A Dual Method For Backward Stochastic Differential Equations with Application to Risk Valuation

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Abstract

We propose a numerical recipe for risk evaluation defined by a backward stochastic differential equation. Using dual representation of the risk measure, we convert the risk valuation to a stochastic control problem where the control is a certain Radon-Nikodym derivative process. By exploring the maximum principle, we show that a piecewise-constant dual control provides a good approximation on a short interval. A dynamic programming algorithm extends the approximation to a finite time horizon. Finally, we illustrate the application of the procedure to risk management in conjunction with nested simulation.

Keywords: Dynamic Risk Measures, Forward–Backward Stochastic Differential Equations, Stochastic Maximum Principle, Risk Management

1 Introduction

The main objective of this paper is to present a simple and efficient numerical method for solving backward stochastic differential equations with convex and homogeneous drivers. Such equations are fundamental modeling tools for continuous-time dynamic risk measures with Brownian filtration, but may also arise in other applications.

The key property of dynamic risk measures is time-consistency, which allows for dynamic programming formulations. The discrete time case was extensively explored by Detlfsen and Scandolo [11], Bion-Nadal [5], Cheridito et al. [7, 8], Föllmer and Penner [12], Frittelli and Rosazza Gianin [15], Frittelli and Scandolo [16], Riedel [32], and Ruszczynski and Shapiro [35].

For the continuous-time case, Coquet, Hu, Mémin and Peng [9] discovered that time-consistent dynamic risk measures, with Brownian filtration, can be represented as solutions of Backward Stochastic Differential Equations (BSDE) [30]; under mild growth conditions, this is the only form

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possible. Specifically, the y-part solution of one-dimensional BSDE, defined below, measures the risk of a variable $\xi_T$ at the current time $t$:

$$
Y_t = \xi_T + \int_t^T g(s,Z_s) \, ds - \int_t^T Z_s \, dW_s, \quad 0 \leq t \leq T,
$$

with the driver $g$ being interpreted as a “risk rate.” The $\mathcal{F}_T$-measurable random variable $\xi_T$ is usually a function of the terminal state of a certain stochastic dynamical system.

Inspired by that, Barrieu and El Karoui provided a comprehensive study in [3, 4]; further contributions being made by Delbean, Peng, and Rosazza Gianin [10], and Quenez and Sulem [31] (for a more general model with Levy processes). In addition, application to finance was considered, for example, in [19]. Using the convergence results of Briand, Delyon and Mémin [6], Stadje [39] finds the drivers of BSDE corresponding to discrete-time risk measures.

Motivated by an earlier work on risk-averse control of discrete-time stochastic process [37], Ruszczyński and Yao [38] formulate a risk-averse stochastic control problem for diffusion processes. The corresponding dynamic programming equation leads to a decoupled forward–backward system of stochastic differential equations.

While forward stochastic differential equations can be solved by several efficient methods, the main challenge is the numerical solution of $\xi_T$, where $\xi_T$ represents the future value function. In particular, Zhang [41] and Touzi et al. [40] use backward Euler’s approximation and regression. Such an approach, however, is not well-suited for risk measurement, because it does not preserve the monotonicity of the risk measure. Alternatively, Øksendal [28] directly attacks continuous-time risk-averse control problem with jumps by deriving sufficient conditions. Algorithms based on maximum principle were investigated by Ludwig et al. [23].

Our idea is to derive a recursive method based on risk-averse dynamic programming, so that the approximation becomes a time-consistent coherent risk measure in discrete time.

The paper is organized as follows. In section 2, we quickly introduce the concept of a dynamic risk measure and review its properties. In section 3 we recall the dual representation of a dynamic risk measure and formulate an equivalent stochastic control problem. The optimality condition for the dual control problem, a special form of a maximum principle, are derived in section 4. Sections 5 and 6 estimate the errors introduced by using constant processes as dual controls. In section 7 we present the whole numerical method with piecewise constant dual controls and analyze its rate of convergence. Finally, in section 8 we illustrate the efficacy of our approach on a two-stage risk management model.

2 The Risk Evaluation Problem

Given a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}_{t \in [0,T]}$ generated by $d$-dimensional Brownian motion $\{W_t\}_{t \in [0,T]}$, we consider the following stochastic differential equation:

$$
\frac{dX_t}{dt} = b(t,X_t) \, dt + \sigma(t,X_t) \, dW_t, \quad X_0 = x, \quad t \in [0,T],
$$

with measurable $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, and $\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^d$. We introduce the following notation.
• \( \mathbb{E}[\cdot] := \mathbb{E}[\cdot | \mathcal{F}_t] \);

• \( L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n) \): the set of \( \mathbb{R}^n \)-valued \( \mathcal{F}_t \)-measurable random variables \( \xi \) such that \( \|\xi\|^2 := \mathbb{E}[\|\xi\|^2] < \infty \); for \( n = 1 \), we write it \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \);

• \( \mathcal{H}^{2,n}[t, T] \): the set of \( \mathbb{R}^n \)-valued adapted processes \( Y \) on \([t, T] \), such that \( \|Y\|^2_{\mathcal{H}^{2,n}[t, T]} := \mathbb{E}\left[ \int_t^T |Y_s|^2 \, ds \right] < \infty \); for \( n = 1 \) we write it \( \mathcal{H}^2[t, T] \);

• \( C^k_b([t, T] \times \mathbb{R}^n) \) the space of functions \( f : [t, T] \times \mathbb{R}^n \to \mathbb{R} \) whose derivative up to \( k \)-th, in the meanwhile, all those derivatives are continuous and bounded; for \( f : \mathbb{R}^n \to \mathbb{R} \), we denote by \( C^k_b(\mathbb{R}^n) \);

• \( C_L(B) \): the space of Lipschitz continuous functions \( f : B \to \mathbb{R} \).

We make following assumptions about the drift and volatility terms.

**Assumption 2.1.**

(i) \( |b(\cdot, 0)| + |\sigma(\cdot, 0)| \in \mathcal{H}^2[0, T] \);

(ii) The functions \( b, \sigma \in C^1_b([0, T] \times \mathbb{R}^n) \), the constant \( C > 0 \) denotes the Lipschitz constants

\[
|b(t, x_1) - b(t, x_2)| + |\sigma(t, x_1) - \sigma(t, x_2)| \leq C|x_1 - x_2| \quad \text{a.s.} \\
|b(t, x_1)| + |\sigma(t, x_1)| \leq C|x_1|, \quad \text{a.s.}
\]

(iii) The dimension of Brownian motion and the state process coincide, i.e., \( n = d \), and we also assume

\[
\sigma(t, x)\sigma^T(t, x) \geq \frac{1}{C} \mathbb{I}, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d.
\]

Our intention is to evaluate risk of a terminal cost generated by the forward process \( \Phi(X_T) \),

\[
\rho_{0,T}[\Phi(X_T)],
\]

where \( \Phi \in C_L(\mathbb{R}^n) \) is bounded, and \( \{\rho_{s,t}\}_{0 \leq s \leq t \leq T} \) is a dynamic risk measure consistent with the filtration \( \{\mathcal{F}_t\}_{t \in [0,T]} \). We refer the reader to [29] for a comprehensive discussion on risk measurement and filtration-consistent evaluations.

Special role in the dynamic risk theory is played by \( g \)-evaluations which defined by one-dimensional backward stochastic differential equations of the following form:

\[
-dY_t = g(t, Y_t, Z_t) \, dt - Z_t \, dW_t, \quad Y_T = \Phi(X_T), \quad t \in [0, T],
\]

with \( \rho^{\Phi}_{t,T}[\Phi(X_T)] \) defined to be equal to \( Y_t \). The driver \( g \) is jointly Lipschitz in \((y, z)\), and the process \( g(\cdot, 0, 0) \in \mathcal{H}^2[0, T] \).

As proved in [9] every \( \mathbb{F} \)-consistent nonlinear evaluation that is dominated by a \( g \)-evaluation with \( g = \mu|y| + \nu|z| \) with some \( \nu, \mu > 0 \) is in fact a \( g \)-evaluation for some \( g \); the dominance is understood as follows:

\[
\rho_{0,T}[\xi + \eta] - \rho_{0,T}[\xi] \leq \rho^{\nu\mu}_{0,T}[\eta]
\]

for all \( \xi, \eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \).

\(^1\) When the norm is clear from the context, the subscripts are skipped.
Proposition 2.2. For all $0 \leq t \leq T$ and all $\xi, \xi' \in L_2(\Omega, \mathcal{F}_T, \mathbb{P})$, the following properties hold:

(i) Generalized constant preservation: If $\xi \in L_2(\Omega, \mathcal{F}_T, \mathbb{P})$, then $\rho^g_{s,t}[\xi] = \xi$;
(ii) Time consistency: $\rho^g_{s,t}[\xi] = \rho^g_{r,s}[\rho^g_{r,t}[\xi]]$, for all $0 \leq s \leq t$;
(iii) Local property: $\rho^g_{s,t}[\xi 1_A + \xi' 1_{A'}] = \rho^g_{s,t}\rho^g_{r,t}[\xi] + \rho^g_{s,t}\rho^g_{r,t}[\xi']$, for all $A \in \mathcal{F}_t$.

From now on, shall focus exclusively on $g$-evaluations as dynamic risk measures, and we shall skip the superscript $g$ in $\rho^g$.

The evaluation of risk is equivalent to the solution of a decoupled forward–backward system of stochastic differential equations (2)–(4). An important virtue of this system is its Markov property:

$$\rho_{t,T}[\Phi(X_T)] = v(t, X_t),$$

where $v : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$. We have

$$v(t, x) = \rho^{x}_{t,T}[\Phi(X^{x}_{T})], \quad (t, x) \in [0, T] \times \mathbb{R}^n,$$

where $\{X^{x}_{T}\}$ is the solution of the system (2) restarted at time $t$ from state $x$:

$$dX^{x}_{s} = b(s, X^{x}_{s}) \, ds + \sigma(s, X^{x}_{s}) \, dW_s, \quad s \in [t, T], \quad X^{x}_{t} = x,$$

and $\rho^{x}_{t,T}[\Phi(X^{x}_{T})]$ is the (deterministic) value of $Y^{x}_{T} \in$ the backward equation (4) with terminal condition $\Phi(X^{x}_{T})$.

Numerical methods for solving forward equations are very well understood (see, e.g., [18]). We focus, therefore, on the backward equation (4). So far, a limited number of results are available for this purpose. The most prominent is the Euler method with functional regression (see, e.g., [24, 25, 40, 41, 42]). Our intention is to show that for drivers satisfying additional coherence conditions, a much more effective method can be developed, which exploits time-consistency, duality theory for risk measures, and the maximum principle in stochastic control.

3 The Dual Control Problem

We further restrict the risk measures under consideration to coherent measures, by making the following additional assumption about the driver $g$.

Assumption 3.1. The driver $g$ satisfies for almost all $t \in [0, T]$ the following conditions:

(i) $g$ is deterministic and independent of $y$, that is, $g : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$, and $g(\cdot, 0) \equiv 0$;
(ii) $g(t, \cdot)$ is convex for all $t \in [0, T]$;
(iii) $g(t, \cdot)$ is positively homogeneous for all $t \in [0, T]$.

Under these conditions, one can derive further properties of the evaluations $\rho^g_{t,T}[\cdot]$ for $t \in [0, T]$, in addition to the general properties of $\mathbb{F}$-consistent nonlinear expectations stated in Proposition 2.2.

Theorem 3.2. Suppose $g$ satisfies Assumption 3.1. Then the dynamic risk measure $\{\rho_{t,T}\}_{0 \leq t \leq T}$ has the following properties:
The convexity of the exponential function yields the postulated bound.

Moreover, a solution to Theorem 3.3. Suppose Assumption 3.1 is satisfied. Then

\[ \rho_{t,r}^g[\varepsilon] = \sup_{\mu \in A_{t,r}} \mathbb{E}[\Gamma_{t,T} \Phi(X^{t,s}_T)], \]  

where \( A_{t,r} \) is the space of \( A_s \)-valued adapted processes on \([t, T]\), and the process \( \{\Gamma_{t,s}\}_{s \in [t,T]} \) satisfies the stochastic differential equation:

\[ d\Gamma_{t,s} = \mu_s \Gamma_{t,s} \, dW_s, \quad s \in [t, T], \quad \Gamma_{t,t} = 1. \]

Moreover, a solution \( \hat{\mu} \) of the optimal control problem (8)-(9) exists.

The following lemma provides a useful estimate.

Lemma 3.4. A constant \( C \) exists, such that for all \( 0 \leq t < s \leq T \) and all \( \{\Gamma_{t,s}\} \) that satisfy (2), we have

\[ ||\Gamma_{t,s} - 1||^2 \leq C(s-t). \]  

Proof. Using Itô isometry, we obtain the chain of relations

\[ ||\Gamma_{t,s} - 1||^2 = \int_t^s ||\mu_r \Gamma_{t,r}||^2 \, dr \leq \int_t^s ||\mu_r||^2 ||\Gamma_{t,r}||^2 \, dr \leq \int_t^s ||\mu_r||^2 (1 + ||\Gamma_{t,r} - 1||^2) \, dr. \]  

If \( K \) is a uniform upper bound on the norm of the subgradients of \( g(r,0) \) we deduce that \( ||\Gamma_{t,r} - 1||^2 \leq \psi_r, r \in [t, T], \) where \( \psi \) satisfies the ODE: \( \frac{d\psi_r}{dr} = K^2(1 + \psi_r) \), with \( \psi_t = 0 \). Consequently,

\[ ||\Gamma_{t,r} - 1||^2 \leq \psi_r = e^{K^2(r-t)} - 1. \]

The convexity of the exponential function yields the postulated bound.  \( \square \)
The dual representation theorem allows us to transform the risk evaluation problem to a stochastic control problem. Our objective now is to approximate the evaluation of risk on a short interval $\Delta$ so that it reduces the functional optimization problem to vector optimization. To proceed, we have to first investigate the corresponding maximum principle of the control problem (8).

4 Stochastic Maximum Principle

In this section, we decipher the optimality conditions of the stochastic control problem (8)–(9). Since only the process $\{\Gamma_{t,s}\}_{s \in [t,T]}$ is controlled, the analysis is rather standard. For completeness, we repeat some important steps here.

Suppose $\hat{\mu}$ is the optimal control; then, for any $\mu \in \mathcal{A}$ and $0 \leq \alpha \leq 1$, we can form a perturbed control function

$$
\mu^\alpha = \hat{\mu} + \alpha(\mu - \hat{\mu}).
$$

It is still an element of $\mathcal{A}$, due to the convexity of the sets $A_s$. The processes $\hat{\Gamma}$, $\Gamma$ and $\Gamma^\alpha$ are the state processes under the controls $\hat{\mu}$, $\mu$, and $\mu^\alpha$, respectively.

We linearize the state equation (9) about $\hat{\Gamma}$ to get, for $s \in [t,T]$,

$$
d\eta_s^\mu = [\hat{\mu}_s \eta_s^\mu + \hat{\Gamma}_t, s(\mu_s - \hat{\mu}_s)] dW_s, \quad \eta_t^\mu = 0. \tag{13}
$$

It is evident that this equation has a unique strong solution. Denote

$$
h_s^\alpha = \frac{1}{\alpha} [\Gamma^\alpha_{t,s} - \hat{\Gamma}_{t,s}] - \eta_s^\mu, \quad s \in [0,T].
$$

The following result justifies the usefulness of the linearized equation (13).

Lemma 4.1.

$$
\lim_{\alpha \to 0} \sup_{0 \leq s \leq T} \|h_s^\alpha\|^2 = 0. \tag{14}
$$

Proof. We first prove that

$$
\lim_{\alpha \to 0} \sup_{t \leq s \leq T} \|\Gamma^\alpha_{t,s} - \hat{\Gamma}_{t,s}\|^2 = 0. \tag{15}
$$

We have

$$
d(\Gamma^\alpha_{t,s} - \hat{\Gamma}_{t,s}) = (\mu^\alpha_s \Gamma^\alpha_{t,s} - \hat{\mu}_s \hat{\Gamma}_{t,s} + \mu^\alpha_s (\Gamma^\alpha_{t,s} - \hat{\Gamma}_{t,s})) dW_s. \tag{16}
$$

By Itô isometry,

$$
\|\Gamma^\alpha_{t,r} - \hat{\Gamma}_{t,r}\|^2 = \int_t^r \|\mu^\alpha_s (\Gamma^\alpha_{t,s} - \hat{\Gamma}_{t,s})\|^2 ds
\leq 2 \int_t^r \|\mu^\alpha_s (\Gamma^\alpha_{t,s} - \hat{\Gamma}_{t,s})\|^2 ds + 2 \int_t^r \|\mu^\alpha_s (\Gamma^\alpha_{t,s} - \hat{\Gamma}_{t,s})\|^2 ds
\leq 2 \int_t^r \|\mu^\alpha_s (\Gamma^\alpha_{t,s} - \hat{\Gamma}_{t,s})\|^2 ds + K \int_t^r \|\Gamma^\alpha_{t,s} - \hat{\Gamma}_{t,s}\|^2 ds,
$$

6
where $K$ is a constant. Since the first integral on the right hand side converges to 0, as $\alpha \to 0$, the Gronwall inequality yields (15).

We can now prove (14). Combining (16) and (13), we obtain the stochastic differential equation for $h^\alpha$:

$$dh^\alpha_s = \left\{ \frac{1}{\alpha} \left[ (\hat{\mu}_s + (\mu_s - \hat{\mu}_s)) \Gamma^\alpha_{t,s} - \hat{\mu}_s \hat{\Gamma}_{t,s} - \hat{\mu}_s \hat{\Gamma}_s^\alpha - \hat{\mu}_s \right] \right\} dW_s$$

$$= \left\{ \frac{1}{\alpha} \hat{\mu}_s \Gamma^\alpha_{t,s} - \hat{\mu}_s \right\} dW_s$$

$$= \left\{ \hat{\mu}_s h^\alpha_s + (\mu_s - \hat{\mu}_s) \Gamma^\alpha_{t,s} - \hat{\mu}_s \right\} dW_s.$$  

Since the processes $\{\hat{\mu}_s\}$ and $\{\mu_s\}$ are bounded, Itô isometry yields again

$$\|h^\alpha_r\|^2 \leq K \int_r^T \|h^\alpha_s\|^2 ds + K \int_r^T \|\Gamma^\alpha_{t,s} - \hat{\Gamma}_{t,s}\|^2 ds,$$

where $K$ is constant. By the Gronwall inequality, using (15), we get the desired result. □

The convergence result above directly leads to the following variational inequality.

**Lemma 4.2.** For any $\mu \in \mathcal{A}$ we have

$$\mathbb{E}[\xi_T \eta^\mu_T] \leq 0.$$  

**Proof.** Since $\hat{\mu}$ is the optimal control,

$$\mathbb{E}[\xi_T (\Gamma^\alpha_{t,T} - \hat{\Gamma}_{t,T})] \leq 0.$$  

Lemma 4.1 leads to

$$\lim_{\alpha \to 0} \mathbb{E}\left[ \xi_T \frac{1}{\alpha} (\Gamma^\alpha_{t,T} - \hat{\Gamma}_{t,T}) \right] = \mathbb{E}[\xi_T \eta^\mu_T] \leq 0,$$

as required. □

We now express the expected value in (17) as an integral, to obtain a pointwise variational inequality (the maximum principle). To this end, we introduce the following backward stochastic differential equation (the adjoint equation):

$$dp_s = -k_s \hat{\mu}_s ds + k_s dW_s, \quad p_T = \xi_T, \quad s \in [t, T],$$  

with $\xi_T = \Phi(X^\mu_T)$. By construction, $\mathbb{E}[\xi_T \eta^\mu_T] = \mathbb{E}[\hat{\mu}_T \eta^\mu_T]$. Applying the Itô formula to the product process $p_s \eta^\mu_s$, we obtain

$$d(p_s \eta^\mu_s) = \left( k_s \hat{\mu}_s + \hat{\mu}_s \right) \left( \hat{\mu}_s \eta^\mu_s + \hat{\mu}_s \right) dW_s + k_s \hat{\mu}_s (\mu_s - \hat{\mu}_s) ds.$$  

If follows that

$$\mathbb{E}[\xi_T \eta^\mu_T] = \mathbb{E}\left[ \int_t^T k_s \hat{\mu}_s (\mu_s - \hat{\mu}_s) ds \right].$$  

(19)
We can summarize our derivations in the following version of the maximum principle. We define the Hamiltonian $H : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$:

$$H(\gamma, \nu, \kappa) = k \gamma \nu.$$  

**Theorem 4.3.** For almost all $s \in [t, T]$, with probability 1,

$$H(\hat{\Gamma}_{t,s}, \hat{\mu}_s, k_s) = \max_{\nu \in \mathcal{A}} H(\hat{\Gamma}_{t,s}, \nu, k_s).$$

**Proof.** For any $\mu \in \mathcal{A}$, we define the set

$$\mathcal{G} = \{ (\omega, s) \in \Omega \times [t, T] : k_s \hat{\Gamma}_{t,s}(\mu_s - \hat{\mu}_s) > 0 \}.$$  

We construct a new control $\mu^* \in \mathcal{A}$:

$$\mu^*_s = \begin{cases} 
\mu_s, & (\omega, s) \in \mathcal{G}, \\
\hat{\mu}_s, & \text{otherwise}.
\end{cases}$$

The measurability and adaptedness of $\mu^*$ can be easily verified. It follows from (17) and (19) that

$$\mathbb{E} \left[ \int_t^T k_s \hat{\Gamma}_{t,s}(\mu^*_s - \hat{\mu}_s) ds \right] \leq 0.$$  

By the construction of $\mu^*$,

$$\iint_\mathcal{G} k_s \hat{\Gamma}_{t,s}(\mu^*_s - \hat{\mu}_s) ds \mathbb{P}(d\omega) \leq 0.$$  

Since the integrand is positive on $\mathcal{G}$, the product measure of $\mathcal{G}$ must be zero. $\square$

### 5 Regularity of the Integrand in the Adjoint Equation

We also make a stronger assumption about the drift and diffusion terms of the forward system, and about the terminal cost function.

**Assumption 5.1.** The functions $b, \sigma, \Phi \in C^2_{b}(\mathbb{R}^n)$, and

$$|\sigma(s, x) - \sigma(t, x)| \leq C|s - t|^\frac{1}{2}$$

for all $s, t \in [0, T]$ and all $x \in \mathbb{R}^n$.

Consider the forward–backward system (7) and (18). The key to our further estimates is the following regularity result about the integrand $\{k_t\}$ in the adjoint equation (18).

**Lemma 5.2.** A constant $C$ exists, such that for all $0 \leq t < s \leq T$, and all $x \in \mathbb{R}^n$,

$$||k_s - k_t|| \leq C|s - t|^\frac{1}{2}.$$  (20)
Proof. The quasilinear parabolic partial differential equation corresponding to the forward–backward system (7)–(18) has the following form (see, e.g. [26, sec. 8.2]),

\[ u_t(t, x) + u_x(t, x) b(t, x) + \frac{1}{2} \text{tr}(u_{xx}(t, x) \sigma(t, x) \sigma^T(t, x)) + u_x(t, x) \sigma(t, x) \mu_t = 0, \]

with the boundary condition \( u(T, x) = \Phi(x) \). Due to the linearity of the driver of (18), the terms with \( u_x \) can be collapsed. Then the equation (21) is the Feynman-Kac equation for

\[ u(s, \tilde{X}_s^l) = \mathbb{E}[\Phi(\tilde{X}_T^l)|F_s], \quad s \in [t, T], \]

where

\[ d\tilde{X}_s^l = [b(s, \tilde{X}_s^l) + \sigma(s, \tilde{X}_s^l) \mu_s] \, ds + \sigma(s, \tilde{X}_s^l) \, dW_s, \quad s \in [t, T], \quad \tilde{X}_t^l = x. \]

Ma and Yong [26] consider it on page 195 in formula (1.12). Under Assumption 5.1, the equation (21) has a classical solution \( u(\cdot, \cdot) \), and then the process

\[ k_s = u_x(s, \tilde{X}_s^l) \sigma(s, \tilde{X}_s^l), \quad s \in [t, T], \]

is the solution of the adjoint equation (18). By [26 Prop. 8.1.1], a process \( H \in \mathcal{H}^{2,n \times n}[t, T] \) exists, such that the process \( G_s = u_x(s, \tilde{X}_s^l) \) satisfies the following \( n \)-dimensional BSDE:

\[ G_t = \Phi_x(\tilde{X}_T^l) + \int_t^T \left[ (b(s, \tilde{X}_s^l) + \sigma(s, \tilde{X}_s^l) \mu_s) G_s + \sigma_x(s, \tilde{X}_s^l) H_s \right] ds - \int_t^T H_s \, dW_s. \]

We obtain the following estimate:

\[
\|k_s - k_t\| = \|u_x(s, \tilde{X}_s^l) \sigma(s, \tilde{X}_s^l) - u_x(t, x) \sigma(t, x)\| \\
\leq \|u_x(s, \tilde{X}_s^l) \sigma(s, \tilde{X}_s^l) - u_x(s, \tilde{X}_s^l) \sigma(t, x) + u_x(s, \tilde{X}_s^l) \sigma(t, x) - u_x(t, x) \sigma(t, x)\| \\
\leq \|u_x(s, \tilde{X}_s^l)\| \|\sigma(s, \tilde{X}_s^l) - \sigma(t, x)\| + \|\sigma(t, x)\| \|G_s - G_t\|. \tag{25}
\]

The first term on the right hand side of (25) can be bounded with the help of Assumption 5.1

\[
\|\sigma(s, \tilde{X}_s^l) - \sigma(t, x)\| \leq \|\sigma(s, \tilde{X}_s^l) - \sigma(s, x)\| + \|\sigma(s, x) - \sigma(t, x)\| \\
\leq C_1 |s - t|^\frac{1}{2} + C_2 \|\tilde{X}_s - x\| \leq C_3 |s - t|^\frac{1}{2},
\]

where \( C_1, C_2, \) and \( C_3 \) are some universal constants. It follows from (24) that

\[ G_T - G_t = -\int_t^T \left[ (b(s, \tilde{X}_s) + \sigma_x(s, \tilde{X}_s) \mu_s) G_s + \sigma_x(s, \tilde{X}_s) H_s \right] ds + \int_t^T H_s \, dW_s. \]

Therefore, the second term on the right hand side of (25) can be bounded as \( \|G_s - G_t\|^2 \leq C_4 |s - t| \).

Integrating these estimates into (25), we obtain (20) with a universal constant \( C \). \qed
6 Error Estimates for Constant Controls on Small Intervals

To reduce an infinite dimensional control problem to a finite dimensional vector optimization, we partition the interval $[0, T]$ into $N$ short pieces of length $\Delta = T/N$, and develop a scheme for evaluating the risk measure (3) by using constant dual controls on each piece. We denote $t_i = i\Delta$, for $i = 0, 1, \ldots, N$.

For simplicity, in addition to Assumption 3.1, we assume that the driver $g$ does not depend on time, and thus all sets $A_t = \partial g(0)$ are the same. We denote them with the symbol $A$; as we shall see later on this is not a major restriction.

If the system's state at time $t_i$ is $x$, then the value of the risk measure (5) is then the optimal value of problem (8). By dynamic programming,

$$v(t_i, x) = \rho_{t_i,t_{i+1}}^x[v(t_{i+1}, X_{t_{i+1}}^{t_i,x})].$$

The risk measure $\rho_{t_i,t_{i+1}}^x[\cdot]$ is defined by problem (8), with terminal time $t_{i+1}$ and the function $\Phi(\cdot)$ replaced by $v(t_{i+1}, \cdot)$. Equivalently, it is equal to $Y_{t_i}^{t_i,x}$, in the corresponding forward–backward system on the interval $[t_i, t_{i+1}]$:

$$dX_i^{t_i,x} = b(s, X_i^{t_i,x}) \, ds + \sigma(s, X_i^{t_i,x}) \, dW_s, \quad X_{t_i}^{t_i,x} = x,$$
$$-dY_i^{t_i,x} = g(Z_i^{t_i,x}) \, ds - Z_i^{t_i,x} \, dW_s, \quad Y_{t_{i+1}}^{t_i,x} = v(t_{i+1}, X_{t_{i+1}}^{t_i,x}).$$

Under Assumption 5.1 the function $v(\cdot, \cdot)$ is the classical solution of the associated Hamilton–Jacobi–Bellman equation:

$$v_t(t, x) + v_x(t, x)b(t, x) + \frac{1}{2} \operatorname{tr}(v_{xx}(t, x)\sigma(t, x)\sigma^T(t, x)) + g(v_x(t, x)\sigma(t, x)) = 0,$$

with the terminal condition $v(T, x) = \Phi(x)$.

Suppose we use a constant control in the interval $[t_i, t_{i+1}]$:

$$\mu_s := \hat{\mu}_{t_i} = \arg \max_{\mu \in A} k_i \nu, \quad \forall s \in [t_i, t_{i+1}],$$

where $(p, k)$ solve the adjoint equation corresponding to (18):

$$dp_s = -k_s \hat{p}_s \, ds + k_s \, dW_s, \quad s \in [t_i, t_{i+1}], \quad p_{t_{i+1}} = v(t_{i+1}, X_{t_{i+1}}^{t_i,x}).$$

We still use $\hat{\Gamma}$ to denote the state evolution under the optimal control, while $\Gamma$ is the process under control $\mu$ defined in (29). It is well-known that the value function $v(\cdot, \cdot)$ of the system (26)–(27) is in $C^2([0, T] \times \mathbb{R}^n)$; see, for example, [42 Thm. 2.4.1]. Therefore, the bounds developed in section 5 remain valid for the processes $(p, k)$ in (30).

Our objective is to show that a constant $C$ exists, independent of $x$, $N$, and $i$, such that the approximation error on the $i$th interval can be bounded as follows:

$$0 \leq \mathbb{E}[v(t_{i+1}, X_{t_{i+1}}^{t_i,x})] \leq C\Delta^\frac{3}{2} + \frac{1}{2} \nu b(t, x).$$

The fact that we do not know $k_i$ will not be essential; later, we shall generate even better constant controls by discrete-time dynamic programming.

We can now derive some useful estimates for the constant control function (29).
Lemma 6.1. A constant $C$ exists, such that for all $x$, $N$ and $i$

$$
\mathbb{E}\left[ v(t_{i+1}, X_{t_{i+1}}^{x,i}) \int_{t_i}^{t_{i+1}} (\hat{\mu}_s - \mu_s) \hat{\Gamma}_{t_{i},s} \, dW_s \right] \leq C \Delta^2. \tag{32}
$$

Proof. From (30) we get:

$$
v(t_{i+1}, X_{t_{i+1}}^{x,i}) = p_{t_i} - \int_{t_i}^{t_{i+1}} k_s \hat{\mu}_s \, ds + \int_{t_i}^{t_{i+1}} k_s \, dW_s. \tag{33}
$$

Then the left hand side of (32) can be written as follows:

$$
\begin{align*}
\mathbb{E}\left[ p_{t_i} - \int_{t_i}^{t_{i+1}} k_s \hat{\mu}_s \, ds + \int_{t_i}^{t_{i+1}} k_s \, dW_s \right] \\
&= -\mathbb{E}\left[ \int_{t_i}^{t_{i+1}} k_s \hat{\mu}_s \, ds \int_{t_i}^{t_{i+1}} (\hat{\mu}_s - \mu_s) \hat{\Gamma}_{t_{i},s} \, dW_s \right] \\
&\quad + \mathbb{E}\left[ \int_{t_i}^{t_{i+1}} k_s \, dW_s \int_{t_i}^{t_{i+1}} (\hat{\mu}_s - \mu_s) \hat{\Gamma}_{t_{i},s} \, dW_s \right]. \tag{34}
\end{align*}
$$

The first term on the right hand side of (34) can be bounded by the Cauchy-Schwarz inequality and the Itô isometry:

$$
\begin{align*}
-\mathbb{E}\left[ \int_{t_i}^{t_{i+1}} k_s \hat{\mu}_s \, ds \int_{t_i}^{t_{i+1}} (\hat{\mu}_s - \mu_s) \hat{\Gamma}_{t_{i},s} \, dW_s \right] \\
&\leq \left( \mathbb{E}\left[ \left( \int_{t_i}^{t_{i+1}} k_s \hat{\mu}_s \, ds \right)^2 \right] \right)^{\frac{1}{2}} \left( \mathbb{E}\left[ \left( \int_{t_i}^{t_{i+1}} (\hat{\mu}_s - \mu_s) \hat{\Gamma}_{t_{i},s} \, dW_s \right)^2 \right] \right)^{\frac{1}{2}} \\
&\leq C_1 \Delta \left( \int_{t_i}^{t_{i+1}} \mathbb{E}\left[ (\hat{\mu}_s - \mu_s)^2 \hat{\Gamma}_{t_{i},s}^2 \right] \, ds \right)^{\frac{1}{2}} \leq C_1 C_2 \Delta^2,
\end{align*}
$$

where $C_1$ and $C_2$ are some constants. The second term on the right hand side of (34) can be evaluated as follows:

$$
\begin{align*}
\mathbb{E}\left[ \int_{t_i}^{t_{i+1}} k_s \, dW_s \int_{t_i}^{t_{i+1}} (\hat{\mu}_s - \mu_s) \hat{\Gamma}_{t_{i},s} \, dW_s \right] &= \mathbb{E}\left[ \int_{t_i}^{t_{i+1}} k_s (\hat{\mu}_s - \mu_s) \hat{\Gamma}_{t_{i},s} \, ds \right] \\
&= \mathbb{E}\left[ \int_{t_i}^{t_{i+1}} (k_s \hat{\mu}_s \hat{\Gamma}_{t_{i},s} - k_s \hat{\mu}_s) \, ds \right] + \mathbb{E}\left[ \int_{t_i}^{t_{i+1}} (k_s - k_s \hat{\Gamma}_{t_{i},s}) \hat{\mu}_s \, ds \right] \\
&= \mathbb{E}\left[ \int_{t_i}^{t_{i+1}} (\sigma_A(k_s \hat{\Gamma}_{t_{i},s}) - \sigma_A(k_s)) \, ds \right] + \mathbb{E}\left[ \int_{t_i}^{t_{i+1}} (k_s - k_s \hat{\Gamma}_{t_{i},s}) \hat{\mu}_s \, ds \right].
\end{align*}
$$

Here, $\sigma_A(z) = \max_{\mu \in A}(z, \mu)$ is the support function of the set $A$.

By Lemma 3.4 a constant $C_3$ exists, such that $||\hat{\Gamma}_{t_{i},s} - 1||^2 \leq C_3 |s - t_i|$. Since the support function is Lipschitz continuous, we can write the following estimate (again, $C_4$ is a sufficiently
large constant)
\[
\mathbb{E}\left[\int_{t_i}^{t_{i+1}} k_t \, dW_t \int_{t_i}^{t_{i+1}} \left(\hat{\mu}_s - \mu_s\right) \, dW_s\right] \\
\leq C_4 \int_{t_i}^{t_{i+1}} \mathbb{E}[|k_s\hat{\Gamma}_{t_i,s} - k_t|] \, ds \\
\leq C_4 \int_{t_i}^{t_{i+1}} \mathbb{E}[|k_s(\hat{\Gamma}_{t_i,s} - 1)| + |k_s - k_t|] \, ds \\
\leq C_4 \left(\int_{t_i}^{t_{i+1}} |\hat{\Gamma}_{t_i,s} - 1|^2 \, ds\right)^{\frac{1}{2}} \left(\int_{t_i}^{t_{i+1}} ||k_s||^2 \, ds\right)^{\frac{1}{2}} + C_4 \left(\int_{t_i}^{t_{i+1}} ||k_s - k_t||^2 \, ds\right)^{\frac{1}{2}} \\
\leq C\Delta^2,
\]
where \(C\) is a sufficiently large constant. In the last step we used Lemma 5.2. \(\square\)

We also have the estimate below:

**Lemma 6.2.** A constant \(C\) exists, such that for all \(x, N,\) and \(i\)
\[
\mathbb{E}\left[\mathcal{V}(t_{i+1}, X_{t_{i+1}}^{i,x}) \int_{t_i}^{t_{i+1}} (\hat{\Gamma}_{t_i,s} - \Gamma_{t_i,s}) \mu_s \, dW_s\right] \leq C\Delta^2. \quad (35)
\]

**Proof.** We proceed as in the proof of the previous lemma. We use (33) and express the left hand side of (35) as follows:

\[
\mathbb{E}\left[\mathcal{V}(t_{i+1}, X_{t_{i+1}}^{i,x}) \int_{t_i}^{t_{i+1}} (\hat{\Gamma}_{t_i,s} - \Gamma_{t_i,s}) \mu_s \, dW_s\right] \\
= -\mathbb{E}\left[\int_{t_i}^{t_{i+1}} k_t \mu_t \, dt \int_{t_i}^{t_{i+1}} (\hat{\Gamma}_{t_i,s} - \Gamma_{t_i,s}) \mu_s \, dW_s\right] \quad (36) \\
+ \mathbb{E}\left[\int_{t_i}^{t_{i+1}} k_t \, dW_t \int_{t_i}^{t_{i+1}} (\hat{\Gamma}_{t_i,s} - \Gamma_{t_i,s}) \mu_s \, dW_s\right].
\]

The first term on the right hand side of (36) can be dealt with by the Cauchy-Schwarz inequality and Itô isometry, exactly as before:

\[
\mathbb{E}\left[\left(\int_{t_i}^{t_{i+1}} k_s \mu_s \, ds\right) \int_{t_i}^{t_{i+1}} (\hat{\Gamma}_{t_i,s} - \Gamma_{t_i,s}) \mu_s \, dW_s\right] \\
\leq \left(\mathbb{E}\left[\int_{t_i}^{t_{i+1}} k_s \mu_s \, ds\right]^2\right)^{\frac{1}{2}} \left(\int_{t_i}^{t_{i+1}} \mathbb{E}[\hat{\Gamma}_{t_i,s}^2] \, ds\right)^{\frac{1}{2}} \leq C_1\Delta^2.
\]

To estimate the second term, consider two controlled state processes:

\[
\hat{\Gamma}_{t_i,s} = 1 + \int_{t_i}^{s} \hat{\mu}_t \, dW_t, \\
\Gamma_{t_i,s} = 1 + \int_{t_i}^{s} \mu_t \, dW_t.
\]
Taking the difference yields,
\[
\hat{R}_{t_i,t} - R_{t_i,t} = \int_{t_i}^{t} (\hat{\mu}_s - \mu_s) \hat{R}_{t_i,s} dW_s + \int_{t_i}^{t} (\hat{R}_{t_i,s} - R_{t_i,s}) \mu_s dW_s. \tag{37}
\]
By Itô isometry,
\[
\mathbb{E}[ (\hat{R}_{t_i,t} - R_{t_i,t})^2 ] \leq C_2 |t - t_i|.
\]
Thus, we can write the bound:
\[
\left| \mathbb{E} \left[ \int_{t_i}^{t} k_s dW_s \int_{t_i}^{t} (\hat{R}_{t_i,s} - R_{t_i,s}) \mu_s dW_s \right] \right| \\
= \left| \mathbb{E} \left[ \int_{t_i}^{t} k_s (\hat{R}_{t_i,s} - R_{t_i,s}) \mu_s ds \right] \right| \\
\leq C_1 \left( \int_{t_i}^{t} \mathbb{E}[k_s^2] ds \right)^{\frac{1}{2}} \left( \int_{t_i}^{t} \mathbb{E}[ (\hat{R}_{t_i,s} - R_{t_i,s})^2 ] ds \right)^{\frac{1}{2}} \leq C \Delta^{\frac{3}{2}}.
\]
where \(C\) is a sufficiently large constant. \(\square\)

We can now compare the value of the functional (8) with the value achieved by a constant control \(\mu\).

**Theorem 6.3.** Suppose Assumptions 2.1, 3.1, and 5.1 are satisfied. Then a constant \(C\) exists, independent on \(x, N\) and \(i\), such that inequality (31) holds.

**Proof.** Using (37), we obtain
\[
\mathbb{E}[v(t_{i+1},X_{t_{i+1}}^i)(\hat{R}_{t_i,t_{i+1}} - R_{t_i,t_{i+1}})] \\
= \mathbb{E} \left[ v(t_{i+1},X_{t_{i+1}}^i) \int_{t_i}^{t_{i+1}} (\hat{\mu}_s - \mu_s) \hat{R}_{t_i,s} dW_s \right] \\
+ \mathbb{E} \left[ v(t_{i+1},X_{t_{i+1}}^i) \int_{t_i}^{t_{i+1}} (\hat{R}_{t_i,s} - R_{t_i,s}) \mu_s dW_s \right].
\]
Combining the estimates from Lemmas 6.1 and 6.2 we obtain the postulated result. \(\square\)

An even smaller error than (31) can be achieved by choosing the best constant control in the interval \([t_i, t_{i+1}]\). For a constant \(\mu_t = \nu\), where \(\nu \in A\), the dual state equation (9) has a closed-form solution, the exponential martingale:
\[
R_{t_i,t} = \exp \left( \nu(W_t - W_{t_i}) - \frac{t - t_i}{2} |\nu|^2 \right).
\]
It follows that an \(O(\Delta^{\frac{3}{2}})\) approximation of the risk measure can be obtained by solving the following simple vector optimization problem:
\[
\tilde{\nu}_{t_{i+1}^i} = \max_{\nu \in A} \mathbb{E} [v(t_{i+1},X_{t_{i+1}}^i) \exp \left( \nu(W_{t_{i+1}} - W_{t_i}) - \frac{\Delta}{2} |\nu|^2 \right)]. \tag{38}
\]
Opposite to (29), we do not need to know \( k_i \) to solve this problem.

By Theorem 6.3, we have:

\[
\nu(t_i, x) - \tilde{\rho}^x_{t_i, t_{i+1}} \left[ \nu(t_{i+1}, X^{i,x}_{t_{i+1}}) \right] \leq C \Delta^{3/2}.
\]  

(39)

By construction, the approximating measure of risk \( \tilde{\rho}^x_{t_i, t_{i+1}} \cdot \) is coherent and satisfies all properties (i)-(iii) of Theorem 3.2.

7 Discrete-Time Approximations by Dynamic Programming

The time-consistency of dynamic risk measure leads to the nested form below:

\[
\rho_{0,T} \left[ \Phi(X_T) \right] = \rho_{t_0, t_1} \left[ \rho_{t_1, t_2} \left[ \ldots \rho_{t_{N-2}, t_{N-1}} \left[ \rho_{t_{N-1}, t_N} \left[ \Phi(X_T) \right] \right] \ldots \right] \right].
\]  

(40)

By using optimal constant dual controls on each interval \([t_i, t_{i+1})\), we may approximate this composition by dynamic programming. For \( i = N \) we define \( \bar{\nu}(x) = \Phi(x) \). Then, for \( i = N - 1, N - 2, \ldots, 0 \), and for \( x \in \mathbb{R}^n \), we restart the diffusion (2) from \( x \) at time \( t_i \) as in (26). Having obtained \( X^{i,x}_{t_{i+1}} \), we can calculate the approximate risk measure (38) on the interval \([t_i, t_{i+1}]:\)

\[
\tilde{\nu}(x) = \tilde{\rho}^x_{t_i, t_{i+1}} \left[ \bar{\nu}_{i+1} (X^{i,x}_{t_{i+1}}) \right] = \max_{\nu \in \mathcal{A}} \mathbb{E} \left[ \tilde{\nu}_{i+1} (X^{i,x}_{t_{i+1}}) \exp \left( \nu(W_{t_{i+1}} - W_{t_i}) - \frac{1}{2} \nu^2 \right) \right].
\]  

(41)

Theorem 7.1. Suppose Assumptions 2.2, 3.1, and 5.1 are satisfied. Then a constant \( C \) exists, such that for all \( N \) and \( x \) we have:

\[
\nu(t_i, x) - \tilde{\nu}(x) \leq C(N - i) \Delta^{3/2}, \quad i = 0, 1, \ldots, N.
\]  

(42)

In particular, \( \nu(0, x) - \tilde{\nu}_0(x) \leq C T \Delta^{3/2} \).

Proof. The result follows by backward induction. It is obviously true for \( i = N \). If it is true for \( i + 1 \), we can easily verify it for \( i \). By the translation property of \( \tilde{\rho}^x_{t_i, t_{i+1}} \cdot \) and (39) we obtain:

\[
\nu(t_i, x) - \tilde{\nu}_i(x) = \nu(t_i, x) - \tilde{\rho}^x_{t_i, t_{i+1}} \left[ \bar{\nu}_{i+1} (X^{i,x}_{t_{i+1}}) \right]
\]

\[
\leq \nu(t_i, x) - \tilde{\rho}^x_{t_i, t_{i+1}} \left[ \nu(t_{i+1}, X^{i,x}_{t_{i+1}}) \right] + C \Delta^{3/2} \leq C(N - i) \Delta^{3/2},
\]

as required.

In practice, the forward process (7) is simulated in an approximate way, for example, by Euler’s method:

\[
X_{t_{i+1}}^{i,x} = x + b(t_i, x) \Delta + \sigma(t_i, x) \Delta W; \Delta W \sim N(0, \sqrt{\Delta} \delta).
\]  

(43)

It is well known that for small \( \Delta \), the error of this Euler scheme is \( O(\Delta^{1/2}) \). Since \( X_{t_{i+1}}^{i,x} \) is a normal random vector, streamlined calculation of the risk measure is possible. Denoting by \( \mathcal{N} \) a standard normal random vector with independent components, we can simplify the calculation of the risk measure in (41) as follows:

\[
\tilde{\nu}_i(x) \approx \max_{\nu \in \mathcal{A}} \mathbb{E} \left[ \bar{\nu}_{i+1} (x + b(t_i, x) \Delta + \sigma(t_i, x) \mathcal{N} \Delta \mathcal{N} - \frac{1}{2} \nu^2) \right].
\]  

(44)

Observe that the same normal random vector \( \mathcal{N} \) is used in both terms of this expression.
Figure 1: The two stage procedure: the $x$-axis is time, the $y$-axis is the stock price.

**Remark 7.2.** Our earlier assumption of time-homogeneity of $g$ is barely a restriction after discretization, because $g$ can be piecewise $\alpha$-Hölder continuous between the grid points. As long as the risk aversion does not change abruptly, the numerical method developed can be easily adapted to the case of a time-dependent driver.

### 8 Application to Risk Management

After the credit crunch, the management of risk is an increasingly important function of any financial institution. The primary goal is to have sufficient capital reserves against potential losses in the future. Such risk management is divided into two stages: *scenario generation* and *portfolio re-pricing*.

Scenario generation refers to the construction of sample paths over a given time horizon. This is also called the *outer stage*, where Monte Carlo simulation is used to generate paths of systems governed by stochastic differential equations. Repricing of portfolio amounts to the computation of the portfolio value at a certain time horizon. The portfolio may consist of derivative securities with nonlinear payoffs that, in conjunction with financial models, require Monte Carlo simulation for this *inner stage* as well (see figure 8). Thus, in real world application, the risk measurement requires calculation of a two-level nested Monte Carlo simulation. Lastly, the risk evaluation is done by a risk measure $\rho$, a functional that maps future random exposure to a real number. Examples of risk measure can be *value at risk*, *conditional value at risk*, *probability of loss*, etc. Such evaluation structure leads to a challenging computation task. Especially, the inner step simulation has to be done for each scenario generated in the first stage. A lot of research has been done to address the computation issue, to name a few, Gordy and Juneja [17], Lee and Glynn [20], Lesnevski *et al.* [21, 22] and Rockafellar and Uryasev [33].

The common objective is measuring the risk of a portfolio of assets at the risk horizon $t = \tau$,
while standing at time 0. We denote the current wealth, i.e., the net present value of portfolio, by \( \mathcal{F}_0 \)-measurable random variable \( X_0 \) (a known quantity). At time \( \tau \), the value of the portfolio is then a \( \mathcal{F}_\tau \)-measurable random variable \( X_\tau \). In almost all real-world applications, we assume a probabilistic model of the evolution of uncertainty between 0 and \( \tau \), for example, a stochastic differential equation. Suppose the outcome \( \Omega \) is a set of possible future scenarios, each of which incorporates sufficient information so as to determine all asset prices at the risk horizon. Then, in each scenario \( \omega \in \Omega \), the portfolio has value \( X_\tau(\omega) \). The \textit{mark-to-market (MTM)} loss of this portfolio at time \( \tau \) in scenario \( \hat{\omega} \in \Omega \) is given by

\[
L(\hat{\omega}) = X_0 - X_\tau(\hat{\omega}).
\]  

(45)

The usual risk measurement is static in the sense that it evaluates the risk of exposure at risk horizon only at current time 0. If one wants to check the risk at an intermediate point of the risk horizon, the whole model has to be re-run. Given the computation efforts of nested simulation, it can be very burdensome. In addition, re-simulating can cause inconsistency of risk evaluation, which is also undesired.

In our work, we use dynamic risk measure and the approximation algorithm (41) proposed in section 7 to measure the risk associated with the portfolio dynamically. In this way, the risk can be monitored continuously and consistently, in other words, for any time instant \( t \) within the risk horizon, the evolution of risk can be traced.

To better illustrate the dynamic risk evaluation, let us consider a specific example, a portfolio consisting of a long position in a single put vanilla option, which expires at time \( T \) and has strike price \( K \). The underlying stock, say \( ABC \), follows a geometric Brownian motion with an initial price \( S_0 \), mean \( \mu \) and volatility \( \sigma \), in other words, under the real-world probability measure \( \mathbb{P} \), its dynamics is given by the following SDE:

\[
\frac{dS_t}{S_t} = \mu \ dt + \sigma \ dW_t, \quad t \in [0, T].
\]  

(46)

Here, \( \{W_t\} \) is \( \mathbb{P} \)-Brownian motion. Let us also set a flat interest rate level \( r \); therefore, under the risk-neutral pricing framework, we have the stock dynamics:

\[
\frac{dS_t}{S_t} = r \ dt + \sigma \ d\bar{W}_t,
\]  

(47)

where \( \bar{W}_t \) is a \( \mathbb{Q} \)-Brownian motion. With these specifications, the initial value of the put can be easily calculated by the \textit{Black-Scholes (BS)} formula, which yields

\[
\mathcal{P}(0, S_0) := BS(0, S_0, \sigma, K, T) = S_0N(d_+(T, S_0)) - KD(0, T)N(d_-(T, S_0));
\]

here \( N \) stands for the cumulative distribution function of normal distribution and

\[
d_+(\tau, x) = \frac{1}{\sigma \sqrt{\tau}} \left[ \ln \frac{x}{K} + (r + \frac{1}{2} \sigma^2)\tau \right],
\]

\[
d_-(\tau, x) = \frac{1}{\sigma \sqrt{\tau}} \left[ \ln \frac{x}{K} + (r - \frac{1}{2} \sigma^2)\tau \right].
\]
Let us fix a risk horizon $\tau$, denote the price of at the risk horizon as $S_\tau(\omega)$. Then, the exposure (MTM) at time $\tau$ is the difference of initial put price $\mathcal{P}(0, S_0)$ and the risk-neutral price of the option at time $\tau$, i.e.,

$$\Phi(S_\tau(\omega)) := \mathcal{P}(0, S_0) - \mathbb{E}^Q_{\tau}\left[(S_\tau - K)^+ \bigg| S_\tau(\omega)\right].$$

(48)

Here, to get $S_\tau(\omega)$, we have to simulate the path of stock under the real-world measure, i.e., according to (46). Then, to compute the right-hand side, we only need to work out the second term; again, it can be computed analytically by the BS formula. It is well known that the loss function $\Phi(\cdot)$ is Lipschitz in the state. We are now in the situation to apply a dynamic risk measure,

$$\rho^g_t[\Phi(S_\tau)] := Y_t, \text{ where } Y_t = \Phi(S_\tau) + \int_t^\tau g(s, Z_s) \, ds - \int_t^\tau Z_s \, dW_s, \, t \in [0, \tau],$$

(49)

which enables us to view the risk at any time $t \in [0, \tau]$.

As for the implementation details, instead of using Monte Carlo simulation, we use a tree model for the outer stage. Our risk evaluation algorithm (44) reduces the functional optimization to vector optimization at every time discretization step. However, since backward induction has to be implemented, the state space also needs to be discretized, which makes the tree structure appealing.

**Remark 8.1.** For a diversified portfolio, we shall do multiple outer loop simulations as well as inner loop simulations, because multiple assets are involved. Also, more sophisticated underlying dynamics is possible, such as stochastic volatility, local volatility model, in which cases Monte Carlo simulation has to be performed. At the risk horizon, all exposures should be netted before the $g$-evaluation through our algorithm for BSDE.

We now present the numerical results based on the following data:

$K = 95, \, T = 0.75, \, S_0 = 100, \, \mu = 0.08, \, \sigma = 0.2, \, r = 0.03, \, \tau = 0.2$.

For risk evaluation, we specify the generator to be:

$$g(z) = \gamma \|\max[zN, 0]\|_p, \, N \sim N(0, 1),$$

where the parameters $\gamma > 0$ and $p \geq 1$ model risk aversion. The corresponding set of ambiguity is then:

$$A = \partial g(0) = \{l \in \mathbb{R}^n_+ : |l|_q \leq \gamma k\},$$

with $1/p + 1/q = 1$, and

$$k = \begin{cases} \frac{1}{\sqrt{\pi}}(2m(2m - 1) \cdots (m + 1))^{1/2m}, & \text{if } p = 2m, \\ \left(2^m \sqrt{2\pi m!}\right)^{1/2m}, & \text{if } p = 2m + 1. \end{cases}$$

Fix $p = 2$, at time 0, given $\mathcal{F}_\tau$-measurable loss $\Phi(\cdot)$ in (48). Table 1 and Figure 2 summarize the valuation when varying the step size and risk tolerance $\gamma$. We can observe convergence of the numerical method, as the step size decreases, uniformly over the whole range of $\gamma$. 

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### Table 1: Risk Valuation Convergence Table

| step size | γ = 0.1 | γ = 0.3 | γ = 0.4 | γ = 0.6 | γ = 0.8 | γ = 1.0 |
|-----------|---------|---------|---------|---------|---------|---------|
| 0.4       | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 0.2       | 0.03407 | 0.23907 | 0.34080 | 0.54207 | 0.73957 | 0.93217 |
| 0.1       | 0.06371 | 0.37300 | 0.52628 | 0.82895 | 1.12492 | 1.41239 |
| 0.08      | 0.07086 | 0.40174 | 0.56573 | 0.88956 | 1.20628 | 1.51403 |
| 0.05      | 0.08261 | 0.44695 | 0.62757 | 0.98446 | 1.33922 | 1.67410 |
| 0.04      | 0.08687 | 0.46282 | 0.64924 | 1.01771 | 1.37878 | 1.73064 |
| 0.02      | 0.09622 | 0.49671 | 0.69544 | 1.08872 | 1.47500 | 1.85268 |
| 0.01      | 0.10165 | 0.51579 | 0.72141 | 1.12877 | 1.52971 | 1.92284 |
| 0.008     | 0.10287 | 0.51998 | 0.72712 | 1.13760 | 1.54184 | 1.93852 |
| 0.005     | 0.10485 | 0.52674 | 0.73632 | 1.15187 | 1.56152 | 1.96407 |
| 0.004     | 0.10557 | 0.52919 | 0.73965 | 1.15705 | 1.56869 | 1.97344 |
| 0.002     | 0.10720 | 0.53465 | 0.74709 | 1.16864 | 1.58483 | 1.99465 |
| 0.001     | 0.10822 | 0.53798 | 0.75163 | 1.17554 | 1.59479 | 2.00786 |
| 0.0008    | 0.10845 | 0.53876 | 0.75269 | 1.17740 | 1.59713 | 2.01099 |
| 0.0005    | 0.10886 | 0.54007 | 0.75447 | 1.18021 | 1.60109 | 2.01630 |
| 0.0004    | 0.10901 | 0.54057 | 0.75515 | 1.18127 | 1.60260 | 2.01833 |

### Table 2: Risk Surface Table

| K, σ | 0.1  | 0.3  | 0.5  | 0.6  | 0.7  | 0.8  | 0.9  | 1    |
|------|------|------|------|------|------|------|------|------|
| 70   | -0.0002 | -0.1479 | -0.0901 | 0.0566 | 0.2612 | 0.5101 | 0.7918 | 1.0962 |
| 80   | -0.0041 | -0.0188 | 0.2444 | 0.4645 | 0.7290 | 1.0279 | 1.3517 | 1.6918 |
| 90   | 0.0737 | 0.3661 | 0.7508 | 1.0114 | 1.3099 | 1.6379 | 1.9869 | 2.3489 |
| 100  | 0.7941 | 1.0211 | 1.4004 | 1.6691 | 1.9782 | 2.3177 | 2.6780 | 3.0506 |
| 110  | 2.4089 | 1.8858 | 2.1561 | 2.4081 | 2.7100 | 3.0475 | 3.4087 | 3.7833 |
| 120  | 3.9179 | 2.8641 | 2.9802 | 3.2007 | 3.4842 | 3.8108 | 4.1655 | 4.5361 |
| 130  | 4.6752 | 3.8600 | 3.8387 | 4.0233 | 4.2831 | 4.5938 | 4.9376 | 5.3003 |

Figure 2: Convergence of the discretization method for γ ranging from 0.1 (low graph) to 1.0 (upper graph); the x-axis represents the number of time steps while the y-axis is the risk estimate.
If we vary the underlying asset’s volatility as well as the strike price of the contract, we can construct the \textit{risk surface}. As Table 2 and Figure 3 show, the risk is plotted against different combinations of volatility $\sigma$ and strike price $K$.

As we can observe, if the stock $ABC$ becomes volatile, the risk of the portfolio should increase, because volatility implies uncertainty. Moreover, since the current stock price is 100, as the strike price increases, the risk also goes up, which indicates being engaged in an out-of-money trade is riskier than at-the-money or in-the-money. Thus, the risk surface constructed coincides with intuition, which validates the risk evaluation approximation.
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