Learning-Based Approaches for Graph Problems: A Survey

Kai Siong Yow∗and Siqiang Luo†
School of Computer Science and Engineering, College of Engineering, Nanyang Technological University, Singapore
April 19, 2022

Abstract

Over the years, many graph problems specifically those in NP-complete are studied by a wide range of researchers. Some famous examples include graph colouring, travelling salesman problem and subgraph isomorphism. Most of these problems are typically addressed by exact algorithms, approximate algorithms and heuristics. There are however some drawbacks for each of these methods. Recent studies have employed learning-based frameworks such as machine learning techniques in solving these problems, given that they are useful in discovering new patterns in structured data that can be represented using graphs. This research direction has successfully attracted considerable amount of attention. In this survey, we provide a systematic review mainly on classic graph problems in which learning-based approaches have been proposed in addressing the problems. We discuss the overview of each framework, and provide analyses based on the design and performance of the framework. Some potential research questions are also suggested. Ultimately, this survey gives a clearer insight and can be used as a stepping stone to the research community in studying problems in this field.

Keywords: graph algorithm, graph optimisation, graph problem, NP-complete problem, learning-based approach, machine learning

1 Introduction

A graph $G = (V, E)$ is a non-linear data structure that contains of a set of vertices (or nodes) $V$ and a set of edges (or links) $E$. Edges can either be undirected or

∗Email: kaisiong.yow@ntu.edu.sg (The author is currently on Leave of Absence from the Department of Mathematics and Statistics, Faculty of Science, Universiti Putra Malaysia.)
†Email: siqiang.luo@ntu.edu.sg
directed, and a symbolic label or a numeric attribute could be associated with each edge in a graph. Graphs can be used to model many aspects in daily life including transportation system, connectivity of the World Wide Web, social networks and scheduling.

Karp [77] showed that there are 21 combinatorial and graph theoretical problems that can be reduced to the boolean satisfiability problem in polynomial time based on Cook’s theorem [31], which implies that these problems are all NP-complete. Out of the 21 problems, ten are decision versions of graph optimisation problems whereas others can be formulated on graphs. Some well known NP-complete problems include 3-satisfiability (3-SAT), clique, vertex cover, graph colouring and partition. More NP-complete problems can also be found in [50]. Due to the nature of these problems, the time required to solve them grows exponentially to the size of the problems, by using any of the currently known algorithm.

Generally, most of the NP-hard graph optimisation problems are tackled by exact algorithms, approximate algorithms and heuristic. Exact algorithms usually solve a problem to optimality based on integer programming. However, they are not suitable in handling large instances due to the scalability issues. Approximate algorithms are generally used to approximate optimal solutions in polynomial time. However, they may not output the exact solutions but only solutions with weak optimality guarantees (which may not even exist sometimes). To solve problems in a more efficient manner, heuristic techniques that sacrifice optimality and accuracy are designed. They are often lack of theoretical guarantees and requires substantial problem-specific research. Since similar combinatorial structures are used to solve the same type of problems, learning-based approaches are introduced in addressing combinatorial optimisation problems in NP-complete.

Machine learning (ML) [100, 7] is one of the artificial intelligence techniques that is used for sophisticated tasks. It is usually used to solve complex multidimensional problems that cannot be solved by numerical reasoning or cannot be described. Machine learning is also useful especially in discovering new patterns in graphs. Hence, it is natural for researchers to apply this learning-based approach in solving NP-hard graph optimisation problems. Machine learning methods use the available past information to make predictions or to improve performances. The models are built based on training data and decisions are made without being explicitly programmed. The applications of ML algorithms include image recognition, fraud detection, games [117] and recommendation systems.

Common approaches in ML are supervised, unsupervised and semi-supervised learning. In a supervised learning, a set of labelled samples is used to make predictions for unseen points. It is commonly used in classification, regression and ranking problems. In an unsupervised learning, only unlabelled samples are used. Instead of responding to feedback, it identifies commonalities in the data and responds according to the presence or absence of such commonalities. Clustering and dimensionality reduction are among the examples of unsupervised learning problems. Semi-supervised learning fall between supervised and unsupervised learning. A set of training samples
that consist of labelled and unlabelled samples are used to predict all unseen points. This approach is beneficial when labelled samples are expensive and unlabelled samples are easily accessible. Another popular approach that is also widely studied is reinforcement learning, where an agent interacts with its environment in discrete time steps, and learns an (nearly) optimal policy to maximise the reward over a course of actions. Note that deep reinforcement learning was shown in achieving outstanding performances in some board games including classic Atari games [99] and Go [117].

The usage of ML techniques in solving combinatorial optimisation problems is increasing in recent years. Bengio et al. [14] commented that models that are formed by combining ML techniques and combinatorial optimisation enhance the training procedures. Due to the natural structure in combinatorial optimisation, symbolic problems are dissected into smaller learning tasks. In addition, the primary ML strategies is to explore decision space and search for the best performing behaviours. Note also that it is rather challenging to design an optimal machine learning algorithm particularly for graph tasks, automated machine learning (AutoML) that combines the advantages of graph machine learning and AutoML becomes a popular research direction. A relevant survey on AutoML can be found in [141] where the focus is on two major topics, hyperparameter optimisation (or tuning) and neural architecture for graph machine learning. In the former, the attention is on how scalable methods are developed whereas in the latter, they compared different methods including search strategies, search spaces and performance estimation.

In this article, we review classic graph problems that have been addressed by using learning-based algorithms, particularly those employ ML techniques. Note that surveys in this area are relatively limited compared to others. Hence, we provide a thorough review to discuss relevant work. For each specific task in a given graph problem, we introduce the learning framework that was proposed. Relevant analyses based on the performance and the design of the model are also given.

We categorise each graph problem according to the nature of the task and the associated approach/method that were used to tackle the problem. These problems are grouped into ten categories. The first eight are classic graph problems discussed in §2, they are clique, partition, subgraph isomorphism, dominating set, travelling salesman problem, graph colouring, vertex cover and maximum cut. The remaining two graph problems that can be found in §3 are routing problems that are modelled by graphs where the focus is on route recommendation and forecast, and community search problems. Note that the routing problem is a generalisation of an NP-complete problem namely the travelling salesman problem, and the community search problem is closely related to the partition problem. We define each problem formally, and discuss the methods that are used to solve the problem in details. Some potential future work are also pointed out at the end of the respective section. An overview of these problems can be found in Table 1.

Our main contributions are summarised as follows:

- We give a systematic review on classic graph problems and some related problems that have been addressed by using learning-based approaches. An overview
of the learning frameworks including their designs are analysed.

• For each problem, we identify its applications and existing approaches that were used in solving the respective task. A brief analysis on their performance is given.

• We point out and suggest some potential problems for further study in this field, which will serve as a stepping stone to explore this area further in order to discover more insightful results.

2 Classic Graph Problems

The graph terminology used in this article is mostly standard, unless stated otherwise. We mainly follow those defined in [36].

In this section, we discuss eight classic graph NP-complete problems that have been tackled by some learning-based approaches. Some of these problems contain more than one task, and some of them have also been addressed by more than one learning-based framework.

2.1 Clique

Let $G = (V, E, W)$ be an undirected graph where $V, E$ and $W$ denote the set of vertices, edges and weight of vertices, respectively. A clique $C$ of $G$ is a subset of vertices such that each pair of vertices are adjacent (or there is an edge between every pair of vertices). A clique can be represented using a binary string $x$ where $x_i = 1$ if the vertex $v_i$ is in the clique. Otherwise, we have $x_i = 0$. The maximum weight clique (MWC) is a clique where the total weight of its vertices is the largest. Let $\bar{G} = (V, \bar{E}, W)$ be the complement graph of $G$ and $x_i \in \{0, 1\}$. For $i \in \{1, 2, \ldots, |V|\}$, the MWC problem can be defined as follows:

$$\max_x \sum_{i=1}^{|V|} w_i x_i$$

such that $\forall (i, j) \in \bar{E}, x_i + x_j \leq 1$.

The MWC problem has many applications. For example, in computer vision [16], genomics [18] and protein structure prediction [96].

Due to the issue of the search space, solving the MWC problem is in fact challenging. A natural way to solve large-scale optimisation problems is to first reduce the problems, either by using exact or greedy methods. In the former, decision variables that should not appeared in the optimal solution are removed according to analytical reasoning [48, 73] whereas in the latter, decision variables are removed according to a measure of fitness [83].
Many algorithms that are designed in solving the MWC problem are either exact [45, 73, 61, 72, 86] or heuristic [106, 19, 131, 42]. Algorithms based on exact solvers are typically hard in scaling to dense graphs of large sizes. The differences are on the branching strategy and the upper bound calculation. Heuristic methods are often more suitable for graphs of small and medium sizes, but are not effective in handling large graphs that have many local optima.

Some recent works employ machine learning methods to address the MWC problem. One recent representative machine-learning based method was proposed by Sun et al. [119]. They employed a greedy approach to remove unpromising decision variables in forming the optimal solution. Ranking-based and correlation-based measures are used to evaluate each decision variable, and a framework based on machine learning namely machine learning for problem reduction (MLPR) is developed. In this framework, problem reduction is modelled as a binary classification problem, and it is trained using easy instances based on a supervised learning approach where the optimal solution is known. The model is then used to predict decision variables that belong to a given hard problem. The approach is evaluated using the MWC problem.

Particularly, they first proposed a random sampling algorithm [119] to construct a feasible solution incrementally, where decision variables are added in a random manner. For the MWC problem, a clique forms a feasible solution and a vertex is a decision variable. The time complexity for the random sampling method is also given. They then quantified each decision variable by using ranking-based and correlation-based measures. The ranking-based measure calculates an accumulated score for each vertex and the correlation-based measure calculates the Pearson correlation coefficient among each vertex. Vertices with scores lower than certain threshold are removed to reduce the graph size.

For MLPR, easy graphs that are used as the training set are solved using an exact solution method. To construct a training set, they extracted features (vertex weight, vertex degree, upper bound on the weight of cliques, graph density and the two ranking-based measures) and labelled each vertex to either 1 or -1, depending on whether the vertex belongs to the optimal solution. By using a support vector machine (SVM) [15], they trained a classification model and used it to predict the label for each vertex in hard instances. Those vertices that are predicted to be 1 are included in the optimal solution. Existing methods are then used to solve the MWC problem based on the reduced graph.

One crucial observation they have is that the optimal solution to the original graph can be generated by solving a reduced graph, especially for sparse graphs. For medium graphs, the statistical measures also perform better than the problem specific features. They also observed that the least number of vertices are needed in the radial basis function kernel to obtain the optimal solution for medium-sized dataset, compared to linear, polynomial and sigmoid kernel functions.

By first preprocessing the data using the proposed problem reduction techniques, they concluded that the performance of the existing methods can be improved. Two exact solvers [73, 72] and two heuristic methods [19, 131] are tested. All the source
codes are available online.

The current learning approach in solving the MWC problem is by first reducing the size of the original problem, and use some existing techniques to obtain the solution. Given that the problem reduction techniques show a promising result in the MWC problem, a natural way to extend this is to test the technique on some other combinatorial optimisation problems. A comprehensive automated problem reduction technique that uses a huge mixed integer programming as an input could also be developed [119]. One could also investigate if additional features could be used when constructing a training set, to further reduce the solution space without deteriorating the accuracy. It is worth to identify what will be the best threshold for the size of the reduced problem in order to achieve the optimal solution in a more efficient manner. A learning framework that can be applied on the original problem can also be designed.

2.2 Partition

Graph partition (also known as community detection) helps to identify relationships among vertices in a given graph. There are many applications in community detection, where it could be used in predicting and identifying missing/false links in a network. It also appears in social network analysis to detect groups that have similar properties. Most of the work in solving community detection problems focus on just vertices and edges of a given graph [85]. However, some algorithms combine multiple aspects of a graph in order to obtain a more accurate result in community detection [123, 103]. We discuss two methods in solving community detection problems related to multi-view graphs and Covid-19 dataset.

A multi-aspect (or multi-view) graph is a collection of graphs that has the same vertex set but different set of edges for each view [59]. Every view in a multi-view graph can be represented by an adjacency matrix. Therefore, a multi-view graph with $k$ views and $n$ vertices can be represented by $k n \times n$ adjacency matrices.

**Problem 2.1** ([59]). Given a multi-view graph and a $p\%$ (p is small) of vertex labels to $r$ communities, find an assignment of all vertices of the graphs to at least one of the $r$ communities.

To address Problem 2.1, Gujral and Papalexakis [59] proposed a semi-supervised multi-aspect community detection (SMACD) method to detect overlapping and non-overlapping communities for multi-view graphs. They assumed that $r$ is given to simplify the problem. The authors claimed that this is the first attempt to incorporate information of multi-view graphs and semi-supervision in solving this problem. A parameter tuning algorithm that does not depend on the partial vertex labels is also developed, so that SMACD is parameter-free. The MATLAB implementation and the synthetic data can be found here, and all the data used for comparison purposes are also available online.

The input of SMACD is the number of communities $r$, a tensor $X \in \mathbb{R}^{n \times n \times k}$ that contains a multi-view graph, and a matrix $Y \in \mathbb{R}^{n \times r}$ that contains vertex as-
signments to communities. The algorithm first computes the sparse decomposition of \( r - 1 \) components, and then assigns each vertex to a community by searching for the community that has the maximum members. They also designed an algorithm, namely non-negative sparse coupled matrix-tensor factorisation (NNSCMTF), by including two extra constraints (non-negativity and latent sparsity constraints) into the coupled matrix-tensor factorisation (CMFT) [2] model. The two proposed constraints are meant to use for community detection.

For the latent sparsity constraint, the sparsity regulariser penalty \( \lambda \) is used in the proposed objective function, given that the function is highly non-convex and hardly to be optimised. Traditionally, the parameter \( \lambda \) is determined via a trial-and-error manner, and it brings some impact to the final result if it is not properly chosen. To address this issue, they introduced automated selection of the sparsity penalty (SelSPF), which will automatically select a suitable \( \lambda \), using the fact that the levels of sparsity in the latent vectors and \( \lambda \) are closely related.

The performance of SMACD is evaluated by using two synthetic [103] and eight real datasets in the literature, where two of the real datasets contain overlapping communities. Eight baselines including GraphFuse [103] are used for comparison purposes. The performance of each method is then evaluated based on three quality measures: adjacent random index (ARI), normalized mutual information (NMI) and purity.

They observed that the performance of SMACD is directly proportional to the degree of supervision, for all three quality measures. A question that arises here is whether a supervised framework should be developed instead to tackle this problem. The effectiveness of SelSPF is also evaluated by comparing it with a brute force approach. The results indicate that \( \lambda \) chosen by both approaches gives similar accuracy. It would be interesting to point out the difference between these two methods particularly in terms of the running time.

Recently, Chaudhary and Singh employed unsupervised machine learning techniques in studying community detection based on Covid-19 dataset [26]. The dataset is obtained from the website of John Hopkins University, which consists of the number of Covid-19 cases of 187 countries from January 22, 2020 to August 15, 2020.

The principal component analysis (PCA) is used to find the most significant variables and to reduce the dimensionality of the dataset, without sacrificing too much information. There are 13 variables in the dataset and the covariance matrix for each variable is generated. They calculated the eigenvalues and eigenvectors based on the covariance matrices. The eigenvector that has the largest eigenvalue is the principal component of the dataset. These eigenvalues reflect the significance of the principal components and are ordered in a descending order. The first eight principal components contribute 99.9% of the cumulative variance, which implies that these eight variables are sufficient for communities detection purposes.

They then applied an unsupervised clustering method, namely \( k \)-means clustering, on the reduced dataset to uncover the communities from the heterogeneous elements. They employed the elbow method to determine the number \( k \) of clusters, where \( k = 6 \)
and $k = 7$ are eventually chosen. The experimental results for $k$-means clustering combined with the PCA are compared with $k$-means clustering without incorporating the PCA. The results indicate that the former provides a more promising result in displaying communities based on the number of cases. A detailed analysis in choosing the value of $k$ should however be discussed. Appropriate comparisons should also be made for a more convincing result.

This work could be extended by considering more attributes to increase the number of variables. Other dimensionality reduction methods could also be used to monitor their effects on the community detection on the relevant datasets [26].

2.3 Subgraph Isomorphism

The subgraph isomorphism (SI) problem takes two graphs as input and determine if one of them contains a subgraph that is isomorphic to the other. The concept of subgraph isomorphism can be applied in many areas. It can be used in modelling and mining social structures for social networks [82], pattern recognition and discovery in databases, discovering protein-protein structures in bioinformatics [98, 6], drug development in chemoinformatics [65] and finding for substructures in knowledge graphs and recommender systems [74, 142]. The subgraph isomorphism problems can be tackled by many algorithms [128, 32, 23], however they all are not suitable for large graphs due to the NP-complete nature. There are several variants of the SI problem including subgraph isomorphism counting and subgraph matching. These two problems involve a high computation cost and they both are discussed in this section.

Active learning [114] is a statistical machine learning method where a learning algorithm interactively queries a user or subject matter expert (SME) for classification of points with the desired outputs. It can be iterated either sequentially (process one unlabelled vertex at a time) or in batch (process a batch of vertices together). It aims to achieve a performance that is comparable to the standard supervised learning scenarios with lesser labelled examples. Ge and Bertozzi [51] proposed a semi-supervised technique based on active learning in studying the subgraph matching problem (SMP), by presenting some case studies for multichannel subgraph matching. In the study, a machine will suggest optimal template target vertices that could probably reduce the solution space.

Problem 2.2 ([51]). Given two multiplex networks, template\footnote{Other terminologies for the terms template and world can also be found in the literature.} $G_t = (V_t, E_t, L_t, C)$ and world $G_w = (V_w, E_w, L_w, C)$, find all the subgraphs of the world that match the template.

Despite the simple definition, the SMP is in fact a very complex problem where the total SI counts goes up to approximately $10^{384}$ [51], even for single channel networks.

Some common approaches [21, 22] used in SI algorithms are tree search and constraint propagation. Tree search maintains a search state when it traverses the tree,
and backtracks when it reaches the end of a branch. Constraint propagation, on the other hand, records world vertices that could possibly match with each template vertex. The possible matches are reduced by repeatedly applying local constraints. These two approaches can be combined in solving the SMP.

To solve the SMP using active learning, the main goal is to identify template vertices that ought to be chosen in order to reduce the search space. Ge and Bertozzi proposed three querying strategies for this task: (1) select the template vertices that has the largest degree centrality measure, (2) select the vertices that has the largest sum of candidates for neighbouring vertices and (3) based on the edge entropy. The edge entropy is defined as

\[ -\sum_i p_i \log(p_i) \]

where \( p_i \) denotes the probability of mapping an edge to its candidate set and \( i \) sums over all edges connected to the vertex. These strategies are carried out after potential candidates are identified based on constraint propagation. Note that constraint propagation is chosen given that a real-time code for SMEs is needed for an active learning method.

Their results suggest additional metrics that involve the graph topology and structure are probably needed. One major concern in reducing the solution space of this problem is about the accuracy where it should be addressed and commented. One could also consider to query world vertices instead of template vertices, where the focus is on the world vertex that has the highest degree or it is one of the candidates of most of the template vertices [51]. The difference in terms of the performance by using the sequential and batch processing could be investigated.

Liu et al. [92] designed a learning method based on deep graph representation to address the subgraph isomorphism counting problem. Their goal is to determine bi-jection mapping functions that count all occurrences of a pattern graph in the original graph, through machine learning methods. However, there are still some challenges where existing datasets that are used in training a graph neural network have limited sizes, compared to other problems such as the shortest path problem [58]. To overcome these challenges, they developed two large-scale datasets for experimental purposes and employed different deep neural network architectures to extract representations of vertices and edges. They then introduced the DIAMNet, a dynamic intermediate attention memory network model, for global counting, which can be used to compute the frequencies of large pattern and data graphs.

In the subgraph isomorphism counting model introduced in [92], a graph can be represented either as a sequence of edges, or vertex features together with a series of adjacency matrices. By using a sequence model (e.g., long short-term memory (LSTM), convolutional neural networks (CNNs), transformer-XL (TXL)) or a graph model (relational graph convolutional network (RGCN), graph isomorphism network (GIN)), the respective representations of a graph and a pattern can be obtained. These representations are inputs of the interaction layer to extract the correlated context between them. The DIAMNet with an external memory is proposed to ad-
dress the efficiency issue in the attention mechanism, when modelling the interactions. To apply the method in large-scale graphs, they employed a gated recurrent network to reduce the time complexity.

The root mean square error and the mean absolute error are used as evaluation metrics. Data and codes are available online. For representation architectures, they observed that graph models have superior performance among most of the sequence models, due to the fact that these graph convolutional operations are designed according to the topology structure. For the interaction layer, DIAMNet performs better than other pooling methods and the attention with memory mechanism (MemAttn) for all representation architectures. Its memory and computational costs grow linearly with the input size. It is also worth noting that DIAMNet helps in extracting the context information of graphs even with moderate performance of the representation layer. These indicate the effectiveness of this technique and it should be considered to apply on other NP-complete problems with appropriate modifications.

Since this framework gives a promising result in the subgraph isomorphism counting problem, one could consider to extend or generalise it particularly to other variants of the SI problem.

2.4 Dominating Set

Let $G = (V, E)$ be a graph. A dominating set of $G$ is a subset $V' \subseteq V$ such that each vertex $u \notin V'$ is adjacent to at least one vertex $v \in V'$. A subset $V' \subseteq V$ is said to be independent (or stable) if for all $u, v \in V'$, vertices $u$ and $v$ are mutually non-adjacent. An independent set is maximal if and only if it is a dominating set. The objective of the minimum independent dominating set problem (MIDS) is to identify an independent dominating set that has the least size. On the other hand, the goal of the minimum weight independent dominating set problem (MWIDS) is to find an independent dominating set $S$ where the sums of the weights of vertices and edges in $S$ are both minimum. The MWIDS is a known NP-complete [24] problem. Its application include wireless sensors and actor networks [4], and multi-document problems [116].

There are many algorithms [34] that can be employed to achieve the objective of the MWIDS. An efficient algorithm that has linear time complexity [25] was designed specifically for series-parallel graphs. Wang et al. [132] proposed to combine a local search algorithm with the LSRR (stands for reinforcement-learning-based repair procedure) to address this problem. They gave three scoring functions in considering both vertex and edge weights, and presented a method to improve local search based on reinforcement learning (RL). Lastly, a repair procedure that uses a frequency technique and RL was proposed to enhance the solution.

The general idea for LSRR algorithm is that vertices with the smallest values will be chosen iteratively to produce an initial solution $S$. While the time limit is not satisfied, a greedy approach and a local search are implemented consecutively to improve $S$, in order to obtain a local optimal solution. After that, the destroy and
repair steps will be executed. The best solution is eventually obtained when the loop
is terminated.

In the algorithm, three scoring functions are proposed for each vertex, to select
vertices that should be added into or removed from the initial solution $S$. The
algorithm then determines the scoring function that should be used according to the
weight properties of both vertices and edges of a given graph. Since the greedy algo-
rithm only searches for feasible solutions (but not infeasible solutions) around the
neighbourhood of $S$, a local search is needed to further improve the solution, with
the aid of a frequency technique.

Note that the proposed reinforcement learning for LSRR is in the heuristic level.
A RL mechanism is designed to select vertices that need to be added into $S$ based
on the searching history, after some vertices in $S$ are removed by using a frequency
technique. The four important concepts in RL, state, action, transition and reward,
are defined accordingly (see [132] for further details). Specifically, the reward function
is divided into positive and negative rewards after the local search is implemented,
depending whether a vertex belongs to the local best solution.

During the local search procedure, some vertices from certain regions may be
searched repeatedly, causing a cycling problem. This issue is addressed by the repair
procedure that consolidates a frequency technique and RL. Vertices that are searched
frequently will be removed (destroy procedure), and those vertices that do not belong
to the local best solution for a long period will be added (repair procedure). Since
the efficiency of the LSRR algorithm is affected by the perturbation probability $p$,
instances with large sizes are tested so that a suitable $p$ value can be obtained. The
efficiency of the algorithm is measured by the percentage relative deviation where a
smaller value indicates a better efficiency. They concluded that $p = 5\%$ gives the best
performance in all but one instances. It is however crucial in identifying the best $p$
value by using a theoretical approach.

For future research, evolutionary algorithms including ant colony optimisation
and genetic algorithms could be combined with the proposed method to improve its
performance. The proposed method could also be modified and extended to some
other combinatorial optimisation problems [132].

### 2.5 Travelling Salesman Problem

The travelling salesman problem (TSP) [88] is a famous combinatorial optimisation
problem, which can be generalised to the vehicle routing problem. The objective of
the TSP is to find a shortest path that visits every city exactly once, and return to
the original city. Typical applications of the TSP include computer wiring, machine
sequencing and scheduling, logistics as well as clustering and ordering in biostatistics.
Many exact, heuristic and approximate algorithms have been proposed over the years.
The best known complexity in solving the TSP using an exact dynamic algorithm
is $O(2^{n} n^2)$ [62]. One of the best exact TSP solvers, Concorde, has been used to
obtain optimal solutions for problems up to 83,900 instances [8]. Three learning-based
approaches are discussed in this section where two are used in addressing ordinary TSP and one is for the decision version of TSP.

Deudon et al. [35] proposed a framework to solve the TSP by extending the neural combinatorial optimization framework. Their framework is based on reinforcement learning, and the well-known 2-opt heuristic (a simple local search algorithm) is employed to improve the current best tour. In their framework, a state represents a partial solution which denotes a sequence of cities visited by far, an action represents the next (unvisited) city that needs to be visited, a new state represents the updated solution and a reward could either be incremental or it comes when a tour is completed. Similar to [12], they used neural networks and the policy gradient to learn the best action when given the current tour. They studied the TSP by assuming that each city has a 2D coordinate \((x, y)\) in a Euclidean space. The neural network is trained by the policy gradient based on a learning rule where a critic is used to reduce the variance of the gradients.

Three datasets TSP20, TSP50 and TSP100 are used for experimental evaluation, and Adam optimiser [80] is used for stochastic gradient descent. Note that the experiments for TSP100 were conducted based on the model that was trained using TSP50, which suggests that the model can be generalised from heuristics to unseen instances. A natural question is that can this be generalised further to larger instances, and how would this affect the accuracy? The results are compared with several baselines including Google OR-Tools, the heuristic of Christofides [27], Lin-Kernighan heuristic and Concorde. Results show that the proposed algorithm gives a promising performance. All their codes are available on GitHub.

Dai et al. [33] designed a framework S2V-DQN (see §2.8 for more details) that combines reinforcement learning and graph embedding to address three combinatorial optimisation problems including the TSP. They generated two types of datasets random and clustered, and used the state-of-the-art solver Concorde [8] to compute the optimal solution. For larger instances, they observed that the model obtains an excellent approximation ratio. When a real-world dataset TSPLIB\(^2\) [108] is used in the experiment, S2V-DQN shows the best performance among all other methods.

The boolean satisfiability (SAT) in a conjunctive normal form has been predicted successfully by using graph neural networks (GNNs) [113], where about 85\% of the accuracy is achieved on SAT instances that have 40 variables. Since SAT can be defined based on just boolean formulae, Prates et al. [105] investigated if the decision variant of the TSP that involves numerical information can be solved by using GNNs. They showed that GNNs that involve very minor supervision can be used to address the problem.

Given a TSP instance with a target cost \(C \in \mathbb{R}\), to prevent the lost of the information about edge weights during the message-passing iterations, apart from the multidimensional vertex embedding, Prates et al. assigned embeddings to edges together with the associated weights. The vertex-to-vertex adjacency matrix \(M_{VV} \in \{0, 1\}^{|V| \times |V|}\) is replaced by an edge-to-vertex adjacency matrix \(M_{EV} \in \{0, 1\}^{|E| \times |V|}\). Since \(C\) is

\(^2\)The data is available on http://elib.zib.de/pub/mp-testdata/tsp/tsplib/tsp/index.html
also needed for the model, it is fed to every edge embedding alongside with its corresponding weight. The stochastic gradient descent algorithm is employed to train the model and the Concorde TSP solver [8] is used to compute the optimal tour costs $C^\ast$. For every graph $G$, two decision instances $X^+ = (G, 1.02C^\ast)$ and $X^- = (G, 0.98C^\ast)$ are produced with the optimal tour cost. This allows the model to predict the TSP decision problem within a small deviation from $C^\ast$.

Their results show that the model can learn faster with a larger deviation from the optimal cost. When the deviation is zero, all acceptance curves are above 50% where they conjectured that a route does exist by default. This behaves in an opposite way to the GNN SAT-solver [113]. The performance of the model is reducing when the model is tested on larger instances before it reaches the lower limit of 50%. Their results also suggested that the model can be generalised to a deviation that is larger than 2% from the optimal tour cost. This should however be justified further if higher deviations will lead to the deterioration of the accuracy.

They also compared the model with two simple heuristics [81], and evaluated them using the true positive rate (TPR). The GNN model outperforms both of these methods by achieving more than 90% TPR when the deviation is at least 4%. All their codes are available online for reproducing purposes. They suggested to train and evaluate the model using a more comprehensive set of real and random graphs, to check if it can be generalised to larger problems.

### 2.6 Graph Colouring

A (proper) colouring of a graph is an assignment of colours to vertices such that two vertices receive different colours if they are adjacent. A $k$-colouring is a colouring such that the number of required colours is at most $k$. A graph is $k$-colourable if it admits a $k$-colouring. The graph colouring problem (GCP) aims to find the minimum $k$ so that a given graph is $k$-colourable. The applications of graph colouring include flow management [9], job scheduling [124], register allocation and timetabling. In this section, we discuss three learning methods in solving GCPs where one of them is the decision version of the GCP.

To perform inference on data represented by graphs, one of the deep learning methods GNN is designed. Motivated by the study where a GNN model is used in addressing TSP [105], Lemos et al. [84] proposed a simple GNN model to solve the decision version of the GCP.

Given a GCP instance that consists of a graph $G$ and a number $C > 2 \in \mathbb{N}$ of colours, each vertex of $G$ is first assigned with the same embedding. Over a uniform distribution, every colour is assigned to a randomly initialised embedding that will later be learned by the model as a trained parameter. The model requires the vertex-to-vertex adjacency matrix $M_{VV} \in \{0, 1\}^{|V| \times |V|}$, and a vertex-to-colour adjacency matrix $M_{VC} \in \{0, 1\}^{|V| \times |C|}$ that links each colour to all vertices. The $M_{VC}$ allows the vertices and colours to interface and update their embeddings. The logit probability that corresponds to the model’s prediction is then computed by a
multilayer perceptron network based on the resulting vertex embeddings.

To train the model, stochastic gradient descent algorithm (implemented via TensorFlow’s Adam optimiser \([80]\)) is used. The loss of the binary cross entropy between the prediction results and the ground truth is defined as a boolean value that indicates if a graph accepts the target colouring. A CSP-Solver\(^3\) is also used to make sure that an undirected graph has a chromatic number \(\chi\). If so, new edges will be added into the graph until the GCP for \(\chi\) cannot be solved by the CSP-Solver. The last two instances will then be added into the respective dataset, to ensure that the dataset consists of hard instances and also balanced.

They compared the performance of their model with Neurosat and two heuristics \([64]\). The performance of the model on unseen instances was evaluated by using 20 instances from COLOR02/03/04 where the sizes of these instances are much larger than the training sets. Their codes are available online. Note that the conflict with the model itself increases when the chromatic number grows. Hence, a study in minimising the conflicts can be conducted in future work.

Another area that is related to the GCP is the grouping problem. Grouping problems that aim to divide a set of items into mutually disjoint subsets are closely related to problems including the GCP \([50]\) and clustering \([3]\). Zhou et al. \([144]\) proposed a general approach, reinforcement learning based local search (RLS), that uses both reinforcement learning and local search techniques to solve grouping problems, which is then verified by the popular GCP. During the search process, they collected and processed information via a machine learning technique to enhance the heuristic algorithms.

In the RLS approach, they assumed that the number \(k\) of groups is given. Let \(n\) be the number of items. They first defined a probability matrix that has size \(n \times k\), where an element \(p_{ij}\) of the matrix represents the probability such that the \(i\)th item \(v_i\) chooses the \(j\)th group \(g_j\) as its group. Initially, every element has an equal probability \(1/k\). By using a group selection strategy at instant \(t\), every item \(v_i\) chooses one suitable group \(g_j\) based on its corresponding probability vector \(p_i(t)\), until a solution \(S_t\) is obtained. To achieve a local optimum, the solution is then enhanced by DB-LS, a descent-based local search algorithm, which is fast and simple in general (the quality of the solution will however be sacrificed). The probability vector is updated and a smoothing technique is applied on the probability vector of each item. The RLS is implemented until a predefined stopping criterion is achieved.

In the model, four selection strategies random, greedy, roulette wheel and hybrid are considered. The observed that each selection strategy brings a big impact to the performance, and the hybrid selection is adopted given that it consists of both greediness and randomness. The DB-LS explores the search space by performing transitions from the current solution to a neighbouring solution iteratively based on a given neighbourhood relation, until no further improvement can be found. They commented that poor-quality local optimal solution \(\hat{S}_t\) is obtained in general, and it could be improved by using a more advanced technique.

\(^3\)https://developers.google.com/optimization/cp/cp_solver
They compared the group for every item $v_i$ in $S_t$ and $\hat{S}_t$. If $v_i$ stays in the original group, the group will be rewarded and its probability vector will be updated. Otherwise, the original group will be penalised and the new group will be compensated. Note that the complexity to update the entire probability matrix is $O(n \times k)$. They then introduced a probability smoothing technique to reduce the group probabilities by identifying old decisions that may mislead the current search. The technique is inspired by clause weighting local search algorithms for SAT, based on forgetting mechanisms [68].

Zhou et al. applied the RLS method on the GCP, and presented experimental results on two benchmark instances DIMACS and COLOR02. They adopted hybrid as the selection strategy as it outperforms the other strategies in terms of the number of iterations and the $k$ values for which the algorithm achieves a legal $k$-colouring. When the reinforcement learning mechanisms are removed, they observed that the performance of the RLS is much better than its variant, especially on hard instances. Similarly, the RLS without the probability smoothing technique is found to converge to the best solution much slower than the ordinary RLS. For DIMACS, they observed that their approach is competitive and able to achieve better results on some instances. For COLOR02, the performance of the RLS is better (if not equal), and it dominates all other baselines.

Due to the poor quality of the DB-LS, they suggested to replace it by other advanced optimisation procedure. A streamlining technique can also be developed to ensure a more effective probability updates. Lastly, the proposed reinforcement technique can also be combined with advanced colouring algorithms for a greater benefit.

Since the optimality of solutions cannot usually be proved using approximate algorithms, the exact algorithms remain useful. Zhou et al. [145] proposed a learning-based exact algorithm, namely the conflict driven clause learning (CDCL), for the GCP using propositional logic. They focused on implicit constraints in between non-adjacent vertices, where these constraints in the $k$-colourability problem can be discovered by encoding them into a SAT problem. Nonetheless, due to the NP-hardness of the GCP, these constraints have very complex relationships. Hence, the CDCL is adapted into a backtracking algorithm, to obtain another algorithm cdclGCP. The cdclGCP uses three procedures in discovering implicit constraints, and then exploit them to reduce the search space.

Random graphs and DIMACS are used for experimental purposes in their study. A conflict is defined either as an empty clause or an empty vertex. In comparing the cdclGCP with a backtracking algorithm backGCP, they observed that there are fewer conflicts in the cdclGCP than in the backGCP, which shows the power of learnt clauses in pruning search. By calibrating the runtimes for the cdclGCP and backGCP, four state-of-the-art exact algorithms for the GCP [63, 110] are compared. They observed that the clause learning is probably less efficient for random graphs, given that the clause learning used in SAT solvers faces the similar issue for random SAT instances. The performance for cdclGCP is however promising for DIMACS graphs.
For future work, they suggested to look for better clause management strategies so that less useful clauses can be removed and relevant clauses can be kept for pruning the search space in the GCP. An approach in combining VCS [63] and CDCL method can also be investigated to solve larger instances in the GCP.

2.7 Vertex Cover

A vertex cover of a graph $G$ is a set of vertices $C \subseteq V(G)$ such that every edge of $G$ is incident with at least one vertex in $C$. The minimum vertex cover (MVC) is a vertex cover that has the least size. This concept is applied in many real-world applications including circuit design, transportation and telecommunication. Existing methods that have been proposed to tackle this problem include evolutionary heuristics [79, 134] and approximate algorithms [107]. We now discuss two learning-based approaches in solving the MVC problem in this section.

A learning automaton [101] is an abstract model of a finite state of a machine that selects its current action based on the past experiences from the environment. There are several learning automata based algorithms that have been proposed in dealing with graph problems including the minimum spanning tree problem [125] and shortest path problem [10]. Mousavian et al. [101] proposed a cellular learning automata based algorithm CLAVC, a combination of both cellular and learning automata, to solve the MVC problem. A cellular learning automaton can be obtained by allocating a learning automaton to every cell of cellular automata. An irregular cellular automaton is a cellular automaton with no prior restriction on the grid structures of its cells.

Mousavian et al. used irregular cellular automaton to model the problem, and each vertex of a given graph is first represented by a cell that is equipped with learning automaton. Each cell will be penalised or rewarded based on some predefined criteria where this process can be done in parallel. The process stops when a predefined threshold is achieved. They conducted experiments using DIMACS benchmark instances and made comparisons with three algorithms [20].

The framework S2V-DQN (see §2.8 for more details) mentioned in §2.5 can also be used to solve the MVC problem. Dai et al. [33] generated Erdős-Rényi [41] and Barabási-Albert [5] graphs for experimental evaluations for this problem. They used IBM CPLEX optimiser to get the optimal solution before their method is compared with other algorithms [102, 12].

In evaluating the approximation ratio, they found that S2V-DQN has an excellent performance against all other methods where its approximation ratio is close to 1 (nearly optimal). It gives similar results when it is generalised to larger instances. In terms of the running time, S2V-DQN provides a much better approximation ratio even though it is somewhat slower compared to other approximate algorithms. For real-world dataset, they used MemeTracker\footnote{http://snap.stanford.edu/netinf/#data} with 960 vertices and 5000 edges, and S2V-DQN outperforms all the methods. They also found that S2V-DQN discovered an algorithm that can be used for the MVC problem where vertices are chosen in order
to get a balance in between degrees of the vertices and connectivity of the remaining graphs.

2.8 Maximum Cut

A cut is a partition of the vertex set of a graph into two disjoint subsets. A cutset of a cut is the set of edges that have one endpoint in each of these subsets. A maximum cut (MAXCUT) is a cut whose size is the maximum among all other cuts. The applications of MAXCUT can be found in theoretical physics and circuit designs.

In order to address the challenges in solving NP-hard combinatorial optimisation problems using learning algorithms, Dai et al. [33] proposed a unique combination of graph embedding and reinforcement learning framework S2V-DQN, which can be applied on and learned a variety of graph problems. The main differences of their proposed solution framework with others are the design pattern, representation and training of the algorithms. They evaluated their framework based on three popular problems: maximum cut, travelling salesman problem (§2.5) and minimum vertex cover (§2.7).

They designed a greedy algorithm with a common formulation so that it can be used to express various combinatorial problems based on different helper functions, cost functions and termination criteria. They also introduced a powerful deep learning architecture, namely structure2vec, to parameterise the evaluation function \( \hat{Q} \) over graphs. Given a current partial solution and for each vertex \( v \), the graph embedding network structure2vec computes a \( p \)-dimensional feature embedding \( \mu_v \). A new embedding will be produced by considering long-range interactions between vertex features and graph characteristics. The vertex embedding \( \mu_v^{(T)} \) contains information about \( T \)-hop neighbourhood if it terminates after \( T \) iterations. These embeddings are used to define \( \hat{Q} \) and their parameters are trained using reinforcement learning.

The framework for reinforcement learning is defined in general, and the details for each of the three problems can be found in [33]. They combined both \( n \)-step Q-learning [120] and fitted Q-iteration to learn the algorithm. The \( n \)-step Q-learning is useful in handling the issue of delayed rewards and the fitted Q-iteration lead to a faster learning convergence.

For MAXCUT, they generated Erdős-Rényi [41] and Barabási-Albert [5] graphs for experimental purposes. The optimal solution is obtained by using IBM CPLEX optimiser. They evaluated the quality of the solutions by using the approximation ratio, and S2V-DQN performs much better than other methods [55, 12]. Their method can also be used for larger instances up to 1200 vertices, where the approximation ratio remains promising.

For real-world datasets, they used one from a library of MVC instances\(^5\) that is publicly available. The results also show that S2V-DQN outperforms all other methods. They discovered a new algorithm through their framework, where vertices are picked so that edges in the existing cutset will not be cancelled. The source codes

---

\(^5\)http://www.optsicom.es/maxcut/#instances
of their framework are also available online.

For future work, the main factor that affects the approximation ratio of the solutions could be investigated and modified, so that S2V-DQN can be generalised to much larger instances.

3 Modern Graph Problems

In this section, we discuss two well studied graph problems that are related to some classic graph problems, where the focus is on routing problems and community search problems.

3.1 Routing Problem

Vehicle routing problem (VRP) is a classic graph problem that determines an optimal route (e.g., the shortest one) for a vehicle to travel from an origin to a various sets of destinations, with some specific constraints. It is a combinatorial optimisation problem and can be solved by using integer programming. It can also be regarded as a generalisation of the TSP (See §2.5). There exist many variants of the VRP, which have direct applications in the industry particularly in the area of transportation. Approaches including exact solution methods and metaheuristics have been proposed in addressing the VRP over the years. We discuss four models (MURAT [87], RNE [66], Trans2Vec [90] and HMTRL [89]) that were proposed for route recommendation and one model (STGCN [138]) for route forecast in this section.

There are two approaches in estimating travel time: (1) path-based method [71, 38] and (2) origin-destination (OD) based method [76, 130]. Route information are required to generate a prediction in the path-based method, which usually incurs a high computation cost and can be error-prone. The OD based method, on the other hand, estimates the travel time without involving the actual route information. However, it remains challenging due to the limited raw input features and the complex spatiotemporal dependency in the road network.

Li et al. [87] proposed a representation learning framework, namely multitask representation learning model for arrival time estimation (MURAT), in estimating the travel time. The framework utilises historical path information during the training phase. They represented an underlying road network using an undirected graph $G = (V, A)$ where $V$ is the vertex set corresponds to links of the road network and $A$ represents connectivity between the links. If links $i$ and $j$ are connected, then $A_{ij} = 1$.

They addressed the following problem:

**Problem 3.1 ([87]).** Given an origin, destination, departure time and $N$ trips, the goal is to estimate the duration using the set of historical trip dataset $X = \{x^{(i)} \mid 1 = 1, 2, \ldots, N\}$ as well as the underlying road network $G$.

Note that data representation is important in learning based models. Representation learning [13] learns representation from data so that useful information from
various tasks can be extracted easily. In this task, they learned representations from three different aspects, that are the raw trip data, the structure of road networks and the spatiotemporal features. Both graph Laplacian regularisation and unsupervised graph embedding are also used to enforce the spatial and temporal smoothness, prior to the representations.

Since the structure of the underlying road network affects the travel time, they used unsupervised graph embedding, for instance Laplacian EigenMap \cite{11} and DeepWalk \cite{104} in learning the representation for each link in order to capture the network structure. The graph Laplacian is added as an additional unsupervised loss as in \cite{11}, where the loss serve as a regulariser so that adjacent links have similar representations. They also constructed temporal and spatial graphs in the embedding space, and embedded each vertex as a fixed length vector.

Various information (e.g., number of turns, travel distance, number of traffic lights) are extracted from the paths as extra input features, which will then be used as auxiliary tasks. They then proposed a multitask learning framework to learn the main task. During the training, the link, spatial and temporal information for each sample \( x^{(i)} \) are embedded into the learned spaces. These representations are input into a deep residual network that generates the prediction for all tasks, together with other numerical features. Then Adam optimiser \cite{80} is used to optimise the embeddings and the weights of the residual network.

They observed that the performance of MURAT is relatively poor without implementing the representation learning. This implies that various path information are crucial for route recommendation. Its performance is also more significant in a dataset that has a better map quality. They also evaluated the performance of MURAT with respect to the travel time and travel distance. It outperforms the selected approaches for majority of the instances.

To study the effect of link embedding, they conducted experiments using five variants of the model. They found that model with supervised embedding and unsupervised pretraining using DeepWalk always give the best result. Four variants of the model are also used to study the effect of spatiotemporal embedding. They observed that the temporal embedding has a higher influence than the spatial embedding. Similarly, three variants are used to evaluate the effect of multitask learning. Their proposed framework achieves the best performance, and the performance can be improved further by adding more auxiliary tasks. They also analysed their model by looking into the effects of model architecture and interpretation.

Given that the proposed framework is flexible, they suggested to integrate the information of real-time traffic into their model. Theoretical justifications for the model can also be studied in the future.

Another problem that is studied frequently in routing problems is the computation of shortest path distances on road networks, given that it can be applied in many practical applications. One of the most classic methods proposed by Dijkstra \cite{37} finds the shortest path by eliminating undesired edges. Many exact algorithms and approximate algorithms such as heuristic contraction hierarchies \cite{53} (an algorithm
based on the exact contraction hierarchies \cite{52}), distance oracles \cite{111}, ALT \cite{56} (A*, landmarks and triangle inequality), arterial hierarchy \cite{146} and k-nearest neighbour (kNN) algorithms \cite{109, 115, 93, 94}, have been proposed over the years. These methods either suffer from huge preprocessing time, or they face scalability issues.

Inspired by some famous vertex embedding methods including DeepWalk \cite{104} and LINE \cite{122}, Huang et al. \cite{66} recently introduced a learning-based road network embedding (RNE) model to approximate shortest path distance, without performing any search. They captured the distances between vertices based on vertex embedding, and applied $L_1$ metric on the embedding vectors to approximate shortest path distances. In the embedding training, a hierarchical method is used, and a fine-tuning method that selects training data of high quality is also designed.

Different from existing vertex embedding methods that capture the social relations of vertices for classification tasks, RNE captures the distances between vertices and focuses on regression tasks. Suppose $G = (V, E, W)$ is a weighted graph. For $d \ll |V|$, they first embedded all vertices that represent road joints in a road network into a $d$-dimensional space $R^d$ as vectors. The $L_p$-distance of the vector pair is used to estimate the shortest path distance between the two vertices. Due to the linearity of $L_1$-distance, they set $p = 1$, which is shown can be used to support $k$NN queries.

During the training phase, they selected a training set that consists of shortest distances between vertex pairs. The loss is minimised by using stochastic gradient descent that has some limitations, and hence a hierarchical embedding model is also proposed to further address them. In the hierarchical embedding model, the road network is iteratively partitioned into subgraphs, and a tree structure that contains the structure information is built. They adopted a well-known multiphase graph partition algorithm \cite{78}, and implemented the graph partitioning in every level, given that its optimal solution is proven to be NP-hard.

In each training phase, different training sample selection strategy is also adopted. For hierarchy embedding, instead of choosing different vertex pairs uniformly from the graph, vertex pairs are chosen uniformly from different subgraph pairs. For vertices embedding, a landmark-based sample selection is proposed so that the relative embedding positions can be controlled. They used distance-based and error-based sample selections in fine-tuning the model, in which a high total error rate and error variance are involved.

Their results show that RNE is effective in shortest path approximation and $L_1$-metric is compatible for network distance representation. The hierarchical embedding is shown to achieve a better performance and speed up the training process. Their model takes reasonable training time and has smaller index size. The percentage error of RNE is also smaller compared to others. With the increase of the distance, the relative error of RNE keeps decreasing given that its absolute error remains.

In intelligent transportation systems, it is crucial to connect various transportation modes such as walking, cycling and public transits, which allow users to use the most appropriate route, by jointly considering several transport modes on a constrained transportation network. Since most recommender systems in transportation focus
on unimodal transport planning, Liu et al. [90] introduced a joint representation learning framework for multimodal transportation recommendation, where the idea is derived from word and network embeddings [97, 104]. There are several challenges in this task: (1) transport heterogeneous in terms of time-efficiency, conveyance, infrastructure, cost and comfortableness, (2) feedback obtain from navigation Apps are limited and implicit, and (3) geospatial locality.

A framework namely Trans2Vec was proposed in addressing these issues, by modelling users, transport modes and origin-destination (OD) pairs into a multimodal transportation graph. To model the heterogeneity of transport modes, an anchor based method was introduced. The interconnections between users and ODs are identified to address both geospatial locality and data sparsity issues.

They constructed a multimodal transportation graph $G$ based on a map database, user demographic attributes and OD POIs information. For $d \ll |V(G)|$, a joint representation learning that learns a $d$-dimensional latent representation $x_v$ for every vertex $v \in V(G)$ is developed. Each transport mode vertex is assigned a discriminative embedding based on an anchor embedding strategy in order to reflect its inherent context information. This helps in allocating each user and OD pair in an appropriate location in the hidden space. A greedy strategy is used to initialise transport mode anchor embeddings where each of the entry has a uniform distribution between $[-\frac{1}{2d}, \frac{1}{2d}]$, after the relevance of each pair of transport mode is measured. Methods in measuring the relevance of OD and users pairs are also designed to refine further their framework. They trained the embeddings using stochastic gradient descent and proposed an effective online recommendation system to tackle this problem.

The framework was evaluated in terms of the overall performance of Trans2Vec, parameter sensitivity, transport mode relevance and the robustness, using large real-world datasets. Trans2Vec and its variant BTrans2Vec show a much better performances compared to others [121, 39] in all but one metrics, which implies that the framework is effective in capturing transport mode preferences. The transport modes can be grouped into two categories where car, bus and taxi are placed in one category whereas walk, bicycle and bus form another category. They also showed that the framework is stable in different OD pairs and groups of users, by first grouping them into four subgroups using $k$-means algorithm.

A common approach in learning based transport mode recommendations [139, 1, 91] based on supervised or unsupervised machine learning techniques is to explicitly extract features from users’ historical data. Some recent studies also employed network embedding [90] and deep learning [143] techniques in their framework. Note that two common patterns that are captured in route representations are spatiotemporal autocorrelations and the semantic coherence that exist in transportation networks and route sequences, respectively. Liu et al. [89] proposed to consolidate representation learning for dynamic graphs and hierarchical multitask learning [140] in their recent study, by developing a new framework, namely hierarchical multitask route representation learning (HMTRL).

In order to capture the spatiotemporal autocorrelations across multiple graphs
using a spatiotemporal module based on graph neural network, they first transformed multimodal transportation networks (MMTN) into time-dependent multi-view graphs (we refer them as multi-view graphs for simplicity). They then designed an attentive learning module for route coherence modelling, before the HMTRL framework is eventually proposed, in which the route representations based on specific transport modes will be obtained. The performance of the framework can also be enhanced by integrating supervision signals in multiple levels.

Multi-view graphs consist of hub-centric graphs (vertices and edges represent transportation hubs and links, respectively) and link-centric graphs (vertices and edges are flips in MMTN), which preserve both the structural integrity and temporal dynamics of MMTN. At each step, GNN is used to capture spatial autocorrelation based on multi-view graphs, where hubs and links that are spatially adjacent tend to be closer in the latent space. Graph convolution operations are also defined to capture \( \ell \)-hop spatial dependencies by repeating the operation \( \ell \) times. For temporal autocorrelation modelling, they employed a simple variant of GNN, namely gated recurrent unit (GRU) \[29\]. Two blocks were also designed in the route representation learning module in which the bi-directional recurrent neural network based route coherence modelling integrates route coherence constraints into both hub and link representations based on historical routes, by using the bi-directional GRU (BiGRU) operation. The second block known as the self-attentive route representation learning employs a self-attention mechanism in transforming routes of arbitrary length into representation vectors for routes of fixed length, by learning the significance scores of hubs and links.

Since auxiliary tasks have been proved to be beneficial in different granularity \[60, 47\], in the HMTRL framework, different auxiliary tasks are introduced, and different layers of neural network are assigned with different tasks. These learning tasks are categorised into the vertex- and route-level MTL. The former corresponds to the representation learning of vertices in multi-view graphs whereas the latter corresponds to the route representation optimisation and recommendation. They used the cross-entropy loss for optimisation, and employed Adam optimiser \[80\] for training purposes.

The data is split into the ratio of 8:1:1 that corresponds to the amount of the training set, validation set and testing set, respectively. The effectiveness of the framework is evaluated by two commonly used metrics Hit@\(k\) and NDCG@\(k\) in recommenders. Their framework is compared with six different methods (two rule-based and six learning methods) \[104, 87, 91\]. The source codes are available online.

The results show that the HMTRL framework outperforms all the six methods under all instances. The ablation studies also demonstrate that the structural and contextual information play a more significant role in this study. The self-attentive operation and the BiGRU also help in improving the performance of the framework. The link related auxiliary tasks are found to be one of the major factors in determining the effectiveness in multimodal transportation recommendations. To achieve a better overall performance, their results suggest to consider mixed routes in the future study.
Traffic forecast is important especially for urban traffic control. Short term predictions can be handled relatively well by using statistical approaches including linear regression. For long term predictions, these methods are however less effective due to the complexity of traffic flows. Instead, dynamical modelling and data-driven methods are used in the existing studies. The former focuses on mathematical and physical tools in formulating traffic problems by computational simulation [129] whereas the latter uses classic statistical (e.g., auto-regressive integrated moving average (ARIMA) [133] and its variants) and machine learning models (e.g., support vector machine (SVM), neural network (NN) and k-nearest neighbours algorithm (kNN)). Machine learning models have higher accuracy and can be used to handle complicated data modellings.

Numerous traffic tasks had recently been solved by deep learning approaches [95, 70]. However, it is still challenging to extract both temporal and spatial features from these dense networks. Yu et al. [138] proposed a deep learning architecture, spatiotemporal graph convolutional networks (STGCN), that contains several spatiotemporal convolutional layers to model temporal and spatial dependencies from graph-structured time series in traffic forecast. The goal is to build a model with complete convolutional structures, to reduce the parameters and improve the training speed.

In the model, they employed graph convolution directly on graph structured data so that meaningful features that appear in the space domain can be extracted. Since the computation cost in graph convolution could be expensive, they applied two approximation strategies, Chebyshev polynomials approximation and first-order approximation, to address this issue. To capture temporal dynamic behaviours of traffic flows, they used entire convolutional structures on time axis, where this representation allows parallel and controllable training procedures. They also constructed spatiotemporal convolutional block (ST-Conv block), which is used to process graph-structured time series. The outputs of the final block in ST-Conv are mapped into a single-step prediction. The final output is obtained and the speed prediction is calculated by applying a linear transformation. The performance of the model is measured by L2 loss.

They evaluated the model using two medium and large scales real-world datasets. They adopted mean absolute errors, root mean squared errors and mean absolute percentage errors as the evaluation metrics. They observed that STGCN achieves the best performance across all the metrics. They also found that machine learning and traditional statistical methods are only suitable to perform short-term forecasts. The model also demonstrates an excellent improvement on short, medium and long term forecasts by modelling spatial topology of the sensors. Their experiments also show that the model converges easily and gives a much faster training speed. They suggested to look into the network structure and parameter settings, and to optimise them further in future work.
3.2 Community Search

Given a graph $G$ and a query vertex $v \in V(G)$, the community search (CS) problem (e.g., [118, 44]) aims to determine the most likely subgraph that contains $v$ and satisfies the connectivity and cohesiveness constraints. Some classical subgraph cohesiveness metrics include $k$-core [112], $k$-truss [30], $k$-clique and $k$-edge-connected component [54]. This problem is closely related to the community detection problem that detects all communities in a given graph by partitioning the whole graph. The difference is that community search is related to either one or multiple query vertices. Community search can be used in many real-world applications [44] including friend recommendation, e-commerce and fraudulent group discovery. Although many CS algorithms have been designed, predefined subgraph patterns are usually adopted in modelling communities which are very inflexible when applied to different application scenarios, leading to some ineffectiveness. To address this issue, learning-based approaches for CS are recently proposed, and two methods that are used to solve interactive and attributed CS problems are discussed in this section. The former can be applied on online social networks whereas the latter is a more complex search problem with additional features.

Gao et al. [49] presented an interactive community search algorithm using GNN (ICS-GNN) to identify communities in an online social network, which captures similarities between vertices simultaneously using the content and structural features. Given an integer $k$, they modelled a community as a subgraph of size $k$ that has the maximum GNN scores (called $k$MG community). Their framework involves multiple rounds of community search and users' feedback are incorporated during the process. They first constructed a subgraph that contains a query vertex from an online social network using breadth-first search. To avoid unrelated vertices during the subgraph construction, they employed an exploration strategy that only includes the edges within the existing vertices. They then trained a GNN model on the subgraph to infer the probability for each vertex in the subgraph. The cross-entropy is used as the loss function in the training. The trained model is then used to infer GNN scores of other vertices. Some approximate methods are then used to locate the $k$MG community according to the GNN scores of vertices due to its NP-hard nature. Particularly, vertices with lower scores that belong to the community are swapped with vertices with higher scores that do not belong to the community, while preserving its connectivity. The swapping process continues until no proper vertex pair is available.

Note that the GNN model is trained with vertices that are labelled by users, which indicate if a vertex should be in a community. Since it is somehow challenging for users to decide a correct label, a ranking loss is also introduced to incorporate implicit feedback from users. To handle the case where a query vertex is also a boundary vertex of a community, they introduced a greedy measure where the global relative benefit of a vertex $u$ is used to determine if $u$ should be added into the community, based on the shortest path from $u$ to any vertex in the community.

The effectiveness and efficiency of the model are evaluated using both offline and
online networks. They observed that ICS-GNN captures the content and structural features by GNN in an effective manner. The ranking loss is also shown to be effective for interactive community search and the global relative benefit produces communities that fit into different distributions.

Another problem that is well studied in CS is the attributed community search problem that aims to find query-dependent communities where each vertex in the communities has a homogeneous attributed value. Most of the existing methods [43, 67] first find a dense community that contains a given query vertex, then shrink the community based on a given attribute function, without considering the correlation between them. Jiang et al. [75] proposed an end-to-end supervised framework, namely query-driven graph convolutional networks (QD-GCN), that encodes the graph topology and vertex attributes simultaneously in solving this problem. Four learning components that are graph encoder, structure encoder, attribute encoder and feature fusion are designed in the model.

The graph encoder integrates both structure and attribute features by encoding the query-independent graph information which makes QD-GCN more robust. It produces a graph embedding which will be used as the input of feature fusion. The structure encoder gives an interface for query vertices and model the graph topology structure information of these vertices, which will produce the query-specific structural embedding. The attribute encoder gives an interface for query attributes and a structure-attribute bipartite graph is constructed to obtain the query-specific attribute embedding. The three embeddings are used in the feature fusion to obtain the final output of the entire model.

For experimental purposes, they evaluated the performance of the model using nine attributed graphs that are available online6, based on two metrics $F_1$ score and Jaccard similarity. The model is compared with both attributed and non-attributed community search algorithms, where the performances of QD-GCN are in fact comparable in both comparisons. This phenomenon may be because of the fusion operation where graph information and query vertices are still transmitted to the attribute encoder. They also observed that the four learning components can drastically improve the effectiveness of the model.

Given that this approach gives a promising result for keyword-based attributed graphs, it might be worth to investigate if it can be modified so that it can be applied on other attributed graphs such as location-based attributed graphs.

4 Conclusions and Future Work

In this article, we give a systematic review on both classic and modern graph problems that have been solved by using learning-based approaches. In the former, we focus on popular graph problems in NP-complete. In the latter, we discuss large-scale routing problems that are modelled by graphs where the emphasis is on route recommendation

6https://linqs.soe.ucsc.edu/data
and forecast, as well as community search problems. We categorize these problems into ten main categories, and review 20 learning frameworks that were used in solving the problems. We also identify the datasets and existing methods that were used for each graph problem. The links of the publicly available source codes are given, and they all are summarised in Table 1. Lastly, we point out and suggest some challenges for further research in the respective section.

We now suggest some additional problems that could potentially be investigated, specifically (but not limited to) for mathematicians or computer scientists who work in the relevant areas.

Recall that the framework proposed in [33] can be used to address three combinatorial optimisation problems, which are the travelling salesman problem, vertex cover and maximum cut. We suggest the following question:

**Problem 4.1.** Given that problems in NP-complete can be reduced from one to the other, can the framework in [33] be extended by modifying the relevant algorithms/functions so that it can be used on some other classic graph problems, especially those in NP-complete?

A natural way for this extension is to first start from the clique and independent set problems, given that together with the vertex cover problem, they form a trio in which their solutions can be identified from the other by appropriate modifications.

Since there are many other NP-complete problems that have not been solved using a learning-based approach, one natural question is:

**Problem 4.2.** Does there exist a more general learning-based framework that can be used to tackle most of the NP-complete problems?

Based on Cook’s theorem [31], we know that boolean satisfiability problem is NP-complete. Since every problem in NP-complete can be reduced to this problem, a learning-based framework could first be designed in addressing the problem, before it is extended to other NP-complete problems.

The Tutte polynomial [126, 127, 17, 40] plays an important role in graph theory and it specialises to other polynomials including the chromatic polynomial, Jones polynomial, flow polynomial and reliability polynomial. This two-variable polynomial also gives us a variety of information about the enumeration of various substructures of undirected graphs. For instance, the number of forests, spanning subgraphs and acyclic orientations of a graph.

Computing the Tutte polynomial \( T(G; x, y) \) of a graph \( G \) is \#P-hard except on one special curve \((x - 1)(y - 1) = 1\), and some special points

\[(x, y) = (1, 1), (-1, -1), (0, -1), (-1, 0), (i, -i), (-i, i), (j, j^2), (j^2, j)\]

where \( j = e^{2\pi i/3} \) [69, 46]. Therefore, we suggest the following problems:

**Problem 4.3.** Does any learning-based approach can be adopted in computing the Tutte polynomial for undirected graphs?
It would probably helpful to tackle this problem by identifying the Tutte polynomials of some small graphs that are commonly appeared during the process, which will then serve as a training set in the model.

Note that the structure of directed graphs is usually more complex than undirected graphs, and there are several Tutte-like polynomials for directed graphs that have been defined including the greedoid polynomial [57, 136], cover polynomial [28] and Tutte invariants [137, 135].

**Problem 4.4.** Does a learning-based framework can also be designed in computing polynomials for directed graphs?

By computing polynomials for undirected and directed graphs based on Problems 4.3 and 4.4, it would also be interesting to identify if there exists any relationship between the Tutte polynomials for undirected graphs and polynomials for directed graphs. If such a generalisation exists, it would certainly be helpful to boost further the research findings particularly for the evaluations of polynomials for directed graphs, given the well developed theory of the Tutte polynomial.

5 Acknowledgment

The first author is most grateful to Universiti Putra Malaysia for granting him Leave of Absence during his time at Nanyang Technological University as a Postdoctoral Research Fellow. Siqiang Luo is supported by Singapore AcRF Tier 1 (RG18/21), and in part by AcRF Tier 1 Seed Funding (RS05/21) and NTU startup grant.

References

[1] A. Abedalla, A. Fadel, I. Tuffaha, H. Al-Omari, M. Omari, M. Abdullah, M. Al-Ayyoub. MTRecS-DLT: Multi-Modal Transport Recommender System using Deep Learning and Tree Models. In 2019 Sixth International Conference on Social Networks Analysis, Management and Security (SNAMS), pp. 274–278, 2019. doi:10.1109/SNAMS.2019.8931864

[2] E. Acar, T. G. Kolda and D. M. Dunlavy. All-at-once optimization for coupled matrix and tensor factorizations, preprint, 2011. arXiv:1105.34226.

[3] L. E. Agustín-Blas, S. Salcedo-Sanz, S. Jiménez-Fernández, L. Carro-Calvo, J. Del Ser and J. A. Portilla-Figueras. A new grouping genetic algorithm for clustering problems. Expert Systems with Applications, 39(10): 9695–9703, 2012. doi:10.1016/j.eswa.2012.02.149

[4] I. F. Akyildiz and I. H. Kasimoglu. Wireless sensor and actor networks: research challenges. Ad hoc networks, 2(4): 351–367, 2004. doi:10.1016/j.adhoc.2004.04.003
[5] R. Albert and A. L. Barabási. Statistical mechanics of complex networks. *Reviews of modern physics*, 74(1): 47, 2002.

[6] N. Alon, P. Dao, I. Hajirasouliha, F. Hormozdiari and S. C. Sahinalp. Biomolecular network motif counting and discovery by color coding, *Bioinformatics*, 24(13): i241–i249, 2008. doi:10.1093/bioinformatics/btn163

[7] E. Alpaydin. *Introduction to machine learning (4th edition)*. MIT Press, 2020.

[8] D. Applegate, R. Bixby, V. Chvátal and W. Cook. Concorde TSP Solver, 2006. http://www.math.uwaterloo.ca/tsp/concorde/

[9] N. Barnier, P. Brisset. Graph Coloring for Air Traffic Flow Management. *Annals of Operations Research*, 130: 163–178, 2004. doi:10.1023/B:ANOR.0000032574.01332.98

[10] H. Beigy and M. R. Meybodi. Utilizing distributed learning automata to solve stochastic shortest path problems. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 14(05): 591–615, 2006. doi:10.1142/S0218488506004217

[11] M. Belkin and P. Niyogi. Laplacian eigenmaps and spectral techniques for embedding and clustering. *Advances in neural information processing systems*, pp. 585–591, 2001.

[12] I. Bello, H. Pham, Q. V. Le, M. Norouzi and S. Bengio. Neural combinatorial optimization with reinforcement learning, *preprint*, 2016. arXiv:1611.09940.

[13] Y. Bengio, A. Courville and P. Vincent. Representation Learning: A Review and New Perspectives. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 35(8): 1798–1828, 2013. doi:10.1109/TPAMI.2013.50

[14] Y. Bengio, A. Lodi and A. Prouvost. Machine learning for combinatorial optimization: a methodological tour d’horizon. *European Journal of Operational Research*, 290(2): 405–421, 2021. doi:10.1016/j.ejor.2020.07.063

[15] B. E. Boser, I. M. Guyon and V. N. Vapnik. A training algorithm for optimal margin classifiers. In *Proceedings of the fifth annual workshop on Computational learning theory (COLT ’92)*, pp. 144–152, 192. doi:10.1145/130385.130401

[16] W. Brendel and S. Todorovic. Segmentation as maximum-weight independent set. *Proceedings of the 23rd International Conference on Neural Information Processing Systems*, pp. 307–315, 2010.

[17] T. H. Brylawski and J. Oxley. The Tutte polynomial and its applications. *Matroid applications, Encyclopedia Math. Appl.*, Cambridge University Press, Cambridge, 40, 123–225, 1992.
[18] S. Butenko and W. E. Wilhelm. Clique-detection models in computational biochemistry and genomics. *European Journal of Operational Research*, 173(1): 1–17, 2006. doi:10.1016/j.ejor.2005.05.026

[19] S. Cai and J. Lin. Fast Solving Maximum Weight Clique Problem in Massive Graphs. *25th International Joint Conference on Artificial Intelligence (IJCAI 2016)*, pp. 568–574, 2016.

[20] S. Cai, K. Su and A. Sattar. Local search with edge weighting and configuration checking heuristics for minimum vertex cover. *Artificial Intelligence*, 175(9-10): 1672–1696, 2011. doi:10.1016/j.artint.2011.03.003

[21] V. Carletti, P. Foggia, A. Saggese and M. Vento. VF2 Plus: An Improved version of VF2 for Biological Graphs. In *International Workshop on Graph-Based Representations in Pattern Recognition*, pp. 168–177, 2015. doi:10.1007/978-3-319-18224-7_17

[22] V. Carletti, P. Foggia, A. Saggese and M. Vento. Introducing VF3: A New Algorithm for Subgraph Isomorphism. In *International Workshop on Graph-Based Representations in Pattern Recognition*, pp. 128–139, 2017. doi:10.1007/978-3-319-58961-9_12

[23] V. Carletti, P. Foggia, A. Saggese and M. Vento. Challenging the Time Complexity of Exact Subgraph Isomorphism for Huge and Dense Graphs with VF3. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 40(4): 804–818, 2017. doi:10.1109/TPAMI.2017.2696940

[24] G. J. Chang. The weighted independent domination problem is NP-complete for chordal graphs. *Discrete Applied Mathematics*, 143(1-3): 351-352, 2004. 10.1016/j.dam.2003.05.004

[25] S. C. Chang, J. J. Liu and Y. L. Wang. The weighted independent domination problem in series-parallel graphs. *Intelligent Systems and Applications*, 274: 77–84, 2015.

[26] L. Chaudhary, B. Singh. Community detection using unsupervised machine learning techniques on COVID-19 dataset. *Soc. Netw. Anal. Min.*, 11:28, 2021. doi:10.1007/s13278-021-00734-2

[27] N. Christofides. *Worst-case analysis of a new heuristic for the travelling salesman problem*. Carnegie-Mellon Univ Pittsburgh Pa Management Sciences Research Group, 1976.

[28] F. R. K. Chung and R. L. Graham. On the cover polynomial of a digraph. *J. Combin. Theory Ser. B*, 65: 273–290, 1995.

[29] J. Chung, C. Gulcehre, K. Cho and Y. Bengio. Empirical evaluation of gated recurrent neural networks on sequence modeling, *preprint*, 2014. arXiv:1412.3555
[30] J. Cohen. Trusses: Cohesive subgraphs for social network analysis. *National security agency technical report*, 16(3.1), 2008.

[31] S. A. Cook. The complexity of theorem-proving procedures. In *Proceedings of the third annual ACM symposium on Theory of computing (STOC ’71)*, pp. 151–158, 1972. doi:10.1145/800157.805047

[32] L. P. Cordella, P. Foggia, C. Sansone and M. Vento. A (sub)graph isomorphism algorithm for matching large graphs. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 26(10): 1367–1372, 2004. doi:10.1109/TPAMI.2004.75

[33] H. Dai, E. B. Khalil, Y. Zhang, B. Dilkina and L. Song. Learning combinatorial optimization algorithms over graph, *preprint*, 2017. arXiv:1704.01665

[34] P. P. Davidson, C. Blum and J. A. Lozano. The weighted independent domination problem: Integer linear programming models and metaheuristic approaches. *European Journal of Operational Research*, 265(3): 860-871, 2018. 10.1016/j.ejor.2017.08.044

[35] M. Deudon, P. Courr´e, A. Lacoste, Y. Adulyasak, L. M. Rousseau. Learning Heuristics for the TSP by Policy Gradient. *International conference on the Integration of Constraint Programming, Artificial Intelligence, and Operations Research. CPAIOR 2018*, pp. 170–181, Springer, Cham, 2018. doi:10.1007/978-3-319-93031-2_12

[36] R. Diestel. *Graph Theory (4th edn.)*. Springer, New York, 2010.

[37] E. W. Dijkstra. A note on two problems in connexion with graphs. *Numerische mathematik*, 1(1): 269–271, 1959.

[38] Z. Ding, B. Yang, R. H. G¨uting and Y. Li. Network-Matched Trajectory-Based Moving-Object Database: Models and Applications. *IEEE Transactions on Intelligent Transportation Systems*, 16(4): 1918–1928, 2015. doi:10.1109/TITS.2014.2383494

[39] Y. Dong, N. V. Chawla and A. Swami. Metapath2vec: Scalable Representation Learning for Heterogeneous Networks. In *Proceedings of the 23rd ACM SIGKDD International Conference on Knowledge Discovery and Data Mining (KDD ’17)*, pp. 135–144, 2017. doi:10.1145/3097983.3098036

[40] J. A. Ellis-Monaghan and C. Merino. Graph polynomials and their applications I: The Tutte polynomial. In: *Structural Analysis of Complex Networks*, Birkh¨auser/Springer, New York, pp. 219-255, 2011.

[41] P. Erdős and A. Rényi. On the evolution of random graphs. *Publ. Math. Inst. Hung. Acad. Sci.*, 5(1): 17–60, 1960.
[42] Y. Fan, N. Li, C. Li, Z. Ma, L. J. Latecki and K. Su. Restart and random walk in local search for maximum vertex weight cliques with evaluations in clustering aggregation. *26th International Joint Conference on Artificial Intelligence (IJCAI 2017)*, pp. 622–630, 2017. doi:10.24963/ijcai.2017/87

[43] Y. Fang, R. Cheng, S. Luo and J. Hu. Effective community search for large attributed graphs. *Proc. VLDB Endow.*, 9(12): 1233–1244, 2016. doi:10.14778/2994509.2994538

[44] Y. Fang, X. Huang, L. Qin, Y. Zhang, W. Zhang, R. Cheng and X. Lin. A survey of community search over big graphs. *The VLDB Journal*, 29(1): 353–392, 2020. doi:10.1007/s00778-019-00556-x

[45] Z. Fang, C. M. Li and K. Xu. An exact algorithm based on maxsat reasoning for the maximum weight clique problem. *Journal of Artificial Intelligence Research* 55: 799–833, 2016. doi:10.1613/jair.4953

[46] G. E. Farr. Tutte-Whitney polynomials: some history and generalizations. In: *Combinatorics, Complexity and Chance*, Oxford University Press, pp. 28–52, 2007.

[47] H. Fei, S. Tan and P. Li. Hierarchical Multi-Task Word Embedding Learning for Synonym Prediction. In *Proceedings of the 25th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining (KDD ’19)*, pp. 834–842, 2019. doi:10.1145/3292500.3330914

[48] F. V. Fomin, F. Grandoni and D. Kratsch. A measure & conquer approach for the analysis of exact algorithms. *Journal of the ACM (JACM)* 56(5): pp. 1–32, 2009. doi:10.1145/1552285.1552286

[49] J. Gao, J. Chen, Z. Li and J. Zhang. ICS-GNN: Lightweight interactive community search via graph neural network. *Proc. VLDB Endow.*, 14(6): 1006–1018, 2021. doi:10.14778/3447689.3447704

[50] M. R. Garey and D. S. Johnson. *Computers and intractability: A guide to the theory of NP-completeness*. W. H. Freeman and Co., San Francisco, USA, 1979.

[51] Y. Ge and A. L. Bertozzi. Active Learning for the Subgraph Matching Problem. *2021 IEEE International Conference on Big Data (Big Data)*, pp. 2641–2649, 2021. doi:10.1109/BigData52589.2021.9671760

[52] R. Geisberger, P. Sanders, D. Schultes and D. Delling. Contraction Hierarchies: Faster and Simpler Hierarchical Routing in Road Networks. In *International Workshop on Experimental and Efficient Algorithms*, pp. 319-333, 2008. doi:10.1007/978-3-540-68552-4_24
[53] R. Geisberger and D. Schieferdecker. Heuristic contraction hierarchies with approximation guarantee. In *Third Annual Symposium on Combinatorial Search*, pp. 31–38, 2010.

[54] A. Gibbons. *Algorithmic graph theory*. Cambridge University Press, 1985.

[55] M. X. Goemans and D. P. Williamson. Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. *J. ACM*, 42(6): 1115–1145, 1995. doi:10.1145/227683.227684

[56] A. V. Goldberg and C. Harrelson. Computing the shortest path: A* search meets graph theory. In *SODA*, 5:156–165, 2005.

[57] G. P. Gordon and E. W. McMahon. A greedoid polynomial which distinguishes rooted arborescences. *Proc. Amer. Math. Soc.*, 107(2): 287–298, 1989.

[58] A. Graves, G. Wayne, M. Reynolds et al. Hybrid computing using a neural network with dynamic external memory. *Nature*, 538, 471–476, 2016. doi:10.1038/nature20101

[59] E. Gujral and E. E. Papalexakis. SMACD: Semi-supervised multi-aspect community detection. In *Proceedings of the 2018 SIAM International Conference on Data Mining*, pp. 702–710, 2018. doi:10.1137/1.9781611975321.79

[60] K. Hashimoto, C. Xiong, Y. Tsuruoka and R. Socher. A Joint Many-Task Model: Growing a Neural Network for Multiple NLP Tasks. In *Proceedings of the 2017 Conference on Empirical Methods in Natural Language Processing*, pp. 1923–1933, 2017.

[61] E. Hébrard and G. Katsirelos. Conflict Directed Clause Learning for the Maximum Weighted Clique Problem. *27th International Joint Conference on Artificial Intelligence (IJCAI 2018)*, pp. 1316–1323, 2018. https://hal.laas.fr/hal-01873485

[62] M. Held and R. M. Karp. A dynamic programming approach to sequencing problems. *Journal of the Society for Industrial and Applied Mathematics*, 10(1): 196–210, 1962. doi:10.1137/0110015

[63] F. Herrmann and A. Hertz. Finding the chromatic number by means of critical graphs. *ACM J. Exp. Algorithmics*, 7(10): 1–9, 2002. doi:10.1145/944618.944628

[64] A. Hertz and D. de Werra. Using tabu search techniques for graph coloring. *Computing* 39: 345–351, 1987. doi:10.1007/BF02239976

[65] J. Huan, W. Wang and J. Prins. Efficient mining of frequent subgraphs in the presence of isomorphism. *Third IEEE International Conference on Data Mining*, pp. 549–552, 2003. doi:10.1109/ICDM.2003.1250974
[66] S. Huang, Y. Wang, T. Zhao and G. Li. A Learning-based Method for Computing Shortest Path Distances on Road Networks. *IEEE 37th International Conference on Data Engineering (ICDE)*, pp. 360–371, 2021. doi:10.1109/ICDE51399.2021.00038

[67] X. Huang and L. V. S. Lakshmanan. Attribute-driven community search. *Proc. VLDB Endow.*, 10(9): 949–960, 2017. doi:10.14778/3099622.3099626

[68] F. Hutter, D. A. D. Tompkins and H. H. Hoos. Scaling and Probabilistic Smoothing: Efficient Dynamic Local Search for SAT. In *Principles and Practice of Constraint Programming–CP 2002*, pp. 233–248, 2002. doi:10.1007/3-540-46135-3_16

[69] F. Jaeger, D. L. Vertigan and D. J. A. Welsh. On the computational complexity of the Jones and Tutte polynomials. *Mathematical Proceedings of the Cambridge Philosophical Society*, 108:35–53, 1990.

[70] Y. Jia, J. Wu and Y. Du. Traffic speed prediction using deep learning method. *IEEE 19th International Conference on Intelligent Transportation Systems (ITSC)*, pp. 1217–1222, 2016. doi:10.1109/ITSC.2016.7795712

[71] Z. Jia, C. Chen, B. Coifman and P. Varaiya. The PeMS algorithms for accurate, real-time estimates of g-factors and speeds from single-loop detectors. *IEEE Intelligent Transportation Systems. Proceedings*, pp. 536–541, 2001. doi:10.1109/ITSC.2001.948715

[72] H. Jiang, C. M. Li, Y. Liu and F. Manyà. A two-stage maxsat reasoning approach for the maximum weight clique problem. *Proceedings of the AAAI Conference on Artificial Intelligence*, 32(1): 1338–1346, 2017.

[73] H. Jiang, C. M. Li and F. Manyà. An Exact Algorithm for the Maximum Weight Clique Problem in Large Graphs. *Proceedings of the AAAI Conference on Artificial Intelligence*, 31(1): 830–838, 2017. https://ojs.aaai.org/index.php/AAAI/article/view/10648

[74] H. Jiang, Y. Song, C. Wang, M. Zhang and Y. Sun. Semi-supervised Learning over Heterogeneous Information Networks by Ensemble of Meta-graph Guided Random Walks. *26th International Joint Conference on Artificial Intelligence (IJCAI 2017)*, pp. 1944–1950, 2017.

[75] Y. Jiang, Y. Rong, H. Cheng, X. Huang, K. Zhao and J. Huang. QD-GCN: Query-Driven Graph Convolutional Networks for Attributed Community Search, *preprint*, 2021. arXiv:2104.03583

[76] I. Jindal, X. Chen, M. Nokleby and J. Ye. A unified neural network approach for estimating travel time and distance for a taxi trip, *preprint*, 2017. arXiv:1710.04350
[77] R. M. Karp. Reducibility among combinatorial problems. In *Complexity of computer computations*, pp. 85–103, Springer, Boston, MA, 1972. doi:10.1007/978-1-4684-2001-2_9

[78] G. Karypis and V. Kumar. Analysis of Multilevel Graph Partitioning. *Proceedings of the 1995 ACM/IEEE Conference on Supercomputing*, pp. 29–29, 1995.

[79] S. Khuri and T. Bäck. An evolutionary heuristic for the minimum vertex cover problem. In *Genetic Algorithms within the Framework of Evolutionary Computation–Proc. of the KI-94 Workshop*, pp. 86–90, 1994.

[80] D. P. Kingma and J. Ba. Adam: A method for stochastic optimization, *preprint*, 2014. arXiv:1412.6980

[81] S. Kirkpatrick, C. D. Gelatt Jr and M. P. Vecchi. Optimization by simulated annealing. *Science*, 220(4598): 671–680, 1983. doi:10.1126/science.220.4598.671

[82] M. Kuramochi and G. Karypis. GREW - A scalable frequent subgraph discovery algorithm. *Fourth IEEE International Conference on Data Mining (ICDM’04)*, pp. 439–442, 2004. doi:10.1109/ICDM.2004.10024

[83] S. Lamm, P. Sanders, C. Schulz, D. Strash and R. F. Werneck. Finding near-optimal independent sets at scale. In *2016 Proceedings of the Eighteenth Workshop on Algorithm Engineering and Experiments (ALENEX)*, pp. 138–150, 2016. doi:10.1137/1.9781611974317.12

[84] H. Lemos, M. Prates, P. Avelar and L. Lamb. Graph Colouring Meets Deep Learning: Effective Graph Neural Network Models for Combinatorial Problems. *2019 IEEE 31st International Conference on Tools with Artificial Intelligence (ICTAI)*, pp. 879–885, 2019. doi:10.1109/ICTAI.2019.00125

[85] J. Leskovec, K. J. Lang and M. Mahoney. Empirical comparison of algorithms for network community detection. In *Proceedings of the 19th international conference on World wide web (WWW ’10)*, pp. 631–640, 2010. doi:10.1145/1772690.1772755

[86] C. M. Li, Y. Liu, H. Jiang, F. Manyà and Y. Li. A new upper bound for the maximum weight clique problem. *European Journal of Operational Research*, 270(1): 66–77, 2018. doi:10.1016/j.ejor.2018.03.020

[87] Y. Li, K. Fu, Z. Wang, C. Shahabi, J. Ye and Y. Liu. Multi-task Representation Learning for Travel Time Estimation. In *Proceedings of the 24th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining (KDD ’18)*, 1695–1704, 2018. doi:10.1145/3219819.3220033

[88] S. Lin and B. W. Kernighan. An Effective Heuristic Algorithm for the Traveling-Salesman Problem. *Operations Research*, 21(2):498–516, 1973. doi:10.1287/opre.21.2.498
[89] H. Liu, J. Han, Y. Fu, J. Zhou, X. Lu and H. Xiong. Multi-modal transportation recommendation with unified route representation learning. Proc. VLDB Endow., 14(3): 342–350, 2021. doi:10.14778/3430915.3430924

[90] H. Liu, T. Li, R. Hu, Y. Fu, J. Gu and H. Xiong. Joint Representation Learning for Multi-Modal Transportation Recommendation. Proceedings of the AAAI Conference on Artificial Intelligence, 33(1): 1036–1043, 2019. doi:10.1609/aaai.v33i01.33011036

[91] H. Liu, Y. Tong, P. Zhang, X. Lu, J. Duan and H. Xiong. Hydra: A Personalized and Context-Aware Multi-Modal Transportation Recommendation System. In Proceedings of the 25th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining (KDD ’19), pp. 2314–2324, 2019. doi:10.1145/3292500.3330660

[92] X. Liu, H. Pan, M. He, Y. Song, X. Jiang and L. Shang. Neural Sub-graph Isomorphism Counting. Proceedings of the 26th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining, 1959–1969, 2020. doi:10.1145/3394486.3403247

[93] S. Luo, B. Kao, G. Li, J. Hu, R. Cheng and Y. Zheng. TOAIN: A throughput optimizing adaptive index for answering dynamic kNN queries on road networks. Proc. VLDB Endow., 11(5): 594–606, 2018. doi:10.1145/3177732.3177736

[94] S. Luo, B. Kao, X. Wu and R. Cheng. MPR - A Partitioning-Replication Framework for Multi-Processing kNN Search on Road Networks, IEEE 35th International Conference on Data Engineering (ICDE), pp. 1310–1321, 2019. doi:10.1109/ICDE.2019.00119

[95] Y. Lv, Y. Duan, W. Kang, Z. Li and F. Wang. Traffic Flow Prediction With Big Data: A Deep Learning Approach. IEEE Transactions on Intelligent Transportation Systems, 16(2): 865–873, 2015. doi:10.1109/TITS.2014.2345663

[96] F. Mascia, E. Cilia, M. Brunato and A. Passerini. Predicting Structural and Functional Sites in Proteins by Searching for Maximum-weight Cliques. Proceedings of the AAAI Conference on Artificial Intelligence, 24(1): 1274–1279. https://ojs.aaai.org/index.php/AAAI/article/view/7495

[97] T. Mikolov, I. Sutskever, K. Chen, G. S. Corrado and J. Dean. Distributed representations of words and phrases and their compositionality. Advances in neural information processing systems, 26, 2013.

[98] R. Milo, S. Shen-Orr, S. Itzkovitz, N. Kashtan, D. Chklovskii and U. Alon. Network motifs: simple building blocks of complex networks. Science, 298(5594): 824–827, 2002. doi:10.1126/science.298.5594.824
[99] V. Mnih, K. Kavukcuoglu, D. Silver et al. Human-level control through deep reinforcement learning. *Nature*, 518, 529–533, 2015. doi:10.1038/nature14236

[100] M. Mohri, A. Rostamizadeh and A. Talwalkar. *Foundations of machine learning*. MIT Press, 2018.

[101] A. Mousavian, A. Rezvanian and M. R. Meybodi. Cellular learning automata based algorithm for solving minimum vertex cover problem. *2014 22nd Iranian Conference on Electrical Engineering (ICEE)*, pp. 996–1000, 2014. doi:10.1109/IranianCEE.2014.6999681

[102] C. H. Papadimitriou and K. Steiglitz. *Combinatorial optimization: algorithms and complexity*. Courier Corporation, 1998.

[103] E. E. Papalexakis, L. Akoglu and D. Ience. Do more views of a graph help? Community detection and clustering in multi-graphs. *Proceedings of the 16th International Conference on Information Fusion*, pp. 899–905, 2013. https://ieeexplore.ieee.org/abstract/document/6641090

[104] B. Perozzi, R. Al-Rfou and S. Skiena. DeepWalk: online learning of social representations. In *Proceedings of the 20th ACM SIGKDD international conference on Knowledge discovery and data mining (KDD ’14)*, pp. 701–710, 2014. doi:10.1145/2623330.2623732

[105] M. Prates, P. H. C. Avelar, H. Lemos, L. C. Lamb, and M. Y. Vardi. Learning to Solve NP-Complete Problems: A Graph Neural Network for Decision TSP. *Proceedings of the AAAI Conference on Artificial Intelligence*, 33(1): 4731–4738, 2019. doi:10.1609/aaai.v33i01.33014731

[106] W. Pullan. Approximating the maximum vertex/edge weighted clique using local search. *J Heuristics*, 14:117–134, 2008. doi:10.1007/s10732-007-9026-2

[107] W. Pullan. Optimisation of unweighted/weighted maximum independent sets and minimum vertex covers. *Discrete Optimization*, 6(2): 214–219, 2009. doi:10.1016/j.disopt.2008.12.001

[108] G. Reinelt. TSPLIB—A Traveling Salesman Problem Library. *ORSA Journal on Computing*, 3(4): 376–384, 1991. doi:10.1287/ijoc.3.4.376

[109] H. Samet, J. Sankaranarayanan and H. Alborzi. Scalable network distance browsing in spatial databases. In *Proceedings of the 2008 ACM SIGMOD International Conference on Management of Data (SIGMOD ’08)*, pp. 43–54, 2008. doi:10.1145/1376616.1376623

[110] P. San Segundo. A new DSATUR-based algorithm for exact vertex coloring. *Computers & Operations Research*, 39(7): 1724–1733, 2012. doi:10.1016/j.cor.2011.10.008
[111] J. Sankaranarayanan and H. Samet, Query Processing Using Distance Oracles for Spatial Networks. In IEEE Transactions on Knowledge and Data Engineering, 22(8):1158–1175, 2010. doi:10.1109/TKDE.2010.75

[112] S. B. Seidman. Network structure and minimum degree. Social Networks, 5(3):269–287, 1983. doi:10.1016/0378-8733(83)90028-X

[113] D. Selsam, M. Lamm, B. Bünz, P. Liang, L. de Moura and D. L. Dill. Learning a SAT solver from single-bit supervision, preprint, 2018. arXiv:1802.03685

[114] B. Settles. Active learning literature survey. Computer Sciences Technical Report 1648, University of Wisconsin–Madison, 2009.

[115] B. Shen, Y. Zhao, G. Li, W. Zheng, Y. Qin, B. Yuan and Y. Rao. V-Tree: Efficient kNN Search on Moving Objects with Road-Network Constraints. In 2017 IEEE 33rd International Conference on Data Engineering (ICDE), pp. 609–620, 2017. doi:10.1109/ICDE.2017.115

[116] C. Shen and T. Li. Multi-document summarization via the minimum dominating set. In Proceedings of the 23rd International Conference on Computational Linguistics (Coling 2010), pp. 984–992, 2010. https://aclanthology.org/C10-1111.pdf

[117] D. Silver, J. Schrittwieser, K. Simonyan et al. Mastering the game of Go without human knowledge. Nature, 550, 354–359, 2017. doi:10.1038/nature24270

[118] M. Sozio and A. Gionis. The community-search problem and how to plan a successful cocktail party. In Proceedings of the 16th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining (KDD ’10), pp. 939–948, 2010. doi:10.1145/1835804.1835923

[119] Y. Sun, X. Li and A. Ernst. Using Statistical Measures and Machine Learning for Graph Reduction to Solve Maximum Weight Clique Problems. IEEE Transactions on Pattern Analysis and Machine Intelligence, 43(5):1746–1760, 2021. doi:10.1109/TPAMI.2019.2954827

[120] R. S. Sutton and A. G. Barto. Reinforcement learning: An introduction, MIT press, 2018.

[121] J. Tang, M. Qu and Q. Mei. PTE: Predictive Text Embedding through Large-scale Heterogeneous Text Networks. In Proceedings of the 21th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining (KDD ’15), pp. 1165–1174, 2015. doi:10.1145/2783258.2783307

[122] J. Tang, M. Qu, M. Wang, M. Zhang, J. Yan and Q. Mei. LINE: Large-scale Information Network Embedding. In Proceedings of the 24th International Conference on World Wide Web (WWW ’15), pp. 1067–1077, 2015. doi:10.1145/2736277.2741093
[123] W. Tang, Z. Lu and I. S. Dhillon. Clustering with Multiple Graphs. *Ninth IEEE International Conference on Data Mining*, pp. 1016–1021, 2009. doi:10.1109/ICDM.2009.125

[124] S. Thevenin, N. Zufferey and J. Y. Potvin. Graph multi-coloring for a job scheduling application. *Discrete Applied Mathematics*, 234: 218–235, 2018. doi:10.1016/j.dam.2016.05.023

[125] J. A. Torkestani and M. R. Meybodi. Learning automata-based algorithms for solving stochastic minimum spanning tree problem. *Applied Soft Computing*, 11(6): 4064–4077, 2011. doi:10.1016/j.asoc.2011.02.017

[126] W. T. Tutte. A contribution to the theory of chromatic polynomials. *Can. J. Math.*, 6: 80–91, 1954. doi:10.4153/CJM-1954-010-9

[127] W. T. Tutte. On dichromatic polynomials. *J. Combin. Theory*, 2: 301–320, 1967. doi:10.1016/S0002-9800(67)80032-2

[128] J. R. Ullmann. An Algorithm for Subgraph Isomorphism. *Journal of the ACM*, 23(1): 31–42, 1976. doi:10.1145/321921.321925

[129] E. I. Vlahogianni. Computational intelligence and optimization for transportation big data: challenges and opportunities. In *Engineering and Applied Sciences Optimization*, pp. 107–128, 2015. doi:10.1007/978-3-319-18320-6_7

[130] H. Wang, X. Tang, Y. H. Kuo, D. Kifer and Z. Li. A Simple Baseline for Travel Time Estimation using Large-scale Trip Data. *ACM Trans. Intell. Syst. Technol.*, 10(2): 1–22, 2019. doi:10.1145/3293317

[131] Y. Wang, S. Cai and M. Yin. Two efficient local search algorithms for maximum weight clique problem. In *Thirtieth AAAI Conference on Artificial Intelligence*, pp. 805–811, 2016.

[132] Y. Wang, S. Pan, C. Li and M. Yin. A local search algorithm with reinforcement learning based repair procedure for minimum weight independent dominating set. *Information Sciences*, 512: 533–548, 2020. doi:10.1016/j.ins.2019.09.059

[133] B. M. Williams and L. A. Hoel. Modeling and forecasting vehicular traffic flow as a seasonal ARIMA process: Theoretical basis and empirical results. *Journal of transportation engineering*, 129(6): 664–672, 2003. doi:10.1061/(ASCE)0733-947X(2003)129:6(664)

[134] X. Xu and J. Ma. An efficient simulated annealing algorithm for the minimum vertex cover problem. *Neurocomputing* 69(7-9): 913–916, 2006. doi:10.1016/j.neucom.2005.12.016

[135] K. S. Yow. *Tutte-Whitney Polynomials for Directed Graphs and Maps*. PhD Thesis, Monash University, 2019.
[136] K. S. Yow, K. J. Morgan and G. E. Farr. Factorisation of greedoid polynomials of rooted digraphs. *Graphs and Combinatorics*, 37(6): 2245–2264, 2021. doi:10.1007/s00373-021-02347-0

[137] K. S. Yow, G. E. Farr and K. J. Morgan. Tutte invariants for alternating dimaps, *preprint*, 2018. arXiv:1803.05539

[138] B. Yu, H. Yin, Z. Zhu. Spatio-temporal graph convolutional networks: A deep learning framework for traffic forecasting. *27th International Joint Conference on Artificial Intelligence (IJCAI 2018)*, pp. 3634–3640, 2018. doi:10.24963/ijcai.2018/505

[139] J. Yuan, Y. Zheng, C. Zhang, W. Xie, X. Xie, G. Sun and Y. Huang. T-drive: driving directions based on taxi trajectories. In *Proceedings of the 18th SIGSPATIAL International Conference on Advances in Geographic Information Systems (GIS ’10)*, pp. 99–108, 2010. doi:10.1145/1869790.1869807

[140] Y. Zhang and Q. Yang. A survey on multi-task learning, *preprint*, 2017. arXiv:1707.08114

[141] Z. Zhang, X. Wang and W. Zhu. Automated machine learning on graphs: A survey, *preprint*, 2021. arXiv:2103.00742

[142] H. Zhao, Q. Yao, J. Li, Y. Song and D. L. Lee. Meta-Graph Based Recommendation Fusion over Heterogeneous Information Networks. In *Proceedings of the 23rd ACM SIGKDD International Conference on Knowledge Discovery and Data Mining (KDD ’17)*, pp. 635–644, 2017. doi:10.1145/3097983.3098063

[143] H. Zhou, Y. Zhao, J. Fang, X. Chen and K. Zeng, Hybrid route recommendation with taxi and shared bicycles. *Distrib Parallel Databases*, 38:563–583, 2020. doi:10.1007/s10619-019-07282-x

[144] Y. Zhou, J. K. Hao and B. Duval. Reinforcement learning based local search for grouping problems: A case study on graph coloring. *Expert Systems with Applications*, 64: 412–422, 2016. doi:10.1016/j.eswa.2016.07.047

[145] Z. Zhou, C. M. Li, C. Huang and R. Xu. An exact algorithm with learning for the graph coloring problem. *Computers & operations research*, 51: 282–301, 2014. doi:10.1016/j.cor.2014.05.017

[146] A. D. Zhu, H. Ma, X. Xiao, S. Luo, Y. Tang and S. Zhou. Shortest path and distance queries on road networks: towards bridging theory and practice. In *Proceedings of the 2013 ACM SIGMOD International Conference on Management of Data (SIGMOD ’13)*, pp. 857–868, 2013. doi:10.1145/2463676.2465277

39
| Problem                  | Task                                      | Approach       | Method                | Code  |
|-------------------------|-------------------------------------------|----------------|-----------------------|-------|
| **Clique**              | Maximum weight clique                     | Supervised     | MLPR [119]            | Link  |
| **Partition**           | Community detection                       | Semi-supervised| SMACD [59]            | Link  |
|                         | Community detection                       | Unsupervised   | K-means [26]          | -     |
| **Subgraph Isomorphism**| Subgraph matching                         | Active (Semi-supervised) | Ge et al. [51] | -     |
|                         | Subgraph isomorphism counting             | Deep           | DIAMNet [92]          | Link  |
| **Dominating Set**      | Minimum weight independent dominating set | Reinforcement  | LSRR [132]            | -     |
| **Travelling Salesman Problem** | TSP                                      | Reinforcement  | Deudon et al. [35] | Link  |
|                         | Decision TSP                              | Graph Neural Network | GNN [105]  | Link  |
| **Graph Colouring**     | Decision GCP                              | Graph Neural Network | GNN [84]  | Link  |
|                         | Graph colouring                           | Reinforcement  | RLS [144]             | -     |
|                         | Graph colouring                           | Clause         | cdlGCP [145]          | -     |
| **Vertex Cover**        | Minimum vertex cover                      | Cellular       | CLAVC [101]           | -     |
|                         | Minimum vertex cover                      | Reinforcement  | S2V-DQN [33]          | Link  |
| **Maximum Cut**         | Maximum cut                               | Reinforcement  | S2V-DQN [33]          | Link  |
| **Routing Problem**     | Route recommendation                      | Supervised     | MURAT [87]            | -     |
|                         | Route forecast                            | Deep           | STGCN [138]           | -     |
| **Community Search**    | Interative CS                             | Supervised     | ICS-GNN [49]          | -     |
|                         | Attributed CS                             | Supervised     | QD-GCN [75]           | -     |

Table 1: An overview of graph problems and the respective learning-based approaches