A CLT for the $L^2$ moduli of continuity
of local times of Lévy processes

Michael B. Marcus  Jay Rosen*

June 25, 2009

Abstract

Let $X = \{X_t, t \in \mathbb{R}_+\}$ be a symmetric Lévy process with local
time $\{L^x_t; (x,t) \in \mathbb{R}^1 \times \mathbb{R}_+^1\}$. When the Lévy exponent $\psi(\lambda)$ is
regularly varying at infinity with index $1 < \beta \leq 2$ and satisfies
some additional regularity conditions

$$
\sqrt{h\psi^2(1/h)} \left\{ \int (L^x_{1+h} - L^x_1)^2 dx - E \left( \int (L^x_{1+h} - L^x_1)^2 dx \right) \right\}
\Rightarrow (8c_{\beta,1})^{1/2} \eta \left( \int (L^x_1)^2 dx \right)^{1/2},
$$

as $h \to 0$, where $\eta$ is a normal random variable with mean zero
and variance one that is independent of $L^x_t$, and $c_{\beta,1}$ is a known
constant.

1 Introduction

In [3], with X. Chen and W. Li, we obtain a central limit theorem for
the $L^2$ modulus of continuity of local times of Brownian motion. In this

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*The research of both authors was supported, in part, by grants from the National
Science Foundation and PSC-CUNY.

Key words and phrases: Central Limit Theorem, $L^2$ moduli of continuity, local
time, Lévy process.

AMS 2000 subject classification: Primary 60F05, 60J55, 60G51.
paper, this result is extended to symmetric stable processes with index 
1 < \beta < 2 and, in one respect, to a much larger class of Lévy processes.

Let \( X = \{X_t, t \in R_+\} \) be a symmetric stable process of index 1 < \beta < 2. We normalize \( X \) so that 
\[
E(e^{i\lambda X_t}) = e^{-|\lambda|^\beta t}.
\]
Set 
\[
c_{\beta,0} = \frac{2}{\pi} \int_0^\infty \frac{\sin^2 p/2}{p^{\beta}} dp 
\]
and 
\[
c_{\beta,1} = \frac{16}{\pi} \int_0^\infty \frac{\sin^4 p/2}{p^{2\beta}} dp. \tag{1.3}
\]
Let \( \{L^x_t; (x,t) \in R^1 \times R^1_+\} \) denote the local time of \( X \) and define 
\[
\alpha_t = \int (L^x_t)^2 dx. \tag{1.4}
\]

An integral sign without limits is to be read as \( \int_{-\infty}^\infty \).

**Theorem 1.1** Let \( \{L^x_t; (x,t) \in R^1 \times R^1_+\} \) be the local time of the symmetric stable process \( X \) of index 1 < \beta \leq 2. For each fixed \( t \in R^1_+ \), 
\[
\lim_{h \downarrow 0} \int \frac{(L^x_{t+h} - L^x_t)^2 dx - 4c_{\beta,0} t h^{-1}}{h(2\beta-1)/2} \overset{\mathcal{L}}{\to} (8c_{\beta,1})^{1/2} \sqrt{\alpha_t} \eta \tag{1.5}
\]
as \( h \to 0 \), where \( \eta \) is a normal random variable with mean zero and variance one that is independent of \( \alpha \). Equivalently 
\[
\lim_{t \to \infty} \int \frac{(L^x_{t+1} - L^x_t)^2 dx - 4c_{\beta,0} t}{t(2\beta-1)/2} \overset{\mathcal{L}}{\to} (8c_{\beta,1})^{1/2} \sqrt{\alpha_1} \eta \tag{1.6}
\]
as \( t \to \infty \).

In \( \llbracket \) we show that under the hypotheses of Theorem 1.1 for all \( t \in R_+ \), 
\[
\lim_{h \downarrow 0} \int \frac{(L^x_{t+h} - L^x_t)^2}{h^{\beta-1}} dx = 4c_{\beta,0} t \quad \text{a. s.,} \tag{1.7}
\]
and in \( L^1 \), so (1.5) does have the form of a central limit theorem.

Theorem 1.1 with \( \lambda = 2 \), is actually a central limit theorem for the local time of \( B(2t) \) where \( B(t) \) is Brownian motion. This is because the
characteristic function of a canonical 2–stable process is not the same as
the characteristic function of Brownian motion, as one can see in (1.1). The
analogue of (1.5) for Brownian motion is given in [3, Theorem 1.1]. (Naturall,
the constants are different.) The proofs in [3] have a superficial resemblance
to the proofs in this paper, but use many special properties of Brownian motion,
particularly the scaling property. While this type of proof could be extended to
cover symmetric stable processes, we do not see how to extend them to the much
larger class of Lévy processes considered in this paper, for which there is no
scaling property.

The equivalence of (1.5) and (1.6) follows from the scaling relationship
for stable processes, see e.g. [7, Lemma 10.5.2],

\[ \{ L^x_{t/h^\beta}; (x,t) \in R^1 \times R^1_+ \} \equiv \{ L^{h^x}_{ht^\beta}; (x,t) \in R^1 \times R^1_+ \} \] (1.8)

which implies that

\[ \int (L^x_{t+h} - L^x_t)^2 \, dx \leq h^{2\beta-1} \int (L^{x+1}_{ht^\beta} - L^{x}_{ht^\beta})^2 \, dx \] (1.9)

and

\[ \alpha_t \leq t^{(2\beta-1)/\beta} \alpha_1. \] (1.10)

In fact, because of (1.8), to prove Theorem 1.1 it suffices to prove (1.5) with \(t = 1\). In this paper we do this for a much larger class of
symmetric Lévy processes than symmetric stable processes. We extend
the definition on page 2 so that now \(X = \{ X_t, t \in R_+ \}\) is a symmetric
Lévy process with

\[ E \left( e^{i\lambda X_t} \right) = e^{-\psi(\lambda)t} \] (1.11)

where \(\psi(\lambda)\) satisfies

1. \(\psi(\lambda)\) is regularly varying at infinity with index \(1 < \beta \leq 2\);
   (1.12)

2. \(\psi\) is twice differentiable almost surely and there exist constants
   \(D_1, D_2 < \infty\) such that for all \(\lambda \geq 1\)
   \[ \lambda |\psi'(\lambda)| \leq D_1 \psi(\lambda) \quad \text{and} \quad \lambda^2 |\psi''(\lambda)| \leq D_2 \psi(\lambda) \quad \text{a.s.} \] (1.13)

and

\[ \int_0^1 (\psi'(\lambda))^2 \, d\lambda < \infty, \quad \int_0^1 |\psi''(\lambda)| \, d\lambda < \infty. \] (1.14)
\[ L^2 \text{ moduli of continuity of local times} \]

3. \[ \int_0^1 \frac{\psi(\lambda)}{\lambda} \, d\lambda < \infty \] (1.15)

(This last condition is very weak.)

The next theorem is the main result in this paper

**Theorem 1.2** Let \( \{L^x_t; (x,t) \in \mathbb{R}^1 \times \mathbb{R}^1_+\} \) be the local time of the symmetric Lévy process \( X \) whose Lévy exponent \( \psi(\lambda) \) satisfies (1.12)–(1.15). Then

\[
\sqrt{h} \psi(1/h) \left\{ \int (L^{x+h}_t - L^x_t)^2 \, dx - E \left( \int (L^{x+h}_t - L^x_t)^2 \, dx \right) \right\} \xrightarrow{L^2} (8c_{\beta,1})^{1/2} \sqrt{\alpha_1} \eta, \quad (1.16)
\]

as \( h \to 0 \). (Here \( \alpha_1 \) is as defined in (1.4) but for the local time of \( X \).)

Theorem 1.2 tells us something about Theorem 1.1 that isn’t obvious from the statement of Theorem 1.1. For a symmetric stable process of index \( \beta \), \( \psi(p) = |p|^{\beta} \) for all \( p \). However we see from (1.13) and (1.14) that (1.5) in Theorem 1.1 depends primarily on the fact that

\[
\lim_{|p| \to \infty} \frac{\psi(p)}{|p|^{\beta}} = C \quad (1.17)
\]

for some finite, nonzero constant \( C \), i.e. the behavior of \( \psi(\cdot) \) at zero is not relevant. The conditions in (1.14) are very weak.

It is interesting to consider the following rearrangements of (1.5) and (1.6):

\[
\int \left( \frac{L^{x+h}_t - L^x_t}{h^{1/2}} \right)^2 \, dx - 4c_{\beta,0} t \xrightarrow{L^2} (8c_{\beta,1})^{1/2} \sqrt{\alpha_1} \eta, \quad (1.18)
\]

as \( h \to 0 \), and

\[
t^{1/(2\beta)} \left( \frac{1}{t} \int (L^{x+1}_t - L^x_t)^2 \, dx - 4c_{\beta,0} \right) \xrightarrow{L^2} (8c_{\beta,1})^{1/2} \sqrt{\alpha_1} \eta, \quad (1.19)
\]

as \( t \to \infty \). Written this way, (1.18), with \( h^{1/2} \) in the denominator, looks more like a classical CLT and (1.19) displays the behavior of the long time average of the \( L^2 \) norm of the increments of the local time.
The critical ingredient in the proof of Theorems 1.1 is Lemma 3.1 which gives moments for the $L^2$ norm of increments of local times of Lévy processes satisfying (1.12)–(1.15). In Section 2 we state Lemmas 2.1–2.4 which give properties of $p_t(x)$, $\Delta^h p_t(x)$ and $\Delta^h \Delta^h p_t(x)$, where $p_t(x)$ is the transition probability density of these Lévy processes. The lemmas are used in Section 3 in the proof of Lemma 3.1. They are proved in Section 5. Theorem 1.2 is proved in Section 4. In its proof we need good asymptotic estimates of the mean and variance of

$$\int (L_t^{x+h} - L_t^x)^2 \, dx.$$  

These are stated in Section 4 and proved in Section 6. Theorem 1.1 is just a special case of Theorem 1.2 with obvious extensions which are possible for stable processes because of the scaling property of local times of stable processes. The few remarks that show how Theorem 1.1 follows from Theorem 1.2 are given at the end of Section 6. In Section 7, which is an appendix, we prove the version of Kac Moment Formula that we use in this paper.

2 Estimates for the probability densities of certain Lévy processes

Let $p_s(x)$ denote the density of the symmetric Lévy process $X = \{X_s, s \in \mathbb{R}_+\}$ with Lévy exponent $\psi(p)$ as described in (1.11). Let $\Delta^h_x$ denote the finite difference operator on the variable $x$, i.e.

$$\Delta^h_x f(x) = f(x + h) - f(x).$$  

We write $\Delta^h$ for $\Delta^h_x$ when the variable $x$ is clear.

The following lemmas provide the main estimates we use in this paper. Their proofs are given in Section 5.

**Lemma 2.1** Let $X$ be a symmetric Lévy process with Lévy exponent $\psi(\lambda)$ that is regularly varying at infinity with index $1 < \beta \leq 2$ and satisfies (1.13) and (1.14). Let $p_s(x)$ denote the transition probability density of $X$. Then

$$p_s(x) \leq C \frac{\psi^{-1}(1/s)}{1 + x^2}, \quad \forall x \in \mathbb{R}^1, s \in (0, 1];$$  

(2.2)
\[ u(x) := \int_0^1 p_s(x) \, ds \leq \frac{C}{1 + x^2}, \quad \forall x \in \mathbb{R}^1; \quad (2.3) \]

\[ \int \int_0^t p_s(x) \, ds \, dx = t, \quad (2.4) \]

and for all \( h \) sufficiently small

\[ v(x) := \int_0^1 \left| \Delta^h p_s(x) \right| \, ds \leq C \left( \frac{1}{h \psi(1/h)} \wedge \frac{h}{|x|} \wedge \frac{h}{x^2} \right) \quad (2.5) \]

\[ \int v(x) \, dx = O(h \log 1/h), \quad (2.6) \]

and

\[ \int v^p(x) \, dx = O \left( \frac{h}{h^{p-1} \psi^{p-1}(1/h)} \right), \quad p \geq 2, \quad (2.7) \]

as \( h \to 0 \). In addition,

\[ w(x) := \int_0^1 \left| \Delta^h \Delta^{-h} p_s(x) \right| \, ds \leq C \left( \frac{1}{h \psi(1/h)} \wedge \frac{1}{\psi(1/h)|x|} \wedge \frac{h^2}{|x|^2} \right); \quad (2.8) \]

\[ \int w(x) \, dx = O \left( \frac{\log(1/h)}{\psi(1/h)} \right); \quad (2.9) \]

\[ \int w^2(x) \, dx = O \left( \frac{1}{h \psi^2(1/h)} \right); \quad (2.10) \]

\[ \int_{|x| \geq u} w^2(x) \, dx \leq O \left( \frac{1}{u \psi^2(1/h)} \right), \quad (2.11) \]

as \( h \to 0 \).

**Lemma 2.2** Under the same hypotheses as Lemma 2.1, set

\[ c_{\psi,h,1} = \int \left( \int \left( \Delta^h \Delta^{-h} p_s(x) \right) \, ds \right)^2 \, dx. \quad (2.12) \]

Then

\[ \lim_{h \to 0} h \psi^2(1/h)c_{\psi,h,1} = c_{\beta,1} \quad (2.13) \]

and

\[ h \psi^2(1/h) \left( c_{\psi,h,1} - \int \left( \int_0^{\sqrt{h}} \Delta^h \Delta^{-h} p_s(x) \, ds \right)^2 \, dx \right) = O(h^{1/2}). \quad (2.14) \]
Remark 2.1 We allow $\psi$ to be regularly varying at infinity with index 2, but note that that because $\psi$ is the Lévy exponent of a symmetric Lévy process
\[ \psi(\lambda) = O(\lambda^2) \quad \text{as} \quad \lambda \to \infty. \] (2.15)
(See, e.g., [7, Lemma 4.2.2] and then include Brownian motion.)

Lemma 2.3 Under the same hypotheses as Lemma 2.1,
\[ \sup_{\delta \leq r \leq 1} p_r(0) \leq C \left( \psi^{-1}(1/\delta) \lor 1 \right) ; \] (2.16)
\[ \sup_{\delta \leq r \leq 1} |\Delta^h p_r(0)| \leq \frac{C}{\delta^3} h^2 ; \] (2.17)
and
\[ \sup_{\delta \leq r \leq 1} |\Delta^h \Delta^{-h} p_r(0)| \leq \frac{C}{\delta^3} h^2 . \] (2.18)

Lemma 2.4 Let $0 < \delta < 1$, then, under the hypotheses of Theorem 1.2 for
\[ \bar{u}_\delta(x) := \sup_{\delta \leq r \leq 1} p_r(x), \quad \bar{v}_\delta(x) := \sup_{\delta \leq r \leq 1} |\Delta^h p_r(x)|, \] (2.19)
\[ \text{and} \quad \bar{w}_\delta(x) := \sup_{\delta \leq r \leq 1} |\Delta^h \Delta^{-h} p_r(x)|, \]
we have
\[ \bar{v}_\delta(x) \leq C \psi^{-1}(1/\delta) \left( 1 \lor \frac{1}{x^2} \right), \] (2.20)
\[ \bar{v}_\delta(x) \leq \frac{C}{\delta^3} h \left( 1 \lor \frac{1}{x^2} \right), \] (2.21)
and
\[ \bar{w}_\delta(x) \leq \frac{C}{\delta^3} h^2 \left( 1 \lor \frac{1}{x^2} \right) . \] (2.22)

In addition
\[ \int \bar{u}_\delta(x) \, dx \leq C \psi^{-1}(1/\delta), \quad \int (\bar{v}_\delta(x))^2 \, dx \leq C \left( \psi^{-1}(1/\delta) \right)^2, \] (2.23)
\[ \int \bar{v}_\delta(x) \, dx \leq \frac{C}{\delta^3} h, \quad \int \bar{v}_\delta^2(x) \, dx \leq \frac{C}{\delta^6} h^2, \] (2.24)
\[ \int \bar{w}_\delta(x) \, dx \leq \frac{C}{\delta^3} h^2, \quad \int \bar{w}_\delta^2(x) \, dx \leq \frac{C}{\delta^6} h^4, \] (2.25)
as $h \to 0$. 

3 Moments of increments of local times.

Let
\[ I_{j,k,h} := \int (L^{x+h} - L^x) \circ \theta_j (L^{x+h} - L^x) \circ \theta_k \, dx \] (3.1)
and
\[ \alpha_{j,k} := \int L^x \circ \theta_j L^x \circ \theta_k \, dx. \] (3.2)

The goal of this section is to establish the following lemma.

Lemma 3.1 Let \( m_{j,k}, 0 \leq j < k \leq K \) be positive integers with \( \sum_{j,k=0, j<k}^K m_{j,k} = m \). If all the integers \( m_{j,k} \) are even, then for some \( \epsilon > 0 \)
\[
E \left( \prod_{j,k=0, j<k}^K (I_{j,k,h})^{m_{j,k}} \right) = \prod_{j,k=0, j<k}^K \frac{(2n_{j,k})!}{2^{n_{j,k}}(n_{j,k}!)} \left( 4c_{\psi,h,1} \right)^{n_{j,k}} \cdot E \left( \prod_{j,k=0, j<k}^K (\alpha_{j,k})^{n_{j,k}} \right) + O \left( h^{(2\beta-1)m+n+\epsilon} \right), \] (3.3)

where \( n_{j,k} = m_{j,k}/2 \) and \( n = m/2 \). If any of the \( m_{j,k} \) are odd, then
\[
E \left( \prod_{j,k=0, j<k}^K (I_{j,k,h})^{m_{j,k}} \right) = O \left( h^{(2\beta-1)m/2+\epsilon} \right). \] (3.4)

In (3.3) and (3.4) the error terms may depend on \( m \), but not on the individual terms \( m_{j,k} \).

Proof We can write
\[
E \left( \prod_{j,k=0, j<k}^K (I_{j,k,h})^{m_{j,k}} \right) = E \left( \prod_{j,k=0, j<k}^K \prod_{i=1}^{m_{j,k}} \left( \int (\Delta_{x_{j,k,i}}^h L^x_{j,k,i} \circ \theta_j) (\Delta_{x_{j,k,i}}^h L^x_{j,k,i} \circ \theta_k) \, dx_{j,k,i} \right) \right) \] (3.5)
\[ L^2 \text{ moduli of continuity of local times} \]

\[ \int \left\{ \prod_{j,k=0}^{K} \prod_{i=1}^{m_{j,k}} \Delta_{x_{j,k,i}}^{h,j} \Delta_{x_{j,k,i}}^{h,k} \right\} E \left( \prod_{j,k=0}^{K} \prod_{i=1}^{m_{j,k}} \left( (L_{1}^{x_{j,k,i}} \circ \theta_{j}) (L_{1}^{x_{j,k,i}} \circ \theta_{k}) \right) \right) \]

\[ \prod_{j,k=0}^{K} \prod_{i=1}^{m_{j,k}} dx_{j,k,i}, \]

where the notation \( \Delta_{x_{j,k,i}}^{h,j} \) indicates that we apply the difference operator \( \Delta_{x_{j,k,i}}^{h} \) to \( L_{1}^{x_{j,k,i}} \circ \theta_{j} \). Note that there are \( 2m \) applications of the difference operator \( \Delta_{x}^{h} \).

Consider

\[ E \left( \prod_{j,k=0}^{K} \prod_{i=1}^{m_{j,k}} \left( (L_{1}^{x_{j,k,i}} \circ \theta_{j}) (L_{1}^{x_{j,k,i}} \circ \theta_{k}) \right) \right). \quad (3.6) \]

We collect all the factors containing \( \theta_{l} \) and write

\[ E \left( \prod_{j,k=0}^{K} \prod_{i=1}^{m_{j,k}} \left( (L_{1}^{x_{j,k,i}} \circ \theta_{j}) (L_{1}^{x_{j,k,i}} \circ \theta_{k}) \right) \right) \]

\[ \quad = E \left( \prod_{l=0}^{K} \left\{ \prod_{j=0}^{l-1} \prod_{i=1}^{m_{j,l}} L_{1}^{x_{j,l,i}} \right\} \left( \prod_{k=l+1}^{K} \prod_{i=1}^{m_{l,k}} L_{1}^{x_{l,k,i}} \right) \circ \theta_{l} \right) \]

\[ = E \left( \prod_{l=0}^{K} H_{l} \circ \theta_{l} \right), \]

where

\[ H_{l} = \left( \prod_{j=0}^{l-1} \prod_{i=1}^{m_{j,l}} L_{1}^{x_{j,l,i}} \right) \left( \prod_{k=l+1}^{K} \prod_{i=1}^{m_{l,k}} L_{1}^{x_{l,k,i}} \right). \quad (3.8) \]

By the Markov property

\[ E \left( \prod_{l=0}^{K} H_{l} \circ \theta_{l} \right) = E \left( H_{0} E_{X_{1}}^{X_{1}} \left( \prod_{l=1}^{K} H_{l} \circ \theta_{l-1} \right) \right). \quad (3.9) \]

Let

\[ m_{l} = \sum_{k=l+1}^{K} m_{l,k} + \sum_{j=0}^{l-1} m_{j,l}, \quad l = 0, \ldots, K - 1, \quad (3.10) \]
and note that $m_l$ is the number of local time factors in $H_l$.

Let

$$f(y) = E^y \left( \prod_{l=1}^{K} H_l \circ \theta_{l-1} \right).$$

(3.11)

It follows from Kac’s Moment Formula, Theorem 7.1, for any $z \in \mathbb{R}^1$

$$E^z \left( \prod_{l=0}^{K} H_l \circ \theta_l \right) = E^z (H_0 f(X_1))$$

(3.12)

$$= \sum_{\pi_0} \int \left\{ \sum_{q=1}^{m_0} \frac{1}{\rho_{0,q}} \leq 1 \right\} p_{\pi_0} \left( x_{\pi_0(1)} - z \right) \prod_{q=2}^{m_0} p_{\pi_0,q} \left( x_{\pi_0(q)} - x_{\pi_0(q-1)} \right)$$

$$\left( \int p_{\left( 1 - \sum_{q=1}^{m_0} \rho_{0,q} \right)} (y - x_{\pi_0(m_0)}) E^y \left( H_1 g(X_2) \right) dy \right) \prod_{q=1}^{m_0} d\rho_{0,q},$$

where the sum runs over all bijections $\pi_0$ from $[1, m_0]$ to

$$I_0 = \bigcup_{k=1}^{K} \{(0, k, i), 1 \leq i \leq m_0,k\}.$$  

(3.13)

Clearly, $I_0$ is the set of subscripts of the terms $x$. appearing in the local time factors in $H_0$.

By the Markov property

$$f(y) = E^y \left( H_1 E^{X_2} \left( \prod_{l=2}^{K} H_l \circ \theta_{l-2} \right) \right)$$

(3.14)

$$:= E^y \left( H_1 g(X_2) \right).$$

Therefore, by (3.9)–(3.14), for any $z' \in \mathbb{R}^1$

$$E^{z'} \left( \prod_{l=0}^{K} H_l \circ \theta_l \right)$$

(3.15)

$$= E^{z'} \left( H_0 E^{X_1} (H_1 g(X_2)) \right)$$

$$= \sum_{\pi_0} \int \left\{ \sum_{q=1}^{m_0} \frac{1}{\rho_{0,q}} \leq 1 \right\} p_{\pi_0} \left( x_{\pi_0(1)} - z' \right) \prod_{q=2}^{m_0} p_{\pi_0,q} \left( x_{\pi_0(q)} - x_{\pi_0(q-1)} \right)$$

$$\left( \int p_{\left( 1 - \sum_{q=1}^{m_0} \rho_{0,q} \right)} (y - x_{\pi_0(m_0)}) E^y \left( H_1 g(X_2) \right) dy \right) \prod_{q=1}^{m_0} d\rho_{0,q}$$
\[ E^{z'} \left( H_0 E^{X_1} \left( H_1 g(X_1) \right) \right) \]

\[ = \sum_{\pi_0, \pi_1} \int_{\{q=1 \leq m_0 \leq 1 \}} p_{r_0,1} (x_{\pi_0(1)} - z') \prod_{q=2}^{m_0} p_{r_0,q} (x_{\pi_0(q)} - x_{\pi_0(q-1)}) \]

\[ \sum_{\pi_1} \int_{\{q=1 \leq m_1 \leq 1 \}} p_{r_1,1} (x_{\pi_1(1)} - y) \prod_{q=2}^{m_1} p_{r_1,q} (x_{\pi_1(q)} - x_{\pi_1(q-1)}) \]

\[ \left( \int p_{r_1,q} (y' - x_{\pi_1(m_1)}) g(y') dy' \right) \prod_{q=1}^{m_1} dr_{1,q} dy \prod_{q=1}^{m_0} dr_{0,q} \]

where the second sum runs over all bijections \( \pi_1 \) from \([1, m_1]\) to \([K] \).

As above, \( I_1 \) is the set of subscripts of the terms \( x \cdot \) appearing in the local time factors in \( H_1 \).

We now use the Chapman-Kolmogorov equation to integrate with respect to \( y \) to get

\[ E^{z'} \left( H_0 E^{X_1} \left( H_1 g(X_1) \right) \right) \]

\[ = \sum_{\pi_0, \pi_1} \int_{\{q=1 \leq m_0 \leq 1 \}} p_{r_0,1} (x_{\pi_0(1)} - z') \prod_{q=2}^{m_0} p_{r_0,q} (x_{\pi_0(q)} - x_{\pi_0(q-1)}) \]

\[ \int_{\{q=1 \leq m_1 \leq 1 \}} p_{r_1,1} (x_{\pi_1(1)} - x_{\pi_0(m_0)}) \]

\[ \prod_{q=2}^{m_1} p_{r_1,q} (x_{\pi_1(q)} - x_{\pi_1(q-1)}) \]

\[ \left( \int p_{r_1,q} (y' - x_{\pi_1(m_1)}) g(y') dy' \right) \prod_{q=1}^{m_1} dr_{1,q} dy \prod_{q=1}^{m_0} dr_{0,q}. \]

Iterating this procedure, and recalling (3.17) we see that

\[ E \left( \prod_{j,k} \prod_{i=1}^{m_{j,k}} \left( (L^{x_j,k,i} \circ \theta_j) (L^{x_j,k,i} \circ \theta_k) \right) \right) \]

\[ = \sum_{\pi_0, \ldots, \pi_K} \prod_{l=0}^{K} \int_{\{q=1 \leq m_l \leq 1 \}} p_{r_l,1} (x_{\pi_l(1)} - x_{\pi_{l-1}(m_{l-1})}) \]

\[ \prod_{q=2}^{m_l} p_{r_l,q} (x_{\pi_l(q)} - x_{\pi_l(q-1)}) \prod_{q=1}^{m_l} dr_{l,q}, \]
where $\pi^{-1}(m_{-1}) := 0$ and $1 - \sum_{q=1}^{m_{-1}} r_{-1,q} := 0$. In (3.18) the sum runs over all $\pi_0, \ldots, \pi_K$ such that each $\pi_l$ is a bijection from $[1, m_l]$ to

$$I_l = \bigcup_{j=0}^{l-1} \{(j, l, i), 1 \leq i \leq m_{j+l}\} \bigcup_{k=l+1}^{K} \{(l, k, i), 1 \leq i \leq m_{k+l}\}. \quad (3.19)$$

As in the observations about $I_0$ and $I_1$, we see that $I_l$ is the set of subscripts of the terms $x \cdot \text{terms appearing in the local time factors in } H_l$. Since there are $2m$ local time factors we have that $\sum_{l=0}^{K} m_l = 2m$.

We now use (3.18) in (3.5) and continue to develop an expression for the left-hand side of (3.5). Let $B$ to denote the set of $K + 1$ tuples $\pi = (\pi_0, \ldots, \pi_K)$ of bijections described in (3.19). Clearly

$$|B| = \prod_{l=0}^{K} m_l! \leq (2m)!. \quad (3.20)$$

Also, similarly to the way we obtain the first equality in (3.7), we see that

$$\prod_{j,k=0}^{K} \prod_{j<k}^{m_{j,k}} \Delta h_{j} x_{j,k,i} \Delta h_{k} x_{j,k,i} = \prod_{l=0}^{K} \prod_{q=1}^{m_l} \Delta h_{l} x_{\pi_l(q)}. \quad (3.21)$$

Consequently

$$E \left( \prod_{j,k=0}^{K} (I_{j,k,h})^{m_{j,k}} \right) = \sum_{\pi_0, \ldots, \pi_K} \int \widehat{T}_h(x; \pi) \prod_{j,k,i} dx_{j,k,i} \quad (3.22)$$

where we take the product over $\{0 \leq j < k \leq K, 1 \leq i \leq m_{j,k}\}, \pi \in B$ and

$$\widehat{T}_h(x; \pi) \quad (3.23)$$

$$= \prod_{l=0}^{K} \prod_{q=1}^{m_l} \Delta x_{\pi_l(q)} \left( \sum_{q=1}^{m_l} r_{l,q} \leq 1 \right) \prod_{q=1}^{m_l} p_{r_{l,q}} \left( x_{\pi_l(q)} - x_{\pi_l(q-1)} \right) \prod_{q=2}^{m_l} p_{r_{l,q}} \left( x_{\pi_l(q)} - x_{\pi_l(q-1)} \right) \prod_{q=1}^{m_l} dr_{l,q}. \quad (3.24)$$

We continue to rewrite the right-hand side of (3.22).
In (3.23), each difference operators, say $\Delta^h_u$ is applied to the product of two terms, say $p \cdot (u-a)p \cdot (u-b)$, using the product rule for difference operators we see that

$$\Delta^h_u \{p \cdot (u-a)p \cdot (u-b)\}$$

$$= \Delta^h_u p \cdot (u-a)p \cdot (u+h-b) + p \cdot (u-a)\Delta^h_u p \cdot (u-b) \quad (3.24)$$

Consider an example of how the term $\Delta^h_a \Delta^h_u p \cdot (u-a)$ may appear. It could be by the application

$$\Delta^h_a \left(\Delta^h_u p \cdot (u-a)p \cdot (v-a)\right), \quad (3.25)$$

in which we take account of the two terms to which $\Delta^h_a$ is applied. Using the product rule in (3.24) we see that (3.25)

$$= \left(\Delta^h_a \Delta^h_u p \cdot (u-a)\right) p \cdot (v-(a+h)) + \Delta^h_u p \cdot (u-a)\Delta^h_a p \cdot (v-a). \quad (3.26)$$

Consider one more example

$$\Delta^h_a \left(\Delta^h_u p \cdot (u-a)\Delta^h_u p \cdot (v-a)\right)$$

$$= \left(\Delta^h_a \Delta^h_u p \cdot (u-a)\right) \Delta^h_u p \cdot (v-(a+h)) + \Delta^h_a p \cdot (u-a)\Delta^h_u \Delta^h_a p \cdot (v-a). \quad (3.27)$$

Note that in both examples the arguments of probability densities with two difference operators applied to it does not contain an $h$. This is true in general because the difference formula, (3.24), does not add an $h$ to the argument of a term to which a difference operator is applied. Otherwise we may have a $\pm h$ added to the arguments of probability densities to which one difference operator is applied, as in (3.27), or to the arguments of probability densities to which no difference operator is applied, as in (3.26).

Based on the argument of the preceding paragraph we write (3.23) in the form

$$E \left( \prod_{j,k=0}^{j<k} (I_{j,k,h})^{m_{j,k}} \right) = \sum_a \sum_{\pi_0,\ldots,\pi_K} \int T_h'(x; \pi, a) \prod_{j,k,i} dx_{j,k,i}, \quad (3.28)$$
where

\[
T_h'(x; \pi, a) = \prod_{l=0}^{K} \int_{\mathcal{R}_l} \left( (\Delta^h_{x_{\pi_1(l)}})^{a_1(l,1)} (\Delta^h_{x_{\pi_1(m_l-1)}})^{a_2(l,1)} \right) \cdot \prod_{q=2}^{m_l} \left( (\Delta^h_{x_{\pi_1(q)}})^{a_1(l,q)} (\Delta^h_{x_{\pi_1(m_l-1)}})^{a_2(l,q)} \right) \prod_{r_1, q=1}^{m_l} dr_{l,q},
\]

(3.29)

and where \( \mathcal{R}_l = \{ \sum_{q=1}^{m_l} r_{l,q} \leq 1 \} \). In (3.28) the first sum is taken over all \( a = (a_1, a_2) : \{(l, q), 0 \leq l \leq K, 1 \leq q \leq m_l \} \mapsto \{0, 1\} \times \{0, 1\} \) (3.30)

with the restriction that for each triple \( j, k, i \), there are exactly two factors of the form \( \Delta^h_{x_{j,k,i}} \), each of which is applied to one of the terms \( p_r^z(\cdot) \) that contains \( x_{j,k,i} \) in its argument. This condition can be stated more formally by saying that for each \( l \) and \( q = 1, \ldots, m_l - 1 \), if \( \pi_1(q) = (j, k, i) \), then \( \{a_1(l, q), a_2(l, q+1)\} = \{0, 1\} \) and if \( q = m_l \) then \( \{a_1(l, m_l), a_2(l+1, 1)\} = \{0, 1\} \). (Note that when we write \( \{a_1(l, q), a_2(l, q+1)\} = \{0, 1\} \) we mean as two sets, so, according to what \( a \) is, we may have \( a_1(l, q) = 1 \) and \( a_2(l, q + 1) = 0 \) or \( a_1(l, q) = 0 \) and \( a_2(l, q + 1) = 1 \) and similarly for \( \{a_1(l, m_l), a_2(l + 1, 1)\} \).) Also, in (3.29) we define \( (\Delta^h_{x})^0 = 1 \) and \( (\Delta^h_{0}) = 1 \).

In (3.29), \( p_r^z(z) \) can take any of the three values \( p_r(z), p_r(z + h), \) or \( p_r(z - h) \). (We must consider all three possibilities.) Finally, it is important to emphasize that in (3.29) each of the difference operators is applied to only one of the terms \( p_r^z(\cdot) \).

To avoid confusion caused by the ambiguity of \( p_r^z \), we first analyze

\[
\sum_a \sum_{\pi_0, \ldots, \pi_K} \int T_h(x; \pi, a) \prod_{j,k,i} dx_{j,k,i},
\]

(3.31)

where

\[
T_h(x; \pi, a) = \prod_{l=0}^{K} \int_{\mathcal{R}_l} \left( (\Delta^h_{x_{\pi_1(l)}})^{a_1(l,1)} (\Delta^h_{x_{\pi_1(m_l-1)}})^{a_2(l,1)} \right) \cdot \prod_{q=2}^{m_l} \left( (\Delta^h_{x_{\pi_1(q)}})^{a_1(l,q)} (\Delta^h_{x_{\pi_1(m_l-1)}})^{a_2(l,q)} \right) \prod_{r_1, q=1}^{m_l} dr_{l,q},
\]

(3.32)
The difference between $T_h(x; \pi, a)$ and $T'_h(x; \pi, a)$ is that in the former we replace $p^\sharp$ by $p$. (I.e. we set $h=0$ in the arguments of the $p^\sharp$ terms in (3.29).) At the conclusion of this proof we show that both (3.31) than (3.28) have the same asymptotic limit as $h$ goes to zero.

We first obtain (3.3). Let $m=2n$, since $m_{j,k}=2n_{j,k}$, $m_l=2n_l$ for some integer $n_l$. (Recall (3.10)). To begin we consider the case in which $a=e$, where

\[ e(l, 2q) = (1, 1) \quad \text{and} \quad e(l, 2q-1) = (0, 0) \quad \forall q. \quad (3.33) \]

When $a=e$ we have

\[
T_h(x; \pi, e) = \prod_{l=0}^{K} \int_{\mathbb{R}^l} \left(p_{1-\sum_{q=1}^{m_{l-1}} r_{l-1,q}+r_{l,1}} \left(x_{\pi_l(1)} - x_{\pi_{l-1}(m_{l-1})}\right) \right. \\
\left. \prod_{q=2}^{n_l} p_{r_{l,2q-1}} \left(x_{\pi_l(2q-1)} - x_{\pi_l(2q-2)}\right) \right) \prod_{q=1}^{m_l} \Delta^h \Delta^{-h} p_{r_{l,2q}} \left(x_{\pi_l(2q)} - x_{\pi_l(2q-1)}\right) \prod_{q=1}^{m_l} dr_{l,q}. \tag{3.34}
\]

Here we use the following notation: $\Delta^h p(u-v) = p(u-v+h) - p(u-v)$, i.e., when $\Delta^h$ has no subscript, the difference operator is applied to the whole argument of the function. In this notation,

\[
\Delta^h \Delta^{-h} p(u-v) = \Delta^h \Delta^{-h} p(u-v). \tag{3.35}
\]

3.1 a = e, with all cycles of order two

Consider the multigraph $G_\pi$ with vertices $\{(j,k,i), \; 0 \leq j < k \leq K, \; 1 \leq i \leq m_{j,k}\}$. Assign an edge between the vertices $\pi_l(2q-1)$ and $\pi_l(2q)$ for each $0 \leq l \leq K$ and $1 \leq q \leq n_l$. Each vertex is connected to two edges. To see this suppose that $\pi_l(2q) = \{(j,k,i)\}$, with $j = l$ and $k = l' \neq l$, then there is a unique $q'$ such that $\pi_{l'}(2q')$ or $\pi_{l'}(2q'-1)$ is equal to $\{(j,k,i)\}$. Therefore all the vertices lie in some cycle. Assume that there are $S$ cycles. We denote them by $C_s, s=1,\ldots,S$. Clearly, it is possible to have cycles of order two, in which case two vertices are connected by two edges.

It is important to note that the graph $G_\pi$ does not assign edges between $\pi_l(2q)$ and $\pi_l(2q+1)$, although these vertices may be connected by the edge assigned between $\pi_{l'}(2q'-1)$ and $\pi_{l'}(2q')$ for some $l'$ and $q'$. 

We estimate (3.32) by breaking the calculation into two cases. In this section we consider the case when \( a = e \) and all the cycles of \( G_\pi \) are of order two. In Section 3.2 we consider the cases when \( a = e \) and not all the cycles of \( G_\pi \) are of order two, and when \( a \neq e \).

Let \( \mathcal{P} = \{(\gamma_{2v-1}, \gamma_{2v}), 1 \leq v \leq n\} \) be a pairing of the \( m \) vertices 

\[
\{(j, k, i), 0 \leq j < k \leq K, 1 \leq i \leq m_{j,k}\}
\]

of \( G_\pi \), that satisfies the following special property: whenever \((j, k, i)\) and \((j', k', i')\) are paired together, \( j = j' \) and \( k = k' \). Equivalently,

\[
\mathcal{P} = \bigcup_{j,k=0}^{K} \mathcal{P}_{j,k}
\]

where each \( \mathcal{P}_{j,k} \) is a pairing of the \( m_{j,k} \) vertices 

\[
\{(j, k, i), 1 \leq i \leq m_{j,k}\}.
\]

We refer to such a pairing \( \mathcal{P} \) as a special pairing and denote the set of special pairings by \( \mathcal{S} \).

Given a special pairing \( \mathcal{P} \in \mathcal{S} \), let \( \pi \) be such that for each \( 0 \leq l \leq K \) and \( 1 \leq q \leq n_l \),

\[
\{\pi_l(2q - 1), \pi_l(2q)\} = \{\gamma_{2v-1}, \gamma_{2v}\}
\]

for some, necessarily unique, \( 1 \leq v \leq n_l \). In this case we say that \( \pi \) is compatible with the pairing \( \mathcal{P} \) and write this as \( \pi \sim \mathcal{P} \). (Recall that when we write \( \{\pi_l(2q - 1), \pi_l(2q)\} = \{\gamma_{2v-1}, \gamma_{2v}\} \), we mean as two sets, so, according to what \( \pi_l \) is, we may have \( \pi_l(2q - 1) = \gamma_{2v-1} \) and \( \pi_l(2q) = \gamma_{2v} \) or \( \pi_l(2q - 1) = \gamma_{2v} \) and \( \pi_l(2q) = \gamma_{2v-1} \).) Clearly

\[
|\mathcal{S}| \leq \frac{(2n)!}{2^n n!}
\]

the number of pairings of \( m = 2n \) objects.

Let \( \pi \in \mathcal{B} \) be such that \( G_\pi \) consists of cycles of order two. It is easy to see that \( \pi \sim \mathcal{P} \) for some \( \mathcal{P} \in \mathcal{S} \). To see this note that if \( \{(j, k, i), (j', k', i')\} \) form a cycle of order two, there must exist \( l \) and \( l' \) with \( l \neq l' \) and \( q \) and \( q' \) such that both \( \{(j, k, i), (j', k', i')\} = \{\pi_l(2q -
$L^2$ moduli of continuity of local times

1, $\pi_t(2q)$} and \{(j, k, i), (j', k', i')\} = \{\pi_t'(2q' - 1), \pi_t'(2q')\}. This implies that $j = j'$, $k = k'$ and \{j, k\} = \{l, l'\}. Furthermore, by (3.37) we have

\[
\{\pi_t(2q - 1), \pi_t(2q)\} = \{\pi_t'(2q' - 1), \pi_t'(2q')\} = \{\gamma_{2q-1}, \gamma_{2q}\} \quad (3.39)
\]

When $\pi \sim \mathcal{P}$ and all cycles are of order two we can write

\[
\prod_{l=0}^{K} \prod_{q=1}^{n_l} \Delta^h \Delta^{\frac{h}{2}} p_{r_{l,2q}}(x_{\pi_t(2q)} - x_{\pi_t(2q-1)})
\]

(3.40)

where $r_{2q}$ and $r'_{2q}$ are the rearranged indices $r_{l,2q}$ and $r'_{l,2q}$. We also use the fact that $\sum_{l=0}^{K} n_l = 2n$.

For use in (3.40) below we note that

\[
\int_0^1 \int_0^1 |\Delta^h \Delta^{-h} p_{r_{2q}}(x_{\gamma_{2q}} - x_{\gamma_{2q-1}}) | \Delta^h \Delta^{-h} p_{r'_{2q}}(x_{\gamma_{2q}} - x_{\gamma_{2q-1}}) \, dr_{2q} \, dr'_{2q}
\]

(3.41)

(see (2.8).)

We want to estimate the integrals in (3.31). However, it is difficult to integrate $T_h(x; \pi, e)$ directly, because the variables,

\[
\{x_{\pi_t(1)} - x_{\pi_t-1(m_l-1)}, x_{\pi_t(2q-1)} - x_{\pi_t-2(q-2)}, x_{\pi_t(2q)} - x_{\pi_t-1(2q-1)}; \quad l \in [0, K], q \in [1, n_l]\},
\]

are not independent. We begin the estimation by showing that over much of the domain of integration, the integral is negligible, asymptotically, as $h \to 0$. To begin, we write

\[
1 = \prod_{v=1}^{n} \left( 1_{\{x_{\gamma_{2q}} - x_{\gamma_{2q-1}} \leq \sqrt{n}\}} + 1_{\{x_{\gamma_{2q}} - x_{\gamma_{2q-1}} \geq \sqrt{n}\}} \right) \quad (3.42)
\]

and expand it as a sum of $2^n$ terms and use it to write

\[
\int T_h(x; \pi, e) \prod_{j,k,i} dx_{j,k,i}
\]

(3.43)

\[
\int \prod_{v=1}^{n} \left( 1_{\{x_{\gamma_{2q}} - x_{\gamma_{2q-1}} \leq \sqrt{n}\}} \right) T_h(x; \pi, e) \prod_{j,k,i} dx_{j,k,i} + E_{1,h}.
\]
We now show that
\[
E_{1,h} = O \left( h^{1/2} \left( \frac{1}{h^{1/2}(1/h)} \right)^n \right) .
\] (3.44)

Note that every term in \( E_{1,h} \) can be written in the form
\[
B_h(\pi, e, D) := \int \prod_{v=1}^n 1_{D_v} T_h(x; \pi, e) \prod_{j,k,i} dx_{j,k,i} \] (3.45)
where each \( D_v \) is either \( \{ |x_{\gamma_v} - x_{\gamma_{v-1}}| \leq \sqrt{h} \} \) or \( \{ |x_{\gamma_v} - x_{\gamma_{v-1}}| \geq \sqrt{h} \} \), and at least one of the \( D_v \) is of the second type.

Consider (3.45) and the representation of \( T_h(x; \pi, e) \) in (3.34). We take absolute values in the integrand in (3.34) and take all the integrals with \( r \) between 0 and 1 and use (3.41) followed by (2.3) to get
\[
|B_h(\pi, e, D)| \leq \int \prod_{v=1}^n 1_{D_v} w^2(x_{\gamma_v} - x_{\gamma_{v-1}}) \prod_{l=0}^{K} u(x_{\pi l (1)} - x_{\pi l (m_l - 1)}) \prod_{q=2}^{n_l} u(x_{\pi l (2q-1)} - x_{\pi l (2q-2)}) \prod_{j,k,i} dx_{j,k,i} .
\] (3.46)

We now take
\[
\{ x_{\gamma_v} - x_{\gamma_{v-1}}, \ v = 1, \ldots, n \}
\] (3.47)
and an additional \( n \) variables from the \( 2n \) arguments of the \( u \) terms,
\[
\bigcup_{l=0}^{K} \{ x_{\pi l (1)} - x_{\pi l (m_l - 1)}, x_{\pi l (2q-1)} - x_{\pi l (2q-2)}, q = 2, \ldots, n_l \}
\] (3.48)
so that the chosen \( 2n \) variables generate the space spanned by the \( 2n \) variables \( \{ x_{j,k,i} \} \). There are \( n \) variables in (3.48) that are not used. We bound the functions \( u \) of these variables by their sup norm, which by (2.3) is finite. Then we make a change of variables and get that
\[
|B_h(\pi, e, D)| \leq C \int \prod_{v=1}^n 1_{D_v} w^2(y_v) \prod_{v=n+1}^{2n} u(y_v) \prod_{v=1}^{2n} dy_v \] (3.49)
\[
\leq C \int \prod_{v=1}^n 1_{D_v} w^2(y_v) \prod_{v=1}^{n} dy_v,
\]
\[
= O \left( h^{1/2} \left( \frac{1}{h^{1/2}(1/h)} \right)^n \right) .
\]
Here we use \((2.3)\) to see that the integrals of the \(u\) terms is finite. Then we use \((2.10)\) and \((2.11)\) to obtain \((3.44)\). (Note that it is because at least one of the \(D_u\) is of the second type that we can use \((2.11)\).)

We now study

\[
\int \prod_{v=1}^{n} \left(1_{\{|x_{\gamma_{2v}}-x_{\gamma_{2v-1}}| \leq \sqrt{\pi}\}}\right) T_h(x;\pi,e) \prod_{j,k,i} dx_{j,k,i}. \tag{3.50}
\]

Recall that for each \(0 \leq l \leq K\) and \(1 \leq q \leq n_l\), \(\{\pi_l(2q-1), \pi_l(2q)\} = \{\gamma_{2v-1}, \gamma_{2v}\}\) for some \(1 \leq v \leq n\). We identify these relationships by setting \(v = \sigma_l(q)\) when \(\{\pi_l(2q-1), \pi_l(2q)\} = \{\gamma_{2v-1}, \gamma_{2v}\}\), and sometimes write this last term as \(\{\gamma_{2\sigma_l(q)-1}, \gamma_{2\sigma_l(q)}\}\).

For \(q \geq 2\) we write

\[
p_{\pi_l,2q-1}(x_{\pi_l(2q-1)} - x_{\pi_l(2q-2)}) = p_{\pi_l,2q-1}(x_{\gamma_{2\sigma_l(q)-1}} - x_{\gamma_{2\sigma_l(q)-2}}) + \Delta h_{l,q} p_{\pi_l,2q-1}(x_{\gamma_{2\sigma_l(q)-1}} - x_{\gamma_{2\sigma_l(q)-2}}), \tag{3.51}
\]

where \(h_{l,q} = (x_{\pi_l(2q-1)} - x_{\gamma_{2\sigma_l(q)-1}}) + (x_{\gamma_{2\sigma_l(q)-1}} - x_{\pi_l(2q-2)})\). When \(q = 1\) we can make a similar decomposition

\[
p_{1-\sum_{q=1}^{n_l-1} r_{l-1,q} + r_{l,1}}(x_{\pi_{l-1}(1)} - x_{\pi_{l-1}(m_{l-1})}) = p_{1-\sum_{q=1}^{n_l-1} r_{l-1,q} + r_{l,1}}(x_{\gamma_{2\sigma_{l-1}+1}} - x_{\gamma_{2\sigma_{l-1}1}}) + \Delta h_{l,1} p_{1-\sum_{q=1}^{n_l-1} r_{l-1,q} + r_{l,1}}(x_{\gamma_{2\sigma_{l-1}+1}} - x_{\gamma_{2\sigma_{l-1}1}}), \tag{3.52}
\]

where \(h_{l,1} = (x_{\pi_{l-1}(1)} - x_{\gamma_{2\sigma_{l-1}1}}) + (x_{\gamma_{2\sigma_{l-1}1}} - x_{\pi_{l-1}(m_{l-1})})\). Note that because of the presence of the term \(\prod_{v=1}^{n} \left(1_{\{|x_{\gamma_{2v}}-x_{\gamma_{2v-1}}| \leq \sqrt{\pi}\}}\right)\) in the integral in \((3.50)\) we need only be concerned with values of \(|h_{l,q}| \leq 2\sqrt{\pi}\), for \(0 \leq l \leq K\) and \(1 \leq q \leq n_l\).

For \(q = 1, \ldots, n_l\), \(l = 0, \ldots, K\), we substitute \((3.51)\) and \((3.52)\) into the term \(T_h(x;\pi,e)\) in \((3.50)\), (see also \((3.51)\)), and expand the products so that we can write \((3.50)\) as a sum of \(2\sum_{l=0}^{n_l} \) terms, which we write as

\[
\int \prod_{v=1}^{n} \left(1_{\{|x_{\gamma_{2v}}-x_{\gamma_{2v-1}}| \leq \sqrt{\pi}\}}\right) T_h(x;\pi,e) \prod_{j,k,i} dx_{j,k,i} \tag{3.53}
\]

\[
= \int \prod_{v=1}^{n} \left(1_{\{|x_{\gamma_{2v}}-x_{\gamma_{2v-1}}| \leq \sqrt{\pi}\}}\right) T_{h,1}(x;\pi,e) \prod_{j,k,i} dx_{j,k,i} + E_{2,h},
\]

where \(E_{2,h}\) is a function of \(h\).
where

\[ T_{h,1}(x; \pi, e) = \prod_{l=0}^{K} \int_{R_{l}} \left( \prod_{q=1}^{m_l} p_{\left(1-\sum_{q=1}^{r_{1-1,q}}+r_{1,1}\right)} (x_{\gamma_{2q_l}(1)-1} - x_{\gamma_{2q_l-1}(n_{l-1})-1}) \right) \]

\[ \prod_{q=2}^{n_l} p_{r_{l,2q-1}} (x_{\gamma_{2q_l(q)-1}} - x_{\gamma_{2q_l(q-1)-1}}) \]

\[ \prod_{q=1}^{n_l} \Delta^h \Delta^h p_{r_{l,2q}} (x_{\pi_l(2q)} - x_{\pi_l(2q-1)}) \prod_{q=1}^{m_l} dr_{l,q}. \]

Using (3.40) we can rewrite this as

\[ T_{h,1}(x; \pi, e) = \int_{\mathcal{R}_0 \times \cdots \times \mathcal{R}_K} \left( \prod_{l=0}^{K} \left( \prod_{q=1}^{m_l} p_{\left(1-\sum_{q=1}^{r_{1-1,q}}+r_{1,1}\right)} (x_{\gamma_{2q_l}(1)-1} - x_{\gamma_{2q_l-1}(n_{l-1})-1}) \right) \right) \]

\[ \prod_{q=2}^{n_l} p_{r_{l,2q-1}} (x_{\gamma_{2q_l(q)-1}} - x_{\gamma_{2q_l(q-1)-1}}) \]

\[ \left( \prod_{v=1}^{n} \Delta^h \Delta^h p_{r_{2v}} (x_{\gamma_{2v}} - x_{\gamma_{2v-1}}) \Delta^h \Delta^h p'_{r_{2v}} (x_{\gamma_{2v}} - x_{\gamma_{2v-1}}) \right) \]

\[ \prod_{l=0}^{K} \prod_{q=1}^{m_l} dr_{l,q}, \]

where \( r_{2v} \) and \( r'_{2v} \) are the rearranged indices \( r_{l,2q} \) and \( r_{l,2q'} \). Since the variables \( x_{\gamma_{2v}}, v = 1, \ldots, n, \) occur only in the last line of (3.55), we make the change of variables \( x_{\gamma_{2v}} - x_{\gamma_{2v-1}} \rightarrow x_{\gamma_{2v}} \) and \( x_{\gamma_{2v-1}} \rightarrow x_{\gamma_{2v-1}} \) and get that

\[ \int T_{h,1}(x; \pi, e) \prod_{j,k,i} dx_{j,k,i} \]

\[ = \int \int_{\mathcal{R}_0 \times \cdots \times \mathcal{R}_K} \left( \prod_{l=0}^{K} \left( \prod_{q=1}^{m_l} p_{\left(1-\sum_{q=1}^{r_{1-1,q}}+r_{1,1}\right)} (x_{\gamma_{2q_l}(1)-1} - x_{\gamma_{2q_l-1}(n_{l-1})-1}) \right) \right) \]

\[ \prod_{q=2}^{n_l} p_{r_{l,2q-1}} (x_{\gamma_{2q_l(q)-1}} - x_{\gamma_{2q_l(q-1)-1}}) \]

\[ \left( \prod_{v=1}^{n} \Delta^h \Delta^h p_{r_{2v}} (x_{\gamma_{2v}}) \Delta^h \Delta^h p'_{r_{2v}} (x_{\gamma_{2v}}) \right) \prod_{l=0}^{K} \prod_{q=1}^{m_l} dr_{l,q} \prod_{j,k,i} dx_{j,k,i}. \]
Since the variables $x_{\gamma_2^v}$, $v = 1, \ldots, n$ occur only in the last line of (3.56) and the variables $x_{\gamma_2^{v-1}}$, $v = 1, \ldots, n$ occur only in the second and third lines of (3.56), we can write (3.56) as

$$
\int T_{h,1}(x; \pi, e) \prod_{j,k,i} dx_{j,k,i} \quad (3.57)
$$

$$
= \int_{\mathbb{R}_0 \times \cdots \times \mathbb{R}_K} \left( \prod_{l=0}^{K} \left( p_{(1-\sum_{q=1}^{m_l} r_{l-1,q})}^l (x_{\gamma_2^{r_l(1)-1}} - x_{\gamma_2^{r_l(n_l-1)-1}}) \right) \prod_{v=1}^{n} dx_{\gamma_2^v-1} \right) \left( \prod_{v=1}^{n} \int \Delta^h \Delta^{-h} p_{r_{2v}}^l (x_{\gamma_2^v}) \Delta^h \Delta^{-h} p_{r_{2v}^l} (x_{\gamma_2^v}) dx_{\gamma_2^v} \right) K \prod_{l=0}^{K} \prod_{q=1}^{m_l} dr_{l,q}. \quad (3.57)
$$

Note that we also use Fubini’s Theorem which is justified since the absolute value of the integrand is integrable, (as we point out in the argument preceding (3.46)). (In the rest of this section use Fubini’s Theorem frequently for integrals like (3.57) without repeating the explanation about why it is justified.)

Analogous to (3.43) we note that

$$
\int T_{h,1}(x; \pi, e) \prod_{j,k,i} dx_{j,k,i} \quad (3.58)
$$

$$
= \int_1^n \left( \prod_{v=1}^{n} \left( 1_{\{ |x_{\gamma_2^v} - x_{\gamma_2^{v-1}}| \leq \sqrt{h} \}} \right) T_{h,1}(x; \pi, e) \prod_{j,k,i} dx_{j,k,i} + \tilde{E}_{1,h}, \quad (3.58)
$$

where $\tilde{E}_{1,h} = O \left( \left( \frac{1}{h\psi^2(1/h)} \right)^n \right)$. The proof of (3.58) is the same as the proof of (3.44).

We now show that

$$
E_{2,h} = O \left( \left( \frac{1}{h\psi(1/h)} \right)^{1/2} \left( \frac{1}{h\psi^2(1/h)} \right)^n \right). \quad (3.59)
$$

To see this note that the terms in $E_{2,h}$ are of the form

$$
\int_1^n \left( \prod_{v=1}^{n} \left( 1_{\{ |x_{\gamma_2^v} - x_{\gamma_2^{v-1}}| \leq \sqrt{h} \}} \right) \right) \quad (3.60)
$$
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\[
\prod_{l=0}^{K} \int_{\mathcal{R}_l} \tilde{p}_{1-\sum_{q=1}^{m_l-1} r_{l-1,q} + r_{l,1}} \left( x_{\gamma_{2\sigma_l(1)}-1} - x_{\gamma_{2\sigma_{l-1}(n_{l-1})-1}} \right) \\
\prod_{q=2}^{n_l} \bar{p}_{r_{l,2q-1}} \left( x_{\gamma_{2\sigma_l(q)-1}} - x_{\gamma_{2\sigma_{l(q-1)-1}}} \right) \\
\prod_{q=1}^{n_l} \Delta^h \Delta^{-h} \bar{p}_{r_{l,2q}} \left( x_{\pi_l(2q)} - x_{\pi_l(2q-1)} \right) \\
\prod_{q=1}^{m_l} dr_{l,q} \prod_{j,k,i} dx_{j,k,i};
\]

where \( \tilde{p}_{r_{l,2q-1}} \) is either \( p_{r_{l,2q-1}} \) or \( \Delta^h \bar{p}_{r_{l,2q-1}} \). Furthermore, at least one of the terms \( \tilde{p}_{r_{l,2q-1}} \) is of the form \( \Delta^h \bar{p}_{r_{l,2q-1}} \).

As in the transition from (3.45) to (3.46) we bound the absolute value of (3.60) by

\[
\int \prod_{v=1}^{n} \left( 1 \{ |x_{\gamma_v} - x_{\gamma_v-1}| \leq \sqrt{\pi} \} \right) u^2(x_{\gamma_v} - x_{\gamma_v-1}) 
\]

where each \( \bar{u} \) is either of the form \( u \) or \( v \), in Lemma 2.1, and where, obviously, the \( h \) in (2.5) is \( h_{l,q} \). Furthermore, we have \( J \) terms of the type \( v \), for some \( J \geq 1 \). It follows from (2.5), the regular variation of \( \psi \) and the fact that \( |h_{l,q}| \leq 2\sqrt{h} \), that

\[
v(\cdot) \leq C \left( \frac{1}{h\psi(1/h)} \right)^{1/2} \frac{1}{1 + x^2}
\]

Using this and (2.3) we can bound the integral in (3.61) by

\[
C \left( \frac{1}{h\psi(1/h)} \right)^{J/2} \int \prod_{v=1}^{n} u^2(x_{\gamma_v} - x_{\gamma_v-1}) 
\]

where all the terms \( \bar{u}(y) = (1 + y^2)^{-1} \).

Since the variables \( x_{\gamma_v}, \nu = 1, \ldots, n \), occur only in the \( u \) terms in (3.63) and the variables \( x_{\gamma_v-1}, \nu = 1, \ldots, n \) occur only in the \( \bar{u} \) terms in
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(3.63), (refer to the change of variables arguments in (3.56) and (3.57)), we can write (3.63) as

\[ C \left( \frac{1}{h \psi(1/h)} \right)^{1/2} \int \left( \prod_{l=0}^{K} \mathfrak{p}(x_{\sigma_l} - x_{\sigma_{l+1}}) \right) \prod_{q=2}^{n_l} \prod_{v=1}^{n} \prod_{v=1}^{w^2} \prod_{v=1}^{dx_{\gamma_{2v}}}. \] (3.64)

As we have been doing we extract a linearly independent set of variables from the arguments of the \( \mathfrak{p} \) terms. The other \( \mathfrak{p} \) terms we bound by one. Then we make a change of variables and integrate the remaining \( \mathfrak{p} \) terms and the \( w^2 \) terms using (2.3) and (2.10). Since \( J \geq 1 \), we get (3.59).

Since \( \psi \) is regularly varying with index \( \beta > 1 \) we see that there exists an \( \epsilon > 0 \) such that

\[ E_{1,h} + E_{2,h} + \tilde{E}_{2,h} = O \left( h^{(2\beta-1)n+\epsilon} \right). \] (3.65)

Therefore, it follows from (3.43), (3.53) and (3.58) that

\[ \int \mathcal{T}_h(x; \pi, e) \prod_{j,k,i} dx_{j,k,i} \] (3.66)

\[ \int \mathcal{T}_{h,1}(x; \pi, e) \prod_{j,k,i} dx_{j,k,i} + O \left( h^{(2\beta-1)n+\epsilon} \right). \]

Let \( \mathcal{R}_l(s) = \{ \sum_{q=1}^{n_l} r_{l,2q-1} \leq 1 - s \} \) and \( \tilde{\sigma}_l(q) := \gamma_{2\sigma_l(q)} - 1 \). We define

\[ F(\tilde{\sigma}, s_0, \ldots, s_K) \]

\[ = \int \left( \int_{\mathcal{R}_0} \prod_{l=0}^{K} P_{1-\sum r_{l,2q-1}-s_{l-1}+r_{l,1}} \right) \prod_{q=2}^{n_l} \prod_{q=1}^{n_l} \left( x_{\tilde{\sigma}_l} - x_{\tilde{\sigma}_{l+1}} \prod_{q=2}^{n_l} p_{r_{1,2q-1}} \right) dx, \]

where \((1 - \sum_{q=1}^{n_l} r_{1,2q-1} - s_{l-1}) := 0\) and \( \tilde{\sigma}_{l}(n_{l-1}) := 0 \). Here the generic term \( dx \) indicates integration with respect to all the variables \( x \) that appear in the integrand.
Since \( \tilde{\sigma}(q) = \gamma_{2\sigma_l(q)-1} \) we can also write (3.67) as
\[
F(\tilde{\sigma}, s_0, \ldots, s_K) = \int \left( \int_{\tilde{R}_0(s_0)} \times \cdots \times \tilde{R}_K(s_K) \right) \prod_{l=0}^{K} p_{1-\sum_{q=1}^{n_l-1} r_{l-1,2q-1} - s_l} \cdot n_l \prod_{q=2}^{n_l} \int dx_{\gamma_{2\sigma_l(q)-1} - x_{\gamma_{2\sigma_l(q)-1} - 1}} \right) dx,
\]
\( x_{\gamma_{2\sigma_l(q)-1} - 1} := 0 \).

Consider the mappings \( \tilde{\sigma}_l \) that are used in (3.67). Recall that \( \sigma_l(q) \) is defined by the relationship
\[
\{ \pi_l(2q-1), \pi_l(2q) \} = \{ \gamma_{2\sigma_l(q)-1}, \gamma_{2\sigma_l(q)} \}.
\]
Therefore, since \( \tilde{\sigma}_l(q) = \gamma_{2\sigma_l(q)-1} \) we can have that either \( \tilde{\sigma}_l(q) = \pi_l(2q-1) \) or \( \tilde{\sigma}_l(q) = \pi_l(2q) \). However, since the terms \( \tilde{\sigma}_l(q) \) are subscripts of the terms \( x \), all of which are integrated, it is more convenient to define \( \tilde{\sigma}_l \) differently.

Recall that \( P_j(k) \), (see (3.36)), is a union of pairings \( P_{j,k} \) of the \( m_{j,k} \) vertices
\[
\{(j,k,i), 1 \leq i \leq m_{j,k}\}.
\]
Each \( P_{j,k} \) consists of \( n_{j,k} \) pairs, that can ordered arbitrarily. If \( \{ \gamma_{2\sigma_l(q)-1}, \gamma_{2\sigma_l(q)} \} \) is the \( i \)-th pair in \( P_{j,k} \), we set \( \tilde{\sigma}_l(q) = (j,k,i) \). (Necessarily, \( l \) will be either \( j \) or \( k \), as we point out in the paragraph containing (3.39)). Thus, each \( \tilde{\sigma}_l \) is a bijection from \([1,n_l]\) to
\[
I_l = \bigcup_{k=l+1}^K \{(l,k,i), 1 \leq i \leq n_{l,k}\} \bigcup_{j=0}^{l-1} \{(j,l,i), 1 \leq i \leq n_{j,l}\}.
\]
Let \( \tilde{B} \) denote the set of \( K+1 \) tuples, \( \tilde{\sigma} = (\tilde{\sigma}_0, \ldots, \tilde{\sigma}_K) \) of such bijections. Note that with this definition of \( \tilde{\sigma}(q) \) (3.67) remains unchanged since we have simply renamed the variables of integration.

By interchanging the elements in any of the \( 2n \) pairs \( \{ \pi_l(2q-1), \pi_l(2q) \} \) we obtain a new \( \pi' \sim P \). In fact we obtain \( 2^{2n} \) different permutations \( \pi \), in this way, all of which are compatible with \( P \), and all of which give the same \( \tilde{\sigma} \) in (3.67). Furthermore, by permuting the pairs
\{\pi_l(2q - 1), \pi_l(2q)\}, 1 \leq q \leq n_l, for each l, we get all the possible permutations \( \tilde{\pi} \sim \mathcal{P} \), and these give all possible mappings \( \tilde{\sigma} \in \tilde{\mathcal{B}} \). Note that 
\[ |\tilde{\mathcal{B}}| = \prod_{l=0}^{K} n_l! \leq (2n)! \]

Consider (3.68). Since 
\[ x_{\gamma_{2q-1(n-1)-1}} = x_{\gamma_{2q0-1}} = 0 \]
appears alone as the argument of one of the density functions. Therefore we can extract a linearly independent set from the arguments of the densities that spans the space spanned by all the arguments of the densities. We use (2.2) to bound the density functions with arguments that are not in the spanning set by 
\[ C\psi^{-1}(1/s) \]
and integrate them with respect to the time variables. Since the time variables are bounded, all this contributes only some constant. With what is left we can make a change of variables and use (2.2) again to see that
\[ F(\tilde{\sigma}, s_0, \ldots, s_K) \leq C, \quad (3.70) \]
for some constant depending only on \( m \).

Let \( \tilde{\mathcal{R}}_l = \{\sum_{q=1}^{n_l} r_{l,2q} \leq 1\} \). We break up the integration over \( \mathcal{R}_l \)
into integration over \( \tilde{\mathcal{R}}_l(s) \) and \( \tilde{\mathcal{R}}_l \) in (3.57) and (3.68). If one carefully examines the time indices in (3.32) and (3.67) and uses Fubini’s Theorem, one sees that
\[ \int T_h(x; \pi, e) \prod_{j,k,i} dx_{j,k,i} \]
\[ = \int_{\tilde{\mathcal{R}}_0 \times \cdots \times \tilde{\mathcal{R}}_K} F(\tilde{\sigma}, \sum_{q=1}^{n_0} r_{0,2q}, \ldots, \sum_{q=1}^{n_K} r_{K,2q}) \]
\[ \prod_{i=1}^{n} \left( \int (\Delta^h \Delta^{-h} p_{r_i}(x)) \left( \Delta^h \Delta^{-h} p_{r'_i}(x) \right) dx \right) \prod_{i=1}^{n} dr_i dr'_i. \]
The variables \( \{r_i, r'_i \mid i = 1, \ldots, n\} \) are simply a relabeling of the variables \( \{r_{l,2q} \mid 0 \leq l \leq K, 1 \leq q \leq n_l\} \). (The exact form of this relabeling does not matter in what follows.) Here, as always, we set \( p_r(x) = 0 \), if \( r \leq 0 \).

By Parseval’s Theorem,
\[ \int (\Delta^h \Delta^{-h} p_r(x)) \left( \Delta^h \Delta^{-h} p_{r'}(x) \right) dx \]
\[ = \frac{1}{2\pi} \int |2 - e^{ih} - e^{-ih}|^2 e^{-r\psi(p)} e^{-r'\psi(p)} dp \geq 0. \]
Using this, (3.70) and Fubini’s Theorem, we see that

\[
\int_{(\hat{R}_0 \times \cdots \times \hat{R}_K) \cap ([0, \sqrt{h}]^{2n})^c} F(\bar{\sigma}, \sum_{q=1}^{n_0} r_{0,2q}, \ldots, \sum_{q=1}^{n_K} r_{K,2q})
\]

\[
\prod_{i=1}^{n} \left( \int (\Delta^h \Delta^{-h} p_{r_i}(x)) \left( \Delta^h \Delta^{-h} p_{r'_i}(x) \right) \, dx \right) \prod_{i=1}^{n} dr_i \, dr'_i
\]

\[
\leq C \int_{[0, \sqrt{h}]^{2n}} \prod_{i=1}^{n} \left( \int (\Delta^h \Delta^{-h} p_{r_i}(x)) \, dx \right) \prod_{i=1}^{n} dr_i \, dr'_i
\]

\[
\leq C \left( \int \left( \int (\Delta^h \Delta^{-h} p_{r}(x)) \, dx \right)^2 \right)^{n-1}
\]

\[
\int \left\{ \int_{0}^{\infty} \int_{\sqrt{h}}^{\infty} (\Delta^h \Delta^{-h} p_{r_i}(x)) \left( \Delta^h \Delta^{-h} p_{r'_i}(x) \right) \, dr_i \, dr'_i \right\} \, dx
\]

\[
= Cc_{\psi,h,1}^{-1} \int \left\{ \int_{0}^{\infty} \int_{\sqrt{h}}^{\infty} (\Delta^h \Delta^{-h} p_{r_i}(x)) \left( \Delta^h \Delta^{-h} p_{r'_i}(x) \right) \, dr_i \, dr'_i \right\} \, dx,
\]

by (2.12). The integral in the final line of (3.73)

\[
\leq c_{\psi,h,1} - \int \left( \int_{0}^{\sqrt{h}} \Delta^h \Delta^{-h} p_{s}(x) \, ds \right)^2 \, dx.
\]

Therefore, it follows from Lemma 2.2 that the first integral in (3.73) is

\[
O(h^{(2\beta-1)n+\epsilon}),
\]

for some \(\epsilon > 0\).

Since \((\hat{R}_0 \times \cdots \times \hat{R}_K) \supseteq [0, \sqrt{h}]^{2n}\) for \(2n\sqrt{h} \leq 1\), it follows from (3.71) and the preceding sentence, that

\[
\int T_h(x; \pi, e) \prod_{j,k,i} dx_{j,k,i}
\]

\[
= \int_{[0, \sqrt{h}]^{2n}} F(\bar{\sigma}, \sum_{q=1}^{n_0} r_{0,2q}, \ldots, \sum_{q=1}^{n_K} r_{K,2q}) \prod_{i=1}^{n} \left( \int (\Delta^h \Delta^{-h} p_{r_i}(x)) \right)
\]

\[
\left( \Delta^h \Delta^{-h} p_{r'_i}(x) \right) \, dx \prod_{l=0}^{K} \prod_{q=1}^{n_l} dr_{l,2q} + O(h^{(2\beta-1)n+\epsilon}).
\]

We use the next lemma which is proved in Subsection 3.3.

**Lemma 3.2** For any fixed \(m\) and \(s_0, \ldots, s_K \leq m\sqrt{h}\), there exists an \(\epsilon > 0\) such that for all \(h > 0\), sufficiently small,

\[
|F(\bar{\sigma}, s_0, \ldots, s_K) - F(\bar{\sigma}, 0, \ldots, 0)| \leq Ch^{\epsilon}.
\]

(3.76)
Proof of Lemma 3.1 continued It follows from (3.75) and Lemmas 3.2 and 2.2, that
\[
\int T_h(x; \pi, e) \prod_{j,k,i} dx_{j,k,i} \quad (3.77)
\]
\[
= F(\tilde{\sigma}, 0, \ldots, 0) \int_{[0, \sqrt{h}^n]} \prod_{i=1}^n \left( \int (\Delta^h \Delta^{-h} p_{r_i}(x)) \right) \left( \Delta^h \Delta^{-h} p_{r_i}(x) \right) dx \prod_{l=0}^K \prod_{q=1}^{n_l} dr_{l,2q} + O(h^{(2\beta-1)n+\epsilon})
\]
\[
= (c_{\psi, h, 1})^n F(\tilde{\sigma}, 0, \ldots, 0) + O(h^{(2\beta-1)n+\epsilon}). \quad (3.78)
\]
We now use the notation introduced in the paragraph containing (3.69), and the fact that there are \(2^{2n}\) permutations that are compatible with \(P\), to see that
\[
\sum_{\pi \sim P} \int T_h(x; \pi, e) \prod_{j,k,i} dx_{j,k,i} \quad (3.79)
\]
\[
= (4c_{\psi, h, 1})^n \sum_{\tilde{\sigma} \in B} F(\tilde{\sigma}, 0, \ldots, 0) + O(h^{(2\beta-1)n+\epsilon}).
\]
Since \(|B| \leq (2n)!\), we see that the error term only depends on \(m\), (recall that \(m = 2n\)). Consider (3.79) and the definition of \(F(\tilde{\sigma}, 0, \ldots, 0)\) in (3.67) and use (3.18), with \(m_{j,k}\) replaced by \(n_{j,k}\), to see that
\[
\sum_{\pi \sim P} \int T_h(x; \pi, e) \prod_{j,k,i} dx_{j,k,i} \quad (3.80)
\]
\[
= (4c_{\psi, h, 1})^n E \left( \prod_{j,k=0}^K (\alpha_{j,k})^{n_{j,k}} \right) + O(h^{(2\beta-1)n+\epsilon}).
\]
Recall the definition of \(S\), to set of special pairings, given in the first paragraph of this subsection. Since there are \((2n)!_{n_{j,k}}\) pairings of the \(2n_{j,k}\) elements \(\{1, \ldots, m_{j,k}\}\) (recall that \(m_{j,k} = 2n_{j,k}\)), we see that when we sum over all the special pairings we get
\[
\sum_{P \in S} \sum_{\pi \sim P} \int T_h(x; \pi, e) \prod_{j,k,i} dx_{j,k,i} \quad (3.81)
\]
It follows from (3.38) that the error term, still, only depends on \( m \).

The right-hand side of (3.81) is precisely the desired expression in (3.3). Therefore, to complete the proof of Lemma 3.1, we show that for all the other possible values of \( a \), the integral in (3.28) can be absorbed in the error term.

### 3.2 \( a = e \) but not all cycles are of order two or \( a \neq e \)

**Lemma 3.3** Suppose that \( a = e \) but not all cycles are of order two or \( a \neq e \). Then

\[
\int T_h(x; \pi, a) \prod_{j,k,i} dx_{j,k,i} = O \left( \frac{h^\epsilon}{\psi^2(1/h)} \right)^n, \tag{3.82}
\]

for some \( \epsilon > 0 \).

In the rest of this section we ignore all factors of \( \log 1/h \).

**Proof** Consider the basic formula (3.32). Since we only need an upper bound, we take absolute values in the integrand and extend the time integral to \([0, 1]\), as we have done several times above. We take the time integral and get an upper bound for (3.32) involving the terms \( u, \psi \), and \( w \). Since \( a \neq e \), the number of \( w \) terms is less than \( 2n \).

We obtain (3.82) by dividing the \( u, \psi \), and \( w \) terms in \( T_h(x; \pi, a) \) into sets. Clearly, if a set contains \( k \) terms of the form \( w \) and \( k' \) terms of the form \( \psi \), there are \( 2k + k' \) difference operators \( \Delta_h \) associated with this set. There are no difference operators associated with sets of \( u \) terms.

Consider a set of two \( w \) terms that lies in a cycle of order two. There are four difference operators \( \Delta_h \) associated with this set. We show this set contributes a bound to (3.82) that is

\[
O \left( \frac{1}{\psi^2(1/h)} \right). \tag{3.83}
\]

(By contributes a bound we mean that this is what we get after we make an appropriate change of variables and integrate out the \( w \) terms in this
set.) Thus we may say that each difference operator in a cycle of order two contributes a bound of
\[ O \left( \left( \frac{1}{h^2(1/h)} \right)^{1/4} \right). \]  
(3.84)

We show that any set that has \( k > 0 \) associated difference operators except for a set of two \( w \) terms that forms a cycle of order two contributes a bound that is
\[ O \left( \left( \frac{1}{h^2(1/h)} \right)^{k/4} \right) h^\epsilon, \]  
(3.85)
for some \( \epsilon > 0 \).

There are \( 4n \) difference operators \( \Delta^h \), in \( T_h(x; \pi, a) \). Consequently unless the graph associated with \( T_h(x; \pi, a) \) consists solely of cycles of order two, we obtain (3.82).

As we construct the sets of \( u, v \) and \( w \) terms, we also choose a collection \( I \cup I' \) of \( m \) terms with arguments that are linearly independent. To bound the contribution of each set we bound all the terms not in \( I \cup I' \) by their supremum, and, after changing variables, integrate the terms in \( I \cup I' \). Using (2.5), (2.6), (2.8) and (2.9) we verify the bounds given in the preceding paragraph. (Actually, there is an exceptions to this rule which we also deal with.)

This is how we divide the \( u, v \) and \( w \) terms into sets. For each \( \pi \) and \( a \) we define a multigraph \( G_{\pi,a} \) with vertices \( \{(j,k,i) : 0 \leq j < k \leq K, 1 \leq i \leq m_{j,k}\} \), and an edge between the vertices \( \pi_l(q-1) \) and \( \pi_l(q) \) whenever \( a(l,q) = (1,1) \). This graph divides the \( w \) terms into cycles and chains. Assume that there are \( S \) cycles. We denote them by \( C_s = \{ \phi_{s,1}, \ldots, \phi_{s,l(s)} \} \), written in cyclic order, where the cycle length \( l(s) = |C_s| \geq 1 \) and \( 1 \leq s \leq S \). For each \( 1 \leq s \leq S \) we take the set of \( l(s) \) terms
\[ G_{\pi,a}^{\text{cycle}} = \{ w(x_{\phi_{s,2}} - x_{\phi_{s,1}}), \ldots, w(x_{\phi_{s,l(s)}} - x_{\phi_{s,l(s)-1}}), w(x_{\phi_{s,1}} - x_{\phi_{s,l(s)}}) \}. \]  
(3.86)

Let
\[ y_{\phi_{s,i}} = x_{\phi_{s,i}} - x_{\phi_{s,i-1}}, \quad i = 2, \ldots, l(s). \]  
(3.87)
It is easy to see that \( \{ y_{\phi_{s,i}} \mid i = 2, \ldots, l(s) \} \) are linearly independent. We put the corresponding \( w \) terms, \( w(x_{\phi_{s,2}} - x_{\phi_{s,1}}), \ldots, w(x_{\phi_{s,l(s)}} - x_{\phi_{s,l(s)-1}}) \)
into $\mathcal{I}$. (On the other hand, since
\[
\sum_{i=2}^{l(s)} y_{\phi_{s,i}} = -(x_{\phi_{s,1}} - x_{\phi_{s,l(s)}}),
\] (3.88)
we see that we can only extract $l(s) - 1$ linearly independent variables from the $l(s)$ arguments of $w$ for a given $s$.) A cycle of length 1 consists of a single point $\phi_{s,1} = \phi_{l(s),1}$ in the graph, so in this case
\[
\mathcal{G}_s^{\text{cycle}} = \{w(0)\}. \tag{3.89}
\]
We explain below how this can occur. Obviously, $w(0)$ is not put into $\mathcal{I}$.

Next, suppose there are $S'$ chains. We denote them by $C'_s = \{\phi'_{s,1}, \ldots, \phi'_{s,l'(s)}\}$, written in order, where $l'(s) = |C'_s| \geq 2$ and $1 \leq s \leq S'$. Note that there are $l'(s) - 1$, $w$ terms corresponding to $C'_s$. Then for each $1 \leq s \leq S'$ we form the set of $l'(s) + 1$ terms
\[
\mathcal{G}_s^{\text{chain}} = \{v(x_{\phi'_{s,1}} - x_{a(s)}), w(x_{\phi'_{s,2}} - x_{\phi'_{s,1}}), \ldots,
\]
\[
\cdots, w(x_{\phi'_{s,l'(s)}} - x_{\phi'_{s,l'(s)-1}}), v(x_{b(s)} - x_{\phi'_{s,l'(s)}})\}\]
where $v(x_{\phi'_{s,1}} - x_{a(s)})$ is the unique $v$ term associated with $\Delta^{x_{\phi'_{s,1}}}_h$, and similarly, $v(x_{b(s)} - x_{\phi'_{s,l'(s)}})$ is the unique $v$ term associated with $\Delta^{x_{\phi'_{s,l'(s)}}}_h$.
(This deserves further clarification. There may be other $v$ terms containing the variable $x_{\phi'_{s,1}}$. But there is only one $v$ term of the form
\[
\int_0^1 |\Delta^{x_{\phi'_{s,1}}}_h p_s(x_{\phi'_{s,1}} - u)| \, ds \tag{3.91}
\]
where $u$ is some other $x$ variable which we denote by $x_{a(s)}$. This is because one operator $\Delta^{x_{\phi'_{s,1}}}_h$ is associated with $w(x_{\phi'_{s,2}} - x_{\phi'_{s,1}})$ and there are precisely two operators $\Delta^{x_{\phi'_{s,1}}}_h$ in (3.82).

It is easy to see that variables $y_{\phi'_{s,i}} = x_{\phi'_{s,i}} - x_{\phi'_{s,i-1}}$, $i = 2, \ldots, l(s)$, are linearly independent. We put the $w$ terms, $w(x_{\phi'_{s,2}} - x_{\phi'_{s,1}}), \cdots, w(x_{\phi'_{s,l'(s)}} - x_{\phi'_{s,l'(s)-1}})$ into $\mathcal{I}$. We leave the $v$ terms in $\mathcal{G}_s^{\text{chain}}$ out of $\mathcal{I}$.

At this stage we emphasize that the terms we have put in $\mathcal{I}$ from all cycles and chains have linearly independent arguments. If fact, the set
\[
}\]
of $x$'s appearing in the different chains and the cycles are disjoint. This is obvious for the cycles and the interior of the chains since there are exactly two difference operators $\Delta_h^x$ for each $x$. It also must be true for the endpoints of the chains, since if this is not the case they could be made into larger chains or cycles.

For the same reason, if a $v$ term involving $\Delta_h^{x'}$ is not in any of the sets of chains, then $x'$ will not appear in the arguments of the terms that are put in $\mathcal{I}$ from all the cycles and chains.

Suppose, after considering the $w$ terms and the $v$ terms associated with the chains of $w$ terms, that there are $p$ pairs of $v$ terms left, each pair corresponding to difference operators $\Delta_h^{z_j}, j = 1, \ldots, p$. ($p$ may be 0). Let

$$\mathcal{Z} := \{z_1, \ldots, z_p\} \quad (3.92)$$

A typical $v$ term is of the form

$$v^{(j)}(z_j - u_{j'}) := v(z_j - u_{j'}) = \int_0^1 |\Delta_h^{z_j} p_i (z_j - u_{j'})| \, dt. \quad (3.93)$$

where $u_{j'}$ is some $x$ term. We use the superscript $(j)$ is to keep track of the fact that this $v$ term is associated with the difference operator $\Delta_h^{z_j}$. We distinguish between the variables $z_j$ and $u_{j'}$ by referring to $z_j$ as a marked variable. Note that if $u_{j'}$ is also in $\mathcal{Z}$, say $u_{j'} = z_k$, then $u_{j'}$ is also a marked variable but in a different $v$ term. (In this case, in $v^{(k)}(z_k - u_{k'})$, where $u_{k'}$ is some other $x$ variable.)

Thus $\mathcal{Z}$ is the collection of marked variables. Consider the corresponding terms

$$v^{(j)}(z_j - u_j) \quad \text{and} \quad v^{(j)}(z_j - v_j), \quad j = 1, \ldots, p \quad (3.94)$$

where $u_j$ and $v_j$ represent whatever terms $x$ and $x'$ are coupled with the two variables $z_j$.

There may be some $j$ for which $u_j$ and $v_j$ in (3.94) are both in $\mathcal{Z}$. Choose such a $j$. Suppose $u_j = v_j = z_k$. We set

$$\mathcal{G}_j^{\mathcal{Z},1} = \{v^{(j)}(z_j - z_k), v^{(j)}(z_j - z_k), v^{(k)}(z_k - u_k), v^{(k)}(z_k - v_k)\} \quad (3.95)$$

and put $v^{(j)}(z_j - z_k)$ into $\mathcal{I}$. Here $u_k$ and $v_k$ are whatever two variables appear with the two marked variables $z_k$. 
On the other hand, suppose \( u_j \) and \( v_j \) are both in \( \mathcal{Z} \) but \( u_j = z_k \) and \( v_j = z_l \) with \( k \neq l \). We set
\[
G^2_j = \{ v^{(j)}(z_j - z_k), v^{(j)}(z_j - z_l), v^{(k)}(z_k - u_k), v^{(k)}(z_k - v_k), v^{(l)}(z_l - u_l), v^{(l)}(z_l - v_l) \}
\]
and put both \( v^{(j)}(z_j - z_k) \) and \( v^{(j)}(z_j - z_l) \) into \( \mathcal{I} \).

We then turn to the elements in \( \mathcal{Z} \) which have not yet appeared in the arguments of the terms that have been put into \( \mathcal{I} \). If there is another \( j' \) for which \( u_j' \) and \( v_j' \) are both in \( \mathcal{Z} \), choose such a \( j' \) and proceed as above. If there are no longer any such elements in \( \mathcal{Z} \), choose some remaining element, say, \( z_i \). Set
\[
G^3_i = \{ v^{(i)}(z_i - u_i), v^{(i)}(z_i - v_i) \}
\]
and if \( u_i \not\in \mathcal{Z} \), place \( v^{(i)}(z_i - u_i) \) into \( \mathcal{I} \). If \( u_i \in \mathcal{Z} \), so that \( v_i \not\in \mathcal{Z} \), place \( v^{(i)}(z_i - v_i) \) into \( \mathcal{I} \).

We then continue until we have exhausted \( \mathcal{Z} \). We form a final set \( G^u \) which contains all the \( u \) terms, so that all \( u, v \) and \( w \) terms have been divided into sets.

It is possible that there are no cycles of length one. We show how we get (3.82) in this case.

We have constructed \( \mathcal{I} \) so that all its members have linearly independent arguments. However, \( \mathcal{I} \) may contain less than \( m \) terms. We simply add to \( \mathcal{I} \) a set \( \mathcal{I}' \) of enough of the remaining \( u \) and \( v \) terms so that \( \mathcal{I} \cup \mathcal{I}' \) has \( m \) terms, whose arguments span \( R^{2n} \), the space spanned by the original \( x \) terms. (It follows from (3.88) that no further \( w \) terms can be added to \( \mathcal{I}' \)). We bound the \( v \) terms in \( \mathcal{I}' \) as follows:
\[
|v(x' - x'')| \leq \frac{C}{h^2(1/h)(1 + (x' - x'')^2)}.
\]

We then make a change of variables setting the arguments of the terms in \( \mathcal{I} \cup \mathcal{I}' \) equal to \( y_1, \ldots, y_m \) and bound the \( v \) terms not in \( \mathcal{I} \cup \mathcal{I}' \) by \( C(h^2(1/h))^{-1} \) and the \( u \) terms not in \( \mathcal{I} \cup \mathcal{I}' \) by \( C \). Finally we integrate. We have \( m \) one dimensional integrals which we bound by (2.6) for the \( v \) terms in \( \mathcal{I} \), by \( C(h^2(1/h))^{-1} \) for the \( v \) terms in \( \mathcal{I}' \), and by (2.3) for \( w \) terms in \( \mathcal{I} \). The integrals of the \( u \) terms in \( \mathcal{I} \) we bound by a constant; (see (2.3)).
Clearly $G^u$ gives a bounded contribution. We now show that (3.85) holds for all other sets of $v$ and $w$ terms, with the exception of sets of $w$ terms in cycles of length 2.

Consider first $G^\text{cycle}$ for a cycle of lengths $l(s)$. We integrate the $l(s) - 1$, $w$ terms which were put in $I$ and bound the remaining $w$ term by $C(h\psi(1/h))^{-1}$ to obtain the bound

\[
C \left( \frac{1}{\psi(1/h)} \right)^{l(s)-1} \frac{1}{h\psi(1/h)} = C \left( \frac{1}{\psi(1/h)} \right)^{l(s)-2} \frac{1}{h\psi^2(1/h)}.
\] (3.99)

Since

\[
\frac{1}{\psi(1/h)} = h^{1/2} \left( \frac{1}{h\psi^2(1/h)} \right)^{1/2}
\] (3.100)

(3.99) is bounded by

\[
C \left\{ h^{(l(s)-2)/2} \right\} \left( \frac{1}{h\psi^2(1/h)} \right)^{l(s)/2}.
\] (3.101)

Since a cycle of length $l(s)$ involves $2l(s)$ difference operators $\Delta_h$, and $l(s)/2 = 2l(s)/4$, we are in the situation of (3.85), unless all cycles are of order two. (This shows, incidentally, that when $a = e$, (3.82) holds unless all cycles are of order two.)

Consider next $G^\text{chain}$. Recall that there are $l'(s) - 1$, $w$ terms in a chain, where $l'(s) \geq 2$. We have put all $l'(s) - 1$ terms $w$ in $I$, and we can bound their integrals by

\[
C \left( \frac{1}{\psi(1/h)} \right)^{l'(s)-1}.
\] (3.102)

In addition there are two $v$ terms in $G^\text{chain}$. The ones not in $I'$ can be bounded by $C(h\psi(1/h))^{-1}$ and the ones in $I'$ are bounded by (3.98), which after integration also contributes $C(h\psi(1/h))^{-1}$. Thus we obtain the following bound for for $G^\text{chain}$:

\[
C \left( \frac{1}{\psi(1/h)} \right)^{l'(s)-1} \left( \frac{1}{h\psi(1/h)} \right)^2 = C \left( \frac{1}{\psi(1/h)} \right)^{l'(s)-3} \left( \frac{1}{h\psi^2(1/h)} \right)^2
\] (3.103)

\[
\leq C h^{(l'(s)-3)/2} \left( \frac{1}{h\psi^2(1/h)} \right)^{1/2} \left( \frac{1}{h\psi^2(1/h)} \right)^{l'(s)/2}.
\]
Note that each chain of length \( l'(s) \) together with the two \( v \) terms associated with the end points involves \( 2l'(s) \) difference operators \( \Delta_h \). Clearly if \( l'(s) \geq 3 \) we are in the situation of (3.85). This holds even for chains of length \( l'(s) = 2 \) since

\[
h^{(2-3)/2} \left( \frac{1}{h\psi^2(1/h)} \right)^{1/2} = \frac{1}{h\psi(1/h)}.
\]

(3.104)

Note that the \( v \) terms that were not initially in \( I \) contribute a bound of \( C(h\psi(1/h))^{-1} \), whether or not they are placed in \( I' \). We continue to use this fact below without commenting on it further.

We next consider \( G_j^{Z,1} \). We integrate the one \( v \) term in \( I \) and any that are in \( I' \) and bound the remaining ones. This gives a bound of

\[
\frac{1}{h^2\psi^3(1/h)} = \frac{1}{h\psi(1/h)} \left( \frac{1}{h\psi^2(1/h)} \right).
\]

(3.105)

Since \( G_j^{Z,1} \) involves four \( \Delta_h \) operators we are in the situation of (3.85).

For \( G_j^{Z,2} \) we integrate two \( v \) terms in \( I \) and any that are in \( I' \) and bound the remaining ones. This gives a bound of

\[
\left( \frac{1}{h\psi^2(1/h)} \right)^2 = \left( \frac{1}{h\psi^2(1/h)} \right)^{1/2} \left( \frac{1}{h\psi^2(1/h)} \right)^{3/2};
\]

(3.106)

Since \( G_j^{Z,2} \) involves six \( \Delta_h \) operators we are in the situation of (3.85).

Finally, for \( G_j^{Z,3} \) we integrate the one \( v \) term in \( I \) and the other if it is in \( I' \). Otherwise we bound it. This gives a bound of

\[
\frac{1}{\psi(1/h)} = h^{1/2} \left( \frac{1}{h\psi^2(1/h)} \right)^{1/2};
\]

(3.107)

Since \( G_j^{Z,3} \) involves two \( \Delta_h \) operators we are in the situation of (3.85).

This shows that if \( a \) and the partition \( \pi \) does not generate exclusively \( w \) terms in cycles of order two and are such that there are no cycles of length one, then (3.82) holds.

We now remove the restriction that \( a \) and \( \pi \) does not give rise to cycles of length one. The only way this anomaly can occur is in terms of the type

\[
\Delta_h^2 \Delta_h^{-2} p_{(1-\sum_{q=1}^{n_{i-1}} r_{i-1,q})+r_{i,1}} (x_{r_{2q-1(i-1)}-1} - x_{r_{2q-1(i-1)-1}}) \]

(3.108)
when $\gamma_{2\sigma_l(1)} - \gamma_{2\sigma_l-1(n_l-1)}$. Note that in this case
\[ \int_0^t \Delta^h \Delta^{-h} p_s(x, 2\gamma_{2\sigma_l(1)} - x, 2\gamma_{2\sigma_l-1(n_l-1)})\, ds = w(0). \quad (3.109) \]
This is what we call a cycle of length one. In this case we have
\[ \Delta^h \Delta^{-h} p_s(1 - \sum_{q=1}^{m_l-1} r_{l-1,q} + r_{l,1}, 0) = -2\Delta^h p_s(1 - \sum_{q=1}^{m_l-1} r_{l-1,q} + r_{l,1}, 0). \quad (3.110) \]

We now show how to deal with (3.110). We return to the basic formulas (3.31) and (3.32). We obtain an upper bound for (3.32) by taking the absolute value of the integrand. However, we do not, initially extend the region of integration with respect to time. Instead we proceed as follows: Let $l'$ be the largest value of $l$ for which (3.110) occurs. We extend the integral with respect to $r_{l,q}$ for all $l > l'$, and also for $l = l'$ and $q > 1$, and bound these integrals with terms of the form $u$, $v$ and $w$. We then consider the integral of the term in (3.110) with respect to $r_{l',1}$.

Clearly
\[ \int_0^1 |\Delta^h p_s(1 - \sum_{q=1}^{m_{l'-1}} r_{l'-1,q} + r_{l',1}, 0)|\, dr_{l',1} \leq \int_1^2 |\Delta^h p_s(0)|\, ds \]
If $\sum_{q=1}^{m_{l'-1}} r_{l'-1,q} \leq 1/2$ this last integral
\[ \leq \int_{1/2}^2 |\Delta^h p_s(0)|\, ds \leq Ch^2 \quad (3.112) \]
by (2.17). Since we have only used two $\Delta^h$ operators we are in the situation of (3.85).

If $\sum_{q=1}^{m_{l'-1}} r_{l'-1,q} \geq 1/2$ then for some $q'$ we have $r_{l'-1,q'} \geq 1/2m$. Note that the variable $r_{l'-1,q'}$ appears in (3.108) and in only one other term. If $q' > 1$, then using the fact that $r_{l'-1,q'} \geq 1/2m$, we use one of the bounds in Lemma 2.4, to bound a term which in the non-exceptional case would be $u$, $v$ or $w$, or their integrals with respect to $x$, by $\Pi_{1/2m}$, $\Pi_{1/2m}$ or $\Pi_{1/2m}$, or their integrals with respect to $x$. One sees from Lemma 2.1 that we don’t loose anything in comparison with the non-exceptional case. The case $r_{l'-1,1} \geq 1/2m$ and $\gamma_{2\sigma_{l'-1}(1)-1} \neq \gamma_{2\sigma_{l'-2}(n_{l'-2})-1}$ is handled the same way.

On the other hand if $r_{l'-1,1} \geq 1/2m$ and $\gamma_{2\sigma_{l'-1}(1)-1} = \gamma_{2\sigma_{l'-2}(n_{l'-2})-1}$, we use Lemma 2.3 to get the same bound of $Ch^2$. 

$L^2$ moduli of continuity of local times
After completing the procedure described in the previous two paragraphs we integrate in (3.32) with respect to $r_{\nu-1,q'}$ and $r_{\nu,1}$, since these variables now appear only in the term in (3.108). What we are left with is bounded by

$$\int_{1/2m}^{1} \int_{0}^{1} \left| \Delta^h p_{(1-\sum_{q=1}^{\mu_{\nu-1}} r_{\nu-1,q') + r_{\nu,1} - 1} q' + 1, r_{\nu-1,q'})} \right| \, dr_{\nu,1} \, dr_{\nu-1,q'} \quad (3.113)$$

Let $\alpha = 1 - \sum_{q \neq q'} r_{\nu-1,q}$. We make the change of variables $r = r_{\nu,1}$ and $s = -r_{\nu-1,q'} + \alpha$ to get that (3.113)

$$\leq \int_{0}^{1} \int_{0}^{1} \left| \Delta^h p_{r+s}(0) \right| \, dr \, ds \quad (3.114)$$

$$\leq \int_{0}^{2} r |\Delta^h p_{r}(0)| \, dr \leq C \int_{0}^{2} r \left( \int \sin^2(p h) e^{-r \psi(p)} \, dp \right) \, dr$$

$$\leq Ch^\beta \int_{0}^{2} \left( \int \frac{p^\beta e^{-r \psi(p)}}{1 + \psi^2(p)} \, dp \right) \, dr = O(h^\beta).$$

Since

$$h^\beta = h^{\beta+1/2} \psi(1/h) \left( \frac{1}{h \psi^2(1/h)} \right)^{1/2} \quad (3.115)$$

we are once again in the situation of (3.85).

We then apply a similar procedure for each $l$ in decreasing order, skipping those for which (3.110) occurs, if they were already bounded by the procedure described in the paragraph preceding the one containing (3.113). Thus we see that cycles of length one are in the situation of (3.85). We proceed to deal with remaining terms as we did when we assumed that there were no cycles of length one and see that (3.82) holds. This completes the proof of Lemma 3.3.

It follows from (3.81) and Lemma 3.3 that when $m$ is even

$$\sum_{a} \sum_{\pi_0, \ldots, \pi_K} \int T_h(x; \pi, a) \prod_{j,k,i} \, dx_{j,k,i} \quad (3.116)$$

$$= \prod_{j,k=0}^{K} \frac{(2n_{j,k})!}{2^{n_{j,k}, n_{j,k}}} (4c_{\psi, h, 1})^{n_{j,k}} E \left\{ \prod_{j,k=0}^{K} (\alpha_{j,k})^{n_{j,k}} \right\} + O\left(h^{(2\beta-1)n+\epsilon}\right).$$
We now show that we get the same estimates when $T_h(x; \pi, a)$ is replaced by $T_h'(x; \pi, a)$; (see (3.29) and (3.31)).

We point out, in the paragraph containing (3.27) that terms of the form $\Delta^h \Delta^{-h} p_r^\sharp$ in (3.29) are always of the form $\Delta^h \Delta^{-h} p_r$. Therefore, in showing that (3.29) and (3.31) have the same asymptotic behavior as $h \to 0$ we need only consider how the proof of (3.116) must be modified when the arguments of the density functions with one or no difference operators applied is effected by adding $\pm h$.

It is easy to see that the presence of these terms has no effect on the integrals that are $O \left( h^{(2j-1)m+\epsilon} \right)$ as $h \to 0$. This is because in evaluating these expressions we either integrate over all of $R^1$ or else use bounds that hold on all of $R^1$. Since terms with one difference operator only occur in these estimations, we no longer need to be concerned with them.

Consider the terms with no difference operators applied to them, now denoted by $p_r^\sharp$. So, for example, instead of $F(\bar{\sigma}, 0, \ldots, 0)$ on the right-hand side of (3.77), we now have

$$\int \left( \int_{\bar{R}_o(0) \times \cdots \times \bar{R}_K(0)} \prod_{l=0}^K p_r^\sharp \left( 1 - \sum_{q=1}^{s_l-1} r_l-1, 2q-1 - s_{l-1} + r_l, 1 \right) \right) \left( x_{\sigma_l(1)} - x_{\sigma_{l-1}(n_{l-1})} \right)^{n_l} p_r \left( x_{\sigma_l(q)} - x_{\sigma_l(q-1)} \right) dx.$$  \hspace{1cm} (3.117)

Suppose that $p_r^\sharp(y_{\sigma(i)} - y_{\sigma(i-1)}) = p_r(y_{\sigma(i)} - y_{\sigma(i-1)} \pm h)$. We write this term as

$$p_r^\sharp(y_{\sigma(i)} - y_{\sigma(i-1)}) = p_r(y_{\sigma(i)} - y_{\sigma(i-1)}) + \Delta^\pm h p_r(y_{\sigma(i)} - y_{\sigma(i-1)}).$$  \hspace{1cm} (3.118)

Substituting all such terms into (3.117) and expanding we get (3.116) and many other terms with at least one $p_r(y_{\sigma(i)} - y_{\sigma(i-1)})$ replaced by $\Delta^\pm h p_r(y_{\sigma(i)} - y_{\sigma(i-1)})$. In this case simply take these terms, extend their integrals to $[0, 1]$ and bound them as in (2.5). Then follow the procedure in the paragraph containing (3.70) to deal with the remaining terms and the functions $1/(1 + (y_{\sigma(i)} - y_{\sigma(i-1)})^2)$. In this the integral in (3.117) is bounded by $C(1/(h^j(1/h)))^2$, where $j$ is the number of terms that have the difference operator applied. Thus we see that replacing $T_h(x; \pi, a)$ by $T'_h(x; \pi, a)$ does not change (3.116) when $m$ is even.

When $m$ is odd we can not construct a graph with all cycles of order 2. Therefore, we are not in the situation covered by Section 3.1. Moreover,
in Section 3.2 we never use the fact that $m$ is even. We actually obtain (3.82) with $n$ replaced by $m/2$, which is what we assert in (3.4). This also holds when $p$ is replaced by $p^\ast$ for the reasons given in the preceding two paragraphs.

3.3 Proof of Lemma 3.2

For any $A \subseteq [0, 3]^n$ we set

$$F_A = \int \left\{ \int_A \prod_{l=0}^K p_{r_{l,1}}(x_{\bar{\sigma}_1(1)} - x_{\bar{\sigma}_1(n_l-1)}) \right. \left. \prod_{q=2}^{n_l} p_{r_{l,2q-1}}(x_{\bar{\sigma}_1(q)} - x_{\bar{\sigma}_1(q-1)}) \prod_{l=0}^K \prod_{q=1}^{n_l} dr_{l,2q-1} \right\}^{\frac{n_l}{n}} \prod_{q=1}^{n_l} dx_{\bar{\sigma}_1(q)}.$$  

Then by Hölder’s inequality, for any $1/a + 1/b = 1$

$$\left\{ \int_A \prod_{l=0}^K p_{r_{l,1}}(x_{\bar{\sigma}_1(1)} - x_{\bar{\sigma}_1(n_l-1)}) \right. \left. \prod_{q=2}^{n_l} p_{r_{l,2q-1}}(x_{\bar{\sigma}_1(q)} - x_{\bar{\sigma}_1(q-1)}) \prod_{l=0}^K \prod_{q=1}^{n_l} dr_{l,2q-1} \right\}^{\frac{n_l}{n}} \prod_{q=1}^{n_l} dx_{\bar{\sigma}_1(q)} \leq |A|^{1/a} \left\{ \int_{[0,3]^n} \prod_{l=0}^K p_{r_{l,1}}^b(x_{\bar{\sigma}_1(1)} - x_{\bar{\sigma}_1(n_l-1)}) \right. \left. \prod_{q=2}^{n_l} p_{r_{l,2q-1}}^b(x_{\bar{\sigma}_1(q)} - x_{\bar{\sigma}_1(q-1)}) \prod_{l=0}^K \prod_{q=1}^{n_l} dr_{l,2q-1} \right\}^{1/b},$$

where $|A|$ denotes the volume of $A$ in $R^n_\ast$.

Since $\beta > 1$ we can choose a $1 < b < \beta$ such that

$$\int_0^3 \left( \psi^{-1}(1/s) \right)^b ds \leq C.$$  

Therefore it follows from (2.2) that

$$\int_0^3 p_r^b(x) dr \leq C \frac{1}{1+x^2}.$$  

(3.122)
Thus there exists a finite constant $C$, depending only on $n$ and $b$, that is independent of $A$, such that

$$F_A \leq C|A|^{1/a}. \quad (3.123)$$

It follows from (3.67), paying special attention to the time variable of $p$ in the second line, that

$$F(\sigma, s_0, \ldots, s_K) = F_{A_{s_0, \ldots, s_K}} \quad (3.124)$$

where

$$A_{s_0, \ldots, s_K} = \left\{ r \in \mathbb{R}^n : \sum_{\lambda=0}^{l-1} \left(1 - \sum_{q=1}^{n_\lambda} r_{\lambda,2q-1} - s_\lambda\right) \leq \sum_{q=1}^{n_l} r_{l,2q-1} \right\} \quad (3.125)$$

In particular

$$A_{0, \ldots, 0} = \left\{ r \in [0,3]^n : \sum_{\lambda=0}^{l-1} \left(1 - \sum_{q=1}^{n_\lambda} r_{\lambda,2q-1}\right) \leq \sum_{q=1}^{n_l} r_{l,2q-1} \right\} \quad (3.126)$$

Let $\phi_l(r) = \sum_{l=0}^{l} (1 - \sum_{q=1}^{n_\lambda} r_{\lambda,2q-1})$. We have

$$A_{s_0, \ldots, s_K} \Delta A_{0, \ldots, 0} \quad (3.127)$$

(Note that the first union are the points in $A_{s_0, \ldots, s_K}$ that are not in $A_{0, \ldots, 0}$ and the second union are the points in $A_{0, \ldots, 0}$ that are not in $A_{s_0, \ldots, s_K}$.)

Since for fixed $a \geq b \geq 0$

$$\left\{ r \in [0,3]^n : a - b \leq \sum_{q=1}^{n_l} r_{l,2q-1} \leq a \right\} \leq Cb^n \quad (3.128)$$
we have that

\[ |A_{s_0,\ldots,s_K} \Delta A_{0,\ldots,0}| \leq CK \left( \sum_{\lambda=0}^{K} s_\lambda \right)^{2n} \]  \tag{3.129}

\[ \leq CK^{m+1} \left( \max_{0 \leq \lambda \leq K} s_\lambda \right)^m, \]

when \( \max_{0 \leq \lambda \leq K} s_\lambda \) is sufficiently small. Let \( K' \) be the cardinality of \( \{l | n_l > 0\} \). It is easy to see that we have actually proved (3.129) with \( K \) replaced by \( K' \). Since \( K' \leq m \), the last line in (3.129) can be written in terms of \( \{s_\lambda\} \) and \( m \). Lemma 3.2 follows from (3.123) and (3.129).

\section{4 Proof of Theorem 1.2}

The proof of Theorem 1.2 follows easily from the preliminary material in Lemmas 4.2–4.5.

Let

\[ I_{j,k,l,h} := \int \left( L_{1/l}^{x+h} - L_{1/l}^{x} \right) \circ \theta_{j/l} \left( L_{1/l}^{x+h} - L_{1/l}^{x} \right) \circ \theta_{k/l} \, dx \]  \tag{4.1}

and

\[ \tilde{I}_{l,h} := \sum_{j,k=0}^{t-1} I_{j,k,l,h}. \]  \tag{4.2}

Using the additivity property of local time we can write

\[ L_1^{x} = \sum_{j=0}^{t-1} L_{1/l}^{x} \circ \theta_{j/l}. \]  \tag{4.3}

so that

\[ \int (L_{1/l}^{x+h} - L_1^{x}) \left( L_{1/l}^{x+h} - L_1^{x} \right) \, dx = \sum_{j,k=0}^{t-1} I_{j,k,l,h} = 2\tilde{I}_{l,h} + \sum_{j=0}^{t-1} I_{j,j,l,h} \]  \tag{4.4}

Similarly, let

\[ \alpha_{j,k,l} := \int L_{1/l}^{x} \circ \theta_{j/l} L_{1/l}^{x} \circ \theta_{k/l} \, dx. \]  \tag{4.5}
and
\[ \tilde{\alpha}_t := \sum_{j, k=0}^{l-1} \alpha_{j,k,l}, \] (4.6)

Thus
\[ \alpha_1 = \int L_1^x L_1^x \ dx = \sum_{j, k=0}^{l-1} \alpha_{j,k,l} = 2\tilde{\alpha}_t + \sum_{j=0}^{l-1} \alpha_{j,j,l}. \] (4.7)

The main ingredient in the proof of Theorem 1.2 is Lemma 4.2 below. We use the following lemma in its proof.

**Lemma 4.1** Under the hypotheses of Theorem 1.2, for \( \alpha_t \) as defined in (1.4), there exists an \( \epsilon > 0 \) and a \( t_0 := t_0(\epsilon) \) such that for all integers \( n > 0 \)
\[ \| \alpha_t \|_n \leq C_n t^2 \psi^{-1}(1/t), \] (4.8)
for all \( 0 < t \leq t_0 \) and some constant \( C_n \) depending only on \( n \). (Note that \( \alpha_t \) is given in (1.4) and \( \| \cdot \|_n := (E(\cdot)^n)^{1/n} \) is defined in (1.4).

In addition
\[ \lim_{t \to \infty} E(\tilde{\alpha}_t)^n = E(\alpha_1/2)^n. \] (4.9)

**Proof** By Kac’s moment formula
\[ E \{(\alpha_t)^n\} = E \left( \left( \int (L_t^x)^2 \ dx \right)^n \right) \] (4.10)
\[ = 2^n \sum_\pi \int \int \left( \sum_{i=1}^{2n} r_i \leq t \right) \prod_{i=1}^{2n} p_{r_i}(x_{\pi(i)} - x_{\pi(i-1)}) \prod_{i=1}^{2n} d\tilde{r}_i \prod_{i=1}^{n} dx_i, \]
where the sum runs over all maps \( \pi : [1, 2n] \to [1, n] \) with \( |\pi^{-1}(i)| = 2 \) for each \( i \). The factor \( 2^n \) comes from the fact that we can interchange each \( x_{\pi(i)} \) and \( x_{\pi(i-1)} \), \( i = 1, \ldots, 2n \).

It is not difficult to see that we can find a subset \( J = \{i_1, \ldots, i_n\} \subseteq [1, 2n] \), such that each of \( x_1, \ldots, x_n \) can be written as a linear combination of \( y_j := x_{\pi(i_j)} - x_{\pi(i_{j-1})}, \ j = 1, \ldots, n \). For \( i \in J^c \) we use the bound \( p_{r_i}(x_{\pi(i)} - x_{\pi(i-1)}) \leq p_{r_i}(0) \), then change variables and integrate out the \( y_j \), to see that
\[ \int \left( \prod_{i=1}^{2n} \int_0^t p_{r_i}(x_{\pi(i)} - x_{\pi(i-1)}) \ dr_i \right) \prod_{i=1}^{n} dx_i \] (4.11)
\[ L^2 \text{ moduli of continuity of local times} \]

\[ \leq C \left( \int_0^t p_r(0) \, dr \right)^n \left( \prod_{i \in J} \int_0^t p_r(x_{\pi(i)} - x_{\pi(i-1)}) \, dr_i \right) \prod_{i=1}^n dx_i \]

\[ \leq C \left( \int_0^t p_r(0) \, dr \right)^n \left( \prod_{i \in J} \int_0^t p_r(y_i) \, dr_i \, dy_i \right) \]

\[ = Ct^n \left( \int_0^t p_r(0) \, dr \right)^n \leq C \left( t^2 \psi^{-1}(1/t) \right)^n, \]

where we use (2.4) and (5.7) for the last line. This gives (4.8).

To obtain (4.9) we use (4.8) to see that for sufficiently large,

\[ \|2\tilde{\alpha}_n - \alpha_1\|_n \leq \|2\tilde{\alpha}_n - \alpha_1\|_n = \| \sum_{j=0}^{\ell-1} \alpha_{i,j,l} \|_n \]

\[ \leq \ell \| \alpha_{0,0,l} \|_n = \ell \| \alpha_{1/l} \|_n \]

\[ \leq C \psi^{-1}(l). \]

Lemma 4.2 Under the hypotheses of Theorem 1.2 and with \( l = l(h) = \lceil \log 1/h \rceil \),

\[ \lim_{h \to 0} 2\sqrt{h \psi^2(1/h) I_{t,h}} \Rightarrow (8c_{\beta,1})^{1/2} \sqrt{\alpha_1} \eta. \]

Proof We show that for each \( m \)

\[ \lim_{h \to 0} E \left( \left( 2\sqrt{h \psi^2(1/h) I_{t,h}} \right)^m \right) = \begin{cases} \frac{(2n)!}{2^n n!} (8c_{\beta,1})^n E \{ (\alpha_1)^n \} & \text{if } m = 2n \\ 0 & \text{otherwise}. \end{cases} \]

(4.14)

It follows from [4] (6.12) that for the \( \beta \)-stable process, with \( \beta > 1 \),

\[ E \left\{ \left( \int (L_t^2)^2 \, dx \right)^n \right\} \leq C^m ((2n)!)^{1/(2\beta)}. \]

(4.15)

When \( \psi \) is regularly varying at infinity with index \( \beta \), for all \( \epsilon > 0 \), there exists a constant \( D = D_\epsilon \) such that

\[ \int_0^\infty e^{-s\psi(p)} \, dp \leq C \left( 1 + \int_1^\infty e^{-sDp^{\beta-\epsilon}} \, dp \right). \]

(4.16)
Using this, and the same proof as in [4], one can show that (4.15) holds, with $\beta$ replaced by $\beta - \epsilon$ for any $\epsilon > 0$, when $\psi$ is regularly varying at infinity with index $\beta$. Therefore, since $\sqrt{(2^n)!} \leq 2^n n!$, the right-hand side of (4.14), which is the $2^n$-th moment of $(8c_{\psi,1})^{1/2} \sqrt{\alpha_1 \eta}$, is bounded above by $C^n ((2^n)!)^{(\beta+1-\epsilon)/(2(\beta-\epsilon))}$. This implies that the weak limit $(8c_{\psi,1})^{1/2} \sqrt{\alpha_1 \eta}$ is determined by its moments; (see [5, p. 227-228]). Therefore, by the method of moments, (4.13) follows from (4.14).

We now obtain (4.14). Considering (2.13) and (4.9) it suffices to show that

$$E((\tilde{I}_{l,h})^m) = \begin{cases} \frac{(2n)!}{2^n n!} (4c_{\psi,h,1})^n E((\tilde{\alpha}_l)^n) & m = 2n \\ O(h^\epsilon (h\psi^2(1/h))^{-n}) & \text{otherwise.} \end{cases}$$

(4.17)

Using the multinomial theorem we have

$$E((\tilde{I}_{l,h})^m) = \sum_{\tilde{m} \in \mathcal{M}} \left( \frac{m!}{\prod_{j,k=0}^{l-1} (m_{j,k})!} \right) E \left( \prod_{j,k=0}^{l-1} (I_{j,k,l,h})^{m_{j,k}} \right)$$

(4.18)

where

$$\mathcal{M} = \left\{ \tilde{m} = \{m_{j,k}, 0 \leq j < k \leq l-1\} \mid \sum_{j,k=0}^{l-1} m_{j,k} = m \right\}.$$  

(4.19)

We now use Lemma 3.1 to compute the expectation on the right-hand side of (4.18). Even though Lemma 3.1 is proved for time intervals of length 1, (see 3.1), it is straightforward to check that it holds for any fixed time interval, if the term $\alpha_{j,k}$, in (3.2), is altered to reflect the new length. Therefore, for some $\epsilon > 0$

$$E((\tilde{I}_{l,h})^m) = \sum_{\tilde{m} \in \mathcal{M}} \left( \frac{m!}{\prod_{j,k=0}^{l-1} (m_{j,k})!} \right) \prod_{j,k=0}^{l-1} \frac{(2n_{j,k})!}{2^{n_{j,k}} (n_{j,k})!} (4c_{\psi,h,1})^{n_{j,k}} E \left( \prod_{j,k=0}^{l-1} (\alpha_{j,k,l})^{n_{j,k}} \right) + O \left( m^\epsilon h^{(2\beta-1)n+\epsilon} \right).$$

(4.20)
when \( m_{j,k} = 2n_{j,k} \) for all \( j \) and \( k \), and is \( O\left(l^m h^{(2\beta - 1)n + \epsilon}\right) \) if any of the \( m_{j,k} \) are odd. Here we use the fact that

\[
\sum_{m \in \mathcal{M}} \left( \frac{m!}{\prod_{j<k}^{l-1} m_{j,k}!} \right) = l^m \tag{4.21}
\]

to compute the error term. (Lemma 3.1 is for a fixed partition of \( m \). Here we include the factor \( l^m \), to account for the number of possible partitions. Note that \( l \) is a function of \( h \).)

When \( m_{j,k} = 2n_{j,k} \) for all \( j \) and \( k \),

\[
\left( \frac{m!}{\prod_{j<k}^{l-1} m_{j,k}!} \right) \prod_{j<k}^{l-1} \frac{(2n_{j,k})!}{2^n n_{j,k}!} = \frac{(2n)!}{2^n n!} \prod_{j<k}^{l-1} \frac{n!}{n_{j,k}!}. \tag{4.22}
\]

Using this in (4.20) we get

\[
E\left((\tilde{I}_{l,h}^m)^m\right) = \frac{(2n)!}{2^n n!} (4c_{\psi,h,1})^n \sum_{\mathcal{N}} \left( \frac{n!}{\prod_{j<k}^{l-1} n_{j,k}!} \right) E\left\{ \prod_{j<k}^{l-1} (\alpha_{j,k,l})^{n_{j,k}} \right\}
+ O\left(l^m h^{(2\beta - 1)n + \epsilon}\right),
\]

where \( \mathcal{N} \) is defined similarly as \( \mathcal{M} \). Using the multinomial theorem as in (4.18) we see that the sum in (4.23) is equal to \( E\{ (\tilde{\alpha})^n \} \), which completes the proof of (4.17). \( \square \)

The next three lemmas give estimates for the mean and variance of \( \int (L_{x+h}^+ - L_x^+)^2 \, dx \). They are proved in Section 6.

Let

\[
c_{\psi,h,0} := \int_0^\infty (p_s(0) - p_s(h)) \, ds. \tag{4.24}
\]

**Lemma 4.3** Under the hypotheses of Theorem 1.2,

\[
\lim_{h \to 0} h \psi(1/h) c_{\psi,h,0} = c_{\beta,0}. \tag{4.25}
\]
Lemma 4.4 Under the hypotheses of Theorem 1.2, for small $h$ and $t(h) = 1/(\log 1/h)$,

$$E \left( \int (L_{x+}^{z+h} - L_{x}^{z})^2 \, dx \right) = 4c_{\psi,h,0}t + O\left(g(h,t)\right)$$

(4.26)
as $h \to 0$, where

$$g(h,t) = \begin{cases} 
  h^2 t^2 (\psi^{-1}(1/t))^3 & 3/2 < \beta \leq 2 \\
  h^2 L(1/h) & \beta = 3/2 \\
  (h\psi^2(1/h))^{-1} & 1 < \beta < 3/2
\end{cases}$$

(4.27)

and $L(\cdot)$ is some function that is slowly varying at infinity. Also

$$\text{Var} \left( \int (L_{x+}^{z+h} - L_{x}^{z})^2 \, dx \right)$$

\leq C \left( \frac{tg(h,t)}{h^2 \psi(1/h)} + \frac{t^2 \psi^{-1}(1/t)}{h^2 \psi^2(1/h)} + \frac{Ct}{h^{3/2} \psi^{5/2}(1/h)} + \frac{Ct \log 1/h}{h^2 \psi^3(1/h)} \right).$$

(4.28)

The proof of this lemma shows that we can take any function $t := t(h)$ such that $\psi^{-1}(1/t) << 1/h$ and $\lim_{h \to 0} t(h) = 0$.

Lemma 4.5 Under the hypotheses of Theorem 1.2,

$$E \left( \int (L_{x+}^{z+h} - L_{x}^{z})^2 \, dx \right) = 4c_{\psi,h,0} + O\left(\overline{g}(h)\right)$$

(4.29)
as $h \to 0$, where

$$\overline{g}(h) = \begin{cases} 
  h^2 & 3/2 < \beta \leq 2 \\
  h^2 L(1/h) & \beta = 3/2 \\
  (h\psi^2(1/h))^{-1} & 1 < \beta < 3/2
\end{cases}$$

(4.30)

and $L(\cdot)$ is slowly varying at infinity.
Proof of Theorem 1.2  We use (4.4) with \( l = [\log 1/h] \). Since \( I_{j,j,l,h} \), \( 0 \leq j \leq l - 1 \), are independent and identically distributed, \( E(I_{j,j,l,h}) = E(I_{0,0,l,h}) \), for all \( j = 0, \ldots, l - 1 \). Consequently,

\[
\int (L^{x+h}_1 - L^x_1)^2 \, dx - E \int (L^{x+h}_1 - L^x_1)^2 \, dx = 2\tilde{I}_l + \sum_{j=0}^{l-1} (I_{j,j,l,h} - E(I_{j,j,l,h}))
\]

\[
+ lE(I_{0,0,l,h}) - E \int (L^{x+h}_1 - L^x_1)^2 \, dx.
\]

We show immediately below that

\[
\lim_{h \to 0} \sqrt{h} \psi^2(1/h) \left( lE(I_{0,0,l,h}) - E \int (L^{x+h}_1 - L^x_1)^2 \, dx \right) = 0.
\]

In addition, using again the property that \( I_{j,j,l,h} \), \( 0 \leq j \leq l - 1 \) are independent and identically distributed,

\[
\text{Var} \left( \sqrt{h} \psi^2(1/h) \sum_{j=0}^{l-1} (I_{j,j,l,h} - E(I_{j,j,l,h})) \right) = lh \psi^2(1/h) \text{Var} (I_{0,0,l,h})
\]

We also show below that

\[
\lim_{h \to 0} lh \psi^2(1/h) \text{Var} (I_{0,0,l,h}) = 0.
\]

Using (4.32)–(4.34) and Lemma 4.2 we get (1.16).

Using (4.26) and (4.29) on the expectations in (4.32), and recalling that \( l = 1/t \), we see that

\[
lE(I_{0,0,l,h}) - E \int (L^{x+h}_1 - L^x_1)^2 \, dx = O(g(h,t)/t) + O(\mathcal{F}(h))
\]

It is easy to verify that (4.32) holds.

Showing that (4.34) holds is a little more subtle so we provide some details. We first consider the last three terms in (4.28) and multiply them by \( lh \psi^2(1/h) = h \psi^2(1/h)/t \) as in (4.33). The first of these is

\[
\frac{h \psi^2(1/h)}{t} \frac{t^2 \psi^{-1}(1/t)}{h \psi^2(1/h)} = t \psi^{-1}(1/t).
\]
This last function is regularly varying as $t \to 0$ with index $1 - 1/\beta$ which is positive by hypothesis.

The next term is

$$
\frac{h\psi^2(1/h)}{t} \frac{t}{h^{3/2}\psi^{5/2}(1/h)} = \frac{1}{h^{1/2}\psi^{1/2}(1/h)}.
$$

(4.37)

Here $(h^{1/2}\psi^{1/2})^{-1}$ is regularly varying as $h \to 0$ with index $(\beta - 1)/2$ which is positive.

The third of the last three terms is

$$
\frac{h\psi^2(1/h)}{t} \frac{t \log 1/h}{h^2\psi^3(1/h)} = \frac{\log 1/h}{h\psi(1/h)}.
$$

(4.38)

Here $(\log 1/h)(h\psi(1/h))^{-1}$ is regularly varying as $h \to 0$ with index $(\beta - 1)$ which is positive. Thus (4.34) holds for these three terms.

We now consider

$$
\frac{h\psi^2(1/h)}{t} \frac{tg(h,t)}{h\psi(1/h)} = g(h,t)\psi(1/h).
$$

(4.39)

We use (4.27) to see that when $\beta > 3/2$ this is equal to

$$
t^2(\psi^{-1}(1/t))^3h^2\psi(1/h).
$$

(4.40)

Here we note that $t^2(\psi^{-1}(1/t))^3$ is regularly varying at zero with index $2 - (3/\beta)$ which is positive since $\beta > 3/2$. In addition by (2.15), $\lim_{h \to 0} h^2\psi(1/h) < \infty$.

When $\beta = 3/2$, (4.39) is equal to

$$
h^2L(1/h)\psi(1/h).
$$

(4.41)

This function is regularly varying at zero with index $2 - (3/2)$.

When $\beta < 3/2$, (4.39) is equal to $(h\psi(1/h))^{-1}$, which is regularly varying at zero with index $\beta - 1$. Thus we have verified (4.34). This completes the proof. \qed
5 Proofs of Lemmas 2.1–2.4

Since the Lévy processes, \( X \), that we are concerned with satisfy

\[
\int \frac{1}{1 + \psi(p)} \, dp < \infty \tag{5.1}
\]

it follows from the Riemann Lebesgue Lemma that they have transition probability density functions, which we designate as \( p_s(\cdot) \). Taking the inverse Fourier transform of the characteristic function \( X_s \), and using the symmetry of \( \psi \), we see that

\[
p_s(x) = \frac{1}{2\pi} \int e^{ipx} e^{-s\psi(p)} \, dp \tag{5.2}
\]

\[
= \frac{1}{\pi} \int_0^\infty \cos px e^{-s\psi(p)} \, dp.
\]

We begin with a technical lemma.

**Lemma 5.1** Let \( X \) be a symmetric Lévy process with Lévy exponent \( \psi(\lambda) \) that is regularly varying at infinity with index \( 1 < \beta \leq 2 \) and satisfies (1.13). Then for any \( r \geq 0 \) and \( t \geq 0 \)

\[
\int_0^t s^r e^{-s\psi(p)} \, ds \leq C \left( t \wedge \frac{1}{\psi(p)} \right)^{r+1} \leq \frac{2Ct^{r+1}}{1 + (t\psi(p))^{r+1}}; \tag{5.3}
\]

\[
\int_0^\infty \psi^r(p) \left( \int_0^1 s^r e^{-s\psi(p)} \, ds \right) \, dp \leq C; \tag{5.4}
\]

\[
\int |\sin(hp)| \psi^r(p) \left( \int_0^1 s^r e^{-s\psi(p)} \, ds \right) \, dp \leq \frac{C}{h\psi(1/h)}; \tag{5.5}
\]

and

\[
\int_0^1 (p_s(0) - p_s(h)) \, ds \leq C \frac{1}{h\psi(h/h)} \tag{5.6}
\]

as \( h \to 0 \).

In addition for all \( t \leq 1 \) and all \( y \in \mathbb{R}^1 \)

\[
\int_0^t p_s(y) \, ds \leq Ct\psi^{-1}(1/t). \tag{5.7}
\]
Proof The first part of the bound in the first inequality in (5.3) comes from taking $e^{-s\psi(p)} \leq 1$; the second from letting $t = \infty$. The second inequality in (5.3) is trivial.

Note that for any $y > 0$

$$y^r \int_0^1 s^r e^{-sy} ds = \frac{1}{y} \int_0^y s^r e^{-s} ds.$$ (5.8)

Consequently

$$y^r \int_0^1 s^r e^{-sy} ds \leq \left( \sup_{s \geq 0} x^r e^{-x} \right) \wedge \left( \frac{1}{y} \int_0^\infty s^r e^{-s} ds \right) \leq C \left( 1 \wedge \frac{1}{y} \right) \leq 2C \frac{1}{1 + y}. \quad (5.9)$$

Using this it is easy to see that

$$\int \psi^r(p) \int_0^1 s^r e^{-s\psi(p)} ds \, dp \leq C \int \left( 1 \wedge \frac{1}{\psi(p)} \right) \, dp \leq C \int_0^1 1 \, dp + C \int_1^\infty \frac{1}{\psi(p)} \, dp.$$ (5.10)

which gives (5.4).

Similarly we obtain (5.5),

$$\int_0^\infty |\sin(hp)| \psi^r(p) \int_0^1 s^r e^{-s\psi(p)} ds \, dp \leq C \int \left( 1 \wedge \frac{1}{\psi(p)} \right) \, dp \leq C \int_0^\infty \frac{hp \wedge 1}{1 + \psi(p)} \, dp \leq C \left( h \int_0^{1/h} \frac{p}{1 + \psi(p)} \, dp + \int_{1/h}^\infty \frac{1}{1 + \psi(p)} \, dp \right) \leq \frac{C}{h\psi(1/h)}. \quad (5.11)$$

(In (5.11) we use the regular variation of $\psi$ at infinity. We continue to do so throughout the rest of this paper without further comment.)

For (5.6) we first note that by (5.2)

$$p_s(0) - p_s(h) = \frac{1}{\pi} \int_0^\infty (1 - \cos ph) e^{-s\psi(p)} \, dp \quad (5.12)$$

$$= \frac{2}{\pi} \int_0^\infty \sin^2 ph/2 e^{-s\psi(p)} \, dp.$$
Therefore by Fubini’s Theorem and (5.3),

\[
\int_0^1 (p_s(0) - p_s(h)) \, ds = \frac{2}{\pi} \int_0^\infty \sin^2 ph/2 \int_0^1 e^{-s\psi(p)} \, ds \, dp \\
\leq C \int_0^\infty \left( 1 \wedge \frac{p^2h^2}{2} \right) \left( 1 \wedge \frac{1}{\psi(p)} \right) \, dp \\
\leq Ch^2 \int_0^{1/h} \frac{p^2}{\psi(p)} \, dp + C \int_{1/h}^\infty \frac{1}{\psi(p)} \, dp \leq C \frac{1}{h\psi(1/h)}.
\]

For (5.7) we use (5.3) to see that

\[
\int_0^t p_s(y) \, ds \leq \frac{1}{2\pi} \int_0^t \int e^{-s\psi(p)} \, dp \, ds \\
\leq C \int_0^\infty \left( t \wedge \frac{1}{\psi(p)} \right) \, dp \\
\leq C \left( t\psi^{-1}(1/t) + \int_{\psi^{-1}(1/t)}^\infty \frac{1}{\psi(p)} \, dp \right) \\
\leq C t\psi^{-1}(1/t).
\]

**Proof of Lemma 2.1**  We first note that

\[
p_s(x) \leq C \left( \psi^{-1}(1/s) \vee 1 \right).
\]

Refer to (5.2). It is obvious that for \( s \geq 1 \), \( p_s(x) \leq C \), for all \( x \). In addition,

\[
p_s(x) \leq \frac{1}{\pi} \psi^{-1}(1/s) + \frac{1}{\pi} \int_{\psi^{-1}(1/s)}^\infty e^{-s\psi(p)} \, dp.
\]

Also, for all \( s \) sufficiently small, the last integral is equal to

\[
\int_1^\infty e^{-u} \, d\psi^{-1}(u/s) < \int_1^\infty \psi^{-1}(u/s) \, e^{-u} \, du
\]

by integration by parts, where we drop a negative term. The final integral in (5.17)

\[
\leq \psi^{-1}(1/s) \int_1^\infty \frac{\psi^{-1}(u/s)}{\psi^{-1}(1/s)} e^{-u} \, du \\
\leq \psi^{-1}(1/s)K \int_1^\infty u^{1/\beta+\delta} \, e^{-u} \, du \leq C \psi^{-1}(1/s),
\]
for all $\delta > 0$; where the constant $K$ depends on $\delta$. (See e.g. [2, Theorem 1.5.6].) Thus we get (5.15).

By integration by parts

\[
p_s(x) = \frac{1}{\pi x} \int_0^\infty e^{-\psi(p)} d(sin px)
\]

(5.19)

\[
= -\frac{1}{\pi x} \int_0^\infty \sin px \left( \frac{d}{dp} e^{-\psi(p)} \right) dp
\]

\[
= -\frac{1}{\pi x^2} \int_0^\infty \cos px \left( \frac{d^2}{dp^2} e^{-\psi(p)} \right) dp.
\]

Furthermore

\[
\frac{d^2}{dp^2} e^{-\psi(p)} = \left( s^2 (\psi'(p))^2 - s \psi''(p) \right) e^{-\psi(p)}.
\]

(5.20)

Therefore, by (1.14)

\[
\left| \int_0^1 \cos px \left( \frac{d^2}{dp^2} e^{-\psi(p)} \right) dp \right| \leq C \left( \int_0^1 \left( (\psi'(p))^2 + |\psi''(p)| \right) dp \right) \leq C.
\]

(5.21)

(We use (1.14) repeatedly in the rest of the paper without comment.) In addition, by (1.13), for all $s$ sufficiently small

\[
\left| \int_1^\infty \cos px \left( \frac{d^2}{dp^2} e^{-\psi(p)} \right) dp \right| \leq C \int_1^\infty \frac{1}{p^2} \left( \psi^2(p)s^2 e^{-\psi(p)} + s \psi(p)e^{-\psi(p)} \right) dp
\]

\[
\leq C \int_1^\infty \frac{1}{p^2} dp \leq C,
\]

(5.22)

since $\sup_{x \geq 0} x^r e^{-x} \leq C$. Using (5.15) and (5.19)–(5.22) we get (2.2).

The inequality in (2.3) follows immediately from (2.2).

The equality in (2.4) is trivial since $\int p_s(x) dx = 1$.

Note that

\[
\Delta^h p_s(x) = p_s(x + h) - p_s(x)
\]

(5.23)

\[
= \frac{1}{\pi} \int_0^\infty (\cos p(x + h) - \cos px) e^{-s\psi(p)} dp
\]
\[ L^2 \text{ moduli of continuity of local times} \]

\[ = -\frac{2}{\pi} \int_0^\infty \cos(px) \sin^2(hp/2) e^{-s\psi(p)} \]
\[ - \frac{1}{\pi} \int_0^\infty \sin(px) \sin(hp) e^{-s\psi(p)} \, dp \]

and

\[ \Delta^h \Delta^{-h} p_s(x) = 2p_s(x) - p_s(x + h) - p_s(x - h) \]
\[ = \frac{4}{\pi} \int_0^\infty \cos(px) \sin^2(hp/2) e^{-s\psi(p)} \, dp. \]

Thus

\[ \Delta^h p_s(x) = -\frac{1}{2} \Delta^h \Delta^{-h} p_s(x) - \frac{1}{\pi} \int_0^\infty \sin(px) \sin(hp) e^{-s\psi(p)} \, dp. \]

We now note that

\[ \sup_x \int_0^1 |\Delta^h p_s(x)| \, ds \leq \frac{C}{h\psi(1/h)} \]

and

\[ \sup_x \int_0^1 |\Delta^h \Delta^{-h} p_s(x)| \, ds \leq \frac{C}{h\psi(1/h)}. \]

To obtain (5.27) we use (5.24) to see that

\[ \sup_x \int_0^1 |\Delta^h \Delta^{-h} p_s(x)| \, ds \leq \frac{4}{\pi} \int_0^1 \int_0^\infty \sin^2(hp/2) e^{-s\psi(p)} \, dp \, ds. \]

Using the calculation in (5.13) we get (5.27).

To obtain (5.26) we note that by (5.3), similarly to (5.13)

\[ \sup_x \int_0^1 \left| \int_0^\infty \sin(px) \sin(hp) e^{-s\psi(p)} \, dp \right| \, ds \]
\[ \leq C \int_0^\infty \left( 1 \wedge \frac{ph}{2} \right) \left( 1 \wedge \frac{1}{\psi(p)} \right) \, dp \]
\[ \leq C \left( h \int_0^{1/h} \frac{p}{1 + \psi(p)} \, dp + \int_0^\infty \frac{1}{\psi(p)} \, dp \right) \leq C \frac{1}{h\psi(1/h)}. \]

Thus (5.26) follows from (5.25), (5.27) and (5.29).

We now show that

\[ \Delta^h \Delta^{-h} p_s(x) = \frac{8 K}{\pi x^2} \]

(5.30)
where

\[ K = K(s, x, h) := \int_0^\infty \sin^2(px/2) \left( \sin^2(hp/2) e^{-s\psi(p)} \right)^{\prime\prime} \, dp. \] (5.31)

To get this we integrate by parts in (5.24),

\[ \int_0^\infty \cos px \sin^2(hp/2) e^{-s\psi(p)} \, dp \] (5.32)

\[ = \frac{1}{x} \int_0^\infty \sin^2(hp/2) e^{-s\psi(p)} \, d(\sin px) \]

\[ = -\frac{1}{x} \int_0^\infty \sin px \left( \sin^2(hp/2) e^{-s\psi(p)} \right)' \, dp \]

\[ = -\frac{1}{x} \int_0^\infty \left( \sin^2(hp/2) e^{-s\psi(p)} \right)' \, d \left( \int_0^p \sin rx \, dr \right) \]

\[ = -\frac{1}{x^2} \int_0^\infty \left( \sin^2(hp/2) e^{-s\psi(p)} \right)' \, d \left( 1 - \cos px \right) \]

\[ = \frac{2}{x^2} \int_0^\infty \sin^2(px/2) \left( \sin^2(hp/2) e^{-s\psi(p)} \right)^{\prime\prime} \, dp. \]

Let \( g(p) = e^{-s\psi(p)} \) and note that

\[ \left( 2 \sin^2(hp/2) e^{-s\psi(p)} \right)' = g(p)h \sin hp + 2g'(p) \sin^2(hp/2) \] (5.33)

and

\[ \left( 2 \sin^2(hp/2) e^{-s\psi(p)} \right)^{\prime\prime} \]

\[ = g(p)h^2 \cos hp + 2g'(p)h \sin hp + 2g''(p) \sin^2(hp/2). \] (5.34)

Substituting (5.34) in (5.32) we write \( K = I + II + III \). Using (5.3) we see that

\[ \int_0^1 |I| \, ds = h^2 \int_0^1 \left| \int_0^\infty \cos hp \sin^2(px/2) e^{-s\psi(p)} \, dp \right| \, ds \] (5.35)

\[ \leq h^2 \int_0^\infty \left( \int_0^1 e^{-s\psi(p)} \, ds \right) \, dp \]

\[ \leq C h^2 \int_0^1 \frac{1}{1 + \psi(p)} \, dp = O (h^2) . \]

Then using (1.13), (1.14) and (5.4) with \( r = 1 \) we get

\[ \int_0^1 |II| \, ds = 2h \int_0^1 \left| \int_0^\infty \sin hp \sin^2(px/2) g'(p) \, dp \right| \, ds \] (5.36)
\( L^2 \) moduli of continuity of local times

\[
\begin{align*}
&\leq 2h \int_0^\infty |\sin(hp) \psi'(p)| \left( \int_0^1 s e^{-s\psi(p)} \, ds \right) \, dp \\
&\leq C h^2 \int_0^\infty |p \psi'(p)| \left( \int_0^1 s e^{-s\psi(p)} \, ds \right) \, dp \\
&\leq C h^2 \left( C_1 + \int_1^\infty \psi(p) \left( \int_0^1 s e^{-s\psi(p)} \, ds \right) \, dp \right) = O \left( h^2 \right).
\end{align*}
\]

Similarly, and also using (5.4) with \( r = 1 \) we get

\[
\begin{align*}
\int_1^0 |III| \, ds &= 2 \int_0^1 \left| \int_0^\infty \sin^2(hp/2) \sin^2(px/2) g''(p) \, dp \right| \, ds \\
&\leq C h^2 \int_0^\infty p^2 \left( \int_0^1 \left( s|\psi''(p)| + s^2|\psi'(p)|^2 \right) e^{-s\psi(p)} \, ds \right) \, dp \\
&\leq C h^2 \left\{ \int_0^1 p^2 \left( |\psi''(p)| + |\psi'(p)|^2 \right) \, dp \\
&\quad + \int_1^\infty \left( \int_0^1 \left( s\psi(p) + s^2\psi^2(p) \right) e^{-s\psi(p)} \, ds \right) \, dp \right\} \\
&= O \left( h^2 \right). \quad (5.37)
\end{align*}
\]

Combining (5.35) with (5.37) we get the third bound in (2.8). The first bound in (2.8) follows from (5.27).

To get the second bound in (2.8) we use (5.24) and the third integral in (5.32) to see that

\[
\Delta^h \Delta^{-h} p_s(x) = -\frac{4 \, L}{\pi \, x} \quad (5.38)
\]

where

\[
L = L(s, x, h) := \int_0^\infty \sin px \left( \sin^2(hp/2) e^{-s\psi(p)} \right)' \, dp. \quad (5.39)
\]

Using (5.33), (1.13), (1.14) and (5.5) with \( r = 0 \) and 1, we see that

\[
\int_0^1 |L| \, ds \leq C \int_0^1 \left( h \int_0^\infty |\sin hp| \, g(p) \, dp + \int_0^\infty \sin^2 \left( hp/2 \right) |g'(p)| \, dp \right) \, ds \\
\leq C h \int_0^\infty |\sin hp| \int_0^1 e^{-s\psi(p)} \, ds \, dp \\
+ C h \left( C_1 + \int_1^\infty |\sin \left( hp/2 \right) p\psi'(p)| \int_0^1 s e^{-s\psi(p)} \, ds \, dp \right)
\]
\[
\begin{align*}
\leq O\left(\frac{1}{\psi(1/h)}\right) &+ Ch \int_0^\infty |\sin (hp/2)| \psi(p) \int_0^1 s e^{-s\psi(p)} \, ds \, dp \\
\leq O\left(\frac{1}{\psi(1/h)}\right). 
\end{align*}
\]

Thus we get the second bound on the right–hand side of (2.8). This completes the proof of (2.8).

To prove (2.5) we first note that by (5.25) it is less than \( w(x)/2 \) plus

\[
C \int_0^1 \left| \int_0^\infty \sin(px) \sin(hp) e^{-s\psi(p)} \, dp \right| \, ds 
\]  
(5.41)

Integrating by parts twice we obtain

\[
\begin{align*}
\int_0^\infty \sin(px) \sin(hp) e^{-s\psi(p)} \, dp \\
&= -\frac{1}{x} \int_0^\infty \sin(hp) e^{-s\psi(p)} \, d(\cos px) \\
&= \frac{1}{x} \int_0^\infty \cos px \left( \sin(hp) e^{-s\psi(p)} \right) \, dp \\
&= \frac{1}{x^2} \int_0^\infty \left( \sin(hp) e^{-s\psi(p)} \right) \, d(\sin px) \\
&= -\frac{1}{x^2} \int_0^\infty \sin px \left( \sin(hp) e^{-s\psi(p)} \right)'' \, dp.
\end{align*}
\]  
(5.42)

Note that

\[
\left( \sin(hp) e^{-s\psi(p)} \right)' = (h \cos hp - \sin hp(s \psi'(p)) e^{-s\psi(p)} 
\]  
(5.43)

Thus the left hand side of (5.42) is bounded by \( \frac{J}{x} \) where

\[
J = J(s, x, h) := \int_0^1 \left| \int_0^\infty \cos px \left( \sin(hp) e^{-s\psi(p)} \right) \, dp \right| \, ds. 
\]  
(5.44)

We write

\[
J \leq J_1 + J_2 
\]  
(5.45)

where

\[
\begin{align*}
|J_1| &\leq h \int_0^1 \int_0^\infty |\cos px \cos(hp)| e^{-s\psi(p)} \, dp \, ds \\
&\leq Ch \int_0^\infty \frac{1}{1 + \psi(p)} \, dp \leq C'h 
\end{align*}
\]  
(5.46)
and using (1.13), (1.14) and (5.4)

\[ |J_2| \leq \int_0^1 \int_0^\infty |\cos px \sin(hp)| |\psi'(p)| se^{-s\psi(p)} \, dp \, ds \quad (5.47) \]

\[ \leq \ h \int_0^1 \int_0^\infty p|\psi'(p)| se^{-s\psi(p)} \, dp \, ds \]

\[ \leq \ h \int_0^1 |\psi'(p)| \int_0^1 se^{-s\psi(p)} \, ds \, dp \]

\[ + C h \int_0^1 \int_1^\infty \psi(p) se^{-s\psi(p)} \, dp \, ds \leq C' h. \]

Therefore

\[ \frac{J}{|x|} \leq C \frac{h}{|x|}. \quad (5.48) \]

In addition (5.41) is \( \frac{G}{x^2} \) where

\[ G = G(x, h) := \int_0^1 \int_0^\infty \sin px \left( \sin(hp) e^{-s\psi(p)} \right)'' \, dp \, ds. \quad (5.49) \]

Since

\[ \left( \sin(hp) e^{-s\psi(p)} \right)'' \]

\[ = \left( -h^2 \sin hp + 2hs \cos hp \psi'(p) - \sin hp(s \psi''(p) - s^2(\psi'(p))^2) \right) e^{-s\psi(p)}, \]

we can write

\[ G \leq G_1 + G_2 + G_3. \quad (5.51) \]

Using (5.50) and (5.4) we get

\[ |G_1| = h^2 \int_0^1 \int_0^\infty \sin px \left( \sin(hp) e^{-s\psi(p)} \right) \, dp \, ds \quad (5.52) \]

\[ \leq C h^2 \int_0^\infty \int_0^1 e^{-s\psi(p)} \, ds \, dp \leq C h^2. \]

Using (1.13), (1.14) and (5.4) we see that

\[ |G_2| = 2h \int_0^1 \int_0^\infty \sin px \cos hp \left( \psi'(p) se^{-s\psi(p)} \right) \, dp \, ds \quad (5.53) \]

\[ \leq 2h \int_0^1 \int_0^\infty |\psi'(p)| se^{-s\psi(p)} \, dp \, ds \]

\[ \leq 2h \left( C_1 + \int_1^\infty p|\psi'(p)| \left( \int_0^1 se^{-s\psi(p)} \, ds \right) \, dp \right) \]

\[ \leq C h \left( C_1 + \int_1^\infty \psi(p) \left( \int_0^1 se^{-s\psi(p)} \, ds \right) \, dp \right) \leq C h. \]
Similarly

$$\left| G_3 \right| \leq h \int_0^\infty p \left( \int_1^1 \left( s |\psi''(p)| + s^2 (\psi'(p))^2 \right) e^{-s\psi(p)} ds \right) dp$$

$$\leq C h \left\{ C_1 + \int_1^\infty p^2 \left( \int_0^1 \left( s^2 |\psi''(p)| + s^2 (\psi'(p))^2 \right) e^{-s\psi(p)} ds \right) dp \right\}$$

Thus we see that for all $|x| > 0$

$$G \leq Ch,$$  \hspace{1cm} (5.55)

for some $C < \infty$ independent of $|x|$. Combining (5.26), (5.48) and (5.55) and taking into account the value of $w(x)$, we get (2.5).

For (2.6) we use (2.5) to see that

$$\int \left( \int_0^1 |\Delta h p_s(x)| ds \right) dx \leq C \left( \int_0^a \frac{1}{h\psi(1/h)} dx + h \int_a^1 \frac{1}{x} dx + h \int_1^\infty \frac{1}{x^2} dx \right),$$

Set $a = a(h) = h^2 \psi(1/h)$. For Lévy processes excluding Brownian Motion, $\lim_{h \to 0} h^2 \psi(1/h) = 0$; (see [7, Lemma 4.2.2]), and we can estimate (5.56) to obtain (2.6). For Brownian Motion take $a = 1$ in (5.56) to obtain (2.6).

Similarly, to obtain (2.7) we use (2.5) to get

$$\int \left( \int_0^1 |\Delta h p_s(x)| ds \right)^p dx \leq C \left( \int_0^a \frac{1}{h^p \psi(1/h)} dx + h^p \int_a^\infty \frac{1}{x^p} dx \right)$$

$$\leq C \left( \frac{a}{h^p \psi(1/h)} + \frac{h^p}{a^{p-1}} \right).$$

For (2.10) we use (2.8) to see that

$$\int \left( \int_0^1 |\Delta h \Delta^{-h} p_s(x)| ds \right)^2 dx$$

$$\leq \kappa$$

$$\leq \kappa.$$
\[
\begin{aligned}
&= \int_0^h \left( \int_0^1 |\Delta^h \Delta^{-h} p_s(x)| \, ds \right)^2 \, dx \\
&\quad + \int_h^\infty \left( \int_0^1 |\Delta^h \Delta^{-h} p_s(x)| \, ds \right)^2 \, dx \\
&\leq \frac{C}{h\psi^2(1/h)} + \frac{C}{\psi^2(1/h)} \int_h^\infty \frac{1}{x^2} \, dx = O \left( \frac{1}{h\psi^2(1/h)} \right).
\end{aligned}
\]

The inequality in (2.11) follows similarly,

\[
\int_u^\infty \left( \int_0^1 |\Delta^h \Delta^{-h} p_s(x)| \, ds \right)^2 \, dx \leq \frac{C}{\psi^2(1/h)} \int_u^\infty \frac{1}{x^2} \, dx = \frac{C}{u\psi^2(1/h)}.
\]

To obtain (2.9) we use (2.8) to see that

\[
\begin{aligned}
&\int \int_1^0 |\Delta^h \Delta^{-h} p_s(x)| \, ds \, dx \\
&\quad = \int_0^h \int_0^1 |\Delta^h \Delta^{-h} p_s(x)| \, ds \, dx + \int_h^1 \int_0^1 |\Delta^h \Delta^{-h} p_s(x)| \, ds \, dx \\
&\quad + \int_1^\infty \int_0^1 |\Delta^h \Delta^{-h} p_s(x)| \, ds \, dx \\
&\leq \frac{C}{h\psi(1/h)} \int_0^h 1 \, dx + \frac{C'}{\psi(1/h)} \int_1^h \frac{1}{|x|} \, dx + Ch^2 \int_1^\infty \frac{1}{|x|^2} \, dx \\
&\leq \frac{C}{\psi(1/h)} + \frac{C \log 1/h}{\psi(1/h)} + Ch^2.
\end{aligned}
\]

\[\square\]

**Proof of Lemma 2.2** Using \(2 - e^{iph} - e^{-iph} = 4 \sin^2(hp/2)\) we see that

\[
\begin{aligned}
\int_0^\infty \Delta^h \Delta^{-h} p_t(x) \, dt &= \frac{1}{2\pi} \int_0^\infty \int e^{-ipx} (2 - e^{iph} - e^{-iph})e^{-t\psi(p)} \, dp \, dt \\
&= \frac{4}{2\pi} \int e^{-ipx} \sin^2(hp/2) \int_0^\infty e^{-t\psi(p)} \, dt \, dp \\
&= \frac{4}{2\pi} \int e^{-ipx} \frac{\sin^2(hp/2)}{\psi(p)} \, dp.
\end{aligned}
\]

It follows from Parseval’s Theorem that

\[
c_{\psi,h,1} = \int \left( \int_0^\infty \Delta^h \Delta^{-h} p_t(x) \, dt \right)^2 \, dx = \frac{8}{\pi} \int \frac{\sin^4(hp/2)}{\psi^2(p)} \, dp.
\]
Using this we write

\[ h\psi^2(1/h)c_{\psi,h,1} = \frac{16}{\pi} \int_0^\infty \left( \frac{\sin^2(p/2)}{\psi(p/h)/\psi(1/h)} \right)^2 dp \]  

(5.63)

For a fixed \( 0 < a < 1 \),

\[ \int_0^a \left( \frac{\sin^2 p/2}{\psi(p/h)/\psi(1/h)} \right)^2 dp = h\psi^2(1/h) \int_0^{a/h} \left( \frac{\sin^2(ph/2)}{\psi(p)} \right)^2 dp \]

\[ \leq \frac{h^5 \psi^2(1/h)}{4} \int_0^{a/h} \frac{p^4}{\psi^2(p)} dp. \]  

(5.64)

For any \( \epsilon > 0 \) we can find an \( h_0 > 0 \), such that for all \( 0 < h \leq h_0 \), the last line above

\[ \leq \frac{(1 + \epsilon)h^5 \psi^2(1/h)}{4(5 - 2\beta)} \frac{(a/h)^5}{\psi^2(a/h)} \leq \frac{a^{5-2\beta}}{2(5 - 2\beta)}. \]  

(5.65)

Note that for any \( \epsilon' > 0 \) and \( p \geq a > 0 \), we can find an \( h_0' > 0 \), such that for all \( 0 < h \leq h_0' \leq h_0 \),

\[ \frac{\psi^2(1/h)}{\psi^2(p/h)} \leq C \max \left( \frac{1}{p^{2\beta-\epsilon'}}, \frac{1}{p^{2\beta+\epsilon}} \right). \]  

(5.66)

(See [2, Theorem 1.5.6].) Therefore, it follows from the Dominated Convergence Theorem that

\[ \lim_{h \to 0} \int_0^\infty \left( \frac{\sin^2 p/2}{\psi(p/h)/\psi(1/h)} \right)^2 dp = \int_0^\infty \sin^4 p/2 \frac{dp}{p^{2\beta}}. \]  

(5.67)

Since (5.64), (5.65) and (5.67) hold for all \( a > 0 \) sufficiently small, we get (2.13).

We now consider (2.14). Just as we obtained (5.61) and (5.62) we see that

\[ \int_{[0,\sqrt{h}]^2} \left( \Delta^h \Delta^{-h} p_r(x) \right) \left( \Delta^h \Delta^{-h} p_{r'}(x) \right) dx \, dr \, dr' \]

\[ = \frac{8}{\pi} \int \frac{\sin^4(ph/2)}{\psi^2(p)} \left( 1 - e^{-\sqrt{h}\psi(p)} \right)^2 dp. \]  

(5.68)
We show below that
\[ h \psi^2(1/h) \int \frac{\sin^4(ph/2)}{\psi^2(p)} e^{-\sqrt{h}\psi(p)} \, dp = O(h^{1/2}), \quad (5.69) \]
which proves (2.14).

To obtain (5.69) we note that
\[ h \psi^2(1/h) \int \frac{\sin^4(ph/2)}{\psi^2(p)} e^{-\sqrt{h}\psi(p)} \, dp \]
\[ = h \psi^2(1/h) \int_{0 \leq |p| \leq 1} \frac{\sin^4(ph/2)}{\psi^2(p)} e^{-\sqrt{h}\psi(p)} \, dp \]
\[ + h \psi^2(1/h) \int_{1 \leq |p| \leq 1/h} \frac{\sin^4(ph/2)}{\psi^2(p)} e^{-\sqrt{h}\psi(p)} \, dp \]
\[ + h \psi^2(1/h) \int_{|p| \geq 1/h} \frac{\sin^4(ph/2)}{\psi^2(p)} e^{-\sqrt{h}\psi(p)} \, dp \]
\[ \leq C h^5 \psi^2(1/h) \int_{0 \leq |p| \leq 1} \frac{p^4}{\psi^2(p)} \, dp \]
\[ + C h^5 \psi^2(1/h) \int_{1 \leq |p| \leq 1/h} \frac{p^4}{\psi^2(p)} \frac{1}{\sqrt{h}\psi(p)} \, dp \]
\[ + C h \psi^2(1/h) e^{-\sqrt{h}\psi(1/h)} \int_{|p| \geq 1/h} \frac{1}{\psi^2(p)} \, dp, \]
where, in the next to last line of (5.70), we use the fact that for \( s \geq 0 \),
\[ e^{-s} \leq (\sup_{s \geq 0} se^{-s})/s. \]
It is obvious that the first and last integral in the last inequality in (5.70) is \( O(\sqrt{h}) \). As for the second integral, if \( 1 < \beta < 5/3 \)
\[ h^5 \psi^2(1/h) \int_{1 \leq |p| \leq 1/h} \frac{p^4}{\psi^2(p)} \frac{1}{\sqrt{h}\psi(p)} \, dp \leq C \frac{1}{h^{1/2}\psi(1/h)} = O(\sqrt{h}); \]
\[ (5.71) \]
if \( 5/3 < \beta \leq 2 \)
\[ h^5 \psi^2(1/h) \int_{1 \leq |p| \leq 1/h} \frac{p^4}{\psi^2(p)} \frac{1}{\sqrt{h}\psi(p)} \, dp \leq C \frac{h^5 \psi^2(1/h)}{h^{1/2}} = O(\sqrt{h}), \]
\[ (5.72) \]
where we use Remark 2.1 when \( \beta = 2 \). When \( \beta = 5/3 \)
\[ h^5 \psi^2(1/h) \int_{1 \leq |p| \leq 1/h} \frac{p^4}{\psi^2(p)} \frac{1}{\sqrt{h}\psi(p)} \, dp \leq C \frac{L(1/h)}{h^{1/2}\psi(1/h)} \leq O(h) \]
\[ (5.73) \]
for some function $L(\cdot)$ that is slowly varying at infinity. This gives us (5.69).

**Lemma 5.2** For $r \geq 0$

\[
\sup_{\delta \leq s \leq 1} s^r e^{-s\psi(p)} \leq C \left( 1 \wedge \frac{1}{\psi^r(p)} \right) \leq \frac{2C}{1 + \psi^r(p)}, \tag{5.74}
\]

and for $k > 0$

\[
\sup_{\delta \leq s \leq 1} s^r e^{-s\psi(p)} \leq \sup_{\delta \leq s \leq 1} \frac{s^{r+k}}{\delta^k} e^{-s\psi(p)} \leq \frac{1}{\delta^k} \frac{2C}{1 + \psi^{r+k}(p)}. \tag{5.75}
\]

**Proof** The first inequality in (5.74) follows from the facts that $y^r e^{-y} \leq C$ and, of course, $\sup_{\delta \leq s \leq 1} s^r e^{-s\psi(p)} \leq 1$. The second inequality in (5.74) is elementary. The inequality in (5.75) follows from (5.74).

**Proof of Lemma 2.3** The inequality in (2.16) follows immediately from (2.2).

By (5.75) with $r = 0$ and $k = 3$

\[
\sup_{\delta \leq s \leq 1} |\Delta^h p_s(0)| = \sup_{\delta \leq s \leq 1} \frac{1}{\pi} \int_0^\infty \sin^2(ph/2) e^{-s\psi(p)} \, dp \leq \frac{h^2}{2\pi} \int_0^\infty p^2 e^{-s\psi(p)} \, dp \leq C \frac{h^2}{\delta^3} \int_0^\infty \frac{p^2}{1 + \psi^3(p)} \, dp \leq \frac{C}{\delta^3} h^2,
\]

Note that $\Delta^h p_r(0) = p_r(h) - p_r(0) < 0$ and $\Delta^h \Delta^{-h} p_r(0) = 2(p_r(0) - p_r(h))$. Thus (2.18) follows immediately from (2.17).

**Proof of Lemma 2.4** The inequality in (2.20) follows immediately from (2.2).

To obtain (2.22) consider the material in the proof of Lemma 2.1 from (5.30) to the statement that $K = I + II + III$. Now, instead of integrating $I$, $II$ and $III$ we take their supremum as $\delta \leq s \leq 1$. We
have

$$\sup_{\delta \leq s \leq 1} |I| \leq h^2 \sup_{\delta \leq s \leq 1} \left| \int_0^\infty \cos hp \sin^2(px/2)e^{-s\psi(p)} \, dp \right|$$  \hspace{1cm} (5.77)

$$\leq h^2 \sup_{\delta \leq s \leq 1} \int_0^\infty e^{-s\psi(p)} \, dp$$

$$\leq h^2 \frac{C}{\delta} \int_0^1 \frac{1}{1 + \psi(p)} \, dp \leq \frac{C}{\delta} h^2,$$

where we use (5.75) with $r = 0$ and $k = 1$.

$$\sup_{\delta \leq s \leq 1} |II| = 2 \sup_{\delta \leq s \leq 1} h \left| \int_0^\infty \sin hp \sin^2(px/2)g'(p) \, dp \right|$$  \hspace{1cm} (5.78)

$$\leq 2h \sup_{\delta \leq s \leq 1} \int_0^\infty |\sin hp| \psi'(p) e^{-s\psi(p)} \, dp$$

$$\leq C h^2 \sup_{\delta \leq s \leq 1} \int_0^\infty |p \psi'(p)| e^{-s\psi(p)} \, dp$$

$$\leq C h^2 \sup_{\delta \leq s \leq 1} \left( C_1 + \int_1^\infty \psi(p) e^{-s\psi(p)} \, dp \right)$$

$$\leq \frac{C}{\delta} h^2 \left( C_1 + \int_1^\infty \frac{\psi(p)}{1 + \psi^2(p)} \, dp \right),$$

where we use (5.75) with $r = 1$ and $k = 1$. Similarly, but with $r, k = 0, 1$ and $r, k = 2, 1$

$$\sup_{\delta \leq s \leq 1} |III| \leq \sup_{\delta \leq s \leq 1} \left| \int_0^\infty \sin^2(hp/2) \sin^2(px/2)g''(p) \, dp \right|$$

$$\leq C h^2 \sup_{\delta \leq s \leq 1} \int_0^\infty p^2 \left( |s \psi''(p)| + s^2 |\psi'(p)|^2 \right) e^{-s\psi(p)} \, dp$$

$$\leq C h^2 \left\{ \int_0^1 p^2 \left( |\psi''(p)| + |\psi'(p)|^2 \right) \, dp \right\}$$

$$+ \sup_{\delta \leq s \leq 1} \int_1^\infty \left( s \psi(p) + s^2 \psi^2(p) \right) e^{-s\psi(p)} \, dp \right\}$$

$$\leq C h^2 \left\{ C_1 + \sup_{\delta \leq s \leq 1} \int_1^\infty \left( s \psi(p) + s^2 \psi^2(p) \right) e^{-s\psi(p)} \, dp \right\}$$

$$\leq \frac{C}{\delta} h^2 \left\{ C_1 + \int_1^\infty \frac{\psi(p)}{1 + \psi^2(p)} \, dp + \int_1^\infty \frac{\psi^2(p)}{1 + \psi^3(p)} \, dp \right\}$$

Combining (5.77)–(5.79) with (5.30) we get the second bound in (2.22).
The first bound on the right-hand side of (2.22) follows from (2.18) since,

\[ \sup_{\delta \leq r \leq 1} \Delta_h \Delta^{-h}_r(x) \leq \sup_{\delta \leq r \leq 1} \Delta_h \Delta^{-h}_r(0), \quad (5.80) \]

(see (5.24).)

To get the second bound on the right-hand side of (2.21) consider the material in the paragraph containing (5.41). For our purposes here we need to obtain

\[ \sup_{\delta \leq s \leq 1} \left| \int_0^\infty \sin(px) \sin(hp) e^{-r\psi(p)} dp \right| \quad (5.81) \]

Integrating by parts twice as in (5.42) we see that (5.81) is bounded by

\[ \sup_{\delta \leq s \leq 1} \left| \frac{1}{x^2} \int_0^\infty \sin px \left( \sin(hp) e^{-s\psi(p)} \right)'' dp \right| \quad (5.82) \]

Thus we have to take \( \sup_{\delta \leq s \leq 1} \) of the terms in (5.52)–(5.54), but without the integral on \( s \). It is easy to see that we get the same bounds as in (5.52)–(5.54) but with the factor 1/\( \delta \) as in (5.77)–(5.79).

By (5.28), (2.21) is bounded by (2.18) plus

\[ Ch \int pe^{-\delta\psi(p)} dp \quad (5.83) \]

\[ \leq Ch \left( \int_0^{\psi^{-1}(1/\delta)} p dp + \frac{1}{\delta^2} \int_0^{\psi^{-1}(1/\delta)} \frac{p}{\psi^2(p)} dp \right) \]

\[ \leq Ch(\psi^{-1}(1/\delta))^2 \leq C \frac{h}{\delta^2}. \]

(For the second integral in the middle line of (5.83) see the comment following (5.70).) This gives the first bound on the right-hand side of (2.21).

The inequalities in (2.23)–(2.25) follow easily from (2.20)–(2.22). \( \square \)

6 Proofs of Lemmas 4.3–4.5

Proof of Lemma 4.3

By (5.2)

\[ h\psi(1/h)c_{\psi,h,0} = \frac{h\psi(1/h)}{\pi} \int_0^\infty \frac{1 - \cos(ph)}{\psi(p)} dp \quad (6.1) \]
\[ L^2 \text{ moduli of continuity of local times} \quad 64 \]

\[ \frac{2h\psi(1/h)}{\pi} \int_0^\infty \frac{\sin^2(ph/2)}{\psi(p)} \, dp \]

\[ = \frac{2}{\pi} \int_0^\infty \frac{\sin^2(p/2)}{\psi(p/h)\psi(1/h)} \, dp. \]

Compare this to (5.63). Following the proof of (2.13), from (5.64) to (5.67), with obvious modifications, we get (4.25).

**Remark 6.1** We note that by (6.1), for symmetric stable processes of index \( \beta \), we get equality in (4.25), namely

\[ c_{\psi,h,0} = h^{\beta-1}c_{\beta,0}. \quad (6.2) \]

**Proof of Lemma 4.4** We show in [3, (8.3)-(8.4)] that by the Kac moment formula, for \( 0 < t \leq 1 \),

\[ E \left( \int (L_t^{x+h} - L_t^x)^2 \, dx \right) = 4 \int_0^t (t-r) (p_r(0) - p_r(h)) \, dr \]

\[ = \frac{8}{\pi} \int_0^\infty \sin^2(hp/2) \int_0^t (t-r)e^{-r\psi(p)} \, dr \, dp. \quad (6.3) \]

(In truth we show it for \( t = 1 \) but it is obvious that it holds for any \( t \).)

Note that

\[ \int_0^t (t-r)e^{-r\psi(p)} \, dr = \frac{t}{\psi(p)} - \frac{1 - e^{-t\psi(p)}}{\psi^2(p)}. \quad (6.4) \]

By (6.1)

\[ \frac{8t}{\pi} \int_0^\infty \frac{\sin^2(ph/2)}{\psi(p)} \, dp = 4c_{\psi,h,0}t. \quad (6.5) \]

This gives the dominant term in (4.26). The absolute value of the remainder is

\[ \frac{8}{\pi} \int_0^\infty \frac{\sin^2(ph/2)}{\psi^2(p)} \left( 1 - e^{-t\psi(p)} \right) \, dp \leq \frac{8}{\pi} \int_0^\infty \frac{\sin^2(ph/2)}{\psi^2(p)} \left( 1 \wedge t\psi(p) \right) \, dp. \quad (6.6) \]

We break this last integral into three parts and see that it is bounded by

\[ C \left( h^2t \int_0^{1/(1/t)} \frac{p^2}{\psi(p)} \, dp + h^2 \int_{1/(1/t)}^{1/h} \frac{p^2}{\psi^2(p)} \, dp + \int_{1/h}^\infty \frac{1}{\psi^2(p)} \, dp \right) \quad (6.7) \]
We have
\[ h^2 \int_0^{\psi^{-1}(1/t)} \frac{p^2}{\psi(p)} \, dp \leq C h^2 t^2 \left( \psi^{-1}(1/t) \right)^3, \]  
(6.8)
(Since \( \lim_{p \to 0} \psi(p)/p^2 > 0 \) this integral is finite; see [7, Lemma 4.2.2]).

In addition
\[ \int_{1/h}^{\infty} \frac{1}{\psi^2(p)} \, dp \leq C \frac{1}{h \psi^2(1/h)} \]  
(6.9)
If \( \beta > 3/2 \)
\[ h^2 \int_{\psi^{-1}(1/t)}^{1/h} \frac{p^2}{\psi^2(p)} \, dp \leq C h^2 t^2 \left( \psi^{-1}(1/t) \right)^3. \]  
(6.10)
If \( \beta = 3/2 \)
\[ h^2 \int_{\psi^{-1}(1/t)}^{1/h} \frac{p^2}{\psi^2(p)} \, dp \leq C h^2 L(1/h) \]  
(6.11)
for some function \( L(\cdot) \) that is slowly varying at infinity. If \( \beta < 3/2 \)
\[ h^2 \int_{\psi^{-1}(1/t)}^{1/h} \frac{p^2}{\psi^2(p)} \, dp \leq C \frac{1}{h \psi^2(1/h)}. \]  
(6.12)
Using \((6.6)-(6.12)\) we get \((4.27)\).

Let
\[ Z = \int \left( L_{t+1}^x - L_t^x \right)^2 \, dx. \]  
(6.13)
We get an upper bound for the variance of \( Z \) by finding an upper bound for \( EZ^2 \) and using \((4.26)\) to estimate \((EZ)^2 \). We proceed as in the beginning of the proof of Lemma 3.1 however there are enough differences that it is better to repeat some of the arguments.

By the Kac Moment Theorem
\[ E \left( \prod_{i=1}^2 \left( \Delta_{x_i}^h L_{t_i}^x \right) \left( \Delta_{y_i}^h L_{t_i}^y \right) \right) \]  
(6.14)
\[ = \prod_{i=1}^2 \left( \Delta_{x_i}^h \Delta_{y_i}^h \right) \sum_{\sigma} \prod_{i=1}^4 p_{r_i} (\sigma(i) - \sigma(i - 1)) \prod_{i=1}^4 dr_i \]  
where the sum runs over all bijections \( \sigma : [1, 4] \mapsto \{x_i, y_i, 1 \leq i \leq 2\} \) and we take \( \sigma(0) = 0 \). We rewrite \((6.14)\) so that each \( \Delta_i^h \) applies to a
single $p$ factor and then set $y_i = x_i$ and then integrate with respect to $x_1, \ldots, x_m$ to get
\[
E \left( \left( \int (L^{x+h}_t - L^x_t)^2 \, dx \right)^2 \right) \tag{6.15}
\]
\[
= 4 \sum_{\pi, a} \int \int \left\{ \sum_{i=1}^4 r_i \leq t \right\} \prod_{i=1}^4 \left( \frac{\Delta h_{\pi(i)}}{\Delta x_{\pi(i-1)}} \right)^{a_1(i)} 
\prod_{i=1}^2 \left( \frac{\Delta h_{\pi(i-1)}}{\Delta x_{\pi(i-1)}} \right)^{a_2(i)} 
\prod_{i=1}^4 p^\sharp_{r_i}(x_{\pi(i)} - x_{\pi(i-1)}) \prod_{i=1}^2 dx_i,
\]
as in (3.29). As we did following (3.29) we continue the analysis with $p^\sharp$ replaced by $p$.

In (6.15) the sum runs over all maps $\pi : [1, 4] \mapsto [1, 2]$ with $|\pi^{-1}(i)| = 2$ for each $i$ and over all $a = (a_1, a_2) : [1, \ldots, 4] \mapsto \{0, 1\} \times \{0, 1\}$ with the property that for each $i$ there are exactly two factors of the form $\Delta^h_{x_i}$. The factor 4 comes from the fact that we can interchange each $y_i$ and $x_i$, $i = 1, 2$. As usual we take $\pi(0) = 0$.

Note that in (6.15) it is possible to have ‘bound states’, that is values of $i$ for which $\pi(i) = \pi(i - 1)$. We first consider the terms in (6.15) with two bound states. There are two possible maps. They are ($\pi(1), \pi(2), \pi(3), \pi(4)) = (1, 1, 2, 2)$ and ($\pi(1), \pi(2), \pi(3), \pi(4)) = (2, 2, 1, 1)$. The terms in (6.15) for the map ($\pi(1), \pi(2), \pi(3), \pi(4)) = (1, 1, 2, 2$) are of the form
\[
\prod_{i=1}^4 \left( \frac{\Delta h_{\pi(i)}}{\Delta x_{\pi(i-1)}} \right)^{a_1(i)} 
\prod_{i=1}^2 \left( \frac{\Delta h_{\pi(i-1)}}{\Delta x_{\pi(i-1)}} \right)^{a_2(i)} 
\prod_{i=1}^4 p^\sharp_{r_i}(x_{\pi(i)} - x_{\pi(i-1)}), \tag{6.16}
\]
where the density terms have the form
\[
p^\sharp_{r_1}(x_1)p_{r_2}(y_1 - x_1)p_{r_3}(x_2 - y_1)p_{r_4}(y_2 - x_2), \tag{6.17}
\]
and where $y_i - x_i = 0$. The value of the integrals of the terms in (6.16) depend upon how the difference operators are distributed. In many cases the integrals are equal to zero. For example suppose we have
\[
\Delta^h_{x_1}p^\sharp_{r_1}(x_1)\Delta^h_{x_1}p^\sharp_{r_2}(0)\Delta^h_{x_2}p_{r_3}(x_2 - x_1)\Delta^h_{x_2}p_{r_4}(0), \tag{6.18}
\]
which we obtain by setting $y_1 = x_1$. (Note that $\Delta^h_{x_1}p^\sharp_{r_2}(0)$ should be interpreted as $\Delta^h_{x_1}p_{r_2}(x_1 - y_1)$ or $\Delta^h_{x_1}p_{r_2}(y_1 - x_1)$). Written out this term
is

\[(p_{r_1}(x_1 + h) - p_{r_1}(x_1)) \Delta_x^h p_{r_2}(0) \]  
\[(p_{r_3}(x_2 - x_1 + h) - p_{r_3}(x_2 - x_1)) \Delta_x^h p_{r_4}(0)\]  

(6.19)

By a change of variables one sees that the integral of this term with respect to \(x_1\) and \(x_2\) is zero.

The only non-zero integrals in (6.16) comes from

\[p_{r_1}(x_1)\Delta^h \Delta^{-h} p_{r_2}(0)p_{r_3}(x_2 - x_1)\Delta^h \Delta^{-h} p_{r_4}(0).\]  

(6.20)

(Similar to the above \(\Delta^h \Delta^{-h} p_{r_2}(0)\) is \(\Delta_x^h \Delta^{-h} y_1 p_{r_2}(x_1 - y_1)\) where \(y_1 = x_1\).)

The integral of this term with respect to \(x_1\) and \(x_2\) is

\[\Delta^h \Delta^{-h} p_{r_2}(0)\Delta^h \Delta^{-h} p_{r_4}(0).\]  

(6.21)

We get the same contribution when \((\pi(1), \pi(2), \pi(3), \pi(4)) = (2, 2, 1, 1)\).

Consequently, the contribution to (6.15) of maps with two bound states is

\[8 \int_{\sum_{i=1}^4 r_i \leq t} \Delta_x^h \Delta_x^{-h} p_{r_2}(0) \Delta_x^h \Delta_x^{-h} p_{r_4}(0) \prod_{i=1}^4 dr_i \]  

(6.22)

\[= 32 \int_{\sum_{i=1}^4 r_i \leq t} (p_{r_2}(0) - p_{r_2}(h)) (p_{r_4}(0) - p_{r_4}(h)) \prod_{i=1}^4 dr_i \]  

\[= 16 \int_{u+v \leq t} (t - u - v)^2 (p_u(0) - p_u(h)) (p_v(0) - p_v(h)) \, du \, dv. \]  

\[\leq 16t^2 \left( \int_0^\infty (p_u(0) - p_u(h)) \, du \right)^2 = (4c_{\psi,h,0} t)^2, \]  

see (4.24).

We next consider the contribution from terms with exactly one bound state. These come from maps of the form \((\pi(1), \pi(2), \pi(3), \pi(4)) = (1, 2, 2, 1)\) or \((\pi(1), \pi(2), \pi(3), \pi(4)) = (2, 1, 1, 2)\). These terms give non-zero contributions of the form

\[Q_2 := \int_{\sum_{i=1}^4 r_i \leq t} p_{r_1}(x) \Delta_x^h p_{r_2}(y - x) \Delta_y^h \Delta_y^{-h} p_{r_3}(0) \Delta_x^h p_{r_4}(x - y) \prod_{i=1}^4 dr_i \, dx \, dy \]  

(6.23)
\[ L^2 \text{ moduli of continuity of local times} \]

\[ = \int \int_{\{\sum_{i=1}^4 r_i \leq t\}} \Delta^{-h} p_{r_2}(y) \Delta^h \Delta^{-h} p_{r_3}(0) \Delta_{y}^{-h} p_{r_4}(y) \prod_{i=1}^4 dr_i \ dy; \]

\[ \text{Q}_3 := \int \int_{\{\sum_{i=1}^4 r_i \leq t\}} p_{r_1}(x) \Delta^h \Delta^{-h} p_{r_2}(y-x) \Delta_{y}^h \Delta^{-h} p_{r_4}(x-y) \prod_{i=1}^4 dr_i \ dx \ dy \] (6.24)

and

\[ \text{Q}_4 := \int \int_{\{\sum_{i=1}^4 r_i \leq t\}} p_{r_1}(x) \Delta^h \Delta^{-h} p_{r_2}(y-x) \Delta_{y}^h \Delta^{-h} p_{r_3}(0) \Delta_{x}^h \Delta^{-h} p_{r_4}(x-y) \prod_{i=1}^4 dr_i \ dx \ dy \] (6.25)

For further explanation consider \( \text{Q}_2 \). This arrangement comes from the sequence \((x_1, y_2, x_2, y_1)\). The expression it is equal to comes by making the change of variables, \(y - x \rightarrow y\) and then integrating with respect to \(x\).

Integrating and using (2.8) we see that

\[ Q_2 \leq t \left( \int_0^1 |\Delta^h \Delta^{-h} p_s(0)| \ ds \right) \int \left( \int_0^t \Delta^{-h} p_r(y) \ dr \right)^2 \ dy \] (6.26)

\[ \leq \frac{Ct}{h\psi(1/h)} \int \left( \int_0^t \Delta^{-h} p_r(y) \ dr \right)^2 \ dy. \]

Here we use the fact that \( \int \Delta^{-h} p_{r_2}(y) \Delta^{-h} p_{r_4}(y) \ dy \geq 0 \) to extend the region of integration with respect to \(r_2\) and \(r_4\). By Parseval’s Theorem and (5.3)

\[ \int \left( \int_0^t \Delta^{-h} p_r(y) \ dr \right)^2 \ dy \] (6.27)

\[ = \frac{1}{2\pi} \int |1 - e^{iph}|^2 \left( \int_0^t e^{-r\psi(p)} \ dr \right)^2 \ dp \]

\[ \leq \frac{8}{\pi} \int \sin^2(hp/2) \left( t \wedge \frac{1}{\psi(p)} \right)^2 \ dp. \]
Similar to the transition between (6.6) and (6.7) the last integral is bounded by
\[ C \left( h^2 t^2 \int_0^{\psi^{-1}(1/t)} p^2 \, dp + h^2 \int_0^{1/h} \frac{p^2}{\psi^2(p)} \, dp + \int_1^{\infty} \frac{1}{\psi^2(p)} \, dp \right). \] (6.28)

Note that
\[ h^2 t^2 \int_0^{\psi^{-1}(1/t)} p^2 \, dp \leq C h^2 t^2 \left( \psi^{-1}(1/t) \right)^3. \] (6.29)

This bound is the right hand side of (6.8). Bounds for the other integrals are given in (6.9)–(6.12). Since the bounds in (6.8)–(6.12) give (4.27), we see that
\[ Q_2 \leq C t g(h, t) \frac{h^2}{\psi^2(1/h)}. \] (6.30)

To obtain a bound for \( Q_3 \) we use (2.10) and (5.7) to see that it is bounded in absolute value by
\[ t \left( \int_0^t p_s(0) \, ds \right) \int \left( \int_0^{1} |\Delta^h \Delta^{-h} p_r(y)\,| \, dr \right)^2 \, dy \leq C t^2 \psi^{-1}(1/t). \] (6.31)

Integrating \( Q_4 \) and using the Cauchy-Schwarz Inequality we see that it is bounded in absolute value by
\[ t \left| \int_0^1 \Delta^h p_r(0) \, dr \right| \left( \int \left| \int_0^{1} \Delta^h \Delta^{-h} p_r(y) \, dr \right|^2 \, dy \int \left| \int_0^{1} \Delta^{-h} p_r(y) \, dr \right|^2 \, dy \right)^{1/2}. \] (6.32)

By (2.5), (2.10) and (2.7) we get
\[ Q_4 \leq \frac{C t}{h^{3/2} \psi^{5/2}(1/h)}. \] (6.33)

Finally, we consider the contribution from terms in (6.15) with no bound states. These have to be from \( \pi \) of the form \((\pi(1), \pi(2), \pi(3), \pi(4)) = (1, 2, 1, 2)\) or of the form \((\pi(1), \pi(2), \pi(3), \pi(4)) = (2, 1, 2, 1)\). They give contributions of the form
\[ Q_5 := \int \int \left\{ \sum_{i=1}^4 p_{r_1}(x) \Delta_y^h p_{r_2}(y-x) \Delta_y^h \Delta_y^h p_{r_3}(x-y) \Delta_y^h p_{r_4}(y-x) \prod_{i=1}^4 dr_i \, dx \, dy \right\}. \] (6.34)
and

\[ Q_6 := \int \int_{\{ \sum_{i=1}^4 r_i \leq t \}} pr_1(x) \Delta_x^h \Delta_y^h p_{r_2}(y-x) p_{r_3}(x-y) \Delta_x^h \Delta_y^h p_{r_4}(x-y) \prod_{i=1}^4 dr_i \, dx \, dy \]  

(6.35)

\[ = \int \int_{\{ \sum_{i=1}^4 r_i \leq t \}} \Delta_x^h \Delta_y^h p_{r_2}(y) p_{r_3}(y) p_{r_4}(y) \prod_{i=1}^4 dr_i \, dy. \]  

Clearly

\[ Q_5 \leq t \int \left( \int_0^1 |\Delta_x^h p_r(y)| \, dr \right) \left( \int_0^1 |\Delta_y^h p_r(y)| \, dr \right) dy. \]  

(6.36)

Using (2.5), and (2.9) we see that

\[ Q_5 \leq Ct \frac{\log 1/h}{h^2 \psi_3(1/h)}. \]  

(6.37)

The term \( Q_6 \) is bounded the same way we bounded \( Q_3 \) and has the same bound.

It follows from (4.26), Lemma 4.3 and (6.22) that

\[ \text{Var} \, Z \leq C \left( \sum_{j=2}^6 |Q_j| + \left( \frac{tg(h,t)}{h\psi(1/h)} \right) \right) \]  

(6.38)

as \( h \to 0 \), since \( g(h,t) < t/(h\psi(1/h)) \). (We need a large constant because expressions for \( Q_j, j = 2, \ldots, 6 \) occur many ways, according to combinatorics of the distribution of the difference operators.)

We leave it to the reader to verify that replacing \( p \) by \( p^2 \) only adds error terms that do not change (4.29) and (4.30).

\[ \square \]

Proof of Lemma 4.5 Use (6.3)–(6.7) with \( \psi^{-1}(1/t) \) replaced by 1. In place of (6.8) we have

\[ h^2 \int_0^1 \frac{p^2}{\psi(p)} \, dp \leq Ch^2. \]  

(6.39)
(Since \( \lim_{p \to 0} \psi(p)/p^2 > 0 \) this integral is finite; see [7, Lemma 4.2.2]).

In place of (6.10) we have, if \( \beta > 3/2 \)

\[
    h^2 \int_1^{1/h} \frac{p^2}{\psi^2(p)} \, dp \leq C h^2. \tag{6.40}
\]

The statements in (6.11) and (6.12) remain the same when \( \psi^{-1}(1/t) \) replaced by 1. With these changes the proof of (4.27) gives (4.29).

**Proof of Theorem 1.1**

We note that by Remark 6.1 and Lemma 4.5, for symmetric stable processes of index \( \beta \),

\[
    E \left( \int (L_{x+h}^2 - L_x^2) \, dx \right) = 4c_{\beta,0} h^{\beta-1} + O(\overline{g}(h)) \tag{6.41}
\]

Since

\[
    \lim_{h \to 0} \sqrt{h} \psi^2(1/h) \overline{g}(h) = 0 \tag{6.42}
\]

we get (1.5) with \( t = 1 \). As we remark in the paragraph proceeding (1.10) this suffices to prove Theorem 1.1 as stated.

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## 7 Appendix: Kac Moment Formula

Let \( X = \{X_t, t \in R_+\} \) denote a symmetric Lévy process with continuous local time \( L = \{L^x_t; (x,t) \in R^1 \times R_+\} \). Since \( L \) is continuous we have the occupation density formula,

\[
    \int_0^t g(X_s) \, ds = \int g(x) L_x^t \, dx, \tag{7.1}
\]

for all continuous functions \( g \) with compact support. (See, e.g. [7, Theorem 3.7.1].)

Let \( f(x) \) be a continuous function on \( R^1 \) with compact support with \( \int f(x) \, dx = 1 \). Let \( f_{\epsilon,y}(x) := \frac{1}{\epsilon} f \left( \frac{x-y}{\epsilon} \right) \). I.e., \( f_{\epsilon,y}(x) \) is an approximate \( \delta \)-function at \( x \). Set

\[
    L_{t,\epsilon}^x = \int_0^t f_{t,x}(X_s) \, ds. \tag{7.2}
\]

It follows from (7.1) that

\[
    L_t^x = \lim_{\epsilon \to 0} L_{t,\epsilon}^x \quad \text{a. s.} \tag{7.3}
\]

Let \( p_t(x,y) \) denote the probability density of \( X_t \).
Theorem 7.1 (Kac Moment Formula) For any fixed $0 < t < \infty$, bounded continuous $g$, and any $x_1, \ldots, x_m, z \in \mathbb{R}^1$,

$$E^z \left( \prod_{i=1}^m L_t^{x_i} g(X_t) \right) = \sum_{\pi} \int_{\{\sum_{j=1}^m r_j \leq t\}} \prod_{j=1}^m p_{r_j}(x_{\pi(j-1)}, x_{\pi(j)}) \left( \int p_{t-r_m}(x_{\pi(m)}, y) g(y) \, dy \right) \prod_{j=1}^m dr_j,$$

(7.4)

where the sums run over all permutations $\pi$ of $\{1, \ldots, m\}$ and $\pi(0) := 0$ and $x_0 := z$.

Proof Let

$$F_t(x_1, \ldots, x_m) = \int_{\{\sum_{j=1}^m r_j \leq t\}} \prod_{j=1}^m p_{r_j}(x_{j-1}, x_j) \left( \int p_{t-r_m}(x_{\pi(m)}, y) g(y) \, dy \right) \prod_{j=1}^m dr_j,$$

(7.5)

Then

$$E^z \left( \prod_{i=1}^m L_t^{x_i} g(X_t) \right) = \sum_{\pi} \int_{\{0 \leq t_{\pi(1)} \leq \ldots \leq t_{\pi(m)} \leq t\}} E^z \left( \prod_{j=1}^m f_{\epsilon, x_j}(X_{t_{\pi(j)}}, g(X_t)) \right) \prod_{j=1}^m dt_{\pi(j)}$$

$$= \sum_{\pi} \int_{\{0 \leq t_1 \leq \ldots \leq t_m \leq t\}} E^z \left( \prod_{j=1}^m f_{\epsilon, x_{\pi(j)}}(X_{t_j}) \right) \prod_{j=1}^m dt_j$$

$$= \sum_{\pi} \int \int_{\{\sum_{j=1}^m r_j \leq t\}} \prod_{j=1}^m f_{\epsilon, x_{\pi(j)}}(y_j) p_{r_j}(y_{j-1}, y_j) \left( \int p_{t-r_m}(y_m, y) g(y) \, dy \right) \prod_{j=1}^m dr_j \, dy_j$$

(7.7)

$$= \sum_{\pi} \int F_t(y_0, \ldots, y_m) \prod_{j=1}^m f_{\epsilon, x_{\pi(j)}}(y_j) \, dy_j$$

where $y_0 := z$.

Since the integrand in (7.5) is dominated by $(2\pi)^{-m/2} \prod_{j=1}^m r_j^{-1/2}$ it follows from the Dominated Convergence Theorem that $F_t(x_1, \ldots, x_m)$
is a continuous function of \((x_1, \ldots, x_m)\) for all \(0 \leq t < \infty\) and all \(m\).

It then follows immediately from (7.6) and the fact that \(\prod_{j=1}^m f_{\epsilon, x_{\pi(j)}}(y_j)\) has compact support that
\[
\lim_{\epsilon \to 0} E \left( \prod_{i=1}^m L_{t,\epsilon}^{x_i} g(X_t) \right) = \sum_\pi F_t(x_{\pi(0)}, x_{\pi(1)}, \ldots, x_{\pi(m)}). \quad (7.8)
\]

A repetition of the above proof shows that \(E \left( \left( \prod_{i=1}^m L_{t,\epsilon}^{x_i} \right)^2 \right)\) is bounded uniformly in \(\epsilon > 0\). This fact and (7.3) show that
\[
\lim_{\epsilon \to 0} E \left( \prod_{i=1}^m L_{t,\epsilon}^{x_i} g(X_t) \right) = E \left( \prod_{i=1}^m L_{t}^{x_i} g(X_t) \right). \quad (7.9)
\]

Obviously (7.8) and (7.9) imply (7.4). \(\square\)

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Jay Rosen \hspace{2cm} Michael Marcus
Department of Mathematics \hspace{2cm} Department of Mathematics
College of Staten Island, CUNY \hspace{2cm} City College, CUNY
Staten Island, NY 10314 \hspace{2cm} New York, NY 10031
jrosen30@optimum.net \hspace{2cm} mbmarcus@optonline.net