The Weyl Double Copy for Gravitational Waves

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We establish the status of the Weyl double copy relation for radiative solutions of the vacuum Einstein equations. We show that all type N vacuum solutions, which describe the radiation region of isolated gravitational systems with appropriate fall-off for the matter fields, admit a degenerate Maxwell field that squares to give the Weyl tensor. The converse statement also holds, i.e. if there exists a degenerate Maxwell field on a curved background, then the background is type N. This relation defines a scalar that satisfies the wave equation on the background. We show that for non-twisting radiative solutions, the Maxwell field and the scalar also satisfy the Maxwell equation and the wave equation on Minkowski spacetime. Hence, non-twisting solutions have a straightforward double copy interpretation.

The discovery of gravitational waves [1] one hundred years after Einstein formulated his general theory of relativity has led to an exciting new area of gravitational physics with possible important prospects for observational astrophysics; a development that has been anticipated eagerly for half a century [2]. An important theoretical breakthrough in this direction will include an efficient and cost-effective method of generating gravitational wave templates; waveforms computed from the theory to be compared with observed waveforms [3, 4]; see [5–9] for a recent reviews. Amongst the myriad approaches proposed to facilitate the easier and less time-consuming generation of templates is one [10, 11] based on techniques adapted from string theory and supergravity scattering amplitude calculations, in particular the double copy method [12–15], which describes gravitational amplitudes as a kind of inner product of gauge theory amplitudes (hence “double copy”).

While initially found at the level of scattering amplitude relations, the double copy also exists at the level of classical solutions, including beyond perturbation theory for certain classes of spacetimes. One class of solutions for which a double copy relation exists is (multi) Kerr-Schild solutions, which can be thought of as exact perturbative (around Minkowski) gravitational solutions [16, 17]. The correspondence between the double copy relations for scattering amplitudes and for classical solutions has been verified in various works [18–22]; see [26, 27] for earlier ideas in this direction. Of particular interest in the present paper is the Weyl double copy relation that exists for vacuum type D solutions and pp-waves [28–30]. This relation is best expressed in spinor language [31]. In the type D case, it can be shown that the Weyl spinor $\Psi_{ABCD} = (-2\Phi^2)^{-1/4} \Phi_{AB} \Phi_{CD}$ with $\Phi_{AB}$ a non-degenerate Maxwell spinor and $\Phi^2 \equiv \Phi_{AB} \Phi_{AB}$. Of particular significance is the fact that the Maxwell spinor also solves the Maxwell equation on Minkowski spacetime [30]. Furthermore, $\Phi^{1/2}$ solves the wave equation on Minkowski spacetime. What lies behind these relations is the existence of the well-known hidden symmetry for type D vacuum solutions as expressed by the existence of a Killing 2-spinor [28–32]. See [33–46] for related works.

In this paper, we extend the curved Weyl double copy relation to all type N vacuum solutions, which describe the radiation region of isolated gravitational systems. In particular, we show that $\Psi_{ABCD} = S^{-1} \Phi_{(AB} \Phi_{CD)}$ with $\Phi_{AB}$ a degenerate Maxwell spinor and $S$ some scalar that in particular satisfies the wave equation on the curved background. For non-twisting radiative spacetimes, the Maxwell field and the scalar field also solve the Maxwell equation and the wave equation, respectively, on Minkowski spacetime. This establishes the Weyl double copy in the sense of [30] for this large class of spacetimes. Notice that, while the double copy for scattering ampliti-
tudes involves two copies of non-Abelian gauge theory, the first step in that procedure is to consider the double copy of the asymptotic states, which for linearised gauge theory are solutions to the Maxwell equation. The fact that certain exact gravity solutions can be interpreted as a double copy of a Maxwell field means that they should be interpreted as coherent states, an exact extension of the linearised asymptotic states in scattering amplitudes. For twisting spacetimes, the Maxwell field and the scalar depend generically on the metric functions. Hence, they are solutions only on the curved spacetime. However, the standard double copy interpretation applies at the linearised level. This may be indicative of the fact that twisting solutions have an intrinsic non-Abelian nature.

As a necessary step in applying the classical double copy tools to gravitational wave physics, we provide a systematic understanding of the status of the double copy for scattering amplitudes. In particular, the construction does not lead to a unique solution of the Maxwell equation. For non-twisting solutions, it would be interesting to consider whether such a relation exists. Expanding out the Bianchi identity by substituting gives two equations:

\[ o_A \nabla A^A \log \Psi_4 + 4 o_A B^B \nabla A^A \Phi_{AB} - \ell A^B \nabla A^A \Phi_{OB} = 0 \]
\[ o_A B^B \nabla A^A \Phi_{OB} = 0 \]

The Maxwell 2-spinor is degenerate, which means that the electromagnetic field is null, i.e. the electric and magnetic fields are perpendicular and of equal magnitude. An example of a null electromagnetic field is that of a plane electromagnetic wave in flat spacetime. Now we must consider whether such a relation exists. Expanding out the Bianchi identity by substituting gives two equations:

\[ o_A \nabla A^A \log \Psi_4 = \frac{\ell}{\Psi_4} \phi_2^2 \]
\[ o_A B^B \nabla A^A \Phi_{OB} = 0 \]

There is a clear structure in equations, where the coefficient of the middle term is the rank of the respective spinor. Equation translates to

\[ \ell \cdot \nabla \log S - \rho = 0, \quad m \cdot \nabla \log S - \tau = 0 \]

where \((\ell, n, m, \tilde{m})\) form an NP null frame. A simple calculation shows that the integrability condition on equations is satisfied, which means that they are simple integral equations that can always be solved. Thus, we are guaranteed the existence of a scalar \(S\) satisfying these equations, which then gives a Maxwell field \(\Phi_2 = \sqrt{\Psi_4} S\). In tensor language, this Maxwell spinor translates to a field strength (called the 'single copy') of the form

\[ F = \Phi_2 \ell^\rho \wedge m^\tau + \tilde{\Phi}_2 \ell^\tau \wedge \tilde{m}^\rho. \]
This establishes the curved Weyl double copy relation for type N vacuum solutions.

Furthermore, it is simple to show using [3] that $S$ solves the wave equation

$$\Box S = \nabla_A \nabla^A S = 2 \sigma_{AB} \nabla^A \nabla^B S = 0.$$  \hspace{1cm} (11)

The real scalar field in the double copy construction (called the ‘zero-th copy’) is the real part of $S$.

These results mirror those that exist for type D solutions. In order to investigate whether the Maxwell field and the scalar field also satisfy the equations of motion on Minkowski spacetime, we investigate the different classes of type N solutions in turn.

**Type N Vacuum Solutions**

Type N vacuum solutions are classified in terms of the optical properties of the congruence generated by the PND. We have that $\kappa = \sigma = 0$; the properties that remain are parametrised by $\rho = -(\Theta + i \omega)$ with $\Theta$ denoting the expansion of the congruence and $\omega$ denoting its twist. The different cases lead to three distinct classes of solutions:

- **Kundt solutions**: $\Theta = 0$, which implies that $\omega = 0$ [51].
- **Robinson-Trautman solutions**: $\Theta \neq 0$, $\omega = 0$.
- **Twisting solutions**: $\Theta \neq 0$, $\omega \neq 0$.

Choosing a null frame for which $\ell$ is the PND, so that $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$, we consider each case separately.

**Kundt solutions**

There are two kinds of type N Kundt solutions, both corresponding to plane-fronted wave solutions [52]. Plane-fronted waves with parallel propagation ($p$-waves) are given by the metric

$$ds^2 = -2d\tau (d\nu + H d\tau) + 2 dz d\bar{z},$$  \hspace{1cm} (12)

with $H(u, z, \bar{z}) = f(u, z) + \bar{f}(u, \bar{z})$ for general functions $f$. Choosing

$$\ell = \partial_\nu, \hspace{0.5cm} n = \partial_u - H \partial_\nu, \hspace{0.5cm} m = \partial_z,$$  \hspace{1cm} (13)

one has $\rho = \tau = 0$ and so [19] implies $S = S(u, \bar{z})$, while the Weyl scalar $\Psi = \partial^2 f$, so [53] implies that

$$\Phi_2 = \sqrt{-\partial^2 f} S(u, \bar{z}).$$  \hspace{1cm} (14)

The other class of plane-fronted waves is given by

$$ds^2 = -2d\tau \left( dv + W dz + \bar{W} d\bar{z} + H d\tau \right) + 2dz d\bar{z},$$  \hspace{1cm} (15)

with $W(v, z, \bar{z}) = -2v(z + \bar{z})^{-1}$ and

$$H(u, v, z, \bar{z}) = \left[ f(u, z) + \bar{f}(u, \bar{z}) \right] (z + \bar{z}) - \frac{\nu^2}{(z + \bar{z})^2};$$

again $f(u, z)$ is arbitrary. Choosing

$$\ell = \partial_v, \hspace{0.5cm} n = \partial_u - (H + \bar{W}) \partial_\nu + \bar{W} \partial_z + W \partial_{\bar{z}}, \hspace{0.5cm} m = \partial_{\bar{z}},$$

one has $\rho = 0$, $\tau = 2\beta = -(z + \bar{z})^{-1}$, so [19] gives $S = \zeta(u, \bar{z})/(z + \bar{z})$. The Weyl scalar $\Psi_4 = (z + \bar{z})^2 \partial^2 \bar{f}$, so [53] implies that

$$\Phi_2 = \sqrt{-\partial^2 \bar{f}} \zeta(u, \bar{z}).$$  \hspace{1cm} (16)

Given that the only non-zero components of $F_{\mu\nu}$ are for $\mu\nu = [u\bar{z}]$ and $[u\bar{z}]$, the simple form of the relevant components of $g^{\mu\nu}$ and the fact that $g = 1$ give

$$\nabla_\nu F_{\mu\nu} = \frac{1}{\sqrt{|g|}} \partial_\nu \left( \sqrt{|g|} g^{\mu\rho} g^{\sigma\nu} F_{\rho\sigma} \right) = \partial_\nu (\eta^{\mu\rho} \eta^{\sigma\nu} F_{\rho\sigma}) = 0.$$  \hspace{1cm} (17)

On the other hand, $S$ does not depend on $f(u, z)$ or $\bar{f}(u, \bar{z})$, meaning that it must solve the wave equation on any member of the family. In particular, it solves the wave equation on Minkowski spacetime. This implies that the Maxwell and the scalar fields also satisfy their equations on Minkowski spacetime, establishing the Weyl double copy for type N Kundt solutions.

**Robinson-Trautman solutions**

Type N Robinson-Trautman solutions take the form [52]

$$ds^2 = -H du^2 - 2du dr + \frac{2r^2}{P^2} dz d\bar{z},$$  \hspace{1cm} (18)

with $H(u, r, z, \bar{z}) = k - 2r \partial_u \log P$ (where $k = 0, \pm 1$) and $2P^2 \partial_r \partial_u \log P(u, z, \bar{z}) = k$. Choosing

$$\ell = \partial_r, \hspace{0.5cm} n = \partial_u - \frac{1}{2} H \partial_r, \hspace{0.5cm} m = -\frac{P}{r} \partial_z,$$  \hspace{1cm} (19)

one has $\rho = -r^{-1}$, $\tau = 0$, so [19] gives $S = -\zeta(u, \bar{z})/r$. Now $\Psi_4 = -\frac{P^2}{r} \partial_u (\partial^2 \Psi)$, so [53] determines that

$$\Phi_2 = \frac{P}{r} \sqrt{\zeta(u, \bar{z})} \partial_u (\partial^2 \Psi/P).$$  \hspace{1cm} (20)

As an example, consider Robinson-Trautman solutions with $k = 0$ in [18]. Writing $P = e^W$ we have $\partial_u \partial_{\bar{z}} W = 0$ and hence $W = w(u, z) + \bar{w}(u, \bar{z})$, implying that $\Psi_4 = -P^2/r \partial_u [\partial_z \bar{w}(u, \bar{z}) + \partial_{\bar{z}} \bar{w}(u, \bar{z})]$. We can obtain type N solutions of the Maxwell equation in the Robinson-Trautman background by taking

$$A = \gamma(u, z, \bar{z}) du,$$  \hspace{1cm} (21)
where $\partial_x \partial_z \gamma = 0$ and hence $\gamma = h(u, z) + \bar{h}(u, \bar{z})$. Thus from (11) we have $\Phi_2 = -P/r \partial \bar{h}(u, \bar{z})$. Plugging into (5) we have

$$\partial_u [\partial^2 \bar{w}(u, \bar{z}) + (\bar{\partial} \bar{w}(u, \bar{z}))^2] = -\frac{1}{rS} (\partial \bar{h}(u, \bar{z}))^2,$$

(22)

and so indeed we have that $S = -\zeta/r$, where $\zeta$ is a function only of $u$ and $\bar{z}$, as required in the general result stated above.

As with Kundt solutions, the only non-zero components of $F_{\mu \nu}$ are for $\mu \nu = [uz]$ and $[u\bar{z}]$. As before, using the fact that $\sqrt{|g|} = r^2/P^2$ and the relevant components of $g^{\mu \nu}$, it can be shown that (17) holds. Once again, $S$ is independent of $P$ and solves the wave equation on any member of the family (18), including Minkowski. Hence, both $F_{\mu \nu}$ and $S$ satisfy their equations also on the flat background, establishing the Weyl double copy for Robinson-Trautman solutions.

**Twisting solutions**

Type N solutions with non-vanishing twist are more complicated, with only one explicit solution known (54). The general metric is given by (55).

$$ds^2 = -2(du + Ldz + \bar{L}d\bar{z}) [dr + Wdz + \bar{W}d\bar{z}] + \frac{2}{P^2 |\rho|^2} dz d\bar{z},$$

(23)

with $K = 2P^2 \Re \left[ \partial (\partial \log P - \partial_u L) \right]$. There exists a residual gauge freedom to choose $P = 1$, but we shall not yet impose this choice. The solution is determined by the complex scalar $L$, which satisfies

$$\Sigma K + P^2 \Re [\partial \bar{\partial} \Sigma - 2 \partial_u \bar{L} \partial \Sigma - \bar{\Sigma} \partial \bar{\partial} \bar{L}] = 0, \quad \partial I = 0,$$

and $\partial_u I \neq 0$, with $I = \bar{\partial} (\partial \log P - \partial_u L) + (\bar{\partial} \log P - \partial_u \bar{L})^2$. Choosing

$$\ell = \partial_r, \quad n = \partial_u - H \partial_r, \quad m = -P \bar{\rho} (\partial - W \partial_r),$$

(24)

$\rho$ is as defined above, while $\tau = 0$. Equation (9) then implies that $S = \rho \chi(u, z, \bar{z})$, with $\chi$ satisfying

$$\partial \chi - \partial_u L \chi = 0.$$

(25)

Defining new coordinates $(v, w) = (I, z)$, the above equation can be solved using the method of characteristics ($I = $ constant correspond to the characteristics)

$$\chi(v, w) = \zeta(I) e^{\int \left[ \frac{(\partial_w w)(v,w')}{2} \right] dw'},$$

(26)

with $\zeta(I)$ arbitrary. The Weyl scalar $\Psi_4 = \rho P^2 \partial_u I$, and so (5) implies

$$\Phi_2 = \rho P \sqrt{\partial_u I} \chi(u, z, \bar{z}).$$

(27)

Only one twisting type N solution, found by Hauser (54), is known explicitly. The metric functions are given by

$$P = (z + \bar{z})^{3/2} f(t), \quad t = \frac{u}{(z + \bar{z})^2}, \quad L = 2i(z + \bar{z}),$$

where $f$ satisfies $16(1 + t^2)f''(t) + 3f(t) = 0$, which is a hypergeometric equation, and $I$ turns out to be given by

$$I = \frac{3}{2[(z + \bar{z})^2 - iu]}. $$

(28)

The solution to (9) is

$$S = \rho \zeta(I),$$

(29)

where $\zeta(I)$ is arbitrary. As expected, this is consistent with the general result (26). The Weyl scalar is $\Psi_4 = (2i/3) \rho P^2 I^2$, implying that

$$\Phi_2 = \rho P I \sqrt{\frac{2i \zeta(I)}{3}}.$$

(30)

As a further remark about the twisting type N solutions, we note that if the gauge freedom to set $P = 1$ is employed, the metric is specified purely in terms of the function $L(u, z, \bar{z})$, and the type N and Ricci flat conditions may be succinctly condensed down to just

$$\partial I = 0, \quad \Im (\bar{\partial} \partial \partial L) = 0, \quad \text{where } I = -\partial_u \partial L.$$  

(31)

The Weyl curvature is given by $\Psi_4 = \rho \partial_u I$.

In contrast to non-twisting solutions, the second equality in (17) does not hold for twisting solutions. Therefore, while there is a curved Weyl double copy relation, in this case it does not translate to a relation where the Maxwell field and the scalar can be thought of as Minkowski fields, unless we consider all the fields (gravity, Maxwell and scalar) at the linearised level.

**NON-UNIQUENESS**

In all the cases above, neither the Maxwell field nor the scalar field are uniquely determined. They are fixed only up to an arbitrary function of some of the coordinates, which we are free to choose. This contrasts with the Weyl double copy for vacuum type D solutions, for which, in a spinor basis adapted to the principal null directions, we have $S^3 \propto (\Phi_2)^{3/2} \propto \Psi_4$, where the proportionality is up to complex parameters (55); hence the Maxwell and scalar fields are functionally fixed. This feature is related
to the fact that vacuum type D spacetimes are fully determined up to a few parameters, whereas vacuum type N spacetimes (of any class, as seen above) have functional freedom. By analogy, there is additional freedom in the Maxwell and scalar fields in the curved background.

In considering a special choice, we may ask whether it is possible to choose $\Phi_2$ and $S$ to be given by specific powers of $\Psi_4$, as in the type D case, i.e. there exists some constant $\alpha$ such that $\Phi_2 \propto (\Psi_4)^\alpha$ and $S \propto (\Psi_4)^{2-\alpha}$. The functional dependence of the results above implies that this possibility holds only for Kundt solutions. For pp-waves, the power is actually undetermined, i.e. the relation above holds for any $\alpha$. A simple choice is $\alpha = 1/2$, where $S$ is constant, and in fact this choice implies that Maxwell plane waves double copy to gravitational plane waves ($\Phi_2$ and $\Psi_4$ are functions of $\alpha$ only). For the other plane-fronted Kundt solutions, such a relation is possible for $\alpha = 0$, in which case $S \propto (\Psi_4)^{-1}$. Analogously simple choices for the other type N classes are: $S \propto 1/r$ for Robinson-Trautman solutions and $S \propto \rho$ for twisting solutions.

Interestingly, pp-waves are the only type N solutions admitting a Killing 2-spinor [29], another feature that they share with type D solutions.

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See Theorem 31.2 of [53].

See Theorem 28.1 and Section 28.1.2 of [58].

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