Equalization with Expectation Propagation at Smoothing Level

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Abstract—In this paper we propose a smoothing turbo equalizer based on the expectation propagation (EP) algorithm with quite improved performance compared to the Kalman smoother, at similar complexity. In scenarios where high-order modulations or large memory channels are employed, the optimal BCJR algorithm is computationally unfeasible. In this situation, low-cost but suboptimal solutions, such as the linear minimum mean square error (LMMSE), are commonly used. Recently, EP has been proposed as a tool to improve the Kalman smoothing performance. In this paper we review these solutions to apply the EP at the smoothing level, rather than at the forward and backwards stages. Also, we better exploit the information coming from the channel decoder in the turbo equalization schemes. With these improvements we reduce the computational complexity, speed up convergence and outperform previous approaches. We included some simulation results to show the robust behavior of the proposed method regardless of the scenario, and its improvement in terms of performance in comparison with other EP-based solutions in the literature.

Index Terms—Expectation propagation (EP), MMSE, low-complexity, turbo equalization, ISI, Kalman, smoothing.

I. INTRODUCTION

The task of a soft equalizer is to mitigate the intersymbol interference (ISI) introduced by the channel, providing a probabilistic estimation of the transmitted symbols given a set of observations. After the equalizer, a channel decoder can help to detect and correct errors if the transmitted message has been protected with some redundancy [1]. These channel decoders highly benefit from soft estimations provided by the equalizer rather than hard decisions. In addition, the equalization can be refined with the help of the information at the output of the channel decoder with turbo equalization [2]–[4].

The BCJR provides a maximum a posteriori (MAP) estimation [5]. However, the BCJR solution becomes intractable in terms of complexity as the memory and/or the constellation size grow. In this scenario, some approximated BCJR solutions, such as [6]–[9], can be employed. These solutions reduce the complexity by just exploring a reduced number of states. However, their performance is quite dependent on the channel realization and they degrade if the number of surviving states does not grow accordingly with the complexity of the scenario. A comparison between them can be found in [10], [11].

A different and extended approximate solution is the linear minimum mean square error (LMMSE). An equalizer based on the LMMSE algorithm can be implemented, among others, as a block [12], a Wiener-filter type [4], [13]–[15] or a Kalman smoothing [16] approach. The Kalman smoothing solution exhibits the performance of the block LMMSE with linear complexity in the length of the transmitted word and cubic in the memory of the channel. It proceeds forwards by computing the transition probabilities from consecutive states through a trellis representation. Then, the same procedure is run backwards. Finally, it merges both procedures into a smoothing approach, providing an approximation to the posterior of each transmitted symbol.

However, the LMMSE estimation is far from optimal. It is a linear solution in the observation, where the discrete priors are approximated by Gaussian distributions whose mean and variance are set to the ones of the discrete distribution at the output of the channel decoder. When this information is not available, they are set to zero mean Gaussian distributions of variance equal to the energy of the constellation. The expectation propagation (EP) algorithm exhibits a structure similar to that of the LMMSE, but the priors depend on the observation, hence being non-linear. The set of Gaussian distributions used to approximate the priors is estimated iteratively, with the aim of better approximating the posterior distribution of the transmitted symbols. The EP computes the mean and variance of these Gaussians by matching the moments of the approximated posterior with the ones of the true posterior, including the discrete probability mass functions (pmf) of the priors. The EP has already been applied to multiple-input multiple-output (MIMO) detection [17]–[20], low-density parity-check (LDPC) channel decoding [21], [22] and standalone/turbo equalization [10], [11], [23]–[25], among others.

In [10], a block implementation of an EP-based equalizer is proposed, whose complexity is quadratic in the frame length. This implementation is revised in [23] to propose an optimized version for turbo equalization. Since the block implementation can be intractable for long frames, a filtered implementation based on the Wiener-filter behavior is also proposed in [23] for turbo equalization. It allows a reduction in complexity which is linear in the frame length and quadratic in the size of the window used. However, the Wiener-filter implementation only uses the observations within the window to obtain an estimation for the transmitted symbols, which can degrade the performance in comparison with its block counterpart. To avoid the inversion of large matrices that are needed in the block EP-based solution without degrading the performance.
as in the Wiener-filter proposal, a smoothing equalizer based on the EP algorithm can be employed.

In this framework of smoothing equalizers, in [24], [25] the EP was applied to better exploit the information fed back from the channel decoder in turbo equalization. However, this approach degrades when used for higher order modulations since no suitable control of negative variances is used, a minimum allowed variance is not set and no damping procedure is included. Also, this equalizer is just designed for turbo equalization, boiling down to the LMMSE for standalone equalization. In [11] the authors proposed a self-iterated EP of the block EP equalizer [10], a state involving a number of smoothing distributions. To get a performance similar to that of the EP was applied to better exploit the information fed back as in the Wiener-filter proposal, a smoothing equalizer based on a smoothing distribution of the current state, \( \text{Proj}_{\mathcal{P}}[q_{\cdot}] \). The expression \( \text{Proj}_{\mathcal{P}}[q_{\cdot}] \) is the projection of the distribution given as an argument, \( q_{\cdot} \), onto the equalizer, then \( \mathcal{P} \) is a smoothing equalizer. This codeword is modulated into the vector of symbols \( \mathbf{u} = [u_1, ..., u_L]^\top \), where each symbol belongs to a complex \( M \)-ary constellation with alphabet \( \mathcal{A} \) and mean transmitted symbol energy \( E_s \). This vector of symbols is transmitted over a channel with weights \( \mathbf{h} = [h_1, ..., h_L] \) and memory \( L = L - 1 \) and it is corrupted with additive white Gaussian noise (AWGN) with known noise variance, \( \sigma_n^2 \). The received signal is denoted as \( \mathbf{y} = [y_1, ..., y_{L-1}]^\top \) with entries given by

\[
y_k = \sum_{j=1}^{L} h_j u_{k-j+1} + w_k = \mathbf{h}^\top \mathbf{s}_k + w_k, \tag{1}
\]

where \( w_k \sim \mathcal{N}(0, \sigma_n^2) \), \( \mathbf{s}_k = [u_{k-L}, ..., u_k]^\top \), and which will be hereafter referred to as the state and \( u_k = 0 \) for \( k < 1 \) and \( k > N \). The received signal is processed by the equalizer. It estimates the posterior probability of the transmitted symbol vector \( \mathbf{u} \) given the whole vector of observations \( \mathbf{y} \) as

\[
p(\mathbf{u}|\mathbf{y}) \propto p(\mathbf{y}|\mathbf{u}) \prod_{k=1}^{N} p(u_k) \tag{2}
\]

where \( p(u_k) \) is the available information on the priors. If the output of the channel decoder is available and fed back to the equalizer, then \( p(u_k) = p_E(u_k) \). Otherwise, a uniform discrete prior is used, which is equivalent to assuming equiprobable symbols. The equalizer and the channel decoder usually exchange extrinsic information. An extrinsic distribution, \( p_E(u_k|\mathbf{y}) \), is computed at the output of the equalizer. The extrinsic distributions are demapped and given to the channel decoder as extrinsic log-likelihood ratios (LLRs), \( L_E(b_i) \). Finally, the decoder computes extrinsic LLRs, \( L_D(b_i) \), which are mapped back onto the \( M \)-ary modulation and fed back to the equalizer.

III. FROM THE BCJR TO THE LMMSE

A. BCJR

In this section, we review the formulation for exact inference in equalization from a Kalman smoothing point of view. In the forward stage of the BCJR, at step \( k \) the posterior probability distribution of the current state, \( \mathbf{s}_k \), is computed given the observations up to time \( k \), \( \mathbf{y}_{1:k} \). This posterior, \( p(\mathbf{s}_k|\mathbf{y}_{1:k}) \), is proportional to the product of the Gaussian likelihood of the current observation, \( p(y_k|\mathbf{s}_k) \), and the predicted state, \( p(\mathbf{s}_k|\mathbf{y}_{1:k-1}) \), i.e.,

\[
p(\mathbf{s}_k|\mathbf{y}_{1:k}) \propto p(y_k|\mathbf{s}_k) \prod_{i=1}^{L} p(u_k)p(s_{k-1}^{\top}|y_{1:k-1}) \tag{3}
\]

where \( p(y_k|\mathbf{s}_k) \) is given by the channel model in (1) and \( p(s_{k-1}^{\top}|y_{1:k-1}) \) denotes the marginal of \( p(s_{k-1}|y_{1:k-1}) \) over its first entry, i.e., \( u_{k-1} \). Similarly, and in parallel, a backward procedure can be run following the same formulation explained above by just left-right flipping the channel, received and transmitted vectors [11]. As a result, the distribution

\[\text{approximately twice the channel length.} \]
\begin{equation}
p(s_k|y_{k:3+L}) \propto \frac{p(s_k|y_{1:k})p(s_k|y_{k:3+L})}{\prod_{i=k-L}^{k} p(u_i)}.
\end{equation}

The computation of these probabilities has complexity $O(t^4)$. For large states, i.e., high L, and/or large constellations sizes, \( t \), it becomes unaffordable due to the large number of combinations to be checked to retain the maximum value.

**B. Kalman Smoother**

Rather than using the true distribution in (4), in the smoothing Kalman filtering we approximate $p(s_k|y_{1:k})$, $p(s_k|y_{k:3+L})$ and $p(u_i)$ by Gaussian distributions, as follows.

The LMMSE approximates the discrete true prior, $p(u_k)$, with a Gaussian distribution, $t_k(u_k)$, as

\begin{equation}
t_k(u_k) = \text{Proj}_C[p(u_k)] \sim \mathcal{N}(\mu_{t_k}, \sigma_{t_k}^2).
\end{equation}

If no feedback is available from the channel decoder, these moments are initialized to $\mu_{t_k} = 0$ and $\sigma_{t_k}^2 = E_s$.

The approximation above yields the following Gaussian approximation of the posterior in (3)

\begin{equation}
q^F(s_k) \propto p(y_k|s_k)t_k(u_k)q^F(s_{k-1}) \sim \mathcal{N}(\mu_k^{F}, \Sigma_k^{F})
\end{equation}

where

\begin{equation}
\mu_k^{F} = \Sigma_k^{F} \left( \sigma_w^{-2}h^Hy_k + \left[ \Sigma_{k-1}^{-1} \mu_{k-1} \right] \right).
\end{equation}

\begin{equation}
\Sigma_k^{F} = \left( \sigma_w^{-2}h^Hh + \left[ \Sigma_{k-1}^{-1} \Sigma_{k-1}^{-1} \right] \right)^{-1}
\end{equation}

and the superscript $^F$ denotes the forward procedure. Vector $\mu_{k-1}^{F}$ is $\mu_{k-1}^{F}$ without its first entry and $\Sigma_{k-1}^{F}$ is the submatrix defined as $\Sigma_{k-1}^{F}$, with its first row and column removed. This process is repeated for $k = 1, \ldots, N + L$. Note that the forward recursion only uses the observations $y_1, \ldots, y_k$ when estimating the posterior distribution in (6), ignoring the following $y_{k+1}, \ldots, y_{k+L}$. The computational complexity of this procedure is dominated by the inversion of the $L \times L$ matrix in (8) along $N$ iterations, yielding a final complexity of $O(NL^3)$.

In the backwards step, the distribution $p(s_k|y_{k:3+L})$ is approximated by the following posterior Gaussian distribution,

\begin{equation}
q^B(s_k) \sim \mathcal{N}(\mu_k^{B}, \Sigma_k^{B}),
\end{equation}

where note that $y_1, \ldots, y_{k-1}$ are not involved.

Finally, in the smoothing step both approximations in (6) and (9) are merged into just one posterior approximation of (4) as

\begin{equation}
q(s_k) = \frac{q^F(s_k)q^B(s_k)}{\prod_{i=k-L}^{k} t_i(u_i)} \sim \mathcal{N}(\mu_k, \Sigma_k)
\end{equation}

where

\begin{equation}
\mu_k = \Sigma_k^{-1} \left( \Sigma_k^{F} \mu_k^{F} + (\Sigma_k^{B})^{-1} \mu_k^{B} - \sigma_w^{-2}h^Hy_k - C_{t_k}^{-1} \mu_{t_k} \right),
\end{equation}

\begin{equation}
\Sigma_k = \left( \Sigma_k^{F} + (\Sigma_k^{B})^{-1} - \sigma_w^{-2}h^Hh - C_{t_k}^{-1} \right)^{-1},
\end{equation}

and

\begin{equation}
\mu_{t_k} = [\mu_{t_k}, \ldots, \mu_{t_k}]^T \text{ and } C_{t_k} = \text{diag}([\sigma_{t_k}^2, \ldots, \sigma_{t_k}^2]).
\end{equation}

The computational complexity of this smoothing step is given by $O(NL^3)$. Hence, the final complexity of the algorithm is $O(2NL^3)$, i.e., the computational complexity of the forward/backward steps (that can be run in parallel) and the one of the smoothing step.

In turbo equalization, the probabilities at the output of the channel decoder, $p_D(u_k)$, are used as priors, $p(u_k)$. Also, we usually handle the extrinsic probabilities to the channel decoder, which can be obtained as

\begin{equation}
q_E(u_k) = \frac{q(u_k)}{t_k(u_k)} \sim \mathcal{N}(\mu_{E_k}, \sigma_{E_k}^2)
\end{equation}

where $q(u_k) \sim \mathcal{N}(\mu_k, \sigma_k^2)$ is the marginal of (10) and

\begin{equation}
\mu_{E_k} = \frac{\mu_k \sigma_{E_k}^2 - \mu_{t_k} \sigma_{t_k}^2}{\sigma_{E_k}^2 - \sigma_{t_k}^2},
\end{equation}

\begin{equation}
\sigma_{E_k}^2 = \frac{\sigma_{t_k}^2 \sigma_{E_k}^2}{\sigma_{E_k}^2 - \sigma_{t_k}^2}.
\end{equation}

The whole procedure is summarized in Algorithm 1.

**IV. EP AT SMOTHING LEVEL**

The EP [26]–[29] is a Bayesian machine learning technique to approximate an intractable probability density function (pdf), such as (2), with an exponential family. In this paper we use it to estimate the factors, $t_k(u_k)$, that minimize the Kullback-Leibler (KL) divergence between the discrete and the approximated posterior. In particular, we propose the factors in (5) to be estimated iteratively. At iteration $\ell$ these factors are given by the Gaussian probabilities

\begin{equation}
t_k^{[\ell]}(u_k) \sim \mathcal{N}(\mu_k^{[\ell]}, \sigma_k^{2[\ell]})
\end{equation}

Fig. 1: System model.
The minimization of the KL divergence amounts to matching the moments,

\[ q_E^{[\ell]}(uk) p(uk) \leftrightarrow q_E^{[\ell]}(uk) t_k^{[\ell+1]}(uk) \]  

(17)

where \( q_E^{[\ell]}(uk) \) is given by (13) with moments \( \mu_k^{[\ell]} \) and \( \sigma_{E_k}^{[\ell]} \) and with \( t_k^{[\ell]}(uk) \) replaced by \( t_k^{[\ell]}(uk) \). One iteration of this procedure is detailed in Algorithm 2. Since this algorithm suffers from instabilities and negative variances, we introduced a damping (with factor \( \beta \)), minimum allowed variance (\( \epsilon \)) and control of negative variances at every iteration.

The full algorithm, described in Algorithm 3, initializes in Step 1) the approximating factors, \( t_k^{[1]}(uk) \), to the values given by the LMMSE, i.e., following (5). Then it runs Algorithm 1 and Algorithm 2 along S iterations to refine these terms. In Algorithm 1, using \( t_k^{[\ell]}(uk) \), we compute the full approximations \( q^{[\ell]}(sk) \), their marginals, \( q^{[\ell]}(uk) \), and the extrinsic probabilities, \( q_E^{[\ell]}(uk) \). Then we apply the moment matching in Algorithm 2 to re-estimate the approximating factors as \( t_k^{[\ell+1]}(uk) \).

We denote this proposal as Kalman smoothing expectation propagation (KSEP) turbo equalizer. Its computational complexity is \( \mathcal{O}(S + 1)2ML^3 \), i.e., the complexity of computing the Kalman smoother plus running S times the EP algorithm. Note that the EP approach is applied once the smoothing step is performed.

In the turbo equalization, we run Algorithm 3 along \( T + 1 \) iterations. The first time Algorithm 3 is run, the inputs \( p(uk) \) are initialized to uniform discrete values. Then, after every turbo iteration, they are set to the extrinsic probabilities provided by the channel decoder, \( p_D(uk) \). Following the guidelines in [23], we propose to set the parameters to: \( S = 3, \beta = \min(\exp(t/1.5)/10, 0.7) \), where \( t \in [0, T] \) is the number of the current turbo iteration and \( \epsilon = 10^{-8} \).

### A. Discussion about EP-based proposals

The block expectation propagation (BEP) equalizer is proposed in [23], where the EP parameters are revised and optimized for turbo equalization. However, it requires to invert a matrix of size \( N \times N \), yielding a quadratic complexity in the frame length. This complexity can be reduced by means of a smoothing implementation, such as belief propagation expectation propagation (BP-EP) [24] or SEP [11]. The BP-EP uses a window of size \( L \) and applies the EP to better approximate with Gaussians the discrete information at the output of the channel decoder. On the other hand, the SEP exhibits a much better performance than the BP-EP in terms of BER.
but it has a higher computational complexity and it does not properly handle the information coming from the channel decoder. The SEP assumes that the true discrete priors used during the moment matching procedure of the EP algorithm are uniform, even after the feedback from the channel decoder. It applies the EP over the forward and backward Kalman filters, i.e., the EP update in the SEP is computed with just part of the observations and the performance is degraded. To overcome these problems, a window larger than the channel length, of size \(2L-1\), is used. This yields a complexity of \(O((S+1)N(2L-1)^3)\) for the forward/backward steps plus a complexity of \(O(N(2L-1)^3)\) due to the smoothing step. Since \(S = 10\) iterations of the moment matching algorithm are needed, the final computational complexity is of \(O(12N(2L-1)^3)\) or, approximately, \(O(96NL^3)\) if we just keep the cubic term in \(L\).

In contrast, and as explained above, the novel KSEP proposal applies the EP at the smoothing level, where information from the whole observation vector is exploited. This fact allows to reduce the dimensions of the matrices to invert to be \(L \times L\). Also, it is optimized for turbo equalization, setting the priors used in the moment matching to the non-uniform distributions provided by the channel decoder. The EP parameters are optimized for turbo equalization, needing only \(S = 3\) iterations of the moment matching to achieve convergence. This yields a complexity of \(O(8NL^3)\), i.e., it is of the same order as the one of the BP-EP or the Kalman smoother, with \(O(2NL^3)\). In Table I we include a comparison in terms of complexity (per turbo loop) between the EP-based equalizers in [11], [23], [24] and our proposal.

| Algorithm       | Complexity          |
|-----------------|---------------------|
| Kalman Smoother | \(O(2NL^3)\)        |
| BEP [23]        | \(O(4HL^2)\)        |
| BP-EP [24]      | \(O(2NL^3)\)        |
| SEP [11]        | \(O(90SL^3)\)       |
| KSEP            | \(O(8SL^3)\)        |

**TABLE I:** Complexity comparison between EP-based equalizers per turbo iteration.

V. EXPERIMENTAL RESULTS

In this section we compare the performance in terms of BER of our proposal, KSEP, with the block LMMSE algorithm and other EP-based proposals found in the literature, such as the BEP [23], SEP [11] and BP-EP [24]. We use different modulations and lengths for the channel. The results are averaged over 100 random channels and \(10^4\) random encoded words of length \(V = 4096\) (per channel realization). The parameters of the KSEP are set to the same values as given in [23] for the BEP: \(S = 3\), \(T = 5\) turbo iterations, \(\beta = \min(\exp(t/1.5)/10,0.7)\), where \(t \in [0,T]\) is the number of the current turbo iteration and \(\epsilon = 10^{-8}\). Each channel tap is independent and identically Gaussian distributed with zero mean and variance equal to \(1/L\). The absolute value of LLRs given to the decoder is limited to 5 in order to avoid very confident probabilities. A (3,6)-regular LDPC of rate 1/2 is used, decoded with a belief propagation algorithm with a maximum of 100 iterations.

In Fig. 2 we show the BER for a 4-PAM constellation and random channels of \(L = 5\) real-valued taps. As can be observed, the BP-EP [24] exhibits the highest error in terms of BER, even compared to the LMMSE. Note that the BP-EP does not properly control the negative variances and does not include any damping procedure or minimum allowed variance. The BEP and KSEP share the same performance. These two proposals have gains close to 2 dB with respect to the LMMSE and 2 dB with respect to BP-EP. The performance of the SEP degrades when comparing with BEP and KSEP. Although the SEP is not optimized for turbo equalization it improves the LMMSE in 0.75 dB.

In Fig. 3 we increase the order of the constellation and the length of the channel. In particular, we show the BER for random channels of \(L = 7\) complex-valued taps and a 64-QAM constellation. We decided not to include the BP-EP since, as discussed in the previous experiment in Fig. 2, its results degrade when using multilevel modulations. We plotted the performance after different number of turbo iterations. Specifically, for standalone equalization (a) and for turbo equalization after \(T = 2\) (b) and \(T = 5\) (c) iterations. We observe that in these three cases the KSEP and BEP have the same performance. In Fig. 3 (a), it can be seen that SEP is slightly better than BEP and KSEP. The reason is that it is run with \(S = 10\) iterations, while BEP and KSEP reduce it to \(S = 3\) since they have been designed for turbo equalization. The KSEP achieves the performance of the SEP in Fig. 3 (a) by just increasing its number of EP iterations to \(S = 6\). In Fig. 3 (b), where the BER after \(T = 2\) turbo iterations is depicted, both BEP and KSEP quite outperform the SEP performance. The reason is that, in addition to exploiting the whole vector of observations when applying EP -in contrast to SEP-, they are properly handling the discrete information returned by the channel decoder. Specifically, they have gains of 3 dB and 4.5 dB with respect to the SEP and LMMSE, respectively. In Fig. 3 (c), the SEP shows better performance than the LMMSE but it is quite far from KSEP and BEP. Specifically, KSEP and BEP improve the performance of SEP...
in 4 dB.

In Fig. 4 we reproduce the scenario in Fig. 3 with a higher order modulation, a 128-QAM. Again, for the standalone equalization case showed in Fig. 4 (a), the performance of SEP is slightly better than the one of KSEP/BEP, although it has been checked that they achieve the SEP performance by increasing the number of iterations to $S = 6$. For the turbo equalization case in Fig. 4 (b)-(c), the SEP improves the LMMSE but its performance is quite far from the one of the KSEP and BEP. Specifically, KSEP and BEP have gains of 5 dB in comparison with the LMMSE in Fig. 4 (b) and gains of almost 6 dB in Fig. 4 (c). Similarly to Fig. 2 and Fig. 3, the SEP is in between BEP/KSEP and LMMSE.

VI. CONCLUSIONS

In this paper, we propose a novel Kalman smoothing EP-based equalizer, the KSEP, that outperforms previous smoothing proposals found in the literature [11], [24]. The KSEP first runs a forward and backward Kalman filter and then merges both approximations into a smoothing one, where the EP algorithm is applied. It solves the problems of previously proposed EP-based equalizers. Firstly, it avoids the degradation in the performance for high-order modulations of the BP-EP [24] approach. Secondly, in contrast to SEP, it applies EP at the smoothing level exploiting information from the whole observation vector and yielding better convergence properties. Thirdly, rather than setting uniform priors in the moment matching procedure as the SEP, it properly handles the information coming from the channel decoder within the moment matching procedure. Also, it reduces the size of the matrices involved in the estimations in comparison to SEP therefore reducing the computational complexity. As illustrated in the experimental section, we have a remarkable gain in performance compared to previous equalizers based in EP and smoothing. The KSEP achieves the same performance as its block counterpart, the BEP, as the Kalman smoother exhibits the same performance as the block LMMSE. Besides, the computational complexity yields $O(8NL^2)$, i.e., it is of the same order as the one of the BP-EP or the Kalman smoother, with $O(2NL^2)$.

APPENDIX

PROOF OF (4)

From (2), the exact posterior probability distribution is

$$p(u|y) \propto p(y|u)p(u) = \prod_{k=1}^{n+L} p(y_k|s_k) \prod_{k=-L+1}^{n+L} p(u_k),$$

(24)

We aim at computing $p(s_k|y)$ by using $p^F(s_k|y_{1:k})$ from the forward step and $p^B(s_k|y_{k+3:L})$ from the backward step. Accordingly, we split (24) into two parts: received symbols until time $k$ and after time $k$,

$$p(u|y) \propto p(y_{1:k}|u_{-L+1:k})p(u_{-L+1:k}) \prod_{i=k+1}^{n+L} p(y_i|s_i)p(u_i)$$

$$\propto p(u_{-L+1:k}|y_{1:k}) \prod_{i=k+1}^{n+L} p(y_i|s_i)p(u_i).$$

(25)

Fig. 3: BER along $E_b/N_0$ for for BEP ($\times$) [23], SEP ($\circ$) [11], KSEP ($\diamond$) and LMMSE ($\triangledown$) turbo equalizers, 64-QAM and averaged over 100 random channels with $L = 7$ complex taps.

We now marginalize $p(u|y)$ over $u_{-L+1:k-L-1}$ simplifying the first factor, from the forward description in (3),

$$p(u_{k-L:s+L}|y) \propto p(s_k|y_{1:k}) \prod_{i=k+1}^{n+L} p(y_i|s_i)p(u_i).$$

(26)

To rewrite the last term of (26) into the posterior $p(u_{k-L:s+L}|y_{k:s+L})$, the following factors are missing

$$p(y_k|s_k) \prod_{i=k-L}^{k-1} p(u_i).$$

(27)
By multiplying and dividing (26) by (27), we get the equivalent distribution

\[
p(u_{k-L:k:L}) \propto p(s_k|y_{1:k})p(u_{k-L:k:L}|y_{k:L}}.
\]

Then after marginalizing (28) over the last symbols \(u_{k+1:k} \), it yields

\[
p(s_k|y) \propto \frac{p(s_k|y_{1:k})p(s_k|y_{k:L}}}{p(y_k|s_k) \prod_{i=k-L} p(u_i)}.
\]

Fig. 4: BER along \(E_b/N_0\) for for BEP (○) [23], SEP (○) [11], KSEP (×) and LMMSE (△) turbo equalizers, 128-QAM and averaged over 100 random channels with \(L = 7\) complex taps.

REFERENCES

[1] J. G. Proakis, *Digital Communications*, 5th ed. New York, NY: McGraw-Hill, 2008.

[2] C. Douillard, M. Jezequel, C. Berrou, P. Didier, and A. Picart, “Iterative correction of intersymbol interference: turbo-equallization,” *European Trans. on Telecommunications*, vol. 6, no. 5, pp. 507–512, Sep 1995.

[3] C. Berrou, *Codes and turbo codes*, ser. Collection IRIS. Springer Paris, 2010.

[4] M. Tüchler and A. Singer, “Turbo equalization: An overview,” *IEEE Trans. on Information Theory*, vol. 57, no. 2, pp. 920–952, Feb 2011.

[5] L. Bahl, J. Cocke, F. Jelinek, and J. Raviv, “Optimal decoding of linear codes for minimizing symbol error rate (corresp.),” *IEEE Trans. on Information Theory*, vol. 20, no. 2, pp. 284–287, 1974.

[6] V. Franz and J. Anderson, “Concatenated decoding with a reduced-search BCJR algorithm,” *IEEE Journal on Selected Areas in Communications*, vol. 16, no. 2, pp. 186–195, Feb 1998.

[7] G. Colavolpe, G. Ferrari, and R. Raheli, “Reduced-state BCJR-type algorithms,” *IEEE Journal on Selected Areas in Communications*, vol. 19, no. 5, pp. 848–859, May 2001.

[8] M. Sikora and D. Costello, “A new SISO algorithm with application to turbo equalization,” in *Proc. IEEE International Symposium on Information Theory (ISIT)*, Sep 2005, pp. 2031–2035.

[9] D. Fentonari, A. Barbieri, and G. Colavolpe, “Reduced-complexity BCJR algorithm for turbo equalization,” *IEEE Trans. on Communications*, vol. 55, no. 12, pp. 2279–2287, Dec 2007.

[10] I. Santos, J. J. Murillo-Fuentes, R. Boloix-Tortosa, E. Arias-de-Reyna, and P. M. Olmos, “Expectation propagation as turbo equalizer in ISI channels,” *IEEE Trans. on Communications*, vol. 65, no. 1, pp. 360–370, Jan 2017.

[11] I. Santos, J. J. Murillo-Fuentes, E. Arias-de-Reyna, and P. M. Olmos, “Probabilistic equalization with a smoothing expectation propagation approach,” *IEEE Trans. on Wireless Communications*, vol. 16, no. 5, pp. 2950–2962, May 2017.

[12] K. Muranov, “Survey of MMSE channel equalizers,” University of Illinois, Chicago, Tech. Rep., 2010.

[13] M. Tüchler, R. Koetter, and A. Singer, “Turbo equalization: principles and new results,” *IEEE Trans. on Communications*, vol. 50, no. 5, pp. 754–767, May 2002.

[14] M. Tüchler, A. Singer, and R. Koetter, “Minimum mean square error equalization using a priori information,” *IEEE Trans. on Signal Processing*, vol. 50, no. 3, pp. 673–683, Mar 2002.

[15] R. Koetter, A. Singer, and M. Tüchler, “Turbo equalization,” *IEEE Signal Processing Magazine*, vol. 21, no. 1, pp. 67–80, Jan 2004.

[16] S. Park and S. Choi, “Iterative equalizer based on Kalman filtering and smoothing for MIMO- ISI channels,” *IEEE Transactions on Signal Processing*, vol. 63, no. 19, pp. 5111–5120, Oct 2015.

[17] J. Céspedes, P. M. Olmos, M. Sánchez-Fernández, and F. Pérez-Cruz, “Probabilistic MIMO symbol detection with expectation consistency approximate inference,” *IEEE Transactions on Vehicular Technology*, vol. 67, no. 4, pp. 3481–3494, April 2018.

[18] I. Santos, M. M. Olmos, M. Sánchez-Fernández, and F. Pérez-Cruz, “Expectation propagation detection for high-order high-dimensional MIMO systems,” *IEEE Trans. on Communications*, vol. 62, no. 8, pp. 2840–2849, Aug 2014.

[19] I. Santos and J. J. Murillo-Fuentes, “EP-based turbo detection for MIMO receivers and large-scale systems,” *IEEE transactions on vehicular technology*, 2018. Under review. [Online]. Available: http://arxiv.org/abs/1805.05065

[20] M. Senst and G. Ascheid, “How the framework of expectation propagation yields an iterative IC-LMMSE MIMO receiver,” in *Proc. IEEE Global Telecommunications Conference (GLOBECOM)*, Dec 2011, pp. 1–6.

[21] P. M. Olmos, J. J. Murillo-Fuentes, and F. Pérez-Cruz, “Tree-structure expectation propagation for LDPC decoding over the BEC,” *IEEE Transactions on Information Theory*, vol. 59, no. 6, pp. 3354–3377, June 2013.

[22] L. Salmamana, P. M. Olmos, F. Pérez-Cruz, and J. J. Murillo-Fuentes, “Tree-structured expectation propagation for LDPC decoding over BMS channels,” *IEEE Trans. on Communications*, vol. 61, no. 10, pp. 4086–4095, Oct 2013.

[23] I. Santos, J. J. Murillo-Fuentes, E. Arias-de-Reyna, and P. M. Olmos, “Turbo EP-based equalization: A filter-type implementation,” *IEEE Transactions on Communications*, vol. 66, no. 9, pp. 4259–4270, Sept 2018.
[24] P. Sun, C. Zhang, Z. Wang, C. Manchon, and B. Fleury, “Iterative receiver design for ISI channels using combined belief- and expectation-propagation,” IEEE Signal Processing Letters, vol. 22, no. 10, pp. 1733–1737, Oct 2015.

[25] J. Hu, H. Loeliger, J. Dauwels, and F. Kschischang, “A general computation rule for lossy summaries/messages with examples from equalization,” in Proc. 44th Allerton Conf. Communication, Control, and Computing, Sep 2006, pp. 27–29.

[26] T. P. Minka, “A family of algorithms for approximate Bayesian inference,” Ph.D. dissertation, Massachusetts Institute of Technology, 2001.

[27] T. Minka, “Expectation propagation for approximate Bayesian inference,” in Proc. 17th Conference on Uncertainty in Artificial Intelligence (UAI), 2001, pp. 362–369.

[28] M. Seeger, “Expectation propagation for exponential families,” Univ. Calif., Berkeley, CA, USA, Tech. Rep., 2005.

[29] C. M. Bishop, Pattern Recognition and Machine Learning (Information Science and Statistics). Secaucus, NJ, USA: Springer-Verlag, New York, 2006.