On the flat strong discontinuities in incompressible polymeric liquids

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Abstract. We studied the discontinuous stationary solutions for the rheological mesoscopic modified model of Pokrovskii-Vinogradov, which describes the dynamics of liquid polymers. The Rankine-Hugoniot conditions for the model were introduced. We justified the existence of stationary solutions with flat surface of strong discontinuity for the case of constant velocity direction across the discontinuity and for the case with change of direction (rotating discontinuity). The stability of such solutions was also considered. For linearized equations of the model we posed the eigenvalue problem for partial solutions with unlimited grow in time. It was shown that such solutions exist in anisotropic case which means the stationary solutions with flat discontinuity are unstable within the given model.

1. Introduction

Polymeric solutions and melts are the liquids consist of very long and complex shaped macromolecules which provides the viscoelasticity of such liquids. The large number of various mathematical models was introduced to describe the dynamics of polymeric liquids. Globally these models could be divided into two large classes. First are phenomenological models [1, 2] which describe the liquid at the macroscopic scale by general laws derived experimentally. This approach provides the models consists of relatively simple equations but it is difficult to use this kind of models for applications with real polymers. Second class contains the microscopic (or statistical) models. This kind of models describe the shape of separate molecules and obtain average properties of the liquid by the methods of mathematical statistics [3, 4, 5]. Ideally this approach allows the one to take more accurate results since the complex behavior of liquid polymers is caused by the molecular structure. But these models usually require a number of not obvious assumptions and also they consists of pretty bulky and complex mathematical relations.

In this paper we will use so-called generalized rheological model of Pokrovskii-Vinogradov [6, 7], which in some sense combines two approaches described above. This model utilize the mesoscopic approach which describes the movement of single molecule in anisotropic liquid. The properties of this liquid determined phenomenologically with the aim of represent the interactions of this molecule with others molecules of polymer and solvent. For simple flows it was shown that the model provides qualitatively accurate results for real flows of solutions and melts of the polymers with various molecular masses and concentrations.

The differential equations of this model were studied in [8]. The goal of the current paper is to study the discontinuous solutions of the model. The strong discontinuities without the
flow through the discontinuity in liquid polymers were described in [9]. Also there could be the regimes with the flow through the discontinuity (see [10] for example).

This paper is focused on the properties of the strong discontinuity with the flow of polymer liquid through it. The examples of piecewise constant solutions are shown in section 2. In section 3 we introduce the partial solutions with unlimited grow in time for linearized problem. The existence of such solutions means instability of the solutions with flat discontinuity within current model.

2. Model description. Rankine-Hugoniot conditions for polymeric liquid

In [8] there was introduced a dimensionless mathematical model which describes the flow of incompressible viscoelastic polymeric fluid. It consists of the following equations:

\[ \text{div} \mathbf{u} = 0, \]
\[ \frac{d \mathbf{u}}{dt} + \nabla p = \frac{1}{\text{Re}} \text{div} \Pi, \]
\[ \frac{da_{11}}{dt} = -2A_1 u_x - 2a_{12} u_y + L_{11} = 0, \]
\[ \frac{da_{12}}{dt} = -A_1 v_x - A_2 u_y + K_I a_{12} = 0, \]
\[ \frac{da_{22}}{dt} = -2a_{12} v_x - 2A_2 v_y + L_{22} = 0. \]

Here \( t \) is the time, \( \mathbf{u} = (u, v) \) is the velocity vector in Cartesian coordinates \( x, y \); \( p \) is the pressure, \( a_{ij}, i, j = 1, 2 \) are the components of the symmetrical anisotropy tensor \( \Pi \); \( L_{ii} = K_I a_{ii} + \beta (a_{11}^2 + a_{22}^2) \), \( i = 1, 2 \), \( K_I = W^{-1} + \bar{K}I/3 \), \( I = a_{11} + a_{22} \), \( \bar{K} = K_I + \beta I = W^{-1} + kI/3 \), \( k = k - \beta, \bar{k} = k + 3\beta > 0, A_i = W^{-1} + a_{ii}, i = 1, 2; k, \beta \) \( (0 < \beta < 1) \) are the phenomenological parameters of the model (see [6]), \( \text{Re} = \rho u_H l/\eta_0 \) is the Reynolds number, \( \rho \) = const is the density, \( W = \eta_0 u_H l \) is the Weissenberg number, \( \eta_0, \tau_0 \) are the initial values of the shear viscosity and relaxation time, \( l \) is the length scale and \( u_H \) is the velocity scale,

\[ \frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{u}, \nabla). \]

The Rankine-Hugoniot conditions for this model were introduced in [8]:

\[ f_t[u] - [u^2 + p - \alpha_1] + f_y [uv - \alpha_{12}] = 0, \]
\[ f_t[v] - [uv - \alpha_{12}] + f_y [v^2 + p - \alpha_2] = 0, \]
\[ f_t[u^2 + \alpha_1] - [u(u^2 - \alpha_1)] + f_y [v(u^2 + \alpha_1) - 2\alpha_{12} u] = 0, \]
\[ f_t[uv + \alpha_{12}] - [uv^2 - \alpha_1)] + f_y [u(v^2 - \alpha_2)] = 0, \]
\[ f_t[v^2 + \alpha_2] - [u(v^2 + \alpha_2) - 2\alpha_{12} v] + f_y [v(v^2 - \alpha_2)] = 0, \]
\[ - [\Omega^{(x)}] + f_y [\Omega^{(y)}] = 0. \]

Here \( f(t, x, y) = f(t, y) - x = 0 \) is the equation of the discontinuity front in Cartesian coordinates, \( [F] = F - F_{\infty} \), where \( F, F_{\infty} \) are the values of \( F \) to the right \( (\bar{f} \to -0) \) and to the left \( (\bar{f} \to +0) \) of the discontinuity; \( \alpha_i = \alpha_{ii} + \kappa^2 = A_i/\text{Re}, \alpha_{ij} = \alpha_{ij}/\text{Re}, i, j = 1, 2, \kappa^2 = 1/(\text{Re}W) \); \( \Omega^{(x)} = p_x - (\alpha_{11})_x - (\alpha_{12})_y, \Omega^{(y)} = p_y - (\alpha_{12})_x - (\alpha_{22})_y \).

The constant solutions of (2.1) has the following form:

\[ u = \hat{u} = \text{const}, \quad v = \hat{v} = \text{const}, \quad \alpha_{11} = \hat{\alpha}_{11} = \text{const}, \]
\[ \alpha_{12} = \hat{\alpha}_{12} = \text{const}, \quad \alpha_{22} = \hat{\alpha}_{22} = \text{const}, \quad p = \hat{p} = \text{const} \]
2.1. From the conditions

\[ \dot{a}_{22} = \dot{\rho}(\cos \hat{\alpha} - 1), \quad \dot{a}_{11} = -\dot{\rho}(\cos \hat{\alpha} + 1), \]
\[ \dot{a}_{12} = \dot{\rho}\sin \hat{\alpha}, \quad -\frac{\pi}{2} \leq \hat{\alpha} \leq \frac{\pi}{2}, \quad \dot{\rho} = \frac{3k^2}{2k}. \]  

(2.4)

Let us take two constant solutions: \( \hat{u}_\infty = 1 \) (we can achieve it by choosing the \( u_H \)), \( \hat{v}_\infty = \tan \hat{\varphi}_\infty, \dot{a}_{11}\infty, \dot{a}_{12}\infty, \dot{a}_{22}\infty, \rho_\infty \) for \( x < 0 \) and \( \hat{u} > 0, \hat{v} = \hat{u}\tan \hat{\varphi}, \dot{a}_{11}, \dot{a}_{12}, \dot{a}_{22}, \rho \) for \( x > 0 \) separated by discontinuity \( x = 0 \) with conditions (2.2):

\[ [\hat{u}^2 + \dot{\rho} - \dot{a}_1] = \hat{u}^2 + \dot{\rho} - \dot{a}_1 - (1 + \rho_\infty - \dot{a}_\infty) = 0, \]
\[ [\hat{u}\dot{\hat{u}} - \dot{a}_{12}] = 0, \]
\[ [\hat{v}(\hat{u}^2 - \dot{a}_1)] = 0, \]
\[ [\hat{v}(\hat{v}^2 - \dot{a}_1)] = 0, \]
\[ [\hat{u}(\hat{v}^2 + \dot{a}_2) - 2\hat{\varphi}\dot{\hat{a}}_{12}] = 0, \]  

(2.5)

where \( \dot{a}_1 = \dot{\rho}(\hat{t} - \cos \hat{\alpha}) > 0, \dot{a}_2 = \dot{\rho}(\hat{t} + \cos \hat{\alpha}) > 0, \dot{a}_{12} = \dot{\rho}\sin \hat{\alpha}, \dot{a}_{1}\infty = \rho_\infty(\hat{t}_\infty - \cos \hat{\varphi}_\infty) > 0, \]
\[ \dot{a}_{2}\infty = \rho_\infty(\hat{t}_\infty + \cos \hat{\varphi}_\infty) > 0, \dot{a}_{12}\infty = \rho_\infty \sin \hat{\varphi}_\infty, \hat{\varphi}_\infty = \frac{3k^2}{2k}, \hat{t}_\infty = (2k - 3)/3, \]
\[ \hat{t}_\infty = (2k - 3)/3, -\frac{\pi}{2} \leq \hat{\varphi}, \hat{\varphi}_\infty \leq \frac{\pi}{2}. \]

Later we will assume that \( \hat{t}, \hat{t}_\infty > 1 \).

Remark 2.1. From the conditions \( t, \hat{t}_\infty > 1 \) we have the inequalities \( \dot{a}_1\dot{a}_2 - \dot{a}_{12}^2 > 0, \dot{a}_{1}\infty\dot{a}_{2}\infty - \dot{a}_{12}\infty > 0. \) In [8] it was shown that these inequalities along with conditions \( \dot{a}_1 > 0, \dot{a}_{1}\infty > 0, \dot{a}_{2}\infty > 0 \) provide the t-hyperbolicity of (2.1) for given pressure \( p, \rho_\infty \).

From (2.5) it follows that

\[ [\tan \hat{\varphi}](1 - \dot{a}_{12}\infty) = 0, \]  

(2.6)

so we have two possible options: either \( [\tan \hat{\varphi}] = 0 \), which means that velocity vector does not change the direction through the discontinuity, or \( \dot{a}_{12}\infty = 1 \).

Let us consider the first case \( ([\tan \hat{\varphi}] = 0) \). Then \( \hat{v} = \hat{u}\tan \hat{\varphi}_\infty = \hat{u}\hat{v}_\infty \) and from (2.5) we get

\[ \sin \hat{\alpha} = \hat{\lambda}\dot{q}_\infty + \hat{b}, \]
\[ \cos \hat{\alpha} = \hat{\alpha}\cos \hat{\varphi}_\infty + \hat{c}, \]
\[ (1 + \hat{v}_\infty^2)(\hat{u}^3 - 1) = 2\hat{\rho}(\hat{u}\hat{t} - \hat{\lambda}\hat{t}_\infty) + 2\hat{\rho}_\infty(\hat{u} - 1)(\hat{v}_\infty - \hat{\rho}_\infty\dot{q}_\infty), \]  

(2.7)

where \( \hat{\lambda} = \hat{\rho}_\infty/\hat{\rho}, \hat{\alpha} = \hat{\lambda}/\hat{u}, \hat{b} = \hat{v}_\infty(\hat{u}^2 - 1)/\hat{\rho}, \hat{c} = (1 - \hat{v}_\infty^2)/\hat{\rho}_\infty + \hat{t} - \hat{t}_\infty, \dot{q}_\infty = \sin \hat{\varphi}_\infty. \) From first two equations of (2.7) follows the equation for \( \dot{q}_\infty \) and \( \hat{u} \) with parameters \( t, \hat{t}_\infty, \hat{v}_\infty, \hat{\rho}, \hat{\rho}_\infty \):

\[ \sqrt{1 - (\hat{\lambda}\dot{q}_\infty + \hat{b})^2} = \hat{\lambda}\sqrt{1 - \hat{q}_\infty^2} + \hat{c}, \quad |\hat{\lambda}\dot{q}_\infty + \hat{b}| \leq 1. \]  

(2.8)

So the constant solution with strong discontinuity at \( x = 0 \) has to satisfy the last equation from (2.7) and equation (2.8) for given \( t, \hat{t}_\infty, \hat{v}_\infty, \hat{\rho}, \hat{\rho}_\infty \) with all inequalities shown above. The Figures 1 and 2 illustrates some examples of such solutions.

Now let us study the case of \( [\tan \hat{\varphi}] \neq 0 (\dot{a}_{12}\infty = 1) \) in (2.6). We will call this kind of discontinuity as rotating shock similar to the rotating shocks in magnetohydrodynamics [11]. From (2.5) it follows that

\[ \hat{\varphi} = \frac{\rho\sin \hat{\varphi} - \hat{B}}{\dot{a}_1}, \quad \hat{B} = \rho_\infty \sin \hat{\varphi}_\infty - \tan \hat{\varphi}_\infty, \]
\[ \dot{a}_1 = \frac{(\hat{B}^2 + \rho_\infty^2)(\hat{t}_\infty^2 - 1))^2}{C^2}, \quad \hat{C} = \dot{a}_{2}\infty + \hat{B}^2 - \rho_\infty^2 \sin^2 \hat{\varphi}_\infty. \]

The examples of such solutions are also shown on the Figures 3 and 4.
Figure 1. The area of \( \hat{\rho} \) and \( \hat{\rho}_\infty \) for which the solution exists (black area). Here \( \hat{t} = 6, \hat{t}_\infty = 5, \hat{v}_\infty = 0 \).

Figure 2. The area of \( \hat{\rho} \) and \( \hat{\rho}_\infty \) for which the solution exists. Black zone – two solutions, gray zone – one solution. Here \( \hat{t} = 6, \hat{t}_\infty = 5, \hat{v}_\infty = 1 \).

Figure 3. The area of \( \hat{\rho} \) and \( \hat{\rho}_\infty \) for which the rotating shock exists (black area). Here \( \hat{t} = 1.05, \hat{t}_\infty = 1.5, \hat{v}_\infty = 0 \).

Figure 4. The area of \( \hat{\rho} \) and \( \hat{\rho}_\infty \) for which the rotating shock exists. Black zone – two solutions, gray zone – one solution. Here \( \hat{t} = 1.05, \hat{t}_\infty = 1.01, \hat{v}_\infty = 0.5 \).

3. Stability of discontinuous solutions. Eigenvalue problem

Let us show the system of linear equations which is obtained by linearization of foregoing mathematical model on the constant solution (2.3), (2.4) with discontinuity \( x = 0 \). As before, all values with subscript “\( \infty \)” corresponds to the zone \( x < 0 \). We will now stick to the case \( \hat{v} = \hat{v}_\infty = 0 \). The linearized equations are:

\[
U_t + B U_x + C^{(0)} U_y + R^{(0)} U + F^{(0)} = 0,
\]

\[
\Delta_{x,y} \Omega = (\alpha_{22} - \alpha_{11})_{yy} + 2(\alpha_{12})_{xy},
\]
\[ t > 0, \ x \in R_+^1 \cup R_-^1, \ y \in R^1. \] Here:

\[
U(t, x, y) = \begin{cases}
U = \begin{pmatrix} u \\ v \\ \alpha_{11} \\ \alpha_{12} \\ \alpha_{22} \end{pmatrix}, & x \in R_+^1, \\
U_\infty, & x \in R_-^1;
\end{cases}
\]

\[ R_+^1 = \{ x | x > 0 \}, \ R_-^1 = \{ x | x < 0 \}, \ u, v (u_\infty, v_\infty) \] are the small perturbations of the velocity vector; \( \alpha_{ij} (\alpha_{ij\infty}) = a_{ij}/\text{Re} (= a_{ij\infty}/\text{Re}_\infty), \ i, j = 1, 2; \ a_{ij} (a_{ij\infty}), \ i, j = 1, 2 \) are the small perturbations of the symmetrical anisotropy tensor;

\[
B = \begin{cases}
B = \hat{u} I_5 + B(0), & x \in R_+^1, \\
B_\infty = \hat{u}_\infty I_5 + B(0)_\infty, & x \in R_-^1;
\end{cases}
\]

\[
\Omega(t, x, y) = \begin{cases}
\Omega = p - \alpha_{11}, & x \in R_+^1, \\
\Omega_\infty = p_\infty - \alpha_{11\infty}, & x \in R_-^1;
\end{cases}
\]

\( p (p_\infty) \) are the small perturbations of the pressure,
the matrices and vectors $U_\infty$, $B_\infty$, $C^{(0)}_\infty$, $R^{(0)}_\infty$, $F^{(0)}_\infty$ have similar form with the $U$, $B$, $C^{(0)}$, $R^{(0)}$, $F^{(0)}$ respectively, but with “$\infty$” subscript for all components,

$$R_{33}(R_{33}\infty) = -\hat{\rho} \left( \frac{\kappa}{3} + (\kappa + 6\beta) \cos \hat{\alpha} \right) \text{Re}/3 = -\hat{\rho}_\infty \left( \frac{\kappa_\infty}{3} + (\kappa_\infty + 6\beta_\infty) \cos \hat{\alpha}_\infty \right) \text{Re}_\infty/3,$$

$$R_{34}(R_{34}\infty) = 2\beta \hat{\rho} \sin \hat{\alpha} \text{Re} = 2\beta_\infty \hat{\rho}_\infty \sin \hat{\alpha}_\infty \text{Re}_\infty,$$

$$R_{35}(R_{35}\infty) = \text{Re}_\infty \hat{\alpha}_{12}/3,$$

$$R_{53}(R_{53}\infty) = \text{Re}_\infty \hat{\alpha}_{22}/3.$$

Remark 3.1. We will assume that solutions $\|U\| = (U,U)^{1/2}$, $|\Omega|$ of the system (3.1), (3.2) are bounded at $|x| \to \infty$.

The initial data should satisfy the equation (3.2) and the incompressibility condition (see [8])

$$u_x + v_y = 0;\quad (u_0)_x + (v_0)_y = 0.$$

Linearization of (2.2) at constant solution gives the following boundary conditions at $x = 0$ for (3.1), (3.2):

$$f_i[\hat{u}] = [\Omega + 2\hat{\alpha}u],$$

$$f_i[\hat{\alpha}] = [u\hat{\alpha} + \hat{\alpha}u - 2\hat{\alpha}_{12}v],$$

$$f_i[\hat{u}^2 + \hat{\alpha}] = [(3\hat{u}^2 - \hat{\alpha})u - \hat{\alpha}\hat{\alpha}11] + 2\hat{\alpha}\hat{\alpha}_{12}[\hat{u}]f_y,$$

$$[\hat{\alpha}_{12}] = [\hat{u}^2](\frac{v}{u} + f_y) + 2[\hat{\rho} \cos \hat{\alpha}]f_y,$$

$$[\frac{v}{u}] = 0,$$

$$[\Omega_x] = [\hat{\alpha}_{12}]y.$$
the system (3.1) at \( x > 0 \) will be such that (the case \( x < 0 \) can be studied the same way):

\[
Z_t + (\tilde{u} I_5 + D) Z_x + \Lambda Z_y + \tilde{R} Z + T^{-1} F^{(0)} = 0.
\]  

(3.5)

Here \( \Lambda = T^{-1} C^{(0)} T, \tilde{R} = T^{-1} R^{(0)} T \). Besides from (3.4) it follows that

\[
\begin{align*}
\lambda f &= \frac{z_4 + \chi}{\chi}, \\
\alpha_{11} &= 2 \lambda (z_1 - z_2 + z_3), \\
\alpha_{12} &= \frac{z_5 - z_4}{\chi}, \quad \alpha_{22} = z_1 + \nu \frac{\tilde{a}_{12}}{\chi}.
\end{align*}
\]

(3.6)

We are looking for the solutions of the following form for the system (3.5), (3.2) (we will drop the tilde above the variables later on):

\[
\begin{align*}
Z(t, x, y) &= Z(x) \exp(\lambda t + i \omega y), \quad x > 0, \\
Z_\infty(t, x, y) &= \tilde{Z}_\infty(x) \exp(\lambda t + i \omega y), \quad x < 0, \\
\Omega(t, x, y) &= \tilde{\Omega}(x) \exp(\lambda t + i \omega y), \quad x > 0, \\
\Omega_\infty(t, x, y) &= \tilde{\Omega}_\infty(x) \exp(\lambda t + i \omega y), \quad x < 0, \\
f(t, y) &= \tilde{f} \exp(\lambda t + i \omega y),
\end{align*}
\]

where \( \lambda = \eta + i \omega_0; \omega_0, \omega \in R^1, \tilde{f} \) is a constant. From (3.7), (3.5), (3.2), (3.3) it follows that:

\[
\begin{align*}
(\lambda I_5 + i \omega \Lambda + \tilde{R}) Z + (\tilde{u} I_5 + D) Z' + \Gamma &= 0, \\
\Omega'' - \omega^2 \Omega &= i \omega \chi \nu' - \omega^2 (z_1 + \frac{\nu \tilde{a}_{12}}{\chi} - 2 \lambda^2 (z_1 - z_2 + z_3)), \quad x > 0; \\
(\lambda I_5 + i \omega \Lambda_\infty + \tilde{R}_\infty) Z_\infty + (I_5 + D_\infty) Z'_\infty + \Gamma_\infty &= 0, \\
\Omega''_\infty - \omega^2 \Omega_\infty &= i \omega \chi_\infty \nu'_\infty - \\
&- \omega^2 (z_{1\infty} + \frac{\nu_\infty \tilde{a}_{12\infty}}{\chi_\infty} - 2 \lambda^2 (z_{1\infty} - z_{2\infty} + z_{3\infty})), \quad x < 0; \\
\lambda f[\tilde{u}] &= [\Omega + 2 \tilde{u} z_3], \\
\lambda f[\tilde{a}_2] &= [\tilde{a}_2 z_3 + \tilde{u} (z_1 + \frac{\nu \tilde{a}_{12}}{\chi}) - \tilde{a}_2 \sigma], \\
\lambda f[\tilde{a}_2 + \lambda^2] &= [(3 \tilde{u}^2 - \chi) z_3 - 2 \chi^2 \tilde{u} (z_1 - z_2 + z_3)] + 2 i \omega \tilde{a}_{12} [\tilde{u}] f, \\
[\frac{\lambda}{2 \tilde{u}}] &= [\tilde{u}] (\frac{\sigma}{2 \tilde{u}} + i \omega f) + 2 i \omega [\tilde{\rho} \cos \tilde{a}] f, \\
[\frac{\sigma}{2 \tilde{u}}] &= 0, \\
[\Omega] &= i \omega [\frac{\chi}{2 \nu}], \quad x = 0.
\end{align*}
\]

(3.10)

Here

\[
\Gamma = \begin{pmatrix} 0 & 0 \\ \Omega' & \Omega' \\ i \omega \Omega & i \omega \Omega \end{pmatrix}, \quad \Gamma_\infty = \begin{pmatrix} 0 & 0 \\ \Omega'_\infty & \Omega'_\infty \\ i \omega \Omega_\infty & i \omega \Omega_\infty \end{pmatrix}
\]
We assume that $\omega > 0$ (the case $\omega < 0$ can be studied the same way). Let us introduce these new variables

$$X = \omega x, \quad z_1 = i\omega \xi_1, \quad z_2 = i\omega \xi_2, \quad z_3 = i\omega \xi_3, \quad \sigma_2 = \xi_2 - \frac{1 + 2\hat{\alpha}_2}{2\hat{\alpha}_2} \xi_1 - \xi_3,$$

$$\Omega = i\omega Q, \quad \mathcal{P} = \frac{u\sigma - \chi\nu}{2\omega}, \quad \lambda_0 = \frac{\lambda}{\omega},$$

$$z_{1\infty} = i\omega \xi_{1\infty}, \quad z_{2\infty} = i\omega \xi_{2\infty}, \quad z_{3\infty} = i\omega \xi_{3\infty}, \quad \sigma_{2\infty} = \xi_{2\infty} - \frac{1 + 2\hat{\alpha}_{2\infty}}{2\hat{\alpha}_{2\infty}} \xi_{1\infty} - \xi_{3\infty},$$

$$\Omega_{\infty} = i\omega Q_{\infty}, \quad \mathcal{P}_{\infty} = \frac{\sigma_{\infty} - \chi_{\infty}\nu_{\infty}}{2\omega}, \quad \frac{d}{dX} = \frac{1}{d\omega}dX,$$

where $\hat{\alpha}_2 = \hat{l} + \hat{\rho}\cos\hat{\alpha} > 0$, $\hat{\alpha}_{2\infty} = \hat{l}_{\infty} + \hat{\rho}\cos\hat{\alpha}_{\infty} > 0$. Then (3.8)-(3.10) transforms to:

$$\frac{dV}{dX} = AV, \quad X > 0, \quad (3.11)$$

$$\frac{dV_{\infty}}{dX} = A_{\infty}V_{\infty}, \quad X < 0, \quad (3.12)$$

$$[Q + 2\hat{u}\xi_3] + i\lambda_0[\hat{u}] = 0,$$

$$[\hat{u}\xi_1 + \hat{\alpha}_2\xi_3 + \frac{2\hat{\alpha}_2}{\lambda_0} + \frac{2\Delta\hat{\delta}}{\lambda_0} - i\lambda_0[\hat{\alpha}_2] = 0,$$

$$[\chi^2\hat{u}\xi_1 + 2\chi^2\hat{u}\sigma_2 + (3\hat{u}^2 - \chi^2)(\xi_3) + f(2\hat{u}_12[\hat{u}] + i\lambda_0[\hat{u}^2 + \chi^2]) = 0,$$

$$[\mathcal{P}] + i\lambda_0[\hat{u}]^2 + 2\hat{\rho}\cos\hat{\alpha} = 0,$$

$$[\xi'_3] = 0,$$

$$[\lambda_0\xi_3 + \hat{u}\xi'_3] = 0, \quad X = 0.$$

Here

$$V = (\xi_1, \sigma_2, \mathcal{P}, Q, \xi'_3, \xi_3)^T, \quad V_{\infty} = (\xi_{1\infty}, \sigma_{2\infty}, \mathcal{P}_{\infty}, Q_{\infty}, \xi'_{3\infty}, \xi_{3\infty})^T,$$

$$A = \begin{pmatrix} -\hat{\lambda} & 0 & 0 & 0 & -\frac{2\hat{\alpha}_2}{\lambda_0} & -\frac{2\hat{\alpha}_2}{\chi_{\infty}} \\
0 & -\hat{\lambda} & 0 & 0 & 0 & 0 \\
-\frac{\chi^2 + \hat{\alpha}_2}{\alpha_{\infty}} & -\frac{2\chi^2}{\alpha_{\infty}} & -\frac{2\chi^2}{\chi_{\infty}} & 1 & \lambda_0 & -\frac{2\hat{\alpha}_2}{\chi_{\infty}} & 0 \\
0 & 0 & -1 & 0 & -\frac{2\hat{u}}{\alpha_{\infty}} & -\frac{2\hat{u}}{\chi_{\infty}} \\
\frac{\chi^2 + \hat{\alpha}_2}{\alpha_{\infty}} & \frac{\chi^2}{\alpha_{\infty}} & -\frac{\lambda_0}{\alpha_{\infty}} & \frac{2\hat{u}}{\alpha_{\infty}} & -\frac{\hat{u}}{\alpha_{\infty}} & -\frac{2\lambda_0}{\alpha_{\infty}} + \frac{2\hat{u}^2}{\alpha_{\infty}} & \frac{\hat{u}^2}{\alpha_{\infty}} & \frac{\hat{u}^2}{\alpha_{\infty}} \end{pmatrix},$$

$$A_{\infty} = \begin{pmatrix} -\lambda_0 & 0 & 0 & 0 & 0 & -2\hat{\alpha}_{2\infty} & -\frac{2\hat{\alpha}_{2\infty}}{\chi_{\infty}} \\
0 & -\lambda_0 & 0 & 0 & 0 & 0 \\
-\frac{\chi_{\infty}^2 + \hat{\alpha}_{2\infty}}{\alpha_{\infty}} & -\frac{2\chi_{\infty}^2}{\alpha_{\infty}} & -\frac{2\chi_{\infty}^2}{\chi_{\infty}} & 1 & \lambda_0 & -\frac{2\hat{\alpha}_{2\infty}}{\chi_{\infty}} & 0 \\
0 & 0 & -1 & 0 & -2 & -\lambda_0 \\
\frac{\chi_{\infty}^2 + \hat{\alpha}_{2\infty}}{\alpha_{\infty}} & \frac{\chi_{\infty}^2}{\alpha_{\infty}} & -\frac{\lambda_0}{\alpha_{\infty}} & \frac{2\hat{u}}{\alpha_{\infty}} & -\frac{\hat{u}}{\alpha_{\infty}} & -\frac{2\lambda_0}{\alpha_{\infty}} + \frac{2\hat{u}^2}{\alpha_{\infty}} & \frac{\hat{u}^2}{\alpha_{\infty}} & \frac{\hat{u}^2}{\alpha_{\infty}} & \frac{\hat{u}^2}{\alpha_{\infty}} \end{pmatrix},$$

$$\xi'_3 = \frac{d\xi_3}{dX}, \quad \xi'_{3\infty} = \frac{d\xi_{3\infty}}{dX}, \quad \hat{\lambda} = \frac{\lambda_0}{\hat{u}},$$
\[ \delta = \alpha_{12}i, \quad \delta_{\infty} = \alpha_{12\infty}i \quad (\delta_{\infty} = \delta, \text{see } (2.5)), \quad \Delta = \hat{u}^2 - \chi^2, \quad \Delta_{\infty} = 1 - \chi_{\infty}^2 \quad (\text{the signs } \Delta \text{ and } \Delta_{\infty} \text{ are the same, see } (2.5)). \]

Remark 3.2. In (3.11)-(3.13) we neglected the matrices \( \hat{R}, \hat{R}/\omega \) since their elements has the order \(1/\omega W, 1/\omega W_{\infty} \), i.e. for the large parameters \( \omega \) (for shortwave perturbations) and for large \( W, W_{\infty} \) the elements of the matrices are small.

Let us look for the solutions of (3.11), (3.12) such that:

\[ V(X) = e^{XA}V(0), \quad X > 0, \]
\[ V_\infty(X) = e^{XA_\infty}V_\infty(0), \quad X < 0, \quad (3.14) \]

where \( V(0), V_\infty(0) \) has to be determined. There exists matrices \( \hat{T} \) and \( \hat{T}_\infty \) such that

\[ A\hat{T} = \hat{T}\text{diag}(\hat{\lambda}, -\hat{\lambda}, q^+, q^-, -1, -1), \]
\[ A_\infty\hat{T}_\infty = \hat{T}_\infty\text{diag}(\hat{\lambda}_0, -\hat{\lambda}_0, q^+_\infty, q^-_\infty, -1, -1) \quad \text{for } \Delta > 0 (\Delta_{\infty} > 0); \]

\[ A\hat{T} = \hat{T}\text{diag}(\hat{\lambda}, -\hat{\lambda}, q^+, q^-, -1, q^+), \]
\[ A_\infty\hat{T}_\infty = \hat{T}_\infty\text{diag}(\hat{\lambda}_0, -\hat{\lambda}_0, q^+_\infty, q^-_\infty, -1, q^+_\infty) \quad \text{for } \Delta < 0 (\Delta_{\infty} < 0). \quad (3.15, 3.16) \]

Here

\[
\begin{align*}
q^+_\infty &= -\frac{\hat{\lambda}_0 \pm \hat{\lambda}_{12}}{\Delta_{\infty}} + i(\frac{\hat{\lambda}_{12} - \hat{\lambda}_{12\infty}}{\Delta_{\infty}}), \\
q^+ &= -\frac{\hat{\lambda}_0 \pm r}{\Delta}, + i(\frac{\hat{\lambda}_{12\infty} - \hat{\lambda}_0 \pm B/2r_{\infty}}{\Delta_{\infty}} \quad \text{for } \Delta > 0; \\
q^-_\infty &= \frac{\hat{\lambda}_0 \pm \hat{\lambda}_{12\infty}}{\Delta_{\infty}} + i(\frac{\hat{\lambda}_{12\infty} - \hat{\lambda}_0 \pm B/2r_{\infty}}{\Delta_{\infty}} \quad \text{for } \Delta < 0; \\
q^- &= -\frac{\hat{\lambda}_0 \pm r_{\infty}}{\Delta_{\infty}} + i(\frac{\hat{\lambda}_{12\infty} - \hat{\lambda}_0 \pm B/2r_{\infty}}{\Delta_{\infty}} \\
\sqrt{A + iB} &= r + iB/2r, \quad r = \sqrt{(A^2 + B^2)^{1/2} + A/2,}
\end{align*}
\]

\( \lambda_0 = \eta_0 + i\zeta_0, \eta_0 = \eta/\omega, \zeta_0 = \omega_0/\omega; A = \chi^2(\eta_0^2 - \zeta_0^2) - \Delta\hat{\alpha}_2 - \hat{\alpha}_{12}^2 + 2\hat{\alpha}_0\hat{\alpha}_{12}, B = 2(\chi^2\zeta_0 - \hat{u}\hat{\alpha}_{12})\eta_0, \]

\( A_{\infty} = \chi^2(\eta_0^2 - \zeta_0^2) - \Delta_{\infty}\hat{\alpha}_{2\infty} - \hat{\alpha}_{12\infty}^2 + 2\hat{\alpha}_0\hat{\alpha}_{12\infty}, B_{\infty} = 2(\chi^2\zeta_0 - \hat{u}\hat{\alpha}_{12})\eta_0, A_{\infty} = \chi^2(\eta_0^2 - \zeta_0^2) + |\Delta_{\infty}|\hat{\alpha}_{2\infty} - \hat{\alpha}_{12\infty} + 2\hat{\alpha}_0\hat{\alpha}_{12\infty}, B_{\infty} = 2(\chi^2\zeta_0 - \hat{u}\hat{\alpha}_{12})\eta_0 \quad \text{for } \Delta < 0. \]

From (3.15), (3.16) the (3.14) could be rewritten as:

\[ V(X) = T \begin{pmatrix} e^{XA^-} & 0 \\ 0 & e^{X} \end{pmatrix} C = T \begin{pmatrix} e^{XA^-} & e^{X}C^- \\ 0 & e^{X}C^+ \end{pmatrix}, \quad X > 0; \]

\[ V_\infty(X) = T_\infty \begin{pmatrix} e^{XA_\infty} & e^{X}C^-_\infty \\ e^{X}C^+_\infty & e^{X}C^+_\infty \end{pmatrix}, \quad X < 0 \quad \text{for } \Delta > 0; \]

\[ V(X) = T \begin{pmatrix} e^{XA^-} & 0 \\ 0 & e^{X} \end{pmatrix} \]

\[ V_\infty(X) = T_\infty \begin{pmatrix} e^{XA_\infty} & e^{X}C^-_\infty \\ e^{X}C^+_\infty & e^{X}C^+_\infty \end{pmatrix}, \quad X < 0 \quad \text{for } \Delta < 0. \]
where $A^- = \text{diag}(-\tilde{\lambda}, -\tilde{\lambda}, q^+, q^-, -1)$, $A^-_{\infty} = \text{diag}(-\lambda_0, -\lambda_0, q^+_\infty, q^-_\infty, -1)$ for $\Delta > 0$; $A^- = \text{diag}(-\tilde{\lambda}, -\tilde{\lambda}, q^-, 1)$, $A^-_{\infty} = \text{diag}(-\lambda_0, -\lambda_0, q^+_\infty, 1)$, $A^+ = \text{diag}(q^+, 1)$, $A^+_{\infty} = \text{diag}(q^+_\infty, 1)$ for $\Delta < 0$;

$$C = \begin{pmatrix} C^- & C^+ \\ C_{\infty}^- & C_{\infty}^+ \end{pmatrix} = T^{-1} V(0), \quad C^- = \begin{pmatrix} C_1 \\ \vdots \\ C_5 \end{pmatrix},$$

$$C_{\infty} = \begin{pmatrix} C^-_{\infty} & C^+_{\infty} \end{pmatrix} = T_{\infty}^{-1} V_{\infty}(0), \quad C^-_{\infty} = \begin{pmatrix} C_{1\infty} \\ \vdots \\ C_{5\infty} \end{pmatrix} \text{ for } \Delta > 0;$$

$$C = \begin{pmatrix} C^- & C^+ \end{pmatrix} = T^{-1} V(0), \quad C^- = \begin{pmatrix} C_1 \\ \vdots \\ C_4 \end{pmatrix}, \quad C^+ = \begin{pmatrix} C_5 \\ C_6 \end{pmatrix},$$

$$C_{\infty} = \begin{pmatrix} C^-_{\infty} & C^+_{\infty} \end{pmatrix} = T_{\infty}^{-1} V_{\infty}(0), \quad C^-_{\infty} = \begin{pmatrix} C_{1\infty} \\ \vdots \\ C_{4\infty} \end{pmatrix}, \quad C^+_{\infty} = \begin{pmatrix} C_{5\infty} \\ C_{6\infty} \end{pmatrix} \text{ for } \Delta < 0.$$

Then for $\text{Re}\lambda_0 > 0$ we will assume the following form for the solutions of (3.11), (3.12):

1) for $\Delta > 0$

$$V(X) = T \begin{pmatrix} e^{XA^-} & C^- \end{pmatrix}, \quad X > 0; \quad V_{\infty}(X) = T_{\infty} \begin{pmatrix} 0 \\ e^{X^+} \end{pmatrix}, \quad X < 0;$$

2) for $\Delta < 0$

$$V(X) = T \begin{pmatrix} e^{XA^-} & C^- \end{pmatrix}, \quad X > 0; \quad V_{\infty}(X) = T_{\infty} \begin{pmatrix} 0 \\ e^{X_{\infty}^+} \end{pmatrix}, \quad X < 0.$$

The boundary conditions (3.13) could be rewritten such that (we will assume that $f = 0$):

$$BV(0) = B_{\infty} V_{\infty}(0).$$

Here

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 2\hat{u} \\ \hat{u} & 0 & 2\tilde{\alpha}_1 \hat{u} & 0 & \frac{2\tilde{\alpha}_1}{\chi^2} \hat{u} & \alpha_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3\hat{u}^2 - \chi^2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\hat{u} & \lambda_0 \\ \hat{u} & 0 & 0 & 0 & 0 & \hat{u} \end{pmatrix}, \quad B_{\infty} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 2 \\ 1 & 0 & \frac{2\tilde{\alpha}_1}{\chi^2} & 0 & \frac{2\tilde{\alpha}_1\Delta_{\infty}}{\chi_{\infty}} & \alpha_{2\infty} \\ 0 & 0 & 2\chi^2_{\infty} & 0 & 0 & 0 \\ \frac{\hat{u}^2}{\lambda_{\infty}} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_0 \end{pmatrix}.$$ 

From (3.17), (3.18) we can rewrite (3.19) as

$$BT \begin{pmatrix} C^- \\ 0 \end{pmatrix} = B_{\infty} T_{\infty} \begin{pmatrix} 0 \\ C^+_{\infty} \end{pmatrix}, \quad \Delta > 0;$$

$$BT \begin{pmatrix} C^- \\ 0 \end{pmatrix} = B_{\infty} T_{\infty} \begin{pmatrix} 0 \\ C^+_{\infty} \end{pmatrix}, \quad \Delta < 0.$$
From the following notations

\[
\mathcal{B}T = \begin{pmatrix}
D^- \\
\vdots \\
5 \text{ columns} \\
\end{pmatrix}, \quad \mathcal{B}_\infty T_\infty = \begin{pmatrix}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array} \\
\end{pmatrix}
\]

\[
\mathcal{B}T = \begin{pmatrix}
D^- \\
\vdots \\
4 \text{ columns} \\
\end{pmatrix}, \quad \mathcal{B}_\infty T_\infty = \begin{pmatrix}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array} \\
\end{pmatrix}
\]

the formula (3.20) will get the form:

\[
\mathcal{D} \begin{pmatrix}
C^- \\
\Delta \\
C^+ \\
\end{pmatrix} = \begin{pmatrix}
D^- & D_\infty^+ \\
\vdots & \vdots \\
\end{pmatrix} \begin{pmatrix}
C^- \\
\Delta \\
C^+ \\
\end{pmatrix} = 0, \quad \Delta > 0;
\]

\[
\mathcal{D} \begin{pmatrix}
C^- \\
\Delta \\
C^+ \\
\end{pmatrix} = \begin{pmatrix}
D^- & D_\infty^- \\
\vdots & \vdots \\
\end{pmatrix} \begin{pmatrix}
C^- \\
\Delta \\
C^+ \\
\end{pmatrix} = 0, \quad \Delta < 0.
\]

Let us assume that Lopatinskii condition is not satisfied: for some \(\lambda_0\), \(\text{Re}\lambda_0 > 0\) we have \(\det \mathcal{D} = 0\). Then there exist a non-trivial partial solutions of (3.7), growing with time (which means an instability of flat discontinuity \(x = 0\)). After the bulky calculations it could be shown that the \(\det \mathcal{D}\) is (the vertical line here means the line breaking of long matrix which does not fit the page width).

\[
\det \mathcal{D} = -\lambda_0 (\tilde{\lambda} - 1) \det
\begin{pmatrix}
A^+ & A^- \\
L^+ - \tilde{\lambda}r^+ & L^- - \tilde{\lambda}r^- \\
q^+ (q^+ + \tilde{\lambda}) & q^- (q^- + \tilde{\lambda}) \\
q^+ + \tilde{\lambda} & q^- + \tilde{\lambda} \\
\end{pmatrix}
\begin{pmatrix}
A_{1\infty} \\
L_{1\infty} - \tilde{\lambda}r_{1\infty} \\
-1 \\
1 \\
\end{pmatrix} = 0, \quad \Delta > 0; \quad \text{(3.21)}
\]

\[
\det \mathcal{D} = -\lambda_0 (\tilde{\lambda} - 1) \det
\begin{pmatrix}
A^- & A_{-1} \\
L^- - \tilde{\lambda}r^- & B_{-1} \\
q^- (q^- + \tilde{\lambda}) & -1 \\
q^- + \tilde{\lambda} & 1 \\
\end{pmatrix}
\begin{pmatrix}
A_{1\infty}^+ \\
L_{1\infty}^+ - \tilde{\lambda}r_{1\infty}^+ \\
\hat{u}q_{\infty}^+ (q_{\infty}^+ + \lambda_0) \\
\frac{\Delta_0 + 1}{\lambda_0} (\lambda_0 + 1 - \hat{u}^2) \\
\end{pmatrix} = 0, \quad \Delta < 0. \quad \text{(3.22)}
\]
Here $A^\pm = b^\pm + a^\pm - \tilde{u}r^\pm (1+\bar{\lambda}^2)$, $A^+_\infty = b^+_\infty + a^+_\infty - \tilde{u}r^+_\infty (1+\bar{\lambda}^2)$, $A^-_\infty = b^-_\infty + a^-_\infty - \tilde{u}r^-_\infty (1+\bar{\lambda}^2)$.

From (3.21), (3.22) it follows that for $\lambda = 1$ ($\lambda_0 = \tilde{u}$) (for example) the one could find the partial solution of the form (3.7) which grows in time. Therefore the flat strong discontinuity with overflow of polymeric liquid is unstable for any $\Delta$ ($\Delta \gtrsim 0$) at the presence of the anisotropy ($\tilde{\alpha}_{12} \neq 0$).

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