Disproof of Joy Christian’s “Disproof of Bell’s theorem”

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Four critical elementary mathematical mistakes in Joy Christian’s counterexample to Bell’s theorem are presented. Consequently, Joy Christian’s hidden variable model cannot reproduce any quantum mechanics results and cannot be used as a counterexample to Bell’s theorem. The mathematical investigation is followed by a short discussion about the possibility to construct other hidden variable theories.

Introduction

In a series of papers [1–8], Joy Christian claims to have constructed a local and realistic hidden variable theory able to reproduce quantum mechanics’ correlations in the case of an EPR-Bohm experiment with spin one half particles. Joy Christian’s model was received with serious criticisms [9, 10, 11, 12, 13], but no single analysis could provide a decisive argument against it. Interestingly, the first comment came very close to uncover the first mathematical problem discussed below. In his reply Joy Christian stated: “With hindsight, however, it would have been perhaps better had I not left out as an exercise an explicit derivation of the CHSH inequality in Ref. [1]. Let me, therefore, try to rectify this pedagogical deficiency here.”. This reply unfortunately managed to discourage other people from trying to check the validity of the mathematical results which are actually very easy to find.

Here are some key statements from subsequent critics’ replies showing that the mathematical correctness was not the main focus of the criticisms:

“the model formally manages to reproduce some quantum theoretical expectation values correctly” [10]

“and here we will assume that the content of these papers is correct” [11]

“The first comment on Christian's paper, [9] by Marcin Pawłowski, is also different from the current paper. That comment seems to say that in Clifford-algebra-valued hidden variable theory it is unable to derive Bell’s inequalities. This is not true since they are indeed derivable, as is explicitly shown in [2].” [12]

“By providing an explicit factorizable model, Joy Christian’s example only disproves the importance of Bell’s theorem as an argument against contextual hidden variable theories.” [13].

In the meantime, a new challenge to Joy Christian’s work came from informal physics blogs: if Joy Christian’s claim of a local realistic theory able to reproduce quantum correlations is right, then a computer simulation on a classical computer would be possible. The author of this paper proceed to do just that: model Joy Christian’s theory on a computer to clarify its claims of local realism. In order to translate the model to computer code, all results had to be systematically double checked and this led to the big surprise of finding out the elementary but critical mathematical mistakes presented below.

Error 1: losing part of the correlation result by incorrect averaging

In his first paper [1], Joy Christian introduces his hidden variable \( \mu \) as a random handedness of the basis for his geometric algebra basis. If \( \{ e_1, e_2, e_3 \} \) is a set of fixed orthonormal vectors, the hidden variable \( \mu \) is defined as follows

\[
\mu \triangleq \pm I = \pm e_1 e_2 e_3 = \pm e_1 \wedge e_2 \wedge e_3 \tag{1}
\]

(see Eq. 14 of Ref. [1])

Then Joy Christian proceeds on computing the Clifford product of two bivectors \( \mu \cdot a, \mu \cdot b \) obtaining in Eq. 17:

\[
(\mu \cdot a)(\mu \cdot b) = -a \cdot b - a \wedge b \tag{2}
\]

which is correct (please note that the plus and minus factors in the two \( \mu \)'s cancel each other out and the expression is equivalent with \((I \cdot a)(I \cdot b))\). However, the last line of Eq. 17 is incorrect. By a well-known identity (Hodge duality):

\[
a \wedge b = I \cdot (a \times b) \tag{3}
\]

it is now easy to see that the last line of Eq. 17 is incorrect when \( \mu = -I \). Because \( I \) is incorrectly replaced by \( \mu \) and gains an illegal minus sign when \( \mu = -I \), this leads to an incorrect canceling of the \( a \wedge b \) term when averaging over all \( \mu \)'s in the oriented vector manifold \( \mathcal{V}_3 \).

In Ref. [9], Pawłowski criticized Christian’s proposal for the presence of Clifford Algebra valued observables if we are to get a scalar in the RHS of the CHSH inequality. Joy Christian responded [2] by stating that the average on \( \mathcal{V}_3 \) removes the \( a \wedge b \) element. However, Eq. 3 of Ref. [2] is incorrect just like the last line of Eq. 17 in Ref. [1].

Even without spelling in detail the error, it is easy to see that the exterior product term should not vanish on any handedness average because handedness is just a paper convention on how to consistently make computations. For example one can apply the same incorrect argument to complex numbers because there is the same
freedom to choose the sign of $\sqrt{-1}$ based on the two dimensional coordinate handedness in this case. Then one can compute the average of let’s say $z = 3 + 2i$ for a fair coin random distribution of handedness and arrive at the incorrect answer: $< z > = 3$ instead of $< z > = z$.

One advertised strength of geometric algebra is the ability to make computations in a coordinate-free fashion. If breaking up an object in its components and performing an average results in elimination of some components, then we are guaranteed that the operation is mathematically illegal.

The same mistake is present in the minimalist paper \cite{7} in Eq. 4. Given a fixed bivector basis $\{ \beta_1, \beta_2, \beta_3 \}$, Eq. 3 of Ref. \cite{7} is correct. The corresponding $\lambda$-dependent basis product however should be the same as Eq. 3 of Ref. \cite{7} because $\beta_i(\lambda)\beta_j(\lambda) = \beta_i\beta_j \lambda^2 = \beta_i\beta_j$ and not gain an illegal $\lambda$ term for the cross product.

So how it was possible to have such an elementary mistake undetected? Most of Joy Christian’s papers suffer from a convention ambiguity: in some cases the computations are done using the $\mu = \pm I$ convention with $I$ arising from a fixed basis, while in others the computations are done using the $\mu = I$ convention (indefinite Hodge duality) which means that $I$ is the current trivector and does not arise from a fixed basis. Illegally mixing the conventions during one computation yields the supposedly agreement with quantum mechanics.

Another way this mistake can arise can be seen in Eqs. 23 and 24 of Ref. \cite{8}. In there Eq. 23 is correct and Eq. 24 is derived by switching $I$ to $-I$ for a change of handedness. It is true that changing handedness (or equivalent performing a reflection) changes the sign of the pseudoscalar $I$, but what is incorrect in Eq. 24 is that $a \times b$ is a pseudovector who should change sign as well. The corrected Eq. 24 should be:

$$(-I \cdot a)(-I \cdot b) = -a \cdot b - (-I)(-(a \times b)) \quad (4)$$

The same mechanism for producing the error probably occurred in deriving Eq. 4 of Ref. \cite{8} due to an incorrect replacement of $\beta(\lambda)$ in Eq. 3 without appropriately switching the sign of the Levi-Civita pseudotensor.

Also there are two additional physics objections as well. First, associating a hidden variable to an abstract computation convention is completely unphysical. Second, by doing it so, the theory predicts the same correlation and all the four experimental outcomes $(+,+),(-,-),(+,-),(-,+)$ are mathematically incorrect.

Error 2: Isotropically weighted averages of non-scalar part of correlations and measurement outcomes cannot be both zero

Specifically, Eq. 18 and Eq. 19 of paper \cite{1} cannot be both right. Let us count how many factors of $\mu$ are in Eq. 18 and Eq. 19. In Eq. 18 there are two factors of $\mu$, one from $\mu \cdot n$ and the other from $d\rho(\mu)$. Eq. 19 has three factors of $\mu$ from $\mu \cdot a$, $\mu \cdot b$, and $d\rho(\mu)$. As the factors are even and odd, integrating on a manifold $\mathcal{V}_3$ where $\mu$ changes signs evenly, only one of the two equations can be zero. Expanding the $(\mu \cdot a)(\mu \cdot b)$ term and using Hodge duality it follows that isotropically weighted averages of non-scalar part of correlations and measurement outcomes cannot be both zero.

By the prior error we already know Eq. 19 is incorrect and the statement that Eq. 18 and Eq. 19 cannot be both right is not a surprise. The new content of this error is that fixing Eq. 19 by any hypothetical generalization of the manifold $\mathcal{V}_3$ breaks Eq. 18. Both Eqs. 18 and 19 are needed to be right if the model is to reproduce experimental results. Therefore the handedness mistakes conclusively rule out both Joy’s model and all its potential Clifford algebra generalizations.

Error 3: Illegal limit for a bivector equation

In Ref. \cite{5}, Joy Christian attempts to implicitly answer Holman’s criticisms \cite{10} that the final answer in Eq. 19 of Ref. \cite{1} has a wrong sign and that the outcome of the experiments is always the same resulting in perfect correlations. Joy Christian first seems to agree that the outcome for any pairs of experimental results is the same, but then tries to prove the opposite in an explanation marred by mathematical mistakes.

The discussion of this problem takes place around Eqs. 42-46 of that paper. Citing Joy Christian: “Furthermore, we have taken the randomness $\mu = +I$ or $-I$ shared by Alice and Bob to be the initial orientation (or handedness) of the entire physical space, or equivalently that of a 3-sphere. Consequently, once $\mu$ is given as an initial state, the polarizations along all directions chosen by Alice and Bob would have the same value, because $\mu$ completely fixes the sense of bivectors $\mu \cdot n$ belonging to $S^2 \subset S^3$, regardless of direction.” In other words, the correlation of the experimental outcome is always $+1$ contradicting quantum mechanics predictions. Up to this point Joy Christian is correct.

Still, Joy Christian continues: “However, and this is an important point, the polarization $(+\mu \cdot a)$ observed by Alice is measured with respect to the analyzer $(-I \cdot a)$, whereas the polarization $(+\mu \cdot b)$ observed by Bob is measured with respect to the analyzer $(+I \cdot b)$.”

It will be shown below in this and next section that the mathematical arguments supporting this (in an attempt to produce both the minus cosine correlation and all the four experimental outcomes $(+,+),(-,-),(+,-),(-,+)$) are mathematically incorrect.

First let us note that the change in sign between Alice and Bob is illegal because they both use the same kind of apparatus during measurement and swapping them should not change anything. Therefore we could stop the analysis here because Joy Christian’s model is obviously wrong as it does not respect this basic symmetry. However let’s follow along Joy Christian’s argument and
discover where the mistakes occur.

To solve the problem, Joy Christian considers two almost parallel vectors instead of one at each detector: \( aa’ \) for Alice and \( bb’ \) for Bob. The two vectors form a bivector and starting from an aligned vector configuration (\( a \) aligned with \( b \) and \( a’ \) aligned with \( b’ \)) the goal is to move the second pair (\( bb’ \)) at Bob’s location in the final detector position using a rotor. In general, a rotor can be expressed as follows:

\[
R = \cos \Omega + \hat{B} \sin \Omega
\]  

with \( \hat{B} \) a unit bivector defined as:

\[
\hat{B} = \frac{m \wedge n}{\sin \Omega}
\]

where \( m \) and \( n \) are two unit vectors and \( \Omega \) is the angle between them. In the case of Eq. 45 of Ref. 5, the unit bivector corresponds to an axial vector \( c \) of unit norm:

\[
c = \frac{a \times a’}{|a \times a’|}
\]

Then at the end of the rotation, after applying a simple trigonometric identity composing bivectors (multiplying two exponential expressions corresponding to the \( a \wedge a’ \) and the \( a \wedge b \) bivectors) Joy Christian takes the limit \( a \rightarrow a’ \) and claims that this makes the bivector component of the final result zero as \( a \wedge a’ \) becomes zero. The end result is that only the cosine factor survives the operation therefore a rotation by “parallel transport” in a “twisted manifold” allows to recover the cosine of the angle between Alice and Bob in their correlation as predicted by quantum mechanics.

The limit operation above is mathematically illegal. To see why, recall that the axial vector \( c \) is of unit norm and as such it is normalized by the sine of the angles between \( a \) and \( a’ \). As \( a’ \) approaches \( a \), the wedge product goes to zero as sine of the angle, but the denominator goes to zero by the same sine of the angle factor. As such the two sines cancel each other and the bivector maintains its magnitude. This is nothing but a restatement of the geometric algebra fact that the magnitude of a bivector does not depend on the shape of the parallelepipeds defining it. But maybe there is a discontinuity at the limit when \( a = a’ \) and the bivector \( \hat{B} \) does become zero. We can see that this is not the case as follows. Suppose \( a, a’, b, b’ \) are four unit vectors in the same plane, and vector \( c \) is a unit vector orthogonal to this plane:

\[
c = \frac{a \times a’}{|a \times a’|} = \frac{a \times b}{|a \times b|}
\]

let \( \epsilon \Omega \) be the angle between \( a \) and \( a’ \), and \((1-\epsilon)\Omega\) the angle between \( a’ \) and \( b \). If \( \hat{B} \) is the unit bivector

\[
\hat{B} = B_{aa’} = B_{a’b} = Ic = \frac{a \wedge a’}{\sin(\epsilon \Omega)} = \frac{a’ \wedge b}{\sin((1-\epsilon)\Omega)}
\]

then Joy Christian’s incorrect argument is as follows:

\[
\cos \Omega + \hat{B} \sin \Omega = R(ab) = R(aa’)R(a’b) = 
\]

\[
(\cos(\epsilon \Omega) + \hat{B} \sin(\epsilon \Omega))(\cos((1-\epsilon)\Omega) + \hat{B} \sin((1-\epsilon)\Omega)) =
\]

\[
\cos(\epsilon \Omega) \cos((1-\epsilon)\Omega) - \sin(\epsilon \Omega) \sin((1-\epsilon)\Omega) +
\]

\[
B_{aa’}(\sin(\epsilon \Omega) \cos((1-\epsilon)\Omega) + \cos(\epsilon \Omega) \sin((1-\epsilon)\Omega)) =
\]

\[
\cos \Omega
\]

The last equality comes from taking the exact limit \( a = a’ \) which makes \( B_{aa’} \) vanish due to \( a \wedge a = 0 \). Comparing the first and last rows, it is clear that there is no discontinuity even when \( a \) is strictly \( a’ \) and the limit result is illegal.

It is also easy to see the mistake another way. Just compare Eq. 46 with the definition of the rotor following Eq. 45. In line two of Eq. 46 the term following the rotor computes to \(-\lambda\) and the final answer in Eq. 46 up to the \(-\lambda\) factor should be the entire value of the rotor and not just its cosine part.

Error 4: Incorrect parallel transport rotor direction

In the problem above Joy Christian attempts to eliminate a bivector by an illegal limit. Taking the limit correctly still results in the wrong result because the first line of Eq. 46 should be equal with the last line of the same equation. The error is in using the incorrect rotor to perform the parallel transport. In geometric algebra any object \( G \) transforms under a rotation by a rotor \( R \) as \( G \rightarrow R^\dagger GR \) with \( R = ab \) and \( R^\dagger = ba \). The angle of rotation is double the angle between vectors \( a \) and \( b \). This formula is completely general and works for scalars, vectors, bivectors, pseudoscalars, or any of their linear combinations. Let us try to apply this general formula to Eq. 46. \( R_{ab} \) reads:

\[
R_{ab} = \exp\{(I \cdot c)\theta_{ab}/2\} = \cos(\theta_{ab}/2) + (I \cdot c)\sin(\theta_{ab}/2)
\]

and the correct computation in Eq. 46 is:

\[
\lim_{\epsilon \rightarrow 0}[\epsilon (I \cdot b)(+\mu \cdot b’)] = -\lambda
\]

\[
= \lim_{a’ \rightarrow a}[R_{ab}^\dagger((I \cdot a)(+\mu \cdot a’))]R_{ab}
\]

\[
= \lim_{a’ \rightarrow a}\{R_{ab}^\dagger(-\lambda)(\exp\{(I \cdot c)\theta_{aa’}\})\}R_{ab}
\]

\[
= \lim_{a’ \rightarrow a}\{R_{ab}R_{ab}[(I \cdot a)(+\mu \cdot a’)]\}
\]

\[
= \lim_{a’ \rightarrow a}[\epsilon (I \cdot a)(+\mu \cdot a’)] = -\lambda
\]

The final result after parallel transport is still \(-\lambda\) because \( \lambda \) is a scalar. The fourth line in the equation above comes from the fact that vector \( c \) commutes with itself. There is another way to understand why the final result was not changed even when the limit is not taken.
A bivector is an oriented surface characterized only by direction, magnitude, and sense of rotation. The vectors $a, a', b, b'$ are in the same plane and the bivector $((+I \cdot a)(+\mu \cdot a'))$ and $((+I \cdot b)(+\mu \cdot b'))$ are actually identical because they have the same orientation, magnitude, and sense of rotation. Rotating the pair $aa'$ into $bb'$ by the angle $\theta_{ab}$ preserves the orientation, magnitude, and sense of rotation.

The second mistake in Eq. 46 (when computing the limit correctly) comes from applying incorrectly the law of rotation for rotors. This formula is different than the general multivector formula and it applies only for rotors. Specifically the mistake in Joy Christian’s paper is in the orientation of his rotor $\mathcal{R}$: instead of rotating around the vector $c$ (or equivalently $a \times a'$), the correct direction is $(a \times a') \times (b \times b')$. This is the correct direction because we are trying to rotate an $a \wedge a'$ bivector with orientation $a \times a'$ into a $b \wedge b'$ bivector with orientation $b \times b'$ and the rotation needs to align the two directions. Therefore the correct rotation direction is $(a \times a') \times (b \times b')$. With this correct direction one can prove that the rotor law of rotation and the multivector law of rotation produce the same result (for example one can use a tedious brute force expansion of the two formulae and show they are identical in components). Applied to the discussion in the paper, since $(a \times a')$ is parallel with $(b \times b')$ the rotor reduces to identity in this case.

Computed correctly with the right limit and the right rotor, we can now see that the outcome at Alice’s and Bob’s detectors is always the same and the results are completely correlated contradicting quantum mechanics. The reason is that the outcome results are nothing but the negative of the local bivectors’ magnitude - a fixed value regardless of direction. Holman’s analysis [10] is therefore proven correct: “Because $\mu$ is a local deterministic hidden variable, its value cannot depend on the choice made by the experimenter. If this value is left unchanged by the first measurement, performing the second measurement in the $e_x$-direction would result in “spin up” in this direction with certainty, in contradiction with the usual quantum predictions.”. Flipping of the signs on Alice’s and Bob’s experimental outcomes is also without merit.

**Conclusion**

Any of the four mathematical mistakes presented above can reject Joy Christian’s claims of a disproval of Bell’s theorem by counterexample. Another error is that computing Eq. 16 of Ref. [5] yields Eq. 3 and not Eq. 15 as claimed: “we believe the experiment will vindicate prediction (15) and refute prediction (3).”. This was proven both analytically and by computer simulation. Since this impacts only a particular claim of a paper and not the viability of the whole research program it was not presented here. (A similar computer simulation was carried out earlier by Stephen Lee [14], but the source code was not made publicly available.) After those mistakes were found the systematic checking of Joy Christian’s mathematical claims was stopped.

Joy Christian’s model is not correct, but can Bell’s theorem be invalidated by another non-commutative “beables” theory? Two theorems by Clifton [15] answer this in the negative for non-contextual hidden variable theories and for relativistic quantum field theories with bounded energy. During computer simulations, several other non-commutative models which correctly realize the minus cosine correlations were discovered. However, the simulation also showed that any transition from non-commutative beables to discrete experimental outcomes destroys this correlation and yields the classical correlations as expected from Bell’s theorem. It is therefore essential for a hidden variable model to predict both quantum correlations and the experimental outcomes. Any commutative beable hidden variable theory is ruled out by Bell’s theorem. Any non-commutative non-contextual beable hidden variable theory is ruled out by Clifton’s analysis. The only remaining way out is to construct a non-commutative contextual beable hidden variable theory. But contextual hidden variable theories are not really considered physical theories as no experimental evidence ever backed them out. It is debatable if Bell’s theorem is important to rule out some contextual hidden variable theories as well, or only non-contextual ones. Bell’s theorem is not the only result ruling out non-contextual hidden variable theories, but only Bell’s result is robust enough (because involves an inequality) to be put to an experimental test.

**Acknowledgement**

I want to thank Cristi Stoica for his help in computer modeling of Joy Christian’s proposed experiment in Ref. [2] and for useful discussions.

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