Generalized Dirichlet to Neumann operator on invariant differential forms and equivariant cohomology

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Abstract

In recent work, Belishev and Sharafutdinov show that the generalized Dirichlet to Neumann (DN) operator $\Lambda_{\text{cl}}$ on a compact Riemannian manifold $M$ with boundary $\partial M$ determines de Rham cohomology groups of $M$. In this paper, we suppose $G$ is a torus acting by isometries on $M$. Given $X$ in the Lie algebra of $G$ and the corresponding vector field $X_M$ on $M$, Witten defines an inhomogeneous coboundary operator $d + \iota_{X_M}$ on invariant forms on $M$. The main purpose is to adapt Belishev-Sharafutdinov’s boundary data to invariant forms in terms of the operator $d + \iota_{X_M}$ in order to investigate to what extent the equivariant topology of a manifold is determined by the corresponding variant of the DN map. We define an operator $\Lambda_{X_M}$ on invariant forms on the boundary which we call the $X_M$-DN map and using this we recover the $X_M$-cohomology groups from the generalized boundary data $(\partial M, \Lambda_{X_M})$. This shows that for a Zariski-open subset of the Lie algebra, $\Lambda_{X_M}$ determines the free part of the relative and absolute equivariant cohomology groups of $M$. In addition, we partially determine the ring structure of $X_M$-cohomology groups from $\Lambda_{X_M}$.

Keywords: Algebraic Topology, equivariant topology, equivariant cohomology, cup product (ring structure), group actions, Dirichlet to Neumann operator.

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1. Introduction

The classical Dirichlet-to-Neumann (DN) operator $\Lambda_{\text{cl}} : C^\infty(\partial M) \longrightarrow C^\infty(\partial M)$ is defined by $\Lambda_{\text{cl}} \theta = \partial \omega / \partial \nu$, where $\omega$ is the solution to the Dirichlet problem

$$\Delta \omega = 0, \quad \omega|_{\partial M} = \theta$$

and $\nu$ is the unit outer normal to the boundary. In the scope of inverse problems of reconstructing a manifold from the boundary measurements, the following question is of great theoretical and applied interest [7]: to what extent are the topology and geometry of $M$ determined by the DN operator?

In this paper we are interested in the equivariant topology analogue of this question. Much effort has been made to address this (non-equivariant) question. For instance, in the case of a two-dimensional manifold $M$ with a connected boundary, an explicit formula is obtained which expresses the Euler characteristic of $M$ in terms of $\Lambda_{\text{cl}}$ and the Euler characteristic completely determines the topology of $M$ in this case [3]. In the three-dimensional case [5], some formulas are obtained which express the Betti numbers $b_i(M)$ and $b_2(M)$ in terms of $\Lambda_{\text{cl}}$ and $X : C^\infty(T(\partial M)) \longrightarrow C^\infty(T(\partial M))$. This culminates in recent work of Belishev and Sharafutdinov [7] who prove that the real additive de Rham cohomology of a compact, connected, oriented smooth Riemannian manifold $M$ of dimension $n$ with boundary is completely determined by its boundary data $(\partial M, \Lambda)$ where $\Lambda : \Omega^k(\partial M) \longrightarrow \Omega^{n-k-1}(\partial M)$ is a generalization of the classical Dirichlet-to-Neumann operator $\Lambda_{\text{cl}}$ to the space of
differential forms. More precisely, they define the DN operator $\Lambda$ as follows: given $\theta \in \Omega^k(\partial M)$, the boundary value problem
\[ \Delta \theta = 0, \quad i^* \partial \theta = 0, \quad i^* (\partial \theta) = 0 \] (1.1)
is solvable and the operator $\Lambda$ is given by the formula $\Lambda \theta = i^* (\partial \theta)$, where $i^*$ is the pullback by the inclusion map $i: \partial M \hookrightarrow M$. Here $\partial$ is the formal adjoint of $d$ relative to the $L^2$-inner product
\[ \langle \alpha, \beta \rangle = \int_M \alpha \wedge \beta \]
which is defined on $\Omega^k(M)$, and $\ast : \Omega^k \to \Omega^{n-k}$ is the Hodge star operator.

More concretely, there are two distinguished finite dimensional subspaces of $H^k(M) = \ker d \cap \ker \delta \subset \Omega^k(M)$, whose elements are called Dirichlet and Neumann harmonic fields respectively, namely
\[ H^k_D(M) = \{ \lambda \in H^k(M) \mid i^* \lambda = 0 \}, \quad H^k_N(M) = \{ \lambda \in H^k(M) \mid i^* \lambda = 0 \}. \]
The dimensions of these spaces are given by: $\dim H^k_D(M) = \dim H^k_N(M) = \beta_k(M)$, where $\beta_k(M)$ is the $k$th Betti number. They prove the following theorem:

**Theorem 1.1 (Belishev-Sharafutdinov [1])** For any $0 \leq k \leq n-1$, the range of the operator
\[ \Lambda + (-1)^{nk+k+\sigma} d \Delta^{-1} d : \Omega^k(\partial M) \to \Omega^{n-k-1}(\partial M) \]
is $i^* H^k_N(\partial M)$.

Since $i^* H^k_N(\partial M) \cong H^k_N(\partial M) \cong H^k(M)$, it follows that $(\Lambda + (-1)^{nk+k+1} d \Delta^{-1} d) \Omega^{n-k-1}(\partial M) \cong H^k(M)$. Using, Poincaré-Lefschetz duality, $H^k(M) \cong H^{n-k}(\partial M, \partial M)$, the theorem immediately implies that the data $(\partial M, \Lambda)$ determines both the absolute and relative de Rham cohomology groups.

In addition, they present the following theorem which gives the lower bound for the Betti numbers of the manifold $\mathcal{M}$ and its boundary through the DN operator $\Lambda$.

**Theorem 1.2 (Belishev-Sharafutdinov [1])** The kernel of $\Lambda$ contains the space $\mathcal{E}(\partial M)$ of exact forms and for each $k$,
\[ \dim \ker \Lambda^k / \mathcal{E}^k(\partial M) \leq \min \{ \beta_k(\partial M), \beta_k(M) \} \]
where $\beta_k(\partial M)$ and $\beta_k(M)$ are the Betti numbers, and $\Lambda^k$ is the restriction of $\Lambda$ to $\Omega^k(\partial M)$.

At the end of their paper, they posed the following problem: can the multiplicative structure of the cohomologies be recovered from the data $(\partial M, \Lambda)$?

To give a partial answer to this question, Shonkwiler [2] Sec. 5.3] defines the map
\[ (\phi, \psi) \mapsto (-1)^k \Lambda(\phi \wedge \Lambda^{-1} \psi), \quad \forall (\phi, \psi) \in i^* H^k_N(M) \times i^* H^k_D(M). \] (1.2)
More precisely, by using the classical wedge product between the differential forms, he considers the mixed cup product between the absolute cohomology $H^k(M, \mathbb{R})$ and the relative cohomology $H^k(\partial M, \partial M, \mathbb{R})$, i.e.
\[ \cup : H^k(M, \mathbb{R}) \times H^k(\partial M, \partial M, \mathbb{R}) \to H^{k+1}(\partial M, \partial M, \mathbb{R}) \]
and then he restricts the second argument to come from the boundary subspace. This subspace is defined by DeTurck and Gluck [3] as the subspace of $H^k(M, \partial M)$ consisting of exact forms which satisfy the Dirichlet boundary condition (i.e. $i^*$ of these exact forms are zero). Shonkwiler then presents the following partial answer to Belishev and Sharafutdinov’s question:

**Theorem 1.3 (Shonkwiler [2])** The boundary data $(\partial M, \Lambda)$ completely determines the mixed cup product in terms of the map (1.2) when the relative cohomology class is restricted to belong to the boundary subspace.
Equivariant setting. We briefly review some notation and results from [2]. Let $M$ be a compact, oriented, smooth Riemannian manifold with boundary and suppose $G$ is a torus acting by isometries on $M$. Denote by $\Omega^k_G$ the $k$-forms invariant under the $G$-action. Given $X$ in the Lie algebra $\mathfrak{g}$ of $G$ and corresponding vector field $X_M$ on $M$, consider Witten’s coboundary operator $dx_M = d + \iota_X$. This operator is no longer homogeneous in the degree of the smooth invariant form on $M$; if $\omega \in \Omega^k_G$, then $dx_M \omega \in \Omega^{k+1}_G \oplus \Omega^{k-1}_G$. Note then that $dx_M : \Omega^+_G \to \Omega^-_G$, where $\Omega^+_G$ is the space of invariant forms of even (+) or odd (−) degree. Let $\delta_{x_M}$ be the adjoint of $dx_M$ and define the resulting Witten-Hodge-Laplacian to be $\Delta_{x_M} = (dx_M + \delta_{x_M})^2 = dx_M \delta_{x_M} + \delta_{x_M} dx_M$.

Because the forms are invariant, it is easy to see that $\delta_{x_M}^2 = 0$ (see [2] for details). In this setting, we define two types of $X_M$-cohomology, the absolute $X_M$-cohomology $H^{\pm}_{X_M}(M)$ and the relative $X_M$-cohomology $H^{\pm}_{\partial X_M}(M, \partial M)$. The first is the cohomology of the complex $(\Omega^*_G, dx_M)$, while the second is the cohomology of the subcomplex $(\Omega^*_G, d, \delta_{x_M})$, where $\omega \in \Omega^*_G$ if it satisfies $i^* \omega = 0$ (the $D$ is for Dirichlet boundary condition). One also defines $\Omega^*_G(N, M) = \{ \alpha \in \Omega^*_G(N, M) | i^* (\omega) = 0 \}$ (Neumann boundary condition). Clearly, the Hodge star $*$ provides an isomorphism $\Omega^*_G(N, M) \cong \Omega^*_{N, G}(M)$, where we write $n \equiv \pm \mod 2$ resulting from subtracting an even/odd number from $n$. Furthermore, because $dx_M$ and $i^*$ commute, it follows that $dx_M$ preserves the Dirichlet boundary conditions while $dx_M$ preserves Neumann boundary conditions. Because of boundary terms, the null space of $\Delta_{x_M}$ does not coincide with the closed and co-closed forms in Witten’s sense. Elements of $\ker \Delta_{x_M}$ are called $X_M$-harmonic forms while the $\omega$ which satisfy $dx_M \omega = \delta_{x_M} \omega$ are $X_M$-harmonic fields; it is clear that every $X_M$-harmonic field is an $X_M$-harmonic form, but the converse is false. The infinite dimensional space of $X_M$-harmonic fields is denoted $H^\infty_{X_M}(M)$, so we have $H^\infty_{X_M}(M) \subset \ker \Delta_{x_M}$. Two useful finite dimensional subspaces of $H^\infty_{X_M}(M)$ are the Dirichlet and Neumann $X_M$-harmonic fields, respectively: $H_{X_M,D}(M)$ and $H_{X_M,N}(M)$. There are therefore two different candidates for $X_M$-harmonic representatives when the boundary is present. This construction firstly leads us to present the $X_M$-Hodge-Morrey decomposition theorem which states that

$$\Omega^*_G(M) = \mathcal{E}^\pm_{X_M}(M) \oplus \mathcal{C}^\pm_{X_M}(M) \oplus H^\pm_{X_M}(M)$$

(1.3)

where $\mathcal{E}^\pm_{X_M}(M) = \{ dx_M \alpha | \alpha \in \Omega^*_G \}$ and $\mathcal{C}^\pm_{X_M}(M) = \{ \delta_{x_M} \beta | \beta \in \Omega^*_{G,N} \}$. This decomposition is orthogonal with respect to the $L^2$-inner product given above.

In addition, in [2] we present an $X_M$-Friedrichs Decomposition Theorem which states that

$$H^\pm_{X_M}(M) = H^\pm_{X_M,D}(M) \oplus H^\pm_{X_M,\text{co}}(M)$$

and

$$H^\pm_{\partial X_M}(M) = H^\pm_{X_M,D}(M) \oplus H^\pm_{X_M,\text{co}}(M)$$

(1.4)

where $H^\pm_{X_M,\text{ex}}(M) = \{ \xi \in H^\pm_{X_M}(M) | \xi = dx_M \sigma \}$ and $H^\pm_{X_M,\text{co}}(M) = \{ \eta \in H^\pm_{X_M}(M) | \eta = \delta_{x_M} \alpha \}$. Together these give the orthogonal $X_M$-Hodge-Morrey-Friedrichs decompositions [2],

$$\Omega^*_G(M) = \mathcal{E}^\pm_{X_M}(M) \oplus \mathcal{C}^\pm_{X_M}(M) \oplus H^\pm_{X_M,D}(M) \oplus H^\pm_{X_M,\text{co}}(M)$$

(1.5)

$$= \mathcal{E}^\pm_{X_M}(M) \oplus \mathcal{C}^\pm_{X_M}(M) \oplus H^\pm_{X_M,D}(M) \oplus H^\pm_{\partial X_M}(M).$$

The two decompositions are related by the Hodge star operator. The orthogonality of (1.3)–(1.5) follows from Green’s formula for $dx_M$ and $\delta_{x_M}$ which states

$$\langle dx_M \alpha, \beta \rangle = \langle \alpha, \delta_{x_M} \beta \rangle + \int_\partial M i^* (\alpha \wedge \ast \beta)$$

(1.6)

for all $\alpha, \beta \in \Omega_*$. The consequence for $X_M$-cohomology is that each class in $H^\pm_{X_M}(M)$ is represented by a unique $X_M$-harmonic field in $H^\pm_{X_M,N}(M)$, and each relative class in $H^\pm_{X_M}(M, \partial M)$ is represented by a unique $X_M$-harmonic field in $H^\pm_{\partial X_M,D}(M)$. We also elucidate in [2] the connection between the $X_M$-cohomology groups and the free part of the relative and absolute equivariant cohomology groups.

The $X_M$-Hodge-Morrey-Friedrichs decompositions (1.5) of smooth invariant differential forms gives us insight to create boundary data which is a generalization of Belishev and Sharafutdinov’s boundary data on $\Omega^*_G(\partial M)$.

In this paper, we take a topological approach, looking to determine the $X_M$-cohomology groups and the free part of the equivariant cohomology groups from the generalized boundary data. To this end, in Section 2 we prove that the
concrete realizations $\mathcal{H}^±_{X_M,N}(M)$ and $\mathcal{H}^±_{X_M,D}(M)$ of the absolute and relative $X_M$-cohomology groups respectively meet only at the origin and in Section 3 we define the $X_M$-DN operator $\Lambda_{X_M}$ on $\Omega^2_D(\partial M)$, the definition involves showing that certain boundary value problems are solvable. The definition of $\Lambda_{X_M}$ represents a generalization of Belishev and Sharafutdinov’s DN-operator $\Lambda$ on $\Omega^2_D(\partial M)$ in the sense that when $X = 0$, we have $\Lambda_0 = \Lambda$. Finally, in the remaining sections, we explain how the boundary data $(\partial M, \Lambda_{X_M})$ encodes more information about the equivariant algebraic topology of $M$ than does the boundary data $(\partial M, \Lambda)$ on $\partial M$. Hence, these results contribute to explain to what extent the equivariant topology of the manifold in question is determined by the $X_M$-DN map $\Lambda_{X_M}$.

Throughout this paper, when arguments follow closely the corresponding arguments in the non-equivariant setting we refer to the original argument and omit the details. These details can be found in the first author’s thesis [1].

2. Main results

Throughout we let $M$ be a compact, connected, oriented, smooth Riemannian manifold with boundary and we suppose $G$ is a torus acting by isometries on $M$. Given $X$ in the Lie algebra $\mathfrak{g}$ and corresponding vector field $X_M$ on $M$, one defines Witten’s inhomogeneous coboundary operator $d_{X_M} = d + i_{X_M} : \Omega^2_g \to \Omega^2_g$ and the resulting $X_M$-harmonic and fields and forms as described in the introduction.

An important classical result is that any harmonic field satisfying both Neumann and Dirichlet boundary conditions (so one vanishing on the boundary) is necessarily zero: see Theorem 3.4.4 in [11] or Lemma 2 in [8].

Theorem 2.1 If an $X_M$-harmonic field $\lambda \in \mathcal{H}^±_{X_M}(M)$ vanishes on the boundary $\partial M$, then $\lambda \equiv 0$, i.e.

$$\mathcal{H}^±_{X_M,N}(M) \cap \mathcal{H}^±_{X_M,D}(M) = \{0\} \quad (2.1)$$

The proof consists in showing that a harmonic field which is both Neumann and Dirichlet has a zero of infinite order at every boundary point and then applying the Strong Unique Continuation Theorem below. However, the proof that there are zeros of infinite order in [11, 8] does not appear to extend to our present setting, so we give a different argument, based on Hadamard’s lemma, and which is also valid in the classical case.

First, we state the Strong Unique Continuation Theorem, due to Aronszajn [3], Aronszajn, Krzywicki and Szarski [4]. In [10], Kazdan writes this theorem in terms of Laplacian operator $\Delta$ but he mentions that it is still valid for any operator having the diagonal form $P = \Delta + I$ lower-order terms, where $I$ is the identity matrix. Hence, one can state this theorem in terms of diagonal form operator by the following form:

Theorem 2.2 (Strong Unique Continuation Theorem [10]) Let $\overline{M}$ be a Riemannian manifold with Lipschitz continuous metric, and let $\omega$ be a differential form having first derivatives in $L^2$ that satisfies $P(\omega) = 0$ where $P$ is a diagonal form operator. If $\omega$ has a zero of infinite order at some point in $\overline{M}$, then $\omega$ is identically zero on $\overline{M}$.

Proof of Theorem 2.1 Suppose $\lambda \in \mathcal{H}^±_{X_M,N}(M) \cap \mathcal{H}^±_{X_M,D}(M)$, then $\lambda$ is smooth by using the results of [2]. Since $\mathfrak{g}^{*} \lambda = \mathfrak{g}^* \lambda = 0$ then $\lambda \equiv 0$ on $M$ and we have that $(\iota_{X_M} \lambda)_{|\partial M} \equiv 0$ as well.

The proof is local so we can consider $M$ to be the upper half space in $\mathbb{R}^n$ with $\partial M = \mathbb{R}^{n-1}$. Since the metric, the differential form $\lambda$ and the vector field $X_M$ are given in the upper half space, we can extend them from there to all of $\mathbb{R}^n$ by reflection in $\partial M = \mathbb{R}^{n-1}$. The resulting objects are: the extended metric, which will be Lipschitz continuous [8]; we extend $\lambda$ to all of $\mathbb{R}^n$ by making it odd with respect to reflection in $\mathbb{R}^{n-1}$ and extend $X_M$ to all of $\mathbb{R}^n$ by making it even with respect to reflection in $\mathbb{R}^{n-1}$ and the extended $X_M$ will be a Lipschitz continuous vector field. But the original $\lambda$ satisfies $\lambda \equiv 0$ on $\mathbb{R}^n$ and $d_{X_M} \lambda = \delta_{X_M} \lambda = 0$ on the upper half space, hence the extended one will be of class $C^1$ and satisfy $d_{X_M} \lambda = \delta_{X_M} \lambda = 0$ on $\mathbb{R}^n$, i.e., the extended $\lambda$ satisfies $P(\lambda) = \Delta_{X_M} \lambda = 0$ on all of $\mathbb{R}^n$ where the operator $\Delta_{X_M}$ has diagonal form, i.e. $P = \Delta_{X_M} + \Delta + I$ lower-order terms. So far, we have satisfied the first condition of Theorem 2.2.

Now, we need to satisfy the remaining hypotheses of Theorem 2.2. Let $x = (x', x_n)$ be a coordinate chart where $x' = (x_1, x_2, \ldots, x_{n-1})$ is a chart on the boundary $\partial M$ and $x_n$ is the distance to the boundary. In these coordinates $x_n > 0$ in $M$ and $\partial M$ is locally characterized by $x_n = 0$. These coordinates are called boundary normal coordinates and the Riemannian metric in these coordinates has the form $\sum_{m,r=1}^{n-1} h_{m,r}(x) dx^m \otimes dx^r + dx^r \otimes dx^r$. 

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Now consider a neighborhood of $p \in \partial M$ where the boundary normal coordinates are well defined. We can write $\lambda = \alpha + \beta \wedge dx_n$ where $\alpha = \Sigma f_k(x)dx^i, \beta = \Sigma g_k(x)dx^j$ and $I_1, I_2 \subset \{1, 2, \ldots, n-1\}$. Our goal is to prove that all the partial derivatives of the coefficients of $\lambda$ (i.e. of $f_k(x)$ and $g_k(x)$) vanish at $p \in \partial M$. Now, $\lambda|_{\partial M} = 0$ which implies that $f_k(x, x^0) = g_k(x, x^0) = 0$. Hence, we can apply Hadamard’s lemma to $f_k(x)$ and $g_k(x)$ and write $f_k(x) = x_n \tilde{f}_k(x)$ and $g_k(x) = x_n \tilde{g}_k(x)$ for some smooth functions $\tilde{f}_k(x)$ and $\tilde{g}_k(x)$. Moreover, these representations for $f_k(x)$ and $g_k(x)$ imply that all the higher partial derivatives of $f_k(x)$ and $g_k(x)$ with respect to each of the $x$-coordinates (i.e. except the normal direction coordinate $x_n$) at the point $p$ are zero.

Therefore, we only need to prove that all the higher partial derivatives of $f_k(x)$ and $g_k(x)$ in the normal direction are zero to deduce that the Taylor series of $f_k(x)$ and $g_k(x)$ around $x_n = 0$ are zero. The proof of this is by contradiction.

Suppose the Taylor series of $f_k(x)$ and $g_k(x)$ around $x_n = 0$ are not zero at $p \in \partial M$ which means that there exist the largest positive integer numbers $k$ and $j$ such that $f_k(x) = \tilde{f}_k(x)$ and $g_k(x) = \tilde{g}_k(x)$ where $f_k(x', 0) \neq 0$ and $g_k(x', 0) \neq 0$ for some $J_1, J_2$. Thus, we can always write $\lambda$ in the following form $\lambda = x_n^2 \tau + x_n^2 \rho \wedge dx_n$ where the differential forms $\tau$ and $\rho$ do not contain $dx_n$. Applying $d\lambda_{x=0} = 0$, we get

$$0 = d\lambda_{x=0} = 2kx_n^{-1}dx_n \wedge \tau + x_n^2 d\tau + x_n^2 \rho \wedge dx_n + x_n^2 t_{xy}\tau + x_n^2 t_{xy}\rho \wedge dx_n.$$

Now, reducing this equation modulo $x_n^2$ we conclude that the term $x_n^2(d\rho \wedge dx_n + t_{xy}\rho \wedge dx_n) \equiv 0$ modulo $x_n^2$ because the term $kx_n^{-1}dx_n \wedge \tau \neq 0$ modulo $x_n^2$ and as a consequence, we infer that $k > j$.

Similarly, we can calculate $\delta x_n \lambda = -\tau \wedge (dx \wedge x_n \rho \wedge dx_n) \equiv 0$ using the Riemannian metric above. It suffices to use $dx \wedge x_n \lambda = x_n^2 \xi \wedge dx_n + x_n^2 \zeta$ for differential forms $\xi$ and $\zeta$ which do not contain $dx_n$ (both of them will contain many of the coefficients $h_{mn}(x)$). Hence, we get

$$0 = d\lambda + x_n \lambda = x_n^2 \delta \xi \wedge dx_n + x_n^2 \delta \zeta \wedge dx_n + x_n^2 t_{xy}\xi \wedge dx_n + x_n^2 t_{xy}\zeta.$$

Reducing this equation modulo $x_n^2$ for the same reason above but replacing $k$ by $j$, we can infer that $k < j$. But this is a contradiction, so there are no such largest positive integers $k$ and $j$. Hence, the Taylor series for the coefficients $f_k(x)$ and $g_k(x)$ around $x_n = 0$ must be zero at $p \in \partial M$. It means that all the higher partial derivatives of $f_k(x)$ and $g_k(x)$ vanish at all points of the boundary $\partial M$. Thus, this facts are enough to show all mixed partial derivatives including $x_n$ also vanish at the boundary. Hence, $\lambda$ has a zero of infinite order at $p \in \partial M$.

The remaining possibility of one of the coefficients $f_k(x)$ and $g_k(x)$ having finite order and the other infinite order in $x_n$ follows from the same argument as above.

Thus, $\lambda$ satisfies all the hypotheses of the strong Unique Continuation Theorem (2.2) so must be zero on all of $\mathbb{R}^n$. Since $M$ is assumed to be connected, $\lambda$ must be identically zero on all of $M$.

As a consequence of Theorem 2.1, we obtain the following.

**Corollary 2.3**

1. The space of $X_M$-harmonic fields can be written as a (not direct) sum:

$$H^{\pm}_{X_M}(M) = H^{\pm}_{X_M,cs}(M) + H^{\pm}_{X_M,co}(M).$$

2. The trace map $i^* : H^{\pm}_{X_M,N}(M) \to i^* H^{\pm}_{X_M,N}(M)$ is an isomorphism.

3. The map $f : i^* H^{\pm}_{X_M,N}(M) \to H^{\pm}_{X_M}(M)$ defined by $f(i^* \lambda_N) = [\lambda_N]$ for $\lambda_N \in H^{\pm}_{X_M,N}(M)$ is an isomorphism.

4. The map $h : i^* H^{\pm}_{X_M,N}(M) \to H^{\pm}_{X_M}(M, \partial M)$ defined by $h(i^* \lambda_N) = [\lambda_N]$ for $\lambda_N \in H^{\pm}_{X_M,N}(M)$ is an isomorphism.

**Proof:**

1. This follows by applying Theorem 2.1 and the $X_M$-Friedrichs Decomposition (1.2).

2. It is clear that $i^*$ is surjective and it follows from Theorem 2.1 that it is injective.

3. $f$ is a well-defined map because $\ker i^* = \{0\}$. Furthermore, $f$ is a bijection because there exists a unique Neumann $X_M$-harmonic field in any absolute $X_M$-cohomology class (Corollary 3.17 of [2]) hence part (3) holds.

4. This follows from part (3) by using $X_M$-Poincaré-Lefschetz duality of [2] (i.e. $H^{\pm}_{X_M}(M) \cong H^{\pm}_{X_M}(M, \partial M)$).
3. $X_M$-DN operator

Before defining this operator, we first need to prove the solvability of a certain boundary value problem (3.1). The proof depends on the main results in [2] and there is not any corresponding statement of it in [11]. When $X = 0$, this gives an independent proof of the solvability of Belishev and Sharafutdinov’s BVP (1.1). Theorem 3.1 represents the keystone to defining the $X_M$-DN operator and then to exploiting a connection between this $X_M$-DN operator and $X_M$-cohomology via the Neumann $X_M$-trace space $i^*\mathcal{H}_X^N(M)$.

**Theorem 3.1** Given $\theta \in \Omega_G^+(\partial M)$ and $\eta \in \Omega_G^+(M)$, then the BVP

$$
\begin{align*}
\Delta_{X_M} \omega &= \eta \text{ on } M \\
i^* \omega &= \theta \text{ on } \partial M \\
i^*(\delta_{X_M} \omega) &= 0 \text{ on } \partial M.
\end{align*}
$$

(3.1)

is solvable for $\omega \in \Omega_G^+(M)$ if and only if

$$
\langle \eta, \kappa\delta \rangle = 0, \quad \forall \kappa\delta \in \mathcal{H}_{X_M,D}^+(M)
$$

(3.2)

The solution of BVP (3.1) is unique up to an arbitrary Dirichlet $X_M$-harmonic field $\mathcal{H}_{X_M,D}^+(M)$.

**Proof:** Suppose BVP (3.1) has a solution. Then one can easily show that condition (3.2) holds by using Green’s formula (1.6).

Now suppose $\eta \in \Omega_G^+(\partial M)$ satisfies $\langle \eta, \kappa\delta \rangle = 0, \forall \kappa\delta \in \mathcal{H}_{X_M,D}^+(M)$ (i.e. $\eta \in \mathcal{H}_{X_M,D}^+(M)^\perp$). Since $\theta \in \Omega_G^+(\partial M)$, we can construct an extension $\omega_1 \in \Omega_G^+(M)$ of the differential form $\theta \in \Omega_G^+(\partial M)$ such that

$$
i^* \omega_1 = \theta, \quad \omega_1 = \delta_{X_M} \beta_{\omega_1} + \lambda_{\omega_1} \in C_{X_M}^+(M) \oplus \mathcal{H}_{X_M}^+(M).
$$

But $\Delta_{X_M} \omega_1 = \delta_{X_M} d_{X_M} \delta_{X_M} \beta_{\omega_1}$, so (1.6) implies that $\Delta_{X_M} \omega_1 \in \mathcal{H}_{X_M,D}^+(M)^\perp$ as well. Hence, $\eta - \Delta_{X_M} \omega_1 \in \mathcal{H}_{X_M,D}^+(M)^\perp$.

We now apply Proposition 3.8 of [2] which for smooth invariant forms states that for each $\mathbf{\eta} \in \mathcal{H}_{X_M,D}^+(M)^\perp$ there is a unique smooth differential form $\mathbf{\omega} \in \Omega_G^+ \cap \mathcal{H}_{X_M,D}^+(M)^\perp$ satisfying the BVP (3.1) but with $\eta = \mathbf{\eta}$ and $\theta = 0$.

Since $\eta - \Delta_{X_M} \omega_1 \in \mathcal{H}_{X_M,D}^+(M)^\perp$ is smooth, it follows from this that there is a unique smooth differential form $\omega_2 \in \Omega_G^+ \cap \mathcal{H}_{X_M,D}^+(M)^\perp$ which satisfies the BVP

$$
\begin{align*}
\Delta_{X_M} \omega_2 &= \eta - \Delta_{X_M} \omega_1 \text{ on } M \\
i^* \omega_2 &= 0 \text{ on } \partial M \\
i^*(\delta_{X_M} \omega_2) &= 0 \text{ on } \partial M.
\end{align*}
$$

(3.3)

Now, let $\omega_2 = \omega - \omega_1$, then the BVP (3.3) turns into the BVP (3.1). Hence, there exists a solution to the BVP (3.1) which is $\omega = \omega_1 + \omega_2$, where the uniqueness of $\omega$ is up to an arbitrary Dirichlet $X_M$-harmonic field.

**Definition 3.2** ($X_M$-DN operator $\Lambda_{X_M}$) We consider on $M$ the BVP (3.1) with $\eta = 0$, i.e.

$$
\begin{align*}
\Delta_{X_M} \omega &= 0 \text{ on } M \\
i^* \omega &= \theta \text{ on } \partial M \\
i^*(\delta_{X_M} \omega) &= 0 \text{ on } \partial M
\end{align*}
$$

(3.4)

then by Theorem 3.1 BVP (3.4) is solvable and the solution is unique up to an arbitrary Dirichlet $X_M$-harmonic field $\kappa\delta \in \mathcal{H}_{X_M,D}^+(M)$. We can therefore define the $X_M$-DN operator $\Lambda_{X_M} : \Omega_G^+(\partial M) \longrightarrow \Omega_G^{0,+}(\partial M)$ by

$$
\Lambda_{X_M} \theta = i^*(d_{X_M} \omega).
$$

Note that taking $d_{X_M} \omega$ eliminates the ambiguity in the choice of the solution $\omega$ which means $\Lambda_{X_M} \theta$ is well-defined.
The results above and those in [2] provide the basic ingredients needed to extend by analogy the results in [7] and some of the results in [13] on the ring structure to the context of $X_M$-cohomology and the $X_M$-DN map. However, some results in Sections 4 and 6 are different and are specified here. We therefore omit the proof of the results below; full details are given in the first author’s thesis [1].

**Proposition 3.3**

1. Let $i^* \mathcal{H}^0_{X_M}(M) = \mathcal{E}_X^0(\partial M) + i^* \mathcal{H}_{X_M}^0(M)$, where $\mathcal{E}_X^0(\partial M) = \{ d_{X_M} \alpha \mid \alpha \in \Omega^*_G(\partial M) \}.$

2. The operator $\Lambda_{X_M}$ is nonnegative in the sense that the integral $\int_{\partial M} \theta \wedge \Lambda_{X_M} \theta$ is nonnegative for any $\theta \in \Omega^*_G(\partial M)$.

3. Let $\omega \in \Omega^*_G(M)$ be a solution to the BVP (3.4) where $\theta \in \Omega^*_G(\partial M)$ is given. Then $d_{X_M} \omega \in \mathcal{H}^0_{X_M}(M)$ and $\delta_{X_M} \omega = 0$.

4. $\ker \Lambda_{X_M} = \text{Ran} \Lambda_{X_M} = i^* \mathcal{H}_{X_M}(M)$, where $\mathcal{H}_{X_M} = \mathcal{H}_{X_M}^+ \oplus \mathcal{H}_{X_M}^-.$

5. The operator $\Lambda_{X_M}$ satisfies the following relations:

$$\Lambda_{X_M} d_{X_M} = 0, \quad d_{X_M} \Lambda_{X_M} = 0, \quad \Lambda_{X_M}^2 = 0.$$

In this corollary, we introduce the $X_M$-Hilbert transform $T_{X_M}$ which is of course the analogue of the usual Hilbert transform (see Section 5 in [7]) and it will be used in Section 6.

**Corollary 3.4** The operator $T_{X_M} := d_{X_M} \Lambda_{X_M}^{-1} i^* \mathcal{H}_{X_M}(M)$ is well-defined; i.e. the equation $\phi = \Lambda_{X_M} \theta$ has a solution $\theta$ for any $\phi \in i^* \mathcal{H}_{X_M}(M)$, and $d_{X_M} \theta$ is uniquely determined by $\phi = \Lambda_{X_M} \theta$. In particular, $T_{X_M} : i^* \mathcal{H}_{X_M}(M) \to i^* \mathcal{H}_{X_M}(M)$ and the operator $d_{X_M} \Lambda_{X_M}^{-1} d_{X_M} : \Omega_G(\partial M) \to \Omega_G(\partial M)$ is well-defined.

The above construction and the results in [2] provide the essential ingredients needed to extend Theorem 4.2 of [7] (our Theorem 1.1) to the present context:

**Theorem 3.5** The Neumann $X_M$-trace spaces $i^* \mathcal{H}^0_{X_M}^+(M)$ can be completely determined from the boundary data $(\partial M, \Lambda_{X_M})$. In particular,

$$(\Lambda_{X_M} - (-1)^{n+1} d_{X_M} \Lambda_{X_M}^{-1} d_{X_M}) \Omega^*_G(\partial M) = i^* \mathcal{H}^0_{X_M}^-(M).$$

**4. $\Lambda_{X_M}$ operator, $X_M$-cohomology and equivariant cohomology**

The following result is an extension of Theorem 1.2 to $X_M$-cohomology. We relate the dimension of $\mathcal{H}^0_{X_M}^\pm(M)$ with the kernel of $\Lambda_{X_M}$ as follows:

**Theorem 4.1** Let $\Lambda_{X_M}^\pm$ be the restriction of $X_M$-DN operator to $\Omega^*_G(\partial M)$. Then $\mathcal{E}_{X_M}^\pm(\partial M) \subseteq \ker \Lambda_{X_M}^\pm$ and

$$\dim \ker \Lambda_{X_M}^\pm / \mathcal{E}_{X_M}^\pm(\partial M) \leq \min \{ \dim \mathcal{H}_{X_M}^+ (\partial M), \dim \mathcal{H}_{X_M}^- (M) \}.$$

Moreover, if every component of $F = N(X_M)$ has a boundary then

$$\max \{ \dim \ker \Lambda_{X_M}^\pm / \mathcal{E}_{X_M}^\pm(\partial M), \dim \ker \Lambda^\pm / \mathcal{E}^\pm(\partial F') \} \leq \min \{ \dim \mathcal{H}_{X_M}^+ (\partial M), \dim \mathcal{H}_{X_M}^- (M) \}.$$

The proof of the first part follows the proof of Theorem 1.2 so we omit it (details are given in [1]). The second part follows by applying Theorem 1.2 to $F'$. It moreover refers implicitly to a possible relation between the dimensions of $\ker \Lambda_{X_M}^\pm / \mathcal{E}_{X_M}^\pm(\partial M)$ and $\ker \Lambda^\pm / \mathcal{E}^\pm(\partial F')$ which needs to be discovered. This idea and others are under investigation in [1] which will help to extend many of the results of [13] to the style of $X_M$-cohomology.

To relate these inequalities to equivariant cohomology, one uses a result in [2] (essentially due to Atiyah and Bott), which asserts that if $F' = F := \text{Fix}(G, M)$, then $\dim \mathcal{H}_{X_M}^+ (M) = \text{rank} \mathcal{H}_{X_M}^+ (M)$—see, Theorem 6.1 below. Hence we conclude that under this assumption, the right hand side of the inequalities above can be replaced by $\min \{ \text{rank} \mathcal{H}_{X_M}^+ (\partial M), \text{rank} \mathcal{H}_{X_M}^- (M) \}$. 

7
5. Recovering $X_M$-cohomology from the boundary data $(\partial M, \Lambda_{X_M})$

In this section, we continue extending the results of Belishev-Sharafutdinov and Shonkwiler’s Theorem \[\text{(1.3)}\] on recovering the de Rham cohomology groups and ring structure from the boundary data $(\partial M, \Lambda)$, to the context of absolute and relative $X_M$-cohomology and their concrete realizations $H^\pm_{X_M}(M)$ and $H^\pm_{X_M, \partial}(M)$.

5.1. Recovering the long exact $X_M$-cohomology sequence of $(M, \partial M)$

We show that the data $(\partial M, \Lambda_{X_M})$ determines the long exact $X_M$-cohomology sequence of the pair $(M, \partial M)$.

Since the vector field $\partial_M$ which we are considering is always tangent to the boundary $\partial M$, we can also define $X_M$-cohomology on $\partial M$, that is $H^\pm_{X_M}(\partial M)$. Hence, from the definitions of the absolute and relative $X_M$-cohomology, we have the following exact $X_M$-cohomology sequence of the pair $(M, \partial M)$ as follows:

$$\cdots \to H^+_X(M, \partial M) \xrightarrow{\partial^*} H^+_{X_M}(M) \xrightarrow{\rho^*} H^+_X(M, \partial M) \xrightarrow{\iota^*} H^+_X(M, \partial M) \xrightarrow{\partial^*} H^+_X(M, \partial M) \xrightarrow{\rho^*} \cdots \quad (5.1)$$

However, Theorem \[\text{(5.5)}\] proves that we can determine the space $\iota^*H^\pm_{X_M,N}(M)$ from the boundary data and Corollary \[\text{(2.3)}\] gives $\iota^*H^\pm_{X_M,N}(M) \cong H^\pm_{X_M}(M)$ and $\iota^*H^\pm_{X_M,N}(M) \cong H^\pm_{X_M}(M, \partial M)$. This says that the additive absolute and relative $X_M$-cohomology are completely determined by $(\partial M, \Lambda_{X_M})$.

So, if the boundary data $(\partial M, \Lambda_{X_M})$ is given, we can construct the sequence

$$\cdots \xrightarrow{\overline{\tau}} \iota^*H^{n-\pm}_{X_M,N}(M) \xrightarrow{\overline{\tau}} \iota^*H^{\pm}_{X_M,N}(M) \xrightarrow{\tau} H^{\pm}_{X,M}(\partial M) \xrightarrow{\overline{\tau}} \iota^*H^{\pm}_{X_M,N}(M) \xrightarrow{\overline{\tau}} \cdots \quad (5.2)$$

where we define the operators of sequence (5.2) by the following formulas:

1. $\overline{\tau} \theta = [\theta]|_{(X_M, \partial M)}$; if $\theta \in \iota^*H^{\pm}_{X_M,N}$ then $\theta$ is $X_M$-closed because $\iota^*$ and $d_{X_M}$ commute.
2. Using Corollary \[\text{(3.4)}\] we set, $\overline{\tau} \theta = - (1)^{n+1}T_{X_M} \theta$, $\forall \theta \in \iota^*H^{\pm}_{X_M,N}$.
3. Let $\theta \in \Omega_G(\partial M)$ be $X_M$-closed. Based on Theorem \[\text{(5.5)}\] $\Lambda_{X_M} \theta = (\Lambda_{X_M} - (1)^{n+1}d_{X_M} \Lambda_{X_M} d_{X_M}) \theta$. Hence, we set $\overline{\tau} \theta = (1)^{n+1}X_M \theta$, $\forall \theta |_{(X_M, \partial M)} \in H^\pm_{X_M}(\partial M)$.

More concretely, our goal is then to recover sequence \[\text{(5.1)}\] from sequence \[\text{(5.2)}\]. It means that we should prove that the following diagram \[\text{(5.3)}\] is commutative.

\[
\begin{array}{cccccccc}
\cdots & \xrightarrow{\overline{\tau}} & \iota^*H^{n-\pm}_{X_M,N}(M) & \xrightarrow{\overline{\tau}} & \iota^*H^{\pm}_{X_M,N}(M) & \xrightarrow{\tau} & H^{\pm}_{X,M}(\partial M) & \xrightarrow{\overline{\tau}} & \iota^*H^{\pm}_{X_M,N}(M) & \xrightarrow{\overline{\tau}} & \cdots \\
& & h \downarrow & & f \downarrow & & i \downarrow & & h \downarrow & \\
\cdots & \xrightarrow{\tau} & H^{\pm}_{X,M}(M, \partial M) & \xrightarrow{\rho} & H^{\pm}_{X,M}(M) & \xrightarrow{\iota} & H^{\pm}_{X,M}(M, \partial M) & \xrightarrow{\tau} & H^{\pm}_{X,M}(M, \partial M) & \xrightarrow{\rho} & \cdots \\
\end{array}
\]

(5.3)

where $i$ is the identity operator while $f$ and $h$ are given in Corollary \[\text{(2.3)}\] Indeed, one can prove the commutativity of the diagram by a method similar to that given in [2] but in terms of the operators $d_{X_M}$ and $\delta_{X_M}$, see [1] for details.

Actually, the above construction proves that the data $(\partial M, \Lambda_{X_M})$ recovers sequence \[\text{(5.1)}\] of the pair $(M, \partial M)$ up to an isomorphism (i.e. up to the maps $f$ and $h$) from the sequence \[\text{(5.3)}\].

5.2. Recovering the ring structure of the real $X_M$-cohomology.

We consider the following question: can the multiplicative ring structure of the real absolute and relative $X_M$-cohomology be recovered from the boundary data $(\partial M, \Lambda_{X_M})$?

First of all, we consider the mixed cup product $\bar{\cup}$ between the absolute and relative $X_M$-cohomology as follows:

$$\bar{\cup} : H^\pm_{X_M}(M) \times H^\pm_{X_M}(M, \partial M) \to H^\pm_{X_M}(M, \partial M)$$

$$[\alpha]|_{(X_M, \partial M)} \bar{\cup} [\beta]|_{(X_M, \partial M)} = [\alpha \wedge \beta]|_{(X_M, \partial M)}.$$
It is easy to check that \( \sqcup \) is a well-defined map. In addition, in [2] we prove that any absolute or relative \( X_M \)-cohomology classes contain a unique Neumann or Dirichlet \( X_M \)-harmonic field respectively. Hence, we can regard any absolute (relative) \( X_M \)-cohomology class as a Neumann (Dirichlet) \( X_M \)-harmonic field. But \( [\alpha]_{X_M,\partial M} \sqcup [\beta]_{X_M,\partial M} = [\alpha \wedge \beta]_{X_M,\partial M} \) is a relative \( X_M \)-cohomology class, so there exists a unique Dirichlet \( X_M \)-harmonic field \( \eta \in H^\pm_{X_M,D}(M) \) such that \( [\alpha \wedge \beta]_{X_M,\partial M} = [\eta]_{X_M,\partial M} \), i.e.

\[
\alpha \wedge \beta = \eta + d_{X_M} \xi \in H^\pm_{X_M,D}(M) \oplus \mathcal{E}^\pm_{X_M}(M),
\]

for some \( \xi \in \Omega^+_{G,D}(M) \). However, it follows from Corollary 2.3 that

\[
H^\pm_{X_M,D}(M) \cong H^{\alpha-\beta}_{X_M,N}(M) \cong i^* H^{\alpha-\beta}_{X_M,N}(M)
\]

According to our illustrations above we know that an absolute \( X_M \)-cohomology class \( [\alpha]_{X_M,M} \in H^\pm_{X_M}(M) \) and relative \( X_M \)-cohomology classes \( [\beta]_{X_M,\partial M}, [\alpha \wedge \beta]_{X_M,\partial M} \in H^\pm_{X_M}(M, \partial M) \) are represented by the Neumann \( X_M \)-harmonic field \( \alpha \in H^\pm_{X_M,N}(M) \) and the Dirichlet \( X_M \)-harmonic fields \( \beta, \eta \in H^\pm_{X_M,D}(M) \) respectively, such that they correspond, respectively, to forms on the boundary by setting

\[
\phi = i^* \alpha \in i^* H^\pm_{X_M,N}(M), \quad \psi = i^* \beta \in i^* H^{\alpha-\beta}_{X_M,N}(M), \quad \vartheta = i^* \eta \in i^* H^{\alpha-\beta}_{X_M,N}(M)
\]

Following [12], our answer to the above question will only be partial, in the sense that we will not consider all the classes of the relative \( X_M \)-cohomology, but will just consider the boundary portion, denoted \( BH^\pm_{X_M,D}(M, \partial M) \), of \( H^\pm_{X_M,D}(M, \partial M) \). This boundary subspace is defined to be \([2]\).

\[
BH^\pm_{X_M,D}(M, \partial M) = \text{im}[\partial^* : H^\pm_{X_M,D}(\partial M) \to H^\pm_{X_M,D}(M, \partial M)].
\]

Here \( \partial^* \) is the standard construction in the long exact sequence (5.1): given an \( X_M \)-closed form \( \lambda \) on \( \partial M \), let \( \check{\lambda} \) be an extension form on \( M \). Then \( d_{X_M} \check{\lambda} \) defines a well-defined element of \( H^\pm_{X_M,D}(M, \partial M) \) denoted \( \partial^* \lambda \). This boundary portion is therefore the image of \( H^\pm_{X_M,D}(\partial M) \) inside \( H^\pm_{X_M,D}(M, \partial M) \) in this long exact sequence.

In [2], we prove that \( H^\pm_{X_M,D}(M, \partial M) \cong H^\pm_{X_M,N}(M)\). Hence, on translation into the language of \( X_M \)-harmonic fields, we can identify

\[
BH^\pm_{X_M,D}(M, \partial M) \cong BH^\pm_{X_M,D}(M)
\]

where \( BH^\pm_{X_M,D} \) is called the boundary subspace of \( H^\pm_{X_M,D}(M) \). Clearly, Hodge star \( * \) gives

\[
BH^{\alpha-\beta}_{X_M,D}(M) := * BH^\pm_{X_M,D}(M)
\]

where \( BH^{\alpha-\beta}_{X_M,D}(M) \) is the boundary subspace of \( H^{\alpha-\beta}_{X_M,D}(M) \). Using this fact together with Corollary 2.3 we conclude that \( BH^\pm_{X_M,D}(M, \partial M) \cong i^* BH^\pm_{X_M,D}(M, \partial M) \).

The above constructions allow us to extend Shonkwiler’s map [12] to the context of \( X_M \)-in order to define the following map with notation as above:

\[
\sqcup : i^* H^\pm_{X_M,N}(M) \times i^* H^{\alpha-\beta}_{X_M,N}(M) \to H^\pm_{X_M,D}(M, \partial M)
\]

\[
\phi \sqcup \psi = \Lambda_{X_M}(\pm \phi \wedge \Lambda_{X_M}^{-1} \eta), \quad \eta \in H^\pm_{X_M,D}(M, \partial M)
\]

By using the same method as [12] together with definition 3.2 we deduce that \( \sqcup \) is well-defined. Now, we can extend Shonkwiler’s Theorem 1.3 to the style above.

**Theorem 5.1** The boundary data \( (\partial M, \Lambda_{X_M}) \) completely determines the mixed cup product structure of the \( X_M \)-cohomology when the relative \( X_M \)-cohomology classes come from the boundary subspace. I.e. if \( (\alpha, \beta) \in H^\pm_{X_M,N}(M) \times BH^\pm_{X_M,D}(M) \) such that \( \alpha \wedge \beta = \eta + d_{X_M} \xi \in H^\pm_{X_M,D}(M) \oplus \mathcal{E}^\pm_{X_M}(M) \) then

\[
i^* \eta = \Lambda_{X_M}(\pm \phi \wedge \Lambda_{X_M}^{-1} \psi)
\]
where $\phi = i^* \alpha$ and $\psi = i^* \beta$. In fact one shows the commutativity of the following diagram,

\[
\begin{array}{ccc}
 i^* H^\pm_{\partial X_M}[M] \times i^* B H^\pm_{X_M}[M] & \xrightarrow{\quad \pi_X \circ \ n \quad} & i^* B H^\pm_{X_M}[M] \\
 \downarrow (f, h) & & \downarrow h \\
 H^\pm_{\partial X_M}[M] \times B H^\pm_{X_M}[M, \partial M] & \xrightarrow{\quad \pi \quad} & B H^\pm_{X_M}[M, \partial M],
\end{array}
\]

where $f$ and $h$ are given in Corollary 2.3.

**PROOF:** Our goal is to show that $\forall (\phi, \psi) = (i^* \alpha, i^* \star d_{\partial X_M} \beta_1) \in i^* H^\pm_{\partial X_M}[M] \times i^* B H^\pm_{X_M}[M]$ one has

\[
(h \circ \pi_X)(i^* \alpha, i^* \star d_{\partial X_M} \beta_1) = ((\pi \circ (f, h))(i^* \alpha, i^* \star d_{\partial X_M} \beta_1)).
\]  

The left-hand side gives

\[
h((\pi_X)(i^* \alpha, i^* \star d_{\partial X_M} \beta_1)) = h(A_{X_M}(\pm \phi \wedge A^{-1}_{X_M} \psi))
\]  

while the right-hand side together with eq. 5.4 and Corollary 2.3 give

\[
\forall ((f \circ \pi_X)(i^* \alpha, h(i^* \star d_{\partial X_M} \beta_1))) = \forall ([\alpha](X_M, M), [\star \star d_{\partial X_M} \beta_1](X_M, M, \partial M)) = [\star \star (\alpha \wedge d_{\partial X_M} \beta_1)](X_M, M, \partial M) = [\star \star \eta](X_M, M, \partial M) = h(i^* \star \eta).
\]

We now need to prove that the right-hand sides of equations 5.8 and 5.9 are equal. This will be the case if

\[
i^* \star \eta = A_{X_M}(\pm \phi \wedge A^{-1}_{X_M} \psi).
\]

The method of Shonkwiler [12] used to prove Theorem 1.3 extends to our setting by combining with results in [2], such as the $X_M$-Hodge-Morrey decomposition theorem (full details are given in [1]).

6. Conclusions

1. The key point used to recover the free part of the relative and absolute equivariant cohomology groups from the boundary data $(\partial M, A_{X_M})$ is the following theorem which is essentially Atiyah and Bott’s localization theorem.

**Theorem 6.1 ([2])** Let $X \in g$ (the Lie algebra of $G$) and let $F' = N(X_M)$. The inclusion $j_X : F' \hookrightarrow M$ induces the following isomorphisms

1. $H^\pm_{X_M}(M) \cong H^\pm(F').$
2. $H^\pm_{X_M}(M, \partial M) \cong H^\pm(F', \partial F').$

Moreover, if $N(X_M) = F := \text{Fix}(G, M)$ then $\dim H^\pm(F, \partial F) = \text{rank} H^\pm_G(M, \partial M)$ and $\dim H^\pm(F) = \text{rank} H^\pm_G(M)$.

Now, combining the above theorem with Theorem 5.5 and Corollary 2.3 we deduce

**Theorem 6.2**

\[
H^\pm_{X_M}(M, \partial M) \cong (A_{X_M} - (\pm 1)^{n+1} d_{X_M} A^{-1}_{X_M} d_{X_M}) \Omega G^\pm(\partial M) \cong H^\pm(F', \partial F')
\]

and

\[
H^\pm_{X_M}(M) \cong (A_{X_M} - (\pm 1)^{n+1} d_{X_M} A^{-1}_{X_M} d_{X_M}) \Omega G^{n-\pm}(\partial M) \cong H^\pm(F').
\]
Since the Neumann $X_M$-harmonic fields are uniquely determined by their Neumann $X_M$-trace (Corollary 2.3), which is in turn determined by the boundary data $(\partial M, \Lambda_{X_M})$ (Theorem 3.3), this means we can conclude, by using $X_M$-Poincaré-Lefschetz duality of [2], that we can realize the relative and absolute $X_M$-cohomology groups (and hence in some sense the free part of the relative and absolute equivariant cohomology groups) as particular subspaces of invariant differential forms on $\partial M$ and they are not just determined abstractly from the generalized boundary data.

(2) We can apply Theorem 4.1 to the manifolds $F' = N(X_M)$ with boundary $\partial F'$. Since $G$ acts on $F'$ the induced action on each $H^{\pm}(F')$ is trivial. Now, we can use Theorem 5.2 to exploit the connection between Belishev and Sharafutdinov’s boundary data on $\partial F'$ (i.e. $(\partial F', \Lambda)$) and ours on $\partial M$ (i.e. $(\partial M, \Lambda_{X_M})$). More concretely, we have the following.

**Theorem 6.3** If every component of $F'$ has a boundary, then

$$\left\{ \Lambda_{X_M} - (\mp 1)^{n+1} dM \Lambda^{-1}_{X_M} dM \right\} \Omega^\pm_G(\partial M) \cong \left\{ \Lambda - (\mp 1)^{n+1} d\Lambda^{-1} \right\} \Omega^\pm(\partial F').$$

This means that the boundary data $(\partial F', \Lambda)$ can be determined from the boundary data $(\partial M, \Lambda_{X_M})$ and vice versa. In this setting, it follows that since the de Rham cohomology groups of $(F', \partial F')$ are determined by $(\partial F', \Lambda)$ (Theorem 4.1), then the $\pm$ de Rham cohomology groups of $(F', \partial F')$ are also determined by $(\partial M, \Lambda_{X_M})$.

(3) When $M$ has no boundary, Witten proves in [13] that $H^{\pm}_{K}(M) \cong H^{\pm}(F')$ where $K$ is a Killing vector field (our $X_M$) on $M$ and he shows how the $K$-cohomology and the isomorphism above are useful in quantum field theory and other mathematical and physical applications. However, when $\partial M \neq \emptyset$, the extended isomorphism is provided by Theorem 6.1 above which gives insight that the extension for other results of Witten [13] are possible. In this light, Theorem 6.2 suggests that $\Lambda_{X_M}$ may also be relevant to quantum field theory and following Witten, possibly to other mathematical and physical interpretations. This shows that $\Lambda_{X_M}$ may be interesting in its own right.

Finally, it is worth considering the following topological problem: Can the torsion part of the absolute and relative equivariant cohomology groups be completely recovered from the boundary data $(\partial M, \Lambda_{X_M})$? (Here torsion is meant as a module over the ring of polynomials on $g$—the standard Cartan model: some torsion information is available from Theorems 6.1 and 6.2 when $X$ is in an isotropy subalgebra, but not all.) Answering this question will indeed complete the picture of the role the boundary data $(\partial M, \Lambda_{X_M})$ plays in the story of the equivariant cohomology of manifolds.

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