

Integrability Properties of Functions with a Given Behavior of Distribution Functions and Some Applications

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Received October 10, 2018; revised November 1, 2018; accepted November 5, 2018

Abstract—We establish that if the distribution function of a measurable function \( v \) defined on a bounded domain \( \Omega \subset \mathbb{R}^n \) \((n \geq 2)\) satisfies, for sufficiently large \( k \), the estimate 
\[ \text{meas}\{ |v| > k \} \leq k^{-\alpha} \varphi(k)/\psi(k), \]
where \( \alpha > 0, \varphi : [1, +\infty) \to \mathbb{R} \) is a nonnegative nonincreasing measurable function such that the integral of the function \( s \to \varphi(s)/s \) over \([1, +\infty)\) is finite, and \( \psi : [0, +\infty) \to \mathbb{R} \) is a positive continuous function with some additional properties, then 
\[ |v|^\alpha \psi(|v|) \in L^1(\Omega). \]
In so doing, the function \( \psi \) can be either bounded or unbounded. We give corollaries of the corresponding theorems for some specific ratios of the functions \( \varphi \) and \( \psi \).

In particular, we consider the case where the distribution function of a measurable function \( v \) satisfies, for sufficiently large \( k \), the estimate 
\[ \text{meas}\{ |v| > k \} \leq Ck^{-\alpha} (\ln k)^{-\beta} \]
with \( C, \alpha > 0 \) and \( \beta \geq 0 \). In this case, we strengthen our previous result for \( \beta > 1 \) and, on the whole, we show how the integrability properties of the function \( v \) differ depending on which interval, \([0, 1]\) or \((1, +\infty)\), contains \( \beta \). We also consider the case where the distribution function of a measurable function \( v \) satisfies, for sufficiently large \( k \), the estimate 
\[ \text{meas}\{ |v| > k \} \leq Ck^{-\alpha} (\ln \ln k)^{-\beta} \]
with \( C, \alpha > 0 \) and \( \beta \geq 0 \). We give examples showing the accuracy of the obtained results in the corresponding scales of classes close to \( L^\alpha(\Omega) \). Finally, we give applications of these results to entropy and weak solutions of the Dirichlet problem for second-order nonlinear elliptic equations with right-hand side in some classes close to \( L^1(\Omega) \) and defined by the logarithmic function or its double composition.

Keywords: integrability, distribution function, nonlinear elliptic equations, right-hand side in classes close to \( L^1 \), Dirichlet problem, weak solution, entropy solution.

DOI: 10.1134/S0081543820020091

INTRODUCTION

By definition (see, for instance, [1, 2]), the distribution function of a measurable function \( v \) defined on a bounded domain \( \Omega \subset \mathbb{R}^n \) \((n \geq 2)\) is the correspondence \( s \to \text{meas}\{ |v| > s \}, s \geq 0 \).

An estimate of the values of the distribution function makes it possible to establish a certain integrability on \( \Omega \) of the original function or a function depending on it. The study of this question is of interest, in particular, to clarify the integrability properties of solutions of elliptic equations and variational inequalities with right-hand side in the space \( L^1(\Omega) \) or classes close to \( L^1(\Omega) \). For instance, it was shown in [2] that, for \( k > 0 \), the distribution function of the entropy solution \( u \) of

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the Dirichlet problem for a second-order nonlinear elliptic equation with right-hand side \( f \in L^1(\Omega) \) satisfies an estimate of the form
\[
\text{meas}\{|u| > k\} \leq Ck^{-\alpha}
\]
with positive constants \( C \) and \( \alpha \), the first of which depends on the space dimension \( n \), an exponent \( p \) characterizing the growth of the coefficients of the equation, and the norm of the function \( f \) in \( L^1(\Omega) \) and the second depends only on \( n \) and \( p \). A similar estimate was also established for the gradient of the entropy solution. The obtained estimates imply the integrability of certain powers of the moduli of the entropy solution and its gradient. In particular, it follows from the estimate (0.1) that \( u \in L^\lambda(\Omega) \) for any \( \lambda \in (0, \alpha) \). In addition, in the case \( p > 2 - 1/n \), the established estimates provide the belonging of the entropy solution to the Sobolev spaces with any exponent less than a limit one.

If, for sufficiently large \( k \), the distribution function of a measurable function \( v : \Omega \to \mathbb{R} \) satisfies the estimate
\[
\text{meas}\{|v| > k\} \leq Ck^{-\alpha}(\ln k)^{-\beta},
\]
where \( C, \alpha > 0 \) and \( \beta > 1 \), or the more general estimate
\[
\text{meas}\{|v| > k\} \leq k^{-\alpha}\varphi(k),
\]
where \( \alpha > 0 \) and \( \varphi : [1, +\infty) \to \mathbb{R} \) is a nonnegative nonincreasing measurable function such that the integral of the function \( s \to \varphi(s)/s \) over \([1, +\infty)\) is finite, then, as shown, for instance, in [3, 4], the function \( v \) belongs to \( L^\alpha(\Omega) \). Using these results, in the mentioned papers, we established conditions on the right-hand side of a second-order nonlinear elliptic equation under which the entropy solution of the corresponding Dirichlet problem and its gradient belong to some limit Lebesgue spaces.

Based on the result proved in [4] that a measurable function \( v \) belongs to the space \( L^\alpha(\Omega) \) if its distribution function satisfies the estimate (0.3) with the above function \( \varphi \), in the present paper, we study the case where, for sufficiently large \( k \), the distribution function of a measurable function \( v \) satisfies the estimate
\[
\text{meas}\{|v| > k\} \leq \frac{\varphi(k)}{k^{\alpha}\psi(k)}
\]
with \( \alpha > 0 \), a function \( \varphi \) as above, and a positive continuous function \( \psi \) on \([0, +\infty)\) having some additional properties. It turns out that, in this case, the inclusion \(|v|^\alpha \psi(|v|) \in L^1(\Omega)\) holds, and the function \( \psi \) can be either bounded or unbounded. The corresponding results are proved in Section 1 (see Theorems 1 and 2). Consequences of these general results for some specific ratios of the functions \( \varphi \) and \( \psi \) in the estimate (0.4) are given in Section 2. In particular, we consider the case where the distribution function of a measurable function \( v \) satisfies the estimate (0.2) with \( C, \alpha > 0 \) and \( \beta \geq 0 \). In so doing, we strengthen the result obtained in [3] for \( \beta > 1 \) and, on the whole, show how the integrability properties of the function \( v \) differ depending on which interval, \([0, 1]\) or \((1, +\infty)\), contains \( \beta \) (see Corollaries 1 and 2). We also consider the case where the distribution function of a measurable function \( v \) satisfies the estimate \( \text{meas}\{|v| > k\} \leq Ck^{-\alpha}(\ln \ln k)^{-\beta} \) with \( C, \alpha > 0 \) and \( \beta \geq 0 \) (see Corollaries 3 and 4). In Section 3, we give examples showing the accuracy of the results of the previous section in the corresponding scales of classes close to \( L^\alpha(\Omega) \). The closeness of a class of functions \( K \) to the space \( L^\alpha(\Omega) \) is understood as the validity of one of the following conditions: (i) \( K \subset L^\alpha(\Omega) \) and \( K \not\subset L^{\alpha+\epsilon}(\Omega) \) for any \( \epsilon > 0 \); (ii) \( K \not\subset L^\alpha(\Omega) \) and \( K \subset L^{\alpha-\epsilon}(\Omega) \) for any \( \epsilon \in (0, \alpha) \). Finally, in Section 4, we give applications of the results of Section 2
to entropy and weak solutions of the Dirichlet problem for second-order nonlinear elliptic equations 
with right-hand side in some classes close to $L^1(\Omega)$ and defined by the logarithmic function or 
its double composition. As a result, we strengthen known and obtain new results on the integrability 
properties of the moduli of the specified solutions.

1. GENERAL THEOREMS

Let $n \in \mathbb{N}$, $n \geq 2$, and let $\Omega$ be a bounded open set in $\mathbb{R}^n$.

We give a proposition on which the subsequent theorems are based.

**Proposition 1.** Let $v : \Omega \to \mathbb{R}$ be a measurable function, and let $\varphi : [1, +\infty) \to \mathbb{R}$ be a 
nonnegative nonincreasing measurable function. Let $\alpha > 0$ and $k_0 \geq 1$. Assume that the following 
conditions are satisfied:

(a) $\int_{1}^{+\infty} \frac{\varphi(s)}{s} ds < +\infty$;

(b) $\forall k \geq k_0$ $\text{meas}\{|v| > k\} \leq k^{-\alpha}\varphi(k)$.

Then $v \in L^\alpha(\Omega)$.

Essentially, the stated proposition coincides with Lemma 2.1 in [4] and does not require a 
separate proof.

We pass to the statement and proof of theorems.

**Theorem 1.** Let $v : \Omega \to \mathbb{R}$ be a measurable function, let $\varphi : [1, +\infty) \to \mathbb{R}$ be a nonnegative nonincreasing 
measurable function, and let $\psi : [0, +\infty) \to \mathbb{R}$ be a positive continuous function. Let $\alpha > 0$, $k_0 \geq 1$, and $\sigma > 0$. Assume that the following conditions are satisfied:

(a) $\int_{1}^{+\infty} \frac{\varphi(s)}{s} ds < +\infty$;

(b) $s^\alpha \psi(s) \to +\infty$ as $s \to +\infty$;

(c) if $k_0 \leq s \leq t$, then $s^\alpha \psi(s) \leq t^\alpha \psi(t)$;

(d) if $s > k_0$ and $t = s[\psi(s)]^{1/\alpha} \geq 1$, then $\varphi(s) \leq \sigma \varphi(t)$;

(e) $\forall k \geq k_0$ $\text{meas}\{|v| > k\} \leq \frac{\varphi(k)}{k^\alpha \psi(k)}$.

Then $|v|^\alpha \psi(|v|) \in L^1(\Omega)$.

**Proof.** Let $w : \Omega \to \mathbb{R}$ be a function such that

$$w(x) = \begin{cases} 
  k_0[\psi(k_0)]^{1/\alpha} & \text{if } |v(x)| \leq k_0, \\
  |v(x)||\psi(|v(x)|)]^{1/\alpha} & \text{if } |v(x)| > k_0.
\end{cases}$$

It is clear that the function $w$ is measurable. We show that $w \in L^\alpha(\Omega)$. Define $k_* = 1 + k_0[\psi(k_0)]^{1/\alpha}$ and fix $k \geq k_*$. It is easy to see that

$$k_0^\alpha \psi(k_0) < k^\alpha. \quad (1.1)$$

By condition (b), there exists $k_1 > k_0$ such that

$$k_1^\alpha \psi(k_1) > k^\alpha. \quad (1.2)$$

In view of the continuity of the function $\psi$ and inequalities (1.1) and (1.2), there exists $k_2 \in (k_0, k_1)$ such that

$$k_2^\alpha \psi(k_2) = k^\alpha. \quad (1.3)$$
Then, by condition (d), we have
\[ \varphi(k_2) \leq \sigma \varphi(k). \] (1.4)

Next, let \( x \in \{|w| > k\} \). Hence,
\[ w(x) > k. \] (1.5)

Therefore, it follows from the definition of the function \( w \) and the inequality \( k \geq k_\ast \) that \( |v(x)| > k_0 \).

Then, by the definition of the function \( w \), we have \( w(x) = |v(x)|[\psi(|v(x)|)]^{1/\alpha} \). This and (1.5) imply the inequality
\[ |v(x)|^\alpha \psi(|v(x)|) > k^\alpha. \] (1.6)

Assume that \( |v(x)| \leq k_2 \). Then, by condition (c), we have \( |v(x)|^\alpha \psi(|v(x)|) \leq k_2^\alpha \psi(k_2) \). Therefore, taking into account (1.3), we obtain the inequality \( |v(x)|^\alpha \psi(|v(x)|) \leq k^\alpha \), which contradicts (1.6).

The obtained contradiction proves that \( |v(x)| > k_2 \). Hence, \( x \in \{|v| > k_2\} \). Thus, \( \{|w| > k\} \subset \{|v| > k_2\} \). Then
\[ \text{meas}\{|w| > k\} \leq \text{meas}\{|v| > k_2\}. \] (1.7)

Using condition (e), equality (1.3), and inequality (1.4), we obtain
\[ \text{meas}\{|v| > k_2\} \leq \frac{\varphi(k_2)}{k_2^\alpha \psi(k_2)} = \frac{\varphi(k_2)}{k^\alpha} \leq \frac{\sigma \varphi(k)}{k^\alpha}. \]

This and (1.7) imply that
\[ \text{meas}\{|w| > k\} \leq \sigma k^{-\alpha} \varphi(k). \] (1.8)

Thus, for any \( k \geq k_\ast \), inequality (1.8) holds. Now, taking into account condition (a), we deduce from Proposition 1 that \( w \in L^\alpha(\Omega) \). Therefore, \( |v| \psi(|v|) \in L^1(\Omega) \). □

The difference between the statement of the following theorem and the statement of Theorem 1 is that condition (d) of the first theorem is replaced by the requirement of boundedness of the function \( \psi \).

Theorem 2. Let \( v : \Omega \to \mathbb{R} \) be a measurable function, let \( \varphi : [1, +\infty) \to \mathbb{R} \) be a nonnegative nonincreasing measurable function, and let \( \psi : [0, +\infty) \to \mathbb{R} \) be a positive bounded continuous function. Let \( \alpha > 0 \) and \( k_0 \geq 1 \). Assume that the following conditions are satisfied:

(a) \( \int_1^{+\infty} \frac{\varphi(s)}{s} ds < +\infty; \)
(b) \( s^\alpha \psi(s) \to +\infty \) as \( s \to +\infty; \)
(c) if \( k_0 \leq s \leq t \), then \( s^\alpha \psi(s) \leq t^\alpha \psi(t); \)
(d) \( \forall k \geq k_0 \) \( \text{meas}\{|v| > k\} \leq \frac{\varphi(k)}{k^\alpha \psi(k)}. \)

Then \( |v| \psi(|v|) \in L^1(\Omega) \).

Proof. By the boundedness of the function \( \psi \), there exists \( c > 0 \) such that
\[ \forall s \in [0, +\infty) \quad \psi(s) \leq c. \] (1.9)

We define
\[ \varphi_1 = \frac{1}{c} \varphi, \quad \psi_1 = \frac{1}{c} \psi. \]

It is clear that the function \( \varphi_1 \) is nonnegative, nonincreasing, and measurable and the function \( \psi_1 \) is positive and continuous. By condition (a), we have
and, in view of conditions (b) and (c), the following assertions hold: \( s^\alpha \psi_1(s) \to +\infty \) as \( s \to +\infty \); if \( k_0 \leq s \leq t \), then \( s^\alpha \psi_1(s) \leq t^\alpha \psi_1(t) \). Next, let \( s > k_0 \) and \( t = s[\psi_1(s)]^{1/\alpha} \geq 1 \). By (1.9), we have \( t \leq s \). This and the fact that the function \( \varphi_1 \) is nonincreasing imply the inequality \( \varphi_1(s) \leq \varphi_1(t) \). Finally, by condition (d), for any \( k \geq k_0 \), we have
\[
\text{meas}\{|v| > k\} \leq \frac{\varphi_1(k)}{k^\alpha \psi_1(k)}.
\]

We now deduce from Theorem 1 that \( |v|^\alpha \psi_1(|v|) \in L^1(\Omega) \). Therefore, \( |v|^\alpha \psi(|v|) \in L^1(\Omega) \). 

2. COROLLARIES

We give corollaries of the general Theorems 1 and 2 for some specific ratios of the functions \( \varphi \) and \( \psi \).

**Corollary 1.** Let \( v : \Omega \to \mathbb{R} \) be a measurable function. Let \( C > 0, \alpha > 0, \) and \( \beta \in [0, 1] \). Assume that
\[
\forall k \geq e \quad \text{meas}\{|v| > k\} \leq Ck^{-\alpha}(\ln k)^{-\beta}.
\] (2.1)
Then, for any \( \gamma > 1 - \beta \), we have \( |v|^\alpha \psi_1\left(|v|\right)\frac{1}{\gamma} \in L^1(\Omega) \).

**Proof.** We fix \( \gamma > 1 - \beta \), and let \( \varphi : [1, +\infty) \to \mathbb{R} \) be the function such that
\[
\varphi(s) = \begin{cases} C & \text{if } 1 \leq s < e, \\ C(\ln s)^{-\beta-\gamma} & \text{if } s \geq e. \end{cases}
\]
It is clear that the function \( \varphi \) is nonnegative, nonincreasing, and measurable.

Let \( \psi : [0, +\infty) \to \mathbb{R} \) be the function such that
\[
\psi(s) = \begin{cases} 1 & \text{if } 0 \leq s < e, \\ (\ln s)^{-\gamma} & \text{if } s \geq e. \end{cases}
\]
It is clear that the function \( \psi \) is positive, bounded, and continuous.

We define \( k_0 = \max\{e, e^{\gamma/\alpha}\} \) and show that conditions (a)–(d) of Theorem 2 are satisfied.

Since \( \beta + \gamma > 1 \), for an arbitrary \( N > e \), we obtain
\[
\int_e^N \frac{\varphi(s)}{s} ds = \frac{C}{1 - \beta - \gamma}(\ln s)^{1-\beta-\gamma}|N|_e \leq \frac{C}{\beta + \gamma - 1}.
\]
Hence,
\[
\int_1^{+\infty} \frac{\varphi(s)}{s} ds < +\infty.
\]
We further note that, for any $\lambda, s > 0$, the inequality $\lambda \ln s < s^\lambda$ holds. Using this, for an arbitrary $s \geq e$, we obtain $s^\alpha \psi(s) = s^\alpha (\ln s)^{-\gamma} \geq \left( \frac{\alpha}{2\gamma} \right)^\gamma s^{\alpha/2}$. Therefore, $s^\alpha \psi(s) \rightarrow +\infty$ as $s \rightarrow +\infty$.

Let $h : (1, +\infty) \rightarrow \mathbb{R}$ be the function such that $h(s) = s^\alpha (\ln s)^{-\gamma}$ for any $s \in (1, +\infty)$. We have $h'(s) = s^{\alpha-1} (\ln s)^{-\gamma-1} (\alpha \ln s - \gamma)$ for any $s \in (1, +\infty)$. This implies that $h' \geq 0$ in $[k_0, +\infty)$. Therefore, if $k_0 \leq s \leq t$, then $s^\alpha \psi(s) \leq t^\alpha \psi(t)$.

Finally, by (2.1) and the definition of the functions $\varphi$ and $\psi$, for any $k \geq k_0$, the following inequality holds:

$$\text{meas}\{|v| > k\} \leq \frac{\varphi(k)}{k^\alpha \psi(k)}.$$

Thus, conditions (a)–(d) of Theorem 2 are satisfied. Hence, by this theorem, $|v|^\alpha \psi(|v|) \in L^1(\Omega)$. Consequently, $|v|^\alpha [\ln(2 + |v|)]^{-\gamma} \in L^1(\Omega)$.

**Corollary 2.** Let $v : \Omega \rightarrow \mathbb{R}$ be a measurable function. Let $C > 0$, $\alpha > 0$, and $\beta > 1$. Assume that

$$\forall k \geq e \quad \text{meas}\{|v| > k\} \leq Ck^{-\alpha} (\ln k)^{-\beta}. \quad (2.2)$$

Then, for any $\gamma \in (0, \beta - 1)$, we have $|v|^\alpha [\ln(1 + |v|)]^{-1} \in L^1(\Omega)$.

**Proof.** We fix $\gamma \in (0, \beta - 1)$, and let $\varphi : [1, +\infty) \rightarrow \mathbb{R}$ be the function such that

$$\varphi(s) = \begin{cases} C & \text{if } 1 \leq s < e, \\ C(\ln s)^{\gamma-\beta} & \text{if } s \geq e. \end{cases}$$

It is clear that the function $\varphi$ is nonnegative, nonincreasing, and measurable.

Let $\psi : [0, +\infty) \rightarrow \mathbb{R}$ be the function such that

$$\psi(s) = \begin{cases} 1 & \text{if } 0 \leq s < e, \\ (\ln s)^\gamma & \text{if } s \geq e. \end{cases}$$

It is clear that the function $\psi$ is positive and continuous.

Define $k_0 = e$ and $\sigma = (1 + \gamma/\alpha)^{\beta-\gamma}$. Now, we note that conditions (a)–(e) of Theorem 1 are satisfied. Indeed, condition (a) of Theorem 1 is satisfied because $\beta - \gamma > 1$. The fulfillment of conditions (b) and (c) of Theorem 1 is obvious. Next, let $s > k_0$ and $t = s[\psi(s)]^{1/\alpha} \geq 1$. Consequently, $t^\alpha = s^\alpha (\ln s)^\gamma \leq s^{\alpha+\gamma}$. Then $\alpha \ln t \leq (\alpha + \gamma) \ln s$. Hence, $(\ln s)^{\gamma-\beta} \leq (1 + \gamma/\alpha)^{\beta-\gamma} (\ln t)^{\gamma-\beta}$ and $\varphi(s) \leq \sigma \varphi(t)$. Thus, condition (e) of Theorem 1 is satisfied. Finally, by (2.2) and the definition of the functions $\varphi$ and $\psi$, condition (e) of Theorem 1 is satisfied. We now deduce from this theorem that $|v|^\alpha \psi(|v|) \in L^1(\Omega)$. Consequently, $|v|^\alpha [\ln(1 + |v|)]^{-1} \in L^1(\Omega)$.

We note that if the conditions of Corollary 2 are satisfied, then, according to Lemma 2 in [3] with equivalent conditions, we have only the inclusion $v \in L^\alpha(\Omega)$. Thus, the conclusion of Corollary 2 is stronger than the conclusion of the mentioned lemma in [3].

**Corollary 3.** Let $v : \Omega \rightarrow \mathbb{R}$ be a measurable function. Let $C > 0$, $\alpha > 0$, and $\beta \in [0,1]$. Assume that

$$\forall k \geq 3 \quad \text{meas}\{|v| > k\} \leq Ck^{-\alpha} (\ln \ln k)^{-\beta}. \quad (2.3)$$

Then, for any $\gamma > 1 - \beta$, we have $|v|^\alpha [\ln(2 + |v|)]^{-1} [\ln(3 + |v|)]^{-\gamma} \in L^1(\Omega)$.
**Proof.** We fix $\gamma > 1 - \beta$, and let $\varphi : [1, +\infty) \to \mathbb{R}$ be the function such that

$$
\varphi(s) = \begin{cases} 
C(\ln 3)^{-1}(\ln \ln 3)^{-\beta - \gamma} & \text{if } 1 \leq s < 3, \\
C(\ln s)^{-1}(\ln \ln s)^{-\beta - \gamma} & \text{if } s \geq 3.
\end{cases}
$$

It is clear that the function $\varphi$ is nonnegative, nonincreasing, and measurable.

Let $\psi : [0, +\infty) \to \mathbb{R}$ be the function such that

$$
\psi(s) = \begin{cases} 
(\ln 3)^{-1}(\ln \ln 3)^{-\gamma} & \text{if } 0 \leq s < 3, \\
(\ln s)^{-1}(\ln \ln s)^{-\gamma} & \text{if } s \geq 3.
\end{cases}
$$

It is clear that the function $\psi$ is positive, bounded, and continuous.

We define $k_0 = \max\{e^x, e^{(1+\gamma)/\alpha}\}$. Taking into account (2.3), similarly to the proof of Corollary 1, we find that conditions (a)–(d) of Theorem 2 are satisfied. Therefore, by this theorem, $|\psi|^{\alpha} \psi(v) \in L^1(\Omega)$. Consequently, $|\psi|^\alpha[\ln(2 + |v|)]^{-1}[\ln(3 + |v|)]^{-\gamma} \in L^1(\Omega)$. \hfill $\square$

**Corollary 4.** Let $v : \Omega \to \mathbb{R}$ be a measurable function. Let $C > 0$, $\alpha > 0$, and $\beta > 1$. Assume that

$$
\forall k \geq 3 \quad \text{meas}\{v > k\} \leq Ck^{-\alpha}(\ln \ln k)^{-\beta}.
$$

Then, for any $\gamma \in (0, \beta - 1)$, we have $|v|^{\alpha}[\ln(2 + |v|)]^{-1}[\ln(3 + |v|)]^{\gamma} \in L^1(\Omega)$. \hfill $\square$

**Proof.** We fix $\gamma \in (0, \beta - 1)$, and let $\varphi : [1, +\infty) \to \mathbb{R}$ be the function such that

$$
\varphi(s) = \begin{cases} 
C(\ln 3)^{-1}(\ln \ln 3)^{\gamma - \beta} & \text{if } 1 \leq s < 3, \\
C(\ln s)^{-1}(\ln \ln s)^{\gamma - \beta} & \text{if } s \geq 3.
\end{cases}
$$

It is clear that the function $\varphi$ is nonnegative, nonincreasing, and measurable.

Let $\psi : [0, +\infty) \to \mathbb{R}$ be the function such that

$$
\psi(s) = \begin{cases} 
(\ln 3)^{-1}(\ln \ln 3)^{\gamma} & \text{if } 0 \leq s < 3, \\
(\ln s)^{-1}(\ln \ln s)^{\gamma} & \text{if } s \geq 3.
\end{cases}
$$

It is clear that the function $\psi$ is positive, bounded, and continuous.

We define $k_0 = \max\{e^x, e^{(1+\gamma)/\alpha}\}$. Taking into account (2.4), similarly to the proof of Corollary 1, we find that conditions (a)–(d) of Theorem 2 are satisfied. Therefore, by this theorem, $|\psi|^{\alpha} \psi(\Omega) \in L^1(\Omega)$. Consequently, $|\psi|^\alpha[\ln(2 + |v|)]^{-1}[\ln(3 + |v|)]^{\gamma} \in L^1(\Omega)$. \hfill $\square$

Further, we introduce the following classes of functions. For any $\alpha > 0$ and $\gamma \geq 0$, we define

$$
K_{1-}^{\alpha, \gamma}(\Omega) = \{ v : \Omega \to \mathbb{R} : v \text{ is measurable and } |v|^\alpha[\ln(2 + |v|)]^{-\gamma} \in L^1(\Omega) \},
$$

$$
K_{2-}^{\alpha, \gamma}(\Omega) = \{ v : \Omega \to \mathbb{R} : v \text{ is measurable and } |v|^\alpha[\ln(2 + |v|)]^{-1}[\ln(3 + |v|)]^{-\gamma} \in L^1(\Omega) \}.
$$

For any $\alpha > 0$ and $\gamma > 0$, we define

$$
K_{1+}^{\alpha, \gamma}(\Omega) = \{ v : \Omega \to \mathbb{R} : v \text{ is measurable and } |v|^\alpha[\ln(1 + |v|)]^{\gamma} \in L^1(\Omega) \},
$$

$$
K_{2+}^{\alpha, \gamma}(\Omega) = \{ v : \Omega \to \mathbb{R} : v \text{ is measurable and } |v|^\alpha[\ln(2 + |v|)]^{-1}[\ln(3 + |v|)]^{\gamma} \in L^1(\Omega) \}.
$$

For any fixed $\alpha > 0$, these classes of functions are close to the space $L^\alpha(\Omega)$ in the sense specified in the Introduction. Using them, the results of this section can be expressed as follows:
(i) if the conditions of Corollary 1 are satisfied, then \( v \in K_{1,\gamma}^{\alpha}(\Omega) \) for any \( \gamma > 1 - \beta \);
(ii) if the conditions of Corollary 2 are satisfied, then \( v \in K_{1,\gamma}^{\alpha}(\Omega) \) for any \( \gamma \in (0, \beta - 1) \);
(iii) if the conditions of Corollary 3 are satisfied, then \( v \in K_{2,\gamma}^{\alpha}(\Omega) \) for any \( \gamma > 1 - \beta \);
(iv) if the conditions of Corollary 4 are satisfied, then \( v \in K_{2,\gamma}^{\alpha}(\Omega) \) for any \( \gamma \in (0, \beta - 1) \).

3. EXAMPLES

We consider examples showing the accuracy of the results of the previous section in the corresponding scales of the classes introduced above.

**Example 1.** Let \( \Omega = \{ x \in \mathbb{R}^n : |x| < 1/e \} \), let \( \alpha > 0 \), and let \( \beta \geq 0 \). Let \( v : \Omega \to \mathbb{R} \) be the function such that

\[
v(x) = \begin{cases} |x|^{-n/\alpha} \left( \ln \frac{1}{|x|} \right)^{-\beta/\alpha} & \text{if } x \in \Omega \setminus \{0\}, \\ 0 & \text{if } x = 0. \end{cases}
\]

It is clear that the function \( v \) is measurable. We fix \( k \geq e \), and let \( x \in \{ |v| > k \} \). Obviously, \( x \in \Omega \setminus \{0\} \). By these inclusions and the definition of the function \( v \), we have

\[
|x|^{-n/\alpha} \left( \ln \frac{1}{|x|} \right)^{-\beta/\alpha} > k, \quad \ln \frac{1}{|x|} > 1.
\]

Hence, \( |x|^{-n/\alpha} > k \). Then \( \ln k < (n/\alpha) \ln(1/|x|) \). Using this inequality and the first inequality in (3.1), we find that \( |x|^n < (n/\alpha)^\beta k^{-\alpha} (\ln k)^{-\beta} \). Therefore, in view of the arbitrariness of \( x \in \{ |v| > k \} \), the set \( \{ |v| > k \} \) is contained in the \( n \)-dimensional ball centered at zero of radius \( (n/\alpha)^\beta k^{-\alpha} (\ln k)^{-\beta} \). Consequently, by the arbitrariness of \( k \geq e \), we conclude that

\[
\forall k \geq e \quad \text{meas} \{ |v| > k \} \leq \omega_n (n/\alpha)^\beta k^{-\alpha} (\ln k)^{-\beta},
\]

where \( \omega_n \) is the measure of the unit ball in \( \mathbb{R}^n \).

Further, let \( w : \Omega \to \mathbb{R} \) be the function such that

\[
w(x) = \begin{cases} |x|^{-n} \left( \ln \frac{1}{|x|} \right)^{-1} & \text{if } x \in \Omega \setminus \{0\}, \\ 0 & \text{if } x = 0. \end{cases}
\]

It is easy to see that \( w \not\in L^1(\Omega) \).

Assume that \( \beta \in [0,1] \). Taking into account (3.2), we deduce from Corollary 1 that, for any \( \gamma > 1 - \beta \), the inclusion \( |v|^\alpha [\ln(2 + |v|)]^{-\gamma} \in L^1(\Omega) \) holds. However, if \( \gamma = 1 - \beta \), then \( |v|^\alpha [\ln(2 + |v|)]^{-\gamma} \not\in L^1(\Omega) \). Indeed, let \( x \in \Omega \setminus \{0\} \) and \( |x| \leq (1/e)^{\alpha/(n-1)} \). We have \( |x|^{-\alpha/n} \geq e \) and, consequently, \( |v(x)| \leq e \). Then \( \ln(2 + |v(x)|) \leq 2 \ln |v(x)| \). In addition, by the definition of the function \( v \), we have \( \ln |v(x)| \leq (n/\alpha) \ln(1/|x|) \). The last two inequalities imply that \( w(x) \leq (2n/\alpha)^{1-\beta} |v(x)|^{\alpha \ln(2 + |v(x)|)^{\alpha / (1-\beta)}} \). This estimate and the property \( w \not\in L^1(\Omega) \) lead to the conclusion that \( |v|^\alpha [\ln(2 + |v|)]^{\alpha / (1-\beta)} \not\in L^1(\Omega) \). The obtained result shows that the inequality \( \gamma > 1 - \beta \) in the statement of Corollary 1 cannot be replaced by the inequality \( \gamma \geq 1 - \beta \) without violating the resulting inclusion.

Now, let \( \beta > 1 \). Taking into account (3.2), we deduce from Corollary 2 that, for any \( \gamma \in (0, \beta - 1) \), the inclusion \( |v|^\alpha (\ln(1 + |v|))^\gamma \in L^1(\Omega) \) holds. At the same time, \( |v|^\alpha (\ln(1 + |v|))^{\beta - 1} \not\in L^1(\Omega) \).
Indeed, let \( x \in \mathbb{R}^n \) and \( 0 < |x| < e^{-4\beta^2} \). We have \( \ln(1/|x|) \geq 4\beta^2 \) and, consequently, \( \ln(\ln(1/|x|)) \leq (1/\beta) \ln(1/|x|) \). Then \( |v(x)| = (n/\alpha) \ln(1/|x|) - (\beta/\alpha) \ln(\ln(1/|x|)) \geq (1/\alpha) \ln(1/|x|) \). Hence, \( w(x) \leq \alpha^{\beta-1} |v(x)|^{\alpha} [\ln(1 + |v(x)|)]^{\beta-1} \). This estimate and the property \( w \not\in L^1(\Omega) \) lead to the conclusion that \( |v(x)|^{\alpha} [\ln(1 + |v(x)|)]^{\beta-1} \not\in L^1(\Omega) \). The obtained result shows that the inclusion \( \gamma \in (0, \beta - 1) \) in the statement of Corollary 2 cannot be replaced by the inclusion \( \gamma \in (0, \beta - 1] \) without violating the resulting inclusion.

The considered example demonstrates that the conclusion of Corollary 1 is accurate in the scale of classes \( K_{\alpha, \gamma}^1(\Omega) \), \( \gamma \geq 0 \), and the conclusion of Corollary 2 is accurate in the scale of classes \( K_{\alpha, \gamma}^1(\Omega) \), \( \gamma > 0 \).

**Example 2.** Let \( \Omega = \{ x \in \mathbb{R}^n : |x| < 1/3^e \} \), let \( \alpha > 0 \), and let \( \beta \geq 0 \). Let \( v : \Omega \to \mathbb{R} \) be the function such that

\[
v(x) = \begin{cases} |x|^{-n/\alpha} \left( \ln \frac{1}{|x|} \right)^{-\beta/\alpha} & \text{if } x \in \Omega \setminus \{0\}, \\ 0 & \text{if } x = 0. \end{cases}
\]

It is clear that the function \( v \) is measurable. We fix \( k > e^3 \), and let \( x \in \{|v| > k\} \). Obviously, \( x \in \Omega \setminus \{0\} \). By these inclusions and the definition of the function \( v \), we have

\[
|x|^{-n/\alpha} \left( \ln \frac{1}{|x|} \right)^{-\beta/\alpha} > k, \quad \ln \frac{1}{|x|} > 1. \tag{3.3}
\]

Hence, \( |x|^{-n/\alpha} > k \). Then \( \ln k < (1 + |\ln(n/\alpha)|) \ln(1/|x|) \). Using this inequality and the first inequality in (3.3), we find that \( |x|^{-n} < (1 + |\ln(n/\alpha)|)^{\beta-k-\alpha} (\ln k)^{-\beta} \). The obtained estimate implies that \( \text{meas}\{ |v| > k \} \leq \omega_n (1 + |\ln(n/\alpha)|)^{\beta-k-\alpha} (\ln k)^{-\beta} \). In the case \( 3 \leq k \leq e^3 \), without using the definition of the function \( v \), we obtain \( \text{meas}\{ |v| > k \} \leq \text{meas} \Omega = \omega_n e^{-3n} \leq \omega_n e^{3(\alpha-n)} (\ln 3)^{\beta-k-\alpha} (\ln k)^{-\beta} \). In view of the above, we conclude that

\[
\forall k \geq 3 \quad \text{meas}\{ |v| > k \} \leq \omega_n e^{3\alpha} (\ln 3 + |\ln(n/\alpha)|)^{\beta-k-\alpha} (\ln k)^{-\beta}. \tag{3.4}
\]

Further, let \( w : \Omega \to \mathbb{R} \) be the function such that

\[
w(x) = \begin{cases} |x|^{-n} \left( \ln \frac{1}{|x|} \right)^{-1} \left( \ln \frac{1}{|x|} \right)^{-1} & \text{if } x \in \Omega \setminus \{0\}, \\ 0 & \text{if } x = 0. \end{cases}
\]

It is easy to see that \( w \not\in L^1(\Omega) \).

Assume that \( \beta \in [0, 1] \). Taking into account (3.4), we deduce from Corollary 3 that, for any \( \gamma > 1 - \beta \), the inclusion \( |v|^\alpha [\ln(2 + |v|)]^{-\gamma} \ln(3 + |v|)^{-\gamma} \in L^1(\Omega) \) holds. However, if \( \gamma = 1 - \beta \), then \( |v|^\alpha [\ln(2 + |v|)]^{-\gamma} \ln(3 + |v|)^{-\gamma} \not\in L^1(\Omega) \). Indeed, let \( x \in \Omega \setminus \{0\} \) and \( |x| \leq e^{-ax/(n-1)} \). Since \( \beta \leq 1 \) and \( \ln(1/|x|) < 1/|x| \), by the definition of the function \( v \), we have \( |v(x)| \geq |x|^{-n/(n-1)\alpha} \geq e^c \). Consequently, \( \ln(2 + |v(x)|) \leq 2 \ln |v(x)| \) and \( \ln(3 + |v(x)|) \leq 2 \ln \ln |v(x)| \). Taking into account these inequalities, we obtain

\[
|v(x)|^{\alpha} [\ln |v(x)|]^{-1} [\ln \ln |v(x)|]^{-(1-\beta)} \leq 4|v(x)|^{\alpha} [\ln(2 + |v(x)|)]^{-1} [\ln \ln(3 + |v(x)|)]^{-(1-\beta)}. \tag{3.5}
\]

In addition, since \( 0 < |x| < 1/e^3 \), we have \( \ln(1/|x|) > 1 \). Therefore, as follows from the definition of the function \( v \), \( \ln |v(x)| \leq (n/\alpha) \ln(1/|x|) \) and \( \ln \ln |v(x)| \leq (1 + |\ln(n/\alpha)|) \ln(1/|x|) \). Using
these inequalities, we find that \( w(x) \leq (n/\alpha)(1 + |\ln(n/\alpha)|)|v(x)|^\alpha |\ln |v(x)||^{-1} |\ln |v(x)||^{-(1-\beta)}. \) This and (3.5) imply the inequality
\[
w(x) \leq (4n/\alpha)(1 + |\ln(n/\alpha)|)|v(x)|^\alpha |\ln (2 + |v(x)||)^{-1} |\ln |v(x)||^{-(1-\beta)},
\]
The obtained estimate along with the property \( w \notin L^1(\Omega) \) leads to the conclusion that the function \( |v|^\alpha |\ln (2 + |v||)^{-1} |\ln |v(x)||^{-(1-\beta)} \) does not belong to \( L^1(\Omega) \). The established result shows that the inequality \( \gamma > 1 - \beta \) in the statement of Corollary 3 cannot be replaced by the inequality \( \gamma \geq 1 - \beta \) without violating the resulting inclusion.

Now, let \( \beta > 1 \). Taking into account (3.4), we deduce from Corollary 4 that, for any \( \gamma \in (0, \beta - 1) \), the inclusion \( |v|^\alpha |\ln (2 + |v||)^{-1} |\ln |v(x)||^{\gamma} \in L^1(\Omega) \) holds. However, if \( \gamma = \beta - 1 \), then this inclusion does not hold. Indeed, we define \( \lambda = \max\{4\beta^2, \alpha \} \), and let \( x \in \mathbb{R}^n \) and \( 0 < |x| \leq e^{-\lambda} \). We have \( \ln(1/|x|) \geq 4\beta^2 \) and, consequently, \( \ln \ln(1/|x|) \leq (1/\beta) \ln(1/|x|) \). Then, by the definition of the function \( v \) and the inequality \( |x| \leq e^{-\alpha} \), we have
\[
\ln |v(x)| = \frac{n}{\alpha} \ln \frac{1}{|x|} - \frac{\beta}{\alpha} \ln \ln \frac{1}{|x|} > \frac{1}{\alpha} \ln \frac{1}{|x|} \geq e.
\]
Therefore,
\[
\ln \frac{1}{|x|} \leq (1 + |\ln \alpha|) \ln |v(x)|, \quad \ln (2 + |v(x)||) \leq 2 \ln |v(x)|. \tag{3.6}
\]
In addition, by the definition of \( v \) and the inequality \( \ln \ln(1/|x|) > 1 \), we have \( |v(x)| \leq |x|^{-n/\alpha} \) and, consequently, \( |v(x)| \leq (n/\alpha) \ln(1/|x|) \). This and inequalities (3.6) imply that \( w(x) \leq (2n/\alpha)(1 + |\ln \alpha|)^{\beta - 1} |v(x)|^\alpha |\ln (2 + |v(x)||)^{-1} |\ln |v(x)||^{\beta - 1} \). The obtained estimate and the property \( w \notin L^1(\Omega) \) lead to the conclusion that \( |v|^\alpha |\ln (2 + |v||)^{-1} |\ln |v(x)||^{\beta - 1} \notin L^1(\Omega) \). The established result shows that the inclusion \( \gamma \in (0, \beta - 1) \) in the statement of Corollary 4 cannot be replaced by the inclusion \( \gamma \in (0, \beta - 1) \) without violating the resulting inclusion.

The considered example demonstrates that the conclusion of Corollary 3 is accurate in the scale of classes \( K_{2, \alpha}^{\alpha, \gamma}(\Omega), \gamma \geq 0 \), and the conclusion of Corollary 4 is accurate in the scale of classes \( K_{2, \alpha}^{\alpha, \gamma}(\Omega), \gamma > 0 \).

We note that, despite the obtained accuracy of the conclusions of Corollaries 1–4 in the specified scales of classes of functions, these conclusions can be strengthened. For instance, if the conditions of Corollary 2 are satisfied, then \( |v|^\alpha |\ln (1 + |v||)^{\beta - 1} |\ln |v(x)||^{\beta - 1} \in L^1(\Omega) \) for any \( \lambda > 1 \).

4. APPLICATIONS

We can specify a number of applications of the results of Sections 1 and 2 to the study of the integrability properties of solutions of elliptic equations and variational inequalities with right-hand side in classes close to \( L^1(\Omega) \). However, in this section, we restrict ourselves to several applications of the results of Section 2 to entropy and weak solutions of the Dirichlet problem for second-order nonlinear elliptic equations with right-hand side in some classes close to \( L^1(\Omega) \) and defined by the logarithmic function or its double composition. Other applications of the results of this paper will be given in our forthcoming publications.

Let \( p \in (1, n) \). Let \( c_1, c_2 > 0 \), let \( g \in L^{p/(p-1)}(\Omega) \) with \( g \geq 0 \) in \( \Omega \), and let \( a_i : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R} \) be a Carathéodory function for any \( i \in \{1, \ldots, n\} \). We assume that, for almost all \( x \in \Omega \) and any \( \xi \in \mathbb{R}^n \),
\[
\sum_{i=1}^{n} |a_i(x, \xi)| \leq c_1 |\xi|^{p-1} + g(x), \quad \sum_{i=1}^{n} a_i(x, \xi) \xi_i \geq c_2 |\xi|^p.
\]
In addition, we assume that, for almost all \( x \in \Omega \) and any \( \xi, \xi' \in \mathbb{R}^n, \xi \neq \xi' \),

\[
\sum_{i=1}^{n} [a_i(x, \xi) - a_i(x, \xi')] (\xi_i - \xi'_i) > 0.
\]

Let \( f \in L^1(\Omega) \). We consider the following Dirichlet problem:

\[
-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x, \nabla u) = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega.
\] (4.1)

**Definition 1.** A weak solution of problem (4.1) is a function \( u \in \tilde{W}^{1,1}(\Omega) \) such that:

(i) for any \( i \in \{1, \ldots, n\} \), we have \( a_i(x, \nabla u) \in L^1(\Omega) \);

(ii) for any function \( v \in C^0_0(\Omega) \),

\[
\int_\Omega \left\{ \sum_{i=1}^{n} a_i(x, \nabla u) D_i v \right\} dx = \int_\Omega f v dx.
\]

We note that if \( p > 2 - 1/n \), then, according to Theorem 1 in [5], there exists a weak solution of problem (4.1) belonging to \( \tilde{W}^{1,\lambda}(\Omega) \) for any \( \lambda, 1 \leq \lambda < n(p - 1)/(n - 1) \).

Further, for any \( k > 0 \), let the function \( T_k : \mathbb{R} \rightarrow \mathbb{R} \) be defined as follows:

\[
T_k(s) = \begin{cases} 
  s & \text{if } |s| \leq k, \\
  k \text{sgn } s & \text{if } |s| > k.
\end{cases}
\]

We denote by \( \tilde{T}^{1,p}(\Omega) \) the set of all functions \( v : \Omega \rightarrow \mathbb{R} \) such that \( T_k(v) \in \tilde{W}^{1,p}(\Omega) \) for any \( k > 0 \).

Note that \( \tilde{W}^{1,p}(\Omega) \subset \tilde{T}^{1,p}(\Omega) \) and the set \( \tilde{T}^{1,p}(\Omega) \setminus L^1(\Omega) \) is nonempty. For arbitrary \( v : \Omega \rightarrow \mathbb{R} \) and \( x \in \Omega \), we define \( k(v, x) = \min\{l \in \mathbb{N} : |v(x)| \leq l\} \).

**Definition 2.** Let \( v \in \tilde{T}^{1,p}(\Omega) \) and \( i \in \{1, \ldots, n\} \). Then \( \delta_i v \) is the function on \( \Omega \) such that \( \delta_i v(x) = D_i T_{k(v, x)}(v)(x) \) for any \( x \in \Omega \).

Note that if \( v \in \tilde{T}^{1,p}(\Omega) \) and \( i \in \{1, \ldots, n\} \), then \( D_i T_k(v) = \delta_i v \times 1_{|v| \leq k} \) a.e. in \( \Omega \) for any \( k > 0 \) (see [4, Proposition 1.3]). Therefore, if \( v \in \tilde{W}^{1,p}(\Omega) \), then \( \delta_i v = D_i v \) a.e. in \( \Omega \) for any \( i \in \{1, \ldots, n\} \).

**Definition 3.** If \( v \in \tilde{T}^{1,p}(\Omega) \), then \( \delta v \) is the mapping from \( \Omega \) to \( \mathbb{R}^n \) such that, for any \( x \in \Omega \) and any \( i \in \{1, \ldots, n\} \), we have \( (\delta v(x))_i = \delta_i v(x) \).

Note that if \( u \in \tilde{T}^{1,p}(\Omega), v \in \tilde{W}^{1,p}(\Omega) \cap L^\infty(\Omega), k > 0, \) and \( i \in \{1, \ldots, n\} \), then the function \( a_i(x, \delta u)(\delta_i u - \delta_i v) \) is summable on the set \( \{|u - v| < k\} \).

**Definition 4.** An entropy solution of problem (4.1) is a function \( u \in \tilde{T}^{1,p}(\Omega) \) such that, for any \( v \in C_0^\infty(\Omega) \) and any \( k > 0 \),

\[
\int_{\{|u - v| < k\}} \left\{ \sum_{i=1}^{n} a_i(x, \delta u)(\delta_i u - \delta_i v) \right\} dx \leq \int_\Omega f T_k(u - v) dx.
\]
By Theorem 6.1 in [2], there exists a unique entropy solution of problem (4.1). We also note that if \( p > 2 - 1/n \) and \( u \) is the entropy solution of problem (4.1), then \( u \) is a weak solution of this problem (see, for instance, [2, 4]).

Define \( q = n(p - 1)/(n - p) \). By Lemmas 3.1 and 4.1 in [2], the entropy solution of problem (4.1) belongs to \( L^q(\Omega) \) for any \( \lambda \in (0, q) \). At the same time, the entropy solution of problem (4.1), in general, does not belong to the space \( L^q(\Omega) \). In this connection, see, in particular, [6, Example 1.4.2]. In the case where \( p \geq 2 - 1/n \) and \( f \ln(1 + |f|) \in L^1(\Omega) \), the existence of a weak solution of problem (4.1) belonging to \( L^q(\Omega) \) follows from the results of [5, 7]. Stronger statements on solutions of problem (4.1) follow from the results obtained in [4]. In particular, by Theorem 3.1 in [4], the entropy solution of problem (4.1) belongs to \( L^q(\Omega) \) if, for some \( \lambda > (n - p)/n \),

\[
f|\ln(1 + |f|)|^\lambda \in L^1(\Omega). \tag{4.2}
\]

In addition, it follows from the same theorem that if \( p > 2 - 1/n \) and, for some \( \lambda > (n - p)/n \), inclusion (4.2) holds, then there exists a weak solution of problem (4.1) belonging to \( L^q(\Omega) \). If \( p = 2 - 1/n \) and, for some \( \lambda > (n - 1)/n \), inclusion (4.2) holds, then the existence of a weak solution of problem (4.1) belonging to \( L^q(\Omega) \) is established using Theorem 3.2 in [4]. Below, we show that the fulfillment of inclusion (4.2) with \( \lambda > (n - p)/n \) provides a stronger integrability of solutions of problem (4.1) than \( L^q \)-integrability. This is a consequence of the consideration of the general case where \( f \) satisfies inclusion (4.2) with \( \lambda > 0 \). We also study the case where \( f|\ln(e + |f|)|^\lambda \in L^1(\Omega) \) with \( \lambda > 0 \). For these purposes, we need the following auxiliary result.

**Proposition 2.** Let \( u \) be the entropy solution of problem (4.1). Let \( \gamma > 0 \) and \( k > 0 \). Then

\[
\operatorname{meas}\{|u| \geq k\} \leq ck^{-q}\left(k^{\gamma p/(p - 1) - 1} + \int_{\{|f| \geq k^\gamma\}} |f|dx\right)^{n/(n-p)},
\]

where \( c \) is a positive constant depending only on \( n, p, \operatorname{meas}\Omega, \) and \( c_2 \).

This proposition is a consequence of Lemma 2.3 in [4].

We pass to the results following from Corollaries 1–4 and Proposition 2.

**Proposition 3.** Let \( u \) be the entropy solution of problem (4.1). Then, for any \( \gamma > 1 \), we have \( |u|^q|\ln(2 + |u|)|^{-1}|\ln \ln(3 + |u|)|^{-\gamma} \in L^1(\Omega) \).

**Proof.** We define \( \beta = 0 \), and let \( k \geq 3 \). By Proposition 2, the following inequality holds: \( \operatorname{meas}\{|u| > k\} \leq c(1 + \|f\|_{L^1(\Omega)})^{n/(n-p)k^{-q}|\ln k|^{-\beta}}. \) Therefore, by Corollary 3, for any \( \gamma > 1 \), we have \( |u|^q|\ln(2 + |u|)|^{-1}|\ln \ln(3 + |u|)|^{-\gamma} \in L^1(\Omega) \).

Since, for \( p > 2 - 1/n \), the entropy solution of problem (4.1) is a weak solution of this problem, we deduce from Proposition 3 the following result.

**Proposition 4.** Let \( p > 2 - 1/n \). Then there exists a weak solution \( u \) of problem (4.1) such that, for any \( \gamma > 1 \), we have \( |u|^q|\ln(2 + |u|)|^{-1}|\ln \ln(3 + |u|)|^{-\gamma} \in L^1(\Omega) \).

Note that Theorem 2.1 in [8] gives the existence of a weak solution \( u \) of problem (4.1) such that \( |u|^q|\ln(2 + |u|)|^{-\gamma} \in L^1(\Omega) \) for any \( \gamma > n/(n-p) \), and, according to Corollary 4.5 in [9], if \( u \) is the entropy solution of problem (4.1), then \( |u|^q|\ln(2 + |u|)|^{-n/(n-p)}|\ln \ln(3 + |u|)|^{-\gamma} \in L^1(\Omega) \) for any \( \gamma > n/(n-p) \). As seen, Propositions 3 and 4 are stronger than the mentioned results from [8, 9] in application to solutions of problem (4.1).
Proposition 5. Let \( \lambda > 0 \) and \( f[\ln(1 + |f|)]^{\lambda} \in L^1(\Omega) \). Let \( u \) be the entropy solution of problem (4.1). Then
\[
\forall k \geq e \quad \text{meas}\{|u| > k\} \leq C k^{-q}(\ln k)^{-\lambda n/(n-p)},
\]
where \( C \) is a positive constant depending only on \( n, p, \) meas \( \Omega, c_2, \lambda, \) and the norm of the function \( f[\ln(1 + |f|)]^{\lambda} \) in \( L^1(\Omega) \).

Proof. We define
\[
\gamma = \frac{p-1}{2p}, \quad c = \frac{1}{\gamma} \int_{\Omega} |f[\ln(1 + |f|)]^{\lambda}| dx.
\]
It follows from Proposition 2 that
\[
\forall k > 0 \quad \text{meas}\{|u| > k\} \leq c k^{-q} \left( k^{-1/2} + \int_{\{|f| \geq k^{\gamma}\}} |f| dx \right)^{n/(n-p)}.
\]
We now define \( C = c((2\lambda)^{\lambda} + c_0)^{n/(n-p)} \) and fix an arbitrary \( k \geq e \). For any \( x \in \{|f| \geq k^{\gamma}\} \), we have \( |f(x)| \leq (\gamma \ln k)^{-\lambda} |f(x)||\ln(1 + |f(x)|)|^{\lambda} \). Consequently,
\[
\int_{\{|f| \geq k^{\gamma}\}} |f| dx \leq c_0(\ln k)^{-\lambda}.
\]
From this and (4.3), taking into account that \( (\ln k)^{\lambda} < (2\lambda)^{\lambda} k^{1/2} \), we deduce the inequality \( \text{meas}\{|u| > k\} \leq C k^{-q}(\ln k)^{-\lambda n/(n-p)} \).

Using Corollary 1 and Proposition 5, we obtain the following results.

Proposition 6. Let \( \lambda \in (0, (n-p)/n] \) and \( f[\ln(1 + |f|)]^{\lambda} \in L^1(\Omega) \). Let \( u \) be the entropy solution of problem (4.1). Then, for any \( \gamma > 1 - \lambda n/(n-p) \), we have \( |u|^q [\ln(2+|u|)]^{-\gamma} \in L^1(\Omega) \).

Proposition 7. Let \( p > 2 - 1/n \). Let \( \lambda \in (0, (n-p)/n] \) and \( f[\ln(1 + |f|)]^{\lambda} \in L^1(\Omega) \). Then there exists a weak solution \( u \) of problem (4.1) such that, for any \( \gamma > 1 - \lambda n/(n-p) \), we have \( |u|^q [\ln(2+|u|)]^{-\gamma} \in L^1(\Omega) \).

Using Corollary 2 and Proposition 5, we come to the following results.

Proposition 8. Let \( \lambda > (n-p)/n \) and \( f[\ln(1 + |f|)]^{\lambda} \in L^1(\Omega) \). Let \( u \) be the entropy solution of problem (4.1). Then, for any \( \gamma \in (0, \lambda n/(n-p) - 1) \), we have \( |u|^q [\ln(1 + |u|)]^\gamma \in L^1(\Omega) \).

Proposition 9. Let \( p > 2 - 1/n \). Let \( \lambda > (n-p)/n \) and \( f[\ln(1 + |f|)]^{\lambda} \in L^1(\Omega) \). Then there exists a weak solution \( u \) of problem (4.1) such that, for any \( \gamma \in (0, \lambda n/(n-p) - 1) \), we have \( |u|^q [\ln(1 + |u|)]^\gamma \in L^1(\Omega) \).

In addition to these propositions, we have the following result.

Proposition 10. Let \( p = 2 - 1/n \). Let \( \lambda > (n-1)/n \) and \( f[\ln(1 + |f|)]^{\lambda} \in L^1(\Omega) \). Then there exists a weak solution \( u \) of problem (4.1) such that, for any \( \gamma \in (0, \lambda n^2/(n-1)^2 - 1) \), we have \( |u|^{n/(n-1)} [\ln(1 + |u|)]^\gamma \in L^1(\Omega) \).

Proof. Let \( \tilde{f} : [0, +\infty) \to \mathbb{R} \) be the function such that, for any \( s \in [0, +\infty) \),
\[
\tilde{f}(s) = \int_{\{|f| \geq s\}} |f| dx.
\]
For any $s > 1$, we have $\tilde{f}(s) \leq (\ln s)^{-\lambda}(\ln(1 + |f|))^{\lambda}L^1(\Omega)$. This and the inequality $\lambda > (n - 1)/n$ imply that

$$\int_1^{+\infty} \frac{1}{s} [\tilde{f}(s)]^{n/(n-1)} ds < +\infty.$$ 

Then, by Theorem 3.2 in [4], for the entropy solution $u$ of problem (4.1), the inclusion $|\delta u| \in L^{n(p-1)/(n-1)}(\Omega)$ holds. Hence, taking into account the equality $p = 2 - 1/n$, we get $|\delta u| \in L^1(\Omega)$. Therefore (see [4, p. 1888]), $u$ is a weak solution of problem (4.1). From Proposition 8 and the equalities $p = 2 - 1/n$ and $q = n(p - 1)/(n - p)$, we deduce that, for any $\gamma \in (0, \lambda n^2/(n - 1)^2 - 1)$, the following inclusion holds: $[u^{n/(n-1)}(\ln(1 + |u|))]^\gamma \in L^1(\Omega)$. \hfill \Box

We now consider the case of weaker integrability of the function $f$.

**Proposition 11.** Let $\lambda > 0$ and $f[\ln\ln(e + |f|)]^\lambda \in L^1(\Omega)$. Let $u$ be the entropy solution of problem (4.1). Then

$$\forall k \geq 3 \quad \text{meas}\{|u| > k\} \leq Ck^{-q(\ln\ln k)^{-\lambda n/(n-p)},}$$

where $C$ is a positive constant depending only on $n$, $p$, $\text{meas}\Omega$, $c_2$, $\lambda$, and the norm of the function $f[\ln\ln(e + |f|)]^\lambda$ in $L^1(\Omega)$.

**Proof.** We define

$$\gamma = \frac{p - 1}{2p}, \quad c_0 = 2^\lambda \int_\Omega |f[\ln\ln(e + |f|)]^\lambda dx$$

and fix an arbitrary $k \geq e^{1/\gamma^2}$. Taking into account the inequality $2\ln(1/\gamma) \leq \ln\ln k$, for any $x \in \{|f| \geq k \gamma\}$, we have $|f(x)| \leq (\ln\ln k)^{-\lambda^2} |f(x)||\ln\ln(e + |f(x)|)|^\lambda$. Consequently,

$$\int_{\{|f| \geq k \gamma\}} |f| dx \leq c_0(\ln\ln k)^{-\lambda}.$$ 

From this and Proposition 2, we deduce that

$$\text{meas}\{|u| > k\} \leq C[(2\lambda)^\lambda + c_0]^{n/(n-p)} k^{-q(\ln\ln k)(-\lambda n/(n-p))}. \quad (4.4)$$

Thus, if $k \geq e^{1/\gamma^2}$, then inequality (4.4) holds. If $3 \leq k < e^{1/\gamma^2}$, we have

$$\text{meas}\{|u| > k\} \leq \text{meas}\Omega = (\text{meas}\Omega) k^q(\ln\ln k)^{\lambda n/(n-p)} k^{-q(\ln\ln k)(-\lambda n/(n-p))}$$

$$\leq (\text{meas}\Omega) e^{(q + \lambda n/(n-p))/\gamma^2} k^{-q(\ln\ln k)(-\lambda n/(n-p)).}$$

As a result, we come to the required conclusion. \hfill \Box

Using Corollary 3 and Proposition 11, we obtain the following results.

**Proposition 12.** Assume that $\lambda \in (0, (n - p)/n)$, $f[\ln\ln(e + |f|)]^\lambda \in L^1(\Omega)$, and $u$ is the entropy solution of problem (4.1). Then $[u^{q[\ln(2 + |u|)]^{-1}[\ln\ln(3 + |u|)]^{-\gamma}}] \in L^1(\Omega)$ for any $\gamma > 1 - \lambda n/(n-p)$.

**Proposition 13.** Assume that $p > 2 - 1/n$, $\lambda \in (0, (n - p)/n)$, and $f[\ln\ln(e + |f|)]^\lambda \in L^1(\Omega)$. Then there exists a weak solution $u$ of problem (4.1) such that, for any $\gamma > 1 - \lambda n/(n-p)$, we have $[u^{q[\ln(2 + |u|)]^{-1}[\ln\ln(3 + |u|)]^{-\gamma}}] \in L^1(\Omega)$. 

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Using Corollary 4 and Proposition 11, we obtain the following results.

**Proposition 14.** Let \( \lambda > (n - p)/n \) and \( f[\ln \ln(e + |f|)]^\lambda \in L^1(\Omega) \). Let \( u \) be the entropy solution of problem (4.1). Then \( |u|^q[\ln(2 + |u|)]^{-1}[\ln \ln(3 + |u|)]^\gamma \in L^1(\Omega) \) for any number \( \gamma \) in \((0, \lambda n/(n - p) - 1)\).

**Proposition 15.** Let \( p > 2 - 1/n \). Let \( \lambda > (n - p)/n \) and \( f[\ln \ln(e + |f|)]^\lambda \in L^1(\Omega) \). Then there exists a weak solution \( u \) of problem (4.1) such that, for any \( \gamma \in (0, \lambda n/(n - p) - 1) \), we have \( |u|^q[\ln(2 + |u|)]^{-1}[\ln \ln(3 + |u|)]^\gamma \in L^1(\Omega) \).

**FUNDING**

This work was partially supported by the Russian Academic Excellence Project (agreement no. 02.A03.21.0006 of August 27, 2013, between the Ministry of Education and Science of the Russian Federation and Ural Federal University).

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Translated by A. Kovalevsky