Dynamical Transition from a Naked Singularity to a Black Hole

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We show that a Reissner-Nordström (RN) black hole can be formed by dropping a charged thin dust shell onto a RN naked singularity. This is in contrast to the fact that a RN naked singularity is prohibited from forming by dropping a charged thin dust shell onto a RN black hole. This implies the strong tendency of the RN singularity to be covered by a horizon in favour of cosmic censorship. We show that an extreme RN black hole can also be formed from a RN naked singularity by the same process in a finite advanced time. We also discuss the evolution of the charged thin dust shells and the causal structure of the resultant spacetimes.

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I. INTRODUCTION

The weak cosmic censorship conjecture states that any spacetime singularity caused by a gravitational collapse must be hidden inside an event horizon [1]. This weak cosmic censorship conjecture plays one of the important roles as a guide to establish new physics beyond general relativity. Many attempts have been so far performed to verify the weak cosmic censorship conjecture by either gedanken experiments or numerical analysis.

In gedanken experiments, by plunging a test particle with a sufficiently large electric charge and/or angular momentum into an extreme black hole, they study whether the black hole turns into a naked singularity.

One early gedanken experiment showed that the weak cosmic censorship is preserved because a spinning test body cannot be plunged into an extreme Kerr-Newman black hole, thereby preventing the formation of a naked singularity [2]. The related studies are presented in [3–5]. While it has also been shown that one can make a nearly extreme black hole to jump over the extremality by tossing into it a test body composed of reasonable matter satisfying the energy conditions [6–8]. In order to justify these results, however, we have to take into account the self-energy of a test body and its back reaction [9, 10]. It has been proven that no gedanken experiments or numerical analysis.

From this fact, a relativistic object may eventually be charged particles but repel positively charged ones. Even if some counterexamples exist, it may be possible that the weak cosmic censorship conjecture would be effectively preserved, if a temporal naked singularity will evolve into a non-singular physical object in a short time. Then the following natural question arises: When there exists a naked singularity, does it evolve into a black hole via some physical process?

An overcharged Reissner-Nordström spacetime is also interesting in the context of ultrahigh energy collision of particles and shells [21, 22]. In particular, a Reissner-Nordström spacetime slightly more charged than extremality is capable of accelerating particles and shells so that the centre-of-mass energy of two colliding such objects can be very high and such high-energy reactions can be exposed to observations in the absence of horizons.

The result obtained in this paper indicates the instability of such spacetimes against small perturbations.

In this paper, in order to clarify the possibility of the transition from a naked singularity to a black hole, we analyze a Reissner-Nordström spacetime with a charged thin shell, which provides the exact solution of the Einstein equations including the back reaction. Since the electric force is much stronger than that of gravitation, a positively charged relativistic object will attract negatively charged particles but repel positively charged ones. From this fact, a relativistic object may eventually become natural. Although we do not show here, it can be seen from a simple calculation that by plunging charged test particles into a Reissner-Nordström naked singularity, it is possible to form an event horizon. However, to find what really happens, the inclusion of the self-energy of the test particles or their back reaction is es-
sential. Hence we shall analyze the gravitational collapse of a charged thin shell (or a singular hypersurface layer), which solves the Einstein equations. This setting will yield an appropriate analysis.

The plan of this paper is as follows. In Sec. II we present the equations of motion of a charged thin shell. We present our analysis in Sec. III, Sec. IV, and Sec. V. The gravitational collapse of the charged thin shell to a naked singularity leads to the formation of an event horizon. This supports the effective validity of the weak cosmic censorship conjecture. The evolution history of the charged thin shell is shown using the Penrose diagram which provides a global structure of the present dynamical spacetime, i.e., a black hole is formed by the gravitational collapse of a charged thin shell onto a naked singularity. We summarize our results and provide some remarks in Sec. VI. We use the geometrical units, i.e., $G = c = 1$.

II. EQUATIONS OF MOTION

We consider a spherically symmetric spacetime with an infinitely thin charged shell, described by a timelike hypersurface $\Sigma$, which divides the spacetime $(M, g)$ into two regions—the inside and outside. From the Birkhoff theorem, each region is described by the Reissner-Nordström metric:

$$ds^2 = -f_\pm dt^2 + \frac{1}{f_\pm} dr^2 + r^2 d\Omega^2,$$

where

$$f_\pm := 1 - \frac{2M_\pm}{r} + \frac{Q_\pm^2}{r^2},$$

and $d\Omega^2$ denotes a metric on a unit two-sphere. Here, we have used the subscript $\pm$ for the variables in the exterior and interior regions, respectively. Since the circumference radius $r$ is continuous even at the shell surface, we do not use the subscript $\pm$.

$M$ and $Q$ denote the mass and electric charge of the Reissner-Nordström spacetime, respectively. When $M \geq |Q|$, the spacetime describes a black hole. The case of $M = |Q|$ is referred to as an extreme Reissner-Nordström black hole. While, if $M < |Q|$, then the spacetime describes a naked singularity.

In our setting, since we will discuss a transition from naked singularity to a black hole, we assume that the masses and electric charges in the interior and exterior geometries satisfy the condition $M_- < |Q_-|$ and $M_+ \geq |Q_+|$, respectively.

The singular hypersurface $\Sigma$ is described by two-dimensional coordinates $(t_\pm, r)$ with the parameteric equation such that $t_\pm = T_\pm(t)$ and $r = R(t)$, where $t$ is the proper time of the shell. The induced metric $h_{ij}$ on $\Sigma$ is given by

$$ds_\Sigma^2 := h_{ij} dy^i dy^j = -d\tau^2 + R(\tau)^2 d\Omega^2,$$

with the intrinsic coordinates $y^i = (\tau, \theta, \phi)$ of the hypersurface $\Sigma$. The four velocities of the thin shell in both interior and exterior coordinates are given by

$$u_\pm^i = T_\pm^i \partial_\pm + \mathcal{R} \partial_\pm,$$

where the overdot denotes differentiation with respect to the proper time $\tau$. Since the four velocities in both coordinates are normalized as $u_\pm^i u_\pm_{\pm \mu} = -1$, we find

$$T_\pm = \frac{\epsilon_\pm \sqrt{R^2 + f_\pm}}{f_\pm},$$

where the signs $\epsilon_\pm = 1$ or $-1$, which represents the sign of $T_\pm f_\pm$. We do not assume that $T_\pm$ is always positive since we will also consider the interior region of the black hole spacetime ($f_+ < 0$). It is important to fix the direction of normal vectors of the thin shell, which explicit forms we will see below. Note that the time coordinates $t_\pm$ are not continuous at the shell radius $r = R$.

The induced metric $h_{ij}$ is continuous at $\Sigma$, i.e.,

$$[h_{ij}] = 0,$$

where the square brackets $([\ ]_\pm)$ denote the difference between the variables evaluated on the both sides of $\Sigma$, i.e., $[X]_\pm := X_+ - X_-$. This junction condition fixes the relation between the two time coordinates, $t_-$ and $t_+$, at $\Sigma$.

The unit normal vectors of the thin shell in both coordinates are given by

$$n_{\pm \mu} dx^\mu = \epsilon_\pm \left(-\mathcal{R} dt_\pm + T_\pm dr_\pm\right),$$

where the overall signs $\epsilon_\pm = 1$ or $-1$, which show the possible directions of the normals appear in the equations of motion through the calculations for the extrinsic curvature. This normal vector must satisfy $u_\pm^i n_{\pm \mu} = 0$ and $n_{\pm \mu} n_{\pm \mu} = 1$.

The stress energy tensor $S_{ij}$ of the thin shell determines the jump of the extrinsic curvature $K_{ij}$ of $\Sigma$ as

$$[K_{ij}]_\pm - [K]_\pm h_{ij} = -8\pi S_{ij},$$

where $K = h^{ij} K_{ij}$ is the trace of the extrinsic curvature. From the difference of the Hamiltonian constraint equations on both sides of $\Sigma$ with the junction condition, we find

$$S_{ij} \{K^{ij}\}_\pm = [T_{\mu \nu} n^\mu n^\nu]_\pm,$$

where $T^{\mu \nu}$ is the energy-momentum tensor associated with spacetime. The bracket $\{\}$ denotes the mean value of the variables evaluated on both sides of $\Sigma$, i.e., $\{X\}_\pm := (X_+ + X_-)/2$. Using the momentum constraint, we find the following conservation equation of the shell energy-momentum:

$$D_i S^i_j = -[T_{\mu \nu} e_j^\mu n^\nu]_\pm,$$
In what follows, we assume that the charged thin shell is composed of pressureless matter fluid, i.e.,

$$S^{ij} = \sigma u^i u^j ,$$  

(11)

where $\sigma$ and $u^i$ are the surface mass density and the velocity field of the charged thin shell. Each component of the junction condition \cite{8} reads as follows:

$$\begin{cases}
\left[ \frac{\epsilon \epsilon_\perp}{\sqrt{R^2 + f}} \left( \dot{R} + \frac{f'}{2} \right) + \frac{\epsilon \epsilon_\parallel \sqrt{R^2 + f}}{R} \right] = 0 , \\
\left[ -\frac{\epsilon \epsilon_\parallel \sqrt{R^2 + f}}{R} \right] = 4\pi \sigma .
\end{cases}$$  

(13)

The energy-momentum tensor $T^{\mu\nu}$ associated with the Reissner-Nordstrom spacetime is

$$-T_{t}^{\ t} = -T_{\sigma}^{\ \sigma} = \frac{Q^2}{8\pi R^4} ,$$  

(14)

and the other components vanish. Relation (19) gives

$$\left\{ \frac{\epsilon \epsilon_\parallel \sqrt{R^2 + f}}{R} \left( \dot{R} + \frac{1}{2} \frac{df}{dr} \right) \right\} = 0 .$$  

(15)

For the comoving coordinates employed in the induced metric $(u^i \partial_i = \partial_r)$, Eq. (10) is integrated as follows:

$$m := 4\pi R^2 \sigma = \text{constant} ,$$  

(16)

where $m$ is the conserved rest mass of the charged thin shell.

From the basic equations (12), (13), (15), and (16), we find the equation of motion as

$$\sqrt{R^2 + f_+} = \epsilon_+ \epsilon_+ F(R) ,$$  

(17)

where

$$F(R) := \frac{R}{2m} (f_- - f_+) - \frac{m}{2R} ,$$  

(18)

and the possible range of the radius of the charged thin shell is restricted as

$$\epsilon_+ \epsilon_+ F(R) \geq 0 ,$$  

(19)

and

$$\epsilon_- \epsilon_- \left( F(R) + \frac{m}{R} \right) \geq 0 .$$  

(20)

Furthermore, the radius of the charged thin shell is restricted as

$$F^2(R) - f_+(R) \geq 0 .$$  

(21)

The energy and electric charge of the charged thin shell are denoted $E$ and $q$ respectively. When we discuss the gravitational collapse of the charged thin shell, the equation of motion will be described by the following physical parameters; $M_-, Q_-, E, q, \text{and } m$. The energy and electric charge of the charged thin shell have relations with the masses and electric charges of the interior and exterior metrics as

$$E = M_+ - M_- ,$$  

(22)

$$q = Q_+ - Q_- .$$  

(23)

To simplify the problem, we assume that $\dot{T}_\pm f_\pm$ are positive. The validity of these assumptions will be explained at the end of Sec. II Thus, the signs are set to be positive $\epsilon_\pm = 1$. We further assume that the overall signs of the normal vectors of the thin shell are positive $\epsilon_\pm = 1$. Under the setting $\epsilon_\pm = 1$, the assumptions $\epsilon_\pm = 1$ indicate that the normal vectors point to a direction in which the radial coordinate increases. We assume that the rest mass $m$ of charged thin shell is positive. Then, Eqs. (19) and (20) reduce to a single restriction

$$F(R) \geq 0 .$$  

(24)

At this juncture, we shall summarize our assumptions about the parameter region. First, we assume that the energy $E$ of the charged thin shell is positive. We also assume that the electric charge $Q_-$ of the interior metric is positive without loss of generality. In order to further simplify our problem, we restrict the rest mass $m$ and the electric charge $q$ of the charged thin shell to the following parameter region for given mass $M_-$ and electric charge $Q_-$ of the interior spacetime. These assumptions are written as

$$0 < m < Q_- ,$$  

(25)

and

$$-Q_- - \sqrt{Q_-^2 - m^2} \leq q \leq -Q_- + \sqrt{Q_-^2 - m^2} .$$  

(26)

Equation (17) is rewritten in the form of one-dimensional particle motion \cite{23,27}:

$$\dot{R}^2 + f_+(R) = F^2(R) .$$  

(27)

Since Eqs. (25) and (26) are satisfied, Eqs. (17) and (27), both equations of motion, are equivalent. By virtue of Eqs. (25) and (26), function $F$ becomes positive for all $R > 0$. From our ansatz $M_+ \geq |Q_+|$, we have the following condition as well:

$$|Q_- + q| \leq M_- + E .$$  

(28)

Substituting Eq. (17) into Eq. (14) yields the following equation:

$$\dot{T}_+ f_+ = \epsilon_+ F .$$  

(29)

If Eqs. (25) and (26) are satisfied, $\dot{T}_+ f_+ \neq 0$. In such parameter regions, the sign of $\dot{T}_+$ in the region $f_+ < 0$ does not change. Therefore it is possible to assume $\dot{T}_+ f_+ > 0$ in the region $f_+ < 0$. Also, since $f_-$ is always positive, hypersurfaces $(t_- = \text{constant})$ are spacelike. It is possible to assume that $\dot{T}_- f_- > 0$ holds.
We see Reissner-Nordström black hole formation via the gravitational collapse of a charged thin shell in a Reissner-Nordström naked singularity. We investigate the gravitational collapse of a charged thin shell in a region $r = 0$ and analyze Eq. (17) to understand the radial motion of the thin shell.

The sign $\varepsilon_\tau$ defined on the domain, which is determined by Eq. (21). Let $R(\tau) = r_0$, where

\[ V := \varepsilon_\tau \sqrt{F^2 - f_+} , \]

with $\tau_0$ and $r_0$ as the initial values. The function $V$ is defined on the domain, which is determined by Eq. (21). The sign $\varepsilon_\tau$ changes when the shell reaches a radius $r$-turning point. Since the function $V$ does not explicitly depend on the proper time $\tau$, this is a normal autonomous system. Let $I$ be an open interval, s.t.,

\[ I = \left\{ r \mid (F^2 - f_+) (r) > 0 \right\} . \]

We can see that $I$ is simply connected. For the initial condition given by Eq. (21), there is a unique maximally extended solution $R(\tau)$ in the range $I$. Choosing $\varepsilon_\tau = -1$, $R(\tau)$ is a strictly monotonically decreasing function in the domain $(\tau_1, \tau_2)$, where we can see

\[ \lim_{\tau \uparrow \tau_1} R(\tau) = \sup I , \]

\[ \lim_{\tau \downarrow \tau_2} R(\tau) = \inf I . \]

Note that $\tau_1$ may be $-\infty$ and $\tau_2$ may be $+\infty$.

The existence of zeros $r_\epsilon$ and $r_c$ ($r_\epsilon \geq r_c$), of $f_+$ is guaranteed by Eq. (25). If the thin shell can reach this radius $r_\epsilon$ and go beyond, a Reissner-Nordström black hole is formed. This is confirmed later by analyzing the global structure of the spacetime. By a simple calculation we find that $(F^2 - f_+) (r_\epsilon) > 0$ and $(F^2 - f_+) (r_c) > 0$. This means that $\{ r_\epsilon, r_c \} \subset I$.

The condition that there exists the zero $r_{\text{sup}}$ of function $V$, which is larger than $r_\epsilon$, is equivalent to the condition that $(F^2 - f_+)$ in the limit of infinity is negative. In fact, the graph of $(F^2 - f_+)$ can have only one extremal point, and it is concave if there is an extreme. The condition can be written as follows:

\[ m > E . \]

From these logics, we will take the initial value $r_0$ in the following range:

\[ r_\epsilon < r_0 < \begin{cases} r_{\text{sup}} & \text{if } m > E , \\ \infty & \text{if } m \leq E . \end{cases} \]
Then the orbit \( \mathcal{O} \) of the maximal set of the solutions of the initial value problem must include \( \{r_e, r_c\} \),
\[
\{r_e, r_c\} \subset \mathcal{O}(r_0) := \{ R(\tau) \mid \tau \in (\tau_1, \tau_2) \} .
\]
For the initial radius \( r_0 \) to be arbitrarily large, we find from Eq. \( (37) \) that the rest mass must be less than or equal to the energy, \( m \leq E \).

The condition that there exists the zero \( r_{\text{inf}} \) of function \( V \), which is smaller than \( r_c \), is equivalent to the condition that \( \left( F^2 - f_+ \right) \) in the limit of \( r = 0 \) is negative. It can be written as follows:
\[
m > |q|. \tag{39}
\]

In Sec. \( \text{II} \) the equations of motion of the thin shell were described in the spherical coordinates. As is widely known, the Reissner-Nordström black hole can be maximally extended. The Reissner-Nordström naked singularity is already inextendible in the spherical coordinates. Trajectories of the thin shell will be analyzed in terms of the ingoing and outgoing Eddington-Finkelstein coordinates. In order to make it easier to confirm the formation of the event horizon, we analyze the gravitational collapse of the thin shell using the coordinate systems of the exterior region.

The global structure of the spacetime is classified based on the trajectory of the thin shell. We focus on the existence of the \( r \)-turning points that determine whether a trajectory reaches the singularity or infinity. From Eqs. \( (36) \) and \( (39) \), the thin shell can have four types of trajectories:

- Crash if \( m \leq \min\{|q|, E\} \),
- Clap if \( E < m \leq |q| \),
- Flyby if \( |q| < m \leq E \),
- Bound if \( \max\{|q|, E\} < m \).

Before further explaining the details of these types, we shall define some terminology. The black hole region of the spacetime is the set of all events that do not belong to the causal past of the future null infinity: \( \mathcal{B} := \mathcal{M} \setminus \mathcal{J}^- (\mathcal{I}^+) \). The future event horizon is defined as the boundary of the black hole region: \( \mathcal{H}^+ := \partial \mathcal{B} \). In the same sense, the past event horizon is defined as follows: \( \mathcal{H}^- := \partial \mathcal{W} \) where \( \mathcal{W} := \mathcal{M} \setminus \mathcal{J}^+ (\mathcal{I}^-) \).

IV. SUBEXTREME BLACK HOLE: \( |Q_+| < M_+ \)

Here, we consider the case \( |Q_+ + q| < M_+ + E \). In particular, for \( Q_+ + q > 0 \), we clarify whether the gravitational collapse of a weakly charged thin shell in a slightly overcharged interior object occurs. The case \( Q_+ + q = 0 \) is described at the end of this section.
A crash trajectory of the thin shell is shown in Fig. 4. The thin shell collapses from region I of the ingoing Eddington-Finkelstein patch, crosses the horizons, enters region III, and hits the curvature singularity \( r = 0 \). Such trajectories are realized for \( m \leq \min\{|q|, E\} \). The shell leaves region \( \Sigma \) through the horizon \( \Sigma^+ \) in the ingoing Eddington-Finkelstein patch. The shell does not intersect the bifurcation two-sphere. Thus the thin shell hits the curvature singularity \( r = 0 \). Note that the trajectory does not intersect the bifurcation two-sphere.

A bound trajectory of the thin shell is shown in Fig. 5. The thin shell leaves through the horizon \( \Sigma^- \) entering region II' of an outgoing Eddington-Finkelstein patch. The shell leaves \( \Sigma^- \) through the horizon \( \Sigma^- \) in the ingoing Eddington-Finkelstein patch. Such trajectories are realized for \( \min\{|q|, E\}, E < m \leq \min\{|q|, E\} \). The thin curve is the clap trajectory of the thin shell.

A flyby trajectory of the thin shell is shown in Fig. 5. The thin shell passes through regions I, II, III, and II', but does not hit the curvature singularity \( r = 0 \). This property of the trajectory is sometimes called oscillation [17]. Such trajectories are realized for \( \max\{|q|, E\} < m \). The shell crosses the horizon \( \Sigma^+ \), transverses region II, and leaves there through \( r = r_{\text{e}} \). The shell enters region III. Then the trajectory has its r-turn at \( r = r_{\text{inf}} \) and enters region II' of the outgoing patch containing region III. The shell crosses \( \Sigma^- \) into region I and goes on out to \( r = \infty \).

A bound trajectory of the thin shell is shown in Fig. 6. The thin shell collapses from region I of the ingoing Eddington-Finkelstein patch, crosses the horizons, enters region III, and hits the curvature singularity \( r = 0 \). Such trajectories are realized for \( m \leq \min\{|q|, E\} \). The shell leaves region \( \Sigma \) through the horizon \( \Sigma^+ \) in an ingoing Eddington-Finkelstein patch. The shell crosses the horizon \( \Sigma^- \) in the ingoing Eddington-Finkelstein patch. The shell does not intersect the bifurcation two-sphere. Thus the thin shell hits the curvature singularity \( r = 0 \). Note that the trajectory does not intersect the bifurcation two-sphere.

A flyby trajectory of the thin shell is shown in Fig. 6. The thin shell passes through regions I, II, III, and II', but does not hit the curvature singularity \( r = 0 \). This property of the trajectory is sometimes called oscillation [17]. Such trajectories are realized for \( \max\{|q|, E\} < m \). The shell crosses the horizon \( \Sigma^+ \), transverses region II, and leaves there through \( r = r_{\text{e}} \). The shell enters region III. Then the trajectory has its r-turn at \( r = r_{\text{inf}} \) and enters region II' of the outgoing patch containing region III. The shell crosses \( \Sigma^- \) into region I and goes on out to \( r = \infty \).

A bound trajectory of the thin shell is shown in Fig. 7. The thin shell leaves through the horizon \( \Sigma^- \) entering region III in the ingoing Eddington-Finkelstein patch. Such trajectories are realized for \( \min\{|q|, E\}, E < m \leq \min\{|q|, E\} \). The thin curve is the clap trajectory of the thin shell.

A bound trajectory of the thin shell is shown in Fig. 7. The thin shell leaves through the horizon \( \Sigma^- \) entering region III in the ingoing Eddington-Finkelstein patch. Such trajectories are realized for \( \min\{|q|, E\}, E < m \leq \min\{|q|, E\} \). The thin curve is the clap trajectory of the thin shell.

A bound trajectory of the thin shell is shown in Fig. 8. The thin shell leaves through the horizon \( \Sigma^- \) entering region III in the ingoing Eddington-Finkelstein patch. Such trajectories are realized for \( \min\{|q|, E\}, E < m \leq \min\{|q|, E\} \). The thin curve is the clap trajectory of the thin shell.

V. EXTREME BLACK HOLE: \( |Q_+| = M_+ \)

In this Sec. V we consider the case \( 0 < |Q_+ + q| = M_+ + E \). The extreme Reissner-Nordström black hole can be formed via the gravitational collapse of a charged thin shell at a finite advanced time. The characteristics of the gravitational collapse of the thin shell are same as the aforementioned results of the subextreme case. Since the formation of the extreme black hole is related to the third law of black hole thermodynamics, we confirm in detail that the advanced time for the formation of the event horizon is finite. \( T_+ (\tau) \) diverges at the proper time \( \tau_\text{e} \).
where \( \tau_e := R^{-1}(r_e) \). We take the limit as \( \tau \to \tau_e \) of the advanced time \( \tau \),

\[
v(\tau) = \int_{\tau_0}^{\tau} \dot{T}_+ d\tau + \int_{\tau_0}^{R(\tau)} \frac{1}{f_+} d\tau.
\]

(40)

To see the improper integral \( v(\tau_e) \) is finite, we check the order of the integrand of \( v \). As \( \tau \to \tau_e \), \( f_+ \) approaches zero and \( F \) does not.

\[
\frac{\dot{T}_+}{R} + \frac{1}{f_+} = -\frac{1}{2F^2} + O(f_+) \text{ as } \tau \to \tau_e.
\]

(41)

Consequently, \( v(\tau) \) can be extended to \( \tau_e \) in the ingoing Eddington-Finkelstein patch. The advanced time \( v(\tau_e) \) for the formation of the extreme black hole is finite. The Penrose diagrams of the spacetimes are shown in Fig. 9, Fig. 10, and Fig. 11.

The third law was first formulated in the following form: It is impossible by any procedure, no matter how idealized, to reduce the surface gravity to zero by a finite sequence of operations \([28]\). The third law in this form does not, so far, have a proof.

Since it is difficult to define the meaning of a “finite sequence of operations,” the following version of the third law is proved and expressed in the formulation: A nonextremal black hole cannot become extremal (i.e., lose its trapped surfaces) at a finite advanced time in any continuous process in which the stress-energy tensor of accreted matter stays bounded and satisfies the weak energy condition in a neighborhood of the outer apparent horizon \([22]\). The third law of this formulation assumes subextreme black holes as an initial object, so the results in this Sec. VI are not counterexamples but provides us with interesting examples with superextreme spacetimes as an initial object. In fact, it can be shown that an extremal black hole can be formed in a finite advanced time even from a Minkowski spacetime by dropping a charged thin dust shell \([1]\).

VI. CONCLUDING REMARKS

It is intuitive that the Reissner-Nordström black hole can be charged up by dropping a charged test particle of the same sign, but it is impossible further in the extreme case. This fact has also been confirmed for various higher dimensional black holes, and it is said that black holes die hard \([13]\).

In this paper, we have discussed the formation of a black hole via the gravitational collapse of a charged thin dust shell in a Reissner-Nordström naked singularity. We have revealed the existence of a transition from a naked singularity to a black hole.

Since our system is symmetric under the time reversal, there is a reverse process of the motion. Does the transition from the Reissner-Nordström black hole to the Reissner-Nordström naked singularity occur by dropping a charged thin dust shell? It is said that this transition does not occur unless the matter energy density of the thin shell is negative \([14, 30]\).

In fact, the real reverse process is the transition from the Reissner-Nordström naked singularity to the Reissner-Nordström black hole by emitting a charged thin dust shell and this process is certainly possible. It should be important to elucidate the relationship between the time reversal symmetry in a system and the reversibility of a transition in a spacetime. It is a way to approach the essence of the weak cosmic censorship conjecture.

Furthermore, in this paper, we saw that an extreme black hole can be formed in a finite advanced time by dropping a charged thin dust shell onto the Reissner-Nordström naked singularity. The version of the third law that has been proved \([27]\) is compatible with our result because it assumes a subextreme black hole as the initial object. Thus, the weak cosmic censorship conjecture and the third law are closely related, and the validity of the third law is very sensitive to its formulation.

1 T.H. is grateful to M. Kimura for pointing out this fact.
In future work, we would like to consider how generally we can describe in what way the naked singularity is prohibited. In order to prove that something is prohibited from forming, it is necessary to make clear what kind of processes are allowed. From this point of view, it seems to be worthwhile to consider the elaborate formulation of the cosmic censorship conjecture based on the findings in the present paper.

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