Monochromatic paths in 2-edge-coloured graphs and hypergraphs

Maya Stein*

Abstract

We answer a question of Gyárfás and Sárközy from 2013 by showing that every 2-edge-coloured complete 3-uniform hypergraph can be partitioned into two monochromatic tight paths of different colours. We also give a lower bound for the number of tight paths needed to partition any 2-edge-coloured complete r-partite r-uniform hypergraph. Finally, we show that any 2-edge-coloured complete bipartite graph has a partition into a monochromatic cycle and a monochromatic path, of different colours, unless the colouring is a split colouring.

1 Introduction

Monochromatic partitions and covering problems are a part of Ramsey theory in the wider sense, and the topic has seen increased activity in the last decade. As in the classical Ramsey problem, one wishes to find certain monochromatic subgraphs in a graph $G$ whose edges are coloured with two colours. Often this is the complete graph on $n$ vertices, $K_n$. Instead of just one monochromatic copy as in Ramsey’s theorem, in monochromatic partitioning problems we aim to find a collection of such copies that together cover the whole vertex set of the host graph.

At the centre of this area lies an observation by Gerencsér and Gyárfás [7], which states that in any 2-colouring of the edges of $K_n$ there are two disjoint monochromatic paths, of different colours, that together cover the vertex set of $K_n$. If we allow each of $\emptyset$, a vertex, an edge to count as a cycle, then the same statement holds substituting

*Department for Mathematical Engineering and Center for Mathematical Modeling, University of Chile. Support by ANID Regular Grant 1221905, by FAPESP-ANID Investigación Conjunta grant 2019/13364-7, and by ANID PIA CMM FB210005.
the two paths with two cycles: this was conjectured by Lehel in the 1970’s (see [2]), and resolved by Bessy and Thomassé in 2010 [4], with earlier asymptotic results in [1, 17].

Generalisations to hypergraphs have been considered, where several notions of cycles and paths exist, and there are some results for loose, ℓ- and Berge-paths and cycles (see e.g. [6, 9, 12, 13, 16]), but for tight paths and cycles, not much is known. A tight path in an r-uniform hypergraph is an ordered sequence of vertices such that each r consecutive vertices form an edge. Tight cycles are defined analogously on cyclic sequences of vertices. Usually, sets of size between 1 and r − 1 are allowed as tight paths or cycles, although in this paper, this is not needed, unless n ≤ 5. We will allow a tight path to be empty, i.e., to have no edges or vertices.

In 2013, Gyárfás and Sárközy [13] suggested the following problem: ‘decide whether there is a partition into a red and a blue tight path in every 2-coloring of a complete 3-uniform hypergraph’. The question was reiterated in Gyárfás’s survey [9], but no advances were made until 2019, when Bustamante, Hán, and the author [5] proved that in any 2-edge-colouring of the 3-uniform complete graph $K^{(3)}_n$, two tight monochromatic cycles of different colours partition almost all its vertices (all but $o(n)$ vertices, to be precise). As tight cycles contain spanning tight paths, this answers Gyárfás and Sárközy’s question asymptotically.

Here, we solve the problem posed by Gyárfás and Sárközy [13] exactly.

**Theorem 1.** For any $n \in \mathbb{N}$ and any 2-colouring of the edges of the 3-uniform complete hypergraph $K^{(3)}_n$, there are two monochromatic vertex-disjoint tight paths, of different colours, that together cover all the vertices.

It is not possible to improve Theorem 1 by replacing the tight paths with tight cycles: Lo and Pfenninger [15] exhibit colourings of arbitrarily large $K^{(r)}_n$, for $k \geq 3$, which do not admit a partition into two tight cycles of different colours. However, it has been announced (see [15]) that a partition into two tight monochromatic cycles, possibly of the same colour, does exist for the case $r = 3$.

Lo and Pfenninger [15] also show an analogue of the result from [5] for 4-uniform hypergraphs (i.e., they show that for any 2-edge-colouring of the 4-uniform complete graph $K^{(4)}_n$, two tight monochromatic cycles of different colours partition almost all the vertices), and indicate they believe their result should generalise to other uniformities. We believe that Theorem 1 should generalise to arbitrary uniformities as well:

**Conjecture 2.** For any $r, n \in \mathbb{N}$ and any 2-colouring of the edges of the $r$-uniform complete hypergraph $K^{(r)}_n$, there are two monochromatic vertex-disjoint tight paths, of different colours, that together cover all the vertices.
In a variant of the original problem in graphs, the underlying complete graph is replaced with the complete balanced bipartite graph $K_{n,n}$. Pokrovskiy \cite{Pokrovskiy2018} showed that these graphs can be partitioned into two monochromatic paths, unless the colouring is a *split colouring*, that is, a colouring where each colour induces the disjoint union of two complete bipartite graphs. (It is easy to see that if these complete bipartite graphs are sufficiently unbalanced, a partition into two monochromatic paths becomes impossible. On the other hand, a partition into three monochromatic paths is always possible in a split colouring.)

**Theorem 3** (Pokrovskiy \cite{Pokrovskiy2018}). *Any 2-colouring of the edges of $K_{n,n}$ that is not a split colouring allows for a partition of $V(K_{n,n})$ into two monochromatic paths of different colours.*

Gyárfás and Lehel \cite{Gyarfas1986,Gyarfas1988} proved earlier that a partition of all but one vertex of $K_{n,n}$ exists.

We give a shorter proof of Pokrovskiy’s result, Theorem 3, and improve it to a partition into a path and a cycle, as follows.

**Theorem 4.** *Any 2-colouring of the edges of $K_{n,n}$ that is not a split colouring allows for a partition of $V(K_{n,n})$ into a monochromatic path and a monochromatic cycle, of different colours.*

It is not possible to obtain a partition into two cycles, or to choose the colour of the cycle in Theorem 4. This can be seen by considering a split colouring of $K_{n,n}$ with colours blue and red, and then recolouring in blue one of the red edges. If the sizes of the classes of the split colouring are chosen sufficiently different, then no partition into two monochromatic cycles exists. Moreover, given any partition into a monochromatic path and a monochromatic cycle, it is not difficult to see that the monochromatic path has to use the recoloured edge, and is therefore blue. However, our proof will show that in certain situations it is possible to choose the colour of the cycle (see Corollary 15).

Finally, we return to $r$-uniform hypergraphs, now for arbitrary uniformities $r$. Having studied the problem for the graphs $K_n$ and $K_{n,n}$, and for the hypergraph $K_n^{(r)}$, it seems natural to consider a multipartite hypergraph variant as well. We propose to replace the underlying complete $r$-uniform hypergraph with $K_{r\times n}^{(r)}$, the complete balanced $r$-partite $r$-uniform hypergraph with $n$ vertices in each partition class. It is not clear whether there is a function $f(r)$ such that for any 2-colouring of the edges of this hypergraph there is a partition into $f(r)$ monochromatic tight paths. But if such a function $f(r)$ exists, then $f(r) \geq r + 1$, as the following result shows.
Proposition 5. For all \( n, r \in \mathbb{N} \) with \( n \geq 3^{r+2} \) and \( r \geq 2 \), one can colour the edges of \( K_{r \times n}^{(r)} \) with two colours so that \( V(K_{r \times n}^{(r)}) \) cannot be partitioned into less than \( r + 1 \) monochromatic tight paths.

Observe that for \( r = 2 \), the bound from Proposition 5 coincides with the bound coming from split colourings, which is 3. To prove the proposition, we introduce a generalisation of the split colouring to hypergraphs.

The rest of the paper is organised as follows. Section 2 contains the proof of Theorem 1, showing the slightly stronger Proposition 7. Section 3 is dedicated to \( r \)-partite \( r \)-uniform hypergraphs and split colourings, and Proposition 5 is proved. In Section 4, we give the brief proof of Theorem 4 based on Theorem 3, then offer an alternative shorter proof of Theorem 3, and finally present a quick application of Theorem 4 to 3-edge-colourings of complete and complete bipartite graphs.

2 Two tight paths for \( K_n^{(3)} \)

In this section we show Theorem 1 thus answering the question of Gyárfás and Sárközy from [13]. Similarly as in the proof for graphs from [7], the starting point for our proof is a path, in our case a tight path, that switches colours at most once and has maximal length with this property. Let us define this object in precise terms.

Definition 6 (Tight bicoloured path). Given any 2-colouring of the edges of the 3-uniform complete hypergraph \( K_n^{(3)} \), we call a tight path \( P = v_1v_2\ldots v_k \) bicoloured if there is an index \( \ell \in \{1, \ldots, k-1\} \) such that each edge \( v_{i-1}v_iv_{i+1} \) with \( 2 \leq i \leq \ell \) has the same colour, and all other edges of \( P \) have the other colour. Call \( v_\ell \) the turning point of \( P \).

Note that monochromatic paths are allowed in Definition 6, namely, a bicoloured path with turning point \( v_\ell \) is monochromatic if and only if \( \ell = 1 \) or \( \ell = k-1 \). As another example, if we have two differently coloured edges \( v_1v_2v_3 \) and \( v_2v_3v_4 \), then they form a bicoloured path \( v_1v_2v_3v_4 \) with turning point \( v_2 \).

Clearly, every bicoloured tight path \( P \) contains two disjoint monochromatic paths \( P_1, P_2 \) of distinct colours (and moreover, if \( P \) has at least 6 vertices, then each of \( P_1, P_2 \) is either empty or has at least an edge). So in order to prove Theorem 1 it suffices to show the following proposition.

Proposition 7. For any \( n \in \mathbb{N} \) and any colouring of the edges of the 3-uniform complete hypergraph \( K_n^{(3)} \), there is a spanning bicolored tight path.
Note that Proposition 7 is slightly stronger than Theorem 1 in the sense that two monochromatic tight paths of distinct colours can not always be concatenated to a bicoloured path.

**Proof of Proposition 7.** We take a longest bicoloured tight path $P$ in $K_3^{(3)}$. Say $P = v_1v_2\ldots v_k$, with $v_1v_2v_3$ red, and with turning point $v_\ell$ (for some $\ell \in \{1, \ldots, k - 1\}$).

If $k = n$, we are done, so we assume $k < n$. Then there is a vertex $w$ which is not on $P$. Because of the maximality of $P$, we know that $w$ cannot be added to $P$ via the edge $wv_1v_2$ or via the edge $v_{k-1}v_kw$. Therefore, $P$ is not monochromatic and thus

$$1 < \ell < k - 1.$$  

Note that reversing $P$ makes $v_{\ell+1}$ the turning point of the reversed bicoloured tight path. This means that we can invoke colour symmetry to be able to assume that

the edge $v_\ell v_{\ell+1}w$ is red.  

We claim that

$v_{\ell+1}wv_{\ell+2}$ is blue.  

In order to see (2) by contradiction, assume that $v_{\ell+1}wv_{\ell+2}$ is red. Consider the tight path $P' := v_1v_2\ldots v_\ell v_{\ell+1}wv_{\ell+2}\ldots v_{k-1}v_k$. If $\ell = k - 2$, then by (1), $P'$ is monochromatic (and thus bicoloured). If $\ell < k - 2$, then, because of (1), no matter which colour the edge $wv_{\ell+2}v_{\ell+3}$ has, $P'$ is a bicoloured tight path. It is also longer than $P$. This is a contradiction, as $P$ was chosen as a maximum length bicoloured tight path. Hence we proved (2).

Next, we claim that

$v_1wv_{\ell+1}$ is red.  

Indeed, in order to see (3), it suffices to consider the tight path

$P'' := v_\ell v_{\ell-1}\ldots v_2v_1wv_{\ell+1}v_{\ell+2}\ldots v_k$.

If the edge $v_1wv_{\ell+1}$ was blue, then by (2), and no matter which colour the edge $v_2v_1w$ has, $P''$ is a bicoloured tight path on $V(P) \cup \{w\}$. This contradicts the maximality of $P$. Thus we proved (3).

Similarly, by considering the tight path $v_1v_2\ldots v_\ell v_{\ell+1}wv_{k-1}\ldots v_{\ell+2}$, which cannot be bicoloured, we can use (1) to see that

the edge $v_{\ell+1}wv_k$ is blue.  

5
Now, consider the edge $v_1wv_k$. If this edge is red, then because of (1) and (3), the tight path

$$v_2v_3\ldots v_\ell v_{\ell+1}wv_1v_kv_{k-1}\ldots v_\ell v_{\ell+2}$$

is bicoloured (no matter which colour the edge $v_1v_kv_{k-1}$ is). Similarly, if $v_1wv_k$ is blue, then by (1) and (2),

$$v_\ell v_{\ell-1}\ldots v_2v_1v_kwv_{\ell+1}v_{\ell+2}\ldots v_{k-2}v_{k-1}$$

is a bicoloured tight path no matter the colour of $v_2v_1v_k$. In either case, we obtain a contradiction to the maximality of the bicoloured tight path $P$, which concludes the proof. 

\section{Split colourings}

\subsection{Split colourings in hypergraphs}

In this section, we consider 2-colourings of the edges of the $r$-partite $r$-uniform hypergraph with $n$ vertices in each partition class, denoted by $K^{(r)}_{r\times n}$. For graphs, the following definition coincides with the usual definition of split colourings.

\begin{definition}[Hypergraph split colourings] Let $V_1, V_2, \ldots, V_r$ be the partition classes of $K^{(r)}_{r\times n}$. For each $i$, partition $V_i$ into two non-empty sets $V^1_i, V^2_i$. Colour each edge having an even number of vertices in $\bigcup_{1 \leq i \leq r} V^1_i$ in red, and colour all other edges blue. Any colouring obtained in this way is called a split colouring.
\end{definition}

We now give the proof of Proposition 5.

\begin{proof}[Proof of Proposition 5] Consider a split colouring of $K^{(r)}_{r\times n}$ with classes $V^1_1, V^2_1, V^1_2, V^2_2, \ldots, V^1_r, V^2_r$, where for $1 \leq i \leq r$, the class $V^1_i$ has size $3^i$, and the class $V^2_i$ has size $n - 3^i$. Let $\mathcal{P}$ be a set of disjoint monochromatic tight paths that cover all the vertices.

Observe that because of the structure of the split colouring, each path in $\mathcal{P}$ can only meet at most one of $V^1_i, V^2_i$, for each $1 \leq i \leq r$, and so,

for each $P \in \mathcal{P}$ and each $i \in [r]$, one of $V^1_i \cap V(P), V^2_i \cap V(P)$ is empty. \hfill (5)

Our aim is to show that $|\mathcal{P}| \geq r + 1$. For this, we let $\mathcal{P}_i$ denote the set of all paths in $\mathcal{P}$ that meet $V^1_i \cup \ldots \cup V^1_i$ (in particular $\mathcal{P}_0 = \emptyset$), set $V_0 := \emptyset$, and use induction on $i$ to show that for $0 \leq i \leq \ell \leq r$,

at most $\sum_{j=1}^i |V^1_j| + i$ vertices in $V_i$ are covered by paths from $\mathcal{P}_i$. \hfill (6)
This is trivially true for $i = 0$ and all $0 \leq \ell \leq r$. For $i \geq 1$, because of (4), the paths in $\mathcal{P}_i \setminus \mathcal{P}_{i-1}$ cannot meet $V^2_i$, and therefore, by induction, the number of vertices in $V^i_\ell$ covered by paths from $\mathcal{P}_i$ is at most $|V^i_1| + 1 + \sum_{j=1}^{i-1} |V^j_1| + i - 1 = \sum_{j=1}^{i} |V^j_2| + i$. This proves (6).

Now, observe that for all $1 \leq i \leq r$,

$$|V^i_1| = 3^i > \sum_{j=1}^{i-1} 3^j + i - 1 = \sum_{j=1}^{i-1} |V^j_1| + i - 1.$$ 

So, by (3) for $\ell = i$, we know that $\mathcal{P}_i \setminus \mathcal{P}_{i-1} \neq \emptyset$ for all $1 \leq i \leq r$. In particular, $|\mathcal{P}_r| \geq r$.

Moreover, by (6) for $\ell = i = r$, the paths from $\mathcal{P}_r$ cover at most

$$\sum_{j=1}^{r} |V^j_1| + r = \sum_{j=1}^{r} 3^j + r \leq 3^{r+2} - 3^r - 1 \leq n - 3^r - 1 = |V^2_r| - 1$$

vertices in $V_r$, where we used the fact that by assumption, $n \geq 3^{r+2}$. As $V^2_r \subseteq V_r$ has to be covered by paths from $\mathcal{P}$, we deduce that $|\mathcal{P}| \geq |\mathcal{P}_r| + 1 \geq r + 1$, which is as desired. 

### 3.2 Split colourings in graphs

For graphs, it is known that split colourings can be characterised by the existence of certain substructures. Pokrovskiy [18] showed that a 2-colouring of $K_{n,n}$ is split if and only if all vertices see both colours and there is no spanning monochromatic component. In Lemma 11 below, we give a new characterization of colourings that are not split colourings or one of their close relatives: monochromatic colourings or $V$-colourings. The latter class is defined now:

**Definition 9 (V-colouring).** Call a colouring of the edges of $K_{n,n}$ a V-colouring if it has the property that each of the colours spans a complete bipartite graph.

Note that the set of all vertices seeing both colours coincides with one of the partition classes of $K_{n,n}$, giving the colouring the form of the letter V.

For the characterisation in Lemma 11, the following notation will be convenient.

**Definition 10 (Bicoloured cycle, good cycle).** In a 2-edge-coloured $K_{n,n}$, call a cycle $C = v_1 v_2 \ldots v_k v_1$ bicoloured if for some $\ell \in \{2, \ldots, k\}$ each edge $v_i v_{i+1}$ with $1 \leq i \leq \ell - 1$ has one colour, and all other edges have the other colour. Call $v_1$ and $v_\ell$ the turning points of $C$. If $v_1$ and $v_\ell$ lie in distinct partition classes, call $C$ good.
Note that, in contrast with Definition 6, monochromatic cycles are not bicoloured.

**Lemma 11.** Given a colouring $c$ of the edges of $K_{n,n}$, the following are equivalent:

(a) $c$ is either a split colouring, or a $V$-colouring, or monochromatic;

(b) there are no good cycles for $c$;

(c) there is no good $C_4$ for $c$.

**Proof.** We only show that (c) implies (a), as the implications (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are straightforward. Unless $c$ is monochromatic, there is a vertex $v$ such that both sets $X^r := \{w \in N(v) : vw \text{ is red}\}$ and $X^b := \{w \in N(v) : vw \text{ is blue}\} = N(v) \setminus X^r$ are non-empty. Because of (c), each vertex from the partition class that contains $v$ is monochromatic to each of $X^r$, $X^b$, but not to $X^r \cup X^b$. So $c$ is a split colouring or a $V$-colouring.

## 4 Bipartite graphs

In Section 4.1 we prove Theorem 4, and in Section 4.2 we give a new proof of Theorem 3. We will need Definition 10 and Lemma 11 from Section 3.2. In Section 4.3, we show an application of Theorem 4 to 3-edge-colourings.

### 4.1 Partition into a path and a cycle

We start by observing that clearly, in any 2-edge-coloured $K_{n,n}$, every bicoloured cycle $C$ contains two disjoint monochromatic paths of distinct colours spanning the same set of vertices as $C$. Also, any two disjoint monochromatic paths of distinct colours can be transformed into a bicoloured or monochromatic cycle, as follows.

**Lemma 12.** Let $P_1$ and $P_2$ be disjoint monochromatic paths of distinct colours in an 2-edge-coloured $K_{n,n}$ such that $V(K_{n,n}) = V(P_1) \cup V(P_2)$. Then there is a spanning cycle that is either bicoloured or monochromatic.

**Proof.** We can assume that $n \geq 2$ and that $P_1 = v_1 \ldots v_\ell$ has at least two vertices. If $P_2$ is empty, $v_1$ and $v_\ell$ are in opposite partition classes, and we can add the edge $v_\ell v_1$ to $P_1$ to obtain the desired cycle. If $P_2$ has only one vertex $w$, the edges $v_\ell w$ and $wv_1$ exist, and can be added to $P_1$ with the same outcome.

So assume $P_2 = w_1 \ldots w_k$ with $k \geq 2$. Observe that one of the endpoints of $P_1$, say $v_1$, does not lie in the same partition class as $w_1$ (otherwise the three vertices
\(v_1, v_\ell, w_1\) are in the same partition class which then would be larger than the other partition class, but both have size \(n\). Similarly, also \(v_\ell, w_k\) are in opposite partition classes and, therefore, we can transform the paths \(P_1, P_2\) into a bicoloured cycle by adding the edges \(v_1w_1\) and \(v_\ell w_k\). 

From the observation above and Lemma 12 it follows that Theorem 3 is equivalent to the following.

**Lemma 13.** Every 2-edge-coloured \(K_{n,n}\) has a spanning bicoloured or monochromatic cycle, unless the colouring is split.

Now Theorem 4 follows immediately from Lemma 13 and the next lemma.

**Lemma 14.** If a 2-edge-coloured \(K_{n,n}\) has a spanning bicoloured cycle, then it has a partition into a monochromatic cycle and a monochromatic path, of distinct colours.

**Proof.** Among all spanning bicoloured cycles, we choose a cycle \(C\) having the maximum number of red edges. Say \(C = v_1v_2\ldots v_kv_1\), with \(v_1v_2\) red, and with turning points \(v_1, v_\ell\). If \(C\) is good, we are done, since the edge \(v_1v_\ell\) generates a monochromatic cycle with one of the two paths \(v_2v_3\ldots v_{\ell-1}\) or \(v_{\ell+1}v_{\ell+2}\ldots v_k\), while the other path is monochromatic in the other colour. So assume \(C\) is not good, that is, vertices \(v_1, v_\ell\) are in the same partition class. In particular, \(v_1v_\ell+1\) and \(v_\ell v_k\) are edges.

If the edge \(v_1v_{\ell+1}\) is red, then \(k > \ell + 1\) (as \(v_kv_1\) is blue) and the bicoloured cycle \((C - v_kv_1 - v_\ell v_{\ell+1}) \cup v_1v_{\ell+1} \cup v_\ell v_k\) has more red edges than \(C\). So \(v_1v_{\ell+1}\) is blue, and \(K_{n,n}\) decomposes into the red path \(v_2\ldots v_\ell\) and the blue cycle on the remaining vertices (the blue cycle may be just an edge, which is allowed).

Observe that one can easily modify the last lines of the proof of Lemma 14, arguing that we either have a red path and a blue cycle or both edges \(v_1v_{\ell+1}\), \(v_\ell v_k\) are blue, giving a spanning bicoloured cycle that is not good and has more red edges than \(C\). So, if at the beginning \(C\) is chosen only among the bicoloured spanning cycles that are not good (if such cycles exist), then we can choose the colour of the cycle.

**Corollary 15.** Let the edges of \(K_{n,n}\) be coloured with red and blue. If there is a spanning bicoloured cycle that is not good, then there is a partition of \(K_{n,n}\) into a red path and a blue cycle.
4.2 Alternative proof of Theorem 3

We give an alternative proof of Theorem 3 by proving Lemma 13. We start by excluding a certain structure outside any longest good cycle. Call a $C_4$ balanced if it has exactly two edges of each colour.

Lemma 16. If $c$ is a 2-edge-colouring of $E(K_{n,n})$ and $C = v_1 \ldots v_k v_1$ is a longest good cycle for $c$, then the restriction $c'$ of $c$ to $K_{n,n} - V(C)$ has no balanced $C_4$.

Proof. Say $C$ has turning points $v_1 \in V_1, v_\ell \in V_2$, and $v_1 v_2$ is red. For contradiction, assume $c'$ has a balanced $C_4$. Then, after exchanging the roles of $V_1$ and $V_2$ if needed, there are $x_i, y_i \in V_i \setminus V(C)$, for $i = 1, 2$, with $x_1 x_2$ red, $y_1 x_2$ blue, and exactly one of $x_1 y_2, y_1 y_2$ red. We will split our proof accordingly. But first note that by colour symmetry we can assume $v_1 x_2$ is red. As the cycle $C - v_1 v_2$ is longer than $C$, it is not good. So $x_1 v_k$ is red, and similarly, $x_1 v_2$ is blue.

Case 1: $y_1 y_2$ is red and $x_1 y_2$ is blue. If $v_1 y_2$ is red, then as above, also $y_1 v_k$ is red. Moreover, $\ell < k$ (replace $v_1 v_2 = v_1 v_3$ with $v_1 x_1 y_2 v_1$ otherwise). Replace $v_{k-1} y_1 v_1$ with $v_{k-1} y_1 v_1 x_1 y_2 v_1$ to see that $y_1 v_{k-1}$ is blue. The good cycle $(C - v_k) \cup v_{k-1} y_1 x_1 y_2 v_1$ is longer than $C$, a contradiction. So $v_1 y_2$ is red. Recall that also $x_1 v_2$ is blue. If $y_1 v_\ell$ is red then so is $y_2 v_{\ell + 1}$ (in particular $\ell \neq k$), and the good cycle $(C - v_1 v_{\ell + 1} - v_k v_1) \cup v_2 y_1 y_2 v_{\ell + 1} \cup v_k x_1 x_2 v_1$ is longer than $C$, a contradiction. So $y_1 v_\ell$ is blue. By maximality of $C$ also $x_2 v_{\ell - 1}$ is blue (and thus $\ell \neq 2$), and $(C - v_1 v_2 - v_{\ell - 1} v_\ell) \cup v_1 y_2 x_1 v_2 \cup v_{\ell - 1} x_2 y_1 v_\ell$ is a longer good cycle, a contradiction.

Case 2: $x_1 y_2$ is red and $x_1 y_2$ is blue. If $y_1 v_\ell$ is blue, then by maximality of $C$, so is $y_2 v_{\ell - 1},$ and $(C - v_k v_1 - v_{\ell - 1} v_\ell) \cup v_k x_1 x_2 v_1 \cup v_k y_1 y_2 v_{\ell - 1}$ is a longer good cycle, so $y_1 v_\ell$ is red. Note that $\ell \neq k$, as otherwise $(C - v_1 v_2 - v_k y_1 x_2 v_1) \cup v_2 y_1 v_{\ell + 1} \cup v_k x_1 x_2 v_1$ is longer than $C$. Also, $v_{\ell + 1} y_2$ is red, as otherwise $(C - v_k v_1 - v_{\ell + 1} v_\ell) \cup v_k x_1 x_2 v_1 \cup v_k y_1 y_2 v_{\ell + 1}$ is a longer good cycle. Now, $v_k y_1$ is blue, as otherwise $(C - v_k v_1 - v_{\ell + 1} v_\ell) \cup v_k y_1 v_{\ell + 1} \cup v_{\ell + 1} x_2 y_1 v_{\ell + 1}$ is a longer good cycle. Consider $(C - v_k v_1 - v_{\ell + 1} v_\ell) \cup v_{\ell + 1} x_2 v_1 \cup v_{\ell - 1} y_1 v_{\ell + 1}$ to see that $v_{\ell - 1} y_2$ is blue. If $x_1 v_\ell$ is blue, add $v_1 x_1 x_2 v_1$ and $v_{\ell - 1} x_2 y_1 v_\ell$ to $C - v_1 - v_{\ell - 1} v_\ell$ for a longer good cycle, and if $x_1 v_\ell$ is red, replace $v_1 v_k$ and $v_{\ell + 1} v_\ell$ with $v_k y_1 x_2 v_1$ and $v_{\ell - 1} y_2 v_{\ell + 1}$ for a longer good cycle, a contradiction.

Now we determine which colourings avoid balanced $C_4$ s. Call an edge-coloring $c$ of $K_{n,n}$ near-monochromatic if there is a colour that appears on at most one edge.

Lemma 17. Any edge-colouring $c$ of $K_{n,n}$ having no balanced $C_4$ is near-monochromatic.

Proof. Assume $c$ is not monochromatic. Then some vertex $v$ sees both colours. Let $V_1, V_2$ be the partition classes of $K_{n,n}$, and use symmetry to assume that $v \in V_1$, and that $|R_v| \geq |B_v| \geq 1$, where $R_v := \{u \in V_2 : uv \text{ is red}\}$ and $B_v := V_2 \setminus R_v$.
If $|R_v| = 1$, then $|B_v| = 1$, and since $c$ has no balanced $C_4$, it is near-monochromatic.

So assume $|R_v| \geq 2$. Then, repeatedly using the fact that there is no balanced $C_4$, we see that each $x \in V_1 \setminus \{v\}$ sends at least one red edge to $R_v$, and thus only red edges to $B_v$, and therefore, only red edges to $R_v$. In particular $|B_v| = 1$ (otherwise $v, x$ and any two vertices from $B_v$ span a balanced $C_4$). Thus $c$ has exactly one blue edge (from $v$ to $B_v$), and is therefore near-monochromatic.

We are ready to prove Lemma 13.

Proof of Lemma 13. We can assume the colouring $c$ is neither a V-colouring nor monochromatic, as otherwise it is easy to find a spanning bicoloured or monochromatic cycle. By Lemma 11 $c$ has a good cycle, let $C_1$ be a largest such cycle.

For any good cycle $C$ in $c$, let $c(C)$ denote the restriction of $c$ to $K_{n,n} - V(C)$. By Lemmas 10 and 11 $c(C_1)$ is near-monochromatic, and thus has a monochromatic spanning path, say in red. Among all good cycles $C$ with the property that $c(C)$ has a red spanning path, choose $C_2$, maximising the number of blue edges. Say $C_2 = v_1 \ldots v_kv_1$ with $v_1v_2$ red and turning points $v_1$ and $v_k$, and let $P = x_1 \ldots x_h$ be a red spanning path in $c(C_2)$. As $v_1$ and $v_k$ are in distinct partition classes, one of them, say $v_1$, is in the same partition class as $x_1$. If both $x_1v_1, v_1x_h$ are blue, the cycle $v_1x_h \cup P \cup x_1v_1v_{h+1}\ldots v_kv_1$ contradicts our choice of $C_2$. So at least one of the two edges, say $x_1v_1$, is red. Then $(C_2 - v_tv_{t+1}) \cup v_1x_1Px_hv_{t+1}$ is a spanning bicolored cycle.

4.3 Three colours

We now turn to colourings of complete and complete bipartite graphs with three colours. Then, it is known that the edges of $K_n$ cannot always be partitioned into three paths of distinct colours, so one drops this condition. In any 3-edge-colouring of $K_n$ there is a partition into three monochromatic paths [18] and a partition of almost all vertices into three monochromatic cycles [11]. Pokrovskiy [18] found 3-colourings that do not admit a partition of all vertices into three monochromatic cycles, however, he conjectures [19] that three cycles and one vertex are always sufficient.

In 3-edge-colourings of the complete bipartite graph $K_{n,n}$, almost all vertices can be partitioned into five monochromatic cycles [14], while folklore 3-colour split colourings [2] show that a partition into fewer than five monochromatic cycles or paths is not always possible. It is conjectured that five is indeed the optimal number for

1A 3-colour split colouring is obtained from a proper edge colouring of $K_{3,3}$ by replacing each vertex with a set of vertices and each edge with a complete bipartite graph in the colour of the edge.
partitions into monochromatic paths [18]. For more details and variants with more colours, see [9].

In [18, 19], Pokrovskiy shows how the following two lemmas for 2-edge-colourings can be used to obtain partitions of 3-edge-coloured graphs.

**Lemma 18.** [3, 18] For any red-blue colouring of the edges of $K_n$ there is a partition of the vertices into a red path and a blue balanced complete bipartite graph.

**Lemma 19.** [19] For any red-blue colouring of the edges of $K_{n,n}$ there is a partition of the vertices into a red path and two blue balanced complete bipartite graphs.

We can use Pokrovskiy’s approach from [19] together with our Theorem 4 to obtain partitions of a 3-edge-coloured $K_n$ or $K_{n,n}$ into a mix of monochromatic cycles and paths. For $K_n$, we start with any 3-colouring of its edges, say in colours red, blue and green. We apply Lemma 18 with blue and green merged, which gives a red path and a balanced complete bipartite graph that only uses blue and green. To this graph we apply Theorem 4 to find a partition into a monochromatic path and a monochromatic cycle, unless the graph is split-coloured, in which case we easily find a partition into three monochromatic cycles. In total, we obtain a partition of $K_n$ into either two monochromatic paths and one monochromatic cycle, or one monochromatic path and three monochromatic cycles.

Applying Lemma 19 and Theorem 4 in the same way we obtain a partition of $K_{n,n}$ into either three monochromatic paths and two monochromatic cycles, or into two monochromatic paths and four monochromatic cycles, or into one monochromatic path and six monochromatic cycles. We can reduce the number of elements of the partition by analysing the edges between the two balanced blue-green complete bipartite graphs given by Lemma 19.

**Lemma 20.** For any 3-edge-colouring of $K_{n,n}$ there is a partition of the vertices into either three monochromatic paths and two monochromatic cycles or into two monochromatic paths and four monochromatic cycles.

**Proof.** Say colours are red, blue, green. Lemma 19 gives a red path and four sets $A_1, B_1 \subseteq V_1$ and $A_2, B_2 \subseteq V_2$ (where $V_1, V_2$ are the partition classes of $K_{n,n}$), with $|A_1| = |A_2| \leq |B_1| = |B_2|$, and no red edge between $A_1$ and $A_2$ or between $B_1$ and $B_2$. We can assume the colourings in green and blue on $A_1, A_2$ and $B_1, B_2$ are both split, as by Theorem 4 any non-split colouring allows for a partition into a monochromatic cycle and a monochromatic path, while a split colouring allows for a partition into three monochromatic cycles, giving a total outcome which is as desired for the lemma.
First assume there is a non-red edge \( e \) between \( A_j \) and \( B_{3-j} \) for \( j = 1 \) or \( j = 2 \). Say \( e \) is blue. Then we can find a blue path through \( e \) such that the remainder of each of \( A_1 \cup A_2 \) and \( B_1 \cup B_2 \) is a balanced complete bipartite graph with a \( V \)-colouring and can thus be partitioned into two monochromatic cycles. This gives a total of two paths and four cycles, which is as desired.

On the other hand, if all edges between \( A_1, B_2 \), and between \( B_1, A_2 \) are red, then there are two disjoint red cycles covering all of \( A_1 \cup A_2 \) and part of \( B_1 \cup B_2 \). Now either the remainder \( B \) of \( B_1 \cup B_2 \) is split, or we can apply Theorem 4. In both cases \( B \) partitions into at most two monochromatic cycles and a monochromatic path. Together with the red path and the two red cycles, we obtain the desired partition. 

References

[1] Allen, P., Covering two-edge-coloured complete graphs with two disjoint monochromatic cycles, *Combin. Probab. Comput.*, 17 (2008), 471–486.

[2] Ayel, J., Sur l’existence de deux cycles supplémentaires unicolores, disjoints et de couleurs différentes dans un graphe complet bicolore, *Doctoral dissertation, Université scientifique et médicale de Grenoble* (1979).

[3] Ben-Eliezer, I., Krivelevich, M. and Sudakov, B. The size Ramsey number of a directed path. *J. Combin. Theory, Ser. B*, 102 (2012), 743–755.

[4] Bessy, S. and Thomassé, S., Partitioning a graph into a cycle and an anticycle, a proof of Lehel’s conjecture, *J. Combin. Theory, Ser. B*, 100(2) (2010), 176–180.

[5] Bustamante, S., Hán, H. and Stein, M., Almost partitioning 2-colored complete 3-uniform hypergraphs into two monochromatic tight cycles. *J. Graph Theory*, 91(1):5–15, 2019.

[6] Bustamante, S. and Stein, M., Partitioning 2-coloured complete k-uniform hypergraphs into monochromatic \( t \)-cycles, *European J. of Combin.*., Volume 71, 2018, 213–221.

[7] Gerencsér, L. and Gyárfás, A., On Ramsey-type problems. *Annales Univ. Eötvös Section Math.*, 10 (1967), 167–170.

[8] Gyárfás, A. Vertex coverings by monochromatic paths and cycles, *J. Graph Theory*, 7 (1983), 131–135.
[9] Gyárfás, A., Vertex covers by monochromatic pieces - A survey of results and problems, *Discrete Math.*, **339**(7) (2016), 1970–1977.

[10] Gyárfás, A. and Lehel, J. A Ramsey-type problem in directed and bipartite graphs, *Period. Math. Hungar.*, **3** (1973), 299–304.

[11] Gyárfás, A., Ruszinkó, M., Sárközy, G. N. and Szemerédi, E. Partitioning 3-Colored Complete Graphs into Three Monochromatic Cycles. *Electron. J. Combin.*, **18**(1) (2011), P53.

[12] Gyárfás, A. and Sárközy, G., Monochromatic loose-cycle partitions in hypergraphs, *Electron. J. Combin.*, **21** (2014), P2.36.

[13] Gyárfás, A. and Sárközy, G., Monochromatic path and cycle partitions in hypergraphs, *Electron. J. Combin.*, **20**(1) (2013), P18.

[14] Lang, R. Schaudt, O. and Stein, M. Almost partitioning a 3-edge colored $K_{n,n}$ into 5 monochromatic cycles. *SIAM J. on Disc. Math.*, 31-2 (2017), 1374–1402.

[15] Lo, A. and Pfenninger, V., Towards Lehel’s conjecture for 4-uniform tight cycles. *Electron. J. Combin.*, **30**(1) (2023), P1.13.

[16] Lu, C., Wang, B. and Zhang, P., Monochromatic loose path partitions in k-uniform hypergraphs. *Discret. Math.*, **340** (2017), 2789–2791.

[17] Łuczak, T., Rödl, V. and Szemerédi, E., Partitioning two-coloured complete graphs into two monochromatic cycles, *Combin. Probab. Comput.*, **70**(4) (1998), 423–436.

[18] Pokrovskiy, A., Partitioning edge-coloured complete graphs into monochromatic cycles and paths, *J. Combin. Theory, Ser. B*, **106** (2014), 70–97.

[19] Pokrovskiy, A., Partitioning a graph into a cycle and a sparse graph, *Discret. Math.*, **346**(1) (2023), 113161.