Embedded desingularization of toric varieties.

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Abstract

We present a new method to achieve an embedded desingularization of a toric variety. Let \( W \) be a regular toric variety defined by a fan \( \Sigma \) and \( X \subset W \) be a toric embedding. We construct a finite sequence of combinatorial blowing-ups such that the final strict transforms \( X' \subset W' \) are regular and \( X' \) has normal crossing with the exceptional divisor.

Introduction

Fix a polynomial ring, a toric (not necessarily normal) variety is defined by a prime ideal generated by binomials. Such varieties can be considered as combinatorial, in fact all the information they carry can be expressed in terms of combinatorial objects. This gives a way of computing geometric invariants of the toric variety. There are also many applications of the theory of toric varieties, see for example [Cox97]. For an introduction to toric varieties see [Dan78], [Oda88] or [Ful93].

This paper is devoted to construct an algorithm of desingularization of toric varieties and log-resolution of binomial ideals. Given a binomial prime ideal, corresponding to a toric variety \( X \), we construct a sequence of combinatorial blowing-ups such that the strict transform of the variety \( X' \) is non singular and has normal crossings with the exceptional divisor. Our algorithm is valid if the ground field is perfect of any characteristic.

In [GPT02] an algorithm of embedded desingularization of toric varieties is given. In this paper the authors construct a toric map \( X' \to X \), which provides an embedded desingularization of \( X \). If \( X \subset W \) is a toric embedding, they construct a refinement of the fan of \( W \) depending on the embedding, see also [Tei04]. By [DCP83, DCP85] such a toric morphism \( X' \to X \) may be dominated by a sequence of blowing-ups along regular centers.

There is another desingularization method of toric varieties [BM06], producing a sequence of blowing-ups along non singular centers. This method is defined in terms of the Hilbert function of the variety.

In [Bla08, Bla09] an algorithm of log-resolution of binomial ideals is constructed based on the computation of an ordering function \( E\text{-ord} \). The function \( E\text{-ord} \) is the order of the ideal defining the variety along a normal crossing divisor \( E \). But this log-resolution algorithm depends on a choice of a Gröbner basis of the ideal.

The algorithm presented here depends on the ordering function \( E\text{-ord} \) and a codimension function \( H\text{codim} \). This algorithm does not depend on any choice and it can be implemented at the computer and we expect to have a working implementation shortly.

On the other hand, we prove an equivalence of the geometric notion of transversality of a variety with respect to a normal crossing divisor and a new notion of transversality of \( \mathbb{Z} \)-modules. Then we are able to translate geometric notions to combinatorial terms.

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This can be considered as a first step on a more ambitious program translating notions from toric varieties, in terms of dual cones and fans, to notions in terms of the binomial equations of the variety.

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The paper is structured as follows: First we recall known facts on toric varieties. There is a bijection between toric varieties and saturated $\mathbb{Z}$-submodules of $\mathbb{Z}^n$. We define a notion of transversality of $\mathbb{Z}$-submodules which will be equivalent to the usual notion of transversality of the toric variety with respect to a normal crossing divisor. Given an affine toric variety $X$, we will prove also the existence of a minimal regular toric variety $V$ transversal to $E$ (the normal crossing divisor) and containing $X$. This minimal embedding $X \subset V$ will define a function $H_{\text{codim}}$ which is the first coordinate of our resolution function. The rest of the coordinates of this resolution function comes from the process of $E$-resolution constructed in [Bla08] and [Bla09]. In the last section we will construct the algorithm of embedded desingularization of a toric variety $X \subset W$, where $W$ is a smooth toric variety.

Note 0.1. Fix a perfect field $k$. We will denote the affine space of dimension $n$ as usual $\mathbb{A}^n = \text{Spec}(k[x_1,\ldots,x_n])$. The torus of dimension $n$ is $\mathbb{T}^n = \text{Spec}(k[x_1^\pm,\ldots,x_n^\pm])$. Note that $\mathbb{A}^n \setminus \mathbb{T}^n$ is a union of $n$ hypersurfaces having only normal crossings. Let $E$ be a set of hypersurfaces such that every $H \in E$ is an irreducible component of $\mathbb{A}^n \setminus \mathbb{T}^n$. The set $E$ corresponds to a subset of indexes $E \subset \{1,\ldots,n\}$. We set

$$\mathbb{T}\mathbb{A}^n_E = \mathbb{A}^n \setminus \bigcup_{H \notin E} H = \text{Spec} \left( k[x_1,\ldots,x_n]_{\prod_{i \notin E} x_i} \right)$$

where $\text{Spec}(k[x_1,\ldots,x_n]_{\prod_{i \notin E} x_i})$ is the localization of the polynomial ring with respect to the product $\prod_{i \notin E} x_i$.

Note that $E$ is a set of hypersurfaces of $\mathbb{T}\mathbb{A}^n_E$ having only normal crossings.

A morphism $\mathbb{T}^n \to \mathbb{T}^1$ which is a group homomorphism is called a character. For example if $a \in \mathbb{Z}^n$ then the morphism defined by $T \to X_1^{a_1} \cdots X_n^{a_n}$ is a character. In fact, every character of $\mathbb{T}^n$ is as above [CLS, Hum75, ES96]. So that the group of characters of a $n$-dimensional torus is a free abelian group of rank $n$.

Note 0.2. There is a bijection between morphisms $\mathbb{T}^d \to \mathbb{T}^n$ which are also group homomorphisms and homomorphisms $\mathbb{Z}^n \to \mathbb{Z}^d$ of $\mathbb{Z}$-modules [Hum75]. Moreover, closed reduced immersions $\mathbb{T}^d \to \mathbb{T}^n$ correspond to surjective homomorphisms $\mathbb{Z}^n \to \mathbb{Z}^d$.

1. **Affine toric varieties.**

We recall some basic definitions and well known results on toric varieties.

**Definition 1.1.** An affine toric variety is an affine variety $X$ of dimension $d$, such that $X$ contains the torus $\mathbb{T}^d$ as a dense open set and the action of the torus extends to an action of $\mathbb{T}^d$ to $X$.

Theorems 1.2 and 1.3 are well known results.

**Theorem 1.2.** [CLS, MS05, ES96] Let $X$ be a scheme of dimension $d$. TFAE:
1. $X$ is an affine variety.

2. $X \cong \text{Spec}(k[t^{a_1}, \ldots, t^{a_n}])$, where $a_1, \ldots, a_n \subset \mathbb{Z}^d$.

3. $X \subset \mathbb{A}^n$ and $I(X)$ is prime and generated by binomials.

**Theorem 1.3.** Let $X$ be an affine toric variety of dimension $d$. $X$ is regular if and only if $X \cong TA^d_E$ for some $E \subset \{1, \ldots, d\}$.

**Definition 1.4.** An affine toric embedding is a reduced closed subscheme $X \subset W$ where:

- $W$ is a regular affine toric variety.
- $X$ is an affine toric variety.
- The inclusion is toric, which means that one has a group homomorphism from the torus of $X$ to the torus of $W$.

Toric varieties are related to $\mathbb{Z}$-submodules of $\mathbb{Z}^n$.

**Definition 1.5.** Let $M \subset \mathbb{Z}^n$ be a $\mathbb{Z}$-module. The saturation of $M$ is:

$$\text{Sat}(M) = \{\alpha \in \mathbb{Z}^n \mid \lambda\alpha \in M \text{ for some } \lambda \in \mathbb{Z}\}$$

We say that a $\mathbb{Z}$-module $L \subset \mathbb{Z}^n$ is saturated if $\text{Sat}(L) = L$.

Note that $L \subset \mathbb{Z}^n$ is saturated if and only if the quotient $\mathbb{Z}^n/L$ is a free $\mathbb{Z}$-module.

Note also that $M \otimes \mathbb{Q} = \text{Sat}(M) \otimes \mathbb{Q}$.

The following theorem is based on known results.

**Theorem 1.6.** [ES96] Let be $n \in \mathbb{N}$, $r \leq n$ and $d \leq n$. There is a bijection correspondence between the following sets:

1. The set of affine toric embeddings $X \subset TA^n_d$, with $d = \dim X$.
2. The set of closed and reduced immersions $\mathbb{T}^d \to \mathbb{T}^n$.
3. The set of surjective homomorphisms of $\mathbb{Z}$-modules, $\mathbb{Z}^n \to \mathbb{Z}^d$.
4. The set of saturated $\mathbb{Z}$-submodules $L \subset \mathbb{Z}^n$ of rank $n - d$.

### 2 Sublattices of $\mathbb{Z}^n$

In this section we introduce the notion of transversality of a $\mathbb{Z}$-module with respect to a subset $E \subset \{1, \ldots, n\}$. We will prove that there exists always a maximal transversal submodule of any saturated $\mathbb{Z}$-submodule of $\mathbb{Z}^n$.

**Lemma 2.1.** Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ with $\alpha \neq 0$.

The following are equivalent:

1. $\gcd\{\alpha_1, \ldots, \alpha_n\} = 1$.
2. The $\mathbb{Z}$-module $(\alpha)$ is saturated (1.5).

Moreover assume that $\alpha_{m+1} = \cdots = \alpha_n = 0$ for some $m \leq n$ and that $(\alpha)$ is saturated, then there is a surjective homomorphism $\psi : \mathbb{Z}^n \to \mathbb{Z}^{n-1}$ with $\ker \psi = (\alpha)$ and such that $\psi(e^{(n)}_j) = e^{(n-1)}_{j-1}$ for $j = m+1, \ldots, n$, where $e^{(n)}_j \in \mathbb{Z}^n$ is the $j$-th element of the canonical base of $\mathbb{Z}^n$. 

Proof. The equivalence of (1) and (2) is an easy exercise.
The last assertion follows from the fact that the Smith normal form of the column matrix $\alpha$ is $(1,0,\ldots,0)$. There is a (non unique) invertible integer matrix $P$ such that $P\alpha = (1,0,\ldots,0)$.
In fact if $\alpha_{m+1} = \cdots = \alpha_n = 0$ then $P$ may be chosen as follows

$$P = \left( \begin{array}{cc} P' & 0 \\ 0 & I_{n-m} \end{array} \right)$$

Set $A$ the $(n-1) \times n$ matrix obtained by deleting the first row of $P$. The matrix $A$ defines the required homomorphism $\mathbb{Z}^n \to \mathbb{Z}^{n-1}$.

**Definition 2.2.** Let $E \subset \{1,\ldots,n\}$ be a subset. We define $\mathbb{Z}^n_E$ and $\mathbb{Z}^n_{E^+}$ to be

$$\mathbb{Z}^n_E = \{(\alpha_1,\ldots,\alpha_n) \in \mathbb{Z}^n \mid \alpha_j \geq 0 \ \forall j \in E\}$$

$$\mathbb{Z}^n_{E^+} = \{(\alpha_1,\ldots,\alpha_n) \in \mathbb{Z}^n \mid \alpha_j > 0 \ \forall j \in E\}$$

**Definition 2.3.** Let $M \subset \mathbb{Z}^n$ be a $\mathbb{Z}$-submodule and $E \subset \{1,\ldots,n\}$ be a subset.
We say that $M$ is weak-transversal to $E$ if $M$ admits a system of $\mathbb{Z}$-generators $\alpha_1,\ldots,\alpha_\ell$ with $\alpha_j \in \mathbb{Z}^n_E$ (2.2).
We say that a $\mathbb{Z}$-module $L \subset \mathbb{Z}^n$ is transversal to $E$ if it is weak-transversal to $E$ and $L$ is saturated.

**Definition 2.4.** Let $M \subset \mathbb{Z}^n$ be a $\mathbb{Z}$-submodule and $E \subset \{1,\ldots,n\}$ be a subset. Consider $pr_j : \mathbb{Z}^n \to \mathbb{Z}$ the $j$-projection, $j = 1,\ldots,n$. Set

$$E_M = \{j \in E \mid pr_j(M) \neq 0\}$$

**Remark 2.5.** The set $E_M$ depends on the module $M$, but we can reduce the study of transversality of $M$ to the smaller subset $E_M$. Note that for any generator system $\alpha_1,\ldots,\alpha_\ell$ of $M$ we have that $\alpha_{i,j} = 0$ for all $i = 1,\ldots,\ell$ and any $j \in E \setminus E_M$.

Propositions 2.6 and 2.7 come from discussions with Ignacio Ojeda.

**Proposition 2.6.** Let $M \subset \mathbb{Z}^n$ be a $\mathbb{Z}$-submodule and $E \subset \{1,\ldots,n\}$ be a subset.
The $\mathbb{Z}$-module $M$ is weak transversal to $E$ (2.3) if and only if $M$ is weak transversal to $E_M$.

*Proof.* One implication is obvious. Assume that $M$ is weak transversal to $E_M$. So that there exists a generator system $\alpha_1,\ldots,\alpha_\ell$ such that $\alpha_{i,j} \geq 0$ for $i = 1,\ldots,\ell$ and $j \in E \setminus E_M$. The result follows from remark 2.5.

**Proposition 2.7.** Let $M \subset \mathbb{Z}^n$ be a $\mathbb{Z}$-submodule and $E \subset \{1,\ldots,n\}$ be a subset.
The $\mathbb{Z}$-module $M$ is weak transversal to $E$ if and only if there is $\gamma \in M \cap \mathbb{Z}^n_{E^+_M}$.

*Proof.* Assume that $M$ is weak transversal to $E$. There is a generator system $\alpha_1,\ldots,\alpha_\ell$ of $M$ with $\alpha_i \in \mathbb{Z}^n_{E_i}$, $i = 1,\ldots,\ell$.
Note that for any $j \in E_M$ there is an index $i \in \{1,\ldots,\ell\}$ such that $\alpha_{i,j} > 0$. Set $\gamma = \alpha_1 + \cdots + \alpha_\ell$ and it is clear that $\gamma \in \mathbb{Z}^n_{E^+_M}$.

Conversely, assume that there is $\gamma \in M \cap \mathbb{Z}^n_{E^+_M}$. Consider $\beta_1,\ldots,\beta_\ell$ a generator system of $M$. There are integers $a_1,\ldots,a_\ell \in \mathbb{Z}$ with $\gamma = a_1\beta_1 + \cdots + a_\ell\beta_\ell$. We may assume that $\gcd\{a_1,\ldots,a_\ell\} = 1$. Note that we may complete $\gamma$ to a generator system of $M$, say, $\gamma,\gamma_2,\ldots,\gamma_\ell$. This is a consequence of the fact that the Smith normal form of the row matrix $(a_1,\ldots,a_\ell)$ is $(1,0,\ldots,0)$.

Now we may choose positive integers $\lambda_2,\ldots,\lambda_\ell$ such that $\gamma_i + \lambda_i\gamma \in \mathbb{Z}^n_E$. Set $a_1 = \gamma$ and $\alpha_i = \gamma_i + \lambda_i\gamma$, $i = 2,\ldots,\ell$. It is clear that $\alpha_1,\ldots,\alpha_\ell$ is a generator system of $M$ and $\alpha_i \in \mathbb{Z}^n_{E_M}$, $i = 1,\ldots,\ell$. In fact we may assume that $\alpha_i \in \mathbb{Z}^n_{E^+_M}$.

\[ \square \]
Proposition 2.8. Let $E \subset \{1, \ldots, n\}$ and let $M \subset \mathbb{Z}^n$ be a $\mathbb{Z}$-module.

If $M$ is weak-transversal to $E$ then $\text{Sat}(M)$ is transversal to $E$.

Proof. Note that $E_M = E_{\text{Sat}(M)}$. Then proposition 2.8 is a direct consequence of 2.6 and 2.7.

Proposition 2.9. Let $E \subset \{1, \ldots, n\}$ be a subset. Let $L \subset \mathbb{Z}^n$ be a saturated $\mathbb{Z}$-submodule.

There exists a unique $\mathbb{Z}$-module $L_0$ such that

- $L_0 \subset L$,
- $L_0$ is transversal to $E$,
- If $L'_0 \subset L$ and $L'_0$ is transversal to $E$ then $L'_0 \subset L_0$.

Proof. Consider all $\mathbb{Z}$-submodules $\{M_{\lambda}\}_{\lambda \in \Lambda}$ such that $M_{\lambda} \subset L$ and $M_{\lambda}$ weak-transversal to $E$ for every $\lambda \in \Lambda$.

Set $M = \sum_{\lambda \in \Lambda} M_{\lambda}$. Note that $M$ is weak-transversal to $E$ and $M \subset L$.

By 2.8 $L_0 = \text{Sat}(M)$ is transversal to $E$ and we have also that $M \subset L_0 \subset L$. In fact by construction $M = L_0$ and it is the biggest $\mathbb{Z}$-module with this property.

3 Affine toric varieties and transversality.

In this section we will prove the equivalence of the new notion of transversality of $\mathbb{Z}$-submodules (definition 2.3) and the geometric usual notion of transversality of a variety with respect to a normal crossing divisor.

Let $W$ be a regular affine toric variety of dimension $n$. It follows from (1.3) that $W \cong \mathbb{T} \mathcal{A}_E^n$ (notation as in 0.1). Recall that $E$ is a set of regular hypersurfaces in $W$ having only normal crossings. Using the isomorphism $W \cong \mathbb{T} \mathcal{A}_E^n$ we may identify $E$ with a set $E \subset \{1, \ldots, n\}$.

With this identification $W = \mathbb{T} \mathcal{A}_E^n = \text{Spec}(k[x_1, \ldots, x_n]_{\prod_{i \in E} x_i})$.

The variety $W = \mathbb{T} \mathcal{A}_E^n$ has a distinguished point $\xi_0 \in W$

$$\xi_0 \in \bigcap_{H \in E} H$$

with coordinates $\xi_0 = (\xi_{0,1}, \ldots, \xi_{0,n})$ where $\xi_{0,i} = 0$ if $i \in E$ and $\xi_{0,i} = 1$ if $i \notin E$.

Definition 3.1. [Bla08, Bla09] Let $W$ be a regular affine toric variety of dimension $n$ and let $J \subset \mathcal{O}_W$ be a sheaf of ideals. For any point $\xi \in W$ consider $E_\xi$ the intersection of all hypersurfaces $H \in E$ with $\xi \in H$:

$$E_\xi = \bigcap_{\xi \in H \in E} H$$

The ideal $I(E_\xi) \subset \mathcal{O}_W$ is generated by all the equations of hypersurfaces $H$ with $\xi \in H \in E$.

We define the function $E\text{-ord}(J) : W \to \mathbb{N}$ as follows:

$$E\text{-ord}(J)(\xi) = \max\{b \in \mathbb{N} \mid J \subset I(E_\xi)^b\}$$

where $J$ is an ideal in $W$.

Note that the function $E\text{-ord}(J)$ is constant along the strata defined by $E$. In fact $E\text{-ord}(J)(\xi)$ is the (usual) order of the ideal $J$ at the generic point of $E_\xi$. The function $E\text{-ord}(J) : W \to \mathbb{N}$ is upper-semi-continuous, see [Bla08, Bla09] for a proof and more details.
Definition 3.2. An ideal \( J \subset \mathcal{O}_W, W = \mathbb{T}^n_E \), is binomial if \( J \) can be generated, as ideal, by binomials: \( x^\alpha - x^\beta \), with \( \alpha, \beta \in \mathbb{N}^n \).

Lemma 3.3. Let \( J \subset \mathcal{O}_W \) be a binomial ideal, \( W = \mathbb{T}^n_E \), and let \( \xi_0 \in W \) be the distinguished point. Then \( \xi_0 \in \text{Max} \text{E-ord}(J) = \{ \xi \in W \mid \text{E-ord}(J)(\xi) = \max \text{E-ord}(J) \} \).

Proof. Note that \( E_{\xi_0} \subset E \) for any \( \xi \in W \).

The following definition is general for any variety:

Definition 3.4. Let \( X \subset W \) be an embedded variety and let \( E \) be a set of regular hypersurfaces of \( W \) having only normal crossings. We say that \( X \) is transversal to \( E \) at a point \( \xi \in X \) if there is a regular system of parameters of \( \mathcal{O}_{W,\xi}, x_1, \ldots, x_n \in \mathcal{O}_{W,\xi} \), such that

- \( I(X)_\xi = (x_1, \ldots, x_r) \) for some \( r \leq n \) and
- For all \( H \in E \) with \( \xi \in H \), then \( I(H)_\xi = (x_i) \) for some \( i \) with \( r < i \leq n \).

Consider \( W = \mathbb{T}^n_E \), for some \( E \subset \{1, \ldots, n\} \). The derivatives with poles along \( E \) is a free \( \mathcal{O}_W \)-module of rank \( n \) and a natural basis of this module is

\[
x_i^{\epsilon_i} \frac{\partial}{\partial x_i} \quad i = 1, \ldots, n
\]

where \( \epsilon_i = 0 \) if \( i \in E \) and \( \epsilon_i = 1 \) if \( i \notin E \).

Lemma 3.5. Let \( X \subset W = \mathbb{T}^n_E \) be an affine toric embedding with \( d = \dim(X) \). Consider any set of binomial generators of the ideal \( I(X) \subset \mathcal{O}_W \)

\[
I(X) = (x^{\alpha_1}_1 - x^{\alpha_1}, \ldots, x^{\alpha_\ell}_m - x^{\alpha_m}) = (f_1, \ldots, f_m)
\]

Fix a point \( \xi \in X \). The variety \( X \) is transversal to \( E \) at the point \( \xi \) (3.4) if and only if the jacobian matrix:

\[
\left( x_i^{\epsilon_i} \frac{\partial f_j}{\partial x_i} \right)_{i,j}
\]

has rank \( n - d \) at the point \( \xi \).

Proof. This lemma is a direct consequence of a general fact on algebraic varieties. \( \square \)

Proposition 3.6. Let \( X \subset W = \mathbb{T}^n_E \) an affine toric embedding. If \( \max \text{E-ord}(I(X)) > 0 \) then \( X \) is not transversal to \( E \).

Proof. It follows from 3.5 and 3.3. At the distinguished point \( \xi_0 \), the jacobian matrix in lemma 3.5 is zero modulo the maximal ideal at \( \xi_0 \). \( \square \)

Theorem 3.7. Let \( V \subset W = \mathbb{T}^n_E \) be an affine toric embedding. They are equivalent:

1. \( V \) is transversal to \( E \).
2. The ideal \( I(V) \) is generated by hyperbolic equations

\[
I(V) = (x^{\alpha_1} - 1, \ldots, x^{\alpha_\ell} - 1)
\]

where \( \ell = n - \dim V, \alpha_1, \ldots, \alpha_\ell \in \mathbb{Z}_E^n \) and they generate a saturated lattice of rank \( \ell \).
Proof. Set $\ell = n - \dim V$. Let $\alpha_1, \ldots, \alpha_\ell \in \mathbb{Z}_E^n$ be such that $\alpha_1, \ldots, \alpha_\ell$ they generate a saturated lattice of rank $\ell$. Assume that $I(V) = (x^{\alpha_1} - 1, \ldots, x^{\alpha_\ell} - 1)$. Consider the jacobian matrix (3.5)

$$
\begin{pmatrix}
\frac{\partial}{\partial x_i} (x^{\alpha_j} - 1)
\end{pmatrix} = 
\begin{pmatrix}
\alpha_{1,1} x^{\alpha_1} & \cdots & \alpha_{\ell,1} x^{\alpha_\ell} \\
\vdots & & \vdots \\
\alpha_{1,n} x^{\alpha_1} & \cdots & \alpha_{\ell,n} x^{\alpha_\ell}
\end{pmatrix}
$$

Note that the rank of this matrix at any point $\xi \in V$ is the rank of the matrix $(\alpha_1 | \cdots | \alpha_\ell)$ having $\alpha_i$ as columns. And the rank of this matrix is $\ell$ (independently of the characteristic of the ground field $k$). So that $V$ is transversal to $E$.

Conversely assume that $V$ is transversal to $E$. We may assume that $E = \{r + 1, \ldots, n\}$. Let show first the codimension one case: $\ell = 1$. So that $I(V) = (x^{\alpha_1} - x^{\alpha_1'})$ where $\alpha_1 = \alpha_1' + \alpha_1$ and $\alpha_1, \alpha_1' \in \mathbb{Z}^r \times \mathbb{N}^n - r$. By 3.6 we have that max $E$-ord($I(V)$) = 0 so that we may assume that $I(V) = (x^{\alpha_1} - 1)$ with $\alpha_1 \in \mathbb{Z}^r \times \mathbb{N}^n - r$. Since $V$ is toric, we may assume that gcd$\{\alpha_1, \ldots, \alpha_1, n\} = 1$. Lemma 2.1 gives the result. We now prove the general case, codimension $\ell > 1$. By 3.6 there is a hyperbolic equation $x^{\alpha_1} - 1 \in I(V)$. Since $V$ is toric, we may assume that gcd$\{\alpha_1, \ldots, \alpha_1, n\} = 1$. Set $W_1$ the toric hypersurface defined by $x^{\alpha_1} - 1$. After reordering the last $n - \ell$ coordinates we may assume that $\alpha_{1,m+1} = \cdots = \alpha_1,n = 0$ and $\alpha_{1,i} > 0$ for $i = r + 1, \ldots, m$, where $r \leq m$. We do not assume anything on $\alpha_{1,1}, \ldots, \alpha_{1,r}$.

Let $\psi : \mathbb{Z}^n \to \mathbb{Z}^r_{\ell} \times \mathbb{N}^n - r_{\ell}$ be the homomorphism given by lemma 2.1. Note that $W_1 \cong \mathbb{T}^{m-1} \times \mathbb{A}^{n-m}$.

We have $V \subset W_1$ and by induction there are $\beta_2, \ldots, \beta_\ell \in \mathbb{Z}^{m-1} \times \mathbb{N}^{n-m}$ such that the ideal of $V$ in $W_1$ is generated by $y^{\beta_2} - 1, \ldots, y^{\beta_\ell} - 1$, and $\beta_2, \ldots, \beta_\ell$ generate a saturated lattice in $\mathbb{Z}_{\ell}^{m-1}$ of rank $\ell - 1$. Let $\beta_2, \ldots, \beta_\ell \in \mathbb{Z}^n$ such that $\psi(\beta_i) = \beta_i$ for $i = 2, \ldots, \ell$. We have that $\alpha_1, \beta_2, \ldots, \beta_\ell$ generate a saturated lattice of rank $\ell$. It is clear that $\beta_i \in \mathbb{Z}^{m-1} \times \mathbb{N}^{n-m}$, but in general $\beta_i \not\in \mathbb{Z}^r \times \mathbb{N}^n - r$. Since $\alpha_{1,i} > 0$ for $i = r + 1, \ldots, m$ there are natural numbers $\lambda_2, \ldots, \lambda_\ell$ such that $\alpha_i = \beta_i + \lambda_i \alpha_1 \in \mathbb{Z}^r \times \mathbb{N}^n - r$, $i = 2, \ldots, \ell$. And $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ generate the same lattice. Finally we have the equality of ideals

$$I(V) = (x^{\alpha_1} - 1, x^{\alpha_2} - 1, \ldots, x^{\alpha_\ell} - 1)$$

\[ \square \]

Lemma 3.8. Let $J = (x^{\beta_1} - 1, \ldots, x^{\beta_s} - 1)$ be an ideal generated by hyperbolic binomials, $\beta_i \in \mathbb{Z}_{E,V}^n$, $i = 1, \ldots, s$. Assume that $J$ is a prime ideal. If $\gamma \in \mathbb{Z}^n$ then $x^{\gamma^+} - x^{\gamma^-} \in J$ if and only if $\gamma$ belongs to the $\mathbb{Z}$-module generated by $\beta_1, \ldots, \beta_s$ in $\mathbb{Z}^n$.

Proof. The $\mathbb{Z}$-module $L$ generated by $(\beta_1, \ldots, \beta_s)$ is associated to the toric variety defined by $J$.

It is well known that $\beta \in L$ if and only if $x^{\beta^+} - x^{\beta^-} \in J$. \[ \square \]

Proposition 3.9. Let $V \subset \mathbb{T} \mathbb{A}^n_E$ be an affine toric embedding. Set $L \subset \mathbb{Z}^n$ be the lattice associated to $V$.

$V$ is transversal to $E$ if and only if $L$ is transversal to $E$.

Proof. It is a consequence of 3.7 and 3.8.

\[ \square \]

Theorem 3.10. Let $X \subset W = \mathbb{T} \mathbb{A}^n_E$ be an affine toric embedding.

There is a unique toric variety $V$ such that the embeddings $X \subset V \subset W$ are toric and $V$ is the minimum toric variety containing $X$ and transversal to $E$.

Proof. It follows from 2.9 and 3.9. \[ \square \]
4 Embedded toric varieties.

In the previous sections we have reduced to the case of affine toric varieties. We generalize here to (non affine) toric varieties, and we define the first coordinate of our resolution function.

Let $W$ be a regular toric variety defined by a fan $\Sigma$ [Ful93]. Let $T \subset W$ be the torus of $W$, which is open and dense in $W$. Set $E$ the simple normal crossing divisor given by $W \setminus T$.

For every $\sigma \in \Sigma$ the open set $W_\sigma \subset W$ is an affine toric variety, so that $W_\sigma \cong \mathbb{A}^{n_\sigma}_E$.

**Definition 4.1.** A toric embedding is a closed subscheme $X \subset W$ such that for every $\sigma \in \Sigma$ if $X_\sigma = X \cap W_\sigma$ then $X_\sigma \subset W_\sigma$ is an affine toric embedding (1.4).

For every $\sigma \in \Sigma$ there is a unique toric affine variety $V_\sigma \subset W_\sigma$ transversal to $E$ and such that $X_\sigma \subset V_\sigma$ (theorem 3.10).

In fact, this toric affine variety $V_\sigma$ is a regular toric affine variety.

**Remark 4.2.** Note that for any $\xi \in W$, there is a unique $\sigma \in \Sigma$ such that $\xi \in W_\sigma$ and the affine open set $W_\sigma$ is minimum with this property. In fact $\xi$ belongs to the orbit of the distinguished point of $W_\sigma$.

**Definition 4.3.** Let $\xi \in X \subset W$ be a point. Let $X_\sigma$ be the minimum affine open set containing the point $\xi$.

The hyperbolic codimension of $X$ at $\xi$ is

$$
\text{Hcodim}(X)(\xi) = \dim V_\sigma - \dim X
$$

where $V_\sigma \subset W_\sigma$ is the minimum toric affine variety such that $V_\sigma \supset X_\sigma$ and it is transversal to $E$ (3.10).

**Remark 4.4.** The hyperbolic codimension $\text{Hcodim}(X)$ (4.3) can be understood as a toric embedding dimension. The number $\text{Hcodim}(X)$ at $\xi$ is the minimum dimension of a regular toric variety $V$ including $X$.

In the case $V_\sigma = W_\sigma$, then $\text{Hcodim}(X)(\xi) = \text{codim}_W(X)$, the codimension of $X$ in $W$.

**Remark 4.5.** [BM06] Let $\Delta \in \Sigma$ be an element of the fan $\Sigma$ defining the regular variety $W$.

The cone $\Delta$ defines a smooth closed subvariety $Z_\Delta \subset W$ as follows:

The toric variety $W$ is covered by affine toric varieties $W_\sigma$ with $\sigma \in \Sigma$. So that $Z_\Delta$ is covered by affine pieces $(Z_\Delta)_\sigma = Z_\Delta \cap W_\sigma$, $\sigma \in \Sigma$.

If $\Delta$ is not a face of $\sigma$ then $(Z_\Delta)_\sigma = \emptyset$.

If $\Delta$ is a face of $\sigma$, note that $W_\Delta \subset W_\sigma$ is an open inclusion. Then $(Z_\Delta)_\sigma$ is the (closure) of the orbit of the distinguished point of $W_\Delta$.

The smooth closed center $Z_\Delta$ we will say that it is a *combinatorial center* of $W$. In fact note that at every affine chart $W_\sigma \cong \mathbb{A}^{n_\sigma}_E$, for some $E$, the combinatorial center $Z_\Delta \cap W_\sigma$ is defined by some coordinates $x_i$ with $i \in E$.

**Remark 4.6.** Note that if $X \subset W \cong \mathbb{A}^{n}_E$ is an affine toric embedding and $Z_\Delta \subset W$ is a combinatorial center, then the strict transforms $X' \subset W'$ give an affine toric embedding.

**Proposition 4.7.** Let $\Delta \in \Sigma$ and $Z_\Delta \subset W$ the combinatorial center associated to $\Delta$ (4.5).

Let $W' \to W$ be the blow-up with center $Z_\Delta$. Set $X' \subset W'$ the strict transform of $X$. If $\xi' \in X'$ then

$$
\text{Hcodim}(X')(\xi') \leq \text{Hcodim}(X)(\xi)
$$

where $\xi'$ maps to $\xi$. 

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Proof. Let $W_\sigma$ be the minimum affine open set of $W$ containing the point $\xi$ and let $V_\sigma$ be the minimum toric affine variety in $W_\sigma$ such that $V_\sigma \supset X_\sigma$ and it is transversal to $E$ (3.10). Let $W'_\sigma$, with $\sigma' \in \Sigma'$, be the minimum affine open set of $W'$ containing the point $\xi'$. Let $V'_\sigma$, be the minimum toric affine variety such that $V'_\sigma \supset X'_\sigma$, and it is transversal to $E'$. Note that $X'_\sigma \subset (X_\sigma)'$ is an open immersion, where $(X_\sigma)' \subset X'$ is the strict transform of $X_\sigma \subset X$.

Let $(V_\sigma)'$ be the strict transform of $V_\sigma$. Note that $(V_\sigma)'$ is smooth and transversal to $E'$. So that $(V_\sigma)' \cap W'_\sigma \supset V'_\sigma$. And the result follows from the last inclusion. 

5 E-resolution of binomial ideals.

In [Bla08, Bla09] were given some notions in terms of binomial basic objects along $E$, where $E$ was a normal crossing divisor in the ambient space $W$. In terms of the E-ord (3.1) one may construct a sequence of combinatorial blowing-ups such that the transform of a given binomial ideal has maximal $E$-order equal to zero.

We remind here the main results. For more details on the several constructions and proofs see [Bla08] or [Bla09]. All these notions work for general binomial ideals, without any restriction.

Note 5.1. Using this structure of binomial basic object along $E$, and the language of mobiles (see [EH02]), it is possible to construct a resolution function involving the $E$-order of certain ideals computed by induction on the dimension of $W$.

Note 5.2. Roughly speaking, given $(W, (J, c), H, E)$, where $J$ is a binomial ideal, $c$ is a positive integer, and $H$ is the set of exceptional hypersurfaces, by induction on the dimension of $W$, construct ideals $J_i$ defined in local flags $W = W_n \supset W_{n-1} \supset \cdots \supset W_i \supset \cdots \supset W_1$, and then objects $(W_i, (J_i, c_{i+1}), H_i, E_i)$ in dimension $i$, where each $E_i = W_i \cap E, H_i = W_i \cap H$. The integer numbers $c_{i+1}$ are computed as the $E$-order of certain ideals $P_{i+1}$ coming from the previous dimension $i+1$, that is $c_{i+1} = \max E$-ord$(P_{i+1})$ is the critical value in dimension $i$. Denote $c_{n+1} = c$.

If the $E$-singular locus of $(J_i, c_{i+1})$ is non empty, then factorize the ideal $J_i = M_i \cdot I_i$, where each ideal $M_i$ is defined by a normal crossings divisor supported by the current exceptional locus $H_i$.

Note 5.3. To make this induction on the dimension of $W$, in [Bla08, Bla09] it is proved the existence of hypersurfaces of $E$-maximal contact at any stage of the resolution process. This hypersurfaces are always coordinate hyperplanes, and produce a combinatorial center to be blown up. Combinatorial centers are convenient to preserve the binomial structure of the ideal after blow-up.

Definition 5.4. A binomial basic object along $E$ is a tuple $B = (W, (J, c), H, E)$ where

- $W$ is a regular toric variety defined by a fan $\Sigma$.
- $E$ is the simple normal crossing divisor given by $W \setminus T$, where $T \subset W$ is the torus of $W$.
- $(J, c)$ is a binomial pair, this means that $J \subset \mathcal{O}_W$ is a coherent sheaf of binomial ideals with respect to $E$, and $c$ is a positive integer number. Note that for any $\sigma \in \Sigma$ the sheaf of ideals $J$ restricted to the open affine subset $W_\sigma \subset W$ is a binomial ideal $J \neq 0$ in $k[x_1, \ldots, x_n]_{\prod_{i \in E} x_i}$.
- $H \subset E$ is a set of normal crossing regular hypersurfaces in $W$. 

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Definition 5.5. Let $J \subset \mathcal{O}_W$ be a binomial ideal, $c$ a positive integer. We call $E$-singular locus of $J$ with respect to $c$ to the set,

$$E\text{-Sing}(J, c) = \{ \xi \in W/ \ E\text{-ord}_\xi(J) \geq c \}.$$ 

Remark 5.6. The $E$-singular locus is a closed subset of $W$.

Definition 5.7. Let $J \subset \mathcal{O}_W$ be a binomial ideal. Let $\xi \in W$ be a point such that $E\text{-ord}_\xi(J) = \max E\text{-ord}(J) = \theta_E$. A hypersurface $V$ is said to be a hypersurface of $E$-maximal contact for $J$ at the point $\xi$ if

- $V$ is a regular hypersurface, $\xi \in V$,

- $E\text{-Sing}(J, \theta_E) \subseteq V$ and their transforms under blowing up with a combinatorial center $Z_\Delta \subset V$ also satisfy $E\text{-Sing}(J', \theta_E) \subseteq V'$, whereas the $E$-order, $\theta_E$ remains constant.

That is, $E\text{-ord}_\xi(J') = E\text{-ord}_\xi(J)$, where $J'$ is the controlled transform of $J$ and $V'$ is the strict transform of $V$.

Remark 5.8. The controlled transform of $J$ is the ideal $J' = I(Y')^{-\theta_E} \cdot J^*$ where $Y'$ is the exceptional divisor and $J^*$ is the total transform of $J$ under blowing up.

Proposition 5.9. Let $J \subset \mathcal{O}_W$ be a binomial ideal. There exists a hypersurface of $E$-maximal contact for $J$.

Definition 5.10. Let $(W, (J, c), H, E)$ be a binomial basic object along $E$. For all point $\xi \in E\text{-Sing}(J, c)$ the resolution function $E\text{-inv}_{(J, c)}$ will have $n$ components with lexicographical order, and will be of one of the following types:

$$E\text{-inv}_{(J, c)}(\xi) = \left\{ \begin{array}{l}
\left( \frac{E\text{-ord}_\xi(I_n)}{c_n}, \frac{E\text{-ord}_\xi(I_{n-1})}{c_{n-1}}, \ldots, \frac{E\text{-ord}_\xi(I_1)}{c_1} \right), \infty, \infty, \ldots, \infty \right) \quad (a) \\
\left( \frac{E\text{-ord}_\xi(I_n)}{c_n}, \frac{E\text{-ord}_\xi(I_{n-1})}{c_{n-1}}, \ldots, \frac{E\text{-ord}_\xi(I_1)}{c_1} \right), \Gamma(\xi), \infty, \ldots, \infty \right) \quad (b) \\
\left( \frac{E\text{-ord}_\xi(I_n)}{c_n}, \frac{E\text{-ord}_\xi(I_{n-1})}{c_{n-1}}, \ldots, \frac{E\text{-ord}_\xi(I_1)}{c_1} \right) \quad (c)
\end{array} \right.$$ 

where the ideals $I_i$ and the integer numbers $c_i$ are as in note 5.2.

In the case $J_i = 1$, for some $i < n$, define $(E\text{-inv}_{(J, c)}(\xi), \ldots, E\text{-inv}_{(J, c)}(\xi)) = (\infty, \ldots, \infty)$ in order to preserve the number of components.

If $E\text{-ord}_\xi(I_i) = 0$, for some $i < n$, then $E\text{-inv}_{(J, c)}(\xi) = \Gamma(\xi)$, where $\Gamma$ is the resolution function corresponding to the monomial case, see [EV00, BEV05]. And complete the resolution function with the needed number of $\infty$ components.

Note 5.11. The $E\text{-inv}_{(J, c)}$ function is an upper-semi-continuous function, see [Bla08, Bla09]. We also denote $E\text{-inv}_{(J, c)}(\xi) = E\text{-inv}_{(J, c)}(\xi)$.

As a consequence of the upper-semi-continuity of the $E\text{-inv}_{(J, c)}$ function,

$$\text{Max}(E\text{-inv}_{(J, c)}) = \{ \xi \in E\text{-Sing}(J, c) | E\text{-inv}_{(J, c)}(\xi) = \max E\text{-inv}_{(J, c)} \}$$

is a closed set. In fact, it is the center of the next blow-up.

It can be proven that the $E\text{-inv}_{(J, c)}$ function drops lexicographically after blow-up, [Bla08, Bla09].
Theorem 5.17. Let \((W, (J, c), H, E)\) be a binomial basic object along \(E\). Let \(W \xrightarrow{\pi} W'\) be a blow-up with combinatorial center \(Z_\Delta = \text{Max}(E-\text{inv}_{(J,c)})\) then

\[
E-\text{inv}_{(J,c)}(\xi) > E-\text{inv}_{(J',c)}(\xi')
\]

where \(\xi \in Z_\Delta, \xi' \in Y' = \pi^{-1}(Z_\Delta), \pi(\xi') = \xi\).

The function \(E-\text{inv}_{(J,c)}\) is the resolution function associated to the binomial basic object along \(E\) given by \((W, (J, c), H, E)\), and \(E-\text{inv}_{(J',c)}\) corresponds to its transform by the blow-up \(\pi\), \((W', (J', c), H', E')\).

Proof. See [Bla08, Bla09]. □

Remark 5.13. The \(E-\text{inv}_{(J,c)}\) function provides a \(E\)-resolution of the binomial basic object along \(E, (W, (J, c), H, E)\).

Definition 5.14. Let \((W, (J, c), H, E)\) be a binomial basic object along \(E\), where \(E = \{E_1, \ldots, E_r\}\), with \(r \leq n = \dim W\). Let \(H = \{H_1, \ldots, H_s\} \subset E\) be the set of exceptional divisors, for some \(s \leq r\).

We define a transformation of the binomial basic object

\[
(W, (J, c), H, E) \leftarrow (W', (J', c), H', E')
\]

by means of the blowing up \(W \xrightarrow{\pi} W'\), in a center \(Z \subset E-\text{Sing}(J, c)\), with

- \(H' = \{H'_1, \ldots, H'_s, Y'\}\) where \(H'_i, i = 1, \ldots, s\), is the strict transform of \(H_i\) and \(Y'\) is the exceptional divisor in \(W'\).
- \(E' = \{E'_1, \ldots, E'_r, Y'\}\) where \(E'_i, i = 1, \ldots, r\), is the strict transform of \(E_i\) and \(Y'\) is the exceptional divisor in \(W'\).
- \(J' = I(Y')^{-c} \cdot J^*\) is the controlled transform of \(J\), where \(J^*\) is the total transform of \(J\).

Definition 5.15. A sequence of transformations of binomial basic objects

\[
(W^{(0)}, (J^{(0)}, c), H^{(0)}, E^{(0)}) \leftarrow (W^{(1)}, (J^{(1)}, c), H^{(1)}, E^{(1)}) \leftarrow \cdots \leftarrow (W^{(N)}, (J^{(N)}, c), H^{(N)}, E^{(N)})
\]

is a \(E\)-resolution of \((W^{(0)}, (J^{(0)}, c), H^{(0)}, E^{(0)})\), or simply a \(E\)-resolution of the pair \((J^{(0)}, c)\), if \(E-\text{Sing}(J^{(N)}, c) = \emptyset\).

Note 5.16. This \(E\)-resolution function, the \(E-\text{inv}_{(J,c)}\), works for general binomial ideals, without any restriction, for more details see [Bla08, Bla09].

The \(E\)-resolution constructed in this way is independent of the choice of coordinates.

In [Bla08, Bla09] was proved that one may use this \(E\)-resolution in order to construct an algorithm of log-resolution of binomial ideals and embedded desingularization of binomial varieties. But this algorithm of log-resolution depends on a choice of a Gröbner basis of the original binomial ideal.

In the next section we will construct an algorithm (6.2) of embedded desingularization of toric varieties which is independent of the choice of coordinates.

Theorem 5.17. Let \(J\) be a binomial ideal. If \(E-\text{Sing}(J, c) \neq \emptyset\) then there exists a \(E\)-resolution of \((J, c)\).

Proof. The \(E\)-resolution of \((J, c)\) is given by the \(E-\text{inv}_{(J,c)}\) function, such that \(E-\text{Sing}(J^{\gamma}, c) = \emptyset\), this means \(\max(E-\text{ord}(J^{\gamma})) = 0\). □

Remark 5.18. At any stage of the \(E\)-resolution process, the \(E-\text{inv}_{(J,c)}\) function determines the next combinatorial center to be blown-up \(Z_\Delta = \text{Max}(E-\text{inv}_{(J,c)})\), or equivalently, the cone \(\Delta, (4.5)\).
6 Embedded desingularization.

Now we construct an algorithm of embedded desingularization of toric varieties. This algorithm is defined in terms of the function \( \text{Hcodim} \) and a \( E \)-resolution of a suitable ideal, depending on the function \( E \)-inv.

**Proposition 6.1.** Let \( X \subset W \) be a toric embedding and let \( \xi \in X \) be a point. Let \( X_\sigma \) be the minimum affine open set containing the point \( \xi \) and let \( V_\sigma \) be the minimum toric affine variety such that \( V_\sigma \supset X_\sigma \) and it is transversal to \( E \).

At any affine open subset, \( X_\sigma \subset V_\sigma \subset W_\sigma \). Let \( W_\sigma \xrightarrow{\pi} W'_\sigma \) be a blow-up with center \( Z_\Delta \) then

\[
(\text{Hcodim}(X)(\xi), E^{-\text{inv}}(I_{V_\sigma}(X_\sigma), c)) > (\text{Hcodim}(X')(\xi'), E^{-\text{inv}}(I_{V'_\sigma}(X'_{\sigma'}), c))
\]

where \( \xi \in Z_\Delta, \xi' \in Y' = \pi^{-1}(Z_\Delta), \pi(\xi') = \xi, I_{V_\sigma}(X_\sigma) \) is the ideal of \( X \) in \( V \), and \( c = \max E\text{-ord}(I_{V_\sigma}(X_\sigma)) \).

**Proof.** It follows from proposition (4.7) and lemma (5.12).

**Algorithm 6.2.** Let \( X \subset W \) be a toric embedding and let \( \xi \in X \) be a point. Let \( W_\sigma \) be the minimum affine open set containing the point \( \xi \) (4.2).

1. Compute \( V_\sigma \), the minimum toric affine variety such that \( V_\sigma \supset X_\sigma \) and it is transversal to \( E \) (3.10).

2. Set \( \text{Hcodim}(X)(\xi) = \dim V_\sigma - \dim X \).
   
   - If \( \text{Hcodim}(X)(\xi) > 0 \) then compute \( E^{-\text{inv}}(I_{V_\sigma}(X_\sigma), 1) \), here we set \( c = 1 \). This determines \( Z_\Delta \). Go to step (3).
   
   - If \( \text{Hcodim}(X)(\xi) = 0 \) the algorithm stops. Note that in this case \( V_\sigma = X_\sigma \subset X \).

3. Perform the blow-up with center \( Z_\Delta \) and go to step (1).

Correctness of the algorithm follows by construction (4.3 and 4.7). Termination of the algorithm follows from (6.1) and (5.17).

**Remark 6.3.** The algorithm of embedded desingularization given in [Bla08] depends on the choice of a system of coordinates. Note that the new algorithm (6.2) given here does not depend on the coordinates election.

**Theorem 6.4. Embedded desingularization**

Let \( X \subset W \) be a toric embedding (4.1). Let \( E \) be the simple normal crossing divisor given by \( W \setminus T \), where \( T \subset W \) is the torus of \( W \).

There exists a sequence of transformations of pairs

\[
(W, E) \leftarrow (W^{(1)}, E^{(1)}) \leftarrow \cdots \leftarrow (W^{(N)}, E^{(N)})
\]

which induces a proper birational morphism \( \Pi : W^{(N)} \rightarrow W \) such that

1. The restriction of this morphism \( \Pi \) to the regular locus of \( X \) along \( E \), defines an isomorphism

\[
\text{Reg}_E(X) \cong \Pi^{-1}(\text{Reg}_E(X)) \subset W^{(N)}
\]

   where \( \text{Reg}_E(X) = \{ \xi \in X \mid X \text{ is regular at } \xi \text{ and has normal crossings with } E \} \).

2. \( X^{(N)} \), the strict transform of \( X \) in \( W^{(N)} \), is regular and has normal crossings with the exceptional divisors \( E^{(N)} \).
Proof. It follows from correctness and termination of algorithm 6.2.

The embedded desingularization of theorem 6.4 can be implemented, since the key points are to determine the stratum \( E_\xi \) (or the open subset \( W_\sigma \)) where the hyperbolic codimension is maximum, and then to compute a combinatorial blow-up, that can be easily encoded in the computer.

**Remark 6.5.** Theorem (6.4) may be used to achieve a log-resolution of a toric ideal. With the notation of theorem (6.4), let \( W^{(N+1)} \rightarrow W^{(N)} \) be the blowing up with center \( X^{(N)} \), which is a permissible center. Note that the total transform of \( I(X)\mathcal{O}_{W^{(N+1)}} \) is locally a monomial ideal (generated by monomials) and we may use an algorithm of log-resolution of monomial ideals as in [Gow05] or [BM06]. So that log-resolution of the ideal \( I(X) \) follows from theorem (6.4) and log-resolution of monomial ideals.

**Example 6.6.** Let \( X \subset W = \mathbb{A}^4 \) be a toric embedding and let \( \xi \in X \) be a point. Let \( E = \{ V(x), V(y), V(z), V(w) \} \) be the simple normal crossing divisor given by \( W \setminus T \). The toric variety \( X \) is given by the equations

\[
X = \{ x^2 - y^3 = 0 \} \cap \{ xyz - w^2 = 0 \}.
\]

The singular locus of the surface \( X \) is the \( z \)-axis. Compute the hyperbolic codimension at some points of \( X \), for example:

- If \( \xi \notin E \), then in a neighborhood of \( \xi \), \( W_\sigma = \text{Spec}(k[x^\pm, y^\pm, z^\pm, w^\pm]) \) and \( E_\xi = \emptyset \). The minimum toric affine variety \( V_\sigma \) such that \( V_\sigma \supset X_\sigma \) and it is transversal to \( E_\xi \) is \( V_\sigma = X_\sigma \). Then \( \text{Hcodim}(X)(\xi) = 0 \).
- If \( \xi \notin E \), then in a neighborhood of \( \xi \), \( W_\sigma = \text{Spec}(k[x^\pm, y^\pm, z^\pm, w^\pm]) \) and \( E_\xi = \emptyset \). The minimum toric affine variety \( V_\sigma \) such that \( V_\sigma \supset X_\sigma \) and it is transversal to \( E_\xi \) is \( V_\sigma = X_\sigma \). Then \( \text{Hcodim}(X)(\xi) = 0 \).
- Set \( (V(x))^c = W \setminus V(x) \) be the complement of \( V(x) \). If \( \xi \in (V(x))^c \cap (V(y))^c \cap (V(z))^c \cap V(w) \), then \( W_\sigma = \text{Spec}(k[x^\pm, y^\pm, z^\pm, w^\pm]) \) and \( E_\xi = V(z) \cap V(w) \). It is clear that \( V_\sigma = \{ x^2 - y^3 = 0 \} \) and therefore \( \text{Hcodim}(X)(\xi) = 3 - 2 = 1 \).
- Assume \( \xi \) is the distinguished point of \( W_\sigma \), in this case \( V_\sigma = W_\sigma \) and \( \text{Hcodim}(X)(\xi) = 4 - 2 = 2 \).
- If \( \xi \) is the origin, \( W_\sigma = \text{Spec}(k[x, y, z, w]) \) and \( E_\xi = V(x) \cap V(y) \cap V(z) \cap V(w) \). The minimum toric affine variety \( V_\sigma = W_\sigma \) and \( \text{Hcodim}(X)(\xi) = 4 - 2 = 2 \).

It is easy to check that the hyperbolic codimension \( \text{Hcodim} \) attains its highest value along the \( z \)-axis.

If one computes the whole resolution function \( (\text{Hcodim}, E^{-\text{inv}}) \) (here we set \( c = 1 \)) then its maximum value is

\[
\max(\text{Hcodim}(X), E^{-\text{inv}}) = (\text{Hcodim}(X)(0), E^{-\text{inv}}(I_{V_\sigma}(X_\sigma), 1)) = (2, 2, 1, 3/2, 2)
\]

and it is achieved at the origin, which is the first center to be blown-up.

We denote as \( x\text{-th chart} \) the chart where we divide by \( x \). For simplicity, we will denote each \( y/z, z/w, w/x \) again as \( y, z, w \).

At the \( x \)-th chart, the controlled transform of the ideal \( I(X) \) is

\[
I(X)' = x^{-1} \cdot (x^2 - x^3y^3, x^3yz - x^2w^2) = x \cdot (1 - xy^3, xyz - w^2).
\]

If \( \eta' \) is a point that maps to the origin then \( \eta' \notin X' \) and it lies in the first exceptional divisor \( V(x) \). Consider \( \xi' \in X' \cap V(z) \cap V(w) \), the affine chart \( W_\sigma' \) associated to \( \xi' \) as in (4.2) is
\[ W_{\sigma'} = \text{Spec}(k[x^\pm, y^\pm, z, w]) \text{ and } E'_\xi = V(z) \cap V(w). \] The minimum toric affine variety \( V_{\sigma'} \) containing \( X' \) is \( V_{\sigma'} = \{ xy^3 - 1 = 0 \} \) and \( \text{Hcodim}(X')(\xi') = 3 - 2 = 1. \) The maximum value of the resolution function is

\[
\text{max}(\text{Hcodim}(X'), E^{-\text{inv}}) = (\text{Hcodim}(X')(\xi'), E^{-\text{inv}}(I_{V'_{\sigma'}}(X'_{\sigma'}), 1)) = (1, 1, 2, \infty, \infty)
\]

and it is reached along \( Z' = \{ z = 0 \} \cap \{ w = 0 \} \cap \{ xy^3 - 1 = 0 \}. \) Inside \( V'_{\sigma'} \), the center is given by coordinates, \( Z'_\Delta = \{ z = 0 \} \cap \{ w = 0 \}, \) which is the next combinatorial center to be blown-up. After the blow-up at \( Z'_\Delta \), we consider the \( w \)-th chart

\[ I(X)' = w^{-1} \cdot (xyzw - w^2) = (xyz - w) \mod I(V'_{\sigma'}), \]

this means \( I(X)' = (xy^3 - 1, xyz - w). \)

Let \( \xi'' \in X'' \) mapping to \( \xi'. \) At this stage of the resolution process, the maximum value of the resolution function is

\[
\text{max}(\text{Hcodim}(X''), E^{-\text{inv}}) = (\text{Hcodim}(X'')(\xi''), E^{-\text{inv}}(I_{V''_{\sigma'}}(X''_{\sigma'}), 1)) = (1, 1, 1, \infty, \infty)
\]

and it is reached along \( Z'' = \{ z = 0 \} \cap \{ w = 0 \} \cap \{ xy^3 - 1 = 0 \}. \) After the blowing-up at \( Z''_\Delta \) we obtain two charts and for both \( \text{max } \text{Hcodim}(X'''') = 0. \) And \( X''' \) is regular and transversal to \( E'''\).

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