Extrapolation of the solar magnetic field within the potential-field approximation from full-disk magnetograms

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Abstract. This paper is concerned with the Laplace boundary-value problem with the directional derivative, corresponding to the specific nature of measurements of the longitudinal component of the photospheric magnetic field. Boundary conditions are specified by a distribution on the sphere of projection of the magnetic field vector unto a given direction. It is shown that the solution of this problem exists in the form of a spherical harmonic expansion, and uniqueness of this solution is proved. A conceptual sketch of numerical determination of the harmonic series coefficients is given. The field of application of the method is analyzed having regard to the peculiarities of actual data. Finally, we present differences in results derived from extrapolating the magnetic field from a synoptic map and a full-disk magnetogram.

1. Introduction

Until the present time the practical potential-field extrapolation of solar magnetic fields has been based on several thoroughly studied Laplace boundary-value problems when applied to two cases of geometry: spherics, and planar geometry. The spherics, as a rule, is used in investigating large-scale long-lived global fields. The planar geometry is exploited to study magnetic active regions occupying small areas on the solar surface.

When employing the spherics, two approaches are distinguished to specifying boundary conditions on the solar surface. One is that it is specified the synoptic distribution of a so-called component $B_l$ (a projection of the magnetic field measured along the line-of-sight direction is assigned to each point of the surface at the time when a given point passes through the central meridian (Altschuler et al., 1977) and (Hoeksema, 1984)). In the other approach, a normal component $B_r$ (Wang, Y.-M. and N.R. Sheeley, 1992)) is specified, which is constructed from $B_l$ assuming that the full vector of the magnetic field is radial. The latter case corresponds to the setting of a classical Neumann boundary-value problem, and the former case refers to a particular variant of the boundary-value problem with the directional derivative (the direction of the derivative with respect to a normal varies according to the heliographic latitude). Data of daily magnetograms do not satisfy these conditions. They are only approximately satisfied by points lying in the neighborhood of the central meridian. Synoptic
maps when generated from series of daily magnetograms accumulated during one solar rotation provide input information corresponding to
the above-mentioned boundary conditions.

Using the planar geometry, unlike the spherics, provides insight into the three-dimensional spatial structure at the time of measurement
thus enabling one to track the temporal evolution of magnetic active
regions. This approach has two important limitations. In the first place, the region under study must be sufficiently small; secondly, it must lie
in the middle of the solar disk where the normality approximation of the measured component of field is valid. Boundary conditions in this
case correspond to the Neumann problem. A variant of the problem
with a constant slope of the derivative (for noncentral regions) can
also be used but at a still greater sacrifice in the size of the region if
variations in the line-of-sight slope to a normal inside the region are to
be neglected.

This paper offers the solution of a Laplace boundary-value problem
(in the spherics) with boundary conditions corresponding to measure-
ments of the longitudinal magnetic components throughout the disk
(data from daily magnetograms). Such a setting of the problem con-
forms to a Laplace problem with a variable directional derivative (by
choosing a rigorously specified projection direction corresponding to the
direction of the line of sight existing at a given time of measurement,
the direction, with respect to which a derivative of the scalar magnetic
potential is taken, changes relative to a normal on the boundary spheri-
cal surface). This let to use completely longitudinal magnetogram data
which are the most sensitive and generally applied. We demonstrate
the existence of a strictly definite harmonic expansion of the potential
field in terms of spherical eigenfunctions, corresponding to the above
boundary conditions. The construction of such a solution is based on
the combined use of Galerkin’s projection method and Trefftz’ method
(the solutions are sought in a class of solutions of the Laplace equation).
The proof of the existence and uniqueness of the solution is constructed.
We present and analyze results of numerical tests simulating the extrap-
olation of actual magnetograms. The limitations of the approach used
in this study are discussed, which are associated with the resolution of the
magnetograms employed and with the uncertainty of specifying the
magnetogram of the averted side of the Sun (”back magnetogram”).
A comparison is made of results derived from extrapolating from a
synoptic magnetogram and from a full-disk magnetogram.
2. Statement of the boundary-value problem

Assume that a force-free currentless approximation holds either throughout the space above the photospheric surface \( S_p \), or between the photosphere and some spherical surface \( S_s \) (source surface). Consider a boundary-value problem

\[
\mathbf{B}(\mathbf{r}) = -\nabla \Psi(\mathbf{r}) \tag{1}
\]

\[
\Delta \Psi(\mathbf{r}) = 0, \mathbf{r} \in \Omega, \partial \Omega = S \tag{2}
\]

\[-\mathbf{d} \cdot \nabla \Psi(\mathbf{r}) = B_d(\theta, \varphi) \quad \text{for} \quad \mathbf{r} \in S_p \tag{3}
\]

\[
\begin{cases}
  a) \quad \Psi(\mathbf{r}) = 0 & \text{for} \quad \mathbf{r} \in S_s, \quad S = S_p \cup S_s, \\
  b) \quad \Psi(\mathbf{r}) \propto O(1/r), \quad r \to \infty, \quad S = S_p
\end{cases} \tag{4}
\]

Here \( \Psi \) is the scalar potential of the magnetic field; \( \mathbf{d} \) is a unit vector directed along the line of sight on to the observer; and \( B_d(\theta, \phi) \) is the distribution of the magnetic component specified over the entire spherical surface. The distribution \( B_d(\theta, \phi) \) corresponds to two magnetograms of the visible and averted sides of the Sun (it is assumed that there is a change of sign in the case of the back magnetogram). Different ways of specifying boundary conditions on the averted side will be discussed below. Two methods of specifying boundary conditions on the outer surface (4)-a) and (4)-b) correspond to the most frequently used typical settings. Condition (4)-a) corresponds to the magnetic field radiality, and condition ((Altschuler et al., 1977)) (4)-b) refers to a regular (at infinity) solution (such a setting is frequently used in treatments of more sophisticated, combined magnetic field models involving a potential-field model, either as a part of a general model ((Zhao and Hoeksema, 1994)), or as the initial approximation of a general model ((Aly and Seehafer, 1993) and (Amary et al, 1997)).

We shall pursue the solution of this boundary-value problem in a standard form of harmonic expansion in terms of eigensolutions of the Laplace equation written in a spherical coordinate system. The spherical coordinate system \((r, \theta, \phi)\) is chosen here to conform to a Cartesian one, in which the axis \( z \) coincides with the direction \( \mathbf{d} \). Such a choice, as will be shown later in the text, simplifies greatly the solution of this problem. For the sake of definiteness, the axis \( x \) is assumed to lie in the plane produced by the axis \( z \) and the heliographic axis \( z' \) (from here on we shall use primed symbols related to the heliographic coordinate system) directed northward; the axis \( y \) is taken to correspond to a right-hand coordinate system.
3. Method of solving the boundary-value problem

As the starting point, we use the following form of harmonic expansion of the scalar magnetic potential:

\[ \Psi(r, \theta, \phi) = R \sum_{l=1}^{\infty} \sum_{m=-l}^{l} c_l^m f_l(r) \tilde{P}_l^{|m|} (\cos \theta) e^{im\phi}, \]  

(5)

where \( R \) is the solar radius, \( c_l^m \) stands for the sought-for complex coefficients of expansion

\[ c_l^{-m} = c_l^m, \]  

(6)

\[ \tilde{P}_l^m = P_l^m / \sqrt{2\pi w_l^m}, P_l^m \] – represents the Legendre functions ((Abramowitz and Stegun 1964)), and  

\[ w_l^m = \int_{-1}^{1} [P_l^m(u)]^2 du = (l + 1/2)^{-1}(l + m)!/(l - m)! \]  

(7)

The expression (5) is an exact solution if the summation with respect to the index \( l \) is made ad infinitum, or, a finite approximation, if the summing is limited to the value of \( L \) (the main index of expansion). As is customary, the contribution of the monopole term of expansion \( l = 0 \) is neglected in this case. Let the dependencies \( f_l(r) \) be defined for two cases of upper boundary conditions:

\[ f_l(r) = \begin{cases} \frac{1}{a_l r^{l-1}} (a_l r^{l-1} - \tilde{\tau}) & \text{for case (4) - a} \\ \frac{1}{\tau^{l-1}} & \text{for case (4) - b} \end{cases}, \]  

(8)

where \( a_l = (R_s/R)^{2l+1}, \tau = r/R, \) and \( R_s \) is the source radius. In the form (5), each term of expansion satisfies the Laplace equation (2) and the upper boundary conditions (4). We seek the solution for (5) in a class of square-integrable functions on a sphere \( R \), which is equivalent to the boundedness condition of infinite sums of the form:

\[ \sum_{l=1}^{\infty} \sum_{m=-l}^{l} |c_l^m|^2 < \infty \]  

(9)

In accordance with the designations used here, we substitute (5) into (3), multiply the resulting equality by \( \tilde{P}_l^{|m|} (\cos \theta) e^{-im\phi} \cos \theta \sin \theta d\phi d\theta \) and write the system of equalities in the form of equalities of integrals over angular coordinates.
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\[ \int_0^{2\pi} \int_0^\pi B_d(\theta, \varphi) \tilde{P}_l^m(\cos \theta) e^{-im\varphi} \cos \theta d\phi d\theta \]

\[ = \sum_{l=1}^{\infty < L} c_k^m \int_{-1}^1 \left[ g_k u^2 \tilde{P}_k^m(u) \tilde{P}_l^m(u) + u(u^2 - 1) \frac{d\tilde{P}_k^m}{du} \tilde{P}_l^m(u) \right] du \] (10)

Here \( u = \cos \theta \) and \( g_k = -\frac{d}{dr}r_{(1)} > 0 \). Using the orthogonality property of Legendre polynomials and the known recurrence relations for expressions of the form \( uP_m^m \) and \((u^2 - 1)dP_m^m/du\) ((Abramowitz and Stegun 1964)), and upon introducing the designation for the left-hand side of the equalities in (10) \( b_l^m \) (these quantities will be called the weight coefficients), the expression (10) can be brought into the form

\[ b_l^m = a_{l-2}^m + a_0^m l^m + a_{l+2}^m, \] (12)

where

\[ a_{l-2}^m = \frac{w_{l-1}^m}{w_{l-2}^m w_l^m} \frac{(l-m-1)(l+m)}{(2l-3)(2l+1)} (g_{l-2} + l - 2) \] (13)

\[ a_0^m = \left\{ \begin{array}{c} \frac{w_{l+1}^m}{w_l^m} \frac{(l-m-1)^2}{2l+1} (g_l + l) + \frac{w_{l-1}^m}{w_l^m} \frac{(l+m)^2}{2l+1} (g_{l-1} - l - 1) \\ \end{array} \right\} \]

\[ a_{l+2}^m = \frac{w_{l+1}^m}{w_{l+2}^m w_l^m} \frac{(l-m+1)(l+m+2)}{(2l+5)(2l+1)} (g_{l+2} - l - 3) \]

In view of the property (6), for determining the coefficients \( c_l^m \), it will suffice to exploit equation (12) for positive \( m \) only. The equalities (12) break down into \( m \) independent systems of equations, each of which is representable in a matrix form (for the finite expansion up to \( L \))

\[ A_{ij}^m C_j^m = B_i^m, \] (14)

where

\[ A_{ij}^m = \delta_{j-2}^i a_{l-2}^m 2j^m + \delta_{j}^i a_0^m 2j^m + \delta_{j-1}^i a_{l+2}^m 2j^m, \] (15)

\[ C_j^m = c_{l-2}^m 2j^m, B_i^m = b_{l-2}^m 2j^m, 1 \leq i, j \leq (L-m)/2 + 1, \delta_{ij}^m = \left\{ \begin{array}{c} 1, i = j \\ 0, i \neq j \end{array} \right\} \]

For each value of \( m \), the matrices \( \tilde{A}^m \) in (14) are of the tridiagonal form and are always nonsingular. The numerical solution of matrix equations of such a form presents no special problems. Noteworthy are
two factors which reduced the search for the desired coefficients to a simple scheme. This implies the selection of a special coordinate system and the inclusion of the term \( \cos(\theta) \) in the integrals of (10). A similar procedure, which involves multiplying by an additional function \( \sin(\theta') \), was implemented in a good-performance method of reconstructing the magnetic field from the component \( B_l \) ([Hoeksema, 1984]), a component of the magnetic field along the line of sight at the time of central meridian passage by the point of its determination. Searching the harmonic expansion in [Hoeksema, 1984] reduces to equations similar to (15), with coefficients having the same qualitative character of behavior depending on its indices. Although test calculations discussed below fully justify the new constructive approach and make an impressive case in favor of its correspondence to the boundary-value problem under consideration, we wish to give a more rigorous mathematical rationale to the proposed method.

4. Existence and uniqueness of the solution of the boundary-value problem

To prove the existence, it will suffice (assuming a square-integrability of the distributions \( \cos(\theta)B_d \)) to show a uniform convergence in the indices \( L \) and \( m \) of the inverse matrix norms \( (\hat{A}^m)^{-1} \) involved in the expression (14). We confine ourselves to case (4)-b) of upper boundary conditions where the structure of the matrices \( \hat{A}^m \) becomes triangular (the elements above the principal diagonal, corresponding to the coefficients \( a_+ \), are zero). For triangular matrices, the elements of the principal diagonal are known to be their eigenvalues. Since the norm of any nonsingular inverse matrix is \( |\lambda_{\min}|^{-1/2} \) (\( \lambda_{\min} \) being the least eigenvalue of the initial matrix), our problem implies analyzing minimum elements of the principal diagonals of the matrices under consideration. Diagonal elements for case (4)-b) of upper boundary conditions have the form:

\[
a^{m,l}_0 = \frac{(2k-1)^2}{2k+1} 2m + 1 + 2k, \quad (l - m) = 2k = 0, 1, 2, \ldots
\]  

(16)

Straightforward examination of formula (16) reveals that the value of \( a^{m,l}_0 \) independently on \( m \) for any \( L \) is minimum when \( k = 1 \), i.e.

\[
\lambda^m_{\min} = \frac{12m + 3}{32m + 7} \geq \lambda_{\min} = \lambda^0_{\min} = \frac{1}{7}
\]  

(17)

A uniform convergence of the desired norms both in the index \( L \) and in the index \( m \) immediately follows from (17) (the norm of a full
operator, which transforms the entire infinite set of the coefficients $c$ with positive indices $m$ to an infinite set of the coefficients $b$, is $\sqrt{7}$.

Thus the existence of the solution of the boundary-value problem with upper boundary conditions of the form (4)-b) is proved. In view of the completeness of the functional space considered on a boundary sphere, the uniqueness of the expansion (5) follows automatically (for $r = R$), with coefficients satisfying equations (12). The uniqueness of the resulting solution throughout the region $\Omega$, in turn, follows from the uniqueness of a regular (at infinity) solution of the Dirichlet problem.

In the case of boundary conditions of the form (4)-a) the elements $a_+$ from (13) are not equal to 0, but with respect to elements of the other diagonals, the character of behavior of which remains the same, they are small as the index $l$ tends to infinity. Therefore, in this case, too, the uniform convergence of each finite $m$ in $L$ seems to occur. It has not yet been possible to find an explicit index expression in $m$ for a minimum eigenvalue or for any lower edge of a set of eigenvalues in order to prove a uniform convergence in this index. It will be assumed that there exists at least one solution satisfying the boundary-value problem formulated. In this case it is reasonably straightforward matter to prove its uniqueness. Let us consider this proof.

Assume that there exists a second solution, and let the difference of two solutions be designated by $\chi$. Then, for $\chi$, with the boundary conditions (4)-a) remaining the same, the boundary conditions (3) become homogeneous. For uniqueness, it will suffice to show that the solution of $\chi$ can be only trivial. Consider the inequality:

$$I = \int_{\Omega} (\nabla (d \cdot \nabla \chi)) \cdot (\nabla (d \cdot \nabla \chi)) dV \geq 0. \quad (18)$$

For $I$, we write the following obvious chain of equalities:

$$I = \int_{\Omega} \nabla [(d \cdot \nabla \chi) \cdot \nabla (d \cdot \nabla \chi)] d\mathbf{V} - \int_{\Omega} (d \cdot \nabla \chi) \cdot \Delta (d \cdot \nabla \chi) d\mathbf{V} \quad (19)$$

$$= \int_{S_p \cup S_s} (d \cdot \nabla \chi) (n \cdot \nabla (d \cdot \nabla \chi)) ds = \int_{S_s} (d \cdot \nabla \chi) \frac{\partial}{\partial r} \nabla (d \cdot \nabla \chi) ds$$

Next, we substitute into (19) the expression for a unit vector $d$ in terms of coordinate unit vectors of a spherical coordinate system $d = e_z = \cos \theta e_r - \sin \theta e_\theta$, and extend the chain of equalities:

$$I = \int_{S_s} \left( \cos \theta \chi_r - \frac{\sin \theta}{r} \chi_\theta \right) \left( \cos \theta \chi_{rr} - \frac{\sin \theta}{r} \chi_{r\theta} + \frac{\sin \theta}{r^2} \chi_\theta \right) d\mathbf{s} \quad (20)$$

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\[ I = \int_{-\pi}^{\pi} \int_{0}^{2\pi} \cos \theta \chi_r (2 \cos \theta \chi_r + \sin \theta \chi_{r\theta}) \, d\phi \, d\theta \] (21)

It is used in (20) that angular partial derivatives of any order of the scalar potential are zero on the surface \( S_s \) according to the equality \( \chi (R_s, \theta, \phi) = 0 \). The dependence \( \chi_{rr} \) is expressed in terms of the first derivative \( \chi_r \) from the Laplace equation written in the spherical coordinate system. Furthermore, on performing a change of variables \( u = \cos \theta \) and introducing the designation \( Q(u) = \int_{0}^{2\pi} (\chi_r)^2 \, d\phi \), we continue transforming the integral:

\[ I = -R_s \int_{-1}^{1} \left[ 2u^2 Q - \frac{1}{2} (1 - u^2) uQ_u \right] \, du \] (22)

\[ = -R_s \int_{-1}^{1} Q \left[ 2u^2 + \frac{1}{2} \frac{d}{du} (1 - u^2) u \right] \, du = -\frac{R_s}{2} \int_{-1}^{1} Q (2u^2 + 1) \, du \]

The expression under the integral sign in (22), in view of the definition of \( Q \), is always positive; whence it follows that \( I \leq 0 \). The last inequality is compatible with the inequality (18) only when the integral \( I \) is zero. The following logical chain holds:

\[ I = 0 \Rightarrow \{ [d \cdot \nabla \chi \equiv \text{const}] \land [d \cdot \nabla \chi \equiv 0_{|S_p}] \} \Rightarrow \{ d \cdot \nabla \chi \equiv 0 \} \]

\[ \Rightarrow \{ [\chi(r) = \chi(x, y)] \land [\chi = 0_{|S_s}] \} \Rightarrow \chi \equiv 0 \]

Thus the uniqueness theorem is proved.

5. The method of recurrence calculation of weight coefficients

Preparatory to discussing the practical implementation of the proposed method and results of tests calculations, we will first give some attention to the relevant procedure of calculating weight coefficients. This procedure provides essentially a high accuracy in numerical tests and in research work with particular data. For the time being, we leave...
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aside the problem of populating with data the back side of the solar sphere. Imagine the situation where we have two magnetograms of the visible and invisible sides of the Sun, representing given values of the magnetic component at nodes of a rectangular grid on the picture plane. In the initial stage we scale the input data to a rectangular grid of the plane \((\theta, \phi)\), having a density which at least does not deteriorate the resolution of the input data. Such a translation of the data can be performed, for example, on the basis of a spline-representation of the input data in the picture plane. Next, we determine the continuous distribution \(B_d(\theta, \phi)\) as an approximation of the grid values by cubic local splines. The resulting distribution is continuous up to the second partial derivatives with respect to the indicated arguments. Let the full integrals of weight coefficients defined by formula (10) be divided into elementary sums of integrals over separate small rectangular regions corresponding to the size of the initial grid subdivision. As the function \(B_d(\theta, \phi)\) is represented by cubic polynomials inside of each elementary region, we shall have to evaluate integrals of the form

\[
\left( \int_{\phi_i}^{\phi_{i+1}} \phi^\alpha e^{-i m \phi} d\phi \right) \left( \int_{\theta_j}^{\theta_{j+1}} \theta^\beta P_l^m(\cos \theta) \cos \theta \sin \theta d\theta \right), \quad \alpha, \beta = 0, 1, 2, 3.
\]

(23)

The expression (23) involves two types of integrals. Integrals over \(\phi\) have straightforward analytical expressions. Integrals over \(\theta\) are also expressible analytically. With low indices \(l\) and \(m\) these integrals can be calculated using the representation of Legendre polynomials in power polynomials. With larger indices, however, such an approach leads to a substantial accumulation of errors (as in the case of a calculation of the polynomials themselves) and makes evaluations of the integrals impossible. It is known that a most appropriate method of calculating Legendre polynomials is to calculate them by recurrence formulas (see, for example, (Altschuler et al., 1977)). For the integrals appearing in (23), (by a corresponding formal integration of recurrence equations for Legendre polynomials) it is possible to obtain the respective recurrence relations which make it possible to perform their calculations as effectively as when calculating the polynomials. The above method of calculating the weight coefficients (henceforth referred to as the recurrence analytical integration) gives no Gibbs effect and, as a result, with minimum computational input, leads to solutions corresponding with a high accuracy on the lower boundary to the initial spline-representation of the grid data, as well as furnishing the opportunity to carry out, without perceptible losses, multiple reconstructions of the magnetic field.
6. Analysis of test calculations, and of calculations simulating the handling of real magnetograms

The pre-assigned a "standard" magnetic field in the form of the expansion (5) was used to analyze the performance of the method proposed above when specifying, as a boundary condition, the data of real daily magnetograms. To construct a "standard" expansion we used an arbitrary magnetic synoptic map from Kitt Peak, specified on a grid with 1-degree resolution. Next, in the heliographic coordinate system \((r', \theta', \phi')\) using the method reported in (Hoeksema, 1984) for solving the boundary-value problem with a given component \(B_l\), we determined the coefficients of the expansion (5) up to \(L = 90\). To model the operation with real differently-resolved magnetograms in the expansion of the standard magnetic field, the value of \(L\) was decreased simply by discarding redundant expansion terms. In doing this, proper account was taken of the correspondence of the typical scales of the last expansion terms to the resolution being analyzed. Main attention was given to three typical scales corresponding to the resolution of magnetograms from Stanford (\(\sim 15^\circ\)), Mt.Wilson (\(\sim 4^\circ\)), and Kitt Peak (\(\sim 1^\circ\), lowres). The standard field formed the basis for constructing "model daily magnetograms" for a subsequent extrapolation. Subsequent expansions were performed up to the same number of terms. Final comparisons were made both with values of the full magnetic vector and the standard field potential at different heights and with values of the standard expansion coefficients of (5). For the latter, the resulting field was re-expanded: the distribution of \(B_r\) on the surface \(S_p\) was calculated and reconstructed from a normal component in the heliographic coordinate system. Precisely this transition from the coordinate system \((r, \theta, \phi)\) to the system \((r', \theta', \phi')\) will be assumed throughout in what follows where the comparison of harmonic coefficients is implied. Note that in the final comparison there was no point in making comparisons with the standard model pointwise, because each re-expansion of the magnetic field employed the method of approximation at nodes by local splines which yield inaccurate values at nodes of the initial grid, while when different coordinate systems are used, the nodes themselves are inaccurate. The key criterion involved comparing the configurations of contour lines of the components under consideration and their location. Results of subsequent extrapolations were considered "equivalent" to the starting model if contour lines were impossible to distinguish visually from those given by the standard magnetic field and if the difference in positions of the contour lines did not exceed the initial resolution of the fields that were reconstructed. Comparisons of harmonic coefficients were made separately only up to \(l = 9\), while a complete survey involved comparing plots of linear
characteristics of the spectrum from the index \( l \) (for each \( l \), quadratic sums of coefficients with different \( m \) were calculated). In this case, results were considered equivalent if spectral lines merged together over the entire range of \( l \). Particular attention was given to the comparison of neutral lines on the source surface and to positions of their intersection of the equator. As is known, neutral lines and intersection points, a calculation of which is of great practical significance, are the most sensitive to minor distortions of large-scale background magnetic fields.

Test I. The standard magnetic field model was used to calculate (omitting the phase of constructing magnetograms) uniformly in coordinates on a sphere \((R, \theta, \phi)\) the distribution of the \(B_d\)-component. The same was done for the \(B_l\)- and \(B_r\)-components. Reconstructions were performed from \(B_d\), \(B_l\) and \(B_r\), respectively. All cases showed a complete equivalence for all of the above-mentioned criteria.

Conclusion: Mathematically, all methods of specifying boundary conditions for the reconstruction are equivalent in regard to the degree of accuracy – quite a natural result.

Test II. Information contained in real magnetograms is given in the form of values of the component \(B_d\) which are uniformly distributed in the picture plane. When nodal points are projected onto the surface of a sphere, as they approach the limb, the resolution of information is deteriorated substantially (the distance between adjacent nodes on the surface of the sphere increases, which is equivalent to a loss of information). The tests outlined below were carried out with the purpose of assessing the influence of this property on the reconstruction of magnetic fields. Note that, when extrapolating from \(B_l\), the influence of this property is unimportant in consequence of the use of synoptic magnetograms constructed on the basis of sampling measurements only in the neighborhood of the central meridian.

The standard magnetic field was used to preliminarily calculate a direct and back artificial magnetogram with the structure of the data which fully corresponded to the structure of data recording in real magnetograms. Subsequently, these magnetograms were treated as input material for extrapolating the magnetic field. The results obtained showed that an extrapolation on the basis of proposed method eliminates unfortunately the possibility of using magnetograms corresponding to the resolution of the Stanford magnetograms. Differences from a true model of the magnetic field, associated with a nonuniform distribution of information across the solar disk, in this case lead to an irreplaceable distortion of the resulting model for all criteria. Such distortions are also distinguishable for Mt.Wilson magnetograms, but they are significantly smaller, and, in principle, corresponding real magnetograms can be used. An extrapolation from artificial magnetograms
corresponding to Kitt Peak magnetograms, showed very good results. Nonuniformity effects of data resolution disappeared almost totally in all their manifestations. The final model in this case is equivalent to a true model, both in the comparison of contour lines and of expansion coefficients. Note that they are in either case difficult to distinguish from each other in comparisons of images of the $B_d$-components of the initial standard magnetogram and of a newly reconstructed one. However, when full images of any magnetic component is developed onto a rectangle in the plane $(\theta', \phi')$, if nonuniformity effects of data resolution are essential, one can see a clear boundary separating the direct and back magnetograms, as well as a distortion zone around it. For the Kitt Peak case such a boundary was virtually undetectable, and on the source surface all contour lines merged totally together.

Conclusion: There exist limitations caused by the resolution of the magnetograms used. The proposed method produces faithful results starting with the resolution of Kitt Peak magnetograms and higher.

7. The problem of specifying a back magnetogram, and results of calculations from real magnetograms

First we present some qualitative considerations regarding the consequences caused by the scarcity of data corresponding to the back magnetogram. On the photospheric surface the influence of the back side on the main strong small-scale field can have a substantial effect only in the neighborhood of the limb corresponding to the field scale. With an increase in the altitude, weaker components of a larger-scale field starts to play a crucial role. Therefore, the region (henceforth referred to as the confidence region), which is the least subjected to the influence of the back side, may be represented as a region bounded by a dome-shaped surface with the height $R$. If the purpose of research is limited only to examinations inside such a region, then it is in principle not very important how the back magnetogram is filled. It is possible, for example, to assume zero boundary conditions or to symmetrically represent the values of the magnetogram being analyzed.

If it is proposed to most fully represent the real boundary conditions of the back side in order to minimize the distortions near the confidence region boundary or to ascend significantly above this region to the source surface, then the first natural possibility would be to endeavor to replenish information for the back side using a corresponding synoptic magnetogram (additionally there exists the hopeful way of using of the vector magnetograms but it connects to principally another method – the regularization method (Gary, 1996), the discussion of this possi-
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bility is beyond the scope of this work). For instance, it is possible to reconstruct a full model of the magnetic field from the $B_r$-component, and to calculate a back magnetogram. In a sense, such a procedure is justified. In this case main attention is directed to the large-scale field which is relatively little affected by substantial changes during a solar rotation. Therefore, one would expect a good correspondence between large-scale fields on the source surface obtained as a result of extrapolations from synoptic data and magnetogram data. Also, it would be desirable to derive new information from the new structural elements associated with the current state of the field. In case of favorable results, it would be possible for us to be able, in particular, to predict changes in solar wind characteristics. Already first calculations of this kind revealed two characteristic types of distortions. One is the limb distortion (a pronounced small but intense band of distortions at the interface of the magnetograms). It is apparent that this type is associated with the impossibility of reconciling (at the interface) of the direct and back magnetograms constructed as indicated above. The other type is the distortion "plateau". This type is distinguished only in the analysis of a rectangular image of the structure of the $B_r$-component on the source surface. It is particularly well seen in the three-dimensional image of the reconstructed distribution of $B_r (R_s, \theta', \phi')$. In this case it is clearly seen that two regions corresponding to the direct and back magnetograms are detached from each other and uniformly displaced from each other along the functional axis $B_r$. Furthermore, if this gap is artificially eliminated, then the two regions corresponding to the positions of the direct and back magnetograms would gradually go over into each other, and the structure of $B_r (R_s, \theta', \phi')$ would always agree well with that obtained by reconstructing the magnetic field from a full synoptic map (from the component $B_l$). The gap does typically not exceed 10% of a maximum value of $B_r$ and can differ in sign. The plateau effect is unseen on the lower surface because it in its magnitude corresponds to very weak values of the field. Conceivably this effect represents an inaccuracy of specifying the zero value of the source magnetograms. This effect is extremely complicated to eliminate. If we try to eliminate the gap merely by the addition of a small constant value to the data of the direct or back magnetograms, then one is left with the possibility that the data from both magnetograms are shifted by the same amount. This involves a relatively complicated transformation of the monopole expansion term which we discard. In such procedures, it varies and can transform to other expansion terms which are taken into account. Each such fitting entails a new reconstruction of fields, which demands much computational time. Numerical calculations showed that it is not always possible to achieve a correspondence between results of
reconstructions from synoptic data and full-disk data. In the case of such global structures of $B_r (R_s, \theta', \phi')$, the arrangement of neutral lines can differ substantially. The positions of important equator intersecting points of neutral lines are most strongly responsive to such modifications. Intersection points are almost always coincident in comparative calculations of $B_l$-based reconstructions from Stanford and Kitt Peak magnetograms. Since their calculations are based on totally different measurements in regard to both resolution and time, these points may be thought of as being stable in a sense and reflecting the physical state of large-scale fields which is eventually associated with current conditions of the interplanetary field at the Earth’s orbit. It is therefore very important to achieve a relative stability of these points in calculations from daily magnetograms.

A next step involved an attempt to determine the back magnetogram using, instead of a synoptic magnetogram, some set of daily magnetograms covering the whole surface of the Sun over a time interval incorporating the magnetogram under consideration. The principle of construction is as follows. To determine the $B_d$-component at an arbitrary point on the solar surface, a set of magnetograms containing this point was used. It was assumed that the axis of solar rotation is normal to the plane of the Earth’s orbit. Within such an approximation, for a particular point it is possible to determine the desired $B_d$-component based on two components from two arbitrary magnetograms for the same point. Finally, $B_d$ at the desired point was determined by an average over the values calculated from all possible pairs of magnetograms. Thus a full synoptic map of the $B_d$-component was generated. Interestingly, when a full $B_d$-synoptic magnetogram was generated from a set of magnetograms appearing in the file heading of a usual $B_l$-synoptic map, all calculations for these two magnetograms by respective methods yielded virtually identical results from comparisons on both the lower and upper surfaces. In particular, the most sensitive neutral lines merged entirely together. This showed that the two types of synoptic magnetograms are virtually identical to each other as regards their information content. Naturally, the new maps contained neither limb- nor plateau-distortions. It is clear that the above-mentioned uncertainties of zero counts of the data from daily magnetograms are averaged when generating synoptic maps of either type and give no perceptible effects. When a part of the $B_d$-synoptic magnetogram was substituted for by the data from the daily magnetogram, limb distortions were vanishingly small. This was rather surprising and showed that the effort made in this area was not useless and that the possibilities for a further improvement of the situation have not been exhausted. Unfortunately, although the plateau-distortions decreased slightly, they were still large enough
to be able to affect the arrangement of intersection points. The zero uncertainty of a direct magnetogram is not fully compensated by an averaged back magnetogram. Neutral lines of the source can also sometimes be affected markedly when the number of daily magnetograms used to construct a $B_d$-synoptic magnetogram is varied. Thus the solution of the problem treated here rests on the insufficiently good quality of daily magnetograms employed. Kit Peak daily magnetograms made available for general use are essentially meant for research into strong fields; moreover, they are not the initial product of measurements. Initial measurements are with significantly higher resolution. It is hoped that not all possibilities of improving the determination of the zero count have been exhausted, both in regard to initial the measurements and at the stage of their conversion to final usable data. Unfortunately, this author has no technical way of handling raw measurements in order to continue analysis along this line.

To illustrate the performance of the new method we give one example of a calculation of open magnetic field regions from a daily magnetogram. By the open magnetic field region is meant an area of the solar surface, with all outgoing field lines reaching the source surface. To identify such region involves calculating all field lines starting from points which are uniformly distributed on the source surface. Crossings of field lines with the photosphere give images of such regions. Figs. 1 and 2 present two calculations from a daily magnetogram (Fig. 1) and from a corresponding synoptic $B_t$-magnetogram (Fig. 2). Footpoints of open field lines here are shown by circles superposed on soft X-ray images of the Sun. The light tone corresponds to coronal hole regions usually associated with regions of open field lines. Calculations from the daily magnetogram are an obvious advantage. The situation illustrated here is typical of the other cases considered in this study. Calculations from daily magnetograms, as a rule, adequately depict the structure of most visible coronal holes, which is a great improvement over synoptic calculations. An important point is that calculations depend only slightly on the method of specifying the back magnetogram. The dependence on "back" boundary conditions has a substantial influence only on the position of the upper ends of field lines issuing out of open regions; furthermore, the field line configuration starts to change noticeably only at altitudes larger than the solar radius.

8. Conclusions

Main results and conclusions may be summarized as follows.
A new method has been developed for extrapolating the solar magnetic fields from daily full-disk magnetograms.

A mathematical rationale for the new method was given.

An analysis was made of the applicability of the method, with due regard for the particular characteristics of existing data.

The most suitable technique has been found for specifying the boundary conditions on the back side of the Sun.

It has been shown that this method can be used successfully only when high-resolution daily magnetograms are employed (with the exception of Stanford magnetograms).

Calculations confirmed that in the confidence region defined above, one can expect a good correspondence between calculated and real magnetic field structures.

In contrast to other techniques, this method makes it possible to describe current conditions of a global magnetic structure (not averaged over time and not local).

The analysis made in this paper has revealed hidden limitations of high-resolution magnetograms which manifest themselves when reconstructing the magnetic structure on the source surface. It may be concluded that with the proviso that they are eliminated or their influence is adequately reduced, it is possible to predict a change of sign of the near-terrestrial interplanetary magnetic field from the reconstructed picture of the radial magnetic field on the source surface.

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Figure captions

Figure 1. Calculation of regions of open field lines from the daily magnetogram of November 27, 1993, $L = 30$, SXT-27/11/93

Figure 2. Calculations of regions of open field lines from synoptic magnetogram CR-1876, $L = 30$, SXT-27/11/93

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