POSITIVE NEIGHBORHOODS OF RATIONAL CURVES

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ABSTRACT. We study neighborhoods of rational curves in surfaces with self-intersection number 1 that can be linearised.

1. INTRODUCTION

Let us consider a holomorphic embedding of the complex projective line into a complex surface. The first invariant we can attach to this embedding is the self-intersection number of the image. When \( C^2 > 0 \), there is a big moduli space of germs of neighborhoods \((S, C)\), see \([3, 5]\). In the paper \([2]\) the authors give an extensive description of these embeddings when the self-intersection number is equal to 1. In particular they describe completely the embeddings that are equivalent to the one of the complex projective line as a line of the projective plane. Our interest here is to present a geometric approach to this case using holomorphic foliations, which perhaps may be useful to study other situations.

We take then a smooth rational curve \( C \) embedded in a complex surface \( S \) with self-intersection number \( C \cdot C = 1 \). Let \( H \) be a line in the complex projective plane \( \mathbb{CP}^2 \). One of the results of \([2]\) is the following:

**Theorem.** Suppose that there exist three different holomorphic fibrations over \( C \) in \( S \). Then a neighborhood of \( C \) is holomorphically equivalent to a neighborhood of \( H \) in \( \mathbb{CP}^2 \).

This is the result we study here using a different viewpoint.

We remark that any holomorphic fibration over \( H \) in \( \mathbb{CP}^2 \) is in fact a linear one, that is, defined by a pencil of lines of \( \mathbb{CP}^2 \) with the base point lying outside \( H \). In fact, a holomorphic fibration extends to a foliation on \( \mathbb{P}^2 \) which, being totally transverse to \( H \), must be of degree 0.

Our idea to prove the Theorem is to analyse the curve of tangencies between two fibrations over \( C \). When this curve is a common fiber, we construct a holomorphic diffeomorphism from a neighborhood of \( C \) to a neighborhood of \( H \). The other possibilities are: (i) the curve of tangencies is \( C \) itself; (ii) the curve of tangencies is transverse to \( C \) but it is not a common fiber. What we show is that in the presence of three different fibrations, necessarily two of them have a common fiber.

It should be mentioned that in \([2]\) the authors describe also the possibility of existence of fibrations over rational curves of positive self-intersection number. We explain this point in Section 3.

We are grateful to Frank Loray for innumerous discussions around this subject.
2. CURVES OF TANGENCIES

For the reader’s convenience, we present a short description of the curve of tangencies between two fibrations of \( S \) over \( C \).

Let us take a regular foliation \( F \) of the surface \( S \) generically transverse to the (smooth) curve \( C \). We will use the notation:

1. \( N_F \) is the normal bundle of \( F \).
2. \( \text{tang}(F, C) \) is the tangency locus between \( F \) and \( C \).
3. If \( G \) is another foliation of \( S \) generically transverse to \( C \), then \( \text{tang}(F, G) \) is the divisor of tangencies between \( F \) and \( G \).
4. \( K_S \) is the canonical bundle of \( S \).
5. \( \chi(C) \) is the Euler characteristic of \( C \).

According to [1], we have:

\[
N_F \cdot C = \chi(C) + \text{tang}(F, C) \quad (N_G \cdot C = \chi(C) + \text{tang}(G, C))
\]

and

\[
\text{tang}(F, G) \cdot C = N_F \cdot C + N_G \cdot C + K_S \cdot C
\]

We are interested in the case where \( F \) and \( G \) are fibrations over the rational curve \( C \) which has self-intersection number equal to 1. From the formulae above we get then \( N_F \cdot C = N_G \cdot C = 2 \). The adjunction formula gives

\[
-K_S \cdot C = \chi(C) + C \cdot C = 3
\]

We get then \( \text{tang}(F, G) \cdot C = 1 \). Therefore the possibilities are

- a (smooth, connected) curve of tangencies transversal to \( C \)
- \( \text{tang}(F, G) = C \)

In both cases, the order of tangency between \( F \) and \( G \) is 1 along the curve of tangencies.

3. THE CASE OF A COMMON FIBER

We consider now the first of the possibilities above with the curve of tangencies as a common fiber. We have then two fibrations \( F \) and \( G \) over the curve \( C \) with a common fiber \( L \); we put \( C \cap L = \{ a \} \). We choose a line \( H \subset \mathbb{CP}(2) \) and fix a biholomorphism \( \phi : C \to H \). We take also two linear fibrations \( F' \) and \( G' \) in \( \mathbb{CP}(2) \) which are transverse to \( H \) and have a common fiber passing through \( \phi(a) \).

**Proposition 1.** There exists a neighborhood of \( C \) in \( S \) which is biholomorphic to a neighborhood of \( H \) in \( \mathbb{CP}(2) \).

**Proof.** 1) To each point \( p \in S \setminus C \) close to \( C \) we associate the points \( \pi_F(p) \in C \) and \( \pi_G(p) \in C \) as the points where the fibers of \( F \) and \( G \) passing through \( p \) intersect \( C \). We define \( \Phi(p) \in \mathbb{CP}(2) \) as the point where the lines of \( F' \) and \( G' \) through \( \phi(\pi_F(p)) \) and \( \phi(\pi_G(p)) \) meet in \( \mathbb{CP}(2) \) (later on we will need to make a choice for these pencils in \( \mathbb{CP}(2) \)). It is easy to see that \( \Phi \) is a biholomorphism between neighborhoods of \( S \setminus \{ a \} \) and \( H \setminus \phi(a) \). Let us show now that \( \Phi \) extends along \( L \) as a holomorphic map.

Let us take coordinates \((x, y)\) around the point \( a \in C \) such that \((0, 0) \in \mathbb{C}^2\) corresponds to \( a \), \( y = 0 \) parametrizes \( C \) and \( F \) is defined as \( dx = 0 \) (the common
fiber $L$ becomes $x = 0)$. The fibration $\mathcal{G}$ is therefore defined by the set of curves

$$x = x_0 + A(x_0, y)$$

where $x_0 \in \mathbb{C}$ is close to 0 and $A(0, y) \equiv 0$. In $\mathbb{C}P(2)$ we use affine coordinates $(X, Y)$ as to have the pencil $\mathcal{F}'$ given by $dX = 0$ and the pencil $\mathcal{G}$ given by the set of lines issued from the point $(0, 1)$. Without any loss of generality we may assume that $\phi(x, 0) = (x, 0)$. A simple computation shows that

$$\Phi(x, y) = (x, 1 - \frac{x}{x_0}) = (x, -\frac{A(x_0, y)}{x_0})$$

where $(x_0, 0)$ is the point of intersection of the $\mathcal{G}$-fiber through $(x, y)$ with the $x$-axis.

When $x \to 0$ (so $x_0 \to 0$), $\Phi(x, y)$ converges to $(0, -\frac{\delta A}{\delta x_0}(0, y))$. Consequently, by Riemann’s extension theorem, $\Phi$ extends holomorphically to the fiber $L$.

2) In order to finish the proof of Proposition 1, we must prove that $\Phi$ is invertible. This is the place where we need to make a choice of the linear fibrations $\mathcal{F}'$ and $\mathcal{G}'$. First of all we notice that the restrictions of tangent spaces $TS|_C$ and $TCP(2)|_H$ are isomorphic, that is, for each $p \in C$ there exists a linear isomorphism $U_p$ from $TS|_C(p)$ to $TCP(2)|_H(\phi(p))$ which is holomorphic in $p$. We define then in $\mathbb{C}P(2)$ the linear fibrations (over $H$) $\mathcal{F}'$ and $\mathcal{G}'$ such that their fibers over $\phi(a)$ are $U_p(T\mathcal{F}_p)$ and $U_p(T\mathcal{G}_p)$ respectively. Finally we take a holomorphic diffeomorphism $\Psi$ from a neighborhood of $a \in C$ to a neighborhood of $(0, 0) \in \mathbb{C}^2$ such that $d\Psi|_C(p) = U_p$ for $p$ close to $a$. In other words, we have now local coordinates $(x, y)$ around $a \in C$ with the properties: i) $C$ is given by $y = 0$; ii) $a$ corresponds to $(0, 0)$ and iii) the fibrations $\mathcal{F}$ and $\mathcal{G}$ have tangent spaces along the $x$-axis (near $(0, 0)$) equal to the linear fibers of $\mathcal{F}'$ and $\mathcal{G}'$.

In order to simplify the exposition, we will assume from now on that $\mathcal{F}$ is given by $dx = 0$, $\mathcal{F}'$ is given by $dX = 0$, $\phi(x, 0) = (x, 0)$ and that $\mathcal{G}'$ has $(0, 1)$ as base point. We proceed then to compute the derivative of $\Phi(x, y) = (x, Y)$ at $(0, 0) \in \mathbb{C}^2$ along the $y$-axis. Let $u(x)$ denote the slope of $\mathcal{G}$ (or $\mathcal{G}'$) at the point $(x, 0)$; $u$ is a holomorphic function with $u(0) = \infty$. Let $v(x, y)$ be the point in the $x$-axis which belongs to the $\mathcal{G}$-fiber through $(x, y)$. We have then

$$\frac{y}{x - v(x, y)} \to u(v(x, y))$$

when $y \to 0$ along the same fiber.
But since 
\[ \frac{Y}{x - v(x, y)} = u(x, y) \]
we see that \( \frac{Y}{y} \to 1 \) when \( y \to 0 \) along the \( \mathcal{G} \)-fiber over \((x, y)\). When we make \( x \to 0 \), we still have \( \frac{Y}{y} \to 1 \), which shows that \( d\Phi(0, 0) \) restricted to the \( y \)-axis is the identity. \( \square \)

Before closing this Section, let us turn to the situation \((2)\) where the rational curve has positive self-intersection number.

**Proposition 2.** If \( C^2 > 1 \) and there exist two fibrations \( \mathcal{F} \) and \( \mathcal{G} \) transverse to it then \( C^2 = 2 \) and there is a neighborhood of \( C \) in \( S \) which is equivalent to a neighborhood of the diagonal \( \Delta \) in \( \mathbb{P}^1 \times \mathbb{P}^1 \). In particular, when \( C \cdot C = 2 \), there are at most two fibrations transverse to it.

**Proof.** From the formula of Section 2 we see that \( \text{tang}(\mathcal{F}, \mathcal{G}) \cdot C = 2 - C \cdot C \). We conclude that \( 2 - C \cdot C \geq 0 \), so that we have either \( C \cdot C = 1 \) or \( C \cdot C = 2 \). In this last case, we conclude also that the two fibrations are transverse to each other and to \( C \) as well. We look now to the diagonal \( \Delta \) inside \( \mathbb{P}^1 \times \mathbb{P}^1 \) (of course \( \Delta \cdot \Delta = 2 \)); let \( \mathcal{F} \) and \( \mathcal{G} \) be the horizontal and vertical fibrations to \( \Delta \). We may construct as in the first part of the Proof of Proposition 1 the map \( \Phi \) from a neighborhood of \( C \) to a neighborhood of \( \Delta \) sending \( C \) to \( \mathcal{F} \) to \( \mathcal{F} \) and \( \mathcal{G} \) to \( \mathcal{G} \). This map is a diffeomorphism, since it is invertible. \( \square \)

4. **Disjoint Curves of Tangencies**

The first case we analyse in order to prove Theorem 1 is when we have two pairs of fibrations, say \((\mathcal{F}, \mathcal{G})\) and \((\mathcal{F}, \mathcal{H})\), with disjoint lines of tangencies \( L_{\mathcal{F}, \mathcal{G}} \) and \( L_{\mathcal{F}, \mathcal{H}} \) transverse to \( C \) and to the respective pair of fibrations. Let us consider the map defined in the beginning of the proof of Proposition 1 (denoted here by \( \Phi_{\mathcal{F}, \mathcal{G}} \)). It conjugates the pair \((\mathcal{F}, \mathcal{G})\) to a pair of linear pencils \((\mathcal{F}', \mathcal{G}')\), which we take in affine coordinates as \( dX = 0 \) and \( (Y - 1)dX - XdY = 0 \) in \( \mathbb{C}P(2) \). We describe such a map in a neighborhood of \( \{ Q \} = L_{\mathcal{F}, \mathcal{G}} \cap C \).

We consider local coordinates \((x, y_1)\) such that: i) \((0, 0)\) corresponds to \( Q \in C \) and \( y_1 = 0 \) to \( C \); ii) \( \mathcal{F} \) is defined by \( dx = 0 \). Let \( y_1 = (x, y) \) be the equation of \( L_{\mathcal{F}, \mathcal{G}} \).

**Lemma 1.** There exists a local diffeomorphism \( H(x, y_1) = (x, y) = (x, h(x, y_1)) \) such that \( h(x, 0) = 0 \) and transforms the pair \((dx = 0, \mathcal{G})\) to the pair
\[ (dx = 0, \ d(x - y(y - 2x)) = 0) \]

**Proof.** We start assuming that \( h(x, 0) = 0 \); the fiber of \( \mathcal{G} \) passing through each \((x, 0)\) cuts the \( y_1 \)-axis in two points \( \{A^+(x), A^-(x)\} \) (with \( A^+(0) = A^-(0) = 0 \)), defining then an holomorphic involution \( I_0 \) that sends one point to the other (and \( A(0) \) is fixed). The same is true for the fibration \( d(x - y(y - 2x)) = 0 \) in relation to the \( y \)-axis: the fiber of \( d(x - y(y - 2x)) = 0 \) through \((x, 0)\) cuts the \( y \)-axis in the points \( \{a^+(x), a^-(x)\} \) \( (a^+(0) = a^-(0) = 0) \); the associated involution \( i_0 \) is conjugated to \( I_0 \) by a holomorphic diffeomorphism \( h_0(y_1) \) that satisfies \( h_0(0) = 0 \).
We may now extend holomorphically \( h_0(y_1) \) to \( h_x(y_1) \) in order to conjugate the involutions \( I_x \) and \( i_x \) that act in a close vertical (with \( y_1 = l_{F,G}(x) \) as the fixed point of each involution).

There is in fact noting special in the choice of \( d (x - y(y - 2x)) = 0 \) to represent the fibration \( G \); it makes easier the analysis of \( \Phi_{F,G} \). For example, the curve \( L_{F,G} \) becomes the diagonal \( y = x \).

As before we assume that the biholomorphism \( \phi \) is \( \phi(x,0) = (x,0). \) Then \( \Phi_{F,G} \) is the rational map

\[
\Phi_{F,G} = (X,Y) = \left( x, 1 - \frac{x}{x - y(y - 2x)} \right) = \left( x, \frac{-y(y - 2x)}{x - y(y - 2x)} \right)
\]

Let us consider in \( CP(2) \) the pencil \( P \) with center at some point \( P \in H, P \neq Q \) and take its pull-back \( \Phi_{F,G}^*(P) \); it is a foliation \( L \) defined in a neighborhood of \( C \) with a radial singularity at the point \( \Phi_{F,G}^{-1}(P) \) and a singularity at \( Q \) of the form

\[
2x(y - x)d y - y^2 d x + A(x,y)d y + yB(x,y)d x = 0
\]

where \( A(x,y) = \sum_{i+j\geq 3} a_{ij}x^iy^j \) and \( B(x,y) = \sum_{i+j\geq 2} b_{ij}x^iy^j. \) We remark the following fact that will be important for us: the curve of tangencies between \( L \) and a foliation defined in a neighborhood of \( Q \) by \( d y - y(1 + C(x,y))d x = 0 \)

\[
(C(x,y) = \sum_{i+j\geq 1} c_{ij}x^iy^j)
\]

is, besides \( y = 0, \) the curve

\[
F(x,y) = (2x^2 - A)(1 + C) - B + y(1 - 2x(1 + C)) = 0
\]

We have then the intersection number at \( Q: (F \cdot C)_Q \geq 2. \)

**Proposition 3.** There are no fibrations \( F,G \) and \( H \) such that \( L_{F,G} \) and \( L_{F,H} \) are disjoint curves transverse to \( C \) and to the respective fibrations.

**Proof.** We consider the pair of foliations \( L, L' \) constructed as we indicated above \( (L') \) is obtained as the pull-back by \( \Phi_{F,H} \) of the pencil in \( CP(2) \) with center at the same point \( P; \) it has a singularity at the point \( Q' = L_{F,H} \cap C. \) When we compute \( tang(L, L') \cdot C, \) we have contributions at \( Q \) and \( Q' \) (2 at least for each point, as we remarked before) and at least 1 for the point \( \Phi_{F,G}^{-1}(P) = \Phi_{F,H}^{-1}(P) \) (the \( F \)-fiber through this point is a common separatrix of \( L, L' \)); there is also the contribution of \( C \cdot C = 1. \) All in all we get \( tang(L, L') \geq 6. \)

But we have

\[
tang(L, L') \cdot C = N_L \cdot C + N_{L'} \cdot C + K_S \cdot C
\]

From [1] Chapter 2.2 we have \( N_L \cdot C = Z(L, C) + C \cdot C, \) where \( Z(L, C) \) is the number of singularities of \( L \) in \( C \) counted with multiplicities. It follows that \( N_L \cdot C = 4; \) in the same way \( N_{L'} \cdot C = 4. \) Finally, since \( K_S \cdot C = -3, \) we arrive at \( tang(L, L') = 5, \) a contradiction.

**Remark 1.** Let us consider a pair of fibrations \( (F,G) \) whose line of tangencies \( L_{F,G} \) is transverse to \( C \) at the point \( Q \) and is not a common fiber. It may happen that \( L_{F,G} \) is tangent to the fibers of \( F \) and \( G \) at \( Q, \) but it is transverse to the fibers of \( F \) and \( G \) at any point close to but different from \( Q. \)
5. Intersecting Curves of Tangencies

We consider now the case where we have curves of tangencies $L_{\mathcal{F},\mathcal{G}}$ and $L_{\mathcal{F},\mathcal{H}}$ transverse to $C$ and to the respective pair of fibrations, and have a common point $Q \in C$. We proceed as in Section 4 in order to define foliations $\mathcal{L}$ and $\mathcal{L}'$ which leave $C$ invariant and have singularities at the point $Q$ and radial singularities at another point of $C$. The same computation we did at the end of Proposition 1 informs us that $\text{tang}(\mathcal{L}, \mathcal{L}') \cdot C = 5$.

We will show that a direct computation produces a different result. Let us then analyse $\text{tang}(\mathcal{L}, \mathcal{L}') \cdot C$ at the point $Q$.

Let us consider a local coordinate $(x, y)$ around $Q$ such that $(0,0)$ corresponds to $Q$, $C$ becomes $y_1 = 0$ and $\mathcal{F}$ is written as $dx = 0$; we still write $\mathcal{G}$ and $\mathcal{H}$ for the two other fibrations in these coordinates. From Lemma 1 we know that there exist two local diffeomorphisms $H(x, y), H'(x, y)$ that transform the pairs $(dx = 0, \mathcal{G})$ and $(dx = 0, \mathcal{H})$ to the pairs

$$(d x = 0, \ d(x − y(y − 2x)) = 0)$$

and

$$(d x = 0, \ d(x − y'(y' − 2x)) = 0)$$

respectively.

Now we take $\mathcal{L}$ and $\mathcal{L}'$ as the pull-backs by these maps $\Phi_{\mathcal{F},\mathcal{G}}$ and $\Phi_{\mathcal{F},\mathcal{H}}$ of the radial pencil $\mathcal{P}$ in $\mathbb{P}^2$ with center in some $P \in \mathcal{H}$, $P \neq (0,0)$. The foliation $\mathcal{L}$ has a singularity at $Q$ given by $2x(y − x)d y − y^2 d x + h.o.t = 0$ and another singularity at $\Phi_{\mathcal{F},\mathcal{G}}^{-1}(P)$ of radial type; analogously for $\mathcal{L}'$ in the coordinates $(x, y')$.

Let us analyse the tangencies between $\mathcal{L}$ and $\mathcal{L}'$ at the point $Q$, or the tangencies at $(0,0)$ between $2x(y − x)d y − y^2 d x + h.o.t. = 0$ and the pull-back of $2x(y' − x)d y' − y'^2 d x + h.o.t. = 0$ by $L = H' \circ H^{-1}$. We write $L(x, y) = (x, yC(x, y))$ and $L_t(x, y) = (x, C_t(x, y)) = (x, y(C(x, y) + tx))$, for $t$ close to 0; then we have $L_t(x, y) = (x, y(a + tx + \sum_{i \geq 1} a_i(x)y^i))$ with $a \neq 0$. As before, we are interested in the components of the curve of tangencies different from $C$.

We now look for the tangencies between $\mathcal{L} : 2x(y − x)d y − y^2 d x + h.o.t. = 0$ and the pull-back $\mathcal{L}'_1$ of $2x(y' − x)d y' − y'^2 d x + h.o.t. = 0$ by $L_1$; we find a curve given by $(4x^2t^2 + \ldots )y + y^2(\ldots ) = 0$, so we get at least 4 when we compute $\text{tang}(\mathcal{L}, \mathcal{L}') \cdot C$ at $Q$ without taking $C$ into account. It follows that when we go to the limit $t = 0$ we have also a contribution of at least 4 for $\text{tang}(\mathcal{L}, \mathcal{L}') \cdot C$ at $Q$ without taking $C$ into account. Including the contributions of $C$ and of the singularities of $\mathcal{L}$ and $\mathcal{L}'$ corresponding to $\Phi_{\mathcal{F},\mathcal{G}}^{-1}(P) = \Phi_{\mathcal{F},\mathcal{H}}^{-1}(P)$ (which have a common separatrix), we finally arrive at $\text{tang}(\mathcal{L}, \mathcal{L}') \cdot C \geq 6$, a contradiction. Therefore we may state:

**Proposition 4.** There are no fibrations $\mathcal{F}, \mathcal{G}$ and $\mathcal{H}$ such that $L_{\mathcal{F},\mathcal{G}}$ and $L_{\mathcal{F},\mathcal{H}}$ are intersecting curves transverse to $C$ and to the respective pair of fibrations.

6. Proof of the Theorem

A basic property of an embedded rational curve $C$ with self-intersection number equal to 1 is that it can be deformed along a family of smooth rational curves of self-intersection numbers also equal to 1 in some neighborhood of $C$, see [6] and also [4].
We can now prove the Theorem by contradiction. Given three fibrations $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ over $C$, we have two possibilities: i) either the three fibrations are tangent along $C$ or ii) there are two pairs which have lines of tangency transverse to $C$ without being a common fiber of the respective pair (otherwise Proposition 1 applies). We then replace $C$ by a close smooth rational curve $C'$ as explained above; the lines of tangencies will become transverse to the fibrations by Remark 1 (in the first case, the lines of tangencies coincide with $C$, of course). We are now in a setting where Propositions 3 and 4 may be applied to reach the contradiction.

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