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On uniqueness of large solutions of nonlinear parabolic equations in nonsmooth domains

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Abstract We study the existence and uniqueness of the positive solutions of the problem (P):
\[ \partial_t u - \Delta u + u^q = 0 \quad (q > 1) \] in \( \Omega \times (0, \infty) \), \( u = \infty \) on \( \partial \Omega \times (0, \infty) \) and \( u(., 0) \in L^1(\Omega) \), when \( \Omega \) is a bounded domain in \( \mathbb{R}^N \). We construct a maximal solution, prove that this maximal solution is a large solution whenever \( q < N/(N - 2) \) and it is unique if \( \partial \Omega = \partial \Omega_c \). If \( \partial \Omega \) has the local graph property, we prove that there exists at most one solution to problem (P).

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1 Introduction

Let \( q > 1 \) and let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with boundary \( \partial \Omega := \Gamma \). It has been proved by Keller [5] and Osserman [11] that there exists a maximal solution \( \overline{u} \) to the stationary equation

\[ -\Delta u + |u|^{q-1}u = 0 \quad \text{in} \ \Omega. \] (1.1)

When \( 1 < q < N/(N - 2) \) this maximal solution is a large solution in the sense that

\[ \lim_{\rho(x) \to 0} \overline{u}(x) = \infty \] (1.2)

where \( \rho(x) = \text{dist} (x, \partial \Omega) \). Furthermore Véron proves in [12] that \( \overline{u} \) is the unique large solution whenever \( \partial \Omega = \partial \Omega_c \). When \( q \geq N/(N - 2) \) his proof of uniqueness does not apply. Marcus and Véron prove in [7] that, there exists at most one large solution, provided \( \partial \Omega \) is locally the graph of a continuous function. The aim of this article is to extend these questions to the parabolic equation

\[ \partial_t u - \Delta u + |u|^{q-1}u = 0 \quad \text{in} \ \Omega \times (0, \infty). \] (1.3)

We are interested into positive solutions which satisfy

\[ \lim_{t \to 0} u(., t) = f \quad \text{in} \ L^1_{\text{loc}}(\Omega), \] (1.4)
where \( f \in L^1_{\text{loc}}(\Omega) \) and

\[
\lim_{(x,t) \to (y,s)} u(x,t) = \infty \quad \forall (y,s) \in \Gamma \times (0,\infty). \tag{1.5}
\]

Notice that if the initial and boundary conditions are exchanged, i.e. \( u(.,t) \) blows-up when \( t \to 0 \) and coincides with a locally integrable function on \( \Gamma \times (0,\infty) \), this problem is associated with the study of the initial trace, and much work has been done by Marcus and Véron [9] in the case of a smooth domain. In particular they obtain the existence and uniqueness when \( q \) is subcritical, i.e. \( 1 < q < 1 + 2/N \).

In this article we prove two series of results:

**Theorem A** Assume \( q > 1 \) and \( \Omega \) is a bounded domain. Then for any \( f \in L^1_{\text{loc}}(\Omega) \) there exists a maximal solution \( \overline{u}_f \) to problem (2.5) satisfying (1.4). If \( 1 < q < N/(N-2) \), \( \overline{u}_f \) satisfies (1.5). At end, if \( 1 < q < N/(N-2) \) and \( \partial \Omega = \partial \Omega^c \), \( \overline{u}_f \) is the unique solution of the problem which satisfies (1.5).

The proof of uniqueness is based upon the construction of self-similar solutions of (2.5) in \( \mathbb{R}^N \setminus \{0\} \times (0,\infty) \), with a persistent strong singularity on the axis \( \{0\} \times (0,\infty) \) and a zero initial trace on \( \mathbb{R}^N \setminus \{0\} \). This solution, which is studied in Appendix, is reminiscent of the very singular solution of Brezis, Peletier and Terman [2], although the method of construction is far different. The uniqueness is a delicate adaptation to the parabolic framework of the proof by contradiction of [12].

**Theorem B** Assume \( q > 1 \), \( \Omega \) is a bounded domain and \( \partial \Omega^c \) is locally a continuous graph. Then for any \( f \in L^1_{\text{loc}}(\Omega) \) there exists at most one solution to problem (2.5) satisfying (1.4) and (1.5).

For proving this result, we adapt the idea which was introduced in [7] of constructing local super and subsolutions by small translations of the domain, but the non-uniformity of the boundary blow-up creates an extra-difficulty. In an appendix we study a self-similar equation which plays a key-role in our construction,

\[
\begin{cases}
H'' + \left( \frac{N-1}{r} + \frac{r}{2} \right) H' + \frac{1}{q-1} H - |H|^{q-1} = 0 \\
\lim_{r \to 0} H(r) = \infty \tag{1.6} \\
\lim_{r \to \infty} r^{2/(q-1)} H(r) = 0.
\end{cases}
\]

We prove the existence and the uniqueness of the positive solution of (1.6) when \( 1 < q < N/(N-2) \) and we give precise asymptotics when \( r \to 0 \) and \( r \to \infty \).

This article is organised as follows: 1- Introduction. 2- The maximal solution 3- The case \( 1 < q < N/(N-2) \). 4- The local continuous graph property. 5- Appendix.

## 2 The maximal solution

In this section \( \Omega \) is an open domain of \( \mathbb{R}^N \), with a compact boundary \( \Gamma := \partial \Omega \). If \( G \) is any open subset of \( \mathbb{R}^N \) and \( 0 < T \leq \infty \), we denote \( G_T^G := G \times (0,T) \). If \( f \in L^1_{\text{loc}}(\Omega) \), we
consider the problem
\[
\begin{aligned}
\partial_t u - \Delta u + |u|^{q-1}u &= 0 \quad \text{in } Q^\Omega_{\infty} \\
\lim_{t \to 0} u(., t) &= f(.,) \quad \text{in } L^1_{\text{loc}}(\Omega) \\
\lim_{(x,t) \to (y,s)} u(x, t) &= \infty \quad \forall (y, s) \in \Gamma \times (0, \infty).
\end{aligned}
\] (2.1)

By the next result, we reduce the lateral blow-up condition by a locally uniform one in which we set \(\rho(x) = \text{dist}(x, \Gamma)\).

**Lemma 2.1** The following two conditions are equivalent
\[
\lim_{(x,t) \to (y,s)} u(x, t) = \infty \quad \forall (y, s) \in \Gamma \times (0, \infty) \quad (2.2)
\]
and
\[
\lim_{\rho(x) \to 0} u(x, t) = \infty \quad \text{uniformly on } [\tau, T], \quad (2.3)
\]
for any \(0 < \tau < T < \infty\).

**Proof.** It is clear that (2.3) is equivalent to the fact that (2.2) holds uniformly on \(\Gamma \times [\tau, T]\). By contradiction, we assume that (2.2) does not hold uniformly for some \(T > \tau > 0\). Then there exists \(\beta > 0\) such that for any \(\delta > 0\), there exist two couples \((y_\delta, s_\delta) \in \Gamma \times [\tau, T]\) and \((x_\delta, t_\delta) \in \Omega \times [\tau, T]\) such that
\[
|x_\delta - y_\delta| + |t_\delta - s_\delta| \leq \delta \quad \text{and} \quad u(x_\delta, t_\delta) \leq \beta. \quad (2.4)
\]
Taking \(\delta = 1/n, n \in \mathbb{N}^*\), we can assume that \(\{\delta\}\) is discrete and that \(y_\delta \to y \in \Gamma\) and \(s_\delta \to s \in [\tau, T]\). Thus \(x_\delta \to y\) and \(t_\delta \to s\). Therefore (2.4) contradicts (2.2). \(\square\)

**Theorem 2.2** For any \(q > 1\) and \(f \in L^1_{\text{loc}}(\Omega)\), there exists a maximal solution \(u := \overline{u}_f\) of
\[
\partial_t u - \Delta u + |u|^{q-1}u = 0 \quad \text{in } Q^\Omega_{\infty} \quad (2.5)
\]
which satisfies
\[
\lim_{t \to 0} u(., t) = f(.,) \quad \text{in } L^1_{\text{loc}}(\Omega). \quad (2.6)
\]

**Proof.** Let \(\Omega_n\) be an increasing sequence of smooth bounded domains such that \(\overline{\Omega}_n \subset \Omega_{n+1} \subset \Omega\) and \(\cup \Omega_n = \Omega\). For each \(n\) let \(u_{n,f}\) be the increasing limit when \(k \to \infty\) of the \(u_{n,k,f}\) solution of
\[
\begin{aligned}
\partial_t u_{n,k, f} - \Delta u_{n,k, f} + u_{n,k, f}^q &= 0 \quad \text{in } Q^\Omega_{\infty}^{2n} \\
\text{\(u_{n,k, f}(x, t) = k\)} &\quad \text{in } \partial\Omega_n \times (0, \infty) \\
\text{\(u_{n,k, f}(x, 0) = f_{\chi \Omega_n}\)} &\quad \text{in } \Omega_n.
\end{aligned} \quad (2.7)
\]
By the maximum principle and a standard approximation argument \(n \to u_{n,k,f}\) is decreasing thus \(n \to u_{n,f}\) too. The limit \(\overline{u}_f\) of the \(u_{n,f}\) satisfies (2.5) and (2.6). It is independent of the exhaustion \(\{\Omega_n\}\) of \(\Omega\). Let \(u\) be a positive solution of (2.5) in \(Q^\Omega_{\infty}\) which satisfies (2.6). Since the initial trace of \(u\) is a locally integrable function, \(u^q \in L^1_{\text{loc}}(\Omega \times [0, \infty))\). By
Fubini we can assume that, for any \( n, u \in L^1_{loc}(\partial \Omega_n \times [0, \infty)) \). Because \((u - u_{n,k,f})_+ \leq u\) and tends to 0 when \( k \to \infty \), it follows by Lebesgue’s theorem that
\[
\lim_{k \to \infty} \|(u - u_{n,k,f})_+\|_{L^1(\partial \Omega_n \times (0,T))} = 0 \quad \forall T > 0.
\]
Applying the maximum principle in \( \Omega_n \times (0, \infty) \) yields to
\[
u \leq \lim_{k \to \infty} u_{n,k,f} = u_{n,f} \implies u \leq \lim_{n \to \infty} u_{n,f} = \pi_f.
\]

**Theorem 2.3** For any \( q > 1 \) and \( f \in L^1_{loc}(\Omega) \), there exists a minimal nonnegative solution \( \pi_f \) of (2.5) in \( Q^0_\infty \) which satisfies (2.6).

**Proof.** The scheme of the construction is similar to the one of \( \pi_f \): with the same exhaustion \( \{\Omega_n\} \) of \( \Omega \), we consider the solution \( u_{n,0,f} \) solution of
\[
\begin{aligned}
\partial_t u_{n,0,f} - \Delta u_{n,0,f} + u_{n,0,f}^q &= 0 \quad \text{in } Q^0_\infty \Omega_n, \\
u_{n,0,f}(x,t) &= 0 \quad \text{in } \partial \Omega_n \times (0, \infty) \quad (2.8) \\
u_{n,0,f}(x,0) &= f\chi_{\Omega_n} \quad \text{in } \Omega_n.
\end{aligned}
\]
By the maximum principle, \( n \mapsto u_{n,0,f} \) is increasing and dominated by \( \pi_f \). Therefore it converges to some solution \( \pi_f \) of (2.5), which satisfies (2.6) as \( u_{n,0,f} \) and \( \pi_f \) do it. Using the same argument as in the proof of Theorem 2.2 there holds \( u_{n,0,f} \leq u \) in \( Q^0_\infty \) for a suitable exhaustion. Thus \( \pi_f \leq u \). \( \square \)

**Remark.** Because of the lack of regularity of \( \partial \Omega \), there is no reason for \( \pi_f \) (resp \( u_f \)) to tend to infinity (resp. zero) on \( \partial \Omega \times (0, \infty) \).

The next statement will be very useful for proving uniqueness results.

**Theorem 2.4** Assume \( q > 1 \), \( f \in L^1_{loc}(\Omega) \) and \( u_f \) is a nonnegative solution of (2.5) satisfying (2.6). Then there exists a nonnegative solution \( u_0 \) of (2.5) satisfying
\[
\lim_{t \to 0} u_0(.,t) = 0 \quad \text{in } L^1_{loc}(\Omega),
\]
such that
\[
0 \leq u_f - u_0 \leq u_0 \leq u_f, \quad (2.9)
\]
and
\[
0 \leq \pi_f - u_f \leq \pi_f - u_0. \quad (2.10)
\]

**Proof.** Step 1: construction of \( u_0 \). The function \( w = u_f - u_0 \) is a nonnegative subsolution of (2.5) which satisfies
\[
\lim_{t \to 0} w(.,t) = 0 \quad \text{in } L^1_{loc}(\Omega).
\]
Using the above considered exhaustion of \( \Omega \), we denote by \( v_n \) the solution of
\[
\begin{aligned}
\partial_t v_n - \Delta v_n + v_n^q &= 0 \quad \text{in } Q^0_\infty \Omega_n, \\
v_n(x,t) &= u_f - u_0 \quad \text{in } \partial \Omega_n \times (0, \infty) \quad (2.11) \\
v_n(x,0) &= 0 \quad \text{in } \Omega_n.
\end{aligned}
\]

By the maximum principle

\[ u_f - u_f \leq v_n \leq u_f \quad \text{in} \ Q_\infty^\Omega. \]

Therefore \( v_{n+1} \geq v_n \) on \( \partial \Omega \times (0, \infty) \); this implies that the same inequality holds in \( Q_\infty^\Omega \). If we denote by \( u_0 \) the limit of the \( \{v_n\} \), it is a solution of (2.5) in \( Q_\infty^\Omega \). For any compact \( K \in \Omega \), there exists \( n_K \) and \( \alpha > 0 \) such that \( \text{dist}(K, \Omega_n^c) \geq \alpha \) for \( n \geq n_K \) therefore \( v_n \) remains uniformly bounded on \( K \) by Brezis-Friedman estimate [3]. Thus the local equicontinuity of the \( v_n \) (consequence of the regularity theory for parabolic equations) implies that \( u_0 \) satisfies (2.9).

Step 2: proof of (2.11). We follow a method introduced in [8] in a different context. For \( n \in \mathbb{N} \) and \( k > 0 \) fixed, we set

\[ Z_{f,n} = u_{f,n} - u_f \quad \text{and} \quad Z_{0,n} = u_{0,n} - u_0, \]

where we assume that the \( n \) are chosen such that \( u_f, u_0 \in L^1_{\text{loc}}(\partial \Omega_n \times [0, \infty)) \), and

\[ \phi(r, s) = \begin{cases} 
    r^q - s^q & \text{if } r \neq s \\
    0 & \text{if } r = s.
\end{cases} \]

By convexity,

\[ \begin{cases} 
    r_0 \geq s_0, \ r_1 \geq s_1 \\
    r_1 \geq r_0, \ s_1 \geq s_0
\end{cases} \implies \phi(r_1, s_1) \geq \phi(r_0, s_0).
\]

Therefore

\[ \phi(u_{f,n}, u_f) \geq \phi(u_{0,n}, u_0) \quad \text{in} \ Q_\infty^\Omega, \]

and

\[ 0 = \partial_t(Z_{f,n} - Z_{0,n}) - \Delta(Z_{f,n} - Z_{0,n}) + u^q_{f,n} - u^q_f - u^q_{0,n} + u^q_0 \\
= \partial_t(Z_{f,n} - Z_{0,n}) - \Delta(Z_{f,n} - Z_{0,n}) + \phi(u_{f,n}, u_f)(Z_{f,n} - Z_{f,n} - \phi(u_{0,n}, u_0)Z_{0,n}, \]

which implies

\[ \partial_t(Z_{f,n} - Z_{0,n}) - \Delta(Z_{f,n} - Z_{0,n}) + \phi(u_{f,n}, u_f)(Z_{f,n} - Z_{0,n}) \leq 0. \]

But \( Z_{f,n} - Z_{0,n} = 0 \) in \( \Omega_n \times \{0\} \) and

\[ \int_0^\infty \int_{\partial \Omega_n} |Z_{f,n} - Z_{0,n}| dS dt = 0 \]

by approximations. By the maximum principle \( Z_{f,n,k} - Z_{0,n,k} \leq 0 \). Letting \( n \to \infty \) yields to

\[ u_f - u_f \leq u_0 - u_0, \]

which ends the proof. \( \square \)
3 The case $1 < q < N/(N - 2)$

In this section we assume that $\Omega$ is a domain of $\mathbb{R}^N$ with a compact boundary. We first prove that the maximal solution is a large solution

**Theorem 3.1** Assume $1 < q < N/(N - 2)$ and $f \in L^1_{\text{loc}}(\Omega)$. Then the maximal solution $\overline{u}_f$ of (2.5) in $Q^1_f$ which satisfies (2.6) satisfies also (2.3).

**Proof.** In Appendix we construct the self-similar solution $V := V_N$ of (2.5) in $Q^\infty_f \setminus \{0\}$ which has initial trace zero in $\mathbb{R}^N \setminus \{0\}$ and satisfies

$$\lim_{|x| \to 0} V_N(x,t) = \infty,$$

locally uniformly on $[\tau, \infty)$, for any $\tau > 0$. Furthermore $V_N(x,t) = t^{-1/(q-1)} H_N(|x|/\sqrt{t})$. If $a \in \partial \Omega$, the restriction to $\Omega_a$ of the function $V_N(x-a,t)$ is bounded from above by $u_{n,f}$.

Letting $n \to \infty$ yields to

$$V_N(x-a,t) \leq \overline{u}_f(x,t) \quad \forall (x,t) \in Q^\infty_f. \quad (3.1)$$

If we consider $x \in \Omega$ and denote by $a_x$ a projection of $x$ onto $\partial \Omega$, there holds

$$t^{-1/(q-1)} H_N(\rho(x)/\sqrt{t}) = V_N(x-a_x,t) \leq \overline{u}_f(x,t). \quad (3.2)$$

Using (5.2), we derive that $\mathcal{P}_f$.

**Theorem 3.2** Assume $1 < q < N/(N - 2)$, $f \in L^1_{\text{loc}}(\Omega)$ and $\partial \Omega = \partial \mathcal{M}^f$. Then $\overline{u}_f$ is the unique solution of (2.5) in $Q^1_f$ which satisfies (2.6) and (2.3).

**Proof.** Assume that $u_f$ is a solution of (2.5) in $Q^1_f$ such that (2.6) and (2.3) hold. By Theorem 2.4 there exists a positive solution $u_0$ with zero initial trace such that

$$0 \leq u_f - u_0 \leq \overline{u}_f \quad (3.3)$$

and (2.11) are satisfied. Since $\overline{u}_f(x,t) \leq ((q-1)t)^{-1/(q-1)}$ (notice that this last expression is the maximal solution of (2.5) in $Q^\infty_{\mathcal{M}}$), the function $u_0$ satisfies also (2.3). Therefore, it is sufficient to prove that $\mathcal{P}_0 = u_0 := u$.

**Step 1: bilateral estimates.** Since $\partial \Omega = \partial \mathcal{M}^f$, for any $a \in \partial \Omega$, there exists a sequence $\{a_n\} \subset \mathcal{M}^f$ converging to $a$. If $u$ is any solution of (2.5) in $Q^1_f$ which satisfies (2.3) and (2.9), there holds

$$V_N(x-a_n,t) \leq u(x,t) \implies V_N(x-a,t) \leq u(x,t).$$

In particular, if $a = a_x$, we see that $u$ satisfies (3.2). In order to obtain an estimate from above we consider for $r < \rho(x)$ the solution $(y,t) \mapsto u_{x,r}(y,t)$ of

$$\begin{cases}
\partial_t u_{x,r} - \Delta u_{x,r} + u_{x,r}^q = 0 & \text{in } Q^R_{\mathcal{M}}(x) \\
|y,t| \to (z,0) u_{x,r}(y,t) = 0 & \forall z \in B_r(x) \\
|z| \to \infty u_{x,r}(x,t) = \infty & \text{locally uniformly on } [\tau, \infty), \text{for any } \tau > 0
\end{cases} \quad (3.4)$$

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Then
\[ \varpi_0(y, t) \leq u_{x,r}(y, t) \iff \varpi_0(y, t) \leq u_{x,\rho(x)}(y, t) \quad \forall (y, t) \in Q^B_{\rho(x)}(x). \]

In particular, with \( u_{0,r} = u_r \),
\[ \varpi_0(x, t) \leq u_{\rho(x)}(0, t) = (\rho(x))^{-2/(q-1)}u_1(0, t/(\rho(x))^2). \]

Therefore
\[ t^{-1/(q-1)}H_N(\rho(x)/\sqrt{t}) \leq u(x, t) \leq \varpi_0(x, t) \leq (\rho(x))^{-2/(q-1)}u_1(0, t/(\rho(x))^2). \quad (3.5) \]

The function \( s \mapsto u_1(0, s) \) is increasing by the same argument as the one of Corollary 1.3 and bounded from above by the unique solution \( P \) of
\[ \begin{cases} -\Delta P + P^q = 0 & \text{in } B_1 \\ \lim_{|x| \to 1} P(x) = \infty. \end{cases} \quad (3.6) \]

Therefore it converges to \( P \) locally uniformly in \( B_1 \) and \( \lim_{s \to \infty} u_1(0, s) = P(0) \). Thus
\[ t/(\rho(x))^2 \to \infty \implies (\rho(x))^{-2/(q-1)}u_1(0, t/(\rho(x))^2) \approx P(0)(\rho(x))^{-2/(q-1)}. \quad (3.7) \]

On the other hand, if \( t/(\rho(x))^2 \to \infty \), equivalently \( \rho(x)/\sqrt{t} \to 0 \),
\[ t^{-1/(q-1)}H_N(\rho(x)/\sqrt{t}) \approx \lambda_{N,q}t^{-1/(q-1)}(\rho(x)/\sqrt{t})^{-2(q-1)} = \lambda_{N,q}(\rho(x))^{-2/(q-1)}, \quad (3.8) \]
by (5.4).

Next, in order to obtain an estimate from above of \( u_1(0, s) \) when \( s \to 0 \), we compare \( u_1 \) to a solution \( u_\Theta \) of (2.5) in \( Q^B_{\infty} \), where \( \Theta \) is a polyhedra inscribed in \( B_1 \); this polyhedra is a finite intersection of half spaces \( \Gamma_i \) containing \( \Pi \). In each of the half space \( \Gamma_i \), with boundary \( \gamma_i \), we can consider the solution \( W_1 \) of (2.5) in \( Q^B_{\infty} \) which tends to infinity on \( \gamma_i \times (0, \infty) \) and has value 0 on \( \Gamma_i \times \{0\} \). This solution depends only on the distance to \( \gamma_i \) and \( t \). Thus it is expressed by the function \( V_1 \) defined in Proposition 5.4 when \( N = 1 \). Moreover, since a sum of solutions is a super solution,
\[ u_1 \leq u_\Theta \leq \sum_i W_i \iff u_1(0, s) \leq \sum_i H_1(\text{dist}(0, \gamma_i)/\sqrt{s}). \quad (3.9) \]

We can choose the hyperplanes \( \gamma_i \) such that for any \( \delta \in (0, 1) \), there exists \( C_\delta \in \mathbb{N} \) such that
\[ u_1(0, s) \leq C_\delta H_1((1 - \delta)/\sqrt{s}). \quad (3.10) \]

Using (5.3) we derive
\[ u(x, t) \geq c_N(\rho(x))^{2/(q-1)-N/2(1/(q-1))}e^{-(\rho(x))^2/4t}, \]
when \( \rho(x)/\sqrt{t} \to \infty \), and
\[ \varpi_0(x, t) \leq CH_1((1-\delta)\rho(x)/\sqrt{t}) \leq C(1-\delta)^{2/(q-1)-1}(\rho(x))^{2/(q-1)-1}t^{1/2-1/(q-1)}e^{-(1-\delta)\rho(x)^2/4t}. \]
Therefore, there exists $\theta > 1$ such that
\[
\overline{u}_0(x, t) \leq C(\rho(x))^{2/(q-1) - N} t^{N/2 - 1/(q-1)} e^{-(\rho(x))^2/4\theta t} \leq Cu(x, \theta t),
\]  
(3.11)
when $\rho(x)/\sqrt{t} \to \infty$. Finally, when $m^{-1} \leq \rho(x)/\sqrt{t} \leq m$ for some $m > 1$, (3.5) shows that $\rho(x)^{-2/(q-1)} u(0, t/(\rho(x))^2)$ and $t^{-1/(q-1)} H_N(\rho(x)/\sqrt{t})$ are comparable. In conclusion, there exist constants $C > P(0)/\lambda_{N,q} > 1$ and $\theta > 1$ such that
\[
u(x, t) \leq \overline{u}_0(x, t) \leq C u(x, \theta t) \quad \forall (x, t) \in Q^\Omega_\infty.
\]  
(3.12)

**Step 2: End of the proof.** Let $\tau > 0$ and $C' > C$ be fixed. The function
\[
t \mapsto u_r(x, t) := C' u(x, t + \theta \tau)
\]  
is a supersolution of (2.5) in $\Omega \times (0, \infty)$ which satisfies $u_r(x, 0) = C' u(x, \theta \tau) > \overline{u}_0(x, \tau)$ by (3.1). Furthermore,
\[
C' u(x, t + \theta \tau) \geq C'(t + \theta \tau)^{-1/(q-1)} H_N(\rho(x)/\sqrt{t + \theta \tau}) = C' \lambda_{N,q} (1 + o(1)) (\rho(x))^{-2/(q-1)},
\]  
as $\rho(x) \to 0$, locally uniformly for $t \in [0, \infty)$. Similarly,
\[
\overline{u}_0(x, t + \tau) \leq (\rho(x))^{-2/(q-1)} u_0(0, t + \tau)/\rho(x)^2) = P(0) (1 + o(1)) (\rho(x))^{-2/(q-1)},
\]  
as $\rho(x) \to 0$, and also locally uniformly for $t \in [0, \infty)$. Therefore $(\overline{u}_0(x, t) - u_r(x, t))_+$ vanishes in a neighborhood of $\partial \Omega \times [0, T]$ for any $T > 0$. By the maximum principle
\[
u(x, t) \geq \overline{u}_0(x, t) \quad \forall (x, t) \in \Omega \times (0, \infty).
\]  
Letting $\tau \to 0$ and $C' \to C$ yields to
\[
u(x, t) \leq \overline{u}_0(x, t) \leq C u(x, t) \quad \forall (x, t) \in Q^\Omega_\infty.
\]  
(3.13)
The conclusion of the proof is contradiction, following an idea introduced in [12] and developed by [12] in the elliptic case. We assume $u \neq \overline{u}_0$, thus $u < \overline{u}_0$. By convexity the function
\[
w = u - \frac{1}{2C}(\overline{u}_0 - u)
\]  
is a supersolution and $w < u$. Moreover $w > w' := ((1 + C)/2C) u$ and $w'$ is a subsolution. Consequently, there exists a solution $u_1$ of (2.5) which satisfies
\[
w' < u_1 \leq w \implies \overline{u}_0 - u_1 \geq (1 + K^{-1})(\overline{u}_0 - u) \quad \text{in } Q^\Omega_\infty.
\]  
(3.14)
Notice that $u_1$ satisfies (2.9) and (2.3), therefore it satisfies (3.13) as $u$ does it. Replacing $u$ by $u_1$ and introducing the supersolution
\[
w_1 = u_1 - \frac{1}{2C}(\overline{u}_0 - u_1)
\]  
and the subsolution $w'_1 := ((1 + C)/2C)u_1$ we see that there exists a solution $u_2$ of (2.5) such that
\[
w'_1 < u_2 \leq w_1 \implies \overline{u}_0 - u_2 \geq (1 + K^{-1})^2 (\overline{u}_0 - u) \quad \text{in } Q^\Omega_\infty.
\]  
(3.15)
By induction, we construct a sequence of positive solutions $u_k$ of (2.5), subject to (2.9) and (2.3) such that
\[ u_0 - u_k \geq (1 + K^{-1})^k (u_0 - u) \text{ in } Q^\Omega_\infty. \] (3.16)
This is clearly a contradiction since $(1 + K^{-1})^k \to \infty$ as $k \to \infty$ and $u_0$ is locally bounded in $Q^\Omega_\infty$. □

4 The local continuous graph property

In this section, we assume that $\partial \Omega$ is compact and is locally the graph of a continuous function, which means that there exists a finite number of open sets $\Omega_j$ ($j = 1, ..., k$) such that $\Gamma \cap \Omega_j$ is the graph of a continuous function. Our main result is the following

**Theorem 4.1** Assume $q > 1$ and $f \in L^1_{\text{loc}}(\Omega)$. Then there exists at most one positive solution of (2.5) in $Q^\Omega_\infty$ satisfying (2.6) and (2.3).

Suppose $u_f$ satisfies (2.5) in $Q^\Omega_\infty$ satisfying (2.6) and (2.3), then clearly the maximal solution $u_f$ endows the same properties. In order to prove that $u_f = u_f$, we can assume that $f = 0$ by Theorem 2.4. We denote by $u$ this large solution with zero initial trace. We consider some $j \in \{1, ..., k\}$, perform a rotation, denote by $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ the coordinates in $\mathbb{R}^N$ and represent $\Gamma \cap \Omega_j$ as the graph of a continuous positive function $\phi$ defined in $C = \{x' \in \mathbb{R}^{N-1} : |x'| \leq R\}$. We identify $C$ with $\{x = (x', 0) : |x'| \leq R\}$ and set
\[ G_R = \{x \in \mathbb{R}^N : |x'| < R, 0 < x_N < \phi(x')\}. \]
We can assume that $\overline{G}_R \subset \Omega \cup \Gamma_1$,
\[ \inf\{\phi(x') : x' \in C\} = R_0 > 0 \quad \text{and} \quad \sup\{\phi(x') : x' \in C\} = R_1 > R_0. \]
For $\sigma > 0$, small enough, we consider $\phi_\sigma \in C^\infty(C)$ satisfying
\[ \phi(x') - \sigma/2 \leq \phi_\sigma(x') \leq \phi(x') + \sigma/2 \quad \forall x' \in C, \]
and set
\[ G_{\sigma,R} = \{x \in \mathbb{R}^N : |x'| < R, 0 < x_N < \phi_\sigma(x') - \sigma\} \]
and
\[ G'_{\sigma,R} = \{x \in \mathbb{R}^N : |x'| < R, 0 < x_N < \phi_\sigma(x') + \sigma\}. \]
The upper boundaries of $G_\sigma$ and $G'_\sigma$ are defined by
\[ \Gamma_{1,\sigma} = \{x = (x', \phi_\sigma(x') - \sigma) : x' \in C\}, \]
\[ \Gamma'_{1,\sigma} = \{x = (x', \phi_\sigma(x') + \sigma) : x' \in C\}, \]
\[ \Gamma_{2,\sigma} = \{x = (x', \phi_\sigma(x')) : x' \in C\}. \]
and the remaining boundaries are
\[ \Gamma_{2,\sigma} = \{ x = (x', x_N) : x' \in \partial C, 0 \leq x_N \leq \phi_\sigma(x') - \sigma \}, \]
\[ \Gamma'_{2,\sigma} = \{ x = (x', x_N) : x' \in \partial C, 0 \leq x_N \leq \phi_\sigma(x') + \sigma \}. \]

In order to have the monotonicity of the domains, we can also assume
\[ \phi_\sigma(x') - \sigma < \phi_{\sigma'}(x') - \sigma' < \phi_\sigma(x') + \sigma' < \phi_{\sigma'}(x') + \sigma \quad \forall 0 < \sigma' < \sigma \quad \forall x' \in C, \quad (4.1) \]

thus, under the condition \(0 < \sigma' < \sigma\),
\[ G_{\sigma,R} \subset G_{\sigma',R} \subset G_{\sigma,R} \subset G'_{\sigma,R}. \quad (4.2) \]

The localization procedure is to consider the restriction of \( u \) to \( Q^{G_R}_{\infty} := G_R \times (0, \infty) \), thus \( u \) is regular in \( G_R \cup \Gamma_2 \times [0, \infty) \) and satisfies
\[ \lim_{x_N \to \phi(x')} u(x', x_N, t) = \infty, \quad (4.3) \]
uniformly with respect to \((x', t) \in C \times [\tau, T]\), for any \(0 < \tau < T\). We construct \( v_\sigma \) as solution of
\[ \partial_t v_\sigma - \Delta v_\sigma + v_\sigma^q = 0 \quad \text{in} \quad Q^{G_\sigma}_{\infty} := G_\sigma, R \times (0, \infty), \quad (4.4) \]
subject to the initial condition
\[ \lim_{t \to 0} v_\sigma(x, t) = 0 \quad \text{locally uniformly in} \ G_\sigma, R, \quad (4.5) \]
and the boundary conditions
\[ \lim_{x_N \to \phi_\sigma(x') - \sigma} v_\sigma(x', x_N, t) = \infty \quad \forall (x', t) \in C \times (0, \infty], \quad (4.6) \]
uniformly on any set \( K \times [\tau, T]\), where \( T > \tau > 0 \) and \( K \) is a compact subset of \( C \) and
\[ v_\sigma(x, t) = 0 \quad \forall (x, t) \in \Gamma_2, \sigma \times [0, \infty). \quad (4.7) \]

We also construct \( w_\sigma \) as solution of
\[ \partial_t w_\sigma - \Delta w_\sigma + w_\sigma^q = 0 \quad \text{in} \quad Q^{G_\sigma'}_{T} := G'_{\sigma,R} \times (0, \infty), \quad (4.8) \]
subject to the initial condition
\[ \lim_{t \to 0} w_\sigma(x, t) = 0 \quad \text{locally uniformly in} \ G'_{\sigma,R}, \quad (4.9) \]
and the boundary conditions
\[ \left\{ \begin{array}{ll} (i) & w_\sigma(x, t) = 0 \quad \forall (x, t) \in \Gamma'_{1, \sigma} \times [0, T], \\ (i') & \lim_{(x,s) \to (y,t)} w_\sigma(x, t) = \infty \quad \forall (y, s) \in \Gamma'_{2, \sigma} \times [0, T]. \end{array} \right. \quad (4.10) \]

The functions \( v_\sigma \) and \( w_\sigma \) inherit the following properties in which the local graph property plays a fundamental role, allowing translations of the truncated domains in the \( x_N \)-direction.
Lemma 4.2  For $\sigma > \sigma' > 0$ there holds

$$v_{\sigma'} \leq v_{\sigma} \quad \text{in } Q^G_{\infty, n},$$  \hspace{1cm} (4.11)

$$w_{\sigma'} \leq w_{\sigma} \quad \text{in } Q^G_{\infty, n},$$  \hspace{1cm} (4.12)

(i)  \quad v_{\sigma}(x', x_N - 2\sigma, t) \leq u(x', x_N, t) \quad \text{in } Q^G_{\infty, n}

(ii)  \quad u(x', x_N, t) \leq v_{\sigma}(x, t) + w_{\sigma}(x, t) \quad \text{in } Q^G_{\infty, n}.\hspace{1cm} (4.13)

Proof.  The inequalities (4.11) and (4.12) are the direct consequence of the fact that the domains $G_{\sigma, R}$ and $G'_{\sigma', R}$ are Lipschitz and the functions $v_{\sigma}$ and $w_{\sigma}$ are constructed by approximations of solutions of (2.5) with bounded boundary data. For proving (4.13)-(i), we compare, for $\tau > 0$, $u(x, t - \tau)$ and $v_{\sigma}(x', x_N - 2\sigma, t)$ in $Q^G_{\infty, n}$. Because $u$ satisfies (2.3), and $v_{\sigma}(x', x_N - 2\sigma, 0) = 0$ in $G_R$, (4.13)-(i) follows by the maximum principle. The proof of (4.13)-(ii) needs no translation, but the fact that the sum of two solutions is a supersolution.

Corollary 4.3  There exist $v_0 = \lim_{\sigma \to 0} v_{\sigma}$ and $w_0 = \lim_{\sigma \to 0} w_{\sigma}$ and there holds

$$v_0 \leq u \leq v_0 + w_0 \quad \text{in } Q^G_{\infty, n}.\hspace{1cm} (4.14)$$

Moreover, the functions $t \mapsto v_0(x, t)$ and $t \mapsto w_0(x, t)$ are increasing on $(0, \infty)$, $\forall x \in G_R$.

Proof.  The first assertion follows from (4.11)-(4.12), and (4.14) from (4.13). Since $v_0$ is the limit, when $\sigma \to 0$ of $v_{\sigma}$ which satisfy equation (4.1) in $Q^G_{T, \infty, n}$, initial condition (4.5) and boundary conditions (4.6), (4.7), it is sufficient to prove the monotonicity of $t \mapsto v_{\sigma}(., t)$. Moreover $v_{\sigma}$ is the limit, when $k$ tends to infinity of the $v_{k, \sigma}$ solutions of (2.5) in $Q^G_{T, \infty, n}$, which satisfy the same boundary conditions as $v_{\sigma}$ on $\Gamma_{2, \sigma} \times [0, T]$, the same zero initial condition and

$$\lim_{x_N \to \phi(x') - \sigma} v_{k, \sigma}(x', x_N, t) = k.$$

For $\tau > 0$, we define $V_{\tau}$ by $V_{\tau}(x, t) = (v_{k, \sigma}(x, t) - v_{k, \sigma}(x, t + \tau))\cdot$. Because $\partial G_{\sigma, R}$ is Lipschitz and $V_{\tau}$ is a subsolution of (2.5) which vanishes on $\partial G_{\sigma, R} \times [0, T]$ and at $t = 0$, it is identically zero. This implies $v_{k, \sigma}(x, t) \leq v_{k, \sigma}(x, t + \tau)$, and the monotonicity property of $v_0$, by strict maximum principle and letting $\sigma \to 0$. The proof of the monotonicity of $w_0$ is similar.

The key step of the proof is the following result.

Proposition 4.4  Let $\epsilon, \tau > 0$. Then there exists $\delta_\epsilon > 0$ such that, if we denote

$$G_{\delta, R'} = \{ x \in (x', x_N) : |x' - x| < R' \text{ and } \phi(x') - \delta \leq x_N < \phi(x') \},$$

there holds, for $R' < R/\sqrt{N - 1}$,

$$w_0(x, t) \leq \epsilon v_0(x, t + \tau) \quad \forall (x, t) \in Q^G_{\infty, n}.\hspace{1cm} (4.15)$$
Proof. Using the result in Appendix, we recall that $V := V_1$ is the unique positive and self-similar solution of the problem
\[
\begin{align*}
\partial_t V - \partial_zz V + V^q = 0 & \quad \text{in } \mathbb{R}_+ \times \mathbb{R}_+ \\
\lim_{t \to 0} V(z, t) &= 0 \quad \forall z > 0 \\
\lim_{z \to 0} V(z, t) &= \infty \quad \forall t > 0,
\end{align*}
\]
and it is expressed by $V_1(z, t) = t^{1/(q-1)} H_1(x/\sqrt{t})$, where $H_1$ satisfies (5.2)-(5.3) with $N = 1$. We set $R_N = R/\sqrt{N-1}$ so that
\[
C_\infty := \{x' = (x_1, \ldots, x_{N-1}) : \sup_{j \leq N-1} |x_j| < R_N\} \subset \{x' : |x'| \leq R\}
\]
and we define
\[
\tilde{w}(x, t) = W(x_N, t) + \sum_{j=1}^{N-1} (W(x_j - R, t) + W(R - x_j, t)).
\]

The function $\tilde{w}$ a super solution in $\Theta \times \mathbb{R}_+$ where $\Theta := \{(x', x_N) : x' \in C_\infty, x_N > 0\}$ which blows up on
\[
\{x : x_N = 0, \sup_{j \leq N-1} |x_j| \leq R\} \cup \{x : x_N \geq 0, x_j = \pm R\}.
\]
Therefore $w_0 \leq \tilde{w}$ in $Q^{G_{\infty}}_{T}$. Moreover $\tilde{w}(x, t) \to 0$ when $t \to 0$, uniformly on
\[
G_{\alpha, R'} := \{x = (x_1, x_2) : |x_1| \leq R', \alpha \leq x_2 \leq \phi(x_1)\},
\]
for any $\alpha \in (0, R_0]$ and $R' \in (0, R_N)$. Since for any $\tau > 0$, $v_0(x, t + \tau) \to \infty$ when $\rho(x) \to 0$, locally uniformly on $[0, \infty)$, and $\tilde{w}(x, t)$ remains uniformly bounded on $Q^{G_{\infty}}_{\infty, R'}$, for any $\delta > R_0$, it follows that for any $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that
\[
w_0(x, t) \leq \tilde{w}(x, t) \leq \epsilon v_0(x, t + \tau) \quad \forall (x, t) \in Q^{G_{\infty}}_{\infty, R'}.
\]
\[\square\]

Proof of Theorem 4.1. Assume $u$ is a solution of (2.5) satisfying (2.6) and (2.3). Then there holds in $Q^{G_{\infty}}_{\infty, R'}$,
\[
v_0(\,, t) \leq u(\,, t) \leq v_0(\,, t) + \epsilon v_0(\,, t + \tau).
\]
(4.17)

Therefore
\[
v_0(\,, t + \tau) \leq u(\,, t + \tau) \leq v_0(\,, t + \tau) + \epsilon v_0(\,, t + 2\tau),
\]
from which follows
\[
(1 + \epsilon)u(\,, t + \tau) \geq (1 + \epsilon)v_0(\,, t + \tau) \geq v_0(\,, t) + \epsilon v_0(\,, t + \tau)
\]

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since \( t \mapsto v_0(\cdot, t) \) is increasing by Corollary 4.3. The maximal solution \( \overline{u}_0 \) satisfies (4.17) too; consequently the following inequality is verified in \( Q_{\infty}^{G_{ij}}, w' \),
\[
(1 + \epsilon)u(\cdot, t + \tau) \geq \overline{u}_0(\cdot, t),
\]
(4.18)
Since \( \partial \Omega \) is compact, there exists \( \delta^* > 0 \) such that (4.18) holds whenever \( t \in [0, T] \) \( (T > 0 \) arbitrary) and \( \rho(x) \leq \delta^* \). Furthermore
\[
\lim_{t \to 0} \max_{\partial \Omega} \{ (\overline{u}_0(x, t) - (1 + \epsilon)u(x, t + \tau))_+ : \rho(x) \geq \delta^* \} = 0
\]
because of (2.6). Since \( (\overline{u}_0(x, t) - (1 + \epsilon)u(x, t + \tau))_+ \) is a subsolution, which vanishes at \( t = 0 \) and near \( \partial \Omega \times [0, T] \), it follows that (4.18) holds in \( Q_{\infty}^{G_\tau} \). Letting \( \epsilon \to 0 \) and \( \tau \to 0 \) yields to \( u \geq \overline{u}_0 \).

**Remark.** The existence of large solutions when \( q \geq N/(N - 2) \) is a difficult problem as it is already in the elliptic case. We conjecture that the necessary and sufficient conditions, obtained by Dhersin-Le Gall when \( q = 2 \) and Labutin in the general case \( q > 1 \), and expressed by mean of a Wiener type criterion involving the \( C_{2,q}' \)-Bessel capacity, are still valid. As in [2], it is clear that if \( \partial \Omega \) satisfies the exterior segment property and \( 1 < q < (N - 1)/(N - 3) \), then \( \overline{u}_0 \) is a large solution.

5 Appendix

The proof of this result is based upon the existence of solution of (2.5) in \( Q_{\infty}^{G_{ij}} \) with a persistent singularity on \( [0] \times [0, \infty) \).

**Proposition 5.1** For any \( q > 1 \), there exists a unique positive function \( V := V_N \) defined in \( \mathbb{R}_+ \times \mathbb{R}_+ \) satisfying, for any \( \tau > 0 \)
\[
\begin{cases}
\partial_t V - \Delta V + V^q = 0 & \text{in } Q_{\infty}^{G_{ij}} \\
\lim_{(x,t) \to (y,0)} V(x,t) = 0 & \forall y \in \mathbb{R}_+ \setminus \{0\} \\
\lim_{|x| \to 0} V(x,t) = \infty & \text{locally uniformly on } [\tau, \infty), \text{ for any } \tau > 0
\end{cases}
\]
(5.1)
Then \( V_N(x,t) = t^{-1/(q-1)}H_N(|x|/\sqrt{t}) \), where \( H := H_N \) is the unique positive function satisfying
\[
\begin{cases}
H'' + \left( \frac{N - 1}{r} + \frac{q}{2} \right) H' + \frac{1}{q - 1} H - H^q = 0 & \text{in } \mathbb{R}_+ \\
\lim_{r \to 0} H(r) = \infty \\
\lim_{r \to \infty} r^{2/(q-1)} H(r) = 0
\end{cases}
\]
(5.2)
Furthermore there holds
\[
H_N(r) = c_N q r^{2/(q-1) - N} - r^{q-1} (1 + O(r^{-2})) \quad \text{as } r \to \infty,
\]
(5.3)
and
\[
H_N(r) = \lambda_N q r^{-2/(q-1)} (1 + O(r)) \quad \text{as } r \to 0,
\]
(5.4)
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Proof. If we assume $1 < q < N/(N-2)$, the $C_{2,q'}$ parabolic capacity of the axis $\{0\} \times \mathbb{R} \subset \mathbb{R}^{N+1}$ is positive, therefore there exists a unique solution $u := u_\mu$ to the problem

$$\partial_t u - \Delta u + |u|^{q-1} u = \mu \quad \in \mathbb{R}^N \times \mathbb{R},$$

(5.5)

(see [1]) where $\mu$ is the uniform measure on $\{0\} \times \mathbb{R}+$ defined by

$$\int \zeta d\mu = \int_0^\infty \zeta(0,t) dt \quad \forall \zeta \in C_0^\infty(\mathbb{R}^{N+1}).$$

If we denote $T_\ell[u](x, t) = \ell^{2/(q-1)} u(\ell x, \ell^2 t)$ for $\ell > 0$, then $T_\ell$ leaves the equation (2.5) invariant, and $T_\ell[u_\mu] = u_{\ell^{2/(q-1) - N} \mu}$. If we replace $\mu$ by $k \mu$ ($k > 0$), we obtain

$$T_\ell[u_{k \mu}] = u_{\ell^{2/(q-1) - N} k \mu}.$$  

(5.6)

Moreover, any solution of (2.5) in $\mathbb{R}^N \setminus \{0\} \times \mathbb{R}+$ which vanishes on $\mathbb{R}^N \setminus \{0\} \times \{0\}$ is bounded from above by the maximum solution $u := U$ of

$$-\Delta u + u^q = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}.$$  

(5.7)

This is obtained by considering the solution $U_\epsilon$ of

$$\begin{cases}
-\Delta u + u^q = 0 \quad \text{in} \quad \mathbb{R}^N \setminus B_\epsilon \\
\lim_{|x| \to \epsilon} u(x) = \infty.
\end{cases}$$

(5.8)

Actually,

$$U(x) := \lim_{\epsilon \to 0} U_\epsilon(x) = \lambda_{N,q} |x|^{-2/(q-1)}$$

with

$$\lambda_{N,q} := \left[ \frac{2}{q-1} \left( \frac{2q}{q-1} - N \right) \right]^{1/(q-1)},$$

an expression which exists since $1 < q < N/(N-2)$. If we let $k \to \infty$ in (5.6), using the monotonicity of $\mu \mapsto u_\mu$, we obtain that $u_{k \mu} \to u_\infty$, $u_\infty \leq U$ and

$$T_\ell[u_{k \mu}] = u_{\ell^{2/(q-1) - N} k \mu} = u_{\infty} \quad \forall \ell > 0.$$  

(5.10)

This implies that $u_\infty$ is self-similar, that is

$$u_\infty(x,t) = t^{-1/(q-1)} h(x/\sqrt{t}).$$

Furthermore, $h(.)$ is positive and radial as $x \mapsto u_\mu(x, t)$ is, and it solves

$$h'' + \left( \frac{N-1}{r} + \frac{r}{2} \right) h' + \frac{1}{q-1} h - h^q = 0 \quad \text{in} \quad \mathbb{R}_+.$$  

(5.11)

Since $u_\mu(x, 0) = 0$ for $x \neq 0$, the a priori bounds $u_{k \mu} \leq U$, the equicontinuity of the $\{u_{k \mu}\}_{k>0}$ implies that $u_\infty(x, 0) = 0$ for $x \neq 0$; therefore

$$\lim_{r \to \infty} r^{2/(q-1)} h(r) = 0.$$  

(5.12)
The same argument as the one used in the proof of Corollary 4.3 implies that \( t \mapsto u_\mu(x, t) \) is increasing, therefore \( \lim_{x \to 0} u_\mu(x, t) = \infty \) for \( t > 0 \). This implies \( \lim_{r \to 0} h(r) = \infty \). Then the proof of (5.3) follows from [10, Appendix]. When \( r \to 0 \), \( h \) could have two possible behaviours [3]:

(i) either

\[
h(r) = \lambda_{N,q} r^{-2/(q-1)} (1 + O(r)), \tag{5.13}
\]

(ii) or there exists \( c \geq 0 \) such that

\[
h(r) = cm_N(r)(1 + O(r)), \tag{5.14}
\]

where \( m_N(r) \) is the Newtonian kernel if \( N \geq 2 \) and \( m_1(r) = 1 + o(1) \).

If (ii) were true with \( c > 0 \) (the case \( c = 0 \) implying that \( h = 0 \) because of the behavior at \( \infty \) and maximum principle), it would lead to

\[
u_\infty(x) = c|x|^{2-N}t^{N-2/(q-1)}(1 + o(1)) \quad \text{as } x \to 0, \tag{5.15}
\]

for all \( t > 0 \). Therefore

\[
\int_\epsilon^T \int_{B_1} u_{k\mu}^q \, dx \, dt < C(\epsilon), \tag{5.16}
\]

for any \( \epsilon > 0 \) and \( k \in (0, \infty) \). We write (5.5) under the form

\[
\partial_t u_{k\mu} - \Delta u_{k\mu} = g_k + k\mu
\]

where \( g_k = -u_{k\mu}^q \), then \( u_{k\mu} = u'_{k\mu} + u''_{k\mu} \), where

\[
\partial_t u'_{k\mu} - \Delta u'_{k\mu} = k\mu
\]

and

\[
\partial_t u''_{k\mu} - \Delta u''_{k\mu} = g_k.
\]

By linearity \( u'_{k\mu} = ku'_{\mu} \). Because of (5.16) \( u'_{k\mu} \) remains uniformly bounded in \( L^1(B_1 \times (\epsilon, T)) \). This clearly contradicts \( \lim_{k \to \infty} u'_{k\mu} = \infty \). Thus (5.4) holds. The proof of uniqueness is an easy adaptation of [3, Lemma 1.1]: the fact that the domain is not bounded being compensated by the strong decay estimate (5.3). This unique solution is denoted by \( V_N \) and \( h = H_N \). □

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