On the set of uniquely decodable codes with a given sequence of code word lengths

Adam Woryna

Silesian University of Technology, Institute of Mathematics, ul. Kaszubska 23, 44-100
Gliwice, Poland

Abstract

For every natural number \( n \geq 2 \) and every finite sequence \( L \) of natural numbers, we consider the set \( UD_n(L) \) of all uniquely decodable codes over an \( n \)-letter alphabet with the sequence \( L \) as the sequence of code word lengths, as well as its subsets \( PR_n(L) \) and \( FD_n(L) \) consisting of, respectively, the prefix codes and the codes with finite delay. We derive the estimation for the quotient \( |UD_n(L)|/|PR_n(L)| \), which allows to characterize those sequences \( L \) for which the equality \( PR_n(L) = UD_n(L) \) holds. We also characterize those sequences \( L \) for which the equality \( FD_n(L) = UD_n(L) \) holds.

Keywords: uniquely decodable code, prefix code, code with finite delay, Kraft’s procedure, Sardinas-Patterson algorithm

2010 MSC: 68R15, 68W32, 94A45, 94A55, 20M35

1. Preliminaries and the statement of the results

Let \( X \) be an alphabet with \( n := |X| \geq 2 \) letters. We refer to a finite sequence
\[
C = (v_1, \ldots, v_m), \quad m \geq 1
\]
of words over \( X \) as a code and to the words \( v_i \in X^* \) \((1 \leq i \leq m)\) as the code words. In particular, our convention differs a bit from the more usual one, where codes are considered as sets of words rather than sequences of words. The code \( C \) is called uniquely decodable if for all \( l, l' \geq 1 \) the equality \( v_{i_1}v_{i_2} \ldots v_{i_l} = v_{j_1}v_{j_2} \ldots v_{j_{l'}} \) with \( 1 \leq i_1, j_1 \leq m \) \((1 \leq t \leq l, 1 \leq t' \leq l')\) implies \( l = l' \) and \( i_t = j_t \) for every \( 1 \leq t \leq l \). Thus every uniquely decodable code must be an injective sequence of non-empty words. In the algebraic language, one could say that the code \( C \) is uniquely decodable if and only if the monoid generated by the set \( \{v_1, \ldots, v_m\} \) (with concatenation of words as the monoid operation) is a free monoid of rank \( m \) freely generated by this set, or that this set is an \( m \)-element basis for this monoid. If for all \( 1 \leq i, j \leq m \) the condition: \( v_i \) is a prefix (initial segment) of \( v_j \) implies \( i = j \), then \( C \) is called a prefix code.

Email address: adam.woryna@polsl.pl (Adam Woryna)
The prefix codes are the most useful examples of uniquely decodable codes and, in a sense, they are universal for all uniquely decodable codes. Namely, according to the Kraft-McMillan theorem [5], for every finite sequence $L = (a_1, \ldots, a_m)$ of natural numbers the following three statements are equivalent: (1) there exists a uniquely decodable code $C = (v_1, \ldots, v_m)$ with the sequence $L$ as the sequence of code word lengths, i.e. $|v_i| = a_i$ for every $1 \leq i \leq m$; (2) there exists a prefix code $C' = (v'_1, \ldots, v'_m)$ with the sequence $L$ as the sequence of code word lengths; (3) the inequality $\sum_{i=1}^{m} n^{-a_i} \leq 1$ holds.

Uniquely decodable codes of length $m \leq 2$ are exceptional, as every such a code has finite delay [2]. Recall that a code $C$ has finite delay if there is a number $t$ with the following property: picking up the consecutive letters of an arbitrary word $u \in X^*$ which can be factorized into the code words, it is enough to pick up at most $t$ first letters of $u$ to be sure which code word begins $u$ (see also [3]). The smallest number $t$ with this property is called the delay of the code $C$. If such a number does not exist, then we say that the code has infinite delay. Obviously, every prefix code has finite delay (which is not greater that the maximum length of a code word) and every code with finite delay must be uniquely decodable. It turns out (see Section 6.1.2 in [4] and Proposition 6.1.9 therein) that a code $C = (v_1, \ldots, v_m)$ has infinite delay if and only if there is an infinite word $u \in X^\omega$ and two factorizations

$$u = v_{i_1} v_{i_2} v_{i_3} \ldots, \quad u = v_{j_1} v_{j_2} v_{j_3} \ldots$$

into code words such that $v_{i_1} \neq v_{j_1}$. If $m \geq 3$, then there are uniquely decodable codes of length $m$ which have infinite delay.

**Example 1.** The code $C = (10, 100, 000)$ has infinite delay because of the following two factorizations of the infinite word $u = 10^\omega$ into the code words:

$$10 - 000 - 000 - 000 - \ldots, \quad 100 - 000 - 000 - 000 - \ldots.$$ 

The code $C$ is also uniquely decodable, as its reverse $C^R = (01, 001, 000)$ is a prefix code (we use the well known fact that a code is uniquely decodable if and only if its reverse is uniquely decodable).

For every finite sequence $L$ of natural numbers we denote by $UD_n(L)$ the set of all uniquely decodable codes over the alphabet $X$ with the sequence $L$ as the sequence of code word lengths. We also consider the subset $PR_n(L) \subseteq UD_n(L)$ of all prefix codes and the subset $FD_n(L) \subseteq UD_n(L)$ of all codes with finite delay. Thus, we have the inclusions $PR_n(L) \subseteq FD_n(L) \subseteq UD_n(L)$ and the set $UD_n(L)$ is non-empty if and only if the set $PR_n(L)$ is non-empty. If $L$ is constant, then each code in $UD_n(L)$ is a block code and we obviously have in this case: $PR_n(L) = UD_n(L)$. As we mentioned above, if the length of $L$ is 1 or 2, then $FD_n(L) = UD_n(L)$. 

2
The aim of this work is to characterize those sequences $L$ for which the equality $PR_n(L) = UD_n(L)$ holds, as well as those sequences $L$ for which $FD_n(L) = UD_n(L)$. For the first characterization, we modify the Kraft’s procedure ([3]) describing the construction of an arbitrary prefix code $C \in PR_n(L)$. This allows us to obtain the following estimation for the quotient $|UD_n(L)|/|PR_n(L)|$ in the case when $L$ is non-constant.

**Theorem 1.** Let $L$ be a non-constant sequence such that the set $UD_n(L)$ is non-empty. Then we have

$$\frac{|UD_n(L)|}{|PR_n(L)|} \geq 1 + \frac{r_a r_b}{|PR_n((a, b))|},$$

where $a$ and $b$ are arbitrary two different values of $L$ and $r_a$ (resp. $r_b$) is the number of those elements in $L$ which are equal to $a$ (resp. to $b$).

As a direct consequence of the above inequality, we obtain the following result.

**Theorem 2.** If the set $UD_n(L)$ is non-empty, then the statements are equivalent:

(i) $UD_n(L) = PR_n(L)$,

(ii) $L$ is constant.

For the second characterization, we involve the Sardinas-Patterson algorithm ([6]) and obtain the following theorem.

**Theorem 3.** If the set $UD_n(L)$ is non-empty, then the statements are equivalent:

(i) $FD_n(L) = UD_n(L)$,

(ii) the length of $L$ is not greater than 2 or, after reordering the elements of $L$, we have $L = (a, a, \ldots, a, b)$, where $a \mid b$.

2. The Kraft’s procedure for prefix codes

Let $L$ be a finite sequence of natural numbers. We now present the Kraft’s method for the construction of an arbitrary code $C \in PR_n(L)$ ([3]), which can be used in deriving the formula for the number of elements in the set $PR_n(L)$.

Let $\tilde{L} := \{\nu_1, \nu_2, \ldots, \nu_l\}$ be the set of values of the sequence $L$ ordered from the smallest to the largest, i.e. $\nu_1 < \nu_2 < \ldots < \nu_l$ and let $r_{\nu_i}$ ($1 \leq i \leq l$) be the number of those elements in $L$ which are equal to $\nu_i$.

To construct an arbitrary code $C \in PR_n(L)$ we proceed as follows. As the code words of length $\nu_1$, we choose arbitrarily $r_{\nu_1}$ words among all the words of length $\nu_1$. This can be done in $\binom{n}{r_{\nu_1}}$ ways. Next, we must arrange the chosen words in $r_{\nu_1}$ available positions of the sequence $C$, which can be done in $r_{\nu_1}!$
ways. For the construction of the code words of length $\nu_2 > \nu_1$, we can use the remaining $n^{\nu_1} - r_{\nu_1}$ available words of length $\nu_1$ as possible prefixes; for the final segments, we can take arbitrary words of length $\nu_2 - \nu_1$. Consequently, the number of ways to construct the code words of length $\nu_2$ is equal to

$$\left(\frac{n^{\nu_2 - \nu_1} \cdot (n^{\nu_1} - r_{\nu_1})}{r_{\nu_2}}\right).$$

Finally, as before, we arrange the chosen words in the sequence $C$, which can be done in $r_{\nu_2}$ ways.

By continuing this reasoning, we see that for every $1 \leq i < l$ the code words of length $\nu_i$ can be chosen arbitrarily among the words of length $\nu_i$ which do not have as a prefix any previously chosen code word. If $N_i$ denotes the number of such available words, then we have

$$N_1 = n^{\nu_1}, \quad N_{i+1} = n^{\nu_i+1-\nu_i}(N_i - r_{\nu_i}), \quad 1 \leq i < l.$$

Hence, for each $1 \leq i \leq l$ the code words of length $\nu_i$ can be constructed and arranged in the sequence $C$ in $\left(\frac{N_i}{r_{\nu_i}}\right)$ ways. Consequently, we obtain the following formula for the cardinality of the set $PR_n(L)$:

$$|PR_n(L)| = \prod_{i=1}^{l} \left(\frac{N_i}{r_{\nu_i}}\right) r_{\nu_i}!.$$  \hspace{1cm} (1)

In particular, if $PR_n(L) \neq \emptyset$, then $n^{\nu_i} \geq N_i \geq r_{\nu_i} \geq 1$ for all $1 \leq i \leq l$, which implies: $N_i > r_{\nu_i}$ for $1 \leq i < l$.

**Example 2.** Let $a, b \geq 1$ be natural numbers. If $C = (v, w)$ and $|v| = a$, $|w| = b$, then in the case $a = b$ we have: $\tilde{L} = \{a\}$ and $r_a = 2$, and in the case $a \neq b$ we have: $\tilde{L} = \{a, b\}$ and $r_a = r_b = 1$. Hence, by formula (1), we obtain:

$$|PR_n((a, b))| = n^{a+b} - n^{\max(a,b)}.$$

The last formula can also be derived directly by the definition of a prefix code, that is without using (1).

3. The sets $UD_n(L)$ for particular sequences $L$

The situation is much more complicated if we want to obtain the formula for the number of elements in the set $UD_n(L)$. Nowadays, there are various algorithms testing the unique decodability of a code. We can use them and try to obtain the formula for $|UD_n(L)|$ in some particular cases of the sequence $L$. In this section, we make the calculations for an exemplary sequence of length three, as well as for the sequences from Theorem 3. Our calculations simultaneously provide the full characterization of the corresponding sets $UD_n(L)$.

The calculations are based on the Sardinas-Patterson algorithm (6), which claims that a code $C$ is uniquely decodable if and only if $C$ is an injective
sequence of non-empty words and \( D_i \cap D_0 = \emptyset \) for all \( i \geq 1 \), where the sets \( D_i \) (\( i \geq 0 \)) are defined recursively as follows: \( D_0 \) is the set of the code words, and for \( i \geq 1 \) the set \( D_i \) is the set of all non-empty words \( w \in X^* \) which satisfy the following condition: \( D_{i-1}w \cap D_0 \neq \emptyset \) or \( D_0 w \cap D_{i-1} \neq \emptyset \), where \( D_i w := \{ vw : v \in D_i \} \).

3.1. The sequence \( L = (2, 3, 3) \)

At first, let us assume that the unique code word of length two consists of two different letters. So, let \( (xy, w, v) \) be a code such that \( x, y \in X \), \( x \neq y \), and \( w, v \in X^3 \), where \( w \neq v \). We have three possibilities: (1) \( (w, v) = (xyz, txy) \) for some \( z, t \in X \), (2) \( (w, v) = (zyy, xty) \) for some \( z, t \in X \), (3) the word \( xy \) is neither a prefix of \( w \) nor a prefix of \( v \) or it is neither a suffix (final segment) of \( w \) nor a suffix of \( v \). In the third case, we obviously have \( (xy, w, v) \in UD_n(L) \).

In the case (1), we have: \( (xy, w, v) = (xy, xyz, txy) \). Now, if \( (z, t) = (x, y) \), then \( (xy)^3 = wv \), and hence \( (xy, w, v) \notin UD_n(L) \). So, let us assume that \( (z, t) \neq (x, y) \). We have now four possibilities: \( z \notin \{ x, t \} \), or \( z = x \neq t \), or \( z = t \neq x \), or \( z = x = t \). If \( z \notin \{ x, t \} \), then \( D_1 = \{ z \} \), \( D_2 = \emptyset \) and hence \( (xy, w, v) \in UD_n(L) \). If \( z = x \neq t \), then \( t \neq y \) and hence \( D_1 = \{ x \} \), \( D_2 = \{ y, yx \} \). \( D_3 = \emptyset \), which implies \( (xy, w, v) \in UD_n(L) \). If \( z = t \neq x \), then \( D_1 = \{ t \} \), \( D_2 = \{ xy \} \), which implies \( (xy, w, v) \notin UD_n(L) \). If \( z = x = t \), then \( D_1 = \{ x \} \), \( D_2 = \{ x, yx, xy \} \) and hence \( (xy, w, v) \notin UD_n(L) \). Thus in the case (1), we obtain: \( (xy, w, v) \notin UD_n(L) \) if and only if \( (w, v) \in UD_n(L) \).

In the other case (2), by taking the reverse of a code \( (xy, w, v) \) and using the same reasoning, we also obtain that there are exactly \( n + 1 \) codes \( (xy, w, v) \) which are non-uniquely decodable. In the case (2), \( D_1 \) is the set of the code words \( (xy, w, v) \) which are non-uniquely decodable. Hence, if \( x \neq y \), then the number of elements in the set

\[
C(x, y) := \{(xy, w, v) : w, v \in X^3 \} \cap UD_n(L)
\]

is equal to

\[
|C(x, y)| = n^3(n^3 - 1) - 2(n + 1).
\]

We now calculate for a fixed \( x \in X \) the number of elements in the set

\[
C(x) := \{(xx, w, v) : w, v \in X^3 \} \cap UD_n(L).
\]

For any \( w, v \in X^3 \setminus \{xxx\} \) with \( w \neq v \) there are two cases: (1) \( xx \) is both the prefix of at least one of the words \( w, v \) and the suffix of at least one of the words \( w, v \), (2) \( xx \) is neither a prefix of \( w \) nor a prefix of \( v \) or it is neither a suffix of \( w \) nor a suffix of \( v \). In the second case, we have \( (xx, w, v) \in UD_n(L) \).

In the first case, we have two possibilities: (1a) \( (xx, w, v) = (xx, xyz, zxx) \) or (1b) \( (xx, w, v) = (xx, yxx, zxx) \) for some \( y, z \in X \setminus \{ x \} \). Both in the case (1a) and in the case (1b) we have: if \( y = x \), then for the code \( (xx, w, v) \) we obtain: \( xx \in D_2 \) and hence \( (xx, w, v) \notin UD_n(L) \). If \( y \neq z \), then \( D_1 \subseteq \{ y, z \} \) and \( D_2 = \emptyset \), and hence \( (xx, w, v) \in UD_n(L) \). Thus the number of all codes of the form \( (xx, w, v) \in C(x) \) satisfying (1) is equal to \( 2((n - 1)^2 - (n - 1)) \), and the number
of all codes \((xx, w, v) \in C(x)\) satisfying (2) is equal to \((n^3 - 1)(n^3 - 2) - 2(n-1)^2\).

Hence

\[ |C(x)| = (n^3 - 1)(n^3 - 2) - 2(n-1). \]

Finally, we obtain

\[ |UD_n(L)| = \sum_{x,y \in X, x \neq y} |C(x, y)| + \sum_{x \in X} |C(x)| = n(n-1)(n^6 + n^5 - n^4 - 2n^2 - 6). \]

For comparison, we obtain by the formula (1):

\[ |PR_n(L)| = n(n-1)(n^6 + n^5 - n^4 - 2n^3 - n^2). \]

3.2. The sequences of the form \(L = (a, \ldots, a, b)\), where \(a \mid b\)

Let \(L = (a, \ldots, a, b)\) be a sequence of length \(m > 1\) such that \(a \mid b\). If \(a = b\), then \(L\) is constant and hence \(UD_n(L) = PR_n(L)\). Let us assume that \(q := b/a > 1\). If \(C \in UD_n(L)\), then obviously the code \(C\) must be of the form \((v_1, \ldots, v_{m-1}, w)\) for some pairwise different words \(v_i\) \((1 \leq i \leq m-1)\) of length \(a\) and the word \(w\) of length \(b\) which is not of the form \(v_{j_1}v_{j_2} \ldots v_{j_q}\) for some \(j_i \in \{1, \ldots, m-1\}, i = 1, 2, \ldots, q\). Conversely, let us assume that \(C\) is an arbitrary code of the form \((v_1, \ldots, v_{m-1}, w)\), where the words \(v_i, w\) are as above. We show that \(C \in UD_n(L)\). Indeed, since \(|w| = qa\), we have \(w = w_1 \ldots w_q\) for some words \(w_i\) \((1 \leq i \leq q)\) each of length \(a\). Let \(1 \leq i_0 \leq q\) be the smallest index such that \(w_{i_0} \notin \{v_1, \ldots, v_{m-1}\}\). For \(1 \leq i < i_0\) let us consider the word \(w_i := w_{i+1} \ldots w_q\). Because of the minimality of \(i_0\), none of the words \(u_i\) \((1 \leq i < i_0)\) is a prefix of \(w\). Hence for \(1 \leq i < i_0\) we have \(D_i = \{u_i\}\) and for \(i \geq i_0\) we have \(D_i = \emptyset\). Thus \(D_i \cap D_0 = \emptyset\) for each \(i \geq 1\), and hence \(C \in UD_n(L)\). Now, by easy calculation, we obtain the following formula:

\[ |UD_n(L)| = n^a(n^a - 1) \ldots (n^a - m + 2)(n^b - (m - 1)^{b/a}). \]

For comparison, we have by (1):

\[ |PR_n(L)| = n^a(n^a - 1) \ldots (n^a - m + 2)(n^b - (m - 1)n^{b-a}). \]

In particular, the above formula for \(|UD_n(L)|\) also works in the case \(a = b\).

4. The proofs of the main results

In this section we derive our main results.

**Theorem 1.** Let \(L\) be a non-constant sequence such that the set \(UD_n(L)\) is non-empty. Then we have

\[
\frac{|UD_n(L)|}{|PR_n(L)|} \geq 1 + \frac{r_a r_b}{|PR_n((a, b))|},
\]

where \(a\) and \(b\) are arbitrary two different values of \(L\) and \(r_a\) (resp. \(r_b\)) is the number of those elements in \(L\) which are equal to \(a\) (resp. to \(b\)).
Proof. We will use the notations as in Section 2 i.e. by $\overline{L} := \{\nu_1, \ldots, \nu_l\}$, we denote the set of values of the sequence $L$ ordered from the smallest to the largest, i.e. $\nu_1 < \nu_2 < \ldots < \nu_l$ and by $r_{\nu_i}$ ($1 \leq i \leq l$) we denote the number of those elements in $L$ which are equal to $\nu_i$. Without losing generality, we can assume that $a < b$. Let $i_0, i_1 \in \{1, \ldots, l\}$ be indices corresponding to the values $a, b \in \overline{L}$, i.e. $\nu_{i_0} = a$, $\nu_{i_1} = b$. Let us fix two different letters $0, 1 \in \mathbb{X}$ and let $PR_{n,a,b}(L)$ be the subset of $PR_n(L)$ consisting of prefix codes with the words $w_a := 0^{a-1}1$, $w_b := 0^{b-1}$ as code words.

An arbitrary code $C \in PR_{n,a,b}(L)$ can be constructed as follows. At first, for every $1 \leq i < i_0$, we choose the code words of length $\nu_i$ and arrange them in the sequence $C$ in the same way as in the Kraft’ procedure keeping only in mind not to choose the “zero” word $0^{\nu_i}$. Thus for every $1 \leq i < i_0$ the number of available words for the code words of length $\nu_i$ is equal to $N_i - 1$ and hence, the number of ways to construct these code words and arrange them in the sequence $C$ is equal to

$$\left(\frac{N_i - 1}{r_{\nu_i}}\right)_{r_{\nu_i}}! = \frac{N_i}{r_{\nu_i}} \cdot \left(\frac{N_i}{r_{\nu_i}}\right)_{r_{\nu_i}}! = n^{\nu_i - \nu_{i_0} + 1} \cdot \left(\frac{N_i}{r_{\nu_i}}\right)_{r_{\nu_i}}!.$$

Note that for $1 \leq i < i_0$ we have $N_i > r_{\nu_i}$, and hence the above number is indeed positive.

For the construction of the code words of length $\nu_{i_0} = a$, we also remember that $0^a$ can not be a code word. Beside of that, the word $w_a = 0^{a-1}1$ must be a code word. Hence, we need to choose $r_a - 1$ words of length $a$ among all $N_{i_0} - 2$ available words, and next to arrange the chosen words together with the word $w_a$ in $r_a$ available positions in the sequence $C$. Thus the number of ways to construct the code words of length $a$ and arrange them in $C$ is equal to

$$\left(\frac{N_{i_0} - 2}{r_a - 1}\right)_{r_a}! = \frac{r_a}{N_{i_0} - 1} \cdot n^{\nu_{i_0} - \nu_{i_0} + 1} \cdot \left(\frac{N_{i_0}}{r_a}\right)_{r_a}!.$$

Since $i_0 < l$, we have $N_{i_0} > r_a$ and hence this number is indeed positive.

In the next step, we construct for every $i_0 < i < i_1$ the code words of length $\nu_i$. We are still restricted to the words different from $0^a$ and hence, the number of ways to do this is equal to

$$\left(\frac{N_i - 1}{r_{\nu_i}}\right)_{r_{\nu_i}}! = n^{\nu_i - \nu_{i_0} + 1} \cdot \left(\frac{N_i}{r_{\nu_i}}\right)_{r_{\nu_i}}!.$$

For the construction of the code words of length $\nu_{i_1} = b$, we must remember that the word $w_b = 0^{b-1}1$ is a code word. But now, we can choose the “zero” word $0^b$ as a code word. Hence, we need to choose $r_b - 1$ words among $N_{i_1} - 1$ available words. In consequence, the number of ways to construct the code words of length $b$ and arrange them in the sequence $C$ is equal to

$$\left(\frac{N_{i_1} - 1}{r_b - 1}\right)_{r_b}! = \frac{r_b}{N_{i_1}} \cdot \left(\frac{N_{i_1}}{r_b}\right)_{r_b}!.$$
Since \( N_{i_1} \geq r_b \), this number is indeed positive.

In the final step, we construct for every \( i_1 < i \leq l \) the code words of length \( \nu_i \). This construction can be done in \( \left( \frac{N_i}{r_{\nu_i}} \right) \nu_i ! \) ways, as we can follow exactly in the same way as in the Kraft’s procedure.

As a result of the above procedure, we see that the number of ways to construct an arbitrary code from the set \( PR_{n,a,b}(L) \) is equal to

\[
\prod_{1 \leq i < i_1} \left( \frac{n^{\nu_{i-\nu_{i+1}}} \cdot N_{i+1}}{N_i} \right) \cdot r_a \cdot \frac{r_b}{N_{i_0} - 1} \prod_{i=1}^{i_1} \left( \frac{N_i}{r_{\nu_i}} \right) \nu_i ! = \frac{r_a r_b}{n^b (N_{i_0} - 1)} |PR_n(L)|.
\]

Hence, we obtain

\[
|PR_{n,a,b}(L)| = \frac{r_a r_b}{n^b (N_{i_0} - 1)} |PR_n(L)|.
\]

By the inequality \( N_{i_0} \leq n^{\nu_{i_0}} = n^a \), we have:

\[
|PR_{n,a,b}(L)| \geq \frac{r_a r_b}{n^b (n^a - 1)} |PR_n(L)| = r_a r_b \frac{|PR_n(L)|}{|PR_n((a,b))|}.
\]

To finish the proof, it suffices to observe that if \( C \in PR_{n,a,b}(L) \), then for the reverse \( C^R \) we have \( C^R \in UD_n(L) \setminus PR_n(L) \). Indeed, the words \( (w_a)^R \), \( (w_b)^R \) are code words in \( C^R \) and the word \( (w_a)^R = 1(0^{a-1}) \) is a prefix of the word \( (w_b)^R = 1(0^{b-1}) \). Since for the arbitrary codes \( C_1, C_2 \) we have \( C_1 = C_2 \Leftrightarrow C^R = C^R \), we conclude the equality

\[
|PR_{n,a,b}(L)| = \{|C^R : C \in PR_{n,a,b}(L)|\}.
\]

In consequence, we obtain

\[
|UD_n(L)| \geq |PR_n(L)| + |PR_{n,a,b}(L)| \geq |PR_n(L)| \left( 1 + \frac{r_a r_b}{|PR_n((a,b))|} \right).
\]

□

**Theorem 3.** If the set \( UD_n(L) \) is non-empty, then the statements are equivalent:

(i) \( FD_n(L) = UD_n(L) \),

(ii) the length of \( L \) is not greater than 2 or, after reordering the elements of \( L \), we have \( L = (a,a,\ldots,a,b) \), where \( a \mid b \).

**Proof.** At first, we show the implication (ii) \( \Rightarrow \) (i). If \( L \) has the length at most 2, then according to [2], every code in \( UD_n(L) \) has finite delay. If \( L = (a,a,\ldots,a,b) \), where \( a \mid b \), then we have two possibilities: \( a = b \) or \( a \not\mid b \). In the first case \( L \) is constant and then each \( C \in UD_n(L) \) is a prefix code, which implies that \( C \) has finite delay.
If \( a \neq b \), then each \( C \in UD_n(L) \) also has finite delay. To show this, let us assume that we have picked up the first \( b \) letters of a word \( u \in X^* \), for which we only know that it is factorizable into code-words. Let \( w \) be the prefix of \( u \) of length \( b \). We have two possibilities: \( w \) is not a code word or \( w \) is a code word.

In the first case, since there are no code words longer than \( b \) and all the code words shorter than \( b \) have the length \( a \), the prefix of length \( a \) in the word \( u \) must be a code word and \( u \) begins with this code word.

In the second case, \( w \) is the code word with which the word \( u \) starts. To show this, let us suppose contrary that \( u \) does not begin with \( w \). Since \( u \) is the only code word of length \( b \) and all the other code words have the length \( a < b \), there must be the maximum number \( k \geq 1 \) such that the word \( w_1 \ldots w_k \) is a prefix of \( w \), where each \( w_i \) is a code word of length \( a \). Now, if \( w_1 \ldots w_k = w \), then \( C \) would not be uniquely decodable. So, let us assume that \( w_1 \ldots w_k \) is a proper prefix of \( w \). In particular, we obtain \( ka < b \). Hence there is a code word \( v \) such that \( w_1 \ldots w_k v \) is a prefix of \( u \). Now, if \( |v| = a \), then in view of the inequality \( ka < b \) and the divisibility \( a \mid b \), we would have \( (k + 1)a \leq b \) and consequently, the word \( w_1 \ldots w_k v \) would be a prefix of \( w \), contrary to the maximality of \( k \).

Hence \( |v| = b \), which implies \( v = w \). Thus \( w \) must be a prefix of \( w_1 \ldots w_k w \). But then the divisibility \( a \mid b \) implies the equality \( w = (w_1 \ldots w_k)^sw_1 \ldots w_k \) for some \( s \geq 0, 1 \leq k' < k \), and again we have a contradiction with the assumption that \( C \) is uniquely decodable. Thus in each case it is enough to pick up at most \( b \) letters of the word \( u \) to know which code word begins this word.

To show (i) \( \Rightarrow \) (ii) let us assume that \( L \) does not satisfy the condition (ii). We must show that there is a code \( C \in UD_n(L) \) with infinite delay. The sequence \( L \) has the length at least three and \( L \) is not constant. Let \( a \) and \( b \) be the two smallest values of \( L \) and let us assume that \( a < b \). Let us define in the same way as in the proof of Theorem [1] the set \( \bar{L} = \{v_1, \ldots, v_l\} \) of the values of \( L \), the sequence \( (r_{\nu_i})_{1 \leq i \leq l} \), the words \( w_a = 0^{a-1}1, w_b = 0^{b-1}1 \), and the subset \( PR_{n,a,b}(L) \subseteq PR_n(L) \). In particular, we have: \( \nu_1 = a, \nu_2 = b \).

If \( r_1 > 1 \), then we can use the construction of the code described in the proof of Theorem [1] and obtain a code \( C \in PR_{n,a,b}(L) \) such that one of its code words of length \( b \) is \( 0^b \). Since \( C^R \in UD_n(L) \) and the words \( 1(0^{a-1}), 1(0^{b-1}), 0^{b} \) are code words in \( C^R \), we see, by analogy to Example [1] that \( C^R \) has infinite delay.

If \( r_1 = 1 \) and \( L \) has at least three values, then there is the smallest \( 1 \leq i_0 \leq l \) such that \( \nu_{i_0} > b \). Similarly as in the previous case, we can use the construction from the proof of Theorem [1] and construct a code \( C \in PR_{n,a,b}(L) \), such that \( 0^{\nu_{i_0}} \) is one of the code words. Then the words \( 1(0^{a-1}), 1(0^{b-1}) \) and \( 0^{\nu_{i_0}} \) are code words in \( C^R \in UD_n(L) \), and similarly as above, we obtain that \( C^R \) has infinite delay.

The last case is when \( r_1 = 1 \) and the only values of \( L \) are \( a \) and \( b \). Since \( L \) does not satisfy the condition (ii), we obtain \( a \mid b \). Let \( \eta \in \{1, \ldots, a - 1\} \) be the remainder from the division of \( b - a \) by \( a \). Then we have \( b - a = qa + \eta \) for some integer \( q \geq 0 \). Since \( L \) has the length at least three, we have \( 2 \leq r_a < a^0 \). Thus, there is an injective code \( C \) with the sequence \( L \) as the sequence of code word lengths and such that the words \( 1^n0^{b-a}, 1^a, 0^a \) are the code words and the word \( 1^a - n0^n \) is not a code word. Then the infinite word \( 1^a0^\infty \) has two factorizations.
into code words:

\[1^a - 0^a - 0^a - \ldots,\]
\[1^a0^{b-a} - 0^a - 0^a - \ldots.\]

Thus it is enough to show that \(C\) is uniquely decodable. For this aim, we need to show that \(D_i \cap D_0 = \emptyset\) for all \(i \geq 1\), where the sets \(D_i\) \((i \geq 0)\) are constructed according to the Sardinas-Patterson algorithm, i.e. \(D_0\) is the set of the code words, and for \(i \geq 1\) the set \(D_i\) consists of all non-empty words \(w \in X^*\) for which the following condition holds: \(D_{i-1}w \cap D_0 \neq \emptyset\) or \(D_0w \cap D_{i-1} \neq \emptyset\).

Let \(S\) be the set of all non-empty words which are proper suffixes (final segments) of the code words. Obviously, every word in \(S\) is shorter than \(b\). Hence the intersection \(S \cap D_0\) contains only the code words of length \(a\) which are the proper suffixes of the other code words. Since \(1^a0^{b-a}\) is the only code word of length greater than \(a\), the set \(S \cap D_0\) consists of the code words of length \(a\) which are suffixes of the code word \(1^a0^{b-a}\). Thus, if \(b-a > a\), then \(S \cap D_0 = \{0^a\}\). If \(b-a < a\), then \(\eta = b-a\), and hence \(S \cap D_0 = \emptyset\), as the suffix of length \(a\) in the code word \(1^a0^{b-a}\) is \(1^{2a-b}0^{b-a} = 1^a \eta 0^n\), which, by our assumption, is not a code word. Hence, we obtain:

\[S \cap D_0 = \begin{cases} \emptyset, & \text{if } b-a < a, \\ \{0^a\}, & \text{if } b-a > a. \end{cases}\] (2)

**Lemma 1.** For every \(i \geq 1\) the inclusion \(D_i \subseteq S\) holds.

**Proof (of Lemma 1).** By the definition of \(D_1\), we have \(D_1 = \{0^{b-a}\} \subseteq S\). Let us assume inductively that \(D_i \subseteq S\) for some \(i \geq 1\). Let \(w \in D_{i+1}\) be an arbitrary word. Then \(D_iw \cap D_0 \neq \emptyset\) or \(D_0w \cap D_i \neq \emptyset\). In the first case, we have \(vw \in D_0\) for some nonempty word \(v \in X^*\), i.e. \(w\) is a proper suffix of the code word \(vw\), and hence \(w \in S\). In the second case, we have \(vw \in D_i\) for some \(v \in X^*\). By the inductive assumption, we obtain \(vw \in S\), i.e. \(w\) is a proper suffix of a code word, and hence \(w\) is also a proper suffix of this code word. Thus \(w \in S\) and consequently, we have \(D_{i+1} \subseteq S\). \(\square\)

Suppose now that \(D_i \cap D_0 \neq \emptyset\) for some \(i \geq 1\). Since \(D_i \subseteq S\), we obtain by (2):

\[D_i \cap D_0 = \{0^a\}.\]

Consequently, there is the smallest number \(i \geq 1\) such that \(0^{\lambda a} \in D_i\) for some integer \(\lambda \geq 1\). Since \(D_1 = \{0^{b-a}\}\) and \(a \nmid b\), we have \(i \geq 2\). By the definition of the set \(D_i\) we have \(D_{i-1}0^{\lambda a} \cap D_0 \neq \emptyset\) or \(D_00^{\lambda a} \cap D_{i-1} \neq \emptyset\). In the first case, we obtain that \(v0^{\lambda a}\) is a code word for some \(v \in D_{i-1}\). Since \(|v0^{\lambda a}| > a\), it must be \(v0^{\lambda a} = 1^a0^{b-a}\), and hence \(v = 1^{a(b-(\lambda+1))}a\). Since \(v \in S\) and \(|v| > a\), the word \(v\) must be a suffix of the code word \(1^a0^{b-a}\) and we obtain a contradiction because the word \(1^a0^{b-(\lambda+1)a}\) is not a suffix of \(1^a0^{b-a}\).

In the second case, we have \(v0^{\lambda a} \in D_{i-1}\) for some code word \(v \in D_0\). Since \(D_{i-1} \subseteq S\) and \(|v0^{\lambda a}| > a\), the word \(v0^{\lambda a}\) must be a suffix of the code word \(1^a0^{b-a}\). Since \(v\) is a code word, we obtain \(|v| = a\). Thus \(v\) must be of the form \(0^a\) or \(1^a\) or \(1^a0^{b}\) for some integer \(0 < \gamma \leq a-1\). If \(v = 0^a\), then \(0^{(\lambda+1)a} = v0^{\lambda a} \in D_{i-1}\) and we obtain a contradiction with the minimality of
i. If \( v = 1^a \), then the word \( v0^\lambda a = 1^a0^\lambda a \) is a suffix of the code word \( 1^a0^{b-a} \); consequently, it must be \( \lambda a = b - a \), and again we have a contradiction with \( a \nmid b \). Hence, it must be \( v = 1^a0^\gamma \) for some integer \( 0 < \gamma \leq a - 1 \). But then \( v0^\lambda a = 1^{a-\gamma}0^{\gamma+\lambda} a \). Consequently, the word \( 1^{a-\gamma}0^{\gamma+\lambda} a \) is a suffix of the code word \( 1^a0^{b-a} \). In particular, we obtain \( \gamma + \lambda a = b - a \). But, since \( b - a = qa + \eta \) and \( 0 < \eta \leq a - 1 \), we obtain \( \gamma = \eta \) and \( \lambda = q \). Thus \( v = 1^{a-\eta}0^\eta \) and we have a contradiction with the assumption that \( 1^{a-\eta}0^\eta \) is not a code word. Consequently \( D_i \cap D_0 = \emptyset \) for every \( i \geq 1 \). Thus \( C \in UD_n(L) \), which completes the proof of Theorem 3. \qed

References

[1] J. Berstel, D. Perrin, Theory of codes Pure and Applied Mathematics, vol. 117. Academic Press Inc., Orlando, FL, 1985.

[2] R. V. Book, C. Kwan, On uniquely decipherable codes with two codewords, IEEE Trans. Comput. 29 (1980) 324-325.

[3] L. G. Kraft (1949), A device for quantizing, grouping, and coding amplitude modulated pulses, Cambridge, MA: MS Thesis, Electrical Engineering Department, Massachusetts Institute of Technology.

[4] M. Lothaire, Algebraic Combinatorics on Words, Encyclopedia of Mathematics and its Applications, vol. 90. Cambridge University Press, Cambridge (2002).

[5] B. McMillan, Two inequalities implied by unique decipherability, IEEE Trans. Information Theory 2 (4): 115116, (1956).

[6] A. Sardinas, G. W. Patterson, A necessary and sufficient condition for the unique decomposition of coded messages, Convention Record of the I.R.E., 1953 National Convention, Part 8: Information Theory, pp. 104108.