GEOMETRIZATION OF ALMOST EXTREMAL REPRESENTATIONS IN $\text{PSL}_2\mathbb{R}$

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Abstract. Let $S$ be a closed surface of genus $g$. In this paper we investigate the relationship between hyperbolic cone-structure on $S$ and representations of the fundamental group into $\text{PSL}_2\mathbb{R}$. We consider surfaces of genus greater than $g$ and we show that, under suitable conditions, every representation $\rho : \pi_1 S \to \text{PSL}_2\mathbb{R}$ with Euler number $\mathcal{E}(\rho) = \pm(\chi(S) + 1)$ arises as holonomy of a hyperbolic cone-structure $\sigma$ on $S$ with a single cone point of angle $4\pi$. From this result, we derive that for surfaces of genus 2 every representation with $\mathcal{E}(\rho) = \pm 1$ arises as the holonomy of some hyperbolic cone-structure.

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1. Introduction

1.1. About the problem. A hyperbolic cone-structure on an oriented surface $S$ is a geometric structure locally modeled on the hyperbolic plane, with its group of orientation-preserving isometries $\text{PSL}_2\mathbb{R}$. Any hyperbolic structure induces in a natural way a holonomy representation $\rho : \pi_1 S \to \text{PSL}_2\mathbb{R}$, that encodes geometric data about the structure; but what can we say about the reverse problem? More precisely:

which representations of a surface group into $\text{PSL}_2\mathbb{R}$ are holonomy representations?

The reverse problem to recover a hyperbolic cone-structure from a given representation $\rho$ is arduous, longer and not always possible. In [22], Tan gives an example of a representation that does not arise as the holonomy of a hyperbolic cone-structure (see also 2.6 below). For this reason, we will say that a representation $\rho$ is
geometrizable by a hyperbolic cone-structure (or briefly geometricizable), if it arises as the holonomy of a hyperbolic cone-structure on $S$. For a closed surface $S$ with $\chi(S) < 0$, every representation $\rho : \pi_1 S \rightarrow \operatorname{PSL}_2 \mathbb{R}$ determines an Euler number $E(\rho)$ (we discuss the Euler number in more detail below, see 3). The Euler number $E(\rho)$ satisfies the so-called Milnor-Wood inequality, that is $|E(\rho)| \leq -\chi(S)$; and parametrizes the connected components of the $\operatorname{PSL}_2 \mathbb{R}$-character variety $\mathcal{X}(S)$. In [9], Goldman showed that every representation with $E(\rho) = -\chi(S)$ arises as the holonomy of a complete hyperbolic structure on $S$. For the other values of $E(\rho)$, it is not yet clear which are holonomy representations. So far as we know, it is still an open question whether the set of holonomy representations is dense among representations of Euler class $|k| < -\chi(S)$.

In [5], we were interested in purely hyperbolic representations, i.e. representations whose image consists only of hyperbolic elements other than the identity. In this work we consider another class of representations of major interest, namely almost extremal representations, i.e. representations such that $E(\rho) = \pm (\chi(S) + 1)$ (hence the reason of such name). We may immediately rule out elementary representations from our interests because they have Euler number zero (see [9]). For this reason, in the sequel, we will consider only non-elementary representations.

It was conjectured that every almost extremal representation arises as the holonomy of hyperbolic cone-structure with one cone point of angle $4\pi$. Mathews took into account this problem in the following series of papers [16],[17],[18], which are extracted from his Honor dissertation [15]. In [17], he proves the following Theorem (see also 5.5 in section 5 below).

**Theorem** (Mathews 2011). Let $S$ be a closed surface of genus $g \geq 2$. Then almost every representation $\rho : \pi_1 S \rightarrow \operatorname{PSL}_2 \mathbb{R}$ with $E(\rho) = \pm (\chi(S) + 1)$, which sends a non-separating curve $\gamma$ on $S$ to an elliptic is the holonomy of a hyperbolic cone-structure on $S$ with one cone point of angle $4\pi$.

Our work starts with this known result. Since we may introduce a measure on the character variety as we will describe below (see 5.1), we may note that this statement makes sense. Here will show the following stronger result.

**Theorem 5.1:** Let $S$ be a closed surface of genus $g \geq 2$. Then every representation $\rho : \pi_1 S \rightarrow \operatorname{PSL}_2 \mathbb{R}$ with $E(\rho) = \pm (\chi(S) + 1)$, which sends a non-separating simple curve $\gamma$ on $S$ to a non-hyperbolic element is the holonomy of a hyperbolic cone-structure on $S$ with one cone point of angle $4\pi$.

By this theorem, the geometrization of almost extremal representations problem is reduced on finding a simple closed curve with non-hyperbolic holonomy. So far, we do not know under which conditions (if any) a non-Fuchsian representation (which may be not almost extremal) sends a simple closed curve to a non-hyperbolic element. This problem is known in the literature as Bowditch question or Bowditch conjecture. By recent works [12] and [13] of Marché and Wolff, the Bowditch question is known to be true in genus two case. In particular, they show that every almost extremal representation (i.e. $E(\rho) = \pm 1$) sends a simple curve to a non-hyperbolic element (see [13, Theorem 1.4]). However, we do not know a priori if such curve is separating or not. Even in [17], Mathews showed the following result, very particular to the genus 2 case.

**Theorem** (Mathews 2011). Let $S$ be a closed surface of genus 2, and let $\rho : \pi_1 S \rightarrow \operatorname{PSL}_2 \mathbb{R}$ be a representation with $E(\rho) = \pm 1$. Suppose $\rho$ sends a separating curve $\gamma$ on $S$ to a non-hyperbolic element. Then $\rho$ arises as the holonomy of a hyperbolic cone-structure on $S$ with one cone point of angle $4\pi$.

Combining the main theorem 5.1 with 5.26 we will derive the following corollary.

**Corollary 5.2:** Let $S$ be a closed surface of genus 2. Then any representation $\rho : \pi_1 S \rightarrow \operatorname{PSL}_2 \mathbb{R}$ with $E(\rho) = \pm 1$ is geometricizable by a hyperbolic cone-structure with one cone point of angle $4\pi$.

Our strategies rely on the existence of a simple closed curve with non-hyperbolic, but we do not know if such curve exists in general. The following question naturally arises.

**Question.** For a general surface, does every representation $\rho$ with Euler number $E(\rho) = \pm (\chi(S) + 1)$ arise as the holonomy of a hyperbolic cone-structure?

Recently, during a conversation with the author, Bertrand Deroin announced his proof, in collaboration with Nicolas Tolozan, of the fact that every representation of the fundamental group of a closed and oriented genus $g$ surface with Euler number $E(\rho) = \pm (\chi(S) + 1)$ arises as the holonomy of a hyperbolic cone-structure with a single cone point of angle $4\pi$.

1.2. **Structure of the paper.** This paper is organized as follow. Section 2 contains the necessary background material in order to tackle the main parts of this work. This material includes, in particular, the basic definitions
about hyperbolic cone-structure and holonomy representation. We discuss about the geometry of hyperbolic transformations, the Lie groups $\text{PSL}_2\mathbb{R}$ and $\widetilde{\text{PSL}}_2\mathbb{R}$ and the relationship between trace and commutator. In section 3, we discuss about the Euler class, giving both the geometrical and algebraic definition. Section 4 contains some generalities about the character variety and, in more detail, the character variety of the punctured torus. We discuss about virtually abelian representations and their characterization and about the action of the mapping class group on each strata of the character variety of the punctured torus. Finally, in section 5 we prove the main theorem 5.1 and the corollary 5.2. In particular, we give a brief description of the character variety of a closed surface with genus $g \geq 2$ in 5.1, and in paragraphs 5.3 and 5.4 we explain why we can remove the "almost every condition" from 5.5. Finally the subsection 5.5 and 5.6 contains respectively the proof of 5.1 and 5.2. At the end of the work, we have added an appendix about the flexibility of the hyperbolic cone-structure. Unlike the Fuchsian case, there is no a bijective correspondence between hyperbolic cone-structure and holonomy representations. More precisely, the same representation $\rho$ arises as the holonomy of uncountably many non-isomorphic cone structure on $S$.

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2. Some hyperbolic geometry

Let $S$ be a closed, connected and orientable surface. We will denote by $\mathbb{H}^2$ the hyperbolic plane and by $\text{PSL}_2\mathbb{R}$ its group of isometries acting by M"obius transformations

$$\text{PSL}_2\mathbb{R} \times \mathbb{H}^2 \to \mathbb{H}^2, \quad \left( \frac{a}{c} \right) z \mapsto \frac{az + b}{cz + d}$$

2.1. Hyperbolic cone-structures. We are going to define the main structure we are interested in, that is hyperbolic cone-structures. For our purposes, we only need to define hyperbolic cone-structures in dimension 2, though the following definition has obvious generalizations to higher dimensions and also other types of geometries. The curious reader may be seen [4] for further details.

Definition 2.1 (Hyperbolic cone-structure). A hyperbolic cone-structure $\sigma$ on a 2-manifold $S$ is the datum of a triangulation of $S$ and a metric, such that

1. the link of each simplex is piecewise linear homeomorphic to a circle, and
2. the restriction of the metric to each simplex is isometric to a geodesic simplex in hyperbolic space.

Hence a 2-dimensional hyperbolic cone-structure is a surface obtained by piecing together geodesic triangles in $\mathbb{H}^2$. The definition clearly includes open surfaces and surfaces with possibly geodesic boundary.

Any interior point $p$ of $S$ has a neighborhood locally isometric to $\mathbb{H}^2$, except possibly at some vertices of the triangulation, around which the angles sum to $\theta \neq 2\pi$. Such points are called cone points. The neighborhood of a cone point is isometric to a wedge of angle $\theta$ in the hyperbolic plane, with sides glued (that is a cone). The angle $\theta$ is called the cone angle at $p$ and letting $\theta = 2k + 1$, we define the number $k$ as the order $\text{ord}(p)$ of the cone point at $p$. If $S$ has boundary then this boundary will be piecewise geodesic. There may be vertices on the boundary around which the angles sum to $\theta \neq \pi$. Such points are called corner point and the value of $\theta$ is the corner angle. Letting $\theta = \pi(1 + 2s)$, then $s$ is the order of the corner points. In such a case a corner point has neighborhood isometric to a wedge of angle $\theta$ in $\mathbb{H}^2$ (without sides glued). Singular points of $\sigma$ on $S$ are cone or corner points, whereas any other points are called regular points. Note a cone angle may be any positive real number, in particular, it can be more than $2\pi$ for interior points or greater than $\pi$ for boundary point. In the sequel, we will only consider closed surfaces whose cone points have order $k \in \mathbb{N}$.

We note that a complete hyperbolic structure $\sigma_0$ on $S$ can be seen as hyperbolic cone-structure where all points are regular. Cone points may be considered as points on which the curvature is concentrated; however, topology imposes limits on the allowable cone angles in a 2-dimensional hyperbolic cone-structure which can be deduced from the Gauß-Bonnet theorem. Precisely we have the following result.
Proposition 2.2. Let $S$ be a compact, connected and orientable surface. Any hyperbolic cone-structure $\sigma$ on $S$ with cone and corner points $p_1, \ldots, p_n$ having orders $k_1, \ldots, k_n$ respectively satisfies the following relation

$$\chi(S) + \sum_{i=1}^{n} k_i < 0, \text{ where } k_i = \text{ord}(p_i).$$

Indeed the left hand side is $2\pi$ times the opposite of the hyperbolic area of $S$.

Proof. By definition, $\sigma$ is the datum of a triangulation $\tau$ such that any simplex is isometric to a geodesic triangle on the hyperbolic plane. In particular cone and corner points are vertices of $\tau$.

Suppose $S$ is closed. Multiplying both sides of the relation (2.1) by $2\pi$, it can be rewritten in the following way

$$2\pi \chi(S) - \sum_{i=1}^{n} 2\pi - \theta_i < 0.$$ 

Around any vertex $p$ the cone angle could be:

$$\theta_p = \begin{cases} 2\pi & \text{if } p \text{ is regular}, \\ \theta_i & \text{if } p \text{ is a cone point}. \end{cases}$$

The Euler characteristic of $S$ can be computed by the well-known formula $\chi(S) = V - E + F$, where $V, E, F$ are the numbers of vertices, edges and faces respectively. Since $\tau$ is a triangulation $2E = 3F$, thus the formula becomes $2\chi(S) = 2V - F$. Since any simplex of $\tau$ is a geodesic triangle, we may deduce that $\pi F > \sum_{i=1}^{n} \theta_i$, because the hyperbolic area of a triangle with angles $\alpha, \beta, \gamma$ is $\pi - \alpha - \beta - \gamma$. Hence

$$2\pi \chi(S) = 2\pi V - \pi F < 2\pi V - \sum_{i=1}^{n} \theta_i = \sum_{i=1}^{n} 2\pi - \theta_i.$$ 

If $S$ has geodesic boundary we doubling $S$ (where corner points are identified) to get a closed surface $S'$. Notice that the previous argument applies word-by-word to $S'$ even if some point are neither regular or cone point of angle $2k\pi$ for some $k$. Hence

$$2\pi \chi(S') - \sum_{q \in S'} 2\pi - \theta_q < 0.$$ 

By symmetry we get the desider result.

2.2. Holonomy representation. Let $\tilde{S}$ be the universal cover of $S$ and let $\pi : \tilde{S} \rightarrow S$ be the covering projection. A hyperbolic cone-structure structure $\sigma$ on $S$ can be lifted to a hyperbolic cone-structure $\tilde{\sigma}$ on the universal cover $\tilde{S}$.

Definition 2.3. Let $\sigma$ be a hyperbolic cone-structure on $S$ and $\tilde{\sigma}$ the lifted hyperbolic cone-structure on $\tilde{S}$. A developing map $\text{dev}_\sigma : \tilde{S} \rightarrow \mathbb{H}^2$ for $\sigma$ is a smooth orientation-preserving map, with isolated critical points and such that its restriction to any simplex on $\tilde{S}$ is an isometry.

Developing maps always exist, and are essentially unique; that is two developing maps for a given structure $\sigma$ differ by post-composition with a Möbius transformation. Explicitly a developing map can be constructed starting from a geodesic simplex $\tilde{s}_0$ of $\tilde{\sigma}$. Since it is isometric to a geodesic triangle $T_0 \subset \mathbb{H}^2$, there exists an isometry $\varphi_0 : \tilde{s}_0 \rightarrow T_0 \subset \mathbb{H}^2$. Let $\tilde{s}_1$ be another simplex which is adjacent to $\tilde{s}_0$; that is $\tilde{s}_1$ shares an edge with $\tilde{s}_0$. Then the isometry $\varphi_1 : \tilde{s}_1 \rightarrow T_1 \subset \mathbb{H}^2$ may be adjusted by a Möbius transformation so as to agree on the overlap, gluing to give a map $\tilde{s}_0 \cup \tilde{s}_1 \rightarrow \mathbb{H}^2$. We may iterate this procedure and at the limit we get a developing map for $\sigma$.

Basically, any developing map gives a way to read the geometry of $\sigma$ on the hyperbolic plane, hence post-compose a developing map $\text{dev}_\sigma$ for $\sigma$ with any element of group $\text{PSL}_2\mathbb{R}$ (which is the group of orientation preserving isometries) does not change the informations encoded on the developed image.

Remark 2.4. For hyperbolic cone-structures, the developing map dev turns out to be a branched map. Branch points are given by cone points of the hyperbolic cone-structure $\tilde{\sigma}$ on $\tilde{S}$. Around them, the developing map fails to be a local homeomorphism and the local degree coincides with the order of the cone point.

The developing map $\text{dev}_\sigma : \tilde{S} \rightarrow \mathbb{H}^2$ of hyperbolic cone-structure $\sigma$ has also an equivariance property with respect to the action of $\pi_1S$ on $\tilde{S}$. For any element $\gamma$, the composition map $\text{dev}_\sigma \circ \gamma$ is another developing map for $\sigma$. Thus there exists an element $g \in \text{PSL}_2\mathbb{R}$ such that

$$g \circ \text{dev}_\sigma = \text{dev}_\sigma \circ \gamma.$$ 

The map $\gamma \mapsto g$ defines a homomorphism $\rho : \pi_1S \rightarrow \text{PSL}_2\mathbb{R}$ which is called holonomy representation. The representation $\rho$ depends on the choice of the developing map, however different choices produce conjugated
representations. Hence it makes sense to consider the conjugacy class of \( \rho \), which is usually called holonomy for the structure.

**Definition 2.5.** Let \( S \) be a closed surface of genus \( g \geq 2 \). A representation \( \rho : \pi_1 S \to \text{PSL}_2 \mathbb{R} \) is said to be **Fuchsian** if it arises as holonomy of a complete hyperbolic structure on \( S \). In particular these representations turn out be faithful and discrete.

Goldman shows in [9] that any Fuchsian representation arises as holonomy of a unique complete hyperbolic structure, that is a hyperbolic structure without cone points. However, the picture changes completely as soon as we consider non-complete hyperbolic structure. Although any hyperbolic cone-structure \( \sigma \) on a 2–manifold \( S \) induces a holonomy representation by standard arguments; the reverse problem to recover a hyperbolic geometry starting from a given representation \( \rho \) is much arduous and not always possible as shown in the following example.

**Example 2.6.** The following example is a generalization of Tan’s counterexample (see [22]); which was given for a surface of genus 3.

Let \( S \) be a genus \( g \) surface, obtained by attaching \( h \) handles to a surface of genus \( g - h \), where \( g - h \geq 2 \). We define a representation \( \rho \) in the following way: \( \rho \) is discrete and faithful on the original surface, and trivial on each handle we have attached. In this way \( \rho(\pi_1 S) \) is a discrete subgroup of \( \text{PSL}_2 \mathbb{R} \) and the quotient \( \mathbb{H}^2 / \rho(\pi_1 S) \) is a genus \( g - h \) surface. However \( \rho \) can not be the holonomy of a hyperbolic cone-structure on \( S \).

Suppose now that \( S \) admits a hyperbolic cone-structure \( \sigma \) with holonomy \( \rho \), and consider its developing map \( \text{dev}_\sigma : \tilde{S} \to \mathbb{H}^2 \). Since \( \text{dev}_\sigma \) is a \((\pi_1 S, \rho(\pi_1 S))\)-equivariant map; it passes down to branch map

\[
 f : S \to \rho(\pi_1 S) \backslash \mathbb{H}^2
\]

Consider now the induced map of fundamental groups. This is the same map induced by the map that pinches to a point each handle we have attached before, hence the map \( f \) is homotopic to a pinching map of degree one. Since any branch cover of degree one is just a homeomorphism we found a contradiction, that is \( \rho \) cannot be the holonomy of a hyperbolic cone-structure.

Hence the following definition makes sense.

**Definition 2.7.** A representation \( \rho : \pi_1 S \to \text{PSL}_2 \mathbb{R} \) is said to be **geometrically realizable** if it arises as holonomy of a hyperbolic cone-structure \( \sigma \) on \( S \). Equivalently, a representation \( \rho \) is geometrically realizable if there exists a possibly branched developing map \( \text{dev}_\sigma : \tilde{S} \to \mathbb{H}^2 \) which is \( \rho \)-equivariant.

Of course, Fuchsian representations are geometrically realizable by a unique complete hyperbolic structure, whereas elementary representations are never geometrically realizable by a hyperbolic cone-structure (see 3.3).

**2.3. Geometry of hyperbolic transformations.** In the sequel we shall need to consider the effect of composing several isometries. For the remainder of this section, we have some lemmata about commutators. The begin with the following lemma by Goldman (see [7, Lemma 3.4.5])

**Lemma 2.8.** Let \( g, h \) be hyperbolic transformations. Then the following are equivalent

- \( g, h \) are hyperbolic and their axes cross,
- \( \text{Tr}[g,h] < 2 \).

**Remark 2.9.** Note that although \( g, h \) are only defined up to sign in \( \text{SL}_2 \mathbb{R} \), the commutator is a well-defined element of \( \text{PSL}_2 \mathbb{R} \), and has a well-defined trace (see also 2.4).

**Proof.** Assuming \( \text{Tr}[g,h] < 2 \), we first show that both \( g \) and \( h \) must be hyperbolic. If \( g \) was elliptic, up to conjugation we may assume that \( g \in \text{SO}_2 \mathbb{R} \) and a straightforward computation show that \( \text{Tr}[g,h] = 2 + \sin^2 \theta(a^2 + b^2 + c^2 + d^2 - 2) \geq 2 \). The same holds if \( g \) is parabolic, so \( g \) must be hyperbolic. The same argument shows that \( h \) must be hyperbolic.

The second step is to show that \( \text{Tr}[g,h] < 2 \) if and only if \( \text{Axis}(g) \) and \( \text{Axis}(h) \) cross. Up to conjugation we may assume that the fixed points for \( g \) are \( \pm 1 \), and that the fixed points for \( h \) are \( r, \infty \). Then we write \( \text{Tr}[g,h] \) as function on \( r \), and it easy to see that \( \text{Tr}[g,h] < 2 \) if and only if \( -1 < r < 1 \).

Before consider the other cases, we list some lemmata about the fixed point(s) of a commutator when \( g, h \) are hyperbolic and their axes cross. We denote by \( g^+ \) and \( g^- \) the attractive and repulsive points of a hyperbolic transformation \( g \).

**Lemma 2.10.** Suppose \( g, h \) are hyperbolic and \( \text{Tr}[g,h] < -2 \), so they are hyperbolic and their axes intersect. Then \( \text{Axis}[g,h] \) does not intersect the axis of \( g \) or \( h \). Moreover the fixed points of \([g,h]\) lie on the segment of the circle at infinity between \( g^+ \) and \( h^+ \): \( [g,h]^+ \) is closer to \( g^+ \) and \( [g,h]^− \) is closer to \( h^+ \).
We have also two similar results when \([g, h]\) is parabolic or elliptic.

**Lemma 2.11.** Suppose \(g, h\) are hyperbolic and \(\text{Tr}[g, h] = -2\), so it is parabolic. Then \(\text{Fix}[g, h]\) lies on the segment of the circle at infinity between \(g^+\) and \(h^+\). The sense of rotation is clockwise if the segment from \(g^+\) to \(h^+\) has the clockwise orientation, otherwise the sense is counterclockwise.

**Lemma 2.12.** Suppose \(g, h\) are hyperbolic and \(-2 < \text{Tr}[g, h] < 2\), so it is elliptic. Then \(\text{Fix}[g, h]\) lie in the region determined by \(\text{Axis}(g)\), \(\text{Axis}(h)\) which is bounded by the arc on the circle at infinity between \(g^+\) and \(h^+\). The sense of rotation is clockwise if the segment from \(g^+\) to \(h^+\) has the clockwise orientation, otherwise the sense is counterclockwise.

These lemmata may be proved with a direct computation. Here we offer the following proof by Matelski [14] which is more elegant and revealing. The first arguments of the proof of 2.10 are the same of the proofs of 2.11 and 2.12, hence we may prove the lemmata in a unique way and then discussing case by case.

*Proofs of lemmata 2.10, 2.11 and 2.12.* Let \(2\lambda_r, 2\lambda_h\) be the translation distance of \(g, h\) and let \(p \in \mathbb{H}^2\) be the point of intersection of the axes of \(g\) and \(h\) and let \(e \in \text{PSL}_2\) a half turn around \(p\). So we have that \(ege = g^{-1}\) and the same holds for \(h\), further he preserves \(\text{Axis}(h)\) but reversed its sense. Thus \(he\) is an elliptic element of \(\text{PSL}_2\) and let \(g\) be its fixed point; observe that \(g \in \text{Axis}(h)\) and it lies between \(p\) and \(h^+\) at a distance \(\lambda_h\) from \(p\). Now consider \(geh\), we have that \((geh)^2 = gb(ege)(che) = gbge^{-1}h^{-1} = [g, h]\). So \(geh\) is a hyperbolic, parabolic or an elliptic transformation, if \([g, h]\) is hyperbolic, parabolic or elliptic respectively. Let \(l_1\) be the perpendicular line from \(g\) to \(\text{Axis}(g)\), and denote \(r\) its foot. Let \(l_2\) be the perpendicular line to \(l_1\) passing through \(q\). Let \(s\) be point along \(\text{Axis}(g)\) between \(r\) and \(g^+\) at a distance \(\lambda_q\) from \(r\). Finally let \(l_3\) be the line passing through \(s\) perpendicular to \(\text{Axis}(g)\). Denote by \(R_l\) the reflection respect the line \(l_1\). Then we have \(he = R_l R_1\), and \(g = R_{l_1} R_{l_3}\), so \(geh = R_{l_1} R_{l_3} R_{l_1}\). Now we have the following trichotomy.

1. The axes \(l_2, l_3\) do not intersect in \(\mathbb{H}^2\) nor in the boundary at infinity. In this case \([g, h]\) is hyperbolic and \(\text{Axis}(g, h)\) is the common perpendicular of \(l_2\) and \(l_3\). In particular the fixed points are in the desired order and this conclude the proof of 2.10.
2. If we are not in the first case the axes cross. In particular if \(l_2, l_3\) intersect at the infinity then \([g, h]\) is parabolic and the fixed point is given by the intersection point. Moreover the fixed point is in the desired position and this conclude 2.11.
3. Finally \(l_2, l_3\) intersect at a point \(o\) and \([g, h]\) is elliptic with mixed point \(o\). By construction the fixed point lies in the desired region of \(\mathbb{H}^2\) and this conclude 2.12.

We conclude with other two lemmata that summarize the remaining cases.

**Lemma 2.13.** Let \(g\) be a parabolic element and let \(h\) be any transformation. Suppose that \([g, h]\) have no common fixed point. Then \([g, h]\) is hyperbolic.

**Lemma 2.14.** Let \(g\) be an elliptic transformation with rotation angle \(2\theta\) and let \(p\) its fixed point. Let \(h\) be any transformation not fixing \(p\). Then \([g, h]\) is hyperbolic.

We do not report here the proofs of lemmata 2.13, 2.14 that can be found in [1, Chapter 7].

2.4. The Lie groups \(\text{PSL}_2\) and \(\text{PSL}_2\). Geometrically \(\text{PSL}_2\) is an open solid torus homeomorphic to \(\mathbb{H}^2 \times S^1\). Indeed we may identify \(\text{PSL}_2\) with the unit tangent bundle \(UT_{\mathbb{H}^2}\), which is homeomorphic to \(\mathbb{H}^2 \times S^1\). However this identification depends on a preliminary choice of a basepoint \((p_0, u_0) \in \mathbb{H}^2 \times S^1\), hence is not canonical in general. More precisely we may associate to any element \(g \in \text{PSL}_2\) the point \((g(p_0), g'(u_0))\) and simple arguments show that this correspondence is well defined and bijective.

Of course we have \(\pi_1(\text{PSL}_2) \cong \mathbb{Z}\) and the universal cover \(\tilde{\text{PSL}}_2\) is naturally identified with \(\mathbb{H}^2 \times \mathbb{R}\). By classical covering theory, \(\text{PSL}_2\) may be seen also as the set of paths \(\{c : [0, 1] \rightarrow \text{PSL}_2\} = \{c : [0, 1] \rightarrow \text{UT}_{\mathbb{H}^2}\}\) up to homotopy starting from the basepoint. Roughly speaking any element of the universal cover may be seen as a path with a unit tangent vector attached to any point that changes continuously, regardless of where it start because the basepoint is arbitrary. By construction the projection of \(c \in \text{PSL}_2\) to \(\text{PSL}_2\) is the unique isometry sending the unit tangent vector at \(c(0)\) to the unit tangent vector at \(c(1)\).

Any element \(c \in \text{PSL}_2\) is elliptic, parabolic or hyperbolic accordingly as is its projection. The identity element lifts to an infinite cyclic subgroup generated by \(z\), namely the center of \(\text{PSL}_2\) which is isomorphic to \(\mathbb{Z}\). In particular these lifts correspond to those paths starting and ending at basepoint \((p_0, u_0)\) of the following form

\[c(t) = (p_0, e^{2\pi t c})\]

for some \(n \in \mathbb{Z}\). With this notation \(z = (p_0, e^{2\pi i})\).
Any element \( g \in \text{PSL}_2 \mathbb{R} \) has infinitely many lifts that differ by a power of \( z \). However we may wonder if there is a nicest lift of \( g \) in some sense and the answer turns out to be positive if \( g \) is hyperbolic or parabolic.

Suppose \( g \) is hyperbolic, hence it is a translation along its axis \( \text{Axis}(g) \) by some distance \( d \). Then there exists a unique one parameter subgroup \( c : \mathbb{R} \to \text{PSL}_2 \mathbb{R} \) (with a little abuse of notation) such that \( c(t) \) is a hyperbolic translation along \( \text{Axis}(g) \) of distance \( |t|d \). In particular \( c(0) = \text{id} \) and \( c(1) = g \). Its restriction to \([0, 1] \) gives a unique path in \( \text{PSL}_2 \mathbb{R} \) which we define as the \textit{preferred or simplest} lift of \( g \).

A similar argument works also for parabolic isometries. Indeed if \( g \) is parabolic then it translates along a horocircle \( h \) by some distance \( d \) with respect to the Euclidean metric induced by the hyperbolic one on \( h \). As above there exists a unique one parameter subgroup \( c : \mathbb{R} \to \text{PSL}_2 \mathbb{R} \) such that \( c(t) \) is a parabolic translation along \( h \) of distance \( |t|d \) and \( c(0) = \text{id} \) and \( c(1) = g \). Again its restriction to \([0, 1] \) gives a unique path in \( \text{PSL}_2 \mathbb{R} \) which we consider as preferred lift.

On the other hand the situation changes dramatically when we consider elliptic elements. If \( g \) is an elliptic isometry then there are infinitely many one parameter subgroups \( c : \mathbb{R} \to \text{PSL}_2 \mathbb{R} \) with \( c(1) = g \), and this is reflected by the fact that anyone of them contains the cyclic subgroup generated by \( z \), \emph{i.e.} the center of \( \text{PSL}_2 \mathbb{R} \). Thus a simplest lift does not exists but there are two simplest lifts, which are respectively the simplest counterclockwise lift \( c_1 \) and the simplest clockwise lift \( c_{-1} \).

We denote the set of simplest lift of hyperbolic and parabolic elements by \( \text{Hyp}_0 \) and \( \text{Par}_0 \). For every hyperbolic element \( c \in \text{PSL}_2 \mathbb{R} \) there exists a unique \( m \in \mathbb{Z} \) such that \( z^{-m}c \in \text{Hyp}_0 \), thus we define \( \text{Hyp}_m = z^m \text{Hyp}_0 \). In the same way \( \text{Par}_m = z^m \text{Par}_0 \).

\textbf{Remark 2.15.} We may furthermore divide \( \text{Par}_m \) into two subsets, namely \( \text{Par}^-_m \) and \( \text{Par}^+_m \) of parabolic elements which are clockwise and counterclockwise rotations about a point at infinity respectively. This distinction arises because clockwise rotations about a point at infinity are never conjugated to a counterclockwise rotation of \( \text{PSL}_2 \mathbb{R} \) (even if they are in \( \text{PSL}_2 \mathbb{C} \)).

Finally we define \( \text{Ell}_1 \) the set of simplest counterclockwise lifts of elliptic elements in \( \text{PSL}_2 \mathbb{R} \). Similarly \( \text{Ell}^-_1 \) is the set of simplest clockwise lifts of elliptic elements in \( \text{PSL}_2 \mathbb{R} \). For any \( m > 0 \) we define as in the other case \( \text{Ell}_m = z^m \text{Ell}_1 \) and \( \text{Ell}^-_m = z^{-m+1} \text{Ell}^-_1 \). Since the set \( \text{Ell}_0 \) is not define we have \( \text{Ell}_1 = z \text{Ell}^-_1 \).

\textbf{2.5. Relationship between trace and commutators.} We finally consider commutators of elements in \( \text{PSL}_2 \mathbb{R} \) and we explain briefly the relation with their trace. Any hyperbolic isometry is characterized by its trace. A similar characterization holds also for elements in \( \text{PSL}_2 \mathbb{R} \). Since \( \text{PSL}_2 \mathbb{R} \) is the universal cover of \( \text{PSL}_2 \mathbb{R} \) it covers also \( \text{SL}_2 \mathbb{R} \), hence the notion of \textit{trace} is well-defined in \( \text{PSL}_2 \mathbb{R} \).

\textbf{Lemma 2.16.} Let \( \overline{\text{Tr}} \) be composition of the covering projection \( \text{PSL}_2 \mathbb{R} \to \text{SL}_2 \mathbb{R} \) with the trace function \( \text{Tr}: \text{SL}_2 \mathbb{R} \to \mathbb{R} \). Then it is continuous and

\begin{enumerate}
  \item \( \overline{\text{Tr}}(z^n) = 2(-1)^n \)
  \item \( \overline{\text{Tr}}(\text{Par}_n) = 2(-1)^n \)
  \item \( \overline{\text{Tr}}(\text{Hyp}_n) \) is the open interval \([2, \infty[\) if \( n \) is even or the open interval \(]-\infty, -2[\) if \( n \) is odd.
\end{enumerate}

The proof of this result may be found in [18]. We now consider commutators in \( \text{PSL}_2 \mathbb{R} \). Since different lifts of any element \( g \in \text{PSL}_2 \mathbb{R} \) differ by powers of \( z \), the following lemma immediately follows.

\textbf{Lemma 2.17.} Let \( g, h \in \text{PSL}_2 \mathbb{R} \). Then \( [g, h] \) has a well-defined lift to \( \text{PSL}_2 \mathbb{R} \). That is, any couple of lifts \( \tilde{g}_1, \tilde{h}_1 \) and \( \tilde{g}_2, \tilde{h}_2 \) satisfy \( \tilde{g}_1, \tilde{h}_1 = [\tilde{g}_2, \tilde{h}_2] \).

\textbf{Proof.} Let \( \tilde{g}_2 = z^n \tilde{g}_1 \) and \( \tilde{h}_2 = z^m \tilde{h}_1 \). Since \( z \) commutes with every element of \( \text{PSL}_2 \mathbb{R} \) we notice that

\[ [\tilde{g}_1, \tilde{h}_1] = [\tilde{g}_2, \tilde{h}_2] \]

as desired. \( \square \)

Even if the lift of a commutator \( [g, h] \) is well-defined, it may differ from the simplest lift. More precisely its simplest lift belongs to \( \text{Hyp}_0 \), however for any couples of lifts \( \tilde{g}, \tilde{h} \) there exists an integer \( n \) such that

\[ [\tilde{g}, \tilde{h}] = z^n [g, h] \]

The previous lemma says that this integer does not depend on the choice of the lifts and the next proposition tell us all possible values of \( n \). We state it without proof that can be found in [9, 15, 18, 19, 23].
Proposition 2.18. Let \( g, h \in \text{PSL}_2 \mathbb{R} \), then \([g, h]\) is well-defined and belongs
\[
\left[ \tilde{g}, \tilde{h} \right] \in \{1\} \cup \left( \bigcup_{n=-1}^{1} \text{Hyp}_n \cup \text{Ell}_n \right) \cup \text{Par}_0 \cup \text{Par}_{-1}^+ \cup \text{Par}_1^-
\]
where \( \text{Ell}_0 \) is the empty set for convenience.

Combining 2.16 with 2.18 we get the following corollary.

Corollary 2.19. Let \( g, h \in \text{PSL}_2 \mathbb{R} \) then
\[
1 \quad \text{Tr}[g, h] > 2 \implies [g, h] \in \text{Hyp}_0, \\
2 \quad \text{Tr}[g, h] = 2 \implies [g, h] \in \text{Par}_0, \\
3 \quad \text{Tr}[g, h] \leq -2 \implies [g, h] \in \text{Ell}_1, \\
4 \quad \text{Tr}[g, h] = -2 \implies [g, h] \in \text{Par}_1^+ \cup \text{Par}_1^-, \\
5 \quad \text{Tr}[g, h] < -2 \implies [g, h] \in \text{Hyp}_-1 \cup \text{Hyp}_1.
\]

3. Euler class of representations

Throughout this section, \( S \) will be a compact surface of genus \( g \). For every representation \( \rho : \pi_1 S \to \text{PSL}_2 \mathbb{R} \) we may naturally associate a \( \mathbb{R}P^1 \)-bundle \( F_\rho \) over \( S \) equipped with a flat connection. Explicitly \( F_\rho \) is obtained as the quotient of \( S \times \mathbb{R}P^1 \) by the diagonal action of \( \pi_1 S \); i.e. for any \( \gamma \in \pi_1 S \) and \((p, z) \in S \times \mathbb{R}P^1 \) we have \( \gamma \cdot (p, z) = (\gamma p, \rho(\gamma)(z)) \). The Euler class \( e(\rho) \) of \( \rho \) arises naturally as an obstruction to finding global sections of this bundle.

3.1. Geometric definition of the Euler class. Suppose \( S \) is closed. Let \( \tau \) be a topological triangulation, then a section \( s_0 \) can be easily found on the \( 0 \)-skeleton choosing an element of \( \mathbb{R}P^1 \) above every vertex. This section can be extended to a section \( s_1 \) over the \( 1 \)-skeleton joining the \( 0 \)-sections by paths of \( \mathbb{R}P^1 \)-elements. Since \( \pi_1(\mathbb{R}P^1) = \mathbb{Z} \) there are infinitely many extensions of \( s_0 \) up to homotopy. Over any \( 2 \)-cell \( T \), the section over \( 1 \)-skeleton defines a \( \mathbb{R}P^1 \)-vector field along \( \partial T \), hence a map \( e_T : \partial T \to \mathbb{R}P^1 \) of degree \( d_T \) that corresponds to the number of times the vector field spins along \( \partial T \). We may assign to every \( 2 \)-cell the integer \( d_T \) giving a \( 2 \)-cochain \( e(\rho) \in H^2(S, \mathbb{Z}) \).

In determining \( e(\rho) \) we made different choices as the triangulation \( \tau \) and the \( 1 \)-section over the \( 1 \)-skeleton. Adjustment by a \( 2 \)-coboundary corresponds to altering the amount of spin chosen along each particular edge. Hence the cohomology class of this \( 2 \)-cochain does not depend on the choice of \( 1 \)-section. Moreover it can be seen that this cohomology class does not depend on the cellular decomposition of our surface \( S \). Thus \( e(\rho) \) is a well-defined \( 2 \)-cocycle called Euler class of \( \rho \) of \( F_\rho \). Since \( H^2(S, \mathbb{Z}) \cong \mathbb{Z} \) we can associate to \( e(\rho) \) the integer \( E(\rho) \) using the Kronecker pairing. We define \( E(\rho) \) as the Euler number associates to \( \rho \).

Lemma 3.1. The Euler number satisfies the following equality
\[
E(\rho) = \sum_{T \in \tau} d_T.
\]

Proof. Let \( [S] \) be the fundamental class of \( S \), that is a generator of \( H_2(S, \mathbb{Z}) \). Now \( [S] = [T_1] + \cdots + [T_n] \), then
\[
E(\rho) = e(\rho)[S] = \sum_{T \in \tau} e(\rho)[T] = \sum_{T \in \tau} d_T
\]
where the last equality holds by definition of \( e(\rho) \).

In [23] Wood, based on earlier work by Milnor [19], showed that the Euler number satisfies the following inequality (which is actually known as Milnor-Wood inequality)
\[
|E(\rho)| \leq -\chi(S).
\]

The equality holds as soon as the representation is Fuchsian, that is faithful and discrete, and they always arise as the holonomy of a unique and complete hyperbolic structure.

Theorem 3.2 (Goldman [9]). Let \( S \) be a closed orientable surface with \( \chi(S) < 0 \), and let \( \rho : \pi_1 S \to \text{PSL}_2 \mathbb{R} \). Then \( \rho \) is the holonomy of a complete hyperbolic structure on \( S \) if and only if \( E(\rho) = \pm \chi(S) \).

Now suppose \( \rho \) is a geometric representation, that is \( \rho \) is the holonomy of a hyperbolic cone-structure on \( S \). Let \( p_1, \ldots, p_n \) be the cone points of orders \( k_1, \ldots, k_n \), respectively. The following formula relates the Euler number of \( \rho \) with the Euler characteristic and the orders of the cone points.
Proposition 3.3. Let $\rho : \pi_1 S \rightarrow PSL_2 \mathbb{R}$ be a representation which is the holonomy of a hyperbolic cone-structure on a closed surface $S$. Then Euler number satisfies the identity

$$E(\rho) = \pm \left( \chi(S) + \sum_{i=1}^{n} k_i \right)$$

where the sign depends on the orientation of $S$.

Proof. Among different proofs in literature we use the following argument of Mathews [17]. Let $\tau$ be a hyperbolic triangulation, such that every cone point is a vertex of the triangulation, so we have a simplicial decomposition of $S$ with hyperbolic triangles. There is a $\mathbb{RP}^1$-vector field $V$ on $S$ with one singularity for every vertex, edge and face of $S$. The orders of the singularities are $1 + k_i$ at any vertex (remember that for regular points $k = 0$), $-1$ on every edge, and $1$ on every face. By the Hopf-Poincaré theorem the sum of the indices of the singularities equals the sum of the indices of the singularities, then $\chi(S) + \sum k_i$. Now perturb the vector field so that the singularities lie off the $1$–skeleton. Then the number of times the vector field spins around a triangle $T \in \tau$ is equal to the sum of the indices of singular points of $V$ inside $T$, or its negative, depending on whether the orientation induced by dev is the same as the orientation induced by the fundamental class $[S]$. For now, assume these orientations agree; otherwise all the cohomology classes must be multiplied by $-1$. Hence the spin of $V$ around any triangle $T \in \tau$ is equal to the sum of indices of the singular point of $V$ inside $T$ which is in turn equal to the degree of the map $s_T : \partial T \rightarrow \mathbb{RP}^1$ defined above. By 3.1 the sum of all indices of singular points is equal to $E(\rho)$, hence

$$E(\rho) = \pm \left( \chi(S) + \sum_{i=1}^{n} k_i \right).$$

$\square$

Remark 3.4. As expected if $\rho$ is the holonomy of hyperbolic structure on $S$ without cone points, then every point in $S$ is regular and we found again the above equality $E(\rho) = \pm \chi(S)$.

Remark 3.5. If $\sigma$ is a hyperbolic cone-structure on $S$, the Gauß-Bonnet condition implies that Euler number is never zero. Since the Euler number of elementary representations is always zero (see [9]), they never arise as the holonomy of a hyperbolic cone-structure on $S$.

Remark 3.6. The Euler number of a representation that is the holonomy of a hyperbolic cone-structure is negative because the developing map of a hyperbolic cone-structure is assumed to be orientation-preserving.

Suppose now $S$ has boundary. We may define the relative Euler class in the same way described above, but we first need to define a trivialization over the boundary. In the case of a surface without boundary, it does not matter how we extend the $0$–section along the $1$–skeleton since each edge belongs to two faces, different choices cancel each other out. Here $S$ has boundary, and again it does not matter how we extend the $0$–section over edges lying in the interior of our surface. However each boundary edge belongs to only one face, then here it does matter. Hence the right thing to do is to define a trivialization along the boundary, that is a $1$–section, and extend such section to a $1$–section over the $1$–skeleton.

Let $\gamma \subset \partial S$ be a boundary component and suppose that $\rho(\gamma)$ has non-elliptic holonomy. A special trivialization along $\gamma$ is the datum of a section $s : \gamma \rightarrow F_{\rho,\gamma}$ defined by following a fixed point of $\rho(\gamma) \in \mathbb{RP}^1$ along $\gamma$ using the flat connection associate to $\mathbb{RP}^1$–bundle. Note that a special trivialization exists whenever $\rho(\gamma)$ has non-elliptic holonomy and it does not depend on the choice of the fixed point. Thus the relative Euler class is a $2$–cochain $e(\rho, s) \in H^2(S, \partial S, \mathbb{Z})$, and it measures the obstruction to extend the special trivialization along the boundary over $S$. In the same way, the relative Euler number is an integer $E(\rho, s)$ defined using the Kronecker pairing, and the Milnor–Wood inequality is satisfied as well (for further details see [9]).

Definition 3.7. Let $S$ be a compact connected orientable surface with boundary. We define Fuchsian those representations $\rho : \pi_1 S \rightarrow PSL_2 \mathbb{R}$ such that $|E(\rho, s)| = -\chi(S)$ with respect to the special trivialization $s$.

As in the closed case Fuchsian representations arise as holonomy of a complete hyperbolic structure on $S$, precisely we have the following result which was proved by Goldman in [9] when $S$ has boundary with hyperbolic holonomy and more generally in the non compact case by Mathews in [17] and by Burger-Iozzi and Wienhard in [3].

Theorem 3.8. Let $S$ be a compact connected orientable surface with $\chi(S) < 0$, and let $\rho : \pi_1 S \rightarrow PSL_2 \mathbb{R}$. If $S$ has boundary, assume $\rho$ takes each boundary curve to a non-elliptic element, so the relative Euler class $E(\rho, s)$ is well-defined. Then $\rho$ is the holonomy of a complete hyperbolic structure on $S$ with totally geodesic or cusped boundary components (respectively as each boundary curve is taken by $\rho$ to a hyperbolic or parabolic) if and only if $E(\rho, s) = \pm \chi(S)$. 

Let $S$ be a surface (possibly with boundary) and decompose $S$ in pieces, i.e. subsurfaces; such that any the relative Euler number is well-defined for any piece. Then the Euler number of $\rho$ can be computed in terms of the relative Euler numbers of each piece with respect to the special trivialization along the boundary. More precisely we have the following lemma whose proof is immediate.

**Lemma 3.9.** Let $F_\rho$ be a $\mathbb{RP}^1-$bundle over $S$ with holonomy $\rho$, and let $\{l_k\}$ be a finite family of disjoint simple closed curves in $S$ containing also the boundary curves of $S$. Let $\overline{F}$ be a section of $F$ defined on $\{l_k\}$. Denote by $\{C_j\}$ the family of the closure of the connected components of $S \setminus \{l_k\}$, then

$$E(\rho, \overline{F}|_{\partial C}) = \sum_j E(\rho_{C_j}, \overline{F}|_{\partial C_j})$$

**Proof.** It is sufficient to observe that the spins along any common boundary cancel out so that the relative Euler class is additive. \qed

### 3.2. Algebraic definition of the Euler class.**

There is also an algebraic interpretation of the (possibly relative) Euler class. Let $S$ be a surface with genus $k$ and with $n$ boundary components ($n$ could be eventually zero) and let $\rho : \pi_1 S \to \text{PSL}_2\mathbb{R}$ be a representation such that $\rho(b_i)$ is not elliptic for every $i \in \mathbb{N}$ (if any).

Let $p$ be a base point in $S$, and let $\tilde{\rho}$ be a lift of $\rho$ in its universal cover, then the fundamental group $\pi_1(S,p)$ has the following presentation

$$\langle a_1,b_1,\ldots, a_k, b_k, c_1,\ldots, c_n|[a_1,b_1]\ldots [a_k,b_k]b_1\ldots b_n = 1 \rangle,$$

and defines a fundamental $(4k + n)$-gon in $S$ which is simply connected based at $p$. Set $g_i = \rho(a_i)$, $h_i = \rho(b_i)$ and $c_i = \rho(c_i)$.

Let $(p_0, u_0)$ be a basepoint in $\text{UTHP}^2 \cong \mathbb{H}^2 \times \mathbb{RP}^1$ and draw geodesics between points which are joined by edge in $S$ starting from $p_0$. This gives a $(4k + n)$-gon in $\mathbb{H}^2$ that may be concave, have self-intersection, or even worse it may be degenerate. We may think this point $(p_0, u_0)$ as a 0-section over $p$, indeed the projection to the second factor gives an element of $\mathbb{RP}^1$ that we take as 0-section over $p$. We now extend it to a 1-section in $S$ in the following way. First notice that there is a bijective correspondence between edges of the fundamental $(4k + n)$-gon in $S$ and edges of the respective polygon in $\mathbb{H}^2$ defined as above. We begin extending the 0-section to 1-section along $a_1$, the respective edge in $\mathbb{H}^2$ is the geodesic segment between $p_0$ and $\rho(a_1)(p_0)$. Consider the points $(p_0, u_0)$ and $(g_1(p_0), g_1(u_0))$, where $g_1 = \rho(a_1)$, then any lift $\tilde{g}_1$ of $g_1$ gives a unique path in $\text{UTHP}^2$ (up to homotopy relative to endpoints) of tangent vectors between these endpoints. We take as 1-section along $a_1$ the projection to the second factor of such path in $\text{UTHP}^2$. We can play the same game for the other edges to define a 1-section over 1-skeleton (where along any boundary edge $c_i$ we consider the section given by the special lift of $\rho(c_i)$). Moving anticlockwise around the polygon in $S$, we now obtain a loop in $\text{UTHP}^2$ which is represented by

$$[\tilde{g}_1, \tilde{h}_1] \ldots [\tilde{g}_k, \tilde{h}_k][\tilde{c}_1 \ldots \tilde{c}_n]$$

where $\tilde{g}_i = \tilde{\rho}(a_i)$ and $\tilde{h}_i = \tilde{\rho}(b_i)$ are arbitrarily lifts of $g_i, h_i$ and $\tilde{c}_i = \tilde{\rho}(c_i)$ are the simplest lifts in $\text{PSL}_2\mathbb{R}$.

Since $[a_1,b_1] \ldots [a_k,b_k]c_1 \ldots c_n = 1$ that product is equal to $z^m$ for some $m \in \mathbb{Z}$. Geometrically $m$ is the number of times the tangent vector field spins around the fundamental $(4k + n)$-gon in $S$. We have the following result.

**Proposition 3.10.** Let $S$ be an orientable surface with $\chi(S) < 0$. Let $\rho : \pi_1 S \to \text{PSL}_2\mathbb{R}$ be a representation, and let $\pi(S)$ have the presentation given above, where no $c_i$ is elliptic. The (possibly relative) Euler class $e(\rho)$ takes the fundamental class $[S]$ to $m \in \mathbb{Z}$ where the unique lift of the relator

$$[\tilde{g}_1, \tilde{h}_1] \ldots [\tilde{g}_k, \tilde{h}_k][\tilde{c}_1 \ldots \tilde{c}_n] \in \text{PSL}_2\mathbb{R}$$

is equal to $z^m$.

Then the Euler number of a representation $\rho : \pi_1 S \to \text{PSL}_2\mathbb{R}$ measure also the obstruction to lift it to a representation in $\text{PSL}_2\mathbb{R}$. In particular, a representation $\rho$ lifts to a representation in $\text{PSL}_2\mathbb{R}$ if and only if there exists a nowhere zero section of the associated $\mathbb{RP}^1-$bundle with holonomy $\rho$.

**Remark 3.11.** If $S$ has boundaries a representation $\rho$ lifts to a representation in $\text{PSL}_2\mathbb{R}$ if and only if there exists a nowhere zero section of the associated $\mathbb{RP}^1-$bundle with holonomy $\rho$ with respect to the special trivialization $s$ along the boundaries.

In the sequel, we will work with punctured torus, and we make a strong use of the following result.

**Proposition 3.12.** Let $S$ be a punctured torus and $\rho : \pi_1 S \to \text{PSL}_2\mathbb{R}$ be a representation such that the relative Euler class is well-defined. Then
1- Tr$[g,h] \leq -2$ if and only if $\mathcal{E}(\rho, s) = -1$, 
2- Tr$[g,h] \geq 2$ if and only if $\mathcal{E}(\rho, s) = 0$, 
where $s$ is the special trivialization along the boundary.

Proof. Suppose first Tr$[g,h] \leq -2$, then $[g,h] \in \text{Hyp}_{-1} \cup \text{Par}^{-}_0 \cup \text{Par}^{+}_0$ by 2.19. We may suppose without loss of generality that $[g,h] \in \text{Hyp}_{-1} \cup \text{Par}^{+}_1$, because the other case occurs reverting the orientation of $S$. Since $[g,h]c = 1$ in $\text{PSL}_2\mathbb{R}$, then $c^{-1} = [g,h]$ and its simple lift $\tilde{c}^{-1} \in \text{Hyp}_0 \cup \text{Par}^+ _0$ and the following holds $\tilde{c}^{-1} = \tilde{z}[g,h]$, thus $[g,h]\tilde{c} = \tilde{z}^{-1}$ and $\mathcal{E}(\rho, s) = -1$.

Now suppose Tr$[g,h] \geq 2$, then $[g,h] \in \text{Hyp}_0 \cup \text{Par}_0 \cup \{1\}$ by 2.19. Since $[g,h]c = 1$ in $\text{PSL}_2\mathbb{R}$, then $c^{-1} = [g,h]$ then $[g,h]\tilde{c} = 1$. Thus $\mathcal{E}(\rho, s) = 0$. \hfill \Box

4. Geometry and algebra of punctured torus

Throughout this chapter, let $H$ be a punctured torus. We prefer to use the letter $H$ here, instead of $S$ of $T$, because in the sequel will be useful to think punctured torus as a handle attached to a surface of lower genus than the original surface $S$.

4.1. Generality about the character variety. We start with some general facts about the character variety. Let $S$ be any surface, the representation variety $\text{Hom}(\pi_1 S, \text{SL}_2\mathbb{R})$ is defined as the set of all homomorphisms $\rho : \pi_1 S \longrightarrow \text{SL}_2\mathbb{R}$. Fix a presentation of $\pi_1 S$, then we may associate to any generator a matrix in $\text{SL}_2\mathbb{R}$ such that the matrices satisfy the condition of any relators. Considering the entries of matrices as coordinates variables, the set $\text{Hom}(\pi_1 S, \text{SL}_2\mathbb{R})$ may be seen as the solution set of some polynomial equations, then it is a closed algebraic variety. In general this variety has singularities.

The character $\chi_\rho$ of a representation $\rho$ is the function defined as follows: $\text{Tr} \circ \rho : \pi_1 S \longrightarrow \mathbb{R}$ given by $\text{Tr} \circ \rho (\alpha) = \text{Tr}(\rho(\alpha))$. By using well-known trace relations, is possible to see that the function $\chi_\rho$ is determined by its values at only finitely many elements $\alpha_1, \ldots, \alpha_n$ (see for instance [11]). We may define a function $T : \text{Hom}(\pi_1 S, \text{SL}_2\mathbb{R}) \longrightarrow \mathbb{R}^n$ that sends any representation $\rho \mapsto \left(\text{Tr}(\rho(\alpha_1)), \ldots, \text{Tr}(\rho(\alpha_n))\right)$ and we may define the character variety to be the image of such function, that is $X(S) = T\left(\text{Hom}(\pi_1 S, \text{SL}_2\mathbb{R})\right)$.

If $S$ is the punctured torus, the character variety of representations in $\text{PSL}_2\mathbb{R}$ can be taken as an obvious quotient and it will be described below. The case of closed surface will be consider in the sequel 5.1.

Remark 4.1. There is an action of $\text{SL}_2\mathbb{R}$ on the representation space $\text{Hom}(\pi_1 S, \text{SL}_2\mathbb{R})$ by conjugation. We quotient space may be taught as the moduli space of flat principal $\text{SL}_2\mathbb{R}$-bundle over $S$. In general the quotient space has singularities; however, away from singularities, such quotient space may be identified with the character variety.

4.2. Characters of the punctured torus representations. In this paragraph we deal with the characters of representations $\rho : \pi_1 H \longrightarrow \text{PSL}_2\mathbb{R}$ without considering geometric structures. Let $p \in H$ be a basepoint for the fundamental group and let $(\alpha, \beta)$ be a basis.

Any representation $\rho : \pi_1 H \longrightarrow \text{PSL}_2\mathbb{R}$ is uniquely determined by the images $\rho(\alpha)$ and $\rho(\beta)$. A representation into $\text{PSL}_2\mathbb{R}$ obviously lifts to $\text{SL}_2\mathbb{R}$, and we have two choices, each for the lifts of $\rho(\alpha)$ and $\rho(\beta)$. For now consider $\rho$ as a representation into $\text{SL}_2\mathbb{R}$ and denote $\rho(\alpha) = g$ and $\rho(\beta) = h$.

The character of $\rho$ is determined by the values of $\text{Tr} \circ \rho$ at finitely many elements of $\pi_1 H$. For the punctured torus with $\pi_1 H = \langle \alpha, \beta, \alpha \beta \rangle$, it is sufficient to consider only the three elements $\alpha, \beta, \alpha \beta$. Any word $w$ of $\pi_1 H$ may be written in terms of $\alpha, \beta$ and their inverses, and the trace of $\rho(w)$ can be expressed as a polynomial in $(x, y, z) = (\text{Tr}g, \text{Tr}h, \text{Tr}gh)$.

In our case we have the important relation $$\text{Tr}[g,h] = \text{Tr}^2g + \text{Tr}^2h + \text{Tr}^2gh - \text{Tr} \text{Tr}g \text{Tr}h - 2$$

and hence we define the polynomial $$k(x, y, z) = x^2 + y^2 + z^2 - xyz - 2$$

Any irreducible representation $\rho_1$ defines the same triple $(x, y, z)$ of another representation $\rho_2$, if and only if $\rho_1$ and $\rho_2$ are conjugate. In this case the triple $(\text{Tr}g, \text{Tr}h, \text{Tr}gh)$ defines the pair $g, h \in \text{SL}_2\mathbb{R}$ uniquely up to conjugacy.

A representation $\rho$ is said to be reducible if its image is a set of matrices such that, acting as linear transformations on $\mathbb{C}^2$, leaves invariant a line in $\mathbb{C}^2$. Of course, irreducible representations are those representations which
not reducible. As pointed out by Mathews in [16]; for irreducible representation \( \rho \) it is possible to deduce all the geometry of \( g \) and \( h \) considered as isometries of the hyperbolic plane.

The set of all \((x, y, z) = (\text{Tr}g, \text{Tr}h, \text{Tr}gh)\) is the character variety \( \mathcal{X}(H) \) of the punctured torus \( H \). In [7, Theorem 4.3], Goldman described the character variety \( \mathcal{X}(H) \).

**Theorem 4.2** (Goldman). Given \((x, y, z) \in \mathbb{R}^3\), there exist \( g, h \in \text{SL}_2\mathbb{R} \) such that \((x, y, z) = (\text{Tr}g, \text{Tr}h, \text{Tr}gh)\) if and only if
\[
\text{Tr} [g, h] = Tr^2 g + Tr^2 h + Tr^2 gh - Trg \text{Tr}h \text{Tr}gh - 2 \geq 2
\]
or at least one of \(|x|, |y|, |z| \) is greater than 2.

For representations \( \rho : \pi_1 H \rightarrow \text{PSL}_2 \mathbb{R} \), the character variety may be described starting from the character variety of representations into \( \text{SL}_2 \mathbb{R} \). There are four different ways to lift the couple \( \rho(\alpha), \rho(\beta) \) into \( \text{SL}_2 \mathbb{R} \), which are related by sign changes. Thus we simply take the character variety \( \mathcal{X}(H) \) of representations into \( \text{SL}_2 \mathbb{R} \) modulo the equivalence relation
\[
(x, y, z) \sim (-x, -y, z) \sim (-x, y, -z) \sim (x, -y, -z).
\]
induced by these four possible lifts.

**Remark 4.3.** The notion of reducibility still makes sense in \( \text{PSL}_2 \mathbb{R} \). Indeed any element of \( \text{PSL}_2 \mathbb{R} \) acts on \( \mathbb{C}^2 \) via linear transformations up to a reflection in the origin, hence the Riemann sphere \( \mathbb{CP}^1 \). Thus the idea of an invariant line still makes sense.

**Remark 4.4.** Also representations in \( \text{PSL}_2 \mathbb{R} \) the value of \( k(x, y, z) = \text{Tr}[g, h] \) is well-defined, even if the values of \( x, y, z \) are ambiguous!

Points with \( k(x, y, z) = 2 \) describe reducible representations, which include also abelian representations, as shown by the following lemma.

**Lemma 4.5.** A representation \( \rho : \pi_1 H \rightarrow \text{PSL}_2 \mathbb{R} \) (or \( \text{SL}_2 \mathbb{R} \)) is reducible if and only if the character of \( \rho \) is such that \( k(x, y, z) = 2 \).

From now on we will deal with only irreducible representations. Points with \( k(x, y, z) \neq 2 \) describe irreducible representations, hence describe a conjugacy class of representations precisely. For any \( t \neq 2 \), we define the relative character variety as the space of all representations (up to conjugacy) with \( \text{Tr}[g, h] = t \) by \( \mathcal{X}_t(H) = k^{-1}(t) \cap \mathcal{X}(H) \).

### 4.3. Virtually abelian representations

In this section we consider a special type of representation, namely **virtually abelian representations**. We dedicate an entire paragraph to them because virtually abelian representations will play a crucial role in the next section.

**Definition 4.6.** A representation \( \rho : \pi_1 H \rightarrow \text{PSL}_2 \mathbb{R} \) is said to be virtually abelian if its image contains an abelian subgroup of finite index.

Consider the following set of \( \mathbb{R}^3 \):
\[
V = \{0 \times 0 \times \mathbb{R} \setminus [-2, 2]\} \cup \{0 \times \mathbb{R} \setminus [-2, 2] \times 0\} \cup \{\mathbb{R} \setminus [-2, 2] \times 0 \times 0\}.
\]

Of course \( V \subset \mathcal{X}(H) \). The following result gives a complete characterization of this type of representations.

**Lemma 4.7.** Let \( \rho : \pi_1 H \rightarrow \text{PSL}_2 \mathbb{R} \) be a representation and let \( (\alpha, \beta) \) be any basis of \( \pi_1 H \). Then \( \rho \) is virtually abelian (but not abelian) if and only if \((\text{Tr}g, \text{Tr}h, \text{Tr}gh) \in V \), where \( g = \rho(\alpha) \) and \( h = \rho(\beta) \).

We refer [16, Lemma 4.9] for the proof. This type of representations has also a nice geometric description given by the following lemma.

**Lemma 4.8.** With the same notation of the previous lemma; a representation \( \rho : \pi_1 H \rightarrow \text{PSL}_2 \mathbb{R} \) is virtually abelian representation if and only if two of \( \{g, h, gh\} \) are half-turns about distinct points \( q_1, q_2 \) and the third is a non-trivial translation along the unique axis passing through \( q_1 \) and \( q_2 \).

**Proof.** The sufficient condition follows immediately. Indeed half-turns have trace 0 and a non-trivial translation has trace greater than 2 in magnitude, hence if \( \{g, h, gh\} \) are isometries of the required type, the triple \((\text{Tr}g, \text{Tr}h, \text{Tr}gh) \in V \).

We need to show the necessary condition, and we may suppose that \( \text{Tr}g = 0, \text{Tr}h = 0 \) and \( |\text{Tr}gh| > 2 \) since the other cases are similar. Hence \( g \) and \( h \) are half-turns about two points \( q_1, q_2 \). Suppose \( q_1 = q_2 \) then \( gh = \text{id} \), that is \( \text{Tr}gh = \pm 2 \) hence a contradiction. Since the points \( q_1, q_2 \) are distinct there is a unique geodesic line \( q_1q_2 \).
passing through them. Both $g$ and $h$ preserve such line reversing its orientation. Of course also the composition $gh$ preserve the line $q_1q_2$ maintaining the orientation (because the orientation is reversed two times). Since $gh \neq \text{id}$, we can conclude that it is a non-trivial translation along $q_1q_2$.

What makes virtually abelian representation interesting for our scopes is the following characterization theorem, which was proved by Mathews in [16].

**Theorem 4.9.** Let $H$ be a punctured torus and let $\rho : \pi_1 H \to \text{PSL}_2 \mathbb{R}$ be a representation. The following are equivalent:

1. $\rho$ is the holonomy representation of a hyperbolic cone-structure on $H$ with geodesic boundary, except for at most one corner point, and no interior cone points;
2. $\rho$ is not virtually abelian.

In particular, the corner angles of the hyperbolic cone-structures on the punctured torus described by this theorem range over all of $[0,3\pi]$. Indeed this can be easily checked using the Gauß-Bonnet theorem.

4.4. The action of $\text{MCG}(H)$ on the character variety. We finally consider the action of $\text{Aut}(\pi_1 H)$ on the character variety. The group $\text{Aut}(\pi_1 H)$ acts simply and transitively on the set of bases of $\pi_1 H$. This is equivalent to consider the effect of changing basis $(\alpha_1, \beta_1) \to (\alpha_2, \beta_2)$ on a representation $\rho : \pi_1 H \to \text{PSL}_2 \mathbb{R}$ by pre-composition. The overlying geometry remains unchanged under the action of $\text{Aut}(\pi_1 H)$; even if the character of the representation may change. Since trace is invariant under conjugation, this action descends to an action of

$$\text{Out}(\pi_1 H) = \frac{\text{Aut}(\pi_1 H)}{\text{Im}(\pi_1 H)} \cong \text{MCG}(H).$$

Here $\text{MCG}(H)$ is the mapping class group of $H$, i.e. the group of homeomorphisms of $H$ up to isotopy. We have the following theorem by Nielsen, see [20], [7] and [10].

**Theorem 4.10** (Nielsen). An automorphism $\psi$ of $\pi_1 H = \langle \alpha, \beta \rangle$ takes $[\alpha, \beta]$ to a conjugate of itself or its inverse.

**Remark 4.11.** A similar result holds also for closed surfaces; even if it does not for other surfaces with boundary (see [20] and [21]).

**Remark 4.12.** Viewing the punctured torus as the quotient of Euclidean plane by the action of two linearly independent translations with a lattice removed; we may easily see that the group $\text{MCG}(H)$ is isomorphic to $\text{GL}_2 \mathbb{Z}$.

By the Nielsen’s theorem $[\alpha_1, \beta_1]$ is conjugate to $[\alpha_2, \beta_2]^{\pm1}$, hence $\text{Tr}[g_1, h_1] = \text{Tr}[g_2, h_2]$ and $k(x_1, y_1, z_1) = k(x_2, y_2, z_2)$. That is, the triples $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ lie on the same level set of the polynomial $k$. Hence the action of the mapping class group of the punctured torus, $\text{MCG}(H)$, preserves the level set $k(x, y, z) = t$, that is preserves the relative character variety $\mathcal{X}_t(H)$ for any $t$.

When $t > 2$ any representation is irreducible and a character corresponds uniquely to a conjugacy class of representations. Goldman in [7] proved that there are only two different types of transformations in $\mathcal{X}_t(H)$, namely:

1. **Pants representation:** that is $(x, y, z)$ is the character of discrete representation $\rho : \pi_1 H \to \text{PSL}_2 \mathbb{R}$, which may be considered the holonomy of complete hyperbolic structure on a pair of pants. In particular there are no elliptics in the image of $\rho$. In this case: let $\overline{\rho} : \pi_1 H \to \text{SL}_2 \mathbb{R}$ be any lift of $\rho$, then up to change the basis of $\pi_1 H$ we may suppose that the character $\langle \text{Tr}\overline{\rho}(\alpha), \text{Tr}\overline{\rho}(\beta), \text{Tr}\overline{\rho}(\alpha \beta) \rangle$ lies in the octant $[-\infty,-2]^3$.
2. **Representation with elliptics:** that is $(x, y, z)$ is equivalent to another character with some coordinate in the interval $[-2,2]$. In this case $(x, y, z)$ is the character of a representation $\rho$ which sends a simple closed curve to an elliptic transformation. We denote this subset with $\Omega_t$.

The subset of pants representations in $\mathcal{X}_t(H)$ consists of a disjoint union of wandering domains that appear as soon as $t \geq 18$. Such domains arising from the intersection of $\mathcal{X}_t(H)$ with the Fricke space $\mathcal{F}(P)$ of a pair of pants $P$. We may observe that $\text{MCG}(H) \cong \text{Out}(\pi_1 H)$ preserves $\Omega_t$ and its complement in $\mathcal{X}_t(H)$.

For any $t$, there is a smooth symplectic structure on the space $\mathcal{X}_t(H)$, i.e. a 2–form $\omega_t$, which is closed and non-degenerate (see [8] for further details). Since the relative character variety $\mathcal{X}_t(H)$ is a closed 2–dimensional subspace of $\mathcal{X}(H)$, the symplectic 2–form $\omega_t$ is also an area form. The action of $\text{MCG}(H)$ is somewhat of bizarre, it changes as soon as we consider different ranges of the value $t$ (see [7, Main Theorem]). In particular when $t > 2$ we have the following theorem.
Finally, combining the previous result 5.1 togther with [13, Theorem 1.4], we get the following stronger result in the genus two case.

**Corollary 5.2.** Let S be a closed surface of genus 2. Then any representation $\rho : \pi_1 S \to PSL_2 \mathbb{R}$ with $E(\rho) = \pm 1$ is geometricizable by a hyperbolic cone-structure with one cone point of angle $4\pi$.

Before continuing with the geometrization problem, some preliminaries about the character variety of closed surfaces are in order.

### 5.1. Brief overview about the characters variety of a closed surface

In subsection 4.1 we stated some generalities about the representation and the character varieties for a general surface. After that, we have described the character variety of punctured torus with more details. Now we describe the character variety for a generic closed surface of genus $g \geq 2$. The representation variety describes all homomorphisms $\rho : \pi_1 S \to PSL_2 \mathbb{R}$; and it turns out to be a closed algebraic set. For a closed surface $S$ of genus $g \geq 2$ the representation variety $Hom(\pi_1 S, PSL_2 \mathbb{R})$ is not connected. If we vary a representation continuously, the Euler number $E(\rho)$ changes continuously but, since it is an integer, it remains constant. For closed surfaces, Goldman classified the components of $Hom(\pi_1 S, PSL_2 \mathbb{R})$ completely.

**Theorem 5.3** (Goldman [9]). Let $S$ be a closed surface of genus $g$ at least 2. Then the space $Hom(\pi_1 S, PSL_2 \mathbb{R})$ has $4g - 3$ connected components which are parametrised by the Euler number.

As remarked in 4.1, there is an action of $PSL_2 \mathbb{R}$ on the representation space $Hom(\pi_1 S, PSL_2 \mathbb{R})$ by conjugation. Such action preserves the connected components; hence the quotient space $X(S)$ still has $4g - 3$ connected components parametrized by the Euler number $E(\rho)$, and it may be identified (away from the singularities) with the character variety $X(S)$.

Singular points correspond to elementary representations. Since the action of $PSL_2 \mathbb{R}$ preserves the subset of non-elementary representations, we may consider the subset $Hom^{ne}(\pi_1 S, PSL_2 \mathbb{R})$ of all non-elementary representations. Without elementary representations, the quotient space $X^{ne}(S)$ turns out to be a symplectic smooth manifold of dimension $6g - 6$. It supports a smooth symplectic structure, i.e. a 2-form $\omega_S$ which is closed and non-degenerate, outside the singular locus. By taking an appropriate power of $\omega_S$, we obtain an area form on $X(S)$ hence a measure $\mu_S$.

**Remark 5.4.** Viewing $X(S)$ as a subset of some $\mathbb{R}^{2g}$, away from the singularities $\omega^n$ is some multiple of the standard Euclidean area form, in particular, $\mu_S$ is absolutely continuous with respect to the Lebesgue measure.

We define extremal components, those components parameterized by $|E(\rho)| = -\chi(S)$. These components turn out to be two diffeomorphic copies of the Teichmüller space. Similarly, we define almost extremal components, those components parameterize by $|E(\rho)| = -(\chi(S) + 1)$. Any representation $\rho$ of these components is defined as almost extremal representation.

The curious reader may see [9] for further details about the representation variety and [8] for more details about the symplectic structure on $X(S)$.

### 5.2. Preliminary discussion

We turn back to the geometrization problem. Let $S$ be a closed surface of genus $g \geq 2$, and let $\rho : \pi_1 S \to PSL_2 \mathbb{R}$ be a representation with $E(\rho) = \pm(\chi(S) + 1)$. In [17], Mathews proved the following result.

**Theorem 5.5** (Mathews). Let $S$ be a closed surface of genus $g \geq 2$. Then almost every representation $\rho : \pi_1 S \to PSL_2 \mathbb{R}$ with $E(\rho) = \pm(\chi(S) + 1)$, which sends a non-separating curve $\gamma$ on $S$ to an elliptic is the holonomy of a hyperbolic cone-structure on $S$ with one cone point of angle $4\pi$. 

**Theorem 4.13** (Goldman [7]). For any $t > 2$, the action of $MCG(H)$ on $\Omega_t$ is ergodic.

**Remark 4.14.** We note that $\Omega_t$ coincide with the whole relative character variety for $2 < t < 18$. 

5. Geometricizable representations with $E(\rho) = \pm(\chi(S) + 1)$

Throughout this section let $S$ be a closed surface of genus $g \geq 2$, and let $\rho : \pi_1 S \to PSL_2 \mathbb{R}$ be a representation with $E(\rho) = \pm(\chi(S) + 1)$. In this section we are going to prove the following theorem.

**Theorem 5.1.** Let $S$ be a closed surface of genus $g \geq 3$. Then every representation $\rho : \pi_1 S \to PSL_2 \mathbb{R}$ with $E(\rho) = \pm(\chi(S) + 1)$, which sends a non-separating simple curve $\gamma$ on $S$ to a non-hyperbolic element is the holonomy of a hyperbolic cone-structure on $S$ with one cone point of angle $4\pi$. 

We define extremal components, those components parameterized by $|E(\rho)| = -\chi(S)$. These components turn out to be two diffeomorphic copies of the Teichmüller space. Similarly, we define almost extremal components, those components parameterize by $|E(\rho)| = -(\chi(S) + 1)$. Any representation $\rho$ of these components is defined as almost extremal representation.

The curious reader may see [9] for further details about the representation variety and [8] for more details about the symplectic structure on $X(S)$. 

We define extremal components, those components parameterized by $|E(\rho)| = -\chi(S)$. These components turn out to be two diffeomorphic copies of the Teichmüller space. Similarly, we define almost extremal components, those components parameterize by $|E(\rho)| = -(\chi(S) + 1)$. Any representation $\rho$ of these components is defined as almost extremal representation.
By Goldman’s theorem 5.3, the set of representations with $\mathcal{E}(\rho) = \pm (\chi(S) + 1)$ is formed by two connected components of the representation space, hence characters of these representations form two connected components of character variety $\mathcal{X}(S)$. Since the singular locus in $\mathcal{X}(S)$ is given by only elementary representations and since they have Euler class zero, it follows that these components are smooth and the non-degenerate $2$–form $\omega_{\mathcal{X}}$ is well-defined everywhere and defines a measure $\mu_{\mathcal{X}}$.

Let us denote by $E$ the subset of all representations with $\mathcal{E}(\rho) = \pm (\chi(S) + 1)$ that send a simple non-separating curve to an elliptic. Then the previous claim of 5.5 may be restated in the following way:

\[ E \text{ subset of } \text{all representations with } \mathcal{E}(\rho) = \pm (\chi(S) + 1) \text{ that send a simple non-separating curve to an elliptic} \text{.} \]

Then the previous claim of 5.5 may be restated in the following way:

Let $S$ be a closed surface of genus $g \geq 2$. Then almost every representation $\rho$ in $E$ arises as the holonomy of a hyperbolic cone-structure with a single cone point of angle $4\pi$.

Now, two simple questions may naturally arise.

1. How big is $E$? That is, what is the measure of the set $E$? Let $k$ be an integer such that $|k| \leq \chi(S)$, denote by $\mathcal{M}^k$ the $k^{\text{th}}$ connected component of the character variety $\mathcal{X}(S)$ and by $\mathcal{N}H^k$ the subset of $\mathcal{M}^k$ of all representations that send a simple closed curve (which may be separating or not) to a non-hyperbolic element of $\text{PSL}_2\mathbb{R}$.

By Goldman’s theorem 5.3, the set $\mathcal{N}H^k$ is formed by two connected components of the character variety $\mathcal{X}(S)$.

We define $\rho$ by Goldman’s theorem of character variety $\mathcal{X}$. The components of the representation space, hence characters of these representations form two connected components since they have Euler class zero, it follows that these components are smooth and the non-degenerate $2$–form $\omega_{\mathcal{X}}$ is well-defined everywhere and defines a measure $\mu_{\mathcal{X}}$.

2. Where does the "almost every" condition come from? In the sequel we will consider separately those representations $\rho$ satisfying the following condition: there is a handle $H \subset S$ and a basis $(\alpha, \beta)$ of $\pi_1 H$ such that the induced representation $\rho_H : \pi_1 H \rightarrow \text{PSL}_2\mathbb{R}$ is virtually abelian. In this case, we will say that $\rho$ contains a virtually abelian pair (see also the definition below 5.6). These representations turn out to be pathological in the sense that will be explained below in the next paragraph 5.3. We may note that any such representation sends a simple non-separating curve to an elliptic, hence any such representation belongs to $E$. Set $B$ the subset of all representations that contain a handle $H$ such that the induced representation $\rho_H$ is virtually abelian. In [17, Proposition 6.3], Mathews proved that $\mu(B) = 0$, i.e., $B$ is a subset of measure zero in $E$. As we see below Mathews’ strategy does not apply to representations in $B$, that is his theorem holds for "almost every" representation in $E$. However, also these representations arise as the holonomy of hyperbolic cone-structure with a single cone point of angle $4\pi$ as we show in 5.5.1.

5.3. Representations with virtually abelian pairs are problematic. Let us start with the following definition.

Definition 5.6. We will say that a representation contains a virtually abelian pair if there are two simple non-separating closed loops with intersection number one and their images via $\rho$ is given by two elliptic elements of order 2 with different fixed points. In this case, their commutator is a hyperbolic transformation along the axis passing through their fixed points.

In order to understand the issues of this kind of representations, we need to explain the proof of 5.5. Let $q$ be any point on $S$ and let $\rho : \pi_1(S, q) \rightarrow \text{PSL}_2\mathbb{R}$ be any representations with $\mathcal{E}(\rho) = -1$ and suppose it sends a non-separating curve $\alpha$ to an elliptic. Starting from $\alpha$, we may found a separating curve $\gamma$ that split $S$ into two pieces, namely we may cut off the handle $H$ containing $\alpha$ from $S$.

More precisely: let $\beta$ be a closed non-separating curve such that $i(\alpha, \beta) = 1$ and denote by $\gamma$ their commutator. Since $\alpha$ is elliptic, then by lemma 2.14 the commutator $\gamma$ has hyperbolic holonomy, in particular $\text{Tr}(\gamma) > 2$ by 2.8. Splits $S$ along $\gamma$, denote by $H$ the handle containing $\alpha$ and by $\Sigma$ the remaining part of $S$. Consider their fundamental groups.

We need to take basepoints $q_H \in H$, $q_\Sigma \in \Sigma$ and $q \in S$. There is nothing special to take $q_H$ and $q_\Sigma$ on the boundary of $H$ and $\Sigma$ respectively, whereas the basepoint $q$ can be taken everywhere on $S$.

Consider the following inclusion $\iota_H : \pi_1(H, q_1) \rightarrow \pi_1(S, q_1)$. Let $\delta$ be a path joining $q$ with $q_H$, then we have the following isomorphisms

\[ \iota_H : \pi_1(S, q_1) \rightarrow \pi_1(S, q) \]

We define $\rho_H$ the representation $\rho_H : \pi_1(H, q_1) \rightarrow \text{PSL}_2\mathbb{R}$ by composition of the maps defined above

\[ \rho_H : \pi_1(H, q_1) \rightarrow \pi_1(S, q_1) \rightarrow \pi_1(S, q) \rightarrow \text{PSL}_2\mathbb{R}. \]

Considering the other inclusion $\iota_\Sigma : \pi_1(\Sigma, q_2) \rightarrow \pi_1(S, q_2)$ and applying the same procedure; we may define a representation $\rho_\Sigma : \pi_1(\Sigma, q_2) \rightarrow \text{PSL}_2\mathbb{R}$ in the same way. Since $\rho(\gamma)$ is hyperbolic, the relative Euler numbers $\mathcal{E}(\rho_H, s)$ and $\mathcal{E}(\rho_\Sigma, s)$ are well-defined with respect to the special trivialization defined along $\gamma$. We may immediately note that $\mathcal{E}(\rho_H, s) = 0$ with respect to the special trivialization along the boundary $\gamma$ by the proposition
3.12. Hence we have localized the deficiency of the Euler class of \( \rho \), and, by additivity, the relative Euler class of the representation induced on the other piece is extremal.

Since \( \mathcal{E}(\rho_\Sigma, s) = \chi(\Sigma) \), the representation \( \rho_\Sigma \) is the holonomy of a complete hyperbolic structure with totally geodesic boundary by the theorem 3.8.

Suppose \( \rho \) does not contain virtually abelian pairs, in particular the representation \( \rho_H \) is not virtually abelian and, by Theorem 4.9, it is the holonomy of a hyperbolic cone-structure \( \sigma_H \) on \( H \) with geodesic boundary, except for at most one corner point of angle \( \theta \) and no cone points inside \( H \). Since \( \text{Tr} \rho(\gamma) > 2 \) the angle of the corner point is greater than \( 2\pi \) and does not exceed \( 3\pi \) (see [16, Proposition 5.8]).

The basic idea of Mathews’ proof is to find a hyperbolic cone-structure on \( \Sigma \) with geodesic boundary except for at most one corner point of angle \( \theta \). No cone points inside \( \Sigma \) and holonomy \( \rho_\Sigma \); that fits together with a hyperbolic cone-structure on \( H \) with one corner point of angle \( \theta \) and holonomy \( \rho_H \). If these structures exist, we may identify the corner points and then glue them along their boundary. Topologically the resulting surface turns out to be the original surface \( S \); geometrically we get \( S \) endowed with a hyperbolic cone-structure with one corner point of angle \( 4\pi \) and holonomy \( \rho \).

In order to find a hyperbolic cone-structure \( \sigma_\Sigma \) on \( \Sigma \) with a corner point of angle \( \theta \), we wish to truncate a flares inside the convex core of \( \Sigma \). Unfortunately, such truncation can not be done too far inside the convex core, but we may cut inside the collar of the geodesic boundary. We recall that the collar width \( \text{flares inside the convex core of } \Sigma \). Unfortunatelly, such truncation can not be done too far inside the convex core, and we may cut inside the collar of the geodesic boundary. Topologically the resulting surface turns out to be the original surface \( S \); geometrically we get \( S \) endowed with a hyperbolic cone-structure with one corner point of angle \( 4\pi \) and holonomy \( \rho \).

Let \( p \) be a point inside the collar, consider the geodesic representative of \( \gamma \) based at \( p \) and cut along it to obtain a hyperbolic cone-structure on \( \Sigma \) with one corner point of angle \( \theta \). The developed image \( \tilde{p} \) of \( p \) lies inside the \( w(t) \)-neighbourhood of the axis of \( \rho(\gamma) \). Using classical notion of hyperbolic geometry we may see that the magnitude of \( \theta \) depends only on the distance of the point \( \tilde{p} \) from the axis of \( \rho(\gamma) \); that is the distance of \( p \) from the geodesic boundary of the (unique) complete hyperbolic structure on \( \Sigma \) with holonomy \( \rho_\Sigma \). Hence the possible values of \( \theta \) are bounded from above by some value \( \theta_{\text{max}} \) which depends on the width of the collar and on the value of \( t \) of the trace of \( \rho(\gamma) \).

What remain to do is to find a hyperbolic cone-structure on \( H \) with one corner point of angle \( 4\pi - \theta \) that fits together with \( \Sigma \) endowed with \( \sigma_\Sigma \). Despite theorem 4.9 ensures the existence of hyperbolic cone-structure on \( H \) with holonomy \( \rho_H \), we do not know a priori if \( \rho_H \) may be the holonomy of hyperbolic cone-structure such that the angle of the corner point lies in the range \( [\theta_{\text{max}}, 3\pi] \).

**Remark 5.7.** The condition that \( \rho : \pi_1 S \rightarrow \text{PSL}_2 \mathbb{R} \) does not contain virtually abelian pair is necessary. Indeed when \( \rho \) contains a virtually abelian pair the representation \( \rho_H \) turns out to be virtually abelian, and such representation does not arise as holonomy of a hyperbolic cone-structure on \( H \) by 4.9.

Following Mathews we give the following definition.

**Definition 5.8.** Let \( H \) be a punctured torus, and let \( (\alpha, \beta) \) be a basis for \( \pi_1 (H, q) \), where \( q \) is a point on the boundary of \( H \). Let \( \rho : \pi_1 (H, q) \rightarrow \text{PSL}_2 \mathbb{R} \) be a representation with \( \text{Tr} [g, h] > 2 \), where \( g = \rho(\alpha) \) and \( h = \rho(\beta) \). For any point \( p \) in \( \mathbb{H}^2 \), we define \( \mathcal{P}(g, h; p) \) to be the (possibly degenerate) hyperbolic pentagon given by the following closed polygon

\[
p \rightarrow [g^{-1}, h^{-1}](p) \rightarrow h(p) \rightarrow gh(p) \rightarrow h^{-1}gh(p) \rightarrow p
\]

With the same notation we may state the following lemma (whose the proof may be found in [16, Lemma 3.4]).

**Lemma 5.9.** Let \( \rho_H : \pi_1 (H, q) \rightarrow \text{PSL}_2 \mathbb{R} \) be a representation. The representation \( \rho_H \) is the holonomy of a branched hyperbolic structure on \( H \) with no interior cone points and at most one corner point if and only if there exist a free basis \( (\alpha, \beta) \) of \( \pi_1 (H, q) \) and a point \( p \in \mathbb{H}^2 \) such that \( \mathcal{P}(g, h; p) \) is a non-degenerate pentagon bounding an immersed open disc in \( \mathbb{H}^2 \).

Consider the representation \( \rho_H \) and the basis \( (\alpha, \beta) \) of \( \pi_1 (H, q) \) given by the construction above. To find a hyperbolic cone-structure on \( H \) with holonomy \( \rho_H \) and such that fits with the hyperbolic cone-structure on \( \Sigma \) means to find a basis \( (\alpha', \beta') \) (possibly different to the given one!) and point \( p \) inside the \( w(t) \)-neighbourhood of the axis \( \rho_H (\alpha') \) such that the pentagon \( \mathcal{P}(g', h'; p) \) is a non-degenerate pentagon bounding an immersed open disc in \( \mathbb{H}^2 \), where \( g' = \rho_H (\alpha') \) and \( h' = \rho_H (\beta') \).

**Remark 5.10.** We remember that the change of basis of \( \pi_1 (H, q) \) changes the character of \( \rho_H \), but it has no effect on the geometry on \( H \). This reasoning may be extended to the entire surface \( S \). Changing the basis of the
handle $H$, we change the presentation of the fundamental group $\pi_1(S, q)$ but, as in the case of the punctured torus, the change of basis does not effect on the geometry on $S$. Hence we do not care if we need to change the basis of $\pi_1(H, q)$ in order to find a good hyperbolic cone-structure with holonomy $\rho_H$.

With this spirit, we give the following definition.

**Definition 5.11.** Consider a punctured torus $H$, with a basis $(\alpha, \beta)$ for $\pi_1(H, q)$, where $q$ is a point on the boundary of $H$; and a representation $\rho : \pi_1(H, q) \rightarrow \text{PSL}_2 \mathbb{R}$ with $\text{Tr}[g, h] > 2$, where $g = \rho(\alpha)$ and $h = \rho(\beta)$. Define $\rho$ to be $\varepsilon$–good for a specified orientation of $H$, if there exists a basis $\alpha', \beta'$, of the same orientation as $\alpha, \beta$, and a point $p$ at distance less than $\varepsilon$ from $Ax(\rho([\alpha', \beta'])$, such that the pentagon $\mathcal{P}(g', h'; p)$ is non-degenerate, bounds an embedded disc, and is of the specified orientation.

We will say that a character is $\varepsilon$–good for a specified orientation of $H$ if it is the character of a $\varepsilon$–good representation. Note that since $\text{Tr}[g, h] > 2$ the representation $\rho$ is irreducible, then any character correspond to a unique conjugacy class of representations. Thus

- a character is $\varepsilon$–good $\iff$ all corresponding representations are $\varepsilon$–good.

We define $\varepsilon$–bad representations (characters) those representations (characters) which are not $\varepsilon$–good. We will define $\varepsilon$–bad representations, those representations which $\varepsilon$–bad for any $\varepsilon$.

By the theorem 4.9, any non-virtually abelian representation is a $\varepsilon$–good representation for some $\varepsilon$, whereas virtually abelian representation are $\varepsilon$–bad for any $\varepsilon$ as we see below 5.12. In particular they are $w(t)$–bad for any $t > 2$. On the other hand a non-virtually abelian representation may be $w(t)$–bad even if it is good for other values of $\varepsilon$. Indeed the "goodness" condition is weaker than $w(t)$–goodness condition, because in the second case the point $p$ must lie at a certain specified distance from $Ax(\rho([\alpha', \beta']))$. In the next section we will show that for any $t > 2$ the only $w(t)$–bad representation are virtually abelian. So far we have the following result.

**Lemma 5.12.** Let $\rho : \pi_1(H, q) \rightarrow \text{PSL}_2 \mathbb{R}$ be a virtually abelian representation. Then $\rho$ is $\varepsilon$–bad for any $\varepsilon$.

We recall for convenience the following characterization of virtually abelian representations. Let $G \subset \text{PSL}_2 \mathbb{R}$ be a subgroup generated by two elements $g, h$, then $G$ is virtually abelian (but not abelian) if and only if two of $\{g, h, gh\}$ are half-turns about points $q_1 \neq q_2 \in \mathbb{H}^2$ and the third is a non trivial translation along the axis $q_1q_2$. In particular, their commutator is also a translation along the same axis of $gh$.

**Proof.** We may suppose without loss of generality that $g, h$ are elliptics of order two and $gh$ is hyperbolic. If we take $p$ in $Ax \mathcal{g}h$, then all vertices of $\mathcal{P}(g, h; p)$ lies on the axis and the pentagon does not bound a disc. Thus we may suppose $p$ lies outside the axis, in particular: the points $p, gh(p)$ and $[g, h](p)$ lie in the same side and at the same distance from the axis, whereas the points $bgh^{-1}(p) and h(p)$ lie on the other side. Since the segment $p \rightarrow [g, h](p)$ lies between $Ax \mathcal{g}h$ and the point $gh(p)$ the pentagon $\mathcal{P}(g, h; p)$ does not bound a disc.

5.4. **Bad representations are virtually abelian.** It is natural to ask if there are $w(t)$–bad representations which are not virtually abelian. Let $t > 2$ be a fixed real number. We consider the following subset of the relative character variety $X_t(H)$:

- $\Omega_t$ the subset of characters of representations taking some simple closed curve to an elliptic;
- $B_t$ the set of $w(t)$–bad characters in $\Omega_t$ (where $w(t)$ is the quantity defined above). It turns out to be closed and nowhere dense in $\Omega_t$; and
- $V_t$ the closed subset of characters of virtually abelian representations in $\Omega_t$.

By Lemma 5.12 the following inclusion holds $V_t \subset B_t$. We recall for convenience that the symplectic $2$–form $\omega$ on $\mathcal{X}(H)$ induces on a symplectic form $\omega_t$ on any level set $X_t(H)$. In particular, since $X_t(H)$ is $2$–dimensional, $\omega_t$ turns out to be an area form $\mu_t$ which is invariant with respect to the action of the mapping class group $\text{MCG}(H)$. Finally, since we are considering representations $\rho$ that sends a simple non-separating curve to an elliptic element; any representation $\rho_H$ defined as above is a representation with elliptics; hence it belongs to open set $\Omega_t$.

**Proposition 5.13.** [17, Proposition 6.2] For all $t > 2$, $\mu_t(B_t) = 0$ where $\mu_t$ is the measure induced by $\mu_H$ on the level set $X_t(H)$. That is: $\mu_t$–almost every character in $\Omega_t$ is $w(t)$–good.

The proof of such proposition relies on the following idea. It is always possible to construct explicitly a representation $\rho' : \pi_1 H \rightarrow \text{PSL}_2 \mathbb{R}$ which is $w(t)/2$–good for any $t > 2$ and let $(x^*, y^*, z^*)$ its character. Little perturbation of such character in $X_t(H)$ is still a character of a $w(t)/2$–good representation. The set $V$ of perturbations turns out to be an open subset of $X_t(H)$ of positive measure and since the action of $\Gamma \cong \text{MCG}(H)$ is ergodic on $\Omega_t$ the claim follows because invariant sets have null, or null, measure. For further details, we invite the reader to read [15, 17]. We are going to show the following result.
Proposition 5.14. For all \( t > 2, B_t = V_1 \). Equivalently: if \( \rho \) is a \( w(t) - \text{bad} \) representation, then it is virtually abelian.

**Notation:** for the sake of readability and simplicity we will make a little abuse of notation confusing a character \((x, y, z)\) with the conjugacy class of representations having it as character. Indeed, under our assumptions, any representation we are considering is irreducible, hence any character corresponds to a unique conjugacy class of representation.

Let \( \rho_0 \in \mathcal{X}(H) \) be a non-virtually abelian representation, hence by Theorem 4.9 it is the holonomy of hyperbolic cone-structure on \( H \) with geodesic boundary, except for at most one corner point, and no interior cone points. In particular we can find a basis \((\alpha, \beta)\) for \( \pi_1(H, q) \) and a point \( p \in \mathbb{H}^2 \) such that the pentagon \( \mathcal{P}(g, h; p) \) bounds a disc in \( \mathbb{H}^2 \), where \( g = \rho_0(\alpha) \) and \( h = \rho_0(\beta) \). Clearly \( \rho_0 \) is a \( d-\)good representation where \( d \) is the distance of \( p \) from the axis of \( \rho_0([\alpha, \beta]) \). If \( d < w(t) \) there is nothing to prove, hence suppose that \( \rho_0 \) is not \( w(t) - \text{good} \), i.e. \( w(t) - \text{bad} \), in particular \( w(t)/2 - \text{bad} \). Hence for any basis \((\alpha, \beta)\) and any point \( p \) at distance less than \( w(t)/2 \) from the axis \( \rho_0([\alpha, \beta]) \), the pentagon \( \mathcal{P}(g, h; p) \) does not bounds a disc in \( \mathbb{H}^2 \). Fix a particular basis \((\alpha', \beta')\) and a particular point \( p' \) at distance less than \( w(t)/2 \) from the axis \( \rho_0([\alpha', \beta']) \) and we define the quantity \( \xi(\alpha', \beta'; p') \) as the maximal radius for a Euclidean ball centered in \( p' \) such that any representation \( \rho \) inside such ball satisfy the following condition

\[
\mathcal{P}(\rho(\alpha'), \rho(\beta'); p') \text{ bounds a disc if and only if the pentagon } \mathcal{P}(g', h'; p') \text{ does}
\]

The quantity \( \xi(\alpha', \beta'; p') \) depends on the choices of the basis and the point; and we define the following

\[
\xi = \inf_{(\alpha, \beta) \text{ basis of } \pi_1(H, q); p \in w(t)/2 - \text{neighbourhood of } \text{Axis}_{\rho_0([\alpha, \beta])}} \xi(\alpha', \beta'; p')
\]

From now on, throughout this subsection, we will say that representation \( \rho \) is \( \varepsilon - \text{good with respect to a fixed basis } (\alpha, \beta) \), if there is a point at distance less than \( \varepsilon \) from the axis of \( \rho([\alpha, \beta]) \) such that the pentagon \( \mathcal{P}(\rho(\alpha), \rho(\beta); p) \) bounds a disc. To be good with respect to a fixed basis is an open condition. Indeed, for any sufficiently little perturbation \( p' \) of the point \( p \) the shape of \( \mathcal{P}(\rho(\alpha), \rho(\beta); p) \) does not change; i.e. \( \mathcal{P}(\rho(\alpha), \rho(\beta); p') \) still bounds a disc. Conversely, to be bad with respect to a fixed basis is a closed condition, i.e. the subset of points \( p \) such that \( \mathcal{P}(\rho(\alpha), \rho(\beta); p) \) does not bounds a disc turns out to be a closed subset of \( U \subseteq \mathbb{H}^2 \). For any point \( p \in \partial U \), the pentagon \( \mathcal{P}(\rho(\alpha), \rho(\beta); p) \) is self-intersecting, but whereas some perturbations of \( p \) produce another self-intersecting pentagon, other perturbations produce non-degenerate pentagons.

![Figure 1. A self-intersecting pentagon when \( p \in \partial U \). In this situation, little perturbations of the point \( p \) in a judicious directions produce a non-degenerate pentagon, whereas other perturbations produce a self-intersecting pentagons like in the picture on the right. The same holds if we perturb the character and we maintain the point \( p \) fixed.](image1.png)

![Figure 2. A self-intersecting pentagon when \( p \notin \partial U \). In this situation, every sufficiently little perturbations of the point \( p \) in any directions produce a de-generate pentagon. The same holds if we perturb the character and we maintain the point \( p \) fixed.](image2.png)

Remark 5.15. Since \( \rho_0 \) is \( w(t)/2 - \text{bad} \), the pentagon \( \mathcal{P}(\rho_0(\alpha), \rho_0(\beta); p) \) is self-intersecting for any basis \((\alpha, \beta)\) and any point \( p \) in the \( w(t)/2 - \text{neighborhood of the axis of } \rho([\alpha, \beta]) \). Following the previous remark, any little perturbation \( p' \) of \( p \) maintains the shape of \( \mathcal{P}(\rho_0(\alpha), \rho_0(\beta); p) \), i.e. \( \mathcal{P}(\rho_0(\alpha), \rho_0(\beta); p') \) does not bounds a disc for any \( p' \) sufficiently near to \( p \).

Remark 5.16. Let \( p \notin \partial U \), and consider the pentagon \( \mathcal{P}(\rho_0(\alpha), \rho_0(\beta); p) \). Since \( \rho_0 \) is not virtually abelian, then any little perturbation of the character of \( \rho_0 \) preserves the shape of the pentagon.

We have the following lemma.
Lemma 5.17. $\xi = 0$ if and only if $\rho_0$ is virtually abelian.

Proof of Lemma 5.17. If $\rho_0$ is virtually abelian, then almost every little perturbation of $\rho_0$ is $w(t)$-good; hence $\xi = 0$. Suppose $\xi = 0$, fix a basis $(\alpha, \beta)$ of $\pi_1(H, q)$ and define

$$\xi(\alpha, \beta) = \inf_{p \in w(t)/2-\text{neighbourhood of } \text{Axisp}(\alpha, \beta)} \xi(\alpha, \beta; p).$$

The function $\xi(\alpha, \beta; p)$ depends continuously only on the distance of the point $p$ from the $\text{Axisp}(\alpha, \beta)$, and $\xi(\alpha, \beta)$ turns out to be the minimum value. Suppose there is a particular basis $(\alpha, \beta)$ such that $\xi(\alpha, \beta) = 0$, then we may see that $\rho_0$ is virtually abelian. Indeed, $\rho_0$ is $w(t)/2$-bad with respect to $(\alpha, \beta)$ (because $w(t)$-bad), and any little perturbations of the character changes the shape of the pentagon $\mathcal{P}(\rho_0(\alpha), \rho_0(\beta); p)$, where $p$ is a point at distance such that the minimum is attained.

Suppose that for any basis $(\alpha, \beta)$ of $\pi_1(H, q)$ the following holds $\xi(\alpha, \beta) > 0$. For any basis, consider the open Euclidean ball $B_{\xi(\alpha, \beta)}(\rho_0)$. Then any representation inside $B_{\xi(\alpha, \beta)}(\rho_0)$ is $w(t)/2$-bad with respect to the basis $(\alpha, \beta)$, because $\rho_0$ is. By ergodicity, we claim that almost every representation is $w(t)/2$-bad, because $\mu(B_{\xi(\alpha, \beta)}(\rho_0)) > 0$ for any basis. Indeed, consider the following subspace

$$\mathcal{B} = \bigcap_{(\alpha, \beta) \text{ basis of } \pi_1(H, q)} \text{MCG}(H) \cdot B_{\xi(\alpha, \beta)}(\rho_0).$$

This is a subspace of $\Omega_{t}$ of full measure, because it is a countable intersection of subsets of full measure. We may easily note that

$$\rho \in \mathcal{B} \iff \rho \text{ is } w(t)/2 \text{-bad for any basis } (\alpha', \beta') \in \text{MCG}(H) \cdot \{ \text{conjugacy class of basis of } \pi_1(H, q) \},$$

but since the action of $\text{MCG}(H)$ is transitive on the set $\{ \text{conjugacy class of basis of } \pi_1(H, q) \}$, then $\rho$ is $w(t)/2$-bad with respect to any basis, hence $\rho \in \mathcal{B}$ if and only if it is $w(t)/2$-bad. Since $\mathcal{B}$ has full measure, we get a contradiction with 5.13. Hence there necessarily exists a basis $(\alpha, \beta)$ such that $\xi(\alpha, \beta) = 0$, and then $\rho_0$ is virtually abelian. \qed

Proof of proposition 5.14. Since $\rho_0$ is not virtually abelian by assumption, the quantity $\xi$ must be strictly positive by 5.17. In particular, any representation inside the Euclidean ball $B_{\xi(\alpha, \beta)}(\rho_0)$ is $w(t)$-bad. By ergodicity, almost every character in $\Omega_{t}$ is $w(t)$-bad; hence a contradiction. Thus $\rho_0$ is $w(t)$-good and $\mathcal{B} = \emptyset$, and this conclude the proof of 5.14. \qed

5.5. Proof of the Theorem 5.1. We divide the proof of 5.1 in two parts. In the first one, we consider representations $\rho$ that contain a virtually abelian pair and we show that any such representation arises as the holonomy of hyperbolic cone-structure with a single cone point of angle $4\pi$.

5.5.1. Representations with virtually abelian pairs. In this paragraph, we consider representations with virtually abelian pairs. We have seen in the previous section 5.3 that this kind of representations is problematic in the sense that Mathews' proof does not apply to them. However, using different geometrical techniques, we are going to prove the following proposition.

Proposition 5.18. Let $S$ be a surface of genus $g \geq 2$ and let $\rho : \pi_1 S \rightarrow \text{PSL}_2 \mathbb{R}$ be a representation with $\mathcal{E}(\rho) = \pm (\chi(S) + 1)$ whose image contains a virtually abelian pair. Then $\rho$ is geometrizable by a hyperbolic cone-structure with a cone point of angle $4\pi$.

Proof of proposition 5.18. Up to change the orientation of $S$, we may suppose that $\mathcal{E}(\rho) = \chi(S) + 1$. We divide the proof into three lemmata. In the first one we show the existence of a simple closed separating curve $\gamma$ dividing $S$ in a punctured torus $H$ and surface with boundary $\Sigma$ such that the induced representation $\rho_1 : \pi_1 H \rightarrow \text{PSL}_2 \mathbb{R}$ is virtually abelian. In order to do this, we fix a base point $q \in S$ for the fundamental group $\pi_1(S, q)$.

Lemma 5.19. There exists a simple separating curve with hyperbolic holonomy such that the induced representation on $H$ is virtually abelian.

Proof. Since $\rho$ contains virtually abelian pairs there are two simple non-separating curves $\alpha_1$ and $\beta_1$ based at $q$ such that $g_1 = \rho(\alpha_1)$ and $h_1 = \rho(\beta_1)$ are elliptics of order 2 and their intersection number is one. Their commutator $\gamma$ is a separating simple curve with hyperbolic holonomy and divides $S$ into two pieces and one of them is a punctured torus. We define $H$ as the punctured torus containing $\alpha_1$ and $\beta_1$, and define $\rho_1$ to be the induced representation of $\pi_1(H, q_1)$ via $\rho$, where $q_1$ is the point on the boundary of $H$ that coincide with $q$ on the overall surface. By construction, $\rho_1$ is virtually abelian. \qed

Let $\Sigma$ be the second piece and define $\rho_2$ to be the induced representation of $\pi_1(\Sigma, q_1)$. Let $\alpha_2, \beta_2, \ldots, \alpha_g, \beta_g$ be a basis for $\pi_1(\Sigma, q_2)$ so that $[\alpha_2, \beta_2] \cdots [\alpha_g, \beta_g]$ is homotopic to $\gamma$ but traversed in opposite direction with respect to $[\alpha_1, \beta_1]$.\]
Lemma 5.20. The representation $\rho : \pi_1(\Sigma, q_2) \to \text{PSL}_2\mathbb{R}$ is Fuchsian.

Proof. Consider the basis for $\pi_1(H, q_1)$ and $\pi_1(\Sigma, q_2)$ defined above. Since $[\alpha_1, \beta_1] = \gamma$ and $[\alpha_2, \beta_2] \cdots [\alpha_g, \beta_g] = \gamma^{-1}$, the fundamental group of $S$ has the following standard presentation

$$\pi_1 S = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g | [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = \text{id} \rangle$$

In terms of hyperbolic transformations the relation above becomes, namely $\rho([\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]) = \text{id}$. It lifts to the relation $\rho_p([\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]) = -\text{id}$ in $\text{SL}_2\mathbb{R}$ because $E(\rho) = \chi(S) + 1$. Since $\rho(\alpha_1) = \rho_1(\alpha_1)$ is elliptic, the trace of $\rho([\alpha_1, \beta_1])$ is greater than 2 by 2.8 and 2.14, hence $E(\rho_1, s) = 0$ by 3.12. The relative Euler class $E(\rho_2, s) = \chi(\Sigma)$ by that is $\rho_2$ is a Fuchsian representation by 3.8 and it is the holonomy of a complete hyperbolic structure with geodesic boundary on $\Sigma$. □

![Figure 3. Fundamental domain of a representation with a virtually abelian pairs of a surface of genus 2](image)

Lemma 5.21. There exists a fundamental domain for $\rho$. It turns out to be a pentagon, namely a degenerate octagon with four sides aligned.

Proof. We are now going to construct a fundamental domain for $\rho$. So let $p$ be a point on Axis $\rho_2(\gamma)$. Since $\rho_2$ is Fuchsian we may start from $p$ to define a fundamental domain for $\rho_2$ such that the sum of all inner angles is exactly $\pi$ that turns out to be a $4g-3$-gon in $\mathbb{H}^2$. Observe that $\rho_2(\gamma)p \in \text{Axis } \rho_2(\gamma)$ so the entire segment joining them lies on the axis of $\rho_2(\gamma)$. Now we use the representation $\rho_1$ to divide such segment into four smaller pieces so that the sum of all interior angles is exactly $4\pi$. Gluing the correspondent sides using $\rho$ we get a closed surface of genus $g$ endowed with a hyperbolic cone-structure with exactly one cone point of angle $4\pi$, and this conclude the proof of 5.18 □

We finally glue the correspondent sides using $\rho$ to obtain a closed surface of genus $g$ endowed with a hyperbolic cone-structure with exactly one cone point of angle $4\pi$, and this conclude the proof of 5.18 □

By Theorem 5.5 and Proposition 5.18, we get the following result

Let $S$ be a closed surface of genus $g \geq 2$. Then every representation $\rho : \pi_1 S \to \text{PSL}_2\mathbb{R}$ with $E(\rho) = \pm(\chi(S) + 1)$, which sends a non-separating curve $\gamma$ on $S$ to an elliptic arises as the holonomy of a hyperbolic cone-structure on $S$ with one cone point of angle $4\pi$. □
5.5.2. Representations with a parabolic non-separating curve. In this paragraph we show that if a representation \( \rho \) sends a simple non-separating closed curve to a parabolic element, then, under suitable conditions, it sends a simple closed non-separating curve to an elliptic. First of all, we show that none representation sends a simple closed curve to the identity.

**Lemma 5.22.** Let \( S \) be a surface of genus \( g \geq 2 \) and let \( \rho : \pi_1 S \rightarrow \text{PSL}_2\mathbb{R} \) be a representation such that \( E(\rho) = \pm(\chi(S) + 1) \). Then no simple closed loop is sent to the identity.

**Proof.** Up to change the orientation of \( S \), we suppose that \( E(\rho) = \chi(S) + 1 \). Suppose that \( \alpha \) is a simple curve such that \( \rho(\alpha) = \text{id} \). If \( \alpha \) is a non-separating curve, then let \( \beta \) be any non-separating simple curve such that \( i(\alpha, \beta) = 1 \), and denote by \( \gamma \) their commutator. Of course \( \rho(\gamma) = \text{id} \). Hence suppose that \( \rho \) send a separating simple curve \( \gamma \) to the identity and split \( S \) in a punctured torus \( H \) and a subsurface \( \Sigma \) cutting along \( \gamma \). Consider their fundamental groups and define \( \rho_H \) and \( \rho_\Sigma \) the representations induced by \( \rho \) as described in section 5.3. The relative Euler numbers are well-defined because \( \rho(\gamma) = \text{id} \), and by additivity \( E(\rho_H, s) + E(\rho_\Sigma, s) = E(\rho) = \chi(S) + 1 \). Then \( \rho_\Sigma \) is the holonomy of a complete hyperbolic structure on \( \Sigma \) with totally geodesic or cusped boundary (see 3.8). On the other hand the holonomy of the boundary \( \gamma \) must be hyperbolic or parabolic, then a contradiction. \( \square \)

Suppose \( \rho \) sends a non-separating simple curve \( \alpha \) to a parabolic element, let \( \beta \) be a simple curve such that \( i(\alpha, \beta) = 1 \) and denote by \( \gamma \) their commutator, by the previous lemma \( \beta \) and \( \gamma \) have not trivial holonomy. If \( h = \rho(\beta) \) is elliptic we have done by 5.5. Then we may assume \( h \) as a parabolic or hyperbolic transformation. Since \( g(\alpha) \) is a parabolic transformation, it might share a fixed point with \( h \). In this case the commutator \( \rho(\gamma) \) turns out to be a parabolic transformation by lemma 2.13. The following lemma shows that we can always find a non-separating curve \( \beta \), such that \( i(\alpha, \beta) = 1 \) and \( h = \rho(\beta) \) does not share any fixed point with \( \rho(\alpha) \).

**Lemma 5.23.** Let \( \rho : \pi_1 S \rightarrow \text{PSL}_2\mathbb{R} \) be a representation with \( E(\rho) = \pm(\chi(S) + 1) \). Suppose \( \rho \) sends a non-separating curve \( \alpha \) to a parabolic element. Then there exists a simple non separating curve \( \beta \) such that \( i(\alpha, \beta) = 1 \) and \( \text{Fix}(\rho(\alpha)) \cap \text{Fix}(\rho(\beta)) = \phi \). In particular \( \rho([\alpha, \beta]) \) is hyperbolic.

**Proof.** Let \( \alpha \) and \( \beta' \) two hyperbolic transformations and suppose they share a fixed point \( q \) at the boundary at infinity. Since \( \rho \) is non-elementary (because it has non trivial Euler number), there is a simple curve \( \xi \) such that

- \( \xi \) does not meet \( \alpha \) and \( \beta' \) and
- \( q \) is not a fixed point for \( \xi \).
Take $\xi$ with the orientation so that $\beta = \beta' \xi$ is homotopic to a simple curve, then it does not fix $q$ because $\xi$ does not and $i(\alpha, \beta) = 1$ by construction.

Thus we may assume that $q = \rho(\alpha)$ and $h = \rho(\beta)$ have not a common fixed point and their commutator is hyperbolic by 2.13. We have the following result.

**Proposition 5.24.** Let $S$ be a surface of genus $g \geq 2$ and let $\rho : \pi_1 S \rightarrow \text{PSL}_2 \mathbb{R}$ be a representation with $E(\rho) = \pm(\chi(S) + 1)$. Suppose $\rho$ sends a non-separating simple closed loop $\alpha$ to a parabolic element and there exists a simple closed curve $\beta$ such that $i(\alpha, \beta) = 1$ and $\text{Fix}(\rho(\alpha)) \cap \text{Fix}(\rho(\beta)) \neq \emptyset$. Then $\rho$ arises as the holonomy of hyperbolic cone-structure on $S$ with one cone point of angle $4\pi$.

**Proof of proposition 5.24.** Let $q$ be a point on $S$ and consider $\pi_1(S, q)$, that is we may consider that all curves are based at $q$. Let $\alpha$ be a non-separating curve with parabolic image and $\beta$ a simple non-separating curve such that $i(\alpha, \beta) = 1$ and $\text{Fix}(\rho(\alpha)) \cap \text{Fix}(\rho(\beta)) \neq \emptyset$. Define $\gamma$ their commutator. Since $\gamma$ is a simple closed separating curve, it splits $S$ in two pieces and let $H$ be the one containing $\alpha$. Of course $H$ it is a handle, and it contains also the curves $\beta$, and $\gamma$ as its boundary component. Let $\rho_H : \pi_1(H, q_1) \rightarrow \text{PSL}_2 \mathbb{R}$ be the induced representation of $\pi_1(H, q_1)$ by $\rho$ where $q_1$ is a point on the boundary that coincide with $q$ on the overall surface. The trace of $\rho_H(\gamma)$ is greater than 2 by 2.8, hence the relative Euler class $E(\rho_H, s) = 0$ by 3.12. In particular, the representation $\rho_H$ can be:

1. a representation with elliptics, or
2. a pants representation.

In the first case, the representation $\rho$ arises as the holonomy of a hyperbolic cone-structure by 5.5. In the second case, $\rho_H$ is the holonomy of a complete hyperbolic structure on a pair of pants. Since $\rho_H$ is not a virtually abelian representation, it arises as the holonomy of hyperbolic cone-structure on $H$, without interior cone points and with at most one corner point on $q$ of angle $\theta > 2\pi$. In particular $\rho_H$ is $w(t)$-good, where $t$ is the trace of $\rho_H(\gamma)$, hence there exists a particular basis $(\alpha', \beta')$ of $\pi_1(H, q)$ and a point $p$ at distance less than $w(t)$ from the axis of $\rho_H([\alpha', \beta'])$ such that the pentagon $P(\rho_H(\alpha'), \rho_H(\beta'); p)$ bounds a disc. Note that the value of $\theta$ depends only on the distance $\delta$ of $p$ from the axis of $\rho_H([\alpha', \beta'])$. Define $\Sigma$ as the closure of $S \setminus H$, and let $\rho_\Sigma$ be the representation induces by the inclusion $\pi_1(\Sigma, q) \hookrightarrow \pi_1(S, q)$. We may note that $E(\rho_\Sigma, s) = \chi(S) + 1$ by the additivity of the Euler number. Hence $\rho_\Sigma$ is the holonomy of a complete hyperbolic structure on $\Sigma$ with totally geodesic boundary. Take a point $q_2$ at distance $\delta$ from the geodesic boundary and consider the geodesic representative of the boundary based at $q_2$. It turns out to be a piecewise geodesic boundary with a single corner point of angle $\theta_2 = 4\pi - \theta_1 < 2\pi$. Finally, the hyperbolic cone-structure on $H$ may be glued to the one on $\Sigma$ along their piecewise geodesic boundary and identifying the corner points. This gives a hyperbolic cone-structure on $S$ with a single cone point of angle $4\pi$. Hence the desired result.

By Theorem 5.5 and Propositions 5.18 and 5.24 we get the following

let $S$ be a closed surface of genus $g \geq 2$. Then every representation $\rho : \pi_1 S \rightarrow \text{PSL}_2 \mathbb{R}$ with $E(\rho) = \pm(\chi(S) + 1)$, which sends a non-separating curve $\gamma$ on $S$ to a non-hyperbolic element arises as the holonomy of a hyperbolic cone-structure on $S$ with one cone point of angle $4\pi$.

that is Theorem 5.1.

5.6. **The case of surfaces of genus 2.** From now on; let $S$ be a closed surface of genus 2 and let $\rho : \pi_1 S \rightarrow \text{PSL}_2 \mathbb{R}$ be a representation with $E(\rho) = \pm 1$. Up to change the orientation of $S$, we may suppose that $E(\rho) = -1$. As above, we denote by $M^{-1}$ the connected component of the character variety $X(S)$ of all representations with $E(\rho) = -1$.

Recently, Marché and Wolff proved the following result in [13, Theorem 1.4].

**Theorem 5.25.** Any representation $\rho \in M^{-1}$ sends a simple closed curve to a non-hyperbolic element.

By their theorem, we have the following possibilities:

1. $\rho$ sends a simple curve to the identity;
2. $\rho$ sends a separating simple curve $\gamma$ to an elliptic element;
3. $\rho$ sends a separating simple curve $\gamma$ to a parabolic element;
4. $\rho$ sends a non-separating simple curve $\gamma$ to an elliptic element;
5. $\rho$ sends a non-separating simple curve $\gamma$ to a parabolic element.

Infact, the case 1 does not occur by 5.22. In [17], Mathews give the following result, very specific to genus 2 case.
Theorem 5.26 (Mathews 2011). Let $S$ be a closed surface of genus 2. Let $\rho : \pi_1 S \to \text{PSL}_2 \mathbb{R}$ be a representation with $E(\rho) = \pm 1$. Suppose $\rho$ sends a separating curve $\gamma$ to a non-hyperbolic element. Then $\rho$ arises as the holonomy of a hyperbolic structure on $S$ with one cone point of angle $4\pi$.

By the previous result, the cases 2 and 3 of the list above are completely covered. Theorem 5.5 together with 5.18 imply that any representation $\rho$, which sends a simple non-separating curve to an elliptic, arises as the holonomy of hyperbolic cone-structure on $S$ with one cone point of angle $4\pi$. Finally, by 5.24 any representation that sends a simple non-separating curve to a parabolic arises as the holonomy of hyperbolic cone-structure on $S$ with one cone point of angle $4\pi$. Hence we have the following

let $S$ be a closed surface of genus 2. Then any representation $\rho : \pi_1 S \to \text{PSL}_2 \mathbb{R}$ with $E(\rho) = \pm 1$

arises as the holonomy of hyperbolic cone-structure on $S$ with a single cone point of angle $4\pi$,

that is our main corollary 5.2.

Appendix A. Flexibility of hyperbolic cone-structure

The geometric structures we have constructed are extremely flexible. For a given representation $\rho$, there may be uncountably many non-isometric structures on $S$. For convenience, throughout this appendix we consider a hyperbolic structure as a particular developing map rather than an equivalent class of developing maps. In [9], Goldman showed that every Fuchsian representation, i.e., every representation $\rho$ with $E(\rho) = \chi(S)$ arises as the holonomy of a unique hyperbolic structure. In other words there is a bijection between the following sets

$$\left\{ \begin{array}{l}
\text{complete hyperbolic structures } \sigma \text{ on } S \\
\text{(developing maps)}
\end{array} \right\} \leftrightarrow \left\{ \begin{array}{l}
\text{Fuchsian representations } \rho
\end{array} \right\}$$

Every complete hyperbolic structure induces a representation $\rho$ that encapsulates all geometric data about the structure. Choose a basepoint $q$ on $S$ and fix a basis $\alpha_1, \beta_1, \ldots, \alpha_q, \beta_q$ of $\pi_1(S, q)$ (i.e., we fix a marking on $S$). Since here $\rho$ is a particular representation (rather than an equivalent class of conjugated representations), $\rho$ determines a well-defined point in $\mathbb{H}^2$ from which to begin the developing map, indeed a fundamental domain for the structure in $\mathbb{H}^2$. Such developing map turns out to be a homeomorphism, indeed a global isometry between the universal cover $\tilde{S}$ of $S$ and the hyperbolic plane. However the picture changes completely as soon as we consider hyperbolic cone-structure on $S$, indeed no-Fuchsian representations does not determine a well-defined point from which to begin the developing map.

$$\left\{ \begin{array}{l}
\text{hyperbolic cone-structures } \sigma \text{ on } S \\
\text{(developing maps)}
\end{array} \right\} \leftrightarrow \left\{ \begin{array}{l}
\text{representations } \rho
\end{array} \right\}$$

The solid arrow denotes a complete determination of one object by another, indeed any hyperbolic structure induces a well-defined representation $\rho$. On the other hand, the dash arrow denotes that the choice of the basepoint (from which to begin the developing map) is involved. Different choices produce different hyperbolic cone-structure, i.e., different developing maps equivalent with respect to the same representation $\rho$.

In [22], Tan introduced a surgery called movements of cone points that show more explicitly the flexibility of hyperbolic cone-structure. Let $p$ be a simple cone point for some fixed hyperbolic cone-structure $\sigma$ on $S$ with holonomy $\rho$. Choose a small neighbourhood $U$ of $p$ such that $U$ is contractible and it is mapped by the developing map onto a geometric disc $D$ in $\mathbb{H}^2$ and $U = \text{dev}_\sigma^{-1}(D)$, locally. Now, we may remove $U$ from $S$ and we attach a new disc as follow. Take any point $p'$ in $D$ distinct from $\text{dev}_\sigma(p)$ and join them by a line $l$ lying completely in $D$. Then $l$ lifts to two distinct lines, namely $\tilde{l}_1$ and $\tilde{l}_2$ in $\tilde{U}$ both ending in $p$. Finally, slit $U$ along these two lines and reglue, matching $\tilde{l}_1$ to $\tilde{l}_2$ and $\tilde{l}_1$ to $\tilde{l}_2$. Now, the two lifts of $p'$ are now identified and becomes a cone point, whereas $p$ is now split to two regular points. It is easy to see that the new is isomorphic to that of $U$ and it depends only on the choice of $p'$; hence the new disc can be attached to $S \setminus U$. The resulting structure is a new hyperbolic cone-structure $\sigma'$, different to $\sigma$ (i.e., $\sigma$ and $\sigma'$ are not isomorphic as hyperbolic cone-structure), with the same holonomy. In particular, the developing map $\text{dev}_{\sigma'}$ is a different to $\text{dev}_{\sigma}$. From the construction is clear that $p'$ is the developed image of the cone-point of the new structure $\sigma'$, which is different to the developed image of $p$.

Starting from $\rho$, $\sigma$ is the hyperbolic cone-structure on $S$ with holonomy $\rho$ obtained by choosing the developed image of $p$ as basepoint from which to begin the developing map; whereas $\sigma'$ is the hyperbolic cone-structure on $S$ obtained by choosing $p'$ as basepoint.
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