Relativistic Light Sails

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Abstract

One proposed method for spacecraft to reach nearby stars is by accelerating sails using either solar radiation pressure or directed energy. This idea constitutes the thesis behind the Breakthrough Starshot project, which aims to accelerate a gram-mass spacecraft up to one-fifth the speed of light toward Proxima Centauri in just over two decades. A proposed configuration is to fire an Earth-based array of gigawatt (or greater) lasers onto a gram-mass, microchip-sized satellite that would be accelerated with large mirror-like sails harvesting the momentum of incident photons.1 While such sails were originally conceived with solar radiation in mind, the invention of lasers in the 1960s enabled efficient laser sailing propulsion systems too (Marx 1966; Redding 1967; Forward 1984).

Recently, the Breakthrough Starshot project2 (hereafter simply Starshot) announced plans to develop the technology needed for a laser sail nano-satellite capable of flying to the closest stars within a generation. A proposed configuration is to fire an Earth-based array of gigawatt (or greater) lasers onto a gram-mass, microchip-sized satellite that would be accelerated up to approximately one-fifth the speed of light, reaching Proxima Centauri in just over two decades (see Heller & Hippke 2017 for deceleration schemes).

While a great deal of literature, experiments, and even spacecraft demonstrations of solar sailing exist (Kawaguchi et al. 2008; Mori et al. 2009), Starshot is unique for two main reasons. First, the target speeds are relativistic; thus, classical expressions suitable in the context of solar sails become invalid. Second, Starshot is designed to be ultra-light, which means the mass of the sail cannot be assumed to be infinite, as is typically assumed in relativistic calculations of photon exchanges with a mirror (e.g., see Gjurchinovski 2013).

1 Tsiolkovsky & Zander first discussed this possibility in 1925, as detailed in Zander (1964).
2 http://breakthroughinitiatives.org/Initiative/3

In this work, we first present a simple derivation of the relativistic velocity curve of a light sail in Section 2. We then extend our analysis to consider the effect of imperfect mirrors and subsequent thermal heating of the sail and spacecraft payload in Section 3. We finish with some key conclusions in Section 4 and highlight parts of the calculation requiring further work.

2. Sailing with a Perfect Mirror

2.1. A Single Photon

We begin by considering the simple case of a single photon of frequency $\nu_i$ fired at a normal incident angle toward a perfect mirror—or, equivalently, a light sail—of mass $m$ moving along the same vector as the photon at speed $c\beta_i$, as depicted in Figure 1. The reflection of the photon is assumed to be perfectly elastic, although we relax this assumption in Section 3. The motion of the mirror and the frequency of the photon can be calculated by requiring the conservation of relativistic energy and momentum. First, the system’s total energy (the sum of the photon’s energy and the mirror’s energy) must be conserved before and after the reflection. Using the relativistic expressions, one may write that

$$h\nu_i + \frac{mc^2}{\sqrt{1 - \beta_i^2}} = h\nu_f + \frac{mc^2}{\sqrt{1 - \beta_f^2}}. \tag{1}$$

Similarly, requiring that the system conserves momentum, we find

$$\frac{h\nu_i}{c} + \frac{m\beta_i c}{\sqrt{1 - \beta_i^2}} = -\frac{h\nu_f}{c} + \frac{mc\beta_f}{\sqrt{1 - \beta_f^2}}. \tag{2}$$
Solving Equations (1) and (2) simultaneously and simplifying, we find
\[ \beta_f = \frac{4r^4(1 - \beta_i)^2 + 2r(1 - \beta_i)\sqrt{1 - \beta_i^2} - 2r^2(1 - \beta_i^2)}{1 - 4r^2(1 - \beta_i)(\beta_i - r^2(1 - \beta_i))} \] (3)

and
\[ \nu_f = \nu_i \frac{1 - \beta_i}{1 + \beta_i + 2r\sqrt{1 - \beta_i^2}} \] (4)

where we have defined \( r \) as the photon’s “relative energy” using
\[ r = \frac{\hbar\nu_i}{mc^2} \] (5)

Note that our calculation has ignored the effect of the mirror’s gravity, which in principle imparts a small gravitational frequency shift that is negligible for gram-mass sails.

2.2. Redshift of the Reflected Photon

Equation (4) may also be expressed in terms of the redshift, \( z \), of the reflected photon,
\[ 1 + z = \frac{1 + \beta_i + 2r\sqrt{1 - \beta_i^2}}{1 - \beta_i} \] (6)

which we plot in Figure 2 for several choices of \( r \). Equation (6) reveals that the reflected photon will have a redshift of zero when
\[ \beta_{r,z=0} = -\frac{r}{\sqrt{1 + r^2}} \] (7)

where the negative sign indicates that the mirror is now coming toward the photon.

We also point out that Equation (6) and Figure 2 reveal that the photon becomes redshifted to infinity (i.e., redshifted out of existence) as \( \beta_i \to 1 \). This result implies that, as the mirror moves closer to \( c \), the transfer of the photon’s energy into the

The kinetic energy of the mirror becomes increasingly efficient. This result is verified in Section 2.4.

2.3. Accelerating a Mirror with a Single Photon

In the limit of \( \beta_i \to 0 \)—in other words, an initially stationary mirror—one may write that the mirror will be accelerated up to a speed of
\[ \lim_{\beta_i \to 0} \beta_f = \frac{2r(1 + r)}{1 + 2r(1 + r)} \] (8)

Solving the above for \( \beta_f = \frac{1}{2} \) yields a characteristic relative photon energy, \( r \), necessary to impart relativistic motion as
\[ r_{\text{rel}} = \frac{\sqrt{3} - 1}{2} \approx 0.366 \ldots \] (9)

To first order in \( r \), Equation (8) is simply \( 2r \), which shows that \( r \gtrsim \mathcal{O}(10^{-2}) \) to get to even a few percent of the speed of light.

2.4. Efficiency

In the case of photon sailing, the primary goal of hitting the sail with photons is to propel it in the desired direction. Two useful figures of merit to consider in this context are the kinetic energy and speed of the sail in response to a photon reflection.
Consider first: what is the gain in kinetic energy of the mirror as a function of its initial velocity, $\beta_i$? The change in the kinetic energy of the mirror is most easily expressed by equating it to the total energy lost by the photon,

$$\Delta E_K = h\nu_i - h\nu_f,$$

$$\epsilon_K = \frac{\Delta E_K}{h\nu_i} = 1 - \frac{\nu_f}{\nu_i},$$

(10)

where on the second line we have reexpressed the kinetic energy gain in units of the incident photon’s energy, which can be considered to be the efficiency by which energy is transferred from the photon to the sail. Using Equation (4), we may now write that

$$\epsilon_K = 1 - \frac{1 - \beta_i}{1 + \beta_i + 2r\sqrt{1 - \beta_i^2}}.$$  

(11)

In the limit of $r \to 0$, where the photon’s energy is much less than the rest-mass energy of the sail, we find that

$$\lim_{r \to 0} \epsilon_K = \frac{2\beta_i}{1 + \beta_i},$$

(12)

which is a monotonically increasing function from $\beta_i = 0 \to 1$. This result therefore demands that the fraction of the photon’s energy transferred to the sail as kinetic energy increases as $\beta_i$ increases. In this sense, the sail becomes more efficient once it has gained some initial momentum, verifying the argument given in Section 2.2.

Consider now the velocity change of the sail as a function of $\beta_i$. In the classical framework, the speed ever-increases linearly into the superluminal regime. At low velocities, one may easily show that our expression in Equation (3) reproduces the classical behavior; for example, in the limit of $r \ll 1$, the related expression in Equation (8) simply gives $2r$, as expected. Therefore, the relative velocity change predicted by our formula should decrease at high speeds in order to reproduce an asymptote toward $c$. One can verify this mathematically by writing

$$\beta_f - \beta_i = (2 - \beta_i)\sqrt{1 - \beta_i^2}r + \mathcal{O}[r^2],$$

(13)

which reproduces the correct behavior of a velocity change of $2r$ at low $\beta_i$ and zero velocity change as $\beta_i$ approaches unity.

2.5. Accelerating a Mirror with an Ensemble of Photons

We now consider firing multiple photons at the sail/mirror in order to induce acceleration. In what follows, we ignore the effect of drag forces, such as interstellar dust and even photonic gas drag (Balasanyan & Mkrtchian 2009). We treat each photon as striking the mirror consecutively, leading to a series of small impulses, each of which slightly increases the velocity of the sail.

We set the initial velocity to $\beta_0 = 0$ and then define $\beta_n$, the velocity after the $n$th reflection, as $\beta_f$ from Equation (3), replacing $\beta_i \to \beta_{n-1}$. Writing out the first few terms and simplifying, one may show that $\beta_n$ may be written as

$$\beta_n = 1 - \frac{1}{1 + 2nr(1 + nr)},$$

(14)

which predicts the velocity of the sail as a function of the number of reflections $n$. These experiments find that the formulae are correct to within machine precision, thereby providing a simple formula to predict the velocity curves of relativistic sails.

By rearranging Equation (14) to make $n$ the subject, we are able to write down a simple formula for the number of photons needed to accelerate a sail up to a target relativistic speed, $\beta_{\text{target}}$:

$$n = \frac{1}{2r}\left(\frac{1 + \beta_{\text{target}}}{\sqrt{1 - \beta_{\text{target}}^2}} - 1\right).$$

(15)

or, equivalently, that the total light energy needed to strike the mirror is

$$E_{\text{light}} = \frac{1}{2}mc^2\left(\frac{1 + \beta_{\text{target}}}{\sqrt{1 - \beta_{\text{target}}^2}} - 1\right).$$

(16)

As a practical example, we plot the velocity curve of a Starshot-like sail ($m = 1$ g) accelerating up to $0.2c$ using our relativistic formula in comparison to the nonrelativistic case in Figure 4.

2.6. Previous Literature and Why Einstein’s Formalism Is Wrong at the 10% Level for Starshot

It is instructive to compare our results to those of the preexisting literature. Our solution calculates two distinct quantities: the redshift of a single photon after reflection (Equation (4)) and the resulting velocity change of the sail (Equation (3)), which forms the basis to scale up to an ensemble of photons (Equation (14)).
A photon’s redshift off a relativistic mirror is a classic problem that has been studied by many previous authors, including Einstein himself in his historic paper introducing special relativity (Einstein 1905). The corresponding velocity change of the mirror is less commonly derived, although our derivation finds that the solutions must come as a pair. Of course, for beamed laser sailing, it is this velocity change that is of greatest interest. Before comparing our velocity predictions to the literature, we first consider the redshift result, due to the rich literature of comparisons at our disposal.

We first compare to Gjurchinovski (2013), who provides a pedagogical derivation of a photon incident upon a relativistic mirror at an angle $\alpha$ but under the explicit assumption of an infinitely heavy mirror ($m \to \infty$). By conserving energy and momentum, Gjurchinovski (2013) obtains

$$
\nu_f = \nu_i \left( \frac{1 - 2\beta \cos \alpha + \beta^2} {1 - \beta^2} \right).
$$

As expected, the above is equivalent to our Equation (4) in the case of a normal incident photon ($\alpha = 0$), as was assumed in our work, and the limit of $r \to 0$ (which is equivalent to Gjurchinovski’s assumption of $m \to \infty$).

Another insightful example to compare to (where it cannot be assumed that $m \to 0$) is for Compton scattering, which is essentially the same problem but with the mirror replaced with an electron. For an electron initially at rest ($\beta_i = 0$), the photon’s frequency is shifted to (Equation (7.2) of Rybicki & Lightman 1979)

$$
\nu_f = \frac{\nu_i}{1 + r(1 - \cos \theta)},
$$

where $\theta$ is the scattering angle equal to $\pi$ for an exact reflection back along the original path, as adopted in our work. As expected, Equation (19) is indeed equivalent to our result in Equation (4) for $\beta_i = 0$ and $\theta = \pi$.

Having established the validity of our redshift formula, we now compare it to that being used in the literature of light sails. Of most relevance is the result curated in the “Roadmap to Interstellar Flight,” a comprehensive review by Lubin (2016) that ultimately inspired the Starshot project (Popkin 2017). Lubin (2016) reports that his relativistic solutions come from Kulkarni et al. (2016), who in Equation (1) have

$$
\lambda_f = \lambda_i \gamma^2 (1 + \beta^2),
$$

which is equivalent to Gjurchinovski (2013) for $\alpha = 0$ and to our Equation (4) in the limit of $r \to 0$. Therefore, although it is not explicitly stated in Kulkarni et al. (2016), the authors appear to have tacitly adopted the infinite-mass-sail approximation.

This can be verified by following the description of their derivation, which, unlike this work and Gjurchinovski (2013), uses Lorentz frame transfers rather than balancing conserved quantities. Specifically, the authors first shift the photon to the sail’s frame, then assume that it “is emitted with the same wavelength as it is incident with” before finally transferring back to the original frame. Crucially, this is the same tacit assumption made by Einstein himself in Einstein (1905); in Section 8 of that work, he adopts the same derivation procedure of frame transfers and uses the same intermediate step in the sail’s rest frame of $\nu'' = \nu'$ in his original notation (which indicates that the reflected light’s frequency equals the incident light’s frequency when viewed in the sail’s frame).

It is with some trepidation that we argue here that Einstein, and indeed all subsequent authors adopting this assumption (e.g., Lightman et al. 1975; Galli & Amiri 2012; Gjurchinovski 2013), must be formally wrong. For a sail (or mirror) at rest, the reflected photon cannot have the same frequency as the incident photon without violating the conservation of energy. The photon has reversed momentum, and so the mirror must increase its absolute momentum (from initially zero, since it is defined to be at rest) to conserve total momentum. Since the mirror is now moving, its kinetic energy must have also increased. Therefore, to conserve the total energy of the system, the photon has to lose energy, which it can only do by decreasing in frequency. Ergo, Einstein’s assumption that $\nu'' = \nu'$ violates the conservation of energy (note that this can also be seen by comparison to Compton scattering where this general statement is false; Rybicki & Lightman 1979).

Another way to think about the above is to assume that Einstein is correct and that $\nu'' = \nu'$ and then look at the consequences. The equality $\nu'' = \nu'$ means that the photon has lost no energy when reflecting off a mirror at rest (since $E = h\nu$). If this is true, then by conservation of energy, the mirror cannot have gained any kinetic energy. In other words, the mirror does not move. This simple point demonstrates the falsehood of $\nu'' = \nu'$, since it requires that no matter how many photons are incident upon a mirror initially at rest, it will never move. In other words, $\nu'' = \nu'$ would make the entire concept of light sailing impossible, since objects could never be accelerated away from being initially at rest.
Although formally wrong, one might argue that, practically speaking, this infinite-mass mirror assumption is always extremely well justified. In other words, one might reasonably posit that whether this assumption is imposed or not, the resulting predictions will be nearly identical. Remarkably, this appears to be false. Consider the other half of the solution, the corresponding velocity change of the mirror in response to an ensemble of photons (which we state in Equation (14)). This solution does not appear in Einstein (1905) but is derived in Kulkarni et al. (2016), who use the same derivation framework for $\nu$ as Einstein.

Kulkarni et al. (2016) relate the relativistic velocity of a perfect sail in response to a constant beam of power $P$ fully on the sail for a time $t$ as

$$t' = \frac{1}{6} \left[ \frac{(1 + \beta_f)(2 - \beta_f)}{(1 - \beta_f) \sqrt{1 - \beta_f^2}} - 2 \right],$$

(21)

where we use $t' \equiv (Pt)/(mc^2)$. Although it is not stated in Lubin (2016) or Kulkarni et al. (2016), we may rearrange Equation (21) to solve for $\beta_f$, which leads to a cubic equation with one real root of

$$\beta_f = 1 - \frac{(\kappa - 2 - 6t')}{\kappa} \frac{(1 + (\sqrt{3} i)/2)}{[\kappa^2(-5 + 2\kappa + 6t'(-\kappa - 4 - 6t'))]^{1/3}},$$

(22)

where $\kappa \equiv \sqrt{5 + 12t'(2 + 3t')}$. Although our work does not strictly assume a constant laser illumination, we can write our Equation (14) in this form by using $nr = E_{\text{inc}}/(mc^2) = (Pt)/(mc^2) = t'$. Clearly, the two equations do not agree mathematically, even in this simple case of perfect reflection. As can be seen in Figure 4, the Lubin (2016) and Kulkarni et al. (2016) formula significantly diverges from that of this work, predicting ~10.8% more energy needed to reach $0.2c$ than our formula.

One might reasonably ask at this point, how can it possibly be that assuming $\hbar \ll mc^2$ (the infinite-mass assumption) leads to a 10% difference in velocity predictions? After all, for our fiducial choice of $\lambda = 650$ nm and $m = 1$ g, we have $r \sim 3.4 \times 10^{-33}$, so how can it cause such a big difference as seen in Figure 4? The answer is that if one starts out by assuming $r \to 0$ for a single photon and then builds upon this result to solve for $n$ photons, one has tacitly assumed $nr \to 0$ due to the principle of ensemble equivalence (Section 2.5). In other words, while the error is tiny for a single photon, it accumulates over a very large number of photons ($\sim 10^{31}$), leading to a sizable effect. By assuming $\nu' = \nu$, Einstein overestimates the photon’s energy post-reflection and thus underestimates the sail’s kinetic energy, which is why our velocity curve is above that of Lubin (2016), shown here in Figure 4.

For Starshot, the rest mass of the sail is 90 TJ (for a 1 g sail), and the laser energy required to accelerate to 0.2c is 10 TJ. Accordingly, in this regime, the approximation of an infinite-mass sail comes into tension. It is for this reason that the rest mass of the sail cannot be neglected in such calculations.

3. Imperfect Sails in Thermal Equilibrium

3.1. Overview

Throughout Section 2, we explicitly assume a perfect mirror, one with a reflection coefficient of unity. In such a case, the sail is maximally efficient and thermally stable, absorbing no photons as thermal energy. Accordingly, the time frame over which one fires the photons at the sail is inconsequential, and, in principle, the sail can receive the full jolt of energy in a single laser pulse. In practice, even slight imperfections in the reflectivity will both degrade the rate of acceleration and lead to the sail absorbing thermal photons, potentially leading to a catastrophic failure of the sail and/or electronic payload. Here, we provide a simple derivation of the magnitude of these effects, starting again from the case of a single photon.

3.2. Inelastic Photon Collisions

We begin by considering a single photon that makes an inelastic collision with the mirror. The picture is therefore similar to that depicted in Figure 1, except the final photon is not reflected but absorbed into the mirror, slightly increasing the rest-mass energy of the mirror. As before, we proceed by balancing the energy,

$$h\nu_i + \frac{mc^2}{\sqrt{1 - \beta_i^2}} = \frac{Mmc^2}{\sqrt{1 - \beta_f^2}},$$

(23)

and momentum,

$$\frac{h\nu_i}{c} + \frac{m\beta_ic}{\sqrt{1 - \beta_i^2}} = \frac{Mmc\beta_f}{\sqrt{1 - \beta_f^2}},$$

(24)

in the system, which may be solved for $\beta_f$ and $M$, where $M$ is the relative increase in the rest-mass energy of the sail, giving

$$M = \left( \frac{r(1 - \beta_i^2) + \sqrt{1 - \beta_i^2}}{\sqrt{1 - r^2(1 - \beta_f^2)}} \right) \left( 1 + \beta_i \frac{1 - \beta_i^2 - r^2(3 - \beta_i)(1 - \beta_i)/(1 + \beta_i)}{2r^3(1 - \beta_i^3)/3} \right)^2,$$

(25)

and

$$\beta_f^\text{abs} = \frac{\beta_i + r(1 - \beta_i)(1 - \beta_i^2 - r^2(1 - \beta_i^2))}{1 - r^2(1 - \beta_i^2)}.$$ 

(26)

Note that we now distinguish between the mirror’s velocity from an absorbed and a reflected photon using the superscripts “abs” and “ref,” respectively. Accordingly, comparing Equations (3) and (26), we can verify the classical result that

$$\lim_{r \to 1, \beta_i \to 0} \beta_f^\text{ref} = 2 \lim_{r \to 1, \beta_i \to 0} \beta_f^\text{abs},$$

(27)

which states that a reflected photon imparts twice the momentum as an absorbed photon (which can be seen not to hold in the relativistic regime).

Consider a sail that is accelerated to relativistic speeds exclusively by absorbed photons but maintains a constant temperature via thermal equilibrium. This means that although the mirror’s rest mass temporarily increases after the absorption, it immediately readjusts this excess energy isotropically, thereby returning to a rest mass $m$. Since isotropic reradiation...
of the sail does not affect its velocity (otherwise anything moving and at a nonzero temperature would feel a constant drag/acceleration force), we may use the principle of ensemble equivalence used in Section 2.5 to show that the velocity curve is

$$\beta_{n}^{\text{abs}} = \frac{nr}{1 + nr}. \quad (28)$$

### 3.3. Accounting for Reflectivity

We now need to combine the two cases, reflection and absorption, into a single model described by a reflection coefficient, $R$. In the following, we define $R$ as the fraction of incident photon power that is reflected elastically by the mirror, with the remaining fraction $\bar{R} = (1 - R)$ being absorbed inelastically. For simplicity, we also assume that the reflection coefficient is achromatic.

We first point out that trying to derive this formula in the case of a single photon appears intractable, on the basis that we have two conserved quantities (energy and momentum) but three unknowns (final mirror velocity, final frequency of the photon, and final rest mass of the mirror).

In order to make progress, we adopt the following approximate model. We assume that a single photon can be split into two components: one of energy $\bar{R}h$, which reflects off the sail, and the other of energy $Rh$, which is absorbed. Let us assume that the elastic collision occurs first, followed by the inelastic collision; in each independent collision, we can analytically solve the final state of the system. In the time between this “pair” of photons and the next, we assume that the sail radiates the excess absorbed energy; i.e., it is in thermal equilibrium. Using this model, we can combine the results found in Sections 2.1 and 3.2 to write that, for an initial velocity of $\beta_0 = 0$, the speed after the $n = 1$st incident photon is

$$\beta_{1}^{\text{mix}} = 1 - \frac{1}{1 + r(1 + R + 2Rr)}, \quad (29)$$

where the superscript “mix” on the left-hand side denotes that this velocity change is a mixture model of both elastic and inelastic components. Using our principle of ensemble equivalence (i.e., that a series of photon impacts is equivalent to one cumulative energetic photon collision), we may write that

$$\beta_{n}^{\text{mix}} = 1 - \frac{1}{1 + nr(1 + R + 2Rnr)}. \quad (30)$$

As expected, Equation (30) can be easily demonstrated to reproduce Equation (14) in the limit of $R \to 1$ and Equation (28) in the limit of $R \to 0$.

### 3.4. Numerical Verification

Equation (30) is derived by assuming that each photon can be treated as a pair of dummy photons. Here, we test the validity of this assumption through numerical simulations.

In each simulation, we consider firing $10^5$ incident photons at a mirror where the photon has a probability $R$ of being an elastic photon and $\bar{R}$ of being inelastic. Starting from rest, we numerically compute the velocity curve of the mirror using Equation (3) for elastic collisions and Equation (26) for inelastic collisions. After each inelastic collision, we assume that the mirror reradiates the absorbed energy isotropically before the next photon arrives (i.e., thermal equilibrium), such that the rest mass of the mirror does not evolve.

Since the simulations are intrinsically stochastic via the reflection probabilities, we repeat each simulation 1000 times and take the mean. Because we have assumed a small number of incident photons (just $10^5$), we use several large choices of $r = 10^{-6}$, $10^{-5}$, and $10^{-4}$ in order to accelerate the mirror to relativistic speeds. We set the reflectivity to $R = 0.9$, representing a fairly poorly optimized sail. Comparing to the predictions of Equation (30), we estimate that the expression is accurate to within 0.04% for all reflectivities $R > 0.9$.

### 3.5. Velocity Losses Due to Nonunity Reflectivities

We may now compare the velocity curve predicted by Equation (30) to that of a perfect mirror in Equation (14) in order to quantify the losses due to nonunity reflectivities,

$$\frac{\beta_{n}^{\text{ref}} - \beta_{n}^{\text{mix}}}{\beta_{n}^{\text{ref}}} = \frac{(1 + 2nr)(1 - R)}{2(1 + nr)(1 + nr(1 + R + 2nr))}, \quad (31)$$

which reduces to the classical result of

$$\lim_{r \to 0} \left(\frac{\beta_{n}^{\text{ref}} - \beta_{n}^{\text{mix}}}{\beta_{n}^{\text{ref}}}\right) = \left(1 - \frac{R}{2}\right). \quad (32)$$

Since $nr$ dictates the final velocity of the mirror, we may replace $nr$ with the target velocity, $\beta_{\text{target}}$, and Taylor expand to first order in $\bar{R}$ to yield

$$\left(\frac{\beta_{n}^{\text{ref}} - \beta_{n}^{\text{mix}}}{\beta_{n}^{\text{ref}}}\right) \approx \left(1 - R\right)f\left(1 - \beta_{\text{target}}\right). \quad (33)$$

Using the cumulative energy needed to accelerate a perfect sail to 0.2c, the final speed of the sail is reduced by 4.4% for a 90% reflectivity and 0.044% for a 99.9% reflectivity. We therefore conclude that velocity losses due to imperfect reflectivities are fairly modest and unlikely to be a limiting design constraint on the sail.

### 3.6. Energy and Thermal Requirements

We also consider the energy that is absorbed by the sail thermally. One may rearrange Equation (30) to write that the cumulative number of photons needed to accelerate a sail up to a target velocity $\beta_{\text{target}}$ is given by

$$r_{\text{tot}}(\beta_{\text{target}}, R) = \frac{1}{4R(1 - \beta_{\text{target}})}\left(-\left(1 + R)(1 - \beta_{\text{target}}) + \sqrt{8\beta_{\text{target}}R(1 - \beta_{\text{target}}) + (1 - \beta_{\text{target}})^2(1 + R)^2}\right)\right). \quad (34)$$

where $r_{\text{tot}} = nr$. The above can also be expressed as an energy given by

$$E_{\text{inc}} = mc^2r_{\text{tot}}(\beta_{\text{target}}, R). \quad (35)$$

Note that this is the energy incident upon the sail and does not account for beam losses due to diffraction or scattering between the laser source and the sail. In total, we assume that the sail has absorbed a fraction $\bar{R}$ of this energy as thermal
photon over a time $t$. For sails with finite transmittance, the prescription given here can be easily modified by attenuating the incident energy accordingly. In the sail’s reference frame, this incident energy is received over a dilated time $t'$. Assuming a constant acceleration (or force) applied to the sail initially at rest, the time dilation factor is (Iorio 2005)

$$t' = \frac{\sinh^{-1}(\beta_{\text{arg}})}{\beta_{\text{arg}}}. \quad (36)$$

For $\beta_{\text{arg}} = 0.2$, this time dilation factor is less than a percent and thus, practically speaking, one may simply assume $t' \approx t$.

As was done earlier, we assume that the sail immediately reradiates the absorbed energy isotropically. For the sake of simplicity, we assume that the sail emits this thermal energy as a blackbody over the laser firing time of $t$, such that the total energy emitted by the sail is $2\Delta t \sigma T^4$, where $\Delta t$ is the area of the sail. Note that the sail’s area is not length contracted, since it is normal to the direction of motion. Equating the received and emitted powers and then solving for the sail’s temperature, $T$, we have

$$T^4 = \left(\frac{\Sigma R e^2}{2 \sigma g T}\right)_r(\beta_{\text{arg}}, R), \quad (37)$$

where $\Sigma$ is the effective surface density of the sail, given by $\Sigma \equiv m/A$. Note that Equation (37) refers to the temperature of the sail in the Earth’s frame of reference, not in the sail’s frame of reference, which we ultimately require. Einstein (1907) and Planck (1908) argue that temperature is covariant, given by $T' = T/\gamma$ (where $\gamma = 1 / \sqrt{1 - \beta^2}$), but Ott (1963) challenges this, obtaining the result $T' = T\gamma$. Later, Landsberg (1966, 1967) argues that thermodynamic quantities such as entropy and temperature should not vary between two reference frames, and we adopt this result in our work too, i.e., $T' = T$.

In order to proceed, we assign some parameters appropriate for the Starshot proposal. We choose optimistic but plausible values for the spacecraft mass of $m = 1$ g and a sail area $A = 16$ m$^2$ and assume $\beta_{\text{arg}} = 0.2$. The firing time is varied between several options. We plot the resulting temperature of the spacecraft, given by Equation (37), as a function of absorptivity in Figure 5.

As an example, for $R = 10^{-5}$ absorptivity, which is plausible with optically coated materials (Rempe et al. 1992), temperatures below 300°C (typical of a high-temperature microsystem; Lien et al. 2011; Chiamori et al. 2014) can be maintained over an 8.6 minute firing period. Such a case would require just over 10 TJ of incident energy on the sail, or a constant power of 19.6 GW, giving an average flux on the sail of 1.2 GW m$^{-2}$ for the adopted 16 m$^2$ area.

### 3.7. Diffraction Losses

The rapid acceleration of the sail causes it to quickly traverse great distances, which poses at least two challenges for the laser system. First, it may be difficult to maintain accurate pointing on the sail at great distances, particularly if atmospheric turbulence introduces small refractive deviations to the optical path. Second, even if perfect pointing is maintained, diffraction of the laser light can introduce significant losses of beam energy by the time it reaches the target. We tackle this second issue in the following and assume a stable sail riding the beam throughout (see Manchester & Loeb 2017 for details on this point).

Consider a transmitter of diameter $D_T$ producing a laser of wavelength $\lambda$, which strikes its target at a distance of $L$ away from the source. For a diffraction-limited beam, the beam width

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5 These disagreements provide an interesting opportunity for experiment onboard Starshot.
at a distance $L$ will be (Kipping & Teachey 2016)

$$ W_L = \frac{\sqrt{2} L \lambda}{D_T}, \quad \text{(38)} $$

where we assume that the final beam width has diffracted to be much greater than the initial width.

For simplicity, we consider a sail that is circular in projection and a Gaussian beam profile. At a distance $L$, the integrated fraction of the laser power striking the sail will be

$$ F_P = \left( \text{erf} \left( \frac{1}{W_L} \sqrt{\frac{A}{2\pi}} \right) \right)^2. \quad \text{(39)} $$

The above can now be evaluated by replacing $L$ with the corresponding distance expected at some target velocity, $\beta_{\text{arg}}$. To make analytic progress, we assume that the sail undergoes strictly uniform but relativistic acceleration. Accordingly, the distance the sail has traversed is

$$ L = \frac{a_{\text{const}}^2}{1 + \frac{1}{\sqrt{1 - \beta_{\text{arg}}^2}}}, \quad \text{(40)} $$

where $a_{\text{const}}$ is the constant acceleration of the sail as observed in the laser’s reference frame, given by $c \beta_{\text{arg}} / t$. Using the above, the fractional power striking the sail is now

$$ \text{erf}^{-1} \sqrt{F_P} = \sqrt{\frac{\alpha}{4\pi}} \frac{D_T}{c} \left( 1 + \frac{1}{\sqrt{1 - \beta_{\text{arg}}^2}} \right). \quad \text{(41)} $$

One may now replace $t$ in the above with Equation (37) to relate the power loss at a given target velocity as a function of the basic sail properties. Before doing so, we present a quick order-of-magnitude calculation by using $L \approx \frac{1}{3} \beta_{\text{arg}} c t$ (non-relativistic) to give $L \approx 2.9$ au. For a $D_T = 10$ m transmitter at 650 nm wavelength, the beam width will be $W_L \approx 40$ km at a distance $L$, and thus we should expect $(16/40,000)^2 \sim 10^{-7}$ fractional power striking the sail. Using the full equation, we obtain similar results, as depicted in Figure 6 for four possible choices of $D_T$.

Our results imply that any firing time of order hours or greater will lead to very large beam losses of at least a million, increasing the energy demands to 10 EJ or more. To keep energy losses within a factor of 10, a kilometer-sized transmitter could fire for 199 s at a sail with an absorptivity satisfying $\tilde{R} < 3.9 \times 10^{-6}$. These calculations argue that the key technical requirements for Starshot are a $\tilde{R} = 10^{-6}$ sail and kilometer-sized lasers achieving 500 GW power over a firing time of a few minutes.

### 4. Discussion

We have derived an equation for the velocity change of a relativistic moving mirror (or, equivalently, a light sail) in response to an ensemble of normal incident photons, as well as the corresponding redshift of the reflected photons. While our formulae for the cases of a perfectly reflective and perfectly absorptive mirror are exact, our formula for a mirror with a reflectivity in the range $0 < R < 1$ should be treated as an excellent approximation rather than a formal truth, and we suggest that the solution may in fact be intractable without approximation.

Crucially, our expression for the velocity curve differs from that stated in Lubin (2016), which motivates the Starshot project (Popkin 2017). The Lubin (2016) result is derived in Kulkarni et al. (2016), who use Lorentz frame transfers and assume that, in the sail’s rest frame, the frequency of reflection equals that of incidence. We have shown that this assumption, also made by Einstein in his seminal 1905 paper introducing relativity, violates the conservation of energy because the sail must increase its momentum (and thus kinetic energy) in response to the reflection; thus, the photons must lose energy by becoming redshifted. Since this treatment overestimates the photon’s final energy, it also underestimates the sail’s velocity. While the effect is negligible ($< 10^{-3}$) for an individual photon, as tacitly assumed by Einstein, Starshot requires $\sim 10^{31}$ reflections, causing this error to accumulate to a 10% difference in our predicted final velocity versus that of Lubin (2016).

Our equations provide an analytic framework to predict the acceleration of a light sail under solar or laser irradiation up to relativistic speeds, as appropriate for the Starshot project, for example.

A useful result from our work is that the relativistic velocity curve from a large number of incident photons can be described analytically as that of a single photon with the equivalent energy of the ensemble for either elastic or inelastic collisions. This insight, which we have used several times and referred to as the principle for ensemble equivalence for convenience, is demonstrated in Section 2.5 and provides a simple analytic approach for modeling sail response functions.

Additionally, we have discussed how the high levels of incident radiation on the sail, necessary to achieve relativistic speeds, will put thermal stress on the sail and payload. Practically speaking, the ideal sail material should be ultralight, rigid against radiation pressure inhomogeneities, thermally stable up to hundreds of Kelvin, and ultra-reflective. To avoid losing more than a factor of 10 of the laser power through diffraction losses, we find that a kilometer-sized transmitter needs to fire for 3.3 minutes or less, exacerbating the thermal stress on the sail. For such a case, we estimate that the absorptivity needs to be less than $4 \times 10^{-6}$ and able to operate at 300°C (573 K).
There are numerous effects we have ignored that will further influence the design requirements for Starshot. For example, additional beam losses due to scattering through the Earth’s atmosphere will certainly lead to a higher laser power output requirement than that estimated here. In terms of the sail itself, at least three effects we have ignored will influence the sail’s velocity. First, drag forces from interstellar dust and even photonic gas (Balasanyan & Mkrtchian 2009) will act to slowly decelerate the sail. Second, we have assumed that the reflection coefficient is achromatic; but, in reality, man-made highly reflective surfaces, such as dielectric mirrors, are often extremely sensitive to wavelength (Rempe et al. 1992). Third, we have assumed that the sail and spacecraft chassis are in thermal equilibrium from the first incident photon to the last; however, in reality, some of this energy will not be reradiated but instead used to warm up the chassis, potentially leading to material deformations, for example. We highlight these problems for the community for future work.

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Erratum: “Relativistic Light Sails” (2017, AJ, 153, 277)

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We correct an error in the conversion from photon reflection number to time when comparing our paper’s formalism for a relativistic light sail’s velocity (Equation (14)) to that of Lubin (2016) and Kulkarni et al. (2016), as spotted by Kulkarni et al. (2017).

In the original version, we neglected to account for the time delay caused by light travel time effects. Because our derivation is in terms of photon reflection number (Section 2.1 to Section 2.5), these parts are unaffected by the conversion error. Only the purple dashed line on Figure 4 is incorrect as a result of this error, and the text in Section 2.6, which discusses the magnitude of the difference between the two formalisms.

To account for light travel time, one may show that the time of the nth photon reflection on the sail is

\[ t_n = t_0 + \delta t \sum_{j=1}^{n} (1 - \beta_{j-1})^{-1}, \tag{1} \]

where \( t_0 \) is a reference time and \( \delta t \) represents the time between each photon emission (assuming a uniform rate, i.e., constant power). Assuming that the sail begins from rest at time \( t_0 \), we may use Equation (14) to write that

\[ t_n = t_0 + \delta t \left( n + nr(n - 1) + nr^2 \left( \frac{1}{3} - n + \frac{2}{3} n^2 \right) \right). \tag{2} \]

We may now re-arrange the above to make \( n \) the subject and replace the real root of the resulting cubic into Equation (14) to give us a formula for \( \beta \) as a function of time under constant laser power. Our formula, written as a function of time, is compared directly to that of Lubin (2016) and Kulkarni et al. (2016) in the right panel of Figure 1. Although the two equations show close agreement for the fiducial choice of parameters in Figure 1, they are not equivalent—as evident from the residual plot in that figure. Specifically, our formula predicts a slightly higher acceleration, due to the additional recoil accounted for by the photon reflections ignored in the Lubin (2016) formalism.

![Figure 1](https://doi.org/10.3847/1538-3881/aaa461)

Figure 1. Velocity curve for a Starshot-like choice of parameters comparing the difference between the predictions of the literature formula of Lubin (2016) and Kulkarni et al. (2016) and this work. Formulae show close agreement although slight residuals (lower right) are present as a result of the infinite sail mass assumption of Lubin (2016).
Although both formulae are fairly unwieldy when expressed in terms of time, we can take the difference between them (the residuals) and perform a series expansion in $t'$. This leads to the following expression for the difference between the two:

$$\beta[\text{this work}] - \beta[\text{Lubin 2016}] = 2\pi \left(t' - 4t'^2 + \frac{28}{3}t'^3 + \mathcal{O}(t'^4)\right).$$

For a target speed of $0.2c$, this corresponds to a difference of 47 m/s, which would change the arrival time at Proxima b by 8.7 minutes. Given planet b’s orbital velocity, this would cause the planet to be in a different location by 25,000 km, or around four Earth radii.

Aside from the comparison in Figure 4 of the original version, the only other slight change in our results is that the temperatures shown in Figure 5 should be interpreted as the temperature the sail rises to during the early phases of the acceleration (when time lags are minimal). As the sail travels, it will cool due to time lags equating to a dissipative effect on the heating. This does not affect the results though as thermal design constraints must meet the maximum heating stress regardless.

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