NON-EXTENDABILITY OF SEMILATTICE-VALUED MEASURES ON PARTIALLY ORDERED SETS

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Abstract. For a poset $P$ and a distributive $(\vee, 0)$-semilattice $S$, a $S$-valued poset measure on $P$ is a map $\mu: P \times P \to S$ such that $\mu(x, z) \leq \mu(x, y) \vee \mu(y, z)$, and $x \leq y$ implies that $\mu(x, y) = 0$, for all $x, y, z \in P$. In relation with congruence lattice representation problems, we consider the problem whether such a measure can be extended to a poset measure $\overline{\mu}: P \to S$, for a larger poset $\overline{P}$, such that for all $a, b \in S$ and all $x \leq y$ in $\overline{P}$, $\overline{\mu}(y, x) = a \vee b$ implies that there are a positive integer $n$ and a decomposition $x = z_0 \leq z_1 \leq \cdots \leq z_n = y$ in $\overline{P}$ such that either $\overline{\mu}(z_{i+1}, z_i) \leq a$ or $\overline{\mu}(z_{i+1}, z_i) \leq b$, for all $i < n$.

In this note we prove that this is not possible as a rule, even in case the poset $P$ we start with is a chain and $S$ has size $\aleph_1$. The proof uses a “monotone refinement property” that holds in $S$ provided $S$ is either a lattice, or countable, or strongly distributive, but fails for our counterexample. This strongly contrasts with the analogue problem for distances on (discrete) sets, which is known to have a positive (and even functorial) solution.

1. Introduction

In the paper [4], the author proved that for any lattice $K$, any distributive lattice $S$ with zero, and any $(\vee, 0)$-homomorphism $\varphi$ from the $(\vee, 0)$-semilattice $\text{Con}_c K$ of all finitely generated congruences of $K$ to $S$, there are a lattice $L$, a lattice homomorphism $f: K \to L$, and an isomorphism $\alpha: \text{Con}_c L \to S$ such that $\varphi = \alpha \circ \text{Con}_c f$. In the paper [4], J. Tůma and the author proved that for a $(\vee, 0)$-semilattice $S$, this statement characterizes $S$ being a lattice. The proof of this negative result strongly uses the lattice structure of the hypothetical lattice $L$, see the proof of [4] Corollary 1.3.

In the present paper, we show that for a certain semilattice $S$ of cardinality $\aleph_1$, the poset structure alone is sufficient to get a related counterexample. More precisely, for a $(\vee, 0)$-semilattice $S$, a $S$-valued poset measure on a poset $P$ is a map $\mu: P \times P \to S$ such that $\mu(x, z) \leq \mu(x, y) \vee \mu(y, z)$ (triangular inequality) and $x \leq y$ implies that $\mu(x, y) = 0$, for all $x, y, z \in P$. We say that $\mu$ is a $V$-measure, if for all $x \leq y$ in $P$ and all $a, b \in S$, if $\mu(y, x) \leq a \vee b$, then there are a positive integer $n$ and a decomposition $x = z_0 \leq z_1 \leq \cdots \leq z_n = y$ in $P$ such that either $\mu(z_{i+1}, z_i) \leq a$ or $\mu(z_{i+1}, z_i) \leq b$, for all $i < n$. In particular, if $P$ is a lattice and $S = \text{Con}_c P$, then the map $\mu$ defined by $\mu(x, y) = \Theta^+(x, y) = \Theta(y, x \vee y)$ is a $\text{Con}_c P$-valued $V$-measure on $P$.

This yields the following poset analogue of the abovementioned lattice-theoretical problem.

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Problem. Let $S$ be a distributive $(\lor,0)$-semilattice. Does any $S$-valued poset measure on a given poset extend to some $S$-valued poset $V$-measure on a larger poset?

A version of this problem for so-called distances (instead of measures) on discrete sets (instead of posets) is stated in \cite{3}. The answer to this related question turns out to be positive (and easy). More surprisingly, this positive solution can be made functorial.

Nevertheless, we prove in the present paper that the problem above has a negative solution. Unlike what is done in \cite{4}, we do not reach here a characterization of all lattices among distributive $(\lor,0)$-semilattices. Our counterexample, denoted by $\mathcal{F}(\omega_1)$ (see Corollary \ref{cor}), is obtained as an application of a certain “free construction” used by M. Ploščica and J. Tůma in \cite{2}. The semilattice $D$ of \cite{4} Section 2], which is the simplest example of a $(\lor,0)$-semilattice which is not a lattice, does not satisfy the negative property used here. This is because $D$ is countable, while we prove in Proposition \ref{prop} that no countable distributive $(\lor,0)$-semilattice can have the required negative property. On the other hand, in relation to \cite{4} Problem 4], the proof of our counterexample uses very little of the Axiom of Choice (namely, only the axiom of countable choices), while the proof of the negative property of the abovementioned semilattice $D$ established in \cite{4} Corollary 2.4] uses the existence of an embedding from $\omega_1$ into the reals.

2. Basic concepts

For posets (i.e., partially ordered sets) $P$ and $Q$, a map $f: P \to Q$ is isotone, if $x \leq y$ implies that $f(x) \leq f(y)$, for all $x, y \in P$. In addition, we say that $f$ is join-preserving, if for any subset $X$ of $P$, whenever the join $\bigvee X$ of $X$ exists in $P$, $\bigvee f[X]$ exists in $Q$, and $\bigvee f[X] = f(\bigvee X)$. For a subset $X$ of a poset $P$, we shall put $\downarrow X = \{p \in P \mid \exists x \in X \text{ such that } p \leq x\}$, and then $\downarrow a = \downarrow\{a\}$, for all $a \in P$. We say that $X$ is a lower subset of $P$, if $X = \downarrow X$.

A $(\lor,0)$-semilattice $S$ is distributive, if $c \leq a \lor b$ in $S$ implies that there are $x \leq a$ and $y \leq b$ in $S$ such that $c = x \lor y$. A distributive $(\lor,0)$-semilattice $S$ is strongly distributive, if every element of $S$ is the join of a finite set of join-irreducible elements of $S$; equivalently, $S$ is isomorphic to the semilattice of all finitely generated lower subsets of some poset.

We shall denote by $\text{otp} P$ the order-type of a well-ordered set $P$. Hence $\text{otp} P$ is an ordinal. We shall also use standard set-theoretical notation and terminology, referring the reader to \cite{1} for further information. In particular, we shall denote by $\omega_1$ the first uncountable ordinal. A subset $C$ of $\omega_1$ is closed unbounded, if $C$ is unbounded in $\omega_1$ and the join of any nonempty bounded subset of $C$ belongs to $C$. It is well-known that the closed unbounded subsets form a countably complete filterbasis on $\omega_1$, see \cite{1} Lemma 7.4. Hence containing a closed unbounded set is a notion of “largeness” for subsets of $\omega_1$.

3. Free distributive extension of a $(\lor,0)$-semilattice

There are several non-equivalent definitions of what should be the “free distributive extension” of a given $(\lor,0)$-semilattice. The one that we shall use is introduced in \cite{2} Section 2]. Let us first recall the construction.

For a $(\lor,0)$-semilattice $S$, we shall put $\mathcal{E}(S) = \{(u, v, w) \in S^3 \mid w \leq u \lor v\}$. A finite subset $x$ of $\mathcal{E}(S)$ is reduced, if it satisfies the following conditions:
(1) $x$ contains exactly one diagonal triple, that is, a triple of the form $(u, u, u)$; we put $u = \pi(x)$.

(2) $(u, v, w) \in x$ and $(v, u, w) \in x$ implies that $u = v = w$, for all $u, v, w \in S$.

(3) $(u, v, w) \in x \setminus \{(\pi(x), \pi(x), \pi(x))\}$ implies that $u, v, w \not\leq \pi(x)$, for all $u, v, w \in S$.

We denote by $\mathcal{R}(S)$ the set of all reduced subsets of $\mathcal{C}(S)$, endowed with the partial ordering $\leq$ defined by

$$x \leq y \iff \forall (u, v, w) \in x \setminus y, \text{ either } u \leq \pi(y) \text{ or } w \leq \pi(y). \quad (3.1)$$

Furthermore, we shall identify $x$ with the element $\{(x, x, x)\}$ of $\mathcal{R}(S)$, for all $x \in S$. For set-theoretical purists, this can for example be done by replacing $\mathcal{R}(S)$ by the disjoint union of $S$ with the set of non-singletons in $\mathcal{R}(S)$. The disjointness can easily be achieved by a suitable modification of the standard definition of a triple.

We shall use the symbol $\bowtie$ to denote the canonical generators of $\mathcal{R}(S)$, so that

$$\bowtie(u, v, w) = \begin{cases} w, & \text{if either } u = v \text{ or } v = 0 \text{ or } w = 0, \\
0, & \text{if } u = 0, \\
\{(0, 0, 0), (u, v, w)\}, & \text{otherwise}, \end{cases}$$

Observe that the canonical map $\pi: \mathcal{R}(S) \to S$ is isotone and that the restriction of $\pi$ to $S$ is the identity. Furthermore, $\pi(\bowtie(u, v, w)) = 0$, for any non-diagonal $(u, v, w) \in \mathcal{C}(S)$.

The following is an easy consequence of (3.1).

$$(x \leq y) \iff x \leq \pi(y), \quad \text{for all } (x, y) \in S \times \mathcal{R}(S). \quad (3.2)$$

We recall the standard facts established in [2] about this construction.

**Proposition 3.1.**

(1) For any $\langle \lor, 0 \rangle$-semilattice $S$, $\mathcal{R}(S)$ is a $\langle \lor, 0 \rangle$-semilattice, and the inclusion map from $S$ into $\mathcal{R}(S)$ is a $\langle \lor, 0 \rangle$-embedding.

(2) For $\langle \lor, 0 \rangle$-semilattices $S$ and $T$, every $\langle \lor, 0 \rangle$-homomorphism $f: S \to T$ extends to a unique $\langle \lor, 0 \rangle$-homomorphism $\mathcal{R}(f): \mathcal{R}(S) \to \mathcal{R}(T)$ such that $\mathcal{R}(f)(\bowtie(u, v, w)) = \bowtie(f(u), f(v), f(w))$, for all $(u, v, w) \in \mathcal{C}(S)$.

(3) The assignment $S \mapsto \mathcal{R}(S)$, $f \mapsto \mathcal{R}(f)$ is a functor.

The extension $\mathcal{R}(S)$ is defined in such a way that $\bowtie(u, v, w) \leq u$ and $w = \bowtie(u, v, w) \lor \bowtie(v, u, w)$, for all $(u, v, w) \in \mathcal{C}(S)$. Hence, putting $\mathcal{R}^0(S) = S$ and $\mathcal{R}^{n+1}(S) = \mathcal{R}(\mathcal{R}^n(S))$ for each $n$, we obtain that the increasing union $\mathcal{D}(S) = \bigcup (\mathcal{R}^n(S) \mid n < \omega)$ is a distributive $\langle \lor, 0 \rangle$-semilattice, extending $S$. Furthermore, putting $\mathcal{D}(f) = \bigcup (\mathcal{D}^n(f) \mid n < \omega)$ for each $\langle \lor, 0 \rangle$-homomorphism $f$, we obtain that $\mathcal{D}$ is a functor. The proof of the following lemma is straightforward.

**Lemma 3.2.** Let $S$ be a $\langle \lor, 0 \rangle$-semilattice and let $(S_i \mid i \in I)$ be a family of $\langle \lor, 0 \rangle$-subsemilattices of $S$. The following statements hold:

(1) $\mathcal{R}\left(\bigcap_{i \in I} S_i\right) = \bigcap_{i \in I} \mathcal{R}(S_i)$ and $\mathcal{D}\left(\bigcap_{i \in I} S_i\right) = \bigcap_{i \in I} \mathcal{D}(S_i)$.

(2) If $I$ is a nonempty upward directed poset and $(S_i \mid i \in I)$ is isotone, then $\mathcal{R}\left(\bigcup_{i \in I} S_i\right) = \bigcup_{i \in I} \mathcal{R}(S_i)$ and $\mathcal{D}\left(\bigcup_{i \in I} S_i\right) = \bigcup_{i \in I} \mathcal{D}(S_i)$.

**Definition 3.3.** For a $\langle \lor, 0 \rangle$-semilattice $S$ and an element $x \in \mathcal{D}(S)$, we define the rank of $x$, denoted by $\text{rk}_x$, as the least natural number $n$ such that $x \in \mathcal{R}^n(S)$, and the complexity of $x$, denoted by $\|x\|$, by $\|x\| = 0$ if $x \in S$, and

$$\|x\| = \sum (\|u\| + \|v\| + \|w\| + 1 \mid (u, v, w) \in x), \quad \text{for all } x \in \mathcal{D}(S) \setminus S.$$
4. The semilattices \( S(\Lambda) \) and \( \mathcal{F}(\Lambda) \)

For any chain \( \Lambda \), we shall denote by \( S(\Lambda) \) the \( \langle \vee, 0 \rangle \)-semilattice defined by generators \( a, b, \) and \( c_i \), for \( i \in \Lambda \), and relations \( c_i \leq a \vee b \) and \( c_i \leq c_j \), for all \( i \leq j \) in \( \Lambda \). Hence the elements of \( S(\Lambda) \) either belong to \( S(\varnothing) = \{0, a, b, a \vee b\} \) or have the form \( c_i, a \vee c_i, \) or \( b \vee c_i \), for some \( i \in \Lambda \). We shall identify \( S(X) \) with the \( \langle \vee, 0 \rangle \)-subsemilattice of \( S(\Lambda) \) generated by \( S(\varnothing) \cup \{c_i \mid i \in X\} \), for any \( X \subset \Lambda \).

For chains \( X \) and \( Y \), any isotone map \( f : X \to Y \) gives raise to a unique \( \langle \vee, 0 \rangle \)-homomorphism \( S(f) : S(X) \to S(Y) \) fixing \( a \) and \( b \) and sending \( c_i \) to \( c_{f(i)} \), for all \( i \in X \). Of course, the assignment \( \Lambda \mapsto S(\Lambda) \), \( f \mapsto S(f) \) is a functor.

We denote by \( \mathcal{F} = \mathcal{D} \circ S \) the composition of the two functors \( \mathcal{D} \) and \( S \).

The proof of the following lemma is straightforward.

**Lemma 4.1.** Let \( \Lambda \) be a chain and let \( \langle X_i \mid i \in I \rangle \) be a family of subsets of \( \Lambda \). The following statements hold:

1. \( S \left( \bigcap_{i \in I} X_i \right) = \bigcap_{i \in I} S(X_i) \).
2. If \( I \) is a nonempty upward directed poset and \( \langle X_i \mid i \in I \rangle \) is isotone, then \( S \left( \bigcup_{i \in I} X_i \right) = \bigcup_{i \in I} S(X_i) \).

As an easy consequence of Lemmas 3.2 and 4.1, we get the following.

**Lemma 4.2.** Let \( \Lambda \) be a chain. Then for any \( x \in \mathcal{F}(\Lambda) \), there exists a least (with respect to the inclusion) subset \( X \) of \( \Lambda \) such that \( x \in \mathcal{F}(X) \); this subset is finite.

We denote by \( \text{supp}(x) \) the subset given by Lemma 4.2 and we call it the *support* of \( x \).

**Notation 4.3.** For a chain \( \Lambda \), well-ordered subsets \( X \) and \( Y \) of \( \Lambda \) such that \( \text{otp} X \leq \text{otp} Y \), and \( x \in \mathcal{F}(X) \), we set \( x[Y/X] = \mathcal{F}(e_{X,Y})(x) \), where \( e_{X,Y} \) denotes the unique embedding from \( X \) into \( Y \) whose range is a lower subset of \( Y \).

Hence \( x[Y/X] \) belongs to \( \mathcal{F}(Y) \), for all \( x \in \mathcal{F}(X) \).

**Lemma 4.4.** Let \( \Lambda \) be a chain and let \( X, Y \) be well-ordered subsets of \( \Lambda \) such that \( \text{otp} X \leq \text{otp} Y \) and \( X \cap Y \) is a lower subset of both \( X \) and \( Y \). Then \( x[Y/X] = x \), for all \( x \in \mathcal{F}(X \cap Y) \).

**Proof.** As the set \( Z = X \cap Y \) is a lower subset of both \( X \) and \( Y \), the homomorphism \( \mathcal{F}(e_{Z,X}) \) (resp., \( \mathcal{F}(e_{Z,Y}) \)) is the inclusion map from \( \mathcal{F}(Z) \) into \( \mathcal{F}(X) \) (resp., \( \mathcal{F}(Y) \)). In particular, \( x = \mathcal{F}(e_{Z,X})(x) = \mathcal{F}(e_{Z,Y})(x) \). Therefore,

\[
x[Y/X] = \mathcal{F}(e_{X,Y})(x) = \mathcal{F}(e_{X,Y}) \circ \mathcal{F}(e_{Z,X})(x) = \mathcal{F}(e_{Z,Y})(x) = x.
\]

We are now reaching a crucial lemma.

**Lemma 4.5** (Interpolation Lemma). Let \( \Lambda \) be a chain, let \( X, Y \) be finite subsets of \( \Lambda \), and let \( \langle x, y \rangle \in \mathcal{F}(X) \times \mathcal{F}(Y) \). If \( x \leq y \), then either there exists \( z \in \mathcal{F}(X \cap Y) \) such that \( x \leq z \leq y \) or \( (y \not\subset X \text{ and } x = c_{\text{min}(Y \setminus X)} \leq y) \).

**Proof.** We shall denote by \( \pi_k^\sharp \) the canonical map from \( \mathcal{R}^k S(\Lambda) \) onto \( \mathcal{R}^k S(\Lambda) \), for all natural numbers \( k \leq l \). Put \( m = \text{rk} x \) and \( n = \text{rk} y \). Observe that \( \text{supp}(x) \subset X \) and \( \text{supp}(y) \subset Y \). We argue by induction on \( \|x\| + \|y\| \). If either \( \text{supp}(x) \subset Y \) or \( \text{supp}(y) \subset X \) then either \( z = x \) or \( z = y \) belongs to \( \mathcal{F}(X \cap Y) \) and satisfies the inequalities \( x \leq z \leq y \), so we are done. So suppose that \( \text{supp}(x) \not\subset Y \) and \( \text{supp}(y) \not\subset X \). In particular, \( X \not\subset Y \) and \( Y \not\subset X \), and both \( \text{supp}(x) \) and \( \text{supp}(y) \) are nonempty. We put \( \xi = \text{min}(Y \setminus X) \).
Suppose that \( m = n = 0 \), that is, \( x, y \in \mathcal{S}(\Lambda) \). Pick \( i \in \text{supp}(x) \). As \( c_i \leq x \leq y \), we obtain that either \( y = a \lor b \) (a contradiction, as then \( \text{supp}(y) = \emptyset \)) or \( y \in \{ c_i, a \lor c_j, b \lor c_j \} \) for some \( j \geq i \). If \( i = j \), then \( \text{supp}(x) = \text{supp}(y) = \{ i \} \), a contradiction. If \( i < j \), then \( \xi = j \) and so \( c_\xi \leq y \).

Suppose now that \( m < n \). Then \( x \leq y \) means that \( x \leq \pi_m^n(y) \) (use \([8.2]\)). As \( \pi_m^n(y) \) has support contained in \( Y \) and rank at most \( m \), it follows from the induction hypothesis that either \( c_\xi \leq \pi_m^n(y) \) (thus, a fortiori, \( c_\xi \leq y \)) or there exists \( z \in \mathcal{F}(X \cap Y) \) such that \( x \leq z \leq \pi_m^n(y) \) (thus, a fortiori, \( z \leq y \)).

So suppose from now on that \( m > 0 \) (i.e., \( x \not\in \mathcal{S}(\Lambda) \)) and \( m \geq n \). If \( x = \bigvee_{i < k} x_i \) where \( k \geq 2 \) and each \( x_i \) has support contained in \( X \) and complexity less than \( \|x\| \), then we apply the induction hypothesis to each inequality \( x_i \leq y \), for \( i < k \). If \( c_\xi \not\leq y \), then for all \( i < k \), there exists \( \xi_i \in \mathcal{F}(X \cap Y) \) such that \( x_i \leq \xi_i \leq y \). Hence \( x \leq z \leq y \), where \( z = \bigvee_{i < k} x_i \) belongs to \( \mathcal{F}(X \cap Y) \). This reduces the problem to the case where \( x = \bigoplus(u, v, w) \), where \( (u, v, w) \) is a non-diagonal triple of elements of \( \mathcal{R}^{m-1}S(X) \) of complexity less than \( \|x\| \).

If \( m > n \), then, as \( \bigoplus(u, v, w) = x \leq y \) with \( \text{supp}(x) \not\subseteq Y \), \( u, v, w \in \mathcal{R}^{m-1}S(X) \), and \( y \in \mathcal{R}^{m-1}S(Y) \), it follows from \([8.1]\) that either \( u \leq y \) or \( w \leq y \). If, for example, \( u \leq y \), then, by the induction hypothesis, either \( c_\xi \leq y \) (in which case we are done) or there exists \( z \in \mathcal{F}(X \cap Y) \) such that \( u \leq z \leq y \). In the second case, \( x \leq z \leq y \). The argument is similar in case \( w \leq y \).

The remaining case is \( m = n > 0 \). As \( \bigoplus(u, v, w) = x \leq y \) with \( \text{supp}(x) \not\subseteq Y \), \( u, v, w \in \mathcal{R}^{m-1}S(X) \), and \( y \in \mathcal{R}^mS(Y) \), it follows from \([8.1]\) that either \( u \leq \pi_m^{m-1}(y) \) or \( w \leq \pi_m^{m-1}(y) \). If \( u \leq \pi_m^{m-1}(y) \), then, by the induction hypothesis, either \( c_\xi \leq \pi_m^{m-1}(y) \) (thus, a fortiori, \( c_\xi \leq y \)) or there exists \( z \in \mathcal{F}(X \cap Y) \) such that \( u \leq z \leq \pi_m^{m-1}(y) \) (in which case \( x \leq z \leq y \)). The case where \( w \leq \pi_m^{m-1}(y) \) is similar.

\[\square\]

**Lemma 4.6.** Let \( \Lambda \) be a chain and let \( X \) be a nonempty subset of \( \Lambda \) admitting a supremum, say, \( \xi \), in \( \Lambda \). Then \( c_\xi \) is the supremum of \( \{ c_i \mid i \in X \} \) in \( \mathcal{F}(\Lambda) \).

**Proof.** Let \( x \in \mathcal{F}(\Lambda) \) such that \( c_i \leq x \) for all \( i \in X \), we prove that \( c_\xi \leq x \). Put \( n = \text{rk } x \) and \( y = \pi^n_0(x) \). Let \( i \in X \). From \( c_i \leq x \) and \( c_i \in \mathcal{S}(\Lambda) \) it follows that \( c_i \leq y \). This holds for all \( i \in X \), hence, as \( c_\xi \) is clearly the supremum of \( \{ c_i \mid i \in X \} \) in \( \mathcal{S}(\Lambda) \), we obtain that \( c_\xi \leq y \). Therefore, \( c_\xi \leq x \). \[\square\]

Now we can state the main technical result of the paper. It says that \( \{ c_\xi \mid \xi < \omega_1 \} \) is the least non-eventually constant isotope \( \omega_1 \)-sequence in \( \mathcal{F}(\omega_1) \) modulo the closed unbounded filter on \( \omega_1 \).

**Theorem 4.7.** Let \( \sigma = \langle x_\xi \mid \xi < \omega_1 \rangle \) be an isotope \( \omega_1 \)-sequence of elements of \( \mathcal{F}(\omega_1) \). Then either \( \sigma \) is eventually constant or there exists a closed unbounded subset \( C \) of \( \omega_1 \) such that \( c_\xi \leq x_\xi \) for all \( \xi \in C \).

**Proof.** Assume that \( \sigma \) is not eventually constant. We put \( X_\xi = \text{supp}(x_\xi) \) and \( n_\xi = |X_\xi| \), for all \( \xi < \omega_1 \). So \( x_\xi = x_\xi'[X_\xi/n_\xi] \), for some \( x_\xi' \in \mathcal{F}(n_\xi) \). As all sets \( X_\xi \) are finite, it follows from the \( \Delta \)-Lemma (see \[1\] Lemma 22.6)) that there are an uncountable subset \( I \) of \( \omega_1 \) and a finite subset \( X \) of \( \omega_1 \) such that \( X_\xi \cap X_\eta = X \) for all distinct \( \xi, \eta \in I \). We may further assume without loss of generality that there are \( n < \omega_1 \) and \( x \in \mathcal{F}(n) \) such that \( n_\xi = n \) and \( x_\xi' = x \), for all \( \xi \in I \). Hence \( x_\xi = x[x_\xi/n_\xi] \), for all \( \xi \in I \). As \( \sigma \) is isotope but not eventually constant, it follows
that $X$ is a proper subset of $X_\xi$, for all $\xi \in I$. Put $Y_\xi = X_\xi \setminus X$. Define $\rho(\xi)$ as the least element of $Y_\xi$.

For subsets $U$ and $V$ of $\omega_1$, let $U < V$ hold, if $u < v$ for all $(u, v) \in U \times V$. By further shrinking $I$, we might assume that $X < Y_\xi$, for all $\xi \in I$. In particular, observe that $X = X_\xi \cap X_\eta$ is a lower subset of both $X_\xi$ and $X_\eta$, for all $\xi \neq \eta$ in $I$.

Let $\xi < \eta$ in $I$ and suppose that there exists $z \in \mathcal{F}(X)$ such that $x_\xi \leq z \leq x_\eta$. Applying the embedding $\mathcal{F}(e_{X_\xi, X_\eta})$ to the inequality $x[X_\xi/n] \leq z$ and using Lemma 4.4, we obtain the inequality $x[X_\eta/n] \leq z$, so $x_\eta = x[X_\eta/n] = z$, a contradiction since the left hand side has support $X_\eta$ while the right hand side has the smaller support $X$. Therefore, as $x_\xi \leq x_\eta$ and by Lemma 4.4, we obtain the inequality $c_{\rho(\eta)} \leq x_\eta$.

Hence, we may assume that $c_{\rho(\xi)} \leq x_\xi$ for all $\xi \in I$. It follows that

$$c_{\rho(\xi)} \leq x_\xi, \quad \text{for all } \xi < \omega_1, \quad (4.1)$$

where we put

$$\overline{\rho}(\xi) = \bigvee \{\rho(\eta) \mid \eta \in I, \eta < \xi\}, \quad \text{for all } \xi < \omega_1.$$

As the range of $\rho$ is unbounded, so is the range of $\overline{\rho}$. Hence, as $\overline{\rho}$ is a complete join-homomorphism from $\omega_1$ to $\omega_1$, the set $C = \{\xi < \omega_1 \mid \overline{\rho}(\xi) = \xi\}$ is a closed unbounded subset of $\omega_1$. It follows from (4.1) that the inequality $c_\xi \leq x_\xi$ holds for all $\xi \in C$.

The following corollary expresses that $\mathcal{F}(\omega_1)$ fails a certain “monotone refinement property”.

**Corollary 4.8.** There are no positive integer $n$ and no finite collection of isotone $\omega_1$-sequences $(x_{i,\xi} \mid \xi < \omega_1)$ of elements of $\mathcal{F}(\omega_1)$, for $0 \leq i \leq n$, such that

1. $x_{0,\xi} = 0$ and $x_{n,\xi} = c_\xi$, for all large enough $\xi < \omega_1$.
2. $x_{i,\xi} \leq c_\xi$, for all $i \leq n$ and all large enough $\xi < \omega_1$.
3. Either $x_{i+1,\xi} \leq a \lor x_{i,\xi}$ or $x_{i+1,\xi} \leq b \lor x_{i,\xi}$, for all $i < n$ and all $\xi < \omega_1$.

**Proof.** We prove that for all $i \leq n$, there exists $\eta_i < \omega_1$ such that $x_{i,\xi} \leq c_{\eta_i}$ for all $\xi < \omega_1$. We argue by induction on $i$. For $i = 0$ it holds by assumption, with $\eta_0 = 0$. Suppose that $x_{i,\xi} \leq c_{\eta_i}$, for all $\xi < \omega_1$. Let $\xi > \eta_i$. Assume, for example, that $x_{i+1,\xi} \leq a \lor x_{i,\xi}$; so $x_{i+1,\xi} \leq a \lor c_{\eta_i}$. Observing that $c_\xi \not\leq a \lor c_{\eta_i}$, we get that $c_\xi \not\leq x_{i+1,\xi}$. As this holds for all $\xi > \eta_i$ and by Theorem 4.4, we obtain that $(x_{i+1,\xi} \mid \xi < \omega_1)$ is eventually constant, and hence, by (2), below some $c_{\eta_{i+1}}$, therefore completing the induction step.

In particular, for $i = n$, we obtain that the $\omega_1$-sequence $(c_\xi \mid \xi < \omega_1)$ is eventually dominated by the constant $c_{\eta_n}$, a contradiction. \hfill $\Box$

Hence we get a negative extension property for posets.

**Corollary 4.9.** There are a poset measure $\mu : (\omega_1 + 1) \times (\omega_1 + 1) \to \mathcal{F}(\omega_1)$ such that $\mu(\omega_1, 0) = a \lor b$ but there are no poset $P$ containing $\omega_1 + 1$, no poset measure $\overline{\mu} : P \times P \to \mathcal{F}(\omega_1)$ extending $\mu$, no positive integer $n$, and no decomposition $0 = z_0 \leq z_1 \leq \cdots \leq z_n = \omega_1$ in $P$ such that either $\overline{\mu}(z_{i+1}, z_i) \leq a$ or $\overline{\mu}(z_{i+1}, z_i) \leq b$ for all $i < n$. 
Proof. Define \( \mu: (\omega_1 + 1) \times (\omega_1 + 1) \to \mathcal{F}(\omega_1) \) by

\[
\mu(\xi, \eta) = \begin{cases} 
0, & \text{if } \xi \leq \eta, \\
\xi, & \text{if } \eta < \xi < \omega_1, \\
\eta, & \text{for all } \xi, \eta \leq \omega_1.
\end{cases}
\]

It is straightforward to verify that \( \mu \) is a poset measure on \( \omega_1 + 1 \). Suppose that \( P, \mathbf{P}, n, z_0, \ldots, z_n \) satisfy the given conditions. We put \( x_{i,\xi} = \mathbf{P}(\xi, z_{n-i}) \), for all \( \xi < \omega_1 \). It is not hard, using the triangular inequality, to verify that the elements \( x_{i,\xi} \) satisfy the assumptions (1)-(3) of Corollary 4.9, a contradiction. \( \square \)

As the following result shows, more “amenable” semilattices do satisfy a certain “monotone refinement property”.

**Proposition 4.10.** Let \( S \) be a distributive \( (\lor, 0) \)-semilattice. If \( S \) is either a lattice, or strongly distributive, or countable, then for all \( a, b \in S \), every chain \( \Lambda \), and every isotope \( \Lambda \)-sequence \( (c_i \mid i \in \Lambda) \) of elements of \( S \) such that \( c_i \leq a \lor b \) for all \( i \in \Lambda \), then there are isotope \( \Lambda \)-sequences \( (a_i \mid i \in \Lambda) \) and \( (b_i \mid i \in \Lambda) \) of elements of \( S \) such that \( a_i \leq a, b_i \leq b, \) and \( c_i = a_i \lor b_i, \) for all \( i \in \Lambda \).

Proof. If \( S \) is a lattice the conclusion is trivial: put \( a_i = a \land c_i \) and \( b_i = b \land c_i \), for all \( i \in \Lambda \).

Now assume that \( S \) is strongly distributive. Denote by \( C_i \) the (finite) set of all maximal join-irreducible elements of \( S \) below \( c_i \), for all \( i \in \Lambda \). Observe that \( C_i \subseteq \downarrow C_j \), for all \( i \leq j \) in \( \Lambda \). For every finite subset \( I \) of \( \Lambda \), denote by \( X_I \) the set of all families \( \{ (a_i, B_i) \mid i \in I \} \) such that

1. \( A_i \subseteq \downarrow a, B_i \subseteq \downarrow b, \) and \( A_i \cup B_i = C_i, \) for all \( i \in I \).
2. For all \( i \leq j \) in \( I \), \( A_i \subseteq \downarrow A_j \) and \( B_i \subseteq \downarrow B_j \).

We claim that \( X_I \) is nonempty, for every finite subset \( I \) of \( \Lambda \). We argue by induction on \( |I| \). The conclusion is obvious for \( I = \emptyset \). For \( I = \{i\} \), put \( A_i = \downarrow a \land C_i \) and \( B_i = \downarrow b \land C_i \). Now suppose that \( I = \{i\} \cup J \), where \( i < j \) for all \( j \in J \) and \( J \) is nonempty. By induction hypothesis, there exists an element \( \{ (A_k, B_k) \mid k \in J \} \) in \( X_J \). Put \( j = \text{min} J, A_i = \downarrow A_j \land C_i, \) and \( B_i = \downarrow B_j \land C_i \). It is straightforward to verify that \( \{ (A_k, B_k) \mid k \in I \} \) belongs to \( X_I \). This completes the induction step.

It follows that the set \( \Omega_I \) of all families \( \{ (A_i, B_i) \mid i \in \Lambda \} \) of elements of the Cartesian product \( \Omega = \prod (\mathfrak{P}(C_i) \times \mathfrak{P}(C_i)) \mid i \in \Lambda \) (where \( \mathfrak{P}(X) \) denotes the powerset of a set \( X \)) whose restriction to \( I \) belongs to \( X_I \) is nonempty, for every finite subset \( I \) of \( \Lambda \). Endow \( \Omega \) with the product topology of the discrete topologies on all (finite) sets \( \mathfrak{P}(C_i) \times \mathfrak{P}(C_i) \). By Tychonoff’s Theorem, \( \Omega \) is compact. Hence the intersection of all \( \Omega_I \), for \( I \) a finite subset of \( \Lambda \), is nonempty. Let \( \{ (A_i, B_i) \mid i \in \Lambda \} \) be an element of that intersection. Then the collection of all elements \( a_i = \lor A_i \) and \( b_i = \lor B_i \), for \( i \in \Lambda \), satisfies the required conditions.

Assume, finally, that \( S \) is countable. Define an equivalence relation \( \equiv \) on \( \Lambda \) by \( i \equiv j \) iff \( c_i = c_j \), for all \( i, j \in \Lambda \), and denote by \( [i] \) the \( \equiv \)-equivalence class of \( i \), for any \( i \in \Lambda \). Putting \( c_{[i]} = c_i \), makes it possible to replace \( \Lambda \) by \( \Lambda/\equiv \). In particular, as \( S \) is countable, \( \Lambda \) becomes countable as well. Now write \( \Lambda = \bigcup (\Lambda_n \mid n < \omega), \) where \( (\Lambda_n \mid n < \omega) \) is an increasing sequence of finite subsets of \( \Lambda \) with \( |\Lambda_n| = n \), for all \( n < \omega \). Denote by \( Y_n \) the set of all families \( \{ (a_l, b_l) \mid l \in \Lambda_n \} \) such that \( a_i \leq a, b_i \leq b, c_i = a_i \lor b_i \), and \( i \leq j \) implies that \( a_i \leq a_j \) and \( b_i \leq b_j \), for all \( i \leq j \) in \( \Lambda_n \).

Suppose that we are given an element of \( Y_n \) as above, and denote by \( k \) the unique
element of $Y_{n+1} \setminus Y_n$. Suppose, for example, that $\min \Lambda_n < k < \max \Lambda_n$, and denote by $i$ (resp., $j$) the largest (resp., least) element of $\Lambda_n$ below $k$ (resp., above $k$). As $c_k \leq c_j = a_j \lor b_j$, there are $a' \leq a_j$ and $b' \leq b_j$ such that $c_k = a' \lor b'$. Put $a_k = a_i \lor a'$ and $b_k = b_i \lor b'$. Then $\langle \langle a_l, b_l \rangle \mid l \in \Lambda_{n+1} \rangle$ is an element of $Y_{n+1}$. So every element of $Y_n$ extends to an element of $Y_{n+1}$. The proof is even easier in case either $k < \min \Lambda_n$ or $k > \max \Lambda_n$. Hence we have constructed inductively a family $\langle \langle a_i, b_i \rangle \mid i \in \Lambda \rangle$ whose restriction to $\Lambda_n$ belongs to $Y_n$, for all $n < \omega$. Therefore, the elements $a_i$ and $b_i$, for $i \in \Lambda$, are as required.

The “monotone refinement property” described above fails in $\mathcal{F}(\omega_1)$. Indeed, consider the isotone $\omega_1$-sequence $\langle c_\xi \mid \xi < \omega_1 \rangle$ together with the inequalities $c_\xi \leq a \lor b$, for $\xi < \omega_1$. Suppose that $\langle a_\xi \mid \xi < \omega_1 \rangle$ and $\langle b_\xi \mid \xi < \omega_1 \rangle$ are isotone $\omega_1$-sequences in $\mathcal{F}(\omega_1)$ such that $c_\xi = a_\xi \lor b_\xi$ while $a_\xi \leq a$ and $b_\xi \leq b$, for all $\xi < \omega_1$. Set $x_{0,\xi} = 0$, $x_{1,\xi} = a_\xi$, and $x_{2,\xi} = c_\xi$, for all $\xi < \omega_1$. Then the isotone $\omega_1$-sequences $\langle x_{i,\xi} \mid \xi < \omega_1 \rangle$, for $i \in \{0, 1, 2\}$, satisfy (1)–(3) of Corollary 4.8 (with $n = 2$), a contradiction.

Nevertheless we do not know whether the monotone refinement property of $S$ either implies or is implied by the statement that every $S$-valued poset measure extends to a $V$-measure.

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