Discrepancy Minimization in Input-Sparsity Time

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Abstract

A recent work of Larsen [Lar23] gave a faster combinatorial alternative to Bansal’s SDP algorithm for finding a coloring $x \in \{-1, 1\}^n$ that approximately minimizes the discrepancy $\text{disc}(A, x) := \|Ax\|_\infty$ of a general real-valued $m \times n$ matrix $A$. Larsen’s algorithm runs in $\tilde{O}(mn^2)$ time compared to Bansal’s $\tilde{O}(mn^{4.5})$-time algorithm, at the price of a slightly weaker logarithmic approximation ratio in terms of the hereditary discrepancy of $A$ [Ban10].

In this work we present a combinatorial $\tilde{O}(\text{nnz}(A) + n^3)$ time algorithm with the same approximation guarantee as Larsen, which is optimal for tall matrices $m = \text{poly}(n)$. Using a more intricate analysis and fast matrix-multiplication, we achieve $\tilde{O}(\text{nnz}(A) + n^{2.53})$ time, which breaks cubic runtime for square matrices, and bypasses the barrier of linear-programming approaches [ES14] for which input-sparsity time is currently out of reach.

Our algorithm relies on two main ideas: (i) A new sketching technique for finding a projection matrix with short $\ell_2$-basis using implicit leverage-score sampling; (ii) A data structure for faster implementation of the iterative EDGE-WALK partial-coloring algorithm of Lovett-Meka, using an alternative analysis that enables “lazy” batch-updates with low-rank corrections. Our result nearly closes the computational gap between real-valued and binary matrices (set-systems), for which input-sparsity time coloring was very recently obtained [JSS23].

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1 Introduction

Discrepancy is a fundamental notion in combinatorics and theoretical computer science. The discrepancy of a real matrix $A \in \mathbb{R}^{m \times n}$ with respect to a “coloring” vector $x \in \{\pm 1\}^n$ is defined as

$$\text{disc}(A, x) := \|Ax\|_\infty = \max_{j \in [m]} |(Ax)_j|.$$  

This is a natural generalization of the classic combinatorial notion of discrepancy of set-systems, corresponding to binary matrices $A \in \{0, 1\}^{m \times n}$ where rows represent (the indicator vector of) $m$ sets over a ground set of $[n]$ elements, and the goal is to find a coloring $x \in \{\pm 1\}^n$ which is as “balanced” as possible simultaneously on all sets, i.e., to find $\text{disc}(A) = \min_{x \in \{\pm 1\}^n} \|Ax\|_\infty$. Discrepancy minimization has various applications in combinatorics, computational geometry, de-randomization, rounding of integer linear programming and approximation algorithms for NP-hard problems [Cha00], and has been studied for decades in math and TCS ([Mat99]). Most of the history of this problem was focused on the existential question of understanding the minimum possible discrepancy achievable for various classes of matrices. For general set-systems of size $n$ (arbitrary $n \times n$ binary matrices $A$), Spencer’s classic result [Spe85] states that $\text{disc}(A) \leq 6\sqrt{n}$, which is asymptotically better than random coloring ($\Theta(\sqrt{n \log n})$). More recent works have focused on restricted matrix families, such as geometric matrices [Lar23] and sparse matrices ([Ban98, BF81]), showing that it is possible to achieve $o(\sqrt{n})$ discrepancy in these restricted cases. For example, the minimum-discrepancy coloring of $k$-sparse binary matrices (sparse set-systems) turn out to have discrepancy at most $O(\sqrt{k \log n})$ [Ban98].

All of these results, however, are non-constructive—they only argue about the existence of low-discrepancy colorings—which prevents their use in algorithmic applications that rely on low-discrepancy (partial) coloring, such as bin-packing problems [Rot13, HR17]. Indeed, the question of efficiently finding a low-discrepancy coloring, i.e., computing $\text{disc}(A)$, was less understood until recently, and is more nuanced: Charikar, Newman and Nikolov [CNN11] showed it is NP-hard to distinguish whether a matrix has $\text{disc}(A) = 0$ or $\text{disc}(A) = \Omega(\sqrt{n})$, suggesting that $\omega(1)$ approximation is inevitable to achieve fast runtimes. The celebrated work of Bansal [Ban10] gave the first polynomial-time algorithm which achieves an additive $O(\sqrt{n})$-approximation to the optimal coloring of general $n \times n$ matrices, matching Spencer’s non-constructive result. Bansal’s algorithm has approximation guarantees in terms of the hereditary discrepancy [LSV86], denoted $\text{herdisc}(A)$, which is the maximum discrepancy of any submatrix $A'$ of $A$ obtained by deleting a subset of the columns of $A$ (equivalently, deleting a subset of the elements in the ground-set $[n]$). More formally, Bansal [Ban10] gave an SDP-based algorithm that finds a coloring $\bar{x}$ satisfying

$$\text{disc}(A, \bar{x}) = O(\log(n) \cdot \text{herdisc}(A)),$$

for any real matrix $A$, in time $\tilde{O}(m n^{2.5})$ assuming state-of-art SDP solvers [JKL+20, IJS+22]. In other words, if all submatrices of $A$ have low-discrepancy colorings, then it is in fact possible (yet still quite expensive) to find an almost-matching overall coloring.

Building on Bansal’s work, Lovett and Meka [LM15] designed a simpler algorithm for set-systems (binary matrices $A$), running in $\tilde{O}((n + m)^3)$ time. The main new idea of their algorithm, which is also central to our work, was a subroutine for repeatedly finding a partial-coloring via random-walks (a.k.a Edge-Walk), which in every iteration “rounds” a constant-fraction of the

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1If we don’t have a polynomial constructive algorithm, then we can’t approximate solution for bin packing in polynomial. This is common in many problems (in integer programming) where we first relax it to linear programming and then rounding it back to an integer solution.
coordinates of a fractional-coloring to an integral one in \{-1, 1\} (more on this in the next section). Followup works by Rothvoss [Rot17] and Eldan-Singh [ES14] extended these ideas to the real-valued case, and developed faster convex optimization frameworks for obtaining low-discrepancy coloring, the latter requiring \(O(\log n)\) linear programs instead of a semidefinite program [Ban10], assuming a value oracle to \(\text{herdisc}(A)\).\(^2\) In this model, [ES14] yields an \(O^*(\max\{mn+n^3, (m+n)^w\})\) time approximate discrepancy algorithm via state-of-art (tall) LP solvers [BLSS20]. This line of work, however, has a fundamental setback for achieving input-sparsity time, which is a major open problem for (high-accuracy) LP solvers [BCLL18]. Sparse matrices are often the realistic case for discrepancy problems, and have been widely studied in this context as discussed earlier in the introduction. Another drawback of convex-optimization based algorithms is that they are far from being practical due to the complicated nature of fast LP solvers.

Interestingly, in the binary (set-system) case, these limitations have been very recently overcome in the breakthrough work of Jain, Sah and Sawhney [JSS23], who gave an \(\tilde{O}(n + \text{nnz}(A))\)-time coloring algorithm for binary matrices \(A \in \{-1, 1\}^{m \times n}\) with near optimal \(O(\sqrt{m} \log (m/n))\) discrepancy. While their approach, too, was based on convex optimization, their main observation was that an approximate LP solver, using first-order methods, in fact suffices for a logarithmic approximation. Unfortunately, the algorithm of [JSS23] does not extend to real-valued matrices, as their LP is based on a “heavy-light” decomposition of the rows of binary matrices based on their support size. More precisely, generalizing the argument of [JSS23] to matrices with entries in range, say, \([-R, R]\), guarantees that a uniformly random vector would only have discrepancy \(\tilde{O}(\text{poly}(R) \cdot \sqrt{m})\) on the “heavy” rows, and this term would also govern the approximation ratio achieved by their algorithm.

By contrast, a concurrent result of Larsen [Lar23] gave a purely combinatorial (randomized) algorithm which is not as fast, but handles general real matrices, and makes a step toward practical coloring algorithms. Larsen’s algorithm improves Bansal’s SDP from \(O(mn^{4.5})\) to \(\tilde{O}(mn^2 + n^3)\) time, at the price of a slightly weaker approximation guarantee: For any \(A \in \mathbb{R}^{m \times n}\), [Lar23] finds a coloring \(\vec{x} \in \{-1, +1\}^n\) such that \(\text{disc}(A, \vec{x}) = O(\text{herdisc}(A) \cdot \log n \cdot \log^{1.5} m)\).

The last two exciting recent developments naturally raise the question of whether input-sparsity time discrepancy-minimization is achievable for general (real-valued) matrices. In fact, this was one of the main open questions raised in a 2018 workshop on Discrepancy Theory and Integer Programming [Dad18].

| References | Methods | Running Time |
|-----------|---------|--------------|
| [Ban10]   | SDP [JLSW20, HJS+22] | \(mn^{4.5}\) |
| ES14*     | (Step 1: Theorem 1.3) + (Step 2: LP [JSW21]) | \(m^2 + m^{2+1/18}\) |
| ES14*     | (Step 1: Theorem 1.3) + (Step 2: LP [BLSS20]) | \(mn + n^3\) |
| ES14*     | (Step 1: Theorem 1.3) + (Step 2: LP [LS14]) | \(\text{nnz}(A)\sqrt{n + n^{2.5}}\) |
| [Lar23]   | Combinatorial | \(mn^2 + n^4\) |
| Ours (Theorem 1.1) | (Step 1: Theorem 1.3) + (Step 2: Lemma 7.14) | \(\text{nnz}(A) + n^{2.53}\) |

Table 1: Progress on approximate discrepancy-minimization algorithms of real-valued \(m \times n\) matrices \((m \geq n)\). For simplicity, we ignore \(n^{o(1)}\) and \(\text{poly}(\log n)\) factors in the table. \("[ES14]*"\) refers to a (black-box) combination of our Theorem 1.3 with [ES14] and using state-of-art LP solvers for square and tall matrices [LS14, BLSS20, JSW21].

\(^2\)[ES14]’s LP requires an upper-bound estimate on \(\text{herdisc}(A)\) for each of the \(O(\log n)\) sequential LPs, hence standard exponential-guessing seems too expensive: the number of possible “branches” is \(> 2^{O(\log n)}\).
1.1 Our Results

Our main result is a positive answer to the last question. We improve Larsen’s algorithm to near-optimal runtime for tall matrices \( m = \text{poly}(n) \), and to subcubic for square matrices, with the same approximation guarantees up to constant factors.

**Theorem 1.1** (Main result). For any parameter \( a \in [0, 1] \), there is a randomized algorithm that, given a real matrix \( A \in \mathbb{R}^{m \times n} \), runs in time

\[
\widetilde{O}(\text{nnz}(A) + n^\omega + n^{2+a} + n^{1+\omega(1,1,a)-a}) = \widetilde{O}(\text{nnz}(A) + n^{2.53}),
\]

and finds a coloring \( x \in \{-1, +1\}^n \) for which

\[
\text{disc}(A, x) = O(\text{herdisc}(A) \cdot \log n \cdot \log^{1.5} m).
\]

Here, \( \omega \approx 2.37 \) is the exponent of fast matrix multiplication (FMM). Without FMM, our algorithm runs in \( \widetilde{O}(\text{nnz}(A) + n^3) \) and is purely combinatorial.

**Remark 1.2.** Let \( \alpha \) denote the dual exponent of matrix multiplication, i.e., \( 2 = \omega(1,1,\alpha) \), and let \( a \in [0, 1] \) be the tunable parameter of Theorem 1.1. Currently \( \alpha \approx 0.31 \) [GU18]. Note that the running time of our algorithm (when using FMM) has a tradeoff between the additive terms \( n^{2+a} \) and \( n^{1+\omega(1,1,a)-a} \), so it is never better than \( n^{2.5} \), as it is never beneficial to set \( a < 1/2 \).

This tradeoff also means that our runtime is barely sensitive to future improvements in the value of the dual exponent (even \( \alpha \approx 1 \) would improve the exponent of the additive term by merely 0.03). Curiously, a similar phenomenon occurs in recent FMM-based LP solvers [CLS19, JSWZ21].

A central technical component of Theorem 1.1 is the following theorem, which quickly finds a “hereditary projection” matrix [Lar23] (i.e., a subspace s.t the projection of a constant-fraction of the rows of \( A \) to its orthogonal-complement has small \( \ell_2 \) norm), and is of independent interest in randomized linear algebra:

**Theorem 1.3** (Fast Hereditary-Projection, Informal version of Lemma 4.6 and Theorem 4.7). Let \( A \in \mathbb{R}^{m \times n} \) with \( m \geq n \), and let \( d = n/4 \). There is a randomized algorithm that runs in \( \widetilde{O}(\text{nnz}(A) + n^\omega) \) time, and outputs a \( d \times n \) matrix \( V \) such that with probability \( 1 - \delta \):

\[
\begin{align*}
&\|V_{l,*}\|_2 = 1, \forall l \in [d]. \\
&\|(A(I - V^T V))_{j,*}\|_2 = O(\text{herdisc}(A) \log(m/n)), \forall j \in [m].
\end{align*}
\]

The \( \widetilde{O} \) notation hides a \( \log(m/\delta) \) factor.

Theorem 1.3 directly improves Theorem 3 in [Lar23] which runs in \( \widetilde{O}(mn^2) \) time without using FMM, and in \( O(mn^{\omega-1}) \) using FMM. In either case, Larsen pays at least \( mn^{\omega-1} \) time, even when \( A \) is sparse (see discussion in the next section). Theorem 4.7 is in some sense best possible: Reading the input matrix \( A \) requires \( O(\text{nnz}(A)) \) time, and explicitly computing the projection matrix \( P = V^T V \) in Theorem 4.7 requires \( n^\omega \) time. We note that, in the case of binary matrices, the projection-free algorithm of [JSS23] avoids this bottleneck (and hence the \( n^\omega \) term), but for real-valued matrices, all known discrepancy algorithms involve projections [Ban10, LM15, Lar23, Rot17].
2 Technical Overview

2.1 Overview and barriers of Larsen’s algorithm

Larsen’s algorithm [Lar23] is a clever re-implementation of the iterated partial-coloring subroutine of Lovett and Meka [LM15]: In each iteration, with constant probability, this subroutine “rounds” at least half of the coordinates of a fractional coloring \( x \in \mathbb{R}^n \) to \( \{-1, 1\} \). As in [LM15], this is done by performing a random-walk in the orthogonal complement subspace \( V_\perp \) spanned by a set of rows from \( A \) (a.k.a. “Edge-Walk” [LM15]). The key idea in [Lar23] lies in a clever choice of \( V \): Using a connection between the eigenvalues of \( A^\top A \) and herdisc(\( A \)) [Lar17], Larsen shows that there is a subspace \( V \) spanned by \( \leq n/4 \) rows of \( A \), so that the projection of \( A \) onto the complement \( V_\perp \) has small \( \ell_2 \) norm, i.e., every row of \( A(I - V^\top V) \) has norm less than \( O(\text{herdisc}(A) \log(m/n)) \). Larsen shows that computing this projection operator (henceforth \( B^\top B \)) can be done in \( O(mn^2) \) time combinatorially, or in \( \mathcal{T}_{\text{mat}}(m, n, n) = \tilde{O}(mn^{ω-1}) \) time using FMM.

The second part of Larsen’s algorithm (PARTIALCOLORING) is to repeatedly apply the above subroutine (PROJECTTOSMALLROWS) to implement the Edge-Walk of [LM15]: Starting from a partial coloring \( x \in \mathbb{R}^n \), the algorithm first generates a subspace \( V_t \) by calling the aforementioned PROJECTTOSMALLROWS subroutine. Then it samples a fresh random Gaussian vector \( g_t \) in each iteration \( t \in [N] = O(n) \), and projects it to obtain \( g_t = (I - V_t^\top V_t)g_t \). The algorithm then decides whether or not to update \( V_t \): More precisely, it maintains a vector \( u_t \), and gradually updates it to account for large entries of \( x + u_{t+1} \) that have reached \( \approx 1 \) in absolute value (as in [LM15]). When a coordinate \( i \in [n] \) reaches this threshold at some iteration, the corresponding unit vector \( e_i \) is added to \( V_t \) and \( V_t \) is updated to \( V_{t+1} \). This update is necessary to ensure that in future iterations, no amount will be added to the \( i \)-th entry of \( u_t \). At the end of the loop, the algorithm outputs a vector \( x^{\text{new}} = x + u_{N+1} \) with property that for each row \( a_t \) of \( A \), the difference between \( \langle a_t, x \rangle \) and \( \langle a_t, x^{\text{new}} \rangle \) is less than \( O(\text{herdisc}(A) \cdot \log^{1/ω}(m)) \). Since \( V_t \) is constantly changing throughout iterations, each iteration requires an online Matrix-Vector multiplication \( (I - V_t^\top V_t)g_t \). Hence, since there are \( N = O(n) \) iterations, \( O(mn^2 + n^3) \) time is required to implement this part, even if fast-matrix multiplication is allowed – Indeed, the Online Matrix-Vector conjecture [HKNS15] postulates that \( mn^2 \) is essentially best possible for such online problem. Beating this barrier for the PARTIALCOLORING subroutine therefore requires to somehow avoid this online problem, as we discuss in the next subsection.

Below we summarize the computational bottlenecks in Larsen’s algorithm, and then explain the new ideas required to overcome them and achieve the claimed overall runtime of Theorem 1.1.

Implementing the first part of [Lar23] (PROJECTTOSMALLROWS) incurs the following computational bottlenecks, which we overcome in Theorem 1.3:

- **Barrier 1.** Computing the projection matrix \( B_t = A(I - V_t^\top V_t) \in \mathbb{R}^{m \times n} \) explicitly (exactly) already takes \( \mathcal{T}_{\text{mat}}(m, n, n) \) time. \(^3\)

- **Barrier 2.** Computing the \( j \)-th row’s norm \( \|e_j^\top B_t\|_2 \) requires \( \mathcal{T}_{\text{mat}}(m, n, n) \) time.

- **Barrier 3.** Computing \( B_t^\top B_t \in \mathbb{R}^{n \times n} \) requires multiplying an \( n \times m \) with a \( m \times n \) matrix, which also takes \( \mathcal{T}_{\text{mat}}(n, m, n) \) time.

Implementing the second part of [Lar23] (PARTIALCOLORING) incurs the following computational bottlenecks:

\(^3\text{We remark that, } \mathcal{T}_{\text{mat}}(m, n, n) = O(mn^2) \text{ without using FMM, and } \mathcal{T}_{\text{mat}}(m, n, n) = O(mn^{ω-1}) \text{ with using FMM.}\)
Barrier 4. The coloring algorithm requires computing \( \eta = \max_{j \in [m]} \| e_j^\top A (I - V^\top V) \|_2 \), which takes \( T_{\text{mat}}(m, n, n) \) time.

Barrier 5. Each of the \( N = O(n) \) iterations, the algorithm first chooses a Gaussian vector \( g \sim \mathcal{N}(0, I_d) \) and projects it to the orthogonal span of \( V_t \), i.e., \( g_t = (I - V_t^\top V_t) g \). Next, it finds a rescaling factor \( g \) by solving a single variable maximization problem. Finally, the algorithm checks whether \( |\langle a_j, v + g \rangle| \geq \tau, |\langle a_j, v \rangle| < \tau \) for each \( j \in [m] \). The overall runtime is therefore \( mn^2 \).

As mentioned above, the last step is conceptually challenging, as it must be done adaptively (\( V_t \) is being updated throughout iterations), and cannot be batched via FMM (assuming the OMv Conjecture [HKNS15]). One of our key observations is circumventing this step by modifying the algorithm and analysis of Larsen, which allows for batching these gaussian projections and then performing low-rank corrections as needed in the last step. We now turn to explain our technical approach and the main ideas for overcoming these computational bottlenecks.

2.2 Our Techniques

A natural approach to speedup the matrix computations in the above algorithm is to use linear sketching techniques (in the sketch-and-solve paradigm [CW13]) as they enable working with much smaller matrices in the aforementioned steps. However, sketching techniques naturally introduce (spectral) error to the algorithm, which is exacerbated in iterative algorithms. Indeed, a nontrivial challenge is showing that Larsen’s algorithm can be made robust to noise, which requires to modify his analysis in several parts of the algorithm. Another key technical challenge, not present in vanilla sketch-and-solve problems (such as linear-regression, low-rank, tensor, inverse problems [CW13, NN13, RSW16, SWZ17, SWZ19, LSZ19, JSWZ21]) is that we can never afford to explicitly store the projection matrix \( B_t \), as even writing it would already require \( O(mn) \) time. This constraint makes it more challenging to apply non-oblivious sketching tools, in particular approximate leverage-score sampling (LSS, [SS11]), which are key to our algorithm. To break the cubic runtime barrier (which is important for near-square matrices) and circumvent the OMv-hardness of Larsen’s Partial-Coloring subroutine, we re-implement this loop by dropping certain verification steps using concentration arguments, and then draw inspiration from the “lazy updates” framework [Vai89, CLS19], to “guess-and-correct” the gaussian projections in the Edge-Walk algorithm. We now turn to formalize these three main contributions.

2.2.1 Robust Analysis of [Lar23]

Approximate norm-estimation suffices. The original algorithm [Lar23] explicitly calculates the exact norm of each row of \( B_t = A (I - V_t^\top V_t) \). Since the JL lemma guarantees \( \|e_j^\top B_t\|_2 \in (1 \pm \epsilon_0)\|e_j^\top B_t\|_2, \forall j \in [n] \) except with polynomially-small probability \( \delta_0 \) (as the sketch dimension is logarithmic in \( 1/\delta_0 \)), it is not hard to show that, even though our approximation could potentially miss some “heavy” rows, this has a minor effect on the correctness of the algorithm: the rows our algorithm selects will be larger than \((1 - \epsilon_0)\) times a pre-specified threshold, and the ones that are not chosen will be smaller than \((1 + \epsilon_0)\) times the threshold. This allows to set \( \epsilon_0 = \Omega(1) \).

\footnote{The naive computation here would take \( O(n^2) \) time. Later we show how to do it in \( O(n) \) time.}

\footnote{The study of OMv originates from [HKNS15], the [LW17, CKL18] provide surprising upper bound for the problem.
Approximate SVD suffices. Recall that in the ProjectToSmallRows algorithm, expanding the subspace \( V \) iteratively, requires to compute SVD\((B_t^\top B_t)\). Since exact SVD is too costly for us, we wish to maintain an approximate SVD instead. Even though this may result in substantially different eigen-spectrum, we observe that a spectral-approximation of SVD\((B_t^\top B_t)\) suffices for this subroutine, since we only need eigenvalues to be: (i) vertical to the row space of \( A(I - V_t^\top V_t) \); (ii) orthonormal to each other. Our correctness analysis shows that an \( \epsilon_B = \Theta(1) \) spectral approximation will preserve (up to constant factor) the row-norm guarantee of ProjectToSmallRows.

2.2.2 Overcoming Barriers 1-6

Speeding-up the “hereditary-projection” step In the ProjectToSmallRows subroutine [Lar23], the matrix \( B_t \) is used for (i) detecting rows with largest norms; (ii) extracting the largest rows to generate a matrix \( \Bar{B} \); and (iii) computing the eigenvectors of \( \Bar{B}^\top \Bar{B} \). We avoid the explicit representation of matrix \( B_t \) by substituting it with a product of appropriately-chosen sketching matrices. Given the aforementioned robustness-guarantees, Barriers 1 and 2 above can be bypassed straight-forwardly by using a JL sketch \( R \in \mathbb{R}^{n \times \tilde{O}(\epsilon^{-2} \log(1/\delta_0))} \) and maintaining the compressed matrix \( \Hat{B}_t := A(I - V_t^\top V_t) R \). Similarly, detecting rows with large \( \ell_2 \) norm can be done in the sketched subspace (as it preserves norms up to constant (by choosing \( \epsilon_0 = \Theta(1) \) and \( \delta_0 = \delta / \text{poly}(m,n) \)), reducing the time for querying row-norms from \( O(mn^2) \) to \( \tilde{O}(\text{nnz}(A) + n^\omega) \).

Implicit Leverage-Score Sampling To overcome Barrier 3 (finding the eigenvalues of \( \Bar{B}_t^\top \Bar{B}_t \)), we rely on our robust- analysis which guarantees that approximating the (Top-\( k \) where \( k \approx n/\log(m/n) \)) SVD of \( \Bar{B}_t^\top \Bar{B}_t \) preserves the correctness of the algorithm. Such approximation can in principle be done in almost input-sparsity time using standard leverage-score sampling (LSS) [DMIMW12, CW13, NN13]. The main challenge, however, is that in our setting we don’t have explicit access to the input matrix \( B_t = A(I - V^\top V) \) whose rows we wish to sample, so we need to find a way to perform implicit leverage-score sampling w.r.t \( B_t \) in close to \( \sim \text{nnz}(A) \) time (!).
To this end, we design a new LSS algorithm, IMPLICITLEVERAGESCORE (Algorithm 2), which takes as inputs a matrix $A \in \mathbb{R}^{m \times n}$ and an orthonormal basis $V \in \mathbb{R}^{n \times n}$. The algorithm first generates a sparse embedding matrix $S_1 \in \mathbb{R}^{O(n) \times m}$ and uses it to generate the compressed matrix $M := (S_1 \cdot A) \cdot (I - V^T V)$. We compute the QR factorization of $M$, and let $R \in \mathbb{R}^{n \times n}$ denote the R of QR factorization of $M$. Then, the algorithm generates another sparse embedding matrix $S_2 \in \mathbb{R}^{n \times \Theta(\log(m/\delta))}$ and computes the compressed matrix $N := (I - V^T V)RS_2$. The matrix $N$ is used to calculate the (approximate) leverage scores. These steps allow us to essentially carry out the LSS lemma without ever calculating the original matrix $B_t = A(I - V^T V)$. Consequently, IMPLICITLEVERAGESCORE can generate a diagonal sampling matrix $\tilde{D}$ in $O(nnz(A) + n^2)$ time, so that w.h.p $\tilde{B}_t^T \tilde{D} \tilde{B}_t \in (1 \pm \epsilon_B) \cdot \tilde{B}_t^T \tilde{B}_t$ (see Procedure SubSample). Using this subroutine to calculate $\tilde{B}_t = \tilde{D}_t B_t$ gives a fast way to approximate SVD($\tilde{B}_t^T \tilde{B}_t$). Finally, we prove that adding the eigenvectors of $\tilde{B}_t^T \tilde{B}_t$ instead of $\tilde{B}_t^T \tilde{B}_t$ to $V$ will still satisfy the required “oversampling” prerequisite for each row in $B$.  

**Faster Iterative Coloring** Recall that with the approximate small-projection matrix in hand, we still need to overcome Barriers 4 and 5 above. Computing the heaviest row $\eta := \max_{j \in [m]} \|a_j^T(I - V^T V)\|_2$ (Barrier 4) can again be done (approximately) via the JL-sketch (as in Barrier 2), by computing $\eta := \max_{j \in [m]} \|a_j^T(I - V^T V)R\|_2$ using the JL matrix $R$. Since $R$ has only $O(\epsilon_1^{-2} \log(1/\delta_1))$ columns (where choosing $\epsilon_1 = \Theta(1)$ and $\delta_1 = \delta/poly(mn)$ is sufficient) this step reduces from $T_{\text{mat}}(m,n,n)$ time to $O(n^2)$. As discussed earlier, Barrier 5 is a different ballgame and major obstruction to speeding up Larsen’s algorithm, since the verification step

$$\mathcal{E}_\tau := \bigwedge_{j=1}^m (|\langle a_j, v + g \rangle| \geq \tau \land |\langle a_j, v \rangle| < \tau)$$  

(1)

needs to be performed adaptively in each of the $n$ iterations, which appears impossible to perform in $\ll mn^2$ time given the OMv Conjecture [HKNS15]. Fortunately, it turns out we can avoid this verification step using a slight change in the analysis of [Lar23]: Larsen’s analysis [Lar23] shows the event $\mathcal{E}_\tau$ in (1) happens with constant probability. By slightly increasing the threshold to $\tau' = \tau(\delta)$, we can ensure the event $\mathcal{E}_{\tau'}$ happens with probability $1 - \delta$. Setting $\delta$ to a small enough constant so as not to affect the other parts of the algorithm, we can avoid this verification step altogether. At this point, the entire algorithm can be boosted to ensure high-probability of success. Combining these ideas yields a (combinatorial) coloring algorithm that runs in $O(nnz(A) + n^2)$. Next, we turn to explain the new idea required to overcome the cubic term $n^3$, which is important for the near-square case ($m \approx n$).  

**Where does the $n^3$ barrier arise from?** Recall that, in order to simulate the Edge-Walk process, the PARTIALCOLORING algorithm generates, in each iteration, a Gaussian vector $g$ and projects it to $\text{Span}(V)$. This requires a generic Mat-Vec product $(I - V_i^T V_i)g$, which takes $n^2$ time. Since $V_i$ is changing adaptively (depending on whether $|x_i + v_i + g_{i,i}| = 1$ and $|x_i + v_i| < 1$ is satisfied or not for each $i$), and there are $N = O(n)$ iterations, $O(n^3)$ time for this online matrix multiplication problem seems unavoidable given the OMv Conjecture [HKNS15] (Each $e_i$ can be added to $V$ at most once, $O(n)$ Mat-Vec products indeed suffice).
2.3 Beating the Cubic Barrier

We show how to re-implement PARTIALCOLORING using a new data structure for speeding-up the Edge-Walk algorithm, combined with FMM. We now describe the main ingredients of this data structure.

**Precomputing Gaussian Projections.** To overcome the $\sim n^2$ running time for computing the Gaussian projections for $g$, we add a preprocessing phase (INIT subroutine) to the PARTIALCOLORING iterative algorithm, which generates—in advance—a Gaussian matrix $G \in \mathbb{R}^{n \times N}$ and stores the projection of every column of $G$ to the row space of $V$, denoted $\{g_{V_i}\}_{g \in G}$. We also design a QUERY procedure, which outputs any desired output vector $g = (I - V_t^T V_t)g$ on-demand. Since we do not know apriori if the subspace $V_i$ will change (and which coordinates $e_i$ will be added to $V_i$), we use a “Guess-and-Correct” approach: We guess the batch Gaussian projections, and then in iteration $t$, we perform a low-rank corrections via our update and query procedure in data structure. This idea is elaborated in more detail below.

Lazy updates for the past rank-1 sequence. Updating all the projections $\tilde{g}$ stored in the data structure would result in prohibitively expensive runtime. Instead, we use the idea of lazy updates: We divide the columns of $G$ into different batches, each batch having the size $K$. We also
use a counter $\tau_u$ to denote which batch is being used currently, and initialize two counters $k_q$ and $k_u$ to record the times QUERY and UPDATE were called, respectively. Every time we call QUERY or UPDATE, we increment the counter by 1. When either $k_q$ or $k_u$ reaches the threshold $K$, we RESTART the process, and “accumulate” the current updates as well as some clean-up operations for future iterations, adding to $P$ the vectors $w$ which are not present yet, and computes a new batch of $\tilde{g}$’s for future use. This procedure runs in time $O(\mathcal{T}_{\text{mat}}(n, K, n))$, and contributes $\mathcal{T}_{\text{mat}}(n, k, n)$ time to the RESTART procedure.

Now, Recall that at the beginning of the PARTIALCOLORING algorithm, when we initialize the data structure, we compute the first $K$ projections $\tilde{g}_1, \ldots, \tilde{g}_k$. Then we enter the iteration. Recall in INIT, we generate a matrix $P$ which is defined to be $V^TV$ for the input matrix $V$. And we pre-compute a batch of projections onto the row space of $V$. ($\tilde{G} = P \cdot \tilde{G}_{\ast,S}$, where $S = [K]$ at the beginning.) Besides, if there is some rows are added to $V$ during the running of the algorithm, we first find its factor that is vertical to the row space of $V$. Then we rescale this vector to unit and name it $w$. We maintain at most $K$ $w$’s. Then through the running of the algorithm, when we call the QUERY, we just simply select the corresponding row in $\tilde{G}$, denoted as $\tilde{g}$, and output the vector $g = \tilde{g} - \sum_{i=1}^{k_w} w_i \tilde{g}_i$. Recall that the number of $g$ we pre-computed and the number of $w$ we maintained are both limited to be less than $K$, when one of them reach the limit, we call the RESTART procedure. When this condition happens, QUERY and UPDATE will take $\mathcal{T}_{\text{mat}}(K, n, n)$. Thus by applying this lazy update idea, we finally reach the subcubic running time.

The final runtime of PARTIALCOLORING using our data structure is therefore a tradeoff based on the choice of batch size $K$. Denoting $K = n^a$, we get a total runtime of

$$\tilde{O}(\text{nnz}(A) + n^\omega + n^{2+a} + n^{1+\omega(1,1,a)−a}).$$

For the current upper bound on the fast rectangular matrix multiplication function $\omega(\cdot, \cdot, \cdot)$ [GU18], setting $a = 0.529$ yields an optimal overall runtime time of $\tilde{O}(\text{nnz}(A) + n^{2.53})$. We note that even an ideal value of $\omega$ would not improve this result by much, as it would merely enable setting $a = 0.5$ which in turn would translate into an $n^{2.5}$ time algorithm, and this is the limit of our approach.

Rodamap. We organize the paper as follows. In Section 3 we give the preliminaries of the paper, including the notations we use, the definitions and some useful results from prior works. In Section 4 we present our first result, a fast algorithm of projecting matrix into small rows. In Section 5 we give a sparsification tool for the matrix that has pattern $A^\top A$. In Section 6 we present the final algorithm of hereditary minimize. In Section 7 we present the second main result, the fast partial coloring algorithm. In Section 8 we discuss the fast maintaining algorithm we use to boost the running time.

3 Preliminaries

3.1 Notations

For a vector $x \in \mathbb{R}^n$, we use $\|x\|_1$ to denote its entrywise $\ell_1$ norm. We use $\|x\|_\infty$ to denote its entrywise $\ell_\infty$ norm. For a matrix $X$, we use $\|X\|$ to denote the spectral norm. We use $e_i$ to denote the vector where $i$-th location is 1 and everywhere else is 0.

For a matrix $A$, we say matrix $A$ positive semi-definite if $A \succeq 0$, i.e., $x^\top Ax \geq 0$ for all vectors $x$. We say $A \succeq B$, if for all $x \in \mathbb{R}^d$, $x^\top Ax \geq x^\top Bx$.

For a positive semi-definite matrix $A$, let $USU^\top$ be the SVD decomposition of $A$. We use $A^{1/2}$ to denote $US^{1/2}U^\top$. We use $A^{-1/2}$ to denote $US^{-1/2}U^\top$ where $\Sigma^{-1/2}$ is diagonal matrix that $i, i$-entry is $\Sigma_{i,i}^{-1/2}$ if $\Sigma_{i,i} \neq 0$ and $i, i$-th entry is 0 if $\Sigma_{i,i} = 0$. 

For a matrix $A$, we use $\text{nnz}(A)$ to denote the number of non-zeros in matrix $A$.

### 3.2 Definitions

Here, we define $\text{disc}$ and $\text{herdisc}$.

**Definition 3.1 (Discrepancy).** For a real matrix $A \in \mathbb{R}^{m \times n}$ with entries are 0 or 1, and a vector $x \in \{1, -1\}^n$. We define the discrepancy of $A$ to be

$$\text{disc}(A, x) := \|Ax\|_{\infty} = \max_{i \in [m]} |(Ax)_i|,$$

And we define the discrepancy of the matrix $A$ as

$$\text{disc}(A) := \min_{x \in \{1, -1\}^n} \text{disc}(A, x).$$

**Definition 3.2 (herdisc).** For a real matrix $A \in \mathbb{R}^{m \times n}$ with entries are 0 or 1. Let $A$ be defined as the set of matrices obtained from $A$ by deleting some columns from $A$. We define the hereditary discrepancy of $A$ to be

$$\text{herdisc}(A) := \max_{B \in A} \text{disc}(B),$$

where disc is defined as Definition 3.1.

### 3.3 Basic Linear Algebra

**Lemma 3.3.** Given two psd matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$. If

$$(1 - \epsilon)A \preceq B \preceq (1 + \epsilon)A$$

then, we have

$$(1 - \epsilon)\lambda_i(A) \leq \lambda_i(B) \leq (1 + \epsilon)\lambda_i(A).$$

Note that, $\lambda_i(A)$ is the $i$-the largest eigenvalue of $A$.

### 3.4 Prior Results on Herdisc

We state a tool from previous work.

**Theorem 3.4 ([Lar17]).** Given a real matrix $A \in \mathbb{R}^{m \times n}$, let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ be eigenvalues of $A^T A$. For all positive integers $k \leq \min\{m, n\}$, we have

$$\text{herdisc}(A) \geq \frac{k}{2e} \sqrt{\frac{\lambda_k}{mn}}.$$  

### 3.5 Properties of Gaussians

We state two statements about the Gaussian variables.

**Claim 3.5.** Let $t$ be a random variable such that $t \sim \mathcal{N}(0, 1)$. Then for any $\lambda > 0$, we have that

$$\Pr[|t| \leq \lambda] \leq 2 \exp(-\lambda^2/2).$$

**Claim 3.6 (Claim 2 in [Lar23]).** For an arbitrary matrix $V \in \mathbb{R}^{l \times n}$ such that $l \leq n$ and all rows of $V$ form an orthogonal basis. Let $g \in \mathbb{R}$ be a vector which is sampled with $n$ i.i.d. $\mathcal{N}(0, 1)$ distributed coordinates. Then for any arbitrary vector $a \in \mathbb{R}^n$ we have that

$$\langle a, (I - V^T V)g \rangle \sim \mathcal{N}(0, a\|I - V^T V\|^2).$$
3.6 Azuma for Martingales with Subgaussian Tails

Then for martingales with subgaussian tails, we introduce the following Azuma’s inequality. We first define the following martingale difference sequence.

**Definition 3.7** (Martingale difference sequence). For two sequences of random variables \( A = \{A_1, A_2, \ldots \} \) and \( B = \{B_1, B_2, \ldots \} \). If for any integer \( t \in \mathbb{Z}_+ \), \( A_{t+1} \) is a function of \( \{B_1, \ldots, B_t\} \) and the following holds,

\[
\Pr[\mathbb{E}[A_{t+1} | B_1, \ldots, B_t] = 0] = 1,
\]

then we say \( A \) is a martingale difference sequence with respect to \( B \).

Then we introduce the following Azuma’s inequality for martingales with Subgaussian tails.

**Theorem 3.8** (Shamir [Sha11]). Let \( \delta \in (0, 1) \) be an arbitrary failure probability. Let \( A = \{A_1, \ldots, A_T\} \) be a martingale difference sequence with respect to a sequence \( B = \{B_1, \ldots, B_T\} \). Let \( b > 1 \) and \( c > 0 \) be two constants. If for any integer \( t \) and all \( a \in \mathbb{R}_+ \), the following holds,

\[
\max\{\Pr[A_t > a | B_1, \ldots, B_{t-1}], \Pr[A_t < -a | B_1, \ldots, B_{t-1}]\} \leq b \exp(-ca^2).
\]

Then it holds that

\[
\frac{1}{T} \sum_{t=1}^{T} A_t \leq 2 \sqrt{\frac{28b \lg(1/\delta)}{cT}}
\]

with probability at least \( 1 - \delta \).

3.7 Sketching Matrices

**Definition 3.9** (Random Gaussian matrix or Gaussian transform, folklore). Let \( S = \sigma \cdot G \in \mathbb{R}^{n \times m} \) where \( \sigma \) is a scalar, and each entry of \( G \in \mathbb{R}^{n \times m} \) is chosen independently from the standard Gaussian distribution. For any matrix \( A \in \mathbb{R}^{m \times n} \), \( SA \) can be computed in \( O(s \cdot \text{nnz}(A)) \) time.

**Definition 3.10** (AMS [AMS96]). Let \( h_1, h_2, \ldots, h_b \) be \( b \) random hash functions picking from a \( 4 \)-wise independent hash family \( H = \{h : [n] \to \{-1 \sqrt{b}, +1 \sqrt{b}\}\} \). Then \( R \in \mathbb{R}^{b \times n} \) is an AMS sketch matrix if we set \( R_{i,j} = h_i(j) \).

**Definition 3.11** (CountSketch [CCFC02]). Let \( h : [n] \to [b] \) be a random 2-wise independent hash function and \( \sigma : [n] \to \{+1, -1\} \) be a random 4-wise independent hash function. Then \( R \in \mathbb{R}^{b \times n} \) is a CountSketch matrix if we set \( R_{h(i),i} = \sigma(i) \) for all \( i \in [n] \) and other entries to zero.

**Definition 3.12** (Sparse Embedding Matrix I [NN13]). We say \( R \in \mathbb{R}^{b \times n} \) is a sparse embedding matrix with parameter \( s \) if each column has exactly \( s \) non-zero elements being \( \pm 1/\sqrt{s} \) uniformly at random, whose locations are picked uniformly at random without replacement (and independent across columns).

**Definition 3.13** (Sparse Embedding Matrix II [NN13]). Let \( h : [n] \times [s] \to [b/s] \) be a random 2-wise independent hash function and \( \sigma : [n] \times [s] \to \{-1, +1\} \) be a 4-wise independent. Then \( R \in \mathbb{R}^{b \times n} \) is a sparse embedding matrix II with parameter \( s \) if we set \( R_{(j-1)b/s + h(i,j),i} = \sigma(i,j)/\sqrt{s} \) for all \( (i,j) \in [n] \times [s] \) and all other entries to zero.

**Definition 3.14** (CountSketch + Gaussian transform, Definition B.18 in [SWZ19]). Let \( S' = SS' \), where \( S' = S' \) is the CountSketch transform (defined in Definition 3.11) and \( S \in \mathbb{R}^{s \times t} \) is the Gaussian transform (defined in Definition 3.9). For any matrix \( A \in \mathbb{R}^{m \times n} \), \( S'A \) can be computed in \( O(\text{nnz}(A) + nts^{\omega-2}) \) time, where \( \omega \) is the matrix multiplication exponent.
3.8 JL Transform

We define Johnson–Lindenstrauss transform [JL84] as follows:

**Definition 3.15 (JL84).** Let $\epsilon \in (0, 1)$ be the precision parameter. Let $\delta \in (0, 1)$ be the failure probability. Let $A \in \mathbb{R}^{m \times n}$ be a fixed matrix. Let $a_i^\top$ denote the $i$-th row of matrix $A$, for all $i \in [m]$. We say $R$ is an $\epsilon, \delta$-JL transform if with probability at least $1 - \delta$,

$$(1 - \epsilon)\|a_i\|_2 \leq \|Ra_i\|_2 \leq (1 + \epsilon)\|a_i\|_2, \forall i \in [m].$$

It is well-known that random Gaussian matrices an AMS matrices gives JL-transform property.

**Lemma 3.16 (Johnson–Lindenstrauss transform, [JL84]).** Let $\epsilon \in (0, 1)$ be the precision parameter. Let $\delta \in (0, 1)$ be the failure probability. Let $A \in \mathbb{R}^{m \times n}$ be a real matrix. Then there exists an sketching matrix $R \in \mathbb{R}^{\epsilon^{-2}\log(mn/\delta) \times n}$ (defined as Definition 3.9 or Definition 3.10), such that the following holds with probability at least $1 - \delta$,

$$(1 - \epsilon)\|a_i\|_2 \leq \|Ra_i\|_2 \leq (1 + \epsilon)\|a_i\|_2, \forall i \in [m],$$

where for a matrix $A$, $a_i^\top$ denotes the $i$-th row of matrix $A \in \mathbb{R}^{m \times n}$.

The JL Lemma is known to be tight due to [LN17].

3.9 Subspace Embedding

We first define a well-known property called subspace embedding.

**Definition 3.17 (Subspace embedding, [Sar06]).** A $(1 \pm \epsilon)$ $\ell_2$-subspace embedding for the column space of an $m \times n$ matrix $A$ is a matrix $S$ for which for all $x \in \mathbb{R}^n$

$$\|SAx\|_2^2 = (1 \pm \epsilon)\|Ax\|_2^2.$$

Let $U$ denote the orthonormal basis of $A$, then it is equivalent to the all the singular values of $SU$ are within $[1 - \epsilon, 1 + \epsilon]$ with probability $1 - \delta$.

[NN13] shows that $r = O(\epsilon^{-2}n\log^8(n/\delta))$ and column sparsity $O(\epsilon^{-1}\log^3(n/\delta))$ suffices for subspace embedding (See Theorem 9 in [NN13]). Later [Coh16] improves the result to $r = O(\epsilon^{-2}n\log(n/\delta))$ and column sparsity is $O(\epsilon^{-1}\log(n/\delta))$ (see Theorem 4.2 in [Coh16]).

**Lemma 3.18 ([NN13, Coh16]).** Let $A \in \mathbb{R}^{m \times n}$ be a matrix. Let $S \in \mathbb{R}^{r \times m}$ denote the sketching matrix (defined as Definition 3.12). If $r = O(\epsilon^{-2}n\log(n/\delta))$ and the column sparsity of $S$ is $s = O(\epsilon^{-1}\log(n/\delta))$, then $S$ satisfies that with probability $1 - \delta$

$$\|SAx\|_2^2 = (1 \pm \epsilon)\|Ax\|_2^2.$$

Further, $SA$ can be done in $(s \cdot \text{nnz}(A))$ time.

3.10 Concentration Inequality

We state the matrix Chernoff bound as follows:

**Lemma 3.19 (Matrix Chernoff Bound [Tro11]).** Let $X_1, \ldots, X_s$ be independent copies of a symmetric random matrix $X \in \mathbb{R}^{n \times n}$ with $\mathbb{E}[X] = 0$, $\|X\| \leq \gamma$ almost surely and $\|\mathbb{E}[X^\top X]\| \leq \sigma^2$. Let $W = \frac{1}{s} \sum_{i \in [s]} X_i$. For any $\epsilon \in (0, 1)$,

$$\Pr[\|W\| \geq \epsilon] \leq 2n \cdot \exp\left(-\frac{s\epsilon^2}{\sigma^2 + \gamma\epsilon/3}\right).$$

Note the $\|W\|$ is the spectral norm of matrix $W$. 

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3.11 Fast Rectangular Matrix Multiplication

We describe several basic backgrounds about fast rectangular matrix multiplication.

**Definition 3.20.** Given three integers $n_1, n_2, n_3$, we use $T_{\text{mat}}(n_1, n_2, n_3)$ to denote time of multiplying a $n_1 \times n_2$ matrix with another $n_2 \times n_3$ matrix.

It is known that, we have the following fact

**Fact 3.21** ([BCS13, Blä13]).

$$T_{\text{mat}}(n_1, n_2, n_3) = O(T_{\text{mat}}(n_1, n_3, n_2)) = O(T_{\text{mat}}(n_2, n_1, n_3)).$$

For convenient, we also define the $\omega(\cdot, \cdot, \cdot)$ function

**Definition 3.22.** For $a, b, c > 0$, we use $n^{\omega(a,b,c)}$ to denote the time of multiplying an $n^a \times n^b$ matrix with another $n^b \times n^c$ matrix.

By the state-of-the-art fast rectangular matrix multiplication result [GU18], we have

**Lemma 3.23** ([GU18]). We have

- $\omega = \omega(1, 1, 1) = 2.373$.
- $\omega(1, 1, 0.529) = 2.05$.

4 Small Row Projection via Implicit Leverage-Score Sampling

Here in this section, we build our algorithm of projecting a matrix to small rows. In Section 4.1 we present the orthogonalize subroutine which is useful in later algorithms. In Section 4.2 we present the leverage score approximation and fast sampling algorithms. In Section 4.3 we present our projection algorithm together with its analysis.

4.1 Orthogonalize Subroutine

Here in this section, we present the following algorithm named ORTHOGONALIZE, which is just the Gram-Schmidt process. This subroutine is very useful in the construction of the following algorithms.

**Algorithm 1** Algorithm 1 in [Lar23]

1: procedure ORTHOGONALIZE($s \in \mathbb{R}^n, V \in \mathbb{R}^{l \times n}$) $\triangleright$ Lemma 4.1
2: \hspace{1em} $s' \leftarrow (I - V^\top V) s$
3: \hspace{1em} if $s' \neq 0$ then
4: \hspace{2em} add $s'/\|s'\|_2$ as a row of $V$
5: \hspace{1em} end if
6: \hspace{1em} return $V$
7: end procedure

The following lemma gives the running time of the above algorithm, with input vector sparsity time.

**Lemma 4.1.** The algorithm (ORTHOGONALIZE in Algorithm 1) runs $O(\|s\|_0 \cdot n)$ time.
Proof. The running time of Orthogonalize can be divided as follows,

- Line 2 takes time \( O(\|s\|_0 \cdot n) \) to compute \( s' \), where for a vector \( x \in \mathbb{R}^d \), \( \|x\|_0 \) denotes the \( \ell_0 \) norm of \( x \).
- Line 4 takes time \( O(n) \) to compute \( s'/\|s'\|_2 \).

Thus we have the total running time of \( O(\|s\|_0 \cdot n) \).

4.2 Fast Sampling via Implicit LSS

In this section, we present two crucial subroutines. That is, leverage score approximation and a fast sampling algorithm based on it.

4.2.1 Leverage Score Approximation

Here we present the following algorithm, which providing a fast leverage score approximation.

**Algorithm 2 Implicit Leverage Score Computation**

```plaintext
1: procedure ImplicitLeverageScore(A ∈ \( \mathbb{R}^{m \times n} \), V ∈ \( \mathbb{R}^{n \times n} \), ε = Θ(1), δ = Θ(1/n))  ▷ Lemma 4.3
2: \( s_1 \leftarrow \tilde{O}(\epsilon^{-2}n) \)  ▷ The goal of this procedure is to compute a constant approximation
3: Choose \( S_1 \in \mathbb{R}^{s_1 \times m} \) to be sparse embedding matrix  ▷ Definition 3.12
4: Compute \( (S_1 \cdot A) \)  ▷ It takes \( \tilde{O}(\epsilon^{-1}nnz(A)) \) time.
5: Compute \( M \leftarrow (S_1 \cdot A) \cdot (I - V^T V) \)  ▷ It takes \( \tilde{O}(\epsilon^{-2}n^\omega) \) time.
6: Let \( R \in \mathbb{R}^{n \times n} \) denote the R of QR factorization of \( M \)
7: \( s_2 \leftarrow \Theta(\log(m/\delta)) \)
8: Choose \( S_2 \in \mathbb{R}^{n \times s_2} \) to be a JL matrix  ▷ Either Definition 3.9 or Definition 3.10
9: Compute \( N \leftarrow (I - V^T V)R^{-1}S_2 \)  ▷ \( N \in \mathbb{R}^{n \times s_2} \)
10: for \( j = 1 \to m \) do  ▷ \( \sigma_j = \Theta(\sigma), \forall j \in [m] \)
11: Compute \( \bar{\sigma}_j \leftarrow \| (e_j^T A) N \|_2 \)
12: end for
13: return \( \bar{\sigma} \)
end procedure
```

**Definition 4.2 (Leverage score).** Let \( A \in \mathbb{R}^{m \times n} \). The leverage score of \( A \) is a vector \( \sigma(A) \in \mathbb{R}^n \) satisfying

\[
\sigma(A)_i = a_i(A^T A)a_i^T,
\]

where \( a_i \) denote the \( i \)-th row of \( A \).

The above algorithm provides an approximation to all the leverage scores of matrix \( A(I - V^T V) \), see the following lemma.

**Lemma 4.3 (Implicit leverage score).** Given a matrix \( A \in \mathbb{R}^{m \times n} \) and a matrix \( V \in \mathbb{R}^{n \times n} \), there is an algorithm (procedure ImplicitLeverageScore Algorithm 2) that runs in \( \tilde{O}(\epsilon^{-2}(nnz(A) + n^\omega)) \) time and output a vector \( \bar{\sigma} \in \mathbb{R}^m \), such that, \( \bar{\sigma} \) is an approximation of the leverage score of matrix \( A(I - V^T V) \), i.e.,

\[
\bar{\sigma} \in (1 \pm \epsilon) \cdot \sigma(A(I - V^T V)),
\]

with probability at least \( 1 - \delta \). The \( \tilde{O} \) hides the \( \log(n/\delta) \) factors.
Remark 4.4. The only difference between classical statement is, in classical there is one input matrix $A$. In our case, the target matrix is implicitly given in a way $A(I - V^TV)$, and we’re not allowed to formally write down $A(I - V^TV)$. The correctness proof is similar as literature [CW13, NN13].

For our task, we only need an overestimation of the leverage score. We believe that the two-sides error might have future applications, therefore, we provide a two-sided error proof.

Proof. We follow an approach of [Woo14], but instead we use a high precision sparse embedding matrix [NN13] and prove a two-sided bound on leverage score.

Correctness.

Let $S_1$ be a sparse embedding matrix with $s = O(\epsilon_{\sigma}^{-2} n \text{poly}(n/(\epsilon_{\sigma} \delta_{\sigma})))$ rows and column sparsity $O(\epsilon_{\sigma}^{-1} \log(n/\delta))$. We first compute $M := (S_1A) \cdot (I - V^TV)$, then compute the QR decomposition $M = QR$.

Now, let $S_2 \in \mathbb{R}^{n \times s_2}$ matrix with $s_2 = O(\epsilon_{\sigma}^{-2} \log(m/\delta_{\sigma}))$, each entry of $S_2$ is i.i.d. $\mathcal{N}(0, 1/s_2)$ random variables. Then we generate an sketched matrix $N := (I - V^TV)R^{-1}S_2$. Set $\tilde{\sigma}_j = \|\langle e_j^TA \rangle N \|_2^2$ for all $j \in [m]$. We argue $\tilde{\sigma}_j$ is a good approximation to $\sigma_j$.

First, with failure probability at most $\delta_{\sigma}/m$, we have that $\tilde{\sigma}_j \in (1 \pm \epsilon_{\sigma}) \cdot \|e_j^TA(I - V^TV)R^{-1}\|_2^2$ via Lemma 3.16. Now, it suffices to argue that $\|e_j^TA(I - V^TV)R^{-1}\|_2^2$ approximates $\|e_j^TU\|_2^2$ well, where $U \in \mathbb{R}^{m \times n}$ is the left singular vectors of $A(I - V^TV)$. To see this, first observe that for any $x \in \mathbb{R}^n$,

$$
\|A(I - V^TV)R^{-1}x\|_2^2 = (1 \pm \epsilon_{\sigma}) \cdot \|S_1A(I - V^TV)R^{-1}x\|_2^2 = (1 \pm \epsilon_{\sigma}) \cdot \|Qx\|_2^2 = (1 \pm \epsilon_{\sigma}) \cdot \|x\|_2^2,
$$

where the first step follows from Lemma 3.18, and the last step is due to $Q$ has orthonormal columns.

This means that all singular values of $A(I - V^TV)R^{-1}$ are in the range $[1 - \epsilon_{\sigma}, 1 + \epsilon_{\sigma}]$. Now, since $U$ is an orthonormal basis for the column space of $A(I - V^TV)$, $A(I - V^TV)R^{-1}$ and $U$ has the same column space (since $R$ is full rank). This means that there exists a change of basis matrix $T \in \mathbb{R}^{n \times n}$ with $A(I - V^TV)R^{-1}T = U$. Our goal is to provide a bound on all singular values of $T$. For the upper bound, we claim the largest singular value is at most $1 + 2\epsilon_{\sigma}$, to see this, suppose for the contradiction that the largest singular is larger than $1 + 2\epsilon_{\sigma}$ and let $v$ be its corresponding (unit) singular vector. Since the smallest singular value of $AR^{-1}$ is at least $1 - \epsilon_{\sigma}$, we have

$$
\|A(I - V^TV)R^{-1}Tv\|_2^2 \geq (1 - \epsilon_{\sigma})\|Tv\|_2^2 > (1 - \epsilon_{\sigma})(1 + 2\epsilon_{\sigma}) > 1,
$$

however, recall $A(I - V^TV)R^{-1}T = U$, therefore $\|A(I - V^TV)R^{-1}Tv\|_2^2 = \|Uv\|_2^2 = \|v\|_2^2 = 1$, a contradiction.

One can similarly establish a lower bound of $1 - 2\epsilon_{\sigma}$. Hence, the singular values of $T$ are in the range of $[1 - 2\epsilon_{\sigma}, 1 + 2\epsilon_{\sigma}]$. This means that

$$
\|e_j^TA(I - V^TV)R^{-1}\|_2^2 = \|e_j^TU\|_2^2 = (1 \pm 2\epsilon_{\sigma}) \cdot \|e_j^TU\|_2^2 = (1 \pm 2\epsilon_{\sigma}) \cdot \sigma(A(I - V^TV))_j,
$$
as desired. Scaling $\epsilon_\sigma$ by $\epsilon_\sigma/2$ yields the approximation result.

**Running Time.**
We divided the running time of the algorithm into the following lines.

- Line 6 takes time $\tilde{O}(\epsilon_\sigma^{-1} \text{nnz}(A))$ to compute the matrix $M$.
- Line 7 takes time $\tilde{O}(\epsilon_\sigma^{-2} n^\omega)$ to compute the QR decomposition of $M$.
- In Line 10, we can first multiply $R^{-1}$ with $S_2$ in time $\tilde{O}(\epsilon_\sigma^{-2} n^2)$, this gives a matrix of size $n \times s_2$. Multiplying this matrix with $(I - V^T V)$ takes $\tilde{O}(n^\omega)$ time. Note that $R \in \mathbb{R}^{n \times n}$ hence $R^{-1}$ can be computed in $O(n^\omega)$ time.
- Line 11 loop takes in total $\tilde{O}(\text{nnz}(A))$ time to compute the output vector $\tilde{\sigma}$.

Hence, the overall time for computing $\tilde{\sigma}(A(I - V^T V)) \in \mathbb{R}^m$ is $\tilde{O}(\epsilon_\sigma^{-2}(\text{nnz}(A) + n^\omega))$. □

4.2.2 Fast Subsampling

Here we present the fast subsampling algorithm based on the leverage score approximation.

**Algorithm 3** Fast Subsample Algorithm

1: procedure Subsample($B \in \mathbb{R}^{m \times n}, V \in \mathbb{R}^{n \times n}, \epsilon_B, \delta_B$) ▷ Lemma 4.5
2: $\tilde{\sigma} \leftarrow \text{ImplicitLeverageScore}(B, V, \delta_B)$ ▷ Compute an $O(1)$-approximation to the leverage scores of $B(I - V^T V)$, $\tilde{\sigma} \in \mathbb{R}^m$
3: Sample a subset rows of $\widehat{B} = B(I - V^T V)$ with proper re-weighting according to approximate leverage score $\tilde{\sigma}$
4: Let $\widehat{D}$ denote the diagonal sampling matrix such that $\Pr[\widehat{B}^T \widehat{D} \widehat{D} \widehat{B} \in (1 \pm \epsilon_B) \cdot \widehat{B}^T \widehat{B}] \geq 1 - \delta_B$ ▷ $\widehat{D} \in \mathbb{R}^{m \times m}$
5: return $\widehat{D}$
6: end procedure

**Lemma 4.5.** There is an algorithm (Algorithm 3) takes $B \in \mathbb{R}^{m \times n}$ and $V \in \mathbb{R}^{n \times n}$ as inputs and outputs a diagonal sampling matrix $\widehat{D}$ with $\|\widehat{D}\|_0 = O(\epsilon_B^{-2} n \log(n/\delta_B))$ and runs in time $\tilde{O}(\text{nnz}(B) + n^\omega)$.

Here $\tilde{O}$ hides the $\log(n/\delta_B)$ factors. Here $\omega$ is the exponent of matrix multiplication. Currently, $\omega \approx 2.373$.

**Proof.** We first approximately compute the leverage score, i.e., gives an $O(1)$-approximation to all leverage scores via Lemma 4.3. Then run we samples a number of rows according to the leverage scores, using Lemma 5.2, we can show the correctness. □

4.3 Fast Projecting to Small Rows

Here in this section, we present our first main result, i.e., the faster algorithm for projecting a matrix to small rows. We first show the algorithm as follows.
Algorithm 4 Our input sparsity projection algorithm (this improves the Algorithm 2 in [Lar23])

1: procedure FastProjectToSmallRows$\left( A \in \mathbb{R}^{m \times n}, \delta \in (0, 1) \right)$ \Comment{Theorem 4.7}
2: $V_1 \in \mathbb{R}^{n \times n}$ \Comment{Initialize an empty matrix}
3: $T \leftarrow \log(8m/n)$
4: $\delta_0 \leftarrow \Theta(\delta/(mT))$
5: $\epsilon_0 \leftarrow 0.01$
6: $r \leftarrow \epsilon_0^2 \log(1/\delta_0)$
7: $\delta_B \leftarrow \Theta(\delta/T)$
8: $\epsilon_B \leftarrow 0.1$
9: for $t = 1 \rightarrow T$ do
10: \hspace{1em} $l_t \leftarrow t \cdot \frac{n}{8T}$
11: \hspace{1em} $m_t \leftarrow m/2^t$
12: \hspace{2em} /* Ideally, we need to compute $B_t \leftarrow A(I - V_t^T V_t) \in \mathbb{R}^{m \times n}$, but we have no time to do that.*/
13: \hspace{2em} Let $R \in \mathbb{R}^{n \times r}$ denote JL sketching matrix (Either Definition 3.10 or Definition 3.9)
14: \hspace{2em} Compute $B_t \leftarrow A(I - V_t^T V_t)R$ \Comment{Lemma 3.16}
15: \hspace{2em} /* Ideally, we need to compute $\|B_j\|_2$, but we had no time to do that.*/
16: \hspace{2em} Compute $\|B_{t,j}\|_2$ for all $j \in [m]$
17: \hspace{2em} $S_t \subset [m]$ denote the indices of $B_t$ such that it contains $m_{t-1}$ rows of the largest norm
18: \hspace{2em} Let $\bar{D}_t$ denote a sparse diagonal matrix where $(i,i)$-th location is 1 if $i \in S_t$ and 0 otherwise
19: \hspace{2em} /* Ideally, we need to compute $B_t = \bar{D}_t A(I - V_t^T V_t) \in \mathbb{R}^{n \times n}$ be the submatrix obtained from $B_t$
20: \hspace{2em} Note that $\bar{\sigma}_t = \lambda_t(\bar{B}_t^T \bar{B}_t)$ and $\bar{\sigma}_t = \lambda_t(\bar{B}_t^T \bar{B}_t)$
21: \hspace{2em} /* Ideally, we need to compute the eigenvalues $\bar{\sigma}_1 \geq \cdots \geq \bar{\sigma}_n \geq 0$ and corresponding eigenvectors $\bar{u}_1, \cdots, \bar{u}_n$ of $\bar{B}_t^T \bar{B}_t$, but in algorithm we had no time to do that*/
22: \hspace{2em} Let $\bar{u}_1, \cdots, \bar{u}_n \in \mathbb{R}^n$ denote the eigenvectors of SVD decomposition of $\bar{B}_t^T \bar{B}_t = \bar{U} \bar{\Sigma} \bar{U}^T$
23: \hspace{2em} Construct matrix $V_{t+1} \in \mathbb{R}^{n \times n}$ by adding rows vectors $\{\bar{u}_1, \cdots, \bar{u}_n/(8T)\}$ to the bottom
24: \hspace{1em} $V_t \in \mathbb{R}^{n \times n}$
25: \hspace{2em} $B_T \leftarrow A \cdot (I - V_{T+1}^T V_{T+1})$
26: \hspace{2em} Let $r_1, \cdots, r_{n/8}$ be the $n/8$ rows of $B_T$ with largest norm
27: end for
28: for $j = 1 \rightarrow n/8$ do
29: \hspace{1em} $V \leftarrow \text{Orthogonalize}(r_j, V)$ \Comment{Algorithm 1}
30: end for
31: return $V$ \Comment{$V \in \mathbb{R}^{\ell \times n}$ and $\ell \leq n/4$}
32: end procedure

4.3.1 Running time

The goal of this section is to prove running time of Algorithm 4. We have the following lemma.

Lemma 4.6 (Running time of Algorithm 4). The algorithm $\left( \text{FastProjectToSmallRows} \right.$ in Algorithm 4 $)$ runs $\tilde{O}(\text{nnz}(A) + n^\omega)$ time. The succeed probability is $1 - \delta$. Note that $\tilde{O}$ hides $\log(n/\delta)$. 

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Proof. The running time of \textsc{FastProjectToSmallRows} can be divided as follows,

- Run the following lines for $i \in [T]$ times, where $T = O(\log(m/n))$:
  - Line 14 needs to compute $\hat{B}_t$, it takes the following time
    \[ O(\text{nnz}(A)r + n^\omega + n^2r) = \tilde{O}(\text{nnz}(A) + n^\omega) \]
  - Line 18 takes time $\tilde{O}(m)$ to construct $D_t$ by computing the norms, and takes time $\tilde{O}(m)$ to find the largest $m_{t-1}$ of them.
  - Using Lemma 4.5, we can compute $\tilde{D}_t$ in $\tilde{O}(\text{nnz}(A) + n^\omega)$ time
- Line 31 runs \textsc{Orthogonize} for $O(n)$ times, and takes $O(n^2)$ each time.

Adding these together, we have the total running time of
\[ \tilde{O}(\text{nnz}(A) + n^\omega). \]

\[ \square \]

4.3.2 Correctness

Here we present the correctness theorem of the Algorithm 4, and together with its proof.

\textbf{Theorem 4.7} (Correctness of Algorithm 4). For any given $m \times n$ matrix $A$, there is an algorithm that takes $A$ as input and output a matrix $V \in \mathbb{R}^{n/4 \times n}$ such that

- having unit length orthogonal rows
- all rows of $A(I - V^T V)$ have norm at most $O(\text{herdisc}(A) \log(m/n))$.

holds with probability at least $1 - \delta$.

\textit{Proof.} We define
\[ T := \log(8m/n), \]
which is the time of the main loop of Algorithm 4 execute.

We define
\[ C_0 := 1000 \]
to be a constant used later.

And we define
\[ m_t := m/2^t \]
to denote the number of rows in $\overline{B}$ at each iteration. We then continue the proof in the following paragraphs.
The rows of $V$ form an orthonormal basis. To start with, let us show that, the output matrix $V$ form an orthonormal basis with its rows. Since $V_0$ has no rows, the claim is true initially.

We first fix $t \in T$, consider at $t$-th iteration, after constructing $V_{t+1}$ by adding row vectors

$$\{\tilde{u}_1, \ldots, \tilde{u}_{n/(8T)}\}$$

to the bottom of $V_t$ in Line 26.

Note that these are eigenvectors, so they must be orthogonal to each other and for any $u$ of them, $\|u\|_2 = 1$. Moreover, we have that

$$\{\tilde{u}_1, \ldots, \tilde{u}_{n/(8T)}\} \subseteq \text{span}(\tilde{B}_t).$$

We note that, since we define $\tilde{B}_t$ as

$$\tilde{B}_t = \tilde{D}_t A (I - V_t^T V_t),$$

all rows of $\tilde{B}_t$ must be orthogonal to any rows of $V_t \in \mathbb{R}^{l_t \times n}$, and it follows that adding the above set of eigenvectors

$$\{\tilde{u}_1, \ldots, \tilde{u}_{n/(8T)}\}$$

as rows of $V_t \in \mathbb{R}^{l_t \times n}$ maintains the rows of $V_{t+1} \in \mathbb{R}^{l_{t+1} \times n}$ form an orthonormal basis.

After the first for-loop, we claim obviously that, the last for-loop (Line 31) preserves the property of $V$ such that, $V$ form an orthonormal basis with its rows.

Row norm guarantee, proved with induction. We now claim that, after the $t$-th iteration of the first for-loop (Line 9 to Line 27), we have that

$$|\{i \in [m] \mid \|b_i\|_2 \geq (1 + \epsilon_0) \cdot C_0 \cdot T \cdot \text{herdisc}(A)\}| \leq m_t,$$

i.e., there are at most $m_t$ rows in $B_t = A (I - V_t^T V_t)$ having norm larger than $(1 + \epsilon_0) \cdot C_0 \cdot T \cdot \text{herdisc}(A)$.

We first note that, for $t = 0$ this holds obviously.

Then have the following inductive assumption:

After iteration $t - 1$, there are at most $m_{t-1}$ rows have norm larger than

$$(1 - \epsilon_0) \cdot C_0 \cdot T \cdot \text{herdisc}(A).$$

We split the inductive step into the following two parts:

Norm of rows not in $\overline{B}_t$ will not increase. We first fix $t \in [T]$. Then by the inductive assumption and our approximated norm computation, we have that for the $t$‘th iteration, all rows not in $\overline{B}_t$ have norm at most

$$(1 + \epsilon_0) \cdot C_0 \cdot T \cdot \text{herdisc}(A).$$

Since we have

$$\text{span}(AV_t^T V_t) \subseteq \text{span}(V),$$

then the norm of each row in matrix $A (I - V_t^T V_t)$ will never increase after we add new rows to $V$. Thus we have the claim proved.
At most \( m_t \) rows in \( \overline{B}_t \) reach the threshold after adding rows. Now in this paragraph we prove that, after adding

\[
\{\tilde{u}_1, \ldots, \tilde{u}_{t/(8T)}\}
\]

the the bottom of \( V_t \) (Then we name the new matrix to be \( v_{t+1} \)), there are no less than \( m_t \) rows in \( \overline{B}_t(I - V_{t+1}^T V_{t+1}) \in \mathbb{R}^{m_{t-1} \times n} \) having norm larger than

\[
(1 - \epsilon_0) \cdot C_0 \cdot T \cdot \text{herdisc}(A).
\]
Here we note that, before adding the new rows to \( V_t \), since all rows of \( B_t \) are already orthogonal to all rows of \( V_t \), we have the following

\[
B_t = B_t(I - V_t^T V_t)^6.
\]

We now proof it by contradiction. For the sake of contradiction, we assume that, more than \( m_t \) rows in \( B_t(I - V_{t+1}^T V_{t+1}) \) have norm larger than \( C_0 \cdot T \cdot \text{herdisc}(A) \) after adding \( \{\tilde{u}_1, \ldots, \tilde{u}_{n/(8T)}\} \) to the bottom of \( V_t \). We assume that \( V_{t+1} \) has \( \ell_{t+1} \) rows.

Select an arbitrary set of unit vectors \( \{w_1, \ldots, w_{n-\ell_{t+1}}\} \subseteq \mathbb{R}^n \) such that,

\[
\text{span}(w_1, \ldots, w_{n-\ell_{t+1}}) = \mathbb{R}^{\tilde{d}} \setminus \text{span}(V_{t+1}).
\]

Using row norms, we can generate

\[
\tilde{B}_t = \tilde{D}_t A(I - V_t^T V_t).
\]

Using \textsc{Subsample} (Algorithm 3), we can generate a matrix

\[
\tilde{B}_t = \tilde{D}_t A(I - V_t^T V_t).
\]

By Lemma 5.2, we know that \( \tilde{B}_t \) only has roughly \( \|\tilde{D}_t\|_0 = O(\epsilon_B^{-2} n \log(n/\delta_B)) \) rows so that

\[
\text{Pr} \left[ (1 - \epsilon_B)\tilde{B}_t^T \tilde{B}_t \preceq \tilde{B}_t^T \tilde{B}_t \preceq (1 + \epsilon_B)\tilde{B}_t^T \tilde{B}_t \right] \geq 1 - \delta_B
\]

Using Lemma 3.3, the above approximation implies that

\[
\text{Pr} \left[ (1 - \epsilon_B)\lambda_i(\tilde{B}_t^T \tilde{B}_t) \preceq \lambda_i(\tilde{B}_t^T \tilde{B}_t) \preceq (1 + \epsilon_B)\lambda_i(\tilde{B}_t^T \tilde{B}_t), \forall i \in [n] \right] \geq 1 - \delta_B.
\]

We define

\[
Q := B_t(I - V_{t+1}^T V_{t+1}).
\]

And for the approximated metrics, we define

\[
\tilde{Q} := \tilde{B}_t(I - V_{t+1}^T V_{t+1}).
\]

We first note that, all rows of \( Q \) lie in \( \text{span}(w_1, \ldots, w_{n-\ell_{t+1}}) \). Then by the construction of \( \tilde{B} \), we have that, all rows of \( \tilde{Q} \) also lie in \( \text{span}(w_1, \ldots, w_{n-\ell_{t+1}}) \). It follows that

\[
\sum_{k \in [m_t-1]} \sum_{j \in [n-\ell_{t+1}]} \langle \tilde{q}_k, w_j \rangle^2 = \sum_{k \in [m_t-1]} \|\tilde{q}_k\|_2^2 > m_t \cdot ((1 - \epsilon_0) \cdot C_0 \cdot T \cdot \text{herdisc}(A))^2,
\]

where the second step follows from the contradiction assumption.

Taking the average over the vectors \( w_j \in \mathbb{R}^n \), there has to be a \( j \in [n - \ell_{t+1}] \) such that

\[
\sum_{k \in [m_t-1]} \langle \tilde{q}_k, w_j \rangle^2 > m_t \cdot ((1 - \epsilon_0) \cdot C_0 \cdot T \cdot \text{herdisc}(A))^2/n.
\]

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Figure 4: This figure shows the difference of the row selection. Figure (a): Larsen’s algorithm explicitly selected the $m_t$ largest rows from $B_t = A(I - V_t^\top V_t)$, and compute the eigenvectors of $B_t^\top B_t$, whose time cost is expensive. Figure (b): In our design, we use a subsample matrix $\tilde{D}_t$ to generate the matrix $\tilde{B}_t$, who has the eigenvalues close to $B_t$. We use the above subsampling technique to reduce the time cost when computing the SVD decomposition. By using this, we can fast compute the eigenvectors.

We denote $v = w_j$. Thus we have that,
\[
\|\tilde{B}_t v\|_2 \geq (1 - \epsilon_B) \cdot \|\bar{B}_t v\|^2 \\
> (1 - \epsilon_B) \cdot m_t \cdot ((1 - \epsilon_0) \cdot C_0 \cdot T \cdot \text{herdisc}(A))^2 / n, \tag{2}
\]
where the first step follows from Lemma 5.2. and the second step follows from the contradiction assumption that there are more than $m_t$ rows in $\bar{B}_t(I - V_{t+1}^\top V_{t+1})$ with norm more than $(1 - \epsilon_0) \cdot C_0 \cdot T \cdot \text{herdisc}(A)$ and $v$ is a unit vector.

We can upper bound $\|\tilde{B}_t v\|^2_2$,
\[
\|\tilde{B}_t v\|^2_2 = v^\top \tilde{B}_t^\top \tilde{B}_t v \\
\leq \sigma_n/(8T) \\
\leq (1 + \epsilon_B) \cdot \sigma_n/(8T), \tag{3}
\]
where the first step follows from the definition of $\ell_2$ norm, the second step follows from that $v$ is orthogonal to $\tilde{u}_1, \ldots, \tilde{u}_{n/(8T)} \in \mathbb{R}^n$ and, the third step follows from the way we generate $\tilde{B}_t$.

Thus we have that,
\[
\sigma_n/(8T) > \frac{1 - \epsilon_B}{1 + \epsilon_B} \cdot m_t \cdot ((1 - \epsilon_0) \cdot C_0 \cdot T \cdot \text{herdisc}(A))^2 / n
\]
\[ \geq \frac{1}{4} \cdot m_t \cdot (C_0 \cdot T \cdot \text{herdisc}(A))^2 / n, \]

where the first step follows from Eq. (2) and Eq. (3), the second step follows from that we set \( \epsilon_B \in (0, 0.5) \) and \( \epsilon_0 = 0.01 \).

But at the same time we have the following,

\[
\begin{align*}
\text{herdisc}(A) \geq \text{herdisc}(B_t) \\
> n \cdot 16eT \cdot \sqrt{\frac{\frac{1}{4} \cdot m_t \cdot (C_0 \cdot T \cdot \text{herdisc}(A))^2 / n}{m_t \cdot m_t - n}} \\
= 2 \cdot \text{herdisc}(A),
\end{align*}
\]

where the first step follows from that \( B_t \) is a matrix obtained from \( A \in \mathbb{R}^{m \times n} \) by deleting a subset of the rows of \( A \), the second step follows from Theorem 3.4 and \( k = n/(8T) \) here, and the last step follows from simplify the above step.

Thus, we get a contradiction.

By the proof above, we have the claim that after the first for-loop (Line 9 to Line 27 in Algorithm 4), there are at most \( m_T = n/8 \) rows in \( B_T = A(I - V_t^t V_t) \) having norm larger than \( C_0 \cdot T \cdot \text{herdisc}(A) \).

Then we do the second for-loop (Line 31 in Algorithm 4). We notice after that, all these rows lie in \( \text{Span}(V_t) \). Thus we have the desired guarantee of the algorithm.

And by union bound of the above steps, we have the result holds with probability at least \( 1 - \delta \).

Thus we complete the proof.

\[ \Box \]

## 5 Sampling a Batch of Rank-1 Terms

In this section, we give a sparsification tool for the matrix that has pattern \( A^\top A \).

We define our sampling process as follows:

**Definition 5.1 (Sampling process).** Let \( H = A^\top A \). Let \( p_j \geq \beta \cdot \sigma_j(A)/n \), suppose we sample with replacement independently for \( s \) rows of matrix \( A \), with probability \( p_j \) of sampling row \( j \) for some \( \beta \geq 1 \). Let \( j_t \) denote the index of the row sampled in the \( t \)-th trial. Define the generated sampling matrix as

\[
\widetilde{H} := \frac{1}{T} \sum_{t=1}^{T} \frac{1}{p_{j_t}} a_{j_t} a_{j_t}^\top,
\]

where \( T \) denotes the number of the trials.

For our sampling process defined as Definition 5.1, we can have the following guarantees:

**Lemma 5.2 (Sample using Matrix Chernoff).** Let \( \epsilon_0, \delta_0 \in (0, 1) \) be precision and failure probability parameters, respectively. Suppose \( \widetilde{H} \) is generated as in Definition 5.1, then with probability at least \( 1 - \delta_0 \), we have

\[
(1 - \epsilon_0) \cdot H \preceq \widetilde{H} \preceq (1 + \epsilon_0) \cdot H.
\]

Moreover, the number of rows sampled is

\[
T = \Theta(\beta \cdot \epsilon_0^{-2}n \log(n/\delta_0)).
\]
Proof. The proof of this Lemma is follows by designing the sequence of random matrices, then applying the matrix Chernoff bound to get the desired guarantee.

We first define the following vector generated from scaling the rows of $A$,

$$y_j = (A^TA)^{-1/2} \cdot a_j$$

for all $j \in [m]$. And for $i \in [m]$, we define the matrix $Y_j := \frac{1}{p_j} y_j y_j^\top$ generated by $y_j$, and we define $X_j := Y_j - I_n$, where $I_n$ is $n \times n$ identity matrix.

Based on the above vector and matrix definitions, we define the following distributions:

- We define a distribution $y$ for random vector: For $y \in \mathbb{R}^n$, if $y \sim y$, then for each $j \in [m]$, $y = y_j$ with probability $p_j$.

- We define a distribution $Y$ such that, For $Y \in \mathbb{R}^{n \times n}$, if $Y \sim Y$, then for each $j \in [m]$, $Y = Y_j$ with probability $p_j$.

- We define a distribution $X$ such that, For $X \in \mathbb{R}^{n \times n}$, if $X \sim X$, then for each $j \in [m]$, $X = X_j = Y_j - I_n$ with probability $p_j$.

Using $H = A^TA$, we write the $y_j$ as

$$y_j = H^{-1/2} \cdot a_j.$$

We notice that,

$$\sum_{j=1}^{m} y_j y_j^\top = \sum_{j=1}^{m} H^{-1/2} a_j a_j^\top H^{-1/2}$$

$$= H^{-1/2} (\sum_{i=1}^{m} a_i a_i^\top) H^{-1/2}$$

$$= H^{-1/2}(A^TA)H^{-1/2}$$

$$= I_n. \quad (4)$$

where the first step follows from the definition of $y_j$, the second step follows from reorganization, the third step follows from definition of $a_j$, and the last step follows from the definition of $H$.

Besides, we notice the the connection between the norm of $y_j$ and the leverage score of $A$:

$$\|y_j\|^2 = a_j^\top (A^TA)^{-1} a_j$$

$$= \sigma_j(A). \quad (5)$$

Here we denote the index of the row that we sample during $t$-th trial as $j_t$, note that $j_t \in [m]$, for all $t \in [T]$.

Unbiased Estimator. For a matrix $X \sim \mathcal{X}$, here we show that $X$ has the expectation of $0$, we note that

$$\mathbb{E}_{X \sim \mathcal{X}}[X] = \mathbb{E}_{Y \sim \mathcal{Y}}[Y] - I_n$$

$$= (\sum_{j=1}^{m} p_j \cdot \frac{1}{p_j} y_j y_j^\top) - I_n$$

$$= 0.$$
where the first step follows from the definition of $X$, the second step follows from the definition of $Y$ and the definition of expectation, and the last step follows from Eq. (4).

**Upper Bound on** $\|X\|$. To give an upper bound of $\|X\|$, we first provide an upper bound for any $\|X_j\|$, then the upper bound of $\|X\|$ follows immediately. We note that,

$$
\|X\| = \|Y_j - I_n\|
\leq 1 + \|Y_j\|
= 1 + \frac{\|y_jy_j^\top\|}{p_j}
\leq 1 + \frac{n \cdot \|y_j\|^2}{\beta \cdot \sigma_j(A)}
= 1 + \frac{n}{\beta}
$$

where the first step follows from the definition of $X_j$, the second step follows from the definition of $Y_j$ and the definition of expectation, and the last step follows from Eq. (4).

**Bound on** $\|\mathbb{E}[X^\top X]\|$. To upper bound the spectral norm of the matrix, we first provide the upper bound of the expectation of the matrix $X^\top X$. We compute the matrix expectation:

$$
\mathbb{E}_{X_{jt} \sim \mathcal{X}} [X_{jt}^\top X_{jt}]
= I_n + \mathbb{E}_{y_{jt} \sim y} \left[ \frac{y_{jt}y_{jt}^\top}{p_j} \right] - 2 \mathbb{E}_{y_{jt} \sim y} \left[ \frac{y_{jt}y_{jt}^\top}{p_j} \right]
= I_n + \left( \sum_{j=1}^m \frac{\sigma_j(A)}{p_j} y_{jt}y_{jt}^\top \right) - 2I_n
\leq \sum_{j=1}^m \frac{n}{\beta} y_{jt}y_{jt}^\top - I_n
= \left( \frac{n}{\beta} - 1 \right) I_n,
$$

where the first step follows from definition of $X$, the second step follows from Eq. (4), Eq. (5) and the definition of expectation, the third step follows from $p_j \geq \beta \cdot \sigma_j(A)/m$ and the definition of $\ell_2$ norm and the last step follows from Eq. (4) and factorising.

Thus we upper bound the spectral norm of $\mathbb{E}_{y_{jt} \sim y} [X_{jt}^\top X_{jt}]$ as

$$
\| \mathbb{E}_{y_{jt} \sim y} [X_{jt}^\top X_{jt}] \| \leq \frac{n}{\beta} - 1.
$$

**Put things together.** Here we define $W := \sum_{t=1}^T X_{jt}$. We choose the parameter

$$
\gamma = 1 + \frac{n}{\beta}, \quad \sigma^2 = \frac{n}{\beta} - 1
$$

Then, we apply Matrix Chernoff Bound as in Lemma 3.19:

$$
\Pr[\|W\| \geq \epsilon_0]
\leq 2n \cdot \exp \left( - \frac{T\epsilon_0^2}{n/\beta - 1 + (1 + n/\beta)\epsilon_0/3} \right)
$$

27
\[ = 2n \cdot \exp(-T \epsilon_0^2 \cdot \Theta(\beta/n)) \]
\[ \leq \delta_0 \]

where the last step follows from that we choose \( T = \Theta(\beta \cdot \epsilon_0^{-2} n \log(n/\delta_0)) \).

Finally, we notice that

\[
W = \frac{1}{T} \left( \sum_{t=1}^{T} \frac{1}{p_{jt}} y_{jt} y_{jt}^\top - I_n \right)
\]
\[
= H^{-1/2} \left( \frac{1}{T} \sum_{t=1}^{T} \frac{1}{p_{jt}} a_{jt} a_{jt}^\top \right) H^{-1/2} - I_n
\]
\[
= H^{-1/2} \tilde{H} H^{-1/2} - I_n.
\]

where the first step follows from definition of \( W \), the second step follows from \( y_{jt} = H^{-1/2} a_{jt} \), and the last step follows from definition of \( \tilde{H} \).

Since

\[
\| H^{-1/2} \tilde{H} H^{-1/2} - I_n \| \leq \epsilon_0
\]

Thus, we know that

\[
(1 - \epsilon_0) I_n \preceq H^{-1/2} \tilde{H} H^{-1/2} \preceq (1 + \epsilon_0) I_n
\]

which implies that

\[
(1 - \epsilon_0) H \preceq \tilde{H} \preceq (1 + \epsilon_0) H.
\]

Thus, we complete the proof. \qed
6 Discrepancy-Minimization Algorithm

6.1 Correctness

The goal of this is to prove Lemma 6.1.

Lemma 6.1. The algorithm (FastHereditaryMinimize in Algorithm 5) takes a matrix $A \in \mathbb{R}^{m \times n}$ as input and outputs a coloring $x \in \{-1, +1\}^n$ such that

$$\text{disc}(A, x) = O(\text{herdisc}(A) \cdot \log n \cdot \log^{1.5} m).$$

holds with probability $1 - \delta_{\text{final}}$.

Proof. We divide the proof into the following two parts.

Proof of Quality of Answer. By the condition of the algorithm stops, we have that, there is no $i \in [n]$ such that $|x_i| < 1$. And by Lemma 7.4, every time after we call FastPartialColoring (Algorithm 6), herdisc($A, x$) changes at most $O(\text{herdisc}(A) \cdot \log^{1.5} m)$. Note that, the FastPartialColoring is called $O(\log n)$ times, thus we have the guarantee. Thus we complete the proof.

Proof of Succeed Probability. For the FastPartialColoring Algorithm, each time we call it, we have that

- with probability at most $19/20$, we know that it outputs fail.
- with $1/20 - \frac{3}{n} \delta_{\text{final}}$, it outputs a good $x^{\text{new}}$.
- with probability at most $\frac{2}{n} \delta_{\text{final}}$ it outputs a $x^{\text{new}}$ with no guarantee.

As long as we repeat second while loop more than $k = \Theta(\log(n/\delta_{\text{final}}))$ times, then we can promise that with probability $1 - \frac{2}{n} \delta_{\text{final}}$, we obtain a non-final $x^{\text{new}}$.

Then by union bound we have the final succeed probability is

$$1 - \left(\frac{2}{n} \delta_{\text{final}}\right)^k \log n \geq 1 - \delta_{\text{final}}.$$ 

\[ \Box \]

6.2 Running Time

The goal of this section is to prove Lemma 6.2.

Lemma 6.2. For any parameter $a \in [0, \alpha]$, the expected running time of algorithm (FastHereditaryMinimize in Algorithm 5) is

$$\tilde{O}(\text{nnz}(A) + n^\omega + n^{2+a} + n^{1+\omega(1,1,a)-a}).$$

Note that $\omega()$ function is defined as Definition 3.22.

Proof. For the running time of FastHereditaryMinimize, we analyze it as follows.

First we notice that, $|S|$ halves for each iteration of the outer while loop. Thus for each iteration, we can only examine the $x_i$’s for $i \in S$, where $S$ is from the previous iteration. Then, we can maintain $S$ in $O(n)$ time.
Algorithm 5

1: procedure FastHereditaryMinimize($A \in \mathbb{R}^{m \times n}$, $\delta_{\text{final}} \in (0, 0.001)$) \Comment{Lemma 6.1, Lemma 6.2}

2: $x \leftarrow 0_n$

3: while $x \notin \{-1, +1\}^n$ do

4: $S \leftarrow \{i \in [n] \mid |x_i| < 1\}$

5: Let $\overline{x} \leftarrow x_S$ be the coordinates of $x$ indexed by $S$

6: Let $\overline{A} \leftarrow A_{\cdot S} \in \mathbb{R}^{m \times |S|}$ be the columns of $A$ indexed by $S$

7: finished $\leftarrow$ false

8: counter $\leftarrow 0$

9: $k \leftarrow \Theta(\log(n/\delta_{\text{final}}))$

10: while finished = false and counter $< k$ do

11: counter $\leftarrow$ counter + 1

12: $x^\text{new} \leftarrow \text{FastPartialColoring}(\overline{A}, \overline{x}, \delta_{\text{final}}^3/n^3)$ \Comment{Algorithm 6}

13: if $x^\text{new} \neq \text{fail}$ then

14: $x_S \leftarrow x^\text{new}$

15: finished $\leftarrow$ true

16: end if

17: end while

18: if finished = false then

19: return fail

20: end if

21: end while

22: return $x$

23: end procedure

By the same argument, Line 6 takes a total time of $O(\text{nnz}(A))$ to generate the submatrices $\overline{A}$.

Recall we have the $S$ halved each iteration, so $|S| \leq n/2^i$ at $i$’th iteration. Thus we have by Lemma 7.14, Line 12 takes time

$$T_{\text{FPC}} = \tilde{O}(\text{nnz}(A) + n^\omega + n^{2+a} + n^{1+\omega(1,1,a)-a})$$

to call FastPartialColoring.

Let $p$ denote the succeed probability of FastPartialColoring, each time, it takes $T_{\text{FPC}}$ time, so total

$$\text{Expected time} = p \cdot T_{\text{FPC}} + (1-p) \cdot p \cdot 2T_{\text{FPC}} + (1-p)^2 \cdot p \cdot 3T_{\text{FPC}} + \cdots$$

$$= \sum_{i=1}^{\infty} (1-p)^{i-1} p \cdot i \cdot T_{\text{FPC}}$$

Let $f(p) = \sum_{i=1}^{\infty} (1-p)^{i-1} p \cdot i$. Then we have

$$f(p) - (1-p)f(p) = \sum_{i=1}^{\infty} (1-p)^{i-1} pi - \sum_{i=1}^{\infty} (1-p)^{i} pi$$

$$= \sum_{i=0}^{\infty} (1-p)^{i} pi + (1) - \sum_{i=1}^{\infty} (1-p)^{i} pi$$
Thus,\[
\sum_{i=1}^{\infty} (1 - p)^i p \leq p + p \cdot 1/p = 1 + p
\]
Thus,\[
f(p) \leq \frac{1 + p}{p}
\]
Thus the expectation time is \(O(T_{FPC}/p)\).
Since there are at most \(\log n\) iterations, so the expected time is \(O(p^{-1}T_{FPC} \cdot \log n)\).

\[\square\]

### 7 Partial Coloring

#### 7.1 Fast Algorithm for Approximating Norms

Here in this section, we present subroutine used as a tool to get fast approximation to the row norm of a matrix, based on the JL lemma. The goal of this section is to prove Lemma 7.1.

**Lemma 7.1.** Let \(V \in \mathbb{R}^{\ell \times n}\) where \(\ell \leq n\) and \(A \in \mathbb{R}^{m \times n}\). For any accuracy parameter \(\epsilon_1 \in (0,0.1)\), failure probability \(\delta_1 \in (0,0.1)\), let \(r = O(\epsilon_1^{-2} \log(m/\delta_1))\), there is an algorithm (procedure `FastApproxMaxNorm` in Algorithm 7) that runs in \(O((nnz(A) + n^2) \cdot r)\) time and outputs a number \(\eta \in \mathbb{R}\)

\[
\eta \geq (1 - \epsilon_1) \cdot \max_{j \in [m]} \|a_j^\top (I - V^\top V)\|_2
\]

holds with probability \(1 - \delta_1\).

**Proof.** Let \(R \in \mathbb{R}^{n \times r}\) denote a random JL matrix. The running time is from computing
- Computing \(VR\) takes \(n^2 r\) time.
- Computing \(V^\top (VR)\) takes \(n^2 r\) time.
- Computing \(AR\) takes \(nnz(A)r\) time.
- Computing \(A \cdot (V^\top \cdot (VR))\) takes \(nnz(A)r\) time.
- Computing \(\|(AR - AA \cdot (V^\top \cdot (VR)))_{j,*}\|_2\) for all \(j \in [m]\). This takes \(O(mr)\) time.
- Taking the max over all the \(j \in [m]\). This step takes \(O(m)\) time.

The correctness follows from JL Lemma (Lemma 3.16) directly. \[\square\]
Algorithm 6 Input sparsity time partial coloring algorithm

1: procedure FastPartialColoring($A \in \mathbb{R}^{m \times n}, x \in (-1, +1)^n, a \in [0, 1], \delta_{\text{final}} \in (0, 0.01), \delta \in (0, 0.01)$) \hfill \triangleright Lemma 7.4, Lemma 7.14
2: \hspace{1em} $u_1 \leftarrow 0_n$
3: \hspace{1em} $\delta_1 \leftarrow \delta_{\text{final}}^2/n^2$
4: \hspace{1em} $\delta_2 \leftarrow \delta_{\text{final}}^2/n^2$
5: \hspace{1em} $V \leftarrow \text{FastProjectToSmallRows}(A, \delta_1)$ \hfill \triangleright Algorithm 4
6: \hspace{1em} $\eta \leftarrow \text{FastApproxMaxNorm}(A, V, \delta_2)$ \hfill \triangleright Algorithm 7
7: \hspace{1em} $\eta \leq O(\text{herdisc}(A) \log (m/n))$
8: \hspace{1em} $\epsilon \leftarrow \Theta((\log(mn) + n)^{-1/2})$
9: \hspace{1em} $N \leftarrow \Theta(\epsilon^{-2} + n)$
10: \hspace{1em} $\beta \leftarrow \Theta(\epsilon \eta \sqrt{N \log (m/\delta)})$ \hfill \triangleright $\beta \leq O(\text{herdisc}(A) \log^{1.5}(m))$
11: \hspace{1em} Generate a Gaussian matrix $G \in \mathbb{R}^{n \times N}$, where each column is sampled from $\mathcal{N}(0, I_n)$
12: \hspace{1em} Maintain $\text{ds}$ \hfill \triangleright Algorithm 9
13: \hspace{1em} $\text{ds.}$.Init$(V, G, n, n_\alpha)$ \hfill \triangleright Algorithm 9
14: \hspace{1em} for $t = 1 \rightarrow N$ do
15: \hspace{2em} $g_{V_t} \leftarrow \text{ds.QUERY}()$ \hfill \triangleright Algorithm 9, $g_{V_t} = (I - V_t^T V_t) G_{s,t}$
16: \hspace{2em} $\triangleright$ Let $\mu > 0$ be maximal such that $\max\{\|x + u_t + \mu \cdot g_{V_t}\|_{\infty}, \|x + u_t - \mu \cdot g_{V_t}\|_{\infty}\} = 1$
17: \hspace{2em} $\mu \leftarrow \text{FindBoundary}(x + u_t, g_{V_t})$ \hfill \triangleright Algorithm 8
18: \hspace{2em} $\hat{g}_t \leftarrow \min\{\epsilon, \mu\} \cdot g_{V_t}$ \hfill \triangleright $\hat{g}_t \in \mathbb{R}^n$
19: \hspace{2em} for $i = 1 \rightarrow n$ do
20: \hspace{3em} if $|x_i + u_{t,i} + \hat{g}_{t,i}| = 1$ and $|x_i + u_{t,i}| < 1$ then
21: \hspace{4em} We will add $e_i$ into $V$ in an implicit way via data structure $\text{ds}$ \hfill \triangleright The explicit way can be found here Algorithm 1
22: \hspace{4em} $\text{ds.}$.UPDATE$(e_i)$ \hfill \triangleright Algorithm 10
23: \hspace{3em} end if
24: \hspace{2em} end for
25: \hspace{2em} $u_{t+1} \leftarrow u_t + \hat{g}_t$
26: \hspace{2em} if $|\{i \in [n] : |x_i + u_{t+1,i}| = 1\}| \geq n/2$ then
27: \hspace{3em} if $|A_{u_{t+1}}|_{\infty} > \beta$ then
28: \hspace{4em} return fail \hfill \triangleright Case 2 of Lemma 7.4
29: \hspace{3em} else
30: \hspace{4em} return $x_{\text{new}} \leftarrow x + u_{t+1}$ \hfill \triangleright Case 1 of Lemma 7.4
31: \hspace{3em} end if
32: \hspace{3em} end if
33: \hspace{2em} end if
34: \hspace{2em} return fail \hfill \triangleright Case 3 of Lemma 7.4
35: end procedure

7.2 Look for Step Size

In order to find the proper step size in the Partial Coloring algorithm, we propose the following subroutine.

Lemma 7.2. There is an algorithm (Algorithm 8) that takes two vectors $a, b \in \mathbb{R}^n$, and outputs a positive number $\mu \in \mathbb{R}_+$, such that $\mu$ is the number with the largest absolute value satisfying that
\[
\max\{\|\mu \cdot a + b\|_{\infty}, \|\mu \cdot a - b\|_{\infty}\} = 1.
\]
Algorithm 7

1: procedure FastApproxNorm$(A \in \mathbb{R}^{m \times n}, V \in \mathbb{R}^{\ell \times n}, \delta_2 \in (0, 0.01))$ \quad \triangledown Lemma 7.1
2: \quad \text{Let } R \in \mathbb{R}^{n \times r} \text{ denote a random JL matrix with } r = \Theta(\epsilon_1^{-2} \log(m/\delta_2)) \quad \triangledown \text{Either Definition 3.10 or Definition 3.9.}
3: \quad \text{Compute } (I - V^\top V)R \quad \triangledown \text{This takes } O(n^2r) \text{ time}
4: \quad \eta \leftarrow \max_{j \in [\ell]} \|a_j^\top (I - V^\top V)R\|_2 \quad \triangledown \text{This takes } O(\text{nnz}(A)r) \text{ time}
5: \quad \text{return } \eta
6: \quad \text{end procedure}

And runs in time $O(n)$.

Proof. Running Time. The running time of Algorithm 8 can be divided into the following lines,

- Line 2 takes time $O(n)$ to compute the $x_i, y_i$’s for $i \in [n]$.
- Line 3 takes time $O(n)$ to compute the $x_i, y_i$’s for $i \in \{n+1, \ldots, 2n\}$.
- Line 4 takes time $O(n)$ to linear scan the $x_i$’s for $i \in [2n]$.
- Line 5 takes time $O(n)$ to linear scan the $y_i$’s for $i \in [2n]$.

The rest part of the algorithm all runs in an constant time. Thus we have the total running time is $O(n)$.

Correctness. We first prove that, if the algorithm outputs a number $\mu$, then for every $i \in [n]$, it holds that

\[-1 \leq \mu \cdot a_i - b_i \leq 1,\]
\[-1 \leq \mu \cdot a_i + b_i \leq 1.\]

Here we call the feasible region of the above question to be $U$. By the construction of our algorithm we have that,

\[ [x_i, y_i] = \{ \mu \in \mathbb{R} \mid -1 \leq \mu \cdot a_i - b_i \leq 1 \}, \quad \forall i \in [n] \]
\[ \text{and } [x_i, y_i] = \{ \mu \in \mathbb{R} \mid -1 \leq \mu \cdot a_{i-n} - b_{i-n} \leq 1 \}, \quad \forall i \in \{n + 1, \ldots, 2n\} \]

Thus we have that

\[ U = \bigcap_{i \in [2n]} [x_i, y_i]. \]

We note that, if the algorithm outputs a number $\mu$, then it must holds that

\[ u \leq v. \]

Recall that we define $u$ and $v$ to be

\[ u := \max\{x_i\}_{i \in [2n]} \]
\[ v := \min\{y_i\}_{i \in [2n]}. \]
thus we have that
\[ [u, v] = \mathcal{U}. \]
Recall the output number \( \mu \) equals to either \( u \) or \( v \), it must stands that
\[ \mu \in \mathcal{U}. \]
Thus the constraints hold for \( \mu \).
Now we prove that
\[ \max\{\|\mu \cdot a + b\|_\infty, \|\mu \cdot a - b\|_\infty\} = 1. \]
Without loss of generality, we assume \(|u| > |v|\), that is, \( \mu = u \). The case that \(|u| \leq |v|\) is just the same. We first define
\[ j := \{i \in [2n] \mid u = x_i\}. \]
Here we first assume \( j \in [n] \), that is, we have that
\[ |\mu \cdot a_j - b_j| = 1. \]
(For the case that \( j \in \{n + 1, \ldots, 2n\} \), we have \(|\mu \cdot a_j + b_j| = 1\), which is the same to analyse.) Note that, \( \mu \in \mathcal{U} \), so we have that
\[ |\mu \cdot a_i - b_i| \leq 1, \forall i \in [n] \]
and \[ |\mu \cdot a_i + b_i| \leq 1, \forall i \in [n]. \]
Thus we have that
\[ \|\mu \cdot a - b\|_\infty = 1 \]
and \[ \|\mu \cdot a + b\|_\infty \leq 1, \]
which is
\[ \max\{\|\mu \cdot a + b\|_\infty, \|\mu \cdot a - b\|_\infty\} = 1. \]
Now for the case that \( j \in \{n + 1, \ldots, 2n\} \), by a similar analysis we have that
\[ \|\mu \cdot a + b\|_\infty = 1 \]
and \[ \|\mu \cdot a - b\|_\infty \leq 1, \]
which also implies the result that
\[ \max\{\|\mu \cdot a + b\|_\infty, \|\mu \cdot a - b\|_\infty\} = 1. \]
By a same analysis we can have the above result if \( \mu = v \).
Now we prove that, \( \mu = u \) and \( \mu = v \) are the only two case that
\[ \max\{\|\mu \cdot a + b\|_\infty, \|\mu \cdot a - b\|_\infty\} = 1. \]
Suppose for the contradiction that there exists an $t \in \mathbb{R}$ such that $t \neq u$ and $t \neq v$, and it holds that
\[ \max\{\|t \cdot a + b\|_\infty, \|t \cdot a - b\|_\infty\} = 1. \]
(6)

If $t \notin \mathcal{U}$, then there must exists and $i \in [n]$ such that
\[ |t \cdot a_i + b_i| > 1 \]
or\[ |t \cdot a_i - b_i| > 1, \]
which is a violation of the hypothesis (Eq.(6)). Then for the case that $t \in \mathcal{U}$, since $t \neq u$ and $t \neq v$, it must holds that $t \in (u, v)$. Note we define
\[ u := \max\{x_i\}_{i \in [2n]} \]
\[ v := \min\{y_i\}_{i \in [2n]}. \]

It must holds that
\[ t > x_i, \forall i \in [2n], \text{and } t < y_i, \forall i \in [2n]. \]

Then we have that
\[ |t \cdot a_i + b_i| < 1 \]
or\[ |t \cdot a_i - b_i| < 1, \]
which implies that
\[ \max\{\|t \cdot a + b\|_\infty, \|t \cdot a - b\|_\infty\} < 1, \]
which is a violation of the hypothesis (Eq.(6)). Thus we conclude there is no such $t$.

If we define $\mu$ to be one of $u$ or $v$, depending on who has larger absolute value, then $\mu$ is the number with the largest absolute value satisfying that
\[ \max\{\|\mu \cdot a + b\|_\infty, \|\mu \cdot a - b\|_\infty\} = 1. \]

Thus we complete the proof.

**Corollary 7.3.** For the case that $\|b\|_\infty \leq 1$, Algorithm 8 always return a number $\mu$.

**Proof.** If $\|b\|_\infty \leq 1$, then we have that
\[ |b_i| \leq 1, \forall i \in [n]. \]

Without loss of generality we assume $b_i \geq 0$, then $b_i \in [0, 1]$, which implies that
\[ \frac{b_i - 1}{a_i} \leq 0 \]
and\[ \frac{b_i + 1}{a_i} \geq 0. \]

Thus we have that
\[ 0 \in [x_i, y_i], \forall i \in [n]. \]
By a similar analysis we have

\[ 0 \in [x_i, y_i], \forall i \in \{n + 1, \ldots, 2n\}. \]

Thus we have

\[ 0 \in U = \bigcap_{i \in [2n]} [x_i, y_i], \]

which means

\[ U \neq \emptyset \]

So it must holds that

\[ u \leq v. \]

Thus we complete the proof.

\[ \square \]

Algorithm 8

1: \textbf{procedure} FindBoundary\((a \in \mathbb{R}^n, b \in \mathbb{R}^n)\) \hspace{1cm} \triangleright \text{Lemma 7.2}
2: \hspace{1cm} Compute \( n \) intervals \([x_i, y_i]\) such that \( x_i, y_i \) are the two boundaries of \(-1 \leq a_i \mu - b_i \leq 1\),
   e.g., \( x_i, y_i \in \{\frac{b_i - 1}{a_i}, \frac{b_i + 1}{a_i}\} \) and \( x_i \leq y_i \), for all \( i \in [n] \)
3: \hspace{1cm} Compute \( n \) intervals \([x_{n+i}, y_{n+i}]\) such that \( x_i, y_i \) are the two boundaries of \(-1 \leq a_i \mu + b_i \leq 1\),
   e.g., \( x_i, y_i \in \{\frac{-b_i - 1}{a_i}, \frac{-b_i + 1}{a_i}\} \) and \( x_i \leq y_i \), for all \( i \in [n] \)
4: \hspace{1cm} Linear scan the \( x_1, \cdots, x_{2n} \), find the index \( u \) such that \( u \leftarrow \max\{x_i\}_{i \in [2n]} \)
5: \hspace{1cm} Linear scan the \( x_1, \cdots, y_{2n} \), find the index \( v \) such that \( v \leftarrow \min\{y_i\}_{i \in [2n]} \)
6: \hspace{1cm} \textbf{if} \( u \leq v \) \textbf{then}
7: \hspace{2cm} \textbf{if} \( |u| \leq |v| \) \textbf{then}
8: \hspace{4cm} \textbf{return} \( v \)
9: \hspace{2cm} \textbf{else}
10: \hspace{4cm} \textbf{return} \( u \)
11: \hspace{2cm} \textbf{end if}
12: \hspace{2cm} \textbf{else}
13: \hspace{4cm} \textbf{return} \text{fail}
14: \hspace{2cm} \textbf{end if}
15: \hspace{1cm} \textbf{end procedure}

Here we give some figures to illustrate the idea of this algorithm.

7.3 Correctness

The goal of this section is to prove Lemma 7.4, which gives the correctness guarantee of the Algorithm 6.

**Lemma 7.4** (Output Gaurantees of Algorithm 6). Let \( A \in \mathbb{R}^{m \times n} \) be an input matrix and \( x \in (0, 1)^n \) be an input partial coloring. Let \( \delta \in (0, 0.01) \) denote a parameter. Conditioning on the event that \( V \) is good and \( \eta \) is good. We have:

- **Case 1.** The Algorithm 6 outputs a vector \( x^{\text{new}} \in \mathbb{R}^n \) such that with probability at least \( 1/10 - \delta \) the following holds,
\( s = \mu \cdot a_1 - b_1 \)

\( s = \mu \cdot a_1 + b_1 \)

\( s = \mu \cdot a_2 - b_2 \)

\( s = \mu \cdot a_2 + b_2 \)

(a) The lines of the constraints in our example

(b) The feasible intervals

Figure 5: Here we give an example of our Algorithm 8. The algorithm is looking for the intervals to make every constraint \(|\mu \cdot a_i + b_i| \leq 1\) or \(|\mu \cdot a_i - b_i| \leq 1\) hold, so we have \(2n\) lines in total \((n\) is the dimension). Here in the example, we assume \(n = 2\), so we have \(2n = 4\) lines in total. Thus we have 4 intervals, and taking the intersection over them, we get the interval \([u, v]\). Figure (a) shows the lines of the constraints we have, the green area shows the feasible region. Figure (b) shows our idea of taking intersection over the \(2n = 4\) constraints. (Here we set \(a = (1, \frac{15}{7})\) and \(b = (0.7, \frac{3}{5})\).)

- \(\|x^{\text{new}}\|_{\infty} = 1;\)
- \(|\{i \in [n] \mid |x_i^{\text{new}}| = 1\}| \geq n/2;\)
- Let \(a_j^T\) denote the \(j\)-th row of matrix \(A \in \mathbb{R}^{m \times n}\), for each \(j \in [m]\). For all \(j \in [m]\), the following holds

\[|\langle a_j, x^{\text{new}} \rangle - \langle a_j, x \rangle| \leq O(\text{herdisc}(A) \cdot \log(m) \cdot \log^{1/2}(m/\delta)).\]

- **Case 2.** With probability at most \(\delta\), the Algorithm 6 will output fail at Line 28.
- **Case 3.** With probability at most \(9/10\), the Algorithm 6 will output fail at Line 34.

**Proof.** We start with choosing the parameters:

\[N := O(n + \max\{\log(mn), n\}) = O(n + \log(m)),\]

and

\[\epsilon := O(\min\{\log^{-1/2}(mn), n^{-1/2}\}).\]
Proof of Case 1. We first prove that for any iteration \( t \in [N] \), no entry of \( x + v_t + \hat{g}_t \) has absolute value larger than 1. Note that in \( t \)-th iteration, if we find \( i \)-th entry of \( x + v_t + \hat{g}_t \) reaches absolute value of 1, we add \( e_i \) to the matrix \( V_t \) (Line 22). Then in the following iterations, after we get the query \( g \) (We omit the \( t \) here for \( g \) and \( V \)), we have by Lemma 8.4 that, \( g \) is the distance of original Gaussian vector to the row space of \( V \), thus the \( i \)-th entry of \( g \) will be 0. Hence in the following iterations, the entry that we add to \( i \)-th entry of \( v \) will stay 0. Thus we complete the claim.

By the algorithm design, the number entries reach the absolute value 1 keeps increasing, and by the condition check of Line 20, we have that

\[
|\{i \in [n] \mid |x^\text{new}_i| = 1\}| \geq n/2.
\]

Then by Lemma 7.9 we have that

\[
\forall j \in [m], \quad |\langle a_j, g \rangle - \langle a_j, x \rangle| \leq \beta
\]

with probability at least \( 1/10 - \delta \), where we notice that

\[
\beta = O(\eta \min\{\log^{-1/2}(mn), n^{-1/2}\} \sqrt{n} + \log(m) \sqrt{\log(m/\delta)}) = O(\eta \log^{1/2}(m)).
\]

Thus we have the guarantee.

Proof of Case 2. From Lemma 7.9 we know that, there is a failure probability at most \( \delta \) that,

\[
\exists j \in [m], \quad |\langle a_j, g \rangle - \langle a_j, x \rangle| > \beta.
\]

Thus we have have the claim proved.

Proof of Case 3. By Lemma 7.10, the Algorithm 6 returns fail with probability at most 4/5. Thus we complete the proof.

Now if we consider the probability of the returning \( V \) and \( \eta \) are not good, we have the following corollary.

Corollary 7.5. Let \( A \in \mathbb{R}^{m \times n} \) be an input matrix and \( x \in (0, 1)^n \) be an input partial coloring. Let \( \delta_{\text{final}} \in (0, 0.01) \) denote a parameter. We assume \( \delta = 1/100 \) here in the algorithm. Then We have:

- **Case 1.** The Algorithm 6 outputs a vector \( x^\text{new} \in \mathbb{R}^n \) such that with probability at least \( 1/10 - \delta - \delta_{\text{final}} \) the following holds,

  \[
  - \|x^\text{new}\|_\infty = 1;
  - |\{i \in [n] \mid |x^\text{new}_i| = 1\}| \geq n/2;
  - \text{Let } a_j^\top \text{ denote the } j \text{-th row of matrix } A \in \mathbb{R}^{m \times n}, \text{ for each } j \in [m]. \text{ For all } j \in [m], \text{ the following holds}
  \]

  \[
  |\langle a_j, x^\text{new} \rangle - \langle a_j, x \rangle| \leq O(\text{herdisc}(A) \cdot \log(m) \cdot \log^{1/2}(m)).
  \]

- **Case 2.** With probability at most \( \delta + \delta_{\text{final}} \), the Algorithm 6 will output fail at Line 28.

- **Case 3.** With probability at most \( 9/10 + \delta_{\text{final}} \), the Algorithm 6 will output fail at Line 34.

- **Case 4.** With probability at most \( \delta_{\text{final}} \), the Algorithm 6 will output \( x^\text{new} \) with no guarantee.
Proof. Recall Lemma 7.4 is conditioning on $V$ and $\eta$ are good, here we mention the failure probability of these two operations.

By the construction of Algorithm 6, when we call

\begin{align*}
\text{FastProjectoSmallRows}(A, \delta_1)
\end{align*}

at Line 5, we set $\delta_1 = \delta_{\text{final}}/2$.

Also when we call FastApproxMaxNorm($A, \delta, V, \eta$), we set $\delta_2 = \delta_{\text{final}}/2$.

Thus, we know

\begin{align*}
\Pr[V, \eta \text{ good}] & \geq 1 - \delta_{\text{final}}. \tag{7}
\end{align*}

and we use “$V, \eta$ not good” to denote the complement event of “$V, \eta$ good”, then we have

\begin{align*}
\Pr[V, \eta \text{ not good}] & \leq \delta_{\text{final}}. \tag{8}
\end{align*}

**Proof of Case 1.** By Lemma 7.4, we have

\begin{align*}
\Pr[\text{case 1 happens} \mid V, \eta \text{ good}] & \geq 1/10 - \delta. \tag{9}
\end{align*}

By basic probability rule,

\begin{align*}
\Pr[\text{case 1 happens}] &= \Pr[\text{case 1 happens} \mid V, \eta \text{ good}] \cdot \Pr[V, \eta \text{ good}] \\
&+ \Pr[\text{case 1 happens} \mid V, \eta \text{ not good}] \cdot \Pr[V, \eta \text{ not good}] \\
&\geq \Pr[\text{case 1 happens} \mid V, \eta \text{ good}] \cdot \Pr[V, \eta \text{ good}] \\
&\geq (1/10 - \delta) \cdot \Pr[V, \eta \text{ good}] \\
&\geq (1/10 - \delta) \cdot (1 - \delta_{\text{final}}) \\
&\geq 1/10 - \delta - \delta_{\text{final}}.
\end{align*}

where the second step follows from $\Pr[] \geq 0$, the third step follows from Eq. (9), the forth step follows from Eq. (7).

**Proof of Case 2.** Then by Lemma 7.4, we have the case happens with probability at most $\delta$ when conditioning on $V, \eta$ are good, i.e.,

\begin{align*}
\Pr[\text{case 2 happens} \mid V, \eta \text{ good}] & \leq \delta. \tag{10}
\end{align*}

By basic probability rule,

\begin{align*}
\Pr[\text{case 2 happens}] &= \Pr[\text{case 2 happens} \mid V, \eta \text{ good}] \cdot \Pr[V, \eta \text{ good}] \\
&+ \Pr[\text{case 2 happens} \mid V, \eta \text{ not good}] \cdot \Pr[V, \eta \text{ not good}] \\
&\leq \Pr[\text{case 2 happens} \mid V, \eta \text{ good}] + \Pr[V, \eta \text{ not good}] \\
&\leq \delta + \Pr[V, \eta \text{ not good}] \\
&\leq \delta + \delta_{\text{final}}
\end{align*}

where the second step follows from $\Pr[] \leq 1$, the third step follows from Eq. (10), and the forth step follows from Eq. (8).
Proof of Case 3. By Lemma 7.4, we have the case happens with probability at most $9/10$ conditioning on $V$ and $\eta$ is good, i.e.,

$$\Pr[\text{case 3 happens} \mid V, \eta \text{ good}] \leq 9/10. \tag{11}$$

By basic probability rule,

$$\Pr[\text{case 3 happens}] = \Pr[\text{case 3 happens} \mid V, \eta \text{ good}] \cdot \Pr[V, \eta \text{ good}] + \Pr[\text{case 3 happens} \mid V, \eta \text{ not good}] \cdot \Pr[V, \eta \text{ not good}] \leq \Pr[\text{case 3 happens} \mid V, \eta \text{ good}] + \Pr[V, \eta \text{ not good}] \leq 9/10 + \delta_{\text{final}}.$$

where the second step follows from $\Pr[] \leq 1$, the third step follows from Eq. (11), and the last step follows from Eq. (8).

Proof of Case 4. This happens when at least one of the returning $V$ and $\eta$ is not good, and the algorithm returns $x^{\text{new}}$ at Line 30. By union bound we have this happens with probability at most

$$\Pr[\text{case 4 happens}] \leq \delta_{\text{final}}.$$

7.4 Iterative Notations

Here in this section, we introduce some notations to be used in later proofs. We first define the matrix $V_t$ we maintained as follows.

Definition 7.6 (Matrix $V$ of each iteration). For all $t \in [T]$, we define $V_t$ to be the matrix forming an orthonormal basis which implicitly maintained in the MAINTAIN data-structure at the start of $t$-th iteration. For $t = 1$, we define $V_1$ to be generated from FASTPROJECTTOSMALLROWS at Line 5 in Algorithm 6.

We then define the Gaussian random vectors related to the above $V_t$.

Definition 7.7 (Random Gaussian vectors). For all $t \in [T]$, in the $t$-th iteration, We define $g_t \in \mathbb{R}^n$ to be a random Gaussian sampled from $\mathcal{N}(0, 1)$. We define vector $g_{V_t}$ to be generated by query the MAINTAIN data-structure at Line 15 in Algorithm 6. By Lemma 8.4, we have

$$g_{V_t} = (I - V_t^T V_t) \cdot g_t.$$

And we define $\hat{g}_t$ to be the vector rescaled from $g_{V_t}$ at Line 18 in Algorithm 6, that is,

$$\hat{g}_t := \min\{\epsilon, \mu\} \cdot g_{V_t}.$$

Through the whole Algorithm 6, we iteratively maintain a vector $u \in \mathbb{R}^n$ to accumulate the random Gaussian vectors of each iteration. We formally define it here as follows,

Definition 7.8 (Accumulated maintained vector). For all $t \in [T]$, we define the accumulated maintained vector $u_t \in \mathbb{R}^n$ to be as follows

$$u_{t+1} := u_t + \hat{g}_t.$$

For the case that $t = 1$, we define $u_1 = 0_n$. 
7.5 Upper bound for the inner products

Here in this section, the goal is to prove Lemma 7.9, which gives the upper bound of the inner products of the output vector with the rows of input matrix.

**Lemma 7.9** (A variation of Lemma 4 in [Lar23]). Let \( \beta = \Theta(\epsilon \eta \sqrt{N \log(m/\delta)}) \). For any input matrix \( A \in \mathbb{R}^{m \times n} \) and any vector \( x \in (-1,1)^n \), let \( a_i^T \) denote the \( i \)-th row of \( A \). Let \( u_{N+1} \in \mathbb{R}^n \) denote the vector \( v \in \mathbb{R}^n \) in the last iteration of Algorithm 6. Then we have that,

\[
\Pr[\forall i \in [m], |\langle a_i, u_{N+1} \rangle| < \beta] \geq 1 - \delta.
\]

**Proof.** To prove the lemma, we first fix \( i \in [m] \) here. Then for \( t \)-th iteration of the outer for-loop (Line 14 in Algorithm 6).

We let \( g_{V_t} \) be defined as Definition 7.7. For simplicity, sometimes we use \( g_t \) to denote \( g_{V_t} \) if it is clear from text.

For the \( t \)'s that Algorithm 6 returns fail before the \( t \)-th iteration, we define \( g_t = 0 \in \mathbb{R}^n \). We also define a cumulative vector \( u_t := \sum_{i=1}^t g_i \). Then we have that

\[
\langle a_i, u_t \rangle = \sum_{j=1}^{t} \langle a_i, g_j \rangle.
\]

Now we condition on the previous \( t - 1 \) iterations and look into the \( t \)-th iteration, we generate a vector \( g_t \in \mathbb{R}^d \) such that every entry is drawn i.i.d. from \( \mathcal{N}(0,1) \). Then we have for any \( \xi > 0 \), we have

\[
\Pr[|\langle a_i, (I - V_t^T V_t) g_t \rangle| > \xi] \geq \Pr[|\min\{\epsilon, \mu\} \cdot |\langle a_i, (I - V_t^T V_t) g_t \rangle| > \xi]
\]

\[
\geq \Pr[|\langle a_i, g_t \rangle| > \xi],
\]

where the first step follows from \( \min\{\epsilon, \mu\} \leq 1 \), the second step follows from Lemma 8.4. We also have the following

\[
\epsilon \cdot |\langle a_i, (I - V_t^T V_t) g_t \rangle| \sim \mathcal{N}(0, \epsilon^2 \|a_i(I - V_t^T V_t)\|^2_2)
\]

by the linearity of the distribution.

We can bound the variance in the following sense,

\[
\epsilon^2 \|a_i(I - V_t^T V_t)\|^2_2 \leq \epsilon^2 \eta^2
\]

by the definition of \( \eta \).

Let \( \beta > 0 \) denote some parameter. Thus using Claim 3.5, we have the following

\[
\Pr[|\langle a_i, g_t \rangle| > \beta] \leq \Pr[|\epsilon \cdot |\langle a_i, (I - V_t^T V_t) g_t \rangle| > \beta]
\]

\[
\leq 2 \exp(-\epsilon^2 \eta^2 \beta^2/2)
\]

Finally, since the \( g_t \) is independent from \( u_i \)'s for \( i \in [t-1] \), we have that

\[
\mathbb{E}[\langle a_i, g_t \mid u_{t-1}, \ldots, u_1 \rangle] = 0.
\]

This follows by the fact that we define \( \mu \) in Line 17 to be

\[
\mu := \arg \max_{\mu \geq 0} \{\|x + u_t + \mu g_t\|_\infty, \|x + u_t - \mu g_t\|_\infty\} = 1.
\]
Thus we have that, the sequence \( \langle a_i, g_1 \rangle, \ldots, \langle a_i, g_t \rangle \) becomes a martingale difference sequence (Definition 3.7) with respect to \( u_1, \ldots, u_t \). Then by Theorem 3.8 we have that with probability at least \( 1 - \delta \), the following holds

\[
|\langle a_i, u_{N+1} \rangle| = |\sum_{t=1}^{N} \langle a_i, g_t \rangle| \\
\leq 2\epsilon \eta \sqrt{112N \lg(1/\delta)} \\
\leq 100\epsilon \eta \sqrt{N \lg(1/\delta)},
\]

where we set the parameters in Theorem 3.8 to be

\( b = 2 \) and \( c \leq (\epsilon \eta)^{-2}/2 \).

Then we set \( \delta = \delta/m \) and let \( \beta = 100\epsilon \eta \sqrt{N \log(m/\delta)} \), we have

\[
\Pr[|\langle a_i, u_{N+1} \rangle| \leq \beta] \geq 1 - \delta/m.
\]

The above implies that

\[
\Pr[|\langle a_i, u_{N+1} \rangle| > \beta] \leq \delta/m
\]

Applying a union bound over all the \( i \in [m] \), then we complete the proof.

\[ \square \]

### 7.6 Upper Bound the Failure Probability

Here in this section, the goal is to prove Lemma 7.10, which gives the upper bound of the probability that the algorithm returns fail.

**Lemma 7.10** (A variation of Lemma 5 in [Lar23]). *For any input matrix \( A \in \mathbb{R}^{m \times n} \) and any vector \( x \in [0, 1]^n \), the probability that Algorithm 6 returns fail at Line 34 is at most \( 4/5 \).

**Proof.** The basic idea of the proof is that, after many iterations (\( N \) in Algorithm 6), we have the expectation of \( \|x + u\|_2^2 \) large enough. Hence lots of entries of \( x + u \) will reach the absolute value of 1. We denote the number of \( e_i \)'s added to \( V_t \) in Line 22 through the running of Algorithm 6 to be \( R \).

Recall we define \( g_t \) to be generated in Line 15 of Algorithm 6 (Definition 7.7). We notice that, we add \( \epsilon \cdot g_t \) or \( \mu \cdot g_t \) to \( u_t \) in Line 25. Here in this paragraph, we first assume that, we add \( \mu \cdot g \), i.e., \( \mu < \epsilon \). Recall we generate \( \mu \) in Line 17 of Algorithm 6 to be

\[
\mu := \arg \max_{\mu > 0} \{ \max\{\|x + u_t + \mu \cdot g_t\|_\infty, \|x + u_t - \mu \cdot g_t\|_\infty\} = 1 \}.
\]

Thus we have that in Line 20 of Algorithm 6, there must be at least one entry \( i \in [n] \) satisfying that

\[
|x_i + u_{t,i} + g_{t,i}| = 1 \text{ and } |x_i + u_{t,i}| < 1
\]

or

\[
\|x_i + u_{t,i} + g_{t,i}\| = 1 \text{ and } |x_i + u_{t,i}| < 1.
\]
We define $\rho_t$ to denote the probability such that the algorithm never return fail before reaching $t$-th iteration and $\mu < \epsilon$ in this iteration. For any $i \in [n]$ and $t \in [N]$, we define the following event by $E_{i,t}$: the index $i$ satisfies

$$|x_i + u_{t,i} + \mu g_{t,i}| = 1 \text{ and } |x_i + u_{t,i}| < 1$$

in Line 20 of the $t$-th iteration.

By the symmetry of the vector $g_t$ (Since $g_t$ is Gaussian, $g_t$ and $-g_t$ are equally likely conditioned on $V_t$), we have that

$$\{i \in [n] \mid E_{i,t} \text{ holds}\} \geq \rho_t/2.$$

We denote the number of $e_i$’s added to $V_t$ in Line 22 through the running of Algorithm 6 to be $R$. And we denote the number of rows added to $V_t$ by $r_t$. By the analysis above, we have that

$$\mathbb{E}[R] = \mathbb{E}\left[\sum_{t=1}^{N} r_t\right] \geq \sum_{t=1}^{N} \rho_t/2.$$

By the construction of our algorithm, no more than $n$ $e_i$’s will be added to $V$ through the algorithm running, thus we have

$$\sum_{t=1}^{N} \rho_t \leq 2 \mathbb{E}[R] \leq 2n.$$

Now for each iteration $t \in [N]$, we define $g_t$ to be the vector added to $u_t$ at Line 25 of Algorithm 6. Here we note that after $N$ iterations,

$$u_{N+1} = \sum_{t=1}^{N} g_t.$$

And this is the final vector output by the algorithm (if it doesn’t return fail).

We have that:

$$\mathbb{E}\left[\|x + u_{N+1}\|^2_2\right] = \mathbb{E}\left[\|x + \sum_{t=1}^{N} g_t\|^2_2\right]$$

$$= \|x\|^2_2 + \sum_{i=1}^{N} \mathbb{E}[\|g_t\|^2_2]. \tag{12}$$

where the first step follows from the definition of $v$, and the second step follows from $\mathbb{E}[g_t|g_1, \ldots, g_{t-1}] = 0$.

Recall we define the $V_t$ to be the matrix $V$ at the start of $t$-th iteration\(^7\) (Definition 7.6). We denote the original Gaussian before projection by $g_t$ (Definition 7.7).

By Claim 7.11, we have

$$\mathbb{E}[\|g_t\|^2_2] \geq (1 - \delta) \cdot \epsilon^2 (n - n/4 - \mathbb{E}[R]) - \epsilon^2 \sqrt{2\rho_t n}.$$

By Claim 7.12 we have the lower bound of the expectation of $R$,

$$\mathbb{E}[R] \geq 0.63n.$$

\(^7\)Note that, in the design of our algorithm, we maintain this matrix $V$ implicitly in the MAINTAIN data structure.
Using Claim 7.13, we obtain

\[ \Pr[R \geq n/2] > 0.2. \]

When \( R \geq n/2 \), Algorithm 6 must terminate and return the \( x^{\text{new}} \) in Line 30. Thus we have that, the probability of terminating and returning \( \text{fail} \) in Line 34 is at most \( 4/5 \).

Thus we complete the proof. \( \square \)

### 7.7 Lower bound for the vector after projection

Here in this section, we prove the following claim, giving the lower bound of the projected vector \( g_t \).

**Claim 7.11** (Lower Bound for the projection of projected vector \( g_t \)). We define \( R \) to be the total number of \( e_i \)'s added to \( V_t \) in Line 22 through the running of Algorithm 6. We define \( \rho_t \) to denote the probability such that the algorithm never return \( \text{fail} \) before reaching \( t \)-th iteration and \( \mu < \epsilon \) in this iteration. Then we have that

\[ \mathbb{E}[\|g_t\|_2^2] \geq (1 - \delta) \cdot \epsilon^2(n - n/4 - \mathbb{E}[R]) - \epsilon^2 \sqrt{2\rho_t n}. \]

**Proof.** For any \( t \in [N] \), we define the indicators \( F_t \) to be

\[ F_t = 1\{\forall i \in [m] \mid |\langle a_i, u_N \rangle| < \beta\}, \]

and \( Y_t \) to be

\[ Y_t = 1\{\mu < \epsilon\}. \]

Then we have that

\[
\mathbb{E}[\|g_t\|_2^2] = \mathbb{E}[F_tY_t\mu^2 \cdot \|I - V_t^\top V_t\| \cdot g_t\|_2^2 + F_t(1 - Y_t)\epsilon^2 \cdot \|I - V_t^\top V_t\| \cdot g_t\|_2^2]
\]

\[
\geq \mathbb{E}[F_t(1 - Y_t)\epsilon^2 \cdot \|I - V_t^\top V_t\| \cdot g_t\|_2^2]
\]

\[
\geq \epsilon^2 \cdot \mathbb{E}[F_t\|I - V_t^\top V_t\| \cdot g_t\|_2^2 - \epsilon^2 \cdot \mathbb{E}[Y_t\|I - V_t^\top V_t\| \cdot g_t\|_2^2]].
\]

where the first step follows from the definition of \( F_t \) and \( G_t \), the second step follows from \( F_tY_t\mu^2 \cdot \|I - V_t^\top V_t\| \cdot g_t\|_2^2 \leq 0 \), and the last step follows from splitting the terms.

Using Cauchy-Schwartz inequality, we have that

\[
\mathbb{E}[Y_t \cdot \|I - V_t^\top V_t\| \cdot g_t\|_2^2] \leq \sqrt{\mathbb{E}[Y_t^2] \cdot \mathbb{E}[\|I - V_t^\top V_t\| \cdot g_t\|_2^4].}
\]

Recall we define \( Y_t \) to be an indicator, thus we have that \( \mathbb{E}[Y_t^2] = \mathbb{E}[Y_t] \). Since \( I - V_t^\top V_t \) is a projection matrix, we have

\[ \|g_t\|_2 \geq \|I - V_t^\top V_t\| \cdot g_t\|_2. \]  \hspace{1cm} (13)

Thus, the following holds

\[
\mathbb{E}[Y_t \cdot \|I - V_t^\top V_t\| \cdot g_t\|_2^2] \leq \sqrt{\mathbb{E}[Y_t^2] \cdot \mathbb{E}[\|g_t\|_2^4]} - \sqrt{\rho_t \cdot \mathbb{E}[\|g_t\|_2^4]}.
\]
where the first step follows from Eq. (13), the second step follows from the definition of $\rho_t$.

For any $i \in [n]$, we denote the $i$-th entry of $g_t$ by $g_{t,i}$. Note that $g_{t,i} \sim \mathcal{N}(0, 1)$ for all $i \in [n]$ and $t \in [N]$. And all $g_{t,i}$’s are i.i.d. Thus we have that
\[
\mathbb{E}[\|g_t\|^4_2] = \mathbb{E}[(\sum_{i=1}^n g_{t,i}^2)^2] 
\leq \sum_{i=1}^n \sum_{j=1}^n \mathbb{E}[g_{t,i}^2 g_{t,j}^2],
\]
where the first step follows from the definition of $g_t$, the second step follows from that $g_{t,i}$’s are i.i.d. For all the terms that $i \neq j$, we have that
\[
\mathbb{E}[g_{t,i}^2 g_{t,j}^2] = \mathbb{E}[g_{t,i}^2] \cdot \mathbb{E}[g_{t,j}^2] = 1.
\]
For the terms that $i = j$, by the 4-th moment of the normal distribution, we have
\[
\mathbb{E}[g_{t,i}^4] = 3.
\]
Thus we have
\[
\mathbb{E}[\|g_t\|^4_2] \leq n(n-1) + 3n \leq 2n^2.
\]
Hence
\[
\mathbb{E}[Y_t \cdot \| (I - V_t^T V_t) \cdot g_t \|^2_2] \leq \sqrt{2\rho_t n}.
\]
Also we have that
\[
\mathbb{E}[F_t \cdot \| (I - V_t^T V_t) \cdot g_t \|^2_2] = \Pr[F_t = 1] \cdot \mathbb{E}[\| (I - V_t^T V_t) \cdot g_t \|^2_2]
\]
\[
= \Pr[F_t = 1] \cdot (\mathbb{E}[n - \dim(V_t)])
\]
\[
= \Pr[F_t = 1] \cdot (n - \mathbb{E}[\dim(V_t)]),
\]
where the first step follows from the definition of expectation, the second step follows from the property of the vector $(I - V_t^T V_t) \cdot g_t$ (Since $g$ is Gaussian), and the last step follows from the definition of expectation.

Note that for any $t \in [N]$, we have that
\[
\dim(V_t) \leq R + n/4
\]
Thus we have
\[
\mathbb{E}[\dim(V_t)] \leq \mathbb{E}[R] + n/4.
\]
Hence we conclude that
\[
\mathbb{E}[\|g_t\|^2_2] \geq \Pr[F_t = 1] \cdot e^2(n - n/4 - \mathbb{E}[R]) - e^2 \sqrt{2\rho_t n}.
\]
We have by Lemma 7.9 that
\[
\Pr[F_t = 1] \geq 1 - \delta.
\]
Hence we have
\[
\mathbb{E}[\|g_t\|^2_2] \geq (1 - \delta) \cdot e^2(n - n/4 - \mathbb{E}[R]) - e^2 \sqrt{2\rho_t n}.
\]
\[
\square
\]
7.8 Lower on Expectation

Here we prove the following claim, which gives the lower bound of expectation of the unit vectors added to $V$ through the algorithm running.

**Claim 7.12.** We denote the number of $e_i$’s added to $V_t$ in Line 22 through the running of Algorithm 6 to be $R$. We define $\rho_t$ to denote that Algorithm 6 never return fail before $i$-th iteration and $\mu < \epsilon$ in this iteration. We assume $\rho_t \leq 0.005$ and $\delta \leq 0.01$.

If
\[
\mathbb{E}[\|g_t\|_2^2] \geq (1 - \delta) \cdot \epsilon^2 (n - n/4 - \mathbb{E}[R]) - \epsilon^2 \sqrt{2\rho_t n}.
\]

Then we have that
\[
\mathbb{E}[R] \geq 0.63n.
\]

**Proof.** By Assumption this claim, we have
\[
\mathbb{E}[\|g_t\|_2^2] \geq (1 - \delta) \cdot \epsilon^2 (n - n/4 - \mathbb{E}[R]) - \epsilon^2 \sqrt{2\rho_t n}.
\]

Note that we assume that $\rho_t \leq 0.005$ and $\delta \leq 0.01$. Then by the above step we have that
\[
\mathbb{E}[\|g_t\|_2^2] \geq 0.99 \cdot \epsilon^2 (0.698n - \mathbb{E}[R]).
\]

(14)

Hence we have
\[
\mathbb{E}[\|x + u_{N+1}\|_2^2] \geq \|x\|_2^2 + (N - 400n) \cdot 0.99 \cdot \epsilon^2 (0.698n - \mathbb{E}[R])
\]
\[
\geq (N - 400n) \cdot 0.99 \cdot \epsilon^2 (0.698n - \mathbb{E}[R])
\]
\[
= 16 \cdot 0.99 \cdot (0.698n - \mathbb{E}[R]),
\]

where the first step follows from Eq. (12) and Eq. (14), the second step follows from $\|x\|_2^2 \leq 0$, and the last step follows by setting $N = 16\epsilon^{-2} + 400n$.

Notice now that Algorithm 6 will only produce an output $v$ such that $\|x + u_{N+1}\|_\infty \leq 1$. Thus we conclude it happens when $\mathbb{E}[\|x + u_{N+1}\|_2^2] \leq n$. Thus we have
\[
n \geq 16 \cdot 0.99 \cdot (0.698n - \mathbb{E}[R]).
\]

By the above inequality, it holds that
\[
\mathbb{E}[R] \geq (0.698 - 1.02(1/16))n > 0.634n.
\]
7.9 From Expectation to Probability

Claim 7.13. If \( \mathbb{E}[R] \geq 0.62n \) then
\[
\Pr[R \geq n/2] > 0.2.
\]

Proof. Here we define \( Z := n - R \). Then we have the expectation of \( Z \)
\[
\mathbb{E}[Z] = n - \mathbb{E}[R] \leq 0.38n.
\]
where the first step follows from the definition of \( Z \), the second step follows from \( \mathbb{E}[R] \geq 0.62n \).

Then by Markov’s inequality we have that
\[
\Pr[Z > a] \leq \frac{\mathbb{E}[Z]}{a}.
\]
Thus we have
\[
\Pr[Z > n/2] < 2 \cdot 0.38 = 0.76 < 0.8,
\]
which implies that,
\[
\Pr[Z \leq n/2] > 0.2.
\]
Finally we have by \( Z := n - R \) that
\[
\Pr[R \geq n/2] > 0.2.
\]

\( \square \)

7.10 Running Time

The goal of this section is to prove Lemma 7.14.

Lemma 7.14. For any parameter \( a \in [0, 1] \), the algorithm (\textsc{FastPartialColoring} in Algorithm 6) runs in
\[
\tilde{O}(\text{nnz}(A) + n^\omega + n^{2+a} + n^{1+\omega(1,1,a)-a})
\]
time. Note that \( \omega \) is the exponent of matrix multiplication.

Proof. The running time of \textsc{FastPartialColoring} can be divided as follows,

- Line 5 takes time \( \tilde{O}(\text{nnz}(A) + n^\omega) \), by Lemma 4.6.
- Line 6 takes time \( O(\text{nnz}(A) + n^2) \), by Lemma 7.1.
- Line 13 takes time \( O(n^\omega + n^2) \) to initialize the data structure, by Lemma 8.7.
- Run the following for \( N = O(\epsilon^{-2} + n) = O(n + \log m) \) times,
  - Line 15 takes time \( O(n^{1+a}) \) to query \( g \), by Lemma 8.7.

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Line 17 takes time $O(n)$ to find $\mu$ due to Lemma 7.2 (via Algorithm 8).

Line 22 takes time $O(n^{1+a})$ to update the data structure, by Lemma 8.7.

- Over the entire algorithm, we will enter Line 28 at most once. To check the satisfied condition for that only takes $\text{nnz}(A)$ time.

Adding them together and by the tighter time for operations of FastMaintain data structure (Lemma 8.10), we have the total running time of FastPartialColoring of

$$T_{\text{FastPartialColoring}} = \tilde{O}(\text{nnz}(A) + n^\omega + n^{2+a} + n^{\omega(1,1,a)-a}).$$

Therefore, we complete the proof.

\section{Fast maintaining lazy data structure}

\subsection{Main Data Structure}

The goal of this section is to prove Theorem 8.1.

\begin{theorem}
For a input matrix $V \in \mathbb{R}^{\ell \times n}$ with $\ell \leq n$, a matrix $G \in \mathbb{R}^{n \times N}$ and a integer $K > 0$, $N = O(n)$, there exists a data structure $\text{Maintain}$ (Algorithm 9, 10) uses $O(n^2)$ space and supports the following operations:

- $\text{Init}(V \in \mathbb{R}^{\ell \times n}, G \in \mathbb{R}^{n \times N}, K \in \mathbb{N}_+)$: It initializes the projection matrix $P = V^\top V \in \mathbb{R}^{n \times n}$. It stores the matrix $G$. For the first $K$ column vectors in $G$, it computes their projection when apply $P$. It sets counter $k_q$ and $k_u$ to be zero. It sets $\tau_q$ to be zero. This procedure runs in time

$$T_{\text{mat}}(n,n,n) + T_{\text{mat}}(K,n,n).$$

- $\text{Query}()$: It output $(I - V^\top V)g$ where $g \in \mathbb{R}^n$ is the next column vector in $G$ to be used. The running time of this procedure is

  \begin{itemize}
  \item $O(nK)$ time, if $k_q \in [K-1]$;
  \item $T_{\text{mat}}(n,k,n)$ time, if $k_u = K$.
  \end{itemize}

- $\text{Update}(u \in \mathbb{R}^n)$: It takes an 1-sparse vector $u \in \mathbb{R}^n$ as input and maintains $V$ by adding $w$ into next row of $V$ (according to Algorithm 1, $w = \tilde{w}/\|\tilde{w}\|_2$ where $\tilde{w} = (I - V^\top V)u$. It is obvious that $w \perp V$). The running time of this procedure is

  \begin{itemize}
  \item $O(nK)$ time, if $k_u \in [K-1]$;
  \item $T_{\text{mat}}(n,k,n)$ time, if $k_u = K$.
  \end{itemize}

- $\text{Restart}()$: It updates the projection $P = V^\top V$, generates $K$ fresh Gaussian vectors by switching the batch of Gaussian vectors we use by $\tau_q$, and compute their projections when apply $P$. It reset the counter $k_q$ and $k_u$ to be zero. The running time of this procedure

$$T_{\text{mat}}(n,K,n).$$

\end{theorem}

\begin{proof}
It follows from combining Lemma 8.2, Lemma 8.3, Lemma 8.4, Lemma 8.5 and Lemma 8.6.
\end{proof}
Algorithm 9

1: data structure \textbf{Maintain} \Comment{Theorem 8.1}
2: members
3: \hspace{0.5cm} \( P \in \mathbb{R}^{n \times n} \)
4: \hspace{0.5cm} \( V \in \mathbb{R}^{n \times n} \)
5: \hspace{0.5cm} \( G \in \mathbb{R}^{n \times N} \)
6: \hspace{0.5cm} \( \ell \)
7: \hspace{0.5cm} \( \tau_q \) \Comment{total counter for query}
8: \hspace{0.5cm} \( k_u \) \Comment{\( k_u \) is a counter for update process}
9: \hspace{0.5cm} \( k_q \) \Comment{\( k_q \) is a counter for query process}
10: \hspace{0.5cm} \( w_1, \cdots, w_K \in \mathbb{R}^n \) \Comment{This is a list}
11: \hspace{0.5cm} \textbf{end members}
12:
13: \textbf{public:}
14: \hspace{0.5cm} \textbf{procedure} \text{Init}(V \in \mathbb{R}^{\ell \times n}, G \in \mathbb{R}^{n \times N}, K) \Comment{Lemma 8.2}
15: \hspace{1cm} \( k_u \leftarrow 0, k_q \leftarrow 0 \)
16: \hspace{1cm} \( \tau_q \leftarrow 0 \)
17: \hspace{1cm} \( P \leftarrow V^\top V \) \Comment{This takes \( T_{\text{mat}}(n,n,n) \)}
18: \hspace{1cm} \text{Let} \( S = \{1, 2, \cdots, K\} \)
19: \hspace{1cm} \text{Let} \( G \in \mathbb{R}^{n \times N} \) denote \( G_{*,S} \)
20: \hspace{1cm} \text{Compute} \( \tilde{G} = P \cdot G_{*,S} \) and let \( \tilde{g}_1, \cdots, \tilde{g}_K \) denote those new \( K \) vectors \Comment{This takes \( T_{\text{mat}}(K,n,n) \)}
21: \hspace{1cm} \textbf{for} \( i = 1 \rightarrow K \) \textbf{do}
22: \hspace{1.5cm} \( w_i \leftarrow 0_n \)
23: \hspace{1cm} \textbf{end for}
24: \hspace{0.5cm} \textbf{end procedure}
25:
26: \textbf{public:}
27: \hspace{0.5cm} \textbf{procedure} \text{Query}() \Comment{Lemma 8.3, 8.4}
28: \hspace{1cm} \( k_q \leftarrow k_q + 1 \)
29: \hspace{1cm} \( \tau_q \leftarrow \tau_q + 1 \)
30: \hspace{1cm} \textbf{if} \( k_q \geq K \) \textbf{then}
31: \hspace{1.5cm} \text{Restart}()
32: \hspace{1cm} \textbf{end if}
33: \hspace{1cm} \textbf{return} \( g_{k_q} - \tilde{g}_{k_q} - \sum_{i=1}^{k_u} w_i w_i^\top g_{k_q} \) \Comment{This takes \( O(nK) \) time}
34: \hspace{0.5cm} \textbf{end procedure}
35: \textbf{data structure}

8.2 Initialization

The goal of this section is to prove Lemma 8.2.

Lemma 8.2 (Running time for \text{Init}). The procedure \text{Init}(V \in \mathbb{R}^{\ell \times n}, G \in \mathbb{R}^{n \times N}, K \in \mathbb{N}_+) initializes the projection matrix \( P = V^\top V \in \mathbb{R}^{n \times n} \). It stores the matrix \( G \). For the first \( K \) column vectors in \( G \), it computes their projection when apply \( P \). It sets counter \( k_q \) and \( k_u \) to be zero. It sets \( \tau_q \) to be zero. This procedure runs in time

\[ T_{\text{mat}}(n,n,n) + T_{\text{mat}}(K,n,n). \]
Algorithm 10 Update

1: data structure Maintain

2: public:

3: procedure Update(u ∈ R^n)

4: ▷ The input vector u is 1-sparse and the nonzero entry is 1

5:  k_u ← k_u + 1

6:  \( \vec{w} \leftarrow (I - P) - \sum_{i=1}^{k_u-1} w_i w_i^\top \)u

7:  \( w_{k_u} \leftarrow \vec{w} / \|\vec{w}\|_2 \)

8:  if \( k_u \geq K \) then

9:  Restart()

10: end if

11: end procedure

12: end data structure

Proof. The running time of Init can be divided into the following lines,

- Line 17 takes time \( \mathcal{T}_{\text{mat}}(n, n, n) \) to compute matrix \( P \).
- Line 17 takes time \( O(nK) \) to generate matrix \( G \).
- Line 20 takes time \( \mathcal{T}_{\text{mat}}(K, n, n) \) to compute matrix \( \vec{G} \).

Taking these together, we have the total running time is

\[ \mathcal{T}_{\text{mat}}(n, n, n) + \mathcal{T}_{\text{mat}}(K, n, n). \]

Thus, we complete the proof.

8.3 Query

The goal of this section is to prove Lemma 8.3 and Lemma 8.4.

Lemma 8.3 (Running time for QUERY). The running time of procedure Query is

- \( O(nK) \) time, if \( k_q \in [K - 1] \).
- \( \mathcal{T}_{\text{mat}}(n, k, n) \) time, if \( k_u = K \).
Proof. The proof can be splitted into two cases.

For the case that \( k_q \in [K - 1] \), Line 33 takes time \( O(nk_u) = O(nK) \) obviously.

For the case that \( k_q = K \), Line 31 runs in time \( T_{\text{mat}}(n, k, n) \). Thus we complete the proof. \( \square \)

Lemma 8.4 (Correctness for Query). The procedure \( \text{QUERY} \) outputs \( (I - V^\top V)g \) where \( g \in \mathbb{R}^n \) is the next vector in \( G \) to be used.

Figure 6: The decomposition of the output vector by \( \text{QUERY} \). The green composition is the precomputed factor and the blue composition is the new added ones, these two together form the projection of \( g \) onto the row span of \( V \).

Figure 7: The Visualization of the decomposition. Here we denote the matrix of rows encoded to \( P \) by \( V_1 \), and the matrix of rows not encoded yet by \( V_2 \). Thus we have that, \( P = V_1^\top V_1 \). And we assume that, \( \text{Span}(V_1) = \text{Span}(e_x) \), \( \text{Span}(V_2) = \text{Span}(e_y) \). Thus we visualize the idea of the decomposition as above. We divide the projection onto \( \text{Span}(V) \) into the projection onto \( \text{Span}(V_1) \) and the projection onto \( \text{Span}(V_2) \). That is, we have \( V^\top V g = V_1^\top V_1 \cdot g + V_2^\top V_2 \cdot g = \bar{g} + \sum_{i=1}^{k_u} w_i w_i^\top g \).

Proof. We divide the rows of \( V \in \mathbb{R}^{\ell \times n} \), into two parts: the rows that have been encoded into \( P \), and the ones that have not. Without loss of generality, we note the first \( \ell - k_u \) rows are encoded, and the last \( k_u \) rows are not. Thus we have that

\[
(I - V^\top V)g
\]
\[ g - \sum_{i=1}^{\ell} v_i v_i^\top g \]
\[ = g - \sum_{i=1}^{\ell-k_q} v_i v_i^\top g - \sum_{i=1}^{k_u} w_i w_i^\top g \]
\[ = g - \tilde{g} - \sum_{i=1}^{k_u} w_i w_i^\top g, \]

where the first step follows from split the multiplication, the second step follows from split the summation, and the last step follows from the definition of \( \tilde{g} \) and the construction \( P \). Thus we complete the proof.

8.4 Update

The goal of this section is to prove Lemma 8.5.

Lemma 8.5 (Running time for Update). The running time of procedure Update is

- \( O(nK) \) time, if \( k_q \in [K-1] \).
- \( \mathcal{T}_{\text{mat}}(n,k,n) \) time, if \( k_u = K \).

Proof. For the Update procedure, the running time can be divided into the following lines,

- Line 6 takes time \( O(nK) \).
- Line 7 takes time \( O(n) \).
- If \( k \) reaches \( K \), Line 9 takes time \( \mathcal{T}_{\text{mat}}(n,K,n) + \mathcal{T}_{\text{mat}}(K,n,n) \).

Thus we have that, if \( k \) doesn’t reach \( K \), the running time is

\( O(nK) \).

If \( k \) reaches \( K \), the running time is

\( \mathcal{T}_{\text{mat}}(n,K,n) + \mathcal{T}_{\text{mat}}(K,n,n) \).

8.5 Restart

The goal of this section is to prove Lemma 8.6.

Lemma 8.6 (Running time for Restart). The running time of procedure Update is

\( O(\mathcal{T}_{\text{mat}}(K,n,n)) \).

Proof. For the Restart procedure, the running time can be divided as the following lines

- Line 15 takes time \( \mathcal{T}_{\text{mat}}(n,K) \).
• Line 17 takes time $O(nK)$.
• Line 18 takes time $\mathcal{T}_{\text{mat}}(K,n,n)$.

Adding them together, we have the running time is

$$\mathcal{T}_{\text{mat}}(n,K,n) + \mathcal{T}_{\text{mat}}(K,n,n) = O(\mathcal{T}_{\text{mat}}(K,n,n)).$$

8.6 Running Time of the Maintain algorithm

The goal of this section is to prove Lemma 8.7.

Lemma 8.7 (Running time of FastMaintain). Let $N = O(n)$. For any parameter $a \in [0, \alpha]$, let $K = n^a$ where $\alpha$ is the dual exponent of matrix multiplication. For any input $G \in \mathbb{R}^{n \times N}$, and an input $b \in \{0,1\}^{n \times N}$ such that only one entry of each row of $b$ is 1, the procedure FastMaintain (Algorithm 11) runs in time

$$O(n^\omega + n^{2+a} + n^{3-a}).$$

For the current $\omega \approx 2.373$ and $\alpha \approx 0.31$. Due to the above equation, if we choose the $a = \min\{\alpha,1/2\}$, then the running time can be simplified to

$$O(n^{2.5} + n^{3-a}).$$

Remark 8.8. Instead of using $\alpha$, we can use the $\omega(\cdot,\cdot,\cdot)$ function in Table 3 in [GU18]. In Lemma 8.10, we provide a tighter analysis of Lemma 8.7.

Proof. The running time can be divided as follows.

• Line 2 takes time $\mathcal{T}_{\text{mat}}(n,n,n) + \mathcal{T}_{\text{mat}}(K,n,n) = O(n^\omega + n^2)$ to initialize the data structure.

• Run the following lines for $N = O(n)$ times (We ignore the condition of restart here):
  – Line 4 takes time $O(nK) = O(n^{1+a})$ to query for $g$.
  – Line 7 takes time $O(nK) = O(n^{1+a})$ time update the data structure.

The above runs in $O(n^{2+a})$ time in total.

• For the restart condition, we note that, Update is called for $O(n)$ times, so we will call Restart for $O(n/K) = O(n^{1-a})$ times, each time it will take $O(n^2)$ time for restarting. The condition that Query calls Restart is as the same. Thus we have the restart time bounded by $O(n^{3-a})$.

Combining the above together, we have the running time of FastMaintain is

$$O(n^\omega + n^{2+a} + n^{3-a}).$$
Algorithm 11 The purpose of writing down SLOWMAINTAIN is proving the correctness of FASTMAINTAIN. See Lemma 8.9.

1: procedure FastMaintain \( b \in \{0, 1\}^{n \times N}, G \in \mathbb{R}^{n \times N}, n, N \)  
2: \( \text{ds.Init}(V, G, K) \) \text{\Comment{Algorithm 9}}  
3: \( \text{for } t = 1 \rightarrow N \text{ do} \)  
4: \( g_t \leftarrow \text{ds.Query()} \) \text{\Comment{Algorithm 9}}  
5: \( \text{for } i = 1 \rightarrow n \text{ do} \)  
6: \( \text{if } b_{i,t} = 1 \text{ then} \)  
7: \( \text{ds.Update}(e_i) \) \text{\Comment{Algorithm 10}}  
8: \( \text{end if} \)  
9: \( \text{end for} \)  
10: \( \text{return } g_1, \cdots, g_N \)  
11: \( \text{end procedure} \)

14: procedure SlowMaintain \( b \in \{0, 1\}^{n \times N}, G \in \mathbb{R}^{n \times N}, n, N \)  
15: \( \text{for } t = 1 \rightarrow N \text{ do} \)  
16: \( g_t \leftarrow (I - V^T V)G_{s,t} \) \text{\Comment{G_{s,t} is the } t\text{-th column of } G}  
17: \( \text{for } i = 1 \rightarrow n \text{ do} \)  
18: \( \text{if } b_{i,t} = 1 \text{ then} \)  
19: \( \text{Add } e_i \text{ to } V \text{ according to Algorithm 1 and update } V \)  
20: \( \text{end if} \)  
21: \( \text{end for} \)  
22: \( \text{end for} \)  
23: \( \text{return } g_1, \cdots, g_N \)  
24: \( \text{end procedure} \)

8.7 Correctness

The goal of this section is to prove Lemma 8.9.

Lemma 8.9. For any input \( b \in \{0, 1\}^{n \times N} \) and \( G \in \mathbb{R}^{n \times N} \), the \( N \) vectors outputted by procedure FastMaintain (Algorithm 11) are exactly same as the \( N \) vectors outputted by procedure SlowMaintain (Algorithm 11).

Proof. For all \( i \in [N] \), we denote vectors output by the two algorithms by \( g_{i,F} \) for FastMaintain and \( g_{i,S} \) for SlowMaintain.

Induction Hypothesis. For any \( t \in [N] \), we have that \( g_{i,F} = g_{i,S} \) for all \( i \in [t-1] \).

Proof of Base Case. For the case that \( i = 1 \), we have by Lemma 8.4 that

\[
g_{1,F} = (I - V^T V)G_{s,1} = g_{1,S}.
\]

Thus the base case holds.

Proof of Inductive case. For the inductive case, we prove it by the following steps. We first notice that, with the binary matrix \( b \), we have added some vectors into \( V \) through the procedure when it reaches \( t \)-th iteration. We divide it into the following two circumstances:
No vectors added to $V$ after the last query of $g_{t-1}$ Since no vectors are added to $V$ after query of $g_{t-1}$, the $V$ stays the same when querying the $g_t$. Thus by Lemma 8.4, we have that

$$g_{i,F} = (I - V^TV)G_{*,1} = g_{i,S}.$$  

Hence we have the correctness.

There are vectors added to $V$ after the last query of $g_{t-1}$ If there’s vectors added to the matrix $V$, then the FastMaintain calls the Update for some times, and added some vectors into $ds$. Without loss of generality, we assume the Update never calls Restart. Then we have that, new vectors are encoded as $w$ in the $ds$. We divide the rows into

- Rows added before querying $g_{t-1}$, denoted the rows as the first $W_0$ ones.
- Rows added after querying $g_{t-1}$ and before querying $g_t$, denote the set as the last $W_1$ ones.

We denote the Gaussian vector as $g$, Thus we have that query $(I - V^TV)\tilde{g}_t$ as follows,

$$g_{t,S} = (I - V^TV)g_{t,S} = g_{t,S} - \sum_{i \in [W_0]} w_{i,S}w_{i,S}^Tg_{t,S} - \sum_{i=W_0+1}^{W_0+W_1} w_{i,S}w_{i,S}^Tg_{t,S}.$$  

And in the Fastmaintain (Algorithm 11), when we call Query, we divide the rows as the ones encoded into $P$ and the ones not, that is

$$g_{t,F} = g_{t,F} - \tilde{g}_{t,F} - \sum_{i \in [k_u]} w_{i,F}w_{i,F}^Tg_{t,F}.$$  

If we denote the rows have been encoded into $P$ as the first $\ell$ rows. We have that

$$\tilde{g}_{t,F} = \sum_{i \in [\ell]} w_{i,F}w_{i,F}^Tg_{t,F}.$$  

By the construction of the procedures, we have that

$$\ell + k_u = W_0 + W_1,$$

and

$$w_{i,F} = w_{i,S}$$

for each $i \in [\ell + k_u]$. And by the using of the counter $\tau_q$ in the $ds$, we have that

$$g_{t,F} = g_{t,S}.$$  

Then by Lemma 8.4, we have the $t$-th query

$$ds.Q(u) = (I - V^TV)G_{*,t}.$$  

Thus we complete the proof.
8.8 Tighter Running Time Analysis

The time analysis of Lemma 8.7 is not tight. We can further improve it by using function $\omega(\cdot, \cdot, \cdot)$.

**Lemma 8.10** (A Tighter Analysis of Running time of FastMaintain). Let $N = O(n)$. For any parameter $a \in [0, 1]$, let $K = n^a$. For any input $G \in \mathbb{R}^{n \times N}$, and an input $b \in \{0, 1\}^{n \times N}$ such that only one entry of each row of $b$ is 1, the procedure FastMaintain (Algorithm 11) runs in time

$$O(n^{\omega(1,1,1)} + n^{2+a} + n^{1+\omega(1,1,a)-a}).$$

For the current $\omega(\cdot, \cdot, \cdot)$ function (see Table 3 in [GU18]). We can choose $a = 0.529$, then the running time becomes

$$O(n^{2.53}).$$

For the ideal $\omega$, we can choose $a = 0.5$, then the running time becomes $n^{2.5}$.

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