Casimir operators of the exceptional group $F_4$: the chain $B_4 \subset F_4 \subset D_{13}$

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Expressions are given for the Casimir operators of the exceptional group $F_4$ in a concise form similar to that used for the classical groups. The chain $B_4 \subset F_4 \subset D_{13}$ is used to label the generators of $F_4$ in terms of the adjoint and spinor representations of $B_4$ and to express the 26-dimensional representation of $F_4$ in terms of the defining representation of $D_{13}$. Casimir operators of any degree are obtained and it is shown that a basis consists of the operators of degree 2, 6, 8 and 12.

I. INTRODUCTION

Although a general formula exists for the quadratic Casimir operator for any group this is not the case for operators of higher degree. Efficient expressions have been developed over the years for all the Casimir operators of the classical groups, but not for the exceptional groups. Berdjis [1] gives the desired Casimir operators implicitly. Until recently explicit results were available only for $G_2$. The degree 6 Casimir of $G_2$ was given in the work of Hughes and Van der Jeugt [2] by an expression involving 29 terms and in the work by Bincer and Riesselmann [3] by an expression involving 23 terms. These results were obtained using computers and leave something to be desired.

Quite recently I have developed a different approach and obtained for $G_2$ results very much alike to those for the classical groups [4]. Moreover it would seem that the same
approach should work for the other exceptional groups. In the present work I address the
group $F_4$ and leave the $E_{6,7,8}$ for a future paper.

This paper is organized as follows. In the next Sec. after explaining the use of the chain
$B_4 \subset F_4 \subset D_{13}$ I obtain concise expressions for the Casimir operators of $F_4$. These require
the knowledge of the generators of $D_{13}$ projected into $F_4$. To obtain this projection I describe
in the next Sec. the 26-dimensional representation of $F_4$ and then obtain in the following
Sec. the desired projection. In the Conclusion I discuss the quadratic Casimir operator of
$F_4$ and demonstrate that the independent Casimir operators are of degree 2, 6, 8 and 12
(corresponding to the exponents of $F_4$ being 1, 5, 7 and 11).

II. THE CASIMIR OPERATORS OF $D_{13}$ AND $F_4$

My approach makes use of the chain $B_4 \subset F_4 \subset D_{13}$. The subgroup $B_4$ of $F_4$ is used
to label the generators of $F_4$. $F_4$ is embedded in $D_{13}$ because the smallest-dimensional
representation of $F_4$ is 26-dimensional and orthogonal and $D_{13}$ is the orthogonal group in
26 dimensions.

I denote the 36 generators of $B_4$ as $B_\alpha^\beta = -B_\bar{\alpha}^\beta$, with indices ranging from $-4$ to $+4$,
zero included, $\bar{\alpha} \equiv -\alpha$. The hermitian property is expressed in this basis as $B_\alpha^\beta\dagger = B_\bar{\alpha}^\beta$. I
denote the generators of $F_4$ as $B_\alpha^\beta$ and $S^{pqrs}$, corresponding to the decomposition of the $52$
(the adjoint) of $F_4$ into the $36$ and $16$ of $B_4$, where the $36$ is the adjoint, i.e., the $B_\alpha^\beta$, and
the $16$ is the spinor $S^{pqrs} = (S^{pqrs})\dagger, p, q, r, s = \pm$. The $B_4 \subset F_4$ relation is exhibited in the
extended Dynkin diagram

$$
\begin{align*}
\alpha_0 & \quad \alpha_1 & \quad \alpha_2 & \quad \alpha_3 & \quad \alpha_4 \\
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \\
u_1 - u_2 & \quad u_2 - u_3 & \quad u_3 - u_4 & \quad u_4 & -\frac{1}{2}(u_1 + u_2 + u_3 + u_4)
\end{align*}
$$

with $B_4$ obtained by omitting $\alpha_4$ and $F_4$ obtained by omitting $\alpha_0$. That is to say: the
$\alpha_i, \quad 1 \leq i \leq 4$, are the simple roots of $F_4$, while the $\alpha_j, \quad 0 \leq j \leq 3$ are the simple roots
of $B_4$. The information encoded in the Dynkin digram is made explicit by setting $\alpha_0 =$
\[ u_1 - u_2, \alpha_1 = u_2 - u_3, \alpha_2 = u_3 - u_4, \alpha_3 = u_4, \alpha_4 = -\frac{1}{2}(u_1 + u_2 + u_3 + u_4), \] where the \( u_i \) are orthogonal unit vectors.

I denote the generators of \( D_{13} \) as \( D^b_a = -D^\bar{a}_{\bar{b}}, (D^b_a)^\dagger = D^\bar{a}_{\bar{b}}, \) zero excluded. The commutation relations of \( D_{13} \) in this basis are

\[
[D^b_a, D^c_d] = \delta^b_c D^d_a - \delta^d_c D^b_a + \delta^d_b D^a_c - \delta^a_b D^d_c
\]  

(1)

It follows from Eq. (1) that

\[
[D^b_a, (D^k)^c_d] = \delta^b_c (D^k)_a^d - \delta^d_c (D^k)_a^b + \delta^d_b (D^k)_c^a - \delta^a_b (D^k)_c^d
\]  

(2)

where I define the \( k \)th power, \( k \geq 1 \), by

\[
(D^k)^b_a = (D^{k-1})^c_d (D^k)^a_c D^b_c = D^c_a (D^{k-1})^b_c, \quad (D^0)^b_a = \delta^b_a
\]  

(3)

(summation convention understood). It now follows that if I define

\[
C_k(D_{13}) = (D^k)^a_a
\]  

(4)

then these \( C_k \) commute with the generators of \( D_{13} \) and so are Casimir operators of \( D_{13} \) of degree \( k \). Equation (4) provides an elegant expression for the Casimir operators of \( D_{13} \) and is an example of the type of expressions valid for all the classical groups. All this is well-known and goes back to Perelomov and Popov [5]. I remark that the 13 independent Casimirs of this type are of degree \( k = 2s, 1 \leq s \leq 13 \). This is because it follows from the antisymmetry property \( D^b_a = -D^\bar{a}_{\bar{b}} \) that the Casimirs for \( k = \text{odd} \) can be expressed in terms of those for \( k = \text{even} \), and it follows from the Cayley-Hamilton theorem that Casimirs of degree \( k > 26 \) can be expressed in terms of those for \( k \leq 26 \). I note further that the Casimir operator of degree 26 can be expressed in terms of the square of a Casimir of degree 13 [which is not of the form given by Eq. (4)] and so the integrity basis for the Casimirs contains those of degree \( k = 2s, 1 \leq s \leq 12 \), and \( k = 13 \), which agrees with the fact that the degrees \( k \) of the Casimirs in the basis should be equal to the exponents of \( D_{13} \) plus one.

We next observe that under the restriction of \( D_{13} \) to \( F_4 \) the adjoint representation of \( D_{13} \) decomposes thus
\[ 325 = 52 + 273 \]  \hspace{1cm} (5)

where the 325 refers to the adjoint of \( D_{13} \) and the 52 to the adjoint of \( F_4 \). Thus we can express the generators \( D^b_a \) of \( D_{13} \) in terms of the generators of \( F_4 \) and the components of the 273-plet. We now obtain the Casimir operators of \( F_4 \) by observing that they are given by Eq. (4) in which the \( D^b_a \) are replaced by their projections into \( F_4 \), i.e.,

\[ C_k(F_4) = (\tilde{D}^k_a)_a \]  \hspace{1cm} (6)

where

\[ \tilde{D}^b_a = D^b_a |_{273=0} \]  \hspace{1cm} (7)

I mean by Eq. (7) that the projected \( \tilde{D}^b_a \) are given by expressing the \( D^b_a \) in terms of the generators of \( F_4 \) and members of the 273-plet and then setting the contribution of the 273-plet equal to zero.

**THE 26-DIMENSIONAL REPRESENTATION OF \( F_4 \)**

To obtain the projected \( \tilde{D} \) I need to obtain first explicit formulas for the 26-dimensional representation of \( F_4 \).

The generators \( D^b_a \) of \( D_{13} \) are given in the defining 26-dimensional representation as the following 26 \( \times \) 26 matrices:

\[ D^b_a = I_{ab} - I_{ba} \]  \hspace{1cm} (8)

where \( I_{ab} \) is the 26 \( \times \) 26 matrix with matrix elements

\[ (I_{ab})_{jk} = \delta_{aj}\delta_{bk} \]  \hspace{1cm} (9)

with the labels \( j, k \) taking on the same values as \( a, b: -13 \leq j, k \leq 13 \), zero excluded.

The Cartan generators of \( F_4 \) are given in the 26-dimensional representation by the 26 \( \times \) 26 matrices as follows:
\begin{align*}
h_1 &= D_5^5 + D_6^6 - D_7^7 + D_8^8 - D_9^9 - D_{10}^{10} \\
h_2 &= D_3^3 + D_4^4 - D_5^5 - D_6^6 + D_{10}^{10} - D_{11}^{11} \\
h_3 &= \frac{1}{2} \left( D_2^2 - 2D_3^3 + D_4^4 + D_6^6 - D_8^8 + D_9^9 - D_{10}^{10} + D_{11}^{11} - D_{12}^{12} \right) \\
h_4 &= \frac{1}{2} \left( -2D_2^2 + D_3^3 + D_4^4 - D_5^5 - D_6^6 + D_7^7 - D_9^9 + D_{12}^{12} - D_{13}^{13} \right)
\end{align*}

These are precisely the same expressions as were obtained by Patera \[6\] and Ekins and Cornwell \[7\] if I relabel their rows and columns thus: their 1 \(\rightarrow\) mine \(-13\), their 2 \(\rightarrow\) mine \(-12\), . . . , their 13 \(\rightarrow\) mine \(-1\), their 14 \(\rightarrow\) mine +1, . . . , their 26 \(\rightarrow\) mine +13.

Given these explicit matrices for the Cartan generators \(h_i\), the associated generators \(e_i\) and \(f_i = e_i^\dagger\) in the Chevalley basis are found from the equations \[7\]

\[ [e_j, h_k] = A_{kj} e_j, \quad [f_j, e_k] = \delta_{jk} h_k \]  

The summation convention does not apply to Eqs. (14) and \(A\) is the Cartan matrix of \(F_4\):

\[ A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \]  

A solution of Eqs. (14) for the simple generators \(e_i\) is as follows:

\begin{align*}
  e_1 &= D_5^5 + D_6^6 + D_{10}^{10} \\
  e_2 &= D_3^3 + D_4^4 + D_{11}^{11} \\
  e_3 &= 2^{-\frac{1}{2}} \left( D_3^1 + D_3^1 + D_4^7 + D_6^6 + D_9^9 + D_{10}^{11} \right) \\
  e_4 &= 2^{-\frac{1}{2}} \left( zD_2^1 + z^* D_2^1 + D_4^3 + D_5^5 + D_9^7 + D_{12}^{12} \right)
\end{align*}

where

\[ z \equiv e^{i\pi/3} \]  

Except for the renumbering of rows and columns and a different choice of phases, my expressions for \(e_1\) and \(e_2\) are precisely the same as those given by Patera \[6\] and Ekins and
However my expressions for \( e_3 \) and \( e_4 \) differ from the corresponding expressions of those authors. It would seem that they resolved some of the arbitrariness in the solution by demanding that it be real; I require that it display the antisymmetry across the antidiagonal corresponding to the fact that we have an orthogonal representation.

In accordance with my labeling of generators of \( F_4 \) in the \( B_4 \) basis in terms of the adjoint and the spinor of \( B_4 \) I have that the above simple generators \( e_i \) should be labeled as follows:

\[
\begin{align*}
\alpha_1 &= u_2 - u_3 \quad \rightarrow e_1 = B_2^3 \\
\alpha_2 &= u_3 - u_4 \quad \rightarrow e_2 = B_3^4 \\
\alpha_3 &= u_4 \quad \rightarrow e_3 = B_1^0 \\
\alpha_4 &= -\frac{1}{2} (u_1 + u_2 + u_3 + u_4) \rightarrow e_4 = S^{+++}
\end{align*}
\]

Next I form commutators of the simple generators and obtain level one generators

\[
\begin{align*}
\alpha_1 + \alpha_2 &= u_2 - u_4 \rightarrow B_2^0 = [B_2^3, B_3^4] = D_2^5 + D_9^4 - D_{11}^8 \\
\alpha_2 + \alpha_3 &= u_3 \rightarrow B_3^0 = [B_3^4, B_4^0] = 2^{-\frac{1}{2}} \left( D_5^1 + D_6^2 + D_8^4 + D_{11}^4 - D_{12}^{10} \right) \\
\alpha_3 + \alpha_4 &= -\frac{1}{2} (u_1 + u_2 + u_3 - u_4) \rightarrow S^{+++} = [B_4^0, S^{+++}] \sqrt{2} \\
&= -2^{-\frac{1}{2}} \left( z^* D_4^1 + z D_4^1 + D_3^5 - D_8^5 - D_{10}^7 + D_{13}^{11} \right) \quad (21)
\end{align*}
\]

level two generators

\[
\begin{align*}
\alpha_1 + \alpha_2 + \alpha_3 &= u_2 \rightarrow B_2^0 = [B_2^3, B_3^4] = 2^{-\frac{1}{2}} \left( D_5^1 + D_7^1 + D_9^4 - D_{10}^4 - D_{11}^6 + D_{12}^8 \right) \\
\alpha_2 + \alpha_3 + \alpha_4 &= -\frac{1}{2} (u_1 + u_2 - u_3 + u_4) \rightarrow S^{+++} \\
&= [B_3^0, S^{+++}] \sqrt{2} = -2^{-\frac{1}{2}} \left( z^* D_6^1 + z D_6^1 + D_5^2 + D_8^3 - D_{11}^7 - D_{13}^{10} \right) \\
\alpha_2 + 2\alpha_3 &= u_3 + u_4 \rightarrow B_3^1 = [B_3^0, B_3^0] = D_5^2 + D_8^2 + D_{12}^9 \quad (22)
\end{align*}
\]

level three generators

\[
\begin{align*}
\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= -\frac{1}{2} (u_1 - u_2 + u_3 + u_4) \rightarrow S^{+++} = [B_2^0, S^{+++}] \sqrt{2} \\
&= -2^{-\frac{1}{2}} \left( z^* D_9^1 + z D_9^1 + D_7^2 + D_{10}^3 + D_{11}^5 + D_{13}^8 \right) \\
\alpha_1 + \alpha_2 + 2\alpha_3 &= u_2 + u_4 \rightarrow B_2^1 = [B_2^0, B_2^0] = D_7^2 - D_9^6 + D_{10}^2
\end{align*}
\]
\[ \alpha_2 + 2\alpha_3 + \alpha_4 = -\frac{1}{2} (u_1 + u_2 - u_3 - u_4) \rightarrow S^{++--} = [B^0_3, S^{++--}] \sqrt{2} \]
\[ = -2^{-\frac{1}{2}} \left( zD^1_8 + z^*D^1_8 - D^3_6 - D^4_6 + D^7_12 - D^9_13 \right) \quad (24) \]

level four generators
\[ \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 = -\frac{1}{2} (u_1 - u_2 + u_3 - u_4) \rightarrow S^{+-++} = [B^0_2, S^{+-++}] \sqrt{2} \]
\[ = -2^{-\frac{1}{2}} \left( zD^1_{10} + z^*D^1_{10} - D^3_7 - D^5_9 - D^9_{12} + D^6_{13} \right) \]
\[ \alpha_1 + 2\alpha_2 + 2\alpha_3 = u_2 + u_3 \rightarrow B^3_2 = [B^1_3, B^1_2] = D^5_7 + D^4_{12} + D^2_{11} \]
\[ \alpha_2 + 2\alpha_3 + 2\alpha_4 = -u_1 - u_2 \rightarrow B^2_1 = [S^{++--}, S^{++--}] = D^6_4 + D^5_8 + D^7_{13} \quad (25) \]

level five generators
\[ \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 = -u_1 - u_3 \rightarrow B^3_1 = [B^2_1, B^2_2] = D^9_6 - D^2_{10} + D^5_{13} \]
\[ \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 = -\frac{1}{2} (u_1 - u_2 - u_3 + u_4) \rightarrow S^{+-++} = [B^0_3, S^{+-++}] \sqrt{2} \]
\[ = 2^{-\frac{1}{2}} \left( zD^1_{11} + z^*D^1_{11} - D^6_7 - D^5_9 + D^3_{12} - D^4_{13} \right) \quad (26) \]

level six generators
\[ \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 = -u_1 - u_4 \rightarrow B^4_1 = [B^2_1, B^2_2] = D^6_0 + D^2_{11} + D^3_{13} \]
\[ \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4 = -\frac{1}{2} (u_1 - u_2 - u_3 - u_4) \rightarrow S^{+-++} = -[B^0_4, S^{+-++}] \sqrt{2} \]
\[ = 2^\frac{1}{2} \left( z^*D^1_{12} + zD^1_{12} + D^8_7 + D^5_{10} - D^3_{11} - D^2_{13} \right) \quad (27) \]

one level seven generator
\[ \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 = -u_1 \rightarrow B^0_1 = -[S^{+-++}, S^{++--}] \sqrt{2} \]
\[ = 2^\frac{1}{2} \left( D^1_{13} + D^1_{13} + D^8_9 + D^6_{10} - D^4_{11} - D^2_{12} \right) \quad (28) \]

one level eight generator
\[ \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4 = -u_1 + u_4 \rightarrow B^1_4 = [B^0_1, B^0_1] = -D^8_{10} + D^4_{12} - D^3_{13} \quad (29) \]

one level nine generator
\[ \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = -u_1 + u_3 \rightarrow B_3^1 = \left[ B_1^0, B_3^0 \right] = -D_{11}^8 + D_{12}^6 - D_{13}^5 \]  

and one level ten generator

\[ 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = -u_1 + u_2 \rightarrow B_2^1 = \left[ B_1^0, B_2^0 \right] = D_{10}^{11} + D_{12}^9 - D_{13}^7 \]  

Note that the root corresponding to the highest level, Eq. (31), is precisely the negative of \( \alpha_0 \), where \( \alpha_0 \) is the extra root added to the Dynkin diagram of \( F_4 \) to produce the extended Dynkin diagram.

In addition to the above 24 \( e \)-type generators, Eqs. (16)–(31), I have 24 \( f \)-type generators obtained by taking the hermitian conjugate of the above. Thus corresponding to the expressions above for the simple (level zero) lowering generators \( e_i \) I have

\[
\begin{align*}
 f_1 &= e_1^\dagger = B_2^{3\dagger} = B_3^2 = D_5^7 + D_6^9 + D_8^{10} \\
 f_2 &= e_2^\dagger = B_3^{4\dagger} = B_4^3 = D_3^5 + D_4^6 + D_{10}^{11} \\
 f_3 &= e_3^\dagger = B_4^{0\dagger} = B_0^4 = 2\frac{1}{2} \left( D_1^3 + D_4^3 + D_6^4 + D_9^8 + D_{11}^{10} \right) \\
 f_4 &= e_4^\dagger = S^{++++} = S^{----} = 2\frac{1}{2} \left( z^* D_1^2 + z D_4^2 + D_3^4 + D_5^6 + D_7^9 + D_{12}^{13} \right)
\end{align*}
\]  

and so on for the generators in higher levels.

Moreover, for the hermitian Cartan generators I have that the Chevalley and \( B_4 \) bases are related as follows:

\[
\begin{align*}
 h_1 &= [f_1, e_1] = \left[ B_3^2, B_2^3 \right] = B_3^3 - B_2^2 \\
 h_2 &= [f_2, e_2] = \left[ B_4^3, B_3^4 \right] = B_4^4 - B_3^3 \\
 h_3 &= [f_3, e_3] = \left[ B_0^4, B_4^0 \right] = -B_4^4 \\
 h_4 &= [f_4, e_4] = \left[ S^{----}, S^{++++} \right] = \frac{1}{2} \left( B_1^1 + B_2^2 + B_3^3 + B_4^4 \right)
\end{align*}
\]  

or, solving above for the \( B_\alpha^\alpha \) and using Eqs. (10)–(13),

\[
-B_1^1 = h_1 + 2h_2 + 3h_3 + 2h_4 \\
= \frac{1}{2} \left( D_2^2 + D_4^4 + D_6^6 + D_8^8 + D_9^9 + D_{10}^{10} + D_{11}^{11} + D_{12}^{12} + 2D_{13}^{13} \right)
\]
\[-B_2^2 = h_1 + h_2 + h_3 = \frac{1}{2} \left( D_2^2 + D_4^2 + D_6^3 - 2D_7^2 + D_8^2 - D_9^1 - D_{10}^{10} - D_{11}^{11} - D_{12}^{12} \right) \]

\[-B_3^3 = h_2 + h_3 = \frac{1}{2} \left( D_2^4 + D_4^1 - 2D_5^5 - D_6^6 - D_8^8 + D_9^9 + D_{10}^{10} - D_{11}^{11} - D_{12}^{12} \right) \]

\[-B_4^4 = h_3 = \frac{1}{2} \left( D_2^2 - 2D_3^3 - D_4^1 + D_6^6 - D_8^8 + D_9^9 - D_{10}^{10} + D_{11}^{11} - D_{12}^{12} \right) \]  

(34)

**THE PROJECTED GENERATORS $\tilde{D}_A^B$**

Now since the 26 is the defining representation of $D_{13}$, the results above expressing the generators of $F_4$ in the 26-dimensional representation in terms of the generators of $D_{13}$ in the 26-dimensional representation, can be interpreted as giving the generators of $F_4$ in terms of those of $D_{13}$ in any representation. Now then the $\tilde{D}_b^a$, the generators of $D_{13}$ projected into $F_4$, are given by inverting the above equations.

Thus the 13 Cartan generators of $D_{13}$ projected into $F_4$ are given by inverting Eqs. (34):

\[\tilde{D}_1^1 = 0\]

\[\tilde{D}_2^2 = -\frac{1}{6} \left( B_1^1 + B_2^2 + B_3^3 + B_4^4 \right)\]

\[\tilde{D}_3^3 = \frac{1}{3} B_4^4\]

\[\tilde{D}_4^4 = -\frac{1}{6} \left( B_1^1 + B_2^2 + B_3^3 - B_4^4 \right)\]

\[\tilde{D}_5^5 = \frac{1}{3} B_3^3\]

\[\tilde{D}_6^6 = -\frac{1}{6} \left( B_1^1 + B_2^2 - B_3^3 + B_4^4 \right)\]

\[\tilde{D}_7^7 = \frac{1}{3} B_2^2\]

\[\tilde{D}_8^8 = -\frac{1}{6} \left( B_1^1 + B_2^2 - B_3^3 - B_4^4 \right)\]

\[\tilde{D}_9^9 = -\frac{1}{6} \left( B_1^1 - B_2^2 + B_3^3 + B_4^4 \right)\]

\[\tilde{D}_{10}^{10} = -\frac{1}{6} \left( B_1^1 - B_2^2 + B_3^3 - B_4^4 \right)\]

\[\tilde{D}_{11}^{11} = -\frac{1}{6} \left( B_1^1 - B_2^2 - B_3^3 + B_4^4 \right)\]

\[\tilde{D}_{12}^{12} = -\frac{1}{6} \left( B_1^1 - B_2^2 - B_3^3 - B_4^4 \right)\]

\[\tilde{D}_{13}^{13} = -\frac{1}{3} B_1^1\]  

(35)
Perhaps an explanation of how Eq. (35) is obtained is in order. Equations (34) are four equations for four $B^a_\alpha$ in terms of thirteen $D^a_\alpha$ (no summations). In addition there are nine more equations for appropriate components of the 273-plet involving these same thirteen $D^a_\alpha$. This total of 13 equations can be written as follows

$$b_A = U_{AB}d_B$$  \hspace{1cm} (36)

where $1 \leq A, B \leq 13$, where $d_B \equiv D^B_B$ (no summation), $b_A \equiv B^A_A$ (no summation) for $A = 1, 2, 3, 4$, and $b_A$ for $5 \leq A \leq 13$ refers to components of the 273-plet. Inversion of Eq. (36) is achieved by

$$d_A = U^{-1}_{AB}b_B$$  \hspace{1cm} (37)

where the inverse of the $13 \times 13$ matrix $U$ is given by

$$U^{-1} = \frac{1}{3} U^\dagger$$  \hspace{1cm} (38)

where the factor $\frac{1}{3}$ accounts for the difference in the normalization of the $d_A$ and $b_A$. Finally the projected $\tilde{d}_A$ are obtained by setting in Eq. (37) $b_A = 0$ for $5 \leq A \leq 13$.

By proceeding in the same fashion I obtain the 156 generators $\tilde{D}^b_a$ with $a > b$ by inverting the 24 $e$-type equations with the result:

for the 24 $\tilde{D}^b_{13}$ with $13 > b$:

$$\tilde{D}^{12}_{13} = \tilde{D}^{11}_{13} = \tilde{D}^{10}_{13} = \tilde{D}^9_{13} = \tilde{D}^8_{13} = \tilde{D}^7_{13} = \tilde{D}^6_{13} = \tilde{D}^5_{13} = \tilde{D}^4_{13} = \tilde{D}^3_{13} = \tilde{D}^2_{13} = \tilde{D}^1_{13} = \tilde{D}^0_{13} = 0,$$

$$\tilde{D}^7_{13} = -\frac{1}{3}B^1_2, \quad \tilde{D}^6_{13} = -\frac{1}{3}B^1_3, \quad \tilde{D}^5_{13} = -\frac{1}{3}B^1_4,$$

$$\tilde{D}^4_{13} = \tilde{D}^3_{13} = -\frac{1}{3\sqrt{2}}B^0_6, \quad \tilde{D}^2_{13} = -\frac{1}{3\sqrt{2}}S^{+---},$$

$$\tilde{D}^1_{13} = \frac{1}{3}B^1_1, \quad \tilde{D}^0_{13} = -\frac{1}{3\sqrt{2}}S^{+++}, \quad \tilde{D}^5_{13} = \frac{1}{3}B^3_1,$$

$$\tilde{D}^6_{13} = -\frac{1}{3\sqrt{2}}S^{++-}, \quad \tilde{D}^7_{13} = \frac{1}{3}B^2_1, \quad \tilde{D}^8_{13} = -\frac{1}{3\sqrt{2}}S^{+++},$$

$$\tilde{D}^9_{13} = \frac{1}{3\sqrt{2}}S^{+++}, \quad \tilde{D}^{10}_{13} = \frac{1}{3\sqrt{2}}S^{+++},$$

$$\tilde{D}^{11}_{13} = -\frac{1}{3\sqrt{2}}S^{+++}, \quad \tilde{D}^{12}_{13} = \frac{1}{3\sqrt{2}}S^{+++}.$$  \hspace{1cm} (39)
for the 22 $\tilde{D}^9_{12}$ with $12 > |b|$:  

$$
\tilde{D}^9_{12} = \tilde{D}^{10}_{12} = \tilde{D}^8_{12} = \tilde{D}^7_{12} = \tilde{D}^5_{12} = \tilde{D}^3_{12} = \tilde{D}_2^0 = 0,
$$

$$
\tilde{D}^6_{12} = \frac{1}{3}B^1_4, \quad \tilde{D}^6_{12} = \frac{1}{3}B^1_3, \quad \tilde{D}^4_{12} = \frac{1}{3}B^1_4
$$

$$
\tilde{D}^2_{12} = \frac{1}{3\sqrt{2}}B^1_0, \quad \tilde{D}^1_{12} = \frac{z}{3\sqrt{2}}S^{++--}, \quad \tilde{D}^1_{12} = \frac{z^*}{3\sqrt{2}}S^{++--},
$$

$$
\tilde{D}^3_{12} = \frac{1}{3\sqrt{2}}S^{+++}, \quad \tilde{D}^4_{12} = \frac{1}{3}B^3_2, \quad \tilde{D}^5_{12} = \frac{1}{3\sqrt{2}}S^{+++},
$$

$$
\tilde{D}^6_{12} = \frac{1}{3}B^2_4, \quad \tilde{D}^7_{12} = -\frac{1}{3\sqrt{2}}S^{++++}, \quad \tilde{D}^8_{12} = \frac{1}{3\sqrt{2}}B^0_2,
$$

$$
\tilde{D}^9_{12} = \frac{1}{3}B^3_3, \quad \tilde{D}^{10}_{12} = \frac{1}{3\sqrt{2}}B^3_0, \quad \tilde{D}^{11}_{12} = \frac{1}{3\sqrt{2}}B^0_3 \tag{40}
$$

for the 20 $\tilde{D}^9_{11}$ with $11 > |b|$: 

$$
\tilde{D}^9_{11} = \tilde{D}^{10}_{11} = \tilde{D}^8_{11} = \tilde{D}^5_{11} = \tilde{D}^4_{11} = 0,
$$

$$
\tilde{D}^9_{11} = \frac{1}{3}B^1_4, \quad \tilde{D}^8_{11} = \frac{1}{3}B^1_3, \quad \tilde{D}^7_{11} = \frac{1}{3\sqrt{2}}B^1_0,
$$

$$
\tilde{D}^3_{11} = -\frac{1}{3\sqrt{2}}S^{+++}, \quad \tilde{D}^7_{11} = \frac{1}{3}B^4_2, \quad \tilde{D}^7_{11} = \frac{z^*}{3\sqrt{2}}S^{+++},
$$

$$
\tilde{D}^1_{11} = \frac{z}{3\sqrt{2}}S^{+++}, \quad \tilde{D}^2_{11} = \frac{1}{3}B^3_2, \quad \tilde{D}^5_{11} = -\frac{1}{3\sqrt{2}}S^{+++},
$$

$$
\tilde{D}^6_{11} = \frac{1}{3\sqrt{2}}B^2_0, \quad \tilde{D}^7_{11} = \frac{1}{3\sqrt{2}}S^{+++}, \quad \tilde{D}^8_{11} = -\frac{1}{3}B^4_2,
$$

$$
\tilde{D}^9_{11} = \frac{1}{3\sqrt{2}}B^2_0, \quad \tilde{D}^{10}_{11} = \frac{1}{3}B^4_3 \tag{41}
$$

for the 18 $\tilde{D}^{10}_{10}$ with $10 > |b|$: 

$$
\tilde{D}^{10}_{10} = \tilde{D}^9_{10} = \tilde{D}^8_{10} = \tilde{D}^5_{10} = \tilde{D}^3_{10} = \tilde{D}^6_{10} = 0,
$$

$$
\tilde{D}^8_{10} = \frac{1}{3}B^1_4, \quad \tilde{D}^5_{10} = \frac{1}{3\sqrt{2}}S^{+++}, \quad \tilde{D}^6_{10} = -\frac{1}{3}B^3_1
$$

$$
\tilde{D}^1_{10} = -\frac{z^*}{3\sqrt{2}}S^{+++}, \quad \tilde{D}^2_{10} = \frac{z}{3\sqrt{2}}S^{+++}, \quad \tilde{D}^3_{10} = \frac{1}{3}B^4_2,
$$

$$
\tilde{D}^4_{10} = -\frac{1}{3\sqrt{2}}S^{+++}, \quad \tilde{D}^7_{10} = \frac{1}{3\sqrt{2}}B^2_0, \quad \tilde{D}^8_{10} = \frac{1}{3\sqrt{2}}S^{+++},
$$

$$
\tilde{D}^9_{10} = \frac{1}{3}B^2_2, \quad \tilde{D}^{10}_{10} = \frac{1}{3\sqrt{2}}B^0_4 \tag{42}
$$

for the 16 $\tilde{D}^{11}_{9}$ with $9 > |b|$:  

$$
\tilde{D}^{11}_{9} = \tilde{D}^{10}_{9} = \tilde{D}^9_{9} = \tilde{D}^8_{9} = \tilde{D}^7_{9} = \tilde{D}^6_{9} = \tilde{D}^5_{9} = \tilde{D}^4_{9} = \tilde{D}^3_{9} = \tilde{D}^2_{9} = \tilde{D}^1_{9} = \tilde{D}^0_{9} = 0,
$$

$$
\tilde{D}^8_{9} = \frac{1}{3}B^1_4, \quad \tilde{D}^5_{9} = \frac{1}{3\sqrt{2}}S^{+++}, \quad \tilde{D}^6_{9} = -\frac{1}{3}B^3_1
$$

$$
\tilde{D}^1_{9} = -\frac{z^*}{3\sqrt{2}}S^{+++}, \quad \tilde{D}^2_{9} = \frac{z}{3\sqrt{2}}S^{+++}, \quad \tilde{D}^3_{9} = \frac{1}{3}B^4_2,
$$

$$
\tilde{D}^4_{9} = -\frac{1}{3\sqrt{2}}S^{+++}, \quad \tilde{D}^7_{9} = \frac{1}{3\sqrt{2}}B^2_0, \quad \tilde{D}^8_{9} = \frac{1}{3\sqrt{2}}S^{+++},
$$

$$
\tilde{D}^9_{9} = \frac{1}{3}B^2_2, \quad \tilde{D}^{10}_{9} = \frac{1}{3\sqrt{2}}B^0_4 \tag{42}
$$
\[ \tilde{D}_9^8 = \tilde{D}_9^3 = \tilde{D}_9^3 = \tilde{D}_9^5 = \tilde{D}_8^8 = 0, \]
\[ \tilde{D}_9^8 = -\frac{1}{3\sqrt{2}}B_0^1, \quad \tilde{D}_9^6 = \frac{1}{3}B_1^1, \quad \tilde{D}_9^5 = -\frac{1}{3\sqrt{2}}S^{++-}, \]
\[ \tilde{D}_9^4 = \frac{1}{3}B_1^1, \quad \tilde{D}_9^3 = \frac{1}{3\sqrt{2}}S^{+++}, \quad \tilde{D}_9^1 = -\frac{z}{3\sqrt{2}}S^{+++}, \]
\[ \tilde{D}_9 = -\frac{z^*}{3\sqrt{2}}S^{+++}, \quad \tilde{D}_9^2 = \frac{1}{3\sqrt{2}}B_2^0, \quad \tilde{D}_9^4 = \frac{1}{3}B_2^4, \]
\[ \tilde{D}_9^5 = \frac{1}{3}B_3^3, \quad \tilde{D}_9^7 = \frac{1}{3\sqrt{2}}S^{++++} \] (43)

for the 14 \( \tilde{D}_8^6 \) with 8 > |\( b \)|:
\[ \tilde{D}_8^6 = \tilde{D}_8^5 = \tilde{D}_8^4 = \tilde{D}_8^3 = \tilde{D}_8^7 = 0, \]
\[ \tilde{D}_8^7 = -\frac{1}{3\sqrt{2}}S^{+++}, \quad \tilde{D}_8^6 = \frac{1}{3}B_1^2, \quad \tilde{D}_8^5 = -\frac{z^*}{3\sqrt{2}}S^{+++}, \]
\[ \tilde{D}_8^1 = -\frac{z}{3\sqrt{2}}S^{+++}, \quad \tilde{D}_8^2 = \frac{1}{3}B_3^3, \quad \tilde{D}_8^5 = -\frac{1}{3\sqrt{2}}S^{+++}, \]
\[ \tilde{D}_8^4 = -\frac{1}{3\sqrt{2}}B_3^0, \quad \tilde{D}_8^6 = \frac{1}{3\sqrt{2}}S^{+++}, \quad \tilde{D}_8^5 = \frac{1}{3\sqrt{2}}B_4^0 \] (44)

for the 12 \( \tilde{D}_7^2 \) with 7 > |\( b \)|:
\[ \tilde{D}_7^2 = \tilde{D}_7^4 = \tilde{D}_7^6 = \tilde{D}_7^5 = 0, \quad \tilde{D}_7^6 = -\frac{1}{3\sqrt{2}}S^{+++}, \quad \tilde{D}_7^5 = \frac{1}{3}B_3^3, \]
\[ \tilde{D}_7^4 = \frac{1}{3\sqrt{2}}S^{+++}, \quad \tilde{D}_7^3 = \frac{1}{3}B_2^2, \quad \tilde{D}_7^1 = -\frac{1}{3\sqrt{2}}S^{+++}, \]
\[ \tilde{D}_7^1 = \tilde{D}_7^1 = \frac{1}{3\sqrt{2}}B_2^0, \quad \tilde{D}_7^3 = \frac{1}{3}B_2^4, \quad \tilde{D}_7^5 = \frac{1}{3}B_2^3 \] (45)

for the 10 \( \tilde{D}_6^5 \) with 6 > |\( b \)|:
\[ \tilde{D}_6^5 = \tilde{D}_6^4 = \tilde{D}_6^3 = \tilde{D}_6^7 = 0, \quad \tilde{D}_6^4 = -\frac{1}{3}B_1^2, \quad \tilde{D}_6^3 = \frac{1}{3\sqrt{2}}S^{+++}, \]
\[ \tilde{D}_6^1 = -\frac{z}{3\sqrt{2}}S^{+++}, \quad \tilde{D}_6^2 = \frac{1}{3\sqrt{2}}S^{+++}, \quad \tilde{D}_6^2 = \frac{1}{3\sqrt{2}}B_3^0, \]
\[ \tilde{D}_6^4 = \frac{1}{3}B_3^4, \quad \tilde{D}_6^5 = \frac{1}{3\sqrt{2}}S^{++++} \] (46)

for the 8 \( \tilde{D}_5^8 \) with 5 > |\( b \)|:
\[ \tilde{D}_5^8 = \tilde{D}_5^4 = \tilde{D}_5^3 = \tilde{D}_5^7 = 0, \quad \tilde{D}_5^4 = \frac{1}{3\sqrt{2}}S^{+++}, \quad \tilde{D}_5^3 = \frac{1}{3}B_3^3, \]
\[ \tilde{D}_5^2 = -\frac{1}{3\sqrt{2}}S^{+++}, \quad \tilde{D}_5^3 = \tilde{D}_5^3 = \frac{1}{3\sqrt{2}}B_3^0, \]
\[ \tilde{D}_5^3 = \frac{1}{3}B_3^3 \] (47)
for the 6 $\tilde{D}_b^i$ with $4 > |b|$:  

\[ \tilde{D}_4^0 = \tilde{D}_4^3 = 0, \quad \tilde{D}_4^1 = - \frac{z}{3\sqrt{2}} S^{++-}, \quad \tilde{D}_4^1 = - \frac{z^*}{3\sqrt{2}} S^{++-}, \]

\[ \tilde{D}_4^2 = \frac{1}{3\sqrt{2}} B_4^0, \quad \tilde{D}_4^2 = \frac{1}{3\sqrt{2}} S^{+++} \] (48)

for the 4 $\tilde{D}_b^i$ with $3 > |b|$:  

\[ \tilde{D}_3^2 = 0, \quad \tilde{D}_3^2 = - \frac{1}{3\sqrt{2}} S^{+++}, \quad \tilde{D}_3^1 = \tilde{D}_3^1 = \frac{1}{3\sqrt{2}} B_4^0 \] (49)

and finally for the two $\tilde{D}_2^i$ with $2 > |b|$:  

\[ \tilde{D}_2^1 = \frac{z^*}{3\sqrt{2}} S^{+++}, \quad \tilde{D}_2^1 = \frac{z}{3\sqrt{2}} S^{+++} \] (50)

Lastly the 156 $\tilde{D}_b^i$ with $a < b$ are obtained from the results above by hermitian conjugation:  

\[ \tilde{D}_b^a = \tilde{D}_a^b, \quad B_b^a = B_a^b, \quad S^{pqrs} = S^{pqrst} \] (51)

This completes the calculation of the Casimir operators of $F_4$.

**CONCLUSION**

I conclude with two remarks,

1. For $k = 2$ the result of inserting the explicit formulas for the projected $\tilde{D}$, Eqs. (35), (39–51), into Eq. (6) can be simplified into the following formula for the quadratic Casimir operator of $F_4$:  

\[ C_2(F_4) = \tilde{D}_b^a \tilde{D}_b^a = \frac{1}{3} B_\alpha^\beta B_\beta^\alpha + \frac{2}{3} S^{pqrs} S^{pqrs} \] (52)

and I remind the reader that the various subscripts are summed over the following range: $-13 \leq a, b \leq 13$ (zero excluded); $-4 \leq \alpha, \beta \leq 4$ (zero included); $p, q, r, s = \pm$.

The general form of this result for the quadratic Casimir of $F_4$ in the $B_4$ basis was to be expected since the two pieces in Eq. (52) are the only quadratic invariants of the subgroup $B_4$ that can be formed out of the adjoint 36 and the spinor 16 of $B_4$. Thus this result can be viewed as a test of the formalism.
2. Recall that according to Eq. (4) the independent Casimirs are of degree \( k = 2s, \quad 1 \leq s \leq 13. \) Now consider the Cartan part of the Casimirs. If I denote the Cartan part of \( C_k(F_4) \) by \( K_k \) then it follows from Eq. (6) that
\[
K_k = \left( \tilde{D}_a^a \right)^k
\]
(53)

Since \( \tilde{D}_a^a = -\tilde{D}_a^a \) this is manifestly zero for \( k = \) odd. For \( k = \) even Eq. (53) becomes (where I have set \( b_\alpha \equiv B_\alpha^\alpha, \) no summation)
\[
K_k = 2 \sum_{a=1}^{13} \left( \tilde{D}_a^a \right)^k = 2 \cdot 6^{-k} \left\{ (b_1 + b_2 + b_3 + b_4)^k + (b_1 + b_2 + b_3 - b_4)^k + (2b_3)^k \\
+ (b_1 + b_2 - b_3 + b_4)^k + (2b_2)^k + (b_1 + b_2 - b_3 - b_4)^k + (b_1 - b_2 + b_3 + b_4)^k \\
+ (b_1 - b_2 + b_3 - b_4)^k + (b_1 - b_2 - b_3 + b_4)^k + (b_1 - b_2 - b_3 - b_4)^k + (2b_1)^k \right\}
\]
(54)

For \( k = 2 \) Eq. (54) gives
\[
K_2 = \frac{2}{3} \left( b_2^2 + b_2^2 + b_3^2 + b_4^2 \right)
\]
(55)
while for \( k = 4 \) it gives
\[
K_4 = 3^{-3} \left( b_1^2 + b_2^2 + b_3^2 + b_4^2 \right)^2
\]
(56)

This proves that the degree 4 Casimir is not functionally independent of the degree 2 Casimir.

For \( k = 6 \) Eq. (54) gives
\[
K_6 = 2^{-2} \cdot 3^{-5} \left\{ 3 \left[ b_1^6 + b_2^6 + b_3^6 + b_4^6 \right] + 5 \left[ b_1^4 \left( b_2^2 + b_3^2 + b_4^2 \right) + b_2^2 \left( b_1^2 + b_3^2 + b_4^2 \right) \\
+ b_3^2 \left( b_1^2 + b_2^2 + b_4^2 \right) + b_4^2 \left( b_1^2 + b_2^2 + b_3^2 \right) \right] \\
+ 30 \left[ b_1^2 \left( b_2^2 b_3^2 + b_2^2 b_4^2 + b_3^2 b_4^2 \right) + b_2^2 b_3^2 b_4^2 \right] \right\}
\]
(57)

which is functionally independent of the degree 2 Casimir (were it proportional to the cube of the degree 2 Casimir it would have the coefficient of the expression in the first, second and third square bracket in the ratio 1:3:6 instead of the 3:5:30 above).
Continuing along these lines I find that the degree 8 is functionally independent of the
degree 2 and 6, while the degree 10 is dependent:

\[ \mathcal{K}_{10} \sim \mathcal{K}_2 \left\{28\mathcal{K}_2(\mathcal{K}_2^3 - \mathcal{K}_6) + 3\mathcal{K}_8 \right\} \]  \hspace{1cm} (58)

and lastly the degree 12 is independent of those of lower degree. Since all the Casimirs are functions of the four quantities \( b_\alpha^2, 1 \leq \alpha \leq 4 \), I can solve for the \( b_\alpha^2 \) in terms of the independent Casimirs of degree 2, 6, 8 and 12, and consequently all Casimirs of higher degree are necessarily dependent. This completes the demonstration that the independent Casimirs are those of degree equal to the exponents of \( F_4 \) plus one.

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