Chiral symmetry breaking as a consequence of nontrivial spatial topology

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Abstract

A singular configuration of external static magnetic field in the form of a pointlike vortex polarizes the vacuum of quantized massless spinor field in 2+1-dimensional space-time. This results in an analogue of the Bohm-Aharonov effect: the chiral symmetry breaking condensate, energy density and current emerge in the vacuum even in the

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case when the spatial region of nonvanishing external field strength is excluded. The dependence of the vacuum characteristics both on the value of the vortex flux and on the choice of the boundary condition at the location of the vortex is determined.

Keywords: vacuum condensate, chiral symmetry. singular vortex

The dynamical breakdown of chiral symmetry is a topic of persisting interest for a long time (for reviews see, e.g., Refs.[1, 2, 3, 4]). An important role in its study is played by the appropriate ”order parameter” – vacuum condensate

\[ C(x, t) = i\langle \text{vac}|T\bar{\Psi}(x, t)\Psi(x, t)|\text{vac}\rangle. \tag{1} \]

In the background of external classical fields, the vacuum expectation value of the time-ordered product of the fermion field operators takes the form

\[ \langle \text{vac}|T\bar{\Psi}(x, t)\Psi(x', t')|\text{vac}\rangle = -i\langle x, t|(-i\gamma^\mu \nabla_\mu + m)^{-1}|x', t'\rangle, \tag{2} \]

where \( \nabla_\mu \) is the covariant derivative in this background. Thus, in a static background, condensate (1) is reduced to the form

\[ C(x) = -\frac{1}{2}\text{tr}\langle x|\gamma^0 \text{sgn}(H)|x\rangle, \tag{3} \]

where

\[ H = -i\gamma^0 \gamma^j \nabla_j + \gamma^0 m, \tag{4} \]

is the pertinent Dirac Hamiltonian and

\[ \text{sgn}(u) = \begin{cases} 1, & u > 0 \\ -1, & u < 0 \end{cases}. \]
For a certain background field (magnetic or gravitational) configuration, the analysis of Eq.(1) (or its particular form, Eq.(3)) can be carried out with a special attention to the limiting procedure $m \to 0$. If the condensate survives in this limit, then it witnesses chiral symmetry breaking by this configuration. Moreover, as it has been shown in Ref.[5], in the case of quantized spinor fields in the background of a homogeneous magnetic field in 2+1-dimensional space-time, the chiral symmetry breaking condensate emerges irrespectively of all other possible types of interaction among quantized spinor fields. The question that we would like to address in the present Letter is, whether the emergence of the chiral symmetry breaking condensate could be caused by nontrivial topology of the base space manifold?

Nontrivial topology can be achieved, formally, by excluding submanifolds of less dimensions from an initial manifold with trivial topology. For example, deleting a line from a three-dimensional space results in the spatial topology becoming nontrivial with the classification in terms of the winding number: $\pi_1 = \mathbb{Z}$ (here $\pi_1$ is the first homotopy group and $\mathbb{Z}$ is the set of integer numbers). Imposing various boundary conditions at the location of the excluded submanifold, one can study a possibility that some of them will induce the chiral symmetry breaking condensate in the vacuum. However, our freedom to vary boundary conditions is restricted by the natural requirement of their physical meaningfulness. Since we are not to consider here boundary conditions invoking instability and confine ourselves to stationary problems, self-adjointness of the Hamiltonian has to be maintained. It is well known that the free Dirac Hamiltonian is essentially self-adjoint when defined on the domain of regular functions (see, e.g., Ref.[6]). Thus,
we have no choice but to pick the regularity boundary condition in order to ensure self-adjointness, and this yields no gain comparing to the case of trivial topology. One has to do something more, than simply saying about the deletion of a line from a space, in order to achieve nontrivial topology physically. And this doing more means inserting a singular magnetic vortex at the location of the deleted line. Then the Dirac Hamiltonian defined on the domain of regular functions is not self-adjoint any more, which may seem from the first sight to be rather forbidding. However, this obstruction can be overcome by exploring a possibility of self-adjoint extension along the lines of the Weyl-von Neumann theory of self-adjoint operators (see, e.g., Refs.\[7, 8\]). As a result, one arrives at a set of boundary conditions which are compatible with self-adjointness and allow for wave functions being square-integrable but irregular at the location of a deleted line. We shall show that these boundary conditions can induce the chiral symmetry breaking condensate in the vacuum.

Omitting the spatial dimension along a deleted line, one gets a two-dimensional space (plane) with a deleted point (puncture). A static singular magnetic vortex inserted at the puncture is given by the Ehrenberg-Siday-Aharonov-Bohm potential \[9, 10\]

\[
V^1 = -\Phi \frac{x^2}{(x^1)^2 + (x^2)^2}, \quad V^2 = \Phi \frac{x^1}{(x^1)^2 + (x^2)^2},
\]

where \(\Phi\) is the vortex flux in \(2\pi\) units (conventions \(\hbar = c = 1\) are implied) and the location of the puncture is taken as the origin of the Cartesian coordinates \(x^1\) and \(x^2\) on the plane. Specifying the covariant derivative as
\[ \nabla_j = \partial_j + iV_j, \quad j = 1, 2, \quad (6) \]

our aim is to study the polarization of the fermionic vacuum by classical static background (5) in 2+1-dimensional space-time.

As it is known (see, e.g., Ref. [11]), in order to consider the chiral symmetry breaking in 2+1 dimensions, quantized spinor fields have to be assigned to the reducible representation of the Clifford algebra

\[ \Psi = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}, \quad (7) \]

where \( \Psi_+ \) and \( \Psi_- \) are assigned to two inequivalent irreducible \((2 \times 2)\) representations; \( \gamma \) matrices are chosen in the form

\[ \gamma^0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}, \quad (8) \]

where \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) are the Pauli matrices. The algebra is completed by adding the \( \gamma \) matrix corresponding to the missing dimension,

\[ \gamma^3 = i \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad (9) \]

and the \( \gamma^5 \) matrix is

\[ \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = i \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \quad (10) \]

Then the massless Dirac Hamiltonian (Eq.(4) at \( m = 0 \)) in background (5) takes the form
\[ H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \] 

(11)

where

\[ H_\pm = \begin{pmatrix} 0 & e^{-i\varphi}[\partial_r - r^{-1}(i\partial_\varphi + \Phi)] \\ e^{i\varphi}[-\partial_r - r^{-1}(i\partial_\varphi + \Phi)] & 0 \end{pmatrix}, \] 

(12)

\[ r = \sqrt{(x^1)^2 + (x^2)^2} \] and \( \varphi = \arctan(x^2/x^1) \) are the polar coordinates. A single-valued solution to the stationary Dirac equation

\[ H\langle x|E \rangle = E\langle x|E \rangle \] 

(13)

with Hamiltonian (11)–(12) is presented as

\[ \langle x|E \rangle = \sum_{n \in \mathbb{Z}} \begin{pmatrix} f_n^+(r, E)e^{in\varphi} \\ g_n^+(r, E)e^{i(n+1)\varphi} \\ f_n^-(r, E)e^{in\varphi} \\ g_n^-(r, E)e^{i(n+1)\varphi} \end{pmatrix}. \] 

(14)

Decomposing the value of the vortex flux into the integer and fractional parts,

\[ \Phi = \lfloor \Phi \rfloor + \{ |\Phi| \}, \quad 0 \le \{ |\Phi| \} < 1 \] 

(15)

(\( \lfloor u \rfloor \) denotes the integer part of quantity \( u \)), one can note that the case of \( \{ |\Phi| \} = 0 \) is equivalent to the case of trivial topology, i.e. absence of the vortex (\( \Phi = 0 \)).\(^1\)

\(^1\)This confirms once more the general fact that a singular magnetic vortex is physically unobservable at integer values of the vortex flux. It was as far back as 1931 that Dirac used actually this fact to obtain his famous condition for the magnetic monopole quantization.\(^{12}\)
obey the condition of regularity at the puncture \( r = 0 \): the corresponding partial Hamiltonians are essentially self-adjoint when defined on the domain of regular functions. That is only the partial Hamiltonian corresponding to \( n = [\Phi] \), that needs a self-adjoint extension. Similarly to the case of nonzero fermion mass \((m \neq 0)\) \[13, 14, 15\], it can be shown that the self-adjoint extension is parametrized by one real continuous variable \( \Theta \), yielding the following condition for the mode with \( n = [\Phi] \):

\[
\cos\left( \frac{\Theta}{2} + \frac{\pi}{4} \right) \lim_{r \to 0} (\mu r)^{[\Phi]} f^\pm_{[\Phi]} = \mp \sin\left( \frac{\Theta}{2} + \frac{\pi}{4} \right) \lim_{r \to 0} (\mu r)^{1-[\Phi]} g^\pm_{[\Phi]},
\]

where \( \mu > 0 \) is the parameter of the dimension of inverse length, which is introduced just to scale the different irregular behaviour of the \( f \) and \( g \) components; note that, since Eq.\((16)\) is periodic in \( \Theta \) with period \( 2\pi \), all permissible values of \( \Theta \) can be restricted, without a loss of generality, to range \(-\pi \leq \Theta \leq \pi\).

The “minus/plus” sign factor in the right-hand side of Eq.\((16)\) deserves a special comment. For a massless fermion in an irreducible representation, an overall change of sign before the Pauli matrices does not mean a transition to an inequivalent representation, because \( \sigma_3 \) by itself does not appear anywhere, while a change of sign before \( \sigma_1 \) and \( \sigma_2 \) corresponds to a transition to an equivalent representation. This is reflected by the fact that the expressions for \( H_+ \) and \( H_- \) coincide, see Eq.\((12)\). Thus, boundary condition \((16)\) is the only point where a distinction between \( \Psi_+ \) and \( \Psi_- \) arises. Unless we had made this distinction, we would get the case of a reducible representation composed of two equivalent irreducible ones, which simply doubles the case of an irreducible representation considered in detail elsewhere \[16].
the condensate and the corresponding mass term break parity rather than chiral symmetry in that case. On the contrary, boundary condition (16) is invariant under parity transformation

\[ \langle x^1, x^2 | E \rangle \rightarrow i\gamma^1\gamma^3 \langle -x^1, x^2 | E \rangle \]

and is not invariant under chiral transformation

\[ \langle x^1, x^2 | E \rangle \rightarrow \gamma^5 \langle x^1, x^2 | E \rangle, \]

unless

\[ \cos \Theta = 0, \]

when both parity and chiral symmetry are maintained.

Having specified the boundary condition at the puncture \( r = 0 \), an explicit form of a solution to Eq.(13) can be obtained. Using this form, all vacuum polarization effects are determined. In addition to condensate (3), also current

\[ j_\varphi(x) = -\frac{1}{2} \text{tr}(x|\gamma^0(\gamma^2 \cos \varphi - \gamma^1 \sin \varphi) \text{sgn}(H)|x) \]

and energy density

\[ \varepsilon^{\text{ren}}(x) = -\frac{1}{2} \text{tr}(x| |H||x)^{(\text{ren})} \]

are induced in the vacuum. Here, the superscript ”ren” in Eq.(21) denotes the use of a certain regularization and renormalization procedure to get rid of both ultraviolet and infrared divergences, for details see Ref.[16]. Concerning Eq.(20), we can add that the radial component of the vacuum current,
\[ j_r(x) = -\frac{1}{2} \text{tr}(x) \gamma^0 (\gamma^1 \cos \varphi + \gamma^2 \sin \varphi) \text{sgn}(H)|x\rangle, \quad (22) \]

is not induced.

We list below the results (details will be published elsewhere):

\[ C(x) = -\frac{\sin(\|\Phi\| \pi)}{\pi^3 r^2} \int_0^\infty dw \frac{K_{\|\Phi\|}^2(w) + K_{1-\|\Phi\|}^2(w)}{\cosh[(2 \|\Phi\| - 1) \ln(w/\mu r) + \ln A]}, \quad (23) \]

\[ j_φ(x) = \frac{\sin(\|\Phi\| \pi)}{\pi r^2} \left\{ \left( \frac{1}{2} - \|\Phi\| \right)^2 \frac{\cosh[(2 \|\Phi\| - 1) \ln(w/\mu r) + \ln A]}{2 \cos(\|\Phi\| \pi)} - \right. \]

\[ - \frac{2}{\pi^2} \int_0^\infty dw w K_{\|\Phi\|}(w) K_{1-\|\Phi\|}(w) \tanh \left[(2 \|\Phi\| - 1) \ln(w/\mu r) + \ln A \right], \quad (24) \]

\[ ε^{\text{ren}}(x) = \frac{\sin(\|\Phi\| \pi)}{\pi r^3} \left\{ \frac{1}{2} - \|\Phi\| \right\} \left[ \frac{3}{4} - \|\Phi\| (1 - \|\Phi\|) \right] + \]

\[ + \frac{1}{\pi^2} \int_0^\infty dw w^2 [K_{\|\Phi\|}^2(w) - K_{1-\|\Phi\|}^2(w)] \tanh \left[(2 \|\Phi\| - 1) \ln(w/\mu r) + \ln A \right], \quad (25) \]

where

\[ \bar{A} = 2^{1-2\|\Phi\|} \frac{\Gamma(1 - \|\Phi\|)}{\Gamma(\|\Phi\|)} \tan(\frac{\Theta}{2} + \frac{\pi}{4}), \quad (26) \]

\( K_\tau(w) \) is the Macdonald function of order \( \tau \), and \( \Gamma(u) \) is the Euler gamma function.

Since the puncture breaks translational invariance on the plane, the vacuum polarization effects (23) – (25) are not translationally invariant. There remains an invariance with respect to rotations around the puncture, and,
therefore, the vacuum polarization effects depend only on the distance from
the puncture. At large distances they are decreasing by power law:

\[
C(x) = \frac{\sin(\|\Phi\|\pi)}{\pi^2 r^2} \left\{ \begin{array}{ll}
(\mu r)^{2\|\Phi\|-1} A^{-1} \frac{\Gamma\left(\frac{\|\Phi\|}{2} - \frac{\|\Phi\|}{2}\right) \Gamma\left(\frac{\|\Phi\|}{2} - 2\|\Phi\|\right)}{\Gamma\left(1 - \|\Phi\|\right)}, & 0 < \|\Phi\| < \frac{1}{2} \\
(\mu r)^{1-2\|\Phi\|} A^{-1} \frac{\Gamma\left(\|\Phi\| + \frac{1}{2}\right) \Gamma(2\|\Phi\| - \frac{1}{2})}{\Gamma\left(\|\Phi\|\right)}, & \frac{1}{2} < \|\Phi\| < 1
\end{array} \right.,
\]

\[ j_\varphi(x) = \frac{\tan(\|\Phi\|\pi)}{2\pi r^2} \left[ \|\Phi\| - \frac{1}{2} \left( \|\Phi\| - \frac{1}{2} \right) - 1 \right], \quad (27) \]

\[ \varepsilon_{\text{ren}}(x) = \frac{\tan(\|\Phi\|\pi)}{2\pi r^3} \left[ \|\Phi\| - \frac{1}{2} \right] \left[ \frac{1}{3}\|\Phi\|(1 - \|\Phi\|) - \frac{1}{4} + \frac{1}{2}\|\Phi\| - \frac{1}{2} \right]. \quad (29) \]

At half-integer values of the vortex flux, taking into account relation

\[ A|_{\|\Phi\| = \frac{1}{2}} = \tan\left(\frac{\Theta}{2} + \frac{\pi}{4}\right), \]

we get

\[ C(x)|_{\|\Phi\| = \frac{1}{2}} = -\frac{\cos \Theta}{2\pi^2 r^2}, \quad (31) \]

\[ j_\varphi(x)|_{\|\Phi\| = \frac{1}{2}} = -\frac{\sin \Theta}{2\pi^2 r^2}, \quad (32) \]

\[ \varepsilon_{\text{ren}}(x)|_{\|\Phi\| = \frac{1}{2}} = \frac{1}{12\pi^2 r^3}; \quad (33) \]

note the \( \Theta \) independence of the last relation.

Integrating Eq.(23) over the whole plane, we obtain the total vacuum
condensate
\[ C \equiv \int d^2 x \mathcal{C}(x) = - \frac{\text{sgn}_0(\cos \Theta)}{2 \| \Phi \| - 1} \]  

(34)

where

\[ \text{sgn}_0(u) = \begin{cases} 
\text{sgn}(u), & u \neq 0 \\
0, & u = 0 
\end{cases} \]

Thus, the total vacuum condensate is infinite at half-integer values of the vortex flux, unless Eq.(19) holds.

Finally, few comments on the case of absence of the condensate in the vacuum, i.e., when Eq.(19) holds. In this case two of the four components of the wave function (14) become regular for all \( n \): if \( \Theta = \frac{\pi}{2} \), then the \( g_n^\pm \) components are regular, and, if \( \Theta = -\frac{\pi}{2} \), then the \( f_n^\pm \) components are regular. Due to this, scale symmetry, as well as parity and chiral symmetry, is maintained. The nontrivial topology reveals itself in the emergence of the vacuum current and energy density only:

\[ j_\varphi(x) = \frac{\tan(\| \Phi \| \pi)}{2 \pi r^2} (\| \Phi \| - \frac{1}{2}) (\| \Phi \| - \frac{1}{2} \pm 1), \quad \Theta = \pm \frac{\pi}{2}, \]  

(35)

\[ \varepsilon^{\text{ren}}(x) = \frac{\tan(\| \Phi \| \pi)}{2 \pi r^3} (\| \Phi \| - \frac{1}{2}) \left[ \frac{1}{3} \| \Phi \| (1 - \| \Phi \|) - \frac{1}{4} + \frac{1}{2} (\| \Phi \| - \frac{1}{2}) \right]^{\pm}, \]  

\[ \Theta = \pm \frac{\pi}{2}. \]  

(36)

Summarizing, we have completed an exhaustive study of vacuum polarization effects in the background of a singular magnetic vortex (\( \| \Phi \| \neq 0 \)) under boundary condition (16) which ensures self-adjointness of the Hamiltonian and parity conservation. If \( \Theta \neq \pm \frac{\pi}{2} \), then chiral symmetry breaking condensate (23) emerges in the vacuum. One would anticipate that scale symmetry
is broken as well, but this is not the case. Note that the vacuum current and energy density contain scale invariant pieces (terms which are not represented by integrals in Eqs. (24) and (25)). Also, large distance asymptotics of the vacuum current and energy density are scale invariant, see Eqs. (28) and (29). But the most significant point is that, at half-integer values of the vortex flux ($|\Phi| = \frac{1}{2}$), the vacuum condensate, as well as the vacuum current and energy density, becomes scale invariant, see Eqs. (31) – (33); consequently, the total vacuum condensate becomes infinite, see Eq. (34). That is only at non-half-integer values of the vortex flux, that both chiral and scale symmetries are broken.

In conclusion, it should be emphasized once more that magnetic field strength $\partial_1 V_2 - \partial_2 V_1$ corresponding to potential (5) vanishes everywhere on the plane punctured at $x^1 = x^2 = 0$. Thus, we have shown that, in the absence of any background field strength or curvature, chiral symmetry breaking occurs just due to nontrivial topology of space (in the Bohm-Aharonov-effect-like manner). It would be interesting to bring possible types of interaction among quantized fermion fields (either four-fermionic, or electrodynamic) into this context in order to consider the mass gap equation and its implications.

2) In 2+1-dimensional space-time the canonical dimension of the fermion field operator is equal to one in the units of the power of inverse length.
Acknowledgements

I am grateful to H. Leutwyler and V.A. Miransky for stimulating discussions and interesting remarks. The research was supported by the State Foundation for Fundamental Research of Ukraine (project 2.4/320) and the Swiss National Science Foundation (grant CEEC/NIS/96-98/7 IP 051219).

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