Scaling limits for the Lego discrepancy

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Abstract

For the Lego discrepancy with \( M \) bins, which is equivalent with a \( \chi^2 \)-statistic with \( M \) bins, we present a procedure to calculate the moment generating function of the probability distribution perturbatively if \( M \) and \( N \), the number of uniformly and randomly distributed data points, become large. Furthermore, we present a phase diagram for various limits of the probability distribution in terms of the standardized variable if \( M \) and \( N \) become infinite.

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1 Introduction

The importance of the notion of uniformity of point sets in the problem of numerical integration has been stressed in many publications (see e.g. [1, 2, 3]). In general, a discrepancy is a measure of non-uniformity of point sets in an integration region. For a set of \(N\) random points it is a function of \(N\) random variables. For the Lego discrepancy, the integration region is considered to be divided into a number of, say \(M\) bins, so that it becomes a function of \(M\) random variables, namely the number of points in each of the bins. Furthermore, the Lego discrepancy is defined such that it is exactly the \(\chi^2\)-statistic of the binned data points.

In [4], we have given criteria for the asymptotic probability distribution of various quadratic discrepancies to become Gaussian when a certain free parameter becomes infinitely large. This parameter often is the dimension \(s\) of the integration region. In the case of the Lego discrepancy, it is the number of bins \(M\). In [5], it is shown that for the Fourier diaphony a Gaussian limit is obtained when both \(N\) and \(s\) go to infinity such that \(cs/N \to 0\), where \(c\) is some constant larger than 1. This theorem clearly gives more information about the behavior of the probability distribution than the statements of [4], for it relates \(s\) and \(N\), whereas in [4] the limit of \(N \to \infty\) is assumed before considering the behavior with respect to \(s\) or \(M\).

In [4] and [6], we introduced techniques to calculate the \(1/N\) expansion of the moment generating function of the probability distribution of quadratic discrepancies, and applied it to the \(L_2^*\)-discrepancy, the Fourier diaphony (both for \(s = 1\)), and the Lego discrepancy. Such an expansion can be used to calculate corrections to the asymptotic distribution for \(N \to \infty\), but, presented as in [7], it cannot give information about limits if \(M\) or \(s\) become infinite also. In this paper we use the mentioned technique to calculate limits for the Lego discrepancy if \(M\) as well as \(N\) become infinite.

First, we will show that the natural expansion parameter in the calculation of the moment generating function is \(M/N\), and calculate a few terms. We will see, however, that a strict limit of \(M \to \infty\) does not exist, and, in fact, this is well known because the \(\chi^2\)-distribution, which gives the lowest order term in this expansion, does not exist if the number of degrees of freedom becomes infinite. We overcome this problem by going over to the standardized variable, which is obtained from the discrepancy by shifting and rescaling it such that it has zero expectation and unit variance. In fact, it is this variable for which the results in [4] and [5] were obtained. In this paper, we derive similar results for the Lego discrepancy, depending on the behavior of the sizes of the bins if \(M\) goes to infinity. We will see that various asymptotic probability distributions occur if \(M, N \to \infty\) such that \(M^\alpha/N \to \text{constant}\) with \(\alpha \geq 0\). If, for example, the bins become asymptotically equal and \(\alpha > 1/2\), then the probability distribution becomes Gaussian. Notice that this includes limits with \(\alpha < 1\), which is in stark contrast with the rule of thumb that, in order to trust the \(\chi^2\)-distribution, each bin has to contain at least a few, say five (see e.g. [8]), data points. Our result states that, for large \(M\) and \(N\), the majority of bins is allowed to remain empty!
2 The $\chi^2$-statistic as a discrepancy

The $\chi^2$-statistic for $N$ data points distributed over $M$ bins with expected number of points $w_nN$ for bin $n = 1, \ldots, M$ is given by

$$
\chi^2 = \sum_{n=1}^{M} \frac{(\mathcal{N}_n - w_nN)^2}{w_nN},
$$

where $\mathcal{N}_n$ is the number of points in bin $n$. If the data points are distributed truly random, i.e., if they are independent and if the probability for a point to fall in bin $n$ is equal to $1/w_n$, then it is distributed along a multinomial distribution:

$$
P[\chi^2 \leq t] = \sum_{\{\mathcal{N}_m\}} \theta(t - \chi^2) N! \prod_{m=1}^{M} \frac{w^\mathcal{N}_m}{\mathcal{N}_m!},
$$

where the sum is over all configurations $\{\mathcal{N}_m\}$ with $\sum_{m=1}^{M} \mathcal{N}_m = N$. In the limit of an infinite number of data points, this becomes the $\chi^2$-distribution, which has moment generating function

$$
\frac{1}{(1 - 2z)^{M-1}} = G_0(z).
$$

A first step in the interpretation of this statistic as a quadratic discrepancy is the insight that it is quadratic in the variables $\mathcal{N}_n$ and can be characterized by a matrix with elements

$$
\mathcal{B}_{n,m} = \frac{\delta_{n,m}}{w_n} - 1,
$$

so that $\chi^2 = \frac{1}{N} \sum_{n,m} \mathcal{N}_n \mathcal{B}_{n,m} \mathcal{N}_m$. Furthermore, we assume that the bins are disjoint subsets of an integration region, and that the data points $x_k, 1 \leq k \leq N$ are in this integration region. The measure of subset $n$ is then equal to $w_n$. The union of the subsets we call $\mathcal{K}$; it has measure

$$
\sum_{n=1}^{M} w_n = 1.
$$

The number of points $\mathcal{N}_n$ is equal to $\sum_{k=1}^{N} \vartheta_n(x_k)$, where $\vartheta_n$ is the characteristic function of bin $n$, so that

$$
\chi^2 = \frac{1}{N} \sum_{k,l=1}^{N} \sum_{n,m=1}^{M} \vartheta_n(x_k) \mathcal{B}_{n,m} \vartheta_m(x_l) = \frac{1}{N} \sum_{k,l=1}^{N} \mathcal{B}(x_k, x_l) = D_N.
$$

The interpretation of a quadratic discrepancy in the definition following [3, 4] is possible, because the two-point function $\mathcal{B}$ as obtained above integrates to zero with respect to each of its variables. From now on, we will call $D_N$ the Lego discrepancy.
3 Sequences and notation

In the following, we will investigate limits in which the number of bins $M$ goes to infinity. Note that for each value of $M$, we have to decide on the values of the measures $w_n$. They clearly have to scale with $M$, because their sum has to be equal to one. There are, of course, many possible ways for the measures to scale, i.e., many double-sequences $\{w_n^{(M)}, 1 \leq n \leq M, M > 0\}$ of positive numbers with

$$\sum_{n=1}^{M} w_n^{(M)} = 1 \quad \forall M > 0 \quad \text{and} \quad \lim_{M \to \infty} \sum_{n=1}^{M} w_n^{(M)} = 1 .$$

(7)

We, however, want to restrict ourselves to discrepancies in which the relative sizes of the bins stay of the same order, i.e., sequences for which

$$\inf_{n,M} Mw_n^{(M)} \in (0, \infty) \quad \text{and} \quad \sup_{n,M} Mw_n^{(M)} \in (0, \infty) .$$

(8)

It will appear to be appropriate to specify the sequences under consideration by another criterion, which is for example satisfied by the sequences mentioned above. It can be formulated in terms of the objects

$$M_p = \sum_{n=1}^{M} \frac{1}{\left[ w_n^{(M)} \right]^{p-1}} , \quad p \geq 1 ,$$

and is given by the demand that

$$\lim_{M \to \infty} \frac{M_p}{M^{p}} = h_p \in [1, \infty) \quad \forall p \geq 1 .$$

(10)

Within the set of sequences we consider, there are those with for which the bins become asymptotically equal, i.e., sequences with

$$w_n^{(M)} = \frac{1 + \varepsilon_n^{(M)}}{M} \quad \text{with} \quad \varepsilon_n^{(M)} > -1, \quad 1 \leq n \leq M \quad \text{and} \quad \lim_{M \to \infty} \max_{1 \leq n \leq M} |\varepsilon_n^{(M)}| = 0 .$$

(11)

They belong to the set of sequences with $h_p = 1 \forall p \geq 1$, which will allow for special asymptotic probability distributions.

In the following analysis, we will consider functions of $M$ and their behavior if $M \to \infty$. To specify relative behaviors, we will use the symbols "\(\sim\)" , "\(\asymp\)" and "\(<\). The first one is used as follows:

$$f_1(M) \sim f_2(M) \iff \lim_{M \to \infty} \frac{f_1(M)}{f_2(M)} = 1 .$$

(12)

If a limit as above is not necessarily equal to one and not equal to zero, then we use the second symbol:

$$f_1(M) \asymp f_2(M) \iff f_1(M) \sim c f_2(M) , \quad c \in (0, \infty) .$$

(13)
We only use this symbol for those cases in which \( c \neq 0 \). For the cases in which \( c = 0 \) we use the third symbol:

\[
f_1(M) \prec f_2(M) \iff \lim_{M \to \infty} \frac{f_1(M)}{f_2(M)} = 0 \, . \tag{14}
\]

We will also use the \( \mathcal{O} \)-symbol, and do this in the usual sense. We can immediately use the symbols to specify the behavior of \( M_p \) with \( M \), for the criterion of Eq. (10) tells us that

\[
M_p \asymp M^p \, , \tag{15}
\]

and that

\[
M_p \sim M^p \quad \text{if} \quad h_p = 1 \, . \tag{16}
\]

In our formulation, also the number of data points \( N \) runs with \( M \). We will, however, never denote the dependence of \( N \) on \( M \) explicitly and assume that it is clear from now on. Also the upper index at the measures \( w_n \) we will omit from now on.

### 4 Feynman rules

In [6] and [7] we have shown that the moment generating function \( G : z \mapsto \mathbb{E}[e^{zD_N}] \) of the probability distribution of the Lego discrepancy can be written as an integral over \( M \) ordinary degrees of freedom (the bosons) and \( 2N \) Grassmannian degrees of freedom (the fermions). The integral is not unique and contains a gauge freedom. Furthermore, it introduces a natural expansion parameter

\[
g = \sqrt{\frac{2z}{N}} \, , \tag{17}
\]

to calculate the generating function perturbatively. In one particular gauge, the Landau gauge, the terms are equal to the contribution of Feynman diagrams, that can be calculated according to the following rules:

- **boson propagator:** \( n \ldots \ldots \ldots m = B_{n,m} \); **rule 1**
- **fermion propagator:** \( i \rightarrow j = \delta_{i,j} \); **rule 2**
- **vertices:** \( 1 \ldots \ldots \ldots p = -g^p \times \text{convolution} \, , \, p \geq 2 \); **rule 3**

To calculate a term in the expansion series, the contribution of all vacuum diagrams, i.e. all diagrams without external legs, carrying the same power of \( g \) has to be taken into account. In the Landau gauge, only connected diagrams have to be calculated, for

\[
\log G(z) = \text{the sum of the connected vacuum diagrams.} \tag{rule 4}
\]
In the vertices, boson propagators are convoluted as $\sum_{m=1}^{M} w_{m} B_{m,n_1} B_{m,n_2} \cdots B_{m,n_p}$, fermion propagators as $\sum_{j=1}^{N} \delta_{i_1,j} \delta_{j,i_2}$, and then these convolutions are multiplied. As a result of this, the bosonic part of each diagram decouples completely from the fermionic part. The contribution of the fermionic part can easily be determined, for every fermion loop only gives a factor $-N$.

4.1 Rules for the bosonic parts

Because of the rather simple expression (3) for the bosonic propagator, we are able to deduce from the basic Feynman rules some effective rules for the bosonic parts of the Feynman diagrams. Remember that the bosonic parts decouples completely from the fermionic parts. The following rules apply after having counted the number of fermion loops and the powers of $g$ coming from the vertices, and after having calculated the symmetry factor of the original diagram. When we mention the contribution of a diagram in this section, we refer to the contribution apart from the powers of $g$ and symmetry factors. This contribution will be represented by the same drawing as the diagram itself.

The first rule is a consequence of the fact that

$$\sum_{n=1}^{M} w_{n} B_{n_1,m} B_{m,n_2} = B_{n_1,n_2}$$

and states that all vertices with only two legs that do not form a single loop can be removed. The second rule is a consequence of the fact that for any $M \times M$-matrix $f$

$$\sum_{n,m=1}^{M} w_{n} w_{m} f_{n,m} B_{n,m} = \sum_{n=1}^{M} w_{n} f_{n,n} - \sum_{n,m=1}^{M} w_{n} w_{m} f_{n,m} ,$$

and states that the contribution of a diagram is the same as that of the diagram in which a boson line is contracted and the two vertices, connected to that line, are fused together to form one vertex, minus the contribution of the diagram in which the line is simply removed and the vertices replaced by vertices with one boson leg less. This rule can be depicted as follows

By repeated application of these rules, we see that the contribution of a connected bosonic diagram is equal to the contribution of a sum of products of so called daisy diagrams.  

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1 For example $\bigcirc = \bigcirc - \bigcirc = \bigcirc - 2 \bigcirc = \bigcirc - \bigcirc - 2 (\bigcirc \bigcirc - \bigcirc)$
which are of the type

\[ \text{Diagram (20)} \]

They are characterized by the fact that all lines begin and end on the same vertex and form single loops. The contribution of such a diagram is given by

\[
d_p(M) = \sum_{n=1}^{M} w_n B_{n,n}^p = \sum_{q=0}^{p} \binom{p}{q} (-1)^{p-q} M_q = M_p[1 + \mathcal{O}(M^{-1})], \tag{21}
\]

where the last equation follows from Eq. (15). The maximal number of leaves, a product in the sum of daisy diagrams contains, is equal to the number of loops \( L_B \) in the original diagram, so that

\[ \text{the contribution of a diagram with } L_B \text{ boson loops is } M_{L_B}[1 + \mathcal{O}(M^{-1})]. \]

rule 7

The leading order contribution of a diagram with \( L_B \) boson loops is thus of the order of \( M^{L_B} \).

4.2 Extra rule if \( h_p = 1 \)

If \( h_p = 1 \) \( \forall p \geq 1 \), then all kind of cancellations between diagrams occur, because in those cases \( M_p \sim M^p \) \( \forall p \geq 1 \). As a result of this, the contribution of a daisy diagram is \( d_p(M) \sim M^p \), and we can deduce the following rule: the contribution of a diagram that falls apart in disjunct pieces if a vertex is cut, is equal to the product of the contributions of those disjunct pieces times one plus vanishing corrections. Diagrammatically, the rule looks like

\[
\text{Diagram (22)}
\]

In [7] we called discrepancies for which Eq. (22) is exact one-vertex decomposable, and have shown that for those discrepancies only the one-vertex irreducible diagrams contribute, i.e., diagrams that do not fall apart in pieces containing bosonic parts if a vertex is cut. The previous rule tells us that, if \( h_p = 1 \) \( \forall p \geq 1 \), then

\[ \log G(z) \sim \text{sum of all connected one-vertex irreducible diagrams}. \]

rule 8

The connected one-vertex irreducible diagrams we call relevant and the others irrelevant.
5 Loop analysis

We want to determine the contribution of the diagrams in this section, and in order to do that, we need to introduce some notation:

\[
\begin{align*}
L_B &= \text{the number of boson loops} \quad ; \\
L_F &= \text{the number of fermion loops} \quad ; \\
L &= \text{the total number of loops} \quad ; \\
I_B &= \text{the number of bosonic lines} \quad ; \\
I_F &= \text{the number of fermionic lines} \quad ; \\
v &= \text{the number of vertices} \quad ; \\
L_M &= L - L_B - L_F = \text{number of mixed loops} \quad .
\end{align*}
\]

These quantities are in principle functions of the diagrams, but we will never denote this dependence explicitly, for it will always be clear which diagram we are referring to when we use the quantities.

With the foregoing, we deduce that the contribution \(C_A\) of a connected diagram \(A\) with no external legs satisfies

\[
C_A \approx M^{L_B} N^{L_F} g^{2I_B} .
\]

The Feynman rules and basic graph theory tell us that, for connected diagrams with no external legs, \(v = L_F\) and \(L = I_B + I_F - v + 1\), so that

\[
I_B = L - 1 = L_B + L_F + L_M - 1 .
\]

If we furthermore use that \(g = \sqrt{2z/N}\), we find that the contribution is given by

\[
C_A \approx \frac{M^{L_B}}{N^{L_M} N^{L_B - 1}} (2z)^{I_B} .
\]

Notice that this expression does not depend on \(L_F\). Furthermore, it is clear that, for large \(M\) and \(N\), the largest contribution comes from diagrams with \(L_M = 0\). Moreover, we see that we must have \(N = \mathcal{O}(M)\), for else the contribution of higher-order diagrams will grow with the number of boson loops, and the perturbation series becomes completely senseless. If, however, \(N \approx M\), then the contribution of each diagram with \(L_M = 0\) is more important than the contribution of each of the diagrams with \(L_M > 0\). Finally, we also see that the contribution of the \(\mathcal{O}(M^{-1})\)-corrections of a diagram (Eq. (21)) is always negligible compared to the leading contribution of each diagram with \(L_M = 0\). These observations lead to the conclusion that, if \(N\) and \(M\) become large with \(N \approx M\), then the leading contribution to \(\log \hat{G}(z)\) comes from the diagrams with \(L_M = 0\), and that there are no corrections to these contributions. If we assume that \(M/N\) is small, then the importance of these diagrams decreases with the number of boson loops \(L_B\) as \((M/N)^{L_B}\).
5.1 The loop expansion of log $G(z)$

Now we calculate the first few terms in the loop expansion of log $G(z)$. We start with the diagrams with one loop (remember that it is an expansion in boson loops and that we only have to calculate connected diagrams for log $G(z)$). The sum of all 1-loop diagrams with $L_M = 0$ is given by

$$ \frac{1}{2} \quad + \quad \frac{1}{4} \quad + \quad \frac{1}{6} \quad + \quad \cdots = (M-1) \sum_{p=1}^{\infty} \frac{1}{2p} (Ng^2)^p $$

$$ = -\frac{M-1}{2} \log(1-2z) \quad , \quad (33) $$

and this is exactly equal to log $G_0(z)$ as we know it for the Lego-discrepancy (cf.[6]). The fractions before the diagrams are the symmetry factors. From now on, we will always write them down explicitly. To calculate the higher loop diagrams, we introduce the following effective vertex:

$$ p \equiv 1 \quad + \quad \cdots \quad + \quad 1 \quad , \quad (34) $$

and the following partly re-summed propagator:

$$ n \quad m = n \quad m + n \quad m + n \quad m + \cdots $$

$$ = \sum_{p=0}^{\infty} (Ng^2)^p \times n \quad m = \frac{1}{1-2z} \times n \quad m \quad . \quad (35) $$

It is the propagator $G_{n,m}^{(z)}$ from [7]. The contribution of the 2-loop diagrams with $L_M = 0$ is given by

$$ \frac{1}{8} \quad + \quad \frac{1}{8} \quad + \quad \frac{1}{8} \quad + \quad \frac{1}{12} \quad = \quad \left[ \frac{1}{8} \frac{Ng^2 M^2}{(1-2z)^2} + \frac{1}{8} \frac{Ng^2 M^2}{(1-2z)^2} + \frac{1}{8} \frac{(Ng^3)^2(M^2-M^2)}{(1-2z)^3} + \frac{1}{12} \frac{(Ng^3)^2 M^2}{(1-2z)^3} \right] [1 + O(M^{-1})] $$

$$ = \frac{1}{N} \left[ \frac{1}{8} (M^2-M^2) \eta(z)^2 + \left( \frac{5M^2}{24} - \frac{M^2}{8} \right) \eta(z)^3 \right] [1 + O(M^{-1})] \quad , \quad (36) $$

where we define

$$ \eta(z) = \frac{2z}{1-2z} \quad . \quad (37) $$
Notice that the first three diagrams vanish if $h_p = 1 \ \forall p \geq 1$. The contribution of the 3-loop diagrams with $L_M = 0$ is given by

\[
\begin{align*}
\frac{1}{48} + \frac{1}{24} + \frac{1}{8} + \frac{1}{16} + \frac{1}{48} + \frac{1}{12} \\
+ \frac{1}{8} + \frac{1}{16} + \frac{1}{8} + \frac{1}{16} + \frac{1}{12} \\
+ \frac{1}{8} + \frac{1}{16} + \frac{1}{8} + \frac{1}{16} + \frac{1}{12} \\
+ \frac{1}{48} + \frac{1}{16} + \frac{1}{8} + \frac{1}{16} + \frac{1}{24} + \frac{1}{12} \\
\end{align*}
\]

If $h_p = 1 \ \forall p \geq 1$, then only the first four diagrams are relevant, and their contribution $C$ satisfies

\[
C \sim \frac{M^3}{N^2} \left[ \frac{1}{48} \eta(z)^4 + \frac{1}{8} \eta(z)^5 + \frac{5}{48} \eta(z)^6 \right].
\]

\[39\]

6 Various limits

In the previous calculations, $M/N$ was the expansion parameter and the expansion of the generating function only makes sense if it is considered to be small. In fact, a limit in which $M \to \infty$ does not even exist, because the zeroth order term is proportional to $M$.

In order to analyze limits in which $M$ as well as $N$ go to infinity, we can go over to the standardized variable $(D_N - E)/\sqrt{V}$ of the discrepancy, where

\[
E = \mathbb{E}[D_N] = M - 1
\]

\[
V = \mathbb{V}[D_N] = 2(M - 1) + \frac{M^2 - M^2 - 2(M - 1)}{N}.
\]

In terms of the standardized variable, the generating function is given by

\[
\hat{G}(\xi) = \mathbb{E} \left[ e^{\xi \frac{D_N - E}{\sqrt{V}}} \right] = \exp \left( -\frac{E\xi}{\sqrt{V}} \right) G \left( \frac{\xi}{\sqrt{V}} \right).
\]

\[42\]
Instead of the variable $z$, the variable $\xi = z\sqrt{V}$ is considered to be of $O(1)$ in this perspective and the contribution of a diagram changes from (32) to

$$C_A \sim \frac{M^{L_B}}{N^{L_B-N-1}V^{\frac{1}{4}(L_B+L_F+L_M-1)}} (2\xi)^{L_B}.$$  \hfill (43)

In the following we will investigate limits of $M \to \infty$ with, at first instance, the criterion of Eq. (10) as only restriction. The fact that the variance $V$ shows up explicitly in the contribution of the diagrams, forces us to specify the behavior of $M_2$ more precisely. We will take

$$M_2 - M^2 \approx M^\gamma, \quad 0 \leq \gamma \leq 2.$$  \hfill (44)

Notice that $h_2 = 1$ if $\gamma < 2$ and that $h_2$ does not exist if $\gamma > 2$. Furthermore, we cannot read off the natural expansion parameter from the contribution of the diagrams anymore, and have to specify the behavior of $N$. We will only consider limits in which

$$N \approx M^\alpha, \quad \alpha > 0.$$  \hfill (45)

Although they are a small subset of possible limits, those that can be specified by a pair $(\alpha, \gamma)$ show an interesting picture. We will derive the results in the next section, but present them now in the following phase diagram:

![Phase Diagram](image)

It shows the region $S = \{(\alpha, \gamma) \in \mathbb{R}^2 | \alpha \in [0, \infty), \gamma \in [0, 2]\}$ of the real $(\alpha, \gamma)$-plane. In this region, there is a critical line $\ell$, given by

$$\ell = \{(f_\ell(t), t) \in S | t \in [0, 2]\} \quad \text{with} \quad f_\ell(t) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq t \leq \frac{3}{2}, \\ 2 - t & \text{if } \frac{3}{2} \leq t \leq 2. \end{cases}$$  \hfill (46)

It separates $S$ into two regions $T$ and $U$, neither of which contains $\ell$. Our results are the following. Firstly,

in the region $T$, the limit of $M \to \infty$ is not defined. \hfill (47)
In this region, the standardized variable is not appropriate, and we see that there are too many diagrams that grow indefinitely with $M$. Secondly, 

\[ \text{in the region } \mathbf{U}, \text{ the limit of } M \to \infty \text{ gives a Gaussian distribution.} \] \hfill (48)

Because we used the standardized variable, this distribution has necessarily zero expectation and unit variance. Finally, 

\[ \text{on the line } \mathbf{\ell}, \text{ various limits exist, depending on the behaviour of } M_p, p > 2. \] \hfill (49)

One of these limits we were able to calculate explicitly. It appears if $M_p - M^p < M^{p - \frac{1}{2}}$ \forall $p \geq 1$, which is, for example, satisfied in the case of equal binning. In this limit, the generating function is given by 

\[ \log \hat{G}(\xi) = \frac{1}{\lambda^2} \left( e^{\lambda \xi} - 1 - \lambda \xi \right), \quad \lambda = \lim_{M \to \infty} \frac{\sqrt{2M}}{N}. \] \hfill (50)

In Appendix A, we show that the probability distribution $\hat{H}$ belonging to this generating function, which is the inverse Laplace transform, is given by 

\[ \hat{H}(\tau) = \sum_{n \in \mathbb{N}} \delta \left( \tau - \left[ n \lambda - \frac{1}{\lambda} \right] \right) \frac{1}{n!} \left( \frac{1}{\lambda^2} \right)^n \exp \left( -\frac{1}{\lambda^2} \right). \] \hfill (51)

It consists of an infinite number of Dirac delta-distributions, weighed with a Poisson distribution. The delta-distributions reveal the fact that, for finite $N$ and $M$, the Lego discrepancy, and also the $\chi^2$-statistic, can only take a finite number of values, so that the probability density should consist of a sum of delta-distributions. In the usual limit of $N \to \infty$, the discrete nature of the random variable disappears, and the $\chi^2$-distribution is obtained. In our limit, however, the discrete nature does not yet disappear. A continuous distribution is obtained if $\lambda \to 0$, which corresponds with going over from $\alpha = \frac{1}{2}$ to $\alpha > \frac{1}{2}$. Then $\hat{G}(\xi) \to \exp \left( \frac{1}{2} \xi^2 \right)$.

## 7 Derivation of the various limits

We will treat the cases $\gamma = 2$ and $\gamma < 2$ independently.

### 7.1 $\gamma = 2$

We distinguish the three cases $0 < \alpha < 1$, $\alpha = 1$ and $\alpha > 1$.

If $\alpha > 1$, then $V \approx M$, and the contribution $C_A$ of a diagram $A$ satisfies $C_A \approx M^\beta$, with 

\[ \beta = \left( \frac{1}{2} - \alpha \right) L_B - \frac{1}{2} L_F + (\alpha + \frac{1}{2})(1 - L_M). \] \hfill (52)
A short analysis shows that only diagrams with \((L_B, L_F, L_M) = (1, 1, 0)\) or \((L_B, L_F, L_M) = (1, 2, 0)\) give a non-vanishing contribution, and those diagrams are

\[
\frac{1}{2} \delta \text{ } \delta = \frac{1}{2} \frac{N(M - 1)2\xi}{N\sqrt{V}} = \frac{E\xi}{\sqrt{V}} \tag{53}
\]

\[
\frac{1}{4} \text{ } \delta \text{ } = \frac{1}{4} \frac{N^2(M - 1)4\xi^2}{N^2V} = \frac{\xi^2}{2} + \mathcal{O}(M^{-1}) \ . \tag{54}
\]

The first diagram gives a contribution that is linear in \(\xi\) and cancels with the exponent in Eq. (42). This has to happen for every value of \(\alpha\), and as we will see, this diagram will occur always. Notice that the diagrams above are the first two diagrams in the series on the l.h.s of Eq. (33). The logarithm of the generating function becomes quadratic, so that the probability distribution becomes Gaussian.

If \(\alpha = 1\), then again \(V \simeq M\), so that \(\beta = -\frac{1}{2}(L_B + L_F) + \frac{3}{2}(1 - L_M)\), and we have to add the diagrams with \((L_B, L_F, L_M) = (2, 1, 0)\):

\[
\frac{1}{8} \text{ } \delta \text{ } + \frac{1}{8} \text{ } \delta \text{ } = \frac{1}{8} \frac{N(M_2 - M^2)4\xi^2}{N^2V} = \frac{(M_2 - M^2)\xi^2}{2NV} \ . \tag{55}
\]

If \(0 < \alpha < 1\), then \(V \simeq M^{2-\alpha}\) and \(\beta = -\frac{\alpha}{2}L_B - (1 - \frac{\alpha}{2})L_F - (\frac{\alpha}{2} + 1)L_M + \frac{\alpha}{2} + 1\), so that, besides the diagram of Eq. (53), only the diagrams of Eq. (55) give a non-vanishing contribution, and this contribution is equal to \(\xi^2/2\).

### 7.2 \(0 \leq \gamma < 2\)

We can distinguish the two cases \(\gamma - \alpha \leq 1\) and \(\gamma - \alpha > 1\).

#### 7.2.1 \(\gamma - \alpha \leq 1\)

In this case, \(V \simeq M\), and the contribution \(C_A\) of a diagram \(A\) satisfies \(C_A \sim M^\beta\) with

\[
\beta = (\frac{1}{2} - \alpha)L_B - \frac{1}{2}L_F + (\alpha + \frac{1}{2})(1 - L_M) \ . \tag{56}
\]

If \(\alpha < \frac{1}{2}\), then \(\beta\) increases with the number of boson loops \(L_B\), and we are not able to calculate the limit of \(M \to \infty\).

If \(\alpha > \frac{1}{2}\), then the only diagrams that have a non-vanishing contribution are those with \((L_B, L_F, L_M) = (1, 1, 0), (1, 2, 0)\) or \((2, 1, 0)\). These are exactly the diagrams of Eq. (53), Eq. (54) and Eq. (55). Notice, however, that the diagrams of Eq. (53) cancel if \(\gamma - \alpha < 0\): then they are irrelevant. The resulting asymptotic distribution is Gaussian again.

If \(\alpha = \frac{1}{2}\), then \(L_B\) disappears from the equation for \(\beta\), and we obtain a non-Gaussian asymptotic distribution. The diagrams that contribute are those with \((L_F, L_M) = (1, 0)\) or \((2, 0)\). There is, however, only one relevant diagram with \((L_F, L_M) = (1, 0)\), namely the diagram of Eq. (53) that gives the linear term. We have to be careful here, because the other diagrams with \((L_F, L_M) = (1, 0)\) still might be non-vanishing. A short analysis
shows that they are given by the sum of all ways to put daisy diagrams to one fermion loop, and that their contribution is given by

\[ C_1(M) = N \log \left( 1 + \sum_{p=1}^{\infty} \frac{(\frac{1}{2}g^2)^p d_p(M)}{p!} \right). \]  

(57)

We know that, if \( h_p = 1 \), then \( d_p(M) = M^p [1 + \varepsilon_p(M)] \) with \( \lim_{M \to \infty} \varepsilon_p(M) = 0 \), so that

\[ C_1(M) = \frac{1}{2} NM g^2 + N \log \left( 1 + e^{-\frac{1}{2}M g^2} \sum_{p=1}^{\infty} \frac{(\frac{1}{2}M g^2)^p \varepsilon_p(M)}{p!} \right). \]  

(58)

The first term gives the leading contribution; the contribution of the relevant diagram, which consists of a boson loop and a fermion loop attached to one vertex. The second term is irrelevant with respect to the first, but can still be non-vanishing, depending on the behavior of \( \varepsilon_p(M) \). Remember that \( \alpha = \frac{1}{2} \) and \( V \approx M \), so that \( M g^2 = 2 \xi M/(N \sqrt{V}) \to constant \), and we can see that the contribution is only vanishing if

\[ \lim_{M \to \infty} N \varepsilon_p(M) = 0 \quad \forall p \geq 1 \iff M_p - M^p \prec M^{p - \frac{1}{2}} \quad \forall p \geq 1. \]  

(59)

For \( p = 1 \) this relation is satisfied because \( \varepsilon_1(M) = 0 \). For \( p = 2 \) this relation is also satisfied if \( \gamma < \frac{3}{2} \).

If the relation is also satisfied for the other values of \( p \), then the only diagrams that contribute to the generating function are the relevant diagrams with \( (L_F, L_M) = (2, 0) \):

\[ \bullet \circ + \bullet \cdots + \bullet \circ + \cdots, \]  

(60)

where we used the effective vertex \([\text{34}]\) again. The contribution of a diagram of this type with \( p \) boson lines is given by

\[ \frac{1}{2 p!} \left( \frac{2 \xi}{N \sqrt{V}} \right)^p N^2 M_p [1 + \mathcal{O}(M^{-1})] \sim \frac{N^2}{2M} \frac{1}{p!} \left( \frac{2 M \xi}{N \sqrt{V}} \right)^p. \]  

(61)

The factor \( 1/2 p! \) is the symmetry factor of this type of diagram. If we sum the contribution of these diagrams and use that \( V \sim 2M \), we obtain

\[ \log \hat{G}(\xi) \sim \frac{1}{\lambda^2} \left( e^{\lambda \xi} - 1 - \lambda \xi \right), \quad \lambda = \lim_{M \to \infty} \sqrt{2M} \frac{\sqrt{2M \xi}}{N}. \]  

(62)

### 7.2.2 \( \gamma - \alpha > 1 \)

In this case, \( V \approx M^{\gamma - \alpha} \) and the contribution \( C_A \) of a diagram \( A \) satisfies \( C_A \prec M^\beta \) with

\[ \beta = (1 - \frac{3+\alpha}{2})L_B - \frac{2-\alpha}{2} L_F + \frac{3+\alpha}{2} \left( 1 - L_M \right). \]  

(63)
If $\gamma + \alpha < 2$, then $\beta$ increases with the number of boson loops $L_B$, and we are not able to calculate the limit of $M \to \infty$.

If $\gamma + \alpha > 2$, then the only diagrams that have a non-vanishing contribution are those with $(L_B, L_F, L_M) = (1, 1, 0)$, $(1, 2, 0)$ or $(2, 1, 0)$. These are exactly the diagrams of Eq. (53), Eq. (54) and Eq. (55). Notice, however, that the diagrams of Eq. (55) cancel if $\gamma - \alpha < 0$: then they are irrelevant. The resulting asymptotic distribution is Gaussian.

If $\gamma + \alpha = 2$, then $\beta = (\alpha - 1)L_F + 1 - L_M$. Because $\gamma - \alpha > 1$, we have $\alpha < \frac{1}{2}$, and non-vanishing diagrams have $(L_F, L_M) = (1, 0)$. Their contribution is given by the r.h.s. of Eq. (58), the first term of which gives the term linear in $\xi$. The second term is non-vanishing, because $Mg^2 \simeq M^{1-(\gamma+\alpha)/2} \to \text{constant}$ and $N\varepsilon_2(M) \simeq M^{\alpha+\gamma-2} \to \text{constant}$.

8 Conclusions

We have shown that the Lego discrepancy with $M$ bins is equivalent to a $\chi^2$-statistic with $M$ bins. We have presented a procedure to calculate the moment generating function of the probability distribution of the discrepancy perturbatively if $M$ and $N$, the number of uniformly and randomly distributed data points, become large. The natural expansion parameter we have identified to be $M/N$, and we have calculated the first few terms in the series explicitly.

In order to calculate limits in which $N, M \to \infty$, we have introduced the objects of Eq. (10) and restricted the behavior of the size of the bins such that they satisfy Eq. (10). Furthermore, we have gone over to the standardized variable of the discrepancy. For this variable, we have derived a phase diagram, representing the limits specified by Eq. (44) and Eq. (45). We have formulated the results in (47), (48) and (49).

One of these results is that there are non-trivial limits if $N, M \to \infty$ such that $M^\alpha/N \to \text{constant}$ with $\alpha < 1$. This result is in stark contrast with the rule of thumb that, in order to trust the $\chi^2$-distribution, each bin has to contain at least a few data points.

Appendix A

We want to calculate the integral

$$\hat{H}(\tau) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{f_\tau(z)} \, dz, \quad f_\tau(z) = \frac{1}{\lambda^2} \left( e^{\lambda z} - 1 - \lambda z \right) - z\tau. \quad (64)$$

We will make use of the fact that

$$f_\tau \left( z + \frac{2\pi i n}{\lambda} \right) = f_\tau(z) - 2\pi i n \frac{1 + \lambda \tau}{\lambda^2} \quad (65)$$
for all $n \in \mathbb{Z}$, so that

$$
\hat{H}(\tau) = \frac{1}{2\pi i} \sum_{n \in \mathbb{Z}} \int e^{f_r(z)} \, dz = \frac{1}{2\pi i} \sum_{n \in \mathbb{Z}} e^{-2\pi i n \frac{1+\lambda \tau}{\lambda^2}} \int e^{f_r(z)} \, dz .
$$

(66)

Notice that the integral is independent of $n$, so that the sum can be interpreted as a sum of Dirac delta-distributions:

$$
\sum_{n \in \mathbb{Z}} e^{-2\pi i n \frac{1+\lambda \tau}{\lambda^2}} = \sum_{n \in \mathbb{Z}} \delta \left( \frac{1+\lambda \tau}{\lambda^2} - n \right) = \sum_{n \in \mathbb{Z}} \lambda \delta \left( \tau - \left[ n \lambda - \frac{1}{\lambda} \right] \right) .
$$

(67)

These delta-distributions restrict the values that $\tau$ can take. If we use these restrictions and do the appropriate variable substitutions, the remaining integral in (66) can be reduced to

$$
\int e^{f_r(z)} \, dz = e^{-\frac{\lambda}{\lambda^2}} \int_{-\pi i}^{\pi i} \exp \left( \frac{e^\varphi}{\lambda^2} - n \varphi \right) \, d\varphi = e^{-\frac{x^2}{\lambda}} \oint \frac{e^{\frac{w}{w}}}{w^{n+1}} \, dw ,
$$

(68)

where $n \in \mathbb{Z}$ and the contour is closed around $w = 0$. According to Cauchy’s theorem, the final integral is only non-zero if $n \in \mathbb{N}$, and in that case its value is $2\pi i \frac{1}{n!} (\frac{1}{\lambda})^n$. The combination of these results gives Eq. (51).

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