Generalized Lagrangian symmetries depending on higher order derivatives. Conservation laws and the characteristic equation

Giovanni Giachetta¹, Luigi Mangiarotti¹ and Gennadi Sardanashvily²

¹ Department of Mathematics and Informatics, University of Camerino, 62032 Camerino (MC), Italy
² Department of Theoretical Physics, Physics Faculty, Moscow State University, 117234 Moscow, Russia
E-mail: giovanni.giachetta@unicam.it, luigi.mangiarotti@unicam.it and sard@grav.phys.msu.su

Abstract.
Given a finite order Lagrangian $L$ on a fibre bundle, its global generalized symmetries depending on higher order derivatives of dynamic variables are considered. The first variational formula is obtained. It leads both to the corresponding Lagrangian conservation laws and the characteristic equation for generalized symmetries of $L$.

Symmetries of differential equations under transformations of dynamic variables depending on their derivatives have been intensively investigated (see, e.g., [3, 11, 14] for a survey). Following [3, 14], we agree to call them the generalized symmetries. In [14], generalized symmetries of Lagrangian systems and the corresponding conservation laws on a local coordinate chart are described in detail. The recent work [6] turns to the global analysis of first order Lagrangian systems and conservation laws under generalized symmetries depending on first order derivatives, but the symmetry condition is imposed on the Poincaré–Cartan form of a Lagrangian. In analytical mechanics, generalized symmetries and the corresponding conserved quantities (e.g., the Runge–Lenz vector in the Kepler problem) are well known [14]. In application to field theory, let us mention the Lie derivative (the Kosmann lift) of Dirac spinor fields [5, 13, 15], the Poisson sigma model [7], and BRST transformations [6, 7]. The latter however involve the notion of jets of functions on graded manifolds [13, 16] which is beyond the scope of this work. Our goal here is to study the conservation laws in higher order Lagrangian systems on fibre bundles under generalized symmetries depending on derivatives of any finite order.

There are different approaches to the study of Lagrangian conservation laws. We are based on the first variational formula (see [8, 13, 18] for a survey).

Let $Y \to X$ be a smooth fibre bundle coordinated by $(x^\lambda, y^i)$. In Lagrangian formalism on $Y \to X$, an $r$-order Lagrangian is defined as a horizontal density

$$L = \mathcal{L}_\omega, \quad \omega = dx^1 \wedge \cdots \wedge dx^n, \quad n = \dim X,$$

on the $r$-order jet manifold $J^r X$ of sections of $Y \to X$. This manifold is equipped with the adapted coordinates $(x^\lambda, y^i, y^i_\Lambda)$, $0 < |\Lambda| \leq r$, where a multi-index $\Lambda$, $|\Lambda| = k$, denotes...
a collection of numbers \((\lambda_k, \ldots, \lambda_1)\) modulo permutations. By \(\Lambda + \Sigma\) is meant the collection \((\lambda_k \sigma, \ldots, \lambda_1 \sigma)\) modulo permutations. We use the notation

\[
d_{\lambda} = \partial_{\lambda} + \sum_{|A| \geq 0} y^A_{\lambda + A} \partial^A, \quad d_{\Lambda} = d_{\lambda_r} \circ \cdots \circ d_{\lambda_1}, \quad \Lambda = (\lambda_r, \ldots, \lambda_1).
\]

(2)

In order to obtain Noether conservation laws, one considers vector fields

\[
u = u^\lambda (x^\mu) \partial_{\lambda} + u^i (x^\mu, y^j) \partial_i
\]

on \(Y\) projected onto \(X\). They are infinitesimal generators of local one-parameter groups of bundle automorphisms of \(Y \to X\). Their jet prolongation onto \(J^r Y\) read

\[
J^r u = u^\lambda \partial_{\lambda} + u^i \partial_i + \sum_{0 < |A| \leq r} u^A_{\Lambda} \partial^A, \quad u^A_{\Lambda} = d_A (u^i - y^i_{\mu} u^\mu) + y^i_{\mu + \Lambda} u^\mu.
\]

(4)

One says that \(u\) (3) is a (variational) symmetry of a Lagrangian \(L\) if the Lie derivative \(L_{J^r u} L\) along \(J^r u\) vanishes. The first variational formula provides the canonical decomposition of this Lie derivative

\[
L_{J^r u} L = u_V \delta L + d_H (h_0 (J^r u [\Xi_L])),
\]

(5)

where \(\delta L\) is the Euler–Lagrange operator of \(L\), \(\Xi_L\) is a Poincaré–Cartan form of \(L\), \(u_V\) is a vertical part of \(u\), \(d_H\) is the total (horizontal) differential, and \(h_0\) is the horizontal projection (see the definitions below). If \(u\) is a symmetry of \(L\), the first variational formula (5) restricted to the kernel of the Euler–Lagrange operator \(\delta L\) leads to the Noether conservation law

\[
0 \approx d_H (h_0 (J^r u [\Xi_L])).
\]

A vector field \(u\) (3) is said to be a divergence symmetry of \(L\) if the Lie derivative \(L_{J^r u} L\) is a total differential \(d_H \sigma\). Then, the first variational formula (5) on \(\text{Ker} \delta L\) provides the generalized Noether conservation law

\[
0 \approx d_H (h_0 (J^r u [\Xi_L]) - \sigma).
\]

(6)

Note that any divergence symmetry of \(L\) is a symmetry of the Euler–Lagrange operator (i.e., \(L_{J^2 u} \delta L = 0\)) as it follows from the master identity

\[
\delta (L_{J^r u} L) = L_{J^2 u} \delta L
\]

(7)

and the equality

\[
\delta (L_{J^r u} L) = \delta (d_H \sigma) = 0.
\]

(8)

We aim at extending the first variational formula (5) to the above mentioned generalized symmetries (see the formula (26) below) and obtaining the corresponding conservation laws (29). Herewith, the equality (31), similar to (8), gives an equation for divergence symmetries of a Lagrangian \(L\).
It should be emphasized the following.

(i) Infinitesimal generalized symmetries, called generalized vector fields, take the local coordinate form

\[ u = u^\lambda(x^\mu, y^I_j, y^J_\Lambda) \partial_\lambda + u^i(x^\mu, y^I_j, y^J_\Lambda) \partial_i, \]

where \( u^\lambda, u^i \) are local functions on some finite order jet manifold \( J^kY \). Their \( r \)-order jet prolongation \( J^r u \) is given by the formula (4). However, the generalized vector fields (9) fail to be vector fields on a finite order jet manifold. Namely, \( J^r u \) is a derivation of the ring \( C^\infty(J^rY) \) of smooth real functions on the jet manifold \( J^rY \) which takes its values in the ring of smooth real functions on the \((r+k)\)-order jet manifold \( J^{r+k}Y \). In [14], generalized vector fields (9) are locally introduced as formal differential operators. In [6], they are associated to sections of the pull-back bundle \( TY \times_Y J^kY \rightarrow J^kY \). We describe generalized symmetries in the framework of infinite order jet formalism.

(ii) The invariance of a Lagrangian under the transformations (9) imposes rather strong conditions on these transformations. Therefore, one allows generalized symmetries to be the divergence symmetries of a Lagrangian.

Infinite order jet formalism provides a convenient tool for studying Lagrangian systems of unspecified finite order [8, 11, 13, 18, 19].

Finite order jet manifolds make up the inverse system

\[ X \leftarrow Y \leftarrow J^1Y \cdots \leftarrow J^{r-1}Y \leftarrow J^rY \leftarrow \cdots. \]

Its projective limit \( J^\infty Y \), called the infinite order jet space, is endowed with the weakest topology such that surjections \( \pi^\infty_r : J^\infty Y \rightarrow J^rY \) are continuous. This topology makes \( J^\infty Y \) into a paracompact Fréchet manifold [20]. A bundle coordinate atlas \( \{U_Y, (x^\lambda, y^i)\} \) of \( Y \rightarrow X \) yields the manifold coordinate atlas

\[ \{(\pi^\infty_0)^{-1}(U_Y), (x^\lambda, y^I_\Lambda)\}, \quad y^i_{\lambda+i,\Lambda} = \frac{\partial x^\mu}{\partial x^{\mu\lambda}} d_\lambda y^i_\Lambda, \quad 0 \leq |\Lambda|, \]

of \( J^\infty Y \), where \( d_\lambda \) are the total derivatives (2).

Let \( \mathcal{O}_r^* \) denote the graded differential algebra of exterior forms on the jet manifold \( J^rY \). With the inverse system (10), we have the direct system of \( C^\infty(X) \)-modules

\[ \mathcal{O}^*(X) \xrightarrow{\pi^1_*} \mathcal{O}^*(Y) \xrightarrow{\pi^0_*} \mathcal{O}_1^* \cdots \xrightarrow{\pi^{r-1}_*} \mathcal{O}_r^* \rightarrow \cdots \]

where \( \pi^{r+1}_* \) are the pull-back monomorphisms. Its direct limit \( \mathcal{O}_\infty^* \) is a graded differential algebra, whose \( d \)-cohomology is proved to be isomorphic to the de Rham cohomology \( H^*(Y) \) of a fibre bundle \( Y \) [2].

Though \( J^\infty Y \) is not a smooth manifold, one can think of elements of \( \mathcal{O}_\infty^* \) as being objects on \( J^\infty Y \) as follows. Let \( \mathcal{O}_r^* \) be the sheaf of germs of exterior forms on the \( r \)-order jet manifold \( J^rY \), and let \( \mathcal{O}_r^* \) be its canonical presheaf. There is the direct system of presheaves

\[ \mathcal{O}_X^* \xrightarrow{\pi^*_r} \mathcal{O}_0^* \xrightarrow{\pi^1_*} \mathcal{O}_1^* \cdots \xrightarrow{\pi^{r-1}_*} \mathcal{O}_r^* \rightarrow \cdots. \]
Its direct limit $\Sigma^*_\infty$ is a presheaf of graded differential algebras on $J^\infty Y$. Let $\Sigma^*_\infty$ be a sheaf constructed from $\Sigma^*_\infty$. The algebra $Q^*_\infty$ of sections of $\Sigma^*_\infty$ is a graded differential algebra whose elements $\varphi$ possess the following property. For any point $p \in J^\infty Y$, there exist an open neighbourhood $U$ of $p$ and an exterior form $\varphi^{(k)}$ on some finite order jet manifold $J^k Y$ such that $\varphi|_U = \varphi^{(k)} \circ \pi_k|_U$. We agree to call elements of $Q^*_\infty$ the exterior forms of locally finite jet order on $J^\infty Y$. There is the natural monomorphism $O^*_\infty \to Q^*_\infty$ whose image consists of the pull-back onto $J^\infty Y$ of exterior forms on finite order jet manifolds.

Restricted to a coordinate chart (11), elements of $O^*_\infty$ can be written in a coordinate form, where horizontal forms $\{dx^\lambda\}$ and contact 1-forms $\{\theta^\lambda = dy^i - y^i_{\lambda + \Lambda} dx^\lambda\}$ make up local generators of the algebra $O^*_\infty$. There is the canonical decomposition

$$O^*_\infty = \bigoplus_{k,s} O^{k,s}_\infty, \quad 0 \leq k, \quad 0 \leq s \leq n,$$

of $O^*_\infty$ into $O^0_\infty$-modules $O^{k,s}_\infty$ of $k$-contact and $s$-horizontal forms together with the corresponding projections

$$h_k : O^*_\infty \to O^{k,*}_\infty, \quad 0 \leq k, \quad h^s : O^*_\infty \to O^{*,s}_\infty, \quad 0 \leq s \leq n.$$

Accordingly, the exterior differential on $O^*_\infty$ is split into the sum $d = d_H + d_V$ of horizontal and vertical differentials

$$d_H \circ h_k = h_k \circ d \circ h_k, \quad d_H(\varphi) = dx^\lambda \wedge d_\lambda(\varphi),$$

$$d_V \circ h^s = h^s \circ d \circ h^s, \quad d_V(\varphi) = \theta^i_\lambda \wedge \partial^i_\lambda(\varphi), \quad \varphi \in O^*_\infty.$$

In particular, we have the relations

$$d_H \circ h_0 = h_0 \circ d, \quad d_H(dx^\mu) = 0, \quad d_H(\theta^i_\lambda) = dx^\lambda \wedge \theta^i_{\lambda + \Lambda}.$$

Furthermore, one defines the projection $\mathbb{R}$-module endomorphism

$$\varrho = \sum_{k>0} \frac{1}{k^2} h_k \circ h^n, \quad \varrho(\varphi) = \sum_{|\lambda| \geq 0} (-1)^{|\lambda|} \partial^\lambda_i [d_\lambda(\partial^\lambda_i \varphi)], \quad \varphi \in O^{0,n}_\infty, \quad (13)$$

of $O^*_\infty$ such that $\varrho \circ d_H = 0$ and $\varrho \circ d \circ \varrho - \varrho \circ d = 0$ (e.g., [4, 8, 21]). Put $E_k = \varrho(O^{k,n}_\infty)$, $k > 0$. Then, the variational operator on $O^{*\lambda}_\infty$ is defined as the morphism $\delta = \varrho \circ d$. It is nilpotent, and obeys the relation $\delta \circ \varrho - \varrho \circ d = 0$. As a consequence, the graded differential algebra $O^*_\infty$ is split into the so called variational bicomplex. Here, we are concerned only with its subcomplexes

$$0 \to \mathbb{R} \to O_\infty^0 \xrightarrow{d_H} O_\infty^{0,1} \xrightarrow{d_H} \cdots \xrightarrow{d_H} O_\infty^{0,n} \xrightarrow{\delta} E_1 \xrightarrow{\delta} E_2 \to \cdots, \quad (14)$$

$$0 \to O_\infty^{1,0} \xrightarrow{d_H} O_\infty^{1,1} \xrightarrow{d_H} \cdots \xrightarrow{d_H} O_\infty^{1,n} \xrightarrow{\varrho} E_1 \to 0. \quad (15)$$

The first one, called the variational complex, provides the algebraic approach to the calculus of variations in the class of exterior forms of finite jet order. Namely, one can think of an
element \( L \in \mathcal{O}_{\infty}^{0,n} \) as being a finite order Lagrangian, while the variational operator \( \delta \) acting on \( \mathcal{O}_{\infty}^{0,n} \) is the Euler–Lagrange operator

\[
\delta L = \delta_i \mathcal{L} \theta^i \wedge \omega = \sum_{|\Lambda| \geq 0} (-1)^{|\Lambda|} d_{\Lambda} \partial_\Lambda^i \mathcal{L} \theta^i \wedge \omega. \tag{16}
\]

The key point is that the paracompact space \( J^{\infty}Y \) admits a partition of unity by functions \( f \in \mathcal{Q}_{\infty}^0 \) of locally finite jet order [20]. It follows that the sheaves of \( \mathcal{Q}_{\infty}^0 \)-modules on \( J^{\infty}Y \) are fine and acyclic. Therefore, the abstract de Rham theorem on cohomology of a sheaf resolution enables one to obtain the cohomology of the grade \( d \) differential algebra \( \mathcal{Q}_{\infty}^* \)[1, 2, 20]. Furthermore, the \( d \)-, \( d_H \)- and \( \delta \)-cohomology of its subalgebra \( \mathcal{O}_{\infty}^* \) is proved to coincide with that of \( \mathcal{Q}_{\infty}^* \)[9, 17, 18]. As a consequence, one can show that cohomology of the variational complex (14) is isomorphic to the de Rham cohomology of a fibre bundle \( Y \), i.e.,

\[
H^{k<n}(dH; \mathcal{O}_{\infty}^*) = H^{k<n}(Y), \quad H^{k-n}(\delta; \mathcal{O}_{\infty}^*) = H^{k>n}(Y)
\]

[1, 9, 17, 18]. It follows that, in particular, any \( \delta \)-closed form \( L \in \mathcal{O}_{\infty}^{0,n} \) is split into the sum

\[
L = h_0 \varphi + d_H \xi, \quad \xi \in \mathcal{O}_{\infty}^{0,n-1}, \tag{17}
\]

where \( \varphi \) is a closed \( n \)-form on \( Y \). In other words, a finite order Lagrangian \( L \) is variationally trivial iff it takes the form (17).

Similarly, one can show that the complex (15) is exact [9, 17, 18, 20]. Its exactness at the term \( \mathcal{O}_{\infty}^{1,n} \) implies that, if \( g(\phi) = 0, \phi \in \mathcal{O}_{\infty}^{1,n} \), then \( \phi = d_H \xi, \xi \in \mathcal{O}_{\infty}^{1,n-1} \). Since \( g \) is a projection operator, there is the \( \mathbb{R} \)-module decomposition

\[
\mathcal{O}_{\infty}^{1,n} = E \oplus d_H(\mathcal{O}_{\infty}^{1,n-1}). \tag{18}
\]

Given a Lagrangian \( L \in \mathcal{O}_{\infty}^{0,n} \), the decomposition (18) provides its splitting

\[
dL = g(dL) + (1 - g)(dL) = \delta L - d_H(\Xi), \tag{19}
\]

where \( \Xi \in \mathcal{O}_{\infty}^{1,n-1} \). This splitting leads to the first variational formula as follows.

Let us consider derivations \( \upsilon \in \mathfrak{d}\mathcal{O}_{\infty}^0 \) of the ring \( \mathcal{O}_{\infty}^0 \) of smooth functions of finite jet order on \( J^{\infty}Y \). With respect to the atlas (11), they are given by the coordinate expression

\[
\upsilon = \upsilon^\lambda \partial_\lambda + \upsilon^i \partial_i + \sum_{|\Lambda| > 0} \upsilon^i_\Lambda \partial_\Lambda^i, \tag{20}
\]

where components \( \upsilon^\lambda, \upsilon^i, \upsilon^i_\Lambda \) are local smooth functions of finite jet order on \( J^{\infty}Y \) which obey the transformation law

\[
\upsilon^\lambda = \frac{\partial x^\lambda}{\partial x^\mu} \upsilon^\mu, \quad \upsilon^i = \frac{\partial y^\mu}{\partial y^i} \upsilon^\lambda + \frac{\partial y^\mu}{\partial x^\mu} \upsilon^i, \quad \upsilon^i_\Lambda = \sum_{|\Sigma| \leq |\Lambda|} \frac{\partial y^\mu_\Sigma}{\partial x^\mu} \upsilon^i_\Sigma + \frac{\partial y^\mu}{\partial x^\mu} \upsilon^i. \]
The interior product $\nu|\phi$, and the Lie derivative $L_\nu\phi$, $\phi \in O^*_\infty$ are defined in a standard way. An element $\nu \in \partial O^0_\infty$ is said to be a generalized symmetry if the Lie derivative of any contact one-form $\theta \in O^{1,0}_\infty$ along $\nu$ again is a contact form. One can easily justify that $\nu \in \partial O^0_\infty$ is a generalized symmetry iff it is given by the coordinate expression (20) where

$$v^i_\lambda = d_\lambda (v^i - y^i_\mu v^\mu), \quad 0 < |\Lambda|.$$  

For instance, let $\tau$ be a vector field on $X$. Then, the derivation $\tau|((d_H f), f \in O^0_\infty$ of the ring $O^0_\infty$ is a generalized symmetry

$$J^\infty \tau = \tau^\mu d_\mu.$$  

Moreover, any generalized symmetry $\nu$ is brought into the form

$$\nu = \nu_H + \nu_V = \nu^\lambda d_\lambda + (\nu^i \partial_i + \sum \nu^\lambda \partial_i^\lambda), \quad \nu^i = v^i - y^i_\mu v^\mu, \quad \nu^\lambda = d_\lambda \nu^i.$$  

This is the horizontal splitting of $\nu$ with respect to the canonical connection $\nabla = dx^\lambda \otimes d_\lambda$ on the $C^\infty(X)$-ring $O^0_\infty$ [13]. In particular, if $\nu^\lambda = 0$, we have the relation

$$\nu|d_H \phi = -d_H (\nu|\phi), \quad \phi \in O^*_\infty.$$  

Let us consider the Lie derivative

$$L_\nu L = \nu|dL + (d(\nu|L)) = \nu_V|dL + d_H (\nu_H|L) + Ld_V (\nu_H|\omega)$$  

of a Lagrangian $L \in O^0_\infty$ along a generalized symmetry $\nu$ (23). Using the splitting (19) and the equality (24), we come to the desired first variational formula

$$L_\nu L = \nu_V|\delta L - \nu_V|d_H \Xi + d_H (\nu_H|L) + Ld_V (\nu_H|\omega) =$$

$$\nu_V|\delta L + d_H (\nu_V|\Xi + \nu_H|L) + Ld_V (\nu_H|\omega) =$$

$$\nu_V|\delta L + d_H (h_0 (\nu|\Xi_L)) + Ld_V (\nu_H|\omega),$$  

where $\Xi_L$ is some Poincaré–Cartan form of a finite order Lagrangian $L$.

Let a generalized symmetry $\nu$ (23) be a divergence symmetry of a Lagrangian $L$, i.e.,

$$L_\nu L = d_H \sigma, \quad \sigma \in O^{0,n-1}_\infty.$$  

By virtue of the expression (25), this condition implies that a generalized symmetry $\nu$ is projected onto $X$, i.e., its components $\nu^\lambda$ depend only on coordinates on $X$. Then, the first variational formula (26) takes the form

$$d_H \sigma = \nu_V|\delta L + d_H (h_0 (\nu|\Xi_L)).$$  

Restricted to $\text{Ker} \delta L$, it leads to the generalized Noether conservation law

$$0 \approx d_H (h_0 (\nu|\Xi_L) - \sigma).$$
A glance at the expression (25) shows that a generalized symmetry \( \nu \) (23) projected onto \( X \) is a divergence symmetry of a Lagrangian \( L \) iff its vertical part \( \nu_V \) is so. Moreover, \( \nu \) and \( \nu_V \) lead to the same generalized Noether conservation law (29). For instance, if \( \nu = J^\infty \tau \) (22), the first variational formula (26) and the conservation law (29) become tautological.

Note that a Poincaré–Cartan form \( \Xi_L \) of an \( r \)-order Lagrangian \( L = L_\omega \) fails to be uniquely defined unless \( r = 1 \) or \( \dim X = 1 \). It is given by the coordinate expression

\[
\Xi_L = L_\omega + \sum_{s=0}^{r-1} F^\lambda_\mu_\nu \ldots \mu_1 \theta^i_{\mu_s \ldots \mu_1} \wedge \omega_\lambda, \quad \omega_\lambda = \partial_\lambda |\omega|, \quad (30)
\]

where the functions \( c^k_{i \ldots} \), of jet order at most \( 2r-k-1 \), satisfy the condition \( c^k_{i (\nu_k \nu_{k-1}) \ldots \nu_1} = 0 \) and \( c^k_{i} = 0 \) [10, 12]. Any Poincaré–Cartan form (30) can be locally brought into the form where all the functions \( c^k_{i} \) equal zero. This local expression is used in [14]. It is globally valid if either \( L \) is of first order (see [6]) or \( \dim X = 1 \), i.e., in the higher order mechanics.

One can obtain the characteristic equation for divergence symmetries of a Lagrangian \( L \) as follows. Let a generalized symmetry \( \nu \) (23) be projected onto \( X \). Then, the Lie derivative \( L_\nu L \) (25) is a horizontal density. Let us require that it is a \( \delta \)-closed form, i.e.,

\[
\delta(L_\nu L) = 0. \quad (31)
\]

In accordance with the equality (17), this condition is fulfilled iff

\[
L_\nu L = h_0 \varphi + d_\sigma \sigma,
\]

where \( \varphi \) is a closed form on \( Y \). It follows that \( \nu \) is a divergence symmetry of \( L \) at least locally. Thus, the equation (31) enables one to find all divergence symmetries of a given Lagrangian \( L \). Note that the master identity (7) fails to be true for generalized symmetries. There is the local relation

\[
\delta(L_\nu L) = L_\nu \delta L + \sum_{|\Lambda| > 0} (-1)^{|\Lambda|} d_\lambda (\partial^\lambda_\nu \nu \delta_\lambda L dy^k) \wedge \omega
\]

used in [14].

References

[1] Anderson I and Duchamp T 1980 Amer. J. Math. 102 781

[2] Anderson I 1992 Contemp. Math. 132 51

[3] Anderson I, Kamran N and Olver P 1993 Adv. Math. 100 53

[4] Bauderon M 1985 Differential Geometry, Calculus of Variations, and their Applications, Lecture Notes in Pure and Applied Mathematics 100 (New York: Marcel Dekker) 67
[5] Fatibene L, Ferraris M and Francaviglia M 1999 *Gen. Rel. Grav.* **31** 1115

[6] Fatibene L, Ferraris M, Francaviglia M and McLenaghan R 2002 *J. Math. Phys.* **43** 3147

[7] Fulp R, Lada T and Stasheff J 2002 *E-print arXiv.* math.QA/0204079

[8] Giachetta G, Mangiarotti L and Sardanashvily G 1997 *New Lagrangian and Hamiltonian Methods in Field Theory* (Singapore: World Scientific)

[9] Giachetta G, Mangiarotti L and Sardanashvily G 2001 *J. Math. Phys.* **42** 4272

[10] Gotay M 1991 *Mechanics, Analysis and Geometry: 200 Years after Lagrange* (Amsterdam: North Holland) 203

[11] Krasil’shchik I, Lychagin V and Vinogradov A 1985 *Geometry of Jet Spaces and Nonlinear Partial Differential Equations* (New York: Gordon and Breach)

[12] Krupka D 1984 *Proceedings of the Conference on Differential Geometry and Applications, Nova Mesto in Morava, Czechoslovakia, 1983* (Brno: J. E. Purkyne Univ.) 167; *E-print arXiv.* math-ph/0203039

[13] Mangiarotti L and Sardanashvily G 2000 *Connections in Classical and Quantum Field Theory* (Singapore: World Scientific)

[14] Olver P 1998 *Applications of Lie Groups to Differential Equations* (Berlin: Springer)

[15] Sardanashvily G 1998 *J. Math. Phys.* **39** 4874

[16] Sardanashvily G 2001 *Mod. Phys. Lett. A* **16** 1531

[17] Sardanashvily G 2002 *Int. J. Math. Math. Sci.* **30** 39

[18] Sardanashvily G 2002 *E-print arXiv.* math-ph/0203040

[19] Saunders D 1989 *The Geometry of Jet Bundles* (Cambridge: Cambridge Univ. Press)

[20] Takens F 1979 *J. Diff. Geom.* **14** 543

[21] Tulczyjew W 1980 *Differential Geometric Methods in Mathematical Physics*, Lect. Notes in Mathematics **836** (Berlin: Springer) 22