Notes on Scrambling in Conformal Field Theory

Chang Liu and David A. Lowe

Physics Department, Brown University, Providence, RI, 02912, USA

Abstract

The onset of quantum chaos in quantum field theory may be studied using out-of-time-order correlators at finite temperature. Recent work argued that a timescale logarithmic in the central charge emerged in the context of two-dimensional conformal field theories, provided the intermediate channel was dominated by the Virasoro identity block. This suggests a wide class of conformal field theories exhibit a version of fast scrambling. In the present work we study this idea in more detail. We begin by clarifying to what extent correlators of wavepackets built out of superpositions of primary operators may be used to quantify quantum scrambling. Subject to certain caveats, these results concur with previous work. We then go on to study the contribution of intermediate states beyond the Virasoro identity block. We find that at late times, time-ordered correlators exhibit a familiar decoupling theorem, suppressing the contribution of higher dimension operators. However this is no longer true of the out-of-time-order correlators relevant for the discussion of quantum chaos. We compute the contributions of these conformal blocks to the relevant correlators, and find they are able to dominate in many interesting limits. Interpreting these results in the context of holographic models of quantum gravity, sheds new light on the black hole information problem by exhibiting a class of correlators where bulk effective field theory does not predict its own demise.

* chang_liu3@brown.edu
† lowe@brown.edu
I. INTRODUCTION

It has been suggested that quantum theories of gravity exhibit a property known as fast scrambling, where a generic quantum state exhibits global thermalization in a timescale that is logarithmic in the system size \[1\]. It is interesting to explore this idea in the context of holographic theories of gravity dual to conformal field theories, where one may try to extract constraints on the class of conformal field theories with gravity duals.

One simple way to quantify this notion of scrambling is to consider the norm (or equivalently the square) of the commutator of a pair of Hermitian operators $V$ and $W$ at different times. For the purposes of the present paper, we will also consider the system at finite temperature, with inverse temperature $\beta$. This leads to a relation with out-of-time-order correlators

$$-\langle[V(0),W(t)]^2\rangle_\beta = \langle V(0)W(t)W(t)V(0)\rangle_\beta + \langle W(t)V(0)V(0)V(t)\rangle_\beta$$

$$-\langle W(t)V(0)V(t)V(0)\rangle_\beta - \langle V(0)V(t)V(t)V(0)\rangle_\beta - \langle V(0)V(t)V(t)V(0)\rangle_\beta. \quad (1)$$

For sufficiently late times, the first two terms are simply the time-independent disconnected diagram $\langle WW\rangle_\beta\langle VV\rangle_\beta$, while the last two terms are genuine out-of-time-order correlators. For the 2d conformal field theories of interest here, these correlators may be computed by continuing the Euclidean four-point function through the second Riemann sheet \[2\], as we describe in detail later. These terms vary as a function of $t$, unlike the disconnected terms, and from them a scrambling timescale may be extracted. In the following section we describe in more detail the dependence of this timescale on the chosen operators. Briefly, one wishes to choose operators that exhibit the longest scrambling timescale, so one may use this commutator computation as a proxy for asking that the longest timescale a generic state scrambles. There may of course exist special choices of operators with much shorter scrambling times, and likewise special choices with much longer times, such as those that commute with the Hamiltonian.

In order to study these out-of-time-order correlators at finite temperature in conformal field theory we will begin with the Euclidean theory on $S^1 \times \mathbb{R}$. The correlators in this theory may be obtained by a conformal mapping from the complex plane. The circle direction is to be periodically identified with period $\beta$ and corresponds to the imaginary time direction. The spatial direction is then necessarily of infinite extent. For the purposes of the present paper we will study four-point correlators of primary operators, as well as correlators of
wavepackets of such operators. Four-point functions of primaries are expressed in the so-called conformal blocks of the theory. In general, these conformal blocks are not known beyond infinite series expansions. However there has been much progress in the literature on obtaining asymptotic expansions of these conformal blocks in a variety of limits, and we will make extensive use of these results in the following [3].

In holographic theories, the graviton mode is dual to the stress energy tensor of the CFT, which in turn is a Virasoro descendant of the identity operator. Long distance bulk physics should be dominated by the propagation of this mode, so the limit where the identity block dominates the conformal block is of particular interest. Assuming this intermediate Verma module dominates the conformal block of the four-point function [2] (as well as assuming large central charge and large external conformal weight $h_w$) obtained a scrambling time logarithmic in the central charge $c$ of the CFT

$$t_* = \frac{\beta}{2\pi} \log \frac{c}{h_w}$$

suggesting (at least if the result can be continued to values $h_w$ of order 1) that conformal field theories exhibit a version of fast scrambling.

In this paper we will study this problem in more detail. One immediate issue is that primary operators on their own do not exhibit the timescale [2], but rather a thermalization timescale of order $\beta$ or less. However the class of states obtained by acting on the thermal state with a primary is not necessarily a good representative of a generic state, so this is not an immediate contradiction. To proceed we fold the primary operators into wavepackets, and consider optimizing the shape of the wavepacket to obtain the longest thermalization time. When this is done, we find a timescale resembling [2] does indeed emerge. Next we examine the contribution of Verma modules with higher conformal weights to the four-point function. While we find the time-ordered four-point functions respect the familiar late-time decoupling theorems, and can be ignored with respect to the identity block, this is no longer true of the out-of-time-order correlators needed to compute [1]. We compute the contributions of these higher intermediate states, and find these can indeed dominate the commutator even when all the time-ordered correlators have a sensible holographic description in terms of bulk low energy effective field theory. This implies that many of the bulk observables, defined over finite ranges of time, that one might use to probe the black hole information problem, are not accessible using low energy effective field theory. In this sense effective field theory does not predict its own demise.
II. SCRAMBLING AND CFT CORRELATORS

We consider a thermal system described by a conformal field theory living on a spatial real line $x$ with imaginary time $-it$ periodically identified with period $\beta$. We can map this spatially infinite thermal system to a CFT defined on the complex plane $z$ via the exponential map

$$z = \exp \left( \frac{2\pi}{\beta} (x + t) \right).$$

We are interested in computing the 4-point functions that appear in (1) so to this end we consider four pair-wise local operators, inserted at distinct spatial positions as in fig. 1. We therefore have, after conformal mapping

$$
\begin{align*}
  z_1 &= e^{\frac{2\pi}{\beta} x_1} \\
  z_2 &= e^{\frac{2\pi}{\beta} x_2} \\
  z_3 &= e^{\frac{2\pi}{\beta} (x_3 + t)} \\
  z_4 &= e^{\frac{2\pi}{\beta} (x_4 + t)}
\end{align*}
$$

where we are interested in the limit $x_1 \to x_2$, $x_3 \to x_4$ to reproduce the desired commutator.

The spacetime dependence of the conformal blocks appearing in the 4-point function will...
only depend on the cross-ratio \( z = z_{12} z_{34} / z_{13} z_{24} \) (and \( \bar{z} \)) which is easily shown to be

\[
zs = \frac{\sinh \left( \frac{\pi}{\beta} (x_1 - x_2) \right) \sinh \left( \frac{\pi}{\beta} (x_3 - x_4) \right)}{\sinh \left( \frac{\pi}{\beta} (t - x_1 + x_3) \right) \sinh \left( \frac{\pi}{\beta} (t - x_2 + x_4) \right)}.
\]

As discussed in appendix we rescale the 4-point function by the coincident 2-point functions, to scale out the operator norm. The rescaled correlators then depend only on the cross-ratios as in (A.4).

As an example, let us consider the identity conformal block in a large \( c \) limit, where the \( V \) and \( W \) operators have conformal weights \( h_v \) and \( h_w \) respectively. The large \( c \) limit is to be taken with \( h_w / c \) fixed, and \( h_v \ll c \) fixed. The conformal block \( F(z) \) in this limit is computed in \([3, 4]\]

\[
z^{2h_v} F(z) \approx \left[ \frac{z \alpha_w (1 - z)^{(\alpha_w - 1)/2}}{1 - (1 - z)^{\alpha_w}} \right]^{2h_v},
\]

with \( \alpha_w = \sqrt{1 - 24h_w / c} \). The real-time out-of-time-order correlator is obtained by continuing this block to the second Riemann sheet as described in \([2]\) and the leading contribution to the rescaled commutator is

\[
z^{2h_v} F(z) \approx \left[ \frac{e^{-\pi i (\alpha_w - 1) z} \alpha_w (1 - z)^{(\alpha_w - 1)/2}}{1 - e^{-2\pi i \alpha_w (1 - z)^{\alpha_w}}} \right]^{2h_v} \sim \left( \frac{1}{1 - \frac{24\pi i h_w}{cz}} \right)^{2h_v}. \tag{4}
\]

Let us take a limit where \( \epsilon_{12} = x_1 - x_2 \) and \( \epsilon_{34} = x_3 - x_4 \) are much smaller than \( \beta \), and without loss of generality set \( x_1 = 0 \). The cross-ratio is then approximately

\[
z \approx \frac{\pi^2}{\beta^2} \frac{\epsilon_{12} \epsilon_{34}}{\sinh^2 \left( \frac{\pi}{\beta} (t + x_3) \right)}
\]

provided we stay away from light-like separations where \( x_3 \to -t \). As we see the conformal block on the second sheet has a simple limit as \( \epsilon_{12} \) and \( \epsilon_{34} \to 0 \), when \( z \to 0 \), corresponding to the actual computation of the commutator

\[
z^{2h_v} F(z) \approx \left( \frac{cz}{24\pi i h_w} \right)^{2h_v}. \tag{5}
\]

The exponential decay of this quantity indicates the commutator between \( V \) and \( W \) becomes
large after a time of order
\[ t = \frac{\beta}{4\pi \hbar_v} \]
showing rapid thermalization of primary operators on a timescale much shorter than \( (2) \).

However the interesting physical question is whether generic states exhibit some notion of quantum scrambling on a longer timescale. To explore this question in the current context of CFT 4-point functions, we can then try to build more generic deformations of the thermal density matrix by acting with primary operators folded into wavepackets with some characteristic spatial size \( L \). Computing the 4-point function of these wavepackets, one can attempt to vary \( L \) to maximize the convoluted amplitude, then ask what thermalization timescale emerges.

Concretely, we convolute the function (4) with spatial Gaussian wavepackets with width \( L \). We will choose \( t, L \) and the \( x_i \) such that light-like singularities in \( z \) are avoided. In this regime, the resulting integral will be dominated by a saddle point value of \( z \), and the convoluted (rescaled) conformal block may then be well approximated by simply substituting this value into (4). Given the simple form of (4), with a cusp at \( z = 1 \), the optimal value for \( L \) will be the one that makes \( z \) approach 1.

For simplicity let us set \( x_1 + x_2 = x_3 + x_4 = 0 \), and we will build Gaussian wavepackets in the variables \( x_1 - x_2 = \ell_v \) and \( x_3 - x_4 = \ell_w \). To fix \( L \) in terms of \( z \), one is therefore interested in the convolution

\[ z(t, L) = \frac{4}{\pi L^2} \int_0^\infty d\ell_v d\ell_w e^{-\left(\ell_v^2 + \ell_w^2\right)/L^2} \frac{\sinh \left( \frac{\pi}{\beta} \ell_v \right) \sinh \left( \frac{\pi}{\beta} \ell_w \right)}{\sinh \left( \frac{\pi}{\beta} \left( t - \frac{\ell_v}{2} + \frac{\ell_w}{2} \right) \right) \sinh \left( \frac{\pi}{\beta} \left( t + \frac{\ell_v}{2} - \frac{\ell_w}{2} \right) \right)}. \]  

(7)

This formula is justified because the exponential variation of \( z \) with \( \ell_v, \ell_w \) is much more rapid than power law variation of the conformal block with \( z \), so analyzing the convolution of \( z \) alone is sufficient to determine \( \ell_v \) and \( \ell_w \) and subsequently \( L \). The integrand has light-like poles, however for suitable values of \( t \) and \( L \) these contributions to the smeared conformal block can be made negligible. In this limit, the integrand can be well-approximated by simply

\[ z(t, L) \approx \frac{4}{\pi L^2} \int_0^\infty d\ell_v d\ell_w e^{-\left(\ell_v^2 + \ell_w^2\right)/L^2} \frac{2 \sinh \left( \frac{\pi}{\beta} \ell_v \right) \sinh \left( \frac{\pi}{\beta} \ell_w \right)}{\cosh \left( \frac{2\pi}{\beta} \ell_v \right)}. \]
This has saddle points when

\[ l_v \tanh \left( \frac{l_v \pi}{\beta} \right) = \frac{\pi L^2}{2\beta} \]

and likewise for \( l_w \). The positive solutions are to be taken corresponding to the limits of integration in (7). If we then ask that the resulting amplitude (4) is maximized in magnitude, we find that we must choose \( L \sim \beta \) near \( t = 0 \). We choose not to change the shape of the wavepackets at time increases, and impose this condition for all values of \( t \). At the end we find the optimal value of \( z \) is

\[ z_{sad} = \text{sech} \left( \frac{2\pi}{\beta} t \right) \]  

up to constant factors of order 1.

Let us now return to the example of the identity conformal block continued to the second Riemann sheet as considered in [2]. In this case, the saddle point approximation to the (rescaled) convoluted block function is for sufficiently late times

\[ z^{2h_v} \mathcal{F}(z) \approx \left( \frac{1}{1 - \frac{12\pi h_w}{c} e^{\frac{2\pi}{\beta} (t-x)}} \right)^{2h_v} \]

where we have restored dependence on the spatial separation \( x \) of the centers of the wavepackets, and inserted the saddle point approximation value for \( z \) (8) for \( t \gg \beta \). It is helpful to plot this for sample parameters as in fig. 2. As \( t-x \) increases from 0 to

\[ t_s = \frac{\beta}{2\pi} \log \frac{c\sqrt{\log 2}}{12\pi h_v^{1/2} h_w} \]

the conformal block decreases in magnitude by a factor of about 1/2. This thermalization time may be viewed as a proxy for the true scrambling time of the system, and shows the distinctive appearance of the logarithm of the system size. The formula is valid for \( 0 < h_v \ll c \), but ideally one would want to argue this formula continues to hold as \( h_w \) becomes of order 1. Unfortunately it is not yet possible to prove this. We note fig. 2 also shows in the late-time limit the asymptotic form (5) is applicable and the timescale for variation is the much shorter time (6).

The correlator of the wavepackets is given by (9) provided one steers clear of the lightcone singularities in (7) which render the approximation (8) invalid. This is a signature that even the wavepackets of primaries are not ideal representatives of a generic state, and retain
Figure 2. Plot of function $|z^{2h_v} \mathcal{F}(z)| = |F(z(t))| = 1/(1 - 12\pi i h_w \exp(2\pi/\beta (t - \log c - x))]^{2h_v}$ where $c = 10^7$, $h_v = 100$, $h_w = 10$, $\beta = 2\pi$ and $x = 0$. Here $t_e = 7.7$ according to (10). In the right panel, a plot of $\text{Re} F(z(t))$ is shown.

regions of spacetime where thermalization has not yet occurred, outside the light-cone of the wavepacket. Nevertheless for the present purposes, the reduced state inside the light-cone appears to be well-thermalized according to the correlators, so this procedure should yield a good measure of the global scrambling time. Again it remains to be seen whether (10) holds in the case of most physical interest where $h_w$ is of order 1.

**III. HIGHER WEIGHT INTERMEDIATE STATES**

We now turn our attention to the contribution of higher weight intermediate states to the out-of-time order correlators, and will find the surprising result that these may dominate over the identity block in the late-time limit. Again we will assume we are taking $c \gg 1$ with $h_w/c$ fixed and $h_v \ll c$ fixed. In addition we will generalize from the identity block to an intermediate channel with conformal weight $h_p$ fixed as $c \to \infty$.

Our starting point is the formula for the conformal block at next-to-leading order in this large $c$ expansion of (4)

$$
\mathcal{F}(z) = \mathcal{F}_0(z) \left( \frac{1 - (1 - z)^{\alpha_w}}{\alpha_w} \right)^{h_p} \, _2F_1(h_p, h_p, 2h_p, 1 - (1 - z)^{\alpha_w})
$$

where $\, _2F_1(\alpha, \beta; \gamma; z)$ is the Gauss hypergeometric function. To continue this expression to the second sheet we use the hypergeometric function identity (5)

$$
\frac{\Gamma(h)^2}{\Gamma(2h)} \, _2F_1(h, h; 2h; w) = \left( \sum_{k=0}^{\infty} \, _2F_1 \left( k, h + k; 1 - w \right) \frac{(1 - w)^k}{k!} \right) - \log(1 - w) \, _2F_1(h, h; 1; 1 - w)
$$
valid for $|1 - w| < 1$, where $(h)_k$ is the Pochhammer symbol, and $\psi(a)$ is the digamma function. Continuing to the second sheet we then obtain

$$\mathcal{F}_{\Pi}(z) = \mathcal{F}_{0,\Pi}(z) \left( \frac{1 - e^{-i2\pi \alpha_w (1 - z)\alpha_w}}{\alpha_w} \right)^{h_p} \left( 2 F_1 \left( h_p, h_p, 2h_p, 1 - e^{-i2\pi \alpha_w (1 - z)\alpha_w} \right) \right) + 2\pi i \alpha_w \frac{\Gamma(2h_p)}{\Gamma(h_p)^2} 2 F_1(h_p, h_p, 1; e^{-i2\pi \alpha_w (1 - z)\alpha_w}) \right). \tag{11}$$

Expanding for small $h_w/c$ and $z \ll 1$ leads to

$$\mathcal{F}_{\Pi}(z) \sim \mathcal{F}_{0,\Pi}(z) \left( z - \frac{\pi i h_w}{6c} \right)^{h_p} \left( 1 + i \tan(\pi h_p) - 2\pi i z^{1 - 2h_p} \frac{\Gamma(2h_p)}{(2 - 2h_p)\Gamma(h_p)^4 \sin(2\pi h_p)} \right).$$

This ends up being dominated by the last term in the third factor, and in fact grows at late times. Even at early times ($z$ near 1) the last term in (11) dominates over the other term in the third factor for $h_p > 1$. The second factor in (11) rapidly approaches a constant much smaller than 1.

The upshot is the identity block dominates for a finite period of time, however after

$$t_s \approx \frac{\beta}{4\pi} \log \left( \frac{c}{h_w} \right)$$

the higher weight intermediate states take over. This late time sum over intermediate states apparently diverges when considered term by term. This would lead one to conclude the commutator grows initially while dominated by the identity block, but then may again decrease at later times, indicating a lack of true scrambling in the conformal field theory.

One possible way to avoid this conclusion is to demand an infinite tower of higher weight intermediate primaries, such that the apparently divergent sum might be resummed to a finite answer. However in the following section we find contributions for $h_p \gg c$ are actually suppressed. We conclude that even a sparse spectrum of intermediate primaries with weights $1 < h_p \ll c$ are sufficient to destroy or drastically modify the onset of quantum chaos. In light of our previous discussion, this may simply mean such smeared primaries are still not good representatives of generic states, and instead one would need to consider commutators of much more general operators to see the correct timescale for global thermalization, or quantum scrambling. Alternatively, it may happen that only operators dual to black hole
states efficiently scramble, and these must be reflected in a choice of external operators that do not couple at all (or only very weakly) to higher weight primaries, such that the identity block may dominate the out-of-time order correlators.

For conformal field theories with holographic anti-de Sitter gravity duals, the implication of the higher intermediate channels is that the bulk effective field theory breaks down when it is used to compute out-of-time-ordered correlators at finite time. On the other hand, there is no indication of such a breakdown when time-ordered CFT correlators are computed (see also [6, 7]), which correspond to the boundary S-matrix of the bulk theory. To see this we simply note that as higher dimensional operators in CFT$_2$ correspond to interactions of increasing mass scale in AdS$_3$, domination of all intermediate channels with dimension $h_p \geq 1$ means that there would be a dual set of an infinite sequence of interactions in the gravitation theory in AdS$_3$. If these high scale interactions affect the infrared physics of the theory, then the standard decoupling theorems of effective field theory such [3] break down.

Now the usual measurements we perform can be well-approximated by transition amplitudes, built out of time-ordered correlators which may be computed as within effective field theory. It is only the particular set of observables corresponding to out-of-time-order correlators, or norms of commutators that exhibit this peculiar behavior. For the black hole information problem this would seem to imply that contrary to expectations, commutators that measure limits on the causal propagation of information are indeed observables sensitive to the ultra-violet structure of the theory, as long hinted at in perturbative string theory computations [9, 10].

IV. INTERMEDIATE CHANNELS WITH $h_p \gg c$

So far we have only considered intermediate channels with fixed $h_p \ll c$. It is also instructive to perform the same analysis for intermediate channels with $h_p \gg c$ where the limit is $h_p \to \infty$ with $c/h_p$, $h_v/h_p$, and $h_w/h_p$ fixed and small. For this we consider equation (16) in [11],

$$\mathcal{F}(z) = (16q)^{h_p - \frac{c}{h_p}} z^{\frac{c}{h_p} - 2h_v}(1 - z)^{\frac{c}{h_p} - (h_v + h_w)} \theta_3(q)^{\frac{c}{h_p} - 8(h_v + h_w)} H(c, h_p, h_i, q)$$

(12)
where the nome $q = e^{i\pi \tau}$ is related to the cross-ratio $z$ by

$$\tau = i \frac{K'(z)}{K(z)} = i \frac{K(1 - z)}{K(z)}$$

where $K(z)$ is the complete elliptic integral with parameter $z$. Here $H$ is a function that is $1 + O(1/h_p)$ and

$$\theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}. \tag{13}$$

Eq. (12) has a branch cut at $z = 1$ from the $1 - z$ factor which will lead to the same analytic behavior for the intermediate case $h_p \ll c$, which we have previously considered. To see this we expand the nome $q$ around $z = 0$ to obtain

$$q = e^{i\pi \tau} = \frac{z}{16} + \frac{z^2}{32} + \cdots.$$

As $\theta_3(q)$ is regular near $q = 0$, we see that on the principal sheet $F(z)$ goes to zero as $z \to 0$. Therefore the heavy intermediate channels are perfectly suppressed on the first Riemann sheet. Crossing the branch cut $z = 1$ from above, the complete elliptic function $K(z)$ picks up an additional imaginary part:

$$\lim_{\epsilon \to 0^+} K(z + i\epsilon) = K(z) + 2iK(1 - z).$$

Analyticity implies that on the second Riemann sheet the nome is now

$$q = \exp \left[ -\frac{\pi K(1 - z)}{K(z) + 2iK(1 - z)} \right] = \exp \left[ -\frac{\pi}{\frac{K(z)}{K(1-z)} + 2i} \right].$$

To expand this expression near $z = 0$, we use

$$\frac{K(z)}{K(1 - z)} \approx \frac{\pi}{4\log 2 - \log z} + O \left( \frac{z}{\log^2 z} \right)$$

so that

$$q \approx e^{\frac{\pi}{2} + \frac{z^2}{4\log z}}. \tag{14}$$

We then need to expand (13) near $q = i$. The expansion near $q = 1$ is

$$|\theta_3(q)| \approx \left| \frac{\sqrt{\pi}}{\sqrt{1-q}} \right|$$
but we can obtain the expansion near \( q = i \) by using the relation

\[
|\theta_3(q)| = \frac{\sqrt{\pi}}{\sqrt{\log q}} \theta_4 \left( e^{\frac{\pi^2}{\log q}} \right)
\]

and substituting in (14) to give \( \theta_3(q) \) near \( q = i \) as

\[
|\theta_3(q)| \approx \frac{\sqrt{-2 \log z}}{\sqrt{\pi}}.
\]

Assembling the various factors, we find again a dramatic enhancement of the higher weight channel on the second Riemann sheet arising from the behavior (15), compared to the behavior on the principal sheet. However when we compare to the \( h_p = 0 \) expression of the previous section, the \( z^{c/24} \) factor of (12) dominates for small \( z \) so we conclude they do not dominate versus the identity channel (again modulo restrictions on the operator couplings \( C_p \) of (A.2)).

V. CONCLUSIONS

In this paper we discussed the issue of smearing local operators in a thermal CFT and its connection with quantum scrambling. We pointed out that the correct scrambling time should be identified with operators that maximize the timescale of variation of the out-of-time ordered correlator, which may occur well before the asymptotic late-time limit. We then examined a somewhat independent issue, that the higher intermediate states with \( 0 < h_p \ll c \) can have large out-of-time ordered correlators. We discussed the implications of this statement, which is that in the AdS\(_3\) gravity dual the UV dynamics and IR dynamics is no longer decoupled when these observables are computed. This lack of decoupling appears even when the usual time-ordered correlators, or transition amplitudes satisfy the standard decoupling lore. When applied to scattering in AdS\(_3\) black hole backgrounds this implies that the commutators that lead one to conclude information is lost semiclassically, are in fact not computable without a full specification of the ultraviolet physics of the theory. The ordinary bulk effective field theory does not predict its own demise when computing these observables.

As for the appearance of a scrambling time of the form (2) we have found a variant of this expression (10), valid when the identity block dominates. The expression involves a term of the form \( \beta/2\pi \log c \), but other significant terms are also present. If other intermediate primaries appear, with conformal weights fixed in a large \( c \) limit, they will dominate the
late-time behavior and may completely spoil thermalization. It will be very interesting to extend the range of validity of these expressions to determine whether there exist a class of 2d conformal field theories that may be viewed as fast scramblers at finite temperature.

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Appendix: Correlators and Conformal Blocks

Conformal blocks are usually written in terms of 4-point functions after a global \( SL(2, \mathbb{C}) \) conformal transformation has sent generic points in the complex plane to the values 0, \( z \), 1, \( \infty \). Here we briefly unpack the relation between these conformal blocks and 4-point functions for general \( z_i \).

A canonical form for the 4-point function at general \( z_i \) in the complex plane is \[15\]

\[
\left\langle \prod_{i=1}^{4} \mathcal{O}_i(z_i) \right\rangle = f(z, \bar{z}) \prod_{i<j} z_{ij}^{-(h_i+h_j)+h/3} \prod_{i<j} \bar{z}_{ij}^{-(\bar{h}_i+\bar{h}_j)+\bar{h}/3}
\]  \hspace{1cm} (A.1)

where \( z_{ij} = z_i - z_j \), the cross-ratio \( z = z_{12} z_{34}/z_{13} z_{24} \) and \( h = \sum_i h_i \). The conformal block on the other hand is usually defined \[16\] for the special choice \( z_i = 0, z, 1, \infty \). To define the correlator as the point \( z_4 \) moves to infinity we must rescale by a factor of \( z_4^{2h_w} \)

\[
\lim_{z_4 \to \infty} z_4^{2h_w} \bar{z}_4^{2\bar{h}_w} \left\langle \prod_{i=1}^{4} \mathcal{O}_i(z_i) \right\rangle = \sum_p C_{12p} C_{34p} \mathcal{F}(p, z) \bar{\mathcal{F}}(p, \bar{z}).
\]  \hspace{1cm} (A.2)
Comparing the two formulae yields

\[
\lim_{z_4 \to \infty} z_4^{2h_v} z_4^{2h_{\bar{v}}} \left( \prod_{i=1}^{4} O_i(z_i) \right) \bigg|_{z_1=0, z_3=1, z_2=z} = f(z, \bar{z}) (1 - z)^{h/3 - h_2 - h_3} z^{h/3 - h_1 - h_2} \\
\times (1 - \bar{z})^{h/3 - h_2 - h_3} \bar{z}^{h/3 - h_1 - h_2} \\
= \sum_p C_{12p} C_{34p} F(p, z) \bar{F}(p, \bar{z}) \tag{A.3}
\]

and we see the canonical form of the 4-point function involves a nontrivial rescaling of the conformal block by a function of the cross-ratio.

Later when we study the commutator of two operators, \( V \) and \( W \) as a function of time, it will be convenient to factor out the norm of the operators. To accomplish this we compute

\[
\frac{\langle V(z_1)V(z_2)W(z_3)W(z_4) \rangle}{\langle V(z_1)V(z_2) \rangle \langle W(z_3)W(z_4) \rangle} = z^{2h_v} z^{2h_{\bar{v}}} \sum_p C_{12p} C_{34p} F(p, z) \bar{F}(p, \bar{z}) \tag{A.4}
\]

using (A.1) and (A.3). Now the expression for general \( z_i \) is a function only of the cross-ratios.

Finally we note that in performing a coordinate transformation to a different coordinate system, each correlator of primaries transforms by

\[
\left\langle \prod_i O(x_i) \right\rangle = \prod_i \left( \frac{\partial z}{\partial x} \right)^{h_i} \left( \frac{\partial \bar{z}}{\partial \bar{x}} \right)^{\bar{h}_i} \left. \right|_{z=z_i, \bar{z}=\bar{z}_i} \left\langle \prod_i O(z_i) \right\rangle
\]

and these factors cancel in the expression (A.4).

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[12] We clarify that in most mathematical literature, the complete elliptic integral $K$ is defined with the modulus $k$ as the argument. Our $z$ is related to $k$ through $z = k^2$. It is also common for many mathematicians to use the symbol $m$ for our $z$.

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