Abstract Harmonic Analysis on Spacetime.

Kahar El-Hussein
Department of Mathematics, Faculty of Science,
Al Furat University, Dear El Zore, Syria and
Department of Mathematics, Faculty of Arts Science Al Quryyat,
Al-Jouf University, KSA
E-mail kumath@hotmail.com

April 8, 2014

Abstract

Let $G = SL(2, \mathbb{C})$ be the $2 \times 2$ connected complex Lie group and let $P = \mathbb{R}^4 \rtimes SL(2, \mathbb{C})$ be the Poincare group (space time). In mathematics, the Poincaré group (spacetime), named after Henri Poincaré, is the group of isometries of Minkowski spacetime, introduced by Hermann Minkowski. It is a non-abelian Lie group with 10 generators. Space-time, in physical science, single concept that recognizes the union of space and time, posited by Albert Einstein in the theories of relativity (1905, 1916). One of the interesting problems for Mathematicians and Physicists is. Can we do the Fourier analysis on $P$. The purpose of this paper is to define the Fourier transform in order to obtain the Plancherel formula for $G$, and then we establish the Plancherel theorem for Spacetime(Poincare group).

Keywords: Semidirect Product of Two Lie Groups, Spacetime (Poincare Group), Fourier Transform, Plancherel Theorem

AMS 2000 Subject Classification: 43A30&35D 05
1 Introduction.

1. Abstract harmonic analysis is the field in which results from Fourier analysis are extended to topological groups which are not commutative that has connections to, theoretical physics, chemistry analysis, algebra, geometry, and the theory of algorithms. The classical Fourier transform is one of the most widely used mathematical tools in engineering. However, few engineers know that extensions Fourier analysis on noncommutative Lie groups holds great potential for solving problems in robotics, image analysis, mechanics. Engineering applications of noncommutative harmonic analysis brings this powerful tool to the engineering world. The Fourier transform, known in classical analysis, and generalized in abstract harmonic analysis. For a long time, people have tried to construct objects in order to generalize Fourier transform and Pontryagin’s theorem to the non abelian case. However, with the dual object not being a group, it is not possible to define the Fourier transform and the inverse Fourier transform between $G$ and $\hat{G}$. These difficulties of Fourier analysis on noncommutative groups makes the noncommutative version of the problem very challenging. It was necessary to find a subgroup or at least a subset of locally compact groups which were not ”pathological”, or ”wild” as Kirillov calls them [13]. Unfortunately if the group $G$ is no longer assumed to be abelian, it is not possible anymore to consider the dual group (i.e the set of all equivalence classes of unitary irreducible representations). Abstract harmonic analysis on locally compact groups is generally a difficult task. Still now neither the theory of quantum groups nor the representations theory have done to reach this goal. Recently, these problems found a satisfactory solution with the papers [5, 6, 7, 8, 9]. The ways were introduced in these papers will be the business of the expertise in the theory of abstract harmonic analysis, and in theoretical physics. In this paper I will define the Fourier transform on the complex semisimple Lie group $SL(2, \mathbb{C})$, and then the Poincare group(Space time) $\simeq \mathbb{R}^4 \rtimes SL(2, \mathbb{C})$.

2 Fourier Transform and Plancherel Formula on Space Time $SL(2, \mathbb{C})$.

2. The Lorentz discoveries on some invariance of Maxwell’s equations late in the 19th century which were to very become the basis of Albert Ein-
stein’s theory of special relativity. While spacetime can be viewed as a consequence of Einstein’s 1905 theory of special relativity, it was first explicitly proposed mathematically by one of his teachers, the mathematician Hermann Minkowski, in a 1908 essay building on and extending Einstein’s work. His concept of Minkowski space is the earliest treatment of space and time as two aspects of a unified whole, the essence of special relativity, cosmology, and gravitational fields. In the following we use the Iwasawa decomposition of $SL(2,\mathbb{C})$, to define the Fourier transform and to demonstrate Plancherel Formula on the complex semisimple Lie group $SL(2,\mathbb{C})$. Therefore let $G = SL(2,\mathbb{C})$ be the complex Lie group, which is

$$SL(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (a, b, c, d) \in \mathbb{C}^4 \text{ and } ad - bc = 1 \right\}$$

and let $G = KNA$ be the Iwasawa decomposition of $G$, where

$$K = K(G) = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} : (\alpha, \beta) \in \mathbb{C}^2 \text{ and } |\alpha|^2 + |\beta|^2 = 1 \right\}$$

$$N = N(G) = \left\{ \begin{pmatrix} 1 \\ n \\ 1 \end{pmatrix} : n \in \mathbb{C} \right\}$$

$$A = A(G) = \left\{ \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} : a \in \mathbb{R}_+^* \right\}$$

Hence every $g \in G$ can be written as $g = kan \in G$, where $k \in K$, $a \in A$, $n \in \mathbb{C}$. We denote by $L^1(G)$ the Banach algebra that consists of all complex valued functions on the group $G$, which are integrable with respect to the Haar measure of $G$ and multiplication is defined by convolution on $G$, and we denote by $L^2(G)$ the Hilbert space of $G$. So we have for any $f \in L^1(G)$ and $\phi \in L^1(G)$

$$\phi \ast f(X) = \int_G f(Y^{-1}X)\phi(Y) dY$$

The Haar measure $dg$ on $G$ can be calculated from the Haar measures $dn; \, da$ and $dk$ on $N; A$ and $K$; respectively, by the formula

$$\int_G f(g)dg = \int_A \int_N \int_K f(ank) dadnk$$
Keeping in mind that $a^{-2\rho}$ is the modulus of the automorphism $n \to ana^{-1}$ of $N$ we get also the following representation of $dg$

$$\int_G f(g)dg = \int_A \int_N \int_K f(ank)dadndk = \int_N \int_A \int_K f(nak)a^{-2\rho}dndadk \quad (5)$$

Furthermore, using the relation $\int_G f(g)dg = \int_G f(g^{-1})dg$, we receive

$$\int_G f(g)dg = \int_K \int_A \int_N f(kan)a^{2\rho}dndadk \quad (6)$$

where $\rho = $ the dimension of $N$. Let $k$ be the Lie algebra of $K$ and $(X_1, X_2, \ldots, X_m)$ a basis of $k$, such that the both operators

$$\Delta = \sum_{i=1}^m X_i^2 \quad (7)$$

$$D_q = \sum_{0 \leq l \leq q} \left( - \sum_{i=1}^m X_i^2 \right)^l \quad (8)$$

are left and right invariant (bi-invariant) on $K$, this basis exist see [2, p.564]. For $l \in \mathbb{N}$, let $D^l = (1 - \Delta)^l$, then the family of semi-norms $\{\sigma_l, l \in \mathbb{N}\}$ such that

$$\sigma_l(f) = \left( \int_K \left| D^l f(y) \right|^2 dy \right)^{\frac{1}{2}}, \quad f \in \mathcal{C}^\infty(K) \quad (9)$$

define on $\mathcal{C}^\infty(K)$ the same topology of the Frechet topology defined by the semi-norms $\|X^\alpha f\|_2$ defined as

$$\|X^\alpha f\|_2 = \left( \int_K \left| X^\alpha f(y) \right|^2 dy \right)^{\frac{1}{2}}, \quad f \in \mathcal{C}^\infty(K) \quad (10)$$

where $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$, see [2, p.565]

Let $\hat{K}$ be the set of all irreducible unitary representations of $K$. If $\gamma \in \hat{K}$, we denote by $E_\gamma$ the space of representation $\gamma$ and $d$, its dimension then we get

**Definition 2.1.** The Fourier transform of a function $f \in \mathcal{C}^\infty(K)$ is defined as

$$Tf = \int_K f(x)\gamma(x^{-1})dx \quad (11)$$
where $T$ is the Fourier transform on $K$

**Theorem (A. Cerezo) 2.1.** Let $f \in C^\infty(K)$, then we have the inversion of the Fourier transform

$$f(x) = \sum_{\gamma \in \hat{K}} d_{\gamma} tr[Tf(\gamma)] \gamma(x)$$  \hspace{1cm} (12)

$$f(I_K) = \sum_{\gamma \in SO(3)} d_{\gamma} tr[Tf(\gamma)]$$  \hspace{1cm} (13)

and the Plancherel formula

$$\|f(x)\|_2^2 = \int |f(x)|^2 dx = \sum_{\gamma \in \hat{K}} \|Tf(\gamma)\|^2$$  \hspace{1cm} (14)

for any $f \in L^1(K)$, where $I_K$ is the identity element of $K$

**Definition 2.2.** For any function $f \in \mathcal{D}(G)$, we can define a function $\Upsilon(f)$ on $G \times K$ by

$$\Upsilon(f)(g, k_1) = \Upsilon(f)(kna, k_1) = f(gk_1 = f(knak_1)$$  \hspace{1cm} (15)

for $g = kna \in G$, $h \in K$, and $k_1 \in K$. The restriction of $\Upsilon(f) \ast \psi(g, k_1)$ on $K(G)$ is $\Upsilon(f) \ast \psi(g, k_1) \downarrow_{K(G)} = f(nak_1) \in \mathcal{D}(G)$, and $\Upsilon(f)(g, k_1) \downarrow_K = f(g, I_K) = f(kna) \in \mathcal{D}(G)$

**Definition 2.3.** Let $f$ and $\psi$ be two functions belong to $\mathcal{D}(G)$, then we can define the convolution of $\Upsilon(f)$ and $\psi$ on $G \times K$ as

$$\Upsilon(f) \ast \psi(g, k_1) = \int_G \Upsilon(f)(gg_2^{-1}, k_1)\psi(g_2)dg_2$$

$$= \int_K \int_N \int_A \Upsilon(f)(kna_2^{-1}n_2^{-1}k_1^{-1}k_1)\psi(k_2n_2a_2)dk_2dn_2da_2$$

and so we get

$$\Upsilon(f) \ast \psi(g, k_1) \downarrow_{K(G)} = \Upsilon(f) \ast \psi(I_Kna, k_1)$$

$$= \int_K \int_N \int_A f(naa_2^{-1}n_2^{-1}k_1^{-1}k_1)\psi(k_2n_2a_2)dk_2dn_2da_2$$

$$= \Upsilon(f) \ast \psi(na, k_1)$$
Definition 2.4. If \( f \in \mathcal{D}(G) \) and let \( \Upsilon(f) \) be the associated function to \( f \), we define the Fourier transform of \( \Upsilon(f)(g, k) \) by

\[
T\mathcal{F}\Upsilon(f)(I_K, \xi, \lambda, \gamma) = T\mathcal{F}\Upsilon(f)(I_K, \xi, \lambda, \gamma)
\]

\[
= \int_K \int_N \int_A \sum_{\delta \in \hat{K}} d\delta \int_{R^2} \Upsilon(f)(kna, k_1)\delta(k^{-1})dk |a^{-i\lambda}e^{-i(\xi, n)}| \gamma(k_1^{-1})d\delta dk d\lambda d\xi
\]

\[
= \int_N \int_A \int_K f(I_Kna, k_1) a^{-i\lambda}e^{-i(\xi, n)} \gamma(k_1^{-1})d\delta dk d\lambda d\xi
\]

\[
= \int_N \int_A \int_K f(na, k_1) a^{-i\lambda}e^{-i(\xi, n)} \gamma(k_1^{-1})d\delta dk d\lambda d\xi
\]

\[
= \int_N \int_A \int_K f(a^{-1}n^{-1}k^{-1}) a^{-i\lambda}e^{-i(\xi, n)} \gamma(k_1^{-1})d\delta dk d\lambda d\xi
\]  

(16)

where \( F \) is the Fourier transform on \( AN \) and \( T \) is the Fourier transform on \( K \), and \( I_K \) is the identity element of \( K \).

Theorem 2.2. (Plancherel's Formula for the Group \( G \)). For any function \( f \in L^1(G) \cap L^2(G) \), we get

\[
\int |f(g)|^2 dg = \int_K \int_N \int_A |f(kna)|^2 d\delta d\lambda d\xi
\]

\[
= \sum_{\delta \in \hat{K}} \int_{R} \int_{R} \|T\mathcal{F}f(\lambda, \xi, \gamma)\|^2 d\lambda d\xi
\]

\[
= \int_{R} \int_{R} \sum_{\delta \in \hat{K}} \|T\mathcal{F}f(\lambda, \xi, \gamma)\|^2 d\lambda d\xi
\]  

(17)

\[
f(I_AI_NI_K) = \int_{R^2} \int_{R^2} \int_{R^2} \sum_{\gamma \in \hat{K}} d\gamma \int_{R} \int_{R} \|T\mathcal{F}f(\lambda, \xi, \gamma)\| d\lambda d\xi
\]

\[
= \sum_{\gamma \in \hat{K}} \int_{R^2} \int_{R} \int_{R} d\gamma \int_{R} \|T\mathcal{F}f(\lambda, \xi, \gamma)\| d\lambda d\xi = \int_{R^2} \int_{R^2} \sum_{\gamma \in \hat{K}} d\gamma \int_{R} \|T\mathcal{F}f(\lambda, \xi, \gamma)\| d\lambda d\xi
\]

where \( I_A, I_N, \) and \( I_K \) are the identity elements of \( A, N \) and \( K \) respectively, where \( F \) is the Fourier transform on \( AN \) and \( T \) is the Fourier transform on \( K \), and \( I_K \) is the identity element of \( K \).

Proof: First let \( f^\vee \) be the function defined by

\[
f^\vee(kna) = f((kna)^{-1}) = f(a^{-1}n^{-1}k^{-1})
\]  

(19)
Then we have

$$\int |f(g)|^2 \, dg$$

$$= \mp(f) \ast f(I_K I_N I_A, I_{K_1})$$

$$= \int_G \Upsilon(f)(I_K I_N I_A(g_2^{-1}), I_{K_1}) \check{f}(g_2) \, dg_2$$

$$= \int \int \int \Upsilon(f)(a_2^{-1} n_2^{-1} k_2^{-1}, I_K) \check{f}(k_2 n_2 a_2) \, da_2 \, dn_2 \, dk_2$$

$$= \int \int \int f(a_2^{-1} n_2^{-1} k_2^{-1}) f((k_2 n_2 a_2)^{-1}) \, da_2 \, dn_2 \, dk_2$$

$$= \int \int \int |f(a_2 n_2 k_2)|^2 \, da_2 \, dn_2 \, dk_2$$

(20)
Secondly

\[ \Upsilon(f) \ast f(I_K I_N I_A, I_{K_1}) \]
\[ = \int \int R^2 R \mathcal{F}(\Upsilon(f) \ast f)(I_K, \lambda, \xi, \xi, I_{K_1}) d\lambda d\xi \]
\[ = \int \int \sum_{\gamma \in \hat{R}} d_\gamma \sum_{\delta \in \hat{R}} d_\delta \text{tr}[T \mathcal{F}(\Upsilon(f) \ast f)(\delta, \lambda, \xi, \gamma)] d\lambda d\xi \]
\[ = \int \int \sum_{\gamma \in \hat{R}} d_\gamma \text{tr}[\int \mathcal{F}(\Upsilon(f) \ast f)(I_K, \lambda, \xi, k_1) \gamma(k_1^{-1}) dk_1] d\lambda d\xi \]
\[ = \int \int \int \int \sum_{\gamma \in \hat{R}} d_\gamma \text{tr}[\int \mathcal{F}(\Upsilon(f) \ast f)(I_k n a, k_1) \gamma(k_1^{-1}) dk_1] \]
\[ a^{-i\lambda} e^{-i(\xi, n)} d\lambda d\xi \]
\[ = \int \int \int \int \int \sum_{\gamma \in \hat{R}} d_\gamma \text{tr}[\int \int \Upsilon(f)(I_k n a a_2^{-1} n_2^{-1} k_2^{-1}, k_1) \gamma(k_2 n_2 a_2) \gamma(k_1^{-1}) dk_1 dk_2] \]
\[ a^{-i\lambda} e^{-i(\xi, n)} d\lambda d\xi \]
\[ = \int \int \int \int \int \sum_{\gamma \in \hat{R}} d_\gamma \text{tr}[\int \int f(n a a_2^{-1} n_2^{-1} k_2^{-1} k_1) \gamma(k_2 n_2 a_2) \gamma(k_1^{-1}) dk_1 dk_2] \]
\[ a^{-i\lambda} e^{-i(\xi, n)} d\lambda d\xi \]

where

\[ e^{-i(\xi, n)} = e^{-i(\xi, n)} = e^{-i(\xi, n_2(n_1, n_2))} \] (21)

Using the fact that

\[ \int \int \int f(k n a) d a d n d k = \int \int \int f(k a n) a^{-2} d a d n d k \] (22)
and

\[
\int \int \int \int_{\mathbb{R}^2 A N K} f(kn) e^{-i\langle (\xi_1, \xi_2), (n_1, n_2) \rangle} d\alpha d\nu d\xi_1 d\xi_2
\]

\[
= \int \int \int \int_{\mathbb{R}^2 A N K} f(kn) e^{-i\langle (\xi_1, \xi_2), a^{-1}(n_1, n_2) a \rangle} a^{-2} d\alpha d\nu d\xi_1 d\xi_2
\]

\[
= \int \int \int \int_{\mathbb{R}^2 A N K} f(kn) e^{-i\langle a^{-1}(\xi_1, \xi_2) a, (n_1, n_2) \rangle} a^{-2} d\alpha d\nu d\xi_1 d\xi_2
\]

\[
= \int \int \int \int_{\mathbb{R}^2 A N K} f(kn) e^{-i\langle (\xi_1, \xi_2), (n_1, n_2) \rangle} d\alpha d\nu d\xi_1 d\xi_2
\]  \hspace{1cm} (23)

Then we get

\[
\mathcal{F}(f) * f(I_K I_N I_A, I_{K_1})
\]

\[
= \int \int \mathcal{F}(\mathcal{F}(f) * f)(I_K, \lambda, \xi, I_{K_1}) d\alpha d\xi
\]

\[
= \int \int \int \int \int \int \sum_{\gamma \in \tilde{R}} d_\gamma tr[\int \int \mathcal{F}(f)(I_k n a a_2^{-1} n_2^{-1} k_2^{-1} k_1) e^{-i\langle (\xi, n), (\gamma, (n, n)) \rangle} d\alpha d\nu dK dK]
\]

\[
= \int \int \int \int \int \int \sum_{\gamma \in \tilde{R}} d_\gamma tr[\int \int f(na a_2^{-1} n_2^{-1} k_2^{-1} k_1) f(k_2 n a_2) e^{-i\langle (\xi, n), (\gamma, (n, n)) \rangle} d\alpha d\nu dK dK]
\]

\[
= \int \int \int \int \int \int \sum_{\gamma \in \tilde{R}} d_\gamma tr[\int \int f(a a_2^{-1} n n_2^{-1} k_1) f(k_2 n a_2) e^{-i\langle (\xi, n), (\gamma, (n, n)) \rangle} d\alpha d\nu dK dK]
\]

We continue our calculation, we get
\[ \exists (f) * f(I_K I_N I_A, I_{K_1}) = \int \int \int \int \int \int \sum d_\gamma \text{tr} \left[ \int \int \frac{f(\gamma k_1)}{K} \frac{f(\gamma (k_2^{-1}) \gamma (k_1^{-1})}{K} d k_1 d k_2 \right] a^{-i \lambda} e^{-i(\xi, n)} a_2^{-i \lambda} e^{-i(\xi, n^2)} d nda_2^2 d \lambda d \xi \]

\[ = \int \int \int \int \int \int \sum d_\gamma \text{tr} \left[ \int \int \frac{f(\gamma (k_2^{-1}) \gamma (k_1^{-1})}{K} d k_1 d k_2 \right] a^{-i \lambda} e^{-i(\xi, n)} a_2^{-i \lambda} e^{-i(\xi, n^2)} d nda_2^2 d \lambda d \xi \]

\[ = \int \int \int \int \int \int \sum d_\gamma \text{tr} \left[ \int \int \frac{f((a_2^{-1} n_2^{-1} k_2^{-1}) \gamma (k_2^{-1}) \gamma (k_1^{-1})}{K} d k_1 d k_2 \right] a^{-i \lambda} e^{-i(\xi, n)} a_2^{-i \lambda} e^{-i(\xi, n^2)} d nda_2^2 d \lambda d \xi \]

\[ = \int \int \sum d_\gamma T_F f(\lambda, \xi, \gamma) T_F f(\lambda, \xi, \gamma) d \lambda d \xi \]

\[ = \int \int \sum d_\gamma \| T_F(f(\lambda, \xi, \gamma)) \|_2^2 d \lambda d \xi \]

**Remark 2.2.** By the same methods we can establish the Plancherel Formula on the Lie group \( SL(2, \mathbb{R}) \).

### 3 Fourier Transform and Plancherel Formula on Space Time \( \mathbb{R}^4 \rtimes SL(2, \mathbb{C}) \).

3. To define the Fourier transform on **spacetime** (Poincaré group), we need to introduce the Lorentz group \( O(3,1) \). Therefore denote the \( p \times p \) identity matrix by \( I_{p,p} \) and define

\[ I_{p,q} = \begin{pmatrix} I_{p,p} & 0 \\ 0 & -I_{q,q} \end{pmatrix} \]

(24)
If \( n = p + q \), the matrix \( I_{p,q} \) is associated with the non degenerate symmetric bilinear form

\[
\theta_{p,q}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sum_{i=1}^{p} x_i y_i - \sum_{j=1}^{q} x_i y_i
\]

(25)

with associated quadratic form

\[
\theta_{p,q}((x_1, \ldots, x_n)) = \sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{q} x_j^2
\]

(26)

We denote by \( O(p, q) \) the group consisting of all matrices of the form

\[
O(p, q) = \{ A \in GL(n, \mathbb{R}), \ A^t I_{p,q} A = I_{p,q} \}
\]

(27)

which is the group of isometries \( \theta_{p,q} \), where \( A^t \) is the transpose matrix of \( A \). The group \( O(p, q) \) has a sub-group consisting of all matrices with determinant +1 and will be denoted \( SO(p, q) \) that is

\[
SO(p, q) = \{ A \in O(p, q), \ \text{det} \ A = 1 \}
\]

(28)

Since our interest is the Lorentz group, then we will restrict our attention to the case for \( p = 3 \) and \( q = 1 \) consider, i.e. the group \( O(3, 1) \), which is called sometimes the full Lorentz group preserve the quadratic form

\[
\theta_{3,1}((x_1, x_2, x_3, x_4)) = \sum_{i=1}^{3} x_i^2 - x_4^2
\]

(29)

In physics, \( x_1 \) is interpreted as time and written \( t \) and \( x_2, x_3, x_4 \) as coordinates in \( \mathbb{R}^3 \) and written \( x, y, z \). Thus, the Lorentz metric is usually written a \( t^2 - x^2 - y^2 - z^2 \) although it also appears as \( x^2 + y^2 + z^2 - t^2 \). The space \( \mathbb{R}^4 \) with the Lorentz metric is called \( Minkowski space \). It plays an important role in Einstein,s theory of special relativity. The group \( O(3, 1) \) as a manifold has four connected components, we denote by \( SO_+(3, 1) \) the connected component of the identity. The Poincare group (spacetime) is given by

\[
P = \mathbb{R}^4 \ltimes SO_+(1; 3) \text{ or with double-cover}
\]

\[
\tilde{P} = \mathbb{R}^4 \ltimes SL(2; \mathbb{C}) = \mathbb{R}^4 \ltimes G
\]

(30)
The action of $SL(2; \mathbb{C})$ on $\mathbb{R}^4$ is the action of the Lorentz group on Minkowski space. We can define the action of $SL(2; \mathbb{C})$ on Minkowski spacetime by writing a point of spacetime as a two-by-two Hermitian matrix in the form
\[ M = \begin{pmatrix} t + z & x - iy \\ x + y & t - z \end{pmatrix}; \quad (x, y, t, z) \in \mathbb{R}^4 \tag{31} \]
This presentation has the pleasant feature that
\[ \det M = t^2 - x^2 - y^2 - z^2 \tag{32} \]
Therefore, we have identified the space of Hermitian matrices (which is four dimensional, as a real vector space) with Minkowski spacetime in such a way that the determinant of a Hermitian matrix $M$ is the squared length of the corresponding vector in Minkowski spacetime. $SL(2; \mathbb{C})$ acts on the space of Hermitian matrices via the group homomorphism $\rho : SL(2; \mathbb{C}) \to Aut(\mathbb{R}^4)$ defined by:
\[ \rho(g)(M) = gMg^* = M, \quad g \in G \tag{33} \]
where $Aut(\mathbb{R}^4)$ is the group of all automorphisms of $\mathbb{R}^4$, $g^*$ is the Hermitian transpose of $g$, and this action preserves the determinant. Therefore, $SL(2; \mathbb{C})$ acts on Minkowski spacetime by (linear) isometries, and so is isomorphic to a subset of the Lorentz group by the definition of the Lorentz group. Multiplication in $P$ is then given as
\[ (v, g)(v', g') = (v + \rho(g)(v'), gg') = (v + gv', gg') \tag{34} \]
for any $(v, v') \in \mathbb{R}^4 \times \mathbb{R}^4$ and $(g, g') \in G \times G$, where $gv' = \rho(g)(v')$. To define the Fourier transform on spacetime, we introduce the following new group
\[ \text{Definition 3.1.} \] Let $Q = \mathbb{R}^4 \times G \times G$ be the group with law:
\[ X \cdot Y = (v, h, g)(v', h', g') = (v + gv', hh', gg') \tag{35} \]
for all $X = (v, h, g) \in G$ and $Y = (v', h', g') \in G$
\[ \text{Definition 3.2.} \] For any function $f \in \mathcal{D}(P)$, we can define a function $\tilde{f}$ on $Q$ by
\[ \tilde{f}(v, g, h) = f(gv, gh) \tag{36} \]
Remark 3.1. The function $\tilde{f}$ is invariant in the following sense

$$\tilde{f}(q^{-1}v, g, q^{-1}h) = \tilde{f}(v, gq^{-1}, h)$$  \hfill (37)

Theorem 3.1. For any function $\psi \in \mathcal{D}(P)$ and $\tilde{f} \in \mathcal{D}(Q)$ invariant in sense (36), we get

$$\psi \ast \tilde{f}(v, h, g) = \psi \ast_c \tilde{f}(v, h, g)$$  \hfill (38)

and where $\ast$ signifies the convolution product on $P$ with respect the variable $(v, h)$, and $\ast_c$ signifies the convolution product on $A$ with respect the variable $(v, g)$

Proof: In fact we have

$$\psi \ast \tilde{f}(v, h, g)$$

$$= \int_{R^4} \int_{G} \tilde{f}((v', g')^{-1}(v, h, g))\psi(v', g')dv' dg'$$

$$= \int_{R^4} \int_{G} \tilde{f}((g'^{-1}(-v'), g'^{-1})(v, h, g))\psi(v', g')dv' dg'$$

$$= \int_{R^4} \int_{G} \tilde{f}((g'^{-1}(-v'), g'^{-1})(v, h, g))\psi(v', g')dv' dg'$$

$$= \int_{R^4} \int_{G} \tilde{f}((g'^{-1}(v - v'), h, g'^{-1}g))\psi(v', g')dv' dg'$$

$$= \int_{R^4} \int_{G} \tilde{f}(v - v', hg'^{-1}g)\psi(v', g')dv' dg'$$

$$= \psi \ast_c \tilde{f}(v, h, g)$$  \hfill (39)

Definition 3.3. Let $\Upsilon F$ be the function on $P \times K$ defined by

$$\Upsilon F(v, (g, k_1)) = F(v, gk_1)$$  \hfill (40)

and let $h(F)$ be the function on $P$ defined by

$$h(F)(v, g) = F(gv, g)$$  \hfill (41)
Definition 3.4. Let $\psi \in \mathcal{D}(P)$ and $F \in \mathcal{D}(P)$, then we can define a convolution product on the Poincare group $P$ as

$$
\psi * \Upsilon F(v, (g, k_1)) = \int_{\mathbb{R}^4} \int_{G} \Upsilon F(v - v', (gg'^{-1}, k_1))\psi(v', g')dv'dg'
$$

$$
= \int_{\mathbb{R}^4} \int_{K} \int_{N} \int_{A} F(v - v', k\eta(a)^{-1}k_1))\psi(v', k'\eta(a'))dv'dk'dn'da'
$$

where $g = k\eta(a)$ and $g' = k'\eta(a')$

Corollary 3.1. For any function $F$ belongs to $\mathcal{D}(P)$, we obtain

$$
\psi * \Upsilon h(F)(v, (g, k_1)) = \int_{\mathbb{R}^4} \int_{G} \Upsilon h(F)(v - v', (gg'^{-1}, k_1))\psi(v', g')dv'dg'
$$

$$
= \int_{\mathbb{R}^4} \int_{G} \Upsilon h(F)(v - v', (gg'^{-1}, k_1))\psi(v', g')dv'dg'
$$

$$
= \int_{\mathbb{R}^4} \int_{G} h(F)(v - v', gg'^{-1}k_1)\psi(v', g')dv'dg'
$$

$$
= \int_{\mathbb{R}^4} \int_{G} F(gg'^{-1}k_1(v - v'), gg'^{-1}k_1)\psi(v', g')dv'dg'
$$

Corollary 3.2. For any function $F$ belongs to $\mathcal{D}(P)$, we obtain

$$
F * \Upsilon h(F)(0, (I_G, I_{K_1})) = \int_{G} \int_{\mathbb{R}^4} ||f(v, g)||^2dv'dg = ||f||^2_2
$$

Proof: If $F \in \mathcal{D}(P)$, then we get

14
\[ F \ast \mathcal{H}(F)(0, (I_G, I_{K_1})) = \int \int \mathcal{H}(F)[(0 - v), (I_G g^{-1}, I_{K_1})]F(v, g)dgdv \]
\[ = \int \int h(F)[(0 - v), I_Gg^{-1}I_{K_1}]F(v, g)dgdv \]
\[ = \int \int h(F)[(-v), g^{-1}]F(v, g)dgdv \]
\[ = \int \int \tilde{F}[g^{-1}(-v), g^{-1}]F(v, g)dgdv \]
\[ = \int \int F[g^{-1}(-v), g^{-1}]^{-1}F(v, g)dgdv = \int \int F[v, g]F(v, g)dgdv \]
\[ = \int \int \|f(v, g)\|^2 dg dv = \|f\|_2^2 \]

**Definition 3.4.** Let \( f \in \mathcal{D}(P) \), we define its Fourier transform by

\[ \mathcal{F}_{\mathbb{R}^4} T \mathcal{F} f(\eta, \gamma, \xi, \lambda) = \int \int \int \int f(v, kna)e^{-i\langle \eta, v \rangle} e^{-i\gamma(k^{-1})a^{-i\lambda}e^{-i(\xi, n)}}dkd\alpha d\beta d\lambda d\xi dv \]

where \( \mathcal{F}_{\mathbb{R}^4} \) is the Fourier transform on \( \mathbb{R}^4 \), \( kna = g, \eta = (\eta_1, \eta_2, \eta_3, \eta_4) \in \mathbb{R}^4, \)
\( v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4, \) and \( dv = dv_1dv_2dv_3dv_4 \) is the Lebesgue measure on \( \mathbb{R}^4 \)

\[ \langle \eta, v \rangle = \langle (\eta_1, \eta_2, \eta_3, \eta_4), (v_1, v_2, v_3, v_4) \rangle \]
\[ = \eta_1v_1 + \eta_2v_2 + v_3\eta_3 + \eta_4v_4 \quad (44) \]

To obtain the Plancherel formula for the space-time, we refer to [8,9] **Plancherel’s Theorem for spacetime 3.2.** For any function \( f \in L^1(P) \cap L^2(P), \) we get

\[ \int |f(v, g)|^2 dv dg = \int \int \sum d_{\gamma} \| \mathcal{F}_{\mathbb{R}^4} T \mathcal{F} \mathcal{F} F(\eta, \gamma, \lambda) \|^2 d\eta d\lambda d\xi \quad (45) \]
**Proof:** Let \( \Upsilon h(F) \) be the function defined as

\[
\Upsilon h(F)(v; (g, k_1)) = h(F)(v; gk_1)
\]

\[
= F(gk_1v; gk_1) = F(gk_1v; gk_1)^{-1}
\]

(46)

then, we have

\[
F \ast \Upsilon h(F)(0, (I_G, I_{K_1}))
\]

\[
= F \ast \Upsilon h(F)(0, (I_K I_N I_A, I_{K_1}))
\]

\[
= \int \int \int \mathcal{F}_{\mathbb{R}^4} \mathcal{F}[F \ast \Upsilon h(F)]((\eta, (I_K, \xi, \lambda, I_{K_1}))d\eta d\lambda d\xi
\]

\[
= \int \int \int \sum_{\gamma \in \mathbb{K}} d_\gamma tr[\int \mathcal{F}_{\mathbb{R}^4} \mathcal{F}(F \ast \Upsilon h(F))((\eta, (I_K, \xi, \lambda, k_1))\gamma(k_1^{-1})dk_1)]d\eta d\lambda d\xi
\]

\[
= \int \int \int \int \sum_{\gamma \in \mathbb{K}} d_\gamma tr[\int \Upsilon h(F)((v - w), (I_K n a^{-1}, k_1))\gamma(k_1^{-1})dk_1]
\]

\[
= \int \int \int \int \int \int \sum_{\gamma \in \mathbb{K}} d_\gamma tr[\int \Upsilon h(F)((v - w), (n a^{-1} k_2^{-1} k_1))\gamma(k_1^{-1})dk_1]F(w, g_2 e^{-i(n, n)} a^{-i\lambda} e^{-i(\xi, n)} d\gamma d\eta d\lambda d\xi
\]

\[
= \int \int \int \int \int \int \int \sum_{\gamma \in \mathbb{K}} d_\gamma tr[\int \Upsilon h(F)((v, (a k_1))\gamma(k_1^{-1})dk_1]F(w, k_2 n a_2)\gamma(k_2^{-1})dk_2]
\]

\[
e^{-i(\eta, v + w)} a^{-i\lambda} e^{-i(\xi, n + n + 2)} d\gamma d\eta d\lambda d\xi
\]

\[
e^{-i(\eta, v + w)} a^{-i\lambda} e^{-i(\xi, n + n + 2)} d\gamma d\eta d\lambda d\xi
\]

\[
16
\]
Continuing calculation we get

\[
F \ast \mathcal{H} h(F) (0, (I_K I_N I_A, I_{K_1}))
\]

\[
= \int \int \int \int \int \int \sum d, \text{tr} \left[ \int \int (\mathcal{F}) (\text{ank}_1v, \text{ank}_1) F(w, k_2n_2a_2) \gamma(k_1^{-1}) \gamma(k_2^{-1}) dk_1 dk_2 \right]
\]

\[
\int \int \int \int \sum d, \text{tr} \left[ \int \int (\mathcal{F}) (\text{ank}_1v, \text{ank}_1)^{-1} F(w, k_2n_2a_2) \gamma(k_1^{-1}) \gamma(k_2^{-1}) dk_1 dk_2 \right]
\]

\[
\int \int \int \int \sum d, \text{tr} \left[ \int \int (\mathcal{F}) (\text{ank}_1v, \text{ank}_1) \gamma(k_1^{-1}) \gamma(k_2^{-1}) dk_1 dk_2 \right]
\]

\[
\int \int \int \int \sum d, \text{tr} \left[ \int \int (\mathcal{F}) (\text{ank}_1v, \text{ank}_1) \gamma(k_1^{-1}) \gamma(k_2^{-1}) dk_1 dk_2 \right]
\]

\[
\int \int \int \int \sum d, \text{tr} \left[ \int \int (\mathcal{F}) (\text{ank}_1v, \text{ank}_1) \gamma(k_1^{-1}) \gamma(k_2^{-1}) dk_1 dk_2 \right]
\]

\[
\int \int \int \int \sum d, \text{tr} \left[ \int \int (\mathcal{F}) (\text{ank}_1v, \text{ank}_1) \gamma(k_1^{-1}) \gamma(k_2^{-1}) dk_1 dk_2 \right]
\]

where \(0, 0, 0, 0\) is the identity element of the vector group \(\mathbb{R}^4\) and \(I_K I_N I_A\) is the identity element of the complex semisimple Lie group \(SL(2; \mathbb{C}) = KNA\).

\section{Conclusion.}

Combining these results with the results were obtained in [8, 9] help us to define the Fourier transform for the Lie groups \(SL(n, \mathbb{R})\), \(SL(n, \mathbb{C})\), \(GL_+(n, \mathbb{R})\), \(GL_-(n, \mathbb{R})\), and for the affine group \(\mathbb{R}^n \ltimes GL(n, \mathbb{R})\).
References

[1] Chirikjian, G. S., and A. Kyatkin, A., (2000), Engineering Applications in Non-commutative Harmonic Analysis, Johns Hopkins University, Baltimore, Maryland, CRC Press.

[2] A. Cerezo and F. Rouviere, ”Solution elemetaire d’un operator differentielle lineare invariant agauch sur un group de Lie reel compact” Annales Scientiques de E.N.S. 4 serie, tome 2, n°4,p 561-581, 1969.

[3] Einstein, Albert, (1926), ”Space–Time”, Encyclopedia Britannica, 13th ed.

[4] K. El- Hussein., (1989), Operateurs Differentiels Invariants sur les Groupes de Deplacements, Bull. Sc. Math. 2e series 113,. p. 89-117.

[5] K. El- Hussein., (2009), Eigendistributions for the Invariant Differential operators on the Affine Group. Int. Journal of Math. Analysis, Vol. 3, no. 9, 419-429.

[6] K. El- Hussein., (2010), Fourier transform and invariant differential operators on the solvable Lie group G4, in Int. J. Contemp. Maths Sci. 5. No. 5-8, 403-417.

[7] K. El- Hussein., (2011), On the left ideals of group algebra on the affine group, Int. Math Forum, Int, Math. Forum 6, No. 1-4, 193-202.

[8] K. El- Hussein., (2013), Non Commutative Fourier Transform on Some Lie Groups and Its Application to Harmonic Analysis, International Journal of Engineering Research & Technology (IJERT) Vol. 2 Issue 10, 2429- 2442.

[9] K. El- Hussein., (2013), Abstract Harmonic Analysis and Ideals of Banach Algebra on 3-Step Nilpotent Lie Groups. International Journal of Engineering Research & Technology (IJERT), Vol. 2 Issue 11, November

[10] Harish-Chandra; (1952), Plancherel formula for $2 \times 2$ real unimodular group, Proc. nat. Acad. Sci. U.S.A., vol. 38, pp. 337-342.

[11] Harish-Chandra; (1952), The Plancherel formula for complex semisimple Lie group, Trans. Amer. Mth. Soc., vol. 76, pp. 485- 528.
[12] S. Helgason., (2005), The Abel, Fourier and Radon Transforms on Symmetric Spaces. Indagationes Mathematicae. 16, 531-551.

[13] Kirillov, A. A., ed, (1994), Representation Theory and Noncommutative Harmonic Analysis I, Springer-Verlag, Berlin.

[14] Matolcsi., (1994). Spacetime Without Reference Frames. Budapest: Akadémiai Kiad.

[15] Petkov, Vesselin., (2010). Minkowski Spacetime: A Hundred Years Later. Springer. p. 70. ISBN 90-481-3474-9., Section 3.4, p. 70.

[16] W.Rudin., (1962), Fourier Analysis on Groups, Interscience Publishers, New York, NY.

[17] Vaen Deal., A., (2007), The Fourier transform in quantum group theory, preprint (math.RA/0609502 at http://lanl.arXiv.org).

[18] G. Warner., (1970), Harmonic Analysis on Semi-Simple Lie Groups, Springer-verlag Berlin heidelberg New york.