ANALYTICITY FOR SOLUTION OF INTEGRO-DIFFERENTIAL OPERATORS

SIMON BLATT

ABSTRACT. We prove that for a certain class of kernels $K(y)$ that viscosity solutions of the integro-differential equation

$$\int_{\mathbb{R}^n} \left( u(x+y) - 2u(x) + u(x-y) \right) K(y) \, dy = f(x, u(x))$$

are locally analytic if $f$ is an analytic function. This extends results in [1] in which it was shown that such solutions belong to certain Gevrey classes.

CONTENTS

1. Introduction
2. Preliminaries
  2.1. Characterization of Analytic Functions
  2.2. A-Priori Estimates for Non-Local Integro-Differential Operators
  2.3. An Estimate for the Binomial
3. The Essential A-Priori Estimate
4. Proof of the Theorem for $Ku(x) = f(x)$
  4.1. A Recursive Estimate
  4.2. The Conclusion using Cauchy’s Method of Majorants
5. Proof of the Theorem for $Ku(x) = f(x, u(x))$
  5.1. Higher Order Chain Rule
  5.2. Conclusion of the Proof
6. References

1. Introduction

Non-local equations play an important role in so different fields as the modeling of american option prices, geometric repulsive potential, the propagation of flames, and particel physics, where the Boltzman equation and the Kac equation are prominent examples of fractional partial differential equations.

Though in recent years the research on non-local partial differential equations exploded, still quite a lot of very basic questions regarding this type of equations remain open that have long been settled in the classical setting. In this article we address one of these questions: Is the solution to an elliptic fractional partial differential equation with analytic right-hand side analytic?

For classical non-linear partial differential equations this is David Hilbert’s 19th problem. Already shortly after, Bernstein could give an answer in [4] for elliptic equations in two independent variables under the assumption that the solution is already $C^3$ and by

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Petrowsky to systems [18]. Different methods of proof and generalization can be found in [11, 14, 12, 10, 15, 17, 16]

In recent years some results on analyticity for special fractional equations on the whole space $\mathbb{R}^n$ or compact manifolds like $\mathbb{S}^n$ appeared [8, 2, 5]. To the best of the authors knowledge, the findings in [1] are the only attempt to consider analyticity of local solutions to general fractional partial differential equations. They prove that the solution belong to certain Gevrey classes but did not succeed in proving that the solutions are indeed analytic.

Let us formulate the main result of this article. We consider translation invariant kernels $K \in C^\infty(\mathbb{R}^n \setminus \{0\}, (0, \infty))$ close to a kernel of fractional Laplacian type in the sense that

\begin{equation}
\frac{|y|^{s+1}K(y)}{2-s} - a_0 \leq \eta
\end{equation}

for all $y \in \mathbb{R}^n \setminus \{0\}$. Here, $\eta > 0$ is going to be a small constant that will be determined later on.

For such kernels and functions $u \in L^\infty(\mathbb{R}^n, \mathbb{R})$ we define the operator

$$Ku(x) = p.v. \int_{\mathbb{R}^n} (u(x+y) - 2u(x) + u(x-y))K(y)dy.$$  

We will furthermore assume that the kernel satisfies the estimate

\begin{equation}
|\partial_y^\alpha K(y)| \leq C \frac{H^{\alpha||\eta||}}{|y|^{n+s+|\alpha|}} \quad \text{on } B_1(0)
\end{equation}

for all multiindices $\alpha \in \mathbb{N}_0^n$. We will assume without loss of generality that $H \geq 1$. In this short note we will prove the following result.

**Theorem 1.1.** For $s \in (1, 2)$ let us assume that $u \in L^\infty(\mathbb{R}^n, \mathbb{R}) \cap C^\infty(B_1(0))$ is a viscosity solution of the equation

$$Ku(x) = f(x, u(x))$$

for an analytic function $f : B_1(0) \times \mathbb{R} \to \mathbb{R}$. Then $u$ is analytic on $B_1(0)$.

Note that in view of the bootstrapping argument in [3] the assumption $u \in C^\infty(B_1(0))$ is not essential. In contrast to [1] we only consider translation invariant equations here. But this is not the reason why the result stated here is stronger: Unfortunately some of the additional terms coming from $x$-dependence of the kernel seem to be missing in [1, inequality (3.2)] and hence their proof seems to at least have a gap. Though we believe that also these additional terms can be controlled we leave this case for a later paper as this will be technically more involved.

As in [1], we proof Theorem 1.1 combining the classical approaches by Friedman and Morrey with the a-priori estimates for solution in [6]. In contrast to [1] we omit the use of incremental differences and discrete partial integration completely and directly work with partial derivatives and partial integration. The essential new ingredient in our proof is to estimate the terms coming from the long-range interactions of the equation in much more sophisticated way using nested balls.

In Section 2 we gather some known facts and tools for the proof of Theorem 1.1, i.e. a characterization of analyticity, the Schauder estimates of Caffarelli and Silvestre in [6] and an elementary estimate for the binomial. The essential estimate for higher derivative is then derived in Section 3 before we turn to the proof of Theorem 1.1 in Sections 4 and 5. In 4 we give the proof first for the special case that the right-hand side of our equation does not depend on $x$ and not on $u$. We do this for two reasons: To make the presentation as readable as possible and since this special case contains the major new difficulties. We will then see in Section 5 that one can deal with the $u$-dependence by applying a higher order chain rule in a fairly standard way.
2. Preliminaries

2.1. Characterization of Analytic Functions. The following fact is well known.

Theorem 2.1. A function $u : \Omega \to \mathbb{R}$ is analytic on $\Omega$, $\Omega \subset \mathbb{R}^n$ open, if and only if for every compact set $K \subset \Omega$ there are constants $C = C_K, A = A_K < \infty$ such that
\[
\|\nabla^k u\|_{L^\infty(B_1(x))} \leq CA^k k!
\]
for all $k \in \mathbb{N}_0$.

A proof of this theorem can be found in [13].

2.2. A-Priori Estimates for Non-Local Integro-Differential Operators. Caffarelli and Silvestre proved the following remarkable theorem.

Theorem 2.2 ([6, Theorem 61]). Let $s \in (1, 2)$ and $u \in L^\infty(\mathbb{R}^n)$ be a viscosity solution of
\[
Ku(x) = f(x) \text{ on } B_1(0)
\]
for an $f \in L^\infty(B_1(0))$ and let $\eta > 0$ in (1.1) be small enough. Then for all $0 < \alpha < 1 - s$ we have $u \in C^{1,\alpha}(B_{\frac{1}{2}}(0))$ and
\[
\|u\|_{C^{1,\alpha}(B_{\frac{1}{2}}(0))} \leq C (\|f\|_L^\infty(B_{\frac{1}{2}}(0)) + \|u\|_{L^\infty(\mathbb{R}^n)}).
\]

Scaling this result, we immediately get the following.

Theorem 2.3. Let $s \in (1, 2)$ and $u \in L^\infty(\mathbb{R}^n)$ solve
\[
Ku(x) = f(x) \text{ in } B_r(0)
\]
for an $f \in L^\infty(B_r(0))$ and let $\eta > 0$ in (1.1) be small enough. Then for all $0 < \alpha < 1 - s$ we have $u \in C^{1,\alpha}(B_{\frac{1}{2}}(0))$ and
\[
r\|\nabla u\|_{L^\infty(B_{\frac{1}{2}}(0))} + r^{1+\alpha} h_{\text{hol}}(\nabla u) \leq C (r\|f\|_{L^\infty(B_{\frac{1}{2}}(0))} + \|u\|_{L^\infty(\mathbb{R}^n)}).
\]

2.3. An Estimate for the Binomial. We will need the following estimate for the binomial.

Lemma 2.4. We have
\[
\frac{k^k}{(k-l)^{k-l} l^l} \leq (2e)^l \binom{k}{l}
\]
for all $k \in \mathbb{N}, k > l > 0$.

Proof. For $0 < l \leq \frac{k}{2}$ we have
\[
\binom{k}{l} \geq 2^{-l} \frac{k^l}{l!}
\]
and
\[
\frac{k^k}{(k-l)^{k-l} l^l} = \left(\frac{k}{k-l}\right)^{k-l} \frac{k^l}{l^l} = \left(1 + \frac{l}{k-l}\right)^{-l} \frac{k^l}{l^l} \leq e^l \frac{k^l}{l^l}.
\]
Hence,
\[
\frac{k^k}{(k-l)^{k-l} l^l} \leq (2e)^l \binom{k}{l}
\]
if $l \leq \frac{k}{2}$. For $l > \frac{k}{2}$ we get applying the above to $k - l$ instead of $k$
\[
\frac{k^k}{(k-l)^{k-l} l^l} \leq (2e)^{k-l} \binom{k}{k-l} \leq (2e)^{\frac{k}{2}} \binom{k}{\frac{k}{2}}.
\]
\[\square\]
3. The Essential A-Priori Estimate

We use the estimates of Caffarelli and Silvestre to derive the following recursive estimate for derivatives of higher order. To shorten notation we use the shortcuts $B_R = B_R(0)$ and $||u||_A = ||u||_{L^\infty(A)}$ for a subset $A \subset \mathbb{R}^n$. Furthermore, we will use

$$||\nabla^k u||_A := \sup_{|\alpha|=k} ||\partial^\alpha u||_A.$$ 

**Theorem 3.1.** Let $u \in L^\infty(\mathbb{R}^n) \cap C^\infty(\Omega)$ and $f : \Omega \to \mathbb{R}$ be smooth such that $K u = f$ on $\Omega$.

If $x_0 \in \Omega$, $\sigma > 0$, and $k \in \mathbb{N}$ are chosen such that $B_{\sigma(4k+1)}(x_0) \subset \Omega$, then

$$\sigma ||\nabla^{k+1} u||_{B_2(x_0)} \leq C \left( \sigma^2 ||\nabla^k f||_{B_2(x_0)} + ||\nabla^k u||_{B_2(x_0)} \right) + \sigma \sum_{j=1}^{k-1} 6^{-j} \left( \sigma \left( ||H^{j+1}||_{B_2(x_0)} + ||H^{j}||_{B_2(x_0)} \right) \right)$$

Proof. After a suitable translation we can assume that $x_0 = 0$. We first show the statement of the theorem under the addition assumption that $u$ is $C^\infty$ on the complete space $\mathbb{R}^n$ and has compact support. For that we chose $\bar{\eta} \in C^\infty(\mathbb{R}^n, [0, 1])$ such that

$$\bar{\eta} \equiv 1 \text{ on } B_3 \quad \text{and} \quad \bar{\eta} \equiv 0 \text{ on } \mathbb{R}^n \setminus B_4$$

and set

$$\eta(x) = \bar{\eta}(\frac{x}{\sigma}).$$

For $k \in \mathbb{N}$ and $i_1, i_k \in \{0, \ldots, n\}$ we decompose

$$w = \partial_{i_1, \ldots, i_k} u = \partial_{i_1}(\eta \partial_{i_1, \ldots, i_k} u) + \partial_{i_1}(1 - \eta) \partial_{i_1, \ldots, i_k} u = w_1 + w_2.$$  

Applying Theorem 2.3 we get

$$\sigma ||\nabla \partial_{i_1, \ldots, i_k} u||_{B_2(0)} \leq C (\sigma^2 ||Kw_1||_{B_2} + ||w_1||_{B_2})$$

We first note that

$$||w_1||_{B_2} = ||\partial_{i_1}(\eta \partial_{i_1, \ldots, i_k} u)||_{B_2} \leq ||\nabla^k u||_{B_2} + ||\nabla \eta||_{B_2} ||\nabla^{k-1} u||_{B_2}$$

$$\leq ||\nabla^k u||_{B_2} + C \sigma ||\nabla^{k-1} u||_{B_2}.$$

To estimate the first term in (3.1), we use $w_1 = w - w_2$ to get

$$||K w_1||_{B_2} \leq ||K u||_{B_2} + ||K w_2||_{B_2} \leq ||\nabla^k f||_{B_0} + ||K w_2||_{B_0}.$$
and observe that for \( x \in B_{2r} \) we have

\[
|Kw_2(x)| = \left| \int_{\mathbb{R}^n} (w_2(x + y) - 2w_2(x) + w(x - y))K(y)dy \right|
\]

\[
= \left| \int_{\mathbb{R}^n} (w_2(x + y) + w(x - y))K(y)dy \right|
\]

\[
\leq 2 \left| \int_{\mathbb{R}^n} w_2(x + y)K(y)dy \right|
\]

(3.4)

\[
= 2 \left| \int_{\mathbb{R}^n} (1 - \eta(x + y))\partial_{i_1, \ldots, i_l} u(x + y)\partial_{i_k} K(y)dy \right|
\]

\[
\leq \left| \int_{B_{2r}} (1 - \eta(x + y))\partial_{i_1, \ldots, i_l} u(x + y)\partial_{i_k} K(y)dy \right|
\]

\[
+ \left| \int_{\mathbb{R}^n \setminus B_{2r}} (1 - \eta(x + y))\partial_{i_1, \ldots, i_l} u(x + y)\partial_{i_k} K(y)dy \right|
\]

\[
= I_1 + J_1.
\]

To estimate \( I_1 \), we note that due to the properties of \( \eta \) and the triangle inequality \( 1 - \eta(x + y) = 0 \) if \( |y| \leq \sigma \) and hence we get from the properties of \( K \) that

(3.5) \[
I_1 \leq CH\|\nabla^{k-1}u\|_{L^2(B_{2r})} \int_{\mathbb{R}^n \setminus B_{2r}} \frac{1}{|y|^{n+1+2\tau}} dy = C \frac{H}{\sigma^{n+1+2\tau}} \|\nabla^{k-1}u\|_{L^2(B_{2r})}.
\]

For \( J_1 \) we use partial integration to get

\[
J_1 \leq \left| \int_{\mathbb{R}^n \setminus B_{2r}(0)} \partial_{i_1, \ldots, i_l} u(x + y)\partial_{i_1, \ldots, i_l} K(y)dy \right| + \left| \int_{B_{2r}(0)} |\partial_{i_1, \ldots, i_l} u(x + y)||\partial_{i_k} K(y)| dS(y) \right|
\]

\[
\leq CH^2\|\nabla^{k-2}u\|_{L^2(\partial B_{2r})} \int_{\mathbb{R}^n \setminus B_{2r}} \frac{1}{|y|^{n+2+2\tau}} + CH\|\nabla^{k-2}u\|_{L^2(\partial B_{2r})} \int_{\partial B_{2r}} \frac{1}{|y|^{n+1+2\tau}} dS(y)
\]

\[
+ \left| \int_{\mathbb{R}^n \setminus B_{2r}(0)} \partial_{i_1, \ldots, i_l} u(x + y)\partial_{i_1, \ldots, i_l} K(y)dy \right|
\]

\[
\leq \frac{CH^2\|\nabla^{k-2}u\|_{L^2(\partial B_{2r})}}{(6\sigma)^{2+2\tau}} + J_2.
\]

where

\[
J_2 = \left| \int_{\mathbb{R}^n \setminus B_{2r}(0)} \partial_{i_1, \ldots, i_l} u(x + y)\partial_{i_1, \ldots, i_l} K(y)dy \right|
\]

Setting

\[
J_1 = \left| \int_{\mathbb{R}^n \setminus B_{2r}(0)} \partial_{i_1, \ldots, i_l} u(x + y)\partial_{i_1, \ldots, i_l} K(y)dy \right|
\]
we obtain as above using integration by parts and (1.2)

\[
\begin{align*}
J_i & \leq \left| \int_{\mathbb{R}^n \setminus B_{t_i}(0)} \partial_{i_1,\ldots,i_j} u(x + y) \partial_{i_1,\ldots,i_j} K(y) dy \right| \\
& \quad + \left| \int_{B_{t_i}(0)} |\partial_{i_1,\ldots,i_j} u(x + y)||\partial_{i_1,\ldots,i_j} K(y)| dS(y) \right| \\
& \leq C H^{i+1} (1 + 1)! \|\nabla^{k-1} u\|_{B_{1(t_i+1/2)}} \int_{B_{1(t_i+1/2)}} \frac{1}{|y|^{i+1/2}} \\
& \quad + C H^{i+1} \|\nabla^{k-1} u\|_{L^2(B_{1(t_i+1/2)})} \int_{B_{1(t_i+1/2)}} \frac{1}{|y|^{i+1/2}} dS(y) + J_{i+1} \\
& \leq C H^{i+1} (1 + 1)! \|\nabla^{k-i+1} u\|_{B_{1(t_i+1/2)}} \frac{1}{(6r)^{i+1/2}} + J_{i+1}.
\end{align*}
\]

Iterating this estimate yields

\[
J_i \leq C \sum_{j=2}^{k-1} \frac{H^j ! \|\nabla^{k-j} u\|_{L^2(B_{1/2})}}{(6r)^{j+1/2}} + J_k
\]

(3.6)

Together the estimates (3.1) – (3.6) prove the statement of the theorem for all \( u \in C^\infty(\mathbb{R}^n, \mathbb{R}) \) with compact support.

To get the statement for \( u \in L^\infty(\mathbb{R}^n, \mathbb{R}) \cap C^\infty(\Omega, \mathbb{R}) \), we let \( u_m \) be such that

\[
u_m = u \text{ on } B_r,
\]

and

\[
\|u_m\|_{L^\infty} \leq \|u\|_{L^\infty}.
\]

We can then apply what we have proven so far to the function \( u_m \) instead of \( u \) to get

\[
\sigma^|\nabla^{k-i+1} u_m\|_{B_{1/2})} \leq C \left( \sigma^|\nabla^k f_m\|_{B_{1/2}(x_0)} + \|\nabla^k u_m\|_{B_{1/2}(x_0)} \right)
\]

\[
+ \sigma^j \sum_{j=1}^{k-1} \frac{H^j ! \|\nabla^{k-j} u_m\|_{B_{1/2}(x_0)}}{(6r)^{j+1/2}} + \sigma^k \frac{H^j ! \|u_m\|_{L^\infty}}{(6kr)^{k+1/2})}
\]

where \( f_m = Ku_m \). It is obvious that due to the properties of the approximations \( u_m \) we can go to the limit in the inequality and thus obtain the inequality for \( u \) once we have shown that

\[
\|\nabla^k f_m\|_{B_{1/2}(x_0)} \to \|\nabla^k f\|_{B_{1/2}(x_0)}
\]

for \( m \to \infty \). For \( x \in B_{2r} \) and \( \alpha \in \mathbb{N}^n \) with \(|\alpha| = k \) we calculate

\[
\partial^\alpha (Ku_m)(x) = \partial^\alpha K u + \partial^\alpha K v_m
\]

where \( v_m = u_m - u \) and using that \( v_m = 0 \) on \( B_m \)

\[
\partial^\alpha K v_m(x) = \partial^\alpha \int_{\mathbb{R}^n \setminus B_m} v_m(y) (K(y + x) + K(y - x)) dy
\]

\[
= \int_{\mathbb{R}^n \setminus B_m} v_m(y) (\partial^\alpha K(y + x) + \partial^\alpha K(y - x)) dy
\]
Hence, 
\[ \|\nabla^{k}K(v_{m})\|_{B_{2r}} \leq CH^{k}\|\nabla^{k}v_{m}\|_{B_{2r}} \leq CH^{k}\|u\|_{C} \overset{m \to \infty}{\to} 0 \]
and thus 
\[ \|\nabla^{k}Ku_{m}\|_{B_{2r}} \to \|\nabla^{k}Ku\|_{B_{2r}} = \|\nabla^{k}f\|_{B_{2r}}. \]

4. Proof of the Theorem for \( Ku(x) = f(x) \)

Let us first illustrate this method for the special case that \( Ku(x) = f(x) \), i.e. that the righthand side of our equation does not depend on \( u \).

4.1. A Recursive Estimate. Following [1] we define the quantities
\[
N_{k} = \sup_{0 < r < 1} \left( 1 - r^{k+1} \right) \|\nabla^{k}f\|_{L^{\infty}(B_{r})} \quad \text{for} \quad k \geq 0,
\]
\[
M_{k} = \sup_{0 < r < 1} \left( 1 - r^{k} \right) \|\nabla^{k}u\|_{L^{\infty}(B_{r})} \quad \text{for} \quad k \geq 1,
\]
\[
M_{0} = \|\nabla^{0}u\|_{L^{\infty}(B_{1})}. \]

We will deduce the following estimate for these quantities from Theorem 3.1.

**Theorem 4.1.** We have 
\[
M_{k+1} \leq C \left( N_{k} + k \sum_{j=0}^{k} \binom{k}{j} M_{k-j} (2e)^{j} H^{j} \right)
\]
for all \( k \in \mathbb{N}_{0} \) and a constant \( A \).

**Proof.** For \( x_{0} \in B_{1}(0) \) and \( k \in \mathbb{N} \) we apply Theorem 3.1 with \( \sigma = \frac{1-|x_{0}|}{6(k+2)} \) to get
\[
\|\nabla^{k+1}u\|_{B_{r}} \leq C^{\sigma^{-1}} \|\nabla^{k}f\|_{B_{2r}} + \sigma^{-1} \|\nabla^{k}u\|_{B_{r}} + \sigma^{-1} \sup_{r>0} \frac{H^{k} \|u\|_{C}}{(6k\sigma)^{k+s}}
\]
where we use \( B_{r} = B_{r}(x_{0}) \) to shorten notation. Hence,
\[
(1 - |x_{0}|)^{k+1} \|\nabla^{k+1}u(x_{0})\| \leq (1 - |x_{0}|)^{k+1} C^{\sigma^{-1}} \|\nabla^{k}f\|_{B_{2r}} + \sigma^{-1} \|\nabla^{k}u\|_{B_{r}} + \sigma^{-1} \sup_{r>0} \frac{H^{k} \|u\|_{C}}{(6k\sigma)^{k+s}}
\]

We estimate
\[
(1 - |x_{0}|)^{k+1} \|\nabla^{k+1}u\|_{B_{2r}(x_{0})} = (6(k+2)\sigma)^{k+1} \|\nabla^{k}f\|_{B_{2r}(x_{0})} \leq \frac{(6(k+2)\sigma)^{k+1} \|\nabla^{k}f\|_{B_{2r}(x_{0})}}{((6(k+2)\sigma)^{k+1} \sigma^{-1})^{k+s}} N_{k} = \left( \frac{k+2}{k+2 - \frac{1}{3}} \right)^{k+1} N_{k} \leq CN_{k}
\]
as \( s > 1 \) and
\[
0 < \left( 1 + \frac{\frac{1}{3}}{k+2 - \frac{1}{3}} \right)^{k+1} < \left( 1 + \frac{\frac{1}{3}}{k+2} \right)^{k+\frac{1}{3}} \to e^{\frac{1}{3}}.
\]

Similarly,
\[
(1 - |x_{0}|)^{k+1} \|\nabla^{k}u\|_{B_{r}(x_{0})} \leq \frac{(6(k+2)^{k+1})}{(6(k+2) - 4)^{k}} M_{k} \leq C(k+2)M_{k}.
\]
Furthermore, we get for $1 \leq l < k$
\[
(1 - |x_0|)^{k+1} \sigma^{|l-1|} H^l_1 \leq \frac{(6(k + 2))^{k+1}}{(6l + 2 - l - 2)^{k-1}(6l)^{|l-1|}}H^l_1 M_{k-l}
\]
\[
= 6 \frac{(k + 2)^{k+1}}{(k - l + \frac{6l}{k})^{k-1}|l+1|} H^l_1 M_{k-l}
\]
Note that
\[
\frac{(k + 2)^{k+1}}{(k - l + \frac{6l}{k})^{k-1}|l+1|} \leq C \frac{(k + 1)^{k+1}}{(k - l)^{k-1}(l + 1)^{l+1}} \leq C(2e)^{k+1}
\]
In the last step we used Lemma 2.4. Finally,
\[
\frac{(1 - |x_0|)^{k+1}}{\sigma(6k\epsilon)^k} = \frac{(6(k + 2))^{k+1}}{(6k)^k} = 6(k + 2)(1 + \frac{2}{k})^k \leq C6(k + 2).
\]
These estimates show that
\[
(1 - |x_0|)^{k+1} |\nabla^{k+1} u(x_0)| \leq C \left( N_k + k \sum_{i=0}^{k} \left( \begin{smallmatrix} k \\ i \end{smallmatrix} \right) M_{k-i}(2e)^{H} \right)
\]
Taking the supreme over all $x_0 \in B_1(0)$ proves the theorem. \qed

4.2. The Conclusion using Cauchy’s Method of Majorants. We will now conclude the proof of Theorem 1.1 using Cauchy’s method of Majorants.

As being analytic is a local statement, we can assume w.l.o.g that there are constants $C_f, A_f < \infty$ such that
\[
N_k \leq C_f A_f k!
\]
for all $k \in \mathbb{N}_0$. Setting $A := \sup \{A_k, 2eH \}$ Theorem 4.1 tells us that
\[
M_{k+1} \leq C(N_k + k \sum_{i=0}^{k} \left( \begin{smallmatrix} k \\ i \end{smallmatrix} \right) M_{k-i}(2eH)) \leq CA^k k! + Ck \sum_{i=0}^{k} \left( \begin{smallmatrix} k \\ i \end{smallmatrix} \right) M_{k-i} A^i k!
\]
for all $k \in \mathbb{N}_0$. We will show that this recursive estimate implies that $M_k \leq C_kA_k k!$ for suitably chosen constants $C_k, A_k$ by comparing it to the solution of an analytic ordinary differential equation.

For this we put
\[
G(t) := C \sum_{k \in \mathbb{N}_0} A_k k^t
\]
and consider the solution to the initial value problem
\[
\begin{align*}
c'(t) &= G(t) + tG(t)c(t) \\
c(0) &= M_0,
\end{align*}
\]
As near to $t$ we have $1 - tG(t) \neq 0$ we can rewrite this equation as
\[
c'(t) = \frac{2G(t) + tG'(t)}{1 - tG(t)}
\]
near 0. Hence, the above initial value problem has a unique analytic solution on some small time interval $(-\varepsilon, \varepsilon)$. The derivatives $\tilde{M}_k = c^{(k)}(0)$ satisfy
\[
\tilde{M}_k \leq C_k A_k k!
\]
for suitable constants $C_u, A_u$ and the recursive relation

$$
\tilde{M}_{k+1} = C(N_k + k \sum_{i=0}^{k} \binom{k}{i} \tilde{M}_{k-i}(e^H)^i!
$$

Comparing this with (4.1) we deduce by induction that

$$
M_k \leq \tilde{M}_k \leq C_u A_u^k k!.
$$

5. Proof of the Theorem for $Ku(x) = f(x, u(x))$

Let us now move to the case that $K(u) = f(x, u(x))$ in $B_1(0)$.

As in the last section we have

$$
M_{k+1} \leq C \left( N_k + k \sum_{i=0}^{k} \binom{k}{i} \tilde{M}_{k-i}(2e^H)^i! \right)
$$

for all $k \in \mathbb{N}_0$ and a constant $A$ where now

$$
N_k = ||\nabla^k(f(x, u(x)))||_{B_r}.
$$

We introduce the terms

$$
\tilde{M}_k = M_k + 1
$$

and

$$
\tilde{N}_k = ||\nabla^k f||_k
$$

where $K$ is the image of $x \mapsto (x, u(x))$. As being analytic is a local property, we can again assume without loss of generality that

$$
\tilde{N}_k \leq CA_f^k k!
$$

for a constant $A_f < \infty$. We still have

(5.1) $$
\tilde{M}_{k+1} \leq C \left( N_k + k \sum_{i=0}^{k} \binom{k}{i} \tilde{M}_{k-i}(2e^H)^i! \right)
$$

We need a higher order chain rule to estimate $N_k$ in terms of $\tilde{N}_k$ and tilde $M_k$.

5.1. Higher Order Chain Rule.

**Proposition 5.1.** Let $g : \mathbb{R}^{m_1} \to \mathbb{R}^{m_2}$ and $f : \mathbb{R}^{m_2} \to \mathbb{R}$ be two $C^k$-functions. Then for an multiindex $\alpha \in \mathbb{N}^{m_1}$ of length $|\alpha| \leq k$ and $x \in \Omega$ the derivative

$$
\partial^\alpha (f \circ g)(x) = P^\alpha_{m_1,m_2}((\partial^\gamma f(g(x)))_{\gamma \leq |\alpha|}, \{\partial^\gamma g_i\}_{0 \leq \gamma \leq |\alpha|})
$$

where $P^\alpha_{m_1,m_2}$ is a linear combination with positive coefficients of terms of the form

$$
\partial^\gamma_{x_{l_1}x_{l_2}}(g(x))\partial^\gamma_{i_1}g_{i_1} \cdots \partial^\gamma_{i_k}g_{i_k}
$$

with $1 \leq k \leq |\alpha|$ and $|\gamma_1| + \ldots + |\gamma_k| = |\alpha|$.  

For $m_1 = m_2 = 1$ we will use the notation $P^k$ instead of $P^\alpha_{m_1,m_2}$. We leave the easy inductive proof of this statement to the reader. Although very precise formulas of the higher order chain rule were given by Faà di Bruno [9] for the univariate case and by for example Constanini and Savits in [7] for the multivariate case, the above proposition contains all that is needed in our proof.

Let us derive an easy consequences of Proposition 5.1 that allows us in a sense to reduce the multivariate case to the univariate one.

**Lemma 5.2.** For constants $a_{\gamma} = a_{|\gamma|}, b_{|\gamma|} \in \mathbb{R}$ depending only on the length of the multiindex $\gamma$ we have

$$
P^\alpha_{m_1,m_2}((a_{|\gamma|}), (b_{|\gamma|})) = P^{\hat{\alpha}}((a_{|\gamma|}), (b_{|\gamma|})).$$
Proof. Plugging functions $g$ and $f$ of the form
\[ g(x_1, \ldots, x_m) = \tilde{g}(x_1 + \cdots + x_m) \cdot (1, \ldots, 1)' \]
and
\[ f(y_1, \ldots, y_m) = f \left( \frac{y_1 + \cdots + y_m}{m^2} \right) \]
into the higher order chain rule we get from
\[ f \circ g = (\tilde{f} \circ \tilde{g})(x_1 + \ldots + x_m) \]
that
\[ P_{m_1,m_2}^{\alpha_1,\gamma_1}((\partial^\alpha f(g(x)))_{|\gamma_1|\leq m_2}, (\partial^\gamma g(x)))_{|\gamma|\leq m_1} = P_{m_1,m_2}^{\alpha_1,\gamma_1}((\partial^\gamma \tilde{f}(\tilde{g}(x)))_{|\gamma_1|\leq m_2}, (\partial^\gamma \tilde{g})_{|\gamma|\leq m_1}) \]
So for constants $a_\gamma, b_\gamma \in \mathbb{R}$ depending only on the length of the multiindex $\gamma$ we have
\[ P_{m_1,m_2}^{\alpha_1,\gamma_1}([a_\gamma],[b_\gamma]) = P_{m_1,m_2}^{\alpha_1,\gamma_1}((a_\gamma),(b_\gamma)). \]
\[ \square \]

We will use this lemma to estimate $N_k$.

**Lemma 5.3.** We have
\[ N_k \leq CP^k((\tilde{N}_i), (\tilde{M}_i)_{i=0,\ldots,k}). \]

**Proof.** Applying Faa di Bruno’s formula to $f \circ g$ where
\[ g(x) = (x, u(x)) \]
we get
\[ \partial^\gamma(f(x, u)) = P_{n,n+1}^m((\partial^\gamma f), (\partial^\gamma g)) \]
where $P_{n,n+1}^m((\partial^\gamma f), (\partial^\gamma g))$ is a linear combination with positive coefficients of terms of the form
\[ \partial^m_{i_1,\ldots,i_n}f(g(x)) \partial^\alpha g_{i_1} \cdots \partial^\alpha g_{i_m} \]
with $1 \leq m \leq |\alpha|$ and $\gamma_1 + \ldots + \gamma_n = \alpha$. Note that due to the special structure of $g$ we have $\partial^\gamma g_i = 1$ for $i = 1, \ldots, n$ and $|\gamma| = 1$ and $\partial^\gamma g_i = 0$ for $i = 1, \ldots, n$ and $|\gamma| \geq 2$. Hence,
\[ (1 - r)^{|\alpha|+s} \partial^m_{i_1,\ldots,i_n} f(g(x)) \partial^\gamma g_{i_1} \cdots \partial^\gamma g_{i_m} \|_{\partial^r(0)} \leq \| \partial^m_{i_1,\ldots,i_n} f(g(x)) \|_{\partial^r(0)} \tilde{M}_{|\gamma_1|} \cdots \tilde{M}_{|\gamma_n|} \leq \tilde{N}_k \tilde{M}_{|\gamma_1|} \cdots \tilde{M}_{|\gamma_n|}. \]
We hence deduce using Lemma 5.2 that
\[ \| (1 - r)^{|\alpha|+s} \partial^r(f(x, u)) \|_{\partial^r(0)} \leq P^r((\tilde{N}_i), (\tilde{M}_i)_{k=0,\ldots,k}). \]
Applying this estimate for all multiindices $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = k$ proves the claim. \[ \square \]

5.2. **Conclusion of the Proof.** Combining (5.1) with Lemma 5.3 we get
\[ \tilde{M}_{k+1} \leq C(P^k((\tilde{N}_i), (\tilde{M}_i)) + k \sum_{i=0}^k \binom{k}{i} \tilde{M}_{k-i}(2eH)'') \]
\[ \leq C(P^k((A'''), (\tilde{M}_i)) + Ck \sum_{i=0}^k \binom{k}{i} \tilde{M}_{k-i}A''') \]
where again $A := \sup(A, 2eH)$. As above we conclude comparing this with the solution to the initial value problem
\[ \begin{cases} c'(t) = G(c(t)) + (uG(t)c(t))', \\ c(0) = M_0. \end{cases} \]
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(Simon Blatt) DEPARTMENT OF MATHEMATICS, PARIS LODRON UNIVERSITÄT SALZBURG, HELLBRUNNER STRASSE 34, 5020 SALZBURG, AUSTRIA

E-mail address, Simon Blatt: simon.blatt@sbg.ac.at