SMOOTH APPROXIMATIONS FOR FRACTIONAL AND MULTIFRACTIONAL FIELDS

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Abstract. We construct absolute continuous stochastic processes that converge to anisotropic fractional and multifractional Brownian sheets in Besov-type spaces.

1. Introduction

Fractional Brownian motion (fBm) with Hurst parameter $H$ is a continuous centered Gaussian process with the covariance function

$$\mathbb{E}B_tB_s = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right), \quad t, s \geq 0.$$ 

It has stationary increments that exhibit a property of long-range dependence for $H > 1/2$, which makes fBm a popular and efficient model for long-range dependent processes in Internet traffic, stock markets, etc.

For numerous applications one needs a multi-parameter model for a long range dependence. For most of those applications, including image processing, geophysics, oceanography, etc, two parameters are enough.

There are different possibilities to define a two-parameter fractional Brownian motion, or fractional Brownian sheet (fBs). One is so-called isotropic fractional Brownian sheet with the covariance function

$$\mathbb{E}B_tB_s = \frac{1}{2} \left( \|t\|^{2H} + \|s\|^{2H} - \|t-s\|^{2H} \right), \quad t, s \in \mathbb{R}^2.$$ 

As the name suggests, its properties are the same in all directions, which is not the case in many applications, especially in image processing. For such applications a better model is anisotropic fractional Brownian sheet with the covariance function

$$\mathbb{E}B_tB_s = \frac{1}{4} \prod_{i=1,2} \left( t_i^{2H_i} + s_i^{2H_i} - |t_i-s_i|^{2H_i} \right), \quad t, s \in \mathbb{R}^2_+.$$ 

It also has stationary increments in the sense that the distribution of $(B_{t+s} - B_t)$ does not depend on $t$.

But the stationarity of increments of fractional Brownian process and sheet means that the behavior of them is the same at each point, and this substantially restricts the area of their application. In particular, they do not allow one to model situations, where the regularity at a point depends on the point, as well

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as the range of dependence. In view of this, recently several multifractional generalized of fBm and fBs were proposed in order to overcome these limitations such as moving average multifractional Brownian motion (mBm) [2], harmonisable mBm [2], Volterra-type mBm [8], Different multiparameter extensions of mBm were studied in [1] [3] [4].

In this paper, we work in a general setting by considering a Gaussian random field \( B(t) \) in the plane which is continuous almost surely and satisfies the following condition on its increments: for all \( s, t \in [0, T] \times [0, T] \)

\[
E(B(s_1, s_2) - B(s_1, t_2) - B(t_1, s_2) + B(t_1, t_2))^2 \leq C(|t_1 - s_1||t_2 - s_2|)^\lambda,
\]

where \( C > 0 \) and \( \lambda > 1 \) are some constants. This, in particular, includes anisotropic fractional and multifractional Brownian sheets.

Our main goal is to construct approximations for \( B(t) \) in a certain Besov, or fractional Sobolev, spaces, by absolutely continuous fields. This allows one to approximate stochastic integrals with respect to fractional Brownian sheet by usual integrals and consequently, to approximate solutions of stochastic partial differential equations involving fractional noise by solutions of partial differential equations with a random source, which in many aspects are similar to non-random partial differential equations.

The paper is organized as follows. In Section 2 we give necessary definitions concerning Besov spaces. In Section 3 the main result, Theorem 3.1 about the convergence of absolutely continuous approximations of the field \( B(t) \) in Besov space \( W^1 \) in probability, is proved. In Section 4 we study fractional and multifractional Brownian sheets. We show in particular that these fields satisfy the conditions of Theorem 3.1 and can be approximated by absolutely continuous random fields.

2. Definitions and notation

In this section, we define functional spaces which are similar to spaces of Hölder continuous functions. They play an important role in definition and analysis of stochastic integrals with respect to fractional random fields (see e.g. [5] Lemma 2.2.16).

Let \( s, t \in \mathbb{R}_2^2 \), \( s = (s_1, s_2) \), \( t = (t_1, t_2) \). Write \( s < t \) if \( s_1 < t_1 \) and \( s_2 < t_2 \). For \( s < t \) denote \([s, t] = [s_1, t_1] \times [s_2, t_2] \subset \mathbb{R}_2^2 \). Let \( T = (T_1, T_2) \in (0, \infty)^2 \), \( [0, T] = [0, T_1] \times [0, T_2] \). For a function \( f : \mathbb{R}_+^2 \rightarrow \mathbb{R} \) we consider two-parameter increments

\[
\Delta_s f(t) := f(t) - f(s_1, t_2) - f(t_1, s_2) + f(s), \quad s, t \in \mathbb{R}_2^2.
\]

Consider a function \( f : [0, T] \rightarrow \mathbb{R} \). For \( \beta \in (0, 1)^2 \) and \( t \in [0, T] \) we denote

\[
\varphi_1^\beta(f)(t) = \int_0^{t_1} \frac{|f(t) - f(s_1, t_2)|}{(t_1 - s_1)^{\beta_1 + 1}} ds_1,
\]

\[
\varphi_2^\beta(f)(t) = \int_0^{t_2} \frac{|f(t) - f(t_1, s_2)|}{(t_2 - s_2)^{\beta_2 + 1}} ds_2,
\]

\[
\varphi_3^{\beta_1, \beta_2}(f)(t) = \int_{[0, t]} \frac{|\Delta_s f(t)|}{(t_1 - s_1)^{\beta_1 + 1}(t_2 - s_2)^{\beta_2 + 1}} ds,
\]

\[
\varphi_f^{\beta_1, \beta_2}(t) = |f(t)| + \varphi_1^{\beta_1}(f)(t) + \varphi_2^{\beta_2}(f)(t) + \varphi_3^{\beta_1, \beta_2}(f)(t).
\]
Let \( W_{0}^{\beta_{1}, \beta_{2}} = W_{0}^{\beta_{1}, \beta_{2}}([0, T]) \) be a space of measurable functions \( f : [0, T] \to \mathbb{R} \), with
\[
\|f\|_{0, \beta_{1}, \beta_{2}} = \sup_{t \in [0, T]} \|f\|_{\beta_{1}, \beta_{2}}(t) < \infty,
\]
Define also \( W_{1}^{\beta_{1}, \beta_{2}} = W_{1}^{\beta_{1}, \beta_{2}}([0, T]) \) as a space of measurable functions \( f : [0, T] \to \mathbb{R} \), with
\[
\|f\|_{1, \beta_{1}, \beta_{2}} = \sup_{0 \leq s < t \leq T} \left( \frac{|\Delta_{s}f(t)|}{(t - s)^{\beta_{1}}(t - s - 2)^{\beta_{2}}} + \frac{1}{(t - s - 2)^{\beta_{2}}} \int_{s}^{t} \frac{|f_{t-}(u, s_{2}) - f_{t-}(s)|}{(u - s_{1})^{1+\beta_{1}}} du + \frac{1}{(t_{1} - s_{1})^{\beta_{1}}} \int_{s_{2}}^{t_{2}} \frac{|f_{t-}(s_{1}, v) - f_{t-}(s)|}{(v - s_{2})^{1+\beta_{2}}} dv + \int_{[s, t]} \frac{|\Delta_{r}f(r)|}{(r_{1} - s_{1})^{1+\beta_{1}}(r_{2} - s_{2})^{1+\beta_{2}}} dr \right) < \infty,
\]
where \( f_{t-}(s) := f(s) - f(s_{1}, t_{2}) - f(t_{1} - s_{1}, t_{2}) + f(t) \).

3. MAIN RESULT

Let \( \{B_{t}, t \in [0, T]\} \) be a random field which satisfies the following conditions

1. \( B_{t} \) is a Gaussian field;
2. there exists constants \( C > 0 \) and \( \lambda > 1 \) such that for all \( s, t \in [0, T] \)
   \[
   \mathbb{E}(\Delta_{s}B_{t})^{2} \leq C(|t_{1} - s_{1}| |t_{2} - s_{2}|)^{\lambda};
   \]
3. the trajectories of \( B_{t} \) are continuous with probability one.

One example of such field is anisotropic fractional Brownian sheet (see Introduction).

We consider the approximation
\[
B_{t}^{\varepsilon} = \frac{1}{\varepsilon^{2}} \int_{t_{1}}^{t_{1}+\varepsilon} \int_{t_{2}}^{t_{2}+\varepsilon} B_{s} \, ds = \frac{1}{\varepsilon^{2}} \int_{[0,\varepsilon]^{2}} B_{u+t} \, du.
\]

**Theorem 3.1.** For all \( \beta_{1}, \beta_{2} \in (0, \lambda/2) \)
\[
\|B^{\varepsilon} - B\|_{1, \beta_{1}, \beta_{2}} \overset{p}{\to} 0, \quad \varepsilon \to 0 + .
\]

**Proof.** Denote
\[
\Delta B_{t}^{\varepsilon} := B_{t}^{\varepsilon} - B_{t} = \frac{1}{\varepsilon^{2}} \int_{[0,\varepsilon]^{2}} (B_{u+t} - B_{t}) \, du.
\]

Then
\[
\Delta_{s}(\Delta B_{t}^{\varepsilon}) = \Delta B_{s_{1}, t_{2}}^{\varepsilon} - \Delta B_{s_{1}, t_{2}}^{\varepsilon} - \Delta B_{t_{1}, s_{2}}^{\varepsilon} + \Delta B_{t_{1}, s_{2}}^{\varepsilon} = \frac{1}{\varepsilon^{2}} \int_{[0,\varepsilon]^{2}} (B_{u+t} - B_{t}) - B_{u_{1}+s_{1}, u_{2}+t_{2}} + B_{s_{1}, t_{2}} - B_{u_{1}+t_{1}, u_{2}+s_{2}} + B_{t_{1}, s_{2}} + B_{u+s} - B_{s}) \, du.
\]

According to the Cauchy–Schwarz inequality,
\[
\mathbb{E}(\Delta_{s}(\Delta B_{t}^{\varepsilon}))^{2} \leq \frac{1}{\varepsilon^{2}} \int_{[0,\varepsilon]^{2}} \mathbb{E}(B_{u+t} - B_{t})^{2} - B_{u_{1}+s_{1}, u_{2}+t_{2}} + B_{s_{1}, t_{2}}
- B_{u_{1}+t_{1}, u_{2}+s_{2}} + B_{t_{1}, s_{2}} + B_{u+s} - B_{s})^{2} \, du.
\]
Considering (1), we obtain
\[ E(\Delta s(\Delta B_1^T))^2 \leq \frac{2}{\varepsilon^2} \int_{[0,\varepsilon]^2} \left( E(\Delta_{u+s} B_{u+t})^2 + E(\Delta_s B_t)^2 \right) du \]
\[ \leq \frac{2}{\varepsilon^2} \int_{[0,\varepsilon]^2} 2C(|t_1 - s_1| |t_2 - s_2|)^\lambda du = 4C(|t_1 - s_1| |t_2 - s_2|)^\lambda. \]

On the other hand, (2) implies
\[ E(\Delta s(\Delta B_2^T))^2 \leq \frac{4}{\varepsilon^2} \int_{[0,\varepsilon]^2} \left( E(\Delta_{u+s_1,t,t} B_{u+t})^2 + E(\Delta_{t,t} B_{u+t})^2 \right) du \]
\[ + E(\Delta_{u_1,s_2} B_{u_1+t_1,u_2+s_2})^2 + E(\Delta_{s} B_{u_1+s_1,t_2})^2 \right) du \]
\[ \leq \frac{8C}{\varepsilon^2} \int_{[0,\varepsilon]^2} \left( (|t_1 - s_1|^{\lambda} u_2^{\lambda} + u_1^{\lambda} |t_2 - s_2|^{\lambda} \right) du \]
\[ = \frac{8C}{\lambda + 1} \left( (|t_1 - s_1|^{\lambda} + |t_2 - s_2|^{\lambda} \right) \varepsilon^{\lambda}, \]

because, in view of (1),
\[ E(\Delta_{u_1,s_1} B_{u_1+t_1})^2 \leq C |t_1 - s_1|^{\lambda} u_2^{\lambda}, \]
\[ E(\Delta_{t,t} B_{u+t})^2 \leq C u_1^{\lambda} |t_2 - s_2|^{\lambda}, \]
\[ E(\Delta_{u_1,s_2} B_{u_1+t_1,u_2,s_2})^2 \leq C |t_1 - s_1|^{\lambda} u_2^{\lambda}, \]
\[ E(\Delta_{s} B_{u_1+s_1,t_2})^2 \leq C u_1^{\lambda} |t_2 - s_2|^{\lambda}. \]

Let \( \delta \in (0, \lambda - 2\max\{\beta_1, \beta_2\}) \). We study three cases.

Case 1: \(|t_1 - s_1| < \varepsilon\). Based on the estimate (3), we get
\[ E(\Delta s(\Delta B_1^T))^2 \leq 4C(|t_1 - s_1| |t_2 - s_2|)^\lambda < 4CT_2^\delta(|t_1 - s_1| |t_2 - s_2|)^{\lambda - \delta} \varepsilon^\delta. \]

Case 2: \(|t_1 - s_1| \geq \varepsilon, \ |t_2 - s_2| < \varepsilon\). Similarly to the Case 1, we obtain
\[ E(\Delta s(\Delta B_1^T))^2 \leq 4C(|t_1 - s_1| |t_2 - s_2|)^\lambda < 4CT_2^\delta(|t_1 - s_1| |t_2 - s_2|)^{\lambda - \delta} \varepsilon^\delta. \]

Case 3: \(|t_1 - s_1| \geq \varepsilon, \ |t_2 - s_2| \geq \varepsilon\). Based on the estimate (1), we get
\[ E(\Delta s(\Delta B_1^T))^2 \leq \frac{8C}{\lambda + 1} \left( (|t_1 - s_1|^{\lambda} + |t_2 - s_2|^{\lambda} \right) \varepsilon^{\lambda} \]
\[ < 4C (T_1^\delta + T_2^\delta) (|t_1 - s_1| |t_2 - s_2|)^{\lambda - \delta} \varepsilon^\delta. \]

Thus, in all 3 cases we have
\[ E(\Delta s(\Delta B_1^T))^2 < 4C (T_1^\delta + T_2^\delta) (|t_1 - s_1| |t_2 - s_2|)^{\lambda - \delta} \varepsilon^\delta. \]

Since \( \Delta s(\Delta B_1^T) \) has a normal distribution, then for \( p > 0 \) we have
\[ E |\Delta s(\Delta B_1^T)|^p \leq C_1 (|t_1 - s_1| |t_2 - s_2|)^{(\lambda - \delta)p/2} \varepsilon^p \]
\[ \leq C_2 (T_1^\delta + T_2^\delta)^{1/2}(|t_1 - s_1| |t_2 - s_2|)^{(\lambda - \delta)p/2} \varepsilon^p, \]
\[ \text{where } C_1 = C_1(p, T_1, T_2) = 2^{3/2} C_1^{1/2} (\varepsilon^{1/2} (T_1^\delta + T_2^\delta))^{1/2} (1/2), \]

According to the two-parameter Garlasch–Rodemich–Rumsey inequality (7 Theorem 2.1), for all \( p > 0, \alpha_1 > p^{-1}, \alpha_2 > p^{-1} \) there exists a constant \( C_2 = C_2(\alpha_1, \alpha_2, p) > 0 \) such that
\[ |\Delta s(\Delta B_1^T)|^p \leq C_2 |t_1 - s_1|^\alpha_1 p - 1 |t_2 - s_2|^\alpha_2 p - 1 \varepsilon, \]

(6)
where

$$\xi = \int_{[0,T]^2} \frac{|\Delta_s(\Delta B^\varepsilon_t)|^p}{|x_1 - y_1|^\alpha |x_2 - y_2|^\beta} \, dx \, dy.$$ 

We choose $0 < \theta < (\lambda - 2 \max \{\beta_1, \beta_2\} - \delta)/2$, $p = \frac{\alpha}{\theta}$, $\alpha_1 = \alpha_2 = \frac{\lambda - \delta - \theta}{2}$. Then, taking into account (5), we obtain

$$E \xi = \int_{[0,T]^2} \frac{E |\Delta_s(\Delta B^\varepsilon_t)|^p}{|x_1 - y_1|^\alpha |x_2 - y_2|^\beta} \, dx \, dy$$

$$\leq C_1 \varepsilon^{\delta p/2} \int_{[0,T]^2} \frac{|x_1 - y_1|^{1-p(1+\alpha_1)} |x_2 - y_2|^{1-p(1+\alpha_2)} \, dx \, dy}$$

$$= C_1 \varepsilon^{\delta p/2} \int_{[0,T]^2} \, dx \, dy = C_1 T_1^2 T_2^2 \varepsilon^{\delta p/2}.$$ 

Therefore, from (6) we get

$$E \sup_{s,t \in [0,T]} \frac{|\Delta_s(\Delta B^\varepsilon_t)|^p}{|t_1 - s_1| |t_2 - s_2|^{(\lambda - 2\theta - \delta)/2} \varepsilon^{\delta/2}} \leq C_3 \varepsilon^{\delta p/2},$$

where $C_3 = C_1 C_2 T_1^2 T_2^2$.

The last estimate implies that for any $\kappa \in (0, 1)$ there exists $c_\kappa$ such that probability of the event

$$A_\varepsilon := \left\{ \text{for all } s, t \in [0,T] : |\Delta_s(\Delta B^\varepsilon_t)| \leq c_\kappa (|t_1 - s_1| |t_2 - s_2|)^{(\lambda - 2\theta - \delta)/2} \varepsilon^{\delta/2} \right\}$$

is not less than $1 - \kappa$.

As we have chosen $0 < \delta < \lambda - 2 \max \{\beta_1, \beta_2\}$ and $0 < \theta < \frac{\lambda - \delta}{2} - \max \{\beta_1, \beta_2\}$, then $h_1 := \frac{\lambda - \delta}{2} - \beta_1 - \theta > 0$, $h_2 := \frac{\lambda - \delta}{2} - \beta_2 - \theta > 0$.

By the definition of the norm $\| \cdot \|_{1, \beta_1, \beta_2}$, we have

$$\|\Delta B^\varepsilon\|_{1, \beta_1, \beta_2} = \sup_{0 \leq s < t \leq T} \left( \frac{|\Delta_s(\Delta B^\varepsilon_t)|}{(t_1 - s_1)^{\beta_1} (t_2 - s_2)^{\beta_2}} \right)$$

$$+ \frac{1}{(t_2 - s_2)^{\beta_2}} \int_{s_1}^{t_1} \frac{|\Delta_s(\Delta B^\varepsilon_{u,t_2})|}{(u - s_1)^{1+\beta_1}} \, du + \frac{1}{(t_1 - s_1)^{\beta_1}} \int_{s_2}^{t_2} \frac{|\Delta_s(\Delta B^\varepsilon_{t_1,v})|}{(v - s_2)^{1+\beta_2}} \, dv$$

$$+ \int_{[s,t]} \frac{|\Delta_s(\Delta B^\varepsilon_{r,t})|}{(r_1 - s_1)^{1+\beta_1} (r_2 - s_2)^{1+\beta_2}} \, dr.$$
Therefore at the set $A_\varepsilon$

$$\|\Delta B_\varepsilon\|_{1,\beta_1,\beta_2} \leq \sup_{0 \leq s < t \leq T} \left( c_\kappa \varepsilon^{\delta/2}(t_1 - s_1)^{h_1}(t_2 - s_2)^{h_2} + c_\kappa \varepsilon^{\delta/2}(t_2 - s_2)^{h_2} \int_{s_1}^{t_1} (u - s_1)^{h_1-1} \, du \right.$$  

$$+ c_\kappa \varepsilon^{\delta/2}(t_1 - s_1)^{h_1} \int_{s_2}^{t_2} (v - s_2)^{h_2-1} \, dv$$  

$$+ c_\kappa \varepsilon^{\delta/2} \int_{[s,t]} (r_1 - s_1)^{h_1-1} (r_2 - s_2)^{h_2-1} \, dr)$$  

$$= \sup_{0 \leq s < t \leq T} \left( c_\kappa \varepsilon^{\delta/2}(t_1 - s_1)^{h_1}(t_2 - s_2)^{h_2} \left( 1 + \frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_1 h_2} \right) \right)$$  

$$\leq c_\kappa \varepsilon^{\delta/2} T_1 h_1^h T_2 h_2 \left( 1 + \frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_1 h_2} \right) \rightarrow 0, \ \varepsilon \rightarrow 0+.$$  

Then for any $a > 0$

$$\lim_{\varepsilon \rightarrow 0^+} P \left( \|\Delta B_\varepsilon\|_{1,\beta_1,\beta_2} \geq a \right) \leq \kappa,$$  

because for sufficiently small $\varepsilon$ one has $c_\kappa \varepsilon^{\delta/2} T_1 h_1^h T_2 h_2 \left( 1 + \frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_1 h_2} \right) < a$. Hence, when $\kappa \rightarrow 0+$ we have

$$\lim_{\varepsilon \rightarrow 0^+} P \left( \|\Delta B_\varepsilon\|_{1,\beta_1,\beta_2} \geq a \right) = 0. \quad \square$$

**Remark 3.2.** The convergence in probability in the last theorem may be not sufficient for some applications. For example, one may want to use this theorem to get an approximate solution to stochastic PDE with a fractional noise by solving a usual PDE with random force. She is not able of course, to solve for all $\omega$’s and takes some fixed $\omega$. She knows, of course, that there is a subsequence of solutions converging almost surely, but a priori it is not known which subsequence is it. So another subsequence (depending, say, on $\omega$) may converge to something different or there might be several such subsequences. To overcome this problem, we give below a proof that for $\varepsilon_n = 2^{-n}$ one has an almost sure convergence, and give moreover an estimate for the rate of convergence.

Let

$$a(\varepsilon) = \sup_{s,t \in [0,T]} \frac{|\Delta_s(\Delta B_\varepsilon)|^p}{|t_1 - s_1||t_2 - s_2|^p(\Lambda - 2\theta - \delta)/2},$$  

$$b(\varepsilon) = \varepsilon^{\delta p/2} \ln^{1+\gamma} \varepsilon, \ \gamma > 0.$$  

(7) implies that

$$Ea(\varepsilon) \leq C_3 \varepsilon^{\delta p/2}.$$  

Then

$$E \left[ \sum_{n \geq 1} \frac{a(\varepsilon_n)}{b(\varepsilon_n)} \right] = E \left[ \sum_{n \geq 1} \frac{a(2^{-n})}{b(2^{-n})} \right] \leq C \sum_{n \geq 1} \frac{1}{n^{1+\gamma}} \rightarrow 0, \ n \rightarrow \infty.$$
Therefore, 
\[
\frac{a(\varepsilon_n)}{b(\varepsilon_n)} \to 0, \quad n \to \infty, \quad \text{a.s.}
\]

Hence, 
\[
a(\varepsilon_n) \leq C(\omega)b(\varepsilon_n) \quad \text{a.s.}
\]

So for all \(s, t \in [0, T]\) and \(n \geq 1\)
\[
|\Delta_s(\Delta B^n_t)| \leq C^{1/p}(\omega)\varepsilon_n^{\delta/2} \ln^{(1+\gamma)/p} \varepsilon_n(t_1 - s_1)(t_2 - s_2)^{(\lambda - \gamma - \delta)/2} \quad \text{a.s.}
\]

Using the definition of the norm \(\| \cdot \|_{1, \beta_1, \beta_2}\), we obtain
\[
\|\Delta B^n_t\|_{1, \beta_1, \beta_2} \leq C^{1/p}(\omega)T^{\gamma_1}T^{\gamma_2} \left( 1 + \frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_1h_2} \right) 
\times \varepsilon_n^{\delta/2} \ln^{(1+\gamma)/p} \varepsilon_n \to 0, \quad n \to \infty, \quad \text{a.s.}
\]

4. Examples

4.1. Fractional Brownian sheet. Fractional Brownian fields in the plane can be defined in various ways. We consider the so-called anisotropic random fields that possess the fractional Brownian property coordinate-wise (see e.g. Section 1.20).

Definition 4.1. A random field \(\{B^n_t, t \in [0, T]\}\) is called a fractional Brownian sheet with Hurst index \(H = (H_1, H_2) \in (0, 1)^2\) if

1. \(B^n_t\) is a Gaussian field such that \(B^n_t = 0, \quad t \in \partial \mathbb{R}^2\);
2. \(EB^{nH} = 0, \quad EB^nB^n = \frac{1}{4} \prod_{i=1,2} \left( t_i^{2H_i} + s_i^{2H_i} - |t_i - s_i|^{2H_i} \right)\).

This field has a continuous modification. Its increments satisfy the equality
\[
E(\Delta B^n_{t})^2 = |t_1 - s_1|^{2H_1} |t_2 - s_2|^{2H_2}.
\]

Hence, for \(B^n_t\) the inequality (1) holds with \(\lambda = 2 \min \{H_1, H_2\}\). Therefore, according to Theorem 3.1 for \(H_i \in (1/2, 1)\) for all \(\beta_1, \beta_2 \in (0, H_1 \wedge H_2)\) one has a convergence of approximations
\[
\|B^n, \varepsilon - B^n\|_{1, \beta_1, \beta_2} \xrightarrow{p} 0, \quad \varepsilon \to 0+,
\]

where
\[
B^n_{t, \varepsilon} = \frac{1}{\varepsilon^2} \int_{t_1}^{t_1+\varepsilon} \int_{t_2}^{t_2+\varepsilon} B^n_s \, ds.
\]

4.2. Multifractional Brownian sheet. We consider a function
\[
H(t) = (H_1(t), H_2(t)) : [0, T] \to (1/2, 1)^2.
\]

Let \(\mu, \nu\) be constants such that
\[
\frac{1}{2} < \mu < \min_{t \in [0, T]} H_i(t) \leq \max_{t \in [0, T]} H_i(t) < \nu < 1.
\]

Assume that there exist positive constants \(c_1, c_2\) such that for all \(t, s \in [0, T]\)

1. \(H_i(t) - H_i(s) \leq c_1 (|t_1 - s_1| + |t_2 - s_2|)\),
2. \(\Delta_\varepsilon H_i(t) \leq c_2 (|t_1 - s_1| + |t_2 - s_2|)\).


As the following inequality holds $\delta > 0$, the constant $\alpha > 0$ can be chosen such that for all $s, t \in [a, b]$ the following inequality holds

$$E(Y_t - Y_s)^2 \leq C_1 \sum_{i=1}^{2} |t_i - s_i|^{2\mu} \leq 2C_1 ||t - s||^{2\mu}.$$  

As $Y_t$ is a Gaussian field, then for any $\alpha > 0$ there exists $C_2 > 0$ such that

$$E(Y_t - Y_s)^\alpha \leq C_2 ||t - s||^{\alpha\mu}.$$  

Taking $\alpha > 2/\mu$, we obtain that the field $Y_t$ is continuous on $[a, b]$ with probability one according to Kolmogorov theorem. \hfill $\square$

**Theorem 4.4.** There exists a constant $C > 0$ such that for all $s, t \in [0, T]$

$$E(\Delta_s Y_t)^2 \leq C(|t_1 - s_1||t_2 - s_2|)^{2\mu}$$

**Proof.** Denote $s' = (s_1, t_2)$, $t' = (t_1, s_2)$. Then

$$\Delta_s Y_t = Y_t - Y_{t'} + Y_s - Y_{s'} = A_1 + A_2 + A_3 + A_4,$$

where

$$A_1 = B_t^{H(t)} - B_{t'}^{H(t)} + B_s^{H(t)} - B_{s'}^{H(t)},$$

$$A_2 = B_s^{H(t)} - B_{s'}^{H(t)} + B_{s'}^{H(s')} - B_{s'}^{H(s')},$$

$$A_3 = B_{s'}^{H(t)} - B_{s'}^{H(t')} + B_s^{H(s)} - B_s^{H(s')},$$

$$A_4 = B_{t'}^{H(t)} - B_{t'}^{H(t')} - B_s^{H(t)} + B_s^{H(t')}.$$  

Hence,

$$E(\Delta_s Y_t)^2 \leq 4(EA_1^2 + EA_2^2 + EA_3^2 + EA_4^2).$$  

We estimate each of 4 terms. As the random field $B_t^{H(t)}$ is a fractional Brownian motion when $H(t) = \text{const}$, then

$$EA_1^2 \leq C_3 |t_1 - s_1|^{2H(t)}|t_2 - s_2|^{2H(t)} \leq C_4 (|t_1 - s_1||t_2 - s_2|)^{2\mu}.$$  

Consider $A_2$.

$$A_2 = A_{21} + A_{22},$$
where
\[
A_{21} = B_s^{H(t)} - B_s^{H(t)} + B_s^{H_1(s'), H_2(t)} - B_s^{H_1(s'), H_2(t)}, \\
A_{22} = B_s^{H_1(s'), H_2(t)} - B_s^{H_1(s'), H_2(t)} + B_s^{H(s') - B_s^{H(s')} \}
\]

\[
E A_{21}^2 = E \left( B_s^{H(t)} - B_s^{H(t)} + B_s^{H_1(s'), H_2(t)} - B_s^{H_1(s'), H_2(t)} \right)^2
\]
\[
= \int_{R^2} \left( (s_1 - u_1)_{+}^{H_1(t)-\frac{1}{2}} - (u_1)_{+}^{H_1(t)-\frac{1}{2}} \right) \left( t_2 - u_2 \right)_{+}^{H_2(t)-\frac{1}{2}} - (u_2)_{+}^{H_2(t)-\frac{1}{2}} \right)^2 du
\]
\[
= \int_{R} \left( (s_1 - u_1)_{+}^{H_1(t)-\frac{1}{2}} - (u_1)_{+}^{H_1(t)-\frac{1}{2}} \right)^2 du
\]
\[
= \int_{R} \left( (t_2 - u_2)_{+}^{H_2(t)-\frac{1}{2}} - (u_2)_{+}^{H_2(t)-\frac{1}{2}} \right)^2 du
\]

By Lemma 5.2
\[
E A_{21}^2 \leq K_2(H_1(t) - H_1(s'))^2 \cdot K_1 |t_2 - s_2|^{2H(t)}
\]

Taking into account Condition (H1), we get
\[
E A_{21}^2 \leq C_5(|t_1 - s_1| |t_2 - s_2|)^{2\mu}
\]

Now we estimate A_{22}.

\[
E A_{22}^2 = E \left( B_s^{H_1(s'), H_2(t)} - B_s^{H_1(s'), H_2(t)} + B_s^{H(s') - B_s^{H(s')} \}
\]
\[
= \int_{R} \left( (s_1 - u_1)_{+}^{H_1(s')-\frac{1}{2}} - (u_1)_{+}^{H_1(s')-\frac{1}{2}} \right)^2 du
\]
\[
\times \int_{R} \left( (t_2 - u_2)_{+}^{H_2(t)-\frac{1}{2}} - (u_2)_{+}^{H_2(t)-\frac{1}{2}} \right)^2 du
\]

By Lemma 5.2
\[
E A_{22}^2 \leq K_1 s_1^{2H_1(s')} K_3(t_2 - s_2)^{2\mu} (H_2(t) - H_2(s')^2)
\]

Considering the Condition (H1), we obtain
\[
E A_{22}^2 \leq C_6(|t_1 - s_1| |t_2 - s_2|)^{2\mu}
\]

(11) and (12) imply that
\[
E A_{2}^2 \leq C_7(|t_1 - s_1| |t_2 - s_2|)^{2\mu}
\]
Further, by Lemma 5.3
\[ E A_2^2 \leq L \left( \left[ (H_1(t) - H_1(t') + H_1(s) - H_1(s'))^2 + (H_2(t) - H_2(t'))^2 
\right. 
\left. + (H_2(s) - H_2(s'))^2 + ((H_1(t) - H_1(t'))^2 + (H_2(t) - H_2(t'))^2 
\right. 
\left. + (H_1(s) - H_1(s'))^2 + (H_2(s) - H_2(s'))^2 \right) \right], \]
and taking into account the conditions (1) and (2), we get
\[ E A_2^2 \leq C_8(\mid t_1 - s_1 \mid \mid t_2 - s_2 \mid)^{2\mu}. \]

The estimation of \( A_4 \) is analogous to that of \( A_2 \). We have
\[ E A_4^2 \leq C_9(\mid t_1 - s_1 \mid \mid t_2 - s_2 \mid)^{2\mu}. \]
Combining (9), (10), (13), (15), we get (8).
\[ \Box \]

Theorems 4.3 and 4.4 imply that multifractional Brownian sheet \( B_t^{H(t)} \) satisfies the conditions of Theorem 3.1. Therefore,

**Corollary 4.5.** For any \( \beta_1, \beta_2 \in (0, \mu) \)
\[ \left\| B_t^{H(t), \varepsilon} - B_t^{H(t)} \right\|_{1, \beta_1, \beta_2} \overset{p}{\to} 0, \quad \varepsilon \to 0+, \]
where
\[ B_t^{H(t), \varepsilon} = \frac{1}{\varepsilon^2} \int_{t_1}^{t_1+\varepsilon} \int_{t_2}^{t_2+\varepsilon} B_s^{H(s)} \, ds. \]

5. Appendix

In this section we prove some technical lemmas that have been used in the proof of Theorem 4.3.

5.1. **Bounds for fractional Brownian motion.** Let \((\Omega, \mathcal{F}, P)\) be a complete probability space.

**Definition 5.1.** Fractional Brownian motion with Hurst index \( H \in (0, 1) \) is a centered Gaussian process \( \tilde{Z}^H = \{ \tilde{Z}_t^H, t \geq 0 \} \) with stationary increments and the covariance function
\[ E \left( \tilde{Z}_t^H \tilde{Z}_s^H \right) = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right). \]

It is well-known that fractional Brownian motion has a continuous modification and can be represented in the following form.
\[ \tilde{Z}_t^H = C_H \int_{\mathbb{R}} \left[ (t-u)^{H-\frac{3}{2}} - (-u)^{H-\frac{3}{2}} \right] dW_u, \]
where \( W \) is a Wiener process, \( C_H = \frac{16}{(2^H \sin \pi H \Gamma(2H))^{1/2}} \) \( (\text{see [5] Chapter 1.3}) \).

Let \( \frac{1}{2} < \mu < H_{\min} \leq H_{\max} < \nu < 1 \). Consider a family of random variables
\[ Z_t^H = \int_{\mathbb{R}} \left[ (t-u)^{H-\frac{3}{2}} - (-u)^{H-\frac{3}{2}} \right] dW_u = C_H^{-1} \tilde{Z}_t^H, \]
\( t \in [0, T], H \in [\mu, \nu] \).

**Lemma 5.2.** There exist positive constants \( K_1, K_2, K_3 \) such that
(1) for all \( t_1, t_2 \in [0, T] \), \( H \in [H_{\min}, H_{\max}] \)

\[
E \left( Z_{t_1}^H - Z_{t_2}^H \right)^2 \leq K_1 |t_1 - t_2|^{2H}; 
\]

(2) for all \( t \in [0, T] \), \( H_1, H_2 \in [H_{\min}, H_{\max}] \)

\[
E \left( Z_t^{H_1} - Z_t^{H_2} \right)^2 \leq K_2 (H_1 - H_2)^2. 
\]

(3) for all \( t_1, t_2 \in [0, T] \), \( H_1, H_2 \in [H_{\min}, H_{\max}] \)

\[
E \left( Z_{t_1}^{H_1} - Z_{t_2}^{H_1} - Z_{t_1}^{H_2} + Z_{t_2}^{H_2} \right)^2 \leq K_3 (t_1 - t_2)^{2\mu} (H_1 - H_2)^2. 
\]

Proof. (i) By the definition,

\[
E \left( Z_{t_1}^H - Z_{t_2}^H \right)^2 = C_H^{-2} E \left( \tilde{Z}_{t_1}^H - \tilde{Z}_{t_2}^H \right)^2 = C_H^{-2} |t_1 - t_2|^{2H},
\]

which entails (16) because \( C_H^{-2} \) is bounded when \( H \in [\mu, \nu] \).

(ii) The inequality (17) is a corollary of (18).

(iii) We prove (18). Let for definiteness, \( t_2 \leq t_1 \), \( H_1 \leq H_1 \). We can write

\[
E \left( Z_{t_1}^{H_1} - Z_{t_2}^{H_1} - Z_{t_1}^{H_2} + Z_{t_2}^{H_2} \right)^2 \\
= \int_{\mathbb{R}} \left[ (t_1 - u)^{H_1 - \frac{1}{2}} - (t_2 - u)^{H_1 - \frac{1}{2}} - (t_1 - u)^{H_2 - \frac{1}{2}} + (t_2 - u)^{H_2 - \frac{1}{2}} \right]^2 du \\
= I_1 + I_2,
\]

where

\[
I_1 = \int_{-\infty}^{t_2} \left[ (t_1 - u)^{H_1 - \frac{1}{2}} - (t_2 - u)^{H_1 - \frac{1}{2}} - (t_1 - u)^{H_2 - \frac{1}{2}} + (t_2 - u)^{H_2 - \frac{1}{2}} \right]^2 du,
\]

\[
I_2 = \int_{t_2}^{t_1} \left[ (t_1 - u)^{H_1 - \frac{1}{2}} - (t_1 - u)^{H_2 - \frac{1}{2}} \right]^2 du.
\]

By the theorem on finite increments, there exists \( h \in [H_2, H_1] \) such that

\[
I_1 = (H_1 - H_2)^2 \int_{-\infty}^{t_2} \left[ (t_1 - u)^{h - \frac{1}{2}} \ln(t_1 - u) - (t_2 - u)^{h - \frac{1}{2}} \ln(t_2 - u) \right]^2 du \\
= (H_1 - H_2)^2 \int_{-\infty}^{t_2} \left[ \int_{t_2}^{t_1} (v - u)^{h - \frac{1}{2}} \left( 1 + (h - \frac{1}{2}) \ln(v - u) \right) dv \right]^2 du.
\]

First we prove that for any \( h \in [\mu, \nu] \)

\[
\int_{-\infty}^{t_2} \left[ \int_{t_2}^{t_1} (v - u)^{h - \frac{3}{2}} dv \right]^2 du \leq C_3 (t_1 - t_2)^{2\mu},
\]
where $C_3 > 0$ is a constant. Actually

$$
\int_{-\infty}^{t_2} \left[ \int_{t_2}^{t_1} (v-u)^{h-3/2} dv \right]^2 du \\
= \int_{-\infty}^{2t_2-t_1} \left[ \int_{t_2}^{t_1} (v-u)^{h-3/2} dv \right]^2 du + \int_{2t_2-t_1}^{t_2} \left[ \int_{t_2}^{t_1} (v-u)^{h-3/2} dv \right]^2 du \\
\leq \int_{-\infty}^{2t_2-t_1} \left[ \int_{t_2}^{t_1} (t_2-u)^{h-3/2} dv \right]^2 du + \int_{2t_2-t_1}^{t_2} \left[ \int_{t_2}^{t_1} (v-t_2)^{h-3/2} dv \right]^2 du \\
= (t_1 - t_2)^{2h} ((2-2h)^{-1} + (h-1/2)^{-2}) \\
= \left( \frac{t_1 - t_2}{T+1} \right)^{2h} (T+1)^{2h} ((2-2h)^{-1} + (h-1/2)^{-2}) \\
\leq (t_1 - t_2)^{2\mu} (T+1) ((2-2\nu)^{-1} + (\mu-1/2)^{-2}).
$$

It is obvious that for all $\varepsilon, \delta > 0$ there exist positive constants $C_\varepsilon$ and $\bar{C}_\delta$ such that

$$
\begin{align*}
|\ln x| &\leq C_\varepsilon x^{-\varepsilon} & \text{if } 0 < x \leq 1, \\
|\ln x| &\leq \bar{C}_\delta x^\delta & \text{if } x \geq 1.
\end{align*}
$$

First we show that

$$
\int_{-\infty}^{t_2-1} \left[ \int_{t_2}^{t_1} (v-u)^{h-3/2} (1 + (h-1/2) \ln(v-u)) dv \right]^2 du \leq C_4(t_1 - t_2)^{2\mu}.
$$

Applying the inequality (22) with $\delta := (\nu - H_{\text{max}})/2$, we obtain

$$
\begin{align*}
&\int_{-\infty}^{t_2-1} \left[ \int_{t_2}^{t_1} (v-u)^{h-3/2} (1 + (h-1/2) \ln(v-u)) dv \right]^2 du \\
&\leq 2 \int_{-\infty}^{t_2-1} \left[ \int_{t_2}^{t_1} (v-u)^{h-3/2} dv \right]^2 du \\
&\quad + 2(\nu - 1/2)^2 \bar{C}_\delta^2 \int_{-\infty}^{t_2-1} \left[ \int_{t_2}^{t_1} (v-u)^{h-3/2} dv \right]^2 du.
\end{align*}
$$

Using (20), we get (23).

Secondly we prove that

$$
\int_{t_2-1}^{t_2} \left[ \int_{t_2}^{t_1} (v-u)^{h-3/2} (1 + (h-1/2) \ln(v-u)) dv \right]^2 du \leq C_5(t_1 - t_2)^{2\mu}.
$$

Choose $\varepsilon := (\mu - H_{\text{min}})/2$. Then (21) implies that if $u \in [t_2-1, t_2]$, $v \in [t_2, t_1]$ then

$$
|\ln(v-u)| = \left| \ln(T+1) + \ln \left( \frac{v-u}{T+1} \right) \right| \leq \ln(T+1) + C_\varepsilon \left( \frac{v-u}{T+1} \right)^{-\varepsilon}. $$
Therefore

\[
\int_{t_2-1}^{t_2} \left( \int_{t_2}^{t_1} (v-u)^{h-3/2} \left( 1 + \frac{h-1}{2} \ln(v-u) \right) dv \right)^2 du 
\leq 2 \left( 1 + \frac{\nu-1}{2} \ln(T+1) \right)^2 \int_{t_2-1}^{t_2} \left( \int_{t_2}^{t_1} (v-u)^{h-3/2} dv \right)^2 du 
+ 2 \left( \nu-1/2 \right)^2 C^2 \epsilon (T+1)^2 \int_{t_2-1}^{t_2} \left( \int_{t_2}^{t_1} (v-u)^{h-3/2-\epsilon} dv \right)^2 du.
\]

Considering (20), we obtain (24).

Combining (19), (23), (24), we get

\[
I_1 \leq C_6 (t_1-t_2)^2 \mu \left( H_2 - H_1 \right)^2.
\]

It remains to estimate \( I_2 \). By the theorem on finite increments, there exists \( h \in [H_2, H_1] \) such that

\[
I_2 = (H_1-H_2)^2 \int_{t_2}^{t_1} \left( (t_1-u)^{h-1/2} \ln(t_1-u) \right)^2 du.
\]

Choose \( \epsilon : = (\mu - H_{\min})/2 \). Then (21) implies that if \( u \in [t_2, t_1] \) then

\[
|\ln(t_1-u)| = \left| \ln(T+1) + \ln\frac{t_1-u}{T+1} \right| \leq \ln(T+1) + C_\epsilon \left( \frac{t_1-u}{T+1} \right)^{-\epsilon},
\]

which entails

\[
I_2 \leq 2(H_1-H_2)^2 \left( \frac{\ln^2(T+1)}{2h} (t_1-t_2)^{2h} + \frac{C^2(T+1)^2}{2(h-\epsilon)} (t_1-t_2)^{2(h-\epsilon)} \right) 
\leq C_7 (t_1-t_2)^{2\mu} (H_1-H_2)^2.
\]

Now the proof is complete. \( \square \)

5.2. Bounds for integrals. Let \( \frac{1}{2} < \mu < H_{\min} \leq H_{\max} < \nu < 1 \).

Lemma 5.3. Let

\[
f(t, u, h) = (t-u)^{h-1/2} - (-u)^{h-1/2}, \quad t \in [0, T], u \in \mathbb{R}, h \in [H_{\min}, H_{\max}].
\]

Then for all \( t \in [0, T], h \in [H_{\min}, H_{\max}] \)

\[
(25) \quad \int_{\mathbb{R}} |f(t, u, h)|^2 du < +\infty,
\]

\[
(26) \quad \int_{\mathbb{R}} |f'_h(t, u, h)|^2 du < +\infty,
\]

\[
(27) \quad \int_{\mathbb{R}} |f''_h(t, u, h)|^2 du < +\infty.
\]
Proof. We prove (27). Inequalities (25) and (26) are proved in a similar way.

\[
\int_{\mathbb{R}} f_{h}(t, u, h) \, du = \int_{-\infty}^{0} \left[ (t - u)^{-\frac{3}{2}} \ln^2(t - u) - (-u)^{-\frac{3}{2}} \ln^2(-u) \right] \, du \\
+ \int_{0}^{t} (t - u)^{2h-1} \ln^4(t - u) \, du \\
= \int_{-\infty}^{0} \left[ \int_{0}^{t} (v - u)^{-\frac{3}{2}} \left( \left( h - \frac{1}{2} \right) \ln^2(v - u) + 2 \ln(v - u) \right) \, dv \right]^2 \, du \\
+ \int_{0}^{t} (t - u)^{2h-1} \ln^4(t - u) \, du =: I_1 + I_2 + I_3,
\]

where

\[
I_1 = \int_{-\infty}^{-1} \left[ \int_{0}^{t} (v - u)^{-\frac{3}{2}} \left( \left( h - \frac{1}{2} \right) \ln^2(v - u) + 2 \ln(v - u) \right) \, dv \right]^2 \, du,
\]

\[
I_2 = \int_{-1}^{0} \left[ \int_{0}^{t} (v - u)^{-\frac{3}{2}} \left( \left( h - \frac{1}{2} \right) \ln^2(v - u) + 2 \ln(v - u) \right) \, dv \right]^2 \, du,
\]

\[
I_3 = \int_{0}^{t} (t - u)^{2h-1} \ln^4(t - u) \, du.
\]

We study each of three terms.

1. Applying the inequality (22), we obtain

\[
I_1 \leq 2 \left( h - \frac{1}{2} \right)^2 \tilde{C}_{\delta} \int_{-\infty}^{-1} \left[ \int_{0}^{t} (v - u)^{-\frac{3}{2}} \ln^2(v - u) + 2 \ln(v - u) \, dv \right]^2 \, du \\
+ 8 \tilde{C}_{\delta} \int_{-\infty}^{t} \left[ \int_{0}^{t} (v - u)^{-\frac{3}{2}} \ln^2(v - u) + 2 \ln(v - u) \, dv \right] \, du \leq Ct^{2\mu} < \infty
\]

(the last estimate follows from the inequality (20)).

2. Consider \( I_2 \).

\[
I_2 = \int_{-1}^{0} \left[ \int_{0}^{t} (v - u)^{-\frac{3}{2}} \left( \left( h - \frac{1}{2} \right) \ln^2(v - u) + 2 \ln(v - u) \right) \, dv \right]^2 \, du \\
+ \left( h - \frac{1}{2} \right) \ln^2(T + 1) + 2 \ln(T + 1) \right) \, dv \right]^2 \, du.
\]

Using inequality (21), we get

\[
I_2 \leq \int_{-1}^{0} \left[ \int_{0}^{t} (v - u)^{-\frac{3}{2}} \left( C_{\varepsilon}^2 \left( h - \frac{1}{2} \right) \left( T + 1 \right)^{2\varepsilon} (v - u)^{-2\varepsilon} \\
+ 2C_{\varepsilon}(T + 1)^{\varepsilon} (v - u)^{-\varepsilon} + \left( h - \frac{1}{2} \right) \ln^2(T + 1) \\
+ 2 \ln(T + 1) \right) \, dv \right]^2 \, du \leq Ct^{2\mu} < \infty,
\]

where the last estimate follows from the inequality (20).
3. Consider $I_3$.
\[
I_3 = \int_0^t v^{2h-1} \ln^4 v \, dv \leq \int_0^1 v^{2h-1} \ln^4 v \, dv + \int_1^{1/v} v^{2h-1} \ln^4 v \, dv \\
\leq \int_0^1 v^{2h-1} \ln^4 v \, dv + \int_1^{1/v} v^{2h-1} \ln^4 v \, dv < \infty.
\]
Thus, inequality (27) holds.

5.3. Bounds for fractional Brownian sheet. Let $\frac{1}{4} < \mu < H_{\min} \leq H_{\max} < \nu < 1$. We consider a family of random variables
\[
B_t^{H,H'} := \int_{\mathbb{R}^2} \left((t_1 - u_1)^{H-1/2} - (-u_1)^{H-1/2}\right) \\
\times \left((t_2 - u_2)^{H'-1/2} - (-u_2)^{H'-1/2}\right) \, dW_u,
\]
t $\in [0,T]$, $H, H' \in [\mu, \nu]$, $i = 1, 2, 3, 4$, where $W = \{W_s, s \in \mathbb{R}^2\}$ is a Wiener field.

**Lemma 5.4.** There exists a constant $L > 0$ such that for all $t \in [0,T]$, $H_i, H'_i \in [H_{\min}, H_{\max}]$, $i = 1, 2, 3, 4$, the following inequality holds
\[
(28) \quad \mathbb{E} \left( B_t^{H_1,H'_1} - B_t^{H_2,H'_2} + B_t^{H_3,H'_3} - B_t^{H_4,H'_4} \right)^2 \\
\leq L \left( \left|H_1 - H_2 + H_3 - H_4\right|^2 + \left|H'_1 - H'_2 + H'_3 - H'_4\right|^2 \right)^2 \\
+ \left( \left|H_1 - H_2\right|^2 + \left|H'_1 - H'_2\right|^2 + \left|H_3 - H_4\right|^2 + \left|H'_3 - H'_4\right|^2 \right)^2 \\
\times \left( \left|H_1 - H_2\right|^2 + \left|H'_1 - H'_2\right|^2 + \left|H_3 - H_4\right|^2 + \left|H'_3 - H'_4\right|^2 \right) .
\]

**Proof.** Denote
\[
f(t,u,h) = (t-u)^{h-1/2} - (-u)^{h-1/2}, \\
f_1(h) = f(t_1,u_1,h), \quad f_2(h) = f(t_2,u_2,h).
\]

Then
\[
\mathbb{E} \left( B_t^{H_1,H'_1} - B_t^{H_2,H'_2} + B_t^{H_3,H'_3} - B_t^{H_4,H'_4} \right)^2 = \int_{\mathbb{R}^2} F^2(t,u) \, du,
\]
\[
F(t,u) = f_1(H_1)f_2(H'_1) - f_1(H_2)f_2(H'_2) + f_1(H_3)f_2(H'_3) - f_1(H_4)f_2(H'_4).
\]

Consider two cases.

**Case 1:** $(H_1 - H_2)(H_3 - H_4) \geq 0$.

In this case
\[
|H_1 - H_2| \leq |H_1 - H_2 + H_3 - H_4|, \\
|H_3 - H_4| \leq |H_1 - H_2 + H_3 - H_4|.
\]

We have
\[
F(t,u) = F_1(t,u) + F_2(t,u) + F_3(t,u),
\]

where
\[
F_1(t,u) = (f_1(H_1) - f_1(H_2))f_2(H'_2), \\
F_2(t,u) = (f_1(H_3) - f_1(H_4))f_2(H'_3), \\
F_3(t,u) = f_1(H_1)(f_2(H'_2) - f_2(H'_2)) + f_1(H_4)(f_2(H'_3) - f_2(H'_3)).
\]
By the mean value theorem, there exist \( h_1 \in [H_1 \land H_2, H_1 \lor H_2] \) and \( h_2 \in [H_3 \land H_4, H_3 \lor H_4] \) such that
\[
f_1(H_1) - f_1(H_2) = f_1'(h_1)(H_1 - H_2),
\]
\[
f_1(H_3) - f_1(H_4) = f_1'(h_2)(H_3 - H_4).
\]
Applying Lemma 5.3, we get
\[
\int_{\mathbb{R}^2} F_1^2(t,u)du \leq (H_1 - H_2)^2 \int_{\mathbb{R}^2} (f_1'(h_1))^2 du \int_{\mathbb{R}^2} (f_2(H_2))^2 du,
\]
\[
\leq C_1(H_1 - H_2 + H_3 - H_4)^2.
\]
In much the same way, we have
\[
\int_{\mathbb{R}^2} F_2^2(t,u)du \leq C_2(H_1 - H_2 + H_3 - H_4)^2.
\]
Hence,
\[
\int_{\mathbb{R}^2} F^2(t,u) du \leq 3 \left( \int_{\mathbb{R}^2} F_1^2(t,u) du + \int_{\mathbb{R}^2} F_2^2(t,u) du + \int_{\mathbb{R}^2} F_3^2(t,u) du \right)
\]
\[
\leq C_3(H_1 - H_2 + H_3 - H_4)^2 + 3 \int_{\mathbb{R}^2} F_3^2(t,u) du.
\]
Consider the latter term. There are two possible cases.

**Case 1a:** \((H_1' - H_2')(H_3' - H_4') \geq 0.\)

In this case
\[
|H_1' - H_2'| \leq |H_1' - H_2' + H_3' - H_4'|,
\]
\[
|H_3' - H_4'| \leq |H_1' - H_2' + H_3' - H_4'|.
\]
By the mean value theorem, there exist \( h_3 \in [H_1' \land H_2', H_1' \lor H_2'] \) and \( h_4 \in [H_3' \land H_4', H_3' \lor H_4'] \) such that
\[
f_2(H_1') - f_2(H_2') = f_2'(h_3)(H_1' - H_2'),
\]
\[
f_2(H_3') - f_2(H_4') = f_2'(h_4)(H_3' - H_4').
\]
Using Lemma 5.3, we estimate
\[
\int_{\mathbb{R}^2} F_3^2(t,u)du \leq (H_1' - H_2')^2 \int_{\mathbb{R}^2} (f_1(h_3))^2 du \int_{\mathbb{R}^2} (f_2'(h_3))^2 du
\]
\[
\leq C_4|H_1' - H_2' + H_3' - H_4'|^2.
\]

**Case 1b:** \((H_1' - H_2')(H_3' - H_4') < 0.\)

Without loss of generality, we assume that \(|H_1' - H_2'| > |H_3' - H_4'|\) and \(H_1' < H_2'\) (hence, \(H_3' < H_4'\)).

Put \(H_1 = H_2' + H_1' - H_3'\). It is not hard to see that \(\tilde{H}_1 \in [H_1', H_2']\).
\[
F_3(t,u) = F_{31}(t,u) + F_{32}(t,u),
\]
where

\[ F_{31}(t, u) = f_1(H_1) \left( f_2(H'_1) - f_2(\bar{H}_1) \right), \]
\[ F_{32}(t, u) = f_1(H_1) \left( f_2(\bar{H}_1) - f_2(H'_2) \right) + f_1(H_4)(f_2(H'_3) - f_2(H'_4)). \]

By the mean value theorem, there exists \( h_5 \in [H'_1, \bar{H}_1] \) such that

\[ F_{31}(t, u) = f_1(H_1)f'_2(h_5) \left( H'_1 - \bar{H}_1 \right) = f_1(H_1)f'_2(h_5)(H'_1 - H'_2 + H'_3 - H'_4). \]

By Lemma 5.3

\[ \int_{\mathbb{R}^2} F_{31}^2(t, u) du \leq (H'_1 - H'_2 + H'_3 - H'_4)^2 \int_{\mathbb{R}^2} (f_1(H_1))^2 du \int_{\mathbb{R}^2} (f'_2(h_5))^2 du \]
\[ \leq C_5(H'_1 - H'_2 + H'_3 - H'_4)^2. \]

We estimate \( F_{32}(t, u) \).

\[ F_{32}(t, u) = -\int_0^{H'_1 - H'_4} f_1(H_1)f'_2(\bar{H}_1 + x) \, dx \]
\[ + \int_0^{H'_1 - H'_4} f_1(H_4)f'_2(H'_4 + x) \, dx \]
\[ \leq \int_0^{H'_1 - H'_4} \int_{H_1}^{H_4} f'_1(y)f'_2(\bar{H}_1 + x) \, dy \, dx \]
\[ + \int_0^{H'_1 - H'_4} \int_{H_1}^{H_4} f'_1(H_4)f''_2(x + y) \, dy \, dx. \]

By the mean value theorem, there exist \( x_1, x_2 \in [0, H'_3 - H'_4] \), \( y_1 \in [H_1 \wedge H_4, H_1 \vee H_4] \), \( y_2 \in [\bar{H}_1 \wedge H'_1, \bar{H}_1 \vee H'_1] \) such that

\[ \int_0^{H'_1 - H'_4} \int_{H_1}^{H_4} f'_1(y)f'_2(\bar{H}_1 + x) \, dy \, dx \]
\[ = (H_1 - H_4)(H'_3 - H'_4)f'_1(y_1)f'_2(\bar{H}_1 + x_1), \]
\[ \int_0^{H'_1 - H'_4} \int_{H_1}^{H_4} f'_1(H_4)f''_2(x + y) \, dy \, dx \]
\[ = (H'_4 - \bar{H}_1)(H'_3 - H'_4)f'_1(H_4)f''_2(x_2 + y_2), \]
\[ = (H'_4 - H'_2)(H'_3 - H'_4)f'_1(H_4)f''_2(x_2 + y_2). \]

Using Lemma 5.3 we get

\[ \int_{\mathbb{R}^2} F_{32}^2(t, u) du \leq C_6(H'_1 - H'_4)^2(H_1 - H_4)^2 + C_7(H'_3 - H'_4)^2(H'_3 - H'_4)^2. \]

Thus, in the case 1b \[28\] holds.

Case 2: \((H_1 - H_2)(H_3 - H_4) < 0\).

Without loss of generality, we assume that \(|H_1 - H_2| > |H_3 - H_4|\) and \(H_1 < H_2\) (hence, \(H_3 < H_4\)).

Put \(\bar{H}_1 = H_2 + H_4 - H_3\). It is not hard to see that \(\bar{H}_1 \in [H_1, H_2]\).
We have
\[ F(t, u) = G_1(t, u) + G_2(t, u) + G_3(t, u), \]
where
\[ G_1(t, u) = (f_1(H_1) - f_1(\hat{H}_1))f_2(H'_1), \]
\[ G_2(t, u) = (f_1(\hat{H}_1) - f_1(H_2))f_2(H'_1) + (f_1(H_3) - f_1(H_4))f_2(H'_4), \]
\[ G_3(t, u) = f_1(H_2)(f_2(H'_1) - f_2(H'_2)) + f_1(H_3)(f_2(H'_3) - f_2(H'_4)). \]

Terms \( G_1(t, u) \) and \( G_2(t, u) \) are estimated similarly to \( F_{31}(t, u) \) and \( F_{32}(t, u) \) in Case 1b. We have
\[ \int_{\mathbb{R}^2} G_1^2(t, u) du \leq C_8(H_1 - H_2 + H_3 - H_4)^2, \]
\[ \int_{\mathbb{R}^2} G_2^2(t, u) du \leq C_9(H_3 - H_4)^2(H'_1 - H'_4)^2 + C_{10}(H_3 - H_4)^2(H_3 - H_2)^2. \]

It remains to estimate \( G_3(t, u) \). We consider two cases.

Case 2a: \( (H'_1 - H'_2)(H'_3 - H'_4) \geq 0 \).

\( G_3(t, u) \) can be estimated similarly to \( F_3(t, u) \) in Case 1a. We get
\[ \int_{\mathbb{R}^2} G_3^2(t, u) du \leq C_{11}(H'_1 - H'_2 + H'_3 - H'_4)^2. \]

Case 2b: \( (H'_1 - H'_2)(H'_3 - H'_4) < 0 \).

This case can be considered in a similar way to Case 1b. Without loss of generality, we assume that \( |H'_1 - H'_2| > |H'_3 - H'_4| \) and \( H'_1 < H'_2 \). Then we obtain
\[ \int_{\mathbb{R}^2} G_3^2(t, u) du \leq C_{12}(H'_1 - H'_2 + H'_3 - H'_4)^2 + C_{13}(H'_3 - H'_4)^2(H_3 - H_2)^2 + C_{14}(H'_3 - H'_4)^2(H'_3 - H'_2)^2. \]

Thus, now the proof is complete.

\[ \square \]

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