Localization of resonance eigenfunctions on quantum repellers

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We introduce a new phase space representation for open quantum systems. This is a very powerful tool to help advance in the study of the morphology of their eigenstates. We apply it to two different versions of a paradigmatic model, the baker map. This allows to show that the long-lived resonances are strongly scarred along the shortest periodic orbits that belong to the classical repeller. Moreover, the shape of the short-lived eigenstates is also analyzed. Finally, we apply an antiunitary symmetry measure to the resonances that permits to quantify their localization on the repeller.

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Open quantum systems with classical chaotic behavior have been far less explored than their closed counterparts, and as a consequence several of their properties are less known. This situation is in contradiction with their importance in many areas of physics, some of them of very recent development. One of the most significant examples is the study of quantum to classical correspondence [1]. But their applicability range extends to microLasers [2, 3, 4], quantum dots [5], chaotic scattering [6], and more topics. The lack of unitarity at the quantum level, and the complex - and mostly fractal - nature of the repeller sets is at the heart of the difficulties in their understanding.

The classical phase space of open chaotic systems is characterized by the forward and backward trapped sets, i.e., fractal sets of trajectories that stay trapped forever in the future and in the past, respectively. The intersection of both sets is called the repeller. Classical properties of this set are its decay rate $\gamma_{cl}$, various generalized dimensions and the spectral properties of the Perron-Frobenius operator [7]. On the other hand, the quantum evolution is described by a nonunitary operator, whose right and left eigenstates (the so called resonances) are nonorthogonal and the eigenvalues $\lambda_i$ are complex with modulus less than or equal to one ($\nu_i^2 = |\lambda_i|^2 = \exp(-\Gamma_i) \leq 1$, where $\Gamma_i \geq 0$ is the decay rate). The short-lived resonances ($\Gamma_i \gg 1$) are associated with the trajectories that escape from the system before the Ehrenfest time, while the long-lived ones ($\Gamma_i = \mathcal{O}(1)$) are related to the classical forward or backward trapped sets (in [7] they were coined quantum fractal eigenstates). According to the conjectured fractal Weyl law, the number of long-lived resonances $N_r$ can be related to $h$ and the structure of the classical phase space through the expression $N_r \sim h^{-(d-1)}$ ($d$ is the fractal dimension of the classical strange repeller). This law has been checked for a three disk system [8] and some quantum maps [9, 10, 11, 12, 13, 14, 15].

Finer information about the semiclassical limit is provided by the structure of eigenfunctions, about which much less is known. For the open baker map it was found that the probability density averaged for several right eigenstates is supported by the corresponding classical forward trapped set [13]. A simple argument by analogy to Shnirelman’s theorem would favor a weak equidistribution of some single spectral components over the repeller, but of course this leaves ample room for “accidental” localization on the periodic orbits, i.e. for the existence of scars in open quantum systems. Such long-lived scarred modes have been observed in optical microcavities [14], and have been extensively studied in closed systems [10]. Recently, some results for the open cat map have been provided in [14]. But the knowledge we have about this behavior is still rudimentary.

In this letter we show that many long lived resonances of the open baker map, besides the obvious localization on the repeller, indeed show an enhancement on some of its shortest periodic orbits. To this purpose we introduce a phase space representation of the spectral components of the open map that is in complete analogy to the usual Husimi functions of the closed case. We also investigate the shape of short-lived resonances of the open triadic baker map. We find that structures resembling the states of a simple Hamiltonian develop outside of the classical repeller. On the other hand, we have applied a measure of antiunitary symmetry to the eigenstates. By doing this we were able to quantify the localization of a given resonance on the repeller. Moreover, this measure turns out to be simply related to the moduli of the eigenvalues. We believe that these tools could provide an alternative way of identifying long and short-lived resonances, a task of the utmost importance in the road to demonstrate the conjectured Weyl law [10]. In fact, this stands as an open problem, where only for the Walsh quantized open baker map [14, 17] exact results could be obtained.

We consider a closed quantum map $U$, chosen as the quantization of a classical map which is chaotic. The phase space is the unit torus and upon quantization we impose antiperiodic boundary conditions resulting in a Hilbert space of dimension $N$, where $h = (2\pi N)^{-1}$ plays
that coincide with some iteration of the basic Markov transition matrix. Our maps have parity (R) and time reversal (T) symmetries in both its classical and quantum versions.

The map is “opened” as usual by means of a projection operator $\Pi_o$ that defines the escape region in phase space. In this way, $\Pi = 1 - \Pi_o$ composed with the closed map defines the open system. Depending on the symmetries of the projector it is possible to maintain or break in the open map the symmetries of the closed one. The resulting operator, $\tilde{U} = U\Pi$ is not normal and has a null subspace whose dimension is $tr(\Pi_o)$. If we assume that it is diagonalizable, its spectral decomposition in terms of right and left eigenstates is

$$\tilde{U} = \sum_i \lambda_i \frac{|\psi_i^R\rangle\langle\psi_i^L|}{\langle\psi_i^R|\psi_i^L\rangle}$$

where we assume the normalization condition $\langle\psi_i^L|\psi_i^L\rangle = |\langle\psi_i^R|\psi_i^R\rangle| = 1$. As the map is contracting, the spectrum $\lambda_i$ is strictly inside the unit circle. Using the decomposition in Eq.(1) the autocorrelation of a coherent state $|q,p\rangle$ can be written as

$$\langle q,p|\tilde{U}^n|q,p\rangle = \sum_i \lambda_i^n h_i(q,p)$$

in terms of complex phase space distributions

$$h_i(q,p) = \frac{\langle q,p|\psi_i^R\rangle\langle\psi_i^L|q,p\rangle}{\langle\psi_i^R|\psi_i^L\rangle}$$

As the right (left) eigenfunctions are approximately concentrated on the backwards (forward) trapped sets, we expect $h_i(p,q)$ to map out the distribution of the single resonances on the intersection of these sets, i.e. on the repeller. Thus they constitute the ideal tool to study the localization properties and eventually the “scarring” in open systems. Notice that $|h_i(p,q)\rangle \propto \sqrt{\mathcal{H}^R_i(p,q) \mathcal{H}^L_i(p,q)}$, where $\mathcal{H}^R_i, \mathcal{H}^L_i$ are Husimi functions of the $R, L$ eigenstates, and that for unitary systems they reduce to the Husimi functions of the eigenstates. We would like to mention that, instead of considering coherent states we could have chosen any other representation in phase space. Different choices can help to analyze different properties of $|\psi_i^R\rangle\langle\psi_i^L|/|\psi_i^R|^2$.

We apply these ideas to study the localization of resonance eigenfunctions for the open baker map. In its standard version it is defined as $p' = p/2 + [2q]/2, q' = 2q - [2q]$, where $[2q]$ is the integer part. It has a complete symbolic dynamics in terms of two symbols. The quantization is obtained in terms of the anti–periodic, discrete Fourier transform $(G_N)_{j,k} \equiv \exp(-i2\pi/N(j+1/2)(k+1/2))/\sqrt{N}$, as $U = G^N_{G'N} \text{diag}(G_N/2, G_N/2)$ [18, 19]. The advantage for this kind of study is that one can easily engineer openings that coincide with some iteration of the basic Markov partition and thus obtain resonances with a simple symbolic description in terms of a subshift of finite type. For concreteness we take the escape region $\Pi_o$ to be the union of the areas defined by $q_1 \in [0,1/2] \times p_1 \in [0,1]$ and $q_2 \in [1 - 1/2^l, 1] \times p_2 \in [0,1]$ leading to a subshift for which the complete symbolic dynamics is pruned of all trajectories that include the strings $0^l$ and $1^l$. The opening is symmetric with respect to $q = 1/2$ in order to retain the parity symmetry. We will focus on the first non-trivial case of $l = 3$ where $q_1 \in [0,1/8]$, $q_2 \in [7/8,1]$, and the strings 000 and 111 are pruned. The topological entropy $\lambda_T \approx 0.4812$ is easily calculated from the leading eigenvalue of the transition matrix. In Fig. 1 we compare the autocorrelation function in phase space (Eq. (2)) for both the open and the closed versions of the fourth iteration of the map clearly showing the disappearance of the periodic points forbidden by the pruning rules.

Four examples of resonance eigenfunctions are displayed in Fig. 2, using the distribution $h_i(q,p)$. They correspond to a Hilbert space of dimension $N = 320$ and $l = 3$. In panel a) we show the longest-lived resonance for which $|\lambda_0| \approx 0.977$. In this case, the complex function $h(q,p)$ is represented through an intensity (moduli) and color (phase) scale, this latter ranging from blue (0) to red ($2\pi$). It is clear that its values are almost completely real, this being an expected property of an eigenstate whose associated eigenvalue is so close to the unit circle. In panels b), c) and d) we show $|h_i(q,p)|$ for eigenstates corresponding to $|\lambda_0| \approx 0.905, |\lambda_c| \approx 0.856$, and $|\lambda_d| \approx 0.641$ respectively (in these cases we restrict our analysis to the moduli of the distribution). We now focus on the specific shape of resonances. Those in a) and c) clearly show an enhanced probability on the periodic orbits 0011 and 01 respectively, which belong to the repeller. This enhancement on the periodic points and their manifolds is typical of the scarring phenomenon. In
and $(0, 0)$ keep symmetries intact. Finally, in order to complete the map in a clear way, we choose to analyze the triadic baker map with an isolated fixed point at $(0, 0)$. In our point of view, the main advantage of this map is that it has no visible prevailing orbit in its structure. The open version of the map can be obtained in the same way as before, by means of a projector $\Pi$, but this time on the central region in position. This amounts to saying that in the quantum case the left and right thirds of the columns of $B_T$ are set to zero. This opens two thirds of the phase space (where the forbidden region is given by $q \in [0, 1/3]$ and $q \in [2/3, 1]$, satisfying $tr(\Pi) = N/3$ in the quantum case). The repeller of this system is reduced to the point $(q, p) = (1/2, 1/2)$. It is important to underline that this map is hyperbolic, but it is not chaotic. However, it is very suitable in order to unveil the structure of resonances lying outside of the repeller, that in the previously studied version of the baker map have a complicated structure. Fig. 3 shows the six longest-lived resonances for $N = 3^3$. In this case the morphology is very clear. They resemble excited states of the hyperbolic Hamiltonian $H = qp$, centered at the fixed point $(q, p) = (1/2, 1/2)$. This suggests that the longest-lived fraction of the short-lived resonances associated to more complicated repellers could be interpreted (and perhaps theoretically constructed) as convenient superpositions of these simpler structures. This completes a very interesting picture, i.e., one could try to devise a theory of short periodic orbits capable of describing the complete significant quantum information of an open system.

In order to study the eigenstates outside of the repeller in a clear way, we choose to analyze the triadic baker map $B_T$. This map is defined as $(p' = p/3, q' = 3q)$ if $0 \leq q < 1/3$, $(p' = (p+1)/3, q' = 3q-1)$ if $1/3 \leq q < 2/3$, and $(p' = (p+2)/3, q' = 3q-2)$ if $2/3 \leq q < 1$. From our point of view, the main advantage of this map is that it has an isolated fixed point at $(q, p) = (1/2, 1/2)$, far from the discontinuities. The quantization of this map in a $N$-dimensional Hilbert space ($N = 2\pi/h$, divisible by 3) with antisymmetric boundary conditions is given by

$$B_T = G_N^T \begin{pmatrix} G_{N/3} & 0 & 0 \\ 0 & G_{N/3} & 0 \\ 0 & 0 & G_{N/3} \end{pmatrix}$$

The open version of the map can be obtained in the same way as before, by means of a projector $\Pi$, but this time on the central region in position. This amounts to saying that in the quantum case the left and right thirds of the columns of $B_T$ are set to zero. This opens two thirds of the phase space (where the forbidden region is given by $q \in [0, 1/3]$ and $q \in [2/3, 1]$, satisfying $tr(\Pi) = N/3$ in the quantum case). The repeller of this system is reduced to the point $(q, p) = (1/2, 1/2)$. It is important to underline that this map is hyperbolic, but it is not chaotic. However, it is very suitable in order to unveil the structure of resonances lying outside of the repeller, that in the previously studied version of the baker map have a complicated structure. Fig. 3 shows the six longest-lived resonances for $N = 3^3$. In this case the morphology is very clear. They resemble excited states of the hyperbolic Hamiltonian $H = qp$, centered at the fixed point $(q, p) = (1/2, 1/2)$. This suggests that the longest-lived fraction of the short-lived resonances associated to more complicated repellers could be interpreted (and perhaps theoretically constructed) as convenient superpositions of these simpler structures. This completes a very interesting picture, i.e., one could try to devise a theory of short periodic orbits capable of describing the complete significant quantum information of an open system.

Finally, the time reversal symmetry provides us with very interesting results regarding the classification of resonances into long and short-lived. In the classical case this symmetry can be restricted to the orbits belonging to the repeller, since they are not affected by the presence of the escape regions, in our example. In other words, the repeller corresponding to our open system retains
the time reversal invariance of the closed one. However, if we look at resonances in Fig. 2 we immediately notice that in general they are not symmetric with respect to the line $q = p$, this indicating that they are not time reversal invariant (in general left eigenstates cannot be obtained from applying the time reversal operator to the right ones, for example). Then, for the quantum case we can define a measure of the time reversal symmetry for each eigenstate in order to see if this is related to its localization on the repeller. This is accomplished by $	au_i = |\langle \psi_i^R | T | \psi_i^L \rangle|$, where $T = KG$ ($K$ stands for the complex conjugation operator). In the case of maps having a time reversal symmetry $\tau_i = 1$ for any $i$. Results for the standard open baker map can be seen in Fig. 4. We have found that there is a simple relation among the values of this measure and the eigenvalues moduli, given by $\tau_i = |\lambda_i|^{l-1}$. On one hand this confirms that the resonances that have a maximum overlap with the repeller (long-lived ones) show the higher time reversibility. On the other hand, it confirms this quantity as an alternative way to distinguish between long and short-lived resonances. We think that by following this approach a prediction on the number of each kind of eigenstates could be found [20].

In conclusion, we have introduced a very powerful phase space representation for open quantum systems that is in complete analogy with the usual Husimi distribution for closed ones. This allowed us to clearly show that long-lived resonances can be strongly scarred along the shortest periodic orbits that belong to the corresponding classical repeller. Moreover, we have obtained some hints on the structure of the short-lived eigenstates, i.e. those living outside of the invariant set. We believe that short periodic orbits of the repeller play an important role also in this case. Finally, a time reversibility measure for the resonances has been devised. We were able to find a simple relation with the eigenvalues moduli showing that the long-lived resonances have the highest time reversibility. We consider that these tools are very promising in the way to demonstrate the fractal Weyl law for open quantum systems.

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