VIRTUAL CLASSES OF ARTIN STACKS

FLAVIA POMA

Abstract. We construct virtual fundamental classes of Artin stacks over a Dedekind domain endowed with a perfect obstruction theory.

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1. Introduction

Virtual classes of moduli stacks play a central role in enumerative geometry as they represent a major ingredient in the construction of deformation invariants (e.g. Gromov-Witten, Donaldson-Thomas). They were introduced in [5] for Deligne-Mumford stacks; the construction has been applied to develop Gromov-Witten theory of algebraic varieties and Deligne-Mumford stacks, as well as Donaldson-Thomas theory (see [4], [6], [1], [2], [17]).

1.1. The existing construction for DM stacks. A virtual class of an algebraic stack $X$ is a functorial cycle class $[X]^{\text{virt}}$ in the Chow group $A^*(X)$ with rational coefficients. The idea of the construction comes from observing the following two facts of intersection theory of schemes and relies on the correspondence between abelian cone stacks and 2-term complexes of coherent sheaves (Theorem 2.4).

(1) Whenever a scheme $X$ is pure dimensional of pure dimension $N$, we can define a fundamental class in $A_N(X)$ as $[X] = \sum_i m_i [X_i]$, where the sum is over the irreducible components of $X$ and $m_i$ is the multiplicity of $X_i$ in $X$. If $i: X \to M$ is a closed immersion and there exists a closed immersion $j: C_i \to E$ of the normal cone $C_i$ in a vector bundle $\pi: E \to X$ (e.g., if $i$ is regular then the normal sheaf $N_i$ is a vector bundle) then $[X] = 0 [C_i]$, where $0$ is the inverse of $\pi^*$ ([9] 6.2.1).

(2) Let $f: X \to Y$ be a morphism of schemes which factors as $p \circ i$ with $i$ closed immersion and $p$ smooth. If $f$ is a closed immersion then $N_f = N_{i \circ p}$ and $C_f = C_{i \circ p}$, where $T_p$ is the tangent bundle. In general, the morphism $i^*T_p \to N_i$ is not injective ([9] B.7.5).

The idea is to generalize (1) to morphisms of stacks. For this we need to construct an intrinsic normal cone $\mathcal{C}_f$ associated to a morphism $f: \mathcal{X} \to \mathcal{Y}$ of algebraic stacks and a closed immersion $\mathcal{E}_f \to \mathcal{C}_f$ in a vector bundle stack $\mathcal{E}_f$ over $\mathcal{X}$. Locally $f$ factors as $p \circ i$, then we can define $\mathcal{C}_f$ locally as the quotient stack $[C_{i \circ p}]$ and then glue these together. To make the gluing process feasible, we construct an intrinsic normal sheaf $\mathcal{N}_f$ such that $\mathcal{C}_f$ is a subcone of $\mathcal{N}_f$. The stack $\mathcal{N}_f$ is obtained globally via the correspondence mentioned above as the stack associated to the...
cotangent complex $L_i^*$ of $f$ \cite{11, 15, 12}. Finally, the existence of a closed immersion $\mathcal{E}_f \to \mathcal{E}_f$ is reduced, via the correspondence above, to the existence of a perfect obstruction theory.

1.2. Obstructions for Artin stacks. The cotangent complex of an Artin stack has three terms, so that one cannot exploit directly the above correspondence to get the intrinsic normal sheaf. A first step in this direction was done by Francesco Nosedda in his PhD thesis \cite{14}, even though his construction was not completely proven to be intrinsic and therefore may depend on the chosen resolution of the perfect obstruction theory (this point is crucial to prove the functoriality of the virtual fundamental class). To our knowledge, no construction of the virtual fundamental class of an Artin stack has been done so far.

1.3. Virtual classes of Artin stacks. Let $f: \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We define the intrinsic normal sheaf $\mathcal{N}_f$ as the cone stack associated to the truncation $\tau_{[-1,0]}L_i^*$ of the cotangent complex of $f$ \cite{2.9}. We notice that there exists a smooth atlas $u: U \to \mathcal{X}$ such that $f \circ u = p \circ i$ \cite{2.11}, then

$$u^*\tau_{[-1,0]}L_i^* = [\mathcal{X}/\mathcal{Y} \to \ker(i^*\Omega_p \to \Omega_u)],$$

and hence $u^*\mathcal{N}_f = [C/\mathcal{Y} \to \ker(i^*\Omega_p \to \Omega_u)]$. As a consequence we can define the intrinsic normal cone $\mathcal{C}_f$, locally in the lisse-étale topology, as $[C/\mathcal{Y}]$ \cite{2.12}. If $f$ is of Deligne-Mumford type then we can assume $u$ to be étale, therefore $\Omega_u = 0$ and our construction of the intrinsic normal cone above coincides with the one in \cite{3}. Regarding obstruction theories, we prove an infinitesimal criterion based on the deformation theory of morphisms of stacks \cite{3.3}. It should be mentioned that another crucial tool for the construction is intersection theory for Artin stacks: Chow groups for Artin stacks over a field are defined in \cite{10} and we verified that Kresch’s theory naturally extends to stacks over a Dedekind domain. As a consequence we get that Manolache’s construction of the virtual pullback in \cite{13} applies to morphisms of Artin stacks over a Dedekind domain. In \cite{13} Manolache uses the virtual pullback to give a short proof of Costello’s pushforward formula \cite{7} 5.0.1. Here we apply Manolache’s construction to prove the pushforward formula in a more general setting \cite{5.2}.

1.4. Possible applications. The main motivation for this work comes from the following applications: Gromow-Witten invariants of tame Artin stacks and Donaldson-Thomas invariants in presence of semi-stable sheaves. In both cases the relevant moduli stack is algebraic but not Deligne-Mumford, therefore the construction presented here provides an associated virtual fundamental class. We hope to return on these points in a future paper.

Acknowledgements. I would like to thank Prof. Burt Totaro for pointing out a mistake in Remark 2.3 his valuable comments helped me removing from section 2 the assumption of $\mathcal{X}$ admitting a stratification by global quotients.

Notations. All stacks are Artin stacks in the sense of \cite{3}, \cite{11} and are of finite type over a Dedekind domain. If $\mathcal{X}$ is an algebraic stack, we denote by $\mathcal{X}_{\text{lis-ét}}$ the lisse-étale topos of $\mathcal{X}$. If $F$ is a sheaf on $\mathcal{X}_{\text{lis-ét}}$, we denote by $C(F)$ the associated abelian cone over $\mathcal{X}$.

2. The intrinsic normal cone

2.1. Abelian cone stacks. Let $\mathcal{X}$ be an algebraic stack. A cone stack over $\mathcal{X}$ is an algebraic $\mathcal{X}$-stack $\mathcal{C}$ (together with a section and an $\mathcal{A}^1$-action) such that, lisse-étale locally on $\mathcal{X}$, $\mathcal{C}$ is the quotient stack $[\mathcal{C}/E]$ of a cone $C$ by a vector bundle $E$ on $\mathcal{X}$. A morphism of cone stacks is an $\mathcal{A}^1$-equivariant morphism of $\mathcal{X}$-stacks. A 2-isomorphism of cone stacks is an $\mathcal{A}^1$-equivariant 2-isomorphism. A cone stack $\mathcal{C}$ is abelian if, lisse-étale locally on $\mathcal{X}$, $\mathcal{C} \cong [\mathcal{C}/E]$, where $C$ is an abelian cone. An abelian cone stack $\mathcal{C}$ is a vector bundle stack if, lisse-étale locally on $\mathcal{X}$, $\mathcal{C} \cong [\mathcal{C}/E]$, where $C$ is a vector bundle (for further details see \cite{15} 1.8–1.9). We denote by $(\text{AbCones}/\mathcal{X})$ the 2-category of abelian cone stacks over $\mathcal{X}$.
2.2. NOTATION. Let $\mathcal{C}^{[-1,0]}(\text{Coh}(\mathcal{X}_{\text{lisse}}))$ be the category of complexes of coherent sheaves in the topos $\mathcal{X}_{\text{lisse}}$ with cohomology sheaves concentrated in degree $-1$ and $0$. Let $\mathcal{C}_{\text{coh}}^{[-1,0]}(\mathcal{X}_{\text{lisse}})$ be the category of complexes of sheaves in the topos $\mathcal{X}_{\text{lisse}}$ with coherent cohomology concentrated in degrees $-1$, $0$. We can view $\mathcal{C}^{[-1,0]}(\text{Coh}(\mathcal{X}_{\text{lisse}}))$ and $\mathcal{C}_{\text{coh}}^{[-1,0]}(\mathcal{X}_{\text{lisse}})$ as 2-categories, where the 2-morphisms are homotopies ([5] 2); we denote by $\mathcal{D}^{[-1,0]}(\text{Coh}(\mathcal{X}_{\text{lisse}}))$ and $\mathcal{D}_{\text{coh}}^{[-1,0]}(\mathcal{X}_{\text{lisse}})$ the associated derived categories.

2.3. REMARK. Let $U \to \mathcal{X}$ be a smooth affine atlas of $\mathcal{X}$, in particular $U$ has the resolution property. It follows that every complex $E^\bullet$ in $\mathcal{D}^{[-1,0]}(\text{Coh}(\mathcal{X}_{\text{lisse}}))$ can be represented, locally in the lisse-étale topology on $\mathcal{X}$, as a 2-terms complex $[E^1 \to E^0]$ such that $E^0$ is locally free.

2.4. THEOREM. There is an equivalence of categories

$$\hat{h}^! / \hat{h}^* : \mathcal{D}_{\text{coh}}^{[-1,0]}(\mathcal{X}_{\text{lisse}}) \to \text{Ho}(\text{AbCones}/\mathcal{X})$$

between $\mathcal{D}_{\text{coh}}^{[-1,0]}(\mathcal{X}_{\text{lisse}})$ and the homotopy category of $(\text{AbCones}/\mathcal{X})$, induced by the morphism

$$\hat{h} : \mathcal{C}^{[-1,0]}(\text{Coh}(\mathcal{X}_{\text{lisse}}))^{\text{opp}} \to (\text{AbCones}/\mathcal{X})$$

such that $\hat{h}([E^1 \to E^0]) = [\mathcal{C}(E^1)/\mathcal{C}(E^0)]$.

2.5. REMARK. Let $[E^1 \to E^0], [F^1 \to F^0] \in \mathcal{C}^{[-1,0]}(\text{Coh}(\mathcal{X}_{\text{lisse}}))$. If $\mathcal{X} : E^0 \to F^1$ is a homotopy of morphisms $\psi, \varphi : E^\bullet \to F^\bullet$, then the 2-morphism $\hat{h}(\mathcal{X}) : \hat{h}(\psi) \to \hat{h}(\varphi)$ is defined in the following way. For every $\mathcal{X}$-scheme $U$ and every $(P, f) \in \hat{h}(F^\bullet)(U)$, let $\{U_i\}$ be an open cover of $U$ such that $U_i \times_U P \cong U_i \times_{\mathcal{X}} C(F^0)$, then

$$\hat{h}(\mathcal{X})(U, P, f) : \hat{h}(\psi)(U, P, f) \to \hat{h}(\varphi)(U, P, f)$$

is obtained by gluing the isomorphisms

$$U_i \times_{\mathcal{X}} C(E^0) \xrightarrow{(\text{id}_{U_i} \times \mathcal{X}) \circ f_i} U_i \times_{\mathcal{X}} C(F^0),$$

where $\mathcal{X}$ is the morphism of cones induced by $\mathcal{X}$, $f_i = f|_{U_i \times_U P}$ and $p_1, p_2$ are the natural projections. In particular $\hat{h}(\mathcal{X})$ is a 2-isomorphism.

2.6. USEFUL CRITERION. Let $\psi : E^\bullet \to F^\bullet$ be a morphism in $\mathcal{C}^{[-1,0]}(\text{Coh}(\mathcal{X}_{\text{lisse}}))$ that induces an isomorphism $C(F^0) \cong C(F^1)|_{C(F^0)}$ and a surjective morphism

$$C(d_E) + C(\psi^{-1}) : C(F^1) \times_{\mathcal{X}} C(F^0) \to C(F^0),$$

then $\hat{h}(\psi)$ is an isomorphism of cone stacks.

Proof. We can assume that $C(\psi^{-1})$ is surjective (since $\hat{h}(F^\bullet) \cong \hat{h}(F^\bullet \oplus E^1)$). Therefore the statement is implied by the following cartesian diagram

$$\begin{align*}
\text{C}(F^1) \times_{\mathcal{X}} C(F^0) & \twoheadrightarrow C(F^0) \\
\text{C}(F^1) \times_{\mathcal{X}} C(F^1) & \twoheadrightarrow C(F^1) \times_{\mathcal{X}} C(F^0) \to C(F^0) \\
\text{C}(F^0) & \twoheadrightarrow C(F^0) \to \hat{h}(E^\bullet)
\end{align*}$$

(the upper square is cartesian because $C(F^0) = C(F^1) \times_{C(F^0)} C(F^0)$).

2.7. LEMMA. The natural functor

$$\mathcal{D}_{\text{coh}}^{[-1,0]}(\text{Coh}(\mathcal{X}_{\text{lisse}})) \to \mathcal{D}_{\text{coh}}^{[-1,0]}(\mathcal{X}_{\text{lisse}})$$

is an equivalence of categories.
Proof. The statement can be checked locally in the lisse-étale topology on $\mathcal{X}$, therefore we can assume that $\mathcal{X}$ has the resolution property. First we show that the functor is fully faithful. Let $E^\bullet, F^\bullet \in \text{D}^{-1,0}(\text{Coh}(\mathcal{X}_{\text{lis-ét}}))$, we want to show that the canonical map

$$\text{Hom}_{\text{D}^{-1,0}(\text{Coh}(\mathcal{X}_{\text{lis-ét}}))}(E^\bullet, F^\bullet) \rightarrow \text{Hom}_{\text{D}^{-1,0}_{\text{coh}}(\mathcal{X}_{\text{lis-ét}})}(E^\bullet, F^\bullet)$$

is a bijection. We can assume $E^\bullet = [E^{-1} \xrightarrow{d_E} E^0]$ and $F^\bullet = [F^{-1} \xrightarrow{d_F} F^0]$. We can reduce to the case where $E^\bullet$ is a coherent sheaf $E$ (similarly $F^\bullet = F$), using the following distinguished triangle

$$E^{-1} \xrightarrow{d_E} E^0 \rightarrow E^\bullet \xrightarrow{+1} E^{-1}[1].$$

By resolution property, there exists a surjective morphism $\psi: F^0 \rightarrow E$ from a locally free sheaf $P^0$; then $[\ker \psi \rightarrow P^0]$ is a complex of locally free sheaves quasi-isomorphic to $E$ and using the argument above we can reduce to the case where $E^\bullet$ is a locally free sheaf $E$. Let $E^\bullet = E/\mathcal{O}_\mathcal{X}$, then $rkE < rkE$, hence we can reduce to $E = \mathcal{O}_\mathcal{X}$. That is, we are left to show that

$$\text{Hom}_{\text{D}^{-1,0}(\text{Coh}(\mathcal{X}_{\text{lis-ét}}))}(\mathcal{O}_\mathcal{X}, F[n]) \rightarrow \text{Hom}_{\text{D}^{-1,0}_{\text{coh}}(\mathcal{X}_{\text{lis-ét}})}(\mathcal{O}_\mathcal{X}, F[n])$$

is a bijection for every coherent sheaf $F$ and $n = -1, 0$. If $n = -1$, both groups are zero. If $n = 0$ then both sides are $\Gamma(\mathcal{X}, F)$.

Let us show that every complex $E^\bullet \in \text{D}^{-1,0}_{\text{coh}}(\mathcal{X}_{\text{lis-ét}})$ is in the essential image. We can assume $E^\bullet = [E^{-1} \xrightarrow{d_E} E^0]$. We have the following exact sequence of complexes of sheaves

$$0 \rightarrow h^{-1}(E^\bullet)[1] \rightarrow E^\bullet \rightarrow [\text{im } d_E \rightarrow E^0] \rightarrow 0,$$

which induces a distinguished triangle

$$h^{-1}(E^\bullet)[1] \rightarrow E^\bullet \rightarrow [\text{im } d_E \rightarrow E^0] \xrightarrow{[-1]} h^{-1}(E^\bullet)[2].$$

Notice that $[\text{im } d_E \rightarrow E^0] = h^0(E^\bullet)$ in $\text{D}^{-1,0}_{\text{coh}}(\mathcal{X}_{\text{lis-ét}})$. Then we have a distinguished triangle

$$h^{-1}(E^\bullet)[1] \rightarrow E^\bullet \rightarrow h^0(E^\bullet) \xrightarrow{[-1]} h^{-1}(E^\bullet)[2].$$

Since $h^0(E^\bullet)$ and $h^{-1}(E^\bullet)$ are coherent, the morphism $h^0(E^\bullet)[-1] \xrightarrow{[-1]} h^{-1}(E)[1]$ corresponds to a morphism $\psi: h^0(E^\bullet)[-1] \rightarrow h^{-1}(E)[1]$ in $\text{D}^{-1,0}(\text{Coh}(\mathcal{X}_{\text{lis-ét}}))$. Completing $\psi$ to a distinguished triangle in $\text{D}^{-1,0}(\text{Coh}(\mathcal{X}_{\text{lis-ét}}))$ and mapping it to $\text{D}^{-1,0}_{\text{coh}}(\mathcal{X}_{\text{lis-ét}})$, we deduce that $E^\bullet$ is quasi-isomorphic to the mapping cone of $\psi$, hence it is in the essential image. \hfill □

Proof of Theorem 2.4. We need to prove the following facts: (1) $\psi: E^\bullet \rightarrow F^\bullet$ is a quasi-isomorphism if and only if $\hat{h}(\psi)$ is an isomorphism; (2) if $\psi, \varphi: E^\bullet \rightarrow F^\bullet$ are morphisms of complexes and $\xi: \hat{h}(\psi) \rightarrow \hat{h}(\varphi)$ is an $H^1$-equivariant 2-isomorphism then there exists a unique homotopy $\kappa: \psi \rightarrow \varphi$ such that $\xi = \hat{h}(\kappa)$; (3) $h^1/h^0$ is essentially surjective. By Remark 2.3 and Lemma 2.4, we can assume $E^\bullet = [E^{-1} \xrightarrow{d_E} E^0]$ and $F^\bullet = [F^{-1} \xrightarrow{d_F} F^0]$ with $E^0, F^0$ locally free.

For the second statement, we define a morphism $C(\mathcal{X}): C(F^{-1}) \rightarrow C(E^0)$ as follows. Let $T$ be an $\mathcal{X}$-scheme and let $f: T \rightarrow C(F^{-1})$ be a morphism. Then $f$ defines an element $(P = T \times_\mathcal{X} C(F^0), f_P)$ in $\hat{h}(F^\bullet)(T)$. The images $\hat{h}(\psi)(T)(P, f_P)$ and $\hat{h}(\varphi)(T)(P, f_P)$ are trivial, hence $\xi(T)(P, f_P)$ corresponds to a morphism $g: T \rightarrow C(E^0)$. We define $C(\mathcal{X})(T)(f) = g$. Then $C(\mathcal{X})$ induces a homomorphism $\mathcal{X}: E^0 \rightarrow F^{-1}$. Moreover $\xi = \hat{h}(\mathcal{X})$ and $\mathcal{X}$ is unique by construction.

Let us now prove the first statement. Let $G = E^0 \times_{F^0} F^{-1}$, then

$$0 \rightarrow G \rightarrow E^0 \oplus F^{-1} \rightarrow F^0$$

is exact. Notice that $E^0 \oplus F^{-1} \rightarrow F^0$ is surjective if and only if $h^0(\psi)$ is surjective. Let us assume that $h^0(\psi)$ is surjective, then we get an exact sequence of cones

$$0 \rightarrow C(F^0) \rightarrow C(E^0) \times C(F^{-1}) \rightarrow C(G) \rightarrow 0.$$
Applying the useful criterion 2.8 we obtain \([C(F^{-1})/C(F^0)] \cong [C(G)/C(E^0)]\), hence the following diagram

\[
\begin{array}{ccc}
C(G) & \longrightarrow & C(E^{-1}) \\
\downarrow & & \downarrow \\
h^1/h^0(F^\bullet) & \longrightarrow & h^1/h^0(E^\bullet)
\end{array}
\]

is cartesian and in particular \(h^1/h^0(\psi)\) is representable. If moreover \(h^0(\psi)\) is an isomorphism and \(h^{-1}(\psi)\) is surjective, then the morphism \(E^{-1} \to G\) is surjective, hence \(C(G) \to C(E^{-1})\) is a closed immersion, which implies that \(h^1/h^0(\psi)\) is a closed immersion. If \(h^{-1}(\psi)\) is also an isomorphism then \(E^{-1} \to G\) is an isomorphism and so \(C(G) \cong C(E^{-1})\); it follows that \(h^1/h^0(\psi)\) is an isomorphism. Viceversa, if \(h^1/h^0(\psi)\) is representable then the induced morphism on automorphisms of objects is injective. Hence we have that the morphism

\[
C(h^0(\psi)): C(h^0(F^\bullet)) \to C(h^0(E^\bullet))
\]

is a closed immersion, which implies that \(h^0(\psi)\) is surjective. If moreover \(h^1/h^0(\psi)\) is a closed immersion then \(C(G) \to C(E^{-1})\) is a closed immersion, hence \(E^{-1} \to G\) is surjective. It follows that \(h^0(\psi)\) is injective and \(h^{-1}(\psi)\) is surjective. Finally, if \(h^1/h^0(\psi)\) is an isomorphism then \(C(G) \cong C(E^{-1})\), hence \(E^{-1} \cong G\), from which we get that \(h^{-1}(\psi)\) is injective. It follows that \(h^1/h^0\) is fully faithful.

It remains to show that every abelian cone stack \(\mathcal{C}\) over \(\mathcal{X}\) is in the essential image of \(h^1/h^0\). By definition, lisse-étale locally on \(\mathcal{X}\), \(\mathcal{C} \times \cong [C(E^\bullet)/C(E^0)]\), where \(E^{-1}_U\) is a coherent sheaf and \(E^0_U\) is a locally free sheaf. The collection \(\{E^{-1}_U \to E^0_U\}_U\) defines a complex \([E^{-1} \to E^0] \in D^{-1,0}\text{co}h(\mathcal{X}_{\text{lis-ét}}).\) \(\square\)

2.8. Cotangent Complex [11 17.3, [15], [12] 2.2.5]. Let \(f: \mathcal{X} \to \mathcal{Y}\) be a quasi-compact and quasi-separated morphism of algebraic stacks. There exists \(L_f^* \in D^{\leq 0}_{\text{qcoh}}(\mathcal{X}_{\text{lis-ét}})\) such that

1. \(f\) is of Deligne-Mumford type if and only if \(L_f^* \in D_{\text{qcoh}}(\mathcal{X}_{\text{lis-ét}})\);
2. \(f\) is of Deligne-Mumford type and smooth if and only if \(L_f^* = \Omega_f\);
3. for every commutative diagram

\[
\begin{array}{ccc}
\mathcal{X}' & \xrightarrow{f'} & \mathcal{Y}' \\
\downarrow & & \downarrow \psi \\
\mathcal{X} & \xrightarrow{f} & \mathcal{Y}
\end{array}
\]

there exists a morphism \(Lg^*L_f^* \to L_{f'}^*\); if the diagram is cartesian and either \(h\) of \(f\) is flat, this morphism is an isomorphism;
4. given two morphisms of \(S\)-stacks \(\mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} Z\) with \(h = g \circ f\), there exists a natural distinguished triangle

\[
Lg^*L_f^* \to L_h^* \to L_f^* \to Lf^*L_g^*[1];
\]
5. if \(f\) factors as \(\mathcal{X} \xrightarrow{i} M \xrightarrow{p} \mathcal{Y}\) with \(i\) representable and a closed embedding with ideal sheaf \(\mathcal{I}\) and \(p\) of Deligne-Mumford type and smooth, then

\[
\tau_{g-1}L_f^* \cong [\mathcal{I}/\mathcal{I}^2 \to i^*\Omega_p].
\]

2.9. Definition. Let \(f: \mathcal{X} \to \mathcal{Y}\) be a morphism of algebraic stacks. We define the intrinsic normal sheaf of \(f\) as the abelian cone stack \(\mathfrak{N}_f = h^1/h^0(\tau_{-1,0}L_f^*).\)
2.10. Definition. A local embedding of \( f: X \to Y \) is a commutative diagram

\[
\begin{array}{ccc}
U & \xrightarrow{u} & X \\
\downarrow & & \downarrow f \\
M & \xrightarrow{p} & Y
\end{array}
\]

where \( u, p \) are smooth and representable and \( i \) is a closed embedding.

2.11. Remark. Notice that there exists a local embedding of \( f \). More explicitly, let \( V \) be a smooth atlas for \( Y \) and let \( U \) be an affine scheme which is a smooth atlas for \( X \times_Y V \). In particular there exists a closed embedding \( j: U \hookrightarrow \mathbb{A}^n \). Let us set \( M = \mathbb{A}^n_Y \), then \( f \circ u \) factors as \( U \xrightarrow{i} M \xrightarrow{p} Y \), where \( i \) is a closed embedding and \( p \) is smooth and representable. Applying 2.8 (4)–(5) to \( f \circ u = p \circ i \) and snake lemma, we get a quasi-isomorphism

\[
\tau_{[-1,0]}^i \mathcal{L} u^* \mathcal{L}^\bullet \to \tau_{[-1,0]}(\mathcal{L}/\mathcal{L}^2 \to i^* \Omega_p \to \Omega_u)
\]

(\( \mathcal{L} \) being the ideal sheaf corresponding to \( i \)), which, by Theorem 2.4, implies

\[
u^* \mathfrak{N}_f = h^i / h^0 \tau_{[-1,0]}(\mathcal{L}/\mathcal{L}^2 \to i^* \Omega_p \to \Omega_u)) = [\mathcal{N}/(\mathcal{I}^* \tau_p/\mathcal{I}_u)],
\]

where \( \mathcal{N}_i = C(\mathcal{L}/\mathcal{L}^2) \) is the normal sheaf of \( i \).

2.12. Proposition. There exists a unique closed subcone stack \( \mathcal{C}_f \subseteq \mathfrak{N}_f \) such that

\[
u^* \mathcal{C}_f = [\mathcal{C}_i/\mathcal{I}^* \tau_p/\mathcal{I}_u],
\]

for every local embedding of \( f \) (\( \mathcal{C}_i \) being the normal cone of \( i \)). If moreover \( Y \) is purely dimensional of pure dimension \( n \), then \( \mathcal{C}_f \) is purely dimensional of pure dimension \( n \).

Proof. By Remark 2.11 there exists a local embedding \( f \circ u = p \circ i \) of \( f \) and \( \nu^* \mathfrak{N}_f = [\mathcal{N}_i/\mathcal{I}^* \tau_p/\mathcal{I}_u] \).

By 3.2, the action of \( i^* T_p \) on \( \mathcal{N}_i \) leaves \( \mathcal{C}_i \) invariant, hence we can define the quotient stack \([\mathcal{C}_i/\mathcal{I}^* \tau_p/\mathcal{I}_u]\) which is a closed subcone stack of \( \nu^* \mathfrak{N}_f \) (since \( \mathcal{C}_i \) is a closed subcone of \( \mathcal{N}_i \)). Let us show that \([\mathcal{C}_i/\mathcal{I}^* \tau_p/\mathcal{I}_u]\) does not depend on the factorization \( p \circ i \) chosen. Let \( U \xrightarrow{i} M \xrightarrow{p} Y \) be another factorization of \( f \circ u \). It is enough to check \([\mathcal{C}_i/\mathcal{I}^* \tau_p/\mathcal{I}_u]\) as closed substacks of \( \nu^* \mathfrak{N}_f \), where \( U \xrightarrow{i} M \times_Y M \xrightarrow{q} M \) is the induced local embedding.

By Remark 2.11 both \( \mathcal{N}_i \) and \( \mathcal{N}_j \) are smooth atlases of \( \nu^* \mathfrak{N}_f \). We get the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{C}_j & \xrightarrow{\alpha_j} & \mathcal{N}_j \\
\downarrow \phi & & \downarrow \alpha_i \\
\mathcal{C}_i & \xrightarrow{\phi} & \mathcal{N}_i
\end{array}
\]

where the left square is cartesian by 4.2.6; therefore it is enough to show that the inverse images of \([\mathcal{C}_i/\mathcal{I}^* \tau_p/\mathcal{I}_u]\) and \([\mathcal{C}_i/\mathcal{I}^* \tau_q/\mathcal{I}_u]\) in \( \mathcal{N}_j \) are the same. We have

\[
\phi^{-1}(\alpha^{-1}_i([\mathcal{C}_i/\mathcal{I}^* \tau_p/\mathcal{I}_u])) = \phi^{-1}(\mathcal{C}_i) = \mathcal{C}_j = \alpha^{-1}_j([\mathcal{C}_j/\mathcal{I}^* \tau_p/\mathcal{I}_u])
\]

Let \( f \circ u' = p' \circ i' \) be another local embedding of \( f \), with \( U' \to X \). We want to show that the cone stacks \([\mathcal{C}_i/\mathcal{I}^* \tau_p/\mathcal{I}_u]\) and \([\mathcal{C}_i/\mathcal{I}^* \tau_q/\mathcal{I}_u]\) agree on \( V = U \times_X U' \). We can find a closed embedding \( j: V \to M'' \) and smooth morphisms of schemes \( q: M'' \to M, q': M'' \to M' \) such that \( i \circ v = q \circ j, i' \circ v = q' \circ j \), where \( v: V \to U \), \( v': V \to U' \) are the projections. It is enough to show that \( v^*[\mathcal{C}_i/\mathcal{I}^* \tau_p/\mathcal{I}_u] = [\mathcal{C}_i/\mathcal{I}^* \tau_q/\mathcal{I}_u] \). By Remark 2.11

\[
v^* \nu^* \mathfrak{N}_f = [\mathcal{N}_i/\mathcal{I}^* \tau_q/\mathcal{I}_u] = v^*[\mathcal{N}_i/\mathcal{I}^* \tau_p/\mathcal{I}_u]
\]
and the following diagram

\[
\begin{array}{c}
N_j \\ \varphi \downarrow \alpha_i \downarrow \\
v^* N_i
\end{array}
\xrightarrow{\alpha_j} v^* u^* \mathcal{R}_f
\]

is commutative; therefore it is enough to show that the inverse images of \( v^*[C_i/(v^* \tau_{p_{ij}}/\tau_{u_{ij}})] \) and \([C_i/(v^* \tau_{p_{ij}}/\tau_{u_{ij}})]\) in \(N_j\) are the same. By Example 4.2.6 in [9], \( \varphi^{-1}(v^* C_i) = C_j \) and hence

\[\varphi^{-1}(\alpha_i^{-1}(v^*[C_i/(v^* \tau_{p_{ij}}/\tau_{u_{ij}})])) = \varphi^{-1}(v^* C_i) = C_j = \alpha_j^{-1}([C_j/(v^* \tau_{p_{ij}}/\tau_{u_{ij}})]).\]

Finally, let us assume that \( \mathcal{Y} \) is purely dimensional of pure dimension \( n \). Let \( f \circ u = p \circ i \) be a local embedding of \( f \), then \( u^* \mathcal{C}_f = [C_i/(v^* \tau_{p_{ij}}/\tau_{u_{ij}})] \). We can assume that \( M \) is purely dimensional of pure dimension \( m \) and \( p, u \) are smooth of relative dimension \( m - n \) (see the construction in Remark 2.11). By [9], B.6.6, we have that \( C_i \) is purely dimensional of pure dimension \( m \). Moreover, we have the following cartesian diagram

\[
(i^* T_{r_1}/\tau_{u_{ij}}) \times C_i \longrightarrow C_i \\
\downarrow \downarrow \\
C_i \longrightarrow [C_i/(v^* \tau_{p_{ij}}/\tau_{u_{ij}})]
\]

from which we get that \( C_i \rightarrow [C_i/(v^* \tau_{p_{ij}}/\tau_{u_{ij}})] \) is surjective and smooth of relative dimension \( 0 \). It follows that \( u^* \mathcal{C}_f \) is purely dimensional of pure dimension \( m \) and hence \( \mathcal{C}_f \) is purely dimensional of pure dimension \( n \).

2.13. Definition. The closed subcone stack \( \mathcal{C}_f \) of \( \mathcal{R}_f \) is called the intrinsic normal cone of \( f \).

2.14. Proposition. Consider the following commutative diagram of algebraic stacks

\[
\begin{array}{c}
\mathcal{X}' \xrightarrow{f'} \mathcal{Y}' \\
\gamma \downarrow \downarrow h \\
\mathcal{X} \xrightarrow{f} \mathcal{Y}
\end{array}
\]

Then there exists a natural morphism \( \alpha : \mathcal{C}_f' \rightarrow g^* \mathcal{C}_f \) such that

1. if (1) is cartesian then \( \alpha \) is a closed immersion;
2. if moreover the morphism \( h \) is flat then \( \alpha \) is an isomorphism.

Proof. Let \( U \xrightarrow{\gamma} \mathcal{X} \) and \( U' \xrightarrow{i'} \mathcal{X}' \) be smooth affine atlases with a morphism \( v : U' \rightarrow U \) such that \( u \circ v = g \circ i' \). There exist a commutative diagram

\[
\begin{array}{c}
U' \xrightarrow{i'} M' \xrightarrow{p'} Y' \\
\downarrow \tilde{g} \downarrow \tilde{h} \\
U \xrightarrow{i} M \xrightarrow{p} Y
\end{array}
\]

where \( i, i' \) are closed embeddings, \( p, p' \) are representable smooth and \( M' = M \times_{\mathcal{X}} \mathcal{X}' \). If (1) is cartesian, we can take \( U' = U \times_M M' = U \times_{\mathcal{X}} \mathcal{X}' \).

By 2.8 (3), there is an isomorphism \( T' \rightarrow q^* T_p \), which is an isomorphism \( i'^* T_{p'} \rightarrow v^* i^* T_p \). Moreover there exists a morphism \( v^* \Omega_u \rightarrow h^0(L_{gou'}) \rightarrow \Omega_{u'} \) which induces a morphism \( T_{u'} \rightarrow v^* T_u \); if (1) is cartesian, this morphism is an isomorphism. By [9] B.6, there exists a morphism \( \tilde{\alpha} : C_{u'} \rightarrow v^* C_{u} \), induced by the natural map \( \mathcal{J} \otimes_{\mathcal{O}_u} \mathcal{O}_{u'} \rightarrow \mathcal{J} \). If (1) is cartesian, \( \tilde{\alpha} \) is a closed embedding and, if moreover \( h \) is flat, \( \tilde{\alpha} \) is an isomorphism. Summing up, we get a morphism of stacks

\[
u^* \mathcal{C}_f' = [C_{i'}/(v^* \tau_{p_{ij'}/\tau_{u_{ij'}}})] \xrightarrow{\tilde{\alpha}} v^*[C_i/(v^* \tau_{p_{ij}}/\tau_{u_{ij}})] = v^*(u^* \mathcal{C}_f),
\]

which is a closed immersion if (1) is cartesian; if moreover \( h \) is flat then \( \alpha \) is an isomorphism. \( \square \)
3. Perfect obstruction theories

3.1. Definition. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks and let $E^* \in D_{\text{coh}}^{[-1,1]}(\mathcal{X}_{\text{lis-ét}})$. A morphism $\varphi : E^* \to \tau_{\geq -1}L_f^1$ in $D_{\text{coh}}^{[-1,1]}(\mathcal{X}_{\text{lis-ét}})$ is called an obstruction theory for $f$ if $h^1(\varphi)$, $h^0(\varphi)$ are isomorphisms and $h^{-1}(\varphi)$ is surjective.

3.2. Remark. If $(E^*, \varphi)$ is an obstruction theory for $f$ then, by the proof of Theorem 2.4, the morphism $h^*/h^0(\varphi) : \mathcal{R}_f \to h^*/h^0(\tau_{[-1,0]}E^*)$ is a closed immersion.

3.3. Deformation theory of morphisms of algebraic stacks. For the basic definitions of deformation theory we refer to [10]. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over a base scheme $S$. Let $\text{Spec} \overline{X} \to \mathcal{X}$ be a geometric point of $\mathcal{X}$. Let $\Lambda = \check{O}_{\overline{X}}$. Consider the deformation category $h_{\overline{X}}(\Lambda)$ such that, for all $A \in (\Lambda^t/\Lambda)$, the objects of $h_{\overline{X}}(\Lambda)$ are morphisms $g_{\overline{X}} : \text{Spec} A \to \overline{X}$ such that $g_{\overline{X}}|_{\text{Spec} \overline{X}} = \overline{X}$. There is a natural functor $\nu_f : h_{\overline{X}}(\Lambda) \to h_{\overline{Y}}(\Lambda)$ given by the composition with $f$. Let $g_{\overline{X}} \in h_{\overline{X}}(\Lambda)(A)$, $g_{\overline{Y}} \in h_{\overline{Y}}(\Lambda)(A')$ be such that $g_{\overline{Y}}(i) = f \circ g_{\overline{X}}$, where $i : \text{Spec} A \to \text{Spec} A'$. We denote by $S_f$ the set of isomorphism classes of $g'_{\overline{X}} \in h_{\overline{X}}(\Lambda)$ such that $g'_{\overline{X}} \circ i = g_{\overline{X}}$, $f \circ g'_{\overline{X}} = g_{\overline{Y}}$.

3.4. Theorem. Let $L_f^1$ be the cotangent complex of $f$. Then, for every geometric point $\overline{X}$ of $\mathcal{X}$ and for every small extension $\Lambda' \to \Lambda$ in $(\Lambda^t/\Lambda)$, (1) there is a functorial surjective set-theoretical map

$$\text{obj}_f : h_{\overline{X}}(\Lambda)(\overline{X} \times_{\overline{Y}}(A)) \to h^1((L_{\overline{X}}L_f^1)^{\vee}) \otimes I$$

such that $\text{obj}_f(g_{\overline{X}}, g_{\overline{Y}}) = 0$ if and only if $S_f$ is not empty.

(2) if $\text{obj}_f(g_{\overline{X}}, g_{\overline{Y}}) = 0$ then $S_f$ is a torsor under $h^0((L_{\overline{X}}L_f^1)^{\vee}) \otimes I$.

(3) if $\text{obj}_f(g_{\overline{X}}, g_{\overline{Y}}) = 0$ and $g'_{\overline{X}} \in S_f$, then the group of infinitesimal automorphisms of $g'_{\overline{X}}$ with respect to $(g_{\overline{X}}, g_{\overline{Y}})$ is isomorphic to $h^{-1}((L_{\overline{X}}L_f^1)^{\vee}) \otimes I$.

Proof. Let us consider a local embedding $u : U \to \overline{X}$ of $f$ (see Remark 2.11), then the map $\overline{X} \to \overline{Y}$ factors through $u$. Let us set $f_U = f \circ u$. By 2.8 $h^1((L_{\overline{X}}L_f^1)^{\vee}) \cong h^1((L_{\overline{Y}}L_f^1)^{\vee})$. We have the following exact sequence

$$0 \to h^{-1}((L_{\overline{X}}L_f^1)^{\vee}) \to h^0((L_{\overline{X}}L_f^1)^{\vee}) \to h^0((L_{\overline{Y}}L_f^1)^{\vee}) \to h^0((L_{\overline{X}}L_f^1)^{\vee}) \to 0.$$

Moreover, by deformation theory of schemes, we know that there is a functorial exact sequence

$$0 \to h^0((L_{\overline{Y}}L_f^1)^{\vee}) \otimes I \to h_{U,Y}(\Lambda') \to h_{U,Y}(\Lambda) \times_{h_{Y,Y}(\Lambda)} h_{Y,Y}(\Lambda') \text{obj}_{f_U} \to h^1((L_{\overline{Y}}L_f^1)^{\vee}) \otimes I \to 0.$$

For every $g_{\overline{X}} \in h_{\overline{X}}(\Lambda)(A)$, there exists $g_{U,1} \in h_{U,Y}(\Lambda)$ such that $u \circ g_{U,1} = g_{\overline{X}}$. If $g_{U,2} \in h_{U,Y}(\Lambda)$ is another morphism such that $u \circ g_{U,2} = g_{\overline{X}}$ then there exists a unique morphism $g_{U \times U} \in h_{U \times U}(\Lambda)$ such that $u_j \circ g_{U \times U} = g_{U,j}$ for $j = 1, 2$, where $u_j : U \times U \to U$ are the projections. By 2.8 $u_1$ and $u_2$ induces isomorphisms $h^1((L_{\overline{X}}L_f^1)^{\vee}) \cong h^1((L_{\overline{Y}}L_f^1)^{\vee})$. Therefore

$$\text{obj}_{f_U}(g_{U,j}, g_{\overline{Y}}) = \text{obj}_{f_U}(u_j \circ g_{U \times U}, g_{\overline{Y}}) = \text{obj}_{f_U}(g_{U \times U}, g_{\overline{Y}}).$$

Hence, setting $\text{obj}(g_{\overline{X}}, g_{\overline{Y}}) = \text{obj}_{f_U}(g_{U,1}, g_{\overline{Y}})$, we get a well-defined surjective map

$$\text{obj}_f : h_{\overline{X}}(\Lambda)(\overline{X} \times_{\overline{Y}}(A)) \to h^1((L_{\overline{X}}L_f^1)^{\vee}) \otimes I.$$

Notice that $\text{obj}_f(g_{\overline{X}}, g_{\overline{Y}}) = 0$ if and only if $\text{obj}_{f_U}(g_{U,1}, g_{\overline{Y}}) = 0$ for some $g_{U,1} \in h_{U,Y}(\Lambda)$ such that $u \circ g_{U,1} = g_{\overline{X}}$. If $g_{U,1} \in h_{U,Y}(\Lambda')$ is such that $g_{U,j} \circ i = g_{U,j}$, $f_U \circ g_{U,j} = g_{\overline{Y}}$, then setting $g'_{\overline{X}} = u \circ g_{U,1}$ we have $g'_{\overline{X}} \circ i = g_{\overline{X}}$, $f \circ g'_{\overline{X}} = g_{\overline{Y}}$; viceversa, if $g'_{\overline{X}} \in h_{\overline{X}}(\Lambda')$ is such that $g'_{\overline{X}} \circ i = g_{\overline{X}}$, $f \circ g'_{\overline{X}} = g_{\overline{Y}}$, then there exists $g_{U,1}$ such that $u \circ g_{U,1} = g_{\overline{X}} \circ i$, hence there exists $g'_{\overline{X}}$ such that $g'_{\overline{X}} \circ i = g_{\overline{X}}$, $u \circ g'_{\overline{X}} = g_{\overline{Y}}$.
By (2.8) we have the following commutative diagram with exact rows

$$0 \to h^{-1}(\mathcal{L}^\ast_{f^!}) \to h^0((\mathcal{L}^\ast_{f^!})^\lor) \xrightarrow{\rho_u} h^0((\mathcal{L}^\ast_{f^!}U^\lor) \xrightarrow{\sigma_u} h^0((\mathcal{L}^\ast_{f^!})^\lor) \to 0$$

$$0 \to h^{-1}(\mathcal{L}^\ast_{f^!}) \to h^0((\mathcal{L}^\ast_{f^!}U) \xrightarrow{\rho_u} h^0((\mathcal{L}^\ast_{f^!}U^\lor) \xrightarrow{\sigma_u} h^0((\mathcal{L}^\ast_{f^!})^\lor) \to 0$$

where $\tilde{u} = u \circ u_j$. Let us fix $\mathcal{F}_X \in h_{U,X}(A), \mathcal{F}_Y \in h_{Y,(A')}$. Then $\mathcal{F}_U \in h_{U,X}(A), \mathcal{F}_{U \times X} \in h_{U,X \times X}(A)$ such that $f \circ \mathcal{F}_X = \mathcal{F}_Y \circ i$, $u \circ \mathcal{F}_U = \mathcal{F}_Y$, $u_j \circ \mathcal{F}_{U \times X} = \mathcal{F}_U$. Let us assume that $\text{obj}(\mathcal{F}_X, \mathcal{F}_Y) = 0$ and fix $\mathcal{F}_X \in \mathcal{S}_f, \mathcal{F}_U \in \mathcal{S}_f \cup \mathcal{F}_{U \times X} \in \mathcal{S}_{U \times X}$ such that $u \circ \mathcal{F}_U = \mathcal{F}_X, u_j \circ \mathcal{F}_{U \times X} = \mathcal{F}_U$. There is a natural surjective map $\mathcal{S}_{f^!} \to \mathcal{S}_f$ given by composition with $u$. If $\alpha_{f^!, 1}, \alpha_{f^!, 2} \in h^0(\mathcal{L}^\ast_{f^!}U^\lor) \otimes I$ are such that $\sigma_u(\alpha_{f^!, 1}) = \sigma_u(\alpha_{f^!, 2})$, then there exists $\alpha_{U \times X}$ in $h^0(\mathcal{L}^\ast_{f^!}U^\lor) \otimes \mathcal{I}$ such that $\rho_{U^\lor}(\alpha_{U \times X}) = \alpha_{U^\lor}$. Therefore

$$u \circ (\alpha_{U^\lor} \cdot \mathcal{F}_U) = u \circ \left(\alpha_{U \times X} \cdot \mathcal{F}_U\right) = \tilde{u} \circ \left(\alpha_{U \times X} \circ \mathcal{F}_U\right).$$

Hence, given $\alpha_f \in h^0(\mathcal{L}^\ast_{f^!}U^\lor) \otimes I$ and setting $\alpha_f \cdot \mathcal{F}_X = u \circ (\alpha_{U^\lor} \cdot \mathcal{F}_U)$ for some $\alpha_{U^\lor}$, we get a well-defined action. If $g_X \in \mathcal{S}_f$, there exists $g_{U \times X} \in \mathcal{S}_{U \times X}$ such that $u \circ g_{U \times X} = g_X$. Hence there exists $\alpha_{U \times X}$ in $h^0(\mathcal{L}^\ast_{f^!}U^\lor) \otimes I$ such that $\rho_{U^\lor}(\alpha_{U \times X}) = \alpha_{U^\lor}$. Therefore

$$\alpha_f = \sigma_u(\alpha_{U^\lor}) = \sigma_u \circ \rho_{U^\lor}(\alpha_{U \times X}) = \sigma_u \circ \rho_{U^\lor}(\alpha_{U \times X}) = \sigma_u = e_{\mathcal{I}}.$$}

Finally, let $g_X \in h_{U,X}(A), g_{U^\lor} \in h_{U^\lor,Y}(A)$ such that $f \circ g_X = g_{U^\lor} \circ i = g_{U^\lor}, \text{obj}(g_X, g_{U^\lor}) = 0$ and let us fix $\mathcal{F}_X \in \mathcal{S}_f$. We claim that the kernel of the natural homomorphism

$$\eta: \text{Aut}_X(g_X) \to \text{Aut}_{U^{\lor}}(g_{U^\lor}) \xrightarrow{\text{Aut}_{U^\lor}(g_{U^\lor})}$$

is isomorphic to $h^{-1}(\mathcal{L}^\ast_{f^!}) \otimes I$. Let consider $g_Y \in h_{U,Y}(A), g_U \in h_{U,Y}(A')$ such that $g_U \circ i = g_Y, u \circ g_U = g_X$. By deformation theory of schemes, we have the following commutative diagram with exact rows

$$\begin{array}{ccc}
\text{Aut}_U(g_U) & \to & \text{Aut}_U(g_Y) \times \text{Aut}_{U^{\lor}}(g_{U^\lor}) \\
\downarrow \gamma & & \downarrow \rho_u \\
\text{Aut}_U(g_U) & \to & \text{Aut}_U(g_Y) \times \text{Aut}_{U^\lor}(g_{U^\lor})
\end{array}$$

where $\text{Aut}_U(g_U) = \text{Aut}_U(g_Y) = \{\text{id}\}$ since $U$ is a scheme. Moreover, $\alpha_{U^{\lor}}$ (respectively $\alpha_u$) acts trivially on $\mathcal{S}_{U^{\lor}}$ (respectively $\mathcal{S}_u$) if and only if it is in the image of $\omega_{U^{\lor}}$ (respectively $\omega_u$). It follows that $\text{ker} \gamma = h^{-1}(\mathcal{L}^\ast_{f^!}) \otimes I$. The claim follows therefore by noticing that $\text{ker} \eta = \text{ker} \gamma$.

3.5. **Theorem** (Infinitesimal criterion for obstruction theories). A pair $(E^+, \varphi)$ is an obstruction theory for $f$ if and only if, for every geometric point $\mathcal{F}$ of $X$ and for every small extension $A' \to A = \mathcal{X}/U$, for every geometric point $\mathcal{F}$ of $X$ and for every small extension $A' \to A = \mathcal{X}/U$,

1. the obstruction $h^1(\varphi) = h^1(\mathcal{L}^\ast_{f^!}) \otimes I$ vanishes if and only if there exists a morphism $g_X^\prime$ such that $g_X^\prime \circ i = g_X$, $f \circ g_X^\prime = g_{U^\lor}$,

2. if $h^1(\varphi) = 0$ then the set of isomorphism classes of such morphisms $g_X^\prime$ is a torsor under $h^0(\mathcal{L}^\ast_{f^!}) \otimes I$. 


(3) if \( \text{obj}(g_X, g_Y') = 0 \) and \( g_X' \in h_X, \pi(A') \) is such that \( f \circ g_X' = g_Y', g_X' \circ i = g_X \), then the group of infinitesimal automorphisms of \( g_X' \) with respect to \( (g_X, g_Y') \) is isomorphic to \( h^{-1}((L^{E'} X, E')) \otimes I \).

Proof. If \((E^*, \varphi)\) is an obstruction theory for \( f \), the statement follows immediately from Theorem 3.4. Vice versa, let assume that the second part of the statement holds and let show that \( h^0(\varphi), h^1(\varphi) \) are isomorphisms and \( h^{-1}(\varphi) \) is surjective. Since the statement is lisse-étalement local, we can assume that \( X \) is an affine scheme \( \text{Spec} \ A \). Then, by assumptions, for every \( R \)-algebra \( B \) and \( B \)-module \( N \), there is a bijection \( \text{hom}(h^j(L^* X) \otimes B, N) \to \text{hom}(h^j(E^*) \otimes B, N) \) for \( j = 0, 1 \), which implies that \( h^j(\varphi) \) is an isomorphism for \( j = 0, 1 \). We can assume that \( f \) factors as \( X \overset{p}{\to} Y \) with \( i \) a closed embedding with ideal sheaf \( I \) into an affine scheme \( M \) and \( p \) smooth. We can further assume that \( \text{ker}(E_0 \to E_1) \) is locally free, \( E_1 \) is a coherent sheaf, \( E_0^i = 0 \) for \( i \neq 1, 0, -1 \) and \( \varphi^{-1} \) surjective. Then, by Remark 2.11 the complex \( G \to i^* \Omega_p \to \Omega_u \), where \( G \) is the cokernel of \( \text{ker} \varphi^0 \times_{E^0} E_1 \to E_0 \), is quasi-isomorphic to \( E^* \). Therefore we can assume \( E_0 = i^* \Omega_p, E_1 = \Omega_u \) and we have to prove that \( E_1 \to J/J^2 \) is surjective; let \( F \) be its image. Let \( X = \text{Spec} \ A, F^p \subset J \) the inverse image of \( F \), and \( \text{Spec} A' \subset M \) the subscheme defined by \( F^p; \) let \( g: \text{Spec} A \to X \) be the identity. We can extend \( g \) to the inclusion \( g': \text{Spec} A' \to M \).

Let \( \pi: J/J^2 \to J/J^p \) be the natural projection. By assumption \( \pi \) factors via \( E^0 \) if and only if \( g \) extends to a map \( \text{Spec} A' \to X \), if and only if \( \pi \circ \varphi^{-1}: E_1 \to J/J^p \) factors via \( E^0 \). As \( \pi \circ \varphi^{-1} \) is the zero map, it certainly factors. Therefore \( \pi \) also factors. Moreover, the fact that \( \pi \) factors via \( E^0 \) together with \( \pi \circ \varphi^{-1} = 0 \) implies \( \pi = 0 \), hence \( \varphi^{-1}: E_1 \to J/J^p \) is surjective. \( \square \)

3.6. Definition. Let \((E^*, \varphi)\) be an obstruction theory for \( f \). We say that \((E^*, \varphi)\) is perfect (of perfect amplitude contained in \([-1, 0]\)) if, lisse-étale locally over \( X \), it is isomorphic to \([E^{-1} \to E_0 \to E^1]\) with \( E^{-1} \), \( \ker(E_0 \to E^1) \) locally free sheaves over \( X \).

3.7. Remark. An obstruction theory \((E^*, \varphi)\) is perfect if and only if \( h^j/h^0(\tau_{[-1, 0]} E^*) \) is a vector bundle stack over \( X \).

4. Virtual fundamental class

4.1. Let \( f: X \to Y \) be a morphism of algebraic stacks, let us assume that \( Y \) is purely dimensional of pure dimension \( m \) and that \( X \) admits a stratification by global quotients in the sense of [10].

3.5.3. Let \((E^*, \varphi)\) be a perfect obstruction theory for \( f \), we denote by

\[
\mu: \mathcal{E}_f = h^j/h^0(\tau_{[-1, 0]} E^*) \to X
\]

the associated vector bundle stack of rank \( r \). By Remark 3.2 the intrinsic normal cone \( \mathcal{E}_f \) is a closed substack of \( \mathcal{E}_f \). Moreover, by [10] 4.3.2, the flat pullback

\[
\mu^*: A_*(X) \to A_{*+r}(\mathcal{E}_f)
\]

is an isomorphism and we denote the inverse by \( 0^! \).

4.2. Definition. The virtual fundamental class of \( X \) relative to \((E^*, \varphi)\) is the cycle class

\[
[X, E^*]_\text{virt} = 0^! \mathcal{E}_f \in A_*(X).
\]

4.3. Remark. The intrinsic cone \( \mathcal{E}_f \) is purely dimensional of pure dimension \( m \), therefore \([X, E^*]_\text{virt} \in A_{m-r}(X) \) and \( m - r \) is called the virtual dimension of \( X \).

4.4. Proposition (Base-change). Consider the following cartesian diagram of algebraic stacks

\[
\begin{array}{ccc}
X' & \overset{f'}{\longrightarrow} & Y' \\
\downarrow g & & \downarrow h \\
X & \overset{f}{\longrightarrow} & Y
\end{array}
\]
where $\mathcal{Y}$ and $\mathcal{Y}'$ are smooth and purely dimensional of pure dimension $m$, $\mathcal{X}$ and $\mathcal{X}'$ admit stratifications by global quotients. Let $(E^*, \varphi)$ be a perfect obstruction theory for $f$. If $h$ is flat or a regular local immersion (of constant dimension) then

$$h^1[\mathcal{X}, E^*]^{\virt} = [\mathcal{X}', Lg^*E^*]^{\virt}.$$  

**Proof.** Let us notice that $Lg^*E^*$ is a perfect obstruction theory for $f'$. Let $\mathcal{E}_f = h^!h^*(Lg^*E^*)$ and let $0^d$ the inverse of $\mu^*$, where $\mu^*: \mathcal{E}_f \to \mathcal{X}'$. If $h$ is flat then, by Proposition $2.14$, we have $g^*\mathcal{E}_f \cong \mathcal{E}_{f'}$, hence $h^!\mathcal{E}_f = [\mathcal{E}_f] \in A_*(\mathcal{E}_f)$. Therefore we get

$$h^1[\mathcal{X}, E^*]^{\virt} = h^!0^d[\mathcal{E}_f] = 0^d[h^!\mathcal{E}_f] = 0^d[\mathcal{E}_f] = [\mathcal{X}', Lg^*E^*]^{\virt}.$$  

If $h$ is a regular local immersion, let consider $h: \mathcal{X}_h \to \mathcal{Y}'$ and let $0^d: g^*\mathcal{E}_f \to \mathcal{X}_h \times_Y \mathcal{E}_f$ be the zero section. Then $0^d[\mathcal{E}_f] = h^!\mathcal{E}_f] \in A_*(g^*\mathcal{E}_f)$, by definition of $h^!$, and

$$0^!\mathcal{E}_f = 0^!\mathcal{E}_f \in A_*(g^*\mathcal{E}_f).$$

Moreover, by [5] 3.3–3.5, $[\mathcal{E}_f] = \mathcal{E}_f = [\mathcal{E}_f] \in A_*(\mathcal{X}_h \times_Y \mathcal{E}_f)$. Hence $h^!\mathcal{E}_f] = [\mathcal{E}_f] \in A_*(\mathcal{E}_f)$ and one concludes as before.  

**4.5.** Let us consider the cartesian diagram (2) and assume that $h$ is a local complete intersection morphism of stacks with finite unramified diagonal over $\mathcal{Y}$. Let $E^*$ and $E'^*$ be perfect obstruction theories for $f$ and $f'$ respectively. Then $E^*$ and $E'^*$ are compatible over $h$ if there exists a homomorphism of distinguished triangles in $D_{\text{coh}}^{[1,0]}(\mathcal{X}_{h-\text{ét}})$

$$g^*E^* \longrightarrow E^* \longrightarrow f'^*L_h \rightarrow g^*E^*[1]$$

(3)

$$g^*L_X \longrightarrow L_x \longrightarrow L_y \rightarrow g^*L_X[1]$$

**4.6. Proposition (Functoriality).** Let $E^*$ and $E'^*$ be compatible perfect obstruction theories as above. If either $h$ is smooth or $\mathcal{Y}$ and $\mathcal{Y}'$ are smooth, then

$$h^1[\mathcal{X}, E^*]^{\virt} = [\mathcal{X}', E'^*]^{\virt}.$$  

**Proof.** By [5] 2.7, the diagram (3) induces a short exact sequence of vector bundle stacks

$$f'^*\mathcal{E}_f \xrightarrow{\psi^*} g^*\mathcal{E}_f$$

If $h$ is smooth, by [5] 3.14, $\mathcal{E}_f = g^*\mathcal{E}_f \times g^*\mathcal{E}_f$, hence $0^!g^*\mathcal{E}_f = 0^!g^*\mathcal{E}_f = [\mathcal{E}_f] \in A_*(\mathcal{X}')$ and

$$h^1[\mathcal{X}, E^*]^{\virt} = h^1[\mathcal{E}_f] = 0^!g^*\mathcal{E}_f = [\mathcal{X}', E'^*]^{\virt}.$$  

If $\mathcal{Y}$ and $\mathcal{Y}'$ are smooth then $h$ factors as $\mathcal{Y}' \xrightarrow{\Gamma_h} \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{Y}' \rightleftharpoons \mathcal{Y}$, where $\Gamma_h$ is the graph of $h$, which is a regular local immersion, and $\pi$ is smooth. We have the following cartesian diagram

$$\mathcal{X}' \longrightarrow \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{X} \quad f' \quad \mathcal{Y}' \longrightarrow \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{Y}' \longrightarrow \mathcal{Y} \quad f$$

and we consider the obstruction theory $\Omega_{\mathcal{Y}} \oplus E^*$ for $f$. Notice that $\Omega_{\mathcal{Y}} \oplus E^*$ is compatible with $E^*$ over $p$ and with $E'^*$ over $\Gamma_h$. Then, by the first part, we can assume that $h$ is a regular local immersion. By [5] 5.9, $[f'^*\mathcal{E}_f] = \psi^*[\mathcal{E}_f] \in A_*(f'^*\mathcal{E}_f)$ and hence

$$[\mathcal{X}', E^*]^{\virt} = 0^!\mathcal{E}_f \mathcal{E}_f = 0^!f'^*\mathcal{E}_f \mathcal{E}_f = 0^!f'^*\mathcal{E}_f \mathcal{E}_f = 0^!h^!\mathcal{E}_f = 0^!\mathcal{E}_f \mathcal{E}_f = [\mathcal{X}', E'^*]^{\virt}.$$  

□
5. Virtual pullbacks and Costello’s pushforward

5.1. Although not mentioned in [10], one can verify that the theory can be extended to Artin stacks over a Dedekind domain. As a consequence we get that Manolache’s construction of the virtual pullback in [13] is valid for morphisms of Artin stacks over a Dedekind domain.

5.2. Proposition (Costello’s pushforward formula). Let us consider a cartesian diagram

\[
\begin{array}{ccc}
\mathcal{X}_1 & \xrightarrow{f} & \mathcal{X}_2 \\
p_1 \downarrow & & \downarrow p_2 \\
\mathcal{Y}_1 & \xrightarrow{g} & \mathcal{Y}_2
\end{array}
\]

where

(1) $\mathcal{Y}_1, \mathcal{Y}_2$ are Artin stacks over $D$ of the same pure dimension,
(2) $\mathcal{X}_1, \mathcal{X}_2$ are Artin stacks over $D$ with quasi-finite diagonal,
(3) $g$ is a morphism of degree $d$, $f$ is proper,
(4) for $i = 1, 2$, $p_i$ admits perfect obstruction theory $E_i^\bullet$ such that $f^* E_2^\bullet \cong E_1^\bullet$.

Then

\[ f_* [\mathcal{X}_1, E_1^\bullet]_{\text{virt}} = d [\mathcal{X}_2, E_2^\bullet]_{\text{virt}} \]

in each of the following cases

(a) $g$ is projective,
(b) $\mathcal{Y}_1, \mathcal{Y}_2$ are Deligne-Mumford stacks and $g$ is proper,
(c) $\mathcal{Y}_1, \mathcal{Y}_2$ have quasi-finite diagonal and $g$ is proper.

Proof. Since in each of the cases listed above we are able to pushforward along $g$ (see [8] 2.8 for case (c)), the statement follows by the same argument of [13] 5.29, after noticing that non-representable proper pushforward commutes with virtual pullback (this can be shown in the same way as in [13] 4.1).

\[ \square \]

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E-mail address: flavia.poma@gmail.com