Competitive Equilibrium for Dynamic Multiagent Systems: Social Shaping and Price Trajectories

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Abstract—In this article, we consider dynamic multiagent systems (MAS) for decentralized resource allocation. The MAS operate at a competitive equilibrium to ensure a balanced supply and demand for available resources. Example resources include energy in a microgrid and carbon permits in a carbon trading system. First, we investigate the MAS over a finite horizon, where the utility functions of agents are parameterized to incorporate individual preferences. We shape individual preferences through a set of utility functions to guarantee the resource price at a competitive equilibrium remains socially acceptable, i.e., the price is upper-bounded by an affordability threshold. We show this problem is solvable implicitly. Next, we consider quadratic MAS and formulate the social shaping problem as a multiagent linear quadratic regulator (LQR) problem. We propose explicit utility sets using quadratic programming and dynamic programming. Moreover, we propose a numerical algorithm for calculating a tight range of the preference function parameters that guarantee a socially accepted price. We then investigate the properties of a competitive equilibrium over an infinite horizon, and show cases where any competitive equilibrium maximizes the social welfare. We show that for initial conditions sufficiently close to the origin, the social welfare maximization solution constitutes a competitive equilibrium with zero price. We also show that for all feasible initial conditions, there exists a time instant after which the optimal price, corresponding to a competitive equilibrium, becomes zero.

Index Terms—Competitive equilibrium, dynamic programming, linear quadratic regulator (LQR), multiagent systems (MAS), resource allocation, social welfare optimization.

I. INTRODUCTION

EFFICIENT resource allocation is a fundamental problem in the literature that can be addressed by multiagent systems (MAS) approaches [1], [2]. Resource allocation problems are of great importance when the total demand must equal the total supply for safe and secure system operation [3], [4], [5]. Depending on the application, there exist two common approaches: 1) social welfare, where agents collaborate to maximize the utilities of all agents [6]; 2) competitive equilibrium in which agents compete to maximize their individual payoffs [7], [16]. For example, social welfare maximization has applications in edge computing that involves allocating computing resources to the edges of the network [8], [9]. A competitive equilibrium is referred to the pair of resource prices and the allocation profiles that maximize an individual agents payoffs and balances the total supply and demand (i.e., clears the market) [17]. A competitive equilibrium is a well-known solution to allocation problems [10], and can be applied as a solution concept to local energy markets in community microgrids with distributed energy resources [11].

A fundamental theorem in classical welfare economics states that the competitive equilibrium is Pareto optimal, meaning that no agent can deviate from the equilibrium to increase profits without reducing another’s payoff [12], [13], [14], [15]. It is also proved that under some convexity assumptions, a competitive equilibrium maximizes social welfare [16], [17], [18]. Mechanism design is a well-known approach to social welfare maximization [19], [20], [21]. The key point in achieving a competitive equilibrium is efficient resource pricing that depends on the utility of each agent. The corresponding price, however, is not guaranteed to be affordable for all agents. If some participants select their utility functions aggressively to overestimate their potential utility, the price potentially increases to the point that it becomes unaffordable to others who have no alternative but to leave the system [22], [23]. A recent example is the Texas power outage disaster in February 2021, when some citizens who had access to electricity during the period of rolling power outages received inflated (and in some cases, unaffordable) electricity bills for their daily power usage [24].

In this article, we consider MAS with distributed resources and linear time-invariant (LTI) dynamics. For instance, agents can represent households in a community microgrid with rooftop photovoltaic (PV) generation and thermal dynamics, or regions in a carbon permit trading system with carbon permits as distributed resources and economic dynamics. To achieve affordability over a finite horizon, we parameterize the utility
functions of the agents and propose some limits on their personal parameters such that the resource price at a competitive equilibrium never exceeds a given threshold, leading to the concept of social shaping. We design an optimization problem and propose a conceptual scheme, based on dynamic programming, which shows how the social shaping problem is solvable implicitly for general classes of utility functions. Next, the social shaping problem is reformulated for quadratic MAS, leading to a linear quadratic regulator (LQR) problem. Solving the LQR problem using quadratic programming and dynamic programming, we propose two explicit sets for the utility function parameters such that they lead to socially acceptable resource prices. Then, a numerical algorithm based on the bisection method is presented that provides accurate and practical bounds on the utility function parameters, followed by some convergence results.

Over an infinite horizon, we examine the behavior of resource price under a competitive equilibrium. We extend our previous work in [16], which considered a finite horizon, by showing that in the infinite horizon problem, the competitive equilibrium maximizes social welfare if feasibility assumptions are satisfied. We also prove that for initial conditions which are sufficiently close to the origin, the social welfare maximization constitutes a competitive equilibrium with zero price; implying adequate resources are available in the network to meet the demand of all agents. Furthermore, we show for any feasible initial condition, there exists a time instant after which the optimal price, corresponding to the competitive equilibrium, is zero. Next, as a special case, we study quadratic MAS for which the system-level social welfare optimization becomes a classical constrained LQR (CLQR) problem. Finally, we investigate how the results can be extended to tracking problems in which the state is desired to track a nonzero reference point [25].

In a preliminary conference version of this article [26], we discussed the problem of social shaping for dynamic MAS over a finite horizon, where the proofs of the proposed theorems were omitted due to the page limits. In another conference version [27], we discussed the existence of a competitive equilibrium and the behavior of the associated resource price over an infinite horizon. In this article, we extend our prior works in the following ways.

1) We introduce two real-world applications for the problem formulation.
2) We present the complete proofs of the proposed theorems.
3) We investigate extensions to tracking problems.
4) We provide new, and a more extensive series of numerical examples, including a real-world example of a community microgrid.

The rest of this article is organized as follows. In Section II, we introduce the problem formulation and its real-world applications. In Section III, we present a conceptual scheme for solving the social shaping problem over a finite horizon. In Section IV, we address the social shaping problem for quadratic MAS over a finite horizon. In Section V, we examine the existence and properties of a competitive equilibrium over an infinite horizon. Then, we investigate how the results can be extended to tracking problems. Section VI provides numerical examples. Finally, Section VII concludes this article.

Notation: We denote by \( \mathbb{R} \) and \( \mathbb{R}^{\geq 0} \) the fields of real numbers and nonnegative real numbers, respectively. Let \( I \) denote the identity matrix with a suitable dimension. The symbol \( \sigma \) represents a vector with an appropriate dimension whose entries are all 1. We use \( ||\cdot|| \) to denote the Euclidean norm of a vector or its induced matrix norm. Let \( \sigma_{\min} \) and \( \sigma_{\max} \) represent the minimum and maximum eigenvalues of a square matrix, respectively.

II. PROBLEM FORMULATION

A. Dynamic Multiagent Model

Consider a dynamic MAS with \( N \) agents indexed in the set \( \mathcal{N} = \{1, 2, \ldots, N\} \) operating over the time horizon \( T \). Let time steps be indexed in the set \( \mathcal{T} = \{0, 1, \ldots, T-1\} \). Each agent \( i \in \mathcal{N} \) is a subsystem with dynamics represented by

\[
x_i(t + 1) = A_i x_i(t) + B_i u_i(t), \quad t \in \mathcal{T}
\]

where \( x_i(t) \in \mathbb{R}^d \) is the dynamical state, \( x_i(0) \in \mathbb{R}^d \) is the given initial state, and \( u_i(t) \in \mathbb{R}^m \) is the control input. Also, \( A_i \in \mathbb{R}^{d \times d} \) and \( B_i \in \mathbb{R}^{d \times m} \) are fixed matrices. Upon reaching the state \( x_i(t) \) and employing the control input \( u_i(t) \) at time step \( t \in \mathcal{T} \), each agent \( i \) receives the utility \( f_i(x_i(t), u_i(t)) = f_i(x_i(t), \theta_i) : \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R} \), where \( \theta_i \in \Theta \) is a personalized parameter of agent \( i \). The terminal utility achieved as a result of reaching the terminal state \( x_i(T) \) is denoted by \( \phi_i(x_i(T)) = \phi_i(x_i(T), \theta_i) : \mathbb{R}^d \rightarrow \mathbb{R} \). Let \( a_i(t) \in \mathbb{R} \) denote the amount of excess resource generated by agent \( i \), i.e., the resource of agent \( i \) that is in addition to the supply needed to stay at the origin at time \( t \in \mathcal{T} \). The amount of consumed resource by the same agent for control action \( u_i(t) \) is denoted by \( h_i(u_i(t)) : \mathbb{R}^m \rightarrow \mathbb{R}^{\geq 0} \). The total excess resource cost is represented by \( C(t) = \sum_{i=1}^{N} a_i(t) \) such that \( C(t) > 0 \) at each time interval \( t \in \mathcal{T} \). Although \( C(t) \) can be zero in practice, for our technical analysis, we assume that it only takes positive values which can be arbitrarily small.

Agents are interconnected through a network without any external resource supply. They share resources internally at the price \( \lambda_i \in \mathbb{R} \), which denotes the price of unit traded resource across the network at time step \( t \in \mathcal{T} \). The traded resource for agent \( i \) is denoted by a trading decision \( e_i(t) \in \mathbb{R} \) which can never be greater than the net supply, i.e., \( e_i(t) \leq a_i(t) - h_i(u_i(t)) \). Then, each agent \( i \) receives a payoff which consists of the utility from resource consumption and the income \( \lambda_i e_i(t) \) from resource exchange.

B. Competitive Equilibrium

Let \( U_i = (u_i^T(0), \ldots, u_i^T(T-1))^T \) and \( E_i = (e_i(0), \ldots, e_i(T-1))^T \) denote the vector of control inputs and the vector of trading decisions for agent \( i \) over the entire time horizon, respectively. Also, let \( u(t) = (u_1(t), \ldots, u_N(t))^T \) and \( e(t) = (e_1(t), \ldots, e_N(t))^T \) denote the vector of control inputs and the vector of trading decisions associated with all agents at time step \( t \in \mathcal{T} \), respectively. Let \( U = (u(0), \ldots, u^T(T-1))^T \) and \( E = (e^T(0), \ldots, e^T(T-1))^T \) be the vector of all control inputs and the vector of all trading decisions at all time steps,
respectively. Let \( \lambda = (\lambda_0, \ldots, \lambda_{T-1})^T \) denote the vector of resource prices throughout the entire time horizon.

**Definition 1:** A competitive equilibrium for a dynamic MAS is a triplet \((\lambda^*, U^*, E^*)\) which satisfies the following two conditions.

1) Given \( \lambda^* \), the pair \((U^*, E^*)\) maximizes the individual payoff function of each agent, i.e., each \((U_i^*, E_i^*)\) solves the following constrained maximization problem:

\[
\max_{U_i, E_i} \phi(x_i(T); \theta_i) + \sum_{t=0}^{T-1} \left( f(x_i(t), u_i(t); \theta_i) + \lambda^*_t e_i(t) \right)
\]

\[
\text{s.t. } x_i(t+1) = A_i x_i(t) + B_i u_i(t),
\]

\[
e_i(t) \leq a_i(t) - h_i(u_i(t)), \quad t \in T.
\]

2) The optimal trading \( E^* \) balances the total traded resource across the network at each time step, i.e.,

\[
\sum_{i=1}^N e_i^*(t) = 0, \quad t \in T.
\]

The optimization problem in (2) is solved by each agent independently.

**C. Social Welfare Maximization**

In the context of resources that underpin critical services in a modern society, (e.g., electricity that supports critical services including hospitals, water infrastructure, communication networks, and the finance sector) the benefit reaped by the whole community is important, not just the benefit derived from a single agent. Accordingly, it is desirable to find an operating point \((U^*, E^*)\) at which social welfare is maximized. This goal can be attained by solving the following social welfare maximization problem:

\[
\max_{U, E} \sum_{i=1}^N \left( \phi(x_i(T); \theta_i) + \sum_{t=0}^{T-1} f(x_i(t), u_i(t); \theta_i) \right)
\]

\[
\text{s.t. } x_i(t+1) = A_i x_i(t) + B_i u_i(t),
\]

\[
e_i(t) \leq a_i(t) - h_i(u_i(t)), \quad t \in T.
\]

The solution to (3) is obtained from the optimal dual variable \( \nu_i^* \) associated with the balancing equality constraint \( \sum_{i=1}^N e_i^*(t) = 0 \) in (4), such that \( \lambda_i^* = -\nu_i^* \).

**D. Social Shaping Problem**

The optimal price \( \lambda_i^* \) is obtained from the optimal dual variable corresponding to the balancing equality constraint \( \sum_{i=1}^N e_i^*(t) = 0 \) in (4), which depends on the utility functions of agents. If some agents select their utility functions aggressively to overestimate their achievable utility, the price may become extremely high and unaffordable for others. In such cases, agents who find the price unaffordable cannot compete in the market and have no alternative but to leave the system. Consequently, all of the resources will be consumed by a limited number of agents. To improve the affordability of the price for all agents we require a mechanism, called social shaping, which, if implemented, ensures the price is always below an acceptable threshold denoted by \( \lambda^1 \in \mathbb{R}^+ \). The problem of social shaping is addressed for static MAS in [22] and [23]. In this article, we define an extended version of the social shaping problem for dynamic MAS as follows.

**Definition 2 (Social shaping for dynamic MAS):** Consider a dynamic MAS whose agents \( i \in N \) have \( f(\cdot; \theta_i) \) and \( \phi(\cdot; \theta_i) \) as their running utility function and terminal utility function, respectively. Let \( \lambda^1 \in \mathbb{R}^+ \) denote a given price threshold that is accepted by all agents. Find a range \( \Theta \) of personal parameters \( \theta_i \) such that if \( \theta_i \in \Theta \) for \( i \in N \), then we yield \( \lambda_i^* \leq \lambda^1 \) at all time steps \( t \in T \).

**Social Shaping Implementation:** When the social shaping problem becomes solvable, it can be implemented as follows for an MAS with a coordinator.

S1) The coordinator negotiates with agents to reach an agreement on the price threshold \( \lambda^1 \). Then, it collects the system parameters from the agents.

S2) According to the collected information, the coordinator finds a range \( \Theta \) for the personal parameters and broadcasts it to all agents.

S3) Agents select their utility functions with the proposed range \( \Theta \) and send them to the coordinator.

S4) Receiving the preferences of agents, the coordinator solves the optimization problem in (4) to calculate the optimal price, which is then broadcast to agents.

S5) Agents trade their resources based on the proposed price and the optimization problem in (2).

**E. Real-World Applications**

In this section, we provide two real-world applications for our problem formulation.

1) **Community Microgrid:** Consider a community microgrid consisting of \( N \) buildings with rooftop PV generation, without any external power injection from the main grid. Buildings are equipped with air conditioners to keep their temperature at a desired level, forming a group of \( N \) thermostatically controlled loads (TCL). Each TCL, indexed by \( i \in N \), has dynamics represented by \( x_i(t+1) = A_i x_i(t) + B_i u_i(t) + B_i w \), where \( x_i(t) \) is the internal temperature (°C), \( u_i(t) \) is the consumed energy (kWh), and \( w \) reflects the impact of the ambient temperature (°C), and \( B_i \) is a term containing the thermal resistance and capacitance of the TCL [28], [29].
In this context, each building $i \in \mathcal{N}$ represents an agent with excess rooftop PV energy $a_i(t)$ in (kWh) at time $t \in \mathcal{T}$. Buildings are connected through a network to share their excess energy internally at a price $\lambda_i^*(t)$ (USD/kWh) such that the total excess PV generation balances the total demand from TCLs at each time step $t \in \mathcal{T}$. Considering the internal temperature $x_i(t)$ and the consumed energy $u_i(t)$, each household encodes its preferences in a utility function $f_i(x_i(t), u_i(t))$ which represents its satisfaction. In general, if the temperature is close to the desired level (showing comfort) or the amount of consumed energy is relatively low, then the household achieves a high level of satisfaction, and therefore, a high $f_i(\cdot)$. In practice, agents may select their preferences through a smart thermostat interface to make a tradeoff between their comfort and consumed energy [30].

Different preferences lead to different electricity prices $\lambda_i^*$. Assume that all buildings have the same class of utility functions but with different parameters, i.e., each building is associated with the utility $f_i(x_i(t), u_i(t)) = f(x_i(t), u_i(t); \theta_i)$, where $\theta_i$ is the personal parameter (preference) of building $i$ which can be selected independently through a smart thermostat interface. By means of social shaping and by imposing some bounds on the choice of $\theta_i$, the coordinator guarantees that the price for electricity exchange inside the microgrid never exceeds an acceptable threshold, so as to ensure affordability. In this example, we assume that there is no power exchange between the microgrid and the main grid. Specifically, we consider the case of remote areas where microgrids operate in a stand-alone mode, or otherwise, the case where a microgrid is temporarily disconnected from the main supply.

Remark 1: To enhance resilience in the microgrid, we require energy storage devices such as batteries and electric vehicles that are designed to store excess energy for later use when the electricity supply is otherwise limited. Accordingly, energy storage devices introduce additional dynamics within the microgrid. In such cases, our aforementioned approach applies and is extended so that households with multiple-input–multiple-output dynamics corresponding to both TCLs and storage devices are accommodated.

2) Carbon Permit Trading System: The regional integrated model of climate and the economy (RICE) can be formulated as a dynamic multiagent model [31], [32]. Each region captured in the RICE model represents an agent. Each time step $t \in \mathcal{T}$ represents a ten-year period. Let $x_i(t)$ be each region $i$’s economic output (USD) at time step $t$ and $u_i(t)$ be the amount of emission (GtCO$_2$) it emits at time step $t$. In the RICE model, each region’s economic output at time step $t + 1$ depends on its economic output at time step $t$ and the emitted emissions at time step $t$. Thus, each region $i \in \mathcal{N}$ has its nonlinear dynamics $x_i(t+1) = g_i(x_i(t), u_i(t))$. Upon the states and control actions, each region $i$ evaluates its social welfare according to a utility function $f_i(x_i(t), u_i(t))$ capturing the fact that the societal welfare for a region depends on both the economic output and carbon emissions.

A carbon permit trading block scheme has been proposed using the RICE model [33]. Under the scheme, the total permitted global emissions at each time step $t$ is assumed to be less than $C(t)$ (GtCO$_2$). Each region $i$ is assigned with its carbon permit $a_i(t)$ (GtCO$_2$) at time step $t$ with the relationship $\sum_{i \in \mathcal{N}} a_i(t) = C(t)$. Regions are allowed to buy or sell carbon permits at a price $\lambda_i^*$(USD/GtCO$_2$) such that the total carbon permits balance the total demand of emissions to be emitted by each region at each time step $t$.

Different configurations of utility functions yield different prices $\lambda_i^*$ for a unit carbon permit. Suppose that all regions adopt the same class of utility functions $f(\cdot; \theta_i)$ but with different choices of parameters $\theta_i$. For each region $i$, the personal parameter $\theta_i$ balances the cost associated with carbon emission $u_i(t)$, which could come from establishing new industries, against the benefit from the economic output $x_i(t)$. By means of social shaping and by imposing bounds on $\theta_i$ for all $i \in \mathcal{N}$, the price per unit carbon permit is guaranteed to be below a threshold, making it affordable for all regions.

III. CONCEPTUAL SOCIAL SHAPING

In this section, we examine how the social shaping problem of dynamic MAS can be solved conceptually.

Lemma 1: Consider a dynamic MAS. Let $\lambda^* = (\lambda_0^*, \ldots, \lambda_{T-1}^*)$ be the vector of optimal prices corresponding to a competitive equilibrium. Then $\lambda_i^* \geq 0$ for all $t \in \mathcal{T}$.

Proof: See Appendix I.

Remark 2: For social shaping, we can skip the case $\lambda_i^* = 0$, because a zero price is always socially resilient, i.e., below the threshold.

Proposition 2: Consider a dynamic MAS. Let Assumption 1 hold. If $\lambda_i^* > 0$, then the total demand and supply are balanced at time step $t$; that is,

$$\sum_{i=1}^{N} h_i(u_i^*(t)) = C(t).$$

Proof: Since Assumption 1 is satisfied, Proposition 1 holds. Therefore, either the competitive problem in (2) or the social welfare problem in (4) can be considered. Next, consider the competitive optimization problem in (2). Let $s_i(t) \in \mathbb{R}^{\geq 0}$ be the slack variable for agent $i$ at time step $t \in \mathcal{T}$ and $s_i = (s_i(0), \ldots, s_i(T-1))$ be the vector of slack variables for agent $i$ throughout the whole time horizon. We can write the inequality constraint $e_i(t) \leq a_i(t) - h_i(u_i(t))$ in (2) as the equality $e_i(t) = a_i(t) - h_i(u_i(t)) - s_i(t)$ for $t \in \mathcal{T}$. Then, substituting $e_i(t)$ into (2) results in an equivalent form for the optimization problem as

$$\begin{align*}
\max_{U_i, S_i} \phi(x_i(T); \theta_i) &+ \sum_{t=0}^{T-1} f(x_i(t), u_i(t); \theta_i) \\
&+ \sum_{t=0}^{T-1} \lambda_i^* [a_i(t) - h_i(u_i(t)) - s_i(t)] \\
\text{s.t.} \quad &x_i(t+1) = A_i x_i(t) + B_i u_i(t), \\
&\quad s_i(t) \in \mathbb{R}^{\geq 0}, \quad t \in \mathcal{T}.
\end{align*}$$

Since $\lambda_i^* > 0$, the resulting objective function is strictly decreasing with respect to $s_i(t)$. Consequently, the optimal slack variable maximizing the objective function is $s_i^*(t) = 0$, meaning that the associated inequality constraint is active; that is, $e_i^*(t) = a_i(t) - h_i(u_i(t)) - s_i^*(t) = 0$.
The summation of the resulting active constraint over $i$, from 1 to $N$, along with the balancing equality $\sum_{i=1}^{N} e_i^*(t) = 0$ in (3), yields $\sum_{i=1}^{N} h_i(u_i^*(t)) = C(t)$.

The next result shows the social shaping problem is solvable implicitly. We denote $x(t) = (x_1^T(t), \ldots, x_N^T(t))^T$, $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$, and $C = (C(0), C(1), \ldots, C(T-1))^T$.

**Theorem 1:** Consider the dynamic MAS introduced in Section II-A. Let Assumption 1 hold. Suppose $f(\cdot; \theta_i), \phi(\cdot; \theta_i)$, and $h_i(\cdot)$ are continuously differentiable. Let $\lambda^i \in \mathbb{R}^>0$ represent the given price threshold accepted by all agents. Then, for each time step $k \in \mathcal{T}$, there exists a function $g_k(x(0); C; \theta)$ which represents the maximum optimal price. Any set $\Theta$ satisfying $\max_{\theta \in \Theta} g_k(x; \theta) \leq \lambda^i$ for all $k \in \mathcal{T}$ solves the social shaping problem.

**Proof:** Since Assumption 1 is satisfied, Proposition 1 holds. In this article, we focus on the competitive optimization problem in (2). According to Lemma 1 and Remark 2, it is sufficient to only examine $\lambda^i > 0$. Following from Proposition 2, the total demand and supply are balanced at time step $t$, meaning that $\sum_{i=1}^{N} h_i(u_i^*(t)) = C(t)$; additionally, we have $s^*_i(t) = 0$.

Substituting $s_i(t) = 0$ into (6) yields an equivalent form for the competitive optimization problem in (2) as

$$\max u_i \phi(x_i(T); \theta_i) + \sum_{t=0}^{T-1} \left( f(x_i(t), u_i(t); \theta_i) + \lambda^i_1 [a_i(t) - h_i(u_i(t))] \right) \quad \text{s.t.} \quad x_i(t+1) = A_i x_i(t) + B_i u_i(t), \quad t \in \mathcal{T} \quad (7)$$

which is valid for $\lambda^i > 0$, for all $t$. Please note that even if $\lambda^i = 0$ then (6) can be written in the form of (7), although the equality in (5) turns into the inequality $\sum_{i=1}^{N} h_i(u_i^*(t)) \leq C(t)$. This result has no impact, or otherwise, causes no change to the upcoming analysis.

The optimization problem in (7) is an unconstrained optimal control problem which can be solved with dynamic programming. First, introduce the cost-to-go function for agent $i$ from time $k$ to $T$ as

$$J_{k,T}^i(x_i(k), u_i(k), \ldots, u_i(T-1), \lambda^i_1, \ldots, \lambda^i_{T-1}; \theta_i) = \phi(x_i(T); \theta_i) + \sum_{t=k}^{T-1} \left( f(x_i(t), u_i(t); \theta_i) + \lambda^i_1 [a_i(t) - h_i(u_i(t))] \right). \quad (8)$$

Then, the optimal cost-to-go at time $k$ for agent $i$, which is also called the value function, is represented as

$$V_{i,k}(x_i(k), \lambda^i_1, \ldots, \lambda^i_{T-1}; \theta_i) = \max_{u_i(k), \ldots, u_i(T-1)} J_{k,T}^i(x_i(k), u_i(k), \ldots, u_i(T-1), \lambda^i_1, \ldots, \lambda^i_{T-1}; \theta_i) \quad \text{s.t.} \quad x_i(t+1) = A_i x_i(t) + B_i u_i(t), \quad t = k, \ldots, T-1 \quad (9)$$

According to the principle of optimality, we obtain

$$V_{i,T}(x_i(T); \theta_i) = \phi(x_i(T); \theta_i)$$

$$V_{i,T-1}(x_i(T-1), \lambda^i_{T-1}; \theta_i) = \max_{u_i(T-1)} f(x_i(T-1), u_i(T-1); \theta_i) + \lambda^i_{T-1} [u_i(T-1) - h_i(u_i(T-1))] + V_{i,T}(A_i x_i(T-1) + B_i u_i(T-1); \theta_i)$$

$$V_{i,0}(x_i(0), \lambda^i_0, \ldots, \lambda^i_{T-1}; \theta_i) = \max_{u_i(0)} f(x_i(0), u_i(0); \theta_i) + \lambda^i_0 [u_i(0) - h_i(u_i(0))] + V_{i,1}(A_i x_i(0) + B_i u_i(0), \lambda^i_1, \ldots, \lambda^i_{T-1}; \theta_i). \quad (10)$$

To obtain the optimal control at time step $k = 0$, the derivative of the associated objective function with respect to $u_i(0)$ must equal zero; that is,

$$\frac{\partial f(x_i(0), u_i(0); \theta_i)}{\partial u_i(0)} - \lambda^i_0 \nabla h_i(u_i(0)) + \frac{\partial V_{i,1}(A_i x_i(0) + B_i u_i(0), \lambda^i_1, \ldots, \lambda^i_{T-1}; \theta_i)}{\partial u_i(0)} = 0. \quad (11)$$

Proposition 1 implies that such an optimal solution exists, although it might not be unique. We investigate two cases.

1) If the optimal solution is unique, we can write $u^*_i(0)$ as a function denoted by $l^i_0(\cdot; \theta_i)$ and parameterized by $\theta_i$. This function depends on $x_i(0)$ and all $\lambda^i$, where $t \in T$; that is,

$$u^*_i(0) = l^i_0(x_i(0), \lambda^i_0, \ldots, \lambda^i_{T-1}; \theta_i). \quad (12)$$

Substituting (12) into the equality $\sum_{i=1}^{N} h_i(u^*_i(0)) = C(0)$ in (5), we yield $\sum_{i=1}^{N} h_i(l^i_0(x_i(0), \lambda^i_0, \ldots, \lambda^i_{T-1}; \theta_i)) = C(0)$. Similarly, for any other time step $k \in \mathcal{T}$ we achieve $u^*_i(k) = l^i_k(x_i(0), \lambda^i_0, \ldots, \lambda^i_{T-1}; \theta_i)$, and

$$\sum_{i=1}^{N} h_i(l^i_k(x_i(0), \lambda^i_0, \ldots, \lambda^i_{T-1}; \theta_i)) = C(k), \quad k \in \mathcal{T} \quad (13)$$

We aim to obtain $\lambda^* = (\lambda^*_0, \ldots, \lambda^*_{T-1})^T$. According to (13), we have $T$ equations with $T$ variables. Based on Proposition 1, there exists $\lambda^*$ which satisfies (13) although it might not be unique. Denote $\mathcal{G} = \{ \lambda^* \in \mathbb{R}^T : \lambda^* \text{ solves } (13) \}$ as the set of all possible optimal prices. Among different possible optimal prices in $\mathcal{G}$, we consider the maximum one, $\lambda^*_0(\cdot; \theta)$, associated with a fixed $\theta$ at each time step $k \in \mathcal{T}$ which is obtained from (13) as $\lambda^*_0 = g_k(x(0), C; \theta)$. We aim to achieve $\lambda^*_0 \leq \lambda^i$ for all $k \in \mathcal{T}$. For different values of agent preferences $\theta$ we would obtain different optimal maximum prices at each time step $k \in \mathcal{T}$, among which we consider the maximum
one when $\theta_i$ takes values in the set $\Theta_i (or \Theta_i \in \Theta_i^N)$, as

$$\chi_k^{\Theta} := \max_{\theta \in \Theta_i^N} g_k(\cdot ; \theta), \quad k = 0, \ldots, T - 1.$$  \hspace{1cm} (14)

Any set $\Theta_i$ satisfying $\chi_k^{\Theta} \leq \lambda_i^1$ for $k \in \mathcal{T}$ ensures that $\lambda_i^* \leq \lambda_i^1$ for $t \in \mathcal{T}$, and thus, solves the social shaping problem of agent preferences.

2) Consider the case where the optimal solution is not unique. Denote $\nu = (\nu_0^\top, \ldots, \nu_{T - 1}^\top)$ as a vector of optimal dual variables $\nu_i^*$ corresponding to the balancing equality constraints $\sum_{i=1}^N e_i(t) = 0$ in (4) for $t \in \mathcal{T}$. According to Proposition 1, there exists $\lambda^*$ associated with a competitive equilibrium such that $\lambda^* = -\nu^*$, although it might not be unique. Denote $\mathcal{G} = \{\lambda^* \in \mathbb{R}^T : \lambda^* = -\nu^*\}$. Among different possible optimal prices in $\mathcal{G}$, we consider the maximum one, $\lambda_i^*(\cdot ; \theta)$, associated with a fixed $\theta$ at each time step $k \in \mathcal{T}$ which is obtained from optimal dual variables as $\lambda_i^* = g_k(x(0), C; \theta)$. The rest of the proof is the same as case 1.

Next, we discuss social shaping in the context of the February 2021 Texas pricing event [24].

Remark 3: Consider the Texas event [24] as per the Introduction. Specifically, if a socially acceptable price threshold was introduced, then the utility functions of the households would be adjusted according to the proposed social shaping scheme, resulting in less power consumption by those that dominated the market. Accordingly, more households would be able to access the limited energy supply available to meet their primary requirements.

In general, utility functions and energy functions can take various forms. In the next section, we focus on a quadratic formulation, a form widely used in classical optimal control problems.

IV. QUADRATIC SOCIAL SHAPING

In this section, we examine quadratic utility functions and explicitly propose two sets of personal parameters that lead to socially acceptable optimal prices.

Assumption 2: Consider the dynamic MAS introduced in Section II-A. Let $\theta_i = (Q_i, R_i)$, where $Q_i, R_i \in \mathbb{R}^d \times d$, $Q_i = \sum_{i=1}^N P_i^\top \in \mathbb{R}^d \times d$, $Q_i = \sum_{i=1}^N P_i^\top > 0$ and $R_i \in \mathbb{R}^{m \times m}$, $R_i = R_i^\top > 0$. Assume for all $i \in \mathcal{N}$ we have

$$f(x_i(t); u_i(t); \theta_i) = -x_i^\top(t)Q_i x_i(t) - u_i^\top(t)R_i u_i(t)$$

$$\phi(x_i(T); \theta_i) = -x_i^\top(T)Q_i x_i(T)$$

$$h_i(u_i(t)) = u_i^\top(t)H_i u_i(t)$$  \hspace{1cm} (15)

where $H_i \in \mathbb{R}^{m \times m}$, $H_i = H_i^\top > 0$.

Assumption 3: Consider the dynamic MAS in Assumption 2 with an initial state $x_i(0)$ such that $|x_i(0)| \leq \gamma_i$, $\|A_i\| \leq \alpha$, $\|B_i\| \leq \beta$, and $H_i \geq \rho I$ for $i \in \mathcal{N}$. Suppose that $\gamma, \alpha, \beta, \rho \in \mathbb{R}^>^0$.

We aim to solve the following social shaping problem.

Dynamic and Quadratic Social Shaping Problem: Suppose Assumptions 2 and 3 hold. Let $\lambda^* \in \mathbb{R}^>^0$ be the price threshold accepted by all agents, and $\delta_{max} \in \mathbb{R}^>^0$ be an upper bound for the norm of the personal parameter $P_i$. We propose an admissible set for $\delta_{max}$ such that all utility functions satisfying $\|Q_i\| \leq \delta_{max}$ (or $Q_i \leq \delta_{max} I$) lead to socially acceptable resource prices at all time steps, i.e., $\lambda_i^* \leq \lambda_i^1$ for $t \in \mathcal{T}$.

To address this problem, we use two approaches: 1) quadratic programming and 2) dynamic programming.

A. Quadratic Programming Approach

Since Assumption 2 is satisfied, Proposition 1 holds. We examine the competitive optimization problem in (2). According to Lemma 1 and Remark 2, it is sufficient to only study $\lambda_i^* > 0$. According to (7), the optimization problem in (2) can be reformulated as

$$\max_{U_i} \quad -x_i^\top(T)Q_i x_i(T) + \sum_{t=0}^{T-1} \left[ -x_i^\top(t)Q_i x_i(t) - u_i^\top(t)R_i u_i(t) + \lambda_i^1 \left( a_i(t) - u_i^\top(t)H_i u_i(t) \right) \right]$$

s.t. $x_i(t+1) = A_i x_i(t) + B_i u_i(t), \quad t \in \mathcal{T}.$  \hspace{1cm} (16)

Theorem 2: Consider the dynamic MAS described in Assumptions 2 and 3 on the horizon $T$. Suppose $\delta_{max} \in \mathbb{R}^>^0$ is selected from the following set:

$$\mathcal{S} = \left\{ \delta_{max} \in \mathbb{R}^>^0 : \delta_{max} \sum_{t=0}^{T-1} \left[ \gamma \alpha^2 t - k - 1 \right] + \beta \sum_{j=0}^{t-1} \sqrt{\frac{C(j)}{\rho} \alpha^2 j - k - 2} \right\} \frac{\sqrt{C(k)}}{N \beta} \lambda_1^1 \text{ for } \forall k \in \mathcal{T}. \right\}$$  \hspace{1cm} (17)

Then for all quadratic utility functions satisfying $\|Q_i\| \leq \delta_{max}$ (or $Q_i \leq \delta_{max} I$), the resulting optimal price is socially resilient, i.e., $\lambda^* \leq \lambda_i^1$.

Proof: See Appendix II.

In the proof of Theorem 2, the dependency between control decisions at different time steps has been ignored. We now apply a dynamic programming approach that takes this dependency into account by a Riccati difference equation and our approach leads to a less conservative set.

B. Dynamic Programming Approach

Similar to Section IV-A, let $\lambda^* > 0$ and consider the optimization problem in (16) which is equivalent to

$$\max_{U_i} \quad -x_i^\top(T)Q_i x_i(T) + \sum_{t=0}^{T-1} \left[ -x_i^\top(t)Q_i x_i(t) - u_i^\top(t)R_i u_i(t) + \lambda_i^1 \left( a_i(t) - u_i^\top(t)H_i u_i(t) \right) \right]$$

s.t. $x_i(t+1) = A_i x_i(t) + B_i u_i(t), \quad t \in \mathcal{T}.$  \hspace{1cm} (18)

Theorem 3: Consider the dynamic MAS on the time horizon $T$. Let Assumptions 2 and 3 hold. Suppose $\delta_{max} \in \mathbb{R}^>^0$ is selected from the following set:

$$\mathcal{S} = \left\{ \delta_{max} \in \mathbb{R}^>^0 : \delta_{max} \sum_{t=0}^{T-1} \gamma \alpha^2 t - k - 1 \right\} \frac{\sqrt{C(k)}}{N \beta} \lambda_1^1,$$
\[ \delta_{\text{max}} \sum_{i=k+1}^{T} \left[ \gamma \alpha^{2t-k-1} + \beta \sum_{j=0}^{k-1} \sqrt{\frac{C(j)}{\rho}} \alpha^{2t-j-k-2} \right] \leq \frac{\sqrt{C(k)}}{N \beta} \lambda^1 \text{ for } \forall k \in \mathcal{T}, k \neq 0 \right). \] (19)

Then, the resulting \( \lambda^* \) is socially resilient for all utility functions satisfying \( \|Q_i\| \leq \delta_{\text{max}} \) (or \( Q_i \leq \delta_{\text{max}} I \)).

**Proof:** See Appendix III.

**C. Numerical Algorithm**

The two proposed sets in Theorems 2 and 3 are conservative but provide insight into the tradeoff between utility functions' parameters and achieving the price threshold. To obtain more accurate and practical results, we propose a numerical algorithm which provides less conservative bounds on the parameters. The algorithm proceeds based on the bisection method. The bisection method shrinks the interval containing the solution by half at each iteration to finally converge on a solution [34].

**Numerical Social Shaping Problem:** Consider the social welfare problem in (4). Let \( \delta_{\text{max}} \in \mathbb{R}^{>0} \) denote the design parameter. Suppose Assumption 2 holds with \( Q_i = q_i I \), where agents have the freedom to select \( q_i \in (0, \delta_{\text{max}}] \). Assume \( \lambda^1 \) is the price threshold accepted by all agents and \( R_i \) is specified for each \( i \in \mathcal{N} \). We aim to find the upper bound \( \delta_{\text{max}} \) by a numerical approach such that if \( q_i \in (0, \delta_{\text{max}}] \) for \( i \in \mathcal{N} \) then \( \lambda^*_i \leq \lambda^1 \) for \( t \in \mathcal{T} \). Accordingly, the key steps required are illustrated in Algorithm 1.

**Lemma 2:** The function \( \lambda^*(\delta) \) in (20) is monotonically increasing.

**Proof:** When \( \delta \) increases, the domain of agents’ preferences expands respectively. Therefore, the maximum possible price can never decrease.

**Theorem 4:** The auxiliary variable \( L_k \) in Algorithm 1 converges to \( L^* \) for some \( L^* \in (0, d_0) \) when \( k \to \infty \).

**Proof:** See Appendix IV.

In the following, we show that Algorithm 1 can provide a tight upper bound.

**Theorem 5:** Suppose there exists \( \delta^1 > 0 \) such that \( \lambda^*(\delta^1) = \lambda^1 \). Then \( \delta_{\text{max}} \) obtained from Algorithm 1 satisfies \( \lambda^*(\delta_{\text{max}}) = \lambda^1 \).

**Proof:** See Appendix V.

**V. EXTENDING THE HORIZON TO INFINITY**

In this section, we investigate the properties of a competitive equilibrium, especially the resource price, over an infinite horizon.

Similar to the finite horizon case (see Sections II-A and II-B), define \( T, \lambda, U_t, E_t, U, \) and \( E \) such that \( T \to \infty \). Recall that \( u(t) = (u_1(t), \ldots, u_N(t))^T, e(t) = (e_1(t), \ldots, e_N(t))^T, \) and \( x(t) = (x_1(t), \ldots, x_N(t))^T \). Consistent with the concept of competitive equilibrium in microeconomics [17], the infinite-horizon competitive equilibrium can be defined as follows.

**Definition 3:** Let \( X_0 \) be the set of all feasible initial conditions. Given a feasible initial condition \( x(0) \in X_0 \), the triplet \((\lambda^*, U^*, E^*)\) is called an infinite-horizon competitive equilibrium if it meets the following conditions.

1) Under the equilibrium, the payoff of each agent is finite and maximized, i.e., \((U^*_t, E^*_t)\) is an optimizer to

\[ \max_{U_t, E_t} \sum_{t=0}^{\infty} \left[ f_i(x_i(t), u_i(t)) + \lambda^* e_i(t) \right] \] s.t. \[ x_i(t+1) = A_i x_i(t) + B_i u_i(t) \]
\[ e_i(t) \leq a_i(t) - h_i(u_i(t)), \quad t \in \mathcal{T}. \] (21)

2) Under the equilibrium, the traded resource is balanced at each time interval, i.e.,

\[ \sum_{i=1}^{N} e^*_i(t) = 0, \quad t \in \mathcal{T}. \] (22)

Similarly, social welfare can be maximized by finding an operating point \((U^*, E^*)\) which solves the following optimization problem:

\[ \max_{U,E} \sum_{i=1}^{N} \sum_{t=0}^{\infty} f_i(x_i(t), u_i(t)) \] s.t. \[ x_i(t+1) = A_i x_i(t) + B_i u_i(t) \]
\[ e_i(t) \leq a_i(t) - h_i(u_i(t)) \]
\[ \sum_{i=1}^{N} e_i(t) = 0, \quad t \in \mathcal{T}, \quad i \in \mathcal{N}. \] (23)

In our previous work [16, Th. 4], we discussed the equivalence between a competitive equilibrium and a social welfare maximization solution over a finite horizon. Using duality theory,
we proved that under some convexity assumptions, Slater’s condition holds, leading to a zero duality gap between the two problems [16, Th. 4]. For an infinite horizon case, however, Slater’s condition is not well established, giving rise to the complexity of the analysis. In this section, we aim to examine the relation between a competitive equilibrium and a social welfare maximization solution over an infinite horizon, using the asymptotic behavior of the pricing sequence.

### A. MAS With General Cost Functions

**Theorem 6:** Let \((\lambda^*, U^*, E^*)\) be an infinite-horizon competitive equilibrium for a feasible initial condition \(x(0) \in \lambda_0\). Then \((U^*, E^*)\) maximizes social welfare over an infinite horizon.

**Proof:** See Appendix VI.

Next, we provide a partially converse statement for Theorem 6, considering initial conditions sufficiently close to the origin.

**Assumption 4:** 1) For \(i \in N\), suppose \(f_i(\cdot)\) is a negative definite (ND) concave function; 2) \(h_i(\cdot)\) is a nonnegative convex function; 3) both \(f_i(\cdot)\) and \(h_i(\cdot)\) pass through the origin; and 4) there exists a lower bound \(C > 0\) on the excess network supply such that \(C(t) \geq C\) for \(t \in T\).

**Theorem 7:** Let Assumption 4 hold. Suppose \((A_i, B_i)\) is controllable for \(i \in N\). Let \(\lambda_0\) be the set of all feasible initial conditions for (21) and (23). Then there exists \(X_0 \subseteq \lambda_0\) such that

1) the set \(X_0\) has a nonempty interior containing the origin;
2) for any \(x(0) \in X_0\), any social welfare maximization solution \((U^*, E^*)\) is part of a competitive equilibrium with \(\lambda^* = 0\) over the infinite horizon.

**Proof:** See Appendix VII.

The next result shows that the price becomes zero after a finite time for any feasible initial condition.

**Theorem 8:** Let Assumption 4 hold. Suppose \(x(0) \in \lambda_0\). Assume \((\lambda^*, U^*, E^*)\) is a competitive equilibrium associated with \(x(0)\). Then there exists a time step \(\bar{T}(x(0))\) such that \(\lambda^*_t = 0\) for \(t > \bar{T}(x(0))\).

**Proof:** See Appendix VIII.

**Remark 4:** The decaying behavior of price in Theorem 8 results from the assumption that the initial conditions are feasible, so the system is stabilizable and \(\lim_{t \rightarrow \infty} x_i(t) = \lim_{t \rightarrow \infty} u_i(t) = 0\). Consequently, after a finite time, the available resources would be more than the required demand to steer the system to the equilibrium point, leading to a zero price. Considering a carbon permit trading system, the decaying behavior of price implies that the environmental policy will be effective in the long term, as the supply of carbon permits will eventually exceed the demand.

### B. MAS With Quadratic Cost Functions

In what follows, we examine a quadratic case and provide analytical results that serve as a benchmark.

Consider \(f_i(\cdot) = f(\cdot; \theta_i)\) and \(h_i(\cdot)\) as in Assumption 2, then the optimization problem (21) can be rearranged as

\[
\begin{align*}
\max_{U, E} & \quad \sum_{t=0}^{\infty} \left[ -x_i(t)^\top Q_i x_i(t) - u_i(t)^\top R_i u_i(t) + \lambda_i^* e_i(t) \right] \\
\text{s.t.} \quad & \quad x_i(t+1) = A_i x_i(t) + B_i u_i(t) \\
& \quad e_i(t) \leq a_i(t) - u_i(t)^\top H_i u_i(t), \quad t \in T
\end{align*}
\]

with the balancing equality constraint \(\sum_{i=1}^{N} e_i(t) = 0\). Also, the social welfare maximization problem (23) becomes

\[
\begin{align*}
\max_{U} & \quad \sum_{i=1}^{N} \sum_{t=0}^{\infty} \left[ -x_i(t)^\top Q_i x_i(t) - u_i(t)^\top R_i u_i(t) \right] \\
\text{s.t.} \quad & \quad x_i(t+1) = A_i x_i(t) + B_i u_i(t) \\
& \quad e_i(t) \leq a_i(t) - u_i(t)^\top H_i u_i(t) \\
& \quad \sum_{i=1}^{N} e_i(t) = 0, \quad t \in T, \quad i \in N.
\end{align*}
\]

Recall that \(x(t)\) and \(u(t)\) are the vectors incorporating the states and control inputs of all agents at time step \(t \in T\), respectively. Denote \(A = \text{blockdiag}\{A_1, \ldots, A_N\}\), \(B = \text{blockdiag}\{B_1, \ldots, B_N\}\), and \(H = \text{blockdiag}\{H_1, \ldots, H_N\}\). In addition, denote \(Q = \text{blockdiag}\{Q_1, \ldots, Q_N\}\) and \(R = \text{blockdiag}\{R_1, \ldots, R_N\}\). According to Lemma 3 in Appendix VII, instead of social welfare maximization (25), we can examine

\[
\begin{align*}
\max_{U} & \quad \sum_{t=0}^{\infty} \left[ -x(t)^\top Q x(t) - u(t)^\top R u(t) \right] \\
\text{s.t.} \quad & \quad x(t+1) = A x(t) + B u(t) \\
& \quad u(t)^\top H u(t) \leq C(t), \quad t \in T
\end{align*}
\]

which is a CLQR problem. Suppose \(P \in \mathbb{R}^{N \times N}\) is a solution to the discrete algebraic Riccati equation

\[
P = A^\top PA + Q - A^\top PB(2B^\top PB + R)^{-1}B^\top PA.
\]

Define \(K = -(R + B^\top PB)^{-1}B^\top PA\). Select

\[
X_0 = \left\{ x(0) \in \mathbb{R}^{Nd} : \|x(0)\|^2 \leq \frac{C\sigma_{\min}(P)}{\sigma_{\max}(P)\sigma_{\max}(K^\top HK)} \right\}.
\]

Next, we expand Theorem 7 exclusively for quadratic MAS.

**Theorem 9:** Suppose \(f_i(\cdot) = f(\cdot; \theta_i)\) and \(h_i(\cdot)\) as in Assumption 2. Let \((A, B)\) be controllable, and \(C(t) \geq C > 0\) for \(t \in T\). If \(x(0) \in X_0\) in (27), then the following statements hold.

1) The social welfare maximization problem (25) is feasible with the following optimal control input:

\[
u^*(t) = Kx(t) = -(R + B^\top PB)^{-1}B^\top PAx(t), \quad t \in T.
\]

2) The optimal price associated with the competitive equilibrium is \(\lambda^*_t = 0\) for \(t \in T\).

**Proof:** See Appendix IX.

**Remark 5:** The zero price in Theorems 7 and 9 results from the assumption that the initial conditions are close enough to
the equilibrium point, where surplus resources are available to stabilize the system. Such assumptions on the initial conditions imply that when the MAS is operating under good enough states \( x(0) \in X_0 \), everyone benefits from such resource sharing at zero cost.

**C. Tracking Problem**

In standard regulation problems, it is desirable to steer the state to zero. However, in real-world applications, the state often follows a reference \( \bar{x} \), which is a well-known class of tracking problems. We can convert a tracking problem into a standard regulation problem by change of variables \( x_{i,e}(t) = x_i(t) - \bar{x} \) and \( u_{i,e}(t) = u_i(t) - \bar{u}_i \), where \( \bar{x} \) is the desired state and \( \bar{u}_i \) is the steady-state control input (see [25, Example 10.3]). In this section, we examine extensions to our proposed results in the context of tracking problems.

Consider the following modified optimization problem instead of (24) to obtain a competitive equilibrium:

\[
\max_{U^i,E^i} \sum_{t=0}^{\infty} \left[ -x^T_{i,e}(t)Q_i x_{i,e}(t) - u^T_{i,e}(t)R_i u_{i,e}(t) + \lambda_i^* e_i(t) \right]
\]

s.t. \( x_{i,e}(t+1) = A_i x_{i,e}(t) + B_i u_{i,e}(t) \)

\[ e_i(t) \leq a_i(t) - (u_{i,e}(t) + \bar{u}_i)^T H_i (u_{i,e}(t) + \bar{u}_i), t \in T \tag{29} \]

where \( U^i = (u_{i,e}^T(0), \ldots, u_{i,e}^T(\infty))^T \). The inequality constraint in (29) can be rearranged as \( e_i(t) \leq a_i(t) - h_i, e_i(u_{i,e}(t)) \), where \( a_i(t) = a_i(t) - u_{i,e}(t) + \bar{u}_i \) and \( h_i, e_i(u_{i,e}(t)) = u_{i,e}^T(t)H_i u_{i,e}(t) + 2u_{i,e}^T H_i u_{i,e}(t) \). With this change of variables, (29) can be represented as the standard form (2).

To be consistent with our previous assumptions, let \( \sum_{t=1}^{N} a_{i,e}(t) > 0 \), i.e., \( \sum_{t=1}^{N} a_{i,e}(t) > \sum_{t=1}^{N} u_{i,e}^T H_i u_{i,e} \). Note that \( h_{i,e}(u_{i,e}(t)) \) is not a nonnegative function, which causes no issue with the presented results. The nonnegativity of \( h_{i,e}(-) \) in Assumption 1 corresponds to the physical interpretation that it represents consumed resources. The social welfare maximization problem can be defined in a similar manner.

Accordingly, the community microgrid problem, introduced in Section II-E1, can be formulated as a tracking problem in which \( \bar{x} \) represents the desired temperature (°C).

**VI. ILLUSTRATIVE EXAMPLES**

In this section, we demonstrate the theoretical results with two synthetic examples and a real-world example of a community microgrid with TCLs.

**A. Social Shaping Over a Finite Horizon**

1) Example 1: Consider a quadratic MAS consisting of three agents operating over the time horizon \( T = 6 \), with the following state-space matrices:

\[
A_1 = \begin{bmatrix} 0.4 & -0.1 & 0.2 \\ 0.2 & 0.3 & 0.1 \\ 0.3 & -0.1 & -0.2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}
\]

\[
A_2 = \begin{bmatrix} -0.1 & 0.2 & -0.3 \\ 0.3 & 0.4 & -0.1 \\ -0.1 & 0.2 & -0.7 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}
\]

\[
A_3 = \begin{bmatrix} 0.5 & -0.2 & 0.3 \\ -0.4 & 0.9 & 0.3 \\ 0.5 & 0.3 & -0.8 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}
\]

and the initial states \( x_1(0) = (25, 35, 75)^T, x_2(0) = (40, 50, 70)^T, \) and \( x_3(0) = (50, 80, 90)^T \). In addition, suppose agents have excess resources \( a_1(t) = -\sin(\frac{\pi}{4} t) + 1.2, a_2(t) = -2\sin(\frac{\pi}{4} t) + 2.2, \) and \( a_3(t) = 0 \). The total excess network generation is \( C(t) = \sum_{i=1}^{3} a_i(t) = -3 \sin(\frac{\pi}{4} t) + 3.4 \).

Considering Assumption 2, agents have \( R_1 = R_2 = R_3 = 0.3I \), and

\[
H_1 = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}, \quad H_3 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}
\]

Let \( \lambda^1 = 20 \). We aim to design \( Q_i \) for \( i \in N \) such that \( \lambda_i^* \leq \lambda^1 \) at all time steps.

**Analytical Approach:** Using Theorems 2 and 3, the upper bound \( \delta_{\max} \) for the parameterized \( Q_i \), is obtained as \( 0.00017 \) and \( 0.00018 \), receptively. Theorem 3 provides a larger upper bound compared to Theorem 2. Therefore, we assign \( \delta_{\max} = 0.00018 \) and select \( Q_i = 0.00018I \) for \( i \in N \). We solve the social welfare maximization in (4) to obtain \( -\lambda_i^* \) as the optimal dual variable associated with the equality constraint \( \sum_{i=1}^{N} e_i(t) = 0 \) for \( t \in T \). The optimal prices \( \lambda_i^* \) are depicted in Fig. 1(a). In Fig. 1(a), we observe that the maximum value of the optimal price is 0.08 which is much less than the price threshold \( \lambda^1 = 20 \), and thus, socially acceptable. This confirms that

\( \lambda^1 = 20 \). The price threshold is selected arbitrarily. It can be any other positive number.

1The price threshold is selected arbitrarily. It can be any other positive number.
Theorems 2 and 3 are valid although they provide conservative results.

**Numerical Approach**: We run Algorithm 1 for 30 steps with the choice of \( d_0 = 1 \) which is sufficiently large and satisfies \( \lambda^*(d_0) = 835.9 > \lambda^1 \). The algorithm converges to \( \delta_{\text{max}} = 0.024 \) which validates Theorem 4. Selecting \( Q_i = 0.024 \) for \( i \in \mathcal{N} \), the optimal prices are obtained and illustrated in Fig. 1(b). In Fig. 1(b), we observe the optimal prices are less than or equal to 20. The maximum value of the price over the entire horizon is \( \lambda^* = 20 \), occurring at time step \( t = 3 \). This validates Theorem 5 and shows that the numerical algorithm provides a tight upper bound for the personalized parameter \( Q_i \), and thus, works well in practice.

In contrast, we select \( Q_i = 0.025 \) for \( i \in \mathcal{N} \), which is outside the sets proposed in Theorems 2 and 3, and Algorithm 1. The obtained optimal price at \( t = 3 \) is \( \lambda^* = 20.8 \) which exceeds the threshold. This is consistent with our expectations. Note that if \( \delta_{\text{max}} \) is less than 0.024 from Algorithm 1 but higher than 0.00018 from Theorem 3, then optimal prices will not exceed the threshold, highlighting the conservativeness of the analytical bounds.

**2) Example 2 (A Community Microgrid)**: Consider a community microgrid consisting of ten buildings and TCLs whose dynamics are represented by \( x_i(t+1) = 0.875x_i(t) + 3.125u_i(t) + 0.125 \) with the initial internal temperature \( x_i(0) = 3 \) (in °C) for \( i \in \mathcal{N} \). The dynamics are derived from discretizing the continuous dynamics in [29] with the sampling time \( \Delta = 0.5 \) (in h). Buildings have excess rooftop PV generation (in kWh) as \( a_1(t) = -a_2(t) = -a_3(t) = -a_4(t) = -a_5(t) = -a_6(t) = -a_7(t) = -a_8(t) = -a_9(t) = -a_{10}(t) = -3\sin(\frac{\pi}{30}t) + 3.3. \) The consumed energy (in kWh) as the result of taking the control action \( u_i(t) \) is \( h_i(u_i(t)) = |u_i(t)|\Delta \) for \( i \in \mathcal{N} \). Suppose agents have quadratic utility functions as in Assumption 2 with the choice of \( \tilde{R}_i = 0.1 \) for \( i \in \mathcal{N} \). Let \( \lambda^1 = 2 \) (in $/kWh). Suppose the ambient temperature is 1 (in °C) and the time horizon is \( T = 50 \). The internal temperature is desired to track a target temperature of 25 (in °C).

We consider a tracking problem similar to (29) with the difference that the inequality constraint is piecewise linear in terms of the control input because \( h_i(u_i(t)) \) is piecewise linear. The coordinator runs Algorithm 1 for 30 steps with the choice of \( d_0 = 1 \) which is sufficiently large and satisfies \( \lambda^*(d_0) = 371.6 > \lambda^1 \) (in $/kWh). The algorithm converges to \( \delta_{\text{max}} = 0.0046 \). In the following, we investigate two scenarios on the choice of personal parameters.

**Scenario I (Personal Parameters Inside the Acceptable Range)**: Agents select \( Q_i = 0.0046 \) for \( i \in \mathcal{N} \) and send it to the coordinator. The coordinator calculates the optimal prices as depicted in Fig. 2, which are less than or equal to \( \lambda^1 = 2 \) (in $/kWh). The maximum value of the price over the entire horizon is \( \lambda^* = 2 \) (in $/kWh), which is consistent with Theorem 5. The state of the last building is depicted in Fig. 3. In Fig. 3, we observe the internal temperature tracking the desired temperature of 25 (in °C) after 30 time steps.

**Scenario II (Personal Parameters Outside the Acceptable Range)**: Suppose the last agent is allowed to select its preference outside the proposed set, e.g., \( Q_{10} = 0.1 \), while others respect the set and select \( Q_i = 0.0046 \) for \( i \in \{1, \ldots, 9\} \). The optimal prices and the state trajectory of the last building are depicted in Figs. 2 and 3, respectively. Fig. 3 illustrates that in this scenario, the last building achieves its desired temperature more effectively compared to Scenario I, which is at the expense of making the price socially unacceptable during time intervals \( t = 16–19 \) (see Fig. 2). This is why in practice, the coordinator does not allow the agents to select their parameters outside the prescribed range.

**B. Competitive Equilibrium Over an Infinite Horizon**

1) **Example 3**: Consider a dynamic MAS with three agents with the following state matrices:

\[
A_1 = \begin{bmatrix}
1.1 & -0.5 & 1.8 \\
-0.4 & 0.6 & 0.7 \\
-0.3 & 0.7 & -0.6
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
0.4 & 1.2 & -0.1 \\
-0.8 & -1.3 & 0.6 \\
0.1 & 0.7 & 0.5
\end{bmatrix}, \quad A_3 = \begin{bmatrix}
0.6 & -1.2 & 0.9 \\
-1.4 & 0.7 & 0.3 \\
-1.5 & 0.7 & 0.1
\end{bmatrix},
\]

(32)

Suppose \( B_i, R_i, \) and \( H_i \) are selected according to Section VI-A 1 and \( Q_i = 0.005I \) for \( i \in \mathcal{N} \). Suppose agents provide excess supply resources \( a_1(t) = 1, a_2(t) = 1.8, \) and \( a_3(t) = 0, \) which leads to the total excess network supply \( C(t) = \sum_{i=1}^{3} a_i(t) = 2.8. \) The forward invariant set \( X_0 \) in (27) represents an L2-ball
and the optimal trading decision are not, and for $X$ and $(27)$, to obtain $E_t \in \{i, j\}$, with the competitive associated with the competitive $(U)$ are not the same, which is because the initial conditions $U$ and $\lambda$ as the Lagrange $(28)$ and denote it as $\lambda_t^i$. In addition, the values of optimal prices $E_t^i$ for $t > 0$, in Case II, Example 3.

Fig. 5. Decaying behavior of optimal price over time intervals, in Case I, Example 3.

of radius 0.21 centered at the origin. In the following, we study two cases on the choice of initial conditions.

Case I (Initial Conditions Outside the Invariant Set $X_0$): Suppose agents have initial conditions, as in Section VI-A 1, which lie outside the set $X_0$ in (27). Let $U^t_{C,ij}$ and $U^t_{S,ij}$ denote the $j$th optimal inputs of agent $i$ which solve the competitive problem (24) and the social welfare maximization (25), respectively. Similarly, denote $E^t_{C,ij}$ and $E^t_{S,ij}$ as the optimal trading decisions of agent $i$ associated with the competitive equilibrium and the social welfare maximization solution, respectively. Similar to the finite horizon case in [16], we solve social welfare maximization (25) to obtain $-\lambda_t^i$ as the Lagrange multiplier of the equality constraint $\sum_{i=1}^N e_i(t) = 0$. In addition, we acquire the associated optimal solution $(U^t_{S,ij}, E^t_{S,ij})$, where $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$, as depicted in Fig. 4. Using the obtained optimal price $\lambda_t^i$, we solve the optimization problem (24) to reach the competitive equilibrium $(U^t_{C,ij}, E^t_{C,ij})$ for $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$, as illustrated in Fig. 4. According to Fig. 4, the competitive equilibrium and the solution of the social welfare maximization coincide, validating Theorem 6. In addition, the values of optimal prices $\lambda_t^i$ over time intervals are depicted in Fig. 5, which indicates $\lambda_t^i = 0$ for $t > 35$. This validates Theorem 8.

Case II (Initial Conditions Inside the Invariant Set $X_0$): Suppose agents have initial conditions $x_1(0) = (0.07, 0.05, 0.08)^\top$, $x_2(0) = (0.1, 0.06, 0.03)^\top$, and $x_3(0) = (0.05, 0.02, 0.1)^\top$, that satisfy $x(0) \in X_0$ in (27). Using the same procedure as in Case I, we obtain the optimal control input $U^t_{S,ij}$ for $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$. In addition, we compute the linear feedback law in (28) and denote it as $U^t_{K,ij}$, representing the $j$th control input of agent $i$, for $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$. Both $U^t_{S,ij}$ and $U^t_{K,ij}$ are depicted in Fig. 6, which illustrates that the two control policies are the same. In Fig. 6, we present optimal prices over respective time intervals, indicating $\lambda_t^i = 0$ at all time intervals. Our results validate Theorems 7 and 9.

Remark 6: In Case I, obtaining $U^t_{K,ij}$, we note that $U^t_{S,ij}$ and $U^t_{K,ij}$ are not the same, which is because the initial conditions are outside the set $X_0$, so the assumptions of Theorem 9 are not satisfied. This is consistent with our expectations.

VII. CONCLUSION

In this article, we investigated the properties of a competitive equilibrium, in particular the price trajectory, in a self-sustained dynamic MAS. First, we focused on a finite horizon case and showed the social shaping problem is solvable both implicitly (for general classes of utility functions) and explicitly (for quadratic MAS). Furthermore, we presented a numerical algorithm which provides more accurate results compared to the proposed analytical solutions. Second, we studied a competitive equilibrium over an infinite horizon. We examined the relationship between a competitive equilibrium and the solution of the social welfare maximization problem. In addition, we investigated the decaying behavior of the price trajectory which depends on the system’s initial state. Finally, we focused on

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quadratic MAS and the associated CLQR problem to obtain explicit results validating the previous findings. In future work, potential extensions to systems with nonlinear dynamics in the presence of disturbances and uncertainties are possible, leading to the concept of robust social shaping. Furthermore, incorporating additional physical constraints to the framework (e.g., voltage constraints and line capacity constraints in a community microgrid, or bounds on carbon emissions in the carbon permit trading system) are also possible. Future work might also consider developing consensus-based algorithms to determine the price threshold.

**APPENDIX I**

**PROOF OF LEMMA 1**

The proof is straightforward following [16, Proposition 2]. By contradiction, suppose there exists a time step $t \in T$ at which $\lambda^*_i < 0$. Then, the objective function in (2) is strictly decreasing with respect to $e_i(t)$. Since $e_i(t)$ is unbounded below, there cannot be a finite $e_i(t)$ which solves (2). This contradicts the definition of competitive equilibrium. Therefore, it follows that $\lambda^*_i \geq 0$ for all $t \in T$.

**APPENDIX II**

**PROOF OF THEOREM 2**

Considering the equality in (5), we obtain

$$\sum_{i=1}^{N} u_i^T(t)H_i u_i(t) = C(t), \quad t \in T. \quad (33)$$

Furthermore, the following inequality holds:

$$\sigma_{\min}(H_i) \|u_i^*(t)\|^2 \leq u_i^T(t)H_i u_i(t). \quad (34)$$

In addition, since $u_i^T(t)H_i u_i(t) \geq 0$, the equality in (33) yields $u_i^T(t)H_i u_i(t) \leq C(t)$. Following from (34), we obtain

$$\sigma_{\min}(H_i) \|u_i^*(t)\|^2 \leq C(t). \quad (35)$$

According to Assumption 3, we have $H_i \succeq \rho I$, meaning that $\sigma_{\min}(H_i) \geq \rho$. Consequently, inequality (35) results in

$$\|u_i^*(t)\| \leq \sqrt{\frac{C(t)}{\rho}}. \quad (36)$$

In addition, from the dynamical equation in (16), we obtain

$$x_i(t) = A_i^e(t)0 + \sum_{j=0}^{t-1} A_i^{t-j-1}B_i u_i(j), \quad t \in \{1, 2, \ldots, T\}. \quad (37)$$

Substituting (37) into (16) yields an unconstrained optimization problem, in which the only decision variable is $U_i$. The associated objective function $J$ is

$$J = \sum_{t=0}^{T-1} \left[ -u_i^T(t)R_i u_i(t) + \lambda^*_i \left( a_i(t) - u_i^T(t)H_i u_i(t) \right) \right]. \quad (38)$$

Let $k$ denote a time interval indexed in $T$. Setting $\frac{\partial J}{\partial u_i(k)} = 0$, applying some matrix manipulations, and using (33), we obtain

$$\lambda^*_k = -\frac{1}{C(k)} \sum_{t=k+1}^{T} \left[ \sum_{i=1}^{N} (A_i^{t-k-1}B_i u_i(k))^T Q_i \left( A_i^e x_i(0) + \sum_{j=0}^{t-1} A_i^{t-j-1}B_i u_i(j) \right) - \frac{1}{C(k)} \sum_{i=1}^{N} u_i^T(k)R_i u_i(k) \right]. \quad (39)$$

Since $Q_i > 0$, we obtain

$$- (A_i^{t-k-1}B_i u_i(k))^T Q_i A_i^{t-k-1}B_i u_i(k) \leq 0. \quad (40)$$

Similarly, $R_i > 0$, and $-u_i^T(k)R_i u_i(k) \leq 0$. Next, we seek an upper bound for $\lambda^*_k$. First, let these two terms equal zero. Then use Assumption 3 and the inequality in (36), and by substitution into the norm of (39), we yield

$$\lambda^*_k \leq \frac{1}{C(k)} \sum_{t=k+1}^{T} \left[ N \alpha^{2t-k-1} \beta \sqrt{\frac{C(k)}{\rho}} \delta_{\max} \gamma \right]$$

$$+ N \beta^2 \sqrt{\frac{C(k)}{\rho}} \delta_{\max} \sum_{j=0}^{t-1} \alpha^{2t-j-k-1} \sqrt{\frac{C(j)}{\rho}} \right]. \quad (41)$$

By assumption, considering (17), the right-hand side of (41) is less than or equal to $\lambda^1$. Therefore, we obtain $\lambda^*_k \leq \lambda^1$.

**APPENDIX III**

**PROOF OF THEOREM 3**

Similar to the previous section, (33) and (36) are satisfied. Since $\lambda^*_i > 0$, we obtain $R_i + \lambda^*_i H_i > 0$. Consequently, the optimization problem in (18) is a standard LQR problem. Therefore, at each time step $k \in T$, the optimal control solution is obtained as $u_i^*(k) = -(B_i^T P_{i,k+1} B_i + (R_i + \lambda^*_i H_i))^{-1}B_i^T P_{i,k+1} A_i x_i(k)$, where

$$P_{i,k} = A_i^e P_{i,k+1} A_i + Q_i - A_i^e P_{i,k+1} B_i (B_i^T P_{i,k+1} B_i + (R_i + \lambda^*_i H_i))^{-1}B_i^T P_{i,k+1} A_i. \quad (42)$$

The Riccati difference equation in (42) is initialized with $P_{i,T} = Q_i$, and is solved backward from $k = T - 1$ to $k = 0$. In addition, since the last term on the right-hand side of (42) is negative semidefinite, we obtain $P_{i,k} \leq A_i^e P_{i,k+1} A_i + Q_i$, and therefore, $\|P_{i,k}\| \leq \alpha^2 \|P_{i,k+1}\| + \|Q_i\|$. Starting from $k = T - 1$, we obtain

$$\|P_{i,T-1}\| \leq (\alpha^2 + 1) \|Q_i\|$$

$$\vdots$$

$$\|P_{i,T-p}\| \leq (\alpha^2 p + \alpha^{2(p-1)} + \cdots + \alpha^2 + 1) \|Q_i\|. \quad (43)$$

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To find the optimal control input, we start from $k = 0$ and proceed in a forward manner. In the first step, we obtain

$$u^*_i(0) = -(B_i^T P_{i,1} B_i + R_i + \lambda_0^* H_i)^{-1} B_i^T P_{i,1} A_i x_i(0).$$

(44)

Premultiply (44) by $u_i^T(0) (B_i^T P_{i,1} B_i + R_i + \lambda_0^* H_i)$ and then taking the summation over $i$, where $i \in \{1, \ldots, N\}$, we obtain

$$\sum_{i=1}^{N} u_i^T(0) (B_i^T P_{i,1} B_i + R_i + \lambda_0^* H_i) u_i^*(0)$$

$$= - \sum_{i=1}^{N} u_i^T(0) B_i^T P_{i,1} A_i x_i(0).$$

(45)

Substitute (33) into (45), and yield

$$\lambda_0^* = - \frac{1}{C(0)} \sum_{i=1}^{N} u_i^T(0) (B_i^T P_{i,1} B_i + R_i + \lambda_0^* H_i) u_i^*(0)$$

$$- \frac{1}{C(0)} C(0) \sum_{i=1}^{N} u_i^T(0) (B_i^T P_{i,1} B_i + R_i) u_i^*(0).$$

(46)

To obtain an upper bound for $\lambda_0^*$, we omit the second term on the right-hand side of (46) which is always nonpositive.

In addition, using norm properties and considering Assumption 3 and the inequalities in (36) and (43), we obtain

$$\lambda_0^* \leq \frac{N}{C(0)} \sqrt{\frac{C(0)}{\rho}} \beta \gamma \delta \sum_{t=1}^{T} \alpha^{2t-1}.$$

(47)

Moving forward to the time step $k > 0$, we obtain

$$\lambda_k^* \leq \frac{N}{\sqrt{C(k)\rho}} \beta \delta \sum_{t=k+1}^{T} \left[ \gamma \alpha^{2t-k-1} + \beta \sum_{j=k+1}^{t-1} \sqrt{\frac{C(j)}{\rho}} \alpha^{2t-j-k-2} \right].$$

(48)

By assumption, considering (19), the right-hand side of (47) and (48) is less than or equal to $\lambda^*$, which confirms $\lambda_k^* \leq \lambda^*$ for $k \in \mathcal{T}$.

**APPENDIX IV**

**PROOF OF THEOREM 4**

From the update rules in Algorithm 1, we obtain

$$b_{k+1} \geq b_k, \quad d_{k+1} \leq d_k$$

(49)

and

$$b_0 \leq b_k < L_k < d_k \leq d_0.$$  

(50)

Inequalities in (49) and (50) imply that $b_k$ is monotonically increasing and bounded above by $d_0 = d_g$. Similarly, $d_k$ is monotonically decreasing and bounded below by $b_0 = 0$. Consequently, $b_k$ and $d_k$ converge when $k \to \infty$.

In addition, from the algorithmic steps, we obtain $|b_k - d_k| = 0.5^k d_g$. Therefore,

$$\lim_{k \to \infty} |b_k - d_k| = 0.$$  

(51)

Considering (49), (50), and (51), we conclude

$$\lim_{k \to \infty} b_k = \lim_{k \to \infty} d_k = \lim_{k \to \infty} L_k = L^*$$

(52)

where $b_k < L^* < d_k$ at each iteration, and $L^* \in (0, d_0)$.

**APPENDIX V**

**PROOF OF THEOREM 5**

If the algorithm stops when $\lambda_k = \lambda^*$, we end with $\delta_{\text{max}} = L_k$ and $\lambda^*(\delta_{\text{max}}) = \lambda^*$. Otherwise, we obtain $\delta_{\text{max}} = \lim_{k \to \infty} L_k = L^*$. By contradiction, suppose $\lambda^*(L^*) \neq \lambda^*$ leading to $L^* \neq \delta^*$. First, consider $L^* < \delta^*$. According to (52), we obtain

$$\exists l > 0 \text{ s.t. } d_k < \delta^* \text{ for } \forall k \geq l.$$  

(53)

Following Lemma 2, we obtain $\lambda^*(d_k) \leq \lambda^*(\delta^*) = \lambda^*$, which contradicts the assumptions in Algorithm 1. If $L^* > \delta^*$, a similar analysis can be made for $b_k$ which leads to a contradiction. Consequently, it follows that $\lambda^*(L^*) = \lambda^*(\delta_{\text{max}}) = \lambda^*$.

**APPENDIX VI**

**PROOF OF THEOREM 6**

As the competitive equilibrium exists for $x(0) \in X_0$, the corresponding optimization problem in (21) is feasible. In addition, the optimal price $\lambda^*$ and the optimal solution $(U^*, E^*)$ are well defined over the infinite horizon. So, we merge the optimization problems in (21) associated with all agents $i \in \mathcal{N}$ to obtain

$$\max_{U, E} \sum_{i=1}^{N} \sum_{t=0}^{\infty} f_i(x_i(t), u_i(t)) + \sum_{i=1}^{N} \lambda_i^* \sum_{t=0}^{\infty} e_i(t)$$

subject to

$$x_i(t + 1) = A_i x_i(t) + B_i u_i(t)$$

and

$$e_i(t) \leq a_i(t) - h_i(u_i(t)), \quad t \in \mathcal{T}, \quad i \in \mathcal{N}.$$  

(54)

such that $\sum_{i=1}^{N} e_i^*(t) = 0$. Clearly, the solution $(U^*, E^*)$ which solves (54) is also a solution to (23). In addition, both (23) and (54) have the same optimal value which is finite. Hence, social welfare maximization (23) is feasible for $x(0) \in X_0$ with the optimal solution $(U^*, E^*)$.

**APPENDIX VII**

**PROOF OF THEOREM 7**

**A. Preliminary Lemma**

In the proof of Theorem 7, we use the following lemma.

**Lemma 3:** Let the social welfare maximization problem be feasible for $x(0) \in X_0$. Then (23) can be solved through the following constrained optimal control problem:

$$\max_{U} \sum_{t=0}^{\infty} \sum_{i=1}^{N} f_i(x_i(t), u_i(t))$$

subject to

$$x_i(t + 1) = A_i x_i(t) + B_i u_i(t)$$

and

$$\sum_{i=1}^{N} h_i(u_i(t)) \leq C(t), \quad t \in \mathcal{T}, \quad i \in \mathcal{N}.$$  

(55)
Proof: Feasibility of (23) leads to the feasibility of (55). Suppose \( U^* \) is a solution to (55). Decomposing \( U^* \) to \( u^*_i(t) \) for \( t \in \mathcal{T} \) and \( i \in \mathcal{N} \), we define
\[
e^*_i(t) = a_i(t) - h_i(u^*_i(t)) + \frac{1}{N} \left( \sum_{i=1}^{N} h_i(u^*_i(t)) - \sum_{i=1}^{N} a_i(t) \right)
\]
which satisfies
\[
e^*_i(t) \leq a_i(t) - h_i(u^*_i(t)) \leq \frac{N}{N} \sum_{i=1}^{N} e^*_i(t) = 0.
\]
Constructing \( E^* \) based on \( e^*_i(t) \) defined in (56), we conclude \( (U^*, E^*) \) solves (23).

B. Proof of the Theorem

According to Lemma 3, we examine (55). The time-varying inequality constraint \( \sum_{i=1}^{N} h_i(u_i(t)) \leq C(t) \) in (55) represents a constraint set \( \Omega_t \subseteq \mathbb{R}^{Nn} \) on control inputs, containing the origin in its interior. Similarly, the time-invariant inequality \( \sum_{i=1}^{N} h_i(u_i(t)) < C \) represents a constraint set \( \Omega \subseteq \mathbb{R}^{Nn} \), containing the origin in its interior. Since \( C(t) \geq C \) for all \( t \), there holds \( \Omega \subseteq \Omega_t \). Therefore, without loss of generality, we replace the time-varying inequality constraint \( \sum_{i=1}^{N} h_i(u_i(t)) \leq C(t) \) in (55) with the time-invariant constraint \( \sum_{i=1}^{N} h_i(u_i(t)) < C \). For \( i \in \mathcal{N} \), consider all controllers restrained by \( \|u_i(t)\|_\infty \leq \epsilon \) for sufficiently small \( \epsilon > 0 \) such that the controllers lie in the interior of the constraint set \( \Omega \). According to [35], the set of all feasible initial conditions that can be steered to the origin in a finite time when control inputs satisfy saturation constraints \( \|u_i(t)\|_\infty \leq \epsilon \) has a nonempty interior containing the origin; to be consistent with the literature, we denote this set of initial conditions as the null controllable region \( \mathcal{C} \). Since \( f_i(\cdot) \) is a negative definite function passing through the origin, the problem in (55) is feasible for initial conditions \( x(0) \in \mathcal{C} \). Now select a neighborhood of the origin \( \mathcal{X}_0 \subseteq \mathcal{C} \) such that all \( x(0) \in \mathcal{X}_0 \) satisfy \( \|x(0)\| \leq \epsilon \). If \( \epsilon \) is selected sufficiently small then the optimal control \( u^*_i(t) \) which solves (55) lies in the set \( \mathcal{C} \), i.e., \( \sum_{i=1}^{N} h_i(u^*_i(t)) < C \) for \( t \in \mathcal{T} \). This is because \( h_i(\cdot) \) is convex in an open neighborhood of the origin, and therefore, continuous [36]. Hence, the optimal control input for (55) would be the solution to the following unconstrained optimal control problem:
\[
\max_{U^*} \sum_{t=0}^{\infty} \sum_{i=1}^{N} f_i(x_i(t), u_i(t))
\]
\[
s.t. \quad x_i(t+1) = A_i x_i(t) + B_i u_i(t), \quad t \in \mathcal{T}, \ i \in \mathcal{N}.
\]
For more insights into the sets, see Fig. 7. The problem in (58) is separable, so \( U^* \) is also a solution to
\[
\max_{U^*} \sum_{t=0}^{\infty} f_i(x_i(t), u_i(t))
\]
\[
s.t. \quad x_i(t+1) = A_i x_i(t) + B_i u_i(t), \quad t \in \mathcal{T}, \ i \in \mathcal{N}.
\]

\[\text{Fig. 7. Three feasible sets in the proof of Theorem 7.}\]

\[
s.t. \quad x_i(t+1) = A_i x_i(t) + B_i u_i(t), \quad t \in \mathcal{T}
\]
\[
\text{for } i \in \mathcal{N}. \text{ Define } e^*_i(t) \text{ as in (56) which leads to (57). Denoting } E^*_i(t) = (e^*_i(0), \ldots , e^*_i(\infty))^\top \text{ for } i \in \mathcal{N}, \text{ we conclude } (U^*, E^*) \text{ is an optimizer for the optimization problem in (21) under } \lambda^*_i = 0 \text{ such that (22) is satisfied. Consequently, } (U^*, E^*), \text{ along with } \lambda^*_i = 0 \text{ for } t \in \mathcal{T}, \text{ constitutes a competitive equilibrium.}
\]

APPENDIX VIII

PROOF OF THEOREM 8

As the competitive equilibrium exists for \( x(0) \in \mathcal{X}_0 \), the optimal price \( \lambda^* \) and the optimal solution \( (U^*, E^*) \) are well-defined over the infinite horizon. According to Theorem 6, \( (U^*, E^*) \) maximizes social welfare. Considering Lemma 3, we examine (55). Since the problem is feasible, the optimal value is finite. Considering that \( f_i(x_i(t), u_i(t)) \) is a negative definite function, we obtain \( \lim_{t \to \infty} f_i(x_i(t), u_i(t)) = 0 \). Concavity of \( f_i(\cdot) \) in an open neighborhood of the origin implies its continuity. As \( f_i(\cdot) \) is a negative definite concave function passing through the origin, we obtain \( \lim_{t \to \infty} f_i(x_i(t), u_i(t)) = 0 \) for \( i \in \mathcal{N} \). Similarly, \( h_i(\cdot) \) is continuous in an open neighborhood of the origin. Therefore, there exists a finite time \( \bar{T}(x(0)) \) such that \( \sum_{i=1}^{N} h_i(u^*_i(t)) < C(t) \) for \( t > \bar{T}(x(0)) \). In the remainder of this proof, by \( \bar{T} \) we mean \( \bar{T}(x(0)) \). Decompose \( U^* \) to \( u^*(t) \) for \( t \in \mathcal{T} \). Applying the first \( T+1 \) optimal control inputs \( u^*(0), \ldots , u^*(T) \) to the system dynamic, we reach \( x(T+1) \). Denote \( U^*_{T+1} = (u^*\top(T+1), \ldots , u^*\top(\infty))\top \). Based on the principle of optimality, \( U^*_{T+1} \) solves the unconstrained optimal control problem in (58) starting from \( t = T + 1 \) with the initial condition \( x(T+1) \). Also, \( U^*_{T+1} \) solves (59) starting from \( t = T + 1 \) with the initial condition \( x(T+1) \) for \( i \in \mathcal{N} \). According to the proof of Theorem 7, the solution to (59) constitutes a competitive equilibrium with zero price. Therefore, \( \lambda^*_i = 0, \) for \( t > \bar{T} \).

APPENDIX IX

PROOF OF THEOREM 9

1) Consider \( \beta > 0 \). Any sublevel set
\[
\mathcal{X} = \{ x \in \mathbb{R}^d : x^\top P x \leq \beta \}
\]
is forward invariant for the closed-loop system \( x(t+1) = (A + BK)x(t) \) [37]. If \( \beta \) is selected such that
the control input \( u(t) = Kx(t) \) satisfies the inequality constraint in (26) for all \( x(t) \in X \), then the origin is locally asymptotically stable and the region of attraction contains \( X \) for the CLQR problem (26). Considering \( u(0) = Kx(0) \), we obtain \( u^\top(0)Hu(0) = x^\top(0)K^\top HKx(0) \). In addition, the following inequality holds:

\[
\sigma_{\min}(K^\top HK) \|x(0)\|^2 \leq x^\top(0)K^\top HKx(0) \leq \sigma_{\max}(K^\top HK) \|x(0)\|^2.
\]

(61)

According to (27), we have \( \sigma_{\max}(K^\top HK)\|x(0)\|^2 \leq C\sigma_{\min}(P)/\sigma_{\max}(P) \). Consequently, (61) yields

\[
u^\top(0)Hu(0) \leq \frac{C\sigma_{\min}(P)}{\sigma_{\max}(P)} \leq C
\]

(62)

which satisfies the inequality constraint in (26). In addition, the set \( X_0 \) in (27) is a subset of the forward invariant set in (60) with the choice of \( \beta = C\sigma_{\min}(P)/\sigma_{\max}(P) \). Since \( u(0) = Kx(0) \) satisfies the inequality constraint and \( x(0) \) lies in the invariant set \( X \) in (60), then \( x(1) \) also lies in the invariant set \( X \) in (60), i.e., \( x^\top(1)Px(1) \leq \sigma_{\min}(P)/\sigma_{\max}(K^\top HK) \). Considering \( \sigma_{\min}(P)\|x(1)\|^2 \leq x^\top(1)Px(1) \), we yield \( \|x(1)\|^2 \leq C\sigma_{\max}(K^\top HK) \). Then, the feedback law \( u(1) = Kx(1) \) leads to \( u^\top(1)Hu(1) \leq \sigma_{\max}(K^\top HK)\|x(1)\|^2 \leq C \), which satisfies the inequality constraint. Using the same logic, all future states \( x(t) \) lie in the invariant set \( X \) in (60), and all future control inputs \( u(t) = Kx(t) \) satisfy the inequality constraint. So, with the initial conditions in (27), we treat (26) as an unconstrained LQR problem. Since \( (A, B) \) is controllable, \( Q > 0 \), and \( R > 0 \), the optimization problem (26) is feasible with the optimal solution (28). According to Lemma 3, the social welfare maximization problem (25) is also feasible and the optimal control is as (28).

2) Follows directly from Theorem 7.

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