Driven Quantum Cyclotron with One Electron or Positron

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The first quantum calculation is presented for a harmonic detection oscillator coupled to a one-particle quantum cyclotron – one trapped electron or positron that occupies only its lowest cyclotron and spin states. The calculation is used to investigate new measurement methods that could circumvent and minimize the detector backaction that limited past measurements of these moments. New methods that allow new measurements are urgently needed because there is now an intriguing 2.4 standard deviation discrepancy between the most precise prediction of the standard model of particle physics, and the most accurate measurement of a property of an elementary particle.

I. MOTIVATION AND OVERVIEW

An intriguing 2.4 standard deviation discrepancy\textsuperscript{[1]} has arisen between the Standard Model’s most precise prediction and the measured value (Fig. 1). Measurements now\textsuperscript{[2–4]} determine the electron magnetic moment in Bohr magnetons (\(\frac{\mu}{\mu_B}\)) to 3 parts in \(10^{13}\) – the most precisely determined property of an elementary particle. The SM prediction requires Dirac theory, quantum electrodynamics, hadronic and weak interaction contributions\textsuperscript{[5]}. The theory requires as input an independently measured value of the fine structure constant, currently provided by the measured Rydberg constant\textsuperscript{[6–8]}, measured mass ratios\textsuperscript{[9, 10]} and a measured atom recoils of Rb\textsuperscript{[11]} and Cs\textsuperscript{[12]}. The part in \(10^{12}\) agreement between SM prediction and measurement that stood for years gave way as a result of a more precise measurement of the latter. The discrepancy triggered new theoretical investigations into possible physics beyond the SM\textsuperscript{[13–17]}.

A one-particle, quantum cyclotron is at the heart of the measurements. A single electron, suspended indefinitely in a Penning trap, is cooled enough so that it initially occupies only one of the two stable cyclotron ground states, one with spin down and one with spin up (Fig. 2). Transitions are driven between these states and a third – the first excited cyclotron state with spin down. The state of the quantum cyclotron is detected after the drives are turned off using quantum jump spectroscopy. The angular cyclotron and anomaly drive frequencies, \(\omega_c\) and \(\omega_a\), that produce one-quantum transitions, determine the magnetic moment in Bohr magnetons,

\[
\pm \frac{\mu}{\mu_B} = 1 + \frac{\omega_a}{\omega_c} = \frac{g}{2}.
\]

The plus and minus signs are for the positron and electron, and the \(g\) value divided by 2 is another name for the ratio of moments. The frequency \(\omega_c\) is the electron cyclotron frequency. The anomaly frequency \(\omega_a = \omega_s - \omega_c\) is the difference between the electron spin precession frequency \(\omega_s\) and its cyclotron frequency.

The measurement sequence considered here starts with preparing the quantum cyclotron in the desired cyclotron ground state. A cyclotron or anomaly drive is applied to make one-quantum transitions. The time evolution of the system during this time is calculated in this work. After the drive is turned off, the state of the quantum cyclotron is measured to determine whether the drive produced a transition. This state measurement is well understood\textsuperscript{[18]}. Repeated trials determine the probability of making a quantum jump as a function of the drive frequencies. The cyclotron and anomaly frequencies are determined from these resonance lineshapes.

Determining the state of the quantum cyclotron requires coupling it to a detector. The quantum nondemolition (QND) coupling investigated here couples the quantum cyclotron to a harmonic detection motion whose oscillation frequency can be measured precisely. This is the axial oscillation of the trapped electron along the direction of the magnetic field of the Penning trap. A small “magnetic bottle” gradient\textsuperscript{[19]} produces the coupling. It measurably shifts the axial frequency when the cyclotron or spin energy changes. This is a QND coupling in that repeatedly detecting the cyclotron and spin state via axial frequency shifts does not change the cyclotron and spin state.

The cost of the QND coupling is an unavoidable detector backaction that arises during the time that the drives

\[\text{ppt} = 10^{-12}\]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Comparison of the measured electron magnetic moment\textsuperscript{[3]} with the standard model predictions\textsuperscript{[5]}.}
\end{figure}

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are applied. It makes the drive frequencies at which
driven cyclotron and anomaly transitions are observed
depend upon the energy of the axial, detection oscilla-
tion. The undesired backaction turns any distribution
of axial oscillator states into a broadening of cyclotron
and anomaly lineshapes. When the cyclotron and
anomaly drives are on, the axial states are kept as
close to their ground state as possible. Nonetheless,
the broadening that remains makes it difficult to accurately
extract the needed cyclotron and anomaly frequencies.
This significant challenge has limited the precision and
accuracy of measurements.

The detection backaction studied here pertains to the
time evolution of the quantum cyclotron during the time
that the driving forces are applied, before the detection
of the state of the quantum cyclotron. The study requires
and relies upon the first solution to a quantum master
equation for the combined cyclotron-spin-axial system.
The axial detection motion is quantized and coupled to
a thermal reservoir. Steady-state solutions to the master
equation are found for weak driving forces. The master
equation is also integrated directly (by simultaneously
solving 450 differential equations) to investigate the ap-
proach to the steady states, and to investigate the effect
of stronger driving forces. Previous calculations \[20, 21\]
also found a steady state for a weak drive, but did so by
assuming the detection motion was a classical oscillation
undergoing Brownian motion. The quantum calculation
is needed to describe the lower temperatures and reduced
damping that now seem to be possible in experiments.

Resonance lineshapes that are needed to determine
cyclotron and anomaly frequencies are predicted to ex-
clude the possibility of new methods and measurements
\[1\]. The dependence of cyclotron and anomaly lineshapes
upon the temperature, coupling strength, and two damping
rates are studied to identify promising measurement
possibilities. One important prediction (announced in
\[22\]), is discussed here in detail. It is the possibility of
resolving the cyclotron excitations that will occur dur-
ing the time that the electron spends in its lowest axial
quantum state, from those that occur while the electron
spends more time in excited axial states. The summary
conclusion is that electron and positron magnetic mo-
moment measurements with ten times lower uncertainties
seem possible.

Details of the quantum system are given in Sec. \[II\]. The
Hamiltonian of the system and the master equation of
the system is presented in Sec. \[III\]. Calculations of single
photon excitations of cyclotron and anomaly transition
are given in Sections \[IV\] and \[V\] respectively. Sec. \[VI\] uses
the quantum calculation to predict and new measurement
possibilities.

II. QUANTUM CYCLOTRON

Before we turn to the calculation that is the heart of
this work, we briefly introduce the one-electron quantum
cyclotron that would be at the heart of the approach be-
ing investigated here. An electron or positron in a Pen-
ning trap is confined within a spatially uniform magnetic
field \(B_0\), along with an electrostatic quadrupole po-
tential. The possibility to use only the ground and first
excited cyclotron states of a single isolated electron has
already been demonstrated \[3\]. The two lowest levels of
the quantum cyclotron are separated by an energy \(\hbar\omega_c\),
where \(\omega_c\) is the angular cyclotron frequency discussed
above. The spin up (quantum number \(m_s = 1/2\)) and
spin down (\(m_s = -1/2\)) states are separated in energy
by \(\hbar\omega_s\), where \(\omega_s\) is the spin precession frequency
introduced above. This one-particle quantum cyclotron has a
Hamiltonian
\[
H = \frac{1}{2}\hbar\omega_s (a_s^\dagger a_s - a_s a_s^\dagger) + \hbar\omega_c(a_c^\dagger a_c + \frac{1}{2}).
\]
(2)

The spin raising and lowering operators are
\[
a_s^\dagger |\uparrow\rangle = |\uparrow\rangle,
\]
(3)
and \(a_s^\dagger\) and \(a_s\) are harmonic raising and lowering oper-
ators for the cyclotron motion \[24\].

An electrostatic quadrupole potential added to the
magnetic field makes a Penning trap that can hold an
electron or positron indefinitely when the vacuum is good
enough \[25\]. The electron (of charge \(-e\) and mass \(m\)) osci-
lates along the magnetic field direction in a harmonic
oscillator potential energy,
\[
W(z) = \frac{1}{2}m\omega_z^2 z^2
\]
(4)
and \(\omega_z\) is the angular axial oscillation frequency. The
electrostatic quadrupole shifts the cyclotron frequency
slightly in a well understood way \[24, 26\] that can be
neglected for the purposes of this calculation.

This axial motion is used to make quantum nondemo-
lition (QND) measurements of one-quantum spin and cy-
clotron transitions. To accomplish this, a small magnetic
torque for the cyclotron \[24\], is added to the spatially uniform
magnetic field, \(B_0\), of the Penning trap. The cyclotron
and spin frequencies in Eq. \[3\] both acquire a small \(z^2\)
dependence,
\[
\begin{align*}
\omega_c(z) &= \omega_c + \frac{eB_2}{m} z^2 \\
\omega_s(z) &= \omega_s + \frac{g eB_2}{m} z^2
\end{align*}
\]
(5)
(6)
The addition modifies the axial trapping potential and
shifts the frequency of the axial oscillation. A QND de-
tection of a one-quantum cyclotron excitation is possible
because it shifts the axial frequency from \(\omega_z\) to \(\omega_z + \delta_c\),
with
\[
\delta_c = \frac{eB_2}{m} \frac{\hbar}{m\omega_z} \approx 2\pi \times (3 \text{ Hz})
\]
(7)
\[24\], without changing the cyclotron state. The shift is
just large enough to be detectable. The relative shift is
\(\delta_c/\omega_z = 1.5 \times 10^{-8}\) for demonstrated experimental values
\[ B_2 = 1500 \text{ T/m}^2 \text{ and } \omega_z/(2\pi) = 200 \text{ MHz}. \]

This bottle shift can be decreased in two ways – by decreasing the magnetic gradient \( B_2 \) or by increasing the axial frequency, \( \omega_z \). Since a next generation experiment [1] uses \( B_2 = 660 \text{ T/m}^2 \), we choose the intermediate value \( B_2 = 1200 \text{ T/m}^2 \) for the illustrations in this paper.

The magnetic gradient is unfortunately also responsible for a backaction that broadens the range of frequencies over which a cyclotron excitation can occur. A one quantum axial excitation within the magnetic bottle gradient shifts the cyclotron frequency by the same \( \delta_c \). A thermal distribution over \( n_z \) axial states (Eq. [22]) thus makes the cyclotron frequency fluctuate over a spread of frequencies that is of order \( n_z \delta_c \).

Two relativistic shifts must be mentioned, both arising from the “relativistic mass increase.” The largest,

\[ \delta_r = \frac{\hbar \omega_c}{mc^2} \approx -2\pi \times (180 \text{ Hz}). \tag{8} \]

It is only a 1 part in \( 10^9 \) shift of the cyclotron frequency per cyclotron quantum, but it is a large shift compared to the experimental precision that can be attained. The cyclotron frequency between the ground and first excited cyclotron states with spin down shift by half of this amount because of the zero-point energy of the cyclotron motion. The shift is thus extremely important, but for the purposes of this calculation it can simply be absorbed into \( \omega_c \). The second relativistic shift,

\[ \delta_{cr} = \frac{\hbar \omega_c}{2mc^2\omega_z} \approx -2\pi \times (0.12 \text{ Hz}), \tag{9} \]

is about 1000 times smaller. It comes from coupling the cyclotron frequency to the axial energy. This coupling has much the same effect in coupling the motions to allow QND detection as does a magnetic bottle [24]. It also produces a corresponding backaction. This relativistic coupling is neglected here because it is 25 times smaller than the coupling caused by the magnetic bottle gradient considered above.

A spin flip shifts the angular axial frequency by \( \delta_s = (g/2)\delta_c \). This is nearly the same size as the corresponding cyclotron frequency shift because \( g/2 \) differs from 1 by only a part in 1000, and experiments are not able to resolve these two shifts from each other. The frequency difference \( \omega_a = \omega_s - \omega_c \) is measured, rather than measuring \( \omega_s \)

\[ \approx \omega_a \approx \omega_c \approx \omega_s. \]

Accordingly, the thousand times smaller difference of the small shifts, \( \delta_a = \delta_s - \delta_c \), is important in the result of this calculation.

Table [I] gives the typical trapped electron frequencies, damping rates, and quantum number used in this calculation. The spin and cyclotron frequencies are for an electron in a \( B = 5.3 \text{ T} \) magnetic field. The cyclotron damping rate is for the first excited state radiating spontaneous emission to return to the ground state, \( n_z = 0 \). The radiation rate is modified by the resonant modes of a cylindrical trap cavity [4, 27, 28]. The spin up cyclotron ground state radiates with a time constant so long that it is essentially stable.

The axial frequency depends upon the trap size and the applied trapping potential [29, 30]. Its damping rate depends upon the quality factor and inductive reactance of the damping and detection circuit to which it is coupled [31]. The maximal radiation rate in the table applies during particle detection. For this calculation, we assume that this rate is electronically reduced by a factor of more than 100 during the time that spin and cyclotron transitions are driven. The average quantum number is for thermal equilibrium with a circuit kept at 0.1 K by a dilution refrigerator [3].

The magnetron orbit of a trapped particle is important experimentally but not for this calculation. It is a motion at a much lower frequency. The average quantum number in the table pertains for the sideband cooling limit [24], and its radiation damping rate is completely negligible. The broadening due to magnetron motion is smaller than that due to axial motion by a factor of \( \omega_m/\omega_z \approx 1/1000 \), and we drop the magnetron motion term to simplify the calculation.

### Table I. Typical frequencies, damping rates, and quantum number of a particle in a Penning trap [3]

| Angular frequency or rate | Frequency (Hz) | Damping time (s) | Quantum number |
|---------------------------|----------------|-----------------|----------------|
| spin                      | \( \omega_s/2\pi \approx 148.5 \text{ GHz} \) | \( \gamma_s^{-1} \approx 10^6 \text{ s} \) | \( n_s = \pm \frac{1}{2} \) |
| cyclotron                 | \( \omega_c/2\pi \approx 148.3 \text{ GHz} \) | \( \gamma_c^{-1} \approx 5 \text{ s} \) | \( n_c = 0 \) |
| axial                     | \( \omega_a/2\pi \approx 200 \text{ MHz} \) | \( \gamma_a^{-1} \approx 0.2 \text{ s} \) | \( n_a = 10 \) |
| magnetron anomaly         | \( \omega_m/2\pi \approx 133 \text{ kHz} \) | \( \gamma_m^{-1} \approx 10^{17} \text{ s} \) | \( n_m = 10 \) |

### Table II. Hierarchy of angular frequencies and rates that are in reach for a new generations of measurements. The numerical values are frequencies in Hz and times in seconds, with \( \delta_s/2\pi = 0.003 \text{ Hz} \) and \( \delta_a^{-1} = 60 \text{ s} \), for example. The \( \gamma_c \) is the smallest value that it can be reduced to in current experiments, as described in the text.

| Angular frequency or rate | Frequency (Hz) | Time constant (s) |
|---------------------------|----------------|------------------|
| \( \delta_s \)            | 0.003          | 60               |
| \( \gamma_c \)            | 0.003          | 60               |
| \( n_s \delta_s \)        | 0.03           | 6                |
| \( \gamma_a \)            | 0.03           | 6                |
| \( n_a \gamma_a \)        | 0.03           | 6                |
| \( \delta_c \)            | 3              | 0.06             |
| \( n_z \delta_c \)        | 30             | 0.006            |

One motivation for this calculation is exploring the new experimental possibility to greatly reduce the axial damping rate while cyclotron and anomaly transitions
are driven. The rate can be electronically switched to the low value in the table just before drives are applied, to make one-quantum anomaly and cyclotron transitions with an electron largely uncoupled from the bath. After the drives are turned off, the damping rate can be electronically switched to a much larger values, as needed to detect the particle state and to damp the axial motion.

III. HAMILTONIAN

The basic Hamiltonian for the quantum cyclotron,

$$H_0 = \hbar \omega_s \left(a_c^\dagger a_c - \frac{1}{2}\right) + \hbar \omega_c \left(a_s^\dagger a_s + \frac{1}{2}\right) + \hbar \omega_z \left(a_z^\dagger a_z + \frac{1}{2}\right) \tag{10}$$

is the sum of independent spin, cyclotron and axial terms. From Eq. (10), a magnetic bottle gradient adds the Hamiltonian term

$$V = \frac{b}{2} \left[\delta_c \left(a_c^\dagger a_c + \frac{1}{2}\right) + \delta_s \left(a_s^\dagger a_s - \frac{1}{2}\right)\right] (a_c^\dagger + a_c)^2. \tag{11}$$

Both terms are written entirely in terms of raising and lowering operators for the spin ($a_s^\dagger$ and $a_s$), cyclotron ($a_c^\dagger$ and $a_c$) and axial ($a_z^\dagger$ and $a_z$) motions. These operators and their relationship to the position and momentum operators are discussed in Ref. [24]. Contributions smaller by order $\omega_z/\omega_c$ are neglected. The magnetron motion of a particle in a Penning trap is neglected on the assumption that it can be cooled to a small radius that does not change during a measurement.

The eigenstates for $H_0$ are direct products of independent cyclotron, spin and axial eigenstates, designated by $|n_c, m_s, n_z\rangle$. These remain the energy eigenstates when the magnetic bottle is added. The energy eigenvalues are

$$E(n_c, m_s, n_z) = \hbar \omega_c \left(n_c + \frac{1}{2}\right) + \hbar \omega_c m_s + \hbar \omega_z \left(n_z + \frac{1}{2}\right) + \hbar \delta_s \left(n_c + \frac{1}{2}\right) + \hbar \delta_s m_s \left(n_z + \frac{1}{2}\right). \tag{12}$$

The magnetic bottle coupling terms are the two terms that depend upon two quantum numbers.

The magnetic bottle (Eq. (10)) is essential for detecting the quantum spin and cyclotron state of the electron. It couples the axial frequency to the cyclotron and spin energy. From Eq. (12) we see that small but observable axial frequency shifts,

$$\Delta\omega_z = (n_c + \frac{1}{2})\delta_c + m_s\delta_s, \tag{13}$$

reveal changes in the cyclotron and spin quantum numbers. The axial motion thus provides QND detection that does not itself change the state of these states.

A detector backaction is a parallel and unavoidable consequence of the QND coupling. Besides enabling detection, the magnetic bottle also couples the axial energy to the cyclotron and spin frequencies, and hence to the anomaly frequency. The simple reason is that axial motion through the magnetic bottle gradient changes the magnetic field in which the cyclotron and spin motions evolve. The resulting shifts,

$$\Delta\omega_c = (n_z + \frac{1}{2})\delta_c \tag{14}$$

$$\Delta\omega_a = (n_z + \frac{1}{2})\delta_a, \tag{15}$$

are a consequence of Eq. (12). Even for completely cooled axial motion, with $n_z = 0$, there is a detection backaction shift of both $\omega_c$ and $\omega_a$ due to the zero-point axial motion. For a thermal distribution of axial states, this detection backaction significantly broadens the observed cyclotron and anomaly resonances, making this calculation necessary. Any distribution of axial states thus spreads out the range of cyclotron (or anomaly) frequencies at which a cyclotron (or anomaly) drive causes one-quantum transitions.

Switching from the Schrödinger picture to the interaction picture transforms away the well-understood spin, cyclotron and axial motions in the absence of a magnetic bottle. Terms that go as $a_z a_z$ and $a_z^\dagger a_z^\dagger$ oscillate rapidly and hence average to zero in the interaction picture. The interaction Hamiltonian $\tilde{V} = e^{iH_0t/\hbar}Ve^{-iH_0t/\hbar}$ is

$$\tilde{V} = \left[h\delta_s \left(a_z^\dagger a_s - \frac{1}{2}\right) + h\delta_c \left(a_z^\dagger a_c + \frac{1}{2}\right)\right] \times \left(a_z^\dagger a_z + \frac{1}{2}\right) \tag{16}.$$

We continue using the time-independent raising and lowering operators from the Schrödinger picture (rather than transforming these to the interaction picture). The interaction picture Hamiltonian has an energy scale set by the tiny bottle shifts, $\delta_c$ and $\delta_a$, rather than by the much larger frequencies $\omega_c$, $\omega_a$ and $\omega_z$.

Figure 2 represents the lowest of these quantum energy levels, with spin down states ($m_s = -1/2$) on the left and spin up states ($m_s = 1/2$) on the right. The lowest of the
infinite ladder of cyclotron states are shown \((n_c = 0, 1)\), as are the lowest three of the infinite ladder of axial states \((n_s = 0, 1, 2)\). For the driving forces we will consider, the electron will essentially occupy only the three cyclotron and spin state combinations

\[
|1, n_z\rangle \approx |n_c = 0, m_s = -\frac{1}{2}, n_z\rangle, \\
|2, n_z\rangle \approx |n_c = 1, m_s = -\frac{1}{2}, n_z\rangle, \\
|3, n_z\rangle \approx |n_c = 0, m_s = +\frac{1}{2}, n_z\rangle, \tag{17}
\]

with \(n_s = 0, 1, \ldots\). These are the basis of time-independent states used for this calculation. The basis is only three states if the axial motion is cooled to its quantum ground state.

Electromagnetic drives that oscillate at angular frequency \(\omega_c + \epsilon_c\) to excite cyclotron transitions, and at angular frequency \(\omega_a + \epsilon_a\) to make anomaly transitions are descried by the Hamiltonians

\[
\begin{align*}
V_c(t) &= \frac{1}{2} \hbar \Omega_c \left[ a_c^\dagger e^{-i(\omega_c + \epsilon_c)t} + a_c e^{i(\omega_c + \epsilon_c)t} \right] \\
V_a(t) &= \frac{1}{2} \hbar \Omega_a \left[ a_a^\dagger e^{-i(\omega_a + \epsilon_a)t} + a_a e^{i(\omega_a + \epsilon_a)t} \right]. \tag{18}
\end{align*}
\]

The Rabi frequencies \(\Omega_c\) and \(\Omega_a\) quantify the drive strengths, and \(\epsilon_c\) and \(\epsilon_a\) are detunings of the drives from resonance. In the interaction picture the drive Hamiltonians are

\[
\begin{align*}
\tilde{V}_c(t) &= \frac{1}{2} \hbar \Omega_c \left[ a_c^\dagger e^{i\gamma_c t} + a_c e^{i\gamma_c t} \right] \\
\tilde{V}_a(t) &= \frac{1}{2} \hbar \Omega_a \left[ a_a^\dagger e^{i\gamma_a t} + a_a e^{i\gamma_a t} \right]. \tag{19}
\end{align*}
\]

An anomaly transition is a simultaneous cyclotron and spin transition. The raising operator for an anomaly transition from \(|2, n_z\rangle\) to \(|3, n_z\rangle\), for example, requires \(a_a^\dagger = a_a^\dagger a_c\), a lowering of the cyclotron state followed by a raising of the spin state. A transition from the spin down ground state to the spin up ground state is accomplished by \(a_a^\dagger a_c^\dagger\).

The axial and cyclotron motions are both coupled to a thermal bath, with damping rates of \(\gamma_z\) and \(\gamma_c\), respectively. An ambient bath temperature of 0.1 K is assumed because it has been demonstrated in experiments [4]. The energy for a one-quantum axial excitation, \(\hbar \omega_z / k_B = 0.01\) K in temperature units, is instead much smaller than 0.1 K. The axial state is thus a Boltzmann distribution with an average quantum number

\[
\bar{n}_z = \left[ \exp \left( \frac{\hbar \omega_z}{k_B T} \right) - 1 \right]^{-1} \approx \frac{k_B T}{\hbar \omega_z} \approx 10. \tag{20}
\]

It may be possible to cool this motion further using cavity sideband cooling [24], but this is not assumed here. A cyclotron excitation requires an energy of \(\hbar \omega_c / k_B = 7.1\) K that is much larger than the 0.1 K bath temperature. The result is that

\[
\bar{n}_c = \left[ \exp \left( \frac{\hbar \omega_c}{k_B T} \right) - 1 \right]^{-1} = 1.2 \times 10^{-32} \approx 0. \tag{21}
\]

The cyclotron motion essentially remains in its \(n_c = 0\) ground state unless an excitation drive is applied.

For an electron or positron coupled to a thermal bath, a density operator must be used. The density operator in the Schrödinger picture, \(\rho\), and the interaction picture, \(\tilde{\rho}\) are related by

\[
\tilde{\rho} = e^{iH_0 t / \hbar} \rho e^{-iH_0 t / \hbar}. \tag{24}
\]

Both \(\rho\) and \(\tilde{\rho}\) can be expanded in the infinite base of time-independent states in Eq. (17). The diagonal elements are the probabilities to be in each basis state. These are invariant under a change between the Schrödinger and interaction pictures. Also invariant are the traces,

\[
P_l = \sum_{n_z=0}^{\infty} \langle l, n_z | \rho | l, n_z \rangle = \sum_{n_z=0}^{\infty} \langle l, n_z | \tilde{\rho} | l, n_z \rangle, \tag{25}
\]

that are the total probabilities to be in the 3 spin and cyclotron states.

The Schrödinger picture density operator, \(\rho\), evolves in time as described by a Lindblad master equation [33, 34],

\[
\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_0 + V_c + V_a, \rho] - \frac{\gamma_c}{2} (a_c^\dagger a_c \rho - 2a_c \rho a_c^\dagger + \rho a_c^\dagger a_c) \\
- \frac{\gamma_a}{2} (a_a^\dagger a_a \rho - 2a_a \rho a_a^\dagger + \rho a_a^\dagger a_a) \\
- \frac{\gamma_z}{2} (\bar{n}_z + 1) (a_a^\dagger a_a \rho - 2a_a \rho a_a^\dagger + \rho a_a^\dagger a_a). \tag{26}
\]

The coherent time evolution is described by the commutator term. The incoherent spontaneous emission from the cyclotron motion (from the first excited cyclotron state to its ground state) is described by the nonlinear terms in line two. (As noted earlier, the heating of the cyclotron motion by the thermal black-body radiation for low temperature surroundings can be neglected.) The coupling of the axial motion and the thermal bath is described by the last two lines. The bath temperatures come in via the average axial quantum number \(\bar{n}_z\) of Eq. (22).

The interaction picture density operator, \(\tilde{\rho}\), evolves as

\[
\frac{d\tilde{\rho}}{dt} = -\frac{i}{\hbar} \left[ \tilde{V} + \tilde{V}_c + \tilde{V}_a, \tilde{\rho} \right] - \frac{\gamma_c}{2} (a_c^\dagger a_c \tilde{\rho} - 2a_c \tilde{\rho} a_c^\dagger + \tilde{\rho} a_c^\dagger a_c) \\
- \frac{\gamma_a}{2} (a_a^\dagger a_a \tilde{\rho} - 2a_a \tilde{\rho} a_a^\dagger + \tilde{\rho} a_a^\dagger a_a) \\
- \frac{\gamma_z}{2} (\bar{n}_z + 1) (a_a^\dagger a_a \tilde{\rho} - 2a_a \tilde{\rho} a_a^\dagger + \tilde{\rho} a_a^\dagger a_a). \tag{27}
\]

As for the Hamiltonian, we use the time-independent, raising and lowering operators. The damping terms transform to have the same form in both pictures. Explicit calculation are done using the interaction picture because it is simpler. \(H_0\) is removed, \(\tilde{V}\) is much smaller than \(V\), and \(\tilde{V}_c + \tilde{V}_a\) varies much less rapidly in time than do \(V_c + V_a\).
IV. DRIVEN CYCLOTRON EXCITATIONS

A. Cyclotron Master Equation

The initial state is a thermal distribution of spin down, cyclotron ground states, \( |1, n_z \rangle \). A weak cyclotron drive, \( V_c, \) produces excited cyclotron states \( |2, n_z \rangle \). This drive provides no mechanism to flip the spin, so the states \( |3, n_z \rangle \) are not populated. For a weak drive, \( \Omega_c \ll \gamma_c \), the probability of a cyclotron excitation is very small. We neglect the possibility of a second cyclotron excitation that follows the first, from the excited state \( |2, n_z \rangle \) to a higher state, because this is much smaller still. The Hermitian density operator for cyclotron excitation,

\[
\hat{\rho} = \hat{\rho}_{11} + \hat{\rho}_{12} + \hat{\rho}_{21} + \hat{\rho}_{22} = \begin{pmatrix} \hat{\rho}_{11} & \hat{\rho}_{12} \\ \hat{\rho}_{21} & \hat{\rho}_{22} \end{pmatrix}
\]

(28)

is the sum of four operators, each defined by

\[
\hat{\rho}_{jk} = \sum_{n_z, n_z'} |j, n_z \rangle \langle j, n_z| \hat{\rho} |k, n_z' \rangle \langle k, n_z' |.
\]

Since \( \hat{\rho} \) is Hermitian, \( \hat{\rho}_{21} = \hat{\rho}_{12}^\dagger \).

The initial density operator at time \( t = 0 \) is diagonal with respect to the axial quantum numbers,

\[
\langle 1, n|\hat{\rho}|1, n \rangle = p_n(T) = \left[ 1 - \exp \left( -\frac{\hbar (\omega_z - \frac{i}{2} \delta_a)}{k_B T} \right) \right] \exp \left( -\frac{n\hbar (\omega_z - \frac{i}{2} \delta_a)}{k_B T} \right)
\]

\[
\approx \left[ 1 - \exp \left( -\frac{\hbar \omega_z}{k_B T} \right) \right] \exp \left( -\frac{n\hbar \omega_z}{k_B T} \right)
\]

(30)

with Boltzmann factors as its nonzero elements. The approximation is nearly exact because \( \delta_a \ll \omega_z \). In the weak drive limit, we would expect this distribution of initial states to remain unchanged.

The probability that the system is excited by one quantum from its spin down, cyclotron ground state,

\[
P_c = \sum_{n_z} \langle 2, n_z|\hat{\rho}|2, n_z \rangle = \text{Tr} [\hat{\rho}_{22}]
\]

(31)

is the sum of the probabilities for excitation to any of the states \( |2, n_z \rangle \). Either the Schrodinger or interaction picture density operator can be used since their diagonal elements are identical.

Determining \( \hat{\rho}_{22} \) requires solving the master equation

\[
\frac{d}{dt} \begin{pmatrix} \hat{\rho}_{11} \\ \hat{\rho}_{21} \\ \hat{\rho}_{22} \end{pmatrix} = -i \begin{pmatrix} a_z^\dagger & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \delta_c & \hat{\rho}_{12} \\ \hat{\rho}_{21} & 0 \end{pmatrix} + \frac{i}{2} \Omega_c \begin{pmatrix} 2 \text{Im}[\hat{\rho}_{21} e^{i\epsilon_z t}] & e^{i\epsilon_z t} (\hat{\rho}_{22} - \hat{\rho}_{11}) \\ e^{-i\epsilon_z t} (\hat{\rho}_{22} - \hat{\rho}_{11}) & 2 \text{Im}[\hat{\rho}_{12} e^{-i\epsilon_z t}] \end{pmatrix}
\]

\[
- \gamma_c \begin{pmatrix} -2\hat{\rho}_{22} & \hat{\rho}_{12} \\ \hat{\rho}_{21} & 2\hat{\rho}_{22} \end{pmatrix}
\]

\[
- \frac{\gamma_c}{2} \hat{\rho}_{22} (a_z a_z^\dagger \hat{\rho} - 2a_z^\dagger \hat{\rho} a_z + \hat{\rho} a_z^\dagger a_z)
\]

\[
- \frac{\gamma_c}{2} \hat{\rho}_{22} (\hat{\rho} (\tilde{n}_z + 1) (a_z^\dagger a_z \hat{\rho} - 2a_z^\dagger \hat{\rho} a_z + \hat{\rho} a_z^\dagger a_z).\]

The first line describes time evolution of the density matrix by \( \hat{V} \). The diagonal terms are 0 because \( |1, n_z \rangle \) and \( |2, n_z \rangle \) are eigenstates of \( \hat{V} \) for the QND measurement. The non-diagonal terms represents the differing bottle shift for \( |1, n_z \rangle \) and \( |2, n_z \rangle \). The second line describes the electromagnetic cyclotron drive. The third term describes synchrotron radiation from the excited cyclotron state at a rate \( \gamma_c \). The fourth and fifth terms arise from the axial damping and reservoir excitation. They do not change \( P_c \) because they do not change either the cyclotron or spin state.

The axial damping terms in the master equation (Eq. (32)) generate no coherence between axial states. Only axially diagonal terms, \( \langle i, n|\hat{\rho}|j, n \rangle \) are nonzero. The transformation

\[
p_{jk;n}(t) = \langle j, n|\hat{\rho}(t)|k, n \rangle e^{it(j-k)\epsilon_z t}.
\]

(33)

makes these coefficients carry all the time dependence. Notice that the probability to be in each of the cyclotron and spin states is given by the trace

\[
P_l = \sum_{n_z=0}^{\infty} \langle l, n_z|p|l, n_z \rangle,
\]

(34)

as well as by Eq. (25), where \( p \) has components \( p_{jk} \). This is because the diagonal matrix elements with \( j = k \) are equal to the those for the density operator in the Schrodinger picture and the interaction picture.

The differential equations after the transformation are

\[
\frac{d}{dt} p_{11;n}(t)
\]

\[
= -[\gamma_z (2\tilde{n}_z + 1) n - \gamma_z \tilde{n}_z] p_{11;n}(t)
\]

\[
+ \gamma_c p_{22;n}(t) - \Omega_c \text{Im} [p_{12;n}]
\]

\[
+ \gamma_z \tilde{n}_z np_{11;n-1}(t) + \gamma_z (\tilde{n}_z + 1)(n + 1) p_{11;n+1}(t)
\]

(35a)

\[
\frac{d}{dt} p_{12;n}(t)
\]

\[
= [i (-\epsilon_c + \delta_c (n + \frac{1}{2}))]
\]

\[
- \frac{\gamma_c}{2} - \gamma_z \tilde{n}_z np_{12;n-1}(t) + \gamma_z (\tilde{n}_z + 1)(n + 1) p_{12;n+1}(t)
\]

(35b)

\[
\frac{d}{dt} p_{22;n}(t)
\]

\[
= -[\gamma_c - \gamma_z (2\tilde{n}_z + 1) n - \gamma_z \tilde{n}_z] p_{22;n}(t)
\]

\[
+ \Omega_c \text{Im} [p_{12;n}]
\]

\[
+ \gamma_z \tilde{n}_z np_{22;n-1}(t) + \gamma_z (\tilde{n}_z + 1)(n + 1) p_{22;n+1}(t)
\]

(35c)

These equations are to be solved for the initial conditions

\( p_{12;n}(0) = p_{22;n}(0) = 0 \) and \( p_{11;n}(0) = p_{n}(T) \).
Equations (35a-35c) is a matrix equation for the vectors $\vec{p}_i(t)$ with components $p_{ni}$,

$$\frac{d}{dt}\vec{p}_{11}(t) = R(0, 0, 0)\vec{p}_{11}(t) - \Omega_z\Im[\vec{p}_{12}(t)] + \gamma_z\vec{p}_{22}(t)$$

(36a)

$$\frac{d}{dt}\vec{p}_{12}(t) = R(\epsilon, \delta, \gamma_c)\vec{p}_{12}(t) - i\frac{\Omega_z}{2}(\vec{p}_{22}(t) - \vec{p}_{11}(t))$$

(36b)

$$\frac{d}{dt}\vec{p}_{22}(t) = R(0, 0, 2\gamma_c)\vec{p}_{22}(t) + \Omega_z\Im[\vec{p}_{12}(t)].$$

(36c)

The non-zero elements of the time-independent matrix are

$$R(\epsilon, \delta, \gamma_c)_{n,n-1} = \gamma_z\bar{n}_z n$$

(37a)

$$R(\epsilon, \delta, \gamma_c)_{n,n} = i[-\epsilon + (n + \frac{1}{2}\delta) - \frac{1}{2}\gamma_c]$$

$$- \gamma_z(2\bar{n}_z + 1) n - \gamma_z\bar{n}_z$$

(37b)

$$R(\epsilon, \delta, \gamma_c)_{n,n+1} = \gamma_z(\bar{n}_z + 1)(n + 1).$$

(37c)

The initial conditions for the vector differential equations above are $\vec{p}_{11}(0) = \vec{p}(T)$ and $\vec{p}_{12}(0) = \vec{p}_{22}(0) = 0$.

B. Steady-State Cyclotron Lineshape

For a weak drive, after transients have died out, there is a steady-state for which driven cyclotron excitation balances the incoherent spontaneous emission of synchrotron radiation. A weak drive, $\Omega_z \ll \gamma_c$ means that the

$$\Tr[p_{11}] = \sum_n p_{11,n}(T) \approx \sum_n p_n(T) = 1$$

(38)

$$\Tr[p_{22}] = \sum_n p_{22,n}(T) \ll 1$$

(39)

so terms involving $\vec{p}_{22}$ are negligibly small compared to those involving $\vec{p}_{11}$. The steady state pertains after a time that is long enough for transients to die out,

$$t \gg \gamma_c^{-1}.$$ 

(40)

The resulting steady state, from Eq. (36) with the time derivatives set to zero and the mentioned approximation is described by

$$R(\epsilon, \delta, \gamma_c)\vec{p}_{12} + i\frac{\Omega_z}{2}\vec{p}(T) = 0$$

(41)

$$R(0, 0, 2\gamma_c)\vec{p}_{22} + \Omega_z\Im[\vec{p}_{12}] = 0.$$ 

(42)

The latter can be simplified because

$$\sum_{n=0}^{\infty} R(0, 0, 2\gamma_c) p_{22,n} = -\gamma_c \Tr[p_{22}],$$

(43)

because axial damping does not change the total population in states $|2, n\rangle$, and because $R(0, 0, 2\gamma_c)$ has a simple structure. The steady state probability for cyclotron excitation by a weak drive is thus

$$P_c = \Tr[p_{22}] = P(\Omega_c, \epsilon_c, \delta_c).$$

(44)

The characteristic lineshape so derived,

$$P(\Omega, \epsilon, \delta) = -\frac{\Omega^2}{2\gamma_c} \Im\left[\sum_{n=0}^{\infty} (iR(\epsilon, \delta, \gamma_c)^{-1}\bar{p}(T))_n\right]$$

(45)

reappears in the next section.

For the limiting case of a $T = 0$ bath, $\bar{n}_z = 0$ and $\bar{p}(T)$ collapses to a single element $p_0(T) = 1$. Only the reciprocal of $R_{00} = -i\epsilon + i\delta/2 - \frac{1}{2}\gamma_c$ contributes to Eq. (45). The steady state lineshape for a weak drive, $P(\Omega_c, \epsilon, \delta)$, thus becomes a Lorentzian,

$$P_0(\Omega, \epsilon, \delta) = \left(\frac{\Omega}{\gamma_c}\right)^2 \frac{(\frac{1}{2}\gamma_c)^2}{(\epsilon - \frac{1}{2}\delta)^2 + (\frac{1}{2}\gamma_c)^2}$$

(46)

in the $T = 0$ limit. The full width at half maximum of this lineshape is $\gamma_c$. The lineshape maximum is shifted by the axial zero point energy to a frequency detuning $\epsilon = \delta/2$. The steady state probability for being excited with a resonant weak drive is $(\Omega/\gamma_c)^2$ - a very small fraction given that the drive is weak.

The symmetric and narrow Lorentzian cyclotron lineshape that pertains for $T = 0$,

$$P_c = P_0(\Omega_c, \epsilon_c, \delta_c),$$

(47)

would be ideal experimentally in some respects. Cavity sideband cooling with a extremely small $\gamma_c$ has been proposed [24] as way to attain this limit. This calculation, however, is an investigation of what can be done for a temperature of 0.1 K, an achieved temperature that is close to but not at this limit.

C. Classical Brownian Motion Lineshape Limit

Before the quantum treatment of the coupled spin, cyclotron and axial system presented above, the calculated lineshape that was compared to experiment [20, 21] assumed the axial detector motion was a classical harmonic oscillation driven by thermal noise. The Brownian motion lineshape that resulted from a weak drive is given in terms of a lineshape function,

$$\chi(\epsilon, \gamma_z, \bar{n}_z) = \frac{4}{\pi} \Re\left[\frac{\gamma'\gamma_z}{(\gamma' + \gamma_z)^2} \sum_{k=0}^{\infty} \left(k + \frac{1}{2}\right) \gamma' + \frac{1}{2} (\gamma_c - \gamma_z) - i\epsilon\right]$$

(48)

(in our notation). The bath temperature $T$ enters via

$$\gamma' = \sqrt{\gamma_z^2 + 4i\gamma_z\bar{n}_z\delta},$$

(49)

since this bath temperature determines $\bar{n}_z$. The steady state pertains when the transition rate $(\pi/2)\Omega^2\chi$ (Eq. (5.19) of [24]) equals the decay rate $\gamma_c \times P(\Omega, \epsilon, \delta)$. Thus

$$P(\Omega, \epsilon, \delta) = \frac{\pi\Omega^2}{2\gamma_c} \chi(\epsilon, \gamma_z, \bar{n}_z)$$

(50)

is the classical, Brownian motion lineshape.
Probability steady state for experimental conditions in Table II. The probability $1/c$ is weak ($\Omega$ illustrates the time evolution for a cyclotron drive that cyclotron excitation can be solved numerically. Figure 4 from Table. III).

larger probability to be in the initial $t$ from the stated boundary conditions at time $t = 0$ to time $t$ for various values of the drive detuning, $\epsilon_c$, as illustrated in Fig. 5. The probability to be in the states $|2, n_z\rangle$ at time $t = 10\gamma_c^{-1}$ is shown for a cyclotron drive that is weak ($\Omega_c = 0.1\gamma_c$ for the realistic experimental

FIG. 3. Comparison of quantum calculation (solid) and classical calculation (dashed) with the different $\gamma_c$'s for weak drive ($\Omega_c = 0.1\gamma_c$) in cyclotron transition. The damping rates $\gamma_z$ for (a-c) are 1000, 100, 10 times the value in Table. II, respectively. Two calculations agree when $n_z\gamma_z > \delta_c$.

D. Discussion of the Quantum Cyclotron Lineshape

The Brownian motion steady-state lineshape approaches with the quantum steady-state lineshape when $n_z\gamma_z \gg \delta_c$, as illustrated in Fig. 3. The 2008 measurement is one example, with $\bar{n}_z\gamma_z \approx 6\delta_c$ (using parameters from Table III).

More generally, the master equation for driven cyclotron excitation can be solved numerically. Figure 4 illustrates the time evolution for a cyclotron drive that is weak ($\Omega_c = 0.1\gamma_c$, resonant ($\epsilon_c = \delta_c/2$) for the realistic experimental conditions in Table III. The probability to be in the $|2, n_z\rangle$ states increases from zero to reach a steady state for $t \gg 1/\gamma_c$. The cyclotron damping time $1/\gamma_c$ sets the scale for the transients to die out. The much larger probability to be in the initial $|1, n_z\rangle$ states stays close to unit probability. The black curve in the figure is this probability with unit probability subtracted out to show the slight decrease needed to conserve probability.

The resonance lineshape for cyclotron excitation is obtained by numerically integrating the master equation from the stated boundary conditions at time $t = 0$ to time $t$ for various values of the drive detuning, $\epsilon_c$, as illustrated in Fig. 5. The probability to be in the states $|2, n_z\rangle$ at time $t = 10\gamma_c^{-1}$ is shown for a cyclotron drive that is weak ($\Omega_c = 0.1\gamma_c$, for the realistic experimental

FIG. 4. Time evolution in response to a weak and resonant cyclotron drive applied for 10 cyclotron damping times, indicated by vertical gray lines. The probability to be in the $|2, n_z\rangle$ states (blue) reaches a steady state after transients die out on a time scale give by the cyclotron damping time, $1/\gamma_c$. The probability to be in the $|1, n_z\rangle$ states is shown in black with unit probability subtracted out.

FIG. 5. Quantum cyclotron lineshape (solid) with clearly resolved axial quantum states (for a weak cyclotron drive with the largest peak normalized to 1) for the quantum calculation (solid), but not for the classical Brownian motion lineshape (dashed). The quantum lineshape is a huge improvement on the lineshape used for the best measurement (dotted).

The series of narrow cyclotron resonances is the first example of axial quantization. It is possible because of the small axial damping that is now possible experimentally. This quantum lineshape is very different than was observed previously, and it is completely inconsistent with the classical cyclotron lineshape, of course. The narrow peaks correspond to resolved quantum states of the axial motion which could not previously be observed. The left peak is for $n_z = 0$, the next for $n_z = 1$, and so on. There are many peaks because the average axial quantum number is $\bar{n}_z = 10$ for
the experimental conditions in Table II. The individual peaks are resolved because two conditions are met. First, $\bar{n}_z \gamma_z \ll \delta_z$, i.e. the width of each axial state, $\bar{n}_z \gamma_z$, is much smaller than the magnetic bottle shift per axial quantum, $\delta_z$. Second, $\gamma_c \ll \delta_z$, i.e. the cyclotron damping width is much smaller than the magnetic bottle shift per axial quantum, $\delta_z$.

The good news from this calculation for potential measurements is how much narrower the $n_z = 0$ resonance peak is compared to the cyclotron lineshape used for the last electron magnetic moment measurement (dotted in Fig. 5 with experimental parameters in Table III). In fact, the linewidth of the $n_z = 0$ peak is only a factor of 3 larger than the cyclotron linewidth, $\gamma_c$ (Fig 6). This is consistent with the indication from Eq. (35c) that the linewidth is of order $\gamma_c + 2\bar{n}_z \gamma_z$. Cavity-inhibition of spontaneous emission makes $\gamma_c$ very small \cite{23}, and there are now experimental methods to bring $\gamma_z$ to the small value in Table II\cite{22}.

\begin{table}[h]
| ang. frequency or rate | frequency (Hz) | time constant (s) |
|------------------------|----------------|-------------------|
| $\delta_a$             | 0.004          | 40                |
| $\gamma_z$             | 1              | 0.16              |
| $\bar{n}_z \delta_a$   | 0.09           | 1.7               |
| $\gamma_c$             | 0.03           | 6                 |
| $\bar{n}_z \gamma_z$   | 23             | 0.007             |
| $\delta_z$             | 4              | 0.04              |
| $\bar{n}_z \delta_z$   | 92             | 0.0017            |
\end{table}

TABLE III. Hierarchy of angular frequencies and rates used on the best completed experiments \cite{22,23}, to be compared with the previous table. The axial temperature was also as low as $\bar{n}_z = 23$. The numerical values are frequencies in Hz and times in seconds.

More good news for possible measurements is that the $n_z = 0$ peak is quite symmetric about its center frequency. This is generally a big help in precisely identifying the center frequency of a resonance. The dotted line in Fig. 6 illustrates the big contrast to the highly asymmetric classical lineshape used for previous measurements.

The small probability, $3.1 \times 10^{-4}$, that a weak cyclotron drive ($\Omega_c = 0.1 \gamma_c$) will make an excitation within 10 cyclotron damping times (53 seconds) is of some concern. However, increasing the cyclotron drive strength to $\Omega_c = \gamma_c$ increases the probability for an excitation to $2.2 \times 10^{-2}$ while increasing the full linewidth from 3 to only 3.6 cyclotron decay widths (solid and dashed curves in Fig. 6). (The master equation had to be integrated directly to examine the effect of power broadening since the steady state solutions apply only in the limit of a weak drive with $\Omega_c < \gamma_c$.) This cyclotron linewidth is narrow enough to make possible magnetic moment measurements that are orders of magnitude more accurate than the current limit (if the anomaly frequency could be determined with a similar accuracy). Because the power broadening is so small, even stronger drives could be used to track a slowly drifting magnetic field\cite{3}.

The offset of the $n_z = 0$ resonance from $\epsilon_z = 0$ to $\epsilon_z = \delta_z/2$ is due to the zero point motion of the quantum axial oscillator. Measuring this peak and its neighbor would determine this offset more accurately than is needed for dramatically improved magnetic moment measurements, since these two peaks are spaced by twice the offset. This could be an imporant new option for precisely measuring the offset.

In summary, this quantum calculation demonstrates the exciting possibility to fully resolve the axial quantum structure in the cyclotron lineshape. With achievable reductions in axial damping in Table II, a cyclotron resonance for a particle in its axial ground state can be fully resolved. This will make it possible to determine the cyclotron frequency (one of two frequencies needed for a magnetic moment measurement) orders of magnitude more precisely. The broad cyclotron linewidth (larger than $\bar{n}_z \delta_z$) that limited past measurements is essentially removed.

V. CALCULATING THE ANOMALY LINESHAPE

A. Anomaly Master Equation

An anomaly drive $V_a$ will transfer population from a thermal distribution of stable, spin-up, cyclotron ground states, $|3, n_z\rangle$ to the unstable states, $|2, n_z\rangle$. These states then decay via the spontaneous emission of synchrotron radiation to the stable spin-down ground states $|1, n_z\rangle$. The attractive feature for measurement is that there is no need to detect an unstable state population before it decays.

The density operator needed to describe anomaly tran-
sitions,
\[
\hat{\rho} \equiv \begin{pmatrix} \hat{\rho}_{22} & \hat{\rho}_{23} \\ \hat{\rho}_{32} & \hat{\rho}_{33} \end{pmatrix},
\] (51)
does not need to include the stable lower states, \(|1, n_z\rangle\), though it must include decay to these states. It has the upper and lower energy states in the same relative matrix locations as in the previous section. What must be calculated is the loss of probability from the initial state during the time that the drive is applied, since this is the probability that a spin-flip transition takes place.

The master equation in the interaction representation is then a lot like Eq. (32), with the indices 1 \to 2 and 2 \to 3,
\[
\frac{d}{dt} \begin{pmatrix} \hat{\rho}_{22} & \hat{\rho}_{23} \\ \hat{\rho}_{32} & \hat{\rho}_{33} \end{pmatrix} = -i \left[ a_z^\dagger a_z + \frac{1}{2} \right] \begin{pmatrix} 0 & -\delta_a \hat{\rho}_{23} \\ \delta_a \hat{\rho}_{32} & 0 \end{pmatrix} - \frac{i}{2} \Omega_a \begin{pmatrix} i2\text{Im}[\hat{\rho}_{32} e^{i\omega t}] & e^{i\omega t} (\hat{\rho}_{33} - \hat{\rho}_{22}) \\ e^{-i\omega t} (\hat{\rho}_{22} - \hat{\rho}_{33}) & i2\text{Im}[\hat{\rho}_{23} e^{-i\omega t}] \end{pmatrix} - \frac{\gamma_c}{2} \begin{pmatrix} 2\hat{\rho}_{22} \hat{\rho}_{33} & 0 \\ 0 & 2\hat{\rho}_{32} \hat{\rho}_{22} \end{pmatrix} - \frac{\gamma_c}{2} \left( a_z a_z^\dagger \hat{\rho} - 2a_z^\dagger \hat{\rho} a_z + \hat{\rho} a_z a_z^\dagger \right) - \frac{\gamma_c}{2} \left( \bar{n}_z + 1 \right) (a_z a_z^\dagger \hat{\rho} - 2a_z^\dagger \hat{\rho} a_z + \hat{\rho} a_z a_z^\dagger) .
\] (52)
The term that is different is the cyclotron damping term that is proportional to \(\gamma_c\). This is because the lower rather than the upper of the two sets of states is unstable. The vanishing element in the matrix comes because the states \(|3, n_z\rangle\) do not decay.

The discussion follows essentially the same steps discussed in the previous section. The differential equations are
\[
\frac{d}{dt} \langle p_{22}\rangle = R(0, 0, 2\gamma_c) \langle p_{22}\rangle - \Omega_a \text{Im} [\langle p_{23}\rangle] \] (53a)
\[
\frac{d}{dt} \langle p_{23}\rangle = R(\epsilon_a, \delta_a, \gamma_c) \langle p_{23}\rangle - i \frac{\Omega_a}{2} (\langle p_{33}\rangle - \langle p_{22}\rangle) \] (53b)
\[
\frac{d}{dt} \langle p_{33}\rangle = R(0, 0, 0) \langle p_{33}\rangle + \Omega_a \text{Im} [\langle p_{23}\rangle] ,
\] (53c)
These equations are to be solved for the initial conditions
\[
\langle p_{33}\rangle(0) = \bar{p}(T) \text{ and } \langle p_{23}\rangle(0) = \bar{p}_{22}(0) = 0.
\]

B. Quasi-Steady-state Solution

Coherent, driven anomaly transitions can balance the incoherent spontaneous emission of synchrotron radiation to produce a quasi-steady state in this case. For a weak drive \(\Omega_a \ll \gamma_c\),
\[
\text{Tr} [p_{33}] = \sum_n p_{33,n}(t) \approx \sum_n p_n(T) = 1 \] (54)
\[
\text{Tr} [p_{22}] = \sum_n p_{22,n}(t) \ll 1
\] (55)
the quasi-steady state pertains in the time range
\[
\gamma_c^{-1} \ll t \ll \gamma_c^{-1} \left( \frac{\gamma_c}{\Omega_a} \right)^2 .
\] (56)
The time must be long enough for transients to die out. It must be short compared to enough that Eq. (55) remains valid. We will justify the value of the upper time limit presently.

During this time interval the time derivatives vanish in Eq. (53). The resulting steady state, from Eq. (53) with the time derivatives set to zero and Eq. (55) inserted, is described by
\[
R(\epsilon_a, \delta_a, \gamma_c) \bar{p}_{22} - \Omega_a \text{Im} [\bar{p}_{23}] = 0
\] (57)
\[
R(0, 0, 2\gamma_c) \bar{p}_{22} - \Omega_a \text{Im} [\bar{p}_{23}] = 0.
\] (58)
Because \(R(0, 0, 2\gamma_c)\) has a simple structure,
\[
\text{Tr} [p_{22}] = -\frac{1}{\gamma_c} \sum_{n=0}^{\infty} (R(0, 0, 2\gamma_c) \bar{p}_{22})_n
\] (59)
When Eqs. (57 58) are substituted,
\[
\text{Tr} [p_{22}] = P(\Omega_a, \epsilon_a, \delta_a)
\] (60)
described by exactly the same function that described the steady state for cyclotron excitation Eq. (45). With anomaly arguments rather than cyclotron arguments, however, the function takes an entirely different shape.

This function does not directly describe a quasi-steady lineshape that can be observed. What reaches a quasi-steady state is the probability to be in the \(|2, n_z\rangle\) states. What is observed is the ever increasing probability to be in the \(|1, n_z\rangle\) states following spontaneous emission. In the time range of Eq. (56), when the anomaly drive is applied for time \(t_d\), the probability of a spin flip is approximately
\[
P_a = (t_d \gamma_c + 1) P(\Omega_a, \epsilon_a, \delta_a).
\] (61)
This overstates the transition probability because the anomaly transition rate changes during the time the anomaly drive is on, but the lineshape is approximately right.

As for the cyclotron lineshape, in the \(T = 0\) limit the quasi-steady state anomaly lineshape for a weak drive becomes a Lorentzian
\[
P_a \approx (t_d \gamma_c + 1) P(\Omega_a, \epsilon_a, \delta_a)
\] (62)
as discussed in the last section. On resonance, the quasi-steady state probability to be in state \(|2, 0\rangle\) at \(T = 0\) is \((\Omega_a / \gamma_c)^2\). This is extremely small for a weak anomaly drive with \(\Omega_a \ll \gamma_c\). For the cases we consider, with temperatures not far from 0, we expect that the rate to transfer population from the initial \(|3, n_z\rangle\) states to the final \(|1, n_z\rangle\) states goes as this small probability times the rate \(\gamma_c\) to decay form \(|2, n_z\rangle\) to \(|1, n_z\rangle\). The population transfer will be small (as needed to have a quasi-steady state) as long as the time is short compared to the inverse of this rate, which gives the upper time limit in Eq. (56).
C. Discussion of the Anomaly Lineshape

The master equation for driven anomaly transitions can also be solved numerically. Figure 7 illustrates the time evolution for an anomaly drive that is weak \( \Omega_a = 0.1 \gamma_c \) and resonant for the realistic experimental conditions in Table II. The probability to be in the \([2, n_z]\) states (blue) increases from zero to reach a quasi-steady state. The cyclotron damping time \( 1/\gamma_c \) sets the scale for the transients to die out. The probability to end up in \([1, n_z]\) states (black) following spontaneous emission from the \([2, n_z]\) states increases continually from zero. The much larger probability to be in the initial \([3, n_z]\) states stays close to unit probability. The red curve in the figure is this probability with unit probability subtracted out to show the slight decrease needed to conserve probability.

![Fig. 7](image)

**FIG. 7.** Time evolution in response to a weak and resonant anomaly drive applied for 10 cyclotron damping times, the latter indicated by vertical gray lines. The probability to be in the \([2, n_z]\) states (blue) reaches a steady state after transients die out on a time scale given by the cyclotron damping time, \( 1/\gamma_c \). The probability to be in the \([1, n_z]\) states after spontaneous emission is shown in black. The probability to be in the initial \([3, n_z]\) states, with unit probability subtracted out, is shown in red.

The resonance lineshape for anomaly transitions is obtained by numerically integrating the master equation from the stated boundary conditions at time \( t = 0 \) to time \( t \) for various values of the drive detuning, \( \epsilon_a \), as illustrated in solid curve in Fig. 8. The probability to be in the \([1, n_z]\) states at time \( t = 10/\gamma_c \) is shown for an anomaly drive that is weak \( \Omega_a = 0.1 \gamma_c \), for the realistic experimental conditions in Table II. The quasi-steady state solution (dashed) overestimates the probability because it takes some time to increase the transition rate to the steady state. However, Fig. 8 shows that it correctly predicts the shape.

The axial quantum states are not resolved in this lineshape for the parameters of Table II. This is because the anomaly frequency shift per axial quantum, \( \delta_a \), is 10 times smaller than the axial width of the \( n_z = 0 \) state, \( n_z \gamma_z \). Moreover, it is also 10 times smaller than the cyclotron damping width, itself. Indeed, the width of the calculated lineshape is \( 2.2 \gamma_z \). The shape is asymmetric with a long tail to higher frequencies because many more axial states above \( n = n_z \) are populated than are states below.

The calculation brings good news for measurements made under the realistic conditions of Table II. The linewidth is much narrower than the much broader lineshape that pertains for the conditions of the best measurement made so far. The “error bar” in the figure corresponds to the ±300 ppt uncertainty (ppt = 1 part in \( 10^{12} \)) of the most accurate measurement to date.

In Fig. 8 the classical Brownian motion lineshape (dotted) is remarkably close to the solution to the master equation obtained by direct integration (solid), much more so than for the cyclotron lineshape. Figure 9 compares quantum and classical calculations with three realizable values of \( \gamma_z \). Because \( n_z \gamma_z \) is higher than \( \delta_a \), the
The damping rates $\gamma_z$ for weak drive are $1000, 100, 10$ times the value in Table II respectively.

axial quantum states are not resolved. For the best measurement [3], with $n_z \gamma_z = 6 \times 10^5 \delta_n$ (Table III), the two calculations predict same lineshape.

D. Temperature and Damping Dependence

We have just discussed how, for the new experimental regimes that are now accessible (Tab. II), greatly improved measurements of the anomaly frequency seem feasible. This section explores additional reductions in anomaly linewidth that may be possible with additional reductions in axial temperature, cyclotron damping rate, and axial damping rate. Fig. 10 shows anomaly lineshapes for a weak drive ($\Omega_a = \gamma_e/10$) for different values of these parameters.

Temperatures of 100 mK (black), 50 mK (dashed) and 25 mK (dotted) are shown. The most accurate measurement was done at 100 mK [3], but dilution refrigerators can reach lower temperatures if the heat load is low enough. Also, cavity-sideband cooling is a possible method to reduce the axial temperature without going to lower apparatus temperatures [24].

The lineshape that uses the cyclotron damping rate from Table II is shown in black, and the blue curves are for a 10 times lower damping rate. The most accurate experiment achieved the low damping rate in the table by using a microwave cavity to suppress the spontaneous emission of synchrotron radiation [28]. A lower loss microwave cavity could further reduce the cyclotron damping rate, though this would also increase the measurement time because it takes several cyclotron damping times for the population in state $|2, n_z\rangle$ to decay to the ground state.

Reducing just the axial temperature reduces the linewidth somewhat. A bigger consequence is that the doing so reduces the asymmetry of the lineshape, which should make it possible to identify the resonance frequency more reliably. Reducing the cyclotron damping as well produced much more line narrowing.

The effects of the axial damping rate have also been investigated. Further reductions in the axial damping rate do not noticeably change any of the curves in Fig. 10.

The possibly to use cavity sideband cooling of the axial motion has been mentioned as a possible route to narrower resonance linewidths [24]. Once the cooling is stopped, the axial motion will reequilibrate at the bath temperature at a rate $\gamma_z$. This is not a steady state, of course, but we can investigate the possibility by directly integrating the master equation. Fig. 11 shows the probability of a spin-flip caused by a weak anomaly drive ($\Omega_a = 0.1 \gamma_c$) applied for a 100 mK temperature bath (solid). For this illustration, the axial motion is ini-
tially assumed to be cooled to the limit of $T = 0$ so that only the lower axial quantum state is initially populated. This causes the linewidth to narrow from ±190 ppt to ±130 ppt. The lineshape also is more symmetric about its center, and the offset frequency is smaller. The drive is applied for time $10/\gamma_c$ in this illustration, which is one axial damping time $1/\gamma_z$. For the parameters we are using for this illustration (Table I), the linewidth gets broader for shorter driving times because of the limited drive duration.

The axial states can be resolved in the anomaly lineshape, just as for the cyclotron lineshape, if the bottle shift $\delta_n$ is large enough. This requires that the damping of the axial state, $n_\gamma \gamma_z$, must be much smaller than bottle shift per axial state, $\delta_n$. It also requires that the cyclotron damping width $\gamma_c$ that broadens the anomaly resonance lines be much smaller than $\delta_n$. In Table I, $\gamma_c$ is actually 10 times larger than $\delta_n$. Thus, resolving the axial quantum structure in the anomaly resonance would require increasing $\delta_n$ by a factor of at least 100 or more. Figure 12 shows the anomaly lineshape with parameters in Table I (solid), 10 times larger $B_2$ (dashed), and 100 times larger $B_2$ (dotted). The axial states can be resolved for 100 times larger magnetic bottle, but the relative precision does not change. Magnetic bottle gradients of the size needed have been produced, but only for Penning traps that are smaller than is otherwise desirable for electron and positron measurements [35–41]. However, the figure shows that resolving the axial quantum states would not help insofar as the linewidth of the lowest resolved peak is a bit bigger than the anomaly linewidth already considered. The linewidth comes from the cyclotron damping $\gamma_c$, and axial broadening $n_\gamma \gamma_z$, both independent of the bottle size $B_2$.

The conclusion of this survey of anomaly lineshapes is that further reductions in their width depend upon the cyclotron damping rate $\gamma_c$ and the temperature of the thermal bath to which the axial motion is coupled.

VI. CONCLUSIONS

A fully quantum calculation of driven transitions of a one-particle quantum cyclotron is presented for the first time. The quantum spin and cyclotron motions of a trapped electron (or positron) in a Penning trap is coupled to a perpendicular axial motion to allow quantum nondemolition (QND) detection of the quantum state. The time evolution for the driven quantum cyclotron is described here by the master equation for coupled quantum oscillators, each of which are coupled in addition to a thermal bath. The master equation is integrated directly, and approximate steady-state solutions are presented, for the case of weak drives. The steady-state calculation for a weak drive used to interpret all measurements so far assumed instead a classical axial motion that was undergoing thermal Brownian motion.

The quantum calculation accurately describes new quantum regimes that can now be experimentally accessed, that cannot be accurately described by the Brownian motion description that it supersedes. It is used here to examine the consequence of reducing the axial damping during the time when one-quantum cyclotron and anomaly transitions are driven - a new experimental possibility [32]. An exciting result is the emergence of extremely narrow quantum resonances that appear within the cyclotron resonance line. These narrow resonances are due to resolved quantum states of the detection oscillator. They are are about 100 time narrower than the broad cyclotron linewidth that was a significant challenge for past measurements. The effect of the backaction of the detector that caused the broad resonance line.
is thereby evaded, even though the electron spends much time in excited axial states. The narrow resonances open the way to measuring electron and positron cyclotron frequencies with unprecedented accuracy.

Two frequencies must be measured to determine the electron or positron magnetic moment, however. Along with the cyclotron frequency, the difference frequency between the spin and cyclotron motion must also be measured. An accurate measurement of this anomaly frequency remains as a big challenge. The axial shifts and broadening are comparable in size for the cyclotron and anomaly resonance lines, but they are a much bigger fraction of the much smaller anomaly frequency. This quantum calculation shows that the quantum structure of the axial motion is very difficult to resolve in the anomaly lineshape, despite the reduced axial damping and a lower axial temperature. However, it also suggests the possibility to observe a much narrower anomaly lineshape. This should enable a magnetic moment measurement whose uncertainty is greatly reduced, perhaps by an order of magnitude. The calculation also shows that further advances should be possible with reduced cyclotron damping and a lower bath temperature.

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