Extension of positive definite functions and Connes’ embedding conjecture

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Abstract

In this paper we formulate a conjecture which is a strengthening of an extension theorem of Bakonyi and Timotin for positive definite functions on the free group on two generators. We prove that this conjecture implies Connes’ embedding conjecture. We prove a weak case of this extension conjecture.

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1 Introduction

1.1 Connes’ embedding conjecture

1.1.1 Statement of the conjecture

If $A$ and $B$ are $C^*$-algebras, we will write $A \otimes_{\text{max}} B$ for the maximal tensor product and $A \otimes_{\text{min}} B$ for the minimal tensor product. For information about tensor products of operator algebras we refer the reader to Chapter 11 of [5]. If $G$ is a countable discrete group, we will write $C^*(G)$ for the full group $C^*$-algebra of $G$. For information about group $C^*$-algebras we refer the reader to Chapter VII of [4]. Let $F$ be the free group on two generators. We now state Connes’ embedding conjecture.

**Conjecture 1.1 (Connes’ embedding conjecture).** We have

$$C^*(F) \otimes_{\text{max}} C^*(F) = C^*(F) \otimes_{\text{min}} C^*(F)$$

The main topic of this paper is a theorem that Conjecture 2.1 in Subsection 2.5 below implies Conjecture 1.1.

Let $F_{\infty}$ be the free group on a countably infinite set of generators. It is well known that $F_{\infty}$ embeds as a subgroup of $F$. Given an identification of $F_{\infty}$ with a subgroup of $F$, we obtain an embedding of $C^*(F_{\infty})$ into $C^*(F)$. This embedding gives rise to a commutative diagram

$$C^*(F_{\infty}) \otimes_{\text{max}} C^*(F_{\infty}) \longrightarrow C^*(F_{\infty}) \otimes_{\text{min}} C^*(F_{\infty})$$

where all the arrows represent $*$-homomorphisms, the vertical arrows are embeddings and Conjecture 1.1 guarantees the bottom arrow is an isomorphism. Thus Conjecture 1.1 implies the following, which is more commonly seen in the literature on Connes’ embedding conjecture.

**Corollary 1.1 (Connes’ embedding conjecture, $F_{\infty}$ version).** We have

$$C^*(F_{\infty}) \otimes_{\text{max}} C^*(F_{\infty}) = C^*(F_{\infty}) \otimes_{\text{min}} C^*(F_{\infty})$$
1.2 Half finite approximation conjecture

1.2.1 Statement of the conjecture

We always assume Hilbert spaces are separable with complex scalars. Given a Hilbert space \( X \), let \( \text{GL}(X) \) be the group of bounded linear operators on \( X \) with bounded inverses and let \( U(X) \) be the group of unitary operators on \( X \).

**Definition 1.1.** Let \( X \) be a Hilbert space. We define a linear representation \( \zeta: F \times F \to \text{GL}(X) \) to be **half finite** if there exist a finite quotient \( \Gamma \) of \( F \) and a linear representation \( \zeta_\bullet: \Gamma \times F \to \text{GL}(X) \) such that \( \zeta \) factors as \( \zeta_\bullet \) precomposed with \( \Pi \times \iota \), where \( \Pi: F \to \Gamma \) is the quotient map and \( \iota \) is the identity map on \( F \). We define \( \zeta \) to be **totally finite** if there exists a finite quotient \( \Lambda \) of \( F \times F \) and a linear representation \( \zeta_\circ: \Lambda \to \text{GL}(X) \) such that \( \zeta \) factors as \( \zeta_\circ \) precomposed with the quotient map from \( F \times F \) to \( \Lambda \).

Conjecture 1.1 will be an easy consequence of the following.

**Conjecture 1.2 (Existence of half finite approximations).** Let \( X \) be a Hilbert space, let \( \rho: F \times F \to U(X) \) be a unitary representation and let \( x \in X \) be a unit vector. Let \( E \) and \( F \) be finite subsets of \( F \) and let \( \epsilon > 0 \). Then there exist a Hilbert space \( Y \), a finite quotient \( \Gamma \) of \( F \), a half finite unitary representation \( \zeta: F \times F \to \Gamma \times F \to U(Y) \) and a unit vector \( y \in Y \) such that

\[
|\langle \rho(g, g')x, x \rangle - \langle \zeta(g, g')y, y \rangle| \leq \epsilon
\]

for all \( g \in E \) and \( g' \in F \).

1.2.2 Totally finite approximations

Conjecture 1.2 asserts that the half finite representations are dense in the unitary dual of \( F \times F \). In [8] it is proved that the representations factoring through finite quotients are dense in the unitary dual of \( F \). Since the tensor product of two such representations is totally finite, given Conjecture 1.1 we can see that the totally finite representations are dense in the unitary dual of \( F \times F \). In other words, Conjecture 1.2 implies the same statement with half finite replaced by totally finite.

1.2.3 Deducing Connes’ embedding conjecture from half finite approximation conjecture

In this subsection we will show how Conjecture 1.2 implies Conjecture 1.1. Assume that Conjecture 1.2 is true. Write \( \| \cdot \|_{\text{max}} \) for the norm on \( C^*(F) \otimes_{\text{max}} C^*(F) \) and \( \| \cdot \|_{\text{min}} \) for the norm on \( C^*(F) \otimes_{\text{min}} C^*(F) \). Fix an element \( \phi \) in the group ring \( \mathbb{C}[F \times F] \) such that \( \|\phi\|_{\text{max}} = 1 \). In order to prove Conjecture 1.1 suffices to show that \( \|\phi\|_{\text{min}} = 1 \). To this end, let \( \sigma > 0 \).

Since \( \|\phi\|_{\text{max}} = 1 \) we can find a Hilbert space \( X \), a unitary representation \( \rho: F \times F \to U(X) \) and a unit vector \( x \in X \) such that \( \|\rho(\phi)x\|^2 \geq 1 - \sigma \). Write

\[
\phi = \sum_{g \in E} \sum_{g' \in F} \alpha_{g,g'}(g, g')
\]
for finite sets \( E, F \subseteq F \) and complex numbers \( (\alpha_{g,g'})_{g \in E, g' \in F} \). Let \( \epsilon > 0 \) be such that

\[
\epsilon \left( \sum_{g \in E} \sum_{g' \in F} |\alpha_{g,g'}|^2 \right) \leq \sigma
\]
Apply Conjecture 1.2 to these parameters with $E$ replaced by $E^{-1}E$ and $F$ replaced by $F^{-1}F$. We obtain $\gamma, \Gamma, \zeta$ and $y$. We have
\[
|\langle \rho(\phi)x, \rho(\phi)x \rangle - \langle \zeta(\phi)y, \zeta(\phi)y \rangle |
\]
\[
= \left| \left( \sum_{g \in E, g' \in E} \sum_{h \in E, h' \in E} \alpha_{g,g'} \rho(g, g') \right) x, \left( \sum_{h \in E, h' \in E} \alpha_{h,h'} \rho(h, h') \right) x \right|
\]
\[
- \left| \left( \sum_{g \in E, g' \in E} \sum_{h \in E, h' \in E} \alpha_{g,g'} \zeta(g, g') \right) y, \left( \sum_{h \in E, h' \in E} \alpha_{h,h'} \zeta(h, h') \right) y \right|
\]
\[
= \left| \sum_{g,h \in E, g', h' \in E} \alpha_{g,g'} \alpha_{h,h'} \langle \rho(g, g')x, \rho(h, h')x \rangle \right|
\]
\[
- \sum_{g,h \in E, g', h' \in E} \alpha_{g,g'} \alpha_{h,h'} \langle \zeta(g, g')y, \zeta(h, h')y \rangle \right|
\]
\[
\leq \sum_{g,h \in E, g', h' \in E} |\alpha_{g,g'}||\alpha_{h,h'}| \langle \rho(h^{-1}g, (h')^{-1}g')x, x \rangle - \langle \zeta(h^{-1}g, (h')^{-1}g')y, y \rangle \right|
\]
\[
\leq \epsilon \sum_{g,h \in E, g', h' \in E} |\alpha_{g,g'}| |\alpha_{h,h'}| \leq \epsilon \left( \sum_{g \in E, g' \in E} |\alpha_{g,g'}| \right)^2 \leq \sigma
\]
Therefore we obtain $||\zeta(\phi)y||^2 \geq ||\rho(\phi)x||^2 - \sigma$ so that $||\zeta(\phi)||^2_{op} \geq 1 - 2\sigma$.

There is a natural commutative diagram
\[
C^*(\mathcal{F}) \otimes_{\max} C^*(\mathcal{F}) \xrightarrow{\max} C^*(\mathcal{F}) \otimes_{\min} C^*(\mathcal{F})
\]
\[
C^*(\Gamma) \otimes_{\max} C^*(\mathcal{F}) \xrightarrow{\max} C^*(\Gamma) \otimes_{\min} C^*(\mathcal{F})
\]
where all the arrows represent surjective *-homomorphisms. Moreover, there are canonical copies of $\phi$ in each of the above algebras. Since $\zeta$ factors through $\Gamma \times \mathcal{F}$, we see that the norm of $\phi$ in the bottom left corner is at least $\sqrt{1 - 2\sigma}$. Since $C^*(\Gamma)$ is finite dimensional, Lemma 11.3.11 in [14] implies the arrow across the bottom of the above diagram is an isomorphism. It follows that the norm of $\phi$ in the bottom right corner is at least $\sqrt{1 - 2\sigma}$ and so $||\phi||_{\min} \geq \sqrt{1 - 2\sigma}$. Since $\sigma > 0$ was arbitrary we obtain $||\phi||_{\min} = 1$ as required. Therefore in order to prove Conjecture 1.2 it suffices to prove Conjecture 1.2.
1.3 Notation

1.3.1 The free group

Fix a pair of free generators $a$ and $b$ for $F$ and endow $F$ with the standard Cayley graph structure corresponding to left multiplication by these generators. If $\Gamma$ is a quotient of $F$ we identify $a$ and $b$ with their images in $\Gamma$. We write $e$ for the identity of $F$. We will also use the symbol $e$ for $2$

We consider the word length associated to $a$ and $b$, which we denote by $|\cdot|$. For $r \in \mathbb{N}$ let $B_r = \{ g \in F : |g| \leq r \}$ be the ball of radius $r$ around $e$. Write $K_r$ for the cardinality of $B_r$.

We define an ordering $\preceq$ on the sphere of radius 1 in $F$ by setting $a \preceq b \preceq a^{-1} \preceq b^{-1}$. From this we obtain a corresponding shortlex linear ordering on all of $F$, which we continue to denote by $\preceq$. For $g \in F$ define $I_g = \bigcup \{ \{ h, h^{-1} \} : h \preceq g \}$. Define a generalized Cayley graph $\text{Cay}(F, g)$ with vertex set equal to $F$ by placing an edge between distinct elements $h$ and $\ell$ if and only if $\ell^{-1}h \in I_g$. Write $g_\uparrow$ for the immediate predecessor of $g$ in $\preceq$ and $g_\downarrow$ for the immediate successor of $g$ in $\preceq$.

1.3.2 Miscellanea

If $z$ and $w$ are complex numbers and $\epsilon > 0$ we will sometimes write $z \approx [\epsilon] w$ to mean $|z - w| \leq \epsilon$.

We write $\mathbb{D}$ for the open unit disk in the complex plane.

If $n \in \mathbb{N}$ we write $[n]$ for $\{1, \ldots, n\}$.

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2 Harmonic analysis on the free group

2.1 The fundamental inequality on the free group

**Definition 2.1.** Let $d \in \mathbb{N}$ and let $F$ be a finite subset of $F$. We define a function $C : F \to \text{Mat}_{d \times d}(\mathbb{C})$ to be positive definite if we have the fundamental inequality

$$\sum_{g, h \in E} \alpha(h)^* C(h^{-1} g) \alpha(g) \geq 0 \quad (2.1)$$

for every subset $E$ of $F$ with $E^{-1}E \subseteq F$ and every function $\alpha : E \to \mathbb{C}^d$.

We define $C$ to be strictly positive definite if $C$ is positive definite and the inequality in (2.1) is saturated only when $\alpha$ is identically 0. We define a function $C : F \to \text{Mat}_{d \times d}(\mathbb{C})$ to be (strictly) positive definite if $C \upharpoonright F$ is (strictly) positive definite for every finite $F \subseteq F$.

A positive definite function on the free group can be thought of as a noncommutative analog of an infinite positive definite Toeplitz matrix.
2.2 The space of normalized strictly positive definite functions

We will always assume the following normalization condition.

**Definition 2.2.** Let $I_d$ denote the $d \times d$ identity matrix. If $C$ is a positive definite function with values in $\text{Mat}_{d \times d}(\mathbb{C})$ whose domain contains $e$, we define $C$ to be **normalized** if $C(e) = I_d$.

If $C$ is normalized then for any fixed $g \in F$ the vectors $\Phi_C(g)_1, \ldots, \Phi_C(g)_d$ are orthonormal. We denote the space of normalized strictly positive definite functions $C : \mathbb{B}_r \to \text{Mat}_{d \times d}(\mathbb{C})$ by $\text{NSPD}(r, d)$. We endow the space of functions from $\mathbb{B}_r$ to $\text{Mat}_{d \times d}(\mathbb{C})$ with the norm

$$||C||_1 = \sum_{g \in \mathbb{B}_r} \sum_{j=1}^d |\Phi_C(g)_j|^2$$

2.3 Realizations of positive definite functions on balls

Note that $\mathbb{B}_r^{-1} \mathbb{B}_r = \mathbb{B}_{2r}$. Therefore if $C \in \text{NSPD}(2r, d)$ we can regard it as a positive definite kernel on the set $\mathbb{B}_r \times [d]$. By Theorem C.2.3 in [2] there exists a Hilbert space $\mathcal{X}(\mathbb{C})$ and a function $\Phi_C : \mathbb{B}_r \to \mathcal{X}(\mathbb{C})$ such that

$$\langle \Phi_C(g)_j, \Phi_C(h)_k \rangle = C(h^{-1}g)_j, k$$

(2.2)

for all $g, h \in \mathbb{B}_r$ and all $j, k \in [d]$. Moreover, we may and will assume that the coordinates of the range of $\Phi_C$ span $\mathcal{X}(\mathbb{C})$. The hypothesis that $C$ is strictly positive definite ensures that the coordinates of the range of $\Phi_C$ will be linearly independent. We will refer to them as the canonical basis for $\mathcal{X}(\mathbb{C})$.

**Definition 2.3.** We say that $(\mathcal{X}(\mathbb{C}), \Phi_C)$ as above is a **realization** of $C$.

We can construct a realization of a positive definite function $C : F \to \text{Mat}_{d \times d}(\mathbb{C})$ in the same way, obtaining a Hilbert space $\mathcal{X}(\mathbb{C})$ and a function $\Phi_C : F \to \mathcal{X}(\mathbb{C})^d$ such that the span of the coordinates of the range of $\Phi_C$ is dense in $\mathcal{X}(\mathbb{C})$.

It is clear that given two realizations of the same positive definite function there exists a natural unitary isomorphism from one realization Hilbert space to the other. This isomorphism transforms a canonical basis vector in one realization to the canonical basis vector in another realization having the same index. If $C : F \to \text{Mat}_{d \times d}(\mathbb{C})$ is positive definite then the function $g \mapsto (\Phi_C(hg)_1, \ldots, \Phi_C(hg)_d)$ is a realization of $\Phi_C$ for any $h \in F$. Thus we may make the following definition.

**Definition 2.4.** Let $d \in \mathbb{N}$ and let $C : F \to \text{Mat}_{d \times d}(\mathbb{C})$ be positive definite. Then any realization of $C$ defines an **associated unitary representation** of $F$ on $\mathcal{X}(\mathbb{C})$ denoted by $\rho_C$ and given by the translation $\rho_C(h)\Phi_C(g)_j = \Phi_C(hg)_j$ for $g, h \in F$ and $j \in [d]$.

2.4 Transport operators and relative energies

**Definition 2.5.** Let $C, D \in \text{NSPD}(2r, d)$. Let $(\mathcal{X}(\mathbb{C}), \Phi_C)$ and $(\mathcal{X}(\mathbb{D}), \Phi_D)$ be realizations of $C$ and $D$ respectively. Define the **transport operator** $t[C, D] : \mathcal{X}(\mathbb{C}) \to \mathcal{X}(\mathbb{D})$ by setting

$$t[C, D] \sum_{g \in \mathbb{B}_r} \sum_{j=1}^d \alpha(g)_j \Phi_C(g)_j = \sum_{g \in \mathbb{B}_r} \sum_{j=1}^d \alpha(g)_j \Phi_D(g)_j$$
for functions \( \alpha : \mathcal{B}_r \to \mathbb{C}^d \). We refer to the square of the operator norm of \( t[C, D] \) as the relative energy of the pair \( (C, D) \) and denote it by \( \varepsilon(C, D) \).

If \( C, D : \mathbb{F} \to \text{Mat}_{d \times d}(\mathbb{C}) \) are strictly positive definite we define the relative energy of the pair \( (C, D) \) to be \( \sup_{r \in \mathbb{N}} \varepsilon(C \upharpoonright \mathcal{B}_r, D \upharpoonright \mathcal{B}_r) \). We continue to denote it by \( \varepsilon(C, D) \). In general we may have \( \varepsilon(C, D) = \infty \). If \( \varepsilon(C, D) < \infty \) then there is a naturally defined transport operator from \( \mathcal{X}(C) \) to \( \mathcal{X}(D) \), which we continue to denote by \( t[C, D] \).

The relevance of Definition 2.5 is that the transport operator between two strictly positive definite functions defined on all of \( \mathbb{F} \) clearly intertwines the associated unitary representations. Thus transport operators will be useful in constructing commuting representations of \( \mathbb{F} \).

### 2.5 Low energy extension conjecture

Our main harmonic analysis conjecture is the following.

**Conjecture 2.1** (Existence of extensions with low energy gain). Let \( r, d \in \mathbb{N} \) and let \( \omega > 0 \). Let \( C_1, \ldots, C_n \) be elements of \( \text{NSPD}(r, d) \). Then for each \( m \in [n] \) there exists a strictly positive definite function \( \mathcal{C}_m : \mathbb{F} \to \text{Mat}_{d \times d}(\mathbb{C}) \) such that such that \( \mathcal{C}_m = \mathcal{C}_m \upharpoonright \mathcal{B}_r \) and such that \( \varepsilon(C_m, C_k) = \varepsilon(\mathcal{C}_m, \mathcal{C}_k) \) for all \( m, k \in [n] \).

In order for Conjecture 2.1 to be plausible we ought to know that any element of \( \text{NSPD}(r, d) \) admits an extension to a positive definite function defined on all of \( \mathbb{F} \). This fact appears as Proposition 4.4 in [1] and as Lemma 25 in [9].

### 3 Proof of Conjecture 1.2 from Conjecture 2.1

In Section 3 we prove Conjecture 1.2 from Conjecture 2.1.

#### 3.1 Unitary approximate conjugacy of representations

**3.1.1 Generalities**

We will use the theory of weak containment of unitary representations of countable discrete groups, for which we refer the reader to Appendix H of [6]. We will say that a unitary representation of a countable discrete group \( G \) is maximal if it weakly contains every other unitary representation of \( G \).

If \( G \) is a countable discrete group, \( \mathcal{X} \) is a Hilbert space and \( \rho : G \to \text{U}(\mathcal{X}) \) is a unitary representation, there is a unique extension of \( \rho \) to a \( * \)-homomorphism from \( C^*(G) \) to the algebra \( \text{B}(\mathcal{X}) \) of bounded operators on \( \mathcal{X} \). We denote this extension by \( \tilde{\rho} \). Let \( \xi : G \to \text{U}(\mathcal{Y}) \) be another unitary representation, potentially on a different Hilbert space. By Theorem F.4.4 in [2], if \( \xi \) is weakly contained in \( \rho \) then \( ||\xi(s)||_{\text{op}} \leq ||\tilde{\rho}(s)||_{\text{op}} \) for all \( s \in C^*(G) \). It follows that if \( \rho \) is a maximal unitary representation then \( \tilde{\rho} \) is injective. We now recall a different notion of approximation for representations.

**Definition 3.1.** Unitary representations \( \rho : G \to \text{U}(\mathcal{X}) \) and \( \xi : G \to \text{U}(\mathcal{Y}) \) are said to be unitarily approximately conjugate if there is a sequence of unitary operators \( u_n : \mathcal{X} \to \mathcal{Y} \) such that for each \( g \in G \) we have

\[
\lim_{n \to \infty} ||u_n^{-1}\xi(g)u_n - \rho(g)||_{\text{op}} = 0.
\]
The following is a special case of Corollary 1.7.5 in [3].

**Theorem 3.1** (Voiculescu). Let $G$ be a countable discrete group. Suppose $\xi$ and $\rho$ are unitary representations of $G$ such that $\tilde{\xi}$ and $\tilde{\rho}$ are injective and such that $\tilde{\xi}(C^*(G))$ and $\tilde{\rho}(C^*(G))$ contain no nonzero compact operators. Then $\xi$ and $\rho$ are unitarily approximately conjugate.

We can now connect weak containment and unitary approximate conjugacy.

**Proposition 3.1.** Suppose $\xi$ and $\rho$ are maximal unitary representations of $F$. Then $\xi$ and $\rho$ are unitarily approximately conjugate.

**Proof of Proposition** 3.1. By Corollary VII.6.7 in [4] the image of an injective representation of $C^*(F)$ contains no nonzero compact operators. Thus Proposition 3.1 follows from Theorem 3.1.

### 3.1.2 Introducing initial data

Consider the group $G = F \times F$. In order to keep a distinction between the factors, we will write $F_\triangleright$ for the left copy and $F_\triangleleft$ for the right copy. We again fix free generators for each copy and endow them with the corresponding word lengths. We will consistently use the letters $g, h$ for elements of $F_\triangleright$ and $g', h'$ for elements of $F_\triangleleft$. If $X$ is a Hilbert space, $\rho: F_\triangleright \times F_\triangleleft \rightarrow GL(X)$ is a linear representation and $j \in \{\triangleright, \triangleleft\}$ we will write $\rho_j$ for the restriction of $\rho$ to $F_j$. We will also write $B_{r,j}$ for the ball of radius $r$ around the identity in $F_j$.

Fix a Hilbert space $X$, a unitary representation $\rho: F_\triangleright \times F_\triangleleft \rightarrow U(X)$ and a unit vector $x \in X$. It is clear that no generality is lost in Conjecture 1.2 if we assume that $\rho$ is maximal and we indeed make this assumption. Fix finite sets $E \subseteq F_\triangleright$, $F \subseteq F_\triangleleft$ and let $r \in \mathbb{N}$ be such that $E \subseteq B_{r,\triangleright}$ and $F \subseteq B_{r,\triangleleft}$. We may assume that $r \geq 5$. Also fix $\epsilon \in (0, 1)$. We have now introduced all the data in the hypotheses of Conjecture 1.2.

Write $L_{r,\epsilon} = \sqrt{\frac{K_r}{\epsilon}}$. Choose $\delta > 0$ such that

$$320L_{r,\epsilon}^{10}K_r\delta \leq \epsilon \quad (3.1)$$

### 3.1.3 Approximate conjugacy with the profinite completion

Let $\hat{F}$ denote the profinite completion of $F$ and let $\mu$ be its Haar probability measure. For each finite quotient $\Lambda$ of $F$, there exists a canonical projection $\Pi_\Lambda: \hat{F} \rightarrow \Lambda$. Writing $1_B$ for the indicator function of a subset $B$ of $\hat{F}$, for each $\lambda \in \Lambda$ we have

$$\left\| 1_{\Pi_\Lambda^{-1}(\lambda)} \right\|_2 = \left( \int_{\hat{F}} \left| 1_{\Pi_\Lambda^{-1}(\lambda)}(\omega) \right|^2 d\mu(\omega) \right)^{\frac{1}{2}}$$

$$= \sqrt{\mu(\Pi_\Lambda^{-1}(\lambda))}$$

$$= \frac{1}{\sqrt{|\Lambda|}}$$

Moreover, if $\lambda$ and $\lambda'$ are distinct elements of $\Lambda$ then the sets $\Pi_\Lambda^{-1}(\lambda)$ and $\Pi_\Lambda^{-1}(\lambda')$ are disjoint, so that $1_{\Pi_\Lambda^{-1}(\lambda)}$ and $1_{\Pi_\Lambda^{-1}(\lambda')}$ are orthogonal in $L^2(\hat{F}, \mu)$. Therefore the set of functions

$$\left\{ \sqrt{|\Lambda|}1_{\Pi_\Lambda^{-1}(\lambda)} : \lambda \in \Lambda \right\}$$
is orthonormal.

The profinite structure of $F$ guarantees that the collection of sets
\[ \{ \Pi^{-1}_\lambda(\lambda) : \lambda \in \Lambda, \Lambda \text{ is a finite quotient of } F \} \]
generates the Borel $\sigma$-algebra of $F$. Therefore we have that the span of the functions
\[ \{ \sqrt{|\Lambda|}1_{\Pi^{-1}_\lambda(\lambda)} : \lambda \in \Lambda, \Lambda \text{ is a finite quotient of } F \} \] (3.2)
is dense in $L^2(F, \mu)$. Choose a sequence $(\Lambda_n)_{n=1}^\infty$ of finite quotients of $F$ such that $\Lambda_n$ is a quotient of $\Lambda_{n+1}$ and such that any finite quotient $\Lambda$ of $F$ is a quotient of $\Lambda_n$ for some $n \in \mathbb{N}$. Write $\Pi_n$ for $\Pi_{\Lambda_n}$. Then the span of the set of functions
\[ \bigcup_{n=1}^\infty \{ \sqrt{|\Lambda_n|}1_{\Pi^{-1}_n(\lambda)} : \lambda \in \Lambda_n \} \] (3.3)
is equal to the span of the set of functions in (3.2). Hence the span of the set of functions in (3.3) is dense in $L^2(F, \mu)$. Moreover, the spans of each of the sets inside the union in (3.3) are increasing.

By considering induced representations, we see that since $\rho$ is a maximal representation of $F_\alpha \times F_\beta$, we must have that $\rho_\alpha$ is a maximal representation of $F_\alpha$. By Theorem 3.1 in [7], the left translation action of $F$ on $(F, \mu)$ is maximal in the order of weak containment among measure preserving actions of $F$. We refer the reader to Chapter 10 of [6] for information on this variant of weak containment, but all we will need to know about it is that Proposition 10.5 and Theorem E.1 in [6] imply that the Koopman representation of a maximal action is a maximal representation. Write $\kappa : F \to \mathbb{U}(L^2(F, \mu))$ for the Koopman representation of the left translation action, so that Proposition 3.1 implies $\rho_\alpha$ and $\kappa$ are unitarily approximately conjugate.

Let $u : X \to L^2(F, \mu)$ be a unitary operator such that
\[ ||u^{-1}\kappa(g)u - \rho_\alpha(g)||_{op} \leq \delta \]
for all $g \in B_{1,\alpha}$.

Consider the vector $ux$. Our previous discussion of (3.3) implies that we can find $n \in \mathbb{N}$ and a function $\alpha : \Lambda_n \to \mathbb{C}$ with
\[ \left| \|ux - \sum_{\lambda \in \Lambda_n} \alpha(\lambda)\sqrt{|\Lambda_n|}1_{\Pi^{-1}_n(\lambda)} \|_2 \right| \leq \delta \]
We may assume $n$ is large enough that the balls of radius $4R$ in $F/\Lambda_n$ are isomorphic to the balls of radius $4R$ in $F$.

The partition $\{ \Pi^{-1}_n(\lambda) : \lambda \in \Lambda_n \}$ of $F$ is permuted by the left translation action of $F$ on $F$ so that $g\Pi^{-1}_n(\lambda) = \Pi^{-1}_n(g\lambda)$ for all $g \in F$ and $\lambda \in \Lambda_n$. Thus we have $\kappa(g)1_{\Pi^{-1}_n(\lambda)} = 1_{\Pi^{-1}_n(g\lambda)}$. Since the vectors
\[ \{ \sqrt{|\Lambda_n|}1_{\Pi^{-1}_n(\lambda)} : \lambda \in \Lambda_n \} \]
are orthonormal and $x$ is a unit vector, we may assume that
\[ \sum_{\lambda \in \Lambda_n} |\alpha(\lambda)|^2 = 1 \]
Enumerate \( \Lambda_n = \{\lambda_1, \ldots, \lambda_d\} \) and for \( j \in [d] \) let

\[
x_j = u^{-1} \sqrt{d} 1_{\Pi_{\gamma}^{-1}(\lambda_j)}
\]

We may assume without loss of generality that \( d \geq R \). Write \( \kappa = u^{-1} \pi u \). We summarize the objects we have just constructed.

- An orthonormal set of vectors \( x_1, \ldots, x_d \) in \( \mathcal{X} \) and an element \( \alpha \in \mathbb{C}^d \) with

\[
\sum_{j=1}^{d} |\alpha_j|^2 = 1
\]

such that

\[
||x - \alpha_1 x_1 - \cdots - \alpha_d x_d|| \leq \delta
\]

- An action \( \sigma : \mathcal{F}_d \to \text{Sym}(d) \).

- A unitary representation \( \kappa : \mathcal{F}_d \to U(\mathcal{X}) \) such that

\[
||\kappa(g) - \rho_\sigma(g)||_{op} \leq \delta
\]

for all \( g \in \mathcal{B}_{1,\varsigma} \) and such that

\[
\kappa(g)x_j = x_{\sigma(g)j}
\]

for all \( g \in \mathcal{F}_d \) and all \( j \in [d] \).

## 3.2 Building a half finite linear representation

### 3.2.1 Proximity between inner products at individual nodes

**Proposition 3.2.** Let \( g', h' \in \mathcal{B}_{r,\rho} \) and let \( \beta, \eta \in \mathbb{C}^d \). Also let \( g \in \mathcal{B}_{1,\varsigma} \) and let \( \varsigma \in \text{Sym}(d) \). We have

\[
\left| \left< \rho_{\omega}(g') \sum_{j=1}^{d} \beta_j x_{\varsigma j}, \rho_{\omega}(h') \sum_{k=1}^{d} \eta_k x_{ck} \right> - \left< \rho_{\omega}(g') \sum_{j=1}^{d} \beta_j x_{\varsigma(g)j}, \rho_{\omega}(h') \sum_{k=1}^{d} \eta_k x_{\varsigma(g)ck} \right> \right| 
\leq 2\delta \|\beta\|_2 \|\eta\|_2
\]

**Proof of Proposition 3.2.** All norms and inner products in the proof of Proposition 3.2 will be in \( \mathcal{X} \). From (3.7) we have

\[
\left| \left< \rho_{\omega}(g') \sum_{j=1}^{d} \beta_j x_{\varsigma j}, \rho_{\omega}(h') \sum_{k=1}^{d} \eta_k x_{ck} \right> - \left< \rho_{\omega}(g') \sum_{j=1}^{d} \beta_j x_{\varsigma(g)j}, \rho_{\omega}(h') \sum_{k=1}^{d} \eta_k x_{\varsigma(g)ck} \right> \right|
= \left| \left< \rho_{\omega}(g') \sum_{j=1}^{d} \beta_j x_{\varsigma j}, \rho_{\omega}(h') \sum_{k=1}^{d} \eta_k x_{ck} \right> - \left< \rho_{\omega}(g') \kappa(g) \sum_{j=1}^{d} \beta_j x_{\varsigma j}, \rho_{\omega}(h') \kappa(g) \sum_{k=1}^{d} \eta_k x_{ck} \right> \right|
\leq \left| \left< \rho_{\omega}(g') \sum_{j=1}^{d} \beta_j x_{\varsigma j}, \rho_{\omega}(h') \sum_{k=1}^{d} \eta_k x_{ck} \right> - \left< \rho_{\omega}(g') \kappa(g) \sum_{j=1}^{d} \beta_j x_{\varsigma j}, \rho_{\omega}(h') \kappa(g) \sum_{k=1}^{d} \eta_k x_{ck} \right> \right|
\]

Write \( x_\beta \) for \( \sum_{j=1}^{d} \beta_j x_{\varsigma j} \) and \( x_\eta \) for \( \sum_{k=1}^{d} \eta_k x_{\varsigma k} \). We compute
Here,

- (3.9) is equal to (3.10) since \( \rho_d \) and \( \rho_B \) commute,
- (3.12) follows from (3.10) - (3.11) since \( \rho_d \) is unitary and therefore (3.10) is 0,
- (3.14) follows from (3.13) - (3.14) since \( \rho_B \) and \( \kappa \) are unitary,
- (3.17) follows from (3.16) since \( x_1, \ldots, x_d \) is orthonormal,
- and (3.18) follows from (3.17) by (3.9).

Proposition 3.2 follows by combining (3.8) with (3.18).

### 3.2.2 Constructing a family of permuted positive definite functions

For \( \zeta \in \text{Sym}(d) \) define a positive definite function \( C_{\zeta} : B_{2r,d} \rightarrow \text{Mat}_{d \times d}(\mathbb{C}) \) by setting

\[
C_{\zeta}((h')^{-1}g')_{j,k} = \langle \rho_B(g) x_{j}, \rho_B(h') x_{k} \rangle
\]

for \( g' \), \( h' \in B_{r,d} \). Also define \( \Delta_r : B_{2r,d} \rightarrow \text{Mat}_{d \times d}(\mathbb{C}) \) by setting

\[
\Delta_r(g') = \begin{cases} 
I_d & \text{if } g' = e \\
0_d & \text{if } g' \in B_{2r} \setminus \{e\}
\end{cases}
\]


where $\mathbf{0}_d$ denotes the $d \times d$ zero matrix. Let $D_\zeta = (1 - \epsilon)C_\zeta + \epsilon\Delta_r$. Note that for any function $\beta : \mathbb{B}_{r,\rho} \rightarrow \mathbb{C}^d$ we have

$$
\left\| \sum_{g' \in \mathbb{B}_{r,\rho}} \sum_{j=1}^d \beta(g')_j \Phi_{D_\zeta}(g')_j \right\|^2
= (1 - \epsilon) \left\| \sum_{g' \in \mathbb{B}_{r,\rho}} \sum_{j=1}^d \beta(g')_j \Phi_{C_\zeta}(g')_j \right\|^2 + \epsilon \left\| \sum_{g' \in \mathbb{B}_{r,\rho}} \sum_{j=1}^d \beta(g')_j \Phi_{\Delta_r}(g')_j \right\|^2
\geq \epsilon \left\| \sum_{g' \in \mathbb{B}_{r,\rho}} \sum_{j=1}^d \beta(g')_j \Phi_{\Delta_r}(g')_j \right\|^2
= \epsilon \left( \sum_{g' \in \mathbb{B}_{r,\rho}} \sum_{j=1}^d |\beta(g')_j|^2 \right)
$$

(3.20) Here, (3.21) follows from (3.20) since the set

$$
\{ \Phi_{\Delta_r}(g')_j : g' \in \mathbb{B}_{r,\rho}, j \in [d] \}
$$

is orthonormal.

### 3.2.3 Establishing bounds on transport operators

**Proposition 3.3.** We have $\epsilon(D_\zeta, D_\gamma) \leq L_{r,\rho}$ for all $\zeta, \gamma \in \text{Sym}(d)$.

**Proof of Proposition 3.3.** Let $\beta : \mathbb{B}_{r,\rho} \rightarrow \mathbb{C}^d$ be such that

$$
y = \sum_{g' \in \mathbb{B}_{r,\rho}} \sum_{j=1}^d \beta(g')_j \Phi_{D_\zeta}(g')_j
$$

is a unit vector in $X(D_\zeta)$. Thus from (3.21) we have

$$
1 \geq \epsilon \left( \sum_{g' \in \mathbb{B}_{r,\rho}} \sum_{j=1}^d |\beta(g')_j|^2 \right)
$$

(3.22)

We compute

$$
||t[D_\zeta, D_\gamma]y|| = \left\| \sum_{g' \in \mathbb{B}_{r,\rho}} \sum_{j=1}^d \beta(g')_j \Phi_{D_\zeta}(g')_j \right\|
\leq \sum_{g' \in \mathbb{B}_{r,\rho}} \left\| \sum_{j=1}^d \beta(g')_j \Phi_{D_\zeta}(g')_j \right\|
$$

(3.23)
Let
\begin{equation}
\text{Proof of Proposition 3.4.}
\end{equation}
Here, (3.24) follows from (3.23) since \( D \) is normalized and (3.26) follows from (3.25) by (3.22).

**Proposition 3.4.** We have \( \varepsilon(D_\varsigma, D_{\sigma(g)_\varsigma}) \leq 1 + 2K_\delta R \) for all \( \varsigma \in \text{Sym}(d) \) and all \( g \in \mathbb{B}_{1,0} \).

**Proof of Proposition 3.4.** Let \( \varsigma \in \text{Sym}(d) \) and \( g \in \mathbb{B}_{1,0} \). Let \( \beta : \mathbb{B}_{r,b} \rightarrow \mathbb{C}^d \) be such that if we write
\[
y = \sum_{g' \in \mathbb{B}_{r,b}} \sum_{j=1}^d \beta(g'_j) \Phi_{D_\varsigma}(g'_j)
\]
then \( y \) is a unit vector in \( \mathcal{X}(D_\varsigma) \). Thus from (3.21), we have
\[
1 \geq \frac{1}{R} \left( \sum_{g' \in \mathbb{B}_{r,b}} \sum_{j=1}^d |\beta(g'_j)|^2 \right)
\]
(3.27)

We compute
\[
||t[D_\varsigma, D_{\sigma(g)_\varsigma}]y||^2 - 1
\]
\[
= \left| \left\langle \sum_{g' \in \mathbb{B}_{r,b}} \sum_{j=1}^d \beta(g'_j) \Phi_{D_{\sigma(g)_\varsigma}}(g'_j), \sum_{h' \in \mathbb{B}_{r,b}} \sum_{k=1}^d \beta(h'_k) \Phi_{D_{\sigma(g)_\varsigma}}(h'_k) \right\rangle - \left\langle \sum_{g' \in \mathbb{B}_{r,b}} \sum_{j=1}^d \beta(g'_j) \Phi_{D_\varsigma}(g'_j), \sum_{h' \in \mathbb{B}_{r,b}} \sum_{k=1}^d \beta(h'_k) \Phi_{D_\varsigma}(h'_k) \right\rangle \right|
\]
(3.28)
\[
= \sum_{g',h' \in \mathbb{B}_{r,b}} \sum_{j,k=1}^d \beta(g'_j) \beta(h'_k) D_{\sigma(g)_\varsigma}((h')^{-1}g')_{j,k}
\]
(3.29)
\[
\leq \sum_{g',h' \in \mathbb{B}_{r,b}} \sum_{j,k=1}^d \beta(g'_j) \beta(h'_k) \left( D_{\sigma(g)_\varsigma}((h')^{-1}g')_{j,k} - D_{\varsigma}((h')^{-1}g')_{j,k} \right)
\]
(3.30)
\[
= (1 - \frac{1}{R}) \sum_{g',h' \in \mathbb{B}_{r,b}} \sum_{j,k=1}^d \beta(g'_j) \beta(h'_k) \left( C_{\sigma(g)_\varsigma}((h')^{-1}g')_{j,k} - C_{\varsigma}((h')^{-1}g')_{j,k} \right)
\]
(3.31)
Apply Conjecture 2.1 to the positive definite functions (3.2.4 Constructing a representation through permutations such that D function from F. Here, θ note that ζ and (3.31) follows from (3.30) since the ∆(3.29) follows from (3.28) using (2.2) and (3.37) follows from (3.36) by (3.27).

(3.33) follows from (3.32) by (3.19)

(3.35) follows from (3.34) by Proposition 3.2

(3.33)

(3.35)

(3.36)

(3.37)

Here,

• (3.29) follows from (3.28) using (2.2)

• (3.31) follows from (3.30) since the Δv components in the definitions of Dζ and Dσ(g)ζ cancel,

• (3.33) follows from (3.32) by (3.19)

• (3.35) follows from (3.34) by Proposition 3.2

• and (3.37) follows from (3.36) by (3.27).

3.2.4 Constructing a representation through permutations

Apply Conjecture 2.1 to the positive definite functions (Dv)v∈V to obtain positive definite functions (D_v)v∈V such that D_v = D_v ⊲ B_v and such that ε(D_v, D_w) = ε(D_v, D_w) for all v, w ∈ V. We regard each D_v as a function from F_v to Mat_{d×d}(C).

Write y_v for X(D_v) and let ζ_v : F_v → U(y_v) be the associated representation of D_v. Define y = ⊕v∈V y_v and ζ = ⊕v∈V ζ_v. Define a representation θ : F_d → GL(y) by setting

\[ \theta(g) = \bigoplus_{v∈V} t[D_v, D_{σ(g)v}] \]

Note that θ factors through the finite group Γ = σ(F_d). Moreover, we have

\[ t[D_v, D_{σ(g)v}]ζ_v = ζ_v,σ(g)v t[D_v, D_{σ(g)v}] \]
for all \( v \in V \) and \( g \in F \). Therefore \( \theta \) commutes with \( \zeta \), so that \( \theta \times \zeta \) is a half finite linear representation of \( G \).

From Proposition 3.3 we see
\[
||\theta(g)|| \leq L_{r,\epsilon}
\]
for all \( g \in F \) and from Proposition 3.4 we see that
\[
||\theta(g)||_{op} \leq 1 + 2K_{r}\delta
\]
for all \( g \in B_{r,\Delta} \).

### 3.3 Repairing the representation to be unitary

#### 3.3.1 Conjugation by an average

In Segments 3.3.1 and 3.3.2 we regard \( \theta \) as a representation of the finite group \( \Gamma \). Define a positive operator \( q \in B(Y) \) by
\[
q = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \theta(\gamma)^* \theta(\gamma).
\]
By applying (??) to \( g^{-1} \) we see that each \( \theta(\gamma) \) is invertible. Hence each operator \( \theta(\gamma)^* \theta(\gamma) \) is strictly positive and so \( q \) is invertible. Define a representation \( \zeta_{\alpha} \) of \( \Gamma \) on \( Y \) by setting
\[
\zeta_{\alpha}(\gamma) = q^{-\frac{1}{2}} \theta(\gamma) q^{-\frac{1}{2}}
\]
for all \( \gamma \in \Gamma \) and all \( g' \in F \). We have that \( \zeta_{\alpha}(g') \) commutes with each \( \theta(\gamma) \). Since \( \zeta_{\alpha}(g') \) is unitary, this implies that \( \zeta_{\alpha}(g') \) commutes with \( \theta(\gamma)^* \) and hence \( \zeta_{\alpha}(g') \) commutes with \( q \). Therefore \( \zeta_{\alpha} \) commutes with \( \zeta_{\alpha} \) and so if we set \( \zeta = \zeta_{\alpha} \times \zeta_{\alpha} \) then \( \zeta \) is a half finite linear representation of \( G \).

We claim that \( \zeta \) is in fact unitary. Write \( I \) for the identity operator on \( Y \). For \( \gamma \in \Gamma \) we have
\[
\zeta_{\alpha}(\gamma)^* \zeta_{\alpha}(\gamma) = \left( q^{-\frac{1}{2}} \theta(\gamma) q^{-\frac{1}{2}} \right)^* \left( q^{-\frac{1}{2}} \theta(\gamma) q^{-\frac{1}{2}} \right)
\]
\[
= q^{-\frac{1}{2}} \theta(\gamma)^* q \theta(\gamma) q^{-\frac{1}{2}}
\]
\[
= q^{-\frac{1}{2}} \theta(\gamma)^* \left( \frac{1}{|\Gamma|} \sum_{\nu \in \Gamma} \theta(\nu)^* \theta(\nu) \right) \theta(\gamma) q^{-\frac{1}{2}}
\]
\[
= q^{-\frac{1}{2}} \left( \frac{1}{|\Gamma|} \sum_{\nu \in \Gamma} \theta(\nu)^* \theta(\nu) \theta(\gamma) \right) q^{-\frac{1}{2}}
\]
\[
= q^{-\frac{1}{2}} \left( \frac{1}{|\Gamma|} \sum_{\nu \in \Gamma} \theta(\nu)^* \theta(\nu) \right) q^{-\frac{1}{2}}
\]
\[
= I
\]
so that \( \zeta_{\alpha}(\gamma) \) is unitary and therefore \( \zeta \) is a unitary representation.
3.3.2 Bounding the spectrum of the average

Proposition 3.5. We have \( \text{spec}(q) \subseteq [L_{r,R}^{-2}, L_{r,R}^2] \).

Proof of Proposition 3.5. Using (??) we see that for any unit vector \( y \in Y \) we have
\[
\langle qy, y \rangle = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \langle \theta(\gamma)^*\theta(\gamma)y, y \rangle = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} ||\theta(\gamma)y||^2 \leq L_{r,R}^2.
\] (3.41)

By applying (??) to \( g^{-1} \) we see that
\[
\inf \{ ||\theta(\gamma)y||^2 : y \in Y \text{ is a unit vector} \} \geq \frac{1}{L_{r,R}^2}.
\] (3.42)

Now suppose \( \lambda \in \text{spec}(q) \). Since \( q \) is self-adjoint, there exists a sequence \( (y_n)_{n=1}^\infty \) of unit vectors in \( Y \) such that \( \lim_{n \to \infty} ||(q - \lambda I)y_n|| = 0 \). This implies that \( \lim_{n \to \infty} \langle (q - \lambda I)y_n, y_n \rangle = 0 \) and so \( \lim_{n \to \infty} \langle qy_n, y_n \rangle \). Thus from (3.41) and (3.42) we have \( L_{r,R}^{-2} \leq \lambda \leq L_{r,R}^2 \).

3.3.3 Estimating the distance to the repaired representation

Proposition 3.6. Suppose \( g \in B_{r,\vartriangle} \). Then \( ||\zeta(g) - \theta(g)||_{op} \leq R^{-1} \).

Proof of Proposition 3.6. Fix \( g \in B_{r,\vartriangle} \). By applying (3.39) to \( g \) and \( g^{-1} \) we see
\[
\frac{1}{1 + 2K_r\delta} I \leq \theta(g)^*\theta(g) \leq (1 + 2K_r\delta)I.
\]

Since \( \theta(g)^*\theta(g) \) is unitarily conjugate to \( \theta(g)\theta(g)^* \) we obtain
\[
\frac{1}{1 + 2K_r\delta} I \leq \theta(g)\theta(g)^* \leq (1 + 2K_r\delta)I.
\]
so that
\[
||\theta(g)\theta(g)^* - I||_{op} \leq 2K_r\delta \quad (3.43)
\]

Since \( q^{-\frac{1}{2}}\theta(g)^*q\theta(g)q^{-\frac{1}{2}} = I \) we have \( \theta(g)^*q\theta(g) = q \). Therefore
\[
||q\theta(g) - \theta(g)q||_{op} = ||q\theta(g) - \theta(g)\theta(g)^*q\theta(g)||_{op} \leq ||I - \theta(g)\theta(g)^*||_{op}||q||_{op}||\theta(g)||_{op} \leq 2||I - \theta(g)\theta(g)^*||_{op}||q||_{op} \leq 2L_{r,R}^2||I - \theta(g)\theta(g)^*||_{op} \leq 4L_{r,R}^2K_r\delta \quad (3.47)
\]

Here,
• (3.35) follows from (3.44) by (3.39) since $e^{2ss} \leq 2$,
• (3.46) follows from (3.45) by Proposition 3.5 since $q$ is self-adjoint,
• and (3.47) follows from (3.46) by (3.43).

Let $z \in \mathbb{C} \setminus \text{spec}(q)$. We compute
\[
\begin{align*}
||&(q - zI)^{-1}\theta(g) - \theta(g)(q - zI)^{-1}||_{\text{op}} \\
&= ||(q - zI)^{-1}\theta(g) - (q - zI)^{-1}(q - zI)\theta(g)(q - zI)^{-1}||_{\text{op}} \\
&\leq ||(q - zI)^{-1}||_{\text{op}}||\theta(g) - (q - zI)\theta(g)(q - zI)^{-1}||_{\text{op}} \\
&= \frac{1}{\text{dist}(z, \text{spec}(q))}||\theta(g) - (q - zI)\theta(g)(q - zI)^{-1}||_{\text{op}} \\
&= \frac{1}{\text{dist}(z, \text{spec}(q))}||\theta(g) - q\theta(g)(q - zI)^{-1} + z\theta(g)(q - zI)^{-1}||_{\text{op}} \\
&= \frac{1}{\text{dist}(z, \text{spec}(q))}||\theta(g) - q\theta(g)q(q - zI)^{-1} + z\theta(g)(q - zI)^{-1} + \theta(g)q(q - zI)^{-1} - q\theta(g)(q - zI)^{-1}||_{\text{op}} \\
&= \frac{1}{\text{dist}(z, \text{spec}(q))}||\theta(g)q(q - zI)^{-1} - q\theta(g)(q - zI)^{-1}||_{\text{op}} \\
&\leq \frac{1}{\text{dist}(z, \text{spec}(q))}||\theta(g)q - q\theta(g)||_{\text{op}} ||(q - zI)^{-1}||_{\text{op}} \\
&= \frac{1}{\text{dist}(z, \text{spec}(q))}||\theta(g)q - q\theta(g)||_{\text{op}} \\
&\leq \frac{4L^2_{r,R}K_r\delta}{\text{dist}(z, \text{spec}(q))} \\
\end{align*}
\]

Here, (3.39) follows from (3.48) by (3.47). Now, let $c : [0, 1] \to \mathbb{C}$ be a simple closed contour with the following properties.

(i) We have $\text{Re}(c(x)) > 0$ for all $x \in [0, 1]$.
(ii) The interval $[L_{r,R}^{-2}, L_{r,R}^2]$ is enclosed by $c$.
(iii) We have $\text{dist}(c(x), [L_{r,R}^{-2}, L_{r,R}^2]) \geq \frac{1}{2}L_{r,R}^{-2}$ for all $x \in [0, 1]$.
(iv) We have $\sup\{|c(x)| : x \in [0, 1]\} \leq 2L_{r,R}^2$
(v) We have $\ell(c) \leq 10L_{r,R}^2$, where $\ell(c)$ denotes the length of $c$.

By Clause (i) we can consistently define a square root function on the image of $c$. Proposition 3.5 together with Clause (ii) in the definition of $c$ implies that $c$ encloses $\text{spec}(q)$. Therefore we can use the holomorphic functional calculus to make the following computation.
\[
\|\theta(g)q^{\frac{1}{2}} - q^{\frac{1}{2}}\theta(g)\|_{\text{op}} \\
= \frac{1}{2\pi} \left\| \theta(g) \left( \int_{0}^{1} c(x)^{\frac{1}{2}}(c(x)I - q)^{-1} \, dx \right) - \left( \int_{0}^{1} c(x)^{\frac{1}{2}}(c(x)I - q)^{-1} \, dx \right) \theta(g) \right\|_{\text{op}} \\
= \frac{1}{2\pi} \left\| \int_{0}^{1} c(x)^{\frac{1}{2}} \left( \theta(g)(c(x)I - q)^{-1} - (c(x)I - q)^{-1}\theta(g) \right) \, dx \right\|_{\text{op}} \\
\leq \frac{\ell(c)}{2\pi} \sup_{0 \leq x \leq 1} \left( |c(x)|^{\frac{1}{2}} \left\| \theta(g)(c(x)I - q)^{-1} - (c(x)I - q)^{-1}\theta(g) \right\|_{\text{op}} \right) \\
\leq 10L_{r,R}^2 \sup_{0 \leq x \leq 1} \left( |c(x)|^{\frac{1}{2}} \left\| \theta(g)(c(x)I - q)^{-1} - (c(x)I - q)^{-1}\theta(g) \right\|_{\text{op}} \right) \\
\leq 20L_{r,R}^3 \sup_{0 \leq x \leq 1} \left( \left\| \theta(g)(c(x)I - q)^{-1} - (c(x)I - q)^{-1}\theta(g) \right\|_{\text{op}} \right) \\
\leq 80L_{r,R}^5 K_r \delta \\
\leq 320L_{r,R}^9 K_r \delta
\]

Here,

- \[3.50\] follows from \[3.50\] by Clause \(\text{(V)}\) in the definition of \(c\),
- \[3.52\] follows from \[3.52\] by Clause \(\text{(IV)}\) in the definition of \(c\),
- \[3.53\] follows from \[3.52\] by \[3.40\],
- and \[3.54\] follows from \[3.53\] by Clause \(\text{(III)}\) in the definition of \(c\).

Now, since \(\text{spec}(q) \subseteq [L_{r,R}^{-\frac{3}{2}}, L_{r,R}^{\frac{3}{2}}]\), the spectral mapping theorem implies that \(\text{spec}(q^{-\frac{1}{2}}) \subseteq [L_{r,R}^{-\frac{3}{2}}, L_{r,R}^{\frac{3}{2}}]\). Since \(q^{-\frac{1}{2}}\) is self-adjoint, this implies \(\|q^{-\frac{1}{2}}\|_{\text{op}} \leq L_{r,R}\). Therefore

\[
\|\zeta_\varphi(g) - \theta(g)\|_{\text{op}} = \|q^{\frac{1}{2}}\theta(g)q^{-\frac{1}{2}} - \theta(g)\|_{\text{op}} \\
= \|q^{\frac{1}{2}}\theta(g)q^{-\frac{1}{2}} - \theta(g)q^{\frac{1}{2}}q^{-\frac{1}{2}}\|_{\text{op}} \\
\leq \|q^{\frac{1}{2}}\theta(g) - \theta(g)q^{\frac{1}{2}}\|_{\text{op}} \|q^{-\frac{1}{2}}\|_{\text{op}} \\
\leq 320L_{r,R}^{10} K_r \delta
\]

Therefore Proposition \[3.6\] follows from \[3.41\].

### 3.4 Finding a witness vector

Define a vector \(y \in \mathcal{Y}\) by setting

\[
y = \frac{1}{d!} \bigoplus_{c \in \text{Sym}(d)} \sum_{j=1}^{d} \alpha_j \Phi_{\varphi}(c) j
\]
Since each $D_\varsigma$ is normalized we have from (5.4) that $y$ is a unit vector. Let $g \in B_{r,\varsigma}$ and let $g' \in B_{r,\varsigma'}$. From Proposition 3.6 we have

$$\langle \zeta(g,g'),y,y \rangle = \langle \zeta_\varsigma(g_\varsigma)(g'),y,y \rangle \approx_\epsilon \langle \theta(g)\zeta_\varsigma(g'),y,y \rangle$$ (3.55)

We have

$$\langle \theta(g)\zeta_\varsigma(g'),y,y \rangle = \frac{e}{d!} \left( \langle \theta(g)\zeta_\varsigma(g'), \bigoplus_{\varsigma \in \text{Sym}(d)} \sum_{j=1}^d \alpha_{\varsigma j} \Phi_{D_\varsigma}(e)_j, \bigoplus_{\varsigma \in \text{Sym}(d)} \sum_{k=1}^d \alpha_{\varsigma k} \Phi_{D_\varsigma}(e)_k \rangle \right) + \frac{1 - e}{d!} \left( \langle \theta(g)\zeta_\varsigma(g'), \bigoplus_{\varsigma \in \text{Sym}(d)} \sum_{j=1}^d \alpha_{\varsigma j} \Phi_{D_\varsigma}(e)_j, \bigoplus_{\varsigma \in \text{Sym}(d)} \sum_{k=1}^d \alpha_{\varsigma k} \Phi_{D_\varsigma}(e)_k \rangle \right)$$ (3.56)

We have

$$\frac{e}{d!} \left| \langle \theta(g)\zeta_\varsigma(g'), \bigoplus_{\varsigma \in \text{Sym}(d)} \sum_{j=1}^d \alpha_{\varsigma j} \Phi_{D_\varsigma}(e)_j, \bigoplus_{\varsigma \in \text{Sym}(d)} \sum_{k=1}^d \alpha_{\varsigma k} \Phi_{D_\varsigma}(e)_k \rangle \right| \leq e$$ (3.57)

From (3.55), (3.59) and (3.57) we have

$$\langle \zeta(g,g'),y,y \rangle \approx_\epsilon \frac{1}{d!} \left( \langle \theta(g)\zeta_\varsigma(g'), \bigoplus_{\varsigma \in \text{Sym}(d)} \sum_{j=1}^d \alpha_{\varsigma j} \Phi_{D_\varsigma}(g'_j), \bigoplus_{\varsigma \in \text{Sym}(d)} \sum_{k=1}^d \alpha_{\varsigma k} \Phi_{D_\varsigma}(e)_k \rangle \right)$$ (3.58)

By construction we have

$$\zeta_\varsigma(g')(e)_j = \Phi_{D_\varsigma}(g'_j)$$ (3.59)

We have

$$\frac{1}{d!} \left( \langle \theta(g) \bigoplus_{\varsigma \in \text{Sym}(d)} \sum_{j=1}^d \alpha_{\varsigma j} \Phi_{D_\varsigma}(g'_j), \bigoplus_{\varsigma \in \text{Sym}(d)} \sum_{k=1}^d \alpha_{\varsigma k} \Phi_{D_\varsigma}(e)_k \rangle \right) = \frac{1}{d!} \sum_{\varsigma \in \text{Sym}(d)} \left( \sum_{j=1}^d \alpha_{\tau(g)-1j} \Phi_{D_\varsigma}(g'_j), \sum_{k=1}^d \alpha_{\varsigma k} \Phi_{D_\varsigma}(e)_k \right)$$ (3.60)

From (3.58), (3.59) and (3.60) we obtain

$$\langle \zeta(g,g'),y,y \rangle \approx_\epsilon \frac{1}{d!} \sum_{\varsigma \in \text{Sym}(d)} \sum_{j,k=1}^d \alpha_{\tau(g)-1j} \alpha_{\varsigma k} \langle \Phi_{D_\varsigma}(g'_j), \Phi_{D_\varsigma}(e)_k \rangle$$ (3.61)

Since $g' \in B_r$ from (??) we have

$$\frac{1}{d!} \sum_{\varsigma \in \text{Sym}(d)} \sum_{j,k=1}^d \alpha_{\varsigma g}^{-1j} \alpha_{\varsigma k} \Phi_{D_\varsigma}(g'_j), \Phi_{D_\varsigma}(e)_k \rangle \approx_\epsilon \frac{1}{d!} \sum_{\varsigma \in \text{Sym}(d)} \sum_{j,k=1}^d \alpha_{\varsigma g}^{-1j} \alpha_{\varsigma k} \Phi_{D_\varsigma}(g'_j), \Phi_{D_\varsigma}(e)_k \rangle$$ (3.62)

From (3.61) and (3.62) we have
\[
\langle \zeta(g, g') y, y \rangle \approx 2e \frac{1}{d!} \sum_{c \in \text{Sym}(d)} \sum_{j, k=1}^{d} \alpha_{\sigma(g)^{-1}c_jx_k} D_c(g')_{j,k} \tag{3.63}
\]

From the construction of \(D\) we have
\[
\frac{1}{d!} \sum_{c \in \text{Sym}(d) \setminus B} \sum_{j, k=1}^{d} \alpha_{\sigma(g)^{-1}c_j} c_k C_c(g')_{j,k} \approx e \frac{1}{d!} \sum_{c \in \text{Sym}(d) \setminus B} \sum_{j, k=1}^{d} \alpha_{\sigma(g)^{-1}c_j} c_k C_c(g')_{j,k} \tag{3.64}
\]

From (3.63) and (3.64) we have
\[
\langle \zeta(g, g') y, y \rangle \approx 3e \frac{1}{d!} \sum_{c \in \text{Sym}(d)} \sum_{j, k=1}^{d} \alpha_{\sigma(g)^{-1}c_j} c_k D_c(g')_{j,k} \tag{3.65}
\]

We have
\[
\frac{1}{d!} \sum_{c \in \text{Sym}(d) \setminus B} \sum_{j, k=1}^{d} \alpha_{\sigma(g)^{-1}c_j} c_k C_c(g')_{j,k} = \frac{1}{d!} \sum_{c \in \text{Sym}(d) \setminus B} \sum_{j, k=1}^{d} \alpha_{\sigma(g)^{-1}c_j} c_k \langle \rho_c(g')_{x_j}, x_k \rangle \tag{3.66}
\]
\[
= \frac{1}{d!} \sum_{c \in \text{Sym}(d)} \left( \rho_c(g') \sum_{j=1}^{d} \alpha_{\sigma(g)^{-1}c_j} x_j \sum_{k=1}^{d} \alpha_k x_k \right) \tag{3.67}
\]
where the equality in (3.66) holds by (3.19). From (3.65) and (3.67) we have
\[
\langle \zeta(g, g') y, y \rangle \approx 3e \frac{1}{d!} \sum_{c \in \text{Sym}(d)} \left( \rho_c(g') \sum_{j=1}^{d} \alpha_{\sigma(g)^{-1}c_j} x_j \sum_{k=1}^{d} \alpha_k x_k \right) \tag{3.68}
\]

By making the changes of variables \(j \mapsto \varsigma^{-1}\sigma(g)j\) in the left sum and \(k \mapsto \varsigma^{-1}k\) in the right sum of (3.68) we obtain
\[
\text{(3.68)} = \frac{1}{d!} \sum_{c \in \text{Sym}(d)} \left( \rho_c(g') \sum_{j=1}^{d} \alpha_{j} x_{\sigma(g)j} \sum_{k=1}^{d} \alpha_k x_k \right) \tag{3.69}
\]
or equivalently
\[
\text{(3.68)} = \left( \rho_c(g') \sum_{j=1}^{d} \alpha_{j} x_{\sigma(g)j} \right) \sum_{k=1}^{d} \alpha_k x_k \tag{3.70}
\]

From (3.70) we have
\[
\left\langle \rho_b(g') \sum_{j=1}^{d} \alpha_j x_{\sigma(g)j}, \sum_{k=1}^{d} \alpha_k x_k \right\rangle = \left\langle \rho_b(g') \kappa(g) \sum_{j=1}^{d} \alpha_j x_j, \sum_{k=1}^{d} \alpha_k x_k \right\rangle \quad (3.71)
\]

From (3.6) we have

\[
\left\langle \rho_b(g') \kappa(g) \sum_{j=1}^{d} \alpha_j x_j, \sum_{k=1}^{d} \alpha_k x_k \right\rangle \approx \epsilon \left\langle \rho_b(g') \rho_c(g) \sum_{j=1}^{d} \alpha_j x_j, \sum_{k=1}^{d} \alpha_k x_k \right\rangle \quad (3.72)
\]

From (3.70), (3.71) and (3.72) we have

\[
\left\langle \zeta(g, g') y, y \right\rangle \approx 4\epsilon \left\langle \rho_b(g') \rho_c(g) \sum_{j=1}^{d} \alpha_j x_j, \sum_{k=1}^{d} \alpha_k x_k \right\rangle \quad (3.73)
\]

From (3.6) and (3.73) we obtain

\[
\left\langle \zeta(g, g') y, y \right\rangle \approx 5\epsilon \left\langle \rho(g, g') x, x \right\rangle
\]

This completes the proof of Conjecture 1.2. In combination with the arguments of Subsection 1.2.3 this completes the proof of Conjecture 1.1.

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