Gauge symmetries and Noether charges in Clebsch-parameterized magnetohydrodynamics

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Abstract

It is shown that the Clebsch parameterization, upon canonizing the Hamiltonian system of ideal fluid or plasma dynamics, converts the Casimir invariants (mass and helicities) in the original noncanonical formulation into the Noether charges pertinent to the gauge symmetries of the parameterization. The problem is addressed in the context of magnetohydrodynamics. The concrete forms of the gauge symmetries of Clebsch parameterization are worked out.

Keywords: Clebsch parameterization, Casimir invariants, magnetohydrodynamics, Hamiltonian system, Noether charge

(Some figures may appear in colour only in the online journal)

1. Introduction

A general Hamiltonian system may have two different kinds of constants of motion (first integrals); the usual ones are caused by symmetries of the Hamiltonian, while the others, so-called Casimir invariants, are independent of a specific choice of Hamiltonian, but are pertinent to the underlying Poisson algebra [1–3]. A Casimir invariant \( C \) is a function (observable) such that \( \{ C, H \} = 0 \) for every \( H \) (here \( \{ , \} \) denotes the Poisson bracket of a Hamiltonian system), i.e., \( C \) belongs to the center of the Poisson algebra. The existence of a nontrivial Casimir invariant \( C \neq \) constant is the determining characteristic of a noncanonical Hamiltonian system (or a degenerate Poisson algebra). Conversely, a canonical Hamiltonian system does not have a Casimir invariant; all constants of motion are of the first kind.

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Interestingly, we may often convert a noncanonical system into a canonical system (cf remark 1). Then, what will become of Casimir invariants? Here we put the method of Clebsch parameterization into perspective (cf appendix A). The noncanonical Hamiltonian system to be examined is that of ideal magnetohydrodynamics (MHD) [4]. A Hamiltonian formulation of fluid/plasma mechanics is generally noncanonical when it is described by Eulerian variables, and has two kinds of Casimir invariants [2, 4–6]; one is the total mass, and the others are helicities. By representing the vector fields (fluid velocity and magnetic field) in terms of Clebsch parameters, we can rewrite the evolution equation as a canonical Hamiltonian system, and provide it with an action principle (cf appendix B). The aim of this work is to elucidate how the original Casimir invariants translate in the canonized system. We will show that they become the first-kind constants of motion corresponding to the gauge symmetries of the Clebsch parameterization [15]. The gauge transformation also conserves the action integral; hence the constants of motion are Noether charges, i.e., the Casimir invariants translate as the Noether charges corresponding to the gauge symmetry of Clebsch parameterization.

We organize this paper as follows: In section 2, we start by reviewing the standard noncanonical Hamiltonian formulation of MHD (section 2.1) and its canonization by Clebsch parameterization (section 2.2). In section 3, we elucidate the symmetry of the canonized system. The f generated by the original Casimir invariant defines a symmetry group (section 3.1); the symmetry turns out to be the gauge symmetry (redundancy) of the Clebsch parameterization (section 3.2), which yields a Noether charge that reproduces the original Casimir invariant (section 3.3). In section 3.4, we derive an extended set of constants of motion by studying a generalized class of symmetries. Section 4 concludes this work with a remark on the Lagrangian formalism and the relabeling symmetry.

Remark 1. (Canonization). There are two different directions toward canonization; one is the ‘reduction’ to eliminate the nontrivial center of the Poisson algebra, and the other is the ‘extension’ to resolve the degeneracy by adding new variables and creating canonical pairs. However, there is no general guarantee of success. For a finite-dimensional system, Darboux’s theorem delineates the ‘local’ landscape of a foliated phase space; separating the null-space of the Poisson operator (matrix), we can endow a symplectic geometry on the leaves of Casimir invariants. Or, adding conjugate variables to each Casimir invariant (viewing the latter as momenta [16]), we can define an extended canonical system. While such reduction or extension is possible locally in a finite-dimensional phase space, singularities of the Poisson operator (where the rank changes) are obstacles, where the topology of the phase space can become highly complex [2, 17]. In an infinite-dimensional phase space, we cannot even count the rank, and very little is known about general (abstract) reduction/extension.

In this work, we explore the route of extension via Clebsch parameterization. Backtracking the route, we can view the noncanonical system as a ‘noncanonical reduction’ of a canonical system of Clebsch parameters (so-called Clebsch reduction; cf appendix A). In a simple setting of Clebsch parameterization, we are led to the grand avenue of Lie–Poisson algebras [2], where the relation between the canonical and noncanonical systems is straightforward. However, the example of an MHD system is not so simple that some nontriviality of extension emerges; shedding light on this profound problem is the motivation of the present study.
2. Hamiltonian formalisms of magnetohydrodynamics

2.1. Ideal magnetohydrodynamics in Eulerian description

The MHD system is described by

$$\partial_t \rho = -\nabla \cdot (\rho V),$$

$$\partial_t V = -(\nabla \times V) \times V - \nabla \left( h + \frac{1}{2} V^2 \right) + \frac{1}{\rho} (\nabla \times B) \times B,$$

$$\partial_t B = \nabla \times (V \times B),$$

where $\rho$ is the density, $V$ is the velocity, $B$ is the magnetic field, and $h$ is the specific enthalpy. All variables are normalized by the Alfvén unit, i.e., the energy densities (thermal $h$, kinetic $rV^2$ and magnetic $mB^2$) are normalized by the representative magnetic energy density $mB_0^2/\mu_0$ ($\mu_0$ is the vacuum permeability). Here assume a barotropic relation $h = h(\rho)$. We consider a smoothly bounded, simply connected domain $\Omega \subset \mathbb{R}^3$. We assume

$$n \cdot V = 0,$$

$$n \cdot B = 0,$$

on the boundary $\partial \Omega$ ($n$ is the normal unit vector onto $\partial \Omega$).

We may cast the MHD equations (1)–(3) in a Hamiltonian form. A general Hamilton’s equation may be written as

$$\partial_t u = J \partial_\rho H,$$

where $u \in X$ (phase space) is a state vector, $J \in \text{End}(X)$ is a Poisson operator, and $H \in C^\infty(X)$ is a Hamiltonian. The adjoint representation of the dynamics reads

$$\partial_t F = \{F, H\},$$

where $F \in C^\infty(X)$ is an observable, and $\{A, B\} = (\partial_\rho A, J \partial_\rho B)$ is the Poisson bracket induced by $J$ (we denote by $(u, v)$ the inner product of the Hilbert space $X$).

For MHD, we define

$$u = \begin{pmatrix} \rho \\ V \\ B \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & -\nabla & 0 \\ -\nabla & -\rho^{-1}(\nabla \times V) \times & \rho^{-1}(\nabla \times \phi) \times B \\ 0 & \nabla \times (\phi \times \rho^{-1}B) & 0 \end{pmatrix},$$

$$H = \int_\Omega \left( \frac{1}{2} \rho V^2 + \rho \mathcal{E}(\rho) + \frac{1}{2} B^2 \right) d^3x,$$

where $\mathcal{E}(\rho)$ is the specific thermal energy (we assume a barotropic relation), $\partial (\rho \mathcal{E})/\partial \rho = h$ is the specific enthalpy. Substituting (8)–(10) into (6) reproduces the MHD equations (1)–(3)\textsuperscript{2}.

\textsuperscript{2} The Poisson operator (9), with the boundary conditions (4) and (5), defines a Poisson bracket [4]. The reader is referred to [2] for examples of different Hamiltonian fluid/plasma systems. There is another interesting example of the Hamiltonian formalism of a two-dimensional incompressible fluid, in which a different set of boundary conditions is involved to formulate a Poisson bracket [18–19].
There are three Casimir invariants (hence, the MHD system is noncanonical):

\[ C_1 = \int_\Omega \rho d^3x, \]
\[ C_2 = \int_\Omega A \cdot B d^3x, \]
\[ C_3 = \int_\Omega V \cdot B d^3x, \]

where \( A = \text{curl}^{-1} B \) is the vector potential (\( \text{curl}^{-1} \) is the inverse operator of curl, which is a self-adjoint operator; see [20]); \( C_1 \) is the total mass, \( C_2 \) is the magnetic helicity, and \( C_3 \) is the cross helicity.

**Remark 2.** (Singular Casimir invariants). There are also ‘singular Casimir invariants’ determining singular leaves on which local Casimir invariants reside [16, 17]. For example, on the unmagnetized \( (B = 0) \) submanifold in the phase space \( X \), the fluid helicity \( \int V \cdot (\nabla \times V) d^3x \) is a local Casimir invariant. While we leave this problem for future investigation, some aspects of singular Casimir invariants are revealed in the light of Clebsch parameterization in [14]; see also appendix B.

### 2.2. Clebsch-parameterized MHD system

We can canonize the MHD system by parameterizing the vector fields \( V \) and \( B \) in terms of Clebsch parameters [10, 14]. Here we invoke a generalized form of Clebsch parameterization such that

\[ U = \nabla a + \sum_{\ell=1}^{\nu} b^{\ell} \nabla c^{\ell}. \]  

When \( \nu = n = \) the dimension of base space (here \( n = 3 \)), (14) gives a complete representation of general \( U \) by \( a, b^{\ell} \) and \( c^{\ell} \) that satisfy appropriate boundary conditions [15] (see also appendix A; in [21], the difficulty of the single \( (\nu = 1) \) Clebsch parameterization is discussed). From now on, we will use the contraction rule on the indexes and omit \( \sum_{\ell=1}^{3} \). Let us put

\[ V = - \nabla \phi_0 - \frac{\mu^{\ell}}{\rho} \nabla \alpha^{\ell} - \frac{\beta^{\ell}}{\rho} \nabla \phi^{\ell}, \]

\[ B = \nabla \frac{\mu^{\ell}}{\rho} \times \nabla \phi^{\ell}. \]

We may rewrite (15) as \( \rho V = - (\rho d\phi_0 + \mu^{\ell} d\alpha^{\ell} + \beta^{\ell} d\phi^{\ell}) \) which reads as a canonical 1-form represented by conjugate variables (fields) \( \phi_0 \) and \( \rho, \alpha^{\ell} \) and \( \mu^{\ell} \), and \( \beta^{\ell} \) and \( \phi^{\ell} \). We find that \( B \) is made up of somewhat strange combinations of parameters, or the momentum \( \rho V \) includes the magnetic terms in skewed combinations. They are, however, rearranged in proper order when we formulate a Hall MHD system; conversely, the MHD is a ‘singular reduction’ of the Hall MHD Poisson manifold [14] (cf appendix B).
To be consistent with the boundary conditions (4) and (5), we demand

\[ \mathbf{n} \cdot \mathbf{V} = - \mathbf{n} \cdot \left( \nabla \phi_0 - \frac{\mu^f}{\rho} \nabla \alpha^f - \frac{\beta^f}{\rho} \nabla \phi^f \right) = 0, \tag{17} \]

\[ \mathbf{n} \times \mathbf{A} = \mathbf{n} \times \left( \frac{\mu^f}{\rho} \nabla \phi^f \right) = 0, \tag{18} \]

where \( \mathbf{A} = \left( \frac{\mu^f}{\rho} \right) \nabla \phi^f \) stands for the vector potential of \( \mathbf{B} \). Evidently, (5) follows from (18).

Denoting

\[ J_e = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J_e^f = \begin{pmatrix} J_e & 0 & 0 \\ 0 & J_e^f & 0 \\ 0 & 0 & J_e^f \end{pmatrix}, \tag{19} \]

we define

\[ u_e = \left( \phi_0, \rho, \alpha^1, \mu^1, \alpha^2, \mu^2, \alpha^3, \mu^3, \phi^1, \beta^1, \phi^2, \beta^2, \phi^3, \beta^3 \right)^T, \tag{20} \]

\[ \mathcal{J}_e = \begin{pmatrix} J_e & 0 & 0 \\ 0 & J_e^f & 0 \\ 0 & 0 & J_e^f \end{pmatrix}, \tag{21} \]

\[ H = \int_0^1 \tau d^3 x \]
\[ = \int_0^1 \left( \frac{1}{2} \rho \left( \nabla \phi_0 + \frac{\mu^f}{\rho} \nabla \alpha^f + \frac{\beta^f}{\rho} \nabla \phi^f \right) \right)^2 \]
\[ + \rho \mathcal{E} + \frac{1}{2} \left( \nabla \frac{\mu^f}{\rho} \times \nabla \phi^f \right) \right)^2 \] \( d^3 x. \) \tag{22} \]

Hamilton’s equation reads as a canonical system

\[ \begin{align*}
\partial_t \phi_0 &= -\mathbf{V} \cdot \nabla \phi_0 + h - V^2/2 - \rho^{-1} \mathbf{J} \cdot \mathbf{A}, \\
\partial_t \rho &= -\nabla \cdot (\rho \mathbf{V}), \\
\partial_t \alpha^f &= -\mathbf{V} \cdot \nabla \alpha^f + \rho^{-1} \mathbf{J} \cdot \nabla \phi^f, \\
\partial_t \mu^f &= -\nabla \cdot (\mu^f \mathbf{V}), \\
\partial_t \phi^f &= -\mathbf{V} \cdot \nabla \phi^f, \\
\partial_t \beta^f &= -\nabla \cdot (\beta^f \mathbf{V}) + \mathbf{J} \cdot \nabla \left( \frac{\mu^f}{\rho} \right),
\end{align*} \tag{23} \]

where \( \mathbf{J} = \nabla \times \mathbf{B} \) is the current density. Upon substitution into (15) and (16), the fields satisfying (23) are shown to solve the MHD equations (1)–(3). By the completeness of the Clebsch parameterization [15] (see also the footnote after equation (3)), every initial condition

The reason why we need \( \nu = 3 \) is seen here. Just for a complete parameterization in the form of (14), we need \( \nu = n - 1 = 2 \) [15]. However, it falls short of representing appropriate boundary conditions. If we put \( \mu^3 = \rho \) and free \( \phi^3 \) as an electromagnetic gauge field, \( \mathbf{A} = \sum_{\ell=0}^2 \left( \frac{\mu^f}{\rho} \right) \nabla \phi^\ell + \nabla \phi^3 \) reads as a \( \nu = 2 \) Clebsch parameterization. Then, the boundary condition (18) gains a gauge freedom pertinent to the freed \( \phi^3 \), which prevents us from constraining \( \delta \phi^f \) properly at the boundary; one may see the problem in calculating \( \partial_r h \) to derive (23). Zakharov and Kuznetsov [21] speculated that \( \nu = 3 \) may be needed by noting the equivalence of the Clebsch parameters and the Lagrangian labels of three-dimensional fluid elements.

\[ \text{J. Phys. A: Math. Theor. 48 (2015) 495501 K Tanehashi and Z. Yoshida} \]
in terms of physical quantities can be expressed by the Clebsch parameters. Therefore, all orbits of MHD system (1)–(3) can be generated by solving the canonized system (23).

We have a Lagrangian (represented by the Clebsch fields)

$$L = \int_{\Omega} \mathcal{L} \, d^3x = \int_{\Omega} \left( \rho \dot{\phi}_0 + \mu' \dot{\phi}_f + \mu' \dot{\phi}^f - \mathcal{H} \right) d^3x,$$

by which the action is written as (denoting $t = x_0$)

$$S = \int_{D} \mathcal{L} \, d^4x,$$

where $D$ is the space–time domain $\mathbb{R} \times \Omega$.

The invariants, $C_1$, $C_2$ and $C_3$ are no longer ‘Casimir invariants’ in the canonized system, while they must be still constants of motion. Definition of how they are conserved in the canonized formalism is the aim of the present exploration. There are two possibilities (corresponding to the two directions of canonization given in remark 1); one is that the Casimir invariants are eliminated from the new phase space, and the other is that they are changed into constants of motion pertinent to some symmetry of the Hamiltonian of the canonized representation. Usually, the former is the case of reduction and the latter is the case of extension. However, the answer is not simple, because Clebsch parameterization can be either extension or reduction. When we have only $\ell = 1$, the incomplete Clebsch parameterization is a reduction to the zero-$C_2$ and zero-$C_3$ leaf, i.e., $C_2$ and $C_3$ are eliminated (see [22] for such a canonical formulation of a neutral fluid; see also an example of in appendix B.3). When we have three $\ell$s (as formulated above), the Clebsch parameterization is an extension (as expected by counting the number of field variables, and as rigorously proved in [15]). The fact that the constancy of the Casimir invariants is independent of the choice of Hamiltonian implies that the symmetry must be that of ‘gauge transformations’, i.e., the group action of the symmetry must not change the physical variables $\rho$, $V$ and $B$. In the next section, we will show the explicit form of the gauge symmetry that derives the Casimir invariants.

3. Gauge symmetries and the corresponding Noether charges

3.1. Symmetry groups generated by original Casimir invariants

Let $\{A, B\}_c = (\partial_a A, J_a \partial_a B)$ denote the canonical bracket. Given a Hamiltonian $H$, a constant of motion is a functional $C$ such that

$$\{ C, H \}_c = 0.$$  \hfill (26)

An infinitesimal transformation (flow) generated by a virtual Hamiltonian $G$ is given by (with small $\epsilon$)

$$\delta u_G = \epsilon J_a \partial_a G.$$  \hfill (27)

If $G$ and the actual Hamiltonian $H$ commute, i.e., if $G$ is a constant of motion, $\delta u_G$ defines a symmetry group: (26) implies

$$H(u_c + \delta u_G) - H(u_c) = 0.$$  \hfill (28)

To simplify the notation, we will omit $\epsilon$ in writing perturbations $\delta u_G$. 

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The symmetry groups generated by the ‘original Casimir invariants’ are

\[ C_1 = \int_{\Omega} \rho \, d^3x \rightarrow \delta \phi_0 = 1, \]  

\[ C_2 = \int_{\Omega} A \cdot B \, d^3x \rightarrow \begin{cases} 
\delta \phi_0 = -\frac{2\mu_f^t}{\rho} \nabla \phi^f \cdot B, \\
\delta \alpha^t = \frac{2}{\rho} \nabla \phi^f \cdot B, \\
\delta \beta^t = 2\nabla \frac{\mu}{\rho} \cdot B, 
\end{cases} \]  

\[ C_3 = \int_{\Omega} V \cdot B \, d^3x \rightarrow \begin{cases} 
\delta \phi_0 = \left( \frac{\mu^t}{\rho} \nabla \alpha^t + \frac{\beta^t}{\rho^2} \nabla \phi^f \right) \cdot B - \frac{\mu^t}{\rho} \nabla \phi^f \cdot \omega, \\
\delta \alpha^t = -\frac{1}{\rho} \nabla \alpha^t \cdot B + \frac{1}{\rho} \nabla \phi^f \cdot \omega, \\
\delta \beta^t = -\nabla \frac{\mu^t}{\rho} \cdot B + \nabla \frac{\mu^t}{\rho} \cdot \omega. 
\end{cases} \]  

where \( \omega = \nabla \times V = -\nabla \frac{\phi^f}{\rho} \times \nabla \alpha^t - \nabla \frac{\mu^t}{\rho} \times \nabla \phi^f. \)

### 3.2. Gauge symmetries of the Clebsch parameterization

Here we show explicitly that the symmetry groups (29)–(31), generated by the original Casimir invariants, are gauge transformations of the Clebsch parameterization, i.e., the physical variables \( \rho, V \) and \( B \) are unchanged by the action of the group.

Under the transformation (29), it is evident that \( \delta \rho = 0 \) and \( \delta B = 0 \), because \( \rho \) and \( B \) are independent of \( \phi_0 \). Also \( \delta V = 0 \), because \( V \) includes \( \phi_0 \) in the form of its gradient.

Under (30), \( \delta \rho = 0 \) and \( \delta B = 0 \) are obvious, because \( \rho \) and \( B \) do not depend on \( \phi_0, \alpha^t \) and \( \beta^t \). We also find that \( \delta V \) vanishes:

\[ \delta V = - \nabla \delta \phi_0 - \frac{\mu^t}{\rho} \nabla \delta \alpha^t - \frac{\beta^t}{\rho} \nabla \phi^f \]

\[ = \nabla \left( \frac{2\mu^t}{\rho^2} \nabla \phi^f \cdot B \right) - \frac{\mu^t}{\rho} \nabla \left( \frac{2}{\rho} \nabla \phi^f \cdot B \right) - \frac{1}{\rho} \left( \frac{2\mu^t}{\rho} \cdot B \right) \nabla \phi^f \]

\[ = \frac{2}{\rho} \left( \nabla \phi^f \cdot B \right) \nabla \frac{\mu^t}{\rho} - \frac{2}{\rho} \left( \nabla \frac{\mu^t}{\rho} \cdot B \right) \nabla \phi^f \]

\[ = \frac{2}{\rho} B \times \left( \nabla \frac{\mu^t}{\rho} \times \nabla \phi^f \right) \]

\[ = \frac{2}{\rho} B \times B = 0. \]
Under (31), \( \delta \rho = 0 \) is evident. \( \delta V \) and \( \delta B \) are calculated as follows:

\[
\delta V = - \nabla \left( \left( \frac{\mu^t}{\rho^2} \nabla \alpha^t + \frac{\beta^t}{\rho^2} \nabla \phi^t \right) \cdot B - \frac{\mu^t}{\rho^2} \nabla \phi^t \cdot \omega \right) \\
+ \frac{1}{\rho} \left( \nabla \frac{\mu^t}{\rho} \cdot B \right) \nabla \alpha^t \\
+ \frac{\mu^t}{\rho} \left( \frac{1}{\rho} \nabla \alpha^t \cdot B - \frac{1}{\rho} \phi^t \cdot \omega \right) \\
+ \frac{1}{\rho} \left( \nabla \frac{\beta^t}{\rho} \cdot B - \nabla \frac{\mu^t}{\rho} \cdot \omega \right) \nabla \phi^t \\
+ \frac{\beta^t}{\rho} \left( \frac{1}{\rho} \nabla \phi^t \cdot B \right)
\]

\[
= \frac{1}{\rho} \left( \nabla \frac{\mu^t}{\rho} \cdot B \right) \nabla \alpha^t - \frac{1}{\rho} \left( \nabla \alpha^t \cdot B \right) \nabla \frac{\mu^t}{\rho} \\
+ \frac{1}{\rho} \left( \nabla \phi^t \cdot \omega \right) \nabla \frac{\mu^t}{\rho} - \frac{1}{\rho} \left( \nabla \frac{\mu^t}{\rho} \cdot \omega \right) \nabla \phi^t \\
+ \frac{1}{\rho} \left( \nabla \frac{\beta^t}{\rho} \cdot B \right) \nabla \phi^t - \frac{1}{\rho} \left( \nabla \phi^t \cdot B \right) \nabla \frac{\beta^t}{\rho}
\]

\[
= \frac{1}{\rho} B \times \left( \nabla \alpha^t \times \nabla \frac{\mu^t}{\rho} + \nabla \phi^t \times \nabla \frac{\beta^t}{\rho} \right) + \frac{1}{\rho} \left( \nabla \frac{\mu^t}{\rho} \times \nabla \phi^t \right)
\]

\[
= \frac{1}{\rho} (B \times \omega + \omega \times B) = 0,
\]

(33)

\[
\delta B = \nabla \frac{\delta \mu^t}{\rho} \times \nabla \phi^t + \nabla \frac{\mu^t}{\rho} \times \nabla \delta \phi^t
\]

\[
= - \nabla \times \left( \frac{2}{\rho} \left( \nabla \frac{\mu^t}{\rho} \cdot B \right) \nabla \phi^t - \frac{2}{\rho} \left( \nabla \phi^t \cdot B \right) \nabla \frac{\mu^t}{\rho} \right)
\]

\[
= \nabla \times \left( \frac{2}{\rho} B \times \left( \nabla \frac{\mu^t}{\rho} \times \nabla \phi^t \right) \right)
\]

\[
= \nabla \times \left( \frac{2}{\rho} B \times B \right) = 0.
\]

(34)

3.3. Noether charges

While a general gauge transformation does not necessarily conserve the action (cf remark 3), the gauge transformations generated by constants of motion must conserve the action because
they are Hamiltonian vectors. To see how the corresponding Noether charges are related to the gauge transformations, let us perform an explicit calculation of the variation of the Lagrangian. By the transformation of (30), we obtain
\[\delta \mathcal{L} = \rho \delta \phi_0 + \mu^f \delta \alpha^f + \delta \beta^f \phi^f \]
\[= - \partial_t (A \cdot B) + \nabla \cdot \left( \frac{\mu^f}{\rho} \phi^f B + \left( A - \nabla \left( \frac{\mu^f}{\rho} \phi^f \right) \right) \times A \right). \tag{35}\]
thus \(\delta \mathcal{L}\) is an exact differential. Similarly, by the transformation of (31), we obtain
\[\delta \mathcal{L} = \rho \delta \phi_0 + \mu^f \delta \alpha^f + \delta \beta^f \phi^f + \delta \alpha^f \phi^f \]
\[= - \partial_t (V' \cdot B) + \nabla \cdot \left( - \left( \frac{\mu^f}{\rho} \alpha^f + \frac{\beta^f}{\rho} \phi^f \right) B + \left( \dot{A} - \nabla \left( \frac{\mu^f}{\rho} \phi^f \right) \right) \times V' \right), \tag{36}\]
where \(V' = V + \nabla \phi_0 = -\frac{\mu^f}{\rho} \nabla \alpha^f - \frac{\beta^f}{\rho} \nabla \phi^f\).

As preparation for deriving the Noether charges, we briefly review the general procedure. We consider a first-order Lagrangian such that \(\mathcal{L} = \int_\Omega \mathcal{L}(q^i, \partial_t q^i) d^3x\), which yields the Euler–Lagrange equation \(\partial_t \mathcal{L} - \partial_x (\partial_{\partial_t} \mathcal{L}) = 0\). Suppose that \(\mathcal{L}\) has a symmetry with respect to a transformation \(q^i \rightarrow q^i + \delta q^i\), i.e., the transformation results in: \(\delta \mathcal{L} = \partial_t N^i q^i\) with some vector \(N^i\). For a solution \(q^i(t, x)\) of the Euler–Lagrange equation, we obtain
\[\delta \mathcal{L} = b q^i \partial_t q^i + \delta (\partial_{\partial_t} q^i) \partial_{\partial_{\partial_t} q^i} \mathcal{L}\]
\[= b q^i \left( \partial_q \mathcal{L} - \partial_x (\partial_{\partial_q} \mathcal{L}) \right) + \partial_t (b q^i \partial_{\partial_q} \mathcal{L})\]
\[= \partial_t \left( b q^i \partial_{\partial_q} \mathcal{L} - \partial_q \mathcal{L} \right). \tag{37}\]
For a perturbation \(\delta q^i\) of a symmetry group, we may write
\[\partial_t \left( b q^i \partial_{\partial_q} \mathcal{L} - \partial_q \mathcal{L} \right) = \partial_t I^c = 0, \tag{38}\]
where \(I^c\) is a Noether current, and \(\int_\Omega I^c d^3x = \int_\Omega (b q^i \partial_{\partial_q} \mathcal{L} - \partial_q \mathcal{L}) d^3x\) is the corresponding Noether charge.

Applying this procedure to the canonized MHD Lagrangian and the gauge symmetries, we obtain
\[\delta \phi_0 = 1 \rightarrow \int_\Omega \rho d^3x = C_1, \tag{39}\]
\[\delta \phi_0 = -\frac{2 \mu^f}{\rho} \nabla \phi^f \cdot B \]
\[\delta \alpha^f = \frac{2 \mu^f}{\rho} \nabla \phi^f \cdot B \rightarrow \int_\Omega A \cdot B d^3x = C_2, \tag{40}\]
\[\delta \beta^f = 2 \nabla \mu^f \cdot B\]
The third Noether charge \( (41) \) is nothing but the cross helicity \( C_3 \); using the boundary condition \( (5) \),

\[
\int_\Omega V' \cdot B \, d^3x - \int_\Omega V \cdot B \, d^3x = \int_\Omega \nabla \phi_0 \cdot B \, d^3x = \int_{\partial \Omega} n \cdot \phi_i B \, d^3x = 0. \tag{42}
\]

We have elucidated a beautiful relation, mediated by the constants \( C_1, C_2 \) and \( C_3 \), between the Eulerian noncanonical formalism and the Clebsch-parameterized canonical formalism; in the former, the constants are the Casimir invariants that foliate the phase space, and in the latter, they are the Noether charges that identify symmetry groups reflecting the redundancy of the Clebsch parameterization (see figure 1).

**Remark 3.** *(Local gauge transformation)* The Clebsch parameterization has a larger (infinite) set of gauge symmetries; the ones that are related to the Noether charges (Casimir invariants) are those of ‘global’ gauge transformatoins. The counterpart, ‘local’ gauge transformations do not yield Noether charges. Let us consider a transformation such that \( u_\epsilon \rightarrow u_\epsilon + \epsilon (t, x) \partial_\mu u_\epsilon C_2 \), where \( \epsilon (t, x) \) is an arbitrary infinitesimal function. Explicitly,

\[
\delta \phi_0 = -2 \epsilon (t, x) \left( \frac{\mu^t}{\rho} \nabla \phi^f \cdot B \right) \delta \alpha^f = 2 \epsilon (t, x) \frac{1}{\rho} \nabla \phi^f \cdot B \quad \text{and} \quad \delta \beta^f = 2 \epsilon (t, x) \frac{1}{\rho} \nabla \phi^f \cdot B. \]

This transformation does not change the physical variables:

\[
\delta V = - \nabla \delta \phi_0 - \frac{\mu^t}{\rho} \nabla \delta \alpha^f - \frac{\delta \beta^f}{\rho} \nabla \phi^f
\]

\[
= \frac{2 \epsilon}{\rho} \left( \nabla \phi^f \cdot B \right) \nabla \frac{\mu^t}{\rho} - \frac{2 \epsilon}{\rho} \left( \nabla \frac{\mu^t}{\rho} \cdot B \right) \nabla \phi^f
\]

\[
= \frac{2 \epsilon}{\rho} B \times B = 0. \tag{43}
\]

\( \delta B \) and \( \delta \rho \) also vanish. However, this transformation changes the action; we obtain

\[
\delta \mathcal{L} = \rho \delta \phi_0 + \mu^t \delta \alpha^f + \delta \beta^f \phi^f
\]

\[
= \epsilon \left( - \partial_\mu (A \cdot B) + \nabla \cdot \left( \frac{\mu^t}{\rho} \phi^f B + \left( A - \nabla \left( \frac{\mu^t}{\rho} \phi^f \right) \right) \times A \right) \right), \tag{44}
\]

which is not an exact differential for a non-constant \( \epsilon (t, x) \).

**3.4. Generalized constants of motion**

The Clebsch parameterization enables us to describe local conservation of mass and helicity. We define two sets of functions \( G_1 \) and \( G_2 \) such that
Figure 1. The relation between Casimir invariants and Noether charges. Left: In the noncanonical formulation in terms of Eulerian physical variables, Casimir invariants foliate the phase space. Orbits are constrained on manifolds that are intersections of the energy contours (level-sets of $H$) and the Casimir leaves (level-sets of $C$). Right: By the Clebsch parameterization, the system is embedded in a larger phase space with redundant degrees of freedom; the Hamiltonian group action generated by the Noether charges delineates the symmetry built in the Clebsch parameterization.

\[ G_1 = \left\{ a \left| \partial_t a + V \cdot \nabla a = 0 \right. \right\}, \quad (45) \]

\[ G_2 = \left\{ \lambda \left| \partial_t \lambda + \nabla \cdot (\lambda V) = 0 \right. \right\}. \quad (46) \]

From (23), $\phi^f \in G_1$ and $\rho, \mu^f \in G_2$. We easily verify

\[ \lambda, \eta \in G_2, s, t \in \mathbb{R} \rightarrow s \lambda + t \eta \in G_2, \quad (47) \]

\[ a, b \in G_1 \rightarrow f(a, b) \in G_1 \text{ for any smooth function } f, \quad (48) \]

\[ \lambda, \eta \in G_2 \rightarrow \frac{\lambda}{\eta} \in G_1, \quad (49) \]

\[ a \in G_1, \lambda \in G_2 \rightarrow a \lambda \in G_2, \quad (50) \]

\[ a, b, c \in G_1 \rightarrow \nabla a \cdot (\nabla b \times \nabla c) \in G_2. \quad (51) \]

When $\lambda \in G_2, \int_{\Omega} \lambda d^3x$ is a constant of motion. From (49), $\sigma^f = \mu^f / \rho \in G_1$ and using (48) and (50), $\rho f (\sigma^f, \phi^f) \in G_2$; hence we find a new constant of motion $\int_{\Omega} \rho f (\sigma^f, \phi^f) d^3x$ ($f$ is an arbitrary smooth function). From (50) and (51), $\sigma^f \nabla \phi^f \cdot (\nabla \sigma^k \times \nabla \phi^l) \in G_2$. Using (47), and summing up the terms such that $i = j$ and $k = l$, we obtain $A \cdot B \in G_2$. From (50), $f (\sigma^f, \phi^f) \in G_2$, thus $\int_{\Omega} f (\sigma^f, \phi^f) A \cdot B d^3x$ is a constant of motion.

Since $f (\sigma^f, \phi^f)$ is a scalar function that is constant along the streamlines of the flow $V$, we may regard it as a weighting function co-moving with the plasma. Choosing $f (\sigma^f, \phi^f)$ to be a support function on a co-moving volume element, $\int_{\Omega} f (\sigma^f, \phi^f) d^3x$ and
\[ \int_\Omega f(\sigma^\ell, \phi^\ell) A \cdot B d^3x \]

measure the local mass and helicity contained in the volume element. These generalized constants of motion are related to the following gauge symmetries:

\[ \begin{align*}
\delta \phi_0 &= f - \frac{\mu^\ell}{\rho^2} f^\ell, \\
\delta \alpha^\ell &= f^\ell, \\
\delta \beta^\ell &= -\rho f^\ell.
\end{align*} \]  

\[ (52) \]

where \( f^\ell \) denotes the derivative of \( f \) with respect to \( \sigma^\ell \), and \( f^\ell_2 \) the one with respect to \( \phi^\ell \). For example, \( \delta V \) under the transformation \( (52) \) is

\[ \delta V = \rho \left( f - \frac{\mu^\ell}{\rho^2} f^\ell \right) + f^\ell_2 \nabla \phi^\ell - \frac{\mu^\ell}{\rho} \nabla f^\ell \\
= \nabla f + f^\ell_2 \nabla \phi^\ell + f^\ell \nabla \frac{\mu^\ell}{\rho} \\
= \nabla f + \nabla f = 0, \]  

and \( \delta \rho = 0 \) and \( \delta B = 0 \) are obvious, thus \( (52) \) is a gauge transformation.

### 4. Conclusion and remarks

We have shown, in the context of MHD, that Casimir invariants of a noncanonical Hamiltonian system are converted, upon canonization by an alternative parameterization (here, Clebsch parameterization) of the state vector, into Noether charges pertinent to the gauge freedom (redundancy) of the new parameterization. While the conclusion delineates a simple relation between different formalisms of dynamical systems, the specific forms of the symmetries, embodying the concrete relation, are rather complicated. Since the Clebsch parameterization implies a nonlinear variable transformation, the symmetries are not apparent (except some simple ones; see [15]). Emphasizing another aspect of the problem, we may say that we could unearth, guided by the known Casimir invariants, a wider class of gauge symmetries of the Clebsch parameterization.

There are some different methods of canonization. A well-known method that applies for various fluid-mechanical systems is the usage of Lagrangian variables to represent the dynamics of fluid elements; the Lagrangian variables label the initial position of each fluid element. Formulating an action by a Lagrangian of Newtonian-mechanical continuum, we obtain canonical systems of Hamilton’s equation of motion [1, 2]; see [23, 24] for the original formulation of the Lagrangian action principle of MHD. In the canonized system of Lagrangian variables, the Casimir invariants are translated differently. The total mass \( C_1 \) and the magnetic helicity \( C_2 \) are now trivially conserved, because they are the integrals of the attributes of fluid elements; the local density and magnetic helicity are not dynamical...
variables, but are constants bound to each fluid element (like charges of particles in particle systems). Only the cross helicity $C_3$ is a nontrivial first integral; it is now the Noether charge pertaining to the relabeling symmetry that represents the arbitrariness of providing fluid elements with Lagrangian labels [12, 25, 26]. Since Clebsch parameters may be viewed as the Eulerian counterparts of Lagrangian labels (or the pull-back along the Cauchy characteristics of the fluid velocity $V$; see [13]), it is natural that the gauge symmetry of the Clebsch parameters (present canonization) and the relabeling symmetry of the Lagrangian labels (Lagrangian canonization) yield the same constant of motion.

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**Appendix A. Clebsch parameterization**

Here we describe a short review of the Clebsch parameterization in the context of Hamiltonian mechanics.

**A.1. Clebsch 2-form**

The original idea of Clebsch [27] was to deform an exact 1-form $dq$ to a non-exact 1-form such that $pdq$. The resultant 2-form $dp \wedge dq$ may be used to represent a vorticity. In three-dimensional space, a vorticity (the curl of a covector) can be parameterized as (using the conventional vector notation)

$$\omega = \sum_{\ell=1}^{\nu} \nabla p^\ell \times \nabla q^\ell.$$  \hfill (A.1)

We need $\nu = 2$ to represent a general exact 2-form $\omega$. Moreover, we need $\nu = 3$ to provide $q^\ell$ and $p^\ell$ with appropriate boundary conditions that are consistent with the boundary conditions of $\omega$ (see [15] for the completeness theorems for the Clebsch parameterization). It is interesting to see the two-dimensional case; a Clebsch 2-form may be written as

$$\omega = [q, p] = \partial_q q \cdot \partial_p - \partial_q p \cdot \partial_p,$$ \hfill (A.2)

which reads as the Poisson bracket of $\mathfrak{sp}(2; \mathbb{R})$.

**Remark 4. (Hodge decomposition).** Notice the difference between the Clebsch parameterization and the Hodge decomposition; the latter writes a differential form as an orthogonal sum $u = d\varphi + \delta w + h(h \in \text{Ker}(d) \cap \text{Ker}(\hat{\delta}))$. For a 1-form $u$, only the second term $\delta w$ can bear a finite vorticity $\delta w$, so it may be compared with the Clebsch 1-from $pdq$. When the space dimension is three, we may identify both 1-forms and 2-forms with three-vectors, and then $\delta w$ reads $\nabla \times w$, just like a Clebsch 2-form $\nabla \times (p \nabla q)$ (notice that $\delta w$ is a 1-form). This identification (which is possible only in three dimensions) is useful when we consider divergence-free (incompressible) fields; we may represent such a 1-form as if it is a 2-form. The projection of a 1-form $u$ onto $\text{Ker}(\hat{\delta})$ ($\delta$ reads as divergence) eliminates the term $d\varphi$ (so-
called Helmholtz decomposition), which means that \( \varphi \) may be regarded as a gauge field for divergence-free fields. See, for example [28], for an application of this gauge transformation to the theory of incompressible fluids.

### A.2. Co-moving Clebsch parameters

With application to fluid mechanical systems, we consider Clebsch parameters that co-move with a fluid. Let \( v \) be a vector representing a fluid velocity in an \( n \)-dimensional space. We evaluate the change of Clebsch parameters induced by \( v \). For convenience, we define the Galilean space–time velocity \( U = (1, v) \) and derivative \( \partial_v = (\partial_v, \nabla) \). We denote by \( L_U \) the space–time Lie derivative. Suppose that scalars \( \zeta^\ell \) (\( \ell = 1, \ldots, m \)) and \( n \)-forms \( \eta^\ell \) (\( \ell = 1, \ldots, m' \)) are constants of motion: \( L_U \zeta^\ell = 0 \) and \( L_U \eta^\ell = 0 \). Then, all quantities produced by exterior algebra of \( \zeta^\ell \) and \( \eta^\ell \) are constants of motion; the mass density \( \rho \) is such an \( n \)-form, and the helicity densities are such 3-forms (see [29] for a differential-geometric characterization of helicities).

The nonlinearity of fluid mechanics is reflected in some Clebsch parameters \( \zeta^\ell \) that modify as functions of Clebsch parameters including the constant ones, creating coupling among all variables: \( L_U \zeta^\ell = F_\ell (\zeta^1, \zeta^2, \ldots, \eta^1, \eta^2, \ldots, \zeta^2, \ldots) \). In the MHD Hamiltonian system (23), \( \phi_0 \), \( \alpha^\ell \) and \( \beta^\ell \) are such variables.

### A.3. Clebsch reduction

Starting from a symplectic phase space \( X_\omega = \text{Span} \{ q^\ell, p^\ell ; \ell = 1, \ldots, \nu \} \) of Clebsch parameters \( \{ q^\ell \) and \( p^\ell \) are scalar functions on \( T^n \); here we omit the issue of boundary conditions), we may consider a ‘reduced’ phase space \( X_\omega = \{ \omega = \sum p^\ell \wedge dq^\ell \} \) of Clebsch 2-forms.

The restriction of the canonical Poisson bracket \( \{ F, G \}_C \) to \( X_\omega \) evaluates as a Lie–Poisson bracket such that

\[
\{ F, G \}_\omega = \{ \partial_\omega F, \delta (\partial_\omega G) \wedge \delta \omega \},
\]

where \( \delta \) is the Hodge dual of \( d^\omega \). Evidently \( F (|\omega|^2) \) is a Casimir invariant. We call \( C_{(1,1)}^\omega (X_\omega) \) a Clebsch reduction of the canonical system \( C_{(1,1)}^\omega (X_C) \). The Hamiltonian vector in \( X_\omega \) generated by \( F (|\omega|^2) \) is a gauge transformation of \( \omega \).

Conversely, we may regard the canonical system \( C_{(1,1)}^\omega (X_C) \) as a canonizing ‘extension’ of the noncanonical system \( C_{(1,1)}^\omega (X_\omega) \). In this perspective, \( X_\omega \) is the phase space of physical quantities, while \( X_\omega \) is the space of artificial parameters.

However, there is a subtlety in calling \( X_C \) an ‘extension’ of \( X_\omega \), because the former can be smaller than the latter; this happens when we assume that \( X_\omega \) contains general 2-forms \( \omega \), but we take insufficient Clebsch pairs to represent all members of \( X_\omega \).

We note that the noncanonical MHD system (and the Hall MHD system given in appendix B) is not a Lie–Poisson type; the complexity is primarily due to the fact that the momentum 1-form \( \rho V \) and the density \( n \)-form \( \rho \) are involved in the noncanonical phase space. The parameter \( \phi_0 \) of the exact part of \( V \) and \( \rho \) constitute the canonical pair; see (15), (16) and (23). As noted in remark 4, however, an incompressible fluid/plasma model may be casted into a Lie–Poisson type.

\footnote{Without regard to the Grassmann algebra of the original Clebsch parameterization, one may consider some Lie algebra \( \{ q, p \} = \omega \) as a generalized Clebsch parameterization of \( \omega \), and apply a similar algebra to define a Lie–Poisson bracket on a reduced phase space [2]. For example the so(3) bracket \( \{ q, p \} = q \times p \) can be used for three-component fields \( q \) and \( p \).}
Appendix B. Hall MHD and neutral fluid

Here we compare the MHD system with a Hall magnetohydrodynamics (HMHD) system and a neutral fluid (NF) system. In the context of Clebsch-parameterized Hamiltonian formalisms, HMHD is at the center of these systems; MHD and NF are brothers derived from HMHD by reduction (the reduction to MHD must be carefully performed to avoid a divergence problem, while the reduction to NF is simple). Interestingly, NF is not a reduction of MHD (although it is in the noncanonical formalism). We draw heavily on [14].

B.1. Clebsch parameterized HMHD system

In terms of Eulerian variables, the HMHD equations read
\[
\begin{align*}
\partial_t \rho & = -\nabla \cdot (\nabla \rho), \\
\partial_t V & = - (\nabla \times V) \times V - \nabla \left( \frac{1}{2} V^2 + \frac{1}{\rho} (\nabla \times B) \times B \right), \\
\partial_t B & = \nabla \times (V \times B),
\end{align*}
\]
where \( V_e = V - \epsilon_h \rho^{-1} J \) is the electron velocity, and \( \epsilon_h \) is the Hall parameter (the ion skin depth normalized by the system size). Boundary conditions are the same as those of MHD. Putting
\[
\begin{align*}
\mathbf{u} & = \begin{pmatrix} \rho \\ V \\ B \end{pmatrix}, \\
\mathcal{J} & = \begin{pmatrix} 0 & -\nabla, & 0 \\ -\nabla - \rho^{-1}(\nabla \times V) \times & \rho^{-1}(\nabla \times \mathbf{a}) \times B & 0 \\ 0 & \nabla \times (\mathbf{a} \times \rho^{-1} \mathbf{B}) - \epsilon_h \nabla \times (\rho^{-1}(\nabla \times \mathbf{a}) \times \mathbf{B}) & \end{pmatrix}, \\
H & = \int_\Omega \left( \frac{1}{2}\rho V^2 + \rho \mathcal{E}(\rho) + \frac{1}{2} B^2 \right) d^3x,
\end{align*}
\]
we can write HMHD in a noncanonical Hamiltonian form (see [30] for the proof of Jaconi’s identity), which has three Casimir invariants:
\[
\begin{align*}
C_1 & = \int_\Omega \rho d^3x, \\
C_2 & = \int_\Omega \mathbf{A} \cdot \mathbf{B} d^3x, \\
C_4 & = \int_\Omega \mathbf{P} \cdot \Omega d^3x,
\end{align*}
\]
where \( \mathbf{P} = V + \epsilon_h^{-1} \mathbf{A} \) is the ion canonical moment and \( \mathbf{\Omega} = \nabla \times \mathbf{P} \). (We must assume another boundary condition \( \mathbf{\Omega} \cdot n = 0 \mid_{\partial \Omega} \) for \( C_3 \) to be a Casimir invariant.)

By Clebsch parameterization
\[
\begin{align*}
\mathbf{P} & = -\nabla \phi_0 - \frac{\mu_1}{\rho} \nabla \phi_1, \\
\end{align*}
\]
we can canonize the HMHD system; with

\[ u_c = \left( \phi_0, \rho, \phi'_1, \mu'_1, \phi'_2, \mu'_2 \right)^T, \]  

\[ \mathcal{J}_c = \begin{pmatrix} J_c & 0 & 0 \\ 0 & J'_c & 0 \\ 0 & 0 & \epsilon_h J'_c \end{pmatrix} \]  

\[ H = \int_{\Omega} \mathcal{H} d^3x = \int_{\Omega} \left( \frac{1}{2} \rho V^2 + \rho \mathcal{E}(\rho) + \frac{1}{2} B^2 \right) d^3x, \]  

we obtain a canonical system

\[ \begin{cases} 
\dot{\phi}_0 = -V \cdot \nabla \phi_0 + h - V^2/2 - \rho^{-1} J \cdot A, \\
\dot{\rho} = -\nabla \cdot (\rho V), \\
\dot{\phi}'_1 = -V \cdot \nabla \phi'_1, \\
\dot{\mu}'_1 = -\nabla \cdot (\mu'_1 V), \\
\dot{\phi}'_2 = -V_c \cdot \nabla \phi'_2, \\
\dot{\mu}'_2 = -\nabla \cdot (\mu'_2 V_c), 
\end{cases} \]  

which is equivalent to HMHD.

The canonized MHD system described in section 2.2 is obtained as the limit \( \epsilon_h \to 0 \) under renormalization such that \( \alpha' = \epsilon_h^{-1}(\phi'_1 - \phi'_1), \beta' = \mu'_1 + \epsilon_h \mu'_2, \mu' = \mu'_2 \) and \( \phi' = \phi'_2 \) [14].

Similar to the case of MHD, the Casimir invariants are converted to the Noether charges pertinent to the gauge symmetries:

\[ C_1 = \int_{\Omega} \rho d^3x \leftrightarrow \delta \phi_0 = 1, \]  

\[ C_2 = \int_{\Omega} A \cdot B d^3x \leftrightarrow \begin{cases} 
\delta \phi_0 = -\frac{2\mu'_1}{\rho} \nabla \phi'_1 \cdot B, \\
\delta \phi'_2 = \frac{2\alpha}{\rho} \nabla \phi'_2 \cdot B, \\
\delta \mu'_1 = 2\epsilon_h \nabla \phi'_1 \cdot B, 
\end{cases} \]  

\[ C_4 = \int_{\Omega} P \cdot \Omega d^3x \leftrightarrow \begin{cases} 
\delta \phi_0 = \frac{2\rho'}{\rho^2} \nabla \phi'_1 \cdot \Omega, \\
\delta \phi' \rho = -\frac{2}{\rho} \nabla \phi'_1 \cdot \Omega, \\
\delta \mu'_1 = -2\nabla \phi'_1 \cdot \Omega, 
\end{cases} \]
B.2. Clebsch parameterized NF system

An NF system is obtained by putting $B = 0$ in (B.4)–(B.6). Then, $C_2$ and $C_3$ become trivial.

Instead, the fluid helicity

$$C_5 = \int_\Omega V \cdot (\nabla \times V) d^3x$$

is a Casimir invariant.

The corresponding canonized system is obtained by just putting $\mu_2^\ell = 0$ and $\phi_2^\ell = 0$ in HMHD, i.e., by eliminating the electromagnetic terms. The fluid velocity is now parameterized as

$$V = -\nabla \phi_0 - \frac{\mu_1^\ell}{\rho} \nabla \phi_1^\ell.$$  \hspace{1cm} (B.20)

The gauge symmetries and corresponding Noether charges are (denoting $\omega = \nabla \times V$)

$$C_1 = \int_\Omega \rho d^3x \leftrightarrow \delta \phi_0 = 1,$$

$$C_5 = \int_\Omega V \cdot \omega d^3x \leftrightarrow \begin{cases} 
\delta \phi_0 = \frac{2\mu_1^\ell}{\rho} \nabla \phi_1^\ell \cdot \omega, \\
\delta \phi_1^\ell = -\frac{2\mu_1^\ell}{\rho} \nabla \phi_1^\ell \cdot \omega, \\
\delta \mu_1^\ell = -2\nabla \phi_1^\ell \cdot \omega.
\end{cases}$$  \hspace{1cm} (B.22)

In tables B1 and B2, we summarize the parameterizations and gauge symmetries.

B.3. Reduction to two-dimensional geometry and single-pair Clebsch form

As noted at the end of section 2.2, representing vector fields by single-pair Clebsch forms yields a reduction onto zero-helicity leaves. Interestingly, it just fits a ‘two-dimensional fluid system’. Let us see a simple example of an NF system. We consider shallow water on $T^2$ (as discussed in appendix A, boundary conditions cause complexity, so we consider a periodic domain). Because of the small scale in the vertical direction (we choose $z$ as the vertical coordinate), we may separate the dynamics on the horizontal plane (spanned by $x$–$y$ coordinates) from those of the vertical direction. By putting $V_z = 0$ and $\partial_z = 0$ in the NF system, we obtain the two-dimensional reduction such that

| System | Clebsch parameterization |
|--------|--------------------------|
| HMHD   | $V = -\nabla \phi_0 - \frac{\mu_1^\ell}{\rho} \nabla \phi_1^\ell - \epsilon_2^1 \frac{\mu_1^2}{\rho} \nabla \phi_2^\ell$ |
|        | $B = \nabla \phi_2^\ell \times \nabla \phi_0^\ell$ |
| NF     | $V = -\nabla \phi_0 - \frac{\mu_1^\ell}{\rho} \nabla \phi_1^\ell$ |
| MHD    | $V = -\nabla \phi_0 - \frac{\mu_1^\ell}{\rho} \nabla \phi_1^\ell - \epsilon_1^1 \frac{\mu_1^2}{\rho} \nabla \phi_2^\ell$ |
|        | $B = \nabla \phi_2^\ell \times \nabla \phi_0^\ell$ |

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Table B2. Original Casimir invariants and corresponding gauge transformations in each system.

| System | Noether Charge | Gauge Transformation |
|--------|----------------|---------------------|
| HMHD   | \( \int \rho \, d^3x \) | \( \delta \phi_0 = 1 \) |
|        | \( \int \mathbf{P} \cdot \mathbf{\Omega} \, d^3x \) | \( \delta \phi_0 = \frac{2\mu}{\rho} \nabla \phi^f \cdot \mathbf{\Omega} \) |
|        | \( \delta \mu^f \) | \( \delta \phi_0 = \frac{\mu}{\rho} \nabla \phi^f \cdot \mathbf{\Omega} \) |
|        | \( \delta \phi_0 \) | \( \delta \mu^f = 2 \nabla \phi^f \cdot \mathbf{\Omega} \) |
|        | \( \delta \phi_0 \) | \( \delta \mu^f = 2 \mathbf{\Omega} \cdot \mathbf{\Omega} \) |
|        | \( \delta \phi_0 \) | \( \delta \mu^f = 2 \mathbf{\Omega} \cdot \mathbf{\Omega} \) |
|        | \( \delta \phi_0 \) | \( \delta \mu^f = 2 \mathbf{\Omega} \cdot \mathbf{\Omega} \) |
| NF     | \( \int \rho \, d^3x \) | \( \delta \phi_0 = 1 \) |
|        | \( \int \mathbf{V} \cdot \mathbf{\omega} \, d^3x \) | \( \delta \phi_0 = \frac{2\mu}{\rho} \nabla \phi^f \cdot \mathbf{\omega} \) |
|        | \( \delta \mu^f \) | \( \delta \phi_0 = - \frac{\mu}{\rho} \nabla \phi^f \cdot \mathbf{\omega} \) |
|        | \( \delta \phi_0 \) | \( \delta \mu^f = -2 \nabla \phi^f \cdot \mathbf{\omega} \) |
| MHD    | \( \int \rho \, d^3x \) | \( \delta \phi_0 = 1 \) |
|        | \( \int \mathbf{V} \cdot \mathbf{B} \, d^3x \) | \( \delta \phi_0 = \left( \frac{\rho}{\mu} \nabla \phi^f + \frac{\mu}{\rho} \nabla \phi^f \right) \cdot \mathbf{B} - \frac{\mu}{\rho} \nabla \phi^f \cdot \mathbf{\omega} \) |
|        | \( \delta \phi_0 \) | \( \delta \phi_0 \) |
|        | \( \delta \mu^f \) | \( \delta \phi_0 \) |
|        | \( \delta \phi_0 \) | \( \delta \mu^f \) |
|        | \( \delta \phi_0 \) | \( \delta \mu^f \) |
|        | \( \delta \phi_0 \) | \( \delta \mu^f \) |
|        | \( \delta \phi_0 \) | \( \delta \mu^f \) |

\[
\mathbf{u} = \begin{pmatrix} \rho \\ V_x \\ V_y \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & -\partial_x & -\partial_y \\ -\partial_x & 0 & \rho^{-1} \omega_z \\ -\partial_y & -\rho^{-1} \omega_z & 0 \end{pmatrix},
\]

(B.23)

where \( \omega_z = \partial_x V_y - \partial_y V_x \). This \( \mathcal{J} \) is the top-left corner 3 \times 3 component of (B.5). In this geometry, we have two classes of Casimir invariants; one is the usual \( C_1 = \int \rho \, d^3x \), and the others are new \( C_6 = \int \rho f (\omega_z / \rho) d^3x \) (\( f \) is an arbitrary smooth function). On the other hand, the fluid helicity \( C_5 = 0 \). Hence the single-pair Casimir parameterization

\[
\mathbf{V} = -\nabla \phi_0 - \frac{\mu}{\rho} \nabla \phi^f,
\]

(B.24)

which automatically yields \( \mathbf{V} \cdot (\nabla \times \mathbf{V}) = 0 \), suffices for the canonization. It is interesting that a new class of Casimir invariant \( C_6 \) resides on the singular leaf of \( C_5 = 0 \); in the
canonized representation, $C_6$ generates a gauge transformation of (B.24) (while that of $C_5$ vanishes); see [16] for a more extended discussion on the hierarchy of Casimir foliations.

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