WEAK-TYPE \((1, 1)\) ESTIMATES FOR STRONGLY SINGULAR OPERATORS

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Abstract. Let \(\psi\) be a positive function defined near the origin such that \(\lim_{t \to 0^+} \psi(t) = 0\). We consider the operator
\[
T_{\theta} f(x) = \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} e^{i\gamma(t)} f(x-t) \frac{dt}{t^{\theta} \psi(t)^{1-\theta}},
\]
where \(\gamma\) is a real function with \(\lim_{t \to 0^+} |\gamma(t)| = \infty\) and \(0 \leq \theta \leq 1\). Assuming certain regularity and growth conditions on \(\psi\) and \(\gamma\), we show that \(T_{1,1}\) is of weak type \((1, 1)\).

1. Introduction and preliminaries

Define, for functions \(f \in C_0^\infty(\mathbb{R})\), the operator
\[
T_{\alpha,\beta} f(x) = \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} e^{\iota t} f(x-t) \frac{dt}{t^\beta},
\]
where \(\alpha > 0\) and \(\beta \geq 1\). The following theorem was proved by Hirschman in [Hir59], and by Fefferman and Stein in [FS72].

Theorem 1.1. Let \(\alpha > 0\) and \(\beta \geq 1\). Whenever \(\alpha + 2 \geq 2\beta\), the following hold.

(1) \(T_{\alpha,\beta}\) extends to a bounded operator on \(L^2(\mathbb{R})\).

(2) If \(\frac{1}{2} - \frac{1}{p} \leq \frac{1}{2} - \frac{\beta-1}{\alpha}\) then \(T_{\alpha,\beta}\) extends to a bounded operator on \(L^p(\mathbb{R})\) for \(1 < p < \infty\).

(3) If \(\frac{1}{2} - \frac{1}{p} > \frac{1}{2} - \frac{\beta-1}{\alpha}\) then \(T_{\alpha,\beta}\) is not a bounded operator on \(L^p(\mathbb{R})\).

The case \(p = 1\) was treated by Fefferman in [Fef70] where he proves the following.

Theorem 1.2. \(T_{1,1}\) is of weak type \((1, 1)\).
Cho and Yang in [CY10] considered the operators

\[ T_{\lambda,\alpha,\beta} f(x) = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 e^{i\lambda t} e^{it-\alpha} f(x-t) \frac{dt}{t^{\beta}}, \]

where \( k \geq 2 \) is an integer, and obtained estimates for the \( L^2 \) norm of \( T_{\lambda,\alpha,\beta} \) as \( \lambda \to \infty \). Namely that

\[ \| T_{\lambda,\alpha,\beta} \| \approx \lambda^{-(\alpha/2-\beta+1)/(\alpha+k)} \] when \( k \) is even, and

\[ \| T_{\lambda,\alpha,\beta} \| \approx \lambda^{-(\alpha/3-\beta+1)/(\alpha+k)} \] when \( k \) is odd.

In the present work, we are interested in proving the analogue of Theorem 1.2 when the oscillating factor is worse than a power. Say, the oscillation could be \( e^{\lambda t}/t \). Such operators were studied by the first author in [FG99]. Given \( 0 \leq \theta \leq 1 \), let

\[ (1.1) \quad T_\theta f(x) = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 e^{i\gamma(t)} f(x-t) \frac{t^\theta}{\psi(t)^{1-\sigma}} dt, \]

where the functions \( \gamma \in C^3((0,1]) \) and \( \psi \in C^2([0,1]) \) satisfy the following assumptions, for some \( t_0, s_0, C > 0 \):

(a.1) \( \gamma, \psi \) and their derivatives \( \gamma', \gamma'' \) and \( \psi' \), are all monotone. We also assume \( \gamma'(t) > 0 \), with \( \gamma' \) decreasing on \((0,1]\).

(a.2) For \( 0 < t < t_0 \),

\[ \left| \frac{\psi'(t)}{\psi(t)} \right| \leq \frac{1}{2} \frac{\gamma''''(t)}{\gamma''(t)}. \] 

(a.3) For \( s > s_0 \),

\[ |\gamma''(\gamma'^{-1}(2s))| \leq C|\gamma''(\gamma'^{-1}(s))|. \]

(a.4) There exist \( \epsilon > 0, A > 1 + \epsilon \) such that

\[ \gamma'(t) \geq A\gamma'((1+\epsilon)t), \]

for \( 0 < t < t_0 \).

(a.5) There exists \( 1/2 < \lambda < 1 \) such that

\[ |\gamma''(t)| \leq C\gamma'(t)^{2\lambda}, \]

for \( 0 < t < t_0 \).

The functions \( \gamma(t) = -t^{1-\sigma} \) and \( \psi(t) = t^{(\sigma+1)/2} \), for \( \sigma > 1 \), satisfy the previous assumptions, and correspond to the operator \( T_{\alpha,\beta} \) above with \( \alpha = \sigma - 1 \) and \( \beta = (\sigma + 1)/2 = (\alpha + 2)/2 \).

The functions \( \gamma(t) = e^{1/t} \) and \( \psi(t) = t^{3/2} e^{-1/2t} \) also satisfy assumptions (a.1)-(a.5), and in this case \( \psi \) is infinitely flat at the origin.

Remark 1.1. Assumption (a.1) implies the existence of the inverse \( \gamma'^{-1}(s) \) of \( \gamma'(t) \) for \( s \geq \gamma'(1) \), a fact that has been used in assumption (a.3).
Remark 1.2. The estimate (1.2) of assumption (a.2) implies
\[ \frac{1}{\psi(t)} \leq C' \sqrt{|\gamma''(t)|} \]
for \(0 < t < t_0\) and a constant \(C'\). This estimate, as discussed in [FG99], is necessary for the operator \(T_0\) to be bounded on \(L^2(\mathbb{R})\).

Remark 1.3. From assumption (a.4), for \(0 < t < t_0\), \(1/t < \gamma'(t)\). Hence, for such \(t\), \(t < \gamma^{-1}(1/t)\), as \(\gamma'\) is decreasing.

Remark 1.4. Assumption (a.4) also implies the estimate
\[ |\gamma''(t)| \geq C'' \frac{\gamma'(t)}{t}, \]
for a constant \(C''\) and \(0 < t < t_0\).

The previous remarks were stated and verified in [FG99], where the following theorem is proved.

**Theorem 1.3.** Suppose \(\gamma\) and \(\psi\) satisfy assumptions (a.1)-(a.5). Then
1. \(T_\theta\) is a bounded operator on \(L^p(\mathbb{R})\) for
\[ \frac{1}{p} = \frac{1 + \theta}{2} \]
and \(0 \leq \theta < 1\), and the operator norm \(\|T_\theta\|_{L^p \to L^p}\) depends only on \(\theta\).
2. \(T_1\) is a bounded operator from \(H^1(\mathbb{R})\) to \(L^1(\mathbb{R})\).

In this work we prove the following theorem.

**Theorem 1.4.** Under assumptions (a.1)-(a.5), \(T_1\) is of weak type \((1, 1)\).

Even though we have a singularity of the form \(1/t\), one cannot apply standard Calderón-Zygmund theory for the operator \(T_1\) because of the oscillating factor \(e^{i\gamma(t)}\). Chanillo and Christ in [CC87] proved weak type \((1, 1)\) estimates for operators with oscillations of the form \(e^{iP(t)}\), where \(P\) is a polynomial. Later, Folch-Gabayet and Wright [FGW12] considered oscillations of the form \(e^{iR(t)}\), where \(R\) is a rational function.

For the proof of Theorem 1.4 we consider, for \(\varepsilon > 0\) and \(\beta \in \mathbb{R}\), the kernel \(K_{\varepsilon, \beta}\) given by

\[ K_{\varepsilon, \beta}(x) = \begin{cases} e^{i\gamma(x)} x^{1+\beta} \psi(x)^{-i\beta} & \varepsilon \leq x \leq 1 \\ 0 & \text{otherwise}, \end{cases} \]

where the functions \(\gamma\) and \(\psi\) satisfy assumptions (a.1)-(a.5). In [FG99], the following properties of \(K_{\varepsilon, \beta}\) were proved.
Lemma 1.5. There exists a constant $C$, independent of $\varepsilon$ and $\beta$, such that, for $\xi \in \mathbb{R}$, $|\xi| > 1$,

$$|\hat{K}_{\varepsilon,\beta}(\xi)| \leq \frac{C(1 + |\beta|)}{\sqrt{\gamma''(\gamma^{-1}(|\xi|))\gamma^{-1}(|\xi|)}},$$

and, for $|\xi| \leq 1$,

$$|\hat{K}_{\varepsilon,\beta}(\xi)| \leq C(1 + |\beta|).$$

Lemma 1.6. There exist $C > 0, \eta > 0$ such that, for $y \in \mathbb{R}$, $|y| < \eta$,

$$(1.4) \quad \int_{|x| \geq 2\gamma^{-1}(1/|y|)} |K_{\varepsilon,\beta}(x - y) - K_{\varepsilon,\beta}(x)|dx \leq C(1 + |\beta|).$$

Lemma 1.5 follows from stationary phase methods, while Lemma 1.6 follows from explicit estimates on the difference in the integral. Both proofs heavily use the assumptions (a.1)-(a.5). See [FG99] for details.

2. PROOF OF THE MAIN THEOREM

In the rest of this paper we will always assume (a.1)-(a.5). Theorem 1.4 will be a consequence of the following theorem.

Theorem 2.1. There exists a constant $A$, independent of $\varepsilon$ and $\beta$, such that, for all $f \in L^1(\mathbb{R})$ and $\alpha > 0$,

$$(2.1) \quad |\{x \in \mathbb{R} : |T_{\varepsilon,\beta}f(x)| > \alpha\}| \leq \frac{A(1 + |\beta|)}{\alpha}||f||_{L^1},$$

where $T_{\varepsilon,\beta}$ is the convolution operator

$$T_{\varepsilon,\beta}f = K_{\varepsilon,\beta} \ast f.$$ 

As estimate (2.1) is uniform in $\varepsilon$ and $\beta$, Theorem 1.4 follows by taking $\varepsilon \to 0$ and $\beta = 0$.

The proof of Theorem 2.1 will use the following extension of a standard Whitney decomposition (cf. [Ste93]).

Lemma 2.2. Let $\Omega \subset \mathbb{R}$ be open and $F = \mathbb{R} \setminus \Omega$. Then there exists a collection of intervals $\{I_k\}$ with disjoint interiors and two constants $C \geq c > 3$, such that $\Omega = \bigcup I_k$ and

$$c\gamma^{-1}(1/|I_k|) \leq \text{dist}(I_k, F) \leq C\gamma^{-1}(1/|I_k|).$$

Note that the distance of each interval to the complement of $\Omega$ is estimated in terms of $\gamma^{-1}$, rather than just to its length.

Proof. Let $\mathcal{M}_k$ be the mesh of dyadic intervals of length $2^{-k}$ in $\mathbb{R}$. For a number $a > 0$ that will be fixed later, let

$$\Omega_k = \{x \in \Omega : a\gamma^{-1}(2^{k+1}) \leq \text{dist}(x, F) \leq a\gamma^{-1}(2^k)\}.$$
If $I \in \mathcal{M}_k$ is such that $I \cap \Omega_k \neq \emptyset$, then
\[ \text{dist}(I, F) \leq a\gamma'^{-1}(2^k) = a\gamma'^{-1}(1/|I|). \]

Now, from assumption (a.4), $\gamma'(t) \geq A\gamma'((1 + \epsilon)t)$. If $l \geq 1$ is such that $4 > A^l \geq 2$, then $\gamma'(t) \geq 2\gamma'((1 + \epsilon)^lt)$, and thus
\[ \gamma'^{-1}(2^k) \leq (1 + \epsilon)^l\gamma'^{-1}(2^{k+1}). \]

Hence
\[ \text{dist}(I, F) \geq a\gamma'^{-1}(2^{k+1}) - |I| \geq \frac{1}{(1 + \epsilon)^l}\gamma'^{-1}(2^k) - |I| \]
\[ > \left(\frac{1}{(1 + \epsilon)^l} - 1\right)\gamma'^{-1}(2^k), \]

since $|I| < \gamma'^{-1}(1/|I|)$ and $|I| = 2^{-k}$. Therefore, if we set $a = 5(1 + \epsilon)^l$, then we have
\[ 4\gamma'^{-1}(1/|I|) \leq \text{dist}(I, F) \leq 20\gamma'^{-1}(1/|I|), \]

because $a < 5A^l < 20$.

As $\bigcup \Omega_k = \Omega$, the lemma follows by taking $\{I_k\}$ as the collection of maximal intervals as above. \qedhere

Remark 2.1. If for each $I_k$ as above we define the interval
\begin{equation}
I_k^* = [y_k - 3\gamma'^{-1}(1/|I_k|), y_k + 3\gamma'^{-1}(1/|I_k|)],
\end{equation}
where $y_k$ is the center of $I_k$, then there exists a fixed $N$ such that at most $N$ intervals $I_j^*$ intersect $I_k^*$. Indeed, if $x \in I_k^*$, then
\[ \gamma'^{-1}(1/|I_k|) \leq \text{dist}(x, F) \leq 23\gamma'^{-1}(1/|I_k|). \]

Note that $|I_k^*| = 6\gamma'^{-1}(1/|I_k|) \geq 6/23 \text{dist}(x, F)$, and that $I_k^*$ is contained in an interval of length $12\gamma'^{-1}(1/|I_k|) \leq 12 \text{dist}(x, F)$ with center $x$. Hence, there can be at most
\[ \frac{12 \text{dist}(x, F)}{6/23 \text{dist}(x, F)} = 46 \]
such intervals, so we can take $N = 46$.

Remark 2.2. From the discussion in Remark 2.1, we see that $I_k^* \subset \Omega$ and, if $50I_k^*$ is the interval with the same center as $I_k^*$ with 50 times its length, then $50I_k^* \setminus \Omega \neq \emptyset$.

Hence $\bigcup I_k^* = \Omega$ and, since each $I_k^*$ intersects at most a finite fixed number of other such intervals,
\begin{equation}
\sum_k |I_k^*| \lesssim |\Omega|.
\end{equation}
Proof of Theorem 2.1. We follow the idea of the proof of Theorem 2' of [Fef70], but we now apply Lemma 2.2 with

\[ \Omega = \{ x \in \mathbb{R} : Mf(x) > \alpha' \} , \]

where \( \alpha' = \alpha/(1 + |\beta|) \) and \( Mf \) is the Hardy-Littlewood maximal function of \( f \). Let \( f = g + b = g + \sum b_k \), where each

\[ b_k(x) = \left( f(x) - \frac{1}{|I_k^*|} \int_{I_k^*} f \right) \chi_{I_k^*}(x) \]

and \( I_k^* \) is as in Remark 2.1. Note that \( \text{supp } b_k \subset I_k^* \), \( \int b_k = 0 \), and we have the estimate

\[ \int |b_k| \leq 2 \int_{I_k^*} |f| \leq 100|I_k^*| \cdot \frac{1}{50|I_k^*|} \int_{50I_k^*} |f| \lessapprox \alpha'|I_k^*| \]

because \( 50I_k^* \setminus \Omega \neq \emptyset \), by Remark 2.2. This implies, by (2.3),

\[ \int |b| \leq \sum_k \int |b_k| \lessapprox \alpha' \sum_k |I_k^*| \lessapprox \alpha' |\Omega| \lessapprox \int |f| , \]

by the Hardy-Littlewood maximal theorem.

As usual, \( |g(x)| \leq \alpha' \) if \( x \notin \Omega \) (by the Lebesgue differentiation theorem) and, if \( x \in I_k^* \),

\[ |g(x)| \leq \frac{1}{|I_k^*|} \int_{I_k^*} |f| \lessapprox \alpha' , \]

By Lemma 1.5, \( T_{\varepsilon, \beta} \) is bounded in \( L^2 \) with norm \( (1 + |\beta|) \), hence

\[ |\{ x \in \mathbb{R} : |T_{\varepsilon, \beta}g(x)| > \alpha \}| \leq \frac{1}{\alpha^2} \int |T_{\varepsilon, \beta}g|^2 \]

\[ \lessapprox \frac{(1 + |\beta|)^2}{\alpha^2} \int |g|^2 \]

(2.4)

\[ \lessapprox \frac{(1 + |\beta|)^2}{\alpha^2} \alpha' \int |g| \lessapprox \frac{1 + |\beta|}{\alpha} ||f||_{L^1} . \]

Now, let \( \phi \in C^\infty(\mathbb{R}) \) be nonnegative, supported in \( \{ x : |x| < 1 \} \) and with \( \int \phi = 1 \). For each \( k \), define

\[ \phi_k(x) = \frac{1}{|I_k|} \phi \left( \frac{x}{|I_k|} \right) , \]

i. e., \( \phi_k \) is the dilation of \( \phi \) with scale \( |I_k| \). Set \( \tilde{b}_k = \phi_k \ast b_k \) and

\[ \tilde{b} = \sum_{|I_k| \leq 1} \tilde{b}_k . \]
Note that, if $|I_k^*| > 1$ and $x \not\in 3I_k^*$, then

$$K_{\epsilon,\beta} * b_k(x) = \int_{I_k^*} K_{\epsilon,\beta}(x - y)b_k(y)dy = 0,$$

because $|x - y| \geq |I_k^*| > 1$ and supp $K_{\epsilon,\beta} \subset (0, 1]$. So, for $x \not\in \bigcup 3I_k^*$, $K_{\epsilon,\beta} * b = \sum_{|I_k^*| \leq 1} b_k$. Thus

$$K_{\epsilon,\beta} * b(x) - K_{\epsilon,\beta} * \tilde{b}(x) = \sum_{|I_k^*| \leq 1} (K_{\epsilon,\beta} * b_k(x) - K_{\epsilon,\beta} * \tilde{b}_k(x))$$

for $x \not\in 3I_k^*$, and if $|I_k^*| \leq 1$,

$$\int_{\mathbb{R} \setminus 3I_k^*} \left| K_{\epsilon,\beta} * b_k(x) - K_{\epsilon,\beta} * \tilde{b}_k(x) \right| dx$$

$$\leq \int_{\mathbb{R} \setminus 3I_k^*} \left| \int_{I_k^*} (K_{\epsilon,\beta}(x - y) - K_{\epsilon,\beta} * \phi_k(x - y))b_k(y)dy \right| dx$$

$$\leq \int_{I_k^*} \int_{\mathbb{R} \setminus 3I_k^*} \left| K_{\epsilon,\beta}(x - y) - K_{\epsilon,\beta} * \phi_k(x - y) \right| dx |b_k(y)|dy$$

$$\leq \int_{I_k^*} \int_{|z| > 2\gamma^{-1}(1/|I_k|)} \left| K_{\epsilon,\beta}(z) - K_{\epsilon,\beta} * \phi_k(z) \right| dz |b_k(y)|dy,$$

because, if $y \in I_k^*$ and $x \not\in 3I_k^*$, then

$$|x - y| \geq |I_k^*| > 2\gamma^{-1}(1/|I_k|).$$

Also

$$\int_{|z| > 2\gamma^{-1}(1/|I_k|)} \left| K_{\epsilon,\beta}(z) - K_{\epsilon,\beta} * \phi_k(z) \right| dz$$

$$= \int_{|z| > 2\gamma^{-1}(1/|I_k|)} \left| K_{\epsilon,\beta}(z) - \int_{|w| < |I_k|} K_{\epsilon,\beta}(z - w)\phi_k(w)dw \right| dz$$

$$\leq \int_{|w| < |I_k|} \int_{|z| > 2\gamma^{-1}(1/|I_k|)} \left| K_{\epsilon,\beta}(z) - K_{\epsilon,\beta}(z - w) \right| dz\phi_k(w)dw$$

$$\lesssim 1 + |\beta|,$$

because $|w| < |I_k|$ implies $2\gamma^{-1}(1/|I_k|) > 2\gamma^{-1}(1/|w|)$, so we obtain the estimate using (1.4).

Thus

$$\int_{\mathbb{R} \setminus \bigcup 3I_k^*} \left| K_{\epsilon,\beta} * b(x) - K_{\epsilon,\beta} * \tilde{b}(x) \right| dx$$

$$\lesssim (1 + |\beta|) \sum_{|I_k^*| \leq 1} \int_{I_k^*} |b_k(y)|dy \lesssim (1 + |\beta|) ||f||_{L^1},$$
and therefore
\[ \{|x \notin \bigcup 3I^*_k : |K_{\varepsilon,\beta} \ast b(x) - K_{\varepsilon,\beta} \ast \bar{b}(x)| > \alpha\| \leq \frac{(1 + |\beta|)}{\alpha} ||f||_{L^1}. \]

Since
\[ \left| \bigcup 3I^*_k \right| \leq 3 \sum_k |I^*_k| \lesssim \frac{1 + |\beta|}{\alpha} ||f||_{L^1}, \]

it remains to prove the estimate
\[ \left| \left\{ x : |K_{\varepsilon,\beta} \ast \bar{b}(x)| > \alpha \right\} \right| \lesssim \frac{1 + |\beta|}{\alpha} ||f||_{L^1}. \]

Write
\[ K_{\varepsilon,\beta} \ast \bar{b}(x) = \sum_{|I^*_k| \leq 1} (1 - \chi_{3I^*_k}(x))K_{\varepsilon,\beta} \ast \bar{b}_k(x) + \sum_{|I^*_k| \leq 1} \chi_{3I^*_k}(x)K_{\varepsilon,\beta} \ast \bar{b}_k(x). \]

It is clear that the second sum is supported in \( \bigcup 3I^*_k \), so it is enough to prove the estimate
\[ \left| \left\{ x : \left| \sum_{|I^*_k| \leq 1} \chi_{x \notin 3I^*_k} K_{\varepsilon,\beta} \ast \bar{b}_k(x) \right| > \alpha \right\} \right| \lesssim \frac{1 + |\beta|}{\alpha} ||f||_{L^1}, \]

which follows from
\[ (2.5) \quad \left\| \sum_{|I^*_k| \leq 1, x \notin 3I^*_k} K_{\varepsilon,\beta} \ast \bar{b}_k(x) \right\|_{L^1} \lesssim (1 + |\beta|) ||f||_{L^1}. \]

To prove \( (2.5) \), we estimate each integral
\[ \int_{x \notin 3I^*_k} |K_{\varepsilon,\beta} \ast \bar{b}_k(x)| dx. \]

Using the fact that \( \int b_k = 0 \) and \( \bar{b}_k = \phi_k \ast b_k \), we see that
\[ K_{\varepsilon,\beta} \ast \bar{b}_k(x) = K_{\varepsilon,\beta} \ast \phi_k \ast b_k(x) = \int_{I^*_k} K_{\varepsilon,\beta} \ast \phi_k(x-y)b_k(y)dy = \int_{I^*_k} (K_{\varepsilon,\beta} \ast \phi_k(x-y) - K_{\varepsilon,\beta} \ast \phi_k(x-y_k))b_k(y)dy. \]
Thus

\[(2.6) \int_{x \notin 3I^*_k} |K_{\varepsilon, \beta} \ast \tilde{b}_k(x)| dx \leq \int_{I^*_k} \int_{x \notin 3I^*_k} \left| K_{\varepsilon, \beta} \ast \phi_k(x - y) - K_{\varepsilon, \beta} \ast \phi_k(x - y_k) \right| dx |b_k(y)| dy.\]

Now

\[K_{\varepsilon, \beta} \ast \phi_k(x - y) - K_{\varepsilon, \beta} \ast \phi_k(x - y_k) = \int_{|z| < |I_k|} \left( K_{\varepsilon, \beta}(x - y - z) - K_{\varepsilon, \beta}(x - y_k - z) \right) \phi_k(z) dz,\]

so the inner integral in (2.6) is estimated by

\[(2.7) \int_{|z| < |I_k|} \int_{x \notin 3I^*_k} \left| K_{\varepsilon, \beta}(x - y - z) - K_{\varepsilon, \beta}(x - y_k - z) \right| dx \phi_k(z) dz.\]

Now, if \(x \notin 3I^*_k\) and \(|z| < |I_k|\),

\[|x - y_k - z| \geq |x - y| - |z| > |I^*_k| - |I_k| > 2\gamma^{-1} \left( \frac{1}{|I_k|} \right) \geq 2\gamma^{-1} \left( \frac{1}{|y - y_k|} \right),\]

and thus (2.7) is estimated by

\[\int_{|z| < |I_k|} \int_{|w| > 2\gamma^{-1} \left( \frac{1}{|y - y_k|} \right)} \left| K_{\varepsilon, \beta}(w) - K_{\varepsilon, \beta}(w - (y_k - y)) \right| dw \phi_k(z) dz \\
\lesssim (1 + |\beta|) \int_{|z| < |I_k|} \phi_k(z) dz = 1 + |\beta|.\]

Therefore

\[\int_{x \notin 3I^*_k} |K_{\varepsilon, \beta} \ast \tilde{b}_k(x)| dx \lesssim (1 + |\beta|) \int |b_k| \lesssim (1 + |\beta|) |a'| I^*_k|,\]

so

\[\left\| \sum_{|I^*_k| \leq 1} K_{\varepsilon, \beta} \ast \tilde{b}_k(x) \right\|_{L^1} \lesssim (1 + |\beta|) \|f\|_{L^1}.\]

We have proved (2.5), and thus completed the proof of Theorem 2.1.

\[\Box\]

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