We start with a standard Lagrangian of a pure $SU(N)$ Yang-Mills theory and respective equations of motion

$$\mathcal{L}_0 = -\frac{1}{4} F^{a}_{\mu\nu} F^{a\mu\nu},$$

$$\left(D^\mu F^a_{\mu\nu}\right)^a = 0,$$

where $(a = 1, 2, .., N)$ denotes the color index, and $(\mu, \nu = r, \theta, \varphi, t)$ are the world indices in the spherical coordinates. In the case of $SU(2)$ QCD one can generalize the static axially-symmetric Dashen-Hasslacher-Neveu (DHN) ansatz [22] by adding a temporal component of the gauge potential

$$A_r^0 = K_1(r, \theta, t), \quad A_3^0 = K_2(r, \theta, t), \quad A_\varphi^0 = K_3(r, \theta, t),$$

$$A_r^1 = K_4(r, \theta, t), \quad A_3^1 = K_5(r, \theta, t).$$

The ansatz leads to a system of five partial differential equations (PDE) which is still invariant under residual $U(1)$ gauge transformations [23,25]

$$K_1' = K_1 \partial_r \lambda, \quad K_2' = K_2 + \partial_\theta \lambda, \quad K_3' = K_3 + \partial_\lambda,$$

$$K_4' = K_4 \cos \lambda + K_5 \sin \lambda, \quad K_5' = K_5 \cos \lambda - K_4 \sin \lambda,$$

where $\lambda(r, \theta, t)$ is an arbitrary gauge function. One can fix the residual symmetry by passing to a singular gauge where the Abelian potential $K_3$ describes static monopoles immersed in the field of dynamical off-diagonal gluons [20]. In the axially-symmetric case the singular gauge is not suitable for numeric solving since the reduced system of PDEs includes three second order hyperbolic type equations and two constraints containing first and second order space derivatives. So we choose a Lorenz gauge by introducing the gauge fixing terms

$$\mathcal{L}_{g.f.} = -\frac{1}{2} r^2 \sin \theta (\partial_r K_1 + \frac{1}{r^2} \partial_\theta K_2 - \partial_\lambda K_3)^2.$$

With this all five equations for $K_i$ become well-defined hyperbolic differential equations which can be solved by using standard numeric recipes. To solve the equations...
we impose boundary conditions which are consistent with local solutions near the boundaries and with finite energy density condition

\[ K_i |_{r=0} = \delta_{i4} C_4, \quad K_i |_{\theta=0, \pi} = 0, \]  

where \( C_4 \) is an arbitrary number. In the asymptotic region, \( r \approx \infty \), the equations admits the following basis solutions in the leading order

\[ \begin{align*}
K_1 |_{r=\infty} &= \frac{1}{r} (a_1(\theta) + b_1(\theta) \sin(Mr) \cos(Mt)), \\
K_{i=2,3,4} |_{r=\infty} &= a_i(\theta) + b_i(\theta) \cos(Mr) \cos(Mt), \\
K_5 |_{r=\infty} &= \frac{1}{r} (a_5(\theta) + b_5(\theta) \cos(Mr) \sin(Mt)),
\end{align*} \]

where the mass scale parameter \( M \) appears due to the presence of scaling invariance of the equations, \( r \rightarrow Mr, \ t \rightarrow Mt \), and it defines a class of conformally equivalent solutions. In further numeric calculations we set \( M = 1 \) without loss of generality. We use a method applied in solving equations for the static sphaleron solution \([24, 25]\). First we decompose the functions \( K_i(r, \theta, t) \) in Fourier series in a consistent manner with the local and asymptotic solutions

\[ K_i = \sum_{n=1,2,\ldots} \tilde{K}^{(n)}_i (r, \theta, t) \cos(nt), \]

where \( a_1(\theta), b_1(\theta), a_i(\theta), b_i(\theta), a_5(\theta), b_5(\theta) \) are the functions \( \tilde{K}^{(n)}_i(r, \theta) \) which are non-vanishing for \( i = 1,2,3,4 \) and odd order \( i = 5 \) modes \( \tilde{K}^{(n)}_i(r, \theta) \). We employ the same method as in the case of \( SU(2) \) QCD and solve the reduced equations for the field modes \( \tilde{K}^{(n)}_i(r, \theta), \tilde{Q}^{(n)}_i(r, \theta), \tilde{S}^{(n)}_i(r, \theta) \) in the two-dimensional numeric domain \((0 \leq r \leq L, 0 \leq \theta \leq \pi)\) up to the fifth order of series decomposition \([7]\). In general, the obtained system of reduced equations describes a wide class of regular stationary axially-symmetric solutions. To find a subclass of Weyl symmetric solutions one can simplify further the system of reduced equations

\[ \begin{align*}
A_1^2 &= K_1, \quad A_2^2 = K_2, \quad A_3^2 = K_4, \quad A_4^2 = K_5, \\
A_1^3 &= Q_1, \quad A_2^3 = Q_2, \quad A_3^3 = Q_4, \quad A_4^3 = Q_5, \\
A_1^7 &= S_1, \quad A_2^7 = S_2, \quad A_3^6 = S_4, \quad A_4^7 = S_5, \\
A_1^4 &= K_3, \quad A_3^4 = K_8,
\end{align*} \]

where the fields \( K, Q, S \) depend on \((r, \theta, t)\). We employ the same method as in the case of \( SU(2) \) QCD and solve the reduced equations for the field modes \( \tilde{K}^{(n)}(r, \theta), \tilde{Q}^{(n)}(r, \theta), \tilde{S}^{(n)}(r, \theta) \) in the two-dimensional numeric domain \((0 \leq r \leq L, 0 \leq \theta \leq \pi)\) up to the fifth order of series decomposition \([7]\). In general, the obtained system of reduced equations describes a wide class of regular stationary axially-symmetric solutions. To find a subclass of Weyl symmetric solutions one can simplify further the system of reduced equations

\( I, U, V \)-type \( SU(2) \) subgroups of \( SU(3) \)

\[ \begin{align*}
A_1^2 &= K_1, \quad A_2^2 = K_2, \quad A_3^2 = K_4, \quad A_4^2 = K_5, \\
A_1^3 &= Q_1, \quad A_2^3 = Q_2, \quad A_3^3 = Q_4, \quad A_4^3 = Q_5, \\
A_1^7 &= S_1, \quad A_2^7 = S_2, \quad A_3^6 = S_4, \quad A_4^7 = S_5, \\
A_1^4 &= K_3, \quad A_3^4 = K_8,
\end{align*} \]
applying the following ansatz

\begin{align}
\tilde{Q}_{1,2,5}^{(n)} &= -\tilde{S}_{1,2,5}^{(n)} = -\tilde{K}_{1,2,5}^{(n)}, \\
\tilde{Q}_{4}^{(n)} &= (-\frac{1}{2} + \frac{\sqrt{3}}{2})\tilde{K}_{4}^{(n)}, \\
\tilde{S}_{4}^{(n)} &= (-\frac{1}{2} - \frac{\sqrt{3}}{2})\tilde{K}_{4}^{(n)},
\end{align}

(9)

where \( n = 1, 3, 5 \) and the even modes \( \tilde{K}_{1}^{(n=2,4)} \) vanish. The ansatz is consistent with the system of equations for the field modes and reduces the number of the equations in each order \((n = 1, 3, 5)\) to four independent PDEs for the basic modes \( \tilde{K}_{1,2,4,5}^{(n)} \). Note that Abelian field strengths corresponding to potentials \( \tilde{K}_{3,8} \) are equal to each other as it takes place in the case of Abelian Weyl symmetric homogeneous magnetic fields [19, 20]. To solve the equations we set the following boundary conditions

\[
\tilde{K}_{1}^{(n)} = c_{1}^{(n)}(\theta), \quad \tilde{K}_{2,4,5}^{(n)} = 0,
\]

(10)

where the initial functions \( c_{1}^{(n)}(\theta) \) are arbitrary functions. We use a non-linear stationary solver based on the Newton iterative method which finds a solution starting with proper chosen initial profile functions. With this one obtains a numeric solution which is presented in Fig. 2. One can observe that SU(2) and SU(3) solutions for the Abelian gauge potentials \( K_{3}, K_{4} \) in Figs. 2c, 3c have the same profiles in the asymptotic region where both of them contain the angle dependent factor \( \sin^{2}\theta \). The presence of this factor implies that a corresponding radial component of the color magnetic field, \( H_{\theta\phi} = \partial_{\theta}K_{3}, \) creates magnetic fluxes through the upper and lower semi-spheres with opposite signs, and a total magnetic flux vanishes identically. Such a behavior is an inherent feature of the non-Abelian theory which admits a solution for a pair of monopole and antimonopole located at one point [27], there is such an analog in the Abelian gauge theory. There is only one known solution in the realistic theories, the sphaleron, which can be treated as a monopole-antimonopole pair [28, 29].

Let us now consider the stability of the vacuum condensates made of monopole pair fields. To verify whether a given classical solution can provide a stable quantum vacuum condensate (in one-loop approximation) it is suitable to apply the quantum effective action formalism. A one-loop effective action is expressed in terms of functional operator logarithms as follows [30, 31]

\[
S^{1\text{loop}}_{\text{eff}} = -\frac{1}{2} \text{Tr} \ln[K_{\mu\nu}^{ab}] + \text{Tr} \ln[M_{\mu\nu}^{ab}],
\]

\[
K_{\mu\nu}^{ab} = -\delta^{ab}\delta_{\mu\nu}\partial_{\rho}^{2} - \delta_{\mu\nu}(\mathcal{D}_{\rho}\mathcal{D}^{\rho})^{ab} - 2f^{acb}\mathcal{F}_{\mu\nu}^{c},
\]

(11)

\[
M_{\mu\nu}^{ab} = -(\mathcal{D}_{\rho}\mathcal{D}^{\rho})^{ab},
\]

where the Wick rotation \( t \to -it \) is performed and \( \mathcal{D}_{\mu}, \mathcal{F}_{\mu\nu} \) are defined with a classical background field \( \mathcal{D}_{\rho}^{\alpha}(t, r, \theta, \phi) \) describe gluon fluctuations. Note that the ghost operator is positively defined and does not produce instability [12]. Substituting interpolation functions for the stationary monopole solutions in the leading order as a background field one can solve the eigenvalue equation [12]. In the case of SU(2) QCD the full eigenvalue spectrum is divided into four sub-spectra corresponding to four factorized systems.
of equations: (I) $\Psi_1^2$, (II) $\Psi_1^4$, $\Psi_3^3$, (III) $\Psi_1^2$, $\Psi_2^2$, $\Psi_3^1$, (IV) $\Psi_1^4$, $\Psi_1^3$, $\Psi_2^1$, $\Psi_2^3$. The lowest eigenvalue is reached by a solution to type II equations, the corresponding eigenfunctions are plotted in Fig. 3. Other type equations have similar structure of the eigenvalue spectrum.

In the case of $SU(3)$ QCD the equations in (12) are not factorized, and one has to solve the full set of thirty two PDEs. Numeric analysis of the solutions for various set of initial amplitudes for the asymptotic background monopole solutions shows that the eigenvalue spectrum is positively defined. The behavior of the lowest eigenvalue at large distance confirms the positiveness of the eigenvalue spectrum, Fig. 4. This proves the stability of the monopole-antimonopole condensate. The stability of the vacuum condensate is confirmed numerically for solutions with asymptotic amplitudes for the Abelian field $K_3$ (or $K_4$) in the interval $0 \leq b_i \leq 2$. For large amplitude values of the background solution negative eigenvalues appear in the spectrum. Remind, that the spherically symmetric monopole solution provides a stable vacuum condensate with asymptotic amplitude values less than a critical one, $a_{cr} \simeq 0.56$ [20]. So that the monopole-antimonopole solutions provide better stability of corresponding vacuum condensates.

To trace the origin of stability of the vacuum condensate let us consider the classical Yang-Mills Lagrangian written in terms of Weyl symmetric fields [20]

$$\mathcal{L}_0 = \sum_{p=1,2,3} \left\{ -\frac{1}{6}(g'_{\mu\nu})^2 - \frac{1}{2} |D_\mu W^p_\nu - D_\nu W^p_\mu|^2 - ig'p W^{\kappa\rho\sigma} W_\mu W^{\kappa\rho\sigma} \right\} - V((W^p_\mu)^4), \quad (13)$$

where $g'_{\mu\nu}$ are Abelian field strength corresponding to the gauge potentials $A_{\mu}^{3,8}$, the complex fields $W^p_\mu$ represent off-diagonal gluons, $V((W^p_\mu)^4)$ is a quartic potential, and the index $"p"$ counts the Weyl symmetric combinations of the gauge potentials. The second and third terms in the Lagrangian includes cubic interaction terms corresponding to anomaly magnetic moment interaction which is precisely the source of the Nielsen-Olesen vacuum instability [12]. Direct substitution of the spherically symmetric $SU(2)$ monopole ansatz given in [20] and axially-symmetric $SU(2)$ DHN ansatz into the Lagrangian implies that such cubic terms remain non-vanishing. It is surprising, in the case of $SU(3)$ QCD despite each cubic contribution in (13) corresponding to $I,U,V$ Weyl combinations does not vanish, their total sum vanishes identically. This reveals the origin of color confinement in QCD, namely, the existence of $SU(3)$ Weyl symmetric classical monopole solution and absence of anomaly magnetic moment interaction terms in the Lagrangian provide stability and gauge invariance of the vacuum monopole condensate. This immediately implies the color confinement in QCD [4]. The absence of a stable monopole condensate in the electroweak theory based on the gauge group $SU(2) \times U(1)$ leads to gauge non-invariant vacuum and, as a consequence, to spontaneous symmetry breaking irrespective of presence of the Higgs potential which is introduced in a pure phenomenological way.

We conclude, the embedded $SU(2)$ stationary monopole solution is not symmetric under the Weyl group transformation, and it represents a saddle point in the space of $SU(3)$ axially-symmetric stationary solutions. Contrary to this the Weyl symmetric $SU(3)$ monopole solution realizes a deepest minimum of the effective potential (at least on the space of axially-symmetric field configurations) and provides a stable vacuum monopole pair condensate. Such a solution can serve as a structural element in further construction of the microscopic theory of the QCD vacuum. This strongly supports the QCD vacuum model as a dual color superconductor as it was conjectured in the seminal papers [2]-[5]. A rich structure of non-linear stationary solutions in QCD, which includes non-linear plane waves, spherical and axially-symmetric monopole like solutions, suggests a novel way to describe non-perturbative phenomena in hadron physics, in particular, in description of glueball spectrum [52, 53]. These issues will be considered in a separate paper.
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