STRICHARTZ ESTIMATES FOR WAVE EQUATION WITH INVERSE SQUARE POTENTIAL

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Abstract. In this paper, we study the Strichartz-type estimates of the solution for the linear wave equation with inverse square potential. Assuming the initial data possesses additional angular regularity, especially the radial initial data, the range of admissible pairs is improved. As an application, we show the global well-posedness of the semi-linear wave equation with inverse-square potential
\[ \partial^2_t u - \Delta u + \frac{a}{|x|^2} u = \pm |u|^{p-1} u \]
for power \( p \) being in some regime when the initial data are radial. This result extends the well-posedness result in Planchon, Stalker, and Tahvildar-Zadeh.

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1. Introduction and Statement of Main Result

The aim of this paper is to study the \( L^q_t(L^r_x) \)-type estimates of the solution for the linear wave equation perturbed by an inverse square potential. More precisely, we shall consider the following wave equation with the inverse square potential

\[ \begin{aligned}
\partial^2_t u - \Delta u + \frac{a}{|x|^2} u &= 0, \\
| t, x \rangle &\in \mathbb{R} \times \mathbb{R}^n, \ a \in \mathbb{R}, \\
| u(t,x)|_{t=0} &= u_0(x), \quad \partial_t u(t,x)|_{t=0} = u_1(x).
\end{aligned} \tag{1.1} \]

The scale-covariance elliptic operator \(-\Delta + \frac{a}{|x|^2}\) appearing in (1.1) plays a key role in many problems of physics and geometry. The heat and Schrödinger flows for the elliptic operator \(-\Delta + \frac{a}{|x|^2}\) have been studied in the theory of combustion [11], and in quantum mechanics [8]. The equation (1.1) arises in the study of the
wave propagation on conic manifolds [4]. We refer the readers to [1, 2, 14, 15] and references therein.

It is well known that Strichartz-type estimates are crucial in handling local and global well-posedness problems of nonlinear dispersive equations. Along this way, Planchon, Stalker, and Tahvildar-Zadeh [14] first showed a generalized Strichartz estimates for the equation (1.1) with radial initial data. Thereafter, Burq, Planchon, Stalker, and Tahvildar-Zadeh [1] removed the radially symmetric assumption in [14] and then obtained some well-posedness results for the semi-linear wave equation with inverse-square potential. The range of the admissible exponents \((q, r)\) for the Strichartz estimates of (1.1) obtained in [1, 14] is restricted under
\[
\frac{4}{q} \leq (n-1)\left(\frac{1}{2} - \frac{1}{r}\right),
\]
which is the same as that of the linear wave equation without potential. Sterbenz and Rodnianski [22] improved the range of the “classical” admissible exponents \((q, r)\) for the linear wave equation with no potential by compensating a small loss of angular regularity.

In this paper, we are devoted to study the Strichartz estimates of the solution of the equation (1.1). By employing the asymptotic behavior of the Bessel function and some fine estimates of Hankel transform, we improve the range of the admissible pairs \((q, r)\) in [1, 14] by compensating a small loss of angular regularity. The machinery we employ here is mainly based on the spherical harmonics expansion and some properties of Hankel transform. As an application of the Strichartz estimates, we obtain well-posedness of (1.1) perturbed by nonlinearity 
\[
|u|^{p-1}u
\]
with power 
\[
p_h < p < p_{\text{conf}}\text{ (defined below)}
\]
in the radial case, which extends the well-posedness result in Planchon et al. [14].

Before stating our main theorems, we need some notations. We say the pair 
\((q, r)\) \(\in\) \(\Lambda\), if \(q, r \geq 2\), and satisfy
\[
\frac{1}{q} \geq \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{r}\right) \text{ and } \frac{1}{q} < (n-1)\left(\frac{1}{2} - \frac{1}{r}\right).
\]
Set the infinitesimal generators of the rotations on Euclidean space:
\[
\Omega_{j,k} := x_j \partial_k - x_k \partial_j,
\]
and define for \(s \in \mathbb{R}\),
\[
\Delta_\theta := \sum_{j<k} \Omega_{j,k}^2, \quad |\Omega|^s = (-\Delta_\theta)^{s/2}.
\]

**Theorem 1.1.** Let \(u\) be a solution of the equation (1.1) with \(a > \frac{1}{(n-1)r} - \frac{(n-2)^2}{4}\).

For any \(\epsilon > 0\) and \(0 < s < 1 + \min\left\{\frac{n-2}{2}, \sqrt{(n-2)^2 + a}\right\}\),

- if \(n \geq 4\), then

\[
\|u(t,x)\|_{L^q_tL^r_x} \leq C_\epsilon \left(\|\langle \Omega \rangle^su_0\|_{H^s} + \|\langle \Omega \rangle^su_1\|_{H^{s-1}}\right),
\]

where \((q, r) \in \Lambda\), and

\[
s = (1 + \epsilon)\left(\frac{2}{q} - (n-1)\left(\frac{1}{2} - \frac{1}{r}\right)\right) \quad \text{and} \quad \bar{s} = n\left(\frac{1}{2} - \frac{1}{r}\right) - \frac{1}{q};
\]

- if \(n = 3\), then

\[
\|u(t,x)\|_{L^q_tL^r_x} \leq C_\epsilon \left(\|\langle \Omega \rangle^su_0\|_{H^{s}} + \|\langle \Omega \rangle^su_1\|_{H^{s-1}}\right),
\]

where \((q, r) \in \Lambda\), and

\[
s = \frac{1}{2} - \frac{1}{r} \quad \text{and} \quad \bar{s} = n\left(\frac{1}{2} - \frac{1}{r}\right) - \frac{1}{q};
\]
where \( q \neq 2 \), \((q, r) \in \Lambda \), and
\[
\bar{s} = (2 + \epsilon) \left( \frac{1}{q} - \left( \frac{1}{2} - \frac{1}{r} \right) \right) \quad \text{and} \quad s = 3 \left( \frac{1}{2} - \frac{1}{r} \right) - \frac{1}{q}.
\]

In addition, the following estimate holds for \( r > 4 \) and \( s = 3 \left( \frac{1}{2} - \frac{1}{r} \right) - \frac{1}{q} \),
\[
\| u(t, x) \|_{L^q_t L^r_x} \leq C \left( \| \Omega \|^{\bar{s}(r)} u_0 \|_{H^s} + \| \Omega \|^{\bar{s}(r)} u_1 \|_{H^{s-1}} \right),
\]
where \( \bar{s}(r) = 1 - \frac{2}{r} \) with \( r \neq \infty \).

**Remark 1.1.**

i). We remark that some of admissible pairs \((q, r)\) in Theorem 1.1 are out of the region ACDO or ACO (in the following figures) obtained in [1, 14].

ii). Our restriction \( a > a_n := \frac{1}{(n-1)^2} - \frac{(n-2)^2}{4} \) is to extend the the range of \((q, r)\) as widely as possible. We remark that \( a_3 = 0 \) and \( a_n < 0 \) for \( n \geq 4 \). Therefore, we recover the result of Theorem 1.5 in Sterbenz [22], which considers \( a = 0 \) and \( n \geq 4 \).

iii). In the extended region \( \Lambda \) (see the below figures), the loss of angular regularity is \( \bar{s} = (1 + \epsilon) \left( \frac{q}{r} - (n-1) \left( \frac{1}{2} - \frac{1}{r} \right) \right) \). When \( n = 3 \), the loss of angular regularity in the line BC is \( \bar{s}(r) > \bar{s} \), since the Strichartz estimate fails at the endpoint \((q, r, n) = (2, \infty, 3)\). It seems that the methods we use here are not available to obtain such estimate at endpoint since Lemma 2.3 and Lemma 2.4 fail at \( r = \infty \). And one might need the wave packet method of Wolff [25] and the argument in Tao [23] to obtain the Strichartz estimate at the endpoint \((q, r, n) = (2, \infty, 3)\) with some loss of angular regularity.

As a consequence of Theorem 1.1 and Corollary 3.9 in [14], we have the following Strichartz estimates for radial initial data:

**Corollary 1.1.** Let \( n \geq 3 \) and \( s < \frac{n}{2} \). Suppose \((u_0, u_1)\) are radial functions, then for \( q, r \geq 2, \frac{1}{q} < \left( n - 1 \right) \left( \frac{1}{2} - \frac{1}{r} \right) \) and \( s = n \left( \frac{1}{2} - \frac{1}{r} \right) - \frac{1}{q} \), the solution \( u \) of the equation (1.1) with \( a > \frac{1}{(n-1)^2} - \frac{(n-2)^2}{4} \) satisfies
\[
\| u(t, x) \|_{L^q_t L^r_x} \leq C \left( \| u_0 \|_{H^s} + \| u_1 \|_{H^{s-1}} \right).
\]
As an application, we obtain some well-posedness result of the following semi-linear wave equation,

\[
\begin{aligned}
\partial_t^2 u - \Delta u + \frac{a}{|x|^2} u &= \pm |u|^{p-1} u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \quad a \in \mathbb{R}, \\
\mathcal{K} u(t, x)|_{t=0} = u_0(x), \quad \partial_t u(t, x)|_{t=0} = u_1(x).
\end{aligned}
\]

In the case of the semi-linear wave equation without potential (i.e. \( a = 0 \)), there are many exciting results on the global existence and blow-up. We refer the readers to \[10\,16\] and references therein. While for the equation (1.6) with \( p \geq p_{\text{conf}} := 1 + \frac{4}{n-2} \) and \( n \geq 3 \), Planchon et al. \[14\] established the global existence when the radial initial data is small in \( \dot{H}^{\infty} \times \dot{H}^{\infty-1} \)-norm with \( s_c := \frac{n}{4} - \frac{2}{p-1} \).

Thereafter, Burq et al. \[1\] removed the radially symmetric assumption on the initial data. As a consequence of Theorem 1.1, we prove the global existence of the solution to the equation (1.6) with \( p_h := 1 + \frac{4p}{(n+1)(n-1)} < p < p_{\text{conf}} \) for small radial initial data \((u_0, u_1) \in \dot{H}^{s_c} \times \dot{H}^{s_c-1} \).

**Theorem 1.2.** Let \( n \geq 3 \) and \( p_h < p < p_{\text{conf}} \). Let \( q_0 = (p-1)(n+1)/2, \ r_0 = (n+1)(p-1)/(2p) \), and

\[
a > \max \left\{ \frac{1}{(n-1)^2} - \frac{(n-2)^2}{4}, \frac{n}{q_0} \left( \frac{n}{q_0} - n + 2 \right), \frac{n}{r_0} \right\}.
\]

Assume \((u_0, u_1)\) are radial functions and there is a small constant \( \epsilon(p) \) such that

\[
\|u_0\|_{\dot{H}^{s_c}} + \|u_1\|_{\dot{H}^{s_c-1}} < \epsilon(p),
\]

then there exists a unique global solution \( u \) to (1.6) satisfying

\[
u \in C_t(\mathbb{R}; \dot{H}^{s_c}) \cap L_t^{q_0}(\mathbb{R} \times \mathbb{R}^n).
\]

**Remark 1.2.** i) The above result extends the well-posedness result in \[14\] from \( p \geq p_{\text{conf}} \) to \( p_h < p < p_{\text{conf}} \).

ii) We remark that the \( L^1 \)-bound of the operator \( \mathcal{K}^{0}_{\lambda,\nu} \), defined below, is the source of our constraint to \( p > p_h \). Inspired by the arguments in Lindblad-Sogge, \[10\,10\] for the usual semi-linear wave equation, if we want to extend the above result to \( p > p_c \), one needs to explore new inhomogeneous Strichartz estimates since the operator \( \mathcal{K}^{0}_{\lambda,\nu} \) is not known as a bounded operator on \( L^1 \). Here \( p_c \) is the positive root of \((n-1)p_c^n - (n+1)p_c - 2 = 0\), and \( p_c \) is called the Strauss’s index.

This paper is organized as follows: In the section 2, we revisit the property of the Bessel functions, harmonic projection operator, and the Hankel transform associated with \(-\Delta + a/|x|^2\). Section 3 is devoted to establishing some estimates of the Hankel transform. In Section 4, we use the previous estimates to prove Theorem 1.1. We show Theorem 1.2 in Section 5. In the appendix, we sketch the proof of Lemma 2.2 by using a weak-type \((1,1)\) estimate for the multiplier operators with respect to the Hankel transform.

Finally, we conclude this section by giving some notations which will be used throughout this paper. We use \( A \lesssim B \) to denote the statement that \( A \leq CB \) for some large constant \( C \) which may vary from line to line and depend on various parameters, and similarly use \( A \ll B \) to denote the statement \( A \leq C^{-1}B \). We employ \( A \sim B \) to denote the statement that \( A \lesssim B \lesssim A \). If the constant \( C \) depends on a special parameter other than the above, we shall denote it explicitly
by subscripts. We briefly write $A + \epsilon$ as $A^+$ for $0 < \epsilon \ll 1$. Throughout this paper, pairs of conjugate indices are written as $p, p'$, where $\frac{1}{p} + \frac{1}{p'} = 1$ with $1 \leq p \leq \infty$.

2. Preliminary

In this section, we provide some standard facts about the Hankel transform and the Bessel functions. We use the oscillatory integral argument to show the asymptotic behavior of the derivative of the Bessel function. The Littlewood-Paley theorems associated to the Hankel transform are collected in this section. Finally we prove a Strichartz estimate for unit frequency by making use of some results in [1].

2.1. Spherical harmonic expansions and the Bessel functions. We begin with the spherical harmonics expansion formula. For more details, we refer to Stein-Weiss [20]. Let

$$\xi = \rho \omega \quad \text{and} \quad x = r \theta \quad \text{with} \quad \omega, \theta \in \mathbb{S}^{n-1}.$$ (2.1)

For any $g \in L^2(\mathbb{R}^n)$, we have the expansion formula

$$g(x) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} a_{k,\ell}(r) Y_{k,\ell}(\theta)$$

where

$$\{Y_{k,1}, \ldots, Y_{k,d(k)}\}$$

is the orthogonal basis of the spherical harmonic space of degree $k$ on $\mathbb{S}^{n-1}$, called $\mathcal{H}^k$, with the dimension

$$d(0) = 1 \quad \text{and} \quad d(k) = \frac{2k + n - 2}{k} C_{n+k-3} \simeq \langle k \rangle^{n-2}.$$ (2.2)

We remark that for $n = 2$, the dimension of $\mathcal{H}^k$ is a constant independent of $k$. We have the orthogonal decomposition

$$L^2(\mathbb{S}^{n-1}) = \bigoplus_{k=0}^{\infty} \mathcal{H}^k.$$ (2.3)

This gives by orthogonality

$$\|g(x)\|_{L^2_{\rho}} = \|a_{k,\ell}(r)\|_{L^2_{\rho}}.$$ (2.4)

By Theorem 3.10 in [20], we have the Hankel transforms formula

$$\hat{g}(\rho \omega) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} 2\pi r^k Y_{k,\ell}(\omega) \rho^{-\frac{n-2}{2}} \int_0^{\infty} J_{k+n-2}(2\pi r \rho) a_{k,\ell}(r) r^{\frac{n}{2}} dr,$$ (2.5)

here the Bessel function $J_k(r)$ of order $k$ is defined by

$$J_k(r) = \frac{(r/2)^k}{\Gamma(k + \frac{1}{2}) \Gamma(1/2)} \int_{-1}^{1} e^{irs} (1 - s^2)^{(2k-1)/2} ds \quad \text{with} \quad k > -\frac{1}{2} \quad \text{and} \quad r > 0.$$ (2.6)

A simple computation gives the estimates

$$|J_k(r)| \leq \frac{C r^k}{2^k \Gamma(k + \frac{1}{2}) \Gamma(1/2)} \left(1 + \frac{1}{k + 1/2}\right),$$ (2.7)
and
\begin{equation}
|J'_k(r)| \leq \frac{C(kr^{k-1} + r^k)}{2^k \Gamma(k + \frac{1}{2}) \Gamma(1/2)} \left(1 + \frac{1}{k + 1/2}\right),
\end{equation}
where \(C\) is a constant and these estimates will be used when \(r \lesssim 1\). Another well
known asymptotic expansion about the Bessel function is
\(J_k(r) = r^{-1/2} \sqrt{\frac{2}{\pi}} \cos(r - \frac{k\pi}{2} - \frac{\pi}{4}) + O_k(r^{-3/2}), \ \text{as } r \to \infty,\)
but with a constant depending on \(k\) (see [20]). As pointed out in [18], if one seeks
a uniform bound for large \(r\) and \(k\), then the best one can do is \(|J_k(r)| \leq Cr^{-\frac{1}{2}}\).
To investigate the behavior of asymptotic on \(k\) and \(r\), we are devoted to Schl"afli’s
integral representation [24] of the Bessel function: for \(r \in \mathbb{R}^+\) and \(k > -\frac{1}{2},\)
\begin{equation}
J_k(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ir \sin \theta - ik \theta} d\theta - \frac{\sin(k\pi)}{\pi} \int_{0}^{\infty} e^{-(r \sinh s + k s)} ds
\end{equation}
\[= J_k(r) - E_k(r).\]
We remark that \(E_k(r) = 0\) for \(k \in \mathbb{Z}^+.\) One easily estimates for \(r > 0\)
\begin{equation}
|E_k(r)| = \left| \frac{\sin(k\pi)}{\pi} \int_{0}^{\infty} e^{-(r \sinh s + k s)} ds \right| \leq C(r + k)^{-1}.
\end{equation}

Next, we recall the properties of Bessel function \(J_k(r)\) in [17, 18], the readers can also refer to [12] for the detailed proof.

**Lemma 2.1** (Asymptotics of the Bessel function). Assume \(k \gg 1.\) Let \(J_k(r)\) be
the Bessel function of order \(k\) defined as above. Then there exist a large constant
\(C\) and small constant \(c\) independent of \(k\) and \(r\) such that:

- when \(r \leq \frac{k}{2}\)
  \begin{equation}
  |J_k(r)| \leq Ce^{-c(k+r)};
  \end{equation}
- when \(\frac{k}{2} \leq r \leq 2k\)
  \begin{equation}
  |J_k(r)| \leq Ck^{-\frac{1}{4}}(k^{-\frac{1}{4}}|r - k| + 1)^{-\frac{1}{4}};
  \end{equation}
- when \(r \geq 2k\)
  \begin{equation}
  J_k(r) = r^{-\frac{1}{2}} \sum_{\pm} a_{\pm}(r)e^{\pm ir} + E(r),
  \end{equation}
where \(|a_{\pm}(r)| \leq C\) and \(|E(r)| \leq Cr^{-1}.

For our purpose, we additionally need the asymptotic behavior of the derivative
of the Bessel function \(J'_k(r).\) It is a straightforward elaboration of the argument of
proving Lemma 2.1 in [12], but we give the proof for completeness.

**Lemma 2.2.** Assume \(r, k \gg 1.\) Then there exists a constant \(C\) independent of \(k\)
and \(r\) such that
\[|J'_k(r)| \leq Cr^{-\frac{1}{2}}.
\]
**Proof.** When \(r \leq \frac{k}{2}\) or \(r \geq 2k\), we apply the recurrence formula [24]
\[J'_k(r) = \frac{1}{2}(J_{k-1}(r) - J_{k+1}(r)),\]
\begin{equation}
(2.8)
\end{equation}
and \begin{equation}
(2.10)
\end{equation} to obtaining
\[|J'_k(r)| \leq Cr^{-\frac{1}{2}}.
\]
When \( k \leq r \leq 2k \), we have by (2.6)

\[
J'_k(r) = J'_k(r) - E'_k(r).
\]

A simple computation gives that for \( r > 0 \)

\[
|E'_k(r)| = \left| \int_0^\infty \frac{\sin(k\pi) e^{-r \sinh s + ks}}{s} \, ds \right| \leq C(r + k)^{-1}. 
\]

Thus we only need to estimate \( \tilde{J}'_k(r) \). We divide two cases \( r > k \) and \( r \leq k \) to estimate it by the stationary phase argument. Let \( \phi_{r,k}(\theta) = r \sin \theta - k\theta \).

**Case 1:** \( k < r \leq 2k \). Let \( \theta_0 = \cos^{-1}(\frac{r}{k}) \), then

\[
\phi_{r,k}(\theta_0) = r \cos \theta_0 - k = 0.
\]

Now we split \( \tilde{J}'_k(r) \) into two pieces:

\[
\tilde{J}'_k(r) = \frac{i}{2\pi} \int_{\Omega_\delta} e^{ir \sin \theta - ik \theta} \sin \theta \, d\theta + \frac{i}{2\pi} \int_{B_\delta} e^{ir \sin \theta - ik \theta} \sin \theta \, d\theta,
\]

where

\[
\Omega_\delta = \{ \theta : |\theta + \theta_0| \leq \delta \}, \quad B_\delta = [-\pi, \pi] \setminus \Omega_\delta \quad \text{with} \quad \delta > 0.
\]

We have by taking absolute values

\[
\left| \frac{1}{2\pi} \int_{\Omega_\delta} e^{ir \sin \theta - ik \theta} \sin \theta \, d\theta \right| \leq C |\sin(\theta_0 + \delta)| \delta.
\]

Integrating by parts, we have

\[
\int_{B_\delta} e^{ir \sin \theta - ik \theta} \sin \theta \, d\theta = \left. \frac{e^{i(r \sin \theta - k \theta)} \sin \theta}{i(r \cos \theta - k)} \right|_{\partial B_\delta} - \int_{B_\delta} \frac{e^{i(r \sin \theta - k \theta)(r - k \cos \theta)}}{i(r \cos \theta - k)^2} \, d\theta,
\]

where \( \partial B_\delta = \{ \pm \pi, \pm \theta_0 + \delta \} \). It is easy to see that

\[
\left| \frac{e^{i(r \sin \theta - k \theta)} \sin \theta}{i(r \cos \theta - k)} \right|_{\partial B_\delta} \leq c |\sin(\theta_0 + \delta)| |r \cos(\theta_0 + \delta) - k|^{-1}.
\]

Since \( r - k \cos \theta > 0 \), we obtain

\[
\left| \int_{B_\delta} \frac{e^{ir \sin \theta - ik \theta}(r - k \cos \theta)}{i(r \cos \theta - k)^2} \, d\theta \right| \leq \int_{B_\delta} \frac{|r - k \cos \theta|}{(r \cos \theta - k)^2} \, d\theta = \frac{\sin \theta}{(r \cos \theta - k)} \left| \partial B_\delta \right| 
\]

\[
\leq c |\sin(\theta_0 + \delta)| \cdot |r \cos(\theta_0 + \delta) - k|^{-1}.
\]

Therefore,

\[
|\tilde{J}'_k(r)| \leq C |\sin(\theta_0 + \delta)| \delta + c |\sin(\theta_0 + \delta)| \cdot |r \cos(\theta_0 + \delta) - k|^{-1}.
\]

We shall choose proper \( \delta \) such that

\[
|\sin(\theta_0 + \delta)| \delta \sim c |\sin(\theta_0 + \delta)| \cdot |r \cos(\theta_0 + \delta) - k|^{-1}.
\]

Noting that \( \cos(\theta_0 + \delta) = \cos \theta_0 \cos \delta \mp \sin \theta_0 \sin \delta \) and the definition of \( \theta_0 \), we get

\[
r \cos(\theta_0 + \delta) - k = k \cos \delta \pm \sqrt{r^2 - k^2} \sin \delta - k.
\]

Since \( 1 - \cos \delta = 2 \sin^2 \frac{\delta}{2} \), one has

\[
|r \cos(\theta_0 + \delta) - k| \sim |k\delta^2 \pm \delta \sqrt{r^2 - k^2}| \quad \text{with small} \ \delta.
\]
On the other hand, we have by $\sin(\theta_0 \pm \delta) = \sin \theta_0 \cos \delta \pm \cos \theta_0 \sin \delta$,

$$\sin(\theta_0 \pm \delta) = \pm \frac{\sqrt{r^2 - k^2}}{r} (1 - \delta^2) \pm \frac{k}{r} \delta.$$

When $|r - k| \leq \frac{k}{4}$, choosing $\delta = Ck^{-\frac{1}{2}}$ with large $C \geq 2$, we have

$$|\sin(\theta_0 \pm \delta)| \cdot |r \cos(\theta_0 \pm \delta) - k|^{-1} \lesssim k^{-\frac{1}{2}} (C^2 - Ck^{-\frac{1}{2}} \sqrt{r^2 - k^2})^{-1} \lesssim k^{-\frac{1}{2}} \lesssim r^{-\frac{1}{2}}.$$

When $|r - k| \geq \frac{k}{4}$, taking $\delta = c(r^2 - k^2)^{-\frac{1}{2}}$ with small $c > 0$, we obtain

$$|\sin(\theta_0 \pm \delta)| \cdot |r \cos(\theta_0 \pm \delta) - k|^{-1} \lesssim |r - k|^\frac{1}{2} r^{-\frac{1}{2}} + r^{-1} + (r^2 - k^2)^{-\frac{1}{2}}(r^2 - k^2)^{-\frac{1}{2}} (c - c^2 k^2 - k^2)^{-\frac{1}{2}} \lesssim k^{-\frac{1}{2}} |r - k|^\frac{1}{2} r^{-\frac{1}{2}} \lesssim r^{-\frac{1}{2}},$$

where we use the fact that $(r^2 - k^2)^{-\frac{1}{2}} \leq (2k)^{-\frac{1}{2}} |r - k|^{-\frac{1}{2}}$ for $k < r$.

**Case 2:** $\frac{k}{2} \leq r \leq k$. When $k - \frac{k}{2} < r < k$, choosing $\theta_0 = 0$ and $\delta = Ck^{-\frac{1}{2}}$ with large $C \geq 2$, it follows from the above argument that

$$|\tilde{J}_k^j(r)| \lesssim |\sin(\theta_0 \pm \delta)| \cdot |r \cos(\theta_0 \pm \delta) - k|^{-1} \lesssim \delta (r^2/2 - |r - k|)^{-1} \lesssim k^{-\frac{1}{2}} \lesssim r^{-\frac{1}{2}}.$$

When $r < k - \frac{k}{2}$, there is no critical point. Hence we obtain

$$|\tilde{J}_k^j(r)| \lesssim |r - k|^2 r^{-\frac{1}{2}} \lesssim r^{-\frac{1}{2}}.$$

Finally, we collect all the estimates to get $|\tilde{J}_k^j(r)| \lesssim r^{-\frac{1}{2}}$. □

Next, we record the two basic results about the modified square function expressions.

**Lemma 2.3** (A modified Littlewood-Paley theorem [18]). Let $\beta \in C_0^\infty(\mathbb{R}^+)$ be supported in $[\frac{1}{2}, 2)$, $\beta_j(\rho) = \beta(2^{-j} \rho)$ and $\sum \beta_j = 1$. Then for any $\nu(k) > 0$ and $1 < p < \infty$, we have

$$\left\| \sum_{j \in \mathbb{Z}} \int_0^\infty (r \rho)^{-\frac{2}{p} - 2} J_{\nu(k)}(r \rho) \cos(t \rho) \beta_{k, \ell}(\rho) \rho^{n-1} \beta_j(\rho) \rho \right\|_{L^p_{r, \rho}}$$

$$\sim \left\| \left( \sum_{j \in \mathbb{Z}} \int_0^\infty (r \rho)^{-\frac{2}{p} - 2} J_{\nu(k)}(r \rho) \cos(t \rho) \beta_{k, \ell}(\rho) \rho^{n-1} \beta_j(\rho) \right)^2 \right\|_{L^p_{r, \rho}}.$$

For the sake of the completeness, we will prove Lemma 2.3 in the appendix by using a weak-type $(1, 1)$ estimate for the multiplier operators with respect to the Hankel transform.

**Lemma 2.4** (Littlewood-Paley-Stein theorem for the sphere, [18, 21, 22]). Let $\beta \in C_0^\infty(\mathbb{R}^+)$ be supported in $[\frac{1}{4}, 4]$ and $\beta(\rho) = 1$ when $\rho \in [1, 2]$. Assume $\beta_j(\rho) = \beta(2^{-j} \rho)$. Then for any $1 < p < \infty$ and any test function $f(\theta)$ defined on $\mathbb{S}^{n-1}$, we have

$$\|f(\theta)\|_{L^p_{\theta}(\mathbb{S}^{n-1})} \sim \left\| \left( \sum_{j=0}^\infty \sum_{k=1}^{d(k)} \sum_{\ell=1}^\infty \beta_j(k) a_{k, \ell}^j \nu_k^j(\theta) \left| a_{k, \ell}^j(\theta) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p_{\theta}(\mathbb{S}^{n-1})},$$

where $d(k)$ is the degree of $\theta$. □
where \( f = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} a_{k,\ell} Y_{k,\ell}(\theta) \).

We conclude this subsection by showing the “Bernstein” inequality on sphere

\[
(2.13) \quad \left\| \sum_{k=2^j}^{2^{j+1}} \sum_{\ell=1}^{d(k)} a_{k,\ell} Y_{k,\ell}(\theta) \right\|_{L^q(S^{n-1})} \leq C_{q,n} 2^{j(n-1)(\frac{q}{2} - \frac{1}{q})} \left( \sum_{k=2^j}^{2^{j+1}} \sum_{\ell=1}^{d(k)} |a_{k,\ell}|^2 \right)^{\frac{1}{2}}
\]

for \( q \geq 2, j = 0, 1, 2, \ldots \).

In fact, since \( \sum_{\ell=1}^{d(k)} |Y_{k,\ell}(\theta)|^2 = d(k)|S^{n-1}|^{-1}, \forall \theta \in S^{n-1} \) (see Stein-Weiss [20]), one has

\[
\left\| \sum_{k=2^j}^{2^{j+1}} \sum_{\ell=1}^{d(k)} a_{k,\ell} Y_{k,\ell}(\theta) \right\|_{L^\infty(S^{n-1})} \leq C \sum_{k=2^j}^{2^{j+1}} \left( \sum_{\ell=1}^{d(k)} |a_{k,\ell}|^2 \right)^{\frac{1}{2}} \left( \sum_{k=2^j}^{2^{j+1}} \sum_{\ell=1}^{d(k)} |Y_{k,\ell}(\theta)|^2 \right)^{\frac{1}{2}} \leq C \sum_{k=2^j}^{2^{j+1}} \sum_{\ell=1}^{d(k)} |a_{k,\ell}|^2 \leq C 2^{j(n-1)} \sum_{k=2^j}^{2^{j+1}} \sum_{\ell=1}^{d(k)} |a_{k,\ell}|^2.
\]

Interpolating this with

\[
\left\| \sum_{k=2^j}^{2^{j+1}} \sum_{\ell=1}^{d(k)} a_{k,\ell} Y_{k,\ell}(\theta) \right\|_{L^2(S^{n-1})} \leq C \sum_{k=2^j}^{2^{j+1}} \sum_{\ell=1}^{d(k)} |a_{k,\ell}|^2
\]

yields (2.13).

2.2. Spectrum of \(-\Delta + \frac{a}{|x|^2}\) and Hankel transform. Let us first consider the eigenvalue problem associated with the operator \(-\Delta + \frac{a}{|x|^2}\):

\[
\begin{cases}
-\Delta u + \frac{a}{|x|^2} u = \rho^2 u & x \in B = \{ x : |x| \leq 1 \}, \\
u(x) = 0 & x \in S^{n-1}.
\end{cases}
\]

If \( u(x) = f(r) Y_k(\theta) \), we have

\[
f''(r) + \frac{n-1}{r} f'(r) + \rho^2 - \frac{k(k+n-2) + a}{r^2} f(r) = 0.
\]

Let \( \lambda = \rho r \) and \( f(r) = \lambda^{-\frac{n-2}{2}} g(\lambda) \), we obtain

\[
(2.14) \quad g''(\lambda) + \frac{1}{\lambda} g'(\lambda) + \left[ 1 - \frac{(k+n-2)^2 + a}{\lambda^2} \right] g(\lambda) = 0.
\]

Define

\[
(2.15) \quad \mu(k) = \frac{n-2}{2} + k, \quad \nu(k) = \sqrt{\mu^2(k) + a} \quad \text{with} \quad a > -(n-2)^2/4.
\]

The Bessel function \( J_{\nu(k)}(\lambda) \) solves the Bessel equation (2.14). And the eigenfunctions corresponding to the spectrum \( \rho^2 \) can be expressed by

\[
(2.16) \quad \phi_{\nu}(x) = (\rho r)^{-\frac{n-2}{2}} J_{\nu(k)}(\rho r) Y_k(\theta) \quad \text{with} \quad x = r\theta,
\]
where

\[(2.17) \quad \left(-\Delta + \frac{a}{|x|^2}\right)\phi_\rho = \rho^2 \phi_\rho.\]

We define the following elliptic operator

\[(2.18) \quad A_{\nu(k)} : = -\partial_r^2 - \frac{n-1}{r} \partial_r + \frac{k(k+n-2)+a}{r^2}
\]
\[-\partial_r^2 - \frac{n-1}{r} \partial_r + \frac{\nu^2(k) - \left(\frac{\alpha-2}{2}\right)^2}{r^2},\]

then \(A_{\nu(k)} \phi_\rho = \rho^2 \phi_\rho.\) Define the Hankel transform of order \(\nu:\]

\[(2.19) \quad (H_\nu f)(\xi) = \int_0^\infty (r \rho)^{-\frac{\alpha-2}{2}} J_\nu(r \rho)f(r \omega) r^{\alpha-1} dr,
\]

where \(\rho = |\xi|, \omega = \xi/|\xi|\) and \(J_\nu\) is the Bessel function of order \(\nu.\) Specially, if the function \(f\) is radial, then

\[(2.20) \quad (H_\nu f)(\rho) = \int_0^\infty (r \rho)^{-\frac{\alpha-2}{2}} J_\nu(r \rho)f(r) r^{\alpha-1} dr.
\]

If \(f(x) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} a_{k,\ell}(r)Y_{k,\ell}(\theta),\) then we obtain by \((2.3)\)

\[(2.21) \quad \hat{f}(\xi) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} 2\pi i^k Y_{k,\ell}(\omega)(H_{\mu(k)} a_{k,\ell})(\rho).
\]

We will also make use of the following properties of the Hankel transform, which appears in \([1, 14]\).

**Lemma 2.5.** Let \(H_\nu\) and \(A_\nu\) be defined as above. Then

(i) \(H_\nu = H_\nu^{-1},\)

(ii) \(H_\nu\) is self-adjoint, i.e. \(H_\nu = H_\nu^*,\)

(iii) \(H_\nu\) is an \(L^2\) isometry, i.e. \(\|H_\nu \phi\|_{L^2} = \|\phi\|_{L^2},\)

(iv) \(H_\nu(A_\nu \phi)(\xi) = |\xi|^2(H_\nu \phi)(\xi),\) for \(\phi \in L^2.\)

Let \(K_{0,\nu} = H_\mu H_\nu,\) then as well as in \([14]\) one has

\[(2.22) \quad A_\mu K_{0,\nu} = K_{0,\nu} A_\nu.
\]

For our purpose, we need another crucial properties of \(K_{0,\nu} = K_{0,\nu}^0\) with \(k = 0:\)

**Lemma 2.6** (The boundness of \(K_{0,\nu}^0,\) \([1, 14]\).) Let \(\nu, \alpha, \beta \in \mathbb{R}, \nu > -1, \lambda = \mu(0) = \frac{\nu^2-2}{2}, -n < \alpha < 2(\nu + 1)\) and \(-2(\nu + 1) < \beta < n.\) Then the conjugation operator \(K_{0,\nu}^0\) is continuous on \(\dot{H}_{\nu,rad}^p(\mathbb{R}^n)\) provided that

\[
\max \left\{0, \frac{\lambda - \nu}{n}, \frac{\beta}{n}\right\} < \frac{1}{p} < \min \left\{\frac{\lambda + \nu + 2}{n}, \frac{\lambda + \nu + 2 + \beta}{n}, 1\right\}
\]

while the inverse operator \(K_{0,\nu}^0\) is continuous on \(\dot{H}_{\nu,rad}^n(\mathbb{R}^n)\) provided that

\[
\max \left\{0, \frac{\lambda - \nu}{n}, \frac{\lambda - \nu + \alpha}{n}\right\} < \frac{1}{q} < \min \left\{\frac{\lambda + \nu + 2}{n}, 1 + \frac{\alpha}{n}, 1\right\}.
\]

We also need the Strichartz estimates for \((1.1)\) in \([1]:\)
Lemma 2.7 (Strichartz estimates). For \( n \geq 2 \), let \( 2 \leq r < \infty \) and \( q, r, \gamma, \sigma \) satisfy
\[
\frac{1}{q} \leq \min \left\{ \frac{1}{2}, \frac{n-1}{2} \left( \frac{1}{2} - \frac{1}{r} \right) \right\}, \quad \sigma = \gamma + \frac{1}{q} - n \left( \frac{1}{2} - \frac{1}{r} \right).
\]
There exists a positive constant \( C \) depending on \( n, a, q, r, \gamma \) such that the solution \( u \) of (1.1) satisfies
\[
\left\| (-\Delta)^{\frac{\sigma}{2}} u \right\|_{L^q_t(L^r_x(\mathbb{R}^n))} \leq C \left( \| u_0 \|_{\dot{H}^\gamma} + \| u_1 \|_{\dot{H}^{\gamma-1}} \right)
\]
provided that when \( n = 2, 3 \)
\[- \min \left\{ \frac{n-1}{2}, \nu(1) - \frac{1}{2}, 1 + \nu(0) \right\} < \gamma < \min \left\{ \frac{n+1}{2}, \nu(1) + \frac{1}{2}, 1 + \nu(0) - 1 \right\},
\]
and when \( n \geq 4 \)
\[- \min \left\{ \frac{n+3}{2(n-1)}, \nu(1) - \frac{n+3}{2(n-1)}, 1 + \nu(0) \right\} < \gamma < \min \left\{ \frac{n+1}{2}, \nu(1) + \frac{1}{2}, 1 + \nu(0) - \frac{1}{q} \right\}.
\]

Next, define the projectors \( M_{jj'} = P_j \tilde{P}_{j'} \) and \( N_{jj'} = \tilde{P}_j P_{j'} \), where \( P_j \) is the usual dyadic frequency localization at \( \xi \sim 2^j \) and \( \tilde{P}_j \) is the localization with respect to \( ( -\Delta + \frac{a}{|x|^2})^{\frac{1}{2}} \). More precisely, let \( f \) be in the \( k \)-th harmonic subspace, then
\[
P_j f = \mathcal{H}_{\mu(k)} \beta_j \mathcal{H}_{\mu(k)} f \quad \text{and} \quad \tilde{P}_j f = \mathcal{H}_{\nu(k)} \beta_j \mathcal{H}_{\nu(k)} f,
\]
where \( \beta_j(\xi) = \beta(2^{-j} |\xi|) \) with \( \beta \in C_0^\infty(\mathbb{R}^+) \) supported in \( [\frac{1}{2}, 2] \). Then, we have the almost orthogonality estimate which is proved in [1].

Lemma 2.8 (Almost orthogonality estimate, [1]). There exists a positive constant \( C \) independent of \( j, j' \), \( k \) such that the following inequalities hold for all positive \( \epsilon_1 < 1 \) \( < \min \left\{ \frac{n+2}{2}, \left( \frac{n-2}{n-1} + a \right)^\frac{1}{2} \right\} \)
\[
\| M_{jj'} f \|_{L^2(\mathbb{R}^n)} \quad \| N_{jj'} f \|_{L^2(\mathbb{R}^n)} \leq C 2^{-\epsilon_1 |j-j'|} \| f \|_{L^2(\mathbb{R}^n)},
\]
where \( f \) is in the \( k \)-th harmonic subspace.

As a consequence of Lemma 2.7 and Lemma 2.8, we have

Lemma 2.9 (Strichartz estimates for unit frequency). Let \( n \geq 3, k \in \mathbb{N} \). Let \( u \) solve
\[
\begin{cases}
(\partial_t^2 - \Delta + \frac{a}{|x|^2}) u = 0, \\
u\big|_{t=0} = u_0(x), \quad u_t\big|_{t=0} = 0,
\end{cases}
\]
where \( u_0 \in L^2(\mathbb{R}^n) \) and
\[
u_0 = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} a_{k,\ell}(r) Y_{k,\ell}(\theta).
\]
Assume that for all \( k, \ell \in \mathbb{N} \), \( \text{supp } [\mathcal{H}_{\nu(k)} a_{k,\ell}] \subset [1, 2] \). Then the following estimate holds for \( a > \frac{1}{(n-1)^2} - \frac{(n-2)^2}{4} \)
\[
\| u(t, x) \|_{L^q_t(L^r_x(\mathbb{R}^n))} \leq C \| u_0 \|_{L^2(\mathbb{R}^n)},
\]
where \( q \geq 2, \frac{1}{q} = \frac{4}{2k} \left( \frac{1}{2} - \frac{1}{r} \right) \) and \( (q, r, n) \neq (2, \infty, 3) \).
Proof. By making use of Lemma 2.7 with $\sigma = 0$, $\gamma = n(\frac{1}{2} - \frac{1}{q}) - \frac{1}{q} = \frac{n+1}{q(n-1)}$, we obtain that $\|u\|_{L_t^1(\mathbb{R}; L^r(\mathbb{R}^n))} \leq C\|u_0\|_{H^s}$. Since $0 < \gamma \leq 1$, we have by Lemma 2.8 with $\epsilon_1 = 1+$,

$$
\|u\|_{L_t^1(\mathbb{R}; L^r(\mathbb{R}^n))} \leq C \left( \sum_{j \in \mathbb{Z}} 2^{2j\gamma} \|P_j u_0\|_{L^2}^2 \right)^{\frac{1}{2}} = C \left( \sum_{j \in \mathbb{Z}} 2^{2j\gamma} \left\| \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} P_j (a_{k,\ell}(r) Y_{k,\ell}(\theta)) \right\|_{L^2}^2 \right)^{\frac{1}{2}}
$$

$$
= C \left( \sum_{j \in \mathbb{Z}} 2^{2j\gamma} \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \left\| P_j \tilde{P}_1 (a_{k,\ell}(r) Y_{k,\ell}(\theta)) \right\|_{L^2}^2 \right) \right)^{\frac{1}{2}}
$$

$$
\leq C \left( \sum_{j \in \mathbb{Z}} 2^{2j\gamma - 2r_1|j|} \|u_0\|_{L^2}^2 \right)^{\frac{1}{2}} \leq C \|u_0\|_{L^2}.
$$

This completes the proof of Lemma 2.9. \qed

3. Estimates of Hankel transforms

In this section, we prove some estimates for the Hankel transforms of order $\nu(k)$. These estimates will be utilized to prove the main results in the next section.

**Proposition 3.1.** Let $k \in \mathbb{N}, 1 \leq \ell \leq d(k)$ and let $\varphi$ be a smooth function supported in the interval $I := [\frac{1}{2}, 2]$. Then

$$
(3.1) \quad \int_0^\infty e^{-t\rho} J_{\nu(k)}(t\rho) b_{k,\ell}(\rho) \varphi(\rho) d\rho \left\| L^2_{\nu(k)(\mathbb{R}; L^2_2(\mathbb{R}; 2\mathbb{R}^n))) \right\| \leq C \min \left\{ R^{\frac{1}{2}}, 1 \right\} \left\| b_{k,\ell}(\rho) \right\|_{L^2_2(I)}.
$$

where $R \in \mathbb{R}$ and $C$ is a constant independent of $R, k$, and $\ell$.

**Proof.** Using the Plancherel theorem in $t$, we have

$$
(3.2) \quad \text{L.H.S. of (3.1)} \leq \left\| \int J_{\nu(k)}(t\rho) b_{k,\ell}(\rho) \varphi(\rho) \right\|_{L^2_2(I)} \left\| b_{k,\ell}(\rho) \right\|_{L^2_2(I)}.
$$

We first consider the case $R \leq 1$. Since $\nu(k) > 0$, one has by (2.4)

$$
(3.3) \quad \text{L.H.S. of (3.1)} \leq \left\| b_{k,\ell}(\rho) \right\|_{L^2_2(I)} \left( \int_R^{2R} \rho^{\nu(k)} \left( \frac{1}{2^{\nu(k)} \Gamma(\nu(k) + \frac{1}{2}) \Gamma(\frac{1}{2})} \right)^{\frac{1}{2}} dr \right) \leq R^{\frac{1}{2}} \left\| b_{k,\ell}(\rho) \right\|_{L^2_2(I)}.
$$

Next we consider the case $R \gg 1$. It follows from (3.2) that (3.1) can be reduced to show

$$
(3.4) \quad \int_R^{2R} |J_k(r)|^2 dr \leq C, \quad R \gg 1,
$$

where the constant $C$ is independent of $k$ and $R$. To prove (3.4), we write

$$
\int_R^{2R} |J_k(r)|^2 dr = \int_{I_1} |J_k(r)|^2 dr + \int_{I_2} |J_k(r)|^2 dr + \int_{I_3} |J_k(r)|^2 dr
$$

where

$$
I_1 = [R, 2R] \cap [0, \frac{k}{2}], \quad I_2 = [R, 2R] \cap [\frac{k}{2}, 2k], \quad \text{and} \quad I_3 = [R, 2R] \cap [2k, \infty].
$$
Using (2.8) and (2.10) in Lemma 2.1, we have
\[(3.5) \int I_{1} |J_{k}(r)|^{2} dr \leq C \int I_{1} e^{-cr} dr \leq Ce^{-cR},\]
and
\[(3.6) \int I_{2} |J_{k}(r)|^{2} dr \leq C.\]
On the other hand, one has by (2.9)
\[(3.7) \int [\frac{1}{2}, 2k] |J_{k}(r)|^{2} dr \leq C \int [\frac{1}{2}, 2k] k^{-\frac{4}{3}}(1 + k^{-\frac{1}{3}} |r - k|)^{-\frac{1}{3}} dr \leq C.
\]
Observing $[R, 2R] \cap [\frac{k}{2}, 2k] = \emptyset$ unless $R \sim k$, we obtain
\[(3.8) \int I_{3} |J_{k}(r)|^{2} dr \leq C.
\]
This together with (3.5) and (3.6) yields (3.1).

**Proposition 3.2.** Let $\gamma \geq 2$ and let $k \in \mathbb{N}, 1 \leq \ell \leq d(k)$. Suppose supp $b_{k,\ell}(\rho) \subset I := [1, 2]$. Then
\[(3.9) \left\| \mathcal{H}_{\nu(k)} \left[ \cos(t\rho) b_{k,\ell}^{0}(\rho) \right](r) \right\|_{L_{x}^{2}(\mathbb{R}; L_{y,n-1d}(\{|R, 2R|\}))} \leq C \min \left\{ R^{-(n+1) + (n-2)\nu(k)} - \frac{n-1}{2}, R^{\frac{n-1}{2}} - \frac{n-2}{2} \right\} \|b_{k,\ell}^{0}(\rho)\|_{L_{x}^{2}(t)},\]
where $R \in 2^{\mathbb{N}}$ and $C$ is a constant independent of $R, k$ and $\ell$.

**Proof.** We first consider the case $R \gg 1$. Using the definition of Hankel transform and the interpolation, we only need to prove
\[(3.10) \left\| \int_{0}^{\infty} e^{-it\rho} J_{\nu(k)}(r\rho) b_{k,\ell}^{0}(\rho) d\rho \right\|_{L_{x}^{2}(\mathbb{R}; L_{y,n-1d}(\{|R, 2R|\}))} \lesssim R^{\frac{1}{2}} \|b_{k,\ell}^{0}(\rho)\|_{L_{x}^{2}},\]
and
\[(3.11) \left\| \int_{0}^{\infty} e^{-it\rho} J_{\nu(k)}(r\rho) b_{k,\ell}^{0}(\rho) \varphi(\rho) d\rho \right\|_{L_{x}^{2}(\mathbb{R}; L_{y,n-1d}(\{|R, 2R|\}))} \lesssim R^{-\frac{n-2}{2}} \|b_{k,\ell}^{0}(\rho)\|_{L_{x}^{2}}.
\]

(3.9) follows from Proposition 5.1. To prove (3.10), it is enough to show that there exists a constant $C$ independent of $k, \ell$ such that
\[(3.12) \left\| \int_{0}^{\infty} e^{-it\rho} J_{\nu(k)}(r\rho) b_{k,\ell}^{0}(\rho) \varphi(\rho) d\rho \right\|_{L_{x}^{2}(\mathbb{R}; L_{y,\gamma}(\{|R, 2R|\}))} \leq C \|b_{k,\ell}^{0}(\rho)\|_{L_{x}^{2}(t)},\]
and
\[(3.13) \left\| \int_{0}^{\infty} e^{-it\rho} J_{\nu(k)}(r\rho) b_{k,\ell}^{0}(\rho) \rho \varphi(\rho) d\rho \right\|_{L_{x}^{2}(\mathbb{R}; L_{y,\gamma}(\{|R, 2R|\}))} \leq C \|b_{k,\ell}^{0}(\rho)\|_{L_{x}^{2}(t)}.
\]
In fact, (3.12) follows from Proposition 3.1 and Lemma 2.2 to showing (3.13).

Secondly, we consider the case \( R \lesssim 1 \). From the definition of Hankel transform, we need to prove

\[
\left\| \int_0^\infty e^{-it\rho} J_{\nu(k)}(r \rho)b_{k,\ell}^0(\rho)(r \rho)^{-\frac{n-2}{2}} \rho^{n-1} \, d\rho \right\|_{L^2_\gamma(B; L^2([0,2R]))} \lesssim R^{\frac{2+\gamma(n-2)}{2}} \| b_{k,\ell}^0(\rho) \|_{L^2_\gamma}.
\]

On the other hand, we have by Proposition 3.1

\[
\left\| \int_0^\infty e^{-it\rho} J_{\nu(k)}(r \rho)b_{k,\ell}^0(\rho)(r \rho)^{-\frac{n-2}{2}} \rho^{n-1} \, d\rho \right\|_{L^2_\gamma(B; L^2([0,2R])))} \lesssim R^{-\frac{n-2}{2}} \| b_{k,\ell}^0(\rho) \|_{L^2_\gamma}.
\]

By interpolation, it suffices to prove the estimate

\[
\left\| \int_0^\infty e^{-it\rho} J_{\nu(k)}(r \rho)b_{k,\ell}^0(\rho)(r \rho)^{-\frac{n-2}{2}} \rho^{n-1} \, d\rho \right\|_{L^2_\gamma(B; L^2([0,2R])))} \lesssim R^{-\frac{n-2}{2} + \nu(k)} \| b_{k,\ell}^0(\rho) \|_{L^2_\gamma}.
\]

Indeed, using Sobolev embedding, we can prove (3.16) by showing

\[
\left\| \int_0^\infty e^{-it\rho} J_{\nu(k)}(r \rho)b_{k,\ell}^0(\rho)(r \rho)^{-\frac{n-2}{2}} \rho^{n-1} \, d\rho \right\|_{L^2_\gamma(B; L^2([0,2R])))} \leq CR^{\frac{1}{2} + \nu(k)} \| b_{k,\ell}^0(\rho) \|_{L^2_\gamma(B; L^2([0,2R])))},
\]

and

\[
\left\| \int_0^\infty e^{-it\rho} J_{\nu(k)}(r \rho)b_{k,\ell}^0(\rho)(r \rho)^{\frac{n-2}{2}} \rho^{n-1} \, d\rho \right\|_{L^2_\gamma(B; L^2([0,2R])))} \leq CR^{\nu(k) - \frac{1}{2}} \| b_{k,\ell}^0(\rho) \|_{L^2_\gamma(B; L^2([0,2R])))}.
\]

These two estimates are implied by (2.4) and (2.5). Therefore, we conclude this proposition.

\[\Box\]

4. PROOF OF THEOREM 1.1

In this section, we use Proposition 3.1 and Proposition 3.2 to prove Theorem 1.1. We first consider the Cauchy problem:

\[
\begin{align*}
(\partial_t - \Delta + \frac{a_k^0}{r^2})u(x,t) &= 0, \\
u(x,0) &= u_0(x), \quad \partial_t u(x,0) = 0.
\end{align*}
\]

We use the spherical harmonic expansion to write

\[
u_0(x) = \sum_{\ell=1}^{d(k)} \sum_{k=0}^{\infty} a_{k,\ell}^0(r) Y_{k,\ell}(\theta).
\]

Then we have the following proposition:

**Proposition 4.1.** Let \( \gamma = \frac{2(n-1)}{n-2} + \) and suppose \( \text{supp} \ (\mathcal{H}_\omega a_{k,\ell}^0) \subset [1,2] \) for all \( k, \ell \in \mathbb{N} \) and \( 1 \leq \ell \leq d(k) \). Then

\[
\|u(x,t)\|_{L^2_{\gamma(n-1)} L^2(\mathbb{R}^{n-1})} \leq C \|u_0\|_{L^2_\gamma}.
\]
Proof. Let us consider the equation (4.1) in polar coordinates. Write \( v(t, r, \theta) = u(t, r\theta) \) and \( g(r, \theta) = u_0(r\theta) \). Then \( v(t, r, \theta) \) satisfies that

\[
\begin{align*}
&\partial_t v - \partial_{rr} v - \frac{n-1}{r} \partial_r v - \frac{1}{r^2} \Delta_{\theta} v + \frac{n}{r^2} v = 0, \\
&v(0, r, \theta) = g(r, \theta), \quad \partial_r v(0, r, \theta) = 0.
\end{align*}
\]

(4.4)

By (4.2), we also have

\[
g(r, \theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} a_{k,\ell}^0(r) Y_{k,\ell}(\theta).
\]

Using separation of variables, we can write

\[
\tilde{v}_{k,\ell}(t, r, \theta) = b_{k,\ell}^0(r), \quad \partial_r \tilde{v}_{k,\ell}(0, r, \theta) = 0,
\]

where \( v_{k,\ell} \) satisfies the following equation

\[
\begin{align*}
&\partial_t v_{k,\ell} - \partial_{rr} v_{k,\ell} - \frac{n-1}{r} \partial_r v_{k,\ell} + \frac{k(k+n-2)+a}{r^2} v_{k,\ell} = 0, \\
v_{k,\ell}(0, r) = a_{k,\ell}^0(r), \quad \partial_r v_{k,\ell}(0, r) = 0.
\end{align*}
\]

(4.5)

for each \( k, \ell \in \mathbb{N} \), and \( 1 \leq \ell \leq d(k) \). From the definition of \( A_{\nu} \), it becomes

\[
\begin{align*}
&\partial_t v_{k,\ell} + A_{\nu(k)} v_{k,\ell} = 0, \\
v_{k,\ell}(0, r) = a_{k,\ell}^0(r), \quad \partial_r v_{k,\ell}(0, r) = 0.
\end{align*}
\]

(4.6)

Applying the Hankel transform to the equation (4.4), we have by Lemma 2.5

\[
\begin{align*}
&\partial_t \tilde{v}_{k,\ell} - \partial_{rr} \tilde{v}_{k,\ell} + \rho^2 \tilde{v}_{k,\ell} = 0, \\
&\tilde{v}_{k,\ell}(0, \rho) = b_{k,\ell}^0(\rho), \quad \partial_r \tilde{v}_{k,\ell}(0, \rho) = 0,
\end{align*}
\]

(4.7)

where

\[
\tilde{v}_{k,\ell}(t, \rho) = (H_{\nu} v_{k,\ell})(t, \rho), \quad b_{k,\ell}^0(\rho) = (H_{\nu} a_{k,\ell}^0)(\rho).
\]

(4.8)

Solving this ODE and using the inverse Hankel transform, we obtain

\[
\begin{align*}
v_{k,\ell}(t, r) &= \int_0^\infty (\rho^2) \frac{J_{\nu(k)}(\rho r)}{\rho} (\cos(t\rho) b_{k,\ell}^0(\rho) \rho^{n-1} d\rho = \\
&= \frac{1}{2} \int_0^\infty (\rho^2) \frac{2J_{\nu(k)}(\rho r)}{\rho} (e^{it\rho} + e^{-it\rho}) b_{k,\ell}^0(\rho) \rho^{n-1} d\rho.
\end{align*}
\]

Therefore, we get

\[
\begin{align*}
u(x, t) &= v(t, r, \theta) \\
&= \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} Y_{k,\ell}(\theta) \int_0^\infty (\rho^2) \frac{2J_{\nu(k)}(\rho r)}{\rho} \cos(t\rho) b_{k,\ell}^0(\rho) \rho^{n-1} d\rho \\
&= \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} Y_{k,\ell}(\theta) H_{\nu(k)} \left[ \cos(t\rho) b_{k,\ell}^0(\rho) \right](r).
\end{align*}
\]

(4.9)

To prove (4.3), it suffices to show

\[
\begin{align*}
\left\| \int_0^\infty \frac{d(k)}{2} \left[ H_{\nu(k)} \left[ \cos(t\rho) b_{k,\ell}^0(\rho) \right] (r) \right]^2 \right\|_{L^2_x([0, \infty))} \leq C \|u_0\|_{L^2_x}.
\end{align*}
\]

(4.10)
Using the dyadic decomposition, we have by $\ell^2 \hookrightarrow \ell^\gamma (\gamma > 2)$

\begin{align}
(4.11) \quad & \left\| \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \left| \mathcal{H}_{\nu(k)} \left[ \cos(tp) b_{k,\ell}^0(\rho) \right] (r) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^2_{r}(\mathbb{R};L^\gamma_{\nu_{n-1}}(\mathbb{R}^+) )}^2 \\
\quad & = \left\| \left( \sum_{R \in 2^k} \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \left| \mathcal{H}_{\nu(k)} \left[ \cos(tp) b_{k,\ell}^0(\rho) \right] (r) \right|^2 \right)^{\gamma} \right)^{\frac{1}{\gamma}} \right\|_{L^2_{r}(\mathbb{R};L^\gamma_{\nu_{n-1}}(\mathbb{R}^+) ))}^2 \\
\quad & \lesssim \left\| \sum_{R \in 2^k} \sum_{k=0}^{\infty} \left| \mathcal{H}_{\nu(k)} \left[ \cos(tp) b_{k,\ell}^0(\rho) \right] (r) \right|^2 \right\|_{L^2_{r}(\mathbb{R};L^\gamma_{\nu_{n-1}}(\mathbb{R}^+) ))}.
\end{align}

By Proposition 3.2 we obtain

\begin{align}
(4.12) \quad & \left\| \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \left| \mathcal{H}_{\nu(k)} \left[ \cos(tp) b_{k,\ell}^0(\rho) \right] (r) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^2_{r}(\mathbb{R};L^\gamma_{\nu_{n-1}}(\mathbb{R}^+) ))}^2 \\
\quad & \lesssim \sum_{R \in 2^k} \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \min \left\{ R^{-\max(1/2, 1/2, 1/2, 1/2)} - \frac{\nu_{n+2}}{2}, R^{-\max(1/2, 1/2, 1/2, 1/2)} - \frac{\nu_{n+2}}{2} \right\} \left\| b_{k,\ell}^0(\rho) \right\|_{L^2_{r}}^2 \\
\quad & \lesssim \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \left\| b_{k,\ell}^0(\rho) \right\|_{L^2_{r}}^2.
\end{align}

Since supp $b_{k,\ell}^0(\rho) \subset [1, 2]$, we have

\begin{align*}
\sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \left\| b_{k,\ell}^0(\rho) \right\|_{L^2_{r}}^2 \lesssim \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \left\| \left( \mathcal{H}_{\nu(k)} a_{k,\ell}^0(\rho) \right) (r) \right\|_{L^2_{\nu_{n-1}}(\mathbb{R}^+) )}^2.
\end{align*}

It follows from Lemma 2.7 that

\begin{align*}
\sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \left\| \left( \mathcal{H}_{\nu(k)} a_{k,\ell}^0(\rho) \right) (r) \right\|_{L^2_{\nu_{n-1}}(\mathbb{R}^+) )}^2 = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \left\| a_{k,\ell}^0(\rho) \right\|_{L^2_{\nu_{n-1}}(\mathbb{R}^+) )}^2 = \| u_0(x) \|_{L^2_{\nu_{n}}(\mathbb{R}^+) )}^2.
\end{align*}

Therefore, we complete the proof of (4.3). \(\square\)

Now we turn to prove Theorem 11. We choose $\beta \in C^\infty_0(\mathbb{R}^+)$ supported in $[\frac{1}{2}, 2]$ such that $\sum_{\nu \in 2^k} \beta(\nu) = 1$ for all $\rho \in \mathbb{R}^+$. Let $\beta_N(\rho) = \beta(\frac{\rho}{N})$ and $\tilde{\beta}_N$ be similar to $\beta_N$. For simplicity, we assume $u_1 = 0$. Then we can write

\begin{align}
(4.13) \quad u(x, t) = \sum_{M \in 2^k} \left\{ Y_{0,1} (\theta) \mathcal{H}_{\nu(0)} \left[ \cos(tp) b_{0,1}^0(\rho) \beta_M(\rho) \right] (r) \right. \\
\quad + \sum_{N \in 2^k} \sum_{k} \tilde{\beta}_N(k) \sum_{\ell=1}^{d(k)} \left( Y_{k,\ell}(\theta) \mathcal{H}_{\nu(k)} \left[ \cos(tp) b_{k,\ell}^0(\rho) \beta_M(\rho) \right] (r) \right) \right\} \\
\quad := u_<(x, t) + u_>(x, t).
\end{align}
Without loss of the generality, it suffices to estimate $u_>(x,t)$. By Lemma 2.3, Lemma 2.4 and the scaling argument, we show that for $2 \leq q, r$ and $r < \infty$

\[(4.14)\]

\[\|u_>(t,x)\|_{L^q_t(L^r_x(\mathbb{R}^n))}^2 \lesssim \sum_{M \in 2^\mathbb{Z}} \sum_{N \in 2^\mathbb{Z}} \left\| \sum_k \hat{\beta}_N(k) \sum_{\ell=1}^{d(k)} \hat{Y}_{k,\ell}(\theta) \mathcal{H}_{\nu(k)} \left[ \cos(t\rho) b_{k,\ell}(\rho) \beta_M(\rho) \right](r) \right\|_{L^q_t(L^r_x(\mathbb{R}^n))}^2 \]

\[\lesssim \sum_{M \in 2^\mathbb{Z}} M^{2(n-\frac{4}{q} - \frac{2}{r})} \sum_{N \in 2^\mathbb{Z}} \left\| \sum_k \hat{\beta}_N(k) \sum_{\ell=1}^{d(k)} \hat{Y}_{k,\ell}(\theta) \mathcal{H}_{\nu(k)} \left[ \cos(t\rho) b_{k,\ell}(M\rho) \beta(\rho) \right](r) \right\|_{L^q_t(L^r_x(\mathbb{R}^n))}^2. \]

**Case 1:** $n \geq 4$. We have by interpolation

\[(4.15)\]

\[\left\| \sum_k \hat{\beta}_N(k) \sum_{\ell=1}^{d(k)} \hat{Y}_{k,\ell}(\theta) \mathcal{H}_{\nu(k)} \left[ \cos(t\rho) b_{k,\ell}(M\rho) \beta(\rho) \right](r) \right\|_{L^q_t(L^r_x(\mathbb{R}^n))} \lesssim \left\| \sum_k \hat{\beta}_N(k) \sum_{\ell=1}^{d(k)} \hat{Y}_{k,\ell}(\theta) \mathcal{H}_{\nu(k)} \left[ \cos(t\rho) b_{k,\ell}(M\rho) \beta(\rho) \right](r) \right\|_{L^{\lambda}_{L^\lambda}}^{\frac{1}{\lambda}} \left\| \sum_k \hat{\beta}_N(k) \sum_{\ell=1}^{d(k)} \hat{Y}_{k,\ell}(\theta) \mathcal{H}_{\nu(k)} \left[ \cos(t\rho) b_{k,\ell}(M\rho) \beta(\rho) \right](r) \right\|_{L^{\frac{1}{\lambda}}_{L^{\frac{1}{\lambda}}}}^{1-\frac{1}{\lambda}}, \]

where

\[(4.16)\]

\[\frac{1}{q} = \frac{\lambda}{2} + \frac{1-\lambda}{\infty}, \quad \frac{1}{r} = \frac{\lambda}{\gamma_0} + \frac{1-\lambda}{2}, \quad \frac{1}{\gamma_0} = \frac{q}{2} \left( \frac{1}{r} - \frac{1}{2} \right). \]

Since $(q, r) \in \Lambda$, one has $\frac{2(n-1)}{n-2} < \gamma_0 \leq \frac{2(n-1)}{n-3}$. By (4.13) and the argument in proving Proposition 4.1 one has

\[(4.17)\]

\[\left\| \sum_k \hat{\beta}_N(k) \sum_{\ell=1}^{d(k)} \hat{Y}_{k,\ell}(\theta) \mathcal{H}_{\nu(k)} \left[ \cos(t\rho) b_{k,\ell}(M\rho) \beta(\rho) \right](r) \right\|_{L^2_t(L^{\frac{2(n-1)}{n-4}}(\mathbb{R}^n))} \lesssim N^{\frac{1}{2} + \frac{d(k)}{2}} \left( \sum_k \left\| \hat{\beta}_N(k) \mathcal{H}_{\nu(k)} \left[ \cos(t\rho) b_{k,\ell}(M\rho) \beta(\rho) \right](r) \right\|_{L^q_t(L^r_x(\mathbb{R}^n)))}^2 \right)^{\frac{1}{2}} \left\| \sum_k \left\| \hat{\beta}_N(k) \mathcal{H}_{\nu(k)} \left[ \cos(t\rho) b_{k,\ell}(M\rho) \beta(\rho) \right](r) \right\|_{L^q_t(L^r_x(\mathbb{R}^n)))}^{2(n-1)} \right)^{\frac{1}{2}}. \]

On the other hand, using the endpoint Strichartz estimate in Lemma 2.4, we have

\[(4.18)\]

\[\left\| \sum_k \hat{\beta}_N(k) \sum_{\ell=1}^{d(k)} \hat{Y}_{k,\ell}(\theta) \mathcal{H}_{\nu(k)} \left[ \cos(t\rho) b_{k,\ell}(M\rho) \beta(\rho) \right](r) \right\|_{L^2_t(L^{\frac{2(n-1)}{n-4}}(\mathbb{R}^n))} \lesssim \left\| \left( \sum_k \left\| \hat{\beta}_N(k) b_{k,\ell}(M\rho) \beta(\rho) \right\|_{L^p}^2 \right)^{\frac{1}{2}} \right\|_{L^2}. \]
Therefore, we obtain by interpolation

$$
\left\| \sum_k \tilde{\beta}_N(k) \sum_{\ell=1}^{d(k)} Y_{k,\ell}(\theta) \mathcal{H}_{V}(k) \left[ \cos(t\rho)b_{k,\ell}^0(M\rho)\beta(\rho) \right](r) \right\|_{L_2^2(\mathbb{R};L_{\infty}^2(\mathbb{R}^n))}^2
\lesssim N^{(n-1)}(\frac{1}{\theta_0} - \frac{n-1}{2n + 3})^+ \left\| \left( \sum_k \sum_{\ell=1}^{d(k)} \tilde{\beta}_N(k) b_{k,\ell}^0(M\rho)\beta(\rho) \right) \right\|_{L_2^2}^2.
$$

(4.19)

By Lemma 2.28 we have

$$
\left\| \left[ \mathcal{H}_{V}(k) a_{k,\ell} \right](r) \right\|_{L_{\infty}^2(\mathbb{R}^{n-1};L_{\infty}^2(\mathbb{R}^n))} = \left\| a_{k,\ell}(\rho) \right\|_{L_{\infty}^2(\mathbb{R}^{n-1};L_{\infty}^2(\mathbb{R}^n))}.
$$

We are in spirit of energy estimate to obtain

$$
\left\| \sum_k \tilde{\beta}_N(k) \sum_{\ell=1}^{d(k)} Y_{k,\ell}(\theta) \mathcal{H}_{V}(k) \left[ \cos(t\rho)b_{k,\ell}^0(M\rho)\beta(\rho) \right](r) \right\|_{L_\infty^2(\mathbb{R};L_{\infty}^2(\mathbb{R}^{n-1});L_2^2(\mathbb{R}^n))}^2
\lesssim \left\| \left( \sum_k \sum_{\ell=1}^{d(k)} \tilde{\beta}_N(k) b_{k,\ell}^0(M\rho)\beta(\rho) \right) \right\|_{L_2^2(\mathbb{R};L_{\infty}^2(\mathbb{R}^{n-1})))}^2
\lesssim \left( \sum_k \sum_{\ell=1}^{d(k)} \tilde{\beta}_N(k) \left\| b_{k,\ell}^0(M\rho)\beta(\rho) \right\|_{L_2^2}^2 \right)^{\frac{1}{2}}.
$$

(4.20)

Combining (4.14), (4.15), (4.19), and (4.20), we have

$$
\left\| u_\geq(t,x) \right\|_{L_\infty^2(\mathbb{R};L_{\infty}^2(\mathbb{R}^n))}^2
\lesssim \sum_{M \in 2^\mathbb{Z}} M^{2(n-\frac{1}{2} - \frac{1}{q})} \sum_{N \in 2^\mathbb{N}} N^{2\lambda(n-1)(\frac{1}{\theta_0} - \frac{n-1}{2n + 3})} \sum_k \sum_{\ell=1}^{d(k)} \tilde{\beta}_N(k) \left\| b_{k,\ell}^0(M\rho)\beta(\rho) \right\|_{L_2^2}^2
\lesssim \sum_{M \in 2^\mathbb{Z}} M^{2(n-\frac{1}{2} - \frac{1}{q})} \sum_{N \in 2^\mathbb{N}} N^{2\lambda(n-1)(\frac{1}{\theta_0} - \frac{n-1}{2n + 3}) + \frac{1}{2}} \sum_k \sum_{\ell=1}^{d(k)} \tilde{\beta}_N(k) \left\| b_{k,\ell}^0(M\rho)\beta(\rho) \right\|_{L_2^2}^2.
$$

(4.21)

By making use of Lemma 2.28 \( s = n(\frac{1}{2} - \frac{1}{q}) - \frac{1}{q}, \) and \( \bar{s} = (1 + \epsilon)(\frac{1}{q} - (n-1)(\frac{1}{2} - \frac{1}{q})), \) we get

$$
\left\| u_\geq(t,x) \right\|_{L_\infty^2(\mathbb{R};L_{\infty}^2(\mathbb{R}^n)))} \lesssim \left\| \Omega \right\|_{H^s} \left\| u_0 \right\|_{H^{\bar{s}}}.
$$

(4.22)

\textbf{Case 2:} \( n = 3. \) Since the endpoint Strichartz estimate fails, the above argument breaks down. By the interpolation, we have

$$
\left\| \sum_k \tilde{\beta}_N(k) \sum_{\ell=1}^{d(k)} Y_{k,\ell}(\theta) \mathcal{H}_{V}(k) \left[ \cos(t\rho)b_{k,\ell}^0(M\rho)\beta(\rho) \right](r) \right\|_{L_\infty^2(\mathbb{R};L_{\infty}^2(\mathbb{R}^n)))}^2
\lesssim \left\| \left( \sum_k \sum_{\ell=1}^{d(k)} \tilde{\beta}_N(k) b_{k,\ell}^0(M\rho)\beta(\rho) \right) \right\|_{L_2^2(\mathbb{R};L_{\infty}^2(\mathbb{R}^n)))}^2
\times \left\| \left( \sum_k \sum_{\ell=1}^{d(k)} \tilde{\beta}_N(k) b_{k,\ell}^0(M\rho)\beta(\rho) \right) \right\|_{L_2^2(\mathbb{R};L_{\infty}^2(\mathbb{R}^n)))}^2.
$$

(4.23)
where

\[(4.24)\quad \frac{1}{q} = \frac{\lambda}{2} + \frac{1 - \lambda}{q_0}, \quad \frac{1}{r} = \frac{\lambda}{4 + \gamma} + \frac{1 - \lambda}{r_0}, \quad \frac{1}{2} = \frac{1}{q_0} + \frac{1}{r_0}, \quad r_0 \neq \infty.\]

By (4.13) and the argument in proving Proposition 4.11 one has

\[(4.25)\quad \left\| \sum_k \tilde{\beta}_N(k) \sum_{\ell=1}^{d(k)} Y_{k,\ell}(\theta) \mathcal{H}_\nu(k) \left( \cos(t\rho) b^{0}_{k,\ell}(M\rho) \beta(\rho) \right)(r) \right\|_{L^2_t(\mathbb{R}; L^\frac{2}{n-1+\frac{\gamma}{4}}(\mathbb{R}^n))} \leq N^{\frac{n+\gamma}{2}} \left\| \left( \sum_k \tilde{\beta}_N(k) \sum_{\ell=1}^{d(k)} \left| \cos(t\rho) b^{0}_{k,\ell}(M\rho) \beta(\rho) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^2_r(\mathbb{R}; L^\frac{2}{n-1+\frac{\gamma}{4}}(\mathbb{R}^n))} \leq N^{\frac{n+\gamma}{2}} \left\| \left( \sum_k \tilde{\beta}_N(k) b^{0}_{k,\ell}(M\rho) \beta(\rho) \right)^2 \right\|^{\frac{1}{2}}_{L^2_r(\mathbb{R}; L^\frac{2}{n-1+\frac{\gamma}{4}}(\mathbb{R}^n))}.
\]

On the other hand, by the Strichartz estimate with \((q_0, r_0)\) in Lemma 2.9 we have

\[(4.26)\quad \left\| \sum_k \tilde{\beta}_N(k) \sum_{\ell=1}^{d(k)} Y_{k,\ell}(\theta) \mathcal{H}_\nu(k) \left( \cos(t\rho) b^{0}_{k,\ell}(M\rho) \beta(\rho) \right)(r) \right\|_{L^2_t(\mathbb{R}; L^\frac{2}{n-1+\frac{\gamma}{4}}(\mathbb{R}^n))} \leq \left\| \left( \sum_k \tilde{\beta}_N(k) b^{0}_{k,\ell}(M\rho) \beta(\rho) \right)^2 \right\|^{\frac{1}{2}}_{L^2_r(\mathbb{R}; L^\frac{2}{n-1+\frac{\gamma}{4}}(\mathbb{R}^n))}\]

This together with (1.14), (1.23) and (4.25) yields that

\[(4.27)\quad \left\| u_{\geq}(t, x) \right\|_{L^2_t(\mathbb{R}; L^\frac{2}{n-1+\frac{\gamma}{4}}(\mathbb{R}^n))} \leq \sum_{\ell=1}^{d(k)} N^{\lambda+} \left\| \tilde{\beta}_N(k) b^{0}_{k,\ell}(M\rho) \beta(\rho) \right\|_{L^2_r(\mathbb{R}; L^\frac{2}{n-1+\frac{\gamma}{4}}(\mathbb{R}^n))} \leq N^{\lambda+} \left\| \tilde{\beta}_N(k) b^{0}_{k,\ell}(M\rho) \beta(\rho) \right\|_{L^2_r(\mathbb{R}; L^\frac{2}{n-1+\frac{\gamma}{4}}(\mathbb{R}^n))}.
\]

Since \(s = n(\frac{1}{2} - \frac{1}{q}) - \frac{1}{q} \), by the scaling argument, Lemma 2.8 we get

\[(4.28)\quad \left\| u_{\geq}(t, x) \right\|_{L^2_t(\mathbb{R}; L^\frac{2}{n-1+\frac{\gamma}{4}}(\mathbb{R}^n))} \leq \left\| (\Omega)^{\frac{s}{2}} u_0 \right\|_{H^s}.\]

Moreover, for \(q = 2, 4 < r < \infty, (4.14)\) and the “Bernstein” inequality (2.13) imply that

\[(4.29)\quad \left\| u_{\geq}(t, x) \right\|_{L^2_t(\mathbb{R}; L^\frac{2}{n-1+\frac{\gamma}{4}}(\mathbb{R}^n))} \leq \sum_{\ell=1}^{d(k)} N^{\lambda+} \left\| \tilde{\beta}_N(k) b^{0}_{k,\ell}(M\rho) \beta(\rho) \right\|_{L^2_r(\mathbb{R}; L^\frac{2}{n-1+\frac{\gamma}{4}}(\mathbb{R}^n))} \leq N^{\lambda+} \left\| \tilde{\beta}_N(k) b^{0}_{k,\ell}(M\rho) \beta(\rho) \right\|_{L^2_r(\mathbb{R}; L^\frac{2}{n-1+\frac{\gamma}{4}}(\mathbb{R}^n))}.
\]

Combining this with (4.22) and (4.28), we complete the proof of Theorem 4.1.
5. Proof of Theorem 1.2

To prove Theorem 1.2, we first use the inhomogeneous Strichartz estimates for the wave equation without potential in [7, 13] and the arguments in [14] to prove an inhomogeneous Strichartz estimates for the wave equation with inverse-square potential.

Proposition 5.1 (Inhomogeneous Strichartz estimates). Let $\Box = \partial_t^2 + A_v$ and let $v$ solve the inhomogeneous wave equation $\Box v = h$ in $\mathbb{R} \times \mathbb{R}^n$ with zero initial data. If $\nu > \max\{\nu_0 - \frac{2}{q_0}, \frac{2}{r_0} - \frac{2}{p_0} - 2\}$, then

\[ \|v\|_{L_t^\nu(\mathbb{R} \times \mathbb{R}^n)} \lesssim \|h\|_{L_t^{\nu_0}(\mathbb{R} \times \mathbb{R}^n)}, \]

where $q_0 = (p-1)(n+1)/2$ and $r_0 = (n+1)(p-1)/(2p)$ with $p_h < p < p_{\text{conf}}$.

Proof. By the continuity property of $\mathcal{K}_{\nu,\lambda}^0$ in Lemma 2.6, it follows that

\[ \|v\|_{L_t^\nu(\mathbb{R} \times \mathbb{R}^n)} \leq \|\mathcal{K}_{\nu,\lambda}^0\|_{q_0 \to q_0} \|\mathcal{K}_{\lambda,\nu}^0 h\|_{L_t^{r_0}(\mathbb{R} \times \mathbb{R}^n)}. \]

Noting that (2.22) with $k = 0$, one has $\mathcal{K}_{\nu,\lambda}^0 h = \mathcal{K}_{\lambda,\nu}^0 v = \square \mathcal{K}_{\lambda,\nu}^0 v$. We recall the inhomogeneous Strichartz estimates, for $\frac{1}{q} - \frac{1}{q_h} = \frac{2}{n+1}$ and $\frac{2n}{n-1} < q < \frac{2(n+1)}{n-1},$

\[ \|u\|_{L_t^p(\mathbb{R} \times \mathbb{R}^n)} \leq C \|(\partial_t - \Delta)u\|_{L_t^{r_0}(\mathbb{R} \times \mathbb{R}^n)}, \]

which was shown by Harmsen [7] and Oberlin [13]. Thus we obtain

\[ \|v\|_{L_t^\nu(\mathbb{R} \times \mathbb{R}^n)} \leq C \|h\|_{L_t^{r_0}(\mathbb{R} \times \mathbb{R}^n)}, \]

where we use the facts that $\frac{1}{r_0} - \frac{1}{q_0} = \frac{2}{n+1}$ and $\frac{2n}{n-1} < q_0 < \frac{2(n+1)}{n-1}. \quad \Box$

Now we are in position to prove Theorem 1.2. Define the solution map

\[ \Phi(u) = \cos \left( t \sqrt{P_o} \right) u_0(x) + \frac{\sin \left( t \sqrt{P_o} \right)}{\sqrt{P_o}} u_1(x) + \int_0^t \frac{\sin \left( (t-s) \sqrt{P_o} \right)}{\sqrt{P_o}} F(u(s,x)) \, ds \]

\[ := u_{\text{hom}} + u_{\text{inh}}, \]

on the complete metric space $X = \{ u : u \in C_t(\dot{H}^{s_\epsilon}) \cap L_t^{q_0}, \|u\|_{L_t^{q_0}} \leq 2C\epsilon \}$ with the metric $d(u_1, u_2) = \|u_1 - u_2\|_{L_t^{q_0}}$, where $P_o = -\Delta + \frac{a}{|x|^2}$ with $a$ satisfying (1.7), and $\epsilon \leq \epsilon(p)$ is as in (1.8).

From Lemma 2.7 and (1.8), we obtain

\[ \|u_{\text{hom}}\|_{C_t(\dot{H}^{s_\epsilon}) \cap L_t^{q_0}} \leq C \left( \|u_0\|_{\dot{H}^{s_\epsilon}} + \|u_1\|_{\dot{H}^{s_\epsilon-1}} \right) \leq C\epsilon. \]

By Proposition 5.1 and the inhomogeneous version of the Strichartz estimates (1.2), one has

\[ \|u_{\text{inh}}\|_{C_t(\dot{H}^{s_\epsilon}) \cap L_t^{q_0}} \leq C \|F(u)\|_{L_t^{r_0}} \leq C \|u\|_{L_t^{q_0}}^p \leq C^2 (C\epsilon)^{p-1} \epsilon \leq C\epsilon. \]

A similar argument as above leads to

\[ \|\Phi(u_1) - \Phi(u_2)\|_{L_t^{q_0}} \leq C (F(u_1) - F(u_2))_{L_t^{r_0}} \leq \frac{C^2 (C\epsilon)^{p-1}}{2} \|u_1 - u_2\|_{L_t^{q_0}} \leq \frac{1}{2} \|u_1 - u_2\|_{L_t^{q_0}}. \]
Therefore, the solution map $\Phi$ is a contraction map on $X$ under the metric $d(u_1, u_2) = \|u_1 - u_2\|_{L^{\infty}_x}$. The standard contraction argument gives the proof.

6. Appendix: The Proof of Lemma 2.3

We will apply the Hörmander’s technique to showing a weak-type $(1, 1)$ estimate for the multiplier operators with respect to the Hankel transform.

The multiplier operators associated with the Hankel transform are defined by

\begin{equation}
[L_j f](r) = \int_0^{\infty}(r \rho)^{\frac{n-2}{2}} J_\nu(r \rho)[\mathcal{H}_\nu f](\rho) \beta_j(\rho) d\omega(\rho), \quad j \in \mathbb{Z}
\end{equation}

where

\begin{equation}
(\mathcal{H}_\nu f)(\rho) = \int_0^{\infty}(r \rho)^{\frac{n-2}{2}} J_\nu(r \rho)f(r) d\omega(r), \quad \text{and } d\omega(\rho) = \rho^{n-1} d\rho.
\end{equation}

Since $\mathcal{H}_\nu = \mathcal{H}_\nu^{-1}$, we have $\mathcal{H}_\nu[L_j f] = \beta_j(\rho)[\mathcal{H}_\nu f]$. We first claim that

\begin{equation}
\left\| \left( \sum_{j \in \mathbb{Z}} |L_j f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\omega)} \sim \left\| \sum_{j \in \mathbb{Z}} L_j f \right\|_{L^p(\omega)} \sim \|f\|_{L^p(\omega)}.
\end{equation}

This implies Lemma 2.3 by choosing $f = \mathcal{H}_\nu[\cos(tp) b_{k,\ell}(\rho)]$. To show (6.3), we need the following

**Lemma 6.1.** Let $f \in L^p(\omega)$, $1 < p < \infty$. Then there exists a constant $C_p$ such that

\begin{equation}
\left\| \left( \sum_{j \in \mathbb{Z}} |L_j f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\omega)} \leq C_p \|f\|_{L^p(\omega)}.
\end{equation}

We postpone the proof for a moment. By duality, one has

\[ \|f\|_{L^p(\omega)} = \sup_{\|g\|_{L^{p'}(\omega)} \leq 1} \int_0^{\infty} f(r) \overline{g(r)} d\omega(r). \]

By Lemma 2.3 we observe that

\[ \int_0^{\infty} f(r) \overline{g(r)} d\omega(r) = \sum_{j, j' \in \mathbb{Z}} \int_0^{\infty} \mathcal{H}_\nu[L_j f](\rho) \mathcal{H}_\nu[L_{j'} g](\rho) d\omega(\rho) \]

\[ = \sum_{j, j' \in \mathbb{Z}} \int_0^{\infty} \beta_j(\rho) \beta_{j'}(\rho) [\mathcal{H}_\nu f](\rho) [\mathcal{H}_\nu g](\rho) d\omega(\rho) \]

\[ \leq C \sum_{j \in \mathbb{Z}} \int_0^{\infty} \beta_j(\rho) \beta_j(\rho) [\mathcal{H}_\nu f](\rho) [\mathcal{H}_\nu \overline{g}(\rho)] d\omega(\rho). \]

This implies that

\begin{equation}
\int_0^{\infty} f(r) \overline{g(r)} d\omega(r) \leq C \sum_{j \in \mathbb{Z}} \int_0^{\infty} |L_j f|(r) |L_j g|(r) d\omega(r).
\end{equation}

Hence by Lemma 6.1 we obtain

\[ \|f\|_{L^p(\omega)} \leq C \sup_{\|g\|_{L^{p'}(\omega)} \leq 1} \left[ \left( \sum_{j \in \mathbb{Z}} |L_j f|^2 \right)^{\frac{1}{2}} \right]_{L^p(\omega)} \left( \sum_{j \in \mathbb{Z}} |L_j g|^2 \right)^{\frac{1}{2}}_{L^{p'}(\omega)} \]

\begin{equation}
\leq C \left[ \left( \sum_{j \in \mathbb{Z}} |L_j f|^2 \right)^{\frac{1}{2}} \right]_{L^p(\omega)}.
\end{equation}
This together with (6.4) gives (6.3). When \( p = 2 \), we have by Lemma 2.5
\[
\left\| \left( \sum_{j \in \mathbb{Z}} |L_j f|^2 \right)^{\frac{1}{2}} \right\|_{L^2(\omega)} = \sum_{j \in \mathbb{Z}} \left| L_j f \right|_{L^2(\omega)}^2 = \int_0^\infty \sum_{j \in \mathbb{Z}} |\beta_j(\rho)|^2 |\mathcal{H}_\nu f|^2 d\omega(\rho) \leq C \| f \|_{L^2(\omega)}.
\]
Define the operator \( S(f) \) by \( f \mapsto \{L_j f\}_{j \in \mathbb{Z}} \), then \( \| S(f) \|_{L^2(\omega, l^2(\mathbb{Z}))} \leq C \| f \|_{L^2(\omega)} \).

To show (6.4), it suffices to prove
\[
\left\| S(f) \right\|_{L^1(\omega, l^2(\mathbb{Z}))} \leq C \| f \|_{L^1(\omega)},
\]
where \( L^1(\omega) \) denotes the weak-\( L^1(\omega) \).

Define the generalized convolution \( f \# g \) as
\[
f \# g(x) = \int_0^\infty (\tau_x f)(y)g(y)d\omega(y), \quad x \in \mathbb{R}^+.
\]
where \( f, g \in L^1(\omega) \), the Hankel translation \( \tau_x f \) is denoted to be
\[
(\tau_x f)(y) = \int_0^\infty K_\nu(x, y, z)f(z)dz, \quad x, y \in \mathbb{R}^+.
\]
and
\[
K_\nu(x, y, z) = \int_0^\infty (xt)^{-\frac{n-2}{2}} J_\nu(xt)(yt)^{-\frac{n-2}{2}} J_\nu(yt)(zt)^{-\frac{n-2}{2}} J_\nu(zt)d\omega(t), \quad x, y, z \in \mathbb{R}^+.
\]
Then \( \mathcal{H}_\nu[f \# g] = \mathcal{H}_\nu(f) \mathcal{H}_\nu(g) \). Moreover, we have \( L_j f = k_j \# f \) with \( k_j = \mathcal{H}_\nu(\beta_j) \).

Taking into account the fact that \( (\tau_x f)(y) = (\tau_y f)(x) \) and Theorem 2.4 in [5], it suffices to prove the Hankel version of the well-known Hörmander condition
\[
\int_{|x-y_0|>2|y-y_0|} (\sum_{j \in \mathbb{Z}} \left| \tau_y k_j(x) - \tau_{y_0} k_j(x) \right|^2)^{\frac{1}{2}} d\omega(x) \leq C,
\]
where \( C \) is independent of \( y, y_0 \). This is implied by
\[
\sum_{j \in \mathbb{Z}} \int_{|x-y_0|>2|y-y_0|} \left| \tau_y k_j(x) - \tau_{y_0} k_j(x) \right| d\omega(x) \leq C,
\]
which can be proved by the arguments in [3, 6].

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