1 Introduction

Let $k$ be a field and $\pi : X \to C$ be a semi-stable fibration of a smooth proper surface over $k$ to a proper smooth curve over $k$. If $k$ is a subfield of the field of complex numbers, $\mathbb{C}$, the following semi-positivity theorem holds.

**Theorem.** (Semi-Positivity Theorem, Xiao) If $\pi : X \to C$ is a fibration of a proper smooth surface to a proper smooth curve over $\mathbb{C}$, then all the quotient bundles of $\pi_*\omega_{X/C}$ are of non-negative degree. [14], p.1

In other words, all the Harder-Narasimhan slopes of $\pi_*\omega_{X/C}$ are non-negative, or equivalently all the Harder-Narasimhan slopes of $R^1\pi_*O_X$ are non-positive. But over a positive characteristic filed, the semi-positivity theorem does not hold in general. Moret-Bailly constructed a non-isotrivial semi-stable fibration of fiber genus 2, $\pi_M : X_M \to \mathbb{P}^1$ such that

$$R^1\pi_M^*O_{X_M} = O(-p) \oplus O(1).$$ [10], p.137

Here $p$ is the characteristic of the base field. $X_M$ is a theta divisor of a principal polarization of a non-isotrivial supersingular abelian surface over $\mathbb{P}^1$. Hence every special fiber of $\pi_M$ is either a supersingular smooth curve of genus 2 or a union of two supersingular elliptic curves which intersect at a point transversally. We can see the $p$-rank of the generic fiber of $\pi_M$ is 0. The main work of this paper is to prove the semi-positivity theorem for a semi-stable fibration over a field of positive characteristic provided with the $p$-rank of the generic fiber is maximal or equivalently the fibration is generically ordinary. Precisely,
Theorem 1. If $\pi : X \rightarrow C$ is generically ordinary semi-stable fibration, then

(a) $\dim H^0(R^1\pi_*O_X) = \dim H^0(R^1\pi_*\mathcal{O}_{X^p^n})$ and $\dim H^1(O_X) = \dim H^1(O_{X^p^n})$ for any $n$.

(b) All the Harder-Narasimhan slopes of $R^1\pi_*(O_X)$ are non-positive.

Here $\pi^p^n : X^p^n \rightarrow C$ is the base change of $\pi : X \rightarrow C$ by the $n$-iterative Frobenius morphism of $C$, $F^n_C : C \rightarrow C$. The proof of the theorem is given in section 2. Let me summarize the idea of the proof of the theorem. When $F_{X/C} : X \rightarrow X^p$ is the relative Frobenius morphism, $F^*_X/C : \mathcal{O}_{X^p} \rightarrow F^*_X/C \mathcal{O}_X$ is injective and the cokernel of this morphism is denoted by $B^1\omega_{X/C}$ or $B^1\omega$. $B^1\omega$ is flat over $O_C$. If $\pi$ is generically ordinary, $H^0(B^1\omega|X_s) = H^1(B^1\omega|X_s) = 0$ for almost all $s \in C$, hence $\pi_*^s B^1\omega = 0$ and $R^1\pi_*^p B^1\omega$ is torsion because $\pi$ is relatively 1-dimensional. Considering the exact sequence of coherent $O_C$-modules,

$$\pi_*^p B^1\omega \rightarrow R^1\pi_*\mathcal{O}_X \rightarrow F^*_C R^1\pi_* \mathcal{O}_X \rightarrow R^1\pi_*^p B^1\omega \rightarrow 0$$

$F^*_C R^1\pi_* O_X$ is a subbundle of $R^1\pi_* O_X$, so $\dim H^0(R^1\pi_* O_X) \geq \dim H^0(F^*_C R^1\pi_* O_X)$. But also we have $\dim H^0(R^1\pi_* O_X) \leq \dim H^0(F^*_C R^1\pi_* O_X)$. Hence $\dim H^0(R^1\pi_* O_X) = \dim H^0(F^*_C R^1\pi_* O_X)$. And by the Leray spectral sequence, $\dim H^1(O_X) = \dim H^1(O_{X^p^n})$.

Repeating this argument we can show

$$\dim H^0(R^1\pi_* O_X) = \dim H^0(R^1\pi_*^p O_{X^{pn}}) \text{ and } \dim H^1(O_X) = \dim H^1(O_{X^{pn}}).$$

The part (b) follows the part (a) and the Riemann-Roch theorem. This is a somewhat interesting phenomenon in a sense that it relates the ordinarity, a Galois theoretic condition, to a numerical property of slopes of vector bundles.

In section 3, as an application of of the main theorem, we will construct a counterexample of Parshin’s expectation on the Miyaoka-Yau inequality. Parshin thought that the failure of the Miyaoka-Yau inequality is related to the non-smoothness of the Picard scheme, so he conjectured that a version of the Miyaoka-Yau inequality holds if the Picard scheme of a given surface of general type is smooth. [11], p.288 We will construct a smooth proper surface of general type over a finite field whose Picard scheme is smooth and $c_1^2 > M c_2$ for any $M > 0$.

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2 Definitions and the Proof of the Main Theorem

Let \( k \) be an algebraically closed field and \( C \) be a projective curve over \( k \).

**Definition 2.1.** \( C \) is (semi-)stable if

1. It is connected and reduced.
2. All the singular points are normal crossing.
3. An irreducible component, which is isomorphic to \( \mathbb{P}^1 \), meets other components in at least 3 (resp. 2) points.

For an arbitrary base scheme, we define a (semi-)stable curve as follows.

**Definition 2.2.** A proper flat morphism of relative dimension 1 of schemes \( \pi : X \rightarrow S \) is a (semi-)stable curve if every geometric fiber of \( \pi \) is a (semi-)stable curve in the sense of definition 2.7.

In this paper, we are mainly concerned with generically smooth semi-stable curves over a proper smooth curve over a field. If \( \pi : X \rightarrow C \) is such a semi-stable fibration over an algebraically closed field \( k \), \( X \) is a proper surface over \( k \) and the singular points of \( X \) are isolated. A singularity of \( X \) is étale locally isomorphic to \( k[x, y, t]/(t^n - xy) \).

If \( \tilde{X} \rightarrow X \) is the minimal blow up of these singularities, the composition \( \tilde{\pi} : \tilde{X} \rightarrow C \) is also a semi-stable fibration.\[1\],p.4 Moreover \( \omega^1_{\tilde{X}/C} \) is isomorphic to the pull back of \( \omega^1_X/C \) by the blow up and \( \tilde{\pi}_*\omega^1_{X/C} = \pi_*\omega^1_{X/C} \).\[13\],p.171 Hence for many purpose, we may assume that \( X \) is a smooth surface over \( k \). From now on, we assume \( X \) is a smooth surface over \( k \) and \( \pi : X \rightarrow C \) is a generically smooth semi-stable fibration unless it is stated otherwise.

**Definition 2.3.** A semi-stable fibration \( \pi : X \rightarrow C \) is isotrivial, if all the special fibers of \( \pi \) are isomorphic.

In particular, an isotrivial fibration \( \pi : X \rightarrow C \) is a smooth fibration. If \( \pi \) is isotrivial, there exists a finite étale cover \( C' \rightarrow C \) such that the base change \( \pi_{C'} : X \times_C C' \rightarrow C' \) is trivial. In particular, if \( \pi \) is isotrivial, \( \deg \pi_*\omega_{X/C} = 0 \).

**Proposition 2.4.** (Szpiro) If \( \pi \) is a non-isotrivial semi-stable fibration, \( \deg \pi_*\omega_{X/C} > 0 \). Equivalently, \( \deg R^1\pi_*\mathcal{O}_X < 0 \).\[13\],p.173

Now assume that \( k \) is a perfect field of positive characteristic \( p \) and that \( X \) is a smooth proper variety defined over \( k \). We have the Frobenius diagram for \( X/k \),

\[
\begin{array}{ccc}
X & \xrightarrow{F_{X/k}} & X^p & \xrightarrow{X} & X \\
\downarrow & & \downarrow & & \downarrow \\
k & & k & & k \\
\end{array}
\]
Here $F_{X/k}$ is the relative Frobenius morphism of $X/k$. When $\Omega_{X/k}$ is the DeRham complex of $X/k$, $F_{X/k*}(\Omega_{X/k})$ is an $O_{X^p}$-linear complex of coherent $O_{X^p}$-modules. The image of $F_{X/k*}\Omega_{X/k}^{i-1} \to F_{X/k*}\Omega_{X/k}^{i}$ is denoted by $B^i\Omega_{X/k}$ or $B^i\Omega$. Each $B^i\Omega$ is a vector bundle on $X^p$.

**Definition 2.5.** $X$ is ordinary (Bloch-Kato ordinary) if $H^i(B^j\Omega_{X/k}) = 0$ for all $i$ and $j$.

There are many equivalent conditions to Bloch-Kato ordinarity [9], p.209. If $X$ is a curve or an abelian variety, $X$ is ordinary if and only if it satisfies the classical definition, that the order of $p$-torsion of the $\text{Pic}_0^0 X/k$ is maximal or that the Frobenius morphism on $H^1(O_X)$ is bijective. If all the integral crystalline cohomologies of $X$, $H^i_{\text{cris}}(X/W)$ are torsion free, $X$ is ordinary if and only if the Newton polygons of $X$ are equal to the Hodge polygons of $X$ for all degrees. Here $W$ is the ring of Witt vectors of $k$.

We can extend the definition of ordinarity to any proper smooth morphism of schemes of characteristic $p$. Assume $f : X \to S$ be a proper and smooth morphism. And let $X^p = X \times_S (S, F_S)$ and $F_{X/S} : X \to X^p$ be the relative Frobenius morphism. The image $B^i_{X/S}$ of $F_{X/S*}\Omega_{X/S}^{i-1} \to F_{X/S*}\Omega_{X/S}^{i}$ is a vector bundle on $X^p$. We define $X/S$ to be ordinary if $R^if_* (B^j_{X/S}) = 0$ for all $i$ and $j$.

Moreover, the notion of ordinarity can be extended to a proper generically smooth morphism to a Spec of a discrete valuation ring with normal crossing on the special fiber. We recall the definition of ordinarity for such a morphism from [7] and [8]. Let $A$ be a discrete valuation ring of positive characteristic and $S = \text{Spec } A$. $s \in S$ is the closed point.

**Definition 2.6.** $f : X \to S$ is locally semi-stable if it is isomorphic to

$$\text{Spec } A[x_1, \cdots, x_n]/(x_1 \cdots x_r - t) \to \text{Spec } A$$

étale locally at a relative singular point, where $t$ is a uniformizer of $A$.

The term “locally semi-stable morphism” is not conventional. Usually such a morphism is just called semi-stable. Here we introduce this definition to avoid a conflict with the former definition of semi-stable curve. Note that the definition of semi-stable curve is little different from that of a locally semi-stable morphism. In the definition of semi-stable curve, it is required that the semi-stable fibration is proper and relatively minimal while in definition 2.6, the morphism should be generically smooth. However if $X \to C$ is a generically smooth relative semi-stable curve over a smooth curve, $X \otimes O_{C,s} \to \text{Spec } O_{C,s}$ is locally semi-stable as per definition 2.6 for each $s \in C$.

Let $X \to S$ be a locally semi-stable morphism and $U \subset X$ be the relative smooth locus and $u : U \to X$ be the inclusion. Then $X \setminus U$ is of codimension at least 2, hence $\omega_{X/S} = u_*\Omega_{U/S}$ is a complex of locally free sheaves on $X$ and $\omega_{X/S}^{1} = \Lambda^1\omega_{X/S}^{1}$. When $X$ is given as $A[x_1, \cdots, x_n]/(x_1 \cdots x_r - t)$ étale locally, $\omega_{X/S}^{1}$ is the free module of rank $n-1$, generated by

$$dx_1/x_1, \cdots, dx_r/x_r, dx_{r+1}, \cdots, dx_n$$
with the relation
\[ \sum_{i=1}^{r} \frac{dx_i}{x_i} = 0. \]

Note that if the relative dimension of \( f \) is \( d \), the highest wedge product \( \omega_X^d \) is the relative dualizing sheaf of \( f : X \to S \).

Let \( X^p \) be the base change of \( X \) by the Frobenius morphism of \( S \) and \( F_{X/S} : X \to X^p \) be the relative Frobenius morphism. Then \( F_{X/S} \cdot \omega_X^\bullet \) is an \( \mathcal{O}_{X^p} \)-linear complex. The image and the kernel of the differentials of the complex \( F_{X/S} \cdot \omega_X^\bullet \) are denoted by \( B_i \omega_X^\bullet \), \( Z_i \omega_X^\bullet \), and \( H^i \omega_X^\bullet \) respectively, and \( H^i \omega_X^\bullet = Z_i \omega_X^\bullet / B_i \omega_X^\bullet \).

**Definition 2.7.** A proper locally semi-stable morphism \( f : X \to S \) is ordinary if
\[ H^j(B_i \omega_X^\bullet|_{X_s}) = 0 \]
for all \( i, j \). Since \( B_i \omega_X^\bullet \) are flat over \( S \), \( f \) is ordinary if and only if \( H^j(X_s, B_i \omega_X^\bullet|_{X_s}) = 0 \) for all \( i, j \) when \( X_s \) is the special fiber. This definition depends on the entire \( X \to S \), and not only on the special fiber. But if \( f \) is smooth, \( f \) is ordinary if and only if the special fiber is ordinary. Moreover if the relative dimension of \( f \) is 1 and the residue field is perfect, \( f \) is ordinary if and only if the Frobenius morphism on \( H^1(\mathcal{O}_{X_s}) \) is bijective, hence the ordinarity depends only on the special fiber.

Since all the above arguments are local on the base \( S \), they are still valid if we replace the base \( S \) by a smooth curve over a perfect field of positive characteristic. Let \( C \) be a smooth curve over a perfect field \( k \). For a vector bundle \( V \) on \( C \), the slope of \( V \) is defined as \( s(V) = \text{deg } V / \text{rank } V \). A vector bundle \( V \) is called semi-stable (resp. stable) if for any proper subbundle \( W \) of \( V \), it is satisfied that \( s(W) \leq s(V) \) (resp. \( s(W) < s(V) \)).
Proposition 2.9. (Harder-Narasimhan) For any vector bundle $V$ on $C$, there exists a unique filtration of $V$ consisting of subbundles of $V$,

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

such that $V_i/V_{i-1}$ is a semi-stable vector bundle of slope $\lambda_i$ and $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. [citeHN,p.220]

This filtration is called the Harder-Narasimhan filtration of $V$ and $\lambda_1, \cdots, \lambda_n$ are called the Harder-Narasimhan slopes of $V$. If the base field is not algebraically closed, the Harder-Narasimhan slopes of $V$ are defined as the Harder-Narasimhan slopes of pullback of the bundle along the base change to an algebraically closed field. When $\pi : X \to C$ is a semi-stable fibration of a proper smooth surface to a proper smooth curve over a subfield of $C$, the semi-positivity theorem states that all the Harder-Narasimhan slope of $R^1\pi_*\mathcal{O}_X$ is non-positive. [14],p.1

Theorem 1. If $\pi : X \to C$ is generically ordinary semi-stable fibration, then

(a) $\dim H^0(R^1\pi_*\mathcal{O}_X) = \dim H^0(R^1\pi_*^{p^n}\mathcal{O}_{X^{p^n}})$ and $\dim H^1(\mathcal{O}_X) = \dim H^1(\mathcal{O}_{X^{p^n}})$ for any $n$.

(b) All the Harder-Narasimhan slopes of $R^1\pi_*(\mathcal{O}_X)$ are non-positive.

Proof. Let $X^p$ be the base change of $X$ by the absolute Frobenius morphism of $C$. There is the Frobenius diagram of $X/C$,

\begin{equation}
\begin{array}{ccc}
X & \xrightarrow{F_{X/C}} & X^p \\
\downarrow \pi^p & & \downarrow \pi \\
C & \xrightarrow{F_C} & C
\end{array}
\end{equation}

(2.1)

We have an exact sequence of coherent $\mathcal{O}_{X^p}$-modules

\begin{equation}
0 \to \mathcal{O}_{X^p} \to F_{X/C}\mathcal{O}_X \to B^1\omega \to 0.
\end{equation}

(2.2)

The long exact sequence via $\pi^p$ for (2.2) is

\begin{equation}
0 \to \mathcal{O}_C \xrightarrow{=} \mathcal{O}_C \to \pi^p(B^1\omega) \to R^1\pi^p(\mathcal{O}_{X^p}) \xrightarrow{F_{X/C}} R^1\pi_*(\mathcal{O}_X) \to R^1\pi^p(B^1\omega) \to 0.
\end{equation}

(2.3)

Since $\pi$ is generically ordinary, the restriction of $\pi^p B^1\omega$ to the ordinary locus in $C$ is 0. But $B^1\omega$ is flat over $\mathcal{O}_C$, so $\pi^p B^1\omega = 0$. Hence in (2.3), $F^*$ is injective and

$$\dim H^0(R^1\pi^p(\mathcal{O}_{X^p})) \leq \dim H^0(R^1\pi_*(\mathcal{O}_X)).$$

On the other hand, because

$$H^0(R^1\pi^p(\mathcal{O}_{X^p})) = H^0(F_C R^1\pi_*\mathcal{O}_X) = H^0(R^1\pi_*(\mathcal{O}_X) \otimes_{\mathcal{O}_C} F_C(\mathcal{O}_C))$$

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and there is an injection $R^1\pi_*\mathcal{O}_X \hookrightarrow R^1\pi_*\mathcal{O}_X \otimes F_{C*}(\mathcal{O}_C)$,

$$\dim H^0(R^1\pi_*\mathcal{O}_X) \leq \dim H^0(F_{C*}^*R^1\pi_*\mathcal{O}_X).$$

Therefore

$$\dim H^0(R^1\pi_*\mathcal{O}_X) = \dim H^0(F_{C*}^*R^1\pi_*\mathcal{O}_X).$$

Since

$$\dim H^1(\mathcal{O}_X) = \dim H^1(\mathcal{O}_C) + \dim H^0(R^1\pi_*\mathcal{O}_X)$$

and

$$\dim H^1(\mathcal{O}_{X^p}) = \dim H^1(\mathcal{O}_C) + \dim H^0(R^1\pi_*^p\mathcal{O}_{X^p}),$$

we have

$$\dim H^1(\mathcal{O}_X) = \dim H^1(\mathcal{O}_{X^p}).$$

We can apply this argument to the relative Frobenius morphism $F_{X^p/C} : X^p \to X^{p+1}$ for any $i$, since $F_{X^p/C} : X^p \to X^{p+1}$ is the base change of the relative Frobenius morphism $F_{X/C} : X \to X^p$ by $F_C^i : C \to C$. Then by the induction, we have

$$\dim H^0(R^1\pi_*\mathcal{O}_X) = \dim H^0(F_{C*}^n R^1\pi_*\mathcal{O}_X)$$

and

$$\dim H^1(\mathcal{O}_X) = \dim H^1(\mathcal{O}_{X^p^n})$$

for any $n$. This proves (a). Now assume $V$ is a subbundle of $R^1\pi_*\mathcal{O}_X$ with a positive degree $d > 0$ and rank $r$. Then $F_{C*}^n V$ is a subbundle of $F_{C*}^n R^1\pi_*\mathcal{O}_X$ with a positive degree $p^n d$ and rank $r$. By the Riemann-Roch theorem on $C$, $\dim H^0(F_{C*}^n V)$ diverges as $n$ goes to infinity. But this is a contradiction to the fact that $\dim H^0(F_{C*}^n R^1\pi_*\mathcal{O}_X)$ is stable. Hence $R^1\pi_*\mathcal{O}_X$ does not have a subbundle of a positive degree, and all the Harder-Narasimhan slopes of $R^1\pi_*\mathcal{O}_X$ are non-positive. \qed

## 2.1 Triviality of slope 0 part of $R^1\pi_*\mathcal{O}_X$

In the proof of Theorem 1, we actually proved that if $\pi$ is generically ordinary, $F_{C*}^n R^1\pi_*\mathcal{O}_X$ does not have a positive Harder-Narasimhan slope for any $n \in \mathbb{N}$. Hence the slope 0 part of $R^1\pi_*\mathcal{O}_X$ is strongly semi-stable. In fact, we can say more about the slope 0 part of $R^1\pi_*\mathcal{O}_X$. For convenience, we will use the following notations.

**Definition 2.10.** For a vector bundle $V$ on $C$, $V_0$ is the slope 0 part of $V$ and $V_-$ is the negative slope part of $V$.

If $\pi$ is generically ordinary, there is the canonical filtration of $R^1\pi_*\mathcal{O}_X$,

$$0 \to (R^1\pi_*\mathcal{O}_X)_0 \to R^1\pi_*\mathcal{O}_X \to (R^1\pi_*\mathcal{O}_X)_- \to 0.$$

**Definition 2.11.** A vector bundle $V$ on $C$ is potentially trivial if there exists a finite étale cover $f : D \to C$ such that $f^*V$ is trivial.
Proposition 2.12. If $\pi : X \to C$ is generically ordinary, $(R^1\pi_*\mathcal{O}_X)_0$ is potentially trivial.

Proof. In the proof of Theorem 1, we saw that

$$F^*_C R^1\pi_*\mathcal{O}_X \hookrightarrow R^1\pi_*\mathcal{O}_X.$$ 

On the other hand, in the canonical exact sequence,

$$0 \to (R^1\pi_*\mathcal{O}_X)_0 \to R^1\pi_*\mathcal{O}_X \to (R^1\pi_*\mathcal{O}_X)_- \to 0,$$

$(R^1\pi_*\mathcal{O}_X)_-$ is an iterative extension of semi-stable vector bundles of negative slopes. Therefore the image of the composition

$$F^*(R^1\pi_*\mathcal{O}_X)_0 \hookrightarrow F^* R^1\pi_*\mathcal{O}_X \hookrightarrow R^1\pi_*\mathcal{O}_X$$

is contained in $(R^1\pi_*\mathcal{O}_X)_0$. Since $(R^1\pi_*\mathcal{O}_X)_0$ and $F^*(R^1\pi_*\mathcal{O}_X)_0$ are of the same rank and the same degree, they should be isomorphic. Now it’s enough to show the following lemma. The author learned this lemma from K.Joshi. The original source of this fact is apparently a letter from Mumford to Seshadri.

Lemma 2.13. If $M$ is a vector bundle on $C$ such that $F^*M \simeq M$ then $M$ is potentially trivial.

Proof. Let $r$ be the rank of $M$. $M$ is a class in $H^1_\text{ét}(GL_r(\mathcal{O}_C))$. If we can show that $M$ is actually in $H^1_\text{ét}(GL_r(\mathbb{F}_p))$, it corresponds to a group homomorphism $\pi_1(C) \to GL_r(\mathbb{F}_p)$. Since $GL_r(\mathbb{F}_p)$ is a finite group, we have a finite étale Galois cover $f : D \to C$, corresponding to the kernel of this group homomorphism. Then $f^*M$ is the trivial vector bundle of rank $r$ on $D$.

We proceed to prove $M$ is a class in $H^1(GL_r(\mathbb{F}_p))$. Let $\{(U_i, g_{ij})\}$ be a trivialization of $M$ and the transition functions. Then $\{(U_i, h_{ij}^{(p)})\}$ is a trivialization of $F^*M$ and transition functions. Here $h_{ij}^{(p)}$ is the matrix obtained by taking the $p$-power of each entry of $g_{ij}$. From the assumption, after replacing $U_i$ by a refinement if necessary, there are $h_i \in GL_r(\mathcal{O}_C)(U_i)$, such that

$$h_i h_{ij}^{(p)} h_j^{-1} = h_{ij}.$$ 

To show $M \in H^1(GL_r(\mathbb{F}_p))$, we should find $k_i \in GL_r(\mathcal{O}_C)(U_i)$ such that

$$(k_i g_{ij} k_j^{-1})^{(p)} = k_i^{(p)} g_{ij}^{(p)} (k_j^{-1})^{(p)} = k_i^{(p)} g_{ij}^{(p)} k_j^{(p)}.$$ 

This is equivalent to

$$k_i^{-1} k_i^{(p)} g_{ij}^{(p)} (k_j^{-1})^{(p)} k_j = g_{ij}.$$ 

Hence the problem is reduced to find $k_i$ satisfying $k_i^{-1} k_i^{(p)} = h_i$. Since it is local in the étale topology, it is enough to find $k_i$ in the strict henselization for a geometric point of
C. Using Hensel’s lemma, it is again reduced to the same problem over an algebraically closed field of characteristic $p$. But by Lang’s Theorem[12], p.76,

$$A \mapsto A^{-1} A^{(p)}$$

is surjective on $GL_r(\bar{k})$. Hence there exists $k_i$ satisfying $k_i^{-1}k_i^{(p)} = h_i$. This proves the claim.

\[\Box\]

**Remark 2.14.** It is natural to expect that if $\pi : X \rightarrow C$ is defined over a field of characteristic 0, $(R^1\pi_*\mathcal{O}_X)_0$ is potentially trivial. But for this problem, we can’t apply the standard reduction argument directly. One reason is that in the reduction situation, we don’t know whether there are infinitely many places at which the reduction is generically ordinary. This obstruction is related to Serre’s ordinary reduction conjecture.

### 3 Counterexample to Parshin’s conjecture

In this section, we will construct a counterexample to Parshin’s conjecture. Let us recall the construction of a counterexample to the Miyaoka-Yau inequality over a field of positive characteristic from [13]. Let $k$ be a perfect field of positive characteristic. $\pi : X \rightarrow C$ is a smooth non-isotrivial fibration of a proper smooth surface to a proper smooth curve over $k$ of fiber genus $g \geq 2$ and of base genus $q \geq 2$. Also set $d = -\deg R^1\pi_*\mathcal{O}_X > 0$. Then

$$c_1^2(X) = 12d + 8(q - 1)(g - 1) \text{ and } c_2(X) = 4(q - 1)(g - 1).$$

When $\pi^{p^n} : X^{p^n} \rightarrow C$ is the base change of $\pi$ by the $n$-iterative Frobenius morphism of $C$, $F^n_C : C \rightarrow C$, $\deg R^1\pi^{p^n}_*\mathcal{O}_{X^{p^n}} = -p^n d$ and

$$c_1^2(X^{p^n}) = 12dp^n + 8(q - 1)(g - 1) \text{ and } c_2(X) = 4(q - 1)(g - 1).$$

For any $M > 0$, if $n$ is sufficiently large,

$$c_1^2(X^{p^n}) > Mc_2(X^{p^n}).$$

**Lemma 3.1.** Suppose that $X$ is a smooth proper surface over $k$ which admits a smooth fibration, $\pi : X \rightarrow C$, to a smooth proper curve $C$ over $k$. If $\pi$ is generically ordinary and Pic $X$ is smooth, then Pic $X^{p^n}$ is smooth for any $n \in \mathbb{N}$ when $X^{p^n} \rightarrow C$ is the base change of $X \rightarrow C$ by $n$-iterative Frobenius morphism $F^n_C : C \rightarrow C$.

**Proof.** Recall the Frobenius diagram

$$\begin{array}{cccccc}
X & \xrightarrow{F_{X/C}} & X^p & \xrightarrow{\alpha} & X \\
\downarrow & & \downarrow & & \downarrow \\
C & \xrightarrow{F_C} & C.
\end{array}$$
Here $\alpha \circ F_{X/C}$ is the absolute Frobenius morphism of $X$ and $F_{X/C} \circ \alpha$ is the absolute Frobenius morphism of $X^p$. Since $\pi$ is smooth, $X^p$ is smooth over $k$. For a smooth proper variety, the Frobenius morphism induces a bijective semi-linear morphism on the rational crystalline cohomologies. Therefore

$$\dim H^i_{\text{cris}}(X/K) = \dim H^i_{\text{cris}}(X^p/K).$$

Here $K$ is the fraction field of the ring of Witt vectors $W = W(k)$ and $H^i_{\text{cris}}(X/K) = H^i_{\text{cris}}(X/W) \otimes K$. In particular,

$$\dim H^1_{\text{cris}}(X/K) = \dim H^1_{\text{cris}}(X^p/K).$$

On the other hand, the $K$-dimension of the crystalline cohomology $H^i_{\text{cris}}(X/K)$ is equal to the $\mathbb{Q}_l$-dimension of the $l$-adic étale cohomology $H^i_{\text{ét}}(X, \mathbb{Q}_l)$, where $l$ is a prime number different from the characteristic of $k$ and $X = X \times_k \bar{k}$. The dimension of $H^i_{\text{ét}}(X, \mathbb{Q}_l)$ is the twice of the dimension of Pic$_X$, so

$$\dim \text{Pic}_X = \dim \text{Pic}_{X^p}.$$

Pic$_X$ is smooth if and only if the dimension of Pic$_X$ is equal to the $k$-dimension of $H^1(\mathcal{O}_X)$. Since $\pi$ is generically ordinary, by Thm 1.(a),

$$\dim H^1(\mathcal{O}_X) = \dim H^1(\mathcal{O}_{X^p}).$$

Hence if Pic$_X$ is smooth, Pic$_{X^p}$ is smooth. By the same argument, Pic$_{X^{p^n}}$ is smooth for any $n$.

**Corollary 3.2.** For any $M > 0$, there is a smooth proper surface of general type $X$ over a finite field whose Picard scheme is smooth and $c_1^2(X) > M c_2(X)$.

**Proof.** By the above lemma, it is enough to give a non-isotrivial generically ordinary smooth fibration $X \to C$ such that Pic$(X)$ is smooth. We will construct such an example by a reduction argument.

Let $F_m$ be the Fermat curve $x^m + y^m + z^m = 0$ over $\mathbb{C}$, with $m > 3$. Denote the genus of $F_m$ by $g$ and notice $g \geq 3$. Let $\mathcal{M}_g$ be the moduli space of smooth proper curves of genus $g$ over $\mathbb{C}$. By [1], p.105, there exists a smooth proper curve $C_0$ in $\mathcal{M}_g$ passing through the point representing $F_m$. Then there is a finite cover $C \to C_0$ and a non-isotrivial smooth fibration $\pi : X \to C$ which induces the composition $C \to C_0 \to \mathcal{M}_g$. Let us choose $s \in C$ such that $X_s = X \times_C k(s) = F_m$. We can take an integral model of $\pi$ with the section $s$ over a noetherian domain of finite type over $\mathbb{Z}$. Explicitly, we can take $A$, an integral domain of finite type over $\mathbb{Z}$, and a smooth fibration $\pi_A : X_A \to C_A$ over $\text{Spec } A$ satisfying

1. $X_A$ and $C_A$ are smooth and proper over $\text{Spec } A$.
2. There is a geometric generic point of $\eta : C \to \text{Spec } A$ such that $\pi_A \times_A \eta$ is isomorphic to $\pi : X \to C$. 


3. There exists a section $S : \text{Spec } A \to C_A$ such that $S \times_A \eta$ corresponds to $s$ with respect to the isomorphism in 2.

4. $S \times_{C_A} X_A$ is isomorphic to the Fermat curve over $\text{Spec } A$.

Since $\text{Spec } A$ is a scheme of finite type over $\mathbb{Z}$, there is a rational point of $\text{Spec } A$ over a number field $F$. Considering the coordinates of this rational point, there is a morphism $\text{Spec } B \to \text{Spec } A$, where $B$ is a localization of $\mathcal{O}_F$, the ring of integers of $F$. By the bases change, we obtain a smooth fibration $\pi_B : X_B \to C_B$ over $\text{Spec } B$. Then for a place $v \in \text{Spec } B$, the fiber of $\pi_v = \pi_B \times k_v$ over $S_v$ is the Fermat curve over the residue field of $v$. The ordinarity of the Fermat curve over a finite field depends only on the characteristic of the field. To be precise, it is ordinary if and only if $p \equiv 1 \mod m$ where $p$ is the characteristic of the finite field.\[\text{Hence at infinitely many places of } \text{Spec } B, \text{ the reduction of } \pi \text{ is generically ordinary. Because } \text{Pic } X_v \text{ is smooth for almost all } v \in \text{Spec } B, \text{ there is a place } v \in \text{Spec } B \text{ such that } \pi_v \text{ is generically ordinary and the Picard scheme of } X_v \text{ is smooth.}\]

**Remark 3.3.** In the above example, while $\dim H^1(\mathcal{O}_{X_{p^n}})$ is stable by the Frobenius base change, $\dim H^0(\Omega^1_{X_{p^n}})$ is strictly increasing\[3, p.94.\] Considering the inequality 

$$c_1^2 \leq 5c_2 + 6\beta_1 + 6(2h^{1,0} - \beta_1),$$

it seems that the Miyaoka-Yau inequality is related to the “correctness” of $h^{1,0}$ rather than that of $h^{0,1}$. The example we have constructed shows that the “correct value” of $h^{0,1}$ does not guarantee the Miyaoka-Yau inequality.

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