Evolution of chiral-odd spin-independent fracture functions in Quantum Chromodynamics.

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Abstract

We construct the evolution equations for the twist-3 chiral-odd spin independent fracture functions in QCD. The Gribov-Lipatov reciprocity relation is fulfilled at the one-loop level for the quasi-partonic two-particle cut vertices only. It is found that the range of the anomalous dimensions matrix is infinite for any given moment of the three-parton fracture function as distinguished from the case of DIS-distributions where the range of the matrix was finite and increases with the number of the moment. In the multicentre limit $N_c \to \infty$ the evolution equation for the quark-gluon-quark correlation function decouples from another equation in the system and becomes homogeneous provided we discard the quark mass effects. This fact provides an opportunity to find its analytic solution explicitly in nonlocal form similarly to the DIS.
1 Introduction.

The deep inelastic scattering (DIS) of leptons on the hadron target is the most effective experimental tool for studying the dynamics of hadron reaction on the parton level which has a firm basis in the quantum field theory provided by light-cone Operator Product Expansion (OPE). The latter gives a strict theoretical ground for separation of two different scales of the underlying process: nonperturbative information is concentrated in the tower of local composite operators while the coefficient function which characterizes the hard interaction process of constituents can be dealt perturbatively. It makes possible the study of logarithmic violation of Bjorken scaling as well as the power suppressed contributions (higher twists) responsible for many subtle phenomena in a polarized scattering. However, there exists an equivalent approach to the analysis of the corresponding quantities which is based on factorization theorems and the evolution equations. In spite of the fact that the latter approach has some difficulties as compared to the former in the study of higher twists, like the loose of the explicit gauge and Lorentz invariance of calculations and also the presence of the overcomplete set of correlation functions, it has the important advantage as being the most close to the intuitive physical picture and the similarity to the parton model. There is another advantage of the latter approach for studying of the higher twist effects from the point of view of experimental capabilities since the OPE provides us for the moments of the structure functions and in order to extract the former one needs to measure the latter in the whole region of the momentum fraction very accurately. Obviously, it is quite difficult task even for next generation colliders. While, with a set of evolution equations at hand, one can find, in principle, the $Q^2$-dependence of the cross section in question by putting the experimental cuts on the region of the attained momentum fractions. This approach can also be used in the situations when the OPE is no longer valid. These are the inclusive production of the hadron in the $e^+e^-$-annihilation, semi-inclusive deep inelastic scattering, Drell-Yan lepton production et ctr.

There is continuous interest in the inclusive production of hadron in hard reaction. This process involves a quark fragmentation function to describe the hadron production from the underlying hard parton scattering. But it differs considerably from the DIS as the short distance expansion is no longer relevant although the given process goes near the light cone. The theoretical basis for strict analyses of the above phenomena is provided by the generalization of the OPE to the time-like region in terms of Mueller’s $\zeta$-space cut vertices, which moments are essentially nonlocal in the coordinate space. Again this approach has all attractive features of the OPE as it provides a consistent framework to account for the higher twist effects as well as it allows to sum up the UV logarithms by using the powerful methods of the renormalization group.
The semi-inclusive hadron production from a quark fragmentation is described in QCD by the specific nonperturbative correlation functions of quark and gluon field operators over the hadron states which can be identified with $\zeta$-space cut vertices. While the behaviour of the latter with respect to the fraction of the parton momentum carried by the hadron is determined by the nonperturbative strong interaction dynamics, the large $Q^2$-scale dependence is governed by perturbation theory only.

In this paper we begin our study of the evolution of the chiral-odd and chiral-even twist-3 fragmentation functions. We begin with nonpolarized twist-3 fracture functions. This is the simplest case with respect to the number of correlation functions involved in mixing under renormalization group evolution. From the phenomenological point of view they appear, for example, in the cross section for semi-inclusive hadron production in the process of measuring the nucleon’s transversity distribution $h_1(x)$ through deep inelastic scattering

$$
\frac{d^4\Delta\sigma}{dx dy d(1/\zeta) d\phi} = \frac{4\alpha^2}{Q^2} \cos \chi \left( 1 - \frac{y}{2} \right) G_1(x, \zeta) + \cos \phi \sin \chi \sqrt{(\kappa - 1)(1 - y)} \left( G_T(x, \zeta) - G_1(x, \zeta) \left( 1 - \frac{y}{2} \right) \right),
$$

where $\kappa = 1 + 4x^2M^2/Q^2$, $y = 1 - E'/E$ and the cross section is expressed in a frame where the lepton beam with energy $E$ defines the $z$-axis and the $x-z$-plane contains the nucleon polarization vector, which has the polar angle $\chi$ and the scattered electron $E'$ has the polar angles $\theta, \phi$. The functions $G_1$ and $G_T$ are expressed in terms of the product of the distribution and fragmentation functions in the following way

$$
G_1(x, \zeta) = \frac{1}{2} \sum_i Q_i^2 g_i^1(x) F^i(\zeta),
$$

$$
G_T(x, \zeta) = \frac{1}{2} \sum_i Q_i^2 \left[ g_i^T(x) F^i(\zeta) + \frac{1}{x} h_i^1(x) I^i(\zeta) \right].
$$

All of them have the expressions in QCD in terms of the light-cone Fourier transformation of correlation functions of fundamental quark and gluon fields over specific hadron states.
\[ \mathcal{F}(\zeta) = \frac{1}{4\zeta} \int \frac{d\lambda}{2\pi} e^{i\lambda \zeta} \langle 0 | \gamma_+ \psi(\lambda n) | h, X \rangle \langle h, X | \bar{\psi}(0) | 0 \rangle. \] (4)

and \( I \) is given by equation (3). The summing over final state hadrons in the definition of the fracture functions is implicit throughout the paper. Note, that the physical regions are different for distribution and fragmentation functions: \( 0 \leq x \leq 1 \) and \( 1 \leq \zeta < \infty \), respectively.

In the present study we address ourselves to the problem of constructing the evolution equations for the function \( I \) which mixes with other cut vertices during the renormalization. The outline of the paper is the following. In section 2 we give the definitions of the \( \zeta \)-space cut vertices which are closed with respect to the renormalization group evolution and discuss the advantages of the light-cone gauge. In the third section we show our technique on an example of abelian gauge theory: we review the renormalization of the theory in the axial gauge and construct corresponding equations. In the section 4 the \( \mathcal{Q}^2 \)-evolution equations are derived for the QCD case. The final section is devoted to the discussion of our results and conclusions. In the appendix we present some useful formulae which make the discussion more transparent.

## 2 Definitions.

It is well known that in order to endow with parton-like interpretation of field theoretical quantities and to get much deeper insight into the corresponding perturbative calculations it is necessary to use ghost-free gauges. Owing to this fact we chose in what follows the light-cone gauge \( B_+ = n^\mu B_\mu = 0 \) for the boson field and make the Sudakov decomposition of the parton four-momentum into transverse and longitudinal components

\[ k^\mu = z p^\mu + \alpha n^\mu + k_\perp^\mu, \] (5)

where \( p \) and \( n \) are null vectors which pick out two different directions on the light cone such that \( p^2 = n^2 = 0, (pn) = 1 \). The advantage of this gauge is that the gluon field operator \( B_\rho \) is related to the field strength tensor \( G_{\rho\sigma} \) via simple relation

\[ B_\mu(\lambda n) = \partial^- G_{+\mu}(\lambda n) = \frac{1}{2} \int_{-\infty}^{\infty} dz \epsilon(\lambda - z) G_{+\mu}(z), \] (6)

so, that the gauge invariant result can be restored after all required calculations have been performed.

For our purposes it is much more suitable to deal with correlation functions listed below\(^2\)

\[ I(\zeta) = \frac{1}{4} \int \frac{d\lambda}{2\pi} e^{i\lambda \zeta} \langle 0 | \psi(\lambda n) | h, X \rangle \langle h, X | \bar{\psi}(0) | 0 \rangle, \] (7)

\(^2\)See discussion of this point in refs. 13, 14.
\[ \mathcal{M}(\zeta) = \frac{1}{4\zeta} \int \frac{d\zeta}{2\pi} e^{i\zeta\gamma_4}\langle 0|m\gamma_4\psi(\Lambda_n)|h, X\rangle\langle h, X|\bar{\psi}(0)|0\rangle, \]  

\[ Z_1^{(1)}(\zeta', \zeta) = \frac{1}{4\zeta} \int \frac{d\zeta}{2\pi} e^{i\zeta\gamma_4}\langle 0|g\gamma_\rho\gamma_4\psi(\Lambda_n)|h, X\rangle\langle h, X|\bar{\psi}(0)B^\perp_\rho(\mu n)|0\rangle, \]  

\[ Z_2^{(1)}(\zeta', \zeta) = \frac{1}{4\zeta} \int \frac{d\zeta}{2\pi} e^{i\zeta\gamma_4}\langle 0|g\gamma_\rho\gamma_4\psi(\Lambda_n)|h, X\rangle\langle h, X|\bar{\psi}(0)B^\perp_\rho(\mu n)|0\rangle, \]  

\[ Z_1^{(2)}(\zeta, \zeta') = \frac{1}{4\zeta} \int \frac{d\zeta}{2\pi} e^{i\zeta\gamma_4\gamma_\rho}\langle 0|g\gamma_\rho\gamma_4\psi(\Lambda_n)|h, X\rangle\langle h, X|\bar{\psi}(0)B^\perp_\rho(\mu n)|0\rangle, \]  

\[ Z_2^{(2)}(\zeta, \zeta') = \frac{1}{4\zeta} \int \frac{d\zeta}{2\pi} e^{i\zeta\gamma_4}\langle 0|B^\perp_\rho(\Lambda_n)|h, X\rangle\langle h, X|\bar{\psi}(0)\gamma_4\gamma_\rho\psi(\Lambda_n)|0\rangle. \]

The quantities determined by these equations form the closed set under the renormalization, however, they are not independent since there is relation between them due to equation of motion for the Heisenberg fermion field operator

\[ I(\zeta) - \mathcal{M}(\zeta) + \int d\zeta' Z_1(\zeta', \zeta) = 0. \]  

Here and in the following discussion we introduce the convention

\[ Z_j(\zeta', \zeta) = \frac{1}{2} \left[ Z_j^{(1)}(\zeta', \zeta) + Z_j^{(2)}(\zeta', \zeta') \right]. \]  

While the former two functions \( I \) and \( \mathcal{M} \) can be made explicitly gauge invariant by inserting the \( P \)-ordered exponential (which is unity in the gauge we have chosen) between the quark fields, the latter can be written in the gauge invariant way introducing the following objects:

\[ \mathcal{R}_1(\zeta', \zeta) = \zeta' Z_1(\zeta', \zeta), \quad \mathcal{R}_2(\zeta, \zeta') = \zeta Z_2(\zeta, \zeta'). \]  

Taking into account eq. (3) it is easy to verify that they are indeed expressed in terms of correlators involving the gluon field strength tensor. The functions \( Z_j^{(1)} \) and \( Z_j^{(2)} \) are related by complex conjugation

\[ \left[ Z_j^{(1)}(\zeta', \zeta) \right]^* = Z_j^{(2)}(\zeta, \zeta'), \quad \left[ Z_j^{(2)}(\zeta, \zeta') \right]^* = Z_j^{(1)}(\zeta', \zeta). \]

Their support properties can be found by applying Jaffe’s recipe [15]. It has been shown that the field operators entering the definition of the correlation functions can be placed in the arbitrary order on the light cone with appropriate sign change according to their statistics. Then taking the particular ordering and saturating the correlation function by the complete set of the physical states we immediately obtain (for definiteness, we consider the function \( Z_1^{(1)} \))

\[ Z_1^{(1)}(\zeta', \zeta) = \frac{1}{4\zeta} \sum_{X,Y} \delta(\zeta - 1 - \zeta_X)\delta(\zeta' - \zeta_Y)\langle 0|\psi|h, X\rangle\langle h, X|\bar{\psi}|Y\rangle\langle Y|B^\perp|0\rangle \]

\[ = \frac{1}{4\zeta} \sum_{X,Y} \delta(\zeta - 1 - \zeta_X)\delta(\zeta' - \zeta_Y)\langle 0|\psi|h, X\rangle\langle h, X|B^\perp|Y\rangle\langle Y|\bar{\psi}|0\rangle. \]
with \( \zeta_X, \zeta_Y \geq 0 \) and we omit the unessential Dirac matrix structure of the vertex. From these equations the restrictions emerge on the allowed values of the momentum fractions: \( 1 \leq \zeta < \infty \), \( 0 \leq \zeta' \leq \zeta \). By analogy one can easily derive similar support properties for other functions.

3 Construction of the evolution equations.

Due to ultraviolet divergences of the momentum integrals in the perturbation theory there is logarithmic dependence of the parton densities on the normalization point. This dependence is governed by the renormalization group. The evolution equations for the leading twist correlation functions determining their \( Q^2 \) dependence can be interpreted in terms of the kinetic equilibrium of partons inside a hadron (for distribution functions) or hadrons inside a parton (for fragmentation function) under the variation of the ultraviolet transverse momentum cut-off \( k \). However, beyond the leading twist the probabilistic picture is lost due to quantum mechanical interference and more general quantities emerge, i.e. multiparticle parton correlation functions, whose scale dependence are determined by Faddeev type evolution equation with pairwise particle interaction \([3, 4]\).

There are two sources of the logarithmic dependence of the correlation functions. The first is the divergences of the transverse momentum integrals of the particles interacting with the vertex and forming the perturbative loop. Another source is the divergences due to the virtual radiative corrections. In the renormalizable field theory the latter are factorized into the renormalization constants of the corresponding Green functions. However, owing to the specific features of the renormalization in the light-like gauge, extensively reviewed in the next section, there is mixing of correlation functions due to the renormalization of field operators. The latter fact is closely related to the matrix nature of renormalization constants of the elementary Green functions in the axial gauge. For example, after renormalization of the fermionic propagator the matrix structure of the bare cut vertex could be changed, in general, since the renormalization matrix acts on the spinor indices of the vertex (see eq. \((57)\)).

In the Leading Logarithmic Approximation (LLA) there is a strong ordering \([6, 7]\) of transverse particle momenta as well as their minus components, so that only the particles entering the divergent virtual block \( \Sigma, \Gamma, \Pi \) or the particles adjacent to the cut vertex can achieve the maximum values of \(|k_\perp|\) and \(\alpha\):

\[
|k_{n\perp}| \ll ... \ll |k_{2\perp}| \ll |k_{1\perp}| \ll \Lambda^2, \\
\alpha_n \ll ... \ll \alpha_2 \ll \alpha_1, \quad (18)
\]
while the plus components of the parton momenta are of the same order of magnitude

\[ z_n \sim \ldots \sim z_2 \sim z_1 \]  

(19)

for \( n \)-rank ladder type diagram.

The radiative corrections to the bare cut vertex can be calculated using the conventional Feynman rules with the following modifications:

- All propagators and vertices on the RHS of the cut are hermitian conjugated to that on the LHS.
- Every time crossing the cut have the propagator \( 1/(k^2 - m^2 + i0) \) replaced by \( -2\pi i\delta(k^2 - m^2) \).
- For each propagator crossing the cut there is a \( \theta \)-function specifying that the energy flow from the LHS to the RHS is positive. (In the infinite momentum frame this is the plus component of the four-momentum.)

These statements complete the rules to handle the cut vertices.

4 Abelian evolution

In this section we show our machinery on a simple example of abelian evolution and generalize it afterwards to the Yang-Mills theory. We start with overfull set of the cut vertices defined by eqs. (7)-(12) and disregard for a moment the relation between them. Then eq. (13) verifies that the evolution equations thus obtained are indeed correct. We have to note that since the observed particle \( h \) is always in the final state some cuts of Feynman diagrams are not allowed and, therefore, we could not obtain the evolution kernel for the cut vertex taking the discontinuity of uncut graph as we are restricted over the limited set of the cuts.

4.1 Renormalization in the light-cone gauge.

The peculiar feature of the light-cone gauge manifest itself in the existence of additional UV divergences of the Feynman graphs which are absent in the usual isotropic gauges. They appear due to particular form of the gluon propagator

\[
D_{\mu\nu}(k) = \frac{d_{\mu\nu}(k)}{k^2 + i0} \\
d_{\mu\nu} = g_{\mu\nu} - \frac{k_\mu n_\nu + k_\nu n_\mu}{k_+} 
\]  

(20)
that possesses an additional power of the transverse momentum $k_\perp$ in the numerator. For our practical aims we limit ourselves with the calculation of the one-loop expressions for the propagators and vertex functions. This is sufficient for reconstruction of the equations in the LLA using the renormalization group invariance.

The unrenormalized fermion Green function is given by the expression

$$G^{-1}(k) = \frac{1}{k - m_0 - \Sigma(k)}, \quad (21)$$

where $\Sigma(k)$ is a self-energy operator. Calculating the latter to the one-loop accuracy with a principal value (PV) prescription \[18\] for the auxiliary pole in the gluon propagator in light-like gauge we get the following result

$$G(k) = (1 - \Sigma_1)U_2^{-1}(k)\frac{1}{k - m}U_1(k), \quad (22)$$

where

$$U_1(k) = 1 - \frac{m}{k_+}\Sigma_2(k)\gamma_+ - \frac{1}{k_+}(\Sigma_2(k) - \Sigma_1)\gamma_+k,$$ \hspace{1cm} (23)

$$U_2(k) = 1 + \frac{m}{k_+}\Sigma_2(k)\gamma_+ + \frac{1}{k_+}(\Sigma_2(k) - \Sigma_1)\gamma_+k,$$ \hspace{1cm} (24)

and

$$\Sigma_1 = \frac{\alpha}{4\pi}\ln \Lambda^2, \quad \Sigma_2(k) = \frac{\alpha}{4\pi}\ln \Lambda^2 \int dz' \frac{z'}{(z - z')}\Theta^0_{11}(z', z' - z). \quad (25)$$

Here $m$ is a renormalized fermion mass related to the bare quantity by the well known relation

$$m_0 = m(1 - 3\Sigma_1). \quad (26)$$

The functions $\Theta^m_{i_1i_2...i_n}$ and some useful relations between them are written down explicitly in the appendix. The renormalization constants are not numbers any more but matrices acting on the spinor indices of fermion field operators.

An abelian Ward identity leads to the equality of the renormalization constants of the gauge boson wave-function and a charge $Z_3 = Z_g$, so, that the corresponding logarithmic dependence on the UV cut-off cancels in their sum in the evolution equation and, therefore, we can neglect the fermion loop insertions into the boson line (in the QCD case this is no longer true).

One can easily calculate the vertex function to the same accuracy. The result is

$$\Gamma_\rho(k_1, k_2) = (1 + \Sigma_1)U_1^{-1}(k_1)G_\rho U_2(k_2), \quad (27)$$

where

$$G_\rho = \gamma_\rho - (k_1 - m)Q_\rho(k_1, k_2)\gamma_+ - \gamma_+Q_\rho(k_1, k_2)(k_2 - m) \quad (28)$$
\[ Q_{\rho}(k_1, k_2) = \Sigma_3(k_1)\gamma_{-\gamma^+\gamma_{\rho}} + \Sigma_3(k_2)\gamma^+_{\rho}\gamma_{-}, \] (29)

Here
\[ \Sigma_3(k_i) = \frac{\alpha}{8\pi} \ln \Lambda^2 \int dz' \frac{(z_i - z')}{z'} \Theta_{111}(z', z' - z_1, z' - z_2). \] (30)

Apart from the graphs we are accounted for there exists an additional UV divergence of the virtual Compton scattering amplitude, however, we do not need its explicit expression for our practical purposes. This completes the consideration of virtual corrections which cause the logarithmic dependence on the UV momentum cut-off of the quantities in question.

4.2 Sample calculation of the evolution kernels

As we have noted above the UV divergences also occur in the transverse-momentum integrals of partons interacting with a bare cut vertex. To extract this dependence properly it is sufficient to separate the perturbative loop from correlation function in question. To this end the latter can be represented in the form of momentum integral in which the integration over the fractional energies of the particles attached to the vertex is removed
\[ \left( \begin{array}{c} \mathcal{I}(\zeta) \\ \mathcal{M}(\zeta) \end{array} \right) = \int \frac{d^4k}{(2\pi)^4} \delta(\zeta - z) \left( \frac{I}{m\gamma_+} \right) F(k), \] (31)

where
\[ F(k) = \int d^4x e^{ikx} \langle 0|\psi(x)|h, X \rangle \langle h, X|\bar{\psi}(0)|0 \rangle. \] (32)

In the same way we can easily write corresponding expressions for the three-particle correlation functions.

Let us consider, for definiteness, the fracture function \( \mathcal{I} \). Simple calculation of the one-loop diagram for the \( 2 \to 2 \) transition in the LLA gives
\[ \mathcal{I}(\zeta)_{\Lambda^2} = g^2 \int \frac{d^4k}{(2\pi)^4} F(k) \int \frac{d^4k''}{(2\pi)^4} \delta(\zeta - z'') \theta(z'' - z) \frac{\delta((k'' - k)^2)}{k''^4} \times \{ -d_{\mu\nu}(k'' - k)\gamma_{\mu}(k'' + m)I(k'' + m)\gamma_{\nu} \} \]
\[ = -\frac{\alpha}{2\pi^2} \int \frac{d^4k}{(2\pi)^4} F(k) \int dz'' \frac{\delta(z'' - \zeta)\theta(z'' - z)}{(z'' - z)} \int d^2k''_1 \int d\alpha'' \frac{\delta(\alpha'' + \frac{k''^2_1}{2(z'' - z)})}{(z'' - z)} \times \left\{ \frac{\alpha''z'' + k''^2_1}{2\alpha''z'' + k''^2_1} - 2m\gamma_+ \left[ \alpha'' + \frac{2\alpha''z'' + k''^2_1}{z'' - z} \right] \right\} \]
\[ = -\frac{\alpha}{2\pi} \ln \Lambda^2 \int \frac{dz}{z} \theta(\zeta - z) \left[ \mathcal{I}(z) - \mathcal{M}(z) \left( 1 + 2\frac{z}{(\zeta - z)} \right) \right]. \] (33)
As long as logarithmic contribution appears when \(|k_\perp|/|k'_\perp| \ll 1\) and \(\alpha/\alpha'' \ll 1\) we expand the integrand in powers of these ratios keeping the terms that do produce the logarithmic divergence. Similarly one can evaluate the transition amplitudes of \(\mathcal{I}\) to the three-particle correlation functions \(\mathcal{Z}_j\):

\[
\mathcal{I}(\zeta) = -\frac{\alpha}{2\pi} \ln \Lambda^2 \int \frac{dz dz' \theta(\zeta - z) \mathcal{Z}_1(z', z)}{(\zeta - z)} \left[ \frac{2}{(\zeta - z)} + \frac{1}{(z - z')} \right]
\]

\[
\mathcal{I}(\zeta) = -\frac{\alpha}{2\pi} \ln \Lambda^2 \int \frac{dz dz' \theta(\zeta - z) \mathcal{Z}_2(z, z')}{(\zeta - z')(\zeta - z - z')}.
\]

Due to the non-quasi-partonic form of the vertex \(\mathcal{I}\) there exists an additional contribution to the evolution equations coming from the contact terms resulting from the cancellation of the propagator adjacent to the quark-gluon and bare cut vertices. As an output the vertex acquires the three-particle piece

\[
\mathcal{I}(\zeta) = \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} \mathcal{Z}_1 (k', k, \rho) \mathcal{G}_\rho (k - k', k) i \Gamma (k - k', k) \delta (\zeta - z) + (c.c.)
\]

\[
= -\frac{\alpha}{2\pi} \ln \Lambda^2 \int dz' \mathcal{Z}_1(z', \zeta) \int \frac{\zeta (z' - z' - z'')}{z''} \mathcal{Z}_1(z', z'' - \zeta, z'' - \zeta + z').
\]

As can be seen eqs. (33) and (34) possess the IR divergences at \(z = \zeta\). They disappear after we account for the virtual radiative corrections (renormalization of the field operators) discussed in the previous subsection. The net result looks like

\[
\Gamma^R = (1 - \Sigma_1) U_1 \Gamma U_1^{-1}, \quad \Gamma = \left( I, \frac{1}{\zeta} m \gamma_+ \gamma, \ g \gamma_\rho \gamma_+ \right).
\]

Assembling all these contributions we come to the evolution equation for \(\mathcal{I}\) given below by eq. (39).

### 4.3 Evolution equations.

Now following the procedure just described it is not difficult to construct the closed set of the evolution equations

\[
\dot{\mathcal{M}}(\zeta) = \frac{\alpha}{2\pi} \int \frac{dz}{z} \theta(\zeta - z) P_{\mathcal{M} \mathcal{M}} \left( \frac{\zeta}{z} \right) \mathcal{M}(z),
\]

\[
\dot{\mathcal{I}}(\zeta) = \frac{\alpha}{2\pi} \int \frac{dz}{z} \theta(\zeta - z) \left\{ P_{\mathcal{M} \mathcal{M}} \left( \frac{\zeta}{z} \right) \mathcal{I}(z) + P_{\mathcal{I} \mathcal{M}} \left( \frac{\zeta}{z} \right) \mathcal{M}(z)
\right.
\]

\[
- \int dz' \left[ P_{\mathcal{I} \mathcal{Z}_1} \left( \frac{\zeta}{z'} \frac{z'}{\zeta} \right) \mathcal{Z}_1(z', z) - \frac{z(\zeta - z)}{(\zeta - z') (z - z')} \mathcal{Z}_2(z, z') \right].
\]
\[ \dot{Z}_1(\zeta', \zeta) = \frac{\alpha}{2\pi} \left\{ \Theta^0_{11}(\zeta', \zeta' - \zeta) \left[ \frac{(\zeta - \zeta')}{\zeta} \mathcal{I} - \mathcal{M} \right] \right. \\
+ \theta(\zeta') \left[ \frac{1}{(\zeta - \zeta')} \mathcal{I} - \frac{1}{\zeta} \mathcal{M} \right] \right. \\
+ \int \frac{dz}{z} \theta(\zeta - z) \left[ P_{Z1} z \left( \frac{\zeta}{z}, \frac{\zeta'}{z} \right) Z_1(\zeta', z) - \frac{z(\zeta - z)^2}{\zeta_z(z - \zeta + \zeta')} Z_2(z, \zeta - \zeta') \right] \\
+ \int dz' \left[ \Theta^0_{111}(\zeta', \zeta - \zeta, \zeta' - \zeta + z') \frac{(\zeta' - \zeta + z')}{\zeta'} Z_1(z', \zeta) \right] \\
+ \theta(\zeta') \left[ \frac{(\zeta' - \zeta)}{\zeta_z'(\zeta' - \zeta)} Z_1(z', \zeta - \zeta') - \theta(\zeta - \zeta') \frac{\zeta' (\zeta - \zeta')}{\zeta_z(\zeta' - \zeta)} Z_2(\zeta', z') \right] \right\}, \tag{40} \]

\[ \dot{Z}_2(\zeta, \zeta') = \frac{\alpha}{2\pi} \left\{ \theta(\zeta - \zeta') \left[ \frac{(\zeta' - \zeta)}{\zeta_z} \mathcal{I} - \frac{1}{\zeta} \mathcal{M} \right] \right. \\
- \int \frac{dz}{z} \theta(\zeta - z) \frac{z(\zeta - z)}{\zeta_z^2} Z_1(z - \zeta, z) \\
+ \int dz' \left[ \theta(\zeta - \zeta') \frac{(\zeta - \zeta') (\zeta - \zeta' + z')}{\zeta_z^2(z')} Z_1(Z', \zeta') \right] \\
- \Theta^0_{11}(\zeta', \zeta' - z') \frac{\zeta'}{(\zeta' - z')} [Z_2(\zeta, z') - Z_2(\zeta, \zeta')] \\
- \Theta^0_{11}(\zeta' - \zeta, \zeta' - z') \frac{(\zeta' - \zeta)}{(\zeta' - z')} [Z_2(\zeta, \zeta') - Z_2(\zeta, \zeta')] \right\} \tag{41} \]

where the dot denotes the derivative with respect to the UV cutoff \( \Lambda^2 \partial / \partial \Lambda^2 \) and splitting functions are given by the following equations

\[ P_{MM}(z) = - \left[ \frac{2}{z(1 - z)} \right]_+ + \frac{1}{z} + 1, \tag{42} \]

\[ P_{TT}(z) = -1 + \frac{1}{2} \delta(z - 1), \tag{43} \]

\[ P_{LM}(z) = - \left[ \frac{2}{z(1 - z)} \right]_+ + \frac{2}{z} + 1, \tag{44} \]

\[ P_{T2}(z, y) = - \left[ \frac{2}{z(1 - z)} \right]_+ + \frac{2}{z} + \frac{1}{y} - \delta(z - 1) \frac{1}{y} \ln(1 - y), \tag{45} \]

\[ P_{Z1z}(z, y) = - \left[ \frac{2}{z(1 - z)} \right]_+ + \frac{2}{z} + \frac{y}{1 - yz} + \delta(z - 1) \left[ \frac{3}{2} - \ln(1 - y) \right]. \tag{46} \]

Now it is an easy task to verify the fulfilment of the equation of motion (13) for the correlation functions as a consistency check of our calculations. By exploiting this relation we exclude \( \mathcal{I} \) from the above set of functions and reduce the system to the basis of independent gauge invariant quantities \( \{ \mathcal{M}, R_j \} \).

An important note is in order now. As distinguished from the DIS case the above eq. (46) has the logarithmic dependence on the ratio of the parton momentum fractions. The
consequence of its presence is obvious. Taking into account the restrictions imposed by eq. (17) we can define the moments of the correlation functions in the following way

\[ M_n = \int_1^\infty \frac{d\zeta}{\zeta^n} M(\zeta), \quad (47) \]

\[ R_m^n = \int_1^\infty \frac{d\zeta}{\zeta^n} \int_0^\zeta d\zeta' \zeta'^m R(\zeta', \zeta). \quad (48) \]

We find for two-particle cut vertex

\[ \dot{M}_n = \frac{\alpha}{2\pi} \left[ -\psi(n + 1) - \psi(n - 1) - 2\gamma_E \right] M_n. \quad (49) \]

where \( \gamma_E \) is a Euler-Masceroni constant and \( \psi(n) = \Gamma'(n)/\Gamma(n) \). And we see the universality of the evolution kernels for the time- and space-like quasi-partonic two-particles cut vertices, i.e. the Gribov-Lipatov reciprocity relation is fulfilled [17]. However, it is impossible to write down the finite system of equations for any moment of the three-parton correlation functions as the logarithm of the ratio of the parton momentum fractions in the evolution kernels leads to the infinite series of the moments as distinguished from the deep inelastic scattering where the range of the anomalous dimension matrix was finite and increases as the moment of the correlation function increases. Therefore, it is not possible to solve the system of equations successively in terms of moments as well as we do not succeed in solving it analytically in a general form. However, in the next section when dealing with the QCD evolution we will find that the system can be reduced to the single equation in the limit of infinite number of colours and there is believe that its solution can be found analytically.

5 Non-abelian evolution.

For the non-abelian gauge theory the equality of the renormalization constants \( Z_g = Z_3 \) no longer holds, so, we should account for the renormalization of the gluon wave-function as well as for the renormalization of charge explicitly. For this purposes to complete the renormalization program outlined in the preceding section we evaluate the gluon propagator to the same accuracy. The result can be written in the compact form

\[ D_{\mu\nu}(k) = \left( 1 + \Pi^{tr}(k) \right) U_{\mu\rho}(k) \frac{d_{\rho\sigma}(k)}{k^2 + i0} U_{\sigma\nu}(k), \quad (50) \]

where

\[ U_{\mu\nu}(k) = g_{\mu\nu} - \frac{1}{2} \Pi^{odd}(k) \frac{k_\mu n_\nu + k_\nu n_\mu}{k^+} \quad (51) \]

and

\[ \Pi^{tr}(k) = 2\frac{\alpha}{4\pi} \ln \Lambda^2 \left\{ C_A \int dz \frac{[z^2 - z\zeta + \zeta^2]^2}{z(z - \zeta)\zeta^2} \Theta^0_{11}(z, z - \zeta) - \frac{N_f}{3} \right\}, \]
\[
\Pi^{\text{odd}}(k) = \frac{\alpha}{4\pi} \ln \Lambda^2 C_A \int dz \frac{[5z \zeta^2 (z - \zeta) + 6z^2 (z - \zeta)^2 + 2\zeta^4]}{z(z - \zeta)\zeta^2} \Theta_{11}(z, z - \zeta) \tag{52}
\]

are the transverse and longitudinal pieces of polarization operator. While the renormalized charge is given by the well known “asymptotic freedom” formula

\[
g_0 = g \left[ 1 + \frac{\alpha}{4\pi} \ln \Lambda^2 \left( \frac{N_f}{3} - \frac{11}{6} C_A \right) \right]. \tag{53}
\]

The second addendum arises from the diagrams with triple-boson interaction vertex. Gathering these contributions together with equations obtained in the previous section (with colour group factors accounted for properly), we come to the final result

\[
\hat{Z}_1(\zeta', \zeta) = \frac{\alpha}{2\pi} C_F \left[ \theta(\zeta') \frac{\zeta'}{\zeta'(\zeta - \zeta')} \mathcal{M}(\zeta - \zeta') - \Theta_{11}(\zeta', \zeta' - \zeta) \frac{\zeta'}{\zeta} \mathcal{M}(\zeta) \right] + \int \frac{dz}{z} \theta(z - \zeta) \left[ P_{Z_1 Z_1} \left( \frac{\zeta'}{z} \right) Z_1(\zeta', z) - (C_F - \frac{C_A}{2}) \frac{z(z - \zeta)^2}{\zeta' (\zeta' - \zeta + \zeta)} Z_2(z, \zeta - \zeta') \right] + \frac{C_A}{2} \left( -2z \frac{\partial}{\partial \zeta'} \int_0^1 dv Z_1(\zeta' - v(\zeta - z), z) \right) + \frac{1}{(\zeta - \zeta')} Z_1(\zeta', \zeta - \zeta') 
\]

\[
\hat{Z}_2(\zeta, \zeta') = \frac{\alpha}{2\pi} \left[ -C_F \theta(\zeta - \zeta') \frac{1}{\zeta'} \mathcal{M}(\zeta) \right] + \int \frac{dz}{z} \theta(z - \zeta) \left[ P_{Z_2 Z_2} \left( \frac{\zeta'}{z} \right) Z_2(\zeta, z) - (C_F - \frac{C_A}{2}) \frac{z(z - \zeta)}{\zeta^2} Z_1(z - \zeta, z) \right] + \frac{C_A}{2} \left( -2z \frac{\partial}{\partial \zeta'} \int_0^1 dv Z_2(z, \zeta' - v(\zeta - z)) \right) + \frac{1}{(\zeta - \zeta')} \left[ Z_2(\zeta, z - \zeta + \zeta') - 2\Theta_{11}(\zeta', \zeta' - \zeta) \frac{\zeta'}{(\zeta' - \zeta')} \left( Z_1(\zeta', \zeta) - Z_1(\zeta', \zeta) \right) \right] \tag{54}
\]
\[
+ \left( C_F - \frac{C_A}{2} \right) \left( \theta(\zeta - \zeta') \frac{(\zeta - \zeta')(\zeta - \zeta' + z')}{\zeta^2 z'} \right) Z_1(z', \zeta')
\]
\[
- \Theta^0_{11}(\zeta', \zeta' - z') \left( \zeta' - \zeta' \right) [Z_2(\zeta, z') - Z_2(\zeta, \zeta')]
\]
\[
- \Theta^0_{11}(\zeta' - \zeta, \zeta' - z') \left( \zeta' - \zeta \right) \left[ Z_2(\zeta, z') - Z_2(\zeta, \zeta') \right] \right) \right) \right) \right) \right) \right) \right) \right) \right)
\]
\[
- \frac{C_A}{2} \theta(\zeta - \zeta') \left( \frac{1}{(\zeta' - \zeta') + \frac{z'}{\zeta'^2}} \right) Z_1(z', \zeta') \right) \right) \right) \right) \right) \right) \right) \right) \right) \right)
\]

(55)

where

\[
P_{21} z_1(z, y) = C_F \left\{ - \left[ \frac{2}{z(1 - z)} \right] + \frac{2}{z} + \delta(z - 1) \left[ \frac{3}{2} - \ln(1 - y) \right] \right\}
\]
\[
+ \left( C_F - \frac{C_A}{2} \right) \frac{y}{1 - yz},
\]
\[
P_{22} z_2(z, y) = \frac{C_A}{2} \left\{ - \left[ \frac{4}{z(1 - z)} \right] + \frac{4}{z} + \frac{1}{1 - yz} \frac{1 + z}{z^2} \right\}
\]
\[
+ \delta(z - 1) \left[ \frac{3}{2} C_F - \frac{C_A}{2} (\ln y + \ln(1 - y)) \right].
\]

(56)

These equations should be supplemented by the equation for the mass cut vertex \( M \) which differs from its abelian analogue \( \{18\} \) only by the group factor \( C_F \).

One can easily observe the significant simplification of the above evolution equations in the limit \( N_c \to \infty \) since \( Z_2 \) decouples from the evolution equation for \( Z_1 \). Therefore, discarding the quark mass cut vertex we obtain homogeneous equation which governs the \( Q^2 \)-dependence of the three-parton correlation function \( Z_1 \). Similar phenomenon has been found in the evolution equations for chiral-even and -odd distribution functions in DIS \( \{10, 19\} \) where it has been observed that for multicolour QCD the momentum fraction carried by gluon in matrix element of quark-gluon operator varies only among the quarks ones and does not exceed the latter. Owing to this feature the solution of approximate equations becomes possible. In the forthcoming paper we will address to this issue in greater details.

6 Conclusion.

We have developed the effective technique for construction of the evolution equations for the twist-3 cut vertices. The physically transparent picture which appears in the light-cone gauge (an essential ingredient of our method) makes the calculations simple. Using this technique we have constructed the basis of chiral-odd non-polarized cut vertices closed under renormalization.
group evolution. The identity provided by the equation of motion for field operators makes the construction of the basis of independent operators trivial.

The most striking difference from the DIS evolution is the appearance of the logarithmic dependence on the parton momentum fractions in the evolution kernels which makes the successive solution of equations in terms of moments impossible. The nonlocality of the cut vertex in coordinate space is essential since even if we start from the local cut vertex it will smeared along the light cone upon the renormalization.

However, an important point is that there is decoupling of three parton correlation function $Z_1$ from $Z_2$. The situation has the closer similarity with DIS where in the limit $N_c \to \infty$ there was a very important simplification as the evolution kernels have been vanishing for contributions with interchanged order of partons on the light cone, i.e. the gluon momentum fraction ranges between those of the quarks only. This property allowed the author of refs. [10, 19] to find the solution of simplified equations exactly on the nonlocal form. In our case the decoupling of $Z_1$ may have the same consequences. This work is in progress now and the results will be published elsewhere.

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A Appendix

In this appendix we present the necessary properties of the $\Theta$-functions

$$
\Theta^m_{i_1 i_2 \ldots i_n}(z_1, z_2, \ldots, z_n) = \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi i} \alpha^m \prod_{k=1}^{n} (\alpha z_k - 1 + i0)^{-i\alpha}.
$$

For our discussion it is enough to have an explicit form of the function

$$
\Theta^0_{11}(z_1, z_2) = \frac{\theta(z_1)\theta(-z_2) - \theta(z_2)\theta(-z_1)}{z_1 - z_2},
$$

since the others can be expressed in its term via relations

$$
\Theta^0_{21}(z_1, z_2) = \frac{z_2}{z_1 - z_2} \Theta^0_{11}(z_1, z_2),
$$

$$
\Theta^0_{111}(z_1, z_2, z_3) = \frac{z_2}{z_1 - z_2} \Theta^0_{11}(z_2, z_3) - \frac{z_1}{z_1 - z_2} \Theta^0_{11}(z_1, z_3),
$$

$$
\Theta^1_{111}(z_1, z_2, z_3) = \frac{1}{z_1 - z_2} \Theta^0_{11}(z_2, z_3) - \frac{1}{z_1 - z_2} \Theta^0_{11}(z_1, z_3).
$$
In the main text we have used the identity

\[ \text{PV} \int dz \frac{\zeta}{(\zeta - z)} \left[ \Theta_{11}^0(z, z - \zeta) + \Theta_{11}^0(\zeta, \zeta - z) \right] = 0. \]  \hspace{1cm} (62)

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