An unreliable $M^{[X]}/G/1$ retrial Queue with multi optional stages of service and delay in repair

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Abstract: Unreliable vacation retrial queue and multi stages of service delay in repair is studied. After completion of the $i^{th}$ ($i=1,2,...,k$) stage of service, the unit may have the option to choose $(i+1)^{th}$ stage of service with probability $\theta_i$, or with $p_i$ may join into orbit to give feedback or may leave the station with probability $q_i = 1-p_i-\theta_i$, $(i=1,2,...k-1)$ and $q_k = 1-p_k$. After service completion if the orbit has no units, server takes a vacation. During repair, the unit waiting in the system to complete the remaining service (delay time) is discussed. We analyzed the system using the method of supplementary variable. Simulation results are given using MATLAB.

1. Introduction
Retrial queueing system with vacations is very useful while dealing with real time situations. The survey on retrial queues by Artelijo et al.[1], Artalejo[2], [3] and Falin et al. [7] is followed to frame this work. Wang et al. [13] have studied the retrial queueing system with single server and second optional services. Recently, Salehiradat al. [11] and Bagyamet al. [4] have discussed about Bernoulli feedback.

Service station breakdowns are very common in queueing systems. Keet al. [9], Choudhury et al. [6] discussed, about two phases of service batch retrial queueing pattern and delaying repair. Chen et al. [5] analyzed the breakdown queues. Wang et al. [13] and Zhang M et al. [14] discussed the vacations in queueing system. Krishnakumar et al. [10] surveyed a queueing systems. This paper finds applications in communications oriented systems and in industrial organizations, etc.

2. Characteristics of the model
2.1 Arrival process
Units arriving the system in batches with Poisson arrival rate $\lambda$. Let $X_k$, the number of units in the $k^{th}$ batch, where $k = 1,2,3,...$ with common distribution $Pr[X_k = n] = \chi_n$, $n = 1,2,3...$ The PGF (probability generating function) of $X$ is $X(z)$. The first and second moments are $E(X)$ and $E(X(X-1))$. 

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2.2 Retrial process:
If there is no space to wait, one from the arriving unit begins service (if the server is free) and rest are waiting in the orbit. If an arriving batch finds the server either busy or on vacation or breakdown, then the batch joins into an orbit. Here, inter-retrial times form an arbitrary distribution \( R(x) \) with corresponding Laplace-Stieltjes transform (LST) \( R'(s) \).

2.3 Service process:
Here, a server gives \( k \) stages of service. The First Stage Service (FSS) is followed by \( \ast \) states of service. The service time \( S \) for \( i=1,2,...,k \) has a distribution (general) function \( S_i(x) \) having LST \( S'_i(s) \) and first and second moments are \( E(S_i) \) and \( E(S_i^2) \), \( i=1,2,...,k \).

2.4 Feedback rule:
After completion of \( i^{th} \) stage of service the customer may go to \( (i+1)^{th} \) stage with probability \( \theta_i \), or may join into the orbit as feedback customer with probability \( p_i \) or leaves the system with probability \( q_i = 1-\theta_i-p_i \) for \( i=1,2,...,k-1 \). If the customer in the last \( k^{th} \) stage may join to the orbit with probability \( p_k \) or leaves the system with probability \( q_k = 1-p_k \). From this model, the service time or the time required by the customer to complete the service cycle is a random variable \( S \) given by \( S = \sum_{i=1}^{k} \Theta_{i-1}S_i \) having the LST \( S^*(s) = \prod_{i=1}^{k} \Theta_{i-1}S'_i(s) \) and the expected value is \( E(S) = \sum_{i=1}^{k} \Theta_{i-1}E(S_i) \), where \( \Theta_0 = \Theta_2 = ... = \Theta_{k-1} \) and \( \Theta_k = 1 \).

2.5 Vacation process:
If the orbit has no units, the server takes a single vacation (simply taking break or secondary job etc.) of random length \( V \). After finishing the vacation, the server is idle to provide service for primary units or units from the orbit. Here, the distribution function \( V(x) \) and LST \( V^*(s) \) with moments \( E(V) \) and \( E(V^2) \).

2.6 Breakdown and repair:
The service station may down at any time with Poisson rate \( \alpha \) where \( i=1,2,...,k \) during service. The unit on service has to wait to complete the remaining service. This waiting time is taken as delay time. The server continues the service for this unit after the repair process. Here, the waiting time is defined as delay time. The delay time \( D_i \) has density function \( D_i(y) \), Laplace-Stieltjes Transform \( D'_i(s) \) and finite \( k^{th} \) moment \( E(D_i^k) \) \( i=1,2,...,k \) and \( k=1,2 \). The repair time \( G_i \) has the distributions function \( G_i(y) \) and LST \( G'_i(s) \) for \( i=1,2,...,k \). Consider various Probability processes involved in the system are mutually exclusive.

In the steady state, let \( R(0) = 0, R(\infty) = 1, S_i(0) = 0, S_i(\infty) = 1, i=1,2,...,k \) are continuous at \( x = 0 \) and \( D_i(0) = 0, D_i(\infty) = 1, G_i(0) = 0, G_i(\infty) = 1 \) are continuous at \( y = 0, (1 \leq i \leq k) \). Let \( R_i(t), S'_i(t), D'_i(t) \) and \( G'_i(t) \) be the elapsed times for retrial, service on \( i^{th} \) stage, delay in repair on \( i^{th} \) stage, repair on \( i^{th} \) stage, \( (1 \leq i \leq k) \) respectively. Now, a random variable at time \( t \),
The Markov process \( \{C(t), N(t); t \geq 0\} \) describes the system state, where \( C(t) \) - the server state and \( N(t) \) - the number in orbit at time \( t \), the functions \( a(x), \mu_i(x), \gamma_i(x), \eta_i(y) \) and \( \xi_i(y) \) are the conditional completion rates for retrial, service, vacation, delay repair and repair respectively (\( 1 \leq i \leq k \)).

\[
a(x)dx = \frac{dR(x)}{1 - R(x)}, \quad \mu_i(x)dx = \frac{dS_i(x)}{1 - S_i(x)}, \quad \gamma_i(x)dx = \frac{dV(x)}{1 - V(x)}, \quad \eta_i(y)dy = \frac{dD_i(y)}{1 - D_i(y)}
\]

and \( \xi_i(y)dy = \frac{dG_i(y)}{1 - G_i(y)} \). Define \( B_i^* = S_i^*S_{i-1}^*...S_1^* \) and \( B_0^* = 1 \). The first moment \( M_{1i} \) and second moment \( M_{2i} \) of \( B_i^* \) are given by

\[
M_{1i} = \lim_{z \to 1} db_i^* [A_i(z)] / dz = \sum_{j=1}^{k} \lambda E(X)E(S_j) [1 + \alpha_j] [E(G_j) + E(D_j)].
\]

\[
M_{2i} = \lim_{z \to 1} d^2 b_i^* [A_i(z)] / dz^2 = \sum_{j=1}^{k} \left[ -M_{1i} \left\{ \frac{\lambda E(X(X - 1)) + \alpha_j [\lambda E(X(X - 1))] [E(G_j) + E(D_j)]}{[-(\lambda E(X))^2 [E(G_j^2) + E(D_j^2)]} \right\}
\right.

\[
\left. + \left(\lambda E(X)E(S_j)\right)^2 E(S_j^2) [1 + \alpha_j] [E(G_j) + E(D_j)] \right)^2 \right] .
\]

where \( A_i(z) = \alpha_i \left( 1 - G_i^*(b(z))D_i^*(b(z)) \right) + b(z) \) and \( b(z) = \lambda \left( 1 - X(z) \right) \)

Let \( \{t_n; n = 1, 2, ...\} \) be the service period ending time or repair period ending time. In this system, \( Z_n = \{C(t_n +), N(t_n +)\} \) forms an embedded Markov chain which is ergodic \( \Leftrightarrow \rho < 1 \), where

\[
\rho = E(X)[1 - R^*(\lambda)] + \sum_{i=1}^{k} \Theta_i \cdot M_{1i} + \sum_{i=1}^{k} \rho \Theta_i - \sum_{i=1}^{k+1} \Theta_i M_{1i} .
\]

3. Steady state probability functions

For the process \( \{N(t), t \geq 0\} \), define the probabilities at time \( t \) as,

- \( P_e(t) \) - Pr the system is empty,
- At time \( t \) and \( n \) customers in the orbit, \( P_e(n,t) \) - Pr an elapsed retrial time \( x \) of the retrial customers,
- \( \Pi_{i,n}(x,t), (1 \leq i \leq k) \) - Pr elapsed service time on the \( i^\text{th} \) stage of the customer under service,
- \( Q_{i,n}(x,t) \) - Pr elapsed vacation time of the customer on vacation,
- \( R_{i,n}(x,y,t), (1 \leq i \leq k) \) - Pr elapsed time for service is and repair is \( y \) on the \( i^\text{th} \) stage,
- \( D_{i,n}(x,y,t), (1 \leq i \leq k) \) - Pr elapsed time for service is and delay repair is on the \( i^\text{th} \) stage.

The stability condition exists for \( t \geq 0, x \geq 0, y \geq 0, n \geq 0 \) for \( i = 1, 2, ... k \).
\[ P_0 = \lim_{t \to \infty} P_0(t), \quad P_i(x) = \lim_{t \to \infty} P_i(x,t), \quad \Pi_{i,n}(x) = \lim_{t \to \infty} \Pi_{i,n}(x,t), \]
\[ Q_n(x) = \lim_{t \to \infty} Q_n(x,t), \quad \Omega_{i,n}(x,y) = \lim_{t \to \infty} \Omega_{i,n}(x,y,t). \text{ for } t \geq 0. \quad R_{i,n}(x,y) = \lim_{t \to \infty} R_{i,n}(x,y,t), \text{ for } t \geq 0. \]

3.1 Steady state equations

The following equations are obtained by the supplementary variable technique for \((i=1,2,\ldots,k)\).

\[ \lambda P_0 = \int_0^\infty Q_0(x) \gamma(x) dx. \quad (1) \]
\[ \frac{dP_n(x)}{dx} = - \lambda P_n(x) - a(x) P_n(x), \quad n \geq 1. \quad (2) \]
\[ \frac{d\Pi_{i,0}(x)}{dx} = - \lambda \Pi_{i,0}(x) - \alpha_i \Pi_{i,0}(x) - \mu_i(x) \Pi_{i,0}(x) + \int_0^\infty \xi_i(y) R_{i,0}(x,y) dy, \quad n = 0. \quad (3) \]
\[ \frac{d\Pi_{i,n}(x)}{dx} = - \lambda \Pi_{i,n}(x) - \alpha_i \Pi_{i,n}(x) - \mu_i(x) \Pi_{i,n}(x) + \lambda \sum_{k=1}^n \gamma_k \Pi_{k,n-k}(x) + \int_0^\infty \xi_i(y) R_{i,n}(x,y) dy, \quad n \geq 1. \quad (4) \]
\[ \frac{dQ_0(x)}{dx} + Q_0(x) [\lambda + \gamma(x)] = 0, \quad n = 0. \quad (5) \]
\[ \frac{dQ_n(x)}{dx} + [\lambda + \gamma(x)] Q_n(x) = \lambda \sum_{k=1}^n \gamma_k Q_{n-k}(x), \quad n = 1,2,\ldots. \quad (6) \]
\[ \frac{d\Omega_{i,0}(x,y)}{dy} + \Omega_{i,0}(x,y)[\lambda + \xi_i(y)] = 0, \quad n = 0. \quad (7) \]
\[ \frac{d\Omega_{i,n}(x,y)}{dy} + \Omega_{i,n}(x,y)[\lambda + \xi_i(y)] = \lambda \sum_{k=1}^n \Omega_{k,n-k}(x,y) \gamma_k, \quad n = 1,2,\ldots. \quad (8) \]
\[ \frac{dR_{i,0}(x,y)}{dy} + R_{i,0}(x,y)[\lambda + \xi_i(y)] = 0, \quad n = 0. \quad (9) \]
\[ \frac{dR_{i,n}(x,y)}{dy} + R_{i,n}(x,y)[\lambda + \xi_i(y)] = \lambda \sum_{k=1}^n R_{i,n-k}(x,y) \gamma_k, \quad n \geq 1. \quad (10) \]

Boundary conditions at \(x = 0\) and \(y = 0\) of the steady state system are

\[ P_n(0) = \sum_{i=1}^k \int_0^\infty \mu_i(x) \Pi_{i,n}(x) dx + \sum_{i=1}^k \int_0^\infty \mu_i(x) \Pi_{i,n-1}(x) dx + \int_0^\infty \gamma(x) Q_n(x) dx, \quad n \geq 1. \quad (11) \]
\[ \Pi_{i,0}(0) = \int_0^\infty a(x) P_i(x) dx + P_0 \lambda \gamma_1, \quad n = 0. \quad (12) \]
\[ \Pi_{i,n}(0) = \int_0^\infty a(x) P_{n+1}(x) dx + \lambda \sum_{k=1}^n \gamma_k \int_0^\infty P_{n-k+1}(x) dx + P_0 \lambda \gamma_{n+1}, \quad n \geq 1. \quad (13) \]
\[ \Pi_{i,n}(0) = \theta_i \int_0^\infty \mu_i(x) \Pi_{i,1,n}(x) dx, \quad n \geq 1, \quad (2 \leq i \leq k). \quad (14) \]
\[ Q_0(0) = \sum_{i=1}^k \int_0^\infty \mu_i(x) \Pi_{i,0}(x) dx, \quad n = 0. \quad (15) \]
\[ Q_n(0) = 0, \quad n = 2,3,\ldots. \quad (16) \]
\[ \Omega_{i,n}(x,0) = \alpha_i[\Pi_{i,n}(x)]\alpha_i, \quad n \geq 1. \quad (17) \]

\[ R_{i,n}(x,0) = \int_0^\infty \eta_i(y)\Omega_{i,n}(x,y)dy, \quad n \geq 0. \quad (18) \]

The normalizing condition is

\[
\left\{ \begin{array}{l}
P_0 + \sum_{n=1}^{\infty} \int p_n(x)dx + \sum_{n=1}^{\infty} \int \Pi_{i,n}(x)dx \\
+ \sum_{n=1}^{\infty} \sum_{i=1}^{k} \int R_{i,n}(x,y)dx dy + \sum_{n=0}^{\infty} \int \Omega_{i,n}(x,y)dy + \sum_{n=0}^{\infty} \int Q_n(x)dx
\end{array} \right\} = 1. \quad (19) \]

The above equations are solved by using generating functions. Multiplying (2) to (18) by \( \sum_{n=0}^{\infty} z^n \) then,

\[ \frac{\partial P(x,z)}{\partial x} = -P(x,z)[\lambda + a(x)]. \quad (20) \]

\[ \frac{\partial \Pi_i(x,z)}{\partial x} + [\lambda(1 - X(z)) + \alpha_i + \mu_i(x)]\Pi_i(x,z) = \int_0^\infty \xi_i(y)R_i(x,y,z)dy. \quad (21) \]

\[ \frac{\partial Q(x,z)}{\partial x} + [\lambda(1 - X(z)) + \gamma(x)]Q(x,z) = 0. \quad (22) \]

\[ \frac{d\Omega_i(x,y,z)}{dy} + [\lambda(1 - X(z)) + \xi_i(y)]\Omega_i(x,y,z) = 0. \quad (23) \]

\[ \frac{dR_i(x,y,z)}{dy} + [\lambda(1 - X(z)) + \xi_i(y)]R_i(x,y,z) = 0. \quad (24) \]

At \( x = 0 \) and \( y = 0 \),

\[ P(0,z) = \sum_{i=1}^{k} \left\{ p_i z + q_i \right\} \int_0^\infty \Pi_i(x,z)\mu_i(x)dx \right\} + \int_0^\infty Q(x,z)\gamma(x)dx - \lambda P_0 - Q_0(0). \quad (25) \]

\[ \Pi_i(0,z) = \frac{1}{z} \int_0^\infty a(x)P(x,z)dx + \lambda \frac{X(z)}{z} \int_0^\infty P(x,z)dx + \frac{\lambda X(z)}{z} P_0. \quad (26) \]

\[ \Pi_i(0,z) = \theta_i \int_0^\infty \mu_{i-1}(x)\Pi_{i-1;n}(x)dx, \quad (2 \leq i \leq k). \quad (27) \]

\[ Q(0,z) = Q_0(0) \quad (28) \]

\[ \Omega_i(x,0,z) = \alpha_i[\Pi_i(x,z)]. \quad (29) \]

\[ R_i(x,0,z) = \int_0^\infty \eta_i(y)\Omega_i(x,y,z)dy, \quad n \geq 0. \quad (30) \]

Solving the equations (20) to (24), it follows that for \( (1 \leq i \leq k) \)
\[ P(x, z) = e^{-\lambda x}[1 - R(x)]P(0, z). \]  
(31)

\[ \Pi_i(x, z) = e^{-A_i(z)x}[1 - S_i(x)]\Pi_i(0, z) \]  
(32)

\[ Q(x, z) = e^{-b(z)x}[1 - V(x)]Q(0, z) \]  
(33)

\[ \Omega_i(x, y, z) = e^{-b(z)y}[1 - D_i(y)]\Omega_i(x, 0, z) \]  
(34)

\[ R_i(x, y, z) = e^{-b(z)y}[1 - G_i(y)]R_i(x, 0, z), \]  
(35)

where, \( A_i(z) = \alpha_i\left(1 - G_i'(b(z))D_i'(b(z))\right) + b(z) \) and \( b(z) = \lambda(1 - X(z)) \).

From (5), \( Q_0(x) = Q_0(0)[1 - V(x)]e^{-b\lambda x}. \)  
(36)

Multiplying (36) by \( \gamma(x) \) on both sides and integrating with respect to \( x \) from 0 to \( \infty \),

from (1), \( Q_0(0) = \frac{\lambda P_0}{V^\ast(\lambda)}. \)  
(37)

From (26) and (31), \( \Pi_i(0, z) = \frac{P(0, z)}{z} - \left[ R^\ast(\lambda)(1 - X(z)) + X(z) \right] \frac{\lambda X(z)}{z} P_0. \)  
(38)

From (32) and (38), \( \Pi_i(0, z) = \Theta_{i-1}\Pi_{i-1}(0, z)\left[B_i^\ast(A_i-1)\right], \)  
(39)

Similarly, \( \Omega_i(x, 0, z) = \alpha_i\Pi_i(0, z)\frac{S_i^\ast[A_i(z)]}{A_i(z)}. \)  
(40)

From (30) and (34), \( R_i(x, 0, z) = \Omega_i(x, 0, z)D_i'(b(z)). \)  
(41)

Using (37) and (39) and (33) in (25), then

\[ P(0, z) = \sum_{i=1}^{k}\left(p_i z + q_i\right)\Pi_i(0, z)\left[S_i^\ast[A_i(z)]\right] + Q(0, z)V^\ast[b(z)] - \lambda P_0 - \frac{\lambda P_0}{V^\ast(\lambda)}. \]  
(42)

\[ P(0, z) = \lambda P_0 \times \left\{ \frac{X(z)\Sigma + z(N(z)-1)}{z - \left[ R^\ast(\lambda)(1 - X(z)) + X(z) \right]} \right\}, \]  
(43)

where \( N(z) = \left[ V^\ast(b(z)) - 1 \right] \frac{1}{V^\ast(\lambda)}. \)

Using (43) in (26), we get, \( \Pi_i(0, z) = \lambda P_0 \left\{ \frac{(N(z)-1)\left[R^\ast(\lambda)(1 - X(z)) + X(z)\right]}{z - \left[ R^\ast(\lambda)(1 - X(z)) + X(z) \right]} + \frac{X(z)}{\omega} \right\}. \)  
(44)
\[ \Pi_i(0, z) = \lambda P_0 \Theta_{i-1} (B^*_i (A_i(z))) \left( \frac{(N(z)-1) \left[ R^*(\lambda)(1-X(z)) + X(z) \right] + X(z)}{z - \left[ R^*(\lambda)(1-X(z)) + X(z) \right] \omega} \right). \] (45)

\[ \Omega_i(x,0,z) = \alpha_i \Theta_{i-1} (B^*_i (A_i(z))) \frac{\Pi_i(0,z)}{A_i(z)}. \] (46)

From (28), we get \[ Q(0,z) = Q_b(0) = \frac{\lambda P_0}{V^*(\lambda)}. \] (47)

Using (44) and (40), we get \[ R_i(x,0,z) = \alpha_i \Theta_{i-1} (B^*_i (A_i(z))) D^*_i (b(z)) \frac{\Pi_i(0,z)}{A_i(z)}. \] (48)

Using Eqn. (31) to Eqn. (35) and Eqn. (43) to Eqn. (48), \[ P(x, z), \Pi_i(x, z), Q(x, z), \Omega_i(x, y, z) \text{ and } R_i(x, y, z) \] are obtained under \( \rho < 1 \) and given below,

\[ P(x, z) = \lambda P_0 \times \frac{X(z) \Sigma + z(N(z)-1)}{z - \left[ R^*(\lambda)(1-X(z)) + X(z) \right] \omega} (1 - R(x)) e^{-\lambda x}. \] (49)

\[ \Pi_i(x, z) = \lambda P_0 \left( \Theta_{i-1} \left[ \frac{(N(z)-1) \left[ R^*(\lambda)(1-X(z)) + X(z) \right] + X(z)}{z - \left[ R^*(\lambda)(1-X(z)) + X(z) \right] \omega} \right) \right]. \] (50)

\[ Q(x, z) = \frac{\lambda P_0}{V^*(\lambda)} \left( 1 - V(x) \right) e^{-b(z)x}. \] (51)

\[ \Omega_i(z) = \alpha_i \Theta_{i-1} \left[ 1 - S_i(z) \right] e^{-A_i(z)x} \left[ 1 - D_i(y) \right] e^{-b_i(y)} \left( B^*_i (A_i(z)) \right) \Pi_i(0,z) \] (52)

\[ R_i(x, y, z) = \alpha_i \Pi_i(0, z) B^*_i \left[ A_i(z) \right] D^*_i (b(z)) \Pi_i(0,z) \left[ 1 - S_i(z) \right] e^{-A_i(z)x} \times \left[ 1 - G_i(y) \right] e^{-b_i(y)} \] (53)

where, \( A_i(z) = \alpha_i \left( 1 - G^*_i (b(z)) D^*_i (b(z)) \right) + b(z) \) and \( b(z) = \lambda \left( 1 - X(z) \right) \).

Next the marginal orbit size distributions due is investigated.

Theorem 3.1. Under \( \rho < 1 \), the stationary distribution of the numbers in the system when server being idle, busy during \( i^{th} \) stage, on vacation, repair on \( i^{th} \) stage (for \( 1 \leq i \leq k \)) are given by

\[ P(z) = \left( 1 - R^*(\lambda) \right) P_0 \times \frac{X(z) \Sigma + z(N(z)-1)}{z - \left[ R^*(\lambda)(1-X(z)) + X(z) \right] \omega}. \] (54)

\[ \Pi_i(z) = \lambda P_0 \Theta_{i-1} (B^*_i (A_i(z))) \left( \frac{1 - S_i(z)}{A_i(z)} \right) \left( \frac{(N(z)-1) \left[ R^*(\lambda)(1-X(z)) + X(z) \right] + X(z)}{z - \left[ R^*(\lambda)(1-X(z)) + X(z) \right] \omega} \right). \] (55)

\[ Q(z) = \frac{P_0 \left( 1 - V^*(b(z)) \right)}{V^*(\lambda)(1-X(z))}. \] (56)
\[ \Omega_j(z) = \frac{\alpha_j \Theta_{i,j} \left( 1 - S^*_i(A(z)) \right) \left( 1 - D^*_i(b(z)) \right)}{A(z)b(z)} B^*_i \left[ -A(z) \right] \Pi_i(0, z). \] (57)

\[ R_i(z) = \frac{\alpha_j \Theta_{i,j} \left( 1 - S^*_i(A(z)) \right) \left( 1 - G^*_i(b(z)) \right)}{A(z)b(z)} B^*_i \left[ -A(z) \right] D^*_i(b(z)) \Pi_i(0, z), \] (58)

where,

\[ P_0 = \left\{ 1 - E(X) \left( 1 - R^*(\lambda) - \omega \right) \right\} \]

\[ \left[ 1 + \frac{N^'(1)}{E(X)} \right] \left[ 1 - \left( 1 - R^*(\lambda) \right) E(X) - \omega \right] + \sum_{i=1}^k \beta \Theta_{i,j} \left( 1 + \alpha_j \left| E(G_j) + E(D_j, i) \right| \right) \left( E(X) + N^'(1) \right) \left( 1 - R^*(\lambda) \right) E(X) \right\}. \] (59)

Proof. Integrating (49) to (53) with respect to x and y, defined the following for \( 1 \leq i \leq k \)

\[ P(z) = \int_0^\infty P(x, z) dx, \quad \Pi_i(z) = \int_0^\infty \Pi_i(x, z) dx, \quad Q(z) = \int_0^\infty Q(x, z) dx, \quad R_i(x, z) = \int_0^\infty R_i(x, y, z) dy, \quad R_i(z) = \int_0^\infty R_i(x, z) dx, \]

\[ \int_0^\infty \Omega_i(x, z) dx = \int_0^\infty \Omega_i(x, y, z) dy, \quad \Omega_j(z) = \int_0^\infty \Omega_j(x, z) dx. \]

Since \( P_0 \) can be determined using (19).

\[ P_0 + P(1) + Q(1) + \sum_{i=1}^k \left( \Pi_i(1) + \Omega_i(1) + R_i(1) \right) = 1 \]

is obtained by setting \( z = 1 \) in (54) to (59).

**Theorem 3.2.** Under \( \rho < 1 \), PGF of the system size and orbit size distribution at stationary point of time is

\[ K(z) = \frac{NR(z)}{Dr(z)}, \] (60)

where \( NR(z) = P_0 \left[ z \right] \left( \sum_{i=1}^k \Theta_{i,j} \left( B^*_i \left[ -A(z) \right] \right) \left( 1 - S^*_i(A(z)) \right) \left( (N(z) - 1) \left( R^*(\lambda)(1 - X(z)) + X(z) \right) + X(z) \right) \right] \]

\[ -N(z) \left( z - R^*(\lambda) + X(z) \left( 1 - R^*(\lambda) \right) \right) \omega \] + \( 1 - X(z) \) \left( \left( X(z) \Sigma + z (N(z) - 1) \right) \left( 1 - R^*(\lambda) \right) \right), \]

\[ Dr(z) = \left[ 1 - X(z) \right] \left( z - R^*(\lambda)(1 - X(z)) + X(z) \right) \Sigma, \]

and

\[ \omega = \sum_{i=1}^k \Theta_{i,j} \left( M_i - \sum_{i=1}^k \Theta_{i,j} \right) + \sum_{i=1}^k \Theta_{i,j} \Sigma. \]

Also

\[ H(z) = \frac{NR(z)}{Dr(z)}, \] (61)
\[
NR(z) = P_0 \left[ \sum_{i=1}^{k} \Theta_{t-1} \left( B'_{t-1} [A_{t-1}(z)] \right) \left[ \left( 1 - S'_t(A_t(z)) \right) \left( N(z)-1 \right) (R^t(\lambda)) + X(z) \left( 1 - R^t(\lambda) \right) \right] \right] \\
-\left[ \left( N(z) - 1 \right) (R^t(\lambda)) + X(z) \left( 1 - R^t(\lambda) \right) \right] \omega + \left[ X(z) \Sigma + (N(z)-1) \left( 1 - R^t(\lambda) \right) \right],
\]

Where \( P_0 \) is given in Eq. (59).

Proof. The statement is obtained by using \( K(z) = P_0 + P(z) + Q(z) + z \sum_{i=1}^{k} \Pi_i(z) + \sum_{i=1}^{k} \Omega_i(z) + \sum_{i=1}^{k} R_i(z) \) and

\[
H(z) = P_0 + P(z) + Q(z) + z \sum_{i=1}^{k} \Pi_i(z) + \sum_{i=1}^{k} \Omega_i(z) + \sum_{i=1}^{k} R_i(z) .
\]

4. Performance measures

Here, the mean numbers in the orbit \( (L_q) \), the mean numbers in the system \( (L_s) \), the mean waiting time in the system \( (W_s) \) and in the queue \( (W_q) \) are required to analyze the model.

**Theorem 4.1.** If the system satisfies \( \rho < 1 \), then the following probabilities of the server state, that is the server is idle during the retrial, busy during \( i^{th} \) stage, on vacation, delaying repair during \( i^{th} \) stage and under repair on \( i^{th} \) stage respectively are obtained.

\[
P = \frac{1 - R^t(\lambda)}{\beta_t} \left( E(X) + N'(1) + \omega - 1 \right).
\]

\[
\Pi_i = \sum_{i=1}^{k} \Pi_i = \frac{1}{\beta_t} \sum_{i=1}^{k} \left\{ \Theta_{t-1} \lambda E(S_i) \right\} \left[ N'(1) + E(X) R^t(\lambda) \right].
\]

\[
Q = \frac{1}{\beta_t} \left[ 1 - E(X) \left( 1 - R^t(\lambda) \right) - \omega \right] \frac{N'(1)}{E(X)}.
\]

\[
\Omega_i = \sum_{i=1}^{k} \Omega_i = \frac{1}{\beta_t} \sum_{i=1}^{k} \alpha_i E(D_i) \left\{ \Theta_{t-1} \lambda E(S_i) \right\} \left[ N'(1) + E(X) R^t(\lambda) \right].
\]

\[
R_i = \sum_{i=1}^{k} R_i = \frac{1}{\beta_t} \sum_{i=1}^{k} \alpha_i E(G_i) \left\{ \Theta_{t-1} \lambda E(S_i) \right\} \left[ N'(1) + E(X) R^t(\lambda) \right].
\]

Proof. The statement followed by using

\[
P = \lim_{z \to 1} P(z), \quad \sum_{i=1}^{k} \Pi_i = \lim_{z \to 1} \sum_{i=1}^{k} \Pi_i(z), \quad \sum_{i=1}^{k} \Omega_i = \lim_{z \to 1} \sum_{i=1}^{k} \Omega_i(z) \text{ and } \sum_{i=1}^{k} R_i = \lim_{z \to 1} \sum_{i=1}^{k} R_i(z).
\]

**Theorem 4.2.** Let \( L_s, L_q, W_s \) and \( W_q \) be the average system size, average orbit size, average waiting time in the system and average waiting time in the orbit respectively, then under \( \rho < 1 \),

\[
L_q = P_0 \left[ \frac{N_q^p(1) D_q^p(1) - D_q^p(1) N_q^p(1)}{3 (D_q^p(1))^2} \right].
\]
where,

\[
Nr^*_q(1) = -2 \left[ \frac{1}{2} \sum_{i=1}^{k} \left( N'(1) + E(X)R'(\lambda) \right) \right] - (E(X))^2 \left[ 1 - R'(\lambda) \right] - 2N'(1) + E(X)(2R'(\lambda) - 1)(\omega - 1),
\]

\[
Dr^*_q(1) = -2E(X)(1 - \rho),
\]

\[
Dr^*_q = 3 \left\{ E(X) \left[ (1 - R'(\lambda)) + 2E(X)\omega + \tau \right] - E(X)(X(1) - \rho) \right\},
\]

\[
Nr^*_q(1) = 3 \left\{ N'(1) + E(X)(X(1)) + E(X)(1 - R'(\lambda))(2N'(1) - 1) \right\} \sum_{i=1}^{k} \Theta_{i-1}M_{2i} + 2M_{2i}M_{1i} - \tau \]

\[
\omega = \sum_{i=1}^{k} \Theta_{i-1}M_{2i} - p_{k}\Theta_{i-1} + \sum_{i=1}^{k} \Theta_{i}M_{1i}.
\]

\[
\tau = \sum_{i=1}^{k} \Theta_{i-1}M_{2i} + \sum_{i=1}^{k} \Theta_{i}M_{1i} - \sum_{i=1}^{k} \Theta_{i}M_{2i}.
\]

and \( \rho = E(X)(1 - R^*(\lambda) - \omega) \).

\[
L_s = \frac{P_0 \left[ Nr^*_q(1)Dr^*_q(1) - Dr^*_q(1)Nr^*_q(1) \right]}{3 \left( Dr^*_q(1) \right)^2},
\]

Where, \( Nr^*_q(1) = Nr^*_q(1) - 6 \sum_{i=1}^{k} \Theta_{i-1}M_{1i} \left( N'(1) + E(X)R^*(\lambda) \right) \).

\[
W_s = \frac{L_s}{\lambda E(X)} \text{ and } W_q = \frac{L_q}{\lambda E(X)}.
\]

Proof: Under \( \rho < 1 \), \( L_s \) is obtained from

\[
L_q = \frac{Nr(z)}{Dr(z)} = \lim_{z \to 1} \frac{d}{dz} H(z) = H'(1) = P_0 \left[ \frac{Nr^*_q(1)Dr^*_q(1) - Dr^*_q(1)Nr^*_q(1)}{3 \left( Dr^*_q(1) \right)^2} \right].
\]

And \( L_s \) is obtained from

\[
L_s = \frac{Nr(z)}{Dr(z)} = \lim_{z \to 1} \frac{d}{dz} K(z) = K'(1) = P_0 \left[ \frac{Nr^*_q(1)Dr^*_q(1) - Dr^*_q(1)Nr^*_q(1)}{3 \left( Dr^*_q(1) \right)^2} \right].
\]

\( W_s \) and \( W_q \) are obtained by Little’s formula, \( L_s = \lambda W_s \) and \( L_q = \lambda W_q \).

4.1 Special case

Single phase, No retrial, No Vacation and No breakdown, No delaying repair
Let $P \{X = 1\} = 1$, $R(\lambda) \rightarrow 1$, $P \{V = 0\} = 1$ and $\alpha = 0$. Our model can be reduced to multi stage M/G/1 queueing system with Bernoulli feedback. The following results agree with Salehirad and Badamchizadeh [12].

$$K(z) = P_0 \left( (1 - S^*_1(A(z))) + \sum_{i=2}^{K} \Theta_{i-1} \left( B^*_i[A_{i-1}(z)] \right) \right) - z \sum_{i=1}^{K} \left( p_i z + q_i \right) \Theta_{i-1} \left( B^*_i[A(z)] \right).$$

5. **Numerical illustration**

Here, some numerical examples are given using MATLAB. The times for retrial, service vacation and repair respectively are exponentially $f(x) = ve^{-\nu x}, x > 0$ for Erlang-2 stage $f(x) = v^2 xe^{-\nu x}, x > 0$ and hyper-exponentially $f(x) = ce^{-v x} + (1-c)v^2 xe^{-\nu x}, x > 0$ distributed. And assume the arbitrary values to the parameters satisfies $\rho < 1$. The computed values of $P_0, P_i, P_\eta, Q$ and $R_i$ for ($i=1, 2, \ldots, K$) respectively are given in the tables. For the effect of $a, p, \gamma$ and $\xi$ are retrial rate, feedback probability, vacation rate and repair rate on FSS respectively graphs are given in Figure 1 to 6.

Table 1 indicates that when $E(X)$ increases, then $P_0$ decreases, $L_q$ and $W_q$ are increasing for the values of $\lambda = 0.5; p_1 = 0.2; \mu_1 = 7; \gamma = 1$; $k = 1; \theta_i = 0.4; a = 5; \eta_I = 7$. Table 2 shows that when $a$ increases, then $P_0$ decreases, $L_q$ and $P$ are increasing for $\lambda = 0.2; p_1 = 0.2; \mu_1 = 7; \gamma = 3; \theta_i = 0.2; p_2 = 0.3; \mu_2 = 10; a_2 = 0.6; \xi = 5; \eta_I = 4; k = 2; \theta_2 = 0.4; J = 2; E(X) = 1$.

| Retrial distribution | Exponential | Erlang – 2 stage | Hyper – Exponential |
|-----------------------|-------------|------------------|---------------------|
| $E(X)$                | $P_0$       | $L_q$            | $W_q$              |
| $0.50$                | 0.5585      | 0.2334           | 0.4669              |
| $0.60$                | 0.5557      | 0.2398           | 0.4796              |
| $0.70$                | 0.5529      | 0.2471           | 0.4943              |
| $0.80$                | 0.5500      | 0.2555           | 0.5111              |
| $0.90$                | 0.5470      | 0.2651           | 0.5301              |

| Retrial rate $a$ | Exponential | Erlang – 2 stage | Hyper – Exponential |
|------------------|-------------|------------------|---------------------|
| $a$              | $P_0$       | $L_q$            | $W_q$              |
| $2.00$           | 0.8559      | 0.1979           | 0.1212              |
| $3.00$           | 0.8565      | 0.1552           | 0.0787              |
| $4.00$           | 0.8568      | 0.1365           | 0.0583              |
| $5.00$           | 0.8570      | 0.1261           | 0.0463              |
| $6.00$           | 0.8571      | 0.1194           | 0.0384              |

Table 1. The effect of Mean batch size $E(X)$ on $P_0, L_q$ and $W_q$
### Table 3. The effect of $p_1$ on $P_0$, $L_q$ and $P$

| Feedback probability | Exponential   | Erlang – 2 stage | Hyper – Exponential |
|----------------------|---------------|-----------------|---------------------|
| $p_1$                | $P_0$ | $L_q$ | $P$ | $P_0$ | $L_q$ | $P$ | $P_0$ | $L_q$ | $P$ |
| 0.10                 | 0.9273 | 0.0410 | 0.0345 | 0.8471 | 0.1068 | 0.0806 | 0.9338 | 0.0370 | 0.0315 |
| 0.20                 | 0.9226 | 0.0536 | 0.0426 | 0.8350 | 0.1499 | 0.1008 | 0.9296 | 0.0481 | 0.0389 |
| 0.30                 | 0.9164 | 0.0717 | 0.0535 | 0.8183 | 0.2203 | 0.1291 | 0.9241 | 0.0641 | 0.0488 |
| 0.40                 | 0.9077 | 0.0998 | 0.0686 | 0.7933 | 0.3489 | 0.1711 | 0.9164 | 0.0885 | 0.0625 |
| 0.50                 | 0.8947 | 0.1479 | 0.0913 | 0.7522 | 0.6295 | 0.2404 | 0.9050 | 0.1296 | 0.0828 |

### Table 4. The effect of $(\gamma)$ on $P_0$, $L_q$ and $Q$

| Vacation rate | Exponential   | Erlang – 2 stage | Hyper – Exponential |
|---------------|---------------|-----------------|---------------------|
| $\gamma$     | $P_0$ | $L_q$ | $Q$ | $P_0$ | $L_q$ | $Q$ | $P_0$ | $L_q$ | $Q$ |
| 5.00          | 0.7707 | 0.4050 | 0.1780 | 0.5083 | 4.1119 | 0.2719 | 0.8086 | 0.2867 | 0.1532 |
| 6.00          | 0.7987 | 0.3716 | 0.1502 | 0.5486 | 3.7027 | 0.2332 | 0.8339 | 0.2625 | 0.1280 |
| 7.00          | 0.8192 | 0.3472 | 0.1299 | 0.5791 | 3.3940 | 0.2040 | 0.8521 | 0.2451 | 0.1099 |
| 8.00          | 0.8348 | 0.3286 | 0.1143 | 0.6030 | 3.1535 | 0.1811 | 0.8658 | 0.2320 | 0.0962 |
| 9.00          | 0.8471 | 0.3140 | 0.1021 | 0.6221 | 2.9612 | 0.1628 | 0.8765 | 0.2210 | 0.0856 |

### Figure 1. $L_q$ versus $a$
Figure 2. $L_q$ verses $\xi$

Figure 3. $P_0$ verses $\eta$
Table 3 shows that when $p_1$ increases, then $P_0$ decreasing, $L_q$ and $P$ are increasing for $\lambda = 0.2$; $\mu_1 = 15$; $\alpha_1 = 0.2$; $\xi_1 = 3$; $\gamma = 5$; $\theta_1 = 0.2$; $\mu_2 = 10$; $\alpha_2 = 0.3$; $\xi_2 = 5$; $\eta_2 = 3$; $\eta_1 = 4$; $k = 1$; $\theta_2 = 0.4$; $E(X) = 2$. Table 4 shows that when $\gamma$ increases, then $P_0$ increases, $L_q$ and $Q$ also decreasing for $\lambda = 0.2$; $p_2 = 0.3$; $\mu_1 = 15$; $\alpha_1 = 0.4$; $\xi_1 = 7$; $\theta_1 = 0.2$; $p_1 = 0.2$; $\mu_2 = 7$; $\alpha_2 = 0.6$; $\xi_2 = 5$; $\eta_2 = 3$; $\eta_1 = 4$; $k = 2$; $\theta_2 = 0.4$; $E(X) = 1$. 

Figure 4. $P_0$ verses $p_1$

Figure 5. $L_q$ verses $\eta_1$ and $\gamma$

Figure 6. $P_0$ verses $a$ and $\gamma$
Figure 1 indicates that $L_q$ decreases for $a$ increases. Figure 2 indicates that $L_q$ decreases for $\xi_1$ increases. Figure 3 indicates that $P_0$ increases for $\eta$ increases. Figure 4 indicates that $P_0$ decreases for $p_1$ increases. Figure 5 indicates that $L_q$ decreases for increasing the values of $\gamma$ and $\eta$. Figure 6 indicates that $P_0$ increases for $a$ and $\gamma$.

6. Conclusion
Unreliable vacation retrial queue and multi stages of service delay in repair with batch arrival policy are meticulously studied. The PGF of the numbers in the system and orbit are found. The performance measures were obtained. $L_s, L_q, W_s$ and $W_q$ are obtained. The mathematical results are validated by simulation results.

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