RESEARCH ANNOUNCEMENTS

THE TETRAGONAL CONSTRUCTION

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1. Preliminaries. Let \( C \) be a nonsingular curve of genus \( g \), and \( \pi: \tilde{C} \rightarrow C \) an unramified double cover. The Prym variety \( P(C, \tilde{C}) \) is by definition \( \text{ker}^0(Nm) \), where \( Nm: J(\tilde{C}) \rightarrow J(C) \) is the norm map, and \( \text{ker}^0 \) is the connected component of 0 in the kernel. By [M] this is a \((g - 1)\)-dimensional, principally polarized abelian variety. Let \( A_g, M_g, R_g \) denote, respectively, the moduli spaces of \( g \)-dimensional principally polarized abelian varieties, curves of genus \( g \), and pairs \((C, \tilde{C})\) as above. \((R_g) \) is a \((2^{2g - 1})\)-sheeted cover of \( M_g \). The Prym map is the morphism

\[
P = P_g: R_g \rightarrow A_{g-1}, \quad (C, \tilde{C}) \mapsto P(C, \tilde{C}).
\]

It is analogous to the Jacobi map \( J = J_g: M_g \rightarrow A_g \) sending a curve to its Jacobian. The main reason for studying \( P \) is that its image in \( R_{g-1} \) is larger than that of \( J \), hence it allows us to handle geometrically a wider class of abelian varieties than just Jacobians. For instance, \( P_g \) is dominant for \( g \leq 6 \) [W] while \( J_g \) is only dominant for \( g \leq 3 \).

The purpose of this announcement is to describe the fibers of \( P \) in the various genera. Our main tool for this is a simple-minded construction which we describe in some detail in paragraph 6. Let us use "\( n \)-gonal" (trigonal, tetragonal, etc.) to describe a pair \((C, f)\) where \( f: C \rightarrow \mathbb{P}^1 \) is a branched cover of degree \( n \) \((3, 4\) respectively). Briefly, our construction takes the data \((C, \tilde{C}, f)\) where \((C, \tilde{C}) \in R_g \) and \((C, f)\) is tetragonal, and returns two new sets of data, \((C_0, \tilde{C}_0, f_0)\) and \((C_1, \tilde{C}_1, f_1)\), of the same type. This procedure is symmetric: starting with \((C_0, \tilde{C}_0, f_0)\) we end up with \((C, \tilde{C}, f)\) and \((C_1, \tilde{C}_1, f_1)\). It is useful due to the following observation.

**Proposition 1.1.** The tetragonal construction commutes with the Prym map:

\[
P(C, \tilde{C}) \approx P(C_0, \tilde{C}_0) \approx P(C_1, \tilde{C}_1).
\]
Remark 1.2. A similar construction was studied by Recillas [R], [DS, III]. He starts with a tetragonal pair \((C, f)\) and produces a triplet \((X, \tilde{X}, g)\) where \((X, g)\) is trigonal and \(P(X, \tilde{X}) \approx J(C)\). This becomes the special case of our construction where \(\tilde{C}\) is taken to be the split double cover of \(C\). The resulting \(C_0, C_1\) are then isomorphic to \(X\) with a \(P^1\) attached (in two different ways) and

\[
P(C_i, \tilde{C}_i) \approx P(X, \tilde{X}) \approx J(C) \approx P(C, \tilde{C}).
\]

2. Genus 6. In [DS] the map \(P : R_6 \to A_5\) was studied at length. The main result was that this map is generically finite, of degree 27.

Theorem 2.1. The fibers of \(P : R_6 \to A_5\) have a structure equivalent to the intersection-configuration of the 27 lines on a cubic surface.

An equivalent formulation is

Corollary 2.2. The Galois group of the field extension \(K(A_5) \subset K(R_6)\) is the Weyl group \(W(E_6)\). (Compare [Ma, Theorem 23.9]).

The theorem limits severely the possible degenerations in a fiber of \(P\). For instance

Corollary 2.3. The ramification locus (in \(R_6\)) is mapped six-to-one to the branch locus (in \(A_5\)).

Proof. A line on a cubic surface \(S\) counts twice if and only if it passes through a double point of \(S\). Through such a point there are six lines. \(\square\)

The proof of the theorem depends on the existence of 5 tetragonal maps, \(f_i (1 \leq i \leq 5)\) on a generic curve \(C\) of genus 6. To each triplet \((C, \tilde{C}, f_i)\) the tetragonal construction associates two others; the ten resulting points of \(P^{-1}P(C, \tilde{C})\) are the ones "incident" to \((C, \tilde{C})\).

The same method allows us to recover the main result of [DS] rather painlessly: we show that starting with \((C, \tilde{C}) \in R_6\), choosing a tetragonal \(f\), applying the tetragonal construction to get \((C_0, \tilde{C}_0, f_0)\), changing the tetragonal \(f_0\) to an \(f_0'\) and repeating the process indefinitely, leads to precisely 27 distinct objects: to the original \((C, \tilde{C})\) are added ten after the first cycle, and only sixteen more after the second cycle. (I.e. each of the five first-generation pairs yields the same set of sixteen second-generation objects!) Therefore \(\text{deg}(P)\) is a multiple of 27. This possible multiplicity is eliminated by checking a degenerate case, where \(C\) is a double cover (branched) of an elliptic curve ("elliptic hyperelliptic").

3. Genus 5. The map \(P_5 : R_5 \to A_4\) turns out, surprisingly, to be more intricate than its higher-genus cousin \(P_6\), and until now has eluded description.
By dimension count, the generic fiber is 2 dimensional; we show that in fact it
is a double cover of a Fano surface.

**Theorem 3.1.** There is a birational isomorphism \( \kappa : \mathbb{A}_4 \rightarrow C \) where \( C \) is
a parameter-space for pairs \((X, \mu)\) consisting of (the isomorphism class of) a
cubic threefold \( X \) together with an "even" point of order two in its intermediate
Jacobian.

**Proposition 3.2.** There is a natural involution \( \lambda : \mathbb{R}_5 \rightarrow \mathbb{R}_5 \) such that
\( \lambda (C, \tilde{C}) \) is related to \((C, \tilde{C})\) by a succession of two tetragonal constructions;
hence \( \mathcal{P} \circ \lambda = \mathcal{P} \).

**Theorem 3.3.** For generic \( A \in \mathbb{A}_4 \), the quotient \( \mathcal{P}^{-1}(A)/\lambda \) is isomorphic
to \( \mathcal{F}(\kappa(A)) \), the Fano surface of lines on the cubic threefold \( \kappa(A) \).

The proofs seem to depend heavily on the results for genus 6 and their
various specializations. As a corollary, we have an explicit parametrization of the
family of (rational equivalence classes of) effective symmetric representatives of
the class \([0]^3/3\) in \( H_2(A, \mathbb{Z}) \). This is twice the class of a curve in its Jacobian,
and the smallest class which is effective on generic \( A \).

4. **Prym-Torelli.** For \( g \leq 4 \) the analysis of \( \mathcal{P}_g \) is fairly easy. It can be
done using nothing but Recillas' trigonal construction (1.2), since any \( A \in \mathbb{A}_{g-1} \)
is Jacobian of a tetragonal curve. In the remaining cases \( g \geq 7 \), \( \mathcal{P}_g \) "ought" to
be injective by dimension count. After some inconclusive work of Tjurin \( \textsc{T} \),
counterexamples to this expected Prym-Torelli theorem were exhibited by Beauville \( \textsc{B}_2 \) for \( g \leq 10 \), using Recillas' construction applied to curves which are
tetragonal in two distinct ways. Using the tetragonal construction we exhibit
counterexamples for all \( g \). Without much justification we make the following

**Conjecture 4.1.** If \( \mathcal{P}(C, \tilde{C}) \approx \mathcal{P}(C', \tilde{C}') \) then \((C', \tilde{C}')\) is obtained from
\((C, \tilde{C})\) by successive applications of the tetragonal construction. In particular,
\( C \) and \( C' \) are tetragonal curves.

5. **Andreotti-Mayer varieties.** In \( \textsc{AM} \), Andreotti and Mayer studied the
Schottky problem of characterizing Jacobians among abelian varieties. Call \( A \in \mathbb{A}_g \) an \( A - M \) variety if its theta divisor \( \theta \) has a \((g - 4)\)-dimensional singular locus,
and let \( N_g \subset \mathbb{A}_g \) be the closure of the locus of \( A - M \) varieties. The main results
of \( \textsc{AM} \) are that \( N_g \) can be explicitly described by equations, and that \( \mathcal{F}(M_g) \) is
an irreducible component of \( N_g \). Perhaps the most spectacular application of
Prym theory was Beauville's refinement of their results \( \textsc{B}_1 \). He obtained a
complete (and lengthy) list of all possible components of \( \mathcal{P}^{-1}(N_g) \), hence, in
principle, a description of \( N_4, N_5 \) (since \( \mathcal{P}_5, \mathcal{P}_6 \) are surjective, when appropriately
compactified). In particular, he showed that $N_4$ has only one irreducible component other than $J(M_4)$.

Using the tetragonal construction, some remarkable coincidences appear in Beauville's list. In fact

**Theorem 5.1.** (1) $N_4$ consists of $J_4$ and another nine-dimensional irreducible component $[B1]$.  
(2) $N_5$ consists of $J_5$ and four irreducible, nine-dimensional loci; three of these parametrize Pryms of elliptic-hyperelliptic curves, and the fourth consists of certain abelian varieties isogenous to a product with an elliptic curve.

(3) For $g > 6$, $N_g \cap \overline{P(R_{g+1})}$ consists of $J_g$, 2 components of Pryms of elliptic-hyperelliptic curves (each $(2g - 1)$ dimensional) and $[(g - 2)/2]$ components of Pryms of reducible curves $C = C_1 \cup C_2$, $(C_1 \cap C_2) = 4$ (each $(3g - 4)$ dimensional).

**Corollary 5.2.** Any $(C, \tilde{C}) \in \mathbb{P}^{-1}(N_g)$ is either tetragonal (or a degeneration of tetragonals) or reducible. The modified Prym-Torelli Conjecture 4.1 holds over $N_g$.

**Conjecture 5.3.** $N_g \subset \overline{P(R_{g+1})}$, hence $N_g$ consists only of the components listed above.

The proof might imitate Andreotti's proof of Torelli's theorem and resurrect Tjurin's work $[T]$ : Given $A \in N_g$, there should be some explicit geometric construction yielding a family of doubly covered tetragonal (or reducible) curves, whose Prym is $A$.

**Corollary 5.4.** For any canonical curve $C \subset \mathbb{P}^{g-1}$, the system of quadrics containing $C$ is spanned by quadrics of rank 4.

**Proof.** A refinement of $[AM]$ shows that the truth of the corollary for a given $C$ depends only on the structure of $N_g$ near $J(C)$; in particular the corollary holds if $J_g$ is the only component of $N_g$ containing $J(C)$. By Conjecture 5.3 and Theorem 5.1, this holds for all $C$ except for hyperelliptics and elliptic-hyperelliptics. A special argument works for these.

**6. The construction.** We sketch the tetragonal construction. Start with an unramified double cover $\pi: \tilde{C} \rightarrow C$ and tetragonal map $f: C \rightarrow \mathbb{P}^1$. Let 
\[ f_* (\pi): f_* (\tilde{C}) \rightarrow \mathbb{P}^1 \]
be the "pushforward" of $\pi: \tilde{C} \rightarrow C$ via $f$. This is a $(16 = 2^4)$-sheeted branched cover. Over $p \in \mathbb{P}^1$, its 16 points correspond to the 16 ways of lifting the quadruple $f^{-1}(p) \subset C$ to a quadruple in $\tilde{C}$. This suggests a convenient way of realizing $f_* (\tilde{C})$ as a curve in $Pic (4)(\tilde{C})$, the Picard variety of line bundles of degree 4: $f_* (\tilde{C})$ is the subvariety parametrizing those effective divisors in $\tilde{C}$ whose norm
(under \( \pi : \tilde{C} \rightarrow C \)) is in the 1-dimensional linear series determined on \( C \) by \( f \).

Note that on the curve \( f_*(\tilde{C}) \) there is a natural involution \( \tau : f_*(\tilde{C}) \rightarrow f_*(\tilde{C}) \). \( \tau \) sends a lifting of \( f^{-1}(p) \) to the complementary lifting, obtained by interchanging the sheets of \( \pi : \tilde{C} \rightarrow C \). (This is induced by the automorphism of \( \text{Pic}^4(\tilde{C}) \) sending a line bundle \( L \) to \( L^{-1} \otimes (f \circ \pi)^* \mathcal{O}_{\mathbb{P}^1}(1) \).) Let \( \tilde{C} \) be the quotient \( f_*(\tilde{C})/\tau \), an 8-sheeted cover of \( \mathbb{P}^1 \).

**Lemma.** \( \tilde{C} \) is reducible: \( \tilde{C} = C_0 \cup C_1 \), each \( C_i \) is a 4-sheeted branched cover of \( \mathbb{P}^1 \). Correspondingly, \( f_*(\tilde{C}) = \tilde{C}_0 \cup \tilde{C}_1 \), where \( \tilde{C}_i \) is acted upon by \( \tau \) with quotient \( C_i \).

**Proof.** Define an equivalence relation \( \sim \) on \( f_*(\tilde{C}) \): \( D_1 \sim D_2 \) if \( f_*(\pi)(D_1) = f_*(\pi)(D_2) \) and \( D_1, D_2 \) have an even number of points (0, 2 or 4) in common. The quotient \( f_*(\tilde{C})/\sim \) is a 2-sheeted branched cover of \( \mathbb{P}^1 \). Clearly it can be branched only where \( f : C \rightarrow \mathbb{P}^1 \) is; but a simple monodromy check shows that at such a point \( f_*(\tilde{C})/\sim \) is locally reducible. (I.e. in going around a branch point, an even number of points of \( \tilde{C} \) are exchanged.) Hence the normalization of \( f_*(\tilde{C})/\sim \) is nowhere ramified over \( \mathbb{P}^1 \), hence consists of two disjoint copies, so \( f_*(\tilde{C}) \) itself is reducible. Finally, \( \tau \) acts on each component separately since it changes an even number (all 4) of the points. Q.E.D.

**Note.** Identifying \( \text{Pic}^4(\tilde{C}) \approx \text{Jac}(\tilde{C}) \), we have that \( f_*(\tilde{C}) \) is contained in the kernel of the norm-homomorphism, which \([M]\) consists of two copies of the Prym variety; \( \tilde{C}_i \) are the intersections with these two components.

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