Vertex-transitive maps with Schläfli type \{3, 7\}

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Abstract

Among all equivelar vertex-transitive maps on a given closed surface \(S\), the automorphism groups of maps with Schläfli types \{3, 7\} and \{7, 3\} allow the highest possible order. We describe a procedure to transform all such maps into 1- or 2-orbit maps, whose symmetry type has been previously studied. In so doing we provide a procedure to determine all vertex-transitive maps with Schläfli type \{3, 7\} which are neither regular or chiral. We determine all such maps on surfaces with Euler characteristic \(-1 \geq \chi \geq -40\).

Keywords: vertex-transitive map, Schläfli type \{3, 7\}, map on a surface.

1 Introduccion

Highly symmetric maps on compact surfaces have attracted attention in recent years. Several efforts have been made to classify such maps in surfaces with given Euler characteristics.

Regular maps (maps with maximal symmetry by reflections) and chiral maps (maps with maximal symmetry by rotations, but no symmetry by reflections) have been the most studied. Regular and chiral maps with Euler characteristic \(-1 \geq \chi \geq -200\) have been classified by Conder in [3]. If we additionally require the so-called diamond condition, regular maps whose automorphism group contains at most 2000 elements were classified as rank 3 abstract polytopes by Hartley in [9], with the exception of those whose automorphism group contains 1024 or 1536 elements.

Edge-transitive maps (maps whose automorphism group acts transitively on their edges) were studied and classified into 14 symmetry types by several authors in [8] and [20]. An atlas of each of the types of edge-transitive maps up to certain Euler characteristic depending on the type was presented by Orbanić in [10].

Vertex-transitive maps have a wider variety of symmetry types than edge-transitive maps. This makes considerably more complicated to determine all such maps on a given
A combinatorial approach to attack the problem of a classification of vertex-transitive maps on surfaces with a given Euler characteristic is outlined in [18, Section 10]. Another approach to this classification, is given by Karabáš and Nedela in [13] and [14], where extra properties are required to the maps. This approach involves actions of groups on surfaces. They also present in [13] an atlas of the so-called Archimedean solids on an orientable surface with Euler characteristic $-2$.

We may restrict ourselves to determine all vertex-transitive maps satisfying the extra condition that all its faces have the same co-degree $p$. The automorphism groups of these maps on a surface with Euler characteristic $-m$ contain at most $4mpq/(pq - 2p - 2q)$ elements, where $q$ is the degree of the vertices. This bound is achieved if and only if the map is regular (see [2, Section 1]). An easy calculation shows that the maps that maximize this bound are those where $p = 3$ and $q = 7$, or $p = 7$ and $q = 3$. Furthermore, if we eliminate these cases, the bound is reduced by a factor of $12/21$.

The most common method to produce an atlas of maps with symmetries consists in an exhaustive search of possible automorphism groups with distinguished generators from which the map can be recovered. In this context, the reduction in the bound described above may play a significant role in classifying vertex-transitive maps with faces of the same size.

As a consequence of the notions of maps on surfaces and automorphisms, there is a variety of algebraic, geometric, topologic and combinatorial techniques to work with these concepts. In this paper we combine the technique mentioned in the paragraph above with operations on maps, which are consistent transformations on all fundamental regions of the map under a certain automorphism group. One advantage of using operations is that we naturally obtain local properties of the map and relations to other maps, that may be difficult to spot otherwise. Additionally, these operations can be applied to higher dimensional objects, when the flags are no longer triangles, but $n$-simplexes.

In Section 2 we recall some definitions and basic results about symmetric maps. We define operations on maps in Section 3. These operations are used in Section 4 to transform vertex-transitive triangulations with seven triangles around each vertex into maps whose symmetry type has been previously studied. This reduction process can be reversed to construct all such maps from maps with a simple symmetry type. We also show that vertex-transitive maps all whose faces are heptagons, and all whose vertices are trivalent, must necessarily be either regular or chiral. In Section 5 we construct three vertex-transitive maps to illustrate the procedure described in Section 4. Finally, in an appendix we list all vertex-transitive triangulations with seven triangles around each vertex in surfaces with Euler characteristic $-1 \geq \chi \geq -40$. 

2
2 Definitions

Throughout, a map is a 2-cell embedding of a finite multigraph (we allow multiple edges) $G$ on a closed surface $S$ without boundary. The connected components of $S$ after removing $G$ are called faces and are homeomorphic to discs. The reader is referred to [18, Section 3] for details. Equivalent definitions using different approaches are given for example in [3], [12], [17], [19] and [21].

A map $M$ on a surface $S$ naturally induces a triangulation on $S$ where the vertices of the triangles are the vertices, midpoints of edges and centres of faces of $M$, with all triangles containing one vertex of each type. These triangles are henceforth called flags. We say that two flags are 0-adjacent (resp. 1- and 2-adjacent) whenever they share a line segment between the midpoint of the edge and the centre of the face (resp. a line segment between the vertex and the centre of the face, and a line segment between the vertex and the midpoint of the edge).

We define the flag graph of a map $M$ on a surface $S$ as the graph whose vertices are the flags of $M$ and two vertices are joined by an edge labelled $i$ whenever the corresponding flags are $i$-adjacent ($i = 0, 1, 2$). Note that the flag graph of $M$ allows a 2-cell embedding on $S$. It is easy to see that the flag graph of a map is always a connected graph without loops where the edges of each color form a matching.

We say that a map is equivalent whenever the co-degree of all faces (number of edges around the faces) is a fixed number $p$ and the degree of all vertices is a fixed number $q$.

An automorphism of a map $M$ is an automorphism of its graph that can be extended to an homeomorphism of the surface. We shall denote the group consisting of all automorphisms of $M$ by $\Gamma(M)$.

We now introduce the type graph of a map, which is a variation of the Delaney-Dress graph described in [6] and [7], with semi-edges instead of loops.

The type graph $\mathcal{T}(M)$ of a map $M$ is the semi-graph (we may allow semi-edges) with edges and semi-edges labelled in $\{0, 1, 2\}$ defined as follows. The vertices of $\mathcal{T}(M)$ are the flag-orbits of $M$ under $\Gamma(M)$, with two of them adjacent by an edge labelled $i$ whenever the flags in one of the orbits are $i$-adjacent to the flags in the other. Whenever a flag in an orbit $O$ is $i$-adjacent to another flag in $O$ we attach a semi-edge labelled $i$ to the vertex corresponding to $O$. Note that the flag graph and the type graph of a map are 3-valent with one edge (or semi-edge) of each color incident to each vertex. In this sense, the type graph of $M$ is the quotient of the flag graph by $\Gamma(M)$. We shall refer to all maps with type graph $\mathcal{T}$ as maps of type $\mathcal{T}$.

As opposite to the flag graphs, we shall consider type graphs with multiple edges.

Whenever $\mathcal{T}$ is a bipartite type graph (not containing semi-edges or odd cycles), every map $M$ of type $\mathcal{T}$ must lie on an orientable surface. In fact, if $(f_1, \ldots, f_s = f_1)$ is a sequence of flags such that $f_i$ is adjacent to $f_{i+1}$ for $i = 1, \ldots, s - 1$ then $s$ is odd, and therefore, the sequence involves an even number of changes of flags. If $\mathcal{T}$ is not bipartite then $M$ may lie either on an orientable or on a non-orientable surface.
In what follows, given the type graph $G$ of a map we denote by $G_i$ the subgraph of $G$ obtained from $G$ by deleting all edges labelled $i$ ($i = 0, 1, 2$).

We note that if $G$ is the type graph of a map, then each connected component of $G_1$ is isomorphic to one of the ones showed in Figure 1.

The number of orbits of vertices (resp. edges, faces) of a map with type graph $G$ coincides with the number of connected components of $G_0$ (resp. $G_1$, $G_2$). This implies that edge-transitive maps (maps where the automorphism group acts transitive on its edges) can have at most 4-orbits on flags, however there is not such a bound for the number of orbits of flags for vertex-transitive maps (maps where the automorphism group acts transitive on its vertices).

A map $M$ whose automorphism group induces $k$ flag orbits is called a $k$-orbit map. In this case $|\Gamma(M)| = |\mathcal{F}(M)|/k$, where $\mathcal{F}$ denotes the set of flags of $M$.

We shall follow the notation of [17] and denote 1-orbit maps as regular maps. A chiral map is a 2-orbit map such that adjacent flags belong to distinct orbits. Note that in [2] regular maps are referred to as reflexible.

The seven types of 2-orbit abstract polyhedra are described in [11] and we shall largely follow its notation. In fact, every finite abstract polyhedron can be seen as a map on a surface, and all maps on surfaces satisfying the so-called diamond condition are also abstract polyhedra. The theory developed in [11] is valid also for maps which are not polyhedra if we ignore the diamond and intersection conditions.

Among the seven types of 2-orbit maps, of particular interest for us are the types denoted by $2_0$ and $2_{01}$ (see Figure 2 for their type graphs). These types are respectively called $2^*ex$ and $2^*$ in [8] and [20], where they consider only maps on orientable surfaces. Every flag of a map of type $2_0$ is 0-adjacent to a flag in the same orbit, and is 1- and 2-adjacent to flags in the other orbit. Similarly, every flag of a map of type $2_{01}$ is 0- and 1-adjacent to flags in the same orbit, and is 2-adjacent to a flag in the other orbit.

The automorphism group of any map of type $2_0$ is generated by an involution $\rho_0$ and a non-involution $\sigma_{12}$, respectively denoted by $\lambda_e$ and $\sigma_f$ in [20]. The automorphism group of any map of type $2_{01}$ is generated by three involutions $\rho_0$, $\rho_1$ and $\rho_{212}$ (the latter denoted by $\alpha_{212}$ in [11]), respectively denoted by $\lambda_e$, $\theta_{uf}$ and $\theta_{uf}\sigma^{-2}_u$ in [20].

Given a type $\mathcal{T}$ of $k$-orbit maps with corresponding type graph $G$ we say that a map $M$ is $\mathcal{T}$-admissible if it admits a labelling of its flags with labels in the vertex set of $G$.
such that

-flags of $M$ labelled $x$ are $i$-adjacent to flags of $M$ labelled $y$ if and only if there is an edge labelled $i$ between the vertices $x$ and $y$ of $G$.

- all flags of $M$ with the same label belong to the same orbit.

Note that $\mathcal{T}$-admissible maps correspond to $\mathcal{T}$-regular maps in [1].

Clearly, if $\mathcal{T}$ is a type of $k$-orbit maps then every $\mathcal{T}$-admissible map is a $k'$-orbit map, for some divisor $k'$ of $k$. In particular, if $\mathcal{T}$ is a type of 2-orbit maps then every $\mathcal{T}$-admissible map is either of type $\mathcal{T}$ or regular.

3 Operations

In Sections 4 and 5 we shall require some operations on maps described next.

An operation $\text{Oper}$ applied to the map $M$ on the surface $S$ yields a map $\text{Oper}(M)$ obtained from $M$ by the following steps.

(a) Divide $S$ in a set of regions $R_1, \ldots, R_k$ isomorphic to discs. Each region inherits a portion of a graph from the map $M$.

(b) Modify the portion of graph on each $R_i$.

(c) Identify the boundaries of the regions $R_i$ in some way determined by $\text{Oper}$ to form a surface $S'$. $\text{Oper}(M)$ will be the induced map on $S'$.

Next we consider the operations dual, Petrial and truncation, as well as two specific operations, called collapsing and rebelting, that can only be applied to certain families of maps. In all these operations each region consists of a collection of flags.

3.1 Dual

The dual operation consists of reinterpreting the centres of faces as vertices and the vertices as centres of faces preserving the same number of edges. This can be done by
considering the same set of flags, but interchanging 0- and 2-adjacencies. Alternatively, we may define it in the following way.

Let $M$ be a map on a surface $S$. For every face $f$ of $M$ consider the region $R_f$ determined by all flags contained in $f$. On each region $R_f$, the portion of the graph consists of a $p$-cycle around its boundary, where $p$ is the co-degree of the face. Replace this portion of graph with a vertex in the interior of the face with $p$ semi-edges which join the midpoint of all original edges to the centre of $f$. Finally, identify the boundaries of the regions $R_f$ in the way they were identified before to obtain the dual $Du(M)$ of $M$. Each edge of $Du(M)$ intersects precisely one edge of $M$ and therefore the edges of these two maps are in a natural one-to-one correspondence.

Note that $Du(M)$ lies on the same surface of $M$. Furthermore, if $M$ is equivelar with Schl"afli type $\{p, q\}$ then $Du(M)$ is equivelar with Schl"afli type $\{q, p\}$.

It is easy to see that $\Gamma(M) \cong \Gamma(Du(M))$ and, therefore, $M$ is a $k$-orbit map if and only if $Du(M)$ is a $k$-orbit map. Clearly $Du(Du(M)) = M$ for every map $M$.

3.2 Petrial

We define the Petrie polygons (also called Petrie paths) of a map $M$ as the faces of $Pe(M)$. The Petrie polygons of $M$ can be visualized in $M$ as zigzags with the property that every two consecutive edges of a Petrie polygon are two consecutive edges of a face; but three consecutive edges of a Petrie polygon belong to the same face only when the vertex between two of the edges has degree 2.

Often the Petrial of a map $M$ is defined in the map with the same vertex and edge set as $M$ with faces given the Petrie polygons of $M$. Alternatively we can define it in the following way.

For each vertex $v$, consider the region $R_v$ consisting of all flags around $v$. Each line-segment between two centres $c_1$ and $c_2$ of faces sharing an edge $e = \{v_1, v_2\}$ in $M$ is shared by the two regions $R_{v_1}$ and $R_{v_2}$. For each such line-segment, identify the corresponding two regions reversing the original local orientation, that is, identifying the line segment in such a way that the point $c_1$ on the region $R_{v_1}$ is identified with the point $c_2$ of the region $R_{v_2}$, and the point $c_2$ on the region $R_{v_1}$ with the point $c_1$ of the region $R_{v_2}$. The resulting map is the Petrial of the map $M$ and is denoted by $Pe(M)$. Note that $Pe(M)$ contains the same vertex and edge set as $M$.

Since the underlying graph of $M$ is invariant under the Petrie operation, $M$ has all vertices with degree $q$ if and only if $Pe(M)$ also does. However, if $Pe(M)$ is equivelar it is not necessarily true that $M$ is equivelar, but it is true that all Petrie polygons of $M$ have the same size.

In general the Petrie operation changes the surface. In fact, the number $v$ of vertices and the number of edges $e$ are preserved under this operation, but the number $f$ of faces needs not be. Therefore the well-known surface invariant Euler characteristic $v - e + f$ of a map $M$ may differ with that of its Petrial. Furthermore, the underlying surface of
Pe(M) may or may not be orientable independently to the orientability of the underlying surface of M.

Clearly Pe(Pe(M)) = M. Furthermore, if M is a vertex-transitive k-orbit map then Pe(M) is also a vertex-transitive k-orbit map.

### 3.3 Truncation

In [17, Section 4.2] (see also [18, Section 3]) the truncation operation is defined in three different ways. Here we present an equivalent definition.

For each vertex v of M consider the region \( R_v \) consisting of all flags around v. The portion of graph on each region consists of a vertex incident to \( q \) semi-edges, where \( q \) is degree of v. Replace v by a \( q \)-cycle in such a way that the new vertices lie in the interior of the old semi-edges and identify the regions \( R_v \) in the original way. The resulting map is the truncation \( Tr(M) \) of M. Note that the edges of \( Tr(M) \) that intersecting two different regions \( R_v \) are in a one-to-one correspondence with the edges of M. We call these edges inherited edges of \( Tr(M) \).

The truncation operation preserves the surface and yields exclusively 3-valent maps. On the other hand, if M is a k-orbit map then \( Tr(M) \) is either a k-, \((3k/2)\)- or \((3k)\)-orbit map (see [17, Proposition 4.5]). Note that \( \Gamma(M) \) is a subgroup of \( Tr(M) \).

### 3.4 Collapsing

Assume that there is an orbit \( \mathcal{F} \) of triangular faces of M with the following two properties:

1. every vertex is incident to either no faces in \( \mathcal{F} \), one face in \( \mathcal{F} \), two faces in \( \mathcal{F} \) sharing an edge, or to three faces in \( \mathcal{F} \) two of which share an edge,

2. each triangle in \( \mathcal{F} \) shares precisely one edge with another face in \( \mathcal{F} \),

3. every triangle in \( \mathcal{F} \) has three distinct vertices, and any two triangles in \( \mathcal{F} \) sharing an edge contain four distinct vertices.

The last item can be reworded as follows. The closure of the region determined by any two triangles in \( \mathcal{F} \) sharing an edge is homeomorphic to a disc.

We define the collapsing operation with respect to \( \mathcal{F} \) as follows. For each pair \( F_1 \) and \( F_2 \) of faces in \( \mathcal{F} \) sharing an edge \( uv \), let \( x \) be the vertex in \( F_1 \setminus F_2 \), and let \( w \) be the vertex in \( F_2 \setminus F_1 \). Delete the triangles \( F_1 \) and \( F_2 \) by identifying the vertices \( w \) and \( x \), the edges \( ux \) and \( uw \), and the edges \( vx \) and \( vw \) (see Figure 3).

In terms of the definition of operation above, we may define the collapsing operation by considering the region \( R_v \) for each vertex v as in the operations Petrial and truncation, and eliminating from it the regions induced by all triangles in \( \mathcal{F} \). If the eliminated region corresponded to two adjacent triangles, the semi-edges incident to v bounding this region are identified. If the eliminated region corresponded to only one triangle then the two
semi-edges delimiting this region become part of the boundary of the new region. The identification rule of the new regions is inherited from that in $M$ except in the part of the boundary arising from the elimination of a triangle $T_1$ in $\mathcal{F}$. By hypothesis $T_1$ is adjacent to precisely one triangle $T_2$ in $\mathcal{F}$. Let $v'$ be the vertex in $T_2$ not contained in $T_1$. Then the part of the boundary of $R_v$ corresponding to the semi-edges bounding $T_1$ is identified to the part of the boundary of $R_{v'}$ arising from the deletion of $T_2$.

Note that Items 1 and 3 guarantee that the collapsing operation preserves the surface.

### 3.5 Rebelting

The rebelting operation can only be applied to maps with vertices of degree 8 whose faces are triangles with the properties that

- some triangles are arranged in belts like the one in Figure 4 such that no edge or face belongs to the intersection of two belts;
- the faces around every vertex are six triangles on belts and two triangles not belonging to belts, arranged in such a way that between the two triangles not belonging to belts there are precisely 3 triangles on belts. In other words, triangles which are not on belts share edges only with triangles on belts, and all triangles sharing edges with triangles on a belt are not on a belt.

The rebelting operation consists of substituting a belt of triangles by a new belt with the same number of (bigger) triangles including one third of each triangle adjacent to the original belt. In Figure 5 the original (dark) belt is substituted by the belt of dotted triangles. Note that the vertices of the original triangulation become midpoints of edges, and the new vertices are the centres of triangles not belonging to belts in the original triangulation.
In terms of the definition of operation at the beginning of the section, we define the collapsing operation as follows. For each triangle \( t \) on a belt let \( e_t \) be the edge of \( t \) shared by \( t \) and a triangle not on a belt. We consider the region \( R_t \) consisting of \( t \) together with the two flags around \( e_t \) not belonging to \( t \). The portion of graph on each region consists of a triangle with precisely two edges in the boundary of the region. We substitute this graph by a vertex \( v_t \) with four semi-edges incident to it. The vertex \( v_t \) is placed in the point of the boundary corresponding to the centre of the triangle not belonging to a belt in the original map. Two of the semi-edges join \( v_t \) through the boundary of \( R_t \) with the two original vertices belonging to \( t \) and to the triangle with centre in \( v_t \). The other ends of the remaining two semi-edges are the midpoints of the two original edges in the boundary of \( R_t \). We identify the regions \( R_t \) in the way they were identified in the original map. Note that the vertex \( v_t \) appears in precisely three regions, and therefore, the resulting map has vertices with degree 9. Furthermore, all faces are triangles.

Clearly the rebelting operation preserves the surface.

4 \( \{7, 3\} \) and \( \{3, 7\} \) vertex-transitive maps

In this section we classify the vertex-transitive maps with Schl"afli type \( \{3, 7\} \) which are neither regular nor chiral in three families. We also describe a procedure to obtain them from regular or 2-orbit maps by means of operations. Additionally we prove that all vertex-transitive maps with Schl"afli type \( \{7, 3\} \) are either regular or chiral.

We shall make use of the following lemmas.
Lemma 4.1. Let $k$ be the number of orbits on flags of a vertex-transitive map with vertices of degree $q$. Then $k$ divides $2q$.

Proof. The orbits of flags around any vertex $v$ induced by the stabilizer of $v$ have the same cardinality. \hfill \Box

Lemma 4.2. Let $M$ be an equivelar map with Schl"afli type $\{p,q\}$. Let $G$ be its type graph and let $C$ be a connected component of $G_2$ (resp $G_0$). Then

- if $C$ is a path, the number of vertices of $C$ divides $p$ (resp $q$),
- if $C$ is a cycle, the number of vertices of $C$ divides $2p$ (resp $2q$).

Proof. The orbits represented in the vertices of $C$ correspond to the orbits containing the flags in some face $F$ of $M$. Among the flags in $F$, the stabilizer of $F$ induces orbits with the same cardinality. Therefore the number of orbits of flags in $F$ must divide $2p$. Furthermore, if $C$ is a path, then there is some orbit containing flags $i$-adjacent to flags in the same orbit for $i = 0$ or $i = 1$ and therefore, the stabilizer of $F$ contains a reflection. Consequently, each orbit $O$ of flags containing flags in $F$ contains at least two flags contained on $F$, moreover, the number of flags in $O \cap F$ is even. This implies the first item.

A dual argument considering the flags around a vertex instead those in a face implies the lemma for $G_0$. \hfill \Box

We are now ready to state our main result about maps with Schl"afli type $\{p,3\}$.

Theorem 4.3. All vertex-transitive maps with Schl"afli type $\{p,3\}$ with $p$ congruent to 1 or 5 modulo 6 are either regular or chiral.

Proof. Let $p$ be congruent to 1 or 5 modulo 6 and $G$ be the type graph of a vertex-transitive map $M$ with Schl"afli type $\{p,3\}$.

By Lemma 4.1, the number of orbits on flags $M$ must be either 1, 2, 3 or 6. The dual version of Lemma 4.2 implies that the number of vertices on $G_0$ is either 1 or 3 if it is a path, or 2 or 6 if it is a cycle. Since $M$ is vertex-transitive, $G_0$ must be connected. On the other hand, Lemma 4.2 implies that the number of vertices on each connected component of $G_2$ divides $2p$, that is, it is either 1, 5 or 2. In the latter case, it must be a cycle of length 2.

It is not hard to see that it is not possible to construct a graph with 6 vertices with the following three properties. The edges with labels 1 and 2 form a spanning cycle, the connected components induced by edges with labels 0 and 2 are among those in Figure 1 and all connected components induced by edges with labels 0 and 1 are either cycles of length 2 or path with five vertices (note that every vertex is adjacent to an edge labeled 1 and an edge labeled 2).
Similarly it can be verified that it is not possible to construct a graph with three vertices satisfying the following three conditions. The edges with labels 1 and 2 form a spanning path, the connected components induced by edges with labels 0 and 2 are among those in Figure 1 and all connected components induced by edges with labels 0 and 1 are either double edges or isolated vertices. Therefore any vertex-transitive map with Schl"afli type \( \{ p, 3 \} \) has at most 2 orbits on flags.

Assume that \( M \) has two orbits on flags. Then \( G \) has only two vertices and an edge labelled \( i \) between them for at least some \( i \). Lemma 4.2 implies that there must also be an edge with label \( i + 1 \) between the two vertices if \( i < 2 \), and an edge with label \( i - 1 \) between the two vertices if \( i > 0 \). This implies that there are edges of all labels between the two vertices and hence the map is chiral. Alternatively we could have argued that the remaining types of 2-orbit maps contain maps with either vertices with even valency, or faces with an even number of edges (see [10]).

The following corollary is a direct consequence of the previous theorem.

**Corollary 4.4.** All vertex-transitive maps with Schl"afli type \( \{ 7, 3 \} \) are either regular or chiral.

Now we turn our attention to vertex-transitive maps with Schl"afli type \( \{ 3, 7 \} \).

Lemma 4.1 implies that any vertex-transitive equivelar map \( M \) with Schl"afli type \( \{ 3, 7 \} \) has either 1, 2, 7 or 14 orbits on flags. Furthermore, similar arguments to those of the proof of Theorem 4.3 show that if \( M \) has two orbits on flags it must be chiral. We first eliminate the possibility of equivelar vertex-transitive maps with Schl"afli type \( \{ 3, 7 \} \) and 7 orbits on flags.

**Proposition 4.5.** There are no 7-orbit vertex-transitive equivelar maps with Schl"afli type \( \{ 3, 7 \} \).

**Proof.** Assume to the contrary that such a map \( M \) exists, then the type graph \( G \) of \( M \) has 7 vertices and \( G_0 \) is an alternating path with 6 edges. On the other hand, as a consequence of Lemma 4.2 every connected component of \( G_2 \) is either a single vertex, two vertices joined by a double edge (one edge of each color), an alternating path with two edges, or an alternating cycle of length 6.

Let \( v \) be the vertex of \( G \) incident to an edge labelled 1 but to no edge labelled 2. Since the connected components induced by all edges with labels 0 and 2 are those in Figure 1 and all vertices different form \( v \) are incident to an edge labelled 2, \( v \) cannot be incident to an edge labelled 0. Therefore the connected component of \( G_2 \) containing \( v \) must be a path with two edges and it must contain the vertex of \( G \) not incident to an edge labelled 1. This fact, together with the allowed connected components of \( G_1 \) forces another edge labelled 0 forming an alternating square of edges labelled 0 and 2, but this in turn induces a connected component of \( G_2 \) different from the ones allowed.
We now describe all types of 14-orbit maps containing vertex-transitive maps with Schl"afli type \{3, 7\}. Let \(M\) be one such map and let \(G\) be its type graph.

Since \(G_0\) is connected and has 14 vertices, it must be an alternating cycle. This implies that every vertex of \(G\) is incident to an edge with label 1. Therefore every connected component of \(G_2\) is either a double edge or a 6-cycle (see Lemma 4.2). By divisibility reasons at least one connected component of \(G_2\) must be a 2-cycle and therefore the graph in Figure 6(A) is a subgraph of \(G\). An exhaustive search for the choices of the vertex sharing an edge labeled 0 with the vertex labelled \(u\) in this figure proves that the only possible type graphs for vertex-transitive 14-orbit maps with Schl"afli type \{3, 7\} are those in (B), (C) and (D) in Figure 6. They correspond to the flag arrangements in Figure 7.

Note that in each of the three cases there exists an orbit of triangles satisfying the conditions to apply the collapsing operation, namely the orbit containing triangles with flags labelled 1, 2, 3, 10, 11, 14 for cases (B) and (C), and labelled 2, 3, 9, 8, 7, 6 for case (D). By applying the collapsing operation we effectively delete all flags on the deleted triangles preserving the remaining flags as well as all symmetries of the map. It may be the case that we gain some extra symmetries and therefore we obtain a \(k\)-orbit map for some divisor \(k\) of 8. The new map will be vertex transitive and admissible to the
respective types defined by the local arrangements of flags in Figure 8. Observe that cases (C) and (D) are equal up to relabeling.

All types in Figure 8 admit the relabeling operation. The triangles with labels 12 and 13 are deleted while the belts formed by the remaining triangles are rearranged to obtain a map with Schlafli type \{3, 9\}. Therefore the new maps are \(k\)-orbit maps for some divisor \(k\) of 6 and are admissible with the local arrangements of flags in Figure 9, where the flags are arbitrarily labelled in the set \{1, \ldots, 6\}. Note that in all cases the new map is not only vertex-transitive, but also face-transitive. Once more the symmetries of the original 14-orbit map are preserved.

Applying the dual to a map admissible with one of the types in Figure 9 we obtain a vertex- and face-transitive map with Schlafli type \{9, 3\}. Applying the Petrial operation to the latter yields a 3-valent map whose Petrie polygons have length 9 and must be admissible with the types described by the local configurations of flags in Figure 10.

Each flag labelled 3 or 6 on Figure 10 can be glued with its 1- and 2-adjacent flags to assemble new flags with vertices on the centres of faces with flags labelled 4 and 5, midpoints of edges on the midpoints of edges with flags labelled 1 and 2 and centres of faces on the centres of faces with flags labelled 1, 2, 3 and 6. This implies that any map \(M'\) admissible with any of the types described by the local configurations in Figure 10 is the truncation of a map \(M\) admissible with type 2\(_{01}\) (case (B)) or with type 2\(_{0}\) (cases
(C) and (D)). Note that edges with flags labelled 1 and 2 are the hereditary edges of $M'$ and appear every three steps in the Petrie polygons. We recall that the edges of $M$ correspond to the hereditary edges of $M'$. Therefore the Petrie polygons with length 9 of $M'$ translate into paths with length 3 in $M$. These paths are 2-zigzags in the sense of the proof of Lemma 7B11 in [15], that is, they traverse an edge with a given local orientation (left or right) and continue by taking the second edge in that direction but changing the local orientation (see Figure 11). In particular, $M$ is a map on a non-orientable surface.

By reversing the process described above we can construct all vertex-transitive maps with Schl"afli type $\{3, 7\}$ which are neither regular nor chiral by determining all maps of type $2_{01}$ satisfying the relation $(\rho_0 \rho_1 \rho_{212})^3 = id$ and all maps of type $2_0$ satisfying the relation $(\rho_0 \sigma_{12}^2)^3 = id$. It is easy to verify that these two relations are the ones determining the length 3 of the 2-zigzags on maps of the given admissibility. Conversely, the truncation of any map with 2-zigzags of size 3 has Petrie polygons with length 9.

The following algorithm produces a vertex-transitive map $M$ with Schl"afli type $\{3, 7\}$ from a map $M$ which is either $2_{01}$-admissible and satisfies the relation $(\rho_0 \rho_1 \rho_{212})^3 = id$ or $2_0$-admissible and satisfies the relation $(\rho_0 \sigma_{12}^2)^3 = id$.

1. Truncate $M$ to obtain a 3-valent map $M' := Tr(M)$ with Petrie polygons of length 9. Let $E_1$ be the set of hereditary edges of $M'$

2. Take the petrial $Pe(M')$ of $M'$ to obtain a map with Schl"afli type $\{9, 3\}$. Since the vertex and edge set of $M'$ and $Pe(M')$ coincide we may think of $E_1$ as a set of edges
of $Pe(M')$. It is not hard to verify from the type graph that, if $M$ is $2_{01}$-admissible, the resulting surface is orientable.

3. Take the dual of $Pe(M')$ to obtain a map $M'' := Du(Pe(M'))$ with Schlafli type $\{3,9\}$. We refer by $E_1'$ the set of edges intersecting those of $E_1$. Then the faces of $M''$ can be arranged in belts of triangles determined by the edges in $E_1'$.

4. Take the inverse of the rebeleting operation on $M''$ with respect to the belts described in the previous item. This produces a map $M^{(3)}$ with Schlafli type $\{3,8\}$ whose faces can be divided in those arranged on belts and those arising from the vertices of $M''$.

5. (a) If $M$ is $2_{01}$-admissible, to construct a map $\overrightarrow{M}$ admissible with the type represented by the flag arrangement (B) in Figure 7, apply the inverse of the collapsing operation in the following way. Label the flags of $M^{(3)}$ as indicated in Figure 8 (B). This can be done by labelling the flags of $M'$ and $M''$ as indicated in Figures 10 B and 11 B respectively as intermediate steps. For each flag $\Phi$ labelled $k$ for some $k \in \{12, 13\}$ consider its defining vertex $v_\Phi$ and edge $e_\Phi$. Let $e_\Phi^{op}$ be the edge incident to $v_\Phi$ other than $e_\Phi$ delimiting a flag labelled $k$ with a vertex at $v_\Phi$. Split the vertex $v_\Phi$ into two vertices $v_1$ and $v_2$ while splitting each of $e_\Phi$ and $e_\Phi^{op}$ in two edges, one of them incident to $v_1$ and the other to $v_2$. Add an edge between $v_1$ and $v_2$.

(b) If $M$ is $2_0$-admissible, to construct a map $\overrightarrow{M}$ admissible with the type represented by the flag arrangement (C) in Figure 7, apply the inverse of the collapsing operation repeating the steps in item (a) replacing the labelling of the flags by the one in Figure 8 (C).

(c) Finally, if $M$ is $2_0$-admissible, to construct a map $\overrightarrow{M}$ admissible with the type represented by the flag arrangement (D) in Figure 7, apply the inverse of the collapsing operation in a similar way to that described in item (a), replacing the labelling of the flags by the one in Figure 8 (D), and considering all flags labelled $k \in \{1,10\}$ instead of those with $k \in \{12,13\}$.

Let $M$ be a $2_{01}$-admissible (resp. $2_0$-admissible) map with $2_{01}$-admissible (resp. $2_0$-admissible) group $G$, and $\overrightarrow{M}$ be the map obtained from $M$ by the algorithm above. Since the Petrial and dual operations preserve the group, and the truncation operation only preserves or increases it, $G$ is also a subgroup of the automorphism group of the map obtained from $M$ by the three first steps of the algorithm. Furthermore, the last two steps of the algorithm effectively add some orbits of flags while preserving the action of $G$ on the old (and new) flags. Therefore $G$ is a subgroup of the map obtained by the algorithm above.

Observe that $M$ and $\overrightarrow{M}$ contain $2|G|$ and $14|G|$ flags respectively, and therefore, $|G|/2$ and $7|G|/2$ edges respectively. This implies that if $\Gamma(\overrightarrow{M}) = G$ then $G$ is a 14-orbit map,
whereas if $G$ is respectively an index 7 or 14 subgroup of $\Gamma(M)$ then $M$ is chiral or regular. By Proposition 4.5, $G$ cannot be an index 2 subgroup of $\Gamma(M)$.

The Euler characteristic of the surface of $M$ is $v - e + f = 14|G|(1/14 - 1/4 + 1/6) = -|G|/6$ which is one third of the number of edges of $M$. Consequently, to obtain all vertex-transitive maps on all surfaces with Euler characteristic $\chi$ with Schl"afl"i type $\{3, 7\}$ which are neither regular nor chiral it suffices to determine all maps with $3\chi$ edges satisfying the conditions required for the algorithm above.

Note, however, that a given $2_{01}$- or $2_0$-admissible map yields two vertex-transitive maps with Schl"afl"i type $\{3, 7\}$ whenever the two possible choices of $k$ in step 5 of the algorithm are essentially different. If these two choices of $k$ produce isomorphic maps, then there is an extra symmetry of the map obtained after step 4. By tracing back this symmetry to $M$ via the operations, we see that the two choices of $k$ are isomorphic if and only if the map $M$ is regular.

Furthermore, a given group $G$ generated by three involutions determines three $2_{01}$-admissible maps depending on the choice of generator to play the role of $\rho_0$. However, in some cases some of these maps may be isomorphic. Consequently, a group $G$ generated by three involutions whose product has order 3 determines $6/t$ maps where $t \in \{1, 2, 3, 6\}$ is the number of isomorphisms of $G$ induced by permutations of its three involutory generators.

To conclude this section we present a theorem that shows that there are infinitely many surfaces that do not admit non-regular vertex-transitive maps with Schl"afl"i type $\{3, 7\}$.

**Theorem 4.6.** Let $M$ be a vertex-transitive map with Schl"afl"i type $\{3, 7\}$ on a surface with Euler characteristic $-p$ for $p$ odd prime. Then $M$ must be regular.

**Proof.** The discussion in Section 4 shows that $M$ must be either regular, chiral or 14-orbit map.

Surfaces with odd Euler characteristic are non-orientable and therefore $M$ cannot be chiral or admissible with the flag-configuration in Figure 7 B.

It remains to show that $M$ cannot be a 14-orbit map admissible with the flag-arrangements in Figure 7 C, D. Assume to the contrary that such a map exists, then it can be obtained by the algorithm in Section 4 from a $2_0$-admissible map with $3p$ edges and $2_0$-admissible group with $6p$ elements. The case $p = 3$ was discarded by an exhaustive search. Now assume that $p \geq 5$. According to [4, Table 16.3] the only groups with order $6p$ are $\mathbb{Z}_{6p}$, $\mathbb{Z}_p \times D_3$, $\mathbb{Z}_3 \times D_p$, $D_{3p}$, $(\mathbb{Z}_p : \mathbb{Z}_q) \times \mathbb{Z}_2$ and $\mathbb{Z}_p : \mathbb{Z}_6$. The last two occur only if and only if $p \equiv 1(\text{mod } 6)$. It is not hard to verify that if a generating set of any of these groups contains only an involution $a$ and another element $b$ then $ab^2$ cannot have order 3. This implies that there are no $2_0$-admissible maps with $3p$ edges and the theorem holds. \hfill \Box

The regular maps N15.1 and N147.1 in [3] lie on surfaces with Euler characteristic $-17$ and $-149$ respectively, both negatives of prime numbers showing that Theorem 4.6 is false if we omit the assumption of non-regularity of $M$.  

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5 Examples

In this section we present in detail three vertex-transitive maps with Schl"afli type \{3, 7\} obtained by the algorithm in Section 4, one for each of the items 5(a), 5(b) and 5(c). We also determine the number of vertex-transitive maps with Schl"afli type \{3, 7\} on surfaces with Euler characteristic $-2$.

Figure 12 illustrates the steps to obtain a vertex-transitive map with Schl"afli type \{3, 7\} admissible with the local arrangements of flags in Figure 7 B from the 201-admissible map in the projective plane consisting of six 2-gons (Figure 12 A). Figure 12 B shows a hemi-prism which is the truncation of the map in Figure 12 A. The Petrial of the hemi-prism is illustrated in Figure 12 C. The surface now is orientable and has Euler characteristic $-2$ (genus 2). Figure 12 D shows the dual of the map in Figure 12. The edges in $E_1$ are indicated in thick dotted lines (see the algorithm in Section 4). The inverse of the relabeling operation applied to the map in Figure 12 D is shown in Figure 12 E. This is the regular map \{3, 8\} *96 in [9] or R2.1 in [3] (see also [17, Figure 8]). Figure 12 F shows another presentation of the map in Figure 12. The edges in thick dotted lines indicate the places where the inverse of the collapsing operation must be applied. Finally, Figure 12 G shows the vertex-transitive map we were looking for.

Figure 13 illustrates the last steps to obtain a vertex-transitive map with Schl"afli type \{3, 7\} admissible with the local arrangements of flags in Figure 7 C from the 20-admissible map in the projective plane with six 2-gons (Figure 12 A). Since steps 1 to 4 of the algorithm are the same as for 201-admissible maps, these steps are illustrated also by Figure 12 B, C, D, E. The edges in dark dotted lines in Figure 13 F indicate where the inverse of the collapsing operation should be applied. Note that these edges are not equivalent to the edges in 12 F. Figure 13 G corresponds to the desired map with Schl"afli type \{3, 7\}.

The two maps obtained so far can be visualized as follows. Consider a belt formed by 12 triangles and embed it on an orientable surface of Euler characteristic $-2$ (genus 2) in such a way that its removal divides the surface in two tori, each of them missing a disk (see Figure 14). On each of these tori embed the map in Figure 14 B, where the outer hexagon is identified with the border of the belt and the inner hexagon is identified as indicated by the arrows to form the torus. If both tori are attached to the belt following the same orientation we obtain the map in Figure 12 G, whereas if the tori are attached with opposite orientations we obtain the map in Figure 13 G.

To obtain a vertex-transitive map with Schl"afli type \{3, 7\} admissible with the local arrangements of flags in Figure 7 D from the 20-admissible map in the projective plane with six 2-gons in Figure 12 A we follow the steps corresponding to the pictures in Figure 12 A–E. Figure 15 F shows the edges where the inverse of the collapsing operation should be applied and the desired map is the one in Figure 15 G.

The map in Figure 12 A and the 201-compatible map on the projective plane consisting of a hexagon and three 2-gons are the only 201-compatible maps with 6 edges. Since the
Figure 12: Construction of a vertex-transitive map with Schlafli type \{3, 7\} on an orientable surface with Euler characteristic \(-2\)
Figure 13: Construction of a vertex-transitive map with Schl"afli type \(\{3, 7\}\) on an orientable surface with Euler characteristic \(-2\)

Figure 14: Vertex-transitive map with Schl"afli type \(\{3, 7\}\) on an orientable surface with Euler characteristic \(-2\)
map in Figure 12 A is regular, it only yields one vertex-transitive map with Schl"afli type \{3, 7\}. The remaining 2_{01}-compatible map with 6 edges is not regular and hence it yields two non-isomorphic vertex-transitive maps with Schl"afli type \{3, 7\} on an orientable surface with Euler characteristic $-2$. Equivalently, the dihedral group with 12 elements $D_6$ is generated by three involutions whose product has order 3; since there is precisely one non-trivial automorphism of $D_6$ permuting these involutory generators, there are precisely three vertex-transitive maps with Schl"afli type \{3, 7\} associated with this group. These three maps correspond to those arising from the map in Figure 12 A and the map on the projective plane with a hexagon and three 2-gons.

The map in Figure 12 A and the hemicube are the only 2_{01}-admissible maps with 6 edges. Since both of them are regular, the maps in Figures 13 G and 15 G, and the two maps arising from the hemicube are the only vertex-transitive maps with Schl"afli type \{3, 7\} arising from 2_{01}-compatible maps with 6 edges. The maps in Figures 13 G and 15 G lie on an orientable surface, whereas the maps arising from the hemicube lie on a non-orientable surface. In both cases the Euler characteristic is $-2$. This implies that there are precisely 5 vertex-transitive maps with Schl"afli type \{3, 7\} on orientable surfaces with Euler characteristic $-2$, and precisely 2 vertex-transitive maps with Schl"afli type \{3, 7\} on non-orientable surfaces with Euler characteristic $-2$.

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Here we present information about all vertex-transitive maps with Schl"afli type \( \{3, 7\} \) on surfaces with Euler characteristic \( -1 \geq \chi \geq -40 \).

Each vertex-transitive map with Schl"afli type \( \{3, 7\} \) on a surface with Euler characteristic \( \chi \) which is neither regular nor chiral can be obtained from either a 201-admissible map with \( -3\chi \) edges satisfying the relation \((\rho_0\rho_1\rho_{212})^3 = id\), or from a 20-admissible map with \( -3\chi \) edges satisfying the relation \((\rho_0\sigma_{12})^3 = id\). This is equivalent to find all groups with order \( -6\chi \) generated by 3 involutions whose product has order 3 and all groups with order \( -6\chi \) generated by an involution and a non-involution such that the product of the former with the square of the latter has order 3.

In the lists of regular and chiral maps \(^3\) there is 1 chiral map and 8 regular maps with Schl"afli type \( \{3, 7\} \), five of the regular maps lie on orientable surfaces and the remaining three on non-orientable surfaces. The regular map \( R3.1 \) and the chiral map \( C17.1 \) can be obtained from the algorithm with the step 5(a), the regular map \( R14.1 \) can be obtained from the algorithm with the step 5(b), and the regular map \( R14.3 \) can be obtained from the algorithm with the step 5(c). The remaining maps in these lists cannot be obtained by the construction above.

As explained at the end of Section \(^3\) a given 201-admissible group \( G \) together with its generators may determine one, two, three or six distinct vertex-transitive maps with Schl"afli type \( \{3, 7\} \), depending on the number of permutations of the involutory generators.
extending to automorphisms of $G$. On the other hand, a given $2_0$-admissible group $G$ with given generators $\rho_0$ and $\sigma_{12}$ may determine one or two vertex-transitive maps with Schl"afli type $\{3,7\}$ depending on whether there is an automorphism of $G$ fixing $\rho_0$ and mapping $\sigma_{12}$ to $\sigma_{12}^{-1}$ (that is, the corresponding $2_{01}$-admissible map is regular). All these possibilities occur in the atlas with the exception of a $2_{01}$-admissible group yielding only two vertex-transitive maps with Schl"afli type $\{3,7\}$.

The following list contains all pairs consisting of a group and a generating set which are subgroups of the automorphism group of a vertex-transitive maps with Schl"afli type $\{3,7\}$ on surfaces with Euler characteristic $-1 \geq \chi \geq -40$. With each such group we indicate one of the corresponding $2_0$- or $2_{01}$-admissible maps according to the lists in [3]. Sometimes the $2_{01}$-admissible map is not regular but can be obtained from a regular map by doubling every edge and interpreting the induced 2-gons as faces. We also mention how many vertex-transitive maps with Schl"afli type $\{3,7\}$ are associated to each group. Notice that for several genera $\chi$ there are no vertex-transitive maps with Schl"afli type $\{3,7\}$ on surfaces with Euler characteristic $\chi$.

**Automorphism groups $G$ of vertex-transitive maps with Schl"afli type $\{3,7\}$ arising from $2_{01}$-admissible maps.**

- $\chi = -2$ ($|G| = 12$)
  - $\rho_0 = (1,2)(3,4)(5,6)$,
  - $\rho_1 = (2,3)(4,5)$,
  - $\rho_{212} = (1,6)(2,5)(3,4)$.
  - 2$_{01}$-admissible map: regular map with six 2-gons on the projective plane.
  - This group yields 3 different 14-orbit vertex-transitive maps with Schl"afli type $\{3,7\}$.

- $\chi = -4$ ($|G| = 24$)
  - $\rho_0 = (1,2)$,
  - $\rho_1 = (2,3)$,
  - $\rho_{212} = (1,2)(3,4)$.
  - 2$_{01}$-admissible map: hemicuboctahedron on the projective plane.
  - This group yields 6 different 14-orbit vertex-transitive maps with Schl"afli type $\{3,7\}$.

- $\chi = -6$ ($|G| = 36$)
  - $\rho_0 = (1,2)$,
  - $\rho_1 = (2,3)$,
  - $\rho_{212} = (1,3)(2,4)$.
  - 2$_{01}$-admissible map: N4.1 in [3].
  - This group yields 2 different 14-orbit vertex-transitive maps and 1 regular map with Schl"afli type $\{3,7\}$. 

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\( \rho_0 = (1, 2)(3, 4)(5, 6), \)
\( \rho_1 = (2, 3)(4, 5), \)
\( \rho_{212} = (1, 3)(2, 4)(5, 6). \)

201-admissible map: dual of N5.2 in [3].
This group yields 3 different 14-orbit vertex-transitive maps with Schläfli type \{3, 7\}.

\( \chi = -8 \ (|G| = 48) \)

\( \rho_0 = (2, 4)(6, 7), \)
\( \rho_1 = (1, 2)(3, 6)(4, 5)(7, 8), \)
\( \rho_{212} = (1, 3)(2, 5)(4, 7)(6, 8). \)

201-admissible map: N10.2 in [3].
This group yields 3 different 14-orbit vertex-transitive maps with Schläfli type \{3, 7\}.

\( \rho_0 = (1, 2)(3, 5)(4, 6)(7, 8), \)
\( \rho_1 = (2, 4)(5, 7), \)
\( \rho_{212} = (1, 3)(2, 5)(4, 6)(7, 8). \)

201-admissible map: N4.2 in [3] with double edges.
This group yields 6 different 14-orbit vertex-transitive maps with Schläfli type \{3, 7\}.

\( \chi = -10 \ (|G| = 60) \)

\( \rho_0 = (3, 4)(5, 6), \)
\( \rho_1 = (1, 2)(3, 4), \)
\( \rho_{212} = (2, 3)(4, 5). \)

201-admissible map: dual of N5.1 in [3].
This group yields 3 different 14-orbit vertex-transitive maps with Schläfli type \{3, 7\}.

\( \rho_0 = (1, 2)(5, 6), \)
\( \rho_1 = (2, 3)(4, 6), \)
\( \rho_{212} = (2, 4)(3, 5). \)

201-admissible map: dual of N6.1 in [3].
This group yields 3 different 14-orbit vertex-transitive maps with Schläfli type \{3, 7\}.

\( \rho_0 = (2, 3)(5, 6), \)
\( \rho_1 = (1, 2)(4, 5), \)
\( \rho_{212} = (2, 4)(3, 5). \)

201-admissible map: N10.6 in [3].
This group yields 3 different 14-orbit vertex-transitive maps with Schläfli type \{3, 7\}.

\( \chi = -16 \ (|G| = 96) \)
\[ \rho_0 = (1, 2)(6, 7), \]
\[ \rho_1 = (1, 3)(2, 4)(5, 7)(6, 8), \]
\[ \rho_{212} = (1, 3)(2, 5)(4, 6)(7, 8). \]

\[2_{01}\text{-admissible map: N10.1 in } [3].\]

This group yields 3 different 14-orbit vertex-transitive maps with Schläfli type \(\{3, 7\}\).

\[ \rho_0 = (1, 2)(3, 6)(4, 5)(7, 12)(8, 14)(9, 15)(10, 11)(13, 16), \]
\[ \rho_1 = (2, 4)(3, 7)(5, 9)(6, 10)(8, 12)(15, 16), \]
\[ \rho_{212} = (1, 3)(2, 5)(4, 8)(6, 11)(7, 13)(9, 16)(10, 15)(12, 14). \]

\[2_{01}\text{-admissible map: N34.6 in } [3].\]

This group yields 3 different 14-orbit vertex-transitive maps with Schläfli type \(\{3, 7\}\).

\[ \rho_0 = (1, 2)(3, 5)(4, 7)(6, 10)(8, 12)(9, 13)(11, 15)(14, 16), \]
\[ \rho_1 = (2, 4)(3, 6)(5, 9)(7, 8)(10, 14)(12, 15), \]
\[ \rho_{212} = (1, 3)(2, 5)(4, 8)(6, 11)(7, 12)(9, 13)(10, 15)(14, 16). \]

\[2_{01}\text{-admissible map: N16.7 in } [3] \text{ with double edges.}\]

This group yields 6 different 14-orbit vertex-transitive maps with Schläfli type \(\{3, 7\}\).

\[ \chi = -18 \ (|G| = 108) \]

\[ \rho_0 = (3, 4)(5, 6)(7, 9), \]
\[ \rho_1 = (2, 3)(4, 5)(6, 8), \]
\[ \rho_{212} = (1, 2)(3, 4)(5, 7)(6, 9). \]

\[2_{01}\text{-admissible map: dual of N11.1 in } [3].\]

This group yields 3 different 14-orbit vertex-transitive maps with Schläfli type \(\{3, 7\}\).

\[ \rho_0 = (1, 2)(3, 6)(4, 8)(5, 9)(7, 12)(10, 11), \]
\[ \rho_1 = (1, 3)(2, 4)(5, 10)(6, 11)(7, 9)(8, 12), \]
\[ \rho_{212} = (1, 4)(2, 5)(3, 7)(6, 10)(8, 11)(9, 12). \]

\[2_{01}\text{-admissible map: N29.3 in } [3].\]

This group yields 1 different 14-orbit vertex-transitive maps with Schläfli type \(\{3, 7\}\).

\[ \rho_0 = (2, 3)(4, 7)(6, 8), \]
\[ \rho_1 = (2, 4)(3, 5)(7, 9), \]
\[ \rho_{212} = (1, 2)(3, 6)(5, 7)(8, 9). \]

\[2_{01}\text{-admissible map: N29.3 in } [3].\]

This group yields 3 different 14-orbit vertex-transitive maps with Schläfli type \(\{3, 7\}\).

\[ \rho_0 = (2, 3)(5, 7)(6, 8), \]
\[ \rho_1 = (2, 4)(3, 5)(7, 9), \]
\[ \rho_{212} = (1, 2)(3, 6)(4, 5)(8, 9). \]

\[2_{01}\text{-admissible map: N29.3 in } [3].\]

This group yields 3 different 14-orbit vertex-transitive maps with Schläfli type \(\{3, 7\}\).
χ = −20 (|G| = 120)

- ρ₀ = (4, 6)(7, 8)(9, 11)(10, 12),
  ρ₁ = (1, 2)(3, 5)(4, 7)(6, 8),
  ρ₂₁₂ = (1, 3)(2, 4)(5, 8)(6, 9)(7, 10)(11, 12).

2₀₁-admissible map: dual of N20.1 in \[3\].
This group yields 3 different 14-orbit vertex-transitive maps with Schlafli type \{3, 7\}.

- ρ₀ = (1, 2)(3, 5)(4, 7)(6, 8)(9, 12)(10, 11),
  ρ₁ = (2, 3)(4, 8)(5, 9)(7, 11),
  ρ₂₁₂ = (2, 4)(3, 6)(5, 10)(7, 9).

2₀₁-admissible map: N30.8 in \[3\].
This group yields 3 different 14-orbit vertex-transitive maps with Schlafli type \{3, 7\}.

- ρ₀ = (2, 4)(6, 9)(7, 11)(8, 12),
  ρ₁ = (1, 2)(3, 6)(5, 8)(7, 10),
  ρ₂₁₂ = (1, 3)(2, 5)(4, 7)(6, 10)(8, 9)(11, 12).

2₀₁-admissible map: dual of N30.11 in \[3\].
This group yields 3 different 14-orbit vertex-transitive maps with Schlafli type \{3, 7\}.

- ρ₀ = (3, 4)(5, 6)(7, 8)(9, 10),
  ρ₁ = (1, 2)(3, 4)(5, 7)(6, 8)(9, 10)(11, 12),
  ρ₂₁₂ = (2, 3)(4, 5)(7, 9)(10, 11).

2₀₁-admissible map: N14.3 in \[3\] with double edges.
This group yields 6 different 14-orbit vertex-transitive maps with Schlafli type \{3, 7\}.

- ρ₀ = (1, 2)(3, 6)(4, 7)(5, 9)(8, 12)(10, 11),
  ρ₁ = (2, 4)(5, 10)(6, 7)(9, 12),
  ρ₂₁₂ = (1, 3)(2, 5)(4, 8)(6, 9)(7, 11)(10, 12).

2₀₁-admissible map: obtained from the group of the hypermap NPH26.3 in \[3\].
This group yields 6 different 14-orbit vertex-transitive maps with Schlafli type \{3, 7\}.

- ρ₀ = (2, 3)(5, 7)(6, 9)(8, 12),
  ρ₁ = (1, 2)(3, 5)(4, 6)(7, 10)(8, 11)(9, 12),
  ρ₂₁₂ = (2, 4)(3, 6)(5, 8)(7, 11).

2₀₁-admissible map: obtained from the group of the hypermap NPH34.9 in \[3\].
This group yields 6 different 14-orbit vertex-transitive maps with Schlafli type \{3, 7\}.

χ = −24 (|G| = 144)

- ρ₀ = (1, 2)(3, 4)(5, 6)(7, 8)(9, 11)(10, 12),
  ρ₁ = (3, 4)(5, 7)(6, 8),
  ρ₂₁₂ = (2, 3)(4, 5)(6, 9)(7, 10)(11, 12).

2₀₁-admissible map: N20.3 in \[3\] with double edges.
This group yields 6 different 14-orbit vertex-transitive maps with Schlafli type \{3, 7\}.
\• \( \rho_0 = (1, 2)(3, 5)(4, 7)(6, 8)(9, 12)(10, 11) \),
\( \rho_1 = (2, 3)(4, 8)(5, 9)(6, 10)(7, 11) \),
\( \rho_{212} = (2, 4)(3, 6)(11, 12) \).

201-admissible map: obtained from the group of the hypermap NPH32.1 in [3].
This group yields 6 different 14-orbit vertex-transitive maps with Schläfli type \( \{3, 7\} \).

\• \( \rho_0 = (1, 2)(3, 5)(4, 6)(7, 9)(8, 11)(10, 12) \),
\( \rho_1 = (2, 3)(4, 5)(6, 8)(7, 10)(9, 11) \),
\( \rho_{212} = (2, 4)(5, 7)(8, 12) \).

201-admissible map: obtained from the group of the hypermap NPH84.1 in [3].
This group yields 6 different 14-orbit vertex-transitive maps with Schläfli type \( \{3, 7\} \).

\[ \chi = -32 \ (|G| = 192) \]

\• \( \rho_0 = (1, 2)(3, 4)(5, 8)(6, 10)(7, 12)(9, 14)(11, 13)(15, 16) \),
\( \rho_1 = (1, 3)(2, 5)(4, 7)(6, 11)(8, 12)(9, 15)(10, 14)(13, 16) \),
\( \rho_{212} = (1, 4)(2, 6)(3, 7)(5, 9)(8, 13)(10, 12)(11, 16)(14, 15) \).

201-admissible map: N34.1 in [3].
This group yields 3 different 14-orbit vertex-transitive maps with Schläfli type \( \{3, 7\} \).

\• \( \rho_0 = (1, 2)(3, 4)(5, 6)(7, 8) \),
\( \rho_1 = (2, 3)(4, 6)(5, 7) \),
\( \rho_{212} = (2, 4)(3, 5)(6, 7) \).

201-admissible map: dual of N42.2 in [3].
This group yields 3 different 14-orbit vertex-transitive maps with Schläfli type \( \{3, 7\} \).

\• \( \rho_0 = (1, 2)(3, 6)(4, 5)(7, 8) \),
\( \rho_1 = (2, 4)(5, 7)(6, 8) \),
\( \rho_{212} = (1, 3)(2, 5)(6, 7) \).

201-admissible map: N58.10 in [3].
This group yields 3 different 14-orbit vertex-transitive maps with Schläfli type \( \{3, 7\} \).

\• \( \rho_0 = (1, 2)(3, 4)(5, 6)(7, 9)(8, 11)(10, 13)(12, 14)(15, 16) \),
\( \rho_1 = (2, 3)(4, 5)(6, 8)(7, 10)(12, 15)(13, 14) \),
\( \rho_{212} = (3, 4)(5, 7)(6, 9)(8, 12)(11, 14)(15, 16) \).

201-admissible map: N22.4 in [3] with double edges.
This group yields 6 different 14-orbit vertex-transitive maps with Schläfli type \( \{3, 7\} \).

\• \( \rho_0 = (1, 2)(3, 5)(4, 7)(6, 8) \),
\( \rho_1 = (2, 3)(4, 8)(6, 7) \),
\( \rho_{212} = (2, 4)(3, 6)(7, 8) \).

201-admissible map: obtained from the group of the hypermap NPH30.4 in [3].
This group yields 6 different 14-orbit vertex-transitive maps with Schläfli type \( \{3, 7\} \).
• \( \rho_0 = (1, 2)(3, 4)(5, 7)(6, 8) \),
\( \rho_1 = (2, 4)(5, 7)(6, 8) \),
\( \rho_{212} = (1, 3)(2, 5)(4, 6) \).

2\_01-\textit{admissible map}: obtained from the group of the hypermap NPH46.5 in [3].
This group yields 6 different 14-orbit vertex-transitive maps with Schl"afli type \{3, 7\}.

• \( \rho_0 = (1, 2)(3, 5)(4, 7)(6, 8)(9, 10)(11, 15)(12, 13)(14, 16) \),
\( \rho_1 = (2, 3)(4, 8)(5, 9)(7, 11)(10, 14)(13, 15) \),
\( \rho_{212} = (2, 4)(3, 6)(5, 10)(8, 12)(9, 13)(11, 16) \).

2\_01-\textit{admissible map}: obtained from the group of the hypermap NPH46.6 in [3].
This group yields a chiral map map (C17.1 in [3]) and 5 different 14-orbit vertex-transitive maps with Schl"afli type \{3, 7\}.

• \( \rho_0 = (1, 2)(3, 5)(4, 6)(7, 10)(8, 12)(9, 14)(11, 16)(13, 15) \),
\( \rho_1 = (2, 3)(5, 7)(6, 9)(8, 13)(10, 11)(12, 14) \),
\( \rho_{212} = (1, 3)(2, 4)(5, 8)(7, 10)(11, 13)(12, 16)(14, 18)(15, 17)(19, 21)(20, 22)(23, 24) \).

2\_01-\textit{admissible map}: obtained from the group of the hypermap NPH58.10 in [3].
This group yields 3 different 14-orbit vertex-transitive maps with Schl"afli type \{3, 7\}.

• \( \rho_0 = (4, 6)(7, 8)(9, 11)(10, 13)(12, 15)(14, 17)(16, 19)(18, 20) \),
\( \rho_1 = (1, 2)(3, 5)(4, 7)(6, 8)(9, 12)(10, 14)(11, 15)(13, 17)(16, 20)(18, 19)(21, 23)(22, 24) \),
\( \rho_{212} = (1, 3)(2, 4)(5, 8)(6, 9)(7, 10)(11, 13)(12, 16)(14, 18)(15, 17)(19, 21)(20, 22)(23, 24) \).

2\_01-\textit{admissible map}: N38.2 in [3].
This group yields 3 different 14-orbit vertex-transitive maps with Schl"afli type \{3, 7\}.

• \( \rho_0 = (1, 2)(3, 6)(4, 7)(5, 9)(8, 15)(10, 17)(11, 18)(12, 19)(13, 20)(14, 21)(16, 22)(23, 24) \),
\( \rho_1 = (2, 4)(5, 10)(6, 11)(7, 13)(9, 16)(12, 15)(18, 23)(19, 21) \),
\( \rho_{212} = (1, 3)(2, 5)(4, 8)(6, 12)(7, 14)(9, 13)(10, 15)(11, 17)(16, 21)(18, 22)(19, 23)(20, 24) \).

2\_01-\textit{admissible map}: N70.5 in [3].
This group yields 3 different 14-orbit vertex-transitive maps with Schl"afli type \{3, 7\}.

• \( \rho_0 = (2, 4)(6, 10)(7, 12)(8, 14)(9, 15)(13, 19)(16, 22)(21, 23) \),
\( \rho_1 = (1, 2)(3, 6)(4, 7)(5, 9)(8, 11)(10, 16)(12, 17)(13, 20)(14, 19)(15, 21)(18, 23)(22, 24) \),
\( \rho_{212} = (1, 3)(2, 5)(4, 8)(6, 11)(7, 13)(9, 10)(12, 18)(14, 15)(16, 23)(17, 24)(19, 21)(20, 22) \).

2\_01-\textit{admissible map}: N78.4 in [3].
This group yields 3 different 14-orbit vertex-transitive maps with Schl"afli type \{3, 7\}.

\textbf{Automorphism groups} \( G \) of vertex-transitive maps on orientable surfaces with Schl"afli type \{3, 7\} arising from 2\_0\_admissible maps.

\( \chi = -2 \ (|G| = 12) \)
• $\rho_0 = (1, 3)(2, 4),$
  $\sigma_{12} = (1, 2, 3, 4)(5, 6, 7).$
  2$_0$-admissible map: regular map with six 2-gons on the projective plane.
  This group yields 2 different 14-orbit vertex-transitive maps with Schläfli type \{3, 7\}.
  \[ \chi = -4 \quad (|G| = 24) \]

• $\rho_0 = (1, 2)(3, 4),$
  $\sigma_{12} = (1, 2, 3)(5, 6).$
  2$_0$-admissible map: N4.1 in \[3\].
  This group yields 2 different 14-orbit vertex-transitive maps with Schläfli type \{3, 7\}.
  \[ \chi = -6 \quad (|G| = 36) \]

• $\rho_0 = (2, 3)(4, 5),$
  $\sigma_{12} = (1, 2)(3, 4, 6, 5).$
  2$_0$-admissible map: dual of N5.2 in \[3\].
  This group yields 2 different 14-orbit vertex-transitive maps with Schläfli type \{3, 7\}.
  \[ \chi = -8 \quad (|G| = 48) \]

• $\rho_0 = (1, 2)(3, 5)(4, 7)(6, 10)(8, 12)(9, 14)(11, 13)(15, 16),$
  $\sigma_{12} = (1, 3, 6, 11)(2, 4, 8, 13, 12, 16, 10, 15, 14, 5, 9, 7).$
  2$_0$-admissible map: N10.2 in \[3\].
  This group yields 2 different 14-orbit vertex-transitive maps with Schläfli type \{3, 7\}.
  \[ \chi = -16 \quad (|G| = 96) \]

• $\rho_0 = (1, 2)(5, 6),$
  $\sigma_{12} = (1, 3)(2, 4, 5, 7, 6, 8).$
  2$_0$-admissible map: N10.1 in \[3\].
  This group yields 2 different 14-orbit vertex-transitive maps with Schläfli type \{3, 7\}.

• $\rho_0 = (1, 2)(3, 5)(4, 7)(6, 10)(8, 12)(9, 14)(11, 13)(15, 19)(16, 21)(17, 20)(18, 24)(22, 27)(23, 28)(25, 29, 32)(2, 4, 8, 13, 12, 18, 25, 24, 14, 5, 9, 15, 20, 19, 26, 31, 30, 21, 10, 16, 22, 17).$
  2$_0$-admissible map: N34.6 in \[3\].
  This group yields 2 different 14-orbit vertex-transitive maps with Schläfli type \{3, 7\}.
  \[ \chi = -18 \quad (|G| = 108) \]

• $\rho_0 = (2, 3)(4, 5)(7, 9)(8, 11)(10, 13)(16, 17),$
  $\sigma_{12} = (1, 2)(3, 4, 6, 8)(5, 7, 10, 14)(9, 12)(11, 15, 13, 16)(17, 18).$
  2$_0$-admissible map: N11.1 in \[3\].
  This group yields 2 different 14-orbit vertex-transitive maps with Schläfli type \{3, 7\}.
• \( \rho_0 = (1, 2)(3, 5)(4, 7)(6, 8)(9, 10)(11, 12), \)
  \(\sigma_{12} = (1, 3, 6, 5, 8, 9, 11, 10, 12, 7, 2, 4).\)
  20-admissible map: N29.3 in [3].
  This group yields 2 different 14-orbit vertex-transitive maps with Schläfli type \(\{3, 7\}\).

• \( \rho_0 = (1, 2)(4, 6)(5, 7)(8, 11)(12, 15)(13, 17), \)
  \(\sigma_{12} = (1, 3, 5, 8, 7, 10, 13, 15, 17, 18, 2, 4)(6, 9, 12, 16, 11, 14).\)
  20-admissible map: N29.4 in [3].
  This group yields 2 different 14-orbit vertex-transitive maps with Schläfli type \(\{3, 7\}\).

• \( \rho_0 = (1, 2)(3, 5)(6, 8)(10, 12)(11, 14)(15, 17), \)
  \(\sigma_{12} = (1, 3, 6, 9, 8, 11, 15, 18, 17, 12, 2, 4)(5, 7, 10, 13, 14, 16).\)
  20-admissible map: N29.5 in [3].
  This group yields 2 different 14-orbit vertex-transitive maps with Schläfli type \(\{3, 7\}\).

\[\chi = -20 \ (|G| = 120)\]

• \( \rho_0 = (3, 4)(5, 6), \)
  \(\sigma_{12} = (1, 2, 3, 5).\)
  20-admissible map: dual of N20.1 in [3].
  This group yields 2 different 14-orbit vertex-transitive maps with Schläfli type \(\{3, 7\}\).

• \( \rho_0 = (1, 2)(3, 5)(6, 8)(9, 10), \)
  \(\sigma_{12} = (1, 3)(2, 4, 6, 9, 12, 5, 7, 10, 8, 11).\)
  20-admissible map: N30.8 in [3].
  This group yields 2 different 14-orbit vertex-transitive maps with Schläfli type \(\{3, 7\}\).

• \( \rho_0 = (1, 2)(3, 4)(5, 7)(6, 9), \)
  \(\sigma_{12} = (2, 3, 5, 8, 7, 9)(4, 6, 10).\)
  20-admissible map: dual of N30.11 in [3].
  This group yields 2 different 14-orbit vertex-transitive maps with Schläfli type \(\{3, 7\}\).

\[\chi = -32 \ (|G| = 192)\]

• \( \rho_0 = (1, 2)(3, 4)(5, 7)(6, 9)(8, 11)(10, 12)(13, 14)(15, 16), \)
  \(\sigma_{12} = (1, 3, 5, 8)(2, 4, 6, 10, 9, 12, 7, 11, 13, 15, 14, 16).\)
  20-admissible map: N34.1 in [3].
  This group yields 2 different 14-orbit vertex-transitive maps with Schläfli type \(\{3, 7\}\).

• \( \rho_0 = (2, 3)(4, 5)(6, 8)(7, 10)(12, 13)(14, 15), \)
  \(\sigma_{12} = (1, 2)(3, 4, 6, 9, 8, 12)(5, 7, 11, 10, 13, 14)(15, 16).\)
  20-admissible map: dual of N42.2 in [3].
  This group yields 2 different 14-orbit vertex-transitive maps with Schläfli type \(\{3, 7\}\).
\[ \rho_0 = (1, 2)(3, 5)(4, 7)(6, 10)(8, 9)(11, 14)(12, 13)(15, 16), \]
\[ \sigma_{12} = (1, 3, 6, 11)(2, 4, 8, 5, 9, 12, 10, 13, 15, 14, 16, 7). \]

2-admissible map: N58.10 in [3].
This group yields 2 different 14-orbit vertex-transitive maps with Schl"afli type \{3, 7\}.

\[ \chi = -40 \ (|G| = 240) \]

\bullet \rho_0 = (3, 4)(5, 6)(7, 8)(11, 12),
\[ \sigma_{12} = (1, 2, 3, 5)(6, 7)(8, 9, 10, 11). \]

2-admissible map: dual of N38.2 in [3].
This group yields 2 different 14-orbit vertex-transitive maps with Schl"afli type \{3, 7\}.

\bullet \rho_0 = (1, 2)(3, 5)(6, 9)(7, 11)(10, 15)(12, 17)(13, 19)(20, 22),
\[ \sigma_{12} = (1, 3, 6, 10)(2, 4, 7, 12, 18, 5, 8, 13, 11, 16, 9, 14, 20, 19, 23, 15, 21, 17, 22, 24). \]

2-admissible map: N70.5 in [3].
This group yields 2 different 14-orbit vertex-transitive maps with Schl"afli type \{3, 7\}.

\bullet \rho_0 = (1, 2)(3, 5)(4, 7)(6, 9)(8, 10)(11, 14)(12, 13)(15, 17)(16, 19)(18, 20),
\[ \sigma_{12} = (1, 3, 6, 5)(2, 4, 8, 11, 10, 13, 9, 12, 15, 18, 17, 7)(14, 16, 20, 19). \]

2-admissible map: N78.4 in [3].
This group yields 2 different 14-orbit vertex-transitive maps with Schl"afli type \{3, 7\}.

Automorphism groups \( G \) of vertex-transitive maps with Schl"afli type \{3, 7\} on non-orientable surfaces (arising from 2-admissible maps).

\[ \chi = -2 \ (|G| = 12) \]

\bullet \rho_0 = (1, 2)(3, 4),
\[ \sigma_{12} = (1, 2, 3). \]

2-admissible map: hemicube (in the projective plane).
This group yields 2 different 14-orbit vertex-transitive maps with Schl"afli type \{3, 7\}.

\[ \chi = -10 \ (|G| = 60) \]

\bullet \rho_0 = (1, 2)(4, 5),
\[ \sigma_{12} = (2, 3, 4, 5, 6). \]

2-admissible map: dual of N10.5 in [3].
This group yields 2 different 14-orbit vertex-transitive maps with Schl"afli type \{3, 7\}.

\[ \chi = -26 \ (|G| = 156) \]

\bullet \rho_0 = (1, 2)(3, 4)(5, 7)(6, 8)(9, 11)(10, 12),
\[ \sigma_{12} = (2, 3, 5, 4, 6, 9, 11, 13, 12, 8, 7, 10). \]

2-admissible map: no reference known of this group or of its associated map.
This group yields 1 regular map (R14.1 in [3]) and 3 different 14-orbit vertex-transitive maps with Schl"afli type \{3, 7\}.  

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• $\rho_0 = (1, 2)(3, 4)(5, 7)(6, 9)(8, 12)(10, 13)$,
  $\sigma_{12} = (2, 3, 5, 8, 9, 12, 4, 6, 10, 13, 7, 11)$.
  $2_0$-admissible map: no reference known of this group or of its associated map.
  This group yields 1 regular maps (R14.3 in [3]) and 3 different 14-orbit vertex-transitive maps with Schl"afli type $\{3, 7\}$.

$\nabla \chi = -28 (|G| = 168)$

• $\rho_0 = (1, 2)(3, 4)(5, 7)(6, 8)$,
  $\sigma_{12} = (1, 3, 5, 2, 4, 6, 7, 8)$.
  $2_0$-admissible map: dual of N23.2 in [3].
  This group yields 2 different 14-orbit vertex-transitive maps with Schl"afli type $\{3, 7\}$.

• $\rho_0 = (1, 2)(3, 4)(5, 7)(6, 8)$,
  $\sigma_{12} = (2, 3, 5, 4, 6, 7, 8)$.
  $2_0$-admissible map: dual of N41.2 in [3].
  This group yields 2 different 14-orbit vertex-transitive maps with Schl"afli type $\{3, 7\}$.

$\nabla \chi = -30 (|G| = 180)$

• $\rho_0 = (1, 2)(3, 5)(6, 9)(7, 10)(11, 12)(16, 17)$,
  $\sigma_{12} = (1, 3, 6)(2, 4, 7, 11, 15, 9, 13, 17, 10, 14, 5, 8, 12, 16, 18)$.
  $2_0$-admissible map: N50.9 in [3].
  This group yields 2 different 14-orbit vertex-transitive maps with Schl"afli type $\{3, 7\}$.

To conclude we show in table I the number of vertex-transitive maps with Schl"afli type $\{3, 7\}$ which are neither regular nor chiral, on surfaces with Euler characteristic $\chi$ for each $-1 \geq \chi \geq -40$. 

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Table 1: Vertex-transitive maps with Schl"afli type \{3, 7\} which are neither regular nor chiral on surfaces of Euler characteristic $-1 \geq \chi \geq -40$. 

| $\chi$ | orientable | non-orientable | Total |
|-------|-------------|----------------|-------|
| −2    | 5           | 2              | 7     |
| −4    | 10          | 0              | 10    |
| −6    | 5           | 0              | 5     |
| −8    | 11          | 0              | 11    |
| −10   | 9           | 2              | 11    |
| −16   | 16          | 0              | 16    |
| −18   | 18          | 0              | 18    |
| −20   | 33          | 0              | 33    |
| −24   | 18          | 0              | 18    |
| −26   | 0           | 6              | 6     |
| −28   | 0           | 4              | 4     |
| −30   | 0           | 2              | 2     |
| −32   | 44          | 0              | 44    |
| −40   | 15          | 0              | 15    |