GEOMETRIC CONTINUED FRACTIONS
AS INVARIANTS
IN THE TOPOLOGICAL CLASSIFICATION
OF ANOSOV Diffeomorphisms
OF TORI

GRISHA KOLUTSKY*

Abstract. We show how an object from the combinatorially geometric version of the analytical number theory, namely geometric continued fractions, appears in the classical smooth dynamics, namely in the problem on the topological classification of Anosov diffeomorphisms of tori.

Contents
1. Introduction. The historical overview 2
2. Basic notations and posing the problem 2
3. From the topological to the linear classification 3
4. Complete solution in the case $n = 2$ 4
5. Notations of geometric continued fractions 4
6. Invariants in the case $n > 2$ 5
7. Algorithms in the case $n > 2$ 6
8. Lagrange’s theorem and its generalizations 9
9. Additional invariants in the case of $SL_n(\mathbb{Z})$-classification 10
10. Recovery of a hyperbolic automorphism from a period 11
11. Acknowledgements 11
References 11

Date: 22 June 2009.

Key words and phrases. Anosov diffeomorphisms, hyperbolicity, topological classification, geometric continued fractions, sails.

Mathematics Subject Classification 2000. Primary 37C15, 11H06, 11J70 Secondary 37D20, 15A36.

*The work exposed here was partially supported by grants 7-01-00017-a and 08-01-00342-a of the Russian Foundation for Basic Research, by the grant No. NSh-3038.2008.1 of the President of Russia for support of leading scientific schools and by the Russian Universities grant No. RNP.2.1.1.5055.
1. Introduction. The historical overview

The problem on topological classification of Anosov diffeomorphisms of $n$-dimensional torus was first considered in 1960s. J. Franks in 1969 and C. Manning in 1973 proved that every Anosov diffeomorphism of $n$-torus, $\mathbb{T}^n$ ($n > 1$), is topologically conjugate to a linear hyperbolic automorphism. It is easy to see that two linear hyperbolic automorphisms are topologically conjugate if and only if they are conjugate by a linear automorphism. So initial problem was reduced to the linear classification of the linear hyperbolic automorphisms of $\mathbb{T}^n$.

There were some algebraic attempts to solve the (reduced) algebraic question. We mention some of them in the Section 3.

In the case $n = 2$ a solution of the last problem goes back to Gauss and Lagrange. A full invariant is a pair — trace of the linear hyperbolic operator and the period of a continued fraction for the slope of an eigenvector of the operator (this statement seems to be well-known in analytical number theory, but we know only one reference [2]). A geometrical interpretation of this invariant is the geometric continued fractions constructed by Klein (for historical survey of the subject see [8]).

Our results presented here are linear classifications of the linear hyperbolic automorphisms of $\mathbb{T}^n$ in the case $n > 2$ using notations of geometric continued fractions, namely Klein’s and Klein-Voronoi’s generalizations (see the Theorem 5 and the Theorem 6).

Recently many good results on the geometric continued fractions and related questions were obtained by different scientists mainly due to interest of V. I. Arnold to this subject. See [8] for further references.

2. Basic notations and posing the problem

In the definitions of Anosov maps and of topological conjugacy we are strictly following the classical notations (see [14]).

Let us recall the principal definition.

Definition 1. An operator $A \in GL_n(\mathbb{Z})$ is said to be linear hyperbolic automorphism (of $\mathbb{T}^n$) if absolute values of all its eigenvalues are not equal to 1.

Next questions are the main topic of our article:

1. How to define if two Anosov diffeomorphisms $A$, $B$ of $\mathbb{T}^n$ are topologically conjugate?
2. Does there exist an effective algorithm, determining $A$, $B$ for the existence of the conjugacy?
What is a complete invariant for the topological classification of Anosov diffeomorphisms on tori?

3. FROM THE TOPOLOGICAL TO THE LINEAR CLASSIFICATION

The following statement was the main step in the investigation on topological classification of Anosov maps on tori.

**Theorem 1.** Every Anosov diffeomorphism $A$ of $T^n$ is topologically conjugate to a linear hyperbolic automorphism of $T^n$, i.e. to the action of $A$ on the homology group $H_1(T^n)$.

For a short proof see [14]. Franks proved it in the assumption that the nonwandering set of $A$ is the whole torus (see [7]). Manning did it in full generality (see [15]).

**Theorem 2.** The topological conjugacy of two linear hyperbolic automorphisms $A$, $B$ of $T_n$ imply the existence of a linear conjugacy, i.e.

$$\exists C \in GL_n(\mathbb{Z}), \text{ s.t. } AC = CB.$$

Its proof can be easily derived from the action of an Anosov diffeomorphism on $H_1(T^n)$.

Therefore the initial problem reduced to the purely algebraic question.

**Problem 1.** Given two linear hyperbolic automorphisms $A$, $B$ of $T_n$. Find if $\exists C \in GL_n(\mathbb{Z})$ s.t. $A = CBC^{-1}$.

For the problem of a more broad conjugacy: $A \sim B$ if and only if $B = CAC^{-1}$ for some $C \in GL_n(\mathbb{C})$ a necessary and sufficient condition is that $A$ and $B$ have the same Jordan Normal Form (JNF). But the condition $C \in GL_n(\mathbb{Z})$ is turned an additional requirement.

This was known already to Gauss. Indeed, in the case $n = 2$ Gauss reduced the question to the question in the theory of binary quadratic forms. The last question was solved by him. For description of the reduction see [2].

Also there were some algebraic attempts to solve Problem 1. We mention here just two of them. Grunewald in 1976 (see [5]) introduced explicit algorithms, that decide if $A, B \in GL_n(\mathbb{Q})$ are conjugate by an element $C \in GL_n(\mathbb{Z})$ or $C \in SL_n(\mathbb{Z})$ and, if such $C$ exists, find one. But these algorithms are "highly exponential" (their complexity is very high) and their geometrical meaning is unclear. Recently Karpenkov suggested to use Hessenberg matrices [9]. His idea is to follow the Gauss reduction theory, which finds special reduced matrices in each conjugacy class. He proved that for a fixed class there exist finitely many reduced Hessenberg matrices. For details see [9].
Our main idea is to use the geometrical interpretation of a complete invariant in the case $n = 2$ and to generalize it to the case $n > 2$ using notations of geometric continued fractions (in the sense of Klein and Klein-Voronoi, all necessary definitions will appear in the Section 5).

4. Complete solution in the case $n = 2$

Let $A$ be a hyperbolic automorphism of $\mathbb{T}^2$, i.e. $A \in GL_2(\mathbb{Z})$ and $\lambda, \mu \in \mathbb{R}$ are its eigenvalues s.t. $|\mu| < 1 < |\lambda|$. Let $v = (x, y)^T$ be an expanding eigenvector, i.e. $Av = \lambda v$, with slope $x/y = \omega_A$ being a quadratic irrationality.

**Theorem 3 (Lagrange).** A decomposition of a real number $\omega$ into a continued fraction is periodical from some place if and only if $\omega$ is a quadratic irrationality.

According to this classical result the continued fraction expansion of $\omega_A$ is periodic:

\[
\omega_A = [a_0; a_1, a_2, \ldots, a_k, a_{k+q}, a_{k+q+1}, \ldots, a_{k+2q}, \ldots] = [a_0; a_1, a_2, \ldots, a_k, (a_{k+1}, \ldots, a_{k+q})],
\]

where $a_{k+iq+j} = a_{k+j}$ for $i \geq 0$, $j = 1, \ldots, q$.

By ”the period” of this continued fraction we shall mean not only $q$, but also the finite sequence of numbers $(a_{k+1}, \ldots, a_{k+q})$ up to a cyclic permutation.

In [2] authors using purely analytic means proved the following statement.

**Theorem 4.** Let $A$ and $B$ be two hyperbolic automorphisms of $\mathbb{T}^2$ with the same JNF, then $A$ is conjugate to $B$ via some $C \in GL_2(\mathbb{Z})$ if and only if continued fraction expansions of $\omega_A$ and $\omega_B$ have the same period (i.e. the same periodic part).

The statement of the Theorem 4 has a very clear and beautiful geometric interpretation in terms of geometric continued fractions.

5. Notations of geometric continued fractions

In this Section we follow Karpenkov’s notations (see [8], [9] and [10]).

A point of $\mathbb{R}^{n+1}$ is said to be integer if all its coordinates are integers. Two sets are called integer-affine (integer-linearly) equivalent if there exists an affine (linear) transformation of $\mathbb{R}^{n+1}$ preserving the lattice of all integer points, and transforming the first set to the second. A plane is called integer if it is integer-affine equivalent to some plane passing through the origin and containing the sublattice of the integer lattice,
A polyhedron (in particular, a triangle or a planar convex polygon) is said to be **integer** if all its vertices are integers.

A segment \(PQ\) is said to be **integer** if its endpoints \((P\) and \(Q\)) are integer. An **integer length** of an integer segment \(PQ\) is the number of integer points contained in the interior of the segment plus one. We denote the integer length by \(Il(PQ)\).

Let \(P, Q\) and \(R\) be three integer points that do not lie in the same straight line. We denote the angle with the vertex at \(Q\) and the rays \(QP\) and \(QR\) by \(\angle PQR\). Also we denote the Euclidean area of a triangle \(PQR\) by \(S_{PQR}\).

**Definition 2.** Consider an arbitrary integer triangle \(PQR\). Then the **integer sine**, \(I\sin(PQR)\) of the angle \(PQR\) is defined by the following formula:

\[
I\sin(PQR) = \frac{2S_{PQR}}{Il(PQ)Il(QR)}.
\]

It is easy to check that the integer sine is a positive integer-valued function.

Consider arbitrary \(n + 1\) hyperplanes in \(\mathbb{R}^{n+1}\) that intersect at a unique point, namely the origin. The complement to the union of these hyperplanes consists of \(2^{n+1}\) open orthants. Let us choose an arbitrary orthant.

**Definition 3.** The boundary of the convex hull of all integer points except the origin in the closure of the orthant is called the **sail** of the orthant. The set of all \(2^{n+1}\) sails is called the **\(n\)-dimensional continued fraction** corresponding to the given \(n + 1\) hyperplanes.

Two \(n\)-dimensional continued fractions are said to be **equivalent** if the union of all sails of the first continued fraction is integer-linear equivalent to the union of all sails of the second continued fraction.

**Definition 4.** An operator in the group \(SL_{n+1}(\mathbb{Z})\) is called an **integer irreducible hyperbolic** if the following conditions holds:

1. the characteristic polynomial of this operator is irreducible over \(\mathbb{Q}\);
2. all its eigenvalues are distinct and real.

Consider some integer irreducible hyperbolic operator \(A \in SL_{n+1}(\mathbb{Z})\). Let us take the \(n\)-dimensional spaces that span all subsets of \(n\) linearly independent eigenvectors of the operator \(A\). The spans of every \(n\) eigenvectors uniquely define \(n + 1\) hyperplanes passing through the origin in
general position. These hyperplanes uniquely define the \(n\)-dimensional (multidimensional) continued fraction associated with \(A\).

Let \(A\) be an integer irreducible hyperbolic operator. Denote by \(\Xi(A)\) the set of all integer operators commuting with \(A\). These operators form a ring with a standard matrix addition and multiplication. Consider the subset of the set \(SL_{n+1}(\mathbb{Z}) \cap \Xi(A)\) that consists of all operators with positive real eigenvalues and denote it by \(\Xi^+(A)\). From the Dirichlet unit element theorem (see [4]) it follows that the subset \(\Xi(A)\) forms a multiplicative Abelian group isomorphic to \(\mathbb{Z}^n\), and that its action is free. Any operator of this group preserves the integer lattice and the union of all \(n+1\) hyperplanes, and hence it takes the \(n\)-dimensional continued fraction onto itself bijectively. (Whenever all eigenvalues are positive, the sails are also taken onto themselves in a one-to-one way.)

The quotient of a sail under this group action is isomorphic to an \(n\)-dimensional torus. By a fundamental domain we mean the union of some faces that contains exactly one face from each equivalence class (with respect to the action of the group \(\Xi(A)\)).

Previous part of this Section was Klein’s version of geometric continued fractions. Now we turn to a Klein-Voronoi’s version.

Consider any real operator \(A\) of \(SL_n(\mathbb{R})\) whose eigenvalues are all distinct. Suppose, that it has \(k\) real eigenvalues \(r_1, \ldots, r_k\) and \(2l\) complex conjugate roots \(c_1, \overline{c}_1, \ldots, c_l, \overline{c}_l\), here \(k + 2l = n\).

Denote by \(T^l(A)\) the set of all real operators commuting with \(A\) such that their real eigenvalues are all unit and the absolute values for all complex eigenvalues equal one. Actually, \(T^l(A)\) is an Abelian group with operation of matrix multiplication.

For a vector \(v\) in \(\mathbb{R}^n\) we denote by \(T_A(v)\) the orbit of \(v\) with respect of the action of the group of operators \(T^l(A)\). If \(v\) is in general position with respect to the operator \(A\) (i.e. it does not lie in invariant planes of \(A\)), then \(T_A(v)\) is homeomorphic to the \(l\)-dimensional torus. For a vector of an invariant plane of \(A\) the orbit \(T_A(v)\) is also homeomorphic to a torus of positive dimension not greater than \(l\), or to a point.

For instance, if \(v\) is a real eigenvector, then \(T_A(v) = \{v\}\). The second example: if \(v\) is in a real hyperplane spanned by two complex conjugate eigenvectors, then \(T_A(v)\) is an ellipse.

Let \(g_i\) be a real eigenvector with eigenvalue \(r_i\) for \(i = 1, \ldots, k\); \(g_{k+2j-1}\) and \(g_{k+2j}\) be vectors corresponding to the real and imaginary parts of some complex eigenvector with eigenvalue \(c_j\) for \(j = 1, \ldots, l\). We consider the coordinate system corresponding to the basis \(\{g_i\}\):

\[OX_1X_2 \ldots X_kY_1Y_2Z_2 \ldots Y_lZ_l.\]
Denote by $\pi$ the $(k+l)$-dimensional plane $OX_1X_2\ldots X_kY_1Y_2\ldots Y_l$. Let $\pi_+$ be the cone in the plane $\pi$ defined by the equations $y_i \geq 0$ for $i = 1, \ldots, l$. For any $v$ the orbit $T_A(v)$ intersects the cone $\pi_+$ in a unique point.

**Definition 5.** A point $p$ in the cone $\pi_+$ is said to be $\pi$-integer if the orbit $T_A(p)$ contains at least one integer point.

Consider all (real) hyperplanes invariant under the action of the operator $A$. There are exactly $k$ such hyperplanes. In the above coordinates the $i$-th of them is defined by the equation $x_i = 0$.

The complement to the union of all invariant hyperplanes in the cone $\pi_+$ consists of $2^k$ connected components. Consider one of them.

**Definition 6.** The convex hull of all $\pi$-integer points except the origin contained in the given connected component is called a factor-sail of the operator $A$. The set of all factor-sails is said to be the factor-continued fraction for the operator $A$.

The union of all orbits $T_A(\ast)$ in $\mathbb{R}^n$ represented by the points in the factor-sail is called the sail of the operator $A$. The set of all sails is said to be the continued fraction for the operator $A$ (in the sense of Klein-Voronoi).

The intersection of the factor-sail with a hyperplane in $\pi$ is said to be an $m$-dimensional face of the factor-sail if it is homeomorphic to the $m$-dimensional disc.

The union of all orbits in $\mathbb{R}^n$ represented by points in some face of the factor-sail is called the orbit-face of the operator $A$.

Integer points of the sail are said to be vertices of this sail.

**Definition 7.** Continued fractions for operators $A$ and $B$ (in the sense of Klein-Voronoi) are said to be equivalent if the union of all sails of the first continued fraction is integer-linear equivalent to the union of all sails of the second continued fraction.

Consider now an operator $A$ in the group $SL_n(\mathbb{Z})$ with irreducible characteristic polynomial. Suppose, that it has $k$ real roots $r_1, \ldots, r_k$ and $2l$ complex conjugate roots: $c_1, \overline{c_1}, \ldots, c_l, \overline{c_l}$, where $k + 2l = n$. In the simplest possible cases $k+l = 1$ any factor-sail of $A$ is a point. If $k+l > 1$, than any factor-sail of $A$ is an infinite polyhedral surface homeomorphic to $\mathbb{R}^{k+l-1}$.

**Definition 8.** The group of all $SL_n(\mathbb{Z})$-operators commuting with $A$ with positive eigenvectors is called the Dirichlet group and denoted by $\hat{\Xi}(A)$. 
The Dirichlet group \( \hat{\Xi}(A) \) takes any sail of \( A \) to itself. By Dirichlet unit theorem the group \( \hat{\Xi}(A) \) is homomorphic to \( \mathbb{Z}^{k+l-1} \) and its action on any sail is free. The quotient of a sail by the action of \( \hat{\Xi}(A) \) is homeomorphic to the \((n-1)\)-dimensional torus. By a fundamental domain of the sail we mean a collection of open orbit-faces such that for any \( \hat{\Xi}(A) \)-orbit of orbit-faces of the sail there exists a unique representative in the collection.

6. INVARIANTS IN THE CASE \( n > 2 \)

The geometric interpretation the Theorem 4 is the following (see [11] for details). A sail for a linear hyperbolic automorphism \( A \) of \( T^2 \) is an infinite polygonal chain \( \ldots, V_{-2}, V_{-1}, V_0, V_1, V_2, \ldots \), where \( V_i \) are integer vertices of the sail.

**Definition 9.** The infinite sequence of positive integers

\[(\ldots, ll(V_{-2}V_{-1}), L \sin(\angle V_{-2}V_{-1}V_0), ll(V_{-1}V_0), L \sin(\angle V_{-1}V_0V_1), ll(V_0V_1), \ldots)\]

is called the LLS-sequence of the sail.

Let \( \omega_A = [a_0; a_1, a_2, \ldots, a_k, (a_{k+1}, \ldots, a_{k+q})] \) be the same as in (I). Then the LLS-sequence of a sail of \( A \) is periodic and its period is equal to \([([a_{k+1}, \ldots, a_{k+q}]) \) up to a cyclic permutation (surely, LLS-sequences of all four sails of \( A \) coincide). For the proof see [11].

Let us generalize the last statement (it would be a multidimensional analogue of the Theorem 4) "automatically" by using notations of Klein’s geometric continued fractions. Unfortunately, its definition restricts us from a linear hyperbolic automorphism to an integer irreducible hyperbolic operators.

Consider two integer irreducible hyperbolic operators \( A, B \in SL_n(\mathbb{Z}) \) (automorphisms of \( T^n \)).

**Theorem 5.** Automorphisms \( A \) and \( B \) are linearly conjugate if and only if they have the same JNF and their \((n-1)\)-dimensional continued fractions are equivalent.

**Remark.** For \( n = 2 \) the equivalence of two continued fractions means the same LLS-sequence in geometric terms or the coincidence of their periods (up to a cyclic permutation) in the analytic language.

**Proof of the Theorem 5.** If \( A \) and \( B \) are linearly conjugate by some operator \( C \in GL_n(\mathbb{Z}) \), then \( C \) realizes the integer-linear equivalence of \((n-1)\)-dimensional continued fractions associated with \( A \) and \( B \) just by the definition: integer points goes to integer points, the convex hull
In every orthant goes to the convex hull in the new orthant, which is the image of the old one, and sails goes to sails.

Inversely, \((n - 1)\)-dimensional continued fractions associated with \(A\) and \(B\) are equivalent, then by the definition there exists a linear transformation of \(\mathbb{R}^n\) preserving the set of all integer points \(C\). Naturally, in this case \(C\) belongs to \(GL_n(\mathbb{Z})\) and \(C\) gives us the sought-for conjugacy.

\[\square\]

**Remark.** In particular all combinatorial data (integer lengths, integer sines, integer volumes of all dimensions (straightforward generalizations of definitions of the integer length and the integer sine) and degrees of all vertices in a fundamental domain) are invariant under an action of a conjugacy.

Of course, assumptions of the Theorem 3 is sufficiently strong. For example, the case of nonreal eigenvalues of a hyperbolic automorphism of \(\mathbb{T}^n\) is not considered. By this reason the Klein-Voronoi’s generalization of continued fractions seems to be more interesting and attractive, than the Klein’s generalization. From the other hand, the Klein-Voronoi’s generalization looks less natural than the Klein’s generalization.

Consider two hyperbolic operators \(A, B \in SL_n(\mathbb{Z})\) with irreducible characteristic polynomials.

**Theorem 6.** Operators \(A\) and \(B\) are linearly conjugate if and only if they have the same JNF and their continued fractions (in the sense of Klein-Voronoi) are equivalent.

**Proof.** If \(A\) and \(B\) are linearly conjugate by some operator \(C \in GL_n(\mathbb{Z})\), then \(C\) realizes an integer-linear equivalence of their continued fractions (in the sense of Klein-Voronoi) just by the definition. Here we need to check, that the action of a conjugacy well coordinates with the sequence of definitions of Klein-Voronoi’s continued fractions only.

Inversely, continued fractions (in the sense of Klein-Voronoi) associated with \(A\) and \(B\) are equivalent, then by the definition there exists a linear transformation of \(\mathbb{R}^n\) preserving the set of all integer points \(C\). Naturally, in this case \(C\) belongs to \(GL_n(\mathbb{Z})\) and \(C\) gives us the sought-for conjugacy. \[\square\]

7. **Algorithms in the case \(n > 2\)**

We mentioned algorithms of Grunewald [5] in the Section 3, but they have big complexity. Karpenkov in his Ph. D. thesis introduced some ”deductive” algorithms, that constructs a fundamental domain for a given geometric continued fraction in the sense of Klein. The
"deductivity" means here that at some moment a human should "help" to the algorithm by looking at an interim result and answer if it is sufficient or not. An application of this "deductive" algorithms for our problems is straightforward. Namely, starting from two arbitrary diffeomorphisms \( f, g \) of \( T^n \), one can get two linear Anosov maps \( A \) and \( B \), which are topologically conjugate to \( f \) and \( g \) respectively. If \( A \) and \( B \) have different JNF than \( f \) and \( g \) are not conjugate. Otherwise, we need to compare fundamental domains of \( A \) and \( B \) for an existence of a conjugacy. It is easy to do it if we had already constructed these fundamental domains — there exists an effective algorithm [8].

Due to results mentioned above we can formulate a following question.

**Problem 2.** Invent an effective algorithm deciding if two hyperbolic toral automorphisms are linearly conjugate or not and finding a conjugating matrix if the answer is positive.

8. **Lagrange’s theorem and its generalizations**

Here we will discuss a classical Lagrange’s theorem about continued fractions and attempts to generalize it.

In 1994 Korkina [12] announced a very natural-looking generalization of this theorem, but her statement appeared to be wrong. German is claiming to have constructed a counterexample, however it was never published. In 2008, German and Lakshtanov [6] presented a correct result, which they considered as a generalization of the Lagrange theorem. The weak points of that generalization is its unreasonable complexity and the fact that its geometrical meaning is unclear. So the following question appears quite naturally.

**Problem 3.** Formulate and prove a simple, natural and correct version of Lagrange’s theorem for geometric continued fractions in the sense of Klein and Klein-Voronoi.

9. **Additional invariants in the case of \( SL_n(\mathbb{Z}) \)-classification**

In the case \( n = 2 \) there is a following difference between classification of hyperbolic toral automorphisms up to conjugacy by \( GL_2(\mathbb{Z}) \) or \( SL_2(\mathbb{Z}) \).

Let \( A \) and \( B \) be linear hyperbolic automorphisms of \( T^2 \), which are linear conjugate by some \( C \in GL_2(\mathbb{Z}) \).

Let \( \omega_A = [a_0; a_1, a_2, \ldots, a_k, (a_{k+1}, \ldots, a_{k+q})] \) be the same as in (1).

In [2] authors proved the following statement.
Theorem 7. 

1. if $q \not\equiv 2$ then $\exists \tilde{C} \in SL_2(\mathbb{Z})$: $A\tilde{C} = \tilde{C}B$.

2. if $q \equiv 2$, $C \in SL_2(\mathbb{Z})$ then this $C$ gives a sought-for conjugacy.

3. if $q \equiv 2$, $C \not\in SL_2(\mathbb{Z})$ then $\not\exists \tilde{C} \in SL_2(\mathbb{Z})$: $A\tilde{C} = \tilde{C}B$.

One can conclude from it, that evenness of the size of a period plays an essential role. We don’t know anything about generalizations of the last theorem to the higher dimensions.

Problem 4. Find a complete additional invariant, which could distinguish classification of hyperbolic toral automorphism under the action of $GL_n(\mathbb{Z})$ and $SL_n(\mathbb{Z})$ for the case $n > 2$.

10. Recovery of a hyperbolic automorphism from a period

One can ask another very natural question, namely the inverse question. What one can say about the existence of hyperbolic toral automorphisms for a given period? If it exists, how many nonconjugate types are?

For the case $n = 2$ there is an explicit answer, for further references see [11]. For $n > 2$ there is only a theorem of Karpenkov (in his Ph. D. thesis) which says, that if for a given fundamental domain provided with combinatorial data (integer lengths, integer sines, integer volumes of all dimensions, degrees of the vertices, etc.) such continued fraction in the sense of Klein exists, then it is unique.

So we can just repeat a classical problem from the theory of geometric continued fractions.

Problem 5. Find a necessary and sufficient conditions for a combinatorial data, that allows us to construct a Klein’s geometric continued fraction, which fundamental domain has exactly this data. The same question for Klein-Voronoi’s continued fractions.

11. Acknowledgements

The author is grateful to D. V. Anosov for the scientific advising, to A. Yu. Zhirov for posing the problem, to O. N. Karpenkov for the introduction to the beautiful world of geometric continued fractions and to A. V. Klimenko for his friendly assistance.

References

[1] V. I. Arnold, *Continued fractions*, M.: Moscow Center of Continuous Mathematical Education, 2002.
References

[2] D. V. Anosov, A. V. Klimenko, G. Kolutsky, On the hyperbolic automorphisms of the 2-torus and their Markov partitions, Preprints of the Max-Planck-Institut fur Mathematik, MPIM2008-54 (2008).

[3] R. Adler, C. Tresser, P. A. Worfolk, Topological conjugacy of linear endomorphisms of the 2-torus, Trans. Amer. Math. Soc., vol. 349 (1997) No. 4, pp. 1633–1652.

[4] Z. I. Borevich, I. R. Shafarevich, Number Theory, N. Y., N.Y.: Academic Press Inc., 1966.

[5] F. J. Grunewald, Solution of the conjugacy problem in certain arithmetic groups, Word problems, || (Conf. on Decision Problems in Algebra, Oxford, 1976), pp. 101–139, Stud. Logic. Foundations Math., 95, North-Holland, Amsterdam-New York, 1980.

[6] O. N. German, E. L. Lakshtanov On a multidimensional generalization of Lagrange’s theorem for continued fractions, Izv. Ross. Akad. Nauk Ser. Mat., vol. 72 (2008), No. 1, pp. 51–66.

[7] J. Franks Anosov diffeomorphisms on tori, Trans. Amer. Math. Soc., vol. 145 (1969), pp. 117–124.

[8] O. N. Karpenkov, Constructing multidimensional period continued fractions in the sense of Klein, arXiv:math/0411.031v3 (2008)

[9] O. N. Karpenkov, Integer conjugacy classes of $SL_3(\mathbb{Z})$ and Hessenberg matrices, arXiv: 0711.0830 (2007).

[10] O. N. Karpenkov, Elementary notions of lattice trigonometry, Math. Scand., vol. 102(2), pp. 161–205, 2008.

[11] O. N. Karpenkov, On determination of periods of geometric continued fractions for two-dimensional algebraic hyperbolic operators, arXiv: 0708.1604 (2007).

[12] E. I. Korkina, La périodicité des fractions continues multidimensionelles, C. R. Ac. Sci. Paris, vol. 319 (1994), pp. 777–780.

[13] A. Khinchin, Continued Fractions, Mineola, N.Y.: Dover Publications, 1997.

[14] A. Katok, B. Hasselblatt. Introduction to the Modern Theory of Dynamical Systems, Cambridge University Press, 1995. (Encyclopedia of Mathematics and Its Applications, Vol. 54)

[15] A. Manning, There are no new Anosov diffeomorphisms on tori, Amer. J. Math., vol. 96 (1974) No. 3, pp. 422–429.

Grisha Kolutsky
Department of the Theory of Dynamical Systems
Faculty of Mechanics and Mathematics
Lomonosov Moscow State University
MSU, GSP, Glavnoe Zdanie, Leninskie Gory
119899 Moscow, Russia
e-mail: kolutsky@mccme.ru