NON-ARCHIMEDEAN STATISTICAL FIELD THEORY

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Abstract. We construct in a rigorous mathematical way interacting quantum field theories on a $p$-adic spacetime. The main result is the construction of a measure on a function space which allows a rigorous definition of the partition function. The advantage of the approach presented here is that all the perturbation calculations can be carried out in the standard way using functional derivatives, but in a mathematically rigorous way.

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1. Introduction

In this article we construct (in a rigorous mathematical way) interacting quantum field theories over a \( p \)-adic spacetime in an arbitrary dimension. We provide a large family of energy functionals \( E(\varphi, J) \) admitting natural discretizations in finite-dimensional vector spaces such that the partition function

\[
Z_{\text{phys}}(J) = \int D(\varphi) e^{-\frac{1}{\kappa_B T} E(\varphi, J)}
\]

can be defined rigorously as the limit of the mentioned discretizations. Our main result is the construction of a measure on a function space such that (1.1) makes mathematical sense, and the calculations of the \( n \)-point correlation functions can be carried out using perturbation expansions via functional derivatives, in a rigorous mathematical way. Our results include \( \varphi^4 \)-theories. In this case, we show that the \( n \)-point correlation functions admit a convergent expansion in the coupling parameter in a certain space of distributions. By the Wick theorem all of the distributions appearing in the mentioned series can be expressed as a sum of products of Green functions, which have singularities. Consequently a renormalization procedure is required, we expect to study the renormalization of the Feynman integrals attached to (1.1) in a forthcoming publication.

From now on \( p \) denotes a fixed prime number. A \( p \)-adic number is a series of the form

\[
x = x_{-k} p^{-k} + x_{-k+1} p^{-k+1} + \ldots + x_0 + x_1 p + \ldots,
\]

with \( x_{-k} \neq 0 \), where the \( x_j \)'s are \( p \)-adic digits, i.e. numbers in the set \( \{0, 1, \ldots, p-1\} \). The set of all possible series of the form (1.2) constitutes the field of \( p \)-adic numbers \( \mathbb{Q}_p \). There are natural field operations, sum and multiplication, on series of the form (1.2), see e.g. [33]. There is also a natural norm in \( \mathbb{Q}_p \) defined as \( |x|_p = p^k \), for a nonzero \( p \)-adic number of the form (1.2). The field of \( p \)-adic numbers with the distance induced by \( |\cdot|_p \) is a complete ultrametric space. The ultrametric (or non-Archimedean) property refers to the fact that \( |x - y|_p \leq \max \{ |x - z|_p, |z - y|_p \} \) for any \( x, y, z \in \mathbb{Q}_p \). We denote by \( \mathbb{Z}_p \) the unit ball, which consists of all series with expansions of the form (1.2) with \( -k \geq 0 \). We extend the \( p \)-adic norm to \( \mathbb{Q}_p^N \) by taking \( ||x||_p = \max_{1 \leq i \leq N} |x_i|_p \), for \( x = (x_1, \ldots, x_N) \in \mathbb{Q}_p^N \).

A fundamental scientific problem is the understanding of the structure of space-time at the level of the Planck scale, and the construction of physical-mathematical models of it.
This problem occurs naturally when trying to unify general relativity and quantum mechanics. In the 1930s Bronstein showed that general relativity and quantum mechanics imply that the uncertainty $\Delta x$ of any length measurement satisfies $\Delta x \geq L_{\text{Planck}} := \sqrt{\frac{\hbar G}{c^3}}$, where $L_{\text{Planck}}$ is the Planck length ($L_{\text{Planck}} \approx 10^{-33}$ cm). This implies that space-time is not an infinitely divisible continuum (mathematically speaking, the spacetime must be a completely disconnected topological space at the level of the Planck scale). Bronstein’s inequality has motivated the development of several different physical theories. At any rate, this inequality implies the need of using non-Archimedean mathematics in models dealing with the Planck scale. In the 1980s, Volovich proposed the conjecture that the space-time at the Planck scale is non-Archimedean, see [48]. This conjecture has propelled a wide variety of investigations in cosmology, quantum mechanics, string theory, QFT, etc., and the influence of this conjecture is still relevant nowadays, see e.g. [1], [7]-[15], [21]-[22], [28]-[32], [34]-[38], [47]-[51].

The space $\mathbb{Q}^N_p$ has a very rich mathematical structure. The axiomatic quantum field theory can be extended to $\mathbb{Q}^N_p$. In [35], we construct a family of quantum scalar fields over a $p$-adic spacetime which satisfy $p$-adic analogues of the G˚arding–Wightman axioms. Since the space of test functions on $\mathbb{Q}^N_p$ is nuclear the techniques of white noise calculus are available in the $p$-adic setting, see e.g. [9], [17], [24], [23]. This implies that a rigorous functional integral approach is available in the $p$-adic framework, see e.g. [18], [42], [43]. In [52], see also [32], Chapter 11, [3]-[4], we introduced a class of non-Archimedean massive Euclidean fields, in arbitrary dimension, which are constructed as solutions of certain covariant $p$-adic stochastic pseudo-differential equations, by using techniques of white noise calculus. In [5], we construct a large class of interacting Euclidean quantum field theories, over a $p$-adic space time, by using white noise calculus. These quantum fields fulfill all the Osterwalder-Schrader axioms, except the reflection positivity. In all these theories the time is a $p$-adic variable. Since $\mathbb{Q}_p$ is not an ordered field, there is no notion of past and future. In certain theories, it is possible to introduce a quadratic form. The orthogonal group of this form plays the role of Lorentz group. Anyway, we do not have a light cone structure, and then this type of theory is also acausal, see [33]. The relevant feature is that the vacuum of all these theories performs fluctuations.

In the case of $\varphi^4$-theories the energy functional $E(\varphi, 0)$ takes the form

$$E(\varphi, 0; \delta, \gamma, \alpha_2, \alpha_4) = \frac{\gamma}{2} \int_{\mathbb{Q}^N_p} \varphi(x) W(\partial, \delta) \varphi(x) \, d^N x + \frac{\alpha_2}{2} \int_{\mathbb{Q}^N_p} \varphi^2(x) \, d^N x$$

$$+ \frac{\alpha_4}{2} \int_{\mathbb{Q}^N_p} \varphi^4(x) \, d^N x,$$

(1.3)

where $\varphi: \mathbb{Q}^N_p \to \mathbb{R}$ is a test function $(\varphi \in \mathcal{D}_\mathbb{R}(\mathbb{Q}^N_p))$, $\delta > N$, $\gamma > 0$, $\alpha_2 \geq 0$, $\alpha_4 \geq 0$, and $W(\partial, \delta) \varphi(x) = \mathcal{F}_{\kappa \to x}^{-1}(A_{w_\delta}(\|\kappa\|)\mathcal{F}_{x \to \kappa}\varphi)$ is pseudo-differential operator, whose symbol has a singularity at the origin.
An interesting observation is that the one-dimensional Vladimirov operator is a special case of the operators \( W (\partial, \delta) \), in this case the action \( E(\varphi, 0; \delta, \gamma, 0, 0) \) appeared in \( p \)-adic string theory, see \[44\], \[50\], \[49\], see also \[15\] and the references therein.

In order to make sense of the partition function attached to \( E(\varphi, 0; \delta, \gamma, \alpha_2, \alpha_4) \), see \[1\], we discretize the fields like in classical QFT. As fields we use test functions \( \varphi \in \mathcal{D}_\mathbb{R} (\mathbb{Q}_p^N) \), which are locally constant with compact support. We have \( \mathcal{D}_\mathbb{R} (\mathbb{Q}_p^N) = \bigcup_{i=1}^\infty \mathcal{D}_\mathbb{R}^i (\mathbb{Q}_p^N) \), where \( \mathcal{D}_\mathbb{R}^i (\mathbb{Q}_p^N) \simeq \mathbb{R}^{\#G_i} \) is a real, finite dimensional vector space consisting of test functions supported in the ball \( B_i^N = \left\{ x \in \mathbb{Q}_p^N; \| x \|_p \leq p^i \right\} \) having the form

\[
(1.4) \quad \varphi (x) = \sum_{i \in G_i} \varphi (i) \Omega \left( p^i \| x - i \|_p \right), \quad \varphi (i) \in \mathbb{R},
\]

where \( G_i \) is a finite set of indices and \( \Omega \left( p^i \| x - i \|_p \right) \) is the characteristic function of the ball \( B_i^N (i) = \left\{ x \in \mathbb{Q}_p^N; \| x - i \|_p \leq p^{-i} \right\} \). Now a natural discretization of partition function \( Z(l) \) is obtained by restricting the fields to \( \mathcal{D}_\mathbb{R}^l (\mathbb{Q}_p^N) \simeq \mathbb{R}^{\#G_i} \) as follows. By identifying \( \varphi \) with the column vector \([\varphi (i)]_{i \in G_i}\), one obtains that

\[
E(\varphi, 0; \delta, \gamma, \alpha_2, 0) = \sum_{i,j \in G_i} p^{-1N} U_{i,j}(l) \varphi (i) \varphi (j),
\]

is a quadratic form in \([\varphi (i)]_{i \in G_i}\), cf. Lemma \[4.2\] and thus taking \( K_BT = 1 \), it is natural to propose that

\[
Z(l) = \int D_l(\varphi) e^{-E(\varphi, 0; \delta, \gamma, \alpha_2, 0)} \text{ def.} \int_{\mathbb{R}^{\#G_i}} e^{-\sum_{i,j \in G_i} p^{-1N} U_{i,j}(l) \varphi (i) \varphi (j)} \prod_{i \in G_i} d\varphi (i),
\]

where \( \prod_{i \in G_i} d\varphi (i) \) is the Lebesgue measure on \( \mathbb{R}^{\#G_i} \), which is a finite dimensional Gaussian integral. We denote the corresponding Gaussian measure as \( \mathbb{P}_l \). The next step is to show the existence of a probability measure \( \mathbb{P} \) such that \( \mathbb{P} = \lim_{l \to \infty} \mathbb{P}_l \) ‘in some sense’. This requires passing to the momenta space and using the Lizorkin space \( \mathcal{L}_\mathbb{R} (\mathbb{Q}_p^N) \subset \mathcal{D}_\mathbb{R} (\mathbb{Q}_p^N) \), resp. \( \mathcal{L}_\mathbb{R}^l (\mathbb{Q}_p^N) \subset \mathcal{D}_\mathbb{R}^l (\mathbb{Q}_p^N) \). The key point is that the operator

\[
\frac{\gamma}{2} W (\partial, \delta) + \frac{\alpha_2}{2}: \mathcal{L}_\mathbb{R} (\mathbb{Q}_p^N) \to \mathcal{L}_\mathbb{R} (\mathbb{Q}_p^N)
\]

has an inverse in \( \mathcal{L}_\mathbb{R} (\mathbb{Q}_p^N) \) for any \( \alpha_2 \geq 0 \). The construction of the measure \( \mathbb{P} \) is made in two steps. In the first step, by using Kolmogorov’s consistency theorem, one shows the existence of a unique probability measure \( \mathbb{P} \) in \( \mathbb{R}^\infty \cup \{ \text{point} \} \) such any linear functional \( f \to \int \mathcal{L}_k (\mathbb{Q}_p^N) f d\mathbb{P}_l \), where \( f \) is a continuous bounded function in \( \mathcal{L}_k^l (\mathbb{Q}_p^N) \), has unique extension of the form \( \int \mathcal{L}_k (\mathbb{Q}_p^N) f d\mathbb{P}_l = \int \mathcal{L}_k^l (\mathbb{Q}_p^N) f d\mathbb{P}_l \), cf. Lemma \[5.1\]. In the second step by using the Gel’fand triple \( \mathcal{L}_\mathbb{R} (\mathbb{Q}_p^N) \hookrightarrow \mathcal{L}_2 (\mathbb{Q}_p^N) \hookrightarrow \mathcal{L}_2' (\mathbb{Q}_p^N) \), where \( \mathcal{L}_2' (\mathbb{Q}_p^N) \) is the topological dual of \( \mathcal{L}_\mathbb{R} (\mathbb{Q}_p^N) \), and the Bochner-Minlos theorem, there exists a probability measure \( \mathbb{P} \) on \( (\mathcal{L}_\mathbb{R} (\mathbb{Q}_p^N), \mathcal{B}) \), that coincides with the probability measure constructed in the first step, cf. Theorem \[5.1\].
For an interaction energy $E_{\text{int}}(\varphi)$ satisfying $\exp(-E_{\text{int}}(\varphi)) \leq 1$, it verifies that

$$\int_{\mathcal{L}_k(\mathbb{Q}_p^N)} e^{-E_{\text{int}}(\varphi)} d\mathbb{P}_l = \int_{\mathcal{L}_k(\mathbb{Q}_p^N)} e^{-E_{\text{int}}(\varphi)} d\mathbb{P} \rightarrow \int_{\mathcal{L}_k(\mathbb{Q}_p^N)} e^{-E_{\text{int}}(\varphi)} d\mathbb{P}$$

as $l \to \infty$. Then a $\mathcal{P}(\varphi)$-theory is given by a cylinder probability measure of the form

$$\frac{1}{\mathcal{L}_k(\mathbb{Q}_p^N)} \int_{\mathcal{L}_k(\mathbb{Q}_p^N)} e^{-E_{\text{int}}(\varphi)} d\mathbb{P}$$

in the space of fields $\mathcal{L}_k(\mathbb{Q}_p^N)$. Notice that $\mathbb{P}$ is a probability measure on $\mathcal{L}_k(\mathbb{Q}_p^N)$, but due to the factor $1_{\mathcal{L}_k}(\varphi)$ our fields are test functions, and not distributions as in [5], [28], see also [3]-[4], [19], and the references therein. Then, the Wick operator $:: \cdot :$ (or Wick regularization) is not required in the definition of $E_{\text{int}}(\varphi)$. This is a very relevant difference with respect to [5], [28]. Here we consider polynomial interactions. The advantage of the approach presented here is that all the perturbation calculations can be carried out in the standard way using functional derivatives, but in a mathematically rigorous way, see Theorem 6.3. However, a renormalization procedure is required. In [5] we construct probability measures for general, interacting QFTs, but using Hida-Kondratiev spaces, which are more bigger than the spaces of distributions used here. However, doing explicit calculations in this very general framework is not easy.

The mathematical framework presented here allows the construction of complex-valued measures of type

$$\frac{1_{\mathcal{L}_k}(\varphi) \exp(\sqrt{-1} \left\{ \frac{\alpha_4}{2} \int_{\mathbb{Q}_p^N} \varphi^4(x) d^N x + \int J(x) \varphi(x) d^N x \right\})}{\int_{\mathcal{L}_k(\mathbb{Q}_p^N)} \exp(\sqrt{-1} \left\{ \frac{\alpha_4}{2} \int_{\mathbb{Q}_p^N} \varphi^4(x) d^N x \right\})} d\mathbb{P}.$$**

Furthermore all the corresponding perturbation expansions can be carried out in the standard form. These measures are obtained from measures of type (1.5) by performing a Wick rotation of type $\varphi \rightarrow \sqrt{-1} \varphi$, see Section 7. The novelty is that this Wick rotation is not performed in spacetime, and thus all these quantum field theories are acausal. More precisely, special relativity is not valid in the spacetime of these theories. However, the vacuum of all these theories perform thermal (resp. quantum) fluctuations, because the Feynman rules are valid, at least formally, in these theories.

The energy functional $E(\varphi, J; \delta, \gamma, \alpha_2, \alpha_4)$, $\varphi \in \mathcal{D}_k(\mathbb{Q}_p^N)$, can be interpreted as the Hamiltonian of a continuous Ising model in the ball $B^N_1$ with an external magnetic field $J$. The Landau-Ginzburg energy functional $E(\varphi, 0; \delta, \gamma, \alpha_2, \alpha_4)$ is non-local, i.e. only long range interactions occur, furthermore, it has $\mathbb{Z}_2$ symmetry ($\varphi \rightarrow -\varphi$). Finally, all the results presented in this article are valid if $\mathbb{Q}_p$ is replaced by any non-Archimedean local field.
2. Basic facts on \( p \)-adic analysis

In this section we fix the notation and collect some basic results on \( p \)-adic analysis that we will use throughout the article. For a detailed exposition on \( p \)-adic analysis the reader may consult \[2\], \[15\], \[47\].

2.1. The field of \( p \)-adic numbers. Throughout this article \( p \) will denote a prime number. Since we have to deal with quadratic forms, for the sake of simplicity, we assume that \( p \geq 3 \) throughout the article. The field of \( p \)-adic numbers \( \mathbb{Q}_p \) is defined as the completion of the field of rational numbers \( \mathbb{Q} \) with respect to the \( p \)-adic norm \( | \cdot |_p \), which is defined as

\[
|x|_p = \begin{cases} 
0 & \text{if } x = 0 \\
p^{-\gamma} & \text{if } x = p^\gamma \frac{a}{b} 
\end{cases}
\]

where \( a \) and \( b \) are integers coprime with \( p \). The integer \( \gamma = \text{ord}_p(x) := \text{ord}(x) \), with \( \text{ord}(0) := +\infty \), is called the \( p \)-adic order of \( x \). We extend the \( p \)-adic norm to \( \mathbb{Q}_p^N \) by taking

\[
||x||_p := \max_{1 \leq i \leq N} |x_i|_p, \quad \text{for } x = (x_1, \ldots, x_N) \in \mathbb{Q}_p^N.
\]

We define \( \text{ord}(x) = \min_{1 \leq i \leq N} \{ \text{ord}(x_i) \} \), then \( ||x||_p = p^{-\text{ord}(x)} \). The metric space \((\mathbb{Q}_p^N, || \cdot ||_p)\) is a complete ultrametric space. As a topological space \( \mathbb{Q}_p \) is homeomorphic to a Cantor-like subset of the real line, see e.g. \[2\], \[47\].

Any \( p \)-adic number \( x \neq 0 \) has a unique expansion of the form

\[
x = p^{\text{ord}(x)} \sum_{j=0}^{\infty} x_j p^j,
\]

where \( x_j \in \{0, 1, 2, \ldots, p-1\} \) and \( x_0 \neq 0 \). By using this expansion, we define the fractional part \( \{x\}_p \) of \( x \in \mathbb{Q}_p \) as the rational number

\[
\{x\}_p = \begin{cases} 
0 & \text{if } x = 0 \text{ or } \text{ord}(x) \geq 0 \\
p^{\text{ord}(x)} \sum_{j=0}^{1-\text{ord}(x)} x_j p^j & \text{if } \text{ord}(x) < 0.
\end{cases}
\]

In addition, any \( x \in \mathbb{Q}_p^N \setminus \{0\} \) can be represented uniquely as \( x = p^{\text{ord}(x)} v(x) \) where \( ||v(x)||_p = 1 \).

2.2. Topology of \( \mathbb{Q}_p^N \). For \( r \in \mathbb{Z} \), denote by \( B_r(a) = \{ x \in \mathbb{Q}_p^N; ||x-a||_p \leq p^r \} \) the ball of radius \( p^r \) with center at \( a = (a_1, \ldots, a_N) \in \mathbb{Q}_p^N \), and take \( B_0(a) := B_r(a) \). Note that \( B_r(a) = B_r(a_1) \times \cdots \times B_r(a_N) \), where \( B_r(a_i) := \{ x \in \mathbb{Q}_p; |x_i - a_i|_p \leq p^r \} \) is the one-dimensional ball of radius \( p^r \) with center at \( a_i \in \mathbb{Q}_p \). The ball \( B_0^N \) equals the product of \( N \) copies of \( B_0 = \mathbb{Z}_p \), the ring of \( p \)-adic integers. We also denote by \( S_r(a) = \{ x \in \mathbb{Q}_p^N; ||x-a||_p = p^r \} \) the sphere of radius \( p^r \) with center at \( a = (a_1, \ldots, a_N) \in \mathbb{Q}_p^N \), and take \( S_r(a) = S_r^N \). We notice that \( S_r^1 = \mathbb{Z}_p^N \) (the group of units of \( \mathbb{Z}_p \)), but \( (\mathbb{Z}_p^\times)^N \subsetneq S_0^N \). The balls and spheres are both open and closed subsets in \( \mathbb{Q}_p^N \). In addition, two balls in \( \mathbb{Q}_p^N \) are either disjoint or one is contained in the other.
As a topological space \((\mathbb{Q}_p^N, || \cdot ||_p)\) is totally disconnected, i.e. the only connected subsets of \(\mathbb{Q}_p^N\) are the empty set and the points. A subset of \(\mathbb{Q}_p^N\) is compact if and only if it is closed and bounded in \(\mathbb{Q}_p^N\), see e.g. [17] Section 1.3, or [2] Section 1.8. The balls and spheres are compact subsets. Thus \((\mathbb{Q}_p^N, || \cdot ||_p)\) is a locally compact topological space.

Since \((\mathbb{Q}_p^N, +)\) is a locally compact topological group, there exists a Haar measure \(d^Nx\), which is invariant under translations, i.e. \(d^N(x + a) = d^N(x)\). If we normalize this measure by the condition \(\int_{\mathbb{Q}_p^N} dx = 1\), then \(d^Nx\) is unique.

**Notation 1.** We will use \(\Omega(p^{-r}||x - a||_p)\) to denote the characteristic function of the ball \(B_r^N(a)\). For more general sets, we will use the notation \(1_A\) for the characteristic function of a set \(A\).

2.3. The Bruhat-Schwartz space. A complex-valued function \(\varphi\) defined on \(\mathbb{Q}_p^N\) is called locally constant if for any \(x \in \mathbb{Q}_p^N\) there exist an integer \(l(x) \in \mathbb{Z}\) such that

\[
\varphi(x + x') = \varphi(x) \quad \text{for any } x' \in B_{l(x)}^N.
\]

A function \(\varphi : \mathbb{Q}_p^N \to \mathbb{C}\) is called a Bruhat-Schwartz function (or a test function) if it is locally constant with compact support. Any test function can be represented as a linear combination, with complex coefficients, of characteristic functions of balls. The \(\mathbb{C}\)-vector space of Bruhat-Schwartz functions is denoted by \(\mathcal{D}(\mathbb{Q}_p^N) := \mathcal{D}\). We denote by \(\mathcal{D}_k(\mathbb{Q}_p^N) := \mathcal{D}_k\) the \(\mathbb{R}\)-vector space of Bruhat-Schwartz functions. For \(\varphi \in \mathcal{D}(\mathbb{Q}_p^N)\), the largest number \(l = l(\varphi)\) satisfying \((2.1)\) is called the exponent of local constancy (or the parameter of constancy) of \(\varphi\).

We denote by \(\mathcal{D}_m^l(\mathbb{Q}_p^N)\) the finite-dimensional space of test functions from \(\mathcal{D}(\mathbb{Q}_p^N)\) having supports in the ball \(B_m^N\) and with parameters of constancy \(\geq l\). We now define a topology on \(\mathcal{D}\) as follows. We say that a sequence \(\{\varphi_j\}_{j \in \mathbb{N}}\) of functions in \(\mathcal{D}\) converges to zero, if the two following conditions hold:

1. there are two fixed integers \(k_0\) and \(m_0\) such that each \(\varphi_j \in \mathcal{D}_{m_0}^{k_0}\);
2. \(\varphi_j \to 0\) uniformly.

\(\mathcal{D}\) endowed with the above topology becomes a topological vector space.

2.4. \(L^p\) spaces. Given \(\rho \in [1, \infty)\), we denote by \(L^\rho := L^\rho(\mathbb{Q}_p^N) := L^\rho(\mathbb{Q}_p^N, d^N x)\), the \(\mathbb{C}\)-vector space of all the complex valued functions \(g\) satisfying \(\int_{\mathbb{Q}_p^N} |g(x)|^\rho d^N x < \infty\). The corresponding \(\mathbb{R}\)-vector spaces are denoted as \(L^\rho := L^\rho(\mathbb{Q}_p^N) = L^\rho(\mathbb{Q}_p^N, d^N x), 1 \leq \rho < \infty\).

If \(U\) is an open subset of \(\mathbb{Q}_p^N\), \(\mathcal{D}(U)\) denotes the space of test functions with supports contained in \(U\), then \(\mathcal{D}(U)\) is dense in \(L^\rho(U)\) defined as

\[
L^\rho(U) = \left\{ \varphi : U \to \mathbb{C}; \|\varphi\|_\rho = \left\{ \int_U |\varphi(x)|^\rho d^N x \right\}^{\frac{1}{\rho}} < \infty \right\},
\]

where \(d^N x\) is the normalized Haar measure on \((\mathbb{Q}_p^N, +)\), for \(1 \leq \rho < \infty\), see e.g. [2] Section 4.3. We denote by \(L^\rho_\mathbb{R}(U)\) the real counterpart of \(L^\rho(U)\).
2.5. The Fourier transform. Set \(\chi_p(y) = \exp(2\pi i \{ y \}_p)\) for \(y \in \mathbb{Q}_p\). The map \(\chi_p(\cdot)\) is an additive character on \(\mathbb{Q}_p\), i.e. a continuous map from \((\mathbb{Q}_p, +)\) into \(S\) (the unit circle considered as multiplicative group) satisfying \(\chi_p(x_0 + x_1) = \chi_p(x_0)\chi_p(x_1)\), \(x_0, x_1 \in \mathbb{Q}_p\). The additive characters of \(\mathbb{Q}_p\) form an Abelian group which is isomorphic to \((\mathbb{Q}_p, +)\). The isomorphism is given by \(\kappa \to \chi_p(\kappa x)\), see e.g. [2, Section 2.3].

Given \(\kappa = (\kappa_1, \ldots, \kappa_N)\) and \(y = (x_1, \ldots, x_N) \in \mathbb{Q}_p^N\), we set \(\kappa \cdot x := \sum_{j=1}^N \kappa_j x_j\). The Fourier transform of \(\varphi \in \mathcal{D}(\mathbb{Q}_p^N)\) is defined as

\[
(\mathcal{F}\varphi)(\kappa) = \int_{\mathbb{Q}_p^N} \chi_p(\kappa \cdot x) \varphi(x) d^N x \quad \text{for} \ \kappa \in \mathbb{Q}_p^N,
\]

where \(d^N x\) is the normalized Haar measure on \(\mathbb{Q}_p^N\). The Fourier transform is a linear isomorphism from \(\mathcal{D}(\mathbb{Q}_p^N)\) onto itself satisfying

\[
(\mathcal{F}(\mathcal{F}\varphi))(\kappa) = \varphi(-\kappa),
\]

see e.g. [2, Section 4.8]. We will also use the notation \(\mathcal{F}_{x \to \kappa} \varphi\) and \(\hat{\varphi}\) for the Fourier transform of \(\varphi\).

The Fourier transform extends to \(L^2\). If \(f \in L^2\), its Fourier transform is defined as

\[
(\mathcal{F}f)(\kappa) = \lim_{k \to \infty} \int_{\|x\|_p \leq p^k} \chi_p(\kappa \cdot x) f(x) d^N x, \quad \text{for} \ \kappa \in \mathbb{Q}_p^N,
\]

where the limit is taken in \(L^2\). We recall that the Fourier transform is unitary on \(L^2\), i.e. \(\|f\|_{L^2} = \|\mathcal{F}f\|_{L^2}\) for \(f \in L^2\) and that (2.2) is also valid in \(L^2\), see e.g. [45, Chapter III, Section 2].

2.6. Distributions. The \(\mathbb{C}\)-vector space \(\mathcal{D}'(\mathbb{Q}_p^N) := \mathcal{D}'\) of all continuous linear functionals on \(\mathcal{D}(\mathbb{Q}_p^N)\) is called the \textit{Bruhat-Schwartz space of distributions}. Every linear functional on \(\mathcal{D}\) is continuous, i.e. \(\mathcal{D}'\) agrees with the algebraic dual of \(\mathcal{D}\), see e.g. [47, Chapter 1, VI.3, Lemma]. We denote by \(\mathcal{D}'_{\mathbb{R}}(\mathbb{Q}_p^N) := \mathcal{D}'_{\mathbb{R}}\) the dual space of \(\mathcal{D}_{\mathbb{R}}\).

We endow \(\mathcal{D}'\) with the weak topology, i.e. a sequence \(\{T_j\}_{j \in \mathbb{N}}\) in \(\mathcal{D}'\) converges to \(T\) if \(\lim_{j \to \infty} T_j(\varphi) = T(\varphi)\) for any \(\varphi \in \mathcal{D}\). The map

\[
\mathcal{D}' \times \mathcal{D} \to \mathbb{C}, \quad (T, \varphi) \to T(\varphi)
\]

is a bilinear form which is continuous in \(T\) and \(\varphi\) separately. We call this map the pairing between \(\mathcal{D}'\) and \(\mathcal{D}\). From now on we will use \((T, \varphi)\) instead of \(T(\varphi)\).

Every \(f \in L^1_{\text{loc}}\) defines a distribution \(f \in \mathcal{D}'(\mathbb{Q}_p^N)\) by the formula

\[
(f, \varphi) = \int_{\mathbb{Q}_p^N} f(x) \varphi(x) d^N x.
\]

Such distributions are called \textit{regular distributions}. Notice that for \(f \in L^2_{\mathbb{R}}, (f, \varphi) = \langle f, \varphi \rangle\), where \(\langle \cdot, \cdot \rangle\) denotes the scalar product in \(L^2_{\mathbb{R}}\).
Remark 1. Let $B(\psi, \varphi)$ be a bilinear functional, $\psi \in \mathcal{D}(Q_p^N), \varphi \in \mathcal{D}(Q_p^M)$. Then there exists a unique distribution $T \in \mathcal{D}'(Q_p^N \times Q_p^M)$ such that

$$ (T, \psi(x) \varphi(y)) = B(\psi, \varphi), \text{ for } \psi \in \mathcal{D}(Q_p^N), \varphi \in \mathcal{D}(Q_p^M), $$

cf. \cite{51} Chapter 1, VI.7, Theorem]

2.7. The Fourier transform of a distribution. The Fourier transform $\mathcal{F}[T]$ of a distribution $T \in \mathcal{D}'(Q_p^N)$ is defined by

$$ (\mathcal{F}[T], \varphi) = (T, \mathcal{F}[\varphi]) \text{ for all } \varphi \in \mathcal{D}(Q_p^N). $$

The Fourier transform $T \to \mathcal{F}[T]$ is a linear (and continuous) isomorphism from $\mathcal{D}'(Q_p^N)$ onto $\mathcal{D}'(Q_p^N)$. Furthermore, $T = \mathcal{F}[\mathcal{F}[T](-\xi)].$

3. $W_\delta$ OPERATORS AND THEIR DISCRETIZATIONS

3.1. The $W_\delta$ operators. Take $\mathbb{R}_+ := \{x \in \mathbb{R}; x \geq 0\}$, and fix a function

$$ w_\delta : Q_p^N \to \mathbb{R}_+ $$

satisfying the following properties:

(i) $w_\delta(y)$ is a radial i.e. $w_\delta(y) = w_\delta(\|y\|_p)$;
(ii) $w_\delta(\|y\|_p)$ is a continuous and increasing function of $\|y\|_p$;
(iii) $w_\delta(y) = 0$ if and only if $y = 0$;
(iv) there exist constants $C_0, C_1 > 0$ and $\delta > N$ such that

$$ C_0 \|y\|_p^\delta \leq w_\delta(\|y\|_p) \leq C_1 \|y\|_p^\delta, \text{ for } y \in Q_p^N. $$

We now define the operator

$$ W_\delta \varphi(x) = \int_{Q_p^N} \frac{\varphi(x-y) - \varphi(x)}{w_\delta(\|y\|_p)} d^Ny, \text{ for } \varphi \in \mathcal{D}(Q_p^N). $$

The operator $W_\delta$ is pseudo-differential; more precisely, if

$$ A_{w_\delta}(\kappa) := \int_{Q_p^N} \frac{1 - x_p(y \cdot \kappa)}{w_\delta(\|y\|_p)} d^Ny, $$

then

$$ W_\delta \varphi(x) = \mathcal{F}_{\kappa \rightarrow x}^{-1} \left[ A_{w_\delta}(\kappa) \mathcal{F}_{x \rightarrow \kappa} \varphi \right] =: W(\partial, \delta) \varphi(x), \text{ for } \varphi \in \mathcal{D}(Q_p^N). $$

The function $A_{w_\delta}(\kappa)$ is radial (so we use the notation $A_{w_\delta}(\kappa) = A_{w_\delta}(\|\kappa\|_p)$), continuous, non-negative, $A_{w_\delta}(0) = 0$, and it satisfies

$$ C_0' \|\kappa\|_p^{\delta-N} \leq A_{w_\delta}(\|\kappa\|_p) \leq C_1' \|\kappa\|_p^{\delta-N}, \text{ for } \kappa \in Q_p^N, $$

cf. \cite{51} Lemmas 4, 5, 8]. The operator $W(\partial, \delta)$ extends to an unbounded and densely defined operator in $L^2(Q_p^N)$ with domain

$$ \text{Dom}(W(\partial, \delta)) = \left\{ \varphi \in L^2; A_{w_\delta}(\|\kappa\|_p) \mathcal{F}\varphi \in L^2 \right\}. $$
In addition:

(i) \((W(\partial,\delta), \text{Dom}(W(\partial,\delta)))\) is self-adjoint and positive operator;

(ii) \(-W(\partial,\delta)\) is the infinitesimal generator of a contraction \(C_0\)-semigroup, cf. [51, Proposition 7].

The evolution equation

\[
\frac{\partial u(x,t)}{\partial t} + W(\partial,\delta) u(x,t) = 0, \quad x \in \mathbb{Q}_p^N, \quad t \geq 0,
\]

is a \(p\)-adic heat equation, which means that the corresponding semigroup is attached to a Markov stochastic process, see [51, Theorem 16].

**Example 1.** An important example of a \(W(\partial,\delta)\) operator is the Taibleson-Vladimirov operator, which is defined as

\[
D^\beta \phi (x) = \frac{1-p^\beta}{1-p^{-\beta-N}} \int_{\mathbb{Q}_p^N} \frac{\phi(x-y) - \phi(x)}{\|y\|_p^{\beta+N}} d^N y = F_{\kappa\rightarrow x}^{-1} \left( \|\kappa\|_p^\beta F_{x\rightarrow \kappa} \right),
\]

where \(\beta > 0\) and \(\phi \in D(\mathbb{Q}_p^N)\), see [51, Section 2.2.7].

The \(W_\delta\) operators were introduced by Chacón-Cortés and Zúñiga-Galindo, see [51] and the references therein. They are a generalization of the Vladimirov and Taibleson operators.

### 3.2. Discretization of \(W_\delta\) operators.

For \(l \geq 1\), we set \(G_l := p^{-l} \mathbb{Z}_p^N / p^l \mathbb{Z}_p^N\) and denote by \(\mathcal{D}_R(\mathbb{Q}_p^N) := \mathcal{D}_R\) the \(\mathbb{R}\)-vector space of all test functions of the form

\[
\varphi(x) = \sum_{i \in G_l} \varphi(i) \Omega(p^l \|x-i\|_p), \quad \varphi(i) \in \mathbb{R},
\]

where \(i\) runs through a fixed system of representatives of \(G_l\), and \(\Omega(p^l \|x-i\|_p)\) is the characteristic function of the ball \(i + p^l \mathbb{Z}_p^N\). Notice that \(\varphi\) is supported on \(p^{-l} \mathbb{Z}_p^N\) and that \(\mathcal{D}_R^l\) is a finite dimensional vector space spanned by the basis

\[
\{ \Omega(p^l \|x-i\|_p) \} \}_{i \in G_l}.
\]

We will identify \(\varphi \in \mathcal{D}_R^l\) with the column vector \([\varphi(i)]_{i \in G_l}\). Furthermore, \(\mathcal{D}_R^l \hookrightarrow \mathcal{D}_R^{l+1}\) (continuous embedding), and \(\mathcal{D}_R = \lim_{l \rightarrow \infty} \mathcal{D}_R^l = \bigcup_{l=1}^{\infty} \mathcal{D}_R^l\).

**Remark 2.** We set

\[
d(l, w_\delta) := \int_{\mathbb{Q}_p^N \setminus B^l_{w_\delta}} \frac{d^N y}{w_\delta(\|y\|_p)}.
\]

By [3.1], \(d(l, w_\delta) < \infty\). Furthermore, we have

\[
p^{(\delta-N)l} C_1 \int_{\mathbb{Q}_p^N \setminus \mathbb{Z}_p^N} \frac{d^N z}{\|z\|_p^{\beta}} \leq d(l, w_\delta) \leq p^{(\delta-N)l} C_0 \int_{\mathbb{Q}_p^N \setminus \mathbb{Z}_p^N} \frac{d^N z}{\|z\|_p^{\beta}},
\]

which implies that \(d(l, w_\delta) \geq Cp^{(\delta-N)l}\) for some positive constant \(C\). In particular, \(d(l, w_\delta) \rightarrow \infty\) as \(l \rightarrow \infty\).
We denote by $W^{(l)}_\delta$ the restriction $W_\delta : \mathcal{D}_R (B^N_l) \to \mathcal{D}_R (B^N_l)$. Take $\varphi \in \mathcal{D}_R (B^N_l)$ is then

$$W^{(l)}_\delta \varphi (x) = \int_{Q^N_p} \frac{\varphi (x - y) - \varphi (x)}{w_\delta (\|y\|_p)} d^N y = \int_{B^N_l} \frac{\varphi (x - y) - \varphi (x)}{w_\delta (\|y\|_p)} d^N y + \int_{Q^N_p \setminus B^N_l} \frac{\varphi (x - y) - \varphi (x)}{w_\delta (\|y\|_p)} d^N y - \left( \int_{Q^N_p \setminus B^N_l} \frac{d^N y}{w_\delta (\|y\|_p)} \right) \varphi (x).$$

**Notation 2.** The cardinality of a finite set $B$ is denoted as $\# B$.

We set

$$A_{i,j} (l) := \begin{cases} \frac{p^{-1} N}{w_\delta (\|i - j\|_p)} & \text{if } i \neq j \\ 0 & \text{if } i = j, \end{cases}$$

and $A := [A_{i,j} (l)]_{i,j \in G_l}$. We denote by $I$ the identity matrix of size $\# G_l \times \# G_l$.

**Lemma 3.1.** The restriction $W^{(l)}_\delta : \mathcal{D}_R^l \to \mathcal{D}_R^l$ is a well-defined linear operator. Furthermore, the following formula holds true:

$$W^{(l)}_\delta \varphi (x) = \sum_{i \in G_l} \left\{ \sum_{j \in G_l} A_{i,j} (l) \varphi (j) - \varphi (i) d (l, w_\delta) \right\} \Omega \left( p^l \|x - i\|_p \right),$$

which implies that $A - d (l, w_\delta) I$ is the matrix of the operator $W^{(l)}_\delta$ in the basis $[3.7]$.

**Proof.** For $x \in i + p^l \mathbb{Z}^N_p$ and for $\varphi (x)$ of the from (3.6), we have

$$W^{(l)}_\delta \varphi (x) = \int_{Q^N_p} \frac{\varphi (y) - \varphi (x)}{w_\delta (\|y - x\|_p)} d^N y = \int_{Q^N_p} \frac{\varphi (j) \Omega (p^l \|y - j\|_p)}{w_\delta (\|y - x\|_p)} d^N y + \int_{Q^N_p} \frac{\varphi (j) \Omega (p^l \|y - i\|_p) - \Omega (p^l \|x - i\|_p)}{w_\delta (\|y - x\|_p)} d^N y$$

$$= \sum_{j \in G_l} \int_{Q^N_p} \frac{\varphi (j) \Omega (p^l \|y - j\|_p)}{w_\delta (\|y - x\|_p)} d^N y + \int_{Q^N_p} \frac{\varphi (i) \left\{ \Omega (p^l \|y - i\|_p) - \Omega (p^l \|x - i\|_p) \right\}}{w_\delta (\|y - x\|_p)} d^N y$$

$$= \sum_{j \in G_l} A_{i,j} (l) \varphi (j) + \int_{Q^N_p \setminus i + p^l \mathbb{Z}^N_p} \frac{\varphi (i) \left\{ \Omega (p^l \|y - i\|_p) - 1 \right\}}{w_\delta (\|y - x\|_p)} d^N y.$$
Now
\[
\int_{\Omega_p \setminus (i+p^\gamma Z_N^N)} \frac{\varphi(i) \Omega(p^\gamma \|y - i\|_p)}{w_\delta(\|y - x\|_p)} d^N y = \int_{\Omega_p \setminus p^\gamma Z_N^N} \frac{\varphi(i) \Omega(p^\gamma \|z\|_p)}{w_\delta(\|z + (i - x)\|_p)} d^N z
\]
\[
= -\varphi(i) \int_{\Omega_p \setminus p^\gamma Z_N^N} \frac{d^N z}{w_\delta(\|z\|_p)}.
\]

4. Energy functionals

4.1. Energy functionals in the coordinate space. For \( \varphi \in \mathcal{D}_R(\Omega_p^N) \), and \( \delta > N, \gamma > 0, \alpha_2 \geq 0 \), we define the energy functional:

\[
(4.1) \quad E_0(\varphi) := E_0(\varphi; \delta, \gamma, \alpha_2) = \frac{\gamma}{4} \iint_{\Omega_p^N \times \Omega_p^N} \frac{(\varphi(x) - \varphi(y))^2}{w_\delta(\|x - y\|_p)} d^N x d^N y + \frac{\alpha_2}{2} \int_{\Omega_p^N} \varphi^2(x) d^N x \geq 0.
\]

Then \( E_0 \) is a well-defined real-valued functional on \( \mathcal{D}_R \). Notice that \( E_0(\varphi) = 0 \) if and only if \( \varphi = 0 \). The restriction of \( E_0 \) to \( \mathcal{D}_R^l \) (denoted as \( E_0^{(l)} \)) provides a natural discretization of \( E_0 \).

Remark 3. The functional

\[
E_m^l(\varphi) := \int_{\Omega_p^N} \varphi^m(x) d^N x \quad \text{for } m \in \mathbb{N} \setminus \{0\}, \varphi \in \mathcal{D}_R^l,
\]

discretizes as

\[
E_m^l(\varphi) = p^{-lN} \sum_{i \in G_l} \varphi^m(i).
\]

Lemma 4.1. For \( \varphi \in \mathcal{D}_R^l \), the following formula holds true:

\[
E_0^{(l)}(\varphi) = p^{-lN} \left( \frac{\gamma}{2} d(l, w_\delta) + \frac{\alpha_2}{2} \right) \sum_{i \in G_l} \varphi^2(i) - \frac{\gamma}{2} p^{-lN} \sum_{i,j \in G_l} A_{i,j}(l) \varphi(i) \varphi(j).
\]

Proof. We set

\[
E_0'(\varphi) := \frac{\gamma}{4} \iint_{\Omega_p^N \times \Omega_p^N} \frac{\{\varphi(x) - \varphi(y)\}^2}{w_\delta(\|x - y\|_p)} d^N x d^N y
\]
\[
= \frac{\gamma}{4} \iint_{\Omega_p^N \times \Omega_p^N} \frac{\left\{ \sum_{i \in G_l} \varphi(i) \left[ \Omega(p^\gamma \|x - i\|_p) - \Omega(p^\gamma \|y - i\|_p) \right] \right\}^2}{w_\delta(\|x - y\|_p)} d^N x d^N y.
\]

Now, by using that for \( i \neq j \),

\[
\Omega(p^\gamma \|x - i\|_p) \Omega(p^\gamma \|y - j\|_p) = 1 \Rightarrow \Omega(p^\gamma \|x - j\|_p) \Omega(p^\gamma \|y - i\|_p) = 0,
\]

\[
\Rightarrow E_0'(\varphi) = \frac{\gamma}{4} \iint_{\Omega_p^N \times \Omega_p^N} \frac{\left\{ \sum_{i \in G_l} \varphi(i) \left[ \Omega(p^\gamma \|x - i\|_p) - \Omega(p^\gamma \|y - i\|_p) \right] \right\}^2}{w_\delta(\|x - y\|_p)} d^N x d^N y.
\]

\[
\Rightarrow E_0'(\varphi) = \frac{\gamma}{4} \iint_{\Omega_p^N \times \Omega_p^N} \frac{\left\{ \sum_{i \in G_l} \varphi(i) \left[ \Omega(p^\gamma \|x - i\|_p) - \Omega(p^\gamma \|y - i\|_p) \right] \right\}^2}{w_\delta(\|x - y\|_p)} d^N x d^N y.
\]
we get that
\[
\left\{ \sum_{i \in G_1} \varphi (i) \left[ \Omega \left( p^l \| x - i \|_p \right) - \Omega \left( p^l \| y - i \|_p \right) \right] \right\}^2 = \\
\sum_{i \in G_1} \varphi^2 (i) \left[ \Omega \left( p^l \| x - i \|_p \right) - \Omega \left( p^l \| y - i \|_p \right) \right] - \\
2 \sum_{i,j \in G_1, i \neq j} \varphi (i) \varphi (j) \Omega \left( p^l \| x - i \|_p \right) \Omega \left( p^l \| y - j \|_p \right).
\]

Therefore
\[
E'_0 (\varphi) = \frac{\gamma}{4} \sum_{i \in G_1} E^{(1)}_i (\varphi) - \frac{\gamma}{2} \sum_{i,j \in G_1, i \neq j} E^{(2)}_{i,j} (\varphi),
\]

where
\[
E^{(1)}_i (\varphi) := \varphi^2 (i) \int_{\mathbb{Q}_p^N \times \mathbb{Q}_p^N} \frac{\left[ \Omega \left( p^l \| x - i \|_p \right) - \Omega \left( p^l \| y - i \|_p \right) \right]^2}{w_{\delta} (\| x - y \|_p)} d^N x d^N y = \\
\varphi^2 (i) \int_{\| x \|_p > p^{-t}} \int_{\| y \|_p \leq p^{-t}} \frac{d^N x d^N y}{w_{\delta} (\| x - y \|_p)} + \varphi^2 (i) \int_{\| x \|_p \leq p^{-t}} \int_{\| y \|_p > p^{-t}} \frac{d^N x d^N y}{w_{\delta} (\| x - y \|_p)} - \\
2 \varphi^2 (i) \int_{\| x \|_p > p^{-t}} \int_{\| y \|_p \leq p^{-t}} \frac{d^N x d^N y}{w_{\delta} (\| x - y \|_p)} = 2p^{-2tN} \varphi (i) d (l, w_{\delta}).
\]

And for \( i, j \in G_1, \) with \( i \neq j, \)
\[
E^{(2)}_{i,j} (\varphi) := \varphi (i) \varphi (j) \int_{\mathbb{Q}_p^N \times \mathbb{Q}_p^N} \frac{\Omega \left( p^l \| x - i \|_p \right) \Omega \left( p^l \| y - j \|_p \right)}{w_{\delta} (\| x - y \|_p)} d^N x d^N y = \\
\frac{p^{-2tN}}{w_{\delta} (\| i - j \|_p)} \varphi (i) \varphi (j).
\]

Consequently,
\[
E'_0 (\varphi) = \frac{\gamma}{2} p^{-tN} d (l, w_{\delta}) \sum_{i \in G_1} \varphi^2 (i) - \frac{\gamma}{2} \sum_{i,j \in G_1, i \neq j} \frac{p^{-2tN}}{w_{\delta} (\| i - j \|_p)} \varphi (i) \varphi (j)
\]
\[
= \frac{\gamma}{2} p^{-tN} d (l, w_{\delta}) \sum_{i \in G_1} \varphi^2 (i) - \frac{\gamma}{2} p^{-tN} \sum_{i,j \in G_1} A_{i,j} (l) \varphi (i) \varphi (j).
\]

(4.2)

The announced formula follows from (4.2) by using Remark 3.

We now set \( U (l) := U = \sum_{i,j \in G_1} U_{i,j} (l), \) where
\[
U_{i,j} (l) := \left( \frac{\gamma}{2} d (l, w_{\delta}) + \frac{\alpha_2}{2} \right) \delta_{i,j} - \frac{\gamma}{2} A_{i,j} (l),
\]
where $\delta_{i,j}$ denotes the Kronecker delta. Notice that $U = \left( \frac{\gamma}{2} d (l, w_\delta) + \alpha_2 \right) I - \frac{\gamma}{2} A$ is the matrix of the operator

$$-rac{\gamma}{2} W_\delta + \frac{\alpha_2}{2}$$

acting on $\mathcal{D}_R^l$, in the basis $(3.7)$, cf. Lemma 3.1

**Lemma 4.2.** With the above notation the following formula holds true:

$$E_0^l (\varphi) = [\varphi (i)]_{i \in G_1}^T p^{-1N} U (l) [\varphi (i)]_{i \in G_1} = \sum_{i,j \in G_1} p^{-1N} U_{i,j} (l) \varphi (i) \varphi (j) \geq 0,$$

for $\varphi \in \mathcal{D}_R^l$, where $U$ is a symmetric, positive definite matrix. Consequently $p^{-1N} U (l)$ is a diagonalizable and invertible matrix.

$$-rac{\gamma}{2} W_\delta (\varphi) (x) + \left( \frac{\gamma}{2} d (l, w_\delta) + \alpha_2 \right) \varphi (x) = J (x).$$

### 4.2. The Fourier transform in $\mathcal{D}^l (\mathbb{Q}_p^N)$. We denote by $\mathcal{D}^l (\mathbb{Q}_p^N) := \mathcal{D}^l$ the $\mathbb{C}$-vector space of the test functions $\varphi \in \mathcal{D} (\mathbb{Q}_p^N)$ having the form: $\varphi (x) = \sum_{i \in G_1} \varphi (i) \Omega \left( p^l \| x - i \|_p \right)$, $\varphi (i) \in \mathbb{C}$. Alternatively, $\mathcal{D}^l$ the $\mathbb{C}$-vector space of the test functions $\varphi \in \mathcal{D} (\mathbb{Q}_p^N)$ satisfying:

1. $\text{supp } \varphi = B^l_N$;
2. for any $x \in B^l_N$, $\varphi |_{x+p^l \mathbb{Z}_p^N} = \varphi (x)$.

Then by using that $\mathcal{F}_{x \to \kappa} \left( \Omega \left( p^l \| x - i \|_p \right) \right) = p^{-lN} \chi_p (i \cdot \kappa) \Omega \left( p^{-l} \| \kappa \|_p \right)$, we get that

$$\hat{\varphi} (\kappa) = p^{-lN} \Omega \left( p^{-l} \| \kappa \|_p \right) \sum_{i \in G_1} \varphi (i) \chi_p (i \cdot \kappa).$$

By using the identity $\Omega \left( p^{-l} \| \kappa \|_p \right) = \sum_{j \in G_1} \Omega \left( p^l \| \kappa - j \|_p \right)$ in (4.4),

$$\hat{\varphi} (\kappa) = \sum_{j \in G_1} \left( p^{-lN} \sum_{i \in G_1} \varphi (i) \chi_p (i \cdot j) \right) \Omega \left( p^l \| \kappa - j \|_p \right)$$

$$=: \sum_{j \in G_1} \hat{\varphi} (j) \Omega \left( p^l \| \kappa - j \|_p \right).$$

Conversely,

$$\varphi (x) = \sum_{j \in G_1} \left( p^{-lN} \sum_{i \in G_1} \hat{\varphi} (i) \chi_p (-i \cdot j) \right) \Omega \left( p^l \| x - j \|_p \right)$$

$$= \sum_{j \in G_1} \varphi (j) \Omega \left( p^l \| x - j \|_p \right).$$

It follows from (4.5)-(4.6) that the Fourier transform is an automorphism of the $\mathbb{C}$-vector space $\mathcal{D}^l$. 
Remark 4. (i) For \( \varphi \in \mathcal{D}_R(\mathbb{Q}_p^N) \), \( \hat{\varphi}(-\kappa) \) and

\[
|\hat{\varphi} (\kappa)|^2 = \sum_{i \in G_i} |\hat{\varphi} (i)|^2 \Omega \left( p^l \| \kappa - i \|_p \right).
\]

(ii) The formulae

\[
\hat{\varphi} (j) = p^{-lN} \sum_{i \in G_i} \varphi (i) \chi_p (i \cdot j), \quad \varphi (j) = p^{-lN} \sum_{i \in G_i} \hat{\varphi} (i) \chi_p (-i \cdot j)
\]

give the discrete Fourier transform its inverse in the additive group \( G_i \).

4.3. Lizorkin spaces of second kind. The space

\[
\mathcal{L} := \mathcal{L}(\mathbb{Q}_p^N) = \left\{ \varphi \in \mathcal{D}(\mathbb{Q}_p^N); \int_{\mathbb{Q}_p^N} \varphi (x) d^N x = 0 \right\}
\]
is called the \( p \)-adic Lizorkin space of second kind. The real Lizorkin space of second kind is \( \mathcal{L}_\mathbb{R} := \mathcal{L}_\mathbb{R}(\mathbb{Q}_p^N) = \mathcal{L}(\mathbb{Q}_p^N) \cap \mathcal{D}(\mathbb{Q}_p^N) \). If

\[
\mathcal{F} \mathcal{L} := \mathcal{F} \mathcal{L}(\mathbb{Q}_p^N) = \left\{ \hat{\varphi} \in \mathcal{D}(\mathbb{Q}_p^N); \hat{\varphi} (0) = 0 \right\},
\]
then the Fourier transform gives rise to an isomorphism of \( \mathbb{C} \)-vector spaces from \( \mathcal{L} \) into \( \mathcal{F} \mathcal{L} \). The topological dual \( \mathcal{L}' := \mathcal{L}'(\mathbb{Q}_p^N) \) of the space \( \mathcal{L} \) is called the \( p \)-adic Lizorkin space of distributions of second kind. The real version is denoted as \( \mathcal{L}'_\mathbb{R} := \mathcal{L}'(\mathbb{Q}_p^N) \).

Let \( A(\partial) \) be a pseudo-differential operator defined as

\[
A(\partial) \varphi (x) = \mathcal{F}^{-1}_{\kappa \rightarrow x} (A(\| \kappa \|_p) \mathcal{F}_{x \rightarrow \kappa} \varphi), \quad \text{for } \varphi \in \mathcal{D}_\mathbb{R}(\mathbb{Q}_p^N),
\]

where \( A(\| \kappa \|_p) \) is a real-valued and radial function satisfying

\[
A(\| \kappa \|_p) = 0 \text{ if and only if } \kappa = 0.
\]

Then, the Lizorkin space \( \mathcal{L}_\mathbb{R} \) is invariant under \( A(\partial) \). For further details about Lizorkin spaces and pseudo-differential operators, the reader may consult [2, Sections 7.3, 9.2].

We now define for \( l \in \mathbb{N} \setminus \{0\} \),

\[
\mathcal{L}^l := \mathcal{L}^l(\mathbb{Q}_p^N) = \left\{ \varphi (x) = \sum_{i \in G_i} \varphi (i) \Omega \left( p^l \| x - i \|_p \right), \varphi (i) \in \mathbb{C}; p^{-lN} \sum_{i \in G_i} \varphi (i) = 0 \right\},
\]

resp. \( \mathcal{L}_\mathbb{R}^l := \mathcal{L}_\mathbb{R}^l(\mathbb{Q}_p^N) = \mathcal{L}^l \cap \mathcal{D}_\mathbb{R}^l \), and

\[
\mathcal{F} \mathcal{L}^l := \mathcal{F} \mathcal{L}^l(\mathbb{Q}_p^N) = \left\{ \hat{\varphi} (\kappa) = \sum_{i \in G_i} \hat{\varphi} (i) \Omega \left( p^l \| \kappa - i \|_p \right), \hat{\varphi} (i) \in \mathbb{C}; \hat{\varphi} (0) = 0 \right\},
\]

By the formulae [4.8], the Fourier transform \( \mathcal{F} : \mathcal{L}^l \rightarrow \mathcal{F} \mathcal{L}^l \) is an automorphism of \( \mathbb{C} \)-vector spaces. The multiplication by the function \( A(\| \kappa \|_p) \) gives rise to a linear transformation from \( \mathcal{L}^l \) onto itself. Consequently, \( A(\partial) : \mathcal{L}^l \rightarrow \mathcal{L}^l \) is a well-defined linear operator.
4.4. Energy functionals in the momenta space. By using (3.2)-(3.4), for \( \varphi \in \mathcal{D}_R \), we have

\[
\int \int_{\mathbb{Q}^N_p \times \mathbb{Q}^N_p} \left\{ \varphi(x) - \varphi(y) \right\}^2 \, d^N x d^N y = 2 \int_{\mathbb{Q}^N_p} \varphi(x) (-W_\delta) \varphi(x) \, d^N x.
\]

Then

\[
E_0(\varphi) = \frac{\gamma}{2} \int_{\mathbb{Q}^N_p} \varphi(x) (-W_\delta) \varphi(x) \, d^N x + \frac{\alpha_2}{2} \int_{\mathbb{Q}^N_p} \varphi^2(x) \, d^N x
\]

\[
= \frac{\gamma}{2} \int_{\mathbb{Q}^N_p} \varphi(x) W(\partial, \delta) \varphi(x) \, d^N x + \frac{\alpha_2}{2} \int_{\mathbb{Q}^N_p} \varphi^2(x) \, d^N x
\]

\[
= \frac{\gamma}{2} \int_{\mathbb{Q}^N_p} A_{w_3}(\|\kappa\|_p) |\hat{\varphi}(\kappa)|^2 \, d^N \kappa + \frac{\alpha_2}{2} \int_{\mathbb{Q}^N_p} |\hat{\varphi}(\kappa)|^2 \, d^N \kappa
\]

\[
= \int_{\mathbb{Q}^N_N} \left( \frac{\gamma}{2} A_{w_3}(\|\kappa\|_p) + \frac{\alpha_2}{2} \right) |\hat{\varphi}(\kappa)|^2 \, d^N \kappa.
\]

Now, for \( \varphi \in \mathcal{D}_R^l \) by using (4.7), we have

\[
E_0(\varphi) = p^{-1N} \sum_{j \in G_1 - \{0\}} \left( \frac{\gamma}{2} A_{w_3}(\|j\|_p) + \frac{\alpha_2}{2} \right) |\hat{\varphi}(j)|^2
\]

\[
+ |\hat{\varphi}(0)|^2 \left\{ \int_{p^2\mathbb{Q}^N_N} \left( \frac{\gamma}{2} A_{w_3}(\|z\|_p) + \frac{\alpha_2}{2} \right) d^N z \right\},
\]

where \( \hat{\varphi}(j) = \hat{\varphi}_1(j) + \sqrt{-1} \hat{\varphi}_2(j) \in \mathbb{C} \). Here we use the alternative notation \( \hat{\varphi}_1(j) = \text{Re}(\hat{\varphi}(j)), \hat{\varphi}_2(j) = \text{Im}(\hat{\varphi}(j)) \) which more convenient for us.

Remark 5. Notice that

\[
\mathcal{F}\mathcal{L}^l_R = \left\{ \hat{\varphi}(\kappa) = \sum_{i \in G_1} \hat{\varphi}(i) \Omega \left( p^j \|\kappa - i\|_p \right); \hat{\varphi}(i) \in \mathbb{C}; \hat{\varphi}(0) = 0, \bar{\varphi}(\kappa) = \hat{\varphi}(-\kappa) \right\},
\]

and that the condition \( \bar{\varphi}(\kappa) = \hat{\varphi}(-\kappa) \) implies that \( \hat{\varphi}_1(-i) = \hat{\varphi}_1(i) \) and \( \hat{\varphi}_2(-i) = -\hat{\varphi}_2(i) \) for any \( i \in G_1 \). This implies that \( \mathcal{F}\mathcal{L}^l_R \) is \( \mathbb{R} \)-vector space of dimension \( \#G_1 - 1 \).

Remark 6. We set \( G_1 \setminus \{0\} := G_1^+ \cup G_1^- \), where the subsets \( G_1^+, G_1^- \) satisfy that

\[
G_1^+ \to G_1^-,
\]

\[
i \to -i
\]

is a bijection. We recall here that \( G_1 \) is a finite additive group. Since \( \#G_1^+ = \#G_1^- \) necessarily \( \#(G_1 \setminus \{0\}) = p^{2N_1} - 1 \) is even, and thus \( p \geq 3 \). Then any function from \( \mathcal{F}\mathcal{L}^l_R \) can be uniquely represented as

\[
\hat{\varphi}(\kappa) = \sum_{i \in G_1^+} \hat{\varphi}_1(i) \Omega_+ \left( p^j \|\kappa - i\|_p \right) + \hat{\varphi}_2(i) \Omega_+ \left( p^j \|\kappa + i\|_p \right),
\]

where

\[
\Omega_+ \left( p^j \|\kappa - i\|_p \right) := \Omega \left( p^j \|\kappa - i\|_p \right) + \Omega \left( p^j \|\kappa + i\|_p \right),
\]

\[
\Omega_+ \left( p^j \|\kappa + i\|_p \right) := \Omega \left( p^j \|\kappa + i\|_p \right) + \Omega \left( p^j \|\kappa - i\|_p \right).
\]
and
\[
\Omega_\pm \left( p^i \| \kappa - i \|_p \right) := \sqrt{-1} \left\{ \Omega \left( p^i \| \kappa - i \|_p \right) - \Omega \left( p^i \| \kappa + i \|_p \right) \right\}.
\]

We take \( \varphi \in L^1_{\mathbb{R}} \), then \( \hat{\varphi}(0) = 0 \), and
\[
E_0^{(l)}(\varphi) = p^{-1N} \sum_{j \in G_i \setminus \{0\}} \left( \frac{\gamma}{2} A_{w^i} (\| j \|_p) + \frac{\alpha_2}{2} \right) (\hat{\varphi}_1^2 (j) + \hat{\varphi}_2^2 (j))
\]

\[
= 2p^{-1N} \sum_{r \in \{1,2\}} \sum_{j \in G_i^+} \left( \frac{\gamma}{2} A_{w^i} (\| j \|_p) + \frac{\alpha_2}{2} \right) \hat{\varphi}_r^2 (j).
\]

By using that \( L^1_{\mathbb{R}} \simeq \mathcal{F}L^1_{\mathbb{R}} \) we get that \( E_0^{(l)} \) is a real-valued functional defined on \( \mathcal{F}L^1_{\mathbb{R}} \simeq \mathbb{R}^{(#G_i^+ - 1)} \).

We now define the diagonal matrix \( B^{(r)} = \left[ B^{(r)}_{i,j} \right]_{i,j \in G_i^+} \), \( r = 1, 2 \), where
\[
B^{(r)}_{i,j} := \begin{cases} \frac{\gamma}{2} A_{w^i} (\| j \|_p) + \frac{\alpha_2}{2} & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}
\]

Notice that \( B^{(1)}_{i,j} = B^{(2)}_{i,j} \). We set
\[
B(l) := B(l, \delta, \gamma, \alpha_2) = \left[ \begin{array}{c} B^{(1)}(l) \\ 0 \end{array} \right]
\]

The matrix \( B = [B_{i,j}] \) is a diagonal of size \( 2 \left( #G_i^+ \right) \times 2 \left( #G_i^+ \right) \). In addition, the indices \( i, j \) run through two disjoint copies of \( G_i^+ \). Then we have the following result:

**Lemma 4.3.** Assume that \( \alpha_2 \geq 0 \). With the above notation the following formula holds true:
\[
E_0^{(l)}(\varphi) := E_0^{(l)}(\hat{\varphi}_1(j), \hat{\varphi}_2(j); j \in G_i^+) = \left[ \begin{array}{c} \hat{\varphi}_1(j) \end{array} \right]_{j \in G_i^+}^T \left[ \begin{array}{c} \hat{\varphi}_1(j) \end{array} \right]_{j \in G_i^+} \geq 0,
\]

for \( \varphi \in L^1_{\mathbb{R}} \simeq \mathcal{F}L^1_{\mathbb{R}} \simeq \mathbb{R}^{(#G_i^+ - 1)} \), where \( 2p^{-1N} B(l) \) is a diagonal, positive definite, invertible matrix.

5. **Gaussian Measures**

We recall that we are taking \( \delta > N, \gamma > 0, \alpha_2 \geq 0 \). We define the partition function attached to the energy functional \( E_0 \) as
\[
\mathcal{Z}(\delta, \gamma, \alpha_2) = \int_{\mathcal{F}L_{\mathbb{R}}(\mathbb{Q}_p^N)} D(\varphi) e^{-E_0(\varphi)},
\]

where \( D(\varphi) \) is a “spurious measure” on \( \mathcal{F}L_{\mathbb{R}}(\mathbb{Q}_p^N) \). For the sake of simplicity we use the notation \( \mathcal{Z} = \mathcal{Z}(\delta, \gamma, \alpha_2) \) wherever possible.
As the discrete version of $\mathcal{Z}(\delta, \gamma, \alpha_2)$ we take

$$
\mathcal{Z}^{(l)}(\delta, \gamma, \alpha_2) := \int_{\mathcal{F}\mathcal{L}_K^l(\mathbb{Q}_N^l)} D_l(\varphi) e^{-E_0(\varphi)}.
$$

We also use the notation $\mathcal{Z}^{(l)} = \mathcal{Z}^{(l)}(\delta, \gamma, \alpha_2)$. Now we define $\mathcal{Z}^{(l)}(\delta, \gamma, \alpha_2)$ as

$$
\mathcal{Z}^{(l)}(\delta, \gamma, \alpha_2) = 
\int_{\mathbb{R}^{(p^{2lN-1})}} \exp\left( - \left[ \frac{\hat{\varphi}_1(j)}{\hat{\varphi}_2(j)} \right] \right)^T \left[ \begin{array}{cc} 2p^{-lN}B(l) & \left[ \frac{\hat{\varphi}_1(j)}{\hat{\varphi}_2(j)} \right] \end{array} \right] \prod_{i \in I_{G_i^+}} d\hat{\varphi}_1(i) d\hat{\varphi}_2(i),
$$

where $\prod_{i \in I_{G_i^+}} d\hat{\varphi}_1(i) d\hat{\varphi}_2(i)$ is the Lebesgue measure on $\mathbb{R}^{(p^{2lN-1})}$.

The integral $\mathcal{Z}^{(l)}$ is the natural discretization of $\mathcal{Z}$. From a classical point of view, one should expect that $\mathcal{Z} = \lim_{l \to \infty} \mathcal{Z}^{(l)}$ in some sense. The goal of this section is to study these matters in a rigorous mathematical way. Our main result is the construction of rigorous mathematical version of the spurious measure $D(\varphi)$.

By Lemma 4.3 $\mathcal{Z}^{(l)}$ is a Gaussian integral, then

$$
\mathcal{Z}^{(l)} = \frac{(2\pi)^{p^{2lN-1}/2}}{\sqrt{\det 4p^{-lN}B(l)}} = \left( \frac{\pi}{2} \right)^{p^{2lN-1}/2} \frac{p^{-lN(p^{2lN-1})}}{\sqrt{\det B}}.
$$

**Definition 1.** We define the following family of Gaussian measures:

$$
d \mathbb{P}_l \left( \left[ \frac{\hat{\varphi}_1(j)}{\hat{\varphi}_2(j)} \right]_{j \in G_i^+} ; \delta, \gamma, \alpha_2 \right) := d \mathbb{P}_l \left( \left[ \frac{\hat{\varphi}_1(j)}{\hat{\varphi}_2(j)} \right]_{j \in G_i^+} \right)
$$

in $\mathcal{F}\mathcal{L}_R^l \simeq \mathbb{R}^{(p^{2lN-1})}$, for $l \in \mathbb{N} \setminus \{0\}$.

Thus for any Borel subset $A$ of $\mathbb{R}^{(p^{2lN-1})} \simeq \mathcal{F}\mathcal{L}_R^l$ and any continuous and bounded function $f : \mathcal{F}\mathcal{L}_R^l \to \mathbb{R}$ the integral

$$
\int_A \left[ \left[ \frac{\hat{\varphi}_1(j)}{\hat{\varphi}_2(j)} \right]_{j \in G_i^+} \right] d \mathbb{P}_l \left( \left[ \frac{\hat{\varphi}_1(j)}{\hat{\varphi}_2(j)} \right]_{j \in G_i^+} \right) =: \int_A f(\hat{\varphi}) d \mathbb{P}_l(\hat{\varphi})
$$

is well-defined.

We define $\mathcal{I} = \cup_{l_0 \in \mathbb{N} \setminus \{0\}} G_i^{l_0}$. Then $\mathcal{I}$ is a countable set. Given a finite subset $J$ of $\mathcal{I}$ there is $l_0 \in \mathbb{N} \setminus \{0\}$ such that $G_i^{l_0}$ is the smallest set of the form $G_i^+$ containing $J$. To each finite subset $J$ of $\mathcal{I}$ we attach a collection of Gaussian random variables

$$
\left[ \left[ \frac{\hat{\varphi}_1(j)}{\hat{\varphi}_2(j)} \right]_{j \in G_i^{l_0}} \right]_{\hat{\varphi}_1(j) = \hat{\varphi}_2(j) = 0, j \in G_i^{l_0} \setminus J}
$$
having joint probability distribution

\[
P_J = P_{l_0} \left( \begin{bmatrix} \hat{\varphi}_1(j)_{j \in G_l^0} \\ \hat{\varphi}_2(j)_{j \in G_l^0} \end{bmatrix} \right) \bigg| \hat{\varphi}_1(j) = \hat{\varphi}_1(0)_{j \in G_l^0 \setminus J}
\]

Notice that \( P_{G_l^0} \left( \begin{bmatrix} \hat{\varphi}_1(j)_{j \in G_l^0} \\ \hat{\varphi}_2(j)_{j \in G_l^0} \end{bmatrix} \right) = P_l \left( \begin{bmatrix} \hat{\varphi}_1(j)_{j \in G_l^0} \\ \hat{\varphi}_2(j)_{j \in G_l^0} \end{bmatrix} \right) \). The family of Gaussian measures \( \left\{ P_J \left( \begin{bmatrix} \hat{\varphi}_1(j)_{j \in G_l^0} \\ \hat{\varphi}_2(j)_{j \in G_l^0} \end{bmatrix} \right) ; J \subset I \right\} \) is consistent, i.e. \( P_J(A) = P_K(A \times R^{#K - #J}) \), for \( J \subset K \), see e.g. [43, Chapter IV, Section 3.1, Lemma 1]. We now apply Kolmogorov’s consistency theorem and its proof, see e.g. [43, Theorem 2.1], to obtain the following result:

**Lemma 5.1.** There exists a probability measure space \((X, \mathcal{F}, P)\) and random variables

\[
\begin{bmatrix} \hat{\varphi}_1(j)_{j \in G_l^0} \\ \hat{\varphi}_2(j)_{j \in G_l^0} \end{bmatrix}, \text{ for } l \in \mathbb{N} \setminus \{0\},
\]

such that \( P_l \) is the joint probability distribution of \( \begin{bmatrix} \hat{\varphi}_1(j)_{j \in G_l^0} \\ \hat{\varphi}_2(j)_{j \in G_l^0} \end{bmatrix} \). The space \((X, \mathcal{F}, P)\) is unique up to isomorphisms of probability measure spaces. Furthermore, for any bounded continuous function \( f \) supported in \( \mathcal{F} L^1_{R_h} \), we have

\[
\int_{\mathcal{F} L^1_{R_h}} f(\varphi) dP_1(\varphi) = \int_{\mathcal{F} L^1_{R_h}} f(\varphi) dP(\varphi).
\]

5.1. **A quick detour into the \( p \)-adic noise calculus.** In this section we introduce a Gelfand triple and construct some Gaussian measures in the non-Archimedean setting.

5.1.1. **A bilinear form in \( D_{R_{p}}(Q_{p}^{N}) \).** For \( \delta > N, \gamma, \alpha_2 > 0 \), we define the operator

\[
D \left( \mathbb{Q}_p^N \right) \rightarrow L^2 \left( \mathbb{Q}_p^N \right)
\]

\[
\varphi \rightarrow \left( \frac{1}{2} W(\partial, \delta) + \frac{\alpha_2}{2} \right)^{-1} \varphi,
\]

where \( \left( \frac{1}{2} W(\partial, \delta) + \frac{\alpha_2}{2} \right)^{-1} \varphi := \mathcal{F}_{\kappa \rightarrow x}^{-1} \left( \frac{\mathcal{F}_{\kappa \rightarrow x} \varphi}{\frac{1}{2} A_{w}((||\kappa||_p) + \frac{\alpha_2}{2})} \right). \)

We define the distribution

\[
G(x) := G(x; \delta, \gamma, \alpha_2) = \mathcal{F}_{\kappa \rightarrow x}^{-1} \left( \frac{1}{\frac{1}{2} A_{w}((||\kappa||_p) + \frac{\alpha_2}{2})} \right) \in D'(Q_{p}^{N}).
\]

By using the fact that \( \frac{1}{\frac{1}{2} A_{w}((||\kappa||_p) + \frac{\alpha_2}{2})} \) is radial and \( (\mathcal{F}(\mathcal{F} \varphi))(\kappa) = \varphi(-\kappa) \) one verifies that

\[
G(x) \in D'_{R_{p}}(Q_{p}^{N}).
\]
Now we define the following bilinear form $\mathbb{B} := \mathbb{B}(\delta, \gamma, \alpha_2)$:

$$\mathbb{B} : \mathcal{D}_R (\mathbb{Q}_p^N) \times \mathcal{D}_R (\mathbb{Q}_p^N) \rightarrow \mathbb{R}$$

$$(\varphi, \theta) \rightarrow \left\langle \varphi, \left(\frac{\gamma}{2} W (\vartheta, \delta) + \frac{\alpha_2}{2}\right)^{-1} \theta \right\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2 (\mathbb{Q}_p^N)$.

**Lemma 5.2.** $\mathbb{B}$ is a positive, continuous bilinear form from $\mathcal{D}_R (\mathbb{Q}_p^N) \times \mathcal{D}_R (\mathbb{Q}_p^N)$ into $\mathbb{R}$.

**Proof.** We first notice that for $\varphi \in \mathcal{D}_R (\mathbb{Q}_p^N)$, we have

$$\mathbb{B}(\varphi, \varphi) = \int_{\mathbb{Q}_p^N} \frac{|\hat{\varphi}(\kappa)|^2 d^N \kappa}{|2 A_{w_3} (\|\kappa\|_p) + \frac{\alpha_2}{2}|} \geq 0.$$ 

Then $\mathbb{B}(\varphi, \varphi) = 0$ implies that $\varphi$ is zero almost everywhere. Since $\varphi$ is a locally constant function, $\mathbb{B}(\varphi, \varphi) = 0$ if and only if $\varphi = 0$.

For $(\varphi, \theta) \in \mathcal{D}_R (\mathbb{Q}_p^N) \times \mathcal{D}_R (\mathbb{Q}_p^N)$, the Cauchy-Schwarz inequality implies that

$$|\mathbb{B}(\varphi, \theta)| \leq \|\varphi\|_2 \left( \int_{\mathbb{Q}_p^N} \frac{|\hat{\varphi}(\kappa)|^2 d^N \kappa}{|2 A_{w_3} (\|\kappa\|_p) + \frac{\alpha_2}{2}|^2} \right)^{\frac{1}{2}} \leq \frac{2}{\alpha_2} \|\varphi\|_2 \|\theta\|_2.$$ 

Now take two sequences in $\mathcal{D}_R (\mathbb{Q}_p^N)$ such that $\varphi_n, \mathcal{D}_R \varphi$ and $\theta_n, \mathcal{D}_R \theta$ with $\varphi, \theta \in \mathcal{D}_R (\mathbb{Q}_p^N)$. We recall that the convergence of these sequences means that there is an positive integer $l$ such that $\varphi_n, \varphi, \theta_n, \theta \in \mathcal{D}_R^l$, and

$$\varphi_n - \varphi \underset{\text{unif.}}{\longrightarrow} 0 \text{ and } \theta_n - \theta \underset{\text{unif.}}{\longrightarrow} 0 \text{ in } p^{-l} \mathbb{Q}_p^N.$$ 

Then

$$\varphi_n (x) - \varphi (x) = \sum_{\mathbf{i} \in G_l} (\varphi_n (\mathbf{i}) - \varphi (\mathbf{i})) \Omega \left(p^l \|x - \mathbf{i}\|_p\right),$$

and

$$\theta_n (x) - \theta (x) = \sum_{\mathbf{i} \in G_l} (\theta_n (\mathbf{i}) - \theta (\mathbf{i})) \Omega \left(p^l \|x - \mathbf{i}\|_p\right)$$

and by (5.2),

$$|\mathbb{B}(\varphi_n - \varphi, \theta_n - \theta)| \leq \frac{2}{\alpha_2} \|\varphi_n - \varphi\|_2 \|\theta_n - \theta\|_2$$

$$\leq \frac{2p^{-lN}}{\alpha_2} \sqrt{\sum_{\mathbf{i} \in G_l} |\varphi_n (\mathbf{i}) - \varphi (\mathbf{i})|^2} \sqrt{\sum_{\mathbf{i} \in G_l} |\theta_n (\mathbf{i}) - \theta (\mathbf{i})|^2}$$

$$\leq \frac{2p^{-lN} \#G_l}{\alpha_2} \left(\max_{\mathbf{i} \in G_l} |\varphi_n (\mathbf{i}) - \varphi (\mathbf{i})| \right) \left(\max_{\mathbf{i} \in G_l} |\theta_n (\mathbf{i}) - \theta (\mathbf{i})| \right) \rightarrow 0$$

as $n \rightarrow \infty$. This fact implies the continuity of $\mathbb{B}$ in $\mathcal{D}_R (\mathbb{Q}_p^N) \times \mathcal{D}_R (\mathbb{Q}_p^N)$. \hfill \(\square\)

In the next sections we only use the restriction of $\mathbb{B}$ to $\mathcal{L}_R (\mathbb{Q}_p^N) \times \mathcal{L}_R (\mathbb{Q}_p^N)$. 

Lemma 5.3. For \( \varphi \in \mathcal{L}_R^l \simeq \mathcal{F} \mathcal{L}_R^l \),
\[
\mathbb{B}_l(\varphi, \varphi) := \mathbb{B}(\varphi, \varphi) = \left[ \begin{array}{c} \hat{\varphi}_1(j) \\ \hat{\varphi}_2(j) \end{array} \right]_{j \in G_i^+}^T 2p^{-lN} B^{-1}(l) \left[ \begin{array}{c} \hat{\varphi}_1(j) \\ \hat{\varphi}_2(j) \end{array} \right]_{j \in G_i^+},
\]
where \( B(l) \) is the matrix defined in (4.9).

Proof. The proof is similar to the proof of Lemma 4.3. We first notice that
\[
(5.3)
\]
By using (4.7), we get that
\[
\mathbb{B}(\varphi, \varphi) = \int_{Q^N_p} \frac{|\hat{\varphi}(\kappa)|^2}{\frac{\alpha}{2}} dN \kappa.
\]
By using (4.7), we get that
\[
(5.3)
\]
Now, the announced formula follows from (5.3).

Given a finite dimensional subspace \( \mathcal{Y} \subset \mathcal{L}_R(Q_p^N) \), we denote by \( \mathbb{B}_\mathcal{Y} \) the restriction of \( \mathbb{B} \) to \( \mathcal{Y} \times \mathcal{Y} \). In the case \( \mathcal{Y} = \mathcal{L}_R^l \), we use the notation \( \mathbb{B}_l \), which agrees with the notation introduced in Lemma 5.3.

Lemma 5.4. Given finite dimensional subspace \( \mathcal{Y} \subset \mathcal{L}_R(Q_p^N) \), there is a positive integer \( l = l(\mathcal{Y}) \) such that \( \mathcal{Y} \subset \mathcal{L}_R^l \simeq \mathcal{F} \mathcal{L}_R^l \), and there is a subset \( \mathcal{J} = \mathcal{J}(\mathcal{Y}) \subset G_i^+ \) such that
\[
(5.4)
\]
Furthermore,
\[
(5.5)
\]

Proof. Since \( \mathcal{L}_R = \bigcup_{l=1}^\infty \mathcal{L}_R^l \) and \( \mathcal{L}_R^l \subset \mathcal{L}_R^m \) for \( m > l \), there is is a positive integer \( l = l(\mathcal{Y}) \) such that \( \mathcal{Y} \subset \mathcal{L}_R^l \). Then there is a subset \( \mathcal{J} \subset G_i^+ \) such that \( \{ \Omega_\pm \left( p^l \| x - i \|_p \right) \}_{i \in \mathcal{J}} \) is a basis of \( \mathcal{Y} \), and so the formula (5.4) holds. The assertion (5.5) follows from (5.3).

Corollary 5.1. The collection \( \{ \mathbb{B}_\mathcal{Y} ; \mathcal{Y} \text{ finite dimensional subspace of } \mathcal{L}_R \} \) is completely determined by the collection \( \{ \mathbb{B}_l ; l \in \mathbb{N} \setminus \{0\} \} \). In the sense that given any \( \mathbb{B}_\mathcal{Y} \), there is an integer \( l \) and a subset \( \mathcal{J} \subset G_i^+ \), the case \( \mathcal{J} = \emptyset \) is included, such that \( \mathbb{B}_\mathcal{Y} = \mathbb{B}_l \restriction \{ \hat{\varphi}_1(j) = 0, \hat{\varphi}_2(j) = 0 ; j \notin \mathcal{J} \} \).

5.1.2. Gaussian measures in the non-Archimedean framework. We recall that \( \mathcal{D}(Q_p^N) \) is a nuclear space, cf. [11] Section 4, and thus \( \mathcal{L}_R(Q_p^N) \) is a nuclear space, since any subspace of a nuclear space is also nuclear, see e.g. [16] Proposition 50.1.

The spaces
\[
\mathcal{L}_R(Q_p^N) \hookrightarrow \mathcal{L}_R^2(Q_p^N) \hookrightarrow \mathcal{L}_R^l(Q_p^N)
\]
form a Gel’fand triple, that is, \( \mathcal{L}_R(Q_p^N) \) is a nuclear space which is densely and continuously embedded in \( \mathcal{L}_R \) (see [2] Theorem 7.4.3]) and \( \| g \|_2^2 = \langle g, g \rangle \) for \( g \in \mathcal{L}_R(Q_p^N) \).
We denote by $\mathcal{B} := \mathcal{B}(\mathcal{L}_R^c (Q_p^N))$ the $\sigma$-algebra generated by the cylinder subsets of $\mathcal{L}_R^c (Q_p^N)$. The mapping
\[
\mathcal{C} : \mathcal{L}_R (Q_p^N) \rightarrow \mathbb{C} \\
f \rightarrow e^{-\frac{1}{2} \mathcal{B}(f,f)}
\]
defines a characteristic functional, i.e. $\mathcal{C}$ is continuous, positive definite and $\mathcal{C}(0) = 1$. The continuity follows from Lemma 5.2. The fact that $\mathcal{B}$ defines an inner product in $L^2 (Q_p^N)$ implies that the functional $\mathcal{C}$ is positive definite.

**Definition 2.** By the Bochner-Minlos theorem, see e.g. [6], [23], [24], there exists a unique probability measure $\mathbb{P} := \mathbb{P}(\delta, \gamma, \alpha_2)$ called the canonical Gaussian measure on $(\mathcal{L}_R (Q_p^N), \mathcal{B})$, given by its characteristic functional as
\[
\int_{\mathcal{L}_R^c (Q_p^N)} e^{\sqrt{-1} (W,f)} d\mathbb{P}(W) = e^{-\frac{1}{2} \mathcal{B}(f,f)}, \quad f \in \mathcal{L}_R (Q_p^N).
\]

We set $(L_{\rho}^\rho) := L^\rho (\mathcal{L}_R^c (Q_p^N), \mathbb{P})$, $\rho \in [1, \infty)$, to denote the real vector space of measurable functions $\Psi : \mathcal{L}_R^c (Q_p^N) \rightarrow \mathbb{R}$ satisfying
\[
\|\Psi\|_{(L_{\rho}^\rho)} = \int_{\mathcal{L}_R^c (Q_p^N)} |\Psi(W)|^\rho d\mathbb{P}(W) < \infty.
\]

5.1.3. **Further remarks on the cylinder measure $\mathbb{P}$.** We set $\mathbb{L} (\varphi) = \exp \frac{-1}{2} \mathcal{B}(\varphi, \varphi)$, for $\varphi \in \mathcal{L}_R$. The functional $\mathbb{L}$ is positive definite, continuous and $\mathbb{L}(0) = 1$. By taking the restriction of $\mathbb{L}$ to a finite dimensional subspace $\mathcal{Y}$ of $\mathcal{L}_R$, one obtains a positive definite, continuous functional $\mathbb{L}_Y(\varphi)$ on $\mathcal{Y}$. By the Bochner theorem, see e.g. [17] Chapter II, Section 3.2, this function is the Fourier transform of a probability measure $\mathbb{P}_Y$ defined in the dual space $\mathcal{Y}' \subset \mathcal{L}_R'$ of $\mathcal{Y}$. By identifying $\mathcal{Y}'$ with $\mathcal{L}_R^c (Q_p^N)/\mathcal{Y}^0$, where $\mathcal{Y}^0$ consists of all linear functionals $T$ which vanish on $\mathcal{Y}$, we get that $\mathbb{P}_Y$ is a probability measure in the finite dimensional space $\mathcal{L}_R^c (Q_p^N)/\mathcal{Y}^0$. The measure $\mathbb{P}$ is constructed from the family of probability measures $\{\mathbb{P}_Y; \mathcal{Y} \subset \mathcal{L}_R$, finite dimensional space $\}$. These measures are compatible and satisfy a suitable continuity condition, and they give rise to a cylinder measure $\mathbb{P}$ in $\mathcal{L}_R^c$. Since $\mathcal{L}_R$ is a nuclear space, this cylinder measure is countably additive. For further details about the construction of the measure $\mathbb{P}$, the reader may consult [17] Chapter IV, Section 4.2, proof of Theorem 1.

Now, by using the formula
\[
\mathbb{L} (\varphi) = \int_{\mathcal{L}_R^c (Q_p^N)/\mathcal{Y}^0} e^{\sqrt{-1} (W,\varphi)} d\mathbb{P}_Y (\varphi) \text{ for } \varphi \in \mathcal{Y},
\]
see [17] Chapter IV, Section 4.1, and the fact that $\mathbb{L} (\varphi) = \exp \frac{-1}{2} \mathcal{B}(\varphi, \varphi)$, for $\varphi \in \mathcal{Y}$, one gets that $\mathbb{P}_Y$ is a Gaussian probability measure in $\mathcal{Y}$, with mean zero, and correlation function $\mathcal{B}$, i.e. if $\mathcal{Y}$ has dimension $n$, then
\[
\mathbb{P}_Y (A) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_A e^{-\frac{1}{2} \mathcal{B}(\psi, \psi)} d\psi,
\]
where \(d\psi\) is the Lebesgue measure in \(\mathcal{Y}\) corresponding to the scalar product \(\mathbb{B}\), and \(A \subset \mathcal{Y}\) is a measurable subset. In conclusion, the cylinder measure \(P\) is uniquely determined by the family of Gaussian measures

\[
\{P_\gamma; \mathcal{Y} \subset \mathcal{L}_R, \text{ finite dimensional space}\},
\]

or equivalently by the sequence

\[
(5.7) \quad \{B_\gamma; \mathcal{Y} \subset \mathcal{L}_R, \text{ finite dimensional space}\},
\]

where \(B_\gamma\) denotes the restriction of the scalar product to \(\mathbb{B}\) to \(\mathcal{Y}\). This is a consequence of the fact that any finite dimensional Gaussian measure, with mean zero, is completely determined by its correlation matrix.

5.2. Existence of a measure on \(\mathcal{F}\mathcal{L}_R^l(\mathbb{Q}_p^N)\).

**Theorem 5.1.** Assume that \(\delta > N, \gamma > 0, \alpha_2 > 0\). (i) The cylinder probability measure \(P = P(\delta, \gamma, \alpha_2)\) is uniquely determined by the sequence \(P_l = P_l(\delta, \gamma, \alpha_2), l \in \mathbb{N} \setminus \{0\}\), of Gaussian measures. (ii) Let \(f : \mathcal{F}\mathcal{L}_R^l(\mathbb{Q}_p^N) \to \mathbb{R}\) be a continuous and bounded function. Then

\[
\lim_{l \to \infty} \int_{\mathcal{F}\mathcal{L}_R^l(\mathbb{Q}_p^N)} f(\varphi) dP_l(\varphi) = \int_{\mathcal{F}\mathcal{L}_R(\mathbb{Q}_p^N)} f(\varphi) dP(\varphi).
\]

**Proof.** (i) We use the notation and results given in Section 5.1.3. By the Corollary 5.1, the sequence \(5.7\) is completely determined by the sequence \(\{P_l; l \in \mathbb{N} \setminus \{0\}\}\), i.e. by the sequence \(\{P_l; l \in \mathbb{N} \setminus \{0\}\}\). Notice that the covariance matrix of \(P_l\) is \(2^{-lN}B^{-1}(l) = 2^{-lN}B_l\), cf. Lemma 5.3. Then the cylinder measure \(P\) is exactly the probability measure announced in Lemma 5.1.

(ii) By using the formula given in Lemma 5.1, for any bounded continuous function \(f\) supported in \(\mathcal{F}\mathcal{L}_R^l\), we have

\[
(5.8) \quad \int_{\mathcal{F}\mathcal{L}_R^l(\mathbb{Q}_p^N)} f(\varphi) dP_l(\varphi) = \int_{\mathcal{F}\mathcal{L}_R(\mathbb{Q}_p^N)} f(\varphi) dP(\varphi).
\]

By the uniqueness of the probability space \((X, \mathcal{F}; P)\) in Lemma 5.1, we can identify the \(\sigma\)-algebra \(\mathcal{F}\) with \(\mathcal{B}(\mathcal{L}_R^l(\mathbb{Q}_p^N))\), the \(\sigma\)-algebra generated by the cylinder subsets of \(\mathcal{L}_R^l(\mathbb{Q}_p^N)\). Then \(\mathcal{F}\mathcal{L}_R^l\) belongs to \(\mathcal{B}(\mathcal{L}_R^l(\mathbb{Q}_p^N))\), and \(\mathcal{F}\mathcal{L}_R = \cup_l \mathcal{F}\mathcal{L}_R^l\) also belongs to \(\mathcal{B}(\mathcal{L}_R(\mathbb{Q}_p^N))\). Now by taking the limit \(l \to \infty\) in \(5.8\), we get the announced formula. \(\square\)

5.3. Further comments on Theorem 5.1. By using the Gel’fand triple,

\[
\mathcal{D}_R(\mathbb{Q}_p^N) \hookrightarrow \mathcal{L}_R^2(\mathbb{Q}_p^N) \hookrightarrow \mathcal{D}_R'(\mathbb{Q}_p^N),
\]

and the fact that \(\mathcal{D}(\mathbb{Q}_p^N)\) is a nuclear space, cf. [11, Section 4], it follows from Lemma 5.2 that

\[
\mathcal{C} : \mathcal{D}_R(\mathbb{Q}_p^N) \to \mathbb{C}, \quad f \to e^{-\mathbb{B}(f,f)}
\]
defines a characteristic functional, then by the Bochner-Minlos theorem, there exists a unique probability measure \( S := S(\delta, \gamma, \alpha_2) \) on \( (\mathcal{D}_R(Q_{p}^{N}), \mathcal{B}_0) \) given by

\[
\int_{\mathcal{D}_R(Q_{p}^{N})} e^{\sqrt{-1}(W, f)} dS(W) = e^{-\frac{1}{2}B(f, f)}, \quad f \in \mathcal{D}_R(Q_{p}^{N}),
\]

where \( \mathcal{B}_0 := \mathcal{B}_0(\mathcal{D}_R(Q_{p}^{N})) \) the \( \sigma \)-algebra generated by the cylinder subsets of \( \mathcal{D}_R(Q_{p}^{N}) \).

Therefore

\[
P = \frac{1_{\mathcal{L}_R(Q_{p}^{N})}S}{\int_{\mathcal{L}_R(Q_{p}^{N})} dS}.
\]

6. Partition functions and generating functionals

In this section we introduce a family of \( \mathcal{P}(\varphi) \)-theories, where

\[
\mathcal{P}(X) = a_3X^3 + a_4X^4 + \ldots + a_{2k}X^{2D} \in \mathbb{R}[X], \quad \text{with} \; D \geq 2,
\]

satisfying \( \mathcal{P}(\alpha) \geq 0 \) for any \( \alpha \in \mathbb{R} \). Notice that this implies that for \( \varphi \in \mathcal{D}_R(Q_{p}^{N}) \) and \( \alpha_4 > 0 \), \( \exp\left(-\frac{\alpha_4}{2} \int \mathcal{P}(\varphi) dN x\right) \leq 1 \). This fact follows from Remark 3. Each of these theories corresponds to a thermally fluctuating field which is defined by means of a functional integral representation of the partition function. All the thermodynamic quantities and correlation functions of the system can be obtained by functional differentiation from a generating functional as in the classical case, see e.g. [26], [39]. In this section, we provide mathematical rigorous definitions of all these objects.

6.1. Partition functions. We assume that \( \varphi \in \mathcal{L}_R(Q_{p}^{N}) \) represents a field that performs thermal fluctuations. We also assume that in the normal phase the expectation value of the field \( \varphi \) is zero. Then the fluctuations take place around zero. The size of these fluctuations is controlled by the energy functional:

\[
E(\varphi) := E_0(\varphi) + E_{\text{int}}(\varphi),
\]

where the first terms is defined in (4.1), and the second term is

\[
E_{\text{int}}(\varphi) := \frac{\alpha_4}{4} \int_{Q_{p}^{N}} \mathcal{P}(\varphi(x)) dN x, \quad \alpha_4 \geq 0,
\]

corresponds to the interaction energy.

All the thermodynamic properties of the system attached to the field \( \varphi \) are described by the partition function of the fluctuating field, which is given classically by a functional integral

\[
Z^{\text{phys}} = \int D(\varphi) e^{-\frac{E(\varphi)}{K_B T}},
\]

where \( D(\varphi) \) is a ‘spurious measure’ on the space of fields, \( K_B \) is the Boltzmann’s constant and \( T \) is the temperature. We use the normalization \( K_B T = 1 \). When the coupling constant \( \alpha_4 = 0 \), \( Z^{\text{phys}} \) reduced to the free-field partition function

\[
Z_0^{\text{phys}} = \int D(\varphi) e^{-E_0(\varphi)}.
\]
It is more convenient to use a normalize partition function $Z^\text{phys}_0$. 

**Definition 3.** Assume that $\delta > N$, and $\gamma, \alpha_2 > 0$. The free-partition function is defined as

$$Z_0 = Z_0(\delta, \gamma, \alpha_2) = \int dP (\varphi).$$

The discrete free-partition function is defined as

$$Z_0^{(l)} = Z_0^{(l)}(\delta, \gamma, \alpha_2) = \int dP_l (\varphi),$$

for $l \in \mathbb{N} \setminus \{0\}$.

By Lemma 5.1, $\lim_{l \to \infty} Z_0^{(l)} = Z_0$. Notice that the term $e^{-E_0(\varphi)}$ is used to construct the measure $P (\varphi)$.

**Definition 4.** Assume that $\delta > N$, and $\gamma, \alpha_2, \alpha_4 > 0$. The partition function is defined as

$$Z = Z(\delta, \gamma, \alpha_2, \alpha_4) = \int e^{-E_{\text{int}}(\varphi)} dP (\varphi).$$

The discrete partition functions are defined as

$$Z^{(l)} = Z^{(l)}(\delta, \gamma, \alpha_2, \alpha_4) = \int e^{-E_{\text{int}}(\varphi)} dP_l (\varphi),$$

for $l \in \mathbb{N} \setminus \{0\}$.

Notice that $e^{-E_{\text{int}}(\varphi)}$ is bounded and (sequentially) continuous in $L_\mathbb{R}$, and consequently in $L_\mathbb{R}^l$ for any $l$. Indeed, take $\varphi_n \overset{D_\mathbb{R}}{\to} 0$, $L_\mathbb{R}$ is endowed with the topology of $D_\mathbb{R}$. Then there is $l$ such that $\varphi_n \in L_\mathbb{R}^l$ for every $n$, and $\varphi_n \overset{\text{unif}_\mathbb{R}}{\to} 0$, i.e.

$$\varphi_n(x) = \sum_{i \in G_l} \varphi^{(n)}(i) \Omega \left(l \|x - i\|_p\right), \text{ and } \max_{i \in G_l} \{\varphi^{(n)}(i)\} \to 0 \text{ as } n \to \infty.$$

Which implies that $E_{\text{int}}(\varphi_n) \to 0$. Again by Lemma 5.1, $\lim_{l \to \infty} Z^{(l)} = Z$.

**6.2. Correlation functions.** From a mathematical perspective a $P(\varphi)$-theory is given by a cylinder probability measure of the form

$$\frac{1_{L_\mathbb{R}} (\varphi) e^{-E_{\text{int}}(\varphi)} dP}{\int_{L_\mathbb{R}} e^{-E_{\text{int}}(\varphi)} dP} = \frac{1_{L_\mathbb{R}} (\varphi) e^{-E_{\text{int}}(\varphi)} dP}{Z},$$

in the space of fields $L_\mathbb{R} (\mathbb{Q}_p^N)$. It is important to mention that we do not require the Wick regularization operation in $e^{-E_{\text{int}}(\varphi)}$ because we are restricting the fields to be test functions.
Definition 5. The m-point correlation functions of a field \( \varphi \in \mathcal{L}_R(\mathbb{Q}_p^N) \) are defined as

\[
G^{(m)}(x_1, \ldots, x_m) = \frac{1}{Z} \int_{\mathcal{L}_R(\mathbb{Q}_p^N)} \left( \prod_{i=1}^{m} \varphi(x_i) \right) e^{-E_{\text{int}}(\varphi)} d\mathbb{P}.
\]

The discrete m-point correlation functions of a field \( \varphi \in \mathcal{L}_R^l(\mathbb{Q}_p^N) \) are defined as

\[
G^{(m)}_l(x_1, \ldots, x_m) = \frac{1}{Z(l)} \int_{\mathcal{L}_R^l(\mathbb{Q}_p^N)} \left( \prod_{i=1}^{m} \varphi(x_i) \right) e^{-E_{\text{int}}(\varphi)} d\mathbb{P}_l,
\]

for \( l \in \mathbb{N} \setminus \{0\} \).

Lemma 6.1. The discrete m-point correlation functions \( G^{(m)}_l(x_1, \ldots, x_m) \) of a field \( \varphi \in \mathcal{L}_R(\mathbb{Q}_p^N) \) are test functions in \( x_1, \ldots, x_m \).

Proof. There is a positive integer \( l = l(\varphi) \) such that \( \varphi \in \mathcal{L}_R^l \) and \( x_1, \ldots, x_m \in B^N_l \). By using that

\[
\varphi(x_i) = \sum_{j \in G_l} \varphi(j) \Omega \left( p^l \| x_i - j \|_p \right);
\]

one gets that \( \prod_{i=1}^{m} \varphi(x_i) \) is a finite sum of terms of the form

\[
\prod_{k=1}^{m} \varphi(j_k) \Omega \left( p^l \| x_k - j_k \|_p \right) =: F(\varphi(j_1), \ldots, \varphi(j_m)) \Theta_l(x_1, \ldots, x_m),
\]

where \( F(\varphi(j_1), \ldots, \varphi(j_m)) \) is a polynomial function defined in \( \mathcal{L}_R^l \), \( j_k \in G_l \), and \( \Theta_l(x) = \Theta_l(x_1, \ldots, x_m) \) is the characteristic function of the polydisc \( B^N_l(j_1) \times \cdots \times B^N_l(j_m) \). Now, by using that \( \exp(-E_{\text{int}}(\varphi)) = \exp(-\frac{\alpha}{4} p^{-lN} \sum_{k=3}^{2D} \sum_{j \in G_l} a_k \varphi^k(j)) \), the correlation function \( G^{(m)}_l(x_1, \ldots, x_m) \) is a finite sum of test functions of the form

\[
\Theta_l(x) \int_{\mathcal{L}_R^l} \left\{ F(\varphi(j_1), \ldots, \varphi(j_m)) \exp(-\frac{\alpha}{4} p^{-lN} \sum_{k=3}^{2D} \sum_{j \in G_l} a_k \varphi^k(j)) \right\} d\mathbb{P}_l =
\]

\[
\Theta_l(x) \int_{\mathcal{L}_R^l} \left\{ F(\varphi(j_1), \ldots, \varphi(j_m)) \exp(-\frac{\alpha}{4} p^{-lN} \sum_{k=3}^{2D} \sum_{j \in G_l} a_k \varphi^k(j)) \right\} d\mathbb{P},
\]

where the convergence of the integrals is guaranteed by the fact that the integrands are bounded functions, cf. Lemma 5.1.

Notice that the pointwise limit \( G^{(m)}(x_1, \ldots, x_m) = \lim_{l \to \infty} G^{(m)}_l(x_1, \ldots, x_m) \) is not a test function due to the fact that \( \Theta_l(x) \) has an arbitrary small exponent of local constancy when \( l \) tends to infinity.
6.3. Generating functionals. We now introduce a current \( J(x) \in \mathcal{L}_\mathbb{R}(\mathbb{Q}_p^N) \) and add to the energy functional \( E(\varphi) \) a linear interaction energy of this current with the field \( \varphi(x) \),

\[
E_{\text{source}}(\varphi, J) := - \int_{\mathbb{Q}_p^N} \varphi(x) J(x)d^N x,
\]

in this way we get a new energy functional

\[
E(\varphi, J) := E(\varphi) + E_{\text{source}}(\varphi, J).
\]

Notice that \( E_{\text{source}}(\varphi, J) = -\langle \varphi, J \rangle \), where \( \langle \cdot, \cdot \rangle \) denotes the scalar product of \( L^2(\mathbb{Q}_p^N) \). This scalar product extends to the pairing between \( \mathcal{L}_\mathbb{R}(\mathbb{Q}_p^N) \) and \( \mathcal{L}_\mathbb{R}(\mathbb{Q}_p^N) \).

**Definition 6.** Assume that \( \delta > N \), and \( \gamma, \alpha_2, \alpha_4 > 0 \). The partition function corresponding to the energy functional \( E(\varphi, J) \) is defined as

\[
\mathcal{Z}(J; \delta, \gamma, \alpha_2, \alpha_4) = \frac{1}{\mathcal{Z}_0(\mathcal{L}_\mathbb{R}(\mathbb{Q}_p^N))} \int_{\mathcal{L}_\mathbb{R}(\mathbb{Q}_p^N)} e^{-E_{\text{int}}(\varphi) + \langle \varphi, J \rangle} d\mathbb{P},
\]

and the discrete versions

\[
\mathcal{Z}^{(l)}(J; \delta, \gamma, \alpha_2, \alpha_4) = \frac{1}{\mathcal{Z}_0^{(l)}(\mathcal{L}_\mathbb{R}(\mathbb{Q}_p^N))} \int_{\mathcal{L}_\mathbb{R}(\mathbb{Q}_p^N)} e^{-E_{\text{int}}(\varphi) + \langle \varphi, J \rangle} d\mathbb{P}_1,
\]

for \( l \in \mathbb{N} \setminus \{0\} \).

For the sake of simplicity we will the notation \( \mathcal{Z}(J) = \mathcal{Z}(J; \delta, \gamma, \alpha_2, \alpha_4) \), \( \mathcal{Z}^{(l)}(J) = \mathcal{Z}^{(l)}(J; \delta, \gamma, \alpha_2, \alpha_4) \).

**Remark 7.** In this section, we need some functionals from the space

\[
(L^p_\mathbb{R}) = L^p(\mathcal{L}_\mathbb{R}(\mathbb{Q}_p^N), d\mathbb{P}), \quad \rho \in [1, \infty),
\]

see Definition 2. Let \( F(X_1, \ldots, X_n) \) be a real-valued polynomial, and \( \xi = (\xi_1, \ldots, \xi_n) \), with \( \xi_i \in \mathcal{L}_\mathbb{R}(\mathbb{Q}_p^N) \) for \( i = 1, \ldots, n \), then the functional

\[
F_\xi(W) := F(\langle W, \xi_1 \rangle, \ldots, \langle W, \xi_n \rangle), \quad W \in \mathcal{L}_\mathbb{R}'(\mathbb{Q}_p^N),
\]

belongs to \((L^p_\mathbb{R}), \rho \in [1, \infty), \) see e.g. [23, Proposition 1.6]. The functional \( \exp C\langle \cdot, \phi \rangle \), for \( C \in \mathbb{R}, \phi \in \mathcal{L}_\mathbb{R} \) belongs to \((L^p_\mathbb{R}), \rho \in [1, \infty), \) see e.g. [23, Proposition 1.7]. The \( \mathbb{R} \)-algebra \( \mathcal{A} \) generated by the functionals \( F_\xi, \exp C\langle \cdot, \phi \rangle \) is dense in \((L^p_\mathbb{R}), \rho \in [1, \infty), \) see e.g. [23, Theorem 1.9].

**Lemma 6.2.** Given \( \varphi \in \mathcal{L}_\mathbb{R}(\mathbb{Q}_p^N), \) \( m \geq 1 \), and \( e_i \geq 0 \) for \( i = 1, \ldots, m \), we define

\[
\mathcal{I}(\varphi) = \int_{(\mathbb{Q}_p^N)^m} \left( \prod_{i=1}^m \varphi^{e_i}(x_i) \right) \prod_{i=1}^m d^N x_i.
\]

Then \( \mathcal{I} \in \mathcal{A} \).
Proof. There is an integer \( l \) such that \( \varphi \in L^p \). By using (6.3), and the fact that the functions \( \Omega \left( p^l \| x_i - j \|_p \right) \), \( j \in G_t \), are orthogonal with respect to the scalar product \( \langle \cdot, \cdot \rangle \) in \( L^2_\R (\mathbb{Q}_p^N) \), we have

\[
\varphi (x_i) = \sum_{j \in G_t} p^l \Omega \left( p^l \| x_i - j \|_p \right) \Omega \left( p^l \| x_i - j \|_p \right),
\]

where \( W_j \in L^0 \left( \mathbb{Q}_p^N \right) \), for \( j \in G_t \). Consequently,

\[
\varphi^{e_i} (x_i) = \sum_{j \in G_t} p^l \Omega \left( p^l \| x_i - j \|_p \right) \Omega \left( p^l \| x_i - j \|_p \right),
\]

and \( \prod_{i=1}^m \varphi^{e_i} (x_i) \) is a finite sum of terms of the form

\[
\left( \prod_{k=1}^m p^l \Omega \left( p^l \| x_k - j_k \|_p \right) \right) \prod_{k=1}^m \Omega \left( p^l \| x_k - j_k \|_p \right),
\]

where \( i_k \in \{1, \ldots, m\} \), \( j_k \in G_t \). Now \( \mathcal{I}(\varphi) \) is a finite sum of terms of the form

\[
\left( \prod_{k=1}^m p^l \Omega \left( p^l \| x_k - j_k \|_p \right) \right) \int_{\mathbb{Q}_p^N} \prod_{k=1}^m \Omega \left( p^l \| x_k - j_k \|_p \right) \prod_{i=1}^m d^N x_i
\]

and therefore \( \mathcal{I} \in \mathcal{A} \).

\( \square \)

**Lemma 6.3.** With the above notation, the following assertions hold true:

(i) \( 1_{L^p_\R} (\varphi) e^{-E_{\text{int}}(\varphi)+(\varphi, J)} \in (L^1_\R) \). In particular, \( Z(J) < \infty \);

(ii) \( \lim_{l \to \infty} \int_{L^p_\R (\mathbb{Q}_p^N)} e^{(\varphi, J)} d\mathbb{P}_l = \int_{L^p_\R (\mathbb{Q}_p^N)} e^{(\varphi, J)} d\mathbb{P}; \)

(iii) \( Z^{(l)}(J) < \infty \) for any \( l \in \mathbb{N} \setminus \{0\} \);

(iv) \( \lim_{l \to \infty} Z^{(l)}(J) = Z(J) \).

**Proof.** (i) The result follows from

\[
\int_{L^p_\R (\mathbb{Q}_p^N)} e^{-E_{\text{int}}(\varphi)+(\varphi, J)} d\mathbb{P}(\varphi) \leq \int_{L^p_\R (\mathbb{Q}_p^N)} e^{(\varphi, J)} d\mathbb{P}(\varphi) \leq \int_{L^p_\R (\mathbb{Q}_p^N)} e^{(W, J)} d\mathbb{P}(W) < \infty,
\]

by using Remark 7.

(ii) For each \( l \in \mathbb{N} \setminus \{0\} \), we take \( \{K_{n_l}\} \) to be an increasing sequence of compact subsets of \( L^p_\R (\mathbb{Q}_p^N) \) having \( L^p_\R (\mathbb{Q}_p^N) \) as its limit. Set

\[
\mathcal{I}^{(l, n)}(J) := \int_{L^p_\R (\mathbb{Q}_p^N)} 1_{K_{n_l}} (\varphi) e^{(\varphi, J)} d\mathbb{P}_l.
\]
Since the integrand $1_{K_{n_l}}(\varphi) e^{(\varphi,J)}$ is continuous and bounded, by Lemma 5.1

$$\mathcal{I}^{(l,n)}(J) = \int_{\mathcal{L}_k^c(\mathbb{Q}_p^N)} 1_{K_{n_l}}(\varphi) e^{(\varphi,J)} \, d\mathbb{P}.$$  

The result follows by the dominated convergence theorem, by taking first the limit $n_l \to \infty$, and then the limit $l \to \infty$, and using the fact that $e^{(\varphi,J)}$ is integrable.

(iii) By Lemma 5.1 and Remark 7

$$\int_{\mathcal{L}_k^c(\mathbb{Q}_p^N)} e^{(\varphi,J)} \, d\mathbb{P} = \int_{\mathcal{L}_k(\mathbb{Q}_p^N)} e^{(\varphi,J)} \, d\mathbb{P} \leq \int_{\mathcal{L}_k(\mathbb{Q}_p^N)} e^{(W,J)} \, d\mathbb{P} < \infty.$$  

We now use that

$$\mathcal{Z}^{(l)}(J) \leq \frac{\mathcal{L}_k(\mathbb{Q}_p^N)}{\mathcal{L}_k^c(\mathbb{Q}_p^N)}.$$  

(iv) It is sufficient to show that

$$\lim_{l \to \infty} \int_{\mathcal{L}_k(\mathbb{Q}_p^N)} e^{-E_{int}(\varphi) + (\varphi,J)} \, d\mathbb{P} = \int_{\mathcal{L}_k(\mathbb{Q}_p^N)} e^{-E_{int}(\varphi) + (\varphi,J)} \, d\mathbb{P}.$$  

This identity is established by using the reasoning given in the second part. \hfill \Box

**Definition 7.** For $\theta \in \mathcal{L}_k(\mathbb{Q}_p^N)$, the functional derivative $D_\theta \mathcal{Z}(J)$ of $\mathcal{Z}(J)$ is defined as

$$D_\theta \mathcal{Z}(J) = \lim_{\epsilon \to 0} \frac{\mathcal{Z}(J + \epsilon \theta) - \mathcal{Z}(J)}{\epsilon} = \left[ \frac{d}{d\epsilon} \mathcal{Z}(J + \epsilon \theta) \right]_{\epsilon = 0}.$$  

**Lemma 6.4.** Let $\theta_1, \ldots, \theta_m$ be test functions from $\mathcal{L}_k(\mathbb{Q}_p^N)$. The functional derivative $D_{\theta_1} \cdots D_{\theta_m} \mathcal{Z}(J)$ exists, and the following formula holds true:

$$(6.4) \quad D_{\theta_1} \cdots D_{\theta_m} \mathcal{Z}(J) = \frac{1}{\mathcal{Z}^m_0} \int_{\mathcal{L}_k(\mathbb{Q}_p^N)} e^{-E_{int}(\varphi) + (\varphi,J)} \left( \prod_{i=1}^m \langle \varphi, \theta_i \rangle \right) \, d\mathbb{P}(\varphi).$$  

Furthermore, the functional derivative $D_{\theta_1} \cdots D_{\theta_m} \mathcal{Z}(J)$ can be uniquely identified with the distribution

$$\mathcal{L}_k^c \left( \left( \mathbb{Q}_p^N \right)^m \right).$$  

Proof. We first compute

$$\left[ \frac{d}{d\epsilon} \mathcal{Z}(J + \epsilon \theta_m) \right]_{\epsilon = 0} = \frac{1}{\mathcal{Z}_0} \lim_{\epsilon \to 0} \int_{\mathcal{L}_k(\mathbb{Q}_p^N)} e^{-E_{int}(\varphi) + (\varphi,J)} \left( \frac{e^{\langle \varphi, \theta_m \rangle} - 1}{\epsilon} \right) \, d\mathbb{P}(\varphi).$$  

We consider the case $\epsilon \to 0^+$, the other limit is treated in a similar way. For $\epsilon > 0$ sufficiently small, by using the mean value theorem,

$$
\frac{e^{\epsilon (\varphi, \theta_m)} - 1}{\epsilon} = \langle \varphi, \theta_m \rangle e^{\epsilon_0 (\varphi, \theta_m)} \quad \text{where} \quad \epsilon_0 \in (0, \epsilon).
$$

Then, by using $e^{-E_{\text{int}}(\epsilon)} \leq 1$ and Remark 7,

$$
e^{-E_{\text{int}}(\epsilon) + \langle \varphi, J \rangle} \left( \frac{e^{\epsilon (\varphi, \theta_m)} - 1}{\epsilon} \right) = \langle \varphi, \theta_m \rangle e^{-E_{\text{int}}(\epsilon) + \langle \varphi, J + \epsilon \theta_m \rangle}
$$

is an integrable function. Now, by applying the dominated convergence theorem,

$$
D_{\theta_m} Z(J) = \left[ \frac{d}{d\epsilon} Z(J + \epsilon \theta_m) \right]_{\epsilon=0} = \frac{1}{Z_0} \int_{\mathcal{L}_B(Q_p^N)} e^{-E_{\text{int}}(\varphi) + \langle \varphi, J \rangle} \langle \varphi, \theta_m \rangle \ d\mathbb{P}(\varphi).
$$

By Remark 7, $e^{-E_{\text{int}}(\varphi) + \langle \varphi, J \rangle} \langle \varphi, \theta_m \rangle \in (L^1_\mathbb{R})$, then, further derivatives can be computed using (6.6).

Finally, formula (6.5) is obtained from (6.4) by using Fubini’s theorem and Remark 1:

$$
D_{\theta_1} \cdots D_{\theta_m} Z(J) = \frac{1}{Z_0} \int_{\mathcal{L}_B(Q_p^N)} e^{-E_{\text{int}}(\varphi) + \langle \varphi, J \rangle} \left\{ \int \cdots \int \prod_{i=1}^m \theta_i (x_i) \varphi (x_i) \prod_{i=1}^m d^N x_i \right\} \ d\mathbb{P}(\varphi).
$$

Remark 8. In an alternative way, one can define the functional derivative $\frac{\delta}{\delta J(y)} Z(J)$ of $Z(J)$ as the distribution from $\mathcal{L}_B(Q_p^N)$ satisfying

$$
\int_{Q_p^N} \theta (y) \left( \frac{\delta}{\delta J(y)} Z(J) \right) (y) \ d^N y = \left[ \frac{d}{d\epsilon} Z(J + \epsilon \theta) \right]_{\epsilon=0}.
$$

Using this notation and formula (6.5), we obtain that

$$
\frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_m)} Z(J) = \frac{1}{Z_0} \int_{\mathcal{L}_B(Q_p^N)} e^{-E_{\text{int}}(\varphi) + \langle \varphi, J \rangle} \left( \prod_{i=1}^m \varphi (x_i) \right) \ d\mathbb{P}(\varphi) \in \mathcal{L}_B((Q_p^N)^m).
$$

Remark 9. Consider the probability measure space $\left( \mathcal{L}_B(Q_p^N), \mathcal{B} \cap \mathcal{L}_B, \frac{1}{Z_0} \mathbb{P} \right)$, where $\mathcal{B} \cap \mathcal{L}_B$ denotes the $\sigma$-algebra generated by the cylinder subsets of $\mathcal{L}_B$. Given $\theta_1, \ldots, \theta_m$ test functions from $\mathcal{L}_B(Q_p^N)$, we attach them the following random variable:

$$
\mathcal{L}_B(Q_p^N) \to \mathbb{R}
$$

$$
\varphi \to \prod_{i=1}^m \langle \varphi, \theta_i \rangle.
$$

The expected value of this variable is given by

$$
D_{\theta_1} \cdots D_{\theta_m} Z(J) \mid_{\epsilon=0} = \frac{1}{Z_0} \int_{\mathcal{L}_B(Q_p^N)} e^{-E_{\text{int}}(\varphi)} \left( \prod_{i=1}^m \langle \varphi, \theta_i \rangle \right) \ d\mathbb{P}(\varphi).
$$
An alternative description of the expected value is given by
\[
\frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_m)} \mathcal{Z}(J) \big|_{J=0} = \frac{1}{Z_0} \int_{\mathcal{L}_R(Q_p^N)} e^{-E_{\text{out}}(\varphi)} \left( \prod_{i=1}^m \varphi(x_i) \right) d\mathbb{P}(\varphi).
\]

As a conclusion we have the following result:

**Proposition 6.1.** The correlation functions \( G^{(m)}(x_1, \ldots, x_m) \in \mathcal{L}'(\mathbb{Q}_p^N)^m \) are given by
\[
G^{(m)}(x_1, \ldots, x_m) = \frac{\mathcal{Z}_0}{Z} \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_m)} \mathcal{Z}(J) \big|_{J=0}.
\]

### 6.4. Free-field theory.

#### 6.4.1. The propagators.

We take \( \delta > N \), and \( \gamma, \alpha_2 > 0 \) as before. For \( J \in \mathcal{L}_R \), the equation
\[
\left( \frac{\gamma}{2} W(\partial, \delta) + \frac{\alpha_2}{2} \right) \varphi_0 = J
\]
has unique solution \( \varphi_0 \in \mathcal{L}_R \). Indeed, \( \varphi_0(\kappa) = \frac{\hat{J}(\kappa)}{\frac{2}{\tau} A_{w^d}(\|\kappa\|_p) + \frac{\alpha_2}{2}} \) is a test function satisfying \( \varphi_0(0) = 0 \). On the other hand, solving equation (6.7) in \( \mathcal{D}'_R \), we have
\[
\varphi_0(x) = F_{\kappa \rightarrow x}^{-1} \left( \frac{1}{\frac{2}{\tau} A_{w^d}(\|\kappa\|_p) + \frac{\alpha_2}{2}} \right) J(x) = G(\|x\|_p) * J(x),
\]
where \( F_{\kappa \rightarrow x}^{-1} \) denotes the Fourier transform from \( \mathcal{D}' \) into \( \mathcal{D}' \), which means that equation (6.7) has a unique solution \( \varphi_0(x) = G(\|x\|_p) * J(x) \) in \( \mathcal{L}_R \), where \( G(\|x\|_p) \) is the ‘standard Green function’. This means that the UV and IF behavior of the propagators are not altered if we use Lizorkin spaces in the construction of \( p \)-adic QFTs.

We now discuss the singular behavior of the Green function in the case of Taibleson-Vladimirov operator:
\[
G(x; \beta, \gamma, \alpha_2) = F_{\kappa \rightarrow x}^{-1} \left( \frac{1}{\frac{2}{\tau} \|\kappa\|_p^\beta + \frac{\alpha_2}{2}} \right),
\]
where \( \beta, \gamma, \alpha_2 > 0 \). In this case \( G(x; \beta, \gamma, \alpha_2) \) is continuous on \( \mathbb{Q}_p^N \setminus \{0\} \). If \( \beta > N \), then \( G(x; \beta, \gamma, \alpha_2) \) is continuous. For \( 0 < \beta \leq N \), \( G(x; \beta, \gamma, \alpha_2) \) is locally constant on \( \mathbb{Q}_p^N \setminus \{0\} \), and
\[
|G(x; \beta, \gamma, \alpha_2)| \leq \begin{cases} 
C \|x\|_p^{\beta-N} & \text{for } 0 < \beta < N \\
C_0 - C_1 \ln \|x\|_p & \text{for } N = \beta,
\end{cases}
\]
for \( \|x\|_p \leq 1 \), where \( C, C_0, C_1 \) are positive constants; \( |G(x; \beta, \gamma, \alpha_2)| \leq C_1 \|x\|_p^{\beta-N} \) as \( \|x\|_p \rightarrow \infty \). Finally, \( G(x; \beta, \gamma, \alpha_2) \geq 0 \) on \( \mathbb{Q}_p^N \setminus \{0\} \), see e.g. [32] Proposition 11.1.

The behavior at the origin of the Green functions considered here depends in an intricate way on the parameters of the QFT considered and on the dimension. This behavior plays a central role in the renormalization of the QFTs presented here. The renormalization will be considered in a forthcoming article.
Theorem 6.1. Set $Z_0(J) := Z(J; \delta, \gamma, \alpha_2, 0)$, then

$$Z_0(J) = N_0' \exp \left\{ \int_{Q_p^N} \int_{Q_p^N} J(x) G(\|x - y\|_p) J(y) d^N x \, d^N y \right\},$$

where $N_0'$ denotes a normalization constant.

Proof. We take $\varphi_0, J \in \mathcal{L}_R$, where $\varphi_0$ is the solution of equation $(6.7)$. We now change variables in $Z_0(J)$ as $\varphi = \varphi_0 + \varphi'$,

$$Z_0(J) = \frac{1}{Z_0} \int_{\mathcal{L}_R(Q_p^N)} e^{\langle \varphi, J \rangle} \, dP = \frac{e^{\langle \varphi_0, J \rangle}}{Z_0} \int_{\mathcal{L}_R(Q_p^N)} e^{\langle \varphi', J \rangle} \, dP' (\varphi')$$

$$= \frac{1}{Z_0} \int_{\mathcal{L}_R(Q_p^N)} e^{\langle \varphi', (\frac{1}{2} W(\vartheta, \delta) + \frac{\alpha_2}{2}) \varphi_0 \rangle} \, dP' (\varphi') e^{\langle G + J, J \rangle}$$

$$= N_0' e^{\langle G + J, J \rangle} = N_0' \exp \left\{ \int_{Q_p^N} \int_{Q_p^N} J(x) G(\|x - y\|_p) J(y) d^N x \, d^N y \right\}.$$

Furthermore, by using $(5.6)$, the characteristic functional of the measure $P'$ is

$$\int_{\mathcal{L}_R(Q_p^N)} e^{\langle T, J \rangle} \, dP' (T) = e^{-\langle T, \varphi_0 \rangle - \frac{1}{2} \langle f, f \rangle}, \quad f \in \mathcal{L}_R (Q_p^N),$$

which means that $P'$ is a Gaussian measure with mean functional $\langle \varphi_0, \cdot \rangle$ and correlation functional $\mathbb{B} (\cdot, \cdot)$. 

The correlation functions $G_0^{(m)}(x_1, \ldots, x_m)$ of the free-field theory are obtained from the functional derivatives of $Z_0(J)$ at $J = 0$:

Theorem 6.2.

$$G_0^{(m)}(x_1, \ldots, x_m) = \left[ \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_m)} Z_0(J) \right]_{J=0}$$

$$= N_0' \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_m)} \exp \left\{ \int_{Q_p^N} \int_{Q_p^N} J(x) G(\|x - y\|_p) J(y) d^N x \, d^N y \right\} |_{J=0}.$$

Remark 10. The random variable $\varphi(x_i)$ corresponds to the random variable $\langle W, \varphi \rangle$, for some $W = W(x_i) \in \mathcal{L}_R (Q_p^N)$, see Remark 9, which is Gaussian with mean zero and variance $\|\varphi\|_2^2$, see e.g. [40] Lemma 2.1.5. Then, the correlation functions $G_0^{(m)}(x_1, \ldots, x_m)$ obey to Wick’s theorem:

(6.8)

$$\frac{1}{Z_0} \int_{\mathcal{L}_R(Q_p^N)} \prod_{i=1}^{m} \varphi(x_i) \, dP = \begin{cases} 0 & \text{if } m \text{ is not even} \\ \sum_{\text{pairings}} \mathbb{E}(\varphi(x_{i_1}) \varphi(x_{j_1}) \cdots \varphi(x_{i_n}) \varphi(x_{j_n})) & \text{if } m = 2n, \end{cases}$$
where
\[ \mathbb{E}(\varphi(x_i) \varphi(x_j)) := \frac{1}{Z_0} \int_{\mathcal{R}(\mathbb{Q}_p^N)} \varphi(x_i) \varphi(x_j) \, d\mathbb{P} \]
and \( \sum_{\text{pairings}} \) means the sum over all \( \frac{(2n)!}{2^n n!} \) ways of writing \( 1, \ldots, 2n \) as \( n \) distinct (unordered) pairs \((i_1, j_1), \ldots, (i_n, j_n)\), see e.g. [42, Proposition 1.2].

For \( n = 2 \), \( G_{0}^{(2)} \) is the free two-point function or the free propagator of the field:
\[
G_{0}^{(2)}(x_1, x_2) = N_0' \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \exp \left\{ \int_{\mathbb{Q}_p^N} \int_{\mathbb{Q}_p^N} J(x)G(\|x - y\|_p)J(y) d^N x \, d^N y \right\} \bigg|_{J=0} = 2N_0' G(\|x_1 - x_2\|_p) \in \mathcal{L}^{\prime}_R(\mathbb{Q}_p^N \times \mathbb{Q}_p^N).
\]

By using Wick’s theorem all the \( 2n \)-point functions can be expressed as sums of products of two-point functions:
\[
G_{0}^{(2n)}(x_1, \ldots, x_2n) = \sum_{\text{pairings}} G(\|x_{i_1} - x_{j_1}\|_p) \cdots G(\|x_{i_n} - x_{j_n}\|_p).
\]

Notice that \( G_{0}^{(2n)}(x_1, \ldots, x_2n) \) is singular at \( x_{i_1} - x_{j_1} = \cdots = x_{i_n} - x_{j_n} = 0 \), where \((i_k, j_k)\) runs over all the possible pairings of the variables \( x_1, \ldots, x_{2n} \). This set is a closed subset of \( \mathbb{Q}_p^{2N} \).

6.5. Perturbation expansions for \( \varphi^4 \)-theories. In this section we assume that \( \mathcal{P}(\varphi) = \varphi^4 \). This hypothesis allow us to provide explicit formulas which completely similar to the classical ones, see e.g. [26, Chapter 2]. At any rate, the techniques presented here can be applied to polynomial interactions of type (6.1).

The existence of a convergent power series expansion for \( Z(J) \) (the perturbation expansion) in the coupling parameter \( \alpha_4 \) follows from the fact that \( \exp(-E_{\text{int}}(\varphi) + \langle \varphi, J \rangle) \) is an integrable function, see Lemma [6.3] (i), by using the dominated convergence theorem, more precisely, we have
\[
Z(J) = Z_0(J) + \sum_{m=1}^{\infty} \frac{1}{m!} \left( -\frac{\alpha_4}{4} \right)^m \int_{\mathcal{R}(\mathbb{Q}_p^N)} \left\{ \int_{\mathbb{Q}_p^N} \left( \prod_{i=1}^{m} \varphi^4(z_i) \right) e^{\langle \varphi, J \rangle} \prod_{i=1}^{m} d^N z_i \right\} \, d\mathbb{P}(\varphi)
\]
\[
= Z_0(J) + \sum_{m=1}^{\infty} Z_m(J),
\]
where
\[
Z_0(J) = \frac{1}{Z_0} \int_{\mathcal{R}(\mathbb{Q}_p^N)} e^{\langle \varphi, J \rangle} \, d\mathbb{P}(\varphi).
\]
In the case \( m \geq 1 \), by using that \( \mathcal{A} \) is an algebra (see Remark 7 and Lemma 6.2), we can apply Fubini’s theorem to obtain that

\[
\mathcal{Z}_m(J) := \frac{1}{Z_0} \frac{m!}{m!} \left( \frac{\alpha_4}{4} \right)^m \int_{\mathcal{L}_k(\mathbb{Q}_p^N)} \left\{ \int_{\mathcal{L}_k(\mathbb{Q}_p^N)} \left( \prod_{i=1}^{m} \varphi^4(z_i) \right)^m e^{(\varphi,J) \prod_{i=1}^{m} d^N z_i} \right\} d\mathbb{P}(\varphi)
\]

\[
= \frac{1}{Z_0} \frac{m!}{m!} \left( \frac{\alpha_4}{4} \right)^m \int_{\mathcal{L}_k(\mathbb{Q}_p^N)} \left\{ \int_{\mathcal{L}_k(\mathbb{Q}_p^N)} \left( \prod_{i=1}^{m} \varphi^4(z_i) \right)^m e^{(\varphi,J) \prod_{i=1}^{m} d^N z_i} \right\} \prod_{i=1}^{m} d^N z_i.
\]

Then

\[
\mathcal{Z}_m(0) = \frac{1}{m!} \left( \frac{\alpha_4}{4} \right)^m \int_{\mathcal{L}_k(\mathbb{Q}_p^N)} G_0^{(4m)}(z_1, z_1, z_1, \ldots, z_m, z_m, z_m, z_m) \prod_{i=1}^{m} d^N z_i,
\]

for \( m \geq 1 \). Therefore from (6.9)-(6.10), with \( J = 0 \), and using \( \mathcal{Z} = \mathcal{Z}(0) \), \( \mathcal{Z}_m(0) := \mathcal{Z}_m \), for \( m \geq 1 \),

\[
\mathcal{Z} = 1 + \sum_{m=1}^{\infty} \mathcal{Z}_m.
\]

Now by using Propositions 6.1, 6.2 and (6.9),

\[
G^{(n)}(x_1, \ldots, x_n) = \frac{Z_0}{Z} \left[ \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} Z(J) \right]_{J=0}^n + \frac{Z_0}{Z} \left[ \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} \sum_{m=1}^{\infty} \mathcal{Z}_m(J) \right]_{J=0}^n
\]

\[
= \frac{Z_0}{Z} G^{(n)}(x_1, \ldots, x_n) + \frac{Z_0}{Z} \left[ \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} \sum_{m=1}^{\infty} \mathcal{Z}_m(J) \right]_{J=0}^n.
\]

Lemma 6.5.

\[
\frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} \sum_{m=1}^{\infty} \mathcal{Z}_m(J) =
\]

\[
= \frac{1}{Z_0} \sum_{m=1}^{\infty} \frac{1}{m!} \left( \frac{-\alpha_4}{4} \right)^m \int_{\mathcal{L}_k(\mathbb{Q}_p^N)} \left\{ \int_{\mathcal{L}_k(\mathbb{Q}_p^N)} \left( \prod_{i=1}^{m} \varphi^4(z_i) \right)^m e^{(\varphi,J) \prod_{i=1}^{m} d^N z_i} \right\} \prod_{i=1}^{m} d^N z_i.
\]

Proof. We recall that by the proof of Lemma 6.2,

\[
\mathcal{J}(\varphi) := \int_{\mathcal{L}_k(\mathbb{Q}_p^N)} \left( \prod_{i=1}^{m} \varphi^4(z_i) \right)^m \prod_{i=1}^{m} d^N z_i
\]
is a finite sum of terms of the form
\[
\left( \prod_{k=1}^{m} p^{l^N e_{ik}} \langle \varphi, W_{j_k} \rangle^{e_{ik}} \right) \prod_{k=1}^{m} \Omega \left( p^l \| x_k - j_k \|_p \right),
\]
then by the definition of \( Z_m(J) \) and Fubini’s theorem, it is sufficient to compute
\[
\frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} \sum_{m=1}^{\infty} \frac{1}{Z_0} \frac{m!}{m!} \left( -\frac{\alpha_4}{4} \right)^{m} \times \int_{\mathcal{L}_k(Q_p^N)} \left\{ \left( \prod_{k=1}^{m} p^{l^N e_{ik}} \langle \varphi, W_{j_k} \rangle^{e_{ik}} \right) e^{\langle \varphi, J \rangle} \right\} d\mathbb{P}(\varphi).
\]

We first establish that
\[
D_{\theta_1} \left\{ \sum_{m=1}^{\infty} \frac{1}{Z_0 m!} \left( -\frac{\alpha_4}{4} \right)^{m} \int_{\mathcal{L}_k(Q_p^N)} \left\{ \left( \prod_{k=1}^{m} p^{l^N e_{ik}} \langle \varphi, W_{j_k} \rangle^{e_{ik}} \right) e^{\langle \varphi, J \rangle} \right\} d\mathbb{P}(\varphi) \right\}
= \sum_{m=1}^{\infty} \frac{1}{Z_0 m!} \left( -\frac{\alpha_4}{4} \right)^{m} \int_{\mathcal{L}_k(Q_p^N)} \left\{ \left( \prod_{k=1}^{m} p^{l^N e_{ik}} \langle \varphi, W_{j_k} \rangle^{e_{ik}} \right) \langle \varphi, \theta_1 \rangle e^{\langle \varphi, J \rangle} \right\} d\mathbb{P}(\varphi),
\]
by using the reasoning given in the proof of Lemma 6.4. Since
\[
\left( \prod_{k=1}^{m} p^{l^N e_{ik}} \langle \varphi, W_{j_k} \rangle^{e_{ik}} \right) \langle \varphi, \theta_1 \rangle e^{\langle \varphi, J \rangle}
\]
is an integrable function, cf. Remark 7, further derivatives can be calculated in the same way. Consequently,
\[
(6.12) \quad D_{\theta_1} \cdots D_{\theta_m} \sum_{m=1}^{\infty} \mathcal{Z}_m(J) = \frac{1}{Z_0} \sum_{m=1}^{\infty} \frac{1}{m!} \left( -\frac{\alpha_4}{4} \right)^{m} \int_{\mathcal{L}_k(Q_p^N)} \left\{ \left( \prod_{i=1}^{n} \langle \varphi, \theta_i \rangle \right) e^{\langle \varphi, J \rangle} J(\varphi) d\mathbb{P}(\varphi) \right\}.
\]
The announced formula follows from (6.12) by Fubini’s theorem.

Now by using (6.11) and Remark 8, we have the following result:

**Theorem 6.3.** Assume that \( \mathcal{P}(\varphi) = \varphi^4 \). The \( n \)-point correlation function of the field \( \varphi \) admits the following convergent power series in the coupling constant:
\[
G^{(n)}(x_1, \ldots, x_n) = \frac{Z_0}{Z} \left\{ G_0^{(n)}(x_1, \ldots, x_n) + \sum_{m=1}^{\infty} G_m^{(n)}(x_1, \ldots, x_n) \right\} \quad \text{in } \mathcal{L}_R'(Q_p^N).
\]
where
\[
G^{(n)}_m(x_1, \ldots, x_n) := \frac{1}{m!} \left( -\frac{\alpha_4}{4} \right)^m \int \left( \frac{Q}{N} \right)^m G^{(n+4m)}_0(z_1, z_1, z_1, \ldots, z_m, z_m, x_1, \ldots, x_n) \prod_{i=1}^m dz_i \in \mathcal{L}_\mathbb{R}' \left( \mathbb{Q}_p^N \right)
\]

It is important to emphasize that formula (6.13) is an equality between distributions ‘with singularities’. The free-field correlation functions \(G^{(n+4m)}_0\) in the sum may now Wick-expanded as in (6.8) into sums over products of propagators \(G^{(2)}_0\). Then, like in the classical case, a renormalization procedure is needed.

7. The Wick rotation

The classical generating functional of \(\mathcal{P}(\varphi)\)-theory with Lagrangian density \(E_0(\varphi) + E_{\text{int}}(\varphi) + E_{\text{source}}(\varphi, J)\) in the Minkowski space is

\[
Z_{\text{phys}}(J) = \frac{\int D(\varphi) e^{\sqrt{-1}(E_0(\varphi) + E_{\text{int}}(\varphi) + E_{\text{source}}(\varphi, J))}}{\int D(\varphi) e^{\sqrt{-1}(E_0(\varphi) + E_{\text{int}}(\varphi))}}.
\]

A natural \(p\)-adic analogue of this function is

\[
Z_C(J) = \left\{ \begin{array}{l}
\int_{\mathcal{L}_\mathbb{R}(\mathbb{Q}_p^N)} e^{\sqrt{-1}(E_{\text{int}}(\varphi) + E_{\text{source}}(\varphi, J))} d\mathbb{P}(\varphi) \\
\int_{\mathcal{L}_\mathbb{R}(\mathbb{Q}_p^N)} e^{\sqrt{-1}(E_0(\varphi) + E_{\text{int}}(\varphi))} d\mathbb{P}(\varphi)
\end{array} \right\}.
\]

Which is a complex-value measure. The key point is that \(e^{\sqrt{-1}(E_0(\varphi) + E_{\text{int}}(\varphi) + E_{\text{source}}(\varphi, J))}\) is integrable, see [23, Theorem 1.9], and then the techniques presented here can be applied to \(Z_C(J)\) and its discrete version

\[
Z_C^{(l)}(J) = \left\{ \begin{array}{l}
\int_{\mathcal{L}_\mathbb{R}(\mathbb{Q}_p^N)} e^{\sqrt{-1}(E_{\text{int}}(\varphi) + E_{\text{source}}(\varphi, J))} d\mathbb{P}_l(\varphi) \\
\int_{\mathcal{L}_\mathbb{R}(\mathbb{Q}_p^N)} e^{\sqrt{-1}(E_0(\varphi) + E_{\text{int}}(\varphi))} d\mathbb{P}_l(\varphi)
\end{array} \right\}, \; l \in \mathbb{N} \setminus \{0\}.
\]

In particular a version Theorem 6.3 is valid for \(Z_C(J)\). To explain the connection of these constructions with Wick rotation, we rewrite (5.6) as follows:

\[
(7.1) \quad \int_{\mathcal{L}_\mathbb{R}(\mathbb{Q}_p^N)} e^{\sqrt{-1}(W,f)} d\mathbb{P}(W) = e^{-|\lambda|^2_\mathbb{R}} R(f), \; f \in \mathcal{L}_\mathbb{R} \left( \mathbb{Q}_p^N \right), \text{ for } \lambda \in \mathbb{C}.
\]

This formula holds true in the case \(\lambda \in \mathbb{R}\). The integral in the right-hand side of (7.1) admits an entire analytic continuation to the complex plane, see [23, Proposition 2.4]. Furthermore,
this fact is exactly the Analyticity Axiom (OS0) in the Euclidean axiomatic quantum field presented in [18, Chapter 6].

A field \( \varphi : \mathbb{Q}^N_p \to \mathbb{R} \) is a function from the spacetime \( \mathbb{Q}^N_p \) into \( \mathbb{R} \) (the target space). We perform a Wick rotation in the target space:

\[
\mathbb{R} \to \sqrt{-1}\mathbb{R}
\]

\[
\varphi \to \sqrt{-1}\varphi.
\]

Then

\[
\int_{\mathcal{L}'_\mathbb{R}^{\prime}(\mathbb{Q}^N_p)} e^{\sqrt{-1}(T,\varphi)} d\mathbb{P}(T) = \int_{\mathcal{L}'_\mathbb{R}^{\prime}(\mathbb{Q}^N_p)} e^{\sqrt{-1}(\sqrt{-1}T,\varphi)} d\mathbb{P}(T) = e^{-\frac{1}{2}B(\varphi,\varphi)}.
\]

Changing variables as \( W = \sqrt{-1}T \), we get

\[
e^{-\frac{1}{2}B(\varphi,\varphi)} = \int_{\sqrt{-1}\mathcal{L}'_\mathbb{R}^{\prime}(\mathbb{Q}^N_p)} e^{\sqrt{-1}(W,\varphi)} d\mathbb{P}'(W).
\]

Therefore, \( \mathbb{P}' \) is a probability measure in \( \sqrt{-1}\mathcal{L}'_\mathbb{R}^{\prime}(\mathbb{Q}^N_p) \) with correlation functional \( B(\cdot,\cdot) \), that can be identified with \( \mathbb{P} \).

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