Space-like surfaces with free boundary in the Lorentz–Minkowski space

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Abstract
We investigate a variational problem in the Lorentz–Minkowski space \( \mathbb{L}^3 \) whose critical points are space-like surfaces with a constant mean curvature and making a constant contact angle with a given support surface along its common boundary. We show that if the support surface is a pseudosphere, then the surface is a planar disc or a hyperbolic cap. We also study the problem of space-like hypersurfaces with free boundary in the higher dimensional Lorentz–Minkowski space \( \mathbb{L}^{n+1} \).

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1. Introduction

It is well known that space-like hypersurfaces with a constant mean curvature in Lorentzian spaces are critical points of the area functional under deformations that keep constantly the enclosed volume by the hypersurface. The utility in physics, especially in general relativity, of constant mean curvature space-like hypersurfaces is that they are convenient to be used as initial data for the Cauchy problem of the Einstein equations and their uniqueness property [8, 12]: in a cosmological spacetime, there exists at most one compact Cauchy hypersurface with a given (nonzero) constant mean curvature and a maximal (i.e. zero mean curvature) compact Cauchy hypersurface that is almost unique. Such hypersurfaces are important because their properties reflect those of the spacetime, such as, for example, the proof of the positive mass conjecture [24]. Another interesting property is that if there is a compact hypersurface with a constant mean curvature \( S \) in a cosmological spacetime, then there exists a foliation associated with \( S \), denoted by \( S(t) \) by compact constant mean curvature hypersurfaces with \( S(0) = S \) and the mean curvature \( S \) varying in a monotone way with respect to \( t \). Thus, a function \( t \) can be defined by the property that its value at each point of a given leaf of the foliation is equal to the mean curvature of that leaf. If this foliation is global and in the case that \( t \) is nonzero, then the function \( t \) provides a global time coordinate, whose gradient is a time-like vector field in the spacetime [5, 9, 10, 18]. This may be of relevance for the problem of time in quantum
gravity [7]. In particular, the problem of existence of such hypersurfaces is central in this theory and it has been treated widely (we only refer to [6] and [23], and references therein). Also, this type of hypersurface becomes asymptotically null as they approach infinity and then they are suitable for studying the propagation of gravitational waves [13, 25].

In this work, we are interested in a modified version of the variational problem. Consider the three-dimensional space \( \mathbb{L}^3 \). Given a three-dimensional region \( W \subset \mathbb{L}^3 \) with smooth boundary \( \partial W \), we consider a compact space-like surface \( M \) that is a critical point of the area among space-like surfaces in \( W \) preserving the volume of the domain bounded by \( M \) and \( \partial W \) with the boundary \( \partial M \) lies on \( \partial W \) and the interior of \( M \) is included in the interior of \( W \). Then, we admit that the deformations \( M_t \) of \( M \) are subject to the constraint that all boundaries \( \partial M_t \) move in the prescribed support surface \( \Sigma = \partial W \). Besides the area of the surface, in the energy functional we have to add a term that represents, with a certain weight \( \lambda \), the energy (area) of the part \( \Omega_1 \) of \( \Sigma \) bounded by \( \partial M_t \). In particular, the induced metric in \( \Sigma \) is non-degenerate.

In this context, the critical points of the energy for any volume-preserving variation are called stationary surfaces and they are characterized by the two properties: (i) the mean curvature is constant and (ii) the contact angle of the surface with the support \( \Sigma \) along its boundary is also constant. Because \( \Sigma \) is non-degenerate, it makes sense to consider the angle between the unit normal vectors of the stationary surface \( M \) and support \( \Sigma \). In the Euclidean space, this type of problem appears in the context of the theory of capillary surfaces, with great influence in many areas of physics and chemistry.

In Lorentzian spaces, this problem is introduced by Alías and Pastor [4] considering the Lorentz–Minkowski space as the ambient space and a space-like plane or a hyperbolic plane as the support surface \( \Sigma \). When the ambient spaces are the other simply connected Lorentzian space forms, namely the de Sitter space and the anti-de Sitter space, similar results have been obtained in [21]. The first kinds of supports are the umbilical surfaces of \( \mathbb{L}^3 \), that is, non-degenerate planes, hyperbolic planes and pseudospheres [20]. As the boundary of a space-like compact surface is a closed space-like curve, in the case that the support surface is a plane, this plane must be space-like.

This paper is motivated by the results that appear in [4]. Among the umbilical surfaces of \( \mathbb{L}^3 \), the remaining case to study is that \( \Sigma \) is a pseudosphere, which is considered in this work, showing

**Theorem 1.1.** The only stationary space-like surfaces in \( \mathbb{L}^3 \) with an embedded and connected boundary and with a pseudosphere as the support surface are the planar discs and the hyperbolic caps.

We point out that it is implicitly assumed in the results of [4, 21] that the boundary of the stationary surface is embedded and connected. For example, in \( \mathbb{L}^3 \) there are pieces of rotational space-like surfaces with constant mean curvature bounded by two concentric circles contained in the same plane and orthogonal to the rotational axis. Of course, these surfaces satisfy the boundary condition on the contact angle (see pictures in [15]). More recently, the first author has studied stationary surfaces in \( \mathbb{L}^3 \) with some assumption on the symmetry of the surface, relating geometric quantities of the surface such as its height, area and volume [16, 17].

The paper is organized as follows. After the preliminaries section, where we present the variational problem, we consider in section 3 our specific setting when the support is a pseudosphere. The fact that the boundary of the stationary surface is a space-like curve and a pseudosphere is a time-like curve makes a slight modification of the variational problem necessary, which will be reformulated in this section. In section 4, we prove theorem 1.1 and finish in section 5 with some discussions of the variational problem in the general \( n \)-dimensional case.
2. Preliminaries

Let $\mathbb{L}^3$ be the three-dimensional Lorentz–Minkowski space, that is, the real vector space $\mathbb{R}^3$ endowed with the Lorentz–Minkowski metric

$$\langle \, , \rangle = (dx_1)^2 + (dx_2)^2 - (dx_3)^2,$$

where $x_1$, $x_2$, and $x_3$ are the canonical coordinates of $\mathbb{R}^3$. Given a connected surface $M$, a smooth immersion $x : M \to \mathbb{L}^3$ is called space-like if the induced metric on $M$ via $x$ is positive definite. Observe that $a = (0, 0, 1)$ is a unit time-like vector field globally defined on $\mathbb{L}^3$, which determines a time-like orientation on $\mathbb{L}^3$. Thus, given a space-like immersion, we can choose a unique unitary time-like normal field $N$ globally defined on $M$ such that $\langle N, a \rangle < 0$. This shows that $M$ is orientable. If $N$ is chosen as above, we say that $N$ is future-directed and the $M$ is oriented by a unit future-directed time-like normal vector field $N$.

Among the examples of space-like surfaces in $\mathbb{L}^3$, we point out here the totally umbilical ones, that is, space-like planes and hyperbolic planes. A space-like plane is given by the set $\{x \in \mathbb{L}^3; \langle x - p, v \rangle = 0, \text{where} \, p \in \mathbb{L}^3 \text{and} \, v \text{is a time-like vector of} \, \mathbb{L}^3\}$. The mean curvature is $H = 0$. After an isometry, a hyperbolic plane $\mathbb{H}^2(p, r)$ centered at $p$ and radius $r > 0$ is given by

$$\mathbb{H}^2(p, r) = \{x \in \mathbb{L}^3; \langle x - p, x - p \rangle = -r^2\}.
$$

This surface has two components: $\mathbb{H}^2_+(p, r) = \{x \in \mathbb{H}^2(p, r); x_3 \geq x_3(p)\}$ and $\mathbb{H}^2_-(p, r) = \{x \in \mathbb{H}^2(p, r); x_3 \leq -x_3(p)\}$. With the future-directed time-like orientation, the mean curvatures of $\mathbb{H}^2_+(p, r)$ and $\mathbb{H}^2_-(p, r)$ are $1/r$ and $-1/r$, respectively.

We state the variational problem. Here, we follow [4] and refer there for more details. Let $\Sigma$ be an embedded and connected non-degenerate surface in $\mathbb{L}^3$ that divides the ambient space $\mathbb{L}^3$ into two connected components denoted by $\mathbb{L}^3_+$ and $\mathbb{L}^3_-$. Let $M$ be a compact surface. In all results of this work, it is assumed that the boundary of $M$ is connected, although this is not necessary to establish the variational problem. Given a space-like immersion $x : M \to \mathbb{L}^3$, an admissible variation of $x$ is a smooth map $X : M \times (-\epsilon, \epsilon) \to \mathbb{L}^3$ such that for each $t \in (-\epsilon, \epsilon)$, the map $X_t : M \to \mathbb{L}^3$ defined by $X_t(p) = X(p, t)$ is a space-like immersion with $X_t(\text{int}(\Sigma)) \subset \mathbb{L}^3_+$ and $X_t(\partial M) \subset \Sigma$ and at the initial time $t = 0$, we have $X_0 = x$. The surface $\Sigma$ is called the support surface. Let $\lambda \in \mathbb{R}$. The energy function $E : (-\epsilon, \epsilon) \to \mathbb{R}$ is defined by

$$E(t) = \int_M dA_t + \lambda \int_{\Omega_t} d\Sigma,$$

where $\Omega_t \subset \Sigma$ is the domain bounded by $X_t(\partial M)$, $dA_t$ denotes the area element of $M$ with respect to the metric induced by $X_t$ and $d\Sigma$ is the area element on $\Sigma$.

The volume function of the variation $V : (-\epsilon, \epsilon) \to \mathbb{R}$ is defined by

$$V(t) = \int_{M \times [0, t]} X^*(dV),$$

where $X^*(dV)$ is the pullback of the canonical volume element $dV$ of $\mathbb{L}^3$. The variation is said to be volume-preserving if $V(t) = V(0) = 0$ for all $t$. The expressions of the first variation formula for the energy $E$ and the volume $V$ are as follows:

$$E'(0) = -2 \int_M \langle H \langle N, \xi \rangle, dA - \int_{\partial M} \langle (\nu, \xi) + \lambda \langle \nu_{\Sigma}, \xi \rangle \rangle ds,$$

$$V'(0) = -\int_M \langle N, \xi \rangle, dA.$$

[4]
Here, $N$ is an orientation on $M$, $H$ is the corresponding mean curvature function, $v$ and $v_\Sigma$ are the unit inward co-normal vectors of $M$ and $\Omega$ along $\partial M$, respectively, and $\xi$ is the variation vector field of the variation $X$:
\[
\xi(p) = \frac{\partial X}{\partial t}(p, 0).
\]
Consider $\tau$ as a unit tangent vector to $\partial M$ and let $\{\tau, v, N\}$ and $\{\tau, v_\Sigma, N_\Sigma\}$ be two orthonormal bases such that $\det(\tau, v, N) = \det(\tau, v_\Sigma, N_\Sigma) = 1$. Denote $\epsilon = 1$ (respectively $-1$) depending on whether $\Sigma$ is time-like (respectively space-like), that is, $\langle N_\Sigma, N_\Sigma \rangle = \epsilon$, where $N_\Sigma$ is a unit normal vector field of $\Sigma$. Recall that if $\partial M$ is space-like, then $\langle v_\Sigma, v_\Sigma \rangle = -\epsilon$. With respect to $\{v_\Sigma, N_\Sigma\}$, $N$ and $v$ are given by
\[
N = -\epsilon \langle N, v_\Sigma \rangle v_\Sigma + \epsilon \langle N, N_\Sigma \rangle N_\Sigma, \tag{4}
\]
\[
v = -\langle N, N_\Sigma \rangle v_\Sigma + \langle N, v_\Sigma \rangle N_\Sigma. \tag{5}
\]
Thus,
\[
\langle v, \xi \rangle = \langle N, v_\Sigma \rangle \langle N_\Sigma, \xi \rangle - \langle N, N_\Sigma \rangle \langle v_\Sigma, \xi \rangle.
\]
Then, (2) is written as
\[
E'(0) = -2 \int_M H \langle N, \xi \rangle \, d\Omega - \int_{\partial M} (\lambda - \langle N, N_\Sigma \rangle) \langle v_\Sigma, \xi \rangle \, ds - \int_{\partial M} \langle N, v_\Sigma \rangle \langle N_\Sigma, \xi \rangle \, ds. \tag{6}
\]
The last term of (6) vanishes since the vector field $\xi$ is tangential to $\Sigma$ along the boundary $\partial M$. Indeed, if $p \in \partial M$, the curve $t \mapsto X(p, t)$ lies on the support surface $\Sigma$ and then its velocity is tangent to $\Sigma$. But at $t = 0$, this velocity is just $\xi(p)$, thus, $\xi(p) \in T_p \Sigma$ and $\langle N_\Sigma, \xi \rangle = 0$ along $\partial M$.

We say that the immersion $x$ is stationary if $E'(0) = 0$ for every admissible volume-preserving variation of $x$. By the method of Lagrange multipliers, there exists $\mu \in \mathbb{R}$ such that $E'(0) + \mu V'(0) = 0$ for any such variation. From (3) and (6), we have
\[
\int_M (2H + \mu) \langle N, \xi \rangle d\Omega + \int_{\partial M} (\lambda - \langle N, N_\Sigma \rangle) \langle v_\Sigma, \xi \rangle \, ds = 0.
\]
A standard argument gives

**Theorem 2.1.** In the above conditions, the immersion $x : M \to \mathbb{L}^3$ is stationary if and only if the mean curvature $H$ is constant and the angle between $M$ and $\Sigma$ along $\partial M$ is constant.

In the case $\Sigma$ is space-like, $N_\Sigma$ is a unit time-like vector. Considering both $N$ and $N_\Sigma$ are oriented by a unit future-directed time-like orientation, the angle $\theta$ is defined as $\lambda = \langle N, N_\Sigma \rangle = -\cosh \theta$. If $\Sigma$ is a time-like surface, $N_\Sigma$ is a unit space-like vector. Assuming again that $M$ is future-directed, the angle between $M$ and $\Sigma$ is $\theta$ such that $\langle N, N_\Sigma \rangle = \sinh \theta$ [22].

**Remark 2.2.**

(i) When $\lambda = 0$, we have the classical problem of a surface with the critical area and with the free boundary in $\Sigma$. In such a case, the intersection between $M$ and $\Sigma$ is orthogonal.

(ii) Our definition of the energy $E$ in (1) is motivated by what occurs in the Euclidean setting when one considers liquid drops resting in some support. In order to define $E(t)$ and $V(t)$; however, it is not necessary that the images of the immersions $X_t$ of the variation lie in one of the two domains determined by $\Sigma$.

(iii) In general, we call a stationary surface $M$ in $\mathbb{L}^3$ supported on a non-degenerate surface $\Sigma$ as a space-like surface with the constant mean curvature whose boundary lies in $\Sigma$ and $M$, and $\Sigma$ makes constant contact angle along the boundary of $M$. 

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3. The case of pseudosphere as the support surface

After an isometry of $\mathbb{L}^3$, a pseudosphere $S^2_1(p, r)$ centered at $p \in \mathbb{L}^3$ and radius $r > 0$ is defined by

$$S^2_1(p, r) = \{x \in \mathbb{L}^3; \langle x - p, x - p \rangle = r^2\}.$$

Recall that $a = (0, 0, 1)$. This surface is time-like with the constant curvature $1/r^2$. Denote the waist of $S^2_1(p, r)$ as $C(p, r) = S^2_1(p, r) \cap \{x_3 = \langle p, a \rangle\}$.

Let $x : M \to \mathbb{L}^3$ be a compact space-like surface immersed into $\mathbb{L}^3$. For a given closed curve $\Gamma \subset S^2_1(p, r)$, we say that $\Gamma$ is the boundary of the immersion $x$ if the restriction map $x|_{\partial M} : \partial M \to \Gamma$ is a diffeomorphism. Because our problem is invariant by homotheties and isometries of the ambient space, without loss of generality we will assume that the support surface is the unit pseudosphere centered at the origin $O$, that is, $S^2_1(O, 1) = S^2_1 = \{x \in \mathbb{L}^3; \langle x, x \rangle = 1\}$ with $C(O, 1) = C$ as its waist. This surface is also known in the literature as the de Sitter surface of $\mathbb{L}^3$. The ambient space $\mathbb{L}^3$ is divided by $S^2_1$ into two domains $\mathbb{L}^3_+$ and $\mathbb{L}^3_-$:

$$\mathbb{L}^3_+ = \{x \in \mathbb{L}^3; \langle x, x \rangle < 1\}, \quad \mathbb{L}^3_- = \{x \in \mathbb{L}^3; \langle x, x \rangle > 1\}.$$

The pseudosphere $S^2_1$ can globally be parametrized by means of a diffeomorphism $F : \mathbb{R} \times C \to S^2_1$ given by $F(t, q) = \gamma_q(t)$, where

$$\gamma_q(t) = \exp_q(ta) = \cosh(t)q + \sinh(t)a$$

is the (future pointing) unitary geodesic orthogonal to $C$ through the point $q \in C$. Recall that $a = (0, 0, 1)$. The orthogonal projection $\pi : S^2_1 \to C$ associates with each $p \in S^2_1$ the unique point $\pi(p)$ such that $F(t, \pi(p)) = p$ for a certain $t$. By the expression of $F$, we deduce

$$\pi(p) = \frac{1}{\sqrt{1 + \langle p, a \rangle^2}}(p + \langle p, a \rangle a), \quad p \in S^2_1.$$

If $u$ is a smooth function defined on $C$, the geodesic graph of $u$ (on $S^2_1$) is the curve given by $\{F(u(q), q); q \in S^2_1\}$. If $p \in S^2_1$ and $v \in T_pS^2_1$, a straightforward computation gives

$$(d\pi)_p(v) = \frac{v - \langle v, a \rangle a}{\sqrt{1 + \langle p, a \rangle^2}} - \frac{\langle p, a \rangle \langle v, a \rangle}{(1 + \langle p, a \rangle)^2}(p + \langle p, a \rangle a).$$

(8)

**Proposition 3.1.** Let $\alpha : S^1 \to S^2_1$ be a closed space-like curve. Then, $\pi \circ \alpha$ is a covering map of $C$. In particular, if $\alpha$ is embedding, then $\alpha(S^1)$ is a geodesic graph on $C$. In particular, in $S^2_1$ there exist no space-like nullhomologous curves.

**Proof.** Define $\psi = \pi \circ \alpha : S^1 \to C$, that is,

$$\psi(s) = \frac{1}{\sqrt{1 + \langle \alpha(s), a \rangle^2}}(\alpha(s) + \langle \alpha(s), a \rangle a).$$

From (8),

$$\langle \psi'(s), \psi'(s) \rangle = \frac{\langle \alpha'(s), \alpha'(s) \rangle}{1 + \langle \alpha(s), a \rangle^2}.$$

Because $\alpha$ is space-like, the map $\psi$ is a local diffeomorphism, and hence $\psi : S^1 \to C$ is a covering map. When $\alpha$ is embedding, then the covering map $\psi$ is one to one, that is, $\psi$ is a global diffeomorphism between $S^1$ and $C$, showing that $\alpha$ is a graph on $C$: $\pi_{\alpha(S^1)} = \psi \circ \alpha^{-1}$. □
Remark 3.2. In the higher dimensional case \((n \geq 2)\), if \(M^n\) is a compact submanifold and \(x : M^n \to S^{n+1}_0\) is a space-like hypersurface in the \((n+1)\)-dimensional de Sitter space \(S^{n+1}_0\), the map \(\psi = \pi \circ x\) is a covering map between \(M^n\) and the \(n\)-sphere \(S^n_0 = S^{n+1}_0 \cap \{x_{n+2} = 0\}\), which is simply connected. Thus, the covering map \(\psi\) is one to one [19].

From proposition 3.1, if \(x : M \to \mathbb{L}^3\) is a space-like immersion of a compact surface \(M\), with \(x(\partial M) = \Gamma\) a closed curve included in \(S^2_1\), then \(\Gamma\) does not bound a domain of \(S^2_1\) and thus the variational problem established in section 2 should be reformulated. With this purpose, let \(\Gamma\) be a space-like embedded curve in \(S^2_1\) and consider the waist \(C\) of \(S^2_1\). Since \(\Gamma\) is homologous to \(\Gamma\), \(\Gamma \cup C\) bounds a domain \(\Omega \subset S^2_1\). Given a variation \(X\) of \(x\), for values \(t\) close to \(t = 0\), \(X_t(\partial M)\) is homologous to \(\Gamma\). Then, in the second term of \(E\) in (1), we replace \(\Omega\) by the domain bounded by \(X_t(\partial M) \cup C\). Let us observe that one can change \(C\) for another curve \(C'\) homologous to \(C\) because the corresponding integral \(\int_{\Sigma} d\Sigma\) changes only by an additive constant, with no consequence in the formula \(E(\omega)\). Therefore, in the case that \(\Sigma\) is a pseudosphere, theorem 2.1 holds in the same terms.

Remark 3.3. We point out that there are no stationary surfaces included in \(\mathbb{L}^1\), and with an embedded boundary \(\Gamma\) in \(S^2_1\), because \(\Gamma\) must be homologous to \(C\), but there are no compact surfaces in \(\mathbb{L}^1\) spanning \(\Gamma\), independently if the surface is space-like or not space-like.

4. Proof of theorem 1.1

From now on, we consider that the boundary \(\Gamma\) of the stationary surface \(M\) is an embedded and connected space-like curve in \(S^2_1\) and thus homologous to \(C\). Replacing \(C\) by another homologous curve \(C'\) if necessary, we can assume that the domain \(\Omega \subset S^2_1\) bounded by \(\Gamma \cup C\) is an embedded surface. Denote by \(v_5\) the unit inward co-normal vector of \(\Omega\) along \(\Gamma\) and \(N_5\) is the Gauss map of \(\Omega\). In particular, for \(p \in \Omega\), \(N_5(p) = p\). Let \(\theta\) be a constant such that \(\langle N, N_5 \rangle = \sinh \theta\). From (4), we have

\[ N = - \cosh \theta v_5 + \sinh \theta N_5, \tag{9} \]
\[ v = - \sinh \theta v_5 + \cosh \theta N_5. \tag{10} \]

From proposition 3.1, we have

**Lemma 4.1.** Let \(\Gamma\) be a closed embedded space-like curve in \(S^2_1\). Then, \(\Gamma\) is a graph on the plane \(P = \{x_3 = 0\}\). Moreover, the orthogonal projection of \(\Gamma\) on \(P\) bounds a simply connected domain.

**Proof.** By proposition 3.1, \(\Gamma\) is a graph of \(S^2_1\) on \(C\). Let \(\Pi : \mathbb{L}^3 \to P\) be the orthogonal projection onto \(P\), that is, \(\Pi(q) = q + \langle q, a \rangle a\). From (7), we obtain \(\Pi(q) = \sqrt{1 + \langle q, a \rangle^2} \pi(q)\). On \(\Gamma\), the map \(\Pi : \Gamma \to P\) is a local diffeomorphism since \(\Gamma\) is space-like: if \(\alpha : S^1 \to \Gamma\) is a parametrization, \(\alpha = \alpha(s)\), we have

\[ |(\Pi \circ \alpha')(s)|^2 = |\alpha'(s)|^2 + \langle \alpha'(s), a \rangle^2 \geq |\alpha'(s)|^2. \]

On the other hand, if there exist two distinct points \(q_1, q_2\) in \(\Gamma\) such that \(\Pi(q_1) = \Pi(q_2)\), the symmetry of \(S^2_1\) implies that \(q_2 = -q_1\). From the above relation between \(\Pi\) and \(\pi\), \(\pi(q_1) = \pi(q_2)\): contradiction. Thus, \(\Pi : \Gamma \to \Pi(\Gamma)\) is a diffeomorphism. \(\square\)

**Lemma 4.2.** Let \(x : M \to \mathbb{L}^3\) be a compact space-like immersion whose boundary \(\Gamma\) is an embedded curve included in \(S^2_1\). Then, \(x(M)\) is a graph on \(P\) and thus \(x(M)\) is a topological disc.
Proof. From the above lemma, there exists a simply connected compact domain \( D \subset P \) such that \( \Gamma \) is a graph on \( \partial D \). If \( \Gamma \) is the boundary of a space-like immersed surface \( M \), then it is known that \( M \) is a graph on \( D \) and thus, a topological disc: see for example [4, lemma 3].

The idea is the following. As in the above proof, the fact that \( \Omega \) is space-like means that \( \Pi : x(M) \to P \) is a local diffeomorphism with \( |(d\Pi)_p(v)|^2 = |v|^2 + \langle v, u \rangle^2 \geq |v|^2 \). The space-like condition on \( M \) implies that \( \Pi(M) = \overline{D} \); in particular, \( \Pi : x(M) \to \overline{D} \) is a covering map. As \( D \) is a simply connected domain, then \( \Pi : M \to \overline{D} \) is a global diffeomorphism. \( \Box \)

We now proceed to show theorem 1.1.

**Proof of theorem 1.1.** Let \( x : M \to \mathbb{L}^3 \) be a stationary surface. By lemma 4.2, we can parametrize \( M \) conformally to the closed unit disc \( \overline{D} = \{ (u, v) \in \mathbb{R}^2; u^2 + v^2 \leq 1 \} \) such that 

\[
(z, \bar{z}) = x(u, v) = E^2, \quad (z, x_u) = 0.
\]

Let \( h_{ij}, i, j = 1, 2 \), be the coefficients of the second fundamental form of \( x : M \to \mathbb{L}^3 \). More precisely,

\[
h_{11} = (x_{uu}, N), \quad h_{12} = h_{21} = (x_{uv}, N), \quad h_{22} = (x_{vv}, N).
\]

We introduce the Hopf quadratic differential \( \phi = \phi(z, \bar{z}) = (h_{11} - h_{22}) - 2i h_{12} \) \( dz^2 \), which is invariant by a conformal coordinate of \( \overline{D} \). The Hopf differential \( \phi \) has two important properties:

(i) \( \phi \) is holomorphic if and only if the mean curvature of the immersion is constant. This is a consequence of the Codazzi equation.

(ii) \( \phi \) vanishes at some point \( p \in M \) if and only if \( p \) is an umbilical point. This occurs because

\[
|h_{11} - h_{22} - 2i h_{12}|^2 = \frac{1}{4\pi^2} (H^2 + K) \geq 0,
\]

where \( K \) is the Gaussian curvature of \( M \).

As a consequence, in a constant mean curvature space-like surface the holomorphicity of \( \phi \) implies that the set of umbilical points coincides with the zeros of a holomorphic differential form. Therefore, either umbilical points are isolated or the immersion is totally umbilical.

On the boundary \( \partial M \), we have \( |z| = 1 \) and then, \( z = e^{i\theta} \). Since \( \partial_r = \frac{1}{2} (\partial_u - i \partial_v) = \frac{1}{2} (\partial_r - i \partial_\theta) \),

\[
\phi = 4\sigma (\partial_r, \partial_\theta) d\zeta^2 = 2\sigma (\partial_r, \partial_\theta) - 2i 2\sigma (\partial_r, \partial_\theta) - 2\sigma (\partial_r, \partial_\theta).
\]

Hence on \( |z| = 1 \), we obtain

\[
\text{Im}(z^2 \phi) = -\sigma (\partial_r, \partial_\theta).
\]

On the other hand, the unit tangent \( t \) and the inward-pointing unit co-normal \( v \) along \( \partial M \) are denoted by

\[
t = E^{-1} \partial_r, \quad v = -E^{-1} \partial_\theta.
\]

We have then \( \text{Im}(z^2 \phi) = E^2 \sigma (t, v) \). If \( \overline{N} \) is the Levi-Civit\`a connection on \( \mathbb{L}^3 \), using (9)–(10) and \( \overline{N} = 0 \), we obtain

\[
\sigma (t, v) = \langle \nabla_t N, v \rangle = \langle \nabla_v N, t \rangle = (\sinh \theta \nabla_t v_S + \cosh \theta \nabla_t N_S, N),
\]

\[
= -\sinh \theta \nabla_t v_S, N + \cosh \theta (t, N) = -\sinh \theta \nabla_t v_S, N,
\]

\[
= \sinh \theta \cosh \theta (t, v_S) - \sinh^2 \theta (t, v_S, N_S),
\]

\[
= \frac{1}{2} \sinh \theta \cosh \theta (t, v_S) + \sinh^2 \theta (v_S, N_S),
\]

\[
= \sinh^2 \theta (v_S, t) = 0.
\]

In other words, the harmonic function \( \text{Im}(z^2 \phi) \) vanishes on \( \partial D \); hence, it must be identically zero in \( \mathbb{D} \). This implies that the holomorphic function \( z^2 \phi \) must be constant in \( \mathbb{D} \). Since at the
origin the value of \( \varepsilon^2 \phi \) is zero, then \( \varepsilon^2 \phi \equiv 0 \). This implies that \( \phi = 0 \) on \( M \) and the immersion is totally umbilical.

We remark that any space-like plane intersects \( S^2_1 \) at the constant angle and that when a hyperbolic plane intersects \( S^2_1 \), this occurs at the constant contact angle too. Indeed, if \( M \) is the plane \( M = \{ x \in \mathbb{L}^3 ; (x - p, v) = 0 \} \), with \( (v, v) = -1 \), \( \langle v, a \rangle < 0 \), then for any \( x \in M \cap S^2_1 \) we have \( N(x) = v \) and \( \nabla_N(x) = x \). Thus, along \( \partial M \), \( \langle N, N_2 \rangle = (p, v) \). In the case that \( M \) is a hyperbolic plane of type \( \mathbb{H}^2_1(p, r) \) or \( \mathbb{H}^2_2(p, r) \), \( N(x) = (x - p)/r \) and then \( \langle N, N_2 \rangle = (1 - r^2 - |p|^2)/(2r) \) along \( \partial M \); hence, the contact angle becomes constant again.

5. Further discussions of the problem in arbitrary dimensions

In this section, we give some remarks about the problem of hypersurfaces with free boundaries in the \((n + 1)\)-dimensional Lorentz–Minkowski space \( \mathbb{L}^{n+1} \). Following [4], we ask whether the totally umbilical hypersurfaces are the only stationary hypersurfaces in \( \mathbb{L}^{n+1} \) whose support hypersurface is an umbilical hypersurface.

In this sense, the conjecture 1 in [4, p 1330] is true. The proof combines the maximum principle and the characterization of constant mean curvature compact space-like hypersurfaces bounded by an \((n - 1)\)-sphere.

**Theorem 5.1.** The only stationary hypersurfaces, whose boundaries are embedded and resting on a space-like hyperplane, are hyperplanar balls and hyperbolic caps.

**Proof.** Let \( x : M^n \to \mathbb{L}^{n+1} \) be a space-like immersion of a compact submanifold \( M \) with the connected boundary. Let \( \Gamma = x(\partial M) \) and assume that \( \Gamma \) is included in the space-like hyperplane \( P = \{ x_{n+1} = 0 \} \). Because \( \Gamma \) is embedded, \( \Gamma \) encloses a simply connected domain \( D \subset P \). The space-like condition of the immersion implies that the orthogonal projection \( \Pi : M \to P \) is a local diffeomorphism. Similar reasoning as in [4, lemma 3] shows that \( x(M) \) is a graph on \( D \); in particular, \( x : M \to \mathbb{L}^{n+1} \) is embedding. Using the maximum principle for the constant mean curvature equation [11], it is known that a graph is included in one side of \( P \). Without loss of generality, we assume that \( M \) lies over \( P \), that is, \( M - \partial M \subset \{ x_{n+1} > 0 \} \).

As a consequence, the surface \( M \) together with \( D \) encloses a domain \( W \subset \mathbb{L}^{n+1} \). Now, we are in a condition to apply the Alexandrov method by the family of parallel vertical hyperplanes [1]. The fact that the angle between \( M \) and \( P \) is constant along \( \Gamma \) makes the Alexandrov process work well because if there is a contact point between boundary points, the condition of the constant contact angle implies that the tangent hyperplanes agree between a point of \( \Gamma \) and its reflected one: see [14] for an example in a more general context. Then, one shows that \( M \) is rotationally symmetric with respect to a straight line \( L \) orthogonal to \( P \). In particular, the boundary \( \Gamma \) is a round \((n - 1)\)-sphere \( S^{n-1} \). In [3] (see [2] in the two-dimensional case) it is proved that the only compact space-like hypersurfaces in \( \mathbb{L}^{n+1} \) spanning \( S^{n-1} \) are umbilical, showing the result.

In the proof, we have first showed that the graph lies in one side of the hyperplane. In this part of the proof, it is not necessary to use that \( M \) is a graph and that \( H \) is constant on \( M \), but that \( H \) does not vanish on the surface.

**Theorem 5.2.** Let \( x : M^n \to \mathbb{L}^{n+1} \) be a space-like immersion of a compact manifold \( M \) whose boundary lies in a hyperplane \( P \). If the (non-necessary constant) mean curvature \( H \) does not vanish, then \( M \) lies in one side of \( P \). If \( H = 0 \) on \( M \), then \( M \) is a compact subset of \( P \).
Proof. First, we point out that we do not know whether $M$ is a graph or not, since we admit the possibility that $x(\partial M)$ is not embedding. But we know that $P$ is a space-like hyperplane because the immersion is space-like and $x(\partial M)$ is a closed submanifold of $P$. Without loss of generality, we suppose that $P$ is the hyperplane $x_{n+1} = 0$. Denote $P_0 = \{x \in \mathbb{R}^{n+1}; \langle x, a \rangle = -r\}$.

As $M$ and $P_1$ are space-like, we consider the future-directed time-like orientations $N$ and $N_t$, respectively, that is, if $a = (0, \ldots, 0, 1)$, $\langle N, a \rangle < 0$ on $M$ and $N_t = a$ for any $t$.

Assume that $H \neq 0$ on $M$. By contradiction, we assume that $M$ has points on both sides of $P$. At the highest point $p$ of $M$ with respect to the plane $P$, let us place $P_1(t)$, where $t_1 = x_{n+1}(p) > 0$. As $N_t(p) = N(p) = a$ and $P_1(t)$ lies above $M$, the maximum principle tells that $0 > H(p)$. Because $H \neq 0$, then we conclude that $H < 0$ on $M$. Similarly, at the lowest point $q$, with $t_2 = x_{n+1}(q) < 0$, we place the hyperplane $P_2$. Now, $M$ lies over $P_2$, and the maximum principle implies $H(q) > 0$: contradiction.

If $H = 0$, the same reasoning as above tells us that $M \subset P$. □

We try to follow the same reasoning for stationary hypersurfaces being the support hypersurface, a hyperbolic and a hyperplane $\mathbb{H}^n$. Following the same ideas as in theorem 5.1, the two ingredients are: (i) show that $M$ lies in one side of $\mathbb{H}^n$ and (ii) use the Alexandrov method to prove that $M$ is rotational, finishing as in theorem 5.1.

Here, we prove the first step but, after showing this fact, we are not able to apply it in a suitable way as the Alexandrov process, because reflections with respect to hyperplanes do not leave the support hypersurface $\mathbb{H}^n$ invariant.

Theorem 5.3. Consider $M^n$ as a compact $n$-dimensional manifold with non-empty boundary and $x : M^n \rightarrow \mathbb{L}^{n+1}$ be a space-like immersion with a (non-necessary constant) mean curvature $H$. Assume that $x(\partial M)$ lies in a hyperbolic plane $\mathbb{H}^n$ and denote by $h > 0$ the mean curvature of $\mathbb{H}^n$ for an appropriate orientation. If $|H| \neq h$, then $M$ lies in one side of $\mathbb{H}^n$. If $|H| = h$, then either $M$ lies included in $\mathbb{H}^n$ or $M$ lies in one side of $\mathbb{H}^n$.

Proof. We use notations similar as in preliminaries in the context $\mathbb{L}^{n+1}$. Without loss of generality, we assume that the support hypersurface is $\mathbb{H}^n = \mathbb{H}^n_{\infty}(O, r)$, $O$ is the origin of coordinates, $r = 1/h$. We consider on $\mathbb{H}^n$ the future-directed time-like orientation $N_0$, that is, $N_0(p) = p$. Consider the foliation of $\mathbb{L}^{n+1}$ given by $\mathbb{H}^n_{t}(a, r), t \in \mathbb{R}$. Let us remark that $\mathbb{H}^n_{t}(Oa, r) = \mathbb{H}^n_{t}$. All these hypersurfaces have constant mean curvature $h > 0$ for the future-directed time-like orientation. Each one of the sides of $\mathbb{H}^n$ is $\{x \in \mathbb{L}^{n+1}; \langle x, x \rangle < -r^2\}$ and $\{x \in \mathbb{L}^{n+1}; \langle x, x \rangle > -r^2\}$.

Suppose that $|H| \neq h$ and by contradiction, we assume that $M$ has points in both sides of $\mathbb{H}^n$. By the compactness of $M$, for $t$ sufficiently close to $-\infty$, $\mathbb{H}^n_{t}(a, r)$ does not intersect $M$. Letting $t \rightarrow 0$, there is the first time $t_1 < 0$ such that $\mathbb{H}^n_{t}(a, r)$ touches $M$ at a (interior) point $q$ but $\mathbb{H}^n_{t_1}(a, r) \cap M = \emptyset$ for $t < t_1$. Let us compare $M$ and $\mathbb{H}^n_{t_1}(a, r)$ at $q$ and we use that the orientations of both hypersurfaces agree at $q$ since both are future-directed orientations. The maximum principle tells that $H(q) > h$. As $|H| \neq h$, then $H > h$ on $M$.

Consider now the other side of $\mathbb{H}^n$. By similar reasoning, there is $t_2 > 0$ such that $\mathbb{H}^n_{t_2}(a, r)$ touches $M$ at a (interior) point $p$ but $\mathbb{H}^n_{t_2}(a, r) \cap M = \emptyset$ for $t > t_2$. Comparing $M$ and $\mathbb{H}^n_{t_2}(a, r)$ at $p$, the maximum principle tells now that $h > H(p)$; contradiction.

In the case when $|H| = h$ and assuming that $M \not\subset \mathbb{H}^n$, the same reasoning implies in the first step that $H(q) > h$, which is not possible. Thus, the case $t_1 < 0$ is not possible. This shows that $M$ lies over $\mathbb{H}^n$. If the number $t_2 > 0$ exists, then we have that $H = -h$. □

The last situation in the proof appears when one consider $\mathbb{H}^n$ as above and $M$ is a hyperbolic cap of $\mathbb{H}^n_{t}(a, r)$, for $t > 0$ sufficiently large.
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