Redistributing Chern numbers of Landau subbands: tight-binding electrons under staggered-modulated magnetic fields

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We investigate the magneto-transport properties of electrons on a square lattice under a magnetic field with the alternate flux strength $\phi \pm \Delta \phi$ in neighboring plaquettes. A new peculiar behavior of the Hall conductance has been found and is robust against weak disorder: if $\phi = \frac{p}{2N} \times 2\pi$ ($p$ and $2N$ are coprime integers) is fixed, the Chern numbers of Landau subbands will be redistributed between neighboring pairs and hence the total quantized Hall conductance exhibits a direct transition by $\pm N e^2/h$ at critical fillings when $\Delta \phi$ is increased from 0 up to a critical value $\Delta \phi_c$. This effect can be an experimental probe of the staggered-flux phase.

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\textbf{Introduction.}—The integer quantum Hall effect (IQHE) is observed in two-dimensional (2D) electron systems, in the presence of a strong perpendicular uniform magnetic field [1]. The phenomenology of IQHE is well explained by Landau quantization and disorder-induced localization. The IQHE electronic states exhibit interesting topological properties [2, 3]. In particular, each state can be labelled by a topologically invariant integer called the Chern number, which is the Hall conductance of this state, in units of $e^2/h$. A state with a nonzero Chern number carries a Hall current and is necessarily extended. In solving the problem regarding how an IQHE state transforms into the insulating state, many authors carried out numerical studies of the tight-binding model in the presence of a uniform magnetic field and a random potential [4, 5, 6, 7]. An interesting behavior has been reported by Sheng and Weng that disorder will induce direct higher IQHE plateau to insulator transition [6]. These theoretical results are in good agreement with several magneto-transport experiments performed in Si metal-oxide-semiconductor field-effect transistors [8] and a 2D hole system confined in a Ge/SiGe quantum well [9].

Recently, an intriguing orbital-current-carrying state in the square lattice, the staggered-flux phase (SFP) which is also known as orbital antiferromagnet or d-density wave (DDW) state [10], regains new attentions since unusual experimental findings in two systems: a pseudogap phenomena (See the review by Timusk et al. [11]) in the underdoped region of high-$T_c$ cuprates [10]; a hidden order in the heavy-fermion compound UR$_2$Si$_2$ [12]. However, a direct experimental observation of the SFP order is difficult. On the other hand, adopting a superconducting wire network decorated with an array of ferromagnets, an artificial system of tight-binding electrons under a staggered-modulated magnetic field has been realized [13], and the Little-Parks oscillations exhibit the edge part of the corresponding Hofstadter spectrum.

Motivated by the above experimental and theoretical studies of the IQHE and the SFP, we investigate the magneto-transport properties of tight-binding electrons on a 2D square lattice in the presence of a staggered-modulated magnetic field which has the alternate flux strength $\phi \pm \Delta \phi$ in neighboring plaquettes. Hofstadter’s butterflies of similar systems have been studied numerically [14, 15]. Here, we find a new peculiar $\Delta \phi$-dependent redistribution behavior of Chern numbers of Landau subbands and propose that this effect be an experimental probe to reveal the possible internal orbital-antiferromagnetic structure of the SFP order in some systems [10, 12].

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{selected_gauge.png}
\caption{(color online). Illustration of the selected gauge. $\phi_1 = \phi + \Delta \phi$ and $\phi_2 = \phi - \Delta \phi$}
\end{figure}

\textbf{Formulation.}—The tight-binding model Hamiltonian is given as follows:

\begin{equation}
H = - \sum_{\langle ij \rangle} t e^{i a_{ij}^t c_i^t c_j} + \text{H.c.} + \sum_i w_i c_i^t c_i
\end{equation}

where the hopping integral $t$ is taken as the unit of energy, $c_i$ ($c_i^t$) is an electron annihilation (creation) operator on site $i$, and $\langle ij \rangle$ refers to two nearest neighboring sites. The magnetic flux per plaquette (the summation of $a_{ij}$ along four links around a plaquette) is given by $\phi_1 = \phi + \Delta \phi$ and $\phi_2 = \phi - \Delta \phi$ alternatively in neighboring plaquettes. $\phi = \frac{p}{2N} \times 2\pi$ ($p$ and $2N$ are coprime
increasing from 0 to 0 jumps of the Hall conductance at [see the corresponding DOS in Fig. 2(b)] centered at the clearly shown, corresponding to four Landau subbands part is shown here). For the uniform case ($\Delta = 0$), the sample considered has the $2N \times 2N$ unit cell for $\phi = \frac{\pi}{2N} \times 2\pi$ and the periodical boundary condition is also adopted.

After the numerical diagonalization of the Hamiltonian [Eq. (1)], the total Hall conductance for the system can be calculated through the standard Kubo formula:

$$\sigma_H(E) = \frac{ie^2}{A_0 h} \sum_{\varepsilon_m < E, \varepsilon_n > E} \frac{(m|v_x|n)(n|v_y|m) - (m|v_y|n)(n|v_x|m)}{(\varepsilon_m - \varepsilon_n)^2}$$

(2)

where $A_0$ is the area of the system, $E$ is the Fermi energy, $\varepsilon_m$ and $\varepsilon_n$ are the corresponding eigenvalues of the eigenstates $|m\rangle$ and $|n\rangle$. The velocity operator is defined as $v_{\tau} = i \sum_{\tau'} (c_{\tau+\tau'}^\dagger c_{\tau'} e^{i\phi_{\tau+\tau'}} - c_{\tau'}^\dagger c_{\tau+\tau'} e^{-i\phi_{\tau+\tau'}})$ with $\tau = \hat{x}$ or $\hat{y}$. And we can rewrite $\sigma_H$ as $\sigma_H(E) = e^2/h \sum_{\varepsilon_m < E} C_m$, where $C_m$ is the Chern number $2$ of the $m$-th subband.

FIG. 2: (color online). The case with $p = 1$, $N = 4$ and $W = 0$. (a) The Hall conductance $\sigma_H$ versus the Fermi energy $E$ for various $\Delta \phi$'s. (b)-(e) The DOS of the cases in (a). The Chern numbers of subbands are shown.

Absence of disorder. (a) An example with $p = 1$ and $N = 4$.—An overall picture for the Hall conductance calculated by Eq. (2) is shown in Fig. 2(a) with the uniform-flux strength $\phi = \frac{1}{3} \times 2\pi$ and various staggered-flux strengths $\Delta \phi = 0.0, 0.1\pi, 0.14\pi, 0.15\pi, 0.2\pi$ (only $E \leq 0$ part is shown here). For the uniform case ($\Delta \phi = 0$), three well-defined IQHE plateaus at $\sigma_H = e^2/h$ ($l = 1, 2, 3$) are clearly shown, corresponding to four Landau subbands [see the corresponding DOS in Fig. 2(b)] centered at the jumps of the Hall conductance at $E \leq 0$. When $\Delta \phi$ increasing from 0 to $0.2\pi$, one sees a systematic evolution of these IQHE plateaus. At $\Delta \phi = 0.1\pi$, the $l = 3$ IQHE plateau narrows from both sides, and the $l = 2$ IQHE plateau narrows from the right side. At $\Delta \phi = 0.2\pi$, the $l = 3$ IQHE plateau disappears and is replaced by a $l = -1$ IQHE plateau. $\Delta \phi = 0.15\pi$ is close to the critical point where the transition from the $l = 3$ IQHE plateau to the $l = -1$ IQHE plateau takes place. All the values of the Hall conductances in Fig. 2(a) fall back to zero when the Fermi energy approaches $E = 0$, due to the particle-hole symmetry.

Such a behavior can be explained through the evolution of the density of states (DOS) depicted in Fig. 2(b)-(e). In Fig. 2(b), (c) ($\Delta \phi = 0, 0.1\pi$), there are four separated subbands (each totally-filled subband contributes $\frac{1}{2}$ to the filling factor) at $E \leq 0$ with the corresponding well-defined four sequential Chern integers $\{+1, +1, +1, -3\}$. In Fig. 2(d) ($\Delta \phi = 0.15\pi$), the upper two subbands merge together and form a pseudogap which makes the corresponding quantization of Hall conductance not well defined. In Fig. 2(e) ($\Delta \phi = 0.2\pi$), the upper two subbands separate again and the sequential Chern integers are redistributed as $\{+1, +1, -3, +1\}$. We can conclude that at the critical filling $\nu_c = \frac{3}{8}$, there is a direct transition of the Hall conductance from the $l = 3$ IQHE plateau to the $l = -1$ IQHE plateau when $\Delta \phi$ is increased from 0 up to a critical value $\Delta \phi_c$.

(b) Redistribution behavior of Chern numbers.—At $p = 1$, we have numerically confirmed that the Hall conductances at weaker flux strengths with $N = 7-36$ all exhibit the similar features as those in Fig. 2(a). Thus in general, given $\phi = \frac{1}{2N} \times 2\pi$, the $N$ sequential Chern integers $\{+1, \ldots, +1, +1, -N + 1\}$ will be redistributed as $\{+1, \ldots, +1, -N + 1, +1\}$ (one can also say that there is a transfer of integer $N$ between the last two Chern numbers) when $\Delta \phi$ is increased from 0 up to a critical value $\Delta \phi_c$. Namely, there is a direct transition of the Hall conductance from the $l = N - 1$ IQHE plateau to the $l = -1$ IQHE plateau) at the critical filling $\nu_c = \frac{1}{2N}$. This is quite different from the global phase diagram proposed by Kivelson, Lee and Zhang [10] for IQHE which predicts that the only allowed transitions are the nearest-neighbor plateau-plateau transitions ($l \rightarrow l \pm 1$), and is also different from the disorder-induced higher IQHE plateau to insulator transition ($l \rightarrow 0$) [6].

In cases with $p > 1$, based on a Diophantine equation, a pattern of subband structure has been found by Yang and Bhatt [7]: the subbands away from the band center form groups with $p$ subbands each, and the total Chern number of each group is $+1$, while the subbands near the band center form a special group with mod($N$, $p$) subbands.

At $p = 3$ and various $N$'s, through numerical calculations, we find a more complicated redistribution behavior of Chern numbers (Table I). For instance, we take the case $N = 8$, when $\Delta \phi$ varies from 0 to $0.2\pi$, the eight sequential Chern numbers
TABLE I: Redistribution behavior of Chern numbers (of the $E \leq 0$ subbands) for $p = 3, 5, 7, 9$ and various $N$’s. Only the last two or three groups are shown for $N \geq 13$.

| $p$ | $N$ | $\Delta \phi$ | Chern numbers of Landau subbands |
|-----|-----|---------------|---------------------------------|
| 0.03 | 0   | $(-5, 11, -5)(-5, 11, -5)(-5, 3)$ |
| 0.1  | 0   | $(3, -5)(-5, 3, 3)(-5, 3)$ |
| 0.2  | 0   | $(3, -5)(-5, 3, 3)(-5, 3)$ |
| 0.08 | 0.00 | $(7, -13, 7)(7, -13, 7)(7, -13, 7)(-3)$ |
| 0.01 | 0.15 | $(7, -13, 7)(7, -13, 7)(7, -13, 7)(-3)$ |
| 0.05 | 0.16 | $(9, -17, 9)(9, -17, 9)(-5, 9)(-5, 9)$ |
| 0.02 | 0.17 | $(9, -17, 9)(9, -17, 9)(-5, 9)(-5, 9)$ |
| 0.01 | 0.18 | $(9, -17, 9)(9, -17, 9)(-5, 9)(-5, 9)$ |
| 0.08 | 0.19 | $(9, -17, 9)(9, -17, 9)(-5, 9)(-5, 9)$ |
| 0.01 | 0.20 | $(9, -17, 9)(9, -17, 9)(-5, 9)(-5, 9)$ |
| 0.05 | 0.21 | $(9, -17, 9)(9, -17, 9)(-5, 9)(-5, 9)$ |
| 0.02 | 0.22 | $(9, -17, 9)(9, -17, 9)(-5, 9)(-5, 9)$ |
| 0.01 | 0.23 | $(9, -17, 9)(9, -17, 9)(-5, 9)(-5, 9)$ |

(c) Scaling properties.—Both the uniform-flux strength $\phi = \frac{c}{2\pi} \times 2\pi$ at $N = 4$–38 and the staggered-flux strength in the previous calculations are relatively high fields. In order to meet the requirements of possible experimental realization, we will reduce the flux strength further. The scaling behavior of $\Delta \phi_e$ with $\phi$ is shown in Fig. 3. For $p \geq 3$, we concentrate on the cases where there are two Chern numbers in the last group and consider the critical filling $\nu = \frac{N-1}{2(N-1)}$. For a fixed $p$, $\Delta \phi_e$ scales with $\phi$ almost linearly. The slopes of the lines with different $p$’s differ from one another. With $\phi$ deceasing, all lines approach the origin point, and therefore $\Delta \phi_e$ approaches zero at the weak-field limit.

FIG. 3: (color online). $\Delta \phi_e$ versus $\phi$ for $p = 1, 3, 5, 7, 9$ and various $N$’s (at the critical filling $\nu = \frac{N-1}{2(N-1)}$)

Presence of disorder.—In realistic solids, impurity or phonon-induced disorder always has effect on the...
magneto-transport properties. We now consider $W = 1$ to see the influence of weak disorder on the critical transition behavior.

As for $p = 1$ and $N = 4$, we can see from Fig. 4 that the presence of weak disorder does not smear out the transition of the Hall conductance from the $l = 3$ IQHE plateau to the $l = -1$ IQHE plateau near the critical filling $\nu = 1/3$.

An experimental probe of the SFP order.— As one application of our theoretical concern, we would take the staggered-flux part $\pm \Delta \phi$ as the possible internal orbital-antiferromagnetic structure of the SFP order in some strongly-correlated electron systems [10, 12], and the uniform-flux part $\phi$ as an external experimental probe. In Fig. 5 at the electron filling $\nu = 15/32$, we fix the value of $\Delta \phi$ and do such a hypothetical experiment by measuring the Hall conductance which varies with $\phi$. The two curves of the Hall conductances with $\Delta \phi = 0$ and $\Delta \phi = 0.1\pi$ exhibit large deviations at many values of $N/p$. Therefore, such an experiment could tell us the characteristic differences between the states with and without the SFP order.

Summary and discussion.— As for the tight-binding electrons on a square lattice under a staggered-modulated magnetic field with the strength $\phi \pm \Delta \phi$, a new peculiar behavior of the Hall conductances has been found: (a) if $\phi = \frac{15}{32} \times 2\pi$ ($p$ and $2N$ are coprime integers) is fixed and $\Delta \phi$ is increased from 0 up to a critical value $\Delta \phi_c$, the Chern numbers of Landau subbands will be redistributed between neighboring pairs; (b) and hence the total quantized Hall conductance exhibits a direct transition by $\pm N e^2/h$ at one or several critical fillings; (c) the neighboring two subbands merge together and form a pseudogap when $\Delta \phi = \Delta \phi_c$; (d) at a fixed $p$, $\Delta \phi_c$ scales with $\phi$ almost linearly; (e) weak disorder does not smear out the peculiar transition of the Hall conductance.

To observe this effect under an accessible uniform magnetic field with $N \sim 1000$, we estimate that both the strength of disorder $W$ and the temperature $k_B T$ should not exceed $\frac{12}{1000}$ (where $4t$ is half of the original band width). Taking $t = 0.01$ eV, the temperature $T$ should be lower than 0.5 K.

As for high-$T_c$ cuprates, the strength of the internal staggered magnetic field is estimated to be about 0.1 T [17]. One may suspect that: will an internal staggered magnetic field survive under a stronger external uniform one? We believe it does, basing on the numerical results: an external uniform magnetic field will suppress the $d$-wave superconducting order, while the SFP order (or equivalently, the DDW order) can exist or even be enhanced in the vortex cores [18, 19].

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