THE GEOMETRY OF FRONTS

KENTARO SAJI, MASAAKI UMEHARA, AND KOTARO YAMADA

Abstract. We shall introduce the singular curvature function on cuspidal edges of surfaces, which is related to the Gauss-Bonnet formula and which characterizes the shape of cuspidal edges. Moreover, it is closely related to the behavior of the Gaussian curvature of a surface near cuspidal edges and swallowtails.

Introduction

Let $M^2$ be an oriented 2-manifold and $f: M^2 \to \mathbb{R}^3$ a $C^\infty$-map. A point $p \in M^2$ is called a singular point if $f$ is not an immersion at $p$. A singular point is called a cuspidal edge or swallowtail if it is locally diffeomorphic to

(1) \[ f_C(u,v) := (u^2, u^3, v) \quad \text{or} \quad f_S(u,v) := (3u^4 + u^2v, 4u^3 + 2uv, v) \]

at $(u,v) = (0,0)$, respectively. These two types of singular points characterize the generic singularities of wave fronts (cf. AGV; for example, parallel surfaces of immersed surfaces in $\mathbb{R}^3$ are fronts), and we have a useful criterion (Fact 1.5; cf. KRSUY) for determining them. It is of interest to investigate these singularities from the viewpoint of differential geometry. In this paper, we shall distinguish two types of cuspidal edges as in Figure 1. More precisely, we shall define the singular curvature function $\kappa_s$ along cuspidal edges. The left-hand figure in Figure 1 is positively curved and the right-hand figure is negatively curved (see Corollary 1.18).

\begin{figure}[h]
\begin{center}
\includegraphics[width=0.5\textwidth]{cusp.png}
\end{center}
\caption{Positively and negatively curved cuspidal edges (Example 1.9).}
\end{figure}

The definition of the singular curvature function does not depend on the orientation nor on the co-orientation of the front and is closely related to the following two Gauss-Bonnet formulas given by Langevin-Lévitt-Rosenberg and Kossowski when $M^2$ is compact:

(2) \[ 2 \deg(\nu) = \chi(M_+) - \chi(M_-) + \#S_+ - \#S_- \quad (\text{LLR}, \text{K1}) \]

(3) \[ 2\pi \chi(M^2) = \int_{M^2} K \, dA + 2 \int_{\text{Singular set}} \kappa_s \, ds \quad (\text{K1}), \]

where $\deg(\nu)$ is the degree of the Gauss map $\nu$, $\#S_+$, $\#S_-$ are the numbers of positive and negative swallowtails respectively (see Section 2), and $M_+$ (resp. $M_-$)
is the open submanifold of $M^2$ to which the co-orientation is compatible (resp. not compatible) with respect to the orientation. In the proofs of these formulas in [LLR] and [K1], the singular curvature implicitly appeared as a form $\kappa_s ds$. (Formula (2) stated in [LLR], and proofs for both (2) and (3) are in [K1].)

Recently, global properties of fronts were investigated via flat surfaces in hyperbolic 3-space $H^3$ ([KUY1, KRSUY]), via maximal surfaces in Minkowski 3-space ([UY]), and via constant mean curvature one surfaces in de Sitter space ([F], see also Lee and Yang [LY]). Such surfaces satisfy certain Osserman type inequalities for which equality characterizes the proper embeddedness of their ends. We also note that Martínez [Mar] investigated global properties of improper affine spheres with singularities, which are related to flat fronts in $H^3$. (See also Ishikawa and Machida [IM].)

The purpose of this paper is to give geometric meaning to the singular curvature function and investigate its properties. For example, it diverges to $-\infty$ at swallowtails (Corollary 1.14). Moreover, we shall investigate behavior of the Gaussian curvature $K$ near singular points. For example, the Gaussian curvature $K$ is generically unbounded near cuspidal edges and swallowtails and will take different signs from the left-hand side to the right-hand side of a singular curve. However, on the special occasions that $K$ is bounded, the shape of these singularities is very restricted: for example, singular curvature is non-positive if the Gaussian curvature is non-negative (Theorem 3.1). A similar phenomena holds for the case of hypersurfaces (Section 5).

The paper is organized as follows: In Section 1, we define the singular curvature, and give its fundamental properties. In Section 2, we generalize the two Gauss-Bonnet formulas (2) and (3) to fronts which admit finitely many corank one “peak” singularities. In Section 3, we investigate behavior of Gaussian curvature. Section 4 is devoted to formulating a topological invariant of closed fronts called the “zig-zag number” (introduced in [LLR]) from the viewpoint of differential geometry. We shall generalize the results of Section 3 to hypersurfaces in Section 5. Finally, in Section 6, we introduce an intrinsic formulation of the geometry of fronts.

Acknowledgements. The authors thank Shyuichi Izumiya, Go-o Ishikawa, Osamu Saeki, Osamu Kobayashi and Wayne Rossman for fruitful discussions and valuable comments.

1. SINGULAR CURVATURE

Let $M^2$ be an oriented 2-manifold and $(N^3, g)$ an oriented Riemannian 3-manifold. The unit cotangent bundle $T^*_1N^3$ has the canonical contact structure and can be identified with the unit tangent bundle $T_1N^3$. A smooth map $f : M^2 \to N^3$ is called a front if there exists a unit vector field $\nu$ of $N^3$ along $f$ such that $L := (f, \nu) : M^2 \to T_1N^3$ is a Legendrian immersion (which is also called an isotropic immersion), that is, the pull-back of the canonical contact form of $T_1N^3$ vanishes on $M^2$. This condition is equivalent to the following orthogonality condition:

$$g(f_*X, \nu) = 0 \quad (X \in TM^2),$$

where $f_*$ is the differential map of $f$. The vector field $\nu$ is called the unit normal vector of the front $f$. The first fundamental form $ds^2$ and the second fundamental form $h$ of the front are defined in the same way as for surfaces:

$$ds^2(X, Y) := g(f_*X, f_*Y), \quad h(X, Y) := -g(f_*X, D_Y\nu) \quad (X, Y \in TM^2),$$

where $D$ is the Levi-Civita connection of $(N^3, g)$. 

We denote by $\mu_{g}$ the Riemannian volume element of $(N^3, g)$. Let $f : M^2 \to N^3$ be a front and $\nu$ the unit normal vector of $f$, and set

\begin{equation}
(1.3) \quad d\hat{A} := f^* (\iota_{\nu} \mu_{g}) = \mu_{g} (f_*, f_u, f_v, \nu) \, du \wedge dv \left( f_u = f_* \left( \frac{\partial}{\partial u} \right), f_v = f_* \left( \frac{\partial}{\partial v} \right) \right),
\end{equation}

called the signed area form, where $(u, v)$ is a local coordinate system of $M^2$ and $\iota_{\nu}$ is the interior product with respect to $\nu \in TN^3$. Suppose now that $(u, v)$ is compatible to the orientation of $M^2$. Then the function

\begin{equation}
(1.4) \quad \lambda(u, v) := \mu_{g} (f_u, f_v, \nu)
\end{equation}

is called the (local) signed area density function. We also set

\begin{equation}
(1.5) \quad dA := |\mu_{g} (f_u, f_v, \nu)| \, du \wedge dv = \sqrt{EG - F^2} \, du \wedge dv = |\lambda| \, du \wedge dv
(E := g(f_u, f_u), F := g(f_u, f_v), G := g(f_v, f_v),)
\end{equation}

which is independent of the choice of orientation-compatible coordinate system $(u, v)$ and is called the (absolute) area form of $f$. Let $M_+$ (resp. $M_-$) be the open submanifolds where the ratio $(d\hat{A})/(dA)$ is positive (resp. negative). If $(u, v)$ is a coordinate system compatible to the orientation of $M^2$, the point $(u, v)$ belongs to $M_+$ (resp. $M_-$) if and only if $\lambda(u, v) > 0$ ($\lambda(u, v) < 0$), where $\lambda$ is the signed area density function.

**Definition 1.1.** Let $f : M^2 \to N^3$ be a front. A point $p \in M^2$ is called a singular point if $f$ is not an immersion at $p$. We call the set of singular points of $f$ the singular set and denote by $\Sigma_f := \{p \in M^2 \mid p$ is a singular point of $f \}$. A singular point $p \in \Sigma_f$ is called non-degenerate if the derivative $d\lambda$ of the signed area density function does not vanish at $p$. This condition does not depend on choice of coordinate systems.

It is well-known that a front can be considered locally as a projection of a Legendrian immersion $L : U^2 \to P(T^*N^3)$, where $U^2$ is a domain in $R^2$ and $P(T^*N^3)$ is the projective cotangent bundle. The canonical contact structure of the unit cotangent bundle $T_0^*N^3$ is the pull-back of that of $P(T^*N^3)$. Since the contact structure on $P(T^*N^3)$ does not depend on the Riemannian metric, the definition of front does not depend on the choice of the Riemannian metric $g$ and is invariant under diffeomorphisms of $N^3$.

**Definition 1.2.** Let $f : M^2 \to N^3$ be a front and $TN^3|_M$ the restriction of the tangent bundle of $N^3$ to $M^2$. The subbundle $E$ of rank 2 on $M^2$ that is perpendicular to the unit normal vector field $\nu$ of $f$ is called the limiting tangent bundle with respect to $f$.

There exists a canonical vector bundle homomorphism

$$\psi : TM^2 \ni X \mapsto f_* X \in E.$$ 

The non-degenerateness in Definition 1.1 is also independent of the choice of $g$ and can be described in terms of the limiting tangent bundle:

**Proposition 1.3.** Let $f : U \to N^3$ be a front defined on a domain $U$ in $R^2$ and $E$ the limiting tangent bundle. Let $\mu : (U; u, v) \to E^* \wedge E^*$ be an arbitrary fixed nowhere vanishing section. Then a singular point $p \in M^2$ is non-degenerate if and only if the derivative $dh$ of the function $h := \mu (\psi(\partial/\partial u), \psi(\partial/\partial v))$ does not vanish at $p$.

**Proof.** Let $\mu_0$ be the 2-form that is the restriction of the 2-form $\iota_{\nu} \mu_{g}$ to $M^2$, where $\iota_{\nu}$ denotes the interior product and $\mu_{g}$ is the volume element of $g$. Then $\mu_0$ is a nowhere vanishing section on $E^* \wedge E^*$, and the local signed area density function $\lambda$ is given by $\lambda = \mu_0 (\psi(\partial/\partial u), \psi(\partial/\partial v))$. 
On the other hand, let \( \mu : (U; u, v) \to \mathcal{E}^* \cap \mathcal{E}^* \) be an arbitrary fixed nowhere vanishing section. Then there exists a smooth function \( \tau : U \to \mathbb{R} \setminus \{0\} \) such that \( \mu = \tau \cdot \mu_0 \) (namely \( h = \tau \lambda \)) and
\[
dh(p) = d\tau(p) \cdot \lambda(p) + \tau(p) \cdot d\lambda(p) = \tau(p) \cdot d\lambda(p),
\]
since \( \lambda(p) = 0 \) for each singular point \( p \). Then \( dh \) vanishes if and only if \( d\lambda \) does as well. \( \square \)

**Remark 1.4.** A \( C^\infty \)-map \( f : U^2 \to M^3 \) is called a *frontal* if it is a projection of isotropic map \( L : U^2 \to T^*_p M^3 \), that is, the pull-back of the canonical contact form of \( T_1 N^3 \) by \( L \) vanishes on \( M^2 \). The definition of non-degenerate singular points and the above lemma do not use the properties that \( L \) is an immersion. So they hold for any frontals.

Let \( p \in M^2 \) be a non-degenerate singular point. Then by the implicit function theorem, the singular set near \( p \) consists of a regular curve in the domain of \( M^2 \). This curve is called the *singular curve* at \( p \). We denote the singular curve by
\[
\gamma : (−\varepsilon, \varepsilon) \ni t \mapsto \gamma(t) \in M^2 \quad (\gamma(0) = p).
\]
For each \( t \in (−\varepsilon, \varepsilon) \), there exists a 1-dimensional linear subspace of \( T_{\gamma(t)} M^2 \), called the *null direction*, which is the kernel of the differential map \( f_* \). A non-zero vector belonging to the null direction is called a *null vector*. One can choose a smooth vector field \( \eta(t) \) along \( \gamma(t) \) such that \( \eta(t) \in T_{\gamma(t)} M^2 \) is a null vector for each \( t \), which is called a *null vector field*. The tangential 1-dimensional vector space of the singular curve \( \gamma(t) \) is called the *singular direction*.

**Fact 1.5** (Criteria for cuspidal edges and swallowtails [KRSUY]). Let \( p \) be a non-degenerate singular point of a front \( f \), \( \gamma \) the singular curve passing through \( p \), and \( \eta \) a null vector field along \( \gamma \). Then

(a) \( p = \gamma(t_0) \) is a cuspidal edge (that is, \( f \) is locally diffeomorphic to \( f_C \) of \( \square \) in the introduction) if and only if the null direction and the singular direction are transversal, that is, \( \det(\gamma'(t), \eta(t)) \) does not vanish at \( t = t_0 \), where \( \det \) denotes the determinant of \( 2 \times 2 \) matrices and where we identify the tangent space in \( T_{\gamma(t_0)} M^2 \) with \( \mathbb{R}^2 \).

(b) \( p = \gamma(t_0) \) is a swallowtail (that is, \( f \) is locally diffeomorphic to \( f_S \) of \( \square \) in the introduction) if and only if
\[
\det(\gamma'(t_0), \eta(t_0)) = 0 \quad \text{and} \quad \left. \frac{d}{dt} \right|_{t=t_0} \det(\gamma'(t), \eta(t)) \neq 0
\]
hold.

For later computation, it is convenient to take a local coordinate system \((u, v)\) centered at a given non-degenerate singular point \( p \in M^2 \) as follows:
- the coordinate system \((u, v)\) is compatible with the orientation of \( M^2 \),
- the \( u \)-axis is the singular curve, and
- there are no singular points other than the \( u \)-axis.

We call such a coordinate system \((u, v)\) an *adapted coordinate system* with respect to \( p \). In these coordinates, the signed area density function \( \lambda(u, v) \) vanishes on the \( u \)-axis. Since \( dx \neq 0 \), \( \lambda_u \) never vanishes on the \( u \)-axis. This implies that
\[
(1.6) \quad \text{the signed area density function } \lambda \text{ changes sign on singular curves},
\]
that is, the singular curve belongs to the boundary of \( M_+ \) and \( M_- \).

Now we suppose that a singular curve \( \gamma(t) \) on \( M^2 \) consists of cuspidal edges. Then we can choose the null vector fields \( \eta(t) \) such that \( \langle \gamma'(t), \eta(t) \rangle \) is a positively
oriented frame field along $\gamma$. We then define the singular curvature function along $\gamma(t)$ as follows:

\begin{equation}
(1.7) \quad \kappa_s(t) := \text{sgn}(d\lambda(\eta)) \frac{\mu_g(\hat{\gamma}''(t), \hat{\gamma}'(t), \nu)}{|\hat{\gamma}'(t)|^3},
\end{equation}

Here, we denote $|\hat{\gamma}'(t)| = g(\hat{\gamma}'(t), \hat{\gamma}'(t))^{1/2}$,

\begin{equation}
(1.8) \quad \hat{\gamma}(t) = f(\gamma(t)), \quad \hat{\gamma}'(t) = \frac{d\hat{\gamma}(t)}{dt}, \quad \text{and} \quad \hat{\gamma}''(t) = D_t \hat{\gamma}'(t),
\end{equation}

where $D$ is the Levi-Civita connection and $\mu_g$ the volume element of $(N^3, g)$.

We take an adapted coordinate system $(u,v)$ and write the null vector field $\eta(t)$ as

\begin{equation}
(1.9) \quad \eta(t) = a(t) \frac{\partial}{\partial u} + e(t) \frac{\partial}{\partial v},
\end{equation}

where $a(t)$ and $e(t)$ are $C^\infty$-functions. Since $(\gamma', \eta)$ is a positive frame, we have $e(t) > 0$. Here,

\begin{equation}
(1.10) \quad \lambda_u = 0 \quad \text{and} \quad \lambda_v \neq 0 \quad \text{(on the u-axis)}
\end{equation}

hold, and then $d\lambda(\eta(t)) = e(t) \lambda_v$. In particular, we have

\begin{equation}
(1.11) \quad \text{sgn}(d\lambda(\eta)) = \text{sgn}(\lambda_v) = \begin{cases} +1 & \text{if the left-hand side of } \gamma \text{ is } M_+, \\ -1 & \text{if the left-hand side of } \gamma \text{ is } M_. \end{cases}
\end{equation}

So we have the following expression: in an adapted coordinate system $(u,v)$,

\begin{equation}
(1.12) \quad \kappa_s(u) := \text{sgn}(\lambda_v) \frac{\mu_g(f_u, f_{uu}, \nu)}{|f_u|^3},
\end{equation}

where $f_{uu} = D_u f_u$ and $|f_u| = g(f_u, f_u)^{1/2}$.

**Theorem 1.6** (Invariance of the singular curvature). The definition of the singular curvature does not depend on the parameter $t$, nor the orientation of $M^2$, nor the choice of $\nu$, nor the orientation of the singular curve.

**Proof.** If the orientation of $M^2$ reverses, then $\lambda$ and $\eta$ both change sign. If $\nu$ is changed to $-\nu$, so does $\lambda$. If $\gamma$ changes orientation, both $\gamma'$ and $\eta$ change sign. In all cases, the sign of $\kappa_s$ is unchanged. \qed

**Remark 1.7.** We have the following expression

\[
\kappa_s = \text{sgn}(d\lambda(\eta)) \frac{\mu_g(\hat{\gamma}'', \nu, \hat{\gamma}' / |\hat{\gamma}'|^2)}{|\hat{\gamma}'|^2} = \text{sgn}(d\lambda(\eta)) \frac{g(\hat{\gamma}'', n)}{|\hat{\gamma}'|^2} \left( n := \nu \times \hat{\gamma}' / |\hat{\gamma}'| \right).
\]

Here, the vector product operation $\times_g$ in $T_x N^3$ is defined by $a \times_g b := *(a \wedge b)$, under the identification $TN^3 \ni X \leftrightarrow g(X, \cdot) \in T^* N^3$, where $*$ is the Hodge $*$-operator. If $\gamma(t)$ is not a singular curve, $n(t)$ is just the conormal vector of $\gamma$. We call $n(t)$ the limiting conormal vector, and $\kappa_s(t)$ can be considered as the limiting geodesic curvature of (regular) curves with the singular curve on their right-hand sides.

**Proposition 1.8** (Intrinsic formula for the singular curvature). Let $p$ be a point of a cuspidal edge of a front $f$, and $(u,v)$ an adapted coordinate system at $p$ such that $\partial / \partial v$ gives the null direction. Then the singular curvature is given by

\[
\kappa_s(u) = \frac{-F_v E_u + 2 E F_{uv} - E E_{uv}}{E^{3/2} \lambda_v},
\]

where $E = g(f_u, f_u), F = g(f_u, f_v), G = g(f_v, f_v)$, and where $\lambda$ is the signed area density function with respect to $(u,v)$.
Proof. Fix \( v > 0 \) and denote by \( \gamma(u) = (u, v) \) the \( u \)-curve. Then the unit vector
\[
n(u) = \frac{1}{\sqrt{E\sqrt{EG-F^2}}} \left( -F \frac{\partial}{\partial u} + E \frac{\partial}{\partial v} \right)
\]
gives the conormal vector such that \( (\gamma'(u), n(u)) \) is a positive frame. Let \( \nabla \) be the Levi-Civita connection on \( \{ v > 0 \} \) with respect to the induced metric \( ds^2 = Edu^2 + 2Fadu + Gdv^2 \), and \( s \) the arclength parameter of \( \gamma(u) \). Then we have
\[
\nabla_{\gamma'(u)} \gamma'(s) = \frac{1}{E} \nabla_{\partial/\partial u} \left( \frac{1}{E} \frac{\partial}{\partial u} \right) \equiv \Gamma_{11}^2 \frac{\partial}{\partial u} \mod \frac{\partial}{\partial u},
\]
where \( \Gamma_{11}^2 \) is the Christoffel symbol given by
\[
\Gamma_{11}^2 = \frac{-FE_u + 2EF_u - EE_v}{2(EG - F^2)}.
\]
Since \( \lambda^2 = EG - F^2 \) and \( g(f_u, n) = 0 \), the geodesic curvature of \( \gamma \) is given by
\[
\kappa_g = g(\nabla_{\gamma'(s)} \gamma'(s), n(s)) = \frac{\sqrt{EG - F^2} \Gamma_{11}^2}{E^{3/2}} = \frac{-FE_u + 2EF_u - EE_v}{|\lambda|E^{3/2}}.
\]
Hence, by Remark \ref{rem:so}, the singular curve of the \( u \)-axis is
\[
\kappa_s = \text{sgn}(\lambda_v) \lim_{v \to 0} \kappa_g = \text{sgn}(\lambda_v) \lim_{v \to 0} \frac{-FE_u + 2EF_u - EE_v}{|\lambda|E^{3/2}}.
\]
It is clear that all of \( \lambda \), \( F \) and \( F_u \) tend to zero as \( v \to 0 \). Moreover, we have
\[
E_v = 2g(D_v f_u, f_u) = 2g(D_u f_v, f_u) = 2 \frac{\partial}{\partial v} g(f_v, f_u) - 2g(f_v, D_u f_u) \to 0
\]
as \( v \to 0 \), and the right differential \( |\lambda|_v \) is equal to \( |\lambda|_v \) since \( \lambda(u, 0) = 0 \). By L'Hospital’s rule, we have
\[
\kappa_s = \text{sgn}(\lambda_v) \frac{-F_v E_u + 2EF_{uv} - EE_v}{|\lambda|_v E^{3/2}} = \frac{-F_v E_u + 2EF_{uv} - EE_v}{\lambda_v E^{3/2}},
\]
which is the desired conclusion. \( \square \)

Example 1.9 (Cuspidal parabolas). Define a map \( f \) from \( \mathbb{R}^2 \) to the Euclidean 3-space \( (\mathbb{R}^3, g_0) \) as
\[
(1.13) \quad f(u, v) = (au^2 + v^2, bv^2 + v^3, u) \quad (a, b \in \mathbb{R}).
\]
Then we have \( f_u = (2au, 0, 1), f_v = (2v, 2bv + 3v^2, 0) \). This implies that the \( u \)-axis is the singular curve, and the \( v \)-direction is the null direction. The unit normal vector and the signed area density \( \lambda = \mu_{\partial_0}(f_u, f_v, \nu) \) are given by
\[
(1.14) \quad \nu = \frac{1}{\delta} \left( -3v - 2b, 2, 2au(3v + 2b) \right), \quad \lambda = v \delta,
\]
where
\[
\delta = \sqrt{4 + (1 + 4a^2u^2)(4b^2 + 12bv + 9v^2)}.
\]
In particular, since \( dv(\partial/\partial v) = \nu_v \neq 0 \) on the \( u \)-axis, \( (f, \nu) : \mathbb{R}^2 \to \mathbb{R}^3 \times S^2 = T_1 \mathbb{R}^3 \) is an immersion, i.e. \( f \) is a front, and each point of the \( u \)-axis is a cuspidal edge. The singular curvature is given by
\[
(1.15) \quad \kappa_s(u) = \frac{2a}{(1 + 4a^2u^2)^{3/2} \sqrt{1 + b^2(1 + 4a^2u^2)}}.
\]
When \( a > 0 \) (resp. \( a < 0 \)), that is, the singular curvature is positive (resp. negative), we shall call \( f \) a cuspidal elliptic (resp. hyperbolic) parabola since the figure looks like an elliptic (resp. hyperbolic) parabola, as seen in Figure \[\text{II}\] in the introduction.

Definition 1.10 (Peaks). A singular point \( p \in M^2 \) (which is not a cuspidal edge) is called a peak if there exists a coordinate neighborhood \( (U; u, v) \) of \( p \) such that
(1) there are no singular points other than cuspidal edges on \( U \setminus \{ p \} \),
(2) the rank of the derivative \( f_*: T_pM \to T_{f(p)}N \) at \( p \) is equal to 1, and
(3) The singular set of \( U \) consists of finitely many regular \( C^1 \)-curves starting at \( p \). The number \( 2m(p) \) of these curves is called the number of cuspidal edges starting at \( p \).

If a peak is a non-degenerate singular point, it is called a non-degenerate peak.

Swallowtails are examples of non-degenerate peaks. A front which admits cuspidal edges and peaks is called a front which admits at most peaks. There are degenerate singular points which are not peaks. Typical examples are cone-like singularities which appear in rotationally symmetric surfaces in \( \mathbb{R}^3 \) of positive constant Gaussian curvature. However, since generic fronts (in the local sense) have only cuspidal edges and swallowtails, the set of fronts which admits at most peaks covers a sufficiently wide class of fronts.

**Example 1.11 (A double swallowtail).** Define a map \( f: \mathbb{R}^2 \to \mathbb{R}^3 \) as
\[
 f(u, v) := (2u^3 - uv^2, 3u^4 - u^2v^2, v).
\]
Then
\[
 \nu = \frac{1}{\sqrt{1 + 4u^2(1 + u^2v^2)}} (-2u, 1, -2u^2v)
\]
is the unit normal vector to \( f \). The pull-back of the canonical metric of \( T\mathbb{R}^3 = \mathbb{R}^3 \times S^2 \) by \( (f, \nu): \mathbb{R}^2 \to \mathbb{R}^3 \times S^2 \) is positive definite. Hence \( f \) is a front. The signed area density function is
\[
 \lambda = (v^2 - 6u^2)\sqrt{1 + 4u^2(1 + u^2v^2)},
\]
and then the singular set is \( \Sigma_f = \{ v = \sqrt{6u} \} \cup \{ v = -\sqrt{6u} \} \). In particular, \( d\lambda = 0 \) at \( (0, 0) \). The first fundamental form of \( f \) is expressed as
\[
 ds^2 = dv^2
\]
at the origin, which is of rank one. Hence the origin is a degenerate peak (see Figure 2).

To analyze the behavior of the singular curvature near a peak, we prepare the following proposition.

**Proposition 1.12 (Boundedness of the singular curvature measure).** Let \( f: M^2 \to (N^3, g) \) be a front with a peak \( p \). Take \( \gamma: [0, \varepsilon) \to M^2 \) a singular curve of \( f \) starting from the singular point \( p \). Then \( \gamma(t) \) is a cuspidal edge for \( t > 0 \), and the singular curvature measure \( \kappa_s ds \) is continuous on \([0, \varepsilon)\), where \( ds \) is the arclength-measure. In particular, the limiting tangent vector \( \lim_{t \to 0} \frac{\gamma'(t)}{\|\gamma'(t)\|} \) exists, where \( \hat{\gamma} = f \circ \gamma \).

**Proof.** Let \( ds^2 \) be the first fundamental form of \( f \). Since \( p \) is a peak, the rank \( ds^2 \) is 1 at \( p \) and then one of the eigenvalues is 0 and the other is not. Hence the eigenvalues of \( ds^2 \) are of multiplicity one on a neighborhood of \( p \). Hence one can choose a local coordinate system \((u, v)\) around \( p \) such that each coordinate curve is tangent to an eigendirection of \( ds^2 \). In particular, we can choose \((u, v)\) such that \( \partial/\partial v \) is the null vector field on \( \gamma \). In such a coordinate system, \( f_v = 0 \) and \( D_uf_v = 0 \).

![Figure 2. A double swallowtail (Example 1.11).](image-url)
hold on $\gamma$. Then the derivatives of $\dot{\gamma} = f \circ \gamma$ are

$$\dot{\gamma}' = u' f_u, \quad D_2 \dot{\gamma}' = u'' f_u + u' D_{1u} f_u \quad \left(\dot{\gamma}' = \frac{d}{dt}\right),$$

where $\gamma(t) = (u(t), v(t))$. Hence

$$\kappa_s = \pm \frac{\mu_g(\dot{\gamma}', D_2 \dot{\gamma}', \nu)}{|\dot{\gamma}'|^3} = \pm \frac{\mu_{g}(f_u, D_{1u} f_u, \nu)}{|u'| |f_u|^3},$$

where $|X|^2 = g(X, X)$ for $X \in TN^3$. Since $ds = |\dot{\gamma}'| dt = |u'| |f_u| dt$ and $f_u \neq 0$, $\kappa_s ds = \pm \frac{\mu_g(f_u, D_{1u} f_u, \nu)}{|f_u|^2} dt$

is bounded.

To analyze the behavior of the singular curvature near a non-degenerate peak, we give another expression of the singular curvature measure:

**Proposition 1.13.** Let $(u, v)$ be an adapted coordinate system of $M^2$. Suppose that $(u, v) = (0, 0)$ is a non-degenerate peak. Then the singular curvature measure has the expression

$$\kappa_s(u) ds = \text{sgn}(\lambda_v) \frac{\mu_g(f_u, f_{uv}, \nu)}{|f_u|^2} du,$$

where $ds$ is the arclength-measure and $f_{uv} := D_{uv} f_v = D_{v} f_u$. In particular, the singular curvature measure is smooth along the singular curve.

**Proof.** We can take the null direction $\eta(u) = a(u)(\partial/\partial u) + e(u)(\partial/\partial v)$ as in (1.3). Since the peak is not a cuspidal edge, $\eta(0)$ must be proportional to $\partial_u$. In particular, we can multiply $\eta(u)$ by a non-vanishing function and may assume that $a(u) = 1$. Then $f_u + e(u)f_v = 0$ and by differentiation we have $f_{uu} + e_{u} f_u + e f_{uv} = 0$, that is,

$$f_u = -e f_v, \quad f_{uu} = -e_{u} f_v - e f_{uv}.$$

Substituting them into (1.12), we have (1.17) using the relation $ds = |\dot{\gamma}'| dt = |f_u| dt$. ☐

**Corollary 1.14** (Behavior of the singular curvature near a non-degenerate peak).

At a non-degenerate peak, the singular curvature diverges to $-\infty$.

**Proof.** We take an adapted coordinate $(u, v)$ centered at the peak. Then

$$\kappa_s(u) = \text{sgn}(\lambda_v) \frac{\mu_g(f_u, f_{uv}, \nu)}{|e(u)| |f_u|^3}.$$  

On the other hand,

$$\mu_g(f_u, f_{uv}, \nu) = \mu_g(f_u, f_{uv}, \nu), \quad \mu_g(f_{uv}, f_u, \nu) = (-\lambda_v) e - \mu_g(f_{uv}, f_u, \nu).$$

Since $f_u(0, 0) = 0$ we have

$$\text{sgn}(\lambda_v) \frac{\mu_g(f_u, f_{uv}, \nu)}{|f_u|^3} \bigg|_{(u, v) = (0, 0)} = -\frac{|\lambda_v(0, 0)|}{|f_u(0, 0)|^3} < 0.$$

Since $e(u) \to 0$ as $u \to 0$, we have the assertion. ☐

**Example 1.15** (The discriminant set of $s^3 + zs^2 + ys + x$). The typical example of peaks is a swallowtail. We shall compute the singular curvature of the swallowtail $f(u, v) = (3u^4 + u^2 v, 4u^3 + 2uv, v)$ at $(u, v) = (0, 0)$ given in the introduction, which is the discriminant set $\{u = 0\}$. Therefore, $F(x, y, z, s) = F_s(x, y, z, s) = 0$ for $s \in \mathbb{R}$ of the polynomial $F := s^3 + zs^2 + ys + x$. Since $f_u \times f_v = 2(6u^2 + v)(1, -u, u^2)$,
the singular curve is \( \gamma(t) = (t, -6t^2) \) and the unit normal vector is given by \( \nu = (1, -u, \dot{u})/\sqrt{1 + u^2 + \dot{u}^2} \). We have
\[
\kappa_s(t) = \frac{\det(\dot{\gamma}', \dot{\gamma}''', \nu)}{|\dot{\gamma}'|^3} = -\frac{\sqrt{1 + t^2 + t^4}}{6|\eta(1 + 4t^2 + t^4)^{3/2}},
\]
which shows the singular curvature tends to \(-\infty\) when \( t \to 0 \).

**Definition 1.16** (Null curves). Let \( f : M^2 \to N^3 \) be a front. A regular curve \( \sigma(t) \) in \( M^2 \) is called a **null curve** of \( f \) if \( \sigma'(t) \) is a null vector at each singular point. In fact, \( \sigma(t) = f(\sigma(t)) \) looks like the curve (virtually) transversal to the cuspidal edge, in spite of \( \sigma' = 0 \), and \( D_t\sigma' \) gives the “tangential” direction of the surface at the singular point.

**Theorem 1.17** (A geometric meaning for the singular curvature). Let \( p \) be a cuspidal edge, \( \gamma(t) \) a singular curve parametrized by the arclength \( t \) with \( \gamma(0) = p \), and \( \sigma(s) \) a null curve passing through \( p = \sigma(0) \). Then the sign of
\[
g(\tilde{\sigma}(0), \dot{\gamma}'(0))
\]
coincides with that of the singular curvature at \( p \), where \( \tilde{\sigma} = f(\sigma), \dot{\gamma} = f(\gamma) \),
\[
\dot{\tilde{\sigma}} = \frac{d\tilde{\sigma}}{ds}, \quad \dot{\gamma}' = \frac{d\gamma}{dt}, \quad \tilde{\sigma} = D_s\left(\frac{d\tilde{\sigma}}{ds}\right), \quad \text{and} \quad \dot{\gamma}'' = D_t\left(\frac{d\gamma}{dt}\right).
\]

**Proof.** We can take an adapted coordinate system \((u, v)\) around \( p \) such that \( \eta := \partial/\partial u \) is a null vector field on the \( u \)-axis. Then \( f_v = f_u \eta \) vanishes on the \( u \)-axis, and it holds that \( f_u := D_v f_u = D_u f_v = 0 \) on the \( u \)-axis. Since the \( u \)-axis is parametrized by the arclength, we have
\[
(1.18) \quad g(f_{uu}, f_u) = 0 \quad \text{on the \( u \)-axis} \quad (f_{uu} = D_u f_u).
\]

Now let \( \sigma(s) = (u(s), v(s)) \) be a null curve such that \( \sigma(0) = (0, 0) \). Since \( \tilde{\sigma}(0) \) is a null vector, \( \dot{u}(0) = 0 \), where \( \dot{=} = d/ds \). Moreover, since \( f_v(0, 0) = 0 \) and \( f_{uv}(0, 0) = 0 \), we have
\[
\tilde{\sigma}(0) = D_v(u f_u + \dot{v} f_v) = \tilde{u} f_u + \dot{v} f_v + \dot{u}^2 D_u f_u + 2u \dot{v} D_u f_v + \dot{v}^2 D_v f_v
\]
\[
= \tilde{u} f_u + \dot{v}^2 D_v f_v = \tilde{u} f_u(0, 0) + \dot{v}^2 f_{vv}(0, 0),
\]
and by (1.18),
\[
g(\tilde{\sigma}(0), \dot{\gamma}''(0)) = g(f_{uu}(0, 0), \tilde{u} f_u + \dot{v}^2 f_{vv}(0, 0)) = \dot{v}^2 g(f_{uu}(0, 0), f_{vv}(0, 0)).
\]

Now we can write \( f_{vv} = a f_u + b (f_u \times g \nu) + c \nu \), where \( a, b, c \in \mathbb{R} \). Then
\[
c = g(f_{vv}, \nu) = g(f_v, \nu)_v - g(f_v, \nu)_v = 0,
\]
\[
b = g(f_{vv}, f_u \times g \nu) = g(f_v, f_u \times g \nu)_v = -\lambda_v,
\]
where we apply the scalar triple product formula \( g(X, Y \times g Z) = \mu_g(X, Y, Z) \) for \( X, Y, Z \in T_f(0, 0)N^3 \). Thus
\[
g(\tilde{\sigma}(0), \dot{\gamma}''(0)) = \dot{v}^2 g(f_{uu}, a f_u - \lambda_v (f_u \times g \nu)) = -\dot{v}^2 \lambda_v g(f_{uu}, f_u \times g \nu)
\]
\[
= \dot{v}^2 \lambda_v \mu_g(\dot{\gamma}', \dot{\gamma}'', \nu) = \dot{v}^2 |\lambda_v| \kappa_s(0).
\]
This proves the assertion. \( \square \)

In the case of fronts in the Euclidean 3-space \( \mathbb{R}^3 = (\mathbb{R}^3, g_0) \), positively curved cuspidal edges and negatively curved cuspidal edges look like cuspidal elliptic parabola or hyperbolic parabola (see Example [1.9] and Figure [11], respectively. More precisely, we have the following:
1.20 (Fronts with Chebyshev net)

Example in $\Sigma$ has constant Gaussian curvature $-1$ if the set $W = M^2 \setminus \Sigma_f$ of regular points is dense in $M^2$ and $f$ has constant Gaussian curvature $-1$ on $W$. Then $f$ is a projection of the Legendrian immersion $L_f: M^2 \to T_1 \sigma^3$, and the pull-back $d\sigma^2 = |df|^2 + |dv|^2$ of the Sasakian metric on $T_1 \sigma^3$ by $L_f$ is flat. Thus for each $p \in M^2$, there exists a coordinate neighborhood $(U; u, v)$ such that $d\sigma^2 = 2(du^2 + dv^2)$. The two different families of asymptotic curves on $W$ are all geodesics of $d\sigma^2$, giving two foliations of $W$. Moreover, they are mutually orthogonal with respect to $d\sigma^2$. Then one can
choose the \( u \)-curves and \( v \)-curves to all be asymptotic curves on \( W \cap U \). For such a coordinate system \((u, v)\), the first and second fundamental forms are

\[
ds^2 = du^2 + 2 \cos \theta\, du\, dv + dv^2, \quad h = 2 \sin \theta\, du\, dv,
\]

where \( \theta = \theta(u, v) \) is the angle between the two asymptotic curves. The coordinate system \((u, v)\) as in (1.20) is called the asymptotic Chebyshev net around \( p \). The sine-Gordon equation \( \theta_{uv} = \sin \theta \) is the integrability condition of (1.20), that is, if \( \theta \) satisfies the sine-Gordon equation, then there exists a corresponding front \( f = f(u, v) \).

For such a front, we can choose the unit normal vector \( \nu \) such that \( f_u \times f_v = \sin \theta \nu \) holds, that is, \( \lambda = \sin \theta \). The singular sets are characterized by \( \theta \in \pi \mathbb{Z} \). We write \( \varepsilon = e^{\pi i \theta} = \pm 1 \) at a singular point. A given singular point is non-degenerate if and only if \( d\theta \neq 0 \). Moreover, the cuspidal edges are characterized by \( \theta_u + \varepsilon \theta_v \neq 0, \theta_u - \varepsilon \theta_v = 0 \) and the swallowtails are characterized by \( \theta_u + \varepsilon \theta_v = 0 \) and \( \theta_{uu} + \theta_{vv} \neq 0 \).

By a straightforward calculation applying Proposition 1.8, we have

\[
\kappa_s = -\varepsilon \frac{\theta_u \theta_v}{|\theta_u - \varepsilon \theta_v|} \quad (\varepsilon = e^{\pi i \theta}).
\]

Recently Ishikawa-Machida [IM] showed that the generic singularities of such fronts are cuspidal edges or swallowtails, as an application of Fact 1.5.

2. The Gauss-Bonnet Theorem

In this section, we shall generalize the two types of Gauss-Bonnet formulas mentioned in the introduction to compact fronts which admit at most peaks.

**Proposition 2.1.** Let \( f : M^2 \to (N^3, g) \) be a front, and \( K \) the Gaussian curvature of \( f \) which is defined on the set of regular points of \( f \). Then \( K\,dA \) can be continuously extended as a globally defined 2-form on \( M^2 \), where \( dA \) is the signed area form as in (1.3).

**Proof.** Let \((u, v)\) be a local coordinate system compatible to the orientation of \( M^2 \), and \( S = (S^1_u)^T \) the (matrix representation of) the shape operator of \( f \) which is defined on the set of regular points \( M^2 \setminus \Sigma_f \). That is, the Weingarten equation holds:

\[
\nu_u = -S^1_u f_u - S^2_u f_v, \quad \nu_v = -S^1_v f_u - S^2_v f_v, \quad \text{where} \quad \nu_u = D_u \nu, \quad \nu_v = D_v \nu.
\]

Since the extrinsic curvature is defined as \( K_{\text{ext}} = \det S \), we have

\[
\mu_g(\nu_u, \nu_v, \nu) = (\det S) \mu_g(f_u, f_v, \nu) = K_{\text{ext}} \lambda,
\]

where \( \lambda \) is the signed area density. Thus,

\[
K_{\text{ext}} \, dA = K_{\text{ext}} \lambda \, du \wedge dv = \mu_g(\nu_u, \nu_v, \nu) \, du \wedge dv
\]

is a well-defined smooth 2-form on \( M^2 \).

By the Gauss equation, the Gaussian curvature \( K \) satisfies

\[
K = c_{N^3} + K_{\text{ext}},
\]

where \( c_{N^3} \) is the sectional curvature of \((N^3, g)\) with respect to the tangent plane. Since \( f_\ast T_p M^2 \subset T_{f(p)} N^3 \) is the orthogonal complement of the normal vector \( \nu(p) \perp \), the tangent plane is well-defined on all of \( M^2 \). Thus \( c_{N^3} \) is a smooth function, and

\[
K \, dA = c_{N^3} \, dA + K_{\text{ext}} \, dA
\]

is a smooth 2-form defined on \( M^2 \). □
Although with the tangent bundle, and the two Gauss-Bonnet formulas are the same.

α

For example, α

(2.3)

κ

are opposite. The singular curvature in the introduction. If the surface is regular, the limiting tangent bundle

Moreover, since the rank of f

Theorem 2.3 (Gauss-Bonnet formulas for compact fronts). Let M^2 be a compact oriented 2-manifold and f: M^2 → R^3 is a front which admits at most peak singularities. Then the singular set coincides with ∂M_+ = ∂M_- and ∂M_+ and ∂M_- are piecewise C^1-differentiable because all singularities are at most peaks, and the limiting tangent vector of each singular curve starting at a peak exists by Proposition 1.12.

Now we suppose that M^2 is compact and f: M^2 → R^3 is a front which admits at most peak singularities. Then the singular set coincides with ∂M_+ = ∂M_- and ∂M_+ and ∂M_- are piecewise C^1-differentiable because all singularities are at most peaks, and the limiting tangent vector of each singular curve starting at a peak exists by Proposition 1.12.

For a given peak p, let α_(+) (resp. α_-(p)) be the sum of all the interior angles of f(M_+) (resp. f(M_-)) at p. Then by definition, we have

α_(+) + α_-(p) = 2π.

Moreover, since the rank of f is one at p, we have (see SUY)

α_(p), α_-(p) ∈ {0, π, 2π}.

Remark 2.2. On the other hand,

K dA = \begin{cases} K d\hat{\alpha} & \text{(on } M_+) , \\ -K d\hat{\alpha} & \text{(on } M_-) \end{cases}

is bounded, and extends continuously to the closure of M_+ and also to the closure of M_- (However, K dA cannot be extended continuously to all of M^2.)

Now we suppose that M^2 is compact and f: M^2 → R^3 is a front which admits at most peak singularities. Then the singular set coincides with ∂M_+ = ∂M_- and ∂M_+ and ∂M_- are piecewise C^1-differentiable because all singularities are at most peaks, and the limiting tangent vector of each singular curve starting at a peak exists by Proposition 1.12.

For a given peak p, let α_(+) (resp. α_-(p)) be the sum of all the interior angles of f(M_+) (resp. f(M_-)) at p. Then by definition, we have

α_(+) + α_-(p) = 2π.

Moreover, since the rank of f is one at p, we have (see SUY)

α_(p), α_-(p) ∈ {0, π, 2π}.

For example, α_(+) = α_-(p) = π when p is a cuspidal edge. If p is a swallowtail, α_(+) = 2π or α_-(p) = 2π. If α_(+) = 2π, p is called a positive swallowtail, and is called a negative swallowtail if α_-(p) = 2π (see Figure 4). Since K dA, K d\hat{\alpha} and κ_α ds are all bounded, we get two Gauss-Bonnet formulas as follows:

\int_{M^2} K d\hat{\alpha} + 2\int_{\Sigma_f} \kappa_\alpha ds = 2\pi \chi(M^2),

(2.5) \int_{M^2} K d\hat{\alpha} - \sum_{p: \text{peak}} (\alpha_(p) - \alpha_-(p)) = 2\pi (\chi(M_+) - \chi(M_-))

hold, where ds is the arclength measure on the singular set.

Remark 2.4. The integral ∫_{M^2} K d\hat{\alpha} is 2π times the Euler number χ_\mathcal{E} of the limiting tangent bundle \mathcal{E} (see [6,3] in Section 6). When N^3 = R^3, χ_\mathcal{E}/2 is equal to the degree of the Gauss map.

Remark 2.5. These formulas are generalizations of the two Gauss-Bonnet formulas in the introduction. If the surface is regular, the limiting tangent bundle \mathcal{E} coincides with the tangent bundle, and the two Gauss-Bonnet formulas are the same.

Proof of Theorem 2.3. Although ∂M_+ and ∂M_- are the same set, their orientations are opposite. The singular curvature κ_α doesn’t depend on the orientation of the singular curve and coincides with the limit of the geodesic curvature if we take the
conormal vector in the positive direction with respect to the velocity vector of the singular curve. Thus we have

\[ (2.6) \quad \int_{\partial M_+} \kappa_s \, ds + \int_{\partial M_-} \kappa_s \, ds = 2 \int_{\Sigma_f} \kappa_s \, ds. \]

Then by the classical Gauss-Bonnet theorem, we have

\[
2 \pi \chi(M_+) = \int_{M_+} K \, dA + \int_{\partial M_+} \kappa_s \, ds + \sum_{p: \text{peak}} \left( \pi m(p) - \alpha_+(p) \right),
\]

\[
2 \pi \chi(M_-) = \int_{M_-} K \, dA + \int_{\partial M_-} \kappa_s \, ds + \sum_{p: \text{peak}} \left( \pi m(p) - \alpha_-(p) \right),
\]

where \(2m(p)\) is the number of cuspidal edges starting at \(p\) (see Definition \[1.10\]). Hence by \[2.6\],

\[
2 \pi \chi(M^2) = \int_{M^2} K \, dA + 2 \int_{\Sigma_f} \kappa_s \, ds,
\]

\[
2 \pi (\chi(M_+) - \chi(M_-)) = \int_{M^2} K \, d\hat{A} - \sum_{p: \text{peak}} \left( \alpha_+(p) - \alpha_-(p) \right),
\]

where we used \[2.6\] and \(\chi(M^2) = \chi(M_+) + \chi(M_-) - \sum_{p: \text{peak}} (m(p) - 1). \)

We shall now define the completeness of fronts and give Gauss-Bonnet formulas for non-compact fronts: As defined in \[KUY2\], a front \(f: M^2 \to N^3\) is called complete if the singular set is compact and there exists a symmetric tensor \(T\) with compact support such that \(ds^2 + T\) gives a complete Riemannian metric on \(M^2\), where \(ds^2\) is the first fundamental form of \(f\). On the other hand, as defined in \[KRSUY\], a front \(f: M^2 \to N^3\) is called weakly complete if the pull-back of the Sasakian metric of \(T_1N^3\) by the Legendrian lift \(L_f: M^2 \to T_1N^3\) is complete. Completeness implies weak completeness.

Let \(f: M^2 \to N^3\) be a complete front with finite absolute total curvature. Then there exists a compact 2-manifold \(\overline{M}^2\) without boundary and finitely many points \(p_1, \ldots, p_k\) such that \(M^2\) is diffeomorphic to \(\overline{M}^2 \setminus \{p_1, \ldots, p_k\}\). We call the \(p_i\)’s the ends of the front \(f\). According to Theorem A of Shiohama \[S\], we define the limiting area growth order

\[ (2.7) \quad a(p_i) = \lim_{r \to \infty} \frac{\text{Area}(B_0(r) \cap E_i)}{\text{Area}(B_{H^2}(r) \cap E_i)}, \]

where \(E_i\) is the punctured neighborhood of \(p_i\) in \(\overline{M}^2\).

**Theorem 2.6** (Gauss-Bonnet formulas for complete fronts). Let \(f: M^2 \to (N^3, g)\) be a complete front with finite absolute total curvature, which has at most peak singularities, and write \(M^2 = \overline{M}^2 \setminus \{p_1, \ldots, p_k\}\). Then

\[ (2.8) \quad \int_{M^2} K \, dA + 2 \int_{\Sigma_f} \kappa_s \, ds + \sum_{i=1}^k a(p_i) = 2 \pi \chi(M^2), \]

\[ (2.9) \quad \int_{M^2} K \, d\hat{A} - \sum_{p: \text{peak}} (\alpha_+(p) - \alpha_-(p)) + \sum_{i=1}^k \varepsilon(p_i) a(p_i) = 2 \pi (\chi(M_+) - \chi(M_-)) \]

hold, where \(\varepsilon(p_i) = 1\) (resp. \(\varepsilon(p_i) = -1\)) if the neighborhood \(E_i\) of \(p_i\) is contained in \(M_+\) (resp. \(M_-\)).
Example 2.7 (Pseudosphere). Define \( f: \mathbb{R}^2 \to \mathbb{R}^3 \) as

\[
f(x, y) := (\text{sech} x \cos y, \text{sech} x \sin y, x - \tan x).
\]

If we set \( \nu := (\tanh x \cos y, \tanh x \sin y, \text{sech} x) \), then \( \nu \) is the unit normal vector and \( f \) is a front whose singular set \( \{ x = 0 \} \) consists of cuspidal edges. The Gaussian curvature of \( f \) is \(-1\), and the coordinate system \((u, v)\) defined as \( x = u - v, y = u + v \) is the asymptotic Chebyshev net (see Example 1.20) with \( \theta = 4 \arctan(\text{sech}(u - v)) \).

Since \( f(x, y + 2\pi) = f(x, y) \), \( f \) induces a smooth map \( f_1 \) from the cylinder \( M^2 = \mathbb{R}^2/(\{0, 2\pi m\}; m \in \mathbb{Z}) \) into \( \mathbb{R}^3 \). The front \( f_1: M^2 \to \mathbb{R}^3 \) has two ends \( p_1, p_2 \) with growth order \( a(p_j) = 0 \). Hence by Theorem 2.6 we have

\[
2 \int_{\Sigma(f)} \kappa_s \, ds = \text{Area}(M^2) = 8\pi.
\]

In fact, the singular curvature is positive.

Example 2.8 (Kuen’s surface). The smooth map \( f: \mathbb{R}^2 \to \mathbb{R}^3 \) defined as

\[
f(x, y) = \frac{1}{1 + 2(1 + 2y^2)e^{2x} + e^{4x}} \left( \frac{4e^x(1 + e^{2x})(\cos y + y \sin y)}{2(1 + 2y^2)e^{2x} + (x - 2)e^{4x}} \right)
\]

is called Kuen’s surface, which is considered as a weakly complete front with the unit normal vector

\[
\nu(x, y) = \frac{1}{1 + 2(1 + 2y^2)e^{2x} + e^{4x}} \left( \frac{8e^{2x}y \cos y - (1 + 2(1 - 2y^2)e^{2x} + e^{4x}) y}{8e^{2x}y \sin y + (1 + 2(1 - 2y^2)e^{2x} + e^{4x}) \cos y} \right),
\]

and has Gaussian curvature \(-1\). The coordinate system \((u, v)\) such that \( x = u - v \) and \( y = u + v \) is the asymptotic Chebyshev net with \( \theta = -4 \arctan(2ye^x/(1 + e^{2x})) \).

Since the singular set \( \Sigma_f = \{ y = 0 \} \cap \{ y = \pm \cosh x \} \) is non-compact, \( f \) is not complete.

Example 2.9 (Cones). Define \( f: \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}^3 \) as

\[
f(x, y) = (\log r \cos \theta, \log r \sin \theta, a \log r) \quad (x, y) = (r \cos \theta, r \sin \theta),
\]

where \( a \neq 0 \) is a constant. Then \( f \) is a front with \( \nu = (a \cos t, a \sin t, -1)/\sqrt{1 + a^2} \).

The singular set is \( \Sigma_f = \{ r = 1 \} \), which corresponds to the single point \((0, 0, 0) \in \mathbb{R}^3 \). That is, all points in \( \Sigma_f \) are degenerate singular points. The image of the singular points is a cone of angle \( \mu = 2\pi/\sqrt{1 + a^2} \) and the area growth order of the two ends are \( 1/\sqrt{1 + a^2} \). Theorem 2.3 cannot be applied to this example because the singularities degenerate. However, this example suggests that it might be natural to define the “singular curvature measure” at a cone-like singularity as the cone angle.

3. Behavior of the Gaussian curvature

Firstly, we shall prove the following assertion, which says that the shape of singular points is very restricted when the Gaussian curvature is bounded.

**Theorem 3.1.** Let \( f: M^2 \to (\mathbb{R}^3, g) \) be a front, \( p \in M^2 \) a singular point, and \( \gamma(t) \) a singular curve consisting of non-degenerate singular points with \( \gamma(0) = p \) defined on an open interval \( I \subset \mathbb{R} \). Then the Gaussian curvature \( K \) is bounded on a sufficiently small neighborhood of \( \gamma(I) \) if and only if the second fundamental form vanishes on \( \gamma(I) \).

Moreover, if the extrinsic curvature \( K_{ext} \) (i.e. the product of the principal curvatures) is non-negative on \( U \setminus \gamma(I) \) for a neighborhood of \( U \) of \( p \), then the singular
The geometry of fronts 15

curvature is non-positive. Furthermore, if \( K_{\text{ext}} \) is bounded below by a positive constant on \( U \setminus \gamma(I) \) then the singular curvature at \( p \) takes a strictly negative value.

In particular, when \( (N^3, g) = (R^3, g_0) \), the singular curvature is non-positive if the Gaussian curvature \( K \) is non-negative near the singular set.

Proof of the first part of Theorem 3.1

We shall now prove the first part of the theorem. Take an adapted coordinate system \((u, v)\) such that the singular point \( p \) corresponds to \((0, 0)\), and write the second fundamental form of \( f \) as

\[
\begin{align*}
(3.1) \quad h &= L du^2 + 2M du \ dv + N \ dv^2 \\
&\quad \left( L = -g(f_u, \nu_u), \quad N = -g(f_v, \nu_v), \quad M = -g(f_\nu, \nu_u) = -g(f_u, \nu_v) \right).
\end{align*}
\]

Since \( f_u \) and \( f_\nu \) are linearly dependent on the \( u \)-axis, \( LN - (M)^2 \) vanishes on the \( u \)-axis as well as the area density function \( \lambda(u, v) \). Then by the Malgrange preparation theorem (see [GG page 91]), there exist smooth functions \( \varphi(u, v) \), \( \psi(u, v) \) such that

\[
(3.2) \quad \lambda(u, v) = \nu \varphi(u, v) \quad \text{and} \quad LN - (M)^2 = \psi(u, v).
\]

Since \( \nu \varphi(u, v) \neq 0 \) holds. Hence \( \varphi(u, v) \neq 0 \) on a neighborhood of the origin.

Firstly, we consider the case \( p \) is a cuspidal edge point. Then we can choose \((u, v)\) so that \( \partial/\partial v \) gives the null direction. Since \( f_v = 0 \) holds on the \( u \)-axis, we have \( M = N = 0 \). By \( (2.1) \) and \( (3.2) \), we have \( K = \psi(u, v)/(\nu \varphi(u, v)^2) \).

Thus the Gaussian curvature is bounded if and only if

\[
L(u, 0)N_v(u, 0) = (LN - (M)^2) \big|_{v = 0} = \psi(u, 0) = 0
\]

holds on the \( u \)-axis. To prove the assertion, it is sufficient to show that \( N_v(u, 0) \neq 0 \).

Since \( \lambda_v = \mu_\varphi(f_u, f_\nu, v) \neq 0 \), \( \{f_u, f_\nu, v\} \) is linearly independent. Here, we have

\[
2g(\nu_v, v) = g(v, v)_v = 0 \quad \text{and} \quad g(\nu_v, f_u)_v = -M = 0.
\]

Thus \( \nu_v = 0 \) if and only if \( g(\nu_v, f_\nu) = 0 \). On the other hand, \( \nu_v(0, 0) \neq 0 \) holds, since \( f \) is a front and \( f_v = 0 \). Thus we have

\[
(3.3) \quad N_v(0, 0) = g(f_v, \nu_v)_v = g(\nu_v, f_\nu) \neq 0.
\]

Hence the first part of Theorem 3.1 is proved for cuspidal edges.

Next we consider the case that \( p \) is not a cuspidal edge point. Under the same notation as in the previous case, \( f_u(0, 0) = 0 \) holds because \( p \) is not a cuspidal edge. Then we have \( M(0, 0) = L(0, 0) = 0 \), and thus the Gaussian curvature is bounded if and only if

\[
L_v(u, 0)N_v(u, 0) = (LN - (M)^2) \big|_{v = 0} = \psi(u, 0) = 0
\]

holds on the \( u \)-axis. Thus, to prove the assertion, it is sufficient to show that \( L_v(0, 0) \neq 0 \). Since \( \lambda_v = \mu_\psi(f_u, f_\nu, v) \) does not vanish, \( \{f_\psi, f_\nu, v\} \) is linearly independent. On the other hand, \( \nu_v(0, 0) \neq 0 \), because \( f \) is a front and \( f_v(0, 0) = 0 \). Since \( g(\nu, v)_v = 0 \) and \( g(\nu_v, f_u)_v = -M = 0 \), we have

\[
(3.4) \quad L_v(0, 0) = g(\nu_v, f_u)_v = g(\nu_v, f_\nu) \neq 0.
\]

Hence the first part of the theorem is proved. \( \square \)

Before proving the second part of Theorem 3.1, we prepare the following lemma:

Lemma 3.2 (Existence of special adapted coordinates along cuspidal edges). Let \( p \) be a cuspidal edge of a front \( f : M^2 \to (N^3, g) \). Then there exists an adapted coordinate system \((u, v)\) satisfying the following properties:

1. \( g(f_u, f_u) = 1 \) on the \( u \)-axis,
2. \( f_v \) vanishes on the \( u \)-axis,
3. \( \lambda_v = 1 \) holds on the \( u \)-axis,
4. \( g(f_\psi, f_u) \) vanishes on the \( u \)-axis, and
5. \( \{f_u, f_\psi, v\} \) is a positively oriented orthonormal basis along the \( u \)-axis.
We shall call such a coordinate system \((u, v)\) a *special adapted coordinate system*. 

**Proof of Lemma 3.2.** One can easily take an adapted coordinate system \((u, v)\) at \(p\) satisfying (1) and (2). Since \(\lambda_v \neq 0\) on the \(u\)-axis, we can choose \((u, v)\) as \(\lambda_v > 0\) on the \(u\)-axis. In this case, \(r := \sqrt{\lambda_v}\) is a smooth function on a neighborhood of \(p\). Now we set 
\[
    u_1 = u, \quad v_1 = \sqrt{\lambda_v(u, 0)} v.
\]

Then the Jacobian matrix is given by 
\[
\frac{\partial (u_1, v_1)}{\partial (u, v)} = \begin{pmatrix} 1 & 0 \\ r'(u) & r(u) \end{pmatrix}, \quad \text{where } r(u) := \sqrt{\lambda_v(u, 0)}.
\]

Thus we have 
\[
(f_{u_1}, f_{v_1})_{|v=0} = (f_u, f_v) \begin{pmatrix} 1 & 0 \\ r'(u) & r(u) \end{pmatrix}_{|v=0} = (f_u, f_v) \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.
\]

This implies that \(f_{u_1} = f_u\) and \(f_{v_1} = 0\) on the \(u\)-axis. Thus the new coordinates \((u_1, v_1)\) satisfy (1) and (2). The signed area density function with respect to \((u_1, v_1)\) is given by \(\lambda_1 := \mu_g(f_{u_1}, f_{v_1}, \nu)\). Since \(f_{v_1} = 0\) on the \(u\)-axis, we have 
\[
(\lambda_1)_{v_1} := \mu_g(f_{u_1}, D_{v_1} f_{v_1}, \nu).
\]

On the other hand, we have 
\[
f_{v_1} = \frac{f_v}{r(u)} \quad \text{and} \quad D_{v_1} f_{v_1} = \frac{D_v f_{v_1}}{r(u)} = \frac{f_{v v}}{r^2} = \frac{f_v}{\lambda_v}
\]
on the \(u_1\)-axis. By (3.5) and (3.6), we have \((\lambda_1)_{v_1} = \lambda_v/\lambda_v = 1\) and have shown that \((u_1, v_1)\) satisfies (1) and (2).

Next, we set 
\[
    u_2 := u_1 + v_1^2 s(u_1), \quad v_2 := v_1,
\]
where \(s(u_1)\) is a smooth function in \(u_1\). Then we have 
\[
\frac{\partial (u_2, v_2)}{\partial (u_1, v_1)} = \begin{pmatrix} 1 + v_1^2 s' & 2v_1 s(u_1) \\ 0 & 1 \end{pmatrix},
\]
and 
\[
\left. \frac{\partial (u_1, v_1)}{\partial (u_2, v_2)} \right|_{v_1=0} = \frac{1}{1 + v_1^2 s'} \begin{pmatrix} 1 & -2v_1 s(u_2) \\ 0 & 1 + v_1^2 s' \end{pmatrix}_{|v_1=0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Thus the new coordinates \((u_2, v_2)\) satisfy (1) and (2). On the other hand, the area density function \(\lambda_2 := \mu_g(f_{u_2}, f_{v_2}, \nu)\) satisfies 
\[
(\lambda_2)_{v_2} = \mu_g(f_{u_2}, f_{v_2}, \nu)_{v_2} = \mu_g(f_{u_2}, D_{v_2} f_{v_2}, \nu).
\]

We have on the \(u_2\)-axis that \(f_{u_2} = f_{u_1}\) and 
\[
f_{v_2} = -\frac{2v_1 s}{1 + v_1^2 s'} f_{u_1} + f_{v_1} 
\]
and 
\[
g(D_{v_2} f_{v_2}) = D_{v_1} f_{v_2} = -\frac{2s}{1 + v_1^2 s'} f_{u_1} + D_{v_1} f_{v_1}.
\]

Thus one can easily check that \((\lambda_2)_{v_2} = 1\) on the \(u\)-axis. By (3.6), we have 
\[
g(f_{u_2}, D_{v_2} f_{v_2}) = -2s + g(f_{u_1}, D_{v_1} f_{v_1}).
\]

Hence, if we set 
\[
s(u_1) := \frac{1}{2} g(f_{u_1}(u_1, 0), (D_{v_1} f_{v_1})(u_1, 0)),
\]
then the coordinate \((u_2, v_2)\) satisfies (1), (2), (3) and (4). Since \(g(f_{v_2}, \nu) = -g(f_{v_2}, \nu) = 0, f_{v_2 v_2}(u_2, 0)\) is perpendicular to both \(\nu\) and \(f_{u_2}\). Moreover, we have on the \(u_2\)-axis 
\[
1 = (\lambda_2)_{v_2} = \mu_g(f_{u_2}, f_{v_2}, \nu)_{v_2} = \mu_g(f_{u_2}, D_{v_2} f_{v_2}, \nu) = g(D_{v_2} f_{v_2}, f_{u_2} \times g(\nu),\)
\]
and can conclude that \( D_{v_2}f_{v_2} \) is a unit vector. Thus \((u_2,v_2)\) satisfies (5) \(\square\)

Using the existence of the special adapted coordinate system, we shall show the second part of the theorem.

**Proof of the second part of Theorem 3.1** We suppose \( K \geq c_{N^2} \), where \( c_{N^2} \) is the sectional curvature of \((N^3,g)\) with respect to the tangent plane. Then by (2.1), \( K_{\text{ext}} \geq 0 \) holds.

If a given non-degenerate singular point \( p \) is not a cuspidal edge, the singular curvature is negative by Corollary (1.14). Hence it is sufficient to consider the case that \( p \) is a cuspidal edge. So we may take a special adapted coordinate system as in Lemma 3.2. We take smooth functions \( \varphi \) and \( \psi \) as in 3.2.

Since \( K \) is bounded, \( \psi(u,0) = 0 \) holds, as seen in the proof of the first part. By the Malgrange preparation theorem again, we may put \( LN - M^2 = v^2\psi_1(u,v) \), and have the expression \( K_{\text{ext}} = \psi_1/\varphi^2 \). Since \( K_{\text{ext}} \geq 0 \), we have \( \psi_1(u,0) \geq 0 \). Moreover, if \( K_{\text{ext}} \geq \delta > 0 \) on a neighborhood of \( p \), then \( \psi_1(u,0) > 0 \). Since \( L = M = N = 0 \) on the \( u \)-axis, we have

\[
0 \leq 2\psi_1(u,0) = (LN - (M)^2)_{vv} = L_v N_v - (M_v)^2 \leq L_v N_v.
\]

Here, \( \{f_{uu},f_{uv},\nu\} \) is an orthonormal basis, and \( g(f_{uu},f_u) = 0 \) and \( L = g(f_{vv},\nu) = 0 \) on the \( u \)-axis. Hence

\[
f_{uu} = g(f_{uu},f_{uv})f_{vv} + g(f_{uv},\nu)\nu = g(f_{uu},f_{uv})f_{vv}.
\]

Similarly, since \( 2g(\nu,\nu) = g(\nu,\nu)_v = 0 \) and \( g(\nu, f_u) = -M = 0 \), we have

\[
\nu_v = g(\nu, f_{vv})f_{uv}.
\]

Since \( \lambda_v = 1 > 0 \) and \( |f_u| = 1 \), the singular curvature is given by

\[
(3.10) \quad \kappa_s = \mu g(f_{uu},f_{uv},\nu) = g(f_{uu},f_{uv})\mu g(f_{uv},\nu) = g(f_{uu},f_{uv}) = \frac{g(f_{uu},\nu_v)}{g(f_{vv},\nu_v)}.
\]

On the other hand, we have on the \( u \)-axis that

\[
-L_v = g(f_u,\nu_u)v = g(f_{uv},\nu_u) + g(f_u,\nu_{uv}) = g(f_u,\nu_{uv}),
\]

because \( g(f_{uu},\nu_u)_v = 0 = -M_u(u,0) = 0 \). Moreover, we have

\[
\nu_{uv} = D_u D_v \nu = D_u D_v \nu + R(f_v, f_u)\nu = D_u D_v \nu = \nu_{uv}
\]

since \( f_v = 0 \), where \( R \) is the Riemannian curvature tensor of \((N^3,g)\). Thus,

\[
L_v = -g(f_u,\nu_{uv}) = -g(f_u,\nu_u)v + g(f_{uv},\nu_v) = M_u + g(f_{uu},\nu_v) = g(f_{uu},\nu_v)
\]

holds. Since we have on the \( u \)-axis that

\[
(5.10) \quad \kappa_s = -\frac{L_v}{N_v} = -\frac{L_v N_v}{N^2 v} \leq 0.
\]

If \( K_{\text{ext}} \geq \delta > 0 \), \( (5.10) \) becomes \( 0 < L_v N_v \), and we have \( \kappa_s < 0 \). \(\square\)

**Remark 3.3.** Let \( f: M^2 \to R^3 \) be a compact front with positive Gaussian curvature. For example, parallel surfaces of compact immersed constant mean curvature surfaces (e.g. Wente tori) give such examples. In this case, we have the following opposite of the Cohn-Vossen inequality by Theorem 2.23:

\[
\int_{M^2} K dA > 2\pi \chi(M^2).
\]

On the other hand, the total curvature of a compact 2-dimensional Alexandrov space is bounded from above by \( 2\pi \chi(M^2) \) (see Machigashira [Mac]). This implies
that a front with positive curvature cannot be a limit of Riemannian 2-manifolds with Gaussian curvature bounded below by a constant. We can give another explanation of this phenomenon as follows: Since $K > 0$, we have $\kappa_s < 0$ and the shape of the surfaces looks like cuspidal hyperbolic parabola. So if the front is a limit of the sequence of immersions $f_n$, the curvature of $f_n$ must converge to $-\infty$.

**Example 3.4 (Fronts of constant positive Gaussian curvature).** Let $f_0: M^2 \rightarrow \mathbb{R}^3$ be an immersion of constant mean curvature 1 and $\nu$ the unit normal vector of $f_0$. Then the parallel surface $f := f_0 - \nu$ gives a front of constant Gaussian curvature 1. If we take isothermal principal curvature coordinates $(u, v)$ on $M^2$ with respect to $f_0$, the first and second fundamental forms of $f$ are given by

$$ds^2 = dz^2 + 2 \cosh \theta d\bar{z} d\bar{z} + d\bar{z}^2 \quad h = 2 \sinh \theta d\bar{z} d\bar{z},$$

where $z = u + iv$ and $\theta$ is a real-valued function in $(u, v)$, which is called the complex Chebyshev net. The sinh-Gordon equation $\theta_{uv} + \theta_{vv} + 4 \sinh \theta = 0$ is the integrability condition. In this case, the singular curve is characterized by $\theta = 0$, and the condition for non-degenerate singular points is given by $d\theta \neq 0$. Moreover, the cuspidal edges are characterized by $\theta_v \neq 0$, and the swallowtails are characterized by $\theta_u \neq 0, \theta_v = 0$ and $\theta_{vv} \neq 0$. The singular curvature on cuspidal edges is given by

$$\kappa_s = -\frac{(\theta_u)^2 + (\theta_v)^2}{4|\theta_v|} < 0.$$ 

The negativity of $\kappa_s$ has been shown in Theorem 3.1. Like the case of fronts of constant negative curvature, Ishikawa-Machida [IM] also showed that the generic singularities of fronts of constant positive Gaussian curvature are cuspidal edges or swallowtails.

Here we should like to remark on the behavior of mean curvature function near the non-degenerate singular points.

**Corollary 3.5.** Let $f : M^2 \rightarrow (N^3, g)$ be a front and $p \in M^2$ a non-degenerate singular point. Then the mean curvature function of $f$ is unbounded near $p$.

**Proof.** The mean curvature function $H$ is given by

$$2H := \frac{EN - 2FM + GL}{EG - F^2} = \frac{EN - 2FM + GL}{2\lambda^2}.$$ 

We may assume that $u$-axis is a singular curve. By applying L’Hospital’s rule, we have

$$\lim_{v \rightarrow 0} H = \lim_{v \rightarrow 0} \frac{E_v N + E N_v - 2F_v M - 2FM_v + G_v L - GL_v}{2\lambda_v}.$$ 

Firstly, we consider the case $(0, 0)$ is a cuspidal edge. Then by the proof of the first part of Theorem 3.1, we have

$$F(0, 0) = G(0, 0) = M(0, 0) = N(0, 0) = G_v(0, 0) = 0.$$ 

Thus

$$\lim_{v \rightarrow 0} H = \lim_{v \rightarrow 0} \frac{E N_v}{2\lambda_v}.$$ 

Since $\lambda(0, 0) = 0$ and $N_v(0, 0) \neq 0$ as shown in the proof of Theorem 3.1, $H$ diverges.

Next, we consider the case that $(0, 0)$ is not a cuspidal edge. When $p$ is not a cuspidal edge, by the proof of the first part of Theorem 3.1, we then have

$$E(0, 0) = F(0, 0) = L(0, 0) = M(0, 0) = E_v(0, 0) = 0, \quad L_v(0, 0) \neq 0.$$ 

Thus

$$\lim_{v \rightarrow 0} H = -\lim_{v \rightarrow 0} \frac{GL_v}{2\lambda_v}.$$ 

diverges, since $\lambda(0, 0) = 0$ and $L_v(0, 0) \neq 0$. \qed
\[ \nu \Omega(-\nu) \]

the half-space containing the singular curve is \( \Omega(-\nu) \)

\[ \Omega(\nu) \]

The outward normal

\[ \nu_0 \]

the vector \( \tau_0 \)

**Figure 5.** The half-space containing the singular curve, Theorem 3.7.

**Generic behavior of the curvature near cuspidal edges.** As an application of Theorem 3.1, we shall investigate the generic behavior of the Gaussian curvature near cuspidal edges and swallowtails in \((\mathbb{R}^3, g_0)\).

We call a given cuspidal edge \( p \in M^2 \) generic if the second fundamental form does not vanish at \( p \). Theorem 3.1 implies that fronts with bounded Gaussian curvature have only non-generic cuspidal edges. In the proof of the theorem for cuspidal edges, \( L = 0 \) if and only if \( f_{uu} \) is perpendicular to both \( \nu \) and \( f_u \), which implies that the osculating plane of the singular curve coincides with the limiting tangent plane, and we get the following:

**Corollary 3.6.** Let \( f: M^2 \to \mathbb{R}^3 \) be a front. Then a cuspidal edge \( p \in M^2 \) is generic if and only if the osculating plane of the singular curve does not coincide with the limiting tangent plane at \( p \). Moreover, the Gaussian curvature is unbounded and changes sign between the two sides of a generic cuspidal edge.

Proof. By (2.1) and (3.2), \( K = \psi / (v\phi^2) \), where \( \psi(0,0) \neq 0 \) if \((0,0)\) is generic. Hence \( K \) is unbounded and changes sign between the two sides along the generic cuspidal edge. \( \square \)

We shall now determine which side has positive Gaussian curvature: Let \( \gamma \) be a singular curve of \( f \) consisting of cuspidal edge points, and let \( \hat{\gamma} = f \circ \gamma \). Define

\[ \kappa_\nu := \frac{g_0(\hat{\gamma}''(\nu))}{|\hat{\gamma}'|^2} \]

on the singular curve, which is independent of the choice of parameter \( t \). We call it the *limiting normal curvature* of the cuspidal edge \( \gamma(t) \). Then one can easily check that \( p \) is a generic cuspidal edge if and only if \( \kappa_\nu(p) \) does not vanish. Let \( \Omega(\nu) \) (resp. \( \Omega(-\nu) \)) be the half-space bounded by the limiting tangent plane such that \( \nu \) (resp. \( -\nu \)) points into \( \Omega(\nu) \) (resp. \( \Omega(-\nu) \)). Then the singular curve lies in \( \Omega(\nu) \) if \( \kappa_\nu(p) > 0 \) and lies in \( \Omega(-\nu) \) if \( \kappa_\nu(p) < 0 \). We call \( \Omega(\nu) \) (resp. \( \Omega(-\nu) \)) the *half-space containing the singular curve* at the cuspidal edge point \( p \). This half-space is in general different from the principal half-space (see Definition 1.19 and Figure 5).

We set

\[ \text{sgn}_0(\nu) := \text{sgn}(\kappa_\nu) \]

\[ = \begin{cases} 
1 & \text{(if } \Omega(\nu) \text{ is the half-space containing the singular curve)} \\
-1 & \text{(if } \Omega(-\nu) \text{ is the half-space containing the singular curve).}
\end{cases} \]

On the other hand, one can choose the *outward normal vector* \( \nu_0 \) near a given cuspidal edge \( p \) as in the middle figure of Figure 5. Let \( \Delta \) be a sufficiently small domain consisting of regular points sufficiently close to \( p \) that lies only to one side
of the cuspidal edge. For a given unit normal vector $\nu$ of the front, we define its sign $\text{sgn}_\Delta(\nu)$ by $\text{sgn}_\Delta(\nu) = 1$ (resp. $\text{sgn}_\Delta(\nu) = -1$) if $\nu$ coincides with the outward normal $v_0$ on $\Delta$. The following assertion holds:

**Theorem 3.7.** Let $f : M^2 \to (\mathbb{R}^3, g_0)$ be a front, $p$ a cuspidal edge and $\Delta$ a sufficiently small domain consisting of regular points sufficiently close to $p$ that lies only to one side of the cuspidal edge. Then $\text{sgn}_\Delta(\nu)$ coincides with the sign of the function $g_0(\hat{\sigma}', \hat{\nu}')$ at $p$, namely

$$
\text{sgn}_\Delta(\nu) = \text{sgn}(g_0(\hat{\sigma}', \hat{\nu}')), \quad \left( t = \frac{d}{ds}, \quad a = D_a \frac{d}{ds} \right),
$$

where $\sigma(s)$ is an arbitrarily fixed null curve starting at $p$ and moving into $\Delta$, and $\hat{\sigma}(s) = f(\sigma(s))$ and $\hat{\nu} = \nu(\sigma(s))$. Moreover, if $p$ is a generic cuspidal edge, then

$$
\text{sgn}_0(\nu) \cdot \text{sgn}_\Delta(\nu)
$$

coincides with the sign of the Gaussian curvature on $\Delta$.

**Proof.** We take a special adapted coordinate system $(u, v)$ as in Lemma 3.2 at the cuspidal edge. The vector $\tau_0 := -f_{vv} = f_u \times \nu$ lies in the limiting tangent plane and points in the opposite direction of the image of the null curve (see Figure 5 right side).

Without loss of generality, we may assume that $\Delta = \{v > 0\}$. The unit normal $\nu$ is the outward normal on $\Delta$ if and only if $g_0(\nu_v, \tau_0) > 0$, namely $N_v = -g_0(f_{vv}, \nu_v) > 0$. Thus we have $\text{sgn}_{\nu > 0}(\nu) = \text{sgn}(N_v)$, which proves (3.12). Since $p$ is generic, we have $\kappa_v(p) \neq 0$ and $\kappa_v(p) = L$ holds. On the other hand, the sign of $K$ on $v > 0$ is equal to the sign of

$$
(LN - (M)^2)_v|_{v=0} = L(u, 0)N_v(u, 0),
$$

which proves the assertion. \qed

**Example 3.8.** Consider again the cuspidal parabola $f(u, v)$ as in Example 1.9. Then $(u, v)$ gives an adapted coordinate system so that $\partial/\partial v$ gives a null direction, and we have

$$
L = g_0(f_u, \nu) = \frac{-2ab}{\sqrt{1+b^4(1+4a^2u^2)}}, \quad N_v = \frac{6}{\sqrt{1+b^4(1+4a^2u^2)}} > 0.
$$

The cuspidal edges are generic if and only if $ab \neq 0$. In this case, let $\Delta$ be a domain in the upper half-plane $\{(u, v) : v > 0\}$. Then the unit normal vector $(1, 1, 1)$ is the outward normal to the cuspidal edge, that is, $\text{sgn}_\Delta(\nu) = +1$. The limiting normal curvature as in (3.11) is computed as $\kappa_v = -ab/(2a^2\sqrt{1+b^4(1+4a^2u^2)})$, and hence $\text{sgn}_0(\nu) = -\text{sgn}(ab)$. Then $\text{sgn}(K) = -\text{sgn}(ab)$ holds on the upper half-plane. In fact, the Gaussian curvature is computed as

$$
K = \frac{-12(ab + 3av)}{v(4 + (1 + 4a^2u^2)(4b^2 + 12bv + 9v^2))}.
$$

On the other hand, the Gaussian curvature is bounded if $b = 0$. Moreover, the Gaussian curvature is positive if $a < 0$. In this case the singular curvature is negative when $a < 0$, as stated in Theorem 3.1.

**Generic behavior of the curvature near swallowtails.** We call a given swallowtail $p \in M^2$ of a front $f : M^2 \to (\mathbb{R}^3, g_0)$ generic if the second fundamental form does not vanish at $p$.

**Proposition 3.9.** Let $f : M^2 \to (\mathbb{R}^3, g_0)$ be a front and $p$ a generic swallowtail. Then we can take a half-space $H \subset \mathbb{R}^3$ bounded by the limiting tangent plane such that any null curve at $p$ lies in $H$ near $p$ (see Figure 7).
THE GEOMETRY OF FRONTS

The half-space containing the singular curve is the closer side of the limiting tangent plane for the left-hand figure, and the farther side for the right-hand figure.

**Figure 6.** The half-space containing the singular curve for generic swallowtails (Example 3.12).

We shall call $H$ the half-space containing the singular curve at the generic swallowtail. At the end of this section, we shall see that the singular curve is in fact contained in this half-space for a neighborhood of the swallowtail (see Figure 6 and Corollary 3.13). For a given unit normal vector $\nu$ of the front, we define the sign $\text{sgn}_0(\nu)$ of it by $\text{sgn}_0(\nu) = 1$ (resp. $\text{sgn}_0(\nu) = -1$) if $\nu$ points (resp. does not point) into the half-space containing the singular curve.

**Proof of Proposition 3.9.** Take an adapted coordinate system $(u, v)$ and assume $f(0, 0) = 0$ by translating in $\mathbb{R}^3$ if necessary. Write the second fundamental form as in (3.1). Since $f_u(0, 0) = 0$, we have $L(0, 0) = M(0, 0) = 0$, and we have the following Taylor expansion:

$$g_0(f(u, v), \nu) = \frac{\nu^2}{2}g_0(f_{uv}(0, 0), \nu(0, 0)) + o(u^2 + v^2) = \frac{1}{2}N(0, 0)v^2 + o(u^2 + v^2).$$

Thus the assertion holds. Moreover we have

$$\text{sgn}(N) = \text{sgn}_0(\nu). \quad \square$$

**Corollary 3.10.** Let $\sigma(s)$ be an arbitrary curve starting at the swallowtail such that $\sigma'(0)$ is transversal to the singular direction. Then

$$\text{sgn}_0(\nu) = \text{sgn}(g_0(\hat{\sigma}'(0), \nu(0, 0)))$$

holds where $\hat{\sigma} = f \circ \sigma$.

We let $\Delta$ be a sufficiently small domain consisting of regular points sufficiently close to a swallowtail $p$. The domain $\Delta$ is called the tail part if $\Delta$ is on the opposite side of the self-intersection of the swallowtail. We define $\text{sgn}_\Delta(\nu)$ by $\text{sgn}_\Delta(\nu) = 1$ (resp. $\text{sgn}_\Delta(\nu) = -1$) if $\nu$ is (resp. is not) the outward normal of $\Delta$. Now we have the following assertion:

**Theorem 3.11.** Let $f : M^2 \to (\mathbb{R}^3, g_0)$ be a front, $p$ a generic swallowtail and $\Delta$ a sufficiently small domain consisting of regular points sufficiently close to $p$. Then the Gaussian curvature is unbounded and changes sign between the two sides along the singular curve. Moreover, $\text{sgn}_0(\nu)\text{sgn}_\Delta(\nu)$ coincides with the sign of the Gaussian curvature on $\Delta$. 

Proof. If we change \( \Delta \) to the opposite side, \( \text{sgn}_\Delta(\nu) \text{sgn}_\Delta(K) \) does not change sign. So we may assume that \( \Delta \) is the tail part. We take an adapted coordinate system \((u, v)\) at the swallowtail and write the null vector field as \( \eta(u) = (\partial/\partial u) + e(u)(\partial/\partial v) \), where \( e(u) \) is a smooth function. Then
\[
f_u(u, 0) + e(u)f_v(u, 0) = 0 \quad \text{and} \quad f_{uv}(u, 0) + e(u)f_{vv}(u, 0) = 0
\]
hold. Since \( u = 0 \) is a swallowtail, \( e(0) = 0 \) and \( e'(0) \neq 0 \) hold, where \( e' = d/du \).

The vector \( f_{uv} \) points toward the tail part \( \Delta \). Thus \( f_v \) points toward \( \Delta \) if and only if \( g_0(f_v, f_{uu}) \) is positive. Since \( f_u = -e(u)f_v \) and \( e(0) = 0 \), we have \( f_{uu}(0, 0) = e'(0)f_u(0, 0) \) and
\[
g_0(f_{uu}(0, 0), f_v(0, 0)) = -e'(0)g_0(f_v(0, 0), f_u(0, 0)) = 0.
\]

Thus \( g_0(f_{uu}(0, 0), f_v(0, 0)) = 0 \) (that is, the tail part is \( v > 0 \)) if and only if \( e'(0) < 0 \).

Changing \( v \) to \(-v\) if necessary, we assume \( e'(0) > 0 \), that is, the tail part lies in \( v > 0 \). For each fixed value of \( u \neq 0 \), we take a curve \( \sigma(s) = (u + \varepsilon s, \varepsilon e(u)s) \) and let \( \hat{\sigma} = f \circ \sigma \). Then \( \sigma \) is traveling into the upper half-plane \( \{v > 0\} \), that is, \( \hat{\sigma} \) is traveling into \( \Delta \). Here, we have
\[
\hat{\sigma}'(0) = e(f_u(u, 0) + e(u)f_v(u, 0)) = 0 \quad \text{and} \quad \hat{\sigma}''(0) = e(f_u + e_f_v)_u + e(e(f_u + e_f_v))|_{v=0}
\]
\[
= e(u)(f_{uv}(u, 0) + e(u)f_{vv}(u, 0)),
\]
where \( e' = d/ds \). In particular, \( \sigma \) is a null curve starting at \((u, 0)\) and traveling into \( \Delta \). Then by Theorem 5.7 we have
\[
\text{sgn}_\Delta(\nu) = \lim_{u \to 0} \text{sgn}(g_0(\hat{\sigma}''(s), \hat{\sigma}'(s))).
\]

Here, the derivative of \( \hat{\nu}(t) = \nu(\sigma(t)) \) is computed as \( \hat{\nu}' = \varepsilon \{\nu_u(u, 0) + e(u)\nu_v(u, 0)\} \).
Since \( e(0) = 0 \), we have
\[
g(\hat{\sigma}''(s), \hat{\sigma}'(s))|_{s=0} = |e(u)|g_0(f_{uv}(u, 0), \nu_u(u, 0)) + |e(u)|^2\varphi(u),
\]
where \( \varphi(u) \) is a smooth function in \( u \). Then we have
\[
\text{sgn}_\Delta(\nu) = \lim_{u \to 0} \text{sgn}(g_0(\hat{\sigma}''(s), \hat{\sigma}'(s))) = \text{sgn}(g_0(f_{uu}(0, 0), \nu_u(0, 0))).
\]

Here, \( L_v(0, 0) = -g_0(f_u, \nu_v) = -g_0(f_{uv}, \nu_u) \) because \( f_u = 0 \), which implies that
\[
\text{sgn}_\Delta(\nu) = \text{sgn}(L_v(0, 0)).
\]

On the other hand, the sign of \( K \) on \( v > 0 \) is equal to the sign of
\[
(LN - (M)^2)|_{v=0} = N(0, 0)L_v(0, 0).
\]

Then 6.13 implies the assertion. \( \square \)

Example 3.12. Let
\[
f_{\pm}(u, v) = \frac{1}{12}(3u^4 - 12u^2v \pm (6u^2 - 12v)^2, 8u^3 - 24uv, 6u^2 - 12v).
\]

Then one can see that \( f_{\pm} \) is a front and \((0, 0)\) is a swallowtail with the unit normal vector
\[
\nu_{\pm} = \frac{1}{\delta}(1, u, u^2 \pm 12(2v - u^2))
\]
\[
\delta = \sqrt{1 + u^2 + 145u^4 + 576v(u - u^2) \pm 24u^2(2v - u^2)}.
\]
In particular, \((u, v)\) is an adapted coordinate system. Since the second fundamental form is \(\pm 24d^2\) at the origin, the swallowtail is generic and \(\text{sgn}_0(\nu_\pm) = \pm 1\) because of (4.13). The images of \(f_\pm\) are shown in Figure 6. Moreover, since \(L_0 = \pm 2\) at the origin, \(\text{sgn}_0(\nu_\pm) = \pm 1\). Then by Theorem 4.11 the Gaussian curvature of the tail side of \(f_+\) (resp. \(f_-\)) is positive (resp. negative).

Summing up the previous two theorems, we get the following:

**Corollary 3.13.** Let \(\gamma(t)\) be a singular curve such that \(\gamma(0)\) is a swallowtail. Then the half-space containing the singular curve at \(\gamma(t)\) converges to the half-space at the swallowtail \(\gamma(0)\) as \(t \to 0\).

4. Zigzag Numbers

In this section, we introduce a geometric formula for a topological invariant called the zigzag number. We remark that Langevin, Levitt and Rosenberg [LLR] gave topological upper bounds of zig-zag numbers for generic compact fronts in \(R^3\). (See Remark 4.3.)

**Zigzag number for fronts in the plane.** First, we mention the Maslov index (see [A]; which is also called the zigzag number) for fronts in the Euclidean plane \((R^2, g_0)\). Let \(\gamma: S^1 \to R^2\) be a generic front, that is, all self-intersections and singularities are double points and 3/2-cusps, and let \(\nu\) be the unit normal vector field of \(\gamma\). Then \(\gamma\) is Legendrian isotropic (isotropic as the Legendrian lift \((\gamma, \nu): S^1 \to T_1 R^2 \simeq R^2 \times S^1\) to one of the fronts in Figure 4(a). The non-negative integer \(n\) is called the rotation number, which is the rotational index of the unit normal vector field \(\nu: S^1 \to S^1\). The number \(k\) is called the Maslov index or zigzag number. We shall give a precise definition and a formula to calculate the number; a 3/2-cusp \(\gamma(t_0)\) of \(\gamma\) is called zig (resp. zag) if the leftward normal vector of \(\gamma\) points to the outside (resp. inside) of the cusp (see Figure 4(b)). We define a \(C^\infty\)-function \(\lambda\) on \(S^1\) as \(\lambda := \det(\gamma', \nu)\), where \(\gamma' = d/dt\). Then the leftward normal vector is given by \((\text{sgn}\lambda)\nu_0\). Since \(\gamma''(t_0)\) points to the inside of the cusp, \(t_0\) is zig (resp. zag) if and only if

\[
\text{sgn}(\lambda' g_0(\gamma'', \nu')) < 0 \quad (\text{resp.} \quad > 0).
\]

Let \(\{t_0, t_1, \ldots, t_l\}\) be the set of singular points of \(\gamma\) ordered by their appearance, and define \(\zeta_j = a\) (resp. \(b\)) if \(\gamma(t_j)\) is zig (resp. zag), and set \(\zeta_0 := \zeta_0 \zeta_1 \ldots \zeta_l\), which is a word consisting of the letters \(a\) and \(b\). The projection of \(\zeta_j\) to the free product \(Z_2 \ast Z_2\) (reduction with the relation \(a^2 = b^2 = 1\)) is of the form \((ab)^k\) or \((ba)^k\). The non-negative integer \(k\) := \(k\) is called the zigzag number of \(\gamma\). We shall give a geometric formula for the zigzag number via the curvature map defined by the second author:

**Definition 4.1 ([U]).** Let \(\gamma: S^1 \to R^2\) be a front with unit normal vector \(\nu\). The curvature map of \(\gamma\) is the map

\[
\kappa_\gamma: S^1 \setminus \Sigma_\gamma \ni t \mapsto [g_0(\gamma', \gamma') : g_0(\gamma', \nu')] \in P^1(R),
\]

where \(\gamma' = d/dt, \Sigma_\gamma \subset S^1\) is the set of singular points of \(\gamma\), and \([ : \) denotes the homogeneous coordinates of \(P^1(R)\).

**Proposition 4.2.** Let \(\gamma\) be a generic front with unit normal vector \(\nu\). Then the curvature map \(\kappa_\gamma\) can be extended to a smooth map on \(S^1\). Moreover, the rotation number of \(\kappa_\gamma\) is the zigzag number of \(\gamma\).
Proof. Let \( t_0 \) be a singular point of \( \gamma \). Since \( \gamma \) is a front, \( \nu'(t) \neq 0 \) holds on a neighborhood of \( t_0 \). As \( \nu' \) is perpendicular to \( \nu \), we have \( \det(\nu, \nu') \neq 0 \). Here, using \( \lambda = \det(\gamma', \nu) \), we have \( \gamma' = -\left(\lambda/\det(\nu, \nu')\right)\nu' \). Hence we have

\[
\kappa_\gamma = [g_0(\gamma', \gamma') : g_0(\gamma', \nu')] = \left[ \lambda^2 : -\frac{\lambda g_0(\nu', \nu')}{\det(\nu, \nu')} \right] = \left[ \lambda : -\frac{g_0(\nu', \nu')}{\det(\nu, \nu')} \right]
\]

well-defined on a neighborhood of \( t_0 \). Moreover, \( \kappa_\gamma(t) = [0 : 1] (= \infty) \) if and only if \( t \) is a singular point. Here, we choose an inhomogeneous coordinate of \([x : y]\) as \( y/x \).

Since \( g_0(\gamma', \nu') = g_0(\gamma'', \nu') \) holds at a singular point \( t_0 \), \( \kappa_\gamma \) passes through \([0 : 1]\) with counterclockwise (resp. clockwise) direction if \( g_0(\gamma'', \nu') > 0 \) (resp. \( < 0 \)), see Figure 7(c).

Let \( t_0 \) and \( t_1 \) be two adjacent zigs, and suppose \( \lambda'(t_0) > 0 \). Since \( \lambda \) changes sign on each cusp, we have \( \lambda'(t_1) < 0 \). Then by [4.1], \( g_0(\gamma'', \nu')(t_0) > 0 \) and \( g_0(\gamma'', \nu')(t_1) < 0 \). Hence \( \kappa_\gamma \) passes through \([0 : 1]\) in the counterclockwise direction at \( t_0 \), and the clockwise direction at \( t_1 \). Thus, this interval does not contribute to the rotation number of \( \kappa_\gamma \). On the other hand, if \( t_0 \) and \( t_1 \) are zig and zag respectively, \( \kappa_\gamma \) passes through \([0 : 1]\) counterclockwise at both \( t_0 \) and \( t_1 \). Then the rotation number of \( \kappa_\gamma \) is \( 1 \) on the interval \([t_0, t_1]\). Summing up, the proposition holds. \( \square \)

**Zigzag number for fronts in Riemannian 3-manifolds.** Let \( M^2 \) be a manifold and \( f: M^2 \to N^3 \) be a front with unit normal vector \( \nu \) into a Riemannian 3-manifold \((N^3, g)\). Let \( \Sigma_f \subset M^2 \) be the singular set, and \( \nu_0 \) be the unit normal vector field of \( f \) defined on \( M^2 \setminus \Sigma_f \) which is compatible with the orientations of \( M^2 \) and \( N^3 \), that is, \( \nu_0 = (f_u \times_g f_v)/|f_u \times_g f_v| \), where \((u, v)\) is a local coordinate system on \( M^2 \) compatible to the orientation. Then \( \nu_0(p) \) is \( \nu(p) \) if \( p \in M_+ \) and \(-\nu(p) \) if \( p \in M_- \).

We assume all singular points of \( f \) are non-degenerate. Then each connected component \( C \subset \Sigma_f \) must be a regular curve on \( M^2 \). Let \( p \in C \) be a cuspidal edge. Then \( p \) is called zig (resp. zag) if \( \nu_0 \) points towards the outward (resp. inward) side of the cuspidal edge (see Figure 4(d)). As this definition does not depend on \( p \in C \), we call \( C \) zig (resp. zag) if \( p \in C \) is zig (resp. zag).

Now, we define the zigzag number for loops on \( M^2 \). Take a null loop \( \sigma: S^1 \to M^2 \), that is, the intersection of \( \sigma(S^1) \) and \( \Sigma_f \) consists of cuspidal edges and \( \sigma' \)
points in the null direction at each singular point. We remark that there exists a null loop in each homotopy class. Let $Z_\sigma = \{t_0, \ldots, t_l\} \subset S^1$ be the set of singular points of $\sigma$ ordered by their appearance along the loop. Define $\zeta_j = a$ (resp. $b$) if $\sigma(t_j)$ is zig (resp. zag), and set $\zeta_\sigma := \zeta_0 \zeta_1 \cdots \zeta_l$, which is a word consisting of the letters $a$ and $b$. The projection of $\zeta_\sigma$ to the free product $\mathbb{Z}_2 \ast \mathbb{Z}_2$ (reduction with the relation $a^2 = b^2 = 1$) is of the form $(ab)^k$ or $(ba)^k$. The non-negative integer $k_\sigma := k$ is called the zigzag number of $\sigma$.

It is known that the zigzag number is a homotopy invariant, and the greatest common divisor $k_f$ of $\{k_\sigma | \sigma$ is a null loop on $M^3\}$ is the zigzag number of $f$ (see [LLR]).

**Remark 4.3** (Langefin-Levitt-Rosenberg’s inequality [LLR]). Let $M^2$ be a compact orientable 2-manifold of genus $g$ and $f : M^2 \to N^3$ a front. When $N^3 = \mathbb{R}^3$, [LLR] proved the following inequality

\begin{equation}
(4.2) \quad a_f + \frac{q_f}{2} \geq \frac{\chi_f}{2} + 1 - g + 2k_f,
\end{equation}

where $a_f$ is the number of the connected components of the singular set $\Sigma_f$, $q_f$ the number of the swallowtails, and half the Euler number of the limiting tangent bundle $\chi_f/2$ is equal to the degree of the Gauss map. Their proof is valid for the general case and (4.2) holds for any $N^3$.

In this section, we shall give a geometric formula for zigzag numbers of loops. First, we define the normal curvature map, similar to the curvature map for fronts in $\mathbb{R}^3$.

**Definition 4.4** (Normal curvature map). Let $f : M^2 \to (N^3, g)$ be a front with unit normal vector $\nu$ and $\sigma : S^1 \to M^2$ a null loop. The normal curvature map of $\sigma$ is the map

$\kappa_\sigma : S^1 \setminus Z_\sigma \ni t \mapsto [g(\sigma', \nu') : g(\sigma', \nu')] \in P^1(\mathbb{R})$,

where $\sigma = f \circ \sigma$, $\nu = \nu \circ \sigma$, $' = d/dt$, $Z_\sigma \subset S^1$ is the set of singular points of $\sigma$, and $[ : ]$ denotes the homogeneous coordinates of $P^1(\mathbb{R})$.

Then we have the following:

**Theorem 4.5** (Geometric formula for zigzag numbers). Let $f : M^2 \to (N^3, g)$ be a front with unit normal vector $\nu$, whose singular points are all non-degenerate, and $\sigma : S^1 \to M^2$ a null loop. Then the normal curvature map $\kappa_\sigma$ can be extended to $S^1$, and the rotation number of $\kappa_\sigma$ is equal to the zigzag number of $\sigma$.

**Proof.** Let $t_0$ be a singular point of $\sigma$, and take a normalized coordinate system $(u, v)$ of $M^2$ on a neighborhood $U$ of $\sigma(t_0)$. Then $f_u = 0$ and $f_v \neq 0$ holds on the $u$-axis, and by the Malgrange preparation theorem, there exists a smooth function $\alpha$ such that $g(f_v, f_v) = v^2 \alpha(u, v)$ and $\alpha(u, 0) \neq 0$. On the other hand, $g(f_v, \nu_v) = -N$ vanishes and $N_v \neq 0$ on the $u$-axis. Hence there exists a function $\beta$ such that $g(f_v, \nu_v) = v \beta(u, v)$ and $\mu(u, 0) \neq 0$. Thus

\begin{equation}
(4.3) \quad \kappa_\sigma = [g(f_v, f_v) : g(f_v, \nu_v)] = [v^2 \alpha(u, v) : v \beta(u, v)] = [\alpha(u, v) : \beta(u, v)]
\end{equation}

can be extended to the singular point $v = 0$. Namely, $\kappa_\sigma(t_0) = [0 : 1] (= \infty)$, where we choose an inhomogeneous coordinate $y/x$ for $[x : y]$. Moreover, $g(\sigma', \sigma') \neq 0$ on regular points, and $\kappa_\sigma(t) = [0 : 1]$ if and only if $t$ is a singular point.

Since $\nu = (\text{sgn} \lambda) \nu_0$, so a singular point $t_0$ is zig (resp. zag) if and only if

$$\text{sgn}(\lambda) \text{sgn}_\Delta(\nu) > 0 \quad (\text{resp.} \; \text{sgn}_\Delta(\nu) < 0),$$

where $\varepsilon$ is a sufficiently small number and $\Delta$ is a domain containing $\sigma(t_0 + \varepsilon)$ which lies only to one side of the cuspidal edge. By Theorem 3.7 $\text{sgn}_\Delta(\nu) = \text{sgn} g(\sigma', \nu')$, where $\sigma'$ is the function defined on $S^1$ by $\sigma'$ and $\nu'$ is the function defined by $\nu$ and $\nu = \nu_0$ at $t_0$. Thus

$$\text{sgn}_\Delta(\nu) = \text{sgn} g(\sigma', \nu'),$$

and the rotation number of $\kappa_\sigma$ is equal to the zigzag number of $\sigma$. Hence $\kappa_\sigma$ is a front with unit normal vector $\nu$ and $\sigma : S^1 \to M^2$ a null loop. Then the normal curvature map $\kappa_\sigma$ is equal to the zigzag number of $\sigma$.
Let \( Z \) (4.4) sgn \( t \) 26 KENTARO SAJI, MASAAKI UMEHARA, AND KOTARO YAMADA

\[
\text{sgn}(\lambda'(\sigma', \nu')) > 0 \quad \text{(resp. < 0)},
\]

where \( \lambda = \lambda \circ \sigma \). Since \( g(\sigma', \nu') = g(\sigma'', \nu') \) holds at singular points, we have

- if \( t_0 \) is zig and \( \lambda'(t_0) > 0 \) (resp. < 0), then \( \kappa_\sigma \) passes through \([0 : 1]\) counterclockwisely (resp. clockwisely).
- if \( t_0 \) is zag and \( \lambda'(t_0) > 0 \) (resp. < 0), then \( \kappa_\sigma \) passes through \([0 : 1]\)
clockwisely (resp. counterclockwisely).

Let \( Z_\sigma = \{t_0, \ldots, t_1\} \) be the set of singular points. Since the function \( \lambda \) has alternative sign on the adjacent domains, \( \lambda'(t_j) \) and \( \lambda'(t_{j+1}) \) have opposite sign. Thus, if both \( t_j \) and \( t_{j+1} \) are zigs and \( \lambda(t_j) > 0 \), \( \kappa_\sigma \) passes through \([0 : 1]\) counterclockwisely (resp. clockwisely) at \( t = t_j \) (resp. \( t_{j+1} \)). Hence the interval \([t_j, t_{j+1}]\) does not contribute to the rotation number of \( \kappa_\sigma \). Similarly, two consecutive zags do not affect the rotation number. On the other hand, if \( t_j \) is zig and \( t_{j+1} \) is zag and \( \lambda(t_j) > 0 \), \( \kappa_\sigma \) passes through \([0 : 1]\) counterclockwisely at both \( t_j \) and \( t_{j+1} \). Hence the rotation number of \( \kappa_\sigma \) on the interval \([t_j, t_{j+1}]\) is 1. Similarly, two consecutive zags increases the rotation number by 1. Hence we have the conclusion. \( \Box \)

5. Singularities of Hypersurfaces

In this section, we shall investigate the behavior of sectional curvature on fronts that are hypersurfaces. Let \( U^n \ (n \geq 3) \) be a domain in \((\mathbb{R}^n; u_1, u_2, \ldots, u_n)\) and \( f: U^n \to (\mathbb{R}^{n+1}, g_0) \)
a front, that is, there exists a unit vector field \( \nu \) (called the unit normal vector) such that \( g_0(f_*X, \nu) = 0 \) for all \( X \in TU^n \) and \( (f, \nu): U^n \to \mathbb{R}^{n+1} \times S^n \) is an immersion. We set

\[
\lambda := \det(f_{u_1}, \ldots, f_{u_n}, \nu),
\]

and call it the signed volume density function. A point \( p \in U^n \) is called a singular point if \( f \) is not an immersion at \( p \). Moreover, if \( d\lambda \neq 0 \) at \( p \), we call \( p \) a non-degenerate singular point. On a sufficiently small neighborhood of a non-degenerate singular point \( p \), the singular set is a \((n-1)\)-dimensional submanifold called the singular submanifold. The 1-dimensional vector space at the non-degenerate singular point \( p \) which is the kernel of the differential map \( (f_*)_p: T_pU^n \to \mathbb{R}^{n+1} \) is called the null direction. We call \( p \in U^n \) a cuspidal edge if the null direction is transversal to the singular submanifold. Then, by a similar argument to the proof of Fact 1.6 in [KRSUY], one can prove that a cuspidal edge is an \( A_2 \)-singularity, that is, locally diffeomorphic to the origin to the front \( f_C(u_1, \ldots, u_n) = (w_1^2, w_2^1, w_3, \ldots, w_n) \).

**Theorem 5.1.** Let \( f: U^n \to (\mathbb{R}^{n+1}, g_0) \ (n \geq 3) \) be a front whose singular points are all cuspidal edges. If the sectional curvature \( K \) at the regular points is bounded, then the second fundamental form on the singular submanifold vanishes. Moreover, if \( K \) is positive everywhere on the regular set, the sectional curvature of the singular submanifold is non-negative. Furthermore, if \( K \geq \delta(>0) \), then the sectional curvature of the singular submanifold is positive.

**Remark 5.2.** The previous Theorem [31] is deeper than this theorem. When \( n \geq 3 \) we can consider sectional curvature on the singular set, but when \( n = 2 \) the singular set is 1-dimensional and so we cannot define the sectional curvature. Rather, one defines the singular curvature instead. We do not define singular curvature for fronts when \( n \geq 3 \).
Proof of Theorem 5.1. Without loss of generality, we may assume that the singular submanifold of \( f \) is the \((u_1, \ldots, u_{n-1})\)-plane, and \( \partial_n := \partial/\partial u_n \) is the null direction. To prove the first assertion, it is sufficient to show that \( h(X, X) = 0 \) for an arbitrary fixed tangent vector of the singular submanifold. By changing coordinates if necessary, we may assume that \( X = \partial_1 = \partial/\partial u_1 \). The sectional curvature \( K(\partial_1 \wedge \partial_n) \) with respect to the 2-plane spanned by \( \{\partial_1, \partial_n\} \) is given by

\[
K(\partial_1 \wedge \partial_n) = \frac{h_{11} h_{nn} - (h_{1n})^2}{g_{11} g_{nn} - (g_{1n})^2} \quad (g_{ij} = g_0(\partial_i, \partial_j), \; h_{ij} = h(\partial_i, \partial_j)),
\]

where \( h \) is the second fundamental form. By the same reasoning as in the proof of Theorem 3.1, the boundedness of \( K(\partial_1 \wedge \partial_n) \) implies

\[
0 = (h_{11} h_{nn} - (h_{1n})^2)_{u_n=0} = h_{11} \left. \frac{\partial h_{nn}}{\partial u_n} \right|_{u_n=0} = h_{11} \left. g_0(D_{u_n} f_{u_n}, \nu_{u_n}) \right|_{u_n=0} = 0.
\]

To show \( h_{11} = h(X, X) = 0 \), it is sufficient to show \( g_0(D_{u_n} f_{u_n}, \nu_{u_n}) \) does not vanish when \( u_n = 0 \). Since \( f \) is a front with non-degenerate singularities, we have

\[
0 \neq \lambda_{u_n} = \det(f_{u_1}, \ldots, f_{u_{n-1}}, D_{u_n} f_{u_n}, \nu),
\]

which implies \( f_{u_1}, \ldots, f_{u_{n-1}}, D_{u_n} f_{u_n}, \) and \( \nu \) are linearly independent when \( u_n = 0 \), and then \( \nu_{u_n} \) can be written as a linear combination of them. Since \( f \) is a front, \( \nu_{u_n} \neq 0 \) holds when \( u_n = 0 \), and we have \( 2g_0(\nu_{u_n}, \nu) = g_0(\nu, \nu)_{u_n} = 0 \), and

\[
g_0(\nu_{u_n}, f_{u_j}) = g_0(\nu, D_{u_n} f_{u_j}) = 0 \quad (j = 1, \ldots, n-1).
\]

Thus we have that \( g_0(D_{u_n} f_{u_n}, \nu_{u_n}) \) never vanishes at \( u_n = 0 \).

Next we show the non-negativity of the sectional curvature \( K_S \) of the singular manifold. It is sufficient to show \( K_S(\partial_1 \wedge \partial_2) \geq 0 \) at \( u_n = 0 \). Since the sectional curvature \( K_{U_n} \) is non-negative, we have

\[
\left. \frac{\partial^2}{(\partial u_n)^2} (h_{11} h_{22} - (h_{12})^2) \right|_{u_n=0} \geq 0,
\]

by the same argument as in the proof of Theorem 5.1. Since the restriction of \( f \) to the singular manifold is an immersion, the Gauss equation yields that

\[
K_S(\partial_1 \wedge \partial_2) = \frac{g_0(\alpha_{11}, \alpha_{22}) - g_0(\alpha_{12}, \alpha_{12})}{g_{11} g_{22} - (g_{12})^2},
\]

where \( \alpha \) is the second fundamental form of the singular submanifold in \( \mathbb{R}^{n+1} \) and \( \alpha_{ij} = \alpha(f_{u_i}, f_{u_j}) \).

On the other hand, since the second fundamental form \( h \) of \( f \) vanishes, \( g_0(\nu_{u_n}, f_{u_j}) = 0 \) holds for \( j = 1, \ldots, n \), that is, \( \nu \) and \( \nu_{u_n} \) are linearly independent vectors. Moreover, we have

\[
\alpha_{ij} = g_0(\alpha_{ij}, \nu) + \frac{1}{|\nu_{u_n}|^2} g_0(\alpha_{ij}, \nu_{u_n}) \nu_{u_n} = h_{ij} \nu + \frac{1}{|\nu_{u_n}|^2} g_0(\alpha_{ij}, \nu_{u_n}) \nu_{u_n} = \frac{1}{|\nu_{u_n}|^2} (h_{ij})_{u_n} \nu_{u_n},
\]

since the second fundamental form \( h \) of \( f \) vanishes and

\[
g_0(\alpha_{ij}, \nu_{u_n}) = g_0(D_{u_j} f_{u_i}, \nu_{u_n}) = (h_{ij})_{u_n} - g_0(D_{u_j} D_{u_i} f_{u_n}, \nu) = (h_{ij})_{u_n}
\]

for \( i, j = 1, \ldots, n-1 \). Thus we have

\[
K_S(\partial_1 \wedge \partial_2) = \frac{1}{g_{11} g_{22} - (g_{12})^2} \left. \frac{\partial^2}{(\partial u_n)^2} (h_{11} h_{22} - (h_{12})^2) \right|_{u_n=0} \geq 0.
\]

□
Example 5.3. We set
\[ f(u, v, w) := (v, w, u^2 + av^2 + bw^2, u^3 + cu^2) : \mathbb{R}^3 \to \mathbb{R}^4, \]
which gives a front with the unit normal vector
\[ \nu = \frac{1}{\delta}(2av(2c + 3u), 2bw(2c + 3u), -2c - 3u, 2), \]
where \( \delta = \sqrt{4 + (3u + 2c)^2(1 + 4a^2v^2 + 4b^2w^2)} \).

The singular set is the \( vw \)-plane and the \( u \)-direction is the null direction. Then all singular points are cuspidal edges. The second fundamental form is given by
\[ H = \delta^{-1}\{6u dv^2 - 2(3u + 2c)(a dv^2 + b dw^2)\}, \]
which vanishes on the singular set if \( ac = bc = 0 \).

On the other hand, the sectional curvatures are computed as
\[
\begin{align*}
K(\partial_u \wedge \partial_v) &= \frac{12a(3u + 2c)}{u\delta^2(4 + (3u + 2c)^2(1 + 4a^2v^2))}, \\
K(\partial_u \wedge \partial_w) &= \frac{12b(3u + 2c)}{u\delta^2(4 + (3u + 2c)^2(1 + 4b^2w^2))},
\end{align*}
\]
which are bounded in a neighborhood of the singular set if and only if \( ac = bc = 0 \). If \( ac = bc = 0 \), \( K \geq 0 \) if and only if \( a \geq 0 \) and \( b \geq 0 \), which implies \( K_S = 4ab(3u + 2c)/\delta^2(\partial_u \wedge \partial_w)^2 > 0 \).

6. INTRINSIC FORMULATION

The Gauss-Bonnet theorem is intrinsic in nature, and it is quite natural to formulate the singularities of wave fronts intrinsically. We can characterize the limiting tangent bundles of the fronts and can give the following abstract definition:

Definition 6.1. Let \( M^2 \) be a 2-manifold. An orientable vector bundle \( E \) of rank 2 with a metric \( \langle \cdot, \cdot \rangle \) and a metric connection \( D \) is called an abstract limiting tangent bundle or a coherent tangent bundle if there is a bundle homomorphism
\[ \psi : TM^2 \to E \]
such that
\[ D_X \psi(Y) - D_Y \psi(X) = \psi([X, Y]) \quad (X, Y \in TM^2). \]

In this setting, the pull-back of the metric \( ds^2 := \psi^* \langle \cdot, \cdot \rangle \) is called the first fundamental form of \( E \). A point \( p \in M^2 \) is called a singular point if the first fundamental form is not positive definite. Since \( E \) is orientable, there exists a skew-symmetric bilinear form \( \mu_p : \mathcal{E}_p \times \mathcal{E}_p \to \mathbb{R} \) for each \( p \in M^2 \), where \( \mathcal{E}_p \) is the fiber of \( E \) at \( p \), such that \( \mu(e_1, e_2) = \pm 1 \) for any orthonormal frame \( \{e_1, e_2\} \) on \( E \).

A frame \( \{e_1, e_2\} \) is called positive if \( \mu(e_1, e_2) = 1 \). A singular point \( p \) is called non-degenerate if the derivative \( d\lambda \) of the function
\[ \lambda := \mu \left( \psi \left( \frac{\partial}{\partial u} \right), \psi \left( \frac{\partial}{\partial v} \right) \right) \]
does not vanish at \( p \), where \( (U; u, v) \) is a local coordinate system of \( M^2 \) at \( p \). On a neighborhood of a non-degenerate singular point, the singular set consists of a regular curve, called the singular curve. The tangential direction of the singular curve is called the singular direction, and the direction of the kernel of \( \psi \) is called the null direction. Then we can define intrinsic cuspidal edges and intrinsic swallowtails according to Fact 4.2. For a given singular curve \( \gamma(t) \) consisting of intrinsic cuspidal edge points, the singular curvature function is defined by
\[ \kappa_s(t) := \text{sgn}(\lambda(\eta)) \hat{k}_p(t), \]
where $\kappa(t) := (D_t \psi(\gamma'(t)), n(t))$ is the limiting geodesic curvature, $n(t) \in \mathcal{E}(\gamma(t))$ is a unit vector such that $\mu(\psi(\gamma'(t)), n(t)) = 1$, and $\eta(t)$ is the null direction such that $(\gamma(t), \eta(t))$ is a positive frame on $M^2$. Then Theorem 6.6 and Proposition 6.8 hold. Let $(U; e_1, e_2)$ be an orthonormal frame field of $\mathcal{E}$ such that $\mu(e_1, e_2) = 1$. Then there exists a unique 1-form $\alpha$ on $U$ such that

$$D_X e_1 = -\alpha(X) e_2, \quad D_X e_2 = \alpha(X) e_1 \quad (X \in TM^2),$$

which is called the connection form. Moreover, the exterior derivative $d\alpha$ does not depend on the choice of a positive frame $(U; e_1, e_2)$ and gives a (globally defined) 2-form on $M^2$. When $M^2$ is compact, the integration

$$(6.3) \quad \chi_\mathcal{E} := \frac{1}{2\pi} \int_{M^2} d\alpha$$

is an integer called the Euler number of $\mathcal{E}$. Let $(U; e_1, e_2)$ be a positive orthonormal frame field of $\mathcal{E}$ and $\gamma(s)$ a curve in $U(\subset M^2)$ such that $\langle \psi(\gamma'(s)), \psi(\gamma'(s)) \rangle = 1$. Let $\varphi(s)$ be the angle of $\psi(\gamma'(s))$ from $e_1(\gamma(s))$. Then we have

$$k_g ds = d\varphi - \alpha.$$

Let $\Delta$ be a triangle with interior angles $A, B, C$. In the interior of $\Delta$, we suppose that there are no singular points and that $\psi^* d\alpha$ is compatible with respect to the orientation of $M^2$. We give an orientation to $\partial \Delta$ such that conormal vectors points into the domain $\Delta$. By using the same argument as in the classical proof of the Gauss-Bonnet Theorem, we get the formulas (2) and (3) in the introduction intrinsically. This intrinsic formulation is meaningful if we consider the following examples:

**Example 6.2 (Cuspidal cross caps).** A map $f: M^2 \to \mathbb{R}^3$ is called a frontal if there exists a unit normal vector field $\nu$ such that $f_* X$ is perpendicular to $\nu$ for all $X \in TM^2$. A frontal is a front if $(f, \nu): M^2 \to \mathbb{R}^3 \times S^2$ is an immersion. A cuspidal cross cap is a singular point locally diffeomorphic to the map $(u, v) \mapsto (u, v^3, uv^3)$ and is a frontal but not a front. In [FSUY], a useful criterion for cuspidal cross caps are given. Though a cuspidal cross cap is not a cuspidal edge, the limiting tangent bundle is well defined and the singular point is an intrinsic cuspidal edge. In particular, our Gauss-Bonnet formulas hold for a frontal that admits only cuspidal edges, swallowtails and cuspidal cross caps, and degenerate peaks like as for a double swallowtail.

**Example 6.3 (Singularities with higher codimensions).** A smooth map $f: M^2 \to \mathbb{R}^n$ defined on a 2-manifold $M^2$ into $\mathbb{R}^n (n > 3)$ is called an admissible map if there exists a map $\nu: M^2 \to G_2(\mathbb{R}^n)$ into the oriented 2-plane Grassman manifold $G_2(\mathbb{R}^n)$, such that it coincides with the Gauss map of $f$ on regular points of $f$. For an admissible map, the limiting tangent bundle is canonically defined and we can apply our intrinsic formulation to it.

A realization problem for abstract limiting tangent bundles is investigated in [SUY]. The realization of first fundamental forms with singularities has been treated in [152].

References

[A] V. I. Arnol’d, Topological Invariants of Plane Curves and Caustics, University Lecture Series 5, Amer. Math. Soc. (1991).

[AGV] V. I. Arnol’d, S. M. Gusein-Zade and A. N. Varchenko, Singularities of Differentiable Maps, Vol. 1, Monographs in Math. 82, Birkhäuser (1985).

[BG] J. W. Bruce and P. J. Giblin, Curves and Singularities, Cambridge University Press (1984).
S. Fujimori, Spacelike CMC 1 surfaces with elliptic ends in de Sitter 3-space, Hokkaido Math. J. 35 (2006), 289–320.

S. Fujimori, K. Saji, M. Umehara and K. Yamada, Singularities of maximal surfaces, preprint, [math.DG/0510366].

M. Golubitsky and V. Guillemin, Stable Mappings and Their Singularities, Graduate Texts in Math. 14, Springer-Verlag (1973).

G. Ishikawa and Y. Machida, Singularities of improper affine spheres and pseudospherical surfaces, preprint, [math.DG/0502154], to appear in International Journ. of Math.

M. Kokubu, W. Rossman, K. Saji, M. Umehara and K. Yamada, Singularities of flat fronts in hyperbolic 3-space, Pacific J. of Math. 221 (2005), 303–351.

M. Kokubu, M. Umehara and K. Yamada, An elementary proof of Small’s formula for null curves in PSL(2, C) and an analogue for Legendrian curves in PSL(2, C), Osaka J. Math. 40 (2003), 697–715.

M. Kokubu, M. Umehara and K. Yamada, Flat fronts in hyperbolic 3-space, Pacific J. Math. 216 (2004), 149–175.

M. Kossowski, The Boy-Gauss-Bonnet theorems for $C^\infty$-singular surfaces with limiting tangent bundle, Annals of Global Analysis and Geometry 21 (2002), 19–29.

M. Kossowski, Realizing a singular first fundamental form as a nonimmersed surface in Euclidean 3-space, J. Geom. 81 (2004), 101–113.

R. Langevin, G. Levitt and H. Rosenberg, Classes d’homotopie de surfaces avec rebroussements et queues d’aronde dans $\mathbb{R}^3$, Canad. J. Math. 47 (1995), 544–572.

S. Lee and S.-D. Yang, A spinor representation for spacelike surfaces of constant mean curvature $-1$ in de Sitter three-space, Osaka J. Math., 43 (2006), 641–663.

Y. Machigashira, The Gaussian curvature of Alexandrov spaces, J. Math. Soc. Japan 50 (1998) 859–878.

A. Martínez, Improper Affine maps, Math. Z. 249 (2005), 755–766.

K. Shiohama, Total curvatures and minimal area of complete open surfaces, Proc. Amer. Math. Soc. 94 (1985), 310–316.

K. Saji, M. Umehara and K. Yamada, Behavior of cuspidal edges at corank one singular points and the realization of intrinsic wave fronts, preprint.

M. Umehara, Geometry of curves and surfaces with singularities, in Mathematics in the 21st Century—Unscaled Peaks of Geometry, edited by R. Miyaoka and M. Kotani, Nihon-Hyoronsha, 2004 (in Japanese).

M. Umehara and K. Yamada, Maximal surfaces with singularities in Minkowski space, Hokkaido Math. J., 35 (2006), 13–40.
tangential half-space
principal half space
m-1 times 2k cusps
$D_+$ \quad \hat{\sigma}$
$D_-$ \hat{\sigma}$
