EXPONENTIAL SUMS AND RIGID COHOMOLOGY

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Abstract. In this article, we prove a comparison theorem between the Dwork cohomology introduced by Adolphson and Sperber and the rigid cohomology. As a corollary, we can calculate the rigid cohomology of Dwork isocrystal on torus.

1. Introduction

Let $p$ be a prime number, $\mathbb{Q}_p$ the field of $p$-adic numbers, $k = \mathbb{F}_q$ the finite field with $q = p^a$ elements. Let $k_i$ be the extension of $k$ of degree $i$ and let $\bar{k}$ be the algebraic closure of $k$. Fix a primitive $p$-th root of unity $\zeta_p$. Let $K_0$ be the unramified extension of $\mathbb{Q}_p(\zeta_p)$ of degree $a$. Let $\Omega$ be the completion of an algebraic closure of $K_0$. Denote by “ord” the additive valuation on $\Omega$ normalized by $\text{ord}(p) = 1$. The norm on $\Omega$ is given by $|u| = p^{-\text{ord}(u)}$ for any $u \in \Omega$.

For a morphism $f : X \to \mathbb{A}_k^n$ with $X$ being a $k$-scheme of finite type of dimension $n$, and the nontrivial additive character $\psi : k \to K_0^\times$ defined by $\psi(t) = \zeta_p^{\text{Tr}_{k_0/\mathbb{F}_p}(t)}$, define exponential sums

$$S_i(X, f) = \sum_{x \in X(k_i)} \psi(\text{Tr}_{k_i/k}(f(x))).$$

The $L$-function is defined by

$$L(X, f, t) = \exp\left(\sum_{i=1}^{\infty} S_i(X, f) t^i / i\right).$$

According to [9, Theorem 6.3], we have

$$L(X, f, t) = \prod_{i=0}^{2n} \det(I - t F^* | H^1_{c, \text{rig}}(X/K_0, f^* \mathcal{L}_\psi))^{(-1)^i+1},$$

where $\mathcal{L}_\psi$ is the Dwork $F$-isocrystal defined over $\mathbb{A}_k^n$ associated to $\psi$ and $F^*$ is the Frobenius endomorphism on the space $H^1_{c, \text{rig}}(X/K_0, f^* \mathcal{L}_\psi)$.

In [2, section 4], Baldassarri and Berthelot compare the Dwork cohomology and the rigid cohomology for singular hypersurfaces. We prove a similar comparison theorem for the complex introduced by Adolphson and Sperber in [1, section 2] to study the exponential sums on the torus $\mathbb{T}_k^n$. Suppose that $f$ is defined by a Laurent polynomial

$$f(x_1, \cdots, x_n) = \sum_{j=1}^{N} a_j x^{w_j} \in k[x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}].$$

Let $\Delta(f)$ be the Newton polyhedron at $\infty$ of $f$ which is defined to be the convex hull in $\mathbb{R}^n$ of the set $\{w_j\}_{j=1}^{N} \cup \{(0, \cdots, 0)\}$ and let $\delta$ be the convex cone generated by $\{w_j\}_{j=1}^{N}$ in $\mathbb{R}^n$. Let $\text{Vol}(\Delta(f))$ be the volume of $\Delta(f)$ with respect to Lebesgue measure on $\mathbb{R}^n$. We say $f$ is nondegenerate with

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respect to $\Delta(f)$ if for any face $\sigma$ of $\Delta(f)$ not containing the origin, the Laurent polynomials $\frac{\partial f}{\partial x_i}$, $i = 1, \cdots, n$ have no common zero in $\bar{k}^n$, where $f_\sigma = \sum_{w_j \in \sigma} a_j x^{w_j}$. Define a weight function on $\delta \cap \mathbb{Z}^n$ by

$$w(u) := \inf \{ c : u \in c\Delta(f), \ c \geq 0 \}.$$

Note that there exists some $M \in \mathbb{Z}_{>0}$ such that $w(\delta \cap \mathbb{Z}^n) \subset \frac{1}{M}\mathbb{Z}_{\geq 0}$.

Consider the Artin-Hasse exponential series:

$$E(t) = \exp \left( \sum_{i=0}^{\infty} \frac{tp^i}{p^i} \right).$$

By [8, Lemma 4.1], the series $\sum_{i=0}^{\infty} \frac{tp^i}{p^i}$ has a zero at $\gamma \in \mathbb{Q}_p(\zeta_p)$ such that $\text{ord} \gamma = 1/(p-1)$ and $\zeta_p \equiv 1 + \gamma \mod \gamma^2$. Set

$$\theta(t) = E(\gamma t) = \sum_{i \geq 0} \lambda_i t^i.
$$

The series $\theta(t)$ is a splitting function in Dwork’s terminology [8, §4a]. In particular, we have $\text{ord} \lambda_i \geq \frac{i}{p-1}$ and $\theta(1) = \zeta_p$.

Fix an $M$-th root $\tilde{\gamma}$ of $\gamma$ in $\bar{\Omega}$. Let $K = K_0(\tilde{\gamma})$, and $\mathcal{O}_K$ the ring of integers of $K$. Let $\tilde{a}_j \in K$ be the Techm"uller lifting of $a_j$ and set

$$\tilde{f}(x) = \sum_{j=1}^{N} \tilde{a}_j x^{w_j} \in K[x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}].$$

For any $b > 0, c \in \mathbb{R}$, consider the following spaces :

$$L(b, c) = \left\{ \sum_{u \in \delta \cap \mathbb{Z}^n} a_u x^u : a_u \in K, \ \text{ord}(a_u) \geq bw(u) + c \right\},$$

$$L(b) = \bigcup_{c \in \mathbb{R}} L(b, c), \quad L_0 = \bigcup_{b > 0} L(b).$$

Set $\gamma_l = \sum_{i=0}^{l} \gamma p^i/p^i, h(t) = \sum_{l=0}^{\infty} \gamma_l t^p$. Then we have

$$\theta(t) = \exp \left( \sum_{l=0}^{+\infty} \frac{\gamma p^i t^l}{p^l} \right) = \exp \left( \gamma t + \sum_{l=1}^{+\infty} (\gamma_l - \gamma_{l-1}) t^p \right) = \exp(h(t) - h(t^p)).$$

Define

$$H(x) = \sum_{j=1}^{N} h(\tilde{a}_j x^{w_j}), \quad F_0(x) = \exp(H(x) - H(x^q)).$$

The estimate $\text{ord}(\lambda_i) \geq \frac{i}{p-1}$ implies that $H(x)$ and $F_0(x)$ are well defined as formal Laurent series. In fact, we have

$$H(x) \in L\left(\frac{1}{p-1}, 0\right), \quad F_0(x) \in L\left(\frac{p}{q(p-1)}, 0\right).$$

Define an operator $\psi_q$ on formal Laurent series by

$$\psi_q \left( \sum_{u \in \mathbb{Z}^n} a_u x^u \right) = \sum_{u \in \mathbb{Z}^n} a_{qu} x^u.$$
Let $\alpha = \psi_q \circ F_0$. Formally, we have

$$
\alpha = \psi_q \circ \exp(H(x) - H(x^q)) = \exp(-H(x)) \circ \psi_q \circ \exp(H(x)).
$$

For $i = 1, \ldots, n$, define operators

$$
E_i = x_i \partial / \partial x_i, \ H_i = E_i H, \ \hat{D}_i = E_i + H_i = E_i + E_i H.
$$

Formally, we have

$$
\hat{D}_i = \exp(-H(x)) \circ E_i \circ \exp(H(x))
$$

for $i = 1, \ldots, n$. Note that $\alpha$ and $\hat{D}_i$ operate on $L^1_{\delta}$. One can show that $\hat{D}_i$ commute with one another. Denote by $K(L^1_{\delta}, \hat{D})$ the Koszul complex on $L^1_{\delta}$ associated to $\hat{D}_1, \ldots, \hat{D}_n$. We have $\alpha \circ \hat{D}_i = q \hat{D}_i \circ \alpha$ for all $i$. This implies that $\alpha$ induces a chain map on $K(L^1_{\delta}, \hat{D})$. Now we can give the main theorem in the present paper.

**Theorem 1.1.** We have an isomorphism

$$
H_{n-i}(K(L^1_{\delta}, \hat{D})) \xrightarrow{\sim} H^{n}_{\text{rig}}(T^\alpha_{K}/K, f^* L_{-\psi})
$$

for each $i$. The endomorphism $\alpha$ on the left corresponds to the endomorphism $F_\ast = q^n(F^\ast)^{-1}$ on the rigid cohomology. Furthermore, $\alpha$ is bijective on each homology group of $K(L^1_{\delta}, \hat{D})$.

As a corollary, we prove the following theorem, which eliminates the condition $p \neq 2$ in [6].

**Theorem 1.2.** Suppose that $f : T^\alpha_{K} \to A^1_{K}$ is nondegenerate with respect to $\Delta(f)$ and that $\dim \Delta(f) = n$. Then

(i) $H^i_{c, \text{rig}}(T^\alpha_{K}, f^* L_{\psi}) = H^i_{\text{rig}}(T^\alpha_{K}, f^* L_{-\psi}) = 0$ if $i \neq n$.

(ii) $\dim H^n_{c, \text{rig}}(T^\alpha_{K}, f^* L_{\psi}) = \dim H^n_{\text{rig}}(T^\alpha_{K}, f^* L_{-\psi}) = n! \text{Vol}(\Delta(f))$.

2. **Comparison Theorem**

By [8, Lemma 4.1], there exists an element $\pi \in \mathbb{Q}_p(\zeta_p)$ such that $\pi^{p-1} + p = 0$ and

$$
\zeta_p \equiv 1 + \pi \mod \pi^2.
$$

Hence $\text{ord}(\pi - \gamma) \geq \frac{2}{p}$ as $\zeta_p - 1 \equiv \pi \equiv \gamma \mod (\zeta_p - 1)^2$. By [8, Lemma 4.1] and the definition of the weight function, we have

$$
G(x) := \exp(\pi (f(x) - f(x^q))) \in L\left(\frac{p-1}{pq}, 0\right).
$$

For $i = 1, \ldots, n$, define operators

$$
D_i = \exp(-\pi f(x)) \circ E_i \circ \exp(\pi f(x)).
$$

We have the Koszul complex $K(L^1_{\delta}, \hat{D})$ and its endomorphism induced by $\alpha_1 = \exp(-\pi \hat{f}(x)) \circ \psi_q \circ \exp(\pi \hat{f}(x))$. Let $R = \exp(H(x) - \pi \hat{f}(x))$. We have

$$
\alpha_1 = R \circ \alpha \circ R^{-1}, \ D_i = R \circ \hat{D}_i \circ R^{-1}
$$

for all $i$.

**Proposition 2.1.** (i) $R, R^{-1} \in L(b_0)$ where $b_0 = \min\left\{\frac{1}{p-1}, \frac{p-1}{p}\right\}$.

(ii) The multiplication by $R$ defines an isomorphism of Koszul complexes $\beta : K(L^1_{\delta}, \hat{D}) \xrightarrow{R} K(L^1_{\delta}, \hat{D})$ and $\alpha_1 \circ \beta = \beta \circ \alpha$. 

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Proof. (ii) follows from (i) and (2.1). Let \( h_1(t) = \sum_{l=1}^{+\infty} \gamma_l t^l \). Then
\[
R = \exp((\gamma - \pi)\hat{f}(x)) \prod_{j=1}^N \exp(h_1(\hat{a}_j x^w)) \cdot L\left(\frac{1}{p-1}, 0\right).
\]
The estimate \( \text{ord}(\gamma - \pi) \geq \frac{2}{p-1} \) implies that
\[
\exp((\gamma - \pi)\hat{f}(x)) = \prod_{j=1}^N \exp((\gamma - \pi)\hat{a}_j x^w) \in L\left(\frac{1}{p-1}, 0\right).
\]
Note that \( \exp(\gamma t^l) = \sum_{k=0}^{+\infty} \frac{\gamma t^l}{x^k} k^p t^l \) and
\[
\text{ord}(\gamma_l) = \text{ord}\left(- \sum_{i=l+1}^{+\infty} \gamma_i^l / p^i \right) \geq \frac{p^l+1}{p-1} - l - 1.
\]
Then
\[
\text{ord}\left(\frac{\gamma_k^l}{k^l}\right) \geq k\left(\frac{p^l+1}{p-1} - l - 1 - \frac{1}{p-1}\right) \geq \frac{p-1}{p} \cdot kp^l
\]
for each \( l \geq 1 \). As the term \( c_i t^{i} \) in the expansion for \( \exp(h_1(t)) = \prod_{l=1}^{+\infty} \exp(\gamma_l t^l) \) only depends on finitely many \( \exp(\gamma_l t^l) \), we have
\[
\text{ord}(c_i) \geq \frac{p-1}{p}, \quad \text{and } \prod_{j=1}^N \exp(h_1(\hat{a}_j x^w)) \in L\left(\frac{p-1}{p}, 0\right).
\]
Hence \( R \in L\left(\frac{1}{p-1}, \frac{p-1}{p}\right) \subset L(b_0) \). The same is true for \( R^{-1} \). \( \square \)

Consider the twisted de Rham complex \( C^\ast(L_{\delta}^\dag, D) \) defined by
\[
C^k(L_{\delta}^\dag, D) = \bigoplus_{1 \leq i_1 < \cdots < i_k \leq n} L_{\delta}^\dag \frac{dx_{i_1}}{x_{i_1}} \wedge \cdots \wedge \frac{dx_{i_k}}{x_{i_k}} \cong (L_{\delta}^\dag)^{\ast}(i)
\]
with the differential \( d : C^k(L_{\delta}^\dag, D) \to C^{k+1}(L_{\delta}^\dag, D) \) given by
\[
d(\omega) = \exp(-\pi \hat{f}) \circ d_\omega \circ \exp(\pi \hat{f})(\omega)
\]
for any \( \omega \in C^k(L_{\delta}^\dag, D) \). Define an endomorphism \( \alpha_1^{(k)} \) of \( C^k(L_{\delta}^\dag, D) \) by
\[
\alpha_1^{(k)} = \bigoplus_{1 \leq i_1 < \cdots < i_k \leq n} q^{n-k} \alpha_1.
\]
The equalities \( \alpha_1 \circ D_i = q D_i \circ \alpha_1 \) imply that \( \alpha_1 \) induces a chain map on \( C^\ast(L_{\delta}^\dag, D) \). For each \( 0 \leq k \leq n \), we have a natural isomorphism
\[
H^k(C^\ast(L_{\delta}^\dag, D)) \simeq H_{n-k}(K(L_{\delta}^\dag, D))
\]
which is compatible with their endomorphisms \( \alpha_1 \).

Let \( \Sigma \) be the fan of all faces of the dual cone of \( \delta \). Let \( X \) be the toric scheme over \( R = \mathcal{O}_K \) associated to \( \Sigma \). By the definition of \( \Sigma \), we have \( X = \text{Spec}(A) \) with \( A = R[x^{\delta \cap Z^n}] \). Set
\[
S = \left\{ \sum_{j=1}^N r_j w_j : 0 \leq r_1, \cdots, r_N \leq 1, \sum_{j=1}^N r_j w_j \in \mathbb{Z}^n \right\}.
\]
Since $S$ is discrete and bounded, it is finite. Set $S = \{s_1, \ldots, s_L\}$. Then $A$ is generated by \{$_x^{s_i}|s_i \in S$\} as an $R$-algebra. More precisely, we have $A \simeq R[y_1, y_2, \ldots, y_L]/I$ by [7, Proposition 1.1.9], where

$$I = \left\{ \sum_{a, b} \left( g_{a}^{0} \cdots y_{L}^{0} - y_{1}^{b_{i}} \cdots y_{L}^{b_{i}} \right) : \sum_{i=1}^{L} a_{i} s_{i} = \sum_{i=1}^{L} b_{i} s_{i}, \ a_{i}, \ b_{i} \in \mathbb{Z}_{\geq 0} \right\}.$$ 

Consider the canonical immersions $X \to A_{R}^L \to \mathbf{P}_{R}^{L}$. Let $\overline{X}$ be the closure of $X$ in $\mathbf{P}_{R}^{L}$. Let $\overline{X}$ be the formal completion of $X$ with respect to the $p$-adic topology. Denote by $X_k, X_K$ the special fiber and the generic fiber of $X$ over $R$, respectively. Denote by $X_{K}^{an}$ the analytic space associated to $X$. By [4, 1.2.4 (ii)], $X_{K}^{an}$ is a strict neighborhood of $|X_k|_{R}$ in $|\overline{X}_{k}|_{\overline{X}}$. Let $V$ be a strict neighborhood of $|X_k|_{\overline{X}}$ in $|\overline{X}_{k}|_{\overline{X}}$ and $E$ a sheaf of $O_{V}$-modules. Define

$$j_{i}^{\ast} E := \lim_{\overline{V}^{\prime}} j_{V^{\prime}}^{\ast} j_{V}^{\ast} E,$$

where the inclusion $j_{V^{\prime}} : V^{\prime} \to V$ runs through the strict neighborhoods of $|X_k|_{\overline{X}}$ contained in $V$.

**Lemma 2.2.** We have an isomorphism $L_{\delta}^{\dagger} \simeq \Gamma(X_{K}^{an}, j_{1}^{\ast} O_{X_{K}^{an}})$ and $H^{i}(X_{K}^{an}, j_{1}^{\ast} (O_{X_{K}^{an}})) = 0$ for all $i \geq 1$.

**Proof.** By [4, Exemples 1.2.4(iii)], $V_{\lambda} := B^{L}(0, \lambda) \cap X_{K}^{an}$ form a cofinal system of strict neighborhoods of $|X_k|_{\overline{X}}$ in $X_{K}^{an}$ when $\lambda \to 1^{+}$, where $B^{L}(0, \lambda)$ is the closed polydisc with radius $\lambda$ of dimension $L$ in $A_{K}^{L, an}$. Let $j_{\lambda}$ be the inclusion $V_{\lambda} \to X_{K}^{an}$ and fix a $\lambda_{0}$. By [5, 1.2.5], we have

$$j_{1}^{\ast} O_{X_{K}^{an}} \simeq j_{\lambda_{0}}^{\ast} j_{1}^{\ast} O_{V_{\lambda_{0}}} \simeq R j_{\lambda_{0}}^{\ast} j_{1}^{\ast} O_{V_{\lambda_{0}}}.$$ 

Note that the inclusion $V_{\lambda} \to V_{\lambda_{0}}$ is affine and $V_{\lambda}$ is affinoid for each $\lambda \leq \lambda_{0}$. We have

$$H^{i}(X_{K}^{an}, j_{1}^{\ast} O_{X_{K}^{an}}) \simeq H^{i}(V_{\lambda_{0}}, j_{1}^{\ast} O_{V_{\lambda_{0}}}) \simeq \lim_{\lambda \to \lambda_{0}} H^{i}(V_{\lambda}, O_{V_{\lambda}}).$$

Hence the second assertion holds. Set

$$W_{L} = \bigcup_{\lambda > 1} \left\{ \sum_{v \in \mathbb{Z}_{>0}} a_{v} y^{v} : a_{v} \in K, \ |a_{v}| \lambda^{|v|} \to 0 \text{ as } |v| \to +\infty \right\}.$$ 

We have $\Gamma(X_{K}^{an}, j_{1}^{\ast} O_{X_{K}^{an}}) = \lim_{\lambda \to \lambda_{1}^{+}} \Gamma(V_{\lambda}, O_{V_{\lambda}}) \cong W_{L}/IW_{L}$. For any $u \in \mathbb{Z}^{n} \cap \delta$, denote the set of solutions $v_{1}s_{1} + \cdots + v_{L}s_{L} = u$ in $\mathbb{Z}_{\geq 0}^{L, an}$ by $S_{u}$. Define a morphism $\chi : W_{L} \to L_{\delta}^{\dagger}$ by

$$\chi \left( \sum_{v \in \mathbb{Z}_{>0}} a_{v} y^{v} \right) = \sum_{u \in \delta \cap \mathbb{Z}^{n}} \left( \sum_{v \in S_{u}} a_{v} \right) x^{u}.$$ 

Since the map $v \mapsto u = v_{1}s_{1} + \cdots + v_{L}s_{L}$ is a linear continuous map, there exists some $B > 0$ such that $|u| \leq B|v|$. We have

$$\left| \sum_{v \in S_{u}} a_{v} \lambda^{B^{-1}|u|} \right| \leq \max_{v \in S_{u}} |a_{v}| \lambda^{|v|}$$

for any $\lambda > 1$. Thus $\chi$ is well defined and it induces a morphism $\tilde{\chi} : W_{L}/IW_{L} \to L_{\delta}^{\dagger}$. It suffices to show $\tilde{\chi}$ is an isomorphism.

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Take \( g(x) = \sum_{u \in \delta \cap \mathbb{Z}^n} b_u x^u \in L(b,c) \) for some \( b > 0, c \in \mathbb{R} \). By the definition of \( S \), for any nonzero element \( u \in \delta \cap \mathbb{Z}^n \), we can find a subset \( s(u) \subset \{1, \cdots, L\} \) such that
\[
u = \sum_{i \in s(u)} v_i s_i, \quad v_i \in \mathbb{Z}_{\geq 1}
\]
and \( w(s_i)^{-1} s_i, w(u)^{-1} u \) lie in the same codimension 1 face of \( \Delta(f) \) for any \( i \in s(u) \). By [1, Lemma 1.9 (c)] and the choice of \( M \), we have
\[
w(u) = \sum_{i \in s(u)} v_i w(s_i) \geq \frac{\sum_{i \in s(u)} |v_i|}{M}.
\]
Set \( g^{(u)} = \prod_{i \in s(u)} g_i^{v_i} \) and \( h(y) = \sum_u b_u g^{(u)} \). We have
\[
\text{ord}(b_u) \geq bw(u) + c \geq \frac{b}{M} |w(u)| + c.
\]
So \( h(y) \in W_L \) and \( \chi(h(y)) = g(x) \). Hence \( \chi \) is surjective.

Take \( f = \sum_v a_v y^v \in W_L \) such that \( \text{ord}(a_v) \geq b|v| + c \) for some \( b, c \in \mathbb{R} \) and \( b > 0 \). If \( \chi(f) = 0 \), we have
\[
\sum_v a_v = 0
\]
for any \( u \). For each \( u \in \delta \cap \mathbb{Z}^n \), take some \( v_u \in S_u \) such that \( |v_u| = \min \{|v| : v \in S_u\} \). By Lemma 2.4 below, we can find \( h_1, \cdots, h_m \in I \) such that for any \( v \in S_u \), there exist \( g_{vu1}, \cdots, g_{vum} \in R[y_1, \cdots, y_L] \) such that
\[
y^v - y^{v_u} = \sum_{j=1}^m g_{vuj} h_j
\]
and \( \deg(g_{vuj}) \leq |v| \) for all \( j \). So
\[
f = \sum_u \sum_{v \in S_u} a_v y^v = \sum_u \sum_{v \in S_u} a_v (y^v - y^{v_u}) = \sum_{j=1}^m h_j \left( \sum_u \sum_{v \in S_u} a_v g_{vuj} \right).
\]
Set \( g_{vuj} = \sum_k g_{vuj}^k y^k \) with \( |k| \leq |v| \) and \( \text{ord}(g_{vuj}^k) \geq 0 \). Define
\[
g_j := \sum_u \sum_{v \in S_u} a_v g_{vuj} = \sum_k y^k \left( \sum_u \sum_{v \in S_u} a_v g_{vuj}^k \right).
\]
We have
\[
\text{ord} \left( \sum_u \sum_{v \in S_u} a_v g_{vuj}^k \right) \geq \min_{v \in S_u} \text{ord}(a_v g_{vuj}^k) \geq \min_{v \in S_u} \text{ord}(a_v) \geq \min_{v \in S_u} b|v| + c \geq b|k| + c.
\]
Hence \( g_j \in W_L \) and \( f \in IW_L \) which imply that \( \tilde{\chi} \) is injective. \( \square \)

Lemma 2.3. For any subset \( J \) of \( \mathbb{Z}_{\geq 0}^n \), there exists a finite subset \( J_0 \) of \( J \) such that \( J \subset \bigcup_{v \in J_0} (v + \mathbb{Z}_{\geq 0}^n) \).

**Proof.** We use induction on \( n \). When \( n = 1 \), it is trivial. Suppose that our assertion holds for any subset of \( \mathbb{Z}_{\geq 0}^n \). Let \( J \) be a subset of \( \mathbb{Z}_{\geq 0}^{n+1} \). Take
\[
J' = \{ v \in \mathbb{Z}_{\geq 0}^n : (a, v) \in J \text{ for some } a \in \mathbb{Z}_{\geq 0} \}.
\]
By the induction hypothesis, there exists a finite subset $J'_0 = \{v'_1, \cdots, v'_r\}$ such that

$$J' \subset \bigcup_{i=1}^{r} \left(v'_i + Z_{\geq 0}^n\right).$$

For each $i$, take some $a_i \in Z_{\geq 0}$ such that $v_i = (a_i, v'_i) \in J$. For any $b \in Z_{\geq 0}$, let

$$J_b = \{(c_1, \cdots, c_{n+1}) \in J : c_1 = b\}.$$  

Using the induction hypothesis again, there exists a finite subset $J_{b, 0}$ such that

$$J_b \subset \bigcup_{v \in J_{b, 0}} \left(v + Z_{\geq 0}^{n+1}\right).$$

Set $a_0 = \max_{1 \leq i \leq r} \{a_i\}$. We have

$$J \subset \left( \bigcup_{b < a_0} J_b \right) \bigcup \left( \bigcup_{i=1}^{r} \left(v_i + Z_{\geq 0}^{n+1}\right) \right) \subset \left( \bigcup_{b < a_0} \bigcup_{v \in J_{b, 0}} (v + Z_{\geq 0}^{n+1}) \right) \bigcup \left( \bigcup_{i=1}^{r} (v_i + Z_{\geq 0}^{n+1}) \right)$$

We can take $J_0 = \left( \bigcup_{b < a_0} J_b \right) \bigcup \{v_1, \cdots, v_r\}$. \hfill \Box

**Lemma 2.4.** There are finitely many elements $h_1, \cdots, h_m \in I$ such that for any $y^a - y^b \in I$, there exist $g_1, \cdots, g_m \in R[y_1, \cdots, y_L]$ such that

$$y^a - y^b = \sum_{i=1}^{m} g_i h_i$$

and $\deg(g_i) \leq \max\{|a|, |b|\}$.

**Proof.** Take $J = \{(a, b) : y^a - y^b \in I\} - \{(0, 0)\}$. By Lemma 2.3 above, there is a finite subset $J_0 = \{(a_1, b_1), \cdots, (a_m, b_m)\} \subset J$ such that $J \subset \bigcup_{v \in J_0} (v + Z_{\geq 0}^{nL})$. Take $h_i = y^{a_i} - y^{b_i}$. We prove the assertion by induction on $|a| + |b|$. When $|a| + |b| = 0$, the assertion is trivial. Take $(a, b) \in J$. Suppose that the assertion holds for any $y^{a'} - y^{b'} \in J$ with $|a'| + |b'| < |a| + |b|$. By the choice of $J_0$, there is some $i_0$ such that $(a, b) \geq (a_{i_0}, b_{i_0})$. We have $|a - a_{i_0}| + |b - b_{i_0}| < |a| + |b|$. By induction hypothesis, there exist $g'_1, \cdots, g'_m \in R[y_1, \cdots, y_L]$ such that

$$y^{a - a_{i_0}} - y^{b - b_{i_0}} = \sum_{i=1}^{m} g'_i h_i$$

and $\deg(g'_i) \leq \max\{|a - a_{i_0}|, |b - b_{i_0}|\}$.

If $|a_{i_0}| \geq |b_{i_0}|$, we have

$$y^a - y^b = y^{a - a_{i_0}} (y^{a_{i_0}} - y^{b_{i_0}}) + y^{b_{i_0}} (y^{a - a_{i_0}} - y^{b - b_{i_0}}).$$

Take $g_{i_0} = y^{a - a_{i_0}} + y^{b_{i_0}} g'_i$ and let $g_i = y^{b_{i_0}} g'_i$ for any $i \neq i_0$. We have

$$y^a - y^b = \sum_{i=1}^{m} g_i h_i$$

and $\deg(g_i) \leq \max\{|a - a_{i_0}|, |b_{i_0}| + \deg(g'_i)\} \leq \max\{|a|, |b|\}$.

If $|b_{i_0}| \geq |a_{i_0}|$, we have

$$y^a - y^b = \sum_{i=1}^{m} g_i h_i$$

and $\deg(g_i) \leq \max\{|a - a_{i_0}|, |b_{i_0}| + \deg(g'_i)\} \leq \max\{|a|, |b|\}$. If $|b_{i_0}| \geq |a_{i_0}|$, we have

$$y^a - y^b = y^{b - b_{i_0}} (y^{a_{i_0}} - y^{b_{i_0}}) + y^{a_{i_0}} (y^{a - a_{i_0}} - y^{b - b_{i_0}}).$$

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Then we conclude as above.

Set

\[ L_0^+ = \bigcup_{r > 1} \left\{ \sum_{u \in \mathbb{Z}^n} a_u x^u : a_u \in K, |a_u| r^{|u|} \to 0 \text{ as } |u| \to +\infty \right\}. \]

Note that \( L_0^+ \subset L_0^+ \). Similarly, we can define the complex \( C^\prime (L_0^+, D) \) and the endomorphism \( \alpha_1 \).

By [3, Proposition 3.4] or [6], we have

**Proposition 2.5.** We have a canonical isomorphism

\[ R\Gamma_{\text{rig}}(T_n^R/K, f^* L_{-\psi}) \cong C^\prime (L_0^+, D) \]

which is compatible with their endomorphisms \( F_* = q^* (F^*)^{-1} \) and \( \alpha_1 \). Furthermore, \( \alpha_1 \) is bijective on each cohomology group of \( C^\prime (L_0^+, D) \).

Let \( \Sigma' \) be a regular refinement of \( \Sigma \) and let \( X' \) be the toric scheme over \( R \) associated to \( \Sigma' \). Denote by \( X'_{\text{K}}', X'_{\text{K}} \) the special fiber and the generic fiber of \( X' \) over \( R \), respectively. Let \( T_n^R \to X' \) be the immersion of the maximal open dense torus, and let \( D' = X' - T_n^R \). Denote by \( X'_{\text{K}} \) the analytic associated to \( X'_{\text{K}} \) (resp. \( D'_{\text{K}} \)). Note that \( X' \) is smooth and \( D' \) is a normal crossing divisor in \( X' \).

Let \( \Omega^{\Sigma}_{X(\text{K})}(\log D_{\text{K}}) \) be the sheaf of differential forms of degree \( k \) with logarithmic poles along \( \Sigma_1 \). We still denote by \( K_{\text{an}}' \) and \( D_{\text{an}} \) the special fiber and the generic fiber of \( X_{\text{K}}' \). We have \( \Omega^{\Sigma}_{X(\text{K})}(\log D_{\text{K}}) \) is a free \( \mathcal{O}_{X_{\text{K}}'} \) module with basis \( \frac{dx_{i_1} \wedge \cdots \wedge dx_{i_k}}{x_{i_1} \cdots x_{i_k}} \), where \( 1 \leq i_1 < \cdots < i_k \leq n \) and \( x_1, \ldots, x_n \) are coordinates of \( T_n^R \). Consider the complex

\[ E := (\Omega_{X'_{\text{K}}}(\log D'_{\text{K}}), d) \]

where the differential is given by \( d = dx + \sum_{i=1}^{n} \pi E_i f \frac{dx_i}{x_i} \). Note that

\[ f \in \Gamma(X'_{\text{K}}(\text{K}), \mathcal{O}_{X'_{\text{K}}}(\text{K})) \cong \Gamma(X'_{\text{K}}', \mathcal{O}_{X'_{\text{K}}'}). \]

Hence the twist de Rham complex \( E \) is well defined. Let \( \Sigma'' \) be a regular fan containing \( \Sigma' \) such that \( |\Sigma''| = \mathbb{R}^n \), let \( X'' \) be the toric scheme over \( R \) associated to \( \Sigma'' \) and let \( \hat{\Sigma}'' \) be its formal completion. In [2, Corollary A.4], taking \( X = \hat{X}'' \), \( Z = X'' - T_n^R \), \( V = X'_{\text{K}} \), \( U = T_n^R \) and \( W = X'_{\text{K}} \), we get a canonical isomorphism

\[ (2.3) \quad R\Gamma(X'_{\text{K}}', j'_{\text{K}}(\log D'_{\text{K}})) \cong R\Gamma_{\text{rig}}(T_n^R/K, \mathcal{L}_{-\psi,K}). \]

Now we can give the proof of **Theorem 1.1.**

**Proof.** Take cones \( \sigma_1, \ldots, \sigma_m \in \Sigma' \) such that the open sets \( U_i \) defined by \( \sigma_i \) form an affine open covering of \( X' \). Let \( j_{i,i} := j_{i_1 \cdots i_k} \) be the inclusion \( U_{\sigma_i} := U_{i_1} \cap \cdots \cap U_{i_k} \subset X' \). Note that \( U_{\sigma} \) is the affine open set defined by the cone \( \sigma_{\hat{\sigma}} := \sigma_{i_1} \cap \cdots \cap \sigma_{i_k} \). By [4, Proposition 2.1.8], we have an exact sequence

\[ 0 \to j^! E → \prod_{i} j_{i,i}^! E → \prod_{i_1 < i_2} j_{i_1,i_2}^! E → \cdots → j_{\Sigma_{\hat{\sigma}}}^! E → 0, \]

where \( j_{\Sigma}^! E = \lim_{\to \nu} j_{\nu}^! j_{\nu,v}^!(E) \) and \( j_{\nu}^! : V \to X'_{\text{K}} \) runs through strict neighborhoods of \( U_{\hat{\Sigma}'}^\infty \) contained in \( X'_{\text{K}} \). We still denote by \( j_{\hat{\Sigma}} \) the inclusion \( U_{\hat{\Sigma}} \subset X'_{\text{K}} \). We have \( j_{\Sigma}^! E = j_{\Sigma}^! (E|_{U_{\hat{\Sigma}'}^\infty}) \) by [4, 1.2.4 (ii) and 2.1.1]. Let \( \hat{\sigma} \) be the dual of \( \sigma \) and \( \hat{\delta} = \delta \). Applying Lemma 2.2 to each \( U_{\hat{\Sigma}}^\infty \), we know that \( RT(X'_{\text{K}}', j^! E) \) can be represented by the total complex of

\[ (2.4) \quad 0 → \prod_{i} \Gamma(U_i^\infty, j^! (E|_{U_i^\infty})) → \cdots → \Gamma(U_{\hat{\Sigma}}^\infty, j^! (E|_{U_{\hat{\Sigma}}^\infty})) → 0. \]
and \( \Gamma(U^K, j^* O_{U^K}) \cong L_{\sigma}^1 \), where

\[
L_{\sigma}^1 = \bigcup_{\lambda > 1} \left\{ \sum_{u \in \sigma \cap \mathbf{Z}^n} A_u x^u : A_u \in K, |A_u| |\lambda| \to \infty \to 0 \right\}
\]

which can be regarded as a subalgebra of \( L_0^1 \). Thus (2.4) can be represented by the bicomplex

(2.5)

\[
0 \to \prod_i C'(L_{\sigma_i}^1) \to \prod_{i_1 < i_2} C'(L_{\sigma_i}^1) \to \cdots \to C'(L_{\sigma_1,\cdots,m}^1) \to 0,
\]

where all \( C'(L_{\sigma_i}^1) \) are regarded as subcomplexes of \( C'(L_0^1, D) \). By Lemma 2.7 below and a spectral sequence argument, \( C'(L_0^1, D) \) can be represented by the total complex of (2.5). Hence

\[ R\Gamma(X^K, j^* E) \cong C'(L_0^1, D). \]

By Proposition 2.5, the isomorphism (2.3) can be represented by the inclusion

\[ C'(L_0^1, D) \hookrightarrow C'(L_0^1, D). \]

By (2.2) and Proposition 2.1, we have

\[ H_{n-i}(K(L_0^1, D)) \cong H_{n-i}^1(T_0^1/K, f^* L_{-\psi}) \]

for each \( i \). The second assertion of Theorem 1.1 follows from (2.2) and Proposition 2.5. \( \square \)

**Lemma 2.6.** Notation as above, we have an exact sequence

(2.6)

\[
0 \to L_{\sigma,R} \to \prod_i L_{\sigma_i,R} \to \prod_{i_1 < i_2} L_{\sigma_i,R} \to \cdots \to L_{\sigma_1,\cdots,m,R} \to 0,
\]

where \( L_{\sigma,R} = R[x^{i R}] \) is regarded as a subalgebra of \( R[x^R] \), for \( \sigma = \sigma_1, \cdots, \sigma_k \). The boundary map

\[
(dg)_{i_1,\cdots,i_{k+1}} = \sum_{s=1}^{k+1} (-1)^{s-1} g_{i_1,\cdots,i_s,i_{s+1}}
\]

for any \( g \in \prod_{i_1 < \cdots < i_k} L_{\sigma_i,R} \).

**Proof.** Consider the Čech complex \( C((U, O_{X'}) \) defined by the affine open covering \( U = \{ U_\sigma \}_{i=1}^m \). We have \( \check{H}^k(U, O_{X'}) \cong H^k(X', O_{X'}) \) for any \( k \). Since \( \phi : X' \to X \) is proper and \( X \) is affine, we have \( H^k \phi_* (O_{X'}) = H^k(X', O_{X'}) \). By [7, Proposition 9.2.5], we have \( R^k \phi_* O_{X'} = 0 \) for any \( k \geq 1 \). Thus (2.6) is exact except at the second position. Since \( \sigma = \cup_{i=1}^m \sigma_i \), we have \( \sigma = \cap_{i=1}^m C_{\sigma_i} \). Thus \( L_{\sigma,R} = \text{Ker}(\prod_i L_{\sigma_i,R} \to \prod_{i_1 < i_2} L_{\sigma_i,R}) \) and then (2.6) is exact. \( \square \)

**Lemma 2.7.** Notation as above, we have an exact sequence

(2.7)

\[
0 \to L_{\sigma}^1 \to \prod_i L_{\sigma_i}^1 \to \prod_{i_1 < i_2} L_{\sigma_i}^1 \to \cdots \to L_{\sigma_1,\cdots,m}^1 \to 0,
\]

where the boundary map is the same as (2.6).
Proof. By the definition of $\Sigma'$, we have $\sigma = \bigcup_{i=1}^{m_i} \sigma_i$. Thus $\delta = \hat{\sigma} = \bigcap_{i=1}^{m_i} \sigma_i$ and $L^\dagger_\delta = \text{Ker}(\prod L^\dagger_{\sigma_i} \to \prod L^\dagger_{\sigma_i})$. Now consider $\prod L^\dagger_{\sigma_i} \xrightarrow{d} \prod L^\dagger_{\sigma_i}$, take $g = (g_{L}) \in \prod L^\dagger_{\sigma_i}$. Suppose that for all $L$, $g_{L} \in L''(b, c)$ for some $b > 0$, $c \in \mathbb{R}$, where

$$L''(b, c) = \left\{ \sum_u A_u x^u : A_u \in K, \text{ord} A_u \geq b|u| + c \right\}.$$ 

Since we always can choose some element $a \in R$ such that $ag_i \in L''(b, 0)$ for all $i$, we treat the case $c = 0$. Now $g_{L}$ can be written as

$$g_{L} = \sum_{k=0}^{+\infty} \pi^k g_{L,k}, \quad g_{L,k} \in R[x^\uparrow L^\dagger \cdot \mathbb{Z}^n],$$

where $g_{L,k}$ is a sum of monomials of degree lying in $[\frac{k}{(b+1)(p-1)}, \frac{k+1}{b(p-1)})$. Note that the degree of polynomial defines a grading on the complex (2.6), and $d$ preserves the grading. Write $g = \sum_{k=0}^{+\infty} \pi^k g_k$, where $g_k = (g_{L,k})$. If $d(g) = 0$, we have $d(g_k) = 0$ for each $k$. By Lemma 2.6, there exists

$$h_k = (h_{L,k}) \in \prod_{i_1 < \cdots < i_{k-1}} L_{\sigma_{L,R}}$$

such that $d(h_k) = g_k$ for each $k$ and we may assume that each $h_{L,k}$ is a sum of monomials of degree lying in $[\frac{k}{(b+1)(p-1)}, \frac{k+1}{b(p-1)})$. Suppose that $h_{L,k} = \sum_u A_{L,k,u} x^u$, we have

$$\text{ord}(\pi^k A_{L,k,u}) \geq \frac{k}{p-1} > \frac{k}{(p-1)\frac{k+1}{b(p-1)}} |u| \geq \frac{b}{2} |u|$$

for any $k \geq 1$. Thus $\sum_{k=0}^{+\infty} \pi^k h_{L,k} \in L''(b, 0)$. Note that $h_{L,0}$ is a polynomial for each $L$. We have

$$h := \sum_{k=0}^{+\infty} \pi^k h_k \in \prod_{i_1 < \cdots < i_{k-1}} L^\dagger_{\sigma_{L}}$$

and $d(h) = g$. Hence (2.7) is exact. \hfill \qed

3. Applications

Define

$$B = \left\{ \sum_{u \in \mathbb{N}^n \mathbb{Z}^n} a_u \gamma^w(u) x^u : a_u \in K, \text{ord} w(u) \to +\infty \right\}.$$ 

We can define the Koszul complexes $K(L(b), \hat{D})$ (0 < $b$ ≤ $\frac{p}{p-1}$) and $K(B, \hat{D})$ with actions of $\alpha$, respectively. They can be regarded as subcomplexes of $K(L^\dagger_\delta, \hat{D})$.

**Proposition 3.1.** For each $i$, the inclusions $B \hookrightarrow L(\frac{1}{p-1}) \hookrightarrow L^\dagger_\delta$ induce a surjective homomorphism $H_i(K(L(b), \hat{D})) \to H_i(K(L^\dagger_\delta, \hat{D}))$.

**Proof.** We use the same method as [6, Corollaire 1.3]. The inclusions $L(\frac{p}{p-1}) \hookrightarrow B \hookrightarrow L^\dagger_\delta$ imply that it suffices to prove the homomorphism $H_i(K(L(\frac{p}{p-1}), \hat{D})) \to H_i(K(L^\dagger_\delta, \hat{D}))$ is surjective. Note that $L^\dagger_\delta = \bigcup_{0 < b \leq \frac{p}{p-1}} L(b)$. By Theorem 1.1 and [5, 3.10], $H_i(K(L^\dagger_\delta, \hat{D}))$ is finite dimensional.

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Thus there exists some \(0 < b \leq \frac{p}{p-1}\) such that \(H_i(K, L(b, \mathcal{D})) \to H_i(K, L_0^1, \mathcal{D}))\) is surjective. Note that \(\psi_q(L(b)) \subset L(qb)\). Consider the following composite
\[
L(b) \xrightarrow{\psi_q} L\left(\min \left\{ \frac{p}{q(p-1)}, b \right\}\right) \xrightarrow{\psi_s} L\left(\min \left\{ \frac{p}{p-1}, qb \right\}\right).
\]
Hence
\[
\alpha(L(b)) \subset L\left(\min \left\{ \frac{p}{p-1}, qb \right\}\right).
\]
So there exists a positive integer \(m\) such that
\[
\alpha^m(H_i(K, L(b, \mathcal{D}))) \subset H_i(K, L(\frac{p}{p-1}), \mathcal{D})).
\]
We have a commutative diagram
\[
\begin{array}{ccc}
H_i(K, L(b, \mathcal{D})) & \longrightarrow & H_i(K, L_0^1, \mathcal{D})) \\
\alpha^m \downarrow & & \alpha^m \downarrow \\
H_i(K, L(\frac{p}{p-1}), \mathcal{D})) & \longrightarrow & H_i(K, L_0^1, \mathcal{D})).
\end{array}
\]
The top horizontal arrow is surjective by the choice of \(b\). The right vertical arrow \(\alpha^m\) is bijective by Theorem 1.1. So the bottom horizontal arrow is surjective.

Let \(X = T_k^r \times A_{k}^{n-r}\) and \(f \in k[x_1, \ldots, x_n, (x_1 \cdots x_r)^{-1}]\).

For any subset \(A \subset S_r = \{r+1, \ldots, n\}\), let
\[
R_A = \{(x_1, \ldots, x_n) : x_i = 0 \text{ for } i \in A\}.
\]
Let \(V_A(f)\) be the volume of \(\Delta(f) \cap R_A^n\) with respect to Lebesgue measure on \(R_A^n\). Set
\[
v_A(f) = \sum_{B \subset A} (-1)^{|B|}(n-|B|)!V_B(f)
\]
and \(\tilde{r} = \dim \Delta(f_{S_r})\). We say that \(f\) is commode with respect to \(S_r\), if for all subsets \(A \subset S_r\), we have \(\dim \Delta(f_A) = \tilde{r} + |S_r - A|\), where \(f_A := \prod_{j \in A} \theta_j \circ f(x)\) and \(\theta_j\) is the map such that \(x_j \mapsto 0\).

**Theorem 3.2.** Let \(X = T_k^r \times A_{k}^{n-r}\), \(f \in \Gamma(X, \mathcal{O}_X)\). Suppose that \(f\) is commode with respect to \(S_r\), nondegenerate with respect to \(\Delta(f)\) and \(\dim \Delta(f) = n\). Then
\[
\text{(i) } H^i_{\text{rig}}(X, f^*\mathcal{L}_\psi) = H^i_{\text{rig}}(X, f^*\mathcal{L}_{-\psi}) = 0 \text{ if } i \neq n,
\]
\[
\text{(ii) } H^i_{\text{rig}}(X, f^*\mathcal{L}_{-\psi}) = \dim H^i_{\text{rig}}(X, f^*\mathcal{L}_{-\psi}) = v_{S_r}(f).
\]

**Proof.** First we consider the case \(r = n\). Since \(X\) is affine, smooth and pure of dimension \(n\), by the Poincaré duality and the trace formula in the rigid cohomology theory, we have
\[
L(T_k^n, f, t) = \prod_{i=0}^{n} \det(I - tq^n(F^*)^{-1}|H^i_{\text{rig}}(T_k^n/K, f^*\mathcal{L}_{-\psi})^{-1})^{i+1}.
\]
When \(f\) is nondegenerate, we have \(H_i(K, \mathcal{D})) = 0\) for \(i \neq 0\) and \(\dim H_0(K, \mathcal{D})) = n! \text{Vol}(\Delta(f))\) by [1, Theorem 2.18]. By Proposition 3.1, we have \(H_i(K, L_0^1, \mathcal{D})) = 0\) for \(i \neq 0\). By Theorem 1.1, we have
\[
H^i_{\text{rig}}(T_k^n, f^*\mathcal{L}_{-\psi}) \cong H_{n-i}(K, L_0^1, \mathcal{D})).
\]

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for $i \neq n$. By [1, (2.5)] and (3.1), we have

$$\det(I - tq^n(F^*)^{-1}|H^n_{\text{rig}}(X/K, f^*{\mathcal L}_-\psi)) = \det(1 - tH_0(\alpha)|H_0(K, (B, \hat{D}))).$$

So

$$\dim H^n_{\text{rig}}(X/K, f^*{\mathcal L}_-\psi) = \dim H_0(K, (B, \hat{D})) = n! \text{Vol}(\Delta(f)).$$

We then use induction on $n - r$. Suppose that Theorem 3.2 holds for $n - (r + 1)$. By [5, 2.3.1] and [10, Theorem 4.1.1], we have a distinguished triangle

$$R\Gamma_{\text{rig}}(X, {\mathcal L}_-\psi, f) \to R\Gamma_{\text{rig}}(U, {\mathcal L}_-\psi, f) \to R\Gamma_{\text{rig}}(Z, {\mathcal L}_-\psi, f_{r+1})[-1] \to$$

where $U = X - V(x_{r+1}) \cong T^{r+1}_k \times A^{n-r-1}_k$ and $Z = V(x_{r+1}) \cong T^r_k \times A^{n-r-1}_k$. One can check that $f, f_{r+1}$ are still commode with respect to $S_{r+1}$, and $f_{r+1}$ is nondegenerate with respect to $\Delta(f_{r+1})$. By the induction hypothesis, (i) holds and

$$\dim H^n_{\text{rig}}(X, {\mathcal L}_-\psi, f) = \dim H^n_{\text{rig}}(U, {\mathcal L}_-\psi, f) - \dim H^{n-1}_{\text{rig}}(Z, {\mathcal L}_-\psi, f_{r+1}) = v_{S_{r+1}}(f) - v_{S_{r+1}}(f_{r+1}) = v_S(f).$$

□

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