Self-organized criticality and interface depinning transitions

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We discuss the relation between self-organized criticality and depinning transitions by mapping sandpile models to equations that describe driven interfaces in random media. This equivalence yields a continuum description and gives insight about various ways of reaching the depinning critical point: slow drive (self-organized criticality), fixed density simulations, tuning the interface velocity (extremal drive criticality), or tuning the driving force. We obtain a scaling relation for the correlation length exponent for sandpiles.

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I. INTRODUCTION

Many systems respond to external perturbations by avalanches which behave intermittently with a power-law distribution of sizes. The paradigm of such self-organized critical (SOC) behavior is the so-called sandpile model [1]. It maintains by an infinitely slow drive a critical steady-state, where the internal dissipation balances the external drive. Candidates for such phenomena include granular piles [2,3], microfracturing processes [4], and earthquakes [5]. Despite many theoretical and numerical investigations a thorough understanding of self-organized criticality is still lacking [6–13]. Fundamental problems which need to be solved involve deriving a continuum theory which would for instance determine the upper critical dimension, above which mean-field theory applies [2,4].

Similar behavior can be found from elastic interfaces driven through random media [14,15]. They undergo a continuous (critical) depinning transition as the external driving force is varied. With increasing force one passes from a phase where the interface is pinned to a depinned phase where the interface moves with a constant velocity. Close to the critical point, the motion of the interface takes place in “bursts” with no characteristic size and the interface develops scaling described by critical exponents. These phenomena can be met in fluids driven through porous media [16], in domain walls in magnets (the Barkhausen effect) [17], in flux lines in type II superconductors [18], and in charge-density waves [19].

In this paper we investigate the connections between self-organized criticality and depinning transitions [14,15,20,21,22]. We first establish a generic, exact relation [20] between sandpile models and driven interfaces which builds upon previous investigations of e.g. a charge-density wave model [23] and a rice-pile model [21]. Specifically, we discuss the Bak-Tang-Wiesenfeld (BTW) [1] model and, as an example, a stochastic sandpile model [24,25] through a mapping to a model for interface depinning with slightly different noise terms.

The mapping enables one to understand the slow-drive criticality used in sandpile simulations in terms of standard concepts for driven interfaces. Using the continuum theory for interface depinning it follows for these sandpile models that the upper critical dimension $d_c$ is 4, and the relevant noise is of quenched type. The connection with interfaces allows us to establish a scaling relation for the correlation length exponent for sandpile models. In addition, we discuss in the interface representation sandpiles driven at fixed density, driven at boundaries, and extremal drive criticality.

II. SANDPIPES

The sandpile models are here defined as follows: to each site of a $d$-dimensional lattice (square in $d = 2$) of size $L^d$ is associated a variable $z_x$ which counts the number of grains on that site. When the number of grains on a site exceeds a critical threshold $z_c > z_c$, the site is active and it topples. This means that 2$d$ grains are removed from that site and given to the 2$d$ nearest neighbors (nn): $z_x \rightarrow z_x - 2d$, $z_{nn} \rightarrow z_{nn} + 1$, $\forall nn$. Sandpiles are usually open such that grains which topple out of the system are lost (in one dimension: $z_x \rightarrow z_x + 1$). It is also possible, as discussed later, to use periodic boundary conditions. When there are no more active sites in the system, one grain is added to a randomly chosen site, and the time and number of topplings till the system again contains no active sites define an avalanche.

In the present work we denote by $\langle z \rangle_{\psi}$ the time averaged size of an avalanche when the initial distribution of non-zero heights is given by $\psi$. We use the symbol $x$ to denote the location of an avalanche.

In the present work the initial distribution of site heights $\psi$ is chosen to correspond to an initial distribution of size $z_x$ of avalanches. It is possible to obtain such a distribution directly from the BTW model [1]. The initial condition for the avalanche is then a single site with $z_x > z_c$. It is also possible to choose other initial conditions. However, for simplicity we choose an initial condition in which all sites have the same initial height $z_x$.

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dissipation of grains balance each other. The boundary conditions (BCs) are essential to obtain criticality and they are usually of the Dirichlet type, \( z = 0 \), such that particles are dissipated to the outside \([14]\). Alternatively, the SOC steady state can be reached by using bulk dissipation and, e.g., periodic BCs \([13]\).

In the SOC steady-state the probability to have avalanches of lifetime \( t \) and size \( s \) follow power-law distributions: \( p(t) = t^{-\gamma} f(t/L^2) \) and \( p(s) = s^{-\tau} f(s/L^0) \), with \( s \sim t^{D/\nu} \) and \( \tau = 1 + (D - 2)/\nu \). Here the size scales as \( s \sim t^D \) and the (spatial) area as \( t^\ell \) (for compact avalanches) with \( \ell \) the linear dimension. The fact that each added grain will perform the order of \( L^2 \) topplings before leaving the system leads to the fundamental result

\[
\langle s \rangle \sim L^2 \tag{1}
\]

independent of dimension \([13]\). Thus, \( \tau = 2 - 2/D \) and \( \tau_c = 1 + (D - 2)/\nu \). Equation (1) yields \( \gamma/\nu = 2 \), where \( \gamma \) describes the divergence of the susceptibility (bulk response to a bulk field) near a critical point, \( \chi = \langle s \rangle \sim \Delta^{-\gamma} \), and \( \nu \) is the (spatial) correlation length exponent, \( \xi \sim \Delta^{-\nu} \). Here \( \Delta = \zeta - \zeta_c \) is the control parameter, \( \zeta = (z_c) \), and the critical value \( \zeta_c = \langle z_c \rangle_{\text{SOC}} \), where this average is taken in the slowly driven SOC steady-state with \( \Delta = 0 \) \([11]\).

### III. INTERFACE DEPINNING

For driven interfaces in random media critical scaling is obtained with a force \( F \) close to a critical value \( F_c \). Depinned interfaces move with a velocity \( v \sim f^\theta \), with \( f = F - F_c \geq 0 \). Pinned interfaces are blocked by pinning paths/manifolds which arise from the quenched disorder environment. Close to criticality, correlations scale as \( x^{2\chi} \), with \( \chi \) the roughness exponent, up to a correlation length \( \xi \sim |f|^{-\nu} \). The characteristic time scale is \( \xi^2 / z \) with \( z \) the dynamic exponent and it follows that \( \theta = \nu (z - \chi) \) \([13,16]\). Near the depinning transition, the simplest choice to describe the dynamics of the interface is the following continuum equation (‘quenched Edwards-Wilkinson’, or linear interface model, LIM) \([13,17]\):

\[
\frac{\partial H}{\partial t} = \nabla^2 H + \eta(x, H) + F. \tag{2}
\]

Here, \( H(x, t) \) measures the height of a given site \( x \) at time \( t \). The quenched noise \( \eta(x, H) \) has correlations given by \( \langle \eta(x, H)\eta(x', H') \rangle = \delta^d(x - x') G(H - H') \), where \( G(H - H') \) decays rapidly, approximated by a delta function for random-field disorder. The critical exponents at the depinning transition have been calculated by \( \varepsilon \)-expansions \([13,16]\) and simulations \([17,20,23]\). The upper critical dimension is \( d_c = 4 \), above which mean-field theory applies \([22]\). Below we will also discuss so-called columnar noise with \( G(H) \equiv 1 \) \([13]\).

The interface equation (2) obeys an invariant so that the static response scales as \( \chi(q, \omega = 0) \sim q^{-2} \), i.e.,

\[
\gamma/\nu = 2. \tag{3}
\]

For forces below \( F_c \), the (bulk) response of the interface triggered by a small increase in \( F \) scales as \( \chi_{\text{bulk}} \equiv d \langle H \rangle/dF \sim (F_c - F)^{-\gamma} \). Right at the critical point one can argue as follows \([14,16]\): the roughness of the interface scales as \( \ell^x \) and assuming that \( \Delta \langle H \rangle \) will scale in the same way it follows

\[
\gamma = 1 + \chi \nu. \tag{4}
\]

This yields \( \chi + 1/\nu = 2 \), i.e., there are only two independent exponents for depinning described by \( \xi \). The standard scaling relations are valid for interfaces with parallel dynamics: all sites with \( \partial H/\partial t > 0 \) are updated in parallel. Note that interfaces with extremal (i.e., one unstable site at a time) and parallel drive have the \textit{same} pinning paths. This manifests the Abelian character of the LIM in that the order in which active sites are advanced does not matter \([24]\).

### IV. MAPPING OF SANDPILE DYNAMICS

Next we will show that the SOC critical behavior can be related exactly to the slowly driven depinning transition in an interface model. Thus, Eqs. (1) and (2) are equivalent and Eq. (4) yields an expression for the correlation length exponent \( \nu \) for sandpiles. The first step is to formulate the stopping of an avalanche in a SOC system as being due to a pinning path for an interface \( H(x, t) \). This field is given in the continuum limit by

\[
H(x, t) = \int_0^t dt' \rho(x, t'), \tag{5}
\]

where the order parameter \( \rho(x, t) \) is the activity (topplings) at site \( x \) at time \( t \), i.e., \( \rho = \dot{H} = v \sim f^\theta \). In words: \( H(x, t) \) counts the number of topplings at site \( x \) up to time \( t \). At the microscopic level this is an exact correspondence between a toppling and the interface advance. A toppling takes place when \( z_x > z_c \), which by the relation

\[
z_x = z_c + \frac{\partial H}{\partial t}, \tag{6}
\]

yields the dynamics \( \partial H/\partial t > 0 \Rightarrow H \to H + 1 \), whereas \( H \) is unchanged at the sites where no toppling takes place. The dynamics of sandpile models thus map to discrete interface equations where an avalanche takes the interface \( H(x, t) \) from one pinning path to the next in the quenched random medium \([14,15,20,23,25]\). Since the interface counts topplings it does not move backwards and thus Eq. (6) effectively reads \( \partial H/\partial t = \theta(z_x - z_c) \), which is the standard discretization for depinning models \([17]\). We are currently investigating the applicability of such discretization procedures to various models \([20,34]\).
Next, we express $z_x$ in terms of $H(x,t)$ for the specific models introduced above. The number of grains $z_x$ on site $x$ is $z_x = N_{in} - N_{out} + F(x,t)$, where $N_{in}$ is the number of grains added to this site from its 2$d$ nearest neighbors (nn) and $N_{out}$ is the number of grains removed from this site due to topplings. The (external) driving force $F(x,t)$ counts the number of grains added from the outside. Since $N_{in} = \Sigma_{nn} H(x_{nn},t)$ and $N_{out} = 2dH(x,t)$ (for details and extensions to other models see [34]), we arrive at

$$\frac{\partial H}{\partial t} = \nabla^2 H - z_c(x,H) + F(x,t), \quad (7)$$

where $\nabla^2 H$ is the discrete Laplacian. The Dirichlet boundary conditions for $z_x$ become $H \equiv 0$ and the dynamics is parallel. Similar connections have been previously discussed for a charge-density wave model [23] and for a boundary driven rice-pile model [10] (see below). In the stochastic model, $z_c(x,H)$ is a random variable which changes after each toppling. Thus $z_c(x,H)$ acts like quenched random point-disorder similar to $\eta(x,H)$ in Eq. (2). The BTW model has $z_c$ equal to a constant. The dissipation needed to reach the SOC state (loss of grains due to topplings) takes place through the BC of $H \equiv 0$. Using strong boundary pinning may thus give rise to the possibility of observing SOC experimentally in systems displaying a depinning transition. We emphasize that the mapping prescription can in principle be applied to any sandpile model. For other, more complicated, toppling rules [33] additional terms like the “Kardar-Parisi-Zhang” nonlinearity $|\nabla H|^2$ may appear.

On the internal (fast) time scale the driving force $F(x,t)$ does not act as a time-dependent noise but as columnar-type disorder. It counts all the grains added to the system by the slow drive, i.e. $F(x,t) \rightarrow F(x,t) + 1$, and thus increases as function of time in an uncorrelated fashion. In the opposite limit when a grain is added (e.g.) each time step (“fast drive”) $F(x,t)$ would correspond to a time-dependent noise [17]. Since $H \equiv 0$ at the boundary and $F$ increases as function of time the steady-state profile of $H$ will be close to a paraboloid or, in one dimension, a parabola (see also [11]). In the steady-state, just after an avalanche, the slowly increasing force $F$ is balanced by the negative curvature $\nabla^2 H$ of the paraboloid such that all sites are pinned ($\partial H/\partial t \leq 0$). This illustrates that the interface effectively is driven by a force equal to the critical force $F_c \equiv z_c - \nabla c$, where $\nabla c$ is the average of $z_c(x,H)$ in the steady state (for the BTW model trivially $z_c = 2d - 1$). Accordingly, the slow drive reaches the depinning critical point by adjusting the dissipation to the driving force such that the velocity (order parameter) is infinitesimal.

The steady-state of the different sandpile models is described by an equation similar to Eq. (7). Thus the exponent relation (1) holds and it is equivalent to Eq. (1) which describes the scaling of the average avalanche size (“susceptibility”). Assuming that a roughness exponent $\chi$ can be defined for sandpile models, one can argue that Eq. (4) is valid also for sandpiles. Furthermore, the upper critical dimension is $d_c = 4$. Note that the ensuing noise will contain a columnar component [23,33,34] due to the random drive $F(x,t)$. The one-dimensional BTW model has a critical force $F_c = 1 - 1 = 0$, which corresponds to the critical point of the columnar-disorder interface model [33]. In $d > 1$, one has $F_c < 0$ which in combination with the fact that the interface by definition cannot move backwards implies that the BTW model displays a more complicated behavior than the columnar models investigated in [23,33]. Note also that avalanches in stochastic models will have a random structure due to the explicit point disorder whereas avalanches in the BTW model show a more regular behavior [20,34].

For the case of the boundary driven one-dimensional rice-pile models [27,28] a similar mapping of the dynamics can be done with an auxiliary field $H(0,t)$ and a drive implemented as $H(0,t) \rightarrow H(0,t) + 1$ [10]. The rice-pile models have Dirichlet BC at $x = 0$ and Neumann BC (reflective) at $x = L$ which yields $s(1) \sim L$. In our picture the boundary drive is $F(1,t) \rightarrow F(1,t) + 1$ and $F(x > 1,t) = 0$. Because of the Neumann BC $H(L,t) = H(L + 1,t)$ the steady state develops a parabolic profile with the left branch pointing up [10].

V. VARIOUS ENSEMBLES

We next consider the more straightforward cases in which sandpiles are studied with periodic boundary conditions (amounting to $H(1) = H(L)$ in one dimension). In such cases the SOC steady state can be tuned into by various approaches. It can be reached by using a carefully tuned bulk dissipation $\epsilon \sim L^{-2}$ [13]. In this case, periodic BCs are also the best since the scaling of the system is not a mixture of boundary and bulk scaling [37]. As above, we arrive at

$$\frac{\partial H}{\partial t} = \nabla^2 H - z_c(x,H) - \epsilon(x,H) + F(x,t) \quad (8)$$

with $H$ periodic. As in Eq. (8), the force $F(x,t)$ is columnar and increases on the slow time scale. The dissipation $\epsilon(x,H)$ takes now into account all the grains removed before the site at $x$ topples. It increases with a (small) probability only when a site topples and this means that $\epsilon$ explicitly depends on $H$. Therefore, a dissipation event effectively corresponds to a shift in the $z_c$ value. Thus, one obtains that the BTW model with bulk dissipation contains a very weak point-disorder component (since the increases in $F$ equal in the statistical sense the increases in $\epsilon$). Though point-disorder is in general expected to be a relevant perturbation, in the infinite system size limit the Larkin length [12,13] associated with the cross-over from columnar behavior diverges and thus the avalanche behavior is not governed by the weak point disorder. By this argument the BTW models with or without bulk...
dissipation are equivalent to the same interface depinning equation (3) in accordance with simulations of the BTW and bulk dissipation models (13). Note that the boundary critical behavior of the BTW model depends on the specific boundary condition: Dirichlet BCs display a different behavior (39), whereas Neumann BCs (reflective) are similar to the bulk. In the case of periodic BCs and bulk dissipation, the \( H \)-field fluctuates around an average flat profile. The terms \( F(x,t) \) and \( \epsilon(x,H) \) will balance each other in the steady state with an average difference such that \( F_c = \zeta_c - \frac{\epsilon}{\nu} < 0 \). For larger dissipation rates the system moves away from the critical point and, in analogy to Eq. (3), the bulk susceptibility scales as \( \chi_{\text{bulk}} \sim 1/\epsilon \sim \xi^{1/\nu} \), with \( \nu = 1/2 \) (13).

The fixed density (or energy) drive previously used in simulations (11) corresponds to a normal driven interface. Thus, the situation is such that \( H(x,t = 0) = 0 \), \( \zeta = L^{-d} \sum x F(x,0) \) with \( F(x,t) = F(x,0) \), and periodic BCs and \( \epsilon(x,H) \equiv 0 \) such that no ‘grains’ are lost. The control parameter \( \Delta = \zeta - \zeta_c \) is varied and criticality is only obtained when \( \Delta = 0 \); note that choosing \( \zeta \) corresponds to using a spatially dependent force \( F(x,0) \) with \( \zeta = \langle F(x,0) \rangle \). Here, the system is not generally in the SOC steady-state but by letting the control parameter \( \Delta \rightarrow 0 \) one reaches the critical point (6,11). The noise is set at the beginning of an avalanche at the columnar values \( F(x,0) \). Depending on the exact nature of the initial configuration one may observe a different dynamic behavior but the steady-state behavior should correspond to the slowly driven case (11).

In “microcanonical” simulations (14) one has dissipation operating on the slow time scale with exactly the same rate as \( F(x,t) \). Thus microcanonical simulations correspond to fixed density simulations with a specific initial configuration: after each avalanche, the time is reset to zero, the force is replaced with \( F \rightarrow F + \nabla^2 H \), and the forces at \( x' \) (\( x'' \)) are increased (decreased) by one unit where \( x' \) and \( x'' \) are randomly chosen sites. Finally the interface is initialized, \( H \equiv 0 \). Since the \( \nabla^2 H \) term does not introduce correlations this new starting condition is equivalent to the fixed density case but with the initial configuration chosen to be in the SOC steady state.

Combining the scaling relations (3) and (4) it follows that

\[
2 + d = D + 1/\nu,
\]

where \( D = d + \chi \). In addition, the average area scales as \( \langle \ell^d \rangle \sim L^{1/\nu} \). These relations are also valid for sandpiles and Eq. (4) provides estimates for \( \nu \): in \( d = 1 \), \( \nu \approx 1.30 \), and in \( d = 2 \), \( \nu \approx 0.78 \). Numerical results yield \( \nu = 1.25(5) \) (\( d = 1 \), stochastic model) (14) and \( \nu = 0.79(4) \) (\( d = 2 \), BTW model) (13). Note, however, that the estimates quoted for \( \nu \) for sandpile models depend on the relation \( D = d + \chi \), which means that the underlying assumption is that the roughness exponent \( \chi \) can be defined for slowly driven sandpile models.

VI. CONCLUSIONS

In summary, we have started from the depinning equation (3) to discuss the continuum description of self-organized critical sandpile models. Thus, their upper critical dimension is \( d_c = 4 \) and a scaling relation for the correlation length exponent \( \nu \) is obtained. We find that the BTW model has columnar disorder \( F \) on the avalanche time scale whereas the stochastic models have explicitly point disorder included. Other models with slightly modified toppling rules (e.g., the Manna model (12)) may or may not belong to the same classes depending on the noise terms arising from the mapping (this we are currently investigating further in (24)). The present approach shows that the relevant noise for sandpiles is ‘quenched’. The physics of sandpiles is such that the random decisions or events (grain deposition, choices for thresholds) are frozen into the dynamics of a site as long as it is stable, and their memory decays only slowly as the activity goes on. A recent field theory for \( \rho(x,t) \) used analogies from systems with absorbing states and assumed that the noise was Reggeon field-theory like (i.e., time-dependent and not quenched) (14). Physically, the effect which is not incorporated in such Gaussian correlations is that the pinning forces along the interface selects a pinning path in the random media which stops the avalanche.

The mapping between interface and sandpile dynamics allows one to characterize the sandpile universality classes by the quenched noise in the interface equations. It also allows to gain novel insight about the previously introduced ways of reaching the depinning critical point: balancing the force with dissipation (slow drive, or self-organized criticality), tuning the average force (as for fixed density sandpiles), tuning the interface velocity (extremal drive criticality), and finally tuning the driving force. This becomes possible because of the diffusive character of interface or sandpile dynamics and because of the Abelian character of the linear interface equation.

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