Abstract: In this paper we show discrepancy bounds for index-transformed uniformly distributed sequences. From a general result we deduce very tight lower and upper bounds on the discrepancy of index-transformed van der Corput-, Halton-, and \((t, s)\) -sequences indexed by the sum-of-digits function. We also analyze the discrepancy of sequences indexed by other functions, such as, e.g., \(\lfloor n^\alpha \rfloor\) with \(0 < \alpha < 1\).

Keywords: discrepancy, uniform distribution, van der Corput-sequence, Halton-sequence, \((t, s)\) -sequence, sum-of-digits function.

1. Introduction

A sequence \((y_n)_{n \geq 0}\) in the unit-cube \([0, 1)^s\) is said to be uniformly distributed modulo one if for all intervals \([a, b) \subseteq [0, 1)^s\) it is true that

\[
\lim_{N \to \infty} \frac{\#\{n : 0 \leq n < N, y_n \in [a, b)\}}{N} = \text{vol}([a, b)).
\]

A quantitative version of (1) can be stated in terms of discrepancy. For an infinite sequence \((y_n)_{n \geq 0}\) in \([0, 1)^s\) its discrepancy is defined as

\[
D_N((y_n)_{n \geq 0}) := \sup_{[a, b) \subseteq [0, 1)^s} \left| \frac{\#\{n : 0 \leq n < N, y_n \in [a, b)\}}{N} - \text{vol}([a, b)) \right|,
\]

where the supremum is extended over all sub-intervals \([a, b)\) of \([0, 1)^s\). For a given finite sequence \(X = (x_1, \ldots, x_M)\) we write \(D_M(X)\) for the discrepancy of \(X\) with

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the obvious adaptions in the above definition. An infinite sequence is uniformly distributed modulo one if and only if its discrepancy tends to zero as $N$ goes to infinity. However, convergence of the discrepancy to zero cannot take place arbitrarily fast. It follows from a result of Roth [28] that for any infinite sequence $(y_n)_{n \geq 0}$ in $[0, 1)$ we have $ND_N((y_n)_{n \geq 0}) \geq c_s (\log N)^{s/2}$ for infinitely many values of $N \in \mathbb{N}$ (by $\mathbb{N}$ we denote the set of positive integers, and we put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$). An improvement of this bound can be obtained from [4].

For the special case $s = 1$, Schmidt [29] (see also [2]) showed that for any infinite sequence $(y_n)_{n \geq 0}$ in $[0, 1)$ we have $ND_N((y_n)_{n \geq 0}) \geq \frac{\log N}{\log 4}$ for infinitely many values of $N \in \mathbb{N}$. This result is best possible with respect to the order of magnitude in $N$. An excellent introduction to this topic can be found in the book of Kuipers and Niederreiter [20] (see also [6, 9, 21, 24]).

Well known examples of uniformly distributed sequences are $(n\alpha)$-sequences (also called Kronecker-sequences, see [9, 20]), van der Corput-sequences and their multivariate analogues called Halton-sequences (see [6, 19, 20, 24]), as well as (digital) $(t, s)$-sequences (see [6, 24]).

In recent years, also the distribution properties of index-transformed uniformly distributed sequences have been studied, especially for the examples mentioned above. In this paper, we mean by an index-transformed sequence of a sequence $(x_n)_{n \geq 0}$ a sequence $(x_{f(n)})_{n \geq 0}$, where $f : \mathbb{N}_0 \to \mathbb{N}_0$. Note that $(x_{f(n)})_{n \geq 0}$ is in general no subsequence of $(x_n)_{n \geq 0}$ since we do not require that $f$ is strictly increasing.

For instance, the distribution properties of index-transformed Kronecker-sequences indexed by the sum-of-digits function were studied in [5, 8, 30, 31]. For this special case, very precise results can be found in [8]. In [7] the well-distribution of index-transformed Kronecker-sequences indexed by $q$-additive functions is considered. Furthermore, in [26] a discrepancy bound for van der Corput-sequences in bases of the form $b = 5^\ell$, $\ell \in \mathbb{N}$, indexed by Fibonacci numbers is shown. The papers [17, 18, 26] deal with index-transformed van der Corput-, Halton-, and $(t, s)$-sequences.

In this paper we are specifically interested in discrepancy bounds for sequences indexed by the $q$-ary sum-of-digits function and related functions and, furthermore, for sequences indexed by “moderately” monotonically increasing sequences, as for example $[n^\alpha]$ with $0 < \alpha < 1$. For an integer $q \geq 2$ and $n \in \mathbb{N}_0$ with base $q$ expansion $n = r_0 + r_1 q + r_2 q^2 + \cdots$ the $q$-ary sum-of-digits function is defined by $s_q(n) := r_0 + r_1 + r_2 + \cdots$.

Previously, it has been shown in [18] that the sequence $(x_{s_q(n)})_{n \geq 0}$, indexed by the $q$-ary sum-of-digits function, where $(x_n)_{n \geq 0}$ denotes the Halton-sequence in co-prime bases $b_1, \ldots, b_s$ is uniformly distributed modulo one. The proof of this result is due to the fact that the sequence generated by the $q$-ary sum-of-digits function is uniformly distributed in $\mathbb{Z}$, see, for example, [12, 27]. In this paper we provide very tight lower and upper bounds on the discrepancy of index-transformed van der Corput-, Halton-, and $(t, s)$-sequences indexed by the sum-of-digits function.
This paper is structured as follows. In Section 2, we provide basic definitions and notation used throughout the subsequent sections. In Section 3, we prove a general theorem (Theorem 1) which will be of great importance in discussing sequences indexed by the sum-of-digits function. In Section 4 we present a concrete application of Theorem 1 which leads to the aforementioned tight bounds on the discrepancy of Halton- and \((t,s)\)-sequences indexed by \(s_q(n)\). Furthermore, we discuss a refinement of these results for van der Corput-sequences. Finally, in Section 5, we deal with discrepancy bounds for sequences which are obtained by certain moderately increasing index sequences, such as, e.g., \(\lfloor n^\alpha \rfloor\) with \(0 < \alpha < 1\).

2. Notation and basic definitions

We first outline the definitions of the sequences studied in this paper, namely van der Corput-, Halton-, and \((t,s)\)-sequences.

Let \(b \geq 2\) be an integer. A van der Corput-sequence \((x_n)_{n \geq 0}\) in base \(b\) is defined by \(x_n = \varphi_b(n)\), where for \(n \in \mathbb{N}_0\), with base \(b\) expansion \(n = a_0 + a_1 b + a_2 b^2 + \cdots\), the so-called radical inverse function \(\varphi_b : \mathbb{N}_0 \to [0,1)\) is defined by

\[
\varphi_b(n) := \frac{a_0}{b} + \frac{a_1}{b^2} + \frac{a_2}{b^3} + \cdots.
\]

It is well known that for any base \(b \geq 2\) the corresponding van der Corput-sequence is uniformly distributed modulo one and that \(ND_N((x_n)_{n \geq 0}) = O(\log N)\), see, for example, [3, 6, 20].

If we choose co-prime integers \(b_1, \ldots, b_s \geq 2\), then \(s\) one-dimensional van der Corput-sequences can be combined to an \(s\)-dimensional uniformly distributed sequence with points \(x_n := (\varphi_{b_1}(n), \ldots, \varphi_{b_s}(n))\) for \(n \in \mathbb{N}_0\). This sequence is called a Halton-sequence and it is known that its discrepancy is of order \((\log N)^s / N\), see [1, 6, 10, 11, 13, 19, 22, 24]. Note that Halton-sequences are a direct generalization of van der Corput-sequences, so van der Corput-sequences can be viewed as one-dimensional Halton-sequences, and indeed Halton-sequences are sometimes also referred to as van der Corput-Halton-sequences (see, e.g., [20]). However, as there will be results in this paper which only hold for the one-dimensional case, it will be useful to explicitly distinguish van der Corput-sequences (which we use for the one-dimensional variant) from Halton-sequences (which we use for the multi-dimensional variant).

Another type of sequences we will be concerned with in this paper are \((t,s)\)-sequences, for the definition of which we need the definition of elementary intervals and \((t,m,s)\)-nets in base \(b\).

For an integer \(b \geq 2\), an elementary interval in base \(b\) is an interval of the form \(\prod_{i=1}^s [a_i b^{-d_i}, (a_i + 1) b^{-d_i}) \subseteq [0,1)^s\), where \(a_i, d_i\) are non-negative integers with \(0 \leq a_i < b^{d_i}\) for \(1 \leq i \leq s\).

Let \(t, m\), with \(0 \leq t \leq m\), be integers. Then a \((t,m,s)\)-net in base \(b\) is a point set \((y_n)_{n=0}^{b^m-1}\) in \([0,1)^s\) such that any elementary interval in base \(b\) of volume \(b^{t-m}\) contains exactly \(b^t\) of the \(y_n\).
Furthermore, we call an infinite sequence \((x_n)_{n \geq 0}\) a \((t, s)\)-sequence in base \(b\) if the subsequence \((x_n)_{n=kb^m}^{(k+1)b^n-1}\) is a \((t, m, s)\)-net in base \(b\) for all integers \(k \geq 0\) and \(m \geq t\). It is known (see, e.g., [6, 23, 24]) that a \((t, s)\)-sequence is particularly evenly distributed if the value of \(t\) is small. In particular, it can be shown that the discrepancy of a \((t, s)\)-sequence in base \(b\) is of order \(b^t(\log N)^s/N\), see, e.g., [6, 23, 24].

A very important sub-class of \((t, s)\)-sequences is that of digital \((t, s)\)-sequences, which are defined over algebraic structures like finite fields or rings. For the sake of simplicity, we restrict ourselves to digital sequences over finite fields \(\mathbb{F}_p\) of prime order \(p\). Again for the sake of simplicity we do not distinguish, here and later on, between elements in \(\mathbb{F}_p\) and the set of integers \(\{0, 1, \ldots, p-1\}\) (equipped with arithmetic operations modulo \(p\)).

For a vector \(c = (c_1, c_2, \ldots) \in \mathbb{F}_p^\infty\) and for \(m \in \mathbb{N}\) we denote the vector in \(\mathbb{F}_p^m\) consisting of the first \(m\) components of \(c\) by \(c(m)\), i.e., \(c(m) = (c_1, \ldots, c_m)\). Moreover, for an \(N \times N\) matrix \(C\) over \(\mathbb{F}_p\) and for \(m \in \mathbb{N}\) we denote by \(C(m)\) the left upper \(m \times m\) submatrix of \(C\).

For \(s \in \mathbb{N}\) and \(t \in \mathbb{N}_0\), choose \(N \times N\) matrices \(C_1, \ldots, C_s\) over \(\mathbb{F}_p\) with the following property. For every \(m \in \mathbb{N}\), \(m \geq t\), and all \(d_1, \ldots, d_s \in \mathbb{N}_0\) with \(d_1 + \cdots + d_s = m - t\), the vectors

\[
\begin{align*}
    \mathbf{c}_1^{(1)}(m), & \quad \mathbf{c}_{d_1}^{(1)}(m), & \quad \mathbf{c}_1^{(s)}(m), & \quad \mathbf{c}_{d_s}^{(s)}(m)
\end{align*}
\]

are linearly independent in \(\mathbb{F}_p^m\). Here \(\mathbf{c}_i^{(j)}\) is the \(i\)-th row vector of the matrix \(C_j\).

For \(n \in \mathbb{N}_0\) let \(n = n_0 + n_1p + n_2p^2 + \cdots\) be the base \(p\) representation of \(n\). For every index \(1 \leq j \leq s\) multiply the digit vector \(n = (n_0, n_1, \ldots)^\top\) by the matrix \(C_j\):

\[
    C_j \cdot n =: (x_{n,j}(1), x_{n,j}(2), \ldots)^\top
\]

(note that the matrix-vector multiplication is performed over \(\mathbb{F}_p\)), and set

\[
    x_n^{(j)} := \frac{x_{n,j}(1)}{p} + \frac{x_{n,j}(2)}{p^2} + \cdots.
\]

Finally set \(x_n := (x_n^{(1)}, \ldots, x_n^{(s)})\). A sequence \((x_n)_{n \geq 0}\) constructed in this way is called a digital \((t, s)\)-sequence over \(\mathbb{F}_p\). The matrices \(C_1, \ldots, C_s\) are called the generator matrices of the sequence.

To guarantee that the points \(x_n\) lie in \([0, 1)^s\) (and not just in \([0, 1]^s\)) we assume that for each \(1 \leq j \leq s\) and \(w \geq 0\) we have \(c_{i,w}^{(j)} = 0\) for all sufficiently large \(v\), where \(c_{i,w}^{(j)}\) are the entries of the matrix \(C_j\) (see [24, p.72, condition (S6)] for more information).

Throughout the paper we use the following notation. For functions \(f, g : \mathbb{N} \to \mathbb{R}\), where \(f \geq 0\), we write \(g(n) = O(f(n))\) or \(g(n) \ll f(n)\), if there exists a \(C > 0\) such that \(|g(n)| \leq Cf(n)\) for all sufficiently large \(n \in \mathbb{N}\). If we would like to stress that the quantity \(C\) may also depend on other variables than \(n\), say \(\alpha_1, \ldots, \alpha_w\), which will be indicated by writing \(\ll_{\alpha_1, \ldots, \alpha_w}\).
3. A general theorem

In this section we present a general result for the discrepancy of sequences of the form \( (x_{g(n)})_{n \geq 0} \), for a particular class of functions \( g : \mathbb{N}_0 \to \mathbb{N}_0 \). Here and in the following, a sequence \( (a_k)_{k \in \mathbb{N}_0} \) is called unimodal if the sequence \( (a_{k+1} - a_k)_{k \in \mathbb{N}_0} \) has exactly one change of sign.

Furthermore, we need the concept of the so-called uniform discrepancy of a sequence. The uniform discrepancy of a sequence \( (x_n)_{n \geq 0} \) in \([0,1]^s\) is defined as

\[
\bar{D}_N((x_n)_{n \geq 0}) := \sup_{k \in \mathbb{N}_0} D_N((x_{n+k})_{n \geq 0}).
\]

**Theorem 1.** Let \( (x_n)_{n \geq 0} \) be an \( s \)-dimensional sequence with uniform discrepancy \( \bar{D}_N = \bar{D}_N((x_n)_{n \geq 0}) \), and let \( f : \mathbb{N}_0 \to \mathbb{R} \) be a non-decreasing function such that \( N \bar{D}_N \leq f(N) \) for \( N \in \mathbb{N}_0 \).

Let \( g : \mathbb{N}_0 \to \mathbb{N}_0 \). Furthermore, let \((N_j)_{j \geq 0} \) be a strictly increasing sequence in \( \mathbb{N} \) with \( 1 = N_0 \), and assume that \((N_j)_{j \geq 0} \) is a divisibility chain, i.e., \( N_0 | N_1, N_1 | N_2, N_2 | N_3 \), etc. Define, for \( k \in \mathbb{N}_0 \),

\[
G_{A,j}(k) := \# \{ n : AN_j \leq n < (A+1)N_j, g(n) = k \}.
\]

Then the following two assertions hold.

1. For \( N \in \mathbb{N} \) with \( N_d \leq N < N_{d+1} \) we have

\[
N \bar{D}_N((x_{g(n)})_{n \geq 0}) \geq \max_{k \in \mathbb{N}_0} G_{0,d}(k).
\]

2. Assume that \( G_{A,j}(k) \) is unimodal in \( k \) for all \( j \in \mathbb{N}_0 \) and all \( A \in \mathbb{N}_0 \), and put

\[
G_j := \max_{k, A \in \mathbb{N}_0} G_{A,j}(k) \quad \text{for} \quad j \in \mathbb{N}_0.
\]

For \( j \in \mathbb{N}_0 \) and \( A \in \mathbb{N}_0 \) let

\[
v_{A,j} := \# \{ k \in \mathbb{N}_0 : g(n) = k \text{ for } AN_j \leq n < (A+1)N_j \}
\]

and put

\[
v_j := \max_{A \in \mathbb{N}_0} v_{A,j}.
\]

Then for \( N \in \mathbb{N} \) with \( N_d \leq N < N_{d+1} \) we have

\[
N \bar{D}_N((x_{g(n)})_{n \geq 0}) \leq \sum_{j=0}^{d} \frac{N_{j+1} - N_j}{N_j} G_j f(v_j).
\]

**Proof.**

1. To show the lower bound choose a non-negative integer \( \kappa \) such that \( \bar{G}_d = G_{0,d}(\kappa) = \max_{k \in \mathbb{N}_0} G_{0,d}(k) \). Then the number of \( n \in \{0, \ldots, N-1\} \) such that \( x_{g(n)} = x_\kappa \) is at least \( \bar{G}_d \) and hence, with an arbitrarily small interval containing \( x_\kappa \) we obtain

\[
D_N((x_{g(n)})_{n \geq 0}) \geq \frac{\bar{G}_d}{N}.
\]
2. To prove the upper bound let

\[ N = a_d N_d + a_{d-1} N_{d-1} + \cdots + a_0 N_0, \]

with \( a_j \in \mathbb{N}_0 \) and

\[ a_j \leq \frac{N_{j+1}}{N_j}; \quad \text{for} \quad j \in \{0, \ldots, d\}. \]

For \( j \in \{0, \ldots, d\} \) and \( \ell \in \{0, \ldots, a_j - 1\} \) we consider the sequence

\[ X_{j,\ell} := (x_{g(AN_j+k)})_{k=0}^{N_j-1} \]

where \( AN_j := a_d N_d + \cdots + a_{j+1} N_{j+1} + \ell N_j \) (strictly speaking, \( A = A(j, \ell) \)). Since \( G_{A,j} \) is unimodal we may assume that for \( AN_j \leq n < (A+1)N_j \) the function \( g(n) \) attains the values

\[ w, w+1, \ldots, w+v, \]

for some \( w \in \mathbb{N}_0 \) and some integer \( v = v_{A,j} \leq v(j) \).

Assume that the value \( w + u_1 \) with \( 0 \leq u_1 \leq v \) is attained most often, the value \( w + u_2 \) with \( 0 \leq u_2 \leq v \) is attained second most often, etc. \( \ldots \), and \( w + u_v \) with \( 0 \leq u_v \leq v \) (indeed, \( u_v \in \{0, v\} \)) is attained least often. If \( w + u_r \) and \( w + u_{r+1} \) are both attained the same number of times, then the order of them is of no relevance.

If we consider the sequence \( X_{j,\ell} \) as a multi-set (i.e., multiplicity of the elements is relevant, but their order is not), then we can decompose \( X_{j,\ell} \) into

\[
\begin{align*}
G_{A,j}(w + u_1) - G_{A,j}(w + u_2) & \quad \text{times} \quad \{x_{w+u_1}\} \\
G_{A,j}(w + u_2) - G_{A,j}(w + u_3) & \quad \text{times} \quad \{x_{w+u_1}, x_{w+u_2}\} \\
G_{A,j}(w + u_3) - G_{A,j}(w + u_4) & \quad \text{times} \quad \{x_{w+u_1}, x_{w+u_2}, x_{w+u_3}\} \\
\vdots \\
G_{A,j}(w + u_{v-1}) - G_{A,j}(w + u_v) & \quad \text{times} \quad \{x_{w+u_1}, x_{w+u_2}, \ldots, x_{w+u_{v-1}}\} \\
G_{A,j}(w + u_v) - G_{A,j}(w + u_{v+1}) & \quad \text{times} \quad \{x_{w+u_1}, x_{w+u_2}, \ldots, x_{w+u_v}\},
\end{align*}
\]

where we formally set \( G_{A,j}(w + u_{v+1}) := 0 \). Note that because of the unimodality of \( G_{A,j}(k) \), for \( r \in \{1, \ldots, v\} \), the sequence \( x_{w+u_1}, x_{w+u_2}, \ldots, x_{w+u_r} \) is a sequence of the form \( x_B, \ldots, x_{B+r-1} \) for some \( B \).

Then, using the assumptions of the theorem and the triangle inequality for the discrepancy (see [20, p. 115, Theorem 2.6]), we obtain

\[
N_j D_{N_j}(X_{j,\ell}) \leq \sum_{r=1}^{v} (G_{A,j}(w + u_r) - G_{A,j}(w + u_{r+1})) r D_r(\{x_{w+u_1}, x_{w+u_2}, \ldots, x_{w+u_r}\}) \leq G_{A,j}(w + u_1) f(v_{A,j}) \leq G_j f(v_j).
\]

Using the triangle inequality for the discrepancy a second time, we finally obtain

\[
ND_N((x_{g(n)})_{n \geq 0}) \leq \sum_{j=0}^{d} a_j G_j f(v_j) \leq \sum_{j=0}^{d} \frac{N_j+1}{N_j} G_j f(v_j).
\]
4. Indexing by the $q$-ary sum-of-digits function

We would now like to show results regarding index-transformed uniformly distributed sequences indexed by the $q$-ary sum-of-digits function. We first discuss an application of the general result in Theorem 1 (Section 4.1) to Halton- and $(t,s)$-sequences, and then show a refined result that applies to the particular case of van der Corput-sequences (Section 4.2).

4.1. Results for Halton- and $(t,s)$-sequences

Let $q \geq 2$ be an integer and $g(n) = s_q(n)$ the $q$-ary sum-of-digits function. For $j \in \mathbb{N}_0$ choose $N_j = q^j$. Then we have

$$G_{0,j}(k) = \# \{ n : 0 \leq n < q^j, s_q(n) = k \}$$

and

$$(1 + x + x^2 + \cdots + x^{q-1})^j = \sum_{k \in \mathbb{N}_0} G_{0,j}(k)x^k,$$

by expanding the polynomial on the left hand side of the latter equation. Hence the sequence $(G_{0,j}(k))_{k \in \mathbb{N}_0}$ is the $j$-fold convolution of the sequence $(1, 1, \ldots, 1, 0, 0, \ldots)$, which implies by [25, Theorem 1] that $G_{0,j}(k)$ is unimodal for sufficiently large $j$. Since any $n \in \mathbb{N}_0$ with $Aq^j \leq n < (A+1)q^j$ can be written as $n = n' + Aq^j$, where $0 \leq n' < q^j$, it follows that $s_q(n) = s_q(n') + s_q(A)$ and hence $G_{A,j}(k) = G_{0,j}(k - s_q(A))$, where we set $G_{0,j}(k - s_q(A)) := 0$ if $k < s_q(A)$. Consequently, $G_{A,j}(k)$ is unimodal for any $A \in \mathbb{N}_0$ and for sufficiently large $j$.

We recall the following lemma from [8].

**Lemma 1 (Drmota and Larcher, [8, Lemma 1]).** For integers $q \geq 2$, $j \geq 1$, and $0 \leq k \leq j(q-1)$ we have

$$G_{0,j}(k) = \frac{q^j}{\sqrt{2\pi j} \sigma_q} \exp \left( -\frac{x^2_{j,k}}{2} \right) \left( 1 + \frac{P_1(x_{j,k})}{\sqrt{j}} + \frac{P_2(x_{j,k})}{j} \right) + O \left( \frac{q^j}{j^2} \right),$$

where $P_1(x)$ and $P_2(x)$ are polynomials, $P_1(x)$ is odd, where $x_{j,k} := \frac{k - \frac{j(q-1)}{2} \sigma_q}{\sigma_q \sqrt{j}}$, and where $\sigma_q := \sqrt{\frac{q^2 - 1}{12}}$. The implied constant in the $O$-notation is uniform for all $k$ and only depends on $q$.

Due to Lemma 1, there exists some $c_q > 0$ such that for sufficiently large $j$ we have $G_{A,j}(k) \leq c_q q^j / \sqrt{j}$, uniformly in $k$ and $A$. Thus we obtain

$$G_j \leq c_q \frac{q^j}{\sqrt{j}} \quad (2)$$
for sufficiently large $j$. On the other hand, for $\bar{k} = \lfloor j^{2^{-1}} \rfloor$ it follows that

$$\max_{k \in \N} G_{0,j}(k) \geq G_{0,j}(\bar{k}) \geq c'_q \frac{q^j}{\sqrt{j}}. \quad (3)$$

Furthermore it is clear that $v_0 = 1$ and $v_j \leq qj$ for all $j \in \N$. As an application of Theorem 1, we obtain the following result.

**Theorem 2.** Let $X := (x_n)_{n \geq 0}$ be an $s$-dimensional sequence such that $mD_m((x_n)_{n \geq 0}) \leq C (\log m)^s$ for all $m \in \N$, where $C$ may depend on $s$ or on the sequence $X$, but not on $m$. Let $q \geq 2$ be an integer. Then there exist $c_q^{(2)}, c_q^{(3)} > 0$, where $c_q^{(3)}$ may also depend on $s$ and $X$, such that

$$\frac{c_q^{(2)}}{\sqrt{\log N}} \leq D_N((x_{sq(n)})_{n \geq 0}) \leq c_q^{(3)} \frac{(\log \log N)^s}{\sqrt{\log N}}.$$

**Proof.** Assume that $q^d \leq N < q^{d+1}$. Then we obtain from Theorem 1 and Equation (3) that

$$D_N((x_{sq(n)})_{n \geq 0}) \geq \frac{c'_q q^d}{N \sqrt{d}} \geq \frac{c_q^{(2)}}{\sqrt{\log N}}.$$

On the other hand, from Theorem 1 and Equation (2),

$$D_N((x_{sq(n)})_{n \geq 0}) \leq \frac{1}{N} \sum_{j=1}^{d} q c_q q^j j^{2^{-1}} C((\log(qj))^s \leq q (\log d)^s \left( \frac{1}{N} \sum_{1 \leq j < d/2} \frac{q^j}{\sqrt{j}} + \frac{1}{N} \sum_{d/2 \leq j \leq d} \frac{q^j}{\sqrt{j}} \right) \leq q (\log \log N)^s \frac{\sqrt{\log N}}{\sqrt{N}}$$

and the result follows.

The general lower bound in Theorem 2 is best possible with respect to the order of magnitude in $N$. This will follow from Theorem 3 below which deals with van der Corput-sequences.

There are several examples of sequences $X$ which satisfy the conditions in Theorem 2 such as Halton- or $(t,s)$-sequences (for a proof of this fact, we refer to Section 6 of this paper). We thus obtain the following corollary.

**Corollary 1.** Let $q \geq 2$ be an integer.

1. Let $(x_n)_{n \geq 0}$ be an $s$-dimensional Halton-sequence in pairwise co-prime bases $b_1, \ldots, b_s$. Then there exist $c_q^{(2)}, c_{q,s,b_1,\ldots,b_s}^{(4)} > 0$ such that

$$\frac{c_q^{(2)}}{\sqrt{\log N}} \leq D_N((x_{sq(n)})_{n=0}^{N-1}) \leq c_{q,s,b_1,\ldots,b_s}^{(4)} \frac{(\log \log N)^s}{\sqrt{\log N}}.$$
2. Let \((x_n)_{n \geq 0}\) be a \((t, s)\)-sequence in base \(b\). Then there exist \(c_q^{(2)}, c_{q,b,s,t}^{(5)} > 0\) such that
\[
\frac{c_q^{(2)}}{\sqrt{\log N}} \leq D_N((x_{s q(n)})_{n \geq 0}) \leq c_{q,b,s,t}^{(5)} \left(\frac{\log \log N}{\sqrt{\log N}}\right)^s .
\]

The result of the first part of Corollary 1 can be improved for the special instance of van der Corput-sequences, as we will show next.

4.2. The van der Corput-sequence indexed by the sum-of-digits function

The following results are based on a general discrepancy estimate which was first presented by Hellekalek [14]. The following definitions stem from [14, 15, 17]. We refer to these references for further information.

For an integer \(b \geq 2\) let \(Z_b = \{z = \sum_{r=0}^{\infty} z_r b^r : z_r \in \{0, \ldots, b-1\}\}\) be the set of \(b\)-adic numbers. \(Z_b\) forms an abelian group under addition. The set \(N_0\) is a subset of \(Z_b\). The Monna map \(\phi_b : Z_b \to [0, 1)\) is defined by
\[
\phi_b(z) := \sum_{r=0}^{\infty} \frac{z_r}{b^r+1}.
\]

Note that the radical inverse function \(\varphi_b\) is nothing but \(\phi_b\) restricted to \(N_0\). We also define the inverse \(\phi_b^+ : [0, 1) \to Z_b\) by
\[
\phi_b^+ \left(\sum_{r=0}^{\infty} \frac{x_r}{b^r+1}\right) := \sum_{r=0}^{\infty} x_r b^r,
\]
where we always use the finite \(b\)-adic representation for \(b\)-adic rationals in \([0, 1)\).

For \(k \in N_0\) we can define characters \(\chi_k : Z_b \to \{c \in \mathbb{C} : |c| = 1\}\) of \(Z_b\) by
\[
\chi_k(z) = \exp(2\pi i \phi_b(k)z) .
\]

Finally, let \(\gamma_k : [0, 1) \to \{c \in \mathbb{C} : |c| = 1\}\) where \(\gamma_k(x) = \chi_k(\phi_b^+(x))\).

For \(b \geq 2\) we put \(\rho_b(0) = 1\) and \(\rho_b(k) = \frac{2}{b^{r+1}\sin(\pi \kappa_r/b)}\) for \(k \in \mathbb{N}\) with base \(b\) expansion \(k = \kappa_0 + \kappa_1 b + \cdots + \kappa_r b^r, \kappa_r \neq 0\).

We have the following general discrepancy bound which is based on the functions \(\gamma_k\).

**Lemma 2.** Let \(g \in \mathbb{N}\). For any sequence \((y_n)_{n \geq 0}\) in \([0, 1)\) we have
\[
D_N((y_n)_{n \geq 0}) \leq \frac{1}{b^g} + \sum_{k=1}^{b^g-1} \rho_b(k) \left| \frac{1}{N} \sum_{n=0}^{N-1} \gamma_k(y_n) \right| .
\]
Proof. For the special case of a prime $b$, this result was shown by Hellekalek [14, Theorem 3.6]. Using [17, Lemma 2.10 and 2.11] it is easy to see that Hellekalek's result can be generalized to the one given in the lemma (cf. [16]).

We show a discrepancy bound for the van der Corput-sequence indexed by the $q$-ary sum-of-digits function for small values of $q$. This result improves on the first part of Corollary 1 for van der Corput-sequences. Moreover, it shows that the general lower bound from Theorem 2 is best possible in the order of magnitude in $N$.

**Theorem 3.** Let $b, q \geq 2$ be integers with $q < 14$, let $(x_n)_{n \geq 0}$ be the van der Corput-sequence in base $b$ and let $(s_q(n))_{n \geq 0}$ be the sequence of the $q$-adic sum-of-digits function. Then we have

$$D_N((x_{s_q(n)})_{n \geq 0}) \ll_{b,q} \frac{1}{\sqrt{\log N}}.$$ 

**Remark 1.** In view of Theorem 2, the upper bound in Theorem 3 is best possible with respect to the order of magnitude in $N$.

Before we give the proof of Theorem 3, we need some preparations and auxiliary results. Writing $e(x) := \exp(2\pi ix)$ for short, we have

$$\frac{1}{N} \sum_{n=0}^{N-1} \gamma_k(x_{s_q(n)}) = \frac{1}{N} \sum_{n=0}^{N-1} e(s_q(n)\phi_b(k)) =: T_k(N).$$

**Lemma 3.** Let $b, q \geq 2$ be integers, let $k \in \mathbb{N}$ and let $(x_n)_{n \geq 0}$ be the van der Corput-sequence in base $b$. Then for any $m \in \mathbb{N}_0$ it is true that

$$|T_k(q^m)| \leq \left(1 - \frac{16(q-1)}{q^2\|\phi_b(k)\|^2}\right)^{m/2},$$

where $\|x\|$ is the distance of a real $x$ to the nearest integer.

**Proof.** First observe that

$$T_k(q^m) = \frac{1}{q^m} \sum_{n_0, \ldots, n_{m-1}=0}^{q-1} e((n_0 + \ldots + n_{m-1})\phi_b(k)) = (T_k(q))^m.$$
We now proceed as in [27]. We use the identities \( \exp(ix) + \exp(-ix) = 2\cos x \) and \( \cos(2x) = 1 - 2\sin^2 x \) to obtain

\[
|T_k(q)|^2 = \frac{1}{q^2} \sum_{n,n'=0}^{q-1} e((n-n')\phi_b(k)) \\
= \frac{1}{q^2} \left( q + \sum_{n,n'=0}^{q-1} (e((n-n')\phi_b(k)) + e(-(n-n')\phi_b(k))) \right) \\
= \frac{1}{q^2} \left( q + 2 \sum_{n,n'=0}^{q-1} \cos (2\pi(n-n')\phi_b(k)) \right) \\
= \frac{1}{q^2} \left( q + 2 \sum_{n,n'=0}^{q-1} (1 - 2\sin^2 (\pi(n-n')\phi_b(k))) \right) \\
= 1 - \frac{4}{q^2} \sum_{n,n'=0}^{q-1} \sin^2 (\pi(n-n')\phi_b(k)) \leq 1 - \frac{4(q-1)}{q^2} \sin^2 (\pi\phi_b(k)) \\
\leq 1 - \frac{16(q-1)}{q^2} \|\phi_b(k)\|^2,
\]

Therefore,

\[
|T_k(q^m)| \leq \left( 1 - \frac{16(q-1)}{q^2} \|\phi_b(k)\|^2 \right)^{m/2}.
\]

We also need the following lemma.

**Lemma 4.** For \( k \in \mathbb{N} \) and any \( N \in \mathbb{N} \) with \( q \)-adic expansion \( N = \sum_{r=0}^{R} a_r q^r \) we have

\[
|T_k(N)| \leq \frac{1}{N} \sum_{r=0}^{R} a_r q^r |T_k(q^r)|.
\]

**Proof.** For \( N = \sum_{r=0}^{R} a_r q^r \),

\[
\{0, \ldots, N-1\} = \bigcup_{r=0}^{R} \{a_R q^R + \cdots + a_{r+1} q^{r+1}, \ldots, a_R q^R + \cdots + a_r q^r - 1\},
\]
and hence

\[ N|T_k(N)| = \left| \sum_{n=0}^{N-1} e^{s_q(n)\phi_b(k)} \right| \]

\[ = \left| \sum_{r=0}^{R} e^{(a_r r + \cdots + a_r r+1)\phi_b(k)} \sum_{n=0}^{a_r r+1} e^{s_q(n)\phi_b(k)} \right| \]

\[ \leq \left| \sum_{r=0}^{R} a_r \sum_{n=0}^{q_r - 1} e^{s_q(n)\phi_b(k)} \right| = \sum_{r=0}^{R} a_r q_r |T_k(q^r)|. \]

We are now ready to give the proof of Theorem 3.

**Proof.** For \( k \in \{b^r, \ldots, b^{r+1} - 1\} \) we have \( \varphi_b(k) = \frac{A_k}{b^{r+1}} \) with \( A_k \in \{1, \ldots, b^{r+1} - 1\} \), where \( A_{k_1} \neq A_{k_2} \) for \( k_1 \neq k_2 \). Hence we obtain from Lemma 3

\[ \sum_{k=1}^{b^r - 1} \rho_b(k)|T_k(q^m)| \leq \sum_{r=0}^{g-1} \frac{2}{b^{r+1}\sin(\pi/b)} \sum_{k=b^r}^{b^{r+1}-1} \left( 1 - \frac{16(q-1)}{q^2} \left\| \frac{A_k}{b^{r+1}} \right\|^{2m/2} \right) \]

\[ \leq \sum_{r=0}^{g-1} \frac{2}{b^{r+1}\sin(\pi/b)} \sum_{a=1}^{b^{r+1}-1} \left( 1 - \frac{16(q-1)}{q^2} \left\| \frac{a}{b^{r+1}} \right\|^{2m/2} \right). \]

For the inner sum we have

\[ \sum_{a=1}^{b^{r+1}-1} \left( 1 - \frac{16(q-1)}{q^2} \left\| \frac{a}{b^{r+1}} \right\|^{2} \right)^{m/2} \]

\[ = \sum_{1 \leq a < b^{r+1}/2} \left( 1 - \frac{16(q-1)}{q^2} \frac{a^2}{b^{2r+2}} \right)^{m/2} \]

\[ + \sum_{b^{r+1}/2 \leq a < b^{r+1}} \left( 1 - \frac{16(q-1)}{q^2} \left( 1 - \frac{a}{b^{r+1}} \right)^2 \right)^{m/2} \]

\[ = \frac{1}{b^{m(r+1)}} \sum_{1 \leq a < b^{r+1}/2} \left( b^{2r+2} - \frac{16(q-1)}{q^2} a^2 \right)^{m/2} \]

\[ + \frac{1}{b^{m(r+1)}} \sum_{b^{r+1}/2 \leq a < b^{r+1}} \left( b^{2r+2} - \frac{16(q-1)}{q^2} (b^{r+1} - a)^2 \right)^{m/2} \]

\[ = \frac{2}{b^{m(r+1)}} \sum_{1 \leq a < b^{r+1}/2} \left( b^{2r+2} - \frac{16(q-1)}{q^2} a^2 \right)^{m/2} + \delta(b) \left( 1 - \frac{4(q-1)}{q^2} \right)^{m/2} , \]

where \( \delta(b) = 0 \) when \( b \) is odd and \( \delta(b) = 1 \) when \( b \) is even.
The assumption $q < 14$ yields $\frac{16(q-1)}{q^2} \geq 1$, and hence
\[
\sum_{a=1}^{b^{r+1}-1} \left( 1 - \frac{16(q-1)}{q^2} \left\| \frac{a}{b^{r+1}} \right\|^2 \right)^{m/2} \leq \frac{2}{b^m(r+1)} \sum_{1 \leq a < b^{r+1}/2} (b^{2r+2} - a^2)^{m/2} + \left( \frac{3}{4} \right)^{m/2}
\]
\[
\leq \frac{2}{b^m(r+1)} \sum_{u=1}^{b^{2r+2}-1} u^{m/2} + \left( \frac{3}{4} \right)^{m/2}
\]
\[
\leq \frac{2}{b^m(r+1)} \int_1^{b^{2r+2}} u^{m/2} \, du + \left( \frac{3}{4} \right)^{m/2}
\]
\[
\leq b_r \frac{b^{2r+2}}{m+1} + \left( \frac{3}{4} \right)^{m/2}
\]
with an implied constant depending only on $b$ and $q$. Therefore
\[
\sum_{k=1}^{b^g-1} \rho_b(k)|T_k(q^m)| \ll_{b,q} \sum_{r=0}^{q-1} \frac{1}{b^{r+1}} \left( b^{2(r+1)} \left( \frac{3}{4} \right)^{m/2} \right) \ll_{b,q} b^g \frac{b^g}{m+1}, \quad (4)
\]
again with implied constants depending only on $b$ and $q$.

Assume that $N = \sum_{r=0}^{R} a_r q^r$. Then, using Lemma 4 and (4), we obtain
\[
\sum_{k=1}^{b^g-1} \rho_b(k)|T_k(N)| \ll_{b,q} \frac{1}{N} \sum_{m=0}^{R} a_m q^m \sum_{k=1}^{b^g-1} \rho_b(k)|T_k(q^m)| \ll_{b,q} b^g \frac{b^g}{m+1}.
\]
Since
\[
\frac{1}{N} \sum_{m=0}^{R} a_m \frac{q^m}{m+1} \leq \frac{1}{N} \sum_{m=0}^{[R/2]} a_m q^m + \frac{1}{N} \sum_{m=[R/2]+1}^{R} a_m \frac{q^m}{m+1}
\]
\[
\leq q^{R/2} \frac{1}{N} + \frac{1}{R} \ll q \frac{1}{\log N}
\]
we obtain
\[
\sum_{k=1}^{b^g-1} \rho_b(k)|T_k(N)| \ll_{b,q} b^g \frac{b^g}{\log N}.
\]
From Lemma 2 it follows that
\[
D_N((x_{s_q(n)})_{n \geq 0}) \ll_{b,q} \frac{1}{b^g} + \frac{b^g}{\log N}.
\]
Choosing $g = \lfloor \log_b \sqrt{\log N} \rfloor$ yields

$$D_N((x_{s_q(n)})_{n \geq 0}) \ll_b q \frac{1}{\sqrt{\log N}}.$$

**Remark 2.** We remark that, in principle, the method of proof based on Lemma 2 cannot only be used for van der Corput-sequences, but also for Halton-sequences in higher dimensions. However, this leads to a discrepancy bound of order $\left( \log N \right)^{-\frac{1}{s+1}}$, which is considerably weaker than the one presented in Theorem 2.

### 5. Other index-transformations

In this section, we would now like to discuss index-transformed Halton- and digital $(t, s)$-sequences indexed by a different kind of sequence than the sum-of-digits function, as, e.g., $(\lfloor n^\alpha \rfloor)_{n \geq 0}$ with $0 < \alpha < 1$. The following theorem provides another general result, namely lower and upper bounds on the discrepancy of sequences indexed by functions which in some sense are “moderately” monotonically increasing.

**Theorem 4.** Let $A \in \mathbb{N}_0$ and write $\mathbb{N}_A := \{A, A+1, A+2, \ldots\}$. Let $f : \mathbb{N}_0 \to \mathbb{N}_A$ be surjective and monotonically increasing. Moreover, define, for $k \in \mathbb{N}_A$,

$$F(k) := \#\{n : n \in \mathbb{N}_0, f(n) = k\}.$$

Under the assumption that $F(k)$ is monotonically increasing in $k$ for sufficiently large $k$, the following three assertions hold.

1. For an arbitrary sequence $(x_n)_{n \geq 0}$ in $[0, 1)^s$ it is true that

$$\frac{F(f(N) - 1)}{N} \leq D_N((x_{f(n)})_{n \geq 0}).$$

2. For a Halton-sequence $(x_n)_{n \geq 0}$ in co-prime bases $b_1, \ldots, b_s$,

$$D_N((x_{f(n)})_{n \geq 0}) \leq C \frac{2F(f(N - 1) + 1)(\log N)^s}{N},$$

where $C$ is a constant independent of $N$.

3. For a digital $(t, s)$-sequence $(x_n)_{n \geq 0}$ over $\mathbb{F}_p$ for prime $p$,

$$D_N((x_{f(n)})_{n \geq 0}) \leq \tilde{C}p^t \frac{2F(f(N - 1) + 1)(\log N)^s}{N},$$

where $\tilde{C}$ is a constant independent of $N$.  
Proof.

1. Let \((x_n)_{n \geq 0}\) be an arbitrary sequence in \([0,1)^s\), and let \(f\) and \(F\) be as in the theorem. If \(f(N) = A\), then, due to the properties of \(f\), we obtain \(F(f(N) - 1) = 0\), so the lower bound on the discrepancy is trivially fulfilled. If, on the other hand, \(f(N) > A\), then it follows by the surjectivity of \(f\) that there exist \(n \in \mathbb{N}_0\) such that \(f(n) = f(N) - 1\). Furthermore, whenever \(n\) is such that \(f(n) = f(N) - 1 < f(N)\), it follows by the monotonicity of \(f\) that \(n < N\). Hence, the value \(f(N) - 1\) occurs \(F(f(N) - 1)\) times among \(f(0), \ldots, f(N - 1)\), and the point \(x_{f(N) - 1}\) is attained \(F(f(N) - 1)\) times in the sequence \(x_{f(0)}, \ldots, x_{f(N - 1)}\). The lower bound follows by considering an arbitrarily small interval containing \(x_{f(N) - 1}\).

2. Without loss of generality, assume \(f(0) = 0\), i.e., \(A = 0\). Furthermore, it is no loss of generality to assume that \(f(1) = 1\) and that \(F(k)\) is monotonically increasing in \(k\) for \(k \geq 0\). Indeed, if this is not the case, we can disregard a suitable number of initial elements \(x_{f(0)}, \ldots, x_{f(N_0)}\), without changing the discrepancy of the first \(N\) points of the sequence \((x_{f(n)})_{n \geq 0}\) by more than \(\frac{N_0}{N}\).

Let \(b_1, \ldots, b_s \geq 2\) be co-prime integers and let \((x_n)_{n \geq 0}\) be the corresponding Halton-sequence. For estimating the discrepancy, we consider an arbitrary interval

\[
I := \prod_{i=1}^s [0, \alpha(i)^{(1)}] \subseteq [0,1)^s,
\]

for some \(\alpha^{(1)}, \ldots, \alpha^{(s)} \in (0,1)\). For each \(i \in \{1, \ldots, s\}\), choose \(m_i\) as the minimal integer such that \(N \leq b_i^{m_i}\). Since \(f(N - 1) \leq N - 1\), the \(i\)-th component \(x_{f(n)}^{(i)}\) of a point \(x_{f(n)}\), \(1 \leq i \leq s\), \(0 \leq n \leq N - 1\), has at most \(m_i\) non-zero digits in its base \(b_i\) representation. From this, it is easily derived that we can restrict ourselves to considering only \(\alpha^{(i)}\) with at most \(m_i\) non-zero digits in their base \(b_i\) expansion, \(1 \leq i \leq s\), as this assumption changes \(D_N((x_{f(n)})_{n \geq 0})\) by a term of order of at most \(N^{-1}\). We can therefore write \(I\) as the disjoint union of intervals

\[
I(j_1, \ldots, j_s) := \prod_{i=1}^s \left[ \sum_{r=1}^{j_i - 1} \frac{\alpha^{(i)}_r}{b_i^r} + \frac{j_i}{b_i^{j_i}}, \sum_{r=1}^{j_i - 1} \frac{\alpha^{(i)}_r}{b_i^r} + \frac{j_i}{b_i^{j_i}} \right],
\]

where \(1 \leq j_i \leq m_i\) for \(1 \leq i \leq s\) and the \(\alpha^{(i)}_r\) represent the base \(b_i\) digits of \(\alpha^{(i)}\). Each of the \(I(j_1, \ldots, j_s)\) can in turn be written as the disjoint union of intervals

\[
\sum_{i=1}^s J(j_i, k_i) := \prod_{i=1}^s \left[ \sum_{r=1}^{j_i - 1} \frac{\alpha^{(i)}_r}{b_i^r} + \frac{k_i}{b_i^{j_i}}, \sum_{r=1}^{j_i - 1} \frac{\alpha^{(i)}_r}{b_i^r} + \frac{k_i + 1}{b_i^{j_i}} \right],
\]

with \(1 \leq j_i \leq m_i\) and \(0 \leq k_i \leq \alpha^{(i)}_{j_i} - 1\). If \(\alpha^{(i)}_{j_i} = 0\), then \(J(j_i, k_i)\) is of zero volume containing no points. Hence we can restrict ourselves to considering only those \(J(j_i, k_i)\) with \(\alpha^{(i)}_{j_i} \geq 1\).
Let now \( i \in \{1, \ldots, s\} \) and \( v \geq 0 \) be fixed. By the construction principle of the points of the Halton-sequence, we see that \( x_v^{(i)} \) is contained in \( J(j_i, k_i) \) if and only if

\[
\begin{pmatrix}
v_0^{(i)} \\
\vdots \\
v_{j_i-2}^{(i)} \\
v_{j_i-1}^{(i)}
\end{pmatrix}
= \begin{pmatrix}
\alpha_i^{(1)} \\
\vdots \\
\alpha_i^{(j_i-1)} \\
k_i
\end{pmatrix},
\]

where the \( v_r^{(i)}, 0 \leq r \leq j_i - 1 \) are the digits of \( v \) in base \( b_i \). Note that (5) has exactly one solution \( (v_0^{(i)}, \ldots, v_{j_i-1}^{(i)}) \mod b_i \). Hence we can identify exactly one remainder \( R^{(i)} \mod b_i \), such that \( x_v^{(i)} \in J(j_i, k_i) \) if and only if \( v \equiv R^{(i)} \mod b_i \). By the Chinese Remainder Theorem, there exists exactly one remainder \( R \mod Q := \prod_{i=1}^s b_i \) such that

\[
x_v \in \prod_{i=1}^s J(j_i, k_i) \quad \text{if and only if} \quad v \equiv R \mod Q.
\]

We now deduce an estimate for the number of points among \( x_{f(0)}, \ldots, x_{f(N-1)} \) that are contained in an interval of the type \( \prod_{i=1}^s J(j_i, k_i) \). For short, we denote this number by \( A(\prod_{i=1}^s J(j_i, k_i)) \).

Note that there exists a number \( \theta = \theta(R, Q, f(N-1)) \in \{0, 1\} \) such that \( 0 = f(0) \leq R + wQ \leq f(N-1) \) if and only if \( w \in \{0, \ldots, \lfloor \frac{f(N-1)}{Q} \rfloor - 1 + \theta \} \), so

\[
A \left( \prod_{i=1}^s J(j_i, k_i) \right) \geq \sum_{w=0}^{\lfloor \frac{f(N-1)}{Q} \rfloor - 1 + \theta} F(R + wQ) \geq \sum_{w=0}^{\lfloor \frac{f(N-1)}{Q} \rfloor - 2 + \theta} F(wQ), \tag{6}
\]

where we used the monotonicity of \( F \). On the other hand, with the same argument,

\[
A \left( \prod_{i=1}^s J(j_i, k_i) \right) \leq \sum_{w=0}^{\lfloor \frac{f(N-1)}{Q} \rfloor - 1 + \theta} F(R + wQ) \leq \sum_{w=1}^{\lfloor \frac{f(N-1)}{Q} \rfloor + \theta} F(wQ). \tag{7}
\]

For the following, let \( K = \lfloor \frac{f(N-1)}{Q} \rfloor + \theta \). Let

\[
\Sigma_A := \sum_{r=0}^{(K-1)Q-1} F(r),
\]

and note that we can write

\[
\Sigma_A = \sum_{w=0}^{K-2} \sum_{r=0}^{Q-1} F(wQ + r) \geq Q \sum_{w=0}^{K-2} F(wQ) = Q \sum_{w=0}^{\lfloor \frac{f(N-1)}{Q} \rfloor - 2 + \theta} F(wQ).
\]
On the other hand, by the definition of $\theta$,

$$\Sigma_A = \sum_{r=0}^{\left\lfloor \frac{f(N-1)}{Q} \right\rfloor - 1 + \theta} F(r) \leq \sum_{r=0}^{f(N-1)-1} F(r) \leq N - 1,$$

from which we conclude that

$$\sum_{w=0}^{\left\lfloor \frac{f(N-1)}{Q} \right\rfloor - 2 + \theta} F(wQ) \leq \frac{N - 1}{Q}.$$  \hspace{1cm} (8)

Moreover, let

$$\Sigma_B := \sum_{r=1}^{KQ} F(r),$$

for which we can derive, in the same way as the corresponding estimate for $\Sigma_A$,

$$\Sigma_B \leq Q \sum_{w=1}^{\left\lfloor \frac{f(N-1)}{Q} \right\rfloor + \theta} F(wQ).$$

Again by the definition of $\theta$,

$$\Sigma_B = \sum_{r=1}^{\left\lfloor \frac{f(N-1)}{Q} \right\rfloor + \theta} F(r) \geq \sum_{r=1}^{f(N-1)} F(r)
\leq \# \{ n \in \mathbb{N}_0 : 0 < f(n) \leq f(N-1) \} = N - 1,$$

where we used that $f(1) = 1$ and that $f$ is monotonically increasing. Consequently,

$$\sum_{w=1}^{\left\lfloor \frac{f(N-1)}{Q} \right\rfloor + \theta} F(wQ) \geq \frac{N - 1}{Q}.$$  \hspace{1cm} (9)

Note, furthermore, that

$$0 \leq \sum_{w=1}^{\left\lfloor \frac{f(N-1)}{Q} \right\rfloor + \theta} F(wQ) - \sum_{w=0}^{\left\lfloor \frac{f(N-1)}{Q} \right\rfloor - 2 + \theta} F(wQ)
\leq F \left( \left( \left\lfloor \frac{f(N-1)}{Q} \right\rfloor - 1 + \theta \right) Q \right)
+ F \left( \left( \left\lfloor \frac{f(N-1)}{Q} \right\rfloor + \theta \right) Q \right)
\leq 2F(f(N-1) + 1).$$  \hspace{1cm} (10)
Combining Equations (6), (9), and (10), and noting that \( \lambda(\prod_{i=1}^{s} J(j_i, k_i)) = \frac{1}{Q} \), gives

\[
\frac{1}{N} A \left( \prod_{i=1}^{s} J(j_i, k_i) \right) - \frac{1}{Q} \geq \frac{1}{N} \sum_{w=0}^{\left\lfloor \frac{f(N-1)}{Q} \right\rfloor + \theta} F(wQ) - \frac{1}{Q} \\
\geq \sum_{w=0}^{\left\lfloor \frac{f(N-1)}{Q} \right\rfloor + \theta} F(wQ) - \frac{2F(f(N-1) + 1)}{N} - \frac{1}{Q} \\
\geq -\frac{2F(f(N-1) + 1)}{N} + \frac{N - 1}{QN} - \frac{1}{Q} \\
\geq -\frac{2F(f(N-1) + 1)}{N} - \frac{1}{NQ}.
\]

In exactly the same way, using (7), (8), and (10), we get

\[
\frac{1}{N} A \left( \prod_{i=1}^{s} J(j_i, k_i) \right) - \frac{1}{Q} \leq \frac{2F(f(N-1) + 1)}{N} + \frac{1}{NQ},
\]

from which we derive

\[
\left| \frac{1}{N} A \left( \prod_{i=1}^{s} J(j_i, k_i) \right) - \frac{1}{Q} \right| \leq \frac{2F(f(N-1) + 1)}{N} + \frac{1}{NQ}.
\]

Finally, note that, by writing \( A(I) \) for the number of points of \( (x_{f(n)})_{n=0}^{N-1} \) in \( I \),

\[
\left| \frac{A(I)}{N} - \lambda(I) \right| \leq \sum_{j_1=1}^{m_1} \cdots \sum_{j_s=1}^{m_s} \sum_{k_1=0}^{\alpha^{(1)}(j_1) - 1} \cdots \sum_{k_s=0}^{\alpha^{(s)}(j_s) - 1} \left| \frac{1}{N} A \left( \prod_{i=1}^{s} J(j_i, k_i) \right) - \lambda \left( \prod_{i=1}^{s} J(j_i, k_i) \right) \right| \\
\leq C \frac{(\log N)^s F(f(N-1) + 1)}{N},
\]

for a suitably chosen constant \( C \), and the result follows.

3. As in Item 2, assume without loss of generality that \( f(0) = 0, f(1) = 1 \), and that \( F(k) \) is monotonically increasing in \( k \) for \( k \geq 1 \). Let \( p \) be a prime and let \( (x_n)_{n \geq 0} \) be a digital \( (t,s) \)-sequence over \( \mathbb{F}_p \). For estimating the discrepancy, we consider an arbitrary interval

\[
I := \prod_{i=1}^{s} [0, \alpha^{(i)}(j_i)) \subseteq [0, 1)^s,
\]

for some \( \alpha^{(1)}, \ldots, \alpha^{(s)} \in (0,1] \). Choose \( m \) as the minimal integer such that \( N \leq p^m \). By a similar argument as for the case of Halton sequences, we
can restrict ourselves to considering only $\alpha^{(i)}$ with at most $m$ non-zero digits $\alpha^{(i)}_1, \ldots, \alpha^{(i)}_m$ in their base $p$ expansion. Moreover, with the same reasoning as in the Halton case, we see that we essentially only need to deal with intervals of the form

$$\prod_{i=1}^{s} J(j_i, k_i) := \prod_{i=1}^{s} \left( \sum_{r=1}^{j_i-1} \frac{\alpha^{(i)}_r}{p^r} + \frac{k_i}{p^{j_i}} \right),$$

with $1 \leq j_i \leq m$ and $0 \leq k_i \leq \alpha^{(i)}_{j_i} - 1$. Again, if $\alpha^{(i)}_{j_i} = 0$, then $J(j_i, k_i)$ is of zero volume containing no points, so we can restrict ourselves to considering only those $J(j_i, k_i)$ with $\alpha^{(i)}_{j_i} \geq 1$.

As for the case of Halton sequences, we would like to derive an upper and a lower bound on the number $A \left( \prod_{i=1}^{s} J(j_i, k_i) \right)$ of points contained in $\prod_{i=1}^{s} J(j_i, k_i)$. To this end, denote the $r$-th row of a generator matrix $C_j, 1 \leq j \leq s$ of $(x_n)_{n \geq 0}$ by $c^{(j)}_r$.

For an integer $v \geq 0$, the point $x_v$ is contained in $\prod_{i=1}^{s} J(j_i, k_i)$ if and only if

$$C \cdot \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{j_1+\cdots+j_s} \end{pmatrix} = A^\top,$$

where $v_0, v_1, v_2, \ldots$ are the base $p$ digits of $v$, where

$$A := (\alpha^{(1)}_1, \ldots, \alpha^{(1)}_{j_1-1}, k_1, \alpha^{(2)}_1, \ldots, \alpha^{(2)}_{j_2-1}, k_2, \ldots, \alpha^{(s)}_1, \ldots, \alpha^{(s)}_{j_s-1}, k_s) \in \mathbb{F}_p^{j_1+\cdots+j_s},$$

and

$$C := (c^{(1)}_1, \ldots, c^{(1)}_{j_1}, c^{(2)}_1, \ldots, c^{(2)}_{j_2}, \ldots, c^{(s)}_1, \ldots, c^{(s)}_{j_s})^\top \in \mathbb{F}_p^{(j_1+\cdots+j_s) \times \mathbb{N}}.$$

Let now $Q := p^{j_1+\cdots+j_s+t}$, let $w \in \mathbb{N}_0$ and consider those $v \geq 0$ with $wQ \leq v \leq (w+1)Q - 1$. For these $v$, the first $j_1+j_2+\cdots+j_s+t$ digits in their base $p$ expansion vary, while all the other digits are fixed. Hence we can write (11) as

$$D_1 \cdot \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{j_1+\cdots+j_s+t} \end{pmatrix} + D_2 \cdot \begin{pmatrix} v_{j_1+\cdots+j_s+t+1} \\ v_{j_1+\cdots+j_s+t+2} \\ \vdots \end{pmatrix} = A^\top,$$

where $C = (D_1|D_2)$ and where $D_1$ is an $(j_1+\cdots+j_s) \times (j_1+\cdots+j_s+t)$-matrix and $D_2$ is an $(j_1+\cdots+j_s) \times \mathbb{N}$-matrix over $\mathbb{F}_p$. 
Due to the fact that \((x_n)_{n \geq 0}\) is a digital \((t,s)\)-sequence, it follows that \(D_1\) has full rank, and hence there are exactly \(p^t\) values \(v\) in \(\{wQ, wQ + 1, \ldots, (w + 1)Q - 1\}\) such that \(x_v\) is contained in \(\prod_{i=1}^s J(j_i, k_i)\).

Now note again that there exists a number \(\theta = \theta(Q, f(N - 1)) \in (0, 1)\) such that \(0 = f(0) \leq wQ \leq f(N - 1)\) if and only if \(w \in \{0, \ldots, \lfloor \frac{f(N - 1)}{Q} \rfloor - 1 + \theta\}\). By our observations above, for each of these \(w \in \{0, \ldots, \lfloor \frac{f(N - 1)}{Q} \rfloor - 1 + \theta\}\) there exist \(p^t\) integers \(R_{w,1}, \ldots, R_{w,p^t} \in \{0, \ldots, Q - 1\}\) such that exactly the points \(x_{R_{w,1} + wQ}, \ldots, x_{R_{w,p^t} + wQ}\) among \(x_{wQ}, x_{wQ + 1}, \ldots, x_{(w+1)Q-1}\) are contained in \(\prod_{i=1}^s J(j_i, k_i)\). Therefore, we can estimate

\[
A \left( \prod_{i=1}^s J(j_i, k_i) \right) \geq \sum_{w=0}^{\lfloor \frac{f(N - 1)}{Q} \rfloor - 2 + \theta} \sum_{z=1}^{p^t} F(R_{w,z} + wQ) \geq p^t \sum_{w=0}^{\lfloor \frac{f(N - 1)}{Q} \rfloor - 2 + \theta} F(wQ),
\]

and

\[
A \left( \prod_{i=1}^s J(j_i, k_i) \right) \leq \sum_{w=0}^{\lfloor \frac{f(N - 1)}{Q} \rfloor - 1 + \theta} \sum_{z=1}^{p^t} F(R_{w,z} + wQ) \leq p^t \sum_{w=1}^{\lfloor \frac{f(N - 1)}{Q} \rfloor + \theta} F(wQ).
\]

In exactly the same way as for a Halton sequence, we obtain, by noting that

\[
\lambda(\prod_{i=1}^s J(j_i, k_i)) = \frac{1}{p^{t_1 + \ldots + t_s}} = \frac{p^t}{Q},
\]

\[
\left| \frac{1}{N} A \left( \prod_{i=1}^s J(j_i, k_i) \right) - \frac{1}{Q} \right| \leq \frac{p^t 2 F(f(N - 1) + 1)}{N} + \frac{p^t}{NQ},
\]

and the result follows.

Examples of functions \(f\) and \(F\) satisfying the assumptions of Theorem 4 are obtained as follows. Let \(g : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+\) be a function that is twice differentiable on \((0, \infty)\), with \(g'(x) > 0\) and \(g''(x) < 0\) for \(x \in (0, \infty)\). Moreover, define \(f(n) := [g(n)]\) for \(n \in \mathbb{N}\). It then easily follows that \(f\) and \(F\) indeed fulfill the assumptions of the theorem and we obtain

\[
F(k + 1) = \left\lfloor g^{-1}(k + 1) \right\rfloor - \left\lfloor g^{-1}(k) \right\rfloor.
\]

We thus obtain the following exemplary corollary to Theorem 4.

**Corollary 2.** Let \(\alpha \in (0, 1)\). Then the following assertions hold.

1. For a Halton-sequence \((x_n)_{n \geq 0}\) in co-prime bases \(b_1, \ldots, b_s\),

\[
\overline{C}_1 \frac{1}{N^{\alpha}} \leq D_N((x_{[n^\alpha]}))_{n \geq 0} \leq \overline{C}_2 \frac{\log N^s}{N^{\alpha}},
\]

where \(\overline{C}_1, \overline{C}_2\) are constants that depend on the sequence and on \(\alpha\), but are independent of \(N\).
2. For a digital \((t,s)\)-sequence \((x_n)_{n\geq 0}\) over \(\mathbb{Z}_p\) for prime \(p\),

\[
\overline{C}_1 \frac{1}{N^\alpha} \leq D_N((x_{[n^\alpha]})_{n\geq 0}) \leq \overline{C}_2 \frac{(\log N)^s}{N^\alpha},
\]

where \(\overline{C}_1, \overline{C}_2\) are constants that depend on the sequence and on \(\alpha\), but are independent of \(N\).

**Proof.** The result follows by combining Theorem 2 with the observation that

\[
c'_\alpha k_{\frac{1}{p^\alpha} - 1} \leq F(k) \leq c_\alpha k_{\frac{1}{p^\alpha} - 1},
\]

with constants \(c'_\alpha, c_\alpha > 0\) that depend on \(\alpha\), but not on \(k\). ■

6. Appendix: Uniform discrepancy

In Corollary 1 we implicitly used the fact that \((t,s)\)-sequences in base \(b\) as well as Halton-sequences in pairwise co-prime bases \(b_1, \ldots, b_s\) have uniform discrepancy of order \((\log N)^s/N\). Since we are not aware of a proof of these facts in the existing literature, we provide one here.

6.1. Uniform discrepancy of \((t,s)\)-sequences in base \(b\)

Assume that \(\Delta_b(t, m, s)\) is a number for which

\[
b^m D_{b^{m\cdot \mathcal{P}}} \leq \Delta_b(t, m, s)
\]

holds for the discrepancy of any \((t, m, s)\)-net \(\mathcal{P}\) in base \(b\).

**Theorem 5.** Let \((x_n)_{n\geq 0}\) be a \((t, s)\)-sequence in base \(b\). Then we have

\[
N \tilde{D}_N((x_n)_{n\geq 0}) \leq (2b - 1) \left( tb^t + \sum_{m=t}^{\lfloor \log_b N \rfloor} \Delta_b(t, m, s) \right).
\]

**Proof.** Let \(k \in \mathbb{N}_0\). We show that

\[
N D_N((x_{n+k})_{n\geq 0}) \leq (2b - 1) \left( tb^t + \sum_{m=t}^{\lfloor \log_b N \rfloor} \Delta_b(t, m, s) \right)
\]

uniformly in \(k \in \mathbb{N}_0\).

For \(N < b^t\), the assertion follows trivially by \(N D_N((x_{n+k})_{n\geq 0}) \leq N\).

Let now \(N \in \mathbb{N}, N \geq b^t\) with \(b\)-adic expansion \(N = a_r b^r + a_{r-1} b^{r-1} + \cdots + a_1 b + a_0\) where \(a_j \in \{0, \ldots, b-1\}\) for \(0 \leq j \leq r\) and \(a_r \neq 0\) (note that \(r \geq t\)). For given \(k \in \mathbb{N}_0\), choose \(\ell \in \mathbb{N}\) such that \((\ell - 1)b^r \leq k < \ell b^r\). Then we can write

\[
k = \ell b^r - (a_{r-1} b^{r-1} + \cdots + a_1 b + a_0) - 1
\]
with some \( d_j \in \{0, \ldots, b - 1\} \) for \( 0 \leq j \leq r - 1 \), and
\[
k = (\ell - 1)b^r + \kappa_{r-1}b^{r-1} + \cdots + \kappa_1 b + \kappa_0
\]
with some \( \kappa_j \in \{0, \ldots, b - 1\} \) for \( 0 \leq j \leq r - 1 \). Note that therefore \( d_j + \kappa_j = (b-1) \) for \( 0 \leq j < r \).

We split up the point set \( \mathcal{P}_{k,N} := \{x_n : k \leq n \leq k + N - 1\} \) in the following way:
\[
\mathcal{P}_{k,N} = \bigcup_{1 \leq d \leq d_0 + 1} \mathcal{P}_{0,d}^1 \bigcup_{1 \leq m \leq \ell - 1} \bigcup_{1 \leq d \leq d_m} \mathcal{P}_{m,d}^1 \bigcup_{1 \leq d \leq d_m} \bigcup_{t \leq m \leq \ell - 1} \bigcup_{0 \leq s \leq \kappa_m + \kappa_m} \mathcal{P}_{m,s}^m \bigcup_{t \leq m \leq \ell - 1} \bigcup_{0 \leq s \leq \kappa_m + \kappa_m} \mathcal{P}_{m,s}^m
\]
where
\[
\mathcal{P}_{0,d}^1 := \{x_{(b^r-d_{r-1}b^{r-1} \cdots -d_{m+1}b^{m+1}-db^m+j) : 0 \leq j < b^m}\},
\mathcal{P}_a^m := \{x_{(b^r-ab^{r-1} \cdots -d_{m+1}b^{m+1}-db^m+j) : 0 \leq j < b^m}\},
\mathcal{P}_{m,s}^m := \{x_{(\ell+a_{r-1})b^r+(\kappa_{r-1}+a_{r-1})b^{r-1}+\cdots+(\kappa_{m+1}+a_{m+1})b^{m+1}+xb^m+j) : 0 \leq j < b^m}\}.
\]
For \( m \leq t - 1 \), we can bound the discrepancy of \( \mathcal{P}_{m,d}^1 \) and \( \mathcal{P}_{m,s}^m \), respectively, by the trivial bound 1. For \( m \geq t \), the point sets \( \mathcal{P}_{m,d}^1 \) and \( \mathcal{P}_{m,s}^m \) are \((t,m,s)\)-nets in base \( b \), and the \( \mathcal{P}_a^m \) are \((t,r,s)\)-nets in base \( b \). From the triangle inequality for the discrepancy we obtain
\[
ND_N(\mathcal{P}_{k,N}) \leq (d_0 + a_0 + \kappa_0 + 1)b^0 + \sum_{m=1}^{t-1} (d_m + a_m + \kappa_m)b^m
\]
\[
+ \sum_{m=t}^{r-1} (d_m + a_m + \kappa_m)\Delta_b(t,m,s) + \max(b_r - 2, 0)\Delta_b(t,r,s)
\]
\[
\leq (2b - 1) + (2b - 2) (t - 1)b^t + \sum_{m=t}^{r-1} \Delta_b(t,m,s) + \max(b - 3, 0)\Delta_b(t,r,s)
\]
\[
\leq (2b - 1) \left( tb^t + \sum_{m=t}^{r} \Delta_b(t,m,s) \right)
\]
and the result follows, since \( r = \lfloor \log_b N \rfloor \).

**Corollary 3.** Let \( (x_n)_{n \geq 0} \) be a \((t,s)\)-sequence in base \( b \). Then we have
\[
N\tilde{D}_N((x_n)_{n \geq 0}) \ll_{s,b} b^t(\log N)^s.
\]

**Proof.** The result follows from Theorem 5 together with the fact that
\[
\Delta_b(t,m,s) \ll_{s,b} b^t m^{s-1}
\]
for \( m \geq t \) (see, for example, [6, 24]).
6.2. Uniform discrepancy of Halton-sequences

Theorem 6. Let \((x)_n\geq 0\) be a Halton-sequence in pairwise co-prime bases \(b_1, \ldots, b_s\). Then we have

\[
N \widetilde{D}_N((x)_n\geq 0) = \frac{1}{s!} \prod_{j=1}^{s} \left( \frac{\lfloor b_j/2 \rfloor \log N}{\log b_j} + s \right) + O((\log N)^{s-1}),
\]

where the implied constant depends on \(b_1, \ldots, b_s\) and \(s\).

Proof. The result follows from an adaption of the proof of [6, Theorem 3.36]. Note that [6, Lemma 3.37] also holds true for \(A(J,k,N,S) := \#\{n \in \mathbb{N} : k \leq n < k + N \text{ and } x_n \in J\}\) instead of \(A(J,N,S) := A(J,0,N,S)\). The rest of the proof of [6, Theorem 3.36] remains unchanged.  

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Address: Peter Kritzer, Gerhard Larcher and Friedrich Pillichshammer: Institut für Finanzmathematik, Johannes Kepler Universität Linz, Altenbergerstr. 69, 4040 Linz, Austria.

E-mail: peter.kritzer@jku.at, gerhard.larcher@jku.at, friedrich.pillichshammer@jku.at

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