When Frictions are Fractional: Rough Noise in High-Frequency Data

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Abstract

The analysis of high-frequency financial data is often impeded by the presence of noise. This article is motivated by intraday transactions data in which market microstructure noise appears to be rough, that is, best captured by a continuous-time stochastic process that locally behaves as fractional Brownian motion. Assuming that the underlying efficient price process follows a continuous Itô semimartingale, we derive consistent estimators and asymptotic confidence intervals for the roughness parameter of the noise and the integrated price and noise volatilities, in all cases where these quantities are identifiable. In addition to desirable features such as serial dependence of increments, compatibility between different sampling frequencies and diurnal effects, the rough noise model can further explain divergence rates in volatility signature plots that vary considerably over time and between assets.

Keywords: Hurst parameter, market microstructure noise, mixed fractional Brownian motion, mixed semimartingales, volatility estimation, volatility signature plot.

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1 Introduction

One of the stylized features of high-frequency financial data is the presence of market microstructure noise (Black 1986). In financial econometrics, the observed (logarithmic) price process $Y$ of an asset is therefore often modeled as a sum

$$Y_t = X_t + Z_t,$$

(1.1)

where $X$, called the efficient price process, reflects the value of the asset according to some economic theory and $Z$ is a microstructure noise process that captures deviations of $Y$ from $X$. Typical noise sources include bid–ask bounces, discreteness of prices, informational asymmetry or transaction costs. As both $X$ and $Z$ are of economic interest but not observable, a major challenge is to develop statistical procedures to disentangle the two based on observations of $Y$ only. For example, given a continuous Itô semimartingale

$$X_t = X_0 + \int_0^t a_s \, ds + \int_0^t \sigma_s \, dB_s,$$

(1.2)

a key quantity of interest is the integrated (price) volatility $C_T = \int_0^T \sigma_s^2 \, ds$ for some finite time horizon $T$. In the absence of noise, estimating $C_T$ is a straightforward matter: given observations $\{X_{i\Delta_n} : i = 1, \ldots, [T/\Delta_n]\}$, the realized variance (RV) defined by $\sum_{i=1}^{[T/\Delta_n]} (\Delta_i^n X)^2$, where $\Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n}$, is a consistent estimator of $C_T$ as $\Delta_n \to 0$ (Andersen et al. 2001, 2003, Barndorff-Nielsen & Shephard 2002).

However, in practice, RV typically explodes as the sampling frequency increases, indicating the presence of noise at high frequencies (see, e.g., the volatility signature plots of Andersen et al. (2000)). In order to construct noise-robust estimators of $C_T$, a common approach in the literature is to model $(Z_t)_{t \geq 0}$ at the observation times $i\Delta_n$ as

$$Z_{i\Delta_n} = \varepsilon_i^n,$$

(1.3)

where for each $n$, $(\varepsilon_i^n)_{i=1}^{[T/\Delta_n]}$ is a discrete time series. Examples for $\varepsilon_i^n$ include rounding noise (Delattre & Jacod 1997, Li & Mykland 2007, Robert & Rosenbaum 2010, 2012, Rosenbaum 2009), white noise (Bandi & Russell 2006, Barndorff-Nielsen et al. 2008, Podolskij & Vetter 2009, Zhang et al. 2005), AR- or MA-type noise (Aït-Sahalia et al. 2011, Da & Xiu 2021, Hansen & Lunde 2006), and certain non-parametric extensions thereof (Jacod et al. 2009, 2017, Li et al. 2020, Li & Linton 2022).

The current paper is motivated by statistical properties found in certain samples of high-frequency financial data that cannot be explained by the aforementioned noise models. For instance, if the noise $Z$ is independent of $X$ and takes the form (1.3), where $\varepsilon = (\varepsilon_i^n)_{i=1}^{[T/\Delta_n]}$ is a stationary time series with a distribution that does not depend on $n$,\(^1\) it is a simple consequence of the law of large numbers (LLN) that

$$\Delta_n^{[T/\Delta_n]} \sum_{i=1} (\Delta_i^n Y)^2 \xrightarrow{p} 2 \operatorname{Var}(\varepsilon)(1 - r(1)),$$

where $r$ is the autocorrelation function (ACF) of $\varepsilon_i^n$; cf. Jacod et al. (2017), Zhang et al. (2005). In particular, the RV of the observed process $Y$ blows up at a rate of $\Delta_n^{-1}$. However,

\(^1\) These assumptions can be substantially relaxed; see Da & Xiu (2021), Jacod et al. (2017).
Figure 1: Volatility signature plot (a) and variance plot (b) for INTC transaction data on November 30, 2007 (top). The same plots on a log–log scale (middle) reveal a divergence rate of $-0.40$ and a shrinkage rate of increments of $0.72$ on this particular day. The histograms (bottom) show the daily divergence rates in volatility signature plots and the daily shrinkage rates of price increments in 2007 INTC transaction data.

as Figure 1 (a) shows, the divergence rate of RV in daily Intel (INTC) transaction data from 2007 is typically much slower (e.g., around $\Delta_n^{-0.40}$ on November 30, 2007). An almost equivalent way of illustrating this observation is to consider variance plots, in which the sample variance of increments of $Y$ is computed as a function of the sampling frequency. In our data sample, we observe shrinking price increments; see Figure 1 (b). By contrast, in the noise model above, the shrinkage rate is 0 since

$$\text{Var}(\Delta_i^n Y) \sim \text{Var}(\varepsilon_i^n - \varepsilon_{i-1}^n) = 2 \text{Var}(\varepsilon)(1 - r(1)),$$

(1.4)

It is, of course, not possible to tell from volatility signature plots or variance plots whether these observations constitute statistically significant findings. To address this issue, we perform four different statistical tests on our data set, the results of which are shown in Figure 2 (a)–(d). Panel (a) shows the histogram of $\log H_{3n}$, where $H_{3n}$ is the Hausman statistic introduced by Aït-Sahalia & Xiu (2019), demonstrating that the presence of noise is highly significant on almost all trading days in the considered sample. Similarly, Jacod
et al. (2017)’s point estimators and 95%-confidence bands for \( \text{Var}(\varepsilon) \) in panel (b) indicate that \( \text{Var}(\varepsilon) \) is significantly different from 0 throughout 2007. This suggests that

\[
\text{Var}(Z_t \Delta_n) = \text{Var}(\varepsilon^n_t) \text{ is bounded away from 0.} \tag{1.5}
\]

Next, panel (c) shows a histogram of the test statistic

\[
\frac{\sum_{i=1}^{[T/\Delta_n]} (\Delta^n_i Y)^2}{\sqrt{2 \sum_{i=1}^{[T/\Delta_n]} (\Delta^n_i Y)^2 (\Delta^n_{i+2} Y)^2}} \left( \frac{\sum_{i=1}^{[T/\Delta_n]} (\Delta^n_i Y + \Delta^n_{i+1} Y)^2}{\sum_{i=1}^{[T/\Delta_n]} (\Delta^n_i Y)^2} - 1 \right),
\]

which is asymptotically \( N(0,1) \) if \( \varepsilon^n_t \) is a (possibly modulated) white noise. The white noise model for \( \varepsilon^n_t \) can thus be safely rejected on each day of our sample in favor of a colored noise model. This is further confirmed by the plots in panels (e) and (f): there is significant second-order autocorrelation in price increments, which would be absent in the case of white noise (cf. Aït-Sahalia et al. (2011)). To obtain an idea of what dependence structure may be appropriate for \( \varepsilon^n_t \), we compute Jacod et al. (2017)’s point estimators and 95%-confidence intervals for the first-order autocorrelation \( r(1) \) of the noise. As panel (d) shows, there is a high correlation between \( \varepsilon^n_t \) and \( \varepsilon^n_{t+1} \), which is not significantly different from 1 on many days.\(^2\) By (1.4), it follows that \( \text{Var}(\varepsilon^n_t - \varepsilon^n_{t-1}) \approx 0 \), which we interprete as

\[
\text{Var}(\Delta^n_t Z) = \text{Var}(\varepsilon^n_t - \varepsilon^n_{t-1}) \rightarrow 0. \tag{1.6}
\]

In conclusion, there is strong empirical evidence that market microstructure noise in our data sample is non-shrinking (because of (1.5)) but with shrinking increments (because of (1.6)).\(^3\) To our best knowledge, all microstructure noise models that have been considered so far in the literature are either non-shrinking with non-shrinking increments (as in (1.3)) or shrinking with (necessarily) shrinking increments (Aït-Sahalia & Xiu 2019, Da & Xiu 2021, Kalnina & Linton 2008).\(^4\) The goal of this work is to fill in this gap.

2 Model

We will now derive our noise model from the desired properties (1.5) and (1.6) and some mild regularity conditions. In what follows, both the noise and the efficient price process are defined on a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) satisfying the usual conditions.

Assumption (C1). The noise process \((Z_t)_{t \geq 0}\) in (1.1) is a stochastic process indexed by a continuous-time parameter \( t \) and is continuous in probability.

While modeling microstructure noise as a continuous-time process is a change of paradigm compared to the existing literature, it is a natural way of implementing the shrinking increments property (1.6) observed in our data. If the noise variables \( Z_t \Delta_n \) do not change

\(^2\) This is not a contradiction to Figure 2 (e), since \( r(1) \) is the autocorrelation of the noise, while the figure shows the autocorrelation of price increments.

\(^3\) We do not claim that this is universally true. In fact, Figure 8 below shows that the classical noise model (1.3) gives a very good fit for INTC transaction data from 1997–2000. This is in line with Aït-Sahalia & Xiu (2019) who found that noise has decreased over time due to improvements in market efficiency.

\(^4\) In Aït-Sahalia et al. (2005), Hansen & Lund (2006), continuous-time noise models are considered for which RV does not diverge.
much on average from \( i \) to \( i + 1 \) (not just in a distributional but in a pathwise sense), this implies some form of continuity (e.g., in probability) between them. Together with the fact that \( \{ i \Delta_n : i, n \in \mathbb{N} \} \) is a dense subset of \([0, \infty)\), the observations \( \{ Z_{i \Delta_n} : i \in \mathbb{N} \} \), at least for large \( n \), essentially determine a continuous-time process \( (Z_t)_{t \geq 0} \).

Remark 2.1. Under Assumption (C1), the noise process is, by definition, compatible between different sampling frequencies, a property that is typically hard to satisfy for colored noise models with non-shrinking increments (see, for example, Section 7.1.2 in Aït-Sahalia & Jacod (2014) or Remark 2.7 in Jacod et al. (2017)).

Assumption (C2). The noise process \( (Z_t)_{t \geq 0} \) is a mean-zero second-order stationary stochastic process.

The (rather strong) stationarity assumption on the noise reduces technicalities in the subsequent exposition and will be relaxed in our final model, where a time-varying and possibly non-stationary stochastic noise volatility is permitted. By the Wold–Karhunen representation of second-order stationary processes (Doob 1953, Chapter XII, Theorem 5.3),
Assumption (C2) implies that, up to deterministic or finite-variation components, $Z$ takes the form
\[ Z_t = Z_0 + \int_0^t g(t-s) \, dM_s \quad (2.1) \]
for some kernel $g \in L^2((0, \infty))$ and some process $(M_t)_{t \geq 0}$ with second-order stationary and orthogonal increments. Let us now consider the variance function
\[ \gamma(t) = \mathbb{E}[(Z_{s+t} - Z_s)^2], \quad t > 0, \quad (2.2) \]
which, by stationarity, does not depend on the value of $s$. Because $Z$ is continuous in probability by Assumption (C1), we have $\gamma(t) \to 0$ as $t \to 0$. Our next assumption quantifies the speed of this convergence.

**Assumption (C3).** As $t \to 0$, we have that $\gamma(t) \sim t^{2H} L(t)$ for some $H \in (0, \frac{1}{2})$ and slowly varying (at $0$) function $L$ that is continuous on $(0, \infty)$.

The condition $H < \frac{1}{2}$ is not restrictive for the purpose of modeling microstructure noise: if $H = \frac{1}{2}$, then $Z$ has the same smoothness as Brownian motion, so in general, there will be no way to discern $Z$ from the efficient price process $X$; if $H > \frac{1}{2}$, then $Z$ is smoother than $X$ and RV remains a consistent estimator of $C_T$.

By a simple covariance computation (Barndorff-Nielsen et al. 2011, Equation (4.14)), Assumptions (C2) and (C3) imply that the ACF of noise increments satisfies
\[ \Gamma^n_r = \text{Corr}(\Delta^n_t Z, \Delta^n_{t+r} Z) \to \Gamma^H_r \quad (2.3) \]
for every $r \geq 0$, where
\[ \Gamma^H_0 = 1 \quad \text{and} \quad \Gamma^H_r = \frac{1}{2} \left( (r+1)^{2H} - 2r^{2H} + (r-1)^{2H} \right), \quad r \geq 1. \quad (2.4) \]
The family of ACFs displayed in (2.4) can therefore be seen as prototypical for the increments of noise processes satisfying Assumptions (C1)–(C3). This observation motivates our final noise model.

**Assumption (Z).** The process $(Z_t)_{t \geq 0}$ is given by
\[ Z_t = Z_0 + \int_0^t g(t-s) \rho_s \, dW_s, \quad t \geq 0, \quad (2.5) \]
where $W$ is a $d$-dimensional standard $\mathbb{F}$-Brownian motion and $(\rho_t)_{t \geq 0}$ is an $\mathbb{F}$-adapted locally bounded $\mathbb{R}^{d \times d}$-valued process. The kernel $g: (0, \infty) \to \mathbb{R}$ is of the form
\[ g(t) = K^H_t t^{H-\frac{1}{2}} + g_0(t) \quad (2.6) \]
for some $H \in (0, \frac{1}{2})$, where
\[ K^H_t = \frac{\sqrt{2H} \sin(\pi H) \Gamma(2H)}{\Gamma(H + \frac{1}{2})} \quad (2.7) \]
is a normalizing constant and $g_0: [0, \infty) \to \mathbb{R}$ is a smooth function with $g_0(0) = 0$. 

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In principle, the function $\gamma$ in (2.2) might satisfy $\gamma(t) \sim t^{2H} L(t)$ with $H = 0$ and $L(t) \to 0$. In this case, $\Gamma^0_r \to \Gamma^0_r = 1_{\{r=0\}} - \frac{1}{2} 1_{\{r=1\}}$, which is exactly the ACF of increments of white noise. Because the case $H = 0$ is special and, at least for white noise, has been extensively studied in the literature, we only consider $H > 0$ in the following.

As $g_0$ is smooth, the kernel $g$ in (2.6) produces exactly the same limiting ACF as in (2.4). We dropped the slowly varying function $L$ to simplify the subsequent analysis (and also because such an extension can hardly be distinguished statistically).

In the special case where $g_0 \equiv 0$ and $\rho_s \equiv \rho$ is a constant, $Z$ is—up to a term of finite variation—simply a multiple of fractional Brownian motion (fBM). If further $X_t = \sigma B_t$ with constant volatility $\sigma$, then the resulting observed process $Y_t = \sigma B_t + \rho Z_t$ is a mixed fractional Brownian motion (mfBM) as introduced by Cheridito (2001). Our model for the observed price process, as the sum of $X$ in (1.2) and $Z$ in (2.5), can be viewed as a non-parametric generalization of mfBM that allows for stochastic volatility in both its Brownian and its noise component. We do keep the parameter $H$, though, which we refer to as the roughness parameter of $Z$ (or $Y$). In analogy with mfBM, we call

$$Y_t = X_t + Z_t = Y_0 + \int_0^t a_s \, ds + \int_0^t \sigma_s \, dB_s + \int_0^t g(t-s)\rho_s \, dW_s, \quad t \geq 0,$$

the observed price process in our model, a mixed semimartingale.

**Remark 2.2.** It is important to note that fractional Brownian motion and other fractional models were also considered as asset price models in the literature, often in the context of long-range dependence; see Mandelbrot (1997), Bayraktar et al. (2004), Bender et al. (2011), Bianchi & Pianese (2018), for example. In those works, it is typically the behavior of the kernel $g$ at $t = \infty$ that is of primary interest, as this determines whether the resulting process has short or long memory. Our interest, by contrast, is the behavior of $g$ around $t = 0$, which governs the local regularity, or roughness, of the noise process $Z$. In fact, on a finite time interval $[0, T]$, there is no way to distinguish between short- and long-range dependence (note that in our model, the behavior of $g$ at $t = \infty$ is not specified by (2.6)). This is why in this work, we explicitly do not call $H$ the Hurst parameter (as this is usually associated with long-range dependence) but rather call it the roughness parameter of $Z$. Of course, for fBM, both interpretations fall together, but for non-parametric generalizations as we consider them in (2.5), this distinction is crucial.

### 2.1 Does microstructure noise exist in continuous time?

In the classical Roll (1984) model of transaction prices, deviations of the observed from the efficient price are due to bid–ask bounces associated to each single trade. This raises the question whether Assumption (Z), which postulates the existence of noise in continuous time, is appropriate. Moreover, another important source of noise is the discreteness of prices (see Harris (1990, 1991) and Delattre & Jacod (1997), Li & Mykland (2007), Robert & Rosenbaum (2010, 2012), Rosenbaum (2009)), which is clearly not satisfied by (2.8).

These seeming contradictions between classical market microstructure theory and our mixed semimartingale model can be resolved by taking into account the time scale at which prices are observed. At low to medium frequency, say, if $\Delta_n \geq 5 \text{ min}$, it is well known that noise is negligible and observed prices essentially behave as semimartingales.\(^5\) As $\Delta_n$ enters

\(^5\) This property can be realized in our model: The size of increments of $Z$ over large time intervals is
Figure 3: Two paths of INTC transaction prices, one from 1997 and one from 2007.

a high-frequency regime, noise becomes noticeable and even dominates when $\Delta_n$ approaches a few seconds. Finally, at ultra-high frequency, eventually all trades are recorded tick by tick and both transaction times and observed prices become discrete.

Without doubt, estimating volatility using tick-by-tick data (see, for example, Jacod et al. (2019), Li et al. (2014), Robert & Rosenbaum (2010, 2012)) necessitates a careful modeling of rounding effects and bid–ask bounces in prices. However, as we can see from Figure 3, prices sampled at 5 seconds in our 2007 INTC data do not show much discreteness or flat periods compared to, for example, a typical price path in 1997. This is in agreement with our previous observation from Figure 1 (b) that price increments are still shrinking at the frequencies we consider (rounding errors would induce a flattening in variance plots). As a result, rounding effects and bid–ask bounces do not seem to be the dominant source of noise in the data we consider.

Next, we give two possible explanations for the existence of microstructure noise in continuous time. Both are related to the very reason why the efficient price $X$ is typically assumed to be a semimartingale. First, according to the fundamental theorem of asset pricing, the absence of arbitrage in an idealized frictionless market implies that prices must be semimartingales (Delbaen & Schachermayer 1994). Real markets, of course, have transaction costs (e.g., bid–ask spreads or commissions). Transaction costs do not only generate trade-specific noise in the form of bid–ask bounces (as in the Roll (1984) model), but have the effect that the absence of arbitrage no longer implies the semimartingale property for prices. For example, both fBM and mfBM (which are special cases of our model) are known to not admit arbitrage in the presence of transaction costs (Cherny 2008, Guasoni et al. 2008, Jarrow et al. 2009). In other words, even if noise due to trading mechanisms is taken away, transaction costs may lead to an additional continuous noise component.

Second, as shown by Aït-Sahalia & Jacod (2020), many microscopic models of tick-by-tick data are compatible (i.e., functionally converge in law to) macroscopic semimartingale models as time is stretched out. In this framework, microstructure noise can be viewed as the difference between the limiting semimartingale process $X$ and the microscopic tick-by-tick observed price process $Y$ (which evolves as a continuous-time but piecewise constant determined by the behavior of the kernel $g_0$ in (2.6) for large $t$, which is not further specified in our model. For instance, if $Z$ is a standard fBM with $H \in (0, \frac{1}{2})$, $Z_{s+t} - Z_s$ is of lower order than $X_{s+t} - X_s$ for large $t$, so the effect of noise is negligible.

\footnote{An important detail: to calculate the variance of increments, we exclude periods of no observations (as they would artificially lower the variance) but include zero returns between identical observed prices.}
process). In this approach, the microstructure noise process \( Z = Y - X \) is, by definition, a continuous-time process. Moreover, since it bridges a microscopic model with a classical white or colored noise as in (1.3) ("\( H = 0 \)") and a noise-free macroscopic model ("\( H = \frac{1}{2} \)"), it seems reasonable to assume a locally fractional nature for \( Z \) with some \( H \in (0, \frac{1}{2}) \).

Finally, let us remark that including both a discrete and a continuous noise component would probably yield the most satisfying solution; but this is beyond the scope of the current paper. Also, a theoretical substantiation of the arguments in the previous paragraph (e.g., by exhibiting a tick-by-tick price model that converges to a mixed semimartingale on an intermediate time scale) remains open and is left to future research.

\section*{2.2 Rough noise versus rough volatility}

In recent years, there has been growing interest in rough volatility models (Gatheral et al. 2018, El Euch & Rosenbaum 2019). In these works, it is the volatility \( \sigma \) that is modeled by a rough stochastic process. In this paper, by contrast, we are concerned with roughness of observed prices, caused by market microstructure noise. It is important to note that roughness on the price level and roughness on the volatility level imply distinct features of asset returns and must therefore be modeled and analyzed separately. For instance, if the observed price is simply \( \int_0^t \sigma_s dB_s \), without noise but with a rough volatility \( \sigma \), we will not see explosion of the RV measure in volatility signature plots. In fact, in the absence of microstructure noise, the asymptotic behavior of RV does \textit{not} depend on the roughness of volatility (Jacod & Protter 2012, Theorem 5.4.2). Therefore, the empirical findings discussed so far and below can neither be explained by nor do they indicate rough volatility. What is true is that the CLTs for our estimators (like many CLTs in high-frequency statistics) all hinge on having a volatility process that is not rougher than Brownian motion; but they do remain consistent irrespective of the roughness of \( \sigma \).

\section*{2.3 The statistical problem and our methodology}

On an abstract level, the statistical problem we are facing in this paper is a deconvolution problem: given a semimartingale signal \( X \) and rough signal \( Z \), how can we recover the two (or certain interesting components of the two, such as volatility) based on observing their sum \( Y = X + Z \). The following result, due to Cheridito (2001) and van Zanten (2007), puts a constraint on the identifiability of the (smoother) semimartingale signal:

\textbf{Proposition 2.3.} Assume that \( Y \) is an mfBM, that is, \( Y = X + Z \) where \( X = \sigma B \) and \( Z = \rho B^H \) for some \( \rho, \sigma \in (0, \infty) \), \( B \) is a Brownian motion and \( B^H \) is an independent fBM with Hurst parameter \( H \in (0, \frac{1}{2}) \). For any \( T > 0 \), the laws of \( (Y_t)_{t \in [0,T]} \) and \( (Z_t)_{t \in [0,T]} \) are mutually equivalent if \( H \in (0, \frac{1}{2}) \) and mutually singular if \( H \in (\frac{1}{2}, 1) \).

In other words, if \( H \in (0, \frac{1}{4}) \), due to the roughness of the noise, there is no way to consistently estimate \( \sigma \) on a finite time interval. This is conceptually similar to the fact that the finite-variation part of a semimartingale cannot be estimated consistently in finite time if there is a Brownian component. We will comment on possible pathways to estimate \( \sigma \) if \( H < \frac{1}{4} \) in Section 7.

\textbf{Remark 2.4.} The case of white noise, which formally corresponds to \( H = 0 \) in terms of roughness, is special in this context: it is rougher than \( Z \) in (2.5), but \( C_T = \int_0^T \sigma_s^2 ds \) can
still be recovered through subsampling (Zhang et al. 2005, Zhang 2006) or pre-averaging (Jacod et al. 2009, Podolskij & Vetter 2009, Hautsch & Podolskij 2013). Indeed, if \( k_n \) is an increasing sequence and \( Z \) is a white noise, then
\[
\sum_{j=0}^{k_n} Y_{i+j} \Delta_n \approx X_i \Delta_n
\]
by the law of large numbers. By contrast, if \( H \in (0, \frac{1}{2}) \), the process \( Z \) in (2.5) is continuous (and so is \( Y \) in (1.1)), which implies that
\[
\sum_{j=0}^{k_n} Y_{i+j} \Delta_n \approx Y_i \Delta_n,
\]
so pre-averaging does not remove the noise part at all! Even worse, if we average over increments of \( Y \), then, by some variance computations (not shown here), this actually removes the semimartingale and not the noise component. Therefore, while classical noise-robust volatility estimators work well if \( Z \) is a modulated white noise, they become inconsistent for \( C_T \) if \( H \in (0, \frac{1}{2}) \).

Against this background, we will first establish a CLT for variation functionals of mixed semimartingales in Section 3 and then use this CLT in Section 4 to derive consistent and asymptotically mixed normal estimators for
\[
\int_0^T \sigma_s^2 ds \quad \text{and} \quad \int_0^T \rho_s^2 ds,
\]
where
\[
\sigma_s^2 = \frac{\Delta^2 n}{\Delta_n^2 Y}, \quad \rho_s^2 = \frac{\Delta^2 n}{\Delta_n^2 Y}.
\]
For semimartingales, this is a well studied topic; see Aït-Sahalia & Jacod (2014) and Jacod & Protter (2012) for in-depth treatments of this subject. For fractional Brownian motion or moving-average processes as in (2.5), the theory is similarly well understood; see Barndorff-Nielsen et al. (2011) and Brouste & Fukasawa (2018). Surprisingly, it turns out that the mixed case is more complicated than the “union” of the purely semimartingale and the purely fractional case. For instance, as we elaborate in Remark 3.4, already for power variations of even order, we may have a large number of higher-order bias terms.

## 3 Central limit theorem for variation functionals

As with most estimators in high-frequency statistics, ours are based on limit theorems for power variations and related functionals. More precisely, given \( L, M \in \mathbb{N} \) and a test function \( f: \mathbb{R}^{d \times L} \rightarrow \mathbb{R}^M \), our goal is to establish a CLT for normalized variation functionals
\[
V^n_f(Y, t) = \sum_{i=1}^{[t/\Delta_n]-L+1} f \left( \frac{\Delta^n_i Y}{\Delta^n_i} \right),
\]
where
\[
\Delta^n_i Y = Y_i \Delta_n - Y_{(i-1)} \Delta_n \in \mathbb{R}^d, \quad \Delta^n_i Y = (\Delta^n_i Y, \Delta_{i+1}^n Y, \ldots, \Delta_{i+L-1}^n Y) \in \mathbb{R}^{d \times L}.
\]

For semimartingales, this is a well studied topic; see Aït-Sahalia & Jacod (2014) and Jacod & Protter (2012) for in-depth treatments of this subject. For fractional Brownian motion or moving-average processes as in (2.5), the theory is similarly well understood; see Barndorff-Nielsen et al. (2011) and Brouste & Fukasawa (2018). Surprisingly, it turns out that the mixed case is more complicated than the “union” of the purely semimartingale and the purely fractional case. For instance, as we elaborate in Remark 3.4, already for power variations of even order, we may have a large number of higher-order bias terms.
3.1 The result

Our CLT will be proved under the following set of assumptions. In what follows, $\|\cdot\|$ denotes the Euclidean norm (in $\mathbb{R}^n$ if applied to vectors and in $\mathbb{R}^{nm}$ if applied to a matrix in $\mathbb{R}^{n \times m}$).

**Assumption (CLT).** The observation process $Y$ is given by the sum of $X$ from (1.2) and $Z$ from (2.5) with the following specifications:

(i) The function $f: \mathbb{R}^{d \times L} \to \mathbb{R}^M$ is even and infinitely differentiable. Moreover, all its derivatives (including $f$ itself) have at most polynomial growth.

(ii) The drift process $a$ is $d$-dimensional, locally bounded and $\mathbb{F}$-adapted. The volatility process $\sigma$ is an $\mathbb{F}$-adapted locally bounded $\mathbb{R}^{d \times d}$-valued process. Moreover, for every $T > 0$, there is $K_1 \in (0, \infty)$ such that for all $s, t \in [0, T]$,

$$
\mathbb{E}\left[1 \wedge \|\sigma_t - \sigma_s\|\right] \leq K_1|t - s|^{\frac{1}{2}}.
$$

(iii) Both $B$ and $W$ are independent $d$-dimensional standard $\mathbb{F}$-Brownian motions.

(iv) The noise volatility process $\rho$ takes the form

$$
\rho_t = \rho_t^{(0)} + \int_0^t \tilde{b}_s \, ds + \int_0^t \tilde{\rho}_s \, d\tilde{W}_s, \quad t \geq 0,
$$

where

(a) $\rho^{(0)}$ is an $\mathbb{F}$-adapted locally bounded $\mathbb{R}^{d \times d}$-valued process such that for all $T > 0$,

$$
\mathbb{E}\left[1 \wedge \|\rho_t^{(0)} - \rho_s^{(0)}\|\right] \leq K_2|t - s|^{\gamma}, \quad s, t \in [0, T],
$$

for some $\gamma \in (\frac{1}{2}, 1]$ and $K_2 \in (0, \infty)$;

(b) $\tilde{b}$ is $d \times d$-dimensional, locally bounded and $\mathbb{F}$-adapted;

(c) $\tilde{\rho}$ is an $\mathbb{F}$-adapted locally bounded $\mathbb{R}^{d \times d \times d}$-valued process (e.g., the $(ij)th$ component of the stochastic integral in (3.3) equals $\sum_{k=1}^d \int_0^t \tilde{\rho}_{ijk}^k \, dW_s^k$) such that for all $T > 0$, there exist $\varepsilon > 0$ and $K_3 \in (0, \infty)$ with

$$
\mathbb{E}\left[1 \wedge \|\tilde{\rho}_t - \tilde{\rho}_s\|\right] \leq K_3|t - s|^{\varepsilon}, \quad s, t \in [0, T].
$$

(d) $\tilde{W}$ is a $d$-dimensional $\mathbb{F}$-Brownian motion that is jointly Gaussian with $(B, W)$.

(v) We have (2.6) with $H \in (0, \frac{1}{2})$ and some $g_0 \in C^\infty([0, \infty))$ with $g_0(0) = 0$.

To describe the CLT for $V^n_t(Y, t)$, we need some more notation. Define $\mu_f$ as the $\mathbb{R}^M$-valued function that maps $v = (v_{k,t,k'}^{t'}) \in (\mathbb{R}^{d \times L})^2$ to $\mathbb{E}[f(Z)]$ where $Z \in \mathbb{R}^{d \times L}$ follows a multivariate normal distribution with mean 0 and $\text{Cov}(Z_{kt}, Z_{k't'}) = v_{kt,k't'}$. Note that $\mu_f$ is infinitely differentiable because $f$ is. Furthermore, if $Z' \in \mathbb{R}^{d \times L}$ is such that $Z$ and $Z'$ are jointly Gaussian with mean 0, covariances $\text{Cov}(Z_{kt}, Z_{k't'}) = \text{Cov}(Z'_{kt}, Z'_{k't'}) = v_{kt,k't'}$ and cross-covariances $\text{Cov}(Z_{kt}, Z'_{k't'}) = q_{kt,k't'}$, we define $\gamma_{f_{m_1}, f_{m_2}}(v, q) = \text{Cov}(f_{m_1}(Z), f_{m_2}(Z'))$ for $m_1, m_2 = 1, \ldots, M$. We further introduce a multi-index notation adapted to the definition of $\mu_f$. For $\chi = (\chi_{k,t,k'}^{t'}) \in \mathbb{N}_0^{(d \times L) \times (d \times L)}$ and $v$ as above, let $|\chi| = \sum_{k,k'=1}^d \sum_{t,t'=1}^L \chi_{k,t,k'}^{t'}$, where

$$
|\chi| = \sum_{k,k'=1}^d \sum_{t,t'=1}^L \chi_{k,t,k'}^{t'}.
$$
Finally, recalling (2.4), we define for all \( k, k' \in \{1, \ldots, d\} \), \( \ell, \ell' \in \{1, \ldots, L\} \) and \( r \in \mathbb{N}_0 \),

\[
\pi_r(s)_{k\ell,k'\ell'} = (\rho_s \rho_s^T)_{kk'} \Gamma_{[\ell-\ell'+r]}, \quad c(s)_{k\ell,k'\ell'} = (\sigma_s \sigma_s^T)_{kk'} \mathbb{1}_{\{\ell=\ell'\}}, \quad \pi(s) = \pi_0(s). \tag{3.6}
\]

The following CLT is our first main result. We use \( \overset{st}{\rightarrow} \) (resp., \( \overset{L^1}{\rightarrow} \)) to denote functional stable convergence in law (resp., convergence in \( L^1 \)) in the space of càdlàg functions \( [0, \infty) \to \mathbb{R} \) equipped with the local uniform topology. In the special case where \( Y \) follows the parametric model of an mBm and the test function is \( f(x) = x^2 \), the CLT was obtained by Dozzi et al. (2015).

**Theorem 3.1.** Grant Assumption (CLT) and let \( N(H) = [1/(2 - 4H)] \). Then

\[
\Delta_n^{-\frac{1}{2}} \left\{ V_f^n(Y, t) - \int_0^t \mu_f(\pi(s)) \, ds 
- \sum_{j=1}^{N(H)} \Delta_n^{j(1-2H)} \sum_{|\kappa|=j} \frac{1}{\chi^j} \int_0^t \partial^\kappa \mu_f(\pi(s))c(s)\chi^s \, ds \right\} \overset{st}{\rightarrow} \mathcal{Z}, \tag{3.7}
\]

where \( \mathcal{Z} = (Z_t)_{t \geq 0} \) is an \( \mathbb{R}^M \)-valued continuous process defined on a very good filtered extension \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) of \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) which, conditionally on \( \mathcal{F} \), is a centered Gaussian process with independent increments and such that the covariance function \( C_t^{m_1 m_2} = \mathbb{E}[Z_t^{m_1} Z_t^{m_2} | \mathcal{F}] \), for \( m_1, m_2 = 1, \ldots, M \), is given by

\[
C_t^{m_1 m_2} = \int_0^t \left\{ \gamma_{f_{m_1} f_{m_2}}(\pi(s), \pi(s)) + \sum_{r=1}^{\infty} \left( \gamma_{f_{m_1} f_{m_2}} + \gamma_{f_{m_2} f_{m_1}} \right)(\pi(s), \pi_r(s)) \right\} \, ds. \tag{3.8}
\]

**Remark 3.2.** In fact, it suffices to require \( f \) be \( 2(N(H)+1) \)-times continuously differentiable with derivatives of at most polynomial growth. A decomposition as in (3.3) is standard for CLTs in high-frequency statistics. But here we need it for \( \rho \) (instead of \( \sigma \)) as the noise process dominates the efficient process in the limit \( \Delta_n \to 0 \). Condition (3.2) on \( \sigma \) is satisfied if, for example, \( \sigma \) is itself a continuous Itô semimartingale.

**Remark 3.3.** Both the LLN limit

\[
V_f(Y, t) = \int_0^t \mu_f(\pi(s)) \, ds \tag{3.9}
\]

and the fluctuation process \( \mathcal{Z} \) originate from the rough process \( Z \). In other words, if \( \sigma \equiv 0 \) (i.e., in the pure fractional case), we would have (3.7) without the \( \sum_{j=1}^{N(H)} \)-expression; see Barndorff-Nielsen et al. (2011). Even if \( \sigma \neq 0 \), in the case where \( H < \frac{1}{4} \), no additional terms are present because \( N(H) = 0 \). This is in line with Proposition 2.3, which states that it is impossible to consistently estimate \( C_t = \int_0^t \sigma_s^2 \, ds \) if \( H < \frac{1}{4} \). If \( H \in \left( \frac{1}{4}, \frac{3}{2} \right) \), the “mixed” terms in the \( \sum_{j=1}^{N(H)} \)-expression will allow us to estimate \( C_t \).

**Remark 3.4.** In the special case \( d = 1 \) and \( f(x) = x^{2p} \) for some \( p \in \mathbb{N} \), (3.7) reads

\[
\Delta_n^{-\frac{1}{2}} \left\{ V_f^n(Y, t) - V_f(Y, t) 
- \sum_{j=1}^{N(H)} \Delta_n^{j(1-2H)} \mu_{2p} \left( \frac{p}{j} \right) \int_0^t \rho_s^{2p-2j} \sigma_s^{2j} \, ds \right\} \overset{st}{\rightarrow} \mathcal{Z},
\]
where \( \mu_{2p} \) is the moment of order \( 2p \) of a standard normal variable. Typically, one is interested in estimating only one of the terms in the sum \( \sum_{j=1}^{N(H)} \) at a time (e.g., \( \int_0^t \sigma^2_s \, ds \) corresponding to \( j = p \)). All other terms (e.g., \( j \neq p \)) have to be considered as higher-order bias terms in this case. The appearance of (potentially many, if \( N(H) \) is large) bias terms for test functions as simple as powers of even order neither happens in the pure semimartingale nor in the pure fractional setting.

Remark 3.5. The following values for \( H \) are special:

\[
\mathcal{H} = \{ \frac{1}{2} - \frac{1}{4n} : n \geq 1 \} = \{ \frac{1}{2}, \frac{3}{8}, \frac{5}{12}, \frac{7}{16}, \ldots \}. \tag{3.10}
\]

Indeed, if \( H \in \mathcal{H} \), then \( N(H) = 1/(2 - 4H) \). In particular, the term in (3.7) that corresponds to \( j = N(H) \) is exactly of order \( \Delta_n^{1/2} \). So in this case, (3.7) can also be viewed as convergence to a non-central mixed normal distribution.

### 3.2 Overview of the proof of Theorem 3.1

In the following, we describe the main difficulties in the proof of Theorem 3.1 and defer the details to the supplementary material. In addition to the usual steps that are common to CLTs in high-frequency statistics, there are two new challenges in the present setting:

(i) The observation process \( Y \) is not a semimartingale (and not even close to one). This is because the rough component \( Z \) dominates the efficient price process \( X \) in the limit as \( \Delta_n \to 0 \) (which cannot be remedied by pre-averaging; see Remark 2.4). In particular, the increments of \( Y \) remain conditionally dependent as \( \Delta_n \to 0 \).

(ii) If \( H \) is close to (but smaller than) \( \frac{1}{2} \), the semimartingale part is only marginally smoother than the noise part. So for the CLT, there will be an intricate interplay between the efficient price process and the noise process.

To overcome the first challenge, we employ a multiscale analysis: by suitably truncating the increments of \( Y \), we can restore, to some degree (not on the finest scale \( \Delta_n \) but on some intermediate scale \( \theta_n \Delta_n \) where \( \theta_n \to \infty \)), asymptotic conditional independence between increments of \( Y \) (see Lemma C.1). This in turn gives \( V_n^\beta(Y, t) \), as a process in \( t \), a semimartingale-like structure on this intermediate scale, which is sufficient for deriving the CLT when we center by appropriate conditional expectations (see (C.6)). However, because increments are still correlated on the finest scale, the limiting process is not the usual one for semimartingales but the one for (modulated) fractional Brownian motion (see (3.8), in particular). Regarding the second challenge above, we find, to our surprise, that the semimartingale component never enters the CLT limit of \( V_n^\beta(Y, t) \) when centered by conditional expectations (see Lemma C.2), no matter how close \( H \) is to \( \frac{1}{2} \). By contrast, it does affect the limit behavior of these conditional expectations (Lemmas C.3–C.9), producing an \( H \)-dependent number of higher-order bias terms that neither appear in the pure semimartingale nor in the pure fractional setting.
4 Estimating the roughness parameter and integrated price and noise volatilities

In this section, we assume \( d = 1 \) for simplicity. We develop an estimation procedure for the roughness parameter of the noise and the integrated price (if \( H > \frac{1}{2} \)) and noise volatilities, that is, for \( H, C_t = \int_0^t \sigma_s^2 \, ds \) and \( \Pi_t = \int_0^t \rho_s^2 \, ds \). To avoid additional bias terms (cf. Remark 3.4), we use quadratic functionals only, that is, we consider \( f_r(x) = x_1x_{r+1} \) for \( x = (x_1, \ldots, x_{r+1}) \in \mathbb{R}^{r+1} \) and \( r \in \mathbb{N}_0 \), and the associated variation functionals \( V_{r,t}^n = V_{0,t}^n = \Delta_n^{1-2H} \sum_{k=1}^{[t/\Delta_n]-r} \Delta_k^n (\Pi_k^n)^{-1/2} Y_{k+r}^n \). Note that \( V_{r,t}^n \) is not a statistic as it depends on the unknown parameter \( H \). Therefore, we introduce \( \hat{V}_{r,t}^n = (\hat{V}_{0,t}^n, \ldots, \hat{V}_{R,t}^n) \), a non-normalized version of \( V_{r,t}^n \) that is a statistic:

\[
\hat{V}_{r,t}^n = \hat{V}_{r,t}^n(Y, t) = \sum_{k=1}^{[t/\Delta_n]-r} \Delta_k^n Y_{k+r}^n, \quad r \in \mathbb{N}_0.
\]

Clearly, \( \Delta_n^{1-2H} \hat{V}_{r,t}^n = V_{r,t}^n \), so our main CLT (Theorem 3.1) immediately yields:

**Corollary 4.1.** Let \( \hat{V}_{r,t}^n = (\hat{V}_{0,r}^n, \ldots, \hat{V}_{R,r}^n) \) for a fixed but arbitrary \( R \in \mathbb{N}_0 \). For \( H \in (0, \frac{1}{2}) \),

\[
\Delta_n^{\frac{1}{2}} \left\{ \Delta_n^{1-2H} \hat{V}_{r,t}^n - \Gamma^H \int_0^t \rho_s^2 \, ds - e_1 \int_0^t \sigma_s^2 \, ds \Delta_n^{1-2H} \mathbf{1}_{[1/2]}(H) \right\} \Rightarrow Z,
\]

where \( \Gamma^H = (\Gamma_0^H, \ldots, \Gamma_R^H) \), \( e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^{1+R} \) and \( Z \) is as in (3.7). The covariance process \( C^H(t) = (C_{ij}^H(t))_{i,j=0,\ldots,R} \) in (3.8) is given by

\[
C_{ij}^H(t) = C_{ij}^H \int_0^t \rho_s^4 \, ds,
\]

\[
C_{ij}^H = \Gamma_{[i-j]} + \Gamma_i^H \Gamma_j^H + \sum_{r=1}^{\infty} \left( \Gamma_r^H \Gamma_{[i-j+r]} + \Gamma_{[r-j]}^H \Gamma_{i+r}^H + \Gamma_r^H \Gamma_{[j-i+r]} + \Gamma_{[r-i]}^H \Gamma_{j+r}^H \right).
\]

As we can see, if \( H \in (\frac{1}{4}, \frac{1}{2}) \), only RV \( (r = 0) \) contains information about \( C_t = \int_0^t \sigma_s^2 \, ds \). But to first order, \( V_{0,t}^n = \Delta_n^{1-2H} \hat{V}_{0,t}^n \), estimates \( \Pi_t = \int_0^t \rho_s^2 \, ds \), the integrated noise volatility. In order to obtain \( C_t \), our strategy is to use \( \hat{V}_{r,t}^n \) for \( r \geq 1 \) to remove the first-order limit of \( \hat{V}_{0,t}^n \). But here is a caveat: both \( \Delta_n^{1-2H} \) and \( \Gamma^H \) contain the unknown parameter \( H \), so we need to estimate \( H \) first.

The most obvious estimator for \( H \) is obtained by calculating the rate of divergence in volatility signature plots, that is, by regressing \( \log \Delta_n \) on \( \log \hat{V}_{0,t}^n \) (see also Rosenbaum (2011) for a more general but related concept). However, as noted by Dozzi et al. (2015) in their Remark 3.1, already in an mfBM model, this regression based estimator only has a logarithmic rate of convergence. Indeed, as our simulation study in Section 5 shows, this estimator systematically overestimates \( H \) unless \( H \) is very close to 0 or \( \frac{1}{2} \). In the pure fractional case, rate-optimal estimators are given by so-called change-of-frequency or autocorrelation estimators (Barndorff-Nielsen et al. 2011, Corcuera et al. 2013). Both extract information about \( H \) by considering the ratio of (different combinations of) \( \hat{V}_{r,t}^n \) for different values of \( r \). For example, the simplest autocorrelation estimator is

\[
\hat{H}_{acf}^n = \frac{1}{2} \left[ 1 + \log_2 \left( \frac{\hat{V}_{1,t}^n}{\hat{V}_{0,t}^n} + 1 \right) \right].
\]
which is based on the fact that \( \hat{V}_{1,t}^n/\hat{V}_{0,t}^n = V_{1,t}/V_{0,t} \to \Gamma_1^H = 2^{2H-1} - 1 \). But due to the bias term that appears in (4.1) when \( r = 0 \), the convergence rate worsens and becomes suboptimal when (4.3) is applied to mixed semimartingales. The first rate-optimal estimator for \( H \) in the case of mfBM was constructed in Theorem 3.2 of Dozzi et al. (2015) by using a variant of (4.3) that cancels the contribution from \( \hat{V}_{0,t}^n \). However, this estimator suffers from a large constant in the asymptotic variance (and another issue that we address in Section 4.2). In fact, in their Remark 3.2, Dozzi et al. (2015) do not recommend using it in practice even though it has a better convergence rate than the estimator based on volatility signature plots.

To do better, our strategy is to use linear combinations of \( \hat{V}_{r,t}^n \) for multiple values of \( r \). To this end, we choose two weight vectors \( a = (a_0, \ldots, a_R) \) and \( b = (b_0, \ldots, b_R) \) in \( \mathbb{R}^{1+R} \) and consider the statistic

\[
\tilde{H}^n = \varphi^{-1} \left( \frac{\langle a, \tilde{V}_t^n \rangle}{\langle b, \tilde{V}_t^n \rangle} \right) \quad \text{with} \quad \varphi(H) = \frac{\langle a, \Gamma^H \rangle}{\langle b, \Gamma^H \rangle},
\]

(4.4)

where \( \langle \cdot, \cdot \rangle \) denotes the standard inner product on \( \mathbb{R}^{1+R} \) and \( a \) and \( b \) are assumed to be such that \( \varphi \) is invertible. The further analysis depends on whether \( H \in (0, \frac{1}{4}) \) or \( H \in (\frac{1}{4}, \frac{1}{2}) \) and, in the latter case, whether \( a_0 = b_0 = 0 \) or at least one of \( a_0 \) and \( b_0 \) is not zero.

### 4.1 Estimation without quadratic variation or if \( H \in (0, \frac{1}{4}) \)

If \( a_0 = b_0 = 0 \), we exclude quadratic variation from our estimation procedure for \( H \). This has the advantage that the term \( c_1\int_0^t \sigma_s^2 \, ds \Delta_t^{1-2H} \) in (4.1) disappears. The same holds true if \( H < \frac{1}{4} \) (even if \( a_0 \) or \( b_0 \) is not zero): there is no asymptotic bias term in (4.1).

**Theorem 4.2.** Assume that \( H \in (0, \frac{1}{4}) \) and choose \( R \in \mathbb{N} \) and \( a, b \in \mathbb{R}^{1+R} \) such that \( \varphi \) from (4.4) is invertible. If \( H \in (\frac{1}{4}, \frac{1}{2}) \), further assume that \( a_0 = b_0 = 0 \).

(i) The estimator \( \tilde{H}^n \) introduced in (4.4) satisfies

\[
\Delta_n^{-\frac{1}{2}} (\tilde{H}^n - H) \overset{st}{\to} \mathcal{N} \left( 0, \text{Var}_{H,0} \frac{\int_0^t \rho_s^4 \, ds}{(\int_0^t \rho_s^2 \, ds)^2} \right),
\]

(4.5)

where \( Z \) is the same as in (4.1) and \( \text{Var}_{H,0} = \text{Var}_{H,0}(R, a, b, H) \) is defined by

\[
\text{Var}_{H,0}(R, a, b, H) = \left( \frac{\varphi^{-1}(\varphi(H))}{\langle b, \Gamma^H \rangle} \right)^2 \{a^T - \varphi(H)b^T\} C^H \{a - \varphi(H)b\}.
\]

(ii) If \( H \in (\frac{1}{4}, \frac{1}{2}) \), choose \( c \in \mathbb{R}^{1+R} \) and define

\[
\tilde{C}_t^n = \left\{ \tilde{V}_{0,t}^n - \frac{\langle c, \tilde{V}_t^n \rangle}{\langle c, \Gamma^H \rangle} \right\} \left( 1 - \frac{c_0}{\langle c, \Gamma^H \rangle} \right)^{-1}.
\]

Then

\[
\Delta_n^{-\frac{1}{2}} \{\tilde{C}_t^n - C_t^n\} \overset{st}{\to} \mathcal{N} \left( 0, \text{Var}_{C} \int_0^t \rho_s^4 \, ds \right),
\]

(4.8)
where

$$Var_C = Var_C(R, a, b, c, H) = u^T C^H u,$$

$$u = \left( e_1 - \frac{c}{\langle c, \Gamma^H \rangle} + \frac{\langle c, \partial_H \Gamma^H \rangle}{\langle c, \Gamma^H \rangle} (\varphi^{-1})'(\varphi(H))(a - \varphi(H)b)(1 - \frac{c_0}{\langle c, \Gamma^H \rangle})^{-1},$$

and $\partial_H \Gamma^H = (\partial_H \Gamma^H_0, \ldots, \partial_H \Gamma^H_R)$ with $\partial_H \Gamma^H_0 = 0$ and

$$\partial_H \Gamma^H_r = \log(r + 1)(r + 1)^{2H} - 2\log(r)r^{2H} + \log(r - 1)(r - 1)^{2H}, \quad r \geq 1. \quad (4.10)$$

(iii) The estimator $\hat{\Pi}^n_t = \Delta_n^{1-2\bar{H}^n}(\hat{V}^n_t)/\langle a, \Gamma^n \rangle$ satisfies

$$\frac{\Delta_n^{-\frac{3}{2}}}{|\log \Delta_n|} \left( \hat{\Pi}^n_t - \Pi_t \right) \xrightarrow{st} \mathcal{N}\left(0, 4 \ Var_{H,0} \int_0^t \rho_s^4 \, ds \right). \quad (4.11)$$

Remark 4.3. To construct $\hat{C}^n_t$, we allow the possibility to choose a new weight vector $c$. Therefore, $a$ and $b$ should be thought of as weights one can choose to, for example, minimize $Var_{H,0}(R, a, b, H)$, while $c$ can then be chosen to minimize $Var_C(R, a, b, c, H)$. Alternatively, one may choose $a$, $b$, and $c$ to minimize $Var_C(R, a, b, c, H)$ directly (if $H > \frac{1}{4}$).

Remark 4.4. According to work in progress by F. Mies (private communication), the rates of $\bar{H}^n$, $\hat{C}^n_t$ and $\hat{\Pi}^n_t$ are optimal in the parametric setting of an mfBM.

In order to obtain feasible CLTs, we replace the unknown quantities in $Var_{H,0}$ and $Var_C$ by consistent estimators thereof. To this end, consider

$$Q^n_t = V^n_t(Y, t) = \Delta_n \sum_{i=1}^{[t/\Delta_n]} \left| \frac{\Delta_n Y}{\Delta_n} \right|^4, \quad \tilde{Q}^n_t = \sum_{i=1}^{[t/\Delta_n]} (\Delta^n Y)^4. \quad (4.12)$$

By Theorem 3.1, we have the LLN

$$Q^n_t \xrightarrow{L^1} 3 \int_0^t \rho_s^4 \, ds. \quad (4.13)$$

Therefore, the following theorem is a direct consequence of Theorem 4.2 and well-known properties of stable convergence in law (Jacod & Protter 2012, Equation (2.2.5)).

**Theorem 4.5.** Grant the assumptions of Theorem 4.2. For (4.15) below, further assume that $H \in (\frac{1}{4}, \frac{3}{2})$. Then

$$\Delta_n^{-\frac{1}{2}}(\bar{H}^n - H) \sqrt{3 \Delta_n (V^n_t)} \xrightarrow{st} \mathcal{N}(0, 1), \quad (4.14)$$

$$\Delta_n^{\frac{1}{2} - 2\bar{H}^n} (\hat{C}^n_t - C_t) \sqrt{3 \Delta_n^{4\bar{H}^n - 1}} \xrightarrow{st} \mathcal{N}(0, 1), \quad (4.15)$$

$$\Delta_n^{-\frac{1}{2}}(\hat{\Pi}^n_t - \Pi_t) \sqrt{\frac{3 \Delta_n^{4\bar{H}^n - 1}}{4 \ Var_{H,0}(R, a, b, \bar{H}^n)\tilde{Q}^n_t}} \xrightarrow{st} \mathcal{N}(0, 1). \quad (4.16)$$
4.2 Estimation with quadratic variation if \( H \in \left( \frac{1}{4}, \frac{1}{2} \right) \)

The estimators based on weight vectors \( a \) and \( b \) with \( a_0 = b_0 = 0 \) were easy to construct but suffer from a serious shortcoming: If the observed price process is simply given by \( Y = \sigma B \) for some constant \( \sigma > 0 \) (i.e., there is no noise), then, by standard CLTs for Brownian motion, the ratio \( \langle a, \hat{V}_t^n \rangle / \langle b, \hat{V}_t^n \rangle \) converges stably in law to the ratio \( Z_1/Z_2 \) of two centered (possibly correlated) normals that are independent of \( B \). In particular, because \( Z_1/Z_2 \) has a density supported on \( \mathbb{R} \), the asymptotic probability that \( \hat{H}_n \) from (4.4) falls into any non-empty open subinterval of \( (0,1) \) is non-zero. So based on \( \hat{H}_n \) only, it is impossible to tell whether there is evidence for rough noise or whether an estimate produced by \( \hat{H}_n \) is simply the result of chance! This shortcoming is shared by the estimator proposed by Dozzi et al. (2015).

To solve this problem, we have to include lag 0 in our estimation of \( H \). If \( H \in \left( \frac{1}{4}, \frac{1}{2} \right) \), this significantly complicates the estimation procedure: By the discussion at the beginning of Section 4, in order to estimate \( C_t \), we need to estimate \( H \) first. At the same time, as Corollary 4.1 shows, using \( \hat{V}_t^n \) to estimate \( H \) induces an asymptotic bias term coming from the \( \int_0^t \sigma_s^2 ds \) term, which can only be corrected with an estimator of \( C_t \). Resolving this circular dependence necessitates a complex iterated estimation procedure for \( H \) and \( C_t \) that we describe in Appendix F. In particular, as \( H \uparrow \frac{1}{2} \), we obtain an increasing number of higher-order bias terms as a result of the interdependence between the \( H \)- and the \( C_t \)-estimators. The final result we obtain after the debiasing procedure described in Appendix F is as follows (for the proof, combine (4.13), Proposition F.5 and Theorem F.6):

**Theorem 4.6.** Assume that \( H \in \left( \frac{1}{4}, \frac{1}{2} \right) \). Choose \( R \geq 1 \), \( m \geq 2 \) and \( a, b, c \in \mathbb{R}^{1+R} \) such that \( b_0 = 0 \) and \( \varphi \) from (4.4) is invertible (now, \( a_0 \) need not be 0 anymore). Further choose \( a^0, b^0 \in \mathbb{R}^{1+R} \) such that \( a_0^0 = b_0^0 = 0 \). The estimators \( \hat{H}_n \), \( \hat{C}_t^n \) and \( \Pi_t^n \), defined in (F.20), (F.22) and (F.23), respectively, satisfy

\[
\Delta_n^{-\frac{1}{2}} (\hat{H}_n - H) \sqrt{\frac{3\Delta_n (\hat{V}_0^n)^2}{\text{Var}(R,a,b,H^n)t^n}} \xrightarrow{\text{st}} \mathcal{N}(0,1),
\]

\[
\Delta_n^{-\frac{1}{2}} (\hat{C}_t^n - C_t) \sqrt{\frac{3\Delta_n (\tilde{H}^n)^{n-1}}{\text{Var}(R,a,b,c,H^n)t^n}} \xrightarrow{\text{st}} \mathcal{N}(0,1),
\]

\[
\Delta_n^{-\frac{1}{2}} (\Pi_t^n - \hat{\Pi}_t^n) \sqrt{\frac{4\Delta_n (\tilde{H}^n)^{n-1}}{\text{Var}(R,a,b,H^n)t^n}} \xrightarrow{\text{st}} \mathcal{N}(0,1),
\]

with \( \tilde{Q}_t^n \) from (4.12) and \( \text{Var}_H \) and \( \text{Var}_C \) from (F.21) and (F.26), respectively.

5 Simulation study

All results reported in this section are based on 5,000 simulations from the mfBM

\[ Y_t = X_t + Z_t = \sigma B_t + \rho B_t^H, \quad t \in [0,T], \]

where \( \sigma = 0.01, \rho = 0.001 \), \( B \) and \( B^H \) are independent and \( T = 1 \) or \( T = 20 \) trading days, each consisting of 6.5 hours or \( n = 23,400 \) seconds. Accordingly, we choose \( \Delta_n = 1/n = \)
1/23,400. The values of $H$ will be taken from the set

$$H \in \{0.05, 0.1, 0.15, 0.2, 0.25, 0.275, 0.3, 0.325, 0.35, 0.375, 0.4, 0.425, 0.45, 0.475\}. \quad (5.1)$$

We also include “$H = 0.5$” (i.e., $\rho = 0$) and “$H = 0$” (i.e., $(B_t^0)_{t \in [0,T]}$ is a standard normal white noise). The choice of the tuning parameters is described in Appendix G.

For $H$, we first compare our estimator $\hat{H}^{n,0} = \tilde{H}^n$ from (4.4), constructed with $a^0$ and $b^0$ from (G.2), with four variants of $\tilde{H}^n$ from (F.20), denoted by $\tilde{H}^{n,i}$ for $i = 0, 1, 2, 3$. For each $i$, $\tilde{H}^{n,i}$ is defined in the same way as $\tilde{H}^n$ in (F.20) except that $N(\tilde{H}^n)$ in (F.14) and (F.15) and $N(\tilde{H}^{k-1}_n)$ in (F.17) are replaced by the fixed number $i$. If $n$ is large,

$$\hat{H}^n = \begin{cases} \hat{H}^{n,0} & \text{if } H \in (0, 0.25), \\ \hat{H}^{n,1} & \text{if } H \in (0.25, 0.375), \end{cases} \quad \tilde{H}^n = \begin{cases} \hat{H}^{n,2} & \text{if } H \in (0.375, 0.417), \\ \hat{H}^{n,3} & \text{if } H \in (0.417, 0.4375) \end{cases} \quad (5.2)$$

with high probability. We do not include four or more correction terms as it becomes increasingly intractable to compute higher-order derivatives of composite functions like $\varphi^{-1}$ and $\psi$ in (F.3) or (F.10). Also, to increase stability, estimates of $H$ are calculated based on $T = 20$ trading days.

As Figure 4 shows, the estimator $\tilde{H}^{n,3}$ has a lower root-mean-square error (RMSE) than $\tilde{H}^{n,i}$, $i = 0, 1, 2$, for most values of $H$. Moreover, $\tilde{H}^{n,0}$ is superior to $\tilde{H}^{n,3}$ in terms of RMSE if $H \leq 0.375$ and inferior to $\tilde{H}^{n,3}$ if $H \geq 0.4$. This is in line with our previous observation that $\tilde{H}^{n,0}$ fails to estimate $H$ if $H = \frac{1}{2}$. In Figure 4, we further consider

- the estimator $\tilde{H}^{n}_{\text{VS}} = \frac{1}{2} (\tilde{\beta}^n_{\text{VS}} + 1)$ based on volatility signature plots, where $\tilde{\beta}^n_{\text{VS}}$ is the slope estimate in a linear regression of $\log \tilde{V}^{n,i}_{0,t}$ on $\log t$ for $i = 1, \ldots, 20$;
• the estimator \( \tilde{H}_{\text{DMS}}^n = \frac{1}{2}(1 + \log_2(\frac{(\hat{V}_{n/4}^n - \hat{V}_{n/2}^n)}{(\hat{V}_{n/2}^n - \hat{V}_{0}^n)}) \) from Dozzi et al. (2015), where \( \log_2 x = \log_2 x \) if \( x > 0 \) and \( \log_2 x = 0 \) otherwise;

• the autocorrelation estimator \( \tilde{H}_{\text{acf}}^n \) from (4.3).

For \( H \leq 0.375 \), the best estimator is \( \tilde{H}_{\text{n},0}^n \). For \( H \geq 0.4 \), the estimator \( \tilde{H}_{\text{n},3}^n \) is similar in performance to the estimators \( \tilde{H}_{\text{VS}}^n, \tilde{H}_{\text{DMS}}^n \) and \( \tilde{H}_{\text{acf}}^n \). Therefore, the best strategy is to combine \( \tilde{H}_{\text{n},0}^n \) and \( \tilde{H}_{\text{n},3}^n \) by using the former if \( H \) is small and the latter if \( H \) is large. We refer to Section 6 for one way of implementing this strategy.

Finally, we study the performance of our volatility estimators. To this end, we implement \( \tilde{C}_{\text{n},0}^n = \hat{C}_{20}^n - \hat{C}_{19}^n \) and \( \Pi_{\text{n},0}^n = \hat{\Pi}_{20}^n - \hat{\Pi}_{19}^n \) from (4.7) and (4.11) on the last of the 20 simulated trading days, using the estimator \( \tilde{H}_{\text{n},0}^n = \hat{H}^n \) from (4.4) that is based on the whole simulated period. Similarly, for \( i = 1, 2, 3 \), we consider \( \tilde{C}_{\text{n},i}^n = \hat{C}_{20}^n - \hat{C}_{19}^n \) and \( \Pi_{\text{n},i}^n = \hat{\Pi}_{20}^n - \hat{\Pi}_{19}^n \) from (F.22) and (F.23) using, instead of \( \hat{H}_{\text{n},i}^n \), the estimator \( \tilde{H}_{\text{n},i}^n \) from above (computed again based on the whole period of 20 simulated days). From Figure 5, we find that \( \tilde{C}_{\text{n},0}^n \) shows a good performance for \( H \in [0.15, 0.4] \), while \( \tilde{C}_{\text{n},3}^n \) performs best for \( H \geq 0.425 \). Together, they cover the whole interval on which \( H \) is identifiable (see Proposition 2.3). For \( \Pi_T \), \( \tilde{\Pi}_{\text{n},0}^n \) works well if \( H \leq 0.325 \) but has a large RMSE otherwise. In Section 7, we comment on possible ways of improving this estimator for large \( H \).

6 Empirical analysis

We apply the estimators from Theorems 4.2 and 4.6 to (logarithmic) INTC transaction data for the whole year of 2007. The data source is the TAQ database. For each trading day in 2007, we collect all trades on the NYSE and NASDAQ from 9:00 am to 4:00 pm Eastern Time. We preprocess the data using the tradesCleanup() function from the R package highfrequency, which follows the recommendations by Barndorff-Nielsen et al. (2009). We sample in calendar time every 5 seconds.

To reduce the variability of the resulting estimates, we calculate, for each trading day from February 1 to December 31, the estimators \( \tilde{H}_{\text{n},3}^n \) and \( \tilde{H}_{\text{n},0}^n \) based on the previous 20 trading days. Afterwards, based on the insights from the simulation study, we calculate an estimate of \( H \) using \( \tilde{H}_{\text{n},3}^n \) if its asymptotic 95%-confidence interval contains 0.5 or is a subset of (0.4, 0.5); otherwise, we report the estimate produced by \( \tilde{H}_{\text{n},0}^n \). Correspondingly,

![Figure 6: Histogram of estimates for H (left) and boxplot of signal-to-noise ratios (right). Each data point corresponds to one company and day. Days where the volatility or the noise volatility estimate is negative are omitted. Outliers in the boxplot are not shown.](image)
we either take $\hat{C}_n,3$ or $\tilde{C}_n,0$ (resp., $\hat{\Pi}_n,3$ or $\tilde{\Pi}_n,0$) to estimate the daily integrated volatility (resp., noise volatility). Figure 6 shows the empirical distribution of the daily estimators of $H$ and a boxplot of the daily signal–to–noise ratios (i.e., of $\hat{C}_n,3/\hat{\Pi}_n,3$ or $\tilde{C}_n,0/\tilde{\Pi}_n,0$). Figure 7 shows the daily $H$-estimates and 95%-confidence intervals throughout the year and the daily volatility and noise volatility estimates for the month of November.

Finally, Figure 8 shows the average daily $H$-estimate for 5s INTC transaction data for each year from 1997 to 2021. While the classical discrete noise model (1.3) is most appropriate until 2000, the rough noise model becomes prevalent between 2001–2010, with a decreasing trend in the roughness of the noise, which is in alignment with the findings of Aït-Sahalia & Xiu (2019). Between 2011–2021, Figure 8 shows estimates close to or even larger than $\frac{1}{2}$. We do not consider this as an indication that the sampled data is noise-free. In fact, for those years, RV still explodes for the majority of days (but at slower rates). At the same time, the observed price increments start to have slightly positive autocorrelations, a property of fractional processes with $H > \frac{1}{2}$, which neither the classical time series model nor the rough noise model can explain. Since our estimators of $H$ combine information from different lags, they largely remain inconclusive in those years. In Appendix H, we carry out a similar analysis for quotes. Here, even in recent years, there is strong evidence of rough noise with $H$ strictly between 0 and $\frac{1}{2}$.

7 Conclusion and future directions

Volatility estimation based on high-frequency return data is often impeded by the presence of market microstructure noise. In this paper, we propose to model microstructure noise as a continuous-time rough stochastic process. In addition to desirable properties such as compatibility between different sampling frequencies, serial dependence of increments
and stochastic volatility for both price and noise, a distinctive feature of these mixed semimartingale models is a non-shrinking noise component with shrinking increments. This property can explain the rich variety of scaling exponents in volatility signature plots and finds strong empirical support in a detailed analysis of transaction and quote data.

Using CLTs for variation functionals and an iterative debiasing procedure, we construct consistent and asymptotically mixed normal estimators for the roughness parameter $H$ of the noise and the integrated price and noise volatilities, whenever these quantities are identifiable. In a simulation study, we find that our estimators of $H$ outperform existing ones in the literature. We further identify estimators for the integrated price volatility $C_T$ that show good performance throughout the region of $H$ in which $C_T$ is identifiable. An interesting open problem is to investigate whether subsampling or pre-averaging techniques can improve our estimators for the noise volatility, which currently work well only when $H$ is not close to $\frac{1}{2}$. As $C_T$ is not identifiable if $H < \frac{1}{4}$, another promising direction is to analyze whether taking a simultaneous small noise limit helps identify price volatility in such cases.

In this first paper, we do not examine the effect of jumps (Aït-Sahalia & Jacod 2009, Jacod & Todorov 2014) or irregular observation times (Barndorff-Nielsen & Shephard 2005, Chen et al. 2020, Jacod et al. 2017, 2019) on our estimators. While the estimators of $H$ and noise volatility from Theorem 4.5 might not be affected by jumps too much, as they do not use quadratic variation, certainly the volatility estimators and all estimators from Theorem 4.6 are. We leave it to future research to develop estimators that are fully robust to jumps and asynchronous sampling. Similarly, the current mixed semimartingale model does not capture rounding effects in observed prices (Aït-Sahalia & Jacod 1997, Robert & Rosenbaum 2010, 2012), which are particularly relevant at the highest sampling frequencies. Including both a discrete and a continuous noise component in a statistical model of high-frequency price dynamics remains a future challenge.

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Supplement to “When Frictions are Fractional: Rough Noise in High-Frequency Data”

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Abstract
This supplement contains the proof of Theorem 3.1 (Appendices A–D), the proof of Theorem 4.2 (Appendix E), the details of the iterative debiasing procedure of Section 4.2 (Appendix F), our choice of tuning parameters for the simulation study (Appendix G) and an empirical analysis of quote data (Appendix H).
A Size estimates

We use the notation from the main paper. In addition, we write $A \lesssim B$ if there is a constant $C$ that is independent of any quantity of interest such that $A \leq CB$. In the following, we repeatedly make use of so-called standard size estimates (cf. Chong (2020c), Appendix D). Under the strengthened hypotheses of Assumption (CLT’), consider for fixed $j, k \in \{1, \ldots, d\}$ and $\ell \in \{1, \ldots, L\}$ an expression like

\[
S_n(t) = \Delta_n^{\frac{1}{2}} \sum_{i=\theta_n+1}^{[T/\Delta_n]} h(\zeta_i^n) \left( \frac{\Delta_n^{i+\ell-1} A^k}{\Delta_n^H} + \frac{1}{\Delta_n^H} \int_{(i+\ell-2)\Delta_n}^{(i+\ell-1)\Delta_n} \left( \sigma_s^{kj} - \sigma_{(i-\theta_n')\Delta_n}^{kj} \right) dB_s^j \right. \\
+ \left. \int_0^\infty \frac{\Delta_n^{i+\ell-1} g(s)}{\Delta_n^H} \left( \rho_s^{kj} - \rho_{(i-\theta_n)\Delta_n}^{kj} \right) 1_{(i-\theta_n)\Delta_n, (i-\theta_n')\Delta_n}(s) dW_s^j \right),
\]

where $\theta_n = [\Delta_n^{-\theta'}], \theta_n' = [\Delta_n^{-\theta''}], \theta_n'' = [\Delta_n^{-\theta'''}]$ and $-\infty \leq \theta', \theta'' < \theta \leq \infty$. In addition, $h$ is a function such that $|h(x)| \lesssim 1 + \|x\|^p$ for some $p > 1$, and $\zeta_i^n$ are random variables with

\[
\sup_{n \in \mathbb{N}} \sup_{i=1, \ldots, [T/\Delta_n]} \mathbb{E}[|\zeta_i^n|^p] < \infty.
\]

For any $q \geq 1$, because $a$ is uniformly bounded by Assumption (CLT’), Minkowski’s integral inequality yields

\[
\mathbb{E}\left[ \left\| \frac{\Delta_n^{i+\ell-1} A}{\Delta_n^H} \right\|^q \right]^{\frac{1}{q}} \leq \frac{1}{\Delta_n^H} \int_{(i+\ell-2)\Delta_n}^{(i+\ell-1)\Delta_n} \mathbb{E}\left[ \left\| a_s \right\|^q \right]^{\frac{1}{q}} ds \lesssim \Delta_n^{1-H}. \tag{A.2}
\]

Similarly, by the Burkholder–Davis–Gundy (BDG) inequality and Assumption (CLT’),

\[
\mathbb{E}\left[ \left\| \frac{1}{\Delta_n^H} \int_{(i+\ell-2)\Delta_n}^{(i+\ell-1)\Delta_n} \left( \sigma_s^{kj} - \sigma_{(i-\theta_n')\Delta_n}^{kj} \right) dB_s^j \right\|^q \right]^{\frac{1}{q}} \lesssim (\theta_n'\Delta_n)^{\frac{1}{2}} \Delta_n^{\frac{1}{2}-H}. \tag{A.3}
\]

Combining Assumption (CLT’) with Lemma B.1, we deduce that

\[
\mathbb{E}\left[ \left\| \int_0^\infty \frac{\Delta_n^{i+\ell-1} g(s)}{\Delta_n^H} \left( \rho_s^{kj} - \rho_{(i-\theta_n)\Delta_n}^{kj} \right) 1_{(i-\theta_n)\Delta_n, (i-\theta_n')\Delta_n}(s) dW_s^j \right\|^q \right]^{\frac{1}{q}} \lesssim (\theta_n\Delta_n)^{\frac{1}{2}} \Delta_n^{\frac{1}{2}} \Delta_n^{\theta'(1-H)} \tag{A.4}
\]

Finally, using Hölder’s inequality to separate $h(\zeta_i^n)$ from the subsequent expression in (A.1), we have shown that

\[
\mathbb{E}\left[ \sup_{t \leq T} |S_n(t)| \right] \lesssim \Delta_n^{\frac{1}{2}} \sum_{i=\theta_n+1}^{[T/\Delta_n]} \left\{ \Delta_n^{1-H} + \Delta_n^{1-H}(\theta_n')^{\frac{1}{2}} + (\theta_n\Delta_n)^{\frac{1}{2}} \Delta_n^{\theta'(1-H)} \right\} \tag{A.5}
\]

The upshot of this example is that the absolute moments of sums and products of more or less complicated expressions can always be bounded term by term: for example, in (A.1),
the terms
\[
\sum_{i=\theta_n+1}^{[t/\Delta_n]} h(\zeta_i^n), \quad \Delta_i^{n+\ell-1} A^k, \quad \int_{(i+\ell-1)\Delta_n}^{(i+\ell)\Delta_n} (\cdots) \, dB^j, \quad \sigma^k_{j} - \sigma^k_{(i-\theta_n)\Delta_n},
\]
\[
\int_0^{(i-\theta_n')\Delta_n} \frac{\Delta_i^{n+\ell-1} g(s)}{\Delta_n^H} (\cdots) \, dW^j, \quad \rho^k_{j} - \rho^k_{(i-\theta_n)\Delta_n}
\]
have sizes (i.e., the $L^2$-moments, for any $q$, are uniformly bounded by a constant times)
\[
\Delta_n^{-1}, \quad 1, \quad \Delta_n, \quad \sqrt{\Delta_n}, \quad (\theta''_n \Delta_n)^{\frac{1}{2}}, \quad \Delta_n^{(1-H)}, \quad (\theta_n \Delta_n)^{\frac{1}{2}},
\]
respectively. The final estimate (A.5) is then obtained by combining these bounds. Clearly, size estimates can be applied to variants of (A.1), too, for example, when the stochastic integral in (A.1) is squared, when we have products of integrals, when $S_n(t)$ is matrix-valued, etc.

Even though size estimates are optimal in general, better estimates may be available in specific cases. One such case occurs when sums have a martingale structure. To illustrate this, let $\mathcal{F}_i^n = \mathcal{F}_{i\Delta_n}$ and consider
\[
S'_n(t) = \Delta_n^{\frac{3}{2}} \sum_{i=1}^{[t/\Delta_n]-L+1} \omega^n_i
\]
with random variables $\omega^n_i$ that are $\mathcal{F}_i^n$-measurable and satisfy $\mathbb{E}[\omega^n_i \mid \mathcal{F}_{i-\theta''_n}] = 0$, where $\theta''_n = [\Delta_n^{-\theta''}]$ for some $0 < \theta'' < 1$. Suppose that $\mathbb{E}[|\omega^n_i|^2]^{1/2} \lesssim \Delta_n^\varpi$ uniformly in $i$ and $n$ for some $\varpi > 0$. Writing
\[
S'_n(t) = \sum_{j=1}^{\theta''_n} S'_{n,j}(t), \quad S'_{n,j}(t) = \Delta_n^{\frac{3}{2}} \sum_{k=1}^{[(t/\Delta_n)-L+1]/\theta''_n} \omega^n_{j+(k-1)\theta''_n},
\]
we observe that each $S'_{n,j}$ is a martingale in $t$ (albeit relative to different filtrations), so the BDG inequality and the triangle inequality yield
\[
\mathbb{E}\left[\sup_{t \leq T} |S'_n(t)|\right] \lesssim (\theta''_n)^{\frac{3}{2}} \Delta_n^\varpi. \quad (A.6)
\]
Very often, $\omega^n_i$ will actually only be $\mathcal{F}_{i+L-1}^n$-measurable. However, a shift by $L$ increments will not change the value of the above estimate. Following Chong (2020b), Section 4, we refer to (A.6) as a martingale size estimate.

B Estimates for fractional kernels

Here we gather some useful results about the kernel $g(t) = K_{H}^{-1} t^{H-1/2}$ introduced in (2.6) (we consider the case $g_0 \equiv 0$ here).

Lemma B.1. Recall the notations introduced in (2.7), (2.4) and (C.2).

(i) For any $k, n \in \mathbb{N},$
\[
\int_0^\infty \Delta_n^k g(t)^2 \, dt = K_{H}^{-2} \left\{ \frac{1}{2H} + \int_1^k \left( t^{H-\frac{1}{2}} - (r-1)^{H-\frac{1}{2}} \right)^2 \, dr \right\} \Delta_n^{2H} \leq \Delta_n^{2H}. \quad (B.1)
\]
(ii) For any $k, \ell, n \in \mathbb{N}$ with $k < \ell$,
\[
\int_{-\infty}^{\infty} \Delta^n_k g(t) \Delta^n_{\ell-k} g(t) \, dt = \Delta^2 H \Gamma_{k-\ell, k}^H \lesssim \Delta^2 H \Gamma_{k-\ell, k}^H,
\]  
where $\Gamma_1^H = \Gamma_1^H$ and $\Gamma_r^H = (r-1)^{-2(1-H)}$ for $r \geq 2$.

(iii) For any $\theta \in (0, 1)$, setting $\theta_n = [\Delta_n^\theta]$, we have for any $i > \theta_n$ and $r \in \mathbb{N}$,
\[
\int_{-\infty}^{(i-\theta_n)\Delta_n} \Delta_i^n g(s) \Delta_i^{n+r} g(s) \, ds \lesssim \Delta^2 H \Delta^{2(1-H)}_n.
\]  

Proof. Let $k \leq \ell$. By direct calculation,
\[
\int_{0}^{\infty} \Delta^n_k g(t) \Delta^n_\ell g(t) \, dt
= \Delta^2 H K^{-2}_H \int_{0}^{k} \left( r^{H-\frac{1}{2}} - (r-1)^{H-\frac{1}{2}+1} \right) \left( (r+(\ell-k))^{H-\frac{1}{2}} - (r+(\ell-k)-1)^{H-\frac{1}{2}} \right) dr,
\]
which shows (B.1) by setting $k = \ell$. Next, let $(B_H^H)_{t \geq 0}$ be a fractional Brownian motion with Hurst index $H$. Then $B^H$ has the Mandelbrot–van Ness representation
\[
B^H_i = K^{-1}_H \int_{\mathbb{R}} \left( (t-s)_{+}^{H-\frac{1}{2}} - (-s)_{+}^{H-\frac{1}{2}} \right) dB_s,
\]
where $B$ is a two-sided standard Brownian motion. Moreover, $\Delta^n_i \Delta^H_i = \int_{\mathbb{R}} \Delta^n_i g(s) \, dB^H_s$ for any $i$. Therefore, by well-known properties of fractional Brownian motion,
\[
\int_{-\infty}^{\infty} \Delta^n_k g(s) \Delta^n_\ell g(s) \, ds = \mathbb{E} [\Delta^n_k B^H \Delta^n_\ell B^H] = \mathbb{E} [B^H_{\Delta_n} B^H_{(\ell-k+1)\Delta_n}] - \mathbb{E} [B^H_{\Delta_n} B^H_{(\ell-k)\Delta_n}]
= \frac{1}{2} \left\{ \Delta^2 H + ((\ell-k+1)\Delta_n)^2 H - ((\ell-k)\Delta_n)^2 H - \Delta^2 H - ((\ell-k)\Delta_n)^2 H + ((\ell-k-1)\Delta_n)^2 H \right\}
= \Delta^2 H \Gamma_{k-\ell, k}^H,
\]
which is the equality in (B.2). Next, use the mean-value theorem twice on $\Gamma_r^H$ in order to obtain for all $r \geq 2$,
\[
\Gamma_r^H = \frac{1}{2} \left( ((r+1)^{2H} - r^{2H}) - \{r^{2H} - (r-1)^{2H}\} \right) \leq \frac{1}{2} (2H) ((r+1)^{2H-1} - (r-1)^{2H-1})
\leq H(2H-1)(r-1)^{2H-2},
\]
which shows the inequality in (B.2). Finally,
\[
\int_{-\infty}^{(i-\theta_n)\Delta_n} \Delta_i^n g(s) \Delta_i^{n+r} g(s) \, ds
= \Delta^2 H K^{-2}_H \int_{\theta_n}^{\infty} \left( t^{H-\frac{1}{2}} - (t-1)^{H-\frac{1}{2}} \right) \left( (t+r)^{H-\frac{1}{2}} - (t+r-1)^{H-\frac{1}{2}} \right) dt
\lesssim \Delta^2 H \int_{\theta_n}^{\infty} \left( t^{H-\frac{1}{2}} - (t-1)^{H-\frac{1}{2}} \right)^2 dt \lesssim \Delta^2 H \int_{\theta_n}^{\infty} (t-1)^{2H-3} dt \lesssim \Delta^2 H \Delta^{(2-2H)}_n,
\]
which yields (B.3).
C Proof of Theorem 3.1

Throughout the proof, by a standard localization argument (cf. Lemma 4.4.9 in Jacod & Protter (2012)), we may and will assume a strengthened version of Assumption (CLT):

Assumption (CLT'). In addition to Assumption (CLT), there is $C > 0$ such that

$$\sup_{(\omega, t) \in \Omega \times (0, \infty)} \left\{ \|a_t(\omega)\| + \|\sigma_t(\omega)\| + \|\rho_t(\omega)\| + \|\tilde{\rho}_t(\omega)\| + \|\tilde{\sigma}_t(\omega)\| \right\} < C.$$ 

Moreover, for every $p > 0$, there is $C_p > 0$ such that for all $s, t > 0$,

$$\mathbb{E}[\|\sigma_t - \sigma_s\|^p]^{\frac{1}{p}} \leq C_p |t - s|^{\frac{1}{2}}, \quad \mathbb{E}[\|\rho_t - \rho_s\|^p]^{\frac{1}{p}} \leq C_p |t - s|^{\gamma},$$

$$\mathbb{E}[\|\tilde{\rho}_t - \tilde{\rho}_s\|^p]^{\frac{1}{p}} \leq C_p |t - s|^\epsilon.$$ (C.1)

Proof of Theorem 3.1. Except for (C.6) below, we may and will assume that $M = 1$. Recalling the decomposition (2.6), since $g_0$ is smooth with $g_0(0) = 0$, we can use the stochastic Fubini theorem (see Protter (2005), Chapter IV, Theorem 65) to write

$$\int_0^t g_0(t - r) \rho_r \, dW_r = \int_0^t \left( \int_r^t g_0(s - r) \, ds \right) \rho_s \, dW_r = \int_0^t \left( \int_0^s g_0(s - r) \rho_r \, dW_r \right) \, ds.$$ 

This is a finite variation process and can be incorporated in the drift process in (2.8). So without loss of generality, we may assume $g_0 \equiv 0$ and $g(t) = K_{i, t}^{-1} t^{H-1/2}$ in the following. Then $Y_t = A_t + M_t + Z_t$, where $A_t = \int_0^t a_s \, ds$ and $M_t = \int_0^t \sigma_s \, dB_s$, and we have $\Delta^n_i Y = \Delta^n_i A + \Delta^n_i M + \Delta^n_i Z$ in the notation of (3.1). Writing $g(t) = 0$ for $t \leq 0$, we also define for all $s, t \geq 0$ and $i, n \in \mathbb{N}$,

$$\Delta^n_i g(s) = g(i \Delta_n - s) - g((i - 1) \Delta_n - s),$$

$$\Delta^n_i g(s) = (\Delta^n_i g(s), \ldots, \Delta^n_{i+L-1} g(s)),$$ (C.2)

such that, in matrix notation,

$$\Delta^n_i Z = \left( \int_0^\infty \Delta^n_i g(s) \rho_s \, dW_s, \ldots, \int_0^\infty \Delta^n_{i+L-1} g(s) \rho_s \, dW_s \right) = \int_0^\infty \rho_s \, dW_s \Delta^n_i g(s).$$

The first step in our proof is to shrink the domain of integration for each $\Delta^n_i Z$. Let

$$\theta \in \left( \frac{1}{4(1 - H)}, \frac{1}{2} \right),$$ (C.3)

which is always possible for $H \in (0, \frac{1}{2})$, and set $\theta_n = \lfloor \Delta_n^{-\theta} \rfloor$. Further define

$$\Delta^n_i Y^{tr} = \Delta^n_i A + \Delta^n_i M + \xi^i_n,$$

$$\xi^i_n = \int_{(i-\theta_n) \Delta_n}^{(i+L-1) \Delta_n} \rho_s \, dW_s \Delta^n_i g(s).$$ (C.4)

Lemma C.1. If $\theta$ is chosen according to (C.3), then

$$\Delta_n^{-\frac{1}{2}} \left\{ V^n_t(Y, t) - \Delta_n \sum_{i=\theta_n+1}^{\lfloor t/\Delta_n \rfloor - L + 1} f \left( \frac{\Delta^i Y^{tr}}{\Delta^i_n} \right) \right\} \overset{L^1}{\rightarrow} 0.$$
The last sum can be further decomposed into three parts:
\[
\Delta_n \sum_{i=\theta_n+1}^{[t/\Delta_n]-L+1} f \left( \frac{\Delta_n Y_{tr}}{\Delta H_n} \right) = V^n(t) + U^n(t) + \frac{1}{\Delta_n} \sum_{i=\theta_n+1}^{[t/\Delta_n]-L+1} \mathbb{E} \left[ f \left( \frac{\Delta_n Y_{tr}}{\Delta H_n} \right) \mid F^n_{i-\theta_n} \right],
\]
where
\[
V^n(t) = \sum_{i=\theta_n+1}^{[t/\Delta_n]-L+1} \Xi^n_i, \quad \Xi^n_i = \Delta_n^2 \left( f \left( \frac{\xi^n_i}{\Delta H_n} \right) - \mathbb{E} \left[ f \left( \frac{\xi^n_i}{\Delta H_n} \right) \mid F^n_{i-\theta_n} \right] \right),
\]
\[
U^n(t) = \Delta_n^2 \sum_{i=\theta_n+1}^{[t/\Delta_n]-L+1} \left\{ f \left( \frac{\Delta_n Y_{tr}}{\Delta H_n} \right) - f \left( \frac{\xi^n_i}{\Delta H_n} \right) - \mathbb{E} \left[ f \left( \frac{\Delta_n Y_{tr}}{\Delta H_n} \right) - f \left( \frac{\xi^n_i}{\Delta H_n} \right) \mid F^n_{i-\theta_n} \right] \right\}.
\]

**Lemma C.2.** For all $H < \frac{1}{2}$, we have that $U^n \overset{L_1}{\longrightarrow} 0$.

In other words, in the limit $\Delta_n \to 0$, the impact of the semimartingale component is negligible, except for its contributions to the conditional expectations in (C.5). As we mentioned above, this is somewhat surprising: It is true that the $L^2$-norm of the semimartingale increment $\Delta_n A + \Delta_n^2 M$, divided by $\Delta_n H$, converges to 0. But the rate $\Delta_n^{1/2-H}$ at which this takes place can be arbitrarily slow if $H$ is close to $\frac{1}{2}$. So Lemma C.2 implies that there is a big gain in convergence rate if one considers the sum of the centered differences $f(\Delta_n Y_{tr}/\Delta H_n) - f(\xi^n_i/\Delta H_n)$. In the proof, we will need for the first time that $f$ has at least $2(N(H) + 1)$ continuous derivatives.

The process $V^n$ only contains the fractional part and is responsible for the limit $\mathcal{Z}$ in (3.7). For the sake of brevity, we borrow a result from Chong (2020a): For each $m \in \mathbb{N}$, consider the sums
\[
V^{n,m,1}(t) = \sum_{j=1}^{J^{n,m}(t)} V_j^{n,m}, \quad V_j^{n,m} = \sum_{k=1}^{m\theta_n} \Xi^n_{(j-1)((m+1)\theta_n+L-1)+k},
\]
\[
V^{n,m,2}(t) = \sum_{j=1}^{J^{n,m}(t) \theta_n+L-1} \sum_{k=1}^{\Xi^n_{(j-1)((m+1)\theta_n+L-1)+m\theta_n+k}},
\]
\[
V^{n,m,3}(t) = \sum_{j=(m+1)\theta_n+L-1}^{[t/\Delta_n]-L+1} \Xi^n_j,
\]
where $J^{n,m}(t) = \left\lfloor \left( \left[ t/\Delta_n \right] - L + 1 \right)/\left( (m+1)\theta_n + L - 1 \right) \right\rfloor$. We then have $V^n(t) = \sum_{i=1}^{3} V^{n,m,i}(t)$. This is very similar to the decomposition on p. 1161 in Chong (2020a). With essentially the same proof, we infer that $V^n(t) \overset{st}{\longrightarrow} \mathcal{Z}$ and, hence,
\[
\Delta_n^2 \left\{ \sum_{i=1}^{[t/\Delta_n]-L+1} f \left( \frac{\Delta_n Y_{tr}}{\Delta H_n} \right) - \sum_{i=\theta_n+1}^{[t/\Delta_n]-L+1} \mathbb{E} \left[ f \left( \frac{\Delta_n Y_{tr}}{\Delta H_n} \right) \mid F^n_{i-\theta_n} \right] \right\} \overset{st}{\longrightarrow} \mathcal{Z},
\]
where $\mathcal{Z}$ is exactly as in (3.7). Therefore, in order to complete the proof of Theorem 3.1, it remains to show that (recall $N(H) = \lfloor 1/(2 - 4H) \rfloor$)
\[
\Delta_n^2 \left\{ \Delta_n \sum_{i=\theta_n+1}^{[t/\Delta_n]-L+1} \mathbb{E} \left[ f \left( \frac{\Delta_n Y_{tr}}{\Delta H_n} \right) \mid F^n_{i-\theta_n} \right] - \int_0^t \mu_f(\pi(s)) \, ds - \sum_{\mathbf{x} \neq \mathbf{0}} \sum_{j=1}^{N(H)} \frac{1}{\lambda_j} \int_0^t \partial^\mathbf{x} \mu_f(\pi(s)) \mathbf{c}(s)^\mathbf{x} \, ds \right\} \overset{L_1}{\longrightarrow} 0.
\]
To this end, we will discretize the volatility processes $\sigma$ and $\rho$ in $\Delta^n Y^{tr}$. The proof is technical (as it involves another multiscale analysis) and will be divided into further smaller steps in Appendix D.

**Lemma C.3.** Assuming (C.3), we have that

$$\Delta_n^{1/2} \sum_{i=\theta_n+1}^{[t/\Delta_n]-L+1} \left\{ E \left[ f \left( \frac{\Delta^n Y^{tr}}{\Delta_n} \right) \right] - \mu_f(\Upsilon^{n,i}) \right\} \xrightarrow{L^1} 0,$$

where $\Upsilon^{n,i} \in (\mathbb{R}^{d \times L})^2$ is defined by

$$(\Upsilon^{n,i})_{k\ell,k^\prime l^\prime} = c((i-1)\Delta_n)_{k\ell,k^\prime l^\prime} \Delta_n^{1-2H} + (\rho(i-1)\Delta_n)^T_{k\ell,k^\prime l^\prime} \int_{(i-\theta_n)\Delta_n}^{(i+L-1)\Delta_n} \frac{\Delta_n^{\ell\ell-1}g(s)\Delta_n^{\ell^\prime l^\prime-1}g(s)}{\Delta_n^{2H}} \, ds. \quad \text{(C.7)}$$

The last part of the proof consists of evaluating

$$\Delta_n^{1/2} \sum_{i=\theta_n+1}^{[t/\Delta_n]-L+1} \mu_f(\Upsilon^{n,i}).$$

This is the place where the asymptotic bias terms arise and which is different from the pure (semimartingale or fractional) cases. Roughly speaking, the additional terms are due to the fact that in the LLN limit (3.9), there is a contribution of magnitude $\Delta_n^{1-2H} c(s)$ coming from the semimartingale part that is negligible on first order but not at a rate of $\sqrt{\Delta_n}$. Expanding $\mu_f(\Upsilon^{n,i})$ in a Taylor sum up to order $N(H)$, we obtain

$$\mu_f(\Upsilon^{n,i}) = \mu_f(\pi((i-1)\Delta_n)) + \sum_{j=1}^{N(H)} \sum_{|\chi|=j} \frac{1}{\chi!} \partial^\chi \mu_f(\pi((i-1)\Delta_n))(\Upsilon^{n,i} - \pi((i-1)\Delta_n))^\chi + \sum_{|\chi|=N(H)+1} \frac{1}{\chi!} \partial^\chi \mu_f(\nu^{n,i}_i)(\Upsilon^{n,i} - \pi((i-1)\Delta_n))^\chi,$$

where $\nu^{n}_i$ is a point between $\Upsilon^{n,i}$ and $\pi((i-1)\Delta_n)$. The next lemma shows two things: first, the term of order $N(H) + 1$ is negligible, and second, for $j = 1, \ldots, N(H)$, we may replace $\Upsilon^{n,i} - \pi((i-1)\Delta_n)$ by $\Delta_n^{1-2H}c((i-1)\Delta_n)$.

**Lemma C.4.** We have that $X^n_1 \xrightarrow{L^1} 0$ and $X^n_2 \xrightarrow{L^1} 0$, where

$$X^n_1(t) = \Delta_n^{1/2} \sum_{i=\theta_n+1}^{[t/\Delta_n]-L+1} \sum_{|\chi|=j} \frac{1}{\chi!} \partial^\chi \mu_f(\pi((i-1)\Delta_n))$$

$$\times \left\{ (\Upsilon^{n,i} - \pi((i-1)\Delta_n))^\chi - \Delta_n^{j(1-2H)} c((i-1)\Delta_n)^\chi \right\}, \quad \text{(C.8)}$$

$$X^n_2(t) = \Delta_n^{1/2} \sum_{i=\theta_n+1}^{[t/\Delta_n]-L+1} \sum_{|\chi|=N(H)+1} \frac{1}{\chi!} \partial^\chi \mu_f(\nu^{n,i}_i)(\Upsilon^{n,i} - \pi((i-1)\Delta_n))^\chi.$$

In a final step, we remove the discretization of $\sigma$ and $\rho$.  

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Lemma C.5. If \( \theta \) is chosen according to (C.3), then
\[
\Delta_n^{-\frac{1}{2}} \left\{ \Delta_n \sum_{i=\lambda_n+1}^{[t/\Delta_n]-L+1} \mu_f((i-1)\Delta_n)) - \int_0^t \mu_f(\pi(s)) \, ds \right\} \overset{L^1}{\rightarrow} 0 \text{ (C.9)}
\]
and
\[
\Delta_n^{-\frac{1}{2}} \left\{ \Delta_n \sum_{i=\theta_n+1}^{[t/\Delta_n]-L+1} \sum_{j=1}^{N(H)} \Delta_n^{j(1-2H)} \sum_{|x|=j} \frac{1}{\chi^j} \partial^x \mu_f((i-1)\Delta_n))c((i-1)\Delta_n)^x - \int_0^t \sum_{j=1}^{N(H)} \sum_{|x|=j} \frac{1}{\chi^j} \partial^x \mu_f(s)) \Delta_n^{j(1-2H)}c(s)^x \, ds \right\} \overset{L^1}{\rightarrow} 0. \text{ (C.10)}
\]

By the properties of stable convergence in law (see Equation (2.2.5) in Jacod & Protter (2012)), the CLT in (3.7) follows by combining Lemmas C.1–C.5.

D Details for the proof of Theorem 3.1

Assumption (CLT') is in force throughout this section.

Proof of Lemma C.1. By the calculations in (A.2)–(A.4), we have \( \mathbb{E}[\|\Delta_n^\theta Y/\Delta_n^H\|^p]^{1/p} \lesssim 1 \) for all \( p \geq 1 \). As \( f \) grows at most polynomially, we see that \( \mathbb{E}[f(\Delta_n^\theta Y/\Delta_n^H)] \) is of size 1. Hence, \( \mathbb{E}[\Delta_n^{\theta/2} \sum_{i=1}^{\theta_n} f(\Delta_n^\theta Y/\Delta_n^H)] \lesssim \Delta_n^{1/2-\theta} \), which implies \( \Delta_n^{\theta/2} \sum_{i=1}^{\theta_n} f(\Delta_n^\theta Y/\Delta_n^H) \to 0 \) in \( L^1 \) since \( \theta < \frac{1}{2} \) by (C.3). As a result, omitting the first \( \theta_n \) terms in the definition of \( V_n^\theta(Y, t) \) does no harm asymptotically. Next, we define

\[
\Lambda_n^\theta = f\left( \frac{\Delta_n^\theta Y}{\Delta_n^H} \right) - f\left( \frac{\Delta_n^\theta Y_Y}{\Delta_n^H} \right), \quad \overline{\lambda}_i^\theta = \Lambda_i^\theta - \mathbb{E}[\Lambda_i^\theta | F_{i-\theta_n}]. \quad \text{(D.1)}
\]

By our choice (C.3) of \( \theta \) and since \( H < \frac{1}{2} \), the lemma is proved once

\[
\mathbb{E} \left[ \sup_{t \leq T} \left| \Delta_n^\theta \sum_{i=\theta_n+1}^{[t/\Delta_n]-L+1} \overline{\lambda}_i^\theta \right| \right] \lesssim \Delta_n^{\theta(\frac{1}{2}-H)} , \quad \text{(D.2)}
\]
\[
\mathbb{E} \left[ \sup_{t \leq T} \left| \Delta_n^\theta \sum_{i=\theta_n+1}^{[t/\Delta_n]-L+1} \mathbb{E}[\Lambda_i^\theta | F_{i-\theta_n}] \right| \right] \lesssim \Delta_n^{\theta(\frac{1}{2}-H)} + \Delta_n^{2(1-H)-\frac{1}{2}} \quad \text{(D.3)}
\]

are established. To this end, let \( \lambda_i^\theta = \Delta_n^\theta Y - \Delta_n^\theta Y_Y / \Delta_n^H = \int_0^{(i-\theta_n)\Delta_n} \rho_s \, dW_s / \Delta_n^\theta \). By Assumption (CLT), we have \( |f(z) - f(z')| \lesssim (1 + \|z\|^{p-1} + \|z'\|^{p-1}) |z - z'| \). In addition, \( \mathbb{E}[f(\Delta_n^\theta Y/\Delta_n^H)] \) is of size 1, so \( \mathbb{E}[\lambda_i^\theta]^2 \lesssim \mathbb{E}[\lambda_i^\theta]^2 \lesssim \mathbb{E}[\lambda_i^\theta]^2 \lesssim \Delta_n^{2(1-H)} \), where we used (A.4) for the last estimation. By construction, \( \overline{\lambda}_i^\theta \) is \( F_{i-\theta_n} \)-measurable and has conditional expectation 0 given \( F_{i-\theta_n} \). Therefore, we can further use an estimate of the kind (A.6) to show that the left-hand side of (D.2) is bounded, up to constant, by \( \sqrt{\theta_n} \Delta_n^{\theta(1-H)} \lesssim \Delta_n^{\theta(1-H)-\theta/2} = \Delta_n^{\theta(1/2-H)} \).
Next, let $\psi^n_i = \sigma(i-\theta_n)\Delta^n B + \int_{(i-\theta_n)}^{(i+L-1)\Delta_n} \rho(i-\theta_n)\Delta_n \, dW_n \Delta^n g(s)$. Since $f$ is smooth, applying Taylor's theorem twice yields $\Lambda^n_i = \Lambda^n_{i,1} + \Lambda^n_{i,2} + \Lambda^n_{i,3}$, where

$$
\Lambda^n_{i,1} = \sum_{|\chi|=1} \partial^{|\chi|} f\left(\frac{\psi^n_i}{\Delta^H_n}\right)(\Lambda^n_i)^\chi, \quad \Lambda^n_{i,2} = \sum_{|\chi|=1} \partial^{|\chi|+1} f(\tilde{\eta}^n_i)\left(\frac{\Delta^n Y - \psi^n_i}{\Delta^H_n}\right)^\chi(\Lambda^n_i)^\chi,
$$

$$
\Lambda^n_{i,3} = \sum_{|\chi|=2} \frac{\partial^{|\chi|} (\eta^n_i)}{\chi!}(\Lambda^n_i)^\chi
$$

and $\chi, \chi' \in \mathbb{N}_{0}^{d \times L}$ are multi-indices and $\eta^n_i$ (resp., $\tilde{\eta}^n_i$) is a point on the line between $\frac{\Delta^n Y}{\Delta^H_n}$ and $\frac{\Delta^n Y - \psi^n_i}{\Delta^H_n}$ (resp., $\frac{\Delta^n Y}{\Delta^H_n}$ and $\frac{\psi^n_i}{\Delta^H_n}$). Accordingly, we split

$$
\Delta^n \sum_{i=\theta_n+1}^{[t/\Delta_n]-L+1} \mathbb{E}[\Lambda^n_i \mid F^n_{i-\theta_n}] = \sum_{j=1}^{3} \mathbb{E}[\Lambda^n_j(t)], \quad \mathbb{L}^n_i(t) = \Delta^n \sum_{i=\theta_n+1}^{[t/\Delta_n]-L+1} \mathbb{E}[\Lambda^n_i \mid F^n_{i-\theta_n}].
$$

Note that $\mathbb{E}[\partial^{|\chi|} f\left(\frac{\psi^n_i}{\Delta^H_n}\right)(\Lambda^n_i)^\chi \mid F^n_{i-\theta_n}] = (\Lambda^n_i)^\chi \mathbb{E}[\partial^{|\chi|} f\left(\frac{\psi^n_i}{\Delta^H_n}\right) \mid F^n_{i-\theta_n}] = 0$ because $\lambda_i^n$ is $F^n_{i-\theta_n}$-measurable, $\psi^n_i$ is centered normal given $F^n_{i-\theta_n}$, and $f$ has odd partial derivatives of first orders (since $f$ is even). It follows that $\mathbb{L}^n_i(t) = 0$ identically. Writing

$$
\mathbb{L}^n_i(s) = \left(\mathbb{I}_{((i-1)\Delta_n,i\Delta_n)}(s), \ldots, \mathbb{I}_{((i+L-2)\Delta_n,(i+L-1)\Delta_n)}(s)\right),
$$

we can decompose $\Delta^n Y - \psi^n_i$ as

$$
\Delta^n A + \int_0^t (\sigma - \sigma(i-\theta_n)\Delta_n) \, dB_n \, \mathbb{L}^n_i(s) + \int_{(i-\theta_n)\Delta_n}^{(i+L-1)\Delta_n} (\rho - \rho(i-\theta_n)\Delta_n) \, dW_n \Delta^n g(s).
$$

By a standard size estimate, it follows that

$$
\mathbb{E}\left[\sup_{t \leq T} \|\mathbb{L}^n_i(t)\|^2\right] \lesssim \left(\Delta^\frac{3}{2} \Delta_n^{-1}\right)\left(\Delta_n^{1-H} + \theta_n^\frac{1}{2}\Delta_n^{1-H} + (\theta_n\Delta_n)^\frac{1}{2}\right) \Delta_n^{\theta(1-H)}
$$

$$
\lesssim \Delta_n^{-\frac{1}{2}} \Delta_n^{\theta(1-H)} (\theta_n\Delta_n)^{\frac{1}{2}} = \Delta_n^{\theta(1-H) - \frac{1}{2}},
$$

proving (D.3) and thus the lemma. \hfill \Box

**Proof of Lemma C.2.** Let $\xi^{n,\text{dis}}_i = \int_{(i-\theta_n)\Delta_n}^{(i+L-1)\Delta_n} \rho(i-\theta_n)\Delta_n \, dW_n \Delta^n g(s)$ and recall the definition of $\xi^n_i$ from (C.4). In a first step, we show that $U^n$ can be approximated by

$$
U^n_i(t) = \Delta^n \sum_{i=\theta_n+1}^{[t/\Delta_n]-L+1} \left\{ f\left(\frac{\sigma(i-\theta_n)\Delta_n B + \xi^{n,\text{dis}}_i}{\Delta^H_n}\right) - f\left(\xi^{n,\text{dis}}_i\right) \right\}
$$

$$
- \mathbb{E}\left[f\left(\frac{\sigma(i-\theta_n)\Delta_n B + \xi^{n,\text{dis}}_i}{\Delta^H_n}\right) \mid F^n_{i-\theta_n}\right].
$$

By (C.1) and a size estimate as in (A.4), the difference $\xi^n_i - \xi^{n,\text{dis}}_i$ is of size $(\theta_n\Delta_n)^{1/2}$. Together with (A.2) and (A.3), we further have that $\Delta^n Y - \sigma(i-\theta_n)\Delta_n B - \xi^{n,\text{dis}}_i$ is of size $\Delta_n + \sqrt{\Delta_n} + (\theta_n\Delta_n)^{1/2}$. By the mean-value theorem, these size bounds imply that

$$
\mathbb{E}\left[\left| f\left(\frac{\Delta^n Y}{\Delta^H_n}\right) - f\left(\frac{\sigma(i-\theta_n)\Delta_n B + \xi^{n,\text{dis}}_i}{\Delta^H_n}\right)\right|^p + \left| f\left(\frac{\xi^n_i}{\Delta^H_n}\right) - f\left(\frac{\xi^{n,\text{dis}}_i}{\Delta^H_n}\right)\right|^p \right] \lesssim (\theta_n\Delta_n)^{\frac{1}{2}}
$$

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for any $p > 0$. Moreover, the $i$th term in the definition of $\overline{U}^n(t)$ is $\mathcal{F}_{i+L−1}^n$-measurable with zero mean conditionally on $\mathcal{F}_{i}^n$. Therefore, employing a martingale size estimate as in (A.6), we obtain $\mathbb{E}[\sup_{t \leq T}|U^n(t) - \overline{U}^n(t)|] \lesssim \sqrt{\theta_n(\Delta_n)^{1/2}} \leq \Delta_n^{1/2-\theta}$, which converges to 0 by (C.3).

Next, because $B$ and $W$ are independent, we can apply Itô’s formula with $\xi^{i,n,\text{dis}}$ as starting point and write

$$f\left(\sigma_{(i-1)\Delta_n}\Delta_n^{H} B + \xi^{i,n,\text{dis}}\right) - f\left(\xi^{i,n,\text{dis}}\right) = \Delta_n^{-H} \sum_{j,k=1}^{d} \sum_{\ell=1}^{L} \int_{(i-\ell-1)\Delta_n}^{(i-\ell)\Delta_n} \frac{\partial}{\partial z_{k\ell}} \left(\frac{\Delta Y_{i}^{n,\text{dis}}(s)}{\Delta_n^{H}}\right) \sigma_{(i-1)\Delta_n}^{kj} dB_{s}^{j} \tag{D.4}$$

where $\Delta Y_{i}^{n,\text{dis}}(s) = \int_{(i-1)\Delta_n}^{s} \sigma_{(i-1)\Delta_n} dB_{r} + \xi^{i,n,\text{dis}}$. Clearly, the stochastic integral is $\mathcal{F}_{i+L−1}^n$-measurable and conditionally centered given $\mathcal{F}_{i}^n$. Therefore, by a martingale size estimate, its contribution to $\overline{U}^n(t)$ is of magnitude $\Delta_n^{1/2-H}$, which is negligible because $H < \frac{1}{2}$. For the Lebesgue integral, we apply Itô’s formula again and write

$$\frac{\partial}{\partial z_{k\ell}} \left(\frac{\Delta Y_{i}^{n,\text{dis}}(s)}{\Delta_n^{H}}\right)\Delta_n^{-H} \sum_{j_2,k_2=1}^{d} \sum_{\ell=1}^{L} \int_{(i-\ell-2)\Delta_n}^{s\wedge(i-\ell)\Delta_n} \frac{\partial^3}{\partial z_{k\ell} \partial z_{k'\ell}} \left(\frac{\Delta Y_{i}^{n,\text{dis}}(r)}{\Delta_n^{H}}\right) \sigma_{(i-1)\Delta_n}^{k_2 j_2} dB_{r}^{j_2}$$

$$+ \frac{\Delta_n^{-2H}}{2} \sum_{k_2,k_2'=1}^{d} \sum_{\ell=1}^{L} \int_{(i-\ell-2)\Delta_n}^{s\wedge(i-\ell)\Delta_n} \frac{\partial^4}{\partial z_{k\ell} \partial z_{k\ell} \partial z_{k'\ell} \partial z_{k'\ell}} \left(\frac{\Delta Y_{i}^{n,\text{dis}}(r)}{\Delta_n^{H}}\right) (\sigma \sigma^T)_{(i-1)\Delta_n} dB_{r}$$

By the same reason as before, the stochastic integral (even after we plug it into the drift in (D.4)) is $\mathcal{F}_{i+L−1}^n$-measurable with zero $\mathcal{F}_{i}^n$-conditional mean and therefore negligible. The Lebesgue integral is essentially of the same form as the one in (D.4). Because $f$ is smooth, we can repeat this procedure as often as we want. What is important, is that we gain a net factor of $\Delta_n^{1-2H}$ in each step (we have $\Delta_n^{2H}$ times a Lebesgue integral over an interval of length at most $\Delta_n$). After $N$ applications of Itô’s formula, the final drift term yields a contribution of size $\sqrt{\theta_n(\Delta_n^{N(1-2H)})}$ to $\overline{U}^n(t)$. As $\theta < \frac{1}{2}$, it suffices to take $N = N(H) + 1$ to make this convergent to 0. \hfill $\square$

**Proof of Lemma C.3.** We begin by discretizing $\rho$ on a finer scale and let

$$\Theta_i^n = \int_{(i-\theta_n)\Delta_n}^{(i+L-1)\Delta_n} \sum_{k=1}^{Q} \rho_{(i-\theta_n)\Delta_n} 1_{(i-\theta_n)\Delta_n} \sum_{l=1}^{L} \left(\Delta_n\Delta_n^{\gamma/q(q)}g(s)\right) dW_s \Delta_n^{\gamma} g(s), \tag{D.5}$$

where $\theta^{(q)} = [\Delta_n^{\theta^{(q)}}]$ for $q = 0, \ldots, Q - 1$, $\theta^{(Q)} = -(L - 1)$ and the numbers $\theta^{(q)}, q = 0, \ldots, Q - 1$ for some $Q \in \mathbb{N}$, are chosen such that $\theta = \theta^{(0)} > \ldots > \theta^{(Q-1)} > \theta^{(Q)} = 0$ and

$$\theta^{(q)} > \frac{\gamma}{1-H} - \frac{\gamma - \frac{1}{2}}{1-H}, \quad q = 1, \ldots, Q. \tag{D.6}$$
where $\gamma$ describes the regularity of the volatility process $\rho(0)$ in (3.3). Because $H < \frac{1}{2}$ and we can make $\gamma$ arbitrarily close to $\frac{1}{2}$ if we want, there is no loss of generality to assume that $\gamma/(1 - H) < 1$. In this case, the fact that a choice as in (D.6) is possible can be verified by solving the associated linear recurrence equation. Defining $\Delta^n Y_{\text{dis}} = \sigma(i-1)\Delta_n \Delta^n B + \Theta^n_i$, we will show in Lemma D.1 below that

$$\Delta^n \sum_{i = \theta_n+1}^{[t/\Delta_n] - L + 1} \left\{ \mathbb{E} \left[ f\left( \frac{\Delta^n Y_{\text{tr}}}{\Delta_n} \right) \mid \mathcal{F}^{n}_{i-\theta_n} \right] - \mathbb{E} \left[ f\left( \frac{\Delta^n Y_{\text{dis}}}{\Delta_n} \right) \mid \mathcal{F}^{n}_{i-\theta_n} \right] \right\} \overset{L^1}{\to} 0. \qquad (D.7)$$

Next, we define another matrix $\Upsilon^n_i \in (\mathbb{R}^{d \times L})^2$ by

$$(\Upsilon^n_i)_{k\ell,k'\ell'} = c((i-1)\Delta_n) \Delta_n \frac{1}{2} + \sum_{q=1}^{Q} \left( \rho_{(i-\theta_n^{(q-1)})\Delta_n}^{T} \rho_{(i-\theta_n^{(q-1)})\Delta_n} \right)_{k,k'}$$

$$\times \int_{(i-\theta_n)\Delta_n}^{(i+L-1)\Delta_n} \Delta_n \frac{1}{2} \left( \frac{1}{(i-\theta_n^{(q-1)})\Delta_n,(i-\theta_n^{(q)})\Delta_n} \right) (s) \, ds. \qquad (D.8)$$

If $c$ and $\rho$ are deterministic, this is the covariance matrix of $\Delta^n Y_{\text{tr}} / \Delta_n$. Also notice that the only difference to $\Upsilon^n_i$ are the discretization points of $\rho$. Next, we show that

$$\Delta^n \sum_{i = \theta_n+1}^{[t/\Delta_n] - L + 1} \left\{ \mathbb{E} \left[ f\left( \frac{\Delta^n Y_{\text{dis}}}{\Delta_n} \right) \mid \mathcal{F}^{n}_{i-\theta_n} \right] - \mu_f(\mathbb{E}[\Upsilon^n_i \mid \mathcal{F}^{n}_{i-\theta_n}]) \right\} \overset{L^1}{\to} 0, \qquad (D.9)$$

where $\mu_f$ is the mapping defined after Assumption (CLT). This will be achieved through successive conditioning in Lemma D.2. Finally, as we show in Lemma D.3, we have

$$\mathbb{E} \left[ \sup_{t \leq T} \left| \Delta^n \sum_{i = \theta_n+1}^{[t/\Delta_n] - L + 1} \left\{ \mu_f(\mathbb{E}[\Upsilon^n_i \mid \mathcal{F}^{n}_{i-\theta_n}]) - \mu_f(\Upsilon^n_i) \right\} \right| \right] \to 0, \qquad (D.10)$$

$$\mathbb{E} \left[ \sup_{t \leq T} \left| \Delta^n \sum_{i = \theta_n+1}^{[t/\Delta_n] - L + 1} \left\{ \mu_f(\Upsilon^n_i) - \mu_f(\Upsilon^n_i) \right\} \right| \right] \to 0, \qquad (D.11)$$

which completes the proof of the current lemma.
Using Hölder’s inequality, the estimates (A.2), (A.3) and (A.4) and the polynomial growth assumption on $\partial^s f$, we see that $\Delta_n^q Y^{\text{dis}} / \Delta_n^H$ is of size one and, since $0 < \theta(q) < \frac{1}{2}$,

$$
\mathbb{E} \left[ \sup_{t \leq T} |Q_2^n(t)| \right] \lesssim \Delta_n^{-\frac{1}{2}} \left( \Delta_n^{2(1-H)} + \Delta_n^{2(1-H)} + \sum_{q=1}^{Q} \Delta_n^{(1-\theta(q-1)) + 2(\theta(q))(1-H)} \right) \rightarrow 0. \quad (D.13)
$$

Next, we further split $Q_{11}^n(t) = Q_{11}^n(t) + Q_{12}^n(t) + Q_{13}^n(t)$ into three terms according to the decomposition (D.12). Using again (A.2) and (A.3), we see that both $Q_{11}^n(t)$ and $Q_{12}^n(t)$ are of size $\Delta_n^{-1/2 + (1-H)} = \Delta_n^{1/2 - H}$. We first tackle the term $Q_{13}^n(t)$, which requires a more careful analysis. Here we need assumption (3.3) on the noise volatility $\rho$. Since $t \mapsto f'_0 \bar{\rho}_s \, d\mathbb{W}_s$ satisfies a better regularity condition than (C.1), we may incorporate the drift term in $\rho^{(0)}$ for the remainder of the proof. Then we further write $Q_{13}^n(t) = Q_{13}^n(t) + Q_{2}^n(t)$ where $Q_{1}^n(t)$ and $Q_{2}^n(t)$ correspond to taking only $\rho^{(0)}$ and $f'_0 \bar{\rho}_s \, d\mathbb{W}_s$ instead of $\rho$, respectively. By (3.4), (A.4) and (D.6), $Q_{1}^n(t)$ is of size

$$
\sum_{q=1}^{Q} \Delta_n^{-\frac{1}{2} + (1-\theta(q-1)) + \theta(q)(1-H)} \rightarrow 0. \quad (D.14)
$$

For $Q_{2}^n(t)$, we write $Q_{2}^n(t) = \sum_{|\chi|=1} (R_{21}^{n,\chi}(t) + R_{22}^{n,\chi}(t) + R_{23}^{n,\chi}(t))$, where, if $\chi_{kt} = 1$,

$$
R_{21}^{n,\chi}(t) = \Delta_n^{\frac{1}{2}} \sum_{i=\theta_n+1}^{[t/\Delta_n]-L+1} d \mathbb{E} \left[ \partial^s f \left( \frac{\Delta_n^q Y^{\text{dis}}}{\Delta_n^H} \right) \sum_{q=1}^{Q} \int_{(i-\theta(q))\Delta_n}^{(i+\theta(q))\Delta_n} \frac{\Delta_n^{n+\ell-1} g(s)}{\Delta_n^H} \right] \times \int_{(i-\theta(q-1))\Delta_n}^{(i-\theta(q))\Delta_n} \left( \frac{\Delta_n^{n+\ell-1} g(s)}{\Delta_n^H} \right) \int_{(i-\theta(q-1))\Delta_n}^{(i-\theta(q))\Delta_n} \frac{\rho^{(1)}_{k,\ell'} \, d\mathbb{W}_{i-\theta_n} - \rho^{(2)}_{(i-\theta(q-1))\Delta_n} \, d\mathbb{W}_{i-\theta_n} \mid \mathcal{F}_{i-\theta_n}^n \right],
$$

$$
R_{22}^{n,\chi}(t) = \Delta_n^{\frac{1}{2}} \sum_{i=\theta_n+1}^{[t/\Delta_n]-L+1} Q \sum_{q=1}^{Q} d \mathbb{E} \left[ \partial^s f \left( \frac{\Delta_n^q Y^{\text{dis}}}{\Delta_n^H} \right) \cdot \partial^s f \left( \frac{\Delta_n^q Y^{\text{dis}}}{\Delta_n^H} \right) \right] \times \int_{(i-\theta(q))\Delta_n}^{(i-\theta(q))\Delta_n} \frac{\Delta_n^{n+\ell-1} g(s)}{\Delta_n^H} \int_{(i-\theta(q))\Delta_n}^{(i-\theta(q))\Delta_n} \frac{\rho^{(1)}_{k,\ell'} \, d\mathbb{W}_{i-\theta_n} - \rho^{(2)}_{(i-\theta(q-1))\Delta_n} \, d\mathbb{W}_{i-\theta_n} \mid \mathcal{F}_{i-\theta_n}^n \right],
$$

$$
R_{23}^{n,\chi}(t) = \Delta_n^{\frac{1}{2}} \sum_{i=\theta_n+1}^{[t/\Delta_n]-L+1} \sum_{q=1}^{Q} d \mathbb{E} \left[ \partial^s f \left( \frac{\Delta_n^q Y^{\text{dis}}}{\Delta_n^H} \right) \cdot \partial^s f \left( \frac{\Delta_n^q Y^{\text{dis}}}{\Delta_n^H} \right) \right] \times \int_{(i-\theta(q))\Delta_n}^{(i-\theta(q))\Delta_n} \frac{\Delta_n^{n+\ell-1} g(s)}{\Delta_n^H} \int_{(i-\theta(q))\Delta_n}^{(i-\theta(q))\Delta_n} \frac{\rho^{(1)}_{k,\ell'} \, d\mathbb{W}_{i-\theta_n} - \rho^{(2)}_{(i-\theta(q-1))\Delta_n} \, d\mathbb{W}_{i-\theta_n} \mid \mathcal{F}_{i-\theta_n}^n \right],
$$

and $\Delta_n^q Y^{\text{dis},q} = \int_{(i-\theta(q))\Delta_n}^{(i+L-1)\Delta_n} \rho^{(1)}_{(i-\theta(q-1))\Delta_n} \, d\mathbb{W}_{n} \Delta_n^{n} g(s)$. Using the BDG and Minkowski integral inequality alternatingly, we obtain, for any $p \geq 2$,

$$
\mathbb{E} \left[ \left( \int_{(i-\theta(q))\Delta_n}^{(i+L-1)\Delta_n} \frac{\Delta_n^{n+\ell-1} g(s)}{\Delta_n^H} \left( \int_{(i-\theta(q))\Delta_n}^{(i+L-1)\Delta_n} \left( \rho^{(1)}_{k,\ell'} \, d\mathbb{W}_{i-\theta_n} - \rho^{(2)}_{(i-\theta(q-1))\Delta_n} \, d\mathbb{W}_{i-\theta_n} \right) \right) \, dW_{s} \right]^{p} \right]^{\frac{1}{p}} \lesssim \left( \int_{(i-\theta(q))\Delta_n}^{(i+L-1)\Delta_n} \frac{\Delta_n^{n+\ell-1} g(s)^2}{\Delta_n^H} \right)^{\frac{1}{2}} \mathbb{E} \left[ \int_{(i-\theta(q))\Delta_n}^{(i+L-1)\Delta_n} \left( \rho^{(1)}_{k,\ell'} \, d\mathbb{W}_{i-\theta_n} - \rho^{(2)}_{(i-\theta(q-1))\Delta_n} \, d\mathbb{W}_{i-\theta_n} \right)^{2p} \right]^{\frac{1}{2}} ds \lesssim \left( \int_{(i-\theta(q))\Delta_n}^{(i+L-1)\Delta_n} \frac{\Delta_n^{n+\ell-1} g(s)^2}{\Delta_n^H} \right)^{\frac{1}{2}} \lesssim \Delta_n^{(\frac{1}{2} + \epsilon') (1-\theta(q-1)) + \theta(q) (1-H)},
$$

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where $\varepsilon'$ is as in (3.5). Thus, $\mathbb{R}^{n,x}_{21} (t)$ is of size $\sum_{q=1}^{Q} \Delta_n^{-\frac{1}{2} + \left(\frac{1}{2} + \varepsilon'\right) (1 - \theta^{(q-1)}) + \theta^{(q)} (1 - H)}$, which is almost the same as (D.14); the only difference is that $\gamma$ is replaced by $\frac{1}{2} + \varepsilon'$. Since we can assume without loss of generality that $\frac{1}{2} + \varepsilon' < \gamma$, the formula (D.6) implies that we have

$$-\frac{1}{2} + \left(\frac{1}{2} + \varepsilon'\right) (1 - \theta^{(q-1)}) + \theta^{(q)} (1 - H) > 0 \text{ for all } q = 1, \ldots, Q,$$

which means that $\mathbb{R}^{n,x}_{21} (t)$ is asymptotically negligible.

Next, using Lemma B.1 (iii) and a similar estimate to the previous display, we see that $(\Theta_i^n - \Delta_n^{q} Y^{\text{dis},q})/\Delta_n^H$ is of size $\Delta_n^{q} (1 - \theta^{(q-1)}) + \Delta_n^{(1 - \theta^{(q-1)})/2}$. Hence, with the two estimates (A.2) and (A.3) at hand, we deduce that $\mathbb{R}^{n,x}_{22} (t)$ is of size

$$\sum_{q=1}^{Q} \Delta_n^{-\frac{1}{2} + \theta^{(q-1)} (1 - H) + \frac{1}{2} (1 - \theta^{(q-1)})} \leq \sum_{q=1}^{Q} \left( \Delta_n^{-\frac{1}{2} - \theta^{(q-1)} (1 - H) + \frac{1}{2} (1 - \theta^{(q-1)})} \right).$$

The last term clearly goes to 0 because $\theta^{(q-1)} < \frac{1}{2}$ by (C.3). Without loss of generality, we can assume that $\gamma > \frac{1}{2}$ is sufficiently close to $\frac{1}{2}$ such that the first term is negligible as well. With this particular value, we then make sure that

$$\frac{\gamma - \frac{1}{2}}{\gamma + \frac{1}{2} - H} < \theta^{(q-1)} < \frac{\gamma - \frac{1}{2}}{\gamma},$$

which, on the one hand, is in line with (D.6) and, on the other hand, guarantees that the second term in the preceding display tends to 0 for all $q = 1, \ldots, Q$.

Finally, to compute $\mathbb{R}^{n,x}_{23} (t)$, we first condition on $\mathcal{F}_{i-\theta^{(q-1)}}$. Because $f$ is even and $\Delta_n^{q} Y^{\text{dis},q}/\Delta_n^H$ has a centered normal distribution given $\mathcal{F}_{i-\theta^{(q-1)}}$, it follows that $\partial^\gamma f (\Theta_i^n / \Delta_n^H)$ is an element of the direct sum of all odd-order Wiener chaoses. At the same time, the double stochastic integrals in $\mathbb{R}^{n,x}_{23} (t)$ belongs to the second Wiener chaos; see Proposition 1.1.4 in Nualart (2006). Since Wiener chaoses are mutually orthogonal, we obtain $\mathbb{R}^{n,x}_{23} (t) = 0$. Because this reasoning is valid for all multi-indices with $|\chi| = 1$, we have shown that $\mathbb{R}^n_{2}(t)$ is asymptotically negligible.

**Lemma D.2.** The convergence (D.9) holds true.

**Proof.** For $r = 0, \ldots, Q$ (where $Q$ is as in Lemma D.1), define

$$\mathbb{Y}^{n,r}_{i} = \int_{(i-\theta^q_\Delta) \Delta_n}^{(i+L-1) \Delta_n} \left( \sum_{q=1}^{r} \rho^q \mathbb{I}_{(i-\theta^q_\Delta) \Delta_n, (\theta^q_\Delta) \Delta_n} (s) \right) dW_s \Delta_n^{q} g(s) / \Delta_n^H,$$

$$\mathbb{Y}^{n,r}_{i} = c((i-1) \Delta_n) \Delta_n^{1-2H} + \sum_{q=r+1}^{Q} (\rho^q)^T \mathbb{I}_{(i-\theta^q_\Delta) \Delta_n} \Delta_n^{q} g(s)^T \Delta_n^H g(s) \Delta_n^H ds.$$

Note that $\mathbb{Y}^{n,r}_{i} \in \mathbb{R}^{d \times L}$, $\mathbb{Y}^{n,r}_{i} \in \mathbb{R}^{(d \times L) \times (d \times L)}$ and that $\mathbb{Y}^{n,Q}_{i} = \Theta_i^n / \Delta_n^H$ by (D.5). In order to show (D.9), we need the following approximation result for each $r = 1, \ldots, Q - 1$:

$$\Delta_n^{\frac{1}{2}} \sum_{i=\theta^q_\Delta+1}^{[\frac{t}{\Delta_n}]-L+1} \mathbb{E} \left[ \mu_f (\mathbb{Y}^{n,r+1}_{i}) (\mathbb{Y}^{n,r+1}_{i}) - \mu_f (\mathbb{Y}^{n,r-1}_{i}) (\mathbb{Y}^{n,r-1}_{i}) \mid \mathcal{F}_{i-\theta^q_\Delta} \right] \xrightarrow{L^1} 0, \quad (D.15)$$
where $\Upsilon_i^{n,r} = \mathbb{E}[\Upsilon_i^{n,r} \mid \mathcal{F}_{i-\theta^{(q)}_n}]$. Let us proceed with the proof of (D.9), taking the previous statement for granted. Defining

$$
\overline{\Upsilon}_i^n = \int_{(i-\theta^{(q)}_n)}^{(i-1)\Delta_n} \sum_{q=1}^Q \rho_i^{(q)} \Delta_n \mathbbm{1}_{((i-\theta^{(q-1)}_n)\Delta_n,(i-\theta^{(q)}_n)\Delta_n)}(s) dW_s \frac{\Delta_i^n g(s)}{\Delta^n_i},
$$

we can use the tower property of conditional expectation to derive

$$
\mathbb{E}\left[f\left(\frac{\Delta_i^n \Upsilon_i^{\text{dis}}}{\Delta^n_i}\right) \mid \mathcal{F}_{i-\theta_n}\right] = \mathbb{E}\left[\mathbb{E}\left[f\left(\frac{\Delta_i^n \Upsilon_i^{\text{dis}}}{\Delta^n_i}\right) \mid \mathcal{F}_{i-1}\right] \mid \mathcal{F}_{i-\theta_n}\right] = \mathbb{E}\left[\mathbb{E}\left[\mu_i f(\overline{\Upsilon}_i^n) \right] \mid \mathcal{F}_{i-\theta_n}\right] + (\rho \rho^T)(i-\theta^{(q-1)}_n) \Delta_n \int_{(i-\theta^{(q)}_n)}^{(i+L-1)\Delta_n} \Delta_i^n g(s) T \Delta_i^n g(s) \frac{\Delta_i^n}{\Delta^n_i} ds \mid \mathcal{F}_{i-\theta^{(q-1)}_n} \mid \mathcal{F}_{i-\theta_n}\right]
$$

Thanks to (D.15), we can replace $\Upsilon_i^{n,Q-1} = \Upsilon_i^{n,Q-1,Q-2}$ in the last line by $\Upsilon_i^{n,Q-1,Q-2}$. We can then further compute

$$
\mathbb{E}\left[\mathbb{E}\left[\mu_i f(\Upsilon_i^{n,Q-1,Q-2}) \mid \mathcal{F}_{i-\theta^{(Q-2)}_n}\right] \mid \mathcal{F}_{i-\theta_n}\right] = \mathbb{E}\left[\mathbb{E}\left[\mu_i f(\Upsilon_i^{n,Q-2,Q-2}) \mid \mathcal{F}_{i-\theta^{(Q-3)}_n}\right] \mid \mathcal{F}_{i-\theta_n}\right].
$$

Again by (D.15), we may replace $\Upsilon_i^{n,Q-2,Q-2}$ by $\Upsilon_i^{n,Q-2,Q-3}$ in (D.16). Repeating this procedure $Q$ times, we obtain $\mu_i f(\Upsilon_i^{n,0}) \mathbb{E}[\Upsilon_i^{n,0} \mid \mathcal{F}_{i-\theta_n}] = \mu_i f(\mathbb{E}[\Upsilon_i^{n,0} \mid \mathcal{F}_{i-\theta_n}])$ in the end, which shows (D.9).

It remains to prove (D.15). For $(u, v) \mapsto \mu_i f(u)(v)$, we use $\partial^u$ to denote differentiation with respect to $u$ (where $\chi' \in \mathbb{R}^{d\times L}$) and $\partial^v$ to denote differentiation with respect to $v$ (where $\chi'' \in \mathbb{R}^{d\times L} \times \mathbb{R}^{d\times L}$). By a Taylor expansion of $\mu_i f(\Upsilon_i^{n,r} + \cdot)$ around the point $(\Upsilon_i^{n,r}, \Upsilon_i^{n,r,r-1})$, the difference inside $\mathbb{E}[\cdot \mid \mathcal{F}_{i-\theta_n}]$ in (D.15) equals

$$
\Delta_{n-1}^{\frac{1}{2}} \sum_{i=\theta_n+1}^{(i-1)\Delta_n} \sum_{\chi''=1}^{|\chi''|=1} \mathbb{E}\left[\partial^u \partial^v \mu_i f(\Upsilon_i^{n,r} + \cdot)(\Upsilon_i^{n,r,r} - \Upsilon_i^{n,r,r-1}) \chi'' \mid \mathcal{F}_{i-\theta_n}\right]
$$

for some $\tau_i^n$ between $\Upsilon_i^{n,r}$ and $\Upsilon_i^{n,r,r-1}$. Write

$$
\mathbb{E}\left[(\rho \rho^T)(i-\theta^{(q-1)}_n) \Delta_n \mid \mathcal{F}_{i-\theta^{(r-1)}_n}\right] = \mathbb{E}\left[(\rho \rho^T)(i-\theta^{(q-1)}_n) \Delta_n \mid \mathcal{F}_{i-\theta^{(r)}_n}\right] + (\rho \rho^T)(i-\theta^{(r-1)}_n) \Delta_n - (\rho \rho^T)(i-\theta^{(r-1)}_n) \Delta_n \mid \mathcal{F}_{i-\theta^{(r-1)}_n}
$$

and note that, because of Assumption (CLT') and the identity

$$
xy - x_0y_0 = y_0(x - x_0) + x_0(y - y_0) + (x - x_0)(y - y_0),
$$

(D.19)
the two conditional expectations on the right-hand side of (D.18) are both of size \((\theta_n^{(r-1)} \Delta_n)^{1/2}\). The same holds true if we replace \(\rho_{(i-\theta_n^{(q-1)})\Delta_n}\) by \(\sigma_{(i-1)\Delta_n}\). Therefore,

\[
\mathbb{E}[\|Y_{i}^{n,r,r} - Y_i^{n,r,r-1}\|^p]^{\frac{1}{p}} \lesssim (\theta_n^{(r-1)} \Delta_n)^{\frac{1}{2}}. \tag{D.20}
\]

Thus, the second expression in (D.17) is of size \(\Delta_n^{-1/2}((\theta_n^{(r-1)} \Delta_n)^{1/2})^2 = \Delta_n^{1/2-\theta(r-1)}\) which goes to 0 as \(n \to \infty\) since all numbers \(\theta^{(r)}\) are chosen to be smaller than \(\frac{1}{2}\); see (D.6).

Next, we expand \(\partial^{\alpha} \mu_f(Y_{i}^{n,r,r})\) around \((0, Y_i^{n,r,r-1})\) and write the first expression in (D.17) as \(S^n_1(t) + S^n_2(t) + S^n_3(t)\), where

\[
S^n_1(t) = \Delta_n^{\frac{1}{2}(\frac{1}{T_n}-\frac{1}{L}+1)} \sum_{i=\theta_n+1}^{T_n} \sum_{|\chi'|=1}^{\left|Y_{i}^{n,r,r-1}\right|} \mathbb{E}[\partial^{\chi'} \mu_f(Y_{i}^{n,r,r-1})(Y_{i}^{n,r,r} - Y_i^{n,r,r-1})\chi' | \mathcal{F}^{n}_{i-\theta_n}],
\]

\[
S^n_2(t) = \Delta_n^{\frac{1}{2}(\frac{1}{T_n}-\frac{1}{L}+1)} \sum_{i=\theta_n+1}^{T_n} \sum_{|\chi'|=1}^{\left|Y_{i}^{n,r,r-1}\right|} \mathbb{E}[\partial^{\chi'} \partial^{\chi''} \mu_f(Y_{i}^{n,r,r-1})(Y_{i}^{n,r} - Y_i^{n,r,r-1})\chi'' | \mathcal{F}^{n}_{i-\theta_n}],
\]

\[
S^n_3(t) = \Delta_n^{\frac{1}{2}(\frac{1}{T_n}-\frac{1}{L}+1)} \sum_{i=\theta_n+1}^{T_n} \sum_{|\chi'|=2}^{\left|Y_{i}^{n,r,r-1}\right|} \frac{1}{\lambda^{t!}} \mathbb{E}[\partial^{\chi'} \partial^{\chi''} \mu_f(\zeta_{n}^{(r)}) (Y_{i}^{n,r,r-1})
\]

\[
\times (Y_{i}^{n,r})\chi'(Y_{i}^{n,r} - Y_i^{n,r,r-1})\chi'' | \mathcal{F}^{n}_{i-\theta_n}],
\]

and \(\zeta_{n}^{(r)}\) is a point between 0 and \(Y_{i}^{n,r,r-1}\). Observe that \(\partial^{\chi''} \mu_f(Y_{i}^{n,r,r-1})\) is \(\mathcal{F}_{i-\theta_n^{(r-1)}}\)-measurable and that \(\mathcal{F}_{i-\theta_n^{(r-1)}}\)-conditional expectation of \(Y_{i}^{n,r,r} - Y_i^{n,r,r-1}\) is 0. Hence,

\[
\mathbb{E}[\partial^{\chi''} \mu_f(Y_{i}^{n,r,r-1})(Y_{i}^{n,r} - Y_i^{n,r,r-1})\chi'' | \mathcal{F}^{n}_{i-\theta_n}] = 0
\]

and it follows that \(S^n_1(t)\) vanishes. Next, by Chong (2020), Equation (D.46), given \(|\chi'| = |\chi''| = 1\), there are \(\alpha, \beta, \gamma \in \{1, \ldots, d\} \times \{1, \ldots, L\}\) such that

\[
\partial^{\chi'} \partial^{\chi''} \mu_f(u) = \frac{1}{2^{|\chi'=\beta|}} \partial u_{\alpha, \beta} v_{\alpha, \beta} f(u) = \frac{1}{2^{|\chi'=\beta|}} \mu_{\alpha, \beta, f}(u) v_{\alpha, \beta} f(u).
\]

If \(u = 0\), since \(f\) has odd third derivatives, we have that \(\mu_{\alpha, \beta, f}(0) = 0\). Therefore, the \(\partial^{\chi'} \partial^{\chi''} \mu_f\)-expression in \(S^n_2(t)\) is equal to 0, so \(S^n_2(t)\) vanishes as well. Finally, we use the generalized Hölder inequality and the estimates (D.20) and (A.4) to see that

\[
\mathbb{E}[\sup_{t \leq T} |S^n_3(t)|] \lesssim \Delta_n^{\frac{1}{2}} \sum_{i=\theta_n+1}^{T_n} \mathbb{E}[\|Y_{i}^{n,r}\|^4]^\frac{1}{2} \mathbb{E}[\|Y_{i}^{n,r} - Y_i^{n,r,r-1}\|^4]^\frac{1}{4}
\]

\[
\lesssim \Delta_n^{-\frac{1}{2}} \Delta_n^{2\theta^{(r)}(1-H)} (\theta_n^{(r-1)} \Delta_n)^{\frac{1}{2}}.
\]

This converges to 0 as \(n \to \infty\) if \(2\theta^{(r)}(1-H) - \frac{1}{2}\theta^{(r-1)} > 0\) for all \(r = 1, \ldots, Q-1\), which is equivalent to \(\theta^{(r)} > \frac{1}{4(1-H)}\theta^{(r-1)}\). Because \(\frac{1}{4(1-H)} < 1\), this condition means that \(\theta^{(r)}\) must not decrease to 0 too fast. By adding more intermediate \(\theta\)'s between \(\theta^{(0)}\) and \(\theta^{(Q-1)}\) if necessary, which does no harm to (D.6), we can make sure this is satisfied.

**Lemma D.3.** The convergences (D.10) and (D.11) hold true.
Proof. By Taylor’s theorem, $\mu_f(\bar{Y}_i^{n,0}) - \mu_f(\bar{Y}_i^{n,0,0})$ is equal to

$$\sum_{|\chi|=1} \partial^\chi \mu_f(\bar{Y}_i^{n,0,0})(\bar{Y}_i^n - \bar{Y}_i^{n,0})\chi + \sum_{|\chi|=2} \frac{1}{\chi!} \partial^\chi \mu_f(\hat{\bar{Y}}_i^n)(\bar{Y}_i^n - \bar{Y}_i^{n,0})\chi \quad (D.21)$$

for some $\hat{\bar{Y}}_i^n$ on the line between $\bar{Y}_i^n$ and $\bar{Y}_i^{n,0}$. The expression $\bar{Y}_i^n - \bar{Y}_i^{n,0}$ contains the difference $(\rho \rho^T)(i-\theta_n q^{-1})_{\Delta_n} - E[ (\rho \rho^T)(i-\theta_n q^{-1})_{\Delta_n} | F_i^{n-1}]$ and a similar one with $\rho(i-\theta_n q^{-1})_{\Delta_n}$ replaced by $\sigma(i-\theta_n q^{-1})_{\Delta_n}$. Inserting $\rho \rho^T$ or $\sigma \sigma^T$ at $i - \theta_n q^{-1}$ artificially (cf. (D.18)), we can use (D.19) and Assumption (CLT) to find that the said difference is of size at most $(\theta_n q^{-1})_{\Delta_n}^{1/2}$. This immediately leads to the bound $\sqrt{\Delta_n} \sum_{i=\theta_n q^{-1}+1}^{\theta_n q^{-1}+1} \sum_{|\chi|=1} \partial^\chi \mu_f(\bar{Y}_i^n)(\bar{Y}_i^n - \bar{Y}_i^{n,0})\chi$. For each $i$, the $|\chi|=1$ expression is $F_i^n$-measurable and has a vanishing conditional expectation given $F_i^{n-1}$. Thus, by a martingale size estimate of the type (A.6), the whole term is of size $\sqrt{\bar{b}_n}(\theta_n q^{-1})_{\Delta_n}^{1/2}$ at most, which tends to $0$ by (C.3). This proves (D.10).

For (D.11), recall $\bar{Y}_i^{n,i}$ from (C.7) and note that the difference $(\bar{Y}_i^{n,i} - \bar{Y}_i^{n,0})_{kk',k'}$ equals

$$\sum_{q=1}^{Q} \left( (\rho \rho^T)(i-\theta_n q^{-1})_{\Delta_n} - (\rho \rho^T)(i-\theta_n q^{-1})_{\Delta_n} \right)_{kk'} \int_{(i-\theta_n q^{-1})_{\Delta_n}}^{(i+1-\theta_n q^{-1})_{\Delta_n}} \frac{\Delta_n^{2H} g(s)}{\Delta_n^{1/2}} ds$$

for all $k, k' = 1, \ldots, d$ and $\ell, \ell' = 1, \ldots, L$. Thus, if we expand

$$\Delta_n^{1/2} \sum_{i=\theta_n q^{-1}+1}^{\theta_n q^{-1}+1} \sum_{|\chi|=1} \partial^\chi \mu_f(\hat{\bar{Y}}_i^n)(\bar{Y}_i^{n,i} - \bar{Y}_i^{n,0})\chi, \quad (D.22)$$

where $\hat{\bar{Y}}_i^n$ is some point between $\bar{Y}_i^{n,i}$ and $\bar{Y}_i^{n,0}$, Hölder’s inequality together with the identity (D.19) as well as the moment and regularity assumptions on $\rho$ shows that the last sum in the above display is of size $\Delta_n^{-1/2} \sum_{q=1}^{Q} (\theta_n q^{-1})_{\Delta_n}^{4\theta_n q^{-1}(1-H)}$, which goes to $0$ as $n \to \infty$; cf. (D.13). Next, recall the decomposition (3.3). As before, we incorporate the drift $t \mapsto \int_0^t \hat{b}_s \, ds$ into $\rho(0)$ so that $\rho = \rho(0) + \rho(1)$ with $\rho(1) = \int_0^t \rho_s \, d\hat{W}_s$. By (D.19),

$$\rho(0)_{i-\theta_n q^{-1}}^{k,k'} - \rho(0)_{i-\theta_n q^{-1}}^{k,k'} = \left( \rho(0)_{i-\theta_n q^{-1}}^{k,k'} - \rho(0)_{i-\theta_n q^{-1}}^{k,k'} \right)_{\Delta_n} + \left( \rho(0)_{i-\theta_n q^{-1}}^{k,k'} - \rho(0)_{i-\theta_n q^{-1}}^{k,k'} \right)_{\Delta_n} + \left( \rho(0)_{i-\theta_n q^{-1}}^{k,k'} - \rho(0)_{i-\theta_n q^{-1}}^{k,k'} \right)_{\Delta_n}$$

The remaining term $\Delta_n^{1/2} \sum_{i=\theta_n q^{-1}+1}^{\theta_n q^{-1}+1} \sum_{|\chi|=1} \partial^\chi \mu_f(\bar{Y}_i^{n,i})(\bar{Y}_i^{n,i} - \bar{Y}_i^{n,0})\chi$ in (D.22) can thus be written as $T_1^n(t) + T_2^n(t) + T_3^n(t)$ according to this decomposition. By Hölder’s inequality and the moment and regularity assumptions on $\rho$, $T_3^n(t)$ is of size at most

$$\Delta_n^{1/2} \sum_{q=1}^{Q} (\theta_n q^{-1})_{\Delta_n}^{2\theta_n q^{-1}(1-H)}, \quad (D.23)$$
which goes to 0 as \( n \to \infty \) as we saw in (D.13). Similarly, thanks to the regularity property (C.1) of \( \rho^{(0)} \), we further obtain \( \mathbb{E}[\sup_{t \leq T_1} |\mathbb{T}_1^n(t)|] \lesssim \Delta_n^{-1/2} \sum_{q=1}^Q \rho^{(1)}(\theta^{(q-1)}(1-H))^{\Delta_n^{2\theta^{(q)}(1-H)})} \), and this also goes to 0 as \( n \to \infty \) by our choice (D.6) of the numbers \( \theta^{(q-1)} \). Finally,

\[
\mathbb{T}_2^n(t) = \Delta_n^{1/2} \sum_{i=0}^{N(H)} \sum_{q=1}^Q \sum_{\|\chi\|=1} \partial^{\|\chi\|} \mu_f(\mathbb{T}_2^n(t)) \times \left\{ \mathbb{E}^{-\Delta_n^{1/2}} g(s) \Delta_n^{2n}\Delta_n^{2H} 1_{(i-\theta^{(q-1)}(1-H)\Delta_n, \Delta_n, \Delta_n))} (s) \right\}^\chi,
\]

where \( \pi_{q,1}^{n,i} = \rho^{(1)}(\theta^{(q-1)}(1-H)\Delta_n - \rho^{(1)}(\theta^{(q-1)}(1-H)\Delta_n)) \). Define \( \mathbb{T}_2^n(t) \) in the same way as \( \mathbb{T}_1^n(t) \) except that in the previous display, \( \mathbb{Y}_{n,i} \) is replaced by \( \mathbb{Y}_{n,1} \), obtained from \( \mathbb{Y}_{n,1} \) by substituting \( (i-\theta^{(q-1)}(1-H)\Delta_n \) for \( (i-1)\Delta_n \) everywhere. By Hölder’s inequality and the regularity assumptions on \( \rho \) and \( \sigma \), \( \mathbb{T}_2^n(t) - \mathbb{T}_2^n(t) \) is of the same size as exhibited in (D.23) and hence asymptotically negligible. Next,

\[
\mathbb{T}_2^n(t) = \Delta_n^{1/2} \sum_{q=1}^Q \sum_{i=0}^{N(H)} \sum_{\|\chi\|=1} \partial^{\|\chi\|} \mu_f(\mathbb{Y}_{q,1}^{n,i}) \times \left\{ \mathbb{E}^{-\Delta_n^{1/2}} g(s) \Delta_n^{2n}\Delta_n^{2H} 1_{(i-\theta^{(q-1)}(1-H)\Delta_n, \Delta_n, \Delta_n))} (s) \right\}^\chi, \tag{D.24}
\]

For fixed \( q \), the part that involves \( \pi_{q,1}^{n,i} - \mathbb{E}[\pi_{q,1}^{n,i} \mid \mathbb{F}_i^{n-\theta^{(q-1)}(1-H)}] \) is a sum where the \( \theta \)th summand is \( \mathbb{F}_i^{n-\theta^{(q-1)}(1-H)} \)-measurable and has, by construction, a zero \( \mathbb{F}_i^{n-\theta^{(q-1)}(1-H)} \)-conditional mean. By a martingale size estimate of the type (A.6), that part is therefore of size

\[
\sum_{q=1}^Q \sqrt{\theta^{(q-1)}(\theta^{(q-1)}(1-H))^{1/2} \Delta_n^{2\theta^{(q)}(1-H)} = \sum_{q=1}^Q \Delta_n^{1/2-\theta^{(q)}(1-H)} \to 0}
\]
as \( n \to \infty \) since all \( \theta^{(q)} < \frac{1}{2} \). Clearly,

\[
\mathbb{E} \left[ \rho^{(1,k\ell)}(\theta^{(q-1)}(1-H)\Delta_n \mid \mathbb{F}_i^{n-\theta^{(q-1)}(1-H)} \right] = \sum_{m=1}^d \mathbb{E} \left[ \rho^{(1,k\ell)}(\theta^{(q-1)}(1-H)\Delta_n \mid \mathbb{F}_i^{n-\theta^{(q-1)}(1-H)} \right] = 0.
\]

Because \( \rho^{(1,k\ell)}(\theta^{(q-1)}(1-H)\Delta_n \) is \( \mathbb{F}_i^{n-\theta^{(q-1)}(1-H)} \)-measurable, we have, in fact, \( \mathbb{E}[\pi_{q,1}^{n,i} \mid \mathbb{F}_i^{n-\theta^{(q-1)}(1-H)}] = 0 \). Therefore, \( \mathbb{T}_2^n(t) \) is asymptotically negligible and the proof of (D.11) is complete. \( \square \)

**Proof of Lemma C.4.** Recall the expressions \( \mathbb{X}_1^n(t) \) and \( \mathbb{X}_2^n(t) \) defined in (C.8). For a given multi-index \( \chi \in \mathbb{N}_0^{(d\times L)\times(d\times L)} \), let \( Q_\chi(x) = x^\chi \) for \( x \in \mathbb{R}^{(d\times L)\times(d\times L)} \), which is a polynomial of degree \( \|\chi\| \). By Taylor’s theorem,

\[
\mathbb{X}_1^n(t) = \Delta_n^{1/2} \sum_{i=0}^{N(H)} \sum_{\|\chi\|=1} \sum_{j=1}^{\|\chi\|} \partial^{\|\chi\|} \mu_f(\pi((i-1)\Delta_n)) \sum_{k-1}^{\|\chi\|} \sum_{\|\chi\|=k} \frac{\Delta_n^{(j-k)(1-2H)}}{\chi!} \partial^{\chi}(c((i-1)\Delta_n)) \mathbb{Y}_1^n - \pi((i-1)\Delta_n) - \Delta_n^{1-2H} c((i-1)\Delta_n) \chi \right\}^\chi. \tag{D.25}
\]
The key term in (D.25) is the expression in braces and we have (recall (3.6) and (2.4))

\[
\Gamma^{n,i} = \pi((i-1)\Delta_n) - \Delta_n^{-1+2H}c((i-1)\Delta_n)
\]

\[
= (\rho \rho^T)(i-1)\Delta_n \left\{ \int_{(i-\theta_n)\Delta_n}^{(i+1-L)\Delta_n} \frac{\Delta_n^2 g(s)T \Delta_n g(s)}{\Delta_n^{2H}} ds - (\Gamma^H|_{t=t'})_{\ell,\ell'} \right\}
\]

\[
= -(\rho \rho^T)(i-1)\Delta_n \int_{-\infty}^{(i-\theta_n)\Delta_n} \frac{\Delta_n^2 g(s)T \Delta_n g(s)}{\Delta_n^{2H}} ds,
\]

because \(\Gamma^H|_{t-t'} = \Delta_n^{-2H} \int_{-\infty}^{\infty} \Delta_n^2 g(s)T \Delta_n g(s) ds\) by (B.2). The size of the last integral is \(\Delta_n^{-2H} (\sup_{t\leq T}|X_n^a(t)|) \lesssim \Delta_n^{-1/2} \Delta_n^{-1/2+2\theta(1-H)} \lesssim \Delta_n^{-1/2+2\theta(1-H)} \to 0\) by (C.3). Using (D.26) and Assumption (CLT'), we further see that the magnitude of \(\Gamma^{n,i} - \pi((i-1)\Delta_n)\) is \(\lesssim \Delta_n^{-1+2H} + \Delta_n^2\theta(1-H)\). Thus, again by Hölder's inequality, we deduce that \(\mathbb{E}[\sup_{t\leq T}|X_n^a(t)|] \lesssim \Delta_n^{-1/2} (\Delta_n^{-1/2-1/2+2\theta(1-H)} + \Delta_n^{1/2+2\theta(1-H)} \to 0\) by the definition of \(N(H)\).

\[\square\]

**Proof of Lemma C.5.** The first convergence (C.9) can be shown analogously to Equation (5.3.24) in Jacod & Protter (2012) and is omitted. For (C.10), we write the left-hand side as \(\sum_{j=1}^{N(H)} Z^n_j(t) - \overline{Z}^n(t)\) where

\[
Z^n_j(t) = \Delta_n^{-\frac{1}{2}+j(1-2H)} \sum_{i=\theta_n+1}^{[t/\Delta_n]-L+1} \left\{ \frac{1}{\chi!} \int_{(i-1)\Delta_n}^{i\Delta_n} (\partial^\chi \mu_f(\pi((i-1)\Delta_n))) c((i-1)\Delta_n) \chi ds \right\},
\]

\[
\overline{Z}^n(t) = \Delta_n^{-\frac{1}{2}} \left( \int_{0}^{\theta_n\Delta_n} + \int_{([t/\Delta_n]-L+1)\Delta_n}^{t} \right) \sum_{j=1}^{N(H)} \frac{1}{\chi!} \int_{(i-1)\Delta_n}^{i\Delta_n} (\partial^\chi \mu_f(\pi(s))) \Delta_n^{j(1-2H)} c(s) \chi ds.
\]

Using the moment assumptions on \(\sigma\) and \(\rho\), since \(t - ([t/\Delta_n]-L+1)\Delta_n \leq L\Delta_n\), we readily see that \(\mathbb{E}[\sup_{t\leq T}|\overline{Z}^n(t)|] \lesssim \Delta_n^{-1/2} (\theta_n\Delta_n + L\Delta_n) \lesssim \Delta_n^{1/2-\theta} + \Delta_n^{1/2} \to 0\).

Let \(j = 1, \ldots, N(H)\) (in particular, everything in the following can be skipped if \(H < \frac{1}{2}\)) and consider, for \(\chi \in \mathbb{N}_0^{(d\times L)\times(d\times L)}\), again the polynomial \(Q_\chi\) introduced in proof of Lemma C.4. Using the mean-value theorem, we can write

\[
Z^n_j(t) = \Delta_n^{-\frac{1}{2}+j(1-2H)} \sum_{i=\theta_n+1}^{[t/\Delta_n]-L+1} \left\{ \frac{1}{\chi!} \int_{(i-1)\Delta_n}^{i\Delta_n} (\partial^\chi \mu_f(\zeta_{n,i}^1) \partial^\chi_2 Q_\chi(\zeta_{n,i}^2) \right\} c((i-1)\Delta_n) \chi ds
\]

for some \(\zeta_{n,i}^1\) and \(\zeta_{n,i}^2\). By Hölder's inequality and Assumption (CLT), we deduce that \(\mathbb{E}[\sup_{t\leq T}|\overline{Z}^n_j(t)|] \lesssim \Delta_n^{1/2+j(1-2H)} \Delta_n^{-1/2} \Delta_n^{1/2} = \Delta_n^{1/2+2H} \to 0\) for any \(H < \frac{1}{2}\).

\[\square\]

**E  Proof of Theorem 4.2**

Since \(\varphi\) is invertible, we can write

\[
H = \varphi^{-1}(\langle a, \Gamma^H \rangle \Pi_t) = G(\langle a, \Gamma^H \rangle \Pi_t, \langle b, \Gamma^H \rangle \Pi_t),
\]

\[
\tilde{H}^n = G(\langle a, \tilde{V}^n_t \rangle, \langle b, \tilde{V}^n_t \rangle) = G(\langle a, V^n_t \rangle, \langle b, V^n_t \rangle), \quad G(x,y) = \varphi^{-1}(x/y).
\]
As \( G \) is infinitely differentiable on \( \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \), we can expand \( \widetilde{H}_n \) in a Taylor sum around \((\langle a, \Gamma^H \rangle \Pi_t, \langle b, \Gamma^H \rangle \Pi_t)\) and obtain

\[
\begin{align*}
\widetilde{H}_n - H &= \sum_{|\alpha|=1} \partial^\alpha G(\langle a, \Gamma^H \rangle \Pi_t, \langle b, \Gamma^H \rangle \Pi_t)(\langle a, V_t^n - \Gamma^H \Pi_t \rangle, \langle b, V_t^n - \Gamma^H \Pi_t \rangle)^\alpha + \mathbb{H}_n, \\
\mathbb{H}_n &= \sum_{|\alpha|=2} \frac{\partial^\alpha G(\alpha^n)}{\chi!}(\langle a, V_t^n \Gamma^H \Pi_t \rangle, \langle b, V_t^n \Gamma^H \Pi_t \rangle)^\alpha,
\end{align*}
\]

(E.2)

where \( \chi \in \mathbb{N}_0^2 \) and \( \alpha^n \) is a point between \((\langle a, V_t^n \rangle, \langle b, V_t^n \rangle)\) and \((\langle a, \Gamma^H \rangle \Pi_t, \langle b, \Gamma^H \rangle \Pi_t)\). By straightforward computations,

\[
\partial^{(1,0)} G(x, y) = (\varphi^{-1})'(x/y)y^{-1} \quad \text{and} \quad \partial^{(0,1)} G(x, y) = -(\varphi^{-1})'(x/y)xy^{-2}.
\]

(E.3)

Therefore, (E.2) becomes

\[
\Delta_n^{\frac{1}{2}} (\widetilde{H}_n - H) = \frac{(\varphi^{-1})'(\varphi(H))}{\langle b, \Gamma^H \rangle \Pi_t} \left\{ \langle a, V_t^n - \Gamma^H \Pi_t \rangle - \varphi(H) \langle b, V_t^n - \Gamma^H \Pi_t \rangle \right\} + \mathbb{H}_n.
\]

(E.4)

Because \( H \in (0, \frac{1}{4}) \) or \( a_0 = b_0 = 0 \), the term inside the braces in the last line can be written as \( \{a^T - \varphi(H)b^T\} \{V_t^n - \Gamma^H \int_0^t \rho_s^2 \, ds - e_1 \int_0^t \sigma_s^2 \, ds \Delta_n^{1-2H} \mathbb{1}_{\left[ \frac{1}{2}, \frac{1}{2} \right]}(H) \} \). Moreover, by Corollary 4.1, the term \( \mathbb{H}_n \) is of magnitude \( \Delta_n \) and hence,

\[
\Delta_n^{\frac{1}{2}} (\widetilde{H}_n - H) \xrightarrow{\text{st}} \frac{(\varphi^{-1})'(\varphi(H))}{\langle b, \Gamma^H \rangle \Pi_t} \{a^T - \varphi(H)b^T\} \mathcal{N} \left( 0, \operatorname{Var}_{H,0} \left( \int_0^t \rho_s^2 \, ds \right) \right),
\]

which proves (4.5).

We now turn to the convergence stated in (4.7) when \( H > \frac{1}{4} \). We decompose

\[
\begin{align*}
V_{0,t} - \frac{\langle c, V_t^n \rangle}{\langle c, \Gamma^H \rangle} &= \left\{ V_{0,t} - \Pi_t \right\} - \frac{\langle c, V_t^n - \Gamma^H \Pi_t \rangle}{\langle c, \Gamma^H \rangle} + \Pi_t \frac{\langle c, \Gamma^H - \Gamma^H \Pi_t \rangle}{\langle c, \Gamma^H \rangle} \\
&= \left\{ V_{0,t} - \Pi_t \right\} - \frac{\langle c, V_t^n - \Gamma^H \Pi_t \rangle}{\langle c, \Gamma^H \rangle} + \Pi_t \frac{\langle c, \partial_H \Gamma^H \Pi_t \rangle}{\langle c, \Gamma^H \rangle} \left( \widetilde{H}_n - H \right) + \mathbb{V}_n,
\end{align*}
\]

(E.5)

\[
\mathbb{V}_n = \frac{1}{2} \Pi_t \frac{\langle c, \partial_H \Gamma^H \Pi_t \rangle}{\langle c, \Gamma^H \rangle} \left( \widetilde{H}_n - H \right)^2,
\]

where \( \partial_H \Gamma^H \) is the second derivative of \( H \mapsto (\Gamma^H_0, \ldots, \Gamma^H_R) \) evaluated at \( H \) and \( \beta^n \) is somewhere between \( H_n \) and \( H \). Since \( c_0 \neq 0 \), the first two terms in the second line of (E.5) are of magnitude \( \Delta_n^{1-2H} \), while the third is of magnitude \( \Delta_n^{1/2} \) by our first result (4.5). Finally, \( \mathbb{V}_n \) is of magnitude \( \Delta_n \), so using Corollary 4.1, we deduce that

\[
\Delta_n^{2H-1} \left\{ V_{0,t} - \frac{\langle c, V_t^n \rangle}{\langle c, \Gamma^H \rangle} \right\} \xrightarrow{\text{p}} C_t - \frac{1}{\langle c, \Gamma^H \rangle} \frac{\langle c, e_1 \rangle}{\langle c, \Gamma^H \rangle} C_t = \left( 1 - \frac{c_0}{\langle c, \Gamma^H \rangle} \right) C_t.
\]

(E.6)
Reusing (E.4) and recalling that $a_0 = b_0 = 0$, we further have that

$$\Delta_n^{-\frac{1}{2}} \left\{ V_{0,t}^n - \frac{\langle c, V_t^n \rangle}{\langle c, \Gamma H^n \rangle} \right\} = \Delta_n^{-\frac{1}{2}} \left\{ V_{0,t}^n - \Pi_t - (1 - \frac{c_0}{\langle c, \Gamma H^n \rangle}) C_t \Delta_n^{2H} \right\}$$

$$= \Delta_n^{-\frac{1}{2}} \left\{ V_{0,t}^n - \Pi_t - C_t \Delta_n^{1-2H} \right\} - \Delta_n^{-\frac{1}{2}} c^T \left\{ V_t^n - \Gamma H \Pi_t - e_1 C_t \Delta_n^{1-2H} \right\}$$

$$+ \Pi_t \frac{\langle c, \partial_H \Gamma H^n \rangle}{\langle c, \Gamma H^n \rangle} \Delta_n^{-\frac{1}{2}} \left\{ \tilde{H}^n - H \right\} + \Delta_n^{-\frac{1}{2}} \varphi^n$$

$$= \left( e_t^T - \frac{c^T}{\langle c, \Gamma H^n \rangle} + \Pi_t \frac{\langle c, \partial_H \Gamma H^n \rangle (\varphi^{-1})(\varphi(H))}{\langle c, \Gamma H^n \rangle \Pi_t} \left\{ a^T - \varphi(H) b^T \right\} \right)$$

$$\times \Delta_n^{-\frac{1}{2}} \left\{ V_t^n - \Gamma H \Pi_t - e_1 C_t \Delta_n^{1-2H} \right\} + \Delta_n^{-\frac{1}{2}} \left( \Pi_t \frac{\langle c, \partial_H \Gamma H^n \rangle}{\langle c, \Gamma H^n \rangle} \right) \left\{ a^T - \varphi(H) b^T \right\} \mathcal{Z}_t.$$

It remains to normalize the left-hand side of (E.6) in order to obtain (4.7):

$$\Delta_n^{-\frac{1}{2} + (2H)} \left\{ \frac{\langle c, \hat{V}_0^n \rangle}{\langle c, \Gamma H^n \rangle} \left( 1 - \frac{c_0}{\langle c, \Gamma H^n \rangle} \right)^{-1} - C_t \right\}$$

$$\overset{st}{\to} \left( 1 - \frac{c_0}{\langle c, \Gamma H^n \rangle} \right)^{-1} \left( e_t^T - \frac{c^T}{\langle c, \Gamma H^n \rangle} + \frac{\langle c, \partial_H \Gamma H^n \rangle (\varphi^{-1})(\varphi(H))}{\langle c, \Gamma H^n \rangle} \left\{ a^T - \varphi(H) b^T \right\} \right) \mathcal{Z}_t$$

$$\sim \mathcal{N} \left( 0, \text{Var}_C \int_0^t \rho^n \, ds \right).$$

Finally, we tackle (4.11). We use the mean-value theorem to decompose

$$\Delta_n^{-\frac{1}{2}} \left( \frac{\langle a, V_t^n \rangle}{\langle a, \Gamma H^n \rangle} - \Pi_t \right) = \Delta_n^{-\frac{1}{2}} \frac{\langle a, V_t^n - \Gamma H \Pi_t \rangle}{\langle a, \Gamma H^n \rangle} - \Pi_t \frac{\langle a, \partial_H \Gamma H^n \rangle}{\langle a, \Gamma H^n \rangle} \Delta_n^{-\frac{1}{2}} \left\{ a^T - \varphi(H) b^T \right\}$$

$$= \left( \frac{a^T}{\langle a, \Gamma H^n \rangle} \right) \Delta_n^{-\frac{1}{2}} \left\{ V_t^n - \Gamma H \Pi_t \right\} - \Pi_t \langle a, \partial_H \Gamma \hat{\beta}^n \rangle \Delta_n^{-\frac{1}{2}} \left\{ \hat{H}^n - H \right\},$$

where $\hat{\beta}^n$ is between $\tilde{H}^n$ and $H$ and therefore satisfies $\hat{\beta}^n \xrightarrow{p} H$. As before, because $\frac{1}{4} < H < \frac{1}{2}$ or $a_0 = b_0 = 0$, we have $V_t^n - \Gamma H \Pi_t = V_t^n - \Gamma H \Pi_t - C_t \Delta_n^{1-2H} 1_{\{1/4 < \}}(H)$. Using Corollary 4.1 and our first result (4.5), we infer that $\Delta_n^{-1/2} \langle a, V_t^n \rangle / \langle a, \Gamma H^n \rangle - \Pi_t$ converges stably in distribution. Applying again the mean-value theorem, this time on the function $H \mapsto \Delta_n^{-2H}$, and recalling the identity $\Delta_n^{-2H} \hat{V}_{r,t}^n = V_{r,t}^n$, we further obtain

$$\Delta_n^{-\frac{1}{2}} \left( \hat{\Pi}^n_t - \Pi_t \right) = \Delta_n^{-\frac{1}{2}} \left( \frac{\langle a, V_t^n \rangle}{\langle a, \Gamma H^n \rangle} - \Pi_t \right) + \Delta_n^{-\frac{1}{2}} \langle a, \hat{V}_t^n \rangle \left\{ \Delta_n^{1-2H} - \Delta_n^{1-2H} \right\}$$

$$= \Delta_n^{-\frac{1}{2}} \left( \frac{\langle a, V_t^n \rangle}{\langle a, \Gamma H^n \rangle} - \Pi_t \right) - 2 \frac{\langle a, \hat{V}_t^n \rangle}{\langle a, \Gamma H^n \rangle} \Delta_n^{1-2H} (\log \Delta_n) \Delta_n^{2(H-\bar{\beta})} \Delta_n^{-\frac{1}{2}} \left\{ \hat{H}^n - H \right\}$$
for another point \( \beta^n \) between \( \hat{H}^n \) and \( H \). By (4.5), \( \beta^n \) converges to \( H \) at a rate of \( \Delta_n^{1/2} \). Therefore, \( \Delta_n^{2(H-\beta^n)} \to 1 \) as \( n \to \infty \). Normalizing by \( \log \Delta_n \), we conclude from (4.5) that

\[
\frac{\Delta_n^{-1/2}}{\log \Delta_n} (\hat{\Pi}_n^t - \Pi_t) = \frac{\Delta_n^{-1/2}}{\log \Delta_n} \left( \left\langle a, V^n_t \right\rangle - \left\langle a, \Gamma H^n \right\rangle \right) - 2 \left( \langle a, V^n_t \rangle \right) \Delta_n^{2(H-\beta^n)} \Delta_n^{-1/2} \{ \hat{H}^n - H \}
\]

\( \xrightarrow{st} \mathcal{N}(0,4 \text{Var}_{H,0} \int_0^t \rho^4 s \, ds) \).

This completes the proof of Theorem 4.2.

\( \square \)

F Estimators based on quadratic variation

F.1 A consistent but not asymptotically normal estimator of \( H \)

To simplify the exposition, we assume that at least one of \( a_0 \) and \( b_0 \) is zero. By symmetry, we shall consider the case where

\[
a_0 \neq 0, \quad b_0 = 0. \tag{F.1}
\]

Also, again to simplify the argument and because this is not really a severe restriction from a statistical point of view, we shall assume that the true value of \( H \) satisfies

\[
H \in \left( \frac{1}{4}, \frac{1}{2} \right) \setminus \mathcal{H}, \tag{F.2}
\]

where \( \mathcal{H} \) is the set from (3.10).

**Proposition F.1.** Let \( H \in \left( \frac{1}{4}, \frac{1}{2} \right) \setminus \mathcal{H} \) and suppose that \( a, b \in \mathbb{R}^{1+R} \) satisfy (F.1) and are such that \( \varphi \) from (4.4) is invertible. Recalling that \( N(H) = [1/(2 - 4H)] \), we further define for \( j = 1, \ldots, N(H) \),

\[
\Phi^n_j = \Phi^n_j(R, a, b, \hat{V}^n_t, \hat{H}^n) = \frac{(-1)^j}{j!} (\varphi^{-1})^{(j)}(\varphi(\hat{H}^n)) \frac{a_0^j}{\langle b, \hat{V}^n_t \rangle^j}. \tag{F.3}
\]

Then \( \hat{H}^n \), as defined in (4.4), satisfies

\[
\Delta_n^{-1/2} \left\{ \hat{H}^n - H + \sum_{j=1}^{N(H)} \Phi^n_j C_t^j \right\} \xrightarrow{st} \mathcal{N} \left( 0, \text{Var}_{H,0} \int_0^t \rho^4 s \, ds \right), \tag{F.4}
\]

where \( \text{Var}_{H,0} \) is defined in (4.6).

For each \( j \), the term \( \Phi^n_j \) is of order \( \Delta_n^{(1-2H)} \). As a result, while \( \hat{H}^n \) is consistent for \( H \), it is affected by many higher-order asymptotic bias terms that depend on \( C_t \). So our next goal is to find consistent estimators of \( C_t \) that we can use to correct \( H \).
F.2 A consistent but not asymptotically normal estimator of $C_t$

With a first estimator of $H$ at hand, we can now construct an estimator of $C_t$ by removing the first-order limit of $\tilde{V}_{0,t}$, hereby replacing $H$ by $\tilde{H}^n$ throughout. Doing so, we have to employ an estimator of $\Pi_t$, the integrated noise volatility. To avoid even more higher-order bias terms, we need one with convergence rate $\sqrt{\Delta n}$. One possibility is to use the estimator $\tilde{\Pi}_t^n$ from Theorem 4.2, constructed from an additional pair of weights $a^0$ and $b^0$ with $a^0_0 = b^0_0 = 0$. Note that even in the noise-free case, where $\rho = 0$, the estimator $\tilde{\Pi}_t^n$ from Theorem 4.2 converges to the desired limit 0 in probability.

**Proposition F.2.** In addition to $a, b \in \mathbb{R}^{1+R}$ satisfying (F.1), choose $a^0, b^0 \in \mathbb{R}^{1+R}$ with $a^0_0 = b^0_0 = 0$ and let

$$\hat{P}_t^n = \langle a^0, \tilde{V}_{t}^n \rangle \langle a^0, \Gamma H^n \rangle, \quad \hat{H}^n_0 = \varphi^{-1} \left( \frac{\langle a^0, \tilde{V}_{t}^n \rangle}{\langle b^0, \tilde{V}_{t}^n \rangle} \right).$$

(F.5)

Further define

$$\tilde{C}_t^{n,1} = \left\{ \tilde{V}_{0,t} - \frac{\langle a, \tilde{V}_{t}^n \rangle}{\langle a, \Gamma H^n \rangle} \right\} \Theta(\tilde{V}_t^n, \tilde{H}_t^n, \tilde{H}_t^{n,0})^{-1},$$

(F.6)

where

$$\Theta(\tilde{V}_t^n, \tilde{H}_t^n, \tilde{H}_t^{n,0}) = \Theta(R, a, b, a^0, b^0, \tilde{V}_t^n, \tilde{H}_t^n, \tilde{H}_t^{n,0})$$

$$= 1 - \frac{a_0}{\langle a, \Gamma H^n \rangle} + \frac{\hat{P}_t^n a_0 \psi'(\varphi(\tilde{H}_t^n))}{\langle b, \tilde{V}_t^n \rangle \langle a, \Gamma H^n \rangle}$$

(F.7)

and

$$\psi(y) = \langle a, \Gamma \varphi^{-1}(y) \rangle, \quad y \in \mathbb{R}. \quad \text{(F.8)}$$

Then, under the assumptions made in Proposition F.1,

$$\Delta_n^{1/2} \left\{ \tilde{C}_t^{n,1} - C_t + \sum_{j=2}^{N(H)} \Psi_j^n C_t^j \right\} \xrightarrow{s.t.} \mathcal{N} \left( 0, \text{Var}_{C,1} \int_0^t \rho_s^4 \, ds \right).$$

(F.9)

where

$$\Psi_j^n = \Psi_j^n (R, a, b, a^0, b^0, \tilde{V}_t^n, \tilde{H}_t^n, \tilde{H}_t^{n,0})$$

$$= \frac{(-1)^j}{j!} \psi^{(j)}(\varphi(\tilde{H}_t^n)) \frac{a_0^2}{\langle a, \Gamma H^n \rangle \langle b, \tilde{V}_t^n \rangle} \hat{P}_t^n \Theta(\tilde{V}_t^n, \tilde{H}_t^n, \tilde{H}_t^{n,0})^{-1} \quad \text{F.10)}$$

for $j = 2, \ldots, N(H)$ and

$$\text{Var}_{C,1} = \text{Var}_{C,1} (R, a, b, H) = u_1^T \mathcal{C}^H u_1. \quad \text{(F.11)}$$

$$u_1 = \left( e_1 - \frac{a}{\langle a, \Gamma H \rangle} + \frac{\psi'(\varphi(H))}{\langle a, \Gamma H \rangle \langle b, \Gamma H \rangle} (a - \varphi(H)b) \right)$$

$$\times \left( 1 - \frac{a_0}{\langle a, \Gamma H \rangle} + \frac{\psi'(\varphi(H))}{\langle a, \Gamma H \rangle \langle b, \Gamma H \rangle} a_0 \right)^{-1},$$

(F.12)

and $\mathcal{C}^H$ is the matrix in (4.2).

Note that $\Psi_j^n$ is of magnitude $\Delta_n^{(j-1)(1-2H)}$. Thus, just as for the initial estimator of $H$, the estimator $\tilde{C}_t^{n,1}$ is consistent but has higher-order bias terms.

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F.3 The first asymptotically normal estimators of $H$ and $C_t$

What is different between the two initial estimators of $H$ and $C_t$ is that in (F.9) the bias terms only hinge on $C_t$, the quantity that $\tilde{C}_t^{n,1}$ is supposed to estimate in the first place. Therefore, we can set up an iteration procedure to correct $\tilde{C}_t^{n,1}$.

**Proposition F.3.** Recall that $N(H) = [1/(2 - 4H)]$ and define

$$\tilde{C}_t^{n,\ell+1} = \tilde{C}_t^{n,1} + \sum_{j=2}^{\ell+1} \Psi_j^n(\tilde{C}_t^{n,\ell-j+2})^j, \quad \ell \geq 0,$$

(F.13)

and

$$\tilde{C}_t^{n,1} = \tilde{C}_t^{n,N(H^n)}.$$

(F.14)

Then we have that

$$\Delta_n^{-1/2} \left\{ \tilde{C}_t^{n,1} - C_t + \sum_{j=2}^{N(H^n)} \Psi_j^n(\tilde{C}_t^{n,N(H^n)-j+1})^j \right\} = \Delta_n^{-1/2} (\tilde{C}_t^{n,1} - C_t) \xrightarrow{st} \mathcal{N} \left( 0, \text{Var}_{C,1} \int_0^t \rho_s^4 s ds \right)$$

with the same $\text{Var}_{C,1}$ as in (F.11).

The corrected estimator $\tilde{C}_t^{n,1}$ is our first consistent and asymptotically mixed normal estimator for $C_t$ in the setting of (F.1). With a bias-free estimator of $C_t$ at hand, we can now proceed to correcting the initial estimator $\tilde{H}_n$ of $H$.

**Proposition F.4.** Recall $\tilde{H}_n$ in (4.4) and define

$$\tilde{H}_1^n = \tilde{H} + \sum_{j=1}^{N(H^n)} \Phi_j^n(\tilde{C}_t^{n,1})^j,$$

(F.15)

with $\Phi_j^n$ as in (F.3). Then

$$\Delta_n^{-1} (\tilde{H}_1^n - H) \xrightarrow{st} \mathcal{N} \left( 0, \text{Var}_{H,1} \frac{\int_0^t \rho_s^4 ds}{(\int_0^t \rho_s^2 ds)^2} \right),$$

where

$$\text{Var}_{H,1} = \text{Var}_{H,1}(R, a, b, H) = w_1^T C^H w_1, \quad w_1 = \frac{(\varphi^{-1})'(\varphi(H))}{\langle b, \Gamma^H \rangle} \left\{ a - \varphi(H)b - a_0 u_1 \right\},$$

and the vector $u_1$ is exactly as in (F.11) and the matrix $C^H$ as in (4.2).

F.4 A multi-step algorithm

Even though $\tilde{C}_t^{n,1}$ and $\tilde{H}_1^n$ from Propositions F.3 and F.4 are rate-optimal and asymptotically bias-free estimators of $C_t$ and $H$, respectively, we can still do better: The estimator $\tilde{C}_t^{n,1}$ is based on the initial estimator $\tilde{C}_t^{n,1}$ from (F.6), which in turn is based on the initial estimator $\tilde{H}^n$ of $H$. Now that we have a better estimator of $H$, namely $\tilde{H}_1^n$, the idea is to use $\tilde{H}_1^n$ to construct an updated estimator, say, $\tilde{C}_t^{n,2}$, of $C_t$. And with this updated estimator of $C_t$, we next update $\tilde{H}_1^n$ to, say, $\tilde{H}_2^n$, which we can then use to update $\tilde{C}_t^{n,2}$ again, and so on. A related approach was used in Li et al. (2020).
Proposition F.5. For \( k = 2, \ldots, m \) where \( m \geq 2 \) is an integer, we define iteratively
\[
\hat{C}^{n,k}_t = \left\{ \hat{V}_{0,t} - \frac{\langle a, \hat{V}^n_t \rangle}{\langle a, \Gamma^{H_{k-1}} \rangle} \right\} \left( 1 - \frac{a_0}{\langle a, \Gamma^{H_{k-1}} \rangle} \right)^{-1}
\] (F.16)
and
\[
\hat{H}^n = \hat{H}^n + \sum_{j=1}^{N(\hat{H}^n_{k-1})} \Phi^n_j(\hat{C}^{n,k}_t)^j.
\] (F.17)
Then
\[
\Delta_n^{-\frac{1}{2}}(\hat{H}^n_k - H) \xrightarrow{\text{st}} \mathcal{N} \left( 0, \text{Var}_{H,k} \frac{\int_0^t \rho_s^4 \, ds}{(\int_0^t \rho_s^2 \, ds)^2} \right),
\] (F.18)
\[
\Delta_n^{\frac{1}{2}}(\hat{C}^{n,k}_t - C_t) \xrightarrow{\text{st}} \mathcal{N} \left( 0, \text{Var}_{C,k} \int_0^t \rho_s^4 \, ds \right),
\] (F.19)
where, for each \( k = 2, \ldots, m \),
\[
\text{Var}_{H,k} = \text{Var}_{H,k}(R, a, b, H) = w_k^T C^H w_k, \quad \text{Var}_{C,k} = \text{Var}_{C,k}(R, a, b, H) = u_k^T C^H u_k,
\]
and
\[
u_k = \left( e_1 - \frac{a}{\langle a, \Gamma^H \rangle} + \frac{\langle a, \partial_H \Gamma^H \rangle}{\langle a, \Gamma^H \rangle} w_{k-1} \right) \left( 1 - \frac{a_0}{\langle a, \Gamma^H \rangle} \right)^{-1},
\]
\[
w_k = \frac{(\varphi^{-1})'(\varphi(H))}{\langle b, \Gamma^H \rangle} (a - \varphi(H)b - a_0 u_k).
\]
Our final estimator of \( H \) is
\[
\hat{H}^n = \hat{H}^n_m.
\] (F.20)

For later references, let us define
\[
\text{Var}_H = \text{Var}_H(R, a, b, H) = \text{Var}_{H,m}(R, a, b, H).
\] (F.21)
The next theorem exhibits our final estimators for \( C_t \) and \( \Pi_t \).

Theorem F.6. Choose \( c \in \mathbb{R}^{1+R} \) and define
\[
\hat{C}^n_t = \left\{ \hat{V}_{0,t} - \frac{\langle c, \hat{V}^n_t \rangle}{\langle c, \Gamma^{H_{n}} \rangle} \right\} \left( 1 - \frac{c_0}{\langle c, \Gamma^{H_{n}} \rangle} \right)^{-1},
\] (F.22)
\[
\hat{\Pi}^n_t = \left\{ \frac{\langle a, \hat{V}^n_t \rangle}{\langle a, \Gamma^{H_{n}} \rangle} - \frac{a_0}{\langle a, \Gamma^{H_{n}} \rangle} \hat{C}^n_t \right\} \Delta_n^{1-2n}.
\] (F.23)
Then
\[
\Delta_n^{-\frac{1}{2}}(\hat{C}^n_t - C_t) \xrightarrow{\text{st}} \mathcal{N} \left( 0, \text{Var}_C \int_0^t \rho_s^4 \, ds \right),
\] (F.24)
\[
\Delta_n^{-\frac{1}{2}}(\hat{\Pi}^n_t - \Pi_t) \xrightarrow{\text{st}} \mathcal{N} \left( 0, 4 \text{Var}_H \int_0^t \rho_s^4 \, ds \right),
\] (F.25)
where
\[
\text{Var}_C = \text{Var}_C(R, a, b, c, H) = u^T C^H u,
\]
\[
u = \left( e_1 - \frac{c}{\langle c, \Gamma^H \rangle} + \frac{\langle c, \partial_H \Gamma^H \rangle}{\langle c, \Gamma^H \rangle} w_m \right) \left( 1 - \frac{c_0}{\langle c, \Gamma^H \rangle} \right)^{-1}.
\] (F.26)
F.5 Proofs

Proof of Proposition F.1. Starting from (E.1), we expand

\[
\Delta_n^{-1/2}(\tilde{H}^n - H) = - \sum_{j=1}^{N(H)} \sum_{|x|=j} \frac{\partial^{x}G(\langle a, V_1^n \rangle, \langle b, V_2^n \rangle)}{\chi !} (-1)^j \\
\times \Delta_n^{-1/2}(\langle a, V_1^n - \Gamma^H \Pi_t \rangle, \langle b, V_2^n - \Gamma^H \Pi_t \rangle)^x - \Pi^n, 
\]

(F.27)

where \( \chi \in \mathbb{N}_0^2 \) and \( \vec{a} \) is a point between \( (\langle a, \Gamma^H \Pi_t \rangle, \langle b, \Gamma^H \Pi_t \rangle) \) and \( (\langle a, V_1^n \rangle, \langle b, V_2^n \rangle) \). In contrast to the proof of (4.5), we expanded \( \tilde{H}^n \) around \( (\langle a, \Gamma^H \Pi_t \rangle, \langle b, \Gamma^H \Pi_t \rangle) \) and not \( (\langle a, V_1^n \rangle, \langle b, V_2^n \rangle) \). We consider the terms where \( \chi = (j, 0) \) for some \( j = 1, \ldots, N(H) \) and where \( \chi = (0, 1) \) separately. In the first case, we have \( \partial^{x}G(x, y) = (\varphi^{-1})^{(j)(j)}(x/y)y^{-j} \) for all \( \chi = (j, 0) \) and \( j \geq 1 \); in the second case, \( \partial^{x}G(x, y) \) was computed in (E.3). With that in mind, and recalling (F.3), we have that

\[
\Delta_n^{-1/2} \left( \tilde{H}^n - H + \sum_{j=1}^{N(H)} \Phi_j C_j \right) \\
= (\varphi^{-1})'(\varphi(\tilde{H}^n)) \frac{a^T - \varphi(\tilde{H}^n)b^T}{\langle b, V_1^n \rangle} \Delta_n^{-1/2} \{ V_1^n - \Gamma^H \Pi_t - e_t C_t \Delta_n^{-2H} \} \\
+ \sum_{j=2}^{N(H)} \frac{(-1)^{j+1}}{j!} (\varphi^{-1})^{(j)}(\varphi(\tilde{H}^n)) \frac{1}{\langle b, V_1^n \rangle} \Delta_n^{-1/2} \{ \langle a, V_1^n - \Gamma^H \Pi_t \rangle^j - a_0^j C_j \Delta_n^{(j-2H)} \Pi_t^j \} \\
- \sum_{j=2}^{N(H)} \sum_{\chi \neq (j, 0)} \frac{\partial^{x}G(\langle a, V_1^n \rangle, \langle b, V_2^n \rangle)}{\chi !} (-1)^j \Delta_n^{-1/2} \{ \langle a, V_1^n - \Gamma^H \Pi_t \rangle, \langle b, V_2^n - \Gamma^H \Pi_t \rangle \}^x \\
- \Pi^n. 
\]

(F.28)

By Corollary 4.1, one can see that \( \Delta_n^{-1/2} \{ \langle a, V_1^n - \Gamma^H \Pi_t \rangle^j - a_0^j C_j \Delta_n^{(j-2H)} \Pi_t^j \} \) is of magnitude \( \Delta_n^{(j-1/2)} \). Thus, the second term on the right-hand side of (F.28) is asymptotically negligible. And so are the third term in (F.28) and \( \Pi^n \): For any \( \chi = (j - i, i) \in \mathbb{N}_0^2 \), Corollary 4.1 and assumption (F.1) imply that \( \Delta_n^{1/2} \{ \langle a, V_1^n - \Gamma^H \Pi_t \rangle, \langle b, V_2^n - \Gamma^H \Pi_t \rangle \}^x \) is of magnitude \( \Delta_n^{(j-i)(1-2H)+i/2-1/2} \) and therefore asymptotically negligible as soon as \( i \geq 1 \) and \( j - i \geq 1 \). Similarly, \( \Pi^n \) is of magnitude at most \( \Delta_n^{N(H)+1(1-2H)-1/2} \), which goes to 0 by the definition of \( N(H) \). Altogether, we obtain by Corollary 4.1 that

\[
\Delta_n^{-1/2} \left( \tilde{H}^n - H + \sum_{j=1}^{N(H)} \Phi_j C_j \right) \xrightarrow{st} (\varphi^{-1})'(\varphi(H)) \frac{a^T - \varphi(H)b^T}{\langle b, \Gamma^H \Pi_t \rangle} \mathcal{Z}_t, 
\]

which concludes the proof.

Proof of Proposition F.2. We start similarly to the proof of (4.8) and decompose

\[
V_{0,t}^n = \frac{\langle a, V_1^n \rangle}{\langle a, \Gamma^H \rangle} = \{ V_{0,t}^n - \Pi_t \} - \frac{\langle a, V_1^n - \Gamma^H \Pi_t \rangle}{\langle a, \Gamma^H \rangle} + \frac{\Pi_t \langle a, \Gamma \Pi_t - \Gamma \Pi_t \rangle}{\langle a, \Gamma^H \rangle}. 
\]

(F.29)
We further analyze the last term in the above display and write

\[
\langle a, \Gamma^H \rangle = K(\langle a, \Gamma^H \rangle, \langle b, \Gamma^H \rangle) = K(\langle a, V_t^n \rangle, \langle b, V_t^n \rangle),
\]

\[
\langle a, \Gamma^H \rangle = K(\langle a, \Gamma^H \rangle, \langle b, \Gamma^H \rangle) = K(\langle a, \Gamma^H \Pi_t \rangle, \langle b, \Gamma^H \Pi_t \rangle),
\]

where \( K(x, y) = \psi(x/y) \) and \( \psi \) is the function from (F.8). We now expand \( \langle a, \Gamma^H \rangle \) in a Taylor sum around the point \( \langle (a, V_t^n), (b, V_t^n) \rangle \) up to order \( N(H) \), singling out the two first-order derivatives as well as the derivatives \( \partial^{(j,0)} \): noting that \( \partial^{(j,0)} K(x, y) = \psi^{(j)}(x/y) y^{-j} \) for \( j \geq 1 \) and \( \partial^{(0,1)} K(x, y) = -\psi'(x/y) xy^{-2} \), we have that

\[
\langle a, \Gamma^H - \Gamma^H \rangle = \psi'(\varphi(\Gamma^H)) \frac{1}{\langle b, V_t^n \rangle} \left( \langle a, V_t^n - \Gamma^H \Pi_t \rangle - \varphi(\Gamma^H) \langle b, V_t^n - \Gamma^H \Pi_t \rangle \right)
\]

\[
- \sum_{j=2}^{N(H)} \frac{(-1)^j}{j!} \psi^{(j)}(\varphi(\Gamma^H)) \frac{1}{\langle b, V_t^n \rangle^j} \langle a, V_t^n - \Gamma^H \Pi_t \rangle^j - J^n,
\]

where

\[
J^n = \sum_{j=2}^{N(H)} \frac{\partial^K}{\chi!} \sum_{\chi \neq (j,0)} (-1)^j \Delta_n^{-\frac{1}{2}} (\langle a, V_t^n - \Gamma^H \Pi_t \rangle, \langle b, V_t^n - \Gamma^H \Pi_t \rangle) \chi
\]

\[
+ \sum_{|\chi|=N(H)+1} \frac{\partial^K}{\chi!} (-1)^{|\chi|} \Delta_n^{-\frac{1}{2}} (\langle a, V_t^n - \Gamma^H \Pi_t \rangle, \langle b, V_t^n - \Gamma^H \Pi_t \rangle) \chi
\]

and \( \tilde{a}^n \) is between \( \langle (a, \Gamma^H \Pi_t), (b, \Gamma^H \Pi_t) \rangle \) and \( \langle (a, V_t^n), (b, V_t^n) \rangle \). Using (F.29) for the first and (F.30) for the second equality, we find that

\[
\Delta_n^{-\frac{1}{2}} \left( \left\{ V_{0,t}^n - \frac{\langle a, V_t^n \rangle}{\langle a, \Gamma^H \rangle} \right\} - \left( 1 - \frac{a_0}{\langle a, \Gamma^H \rangle} + \frac{\Pi_t \psi'(\varphi(\Gamma^H))}{\langle a, \Gamma^H \rangle} a_0 \right) C_t \Delta_n^{1-2H} \right)
\]

\[
+ \frac{\Pi_t}{\langle a, \Gamma^H \rangle} \Delta_n^{-\frac{1}{2}} \left\{ \langle a, \Gamma^H - \Gamma^H \rangle + \sum_{j=2}^{N(H)} \frac{(-1)^j}{j!} \psi^{(j)}(\varphi(\Gamma^H)) \frac{a_0^j}{\langle b, V_t^n \rangle^j} C_t^j \Delta_n^{j(1-2H)} \right\}
\]

\[
= \Delta_n^{-\frac{1}{2}} \left\{ V_{0,t}^n - \Pi_t - C_t \Delta_n^{1-2H} \right\} - \frac{1}{\langle a, \Gamma^H \rangle} \Delta_n^{-\frac{1}{2}} \left\{ \langle a, V_t^n - \Gamma^H \Pi_t \rangle - a_0 C_t \Delta_n^{1-2H} \right\}
\]

\[
+ \frac{\Pi_t}{\langle a, \Gamma^H \rangle} \Delta_n^{-\frac{1}{2}} \left\{ \langle a, \Gamma^H - \Gamma^H \rangle + \sum_{j=2}^{N(H)} \frac{(-1)^j}{j!} \psi^{(j)}(\varphi(\Gamma^H)) \frac{a_0^j}{\langle b, V_t^n \rangle^j} C_t^j \Delta_n^{j(1-2H)} \right\}
\]

\[
= \left\{ e_T^T - \frac{a_T^T}{\langle a, \Gamma^H \rangle} + \frac{\Pi_t \psi'(\varphi(\Gamma^H))}{\langle a, \Gamma^H \rangle} (a_T - \Gamma^H b_T) \right\} \Delta_n^{-\frac{1}{2}} \left\{ V_t^n - \Gamma^H \Pi_t - e_1 C_t \Delta_n^{1-2H} \right\}
\]

\[
- \frac{\Pi_t}{\langle a, \Gamma^H \rangle} \sum_{j=2}^{N(H)} \frac{(-1)^j}{j!} \psi^{(j)}(\varphi(\Gamma^H)) \frac{1}{\langle b, V_t^n \rangle^j} \Delta_n^{-\frac{1}{2}} \left\{ \langle a, V_t^n - \Gamma^H \Pi_t \rangle^j - a_0^j C_t^j \Delta_n^{j(1-2H)} \right\}
\]

\[
- \frac{\Pi_t}{\langle a, \Gamma^H \rangle} \Delta_n^{-\frac{1}{2}} J^n.
\]

For the exact same reasons as explained after (F.27), the term involving \( J^n \) is asymptotically negligible: \( \left\langle (a, V_t^n - \Gamma^H \Pi_t), (b, V_t^n - \Gamma^H \Pi_t) \right\rangle \chi \) is of magnitude \( \Delta_n^{j(1-2H)+i/2} \leq \Delta_n^{3/2-2H} \) if \( |\chi| = 2, \ldots, N(H) \) and \( \chi \neq (j,0) \), and it is of magnitude \( \leq \Delta_n^{N(H)+1(1-2H)} \).
if $|\chi| = N(H) + 1$; in both cases, the exponent is strictly bigger than $\frac{1}{2}$. Moreover, by Corollary 4.1, $\Delta_n^{j(1-2H)}(a, V^n_t - \langle a, H^\alpha \rangle \Pi_t) \xrightarrow{p} \tilde{a}_0^j C_t^j$, which implies that the second term on the right-hand side of (F.31) is of magnitude $\Delta_n^{j(1-1/2)}$ for $j = 2, \ldots, N(H)$. Thus, by Corollary 4.1, the left-hand side of (F.31) converges stably in law to

$$Z_t' = \left\{ e_T - \frac{a_T}{\langle a, H^\alpha \rangle} + \frac{\Pi_t}{\langle a, H^\alpha \rangle} \psi'(\varphi(H)) \left( a - \varphi(H)b_T \right) \right\} Z_t.$$  \hfill (F.32)

Next, we replace $\Pi_t$ in the first two lines of (F.31) by $\Delta_n^{1-2H} \tilde{P}_t^n$, where $\tilde{P}_t^n$ was introduced in (F.5). The resulting difference is given by

$$\Delta_n^{-\frac{1}{2}} \left\{ \Delta_n^{1-2H} \tilde{P}_t^n - \Pi_t \right\} \sum_{j=1}^{N(H)} \frac{(-1)^j}{j!} \psi(j)(\varphi(\tilde{H}^n)) \frac{a_0^j}{\langle b, V^n_t \rangle} C_t^j \Delta_n^{j(1-2H)}.$$  \hfill (F.33)

By the proof of Theorem 4.2 (see (E.7) in particular), $\Delta_n^{-1/2} \{ \Delta_n^{1-2H} \tilde{P}_t^n - \Pi_t \}$ converges stably in distribution. As a consequence, the expression in the previous display converges to 0 in probability as $n \to \infty$. By (F.7), (F.10) and (F.32), it follows that

$$\Delta_n^{2+1-2H} \left( \left\{ V^n_{0,t} - \langle a, V^n_t \rangle \right\} \Delta_n^{2H-1} \Theta(\tilde{V}_t^n, \tilde{H}^n, \tilde{H}^{n,0}) - C_t + \sum_{j=2}^{N(H)} \Psi_j^n C_t^j \right)$$

$$= \Theta(\tilde{V}_t^n, \tilde{H}^n, \tilde{H}^{n,0})^{-1} \Delta_n^{-\frac{1}{2}} \left\{ V^n_{0,t} - \langle a, V^n_t \rangle \right\} - \Theta(\tilde{V}_t^n, \tilde{H}^n, \tilde{H}^{n,0}) C_t \Delta_n^{1-2H}$$

$$+ \sum_{j=2}^{N(H)} \frac{(-1)^j}{j!} \psi(j)(\varphi(\tilde{H}^n)) \frac{a_0^j}{\langle b, V^n_t \rangle} C_t^j \Delta_n^{j(1-2H)}$$

$$\xrightarrow{st} \left( 1 - \frac{a_0}{\langle a, H^\alpha \rangle} + \frac{\psi'(\varphi(H))}{\langle a, H^\alpha \rangle \langle b, H^\alpha \rangle} a_0 \right)^{-1} Z_t' \sim \mathcal{N}(0, \text{Var}_{C,1} \int_0^t \rho_s^4 \text{d}s).$$

The CLT stated in (F.9) is proved.

\[ \square \]

**Proof of Proposition F.3.** We first prove by induction that for $\ell = 0, \ldots, N(H) - 2$, the difference $\bar{C}_t^{n,\ell+1} - C_t$ converges in probability with a convergence rate of $\Delta_n^{(\ell+1)(1-2H)}$. If $\ell = 0$, then $\bar{C}_t^{n,1} = C_t$, so by (F.34),

$$\Delta_n^{2H-1}(\bar{C}_t^{n,1} - C_t) \xrightarrow{p} - \frac{\psi^{(2)}(\varphi(H)) a_0^2}{2 \langle a, H^\alpha \rangle \langle b, H^\alpha \rangle^2} \left( 1 - \frac{a_0}{\langle a, H^\alpha \rangle} + \frac{\psi'(\varphi(H))}{\langle a, H^\alpha \rangle \langle b, H^\alpha \rangle} a_0 \right)^{-1} C_t^2.$$  \hfill (F.34)

Suppose now that $\bar{C}_t^{n,\ell+1} - C_t$ converges at a rate of $\Delta_n^{(\ell+1)(1-2H)}$ for $\ell = 0, \ldots, \ell' - 1$. Decomposing

$$\bar{C}_t^{n,\ell+1} - C_t = \left\{ \bar{C}_t^{n,1} - C_t + \sum_{j=2}^{\ell+1} \Psi_j^n C_t^j \right\} + \sum_{j=2}^{\ell+1} \Psi_j^n \left\{ (\bar{C}_t^{n,\ell'-j+2} - C_t^j) \right\},$$  \hfill (F.35)

we note that the first term on the right-hand side converges at a rate of $\Delta_n^{(\ell'+1)(1-2H)}$ by (F.34). The second term can be rewritten as

$$\sum_{j=2}^{\ell+1} \psi^{(j)}(\bar{C}_t^{n,\ell'-j+2} - C_t^j) = \sum_{j=2}^{\ell+1} \sum_{m=1}^{j} \frac{j!}{(j-m)!} C_t^{j-m} \bar{C}_t^{n,\ell'-j+2} - C_t^m.$$

\[ 27 \]
By assumption, \( \tilde{C}_t^{n,(\ell' - j + 1)+1} - C_t \) is of size \( \Delta_n^{(\ell' - j + 2)(1-2H)} \). Moreover, from (F.10), the product \( \Psi_j^n \Delta_n^{(1-j)(1-2H)} \) converges in probability. Thus, \( \Psi_j^n \) is of magnitude \( \Delta_n^{(j-1)(1-2H)} \) and we conclude that \( \Psi_j^n \left( \tilde{C}_t^{n,(\ell' - j + 2)} - C_t \right) \) is of magnitude \( \Delta_n^{(j-1+\ell'-(j+2))(1-2H)} \leq \Delta_n^{(\ell'+1)(1-2H)} \).

Altogether, \( \tilde{C}_t^{n,(\ell'+1)} - C_t \) is of magnitude \( \Delta_n^{(\ell'+1)(1-2H)} \).

We can now complete the proof of the proposition. By a similar decomposition to (F.35) with \( \ell' = N(H) - 1 \),

\[
\tilde{C}_t^{n,N(H)} - C_t = \left\{ \tilde{C}_t^{n,1} - C_t + \sum_{j=2}^{N(H)} \Psi_j^n C_t^j \right\} + \sum_{j=2}^{N(H)} \Psi_j^n \left\{ (\tilde{C}_t^{n,N(H)} - 1)^j - C_t^j \right\}. \tag{F.37}
\]

We know that \( \tilde{C}_t^{n,N(H)} - 1 - C_t \) is of magnitude \( \Delta_n^{(N(H)-j+1)(1-2H)} \). Therefore, proceeding exactly as in (F.36), we see that the right-hand side of (F.37) times \( \Delta_n^{1/2-2H} \) is of size \( \Delta_n^{(N(H)+1)(1-2H)-1/2} \) which goes to 0 as \( n \to \infty \) since the exponent is positive by the definition of \( N(H) \). So \( \Delta_n^{1/2-2H} \{ \tilde{C}_t^{n,N(H)} - C_t \} \) converges stably to the same distribution as \( \Delta_n^{1/2-2H} \{ \tilde{C}_t^{n,1} - C_t \} \) does. Finally, \( \Delta_n^{1/2} \{ \tilde{C}_t^{n,N(H)} - C_t \} = \Delta_n^{1/2} \{ \tilde{C}_t^{n,N(H)} - C_t \} + \Delta_n^{1/2} \{ \tilde{C}_t^{n,N(H)} - \tilde{C}_t^{n,N(H)} \} \). Since \( \tilde{H}^n \) is a consistent estimator for \( H \) and \( H \not\in \mathcal{H} \), for small enough \( \varepsilon > 0 \) (such that the event \( \{ |\tilde{H}^n - H| \leq \varepsilon \} \subseteq \{ N(H) = N(H) \} \)),

\[
\mathbb{P}(\Delta_n^{1/2} \{ \tilde{C}_t^{n,N(H)} - \tilde{C}_t^{n,N(H)} \} > \varepsilon \) \leq \mathbb{P}(\{ |\tilde{H}^n - H| > \varepsilon \}) \to 0. \tag{F.38}
\]

Thus, the CLT of \( \tilde{C}_t^{n,N(H)} - C_t \) is not affected when \( N(H) \) is replaced by \( N(\tilde{H}^n) \). \( \square \)

**Proof of Proposition F.4.** We first decompose

\[
\Delta_n^{1/2} \left\{ \tilde{H}^n - H + \sum_{j=1}^{N(H)} \Phi_j^n \tilde{C}_t^{n,1}^j \right\} = \Delta_n^{1/2} \left\{ \tilde{H}^n - H + \sum_{j=1}^{N(H)} \Phi_j^n C_t^j \right\} + \Phi_1^n \Delta_n^{1/2} \left\{ \tilde{C}_t^{n,N(H)} - C_t \right\} + \sum_{j=2}^{N(H)} \Phi_j^n \Delta_n^{1/2} \left\{ (\tilde{C}_t^{n,N(H)})^j - (\tilde{C}_t^{n,N(H)})^j \right\} \tag{F.39}
\]

where

\[
\Pi_1^n = \sum_{k=2}^{N(H)} \Phi_1^n \Psi_k^n \Delta_n^{1/2} \left\{ (\tilde{C}_t^{n,N(H) - k + 1})^k - C_t^k \right\} + \sum_{j=2}^{N(H)} \Phi_j^n \Delta_n^{1/2} \left\{ (\tilde{C}_t^{n,N(H)})^j - (\tilde{C}_t^{n,N(H)})^j \right\}.
\]

By the proof of Proposition F.3 and the mean-value theorem, \( (\tilde{C}_t^{n,N(H) - k + 1})^k - C_t^k \) is of size \( \Delta_n^{(N(H) - k + 1)(1-2H)} \) and \( (\tilde{C}_t^{n,N(H)})^j - C_t^j \) is of size \( \Delta_n^{2H-1/2} \). Furthermore, from (F.3),
we see that $\Phi_n^j \Delta_n^{-j(1-2H)}$ converges in probability. Hence, $\Phi_n^j \{(C_t^{m,N(H)})^j - C_t^j\}$ is of size $\Delta_n^{k/2+j(1-2H)}$. Also, $\Psi_n^j$ is of size $\Delta_n^{(k-1)(1-2H)}$, so $\Phi_n^j \Psi_n^j \Delta_n^{-1/2} \{(C_t^{m,N(H)} - k+1)^j - C_t^j\}$ is of size $\Delta_n^{(N(H)+1)(1-2H)-1/2}$. Recall also that $\Delta_n^{-1/2} \{(C_t^{m,N(H)} - \tilde{C}_t^{m,N(H)})$ is negligible by the last part of the proof of Proposition F.3. Altogether, $\Pi_1^a$ is asymptotically negligible.

Now, recalling (F.6), (F.7) and (F.10), we decompose

$$
\Delta_n^{1-2H} \left\{ \tilde{C}_t^{n,1} - C_t + \sum_{k=2}^{N(H)} \Psi_n^k C_t^k \right\}
$$

$$
= \frac{\Delta_n^{1-2H}}{\Theta(V_t^n, H^n, H_n, 0)} \left\{ \left\{ V_{0,t} - \langle a, V_t^n \rangle \right\} \Delta_n^{2H-1} - \Theta(V_t^n, \tilde{H}_n, \tilde{H}_n, 0) C_t \right\}
$$

$$
+ \frac{\Delta_n^{1-2H} \tilde{P}_t^n}{\langle a, \Gamma H_n \rangle} \sum_{j=2}^{N(H)} \left( -1 \right)^j \frac{j!}{\psi(j)(\varphi(\tilde{H}_n))} \frac{a_0^j}{\langle b, V_t^n \rangle} C_t^j \Delta_n^{j(1-2H)}
$$

$$
= \frac{\Delta_n^{1-2H}}{\Theta(V_t^n, H^n, H_n, 0)} \left\{ V_{0,t} - \langle a, V_t^n \rangle \right\} \left( 1 - \frac{a_0}{\langle a, \Gamma H_n \rangle} \right) C_t \Delta_n^{1-2H}
$$

$$
+ \frac{\Pi_t}{\langle a, \Gamma H_n \rangle} \sum_{j=2}^{N(H)} \left( -1 \right)^j \frac{j!}{\psi(j)(\varphi(\tilde{H}_n))} \frac{a_0^j}{\langle b, V_t^n \rangle} C_t^j \Delta_n^{j(1-2H)}
$$

$$
+ \frac{\Delta_n^{1-2H} \tilde{P}_t^n - \Pi_t}{\Theta(V_t^n, H^n, H_n, 0)} \sum_{j=1}^{N(H)} \left( -1 \right)^j \frac{j!}{\psi(j)(\varphi(\tilde{H}_n))} \frac{a_0^j}{\langle b, V_t^n \rangle} C_t^j \Delta_n^{j(1-2H)}.
$$

The last term is asymptotically negligible as already seen in the discussion following (F.33), while the first term on the right-hand side of (F.40) was analyzed in the (F.31). Combining this with (F.28), we continue the computations started in (F.39):

$$
\Delta_n^{1/2} \left\{ \tilde{H}_n - H + \sum_{j=1}^{N(H)} \Phi_n^j \langle \tilde{C}_t^{n,1} \rangle^j \right\}
$$

$$
= \left\{ (\varphi^{-1})'(\varphi(\tilde{H}_n)) \right\} \frac{a^T - \varphi(\tilde{H}_n)b^T}{\langle b, V_t^n \rangle} + \frac{\Phi_n^j \Delta_n^{2H-1}}{\Theta(V_t^n, H^n, H_n, 0)} \left\{ e_1^T - \frac{a^T}{\langle a, \Gamma H_n \rangle} \right\}
$$

$$
+ \frac{\Pi_t}{\langle a, \Gamma H_n \rangle} \sum_{j=1}^{N(H)} \left( -1 \right)^j \frac{j!}{\psi(j)(\varphi(\tilde{H}_n))} \frac{a_0^j}{\langle b, V_t^n \rangle} C_t^j \Delta_n^{j(1-2H)}
$$

$$
= \frac{w_1(\tilde{H}_n, \tilde{H}_n, V_t^n) \Delta_n^{1/2} \{ V_{0,t} - \Gamma H \Pi_t - e_1 C_t \Delta_n^{1-2H} \} + \tilde{\Pi}_1^n}{w_1(\tilde{H}_n, \tilde{H}_n, V_t^n) \Delta_n^{1/2} \{ V_{0,t} - \Gamma H \Pi_t - e_1 C_t \Delta_n^{1-2H} \} + \tilde{\Pi}_1^n},
$$

where

$$
w_1(\tilde{H}_n, \tilde{H}_n, V_t^n) = \frac{(\varphi^{-1})'(\varphi(\tilde{H}_n))}{\langle b, V_t^n \rangle} \left\{ a^T - \varphi(\tilde{H}_n)b^T - a_0 u_1(\tilde{H}_n, \tilde{H}_n, V_t^n) \right\},
$$

$$
u_1(\tilde{H}_n, \tilde{H}_n, V_t^n) = \Theta(V_t^n, \tilde{H}_n, \tilde{H}_n, 0)^{-1}
$$

$$
	imes \left\{ e_1^T - \frac{a^T}{\langle a, \Gamma H_n \rangle} + \frac{\Pi_t}{\langle a, \Gamma H_n \rangle} \sum_{j=1}^{N(H)} \left( -1 \right)^j \frac{j!}{\psi(j)(\varphi(\tilde{H}_n))} \frac{a_0^j}{\langle b, V_t^n \rangle} (a^T - \varphi(\tilde{H}_n)b^T) \right\}.
$$

In $\tilde{\Pi}_1^n$, we have incorporated the last three terms on the right-hand side of (F.28), the last two terms on the right-hand side of (F.31), the last expression in (F.40) as well as $\Pi_1^a$ from
By the discussions following these equations, we know that \( \hat{\Pi}_n \) is asymptotically negligible. Therefore, we obtain
\[
\Delta_n^{-\frac{1}{2}} \left\{ \hat{H}^n - H + \sum_{j=1}^{N(H)} \Phi_j^n (\hat{C}^{(n,1)}_H) \right\} \xrightarrow{st} \frac{w^T_t}{\Pi_t} Z_t \sim \mathcal{N} \left( 0, \text{Var}_{H,1} \left( \int_0^t \rho_s^4 \, ds \right) \right).
\]

To conclude, it remains to observe that this CLT is not affected when \( N(H) \) is replaced by \( N(\hat{H}^n) \) because \( H \not\in \mathcal{H} \); cf. the argument used to show (F.38).

**Proof of Proposition F.5.** For \( k = 2, \ldots, m \), define
\[
u_k(\hat{H}_{k-1}^n, \hat{H}^n, V_t^n) = \left\{ e^T - \frac{a^T}{\langle a, \Gamma \hat{H}_{k-1}^n \rangle} + \Pi_t (a, \partial_H \Gamma H) w_{k-1}(\hat{H}_{k-2}^n, \hat{H}^n, V_t^n) \right\} \left( 1 - \frac{a_0}{\langle a, \Gamma \hat{H}_{k-1}^n \rangle} \right)^{-1},
\]
\[
\nu_k(\hat{H}_{k-1}^n, \hat{H}^n, V_t^n) = \left\{ \frac{(\varphi^{-1})'(\varphi(\hat{H}^n))}{\langle b, V_t^n \rangle} \right\} \left\{ a^T - \varphi(\hat{H}^n) b^T - a_0 \nu_k(\hat{H}_{k-1}^n, \hat{H}^n, V_t^n) \right\}.
\]

In the definition of \( u_2(\hat{H}_{k-1}^n, \hat{H}^n, V_t^n) \), the term \( u_1(\hat{H}_{k-1}^n, \hat{H}^n, V_t^n) \) is replaced by the term \( w_1(\hat{H}_{k-1}^n, \hat{H}^n, V_t^n) \) from (F.42). By induction over \( k \), we are going to show for all \( k = 1, \ldots, m \) that
\[
\Delta_n^{-\frac{1}{2}} (\hat{H}_k^n - H) = \nu_k(\hat{H}_{k-1}^n, \hat{H}^n, V_t^n) \Delta_n^{-\frac{1}{2}} \left\{ V_t^n - \Gamma H \Pi_t - e_1 C_t \Delta_n^{1-2H} \right\} + \hat{\Pi}_k^n \tag{F.43}
\]
for some asymptotically negligible expression \( \hat{\Pi}_k^n \) and that
\[
u_k(\hat{H}_{k-1}^n, \hat{H}^n, V_t^n) \xrightarrow{p} u_k^T, \quad w_k(\hat{H}_{k-1}^n, \hat{H}^n, V_t^n) \xrightarrow{p} \frac{w_k^T}{\Pi_t}, \tag{F.44}
\]
where, for \( k = 1 \), we take the expressions in (F.42) instead. For \( k = 1 \), (F.43) was already shown in (F.41), and (F.44) is obvious, so we may consider \( k \geq 2 \) now and assume (F.43) and (F.44) for \( k - 1 \). In particular,
\[
\Delta_n^{-\frac{1}{2}} \{ \hat{H}_k^n - H \} \xrightarrow{st} \frac{w_{k-1}^T}{\Pi_t} Z_t \sim \mathcal{N} \left( 0, \text{Var}_{H,k-1} \left( \int_0^t \rho_s^4 \, ds \right) \right).
\]

It is straightforward to see that
\[
u_k(\hat{H}_{k-1}^n, \hat{H}^n, V_t^n) \xrightarrow{p} u_k^T \quad \text{and} \quad w_k(\hat{H}_{k-1}^n, \hat{H}^n, V_t^n) \xrightarrow{p} \frac{w_k^T}{\Pi_t}, \tag{F.45}
\]
so we can proceed to showing (F.43) for \( k \). Expanding \( \langle a, \Gamma \hat{H}_{k-1}^n \rangle \) around \( H \) and using the
induction hypothesis, we can find $\beta_{k-1}^n$ between $\widehat{H}_{k-1}^n$ and $H$ such that

\[
\Delta_n^{-\frac{1}{2}} \left\{ V_{0,t}^n - \frac{\langle a, V_t^n \rangle}{\langle a, \Gamma H_{k-1}^n \rangle} - \left( 1 - \frac{a_0}{\langle a, \Gamma H_{k-1}^n \rangle} \right) C_t \Delta_n^{1-2H} \right\} \
= \Delta_n^{-\frac{1}{2}} \left\{ V_{0,t}^n - \Pi_t - C_t \Delta_n^{1-2H} \right\} - \frac{1}{\langle a, \Gamma H_{k-1}^n \rangle} \Delta_n^{\frac{1}{2}} \left\{ \langle a, V_t^n - \Gamma^H \Pi_t \rangle - a_0 C_t \Delta_n^{1-2H} \right\} \\
+ \frac{\Pi_t}{\langle a, \Gamma H_{k-1}^n \rangle} \Delta_n^{-\frac{1}{2}} \langle a, \Gamma \widehat{H}_{k-1}^n - \Gamma^H \rangle \\
= \Delta_n^{-\frac{1}{2}} \left\{ V_{0,t}^n - \Pi_t - C_t \Delta_n^{1-2H} \right\} - \frac{1}{\langle a, \Gamma H_{k-1}^n \rangle} \Delta_n^{\frac{1}{2}} \left\{ \langle a, V_t^n - \Gamma^H \Pi_t \rangle - a_0 C_t \Delta_n^{1-2H} \right\} \quad (F.46) \\
+ \frac{\Pi_t}{\langle a, \Gamma H_{k-1}^n \rangle} \{ w_{k-1}(\widehat{H}_{k-2}^n, \widehat{H}_n, V_t^n) \Delta_n^{-\frac{1}{2}} \left\{ V_t^n - \Gamma^H \Pi_t - e_1 C_t \Delta_n^{1-2H} \right\} + \tilde{w}_k^{n} \} \\
+ \frac{1}{2!} \frac{\Pi_t}{\langle a, \Gamma H_{k-1}^n \rangle} \langle a, \partial_{HH} \Gamma^n \rangle \Delta_n^{-\frac{1}{2}} \{ \widehat{H}_{k-1}^n - H \}^2 \\
= u_k(\widehat{H}_{k-1}^n, \widehat{H}_n, V_t^n) \left( 1 - \frac{a_0}{\langle a, \Gamma H_{k-1}^n \rangle} \right) \Delta_n^{-\frac{1}{2}} \left\{ V_t^n - \Gamma^H \Pi_t - e_1 C_t \Delta_n^{1-2H} \right\} + \tilde{w}_k^n,
\]

where $\tilde{w}_k^n = \Pi_t \langle a, \Gamma \widehat{H}_{k-1}^n \rangle^{-1} \{ \langle a, \partial_{HH} \Gamma^n \rangle \widehat{H}_{k-1}^n + \frac{1}{2} \langle a, \partial_{HH} \Gamma \beta_{k-1}^n \rangle \Delta_n^{-1/2} \{ \widehat{H}_{k-1}^n - H \}^2 \}$. Because $\tilde{w}_k^n \overset{P}{\to} 0$. Recalling (F.16), we infer from (F.46) and (F.45) that $\Delta_n^{1/2-2H}(\widehat{C}_t^{n,k} - C_t) \overset{st}{\to} u_k^T \mathcal{Z}_t$, which is (F.19). Now recall the definitions (F.16) and (F.17). Using (F.28) and the formula $\Phi^n_1 = -\Delta_n^{1-2H}(\varphi^{-1})(\varphi(\widehat{H}^n))a_0/\langle b, V_t^n \rangle$ for the second equality and (F.46) for the third, we obtain

\[
\Delta_n^{-\frac{1}{2}} \{ \widehat{H}_{k}^n - H \} = \Delta_n^{-\frac{1}{2}} \left\{ \widehat{H}_{k}^n - H + \sum_{j=1}^{N(\widehat{H}_{k-1}^n)} \Phi^n_j C_t^j \right\} + \Phi^n_1 \Delta_n^{-\frac{1}{2}} \{ \widehat{C}_t^{n,k} - C_t \} \\
+ \sum_{j=2}^{N(\widehat{H}_{k-1}^n)} \Phi^n_j \Delta_n^{-\frac{1}{2}} \{ (\widehat{C}_t^{n,k})^j - C_t^j \} \quad (F.47) \\
= \frac{(\varphi^{-1})(\varphi(\widehat{H}^n))}{\langle b, V_t^n \rangle} \{ a^T - \varphi(\widehat{H}^n)b^T - a_0 u_k(\widehat{H}_{k-1}^n, \widehat{H}_n, V_t^n) \} \\
\times \Delta_n^{-\frac{1}{2}} \left\{ V_{t}^n - \Gamma^H \Pi_t - e_1 C_t \Delta_n^{1-2H} \right\} + \tilde{w}_k^n \\
= w_k(\widehat{H}_{k-1}^n, \widehat{H}_n, V_t^n) \Delta_n^{-\frac{1}{2}} \left\{ V_t^n - \Gamma^H \Pi_t - e_1 C_t \Delta_n^{1-2H} \right\} + \tilde{w}_k^n.
\]

In the last line, $\tilde{w}_k^n$ contains the last three terms on the right-hand side of (F.28) and

\[
\tilde{w}_k^n \left( 1 - \frac{a_0}{\langle a, \Gamma H_{k-1}^n \rangle} \right)^{-1} + \sum_{j=2}^{N(H)} \Phi^n_j \Delta_n^{-\frac{1}{2}} \{ (\widehat{C}_t^{n,k})^j - C_t^j \} + \Delta_n^{-\frac{1}{2}} \sum_{j=2}^{N(H)\wedge N(\widehat{H}_{k-1}^n)} \Phi^n_j (\widehat{C}_t^{n,k})^j.
\]

The term $\Phi^n_j \Delta_n^{-1/2}((\widehat{C}_t^{n,k})^j - C_t^j)$ is of size $\Delta_n^{(j-1)(1-2H)}$ because $\Phi^n_j$ is of size $\Delta_n^{(1-2H)}$. Also, the last sum goes to 0 in probability by a similar argument to (F.38). Therefore, $\tilde{w}_k^n$
is asymptotically negligible. This together with (F.47) implies (F.43) and our induction argument is complete. From (F.43), we immediately obtain (F.18).

**Proof of Theorem F.6.** The proof of (F.24) and (F.25) is similar to that of (F.19) and (4.11) in Theorem F.5 and 4.2, respectively.

**G  Choosing the tuning parameters**

We fix the number of iterations in the multi-step algorithm of Section F.4 at $m = 50$. In fact, for an overwhelming majority of estimates obtained in the simulation and the empirical analysis of Section 6, a precision of $10^{-5}$ was attained after fewer than 50 steps. We further make the choice $R = 60$, which corresponds to considering quadratic variations with time lags up to one minute. In order to tune the remaining parameters, we want to choose the vectors $a$, $b$, and $c$ in such a way that $(a, b, c) \mapsto \text{Var}_C(R, a, b, c, H)$ is small as possible. Due to the complexity of how $\text{Var}_C$ depends on $a$, $b$, and $c$, we were not able to find (and doubt there is) an analytical expression for the minimizers. In addition, $\text{Var}_C$ depends on $H$, which is unknown. Pretending we knew $H$ for the moment and $H \in (\frac{1}{4}, \frac{1}{2})$, in order to resolve the first issue, we choose

$$a = c = \frac{\Gamma^H - \langle \Gamma^H, b \rangle b}{\| \Gamma^H - \langle \Gamma^H, b \rangle b \|}, \quad b = \frac{\partial H \Gamma^H}{\| \partial H \Gamma^H \|}$$

(G.1)

as initial values (This is a heuristic choice: with these vectors, $\langle c, \partial H \Gamma^H \rangle = 0$ in (F.26) and $\langle u, \Gamma^H \rangle = 0$. Consequently, if $c^0_{0,1}$ and $c^0_{0,2}$ denote the two zeroth-order terms in (4.2), then $u^T c^0_{0,2} u = 0$ and, in $u^T c^0_{0,1} u = \sum_{i,j=0}^R u_i u_j (c^0_{0,1})_{ij}$, the part of the sum where $i = j$ is 0.) Then we run the R function `fminsearch()` from the package pracma to find (local) minimizers $a(H)$, $b(H)$ and $c(H)$ of $(a, b, c) \mapsto \text{Var}_C(R, a, b, c, H)$ from (F.26). Similarly, we obtain $a^0(H)$, $b^0(H)$ and $c^0(H)$ as minimizers of $(a, b, c) \mapsto \text{Var}_C(R, a, b, c, H)$ from (4.9) by taking the same initial weights $b$ and $c$ from (G.1) for $b^0$ and $c^0$ and by choosing $a^0$ as the vector obtained by substituting 0 for the first component of $a$ from (G.1). As $H$ is unknown in this process, we simply take the minimizers at $H_0 = 0.35$, that is,

$$a^0 = a^0(0.35), \quad b^0 = b^0(0.35), \quad c^0 = c^0(0.35),$$

$$a = a(0.35), \quad b = b(0.35), \quad c = c(0.35).$$

(G.2) (G.3)

One could, of course, plug in a consistent estimator of $H$ (e.g., $\hat{H}^n$, computed for some initial choice of $a$, $b$ and $c$), determine the minimizing vectors, use them to construct an update of $\hat{H}^n$, and repeat this procedure. However, such an adaptive scheme of constructing $\hat{H}^n$ makes the weight vectors dependent on the latest estimator of $H$ and therefore changes its asymptotic variance in every step. Unfortunately, we see no way of keeping track of those changes, in particular because we do not know the precise form of how $a(H)$, $b(H)$ and $c(H)$ depend on $H$. It turns out that the variances $\text{Var}_C(R, a^0(H_0), b^0(H_0), c^0(H_0), H)$ and $\text{Var}_C(R, a(H_0), b(H_0), c(H_0), H)$ at other values of $H$ based on the choice $H_0 = 0.35$ turn out to be reasonably close to the $H$-dependent minimal values $\text{Var}_C(R, a^0(H), b^0(H), c^0(H), H)$ and $\text{Var}_C(R, a(H), b(H), c(H), H)$, respectively (no more than 2.1% larger in the former case; no more than 8.1% larger in the latter case for all $H$ in (5.1) except for $H = 0.45$, where the variance based on (G.3) is 2.6 times larger).
Figure 9: Histogram of $H$-estimates. Each data point corresponds to one company and trading day.

Figure 10: Top: Estimates of $H$ with asymptotic 95%-confidence intervals. Bottom row: Volatility (solid line) and noise volatility (dashed line) estimates for May 2019 (AXP) and September 2019 (IBM), respectively.

## H Empirical analysis of quotes

Having studied transaction data in Section 6, we report results about quote data in this section. We consider (logarithmic) mid-quote data for each of the 29 stocks that were constituents of the DJIA index for the whole year of 2019. Using the TAQ database, we collect, for each trading day in 2019, all quotes on the NYSE and NASDAQ from 9:00 am until 4:00 pm Eastern Time and preprocess them using the `quotesCleanup()` function from the R package `highfrequency`. We sample in calendar time every second and follow the rules outlined in Section 6 to compute estimates of $H, C_T$ and $\Pi_T$.

Figure 9 shows the histogram of $H$-estimates, computed for each trading day and each of the 29 examined companies. In contrast to trade data where noise has decreased over
time, rough noise is strongly present in quote data as late as 2019. In Figure 10, we further consider two particular examples, American Express (AXP) and IBM. Both confirm that rough noise is consistently present throughout most parts of the year.

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