TRANSCENDENTAL HOLOMORPHIC MAPS BETWEEN REAL
ALGEBRAIC MANIFOLDS IN A COMPLEX SPACE

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Abstract. We give an example of a real algebraic manifold embedded in a complex space that does not satisfy the Nash-Artin approximation Property. This Nash-Artin approximation Property is closely related to the problem of determining when the biholomorphic equivalence for germs of real algebraic manifolds coincides with the algebraic equivalence. This example is an elliptic Bishop surface, and its construction is based on the functional equation satisfied by the generating series of some walks restricted to the quarter plane.

1. Introduction

An important problem in complex geometry is to classify germs of real analytic manifolds in $\mathbb{C}^N$ up to biholomorphic equivalence (see [BER00] for an introduction to this problem). This problem goes back to Poincaré [Po07], and E. Cartan, for germs real analytic smooth hypersurfaces in $\mathbb{C}^2$ [Ca32], then S. S. Chern and J. K. Moser, for germs real analytic smooth hypersurfaces in $\mathbb{C}^N$ for $N \geq 2$ [CM75], gave a complete description of this classification, when the hypersurfaces are assumed to be Levi nondegenerate. More precisely, S. S. Chern and J. K. Moser first gave a complete classification up to formal biholomorphisms, and then they showed that any formal biholomorphism between Levi nondegenerate real hypersurfaces is necessarily convergent.

Therefore a natural question was to understand when the biholomorphic equivalence and the formal (biholomorphic) equivalence coincide. This question has been widely studied since the work [CM75]; the reader can consult [Mir13] for a general presentation of this problem. The first negative answer to this question has been given in [MW83]: the authors considered a particular example of the germ of a real algebraic smooth surface $(M, 0)$ in $\mathbb{C}^2$ for which there exists the germ of a real analytic smooth surface $(M', 0) \subset (\mathbb{C}^2, 0)$, such that $(M, 0)$ and $(M', 0)$ are formally equivalent but not biholomorphically equivalent. This surface $M$ has the following particular property: its tangent space at any point near the origin is totally real, but its tangent space at the origin is a complex line (in other words it has a CR-singularity at the origin). Such a surface is called a Bishop surface. For quite a while, it remained open to know if, for any germ of a real analytic CR-manifold $(M, 0)$, the formal equivalence class of $(M, 0)$ was the same as its biholomorphic equivalence class. This question has been recently answered in the negative in [KS16].

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In the case of real algebraic manifolds in $\mathbb{C}^N$, one can define the notion of algebraic (biholomorphic) equivalence. One says that two germs of real algebraic manifolds $(M, 0), (M', 0)$ in $\mathbb{C}^N$ are algebraically equivalent if there is a germ of biholomorphic map $h : (\mathbb{C}^N, 0) \rightarrow (\mathbb{C}^N, 0)$ such that $h(M) = M'$, and such that the components of $h(z)$ are algebraic, that is, there are given by algebraic power series. The question to know if the biholomorphic equivalence implies the algebraic equivalence of germs of real algebraic manifolds has first been asked in [BER00]. This is even stated as a conjecture in [BMR02]. Up to now the answer to this question remains completely open (see also [Mir13]), even if it is known to be true in some important cases (as the case of hypersurfaces [BMR02]).

A related problem has been introduced by N. Mir in [Mir12]. He introduced the notion of Nash-Artin approximation Property for a smooth real algebraic set $M$ (see Definition 1 below). The Nash-Artin approximation Property for $M$ implies that the biholomorphic equivalence class of $(M, 0)$ coincides with the algebraic equivalence class of $(M, 0)$. In fact this approximation property is natural since, for a lot of the cases where these two notions of equivalence are known to coincide, the Nash-Artin property is also satisfied. Therefore, a natural question is to know if this approximation property is always satisfied for real algebraic manifolds. In the case of real algebraic CR-manifolds, this is even a conjecture (see [Mir13]).

The aim of this note is to provide examples of germs of real algebraic surfaces in $\mathbb{C}^2$ that do not satisfy the Nash-Artin approximation Property. These examples are constructed from the functional equations satisfied by the generating series of some walks restricted to the quarter plane. These examples are in fact elliptic Bishop surfaces.

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## 2. The Nash-Artin Approximation Property

**Definition 1.** [Mir12] Let $(M, 0) \subset (\mathbb{C}^N, 0)$ be the germ of a real algebraic manifold. We say that $(M, 0)$ has the Nash-Artin approximation Property if

i) for every $N'$ and every real algebraic set $\Gamma \subset \mathbb{C}^N \times \mathbb{C}^{N'}$

ii) for every germ of holomorphic map $h : (\mathbb{C}^N, 0) \rightarrow \mathbb{C}^{N'}$

iii) for every $c \in \mathbb{N}$ such that

$$\text{Graph}(h) \cap (M \times \mathbb{C}^{N'}) \subset \Gamma,$$

there is a holomorphic algebraic map germ $h_c : (\mathbb{C}^N, 0) \rightarrow \mathbb{C}^{N'}$ such that

$$\text{Graph}(h) \cap (M \times \mathbb{C}^{N'}) \subset \Gamma,$$

and $h_c$ agrees with $h$ up to order $c$.

The terminology comes from the Artin approximation Theorem [Art69].

**Remark 2.** Assume that the ideal defining $M$ is generated by the real valued polynomials $f_1(z, \overline{z}), \ldots, f_p(z, \overline{z})$ and the ideal defining $\Gamma$ is generated by the real valued polynomials $g_1(z, w, \overline{z}, \overline{w}), \ldots, g_s(z, w, \overline{z}, \overline{w})$. Here $z = (z_1, \ldots, z_N)$ and $w = (w_1, \ldots, w_{N'})$. Then the condition

$$\text{Graph}(h) \cap (M \times \mathbb{C}^{N'}) \subset \Gamma,$$
is equivalent to the existence of series $k_{j,l}(z,\overline{z})$, for $1 \leq j \leq s$ and $1 \leq l \leq p$, such that

$$(2.1) \quad g_j(z, h(z), \overline{z}, \overline{h(z)}) + \sum_{l=1}^{p} k_{j,l}(z, \overline{z})f_j(z, \overline{z}) = 0 \quad \forall j = 1, \ldots, s.$$  

3. Generating series of walks restricted to the quarter plane

We consider the following situation: We fix a set of steps $S \subset \{-1, 0, 1\}^2\setminus\{(0, 0)\}$. For all $(i, j) \in \mathbb{N}^2$ and $n \in \mathbb{N}$, we denote by $a_{i,j,n}$ the number of walks with steps in $S$ of length $n$, starting at the origin and ending at $(i, j)$, and remaining in the quarter plane $\mathbb{N}^2$. The associated generating series is defined by

$$Q(x, y, t) := \sum_{i,j,n \in \mathbb{N}} a_{i,j,n} x^i y^j t^n \in \mathbb{Z}[[x, y, t]].$$

The reader may consult [BMM09] for a general account on the study of these generating series. We recall here that $Q(x, y, t)$ is the solution of the equation

$$(3.1) \quad \begin{aligned} xy &= K_S(x, y, t)l(x, y, t) + h(x, t) + g(y, t) \end{aligned}$$

The term $xy - t\sum_{(a,b) \in S} x^{a+1}y^{b+1}$ is called the kernel of the equation and is denoted by $K_S(x, y, t)$. In fact, by the division theorem of formal power series (see for instance [Ro1, Example 1.14]), the equation

$$(3.2) \quad xy = K_S(x, y, t)l(x, y, t) + h(x, t) + g(y, t)$$

has a unique solution $(l(x, y, t), h(x, t), g(y, t)) \in \mathbb{C}[[x, t]] \times \mathbb{C}[[x, t]] \times \mathbb{C}[[y, t]]$. In particular we have necessarily

$$l(x, y, t) = Q(x, y, t), \quad h(x, t) = xtQ(x, 0, t), \quad g(y, t) = ytQ(0, y, t).$$

We have the following lemma (for a general account on the Bishop surfaces, see [Bi65], [Mo85] or [HY09] for example):

**Lemma 3.** We denote by $M_S$ the real algebraic surface defined by $K_S(w, \overline{w}, z) = 0$. Then $M_S$ is smooth at the origin if and only if $(-1, -1) \in S$. In this case the germ $(M_S, 0)$ is the germ of a Bishop surface with a Bishop invariant equal to 0.

**Proof.** We have

$$\frac{\partial K_S}{\partial w}(w, \overline{w}, z) = w - z \sum_{(a,b) \in S} (a+1)w^a \overline{w}^{b+1},$$

$$\frac{\partial K_S}{\partial \overline{w}}(w, \overline{w}, z) = w - z \sum_{(a,b) \in S} (b+1)w^{a+1} \overline{w}^b,$$

$$\frac{\partial K_S}{\partial z}(w, \overline{w}, z) = \sum_{(a,b) \in S} w^{a+1} \overline{w}^{b+1}.$$
Thus, $\frac{\partial K_S}{\partial w}(0,0,0) = \frac{\partial K_S}{\partial z}(0,0,0) = 0$. Therefore $M_S$ is smooth at the origin if and only if $\frac{\partial K_S}{\partial w}(0,0,0) \neq 0$, that is, if and only if $(-1,-1) \in S$. In this case, the germ $(M_S,0)$ is also defined by

$$\varphi(w,\overline{w},z) := z - w\overline{w}\left(1 + \sum_{(a,b) \in S \setminus \{(-1,-1)\}} w^{a+1}w^{b+1}\right)^{-1} = 0.$$ 

But $\varphi$ can be expanded as

$$\varphi(w,\overline{w},z) = z - w\overline{w} + E(w,\overline{w})$$

where $E(w,\overline{w})$ is the germ of a real analytic function of vanishing order at least three at the origin. Therefore $(M_S,0)$ is the germ of a Bishop surface with a Bishop invariant equal to 0.

To such a kernel $K_S$ is usually associated a group of birational automorphisms $G(S)$ that preserves $K_S$. By [KR12, Theorem 1], this group is finite if and only if the series $Q_S$ is D-finite (or holonomic). In fact they show the following stronger result: if $G(S)$ is infinite then $Q(x,0,t)$ and $Q(0,y,t)$ are not D-finite. In particular, in this case, $Q(x,0,t)$ and $Q(0,y,t)$ are transcendental convergent power series. There are 56 such walks and their list can be found in [BMM09, Table 4] for instance.

Among these 56 walks with infinite group, only 7 satisfy Lemma 3 while having a Kernel satisfying the following symmetry:

$$(3.3) \quad K_S(w,\overline{w},z) = K_S(\overline{w},w,z).$$

These correspond to the following sets $S$: 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}

For these walks, by symmetry, the solutions $(l,h,g)$ of (3.2) satisfy the relation $h(x,t) = g(x,t)$ and have positive integer coefficients. In fact, it has been recently shown that the generating series corresponding to these 7 types of walks are among those that are $x$-hypertranscendental and $y$-hypertranscendental [DHRS18]. That is, $Q(x,0,t)$ (resp. $Q(0,y,t)$) is not solution, as a function of $x$ (resp. of $y$), of a polynomial differential equation.

4. Example

Let $M_S \subset \mathbb{C}^2_{z,w}$ be the real algebraic manifold defined by $K_S$ where $S$ is one of the sets given in Figure 1. Since

$$\sum_{(a,b) \in S} w^{a+1}w^{b+1} = \sum_{(a,b) \in S} w^{a+1}w^{b+1}$$

by (3.3), we easily check that $M$ is defined by the real algebraic equations:

$$\left\{ \begin{array}{l}
|w|^2 - z \sum_{(a,b) \in S} w^{a+1}w^{b+1} = 0 \\
\text{Im}(z) = 0
\end{array} \right..$$
Let $\Gamma$ be the subset of $\mathbb{C}^2_{z,w} \times \mathbb{C} \times h$ defined by

$$|w|^2 + h + \overline{h} = 0$$

or, equivalently, by

$$|w|^2 + 2 \Re(h) = 0.$$

Then $\Gamma$ is a real algebraic smooth hypersurface. Now let

$$h : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$$

be a germ of (formal) holomorphic map such that $\text{Graph}(h) \cap (M_S \times \mathbb{C}) \subset \Gamma$. By (2.1), this is equivalent to saying that there exist power series $k(z, w, \overline{z}, \overline{w})$ and $l(z, w, \overline{z}, \overline{w})$ such that the following relations are satisfied:

$$|w|^2 + 2 \Re(h(z, w)) + k(z, w, \overline{z}, \overline{w}) + K_S(w, \overline{w}, z)l(z, w, \overline{z}, \overline{w}) = 0.$$  

Equivalently we have

$$w \overline{w} + h(x, w) + \overline{h}(x, \overline{w}) + K_S(w, \overline{w}, x)l(x, w, \overline{w}) = 0.$$  

for some $l(x, w, \overline{w}) \in \mathbb{C}[[x, w, \overline{w}]]$. But (4.1) is exactly (3.2) whose unique solution is the generating series $l$ that counts the number of walks restricted to the quarter plane by the length and by the end point, and whose elementary steps are $E$, $N$, $NE$ and $SW$ (the first one listed on Figure 1). In this case the solution $(l, h)$ have (convergent) transcendental power series components (with real coefficients) as explained before.

This proves that $h$ is the unique germ of a holomorphic map such that

$$\text{Graph}(h) \cap (M_S \times \mathbb{C}) \subset \Gamma$$

and this germ is not algebraic.

This shows that $M_S$ does not satisfy the Nash-Artin approximation Property.

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