The new definition of lattice gauge fields and the Landau gauge

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The Landau gauge fixing algorithm in the new definition of gauge fields is presented. In this algorithm a new solver of the Poisson equations based on the Green’s function method is used. Its numerical performance of the gauge fixing algorithm is presented. Performance of the smeared gauge fixing in SU(3) is also investigated.

1. The Landau gauge fixing algorithm in the new definition of gauge fields

We define the gauge field\cite{1,2} on links as an element of SU(3) Lie algebra as,

\[ e^{A_{x,\mu}} = U_{x,\mu} \]

where \( A_{x,\mu} \) is Hermitian and it is more natural as an averaged gauge field on the link in the continuum theory. We use the analytic method of eigenspace projection operator together with Cardano’s eigenvalue solver for transformation between \( U \) and \( A \). It is remarkable that the definition of gauge field allows the same type of formulation of the lattice Landau or Coulomb gauge condition as the continuum theory, i.e., minimizing condition of \( \|A\|^2 \) along the gauge orbit. The definition \( A = \log U \) requires a proper choice of branch for traceless \( A \), and \( A \) becomes a nonanalytic function of \( U \), and yet \( \|A\|^2 \) still remains a continuous function of \( U \).

This fact justifies the existence of the global minimum of \( \|A\|^2 \) along the gauge orbit (the minimal Landau gauge condition).

The gauge transformation is

\[ e^{A_{x,\mu}^g} = g^\dagger_x e^{A_{x,\mu}} g_{x+\mu} \]

where \( g_x = e^{\epsilon_x} \) and \( \epsilon_x \) is a traceless antihermitian matrix.

When \( \epsilon \) is infinitesimal

\[ \delta A_{x,\mu} = (D\epsilon)_{x,\mu} = f(\text{adj} A_{x,\mu}) \partial \epsilon_{x,\mu} + [A_{x,\mu}, \bar{\epsilon}_{x+\mu}/2] \]

where

\[ f(x) = \frac{x/2}{\tanh(x/2)} \quad \partial \epsilon_{x,\mu} = \epsilon_{x+\mu} - \epsilon_x \]

and

\[ \bar{\epsilon}_{x+\mu}/2 = \frac{1}{2}(\epsilon_{x+\mu} + \epsilon_x) \]

Gribov region \( \Omega \) is defined as a set of local minimum points on the gauge orbit \( F \)

\[ \Delta \|A\|^2 = -2(\partial A|\epsilon) + (\epsilon| - \partial D|\epsilon) + \cdots \]

\[ \Omega = \{ A| - \partial D \geq 0 \quad \partial A = 0 \} \]

The minimal Landau gauge condition defines the fundamental modular region by the absolute minimum along the gauge orbit,

\[ \Lambda = \{ A| \|A\|^2 = \text{Min}_{g} \|A^g\|^2 \}, \quad \Lambda \subset \Omega \]

The equation to solve for the Landau gauge fixing is highly nonlinear,

\[ \partial A^g = 0 \quad \text{where} \quad g = e^{\epsilon} \]

Linearizing this equation, we can solve it iteratively by the Newton method with an over-relaxation factor \( \eta \),

\[ \epsilon = (-\partial D(A))^{-1} \eta \partial A \]

This is a non-Abelian extension of the Fourier acceleration type. We make a perturbation
expansion with respect to $A$ for the inverse of the covariant Laplacian, as
\[ (-\partial D(A))^{-1} \cong \sum_{n=0}^{N_{end}} ((-\partial^2)^{-1} \partial \hat{D})^n (-\partial^2)^{-1} \] (10)
where $\hat{D} = D(A) - \partial$.

Performance check for various $N_{end}$ is given below.

2. The new solver of the Poisson equation

The inversion of the Laplacian is an important problem in numerical analyses, and performances of various methods have been investigated\(^8\). Efficiency of the methods depends on types of the problem posed, i.e., balance of accuracy and CPU time. We report a new trial, use of the Green function. While high accuracy being maintained, direct use of the Green function in a single machine computation requires a work of order $V^2$, and is obviously inefficient in larger size $V$. We devise a method that makes a partial use of the Green function for coarse lattice with a preconditioning for the Poisson equation.

Let us consider d dimensional Poisson equation,
\[ -\partial^2 \phi = 2d(I - A)\phi = \rho, \] (11)
where $A$ stands for an averaging operation on neighboring sites. A preconditioned equaiton is given as
\[ 2d(I - A^2)\phi = (I + A)\rho, \] (12)
where $(I - A^2)$ becomes block-diagonal on even(odd)-site. After solving the even-site equation (12), we use (11) for $\phi$ on odd-site.

The Green function for even(odd)-site equation (11) is defined as
\[ 2d(I - A^2)G = \delta - \frac{1}{V/2}. \] (13)
we found that a good approximant $\hat{G}$ of the Green function $G$ can be given by the Green function $G_c$ on coarse lattice with volume $V/2^d$, as
\[ \hat{G} = (a_0 + a_1 A^2 + a_2 A^4) \delta - c + b A^6 G_c, \] (14)
where $a_0$, $a_1$, $a_2$, $b$ and $c$ are optimized parameters. We can add an extra function $\Delta G$ for a higher accuracy correction. For an accuracy, $10^{-4}$, of the Green function $G$ on $8^3 \times 16$ lattice, numbers of sites for supports of $G_c$ and $\Delta G$ are 256 and 162, respectively, out of 4096 of full even-site. Owing to the high accuracy of the above Green function algorithm, this Poisson equation solver allows the over-relaxation factor $\eta$ to be a larger value, 1.5 to 1.8 in gauge fixing algorithm (9) and (10) and gives a 30% faster performance than that in the usual conjugate gradient solver.

The following Fig. 1 shows processes of gauge fixing in use of the Green function method and the conjugate gradient method.

![Figure 1](image-url)

Figure 1. A comparison of $\log_{10} \text{Max} |\partial A|$ in the conjugate gradient method(dashed) and the Green’s function method(continuous).

3. Numerical performance of the gauge fixing algorithm

We compare performance of the gauge fixing algorithm (9) and (10) for various $A$ perturbation orders $N_{end}$ in Fig. 2, which shows that the non-Abelian extension improves efficiency. Relative CPU time ratios over that of $N_{end} = 3$ case are 1.05 for $N_{end} = 4$ case and 1.91 for $N_{end} = 2$ case.
In general, the Fourier acceleration type of methods accomplish much better performance in ultimate precision over simple norm-minimizing methods, and our method is free of critical slowing down.

4. Smeared gauge fixing in SU(3)

As is well known, the Landau gauge fixing suffers from the problem of Gribov’s copies, that is, the configuration $A^g$ is captured by the plenty of local minima of $\|A^g\|^2$ along the gauge orbits. In order to avoid noise of those Gribov’s copies when measuring, e.g., the gluon propagators etc., the method of gauge fixing to the global minimum of the $\|A^g\|^2$ is necessary. Hetrick and de Forcrand proposed a clever method of smeared gauge fixing\[9\]. They investigated performance of the smeared gauge fixing in case of SU(2), and reported that it works well for large $\beta$. Here we report that it works very well also for SU(3) in case of large $\beta$. We observed that it works perfectly in gauge fixing to the global minimum of $\|A^g\|^2$, by using 50 configurations generated by random gauge transformation from a sample in $\beta = 5$ on $4^3 \times 8$ lattice, all of which are transformed to the unique minimum.

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