UNIQUENESS FOR SOME CLASSES OF PARABOLIC PROBLEMS

F. FEO

ABSTRACT. We prove some uniqueness results for weak solutions to some classes of parabolic Dirichlet problems.

1. INTRODUCTION

In the present paper we investigate the uniqueness of weak solutions to the following class of parabolic Dirichlet problems

\[
\begin{aligned}
&u_t - \text{div} (a(x, t, u, \nabla u)) + H(x, t, \nabla u) + G(x, t, u) = f &\quad &\text{in} & &Q_T = \Omega \times (0, T) \\
&u(x, t) = 0 &\quad &\text{in} & &\partial \Omega \times (0, T) \\
&u(x, 0) = u_0(x) &\quad &\text{on} & &\Omega,
\end{aligned}
\]

where \( \Omega \) is a bounded open subset of \( \mathbb{R}^N \), \( N \geq 2 \), \( p > 1 \) and \( T > 0 \). We assume that \( a : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}, H : Q_T \times \mathbb{R}^N \to \mathbb{R} \) and \( G : Q_T \times \mathbb{R} \to \mathbb{R} \) are Carathéodory functions satisfying the following structural conditions

\[
\begin{aligned}
& a(x, t, s, \xi) \cdot \xi \geq \alpha_1 |\xi|^p, \\
& |a(x, t, s, \xi)| \leq \beta_1 \left[ |s|^{p-1} + |\xi|^{p-1} + a_1(x, t) \right], \\
& (a(x, t, s, \xi) - a(x, t, s, \xi')) (\xi - \xi') > 0 \quad \text{for} \ \xi \neq \xi', \\
& |H(x, t, \xi)| \leq b(x, t) |\xi|^{\gamma},
\end{aligned}
\]

and

\[
\begin{aligned}
& |G(x, t, s)| \leq c(x, t) |s|^\lambda
\end{aligned}
\]

for a.e. \( (x, t) \in Q_T, \forall s \in \mathbb{R}, \forall \xi, \xi' \in \mathbb{R}^N \), where \( \alpha_1 \) and \( \beta_1 \) are positive constants, \( \gamma = p - \frac{N+2}{N+1} \), \( \lambda = p \frac{N+2}{N+1}, a_1 \in L^p(Q_T), b \in L^r(Q_T) \) and \( c \in L^\rho(Q_T) \) with \( r = N+2 \) and \( \rho = \frac{N+2}{N+1} \). Moreover \( f = f_0 - \text{div} F \) with

\[
\begin{aligned}
& f_0 \in L^{\left( \frac{N(N+2)}{N+2} \right)}(Q_T), F \in \left( L^{p'}(Q_T) \right)^N &\quad &\text{and} & &u_0 \in L^2(\Omega).
\end{aligned}
\]

2000 Mathematics Subject Classification. 35K55, 35K20, 35R05.

Key words and phrases. Uniqueness, Parabolic operator, Weak solutions.
We recall that a weak solution\footnote{We refer to \cite{1} for definitions of involved function spaces and parabolic framework.} to Problem (1.1) is a measurable function belonging to $C(0, T; L^2(\Omega)) \cap L^p(0, T; W^{1,p}_0(\Omega))$ such that $\forall t \in (0, T]$ 
\begin{equation}
(1.8) \quad \int_\Omega uv(x, t)dx + \int_{Q_t} [-uv_t + a(u, \nabla u)v + H(\nabla u)v + G(u)v]dxd\tau
\end{equation}
\begin{equation}
= \int_\Omega u_0v(x, 0)dx + \int_{Q_t} (f_0v + F\nabla v)dxd\tau,
\end{equation}
for each $v \in W^{1,2}_0(0, T; L^2(\Omega)) \cap L^p(0, T; W^{1,p}_0(\Omega))$, where $Q_t := \Omega \times (0, t)$.

If the function $a$ does not depend on $u$ and $H \equiv G \equiv 0$, we known there exists a unique weak solution (see e.g. \cite{22}). Here we prove the uniqueness of weak solutions to Problem (1.1) when $H \equiv 0$ and $G \neq 0$ assuming that operator $-\text{div } a(x, t, u, \nabla u)$ is strongly monotone and the functions $a$ and $G$ are Lipschitz continuous with respect to $u$ (that is usual as far as uniqueness result concern). When $p \geq 2$ we assume the principle part is not degenerate, i.e. in the model case $-\text{div } a(x, t, u, \nabla u) = -\text{div}(a_0(x, t, u)(\varepsilon + |\nabla u|^{p-1}\nabla u))$ with $\varepsilon > 0$. In this case we can relax hypothesis on the function $a$ assuming only a locally Lipschitz continuity with respect to $u$ (see Section 2 for details). To our knowledge in literature there are not any existence results for weak solutions to Problem (1.1) when $H \equiv 0$ and $G \neq 0$, then we will give some details (see Proposition 3.1) for convenience of the reader.

When $H \neq 0$ and $G \equiv 0$, the existence of a weak solution to problem (1.1) is investigated in \cite{23}. If $a$ does not depend on $u$ and under Lipschitz continuity on the lower order term $H$, we prove that such a solution is unique.

Our proof of uniqueness adapts the idea of \cite{1} (used also in \cite{18} for an anisotropic elliptic operator) to the evolution case: the main tool is the embedding in the parabolic equation framework. In the case $H \equiv 0$ and $G \neq 0$ the method is improved using Gronwall’s Lemma. Our technique works also in proving some comparison principles.

There is an extensive literature about uniqueness of solution for elliptic equations. We just mention some of these papers: e.g. \cite{2}, \cite{3}, \cite{4}, \cite{7}, \cite{8}, \cite{9}, \cite{12}, \cite{13}, \cite{14} and \cite{19}. For the evolution case some uniqueness results can be found for example in \cite{6}, \cite{20}, \cite{21} and \cite{24} in the framework of weak solutions. When datum $f$ is only integrable, uniqueness of renormalized and entropy solutions is proved for example in \cite{5}, \cite{10}, \cite{11}, \cite{17} and \cite{25}.

2. Statements of results

First we study the case $H \equiv 0$, that is we consider the following class of nonlinear parabolic homogeneous Dirichlet problems
\begin{equation}
(2.1) \quad \begin{cases}
u \quad u_t - \text{div } a(x, t, u, \nabla u) + G(x, t, u) = f & \text{in } Q_T \\
\quad u(x, t) = 0 & \text{in } \partial \Omega \times (0, T) \\
\quad u(x, 0) = u_0(x) & \text{on } \Omega,
\end{cases}
\end{equation}
when (1.2)- (1.4) and (1.6)-(1.7) hold, function $a$ also satisfies the following strong monotony condition
\begin{equation}
(2.2) \quad (a(x, t, s, \xi) - a(x, t, s, \xi')) (\xi - \xi') \geq \alpha (\varepsilon + |\xi| + |\xi'|)^{p-2} |\xi - \xi'|^2
\end{equation}
and the following Lipschitz continuity condition
\begin{equation}
|a(x, t, s, \xi) - a(x, t, s', \xi)| \leq \beta \left[ \phi + |\xi|^{p-1} + (|s| + |s'|)^\theta \right] |s - s'|
\end{equation}
and function $G$ also satisfies the following Lipschitz continuity condition
\begin{equation}
|G(x, t, s) - G(x, t, s')| \leq \varrho |s - s'|
\end{equation}
for some $\varrho \geq 0$ with $\alpha > 0$, $\varepsilon \geq 0$, $\beta, \varrho > 0$ and $\phi \geq 0$.

We investigate separately the case $1 < p < 2$ and $p \geq 2$.

**Theorem 2.1.** Let us assume $1 < p < 2$, (1.2), (1.5), (1.7), (2.2) with $\varepsilon = 0$, (2.3) with $\theta = 0$ and (2.4) hold. Then there exists a unique weak solution to Problem (2.1).

**Theorem 2.2.** Let us assume $p \geq 2$, (1.2), (1.3), (1.5), (1.7), (2.2) with $\varepsilon > 0$, (2.3) with $0 \leq \theta \leq \frac{p(N+2)}{2N}$ and (2.4) hold. Then there exists a unique weak solution to Problem (2.1).

**Remark 2.1.** Theorem 2.1 holds if we replace $|s - s'|$ in (2.3) and (2.4) by $\omega(|s - s'|)$, where $\omega : [0, +\infty] \to [0, +\infty]$ is such that $\omega(s) \leq s$ for $0 \leq s \leq \kappa$ for some $\kappa > 0$.

Analogous generalization holds for Theorem 2.2.

Moreover in order to prove uniqueness results for problems with lower order term $H(x, t, \nabla u)$ we suppose function $a$ does not depend on $u$ and $G \equiv 0$. More precisely we study the following class of nonlinear parabolic homogeneous Dirichlet problems
\begin{equation}
\begin{aligned}
&u_t - \div a(x, t, \nabla u) + H(x, t, \nabla u) = f & \text{in } Q_T \\
u(x, t) = 0 & \text{in } \partial \Omega \times (0, T) \\
u(x, 0) = u_0(x) & \text{on } \Omega,
\end{aligned}
\end{equation}
when (1.2), (1.5) and (1.7) hold. We know (see 23) there exists at least a weak solution to Problem (2.5). As usual we assume the following locally Lipschitz condition on $H$
\begin{equation}
[H(x, t, \xi) - H(x, t, \xi')] \leq h(x, t) (\eta + |\xi| + |\xi'|)^\sigma |\xi - \xi'|
\end{equation}
with $\sigma \in \mathbb{R}, \eta \geq 0$ and $h$ a suitable function.

We investigate separately the case $p < 2$ and $p \geq 2$.

**Theorem 2.3.** Let us assume $\frac{2N}{N+2} \leq p < 2$, (1.2), (1.3), (1.5), (1.7), (2.2) with $\varepsilon = 0$ and (2.6) with $h \in L^\infty(Q_T), \eta > 0$ and $\sigma \leq \frac{p-2}{2}$ hold. Then there exists a unique weak solution to Problem (2.5).

**Theorem 2.4.** Let us assume (1.2), (1.5), (1.7), (2.2) with $\varepsilon > 0$ and (2.6) with $h \in L^r(Q_T)$ for $r \geq N+2, \eta = 0$ and $0 \leq \sigma \leq p \left( \frac{1}{N+2} - \frac{1}{r} \right) + \frac{p-2}{2}$ hold. Then for $2 \leq p \leq \frac{2(N+2)}{(N+2)(N+2)-2}$ there exists a unique weak solution to Problem (2.5).

**Remark 2.2.** If $p > \frac{2(N+2)}{(N+2)(N+2)-2}$, then Theorem 2.4 holds with $0 \leq \sigma \leq \frac{p}{N+2} - \frac{p}{r} + \frac{p-2}{2}$.

**Remark 2.3.** If in Problem (2.5) we add extra term $G(x, t, u)$ that is an increasing function in the variable $u$, the uniqueness of weak solutions can be proved under hypothesis of Theorems 2.3 and 2.4.
The arguments used in the proofs of previous theorems allows us to obtain also some comparison principles.

**Corollary 2.1.** (Comparison principle) In the hypothesis of Theorems 2.1 and 2.2, let us assume $u$ and $v$ are two solutions to Problem (2.1) such that $u(x, 0) \leq v(x, 0)$ a.e. in $\Omega$. Then $u_1 \leq u_2$ a.e. in $Q_T$.

**Corollary 2.2.** (Comparison principle) In the hypothesis of Theorems 2.3 and 2.4, let us assume $u$ and $v$ are two solutions to Problem (2.5) such that $u(x, 0) \leq v(x, 0)$ a.e. in $\Omega$. Then $u_1 \leq u_2$ a.e. in $Q_T$.

3. Operators with a zero order term

In this section we study Problem (2.1) when (1.2)-(1.4) and (1.6)-(1.7) hold.

3.1. Some preliminary results. In order to prove theorems of the previous section we need to recall the following embedding in the parabolic framework.

**Lemma 3.1.** (see Proposition 3.1 of [13]) Let $u \in L^\infty(0, T; L^p(\Omega)) \cap L^p(0, T; W_0^{1, p}(\Omega))$ with $p \geq 1$ and $\rho \geq 1$. Then $u \in L^p(Q_T)$ with $q = \frac{N\rho + p}{N}$ and there exists a constant $C_p$ that depends on $N$ and $p$ such that

$$
\|u\|_{L^q(Q_T)} \leq C_p \left( \sup_{0 < t < T} \|u(\cdot, t)\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(Q_T)} \right).
$$

Moreover it results

$$
\iint_{Q_T} |u|^q \leq C_p^q \left( \sup_{0 < t < T} \int_\Omega |u|^p \right)^{\frac{q}{p}} \iint_{Q_T} |\nabla u|^p.
$$

Moreover in the proof of uniqueness result for Problem (2.1) we need the following version of Gronwall lemma.

**Lemma 3.2.** Let $T > 0$ and let $a, d$ be non-decreasing functions belonging to $L^1_{loc}(\mathbb{R}_+)$, $b \in L^\infty_{loc}(\mathbb{R}_+)$ and $z \in L^1_{loc}(\mathbb{R}_+)$ such that

$$
z(t) \leq a(t) + d(t) \int_0^t b(s) z(s) \, ds \quad \text{for a.e. } t \in [0, T],
$$

then

$$
z(t) \leq a(t) \left[ 1 + d(t) \int_0^t b(s) \exp \left( d(t) \int_0^s b(s) \, ds \right) \right] \quad \text{for a.e. } t \in [0, T].
$$

3.2. Existence of a weak solution. To our knowledge in literature there are not existence results for weak solutions to Problem (2.1).

**Proposition 3.1.** Under assumptions (1.2)-(1.4) and (1.6)-(1.7) there exists at least a weak solution $u \in L^\infty(0, T; L^2(\Omega)) \cap L^p(0, T; W_0^{1, p}(\Omega))$ to Problem (2.1).

The proof is standard but for convenience of reader we will give here some steps. We observe that the coercivity of the operator is guaranteed only if the norm $\|c\|_{L^p(Q_T)}$ is small enough. Then as usual we consider the approximate problems

$$
\begin{cases}
(u_n)_t + \Delta u_n = f_n - \text{div } F & \text{in } Q_T \\
u_n(x, t) = 0 & \text{in } \partial \Omega \times (0, T) \\
u_n(x, 0) = u_0(x) & \text{on } \Omega,
\end{cases}
$$

where $u_n$ is solution to the following problem:

$$
\begin{cases}
(u_n)_t + \Delta u_n = f_n & \text{in } Q_T \\
u_n(x, t) = 0 & \text{in } \partial \Omega \times (0, T) \\
u_n(x, 0) = u_0(x) & \text{on } \Omega,
\end{cases}
$$

for some sequence $(f_n)_{n \in \mathbb{N}}$ of functions such that $f_n \to f$ in $L^p(0, T; L^2(\Omega))$. Then the weak limit of the approximate solutions $u_n$ is a weak solution to the original problem. The details are omitted for brevity.
where \( L_n u = - \text{div} a(x, t, u, \nabla u) + G_n(x, t, u) \), \( G_n(x, t, s) = T_n(G(x, t, s)) \), \( T_n \) is the truncation at level \( \pm n \), defined by
\[
T_n(s) = \max \{-u, \min \{n, s\}\}
\]
and \( \{f_n\}_{n \in \mathbb{N}} \subset L^{p'}(Q_T) \) such that with \( f_n \to f_0 \) strongly in \( L^{q'}(Q_T) \) with \( q = \frac{(N+2)}{N} \). The operator \( L_n \) is pseudomonotone and coercive, then (see e.g. \[22\]) there exists a weak solution \( u_n \in L^\infty(0, T; L^2(\Omega)) \cap L^p(0, T; W^{1,p}_0(\Omega)) \). The following a priori estimate of \( u_n \) holds.

**Lemma 3.3.** Assume that \( (1.2)-(1.4) \) and \( (1.6)-(1.7) \) hold. If \( u_n \) is a weak solution to Problem \((3.4)\), then there exists a constant \( C_0 \) (depending on the data that appear in the structure conditions but not on \( n \)) such that
\[
\sup_{0 < t < T} \|u_n\|_{L^2(\Omega)} + \|\nabla u\|_{L^p(Q_T)} \leq C_0.
\]

**Proof.** Using \( u_n \) as test function in Problem \((3.4)\), under assumptions \((1.2)\) and \((1.6)\) we obtain for \( t \in (0, T) \)
\[
\frac{1}{2} \int_\Omega u_n^2(t) dx + \alpha_1 \int_{Q_T} |\nabla u_n|^p dxd\tau \leq \int_{Q_T} f_n u_n + F \nabla u_n dxd\tau.
\]
Using Hölder inequality, \((3.1)\) and Young inequality we have
\[
\int_{Q_T} (f_0 u_n + F \nabla u_n) dxd\tau \leq \frac{\alpha_1}{p} \|\nabla u\|_{L^p(Q_1)} + \kappa_1 \|F\|_{L^{p'}(Q_1)} + \kappa_2 \|f_n\|_{L^{p'}(Q_1)}
\]
for some positive constant \( \kappa_1, \kappa_2 \) and \( \kappa_3 \) with \( \kappa_3 < \min \{\frac{1}{2}, \frac{\alpha_1}{p}\} \). Using \((3.1)\) and Young inequality, we get
\[
\int_{Q_T} c(x, \tau)|u|^\lambda dxd\tau \leq \|c\|_{L^{p'}(Q_1)} \left( \sup_{0 < \tau < t} \int_\Omega u_n^2(\tau) dx \right)^{\frac{p}{p'} - 1} \left( \int_{Q_T} |\nabla u_n|^p dxd\tau \right)^{\frac{1}{p'}}
\]
for some positive constant \( \kappa_4 \). Using \((3.8)\) and \((3.9)\) in \((3.7)\) and taking the supremum on \((0, t_1)\) for some \( t_1 \leq T \) such that \( \|c\|_{L^{p'}(Q_{t_1})} \) is small enough, we obtain
\[
\sup_{t \in (0, t_1]} \int_\Omega u_n^2(t) dx + \int_{Q_{t_1}} |\nabla u_n|^p dxd\tau \leq \kappa_5 \left[ \int_\Omega u_0^2 dx + \||F\||_{L^{p'}(Q_{t_1})} + 1 \right],
\]
for some positive constant \( \kappa_5 \), since \( \{f_n\}_{n \in \mathbb{N}} \) is bounded in \( L^{p'}(Q_T) \). In order to avoid the assumption on smallness of the norm \( \|c\|_{L^{p'}(Q_T)} \) we split (see also \[22\] and \[16\]) the interval \([0, T]\) in \( M \) small subintervals \((t_i, t_{i+1})\) for \( i = 0, \ldots, M - 1 \) in such a way \( \|c\|_{L^{p'}(Q \times (t_i, t_{i+1}))} \) is small enough. We are able to derive an estimate like \((3.10)\) for small cylinder \( \Omega \times (t_i, t_{i+1}) \). Finally taking the sum of different iterations, \((3.6)\) holds for the inter cylinder \( Q_T \). 
\[ \square \]
Proof of Proposition 3.1. By Lemma 3.3 it follows that \( \{u_n\}_{n \in \mathbb{N}} \) is bounded sequence of \( L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_{0,0}^{1,p}(\Omega)) \). Then it is possible to proceed as in the proof of Theorem 2.2 of [23] to pass to the limit in (3.4) and to conclude the existence of at least a weak solution to Problem (2.1).

We explicitly write only the computation about the boundness of \( G_n(x,t,u_n) \) in \( L^q(Q_T) \). Ended by (1.0), Hölder inequality and (3.1) we have

\[
\int_\Omega G_n(x,t,u_n) \cdot \varphi \, dx \leq \int_\Omega c(x,t)^q |u_n|^\lambda q \, dx + \frac{1}{\lambda q} \int_\Omega c(x,t)^{\lambda q} \, dx \\
\leq \|c\|_{L^p(\Omega)}^{1-\frac{q}{\lambda}} \left[ C_p \left( \sup_{0 < t < T} \|u_n(.,t)\|_{L^2(\Omega)} + \|\nabla u_n\|_{L^p(\Omega)} \right) \right]^{\lambda q}.
\]

By Lemma 3.3 the norm of \( G_n(x,t,u_n) \) in \( L^q(Q_T) \) is bounded by a constant depending on the data that appear in the structure conditions but not on \( n \).

3.3. Uniqueness of weak solutions. In this subsection we prove \textit{ab aburdo} the uniqueness of weak solutions to Problem (2.1).

Proof of uniqueness in the hypothesis of Theorem 2.1. We argue by contradiction. Let us assume that Problem (2.1) admits two different solutions \( u \) and \( v \) and \( D = \{(x,t) \in Q_T : w > 0\} \) has positive measure, where \( w = u - v \). Using \( \varphi = \frac{T_k(w^+)}{k} \) for \( k \in \mathbb{N}, \sup_{D} w^+ \) as test function in the difference of the equations (where \( T_k(\cdot) \) is defined in (3.3)), we obtain for \( t \in (0,T) \)

\[
\int_\Omega w \varphi + \int_\Omega \{ -w \varphi_t + [a(x,t,u,\nabla u) - a(x,t,v,\nabla v)] \nabla \varphi + [c(x,t,u) - c(x,t,v)] \varphi \} = 0.
\]

Let us denote \( \Psi_k(s) = \int_0^s T_k(\sigma) \, d\sigma \). We have that

\[
(3.11) \quad \int_\Omega w \varphi - \int_\Omega w \varphi_t = \frac{1}{k} \int_\Omega \Psi_k(w^+(t))
\]

for \( k > 0 \). By (2.2), (2.3), (2.4) (3.11) we get

\[
(3.12) \quad \frac{k}{2} \int_\Omega \Psi_k(w^+(t)) + \alpha \int_\Omega \frac{|\nabla \varphi|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \\
\leq \beta \int_\Omega \left( \phi + |\nabla v|^{p-1} \right) |\nabla \varphi| + \frac{2}{k} \int_\Omega |w| \varphi,
\]

where \( D_k = \{(x,t) \in D : w^+ < k\} \). Using Young inequality with some \( \delta > 0 \) it follows

\[
(3.13) \quad \int_{Q_t \cap D_k} \left( \phi + |\nabla v|^{p-1} \right) |\nabla \varphi| \leq \frac{\delta (\phi + 1)}{2} \int_{Q_t \cap D_k} \frac{|\nabla \varphi|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \\
+ \frac{\phi}{4\delta} \int_{Q_t \cap D_k} (|\nabla u| + |\nabla v|)^{2-p} + \frac{1}{4\delta} \int_{Q_t \cap D_k} (|\nabla u| + |\nabla v|)^p.
\]

Choosing \( \delta \) small enough, using (3.13) and Young inequality in (4.6) and noticing that \( \Upsilon(s) = 2\Psi_k(s) - sT_k(s) \geq 0 \) for \( s \geq 0 \) (check that \( \Upsilon(s) = 0 \) for \( 0 \leq s \leq k \) and...
UNIQUENESS FOR SOME... 7

(3.14) \[
\frac{1}{k^2} \int_{\Omega} \Psi_k(w^+(t)) + c_1 \int_{Q_t \cap D_k} \frac{|
abla \varphi|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \\
\leq c_2 \left( |D_k \cap Q_t| + \int_{Q_t \cap D_k} \frac{|
abla u| + |\nabla v|}{(|\nabla u| + |\nabla v|)^p} + \frac{1}{k^2} \int_{Q_t} \Psi_k(w^+) \right)
\]

for some positive constants \(c_1, c_2\) independent on \(k\).

Using Gronwall inequality (3.3) and taking the supremum on \(t \in (0, T)\), we get

(3.15) \[
\frac{1}{k^2} \sup_{t \in (0, T)} \int_{\Omega} \Psi_k(w^+(t)) \leq c_2 \left( 1 + Te^T \right) \left( |D_k| + \int_{D_k} (|\nabla u| + |\nabla v|)^p \right). 
\]

It is easy to check that

(3.16) \[
\zeta_1(k) := \left[ |D_k| + \int_{D_k} (|\nabla u| + |\nabla v|)^p \right] \rightarrow 0
\]
when \(k\) goes to zero, then

(3.17) \[
\lim_{k \to 0} \frac{1}{k^2} \sup_{t \in (0, T)} \int_{\Omega} \Psi_k(w^+(t)) = 0.
\]

Recalling that \(\frac{1}{2} |T_k(s)|^2 \leq \Psi_k(s)\), we have

(3.18) \[
\lim_{k \to 0} \sup_{t \in (0, T)} \int_{\Omega} |\varphi|^2 = 0.
\]

Coming back to (3.14), taking the supremum and using (3.17) and (3.16), we obtain

(3.19) \[
\lim_{k \to 0} \int_{D_k} \frac{|
abla \varphi|^2}{(|\nabla u| + |\nabla v|)^{2-p}} = 0.
\]

Moreover inequality (3.2) and Hölder inequality imply

\[
C_1^{-\frac{N+2}{N}} |D \setminus D_k| \leq C_1^{-\frac{N+2}{N}} \left( \sup_{\in (0, T)} \int_{\Omega} |\varphi|^2 \right)^\frac{N}{N+2} \left( \int_{D_k} \frac{|
abla \varphi|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \right)^\frac{N}{N+2} \left( \int_{D_k} \frac{|\nabla \varphi|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \right)^\frac{N}{N+2}.
\]

By (3.16), (3.18) and (3.19) it follows

(3.20) \[
|D| = \lim_{k \to 0} |D \setminus D_k| = 0.
\]

To complete the proof it suffices to replace \(u\) and \(v\).

**Proof of uniqueness in the hypothesis of Theorem 2.2.** Arguing as in the previous proof and taking into account the following extra term

\[
\int_{Q_t \cap D_k} (|u| + |v|)^\theta |\nabla \varphi|,
\]


Using Young inequality with some $\delta > 0$, we have the analogue of (3.13), i.e.

\[
\int_{Q_t \cap D_k} \left[ \phi + |\nabla v|^{p-1} + (|u| + |v|)^\theta \right] |\nabla \varphi|^2 \leq \frac{\delta}{2} \left[ \frac{(\phi + 1)}{\varepsilon^{p-2}} + 1 \right] \int_{Q_t \cap D_k} (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} |\nabla \varphi|^2 + \frac{\phi}{45} |Q_t \cap D_k| + \frac{1}{45} \int_{Q_t \cap D_k} |\nabla v|^p + \frac{1}{45} \int_{Q_t \cap D_k} (|u| + |v|)^{2\theta}.
\]

Choosing $\delta$ small enough in (3.22), inequality (3.21) gets

\[
\int_{\Omega} \Psi_k(w^+(t)) + \alpha \int_{Q_t \cap D_k} (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} |\nabla \varphi|^2 \leq \beta \int_{Q_t \cap D_k} \left[ \phi + |\nabla v|^{p-1} + (|u| + |v|)^\theta \right] |\nabla \varphi| + \frac{2\phi}{k^2} \int_{Q_t} \Psi_k(w^+).
\]

Using Young inequality with some $\delta > 0$, we have the analogue of (3.13), i.e.

\[
\int_{Q_t \cap D_k} \left[ \phi + |\nabla v|^{p-1} + (|u| + |v|)^\theta \right] |\nabla \varphi|^2 \leq \frac{\delta}{2} \left[ \frac{(\phi + 1)}{\varepsilon^{p-2}} + 1 \right] \int_{Q_t \cap D_k} (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} |\nabla \varphi|^2 + \frac{\phi}{45} |Q_t \cap D_k| + \frac{1}{45} \int_{Q_t \cap D_k} |\nabla v|^p + \frac{1}{45} \int_{Q_t \cap D_k} (|u| + |v|)^{2\theta}.
\]

Choosing $\delta$ small enough in (3.22), inequality (3.21) gets

\[
\int_{\Omega} \Psi_k(w^+(t)) + c_1 \int_{Q_t \cap D_k} (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} |\nabla \varphi|^2 \leq c_2 \left[ |D_k| + \int_{D_k} |\nabla v|^p + \int_{D_k} (|u| + |v|)^{2\theta} \right].
\]

for some positive constants $c_1, c_2$ independent on $k$. We remark that since $2\theta \leq \frac{2(N+2)}{N}$

(3.24) $\zeta_2(k) := |D_k| + \int_{D_k} |\nabla v|^p + \int_{D_k} (|u| + |v|)^{2\theta} \rightarrow 0$

when $k$ goes to zero. Arguing as in the next proof we obtain (3.13) and

(3.25) $\lim_{k \rightarrow 0} \int_{D_k} |\nabla \varphi|^2 = 0$.

Moreover inequality (3.1) and Young inequality imply

\[
C_1^{-1} |D \setminus D_k|^{\frac{N+2}{2}} \leq \sup_{t \in (0, T)} \left( \int_{\Omega} |\varphi|^2 \right)^{\frac{1}{2}} + \frac{1}{2} \int_{D_k} |\nabla \varphi|^2 + \frac{1}{2} |D_k|.
\]

By (3.26), (3.16), (3.14) and (3.15) it follows (3.26). Then the assert holds changing the role of $u$ and $v$.

\[\square\]

Remark 3.1. If $G(x, t, u) = c(x)u$ the previous proofs follow easier multiplying equation by $\exp(tc(x))$ and using as test function $\varphi = \frac{T_k((u - v)^+ \exp(tc(x)))}{k}$.

Proof of Corollary 2.1. We can argue as in the previous proofs of uniqueness putting $w = (u - v)^+$.

\[\square\]

4. Operators with a first order term

We known that there exists at least a weak solution to Problem (2.5). Here we have to prove only the uniqueness.
Proof of Theorem 2.3. Let us suppose that \( u \) and \( v \) are two weak solutions to Problem 2.3 belonging to \( L^\infty(0, T, L^2(\Omega)) \cap L^p(0, T, W^{1,p}_0(\Omega)) \) such that \( w = u - v > 0 \) in a subset \( D \subset Q_T \) with \( |D| > 0 \). Let us denote

\[
w_k = \begin{cases} 
  w^+ - k & \text{if } w^+ > k \ 
  0 & \text{otherwise}
\end{cases}
\]

for \( k \in \left(0, \sup_D w^+\right)\). We use \( w_k \) as test function in the difference of the equation:

\[
\int_\Omega w w_k + \int_{Q_T} [-w (w_k)_t + (a(x, t, u, \nabla u) - a(x, t, v, \nabla v)) \nabla w_k]
\leq \int_{Q_T} |H(x, t, \nabla v) - H(x, t, \nabla u)| w_k
\]

for \( t \in (0, T) \). We observe that

\[
\int_\Omega w w_k - \int_{Q_T} w (w_k)_t = \frac{1}{2} \int_\Omega w_k^2(t).
\]

Using (4.2), (2.2) and (2.6) with \( h \in L^\infty(Q_T), \eta > 0 \) and \( \sigma \leq \frac{p-2}{2} \), inequality (4.1) becomes

\[
\frac{1}{2} \int_\Omega w_k^2(t) + \alpha \int_{Q_T} \left( \frac{\nabla w_k}{|\nabla u| + |\nabla v|} \right)^{2-p} \leq \frac{1}{2} \int_{Q_T} \frac{|\nabla w_k| w_k}{(\eta + |\nabla u| + |\nabla v|)^\sigma}.
\]

Taking the supremum on \( t \in (0, T) \) we obtain

\[
\frac{1}{2} \sup_{0 < t < T} \int_\Omega w_k^2 + \alpha \int_{E_k} \left( \frac{\nabla w_k}{|\nabla u| + |\nabla v|} \right)^{2-p} \leq \frac{1}{2} \int_{E_k} \frac{|\nabla w_k| w_k}{(\eta + |\nabla u| + |\nabla v|)^\sigma},
\]

where \( E_k = \left\{ (x, t) \in Q_T : k < w^+ < \sup_D w \right\} \). Since \( \sigma \leq \frac{p-2}{2} \), Young inequality gets

\[
\int_{E_k} \left( \frac{|\nabla w_k| w_k}{(\eta + |\nabla u| + |\nabla v|)^\sigma} \right) \leq \delta \int_{E_k} \left( \frac{|\nabla w_k|}{|\nabla u| + |\nabla v|} \right)^{2-p} + \frac{1}{4\delta \eta^{-2\sigma+p-2}} \int_{E_k} |w_k|^2
\]

for \( \delta > 0 \). Putting (4.4) in (4.3) and choosing \( \delta \) small enough we have

\[
\sup_{0 < t < T} \int_\Omega w_k^2 + \int_{E_k} \left( \frac{|\nabla w_k|}{|\nabla u| + |\nabla v|} \right)^{2-p} \leq c \int_{E_k} |w_k|^2,
\]
where \( c \) is a positive constant independent on \( k \). Using (3.2), Hölder and Young inequalities and (15) we have

\[
\iint_{E_k} |w_k|^2 \leq C_{2N}^{\frac{2N}{N+2}} \left[ \sup_{0 < t < T} \int_{\Omega} |w_k|^2 \right]^{\frac{2}{N+2}} \iint_{E_k} |
abla w_k|^{\frac{2N}{N+2}} \times \left( \iint_{E_k} (|\nabla u| + |\nabla v|)^{(2-p)\frac{4}{N+2}} \right) \leq C_{2N}^{\frac{2N}{N+2}} \frac{2}{N+2} \sup_{0 < t < T} \int_{\Omega} |w_k|^2 + \frac{N}{N+2} \iint_{E_k} (|\nabla u| + |\nabla v|)^{(2-p)\frac{4}{N+2}} \times \left( \iint_{E_k} (|\nabla u| + |\nabla v|)^{(2-p)\frac{4}{N+2}} \right) \leq C_{2N}^{\frac{2N}{N+2}} \frac{N}{N+2} \left( \iint_{E_k} (|\nabla u| + |\nabla v|)^{(2-p)\frac{4}{N+2}} \right) \frac{2}{N+2},
\]

where \( C_{2N}^{\frac{2N}{N+2}} \) is the constant in (3.1). It easily follows that \( 1 \leq c \frac{2N}{N+2} \left( \iint_{E_k} (|\nabla u| + |\nabla v|)^{(2-p)\frac{4}{N+2}} \right) \frac{2}{N+2} \).

Since \( p \geq \frac{2N}{N+2} \), the right-hand side goes to zero when \( k \) goes to \( \sup w^+ \), which is impossible. To complete the proof we have to change the role of \( u \) and \( v \).

**Proof of Theorem 2.4.** We argue as in the proof of Theorem 2.3 obtaining

\[
\frac{1}{2} \sup_{0 < t < T} \int_{\Omega} |w|^2 + \alpha \iint_{E_k} |\nabla w_k|^2 (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} \leq \iint_{E_k} h (|\nabla u| + |\nabla v|)^\sigma |\nabla w_k| w_k.
\]

Let us suppose \( \sigma \geq \frac{p-2}{2} \). Using Hölder inequality and inequality (3.1) we have

\[
\iint_{E_k} h (|\nabla u| + |\nabla v|)^\sigma |\nabla w_k| w_k \leq \left( \iint_{E_k} h (|\nabla u| + |\nabla v|)^\sigma \frac{p-2}{2} \right)^{\frac{N+2}{N+2}} \leq C_2 \left( \iint_{E_k} h (|\nabla u| + |\nabla v|)^\sigma \frac{p-2}{2} \right)^{\frac{N+2}{N+2}} \left( \iint_{E_k} |\nabla w_k|^2 (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} \right)^{\frac{1}{2}} \times \left[ \sup_{0 < t < T} \left( \int |w_k|^2 \right)^{\frac{1}{2}} + \left( \iint_{E_k} |\nabla w_k|^2 \right)^{\frac{1}{2}} \right],
\]
where $C_2$ is the constant in (3.1). Putting (4.7) in (4.6), by Young inequality and some easy computation we obtain

$$
\min \left\{ \frac{1}{2}, \alpha \varepsilon^{p-2} \right\} \left[ \sup_{0 < t < T} \int_{\Omega} |w|^2 + \int_{E_k} |\nabla w_k|^2 \right] \\
\leq \sqrt{2} C_2 \max \left\{ 1, \frac{1}{\varepsilon^{p-2}} \right\} \left( \int_{E_k} \left( h \left( |\nabla u| + |\nabla v| \right)^{\sigma - \frac{p-2}{2}} \right)^{N+2} \right)^{\frac{1}{N+2}} \times \\
\left[ \sup_{0 < t < T} \int_{\Omega} |w_k|^2 + \int_{E_k} |\nabla w_k|^2 (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} \right],
$$

i.e.

$$
1 \leq \frac{\sqrt{2} C_2 \max \left\{ 1, \frac{1}{\varepsilon^{p-2}} \right\} \left( \int_{E_k} \left( h \left( |\nabla u| + |\nabla v| \right)^{\sigma - \frac{p-2}{2}} \right)^{N+2} \right)^{\frac{1}{N+2}}}{\min \left\{ \frac{1}{2}, \alpha \varepsilon^{p-2} \right\}} \\
\times \left[ \sup_{0 < t < T} \int_{\Omega} |w_k|^2 + \int_{E_k} |\nabla w_k|^2 (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} \right].
$$

Since $\frac{N+2}{p} + \frac{\sigma - \frac{p-2}{2}(N+2)}{p} \leq 1$, the right-hand side in (4.8) goes to zero when $k$ goes to sup $D$, which is impossible.

Conversely if $0 \leq \sigma < \frac{p-2}{2}$ as before we have

$$
\int_{E_k} (|\nabla u| + |\nabla v|)^2 |\nabla w_k| w_k \\
\leq \left( \int_{E_k} \left( h \left( |\nabla u| + |\nabla v| \right)^{\sigma - \frac{p-2}{2}} \right)^{N+2} \right)^{\frac{1}{N+2}} \left( \int_{E_k} |\nabla w_k|^2 \right)^{\frac{1}{2}} \left( \int_{E_k} w_k^{2(N+2)} \right)^{\frac{N}{2(N+2)}} \\
\leq C_2 \left( \int_{E_k} \left( h \left( |\nabla u| + |\nabla v| \right)^{\sigma - \frac{p-2}{2}} \right)^{N+2} \right)^{\frac{1}{N+2}} \left( \int_{E_k} |\nabla w_k|^2 \right)^{\frac{1}{2}} \times \\
\left[ \sup_{0 < t < T} \int_{\Omega} |w_k|^2 + \int_{E_k} |\nabla w_k|^2 \right]^{\frac{1}{2}} \left( \int_{E_k} \left( h \left( |\nabla u| + |\nabla v| \right)^{\sigma - \frac{p-2}{2}} \right)^{N+2} \right)^{\frac{1}{N+2}}.
$$

Then putting (4.9) in (4.6) by Young inequality and some easy computation we have

$$
1 \leq \frac{\sqrt{2} C_2 \max \left\{ 1, \frac{1}{\varepsilon^{p-2}} \right\} \left( \int_{E_k} \left( h \left( |\nabla u| + |\nabla v| \right)^{\sigma - \frac{p-2}{2}} \right)^{N+2} \right)^{\frac{1}{N+2}}}{\min \left\{ \frac{1}{2}, \alpha \varepsilon^{p-2} \right\}} \\
\times \left[ \sup_{0 < t < T} \int_{\Omega} |w_k|^2 + \int_{E_k} |\nabla w_k|^2 \right]^{\frac{1}{2}} \left( \int_{E_k} \left( h \left( |\nabla u| + |\nabla v| \right)^{\sigma - \frac{p-2}{2}} \right)^{N+2} \right)^{\frac{1}{N+2}}.
$$

Since $\frac{N+2}{p} + \frac{\sigma - \frac{p-2}{2}(N+2)}{p} \leq 1$, the contradiction follows again. Changing the role of $u$ and $v$, we complete the proof.

Proof of Corollary 2.2. We can argue as in the proof of Theorems 2.3 and 2.4 putting $w = (u - v)^+$. 

REFERENCES

[1] Alvino A., Betta M. P., Mercaldo A., Comparison principle for some classes of nonlinear elliptic equations, Journal of Differential Equations 249, n.12, 3279-3290

[2] Alvino A., Ferone V., Mercaldo A., Sharp a priori estimates for a class of nonlinear elliptic equations with lower order terms, Preprint.

[3] Alvino A., Mercaldo A., Nonlinear elliptic equations with lower order terms and symmetrization methods. Boll. Unione Mat. Ital. (9) 1 (2008), no. 3, 645-661.

[4] Alvino A., Mercaldo A., Nonlinear elliptic problems with L 1 data: an approach via symmetrization methods. Mediterr. J. Math. 5 (2008), no. 2, 173-185.
[5] Andreu F., Mazón J. M., Segura de León S., Toledo J., Existence and uniqueness for a degenerate parabolic equation with $L^1$-data, Trans. Amer. Math. Soc. 351 (1999), n. 1, 285–306.

[6] Artola M., Sur une classe de problèmes paraboliques quasilineaires, Bollettino U.M.I. 6 (1986), 5-B, 51-70.

[7] Barles G., Porretta A., Uniqueness for unbounded solutions to stationary viscous Hamilton-Jacobi equations, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) 5 (2006), 107–136.

[8] Betta M.F., Mercaldo A., Uniqueness results for nonlinear elliptic equations via symmetrization methods, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 21 (2010), 1–14.

[9] Betta M. F., Mercaldo A., Murat F., Porzio M. M., Existence and uniqueness results for nonlinear elliptic problems with a lower order term and measure datum. C. R. Math. Acad. Sci. Paris 334 (2002), n. 9, 757–762.

[10] Blanchard D., Murat F., Renormalized solution for nonlinear parabolic problems with $L^1$ data, existence and uniqueness, Proc. Roy. Soc. Edinburgh Sect. A 127 (1997), 1137–1152.

[11] Blanchard D., Murat F., Redwane H., Existence and uniqueness of a renormalized solution for a fairly general class of nonlinear parabolic problems, J. Differential Equations 177 (2001), 331–374.

[12] Boccardo L., Gallouët T., Murat F., Unicité de la solution de certaines équations elliptiques non linéaires, C. R. Acad. Sci. Paris Ser. I Math. 315 (1992), 1159–1164.

[13] Casado-Díaz J., Murat F., A. Porretta, Uniqueness results for pseudomonotone problems with $p>2$, C. R. Acad. Sci. Paris 344 (2007) 487–492.

[14] Chipot M., Michaille G., Uniqueness results and monotonicity properties for strongly nonlinear elliptic variational inequalities, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 16 (1989), 137–166.

[15] DiBenedetto E., Degenerate parabolic equations, Springer-Verlag, New York (1993).

[16] Di Nardo R., Feo F., Guibé O., Existence result for nonlinear parabolic equations with lower order terms, Anal. Appl. (Singap.) 9 (2011), n. 2, 161–186.

[17] Di Nardo R., Feo F., Guibé O., Uniqueness of renormalized solutions to nonlinear parabolic problems with lower-order terms. Proc. Roy. Soc. Edinburgh Sect. A 143 (2013), n. 6, 1185–1208.

[18] Di Nardo R., Feo F., Existence and uniqueness for nonlinear anisotropic elliptic equations, in press on Archiv der Mathematik.

[19] Guibé O., Mercaldo A., Uniqueness results for noncoercive nonlinear elliptic equations with two lower order terms, Commun. Pure Appl. Anal. 7 (2008), 163–192.

[20] Ladyženskaja O. A., Solonnikov V. A., Ural’ceva N. N., Linear and quasilinear equations of parabolic type (Russian), Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23 American Mathematical Society, Providence, R.I. (1968).

[21] Lieberman G. M., Intermediate Schauder theory for second order parabolic equations II. Existence, uniqueness, and regularity, J. Differential Equations 63 (1986), 32-57.

[22] Lions J.-L, Quelques méthodes de résolution des problèmes aux limites non linéaires (French), Dunod et Gauthier-Villars, Paris 1969.

[23] Porzio M. M., Existence of solutions for some “noncoercive” parabolic equations, Discrete Contin. Dynam. Systems 5 (1999), n. 3, 553–568.

[24] Porzio M. M., Existence, uniqueness and behavior of solutions for a class of nonlinear parabolic problems, Nonlinear Anal. 74 (2011), n. 16, 5359–5382.

[25] Prignet A., Existence and uniqueness of "entropy" solutions of parabolic problems with $L^1$ data, Nonlinear Anal. 28 (1997), n. 12, 1943–1954.

Dipartimento di Ingegneria, University of Naples Parthenope, Centro Direzionale, Isola C4 - 80143 Napoli
E-mail address: filomena.feo@uniparthenope.it