On Hot Bangs and the Arrow of Time in Relativistic Quantum Field Theory

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Dedicated to Rudolf Haag on the occasion of his 80th birthday

Abstract

A recently proposed method for the characterization and analysis of local equilibrium states in relativistic quantum field theory is applied to a simple model. Within this model states are identified which are locally (but not globally) in thermal equilibrium and it is shown that their local thermal properties evolve according to macroscopic equations. The largest space–time regions in which local equilibrium states can exist are timelike cones. Thus, although the model does not describe dissipative effects, such states fix in a natural manner a time direction. Moreover, generically they determine a distinguished space–time point where a singularity in the temperature (a hot bang) must have occurred if local equilibrium prevailed thereafter. The results illustrate how the breaking of the time reflection symmetry at macroscopic scales manifests itself in a microscopic setting.

1 Introduction

Within the framework of relativistic quantum field theory, a general method has been established in [1] for the characterization and analysis of states which are locally close to thermal equilibrium. This novel approach is based on the idea of comparing states locally with the members of a family of thermal reference states consisting of mixtures of global equilibrium states: If a given state happens to coincide at a space–time point \( x \) with some such reference state to some degree of accuracy, \( i.e. \) if the expectation values of a sufficiently large number of distinguished local observables at \( x \) coincide in the two states, the given state is said to be locally in equilibrium at \( x \). One can then determine the thermal properties of the state at \( x \) by noticing that the local observables have an unambiguous macroscopic interpretation in all global equilibrium states. Moreover, the space–time evolution of these thermal properties is directly linked to the microscopic dynamics.
The application of this method to concrete models thus requires the following steps: (1) Determine the global equilibrium states (KMS states) of the theory; their mixtures define the convex set of thermal reference states. (2) Select for each space–time point \( x \) a set of suitable local observables which are sensitive to thermal properties. As was explained in [1], a natural choice consists of the basic observable point fields of the theory as well as of density like quantities which are obtained by taking normal products. (3) Determine the thermal interpretation of these local observables. This is accomplished by computing their expectation values in the global equilibrium states. The result is a family of thermal functions depending on the parameters characterizing the thermal states, such as temperature and chemical potential; they describe certain specific intensive macroscopic properties of the global equilibrium states. (4) After these preparations, one can characterize the local equilibrium states in a given space–time region as follows: at each point of the region, the mean values of the distinguished local observables taken in such a state coincide with those in some thermal reference state. This reference state may vary from point to point. One then ascribes to the local equilibrium state those macroscopic thermal properties of the respective reference states which are fixed by the distinguished thermal functions. The space–time evolution of these properties can be derived directly from the underlying microscopic dynamics.

In the present investigation we apply this scheme to the theory of a free massless scalar field. In view of the simplicity of this model, it may be necessary to comment on the physical significance of such an analysis. It is clear from the outset that a non–equilibrium state in this model will never approach (local) equilibrium because of the lack of interaction. So we will have nothing to say about this intriguing problem. But, appealing to physics, one knows that, once local equilibrium has been reached in a system, there appear in general long–living macroscopic structures, such as hydrodynamical flows, where dissipative effects are of minor importance. Moreover, if the local temperature of a system is sufficiently high and the interaction is asymptotically free, these dissipative effects ought to be small also on dynamical grounds. Thus one may expect that massless free field theory provides some physically meaningful qualitative information about the possible space–time patterns of local equilibrium states in hot relativistic systems.

The present investigation is primarily intended as an illustration of the general ideas expounded in [1], but it offers some surprises. First, we will see that in all local equilibrium states of the model one can determine locally macroscopic properties, such as the particle density (distribution function), without having to rely on an \textit{ad hoc} procedure of coarse graining. Second, we will establish evolution equations for these macroscopic observables, akin to transport equations. These equations, together with further constraints arising from the statistical interpretation of the microscopic theory, imply that the largest space–time regions in which local equilibrium states (not being in global equilibrium) can exist have the form of timelike cones. Phrased differently, the time reflection symmetry is broken in these states, so one can fix with their help a time direction. Finally, it turns out that such states determine generically a unique space–time point where
a singularity in the temperature (a hot bang) must have occurred if the state was in local equilibrium thereafter. An example of such a state was already presented in [1]. The present analysis shows that the appearance of such singularities is a generic feature of the model. Incidentally, it provides a consistent qualitative picture of the space–time patterns of the cosmic background radiation.

The general mathematical framework and scheme for the identification and analysis of local equilibrium states is outlined in the subsequent section. For a more detailed discussion and physical justification of the method we refer to [1]. In Sec. 3 we describe the model and carry out the preparatory steps, indicated above, which are required for the analysis of the local equilibrium states. Section 4 contains the discussion of the space–time behaviour of thermal properties of these states and the derivation of macroscopic equations governing their evolution. In Sec. 5 we show that local (in contrast to global) equilibrium cannot exist in all of space–time and we determine the maximal regions admitting this feature. The paper concludes with a discussion of these results and some interesting perspectives.

2 General framework and scheme

The basic objects in our analysis are local observable fields, such as conserved currents and the stress energy tensor, denoted generically by \( \phi(x) \) (with tensor indices omitted). Mathematically, these fields may be interpreted in different ways. Usually, they are regarded as operator valued distributions, i.e. their space–time averages with test functions \( f \) having compact support in Minkowski space \( \mathbb{R}^4 \),

\[
\phi(f) = \int dx \, f(x) \phi(x),
\]

are assumed to be defined on some common dense and stable domain \( D \) in the underlying Hilbert space \( \mathcal{H} \). The finite sums and products of the averaged field operators generate a \( * \)-algebra \( \mathcal{A} \), called the algebra of observables, consisting of all polynomials \( A \) of the form

\[
A = \sum \phi(f_1)\phi(f_2)\cdots\phi(f_n).
\]

(2.2)

We include also all multiples of the unit operator \( 1 \) in \( \mathcal{A} \). As the fields \( \phi \) are real (being observable), the \( * \)-operation (Hermitian conjugation) on \( \mathcal{A} \) is defined by

\[
A^* = \sum \phi(\overline{f}_n)\cdots\phi(\overline{f}_2)\phi(\overline{f}_1),
\]

(2.3)

where \( \overline{f} \) denotes the complex conjugate of \( f \). The Poincaré transformations \( \lambda \in \mathcal{P}_+^\uparrow \) act on \( \mathcal{A} \) by automorphisms \( \alpha_\lambda \),

\[
\alpha_\lambda(A) = \sum \phi(f_{1,\lambda})\phi(f_{2,\lambda})\cdots\phi(f_{n,\lambda}),
\]

(2.4)

where \( f_\lambda \) is defined by \( f_\lambda(x) = D(\lambda)f(\lambda^{-1}x) \) and \( D \) is a matrix representation of \( \mathcal{P}_+^\uparrow \) corresponding to the tensor character of the respective field \( \phi \). If \( \lambda \) is a pure translation, \( \lambda = (1, a), a \in \mathbb{R}^4 \), we denote the corresponding automorphism by \( \alpha_a \).
The states of the physical systems are described by positive normalized expectation functionals on the algebra $\mathcal{A}$, generically denoted by $\omega$. We recall that these functionals have the defining properties $\omega(c_1 A_1 + c_2 A_2) = c_1 \omega(A_1) + c_2 \omega(A_2)$, $c_1, c_2 \in \mathbb{C}$ (linearity), $\omega(A^* A) \geq 0$ (positivity), and $\omega(1) = 1$ (normalization). As has been explained in the introduction, an important ingredient in the present investigation are the global equilibrium states which are characterized by the KMS condition \[2, 3\]. Given a Lorentz frame, fixed by some positive time-like vector $e \in V_+$ of unit length, $e^2 = 1$, this condition can be stated as follows.

**KMS condition:** A state $\omega_\beta$ satisfies the KMS condition at inverse temperature $\beta > 0$ in the given Lorentz frame if for each pair of operators $A, A' \in \mathcal{A}$ there is some function $h$ which is analytic in the strip $S_\beta = \{ z \in \mathbb{C} : 0 < \text{Im} z < \beta \}$ and continuous at the boundaries such that

$$h(t) = \omega_\beta(A' \alpha_{te}(A)), \quad h(t + i\beta) = \omega_\beta(\alpha_{te}(A) A'), \quad t \in \mathbb{R}. \quad (2.5)$$

In this situation $\omega_\beta$ is called a KMS state.

Any KMS state $\omega_\beta$ describes an ensemble which is in thermal equilibrium in the distinguished Lorentz frame, describing the rest system of the state. As we have to keep track of both, temperatures and rest systems, we combine this information into a four vector $\beta e \in V_+$, again denoted by $\beta$. Moreover, in order to simplify the subsequent discussion, we assume that for any given $\beta \in V_+$ the corresponding KMS state $\omega_\beta$ is unique. As a consequence of this uniqueness assumption, the KMS states transform under Poincaré transformations according to

$$\omega_\beta \circ \alpha^{-1} = \omega_{\Lambda \beta}, \quad (2.6)$$

i.e. they are isotropic in their rest systems and invariant under space-time translations. Next, let us introduce the thermal reference states. They are defined, for any compact subset $B \subset V_+$, as arbitrary mixtures of KMS states $\omega_\beta$ with $\beta \in B$. We emphasize that the restriction to mixtures with compact temperature support $B$ is merely a matter of mathematical convenience in the present context. In general one may admit larger families of reference states with non-compact temperature support. In order to obtain a more explicit description of the reference states and to avoid mathematical subtleties, we assume that the KMS states $\omega_\beta$ are weakly continuous in $\beta$, i.e. all functions

$$\beta \mapsto \omega_\beta(A), \quad A \in \mathcal{A}, \quad (2.7)$$

are continuous. Apart from phase transition points (which are excluded here by our uniqueness assumption), this property is expected to hold quite generally. With this input, the states $\omega_B \in \mathcal{C}_B$ can be represented in the form

$$\omega_B(A) = \int d\rho(\beta) \omega_\beta(A), \quad A \in \mathcal{A}, \quad (2.8)$$

where $\rho$ is a positive normalized measure which has support in $B$. The convex set of all these states will be denoted by $\mathcal{C}_B$ and the elements of $\mathcal{C} = \bigcup C_B$, where the
union extends over all compact subsets \( B \subset V_+ \), will be our reference states for the characterization and analysis of the local equilibrium states.

It is mathematically meaningful and conceptually convenient in the present analysis to consider also the unregularized fields \( \phi(x) \). Because of their singular nature they have to be interpreted in the form sense, however. Let us illustrate this for the KMS states. In view of the invariance of these states under space–time translations, the expectation values \( \omega_\beta(\phi(f)) \) do not depend on the choice of \( f \) as long as the value of the space–time integral of the test function is kept fixed. One may thus extend the thermal reference states to the unsmeared fields \( \phi(x) \) by the formula

\[
\omega_B(\phi(x)) = \omega_B(\phi(f)),
\]

where the space–time integral of \( f \) is put equal to 1. This expression does not depend on \( x \). We denote the linear space generated by all fields \( \phi(x) \) at \( x \) by \( Q_x \) and assume without further mentioning that all states \( \omega \) of interest here can be extended to the spaces \( Q_x \) for \( x \) varying in some region.

It is crucial in the present approach that the local fields \( \phi(x) \) provide the same information about properties of the thermal reference states as certain macroscopic (central) observables. To verify this, we pick a test function \( f \) of compact support which integrates to 1 and put

\[
f_n(x) = n^{-4} f(n^{-1} x - x_n), \quad n \in \mathbb{N},
\]

where \( x_n \) is a sequence of translations tending sufficiently rapidly to spacelike infinity such that the support of \( f_n \) lies in the causal complement of any given bounded region for almost all \( n \). It follows from the uniqueness of the KMS states and their invariance under space–time translations by an application of the mean ergodic theorem that

\[
\omega_\beta(\phi(f_n)^* \phi(f_n)) - \omega_\beta(\phi(\phi)^* \omega_\beta(\phi(0))) \to 0.
\]

Moreover, because of the local commutativity of observables and the support properties of the test functions, \( \phi(f_n) \) commutes for almost all \( n \) with any given \( A \in \mathcal{A} \), \( i.e. \) the operators \( \phi(f_n) \) form a central sequence in \( \mathcal{A} \). Applying standard arguments, it follows that the limit

\[
\Phi = \lim_{n \to \infty} \phi(f_n)
\]

exists in all thermal reference states and defines a macroscopic observable commuting with all elements \( A \in \mathcal{A} \) and the translations. In fact, making use of (2.8), (2.10) and applying the dominated convergence theorem, one obtains

\[
\omega_B(A^* \Phi A) = \lim_{n \to \infty} \omega_B(A^* \phi(f_n) A) = \int d\rho(\beta) \omega_\beta(A^* A) \omega_\beta(\phi(0)).
\]

This relation implies that the (according to (2.7) continuous) function

\[
\beta \mapsto \Phi(\beta) = \omega_\beta(\phi(0)),
\]

called thermal function, determines the central decomposition of \( \Phi \); hence we may identify \( \Phi \) with this function. Such central operators are called macro–observables.
in the following. Since \( \omega_B(\phi(x)) = \omega_B(\phi(0)) = \omega_B(\Phi) \), the expectation values of the local observables \( \phi(x) \) in the thermal reference states can be interpreted in terms of the macro–observables \( \Phi \), as claimed.

Let us now turn to the characterisation of the local equilibrium states. Fixing some suitable subspace \( S_x \subset Q_x \), we say a state \( \omega \) is \( S_x \)-compatible with a thermal interpretation at \( x \) (\( S_x \)-thermal, for short) if there exists some \( \omega_B \in \mathcal{C} \) such that
\[
\omega(\phi(x)) = \omega_B(\phi(x)), \quad \phi(x) \in S_x.
\]
(2.14)

Thus \( \omega \) cannot be distinguished from the thermal reference state \( \omega_B \) by the local observables in \( S_x \). One can therefore consistently assign to it the thermal properties of \( \omega_B \) described by the macro–observables \( \Phi \) corresponding to \( \phi(x) \in S_x \).

In this way the state \( \omega \) acquires at \( x \) a physically meaningful (partial) interpretation in macroscopic terms.

This definition extends in a straightforward manner to states which are locally in equilibrium in (open) regions \( \mathcal{O} \subset \mathbb{R}^4 \). One first identifies the spaces of observables \( S_x, x \in \mathbb{R}^4 \), with the help of the automorphic action of the translations, setting
\[
S_x = \alpha_x(S_0).
\]
(2.16)

This being understood, a state \( \omega \) is said to be \( \mathcal{S}_\mathcal{O} \)-thermal if for each \( x \in \mathcal{O} \) there is some thermal reference state \( \omega_{B_x} \in \mathcal{C}, x \in \mathcal{O}, \) which are weakly integrable in \( x \) and have temperature supports \( B_x \subset \mathcal{V}_x \) for \( x \) varying in any given compact subset of \( \mathcal{O} \). So the functions \( x \mapsto \omega(\Phi)(x) \) are differentiable in the sense of distributions. As a matter of fact, for any test function \( f \) having compact support in \( \mathcal{O} \)
\[
\left| \int dx \ f(x) \omega(\Phi)(x) \right| \leq \| f \|_1 \| \Phi \|_B,
\]
(2.18)
where \( \| f \|_1 \) denotes the \( L^1 \)–norm of \( f \), and we have introduced on the thermal functions (2.13) the seminorms
\[
\| \Phi \|_B = \sup_{\beta \in B} | \Phi(\beta) |.
\]
(2.19)

This statement follows from relations (2.8), (2.13) and (2.14). It will allow us to extend the lifts of local equilibrium states to larger spaces of macro–observables.
3 The model

In the remainder of this article we apply the preceding general scheme to the theory of a free massless scalar field. After a brief synopsis of the model, we proceed according to the steps outlined in the introduction and determine the thermal reference states as well as a distinguished space of local observables needed for the identification and macroscopic interpretation of the local equilibrium states. An example of such a state is given at the end of this section.

The free massless scalar field \( \phi_0(x) \) on \( \mathbb{R}^4 \) is characterized by its field equation and commutation relation

\[
\Box_x \phi_0(x) = 0, \quad [\phi_0(x_1), \phi_0(x_2)] = (2\pi)^{-3} \int dp \, e^{-i(x_1-x_2)p} \varepsilon(p_0) \delta(p^2) \cdot 1, \tag{3.1}
\]

which are to be understood in the sense of distributions. It generates, together with its normal products with reference to the vacuum state (cf. the examples below), a polynomial *-algebra \( \mathcal{A} \). This algebra is stable under the actions of the Poincaré group \( P_+ \), given by \( \alpha_{\Lambda, a}(\phi_0(x)) = \phi_0(\Lambda x + a) \) and the gauge group \( \mathbb{Z}^2 \), given by \( \gamma(\phi_0(x)) = -\phi_0(x) \).

We restrict attention here to states \( \omega \) on \( \mathcal{A} \) which are gauge invariant, i.e. \( \omega \circ \gamma = \omega \), so the respective \( n \)-point functions of \( \phi_0 \) vanish if \( n \) is odd. The simplest examples of this kind are quasifree states. They are completely determined by their two-point functions through the formula

\[
\omega(\phi_0(x_1)\phi_0(x_2)\cdots\phi_0(x_n)) = \begin{cases} \sum_{\text{pairings}} \omega(\phi_0(x_{i_1})\phi_0(x_{i_2})\cdots\omega(\phi_0(x_{i_{n-1}})\phi_0(x_{i_n})) & n \text{ even,} \\ 0 & n \text{ odd.} \end{cases} \tag{3.2}
\]

It is a well known fact that the algebra \( \mathcal{A} \) has a unique gauge invariant KMS state \( \omega_\beta \) for each temperature vector \( \beta \in V_+ \). This state is quasifree, so it is determined by its two-point function given by

\[
\omega_\beta(\phi_0(x_1)\phi_0(x_2)) = (2\pi)^{-3} \int dp \, e^{-i(x_1-x_2)p} \varepsilon(p_0) \delta(p^2) \frac{1}{1 - e^{-\beta p}}. \tag{3.3}
\]

The KMS states \( \omega_\beta \) comply with our continuity assumption (2.7) and fix the convex set \( \mathcal{C} \) of thermal reference states which enters into our definition of local equilibrium states.

Next, we describe the spaces \( \mathcal{S}_x \) of local observables which will be used to analyze the local equilibrium states. They are generated by the unit operator 1, the field \( \phi_0(x) \) and balanced derivatives of its normal ordered square. Introducing the multi-index notation \( \mu = (\mu_1, \mu_2, \ldots, \mu_m) \) and setting \( \partial_\zeta^\mu = \partial_{\mu_1} \cdots \partial_{\mu_m} \), the latter observables are given by

\[
\theta^\mu(x) = \lim_{\zeta \to 0} \partial^\mu_\zeta \left( \phi_0(x + \zeta)\phi_0(x - \zeta) - \omega_\infty(\phi_0(x + \zeta)\phi_0(x - \zeta)) \right), \tag{3.4}
\]
where the limit is approached from spacelike directions $\zeta$ and $\omega_\infty$ denotes the vacuum (i.e. $\beta \wedge \infty$) state. We assume that all states of interest here are in the domains of these forms and note that $\theta^\mu (x) = 0$ if $m$ is odd since $\phi_0(x+\zeta)\phi_0(x-\zeta)$ is even in $\zeta$ as a consequence of locality.

As has been explained in Sec. 2, each $\theta^\mu (x)$ determines some macro–observable $\Theta^\mu$. The corresponding thermal functions have been computed in \cite{1} and are given by

$$\beta \mapsto \Theta^\mu (\beta) = \omega_\beta (\theta^\mu (0)) = c_m \partial_\beta (\beta^2)^{-1}, \quad (3.5)$$

where $c_m = 0$ if $m$ is odd and $c_m = (-1)^{m/2}(4\pi)^m(m+2)!^{-1}B_{m+2}$ if $m$ is even, $B_n$ being the (modulus of the) Bernouilli numbers. As $\square_\beta (\beta^2)^{-1} = 0$ on $V_+$, the observables $\Theta^\mu$ generate only a subspace of the space of all macro–observables. Yet, as we will see, the linear combinations of the corresponding thermal functions are dense in the space of smooth solutions of the wave equation on $V_+$ with respect to the topology induced by the seminorms (2.19). This fact is of relevance since it implies that many macro–observables of interest, such as the entropy density \cite{1} or the particle density, can be approximated in the states considered here with arbitrary precision by linear combinations of the $\Theta^\mu$.

**Lemma 3.1** Let $\Xi$ be any smooth solution of the wave equation on $V_+$, let $B \subset V_+$ be compact and let $\varepsilon > 0$. There are finitely many constants $c_\mu$, $c_\mu'$ such that

$$\| \Xi - \sum c_\mu \Theta^\mu \| _B < \varepsilon \quad \text{and} \quad \| \partial^\nu \Xi - \sum c_\mu' \Theta^{\nu\mu} \| _B < \varepsilon,$$

where $\nu\mu$ denotes the multi index $(\nu, \mu_1, \mu_2, \ldots \mu_m, \nu = 0, 1, 2, 3$.

**Proof.** It is sufficient to establish the statement for closed double cones $B$ of the form $B = \{ (\kappa, 0) + V_+ \} \cap \{ (\kappa^{-1}, 0) - V_+ \}$ for $0 < \kappa < 1$. These regions are stable under the involution $\iota (\beta) \equiv \beta / \beta^2$, $\beta \in V_+$. As is well known, this transformation induces an involution on the smooth solutions of the wave equation on $V_+$, given by $I(\Xi)(\beta) \equiv \iota (\beta)^2 \Xi (\iota (\beta))$. It is continuous relative to the seminorms (2.19),

$$\| I(\Xi) \| _B \leq \kappa^{-2} \| \Xi \| _B, \quad \| \partial^\nu I(\Xi) \| _B \leq 2\kappa^{-3} \| \Xi \| _B + 3\kappa^{-4} \sum \| \partial^\nu' \Xi \| _B.$$

(Similar bounds can be established also for higher derivatives by elementary computations.) Since $I$ is an involution, $\Xi$ and $I(\Xi)$ may be interchanged in these inequalities.

For the proof of the first half of the statement it is convenient to proceed, in a first step, to the functions $l_\mu \Theta^\mu$, where $l_\mu = l_{\mu_1} \ldots l_{\mu_m}$, $l$ being any positive lightlike vector, and, in a second step, to their images $I(l_\mu \Theta^\mu)$ under the involution $I$. According to the preceding remarks it suffices to show that the linear span of the latter functions is dense in the space of all smooth solutions of the wave equation with regard to the seminorms $\| \cdot \| _B$. The functions $I(l_\mu \Theta^\mu)$, resulting from (3.5), have the simple form

$$\beta \mapsto I(l_\mu \Theta^\mu)(\beta) = c_m (l\beta)^m;$$
where $c'_m \neq 0$ if $m$ is even and $c'_m = 0$ if $m$ is odd. So we have to show that suitable linear combinations of the functions $\beta \mapsto (l\beta)^m$ for even $m$ and lightlike $l$ approximate any given smooth solution of the wave equation, uniformly on compact sets $B \subset V_+$.

We make use of the fact that the functions $z \mapsto e^{\pm iz}$ are analytic in the complex half–plane $\mathbb{C}_+ \doteq \{ z \in \mathbb{C} : \text{Re } z > 0 \}$. Thus they can be represented on any compact subset $C \subset \mathbb{C}_+$ by a uniformly convergent power series in $z$, whence $z \mapsto e^{\pm iz} = e^{\pm i\sqrt{z}}$ can be represented by a uniformly convergent power series in $z^2 \in C$. As $\kappa|l| \leq l\beta \leq 2\kappa^{-1}|l|$ for $\beta \in B$ and $l = (|l|, l)$, we see that the functions $\beta \mapsto e^{\pm i l\beta}$ can be represented by uniformly convergent power series involving only even powers $(l\beta)^m$, $\beta \in B$. Moreover, the functions

$$
\beta \mapsto \sum_{\pm} \int d^3l \, f_\pm(l) \, e^{\pm i l\beta},
$$

where $f_\pm$ are absolutely integrable, can be approximated by linear combinations of $\beta \mapsto e^{\pm i l\beta}$, uniformly on compact sets $B$. So we conclude that linear combinations of $\beta \mapsto (l\beta)^m$ for even $m$ and suitable $l$ approximate these functions as well.

It remains to show that the restriction of any smooth solution $\Xi$ of the wave equation to the double cone $B$ admits an integral representation as given above. This can be seen if one extends the smooth Cauchy data of $\Xi$ on the base of $B$ to test functions with compact support in a slightly larger region. Using these data as initial values, one obtains a smooth solution $\Xi_B$ of the wave equation which coincides with $\Xi$ on $B$ according to standard uniqueness results. In view of the regularity properties of its Cauchy data, $\Xi_B$ admits an integral representation as given above. In fact, the corresponding functions $f_\pm$ are, apart from an integrable singularity at the origin, smooth and rapidly decreasing. This completes the proof of the first part of the statement.

The proof of the second part proceeds along similar lines and we therefore indicate only the essential steps. We begin by noting that

$$
\beta \mapsto l_\mu \Theta^\mu (\beta) = \partial^\nu c'_{m+1} (l\beta)^m (\beta^2)^{-m-1},
$$

where $c'_{m+1} \neq 0$ if $m$ is odd. Applying the involution $I$ to $\beta \mapsto (l\beta)^m (\beta^2)^{-m-1}$, we see that it suffices to show that suitable linear combinations of the functions $\beta \mapsto (l\beta)^m$ for odd $m$ and lightlike $l$ approximate any given smooth solution of the wave equation, uniformly on compact sets $B \subset V_+$, and that the same holds true for their respective partial derivatives.

We now use the fact that the functions $z \mapsto (1/\sqrt{z}) \, e^{\pm iz}$ are analytic on $\mathbb{C}_+$. Hence $z \mapsto e^{\pm iz} = (z/\sqrt{z}) \, e^{\pm i\sqrt{z}}$ can be represented, on the domains specified above, by uniformly convergent power series involving only odd powers of $z$. So the functions $\beta \mapsto e^{\pm i l\beta}$ can likewise be represented by uniformly convergent power series involving only odd powers $(l\beta)^m$, $\beta \in B$. In view of the analyticity of these functions, this uniformity of convergence holds also for their respective partial derivatives.
Now if the functions $f_\pm$ in the above integral representation of solutions of the wave equation are sufficiently rapidly decreasing, these integrals can not only be approximated by suitable linear combinations of the functions $\beta \mapsto e^{\pm i\beta}$, uniformly on $B$, but the same holds true also for their respective partial derivatives. Yet, as has been explained, the restrictions of smooth solutions of the wave equation to $B$ can be represented in this desired way, so the proof of the statement is complete.  

In order to illustrate the significance of this result, let us consider the mean density of particles of momentum $p$. From a macroscopic point of view, the corresponding observable consists, for a system in a finite box, of the product of the annihilation and creation operators of a particle of momentum $p$, divided by the volume of the box. Being in a setting describing the thermodynamic limit, this observable is obtained as limit of sequences of elements of $A$ which are bilinear in the basic field $\phi_0$,

$$N_p = \lim_{n} 2|p| \phi_0(f_{p,n})^* \phi_0(f_{p,n}). \quad (3.6)$$

Here $p = (|p|, p)$ and $f_{p,n}(x) \equiv n^{-5/2} g(n^{-1}x_0) h(n^{-1}x) e^{ipx}$, where $g$, $h$ are test functions satisfying $\int dx_0 g(x_0) = 1$ and $\int d^3x |h(x)|^2 = 1$. The limit exists in all thermal reference states and its expectation values can be centrally decomposed similarly to relation (2.12). Thus each $N_p$ is a macro-observable commuting with all elements of $A$. The corresponding thermal function is, as expected, given by Planck’s famous formula for the density of massless particles of momentum $p$ in a thermal equilibrium state,

$$\beta \mapsto N_p(\beta) = (2\pi)^{-3} \frac{1}{e^{\beta p} - 1}. \quad (3.7)$$

As these functions are smooth solutions of the wave equation on $V_+$, we infer from the preceding lemma that, instead of relying on (3.6), one can determine the particle density $N_p$ also locally. More precisely, for given temperature support $B$ there are local observables in $S_x$ which allow to determine the mean values of $N_p$ in all thermal reference states in $C_B$ with arbitrary precision. Since all powers of $N_p$ are represented by smooth solutions of the wave equation as well, the same holds true also for all moments of $N_p$ in these states.

Having thus determined all relevant quantities for the examination of local equilibrium states, let us turn now to their actual analysis. We will consider states $\omega$ which are $S_0$-thermal in regions $O$ of Minkowski space, where $S_x$, $x \in O$, are the spaces of local observables defined above. Taking into account the preceding lemma and relation (2.13), these states can be lifted at each $x \in O$ to all macro-observables $\Xi$ whose thermal functions are smooth solutions of the wave equation on $V_+$. These observables will be called admissible in the following. We are interested in the space–time behaviour of their respective expectation values,

$$x \mapsto \omega(\Xi)(x), \quad x \in O, \quad (3.8)$$

which describe the evolution of these states from a macroscopic point of view.
An interesting example of a local equilibrium state which is $S_{V^+}$-thermal was presented in [1]. This state is quasifree and therefore fixed by its two-point function which, for $x_1, x_2 \in V^+$, has the form

$$\omega_{bhb}(\phi_0(x_1)\phi_0(x_2)) = (2\pi)^{-3} \int dp \, e^{-i(x_1-x_2)_p} \varepsilon(p_0) \delta(p^2) \frac{1}{1-e^{-\eta(x_1+x_2)_p}},$$

(3.9)

where $\eta > 0$ is some parameter. As was shown in [1], the restriction of $\omega_{bhb}$ to $S_x$ coincides with the KMS-state $\omega_\beta(x)$, where $\beta(x) = 2\eta x$, $x \in V^+$. So one obtains for the expectation values of the admissible macro-observables $\Xi$

$$\omega_{bhb}(\Xi)(x) = \Xi(2\eta x), \quad x \in V^+, \quad (3.10)$$

where $\beta \mapsto \Xi(\beta)$ is the thermal function corresponding to $\Xi$. Plugging the observable $T^2$ determining the square of the temperature into this equation (which is legitimate since $\beta \mapsto T^2(\beta) = (\beta^2)^{-1}$ is a smooth solution of the wave equation), one finds that the temperature of the state $\omega_{bhb}$ tends to infinity at the boundary of $V^+$. As a matter of fact, the state $\omega_{bhb}$ describes the space-time evolution of a hot bang at the origin of Minkowski space. This can be seen most clearly by looking at the expectation values of the density operators $N_p$,

$$\omega_{bhb}(N_p)(x) = (2\pi)^{-3} \frac{1}{e^{2\eta p x} - 1}, \quad x \in V^+. \quad (3.11)$$

If $x$ approaches the boundary of the lightcone $V^+$, this expression stays bounded unless $x$ and the four-momentum $p$ become parallel. Thus the bulk of the particles giving rise to the infinite temperature at the boundary of the lightcone originates from its apex. Whence the state $\omega_{bhb}$ provides a qualitative picture of the space-time patterns of the radiation caused by a hot bang. We will see that the occurrence of such singularities is a generic feature of the present model.

4 Evolution equations

We want to demonstrate now how the microscopic dynamics leads to evolution equations for the expectation values of macro-observables in local equilibrium states in the present simple model. We begin by noting that the field equation for $\phi_0$ entails the following equations (in the sense of distributions)

$$\partial_{x^\nu} \partial_{\zeta^\nu} \phi_0(x + \zeta)\phi_0(x - \zeta) = 0, \quad (4.1)$$

$$\Box_x \phi_0(x + \zeta)\phi_0(x - \zeta) = -\Box_{\zeta} \phi_0(x + \zeta)\phi_0(x - \zeta). \quad (4.2)$$

This implies for the local observables $\theta^\mu$, introduced in relation (3.4),

$$\partial_{x^\nu} \theta^\nu \theta^\mu(x) = 0, \quad (4.3)$$

$$\Box \theta^\mu(x) = -\theta^\nu_{\nu} \theta^\mu(x). \quad (4.4)$$
In particular, each $\theta^\mu$ is a conserved tensorial current. The latter relations lead to linear evolution equations for the expectation values of the corresponding macro–observables $\Theta^\mu$ in local equilibrium states. As a matter of fact, the following lemma holds.

**Lemma 4.1** Let $\omega$ be an $S_\varnothing$–thermal state and let $\Xi$ be any admissible macro–observable. Then

$$\partial_\nu \omega(\partial^\nu \Xi)(x) = 0, \quad \Box \omega(\Xi)(x) = 0 \quad \text{for } x \in \varnothing,$$

where $\partial^\nu \Xi$ is the macro–observable with thermal function $\beta \mapsto \partial^\nu \Xi(\beta)$.

**Proof.** For the proof of the first equation we proceed from relation (4.3) which implies for any choice of constants $c^\mu$

$$\partial_\nu \omega(\sum c^\mu \theta^\nu^\mu(x)) = 0.$$

As $\omega$ is $S_\varnothing$–thermal, we also have for $x \in \varnothing$

$$\omega(\sum c^\mu \Theta^\nu^\mu)(x) = \omega(\sum c^\mu \theta^\nu^\mu(x)),$$

and combining these two equations we arrive at

$$\partial_\nu \omega(\sum c^\mu \Theta^\nu^\mu)(x) = 0.$$

The first part of the statement now follows from the continuity property (2.18) of local equilibrium states and Lemma 3.1.

For the second equality we make use of relation (4.4) which implies

$$\beta \mapsto \Theta^\nu^\mu(\beta) = \omega_\beta(\theta^\nu^\mu(x)) = - \omega_\beta(\Box \theta^\mu(x)) = - \Box \omega_\beta(\theta^\mu(x)) = 0,$$

where the latter equality follows from the invariance of the KMS states under space–time translations. Hence $\Theta^\nu^\mu = 0$, so taking into account once more that $\omega$ is $S_\varnothing$–thermal we obtain

$$\Box \omega(\sum c^\mu \Theta^\mu)(x) = \Box \omega(\sum c^\mu \theta^\mu(x)) = \omega(\sum c^\mu \Box \theta^\mu(x))$$

$$= - \omega(\sum c^\mu \theta^\nu^\mu(x)) = - \omega(\sum c^\mu \Theta^\nu^\mu)(x) = 0.$$

As before, the second assertion then follows from the continuity properties of $\omega$ and Lemma 3.1. \hfill \Box

A macro–observable of particular physical interest is the particle density $N_p$. Its evolution in local equilibrium states is described in the following proposition.

**Proposition 4.2** Let $\omega$ be any $S_\varnothing$–thermal state and let $p$ be any positive lightlike vector. Then

$$(p \partial) \omega(N_p)(x) = 0, \quad \Box \omega(N_p)(x) = 0 \quad \text{for } x \in \varnothing.$$
**Proof.** Consider the function $L_p$ on $V_+$,

$$\beta \mapsto L_p(\beta) = (2\pi)^{-3} \ln (1 - e^{-\beta p}).$$

It is a smooth solution of the wave equation and consequently $L_p$ is an admissible macro–observable. Now

$$\partial^{\nu} L_p(\beta) = p^{\nu} (2\pi)^{-3} \frac{1}{e^{\beta p} - 1} = p^{\nu} N_p(\beta).$$

Applying the preceding lemma, we conclude that

$$(p \partial) \omega(N_p)(x) = \partial^{\nu} \omega(p^{\nu} N_p)(x) = \partial^{\nu} \omega(\partial^{\nu} L_p)(x) = 0,$$

as claimed. The second part of the statement follows directly from the lemma. □

The preceding result shows that the local equilibrium situations, described by $\mathcal{S}_0$–thermal states, are compatible with those considered in the context of transport equations, such as the Boltzmann equation for the particle distribution function $x, p \mapsto \overline{N}(x, p)$. In the latter case one characterizes the distribution functions $\overline{N}$ describing local equilibrium by the condition that the collision term in the Boltzmann equation vanishes. In the present approach these distributions are to be identified with $x, p \mapsto \omega(N_p)(x)$, and the first part of the proposition shows that these functions satisfy the collisionless Boltzmann equation as well. From the second part we see that they have to comply with an additional constraint of dynamical origin.

We mention as an aside that the functions $x, p \mapsto \overline{N}(x, p)$ contain all macroscopic information about the local equilibrium states. As a matter of fact, given any non–negative function $\overline{N}$ satisfying in some convex space–time region $\mathcal{O}$ the two equations in the proposition and being sufficiently well–behaved in $p$, one can define a quasifree functional $\varphi$ on the algebra $\mathcal{A}$ of local observables, setting for $x_1, x_2 \in \mathcal{O}$

$$\varphi(\phi_0(x_1)\phi_0(x_2)) = \omega_{\infty}(\phi_0(x_1)\phi_0(x_2)) + \int d^3 p \frac{1}{|p|} \overline{N}((x_1 + x_2)/2, p) \cos ((x_1 - x_2)p).$$

(4.5)

This functional is real and satisfies all linear constraints imposed by the field $\phi_0$, but it need not be positive. Yet if it complies with the latter condition, it defines a local equilibrium state in the region $\mathcal{O}$. As we shall see in the subsequent section, the condition of positivity imposes further stringent constraints.

### 5 Regions of local equilibrium

In this section we want to determine the shape of space–time regions in which local equilibrium is possible. We restrict attention here to convex regions. Given
a state $\omega$, let $O_\omega$ be any maximal (in-extendible) region of this kind in which $\omega$ is $SO-$thermal. As was exemplified in Sec. 3, there are states where $O_\omega$ contains a lightcone. It will turn out that, conversely, any such $O_\omega$ has to be contained in some timelike cone unless the lifts of $\omega$ to the admissible macro–observables coincide at all points of $O_\omega$ with some fixed thermal reference state.

It will be convenient in this analysis to proceed from $\omega$ to suitably regularized states. To this end we pick some non–negative test function $f$ which has support in a small neighbourhood of $0 \in \mathbb{R}^4$ and satisfies $\int dy f(y) = 1$. The corresponding regularized state is given by $\omega_f = \int dy f(y) \omega \circ \alpha_y$, hence $\omega_f \circ \alpha_x = \omega_{f-x}$. As the set $C$ of thermal reference states is convex, $\omega_f$ is $SO-$thermal in some slightly smaller region $O \subset O_\omega$. Thus, taking into account relation (2.18) and Lemma 3.1, we obtain for all admissible macro–observables $\Xi$ and multi indices $\mu$ the bounds

$$| \partial^\mu \omega_f(\Xi)(x)| \leq c^\mu \|\Xi\|_B,$$

uniformly on compact subsets of $O$, where $c^\mu$ are certain constants and $B \subset V_+$ is compact.

We will analyze the states $\omega_f$ on subspaces $\Gamma_p$ of macro–observables, fixed by the positive lightlike vectors $p$. These spaces consist of operators whose thermal functions are of the form $V_+ \ni \beta \mapsto \Gamma(\beta p)$, where $\Gamma$ is smooth on $\mathbb{R}_+$. So they are admissible macro–observables. As their sums, products and adjoints are also of this form, the spaces $\Gamma_p$ are in fact abelian $*$–algebras. Of special interest are their elements $E_p$ and $M_p^n$ given by

$$\beta \mapsto E_p(\beta) = e^{i\beta p}, \quad \beta \mapsto M_p^n(\beta) = (\beta p)^n, \quad n \in \mathbb{N}_0. \quad (5.2)$$

Since $\omega_f$ is a state, its lift $\omega_f(\cdot)(x)$ to the admissible macro–observables at any point $x \in O$ defines a state on each $\Gamma_p$. We collect some pertinent properties of these lifts in the subsequent lemma.

**Lemma 5.1** Let $\omega_f$ be any regularized $SO-$thermal state. Then, for $x \in O$,

(a) $(p \partial) \omega_f(E_p)(x) = 0, \quad \square \omega_f(E_p)(x) = 0,$

(b) $| \omega_f(E_p)(x) | \leq 1,$

(c) $p \mapsto \omega_f(E_p)(x)$ extends to an entire analytic function on $\mathbb{C}^4$ which is smooth in $x$,

(d) $0 \leq \omega_f(M_p^{n+1})(x) \leq (\omega_f(M_p^n)(x))^{n/(n+1)} (\omega_f(M_p^{2n+1})(x))^{1/(n+1)}$, $n \in \mathbb{N}_0.$

**Proof.** Statement (a) follows from Lemma 4.1 and the relation $p' E_p = -i \partial' E_p$. For the proof of (b) we make use of the fact that $\omega_f(\cdot)(x)$ is a state on $\Gamma_p$, hence $| \omega_f(E_p)(x) |^2 \leq \omega_f(E_p^* E_p)(x) = 1$, $x \in O$. In order to establish the existence of the extension in (c), we first extend the operators $E_p$ to admissible operators $E_k$, $k \in \mathbb{C}^4$. Their respective thermal functions are given in proper coordinates by

$$\beta \mapsto E_k(\beta) = (\cos(\beta_0|k|) + ik_0 |k|^{-1} \sin(\beta_0|k|)) e^{-ik\beta}.$$
These functions are entire analytic in $k$ and locally uniformly bounded if $\beta$ varies in compact sets $B \subset V_+$. It thus follows from the continuity properties subsumed in Proposition 5.2 that $k \mapsto \omega_f(E_k)(x)$ provides the desired extension. The remaining statement (d) is a straightforward consequence of Hölder’s inequality, taking into account that the operators $M^n_p$, $n \in \mathbb{N}_0$, are positive elements of $\Gamma_p$ (being represented by positive functions on $V_+$).

After these preparations we can establish now the limitations on the possible shape of regions of local equilibrium, indicated above.

**Proposition 5.2** Let $\omega$ be a state which is $S_{\mathcal{O}_\omega}$-thermal in some convex region $\mathcal{O}_\omega$ containing a lightcone. There are the following alternatives for the lifts $\omega(\cdot)(x)$ of $\omega$ to the admissible macro-observables:

(a) The lifts do not depend on $x \in \mathcal{O}_\omega$, i.e. they all coincide with some fixed thermal reference state.

(b) The lifts depend non-trivially on $x \in \mathcal{O}_\omega$, and there is some timelike simplicial cone (an intersection of characteristic half spaces) containing $\mathcal{O}_\omega$.

**Proof.** As explained above, we proceed from $\omega$ to the regularized states $\omega_f$. Any one of the latter states is $S_{\mathcal{O}}$-thermal in the region $\mathcal{O} = \bigcap_{x \in \text{supp}\,f} \{\mathcal{O}_\omega + x\}$, which likewise is convex and contains some lightcone, say $V_+$ for concreteness. Because of the convexity of $\mathcal{O}$, $V_+ + y \subset \mathcal{O}$ for $y \in \mathcal{O}$. We consider now the functions, $p$ being an arbitrary positive lightlike vector,

$$x \mapsto \overline{E}_p(x) = \omega_f(E_p)(x), \quad x \in \mathcal{O}.$$  

Introducing the coordinates $x_\pm = x_0 \pm e x$ and $x_\perp = x - (ex)e$, where $e = p/|p|$, it follows from part (a) of the preceding lemma that $\overline{E}_p$ does not depend on $x_\pm$ and satisfies $\Delta_{\perp} \overline{E}_p(x) = 0$, $x \in \mathcal{O}$, where $\Delta_{\perp}$ is the Laplacian with respect to $x_\perp$.

Now, given $x_- \in \{y_- : y \in \mathcal{O}\}$ and any $x_\perp \in \mathbb{R}^2$, there exists an $x \in \mathcal{O}$ with these components. For if $y \in \mathcal{O}$, the point $x$ with the components $x_- = y_-$, $x_+ = y_+ + t$, $x_\perp$ is, for sufficiently large $t$, contained in the lightcone $z + V_+ \subset \mathcal{O}$, provided $z \in \mathcal{O}$ and $(y - z)$ is positive timelike. So for fixed $x_- \in \{y_- : y \in \mathcal{O}\}$ one has $\Delta_{\perp} \overline{E}_p(x) = 0$, $x_\perp \in \mathbb{R}^2$. But according to parts (b) and (c) of the preceding lemma, $x \mapsto \overline{E}_p(x)$ is smooth and bounded in modulus by 1, hence it cannot depend on $x_\perp$ in view of the growth properties of non-trivial solutions of the Laplace equation (Harnack’s inequality).

Because of the analyticity properties of $\overline{E}_p$ established in part (c) of the lemma, we can represent this function as power series, $\overline{E}_p(x) = \sum_{m=0}^{\infty} |p|^m c_m(x) e^{\mu}$, where $e = (1, e)$ and all coefficients $c_m(x)$ are smooth in $x$. Moreover, in view of the results obtained in the preceding step, $x \mapsto c_m(x) e^{\mu}$ can depend only on $x_- = ex$ in a non-trivial manner. Thus, by $k$-fold partial differentiation with respect to $x$ and an application of the chain rule, we obtain

$$(y \partial)^k c_m(x) e^{\mu} = (ye)^k \partial_0^k c_m(x) e^{\mu}, \quad x \in \mathcal{O}, \, y \in \mathbb{R}^4.$$
Expressing the products of the components of $e$ in terms of spherical harmonics, it is apparent that, for $k > m$, this equality can only be satisfied if $\partial_0^k c_\mu(x) e^\mu = 0$. So each function $x \mapsto c_\mu(x) e^\mu$ is a polynomial in $ex$ of degree $m$ or less with coefficients which are polynomials in the components of $e$.

Making use of the fact that $\bar{F}_p(x) = \sum_{m=0}^{\infty} \frac{1}{m!^m} \omega_f (M^m_e)(x)$, where $M^m_e$ are the operators introduced in (5.2), we therefore find that each function

$$x \mapsto \omega_f (M^m_e)(x) = (-i)^m m! c_\mu(x) e^\mu, \quad x \in \mathcal{O},$$

is a polynomial in $ex$ of degree $m$ or less. According to part (d) of the preceding lemma these polynomials are non-negative. Moreover, there is some $m$ such that the corresponding polynomial is of first degree in $ex$, unless $x \mapsto \omega_f (M^m_e)(x)$ is constant for all $m \in \mathbb{N}$. To verify this, let $x \mapsto \omega_f (M^k_e)(x)$ be constant for $k = 0, \ldots, m - 1$. It then follows from (d) and the fact that the polynomial degree of $x \mapsto \omega_f (M^{2m-1}_e)(x)$ is at most $2m - 1$ that $x \mapsto \omega_f (M^m_e)(x)$ increases no faster than $|x|^{(2m-1)/m}$ for large $x \in V_+ \subset \mathcal{O}$. Hence $x \mapsto \omega_f (M^m_e)(x)$, being a polynomial, has to be of first degree in $ex$ if it is not constant.

If the second alternative in the statement holds, i.e. if $x \mapsto \omega_f (\Xi)(x)$ depends non-trivially on $x$ for some $\Xi$, there is also some $M^m_e$ for which this is true since the linear span of the latter operators is dense in the space of admissible macro-observables, cf. the proof of Lemma 3.1. Thus, because of the preceding results, we may assume that

$$\omega_f (M^m_e)(x) = P_f(e)(ex) + Q_f(e), \quad x \in \mathcal{O},$$

where $P_f, Q_f$ are polynomials and $P_f$ is not identically zero. But $\omega_f (M^m_e)(x) \geq 0$, $x \in \mathcal{O}$, so

$$P_f(e)(ex) + Q_f(e) \geq 0, \quad x \in \mathcal{O}.$$

Since $V_+ \subset \mathcal{O}$ and $P_f \neq 0$ it follows that $P_f(e) > 0$ for almost all $e$ and $Q_f(e) \geq 0$. Hence $\mathcal{O} \subset \bigcap_e \{x : ex \geq -Q_f(e)/P_f(e)\}$. In particular, $\mathcal{O}$ is contained in some positive timelike simplicial cone consisting of the intersection of four characteristic half-spaces. The same is therefore true for $\mathcal{O}_\omega$ since the support of $f$ is compact.

We mention as an aside that, due to the linearity of $\omega_f$ in $f$ and $\omega_f \circ \alpha_y = \omega_{f-y}$, one has $P_f(e) = \int dy f(y) P(e)$ and $Q_f(e) = \int dy f(y) \left( (ey) P(e) + Q(e) \right)$ for certain polynomials $P, Q$ which do not depend on $f$.

The statement has thus been established for regions $\mathcal{O}_\omega$ containing $V_+$. But the arguments can be carried over to arbitrary regions containing some shifted forward or backward lightcone, so the proof of the proposition is complete. $\square$

The preceding result shows that non-trivial local equilibrium states $\omega$ fix a time direction in the sense that they determine some maximal timelike cone in which local equilibrium can exist. In spite of the fact that the underlying theory is time reflection invariant, an observer who has such a state as a background may consistently take the corresponding cone as an indicator of the future direction.
If he perturbs $\omega$ by some local unitary operation $U$, $\omega_U(\cdot) = \omega(U^* \cdot U)$, he finds that the perturbed state $\omega_U$ coincides with $\omega$ with regard to all measurements performed after the perturbation, i.e., in the future of the space–time support of $U$. This is a consequence of the timelike commutativity of observables (Huyghens’ principle) in the present model. So, in a sense, the model also describes the return to (local) equilibrium. These results show that the arrow of time manifests itself already in the macroscopic space–time patterns of local equilibrium states.

Of particular interest is the situation where $\omega$ describes a non–trivial macroscopic hydrodynamic flow in some timelike cone, i.e., the local rest systems of $\omega$, which can be determined by the admissible macro–observables $\beta \mapsto M_{e}(\beta) = e\beta$, $e$ being positive lightlike, vary from point to point. In this case the arguments given in the preceding proposition imply that the function $x \mapsto \omega(M_{e})(x)$ has to be of the form

$$\omega(M_{e})(x) = c_{\omega} e(x - x_{\omega}),$$ (5.3)

where $c_{\omega}$ is some non–zero (say, positive) constant and $x_{\omega}$ some constant vector. If an observer finds such a behaviour in some bounded space–time region, he would infer that the maximal timelike cone where $\omega$ can be locally in equilibrium is the lightcone $V_{+} + x_{\omega}$ since $\omega(M_{e})(x)$ has to be non–negative. Anticipating that $\omega$ has this property, he would also conclude that the functions $x \mapsto \omega(M_{e})(x)$ vanish at the apex $x_{\omega}$ of this cone, hence the temperature of the state tends to infinity there. Moreover, if $x$ approaches any point on the ray $x_{\omega} + \mathbb{R}_{+} e$ at the boundary of the cone, the time axes of the respective local rest systems of the state bend towards the direction of $e$, indicating that there is a dominant flow of particles originating from $x_{\omega}$. This can be made more explicit by looking at the particle density $x \mapsto \omega(N_{p})(x)$ which can be shown to diverge if $x - x_{\omega}$ and $p$ become parallel. So the observer would be led to interpret the underlying state as the result of a hot bang which has occurred at $x_{\omega}$ and has quickly approached local equilibrium thereafter. Of course, all his conclusions depend on the hypothesis that the partial state which he observes is locally in equilibrium in all of $V_{+} + x_{\omega}$. Whether this is really the case can evidently not be decided by observations made in any finite space–time region.

6 Summary and outlook

In the present investigation we have applied the general formalism for the characterization and analysis of local equilibrium states, established in [1], to a simple model. There are two elemental ideas underlying this approach: (a) local equilibrium states cannot be distinguished from global ones by observations made in small space–time regions and (b) local measurements admit a macroscopic interpretation in all global equilibrium states. The combination of these two basic facts made it possible to interpret within the microscopic setting the space–time patterns of local equilibrium states in macroscopic terms.

The results of our investigation show that this approach not only is physically
meaningful, but it also provides a convenient framework for concrete computations. In the case at hand, we have been able to establish equations for the space–time evolution of macroscopic properties of local equilibrium states, such as the particle density, without having to enter into the difficult dynamical question of how a system is driven towards local equilibrium. No \textit{a priori} hypotheses, such as Boltzmann’s assumption of molecular chaos, were necessary to establish these results. Instead, the intrinsic characterization (a) of local equilibrium was sufficient for their derivation.

The present approach revealed also interesting constraints on the possible macroscopic space–time structure of local equilibrium states. It turned out that local equilibrium can exist only in certain specific space–time regions which generically have the form of lightcones. Such states can therefore be used to distinguish between future and past. Moreover, typically there appear singularities at the apices of these cones which have the interpretation of a hot bang. To the best of our knowledge, these results provide the first example of a “singularity theorem” in a microscopic setting.

The present model of a massless free field exhibits a greatly simplifying feature, however: it is scale invariant, \textit{i.e.} physics looks the same at microscopic and macroscopic scales. For a study of the effects of an inherent scale, one may consider the example of massive free field theory. Indeed, there appear some differences. It is not difficult to show that, in the massive case, non–trivial local equilibrium states having locally a sharp temperature do not exist, their temperature support is typically all of $V_+$. This suggests to enlarge the space $\mathcal{C}$ of thermal reference states accordingly. In this extended framework it is still possible to establish evolution equations for the thermal equilibrium states, in complete analogy to the present results, and to analyze their space–time behaviour. These results will be put on record elsewhere.

It seems attractive to apply, in a further step, the general formalism in \cite{1} to free fields on arbitrary globally hyperbolic space–times. In particular, the shape of regions admitting local equilibrium is of interest there as well. For maximally symmetric spaces there should appear no new conceptual difficulties in such an analysis since the basic ingredients, local observables and “global” equilibrium states for geodesic observers (\textit{i.e.} states which are in equilibrium in the causal closure of the respective geodesics), do exist in these cases. But in generic spacetimes the situation is less clear in view of the fact that such global equilibrium states need not exist. Thus what is required there is a substitute for these states which allows one to identify the local equilibrium situations. For the solution of this problem the generally covariant framework of quantum field theories, proposed in \cite{5}, seems to be of relevance since it allows the unified treatment of a given theory on arbitrary spacetimes. So one may be able to use the existence of global equilibrium states on one spacetime to interpret the local equilibrium properties of states on the others.

We mention as an aside that the present approach to the characterization of equilibrium states may be regarded as a refinement of the ideas underlying the
Hadamard condition \cite{6} which is of fundamental importance in this setting. Yet whereas the Hadamard condition has been invented to characterize locally the folia of all physical states, the condition of local equilibrium is designed to identify within these folia states having a specific physical interpretation. Thus the present ideas seem to be of relevance also for the interpretation of states on arbitrary space–time manifolds.

Returning to the case of Minkowski space, a central issue is the treatment of interacting models where one may study the intriguing question of the approach to equilibrium. In order to indicate the problems appearing in this context, let us consider first a state $\omega$ which is locally thermal on a space of observables generated by balanced derivatives $\theta^\mu(x)$ of the normal ordered squares of the underlying fields \cite{1,7}, similarly to the observables \cite{3,4} used in the present investigation. In the interacting case, these observables no longer satisfy the equations (4.3) and (4.4), which were the basis for the present analysis, there appear additional "source terms" $\sigma^\mu(x)$, $\tau^\mu(x)$,

$$\partial_\nu \theta^\nu^\mu(x) = \sigma^\mu(x), \quad \Box \theta^\mu(x) = \tau^\mu(x).$$

(6.1)

Denoting by $\Theta^\mu$ the macro–observables corresponding to $\theta^\mu(x)$, one thus obtains equations for their evolution in state $\omega$ of the form

$$\partial_\nu \omega(\Theta^\nu^\mu)(x) = \omega(\sigma^\mu(x)), \quad \Box \omega(\Theta^\mu)(x) = \omega(\tau^\mu(x)).$$

(6.2)

The terms appearing on the right hand side of these equations resemble the collision terms in transport equations \cite{8}. In particular, they vanish if $\omega$ is sufficiently close to equilibrium at $x$ since $\sigma^\mu(x)$, $\tau^\mu(x)$ have zero expectation values in all thermal reference states due to their space–time invariance.

In order to establish the approach to (local) equilibrium one thus has to show that these collision terms vanish asymptotically. The present results suggest that, for a given state $\omega$, one may expect such a behaviour at best in some lightcone and, because of the PCT theorem, this cone may be either future or past directed. So one has to identify the convex subsets of states exhibiting the desired behaviour for given time direction. Mixtures of the two types of states may in general not exhibit an ergodic behaviour.

Next, there is the problem of local thermalization of a given state $\omega$ on the space of observables $\theta^\mu(x)$, which was taken for granted in the preceding discussion. Its solution requires a proof that there are asymptotic bounds of the form \cite{11}

$$|\omega(\sum c_\mu \theta^\mu(x))| \leq \| \sum c_\mu \Theta^\mu \|_B,$$

(6.3)

where $\| \cdot \|_B$ denotes any one of the seminorms on the macro–observables, introduced in Sec. 2. Choosing $B$ sufficiently large, it seems feasible to establish such a result without major difficulties for finite dimensional spaces of observables $\theta^\mu(x)$ whose corresponding thermal functions $\Theta^\mu$ are linearly independent. Yet if there exist relations between these functions of the form $\sum c_\mu \Theta^\mu = 0$ which do not have
a counterpart at the microscopic level, one has to understand the mechanism by which the expectation values of the corresponding combinations \(\sum c_\mu \Theta^\mu(x)\) of local observables vanish asymptotically in the state \(\omega\). It is of interest in this context that there exists a large family of states which may be regarded as perturbations of global equilibrium states and which satisfy the above inequality [1].

Once local equilibrium has been reached in a state \(\omega\), either exactly or in the sense that the collision terms in (6.2) may effectively be neglected, one arrives at the same conservation laws and evolution equations for the macro–observables \(\Theta^\mu\) in the state \(\omega\) as in free field theory,

\[
\partial_\nu \omega(\Theta^\mu_\nu)(x) = 0, \quad \Box \omega(\Theta^\mu)(x) = 0.
\] (6.4)

These equations thus have a universal character. Yet in order to be able to interpret them in physical terms it is necessary to determine the underlying thermal functions \(\beta \mapsto \Theta^\mu(\beta)\), which are model dependent. This requires a computation of the thermal two–point functions of the underlying fields. The general form of these functions is known [9] and there exist also various methods for their perturbative computation in models [10, 11], cf. also the novel approach in [12]. It seems therefore possible to determine the macroscopic space–time patterns of local equilibrium states in the presence of interaction by similar methods as in the present investigation. We hope to return to this interesting issue elsewhere.

Let us finally mention that the basic ideas underlying the approach in [1] can also be applied to non–relativistic theories, such as spin systems. Of course, the concept of local observable, as used in the present analysis, is no longer meaningful there, one has to replace it by suitable subspaces of the algebra of all observables. A general characterization of these subspaces is still an open problem. But a preliminary study [13] of the XX–model indicates that certain macroscopic aspects of local equilibrium states, such as the existence of evolution equations, can still be established in such a setting, at least in simple cases. So there is evidence that the general approach to the analysis of local equilibrium states, proposed in [1], provides an efficient framework for the analysis of this important issue.

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