A splitting of the local rigidity of Clifford-Klein forms of homogeneous spaces of completely solvable Lie groups

Yoshinori Tanimura

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Abstract

In this article, we discuss the local rigidity of Clifford-Klein forms of homogeneous spaces of 1-connected completely solvable Lie groups. In fact, we introduce a splitting of the local rigidity: vertical rigidity and horizontal rigidity. By using this splitting, we refine some existing results about the local rigidity and introduce a new approach to Baklouti’s conjecture about the local rigidity.

1 Introduction

1.1 Abstract

For a Lie group $G$, a closed subgroup $H \subset G$ and a discrete subgroup $\Gamma \subset G$, if the natural action $\Gamma \curvearrowright G/H$ is properly discontinuous and free, then the double coset space $\Gamma \backslash G/H$ is called a Clifford-Klein form ([Kob89]). We want to know how the structure of Clifford-Klein form changes if $\Gamma$ is perturbed in $G$. By formulating the classification space of Clifford-Klein forms, the problem is boiled down to the topological structure of $D(\Gamma, G, H)$ and $M(\Gamma, G, H)$ (Definition 2.1).

In this article, we discuss local rigidity of Clifford-Klein forms (Definition 2.2) in the case of $G$ being nilpotent or completely solvable. In the reductive case, it is well-known that there are some cases of structures of Clifford-Klein forms that satisfy local rigidity (The Weil-Selberg-Kobayashi local rigidity theorem) or stronger rigidity (Mostow’s rigidity theorem). On the other hand, in the solvable case, structures of Clifford-Klein forms scarcely satisfy local rigidity. In regards to this, the following conjecture is unsolved.

Conjecture 1.1 ([Bak11, Conjecture 3.1])
Let $G$ be a 1-connected nilpotent Lie group, $H \subset G$ a closed subgroup and $\Gamma \subset G$ a non-trivial discrete subgroup. Then no elements of $\mathcal{R}(\Gamma, G, H)$ are locally rigid.

It is known that this conjecture is true in a special case.

Fact 1.2 ([BK13, Theorem 3.5])
Let $G$ be a 1-connected nilpotent Lie group, $H \subset G$ a closed subgroup and $\Gamma \subset G$ a non-trivial discrete subgroup such that its syndetic hull is not characteristically nilpotent. Then no elements of $\mathcal{R}(\Gamma, G, H)$ are locally rigid.
For example, if a 1-connected nilpotent Lie group $L$ is of dimension more than 1 and 2-step nilpotent, abelian or of dimension lower than 7, then $L$ is not characteristically nilpotent.

In this article, we will refine Fact 1.2 and introduce a new approach to Conjecture 1.1 by splitting local rigidity to vertical rigidity and horizontal rigidity (Definition 3.4). In fact, Fact 1.2 will be made the following theorem.

**Theorem 1.3**

Let $G$ be a 1-connected nilpotent Lie group, $H \subset G$ a closed subgroup and $\Gamma \subset G$ a non-trivial discrete subgroup such that its syndetic hull is not characteristically nilpotent. Then no elements of $\mathcal{R}(\Gamma, G, H)$ are vertically rigid.

### 1.2 Background

Some Clifford-Klein forms describe some geometrically important spaces. So there are many researches being conducted about the classification of Clifford-Klein forms. For example, the Teichmüller theory, researched by Teichmüller, Ahlfors, S. Kobayashi and so on, is identified with the classification of Clifford-Klein forms which describe closed Riemann surfaces. For the classification of Clifford-Klein forms which describe closed hyperbolic manifolds whose dimension is no less than 3, Mostow’s rigidity theorem is well-known. And for the classification of Clifford-Klein forms which describe a certain kind of 3 dimensional Lorentzian manifold, T. Kobayashi solved the Goldman conjecture.

For the classification of Clifford-Klein forms, the main interest is the cases of $G$ being simple, semi-simple or reductive. This is because the cases of $G$ being simple, semi-simple or reductive are often geometrically important. But if we recognize that the classification of Clifford-Klein forms are the abstract problem for Lie groups, there is room to discuss the case of $G$ not being reductive, especially of it being abelian, nilpotent or solvable. From such reasons, some research works have recently been focused on the cases of $G$ being abelian, nilpotent or solvable.

### 2 Preliminaries

Let $G$ be a Lie group, $H \subset G$ a closed subgroup and $\Gamma \subset G$ a discrete subgroup. We define the classification spaces of properly discontinuous and free actions of $\Gamma \curvearrowright G/H$.

**Definition 2.1**

Let $\text{Hom}(\Gamma, G)$ be the space of all homomorphisms of Lie groups with the pointwise-convergence topology and define the two actions $G \curvearrowright \text{Hom}(\Gamma, G) \curvearrowright \text{Aut}(\Gamma)$ by the adjoint action $G \curvearrowright G$ and the natural action $\text{Aut}(\Gamma) \curvearrowright \Gamma$. We define these four spaces.

- **The parameter space** ([Kob93, Remark 3])
  \[
  \mathcal{R}(\Gamma, G, H) := \left\{ \varphi \in \text{Hom}(\Gamma, G) \mid \varphi : \Gamma \hookrightarrow G \text{ is an embedding, and} \right. \\
  \left. \varphi(\Gamma) \curvearrowright G/H \text{ is properly discontinuous and free.} \right\}
  \]
  
- **The Chabauty space**
  \[
  \mathcal{C}(\Gamma, G, H) := \mathcal{R}(\Gamma, G, H)/\text{Aut}(\Gamma).
  \]

- **The deformation space** ([Kob01, Definition 5.3.1])
  \[
  \mathcal{D}(\Gamma, G, H) := G \backslash \mathcal{R}(\Gamma, G, H).
  \]
• The moduli space ([Kob01, Definition 5.3.2])
\[ \mathcal{M}(\Gamma, G, H) := G \backslash \mathcal{R}(\Gamma, G, H) / \text{Aut}(\Gamma). \]

These spaces can be visually summarized as follows.

\[ \mathcal{R}(\Gamma, G, H) \xrightarrow{G\backslash} \mathcal{D}(\Gamma, G, H) \]
\[ \text{/Aut}(\Gamma) \quad \bigcap \bigcap \quad /\text{Aut}(\Gamma) \]
\[ \mathcal{C}(\Gamma, G, H) \xrightarrow{G\backslash} \mathcal{M}(\Gamma, G, H) \]

\( \mathcal{M}(\Gamma, G, H) \) is the space of Clifford-Klein forms with a natural topology. So for \( \varphi \in \mathcal{R}(\Gamma, G, H) \), that \( [\varphi]_\mathcal{M} \in \mathcal{M}(\Gamma, G, H) \) is isolated means that the structure of Clifford-Klein form defined by \( \varphi \) is invariant by perturbation of \( \varphi \). Moreover, since Aut(\( \Gamma \)) is discrete, we can guess the local structure of \( \mathcal{D}(\Gamma, G, H) \) is similar to that of \( \mathcal{M}(\Gamma, G, H) \). So we define local rigidity as the following.

**Definition 2.2 ([Kob93, Remark 3])**
An element \( \varphi \in \mathcal{R}(\Gamma, G, H) \) is said to be **locally rigid** if \( [\varphi]_\mathcal{D} \in \mathcal{D}(\Gamma, G, H) \) is isolated.

**Remark 2.3**
Later, we will discuss the case that \( \Gamma \) is a connected Lie group. In this case, the topology of Hom(\( \Gamma, G \)) will be the compact-open topology.

### 3 A splitting of local rigidity and the going-through map

#### 3.1 1-connected completely solvable Lie groups

Let \( G \) be a 1-connected completely solvable Lie group, \( H \) a closed subgroup of \( G \) and \( \Gamma \) a discrete subgroup of \( G \). As long as the main interest is in the deformation space, we can replace \( \Gamma \) (resp. \( H \)) with a closed connected subgroup \( L \) (resp. \( H \)) of \( G \) by considering the syndetic hull. Since \( L \) is a 1-connected Lie group, \( \mathcal{R}(L, G, H) \subset \text{Hom}(L, G) \cong \text{Hom}(l, g), \text{Aut}(L) \cong \text{Aut}(l) \).

**Remark 3.1**
The idea of using syndetic hulls appeared in [KN06] for the first time. They discussed the case that \( G \) is a specific nilpotent Lie group. A. Baklouti and I. Kédim discussed the general 1-connected nilpotent case in [BK09] and the 1-connected completely solvable case in [BK10].

In the following, we denote \( \mathcal{R}(L, G, H) \) by \( \mathcal{R} \) and respectively \( \mathcal{D}, \mathcal{C} \) and \( \mathcal{M} \) in the same way. The \( G \)-equivalent principal bundle structure in the following proposition is essential.

**Proposition 3.2**
The quotient map \( \mathcal{R} \to \mathcal{C} \) is a principal Aut(\( l \))-bundle.

**Proof.** We denote the set of all injective linear maps from \( l \) to \( g \) by Lin\(^c\)(\( l, g \)). Then, since \( \mathcal{R} \subset \text{Hom}(l, g) \subset \text{Lin}^c(l, g) \) and Aut(\( l \)) \( \subset GL(l) \), \( \mathcal{R} \to \mathcal{C} \) is a restriction of the frame bundle Lin\(^c\)(\( l, g \)) \( \to \text{Lin}^c(l, g) / \text{Aut}(l) \).

Later, we will introduce splitting of local rigidity by using this bundle structure.
3.2 On principal bundles

In this subsection, we discuss in the following condition for simplification of discussion.

Setting 3.3

Let $X$ be a topological space, $G$ and $H$ topological groups and $X$ has a continuous left $G$-action $G \curvearrowright X$ and a continuous right $H$-action $X \curvearrowleft H$. We assume the actions $G \curvearrowright X$ and $X \curvearrowleft H$ are commutative and the quotient map $X \to X/H$ a principal $H$-bundle. And let $G\pi, \pi_H, G\varpi$ and $\varpi_H$ in the following diagram be the quotient maps.

\[
\begin{array}{c}
X \\
\pi_H \\
X/H \\
\end{array} \xrightarrow{G\pi} \xrightarrow{G\varpi} G\backslash X \xrightarrow{\varpi_H} X/H
\]

3.2.1 Splitting of local rigidity

At first, we will split the local rigidity into two conditions.

Definition 3.4

Assume Setting 3.3 and take $x \in X$.

- $x$ is **locally rigid** if $Gx \in G\backslash X$ is isolated.
- $x$ is **vertically rigid** if the $H$-orbit of $Gx \in G\backslash X$ is discrete.
- $x$ is **horizontally rigid** if $GxH \in G\backslash X/H$ is isolated.

Remark 3.5

Assume Setting 3.3. For $x \in X$, the followings are equivalent.

1. $x$ is locally rigid.
2. $x$ is vertically and horizontally rigid.

3.2.2 The going-through map

Next we introduce the going-through map. The going-through map is useful to discuss vertical rigidity. More precisely, we can describe a necessary condition of vertical rigidity by using the going-through map.

Notation 3.6

Assume Setting 3.3. Take a point $x \in X$.

- $G_xH$ (resp. $G_xH$) denotes the isotropy group of the action $G \curvearrowright X/H$ (resp. $G\backslash X \curvearrowleft H$) at $xH$ (resp. $Gx$).
- We define two continuous maps $G\theta_x: G \to X$ and $\theta_{x,H}: H \to X$ by the followings:
  \[
  G\theta_x(g) := gx \ (g \in G), \ \theta_{x,H}(h) := xh \ (h \in H)
  \]

Since the quotient map $H \to X/H$ is a principal $H$-bundle, $\theta_{x,H}: H \to x \cdot H$ is a homeomorphism.
Let $\bar{\theta}_{x, H} : G_x H \setminus H \to G \setminus X$ be the map induced by $G \pi \circ \theta_{x, H} : H \to G \setminus X$. Then the map $\bar{\theta}_{x, H} : G_x H \setminus H \to (Gx) \cdot H$ is bijective and continuous (however not necessarily homeomorphism (See Subsection 3.4)).

**Lemma 3.7**
For $x \in X$, there exists a unique surjective continuous homomorphism $\alpha_x : G_x H \to G_x H$ such that $gx = x\alpha_x(g)$ ($g \in G_x H$).

**Proof.** Since the action $X \acts H$ is free, $\alpha_x$ is unique. And the composition of $G\theta_x |_{G_x H} : G_x H \to x \cdot H$ and $\theta_x^{-1} : x \cdot H \to H$ satisfies the condition of $\alpha_x$.

**Definition 3.8**
We call the map $\alpha_x$ defined in Lemma 3.7 the going-through map of $x$.

In the case of the deformation space, the going-through map is a same map defined in [BK13, Equation 3.2]. By using the going-through map, we obtain the following observation.

We denote the identity component of $H$ by $H_0$.

**Proposition 3.9**
Assume Setting 3.3. If there exists a locally rigid point $x \in X$, then $H_0$ is a subquotient group of $G$.

**Proof.** Since $\alpha_x : G_x H \to G_x H$ is surjective, it is enough to show $H_0 \subset G_x H$. Since the natural map $G_x H \setminus H \to (Gx) \cdot H \subset G \setminus X$ is bijective continuous and $(Gx) \cdot H \subset G \setminus X$ is discrete, $G_x H \setminus H$ is also discrete.

### 3.3 On the deformation spaces

Now we will make use of the notation of Subsection 3.3. By Lemma 3.2, $(G, R, \text{Aut}(l))$ satisfies the condition of $(G, X, H)$ in Setting 3.3. So we can define local rigidity, horizontal rigidity, vertical rigidity and the going-through maps for elements of $R$. And the local rigidity defined here is equivalent to the local rigidity defined in Definition 2.2. By Proposition 3.9, we get the following.

**Theorem 3.10**
Let $G$ be a completely solvable (resp. nilpotent) Lie group, $H \subset G$ a closed subgroup, $\Gamma \subset G$ a discrete subgroup and $L$ a syndetic hull of $\Gamma$. Then if the derivation algebra of $l$ is NOT completely solvable (resp. nilpotent), then no elements of $R(\Gamma, G, H)$ are vertically rigid.

A Lie algebra is called **characteristically nilpotent** if the Lie algebra of its derivations is nilpotent. So Theorem 3.10 is a precise description of Theorem 1.3. The details of characteristically nilpotent Lie algebras are written in [AC01].

**Remark 3.11**
For $\varphi \in R$, the isotropy group $G_{\varphi \text{Aut}(l)} \subset G$ at $\varphi \text{Aut}(l) \in C$ is the normalizer of the Lie subalgebra $\varphi(l) \subset \mathfrak{g}$. In particular, the condition $G_{\varphi \text{Aut}(l)} = G$ is equivalent to that $\varphi(l) \subset \mathfrak{g}$ is an ideal.
3.4 A sufficient condition that $\tilde{\theta}_{x, H}$ is homeomorphism

In Setting 3.3 the map of $\tilde{\theta}_{x, H}$ in Notation 3.6 is NOT homeomorphic in generally. We want a sufficient condition that $\tilde{\theta}_{x, H}$ is homeomorphic.

**Proposition 3.12**

Assume Setting 3.3. Let $A \subset G \backslash X/H$ be a subset, $P := G\varpi^{-1}(A) \subset X/H$, $s: P \to X$ a section of $\pi_H: X \to X/H$ and $H' \subset H$ a closed subgroup. Assume the following conditions.

1. For all $p \in P$ and $g \in G$, there exists $g_0 \in G$ such that $s(gp) = g_0s(p)$.
2. For all $p \in P$, $G_{s(p)}H = H'$.

Then there exists the unique homeomorphism $\Phi_\alpha: A \times (H'\backslash H) \xrightarrow{\cong} \varpi_H^{-1}(A)$ such that $\Phi_\alpha(G\varpi(p), H'h) = \tilde{\theta}_{s(p), H}(H'h)$ ($p \in P$, $h \in H'$).

**Remark 3.13**

If $P$ consists one point, the conditions (1) and (2) are true.

**Proof of Proposition 3.12** Assume the following claims for all element $(p, h) \in P \times H$.

1. For all $g \in G$, there exists $h' \in H'$ such that $gs(p)h = s(gp)h'h$.
2. For all $g \in G, h' \in H'$, there exists $g' \in G$ such that $g's(p)h = s(gp)h'h$.

Then the following equivalent relations are the same.

- The relation defined by two actions $G \equiv P$ and $H' \equiv H$.
- The relation defined by the action $G \equiv P \times H$ induced by the identification $X|_P \cong P \times H$ with respect to $s$.

The proposition follows from this observation immediately.

We will prove above two claims. By the assumption (1), there exists a map $\beta: G \cdot p \to G$ such that $s(gp) = \beta(gp)s(p)$ ($g \in G$).

At first, we prove the claim 1. Let $g \in G$. Since $g\beta(gp)^{-1} \in G_{gp} = G_{s(gp)}H$,

$$gs(p)h = g\beta(gp)^{-1}s(gp)h = s(gp) \cdot \alpha_{s(gp)}(g\beta(gp)^{-1})h.$$ 

Since $\alpha_{s(gp)}(g\beta(gp)^{-1}) \in G_{s(gp)}H = H'$, the claim 1 holds.

Next we prove claim 2. Let $g \in G$ and $h' \in H$. By the surjectivity of the going-through map $\alpha_{s(p)}: G_{s(p)}H \to G_{s(p)}H = H'$, there exists $g_0 \in G_{s(p)}H$ such that $h' = \alpha_{s(p)}(g_0)$. So

$$s(gp)h' = \beta(gp)s(p)\alpha_{s(p)}(g_0)h = \beta(gp)g_0s(p)h.$$ 

This implies the claim 2.

3.5 Examples of vertical rigidity and horizontal rigidity

Ali Baklouti et al. conjectured there are no locally rigid Clifford-Klein forms in the nilpotent case (Conjecture 3.4). However, there are some vertically rigid ones and horizontally rigid ones.
3.5.1 An example of vertical rigidity

At first, we construct a vertically rigid Clifford-Klein forms.

Proposition 3.14

Let \( l \) be a characteristically nilpotent Lie algebra of dimension more than 1, \( g := \text{Der}(l) \ltimes l, L \) (resp. \( G \)) the 1-connected Lie group associated by \( l \) (resp. \( g \)) and \( H := \{ e \} \). Then the natural inclusion \( \varphi : l \hookrightarrow g \) is vertically rigid in \( R \).

To prove this proposition, we use the following fact about characteristic nilpotency.

Fact 3.15 ([LT59, Theorem 1])

For a Lie algebra \( l \) of dimension more than 1, the followings are equivalent.

1. \( \text{Der}(l) \) is nilpotent (i.e. \( l \) is characteristically nilpotent).
2. \( \text{Der}(l) \) consists nilpotent elements.
3. \( \text{Der}(l) \ltimes l \) is nilpotent.

In particular, if \( l \) is characteristically nilpotent, \( \text{Aut}(l)_0 \) is 1-connected.

Proof of Proposition 3.14

Since \( \varphi(l) \subset g \) is an ideal, \([\varphi]_C \in C \) is a fixed point of the action \( G \ltimes C \). So we obtain a homeomorphism \( (\text{Im}(\varphi)) \setminus \text{Aut}(l) \cong [\varphi]_D \cdot \text{Aut}(l) \) by Proposition 3.12.

Under the identification \( \text{Aut}(l) \cong \text{Aut}(L) \), since \( \alpha_\varphi(T, 0) = T \) for all \( (T, 0) \in G = \text{Aut}(l)_0 \ltimes L \), \( \text{Im}(\varphi) = \text{Aut}(l)_0 \). So \([\varphi]_D \cdot \text{Aut}(l) \cong \text{Aut}(l)_0 \setminus \text{Aut}(l) \). It means \( \varphi \in R \) is vertically rigid.

3.5.2 Examples of horizontal rigidity

Next we construct a horizontally rigid Clifford-Klein form. In the following, we denote \( R(L, G, \{ e \}) \) by \( R(L, G) \).

Proposition 3.16

Let \( G \) be a 1-connected completely solvable Lie group. Then \( R(G, G) = \text{Aut}(g), D(G, G) = \text{Out}(g) \) and each \( C(G, G) \) and \( M(G, G) \) consists one point. In particular, all elements of \( R(G, G) \) are horizontally rigid.

Since the proof of this proposition is easy, we omit it. In this example, we consider compact Clifford-Klein forms. On the other hand, there is an example of horizontal rigidity of non-compact Clifford-Klein forms. Let \( n \) be a positive integer, define a linear map \( \sigma_n : \mathbb{R}^n \rightarrow \mathbb{R}^n \) by \( \sigma_n(x_0, \ldots, x_{n-1}) = (0, x_0, \ldots, x_{n-2}) \) and let \( l_n := \mathbb{R}\sigma_n \ltimes \mathbb{R}^n \). Then \( l_n \) is called the \( n + 1 \) dimensional ladder Lie algebra. Let \( L_n \) be the 1-connected Lie group associated to \( l_n \). Then the following proposition can be shown immediately.

Proposition 3.17

If \( n > 3 \), \((0, \mathbb{R}^n) \subset l_n \) is only the \( n \)-dimensional abelian Lie subalgebra. In particular, \( C(\mathbb{R}^n, L_n) \) consists one point and all elements of \( R(\mathbb{R}^n, L_n) \) are horizontally rigid.
References

[AC01] José M. Ancochea and Rutwig Campoamor. Characteristically nilpotent Lie algebras: a survey. *Extracta Math.*, 16(2):153–210, 2001.

[Bak11] Ali Baklouti. On discontinuous subgroups acting on solvable homogeneous spaces. *Proc. Japan Acad. Ser. A Math. Sci.*, 87(9):173–177, 2011.

[BK09] Ali Baklouti and Imed Kédim. On the deformation space of Clifford-Klein forms of some exponential homogeneous spaces. *Internat. J. Math.*, 20(7):817–839, 2009.

[BK10] Ali Baklouti and Imed Kédim. On non-abelian discontinuous subgroups acting on exponential solvable homogeneous spaces. *Int. Math. Res. Not. IMRN*, (7):1315–1345, 2010.

[BK13] Ali Baklouti and Imed Kedim. On the local rigidity of discontinuous groups for exponential solvable Lie groups. *Adv. Pure Appl. Math.*, 4(1):3–20, 2013.

[KN06] Toshiyuki Kobayashi and Salma Nasrin. Deformation of properly discontinuous actions of $\mathbb{Z}^k$ on $\mathbb{R}^{k+1}$. *Internat. J. Math.*, 17(10):1175–1193, 2006.

[Kob89] Toshiyuki Kobayashi. Proper action on a homogeneous space of reductive type. *Math. Ann.*, 285(2):249–263, 1989.

[Kob93] Toshiyuki Kobayashi. On discontinuous groups acting on homogeneous spaces with noncompact isotropy subgroups. *J. Geom. Phys.*, 12(2):133–144, 1993.

[Kob01] Toshiyuki Kobayashi. Discontinuous groups for non-Riemannian homogeneous spaces. In *Mathematics unlimited—2001 and beyond*, pages 723–747. Springer, Berlin, 2001.

[LT59] G. Leger and S. Tōgō. Characteristically nilpotent Lie algebras. *Duke Math. J.*, 26:623–628, 1959.

Graduate School of Mathematical Sciences, The University of Tokyo
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