathcal{P}(\lambda, \lambda_*)$ such that
\[
\mathcal{E}[h^\lambda] < \omega_{n-1} \int_R^R \left( n - 1 + \frac{t^2 H^2(t)}{H^2(\xi)} \right) \frac{n-1}{2} dt = \mathcal{H}[H] = \mathcal{E}[h].
\] (5.16)
For every constant $c > 0$, a similar approach as in the proof of (3.1) gives that
\[
\mathcal{E} \left[ \frac{ch}{|h|^2} \right] = \mathcal{E}[h].
\]
Thus, we assume that
\[
\lim_{|x| \to r} |h(x)| = r_* \quad \text{and} \quad \lim_{|x| \to R} |h(x)| = R_*.
\]
In order to prove the theorem, we use $\mathcal{X}$ to denote the family of increasing continuous mappings from $E$ onto $F$ in $W^{1,n-1}(E)$.

Claim 5.2 There exists $H_0 \in \mathcal{X}$ such that
\[
\mathcal{H}[H_0] = \inf_{H \in \mathcal{X}} \mathcal{H}[H].
\]
We will divide the proof of the claim into three steps, which is based upon the ideas from [5, Theorem 3.3].

**Step 1.** Compactness. Let \( \{H_m\} \) be a minimizing sequence in \( \mathcal{X} \), i.e.,

\[
\lim_{m \to \infty} H[H_m] = \inf_{H \in \mathcal{X}} H[H] =: \mathcal{I}.
\]  

(5.17)

For \( t \in E \) and \( H \in \mathcal{X} \), we see that there exists a positive constant \( C_{3,1} = C_{3,1}(n, E, F) \) such that \( \Lambda \) satisfies the following coercivity condition

\[
\Lambda(t, H, K) = \left(n - 1 + t^2 \frac{K^2}{H^2}\right)^{\frac{n-1}{2}} \geq C_{3,1}|K|^{n-1}.
\]  

(5.18)

Therefore, for \( m \) large enough, we obtain from (5.16)–(5.18) that

\[
\mathcal{I} + 1 \geq H[H_m] = \omega_{n-1} \int_0^R H(t, H_m, \hat{H}_m) dt \geq C_{3,1} \omega_{n-1} \int_0^R |\hat{H}_m|^{n-1} dt.
\]

This, together with the uniformly boundedness of \( H_m \), i.e., \( r_s \leq H_m(t) \leq R_s \) in \( E \), implies that there exists a positive constant \( C_{3,2} = C_{3,2}(n, E, F) \) such that

\[
\|H_m\|_{\mathcal{W}^{1,n-1}(E)} \leq C_{3,2}.
\]

Since \( n - 1 \geq 3 \) and \( H_m \in \mathcal{X} \), we deduce from [5, Exercise 1.4.5] that there exists a subsequence (still denoted by \( H_m \)) such that \( H_m \) converges weakly to a continuous mapping \( H_0 \) (denoted by \( H_m \to H_0 \)) in \( \mathcal{W}^{1,n-1}(E) \) satisfying \( H_0(r) = r_s \) and \( H_0(R) = R_s \) (cf. [5, Pages 36 and 89]).

Moreover, it follows from the fact \( \|H_m\|_{\mathcal{W}^{1,n-1}(E)} \leq C_{3,2} \) and [8, Pages 161 and and 175] that there exists a subsequence of \( H_m \) (still denoted by \( H_m \)) such that \( H_m \to H_0 \) a.e. in \( E \) as \( m \to \infty \), which means that there exists a subset \( E' \subseteq E \) such that \( H_m \to H_0 \) in \( E \setminus E' \) as \( m \to \infty \), where the measure \( m(E') = 0 \). For any \( x, y \in E \setminus E' \) and \( x < y \), it follows that

\[
H_0(x) = \lim_{m \to \infty} H_m(x) \leq \lim_{m \to \infty} H_m(y) = H_0(y).
\]

This, together with the continuity of \( H_0 \) and the fact \( m(E') = 0 \), implies that \( H_0 \) is an increasing mapping in \( E \). Therefore, \( H_0 \in \mathcal{X} \).

**Step 2.** Lower semicontinuity. In the following, we show that \( \lim_{m \to \infty} H[H_m] \geq H[H_0] \) as \( H_m \to H_0 \) in \( \mathcal{W}^{1,n-1}(E) \). It follows from the fact

\[
\partial_{KK} \Lambda(t, H, K) = (n - 1)(n - 3) \frac{t^4 K^2}{H^4} + (n - 1) \frac{t^2 K^2}{H^2} \geq 0
\]

that \( \Lambda(t, H, K) \) is strictly convex in \( K \). Thus,

\[
\Lambda(t, H_m, \hat{H}_m) > \Lambda(t, H_m, \hat{H}_0) + \partial_K \Lambda(t, H_m, \hat{H}_0)(\hat{H}_m - \hat{H}_0).
\]  

(5.19)

Fix \( \delta > 0 \). Since \( H_m \to H_0 \) a.e. in \( E \) as \( m \to \infty \), then Egorov’s theorem asserts that \( H_m \to H_0 \) uniformly in \( E_\delta \) as \( m \to \infty \). Here \( E_\delta \) is a measurable subset of \( E \) with measure \( m(E - E_\delta) \leq \delta \). Now write

\[
F_\delta = \left\{ t \in E : |\hat{H}_0(t)| \leq \frac{1}{\delta} \right\}.
\]
Then \( m(E - F_\delta) \to 0 \) as \( \delta \to 0^+ \). Obviously, \( m(E - G_\delta) \to 0 \) as \( \delta \to 0^+ \), where \( G_\delta = E_\delta \cap F_\delta \). Since \( H_0 \in W^{1,n-1}(E) \), \( H_m(E) = F \) and \( H_m \to H_0 \) a.e. in \( E \), then the Lebesgue dominated convergence theorem tells us that
\[
\lim_{m \to \infty} \int_{G_\delta} \Lambda(t, H_m, \dot{H}_0) dt = \int_{G_\delta} \Lambda(t, H_0, \dot{H}_0) dt.
\]
(5.20)

In the following, we will prove
\[
\lim_{m \to \infty} \int_{G_\delta} \Lambda_K(t, H_m, \dot{H}_0)(\dot{H}_m - \dot{H}_0) dt = 0.
\]
(5.21)

For any \( \varrho \in F = [r_*, R_*] \), it follows from the fact \( |\dot{H}_0(t)| \leq \frac{1}{\varrho} \) in \( G_\delta \) that there exists a constant \( C_{3.3} = C_{3.3}(\varrho, n, E) \) such that \( \Lambda_K(t, \varrho, \dot{H}_0) \leq C_{3.3} \) in \( G_\delta \). Since \( H_m \to H_0 \) uniformly in \( G_\delta \) as \( m \to \infty \), we obtain that \( \Lambda_K(t, H_m, \dot{H}_0) \to \Lambda_K(t, H_0, \dot{H}_0) \) uniformly in \( G_\delta \) as \( m \to \infty \). Therefore, for any \( \varepsilon > 0 \), there exists a positive integer \( M_1 = M_1(\varepsilon, \delta, n, E, F) \) such that for any \( m > M_1 \) and \( t \in G_\delta \),
\[
|\Lambda_K(t, H_m, \dot{H}_0) - \Lambda_K(t, H_0, \dot{H}_0)| < \varepsilon.
\]
Hence, for each \( \delta > 0 \), (5.21) follows from \( \dot{H}_m \to \dot{H}_0 \) in \( L^{n-1}(E) \) and the fact that \( \Lambda_K(t, H_0, \dot{H}_0) \in L^{\frac{n-1}{n}}(E) \).

We deduce from (5.19)–(5.21) that
\[
\lim_{m \to \infty} \mathcal{H}[H_m] \geq \lim_{m \to \infty} \omega_{n-1} \int_{G_\delta} \Lambda(t, H_m, \dot{H}_m) dt \geq \omega_{n-1} \int_{G_\delta} \Lambda(t, H_0, \dot{H}_0) dt.
\]

Let \( \delta \to 0^+ \). We recall the monotone convergence theorem to conclude that
\[
\lim_{m \to \infty} \mathcal{H}[H_m] \geq \omega_{n-1} \int_R \Lambda(t, H_0, \dot{H}_0) dt = \mathcal{H}[H_0].
\]

**Step 3.** Since
\[
\lim_{m \to \infty} \mathcal{H}[H_m] = \inf_{H \in \mathcal{X}} \mathcal{H}[H], \quad \lim_{m \to \infty} \mathcal{H}[H_m] \geq \mathcal{H}[H_0]
\]
and \( H_0 \in \mathcal{X} \), we deduce that \( \mathcal{H}[H_0] = \min_{H \in \mathcal{X}} \mathcal{H}[H] \). The proof of Claim 5.2 is complete.

Next, we will prove that
\[
\mathcal{H}[H_0] = \min_{H \in \mathcal{B}} \mathcal{H}[H].
\]
(5.22)

By [14, Proposition 1.2], the continuity of \( H_0 \) and the fact \( H_0 \in W^{1,n-1}(E) \), we know that \( H_0 \) is absolutely continuous in \( E \). Recall that the fact \( H_0 \) minimizes the functional \( \mathcal{H}[H] \) implies that \( H_0 \) is a solution of
\[
\int_R \left( \frac{\partial}{\partial H} \Lambda(t, H, \dot{H}) \cdot \eta + \frac{\partial}{\partial \dot{H}} \Lambda(t, H, \dot{H}) \cdot \dot{\eta} \right) dt = 0
\]
for all \( \eta \in AC_0(E, \mathbb{R}) \), i.e., \( \eta \) is an absolutely continuous mapping from \( E \) into \( \mathbb{R} \) and we require that there exist \( r < r_1 \leq R_1 < R \) with \( \eta(t) = 0 \) if \( t \) is not contained in \( [r_1, R_1] \) (cf. [8, Page 13]). Since \( \Lambda \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^+ \times (\mathbb{R}^+ \cup \{0\})) \) and \( \Lambda_K > 0 \) in \( \mathbb{R}^+ \times \mathbb{R}^+ \times (\mathbb{R}^+ \cup \{0\}) \), then by [8, Page 17] we know that \( H_0 \in C^\infty(E) \). Hence, [8, Page 16] tells us that \( H_0 \) is a solution of (5.3). By Lemma 5.1, we know that \( H_0 = H_n \in \mathcal{B} \). This, together with Claim 5.2, implies that
\[
\inf_{H \in \mathcal{B}} \mathcal{H}[H] \geq \min_{H \in \mathcal{X}} \mathcal{H}[H] = \mathcal{H}[H_0] \geq \inf_{H \in \mathcal{B}} \mathcal{H}[H].
\]
Therefore, (5.22) is true.

Now, we are ready to finish the proof of the theorem. For any \( \lambda > 0 \), let 
\[ h^0_\lambda(t\xi) = H_0(t)\Phi^\lambda(\xi), \]
where \( t \in E \) and \( \xi \in S^{n-1} \). For any \( \lambda \neq 1 \), by (5.1), (5.2), and (5.22), we obtain that
\[
\min_{h \in R(\lambda, A^*)} \mathcal{E}[h] = \min_{H \in B} \mathcal{H}[H] = \mathcal{H}[H_0] = \mathcal{E}[h^1_0] > \mathcal{E}[h^0_1].
\]
Hence
\[
\inf_{h \in P(\lambda, A^*)} \mathcal{E}[h] \leq \mathcal{E}[h^0_\lambda] < \mathcal{E}[h^1_0] = \min_{h \in R(\lambda, A^*)} \mathcal{E}[h] = \mathcal{H}[H_0] = \mathcal{H}[H_*],
\]
where \( \mathcal{H}[H_*] \) is the constant from (5.5). The proof of the theorem is complete. \( \Box \)

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