SOME REMARKS ON THE STRUCTURE OF FINITE MORSE INDEX
SOLUTIONS TO THE ALLEN-CAHN EQUATION IN $\mathbb{R}^2$

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ABSTRACT. For a solution of the Allen-Cahn equation in $\mathbb{R}^2$, under the natural linear growth energy bound, we show that the blowing down limit is unique. Furthermore, if the solution has finite Morse index, the blowing down limit satisfies the multiplicity one property.

1. Introduction

Let $u \in C^2(\mathbb{R}^2)$ be a solution to the problem

$$\Delta u = W'(u)$$

where $W$ is a standard double-well potential.

Assume the energy grows linearly, i.e. there exists a constant $C > 0$ such that

$$\int_{B_R(0)} \frac{1}{2} |\nabla u|^2 + W(u) \leq CR, \quad \forall R > 0.$$ (1.2)

For $\varepsilon \to 0$, let $u_\varepsilon(x, y) := u(\varepsilon^{-1} x, \varepsilon^{-1} y)$.

By (1.2), we can assume that, up to a subsequence of $\varepsilon \to 0$,

$$\begin{align*}
\varepsilon \|
abla u_\varepsilon \|^2 & \to \mu_1, \\
\frac{1}{\varepsilon} W(u_\varepsilon) & \to \mu_2,
\end{align*}$$

weakly as Radon measures on any compact set of $\mathbb{R}^2$. Denote $\mu = \mu_1 / 2 + \mu_2$ and $\Sigma = \text{spt} \mu$.

We can also assume the matrix valued measures

$$\varepsilon \nabla u_\varepsilon \otimes \nabla u_\varepsilon d x \to [\tau_{\alpha\beta}] \mu_1,$$

where $[\tau_{\alpha\beta}]$, $1 \leq \alpha, \beta \leq 2$, is measurable with respect to $\mu_1$. Moreover, $\tau$ is nonnegative definite $\mu_1$-almost everywhere and it satisfies

$$\sum_{\alpha=1}^{2} \tau_{\alpha\alpha} = 1, \quad \mu_1 - a.e.$$ (4)

By [4], we have the following characterization about the convergence of $u_\varepsilon$:

**Theorem 1.1.**

(i) $u_\varepsilon \to \pm 1$ uniformly on any compact set of $\mathbb{R}^2 \setminus \Sigma$;

(ii) there exists $N \in \mathbb{N}$ and $N$ unit vectors $e_i$, $1 \leq i \leq N$, such that $\Sigma = \bigcup_{i=1}^{N} L_i$, where

$$L_i := \{te_i : t \geq 0\};$$

(iii) $\mu_1 = 2\mu_2 = \sigma_0 \sum_{i=1}^{N} n_i \mathcal{H}^1 |_{L_i}$, where $\sigma_0$ is a constant and $n_i \in \mathbb{N}$;

(iv) $1 - \tau = e_i \otimes e_i$ on $L_i \setminus \{0\}$;

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Theorem 1.3. Let $u$ be a solution of
\begin{equation}
\text{(1.1)}
\end{equation}

In the above, the constant $\sigma_0$ is determined as follows. There exists a function $g \in C^2(\mathbb{R})$ satisfying
\begin{equation}
\begin{cases}
g'' = W'(g), & \text{on } \mathbb{R}, \\
g(0) = 0, \\
\lim_{t \to \pm\infty} g(t) = \pm 1.
\end{cases}
\end{equation}

Moreover, the following identity holds for $g$:
\begin{equation}
g'(t) = \sqrt{2W(g(t))} > 0, \quad \text{on } \mathbb{R}.
\end{equation}

As $t \to \pm\infty$, $g(t)$ converges to $\pm 1$ exponentially. Hence the following quantity is finite:
\begin{equation}
\sigma_0 := \int_{-\infty}^{+\infty} \frac{1}{2} |g'(t)|^2 + W(g(t))dt = \int_{-\infty}^{+\infty} |g'(t)|^2 dt.
\end{equation}

In this theorem, we do not claim the uniqueness of $\Sigma$ and $(n_i)$, because it is obtained by a compactness argument. It may depend on the subsequence of $\varepsilon \to 0$. Our first main result is

**Theorem 1.2.** $\Sigma$ and $(n_1, \cdots, n_N)$ is uniquely determined by $u$.

Next we further assume that $u$ has finite Morse index, i.e. the maximal dimension of linear subspaces of
\begin{equation}
\{\varphi \in C^\infty_0(\mathbb{R}^2) : \int_{\mathbb{R}^2} |\nabla \varphi|^2 + W''(u)\varphi^2 \leq 0\}
\end{equation}
is finite. This is equivalent to the fact that $u$ is stable outside a compact set (see [1]), i.e. there exists a compact set $K$ such that for any $\varphi \in C^\infty_0(\mathbb{R}^2 \setminus K)$,
\begin{equation}
\int_{\mathbb{R}^2} |\nabla \varphi|^2 + W''(u)\varphi^2 \geq 0.
\end{equation}

Our second result is

**Theorem 1.3.** Let $u$ be a solution of (1.1) with finite Morse index. Then in the blowing down limit, $n_i = 1$ for every $i = 1, \cdots, N$.

As in [2], we introduce the following notations. Assume $e_i$ are in clockwise order. For each $i = 1, \cdots, N$, let $L_i^\pm$ be the rays generated by the vector $(e_i + e_{i+1})/2$ and $(e_i + e_{i-1})/2$ respectively (with obvious modification at the end points $i = 1, N$). Denote $\Omega_i$ to be the cone bounded by $L_i^\pm$. Our final result says

**Theorem 1.4.** Let $u$ be a solution of (1.1) in $\mathbb{R}^2$ with finite Morse index, and $\Omega_i$ be defined as above. In each $\Omega_i$, which we assume to be the cone $\{-\lambda_i x < y < \lambda_i x\}$ for two positive constants $\lambda_i$, there exists three constants $C, R_0$ and $t_i$ such that
\begin{equation}
\sup_{-\lambda_i x < y < \lambda_i x} |u(x, y) - g(y - t_i)| \leq Ce^{-\frac{C}{x}}, \quad \forall x > R_0.
\end{equation}

If we have known Theorem 1.3 this theorem will follow from the refined asymptotic result in [2]. Here the point is, we can prove Theorem 1.3 and Theorem 1.4 at the same time. This will be achieved by adapting Gui’s method in [1] to the multiple interfaces setting.

It should be mentioned that it is conjectured that finite Morse index solutions of (1.1) satisfies the energy growth bound (1.2). On the other hand, if a solution satisfies the conclusion of Theorem 1.4 it has finite Morse index (see [3]).
In this paper, a point in $\mathbb{R}^2$ is denoted by $X = (x, y)$.

The organization of this paper is as follows. In Section 2 we prove Theorem 1.2, Theorem 1.3, and Theorem 1.4 is proved in Section 3 at the same time.

2. Uniqueness of the blowing down limit

By direct integration by parts, we get the stationary condition

$$\int_{\mathbb{R}^2} \left[ \frac{1}{2} \mathbf{\nabla} u^2 + W(u) \right] \, dx - DX(\mathbf{\nabla} u, \mathbf{\nabla} u) = 0, \quad \forall \mathbf{x} \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^2).$$

Following [9], this condition implies the existence of a function $U \in C^3(\mathbb{R}^2)$ satisfying

$$\mathbf{\nabla}^2 U = \left[ \frac{u_r^2 - u_u^2 + 2W(u)}{2u_ru_u} \quad \frac{2u_r u_u}{u_r^2 - u_u^2 + 2W(u)} \right].$$

Moreover, by the Modica inequality (see [8])

$$\frac{1}{2} |\mathbf{\nabla} u|^2 \leq W(u), \quad \text{in } \mathbb{R}^2,$$

$U$ is convex. After subtracting an affine function, we can assume $U(0) = 0$ and $\mathbf{\nabla} U(0) = 0$. Hence by the convexity of $U$, $U \geq 0$ in $\mathbb{R}^2$.

**Lemma 2.1.** There exists a constant $C$ such that,

$$U(x, y) \leq C(|x| + |y|), \quad \text{in } \mathbb{R}^2.$$

**Proof.** By definition,

$$\Delta U = 4W(u). \quad (2.1)$$

Then for any $R > 0$,

$$\int_{B_R(x)} \mathbf{U} = \int_0^R \frac{d}{dr} \left( \int_{B_r(x)} \mathbf{U} \right) = \int_0^R \frac{1}{2\pi r} \int_{B_r(x)} 4W(u) \leq CR,$$

where we have used (1.2).

The conclusion follows from this integral bound and the convexity of $U$. \hfill \Box

By this linear growth bound and the convexity of $U$, as $\varepsilon \to 0$,

$$U_\varepsilon(x, y) := \varepsilon U(\varepsilon^{-1} x, \varepsilon^{-1} y) \to U_\infty(x, y)$$

uniformly on compact sets of $\mathbb{R}^2$. Here $U_\infty$ is a 1-homogeneous, nonnegative convex function. By convexity, this limit is independent of subsequences of $\varepsilon \to 0$.

Take a sequence $\varepsilon_i \to 0$ such that the blowing down limit of $u_{\varepsilon_i}$ is $\Sigma = \cup_{i=1}^N \{ t\varepsilon_i : t \geq 0 \}$ and the density on $\{ t\varepsilon_i : t \geq 0 \}$ is $n_{\varepsilon_i}$. Then outside $\Sigma$, by the strict convexity of $W$ near $\pm 1$,

$$|\mathbf{\nabla} u_{\varepsilon_i}(X)|^2 + W(u_{\varepsilon_i}(X)) \leq C e^{-c \varepsilon_i^{-1} \text{dist}(X, \Sigma)}.$$

Because

$$\mathbf{\nabla}^2 U_{\varepsilon_i} = \left[ \frac{\varepsilon_i u_{\varepsilon_i}^2 - \varepsilon_i u_{\varepsilon_i} u_{\varepsilon_i}^2}{2\varepsilon_i u_{\varepsilon_i} u_{\varepsilon_i}} \quad \frac{2\varepsilon_i u_{\varepsilon_i} u_{\varepsilon_i} u_{\varepsilon_i}^2}{\varepsilon_i u_{\varepsilon_i}^2 - \varepsilon_i u_{\varepsilon_i} u_{\varepsilon_i}^2} \right],$$

we also have

$$|\mathbf{\nabla}^2 U_{\varepsilon_i}(X)|^2 \leq C e^{-c \varepsilon_i^{-1} \text{dist}(X, \Sigma)}.$$

Hence $\mathbf{\nabla}^2 U_{\varepsilon_i} \equiv 0$ in $\mathbb{R}^2 \setminus \Sigma$, that is, $U_{\varepsilon_i}$ is linear in every connected component of $\mathbb{R}^2 \setminus \Sigma$. Thus the set $\{ U_{\varepsilon_i} < 1 \}$ is a convex polygon with its vertex points lying on $\Sigma$. Now it is clear that $\Sigma$ is uniquely determined by $U_{\varepsilon_i}$. Since $U_{\varepsilon_i}$ is independent of the choice of subsequences of $\varepsilon \to 0$, $\Sigma$ also does not depend on the choice of subsequences of $\varepsilon \to 0$. 


In a neighborhood of \( \{t \varepsilon \alpha : t \geq 0\} \), written in the \((e_i, e_\alpha)\) coordinates, the matrix valued measure \( \nabla^2 U_{e_i} \, dx_\alpha \) can be written as
\[
\nabla^2 U_{e_i} \, dx_\alpha = \left[\begin{array}{cc}
\varepsilon_i u_{e_i, e_\alpha}^2 - \varepsilon_i u_{e_\alpha, e_i}^2 + \varepsilon_i W(u_{e_i}) & \varepsilon_i u_{e_i, e_\alpha}^2 u_{e_\alpha, e_i}^2 \\
2 \varepsilon_i u_{e_\alpha, e_\alpha} u_{e_i, e_i}^2 - \varepsilon_i u_{e_i, e_\alpha}^2 - \varepsilon_i u_{e_\alpha, e_\alpha}^2 + \frac{2}{\varepsilon_i} W(u) \end{array}\right] \, dx_\alpha.
\]
By Theorem 1.1, after passing to the limit, we obtain that in a neighborhood of \( \{t \varepsilon \alpha : t \geq 0\} \), the limit of \( \nabla^2 U_{e_i} \, dx_\alpha \) equals
\[
\begin{pmatrix}
0 & 0 \\
0 & 2 n_\alpha \sigma_0 H^1_{|t \varepsilon \alpha : t \geq 0|} 
\end{pmatrix}.
\]
Hence across the ray \( \{t \varepsilon \alpha : t \geq 0\} \), \( \nabla U_\infty \) has a jump \( 2 n_\alpha \sigma_0 e_\alpha^+ \). In other words, let \( e^\pm = \nabla U_\infty \) on each side of \( \{t \varepsilon \alpha : t \geq 0\} \), then
\[
e^+ - e^- = 2 n_\alpha \sigma_0 e_\alpha^+.
\]
Thus \( n_\alpha \) is uniquely determined by \( U_\infty \). This proves Theorem 1.2.

3. The multiplicity one property

Since \( u \) is assumed to have finite Morse index, it is stable outside a compact set. Then standard argument using the stable De Giorgi theorem gives the following

**Lemma 3.1.** For any \( X_i = (x_i, y_i) \in u^{-1}(0) \to \infty \),
\[
u_i(x, y) := u(x_i + x, y_i + y)
\]
converges to a one dimensional solution \( g(e \cdot X) \) in \( C^2_{\text{loc}}(\mathbb{R}^2) \), where \( e \) is a unit vector.

Recall the cone \( \Omega_i \) introduced in Section 1. The nodal set of \( u \) in \( \Omega_i \) has the following description.

**Lemma 3.2.** There exists an \( R_i > 0 \) large such that, for each \( i \), in \( \Omega_i \setminus B_{R_i}(0) \), \( \{u = 0\} \)
consists of \( n_i \) curves, which can be represented by the graph of functions defined on \( L_i \),
with its \( C^1 \) norm converging to 0 at infinity.

**Proof.** Take an \( \Omega_i \), which we assume to be \( (-\lambda_+ x < y < \lambda_+ x) \) for two constants \( \lambda_+ > 0 \). \( L_i \)
is assumed to be the ray \( x > 0, y = 0 \). By [10] Theorem 5, for all \( \varepsilon \) small, there exists a constant \( t_\varepsilon \in (-\frac{1}{2}, 1/2) \), such that
\[
\{u_{e_i} = t_\varepsilon\} \cap (B_2 \setminus B_{1/2}) \cap \Omega_i
\]
consists of \( n_i \) curves in the form
\[
y = h^\alpha_\varepsilon(x), \quad \text{for } 1/2 \leq x \leq 2, \quad 1 \leq \alpha \leq n_i,
\]
where \( \|h^\alpha_\varepsilon\|_{C^{1,2}[1/2, 1]} \) is uniformly bounded. By [4], for each \( \alpha \), \( h^\alpha_\varepsilon \) converges to \( 0 \) uniformly on \([1/2, 2] \) as \( \varepsilon \to 0 \).

By Lemma [3.1] for each \( t \in [-3/4, 3/4] \), \( \{u_{e_i} = t\} \)
consists of \( n_i \) curves, in the form
\[
y = h^\alpha_\varepsilon(x, t), \quad \text{for } 1/2 \leq x \leq 2, \quad 1 \leq \alpha \leq n_i,
\]
which lies in an \( O(\varepsilon) \) neighborhood of \( \{u_{e_i} = t_\varepsilon\} \).
Moreover, after a scaling and using Lemma [5.1] we get
\[
\lim_{\varepsilon \to 0} \sup_{1/2 \leq x \leq 2} \left| \frac{d}{dx} h^\alpha_\varepsilon(x, t) \right| = 0.
\]
Rescaling back to \( u \) we conclude the proof. \( \square \)

Now we are in the following situation:
(H1) There are two positive constants $R > 0$ large and $\lambda > 0$.
(H2) The domain $C := \{(x, y) : |y| < \lambda x, x > R\}$.
(H3) $u \in C^2(C)$ satisfies (1.1) in $C$.
(H4) \{u = 0\} consists of $N$ curves $\{y = f_i(x)\}$, $1 \leq i \leq N$, where $f_i \in C^{\infty}[R, +\infty)$

The last condition implies that

$$f_1 < f_2 < \cdots < f_N,$$

$$\lim_{x \to +\infty} f_i(x) = 0, \quad \forall 1 \leq i \leq N.$$

The main goal in this section is to prove

**Theorem 3.3.** We must have $N = 1$. Moreover, there exists a constant $t$ such that

$$|f(x) - f| \leq Ce^{-\frac{t}{\sigma}}.$$

and

$$\sup_{-\lambda < y < \lambda} |u(x, y) - g(y) - t| \leq Ce^{-\frac{t}{\sigma}}.$$

where the constant $C$ depends only on $W$.

Theorem 3.3 and 1.4 follow from this theorem, Theorem 1.1 Theorem 1.2 and Lemma 3.4.

Possibly by a change of sign, assume $u < 0$ in $\{y < f_1(x)\}$.

**Lemma 3.4.** For any $1 \leq i \leq N$ and $t \to +\infty$,

$$u'(x, y) := u(t + x, f_i(t) + y)$$

converges to $g(y)$ in $C^2_{loc}(\mathbb{R}^2)$.

**Proof.** This is a consequence of Lemma 3.1 and Lemma 3.2. Note that $\{u' = 0\} = \{y = f'(x)\}$ where $f'(x) := f_i(x + t) - f_i(t)$. As $t \to +\infty$, $\frac{df_i}{dt}$ converges to $0$ uniformly on any compact set of $\mathbb{R}$. Hence by noting that $f'(0) = 0$, $f''$ also converges to $0$ uniformly on any compact set of $\mathbb{R}$. This implies that the limit $u'^\infty = 0$ on $\{y = 0\}$. From this we see $u_{\infty}(x, y) \equiv g(y)$. Since this limit is independent of subsequences of $t \to +\infty$, we finish the proof. \hfill \Box

**Lemma 3.5.** In $\overline{C}$,

$$1 - u(x, y)^2 \leq Ce^{-c \min(y - f_i(x))}.$$

**Proof.** By the previous lemma, for any $M > 0$, if we have chosen $R$ large enough, $u^2 > 1 - \sigma(M)$ in $\{(x, y) : |y - f_i(x)| > M, \forall i\}$, where $\sigma(M)$ is a constant depending on $M$ satisfying $\lim_{M \to +\infty} \sigma(M) = 0$. By choosing $M$ large (then $\sigma(M)$ can be made small so that $W$ is strictly convex in $(1 - \sigma(M), 1)$), in $\{(x, y) : |y - f_i(x)| > M, \forall i\}$,

$$\Delta W(u) \geq cW(u).$$

From this we deduce the exponential decay

$$W(u) \leq Ce^{-cdist(X, \{(x, y) : |y - f_i(x)| < M\})}.$$

Finally, because $|f'_i(x)| < 1$, the distance to $\{y = f_i(x)\}$ is comparable to $|y - f_i(x)|$. This finishes the proof. \hfill \Box
As a consequence,

\[ 1 - u(x,y)^2 \sim O(e^{-cx}) \quad \text{on } \{ y = \pm Ax \}. \tag{3.1} \]

Another consequence of this exponential decay is:

**Corollary 3.6.** In $C$,

\[ |u_t(x,y)| + |u_{xx}(x,y)| \leq Ce^{-\min_{i,y} \frac{dy}{f_i(x)}}. \]

This follows from standard gradient estimates.

This exponential decay implies that

\[ \int_{-Ax}^{Ax} u_x(x,y)^2 + u_{xx}(x,y)^2 \, dy \leq C, \quad \forall x > R. \tag{3.2} \]

**Lemma 3.7.** For any $1 \leq i \leq N - 1$,

\[ \lim_{x \to +\infty} (f_{i+1}(x) - f_i(x)) = +\infty. \]

**Proof.** By Lemma 3.4 for any $t \to +\infty$,

\[ u'(x,y) := u(x + t, y + f_i(t)) \]

converges uniformly to $g(y)$ on any compact set of $\mathbb{R}^2$.

From this we see, for any $L > 0$, if $t$ is large enough, $u' > 0$ on $\{ x = 0, 0 < y < L \}$ and $u' < 0$ on $\{ x = 0, -L < y < 0 \}$. The conclusion follows from this claim directly. \(\Box\)

**Proposition 3.8.** For any $x > R$,

\[ \int_{-Ax}^{Ax} \frac{u_x^2 - u_{xx}^2}{2} + W(u) \, dy = N\sigma_0 + O(e^{-cx}). \]

**Proof.** This is the Hamiltonian identity, see [3].

First, differentiating in $x$, integrating by parts and using (3.1) leads to

\[ \frac{d}{dx} \int_{-Ax}^{Ax} \frac{u_x^2 - u_{xx}^2}{2} + W(u) \, dy = O(e^{-cx}). \tag{3.3} \]

Next, by Lemma 3.5 for any $\delta > 0$, there exists an $L > 0$ such that for all $x$,

\[ \int_{\{ y \in (-Ax, Ax) \cap [-f_i(x), f_i(x)] \}} \frac{u_x^2 - u_{xx}^2}{2} + W(u) \, dy \leq \delta. \tag{3.4} \]

While for each $i = 1, \cdots, N$, by Lemma 3.4, we have

\[ \lim_{x \to +\infty} \int_{f_i(x) - L}^{f_i(x) + L} \frac{u_x^2 - u_{xx}^2}{2} + W(u) \, dy = \int_{-L}^{L} \frac{1}{2} g'(y)^2 + W(g(y)) \, dy = \sigma_0 + O(\delta), \tag{3.5} \]

where in the last step we have used the exponential convergence of $g$ at infinity.

Combining (3.4) and (3.5), by noting that $\delta$ can be arbitrarily small, we get

\[ \lim_{x \to +\infty} \int_{-Ax}^{Ax} \frac{u_x^2 - u_{xx}^2}{2} + W(u) \, dy = N\sigma_0. \]

The conclusion of this lemma follows by combining this identity and (3.3). \(\Box\)
Proposition 3.9. There exist two constants $L_0 > 0$ and $\mu > 0$ so that the following holds. For any constants $L^+ > L_0$ and $L^- > L_0$ and $v \in H^1(-L^- , L^+)$ satisfying
\[ \int_{-L^-}^{L^+} v(t) g'(t) dt = 0, \] (3.6)
we have
\[ \int_{-L^-}^{L^+} \left( \frac{dv}{dt}(t) \right)^2 + W''(g(t))v(t)^2 dt \geq \mu \int_{-L^-}^{L^+} v(t)^2 dt. \] (3.7)

Proof. Assume by the contrary, there exist $L_j^+ \to +\infty$ and $v_j \in H^1(-L_j^-, L_j^+)$ satisfying
\[ \int_{-L_j^-}^{L_j^+} v_j(t) g'(t) dt = 0, \] (3.8)
and
\[ \int_{-L_j^-}^{L_j^+} v_j(t)^2 dt = 1, \] (3.9)
but
\[ \int_{-L_j^-}^{L_j^+} \left( \frac{dv_j}{dt}(t) \right)^2 + W''(g(t))v_j(t)^2 dt \leq \frac{1}{j}. \] (3.10)

From the last two assumptions we deduce that
\[ \int_{-L_j^-}^{L_j^+} \left( \frac{dv_j}{dt}(t) \right)^2 dt \leq C, \] (3.11)
for some constant $C$ depending only on $\sup |W''|$. Hence the $1/2$-Hölder seminorm of $v_j$ is uniformly bounded. Then by (3.9), $\sup |v_j|$ is also uniformly bounded. Assume $v_j$ converges to $v_\infty$ in $C_{loc}(\mathbb{R})$.

By the exponential decay of $g'$ at infinity, (3.8) can be passed to the limit, which gives
\[ \int_{-\infty}^{+\infty} v_\infty(t) g'(t) dt = 0. \] (3.12)
(3.9) and (3.11) can also be passed to the limit, leading to
\[ \int_{-\infty}^{+\infty} v_\infty(t)^2 + \left( \frac{dv_\infty}{dt}(t) \right)^2 dt \leq C + 1. \] (3.13)

Because $g$ converges to $\pm 1$ at $\pm \infty$ respectively, there exists an $R_2$ such that
\[ W''(g(t)) \geq c_0 := \frac{1}{2} \min\{W''(-1), W''(1)\} > 0, \quad \text{in } |t| \geq R_2. \] (3.14)

Thus for any $R \geq R_2$,
\[ \int_{-R}^{R} \left( \frac{dv_\infty}{dt}(t) \right)^2 + W''(g(t))v_\infty(t)^2 dt \leq \liminf_{j \to +\infty} \int_{-R}^{R} \left( \frac{dv_j}{dt}(t) \right)^2 + W''(g(t))v_j(t)^2 dt \]
\[ \leq \liminf_{j \to +\infty} \int_{-L_j^-}^{L_j^+} \left( \frac{dv_j}{dt}(t) \right)^2 + W''(g(t))v_j(t)^2 dt \]
\[ \leq 0. \]

By (3.13), we can let $R \to +\infty$, which leads to
\[ \int_{-\infty}^{+\infty} \left( \frac{dv_\infty}{dt}(t) \right)^2 + W''(g(t))v_\infty(t)^2 dt \leq 0. \]
Moreover, for any \( v_j \) in \( C_{loc}(\mathbb{R}) \),

\[
\lim_{j \to +\infty} \int_{-R_1}^{R_1} v_j(t)^2 \, dt = 0.
\]

(3.15)

Substituting this into (3.10), by noting (3.14), we get

Proof of Theorem 3.3. \( \square \)

Combining this with (3.15) we get a contradiction with (3.9). Thus under the assumptions (3.8) and (3.9), (3.10) cannot hold. \( \square \)

With these preliminaries, we come to the proof of Theorem 3.3.

**Proof of Theorem 3.3** Given a tuple \( (t_1, \ldots, t_N) \) with \( t_1 < \cdots < t_N \), define

\[
g(y; t_1, \ldots, t_N) = \begin{cases} 
  g(y - t_1), & y < t_1^+ \\
  \min\{g(y - t_1), -g(y - t_2)\}, & t_1^+ < y < t_2^+, \\
  \min\{-g(y - t_2), g(y - t_3)\}, & t_2^+ < y < t_3^+, \\
  \ldots. & 
\end{cases}
\]

In the above,

\[
t_1^+ := \frac{t_1 + t_2}{2}, \quad t_1^- := \frac{t_1 - t_2}{2},
\]

and for simplicity of notation \( t_i^- = -\lambda x \) and \( t_i^+ = \lambda x \).

Note that \( g(y; t_i) \) is continuous, while its derivative in \( t \) has a jump at \( t_i^+ \). (In fact, the left and right derivatives at each \( t_i^+ \) only differ by a sign.)

Next we define

\[
F(x; t_1, \ldots, t_N) := \int_{-\lambda x}^{\lambda x} \left| u(x, y) - g(y; t_1, \ldots, t_N) \right|^2 \, dy.
\]

We divide the proof into three steps.

**Step 1.** As \( x \to +\infty \),

\[
\int_{-\lambda x}^{\lambda x} \left| u(x, y) - g(y; f_i(x)) \right|^2 \, dy \to 0.
\]

This follows from Lemma 3.4 and Lemma 3.5.

**Step 2.** By Step 1,

\[
\lim_{x \to +\infty} F(x; f_1(x), \ldots, f_N(x)) = 0.
\]

Moreover, for any \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that, if \( |t_i - f_i(x)| > \delta \) for some \( i \), then

\[
\liminf_{x \to +\infty} F(x; t_1, \ldots, t_N) \geq \epsilon.
\]

(3.16)

Direct calculations give

\[
\frac{\partial F}{\partial t_i}(x; t_1, \ldots, t_N) = 2(-1)^i \int_{t_i}^{t_i+} \left[ u(x, y) - (-1)^i \right] \frac{g'(y - t_i)}{2} \, dy.
\]

(3.17)

\[
\frac{\partial^2 F}{\partial t_i^2}(x; t_1, \ldots, t_N) = \int_{t_i}^{t_i+} g''(y - t_i) \, dy + 2(-1)^i \int_{t_i}^{t_i+} \left[ u(x, y) - (-1)^i \right] g''(y - t_i) \, dy.
\]

(3.18)
By Step 1, Lemma 3.5 and the exponential decay of $g''$ at infinity, there exists a $\sigma > 0$ such that, for any $(t_1, \cdots, t_N)$ satisfying $|t_i - f_i(x)| < \sigma$, $\frac{d^2 F}{d t_i} (x; t_i) > \sigma$.

Finally, if $|i - j| > 1$, $\left| \frac{d^2 F}{d t_i d t_j} (x; t_i) \right| \leq C e^{-\sigma (t_i - t_j)}$.

Combining this with (3.13) we see $\left[ \frac{d^2 F}{d t_i d t_j} (x; t_i) \right]$ is positively definite for those $(t_1, \cdots, t_N)$ satisfying the condition that $|t_i - f_i(x)|$ is small enough for all $i$.

Combining the above analysis, we see for all $x$ large, there exists a unique tuple $(t_i(x))$ such that

$$F(x; t_i(x)) = \min_{(t_i) \in \mathbb{R}^N} F(x, t_i).$$

Moreover,

$$\lim_{x \to +\infty} |t_i(x) - f_i(x)| = 0, \quad \forall 1 \leq i \leq N. \quad (3.19)$$

By the implicit function theorem, for each $i$, $t_i(x)$ is twice differentiable in $x$.

Lemma 3.7 and (3.19) implies that for any $1 \leq i \leq N - 1$,

$$t_{i+1}(x) - t_i(x) \to +\infty, \quad \text{as } x \to +\infty. \quad (3.20)$$

Let

$$v(x, y) := u(x, y) - g(y; t_i(x)).$$

Clearly

$$\lim_{x \to +\infty} \|v\|_{L^2(-\infty, x)} = \lim_{x \to +\infty} F(x; t_i(x)) = 0. \quad (3.21)$$

In the following we denote $g^* := g(y; t_i(x))$ and

$$g_i(y) := (-1)^{i-1} g(y - t_i(x)), \quad \text{for } y \in (t_i^-, t_i^+).$$

By definition,

$$0 = \frac{\partial F}{\partial t_i} (x; t_i(x)) = 2 \int_{t_i^{-}}^{t_i^{+}} (u - g_i) g_i'. \quad (3.22)$$

Differentiating (3.22) with respect to $x$ leads to

$$\left[ \int_{t_i^{-}}^{t_i^{+}} |g_i'|^2 - (u - g_i) g_i'' \right] t_i'(x) + \int_{t_i^{-}}^{t_i^{+}} u_i g_i' \right| = - \frac{[u(x, t_i'(x)) - g_i(t_i'(x))]}{[g_i'(t_i'(x))]} \frac{t_i'(x)}{2} + \frac{[u(x, t_i'(x)) - g_i(t_i'(x))]}{[g_i'(t_i'(x))]} \frac{t_i'(x) + t_i'(x)}{2}. \quad (3.23)$$

Note that by the result in Step 1 and the exponential decay of $g''$ at infinity,

$$\lim_{x \to +\infty} \int_{t_i^{-}}^{t_i^{+}} (u - g_i) g'' \leq \lim_{x \to +\infty} \left[ \int_{t_i^{-}}^{t_i^{+}} (u - g_i)^2 \right]^{\frac{1}{2}} \left[ \int_{t_i^{-}}^{t_i^{+}} |g_i'|^2 \right]^{\frac{1}{2}} = 0,$$

while by (3.20), there exists a constant $c > 0$ such that

$$\int_{t_i^{-}}^{t_i^{+}} |g_i'|^2 \geq c, \quad \forall x \text{ large}$$. 

By Lemma 3.5 and (3.20), \( u(x, t^*_i(x)) \) and \( g_i(t^*_i(x)) \) all converge to 0 as \( x \to +\infty \). Thus by (3.24) we obtain

\[
\dot{t}_i'(x) = - \frac{\int_{\mathcal{C}_i(x)} u_i g'_i}{\int_{\mathcal{C}_i(x)} |g'_i|^2} + o(1) \sum_{j \neq i} \frac{\int_{\mathcal{C}_j(x)} u_j g'_j}{\int_{\mathcal{C}_j(x)} |g'_j|^2} + O(e^{-c_i}) \to 0, \quad \text{as} \ x \to +\infty.
\]

(3.24)

Differentiating this once again we see \( t''_i(x) \) also converges to 0 as \( x \to +\infty \).

Similar to the calculation in [3] page 927, we have

\[
\int_{\mathcal{C}_i(x)} \left( \frac{u_i^2 - u_i^2}{2} + W(u) \right) - \frac{|g'_i|^2}{2} - W(g_i)
\]

\[
= \int_{\mathcal{C}_i(x)} \left( \frac{u_i^2 - |g'_i|^2}{2} + W(u) - W(g_i) - \frac{u_i^2}{2} \right)
\]

\[
= \int_{\mathcal{C}_i(x)} \left[ W(u) - W(g_i) - \frac{W'(u) + W'(g_i)}{2} (u - g_i) \right]
\]

\[
+ \frac{1}{2} \int_{\mathcal{C}_i(x)} (u - g_i) u_{xx} - u_i^2 \right] + \mathcal{B},
\]

where \( \mathcal{B} \) is the boundary terms coming from integrating by parts. In the above we have used

\[
\int_{\mathcal{C}_i(x)} u_i^2 - |g'_i|^2 = \int_{\mathcal{C}_i(x)} (u_i - g'_i) (u_i + g'_i)
\]

\[
= - \int_{\mathcal{C}_i(x)} (u - g_i) (u_{yy} + g''_i)
\]

\[
+ [u(x, t^*_i(x)) - g_i(t^*_i(x))] [u_i(x, t^*_i(x)) + g'_i(t^*_i(x))]
\]

\[
- [u(x, t^*_i(x)) - g_i(t^*_i(x))] [u_i(x, t^*_i(x)) + g'_i(t^*_i(x))]
\]

\[
= - \int_{\mathcal{C}_i(x)} (u - g_i) \left[ W'(u) + W'(g_i) \right] + \int_{\mathcal{C}_i(x)} u_{xx} (u - g_i)
\]

\[
+ [u(x, t^*_i(x)) - g_i(t^*_i(x))] [u_i(x, t^*_i(x)) + g'_i(t^*_i(x))]
\]

\[
- [u(x, t^*_i(x)) - g_i(t^*_i(x))] [u_i(x, t^*_i(x)) + g'_i(t^*_i(x))].
\]

Summing in \( i \) and using the Hamiltonian identity, we obtain

\[
\int_{-\Delta x}^{\Delta x} u_{xx} (u - g^*) - u_i^2 = \sum_i \int_{\mathcal{C}_i(x)} \left[ (u - g_i) u_{xx} - u_i^2 \right]
\]

\[
= -2 \sum_i \left[ [u(x, t^*_i(x)) - g_i(t^*_i(x))] g'_i(t^*_i(x)) \right] + o(||v||^2)
\]

\[
+ 2 \sum_i \left[ \int_{-\infty}^{+\infty} |g'_i|^2 + \int_{-\infty}^{+\infty} |g'_i|^2 \right] + O(e^{-c_i}).
\]

(3.25)

On the other hand, similar to [3] Eq. (4.35), we have

\[
\int_{\mathcal{C}_i(x)} u_{xx} (u - g_i) = \int_{\mathcal{C}_i(x)} \left( W'(u) - u_{yy} \right) (u - g_i)
\]
Combining this with (3.25), we deduce that
\[ 2 \int H \, dx \, d \int (u - g_i)^2 + W''(g_i) (u - g_i)^2 \]
+ \[ \int \sum_{i}^{n} \left( g_{i}^{\prime\prime} - u_{yy} \right) (u - g_i) + W''(g_i) (u - g_i)^2 \]
= \[ o(\|v\|^2) + \int \sum_{i}^{n} \left| (u - g_i)_y \right|^2 + W''(g_i) (u - g_i)^2 \]
- \[ \left[ u(x, t_i^*(x)) - g_i(t_i^*(x)) \right] \left[ u'_y(x, t_i^*(x)) - g_i'(t_i^*(x)) \right] \]
+ \[ \left[ u(x, t_i^*(x)) - g_i(t_i^*(x)) \right] \left[ u'_y(x, t_i^*(x)) - g_i'(t_i^*(x)) \right] \].

Summing in \( i \) we get
\[ \int_{-\lambda x}^{\lambda x} u_{xx} (u - g^*) = o(\|v\|^2) + \sum_{i}^{n} \left| (u - g_i)_y \right|^2 + W''(g_i) (u - g_i)^2 \]
+ \[ 2 \sum_{i}^{n} \left[ u(x, t_i^*(x)) - g_i(t_i^*(x)) \right] g_i'(t_i^*(x)) + O(e^{-c\lambda x}). \]  

By (3.22) and (3.20), Proposition 3.9 applies to \( u - g_i \) in \( (t_i^*(x), t_i^*(x)) \), which gives
\[ \int \sum_{i}^{n} \left| (u - g_i)_y \right|^2 + W''(g_i) (u - g_i)^2 \geq \mu \int \sum_{i}^{n} (u - g_i)^2 . \]

Hence
\[ \int_{-\lambda x}^{\lambda x} u_{xx} (u - g^*) \geq (\mu + o(1)) \|v\|^2 + 2 \sum_{i}^{n} \left[ u(x, t_i^*(x)) - g_i(t_i^*(x)) \right] g_i'(t_i^*(x)) + O(e^{-c\lambda x}) . \]

Combining this with (3.25), we deduce that
\[ \int_{-\lambda x}^{\lambda x} u_x^2 \geq (\mu + o(1)) \|v\|^2 + 4 \sum_{i}^{n} \left[ u(x, t_i^*(x)) - g_i(t_i^*(x)) \right] g_i'(t_i^*(x)) \]
- \[ 2 \sum_{i}^{n} \left[ \int_{t_i^*(x)}^{\infty} |g_i'|^2 + \int_{-\infty}^{t_i^*(x)} |g_i'|^2 \right] + O(e^{-c\lambda x}). \]

Differentiating \( \|v\|^2 \) twice in \( x \) leads to
\[ \frac{1}{2} \frac{d}{dx} \|v\|^2 = \sum_{i}^{n} \int_{t_i^*(x)}^{\infty} (u - g_i) \left[ u_x + g_i^\prime t_i'(x) \right] \]
\[ = \sum_{i}^{n} \int_{t_i^*(x)}^{\infty} (u - g_i) u_x, \quad \text{by (3.22)} \]
and
\[ \frac{1}{2} \frac{d^2}{dx^2} \|v\|^2 = \sum_{i}^{n} \int_{t_i^*(x)}^{\infty} u_x^2 + u_x g_i^\prime t_i'(x) + u_{xx} (u - g_i) \]
\[ \geq 2 \sum_{i}^{n} \int_{t_i^*(x)}^{\infty} u_x^2 - \frac{3}{2} \sum_{i}^{n} \left( \int_{t_i^*(x)}^{\infty} u_x g_i^\prime \right)^2 \quad \text{by (3.25) and (3.24)} \]
Then by (3.24) and the Cauchy-Schwarz inequality, we get

$$\sum_{i} \left( \int_{\tau_i^f(x)}^{\infty} |g_i'|^2 + \int_{-\infty}^{\tau_i^f(x)} |g_i''|^2 \right) \geq \frac{1}{2} \sum_{i} \int_{\tau_i^f(x)}^{\infty} |g_i''|^2 + \frac{1}{2} \sum_{i} \left[ \int_{\tau_i^f(x)}^{\infty} |g_i'|^2 + \int_{-\infty}^{\tau_i^f(x)} |g_i''|^2 \right]$$

By noting (3.21), from this inequality we deduce that

$$\|v\|^2 \leq Ce^{-cx}, \quad \text{for all } x \text{ large.} \quad (3.31)$$

**Step 3.** Note that

$$g_i(t_i^f(x))g_i'(t_i^f(x)) = \int_{t_i^f(x)}^{\infty} |g_i'|^2 + g_i''$$

because $g_i$ is close to 1 in $(t_i^f(x), +\infty)$ (see (3.20)) and hence $g_i'' = W_1(g_i) < 0$ in this interval. We also have $g_i(t_i^f(x))g_i'(t_i^f(x)) > 0$, because $g_i(t_i^f(x)) > 0$ and $g_i'(t_i^f(x)) > 0$.

Then for all $x$ large, by noting that $g_i(t_i^f(x))$ is close to 1 and $u(x, t_i^f(x)) - g_i(t_i^f(x))$ is close to 0, we obtain

$$\left| u(x, t_i^f(x)) - g_i(t_i^f(x)) \right| g_i'(t_i^f(x)) \leq \frac{1}{2} g_i(t_i^f(x))g_i'(t_i^f(x)) \leq \frac{1}{2} \int_{t_i^f(x)}^{\infty} |g_i'|^2.$$  

Substituting this into (3.25), we get

$$\int_{x}^{\infty} u_x^2 (u - g_i) + o(\|v\|^2) + O(e^{-cx})$$

$$\leq \left[ \int_{x}^{\infty} u_x^2 \right] \|v\|^2 + O(\|v\|^2) + O(e^{-cx}) \quad (3.32)$$

Then by (3.24) and the Cauchy-Schwarz inequality, we get

$$|l_i^f(x)| \leq Ce^{-cx}, \quad \forall i.$$  

Thus for all $1 \leq i \leq N$, $\lim_{x \to +\infty} t_i(x)$ exists and it is finite. By noting (3.19), for each $i$, the limit $\lim_{x \to +\infty} f_i(x)$ also exists. In particular, this limit is finite. Then for all $1 \leq i \leq N - 1$, $\lim_{x \to +\infty} (f_{i+1}(x) - f_i(x))$ also exists and it is finite. However, this is a contradiction with Lemma [3.7] if $N \geq 2$. Hence we must have $N = 1$.

Finally, the exponential convergence of $u(x, \cdot)$ follows from (3.32), and the exponential convergence of $f_i(x)$ follows from this exponential convergence and the (uniform) positive lower bound on $g'$ and $u_j(x, \cdot)$ in the part where $|g| < 1/2$ and $|u| < 1/2$.  

$\square$
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