Ghetto of Venice: Access to the Target Node and the Random Target Access Time

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Abstract

Random walks defined on undirected graphs assign the absolute scores to all nodes based on the quality of path they provide for random walkers. In city space syntax, the notion of segregation acquires a statistical interpretation with respect to random walks. We analyze the spatial network of Venetian canals and detect its most segregated part which can be identified with canals adjacent to the Ghetto of Venice.

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1 Introduction

The spatial network of Venice that stretches across 122 small islands between which the canals serve the function of roads is constituted by 96 canals. In March, 1516 the Government of the Serenissima Repubblica issued special laws, and the first Ghetto of Europe was instituted. It was an area where Jews were forced to live and which they could not leave from sunset to dawn. The Ghetto existed
for more than two and a half centuries, until Napoleon conquered Venice and finally opened and eliminated every gate (1797).

The phenomenon of clustering of minorities, especially that of newly arrived immigrants, is well documented [1] (the reference appears in [2]). Clustering is considering to be beneficial for mutual support and for the sustenance of cultural and religious activities. At the same time, clustering and the subsequent physical segregation of minority groups would cause their economic marginalization. The spatial analysis of the immigrant quarters [2] and the study of London’s changes over 100 years [3] shows that they were significantly more segregated from the neighboring areas, in particular, the number of street turning away from the quarters to the city centers were found to be less than in the other inner-city areas being usually socially barricaded by the railways, canals and industries.

It has been suggested [4] that space structure and its impact on movement are critical to the link between the built environment and its social functioning. Spatial structures creating a local situation in which there is no relation between movements inside the spatial pattern and outside it and the lack of natural space occupancy become associated with the social misuse of the structurally abandoned spaces. It is well known that the urban layout effects on the spatial distribution of crime [5]. It has been noted that crime seems to be highest where the urban grid is most broken up (in effect creating most local segregation), and lowest where the lines are longest, and in fact most integrated. The linear routes through the estate have least burglary, and the most broken up, locally enclosed spaces the most [4].

In the present paper, we analyze the integration/segregation statistics of the Venetian canal network by means of random walks. In the forthcoming section (Sec. 2), we discuss the role of city space syntax for attaining traffic equilibrium in the city transport network. Then, in Sec. 3 we show that random walks establish the Euclidean space structure on undirected graph, in which distances and angles have the clear statistical interpretations well known in quantitative theory of random walks. In Secs. 4 and 5, we analyze Venetian space syntax and detect the Ghetto of Venice as the most segregated part of the city canal network. Then, we conclude in the last section.
2 City space syntax

Studies of urban networks have a long history. In most of researches devoted to the improving of transport routes, the optimization of power grids, and the pedestrian movement surveys the relationships between certain components of the urban texture are often measured along streets and routes considered as edges of a planar graph, while the traffic end points and street junctions are treated as nodes. Such a primary graph representation of urban networks is grounded on relations between junctions through the segments of streets. The usual city map based on Euclidean geometry can be considered as an example of primary city graphs.

The notion of traffic equilibrium had been introduced by J.G. Wardrop in [7] and then generalized in [8] to a fundamental concept of network equilibrium. Given a connected undirected graph \( G(V,E) \), in which \( V \) is the set of nodes and \( E \) is the set of edges, we can define the traffic volume \( f : E \to (0, \infty] \) through every edge \( e \in E \). It then follows from the Perron-Frobenius theorem that the linear equation

\[
f(e) = \sum_{e' \in E} f(e') \exp(-h \ell(e'))
\]

has a unique positive solution \( f(e) > 0 \), for every edge \( e \in E \), for a fixed positive constant \( h > 0 \) and a chosen set of positive metric length distances \( \ell(e) > 0 \). This solution is naturally identified with the traffic equilibrium state of the transport network defined on \( G \), in which the permeability of edges depends upon their lengths. The parameter \( h \) is called the volume entropy of the graph \( G \), while the volume of \( G \) is defined as the sum \( \text{Vol}(G) = \frac{1}{2} \sum_{e \in E} \ell(e) \).

The degree of a node \( v \in V \) is the number of its neighbors in \( G \), \( \text{deg}(v) = k_v \). It has been shown in [6] that among all undirected connected graphs of normalized volume, \( \text{Vol}(G) = 1 \), which are not cycles and \( k_v \neq 1 \) for all nodes, the minimal possible value of the volume entropy,

\[
\min(h) = \frac{1}{2} \sum_{v \in V} k_v \log(k_v - 1)
\]
is attained for the length distances

\[ \ell(e) = \frac{\log \left( \left( k_{i(e)} - 1 \right) \left( k_{t(e)} - 1 \right) \right)}{2 \min(h)}, \tag{3} \]

where \( i(e) \in V \) and \( t(e) \in V \) are the initial and terminal vertices of the edge \( e \in E \) respectively. It is then obvious that substituting (2) and (3) into (1) the operator \( \exp(-h\ell(e)) \) is given by a symmetric Markov transition operator,

\[ f(e) = \sum_{e' \in E} \frac{f(e')}{\sqrt{\left( k_{i(e)} - 1 \right) \left( k_{t(e)} - 1 \right)}}, \tag{4} \]

which rather describes time reversible random walks over edges than over nodes. The flows satisfying (1) with the operator (4) meet the mass conservation property, \( \sum_{i \sim j} f_{ij} = \pi_j, \sum_{j \in V} \pi_j = 1 \). The Eq.(4) unveils the indispensable role Markov’s chains defined on edges play in equilibrium traffic modelling and exposes the degrees of nodes as a key determinant of the transport networks properties.

Wardrop’s traffic equilibrium [7] is strongly tied to the human apprehension of space since it is required that all travellers have enough knowledge of the transport network they use. The human perception of places is not an entirely Euclidean one, but are rather related to the perceiving of the vista spaces (streets and squares) as single units and of the understanding of the topological relationships between these vista spaces, [9]. Decomposition of city space into a complete set of intersecting vista spaces produces a spatial network which we call the dual graph representation of a city. Therein, the relations between streets treated as nodes are traced through their junctions considered as edges.

Dual city graphs are extensively investigated within the concept of space syntax, a theory developed in the late 1970s, that seeks to reveal the mutual effects of complex spatial urban networks on society and vice versa, [10] [11]. Spatial perception that shapes peoples understanding of how a place is organized determines eventually the pattern of local movement which is predicted by the space syntax method with surprising accuracy [12]. The robustness of agreement between centrality of spaces and rush hour movement rates is now supported by a number of similar studies of pedestrian movement in different parts of the world and in an everyday commercial work of the Space Syntax Ltd., [13].
3 Study of undirected graphs by means of random walks

The issues of global connectivity of finite graphs and accessibility of their nodes have always been the classical fields of researches in graph theory. Any graph representation naturally arises as an outcome of categorization, when we abstract a real world system by eliminating all but one of its features and by the grouping of things (or places) sharing a common attribute by classes or categories. For instance, the common attribute of all open spaces in city space syntax is that we can move through them. All elements called nodes that fall into one and the same group $V$ are considered as essentially identical; permutations of them within the group are of no consequence. The symmetric group $S_N$ consisting of all permutations of $N$ elements ($N$ being the cardinality of the set $V$) constitute the symmetry group of $V$. If we denote by $E \subseteq V \times V$ the set of ordered pairs of nodes called edges, then a graph is a map $G(V, E) : E \rightarrow K \subseteq \mathbb{R}_+$ (we suppose that the graph has no multiple edges).

The nodes of $G(V, E)$ are weighted with respect to some measure $m = \sum_{i \in V} m_i \delta_i$, specified by a set of positive numbers $m_i > 0$. The space $\ell^2(m)$ of square-assumable functions with respect to the measure $m$ is the Hilbert space $\mathcal{H}$ (a complete inner product space). Among all linear operators defined on $\mathcal{H}$, those invariant under the permutations of nodes are of essential interest since they reflect the symmetry of the graph. Although there are infinitely many such operators, only those which maintain conservation of a quantity may describe a physical process. The Markov transition operators which share the property of probability conservation considered in theory of random walks on graphs are among them. Laplace operators describing diffusions on graphs meet the mean value property (mass conservation); they give another example [14] studied in spectral graph theory.

Being defined on connected undirected graphs, a Markov transition operator $T$ has a unique equilibrium state $\pi$ (a stationary distribution of random walks) such that $\pi T = \pi$ and $\pi = \lim_{t \to \infty} \sigma T^t$ for any density $\sigma \in \mathcal{H}$ ($\sigma_i \geq 0$, $\sum_{i \in V} \sigma_i = 1$). There is a unique measure $m_\pi = \sum_{i \in V} \pi_i \delta_i$ related to the stationary distribution $\pi$. 
with respect to which the Markov operator $T$ is self-adjoint,

$$\hat{T} = \frac{1}{2} \left( \pi^{1/2} T \pi^{-1/2} + \pi^{-1/2} T^\top \pi^{1/2} \right), \quad (5)$$

where $T^\top$ is the adjoint operator. The orthonormal ordered set of real eigenvectors $\psi_i$, $i = 1 \ldots N$, of the symmetric operator $\hat{T}$ defines a basis in $\mathcal{H}$. In quantitative theory of random walks defined on graphs \[16, 15\] and in spectral graph theory \[17\], the properties of graphs are studied in connection with the eigenvalues and eigenvectors of self-adjoint operators defined on them. In particular, the symmetric transition operator defined on undirected graphs is $\hat{T}_{ij} = 1/\sqrt{k_i k_j}$. Its first eigenvector $\psi_1$ belonging to the largest eigenvalue $\mu_1 = 1$,

$$\psi_1 \hat{T} = \psi_1, \quad \psi_{1,i}^2 = \pi_i, \quad (6)$$

describes the local property of nodes (connectivity), $\pi_i = k_i/2M$, where $2M = \sum_{i \in V} k_i$, while the remaining eigenvectors $\{\psi_s\}_{s=2}^N$ belonging to the eigenvalues $1 > \mu_2 \geq \ldots \mu_N \geq -1$ describe the global connectedness of the graph.

Markov’s symmetric transition operator $\hat{T}$ defines a projection of any density $\sigma \in \mathcal{H}$ on the eigenvector $\psi_1$ of the stationary distribution $\pi$,

$$\sigma \hat{T} = \psi_1 + \sigma^\perp \hat{T}, \quad \sigma^\perp = \sigma - \psi_1, \quad (7)$$
in which $\sigma^\perp$ is the vector belonging to the orthogonal complement of $\psi_1$. Thus, it is clear that any two densities $\sigma, \rho \in \mathcal{H}$ differ with respect to random walks only by their dynamical components, $(\sigma - \rho) \hat{T}^t = (\sigma^\perp - \rho^\perp) \hat{T}^t$ for all $t > 0$. Therefore, we can define a distance between any two densities which they acquire with respect to random walks by

$$\|\sigma - \rho\|^2_T = \sum_{t \geq 0} \langle \sigma - \rho \rvert T^t \rvert \sigma - \rho \rangle. \quad (8)$$

or, in the spectral form,

$$\|\sigma - \rho\|^2_T = \sum_{t \geq 0} \sum_{s=2}^N \mu_s \langle \sigma - \rho \rvert \psi_s \rangle \langle \psi_s \rvert \sigma - \rho \rangle = \sum_{s=2}^N \frac{\langle \sigma - \rho \rvert \psi_s \rangle \langle \psi_s \rvert \sigma - \rho \rangle}{1 - \mu_s}, \quad (9)$$
where we have used Dirac’s bra-ket notations especially convenient in working with inner products and rank-one operators in Hilbert space.

If we introduce in $\mathcal{H}(V)$ a new inner product by

$$
(\sigma, \rho)_T = \sum_{t \geq 0} \sum_{s=2}^{N} \frac{\langle \sigma \mid \psi_s \rangle \langle \psi_s \mid \rho \rangle}{1 - \mu_s}
$$

(10)

for all $\sigma, \rho \in \mathcal{H}(V)$, then (9) is nothing else as

$$
\| \sigma - \rho \|_T^2 = \| \sigma \|_T^2 + \| \rho \|_T^2 - 2 (\sigma, \rho)_T,
$$

(11)

where

$$
\| \sigma \|_T^2 = \sum_{s=2}^{N} \frac{\langle \sigma \mid \psi_s \rangle \langle \psi_s \mid \sigma \rangle}{1 - \mu_s}
$$

(12)

being the squared norm of $\sigma \in \mathcal{H}(V)$ with respect to random walks. We finish the description of the $(N-1)$-dimensional Euclidean space structure associated to random walks by mentioning that given two densities $\sigma, \rho \in \mathcal{H}(V)$, the angle between them can be introduced in the standard way,

$$
\cos \angle (\rho, \sigma) = \frac{(\sigma, \rho)_T}{\| \sigma \|_T \| \rho \|_T}.
$$

(13)

Random walks embed connected undirected graphs into Euclidean space. This embedding can be used in order to compare nodes and to retrace the optimal coarse-graining representations.

Namely, let us consider the density $\delta_i$ which equals 1 at the node $i \in V$ and zero for all other nodes. It takes form $v_i = \pi_i^{-1/2} \delta_i$ with respect to the measure $m_\pi$. Then, the squared norm of $v_i$ is given by

$$
\| v_i \|_T^2 = \frac{1}{\pi_i} \sum_{s=2}^{N} \frac{\psi_{s,i}^2}{1 - \mu_s},
$$

(14)

where $\psi_{s,i}$ is the $i^{th}$-component of the eigenvector $\psi_s$. In theory of random walks [16], the quantity (14) expresses the access time to a target node quantifying the expected number of steps required for a random walker to reach the node $i \in V$ starting from an arbitrary node chosen randomly among all other nodes with respect to the stationary distribution $\pi$. 

7
The Euclidean distance between any two nodes of the graph $G$ established by random walks,

$$K_{i,j} = \|v_i - v_j\|_T^2 = H_{ij} + H_{ji}, \quad (15)$$

is known as the commute time in theory of random walks and equals to the expected number of steps required for a random walker starting at $i \in V$ to visit $j \in V$ and then to return to $i$ again, $[16]$. The expected number of steps a random walker reaches $j$ if starts from $i$ is called the first hitting time (or the access time) $[16]$,

$$H_{ij} = \|v_j\|_T^2 - (v_i, v_j)_T = \sum_{k=2}^{N} \frac{1}{1-\mu_k} \left( \frac{\psi_{k,i}}{\pi_j} - \frac{\psi_{k,i}\psi_{k,j}}{\sqrt{\pi_i \pi_j}} \right). \quad (16)$$

The cosine of an angle calculated in accordance to (13) has the structure of Pearson’s coefficient of linear correlations that reveals its natural statistical interpretation. Correlation properties of flows of random walkers passing by different paths have been remained beyond the scope of previous studies devoted to complex networks and random walks on graphs. The notion of angle between any two nodes in the graph arises naturally as soon as we become interested in the strength and direction of a linear relationship between two random variables, the flows of random walks moving through them. If the cosine of an angle $[13]$ is 1 (zero angles), there is an increasing linear relationship between the flows of random walks through both nodes. Otherwise, if it is close to -1 ($\pi$ angle), there is a decreasing linear relationship. The correlation is 0 ($\pi/2$ angle) if the variables are linearly independent. It is important to mention that as usual the correlation between nodes does not necessary imply a direct causal relationship (an immediate connection) between them.

In the forthcoming sections, we study the average of the first access time $H_{ij}$ (16) with respect to its first and second indices.

4 Detection of ghettos by access to a target node

The Euclidean structure introduced on undirected graphs by random walks can be used in order to investigate city space syntax. Various properties of access times have been recently studied in concern with the traffic flow forecasting $[13]$, in order to model the
wireless terminal movements in a cellular wireless network [19], in a statistical test for the presence of a random walk component in the repeat sales price index models in house prices [20], in the growth modelling of urban conglomerations [21], and in many other works where random walks have been considered directly on the city maps and physical landscapes. In contrast to all previous studies, we use random walks in order to investigate the morphology of urban textures described by dual city graphs. In the context of traditional space syntax studies, the value of access time (the Euclidean norm, (14)) calculated for a node of a dual graph can be naturally interpreted as the expected number of random elementary navigation actions (i.e., the random turns at the junctions between axial lines) required to reach the certain open space in the city.

The access to a target node introduced in theory of random walks [16] is the important global characteristic of the node which is equal to the average of $H_{ji}$ (16) over its first index. It quantifies the expected number of steps required for a random walker to reach the node starting from an arbitrary node chosen randomly among all other nodes of the network with respect to the stationary distribution $\pi$,

$$\sum_{j \in V} \pi_j H_{ji} = \frac{1}{\pi_i} \sum_{k=2}^{N} \frac{\psi_{k,i}^2}{1 - \mu_k} = \| v_i \|^2_T. \quad (17)$$

Being a global characteristic of a node in the graph, the access (17) assigns the absolute scores to all nodes in the network based on the quality of path they provide for random walks. In urban spatial networks, the value given by (17) varies strongly from one open space to another: the norm of a space (street, square, or canal) that can be easily reached (just in a few random syntactic steps) from any other space in the city is minimal, while it could be very large for a statistically segregated street. The relatively isolated groups of nodes and bottlenecks can be visually detected from distribution histograms of the Euclidean norms (see Fig. 1). In particular, it would facilitate the detection of ghettos and urban sprawl[9]. In Fig. 1 we have presented the histogram of empirical distribution of mean access times for the dual graph of Venetian canals. It is interesting to note that the utmost peak on the histogram correspondent from 250 to 300 random syntactic steps represents the canals of the Cannare-

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[9] Urban sprawl is the spreading out of a city and its suburbs at the fringe of an urban area.
gio district surrounding the quarter of Venetian Ghetto. From 16th century, the quarter had been enlarged later to cover the neighboring Ghetto Vecchio and the Ghetto Nuovissimo. As a result a specific Ghetto canal subnetwork arose in Venice weakly connected to the main canals. The norm \(14\) which a node acquires in regard to random walks is a natural statistical centrality measure of the vertex within a graph. Probably, the most important message of space syntax theory is that the local property (connectivity) of city spaces (streets and squares) and their global configuration variable (centrality) are positively related in a city, and this part-whole relationship known as intelligibility \([11]\) is a key determinant of human behaviors in urban environments. An adequate level of intelligibility has been found to encourage peoples way-finding abilities. Intelligibility of Venetian canal network reveals itself quantitatively in the scaling of the norms of nodes with connectivity shown in Fig.2. The three utmost points in the left upper part of the graph displayed on Fig.2 represent the most intelligible structure in the Venetian canal network formed by the Venetian Lagoon, the Giudecca canal, and the Grand Canal. The four points characterized by the worse accessibility levels delineate the Ghetto canal subnetwork segregated statistically from the rest of the network.
Figure 2: The scatter plot of connectivity vs. the norm of a node in the dual graph representation of 96 Venetian canals. The slope of the regression line equals 2.07.

5 Random target access time

The average of access time with respect to its second index, the random target access time \[16\], determines the expected number of steps a random walker needs to reach an arbitrary node of the graph chosen randomly from the stationary distribution \(\pi\) (if a random walk starts in \(i \in V\)). In contrast to \[17\], the value of this average is independent of the starting node \(i \in V\) being a global spectral characteristic of a graph,

\[
\sum_{j \in V} \pi_j H_{ij} = \sum_{k=2}^{N} \frac{1}{1 - \mu_k}.
\] (18)

The latter equation expresses the so called random target identity, \[16\]. Computations of random target access times \[18\] for the dual city graphs show that they are closely tied to the network size (i.e., the numbers of junctions between different streets and canals in a city) (see Fig. 3). The diagram shown on Fig. 3 have been calculated for the dual graph representations of two German organic medieval cities founded shortly after the Crusades and developed within the medieval fortresses (Rothenburg ob der Tauber in Bavaria and the downtown of Bielefeld, nowadays the biggest city of Eastern Westphalia), the street grid in Manhattan (a borough of New York City
Figure 3: The comparative diagram of the random target access times and the sizes of dual graphs for five compact urban patterns. Heights of left pillars correspond to the number of nodes in the dual city graphs. Pillars on right show the random target access times, the expected number of steps a random walker needs to hit a node randomly chosen from the stationary distribution $\pi$.

with an almost regular grid-like city plan), and two city canal networks, in Amsterdam (binding to the delta of the Amstel river, forming a dense canal web exhibiting a high degree of radial symmetry) and in Venice (that stretches across 122 small islands between which the canals serve the function of roads), within the same frame.

6 Conclusion

The properties of random walks defined on the dual graph representation of a transport network is related to the unique equilibrium configuration of not random commodity flows along edges of its primary graph representation that unveils the role Markov’s chain processes and space syntax approach play in the successful traffic modelling. The amazing effectiveness of random walk models in describing complex cooperative phenomena has been often discussed in literature. [22]. In space syntax theory, the concept of random walks (implicitly reckoned by means of integration measures) has been applied to the ”probabilistic” analysis of spatial configurations
and proved its power by the surprisingly accurate predictions of human behavior in cities [12].

We have demonstrated that random walks establish the Euclidean space structure on undirected graphs, in which the notions of distance and angle acquire the clear statistical interpretations. In particular, every open space (a street, a square, or a canal) acquires a norm based on the quality of path it provides for random walks. This norm can be used as a measure of statistical segregation the node is characterized in the given network.

In the present paper, we have canalized space syntax of Venetian’s canals by means of random walks defined on the dual graph representation of the network and detect the segregated part that is identified with the specific canal subnetwork of Venetian Ghetto.

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