Modeling Stochastic Variability in Multiband Time-series Data

Zhirui Hu1 and Hyungsuk Tak2,3,4,5

1 Department of Statistics, Harvard University, Cambridge, MA 02138, USA
2 Center for Astrostatistics, Pennsylvania State University, University Park, PA 16802, USA; tak@psu.edu
3 Department of Statistics, Pennsylvania State University, University Park, PA 16802, USA
4 Department of Astronomy and Astrophysics, Pennsylvania State University, University Park, PA 16802, USA
5 Institute for Computational and Data Sciences, Pennsylvania State University, University Park, PA 16802, USA

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Abstract

In preparation for the era of time-domain astronomy with upcoming large-scale surveys, we propose a state-space representation of a multivariate damped random walk process as a tool to analyze irregularly-spaced multifilter light curves with heteroscedastic measurement errors. We adopt a computationally efficient and scalable Kalman filtering approach to evaluate the likelihood function, leading to maximum $O(k^2n^3)$ complexity, where $k$ is the number of available bands and $n$ is the number of unique observation times across the $k$ bands. This is a significant computational advantage over a commonly used univariate Gaussian process that can stack all multiband light curves in one vector with maximum $O(kn^3)$ complexity. Using such efficient likelihood computation, we provide both maximum likelihood estimates and Bayesian posterior samples of the model parameters. Three numerical illustrations are presented: (i) analyzing simulated five-band light curves for a comparison with independent single-band fits; (ii) analyzing five-band light curves of a quasar obtained from the Sloan Digital Sky Survey Stripe 82 to estimate short-term variability and timescale; (iii) analyzing gravitationally lensed $g$- and $r$-band light curves of Q0957+561 to infer the time delay. Two R packages, Rdrw and timedelay, are publicly available to fit the proposed models.

Unified Astronomy Thesaurus concepts: Astrostatistics (1882); Interdisciplinary astronomy (804); Astrostatistics tools (1887)

Supporting material: data behind figures

1. Introduction

A Gaussian process (GP) is one of the most important data analytic tools in astronomy due to its well-known computational and mathematical conveniences. GPs are especially useful for analyzing astronomical time-series data since they are continuous-time processes accounting for irregular observation cadences. Moreover, a GP’s state-space representation enables modeling of heteroscedastic measurement errors as well. Such analytic advantages have made GPs so popular that it is nearly impossible to list all subfields of astronomy where GPs are useful; 19,483 ApJ articles appear on the webpage of IOPscience and 18,038 MNRAS articles show up on the webpage of MNRAS with the keyword “Gaussian process” (on 2020 February 19). However, it is the case that a multi-output GP, which is suitable for modeling multiband time-series data, has not been well documented in the astronomical literature. This vector-output GP is widely used in other fields, e.g., cokriging or coregionalization in geostatistics (Journel & Huijbregts 1978; Gelfand et al. 2004; Alvarez et al. 2012) and multitask learning in machine learning (Caruana 1997). The key idea is to model dependence among multisource data via a covariance function to take advantage of their dependent structure in making an inference or a prediction.

We propose a state-space representation of a multivariate damped random walk process as a specific class of a multi-output GP. This process is also called a multivariate Ornstein–Uhlenbeck process (Gardiner 2009; Singh et al. 2018) and a vectorized continuous-time autoregressive model with order one, i.e., a vectorized CAR(1) or CARMA(1, 0) (Marquardt & Stelzer 2007). In particular, the proposal is a multivariate generalization of the work of Kelly et al. (2009). They adopt a univariate GP with the Matérn(1/2) covariance function (i.e., damped random walk process) to fit single-filter quasar light curves. Using this single-band model, they investigate associations between model parameters and physical properties of quasars. Following their work, MacLeod et al. (2010), Kozlowski et al. (2010), Kim et al. (2012), and Andrae et al. (2013) show more empirical evidence for such astrophysical interpretations of the model parameters. The proposed multivariate generalization of their analytic tools can incorporate more data from all available bands into one comprehensive model. This enables more accurate inference on such physically meaningful model parameters.

Also, the multivariate aspect of the proposal is essential for studying stochastic variability in active galactic nuclei (AGNs) in the era of the Vera C. Rubin Observatory Legacy Survey of Space and Time (LSST, Ivezić et al. 2019). LSST light curves are supposed to be sparse when only one band is considered. In general, it is challenging to extract information about short-term variability and timescale from sparsely observed single-band light curves. This problem becomes worse if the actual timescale of AGN variability is much shorter than the typical observation cadence in each band. The proposed multiband model, however, can alleviate this issue of sparse sampling. This is because it can take advantage of more data points observed at non-overlapping times in all bands. It will lead to more accurate inference on short-term variabilities and timescales, which in turn will be helpful for investigating AGN variability and light-curve classification.

From a methodological point of view, the proposed method is flexible enough to model various aspects of astronomical multifilter light curves. Above all, the proposed process is a
continuous-time process in a state-space representation, suitable for modeling irregularly-spaced multiband time-series data with heteroscedastic measurement errors. Also, the process does not require the data at each observation time to be a vector of the same length; the number of observations at each observation time can range from one to \( k \), the total number of bands. This feature is desirable because there can be ties in observation times possibly due to rounding. In addition, the lengths of multifold light curves do not need to be the same; a light curve from one band can be longer than other light curves from different bands. Such flexibility makes the proposed process ideal for modeling Sloan Digital Sky Survey (SDSS) and LSST multiband time-series data with heteroscedastic measurement errors.

Moreover, the proposed process has a couple of computational advantages. We adopt a Kalman filtering approach for evaluating the resulting likelihood function with maximum \( O(k^3n) \) complexity, where \( n \) is the total number of unique observation times across the entire \( k \) bands. This is more scalable and efficient than an existing strategy of applying a univariate GP to multifier light curves that are stacked up in a single vector (Zu et al. 2016; Czekala et al. 2017). This is because the univariate GP approach leads to maximum \( O(k^2n^3) \) complexity. Such an efficient likelihood computation makes the following likelihood-based inference efficient; we provide both maximum likelihood estimates and Bayesian posterior samples of model parameters.

Our numerical studies include a simulation study and two realistic data analyses. These show that the proposed process results in more comprehensive inference than independent single-band analyses. The simulation study generates multiband light curves using the proposed model with some fixed parameter values. Then it checks whether the resulting inference successfully recovers these generative parameter values. The next numerical illustration analyzes realistic five-band light curves of a quasar obtained from SDSS Stripe 82 to infer short-term variabilities and timescales. Finally, we apply the proposed process to inferring the time delay between doubly lensed images of Q0957+561, whose light curves are observed in the \( g \) and \( r \) bands.

2. Model Specification

A univariate damped random walk process (Kelly et al. 2009) is defined as

\[
dX(t) = -\frac{1}{\tau}(X(t) - \mu)dt + \sigma dB(t),
\]

where \( X(t) \) denotes the magnitude of an astronomical object at time \( t \in \mathbb{R} \), \( \tau \) is the timescale of the process in days, \( \mu \) is the long-term average magnitude of the process, \( \sigma \) is the short-term variability of the process on the magnitude scale, and \( B(t) \) is the standard Brownian motion.

A multivariate version of the damped random walk process (Gardiner 2009) is defined as

\[
dX(t) = -D^{-1}_\tau(X(t) - \mu)dt + D_\sigma dB(t),
\]

where \( X(t) = (X_1(t), \ldots, X_k(t)) \) is a vector of length \( k \) that denotes magnitudes of the \( k \) bands at time \( t \in \mathbb{R} \), \( D_\tau = \text{diag}(\tau_1, \ldots, \tau_k) \) is a \( k \times k \) diagonal matrix whose diagonal elements are \( k \) timescales with each \( \tau_j \) representing the timescale of the \( j \)th band in days, \( \mu = \{\mu_1, \ldots, \mu_k\} \) is a vector for long-term average magnitudes of \( k \) bands, and \( D_\sigma = \text{diag}(\sigma_1, \ldots, \sigma_k) \) is a \( k \times k \) diagonal matrix whose diagonal elements are short-term variabilities (in magnitudes) of \( k \) bands. Finally, \( B(t) = \{B_1(t), \ldots, B_k(t)\} \) is a vector for \( k \) standard Brownian motions whose pairwise correlations are modeled by correlation parameters \( \rho_{jl}(1 \leq j < l \leq k) \) such that \( dB_j(t)dB_l(t) = \rho_{jl}dt \). These correlations are essentially cross-correlations because they govern the correlations among different continuous-time processes. (The subscripts \( j \) and \( l \) will be numeric hereafter, i.e., the subscripts \( j \) and \( l \) denote the \( j \)th band and \( l \)th band, respectively, not \( j \)-band and \( l \)-band.) We use \( \rho \) to denote a vector of these \( k(k-1)/2 \) correlation parameters.

These correlation parameters are the key to the multiband modeling. If these correlations are set to zeros, then the proposed multiband model is essentially equivalent to a single-band model with an independent assumption across bands. In this case, data from one band do not contribute to the parameter estimation of other bands; see the left panel of Figure 1. This is because a zero correlation between two multivariate Gaussian random variables implies their independence. The proposed multiband model accounts for the dependence between bands by introducing their correlation parameters. This enables sharing of information across multiband data to infer parameters of all bands; see the right panel of Figure 1.

The matrix \( D_\tau \) in (1) does not need to be diagonal as long as it is positive definite (Vatiwupong & Phewchehan 2019). But here we limit it to be diagonal for better interpretability and computational efficiency. If \( D_\tau \) is a general positive definite matrix, timescales become correlated via off-diagonal elements of \( D_\tau \). One downside is that it is difficult to interpret off-diagonal elements of \( D_\tau \). If \( D_\tau \) is diagonal, on the other hand, it is clear that \( \tau_j \) is interpreted as the timescale of the \( j \)th band. Such simple interpretability makes a simulation of the proposal more intuitive because it is enough to specify \( k \) timescales instead of filling out off-diagonal elements of \( D_\tau \). Second, if \( D_\tau \) is diagonal, we can easily calculate a matrix exponent, e.g., \( \exp(-(t-s)D^{-1}_\tau) \) in (2) below, significantly reducing the computational burden with fewer parameters.

The solution of the stochastic differential equation in (1) is Gaussian, Markovian, and stationary (Gardiner 2009, p105),
i.e., given $X(s)$ and for $t \geq s$, 

$$X(t) | X(s), \mu, \sigma, \tau, \rho \sim \text{MVN}_k(\mu + e^{-(t-s)}D_{k}^{-1}(X(s) - \mu), Q(t - s)),$$  

(2)

where $\text{MVN}_k(a, b)$ represents a $k$-dimensional multivariate Gaussian distribution with mean vector $a$ and covariance matrix $b$. The $(j, l)$ entry of the covariance matrix $Q(t - s)$ is defined as 

$$q_{jl} = \frac{\sigma_j \sigma_l \tau_j \tau_l}{\tau_j + \tau_l} (1 - e^{-(t-s)^{-1} \tau_j \tau_l}).$$  

(3)

We evaluate this continuous-time process at $n$ discrete observation times $t = \{t_1, \ldots, t_n\}$. Then the joint probability density function of $X(t) = \{X(t_1), \ldots, X(t_n)\}$ is 

$$f_1(X(t) | \mu, \sigma, \tau, \rho) = \prod_{i=1}^{n} f_2(X(t_i)|X(t_{i-1}), \mu, \sigma, \tau, \rho),$$  

(4)

where $f_2$ denotes the density function of the multivariate Gaussian distribution defined in (2), $t_0 = -\infty$, and the subscript $i$ will be used to distinguish observation times ($i = 1, 2, \ldots, n$).

The observed data $x = \{x_1, \ldots, x_n\}$ are multifilter light curves measured at irregularly-spaced observation times $t$ with known measurement error standard deviations, $\delta = \{\delta_1, \ldots, \delta_n\}$. Since one or more bands can be used at each observation time $t_i$, the length of a vector $x_i$ can be different, depending on how many bands are used at the $i$th observation time. For example, if $g$ and $r$ bands are used at time $t_i$, then $x_i$ is a vector containing two magnitudes from the $g$ and $r$ bands, and $\delta_i$ is a vector of two corresponding measurement error standard deviations. We assume that these observed data are realizations of the latent multifilter light curves $X(t) = \{X(t_1), \ldots, X(t_n)\}$ with known Gaussian measurement error standard deviations $\delta$. That is, for $i = 1, \ldots, n$, 

$$x_i | X(t_i) \sim \text{MVN}_k(X^*(t_i), D^{g}_{k}),$$  

(5)

where $k$ is the number of bands used at observation time $t_i$, and $X^*(t_i)$ denotes a subvector of $X(t_i)$ corresponding to the bands that are used to observe $x_i$. For example, if $g$ and $r$ bands are used for measuring $x_i$ at $t_i$, then $X^*(t_i)$ is a vector of length two ($k_i = 2$) composed of the two elements of $X(t_i)$ corresponding to the $g$ and $r$ bands. The notation $D^{g}_{k}$ denotes a $k_i \times k_i$ diagonal matrix whose diagonal elements are $\delta_i = \{\delta_{i1}, \ldots, \delta_{ik_i}\}$. These observed data $x$ are assumed to be conditionally independent given the latent data $X(t)$. Thus the resulting joint probability density function of the observed data given the latent data is expressed as 

$$h_1(x | X(t)) = \prod_{i=1}^{n} h_2(x_i | X(t_i)), $$  

(6)

where $h_2$ is the multivariate Gaussian density function defined in (5).

We summarize the proposed state-space representation in the following diagram.

Space: $$x_1 \quad x_2 \quad \cdots \quad x_n$$  

State: $$X(t_1) \rightarrow X(t_2) \rightarrow \cdots \rightarrow X(t_n)$$

The arrows represent dependent and conditionally independent relationships. For example, both $X(t_2)$ and $x_1$ depend only on $X(t_1)$, and they are conditionally independent given $X(t_1)$ because there is no direct arrow between $X(t_2)$ and $x_1$. The conditional distributions of the latent magnitudes in the state level are defined in (2), and those of the observed data given the latent magnitudes in the space level are given in (5). The advantage of this state-space approach is that we can model the noisy observations $x$ with known measurement error standard deviations $\delta$, as is done in (5). See also Kelly et al. (2009, 2014) for the state-space representations of univariate CARMA(1, 0) and CARMA($p, q$), respectively, and Section 3 of Durbin & Koopman (2012) for details of state-space representation of GPs.

Consequently, the likelihood function of the model parameters with the latent process integrated out is 

$$L(\mu, \sigma, \tau, \rho) = \int h_1(x | X(t))f_1(X(t) | \mu, \sigma, \tau, \rho) dX(t).$$  

(7)

Here $f_1$ and $h_1$ are defined in (4) and (6), respectively.

3. Computation of the Likelihood Function via Kalman Filtering

Kalman filtering (Kalman 1960) is a well-known technique to evaluate the likelihood function of a state-space model when both state and space models are Gaussian. The proposed process has a Gaussian state-space representation as shown in (2) and (5). Thus, it is natural to adopt Kalman filtering to compute the likelihood function in (7) via a product of $n$-dimensional multivariate Gaussian densities ($i = 1, \ldots, n$). This leads to $O(n^2)$ complexity. The minimum complexity is $O(n)$ when only one band is used at each observation time ($k_i = 1$), and the maximum complexity is $O(nk^2)$ when all of the $k$ bands are used at every observation time ($k_i = k$).

Let $F(t_i)$ denote the natural filtration at time $t_i$, i.e., all of the information about the observed data available until time $t_i$. Using this notation, we define the following predictive mean vector and covariance matrix at $t_{i-1}$: with $\Delta t_i = t_i - t_{i-1}$,

$$\mu = \mu_{i-1} + E(X(t_i) | F(t_{i-1}), \mu, \sigma, \tau, \rho) = \mu + e^{-\Delta t_i D_{g}^{-1}}(\mu_{i-1} - \mu),$$  

$$\Sigma = \text{Cov}(X(t_i) | F(t_{i-1}), \mu, \sigma, \tau, \rho) = e^{-\Delta t_i D_{g}^{-1}}(\Sigma_{i-1} - \mu_{i-1} \mu_{i-1}' \mu_{i-1}) + Q(\Delta t_i).$$  

(8)

We assume that 

$$\mu_{i|0} = \mu \quad \text{and} \quad \Sigma_{i|0} = \{q_{jl} = \frac{\sigma_j \sigma_l \tau_j \tau_l}{\tau_j + \tau_l} (1 - e^{-\Delta t_i \tau_j \tau_l}).$$

Here, each element of the covariance matrix $Q(\Delta t_i)$ in (8) is defined in (3), and the updated mean vector and covariance
matrix (after observing data at \( t_i \)) are
\[
\mu_{ij} = E(X(t_i)|F(t_i), \mu, \sigma, \tau, \rho) = \mu_{ij-1} + \Sigma_{ij-1}^{-1} \left( \Sigma_{ij}^* \right)_{ij-1} \frac{1}{D^2} (x_i - \mu_{ij-1}),
\]
\[
\Sigma_{ij} = \text{Cov}(X(t_i)|F(t_i), \mu, \sigma, \tau, \rho) = \Sigma_{ij-1} - \Sigma_{ij-1}^* \left( \Sigma_{ij}^* \right)_{ij-1} + \frac{1}{D^2} \Sigma_{ij-1}^*.
\]
The notation \( \Sigma_{ij-1}^* \) denotes a submatrix of \( \Sigma_{ij-1} \) restricted to the bands used for observing \( x_i \). \( \Sigma_{ij-1}^* \) is a submatrix of \( \Sigma_{ij-1} \) whose rows correspond to the bands used and whose columns correspond to the entire bands, and \( \Sigma_{ij-1}^* \) is a submatrix of \( \Sigma_{ij-1} \) whose rows correspond to the entire bands and whose columns correspond to the bands used. For example, suppose there are five bands, \( u, g, r, i, z \), and we use \( u \) and \( r \) bands to observe \( x_i \). Then, \( \Sigma_{ij-1}^* \) is a \( 1 \times 2 \) covariance matrix constructed by selecting rows and columns corresponding to \( u \) and \( r \) bands from \( \Sigma_{ij-1} \). \( \Sigma_{ij-1}^* \) is a \( 2 \times 5 \) matrix made by choosing two rows corresponding to \( u \) and \( r \) bands and all columns from \( \Sigma_{ij-1} \). \( \Sigma_{ij-1}^* \) is a \( 5 \times 2 \) matrix built by choosing all rows and two columns corresponding to \( u \) and \( r \) bands from \( \Sigma_{ij-1} \).

Consequently, the likelihood function in (7) can be computed as follows:
\[
L(\mu, \sigma, \tau, \rho) = \prod_{i=1}^{n} p(x_i|F(t_i), \mu, \sigma, \tau, \rho),
\]
where \( p \) is another multivariate Gaussian density of \( x_i|F(t_i), \mu, \sigma, \tau, \rho \sim \text{MVN}((\mu^*_{ij-1}, \Sigma^*_{ij-1} + D^2\delta)) \).

The notation \( \mu^*_{ij-1} \) denotes a subvector of \( \mu_{ij-1} \) restricted only to the bands used to observe \( x_i \).

By definition, the maximum likelihood estimates of the model parameters are the values that jointly maximize \( L(\mu, \sigma, \tau, \rho) \). We use a gradient-free optimization algorithm (Nelder & Mead 1965) to obtain the maximum likelihood estimates.

4. Bayesian Inference
For Bayesian hierarchical modeling, we adopt scientifically motivated, weakly informative, and independent prior distributions on the model parameters (Tak et al. 2017b, 2018a): for \( j = 1, 2, \ldots, k \) and \( j < i \leq k \),
\[
\mu_j \sim \text{Unif}(-30, 30), \quad \tau_j \sim \text{inv-Gamma}(1, 1),
\]
\[
\rho_{ij} \sim \text{Unif}(-1, 1), \quad \sigma_j^2 \sim \text{inv-Gamma}(1, c),
\]
where \( \text{inv-Gamma}(a, b) \) denotes the inverse-Gamma distribution with shape parameter \( a \) and scale parameter \( b \). We assume that each long-term average magnitude \( \mu_j \) is in a reasonably wide range between \(-30 \) and \( 30 \). The correlation parameters are between \(-1 \) and \( 1 \) by definition. Setting up an inverse-Gamma\((a, b)\) prior on an unknown parameter is considered as setting up a soft lower bound, \( a/(b + 1) \), of the unknown parameter (Section 4.2, Tak et al. 2018a). We set up soft lower bounds of \( \tau_j \)'s and \( \sigma_j^2 \)'s to prevent undesirable limiting behaviors of a damped random walk process (Section 2.5, Tak et al. 2017b). Specifically, the observed time series will look like a white-noise process if the timescale of the process is much shorter than the typical observation cadence (i.e., in the limit of timescale going to zero). Thus, we set up a half-day soft lower bound for each \( \tau_j \). This undesirable behavior is also expected if the short-term variability (variance) of the process is much smaller than the typical measurement error variance. To reflect this, the constant \( c \) in the inverse-Gamma prior of \( \sigma_j^2 \) in (11) is set to an arbitrarily small constant, \( 10^{-7} \), so that the soft lower bound of each \( \sigma_j \) is 0.00022.

Let \( q \) be a joint prior density function of \( \mu, \sigma, \tau, \rho \) whose distributions are specified in (11). Then, the resulting full posterior density function \( \pi \) is
\[
\pi(\mu, \sigma, \tau, \rho|x) \propto L(\mu, \sigma, \tau, \rho) \times q(\mu, \sigma, \tau, \rho).
\]
We adopt a Metropolis-Hastings within Gibbs sampler (Tierney 1994) to draw (dependent) posterior samples from the full posterior distribution \( \pi(\mu, \sigma, \tau, \rho|x) \). Initial values of the model parameters are set to their maximum likelihood estimates. Then, it sequentially updates each parameter given the observed data and all the other parameters at each iteration. For example, suppose we have three parameters \( \theta_1, \theta_2, \theta_3 \) to be updated at iteration \( s \) given previously updated values. We sequentially update each parameter as follows:

Given \( (\theta_1^{(s-1)}, \theta_2^{(s-1)}, \theta_3^{(s-1)}) \),
\[
\text{sample } \theta_1^{(s)} \text{ from } \theta_1^{(s-1)}, \quad \text{setting it to } \theta_1^{(s)}.
\]
\[
\text{sample } \theta_2^{(s)} \text{ from } \theta_2^{(s-1)}, \quad \text{setting it to } \theta_2^{(s)}.
\]
\[
\text{sample } \theta_3^{(s)} \text{ from } \theta_3^{(s-1)}, \quad \text{setting it to } \theta_3^{(s)}.
\]
The parentheses superscript indicates at which iteration the value of the parameter is updated. We use a truncated Gaussian distribution between \(-30 \) and \( 30 \) as a proposal distribution for each \( \mu_j \), a truncated Gaussian proposal distribution between \(-1 \) and \( 1 \) for each \( \rho_{ij} \), and a log-normal proposal distribution for each of \( \sigma_j^2 \) and \( \tau_j \). We also use an adaptive Markov chain Monte Carlo procedure (Section 3.2, Tak et al. 2017b) so that proposal scales (or called jumping scales) are automatically tuned.

A tuning-free R package, Rdrw, that can analyze and simulate both single-band and multiband light curves according to the proposed model is publicly available at CRAN.\(^6\)

5. Numerical Illustrations
5.1. A Simulation Study on Five-band Light Curves
We simulate a set of five-band light curves from the proposed multiband model via a two-step procedure. The first step simulates complete data and the second step deletes some of them to be more realistic, e.g., making seasonal gaps. The simulation setting of the first step is as follows: (i) the total number of unique observation times across five bands is set to 300; (ii) the observation cadences are randomly drawn from the Gamma(\( \alpha = 3, \beta = 1 \)) distribution whose mean is 3 days; (iii) the measurement error standard deviations of the \( j \)th band are randomly drawn from the \( N(0.01 + 0.004(j - 1), 0.002^2) \) distribution \( (j = 1, \ldots, 5) \) so that the first band accompanies the smallest measurement error standard deviations and the last band

\(^6\) https://cran.r-project.org/package=Rdrw
Al s o , {m} 180, and each band has 36 observations with heteroscedastic measurement seasonal gaps exist. After the deletion, the total number observation times is involves the largest; (iv) the generative parameter values are \( \mu_j = 17 + 0.5(j - 1) \), \( \sigma_j = 0.01j \), \( \tau_j = 100 + 20j \) for \( j = 1, \ldots, 5 \). Also, \( \rho_{jl} = 1.1^{-|j-l|} \) for \( 1 \leq j < l \leq 5 \) so that the first and last bands are least correlated; (v) given these parameter values, the latent magnitudes \( X(t) \) are generated via (2); (vi) finally, conditioning on these latent magnitudes, the observed magnitudes \( x \) are generated via (5). The R package Rdrw has a functionality to return \( x \) as a \( 300 \times 5 \) matrix.

Given these simulated data in the first step, we remove some observations. We assume that only three consecutive observations occur for each band, and there are three seasonal gaps: (i) for the \( j \)th band, we keep only the following observation numbers: \( \{1, 2, 3\} \) \( + 3(j - 1) + 15(b - 1); b = 1, 2, \ldots, 20 \). For example, the observation numbers to be kept for the second band \( (j = 2) \) are 4, 5, 6, 19, 20, 21, \ldots, 289, 290, 291, and the other observations in the second band are removed. Similarly, the observation numbers for the third band \( (j = 3) \) are 7, 8, 9, 22, 23, 24, \ldots, 292, 293, 294. By this rule, we can keep the 300 observation times in total and each observation time accompanies a measurement from one band \( (k_i = 1 \) for all \( i \) in (10), leading to \( O(n) \) complexity). (ii) Next we create three seasonal gaps by removing 120 observation times out of 300. The observation numbers falling into the three seasonal gaps range from 41 to 80, from 141 to 180, and from 241 to 280. The total number of observations in the final data set is 180. The simulated five-band light curves before and after removing some of the data are displayed in the top and bottom panels of Figure 2, respectively. The heteroscedastic measurement error standard deviations are too small to be displayed. Also, the details of the simulated data are summarized in Table 1.

Using this simulated data set (i.e., after the removal), we fit a univariate damped random walk model (Kelly et al. 2009) to each single-band light curve. And we fit the proposed multivariate damped random walk model to the entire five-band data set. For each fit, we run a Markov chain for 60,000 iterations, which is initiated at the maximum likelihood estimates of the model parameters. We discard the first 10,000 iterations as burn-in, and then we thin the remaining chain by a factor of five, i.e., from length 50,000 to 10,000. We use this thinned Markov chain Monte Carlo sample of size 10,000 to summarize the results. We check the convergence of the Markov chain by computing effective sample sizes as a numerical indication of the autocorrelation function; the higher the effective sample size is, the more quickly the autocorrelation function decreases. The average effective sample size for the five short-term variabilities is 2896 and that for the five timescales is 1866.

The marginal posterior distributions of the short-term variabilities obtained by the proposed multiband model are exhibited in the first column of Figure 3. The second column displays those of timescales. The black solid curves superimposed on the histograms denote the posterior distributions obtained by the single-band model. The vertical dashed lines represent the generative true values. It turns out that the posterior distributions obtained by the proposed multiband model (histograms) tend to have higher peaks and narrower spread around the generative true values than those obtained by the single-band models (solid curves). These results show that the proposed model results in more accurate inferences on the model parameters. This makes intuitive sense because the single-band model accesses only 36 observations in each band, while the multiband model enables sharing of information across bands via their correlations. That means that the multiband model allows all of the 180 observations to contribute to the inference on every model parameter through their dependence.

When it comes to cross-correlation parameters, a comparison between single-band and multiband models is not possible because correlations are available only in the multiband model. We display the marginal posterior distributions of the 10 cross-correlation parameters in Figure 4. It turns out that the true parameter values are located close to the modes of these posterior distributions.

5.2. Five-band Light Curves of an SDSS S82 Quasar

The five-band \((u, g, r, i, z)\) light curves of a quasar used in this illustration are obtained from a catalog of 9258 SDSS

![Figure 2] A simulated data set of five-band light curves before (top) and after (bottom) removing some observations. The deleted data are based on three assumptions: (i) there is only a single-band measurement at each observation time; (ii) three consecutive observations are made for each band; (iii) three seasonal gaps exist. After the deletion, the total number observation times is 180, and each band has 36 observations with heteroscedastic measurement errors (whose standard deviations are too small to be displayed).

| Band | Length | Median Cadence | Median Magnitude | Median SD |
|------|--------|----------------|------------------|-----------|
| 1    | 36     | 3.368          | 17.492           | 0.010     |
| 2    | 36     | 4.044          | 17.979           | 0.014     |
| 3    | 36     | 3.950          | 18.513           | 0.018     |
| 4    | 36     | 4.264          | 18.963           | 0.022     |
| 5    | 36     | 3.547          | 19.723           | 0.026     |

Note. ‘SD’ represents the measurement error standard deviation. Median cadence is in days. The first band data are the brightest with the smallest measurement uncertainty, and the last band data are the faintest with the largest measurement uncertainty.
Stripe 82 quasars that are spectroscopically confirmed (MacLeod et al. 2012).\footnote{http://faculty.washington.edu/ivezic/cmacleod/qso_dr7} The name of the quasar (dbID) is 3078106. We display these five-band light curves in Figure 5. Most of the measurement error standard deviations are too small to be displayed in this figure. There is no tie in observation times, i.e., a single-band magnitude is measured at each observation time \((k_i = 1\) for all \(i\) in (10)), leading to \(O(n)\) complexity for the likelihood computation. The length of each single-band light curve is different: 132 observations in the \(u\) band, 137 in the \(g\) band, 141 in the \(r\) band, 138 in the \(i\) band, and 139 in the \(z\) band. In total, there are 687 observation times across the bands \((n = 687)\); see Table 2 for more details of the data.

We run a Markov chain of length 60,000, discarding the first 10,000 iterations as burn-in. We thin the remaining chain by a factor of five from length 50,000 to 10,000. We use this thinned Markov chain to make an inference on each parameter because the effective sample sizes are satisfactory across all model parameters. The sampling result is summarized in Table 3.

We compare the marginal posterior distributions of the short-term variabilities and timescales obtained by the proposed multiband model with those obtained by the single-band models. Figure 6 displays these marginal posterior distributions of the short-term variabilities in the first column and those of the timescales in the second column. The histograms indicate the posterior distributions obtained by the multiband model, while the superimposed solid black curves represent those obtained by the single-band model.

Overall, the marginal posterior distributions obtained by the multiband model (histograms) tend to have higher peak and narrower spread than those obtained by the single-band models.
Table 2
Details of the Multiband Time-series Data of a Quasar 3078106

| Band | Length | Median Cadence | Median Magnitude | Median SD |
|------|--------|----------------|-----------------|----------|
| u    | 132    | 2.004          | 20.768          | 0.102    |
| g    | 137    | 1.996          | 20.383          | 0.027    |
| r    | 141    | 1.995          | 19.995          | 0.024    |
| i    | 138    | 1.994          | 19.627          | 0.025    |
| z    | 139    | 1.994          | 19.473          | 0.071    |

Note. “SD” represents the measurement error standard deviation. Median cadence is in days.

Table 3
Details of the Posterior Samples of the Model Parameters

| Parameter | Mean   | SD     | 68.3% Credible Interval | ESS |
|-----------|--------|--------|-------------------------|-----|
| µ₁        | 20.780 | 0.043  | (20.739, 20.820)        | 379 |
| µ₂        | 20.347 | 0.036  | (20.312, 20.382)        | 290 |
| µ₃        | 19.934 | 0.034  | (19.902, 19.965)        | 266 |
| µ₄        | 19.616 | 0.029  | (19.588, 19.644)        | 380 |
| µ₅        | 19.471 | 0.032  | (19.442, 19.499)        | 397 |
| log₁₀(σ₁) | -1.887 | 0.094  | (-1.983, -1.792)        | 704 |
| log₁₀(σ₂) | -1.984 | 0.064  | (-2.048, -1.919)        | 697 |
| log₁₀(σ₃) | -2.060 | 0.068  | (-2.129, -1.992)        | 475 |
| log₁₀(σ₄) | -1.985 | 0.071  | (-2.058, -1.913)        | 804 |
| log₁₀(σ₅) | -1.992 | 0.096  | (-2.088, -1.896)        | 800 |
| µ₁₂       | 2.147  | 0.212  | (1.937, 2.344)          | 362 |
| µ₁₃       | 2.206  | 0.153  | (2.061, 2.350)          | 326 |
| µ₁₄       | 2.253  | 0.175  | (2.087, 2.415)          | 253 |
| µ₁₅       | 2.093  | 0.175  | (1.924, 2.260)          | 337 |
| µ₃₁       | 2.097  | 0.212  | (1.898, 2.296)          | 430 |
| µ₃₂       | 0.811  | 0.135  | (0.685, 0.933)          | 301 |
| µ₃₃       | 0.809  | 0.125  | (0.683, 0.929)          | 476 |
| µ₃₄       | 0.871  | 0.086  | (0.799, 0.944)          | 493 |
| µ₃₅       | 0.543  | 0.212  | (0.322, 0.759)          | 199 |
| µ₄₂       | 0.740  | 0.123  | (0.632, 0.853)          | 288 |
| µ₄₃       | 0.742  | 0.155  | (0.601, 0.882)          | 220 |
| µ₄₄       | 0.490  | 0.238  | (0.239, 0.734)          | 194 |
| µ₄₅       | 0.634  | 0.165  | (0.466, 0.795)          | 287 |
| µ₅₂       | 0.686  | 0.182  | (0.503, 0.859)          | 224 |
| µ₅₃       | 0.862  | 0.123  | (0.769, 0.959)          | 493 |

Note. We use “Mean” to indicate the posterior mean, “SD” to represent the posterior standard deviation, and the 68.3% credible interval is based on 15.85% and 84.15% quantiles of the corresponding posterior sample, and “ESS” to denote the effective sample size out of 10,000. Numerical subscripts are indicators of the bands: 1, 2, 3, 4, and 5 correspond to u, g, r, i, and z, respectively.

Figure 6. The results of fitting multiband and single-band models on the realistic data of SDSS S82 quasar 3078106. The marginal posterior distributions of the short-term variabilities obtained by the proposed multiband model are displayed in the first column and those of the timescales are shown in the second column. The superimposed solid black curves represent the marginal posterior distributions of corresponding parameters obtained by the univariate damped random walk (uDRW) model. Overall, the posterior distributions obtained by the multiband model tend to have higher peak and narrower spread. Also, the modal locations of timescales in the second column are more consistent across bands in the multiband model. This indicates a possibility that the timescales of this quasar are similar in all bands, which single-band models do not reveal.

5.3. Estimation of Time Delay between Doubly Lensed Multiband Light Curves of Q0957+561

Estimation of time delay is one of the key factors for estimation of the Hubble constant via time delay cosmography (Liao et al. 2015; Treu & Marshall 2016; Suyu et al. 2017). However, it has been the case that the time delays are estimated only from single-band light curves. The proposed process can be used to estimate time delays among gravitationally lensed multiband light curves of an AGN, as is the case for a single-band model (Dobler et al. 2015; Liao et al. 2015; Tak et al. 2017b). For this purpose, we introduce a few more parameters for the time delays and microlensing adjustment.

We use the proposed multi-filter process to model each multiply lensed image observed in several bands, e.g., \( X_{A}(t), X_{B}(t), X_{C}(t), \) and \( X_{D}(t) \) corresponding to quadruply lensed images, \( A, B, C, \) and \( D. \) (The notation here is consistent with that in Section 2 except for the subscripts that distinguish lensed images.) Taking lensed image \( A \) as an example, the notation \( X_{A}(t) \) represents \( \{X_{A}(t_1), ..., X_{A}(t_n)\} \) and each component \( X_{A}(t_k) \) is a vector of length \( k \) (the number of available bands), i.e., \( X_{A}(t_k) = \{X_{A,1}(t_k), ..., X_{A,n}(t_k)\}. \) The observed data of lensed image \( A \) are \( X_{A} = \{X_{A,1}, ..., X_{A,n}\} \) and the
Consequently, given the time delays \((\Delta_{AB}, \Delta_{AC}, \Delta_{AD})\) and polynomial regression coefficients for microlensing \((\beta_A, \beta_B, \beta_C, \beta_D)\), we can use only one multiband process \(X'_{A}(\cdot)\) to model all of the gravitationally lensed multilens light curves, i.e., \(X'_{A}(t), X'_{A}(t - \Delta_{AB}), X'_{A}(t - \Delta_{AC}),\) and \(X'_{A}(t - \Delta_{AD})\).

Moreover, given the time delays and polynomial regression coefficients for microlensing, we can unify the notation, combining all multiband light curves of the lensed images. Let \(\bar{t} = \{\bar{t}_1, \ldots, \bar{t}_{4n}\}\) be the sorted \(4n\) observation times among \(t, t - \Delta_{AB}, t - \Delta_{AC},\) and \(t - \Delta_{AD}\). The unified notation for the observed data at \(\bar{t}_i\) is defined as follows: for \(i = 1, 2, \ldots, 4n\),

\[
y_i = \begin{cases} 
X_{A,i} & \text{if } \bar{t}_i \in t, \\
X_{B,i} & \text{if } \bar{t}_i \in t - \Delta_{AB}, \\
X_{C,i} & \text{if } \bar{t}_i \in t - \Delta_{AC}, \\
X_{D,i} & \text{if } \bar{t}_i \in t - \Delta_{AD}.
\end{cases}
\]

with measurement error standard deviation

\[
\eta_i = \begin{cases} 
\delta_{A,i} & \text{if } \bar{t}_i \in t, \\
\delta_{B,i} & \text{if } \bar{t}_i \in t - \Delta_{AB}, \\
\delta_{C,i} & \text{if } \bar{t}_i \in t - \Delta_{AC}, \\
\delta_{D,i} & \text{if } \bar{t}_i \in t - \Delta_{AD}.
\end{cases}
\]

The unifying notation for the latent data \(Y(\bar{t}_i)\) is

\[
Y(\bar{t}_i) = \begin{cases} 
X'_{A}(\bar{t}_i) + p_A(\bar{t}_i) & \text{if } \bar{t}_i \in t, \\
X'_{A}(\bar{t}_i) + p_B(\bar{t}_i) & \text{if } \bar{t}_i \in t - \Delta_{AB}, \\
X'_{A}(\bar{t}_i) + p_C(\bar{t}_i) & \text{if } \bar{t}_i \in t - \Delta_{AC}, \\
X'_{A}(\bar{t}_i) + p_D(\bar{t}_i) & \text{if } \bar{t}_i \in t - \Delta_{AD}.
\end{cases}
\]

where \(p_A(\bar{t}_i)\) is a vector of length \(k\) composed of \(\{p_{A,1}(\bar{t}_i), \ldots, p_{A,k}(\bar{t}_i)\}\), and similarly define \(p_B(\bar{t}_i), p_C(\bar{t}_i),\) and \(p_D(\bar{t}_i)\).

Using the unified notation, we specify the distributions for the proposed time delay model. Let us define \(\Delta = \{\Delta_{AB}, \Delta_{AC}, \Delta_{AD}\}\) and \(\beta = \{\beta_A, \beta_B, \beta_C, \beta_D\}\). Then the joint density function of the latent data is

\[
f_\Delta(Y(\bar{t}_i)) \propto \prod_{i=1}^{4n} f_\Delta(Y(\bar{t}_i)|Y(\bar{t}_{i-1}), \sigma, \tau, \rho, \Delta, \beta),
\]

where the conditional distributions for \(f_\Delta\) are defined as

\[
y_i|Y(\bar{t}_i), \Delta, \beta \sim \text{MVN}_k(\mathbf{Y}(\bar{t}_i), \mathbf{D}_\eta^2)
\]

for \(i = 1, 2, \ldots, 4n\). The \((j, l)\) entry of the covariance matrix \(Q(\bar{t}_i - \bar{t}_{i-1})\) is defined in (3). The joint density function of the observed data is

\[
h_1(y|Y(\bar{t}_i), \Delta, \beta) = \prod_{i=1}^{4n} h_2(y_i|Y(\bar{t}_i), \Delta, \beta),
\]

where the conditional distributions for \(h_2\) are

\[
y_i|Y(\bar{t}_i), \Delta, \beta \sim \text{MVN}_k(\mathbf{Y}(\bar{t}_i), \mathbf{D}_\eta^2)
\]

for \(i = 1, 2, \ldots, 4n\). The notation \(\mathbf{Y}(\bar{t}_i)\) denotes a subvector of \(Y(\bar{t}_i)\) corresponding to the bands used for observing \(y_i\) at \(\bar{t}_i\). We also note that the conditions in (16)–(19) include \(\Delta\) and \(\beta\) because without \(\Delta\) and \(\beta\) we cannot define \(y_i\)'s and \(Y(\bar{t}_i)\)'s. The resulting likelihood function of the model parameters with the latent process marginalized out is

\[
L(\sigma, \tau, \rho, \Delta, \beta) = \int f_\Delta(Y(\bar{t}_i)|\sigma, \tau, \rho, \Delta, \beta) \times h_1(y|Y(\bar{t}_i), \Delta, \beta) \, dY(\bar{t}_i),
\]

where \(f_\Delta\) and \(h_1\) are defined in (16) and (18), respectively. The same Kalman filtering procedure in Section 3 is used to calculate this likelihood function. The only difference is that there are twice as many observations (from \(n\) to \(2n\)) for a doubly lensed system and four times more (from \(n\) to \(4n\)) for a quadruply lensed system. The number of unknown parameters when \(k\) bands are used for a lensed images with the \(m\)-order polynomial regression is \(2k + k(k - 1)/2 + (a - 1) + ak(m + 1)\). That is, \(k\) \(\sigma_j\)'s, \(k\) \(\tau_j\)'s, \(k(k - 1)/2\) \(\rho_j\)'s, \(a - 1\) time delays, and \(ak(m + 1)\) polynomial regression coefficients. For example, the number of unknown parameters is 103 when five bands are used (\(k = 5\)) for a quadruply lensed quasar (\(a = 4\)) with a cubic polynomial regression (\(m = 3\)).

For a Bayesian inference, we adopt the same priors for \(\sigma, \tau,\) and \(\rho\) as specified in (11), and weakly informative independent prior distributions for the additional parameters, \(\Delta\) and \(\beta\) \(\Delta\) (Section 2.4, Tak et al. 2017b). Specifically, we assume an independent multivariate Gaussian distribution, \(\text{MVN}_{m+1}(\mathbf{0}_{m+1}, D)\), for each \(\beta_A, \beta_B, \beta_C, \beta_D\) where \(\mathbf{0}_{m+1}\) denotes a vector of \(m + 1\) zeros and \(D\) is a diagonal matrix whose diagonal elements are set to relatively large constants. We also adopt a uniform distribution over the range between \(\min(\bar{t}) - \max(\bar{t})\) and \(\max(\bar{t}) - \min(\bar{t})\) for each of \(\Delta_{BA}, \Delta_{CA},\) and \(\Delta_{DA}\). We use \(q(\sigma, \tau, \rho, \Delta, \beta)\) to denote the joint prior distribution of these unknown model parameters.
parameters. The resulting full posterior distribution of the unknown model parameters is
\[
\pi(\sigma, \tau, \rho, \Delta, \beta) \propto L(\sigma, \tau, \rho, \Delta, \beta) \\
\times q(\sigma, \tau, \rho, \Delta, \beta).
\]

We adopt the same adaptive Metropolis–Hastings within Gibbs sampler to draw posterior samples from this full posterior distribution. This time, it is natural to think of a two-step sampling scheme. First, given the parameters related to the time delay model, i.e., \( \Delta \) and \( \beta \), we can completely determine \( Y \) and \( Y(t) \). Thus, given \( \Delta \) and \( \beta \), we can update the parameters relevant to the multivariate damped random walk process, i.e., \( \sigma, \tau, \) and \( \rho \), as done in Sections 5.1 and 5.2. Second, given the updated parameters, \( \sigma, \tau, \) and \( \rho \), we update the time-delay-related parameters, \( \Delta \) and \( \beta \). Here we use a truncated Gaussian proposal distribution for each \( \Delta \), and we do not need a proposal distribution for \( \beta \) because the conditional posterior distribution of \( \beta \) is a multivariate Gaussian. The sampler iterates these two steps at each iteration. An R package, \texttt{timedelay}, features this two-step update scheme to fit the proposed time delay model on a doubly lensed system observed in two bands.\(^8\) (The package will be updated later for more general cases.)

Using the proposed time delay model and fitting procedure, we estimate the time delay between doubly lensed images of Q0957+561. The data are observed in \( g \) and \( r \) bands and are publicly available (Shalyapin et al. 2012). The details of the data are summarized in Table 4. The \( g \)- and \( r \)-band light curves of lensed image \( A \) are displayed in the top panel of Figure 7, and those of lensed image \( B \) are shown in the bottom panel. The resulting model involves 22 unknown model parameters. To fit the proposed time delay model with a cubic polynomial regression for microlensing \((m = 3)\), we implement a Markov chain of length 60,000, discarding the first 10,000 iterations as burn-in. We thin the remaining chain by a factor of five (from 50,000 to 10,000). We summarize our inferential result using this thinned Markov chain. The effective sample size of the time delay \( \Delta_{AB} \) is 675, leading to its autocorrelation quickly decreasing to zero. As for single-band fits, we adopt a quadratic polynomial regression \((m = 2)\) for the microlensing adjustment because a Markov chain with a cubic order \((m = 3)\) does not converge.

The top panel of Figure 8 shows the marginal posterior distributions of \( \Delta_{AB} \) obtained by fitting two single-band models independently. The solid green curve indicates the marginal posterior distribution of \( \Delta_{AB} \) based only on the \( g \)-band data.\(^8\) https://cran.r-project.org/package=timedelay

Table 4
Details of the \( r \)- and \( g \)-band Time-series Data of Quasar Q0957+561 (Shalyapin et al. 2012)

| Image (Band) | Length | Median Cadence | Median Magnitude | Median SD |
|--------------|--------|---------------|-----------------|---------|
| A (\( r \))  | 132    | 1.978         | 16.983          | 0.012   |
| A (\( g \))  | 142    | 1.881         | 17.205          | 0.016   |
| B (\( r \))  | 132    | 1.978         | 16.966          | 0.012   |
| B (\( g \))  | 142    | 1.881         | 17.144          | 0.016   |

Note. “SD” represents the measurement error standard deviation.

\( B \) is a multivariate Gaussian. The sampler is an R package, \texttt{timedelay}, features this two-step update scheme to fit the proposed time delay model on a doubly lensed system observed in two bands.\(^8\) (The package will be updated later for more general cases.)

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Figure 7. Light curves of doubly lensed quasar Q0957+561 (Shalyapin et al. 2012). The \( g \)- and \( r \)-band light curves of lensed image \( A \) appear in the top panel and those of lensed image \( B \) appear in the bottom panel. Due to strong gravitational lensing, multiband light curves of one image lag behind by the time delay. The \( g \)-band light curve has more fluctuations, which is crucial in estimating the time delay.

(The data used to create this figure are available.)

Figure 8. The solid green curve in the top panel represents the marginal posterior distribution of the time delay \( \Delta_{AB} \) of Q0957+561 obtained by fitting the single-band model to the \( g \)-band data. Similarly the red dashed curve in the top panel is the one from the \( r \)-band data. These single-band fits reveal a couple of possibilities for the time delay. In the bottom panel, however, the posterior distribution of \( \Delta_{AB} \) obtained from the multiband model narrows down the possibility of the time delay by jointly modeling the data in both bands.

The resulting posterior mean of \( \Delta_{AB} \) is 413.392 and its posterior standard deviation is 2.916. The red dashed curve represents the distribution obtained with only the \( r \)-band data, whose posterior mean is 420.597 and standard deviation 8.165. Clearly, the fitting results are not consistent. The \( g \)-band posterior distribution has a mode near 415 days, while the
\(r\)-band posterior distribution shows two modes, one near 415 days and the other near 420 days.

The bottom panel of Figure 8 exhibits the marginal posterior distribution of \(\Delta_{AB}\) obtained by fitting the proposed multiband model to the entire data. The resulting posterior mean and standard deviation of \(\Delta_{AB}\) are 414.324 and 2.307, respectively. It is now clear which mode the data support more. The relative height of the mode near 420 days is not even comparable to that of the mode near 415 days. Also, the peak near 415 days is higher and narrower than before, which can be confirmed by the smaller posterior standard deviation. This makes intuitive sense because the \(g\)-band light curves have more fluctuations, as shown in Figure 7, and thus it is easier to find the time delay by matching those fluctuations. Consequently, the joint model puts more weight on the information from the \(g\)-band, enabling us to narrow the two possibilities down to the highest mode near 415 days.

This feature of sharing information across different bands will be useful for finding time delays from the multiband LSST light curves. This is because single-band LSST light curves are sparsely observed, and thus single-band observations may not exhibit enough fluctuation patterns that are needed to estimate time delays.

### 6. Discussion

The proposed state-space representation of a multivariate damped random walk process can be applied to other subfields of astronomy as well. Photometric reverberation mapping is one such example. Zu et al. (2011) model continuum variability by a univariate GP with a damped random walk covariance function to estimate AGN reverberation time lags. Zu et al. (2016) modify this model to infer a two-band reverberation time lag. They stack up two-band light curves in one vector and apply the univariate GP framework with \(O(n^3)\) complexity. We note that their two-band time-lag model is closely related to the single-band time delay model between doubly lensed light curves. This is because the former reduces to the latter if an indicator function \(I_{\Delta}(t)\), which is one when \(t = \Delta\) and zero otherwise, is used as a transfer function \(\Psi(t)\). This implies that the proposed multivariate process can be useful for improving their reverberating time-lag estimate, as is the case in the estimation of multiband time delay.

We also note that the proposed model can be a parametric alternative to a widely used non-parametric cross-correlation method in multiband data analyses, e.g., Edelson et al. (2015). This is because the proposed multivariate process can be considered as a parametric cross-correlation method since it produces posterior distributions of all pairwise cross-correlation parameters. Its applicability is not limited to correlations among multiband time series of one object. It can be used to find cross-correlations of multiple time-series data of different sources if their timescales are similar enough to make the cross-correlation meaningful.

From a methodological point of view, the proposed model has several advantages over coregionalization (Journel & Huijbregts 1978; Gelfand et al. 2004; Álvarez et al. 2012), although the latter is more flexible. First, unlike the proposed damped random walk, coregionalization does not account for the damped part. This is because it expresses each component of a vector-valued output as a linear function of independent GPs, which corresponds only to the random walk part. Second, the proposal has better interpretability. The dependence among multiple bands is clearly modeled by well-known cross-correlation parameters \(\rho_{ij}\). Also each light curve is modeled by a univariate damped random walk with only three parameters \((\mu_i, \sigma_i, \tau_i)\), i.e., long-term average, short-term variability, and timescale of a latent process. On the other hand, the coefficients of linear functions in coregionalization do not have such intuitive interpretability. Lastly, the proposal is computationally more scalable because typical multi-output Gaussian processes have maximum \(O(n^3k^3)\) complexity.

There are a few limitations of this work. One disadvantage is the computational cost even though it increases linearly with \(O(nk^3)\). The proposed model incorporates more data from all bands and involves more parameters to model their dependence. Thus, the resulting computational cost is inevitably large. For example, the CPU time taken for running the five-band simulation in Section 5.1 is about 26 hr, that for analyzing the SDSS five-band data in Section 5.2 is about 100 hr, and that for fitting the time delay model to the Q0957+561 data in Section 5.3 is about 9 hr on a personal laptop. Streamlining the code implementation, e.g., via Stan (Carpenter et al. 2017) is necessary to make it computationally less burdensome for fitting bigger data of large-scale surveys. Another limitation is that the code for fitting the proposed model is currently available only in R (R Development Core Team 2018), although astronomers are more familiar with Python. Thus, we plan to collaborate with bilingual astronomers or statisticians either to translate the R code to Python code or to develop a wrapper. This may also help to reduce the computational cost.

In addition, it is unclear whether the proposed multivariate damped random walk is vulnerable to well-known limitations of the univariate damped random walk. For example, Mushotzky et al. (2011) and Zu et al. (2013) report empirical evidence that the damped random walk process fails to describe the optical variability of a quasar on a very short timescale. Graham et al. (2014) and Kasliwal et al. (2015) echo their arguments that not all types of AGN variabilities can be described by the damped random walk process. Kozłowski (2016) warns of potential biases in associations between model parameters and physical properties when true underlying processes are not damped random walks. Also, Kozłowski (2017) points out that a timescale estimate can be severely biased if a survey length is shorter than ten times the unknown true decorrelation timescale, which is typically the case in most surveys. Looking into the limiting behavior of the process, Tak et al. (2017b) demonstrate that it fails to fit AGN light curves if the timescale of AGN variability is much smaller than the typical observation cadence in the data. (See also Kozłowski 2017 for simulation studies on this issue.)

It is unclear whether these limitations are equally applicable to a multivariate case. This is mainly because various factors, such as the data quality in each band, can play a role in determining the accuracy of the resulting parameter estimation. In this work, for example, the dependences among multiple bands are modeled by their cross-correlation parameters. Thus, if the data quality is not good enough to estimate these correlations accurately, the proposal may not benefit from the idea of dependent modeling, i.e., sharing information across bands. An extensive simulation study that controls various data quality factors in realistic settings (e.g., Kozłowski 2017) is necessary for understanding the practical limitations of the proposal. We leave this as a future direction to explore.
Several more opportunities to build upon the current work exist. (i) Since multimodality is common in the estimation of time delay, it is promising to incorporate a multimodal Markov chain Monte Carlo sampler. For example, a repelling–attracting Metropolis algorithm has been successfully applied to a single-band time delay model (Tak et al. 2018b). (ii) Next, the proposed model is stationary, meaning that it does not account for outlying observations. Tak et al. (2019) demonstrate that the parameter estimation of a univariate damped random walk process can be severely biased in the presence of a few outliers. As an easy-to-implement solution, they introduce a computational trick that turns the Gaussian measurement errors to Student’s $t$-errors, leading to more robust and accurate inferences. It will be a great improvement if this trick is incorporated into the proposed model fitting procedure. (iii) Also, improving the convergence rate of a Markov chain is important for enhancing computational efficiency. For a heteroscedastic model, it is theoretically shown that a specific data augmentation scheme can expedite the convergence rate significantly (Tak et al. 2020). Although this scheme has not been applied to heteroscedastic time-series data, it is well worth investigating. (iv) Finally, it is necessary to generalize the proposed multiband model further. This is because the current model can describe only a specific type of variability defined by a damped random walk process, i.e., CARMA(1, 0). In a univariate case, however, a general-order CARMA($p$, $q$) model has been widely used due to limitations of the damped random walk process (Kelly et al. 2014; Moreno et al. 2019). Thus, more flexible modeling of AGN variability will be possible by using the general-order multiband CARMA($p$, $q$) model (Marquardt & Stelzer 2007; Schlemm & Stelzer 2012). We note that it is crucial to carefully design the state-space representation of the multiband CARMA($p$, $q$) to account for the heteroscedasticity of astronomical time-series data. We invite interested readers to explore these possibilities.

7. Concluding Remarks

In the era of astronomical big data with large-scale surveys, such as SDSS and LSST, it is important to possess various data analytic tools to handle the resulting multiband data. There are possibly many existing tools that can be more powerful than the one presented in this work if a user is fully aware of their strong and weak points. For example, if fitting smooth curves to multiband time-series data is of interest, various non-parametric curve-fitting methods such as kernel smoothing, spline, wavelet, and local polynomial regression (Loader 1999) are available. However, these non-parametric tools may not necessarily have interpretability that a parametric model can provide. A multivariate GP regression is a flexible way to model various types of variability, e.g., incorporating a quasi-periodic aspect of the Doppler shift in a covariance function (Jones et al. 2017). However, it is well known that a GP regression often involves a prohibitive computational cost because it requires taking an inverse of a covariance matrix. The tool we present in this work will be useful for modeling stochastic variability in irregularly-spaced astronomical time-series data with heteroscedastic measurement errors, though it is not a panacea as described in Section 6. We hope this work will initiate more active methodological research and discussion on multiband data analyses in preparation for the upcoming era of LSST-driven time-domain astronomy.

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**ORCID iDs**

Hyungsuk Tak @ https://orcid.org/0000-0003-0334-8742

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