Unobstructed K-deformations of Generalized Complex Structures and Bihermitian Structures

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Abstract

We introduce “K-deformations” of generalized complex structures on a compact Kähler manifold $M = (X, J)$ with an effective anti-canonical divisor and show that obstructions to K-deformations of generalized complex structures on $M$ always vanish. Applying unobstructed K-deformations and the stability theorem of generalized Kähler structures, we construct deformations of bihermitian structures in the form $(J, J^-, h_t)$ on a compact Kähler surface with a non-zero holomorphic Poisson structure. Then we prove that a compact Kähler surface $S$ admits a non-trivial bihermitian structure with the torsion condition and the same orientation if and only if $S$ has a non-zero holomorphic Poisson structure. Further we obtain bihermitian structures $(J, J^-, h)$ on del Pezzo surfaces, degenerate del Pezzo surfaces and some ruled surfaces for which the complex structure $J$ is not equivalent to $J^-$ under diffeomorphisms.

Contents

1 Introduction 2
2 Unobstructed K-deformations of generalized complex structures 5
3 K-deformations of generalized complex structures in terms of $\text{CL}^2(-D)$ 12
4 Deformations of generalized Kähler structures 13

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1 Introduction

The concept of generalized complex structures were introduced by Nijel Hitchin [17] depending on a simple idea by replacing the tangent bundle on a manifold with the direct sum of the tangent bundle and the cotangent bundle which include both symplectic and complex structures as special cases. Interesting generalized complex structures are arising as hybrid of symplectic and complex structures. An associated notion of generalized Kähler structures consists of two commuting generalized complex structures which yield a generalized metric. It is remarkable that generalized Kähler structures are equivalent to the so-called bihermitian structures [16]. Bihermitian structures on complex surfaces are extensively studied from view points of conformal geometry and complex geometry [1, 3, 8]. Bihermitian structures are also appeared in the Physics as geometric structures on the target space for the supersymmetric σ-model [26]. Main purpose of this paper is to solve remarkable problems in bihermitian geometry by using deformations of generalized Kähler structures.

Let $M = (X, J)$ be a compact Kähler manifold with effective anti-canonical divisor where $X$ is the underlying differential manifold. Then there is the generalized complex structure $\mathcal{J}_{\bar{J}}$ induced from the complex structure $\mathcal{J}$. At first we shall show that there exist certain unobstructed deformations of generalized complex structures of $\mathcal{J}_{\bar{J}}$, which is called $K$-deformations. This is regarded as a generalization of unobstructedness theorem of Calabi-Yau manifolds [5], [34] (see also Miyajima’s result on unobstructed deformations in the case of normal isolated singularity [29].) We apply a unified method as in [11], [12] to a meromorphic $n$-form with a pole along the anti-canonical divisor and show that the obstructions to deformations vanish:
Theorem 2.5 Let $M = (X, J)$ be a compact Kähler manifold of dimension $n$ and we denote by $\mathcal{J}_J$ the generalized complex structure given by $J$. If $M$ has an effective, anti-canonical divisor $D$, then $M$ admits unobstructed $K$-deformations of generalized complex structures $\{\mathcal{J}_t\}$ starting with $\mathcal{J}_0 = J$ which are parametrized by an open set of $H^2(\mathcal{L}(-D)) \cong H^{n,2} \oplus H^{n-1,1} \oplus H^{n-2,0}$.

In the cases of Kähler surfaces, we obtain the following:

Corollary 2.7 Let $S$ be a compact Kähler surface with the complex structure $J$ and a Kähler form $\omega$. If $S$ has an effective, anti-canonical divisor $[D] = -K_S$, then $S$ admits unobstructed $K$-deformations of generalized complex structures parametrized by an open set of the full cohomology group $H^0(S) \oplus H^2(S) \oplus H^4(S)$ of even degree.

The obstruction space of $K$-deformations on compact Kähler surface $S$ is given by $H^{1,2} \oplus H^{0,1} \cong H^1(S, \mathbb{C})$. Thus the obstruction space does not vanish if $b_1(S) \neq 0$. For instance, the product of $\mathbb{CP}^1$ and the elliptic curve $E$ admits unobstructed $K$-deformations, nevertheless the obstruction space does not vanish.

We apply our unobstructed $K$-deformations to construct bihermitian structures on compact Kähler surfaces. A bihermitian structure on a differential manifold $X$ is a triple $(J^+, J^-, h)$ consisting of two complex structure $J^+$ and $J^-$ and a metric $h$ which is a Hermitian metric with respect to both $J^+$ and $J^-$. In this paper we always assume that a bihermitian structure satisfies the torsion condition:

$$-d^c_+ \omega_+ = d^c_- \omega_- = db,$$  \hspace{1cm} (1.1)

where $d^c_\pm = \sqrt{-1} (\bar{\partial}_\pm - \partial_\pm)$ and $\omega_\pm$ denote the fundamental 2-forms with respect to $J^\pm$ and $b$ is a real 2-form. A bihermitian structure on $X$ is non-trivial if there is a point $x \in X$ such that $J^+_x \neq \pm J^-_x$. A bihermitian structure with $J^+_x \neq J^-_x$ for all $x \in X$ is called a strongly bihermitian structure. If $J^+$ and $J^-$ induce the same orientation then $(J^+, J^-, h)$ is a bihermitian structure with the same orientation. A bihermitian structure $(J^+, J^-, h)$ on $X$ is distinct if the complex manifold $(X, J^+)$ is not biholomorphic to the complex manifold $(X, J^-)$. We say a complex manifold $M = (X, J)$ admits a bihermitian structure if there is a bihermitian structure $(J^+, J^-, h)$ on $X$ with $J^+ = J$. We treat the following question in this paper:

Which compact complex surfaces admit nontrivial bihermitian structures?

The question was addressed by Apostolov, Gauduchon and Grantcharov [3]. Kobak [21] gave strongly bihermitian structures on the torus $T^4$. Hitchin constructed (non-strongly)
bihermitean structures on del Pezzo surfaces by using the Hamiltonian diffeomorphisms \cite{19}. Fujiki and Pontecorvo constructed anti-self-dual bihermitean structures on certain non-Kähler surfaces in class VII, especially hyperbolic and parabolic Inoue surfaces by using the Twistor spaces \cite{8}. There is a one to one correspondence between bihermitean structures with the torsion condition (1.1) and generalized Kähler structures \cite{16}. Thus we can obtain bihermitean structures by constructing generalized Kähler structures. Lin and Tolman developed the generalized Kähler quotient construction to obtain examples of generalized Kähler manifolds \cite{27} and the author established the stability theorem of generalized Kähler structures to construct generalized Kähler deformations on a compact Kähler manifold with a holomorphic Poisson structure \cite{13}, \cite{14}. However these generalized Kähler structures do not give a precise answer of the above question since both complex structures of the corresponding bihermitean structures may be deformed.

If we try to obtain deformations of bihermitean structures \((J,J_t^- ,h_t)\) fixing one of complex structures, then we encounter a problem of the obstructions to deformations of generalized complex structures (see \cite{15}). We use K-deformations instead of ones of generalized complex structures in order to overcome the difficulty. Since K-deformations on compact Kähler surfaces are unobstructed, we obtain

**Theorem 6.1** Let \(S = (X,J)\) be a compact Kähler surface with a Kähler form \(\omega\). If \(S\) has a non-zero holomorphic Poisson structure, then \(S\) admits deformations of non-trivial bihermitean structures \((J,J_t^- ,h_t)\) with the torsion condition which satisfies

\[
\left. \frac{d}{dt}J_t^- \right|_{t=0} = -2(\beta + \bar{\beta}) \cdot \omega
\]

and \(J_0^- = J\) and \(h_0\) is the Kähler metric of \((X,J,\omega)\), where \(\beta \cdot \omega\) is the \(T^1_1\)-valued \(\overline{\partial}\) closed form of type \((0,1)\) which gives the Kodaira-Spencer class \([\beta \cdot \omega] \in H^1(S, \Theta)\) of the deformations \\{\(J_t^-\)\}.

It is shown that a non-trivial bihermitean structure with the torsion condition and the same orientation on a compact surface gives a nonzero holomorphic Poisson structure \cite{3}, \cite{19}, (see Proposition 2 and Remark 2 in \cite{1}). Thus it follows from our theorem 6.1 that

**Theorem 6.2** A compact Kähler surface admits non-trivial bihermitean structure with the torsion condition and the same orientation if and only if \(S\) has nonzero holomorphic Poisson structure.

For instance it turns out that all degenerate del Pezzo surface and all Hirzebruch surfaces admit non-trivial bihermitean structures with the torsion condition and the same orientation. Further since there is a classification of Poisson surfaces \cite{4}, \cite{32}, \cite{33}, we obtain all compact Kähler surfaces which admit bihermitean structures with the torsion condition and the same orientation. Let \(T^*\Sigma\) be the cotangent bundle for every Riemannian surface \(\Sigma\) with genus \(g\) and \(S\) the projective space bundle \(\mathbb{P}(T^*\Sigma \oplus \mathcal{O}_{\Sigma})\) with the fibre \(\mathbb{P}^1\). Then
$S$ has an effective divisor $2[E_\infty]$, where $E_\infty$ is the section of $S \to \Sigma$ with intersection number $2 - 2g$. Let $\beta$ be a holomorphic Poisson structure with the zero locus $2[E_\infty]$. Then it turns out that the class $[\beta \cdot \omega] \in H^1(S, \Theta)$ does not vanish. We denote by $X$ the underlying differential manifold of the complex surface $S$. Thus we obtain

**Theorem 8.15** There is a family of distinct bihermitian structures $(J, J^-_t, h_t)$ with the torsion condition and the same orientation on $S := \mathbb{P}(T^*\Sigma \oplus O_\Sigma)$, that is, the complex manifold $(X, J^-)$ is not biholomorphic to $S = (X, J)$ for small $t \neq 0$.

In section 2 we obtain unobstructed K-deformations of generalized complex structures. In section 3 unobstructed K-deformations of generalized complex structures is given by the action of $\mathbb{C}L^2(-D)$ which is necessary for our construction of generalized Kähler structures. In section 4 we recall the stability theorem of generalized Kähler structures and in section 5 we describe a family of sections $\Gamma^\pm(a(t), b(t))$ of $GL(TX)$ which gives deformations of bihermitian structures $(J^+_t, J^-_t, h_t)$. In section 6,7 we shall show our main theorem 6.1.

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## 2 Unobstructed K-deformations of generalized complex structures

Let $M = (X, J)$ be a compact Kähler manifold of dimension $n$, where $X$ is the underlying differential manifold and $J$ is the complex structure on it. Then there is the generalized complex structure $\mathcal{J}$ defined by $J$. We denote by $T^{1,0}$ the holomorphic tangent bundle and $\wedge^{0,1}$ the $C^\infty$ vector bundle of forms of type $(0, 1)$. The vector bundle $\mathcal{L}$ is defined to be the direct sum $T^{1,0} \oplus \wedge^{0,1}$ and we denote by $\wedge^r \mathcal{L}$ the bundle of anti-symmetric tensors of $\mathcal{L}$ with degree $r$ which has the decomposition,

$$\wedge^r \mathcal{L} = \bigoplus_{p+q=r} T^{p,0} \otimes \wedge^{0,q},$$

where $T^{p,0}$ is the bundle of $p$-vectors and $\wedge^{0,q}$ is the bundle of forms of type $(0, q)$. Thus we have the complex $(\wedge^r \mathcal{L}, \partial)$ which is the direct sum of the ordinary $\partial$-complexes $(T^{p,0} \otimes \wedge^{0,*}, \partial)$. The sheaf of smooth sections of $\wedge^r \mathcal{L}$ is denoted by $\wedge^r \mathcal{L}$. We assume that $M$ has an anti-canonical divisor $D$ which is given by the zero locus of a holomorphic section $\beta \in H^0(M, K_M^{-1})$. Thus the canonical line bundle $K_M$ is the dual bundle $-[D]$. We denote by $I_D$ the ideal sheaf of the divisor $D$ which is the sheaf of sections of the canonical line
bundle $K_M$. We define a sheaf $\Lambda^r L(-D)$ by

$$\Lambda^r L(-D)(U) := \{ fa | f \in I_D(U), \ a \in \Lambda^r L(U) \} ,$$

(2.1)

where $U$ is an open set of the manifold $X$. In particular, the sheaf $\Lambda^0 L(-D)$ is given by $I_D \otimes \mathcal{C}^\infty(X)$. We denote by $\Gamma(X, \Lambda^r L(-D))$ the set of global smooth sections of the sheaf $\Lambda^r L(-D)$. (In this paper, for simplicity, we often say a section of a sheaf instead of a global section of a sheaf.) The sheaf $\Lambda^r L(-D)$ is locally free which is the sheaf of smooth sections of a vector bundle $\Lambda^r L(-D)$. Tensoring with $\beta^{-1}$ gives an identification,

$$\Lambda^r L(-D) \cong \Lambda^r L \otimes [-D] \cong \Lambda^r L \otimes K_M$$

Then we have

$$\Lambda^0 L(-D) = \Lambda^{0,0} \otimes K_M$$
$$\Lambda^1 L(-D) = (\Lambda^{0,1} \otimes K_M) \oplus (T^{1,0} \otimes K_M)$$
$$\Lambda^2 L(-D) = (\Lambda^{0,2} \otimes K_M) \oplus (T^{1,0} \otimes \Lambda^{1,0} \otimes K_M) \oplus (T^{2,0} \otimes K_M)$$

Thus we have the subcomplex $(\Lambda^r L(-D), \partial)$ of the complex $(\Lambda^r L, \partial)$. Let $\Omega$ be a meromorphic $n$-form with simple pole along the divisor $D$ which is unique up to constant multiplication since $H^0(M, K \otimes [D]) \cong \mathbb{C}$. Note that the $d$-closed form $\Omega$ is locally written as

$$\Omega|_U = \frac{dz_1 \wedge \cdots \wedge dz_n}{f},$$

by a system of holomorphic coordinates $(z_1, \cdots, z_n)$ on a small open set $U$ and $f \in I_D(U)$. The action of $\Lambda^r L(-D)$ on $\Omega$ is defined by the interior and exterior product which is the spin representation of the Clifford algebra $\text{CL}(T \oplus T^*)$. Then it turns out that this action yields a smooth differential form on $M$. Thus we obtain an identification,

$$\Lambda^r L(-D) \cong \oplus_{p+q=r} \Lambda^{n-p,q},$$

where $\Lambda^{n-p,q}$ is the $\mathcal{C}^\infty$ vector bundle of forms of type $(n-p,q)$.

**Lemma 2.1.** The direct sum $\oplus_r \Lambda^r L(-D)$ is involutive with respect to the Schouten bracket.

**Proof.** A local section $\alpha$ of the sheaf $\Lambda^r L$ is a local section of the sheaf $\Lambda^r L(-D)$ if and only if $\alpha \cdot \Omega$ is a smooth differential form. The Schouten bracket $[\alpha_1, \alpha_2]_S$ for $\alpha_1, \alpha_2 \in \Lambda^r L(-D)$ is given by $[[d, \alpha_1], \alpha_2]_G$, where $[\cdot, \cdot]_G$ denotes the graded bracket, which is the derived bracket construction (see [13], [25]). Thus $[\alpha_1, \alpha_2]_S \cdot \Omega$ is a smooth differential form since $d\Omega = 0$. It follows that $[\alpha_1, \alpha_2]_S$ is a local section of $\Lambda^r L(-D)$. \qed
We define a vector bundle $U^{-n+r}$ by

\[
U^{-n+r} := \bigoplus_{0 \leq p, q \leq n} \wedge^{n-p,q}
\]

Then the identification $\wedge^r L(-D) \cong \bigoplus_{p+q=r} \wedge^{n-p,q}$ gives an isomorphism between complexes,

\[
(\wedge^r L(-D), \bar{\partial}) \cong (U^{-n+\bullet}, \bar{\partial}).
\]

Thus the cohomology group $H^r(\bar{\mathcal{L}}(-D))$ is given by the direct sum of the Dolbeault cohomology groups,

\[
H^r(\bar{\mathcal{L}}(-D)) \cong \bigoplus_{p+q=r} H^{n-p,q},
\]

where $H^{n-p,q}$ is the Dolbeault cohomology group $H^{n-p,q}(M)$. Thus we have

**Proposition 2.2.**

\[
\begin{align*}
H^0(\bar{\mathcal{L}}(-D)) &= H^0(M, K_M) \cong H^{n,0} \\
H^1(\bar{\mathcal{L}}(-D)) &= H^1(M, K_M) \oplus H^0(M, \Theta \otimes K_M) \cong H^{n,1} \oplus H^{n-1,0} \\
H^2(\bar{\mathcal{L}}(-D)) &= H^2(M, K_M) \oplus H^1(M, \Theta \otimes K_M) \oplus H^0(M, \wedge^2 \Theta \otimes K_M) \\
&\cong H^{n,2} \oplus H^{n-1,1} \oplus H^{n-2,0}
\end{align*}
\]

We define vector bundles $E^\bullet$ by

\[
E^{-1} = U^{-n}, \quad E^0 = U^{-n+1}, \\
E^1 = U^{-n} \oplus U^{-n+2}, E^2 = U^{-n+1} \oplus U^{-n+3},
\]

Then we have the complex by using the exterior derivative $d$,

\[
0 \longrightarrow E^{-1} \overset{d}{\longrightarrow} E^0 \overset{d}{\longrightarrow} E^1 \overset{d}{\longrightarrow} E^2 \overset{d}{\longrightarrow} \cdots
\]

The cohomology groups $H^\bullet(E^\bullet)$ of the complex $(E^\bullet, d)$ is given by the direct sum of the Dolbeault cohomology groups,

\[
\begin{align*}
H^{-1}(E^\bullet) &= H^{n,0} \\
H^0(E^\bullet) &= H^{n,1} \oplus H^{n-1,0} \\
H^1(E^\bullet) &= H^{n,2} \oplus H^{n-1,1} \oplus H^{n-2,0} \oplus H^{n,0} \\
H^2(E^\bullet) &= H^{n,3} \oplus H^{n-1,2} \oplus H^{n-2,1} \oplus H^{n-3,0} \oplus H^{n,1} \oplus H^{n-1,0}
\end{align*}
\]

Let $\varepsilon_1$ be a smooth global section of $\wedge^2 \bar{\mathcal{L}}(-D) \oplus \wedge^0 \bar{\mathcal{L}}(-D)$ with $d\varepsilon_1 \cdot \Omega = 0$. For such a section $\varepsilon_1$ of $\wedge^2 \bar{\mathcal{L}}(-D) \oplus \wedge^0 \bar{\mathcal{L}}(-D)$, we shall construct a family of smooth global sections
\[ \varepsilon(t) \text{ of } \bigwedge^0 L(-D) \oplus \bigwedge^2 L(-D) \] which gives deformations of maximal isotropic subbundles \( \{ \mathcal{T}_t \} \) by the Adjoint action

\[ L_t := \text{Ad}_{e^{\varepsilon(t)}} L = \{ E + [\varepsilon(t), E] | E \in L \}. \]  

Then the decomposition \( (T \oplus T^*)^C = L_t \oplus \mathcal{T}_t \) gives an almost generalized complex structure \( \mathcal{J}_t \) whose eigenspaces are \( L_t \) and \( \mathcal{T}_t \), where note that \( \mathcal{T}_t \) is the complex conjugate of \( L_t \). Thus a family of section \( \{ \varepsilon(t) \} \) yields deformations of almost generalized complex structures \( \{ \mathcal{J}_t \} \), where \( t \) is a parameter of the deformation. A family of section \( \varepsilon(t) \) is given in the form of power series,

\[ \tilde{\varepsilon}(t) = \varepsilon_1 t + \varepsilon_2 \frac{t^2}{2!} + \varepsilon_3 \frac{t^3}{3!} + \cdots. \]

If \( \mathcal{J}_t \) is integrable, deformations \( \mathcal{J}_t \) given by a family of smooth global sections \( \varepsilon(t) \) of \( \bigwedge^0 L(-D) \oplus \bigwedge^2 L(-D) \) is called \textit{K-deformations of generalized complex structures}. The structures \( \mathcal{J}_t \) are integrable if and only if the family of global sections \( \varepsilon(t) \) satisfies the Maurer-Cartan equation,

\[ \bar{\partial} \varepsilon(t) + \frac{1}{2} [\varepsilon(t), \varepsilon(t)]_S = 0, \]

where \([\varepsilon(t), \varepsilon(t)]_S \in \Gamma(X, \bigwedge^3 L(-D))\) denotes the Schouten bracket of \( \varepsilon(t) \). The action of \( e^{\varepsilon(t)} \) on \( \Omega \) gives a non-degenerate, pure spinor \( e^{\varepsilon(t)} \cdot \Omega \) which induces the almost generalized complex structure \( \mathcal{J}_t \). It is crucial to solve the following equation,

\[ d e^{\varepsilon(t)} \cdot \Omega = 0, \]

rather that the Maurer-Cartan equation. In fact we have,

\begin{proposition}
If \( \varepsilon(t) \) satisfies the equation \( d e^{\varepsilon(t)} \cdot \Omega = 0 \), then \( \mathcal{J}_t \) is integrable.
\end{proposition}

\begin{proof}
The equation (2.3) is equivalent to the equation

\[ e^{-\varepsilon(t)} \cdot d e^{\varepsilon(t)} \cdot \Omega = 0 \]

(2.4)

Let \( \pi_{U-n+3} \) be the projection to the component \( U^{-n+3} \). Then from [13], we have

\[ \pi_{U-n+3} \left( e^{-\varepsilon(t)} \cdot d e^{\varepsilon(t)} \right) \cdot \Omega = \pi_{U-n+3} d \Omega + \pi_{U-n+3} [d, \varepsilon(t)] \cdot \Omega + \frac{1}{2} [\varepsilon(t), \varepsilon(t)]_S \Omega \]

(2.5)

\[ = \left( \bar{\partial} \varepsilon(t) + \frac{1}{2} [\varepsilon(t), \varepsilon(t)]_S \right) \cdot \Omega = 0 \]

(2.6)

Then it follows that \( \bar{\partial} \varepsilon(t) + \frac{1}{2} [\varepsilon(t), \varepsilon(t)]_S = 0 \) on the complement \( X \setminus D \). Since \( \varepsilon(t) \) is a smooth section on \( X \), we have \( \bar{\partial} \varepsilon(t) + \frac{1}{2} [\varepsilon(t), \varepsilon(t)]_S = 0 \) on \( X \).
\end{proof}
**Theorem 2.4.** Let $M = (X, J)$ be a compact Kähler manifold of dimension $n$. We denote by $\mathcal{J}$ the generalized complex structure given by $J$. We suppose that $M$ has an effective, anti-canonical divisor $D$. Then for every smooth global section $\varepsilon_1$ of $\wedge^2 \mathcal{L}(-D) \boxplus \wedge^0 \mathcal{L}(-D)$ with $d\varepsilon_1 \cdot \Omega = 0$, there is a family of smooth global sections $\varepsilon(t)$ of $\wedge^2 \mathcal{L}(-D) \boxplus \wedge^0 \mathcal{L}(-D)$ such that $\mathcal{J}_t$ defined by $\varepsilon(t)$ is an integrable generalized complex structure and $\frac{d}{dt}\varepsilon(t)|_{t=0} = \varepsilon_1$, where $t$ is a parameter of deformations which is sufficiently small.

**Proof of Theorem 2.4.** We shall construct a family of smooth global sections $\varepsilon(t) = \varepsilon_1 t + \varepsilon_2 \frac{t^2}{2!} + \cdots$ of $\wedge^2 \mathcal{L}(-D) \boxplus \wedge^0 \mathcal{L}(-D)$ for every section $\varepsilon_1$ of $\wedge^2 \mathcal{L}(-D) \boxplus \wedge^0 \mathcal{L}(-D)$ with $d\varepsilon_1 \cdot \Omega = 0$ which satisfies the following,

$$
de^{\varepsilon(t)} \cdot \Omega = 0 \quad (2.7)$$

We denote by $(e^{\varepsilon(t)})_{[i]}$ the $i$-th term of $e^{\varepsilon(t)}$ in $t$. Since both $\varepsilon_1 \cdot \Omega$ and $\Omega$ are $d$-closed, we have $(de^{\varepsilon(t)})_{[i]} \cdot \Omega = (e^{-\varepsilon(t)} de^{\varepsilon(t)})_{[i]} = d\varepsilon_1 \cdot \Omega = 0$. We shall construct $\varepsilon(t)$ by the induction on $t$. We assume that there already exists a set of sections $\varepsilon_1, \cdots, \varepsilon_{k-1}$ of $\wedge^2 \mathcal{L}(-D) \boxplus \wedge^0 \mathcal{L}(-D)$ such that

$$(de^{\varepsilon(t)} \cdot \Omega)_{[i]} = 0, \quad 0 \leq i < k \quad (2.8)$$

The assumption (2.8) is equivalent to the following,

$$(e^{-\varepsilon(t)} d e^{\varepsilon(t)})_{[i]} \cdot \Omega = 0, \quad 0 \leq i < k \quad (2.9)$$

Then $k$-th term is given by

$$(e^{-\varepsilon(t)} d e^{\varepsilon(t)})_{[k]} \cdot \Omega = \sum_{i+j=k} (e^{-\varepsilon(t)})_{[i]} \cdot (de^{\varepsilon(t)})_{[j]} \cdot \Omega = (de^{\varepsilon(t)})_{[k]} \cdot \Omega \quad (2.10)$$

It follows from (2.5) that

$$(e^{-\varepsilon(t)} d e^{\varepsilon(t)})_{[k]} \cdot \Omega = (d\varepsilon(t))_{[k]} \cdot \Omega + \frac{1}{2} ([\varepsilon(t), \varepsilon(t)]_s)_{[k]} \cdot \Omega \quad (2.11)$$

We denote by $\text{Ob}_k$ the non-linear term $\frac{1}{2} ([\varepsilon(t), \varepsilon(t)]_s)_{[k]} \cdot \Omega$. Since $[\varepsilon(t), \varepsilon(t)]_s$ is a section of $\wedge^2 \mathcal{L}(-D) \boxplus \wedge^1 \mathcal{L}(-D)$, $\text{Ob}_k$ is a section of $E^2 = U^{-n+3} \boxplus U^{-n+1}$. It follows from (2.10) that $\text{Ob}_k$ is a $d$-exact differential form. Hence $\text{Ob}_k$ defines the cohomology class $[\text{Ob}_k] \in H^2(E^* \wedge)$ of the complex $(E^*, d)$. Since $M$ is a Kähler manifold, we apply the $\partial \bar{\partial}$-lemma to obtain the injective map $p_2$ from $H^2(E^*)$ to the direct sum of the de Rham cohomology groups. Since $\text{Ob}_k$ is $d$-exact, the image of the class $p_2([\text{Ob}_k]) = 0$. Hence the class $[\text{Ob}_k] \in H^2(E^*)$ vanishes, since the the map $p_2$ is injective. Then the Hodge decomposition of the complex $(E^*, d)$ shows that $\text{Ob}_k = dd^*G(\text{Ob}_k)$, where $d^*$ is the
formal adjoint and $G$ is the Green operator of the complex $(E^\bullet, d)$. Thus there is a unique section $\varepsilon_k$ of $\wedge^2 T(-D) \oplus \wedge^0 T(-D)$ such that
\[
\frac{1}{k!}\varepsilon_k \cdot \Omega = -d^* G(\text{Ob}_k) \in E^1,
\]
since $\wedge^2 T(-D) \oplus \wedge^0 T(-D) \cong E^1 = U^{-n} \oplus U^{-n+2}$. It follows that $\frac{1}{k!}d\varepsilon_k \cdot \Omega = -dd^* G(\text{Ob}_k) = -\text{Ob}_k$. Thus by the induction, we obtain the power series $\varepsilon(t)$ which satisfies the equation $(e^{-\varepsilon(t)} d e^{\varepsilon(t)})_{[k]} = 0$. Thus by the induction, we obtain the power series $\varepsilon(t)$ which satisfies the equation (2.7). As in [13] the power $\varepsilon(t)$ is a convergent series which is smooth. □

Theorem 2.5. Let $M = (X, J)$ be a compact Kähler manifold of dimension $n$ and we denote by $\mathcal{J}$ the generalized complex structure given by $J$. If $M$ has an effective, anti-canonical divisor $D$, then $M$ admits unobstructed $K$-deformations of generalized complex structures $\{J_t\}$ starting with $J_0 = J$ which are parametrized by an open set of $H^2((\mathcal{L}(-D))) \cong H^{n,2} \oplus H^{n-1,1} \oplus H^{n-2,0}$ with the origin, that is, there is a family of smooth global sections $\varepsilon(t)$ of the sheaf $\wedge^2 \mathcal{L}(-D)$ such that $J_t$ defined in (2.2) is an integrable generalized complex structure and $\frac{d}{dt}\varepsilon(t)|_{t=0} = \varepsilon_1$ for every representative $\varepsilon_1$ of $H^2(\mathcal{L}(-D))$ for small $t$, where $t$ is a parameter of deformations.

Proof. Let $\varepsilon_1$ be a representative of the cohomology group $H^2(\mathcal{L}(-D))$. Then $\varepsilon_1 \cdot \Omega$ is a smooth differential form with $\overline{\partial}\varepsilon_1 \cdot \Omega = 0$. Then it follows from the $\overline{\partial}\overline{\partial}$-lemma that there is a function $k_1$ of $\wedge^0 \mathcal{L}(-D)$ which satisfies $d(\varepsilon_1 + k_1) \cdot \Omega = 0$. We put $\tilde{\varepsilon}_1 = \varepsilon_1 + k_1$. Applying the theorem [2.4], we obtain a section $\tilde{\varepsilon}(t)$ of $\wedge^2 \mathcal{L}(-D) \oplus \wedge^0 \mathcal{L}(-D)$ with $d\tilde{\varepsilon}(t) \cdot \Omega = 0$. The section $\tilde{\varepsilon}(t)$ is written as $\tilde{\varepsilon}(t) = \varepsilon(t) + \kappa(t)$, where $\varepsilon(t) \in \wedge^2 \mathcal{L}(-D)$ and $\kappa \in \wedge^0 \mathcal{L}(-D)$. Since $\text{Ad}_{e^{\varepsilon(t)}} = \text{Ad}_{e^{\varepsilon(t)}}$, the section $\varepsilon(t) = \varepsilon_1 t + \frac{1}{2!} \varepsilon_2 t^2 + \cdots$ gives $K$-deformations as we want. □

By taking $\varepsilon(t)$ as a family of global sections of $\wedge^2 \mathcal{L}(-D) \cap (T^{1,0} \otimes \wedge^{0,1})$, we have unobstructed deformations of usual complex structures $J_t$ which is given by the adjoint action $\text{Ad}_{e^{\varepsilon(t)}}$, where $\text{Ad}_{e^{\varepsilon(t)}}$ is a family of sections of $\text{GL}(TX, \mathbb{C})$. Thus we obtain the following corollary, which is already obtained by Miyajima in the case of deformations of a normal isolated singularity [29].

Corollary 2.6. There is a family of deformations of complex structures $\{J_t\}$ starting with $J_0 = J$ which satisfies
\[
\frac{d}{dt}\varepsilon(t)|_{t=0} = \varepsilon_1,
\]
for every representative $\varepsilon_1$ of $H^1(M, \Theta(-D))$.

Proof. We define a sheaf $\wedge^r \mathcal{L}(-D)_{sl}$ by
\[
\wedge^r \mathcal{L}(-D)_{sl}(U) = \{ fa \mid f \in I_D(U), a \in T^{1,0} \otimes \wedge^{0,r-1} \}.
\]
The sheaf $\wedge^r \mathcal{L}(-D)_{\text{SL}}$ is the intersection $\wedge^r \mathcal{L}(-D) \cap (T^{1,0} \otimes \wedge^{0,r-1})$ which is locally free. Then $\wedge^r \mathcal{L}(-D)_{\text{SL}}$ is a sheaf of smooth sections of the vector bundle $\wedge^r \mathcal{L}(-D)_{\text{SL}}$. As before, by the action on the meromorphic form $\Omega$ with a simple pole along the anti-canonical divisor $D$, we have the identification,

$$\wedge^r \mathcal{L}(-D)_{\text{SL}} \cong \wedge^{n-1,r-1}$$

Then we have the complex $(\wedge^r \mathcal{L}(-D)_{\text{SL}}, \partial)$ which is isomorphic to the Dolbeault complex $(\wedge^{n-1,r}, \partial)$. We define vector bundles $E^*_{\text{SL}}$ by

$$E^0_{\text{SL}} = \wedge^{n-1,0}, \quad E^1_{\text{SL}} = \wedge^{n,0} \oplus \wedge^{n-1,1},$$
$$E^2_{\text{SL}} = \wedge^{n,1} \oplus \wedge^{n-1,2}, \quad E^3_{\text{SL}} = \wedge^{n,2} \oplus \wedge^{n-1,3}, \ldots$$

Then we have the complex $(E^*_{\text{SL}}, d)$ with the cohomology group $H^*_{\text{SL}}(E)$,

$$0 \to E^0_{\text{SL}} \xrightarrow{d} E^1_{\text{SL}} \xrightarrow{d} E^2_{\text{SL}} \xrightarrow{d} \cdots$$

Let $\varepsilon_1$ be a representative of $H^1(M, \Theta(-D))$. Then $\varepsilon_1$ is a smooth global section of $\wedge^2 \mathcal{L}(-D)_{\text{SL}}$ with $\partial \varepsilon_1 \cdot \Omega = 0$. Then it follows from the $\partial \overline{\partial}$-lemma that there is a section $\kappa_1$ of $\wedge^0 \mathcal{L}(-D)$ with $d(\varepsilon_1 + \kappa_1) \cdot \Omega = 0$. We put $\tilde{\varepsilon}_1 = \varepsilon_1 + \kappa_1$. For such a section $\tilde{\varepsilon}_1$ of $\wedge^2 \mathcal{L}(-D)_{\text{SL}} \oplus \wedge^0 \mathcal{L}(-D)$ with $d\tilde{\varepsilon}_1 \cdot \Omega = 0$, we shall construct a family of sections $\bar{e}(t)$ of $\wedge^2 \mathcal{L}(-D)_{\text{SL}} \oplus \wedge^0 \mathcal{L}(-D)$ which satisfies

$$d\bar{e}(t) \cdot \Omega = 0,$$

where $\bar{e}(t) = \varepsilon_1 t + \frac{1}{2!} \tilde{\varepsilon}_2 t^2 + \cdots$. As in the proof of the theorem 2.4, we have the obstruction $(\text{Ob}_k)_{\text{SL}}$ which gives the class $[(\text{Ob}_k)_{\text{SL}}] \in H^2(E_{\text{SL}})$. It suffices to show that the class $[(\text{Ob}_k)_{\text{SL}}]$ vanishes. In fact, the $(\text{Ob}_k)_{\text{SL}} \in E^2_{\text{SL}}$ is a $d$-exact differential form. It follows from the Hodge decomposition that the map $p_{\text{SL}}$ from $H^*_{\text{SL}}(E)$ to the direct sum of the de Rham cohomology groups is injective. Since $(\text{Ob}_k)_{\text{SL}}$ is $d$-exact, the image $p_{\text{SL}}^2([(\text{Ob}_k)_{\text{SL}}])$ vanishes. Thus the class $[(\text{Ob}_k)_{\text{SL}}] \in H^2(E_{\text{SL}})$ vanishes also. Hence we have $\bar{e}(t)$ with $d\bar{e}(t) \cdot \Omega = 0$. Let $e(t)$ be the component of $\bar{e}(t)$ of $\wedge^2 \mathcal{L}(-D)_{\text{SL}}$. Since $\text{Ad}_{e(t)} = \text{Ad}_{e(t)}$, we have deformations of generalized complex structures $J_t$ given by $e(t)$.

In the cases of Kähler surfaces, we obtain the following,

**Corollary 2.7.** Let $S$ be a compact Kähler surface with the complex structure $J$ and a Kähler form $\omega$. If $S$ has an effective, anti-canonical divisor $[D] = -K_S$, then $S$ admits unobstructed deformations of generalized complex structures parametrized by an open set of the full cohomology group $H^0(S) \oplus H^2(S) \oplus H^4(S)$ of even degree on $S$.

**Proof of Corollary 2.7.** Since $-K_S$ is effective, then we have the vanishing $H^2(S, \mathcal{O}_S) = H^0(S, K_S) \cong H^0(S, I_D) = \{0\}$. Thus $H^2(S) \cong H^{1,1}$. Then the result follows from the theorem 2.4. \qed
3 K-deformations of generalized complex structures in terms of $\text{CL}^2(-D)$

We denote by $\wedge^r L(-D)$ the complex conjugate of the bundle $\wedge^r \mathcal{T}(-D)$. A section of $\wedge^r L(-D)$ is locally written as $f \alpha$ for $f \in L^1$ and $\alpha \in \wedge^r \mathcal{T}$. Let $(\wedge^2 \mathcal{T}(-D) \oplus \wedge^2 L(-D))^\mathbb{R}$ be the real part of the bundle $\wedge^2 \mathcal{T}(-D) \oplus \wedge^2 L(-D)$, which is the subbundle of $\text{CL}^2$. We define a bundle $\text{CL}^2(-D)$ by

$$\text{CL}^2(-D) := (\wedge^2 \mathcal{T}(-D) \oplus \wedge^2 L(-D))^\mathbb{R} \oplus \wedge^0 \mathcal{T}(-D)$$

**Lemma 3.1.** For small deformations of almost generalized complex structures $\mathcal{J}_t$ given by a family of smooth global sections $\varepsilon(t)$ of $\wedge^2 \mathcal{T}(-D) \oplus \wedge^0 \mathcal{T}(-D)$ as in (2.2), there exists a unique family of global sections $a(t)$ of the bundle $\text{CL}^2(-D)$ such that

$$e^{\varepsilon(t)} \cdot \Omega = e^{a(t)} \cdot \Omega.$$

that is, $\mathcal{J}_t = \text{Ad}_{e^{a(t)}} \mathcal{J}_0$.

Conversely if we have a family of deformations of almost generalized complex structure $\mathcal{J}_t = \text{Ad}_{e^{a(t)}} \mathcal{J}_0$ which is given by the action of a family of global sections $a(t)$ of $\text{CL}^2(-D)$, then there exists a unique family of global sections $\varepsilon(t)$ of $\wedge^2 \mathcal{T}(-D) \oplus \wedge^0 \mathcal{T}(-D)$ such that $\mathcal{J}_t$ is given by the action of $\varepsilon(t)$ and $e^{a(t)} \cdot \Omega = e^{\varepsilon(t)} \cdot \Omega$.

**Proof.** For a section $\varepsilon$ of $\wedge^2 \mathcal{T}(-D) \oplus \wedge^0 \mathcal{T}(-D)$, we have a unique $a \in \Gamma(X, \text{CL}^2(-D))$ such that $e^\varepsilon \cdot \Omega = e^a \cdot \Omega$. Conversely, there is a unique section $\varepsilon$ of $\wedge^2 \mathcal{T}(-D) \oplus \wedge^0 \mathcal{T}(-D)$ such that $e^\varepsilon \cdot \Omega = e^a \cdot \Omega$ for any section $a$ of $\text{CL}^2(-D)$. Then applying the method in [13], we obtain the result.

The operator $e^{-a(t)} \circ d \circ e^{a(t)}$ acting on $K_J = U^{-n}$ is already discussed in [12] which is a Clifford-Lie operator of order 3 whose image is in $U^{-n+1} \oplus U^{-n+3}$.

It is shown in [13] that the almost generalized complex structure $\mathcal{J}_t = \text{Ad}_{e^{a(t)}} \mathcal{J}$ is integrable if and only if the projection to the component $U^{-n+3}$ vanishes, that is,

$$\pi_{U^{-n+3}} e^{-a(t)} \circ d \circ e^{a(t)} \cdot \Omega = 0$$

In particular, $e^{-a(t)} \circ d \circ e^{a(t)} \cdot \Omega = 0$ implies that the $\mathcal{J}_t$ is integrable. We denote by $(e^{-a(t)} \circ d \circ e^{a(t)})_k$ the $k$ th term of $e^{-a(t)} \circ d \circ e^{a(t)}$.

Thus by the theorem [24] and the lemma [3.1] we have the following,

**Proposition 3.2.** Let $M = (X, J)$ be a compact Kähler manifold with a Kähler form $\omega$. We assume that $M$ has an effective, anti-canonical divisor $D$. If there is a set of global sections $a_1, \cdots, a_{k-1}$ of $\text{CL}^2(-D)$ which satisfies

$$(e^{-a(t)} \cdot d e^{a(t)})_i \cdot \Omega = 0, \quad 0 \leq i < k,$$

(3.1)
and \(\|a(t)\|_s \ll C_1 M(t)\), then there is a global section \(a_k\) of \(CL^2(-D)\) which satisfies the followings:

\[ (e^{-a(t)} d e^{a(t)})_{[k]} \cdot \Omega = 0 \]

and \(\|a(t)\|_s \ll C_1 \lambda M(t)\), where \(a(t) = \sum_{i=1}^{\infty} \frac{1}{i!}a_i t^i\) and \(M(t)\) is the convergent series \((t, \delta)\) in section 7 and \(C_1\) is a positive constant and \(\|a(t)\|_s\) denotes the Sobolev norm of \(a(t)\).

(The proof of the inequality \(\|a(t)\|_s \ll C_1 \lambda M(t)\) is already seen in [13], see proposition 1.1 and 1.4 in [15] for more detail).

4 Deformations of generalized Kähler structures

Let \((X, J, \omega)\) be a compact Kähler manifold with an effective anti-canonical divisor \(D\) and \((\mathcal{J}, \mathcal{J}_\psi)\) the generalized Kähler structure induced from \((J, \omega)\) by \(\mathcal{J} = \mathcal{J}_J\) and \(\psi = e^{\sqrt{-1}\omega}\).

Since two generalized complex structures \(\mathcal{J}\) and \(\mathcal{J}_\psi\) are commutative, the generalized Kähler structure \((\mathcal{J}, \mathcal{J}_\psi)\) gives the simultaneous decomposition of \((T \oplus T^*)^C\),

\[ (T \oplus T^*)^C = L^+_J \oplus L^-_J \oplus T^+_J \oplus T^-_J, \]

where \(L^+_J \oplus L^-_J\) is the eigenspace with eigenvalue \(\sqrt{-1}\) with respect to \(\mathcal{J}\) and \(L^+_J \oplus T^-_J\) is the eigenspace with eigenvalue \(-\sqrt{-1}\) with respect to \(\mathcal{J}_\psi\) and \(T^+_J \oplus T^-_J\) denotes the complex conjugate. In [13] [14], the author showed the stability theorem of generalized Kähler structures with one pure spinor, which implies that if there is a one dimensional analytic deformations of generalized complex structures \(\{\mathcal{J}_t\}\) parametrized by \(t\), then there exists a family of non-degenerate, \(d\)-closed pure spinor \(\psi_t\) such that the family of pairs \((\mathcal{J}_t, \psi_t)\) becomes deformations of generalized Kähler structures starting from \((\mathcal{J}_0, \psi_0)\).

As in section 2, small K-deformations \(\mathcal{J}_t\) are given by the adjoint action of a family of sections \(a(t)\) of \(CL^2(-D)\),

\[ \mathcal{J}_t := \text{Ad}_{e^{a(t)}} \mathcal{J}_0. \]

Then we can obtain a family of real sections \(b(t)\) of the bundle \((L^-_{\mathcal{J}_0} \cdot T^+_{\mathcal{J}_0} \oplus T^-_{\mathcal{J}_0} \cdot L^+_{\mathcal{J}_0})^R\) such that \(\psi_t = e^{a(t)}_t e^{b(t)} \psi_0\) is a family of non-degenerate, \(d\)-closed pure spinor. The bundle \(K^1 = U^{0, -n+2}\) is generated by the action of real sections of \((L^-_{\mathcal{J}_0} \cdot T^+_{\mathcal{J}_0} \oplus T^-_{\mathcal{J}_0} \cdot L^+_{\mathcal{J}_0})\) on \(\psi\) (see page 125 in [14] for more detail).

We define a family of sections \(Z(t)\) of \(CL^2\) by

\[ e^{Z(t)} = e^{a(t)} e^{b(t)}. \]

Since \(\text{Ad}_{e^{b(t)}} \mathcal{J}_0 = \mathcal{J}_0\), we obtain \(\mathcal{J}_t = \text{Ad}_{e^{a(t)}} \mathcal{J}_0 = \text{Ad}_{e^{a(t)}} \mathcal{J}_0 = \text{Ad}_{e^{a(t)}} \mathcal{J}_0 = \text{Ad}_{e^{Z(t)}} \mathcal{J}_0\). Then the family of deformations of generalized Kähler structures is given by the action of \(e^{Z(t)}\),

\[ (\mathcal{J}_t, \psi_t) = (\text{Ad}_{e^{Z(t)}} \mathcal{J}_0, e^{Z(t)} \cdot \psi_t). \]
By the similar method as in [14] together with the proposition 3.2 we obtain the following proposition,

**Proposition 4.1.** Let $M = (X, J)$ be a compact Kähler manifold with a Kähler form $\omega$. We assume that $M = (X, J)$ has an anti-canonical divisor $D$. If there is a set of sections $a_1, \ldots, a_{k-1}$ of $CL^2(-D)$ which satisfies

$$
\left(e^{-a(t)} de^{a(t)}\right)_{[i]} \cdot \Omega = 0, \quad 0 \leq i < k,
$$

and $\|a(t)\|_s \ll K_1 M(t)$ for a positive constant $K_1$, then there is a set of real sections $b_1, \ldots, b_k$ of the bundle $(L_{-J_0} \cdot L_{J_0}^+ \oplus L_{-J_0} \cdot L_{J_0}^+)$ which satisfies the following equations:

$$
\left(e^{-Z(t)} de^{Z(t)}\right)_{[k]} \cdot \Omega = 0 \quad (4.1)
$$

$$
\left(de^{Z(t)} \cdot \psi_0\right)_{[i]} = 0, \quad \text{for all } i \leq k \quad (4.2)
$$

$$
\|a(t)\|_{s_k} \ll K_1 \lambda M(t) \quad (4.3)
$$

$$
\|b(t)\|_{s_k} \ll K_2 M(t) \quad (4.4)
$$

where $a_k$ is the section constructed in the proposition 3.2 and $e^{Z(t)} = e^{a(t)} e^{b(t)}$ and $M(t)$ is the convergent series in the proposition 3.2 and a positive constant $K_2$ is determined by $\lambda$ and $K_1$. The constant $\lambda$ in $M(t)$ is sufficiently small which will be suitably selected to show the convergence of the power series $Z(t)$ as in [13].

5 Deformations of bihermitian structures

There is a one to one correspondence between generalized Kähler structures and bihermitian structures with the condition (1.1) [16]. In this section we shall give an explicit description of $\Gamma_t^+$ which gives rise to deformations of bihermitian structures $(J_t^+, J_t^-)$ corresponding to deformations of generalized Kähler structures with one pure spinor $(J_t, \psi_t)$ in section 4. The correspondence is defined at each point on a manifold. The non-degenerate, pure spinor $\psi_t$ induces the generalized complex structure $J_{\psi_t}$. Since $(J_t, J_{\psi_t})$ is a generalized Kähler structure and $J_t$ commutes with $J_{\psi_t}$, we have the simultaneous decomposition of $(T \oplus T^*)^C$ into four eigenspaces as before,

$$
(T \oplus T^*)^C = L_{J_t}^+ \oplus L_{J_t}^- \oplus L_{J_{\psi_t}}^+ \oplus L_{J_{\psi_t}}^-,
$$

where each eigenspace is given by the intersection of eigenspaces of both $J_t$ and $J_{\psi_t}$,

$$
L^-_{J_t} = L_{J_t} \cap L_{\psi_t}, \quad L^+_{J_t} = L_{J_t}^* \cap L_{\psi_t}^-,
$$

$$
L^+_{J_t} = L_{J_t} \cap L_{\psi_t}, \quad L^-_{J_t} = L_{J_t}^* \cap L_{\psi_t}^-.
$$
where $L_{\mathcal{J}_t}$ is the eigenspace of $\mathcal{J}_t$ with eigenvalue $\sqrt{-1}$ and $L_{\psi_t}$ denotes the eigenspace of $\mathcal{J}_{\psi_t}$ with eigenvalue $\sqrt{-1}$. Since $\mathcal{J}_t = \text{Ad}_{e^{Z(t)}}(\mathcal{J}_0) = \text{Ad}_{e^{Z(t)}} \circ \mathcal{J}_0 \circ \text{Ad}_{e^{-Z(t)}}$ and $\mathcal{J}_{\psi_t} = \text{Ad}_{e^{Z(t)}}(\mathcal{J}_\omega)$, we have the isomorphism between eigenspaces,

$$\text{Ad}_{e^{Z(t)}} : \overline{L_{\mathcal{J}_t}^\pm} \rightarrow \overline{L_{\mathcal{J}_t}^\pm}.$$

Let $\pi$ be the projection from $T \oplus T^*$ to the tangent bundle $T$. We restrict the map $\pi$ to the eigenspace $\overline{L_{\mathcal{J}_t}^\pm}$ which yields the map $\pi_t^\pm : \overline{L_{\mathcal{J}_t}^\pm} \rightarrow T^\mathbb{C}$. Let $T_{j_t^\pm}^{1,0}$ be the complex tangent space of type $(1,0)$ with respect to $J_{j_t^\pm}$. Then it follows that $T_{j_t^\pm}^{1,0}$ is given by the image of $\pi_t^\pm$,

$$T_{j_t^\pm}^{1,0} = \pi_t^\pm(\overline{L_{\mathcal{J}_t}^\pm})$$

Since deformations of generalized Kähler structures are given by the action of $e^{Z(t)}$, the ones of bihermitian structures $J_{j_t^\pm}$ should be described by the action of $\Gamma_t^\pm$ of the bundle $GL(T)$ which is obtained from $Z(t)$. We shall describe $\Gamma_t^\pm$ in terms of $a(t)$ and $b(t)$. A local basis of $\overline{L_{\mathcal{J}_0}^\pm}$ is given by

$$\{ \text{Ad}_{e^{\pm\sqrt{-1}\omega}} V_i = V_i \pm \sqrt{-1}[\omega, V_i] \}_{i=1}^n,$$

for a local basis $\{V_i\}_{i=1}^n$ of $T^{1,0}_j$, where we regard $\omega$ as an element of the Clifford algebra and then the bracket $[\omega, V_i]$ coincides with the interior product $i_{V_i}\omega$. It follows that the inverse map $(\pi_0^\pm)^{-1} : T_{j_t^\pm}^{1,0} \rightarrow \overline{L_{\mathcal{J}_t}^\pm}$ is given by the adjoint action of $e^{\pm\sqrt{-1}\omega}$,

$$\text{Ad}_{e^{\pm\sqrt{-1}\omega}} = (\pi_0^\pm)^{-1}. \quad (5.1)$$

We define a map $(\Gamma_t^\pm)^{1,0} : T_{j_t^\pm}^{1,0} \rightarrow T_{j_t^\pm}^{1,0}$ by the composition,

$$(\Gamma_t^\pm)^{1,0} = \pi_t^\pm \circ \text{Ad}_{e^{Z(t)}} \circ (\pi_0^\pm)^{-1} = \pi \circ \text{Ad}_{e^{Z(t)}} \circ \text{Ad}_{e^{\pm\sqrt{-1}\omega}} \quad (5.2)$$

$$(\Gamma_t^\pm)^{1,0}$$

Together with the complex conjugate $(\Gamma_t^\pm)^{0,1} : T_{j_t^\pm}^{0,1} \rightarrow T_{j_t^\pm}^{0,1}$, we obtain the map $\Gamma_t^\pm$ which satisfies $J_{j_t^\pm} = (\Gamma_t^\pm)^{-1} \circ J \circ \Gamma_t^\pm$.

Let $J^*$ be the complex structure on the cotangent space $T^*$ which is given by $\langle J^* \eta, v \rangle = \langle \eta, J v \rangle$, where $\eta \in T^*$ and $v \in T$ and $\langle , \rangle$ denote the coupling between $T$ and $T^*$. We
define a map \( \hat{J}^\pm : T \oplus T^* \rightarrow T \oplus T^* \) by \( \hat{J}^\pm(v, \eta) = v \mp J^* \eta \) for \( v \in T \) and \( \eta \in T^* \). Then \( \Gamma^\pm_t \) is written as

\[
\Gamma^\pm_t = \pi \circ \text{Ad}_{e^{z(t)}} \circ \hat{J}^\pm \circ \text{Ad}_{e^\varphi}
\]

(5.4)

\[
= \pi \circ \text{Ad}_{e^{a(t)}} \circ \text{Ad}_{\varphi(t)} \circ \hat{J}^\pm \circ \text{Ad}_{e^\varphi} \in \text{End}(T),
\]

(5.5)

where note that \( \hat{J}^\pm \circ \text{Ad}_{e^\varphi}(T_{j,0}^{1,0}) = \overline{T_{j,0}^{1,0}} \). The \( k \) th term of \( \Gamma^\pm_t \) is denoted by \( (\Gamma^\pm_t)_{[k]} \) as before. Note that \( (\Gamma^\pm_t)_{[0]} = \text{id}_T \). We also put \( \Gamma^\pm(a(t), b(t)) = \Gamma^\pm_t \).

**Lemma 5.1.** The \( k \) th term \( (\Gamma^\pm_t)_{[k]} \) is given by

\[
(\Gamma^\pm_t)_{[k]} = \frac{1}{k!} \pi \circ (\text{ad}_{a_k} + \text{ad}_{b_k}) \circ \hat{J}^\pm \circ \text{Ad}_{e^\varphi} + \Gamma_k^\pm(a_{<k}, b_{<k})
\]

where the second term \( \Gamma_k^\pm(a_{<k}, b_{<k}) \) depends only on \( a_1, \cdots, a_{k-1} \) and \( b_1, \cdots, b_{k-1} \).

**Proof.** Substituting the identity \( \text{Ad}_{e^{z(t)}} = \text{id} + \text{ad}_{Z(t)} + \frac{1}{2!}(\text{ad}_{Z(t)})^2 + \cdots \), we have

\[
\Gamma^\pm_t = \pi \circ \text{Ad}_{e^{z(t)}} \circ \hat{J}^\pm \circ \text{Ad}_{e^\varphi}
\]

(5.6)

\[
= \pi \circ \left( \sum_{i=0}^{\infty} \frac{1}{i!} \text{ad}_{Z(t)}^i \circ \hat{J}^\pm \circ \text{Ad}_{e^\varphi} \right)
\]

(5.7)

Then \( k \)-th term is given by

\[
(\Gamma^\pm_t)_{[k]} = \pi \circ \left( \text{ad}_{Z(t)} \circ \hat{J}^\pm \circ \text{Ad}_{e^\varphi} \right)_{[k]} + \sum_{i=2}^{k} \pi \circ \left( \frac{1}{i!} \text{ad}_{Z(t)}^i \circ \hat{J}^\pm \circ \text{Ad}_{e^\varphi} \right)_{[k]}
\]

(5.9)

\[
= \frac{1}{k!} \pi \circ (\text{ad}_{a_k} + \text{ad}_{b_k}) \circ \hat{J}^\pm \circ \text{Ad}_{e^\varphi} + \Gamma_k^\pm(a_{<k}, b_{<k}),
\]

(5.10)

where \( \Gamma_k^\pm(a_{<k}, b_{<k}) \) denotes the non-linear term depending \( a_1, \cdots, a_{k-1} \) and \( b_1, \cdots, b_{k-1} \).

\( \square \)

**Lemma 5.2.** Let \( b \) be a section of the bundle \( (L^-_J \cdot \overline{L}^+_{J^*} \oplus \overline{L}^-_{J^*} \cdot L^+_J) \). Then we have

\[
[\pi \circ \text{Ad}_{\varphi} \circ \hat{J}^\pm \circ \text{Ad}_{e^\varphi}, J] = 0 \in \text{End}(T).
\]

**Proof.** For simplicity, we write \( L^\pm \) for \( L^\pm_J \). Note that \( [\pi \circ \hat{J}^\pm \circ \text{Ad}_{e^\varphi}, J] = [\text{id}_T, J] = 0 \) and \( \text{Ad}_{\varphi} = \text{id} + \text{ad}_b + \frac{1}{2!}(\text{ad}_b)^2 + \cdots \). We also recall the image \( \hat{J}^\pm \circ \text{Ad}_{e^\varphi}(T_{j,0}^{1,0}) = (\pi_0^+)^{-1}(T_{j,0}^{1,0}) = \overline{L}^\pm_J \). Since \( b \) is a section \( \overline{L}^+ \cdot L^- \oplus L^+ \cdot \overline{L}^- \), the image \( (\text{ad}_b)^n(\overline{L}^\pm) \) is given by

\[
\begin{align*}
(\text{ad}_b)^n(\overline{L}^+_{J}) &= \overline{L}^+, \quad (n: \text{even}) \\
(\text{ad}_b)^n(\overline{L}^+_{J}) &= \overline{L}^+, \quad (n: \text{odd})
\end{align*}
\]
Since $\pi(\mathbb{L}^\pm) = T_j^{1,0}$, we have $\pi \circ (\text{ad}_b)^n \circ \hat{J}^\pm \circ \text{Ad}_\omega(T_j^{1,0}) = T_j^{1,0}$. Thus $\pi(\text{ad}_b)^n \circ \hat{J}^\pm \circ \text{Ad}_\omega \in \text{End}(T)$ preserves $T_j^{1,0}$. Hence we have

$$\left[ \pi \circ (\text{ad}_b)^n \circ \hat{J}^\pm \circ \text{Ad}_\omega, J \right] = 0.$$ 

Then the result follows. \qed

The tensor space $T \otimes T^*$ defines a subbundle of $\text{CL}^2$. We denote it by $T \cdot T^*$. An element $\gamma \in T \cdot T^*$ gives the endomorphism $\text{ad}_\gamma$ by $\text{ad}_\gamma E = [\gamma, E]$ for $E \in T \oplus T^*$, which preserves the tangent bundle $T$ and the cotangent bundle $T^*$ respectively. We also regard $\text{ad}_\gamma$ as a section of $\text{End}(T)$.

**Lemma 5.3.** Let $\gamma$ be an element of $T \cdot T^*$. Then we have

$$\pi \circ (\text{ad}_\gamma \circ \hat{J}^\pm \circ \text{Ad}_\omega) = \text{ad}_\gamma \in \text{End}(T).$$

**Proof.** For a tangent vector $v \in T$, we have $\text{Ad}_\omega v = v + [\omega, v] = v + \text{ad}_\omega v$. Since the map $\text{ad}_\gamma$ preserves the cotangent $T^*$, we have $\text{ad}_\gamma \circ \hat{J}^\pm \circ \text{ad}_\omega(v) \in T^*$ for all tangent $v \in T$. Thus it follows that $\pi(\text{ad}_\gamma \circ \hat{J}^\pm \circ \text{ad}_\omega) = 0$, since $\pi$ is the projection to the tangent $T$. Thus we obtain the result. \qed

**Lemma 5.4.** We assume that there is a set of sections $a_1, \ldots, a_k$ of $\text{CL}^2(-D)$ and real sections $b_1, \ldots, b_k$ of $(L_{\mathcal{J}_0}^\pm \cdot \mathbb{L}_{\mathcal{J}_0}^\pm \cdot L_{\mathcal{J}_0}^\pm)$ which satisfies the following equations,

$$e^{-Z(t)} \cdot d e^{Z(t)} \cdot [i] \cdot \Omega = 0, \quad 0 \leq \forall i \leq k$$

$$d e^{Z(t)} \cdot [i] \cdot \psi_0 = 0, \quad 0 \leq \forall i \leq k$$

$$[\Gamma^\pm_{\frac{1}{2}}(i), J] = 0, \quad 0 \leq \forall i < k$$

Then the $k$-th term $(\Gamma^\pm_{\frac{1}{2}})[k]$ satisfies

$$\pi_{U^{-n+3}}[d, (\Gamma^\pm_{\frac{1}{2}})[k]] = 0,$$

where $[d, (\Gamma^\pm_{\frac{1}{2}})[k]]$ is an operator from $U^{-n} = K_{\mathcal{J}}$ to $U^{-n+1} \oplus U^{-n+3}$ and $\pi_{U^{-n+3}}$ denotes the projection to the component $U^{-n+3}$.

**Proof.** Since the obstructions to $K$-deformations of generalized complex structures vanishes, we obtain a family of section $\hat{a}(t)$ with $\hat{a}_i = a_i$ for $i = 1, \ldots, k$ such that $\hat{a}(t)$ gives $K$-deformations of generalized complex structures, that is,

$$\pi_{U^{-n+3}} e^{-\hat{a}(t)} \cdot d e^{\hat{a}(t)} \cdot \Omega = 0.$$
The stability theorem of generalized Kähler structures in [13] provides deformations of generalized Kähler structures with one pure spinor:

\[ (\text{Ad}_{e^{2(t)}}, \mathcal{J}_0, e^{\tilde{Z}(t)} \psi_0), \]

where \( e^{\tilde{Z}(t)} = e^{{\tilde{a}(t)}} e^{{\tilde{b}(t)}} \), where \( \tilde{b}(t) \) is a family of real sections with \( \tilde{b}_i = b_i \), for \( i = 1, \cdots, k \). From the correspondence between generalized Kähler structures and bihermitian structures, we have the family of bihermitian structures \((J_\xi^+, J_\xi^-)\) which is given by the action of \( \Gamma^\pm_\xi := \Gamma^\pm_\xi (\tilde{a}(t), \tilde{b}(t)) \) of \( \text{GL}(T) \). Since \( J_\xi^\pm \) is integrable, we have

\[ \pi_{U_{-n+3}} ((\Gamma^\pm_\xi)^{-1} d \Gamma^\pm_\xi) = 0. \] (5.11)

Let \( \Omega \) be a \( d \)-closed meromorphic form of type \((n, 0)\) with a simple pole along \( D \) as before. Then we have

\[ d \Gamma^\pm_\xi \Omega \equiv \Gamma^\pm_\xi E(t) \Omega. \]

Since \( d \Omega = 0 \), the degree of \( E(t) \) is greater than or equal to 1. The condition \([ (\Gamma^\pm_\xi)_{ij}, J] = 0 \), \( (0 \leq i < k) \) implies that \( (\Gamma^\pm_\xi)_{ij} E(t) \Omega \in U_{-n+1} \). Thus we have

\[ d(\Gamma^\pm_\xi)_{ij} \Omega = \sum_{0 \leq i, j < k} (\Gamma^\pm_\xi)_{ij} E(t)_{ij} \Omega \in U_{-n+1} \]

Hence we have \( \pi_{U_{-n+3}} [d, (\Gamma^\pm_\xi)_{ij}] = 0 \). \( \square \)

**Lemma 5.5.** For a section \( a \) of \( \text{CL}^2(-D) \) and every section \( P \) of \( \text{End}(T \oplus T^*) \), we define a section \( \zeta \) of \( T \cdot T^* \) by

\[ a \| \| \sigma \| \| \pi | \| (P \| \| T) \| \| J \]\n
Then \( \zeta \) is a section of \( \text{CL}^2(-D) \), where \( P | T : T \rightarrow T \oplus T^* \) denotes the restriction to the tangent bundle \( T \).

**Proof.** As in section 2, we have

\[ \wedge^2 \mathcal{T}(-D) = (\wedge^{0,2} \{ -D \}) \oplus (T^{1,0} \otimes \wedge^{0,1} \{ -D \}) \oplus (T^{2,0} \otimes \{ -D \}), \]

Thus a section \( \epsilon \in \wedge^2 \mathcal{T}(-D) \) gives \( \pi \circ \text{ad}_a(E) \in T^{1,0}(-D) \) for all \( E \in T \oplus T^* \). Since \( \text{CL}^2(-D) = (\wedge^2 \mathcal{T}(-D) \oplus \wedge^2 \mathcal{L}(-D)) \otimes \wedge^1 \mathcal{L}(T) \), we have \( \pi \circ \text{ad}_a(E) \in T^{1,0}(-D) \oplus T^{0,1}(-D) \) for all \( E \in T \oplus T^* \). Hence \( \pi \circ \text{ad}_a \circ P | T \) is a section of \( (T^{1,0}(-D) \oplus T^{0,1}(-D)) \otimes \wedge^1 \). Taking the bracket, it turns out that \( \pi \circ \text{ad}_a \circ P | T \) is a section of \( (T^{1,0}(-D) \otimes \wedge^{0,1}) \oplus (T^{0,1}(-D) \otimes \wedge^{1,0}) \) which is the subbundle \( \wedge^2 \mathcal{T}(-D) \oplus \wedge^2 \mathcal{L}(-D) \). Thus \( \zeta \) is a real section of \( \wedge^2 \mathcal{T}(-D) \oplus \wedge^2 \mathcal{L}(-D) \subset \text{CL}^2(-D) \). \( \square \)
Lemma 5.6. We define a section \( \zeta_k \) of \( T \cdot T^* \) by

\[
\text{ad}_{\zeta_k} = [(\Gamma^+_t)|_k, J] \in \text{End} \,(T)
\]

for a section \( a(t) \) of \( \text{CL}^2(-D) \) and \( b(t) \in (L_T \cdot T_T^+ \oplus T_T^+ \cdot L_T^-) \), where \( (\Gamma^+_t)|_k := \Gamma^+(a(t), b(t)) |_k \).

Then \( \zeta_k \) is a section of \( \text{CL}^2(-D) \). Further we define \( \gamma_k \) by

\[
\text{ad}_{\gamma_k} := -\frac{1}{(2\sqrt{-1})^2}[\text{ad}_{\zeta_k}, J].
\]

Then \( \gamma_k \in T \cdot T^* \) is also a section of \( \text{CL}^2(-D) \) which satisfies

\[
\text{ad}\zeta_k + [\text{ad}\gamma_k, J] = 0.
\]

Proof. From the description in (5.4),

\[
\Gamma^+_t = \pi \circ \text{Ad}_{e^{a(t)}} \circ \text{Ad}_{e^{b(t)}} \circ \hat{J}^+ \circ \text{Ad}_\omega \quad (5.12)
\]

\[
= \pi \circ (\text{Ad}_{e^{a(t)}} - \text{id}) \circ Q + \pi \circ Q \quad (5.13)
\]

where \( Q := \text{Ad}_{e^{b(t)}} \circ \hat{J}^+ \circ \text{Ad}_\omega \).

From the lemma 5.2 we have \([\pi \circ Q, J] = 0\). Thus we have

\[
[\Gamma^+_t, J] = [\pi \circ (\text{Ad}_{e^{a(t)}} - \text{id}) \circ Q |_T, J]
\]

Since we have \( \text{Ad}_{e^{a(t)}} - \text{id} = \text{ad}_{a(t)} \circ R \), where \( R = \sum_{j=1}^{\infty} \frac{1}{j!} \text{ad}_{a(t)}^{j-1} \). If we set \( P = R \circ Q \), we have

\[
[\Gamma^+_t, J] = [\pi \circ \text{ad}_{a(t)} \circ P |_T, J]
\]

Then it follows from the lemma 5.5 that \( \zeta_k \) is a section of \( \text{CL}^2(-D) \). We decompose \( \text{ad}\zeta_k \) by

\[
\text{ad}\zeta_k = (\text{ad}\zeta_k)' + (\text{ad}\zeta_k)''
\]

where \( (\text{ad}\zeta_k)' \in T^{1,0}(-D) \otimes \Lambda^{0,1} \) and \( (\text{ad}\zeta_k)'' \in T^{0,1}(-D) \otimes \Lambda^{1,0} \). Then the bracket is given by \([\text{ad}\zeta_k, J] = -2\sqrt{-1}(\text{ad}\zeta_k)' + 2\sqrt{-1}(\text{ad}\zeta_k)'' \in \text{CL}^2(-D) \). Thus \( \gamma_k \) is also a section of \( \text{CL}^2(-D) \) which satisfies

\[
\text{ad}\zeta_k + [\text{ad}\gamma_k, J] = 0.
\]

Lemma 5.7. Let \( \Gamma^+_t \) be a section of \( \text{GL}(T) \) given in the lemma (5.4) and \( \zeta_k \) and \( \gamma_k \) be as in the lemma (5.6). Then there is a global function \( \rho_k \) of \( \Lambda^0 \mathcal{E}(-D) \) such that

\[
d\gamma_k \cdot \Omega = d(\rho_k \Omega),
\]

\[
\Box
\]
Proof. The condition \( \pi_{U^{-n+1}}[d, (\Gamma_t^\pm)[k]] = 0 \) in the lemma implies that \((d\Gamma_t^\pm)[k] \cdot \Omega \in U^{-n+1} \). Thus we have that \( d\gamma_k \cdot \Omega \in U^{-n+1} \). Since \( \text{ad}_{\gamma_k} \) is a section of \( \text{GL}(T) \), we see that \( d\gamma_k \cdot \Omega \) is a \( d \)-exact form of type \((n,1)\). Then applying the \( \partial \overline{\partial} \)-lemma, it turns out that \( d\gamma_k \cdot \Omega = d\omega_k \Omega \) for a smooth function \( \omega_k \). Since \( \omega_k \Omega \) is smooth, we have \( \omega_k \) is a global function of \( \wedge^0 \overline{\Omega}(-D) \).

6 Bihermitian structures on compact Kähler surfaces

Let \( S \) be a compact Kähler surface with a Kähler form \( \omega \) with an anti-canonical divisor \( D \). The divisor \( D \) is given as the zero locus of a section \( \beta \in H^0(S, K^{-1}) \cong H^0(S, \wedge^2 \Theta) \). Then the section \( \beta \) is also regarded as a section of \( H^0(S, \wedge^2 \Theta(-D)) \) which is a holomorphic Poisson structure vanishing along the divisor \( D \). The contraction \( \beta \cdot \omega \) of \( \beta \) by \( \omega \) is defined by the commutator \([\beta, \omega]\) which is a \( \overline{\partial} \)-closed \( T^{1,0} \)-valued form of type \((0,1)\). Let \( \Omega \) be the meromorphic 2-form on \( S \) with a pole along the divisor \( D \) with \( \beta \cdot \Omega = 1 \). Then we have

\[
(\beta \cdot \omega) \cdot \Omega = [\beta, \omega] \cdot \Omega = -\omega,
\]

since \( \omega \cdot \Omega = 0 \). Thus \( \beta \cdot \omega \) is a section of \( T^{1,0}(-D) \otimes \wedge^{0,1} \) which gives the class \([\beta \cdot \omega] \in H^1(S, \Theta(-D))\).

Then applying unobstructed deformations in the theorem we obtain the following,

**Theorem 6.1.** Let \( S \) be a compact Kähler surface with complex structure \( J \) and Kähler form \( \omega \). We denote by \( g \) the Kähler metric on the Kähler surface \((S, J, \omega)\). If there is a non-zero holomorphic Poisson structure \( \beta \) on \( S \), then the surface \( S \) admits deformations of bihermitian structures \((J, J_t^{-}, h_t)\) which satisfies \( J_0^{-} = J, h_0 = g \) and

\[
\frac{d}{dt} J_t^{-} \big|_{t=0} = -2(\beta \cdot \omega + \overline{\beta} \cdot \omega),
\]

(6.1)

where \( \beta \cdot \omega \) is the \( \overline{\partial} \) closed \( T^{1,0} \)-valued forms of type \((0,1)\) which gives the Kodaira-Spencer class \(-2[\beta \cdot \omega] \in H^1(S, \Theta)\) of the deformations \( \{J_t^{-}\} \). In particular, if the class \([\beta \cdot \omega] \in H^1(S, \Theta)\) does not vanish, then \((J, J_t^{-}, h_t)\) is a distinct bihermitian structure for small \( t \neq 0 \).

Proof. For a family of sections \( a(t) \) of \( \text{CL}^2(-D) \) and real sections \( b(t) \) of \((\overline{L}_J \cdot L_J^{-} \oplus L^+_J \cdot \overline{L}_J)\), we define a family of section \( Z(t) \) of \( \text{CL}^2 \) by \( e^{Z(t)} = e^{a(t)} e^{b(t)} \), where we denote by \((\overline{L}_J \cdot L_J^{-} \oplus L^+_J \cdot \overline{L}_J)\) the real subbundle of the bundle \((\overline{L}_J \cdot L_J^{-} \oplus L^+_J \cdot \overline{L}_J)\). Since \( b(t) \cdot \Omega = 0 \), we have

\[
de^{Z(t)} \cdot \Omega = de^{a(t)} e^{b(t)} \cdot \Omega = de^{a(t)} \cdot \Omega.
\]
Then since $a(t)$ is a section of $\text{CL}^2(-D)$, it turns out that $de^{Z(t)} \cdot \Omega$ is a smooth differential form on $S$. The action of $Z(t) \in \text{CL}^2$ gives rise to almost bihermitian structures $(J^+_t, J^-_t, h_t)$ with $J^+_0 = J$ and $h_0 = g$.

We shall construct $a(t)$ and $b(t)$ which satisfy the following three equations,

\begin{align*}
de^{Z(t)} \cdot \Omega &= 0 \quad (6.2) \\
de^{Z(t)} \cdot \psi &= 0 \quad (6.3) \\
J^+_t &= J, \quad (6.4)
\end{align*}

As in section 5, the structure $J^\pm_t$ is described by the adjoint action of a section $\Gamma^\pm_t = \Gamma^\pm(a(t), b(t)) \in \text{GL}(T)$. Then the equation $J^+ = J$ is equivalent to $[\Gamma^+_t, J] = [\Gamma(a(t), b(t)), J] = 0$. We denote by $(de^{Z(t)})_{[i]}$ the $i$-th term of $(de^{Z(t)})$ in $t$ and $(\Gamma^+_t)_{[i]}$ is also the $i$-th term of $(\Gamma^+_t)$ in $t$. Thus the three equations are reduced to the following equations for all integer $i \geq 0$:

\begin{align*}
(de^{Z(t)})_{[i]} \cdot \Omega &= 0 \quad (6.5) \\
(de^{Z(t)})_{[i]} \cdot \psi &= 0 \quad (6.6) \\
[\Gamma^+_t]_{[i]}, J] &= 0. \quad (6.7)
\end{align*}

We shall construct our solutions by the induction on $t$.

At first, we set $\hat{a}_1 := \beta + \bar{\beta}$. Then the proposition 4.1 yields a real section $\hat{b}_1 \in (\mathcal{L}^+_\mathcal{J} \cdot \mathcal{L}^-_\mathcal{J} \oplus \mathcal{L}^+_\mathcal{J} \cdot \mathcal{L}^-_\mathcal{J})^\mathbb{R}$ such that

\begin{align*}
d(\hat{a}_1 + \hat{b}_1) \cdot \Omega &= d\beta \cdot \Omega = 0 \quad (6.8) \\
d(\hat{a}_1 + \hat{b}_1) \cdot \psi &= 0, \quad (6.9)
\end{align*}

where we set $\beta \cdot \Omega = 1$. We denote by $\Gamma^+(\hat{a}_1, \hat{b}_1)$ the first term $(\Gamma^+_t)_{[1]}$ in $t$ for $\hat{a}_1, \hat{b}_1$. Then from the lemma 5.1 we have

$$
\Gamma^+(\hat{a}_1, \hat{b}_1) = \pi \circ \left( (\text{ad}_{\hat{a}_1} + \text{ad}_{\hat{b}_1}) \circ \hat{J}^+ \circ \text{Ad}_{\omega} \right) \quad (6.10)
$$

As in the lemma 5.6, we define $\gamma_1 \in T \cdot T^*$ and $\text{ad}_{\gamma_1}$ by

\begin{align*}
\text{ad}_{\gamma_1} &:= \left[ \Gamma^+(\hat{a}_1, \hat{b}_1), J \right] \quad (6.11) \\
\text{ad}_{\gamma_1} &:= \frac{-1}{(2\sqrt{-1})^2} \left[ \text{ad}_{\gamma_1}, J \right] \quad (6.12)
\end{align*}

Then it follows from the lemma 5.6 that $\gamma_1$ is a section of $\text{CL}^2(-D)$ and we have

$$
[\text{ad}_{\gamma_1}, J] + \left[ \Gamma^+(\hat{a}_1, \hat{b}_1), J \right] = \frac{-1}{(2\sqrt{-1})^2} \left[ \left[ \text{ad}_{\gamma_1}, J \right], J \right] + \text{ad}_{\gamma_1} = 0 \quad (6.13)
$$
From the lemma 5.7, we have $d\gamma_1 \cdot \Omega = -d\rho_1 \Omega$, where $\rho_1$ is a function with $\rho_1 \Omega$ is a smooth form, that is, $\rho_1$ is a section of $\bigwedge^0 \mathcal{L}(-D)$. Then we define $a_1$ by

$$a_1 = \hat{a}_1 + \gamma_1 + \rho_1 \quad (6.14)$$

Then we have $da_1 \Omega = 0$. Then applying the proposition 4.1 again, we have a section $b_1$ of $(\mathcal{L}^+ \cdot L^- \oplus L^+_+ \cdot \mathcal{L}^-)_R$ such that $d(a_1 + b_1) \cdot \psi = 0$. From the lemma 5.1 and the lemma 5.3 we have

$$\Gamma^+(a_1, b_1) = \pi \circ \text{ad}_{\hat{a}_1} + \text{ad}_{b_1} \circ \hat{J}^+ \circ \text{Ad}_{\omega} \quad (6.15)$$

$$= \pi \circ \text{ad}_{\hat{a}_1} + \text{ad}_{b_1} \circ \hat{J}^+ \circ \text{Ad}_{\omega} + \text{ad}_{\gamma_1} \quad (6.16)$$

From the lemma 5.2 we have

$$[\pi \circ \text{ad}_{b_1} \circ \hat{J}^+ \circ \text{Ad}_{\omega}, J] = [\pi \circ \text{ad}_{b_1} \circ \hat{J}^+ \circ \text{Ad}_{\omega}, J] = 0$$

Then it follows from (6.13) that

$$[\Gamma^+(a_1, b_1), J] = [\Gamma^+(\hat{a}_1, \hat{b}_1), J] + [\text{ad}_{\gamma_1}, J] = 0, \quad (6.17)$$

Thus we obtain

$$d(a_1 + b_1) \cdot \Omega = 0 \quad (6.18)$$

$$d(a_1 + b_1) \cdot \psi = 0 \quad (6.19)$$

$$[\Gamma^+(a_1, b_1), J] = 0 \quad (6.20)$$

Next we assume that there is a set of sections $a_1, \cdots, a_{k-1}$ of $\text{CL}^2(-D)$ and sections $b_1, \cdots, b_{k-1}$ of $(\mathcal{L}^+ \cdot \mathcal{L}^- \oplus L^+_{\mathcal{L}^-} \cdot \mathcal{L}^-_{-\mathcal{L}^-})_R$ such that

$$\left( de^{Z(t)} \right)_{[i]} \cdot \Omega = 0 \quad (6.21)$$

$$\left( de^{Z(t)} \right)_{[i]} \cdot \psi = 0 \quad (6.22)$$

$$[\Gamma^+(a(t), b(t))_{[i]}, J] = 0, \quad (6.23)$$

for all $0 \leq i < k$, where $\left( de^{Z(t)} \right)_{[i]}$ denotes the $i$-th term of $\left( de^{Z(t)} \right)$ in $t$ and $\Gamma^+(a(t), b(t))_{[i]}$ is the $i$-th term of $\Gamma^+(a(t), b(t))$ for $a(t) = \sum_{j=1}^{k-1} t^j a_j$, $b(t) = \sum_{j=1}^{k-1} t^j b_j$. Then the proposition 4.1 yields a section $\hat{a}_k$ of $\text{CL}^2(-D)$ and a section $\hat{b}_k$ of $(\mathcal{L}^+ \cdot \mathcal{L}^- \oplus L^+_{\mathcal{L}^-} \cdot \mathcal{L}^-_{-\mathcal{L}^-})_R$ such that

$$\left( de^{\hat{Z}(t)} \cdot \Omega \right)_{[k]} = \left( de^{\hat{a}(t)} \cdot \Omega \right)_{[k]} = 0, \quad (6.24)$$

$$\left( de^{\hat{Z}(t)} \cdot \psi \right)_{[k]} = 0 \quad (6.25)$$
where \( \dot{Z}(t) \) is a section of \( \text{CL}^2 \) given by \( e^{\dot{Z}(t)} = e^{\hat{a}(t)} e^{\hat{b}(t)} \) and

\[
\hat{a}(t) = \sum_{j=1}^{k-1} \frac{t^j}{j!} a_j + \frac{t^k}{k!} \hat{a}_k, \quad \hat{b}(t) = \sum_{j=1}^{k-1} \frac{t^j}{j!} b_j + \frac{t^k}{k!} \hat{b}_k
\]

Then for the section \( \Gamma^+(\hat{a}(t), \hat{b}(t)) \) of \( \text{GL}(T) \), as in the lemma 5.6 we define \( \gamma_k \in T \cdot T^* \) and \( \text{ad}_{\zeta_k} \) by

\[
\text{ad}_{\zeta_k} := \left[ \Gamma^+(\hat{a}(t), \hat{b}(t))_{[k]}, J \right] \tag{6.26}
\]

and

\[
\text{ad}_{\eta_k} := \frac{-k!}{(2\sqrt{-1})^2} \left[ \text{ad}_{\zeta_k}, J \right] \tag{6.27}
\]

Then from the lemma 5.6 we see that \( \gamma_k \) is a section of \( \text{CL}^2(-D) \) and we have

\[
\frac{1}{k!} \left[ \text{ad}_{\gamma_k}, J \right] + \left[ \Gamma^+(\hat{a}(t), \hat{b}(t))_{[k]}, J \right] = 0 \tag{6.28}
\]

The lemma 5.7 shows that \( d\gamma_k \cdot \Omega = -d\rho_k \cdot \Omega \) for a global function \( \rho_k \) of \( \wedge^0 \mathbb{T}(-D) \). We define \( a_k \in \text{CL}^2(-D) \) by

\[
a_k := \hat{a}_k + \gamma_k + \rho_k \tag{6.29}
\]

Then we have

\[
\left( de^{\hat{a}(t)} \right)_{[k]} \cdot \Omega = \left( de^{\hat{a}(t)} \right)_{[k]} \cdot \Omega + d(\gamma_k + \rho_k) \cdot \Omega = 0.
\]

Applying the proposition 4.1 again, we have a section \( b_k \) of \( \left( \mathcal{T}^+ \cdot \mathcal{L}^+ \oplus \mathcal{L}^+ \cdot \mathcal{T}^+ \right)_R \) with \( (de^{Z(t)} \cdot \psi)_{[k]} = 0 \), where \( Z(t) = \log \left( e^{a(t)} e^{b(t)} \right) \). As in lemma 5.1 \( (\Gamma^+_t)_{[k]} = \Gamma^+(a(t), b(t))_{[k]} \) satisfies the following,

\[
\left[ \Gamma^+(a(t), b(t))_{[k]}, J \right] = \frac{1}{k!} \left[ \pi \circ (\text{ad}_{a_k} + \text{ad}_{b_k}) \circ \hat{J}^\pm \circ \text{Ad}_{e^\omega}, J \right] + \left[ \hat{\Gamma}^+_k(a_{<k}, b_{<k}), J \right] \tag{6.30}
\]

Substituting (6.29) into (6.30) and using lemma 5.2 and lemma 5.3 we have

\[
\left[ \Gamma^+(a(t), b(t))_{[k]}, J \right] = \frac{1}{k!} \left[ \pi \circ (\text{ad}_{\hat{a}_k} + \text{ad}_{\eta_k}) \circ \hat{J}^\pm \circ \text{Ad}_{e^\omega}, J \right] + \left[ \hat{\Gamma}^+_k(a_{<k}, b_{<k}), J \right]
\]

\[
= \frac{1}{k!} \left[ \pi \circ \text{ad}_{\hat{a}_k} \circ \hat{J}^\pm \circ \text{Ad}_{e^\omega}, J \right] + \left[ \hat{\Gamma}^+_k(a_{<k}, b_{<k}), J \right] + \frac{1}{k!} \left[ \text{ad}_{\eta_k}, J \right]
\]

From lemma 5.1 and lemma 5.2 we also have

\[
\left[ \Gamma^+(\hat{a}(t), \hat{b}(t))_{[k]}, J \right] = \frac{1}{k!} \left[ \pi \circ \text{ad}_{\hat{a}_k} \circ \hat{J}^\pm \circ \text{Ad}_{e^\omega}, J \right] + \left[ \hat{\Gamma}^+_k(a_{<k}, b_{<k}), J \right] \tag{6.31}
\]

Thus from (6.28) we obtain

\[
\left[ \Gamma^+(a(t), b(t))_{[k]}, J \right] = \frac{1}{k!} \left[ \text{ad}_{\gamma_k}, J \right] + \left[ \Gamma^+(\hat{a}(t), \hat{b}(t))_{[k]}, J \right] = 0, \tag{6.32}
\]
where \( a(t) = \sum_{j=1}^{k} \frac{\nu_j}{t} a_j \) and \( b(t) = \sum_{j=1}^{k} \frac{\nu_j}{t} b_j \). Thus \( Z(t) \) satisfies the equations,

\[
(d^2Z(t))_{[k]} \cdot \Omega = 0 \tag{6.33}
\]

\[
(d^2Z(t))_{[k]} \cdot \psi = 0 \tag{6.34}
\]

\[
[ \Gamma(a(t),b(t))_{[k]}, J ] = 0, \tag{6.35}
\]

In section 6, we shall show that the formal power series \( Z(t) \) is a convergent series which is smooth. Then the sections \( a(t) \) and \( b(t) \) give deformations of bihermitian structures \((J_t^+, J_t^-, h_t)\). Finally we shall show that the family of deformations satisfies the equation (6.1) in the theorem 6.1. We already have \([\Gamma^+ (\hat{a}_1, \hat{b}_1), J]\) which implies that \( J_t^+ = J \). From the lemma 5.1 and the lemma 5.2, the 1st term of \( J_t^− \) is given by

\[
[(\Gamma_0^−)_{[1]}, J] = [(\pi \circ (\text{ad}_{\hat{a}_1} + \text{ad}_{\hat{b}_1}) \circ \hat{J}^− \circ \text{Ad}_\omega), J] = [(\text{ad}_{\hat{b}_1} + \pi \circ \text{ad}_{\hat{a}_1} \circ \hat{J}^− \circ \text{Ad}_\omega), J]
\]

Since \( \hat{a}_1 = \beta + \overline{\beta} \), we have \( \pi \circ \text{ad}_{\hat{a}_1}|_T = 0 \). We also have

\[
[\text{ad}_{\hat{b}_1}, J] = [(\pi \circ \text{ad}_{\hat{b}_1} \circ J^* \circ \text{ad}_\omega), J] = -[\Gamma^+ (\hat{a}_1, \hat{b}_1), J].
\]

Thus we obtain

\[
[(\Gamma_0^−)_{[1]}, J] = 2[(\pi \circ \text{ad}_{\hat{b}_1} \circ J^* \circ \text{ad}_\omega), J]
\]

Then we have for a vector \( v \),

\[
2(\pi \circ \text{ad}_{\hat{a}_1} \circ J^* \circ \text{ad}_\omega) v = -2[\beta + \overline{\beta}, [\omega, Jv]]
\]

\[
= -2[[\beta + \overline{\beta}, \omega], Jv] = -2(\beta \cdot \omega + \overline{\beta} \cdot \omega) Jv.
\]

Thus it follows that \( \frac{d}{dt} J_t^− |_{t=0} = [(\Gamma_0^−)_{[1]}, J] = -2(\beta \cdot \omega + \overline{\beta} \cdot \omega) \) and the Kodaira-Spencer class of deformations \( \{ J_t^− \} \) is given by the class \(-2[\beta \cdot \omega] \in H^1(M,\Theta) \). If the class \([\beta \cdot \omega] \in H^1(M,\Theta) \) does not vanish, then the deformations \( \{ J_t^− \} \) is not trivial. Thus \((X, J_t^−)\) is not biholomorphic to \((X, J)\) for small \( t \neq 0 \).

Hence we have the result.

\[ \square \]

**Theorem 6.2.** A compact Kähler surface admits non-trivial bihermitian structure with the torsion condition and the same orientation if and only if \( S \) has nonzero holomorphic Poisson structure.

**Proof.** It is already shown in [3], [18] that a non-trivial bihermitian structure with the torsion condition and the same orientation carries a non-zero holomorphic Poisson structure. It follows from the theorem 6.1 that if a compact Kähler surface \( S \) has a non-zero holomorphic Poisson structure, then \( S \) admits a non-trivial bihermitian structure. Thus the result follows. \[ \square \]
7 The convergence

In order to show the convergence of the power series in section 6, we apply the similar method in [13, 15]. We also use the same notation as in [22]. Let $P(t) = \sum_k P_k t^k$ be a power series in $t$ whose coefficients are sections of a vector bundle on a Riemannian manifold. We denote by $\|P_k\|_s$ the Sobolev norm of the section $P_k$ which is given by the sum of the $L^2$-norms of $i$ th derivative of $P_k$ for all $i \leq s$, where $s$ is a positive integer with $s > 2n + 1$. We put $\|P(t)\|_s = \sum_k \|P_k\|_s t^k$. Given two power series $P(t), Q(t)$, if $\|P_k\| \leq \|Q_k\|$ for all $k$, then we denote it by

$$P(t) \ll Q(t).$$

For a positive integer $k$, if $\|P_i\| \leq \|Q_i\|$ for all $i \leq k$, we write it by

$$P(t) \ll_k Q(t).$$

We also use the following notation. If $P_i = Q_i$ for all $i \leq k$, we write it by

$$P(t) \equiv_k Q(t).$$

Let $M(t)$ be a convergent power series defined by

$$M(t) = \sum_{\nu=1}^{\infty} \frac{1}{16c} \frac{(ct)^\nu}{\nu^2} = \sum_{\nu=1}^{\infty} M_\nu t^\nu,$$

for a positive constant $c$, which is determined later suitably. The key point is the following inequality,

$$M(t)^2 \ll \frac{1}{c} M(t)$$

We put $\lambda = c^{-1}$. Then we also have

$$e^{M(t)} \ll \frac{1}{\lambda} e^\lambda M(t).$$

We will take $\lambda$ sufficiently small which will be determined later. (Note that $\lambda$ gives a change of parameter $t$ by constant multiplication.)

As in the proposition 4.1 if there is a set of sections $a_1, \cdots, a_{k-1}$ of $\mathbb{CL}^2$ which satisfies

$$\pi_{U^{-n+3}} \left( e^{-a(t)} de^{a(t)} \right)_{[i]} = 0, \quad \text{for all } i < k,$$

and $\|a(t)\|_s \ll_{k-1} K_1 M(t)$, then there is a set of real sections $b_1, \cdots, b_k \in (L^{-}_J \cdot \mathbb{T}^{+}_J \oplus \mathbb{T}^{-}_J \cdot L^{+}_J)$ which satisfy the following equations:

$$\pi_{U^{-n+3}} \left( e^{-Z(t)} de^{Z(t)} \right)_{[k]} = 0$$

(7.5)

$$\left( de^{Z(t)} \cdot \psi_0 \right)_{[i]} = 0, \quad \text{for all } i \leq k$$

(7.6)

$$\frac{1}{k!} \|\hat{a}_k\|_s < K_1 \lambda M_k$$

(7.7)

$$\frac{1}{k!} \|\hat{b}_k\|_s < K_2 M_k$$

(7.8)
where \( \hat{a}_k \) is the section in the proposition 3.2 and \( M(t) \) is the convergent series in (7.3) with a constant \( \lambda \). Note that \( K_1 \) is a positive constant and a positive constant \( K_2 \) is determined by \( \lambda, K_1 \). We also have an estimate of \( e^{\gamma(t)} = e^{\alpha(t)} e^{\beta(t)} \) in [13],

\[
\| Z(t) \| \ll_k M(t).
\]

Then \( \gamma_k \) in the lemma 5.6 satisfies

\[
\| \gamma_k \|_s < \| \Gamma_k^+ (a_{<k}, \hat{a}_k, b_{<k}, \hat{b}_k) \|_s \tag{7.9}
\]

\[
< 2 \| \hat{a}_k \|_s + 2 \| \hat{b}_k \|_s + \| \Gamma_k^+ (a_{<k}, b_{<k}) \|_s \tag{7.10}
\]

Recall that \( \Gamma_t^+ = \pi \left( \text{Ad}_{e^{\gamma(t)}} \circ \hat{J}^+ \circ \text{Ad}_{e^{\gamma(t)}} \right) \). Then we have an estimate of the non-linear term \( \| \Gamma_k^+ (a_{<k}, b_{<k}) \|_s \)

\[
\| \Gamma_k^+ (a_{<k}, b_{<k}) \|_s < C \| (e^{\gamma(t)} - Z(t) - 1)_{[k]} \|_s,
\]

where \( C \) denotes a constant. It follows from (7.4) that \( \| (e^{\gamma(t)} - Z(t) - 1)_{[k]} \|_s < C(\lambda)M_k \), where \( C(\lambda) \) satisfies \( \lim_{\lambda \to 0} C(\lambda) = 0 \). Thus we have

\[
\frac{1}{k!} \| \gamma_k \|_s < \frac{2}{k!} \left( \| \hat{a}_k \|_s + \| \hat{b}_k \|_s \right) + C(\lambda)M_k < 2\lambda K_1 M_k + 2K_2 M_k + C(\lambda)M_k.
\]

By using the Hodge decomposition and the Green operator We also have a unique global function \( \rho_k \) of \( \wedge^0 \Omega (-D) \) which satisfies the followings,

\[
d\rho_k \Omega = -d\gamma_k \cdot \Omega \tag{7.11}
\]

\[
\| \rho_k \|_s \leq C_1 \| \gamma_k \|_s, \tag{7.12}
\]

where \( C_1 \) is a constant. Then we obtain

\[
\frac{1}{k!} \| a_k \|_s < \frac{1}{k!} \| \hat{a}_k \|_s + \frac{1}{k!} \| \gamma_k \|_s + \frac{1}{k!} \| \rho_k \|_s
\]

\[
< \frac{1}{k!} \| \hat{a}_k \|_s + \frac{1}{k!} (1 + C_1) \| \gamma_k \|_s
\]

\[
< \lambda K_1 M_k + 2(1 + C_1)(\lambda K_1 M_k + K_2 M_k + C(\lambda)M_k)
\]

We take \( \lambda \) and \( K_2 \) sufficiently small such that \( \lambda K_1 M_k + 2(1 + C_1)(\lambda K_1 M_k + K_2 M_k + C(\lambda)M_k) < K_1 M_k \). Then we obtain

\[
\frac{1}{k!} \| a_k \|_s < \frac{1}{k!} \| \hat{a}_k \|_s + \frac{1}{k!} \| \gamma_k \|_s < K_1 M_k.
\]

Thus our solution \( a(t) \) satisfies that \( \| a(t) \|_s \ll_k K_1 M(t) \) for all \( k \) by the induction. It implies that \( a(t) \) is a convergent series. Applying the proposition 4.1 again, we have \( \| b(t) \|_s \ll_k K_2 M(t) \). Hence \( b(t) \) is also a convergent series. Thus it follows that \( Z(t) \) is a convergent series.
Chapter 8

8 Applications

8.1 Bihermitian structures on del Pezzo surfaces

A del Pezzo surface is by definition a smooth algebraic surface with ample anti-canonical line bundle. A classification of del Pezzo surfaces are well known, they are $\mathbb{CP}^1 \times \mathbb{CP}^1$ or $\mathbb{CP}^2$ or a surface $S_n$ which is the blow-up of $\mathbb{CP}^2$ at $n$ points $P_1, \cdots, P_n$, $(0 < n \leq 8)$. The set of the points $\Sigma := \{P_1, \cdots, P_n\}$ must be in general position to yield a del Pezzo surface. The following theorem is due to Demazure, [6] (see page 27), which shows the meaning of general position,

Theorem 8.1. The following conditions are equivalent:

1. The anti-canonical line bundle of $S_n$ is ample
2. No three of $\Sigma$ lie on a line, no six of $\Sigma$ lie on a conic and no eight of $\Sigma$ lie on a cubic with a double point $P_i \in \Sigma$
3. There is no curve $C$ on $S_n$ with $-K_{S_n} \cdot C \leq 0$.
4. There is no curve $C$ with $C \cdot C = -2$ and $K_{S_n} \cdot C = 0$.

Remark 8.2. If three points lie on a line $l$, then the strict transform $\hat{l}$ of $l$ in $S_3$ is a $(-2)$-curve with $K_{S_3} \cdot \hat{l} = 0$. If six points belong to a conic curve $C$, then the strict transform form $\hat{C}$ of $C$ is again a $(-2)$-curve with $K_{S_6} \cdot \hat{C} = 0$. If eight points $P_1, \cdots, P_8$ lie on a cubic curve with a double point $P_i$, then the strict transform $\hat{C}$ of $C$ satisfies $\hat{C} \sim \pi^{-1}C - 2E_1 - E_2 - \cdots - E_8$, where $E_i$ is the exceptional curve $\pi^{-1}(P_i)$. Then we also have $\hat{C}^2 = -2$ and $K_{S_8} \cdot \hat{C} = 0$.

Let $D$ be a smooth anti-canonical divisor of $S_n$ which is given by the zero locus of a section $\beta \in H^0(S_n, K_{S_n}^{-1})$. Since the anti-canonical bundle $K_{S_n}^{-1}$ is regarded as the bundle of 2-vectors $\wedge^2 \Theta$ and $[\beta, \beta]_{S} = 0 \in \wedge^3 \Theta$ on $S_n$, every section $\beta$ is a holomorphic Poisson structure. On $S_n$, we have the followings,

$$\dim H^1(S_n, \Theta) = \begin{cases} 2n - 8 & (n = 5, 6, 7, 8) \\ 0 & (n < 5) \end{cases}$$

and

$$\dim H^0(S_n, K^{-1}) = 10 - n$$

and

$$H^{1,1}(S_n) = 1 + n.$$

Further we have $H^2(S_n, \Theta) = \{0\}, \ H^1(S_n, \wedge^2 \Theta) \cong H^1(S_n, -K_{S_n}) = \{0\}$. Hence the obstruction vanishes and we have deformations of generalized complex structures parametrized by $H^0(S_n, K_{S_n}^{-1}) \oplus H^1(S_n, \Theta)$.

In particular, if $n \geq 5$, we have deformations of ordinary complex structures on $S_n$.
Proposition 8.3. Let \( D \) be a smooth anti-canonical divisor given by the zero locus of \( \beta \) as above. Then there is a Kähler form \( \omega \) with the class \([\beta \cdot \omega] \neq 0 \in H^1(S_n, \Theta)\).

We also have \( H^2(\mathbb{CP}^1 \times \mathbb{CP}^1, \Theta) = 0 \) and \( H^1(\mathbb{CP}^1 \times \mathbb{CP}^1, -K) = 0 \).

Thus we can apply our construction to every del Pezzo surface. From the main theorem together with the proposition 8.3, we have

Proposition 8.4. Every del Pezzo surface admits deformations of bihermitian structures \((J, J_t^-, h_t)\) with \( J_0^- = J \) which satisfies

\[
\frac{d}{dt} J_t^- \big|_{t=0} = -2(\beta \cdot \omega + \overline{\beta} \cdot \omega),
\]

for every Kähler form \( \omega \) and every holomorphic Poisson structure \( \beta \). Further, a del Pezzo surface \( S_n \) \((n \geq 5)\) admits distinct bihermitian structures \((J, J_t^-, h_t)\), that is, the complex manifold \((X, J_t^-)\) is not biholomorphic to \((X, J)\) for small \( t \neq 0 \).

Note that for small \( t \neq 0 \), \( J_t^- \neq \pm J \). We will give a proof of the proposition 8.3 in the rest of this subsection.

Let \( N_D \) is the normal bundle to \( D \) in \( S_n \) and \( i^* T_{S_n} \) the pull back of the tangent bundle \( T_{S_n} \) of \( S_n \) by the inclusion \( i : D \to S_n \). Then we have the short exact sequence,

\[0 \to T_D \to i^* T_{S_n} \to N_D \to 0\]

and we have the long exact sequence

\[0 \to H^0(D, T_D) \to H^0(D, i^* T_{S_n}) \to H^0(D, N_D) \xrightarrow{\partial} H^1(D, T_D) \to \cdots\]

Since the line bundle \( N_D \) is positive, \( H^1(D, N_D) = \{0\} \) and \( \dim H^0(D, N_D) \) is equal to the intersection number \( D \cdot D = 9 - n \) by the Riemann-Roch theorem. Since \( D \) is an elliptic curve, \( \dim H^1(D, T_D) = \dim H^0(D, T_D) = 1 \). Hence if follows that \( 9 - n \leq \dim H^0(D, i^* T_{S_n}) \leq 10 - n \).

Let \( \mathcal{I}_D \) be the ideal sheaf of \( D \) and \( \mathcal{O}_D \) the structure sheaf of \( D \). Then we have the short exact sequence

\[0 \to \mathcal{I}_D \to \mathcal{O}_{S_n} \to i_* \mathcal{O}_D \to 0\]

By the tensor product, we also have

\[0 \to \mathcal{I}_D \otimes T_{S_n} \to T_{S_n} \to i_* \mathcal{O}_D \otimes T_{S_n} \to 0\]

Then from the projection formula we have

\[H^p(S_n, i_* \mathcal{O}_D \otimes T_{S_n}) \cong H^p(S_n, i_* (\mathcal{O}_D \otimes i^* T_{S_n})) \cong H^p(D, i^* T_{S_n}),\]
for $p = 0, 1, 2$. From (8.3), we have the long exact sequence,

$$H^0(S_n, T_{S_n}) \to H^0(D, i^*T_{S_n}) \to H^1(S_n, \mathcal{I}_D \otimes T_{S_n}) \xrightarrow{j} H^1(S_n, T_{S_n}) \to \cdots$$  (8.4)

Hence we obtain

**Lemma 8.5.** The map $j : H^1(S_n, \mathcal{I}_D \otimes T_{S_n}) \to H^1(S_n, T_{S_n})$ is not the zero map.

**Proof.** We have the exact sequence,

$$\cdots \to H^0(D, i^*T_{S_n}) \to H^1(S_n, \mathcal{I}_D \otimes T_{S_n}) \xrightarrow{j} H^1(S_n, T_{S_n})$$  (8.5)

From the Serre duality with $\mathcal{I}_D = K_{S_n}$, we have $H^0(S_n, \mathcal{I}_D \otimes T_{S_n}) \cong H^2(S_n, \Omega^1_{S_n}) = \{0\}$ and $H^2(S_n, \mathcal{I}_D \otimes T_{S_n}) = H^0(S_n, \Omega^1) = 0$. From the Riemann-Roch theorem, $\dim H^1(S_n, \mathcal{I}_D \otimes T_{S_n}) = n + 1$. Then it follows from (8.2) that

$$\dim H^0(D, i^*T_{S_n}) < \dim H^1(S_n, \mathcal{I}_D \otimes T_{S_n})$$

Note $10 - n < n + 1$ for all $n \geq 5$. Hence the map $j$ is non-zero.

**Remark 8.6.** Since $n \geq 5$, we have $H^0(S_n, T_{S_n}) = \{0\}$. Applying the Serre duality with $K_{S_n} = \mathcal{I}_D$, we have $H^2(S_n, T_{S_n}) \cong H^0(S_n, \mathcal{I}_D \otimes \Omega^1) = 0$. From the Riemann-Roch, we obtain $\dim H^1(S_n, T_{S_n}) = 2n - 8$.

Let $\beta$ be a non-zero holomorphic Poisson structure $S_n$ with the smooth divisor $D$ as the zero locus. Then $\beta$ is regarded as a section of $\mathcal{I}_D \otimes \Lambda^2 \Theta$. Thus the section $\beta \in H^0(S_n, \mathcal{I}_D \otimes \Lambda^2 \Theta)$ gives an identification,

$$\Omega^1 \cong \mathcal{I}_D \otimes T_{S_n}.$$  

Then the identification induces the isomorphism

$$\hat{\beta} : H^1(S_n, \Omega^1) \cong H^1(S_n, \mathcal{I}_D \otimes T_{S_n}).$$

Let $j$ be the map in the lemma\[8.5\]. Then we have the composite map $j \circ \hat{\beta} : H^1(S_n, \Omega^1) \to H^1(S_n, \Theta)$ which is given by the class $[\beta \cdot \omega] \in H^1(S_n, T_{S_n})$ for $[\omega] \in H^1(S_n, \Omega^1)$.

**Proposition 8.7.** The composite map $j \circ \hat{\beta} : H^1(S_n, \Omega^1) \to H^1(S_n, T_{S_n})$ is not the zero map.

**Proof.** Since the map $\hat{\beta}$ is an isomorphism, $\hat{\beta}(\omega)$ is not zero. It follows from lemma\[8.5\] that the map $j$ is non-zero. Hence the composite map $j \circ \hat{\beta}$ is non-zero also. \[\square\]
**Proof.** of lemma 8.3 The set of Kähler class is an open cone in $H^{1,1}(S_n, \mathbb{R}) \cong H^2(S_n, \mathbb{R})$. We have the non-zero map $j \circ \hat{\beta} : H^2(S_n, \mathbb{C}) \cong H^1(S_n, \Omega^1) \to H^1(S_n, \Theta)$ for each $\beta \in H^0(S_n, K^{-1})$ with $\{\beta = 0\} = D$. It follows that the kernel $j \circ \hat{\beta}$ is a closed subspace and the intersection $\ker(j \circ \hat{\beta}) \cap H^2(S_n, \mathbb{R})$ is closed in $H^2(S_n, \mathbb{R})$ whose dimension is strictly less than $\dim H^2(S_n, \mathbb{R})$. Thus the complement in the Kähler cone

$$\{ [\omega] : \text{Kähler class} \mid j \circ \hat{\beta}( [\omega] ) \neq 0 \}$$

is not empty. Thus there is a Kähler form $\omega$ such that the class $[\beta \cdot \omega] \in H^1(S_n, \Theta)$ does not vanish for $n \geq 5$. □

We also remark that our proof of the lemma 8.3 still works for degenerate del Pezzo surfaces.

### 8.2 Vanishing theorems on surfaces

Let $M$ be a compact complex surface with canonical line bundle $K_M$. We shall give some vanishing theorems of the cohomology groups $H^1(M, -K_M)$ and $H^2(M, \Theta)$ on a compact smooth complex surface $M$, which are the obstruction spaces to deformations of generalized complex structures starting from the ordinary one $(X, \mathcal{J}_j)$. The following is practical to show the vanishing of $H^1(M, -K_M)$.

**Proposition 8.8.** Let $M$ be a compact complex surface with $H^1(M, \mathcal{O}_M) = 0$. If $-K_M = m[D]$ for a irreducible, smooth curve $D$ with positive self-intersection number $D \cdot D > 0$ and a positive integer $m$, then $H^1(M, K^n_M) = 0$ for all integer $n$.

The proposition is often used in the complex geometry. For completeness, we give a proof.

**Proof.** Let $I_D$ be the ideal sheaf of the curve $D$. Then we have the short exact sequence, $0 \to I_D \to \mathcal{O}_M \to j_* \mathcal{O}_D \to 0$, where $j : D \to X$. Then we have the exact sequence,

$$H^0(M, \mathcal{O}_M) \to H^0(M, j_* \mathcal{O}_D) \xrightarrow{\delta} H^1(M, I_D) \to H^1(M, \mathcal{O}_M)$$

It follows that the coboundary map $\delta$ is a 0-map. Thus from $H^1(M, \mathcal{O}_M) = 0$, we have $H^1(M, I_D) = H^1(M, -[D]) = 0$. We use the induction on $k$. We assume that $H^1(M, I^n_D) = H^1(M, -k[D]) = 0$ for a positive integer $k$. The short exact sequence $0 \to I_D^{k+1} \to I_D^k \to j_* \mathcal{O}_D \otimes I_D^k \to 0$ induces the exact sequence,

$$H^0(M, j_* \mathcal{O}_D \otimes I_D^k) \to H^1(M, I^{k+1}_D) \to H^1(M, I^k_D).$$

By the projection formula, we have $H^0(M, j_* \mathcal{O}_D \otimes I_D^k) = H^0(D, -k[D]|_D)$. Since $D \cdot D > 0$, it follows that the line bundle $-k[D]|_D$ is negative and then $H^0(D, -k[D]|_D) =$
\( \text{H}^0(M, I_D^k) = 0. \) It implies that \( \text{H}^1(M, I_D^{k+1}) = \text{H}^1(M, -(k + 1)[D]) = 0. \) Thus by the induction, we have \( \text{H}^1(M, -nD) = 0 \) for all positive integer \( n. \) Applying the Serre duality, we have \( \text{H}^1(M, -nD) \cong \text{H}^1(M, (n - m)D) = 0. \) Thus \( \text{H}^1(M, nD) = 0 \) for all integer \( n. \) Then the result follows since \( \text{H}^1(M, K^n) = \text{H}^1(M, -(n + 1)mD) = 0. \)

The author also refer to the standard vanishing theorem. If \( D = \sum_i a_i D_i \) is a \( \mathbb{Q} \)-divisor on \( M, \) where \( D_i \) is a prime divisor and \( a_i \in \mathbb{Q}. \) Let \( [a_i] \) be the round-up of \( a_i \) and \( \lfloor a_i \rfloor \) the round-down of \( a_i. \) Then the fractional part \( \{a_i\} \) is \( a_i - \lfloor a_i \rfloor. \) Then the round-up and the round-down of \( D \) is defined by

\[
[D] = \sum_i [a_i] D_i, \quad \lfloor D \rfloor = \sum_i \lfloor a_i \rfloor D_i
\]

and \( \{D\} = \sum_i \{a_i\} D_i \) is the fractional part of \( D. \) A divisor \( D \) is nef if one has \( D \cdot C \geq 0 \) for any curve \( C. \) A divisor \( D \) is nef and big if in addition, one has \( D^2 > 0. \) We shall use the following vanishing theorem. The two dimensional case is due to Miyaoka and the higher dimensional cases are due to Kawamata and Viehweg

**Theorem 8.9.** Let \( M \) be a smooth projective surface and \( D \) a \( \mathbb{Q} \)-divisor on \( M \) such that

1. \( \text{supp}\{D\} \) is a divisor with normal crossings,
2. \( D \) is nef and big.

Then \( \text{H}^i(M, K_M + [D]) = 0 \) for all \( i > 0. \)

If \( -K_M = mD \) is nef and big divisor where \( D \) is smooth for \( m > 0. \) Then applying the theorem, we have

\[
\text{H}^i(M, -K_M) \cong \text{H}^i(M, K_M - 2K_M) = 0,
\]

for all \( i > 0. \)

Next we consider the vanishing of the cohomology group \( \text{H}^2(M, \Theta). \) Applying the Serre duality theorem, we have

\[
\text{H}^2(M, \Theta) \cong \text{H}^0(M, \Omega^1 \otimes K_M)
\]

If \( -K_M \) is an effective divisor \( [D], \) then \( K_M \) is given by the ideal sheaf \( I_D \) of \( D. \) The short exact sequence: \( 0 \rightarrow \Omega^1 \otimes I_D \rightarrow \Omega^1 \rightarrow \Omega^1 \otimes \mathcal{O}_D \rightarrow 0 \) gives us the injective map,

\[
0 \rightarrow \text{H}^0(M, \Omega^1 \otimes K_M) \rightarrow \text{H}^0(M, \Omega^1).
\]

Hence we have

**Proposition 8.10.** if \( M \) is a smooth surface with effective anti-canonical divisor satisfying \( \text{H}^0(M, \Omega^1) = 0, \) then we have the vanishing \( \text{H}^2(M, \Theta) = 0. \)
8.3 Non-vanishing theorem

**Proposition 8.11.** Let $M$ be a Kähler surface with a Kähler form $\omega$ and a non-zero Poisson structure $\beta \in H^0(M, \Lambda^2 \Theta)$. Let $D$ be the divisor defined by the section $\beta$. If there is a curve $C$ of $M$ with $C \cap \text{supp} \ D = \emptyset$, then the class $[\beta \cdot \omega] \in H^1(M, \Theta)$ does not vanish.

**Proof.** Since $\beta$ is not zero on the complement $M \setminus D$, there is a holomorphic symplectic form $\hat{\beta}$ on the complement. The symplectic form $\hat{\beta}$ gives the isomorphism $\Theta \cong \Omega^1$ on $M \setminus D$ which induces the isomorphism between cohomology groups $H^1(M \setminus D, \Theta) \cong H^1(M \setminus D, \Omega^1)$. Then the restricted class $[\beta \cdot \omega]|_{M \setminus D}$ corresponds to the Kähler class $[\omega]|_{M \setminus D} \in H^1(M \setminus D, \Omega^1)$ under the isomorphism. Since there is the curve $C$ on the complement $M \setminus D$ and $\omega$ is a Kähler form, the class $[\omega]|_{M \setminus D} \in H^1(M \setminus D, \Omega^1)$ does not vanish. Then it follows that the class $[\beta \cdot \omega]|_{M \setminus D}$ does not vanish also. Thus we have that the class $[\beta \cdot \omega] \in H^1(M, \Theta)$ does not vanish.

8.4 Deformations of bihermitian structures on the Hirzebruch surfaces $F_2$

Let $F_2$ be the projective space bundle of $T^*\mathbb{C}P^1 \oplus \mathcal{O}_{\mathbb{C}P^1}$ over $\mathbb{C}P^1$ with fibre $\mathbb{C}P^1$, $F_2 = \mathbb{P}(T^*\mathbb{C}P^1 \oplus \mathcal{O}_{\mathbb{C}P^1})$.

We denote by $E^+$ and $E^-$ the sections of $F_2$ with positive and negative self-intersection numbers respectively. An anti-canonical divisor of $F_2$ is given by $2E^+$, while the section $E^-$ with $E^- \cdot E^- = -2$ is the curve which satisfies $E^+ \cap E^- = \emptyset$. Thus we have the non-vanishing class $[\beta \cdot \omega] \in H^1(F_2, \Theta)$, where $\beta$ is a section of $-K$ with the divisor $2E^+$. (Note that the canonical holomorphic symplectic form $\hat{\beta}$ on the cotangent bundle $T^*\mathbb{C}P^1$ which induces the holomorphic Poisson structure $\beta$. The structure $\beta$ can be extended to $F_2$ which gives the anti-canonical divisor $2[E^+]$.)

**Proposition 8.12.** The class $[\beta \cdot \omega] \in H^1(F_2, \Theta)$ does not vanish for every Kähler form $\omega$ on $F_2$.

**Proof.** The result follows from the proposition 8.11. \qed

On the surface $F_2$, the anti-canonical line bundle of $F_2$ is $2E^+$ and $H^1(F_2, \mathcal{O}_{F_2}) = 0$. Hence from the proposition 8.8 we have the vanishing $H^i(F_2, -K_X) = \{0\}$ for all $i > 0$. Since the surface $F_2$ is simply connected, it follows from the proposition 8.10 that $H^2(F_2, \Theta) = 0$. Hence the obstruction vanishes. It is known that every non-trivial small deformation of $F_2$ is $\mathbb{C}P^1 \times \mathbb{C}P^1$. Thus we have
**Proposition 8.13.** Let \((X, J)\) be the Hirzebruch surface \(F_2\) as above. Then there is a family of deformations of bihermitian structures \((J^+_t, J^-_t, h_t)\) with the torsion condition and the same orientation with \(J^+_t = J^-_0 = J\) such that \((X, J^-_t)\) is \(\mathbb{CP}^1 \times \mathbb{CP}^1\) for small \(t \neq 0\).

Let \(F_e\) be the projective space bundle \(\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-e))\) over \(\mathbb{CP}^1\) with \(e > 0\). There is a section \(b\) with \(b^2 = -e\), which is unique if \(e > 0\). Let \(f\) be a fibre of \(F_e\). Then \(-K\) is given by \(2b + (e + 2)f\), which is an effective divisor. Thus from the proposition 8.10, we have \(H^2(F_e, \Theta) = \{0\}\). Let \(P = \dim H^0(F_e, K^{-1})\) be listed in the table 7.1.1 of [33],

\[
P^{-1}(F_e) = \begin{cases} 
9 & e = 0, 1 \\
9 & e = 2 \\
e + 6 & e \geq 3
\end{cases}
\]

Since \(K\) is given by the ideal sheaf \(I_D\) for the effective divisor \(D = 2b + (e + 2)f\), it follows from the Serre duality that \(H^2(F_e, K^{-1}) = H^0(F_e, I_D) = \{0\}\). Thus applying the Riemann-Roch theorem, we obtain

\[
\dim H^1(F_e, K^{-1}) = e - 3,
\]

for \(e \geq 3\). In the case \(e = 3\), note that \(H^1(F_3, K^{-1}) = H^2(F_3, \Theta) = \{0\}\). From our main theorem, we have

**Proposition 8.14.** The Hirzebruch surface \(F_e\) admits deformations of non-trivial bihermitian structures with the torsion condition and the same orientation \((J, J^-_t, h_t)\) with \(J^-_t \neq \pm J\) for small \(t \neq 0\).

### 8.5 Bihermitian structures on ruled surfaces \(\mathbb{P}(T^*\Sigma_g \oplus \mathcal{O}_{\Sigma_g})\)

We can generalized our discussion of \(F_2\) to the projective space bundle of \(T^*M \oplus \mathcal{O}_M\) over a compact Kähler manifold \(M\). Then we also have the Poisson structure \(\beta\) and as in the proposition 8.11 it is shown that the class \([\beta \cdot \omega]\) does not vanish. Thus we have the deformations of bihermitian structures from the stability theorem [14].

If \(M\) is a Riemannian surface \(\Sigma_g\) of genus \(g \geq 1\), then the projective space bundle is called a ruled surface of degree \(g\). It is known that small deformations of any ruled surface of degree \(g \geq 1\) remain to be ruled surfaces of the same degree. We denote by \(S = (X, J)\) the projective space bundle \(\mathbb{P}(T^*\Sigma \oplus \mathcal{O}_\Sigma)\), where \(X\) is the underlying differential manifold and \(J\) is the complex structure. Applying our main theorem, we have

**Theorem 8.15.** There is family of distinct bihermitian structures \((J, J^-_t, h_t)\) with the torsion condition and the same orientation on \(S := \mathbb{P}(T^*\Sigma \oplus \mathcal{O}_\Sigma)\), that is, the complex manifold \((X, J^-_t)\) is not biholomorphic to \(S = (X, J)\) for small \(t \neq 0\).
8.6 Bihermitian structures on degenerate del Pezzo surfaces

We shall consider the blow-up of $\mathbb{CP}^2$ at $r$ points which are not in general position. We follow the construction as in [6], (see page 36). We have a finite set $\Sigma = \{x_1, \ldots, x_r\}$ and $X(\Sigma)$ obtained by successive blowing up at $\Sigma$,

$$X(\Sigma) \to X(\Sigma_{r-1}) \to \cdots \to X(\Sigma_1) \to \mathbb{CP}^2,$$

At first $X(\Sigma_1)$ is the blow-up of $\mathbb{CP}^2$ at a point $x_1 \in \mathbb{CP}^2$ and we have $\Sigma_i = \{x_1, \ldots, x_i\}$ and $X(\Sigma_{i+1})$ is the blow-up of $X(\Sigma_i)$ at $x_{i+1} \in X(\Sigma_i)$. Let $E_i$ be the divisor given by the inverse image of $x_i \in X(\Sigma_{i-1})$. If $\Gamma$ is an effective divisor on $\mathbb{CP}^2$, one notes that $\text{mult}(x_i, \Gamma)$ the multiplicity of $x_i$ on the proper transform of $\Gamma$ in $X(\Sigma_{i-1})$, and one says that $\Gamma$ passes through $x_i$ if $\text{mult}(x_i, \Gamma) > 0$. Define $\hat{E}_1, \ldots, \hat{E}_r$ by recurrence as follows,

On $X(\Sigma_1)$, one put $\hat{E}_1 = E_1$; on $X(\Sigma_2)$, $\hat{E}_1$ is a proper transform of the previous $E_1$ and one also put $\hat{E}_2 = E_2$; on $X(\Sigma_3)$, $\hat{E}_1$ and $\hat{E}_2$ are the proper transform of previous $\hat{E}_1$ and $\hat{E}_2$ respectively and $\hat{E}_3 = E_3$. Then $\hat{E}_1, \ldots, \hat{E}_r$ are irreducible components of $E_1 + \cdots + E_r$.

We assume that the following condition on $\Sigma$,

(*) For each $i = 1, \ldots, r$, a point $x_i \in X(\Sigma_{i-1})$ does not belong to an irreducible curve $\hat{E}_j$ with self-intersection number $-2$ for $1 \leq j \leq i - 1$.

If a point $x_i \in X(\Sigma_{i-1})$ belongs to an irreducible curve $\hat{E}_j$ with self-intersection number $-2$, then the proper transform of $\hat{E}_j$ becomes a curve with self-intersection number $-3$. If there is a rational curve with self-intersection number $-3$ or less, the anti-canonical divisor of $X(\Sigma)$ is not nef.

**Definition 8.16.** A set of points $\Sigma$ is in *almost general position* if $\Sigma$ satisfies the following:

1. $\Sigma$ satisfies the condition (*)
2. No line passes through 4 points of $\Sigma$
3. No conic passes through 7 points of $\Sigma$

We call $X(\Sigma)$ a degenerate del Pezzo surface if $\Sigma$ is in almost general position. Note that if $\Sigma$ is in general position, $\Sigma$ is in almost general position. In [6], the following theorem was shown,

**Theorem 8.17.** [6] The following conditions are equivalent:

1. $\Sigma$ is in almost general position
(2) The anti-canonical class of $X(\Sigma)$ contains a smooth and irreducible curve $D$.
(3) There is a smooth curve of $\mathbb{CP}^2$ passing all points of $\Sigma$.
(4) $H^1(X(\Sigma), K^n_{X(\Sigma)}) = \{0\}$ for all integer $n$
(5) $-\mathbf{K}_{X(\Sigma)} \cdot C \geq 0$ for all effective curve $C$ on $X(\Sigma)$ and in addition, if $-\mathbf{K}_{X(\Sigma)} \cdot C = 0$, then $C \cdot C = -2$.

Then from (2) there is a smooth anti-canonical divisor on a degenerate del Pezzo surface and we have $H^1(X(\Sigma), \mathcal{O}_X) = 0$. Hence from the proposition 8.8 we have the vanishing $H^i(X(\Sigma), -\mathbf{K}_X) = 0$, for all $i > 0$. A degenerate del Pezzo surface $X(\Sigma)$ satisfies $H^0(X(\Sigma), \Omega^1) = 0$. Then it follows from the proposition 8.10 that $H^2(X(\Sigma), \Theta) = 0$.

Let $X(\Sigma)$ be a degenerate del Pezzo surface which is not a del Pezzo surface, that is, the anti-canonical class of $X(\Sigma)$ is not ample. Then from (5), there is a $(-2)$-curve $C$ with $\mathbf{K}_X(\Sigma) \cdot C = 0$. Then it follows that $C$ is a $\mathbb{CP}^1$. Thus we contract $(-2)$-curves on a degenerate del Pezzo to obtain a complex surface with rational double points, which is called the Gorenstein log del Pezzo surface. Let $\beta$ be a section of $-\mathbf{K}_{X(\Sigma)}$ with the smooth divisor $D$ as the zero set. We denote by $J$ the complex structure of the del Pezzo surface $X(\Sigma)$. From our main theorem, we have

**Theorem 8.18.** A degenerate del Pezzo surface admits deformations of distinct bihermitian structures $(J, J_t^{-}, h_t)$ with $J_0^{-} = J$ and $J_t^{-} \neq \pm J$ for small $t \neq 0$, that is, $\frac{d}{dt} J_t^\perp |_{t=0} = -2(\beta \cdot \omega + \overline{\beta} \cdot \omega)$, and the complex structure $J_t^{-}$ is not equivalent to $J$ of $X(\Sigma)$ under diffeomorphisms for small $t \neq 0$, where $\omega$ is a Kähler form.

**Proof.** If $X(\Sigma)$ is a del Pezzo surface, we already have the result. If $X(\Sigma)$ is not a del Pezzo but a degenerate del Pezzo, we still have $H^2(X(\Sigma), \Theta) = H^1(X(\Sigma), K^{-1}) = \{0\}$. Thus we have deformations of bihermitian structures as in our main theorem. It is sufficient to show that the class $[\beta \cdot \omega]$ does not vanish. Since there is a smooth $(-2)$-curve $C$ with $K \cdot C = 0$, the line bundle $K^{-1}|_C \to C \cong \mathbb{CP}^1$ is trivial. If there is a point $P \in D \cap C$, then $\beta(P) = 0$ and it follows that $\beta|_C \equiv 0$. Since the anti-canonical divisor $D$ is smooth, we have $D = C$. However $D \cdot D = 9 - r \neq 0$ and $D \cdot C = -K \cdot C = 0$. It is a contradiction. Thus $D \cap C = \emptyset$. Then applying the proposition 8.11 we obtain $[\beta \cdot \omega] \neq 0 \in H^1(X(\Sigma), \Theta)$.

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