Noncommutative Linear Sigma Models

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**Abstract**

We examine noncommutative linear sigma models with $U(N)$ global symmetry groups at the one-loop quantum level, and contrast the results with our previous study of the noncommutative $O(N)$ linear sigma models where we have shown that Nambu-Goldstone symmetry realization is inconsistent with continuum renormalization. Specifically we find no violation of Goldstone’s theorem at one-loop for the $U(N)$ models with the quartic term ordering consistent with possible noncommutative gauging of the model. The difference is due to terms involving noncommutative commutator interactions, which vanish in the commutative limit. We also examine the $U(2)$, and $O(4)$ linear sigma models with matter in the adjoint representation, and find that the former is consistent with Goldstone’s theorem at one-loop if we include only trace invariants consistent with possible noncommutative gauging of the model, while the latter exhibits violations of Goldstone’s theorem of the kind seen in the fundamental of $O(N)$ for $N > 2$. 
1 Introduction

Recently field theories on noncommutative spacetime backgrounds have been the subject of intense scrutiny [1]. Part of this motivation stems from the fact that noncommutative $U(N)$ gauge theories arise on D-branes in the presence of a constant NS-NS B-field background, in the zero-slope, field theoretic limit of string theory [2],[3]. A second motivation, independent of string theory, is the question of whether the world we live in is based on a noncommutative spacetime. In order to construct realistic models of particle physics on noncommutative spacetimes, one needs to be sure that noncommutative theories preserve the features that underlie the standard model, including perturbative renormalizability in the presence of spontaneous symmetry breaking[4],[5].

The general scheme for defining field theories with the noncommutative spacetime structure defined by $[\hat{x}^{\mu}, \hat{x}^{\nu}] = i\theta^{\mu\nu}, \theta^{\mu\nu}$ real, constant and antisymmetric, is to invoke Weyl-Moyal correspondence. This has the effect of replacing the underlying noncommutative spacetime with a commutative spacetime at the expense of replacing the ordinary pointwise product of spacetime dependent functions with an infinitely nonlocal star product. The induced momentum space Feynman rules for interaction vertices associated with a given field theory then involve momentum-dependent phases, which generically split a graph (at least at one-loop) into planar and nonplanar parts. The former are identical to the usual commutative graphs (up to a total phase depending only on the external momenta, and a combinatorial reweighting), and in particular possess the usual divergence structure associated with a commutative quantum field theory. The latter, nonplanar components are explicitly finite (at least at one-loop) because of oscillatory damping due to the phases, and replace an ultraviolet divergence with an infrared divergence in the external momenta [6],[7].

Superficially, as a consequence of the finiteness of nonplanar graphs, and of the similar divergence structure of the planar graphs, one might conclude that the renormalization of noncommutative field theories proceeds as in the commutative theory, because the counterterm structure is formally the same. However, as is well-known, the renormalization of spontaneously broken theories, with either underlying global or gauge symmetries, is more subtle because the number of counterterm vertices exceeds the number of renormalization parameters. As a result, the renormalizability of (commutative) spontaneously broken theories hinges in general on intricate graphwise cancellations [4], [5] order by order in perturbation theory. Thus it is of obvious interest to examine whether or not these cancellations persist in noncommutative field theories.

In a previous paper [8] we studied the spontaneous symmetry breaking of a global $O(N)$ symmetry in the noncommutative deformation of the linear sigma model with scalars in the fundamental representation. We found that one-point tadpoles of the sigma at one-loop were insensitive to the noncommutativity because no external momentum flows into the trilinear tadpole vertex. Thus the one-point sigma counterterm is identical to the one in the commutative limit, which in turn fixes the pion mass counterterm to be the same.
as its commutative limit. On the other hand, the planar components of the 1PI graphs contributing to the one-loop pion (inverse) propagator renormalization are re-weighted with respect to the corresponding commutative graphs. As a consequence, there is an unavoidable UV cutoff dependence (for nonzero external momentum) after renormalization, signalling the nonexistence of a continuum limit, and noncommuting UV ($\Lambda_{UV} \to \infty$) and IR ($p \to 0$) limits. Specifically we found that the sum of the 1PI graphs and the counterterm contributing to the pion mass renormalization yielded quadratic and logarithmic UV cutoff $\Lambda$ dependences as:

$$\sum_{1\text{-}\text{loop}} = \frac{\lambda\delta^i_j}{16\pi^2} \left\{ \left[ N(1 - f) + f \right] \Lambda^2 \left( 1 - \frac{1}{1 + \Lambda^2(p \circ p)} \right) - \left( 2 - f \right) m^2_\sigma \log(1 + \Lambda^2(p \circ p)) \right\} + \text{finite} \ast p^2 \quad (1)$$

respectively. Here $N$ is the dimension of the fundamental of $O(N)$, $p \circ q \equiv -p_\mu q^\mu /4$, and $f$ takes into account the two possible quartic orderings for $\pi\pi\pi\pi$ and $\pi\pi\sigma\sigma$ terms:

$$\frac{\lambda}{4} f(\pi^k \ast \pi^k) \ast (\pi^l \ast \pi^l) + \frac{\lambda}{4} (1 - f)(\pi^k \ast \pi^l) \ast (\pi^k \ast \pi^l)$$

$$+ \frac{\lambda}{2} f(\pi^k \ast \pi^k) \ast \sigma \ast \sigma + \frac{\lambda}{2} (1 - f)(\pi^k \ast \sigma) \ast (\pi^k \ast \sigma) \subset \mathcal{L} \quad (2)$$

For nonzero $\theta$ and $p$, the only circumstance under which we can take the continuum limit is when $f = 2$ and $N = 2$, where both logarithmic and quadratic dependences on $\Lambda$ vanish. This corresponds to the Abelian $O(2)$ model, and if written in terms of a complex scalar $\phi$ corresponds precisely to the ordering $\phi^* \phi^* \phi^* \phi$. Otherwise, the conditions $f = N/(N-1)$ and $f = 2$ required for the cancellations of quadratic and logarithmic dependences on $\Lambda$ respectively, cannot be simultaneously satisfied, Goldstone’s theorem fails at the one-loop level, and the continuum limit of the model fails to exist. Thus for general $N > 2$ and for all possible orderings consistent with the global $O(N)$ symmetry, the noncommutative $O(N)$ linear sigma model does not exist in the continuum limit.

Prima facie, this incompatibility of continuum renormalizability with spontaneous symmetry breaking for $O(N)$ linear sigma models appears to present severe difficulties for attempts to make realistic models of particle physics on noncommutative spacetimes. First, it is clear that models with spontaneously broken gauge symmetries must have consistent spontaneously broken global limits (as the gauge couplings vanish); the absence of such a global limit with spontaneous symmetry breaking would preclude its subsequent gauging (at least perturbatively). Second, the standard model of the fundamental interactions (and unified theories which encompass it) depends, for electroweak symmetry breaking, on a complex Higgs doublet. As is well known, resolved into real components, the purely scalar sector of the standard model is $O(4)$ invariant (and not just $SU(2) \times U(1)$ invariant), with the real

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1The interested reader may verify that the $N = 2, f = 2$ case of our general analysis, is precisely the global limit of the Abelian $U(1) \equiv O(2)$ model subsequently considered in [9] as can be easily seen by comparing Lagrangian densities, and Feynman rules; the conclusions of reference [9] agree with our previous analysis [8].
components in the fundamental representation; our previous results then appear to preclude noncommutative deformation of the standard model. We will argue below that this is not necessarily the case. In particular, noncommutative theories with \( N \) complex scalars, \( \Phi^i \) \((i = 1..N, N > 1)\), and with \( U(N) \) invariant self-interactions, are not invariant under an \( O(2N) \) symmetry acting on their real components, due to purely noncommutative commutator interactions arising from the noncommutativity of the spacetime. Thus we will first undertake an analysis for the case of a \( U(N) \) symmetry group with the scalars in the fundamental representation, choosing the quartic invariant \( \Phi^\dagger \Phi \Phi^\dagger \Phi \). The spontaneous breaking of this group to \( U(N-1) \) leaves \( N-1 \) complex pions, and one real pion. We find that the one-loop 1PI graphs contributing to the mass renormalization of the complex pions, like the one-point tadpoles, do not see the noncommutativity at this order, and so Goldstone’s theorem holds. The 1PI one-loop graphs contributing to the mass renormalization of the real pion (which arises through the breaking of the \( U(1) \equiv O(2) \) subgroup of the \( U(N) \)), are split into divergent planar, and finite nonplanar pieces in such a way that Goldstone’s theorem holds at one-loop. The essential difference between the \( U(N) \) models and the corresponding \( O(2N) \) models \((N > 1)\) is the presence of the purely noncommutative commutator interactions in the former.

We will also begin to explore how our present, and previous [8], results might depend on the scalar field representation responsible for spontaneously breaking the symmetry. In particular, we consider both an \( O(4) \) and a \( U(2) \) model, with scalars in the adjoint representation, to see if our previous results depended on our scalars being in the respective fundamentals. For the \( U(2) \) model with matter in the adjoint representation, we will find that Goldstone’s theorem holds if we include only interactions involving a single trace operator, which we will in turn demonstrate are the only ones consistent with noncommutative gauge invariance in the case that we gauge the \( U(2) \) symmetry. In this model Goldstone’s theorem holds due to a notable cancellation of a purely noncommutative graph involving couplings to the \( U(1) \) component of the field. For the \( O(4) \) model with matter in the adjoint, we find violations of Goldstone’s theorem at one-loop of the type found in the \( O(N) \) fundamental representation studied in [8]. Finally we discuss the implications of these results for model building, and comment on the nature of the IR divergences found by [7] in the context of noncommutative theories with matter in the adjoint representation.

2 NC \( U(N) \) Linear Sigma model: Fundamental Rep

In this section we examine Goldstone’s theorem in the noncommutative deformation of the linear sigma model with a global \( U(N) \) symmetry group, and contrast the results with our previously discovered violations of Goldstone’s theorem in the \( O(N) \) linear sigma model.

The noncommutative \( U(N) \) linear sigma model is defined by the Lagrangian density given by

\[
\mathcal{L} = \partial_\mu \Phi^\dagger \ast \partial^\mu \Phi + \mu^2 \Phi^\dagger \ast \Phi - \lambda \Phi^\dagger \ast \Phi \ast \Phi^\dagger \ast \Phi \quad (3)
\]
where $\Phi$ is an $N$-vector of complex fields $\phi_i$ ($i = 1..N$), where the star product is defined as usual by $f(x) \ast g(x) = \exp (i\theta^{\mu\nu}\partial_\mu \partial_\nu) f(y)g(z)|_{y,z\rightarrow x}$, and where we have included the star ordering of the quartic term consistent with noncommutative gauge invariance of a possible gauging of the model (see below). For $\mu^2 > 0$, the symmetry is spontaneously broken to $U(N - 1)$. Throughout the remainder of this paper, we will consider only translationally invariant vacua. By an $SU(N)$ transformation, we can rotate the resulting VEV into the last field of $\Phi$, and by a $U(1)$ rotation we can identify this VEV with a constant shift, $\sigma$, in the real part of this field. Thus we define $\pi_i = \phi_i$ for $i = 1..N - 1$, while $\phi_N = (\sigma + a + i\pi_0)/\sqrt{2}$; there are $N - 1$ complex Goldstone bosons, and one real Goldstone mode. The minimization of the potential for this configuration implies:

$$V(a) = \frac{\mu^2}{2} a^2 - \frac{\lambda}{4} a^4 \longrightarrow a^2 = \frac{\mu^2}{\lambda} \tag{4}$$

Writing (3) in terms of these variables yields:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \pi_0)^2 + \partial_\mu \pi_i^* \partial_\mu \pi_i - \frac{1}{2}(2\mu^2)\sigma^2 - \lambda a \sigma^3 - \lambda a \pi_0^2 \sigma - 2\lambda a \pi_i^* \pi_i \sigma - \lambda \pi_i^* \pi_j \pi_j^* \pi_j - \frac{\lambda}{4}(\sigma^4 + \pi_0^4) - \lambda \pi_0^2 \sigma^2 + \frac{\lambda}{2} \pi_0 \sigma \pi_0 - \lambda \pi_i^* \pi_i (\sigma^2 + \pi_0^2) - \lambda \pi_i^* \pi_i [\sigma, \pi_0] \tag{5}$$

For notational brevity all star products will be suppressed henceforth, unless there is danger of confusion. Furthermore, we will implicitly use the identity

$$\int A_1 \ast \ldots \ast A_n = \int A_{\sigma(1)} \ast \ldots \ast A_{\sigma(n)} \tag{6}$$

(where $\{\sigma(1) \ldots \sigma(n)\}$ represents any cyclic permutation of $\{1 \ldots n\}$), with the understanding that all Lagrangian density terms sit under a spacetime integral. This identity means that quadratic terms in the action, and hence propagators, are identical to their commutative counterparts.

To simplify the discussion relative to that occurring in [8], we will not a priori impose the vanishing of the tadpole as a renormalization condition. Instead we will include the one-point tadpole contributions, and their counterterm directly in calculating the mass renormalization of the pion. In this completely equivalent language, the two counterterms present cancel each other, up to the wavefunction renormalization, so the sum of the one-particle irreducible (1PI) graphs and the one-point tadpole insertions must be automatically finite up to wavefunction renormalization (and for Goldstone’s theorem to hold at one-loop, must vanish in the $p \rightarrow 0$ limit)[10]. Furthermore, to exhibit the essentially algebraic nature of the result, we will expand the non-phase part of the integrands about zero-external momentum, in the cases where there are two propagators in the loop using the Taylor expansion

$$\frac{1}{k^2((p + k)^2 - m^2)} = \frac{1}{k^2(k^2 - m^2)} - p_\mu \frac{2k^\mu}{k^2(k^2 - m^2)^2} + \ldots \tag{7}$$

As Gubser and Sondhi have argued [11], more exotic vacua such as stripe phases are possible in noncommutative theories.
and then note that the $p$-dependent terms yield finite loop-momentum integrals (for all $p$), and vanish as $p \to 0$. We then define the momentum integrals

$$I(m^2) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2}, \quad I_{\theta,p}(m^2) = \int \frac{d^4k}{(2\pi)^4} \frac{\cos(k \times p)}{k^2 - m^2} = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik \times p}}{k^2 - m^2}$$

(8)

where $k \times p = k_\mu \theta^{\mu \nu} p_\nu$.

The vertices for the theory are listed in the Appendix, and the propagators are the usual ones. Dashes denote complex pions, dots the real pion associated with the sigma, and solid lines denote the sigma. The 1PI one-loop graphs contributing to the mass renormalization of the $N-1$ complex pions are

$$\pi_i \quad k \quad \pi_j^* \quad \equiv (a); \quad \pi_i \quad k \quad \pi_j^* \quad \equiv (b)$$

$$\pi_i \quad k \quad \pi_j^* \quad \equiv (c); \quad \pi_i \quad k \quad \pi_j^* \quad \equiv (d)$$

(9)

They are given respectively by

$$(a) = -2i\lambda i \int \frac{d^4k}{(2\pi)^4} \frac{\delta_{kl} \left[ \delta^{ij} \delta^{ kl} e^0 + \delta^{il} \delta^{jk} e^0 \right]}{k^2} = 2N\lambda \delta^{ij} I(0)$$

$$(b) = -2i\lambda \lambda \delta^{ij} \int \frac{d^4k}{(2\pi)^4} \frac{e^0 \cos(0)}{k^2} = \lambda \delta^{ij} I(0)$$

$$(c) = -2i\lambda \lambda \delta^{ij} \int \frac{d^4k}{(2\pi)^4} \frac{e^0 \cos(0)}{k^2 - 2\mu^2} = \lambda \delta^{ij} I(2\mu^2)$$

$$(d) = -2i\lambda \lambda \delta^{ij} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-i \theta(k \times p) e^{-i \theta(p \times -k)}}}{k^2 - 2\mu^2}$$

$$= \frac{4\lambda^2 \lambda^2}{2\mu^2} \delta^{ij} \int \frac{d^4k}{(2\pi)^4} \left[ \frac{1}{k^2 - 2\mu^2} - \frac{1}{k^2} \right] + \delta^{ij} A^\mu p_\mu$$

$$= 2\lambda \delta^{ij} \left[ I(2\mu^2) - I(0) \right] + \delta^{ij} A^\mu p_\mu$$

(10)

where $A^\mu(p)$ is finite for all $p$. Evidently these 1PI graphs do not see the noncommutativity. Meanwhile the one-point tadpoles insertions, as in [8], also do not see the noncommutativity at one-loop. They are given by

$$\pi_i \quad p \quad \pi_j^* \quad \equiv (e); \quad \pi_i \quad p \quad \pi_j^* \quad \equiv (f); \quad \pi_i \quad p \quad \pi_j^* \quad \equiv (g)$$

(11)
Their values are given by

\[
(e) = (-2i\lambda\delta^{ij}) \frac{i}{2\mu^2} (-2i\lambda a\delta^{kk}) iI(2\mu^2) = -2(N - 1)\lambda\delta^{ij} I(0)
\]

\[
(f) = (-2i\lambda\delta^{ij}) \frac{i}{2\mu^2} (-6i\lambda a)\frac{i}{2} I(2\mu^2) = -3\lambda\delta^{ij} I(2\mu^2)
\]

\[
(g) = (-2i\lambda\delta^{ij}) \frac{i}{-2\mu^2} (-2i\lambda a)\frac{i}{2} I(2\mu^2) = -\lambda\delta^{ij} I(0)
\]

where all noncommutative vertices manifestly collapse.

The sum of these seven graphs is equal to zero (modulo the finite term which itself vanishes as \(p \to 0\)), independently of a regulator, and of the noncommutativity, whence Goldstone’s theorem holds at one-loop; the complex pions undergo no mass renormalization. Now consider the one-loop mass renormalization of \(\pi_0\). The 1PI graphs contributing are given by

\[
\pi_0 \quad \pi_0 \quad \pi_0 \equiv (h);
\]

\[
\pi_0 \quad \pi_0 \quad \pi_0 \equiv (i)
\]

\[
\pi_0 \quad \pi_0 \quad \pi_0 \equiv (j);
\]

\[
\pi_0 \quad \pi_0 \quad \pi_0 \equiv (k)
\]

with values given by

\[
(h) = \frac{-2i\lambda}{2} \int \frac{d^4k}{(2\pi)^4} i \left[ 2 \cos^2 \left( \frac{k \cdot p}{2} \right) + 1 \right] = 2\lambda I(0) + \lambda \theta, p(0)
\]

\[
(i) = \frac{-2i\lambda}{2} \int \frac{d^4k}{(2\pi)^4} i \left[ 2 \cos^2(0) - \cos(k \cdot p) \right] = 2\lambda I(2\mu^2) - \lambda \theta, p(2\mu^2)
\]

\[
(j) = -2i\lambda\delta^{kk} \int \frac{d^4k}{(2\pi)^4} \frac{ie^0 \cos(0)}{k^2} = 2(N - 1)\lambda I(0)
\]

\[
(k) = (-2i\lambda a)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i^2 \cos^2 \left( \frac{p \cdot k}{2} \right)}{k^2 [k^2 - 2\mu^2]^2}
\]

\[
= \frac{4\lambda^2 a^2}{2\mu^2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{4} \left( 1 + \cos(k \cdot p) \right) \left[ \frac{1}{k^2 - 2\mu^2} - \frac{1}{k^2} \right] + B^\mu_\ell p_\mu
\]

\[
= \lambda [I(2\mu^2) - I(0)] + \lambda \theta, p(2\mu^2) - \theta, p(0) + B^\mu_\ell p_\mu
\]

where \(B^\mu_\ell(p)\) is finite for all \(p\). Evidently, the nonplanar contributions due to the noncommutativity cancel between these graphs. For completeness, the one-point tadpole insertions are

\[
\pi_0 \quad \pi_0 \quad \pi_0 \equiv (l);
\]

\[
\pi_0 \quad \pi_0 \quad \pi_0 \equiv (m);
\]

\[
\pi_0 \quad \pi_0 \quad \pi_0 \equiv (n)
\]
with values given by

\[(l) = (-2i\lambda a)\frac{i}{-2\mu^2}(-2i\lambda a \delta^{kk})iI(0) = -2(N - 1)\lambda I(0)
\]

\[(m) = (-2i\lambda a)\frac{i}{-2\mu^2}(-6i\lambda a)\frac{i}{2}I(2\mu^2) = -3\lambda I(2\mu^2)
\]

\[(n) = (-2i\lambda a)\frac{i}{-2\mu^2}(-2i\lambda a)\frac{i}{2}I(0) = -\lambda I(0)
\]

(16)

The sum of these seven graphs also vanishes (again modulo the finite \(p\) dependent term, which vanishes as \(p \to 0\)); so that Goldstone’s theorem also holds for the neutral pion of this model.

Let us reflect on these results. First, had we included the other ordering of the quartic term \(\phi_i^* \phi_j^* \phi_i \phi_j\), we would again find violations of Goldstone’s theorem of the type found in \[8\]. Secondly, we contrast these results with those of the general \(O(N)\) model studied in \[8\], where we showed violations of Goldstone’s theorem at one-loop for all orderings consistent with the \(O(N)\) global symmetry (except for the trivial Abelian case). The difference here of course is that we are working with a \(U(N)\) group, which we now show exhibits crucial algebraic differences with the \(O(N)\) case in noncommutative scalar theories.

Matter in the fundamental of \(O(N)\) is described by a real \(N\)-vector of fields, which we denote by \(\Psi\). As such, the invariant term \(\Psi^T \Psi\) merely is the sum of squares of the real components. Then, the expansion of the quartic invariant can yield cross-terms only of the form \(\psi_i \psi_i^* \psi_j \psi_j\), corresponding to the two possible star product orderings of such an invariant. Note that no more than two distinct fields can occur.

On the other hand, the fundamental of \(U(N)\) is described by a complex \(N\)-vector of fields, \(\Phi\). Now however, the quadratic invariant \(\Phi^T \Phi\), written in terms of real fields picks up the commutator of each field’s real part with its imaginary part due to the noncommutativity since

\[(R - iI)(R + iI) = R^2 + I^2 + i[R, I]
\]

(17)

While such commutators in the quadratic term vanish when integrated over spacetime, the quartic invariant now picks up products of such commutators with other fields or commutators which, for \(N > 1\), constitute new interactions between real components of two complex \(\phi\)’s, not present in, and incompatible with, the \(O(2N)\) symmetry.

Let us make this argument manifest. Expanding the quartic term in the \(U(N)\) theory in terms of its real components yields

\[
\phi_i^* \phi_i \phi_j^* \phi_j = (\phi_i R - i\phi_i I)(\phi_j R + i\phi_j I)(\phi_j R - i\phi_j I)(\phi_j R + i\phi_j I)
\]

\[
= \phi_i^2 R^2 + \phi_j^2 R^2 + \phi_i^2 \phi_j^2 + \phi_i^2 \phi_j^2 - [\phi_i R, \phi_i I][\phi_j R, \phi_j I]
\]

\[+ i(\phi_i^2 R^2 + \phi_i^2 I^2)[\phi_j R, \phi_j I] + i(\phi_j^2 R^2 + \phi_j^2 I^2)[\phi_i R, \phi_i I]
\]

(18)
We note that for \( i \neq j \) (which can occur for \( N > 1 \)), the presence of interactions (those involving three or four distinct real fields) which cannot occur in the \( O(2N) \) case by the general argument above. We emphasize this is a purely noncommutative effect\(^3\). The presence of these extra, purely noncommutative interactions is responsible for the differing behaviour of the spontaneously broken phase at the quantum level for these models.

To conclude, we have found that one cannot in general spontaneously break a fundamental representation NC \( O(N) \) linear sigma model, while one can break a fundamental representation NC \( U(N) \) linear sigma model for the noncommutative gauge invariant quartic ordering. This latter theme is one that will arise again in a more dramatic fashion in the adjoint representation model to which we now turn.

3 NC \( U(2) \) Sigma Model: Adjoint Representation

We now examine the status of Goldstone’s theorem in the noncommutative deformation of the linear sigma model with scalars in the adjoint representation of \( U(2) \). There are several reasons for this: first, we wish to compare the results for adjoint representation scalars with our results from the previous section for fundamental scalars, in a tractable case. Secondly, adjoint matter naturally arises in noncommutative world-volume theories on D-branes. Thirdly, grand unified theories embedding the standard model commonly rely on adjoints for the first stage of symmetry breaking.

We write the scalars in the adjoint of \( U(2) \) as

\[
\Phi = \phi_a T^a = \frac{1}{2} \left( \begin{array}{cc} \phi_4 + \phi_3 & \sqrt{2} \phi^* \phi_1 \\ \phi^* \phi_2 & \phi_4 - \phi_3 \end{array} \right)
\]

(19)

where \( T^a \) are the canonical generators of \( U(2) \): \( T^a = \sigma^a / 2 \), for \( a = 1, 2, 3 \) and \( T^4 = I_2 / 2 \). The global \( U(2) \) symmetry transformation acts as

\[
\Phi \rightarrow U \Phi U^\dagger
\]

(20)

and as before does not involve the star product because the symmetry is global. For simplicity we impose invariance under \( \Phi \rightarrow -\Phi \). The Lagrangian density for the global model we consider is defined by

\[
\mathcal{L} = \text{Tr} \left( \partial_\mu \Phi \star \partial^\mu \Phi \right) + \mu^2 \text{Tr} \left( \Phi \star \Phi \right) - \lambda_1 \text{Tr} \left( \Phi \star \Phi \star \Phi \star \Phi \right) - \lambda_2 \left[ \text{Tr} (\Phi \star \Phi) \right]^2
\]

(21)

where we define

\[
\text{Tr} (\Phi^4_{\star}) \equiv \Phi^4_{i} \star \Phi^4_{j} \star \Phi^4_{k} \star \Phi^4_{l}
\]

\[
\left[ \text{Tr} (\Phi \star \Phi) \right]^2_{\star} \equiv \Phi^4_{i} \star \Phi^4_{j} \star \Phi^4_{k} \star \Phi^4_{l}
\]

(22)

\(^3\)For \( i = j \) (or \( N = 1 \)), the last two terms vanish under the spacetime integral, and the product of commutators merely induces the orderings of the \( O(2) \) model studied in \(^8\) with \( f = 2 \).
and where we discuss the remaining, omitted trace invariants and star product orderings at the end of this section.

Let us now consider spontaneous symmetry breaking which occurs for $\mu^2 > 0$ (we take $\lambda_i > 0$). Then $\Phi$ acquires a vacuum expectation value, say $\Phi_0$, and since it is a Hermitian (but not necessarily traceless) matrix, we analyze it by diagonalization to the form

$$\Phi_0 = \frac{1}{2} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

whence the potential becomes

$$V(a, b) = -\frac{\mu^2}{4}(a^2 + b^2) + \frac{\lambda_1}{16}(a^4 + b^4) + \frac{\lambda_2}{16}(a^4 + 2a^2b^2 + b^4)$$

This is minimized for

$$a^2 = b^2 = \frac{\mu^2}{\lambda_1^2 + \lambda_2} \equiv \frac{\mu^2}{\lambda}$$

The states corresponding to $a = b$, which are degenerate in energy with the states corresponding to $a = -b$, and admitted because we are considering $U(2)$ and not simply $SU(2)$, do not reflect spontaneously broken states, because $\Phi_0$ is then proportional to the identity and so manifestly commutes with all of the generators. Furthermore since they correspond to constant shifts in the $U(1)$ component $\phi_4$, they are forbidden by the discrete symmetry. On the other hand, the states corresponding to $a = -b$ do yield spontaneously broken vacua, since they do not commute with the $T^1$ and $T^2$ generators and reflect a vacuum expectation value for the field $\phi_3$.

In notation suggestive of the linear sigma model, we expand around the vacuum $b = -a < 0$ (without any loss of generality), defining $\sigma$ and $\pi$ through

$$\Phi' = \frac{1}{2} \begin{pmatrix} \phi_4 + \sigma & \sqrt{2}\pi^* \\ \sqrt{2}\pi & \phi_4 - \sigma \end{pmatrix} \equiv \Phi - \Phi_0$$

so that $\phi_3 = \sigma + a$. Expanding the scalar potential in terms of these variables yields

$$V = \frac{1}{2}(2\mu^2)\sigma^2 + \frac{1}{2}(\lambda_1 a^2)\phi_4^2 + \frac{\lambda_1 + \lambda_2}{2}\pi^*\pi\pi^*\pi + \frac{\lambda_2}{2}\pi^*\pi^*\pi^*\pi + \frac{\lambda_1 + \lambda_2}{2}(\pi^*\pi + \pi^*\pi^*)\sigma^2 - \frac{\lambda_1}{2}\pi^*\pi\pi\sigma + \frac{\lambda_1 + \lambda_2}{2}(\pi^*\pi + \pi^*\pi^*)\phi_4^2 + \frac{\lambda_1}{2}\pi^*\pi\phi_4\phi_4 + a\lambda(\pi^*\pi + \pi^*\pi^*)\sigma + \frac{\lambda}{4}(\sigma^4 + \phi_4^4) + \lambda a\sigma^3 + \frac{\lambda_1 + \lambda_2}{2}\sigma^2\phi_4^2 + \frac{\lambda_1}{4}\sigma\phi_4\sigma\phi_4 + (\lambda + \lambda_1)a\phi_4^2\sigma + \frac{\lambda_1}{2}[(\pi^*\phi_4\pi\sigma - \pi^*\sigma\pi\phi_4 + a(\pi^*\phi_4 - \pi^*\phi_4))]$$

using $\lambda = \lambda_1/2 + \lambda_2$. 

9
The symmetrized vertices for this theory are listed in the Appendix. In the following, solid lines denote the $\sigma$, dots denote the $\phi_4$, and dashes denote the $\pi$. Excluding the purely noncommutative interactions for separate consideration, there are four 1PI graphs contributing to the mass renormalization of the complex pion (Goldstone mode) in this model:

\[
\begin{align*}
\pi &\quad k &\quad \pi^* \equiv (a) ; &\quad \pi &\quad k &\quad \pi^* \equiv (b) \\
\pi &\quad k &\quad \pi^* \equiv (c) ; &\quad \pi &\quad k &\quad \pi^* \equiv (d)
\end{align*}
\]

with values given by

\[
(a) = -2i(\lambda_1 + \lambda_2) \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2} - 2i\lambda_2 \int \frac{d^4k}{(2\pi)^4} \frac{i \cos^2(\frac{k \times p}{2})}{k^2} \\
= (2\lambda_1 + 3\lambda_2)I(0) + \lambda_2 I_{\theta,p}(0)
\]

\[
(b) = (\lambda_1 + \lambda_2)I(2\mu^2) - \frac{\lambda_1}{2} I_{\theta,p}(2\mu^2)
\]

\[
(c) = (\lambda_1 + \lambda_2)I(\lambda_1 a^2) + \frac{\lambda_1}{2} I_{\theta,p}(\lambda_1 a^2)
\]

\[
(d) = (-2i\lambda a)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{(p + k)^2} \left[ \frac{1}{k^2 - 2\mu^2} - \frac{1}{k^2} \right] \cos^2(\frac{k \times p}{2}) \\
= 4\lambda^2 a^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{2\mu^2} \left[ \frac{1}{k^2 - 2\mu^2} - \frac{1}{k^2} \right] \cos^2(\frac{k \times p}{2}) + C^\mu(p)p_\mu \\
= \lambda \left[I(2\mu^2) - I(0)\right] + \lambda \left[I_{\theta,p}(2\mu^2) - I_{\theta,p}(0)\right] + C^\mu(p)p_\mu
\]

where $C^\mu$ is finite for all $p$.

The one-point tadpole contributions are given by

\[
\begin{align*}
\pi &\quad \pi \equiv (e) ; &\quad \pi &\quad \pi^* \equiv (f) ; &\quad \pi &\quad \pi^* \equiv (g)
\end{align*}
\]

which are respectively given by

\[
(e) = (-2i\lambda a)^2 \frac{i}{-2\mu^2} iI(0) = -2\lambda I(0)
\]

\[
(f) = (-2i\lambda a)^2 \frac{i}{-2\mu^2} (-6i\lambda a) \frac{i}{2} I(2\mu^2) = -3\lambda I(2\mu^2)
\]

\[
(g) = (-2i\lambda a) \frac{i}{-2\mu^2} (-2i(\lambda + \lambda_1) a) \frac{i}{2} I(\lambda_1 a^2)
\]

\[
= -(\lambda + \lambda_1) I(\lambda_1 a^2)
\]

(31)
The sum of these seven graphs is given by
\[
\sum = \frac{\lambda_1}{2} [I(0) - I_{\theta,p}(0)] - \lambda_2 \left[ I(2\mu^2) - I_{\theta,p}(2\mu^2) \right] - \frac{\lambda_1}{2} \left[ I(\lambda_1 a^2) - I_{\theta,p}(\lambda_1 a^2) \right] + C^\mu(p)p_\mu \tag{32}
\]
In the commutative limit \( \theta \to 0 \), this degenerates to the finite term \( C^\mu(p)p_\mu \) (which itself vanishes as \( p \to 0 \)), so the mass counterterm vanishes and this is a demonstration of Goldstone’s theorem for this model. However for nonzero \( \theta \), the \( I(m^2) \) terms are divergent and require regularization, say by an ultraviolet cutoff \( \Lambda \). But there is no counterterm freedom to cancel the \( \Lambda \) dependence, so for nonzero \( p \) and nonzero \( \theta \) we cannot take the continuum limit; that is, UV \((\Lambda \to \infty)\) and IR \((p \to 0)\) limits do not commute.

However, we have (intentionally) neglected a purely noncommutative graph due to the last interaction in (27). The purely noncommutative interaction generated by \((\pi \pi^* - \pi^* \pi)\phi_4\) yields a graphical contribution given by

\[
\pi - p \quad \quad \pi^* \quad \quad k \quad \quad k + p
\]

\[
= (-\lambda_1 a)^2 \int \frac{d^4k}{(2\pi)^4} \frac{\sin(\frac{p \cdot k}{2}) \sin(\frac{-k \cdot p}{2})}{k^2[(p+k)^2 - \lambda_1 a^2]}
\]

\[
= \lambda_1 a^2 \int \frac{d^4k}{(2\pi)^4} \frac{\sin^2(\frac{k \cdot p}{2})}{\lambda_1 a^2} \left[ \frac{1}{k^2 - \lambda_1 a^2} - \frac{1}{k^2} \right] + D_\theta^\mu(p)p_\mu
\]

\[
= \frac{\lambda_1}{2} \int \frac{d^4k}{(2\pi)^4} [1 - \cos(k \cdot p)] \left[ \frac{1}{k^2 - \lambda_1 a^2} - \frac{1}{k^2} \right] + D_\theta^\mu(p)p_\mu
\]

\[
= \frac{\lambda_1}{2} [I(\lambda_1 a^2) - I_{\theta,p}(\lambda_1 a^2)] - \frac{\lambda_1}{2} [I(0) - I_{\theta,p}(0)] + D_\theta^\mu(p)p_\mu \tag{33}
\]

where again \( D_\theta^\mu \) is finite for all \( p \), and vanishes also in the limit \( \theta \to 0 \). Rather unexpectedly, this graph, which manifestly vanishes in the commutative limit, and involves the \( U(1) \) component of the matter field, cancels the \( \lambda_1 \) pieces in (32), leaving behind a residual divergence (for nonzero \( p \)) that depends only on the coupling to the \( \text{Tr}(\Phi^2)^2 \) term in the potential.

However, in the corresponding gauge theory, the term
\[
\text{Tr}(\Phi_i^2) \ast \text{Tr}(\Phi_r^2)
\]

is not gauge invariant even under the spacetime integral. In fact no term involving the product of more than one trace in the adjoint representation is gauge invariant (even under \( \int d^Dx \)) in noncommutative theories for \( N > 1 \). To see this write (34) in terms of its internal indices (first choosing the canonical ordering with respect to the star product) and gauge transform:
\[
[\text{Tr}(\Phi^2)]^2 = \Phi_j^i \ast \Phi_l^i \ast \Phi_k^j \ast \Phi_k^j \\
\to (U_{i1}^i \ast \Phi_l^{i1} \ast U_{j2}^j \ast \Phi_l^{j2} \ast U_{l1}^l) \ast (U_{k1}^k \ast \Phi_l^{k1} \ast U_{l2}^l \ast \Phi_l^{k2} \ast U_{k2}^k)
\]

\[
= U_{i1}^i \ast \Phi_l^{i1} \ast \Phi_l^{j2} \ast U_{k1}^k \ast \Phi_l^{k1} \ast \Phi_l^{k2} \ast U_{k2}^k \tag{35}
\]
The presence of the star product does not allow us to use \( U^j_i \ast U^j_k = \delta^j_k \) on the remaining local \( U \) and \( U^\dagger \) factors which are separated by two factors of \( \Phi \), even if we use the cyclicity property of the star product under the spacetime integral. This is to be contrasted with a single internal index trace term (with canonical internal index ordering), and the commutative limit where the ordering of components is immaterial.

It is clear this argument applies both to the other internal index ordering \( \Phi^j_i \ast \Phi^k_l \ast \Phi^i_j \ast \Phi^l_k \) (whose gauge transformation does not allow the use of \( U \ast U^\dagger = I \) anywhere), and to any product of (internal index) traces in the adjoint representation. Thus if we forbid \([\text{Tr}(\Phi^2)]^2\) from the scalar potential, by regarding the global theory as the limit of a gauge theory, we have no remaining violation of Goldstone’s theorem for this model. Incidentally, this argument also forbids the other terms still allowed by the imposition of the discrete symmetry that we neglected when we wrote the scalar potential for this theory; namely

\[
\text{Tr}(\Phi) \ast \text{Tr}(\Phi^3) , \text{Tr}(\Phi) \ast \text{Tr}(\Phi) , \text{[Tr}(\Phi)]^4 \ast \text{Tr}(\Phi^2) \ast [\text{Tr}(\Phi)]^2
\]

as well as other star product orderings of the \( \text{Tr}(\Phi^4) \) term.

An immediate consequence of the preceding argument is that for \( U(N) \) gauge theories with adjoint scalar matter, the symmetry breaking pattern is restricted to only one of the two possible patterns that would be allowed by the commutative limit of the theory. Specifically, because noncommutative gauge invariance forbids \( \text{Tr}(\Phi^2) \ast \text{Tr}(\Phi^2) \), vacuum stability now requires \( \lambda_1 > 0 \), and thus allows only the breaking pattern \( U(N) \rightarrow U(n_1) \times U(N - n_1) \) (with \( n_1 = N/2, N \) even; or \( n_1 = (N + 1)/2, N \) odd)\[12\], and forbids \( U(N) \rightarrow U(N - 1) \)[12].

This argument has another consequence for noncommutative theories in general. As van Raamsdonk and Seiberg [7] demonstrated in considering scalar theories with scalars represented by \( N \times N \) matrices, all infrared divergences of the type found in [6] are proportional at one-loop to

\[
\text{Tr}(\mathcal{O}_1)\text{Tr}(\mathcal{O}_2)
\]

where \( \mathcal{O}_i \) are operators built out of \( \Phi \). Furthermore we have seen above that an operator of this form \( (\text{Tr}(\Phi^2) \ast \text{Tr}(\Phi^2)) \) appearing in the scalar potential, would induce violations of Goldstone’s theorem by renormalization effects. However the preceding argument indicates that these are precisely the form of operators that are not gauge invariant in an adjoint representation gauge theory. So if we regard these theories as embedded in a corresponding gauge theory where we must forbid such terms, then we would expect that infrared divergences for \( N > 1 \)[9] of the form observed in [7], no longer appear.

\[\text{For the } N = 1 \text{ case considered in } [3], \text{ corresponding to a single scalar, the above argument fails, since the index structure becomes degenerate.}\]
In this section we repeat the analysis of the previous section for the noncommutative \(O(4)\) sigma model in the adjoint representation; again this will allow us to study, in a simple context, scalar representation (in)dependence of our results on Goldstone renormalization, this time in the context of orthogonal symmetry groups.

We consider the classical symmetry breaking \(O(4) \rightarrow U(2)\). Now \(\Phi\) is a real antisymmetric matrix, whence the vacuum state \(\Phi_0\) can be put in standard form

\[
\Phi_0 = \frac{a}{2} \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}
\]

(38)

The scalar potential is given by

\[
V(\Phi) = \frac{\mu^2}{2} \text{Tr}(\Phi^2) + \frac{\lambda_1}{4} [\text{Tr}(\Phi^2)]^2 + \frac{\lambda_2}{4} \text{Tr}(\Phi^4)
\]

(39)

where we note that the sign of the quadratic term is opposite that of the \(U(2)\) model of the previous section because of the antisymmetry (as opposed to Hermiticity) of \(\Phi\), and where we have normalized differently for later convenience (we now assume the canonical internal index ordering with respect to the star product as per the conclusions of the previous section).

Thus the minimization of the potential with respect to the vacuum (38), yields

\[
V(\Phi_0) = -\frac{\mu^2}{2} a^2 + \frac{\lambda_1 a^4}{4} + \frac{\lambda_2 a^4}{16} \rightarrow a^2 = \frac{4\mu^2}{4\lambda_1 + \lambda_2} = \frac{\mu^2}{\lambda_1 + \lambda_2}
\]

(40)

The suitable parametrization of \(\Phi\) relevant to a discussion of spontaneous symmetry breaking is given by

\[
\frac{1}{2} \begin{pmatrix}
0 & (\sigma + a) + \psi & \alpha + \pi_1 & \beta + \pi_2 \\
-(\sigma + a) + \psi & 0 & \pi_2 - \beta & \alpha - \pi_1 \\
-(\alpha + \pi_1) & \beta - \pi_2 & 0 & (\sigma + a) - \psi \\
-(\beta + \pi_2) & \pi_1 - \alpha & \psi - (\sigma + a) & 0
\end{pmatrix}
\]

(41)

where the \(\sigma\) is the field acquiring the VEV, and \(\pi_1, \pi_2\) are the two Goldstone modes. Focusing now on the one-loop mass renormalization of one of the \(\pi\)’s, say \(\pi_1\), the expansion of the potential reads

\[
V = \frac{1}{2} (2\mu^2) \sigma^2 + \frac{1}{2} (\lambda_2 a^2 / 2) \left[ \psi^2 + \alpha^2 + \beta^2 \right] + \frac{1}{4} \left( \lambda_1 + \frac{\lambda_2}{2} \right) \pi_1^4 + \left( \frac{\lambda_1}{2} + \frac{\lambda_2}{4} \right) \pi_2^2 + \\
\alpha^2 + \beta^2 + \psi^2 + \sigma^2 + \frac{\lambda_2}{8} \left[ -\pi_1 \pi_2 \pi_1 \pi_2 + \pi_1 \alpha \pi_1 \alpha + \pi_1 \beta \pi_1 \beta + \pi_1 \psi \pi_1 \psi - \pi_1 \sigma \pi_1 \sigma \right] \\
+ \left( \lambda_1 + \frac{\lambda_2}{4} \right) a \sigma \left[ \sigma^2 + \pi_1^2 + \pi_2^2 \right] + \left( \lambda_1 + \frac{3\lambda_2}{4} \right) a \sigma \left[ \alpha^2 + \beta^2 + \psi^2 \right] + \ldots
\]

(42)
where the ellipsis represents (four-field) terms that do not contribute to the one-loop mass renormalization of \( \pi_1 \). The Feynman rules for these vertices are in the appendix. Now that we have six distinct fields, we simply use dotted lines to denote the \( \pi \)’s, and use solid lines for the other four fields, and instead explicitly label the lines.

The 1PI graphs contributing are

\[
\begin{align*}
\pi_1 \pi_1 \pi_1 & \equiv (a) ; \\
\pi_1 \pi_2 \pi_1 & \equiv (b) ; \\
\pi_1 (\beta ) (\psi) & \equiv (c) ; \\
\pi_1 \sigma \pi_1 & \equiv (d) ; \\
\pi_1 \pi_1 \sigma & \equiv (e) .
\end{align*}
\]

and are given respectively by

\[
\begin{align*}
(a) &= -2i \left( \lambda_1 + \frac{\lambda_2}{4} \right) \frac{i}{2} \int \frac{d^d k}{(2\pi)^4} \frac{1 + \cos^2 \left( \frac{k \cdot p}{2} \right)}{k^2 (p + k)^2} - \left( \lambda_1 + \frac{\lambda_2}{4} \right) \left[ 2I(0) + I_{\theta,p}(0) \right] \\
(b) &= \left( \lambda_1 + \frac{\lambda_2}{2} \right) I(0) - \frac{\lambda_2}{4} I_{\theta,p}(0) \\
(c) &= 3 \left[ \left( \lambda_1 + \frac{\lambda_2}{2} \right) I(2\mu^2) + \frac{\lambda_2}{4} I_{\theta,p}(\lambda_2 a^2/2) \right] \\
(d) &= \left( \lambda_1 + \frac{\lambda_2}{2} \right) I(2\mu^2) - \frac{\lambda_2}{4} I_{\theta,p}(2\mu^2) \\
(e) &= \left[ -2i \left( \lambda_1 + \frac{\lambda_2}{4} \right) a \right]^2 \frac{i^2}{2} \int \frac{d^d k}{(2\pi)^4} \frac{\cos^2 \left( \frac{k \cdot k}{2} \right)}{k^2} \\
&= \frac{2 \left( \lambda_1 + \frac{\lambda_2}{4} \right)^2 a^2}{2\mu^2} \int \frac{d^d k}{(2\pi)^4} \left[ 1 + \cos(p \cdot k) \right] \left[ \frac{1}{k^2 - 2\mu^2} - \frac{1}{k^2} \right] + D^\mu p_\mu \\
&= \left( \lambda_1 + \frac{\lambda_2}{4} \right)^2 \left[ I(2\mu^2) - I(0) \right] + \left( \lambda_1 + \frac{\lambda_2}{4} \right) \left[ I_{\theta,p}(2\mu^2) - I_{\theta,p}(0) \right] + D^\mu p_\mu
\end{align*}
\]

where \( D^\mu \) is finite for all \( p \), and where the factor of three in the third graph originates from having three species of particle with the same contribution to this calculation.

The one-point tadpoles contributions are

\[
\begin{align*}
\pi_1(2) \pi_1 \sigma & \equiv (f) ; \\
\pi_1 \sigma \pi_1 & \equiv (g) ; \\
\pi_1 \alpha(\beta)(\psi) & \equiv (h) .
\end{align*}
\]
with values given by

\[(f) = 2 \times -2i \left( \lambda_1 + \frac{\lambda_2}{4} \right) a \frac{i}{-2\mu^2} (-2i) \left( \lambda_1 + \frac{\lambda_2}{4} \right) a \frac{i}{2} I(0) = -2 \left( \lambda_1 + \frac{\lambda_2}{4} \right) I(0)\]

\[(g) = -2i \left( \lambda_1 + \frac{\lambda_2}{4} \right) a \frac{i}{-2\mu^2} (-6i) \left( \lambda_1 + \frac{\lambda_2}{4} \right) a \frac{i}{2} I(2\mu^2) = -3 \left( \lambda_1 + \frac{\lambda_2}{4} \right) I(2\mu^2)\]

\[(h) = 3 \times -2i \left( \lambda_1 + \frac{\lambda_2}{4} \right) a \frac{i}{-2\mu^2} (-2i) \left( \lambda_1 + \frac{3\lambda_2}{4} \right) a \frac{i}{2} I(\frac{\lambda_2 a^2}{2}) = -3 \left( \lambda_1 + \frac{3\lambda_2}{4} \right) I(\frac{\lambda_2 a^2}{2})\]

\[(46)\]

where the overall factors of two and three in the first and third graphs respectively again come from the multiplicity of particle species with the same contribution.

Thus the total one-loop contribution to the mass renormalization of the \(\pi_1\) (or \(\pi_2\)) in this model is

\[\sum = \frac{\lambda_2}{4} [I(0) - I_{\theta,p}(0)] - \frac{3\lambda_2}{4} \left[ I(\frac{\lambda_2 a^2}{2}) - I_{\theta,p}(\frac{\lambda_2 a^2}{2}) \right] - \lambda_1 \left[ I(2\mu^2) - I_{\theta,p}(2\mu^2) \right] + D^\mu p_\mu (47)\]

Unlike the \(U(2)\) adjoint representation model, there is no purely noncommutative graph that saves us for either quartic invariant, and so again we cannot take the continuum limit (\(\Lambda_{UV} \to \infty\)) and Goldstone’s theorem fails for this model.

## 5 Discussion

To summarize: in noncommutative field theory, \(U(N)\) (\(N > 1\)) linear sigma models with complex scalars in the fundamental representation, do not have \(O(2N)\) global invariance due to noncommutative commutator interactions between the real components, which vanish in the commutative limit. As a result of these commutator interactions, noncommutative linear \(U(N)\) sigma models with fundamental matter can be continuum renormalized while preserving Nambu-Goldstone symmetry realization, at least at one-loop. This contrasts with our previous results [8], where we demonstrated that for noncommutative linear \(O(N)\) sigma models with fundamental matter, continuum renormalization is inconsistent with Nambu-Goldstone symmetry realization already at one loop (except for the degenerate Abelian case \(O(2) \equiv U(1)\)).

To investigate possible scalar representation dependence of these contrasting results, we have considered linear sigma models with adjoint matter. For the adjoint \(U(2)\) linear sigma model, we again find that Nambu-Goldstone symmetry realization survives at one-loop, provided we drop interaction terms (and star product orderings) which would be inconsistent with the gauging of the symmetry; noncommutative restrictions on the allowed operators in a \(U(N)\) gauge theory Lagrangian also act to restrict the allowed symmetry breaking patterns.
For the adjoint $O(4)$ linear sigma model, we find violations of Nambu-Goldstone symmetry realization at one-loop order, as in the fundamental $O(N)$ models. These results suggest that the difference in behaviour is determined by the symmetry group, as opposed to the scalar representation thereof.

Our results for the noncommutative linear $U(N)$ sigma models now open the possibility of building models of the elementary particles and their interactions based on noncommutative non-Abelian theories with spontaneous symmetry breaking. Clearly, to make particle physics models, it is necessary that the spontaneous symmetry breaking be consistent with the renormalization not just of the global limits of these theories, but also with their gaugings; we see two reasons to be sanguine on this point, at least at one-loop. First, the gauging of the $U(1)(=O(2))$ model$^\text{[9]}$ is consistent with spontaneous symmetry breaking for precisely the star orderings uniquely picked out$^\text{[8]}$ by our$^\text{[8]}$ calculation of the Goldstone violating effects in the general noncommutative $O(N)$ fundamental linear sigma model. Second, in our treatment of the non-Abelian $U(2)$ model with adjoint scalars, violations of Nambu-Goldstone symmetry realization vanish when one restricts to the subset of couplings which would be allowed, were the symmetry to be gauged; so the limited evidence suggests that global theories may be a good guide to the behaviour of the local theories, much as in the case of commutative field theories$^\text{[4]}, [5]$. However, to go from models to actual theories would require demonstration of all-order consistency of continuum renormalization of noncommutative theories with spontaneous symmetry breaking. While failure of Nambu-Goldstone symmetry breaking can be demonstrated at one-loop, demonstrating consistency requires an all order analysis; this remains a major open issue in this field.

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A Appendix

A.1 Scalar Potential Feynman Rules, $U(N)$ Fundamental

All momenta flow into the vertices.

$^5$to make the anomalous effects vanish, which can happen only in the Abelian $O(2)$ case.
\[
\begin{align*}
\pi_i^* p_1 & \quad \pi_j p_2 & = & -2i\lambda \left[ \delta^{ij} \delta^{kl} e^{-\frac{i}{2}(p_1 \times p_2 + p_3 \times p_4)} + \delta^{il} \delta^{jk} e^{\frac{i}{2}(p_1 \times p_2 + p_3 \times p_4)} \right] \\
\pi_k^* p_3 & \quad \pi_l p_4
\end{align*}
\]

\[
\begin{align*}
\sigma(\pi_0) p_1 & \quad \sigma(\pi_0) p_2 & = & -2i\lambda \left[ \cos\left(\frac{p_1 \times p_2}{2}\right) \cos\left(\frac{p_3 \times p_4}{2}\right) + \cos\left(\frac{p_1 \times p_3}{2}\right) \cos\left(\frac{p_2 \times p_4}{2}\right) \right. \\
& & & \left. + \cos\left(\frac{p_1 \times p_4}{2}\right) \cos\left(\frac{p_2 \times p_3}{2}\right) \right] \\
\sigma(\pi_0) p_3 & \quad \sigma(\pi_0) p_4
\end{align*}
\]

\[
\begin{align*}
\sigma p_1 & \quad \sigma p_2 & = & -2i\lambda \left[ 2 \cos\left(\frac{p_1 \times p_2}{2}\right) \cos\left(\frac{p_3 \times p_4}{2}\right) - \cos\left(\frac{p_1 \times p_3}{2} + \frac{p_2 \times p_4}{2}\right) \right] \\
\pi_0^* p_3 & \quad \pi_0 p_4
\end{align*}
\]

\[
\begin{align*}
\pi_i^* p_1 & \quad \pi_j p_2 & = & -2i\lambda \delta^{ij} e^{-\frac{i}{2}(p_1 \times p_2) \cos\left(\frac{p_3 \times p_4}{2}\right)} \\
\sigma(\pi_0) p_3 & \quad \sigma(\pi_0) p_4
\end{align*}
\]

\[
\begin{align*}
\pi_i^* p_1 & \quad \pi_j p_2 & = & -2i\lambda \delta^{ij} e^{-\frac{i}{2}(p_1 \times p_2) \sin\left(\frac{p_3 \times p_4}{2}\right)} \\
\sigma p_3 & \quad \sigma p_4
\end{align*}
\]

\[
\begin{align*}
\sigma p_1 & \quad \sigma p_3 & = & -2i\lambda a \left[ \cos\left(\frac{p_1 \times p_2}{2}\right) + \cos\left(\frac{p_1 \times p_3}{2}\right) + \cos\left(\frac{p_2 \times p_3}{2}\right) \right] \\
\sigma p_2 & \quad \sigma p_3
\end{align*}
\]

\[
\begin{align*}
\pi_i^* p_1 & \quad \sigma p_3 & = & -2i\lambda a \delta^{ij} e^{-\frac{i}{2}(p_1 \times p_2) \sin\left(\frac{p_3 \times p_4}{2}\right)} \\
\pi_j p_2 & \quad \sigma p_3
\end{align*}
\]
A.2 Scalar Potential Feynman Rules, $U(2)$ Adjoint

All momenta flow into the vertices.

$$\pi^* p_1 \sigma p_3 = -2i\lambda a \cos \left( \frac{p_1 \times p_2}{2} \right)$$ \hspace{1cm} (55)

$$\pi^* p_3 \pi p_4 = -2i(\lambda_1 + \lambda_2) \cos \left( \frac{p_1 \times p_2}{2} \right) \cos \left( \frac{p_3 \times p_4}{2} \right) - 2i\lambda_2 \cos \left( \frac{p_1 \times p_3}{2} \right) \cos \left( \frac{p_2 \times p_4}{2} \right)$$ \hspace{1cm} (56)

$$\pi^* p_1 \pi p_2 = -2i(\lambda_1 + \lambda_2) \cos \left( \frac{p_1 \times p_2}{2} \right) \cos \left( \frac{p_3 \times p_4}{2} \right) + i\lambda_1 \cos \left( \frac{p_1 \times p_3}{2} \right) + \frac{p_2 \times p_4}{2}$$ \hspace{1cm} (57)

$$\pi^* p_1 \pi p_2 = -2i(\lambda_1 + \lambda_2) \cos \left( \frac{p_1 \times p_2}{2} \right) \cos \left( \frac{p_3 \times p_4}{2} \right) - i\lambda_1 \cos \left( \frac{p_1 \times p_3}{2} \right) + \frac{p_2 \times p_4}{2}$$ \hspace{1cm} (58)

$$\sigma p_3 \sigma p_4 = -2i(\lambda_1 + \lambda_2) \cos \left( \frac{p_1 \times p_2}{2} \right) \cos \left( \frac{p_3 \times p_4}{2} \right) - i\lambda_1 \cos \left( \frac{p_1 \times p_3}{2} \right) + \frac{p_2 \times p_4}{2}$$ \hspace{1cm} (59)

$$\sigma p_3 \sigma p_4 = -2i(\lambda_1 + \lambda_2) \cos \left( \frac{p_1 \times p_2}{2} \right) \cos \left( \frac{p_3 \times p_4}{2} \right) - i\lambda_1 \cos \left( \frac{p_1 \times p_3}{2} \right) + \frac{p_2 \times p_4}{2}$$ \hspace{1cm} (60)

$$\sigma(\phi_4) p_1 \sigma(\phi_4) p_2 = -2i\lambda \left[ \cos \left( \frac{p_1 \times p_2}{2} \right) \cos \left( \frac{p_3 \times p_4}{2} \right) + \cos \left( \frac{p_1 \times p_3}{2} \right) \cos \left( \frac{p_2 \times p_4}{2} \right) \right]$$

$$\sigma(\phi_4) p_3 \sigma(\phi_4) p_4 = -2i\lambda \left[ \cos \left( \frac{p_1 \times p_2}{2} \right) \cos \left( \frac{p_3 \times p_4}{2} \right) + \cos \left( \frac{p_1 \times p_3}{2} \right) \cos \left( \frac{p_2 \times p_4}{2} \right) \right]$$

$$\sigma(\phi_4) p_3 \sigma(\phi_4) p_4 = -2i\lambda \left[ \cos \left( \frac{p_1 \times p_2}{2} \right) \cos \left( \frac{p_3 \times p_4}{2} \right) + \cos \left( \frac{p_1 \times p_3}{2} \right) \cos \left( \frac{p_2 \times p_4}{2} \right) \right]$$

$$\sigma(\phi_4) p_3 \sigma(\phi_4) p_4 = -2i\lambda \left[ \cos \left( \frac{p_1 \times p_2}{2} \right) \cos \left( \frac{p_3 \times p_4}{2} \right) + \cos \left( \frac{p_1 \times p_3}{2} \right) \cos \left( \frac{p_2 \times p_4}{2} \right) \right]$$

$$\sigma(\phi_4) p_3 \sigma(\phi_4) p_4 = -2i\lambda \left[ \cos \left( \frac{p_1 \times p_2}{2} \right) \cos \left( \frac{p_3 \times p_4}{2} \right) + \cos \left( \frac{p_1 \times p_3}{2} \right) \cos \left( \frac{p_2 \times p_4}{2} \right) \right]$$
\[\pi^* p_1 \quad \phi_4 p_2 \quad \pi p_3 \quad \sigma p_4 = -\lambda_1 \sin\left(\frac{p_1 \times p_2}{2} + \frac{p_3 \times p_4}{2}\right)\] (61)

\[\pi^* p_1 \quad \sigma p_3 = -2i\lambda a \cos\left(\frac{p_1 \times p_2}{2}\right)\] (62)

\[\pi^* p_1 \quad \sigma p_3 = -2i\lambda \left[\cos\left(\frac{p_1 \times p_2}{2}\right) + \cos\left(\frac{p_1 \times p_3}{2}\right) + \cos\left(\frac{p_2 \times p_3}{2}\right)\right]\] (63)

\[\phi_4 p_1 \quad \sigma p_3 = -2i(\lambda + \lambda_1) a \cos\left(\frac{p_1 \times p_2}{2}\right)\] (64)

\[\phi_4 p_1 \quad \phi_4 p_3 = -\lambda_1 a \sin\left(\frac{p_1 \times p_2}{2}\right)\] (65)

\[\phi_4 p_1 \quad \phi_4 p_3 = -\lambda_1 a \sin\left(\frac{p_1 \times p_2}{2}\right)\] (66)

### A.3 Scalar Potential Feynman Rules, O(4) Adjoint (Partial)

All momenta flow into the vertices.

\[\pi_1 p_1 \quad \pi_1 p_2 \quad \pi_1 p_3 \quad \pi_1 p_4 = -2i\left(\lambda_1 + \frac{\lambda_2}{4}\right) \left[\cos\left(\frac{p_1 \times p_2}{2}\right) \cos\left(\frac{p_3 \times p_4}{2}\right) + \cos\left(\frac{p_1 \times p_3}{2}\right) \cos\left(\frac{p_2 \times p_4}{2}\right)\right] + \cos\left(\frac{p_1 \times p_4}{2}\right) \cos\left(\frac{p_2 \times p_3}{2}\right)\] (67)
\[ \pi_1 \ p_1 \quad \pi_1 \ p_2 \quad \pi_2 \ p_3 \quad \pi_2 \ p_4 = -2i(\lambda_1 + \frac{\lambda_2}{2}) \cos\left(\frac{p_1 \times p_2}{2}\right) \cos\left(\frac{p_3 \times p_4}{2}\right) + i\frac{\lambda_2}{2} \cos\left(\frac{p_1 \times p_3}{2} + \frac{p_2 \times p_4}{2}\right) \]

(68)

\[ \alpha(\beta)(\psi) \ p_3 \quad \alpha(\beta)(\psi) \ p_4 \quad \pi_1 \ p_1 \quad \pi_1 \ p_2 \]

\[ = -2i(\lambda_1 + \frac{\lambda_2}{2}) \cos\left(\frac{p_1 \times p_2}{2}\right) \cos\left(\frac{p_3 \times p_4}{2}\right) - i\frac{\lambda_2}{2} \cos\left(\frac{p_1 \times p_3}{2} + \frac{p_2 \times p_4}{2}\right) \]

(69)

\[ \sigma \ p_3 \quad \sigma \ p_4 \quad \pi_1 \ p_1 \quad \pi_1 \ p_2 \]

\[ = -2i(\lambda_1 + \frac{\lambda_2}{2}) \cos\left(\frac{p_1 \times p_2}{2}\right) \cos\left(\frac{p_3 \times p_4}{2}\right) + i\frac{\lambda_2}{2} \cos\left(\frac{p_1 \times p_3}{2} + \frac{p_2 \times p_4}{2}\right) \]

(70)

\[ \sigma \ p_3 \quad \sigma \ p_4 \quad \pi_1(2) \ p_1 \quad \pi_1(2) \ p_2 \]

\[ = -2i(\lambda_1 + \frac{\lambda_2}{2}) \cos\left(\frac{p_1 \times p_2}{2}\right) \]

(71)

\[ \sigma \ p_3 \quad \sigma \ p_4 \quad \pi_1(2) \ p_1 \quad \pi_1(2) \ p_2 \]

\[ = -2i(\lambda_1 + \frac{3\lambda_2}{4}) \cos\left(\frac{p_1 \times p_2}{2}\right) \]

(73)

\[ \alpha(\beta)(\psi) \ p_1 \quad \sigma \ p_3 \quad \alpha(\beta)(\psi) \ p_2 \quad \sigma \ p_3 \quad \alpha(\beta)(\psi) \ p_1 \quad \sigma \ p_3 \quad \alpha(\beta)(\psi) \ p_2 \quad \sigma \ p_3 \]

\[ = -2i(\lambda_1 + \frac{\lambda_2}{2}) \cos\left(\frac{p_1 \times p_2}{2}\right) \]

(74)

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