Nonexistence and uniqueness for pure states of ferroelectric six-vertex models

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Funding information
NSF, Grant/Award Number: NSF DMS-1664619; NSF Graduate Research Fellowship, Grant/Award Number: DGE-1144152; Harvard Merit/Graduate Society Term-time Research Fellowship

Abstract
In this paper, we consider the existence and uniqueness of pure states with some fixed slope \((s, t) \in [0, 1]^2\) for a general ferroelectric six-vertex model. First, we show there is an open subset \(\mathcal{H} \subset [0, 1]^2\), which is parameterized by the region between two explicit hyperbolas, such that there is no pure state for the ferroelectric six-vertex model of any slope \((s, t) \in \mathcal{H}\). Second, we show that there is a unique pure state for this model of any slope \((s, t)\) on the boundary \(\partial \mathcal{H}\) of \(\mathcal{H}\). These results confirm predictions of Bukman–Shore from 1995.

MSC (2020)
82B20 (primary)

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1 | INTRODUCTION

1.1 | Preface

A fundamental question in mathematical physics concerns the classification and analysis of pure states (translation-invariant, ergodic Gibbs measures) for statistical mechanical systems. Over the past two decades, a deep understanding in this direction has been achieved for two-dimensional
dimer models. Indeed, the classification aspect of this question was addressed in 2005 by Sheffield [31], who showed that dimer pure states are parameterized by pairs of real numbers, also called slopes, that govern the average gradient for the height function associated with the model. These pure states were then analyzed in 2006 by Kenyon–Okounkov–Sheffield [25], who showed that they come in three types. The first is \textit{frozen}, where the associated height function is essentially deterministic; the second is \textit{gaseous}, where the height variance is nonzero but bounded; and the third is \textit{liquid}, where the height fluctuations diverge logarithmically in the lattice size.

In this paper, we consider the existence and uniqueness of pure states for the six-vertex model. This model, whose definition we will recall more precisely in Section 1.2, involves six positive numbers \((a_1, a_2, b_1, b_2, c_1, c_2)\) that provide weights for local configurations. Associated with these weights is an \textit{anisotropy parameter} \(\Delta = \Delta (a_1, a_2, b_1, b_2, c_1, c_2)\), defined by setting

\[
\Delta = \frac{a_1 a_2 + b_1 b_2 - c_1 c_2}{2 \sqrt{a_1 a_2 b_1 b_2}},
\]

that is believed to dictate both qualitative and quantitative features of the model.

Here, we will be interested in the regime when \(\Delta > 1\), which is known as the \textit{ferroelectric phase} of the six-vertex model. The description of pure states for this six-vertex model differs considerably from its counterpart in the dimer setting. Indeed, even the question of existence for pure states has different answers in these two contexts; in particular, we will see that there are certain slopes for the ferroelectric six-vertex model that are in principle allowable but do not admit pure states.

However, before explaining our results in more detail, let us first outline the predictions in this direction from the physics literature, due to Bukman–Shore [5] in 1995. Associated with any pure state \(\mu\) of a six-vertex model is a \textit{slope} \((s, t)\), where \(s\) and \(t\) denote the probabilities that a given vertical and horizontal edge is occupied under \(\mu\), respectively; as such, we must have \((s, t) \in [0, 1]^2\). It is widely believed that each slope \((s, t) \in (0, 1)^2\) admits at most one pure state for any ferroelectric six-vertex model. In the context of gradient models governed by a simply attractive Gibbs potential (namely, convex potentials that only depend on height differences at neighboring vertices), the stronger statement was shown in [31] that every slope \((s, t) \in (0, 1)^2\) admits a unique pure state. This result was recently generalized by Menz–Tassy [26], who showed it also holds for random Lipschitz height functions satisfying stochastic monotonicity, which can be viewed as a version of the Fortuin–Kasteleyn–Ginibre (FKG) condition for height models. Yet, these results do not apply for the ferroelectric six-vertex model, as it is not stochastically monotone.

In fact, it was predicted in [5, Section 3.5] that there is a ‘lens-shaped’ region \(\mathcal{S} \subset [0, 1]^2\) admitting no pure states for the ferroelectric six-vertex model; we refer to the shaded part of Figure 1 for a depiction. A precise parameterization for this open set \(\mathcal{S}\) was also proposed there, by writing its boundary \(\partial \mathcal{S}\) as the union \(\mathcal{H}_1 \cup \mathcal{H}_2\) of two explicit hyperbolas (given by (1.4) and (1.5) below). In addition, [5] proposed characterizations for how the pure state \(\mu_{s,t}\) of slope \((s, t)\) should qualitatively behave in different regions of \([0, 1]^2\). See also Section 1.2.1 of the work [8] by Corwin–Ghosal–Shen–Tsai for a restatement of these predictions, and also Section 4 of the work [9] by de Gier–Kenyon–Watson where the first (nonexistence) part of the below characterization is predicted for the five-vertex model. In what follows, \(\overline{\mathcal{S}} = \mathcal{S} \cup \partial \mathcal{S}\) denotes the closure of \(\mathcal{S}\).

1. Nonexistence: If \((s, t) \in \mathcal{S}\), then there should be no pure state \(\mu_{s,t}\) of slope \((s, t)\).
2. KPZ States: If \((s, t) \in \partial \mathcal{S}\), then \(\mu_{s,t}\) should exhibit Kardar–Parisi–Zhang (KPZ) behavior.
3. Liquid States: If \((s, t) \in (0, 1)^2 \setminus \overline{\mathcal{S}}\), then \(\mu_{s,t}\) should be liquid.
4. Frozen States: If \((s, t)\) is on the boundary of \([0, 1]^2\), then \(\mu_{s,t}\) should be frozen.
This description of pure states contrasts sharply with its counterpart in the dimer model. Indeed, while frozen and liquid states both appear in dimers, regions of nonexistence (that disconnect the space of liquid states) and KPZ states do not. Let us mention that, although [5] did not precisely predict the sense in which the measures $\mu_{s,t}$ for $(s, t) \in \partial \mathcal{H}$ should exhibit KPZ behavior, systems in the Kardar–Parisi–Zhang universality class are typically characterized by exhibiting fluctuations of order $N^{1/3}$ on a domain of size $N$ [22]. We refer to the surveys [7] of Corwin and [29] of Quastel for a more detailed introduction to this class.

Although it will not be a focus of this paper, let us mention that there are also predictions for the existence and behavior of six-vertex pure states outside of the ferroelectric regime (that is, when $\Delta < 1$). In this context, every slope $(s, t) \in [0, 1]^2$ is believed to admit a unique pure state. For $\Delta \leq \frac{1}{2}$, which comprises all of the antiferroelectric regime $\Delta < -1$ and most of the critical regime $\Delta \in (-1, 1)$, the six-vertex model can be viewed as a discrete gradient model governed by a simply attractive Gibbs potential. Therefore, the results of [31] (see also [26, Section 13.4]) apply and yield the existence of a unique pure state of each slope $(s, t) \in [0, 1]^2$. The qualitative characteristics of these pure states are believed to closely resemble those that appear in dimer models. More specifically, in the critical regime, all pure states should be either liquid or frozen and, in the antiferroelectric regime, all pure states should be liquid, gaseous, or frozen. For more information on these predictions (including phase diagrams for the associated pure states), we refer to Section 9 of the survey [30] by Reshetikhin and Section 4 of that [28] by Palamarchuk–Reshetikhin.

In certain cases, these predictions have been mathematically established. This includes conformal invariance at the free-fermion point $\Delta = 0$ [23]; Gaussian free field fluctuations and precise asymptotics for correlation functions for all states when $|\Delta|$ is sufficiently small [16–18, 24]; delocalization [6, 14, 27], logarithmic variance bounds [11, 12, 19], and rotational invariance [13] (liquid characteristics) in the maximal entropy state of the $(a_1, a_2, b_1, b_2, c_1, c_2) = (1, 1, 1, 1, c, c)$ critical six-vertex model, when $\Delta = 1 - \frac{c^2}{2} \in [-1, \frac{1}{2}]$; and localization (a gaseous characteristic) in the maximal entropy state of the $(1, 1, 1, 1, c, c)$ antiferroelectric six-vertex model, when $\Delta = 1 - \frac{c^2}{2} < -1$. An intriguing open question would be to better understand all pure states of the critical and antiferroelectric six-vertex models in relation to the predicted phase diagrams (described in [28, 30]).

For the ferroelectric six-vertex model, the situation is different. As mentioned above, the results from [26, 31] do not imply existence or uniqueness for its pure states. In fact, except for its fourth
part concerning frozen phases (whose analysis follows directly from definitions), no aspect of the above phase diagram for ferroelectric six-vertex pure states had been mathematically proven until recently. To our knowledge, the only result in this direction concerns its second (KPZ) regime and appeared in [1, Appendix A.2], where a pure state $\mu(s) = \mu_{s,t}$ of any slope $(s, t) \in \partial \mathcal{H}$ was introduced for the ferroelectric six-vertex model. Additionally, [1] established both qualitative and quantitative properties for $\mu(s)$, which are considerably different from those for pure states of tiling models. For instance, it was shown by Kenyon the latter are conformally invariant [23] with Gaussian free field fluctuations [24]. In contrast, the six-vertex pure states $\mu(s)$ are quite anisotropic, exhibiting Baik–Rains fluctuations of exponent $\frac{1}{3}$ along a single direction and Gaussian fluctuations of exponent $\frac{1}{2}$ elsewhere [1]; this is an indication of the KPZ behavior manifested by these states. Essentially nothing (including existence and uniqueness) remains known about the conjecturally liquid states in the third regime of the phase diagram, where $(s, t) \notin \mathcal{H}$.

Our results in this paper are twofold. First, we show that there are no pure states of any slope $(s, t) \in \mathcal{H}$, thereby establishing the first regime of the above phase diagram. Second, we show that there is at most one pure state of any slope $(s, t) \in \partial \mathcal{H}$, thereby showing that the KPZ states introduced in [1] are the unique ones of their slopes.

To establish these results, we will make use of the stochastic six-vertex model, which is one whose six weights are of the form $(1, 1, B_1, B_2, 1 - B_1, 1 - B_2)$. This specialization was first considered by Gwa–Spohn [21] in 1992 as an instance of the six-vertex model whose weights are stochastic and therefore give rise to a Markov process with local update rules. As observed in [21], and also by Borodin–Corwin–Gorin in [4], this feature enables an interacting particle system interpretation of the stochastic six-vertex model; both algebraic and probabilistic ideas developed in the former context can then be adapted to analyze it. One reason as to why the stochastic six-vertex model is prevalent in our setting is that the pure states $\mu(s)$ described above (with slopes on $\partial \mathcal{H}$) in fact serve as its translation-invariant stationary measures. For this reason, these pure states $\mu(s)$ are sometimes referred to as stochastic Gibbs states [8].

So, we first apply a gauge transformation to view a pure state for any ferroelectric six-vertex model as one for some stochastic six-vertex model; the existence of such a transformation is guaranteed by the ferroelectricity of the original model. Next, we introduce the notion of a partition function stochastic lower bound for a pure state $\mu$ of the stochastic six-vertex model, which states the following (see Definition 3.5 below for a more precise formulation). With high probability, the partition function on a large $N \times N$ domain for this model with boundary data sampled under $\mu$ is at least $e^{-\alpha(N^2)}$. The benefit to a measure $\mu$ satisfying this property is that it can be ‘compared’ with a stochastic six-vertex model with free exit data. Indeed, since the partition function induced by the latter is equal to 1, it can be quickly shown (see Lemma 3.8) that any event very unlikely under the free stochastic model is also unlikely under $\mu$.

Our task then reduces to establishing two results. The first (see Proposition 3.6) states that any pure state $\mu$ satisfying a partition function stochastic lower bound must coincide with a stochastic Gibbs state $\mu(s)$ introduced in [1]. The second (see Proposition 3.7) states that any pure state of slope $(s, t) \in \mathcal{H}$ must satisfy a partition function stochastic lower bound. Since the measures $\mu(s)$ all have slopes in $\partial \mathcal{H}$, this shows that no pure state of slope $(s, t) \in \mathcal{H}$ can exist and that it is uniquely given by $\mu(s)$ if $(s, t) \in \partial \mathcal{H}$.

To prove the first of the two results mentioned above, we use one from [2], which states that the local statistics for any stochastic six-vertex model with free exit data are given by a stochastic Gibbs state $\mu(s)$ in the thermodynamical limit. Combining this with the above mentioned comparison between $\mu$ and such a stochastic six-vertex model, this will show $\mu = \mu(s)$. 
To prove the second, we use the property that the partition function of any stochastic Gibbs state on an $N \times N$ domain is likely at least $e^{-O(N)}$ (differing from the more typical $e^{O(N^2)}$ asymptotics expected for liquid states). This can quickly be deduced from the facts that stochastic Gibbs states are stationary measures for the stochastic six-vertex model, that the total partition function for any such model with free exit data is equal to 1, and that there are at most $e^{O(N)}$ choices of exit data. We will then compare the partition function for the pure state $\mu$ of slope $(s, t) \in S$ with that for a stochastic Gibbs state, as follows. First, we introduce a ‘parsification procedure’ that reduces the slope of any pure state, while only reducing its partition function by at most a factor of $e^{o(N^2)}$.

Next, since $(s, t) \in H$, there exists a slope $(s_0, t_0) \in \partial H$ with $s_0 \geq s$ and $t_0 \geq t$ (see Figure 9). We can then interpret the pure state $\mu$, of slope $(s, t)$, as a parsification of the stochastic Gibbs state $\mu(s_0)$, of slope $(s_0, t_0)$. Since the partition function of the latter is at least $e^{-O(N)}$, and since parsification does not decrease partition functions by more than $e^{o(N^2)}$, this will yield the required lower bound of $e^{-o(N^2)}$ for the partition function induced by $\mu$.

Throughout this paper, we let $\mathbb{P}_\mu$ and $\mathbb{E}_\mu$ denote the probability and expectation with respect to any measure $\mu$, respectively. Furthermore, we let $\mathbb{1}_A$ denote the indicator function for $A$.

### 1.2 Gibbs measures for the six-vertex model

A domain $\Lambda \subseteq \mathbb{Z}^2$ is a connected induced subgraph of $\mathbb{Z}^2$. The boundary of $\Lambda$, denoted by $\partial \Lambda \subseteq \Lambda$, is the set of vertices in $\mathbb{Z}^2 \setminus \Lambda$ that are adjacent to some vertex in $\Lambda$, and the closure of $\Lambda$ is defined to be the union $\overline{\Lambda} = \Lambda \cup \partial \Lambda$ of it with its boundary.

We now define six-vertex ensembles on domains $\Lambda \subseteq \mathbb{Z}^2$, to which we begin by introducing arrow configurations. An arrow configuration is a quadruple $(i_1, j_1; i_2, j_2)$ such that $i_1, j_1, i_2, j_2 \in \{0, 1\}$ and $i_1 + j_1 = i_2 + j_2$. We view such a quadruple as an assignment of arrows to a vertex $v \in \Lambda$. Specifically, $i_1$ and $j_1$ denote the numbers of vertical and horizontal arrows entering $v$, respectively; similarly, $i_2$ and $j_2$ denote the numbers of vertical and horizontal arrows exiting $v$, respectively. The fact that $i_1 + j_1 = i_2 + j_2$ means that the numbers of incoming and outgoing arrows at $v$ are equal; this is sometimes referred to as arrow conservation. There are six possible arrow configurations, which are depicted on the left side of Figure 2.

A six-vertex ensemble on $\Lambda$ is an assignment of an arrow configuration to each vertex of $\Lambda$ in such a way that neighboring arrow configurations are consistent; this means that, if $v_1, v_2 \in \Lambda$ are two adjacent vertices, then there is an arrow to $v_2$ in the configuration at $v_1$ if and only if there is an arrow from $v_1$ in the configuration at $v_2$. Observe in particular that the arrows in a six-vertex ensemble form noncrossing up-right directed paths connecting vertices of $\Lambda$, which enter and exit through its boundaries; see the right side of Figure 2 for a depiction.

Let $\mathfrak{C}(\Lambda)$ denote the set of six-vertex ensembles on $\Lambda$. For any $\mathcal{E} \in \mathfrak{C}(\Lambda)$ and subdomain $\Lambda' \subseteq \Lambda$, we let $\mathcal{E}|_{\Lambda'} \in \mathfrak{C}(\Lambda')$ denote the restriction of $\mathcal{E}$ to $\Lambda'$. We refer to Figure 3 for a depiction. A cylinder set in $\mathfrak{C}(\Lambda)$ is one of the form $\{\mathcal{E} \in \mathfrak{C}(\Lambda) : \mathcal{E}|_{\Lambda'} = \mathcal{F}\}$, for some finite subdomain $\Lambda' \subseteq \Lambda$ and six-vertex ensemble $\mathcal{F} \in \mathfrak{C}(\Lambda')$. Assigning to $\mathfrak{C}(\Lambda)$ the $\sigma$-algebra generated by all cylinder sets, let $\mathfrak{P}(\mathfrak{C}(\Lambda))$ denote the space of probability measures on $\mathfrak{C}(\Lambda)$.

For any $\mathcal{E} \in \mathfrak{C}(\Lambda)$ and $(x, y) \in \Lambda$, let $\chi^{(v)}(x, y) = \chi^{(v)}(x, y) \in \{0, 1\}$ denote the indicator for the event that an arrow in $\mathcal{E}$ vertically exits from $(x, y)$. Stated alternatively, letting $(i_1(x, y), j_1(x, y); i_2(x, y), j_2(x, y))$ denote the arrow configuration at $(x, y)$, we set $\chi^{(v)}(x, y) = i_1(x, y + 1) = i_2(x, y)$. Similarly, let $\chi^{(h)}(x, y) = \chi^{(h)}(x, y) = j_1(x + 1, y) = j_2(x, y) \in \{0, 1\}$ denote the indicator for the event that an arrow in $\mathcal{E}$ horizontally exits through $(x, y)$. 
FIGURE 2  The chart to the left shows all six possible arrow configurations, along with the associated vertex weights. An example of a six-vertex ensemble is shown to the right.

| Configuration | $a_1$ | $b_1$ | $c_1$ |
|---------------|-------|-------|-------|
| $(0,0;0,0)$   |       |       |       |
| $(1,0;0,1)$   |       |       |       |
| $(1,0;0,1)$   |       |       |       |
| $(1,1;1,1)$   |       |       |       |
| $(0,1;0,1)$   |       |       |       |
| $(0,1;0,1)$   |       |       |       |

FIGURE 3  Depicted above is the Gibbs property from Definition 1.1

Next fix six real numbers $a_1, a_2, b_1, b_2, c_1, c_2 > 0$. We assign a weight $w(i_1, j_1; i_2, j_2)$ to each of the six possible arrow configurations by setting

$$w(0,0;0,0) = a_1; \quad w(1,0;0,1) = b_1; \quad w(1,0;0,1) = c_1;$$

$$w(1,1;1,1) = a_2; \quad w(0,1;0,1) = b_2; \quad w(0,1;1,0) = c_2,$$

and $w(i_1, j_1; i_2, j_2) = 0$ for any $(i_1, j_1; i_2, j_2)$ not of the above form. We refer to the left side of Figure 2 for a depiction.

Given some domain $\Lambda \subseteq \mathbb{Z}^2$, a six-vertex ensemble $\mathcal{E} \in \mathcal{E}(\Lambda)$, and a vertex $v \in \Lambda$ with some arrow configuration $(i_1, j_1; i_2, j_2) = (i_1(v), j_1(v); i_2(v), j_2(v))$ under $\mathcal{E}$, we define the weight of $v$ with respect to $\mathcal{E}$ to be $w_{\mathcal{E}}(v) = w(i_1, j_1; i_2, j_2)$. If $\Lambda$ is finite, we may define the weight $w(\mathcal{E})$ of $\mathcal{E}$ to be the product of the weights of its vertices, namely,

$$w(\mathcal{E}) = \prod_{v \in \Lambda} w_{\mathcal{E}}(v).$$
The six-vertex model (with free boundary conditions) on \( \Lambda \) is then the probability measure \( \mathbb{P} = \mathbb{P}(a_1, a_2, b_1, b_2, c_1, c_2) \in \mathcal{P}(\mathcal{G}(\Lambda)) \) such that \( \mathbb{P}[\mathcal{E}] = Z^{-1}w(\mathcal{E}) \) for any six-vertex ensemble \( \mathcal{E} \in \mathcal{G}(\Lambda) \), where \( Z = \sum_{\mathcal{E} \in \mathcal{G}(\Lambda)} w(\mathcal{E}) \) is the partition function chosen so these probabilities sum to 1.

This definition no longer applies directly when \( \Lambda \) is infinite, since then the product (1.3) might not converge. In this case, we consider the notion of Gibbs measures, given as follows.

**Definition 1.1.** Let \( \Lambda \subseteq \mathbb{Z}^2 \) denote a domain, and let \( \mu \in \mathcal{P}(\mathcal{G}(\Lambda)) \) denote a measure. We say that \( \mu \) satisfies the Gibbs property, or is a Gibbs measure, for the six-vertex model with weights \( (a_1, a_2, b_1, b_2, c_1, c_2) \) if the following holds for any finite subdomain \( \Lambda' \subseteq \Lambda \). Sample \( \mathcal{E} \in \mathcal{G}(\Lambda) \) consistent with \( \mathcal{H} \), the conditional probability that \( \mathcal{E}_{\Lambda'} = \mathcal{E}' \) is proportional to \( w(\mathcal{E}') \) from (1.3). Stated alternatively, \( \mathbb{P}_\mu[\mathcal{E}_{\Lambda'} = \mathcal{E}'] = Z^{-1}w(\mathcal{E}') \) for any \( \mathcal{E}' \in \mathcal{G}(\Lambda') \) consistent with \( \mathcal{H} \), where \( Z = Z_{\Lambda';\mu;\mathcal{H}} \) denotes the constant so that these probabilities sum to 1. We refer to Figure 3 for a depiction.

Let us set some additional notation when \( \Lambda = \mathbb{Z}^2 \). For any \( u \in \mathbb{Z}^2 \), define the translation map \( \mathfrak{T}_u : \mathbb{Z}^2 \to \mathbb{Z}^2 \) by setting \( \mathfrak{T}_u(v) = v - u \) for each \( v \in \mathbb{Z}^2 \). Then \( \mathfrak{T}_u \) induces a map on both \( \mathcal{G}(\mathbb{Z}^2) \) and \( \mathcal{P}(\mathcal{G}(\mathbb{Z}^2)) \) that translates a six-vertex ensemble by \( -u \); we also refer to them by \( \mathfrak{T}_u \). A measure \( \mu \in \mathcal{P}(\mathcal{G}(\mathbb{Z}^2)) \) is called translation-invariant if \( \mathfrak{T}_u \mu = \mu \), for any \( u \in \mathbb{Z}^2 \). Recalling the horizontal and vertical indicator functions \( \chi^{(v)}(x, y) = \chi^{(v)}(x, y) \) and \( \chi^{(h)}(x, y) = \chi^{(h)}(x, y) \) associated with any six-vertex ensemble \( \mathcal{E} \in \mathcal{G}(\Lambda) \), we define the slope of any translation-invariant Gibbs measure \( \mu \in \mathcal{P}(\mathcal{G}(\mathbb{Z}^2)) \) to be the pair \( (\mathbb{E}_\mu[\chi^{(v)}(0, 0)], \mathbb{E}_\mu[\chi^{(h)}(0, 0)]) \in [0, 1] \times [0, 1] \).

Similarly to as above, an event \( \mathcal{A} \) in the \( \sigma \)-algebra associated with \( \mathcal{G}(\mathbb{Z}^2) \) is called translation-invariant if \( \mathfrak{T}_u \mathcal{A} = \mathcal{A} \), for any \( u \in \mathbb{Z}^2 \). We further call a translation-invariant measure \( \mu \in \mathcal{P}(\mathcal{G}(\mathbb{Z}^2)) \) ergodic if, for any translation-invariant event \( \mathcal{A} \), we have that \( \mathbb{P}_\mu[\mathcal{A}] \in \{0, 1\} \).

A translation-invariant, ergodic Gibbs measure for the six-vertex model on \( \mathbb{Z}^2 \) is called a pure state.

### 1.3 Results

To state our results, we must first explicitly parameterize the ‘lens-shaped’ region \( \mathfrak{S} \) (shaded in Figure 1) where pure states of the ferroelectric six-vertex model should not exist.

So, fix parameters \( a_1, a_2, b_1, b_2, c_1, c_2 > 0 \) such that the \( \Delta \) from (1.1) satisfies \( \Delta > 1 \). We further assume throughout this paper that \( a_1 a_2 > b_1 b_2 \). The results we will state can then be obtained in the opposite regime by ‘complementing’ vertical edges, that is, by replacing the arrow configuration \( (i_1(x, y), j_1(x, y); i_2(x, y), j_2(x, y)) \) at \( (x, y) \) with \( (1 - i_2(x, -y), j_1(x, -y); 1 - i_1(x, -y), j_2(x, -y)) \) (equivalently, by first switching vertical arrows with absences of them and then reflecting the ensemble across the \( x \)-axis). This amounts to changing the six-vertex weights from \( (a_1, a_2, b_1, b_2, c_1, c_2) \) to \( (b_1, b_2, a_1, a_2, c_1, c_2) \) and slope of an associated pure state from \( (s, t) \) to \( (1 - s, t) \). Thus, by applying these symmetries, one can transform from the \( a_1 a_2 > b_1 b_2 \) setting to the \( b_1 b_2 > a_1 a_2 \) one.

Now, define the function \( \mathfrak{h} = \mathfrak{h}(a_1, a_2, b_1, b_2, c_1, c_2) : \mathbb{R}^2 \to \mathbb{R} \) by setting

\[
\mathfrak{h}(x, y) = 2xy\sqrt{\Delta^2 - 1} - \left( \sqrt{\frac{a_1 a_2}{b_1 b_2}} - \Delta + \sqrt{\Delta^2 - 1} \right)x + \left( \sqrt{\frac{a_1 a_2}{b_1 b_2}} - \Delta - \sqrt{\Delta^2 - 1} \right)y,
\] (1.4)
and the region \( \mathcal{H} = \mathcal{H}(a_1, a_2, b_1, b_2, c_1, c_2) \subset [0, 1]^2 \) by setting
\[
\mathcal{H} = \left\{ (s, t) \in (0, 1)^2 : \max \{ \mathfrak{h}(s, t), \mathfrak{h}(t, s) \} < 0 \right\}.
\]

Further let \( \partial \mathcal{H} \) and \( \overline{\mathcal{H}} = \mathcal{H} \cup \partial \mathcal{H} \) denote the boundary and closure of \( \mathcal{H} \), respectively. In particular, defining the hyperbolas
\[
\mathfrak{h}_1 = \left\{ (s, t) \in [0, 1]^2 : \mathfrak{h}(s, t) = 0 \right\}; \quad \mathfrak{h}_2 = \left\{ (s, t) \in [0, 1]^2 : \mathfrak{h}(t, s) = 0 \right\},
\]
we have that \( \partial \mathcal{H} = \mathfrak{h}_1 \cup \mathfrak{h}_2 \). In the context of the ferroelectric six-vertex model, these curves \( \mathfrak{h}_1 \) and \( \mathfrak{h}_2 \) originally appeared as equation (3.38) of [5]. We refer to Figure 1 for a depiction.

Now we can state the following theorem, which will be established in Section 3.2.

**Theorem 1.2.** Fix \( a_1, a_2, b_1, b_2, c_1, c_2 > 0 \) with \( a_1 a_2 > b_1 b_2 \) and \( \Delta > 1 \), and let \( (s, t) \in \overline{\mathcal{H}} \).

1. If \( (s, t) \in \mathcal{H} \), then there does not exist a pure state for the six-vertex model with weights \( (a_1, a_2, b_1, b_2, c_1, c_2) \) and slope \( (s, t) \).
2. If \( (s, t) \in \partial \mathcal{H} \), then there exists a unique pure state for the six-vertex model with weights \( (a_1, a_2, b_1, b_2, c_1, c_2) \) and slope \( (s, t) \).

The measures described in the second part of Theorem 1.2 are reasonably explicit; they were introduced in [1] and will be recalled in Section 2.4 (see also the proof of Corollary 3.4).

The remainder of this paper is organized as follows. In Section 2, we set notation and recall several results that will be used later in this article. Next, in Section 3.2, we establish Theorem 1.2 assuming two results given by Proposition 3.6 and Proposition 3.7. The former is then established in Section 3.4 and the latter in Section 4.

## 2 | MISCELLANEOUS PRELIMINARIES

In this section, we introduce some notation and collect several (primarily known) results that will be used later in this work. We begin in Section 2.1 by describing notation for six-vertex ensembles that will be used throughout this paper. We then in Section 2.2 establish a result for the regularity of boundary data induced by a pure state of the six-vertex model. In Section 2.3, we recall the definition of the stochastic six-vertex model from [21], and in Section 2.4 we recall a family of pure states associated with it from [1]. We then in Section 2.5 describe a result from [2] for the convergence of local statistics in the stochastic six-vertex model with free exit data.

### 2.1 | Paths in six-vertex ensembles

Here we set some notation for paths in a six-vertex ensemble on a rectangular domain, namely, one of the form \([M_1, M_2] \times [N_1, N_2] \subset \mathbb{Z}^2\), for some integers \( M_1 \leq M_2 \) and \( N_1 \leq N_2 \). We set its south, west, north, and east boundaries to be \([M_1, M_2] \times \{N_1 - 1\}, \{M_1 - 1\} \times [N_1, N_2], [M_1, M_2] \times \{N_1 - 1\}, \{M_1 - 1\} \times [N_1, N_2]\).
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For any two points \( z_1 = (x_1, y_1) \in \mathbb{R}^2 \) and \( z_2 = (x_2, y_2) \in \mathbb{R}^2 \), we write \( z_1 \geq z_2 \), or equivalently \( z_2 \leq z_1 \), if \( x_1 \geq x_2 \) and \( y_1 \geq y_2 \). A curve \( p \subset \mathbb{R}^2 \) is nondecreasing if, for any \( z_1, z_2 \in p \) we either have that \( z_1 \geq z_2 \) or \( z_2 \geq z_1 \). Given some domain \( \Lambda \subseteq \mathbb{Z}^2 \), a (directed) path \( p \) on \( \Lambda \) is a continuous, nondecreasing curve in \( \mathbb{R}^2 \) connecting a sequence of adjacent vertices in \( \Lambda \) by edges of \( \mathbb{Z}^2 \), such that no edge of \( p \) connects two vertices in \( \partial \Lambda \). Any compact path \( p \) has a starting point, which is the unique \( z \in p \) such that \( z \leq z' \) for any \( z' \in p \). It also has an ending point, which is the unique \( z \in p \) such that \( z \geq z' \) for any \( z' \in p \). We say \( p \) enters \( \Lambda \) through \( z \) if \( z \) is the starting point for \( p \) and \( z \in \partial \Lambda \). Similarly, \( p \) exits \( \Lambda \) through \( z \) if \( z \) is the ending point for \( p \) and \( z \in \partial \Lambda \).

Given two paths \( p_1 \) and \( p_2 \) on \( \Lambda \), we say that \( p_1 \) lies below \( p_2 \), or equivalently that \( p_2 \) lies above \( p_1 \), if the following two statements hold. First, for any \( (x_1, y_1) \in p_1 \), there exists some \((x_2, y_2) \in p_2 \) such that \( x_1 \geq x_2 \) and \( y_1 \leq y_2 \). Second, for any \( (x_2, y_2) \in p_2 \), there exists some \((x_1, y_1) \in p_1 \) such that \( x_1 \geq x_2 \) and \( y_1 \leq y_2 \). If these two properties are satisfied, then we write \( p_1 \preceq p_2 \). An ensemble of paths \( P = (p_1, p_2, \ldots, p_k) \) is called noncrossing if \( p_1 \preceq p_2 \preceq \cdots \preceq p_k \) and no two paths \( p_i, p_j \in P \) share an edge (although they may share vertices); see Figure 4.

Under this notation, six-vertex ensembles on a rectangular domain \( \Lambda \subset \mathbb{Z}^2 \) are in bijective correspondence with noncrossing path ensembles on \( \Lambda \), each of whose paths enters \( \Lambda \) through a vertex on its south or west boundary and exits \( \Lambda \) through a vertex its north or east boundary. We index the paths in any such ensemble \( P = (p_{-B}, p_{1-B}, \ldots, p_A) \), for some integers \( A = A(P) \geq 0 \) and \( B = B(P) \geq -1 \), so that \( p_{-B}, p_{1-B}, \ldots, p_0 \) enter through the south boundary of \( \Lambda \), and \( p_1, p_2, \ldots, p_A \) enter through the west boundary. In this way, there are \( A + B + 1 \) paths in \( P \), and \( p_1 \) is the bottommost path entering through the west boundary of \( \Lambda \). We refer to Figure 4 for a depiction of the noncrossing path ensemble associated with the six-vertex ensemble on the right side of Figure 2.

For each integer \( i \in [-B, A] \), let \( u_i \in \partial \Lambda \) denote the vertex through which the path \( p_i \) enters \( \Lambda \), and let \( v_i \in \partial \Lambda \) denote the vertex through which \( p_i \) exits \( \Lambda \). We then refer to the \((A + B + 1)\)-tuple \( u = (u_{-B}, u_{1-B}, \ldots, u_A) \subseteq \partial \Lambda \) as the entrance data for \( P \) (or equivalently for the associated six-vertex ensemble) and to the \((A + B + 1)\)-tuple \( v = (v_{-B}, v_{1-B}, \ldots, v_A) \subseteq \partial \Lambda \) as the exit data for \( P \). Boundary data for \( P \) consists of the union \( u \cup v \) of its entrance and exit data.
Let $\mathfrak{C}_u(\Lambda) \subseteq \mathfrak{C}(\Lambda)$ denote the set of six-vertex ensembles on $\Lambda$ with entrance data given by $u$, and let $\mathfrak{C}_{uv}(\Lambda) \subseteq \mathfrak{C}_u(\Lambda)$ denote the set of those with boundary data given by $u \cup v$. The six-vertex model with entrance data $u$ (and free exit data) is the probability measure on $\mathfrak{C}_u(\Lambda)$ that assigns probability $\mathbb{P}[\mathcal{E}] = \mathbb{P}_u,\Lambda[\mathcal{E}] = Z_{u,\Lambda}^{-1} w(\mathcal{E})$ to any $\mathcal{E} \in \mathfrak{C}_u(\Lambda)$, where we recalled the weight $w(\mathcal{E})$ from (1.3) and defined the normalizing constant $Z_{u,\Lambda} = \sum_{\mathcal{E} \in \mathfrak{C}_u(\Lambda)} w(\mathcal{E})$ so that these probabilities sum to 1. Similarly, the six-vertex model with boundary data $u \cup v$ is the probability measure on $\mathfrak{C}_{uv}(\Lambda)$ that assigns probability $\mathbb{P}[\mathcal{E}] = \mathbb{P}_u,\Lambda,v[\mathcal{E}] = Z_{u,\Lambda}^{-1} w(\mathcal{E})$ to any $\mathcal{E} \in \mathfrak{C}_{uv}(\Lambda)$, where $Z_{u,v,\Lambda} = \sum_{\mathcal{E} \in \mathfrak{C}_{uv}(\Lambda)} w(\mathcal{E})$.

2.2 Regularity for boundary data

In this section, we introduce the notion of regularity for boundary data of six-vertex ensembles and show that the boundary data induced by a pure state is likely regular. In what follows, we refer to a subset $I \subseteq \mathbb{Z}^2$ as an interval if it is of the form $I = [A_1, A_2] \times \{B\}$ or $I = \{A\} \times [B_1, B_2]$ for some $A_1, A_2, B, B_1, B_2 \in \mathbb{Z}$ with $A_1 \leq A_2$ or $B_1 \leq B_2$; in the former case, this interval is horizontal, and in the latter case it is vertical. For any interval $I \subseteq \mathbb{Z}^2$, we let $|I|$ denote the number of vertices in $I$. In particular, if $I = [A_1, A_2] \times \{B\}$ or $I = \{A\} \times [B_1, B_2]$, then $|I| = A_2 - A_1 + 1$ or $|I| = B_2 - B_1 + 1$, respectively.

We begin with the following two definitions. The first states that a set is $(R; \eta)$-regular with slope $\rho$ on an interval $J$ if it $\eta$-approximates the Lebesgue measure of density $\rho$ on any subinterval $I \subseteq J$ of length $R$. These second states that boundary data on a rectangular domain $\Lambda$ is regular if it is regular on each of the four boundaries of $\Lambda$.

**Definition 2.1.** Fix real numbers $\eta, \rho \in (0, 1]$ and $R \geq 1$. For any subset $u \subseteq \mathbb{Z}^2$ and finite interval $J \subset \mathbb{Z}^2$ of the form $[A_1, A_2] \times \{B\}$ or $\{A\} \times [B_1, B_2]$, we say $u$ is $(R; \eta)$-regular with slope $\rho$ on $J$ if the following holds. For any interval $I \subseteq J$ with $|I| \leq R$, we have $|I \cap u - \rho|I|| \leq \eta R$.

**Definition 2.2.** Fix real numbers $\eta, s, t \in (0, 1]$ and $R \geq 1$. Let $\Lambda \subset \mathbb{Z}^2$ be a rectangular domain, and let $u \cup v$ be boundary data on $\Lambda$. We say $u \cup v$ is $(R; \eta)$-regular with slope $(s, t)$ if it is $(R; \eta)$-regular with slope $s$ along both the north and south boundaries of $\Lambda$, and it is $(R; \eta)$-regular with slope $t$ along both the west and east boundaries of $\Lambda$.

**Remark 2.3.** Suppose $u \subseteq \mathbb{Z}^2$ is $(R; \eta)$-regular with slope $s$ on some interval $J \subset \mathbb{Z}^2$. Then, for any interval $I \subseteq J$ such that $|I| \geq R$, we have $||u \cap I| - s|I|| < 2\eta|I|$. Indeed this follows from the fact that $I$ can be covered by a family of at most $2R^{-1}|I|$ intervals, each of length bounded above by $R$.

The following lemma states that boundary data induced by a pure state of slope $(s, t)$ is likely regular with this slope.

**Lemma 2.4.** Fix a pair $(s, t) \in (0, 1]^2$ and a translation-invariant, ergodic measure $\mu \in \mathcal{P}(\mathfrak{C}(\mathbb{Z}^2))$ of slope $(s, t)$. For any real number $\eta \in (0, 1)$, there exists a constant $C_0 = C_0(\eta, \mu) > 1$ such that the following holds. Let $R > C_0$ be any real number, and $M, N \in [R, \eta^{-1}R]$ be integers; set $\Lambda = [1, M] \times [1, N] \subset \mathbb{Z}^2$. Sample a random six-vertex ensemble $\mathcal{E} \in \mathfrak{C}(\mathbb{Z}^2)$ under $\mu$, and let $u \cup v$ denote the boundary data induced on $\Lambda$ by the restriction $\mathcal{E}_{\mathbb{Z}^2 \setminus \Lambda}$. Then, with probability at least $1 - \eta$, the boundary data $u \cup v$ is $(R; \eta)$-regular with slope $(s, t)$. 
To establish this lemma, we first require the following one, which provides a law of large numbers for sums of the indicator functions $\chi^{(v)}(u) = \chi^{(v)}_{\mathcal{E}}(u)$ and $\chi^{(h)}(u) = \chi^{(h)}_{\mathcal{E}}(u)$ (from Section 1.2) along horizontal and vertical lines.

**Lemma 2.5.** Let $\mu \in \mathcal{P}(\mathfrak{G}(\mathbb{Z}^2))$ be a translation-invariant, ergodic measure of slope $(s, t) \in (0, 1]^2$, and let $\mathcal{E} \in \mathfrak{G}(\mathbb{Z}^2)$ be a random six-vertex ensemble sampled under $\mu$. For any real number $\delta > 0$, there exists a constant $C_0 = C_0(\delta; \mu) > 1$ such that

$$
\mathbb{P}_\mu \left[ \left| \frac{1}{N} \sum_{x=0}^{N-1} \chi^{(v)}_{\mathcal{E}}(x, 0) - s \right| < \delta \right] \geq 1 - \delta;
\mathbb{P}_\mu \left[ \left| \frac{1}{N} \sum_{y=0}^{N-1} \chi^{(v)}_{\mathcal{E}}(0, y) - t \right| < \delta \right] \geq 1 - \delta,
$$

for any integer $N > C_0$.

**Proof.** Let us only establish the first bound in (2.1), as the proof of the latter is entirely analogous. To that end, first define the map $f : \mathfrak{G}(\mathbb{Z}^2) \to \{0, 1\}^{\mathbb{Z}^2}$ so that, for any six-vertex ensemble $\mathcal{E} \in \mathfrak{G}(\mathbb{Z}^2)$, the value of $f(\mathcal{E})$ at some vertex $u \in \mathbb{Z}^2$ is the indicator $\chi^{(v)}_{\mathcal{E}}(u) \in \{0, 1\}$. Let $\lambda = f_\# \mu$ denote the probability measure on $\{0, 1\}^{\mathbb{Z}^2}$ obtained as the pushforward of $\mu$ under $f$.

Next, let $\lambda_0$ denote the marginal of $\lambda$ on the $x$-axis $\mathbb{Z} \times \{0\}$; in this way, $\lambda_0$ is a probability measure on $\{0, 1\}^\mathbb{Z}$. Since $\mu$ is translation-invariant, as is $\lambda$, and therefore $\lambda_0$ is as well. Hence, since $\lambda_0$ prescribes the law of $(\chi^{(v)}_{\mathcal{E}}(x, 0))_{x \in \mathbb{Z}} \in \{0, 1\}^\mathbb{Z}$ under $\mu$, the strong ergodic theorem implies the almost sure (with respect to $\mu$) existence of the limit

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{x=0}^{N-1} \chi^{(v)}_{\mathcal{E}}(x, 0) = S(\mathcal{E}).
$$

Now, for any $N_1, N_2, y \in \mathbb{Z}$ with $N_1 \leq N_2$ and any $\mathcal{E} \in \mathfrak{G}(\mathbb{Z}^2)$, observe that

$$
\left| \sum_{x=N_1}^{N_2} \chi^{(v)}_{\mathcal{E}}(x, y + 1) - \sum_{x=N_1}^{N_2} \chi^{(v)}_{\mathcal{E}}(x, y) \right| \leq 1.
$$

In particular, it follows for any $k \geq 0$ that

$$
\left| \sum_{x=0}^{N-1} \chi^{(v)}_{\mathcal{E}}(x, 0) - \sum_{x=0}^{N-1} \chi^{(v)}_{\mathcal{E}}(x, k) \right| \leq k.
$$

Summing over $k \in [0, M - 1]$, this gives

$$
\left| \frac{1}{N} \sum_{x=0}^{N-1} \chi^{(v)}_{\mathcal{E}}(x, 0) - \frac{1}{MN} \sum_{y=0}^{M-1} \sum_{x=0}^{N-1} \chi^{(v)}_{\mathcal{E}}(x, y) \right| \leq \frac{1}{MN} \left| \sum_{x=0}^{N-1} \chi^{(v)}_{\mathcal{E}}(x, 0) - \sum_{y=0}^{M-1} \sum_{x=0}^{N-1} \chi^{(v)}_{\mathcal{E}}(x, y) \right| \leq \frac{M}{N}.
$$

Combining this with (2.2), we deduce that

$$
\lim_{M \to \infty} \left( \lim_{N \to \infty} \frac{1}{MN} \sum_{y=0}^{M-1} \sum_{x=0}^{N-1} \chi^{(v)}_{\mathcal{E}}(x, y) \right) = S(\mathcal{E}),
$$

(2.3)
holds almost surely with respect to $\mu$. Then, since
\[
\lim_{M \to \infty} \left( \lim_{N \to \infty} \frac{1}{MN} \left| \delta([0,N-1] \times [0,M-1]) \right| \right) = 0,
\]
the event
\[
\left\{ \lim_{M \to \infty} \left( \lim_{N \to \infty} \frac{1}{MN} \sum_{y=0}^{M-1} \sum_{x=0}^{N-1} \chi^v(x,y) \right) \right\} \in I
\]
is invariant with respect to translations by elements of $\mathbb{Z}^2$, for any interval $I \subseteq [0,1]$.

Therefore, (2.3) and the ergodicity of $\mu$ together imply $\mathbb{P}_\mu[S(\mathcal{E}) \in I] \in \{0, 1\}$, for any $I \subseteq [0,1]$. It follows that there exists some $s_0 \in [0,1]$ such that $S(\mathcal{E}) = s_0$ almost surely under $\mu$. Since $\mu$ has slope $(s, t)$, we have $s_0 = s$, and so the first estimate in (2.1) follows from the almost sure limit (2.2).

As mentioned previously, the proof of the second is very similar and is therefore omitted. □

Now we can establish Lemma 2.4.

Proof of Lemma 2.4. Set $\delta = \frac{\eta^3}{12}$, and let $I \subset \partial \Lambda$ be an interval satisfying $\frac{\eta R}{3} \leq |I| \leq \frac{\eta R}{2}$. If the constant $C_0(\eta, \mu)$ here is chosen to be larger than the $3\eta^{-1}C_0(\delta, \mu)$ from Lemma 2.5, then (2.1) implies
\[
\mathbb{P}_\mu \left[ \left| I \cap (u \cup v) - s|I| \right| < \delta R \right] \geq 1 - \delta, \quad \text{or} \quad \mathbb{P}_\mu \left[ \left| I \cap (u \cup v) - t|I| \right| < \delta R \right] \geq 1 - \delta, \quad (2.4)
\]
if $I$ lies on either the north or south boundary of $\Lambda$, or if $I$ lies on either the east or west boundary of $\Lambda$, respectively.

Next let $I$ be a set of mutually disjoint intervals of lengths between $\frac{\eta R}{3}$ and $\frac{\eta R}{2}$, whose union constitutes the north and south boundaries of $\Lambda$. Similarly, let $J$ be a set of mutually disjoint intervals of lengths between $\frac{\eta R}{3}$ and $\frac{\eta R}{2}$, whose union constitutes the east and west boundaries of $\Lambda$. Defining the event
\[
\mathcal{A} = \left\{ \max_{I \in I} \left| I \cap (u \cup v) - s|I| \right| < \delta R \right\} \cap \left\{ \max_{J \in J} \left| I \cap (u \cup v) - t|I| \right| < \delta R \right\},
\]
observe that boundary data $u \cup v$ is $(R; \eta)$-regular with slope $(s, t)$ on $\Lambda$. Indeed, suppose, for instance, $J$ is an interval on the west boundary of $\partial \Lambda$ of length at most $R$. If $|J| \leq \eta R$, then $|I \cap (u \cup v) - t|I| | \leq |J| \leq \eta R$. If instead $|J| > \eta R$, then $J$ admits a cover by at most $3\eta^{-1}$ intervals in $J$, and so on the event $\mathcal{A}$ we have $|I \cap (u \cup v) - t|I| | \leq 3\eta^{-1}\delta R \leq \eta R$. This establishes the bound $|I \cap (u \cup v) - t|I| | \leq \eta R$ in both cases and therefore verifies the $(R; \eta)$-regularity with slope of $u \cup v$ along the west boundary of $\partial \Lambda$. The proof that $u \cup v$ is $(R; \eta)$-regular (with the appropriate slope) along the other three boundaries is entirely analogous and is therefore omitted.

It therefore suffices to lower bound the probability $\mathbb{P}_\mu[\mathcal{A}]$, which we will do by estimating the sizes of $I$ and $J$. To that end, observe that since the total length of the north and south boundaries of $\Lambda$ is $2M \leq 2\eta^{-1}R$, we have $|I| \leq \frac{6M}{\eta R} \leq 6\eta^{-2}$ and $|J| \leq \frac{6N}{\eta R} \leq 6\eta^{-2}$. Thus, by a union bound over (2.4), we deduce that $\mathbb{P}_\mu[\mathcal{A}] \geq 1 - 12\eta^{-2}\delta = 1 - \eta$, which implies the lemma. □
2.3 Stochastic six-vertex model

Fix real numbers $B_1, B_2 \in (0, 1)$. Introduced in [21], the $(B_1, B_2)$-stochastic six-vertex model is the special case of the six-vertex model whose weights are given by

\[ w(0, 0; 0, 0) = 1; \quad w(1, 0; 1, 0) = B_1; \quad w(1, 0; 0, 1) = 1 - B_1; \]
\[ w(1, 1; 1, 1) = 1; \quad w(0, 1; 0, 1) = B_2; \quad w(0, 1; 1, 0) = 1 - B_2. \] (2.5)

These weights are stochastic, in that they satisfy the property that sum of the weights of all arrow configurations with a fixed set of incoming arrows is equal to 1. Stated alternatively, we have $\sum i_2,j_2 w(i_1,j_1;i_2,j_2) = 1$ for any fixed $(i_1, j_1) \in \{0, 1\} \times \{0, 1\}$.

The choice (2.5) enables a Markovian sampling procedure for the stochastic six-vertex model with fixed entrance data. Let us explain this sampling in more detail for the model on a rectangular domain $\Lambda = [M_1, M_2] \times [N_1, N_2] \subset \mathbb{Z}^2$; we will also permit $\infty \in \{M_2, N_2\}$, allowing $\Lambda$ to be a quadrant. We may assume by translation that $M_1 = N_1$, and so we set $M_2 = M$ and $N_2 = N$.

For each integer $n \in [2, M + N]$, we will define a probability measure $P_n = P_{n,\Lambda}(B_1, B_2)$ on the set of six-vertex ensembles whose vertices are all contained in the subdomain $T_n = \{(x, y) \in \mathbb{Z}^2_{\geq 0} : x + y \leq n\} \cap \Lambda \subseteq \Lambda$. The stochastic six-vertex model on $\Lambda$, denoted by $P = P_{\Lambda}(B_1, B_2)$, will then be set to $P_{M+N}$. For each positive integer $n$, we define $P_{n+1}$ from $P_n$ through the following Markovian update rules (in the case $n = 1$, $T_n$ is empty, and so $P(\mathcal{E}(T_1))$ is empty).

Use $P_n$ to sample a six-vertex ensemble $\mathcal{E}_n$ on $T_n$, which assigns an arrow configuration to each vertex in $\Lambda$ strictly below the diagonal $D_n = \{(x, y) \in \mathbb{Z}^2_{\geq 0} : x + y = n\} \cap \Lambda$. Each vertex on $D_n$ is also given “half” of an arrow configuration, in the sense that it is given the directions of all entering paths but no direction of any exiting path; we refer to the left side of Figure 5 for a depiction. To extend $\mathcal{E}_n$ to an ensemble on $T_{n+1}$, we must ‘complete’ the configurations (specify the exiting paths) at all vertices $v \in D_n$. Any half-configuration can be completed in at most two ways; selecting between these completions is done randomly, according to the probabilities (2.5). All choices are mutually independent.

In this way, we obtain a random ensemble $\mathcal{E}_{n+1}$ on $T_{n+1}$; the resulting probability measure on path ensembles with vertices in $T_{n+1}$ is denoted by $P_{n+1}$. Then define $P = P_{M+N}$ if $M + N$ is finite and $P = \lim_{n \to \infty} P_n$ if $M + N = \infty$. 

FIGURE 5 The sampling procedure for the stochastic six-vertex model is depicted on the left. The Markov property described in Remark 2.6 is depicted on the right.
Remark 2.6. Observe that a random six-vertex ensemble $\mathcal{E} \in \mathfrak{C}(\Lambda)$ sampled under $\mathcal{P}$ satisfies the following Markov property. Let $\Lambda' = [X_1, X_2] \times [Y_1, Y_2] \subseteq \Lambda$ be a rectangular subdomain; let $u = u(\mathcal{E})$ denote the entrance data for $\mathcal{E}_{N}$ on $\Lambda'$; and set $\Xi = ([X_1, \infty) \times [Y_1, \infty)) \cap \Lambda$. Then, conditional on $u$, $\mathcal{E}_{N}$ is independent of $\mathcal{E}_{\Lambda \setminus \Xi}$. We refer to the right side of Figure 5 for a depiction.

2.4 Pure states for the stochastic six-vertex model

In this section, we describe a family of pure states $\mu(\rho)$ for the stochastic six-vertex model. We begin by explaining how to sample the restrictions of these measures to the positive quadrant $\mathbb{Z}^2_{>0}$; the extension of these measures to all of $\mathbb{Z}^2$ is then done through a translation and limiting procedure.

To implement the former task, we require certain entrance data. For any $\rho_1, \rho_2 \in [0, 1]$, double-sided $(\rho_1, \rho_2)$-Bernoulli entrance data is that in which sites on the $y$-axis are independently entrance sites for paths with probability $\rho_1$, and sites on the $x$-axis are independently entrance sites for paths with probability $\rho_2$.

Now, fix $0 < B_1 < B_2 < 1$ and define $\kappa > 1$ and $\varphi : [0, 1] \to [0, 1]$ by

$$\kappa = \kappa_{B_1, B_2} = \frac{1 - B_1}{1 - B_2} > 1; \quad \varphi(z) = \varphi_{B_1, B_2}(z) = \frac{\kappa z}{(\kappa - 1)z + 1}, \quad \text{for any } z \in [0, 1].$$

Further fix $\rho \in [0, 1]$, and consider the stochastic six-vertex model on the nonnegative quadrant with double-sided $(\varphi(\rho), \rho)$-Bernoulli entrance data; denote the associated measure on $\mathfrak{C}(\mathbb{Z}^2_{>0})$ by $\mu_0 = \mu_0(\rho)$. It was shown in [1] that this measure is translation-invariant in the following sense.

Lemma 2.7 [1, Lemma A.2]. Fix $\rho \in [0, 1]$, and sample a six-vertex ensemble $\mathcal{E} \in \mathfrak{C}(\mathbb{Z}^2_{>0})$ randomly under $\mu_0(\rho)$. Then, for any $(x, y) \in \mathbb{Z}^2_{>0}$, the random variables $\{\chi^h(\mathcal{E})(x, y + 1), \chi^h(\mathcal{E})(x, y + 2), \ldots\} \cup \{\chi^v(\mathcal{E})(x + 1, y), \chi^v(\mathcal{E})(x + 2, y), \ldots\}$ are mutually independent. Furthermore, each $\chi^h(\mathcal{E})(x, y)$ and $\chi^v(\mathcal{E})(x, y)$ is a 0-1 Bernoulli random variable with mean $\varphi(\rho)$ and $\rho$, respectively.

Observe, in particular that, if $(x, y) = (0, 0)$, then this is the definition of double-sided $(\varphi(\rho), \rho)$-Bernoulli entrance data for $\mu_0$. The fact that it is also true for any $(x, y) \in \mathbb{Z}^2_{>0}$ allows us to define a family of measures $\{\mu_N\} = \{\mu_N(\rho)\}_{N \geq 0}$ as follows. For each integer $N \geq 1$, let $\mu_N = \mu_N(\rho) = \mathfrak{T}(\rho_N, N) \mu_0$ denote the measure on $\mathbb{Z}^2_{>0}$ formed by translating $\mu_0$ by $(-N, -N)$ (that is, $N$ spaces down and to the left). Due to the translation-invariance of $\mu_0$ from Lemma 2.7, these measures are compatible in the sense that $\mu_M$ is the restriction of $\mu_N$ to $\mathbb{Z}^2_{>0-M}$, for any integers $N \geq M \geq 0$.

Therefore, we can define the limit $\mu = \mu(\rho) = \lim_{N \to \infty} \mu_N(\rho)$ on all of $\mathbb{Z}^2$. By Lemma 2.7, this limit is invariant with respect to any vertical or horizontal shift. Thus, $\mu(\rho)$ is a translation-invariant Gibbs measure for the stochastic six-vertex model on $\mathbb{Z}^2$. The following proposition indicates that it is moreover a pure state.

Proposition 2.8. Fix real numbers $0 < B_1 < B_2 < 1$ and $\rho \in [0, 1]$. Then, the measure $\mu(\rho)$ is a pure state of slope $(\rho, \varphi(\rho))$ for the $(B_1, B_2)$-stochastic six-vertex model.

Remark 2.9. Under the stochastic specialization $(a_1, a_2, b_1, b_2, c_1, c_2) = (1, 1, B_1, B_2, 1 - B_1, 1 - B_2)$ of weights, the condition $h(s, t) = 0$ (recall (1.4)) is equivalent to $t = \varphi(s)$. 
To establish this proposition, we require the following lemma from [2] that states the following. If the boundary data for two stochastic six-vertex models on the quadrant coincide in a union of two intervals on the $x$-axis, then these models can be coupled to coincide in rectangles above these intervals with high probability. Although the below result was stated in [2] in the case of one interval and for the stochastic six-vertex model on the discrete upper half-plane, it is quickly verified that the proof there also applies with multiple intervals for the model on the quadrant.

**Lemma 2.10** [2, Proposition 2.17]. For any $0 < B_1 < B_2 < 1$, there exists a constant $c = c(B_2) > 0$ such that the following holds. Fix two entrance data $u$ and $u'$ on the quadrant $\mathbb{Z}^2_{>0}$, and let $\mathcal{S}$ and $\mathcal{S}'$ denote the $(B_1, B_2)$-stochastic six-vertex models on $\mathbb{Z}^2$ with entrance data $u$ and $u'$, respectively. Further let $K_1, K_2, M, N > 0$ be integers; assume that $N \leq K_1, K_2$ and $N \geq \frac{3M}{1-B_2}$. Denote the intervals $I_1 = [K_1 - N, K_1 + N] \times \{0\}$ and $I_2 = [K_2 - N, K_2 + N] \times \{0\}$, and additionally suppose that $u \cap (I_1 \cup I_2) = u' \cap (I_1 \cup I_2)$.

Randomly sample two six-vertex ensembles $\mathcal{E} \in \mathcal{G}_u(\mathbb{Z}^2_{>0})$ and $\mathcal{E}' \in \mathcal{G}_{u'}(\mathbb{Z}^2_{>0})$ under $\mathcal{S}$ and $\mathcal{S}'$, respectively, and set the domain $\Lambda = ([K_1 - M, K_1 + M] \cup [K_2 - M, K_2 + M]) \times [0, M]$. Then, it is possible to couple $\mathcal{E}$ and $\mathcal{E}'$ in such a way that $\mathcal{E}_\Lambda = \mathcal{E}'_\Lambda$ holds with probability at least $1 - c^{-1} e^{-cM}$.

Now we can establish Proposition 2.8.

**Proof of Proposition 2.8.** As mentioned above, $\mu(\rho)$ is translation-invariant and has slope $(\rho, \varphi(\rho))$ by Lemma 2.7, so it suffices to show that it is ergodic. To that end, recalling the translation map $\mathcal{T}_u : \mathbb{Z}^2 \to \mathbb{Z}^2$ from Section 1.2 and abbreviating $\mathcal{T}_X = \mathcal{T}_{(-X,0)}$, we will show that $\mu(\rho)$ satisfies the following mixing property. For any two events $A_1$ and $A_2$, we have that

$$\lim_{X \to \infty} \mathbb{P}[A_1 \cap \mathcal{T}_X A_2] = \mathbb{P}[A_1] \mathbb{P}[A_2], \tag{2.7}$$

where the probability measure is with respect to $\mu(\rho)$.

Assuming (2.7), we can quickly establish the ergodicity of $\mu(\rho)$. Indeed, letting $\mathcal{A}$ be any translation-invariant event, apply (2.7) with $A_1 = A = A_2$. Then, $\mathcal{T}_X A_2 = A$ by the translation-invariance of $\mathcal{A}$, and so (2.7) yields $\mathbb{P}[A] = \mathbb{P}[A]^2$. Hence, $\mathbb{P}[A] \in \{0, 1\}$, and so $\mu(\rho)$ is ergodic.

Thus, it remains to verify (2.7). To that end, we may assume that $A_1$ and $A_2$ are both cylinder subsets. Then, there exist finite domains $\Lambda, \Lambda' \subseteq \mathbb{Z}^2$ and six-vertex ensembles $\mathcal{G}_1 \subseteq \mathcal{G}(\Lambda)$ and $\mathcal{G}_2 \subseteq \mathcal{G}(\Lambda')$ such that $A_1 = \{\mathcal{E} \in \mathcal{G}(\mathbb{Z}^2) : \mathcal{E}_\Lambda = \mathcal{G}_1\}$ and $A_2 = \{\mathcal{E} \in \mathcal{G}(\mathbb{Z}^2) : \mathcal{E}_{\Lambda'} = \mathcal{G}_2\}$. Letting $R > 0$ be any sufficiently large integer such that $\Lambda, \Lambda' \subseteq [-R, R] \times [-R, R]$, we may then assume that $\Lambda = [-R, R] \times [-R, R]$.

Next, let $N \geq \frac{18R}{1-B_3}$ and $X \geq 6N$ be any two integers. We consider two $(B_1, B_2)$-stochastic six-vertex models on the quadrant $\Gamma = (-2N, \infty) \times (-R, \infty) \subset \mathbb{Z}^2$ with different entrance data. The first, denoted $\mathcal{S}$, has double-sided $(\varphi(\rho), \rho)$-Bernoulli entrance data $u$. Defining the intervals $I_1 = [-N, N] \times \{-R\}$ and $I_2 = [X - N, X + N] \times \{-R\}$, the second, denoted $\mathcal{S}'$, has entrance data $u \cap (I_1 \cup I_2)$ (meaning that we remove all entrance sites from $u$ that are not contained in $I_1 \cup I_2$). Let $\mathcal{E}, \mathcal{E}' \in \mathcal{G}(\Gamma)$ be six-vertex ensembles sampled under $\mathcal{S}$ and $\mathcal{S}'$, respectively.

Observe from Lemma 2.7 that the law of $\mathcal{E}$ is given by the marginal of $\mu(\rho)$ on $\mathcal{G}(\Gamma)$. Thus, we may set the events $A_1$ and $A_2$ in (2.7) by

$$A_1 = \{\mathcal{E}_\Lambda = \mathcal{G}_1\}; \quad A_2 = \{\mathcal{E}_{\Lambda'} = \mathcal{G}_2\}.$$
Let us further define the events $A'_1$ and $A'_2$ by

$$A'_1 = \{ \mathcal{E}' = \mathcal{C}_1 \}; \quad A'_2 = \{ \mathcal{E}' = \mathcal{C}_2 \}.$$ 

Now, the entrance data for $(\mathcal{E}, \mathcal{E}')$ coincide on $I_1 \cup I_2$. Therefore, the $(K_1, K_2, M, N) = (2N, X + 2N, \lfloor \frac{(1-B^2)N}{3} \rfloor, N)$ case of Lemma 2.10 (and the fact that $N \geq \frac{18R}{1-B^2}$) yields the existence of a constant $c = c(B_2) > 0$ such that $\mathcal{E}$ and $\mathcal{E}'$ can be coupled to coincide on $\Lambda \cup \mathcal{T}_X \Lambda$, off of an event of probability at most $c^{-1}e^{-cN}$. In particular, this implies the three bounds

$$|\mathbb{P}[A_1] - \mathbb{P}[A'_1]| \leq c^{-1}e^{-cN}; \quad |\mathbb{P}[\mathcal{T}_X A_2] - \mathbb{P}[\mathcal{T}_X A'_2]| \leq c^{-1}e^{-cN};$$

$$|\mathbb{P}[A_1 \cap \mathcal{T}_X A'_2] - \mathbb{P}[A'_1 \cap \mathcal{T}_X A'_2]| \leq c^{-1}e^{-cN}. \quad (2.8)$$

Next let $B$ denote the event that no path in $\mathcal{E}'$ intersects both $\Lambda$ and $\mathcal{T}_X \Lambda$. Since the $\{X^{(u)}(u)\}$ are mutually independent for $u \in I_1 \cup I_2$ (due to the choice of double-sided Bernoulli boundary data for $u$), it follows from the sampling procedure described in Section 2.3 that the events $A'_1 \cap B$ and $\mathcal{T}_X A'_2 \cap B$ are independent after conditioning on $B$. Hence,

$$\mathbb{P}[A'_1 | B] \mathbb{P}[\mathcal{T}_X A'_2 | B] = \mathbb{P}[A'_1 \cap \mathcal{T}_X A'_2 | B],$$

which implies

$$|\mathbb{P}[A'_1 \cap \mathcal{T}_X A'_2] - \mathbb{P}[A'_1] \mathbb{P}[\mathcal{T}_X A'_2]| \leq 2\mathbb{P}[B^c]. \quad (2.9)$$

To bound $\mathbb{P}[B^c]$, we again apply Lemma 2.10. To that end, let $\mathcal{E}_0 \in \mathfrak{E}(\Gamma)$ denote the six-vertex ensemble on $\Gamma$ that assigns arrow configuration $(0,0;0,0)$ to each vertex, namely, it is the ‘empty’ six-vertex ensemble that has no paths. Then, since $X \geq 6N$, the entrance data for $(\mathcal{E}', \mathcal{E}_0)$ coincide on the interval $I_0 = [2N, 4N] \times \{-R\}$. Thus, Lemma 2.10 implies that it is possible to couple $\mathcal{E}'$ with $\mathcal{E}_0$ on $[3N - R, 3N + R] \times [-R, R]$, off of an event of probability at most $c^{-1}e^{-cN}$. Since if $\mathcal{E}'$ contains no paths in $[3N - R, 3N + R] \times [-R, R]$ then $B$ holds, it follows that $\mathbb{P}[B^c] \leq c^{-1}e^{-cN}$.

Combining this with (2.8), (2.9), and the fact that $\mathbb{P}[A_2] = \mathbb{P}[\mathcal{T}_X A_2]$ (by the translation-invariance of $\mu(\rho)$), we obtain

$$|\mathbb{P}[A_1 \cap \mathcal{T}_X A_2] - \mathbb{P}[A_1] \mathbb{P}[A_2]| \leq 6c^{-1}e^{-cN},$$

whenever $N \geq \frac{18R}{1-B^2}$ and $X \geq 6N$. Letting $X$ tend to $\infty$ and then $N$ tend to $\infty$, we deduce (2.7) and therefore the proposition.

□

2.5 Local statistics in the stochastic six-vertex model

In this section, we provide a convergence of local statistics result for the stochastic six-vertex model with free exit data that states the following. Consider this model on the quadrant $\mathbb{Z}_+^2$ whose entrance data are regular with slope $\rho \in [0, 1]$ on an interval of the positive $x$-axis; then, the local statistics of this model slightly above this interval are approximately given by the measure $\mu(\rho)$ from Section 2.4. This statement follows from results in [2], but we will briefly outline how more precisely.
Lemma 2.11 [2]. Fix real numbers $0 < B_1 < B_2 < 1$ and $\rho, \varepsilon \in (0, 1]$; an integer $k \geq 1$; and a six-vertex ensemble $\zeta \in \mathcal{E}([-k, k] \times [-k, k])$. Then, there exist constants $\delta = \delta(B_1, B_2, \rho, \varepsilon, k) > 0$, $c_0 = c_0(B_1, B_2, \rho, k) > 0$, and $C_0 = C_0(B_1, B_2, \rho, \varepsilon, k) > 1$ such that the following holds for any integers $N > C_0$ and $M \in [\frac{c_0 N}{2}, c_0 N]$.

Let $u$ be some entrance data on the quadrant $\mathbb{Z}^2_{>0}$ that is $(\delta N; \delta)$-regular with slope $\rho$ on the interval $[\frac{N}{2}, \frac{3N}{2}] \times \{0\}$. Sample random six-vertex ensembles $\mathcal{E} \in \mathcal{E}(\mathbb{Z}^2_{>0})$ under the $(B_1, B_2)$-stochastic six-vertex model $\mathbb{S}$ on $\mathbb{Z}^2_{>0}$ with entrance data $u$ and free exit data, and $F \in \mathcal{E}(\mathbb{Z}^2)$ under $\mu(\rho)$. Then,

\[
\left| P_{\mathbb{S}} [\mathcal{E}_{[N-k,N+k] \times [M-k,M+k]} = \zeta] - P_{\mu(\rho)} [F_{[-k,k] \times [-k,k]} = \zeta] \right| < \varepsilon. \tag{2.10}
\]

Proof (Outline). This result will follow from [2, Theorem 1.3] (which establishes (2.10) on a discrete cylinder); [2, Proposition 5.7] (which compares the stochastic six-vertex model on a discrete cylinder with that on the discrete upper half-plane); and Lemma 2.10 above (which enables a comparison between the stochastic six-vertex model on the discrete upper half-plane with that on the positive quadrant). Let us briefly outline how this proceeds.

First define the discrete torus $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$ and discrete cylinder $\mathcal{C}_N;2M = \mathbb{T}_N \times \{1, 2, \ldots, 2M\}$. Denoting the interval $J = (\left[ \frac{N}{2}, \frac{3N}{2} \right] \times \{0\}) \cap \mathbb{Z}^2$, we may identify $J$ with the lower end of $\partial \mathcal{C}_N;2M$; in this way, $u \cap J$ induces entrance data $\mathcal{C}_N;2M$. Let $\mathcal{E}'$ denote random six-vertex ensemble on $\mathcal{C}_N;2M$ sampled from the $(B_1, B_2)$-stochastic six-vertex model $\mathbb{S}'$ on $\mathcal{C}_N;2M$ with this entrance data $u \cap J$ and free exit data. Then, the $\psi = \rho$ case of [2, Theorem 1.3] yields for each real number $\omega > 0$ the existence of constants $\delta = \delta(B_1, B_2, \rho, \varepsilon, \omega, k) > 0$ and $C_1 = C_1(B_1, B_2, \rho, \varepsilon, \omega, k) > 1$ such that

\[
\left| P_{\mathbb{S}'} [\mathcal{E}'_{[N-k,N+k] \times [M-k,M+k]} = \zeta] - P_{\mu(\rho)} [F_{[-k,k] \times [-k,k]} = \zeta] \right| < \frac{\varepsilon}{3}, \tag{2.11}
\]

for any integers $N > C_1$ and $M \in [\omega N, \omega^{-1} N]$, if $u$ is $(\delta N; \delta)$-regular.

Next, let $\mathcal{E}'' \in \mathcal{E}(\mathbb{Z} \times \mathbb{Z}_{>0})$ denote a random six-vertex ensemble sampled under the $(B_1, B_2)$-stochastic six-vertex model $\mathbb{S}''$ on the discrete upper half-plane $\mathbb{Z} \times \mathbb{Z}_{>0}$ with entrance data $u \cap J$. Then, [2, Proposition 5.7] implies the existence of a constant $c_1 = c_1(B_2) > 0$ and a coupling between $\mathcal{E}'$ and $\mathcal{E}''$ such that they coincide on $[N - M, N + M] \times [0, 2M]$ off of an event of probability at most $c_1^{-1} e^{-c_1 M}$, if $M < c_1 N$. Combined with (2.11), this for sufficiently large $N$ yields

\[
\left| P_{\mathbb{S}''} [\mathcal{E}''_{[N-k,N+k] \times [M-k,M+k]} = \zeta] - P_{\mu(\rho)} [F_{[-k,k] \times [-k,k]} = \zeta] \right| < \frac{2\varepsilon}{3}, \tag{2.12}
\]

if $\omega N < M < c_1 N$.

Now, since no paths exist in $\mathcal{E}''$ left of the line $x = N$, we may equivalently view $\mathcal{E}''$ as sampled from the $(B_1, B_2)$-stochastic six-vertex ensemble on the quadrant $\mathbb{Z}^2_{>0}$ with entrance data $u \cap J$. Thus, Lemma 2.10 again yields a coupling between $\mathcal{E}$ and $\mathcal{E}''$ such that they coincide on $[N - M, N + M] \times [0, 2M]$ off of an event of probability at most $c_1 e^{-c_1 M}$, if $M < c_1 N$. This, together with (2.12), implies (2.10).

\[\Box\]

### 3 | COMPARISON WITH THE STOCHASTIC SIX-VERTEX MODEL

In this section, we establish Theorem 1.2, conditional on a certain statement (given by Proposition 3.7). To that end, we begin in Section 3.1 by implementing a gauge transformation to show
that a pure state of any ferroelectric six-vertex model is equivalent to a pure state of a particular stochastic six-vertex model. Then in Section 3.2 we introduce the notion of a partition function stochastic lower bound and state two results, given by Proposition 3.6 and Proposition 3.7, classifying pure states satisfying this bound; we further establish Theorem 1.2 conditional on these two results. In Section 3.3 and Section 3.4, we establish Proposition 3.6; Proposition 3.7 will be established in Section 4.

3.1 Gauge equivalence with a stochastic pure state

We begin with the following result stating that the Gibbs property for the six-vertex model is invariant under a certain gauge transformation of its weights; similar results, with analogous proofs, were showed at the end of [3, Section 3] and [1, Appendix A.1] (see also [15, Section 2]).

Lemma 3.1. Fix real numbers $a_1, a_2, b_1, b_2, c_1, c_2 > 0$ and $r, x, y, z > 0$. If some measure $\mu \in \mathcal{P}(\mathcal{G}({\mathbb{Z}}^2))$ satisfies the Gibbs property for the six-vertex model with weights $(a_1, a_2, b_1, b_2, c_1, c_2)$, then it also does for the six-vertex model with weights $(ra_1, rza_2, ryzb_1, ry^{-1}b_2, rxc_1, rx^{-1}zc_2)$.

Proof. Let $\mathcal{E} \in \mathcal{G}({\mathbb{Z}}^2)$ denote a random six-vertex ensemble on $\mathbb{Z}^2$ sampled under $\mu$. Fix a finite rectangular domain $\Lambda \subset \mathbb{Z}^2$, and condition on the restriction $\mathcal{E}_{\Lambda} = \mathcal{H}$ of $\mathcal{E}$ to the complement of $\Lambda$. Then $\mathcal{H}$ induces boundary conditions, denoted by $\mathbf{u} \cup \mathbf{v} = \mathbf{u}(H) \cup \mathbf{v}(H)$, for the restriction $\mathcal{E}_{\Lambda}$ of $\mathcal{E}$ to $\Lambda$. Set $\mathbf{u} = (u_{-B, u_{1-B}}, \ldots, u_A)$ and $\mathbf{v} = (v_{-B, v_{1-B}}, \ldots, v_A)$, for some integers $A = A(H) \geq 0$ and $B = B(H) \geq -1$.

Now, for any six-vertex ensemble $\mathcal{G} \in \mathcal{G}(\Lambda)$ on $\Lambda$, let $N_1(\mathcal{G}), N_2(\mathcal{G}), N_3(\mathcal{G}), N_4(\mathcal{G}), N_5(\mathcal{G})$, and $N_6(\mathcal{G})$ denote the numbers of vertices in $\Lambda$ whose arrow configurations under $\mathcal{G}$ are given by $(0,0;0,0)$, $(1,1;1,1)$, $(1,0;1,0)$, $(0,1;0,1)$, $(1,0;0,1)$, and $(0,1;1,0)$, respectively. Then, since $\mu$ satisfies the Gibbs property for the six-vertex model with weights $(a_1, a_2, b_1, b_2, c_1, c_2)$, there exists a constant $Z = Z_H > 0$ such that

$$
P(\mathcal{E}_{\Lambda} = \mathcal{G} | \mathcal{E}_{\Lambda} = \mathcal{H}) = Z^{-1}a_1^{N_1}a_2^{N_2}b_1^{N_3}b_2^{N_4}c_1^{N_5}c_2^{N_6}$$

$$
= Z^{-1}r^{-M_1}x^{-M_2}y^{(M_4-M_3)/2}z^{(M_2-M_4)/2}$$

$$
\times (ra_1)^{N_1(rza_2)^{N_2}(ryzb_1)^{N_3}(ry^{-1}b_2)^{N_4}(rxc_1)^{N_5}(rx^{-1}zc_2)^{N_6}},
$$

for any $\mathcal{G} \in \mathcal{G}(\mathbf{u};\mathbf{v})(\Lambda)$ with boundary data consistent with that of $\mathcal{H}$ (namely, $\mathbf{u} \cup \mathbf{v}$). Here, we have abbreviated $N_i = N_i(\mathcal{G})$ for each index $i \in [1, 6]$ and defined the $M_i = M_i(\mathcal{G})$ by

$$
M_1 = N_1 + N_2 + N_3 + N_4 + N_5 + N_6; \quad M_2 = N_5 - N_6;$$

$$
M_3 = 2N_2 + 2N_3 + N_5 + N_6; \quad M_4 = 2N_2 + 2N_4 + N_5 + N_6.
$$

We claim that $M_1, M_2, M_3$, and $M_4$ are all independent of the six-vertex ensemble $\mathcal{G} \in \mathcal{G}(\mathbf{u};\mathbf{v})(\Lambda)$. Indeed, this holds for $M_1$, as it denotes the number of vertices in $\Lambda$. Furthermore, $2N_2 + 2N_3 + N_5 + N_6$ denotes the number of vertical edges in $\Lambda$ occupied by an arrow under $\mathcal{G}$. So, setting $\mathcal{G}(x,y) = y$ for any $(x,y) \in \mathbb{Z}^2$, we have that $M_3 = \sum_{j=-B}^{A}(\mathcal{G}(v_j) - \mathcal{G}(u_j))$, which only depends on $\mathbf{u} \cup \mathbf{v}$. Similarly, $2N_2 + 2N_4 + N_5 + N_6$ denotes the number of horizontal edges in
occupied by an arrow under \( \mathcal{C} \). So, setting \( \mathcal{R}(x, y) = x \) for any \((x, y) \in \mathbb{Z}^2\), we have that 
\[ M_4 = \sum_{j=-B}^{A} \mathcal{R}(u_j) - \mathcal{R}(v_j), \]
which too only depends on \( u \cup v \). Moreover, letting \( k = k(H) \in [-B - 1, A] \) denote the maximal index of a path whose ending point \( v_k \) is on the east boundary of \( \Lambda \) (where we set \( k = -B - 1 \) if no path satisfies this property), we find that 
\[ M_2 = N_5 - N_6 = B - A + k + 1 \]
also is independent of \( \mathcal{C} \in \mathcal{G}_{u \cup v}(\Lambda) \).

Thus, the prefactor \( Z^{-1} r^{-M_1 - x - M_2 / 2} z^{(M_2 - M_3) / 2} \) on the right side of (3.1) only depends on \( H \), and so it follows that \( \mu \) satisfies the Gibbs property for the six-vertex model with weights \((r_1, rz, r yz b_1, ry^{-1} b_2, r x c_1, rx^{-1} z c_2)\) on any rectangular subdomain of \( \mathbb{Z}^2 \). Since any finite domain \( \Lambda \subset \mathbb{Z}^2 \) is a subdomain of a rectangular one, we deduce that \( \mu \) satisfies the Gibbs property for the six-vertex model with weights \((r_1, rz, r yz b_1, ry^{-1} b_2, r x c_1, rx^{-1} z c_2)\).

The following corollary states that we can gauge transform the weights of any ferroelectric six-vertex model to those of a stochastic six-vertex model, while preserving the Gibbs property.

Corollary 3.2. Fix real numbers \( a_1, a_2, b_1, b_2, c_1, c_2 > 0 \), and set \( \Delta \) as in (1.1). Assume that \( \Delta \geq 1 \) and that \( b_1 b_2 < a_1 a_2 \), and define

\[ B_1 = \left( \Delta - \sqrt{\Delta^2 - 1} \right) \sqrt{\frac{b_1 b_2}{a_1 a_2}}; \quad B_2 = \left( \Delta + \sqrt{\Delta^2 - 1} \right) \sqrt{\frac{b_1 b_2}{a_1 a_2}}; \quad (3.2) \]

Then \( \mu \in \mathcal{P}(\mathcal{G}(\mathbb{Z}^2)) \) is a Gibbs measure for the six-vertex model with weights \((a_1, a_2, b_1, b_2, c_1, c_2)\) if and only if it is one for the \((B_1, B_2)\)-stochastic six-vertex model.

Proof. Define the real numbers \( r, x, y, \) and \( z \) by

\[ r = \frac{1}{a_1}; \quad x = \frac{a_1}{c_1} \left( 1 - \left( \Delta - \sqrt{\Delta^2 - 1} \right) \sqrt{\frac{b_1 b_2}{a_1 a_2}} \right); \quad y = \left( \Delta - \sqrt{\Delta^2 - 1} \right) \sqrt{\frac{a_2 b_2}{a_1 b_1}}; \quad z = \frac{a_1}{a_2}, \]

which are all positive due to the bounds

\[ a_1, a_2, b_1, b_2, c_1, c_2 > 0; \quad 0 < \Delta - \sqrt{\Delta^2 - 1} < 1 < \sqrt{\frac{a_1 a_2}{b_1 b_2}}. \]

Now the corollary follows from Lemma 3.1 and the fact that \((1,1,B_1,B_2,1-B_1,1-B_2) = (ra_1, rz a_2, r yz b_1, ry^{-1} b_2, r x c_1, rx^{-1} z c_2)\).

Remark 3.3. Under the choices of \( 0 < B_1 < B_2 < 1 \) in (3.2), recalling \( \mathfrak{h} \) from (1.4) and \( \varphi \) from (2.6), the conditions \( t < \varphi(s) \) and \( t = \varphi(s) \) are equivalent to \( \mathfrak{h}(s, t) < 0 \) and \( \mathfrak{h}(s, t) = 0 \), respectively.

In view of Corollary 3.2 and Remark 3.3, we can now use the measures \( \mu(\rho) \) from Section 2.4 to show existence of a pure state for the ferroelectric six-vertex model with any slope \((s, t) \in \partial \mathcal{S})\).
Corollary 3.4. Fix real numbers \( a_1, a_2, b_1, b_2, c_1, c_2 > 0 \); set \( \Delta \) as in (1.1); assume that \( \Delta > 1 \) and \( a_1 a_2 > b_1 b_2 \); and let \( (s, t) \in \partial \mathcal{S} \). Then, there exists a pure state of slope \( (s, t) \) for the six-vertex model with weights \( (a_1, a_2, b_1, b_2, c_1, c_2) \).

Proof. Since \( (s, t) \in \partial \mathcal{S} \), we have \( (s, t) \in \mathcal{H}_1 \cup \mathcal{H}_2 \); let us first assume \( (s, t) \in \mathcal{H}_1 \), so that \( \mathcal{H}(s, t) = 0 \). Setting \( 0 < B_1 < B_2 < 1 \) as in (3.2) and defining \( \varphi \) as in (2.6), Remark 3.3 implies that the condition \( \mathcal{H}(s, t) = 0 \) is equivalent to \( t = \varphi(s) \). Then, by Proposition 2.8, the measure \( \mu(s) \) from Section 2.4 is a pure state of slope \( (s, \varphi(s)) \) for the \((B_1, B_2)\)-stochastic six-vertex model. Thus, Corollary 3.2 implies it is also a pure state of slope \( (s, t) \) for the six-vertex model with weights \( (a_1, a_2, b_1, b_2, c_1, c_2) \), establishing the corollary if \( (s, t) \in \mathcal{H}_1 \).

If instead \( (s, t) \in \mathcal{H}_2 \), then \( (t, s) \in \mathcal{H}_1 \). Therefore, the above implies that \( \tilde{\mu}(t) \) is a pure state of slope \( (t, s) \) for the six-vertex model with weights \( (a_1, a_2, b_2, b_1, c_2, c_1) \) (where we were permitted to interchange weights in the pairs \( b_2, b_1 \) and \( c_2, c_1 \), since the definitions (1.4) of \( \mathcal{H} \) and (3.2) of \( B_1, B_2 \) only depend on \( (a_1, a_2, b_1, b_2, c_1, c_2) \)). Let \( \tilde{\mu}(t) \in \mathcal{P}(\mathcal{C}(\mathbb{Z}^2)) \) denote the ‘reflection’ of this measure into the line \( y = x \), namely, the law of \( \tilde{\epsilon} \in \mathcal{C}(\mathbb{Z}^2) \) that is the ensemble obtained by first sampling \( \mathcal{E} \in \mathcal{C}(\mathbb{Z}^2) \) under \( \mu(t) \) and then reflecting \( \mathcal{E} \) into the line \( y = x \). Since this reflection changes the six-vertex weights \( (a_1, a_2, b_2, b_1, c_2, c_1) \) to \( (a_1, a_2, b_1, b_2, c_1, c_2) \), and changes any slope \( (s', t') \) of a pure state to \( (t', s') \), \( \tilde{\mu}(t) \) is a pure state of slope \( (s, t) \) for the six-vertex model with weights \( (a_1, a_2, b_1, b_2, c_1, c_2) \). This shows the existence of a pure state for this model of any slope \( (s, t) \in \mathcal{H}_1 \cup \mathcal{H}_2 = \partial \mathcal{S} \), thereby establishing the corollary. \( \square \)

3.2 | Partition function stochastic lower bounds

We begin this section with the following definition for when a pure state of the stochastic six-vertex model has the property that its partition function on an \( N \times N \) domain is likely at least \( e^{-o(N^2)} \).

Definition 3.5. Fix real numbers \( 0 < B_1 < B_2 < 1 \) and a translation-invariant Gibbs measure \( \mu \in \mathcal{P}(\mathcal{C}(\mathbb{Z}^2)) \) for the \((B_1, B_2)\)-stochastic six-vertex model. We say that \( \mu \) satisfies a partition function stochastic lower bound if, for any real number \( \delta > 0 \), there exists a constant \( C_0 = C_0(\delta, \mu) > 1 \) such that the following holds for any integer \( N > C_0 \).

Set \( \Lambda_N = [1, N] \times [1, N] \), and let \( \nu_N \in \mathcal{P}(\mathcal{C}(\mathbb{Z}^2 \setminus \Lambda_N)) \) denote the marginal distribution of \( \mu \) on \( \mathcal{C}(\mathbb{Z}^2 \setminus \Lambda_N) \) (induced by restricting six-vertex ensembles on \( \mathbb{Z}^2 \) to ones on \( \mathbb{Z}^2 \setminus \Lambda_N \)). Any six-vertex ensemble \( \mathcal{H} \in \mathcal{C}(\mathbb{Z}^2 \setminus \Lambda_N) \) that can be obtained as the restriction of one on \( \mathbb{Z}^2 \) induces boundary conditions \( \mathcal{u}(\mathcal{H}) \cup \mathcal{v}(\mathcal{H}) \) on \( \Lambda \). Recalling the weights \( w(\mathcal{E}) \) from (1.3) for the \((B_1, B_2)\)-stochastic six-vertex model and defining the partition function \( Z(\mathcal{H}) = \sum_{\mathcal{E} \in \mathcal{C}(\mathcal{H}, \mathcal{v}(\mathcal{H}))} w(\mathcal{E}) \), we have

\[
P_{\nu_N}\left[Z(\mathcal{H}) \geq e^{-\delta N^2}\right] \geq 1 - \delta.
\]

Before describing the use of this notion, let us state two results. The first, which will be established in Section 3.4, classifies any pure state satisfying a partition function stochastic lower bound as one of the measures \( \mu(\rho) \) from Section 2.4. The second, which will be established in Section 4.3, states that any pure state with a certain slope satisfies a partition function stochastic lower bound.
Proposition 3.6. Fix real numbers $0 < B_1 < B_2 < 1$ and a pair $(s, t) \in (0, 1]^2$. If $\mu \in \mathcal{P}(\mathbb{Z}^2)$ is a pure state of slope $(s, t)$ for the $(B_1, B_2)$-stochastic six-vertex model that satisfies a partition function stochastic lower bound, then $\mu = \mu(s)$.

Proposition 3.7. Fix real numbers $0 < B_1 < B_2 < 1$ and a pair $(s, t) \in (0, 1]^2$. Let $\mu$ be a pure state of slope $(s, t)$ for the $(B_1, B_2)$-stochastic six-vertex model. If $s \leq t \leq \varphi(s)$, then $\mu$ satisfies a partition function stochastic lower bound.

Given Proposition 3.6 and Proposition 3.7, we can quickly establish Theorem 1.2.

Proof of Theorem 1.2 Assuming Proposition 3.6 and Proposition 3.7. By Corollary 3.4, there exists a pure state of any slope $(s, t) \in \partial\mathcal{H}$ for the six-vertex model with weights $(a_1, a_2, b_1, b_2, c_1, c_2)$. Thus, it remains to show that the pure state of this slope is unique, and that no such pure state can exist if $(s, t) \in \mathcal{H}$. We may assume in what follows that $s, t \in (0, 1]$, since otherwise $(s, t) = (0, 0)$, and there is a unique pure state of this slope (which deterministically assigns arrow configuration $(0, 0; 0, 0)$ to each vertex of $\mathbb{Z}^2$).

So, fix $(s, t) \in \mathcal{H}$ with $s, t > 0$, and let $\mu$ be a pure state of slope $(s, t)$ for the six-vertex model with weights $(a_1, a_2, b_1, b_2, c_1, c_2)$. By replacing $\mu$ with its reflection into the line $x = y$ if necessary (as in the proof of Corollary 3.4), we may assume that $t \geq s$. Then, define the weights $0 < B_1 < B_2 < 1$ as in (3.2), and recall the function $\varphi(z) = \varphi_{B_1, B_2}(z)$ from (2.6). By Corollary 3.2, $\mu$ is a pure state for the $(B_1, B_2)$-stochastic six-vertex model with slope $(s, t)$.

Since $(s, t) \in \mathcal{H}$, we have that $h(s, t) \leq 0$. By Remark 3.3, this yields $t \leq \varphi(s)$, and so $s \leq t \leq \varphi(s)$. Hence, Proposition 3.7 implies $\mu$ satisfies a partition function stochastic lower bound. Thus, by Proposition 3.6, $\mu$ is equal to the pure state $\mu(s)$ from Section 2.4, which has slope $(s, \varphi(s)) \in \mathcal{H}_1 \subset \partial\mathcal{H}$. This uniquely determines $\mu$ if $(s, t) \in \partial\mathcal{H}$ and shows that $\mu$ cannot exist if $(s, t) \in \mathcal{H}$, thereby establishing the theorem.

The benefit to translation-invariant Gibbs measures satisfying a partition function stochastic lower bound is that they can be compared to a stochastic six-vertex model with free exit data, which is sometimes more amenable to direct analysis since it is a Markov process. This comparison is made more precise through the following lemma stating that, if $\mu$ satisfies a partition function stochastic lower bound, then an event exponentially unlikely under the stochastic six-vertex model with free exit data is also unlikely under $\mu$.

Lemma 3.8. For any real numbers $\varepsilon, \gamma \in (0, 1]$ and $0 < B_1 < B_2 < 1$, and translation-invariant Gibbs measure $\mu \in \mathcal{P}(\mathbb{Z}^2)$ for the $(B_1, B_2)$-stochastic six-vertex model satisfying a partition function stochastic lower bound, there exists a constant $C = C(\mu, \varepsilon, \gamma, B_1, B_2) > 1$ such that the following holds for any integer $N > C$. Recall $\Lambda_N = [1, N] \times [1, N]$ and $\nu_N \in \mathcal{P}(\mathbb{G}(\mathbb{Z}^2 \setminus \Lambda_N))$ from Definition 3.5; sample $\mathcal{H} \in \mathbb{G}(\mathbb{Z}^2 \setminus \Lambda)$ under $\nu_N$; and denote its boundary data by $u(H) \cup v(H)$.

Further sample random six-vertex ensembles $\mathcal{E} \in \mathbb{G}(\mathbb{Z}^2)$ under $\mu$, and $F \in \mathbb{G}_{u(H)}(\Lambda_N)$ under the $(B_1, B_2)$-stochastic six-vertex model $\mathcal{G}$ on $\Lambda$ with entrance data $u(H)$ and free exit data. Then, for any subset $\mathcal{D} \subset \mathbb{G}(\Lambda_N)$ such that $E_{\nu_N}[\mathbb{P}_\mathcal{G}[F \in \mathcal{D}]] \leq e^{-\gamma N^2}$, we have $\mathbb{P}_\mu[\mathcal{E} \in \mathcal{D}] < \varepsilon$.

Proof. Throughout this proof, we abbreviate $\Lambda = \Lambda_N$, $\nu = \nu_N$ and $u \cup v = u(H) \cup v(H)$ (the last of which is random). We further recall the partition function $Z(H)$ from Definition 3.5, and define
the event
\[ A = A_N(\gamma) = \{ Z(H) \geq e^{-\gamma N^2/2} \}. \]

Then, the fact that \( \mu \) satisfies a partition function stochastic lower bound yields a constant \( C_0 = C_0(\mu, \gamma, \varepsilon) > 0 \) such that \( \mathbb{P}_\nu[A] \geq 1 - \frac{\varepsilon}{2} \) holds whenever \( N > C_0 \). Thus, a union bound gives
\[
\mathbb{P}_\mu[\mathcal{E} \in \mathfrak{D}] \leq \mathbb{P}_\mu[\{\mathcal{E} \in \mathfrak{D} \} \cap A] + \mathbb{P}_\mu[A^c] \leq \mathbb{P}_\mu[\{\mathcal{E} \in \mathfrak{D} \} \cap A] + \frac{\varepsilon}{2}. \tag{3.3}
\]

Next, since \( A \) is measurable with respect to \( \mathcal{E}_{Z^2 \setminus \Lambda} \), we have that
\[
\mathbb{P}_\mu[\{\mathcal{E} \in \mathfrak{D} \} \cap A] = \mathbb{E}_\nu [1_A \mathbb{P}_\mu[\mathcal{E} \in \mathfrak{D}| \mathcal{E}_{Z^2 \setminus \Lambda} = \mathcal{H}]] = \mathbb{E}_\nu \left[ \frac{1}{Z(H)} \sum_{\mathcal{E}' \in \mathfrak{D} \cap \mathfrak{G}_{u,v}(\Lambda)} w(\mathcal{E}') \right] \leq e^{\gamma N^2/2} \mathbb{E}_\nu \left[ \sum_{\mathcal{E}' \in \mathfrak{D} \cap \mathfrak{G}_{u,v}(\Lambda)} w(\mathcal{E}') \right], \tag{3.4}
\]

since \( Z(H) \geq e^{-\gamma N^2/2} \) on \( A \). Next, due to the stochasticity of the six-vertex weights of \( \mathfrak{G} \), we have the deterministic identity \( Z_u(\Lambda) = \sum_{\mathcal{E}' \in \mathfrak{G}_{u,v}(\Lambda)} w(\mathcal{E}') = 1 \). Thus,
\[
\sum_{\mathcal{E}' \in \mathfrak{D} \cap \mathfrak{G}_{u,v}(\Lambda)} w(\mathcal{E}') = \frac{1}{Z_u(\Lambda)} \sum_{\mathcal{E}' \in \mathfrak{D} \cap \mathfrak{G}_{u,v}(\Lambda)} w(\mathcal{E}') = \mathbb{P}_\mathfrak{G}[F \in \mathfrak{D}].
\]

Taking the expectation of this bound with respect to \( \nu \), and also applying (3.4) and the bound \( \mathbb{E}_\nu[\mathbb{P}_\mathfrak{G}[F \in \mathfrak{D}]] \leq e^{-\gamma N^2} \), it follows for \( N \) sufficiently large that
\[
\mathbb{P}_\mu[\{\mathcal{E} \in \mathfrak{D} \} \cap A] \leq e^{\gamma N^2/2} \mathbb{E}_\nu[\mathbb{P}_\mathfrak{G}[F \in \mathfrak{D}]] \leq e^{-\gamma N^2/2} < \frac{\varepsilon}{2}.
\]

This, together with (3.3), implies the lemma. \( \square \)

Now, to establish Proposition 3.6, we must show that \( \mathbb{E}_\mu[\psi] = \mathbb{E}_{\mu(\delta)}[\psi] \) holds, for any local function \( \psi : \mathfrak{G}(Z^2) \to \mathbb{R} \). To that end, we will consider shift-averages of \( \psi \) over a large square grid. First, as Lemma 3.9, we will show the probability that this average differs nonnegligibly from \( \mathbb{E}_{\mu(\delta)}[\psi] \) decays exponentially in the grid size under the stochastic six-vertex model with free exit data. Next, by Lemma 3.8, it will follow that these shift averages likely converge to \( \mathbb{E}_{\mu(\delta)}[\psi] \) under \( \mu \); see Lemma 3.10. Then Proposition 3.6 will follow from the fact that these shift averages under \( \mu \) converge to \( \mathbb{E}_\mu[\psi] \) (by the ergodic theorem).

### 3.3 Shift-averages under the stochastic six-vertex model

Throughout the remainder of this paper, we fix real numbers \( 0 < B_1 < B_2 < 1 \) and will allow constants to depend on them, even when not explicitly mentioned.

Let us set some additional notation that will be used in this section. Suppose that we are given
\[
s \in (0, 1); \quad K, M \in \mathbb{Z}_{\geq 0}; \quad Y \in [0, M] \cap \mathbb{Z}; \quad k \in \mathbb{Z}_{\geq 0}; \quad \mathcal{G} \in \mathfrak{G}([-k, k] \times [-k, k]).
\]
From these parameters, define

\[ X = \left\lceil \frac{M}{2} \right\rceil; \quad N = KM; \quad \Lambda = \Lambda_N = [1, N] \times [1, N] \subseteq \mathbb{Z}^2. \tag{3.5} \]

Now let us define a partition \( \Lambda = \bigcup_{i=1}^{K^2} \Omega_i \) into \( K^2 \) subdomains as follows. For each index \( i \in [1, K^2] \), let \( j = j(i) \in [0, K - 1] \) denote the integer such that \( i = jK + r + 1 \) for some \( r = r(i) \in [0, K - 1] \). Then define the subdomain \( \Omega_i = [rM + 1, rM + M] \times [jM + 1, jM + M] \subseteq \Lambda \) and define the vertex \( z_i = (rM + X, jM + Y) \in \Omega_i \). We refer to Figure 6 for a depiction.

Next, for any six-vertex ensemble \( \mathcal{E} \in \mathfrak{G}(\Lambda) \), we will define indicator functions \( \psi_i(\mathcal{E}) \in \{0, 1\} \) for the event that \( \mathcal{G} \) ‘locally appears around’ \( z_i \) in \( \mathcal{E} \). More specifically, for each index \( i \in [1, K^2] \), define (recalling the translation operator \( \mathfrak{T}_u \) from Section 1.2)

\[ \psi_i(\mathcal{E}) = \psi_i^{(G)}(\mathcal{E}) = 1\left( (\mathfrak{T}_{z_i} \mathcal{E})_{[-k,k] \times [-k,k]} = \mathcal{G} \right). \tag{3.6} \]

Moreover, for any real number \( \eta \in (0, 1) \), let \( \Theta_i(\eta) \) denote the event on which the entrance data for the ensemble \( \mathcal{E}_{\Omega_i} \) is \( (\eta M; \eta) \)-regular with slope \( s \) along the south boundary of \( \Omega_i \).

The following lemma provides an exponential probability concentration estimate for the sum of the \( \psi_i(\mathcal{E}) \) over some index set \( I \subseteq [1, K^2] \) for the stochastic six-vertex model with free exit data (after restricting to the event \( \bigcap_{i \in I} \Theta_i(\eta) \), for sufficiently small \( \eta \)).

**Lemma 3.9.** For any real number \( \varepsilon \in (0, 1] \), there exist constants \( \delta = \delta(\varepsilon, s, k) > 0, c_1(s, k) > 0, c_2 = c_2(\varepsilon) > 0, \) and \( C = C(\varepsilon, s, k) > 1 \) such that the following holds. Adopt the notation above, and assume that \( M > C \) and \( \frac{c_1 M}{2} < Y < c_1 M \). Let \( \eta \in (0, \delta) \) be a real number and \( I \subseteq [1, K^2] \) be a nonempty subset. Further fix some entrance data \( u \) on \( \Lambda \); let \( \mathcal{E} \in \mathfrak{G}_u(\Lambda) \) denote a random six-vertex ensemble sampled under the \((B_1, B_2)\)-stochastic six-vertex model \( \mathfrak{S} \) on \( \Lambda \) with entrance data \( u \) and free exit data; and let \( \mathcal{F} \in \mathfrak{G}(\mathbb{Z}^2) \) denote a random six-vertex ensemble sampled under \( \mu(s) \). Then,

\[ \mathbb{P}_{\mathfrak{S}} \left[ \left\{ \frac{1}{|I|} \sum_{i \in I} \psi_i(\mathcal{E}) - \mathbb{P}_{\mu(s)}[\mathcal{F}_{[-k,k] \times [-k,k]} = \mathcal{G}] > \varepsilon \right\} \cap \bigcap_{i \in I} \Theta_i(\eta) \right] < e^{-c_2 |I|}. \tag{3.7} \]
Proof. Throughout this proof, we set \( \zeta = \mathbb{P}_{\mathcal{M}(s)}[F_{[-k,k] \times [-k,k]} = \mathcal{G}] \) and, for each index \( i \in [1, K^2] \), we let \( u^{(i)} \) denote the (random) entrance data on \( \Omega_i \) for \( \mathcal{E}_{\Theta_i} \). In this way, \( u^{(i)} \) is \((\eta M; \eta)\)-regular along the south boundary of \( \Omega_i \) on the event \( \Theta_i(\eta) \). Thus, Lemma 2.11 yields constants \( \delta = \delta(\varepsilon, s, k) > 0 \), \( c_1 = c_1(s, k) > 0 \), and \( C_1 = C_1(\varepsilon, s, k) > 1 \) such that

\[
1_{\Theta_i(\eta)} \left| \mathbb{P}_{\mathcal{G}} \left[ \psi_i(\mathcal{E}) = 1 | u^{(i)} \right] - \zeta \right| < \frac{\varepsilon}{2},
\]

holds whenever \( M > C_1, \frac{c_1 M}{2} < Y < c_1 M \), and \( \eta < \delta \).

Next, Remark 2.6 implies for any \( i \in [1, K^2] \) that \( \mathcal{E}_i \) is independent of \( \bigcup_{j=1}^{i-1} \mathcal{E}_j \), after conditioning on \( u^{(i)} \). Hence, we obtain from (3.8) that

\[
1_{\Theta_i(\eta)} \left| \mathbb{P}_{\mathcal{G}} \left[ \psi_i(\mathcal{E}) = 1 \left| \bigcup_{j=1}^{i-1} \mathcal{E}_j \right] - \zeta \right| < \frac{\varepsilon}{2}.
\]

This, together with the Chernoff estimate (or, alternatively, the Azuma–Hoeffding inequality) for sums of 0-1 Bernoulli random variables, yields a constant \( c_2 = c_2(\varepsilon) > 0 \) such that

\[
\mathbb{P}_{\mathcal{G}} \left[ \left\{ \sum_{i \in I} \psi_i(\mathcal{E}) - \zeta | I | > \varepsilon | I | \right\} \cap \bigcap_{i \in I} \Theta_i(\eta) \right] < e^{-c_2 |I|},
\]

from which we deduce the lemma. \( \square \)

### 3.4 Proof of Proposition 3.6

The following lemma states that shift-averages of the local functions \( \psi_i \) from (3.6) under a pure state of slope \((s, t)\) satisfying a partition function stochastic lower bound converge to their expectations under \( \mu(s) \).

**Lemma 3.10.** Fix an integer \( k > 0 \); a real number \( \varepsilon > 0 \); a pair \((s, t) \in (0, 1]^2\); and a pure state \( \mu \in \mathcal{P}(\mathcal{G}(\mathbb{Z}^2)) \) of slope \((s, t)\) for the \((B_1, B_2)\)-stochastic six-vertex model that satisfies a partition function stochastic lower bound in the sense of Definition 3.5. Then, there exist \( c = c(s, k) > 0 \) and \( C_1 = C_1(\varepsilon, \mu, k) > 1 \) so that, for any integers \( M > C_1 \) and \( Y \in (\frac{cM}{2}, cM) \), there is a constant \( C_2 = C_2(M, \varepsilon, \mu, k) > 1 \), such that the following holds for any integer \( K > C_2 \).

Fix a six-vertex ensemble \( \mathcal{G} \in \mathcal{G}((-k,k] \times [-k,k]) \); set \( X, N, \) and \( \Lambda \) as in (3.5); and set \( \psi_i \) as in (3.6). Further let \( \mathcal{E} \in \mathcal{G}(\Lambda) \) denote a random six-vertex ensemble sampled under (the marginal on \( \mathcal{G}(\Lambda) \)) \( \mu \), and let \( F \in \mathcal{G}(\mathbb{Z}^2) \) denote a random six-vertex ensemble sampled under \( \mu(s) \). Then,

\[
\mathbb{P}_{\mu} \left[ \left| \frac{1}{K^2} \sum_{i=1}^{K^2} \psi_i(\mathcal{E}) - \mathbb{P}_{\mu(s)}[F_{[-k,k] \times [-k,k]} = \mathcal{G}] \right| > \varepsilon \right] < \varepsilon.
\]
Proof. For any nonempty index set $I \subseteq [1, K^2]$, real number $\eta > 0$, and six-vertex ensemble $\mathcal{E}_0 \in \mathfrak{G}(\Lambda)$, define the event

$$\mathfrak{D}_I(\varepsilon; \eta) = \left\{ \frac{1}{|I|} \sum_{i \in I} \psi_i(\mathcal{E}_0) - P_{\mu(s)}[\mathcal{F}_{[-k,k] \times [-k,k]} = \mathcal{O}] > \frac{\varepsilon}{2} \right\} \cap \bigcap_{i \in I} \Theta_i(\eta),$$

(3.10)

similar to the one appearing in (3.7) (where we have recalled $\Theta_i(\eta)$ from below (3.6)).

Next, recalling the measure $\nu = \nu_N \in \mathcal{P}(\mathfrak{G}(\mathbb{Z}^2 \setminus \Lambda_N))$ from Definition 3.5, sample $\mathcal{H} \in \mathfrak{G}(\mathbb{Z}^2 \setminus \Lambda)$ under $\nu$ and denote its boundary data by $\mathfrak{u}(\mathcal{H}) \cup \mathfrak{v}(\mathcal{H})$. Let $\mathcal{E}' \in \mathfrak{E}(\mathcal{H}(\Lambda))$ denote a random six-vertex ensemble sampled under the $(B_1, B_2)$-stochastic six-vertex model $\mathfrak{G}$ on $\Lambda$ with entrance data $\mathfrak{u}(\mathcal{H})$ and free exit data. Then, Lemma 3.9 yields (after taking the expectation there over $\mathfrak{u}$ with respect to $\nu$) constants $\delta = \delta(\varepsilon, s, k) > 0$, $c = c(s, k) > 0$, $c_0 = c_0(\varepsilon) > 0$, and $C = C(\varepsilon, s, k) > 1$ such that

$$\mathbb{E}_\nu[\mathbb{P}_{\mathfrak{G}}[\mathfrak{D}_I(\varepsilon; \eta)]] < e^{-c_0|I|},$$

(3.11)

for any real number $\eta \in (0, \delta)$; integers $M > C$ and $\frac{cM}{2} < Y < cM$; and nonempty set $I \subseteq [1, K^2]$.

Fix parameters $\eta, M, Y$ satisfying these properties. We would eventually like to bound the probability of the events $\mathfrak{D}_I$ but without the $\Theta_i(\eta)$ appearing in their definitions (3.10). To do this, it will be useful to show an exponential bound for the probability of the union of the $\mathfrak{D}_I(\varepsilon; \eta)$ over all $I$ of sufficiently large size. To that end, further fix a real number $\omega \in (0, \frac{1}{2})$, later to be chosen sufficiently small. Defining the set

$$\mathfrak{I}(\omega) = \{ I \subseteq [1, K^2] : |I| \geq (1 - \omega)K^2 \},$$

taking a union bound in (3.11) over $I \in \mathfrak{I}(\omega)$ yields

$$\mathbb{E}_\nu\left[\mathbb{P}_{\mathfrak{G}}\left[\bigcup_{I \in \mathfrak{I}(\omega)} \mathfrak{D}_I(\varepsilon; \eta)\right]\right] < e^{c_0(\omega - 1)K^2}\left|\mathfrak{I}(\omega)\right| < e^{c_0(\omega - 1)K^2}\left(\frac{4}{\omega}\right)^{2\omega K^2}.$$  

(3.12)

Here, we have used the fact that

$$|\mathfrak{I}(\omega)| = \sum_{r \geq (1 - \omega)K^2} \binom{K^2}{r} = \sum_{r \leq \omega K^2} \binom{K^2}{r} \leq \sum_{r \leq \omega K^2} \left(\frac{4K^2}{r}\right)^r \leq \left(\frac{4}{\omega}\right)^{\omega K^2} \sum_{r \leq \omega K^2} 2^{r-\omega K^2} = 2\left(\frac{4}{\omega}\right)^{\omega K^2},$$

where the third statement holds since $r! \geq \left(\frac{r}{4}\right)^r$ and the fourth holds since

$$\left(\frac{4K^2}{r}\right)^r \leq \frac{1}{2}\left(\frac{4K^2}{r+1}\right)^{r+1} \leq \frac{1}{4}\left(\frac{4K^2}{r+2}\right)^{r+2} \leq \cdots \leq \frac{1}{2^{\omega K^2-r}}\left(\frac{4}{\omega}\right)^{\omega K^2}.$$  

where in each bound we used the fact that $2\left(\frac{4K^2}{r}\right)^r < \left(\frac{4K^2}{r+1}\right)^{r+1}$ for $r' \in [0, \frac{K^2}{2}]$. Selecting $\omega = \omega(\varepsilon, s, k) > 0$ sufficiently small so that

$$c_0\omega + 2\omega \log\left(\frac{4}{\omega}\right) < \frac{c_0}{2},$$
we deduce from (3.12) and the fact that $N = KM$ that

$$
\mathbb{E}_{\gamma} \left[ \mathbb{P}_{\otimes} \left[ \bigcup_{I \in \mathcal{G}(\omega)} \mathcal{D}_I (\varepsilon; \eta) \right] \right] < e^{c_0 (\omega - 1) K^2} |\mathcal{G}(\omega)| < e^{-c_0 N^2 / 2M^2}.
$$

Hence, by the $\gamma = \frac{c_0}{2M^2}$ case of Lemma 3.8, there exists a constant $C_0 = C_0 (M, \varepsilon, \mu, k) > 1$ such that for $K > C_0$ we have

$$
\mathbb{P}_{\mu} \left[ \bigcup_{I \in \mathcal{G}(\omega)} \mathcal{D}_I (\varepsilon; \eta) \right] < \frac{\varepsilon}{2}.
$$

(3.13)

Now, observe if

$$
\left| \frac{1}{K^2} \sum_{i=1}^{K^2} \psi_i (\mathcal{E}) - \mathbb{P}_{\otimes} [F_{[-k,k] \times [-k,k]} = \mathcal{G}] \right| > \varepsilon,
$$

then for any $I \in \mathcal{G}(\omega)$ we have

$$
\left| \sum_{i \in I} \psi_i (\mathcal{E}) - \mathbb{P}_{\mu} [F_{[-k,k] \times [-k,k]} = \mathcal{G}] \right| \geq \left| \sum_{i=1}^{K^2} \psi_i (\mathcal{E}) - K^2 \mathbb{P}_{\mu} [F_{[-k,k] \times [-k,k]} = \mathcal{G}] \right| - 2\omega K^2
$$

$$
\geq (\varepsilon - 2\omega)K^2 \geq (\varepsilon - 2\omega) |I|.
$$

Hence, if we further select $\omega = \omega (\varepsilon, s, k) > 0$ sufficiently small so that $\omega < \frac{\varepsilon}{4}$, then

$$
\left\{ \left| \frac{1}{K^2} \sum_{i=1}^{K^2} \psi_i (\mathcal{E}) - \mathbb{P}_{\otimes} [F_{[-k,k] \times [-k,k]} = \mathcal{G}] \right| > \varepsilon \right\} \cap \bigcup_{I \in \mathcal{G}(\omega)} \bigcap_{i \in I} \Theta_i (\eta) \subseteq \bigcup_{I \in \mathcal{G}(\omega)} \bigcap_{i \in I} \mathcal{D}_I (\varepsilon; \eta),
$$

and so (3.13) yields

$$
\mathbb{P}_{\mu} \left[ \left\{ \left| \frac{1}{K^2} \sum_{i=1}^{K^2} \psi_i (\mathcal{E}) - \mathbb{P}_{\otimes} [F_{[-k,k] \times [-k,k]} = \mathcal{G}] \right| > \varepsilon \right\} \cap \bigcup_{I \in \mathcal{G}(\omega)} \bigcap_{i \in I} \Theta_i (\eta) \right] < \frac{\varepsilon}{2}.
$$

(3.14)

Therefore, by a union bound, it suffices to show for sufficiently large $M$ that

$$
\mathbb{P}_{\mu} \left[ \bigcup_{I \in \mathcal{G}(\omega)} \bigcap_{i \in I} \Theta_i (\eta) \right] \geq 1 - \frac{\varepsilon}{2},
$$

or equivalently that

$$
\mathbb{P}_{\mu} \left[ \sum_{i=1}^{K^2} 1_{\Theta_i (\eta)} \geq (1 - \omega) K^2 \right] \geq 1 - \frac{\varepsilon}{2}.
$$

(3.14)
To that end, first observe that Lemma 2.4 yields a constant $C_1 = C_1(\epsilon, \mu, s, k) > 1$ such that $\mathbb{P}[\Theta_i(\eta)^c] < \frac{\epsilon \omega}{2}$ holds for any index $i \in [1, K^2]$ whenever $M > C_1$ (since $\eta$ and $\omega$ only depend on $\epsilon, s$, and $k$). Thus, a Markov estimate yields

$$\mathbb{P}_\mu \left[ \sum_{i=1}^{K^2} \mathbbm{1}_{\Theta_i(\eta)} \leq (1 - \omega)K^2 \right] = \mathbb{P}_\mu \left[ \sum_{i=1}^{K^2} \mathbbm{1}_{\Theta_i(\eta)^c} \geq \omega K^2 \right] \leq \frac{1}{\omega K^2} \mathbb{E}_\mu \left[ \sum_{i=1}^{K^2} \mathbbm{1}_{\Theta_i(\eta)^c} \right] = \frac{1}{\omega K^2} \sum_{i=1}^{K^2} \mathbb{P}_\mu[\Theta_i(\eta)^c] \leq \frac{\epsilon}{2},$$

which implies (3.14) and therefore the lemma.

Now we can quickly establish Proposition 3.6.

**Proof of Proposition 3.6.** Let $\mathcal{E}, \mathcal{F} \in \mathfrak{C}(\mathbb{Z}^2)$ denote random six-vertex ensembles sampled under $\mu$ and $\mu(s)$, respectively. It suffices to show that, for any integer $k > 0$ and six-vertex ensemble $\mathcal{G} \in \mathfrak{C}([-k, k] \times [-k, k])$, we have

$$\mathbb{P}_\mu[\mathcal{E}_{[-k,k] \times [-k,k]} = \mathcal{G}] = \mathbb{P}_{\mu(s)}[\mathcal{F}_{[-k,k] \times [-k,k]} = \mathcal{G}]. \quad (3.15)$$

To that end, fix $k \in \mathbb{Z}_{>0}$ and $\mathcal{G} \in \mathfrak{C}([-k, k] \times [-k, k])$, and recall the function $\psi_i(\mathcal{E})$ from (3.6). Since $(M \mathbb{Z})^2$ is an amenable group and $\mu$ is invariant under its action, it follows from the pointwise ergodic theorem for amenable group actions (see, for instance, [20, part (ii) of Theorem 3.3]) that, for any integer $M > 0$, the limit

$$H(M) = \lim_{K \to \infty} \frac{1}{K^2} \sum_{i=1}^{K^2} \psi_i(\mathcal{E}),$$

exists almost surely under $\mu$, and its expectation is given by

$$\mathbb{E}_\mu[H(M)] = \mathbb{P}_\mu[\mathcal{E}_{[-k,k] \times [-k,k]} = \mathcal{G}]. \quad (3.16)$$

Now, Lemma 3.10 implies for any $\epsilon > 0$ that

$$\lim_{M \to \infty} \mathbb{P}_\mu \left[ \left| H(M) - \mathbb{P}_{\mu(s)}[\mathcal{F}_{[-k,k] \times [-k,k]} = \mathcal{G}] \right| > \epsilon \right] = 0,$$

from which (3.15) follows by taking expectation and applying (3.16).

## 4 | PARTITION FUNCTION ESTIMATES

In this section, we establish Proposition 3.7, which we do in Section 4.3, after introducing a sparsification procedure in Section 4.1 and an ensemble extension property in Section 4.2 that will be used in its proof. Throughout this section, we fix real numbers $0 < B_1 < B_2 < 1$ and will allow constants to depend on them, even when not explicitly mentioned.
4.1 | \((L; K)\)-Restrictions

In this section, we introduce and describe properties of a certain way of ‘sparsifying’ six-vertex ensembles, which we refer to as \((L; K)\)-restriction. This procedure has the benefit of simultaneously altering the slope of a six-vertex ensemble (see Lemma 4.2), while not reducing the associated partition function on an \(N \times N\) square by more than \(e^{-o(N^2)}\), assuming \(K \gg 1\) (see Proposition 4.3). This will eventually enable us in the proof of Proposition 3.7 to compare partition functions of a pure state of slope \((s, t) \in \mathcal{H}\) to one from Section 2.4 of slope \((s_0, t_0) \in \partial \mathcal{H}\), whose partition function equals to 1 (as it is induced by a stochastic model with free exit data).

This \((L; K)\)-restriction procedure removes \(K - L\) out of every \(K\) consecutive paths in a six-vertex ensemble (and retains the remaining \(L\) ones). This is made more precise through the following definition.

**Definition 4.1.** Fix an integer \(N > 0\); define the domain \(\Lambda_N = [1, N] \times [1, N] \subset \mathbb{Z}^2\); and let \(u \cup v = (u_{-B}, u_{1-B}, \ldots, u_A) \cup (v_{-B}, v_{1-B}, \ldots, v_A)\) be boundary data on \(\Lambda_N\) for a six-vertex ensemble \(E \in \mathcal{G}(\Lambda_N)\). For any integers \(K > 0\) and \(L \in [0, K]\), we define the \((L; K)\)-restriction of \(u \cup v\) to be the boundary data \(u' \cup v' = (u'_{-B}, u'_{1-B}, \ldots, u'_{A}) \cup (v'_{-B}, v'_{1-B}, \ldots, v'_{A})\) obtained by setting \(u'_i \in u'\) and \(v'_i \in v'\) if and only if there exist \(m \in \mathbb{Z}\) and \(r \in [1, L]\) such that 
\[
u'_i = u_{mK+r}\quad \text{and} \quad v'_i = v_{mK+r},
\]
respectively.

Similarly, the \((L; K)\)-restriction \(E' \in \mathcal{G}(\Lambda_N)\) of \(E\) is the six-vertex ensemble defined as follows. Denoting the noncrossing path ensembles associated with \(E\) and \(E'\) by \(P = (p_{-B}, p_{1-B}, \ldots, p_A)\) and \(P' = (p'_{-B}, p'_{1-B}, \ldots, p'_{A})\), respectively, we have \(p'_i \in P'\) if and only if there exist \(m \in \mathbb{Z}\) and \(r \in [1, L]\) such that \(p'_i = p_{mK+r}\). We refer to Figure 7 for a depiction.

The following lemma states that \((L; K)\)-restricting regular boundary data of slope \((s_0, t_0)\) largely preserves its regularity but ‘reduces’ its slope to \((s, t) \approx (\delta s_0, \delta t_0)\), where \(\delta = \frac{L}{K}\).
Lemma 4.2. Fix real numbers $\eta \in (0, 1)$ and $R \geq 1$; two pairs $(s_0, t_0), (s, t) \in (0, 1]^2$; and integers $N \geq K \geq L > 0$. Assume

\[
\frac{s_0 L}{K} - s < \eta; \quad \frac{t_0 L}{K} - t < \eta; \quad R \leq L; \quad \eta < \frac{S_0 t_0}{4}.
\]

Define the domain $\Lambda = \Lambda_N = [1, N] \times [1, N] \subseteq \mathbb{Z}^2$, and fix boundary data $u \cup v$ on $\Lambda$; let $u' \cup v'$ denote the $(L; K)$-restriction of $u \cup v$. If $u \cup v$ is $(R; \eta)$-regular with slope $(s_0, t_0)$, then $u' \cup v'$ is

\[
\left(\frac{2K}{s_0 t_0 \omega}, \frac{4(\eta + \omega)}{s_0 t_0}\right)
\]

-regular with slope $(s, t)$, for any real number $\omega > 0$.

Proof. Let $I \subset \partial \Lambda$ be an interval with $|I| \leq \frac{2K}{s_0 t_0 \omega}$. It suffices to show that if $I$ lies on the north or south boundary of $\Lambda$, then $|I \cap (u' \cup v') - s| \leq 8(\eta + \omega)(s_0 t_0 \omega)^{-1} K$ and, if $I$ lies on the east or west boundary of $\Lambda$, then $|I \cap (u' \cup v') - t| \leq 8(\eta + \omega)(s_0 t_0 \omega)^{-1} K$. Let us assume that $I \subset [1, N] \times \{0\}$ lies on the south boundary of $\Lambda$, as the remaining cases are entirely analogous.

In this case, let $u = (u_{-B}, u_{1-B}, \ldots, u_A)$ and $v = (v_{-B}, v_{1-B}, \ldots, v_A)$; for each integer $i \geq 1$, set $u_i = (0, y_i)$. Further define the intervals $I_m = \{0\} \times [y_{mK+1}, y_{mK+K}] \subset \partial \Lambda$ and $J_m = \{0\} \times [y_{mK+1}, y_{mK+L}] \subseteq I_m$ along the west boundary of $\Lambda$, for each integer $m \geq 0$ for which they exist. Since $u$ is $(R; \eta)$-regular of slope $(s, t)$ and $|I_m| \geq K \geq L \geq R$, by Remark 2.3 we have that $K = |u \cap I_m| \geq (s_0 - 2 \eta)|I_m| \geq \frac{s_0|I_m|}{2}$ (as $\eta < \frac{s_0 t_0}{4}$). Thus, $|I_m| \leq 2s_0^{-1} K$.

Now, we may assume that $|I| \geq \frac{2K}{s_0 t_0 \omega}$, for otherwise $|I \cap (u' \cup v') - s| \leq |I| \leq 8(\eta + \omega)(s_0 t_0 \omega)^{-1} K$. Then $|I| > 4s_0^{-1} K$ so, since $|I_m| \leq 2s_0^{-1} K$, there exist integers $0 \leq m_1 \leq m_2$ such that $\bigcup_{m=m_1}^{m_2} I_m \subseteq I$ and $|I \setminus \bigcup_{m=m_1}^{m_2} I_m| \leq 4s_0^{-1} K$. Since $u'$ is the $(L; K)$-restriction of $u$, this yields

\[
|I \cap (u' \cup v') - t| \leq \left| \bigcup_{m=m_1}^{m_2} (J_m \cap u) \right| - t|I| + 4s_0^{-1} K
\]

\[
\leq (m_2 - m_1 + 1)L - t \left| \bigcup_{m=m_1}^{m_2} I_m \right| + 8s_0^{-1} K, \tag{4.1}
\]

Moreover, by Remark 2.3, the $(R; \eta)$-regularity of $u \cap v$ and the fact that $|I_m \cap u| = K$ imply $|K - t_0|I_m| \leq 2\eta|I_m|$. Summing over $m \in [m_1, m_2]$, we obtain

\[
\left| \bigcup_{m=m_1}^{m_2} I_m \right| - t_0^{-1}(m_2 - m_1 + 1)K \leq 2t_0^{-1} \eta|I|.
\]

Together with (4.1) and the bounds $|L - t_0^{-1} t K| \leq t_0^{-1} \eta K$ and $(m_2 - m_1 + 1)K \leq |I| \leq 2(s_0 t_0 \omega)^{-1} K$, this gives

\[
|I \cap (u' \cup v') - t| \leq (m_2 - m_1 + 1)|L - t_0^{-1} t K| + 2t_0^{-1} t \eta|I| + 8s_0^{-1} K \leq 3t_0^{-1} \eta|I| + 8s_0^{-1} K
\]

\[
\leq 8(\eta + \omega)(s_0 t_0 \omega)^{-1} K,
\]

which, as mentioned above, implies the lemma. □
We next have the following proposition that compares probabilities between two stochastic six-vertex models, the latter of whose boundary data is the \((L; K)\)-restriction of that of the former. Observe in the below that the prefactor \(((1-B_1)(1-B_2))^{4MN}\) appearing on the right side of (4.2) is \(e^{-\alpha(N^2)}\) if \(K \gg 1\), indicating in this case that \((L; K)\)-restriction cannot reduce a partition function by more than \(e^{\alpha(N^2)}\).

**Proposition 4.3.** Fix integers \(N, K > 0\) and \(L \in [0, K]\). Set \(M = \lceil \frac{N}{K} \rceil\), define \(\Lambda = [1, N] \times [1, N] \subseteq \mathbb{Z}^2\), and fix some entrance data \(u\) for a six-vertex ensemble on \(\Lambda\). Let \(u'\) denote the \((L; K)\)-restriction of \(u\), and fix some six-vertex ensemble \(\mathcal{G} \in \mathcal{C}(u')\) on \(\Lambda\) with entrance data \(u'\).

Consider two \((B_1, B_2)\)-stochastic six-vertex models on \(\Lambda\), denoted by \(\mathcal{S}\) and \(\mathcal{S}'\), with entrance data \(u\) and \(u\)', respectively, and both with free exit data. Let \(\mathcal{E} \in \mathcal{E}(\Lambda)\) denote random six-vertex ensembles sampled under \(\mathcal{S}\) and \(\mathcal{E}' \in \mathcal{E}(\Lambda)\) denote the \((L; K)\)-restriction of \(\mathcal{E}\). Then,

\[
\mathbb{P}_{\mathcal{S}}[\mathcal{E}' = \mathcal{G}] \geq ((1-B_1)(1-B_2))^{4MN}\mathbb{P}_{\mathcal{S}}[F = \mathcal{G}]. \tag{4.2}
\]

**Proof.** For each \(n \geq 1\), recall from Section 2.3 the subdomain \(\mathcal{D}_n = \{(x, y) \in \mathbb{Z}^2 \geq 0 : x + y \leq n\} \cap \Lambda\) and diagonal \(\mathcal{D}_n = \{(x, y) \in \mathbb{Z}^2 > 0 : x + y = n\} \cap \Lambda\). We will show for each integer \(n \in [2, 2N]\) that

\[
\mathbb{P}_{\mathcal{S}}[\mathcal{E}'_n = \mathcal{G}_n | \mathcal{E}'_{n-1} = \mathcal{G}_{n-1}] \geq ((1-B_1)(1-B_2))^{2M}\mathbb{P}_{\mathcal{S}}[F_n = \mathcal{G}_n | F_{n-1} = \mathcal{G}_{n-1}]. \tag{4.3}
\]

Given (4.3) we deduce from the facts that \(\mathcal{D}_{2N} = \Lambda\) and that \(\mathcal{E}'_{n} = \mathcal{G}_n\) and \(F_n = \mathcal{G}_n\) both hold deterministically that

\[
\mathbb{P}_{\mathcal{S}}[\mathcal{E}' = \mathcal{G}] = \mathbb{P}_{\mathcal{S}}[\mathcal{E}'_{2N} = \mathcal{G}_{2N}] = \prod_{n=2}^{2N}\mathbb{P}_{\mathcal{S}}[\mathcal{E}'_n = \mathcal{G}_n | \mathcal{E}'_{n-1} = \mathcal{G}_{n-1}] \\
\geq ((1-B_1)(1-B_2))^{4MN}\prod_{n=2}^{2N}\mathbb{P}_{\mathcal{S}}[F_n = \mathcal{G}_n | F_{n-1} = \mathcal{G}_{n-1}] \\
= ((1-B_1)(1-B_2))^{4MN}\mathbb{P}_{\mathcal{S}}[F_{2N} = \mathcal{G}_{2N}] \\
= ((1-B_1)(1-B_2))^{4MN}\mathbb{P}_{\mathcal{S}}[F = \mathcal{G}],
\]

which yields (4.2). Thus, it suffices to establish (4.3).

To that end, we begin with some notation. Denote \(u = (u_{-B}, u_{1-B}, \ldots, u_A)\), and let \(I = [-B, A] \cap \bigcup_{m \in \mathbb{Z}} [mK + 1, mK + L]\). We further define the sets \(R = (r_1, r_2, \ldots, r_x) = [-B, A] \cap \bigcup_{m \in \mathbb{Z}} [mK + L]\) and \(S = (s_1, s_2, \ldots, s_y) = [-B, A] \cap \bigcup_{m \in \mathbb{Z}} [mK + 1]\). In this way, \(R\) constitutes potential indices \(r \in I\) for which \(u_r \in u'\) and \(u_{r+1} \notin u'\), and \(S\) constitutes potential indices \(s \in I\) for which \(u_s \in u'\) and \(u_{s-1} \notin u'\). Observe under this notation that \(x, y \leq 2M\). For the example depicted in Figure 7, we have that \((A, B) = (4, 3), (L, K) = (2, 3), I = (-2, -1, 1, 2, 4), R = (-1, 2), S = (-2, 1, 4), x = 2,\) and \(y = 3\). The remainder of the proof is divided into the following three parts.

(1) **Sampling:** Let us describe a coupled sampling of \((\mathcal{E}_n, \mathcal{E}'_n)\) given \((\mathcal{E}_{n-1}, \mathcal{E}'_{n-1})\) and that \(\mathcal{E}'_{n-1} = \mathcal{G}_{n-1} = F_{n-1}\). To do this, denote the noncrossing path ensemble associated with \(\mathcal{E}\) by
\[ P = (p_{-B}, p_{1-B}, \ldots, p_A) \]. Our conditioning on \((E_{r_{n-1}'}, E_{r_r})\) prescribes for each \(v \in D_n\) all indices \(i \in [-B, A]\) such that \(v \in p_i\) (that is, the indices of all paths passing through \(v\)). Denoting the arrow configurations at any \(v \in T_n\) under \(E_{r_r'}\) and \(E_{r_r'}'\) by \((i_1(v), j_1(v); i_2(v), j_2(v))\) and \((i'_1(v), j'_1(v); i'_2(v), j'_2(v))\), respectively, we have that \((i_1(v), j_1(v)) = (i'_1(v), j'_1(v))\) unless there exists some \(m \notin I\) for which \(v \in p_m\).

For any \(v \in D_n\), let us randomly define \((i_2(v), j_2(v))\) and \((i'_2(v), j'_2(v))\) (given \((i_1(v), j_1(v))\) and \((i'_1(v), j'_1(v))\)) from \((E_{r_{n-1}'}, E_{r_{n-1}}')\) as follows. In the below, all choices over \(v \in D_n\) are mutually independent.

1. If \(v \notin \bigcup_{m \in I} p_m\), then couple \((i_2(v), j_2(v)) = (i'_2(v), j'_2(v))\) under the probabilities from (2.5). Specifically, for any \(i_2, j_2 \in \{0, 1\}\) set \((i_2(v), j_2(v)) = (i'_2(v), j'_2(v))\) with probability \(w(i_1(v), j_1(v); i_2, j_2) = w(i'_1(v), j'_1(v); i_2, j_2)\) (where the latter equality holds since \((i_1(v), j_1(v)) = (i'_1(v), j'_1(v))\) if \(v \notin \bigcup_{m \in I} p_m\).

2. Otherwise, set \((i_2(v), j_2(v))\) and \((i'_2(v), j'_2(v))\) independently, according to the probabilities in (2.5).

This provides a sampling of \((E_{r_{n-1}'}, E_{r_{n-1}}')\) given \((E_{r_{n-1}}, E_{r_{n-1}}')\). Defining for each \(v \in D_n\) the events

\[ Y^{(1)}(v) = \{j'_2(v) \geq i'_1(v)\}; \quad Y^{(2)}(v) = \{i'_2(v) \geq j'_1(v)\} \]

\[ Y_n = \{F_{r_n} = G_{r_n}\} \cap \bigcap_{r \in R} \bigcap_{v \in p_r \cap p_{r+1} \cap D_n} \bigcap_s Y^{(1)}(v) \cap \bigcap_s Y^{(2)}(v) \]

we claim that \(E_{r_n}' = G_{r_n}\) holds on \(Y_n\) and that

\[ P[Y_n] \geq ((1 - B_1)(1 - B_2))^{2M} P_{\varnothing} \left[ F_{r_n} = G_{r_n} | E_{r_{n-1}'}, E_{r_{n-1}}' \right] \]

which would together imply (4.3).

(2) Proof of (4.4): To establish the latter claim (4.4), observe for any \(v \in D_n \cap \bigcup_{r \in R} (p_r \cap p_{r+1})\) or \(v \in D_n \cap \bigcup_{s \in S} (p_{s-1} \cap p_s)\) that

\[ P\left[Y^{(1)}(v) | F_{r_n} = G_{r_n}\right] \geq w(0, 1; 0, 0) = 1 - B_1; \quad P\left[Y^{(2)}(v) | E_{r_{n-1}'}, E_{r_{n-1}}'\right] \geq w(0, 1; 1, 0) = 1 - B_2, \]

since then \((i_2(v), j_2(v))\) and \((i'_2(v), j'_2(v))\) are independent under the coupling described above.

This, together with the above mentioned bounds \(|R| = x < 2M\) and \(|S| = y < 2M\) and the mutual independence between the \(Y^{(1)}(v)\) for \(v \in \bigcup_{m \notin I} (p_m \cap D_n)\), yields (4.4).

(3) Proof that \(E_{r_n}' = G_{r_n}\) on \(Y_n\): Let us restrict to \(Y_n\) and denote the arrow configuration at \(v \in \Lambda\) under \(G\) by \((I_1(v), J_1(v); I_2(v), J_2(v))\); it suffices to show that \((i'_1(v), j'_1(v); i'_2(v), j'_2(v)) = (I_1(v), J_1(v); I_2(v), J_2(v))\), for any vertex \(v \in D_n\). To do this, we consider cases depending on \(v\).

3(a): Assume \(v \notin \bigcup_I p_i\). Then \((I_1(v), J_1(v); I_2(v), J_2(v)) = (0, 0; 0, 0)\), which since \(E_{r_{n-1}'} = G_{r_{n-1}}\) implies \((i'_1(v), j'_1(v)) = (0, 0)\). Thus, we have that \((i'_1(v), j'_1(v); i'_2(v), j'_2(v)) = (0, 0; 0, 0) = (I_1(v), J_1(v); I_2(v), J_2(v))\). It thus remains to consider the case \(v \in \bigcup_I p_i\).

3(b): Assume \(v \in \bigcup_I p_i\) and \(v \notin \bigcup_{m \notin I} p_m\). Then we have that \((i_1(v), j_1(v); i_2(v), j_2(v)) = (I_1(v), J_1(v); I_2(v), J_2(v))\), since \(F_{r_n} = G_{r_n}\) on \(Y_n\) and no path through \(v\) is removed from \(E\) upon passing to its \((L; K)\)-restriction \(F\). Moreover, \((i_1(v), j_1(v); i_2(v), j_2(v)) = (i'_1(v), j'_1(v); i'_2(v), j'_2(v))\), due to the coupling between
(\(i_2(v), j_2(v)\)) and \((i'_2(v), j'_2(v))\) for \(v \not\in \bigcup_{m \in I} p_m\). It therefore again follows that
\[
(i'_1(v), j'_1(v); i'_2(v), j'_2(v)) = (I_1(v), J_1(v); I_2(v), J_2(v)).
\]

3(c): Assume \(v \in \bigcup_{i \in I} p_i \cap \bigcup_{m \not\in I} p_m\). Then, \((i_1(v), j_1(v); i_2(v), j_2(v)) = (1, 1; 1, 1)\), since \(v\) is in the intersection \(p_i \cap p_m\), for some \(i \in I\) and \(m \not\in I\). In particular, there either exists some index \(r \in R\) or \(s \in S\) such that \(v \in p_r \cap p_{r+1}\) or \(v \in p_{s-1} \cap p_s\), respectively. In the former case, \(p_{r+1}\) is removed from \(\Lambda\) when passing to \(\Lambda'\), and so \((I_1(v), J_1(v); I_2(v), J_2(v)) = (1, 0; 0, 1)\); in the latter case, \(p_{s-1}\) is removed from \(\Lambda\) when passing to \(\Lambda'\), and so \((I_1(v), J_1(v); I_2(v), J_2(v)) = (0, 1; 1, 0)\). Since \((i'_1(v), j'_1(v)) = (I_1(v), J_1(v))\), we have in the former case that \((i'_1(v), j'_1(v); i'_2(v), j'_2(v)) = (1, 0; 0, 1)\) on \(Y(1)(v)\), and in the latter case that \((i'_1(v), j'_1(v); i'_2(v), j'_2(v)) = (0, 1; 1, 0)\) on \(Y(2)(v)\). This again implies on \(Y_n\) that \((i'_1(v), j'_1(v); i'_2(v), j'_2(v)) = (I_1(v), J_1(v); I_2(v), J_2(v))\), from which we deduce \(E'_n = G_r\), and therefore the proposition. □

4.2 | Extension of six-vertex ensembles

In this section, we establish the following lemma that provides a condition for when it is possible to 'extend' a six-vertex ensemble on a square to one on a larger square with given boundary data; see the left side of Figure 8 for a depiction. This condition states that the boundary data for these ensembles along the smaller and larger squares are regular with the same slope.

**Lemma 4.4.** Fix real numbers \(\eta, s, t \in (0, 1)\) and integers \(N, W, R > 0\); assume that
\[
\min\{sW, tW\} \geq 50\eta N; \quad R \leq \eta N; \quad 50(s^{-1} + t^{-1})R \leq W \leq N. \tag{4.5}
\]

Define the domains \(\Lambda = \Lambda_{N+2W} = [1, N + 2W] \times [1, N + 2W] \subset \mathbb{Z}^2\) and \(\Lambda' = [W + 1, N + W] \times [W + 1, N + W]\). Let \(u \cup v\) and \(u' \cup v'\) denote boundary data on \(\Lambda\) and \(\Lambda'\) for six-vertex ensembles \(\mathcal{F} \in \mathcal{G}(\Lambda)\) and \(\mathcal{F}' \in \mathcal{G}(\Lambda')\), respectively. If \(u \cup v\) and \(u' \cup v'\) are both \((R; \eta)\)-regular with slope \((s, t)\), then there exists a six-vertex ensemble \(\mathcal{E} \in \mathcal{G}_{u \cup v}(\Lambda)\) such that \(\mathcal{E}_{\Lambda'} = \mathcal{F}'\).
Proof. Let us partition $\Lambda \setminus \Lambda'$ into four subdomains $\Lambda \setminus \Lambda' = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ by setting

$$
\Gamma_1 = [1, W] \times [1, N + W]; \\
\Gamma_2 = [W + 1, N + 2W] \times [1, W]; \\
\Gamma_3 = [N + W + 1, N + 2W] \times [W + 1, N + 2W]; \\
\Gamma_4 = [1, N + W] \times [N + W + 1, N + 2W].
$$

We refer to the right side of Figure 8 for a depiction. We divide the remainder of the proof into four parts.

1) **Notation:** Denote the vertex sequences $u = (u_{-B}, u_{1-B}, \ldots, u_A), v = (v_{-B}, v_{1-B}, \ldots, v_A), u' = (u'_{-B'}, u'_{1-B'}, \ldots, u'_A)$, and $v' = (v'_{-B'}, v'_{1-B'}, \ldots, v'_A)$. Also let $C$ and $D + 1$ denote the numbers of vertices in $v$ on the east and north boundaries of $\Lambda$, respectively. In this way, $A + B + 1 = C + D + 1$ both denote the number of vertices in $v$, and $v_{C-B-1}$ lies on the east boundary of $\Lambda$, but $v_{C-B}$ lies on its north boundary. Similarly, let $C'$ and $D' + 1$ denote the numbers of vertices in $v'$ on the east and north boundaries of $\Lambda'$, respectively.

Moreover, for each index $i \in \{1, 4\}$, let $A_i$ denote the number of vertices in $u$ on the west boundary of $\Gamma_i$; for each $i \in \{2, 3\}$, let $C_i$ denote the number of vertices in $v$ on the east boundary of $\Gamma_i$; and for each $i \in \{1, 4\}$, let $D_i$ denote the number of vertices in $u$ on the north boundary of $\Gamma_i$. For example, under this notation, we have that $u \cap \partial \Gamma_1 = (u_{1-B_1}, u_{2-B_1}, \ldots, u_{A_1})$, $A = A_1 + A_4$, and $B + 1 = B_1 + B_2$. We refer to the right side of Figure 8 for a depiction.

Next, for each $i \in \{1, 2, 3, 4\}$, we define an integer $K_i$ that will denote the number of paths passing between $\Gamma_{i-1}$ and $\Gamma_i$, where we let $\Gamma_0 = \Gamma_4$. To that end, we first set $K_1 = \lfloor sW \rfloor$ and then define $K_2, K_3, K_4$ by the relations

$$
A_1 + B_1 = K_1 + A' + K_2; \\
K_2 + B_2 = C_2 + K_3 + B' + 1; \\
K_3 + C' + K_4 = C_3 + D_3; \\
A_4 + K_1 + D' + 1 = K_4 + D_4.
$$

Observe that the fourth equality in (4.6) is a consequence of the first three, together with the facts that $A' + B' + 1 = C' + D' + 1$ (both equal the number of paths passing through $\Lambda'$) and $A_1 + A_4 + B_1 + B_2 = A + B + 1 = C + D + 1 = C_2 + C_3 + D_3 + D_4$ (all equal the number of paths in $\Lambda$). Under the interpretation for $K_i$ as the number of paths passing between $\Gamma_{i-1}$ and $\Gamma_i$, the four equations (4.6) indicate that the same number of paths enter $\Gamma_i$ as exit $\Gamma_i$, for each $i \in \{1, 2, 3, 4\}$; we again refer to the right side of Figure 8 for a depiction.

2) **Estimates on the $K_i$:** We claim that

$$
sW - 40\eta N \leq K_1, K_3 \leq sW + 40\eta N; \\
tW - 40\eta N \leq K_2, K_4 \leq tW + 40\eta N,
$$

which in particular implies that each of the $K_i$ are positive by (4.5). To establish (4.7), first observe from the $(R; \eta)$-regularity of the boundary data $u \cup v$ and $u' \cup v'$; the bound $R \leq W \leq N$ from (4.5); and Remark 2.3 that

$$
(t - 2\eta)(N + W) \leq A_1, C_3 \leq (t + 2\eta)(N + W); \\
(t - 2\eta)W \leq A_4, C_2 \leq (t + 2\eta)W; \\
(s - 2\eta)(N + W) \leq B_2, D_4 \leq (s + 2\eta)(N + W); \\
(s - 2\eta)W \leq B_1, D_3 \leq (s + 2\eta)W
$$

Then, for each $i \in \{1, 2, 3, 4\}$, we have

$$
A_i \leq (s - 2\eta)(N + W) + \eta N, \\
B_i \leq (s - 2\eta)W + \eta N, \\
C_i \leq (t + 2\eta)(N + W) - \eta N, \\
D_i \leq (t + 2\eta)W - \eta N,
$$

and so

$$
sW - 40\eta N \leq K_1, K_3 \leq sW + 40\eta N; \\
tW - 40\eta N \leq K_2, K_4 \leq tW + 40\eta N.
$$

We refer to the right side of Figure 8 for a depiction.
and

\[(t - 2\eta)N \leq A', C' \leq (t + 2\eta)N; \quad (s - 2\eta)N \leq B' + 1, D' + 1 \leq (s + 2\eta)N.\]  

(4.9)

Since \(W \leq N\) and \(K_1 = [sN]\), we deduce (4.7) from inserting (4.8) and (4.9) into (4.6).

(3) Boundary data on the \(\Gamma_i\): We will define boundary data \(u^{(i)} \cup v^{(i)}\) on the \(\Gamma_i\) in such a way that they each admit a six-vertex ensemble \(\mathcal{E}_i\); are consistent with each other; and are consistent with \(u \cup v\) and \(u' \cup v'\). Then, \(E \in \mathcal{G}_{u,v}(\Lambda)\) will be formed by the union \(P' \cup \bigcup_{i=1}^4 \mathcal{E}_i\). This boundary data \(u^{(i)} \cup v^{(i)}\) will be obtained as the union of the boundary data induced on \(\Gamma_i\) by \(u \cup v\) and \(u' \cup v'\) with another set of \(K_i + K_{i+1}\) vertices \(K^{(i)} \subseteq (\Gamma_{i-1} \cup \Gamma_{i+1}) \cap \partial \Gamma_i\) (where \(\Gamma_5 = \Gamma_1\)). In particular, let \(B = u \cup u' \cup u \cup v'\), and set \(u^{(i)} \cup v^{(i)} = (B \cap \partial \Gamma_i) \cup K^{(i)}\), where \(K^{(i)} = K^{(i;1)} \cup K^{(i;2)}\), and \(K^{(i;1)}\) and \(K^{(i;2)}\) denote the \(K_i\) and \(K_{i+1}\) 'most south' or 'most west' vertices in \(\Gamma_{i-1} \cap \partial \Gamma_i\) and \(\Gamma_{i+1} \cap \partial \Gamma_i\), respectively (see the thick parts of the boundaries between the \(\Gamma_i\) on the right side of Figure 8). More specifically, we set

\[
K^{(1;1)} = \bigcup_{x=1}^{K_1} \{(x, N + W + 1)\}; \quad K^{(1;2)} = \bigcup_{y=1}^{K_2} \{(W + 1, y)\};
\]

\[
K^{(2;1)} = \bigcup_{y=1}^{K_2} \{(W, y)\}; \quad K^{(2;2)} = \bigcup_{x=1}^{K_3} \{(N + W + x, W + 1)\};
\]

\[
K^{(3;1)} = \bigcup_{x=1}^{K_3} \{(N + W + x, W)\}; \quad K^{(3;2)} = \bigcup_{y=1}^{K_4} \{(N + W, N + W + y)\};
\]

\[
K^{(4;1)} = \bigcup_{y=1}^{K_4} \{(N + W + 1, N + W + y)\}; \quad K^{(4;2)} = \bigcup_{x=1}^{K_1} \{(x, N + W)\}.
\]

(4) Existence of a vertex ensemble: Setting \(u^{(i)} \cup v^{(i)} = (B \cap \partial \Gamma_i) \cup K^{(i;1)} \cup K^{(i;2)}\), let us show \(\mathcal{E}_{u^{(i)};v^{(i)}}(\Gamma_i)\) is nonempty for every \(i \in \{1, 2, 3, 4\}\). As the proof for each \(i\) is entirely analogous, we only address the case \(i = 1\). To that end, we use the following fact, which is directly verified by induction on \(m + n + 1\). Let \(\Gamma \subseteq \mathbb{Z}^2\) be a rectangular domain, and let \(j \cup k\) be some boundary data on \(\Gamma\). Denoting \(\mathbf{j} = (j_1, j_2, \ldots, j_m)\) and \(\mathbf{k} = (k_1, k_2, \ldots, k_m)\), the set \(\mathcal{E}_{\mathbf{j};\mathbf{k}}(\Gamma)\) is nonempty if and only if \(k_i \geq j_i\) for each \(i \in [-n, m]\) (where we recall from Section 2.1 that \((x_1, y_1) \succeq (x_2, y_2)\) if \(x_1 \geq y_1\) and \(x_2 \geq y_2\)).

So, setting \(u^{(1)} = (u^{(1)}_{1-B_1}, u^{(1)}_{2-B_1}, \ldots, u^{(1)}_{A_1})\) and \(v^{(1)} = (v^{(1)}_{1-B_1}, v^{(1)}_{2-B_1}, \ldots, v^{(1)}_{A_1})\), it suffices to show that \(u^{(1)}_i \leq v^{(1)}_i\) for each \(i \in [1 - B_1, A_1]\). This holds if \(i \in [1 - B_1, 0]\), since then \(u^{(1)}_i\) lies on the south boundary of \(\Gamma_1\), and if \(i \in [A_1 - K_1 + 1, A_1]\), since then \(v^{(1)}_i\) lies on the north boundary of \(\Gamma_1\). So, we may suppose that \(i \in [1, A_1 - K_1]\), in which case we let \(u^{(1)}_i = (0, w_i)\) and \(v_i = (W + 1, y_i)\); it suffices to show that \(w_i \leq y_i\).

Letting \(j = A_1 - i\), the \((R; \eta)\)-regularity of the boundary data \(u \cup v\) and \(u' \cup v'\) (and Remark 2.3) implies

\[
w_i = w_{A_1-j} \leq N - s^{-1}(j - 2\eta j - R); \quad y_i = y_{A_1-j} \geq N - s^{-1}(j - K_1 + 2\eta j + R).
\]
Thus, the bound $w_i \leq y_i$ follows from (4.7) and the fact that $sW - 40\eta N \geq 8\eta N + 2R \geq 4\eta(N + W) + 2R \geq 4\eta j + 2R$ (which holds by (4.5)). Hence, each $u_i^{(1)} \leq v_i^{(1)}$, so $\mathfrak{C}_{u_i^{(1)}, v_i^{(1)}}(\Gamma_1)$ is nonempty and therefore contains a six-vertex ensemble $\mathfrak{E}_1$.

Similarly, for each $i \in \{2, 3, 4\}$, the set $\mathfrak{C}_{u_i, v_i}(\Gamma_i)$ is nonempty and contains some six-vertex ensemble $\mathfrak{E}_i$. Letting $\mathfrak{E}_i = \mathfrak{E}_i' \cup \bigcup_{i=1}^{4} \mathfrak{E}_i$, we have $\mathfrak{E} \in \mathfrak{C}_{u, v}(\Lambda)$ and $\mathfrak{E}_\Lambda = \mathfrak{F}'$, since the boundary data for the $\mathfrak{E}_i$ are consistent with each other and with $u \cup v \cup u' \cup v'$; this yields the lemma.

\[ \square \]

## 4.3 | Proof of Proposition 3.7

In this section, we establish Proposition 3.7.

**Proof of Proposition 3.7.** Fix a real number $\delta \in (0, 1)$, and let $N > 0$ be an integer, which we will chose to be sufficiently large below. Define the real number $\eta \in (0, 1)$ and integer $R > 0$ by

$$\eta = sB_1 B_2 (1 - B_1)(1 - B_2) \frac{\delta}{650}; \quad R = \left\lfloor \frac{\eta N}{3} \right\rfloor.$$ (4.10)

Moreover, recall from Definition 3.5 the domain $\Lambda = \Lambda_N = [1, N] \times [1, N] \subseteq \mathbb{Z}^2$; the marginal distribution $\nu = \nu_N \in \mathcal{P}(\mathfrak{C}(\mathbb{Z}^2 \setminus \Lambda_N))$ of $\mu$; the boundary data $u(H) \cup v(H)$ on $\Lambda$ induced by any six-vertex ensemble $H \in \mathfrak{C}(\mathbb{Z}^2 \setminus \Lambda)$; and the partition function $Z(H)$ for the $(B_1, B_2)$-stochastic six-vertex model with boundary data $u(H) \cup v(H)$.

Define the event $\mathcal{A}$ on which $u(H) \cup v(H)$ is $(R; \eta)$-regular with slope $(s, t)$. Since Lemma 2.4 implies for sufficiently large $N$ that $\mathbb{P}_\nu[\mathcal{A}] = \mathbb{P}_\mu[\mathcal{A}] \geq 1 - \frac{\eta}{3} \geq 1 - \delta$, it suffices to show that $Z(H) \geq e^{-\delta N^2}$ holds on $\mathcal{A}$, if $N$ is sufficiently large.

Let us briefly outline how we will do this. We will first define a square subdomain of the form $\Lambda' = [W + 1, N - W] \times [W + 1, N - 1] \subseteq \Lambda$ occupying ‘most’ of $\Lambda$. It will then suffice to lower bound the sum of $w(\mathcal{E}')$ over all $\mathcal{E}' \in \mathfrak{C}(\Lambda')$ whose boundary data has approximate slope $(s, t)$. Indeed, since Lemma 4.4 implies that each such $\mathcal{E}'$ admits an extension $\mathcal{E}$ to $\Lambda$ with boundary data $u(H) \cup v(H)$, this would yield an estimate on the sum over all such $w(\mathcal{E})$ and therefore on $Z(H)$. To establish the former lower bound, we will define a pair $(s_0, t_0) \in (0, 1)^2$ such that $t_0 = \varphi(s_0)$ and consider the $(B_1, B_2)$-stochastic six-vertex model on $\Lambda'$ with double-sided $(t_0, s_0)$-Bernoulli entrance data. Conditioning on this entry data, the weight sum of all six-vertex ensembles with this entry data is equal to 1. However, the dominant contribution to this sum arises from ensembles whose boundary data has approximate slope $(s_0, t_0)$ and not $(s, t)$. So, we will consider the weight sum of the $(L; K)$-restrictions these ensembles, whose boundary data will have approximate slope $(s, t)$ if $K$ and $L$ are appropriately chosen. Then the required lower bound will follow from Proposition 4.3 estimating the weight sum of the latter, restricted ensembles in terms of that of the original, unrestricted ones.

To implement this procedure, we begin by introducing the quantities $(s_0, t_0), K, L, \text{ and } W$ used there. So, recalling $\kappa$ from (2.6), define the pair $(s_0, t_0)$ by setting

$$s_0 = \frac{\kappa s - t}{(\kappa - 1)t}; \quad t_0 = \frac{\kappa s - t}{(\kappa - 1)s}.$$ (4.11)
In this way, \((s_0, t_0)\) denotes the point where the line \(\{y = \frac{t}{s}x\}\) (passing through \((0, 0)\) and \((s, t)\)) intersects the curve \(\{(x, y) : y = \varphi(x)\} \cap \mathbb{R}^2\); we refer to Figure 9 for a depiction. Indeed, the facts that \(\frac{t_0}{s_0} = t/s\) and \(t_0 = \varphi(s_0)\) follow from the definitions (2.6) and (4.11) of \(\varphi\) and \((s_0, t_0)\), respectively.

Denoting \(\tilde{\theta} = \frac{s_0}{s}\), we moreover have that

\[
0 < s_0 \leq t_0 \leq 1; \quad 0 < \frac{s}{s_0} = \tilde{\theta} = \frac{t}{t_0} = \frac{(\kappa - 1)st}{\kappa s - t} \leq 1. \tag{4.12}
\]

To verify these, observe since \(t \leq \varphi(s)\) that

\[
0 < (\kappa - 1)st \leq \kappa s - t. \tag{4.13}
\]

This implies the first bound \(s_0 > 0\) in the first statement of (4.12). The bounds \(s_0, t_0 \leq 1\) in that statement follow from (4.11) and the fact that \(t \geq s\), and then \(s_0 \leq t_0\) holds since \(\varphi(z) \geq z\) for each \(z \in (0, 1]\). The second statement of (4.12) again follows from (4.13).

Next, define the integers \(K, L, W, M > 0\) by setting

\[
K = \left\lfloor \frac{s_0^2 t_0^2 \eta R}{16} \right\rfloor; \quad L = \left\lceil \tilde{\theta} K \right\rceil; \quad W = \left\lceil 60 \left(\frac{1}{s} + \frac{1}{t}\right) \eta N \right\rceil; \quad M = \left\lceil \frac{N - 2W}{K} \right\rceil. \tag{4.14}
\]

Further define the domain \(\Lambda' = [W + 1, N - W] \times [W + 1, N - W] \subseteq \Lambda\). We will show that there exists boundary data \(x' \cup y'\) on \(\Lambda'\) that is \((R; \eta)\)-regular with slope \((s, t)\) such that

\[
\sum_{\mathcal{E}' \in \mathcal{E}_{x',y'}(\Lambda')} w(\mathcal{E}') \geq (B_1 B_2 (1 - B_1)(1 - B_2))^{4MN+4N}. \tag{4.15}
\]

Let us establish the proposition assuming (4.15). To that end, first observe for sufficiently large \(N\) that \(\min\{sW, tW\} \geq 50\eta(N - 2W), R \leq \eta(N - 2W), \) and \(50(s^{-1} + t^{-1})R \leq W \leq N - 2W, \) due to the choices (4.14) of \(W\) and (4.10) of \(\eta < \frac{s}{650}\) and \(R\). Thus, since \(x' \cup y'\) and \(u(\mathcal{H}) \cup v(\mathcal{H})\) are both \((R; \eta)\)-regular with slope \((s, t)\) on the event \(\mathcal{A}\), Lemma 4.4 implies for each \(\mathcal{E}' \in \mathcal{E}_{x',y'}(\Lambda')\) the existence of some \(\mathcal{E} \in \mathcal{E}_{u(\mathcal{H}),v(\mathcal{H})}(\Lambda)\) such that \(\mathcal{E}_{\Lambda'} = \mathcal{E}'\). Then, since \(|\Lambda \setminus \Lambda'| \leq 4WN\) and the
weight of any vertex under the \((B_1, B_2)\)-stochastic six-vertex model is at most \(B_1 B_2 (1 - B_1)(1 - B_2)\), we have

\[
w(\mathcal{E}) \geq (B_1 B_2 (1 - B_1)(1 - B_2))^{4W_N} w(\mathcal{E}').
\]

This, together with (4.15), implies on \(A\) that

\[
Z(H) = \sum_{\mathcal{E} \in \mathcal{G}(H; N(H))} w(\mathcal{E}) \geq (B_1 B_2 (1 - B_1)(1 - B_2))^{4W_N} \sum_{\mathcal{E}' \in \mathcal{G}'(\nu')} w(\mathcal{E}') \\
\geq (B_1 B_2 (1 - B_1)(1 - B_2))^{8(M+W)N}.
\]

Then, since (4.10) and (4.14) together yield

\[
W \leq \frac{65\eta N}{st} < B_1 B_2 (1 - B_1)(1 - B_2) \frac{\delta N}{10}; \quad M \leq \frac{N}{K} \leq \frac{96}{(s_0 t_0 \eta)^2},
\]

it follows from (4.16) that

\[
Z(H) \geq (B_1 B_2 (1 - B_1)(1 - B_2))^{8(M+W)N} \geq e^{-\delta N^2},
\]

holds for sufficiently large \(N\) on \(A\), which implies the proposition.

Hence, it suffices to verify the existence of \((R; \eta)\)-regular boundary data \(x' \cup y'\) on \(\Lambda'\) such that (4.15) holds. To that end, for any real numbers \(R_0 > 1\) and \(\eta_0 \in (0, 1)\); pair \((S, T) \in (0, 1)^2\); and rectangular domain \(\Gamma \subseteq \mathbb{Z}^2\), let \(\mathcal{R}(R_0, \eta_0; S, T; \Gamma) \subseteq \mathcal{G}(\Gamma)\) denote the set of six-vertex ensembles on \(\Gamma\) whose boundary data is \((R_0; \eta_0)\)-regular with slope \((S, T)\). Additionally, for any entrance data \(w\) on \(\Gamma\), let \(\mathcal{R}_w(R_0, \eta_0; S, T; \Gamma) = \mathcal{R}(R_0, \eta_0; S, T; \Gamma) \cap \mathcal{G}_w(\Gamma)\) denote the set of six-vertex ensembles in \(\mathcal{R}(R_0, \eta_0; S, T; \Gamma)\) with entrance data given by \(w\).

Now, sample a random six-vertex ensemble \(\mathcal{G} \in \mathcal{G}(\Lambda')\) on \(\Lambda'\) under a \((B_1, B_2)\)-stochastic six-vertex model \(\mathcal{S}\) with \((s_0, t_0)\)-Bernoulli entrance data; denote the (random) entrance data for \(\mathcal{G}\) by \(z\). Then the fact that \(t_0 = \varphi(s_0)\), Proposition 2.8, and Lemma 2.4 together imply for sufficiently large \(N\) that

\[
\mathbb{P}_\mathcal{S}\left[ \mathcal{G} \in \mathcal{P}\left(L, \frac{s_0 t_0 \eta}{8}; s_0, t_0; \Lambda'\right) \right] \geq \frac{1}{2}.
\]

Hence,

\[
\mathbb{E}\left[ \sum_{\mathcal{E} \in \mathcal{P}_z(L, s_0 t_0 \eta / 8; s_0, t_0; \Lambda')} w(\mathcal{E}) \right] = \mathbb{P}_\mathcal{S}\left[ \mathcal{G} \in \mathcal{P}\left(L, \frac{s_0 t_0 \eta}{8}; s_0, t_0; \Lambda'\right) \right] \geq \frac{1}{2},
\]

where the expectation on the left side is with respect to the entrance data \(z\) for \(\mathcal{G}\). In particular, there exists some (deterministic) choice \(x\) of entrance data on \(\Lambda'\) such that

\[
\sum_{\mathcal{E} \in \mathcal{P}_x(L, s_0 t_0 \eta / 8; s_0, t_0; \Lambda')} w(\mathcal{E}) \geq \frac{1}{2}.
\]
Moreover, since $|\partial \Lambda'| \leq 4(N - W) \geq 4N - 4$, there are at most $2^{4N-4}$ possible choices of exit data for any $E \in \mathfrak{P}_x(L; \frac{s_0 t_0 \eta}{8}; s_0, t_0; \Lambda')$. Hence, it follows from (4.17) that there exists some (deterministic) exit data $y$ on $\Lambda'$ such that the boundary data $x \cup y$ is $(L; \frac{s_0 t_0 \eta}{8})$-regular with slope $(s_0, t_0)$ and

$$\sum_{E \in \mathfrak{G}_{x,y}(\Lambda')} w(E) \geq 2^{-4N}. \tag{4.18}$$

Now let $x' \cup y'$ denote the $(L; K)$-restriction of $x \cup y$; we claim that $x' \cup y'$ is $(R; \eta)$-regular with slope $(s, t)$. To see this, observe from (4.14) and the identity $\frac{s}{s_0} = \frac{L}{t_0}$ that $|\frac{s_0 L}{K} - s| < \frac{s_0 t_0 \eta}{8}$ and $|\frac{t_0 L}{K} - t| < \frac{s_0 t_0 \eta}{8}$ hold for sufficiently large $N$. Thus, since $x \cup y$ is $(L; \frac{s_0 t_0 \eta}{8})$-regular, Lemma 4.2 applies (whose $\omega$ and $\eta$ equal to $\frac{s_0 t_0 \eta}{8}$ here) and implies $x' \cup y'$ is $(R; \eta)$-regular with slope $(s, t)$.

Thus, it suffices to verify (4.15). To that end, observe by summing Proposition 4.3 that

$$\sum_{E' \in \mathfrak{G}_{x',y'}(\Lambda')} w(E') \geq ((1 - B_1)(1 - B_2))^{4MN} \sum_{E \in \mathfrak{G}_{x,y}(\Lambda')} w(E),$$

and so (4.15) follows from (4.18) and the fact that $B_1 B_2 (1 - B_1)(1 - B_2) \leq \frac{1}{2}$. \hfill $\Box$

ACKNOWLEDGEMENTS

The author heartily thanks Alexei Borodin for helpful conversations and valuable suggestions on an earlier draft of this paper. The author is also grateful to Ivan Corwin, Jan de Gier, Vadim Gorin, and Richard Kenyon for enlightening conversations. This work was partially supported by NSF grant NSF DMS-1664619, the NSF Graduate Research Fellowship under grant DGE-1144152, and a Harvard Merit/Graduate Society Term-time Research Fellowship.

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