INTEGRAL SUBSCHEMES OF CODIMENSION TWO

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Abstract. In this paper we study the problem of finding all integral subschemes in a fixed even linkage class \( \mathcal{L} \) of subschemes in \( \mathbb{P}^n \) of pure codimension two. To a subscheme \( X \in \mathcal{L} \) we associate two invariants \( \theta_X, \eta_X \). When taken with the height \( h_X \), each of these invariants determines the location of \( X \) in \( \mathcal{L} \), thought of as a poset under domination. In terms of these invariants, we find necessary conditions for \( X \) to be integral. The necessary conditions are almost sufficient in the sense that if a subscheme dominates an integral subscheme and satisfies the necessary conditions, then it can be deformed with constant cohomology to an integral subscheme.

Introduction

In [4], Gruson and Peskine show that an integral arithmetically Cohen-Macaulay curve in \( \mathbb{P}^3 \) has numerical character without gaps. Conversely they show that any admissible sequence without gaps arises as the numerical character of a smooth connected arithmetically Cohen-Macaulay curve. Arithmetically Cohen-Macaulay curves form just one of many even linkage classes of curves in \( \mathbb{P}^3 \). In the present paper, we obtain a similar result for even linkage classes of pure codimension two subschemes in \( \mathbb{P}^n \) which are not arithmetically Cohen-Macaulay.

In the first section we briefly review the linkage theory of codimension two subschemes in \( \mathbb{P}^n \). We recall the notions of \( \mathcal{E} \) and \( \mathcal{N} \)-type resolutions, the cone construction, Rao’s correspondence, and criteria for domination in an even linkage class. We close the section with the theorem that the Lazarsfeld-Rao property holds for any such even linkage class which does not consist of arithmetically Cohen-Macaulay subschemes.

In the second section, we recall Martin-Deschamps and Perrin’s notion of admissible character and the partial ordering of domination on them. We show that there is a bijection between dominations \( \gamma \leq_h \sigma \) and functions \( \eta, \theta : \mathbb{Z} \to \mathbb{N} \) with certain properties (proposition 2.6 and proposition 2.9).

Section three is devoted to adapting the invariants of section two to the geometric situation. Specifically, it is shown that for a subscheme \( X \) in an even linkage class \( \mathcal{L} \) of pure codimension two subschemes in \( \mathbb{P}^n \) which are not arithmetically Cohen-Macaulay, either of the invariants \( \eta_X \) or \( \theta_X \) determine the class of all subschemes that can be deformed with constant cohomology to \( X \) through subschemes in \( \mathcal{L} \).

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In section four, we give a preliminary result towards studying the integral subschemes in an even linkage class. If $X$ is a subscheme, we may define $s_0(X)$ to be the least degree of a hypersurface $S$ on which $X$ lies, $s_1(X)$ to be the least degree hypersurface on which $X$ lies which does not contain $S$, and $t_1(X)$ to be the least degree of a hypersurface which meets $S$ properly. If $X$ is an integral subscheme, then $X$ lies on an integral hypersurface of minimal degree, hence $s_1(X) = t_1(X)$. The main result of this section (theorem 4.7) is a characterization in terms of $\theta$ of which subschemes satisfy this latter property.

In section five, we prove our main results about integral subschemes in a fixed even linkage class. We give necessary conditions for a subscheme $X$ to be integral in terms of the invariants $\theta_X$ and $s_0(X)$ (theorem 5.8). Further, we show that if $X$ is integral, $X \leq Y$, and $Y$ satisfies the necessary conditions, then $Y$ deforms with constant cohomology through subschemes in $L$ to an integral subscheme $Y$ (theorem 5.11). To close out the section, we give several examples of different kinds of behavior of integral subschemes in an even linkage class.

While the results above are good for integral subschemes, the situation is not so simple for smooth connected subschemes (example 5.15). To obtain a result like theorem 5.11 for smooth connected subschemes of codimension two in $\mathbb{P}^n$, more conditions are needed. Even with extra conditions, typical constructions of smooth subschemes won’t work when $n \geq 6$.

The ideas of this paper originated in my PhD Thesis, where the case of curves in $\mathbb{P}^3$ was studied. I would like to thank Robin Hartshorne for a careful reading of that thesis, as his questions and suggestions led to many improvements.

1. LINKAGE THEORY IN CODIMENSION TWO

In this section we review the main results of linkage theory for subschemes in $\mathbb{P}^n$ of codimension two. The main reference for this section is [19], where various results of linkage theory for Cohen-Macaulay subschemes are generalized to subschemes of pure codimension. The main result of importance here is the fact that the Lazarsfeld-Rao property holds for even linkage classes of subschemes of pure codimension two in $\mathbb{P}^n$. We assume that the reader is familiar with the notion of simple linkage and the equivalence relation that it generates (see [8, §4]).

**Definition 1.1.** A sheaf $F$ on $\mathbb{P}^n$ is called *dissocié* if it is a direct sum of line bundles.

**Definition 1.2.** Let $V \subset \mathbb{P}^n$ be a subscheme of codimension 2. An $\mathcal{E}$-type resolution for $V$ is an exact sequence

$$0 \to \mathcal{E} \to \mathcal{Q} \to \mathcal{O}(\to \mathcal{O}_V \to 0)$$

such that $\mathcal{Q}$ is dissocié and $H^1_\ast(\mathcal{E}) = 0$.

**Remark 1.3.** The condition that $H^1_\ast(\mathcal{E}) = 0$ is equivalent to the the condition that $H^0(\mathcal{Q}) \to I_V$ is surjective. In particular, any subscheme $V$ of codimension two has an $\mathcal{E}$-type resolution obtained by sheafifying a free graded surjection onto the ideal $I_V$. The theorem of Auslander and Buchsbaum shows that $V$ is locally Cohen-Macaulay $\iff \mathcal{E}$ is locally free. Further, it can be shown that $V$ is of pure codimension two $\iff Ext^1(\mathcal{E}^\vee, \mathcal{O}) = 0$. This second equivalence generalizes to subvarieties of higher codimension [19, corollary 1.18].
Definition 1.4. If $V \subset \mathbb{P}^n$ is a subscheme of codimension 2, then an $\mathcal{N}$-type resolution for $V$ is an exact sequence

$$0 \to \mathcal{P} \to \mathcal{N} \to \mathcal{O}(\to \mathcal{O}_V \to 0)$$

where $\mathcal{P}$ is dissocié, $\mathcal{N}$ is reflexive, $\mathcal{E}xt^1(\mathcal{N}, \mathcal{O}) = 0$ and $H^1_*(\mathcal{N}^\vee) = 0$.

Remark 1.5. Unlike the situation for $\mathcal{E}$-type resolutions, not every subscheme of codimension two has an $\mathcal{N}$-type resolution. In fact, a subscheme $V \subset \mathbb{P}^n$ of codimension $\geq 2$ has an $\mathcal{N}$-type resolution if and only if $V$ is of pure codimension 2. A similar statement holds in higher codimension [19, corollary 1.20].

Proposition 1.6. Let $V \subset \mathbb{P}^n$ be a subscheme of pure codimension 2 which is contained in a complete intersection $X$ of two hypersurfaces with degrees summing to $d$. Let

$$0 \to \mathcal{G}_2 \to \mathcal{G}_1 \to \mathcal{G}_0(\to \mathcal{O}_V \to 0)$$

be an $\mathcal{E}$-type (respectively $\mathcal{N}$-type) resolution for $V$ and let $F$ be a Koszul resolution for $X$. Then there is a morphism of complexes $\alpha : F \to G$, induced by the inclusion $I_X \to I_V$. The mapping cone of the morphism $\alpha^*(-d)$ gives a resolution for the subscheme $W \subset \mathbb{P}^n$ which is linked to $V$ by $X$. If the induced isomorphism $\mathcal{G}_0^\vee(-d) \to F_0^\vee(-d)$ is split off, the resulting resolution is an $\mathcal{N}$-type (respectively $\mathcal{E}$-type) resolution for $W$.

Proof. This follows from [19, §1].

Remark 1.7. Let $0 \to \mathcal{E}_V \to Q \to \mathcal{O}$ be an $\mathcal{E}$-type resolution for a subscheme $V$ as in the proposition. If $V$ is linked to $W$ and $W$ is linked to $Z$, then applying proposition 1.6 twice produces an $\mathcal{E}$-type resolution for $Z$ such that $\mathcal{E}_Z = \mathcal{E}_V(h) \oplus \mathcal{F}$ where $h \in \mathbb{Z}$ and $\mathcal{F}$ is dissocié. This motivates the definition of stable equivalence and the theorems which follow.

Definition 1.8. Two reflexive sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$ on $\mathbb{P}^n$ are stably equivalent if there exist dissocié sheaves $Q_1, Q_2$ and $h \in \mathbb{Z}$ such that $\mathcal{E}_1 \oplus Q_1 \cong \mathcal{E}_2(h) \oplus Q_2$. This is an equivalence relation among reflexive sheaves on $\mathbb{P}^n$.

Theorem 1.9. There is a bijective correspondence between even linkage classes of purely two-codimensional subschemes of $\mathbb{P}^n$ and stable equivalence classes of reflexive sheaves $\mathcal{E}$ on $\mathbb{P}^n$ such that $H^1_*(\mathcal{E}) = 0$ and $\mathcal{E}xt^1(\mathcal{E}^\vee, \mathcal{O}) = 0$ via $\mathcal{E}$-type resolutions.

Proof. This is [19, theorem 2.11], where the ideas of Rao’s original proof [22] are extended to the case of subschemes which are not locally Cohen-Macaulay.

Theorem 1.10. There is a bijective correspondence between even linkage classes of purely two-codimensional subschemes of $\mathbb{P}^n$ and stable equivalence classes of reflexive sheaves $\mathcal{N}$ on $\mathbb{P}^n$ such that $H^1_*(\mathcal{N}^\vee) = 0$ and $\mathcal{E}xt^1(\mathcal{N}, \mathcal{O}) = 0$ via $\mathcal{N}$-type resolutions.

Proof. This follows from theorem 1.9 and proposition 1.6 once we note that two reflexive sheaves are stably equivalent if and only if their duals are stably equivalent.
Proposition 1.11. Let $\Omega$ be a stable equivalence class of reflexive sheaves on $\mathbb{P}^n$. Let $F_0$ be a sheaf in $\Omega$ of minimal rank. Then for each $G \in \Omega$, there exists $h \in \mathbb{Z}$ and a dissocié sheaf $Q$ such that $G \cong F_0(h) \oplus Q$. In particular, the isomorphism class of $F_0$ is uniquely determined up to twist.

Proof. This is [19, proposition 2.3].

Notation 1.12. Let $L$ be an even linkage class of codimension two subschemes in $\mathbb{P}^n$. If $L$ is associated with the stable equivalence class $\Omega$ via $E$-type resolution (as in theorem 1.9) and $E_0$ is a sheaf of minimal rank for $\Omega$ (as in proposition 1.11), we will say that $L$ corresponds to $[E_0]$ via $E$-type resolution. Similarly, if $L$ is associated to $\Omega$ via $N$-type resolution and $N_0$ is a sheaf of minimal rank for $\Omega$, we will say that $L$ corresponds to $[N_0]$ via $N$-type resolution.

The above theorems give a classification of even linkage classes. In the remainder of this section, we describe the structure of an even linkage class.

Definition 1.13. Let $V \subset \mathbb{P}^n$ be a subscheme of pure codimension two. Let $S$ be a hypersurface of degree $s$ which contains $V$ and let $h$ be an integer. We say that $W$ is obtained from $V$ by an elementary double link of height $h$ on $S$ if there are hypersurfaces $T_1, T_2$ of respective degrees $t_1, t_1 + h$ which meet $S$ properly in such a way that $S \cap T_1$ links $V$ to a subscheme $Z$, which in turn is linked to $W$ by $S \cap T_2$. In the special case when $h \geq 0$ and $T_2 = T_1 \cup H$ where $H$ is a hypersurface of degree $h$, we say that $Y$ is obtained from $X$ by a basic double link of height $h$ on $S$. In either case, we say that the double link has type $(s, h)$.

Proposition 1.14. Suppose that $V \subset \mathbb{P}^n$ is a subscheme of pure codimension two and that $W$ is obtained from $X$ by an elementary double link of type $(s, h)$. If $V$ has an $E$-type (respectively $N$-type) resolution of the form

$$0 \to G_2 \to G_1 \to O \to O_V \to 0,$$

then $W$ has an $E$-type (respectively $N$-type) resolution of the form

$$0 \to G_2(-h) \oplus O(-s - h) \to G_1(-h) \oplus O(-s) \to O \to O_W \to 0.$$

Proof. Applying proposition 1.6 twice, we obtain such a resolution with the extra summand $O(-t_1 - h)$ appearing in the first two terms. Because the same hypersurface $S$ was used for both links, we can take the induced map between these summands to be the identity. Splitting off this summand, we are left with the resolution stated.

Definition 1.15. Let $X, X'$ be subschemes of pure codimension two in $\mathbb{P}^n$. We say that $X'$ dominates $X$ at height $h \geq 0$ if $X'$ can be obtained from $X$ by a sequence of basic double links with heights summing to $h$, followed by a deformation which preserves cohomology and even linkage class. In this case we write $X \leq_h X'$, or simply $X \leq X'$ if $h$ is not specified.

Definition 1.16. If $f : \mathbb{Z} \to \mathbb{N}$ is a function such that $f(n) = 0$ for $n << 0$, then we define the function $f^\#$ by $f^\#(a) = \sum_{n \leq a} f(n)$. 
**Proposition 1.17.** Let $X, Y \subset \mathbb{P}^n$ be subschemes of pure codimension two and let $h \geq 0$ be an integer. Then $X \leq_h Y$ if and only if there exist $N$-type resolutions

$$
0 \to \oplus \mathcal{O}(-m)^{r(m)} \to \mathcal{N} \to \mathcal{I}_X(h_1) \to 0
$$

$$
0 \to \oplus \mathcal{O}(-m)^{s(m)} \to \mathcal{N} \to \mathcal{I}_Y(h_2) \to 0
$$

such that $h_2 - h_1 = h$ and $r^# \geq s^#$.

**Proof.** This is [19, proposition 3.9].

**Corollary 1.18.** If $X \leq_h Y$ and $Y \leq_k Z$, then $X \leq_{h+k} Z$. In particular, domination is transitive.

**Proof.** This is [19, corollary 3.10].

**Definition 1.19.** Let $\mathcal{L}$ be an even linkage class of subschemes of pure codimension two in $\mathbb{P}^n$. We say that $\mathcal{L}$ has the Lazarsfeld-Rao property if $\mathcal{L}$ has a minimal element $X_0$ such that $0 \leq X$ for each $X \in \mathcal{L}$.

**Theorem 1.20.** Let $\mathcal{L}$ be an even linkage class of codimension two subschemes of $\mathbb{P}^n$ corresponding to the stable equivalence class $[N_0]$ via $N$-type resolutions, with $N_0 \neq 0$. Then there exists a function $q : \mathbb{Z} \to \mathbb{N}$ and a subscheme $X_0 \in \mathcal{L}$ such that $X_0 \leq Y$ for all $Y \in \mathcal{L}$ and $X_0$ has an $N$-type resolution of the form

$$
0 \to \oplus \mathcal{O}(-n)^{q(n)} \to N_0 \to \mathcal{O}_{\mathbb{P}^n}(h)(\to \mathcal{O}_{X_0}(h) \to 0).
$$

In particular, $\mathcal{L}$ has the Lazarsfeld-Rao property. $X'_0$ is another such minimal element of $\mathcal{L}$. $X'_0$ has an $N$-type resolution of the same form as that of $X_0 \iff X_0$ deforms to $X'_0$ with constant cohomology through subschemes in $\mathcal{L}$.

**Proof.** This is [19, theorem 3.26].

**Theorem 1.21.** Let $\mathcal{L}$ be an even linkage class corresponding to the stable equivalence class $[N_0]$ via $N$-type resolution, with $N_0 \neq 0$. Let $X_0$ be a minimal subscheme for $\mathcal{L}$. If $S$ is a hypersurface of minimal degree which contains $X_0$, then there exists a hypersurface $T$ containing $X_0$ such that $S \cap T$ links $X_0$ to a subscheme $Y_0$ which is minimal for its even linkage class.

**Proof.** This is a simplified version of [19, theorem 3.31].

2. Domination of Admissible Characters

In this section we recall Martin-Deschamps and Perrin’s notion of admissible characters, and the partial ordering of domination on them [13,V]. In this section we give properties of the partial ordering along with alternative ways of describing it. The relationship between domination of admissible characters and domination of subschemes in $\mathbb{P}^n$ will be explained in the next section.

**Definition 2.1.** A character $\sigma : \mathbb{Z} \to \mathbb{Z}$ which has finite support and satisfies $\sum_{l \in \mathbb{Z}} \sigma(l) = 0$. A character $\sigma$ is said to be admissible if it satisfies

1. $\sigma(l) = 0$ for $l < 0$
2. $\sigma(0) = -1$
3. Setting $s_0(\sigma) = \inf \{l \geq 0 : \sigma(l) \neq -1\}$, we have $\sigma(s_0(\sigma)) \geq 0$
4. Setting $s_1(\sigma) = \inf \{l \geq s_0(\sigma) : \sigma(l) \neq 0\}$, we have $\sigma(s_1(\sigma)) > 0$
Let $\gamma, \gamma'$ be two admissible characters. If $h \geq 0$ is an integer, we say that $\gamma'$ dominates $\gamma$ at height $h$ if

1. $s_0(\gamma) \leq s_0(\gamma') \leq s_0(\gamma) + h$
2. $\gamma'(l) \geq 0$ for $s_0(\gamma') \leq l < s_0(\gamma) + h$
3. $\gamma'(l) \geq \gamma(l - h)$ for $l \geq s_0(\gamma) + h$

In this case we write $\gamma \lesssim h \gamma'$. If $h$ isn’t specified, we simply write $\gamma \lesssim \gamma'$.

**Proposition 2.3.** If $\gamma, \sigma, \tau$ are three admissible characters and $\gamma \leq_h \sigma$ and $\sigma \leq_k \tau$, then $\gamma \lesssim_{h+k} \tau$. In particular, domination gives a partial order relation on the set of admissible characters.

**Proof.** This is [13, V, proposition 2.2].

**Definition 2.4.** Let $f : \mathbb{Z} \to \mathbb{N}$ be a function. then $f_s = \max\{l : f(l) > 0\}$ if such an $l$ exists, otherwise $f_s = -\infty$. Similarly we define $f_a = \min\{l : f(l) > 0\}$ if such an $l$ exists, otherwise $f_a = \infty$.

**Definition 2.5.** Let $f : \mathbb{Z} \to \mathbb{N}$ be a function. We say that $f$ is connected in degrees $\geq a$ (respectively $> a$) if $f(b) > 0$ for some $b > a$ implies that $f(l) > 0$ for all $a \leq l \leq b$ (respectively $a < l \leq b$). Similarly we say that $f$ is connected in degrees $\leq b$ (respectively $< b$) if $f(b) > 0$ for some $a < b$ implies that $f(l) > 0$ for $a \leq l \leq b$ (respectively $a < l < b$). $f$ is said to be connected about an interval $[a, b]$ if $f$ is connected in degrees $\leq b, f$ is connected in degrees $\geq a$ and $f$ is nonzero on $[a, b]$. $f$ is connected if it is connected about the interval $[f_a, f_s]$.

**Proposition 2.6.** Let $\gamma$ be an admissible character, $h \geq 0$ be an integer and $\sigma : \mathbb{Z} \to \mathbb{Z}$ be a function. Then $\sigma$ is an admissible character such that $\sigma \geq_h \gamma$ if and only if the function $\eta$ given by

$$
\eta(l) = \sigma(l) - \gamma(l - h) + \binom{l}{0} - \binom{l - h}{0}
$$

is nonnegative, is connected in degrees $< s_0(\gamma) + h$ and satisfies $\sum \eta(l) = h$. This function is denoted $\eta_{\gamma, \sigma, h}$ or simply $\eta$.

**Proof.** Assume that $\sigma \geq_h \gamma$ is an admissible character and define $\eta$ as above. A simple calculation shows that $-\gamma(l - h) + \binom{l}{0} - \binom{l - h}{0} = 1$ for $0 \leq l < s_0(\gamma) + h$. It follows that $\eta(l) = 0$ for $0 \leq l < s_0(\sigma)$ and that $\eta(l) = \sigma(l) + 1$ for $s_0(\sigma) \leq l < s_0(\gamma) + h$. Since $\sigma(l) \geq 0$ in this last range by definition of domination, we see that $\eta$ is connected in degrees $< s_0(\gamma) + h$. For $l \geq s_0(\gamma) + h$, we have that $\eta(l) = \sigma(l) - \gamma(l - h)$, which is $\geq 0$ for such $l$ by definition of domination. It follows that $\eta(l) \geq 0$ for all $l \in \mathbb{Z}$. Finally, $\sum \binom{l}{0} - \binom{l - h}{0} = h$, so the fact that $\gamma$ and $\sigma$ are characters shows that $\sum \eta(l) = h$.

Conversely, suppose that $\eta : \mathbb{Z} \to \mathbb{N}$ satisfies the conditions of the proposition. If we define $\sigma$ by the formula for $\eta$, the fact that $\sum \eta(l) = h$ shows that $\sigma$ is a character. Since $\eta$ sums to $h$ and is connected in degrees $< s_0(\gamma) + h$, we see that $\eta(l) = 0$ for $l < s_0(\gamma)$. It follows that $\sigma(l) = 0$ for $l < 0$ and $\sigma(l) = -1$ for $0 \leq l < s_0(\gamma)$. In particular, $\sigma$ satisfies the first two conditions for admissibility and $s_0(\sigma) \geq s_0(\gamma)$. Since $\sigma(s_0(\gamma) + h) \geq 0$, we also see that $s_0(\sigma) \leq s_0(\gamma) + h$, hence the first condition for domination holds. The third condition of domination holds because $\sigma(l) - \gamma(l - h) = \eta(l) \geq 0$ for $l \geq s_0(\gamma) + h$. 


To check the second domination condition, we consider two cases. If \( \eta(l) = 0 \) for \( 0 \leq l < s_0(\gamma) + h \), then \( \sigma(l) = -1 \) for \( l \) in this range, so \( s_0(\sigma) = s_0(\gamma) + h \) and the second domination condition holds vacuously. If this is not the case, let \( N = \min \{ l : \eta(l) > 0 \} \). The connectedness condition on \( \eta \) shows that \( \eta(l) > 0 \) for \( N \leq l < s_0(\gamma) + h \), which implies that \( s_0(\sigma) = N \) and \( \sigma(l) \geq 0 \) for \( s_0(\sigma) \leq l < s_0(\gamma) + h \). This checks the second condition, so all three conditions for domination hold.

Finally, we check the last two conditions for admissibility. From the domination conditions, we see that \( \sigma(l) \geq 0 \) for \( s_0(\sigma) \leq l < s_0(\gamma) + h \), which immediately gives the third admissibility condition. Further, we have that \( \sigma(l) \geq 0 \) for \( s_0(\gamma) + h \leq l \leq s_1(\gamma) + h \) and that \( \sigma(s_1(\gamma) + h) > 0 \). It follows that the last admissibility condition holds, finishing the proof.

**Proposition 2.7.** Let \( \gamma \leq_h \tau \) and \( \gamma \leq_k \sigma \) be two domination of admissible characters. Then \( \tau \leq_{k-h} \sigma \) if and only if \( \eta_{\gamma,\sigma}(l) - \eta_{\gamma,\tau}(l - k + h) \geq 0 \) for all \( l \in \mathbb{Z} \), in which case we have the formula

\[
\eta_{\tau,\sigma,k-h}(l) = \eta_{\gamma,\sigma,k}(l) - \eta_{\gamma,\tau,k}(l - k + h).
\]

**Proof.** If \( k \geq h \), then a simple calculation shows gives the formula above. In the case that \( k < h \), we have that \( \eta_{\tau,\sigma,k-h} \) is nonnegative by proposition 2.6, so the forward implication is clear.

Conversely, suppose that the function \( \eta = \eta_{\tau,\sigma,k-h} \) as defined by the formula is nonnegative. It is clear that \( \sum \eta(l) = k - h \), so we have that \( k - h \geq 0 \). For \( l < s_0(\tau) + k - h \) we have that \( \eta_{\gamma,\tau}(l) = 0 \), so \( \eta(l) = \eta_{\tau,\sigma}(l) \) in this range. Since \( \eta_{\tau,\sigma} \) is connected in degrees \( s_0(\tau) + k - h \) (in fact, in degrees \( s_0(\gamma) + k \)), this holds for \( \eta \) as well. We deduce the domination \( \tau \leq_{k-h} \sigma \) from proposition 2.6.

**Remark 2.8.** In the the proof of proposition 2.6, we showed that \( \eta(l) = \sigma(l) + 1 \) for \( s_0(\sigma) \leq l < s_0(\gamma) + h \). If we had used simply \( \sigma(l) \) instead, we would get a second function \( \theta \). It turns out that while \( \eta \) is easy to deal with algebraically, the function \( \theta \) has some nice geometric properties, as we will see in the next section. For \( \theta \) we have the following analogous proposition.

**Proposition 2.9.** Let \( \gamma \) be an admissible character and \( h \geq 0 \) an integer. Then there is a bijection between admissible characters \( \sigma \geq \gamma \) and functions \( \theta : \mathbb{Z} \to \mathbb{N} \) such that \( \sum \theta(l) = m \leq h \) and \( \theta(l) = 0 \) for \( l < s_0(\gamma) + m \). If \( \sigma \geq_h \gamma \) is given and \( \eta \) is the corresponding function from proposition 2.6, then \( \theta \) is given by

\[
\theta(l) = \eta(l) - \begin{pmatrix} 1 - s_0(\sigma) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ l - s_0(\gamma) - h \end{pmatrix}.
\]

This function is denoted \( \theta_{\gamma,\sigma,h} \), or just \( \theta \).

**Proof.** Let \( \sigma \geq_h \gamma \) be the given domination of admissible characters and let \( \eta, \theta \) be the two functions defined via propositions 2.6 and proposition 2.9. As noted in remark 2.8, we have that \( \theta(l) = \sigma(l) \) for \( s_0(\sigma) \leq l < s_0(\gamma) + h \). Since \( \sigma(l) \geq 0 \) for \( l \) in this range, we conclude that \( \theta(l) \geq 0 \) for all \( l \in \mathbb{Z} \). Clearly \( m = \sum \theta(l) \leq \sum \eta(l) = h \). Since \( \eta(l) = 0 \) for \( l < s_0(\sigma) = s_0(\gamma) + m \), this is also true for \( \theta \).
Conversely, suppose that we are given \( \theta \) as in the proposition, with \( m = \sum \theta(l) \). To give the corresponding \( \sigma \geq_h \gamma \), it suffices to produce \( \eta \) with the conditions of proposition 2.6. For \( l \in \mathbb{Z} \), we define

\[
\eta(l) = \theta(l) + \left( l - s_0(\gamma) - m \right) - \left( l - s_0(\gamma) - h \right).
\]

It is easily seen that \( \eta(l) \geq 0 \) for each \( l \in \mathbb{Z} \) and that \( \sum \eta(l) = h \). Because \( \theta(l) = 0 \) for \( l < s_0(\gamma) + m \), the formula shows that \( \eta \) also vanishes in this range. The formula also shows that \( \eta(l) \geq 1 \) for \( s_0(\gamma) + m \leq l < s_0(\gamma) + h \), hence \( \eta \) is connected in degrees \( < s_0(\gamma) + h \).

**Proposition 3.2.** Let \( \gamma \leq_h \tau \) and \( \gamma \leq_k \sigma \) be two dominations of admissible characters. Then \( \tau \leq_{k-h} \sigma \) if and only if \( \theta(l) = \theta_{\gamma,\sigma}(l) - \theta_{\gamma,\tau}(l - k + h) \geq 0 \) for all \( l \in \mathbb{Z} \) and \( \sum \theta(l) \leq k - h \), in which case we have \( \theta_{\tau,\sigma,k-h} = \theta \).

**Proof.** First assume that \( \tau \leq_{k-h} \sigma \). In this case the definition of domination gives that \( h \leq k \) and \( s_0(\tau) \leq s_0(\sigma) \leq s_0(\tau) + k - h \). Since \( s_0(\sigma) = s_0(\gamma) + \sum \theta_{\gamma,\tau}(l) \) and \( s_0(\tau) = s_0(\gamma) + \sum \theta_{\gamma,\tau}(l) \), the second inequality is equivalent to \( 0 \leq \sum \theta(l) \leq k - h \).

It is an easy calculation that \( \theta_{\tau,\sigma} = \theta \). Since \( \theta_{\tau,\sigma} \) is nonnegative, we deduce the forward direction.

Conversely, assume the two conditions. As noted in the preceding paragraph, \( \sum \theta(l) \leq k - h \) is equivalent to \( s_0(\sigma) \leq s_0(\tau) + k - h \). A tedious calculation gives the formula

\[
\eta_{\tau,\sigma}(l) = \theta(l) + \left( l - s_0(\sigma) \right) - \left( l - s_0(\tau) - k + h \right),
\]

which comes as no surprise given the statement of proposition 2.8. Now the fact that \( \theta \) is nonnegative and the inequality \( s_0(\sigma) \leq s_0(\tau) + k - h \) show that \( \eta_{\tau,\sigma} \) is nonnegative. Applying proposition 2.10 gives that \( \tau \leq_{k-h} \sigma \).

3. Geometric Invariants

Propositions 2.6 and 2.9 give two descriptions of the difference between a larger and smaller admissible character. Now we apply this to the geometric situation. In what follows, we will use \( n \)th difference functions quite a bit, so we make the following abbreviation to simplify matters.

**Notation 3.1.** Let \( \mathcal{F} \) be a coherent sheaf on \( \mathbb{P}^n \). We will use the abbreviation \( \Delta^m \mathcal{F} \) to denote the \( n \)th difference function of the cohomology function \( h^0(\mathcal{F}(l)) \). In other words, \( \Delta^m \mathcal{F}(l) = \Delta^m(h^0(\mathcal{F}(l))) \).

**Proposition 3.2.** Let \( X \subset \mathbb{P}^n \) be a subscheme of codimension \( \geq 2 \). Then the function \( \gamma : \mathbb{Z} \rightarrow \mathbb{Z} \) defined by

\[
\gamma(l) = \Delta^n \mathcal{I}_X(l) - \binom{l}{0}
\]

is an admissible character.

**Proof.** Consider the function \( \varphi(l) = h^0(\mathcal{I}_X(l)) - h^0(\mathcal{O}_{\mathbb{P}^n}(l)) \). Clearly \( \varphi(l) = 0 \) for \( l < 0 \). By Serre’s vanishing theorem, we see that \( \varphi(l) = -P_X(l) \) for \( l \gg 0 \), where
$P_X$ is the Hilbert polynomial for $X$. Since $X$ is of codimension $\geq 2$, the Hilbert polynomial has degree $\leq n - 2$, hence we have that $\Delta^{n-1}(P_X(l)) = 0$ for all $l$. It follows that the function $\Delta^{(n-1)}\varphi(l)$ has finite support. To see that $\gamma$ is a character, we note more generally that the first difference of a function of finite support is a character.

To check admissibility, we first note that $\gamma(l) = 0$ for $l < 0$ because this holds for $\varphi$. If we set $s_0 = \min\{l : h^0(\mathcal{I}_X(l)) \neq 0\}$, we see that $\gamma(l) = -1$ for $0 \leq l < s_0$ because $\Delta^n\mathcal{O}_{\mathbb{P}^n}(l) = {l\choose 0}$. Choosing $0 \neq v \in H^0(\mathcal{I}_X(s_0))$ gives rise to an injection $\tau : \mathcal{O}(-s_0) \rightarrow \mathcal{I}_X$, whose image consists of all multiples of an equation for a hypersurface $S$ of minimal degree which contains $X$. Letting $s_1$ be the least degree of a hypersurface $T$ which contains $X$ but is not a multiple of $S$ (such hypersurfaces exist because $X$ has codimension $\geq 2$), we see that $s_1$ is the least twist where $H^0(\tau)$ is not surjective. Since $\Delta^n\mathcal{O}(-s)(l) = {l - s\choose s}$, we have that $\gamma(l) = 0$ for $s_0 \leq l < s_1$ and that $\gamma(s_1) > 0$. This shows that $\gamma$ is an admissible character.

**Definition 3.3.** Let $X \subset \mathbb{P}^n$ be a subscheme of pure codimension two. The admissible character $\gamma(l) = \Delta^n\mathcal{I}_X(l) - {l\choose 0}$ of proposition 3.2 is called the $\gamma$-character of $X$, and is denoted $\gamma_X$. We define $s_0(X) = s_0(\gamma_X)$, $s_1(X) = s_1(\gamma_X)$ and $e(X) = \max\{l : h^{n-2}(\mathcal{O}_{\mathcal{X}_X}(l)) \neq 0\}$. $t_1(X)$ is the least degree of a hypersurface $T \supset X$ such that there is a hypersurface $S \supset X$ of degree $s_0(X)$ which meets $T$ properly. The **higher Rao modules of $X$** are the defined by $M^i_X = H^i_\gamma(\mathcal{I}_X)$ for $1 \leq i \leq n-2$. These are graded modules over the homogeneous coordinate ring $S = H^0(\mathcal{O}_{\mathbb{P}^n})$ of $\mathbb{P}^n$.

**Remark 3.4.** The geometric meaning of the integers $s_0(X)$ and $s_1(X)$ is described in the proof of proposition 3.2. We always have the inequalities $s_0(X) \leq s_1(X) \leq t_1(X)$.

**Remark 3.5.** Let $X$ and $Y$ be two subschemes of $\mathbb{P}^n$ having codimension $\geq 2$. In this case, the condition that $\gamma_X \leq h \gamma_Y$ has a simple formulation in terms of the ideal sheaves. The function $\eta$ of proposition 2.6 can be written

$$\eta(l) = \gamma_Y(l) - \gamma_X(l - h) + {l\choose 0} - {l-h\choose 0} = \Delta^n\mathcal{I}_Y(l) - \Delta^n\mathcal{I}_X(l - h).$$

Applying proposition 2.6, it follows that $\gamma_X \leq h \gamma_Y$ if and only if the function $\Delta^n\mathcal{I}_Y(l) - \Delta^n\mathcal{I}_X(l - h)$ is nonnegative, connected in degrees $< s_0(\gamma_X) + h$ and sums to $h$.

**Proposition 3.6.** Let $X \subset \mathbb{P}^n$ be a subscheme and suppose that $Y$ is obtained from $X$ by an elementary double link of type $(s,h)$. Then $\gamma_X \leq h \gamma_Y$, $M^i_Y \cong M^i_X(-h)$ for each $1 \leq i \leq n - 2$ and $\eta = \eta_{\gamma_X \gamma_Y,h}$ is given by

$$\eta(l) = {l-s\choose 0} - {l-s-h\choose 0}.$$ 

**Proof.** Suppose that $Y$ is obtained from $X$ by an elementary double link of type $(s,h)$. $X$ has an $\mathcal{N}$-type resolution that gives rise to the exact sequence

$$0 \rightarrow \mathcal{P} \rightarrow \mathcal{N} \rightarrow \mathcal{I}_X \rightarrow 0.$$
By proposition 1.14, there is an \( N \)-type resolution for \( Y \) that gives an exact sequence

\[
0 \rightarrow P(-h) \oplus \mathcal{O}(-s - h) \rightarrow N(-h) \oplus \mathcal{O}(-s) \rightarrow I_Y \rightarrow 0.
\]

Twisting the second sequence by \( h \) and adding the trivial summand \( \mathcal{O}(-s + h) \) to the first resolution gives a pair of exact sequences

\[
0 \rightarrow P \oplus \mathcal{O}(-s + h) \rightarrow N \oplus \mathcal{O}(-s + h) \rightarrow I_X \rightarrow 0
\]

\[
0 \rightarrow P \oplus \mathcal{O}(-s) \rightarrow N \oplus \mathcal{O}(-s + h) \rightarrow I_Y(h) \rightarrow 0.
\]

From these sequences we see that \( M_Y^i \cong H^i(N(-h)) \cong M_X^h(-h) \) for \( 1 \leq i \leq n - 2 \). Because the exact sequences above are also exact on global sections, we have that \( \Delta^n I_Y(l) = \Delta^n I_X(l-h) + (l-s) - (l-s-h) \), hence \( \eta(l) = \frac{(l-s)}{0} - \frac{(l-s-h)}{0} \) by remark 3.5. This function is clearly nonnegative and sums to \( h \). The fact that \( \eta \) is connected in degrees \( \leq s_0(X) + h \) follows from the fact that \( s \geq s_0(X) \). It follows from proposition 2.6 that \( \gamma_X \leq_h \gamma_Y \).

**Corollary 3.7.** With the hypotheses of Proposition 3.6, further assume that \( h = 1 \). Then \( \theta = \theta_{\gamma_X, \gamma_Y, 1} \) is given by the following:

1. If \( s = s_0(X) \), then \( \theta(l) = 0 \) for all \( l \in \mathbb{Z} \)
2. If \( s > s_0(X) \), then \( \theta(l) = \frac{(l-s)}{0} - \frac{(l-s-1)}{0} \) for all \( l \in \mathbb{Z} \)

**Proof.** This is a simple consequence of proposition 3.6 and the formula relating \( \theta \) and \( \eta \), once we note that \( s_0(Y) > s_0(X) \iff s > s_0(X) \).

**Theorem 3.8.** Let \( \mathcal{L} \) be an even linkage class of codimension two subschemes of \( \mathbb{P}^n \). Let \( X, Y \in \mathcal{L} \) and let \( l \geq 0 \) be an integer. Then \( X \leq_h Y \) if and only if \( \gamma_X \leq_h \gamma_Y \) and \( M_Y^i \cong M_X^h(-h) \) for each \( 1 \leq i \leq n - 2 \).

**Proof.** If \( Y \) is obtained from \( X \) by a basic double link of type \((s, h)\), then \( \gamma_X \leq_h \gamma_Y \) and \( M_Y^i \cong M_X^h(-h) \) by proposition 3.6. Since both of these conditions are transitive, it is clear that if \( Y \) is obtained from \( X \) by a sequence of basic double links with heights summing to \( h \), then \( \gamma_X \leq_h \gamma_Y \) and \( M_Y^i \cong M_X^h(-h) \). These invariants are fixed under a deformation which preserves cohomology and even linkage class, so we have proved the forward direction.

Conversely, suppose that \( \gamma_X \leq_h \gamma_Y \) and \( M_Y^i \cong M_X^h(-h) \) for \( 1 \leq i \leq n - 2 \). Let \( \mathcal{N}_0 \) be a reflexive sheaf such that \( \mathcal{L} \) corresponds to \([\mathcal{N}_0]\) via \( N \)-type resolutions. Since \( X \) and \( Y \) are in \( \mathcal{L} \), we can use Rao’s correspondence (theorem 1.10) to find \( N \)-type resolutions that give exact sequences

\[
0 \rightarrow P_X \rightarrow Q_X \oplus \mathcal{N}_0 \rightarrow I_X \rightarrow 0
\]

\[
0 \rightarrow P_Y \rightarrow Q_Y \oplus \mathcal{N}_0 \rightarrow I_Y(k) \rightarrow 0
\]

for some \( k \in \mathbb{Z} \). If \( \mathcal{N}_0 \neq 0 \), then some of the higher Rao modules of \( X \) are nonzero, and the condition \( M_Y^i \cong M_X^h(-h) \) shows that \( k = h \). If all the higher Rao modules are zero, then \( \mathcal{N}_0 = 0 \) and we can twist the dissocié sheaves \( P_Y \) and \( Q_Y \) by \( h - k \) to assume \( k = h \). In either case, we can find such resolutions with \( h = k \).

Adding \( Q_Y \) to the first sequence and \( Q_X \) to the second, we obtain exact sequences

\[
0 \rightarrow \oplus \mathcal{O}(-l)^{a(l)} \rightarrow N \rightarrow I_X \rightarrow 0
\]
0 → \oplus \mathcal{O}(−l)^{h(l)} → \mathcal{N} → \mathcal{I}_Y(h) → 0.

The condition \( \gamma_X \leq_h \gamma_Y \) tells us that \( \Delta^n\mathcal{I}_Y(l) \geq \Delta^n\mathcal{I}_X(l−h) \) for all \( l \in \mathbb{Z} \) by remark 3.5. Noting that the two sequences above are exact on global sections, it follows that \( \Delta^n(\oplus \mathcal{O}(−l)^{a(l)}) \geq \Delta^n(\oplus \mathcal{O}(−l)^{b(l)}) \). Since \( \Delta^n\mathcal{O}(−l) = (l)_0 \), this condition is equivalent to the condition that \( a^\#(l) \geq b^\#(l) \) for all \( l \in \mathbb{Z} \). By proposition 1.17, we conclude that \( X \leq_h Y \), finishing the proof.

**Proposition 3.9.** Let \( X \subset \mathbb{P}^n \) be a subscheme of pure codimension two having \( \gamma \)-character \( \gamma_X \) and let \( h \geq 0 \) be an integer. If \( \sigma \) is an admissible character with \( \gamma_X \leq_h \sigma \), then there exists a subscheme \( Y \subset \mathbb{P}^n \) which is obtained from \( X \) by a sequence of basic double links of height one such that \( X \leq_h Y \) and \( \gamma_Y = \sigma \).

**Proof.** We induct on \( h \). The base case is easy, since it is easy to check that if \( \gamma_X \leq_0 \sigma \) is a domination of admissible characters, then in fact \( \gamma_X = \sigma \). It follows that we may take \( Y = X \) in this case.

We now assume that \( h > 0 \). Set \( \eta = \eta_{\gamma_X, \sigma} \). Since \( \eta \) is connected in degrees \( < s_0(X) + h \) by proposition 2.6, we have that \( \eta_0 − h + 1 \geq s_0(X) \), so we can obtain a subscheme \( Z \) from \( X \) by a basic double link of type \( (s, 1) \), where \( s = \eta_0 − h + 1 \). From propositions 2.7 and 3.6 we see that \( \gamma_Z \leq_{h−1} \sigma \). By induction hypothesis we can obtain a subscheme \( Y \geq_{h−1} Z \) which is obtained from \( Z \) by a sequence of basic double links and satisfies \( \gamma_Y = \sigma \). Extending the chain of basic double links by the link from \( Z \) to \( Y \) completes the proof.

**Corollary 3.10.** Let \( \mathcal{L} \) be an even linkage class of subschemes of pure codimension two in \( \mathbb{P}^n \) and let \( X, Y \in \mathcal{L} \). Then there is a bijection between cohomology preserving deformation classes of subschemes \( Y \in \mathcal{L} \) such that \( X \leq_h Y \) and admissible characters \( \sigma \geq_h \gamma_X \).

**Proof.** If \( \{Y_h\}_{h \in \mathbb{H}} \) is such a deformation class, then \( \gamma_{Y_h} \) is a constant admissible character \( \geq_h \gamma_X \). On the other hand, if \( \sigma \geq_h \gamma_X \), then we can apply proposition 3.9 to obtain a subscheme \( Y \geq_h X \) with \( \gamma_Y = \sigma \). If \( Y' \) is another such subscheme, then \( \gamma_Y = \gamma_{Y'} \) and \( M_{Y'}^\# \cong M_Y^\# \). By theorem 3.8. Applying theorem 3.8 again, we have that \( Y \leq_{s_0} Y' \), which is equivalent to saying that there is a deformation from \( Y \) to \( Y' \) through subschemes in \( \mathcal{L} \) with constant cohomology.

**Remark 3.11.** In [2, theorem 3.5.7], Bolondi and Migliore prove essentially the same result in the case that \( X \) is a minimal element for a non-ACM even linkage class of locally Cohen-Macaulay subschemes of codimension two in \( \mathbb{P}^n \).

Let \( \mathcal{L} \) be an even linkage class of pure codimension two subschemes in \( \mathbb{P}^n \) which corresponds to \([\mathcal{N}_0]\) via \( \mathcal{N} \)-type resolution with \( \mathcal{N}_0 \neq 0 \). By theorem 1.20, \( \mathcal{L} \) has a minimal element \( X_0 \). If we apply corollary 3.10 with \( X = X_0 \), we see that the admissible characters \( \sigma \geq \gamma_{X_0} \) index all the cohomology preserving deformation classes of \( \mathcal{L} \). By propositions 2.6 and 2.8, they can also be indexed by functions \( \eta \) and \( \theta \) which satisfy certain conditions.

**Definition 3.12.** Let \( \mathcal{L} \) be an even linkage class of subschemes of codimension two in \( \mathbb{P}^n \). Assume that \( \mathcal{L} \) corresponds to \([\mathcal{N}_0]\) via \( \mathcal{N} \)-type resolutions, with \( \mathcal{N}_0 \neq 0 \). Let \( X_0 \) be a minimal element for \( \mathcal{L} \). Then we define \( s_0(\mathcal{L}) = s_0(X_0), s_1(\mathcal{L}) = s_1(X_0), t_1(\mathcal{L}) = t_1(X_0) \) and \( e(\mathcal{L}) = e(X_0) \).
Remark 3.13. We note that the integers above are well-defined. Indeed, if \(X_0, X'_0\) are minimal elements for \(\mathcal{L}\), then theorem 1.20 shows that \(X_0\) and \(X'_0\) have the same cohomology, and hence \(s_0(X_0) = s_0(X'_0), s_1(X_0) = s_1(X'_0)\) and \(e(X_0) = e(X'_0)\). To see that \(t_1(\mathcal{L})\) is well-defined, we apply theorem 1.21 to see that any minimal subscheme \(X_0\) links to a minimal subscheme \(Y_0\) for the dual linkage class via a hypersurface \(S\) of degree \(s_0(\mathcal{L})\) and another hypersurface \(T\). Since the degrees of \(X_0\) and \(Y_0\) are determined by the even linkage class \(\mathcal{L}\), so is the degree of \(T\). This degree is \(t_1(\mathcal{L})\).

Definition 3.14. Let \(\mathcal{L}, X_0\) be as in definition 3.12. Let \(X \in \mathcal{L}\). We define the height of \(X\) to be the unique nonnegative integer \(h_X\) such that \(M^i_X \cong M^i_{X_0}(-h_X)\) for each \(1 \leq i \leq n - 2\).

Definition 3.15. Let \(\mathcal{L}, X_0\) be as in definition 3.12. Let \(\gamma_0 = \gamma_{X_0}\). Then we define \(\eta_X = \eta_{\gamma_0, \gamma_X, L_X}\), as defined in proposition 2.6. Similarly we define \(\theta_X = \theta_{\gamma_0, \gamma_X, L_X}\), as defined in proposition 2.9.

Proposition 3.16. Let \(\mathcal{L}\) be an even linkage class as in definition 3.12. If \(X \in \mathcal{L}\), then

1. \(s_0(X) = s_0(\mathcal{L}) + \sum \theta_X(l)\)
2. \(s_1(X) = \min\{s_1(\mathcal{L}) + h_X, (\theta_X)_a\}\)
3. \(e(X) = \max\{e(\mathcal{L}) + h_X, (\eta_X)_o - n\}\)

Proof. Suppose that \(X\) is obtained from \(X'\) by a basic double link of type \((s, 1)\). Corollary 3.7 shows that \(\sum \theta_{X', X}(l) = 1\) if \(s > s_0(X')\), and is equal to 0 if \(s = s_0(X')\). Noting that \(e(X) = \max\{l : H^{n-1}(I_X(l)) \neq 0\}\) and looking at the way an \(\mathcal{N}\)-type resolution changes in going from \(X'\) to \(X\) (proposition 1.14), we find that \(e(X) = \max\{e(X') + 1, s - n\}\). Using these two results, we find that if we obtain \(X\) from a minimal element \(X_0\) by a sequence of basic double links of height one, then the first and third formulas above hold. By proposition 3.9 and corollary 3.10, we can find \(\overline{X}\) which has the same cohomology as \(X\) and is obtained from a minimal element by a sequence of basic double links of height one, which proves (1) and (3).

For the second relationship, we combine the formulas for \(\eta\) (proposition 2.6) and \(\theta\) (proposition 2.9) to get

\[
\gamma_X(l) = \gamma_{X_0}(l) - \binom{l}{0} + \binom{l - h_X}{0} + \binom{l - s_0(X)}{0} - \binom{l - s_0(X_0) - h_X}{0} + \theta(l).
\]

If we subtract the \(\theta(l)\) part, we can use the definitions of \(s_0\) and \(s_1\) to see that what remains is equal to 0 for \(s_0(X) \leq l < s_1(X_0) + h_X\) and positive for \(l = s_1(X_0) + h_X\). The second equality follows.

Remark 3.17. Another invariant is described in [2] which also describes the location of a subscheme \(X\) in \(\mathcal{L}\) with respect to a minimal element \(X_0\). This invariant is a sequence of integers \(\{b, g_2, g_3, \ldots, g_r\}\) where \(b \geq 0, g_i \leq g_{i+1}\) and \(r + b = h_X - 1\). For the reader who is familiar with this invariant, we briefly describe how to go back and forth via the invariant \(\theta_X\).

If we know \(\{b, g_2, g_3, \ldots, g_r\}\), then we obtain the function \(\theta_X\) by the rule

\[
\theta_X(l) = \#\{k : g_k + r - k = l\}
\]

In this case we have \(\sum \theta(l) = r - 1\) and \(s_0(X) = s_0(X_0) + r - 1\).
Conversely, given the function $\theta_X$ with $m = \sum \theta_X(l)$, we compute the corresponding invariant $\{b, g_2, g_3, \ldots, g_r\}$ by $b = h_X - m - 1, r = m - 1$ and

$$g_k = r - k + \max\{s : \sum_{l \geq s} \theta(l) > r - k\}.$$ 

**Lemma 3.18.** Let $\mathcal{L}$ be as in definition 3.12 and let $X \in \mathcal{L}$ be a subscheme which links to $Y$ by a complete intersection of hypersurfaces of degrees $s$ and $t$. Letting $s_0 = s_0(\mathcal{L})$ and $t_1 = t_1(\mathcal{L})$, we have that

$$h_X + h_Y = s + t - s_0 - t_1.$$ 

**Proof.** Let $X_0$ and $Y_0$ be a pair of minimal elements for their even linkage classes, which are linked by hypersurfaces of degrees $s_0$ and $t_1$ (see remark 3.13). If $X_0$ has an $\mathcal{N}$-type resolution giving an exact sequence

$$0 \to \mathcal{P}_0 \to \mathcal{N}_0 \to \mathcal{I}_{X_0} \to 0$$

then in applying proposition 1.6 we get an $\mathcal{E}$-type resolution for $Y$ which gives an exact sequence

$$0 \to \mathcal{N}_0^\vee (-s_0 - t_1) \to \mathcal{P}_0^\vee (-s_0 - t_1) \oplus \mathcal{O}(-s_0) \oplus \mathcal{O}(-t_1) \to \mathcal{I}_{Y_0} \to 0.$$ 

$X$ has an $\mathcal{N}$-type resolution which involves $\mathcal{N}_0^\vee (-s_0 - t_1 - h_Y)$ and $Y$ has an $\mathcal{E}$-type resolution which involves $\mathcal{N}_0^\vee (h_X - s - t)$. By looking at the higher Rao modules of $Y$ (which are not all zero because we assumed that $\mathcal{L}$ was not the ACM class), we see that the twists must agree, so we have $h_X - s - t = -s_0 - t_1 - h_Y$.

**Theorem 3.19.** Let $\mathcal{L}, X_0$ be as in definition 3.12. Let $s_0 = s_0(\mathcal{L})$ and $t_1 = t_1(\mathcal{L})$ and let $X \in \mathcal{L}$ be a subscheme which links to $Y$ by a complete intersection of hypersurfaces of degrees $s$ and $t$. Then the functions $\eta_X$ and $\eta_Y$ are related by the formula

$$\eta_X(l) - \eta_Y(s + t - 1 - l) = \begin{pmatrix} l - s \\ 0 \end{pmatrix} + \begin{pmatrix} l - t \\ 0 \end{pmatrix} - \begin{pmatrix} l - s_0 - h_X \\ 0 \end{pmatrix} - \begin{pmatrix} l - t_1 - h_X \\ 0 \end{pmatrix}.$$ 

**Proof.** By theorem 1.21 there exists a minimal element $Y_0$ for the dual linkage class to $\mathcal{L}$ which links to $X_0$ by surfaces of degrees $s_0$ and $t_1$. Conceptually, the situation is represented by a diagram

$$\begin{array}{c} X \xrightarrow{s,t} Y \\ \uparrow \quad \uparrow \\ X_0 \xrightarrow{s_0,t_1} Y_0 \end{array}$$

in which the horizontal arrows denote linkages and the vertical arrows denote dominations. By theorem 1.20, $X_0$ has an $\mathcal{N}$-type resolution which gives an exact sequence

$$0 \to \mathcal{P}_0 \to \mathcal{N}_0 \to \mathcal{I}_{X_0} \to 0.$$
We next apply proposition 1.6 to the $N$-type resolution for $X_0$ to get an $E$-type resolution for $Y_0$, which gives an exact sequence

$$0 \rightarrow N_0^\lor(-s_0 - t_1) \rightarrow P_0^\lor(-s_0 - t_1) \oplus O(-s_0) \oplus O(-t_1) \rightarrow I_{Y_0} \rightarrow 0.$$ 

Now obtain $X$ from $X_0$ by an even number of links. If we apply proposition 1.6 for each link, we get a resolution for $I_X$

$$0 \rightarrow P_0(-k) \oplus P_X \rightarrow N_0(-k) \oplus Q_X$$

where $P_X$ and $Q_X$ are dissocié. In looking at the higher Rao modules, we see that $k = h_Y$. Remark 2.9 shows that $\eta_X(l) = \Delta^nI_X(l) - \Delta^nI_{X_0}(l - h_X)$, which can be rewritten as $\Delta^nQ_X(l) - \Delta^nP_X(l)$ in view of the two resolutions. Similarly, we apply proposition 1.6 while doing an even number of linkages to get from $Y_0$ to $Y$, and obtain a resolution for $I_Y$ of the form

$$0 \rightarrow N_0^\lor(-s_0 - t_1 - h_Y) \oplus P_Y \rightarrow P_0^\lor(-s_0 - t_1 - h_Y) \oplus O(-s_0 - h_Y) \oplus O(-t_1 - h_Y) \oplus Q_Y$$

where $P_Y$ and $Q_Y$ are dissocié. As above, we have $\eta_Y(l) = \Delta^nQ_Y(l) - \Delta^nP_Y(l)$.

Now we apply proposition 1.6 to the linkage between $X$ and $Y$. After using the formula of lemma 3.18 to simplify some expressions, we obtain another resolution for $I_X$ of the form

$$\begin{array}{c}
0 \\
\downarrow \\
\begin{array}{c}
P_0(-h_X) \oplus Q_Y(-s - t) \oplus O(-t_1 - h_X) \oplus O(-s_0 - h_X) \\
\downarrow \\
N_0(-h_X) \oplus P_Y(-s - t) \oplus O(-s) \oplus O(-t)
\end{array}
\end{array}$$

Comparing with the other resolution for $I_X$, we see that $\eta_X(l)$ is given by

$$\begin{align*}
\Delta^nP_Y^\lor(l - s - t) - \Delta^nQ_Y^\lor(l - s - t) + \\
\begin{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
(l - s) \\
0
\end{pmatrix} & (l - t) \\
0 & 0
\end{pmatrix} & (l - s_0 - h_X) \\
0 & 0
\end{pmatrix} - (l - t_1 - h_X).
\end{pmatrix}
\end{align*}$$

Since $P_Y$ and $Q_Y$ are dissocié sheaves of the same rank, we can write

$$-\Delta^nQ_Y^\lor(l - s - t) + \Delta^nP_Y^\lor(l - s - t) = \Delta^nQ_Y(s + t - 1 - l) - \Delta^nP_Y(s + t - 1 - l),$$

which is just $\eta_Y(s + t - 1 - l)$. Substituting this into the formula for $\eta_X$ gives the result.

**Corollary 3.20.** With the same hypotheses as theorem 3.19, assume further that $s = s_0(X)$. Then $\theta_X$ is related to $\eta_Y$ by the formula

$$\theta_X(l) = \eta_Y(s + t - 1 - l) + \begin{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
(l - t) \\
0
\end{pmatrix} & (l - t_1 - h_X).
\end{pmatrix}
\end{pmatrix}
\end{align*}$$

**Proof.** This follows immediately from the formula relating $\theta_X$ and $\eta_X$ of proposition 2.9.
4. Subschemes which satisfy \( s_1 = t_1 \)

In this section we study the subschemes \( X \) in an even linkage class \( \mathcal{L} \) which satisfy \( s_1(X) = t_1(X) \). As in the previous section, we restrict to the case in which \( \mathcal{L} \) is not the class of arithmetically Cohen-Macaulay subschemes. A purely numerical criteria is found for the existence of such a subscheme in terms of the invariant \( \theta \). This is achieved by finding a sharp lower bound for \( t_1(X) \), from which the theorem follows immediately.

We mentioned earlier that we always have the inequality \( s_1(X) \leq t_1(X) \). Before going on to the results in this section, we give an example to show that this inequality can be strict.

**Example 4.1.** Let \( C_0 \) be the disjoint union of two lines in \( \mathbb{P}^3 \) and let \( C \) be an elementary double link from \( C_0 \) of type \((8,1)\). Then one easily computes that \( s_0(C) = s_1(C) = 3 \), but we must have \( t_1(C) > 3 \) because \( C \) has degree 10 and hence cannot be contained in a complete intersection of cubic surfaces. In fact, it can be shown that \( t_1(C) = 8; t_1(C) \leq 8 \) because \( C \) was an elementary double link of type \((8,1)\) from \( C_0 \). To see that \( t_1(C) \geq 8 \), suppose that \( C \) links to a curve \( D \) via surfaces of degrees 3 and \( t \). Then the degree of \( D \) is \( 3t - 10 \), while the height of \( D \) is \( t - 2 \). Since \( D \) is in the even linkage class of two skew lines, we see that when the height of \( D \) is \( t - 2 \), the degree of \( D \) is at least \( 2 + 2(t - 2) = 2t - 2 \). Comparing these two estimates on the degree of \( D \), we see that \( t \geq 8 \).

**Lemma 4.2.** Let \( F_a, F_b \subset \mathbb{P}^n \) be hypersurfaces of degrees \( a \leq b \) which meet properly. If \( F_c \) is any other hypersurface, then there exists a hypersurface \( G_b \) of degree \( b \) which contains \( F_a \cap F_b \) and meets \( F_c \) properly.

**Proof.** Let \( X \) denote the complete intersection \( F_a \cap F_b \) and consider the projective space \( \mathbb{P} = \mathbb{P}H^0(\mathcal{L}_X(b)) \). Let \( f \) be the equation of the hypersurface \( F_c \) and factor \( f \) into its irreducible factors as \( \Pi_i p_i \). Let \([p_i]\subset\mathbb{P}\) denote the linear subspace consisting of all nonzero multiples of \( p_i \) (note that \([p_i]\) may be empty). Then each \([p_i]\) is a proper linear subspace of \( \mathbb{P} \), since for each \( i \) either \( p_i \) does not divide the equation for \( F_b \) or \( p_i \) does not divide the equation for \( F_a \), and hence does not divide some multiple of that equation of degree \( b \). Since these are proper, the union \( \bigcup_{i=1}^r[p_i] \) is a closed proper subspace of \( \mathbb{P} \). It follows that for the general element \( g \in H^0(\mathcal{L}_X(b)) \), none of the \( p_i \) divide \( g \), hence the intersection \( Z(f) \cap Z(g) \) is proper.

**Corollary 4.3.** Let \( X \) be a codimension two subscheme of \( \mathbb{P}^n \) which is contained in surfaces \( F_a, F_b \) of degrees \( a \leq b \) which meet properly. Then \( t_1(X) \leq b \).

**Proof.** Apply lemma 4.2 with \( F_c = S \), \( S \) a surface of degree \( s_0(X) \) which contains \( X \).

**Remark 4.4.** One consequence of corollary 4.3 is that for any subscheme \( X \), the least degree of a complete intersection of two hypersurfaces containing \( X \) is precisely \( s_0(X)t_1(X) \), and that any pair of surfaces giving such a complete intersection have degrees \( s_0(X) \) and \( t_1(X) \).

**Notation 4.5.** For the remainder of this section, we let \( \mathcal{L} \) denote a fixed even linkage class of codimension two subschemes in \( \mathbb{P}^n \) such that \( \mathcal{L} \) corresponds to \([N_0]\) via \( \mathcal{N} \)-type resolution with \( N_0 \neq 0 \). We set \( s_0 = s_0(\mathcal{L}), s_1 = s_1(\mathcal{L}) \) and \( t_1 = t_1(\mathcal{L}) \).
**Theorem 4.6.** Let $X \in \mathcal{L}$ and let $\theta = \theta_X$. Then we have the following sharp lower bounds on $t_1(X)$.

1. If $\theta_0 < t_1 + h_X - 1$, then $t_1(X) \geq t_1 + h_X$.
2. If $\theta_0 \geq t_1 + h_X - 1$, then $t_1(X) \geq \max\{l : \theta(l) \neq 0 \text{ and } \theta(l-1) = 0\}$

**Proof.** Let $s = s_0(X)$ and $t = t_1(X)$. Then there exist hypersurfaces $S, T$ of degrees $s, t$ such that $S \cap T$ links $X$ to another scheme $Y$. Corollary 3.20 gives the formula

$$\theta(l) = \eta_Y(s + t - 1 - l) + \begin{pmatrix} l - t \hline 0 \end{pmatrix} - \begin{pmatrix} l - t_1 - h_X \hline 0 \end{pmatrix}.$$ 

Now suppose that $\theta_0 < t_1 + h_X - 1$ (This includes the case when $\theta = 0$). If $t < t_1 + h_X$, then formula above shows that $\eta_Y(s + t - t_1 - h_X) = -1$, which contradicts the fact that $\eta_Y$ is a nonnegative function. Thus $t \geq t_1 + h_X$.

Now consider the case when $\theta_0 \geq t_1 + h_X - 1$. Since $\eta_Y$ is connected in degrees $< s_0 + h_Y$, $\eta_Y(s + t - 1 - l)$ is connected in degrees $\geq s + t - t_1 - h_Y = s_0 + h_X$. Looking at the formula relating $\theta$ and $\eta_Y$, we conclude that $\theta$ is connected in degrees $\geq t$ (consider the cases $t \leq t_1 + h_X$ and $t > t_1 + h_X$ separately). It follows that $t \geq \max\{l : \theta(l) \neq 0 \text{ and } \theta(l-1) = 0\}$.

To see that these bounds are sharp, we construct examples by induction on $\sum \theta(l)$. If this sum is zero, then we can obtain a subscheme $X'$ by directly linking to a minimal element $Y_0$ for the dual linkage class hypersurfaces of degrees $s_0$ and $t_1 + h_X$. From corollary 4.3 it is clear that $t_1(X') \leq t_1 + h_X$.

Now assume that $\theta \neq 0$. Let $r = \max\{l : \theta(l) \neq 0 \text{ and } \theta(l-1) = 0\}$ and let $w = \theta_0 - r + 1$ (this is the width of the rightmost connected piece of $\theta$). Define $\theta'$ by

$$\theta'(l) = \theta(l + w) - \begin{pmatrix} l - r - w \hline 0 \end{pmatrix} + \begin{pmatrix} l - \theta_0 - 1 \hline 0 \end{pmatrix}.$$ 

Because $\theta$ satisfies the criteria of proposition 2.9 for the height $h_X$, it is easily seen that $\theta'$ does as well for the height $h_X - w$. Hence $\theta' = \theta_X'$, for some subscheme in $\mathcal{L}$ of height $h_X - w$. If we define $r'$ for $X'$ analogously to the definition of $r$ for $X$, we can use our induction hypothesis to assume that $t_1(X') \leq \max\{r', t_1 + h_X - w\}$.

Using proposition 3.6, we see that if $\overline{X}$ is obtained from $X'$ by an elementary double link of type $(r, w)$, then $\overline{X}$ is in the cohomology preserving deformation class of $X$. If $\theta_0 < t_1 + h_X - 1$, then $r < t_1 + h_X - w - 1$ and we can first link $X'$ to $Y'$ by hypersurfaces of degrees $r, t_1 + h_X - w$ and then link $Y$ to $\overline{X}$ by hypersurfaces of degrees $r, t_1 + h_X$. From corollary 4.3, we see that $t_1(\overline{X}) \leq t_1 + h_X$. On the other hand, if $\theta_0 \geq t_1 + h_X - 1$, then $r \geq r'$, and we can use hypersurfaces of degrees $r, s_0(X')$ to link $X'$ to $Y$, and then use hypersurfaces of degrees $r, s_0(X') + w$ to link $Y$ to $\overline{X}$. In this case we have that $r \geq s_0(X')$, so corollary 4.3 shows that $t_1(\overline{X}) \leq r$. In either case, we obtain sharp bounds.

**Theorem 4.7.** Let $X \in \mathcal{L}$. Then $X$ can be deformed with constant cohomology through subschemes in $\mathcal{L}$ to $\overline{X}$ satisfying $s_1(\overline{X}) = t_1(\overline{X})$ if and only if $\theta_X$ is connected about the interval $[s_1 + h_X, t_1 + h_X - 1]$.

**Proof.** Let $\theta = \theta_X$. Supposing that $X \in \mathcal{L}$ satisfies $t_1(X) = s_1(X)$, we first show that $\theta_X$ is connected about $[s_1 + h_X, t_1 + h_X - 1]$. If $\theta = 0$, then proposition 3.16 shows that $s_0(X) = s_0$ and $s_1(X) = s_1 + h_X$. If $s_1 < t_1$, then lemma 3.18 shows
that \( X \) links to a subscheme of negative height, a contradiction. Hence \( s_1 = t_1 \), the interval in empty, and \( \theta \) is connected about \([s_1 + h_X, t_1 + h_X - 1]\), as we wanted.

Now assume that \( \theta \neq 0 \). In this case, proposition 3.16 shows that \( s_1(X) \leq \theta_a \).

If \( \theta_o < t_1 + h_X - 1 \), then from theorem 4.6 we get

\[
s_1(X) \leq \theta_a \leq \theta_o < t_1 + h_X \leq t_1(X)
\]

and hence \( s_1(X) < t_1(X) \), contradiction. We may henceforth assume that \( \theta_o \geq t_1 + h_X - 1 \). If \( \theta \) is not connected, then by the second case of theorem 4.6 we have

\[
s_1(X) \leq \theta_a < \max\{l : \theta(l) \neq 0 \text{ and } \theta(l-1) = 0\} \leq t_1(X),
\]

which also contradicts our assumption, so \( \theta \) is connected. Finally, if \( \theta_a > s_1 + h_X \), then \( s_1(X) = s_1 + h_X < \theta_a \leq t_1(X) \), which also contradicts assumption. We have shown that \( \theta_o \geq t_1 + h_X - 1 \), \( \theta_o \leq s_1 + h_X \), and \( \theta \) is connected. This proves the forward part of the theorem. Now suppose that \( \theta \) is connected about \([s_1 + h_X, t_1 + h_X - 1]\). We will use the sharpness of the bounds in theorem 4.6 to show that \( X \) deforms to \( X \) satisfying \( s_1(X) = t_1(X) \). If \( \theta_o < t_1 - h_X - 1 \), then \( \theta \) can only be connected about \([s_1 + h_X, t_1 + h_X - 1]\) if \( \theta = 0 \) and the interval is empty. In this case, \( s_1 = t_1 \), and using the sharpness of theorem 4.6, we can find \( X \) in the deformation class of \( X \) with \( t_1(X) = t_1 + h_X = s_1 + h_X = s_1(X) \) (this last equality follows from proposition 3.16 and the fact that \( \theta = 0 \)).

Now assume that \( \theta_o > t_1 + h_X - 1 \) (in particular, \( \theta \neq 0 \)). The condition on \( \theta \) shows that \( \theta_o \leq s_1 + h_X \), hence \( s_1(X) = \theta_o \) by proposition 3.16. Using the sharpness of the second case of theorem 4.6, we can find \( X \) in the deformation class of \( X \) such that \( t_1(X) = \theta_a = s_1(X) \). This finishes the proof.

**Corollary 4.8.** With the hypotheses of theorem 4.7, assume that \( X \) links to \( Y \) by hypersurfaces of degrees \( s = s_0(X) \) and \( t = t_1(X) \). If \( \theta = \theta_X \) is the zero function, then so is \( \theta_Y \).

If \( \theta \neq 0 \), then

\[
\theta(l) = \theta_Y(s + t - 1 - l) + \left\lfloor \frac{l - \theta_o}{0} \right\rfloor - \left\lfloor \frac{l - \theta_a - 1}{0} \right\rfloor.
\]

**Proof:** If \( \theta = 0 \), then the formula of corollary 3.20 makes it clear that \( \eta_Y = 0 \) hence \( \theta_Y = 0 \), so we go on to the case where \( \theta \neq 0 \). In this case, theorem 4.7 shows that \( t = s_1(X) = \theta_o \). Corollary 3.20 gives us the formula

\[
\theta(l) = \eta_Y(s + t - 1 - l) + \left\lfloor \frac{l - \theta_a}{0} \right\rfloor - \left\lfloor \frac{l - t_1 - h_X}{0} \right\rfloor.
\]

Now we consider two cases. If \( (\eta_Y)_s \geq s_0 + h_Y, \) then \( \eta_Y = \theta_Y, \eta_Y(s + t - 1 - l) = 0 \) for \( l \geq t_1 + h_X \) (by lemma 3.18), and hence \( \theta_o \leq t_1 + h_X - 1 \). The property of theorem 4.6 shows that \( \theta_a = t_1 + h_X - 1 \), and we deduce the result.

Now we consider the case \( (\eta_Y)_s < s_0 + h_Y \). In this case \( \eta_Y(s + t - 1 - l) \) is nonzero for some \( l \geq t_1 + h_X \) and we must have \( (\eta_Y)_s = s + t - 1 - \theta_o \). Since \( \eta_Y = \theta_Y + 1 \) on the interval \([\eta_Y)_s, s_0 + h_Y - 1 \) and \( t_1 + h_X = s + t - (s_0 + h_Y) \), we can write

\[
\theta_Y(s + t - 1 - l) = \theta_Y(s + t - 1 - l) + \left\lfloor \frac{l - t_1 - h_X}{0} \right\rfloor - \left\lfloor \frac{l - s - t - \theta_o}{0} \right\rfloor.
\]

Substituting in the formula for \( (\eta_Y)_s \) (in terms of \( \theta_o \)) into the above expression gives the formula.
Corollary 4.9. Let $\mathcal{M} \subset \mathcal{L}$ denote the subset of subschemes $X$ such that $s_1(X) = t_1(X)$. Then $\mathcal{M}$ has a minimal element with respect to domination.

Proof. We produce a minimal element for $\mathcal{M}$ as follows. Let $X_0$ be an (absolute) minimal element for $\mathcal{L}$. If we link $X_0$ to a minimal element $Y_0$ of the dual class via hypersurfaces $S, T$ of degrees $s_0, t_1$ (possible by theorem 1.21) and then link $Y_0$ to a subscheme $X_1$ via hypersurfaces $S', T$ of degrees $s_0 + t_1 - s_1, t_1$, the subscheme $X_1$ is obtained from $X_0$ by an elementary double link of type $(t_1, t_1 - s_1)$. From corollary 4.3 it is evident that $t_1(X_1) \leq t_1$. On the other hand, we also find from proposition 3.6 that

$$\eta_{X_1}(l) = \begin{pmatrix} l - t_1 & l - 2t_1 + s_1 \\ 0 & 0 \end{pmatrix}.$$  

Since $(\eta_{X_1})_a = t_1 \geq s_0 + h_{X_1}$, we have $\theta_X = \eta_X$ and proposition 3.16 shows that $s_1(X_1) = t_1$. Thus $s_1(X_1) = t_1(X_1)$ and $X_1 \in \mathcal{M}$.

Let $X \in \mathcal{M}$. Since $s_1(X) = t_1(X)$, theorem 4.7 shows that $\theta_X$ is connected about $[s_1 + h_X, t_1 + h_X - 1]$. Since $\theta_{X_1}$ is the identity function for the interval $[s_1 + h_{X_1}, t_1 + h_{X_1} - 1]$, we easily check that the function

$$\theta(l) = \theta_X(l) - \theta_{X_1}(l - h_X + h_{X_1})$$

is nonnegative. Since $\sum \theta_{X_1}(l) = h_{X_1}$, it is immediate that $\sum \theta(l) \leq h_X - h_{X_1}$. We can now apply proposition 2.10 to see that $X_1 \leq h_X - h_{X_1} X$. Hence each element in $\mathcal{M}$ dominates $X_1$, so $X_1$ is a minimal element for $\mathcal{M}$.

Remark 4.10. The reader might wonder if the subposet $\mathcal{M} \subset \mathcal{L}$ of proposition 4.9 satisfies the Lazarsfeld-Rao property. The answer is no. For example, if $X_1$ is the minimal element of $\mathcal{M}$, let $X_2$ be obtained from $X_1$ by a basic double link of type $(t_1 + h_{X_1} + 1, 1)$. In this case $\theta_{X_2}$ is connected about $[s_1 + h_{X_2}, t_1 + h_{X_2} - 1]$, but it is impossible for $X_2$ to satisfy $s_1(X_2) = t_1(X_2)$ because all hypersurfaces of degrees $< t_1 + h_{X_1} + 1$ which contain $X_2$ share a common hyperplane.

5. Integral Subschemes in an Even Linkage Class

In this section we give a necessary conditions for subschemes to be integral and investigate to what extent these conditions are sufficient. For this section we work in a fixed even linkage class $\mathcal{L}$ of codimension two subschemes of $\mathbb{P}^n$ which corresponds to $[N_0]$ with $N_0 \neq 0$. To simplify notation we set $s_0 = s_0(\mathcal{L}), s_1 = s_1(\mathcal{L}), t_1 = t_1(\mathcal{L})$ and $e = e(\mathcal{L})$. Before working in $\mathcal{L}$, we prove a few technical lemmas to allow for more conceptual proofs of the main results.

Proposition 5.1. Let $X \subset \mathbb{P}^n$ be an integral subscheme of codimension two. Let $t \in \mathbb{Z}$ be an integer such that $t = s_0(X)$ or $t \geq s_1(X)$. Then the general hypersurface $H$ containing $X$ of degree $t$ enjoys the following properties.

1. $H$ is integral.
2. $X$ is not contained in the singular locus of $H$.
3. $X$ is generically Cartier on $H$.

Proof. Let $H \supset X$ be a hypersurface of degree $t = s_0(X)$ whose equation is $h$. If $h = fg$ is reducible, then $X \subset Z(f)$ or $X \subset Z(g)$ because $X$ is integral, which would
contradict the minimality of the degree of $H$. We conclude that $H$ is integral. Now suppose that $X$ is contained in the singular locus of $H$. $X$ is reduced, so it would lie on the hypersurfaces with equations $\partial h/\partial X_i$, where $X_i$ are the homogeneous coordinates. These equations must all be zero, as otherwise the minimality of the degree of $H$ is contradicted. From Euler’s formula $\sum (\partial h/\partial X_i) X_i = th$, we see that $t = 0$, the characteristic of $k$ is $p > 0$, and $h$ is a $p^{th}$ power of another equation $g$. Since $h$ is irreducible, we again reach a contradiction, and conclude that $X$ is not contained in the singular locus of $H$. Thus we have proved the first two conditions in the special case when $t = s_0(X)$.

Now fix a hypersurface $S \supset X$ of degree $s_0(X)$ and let $T$ be a hypersurface of degree $s_1(X)$ which is not a multiple of $S$. If the equation of $T$ is $f$, we again find that $f$ is not reducible, for if $f = gh$, then $X \subset Z(g)$ or $X \subset Z(h)$ would give a hypersurface of degree $< s_1(X)$ which meets $S$ properly. Thus $T$ is integral and $X$ is contained in the complete intersection $S \cap T$. Because $\mathcal{I}_{S/T}(s_1(X))$ is generated by global sections, the general hypersurface $H \supset S \cap T$ of degree $> s_1(X)$ is integral. Since integrality among hypersurfaces of a fixed degree is an open condition, the general hypersurface $H \supset X$ of degree $> s_1(X)$ is integral. This proves the first property.

The set of $h \in \mathbb{P}H^0(\mathcal{I}_X(t))$ such that $X \subset Z(\partial h/\partial X_i)$ for each $i$ is closed. On the other hand, this closed set must be proper, since if $S \supset X$ is a hypersurface of minimal degree, then the general union of $S$ with a nonsingular hypersurface which meets $X$ properly will give a hypersurface whose singular locus does not contain $X$. This checks the second property. If $x \in X \cap T_{\text{reg}}$, then the ideal of $X$ at $x$ is a height one prime ideal in the regular local ring $\mathcal{O}_{T,x}$, hence is principal by Krull’s Haupptidealsatz. This shows that $X$ is generically Cartier on each hypersurface satisfying the second property.

**Proposition 5.2.** Let $X \in \mathcal{L}$ be an integral subscheme with $0 \neq \theta_X = \theta$. Then $\theta_a \leq s_0 + h_X$.

**Proof.** By way of contradiction, assume that $X$ is integral and $\theta_a > s_0 + h_X$. Applying proposition 5.1, we see that $X$ is linked to a subscheme $Y$ by integral hypersurfaces $S,T$ of degrees $s = s_0(X), t = s_1(X)$. Corollary 4.8 shows that $\sum \theta_Y(l) < \sum \theta(l)$, hence $s_0(Y) < s_0(X) = s$. Since $S$ is an integral hypersurface of degree $s$, we must have that $t_1(Y) \leq s$. By theorem 4.7, we have that $t = \theta_a > s_0 + h_X$, hence $s < t_1 + h_Y$ by lemma 3.18, and $t_1(Y) < t_1 + h_Y$. On the other hand, corollary 4.8 shows that $\theta_Y(s + t - 1 - l) = 0$ for $l < \theta_Y$. In particular, this holds for $l \leq s_0 + h_X$, so $\theta_Y(l) = 0$ for $l \geq s + t - 1 - s_0 - h_X = t_1 + h_Y - 1$ and $(\theta_Y)_o < t_1 + h_Y - 1$. Theorem 4.6 now gives that $t_1(Y) \geq t_1 + h_Y$, a contradiction.

Now we prove another necessary condition on integral subschemes, but this one requires more work. In [9], Lazarsfeld and Rao show that if $e(C) + 4 < s_0(C)$ for a curve $C$ in $\mathbb{P}^3$, then $C$ is the unique minimal curve in its even linkage class. I will show that if $X \in \mathcal{L}$ and $e + n + 1 + h_X < s_0(X)$, then $\mathcal{L}$ has a unique minimal element $X_0$, and that $X_0 \subset X$. First I must recall a few results on moduli for subschemes in fixed even linkage classes.

**Proposition 5.3.** Let $X \in \mathcal{L}$ be a subscheme. Then the subset

$$H_X = \{ X' : X \text{ deforms to } X' \text{ through schemes in } \mathcal{L} \text{ with constant cohomology } \}$$
of the Hilbert scheme is irreducible.

Proof. Fixing $X$, we can find $N$ such that $I_X$ is $N$-regular. By the theorem of Castelnuovo-Mumford [17,p.99], the total ideal of $X$ is generated by its homogeneous parts of degree $\leq N$. Letting $Q = \oplus_{l\leq N} \mathcal{O}(-l)^{h^0(I_X(l))}$, we can find an $E$-type resolution for $X$ of the form

$$0 \to E \to Q \to I_X \to 0$$

and each $X' \in H_X$ has an $E$-type resolution of the same form. Let $V$ be the parameter space of all morphisms $\{E \xrightarrow{\varphi} Q\}$. $V$ is a smooth projective variety and comes equipped with a universal morphism $p^* E \xrightarrow{\varphi} p^* Q$, where $p : V \times \mathbb{P}^n \to \mathbb{P}^n$ is the second projection.

Let $W \subset \mathbb{P}^n$ denote the subset where $E$ is locally free. By [6, corollary 1.4] the complement of $W$ is of codimension $\geq 3$. On the open set $V \times W \subset V \times \mathbb{P}^n$ we have the complex

$$0 \to p^* E \xrightarrow{\varphi} p^* Q \xrightarrow{\Lambda^r \varphi \otimes 1} p^* \mathcal{O} \to 0.$$ 

By [17, corollary 4.6], the set $U$ consisting of $v \in V$ such that $0 \to p^* E \otimes k(v) \to p^* Q \otimes k(v) \to p^* \mathcal{O}$ is exact is open. Let $j : U \times W \subset U \times \mathbb{P}^n$ denote the inclusion map. Applying $j_*$ extends the complex to $U \times \mathbb{P}^n$. Let $\mathcal{X}$ be the subscheme defined by the cokernel of the last map in the complex. For each $u \in U$, we have a sequence

$$0 \to \mathcal{E} \xrightarrow{\varphi} Q \to \mathcal{O} \to \mathcal{O}_{X_u} \to 0,$$

which is exact on $W$. Since $\mathcal{E}$, $Q$, and $\mathcal{O}$ are reflexive, and the codimension of $W$ is $\geq 3$, this sequence is exact on $\mathbb{P}^n$. In particular, the Hilbert polynomial of the fibres is constant, and $\mathcal{X}$ is a flat family over $U$. Hence there is an induced map $U \to \mathcal{H}_X$, where $\mathcal{H}_X$ is the Hilbert scheme for subschemes in $\mathbb{P}^n$ with the same Hilbert polynomial as $X$. We take $H_X$ to be the image with induced reduced structure.

**Proposition 5.4.** If $U \subset H_X$ is an open set such that $X_0 \subset X'$ for each $X' \in U$, then $X_0 \subset X'$ for all $X' \in H_X$.

**Proof.** Since $H_X$ is irreducible by proposition 5.3, it suffices to show that the set of $\{X' \in H_X : X_0 \subset X'\}$ is closed. For this, we will show that the corresponding set is closed in $H_X$, the Hilbert scheme for all subschemes of $\mathbb{P}^n$ having the same Hilbert polynomial as $X$. If $\mathcal{F}$ is the flag scheme for all inclusions $X_0 \subset X$, then we have natural projections

$$\mathcal{F} \xrightarrow{p} \mathcal{H}_{X_0}$$

$$\downarrow q \quad \downarrow$$

$$\mathcal{H}_X$$

The inverse image of $\{X_0\}$ under $p$ is closed in $\mathcal{F}$, and the image of this closed set under $q$ is closed in $\mathcal{H}_X$, because $q$ is projective. It follows that the set of interest is closed in $\mathcal{H}_X$, and hence so is its intersection with $H_X$. 
Lemma 5.5. Let $X \subset \mathbb{P}^n \times H \overset{\pi}{\to} H$ be a family of subschemes with constant cohomology from an even linkage class $L$ of codimension two subschemes in $\mathbb{P}^n$. Then the function $t_1(X_h)$ is upper semicontinuous on $H$.

Proof. Since it suffices to show this on each irreducible component of $H$, we can base extend by the irreducible components with reduced induced structure to reduce to the case where $H$ is integral. To show that $t_1(X_h)$ is upper semicontinuous, we must show that for each $h \in H$, there is an open neighborhood $h \in U \subset H$ such that $t_1(X_k) \leq t_1(X_h)$ for each $k \in U$.

Fix $h \in H$, and let $t = t_1(X_h)$. Since the dimension $h^0(I_{X_h}(t))$ is constant for $k \in H$, we see by Grauert’s theorem [6, III, corollary 12.9] that the sheaf $\mathcal{F} = \pi_* (\mathcal{I}_X \otimes q^*(\mathcal{O}(t)))$ is locally free on $H$. Letting $\mathbb{P}\mathcal{F} \overset{f}{\to} H$ be the corresponding vector bundle over $H$, and base extending the universal family by $f$, we obtain a flag scheme for inclusions $X_k \subset T$, where $k \in H$ and $T$ is a hypersurface of degree $t$. We can similarly find a bundle $\mathbb{P}\mathcal{G} \overset{g}{\to} H$ which parametrizes inclusions $X_k \subset S$ with $k \in H$ and $S$ a hypersurface of degree $s$.

By definition of $t_1(X_h)$, we can find hypersurfaces $S, T$ containing $X$ of degrees $s, t$ such that $S \cap T$ has dimension $n - 2$. Since $\mathbb{P}\mathcal{F} \overset{f}{\to} H$ is a vector bundle, we can find an open set $U_t \subset H$ and a local section $\sigma_t : U_t \to f^{-1}U_t$ such that $\sigma_t(h)$ corresponds to the inclusion $X_k \subset T$. In this way we obtain a flat family

$$\tilde{X}_t \subset \tilde{T} \longrightarrow \mathbb{P}^n \times U_t$$

We can obtain a similar family $\tilde{X}_s \subset \tilde{S} \subset \mathbb{P}^n \times U_s \to U_s$ whose fibre at $h$ corresponds to the inclusion $X_h \subset S$.

Letting $U = U_s \cap U_t$ be the intersection of the open sets, we obtain families $\tilde{X} \subset \tilde{T}$ and $\tilde{X} \subset \tilde{S}$ over $U$. The map

$$\tilde{S} \cap \tilde{T} \to U$$

is surjective and the fibre over $h$ has dimension $n - 2$. Now we can use the semicontinuity of the dimension of the fibres of this morphism [6, II, exercise 3.22] to find an open set $h \in V \subset U$ over which the fibres have dimension $\leq n - 2$. Clearly the intersections $\tilde{S}_h \cap \tilde{T}_h$ have dimension $\geq n - 2$, so for $k \in V$ this intersection has dimension exactly $n - 2$, which shows that $t_1(X_k) \leq t$.

Proposition 5.6. Let $\mathcal{L}$ be an even linkage class such that $0 < \delta = s_0 - e - n - 1$. Then $\mathcal{L}$ has a unique minimal element $X_0$. Further, if $Y \in \mathcal{L}'$ satisfies $(\eta_Y)_0 < t_1 + h_Y + \delta$ and $Y$ links to $X$ by hypersurfaces of degrees $< t_1 + h_Y + \delta$, then $X_0 \subset X$.

Proof. For the first statement, we use the same proof as in [9]. One can use a minimal $\mathcal{E}$-type resolution for a dual curve $Y_0$. Upon linking to $X_0$, applying proposition 1.6 to get a resolution for $\mathcal{I}_{X_0}$ and cancelling the two summands corresponding to the hypersurfaces used for the linkage, we obtain an $\mathcal{N}$-type resolution

$$0 \to \mathcal{P} \to \mathcal{N} \to \mathcal{I}_{X_0} \to 0$$
Proposition 5.7. Let \( Y = Y_0 \) be minimal hypersurface of type \((A, 1)\) in \( Y \) by hypersurfaces \( S, T \) of degrees \( s_0, t_1 \) by theorem 1.21. It follows from [7] that there is an exact sequence

\[
0 \rightarrow \mathcal{I}_{S \cup T} \rightarrow \mathcal{I}_{Y_0} \rightarrow \omega_{X_0}(n + 1 - s_0 - t_1) \rightarrow 0.
\]

Twisting by any \( l < t_1 + \delta \) and taking global sections gives an isomorphism

\[
H^0(\mathcal{I}_{S \cup T}(l)) \cong H^0(\mathcal{I}_{Y_0}(l)),
\]

which shows that \( H^0(\mathcal{I}_{Y_0}(l)) \) is generated by \( S \) and \( T \). It follows that if \( F_0 \cap F_0 \cap Y_0 \) is a complete intersection with \( a, b < t_1 + \delta \), then \( F_0 \cap F_0 \cap Y_0 \) is a minimal curve for \( \eta \). Hence we can link \( Y \) to \( X_1 \) by hypersurfaces of degrees \( A, B \) consisting of subschemes \( Y' \) which link by hypersurfaces of degrees \( A, B \) to subschemes \( Y' \in H_{Y_1} \), hence we see that the general \( X \in H_{X_1} \) contains \( X_0 \). By proposition 5.5, we have that all \( X' \in H_{X_1} \) must contain \( X_0 \).

Now we return to \( Y \). Suppose that \( Y \) links to some \( X \) by hypersurfaces of degrees \( a, b \), which are both less than \( t_1 + h_Y + \delta \). Then \( t_1(Y) < t_1 + h_Y + \delta \) by corollary 4.3. This shows that \( Y \) links to subschemes \( X' \) by hypersurfaces of degrees \( A, B + 1 \). Since \( Y \) was in the deformation class of an elementary double link from \( Y_1 \) of type \((A, 1)\), we see that \( X' \in H_{X_1} \), hence \( X_0 \subset X' \). It follows that the general hypersurface \( T \) of degree \( B + 1 \) containing \( Y \) also contains \( Y \cup X_0 \), and hence this is true for all such \( T \) as this condition is closed. In particular, \( X \) must contain \( X_0 \), for otherwise we could find a hypersurface \( F_a \) of degree \( a \) which does not contain \( Y \cup X_0 \), and the union of \( F_a \) with a general hypersurface of degree \( t_1 + h_Y + \delta - 1 - a \) would also not contain \( Y \cup X_0 \), a contradiction.

Proposition 5.7. Let \( X \in \mathcal{L} \). If \( s_0(X) > e + n + 1 + h_X \), then

1. \( \mathcal{L} \) has a unique minimal element \( X_0 \)
2. \( X_0 \subset X \)

Proof. Since \( s_0(X) = s_0 + \sum \theta_X(l) \leq s_0 + h_X \), we have that \( s_0 > e + n + 1 \), so \( \mathcal{L} \) has a unique minimal element \( X_0 \) by proposition 5.7. Set \( \delta = s_0 - e - n - 1 \).

Letting \( s = s_0(X) \) and choosing \( t > \max\{t_1(X), t_1 + h_X\} \), we can link \( X \) to \( Y \) by hypersurfaces of degrees \( s \) and \( t \). The hypothesis on \( t \) and the formula of corollary 3.20 show that \( \eta_Y(l) = 0 \) for \( l \geq t \), so \( (\eta_Y)_o < t \). Further, the hypothesis \( s > e + n + 1 + h_X \) combined with lemma 3.18 show that \( t < t_1 + h_Y + \delta \), so we can apply proposition 5.7 to \( Y \) with \( a = s \) and \( b = t \) to conclude that \( X_0 \subset X \).

Theorem 5.8. Let \( X \in \mathcal{L} \) be integral and assume that \( X \) is not minimal. Then

1. \( \theta_X \) is connected about \([s_0 + h_X, t_1 + h_X - 1]\)
2. \( s_0(X) \leq e + n + 1 + h_X \).
Proof. The first condition is a consequence of theorem 4.7 and proposition 5.2. If \( s_0(X) > e + n + 1 + h_X \), then proposition 5.7 shows that \( \mathcal{L} \) contains a minimal element \( X_0 \), which is contained in \( X \). This contradicts the assumption that \( X \) is integral, unless \( X \) is minimal.

Proposition 5.9. Let \( T \subset \mathbb{P}^n \) be an integral hypersurface, \( Y \subset T \) a generically Cartier generalized divisor. Let \( m \) be an integer such that the linear system \( \mathbb{P}H^0(\mathcal{I}_{Y,T}(m)) \) cuts out a scheme \( Y' \supset Y \) which differs from \( Y \) on a set of codimension \( > 1 \) in \( T \) and that \( H^0(\mathcal{I}_{Y,T}(m-1)) \neq 0 \). Then the general complete intersection of \( T \) with a hypersurface \( Z(f) \supset Y \) of degree \( m \) links \( Y \) geometrically to an integral subscheme \( X \).

Proof. The linear system \( V = H^0(\mathcal{I}_{Y,T}(m)) \) has \( Y' \) as base locus, hence defines a morphism \( T - Y' \rightarrow \mathbb{P}V \). By hypothesis there exists \( 0 \neq s \in H^0(\mathcal{I}_{Y,T}(m-1)) \). Let \( S \subset \mathbb{P}^n \) be a hypersurface whose restriction to \( T \) is the scheme of zeros of \( s \) and let \( W \subset V \) denote the sublinear system of multiples of \( s \). \( W \) gives an embedding of \( T - S \) into \( \mathbb{P}W \), which can be factored as \( T - S \subset \mathbb{P}^n - S \rightarrow \mathbb{P}W \). The first map is a closed immersion and the second is an open immersion, hence the composite map is unramified. This composite map can also be factored \( T - S \rightarrow T - Y' \rightarrow \mathbb{P}V \rightarrow \mathbb{P}W \) where \( \pi \) is a projection from a linear subspace. It follows that \( \sigma \) is unramified when restricted to \( T - S \) and that the dimension of the image of \( \sigma \) is at least two.

We are now in position to apply Jouanolou’s Bertini theorem [8, theorem 6.10]. Let \( H_f \) be a general hyperplane in \( \mathbb{P}V \) which corresponds to \( f \in H^0(\mathcal{I}_{Y,T}(m)) \). By Jouanolou’s Bertini theorem, \( \sigma^{-1}(H_f) \) is geometrically irreducible and \( \sigma^{-1}(H_f) - S \) is reduced. Also, since \( Y' \) is generically Cartier on \( T \) and \( \mathcal{I}_{Y',T}(m) \) is generated by its global sections, the general \( f \in H^0(\mathcal{I}_{Y,T}(m)) \) generates \( \mathcal{I}_{Y',T} \) at its generic points of codimension one in \( T \) and meets \( S - Y' \) in codimension \( > 1 \). Putting these facts together, we find that for general \( f \), \( Z(f) \cap T = Y \cup X \) where \( X \) has no common component with \( S \), is reduced and geometrically irreducible when restricted to \( T - S \). In other words, \( X \) links to \( Y \) geometrically and is integral.

Proposition 5.10. Let \( X \in \mathcal{L} \) be an integral and \( h > 0 \) an integer. Let \( t \leq e(X) + n + 1 + h \) be an integer such that \( t = s_0(C) \) or \( t \geq t_1(C) \). Then the general elementary double link \( X' \) of type \((t,h)\) is integral.

Proof. Let \( t \) be as in the theorem, and let \( T \) be a general surface of degree \( t \) which contains \( X \). Then \( T \) is integral and \( X \) is generically Cartier on \( T \) by proposition 5.1. By proposition 5.10, a general hypersurface \( H \) of large degree \( d \) links \( X \) geometrically to an integral subscheme \( Y \). We have an isomorphism \( \mathcal{I}_Y/\mathcal{I}_{\mathcal{S} \cap T} \cong \omega_X(4-d-t) \). Restricting the ideal sheaves to the surface \( T \) yields the exact sequence

\[ 0 \rightarrow \mathcal{I}_{\mathcal{S} \cap T,T} \rightarrow \mathcal{I}_{Y,T} \rightarrow \omega_X(4-d-t) \rightarrow 0 \]

Note that \( \mathcal{I}_{\mathcal{S} \cap T,T} \cong \mathcal{O}_T(-d) \) and that the map on the left is just multiplication by the equation for \( S \). If we twist this sequence by \( d + h \) we get an exact sequence on global sections

\[ 0 \rightarrow H^0(\mathcal{O}_T(h)) \rightarrow H^0(\mathcal{I}_{Y,T}(d + h)) \rightarrow H^0(\omega_X(n + 1 - t + h)) \rightarrow 0. \]

Since \( t \leq e(X) + n + 1 + h \), the last cohomology group is nonzero, hence not every element of \( V = H^0(\mathcal{I}_{Y,T}(d + h)) \) is a multiple of the equation for \( S \), although
all multiples of this equation by linear forms is contained in $V$. It follows that the map $V \otimes O_T(-d - h) \to O_T$ defines the ideal sheaf of a curve $Y'$ such that $Y \subset Y' \subset Y \cup X$. Further, the second inclusion is proper because not every element of $V$ was a multiple of the equation of $S$. Since $X$ is integral, the proper subscheme $Y' \cap X$ has codimension $> 2$.

Since we also have $H^0(\mathcal{I}_{D,T}(d)) \neq 0$, the hypotheses of proposition 5.10 hold and we find that $Y$ links geometrically via $T$ and a hypersurface $S'$ of degree $d + h$ to a subscheme $X'$ which is integral. This completes the proof.

**Theorem 5.11.** Let $X, Y \in \mathcal{L}$ such that $X$ is integral, $X \leq Y$, and $Y$ satisfies the conclusion of theorem 5.8. Then $Y$ can be deformed with constant cohomology through schemes in $\mathcal{L}$ to $\overline{Y}$ with $\overline{Y}$ integral.

**Proof.** We induct on the relative height $h = h_Y - h_X$. Let $\eta = \eta_{X,Y}, \theta = \theta_{X,Y}$. Define $r = \min\{ l : \eta(l) \neq 0 \text{ and } \eta(l + 1) = 0 \}$ and let $A = r - h_Y + h_X + 1, w = r - \eta_0 + 1$. If $X'$ is obtained from $X$ by an elementary double link of type $(A, w)$, then applying propositions 2.7 and 3.6 shows that $X' \leq Y$. Since $h_Y - h_X = h - w < h$, we can apply the induction hypothesis once we show that $X'$ can be taken integral. For this we will check the conditions of proposition 5.11.

First we check that $A = s_0(X)$ or $A \geq s_1(X)$. If $A < s_0(X)$, then because $s_0(X) \leq (\eta_Y)_a$ we find that $r < (\eta_X)_a + h_Y - h_X - 1$. Proposition 2.7 gives us the formula $\eta_X(l) = \eta_X(l + h_Y - h_X) + \eta(l)$, so the inequality would imply that $\eta_Y$ is not connected, a contradiction, so we must have $A \geq s_0(X)$.

Now assume that $A > s_0(X)$. Then $r > s_0(X) + h_Y - h_X - 1$, which implies that $\theta(r) \neq 0$. Since $\eta(r + 1) = 0$, we also have $\theta(r + 1) = 0$. Now consider the formula of proposition 2.10.

$$\theta_Y(l) = \theta_X(l + h_Y - h_X) + \theta(l)$$

If $\theta_X \neq 0$, then we must have $(\theta_X)_a + h_Y - h_X \leq r + 1$ for $\theta_Y$ to be connected. In this case, theorem 4.7 shows that $s_1(X) \leq r - h_Y + h_X + 1$, which is equivalent to $A \geq s_1(X)$. If $\theta_X = 0$, then since $\theta_Y$ must be connected about $[s_0 + h_Y, t_1 + h_Y - 1]$, we find that $r \geq t_1 + h_Y - 1 \geq s_1 + h_Y - 1$, which shows that $A \geq s_1 + h_X$, which is $\geq s_1(X)$ by proposition 2.11. We conclude that $A = s_0(X)$ or $A \geq s_1(X)$.

Secondly we check that $A \leq e(X) + n + 1 + w$. In view of the formula $e(X) = \max\{ e + h_X, (\eta_X)_a - n \}$, we find that this is equivalent to showing that $\eta_a \leq \max\{ e + n + 1, (\eta_X)_a + 1 - h_X \} + h_Y$. If $\eta_X = 0$, then the fact that $\eta_Y$ is connected about $[e + n + 1 + h_Y, t_1 + h_Y - 1]$ gives this fact. If $\eta_X \neq 0$, the condition on $\eta_X$ from theorem 5.9 gives that $(\eta_X)_a \geq t_1 + h_X - 1$. Since $t_1 \geq e + n + 1$, we need to check that $\eta_a \leq (\eta_X)_a + h_Y - h_X + 1$. If this is not the case, then the formula which relates $\eta, \eta_X$ and $\eta_Y$ shows that $\eta_Y$ is not connected, contradicting the conditions of theorem 5.9. Hence $A \leq e(X) + n + 1 + w$ and our proof is complete.

**Example 5.12.** For any even linkage class which has an integral minimal element, the conditions of theorem 5.8 are both necessary and sufficient for the existence of an integral subscheme. For example, Lazarsfeld and Rao show that if $C$ is a smooth curve, then the general embedding of $C$ in $\mathbb{P}^3$ of large degree will have image which is minimal in its even linkage class.

**Example 5.13.** Let $\mathcal{L}$ be the even linkage class of 4 skew lines which lie on a quadric surface $Q$ in $\mathbb{P}^3$. One can calculate that $s_0 = 2, s_1 = t_1 = 4$, and $e = -1$. 24 SCOTT NOLLET
In this case, it turns out that there are two minimal integral curves. One can be obtained from the 4 lines by taking a general elementary double link of type $(2, 1)$. These curves are of type $(1, 5)$ on the smooth quadric, and hence have smooth connected representatives. The other can be obtained by a general elementary double link of type $(4, 2)$. This can be produced by first linking the 4 skew lines to another set of 4 skew lines via $Q$ and a quartic, and then the second set of 4 skew lines can be geometrically linked to an integral curve by two quartic surfaces via proposition 5.9. In this case, the conditions of theorem 5.8 are both necessary and sufficient.

Example 5.14. Using the construction of Hartshorne and Hirschowitz, one can show that the general union of 10 lines meeting at 9 points generizes to a smooth rational curve of degree 10. In this case, both the rational curve and the union of the 10 lines have seminatural cohomology, and we can compute that $e = -1$ and $s_0 = 5$, so that both of these curves are unique minimal elements in their even linkage classes by proposition 5.6. On the other hand, one even linkage class has a minimal integral curve and the other doesn’t. We conclude that it is not possible to get purely cohomological criteria for the existence of integral subschemes in an even linkage class. In the case of the rational curve, there is just one class with a minimal integral curve of height $h > 0$. It can be obtained from the rational curve by an elementary link of type $(5, 2)$.

Example 5.15. Consider the even linkage class of a double line $C_0$ with arithmetic genus $g = -4$. In [17, theorem 8.2.7], it is shown that the even linkage class of $C_0$ has two minimal integral curves, which can be obtained from $C_0$ as follows. The first can be obtained by first linking $C_0$ to $D_1$ by quadric surfaces, and then linking $D_0$ to $C_1$ by a cubic and quintic surface. In general, the curve $C_2$ produced in this way will be smooth and connected. The second minimal integral curve can be obtained by first linking $C_0$ to $D_2$ by a quadric and quartic surface, and then linking $D_2$ to $C_2$ by a cubic and a sextic surface. $C_2$ can again be taken to be smooth and connected. It is also shown that not every deformation class containing an integral curve contains a smooth connected curve, so we cannot hope to get results like theorem 5.8 and theorem 5.11 for smooth connected curves without adding some extra hypotheses.

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