DISTRIBUTION OF DETERMINANT OF SUM OF MATRICES

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Abstract. Let \( \mathbb{F}_q \) be an arbitrary finite field of order \( q \). In this article, we study \( \det S \) for certain types of subsets \( S \) in the ring \( M_2(\mathbb{F}_q) \) of \( 2 \times 2 \) matrices with entries in \( \mathbb{F}_q \). For \( i \in \mathbb{F}_q \), let \( D_i := \{ x \in M_2(\mathbb{F}_q) : \det(x) = i \} \). Then our results can be stated as follows. First of all, we show that when \( E \) and \( F \) are subsets of \( D_i \) and \( D_j \) for some \( i, j \in \mathbb{F}_q^* \), respectively, we have

\[
\det(E + F) = \mathbb{F}_q,
\]

whenever \( |E||F| \geq 15^2 q^4 \), and then provide a concrete construction to show that our result is sharp. Next, as an application of the first result, we investigate a distribution of the determinants generated by the sum set \( (E \cap D_i) + (F \cap D_j) \), when \( E, F \) are subsets of the product type, i.e., \( U_1 \times U_2 \subseteq \mathbb{F}_q^2 \times \mathbb{F}_q^2 \) under the identification \( M_2(\mathbb{F}_q) = \mathbb{F}_q^2 \times \mathbb{F}_q^2 \). Lastly, as an extended version of the first result, we prove that if \( E \) is a set in \( D_i \) for \( i \neq 0 \) and \( k \) is large enough, then we have

\[
\det(2^k E) := \det\left(\sum_{\text{2k terms}} E\right) \supseteq \mathbb{F}_q^*,
\]

whenever the size of \( E \) is close to \( q^\frac{d}{2} \). Moreover, we show that, in general, the threshold \( q^\frac{d}{2} \) is best possible. Our main method is based on the discrete Fourier analysis.

1. Introduction

Let \( E \) be a finite subset of \( \mathbb{R}^d \), \( d \geq 2 \). The Erdős distinct distances problem is to find the best possible lower bound of the distance set \( \Delta(E) \) in terms of \( |E| \), where \( \Delta(E) \) is defined as

\[
\Delta(E) := \{|x - y| : x, y \in E\}.
\]

In dimension two, Erdős [14] conjectured that \( |\Delta(E)| \gg |E|/\sqrt{\log |E|} \). This was solved up to logarithmic factor by Guth and Katz [11]. Indeed, they proved that \( |\Delta(E)| \gg |E|/\log |E| \). In higher dimensions, it was also conjectured by Erdős [14] that \( |\Delta(E)| \gg |E|^{2/d} \), which has long stayed unsettled. We refer readers to [37,38] for recent developments and partial results on the Erdős distinct distances problem in three and higher dimensions. As a continuous analog of the Erdős distinct distances conjecture, Falconer [9] conjectured that any subset \( E \) of \( \mathbb{R}^d \) of the Hausdorff dimension greater than \( d/2 \) determines a distance set of a positive Lebesgue measure. This conjecture is still open in all dimensions, and, recently, much progress on this problem has been made (see, for example, [33,1,41,12,13,7,10]).

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In the finite field setting, the distance problems turn out to have features of both the Erdős and Falconer distance problems. Bourgain-Tao-Katz [2] studied the finite field Erdős distance problem for the first time. Let $\mathbb{F}_q^d, d \geq 2$, be the $d$-dimensional vector space over a finite field $\mathbb{F}_q$ with $q$ elements. Throughout this paper, we assume that $q$ is an odd prime power. Given two subsets $E, F$ of $\mathbb{F}_q^d$, the distance set, denoted by $\Delta(E, F)$, is defined as

$$\Delta(E, F) := \{\|x - y\| : x \in E, y \in F\},$$

where $\|\alpha\| = \alpha_1^2 + \cdots + \alpha_d^2$ for $\alpha = (\alpha_1, \ldots, \alpha_d)$. The first non-trivial result was obtained by Bourgain-Tao-Katz [2] using arithmetic-combinatorial methods and the connection of the geometric incidence problem of counting distances with sum-product estimates. They showed that if $q \equiv 3 \mod 4$ is a prime and $E$ is a subset of $\mathbb{F}_q^2$ with $|E| = q^{2-\delta}$ for some $0 < \delta < 2$, then there exists a positive number $\epsilon = \epsilon(\delta)$ such that

$$|\Delta(E, E)| \geq |E|^\frac{1}{4} + \epsilon.$$

In their proof of this result, it was not trivial to find an explicit relationship between $\delta$ and $\epsilon$. Furthermore, as pointed out in [25], their result could not be extended over an arbitrary finite field. Indeed, if $q = p^2$ for a prime $p$, then taking $E = \mathbb{F}_p \times \mathbb{F}_p$, we have $|\Delta(E, E)| = p = q^{1/2}$ and $|E| = p^2 = q$. Moreover, if $q \equiv 1 \mod 4$, then there exists $i \in \mathbb{F}_q$ with $i^2 = -1$ so that the set $E = \{(t, it) : t \in \mathbb{F}_q\}$ satisfies that $|E| = q$ and $\Delta(E, E) = \{0\}$.

Over an arbitrary finite field, not necessarily a prime field, it was Iosevich and Rudnev [25] who obtained an explicit lower bound on the size of $\Delta(E, E)$ in terms of the size of $E$. More precisely, they proved that if $E \subseteq \mathbb{F}_q^d$ such that $|E| \geq Cq^{d/2}$ for a sufficiently large constant $C$, then

$$(1.1) \quad |\Delta(E, E)| \gg \min \left\{ q, \frac{|E|}{q^{(d-1)/2}} \right\}. $$

Here, and throughout this paper, $X \gg Y$ means that there is a constant $C$ independent of $q$ such that $CX \geq Y$ and we also write $Y \ll X$ for $X \gg Y$. In addition, $X \sim Y$ is used to indicate that $X \gg Y$ and $Y \gg X$. Shparlinski [36] extended the result (1.1) to the case when $E, F$ are arbitrary subsets of $\mathbb{F}_q^d$,

$$|\Delta(E, F)| > \frac{1}{2} \min \left\{ q, \frac{|E||F|}{q^d} \right\}. $$

Similar results were obtained for generalized distances defined by certain polynomials (see, for example, [23, 30, 40]). In specific ranges of sizes of sets $E, F$ in $\mathbb{F}_q^d$, slightly better lower bounds were given in [6, 28].

Notice that the above Shparlinski’s result implies that if $E, F \subseteq \mathbb{F}_q^d$ with $|E||F| \geq q^{d+1}$, then the distance set $\Delta(E, F)$ contains a positive proportion of all possible distances. This can be considered as a result on a finite field version of the Falconer distance problem.

In view of these examples, Iosevich and Rudnev posed the following problems.

**Problem 1.1** (The Erdős-Falconer distance problem). Let $E, F$ be subsets of $\mathbb{F}_q^d$. How much large sets $E, F$ do we need to assure that the distance set $\Delta(E, F)$ contains a positive proportion of all distances?

Iosevich and Rudnev [25] also raised the following question which calls for much stronger conclusion than in the Erdős-Falconer distance problem.
Problem 1.2 (The Strong Erdős-Falconer distance problem). Let $E, F$ be subsets of $\mathbb{F}_q^d$. What is the smallest exponent $\alpha$ such that if $|E||F| \geq Cq^\alpha$, then the distance set $\Delta(E, F)$ contains all distances?

When $E = F$, Iosevich and Rudnev [25] proved that if $E \subseteq \mathbb{F}_q^d, d \geq 2$, and $|E| \geq 4q^{(d+1)/2}$, then $\Delta(E, E) = \mathbb{F}_q$. The authors in [21] constructed an example to show that the exponent $(d + 1)/2$ in odd dimensions can not be improved without further restrictions. In even dimensions, it is conjectured that any subset $E$ of $\mathbb{F}_q^d$ with $|E| \geq Cq^{d/2}$ determines all distances. This conjecture is open in all even dimensions and the exponent $(d + 1)/2$, due to Iosevich and Rudnev, has not been improved in all even dimensions. There have been recently produced much related results for which we refer to [3, 27].

On the other hand, after Iosevich and Rudnev’s work, the Erdős-Falconer type distance problem has been studied for other geometric objects (see, for instance, [19, 39, 32]). Among other things, a similar question has been addressed in the setting of matrix rings. For an integer $n \geq 2$, let $M_n(\mathbb{F}_q)$ be the set of $n \times n$ matrices with entries in $\mathbb{F}_q$ and $SL_n(\mathbb{F}_q)$ be the special linear group in $M_n(\mathbb{F}_q)$. Ferguson, Hoffman, Luca, Ostafe, and Shparlinski [15] studied the following problem.

Problem 1.3. Let $E$ and $F$ be sets in $M_3(\mathbb{F}_q)$. How large do $E$ and $F$ need to be to guarantee that there exists $(x, y) \in E \times F$ such that $\det(x + y) = 1$?

Ferguson et al. [15] developed a version of the Kloosterman sum over matrix rings to prove that if $|E||F| \geq 2q^{n^2-2}$, then there exist elements $x \in E$ and $y \in F$ such that $\det(x + y) = 1$. In the paper [31], Li and Hu gave an explicit expression of Gauss sum for the special linear group $SL_n(\mathbb{F}_q)$, and as a consequence, they obtained an improvement of Ferguson et al.’s result. More precisely, they showed that if $n = 2$, then the condition $|E||F| \geq Cq^5$ is enough, but in higher dimensional cases, we need $|E||F| \geq Cq^{2n^2-2n}$. Note that a graph theoretic proof of the result for the case $n = 2$ was given recently by Demiroğlu Karabulut [5]. More precisely, she proved that if $|E||F| > 4q^7/(q - 1)^2$, then for every $t \in \mathbb{F}_q^*$ there exists $(x, y) \in E \times F$ such that $\det(x - y) = t$. In Appendix, based on the discrete Fourier analysis, we will give an alternative proof for a similar result of Karabulut but for more accurate size conditions on sets: if $E, F \subseteq M_2(\mathbb{F}_q)$ with $|E||F| > 4q^5$, then we have $\det(E + F) \supseteq \mathbb{F}_q^*$. We refer readers to [4, 16, 17, 18, 26, 34, 35] for recent results in the setting of matrix rings.

1.1. Statement of main results. In this paper, we study Problem 1.3 for $n = 2$ through a discrete Fourier analysis based on an Odot-product. For $i \in \mathbb{F}_q$, recall that $D_i$ is a subset of $M_2(\mathbb{F}_q)$ defined as

$$D_i = \{x \in M_2(\mathbb{F}_q) : \det(x) = i\}.$$  

For $S \subseteq M_2(\mathbb{F}_q)$, let $\det(S)$ denote the set of determinants generated by $S$, i.e.,

$$\det(S) := \{\det(x) \in \mathbb{F}_q : x \in S\}.$$  

The first result of ours is concerned with the sum set $S = E + F$ with a restriction $E \subseteq D_i$ and $F \subseteq D_j$ for $i, j \in \mathbb{F}_q^*$. Namely, we produce an optimal result on Problem 1.3 for the sum set $E + F$.

Theorem 1.4. For $i, j \in \mathbb{F}_q^*$, let $E \subseteq D_i$ and $F \subseteq D_j$. If $|E||F| \geq 15^2q^4$, then we have $\det(E + F) = \mathbb{F}_q$. 

Note that this result should be compared with results of Ferguson et al., Li and Hu, and Karabulutin in the paragraph subsequent to Problem 1.3. In our result, we impose a stronger condition on $E, F$, i.e., $E \subseteq D_i$ and $E \subseteq D_j$, than they did in $[15,31,5]$, while our threshold $q^4$ is much better than those in their results (for $n = 2$).

One can easily construct an example to show that the threshold $q^4$ can not be lower for arbitrary subsets $E, F$ of $M_2(\mathbb{F}_q)$. For instance, let $q = p^2$ for some odd prime $p$ and take

$$E = F = \left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in M_2(\mathbb{F}_q) : x_1, x_2, x_3, x_4 \in \mathbb{F}_p \right\}.$$  

Then $|E| = |F| = q^2$ and $\det(E + F) = \mathbb{F}_p$. This example proposes a conjecture that for any subsets $E, F$ of $M_2(\mathbb{F}_q)$ with $|E||F| \geq Cq^4$ for a large constant $C > 1$, we have $\det(E + F) = \mathbb{F}_q$. Notice that Theorem 1.4 confirms this conjecture (up to a constant) in the specific case when $E \subseteq D_i$ and $F \subseteq D_j$ for $i, j \neq 0$. Then there arises a natural question whether it is possible to improve the threshold $q^4$ in the specific cases. In this paper we show that the threshold $q^4$ can not go lower in general, so Theorem 1.4 is sharp. Indeed, for any non-square number $i$ of $\mathbb{F}_q$, we will construct a set $E \subseteq D_i$ such that $|E| \sim q^2$, but $\det(E + E) \neq \mathbb{F}_q$.

Notice that we have obtained the very explicit constant $15^2$ for the bound in Theorem 1.4. Such an explicit constant is not available in the literature in general, and is one of features this paper owns. It would be interesting to search for a smaller constant than this.

Taking $E = F$, the following corollary follows immediately from Theorem 1.4.

**Corollary 1.5.** Let $i$ be an element of $\mathbb{F}_q^*$ and $E$ be a set in $D_i$. If $|E| \geq 15q^2$, then we have

$$\det(E + E) = \mathbb{F}_q.$$ 

As a motivation for the second result, let us first consider the following simple question to answer. Given two varieties $D_i, D_j$ in $M_2(\mathbb{F}_q)$ for non-zero $i, j \in \mathbb{F}_q$, determine the smallest exponent $\beta$ such that for any sets $E, F \subseteq M_2(\mathbb{F}_q)$ with $|E||F| \geq Cq^\beta$, we have

$$\det((E \cap D_i) + (F \cap D_j)) = \mathbb{F}_q.$$ 

As it stands, the answer for the smallest exponent $\beta$ is 8. To see this, take $E = M_2(\mathbb{F}_q) \setminus D_i$ and $F = M_2(\mathbb{F}_q) \setminus D_j$. Then $(E \cap D_i) + (F \cap D_j)$ is an empty set, and $|E||F| \sim q^8$ (equivalently, $|E||F| \sim q^8$). This example proposes that the smallest exponent $\beta$ can not be less than 8. On the other hand, if we take $E = F = M_2(\mathbb{F}_q)$, then $|E||F| = q^8$ and $\det((E \cap D_i) + (F \cap D_j)) = \mathbb{F}_q$.

However, in our second result we prove that if we work with subsets $E, F$ with some restriction, we obtain a non-trivial result. To explain this, we fix the identification $M_2(\mathbb{F}_q) = \mathbb{F}_q^2 \times \mathbb{F}_q^2$ through the assignment

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \mapsto (x_1, x_2, x_3, x_4).$$

Write $\begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$ for the subset of $M_2(\mathbb{F}_q)$ corresponding to $S_1 \times S_2 \subseteq \mathbb{F}_q^2 \times \mathbb{F}_q^2$. We will say that a subset $S \subseteq M_2(\mathbb{F}_q)$ is of product type if it is written as $S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$ for some $S_1, S_2 \subseteq \mathbb{F}_q^2$. Then as an application of Theorem 1.4 we obtain the following.
Theorem 1.6. Let $E, F \subseteq M_2(\mathbb{F}_q)$ be of product type. If $|E|, |F| \geq Cq^3$ for a sufficiently large constant $C$, then for any $i, j \in \mathbb{F}_q^*$, we have

$$\det((E \cap D_i) + (F \cap D_j)) = \mathbb{F}_q.$$ 

A few words on Theorem 1.6 are in order. First, note that the theorem implies that the subset $E \cap D_i$ is nonempty for any $i \neq 0$. In fact, we will see from Lemma 5.1 that we have $|E \cap D_i| \sim \frac{|E|}{q}$, which can be combined with Theorem 1.4 to deduce Theorem 1.6. Also notice from Theorem 1.6 that if $E = \begin{pmatrix} A & A \\ A & A \end{pmatrix}$ for some $A \subseteq \mathbb{F}_q$ with $|A| \geq Cq^{3/4}$, then for any $i, j \neq 0$, we have

$$\det((E \cap D_i) + (E \cap D_j)) = \mathbb{F}_q.$$ 

We now address an extension of Corollary 1.5. For a fixed $\ell \in \mathbb{N}$ and $E \subseteq M_2(\mathbb{F}_q)$, we define

$$\ell E = E + \cdots + E \subseteq M_2(\mathbb{F}_q).$$

In fact, in this case, the threshold $q^2$ of Corollary 1.5 can be improved whenever $\ell$ becomes larger as the following shows.

Theorem 1.7. Let $k \geq 2$ be an integer and $i$ be an element in $\mathbb{F}_q^*$. If $E \subseteq D_i$ and $|E| \geq Cq^{\frac{6k-5}{6k-4}}$ for a sufficiently large constant $C$, then we have

$$\det(2kE) = \det(E + \cdots + E) \supseteq \mathbb{F}_q^*.$$ 

It follows from Theorem 1.7 that if $k$ is large enough, then $\det(2kE) \supseteq \mathbb{F}_q^*$ whenever the size of $E$ is close to $q^{2k}$. However, in general, one can not expect to go lower than $q^{k^2}$. To see this, let $q = p^2$ for some odd prime $p$, and $E$ be the special linear group $SL_2(\mathbb{F}_p)$. Then it is obvious that $|E| \sim p^3 = q^{3/2}$. But since $2kE$ is a subset of $M_2(\mathbb{F}_p)$ for any $k$, we have $\det(2kE) \subseteq \mathbb{F}_p \subsetneq \mathbb{F}_q$.

Lastly, we would like to say a few words on the exposition of the paper. Unlike in the literature, we elaborated on finding explicit constants $C$ for the bounds in Theorem 1.4 and Corollary 1.5. This asked us to write out almost all details for readers, which had the exposition a bit lengthy, because they have their own distinctions and some subtleties even though some of them look similar.

1.2. Outline of this paper. The remaining parts of this paper are organized to provide the complete proofs of our main theorems. In Section 2, we summarize the background knowledge of the discrete Fourier analysis which will be used as a main tool. In particular, a new operation called the Odot-product is introduced. Section 3 is designed to prove Theorem 1.4 whose sharpness is shown in Section 4. In Section 5, a proof of Theorem 1.6 is given. In Section 6, we obtain a lower bound on the cardinality of the sum of two matrix sets, which will play a crucial role in proving Theorem 1.7. In the final section, we complete the proof of Theorem 1.7.

1.3. Acknowledgement: The authors would like to thank Igor Shparlinski for introducing the paper of Li and Hu [31] to them.
2. Preliminaries

In this section, we review the discrete Fourier analysis and exponential sums. In addition, we introduce the so-called Odot-product on $M_2(F_q)$ and investigate its properties which play a key role in proving our main results.

2.1. Discrete Fourier analysis and exponential sums. Throughout this paper, we will denote by $\chi : F_q \rightarrow S^1$ the canonical additive character of $F_q$. For instance, if $q$ is prime, then we have $\chi(t) = e^{2\pi it/q}$. If $q = p^n$ for some odd prime $p$, then we take $\chi(t) = e^{2\pi i Tr(t)/p}$ for all $t \in F_q$, where $Tr$ denotes the trace function from $F_q$ to $F_p$ defined by

$$Tr(t) = t + t^p + t^{p^2} + \cdots + t^{p^{n-1}} \in F_p.$$ 

Recall that the character $\chi$ enjoys the orthogonality property; for any $m \in F_q^d$, $d \geq 1$,

$$\sum_{x \in F_q^d} \chi(m \cdot x) = \begin{cases} 0 & \text{if } m \neq (0, \ldots, 0) \\ q^d & \text{if } m = (0, \ldots, 0), \end{cases}$$

where $m \cdot x$ denotes the usual dot-product notation. Given a complex-valued function $f$ defined on $F_q^d$, the Fourier transform of $f$ is defined by

$$\hat{f}(m) := q^{-d} \sum_{x \in F_q^d} \chi(-m \cdot x)f(x).$$

The Plancherel theorem in this context says that

$$\sum_{m \in F_q^d} |\hat{f}(m)|^2 = \frac{1}{q^d} \sum_{x \in F_q^d} |f(x)|^2.$$ 

In particular, if $E \subseteq F_q^d$, then

$$\sum_{m \in F_q^d} |\hat{E}(m)|^2 = \frac{|E|}{q^d}.$$ 

Here, throughout this paper, we identify the set $E \subseteq F_q^d$ with the indicator function $1_E$ of the set $E$.

Let $\eta : F_q^* \rightarrow S^1$ be the quadratic character of $F_q^*$, i.e., a group homomorphism defined by $\eta(t) = 1$ if $t$ is a square, and $-1$ otherwise. Recall that the orthogonality property of $\eta$ states that for any $a \in F_q^*$,

$$\sum_{t \in F_q^*} \eta(at) = 0.$$ 

Next, we collect well-known properties of the Gauss sum and the Kloosterman sum. Let us begin by giving the definition of the Gauss sum. The Gauss sum $G_a(\eta, \chi)$ associated with the characters $\chi, \eta$, and an element $a \in F_q^*$ is defined by

$$G_a(\eta, \chi) = \sum_{t \in F_q^*} \eta(t)\chi(at).$$

It is well known that $|G_a(\eta, \chi)| = q^{1/2}$ for all $a \in F_q^*$. Moreover, the value of the Gauss sum for $a = 1$ is explicitly given as follows.
Lemma 2.1. [29, Theorem 5.15] Let $\mathbb{F}_q$ be a finite field with $q = p^n$, where $p$ is an odd prime and $n \in \mathbb{N}$. Then we have

$$G_1(\eta, \chi) = \begin{cases} (-1)^{n-1}q^{\frac{1}{2}} & \text{if } p \equiv 1 \mod 4 \\ (-1)^{n-1}q^{\frac{1}{2}} & \text{if } p \equiv 3 \mod 4. \end{cases}$$

We notice that $\eta(-1) = 1$ if and only if $-1$ is a square number of $\mathbb{F}_q$ (namely, $q \equiv 1 \mod 4$); or equivalently, $\eta(-1) = -1$ if and only if $-1$ is not a square number of $\mathbb{F}_q$ (namely, $q \equiv 3 \mod 4$).

From this fact and Lemma 2.1, it follows that

$$\eta(-1)G_1^2 = q.$$

Hereafter, to use a simple notation, we write $G_1$ for $G_1(\eta, \chi)$.

The following result is a corollary of Lemma 4.3 in [22]. For the reader’s convenience, we provide a proof here.

Lemma 2.2. For $a \in \mathbb{F}_q^*$, $b \in \mathbb{F}_q$, we have

$$\sum_{s \in \mathbb{F}_q^*} \chi(as^2 + bs) = \eta(a)G_1 \chi \left( \frac{b^2}{-4a} \right) - 1.$$

Proof. Since $\chi(0) = 1$, it is enough to prove that

$$\sum_{s \in \mathbb{F}_q} \chi(as^2 + bs) = \eta(a)G_1 \chi \left( \frac{b^2}{-4a} \right).$$

Since $as^2 + bs = a \left( s + \frac{b}{2a} \right)^2 - \frac{b^2}{4a}$, by a change of variables we have

$$\sum_{s \in \mathbb{F}_q} \chi(as^2 + bs) = \sum_{s \in \mathbb{F}_q} \chi(as^2)\chi \left( \frac{b^2}{-4a} \right).$$

Thus, the lemma follows from the observation that if $a \neq 0$, then

$$\sum_{s \in \mathbb{F}_q} \chi(as^2) = \eta(a)G_1.$$  

We will also utilize the following properties of the Gauss sum which can be proved by using a change of variables and properties of the quadratic character $\eta$. For $a, b \neq 0$, we have

$$\sum_{s \in \mathbb{F}_q^*} \eta(as)\chi(bs) = \sum_{s \in \mathbb{F}_q^*} \eta(as^{-1})\chi(bs) = \eta(ab)G_1.$$

We review estimates on the (generalized) Kloosterman sum which can be found in [24, 29]. An estimate of the Kloosterman sum is given by

$$\left| \sum_{t \in \mathbb{F}_q^*} \chi(at + bt^{-1}) \right| \leq 2q^{\frac{1}{2}} \quad \text{for } a, b \in \mathbb{F}_q^*.$$
and an estimate of the generalized Kloosterman sum is given by
\[ \left| \sum_{t \in \mathbb{F}_q^*} \eta(t) \chi(at + bt^{-1}) \right| \leq 2q^{\frac{3}{2}} \quad \text{for} \quad a, b \in \mathbb{F}_q. \]

2.2. **Odot-product and its properties.** In this subsection, we will define the so-called *Odot-product* on the vector space \( M_2(\mathbb{F}_q) = \mathbb{F}_q^4 \), which can be compared with the ordinary inner product on \( \mathbb{F}_q^4 \). Then we will set up a main tool, i.e., a discrete Fourier theoretic machinery for the Odot-product, which is modeled on the well-established (discrete) one for the ordinary inner product.

**Definition 2.3** (Odot-product). For \( x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \), \( y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \in M_2(\mathbb{F}_q) \), define
\[ x \odot y := x_1 y_4 - x_2 y_3 - x_3 y_2 + x_4 y_1. \]

Let us call \( \odot \) the Odot-product on \( M_2(\mathbb{F}_q) \).

For \( x \in M_2(\mathbb{F}_q) \), we will often use the notation \( \|x\|_* \) to denote \( \det(x) \). Namely,
\[ \|x\|_* = \det(x) = x_1 x_4 - x_2 x_3. \]

We collect basic properties of the Odot-product which follow easily from the definitions of the Odot-product and \( \| \cdot \|_* \). We leave the details to readers.

**Lemma 2.4.** Let \( x, y \in M_2(\mathbb{F}_q) \), \( c \in \mathbb{F}_q \). Then the Odot-product \( \odot \) satisfies the followings.
\[
\begin{align*}
x \odot y &= y \odot x, \\
c(x \odot y) &= (cx) \odot y = x \odot (cy), \\
\|cx\|_* &= c^2 \|x\|_*, \\
x \odot x &= 2\|x\|_* \quad \text{and} \quad \|x \pm y\|_* = \|x\|_* + \|y\|_* \pm x \odot y.
\end{align*}
\]

One can check that the following orthogonality of \( \chi \) holds for the Odot-product: for \( m \in M_2(\mathbb{F}_q) \),
\[ \chi(m \odot x) = \begin{cases} 0 & \text{if } m \neq 0, \\
q^4 & \text{if } m = 0. \end{cases} \]

Given a function \( f : M_2(\mathbb{F}_q) \rightarrow \mathbb{C} \), we define
\[ \tilde{f}(x) := \sum_{m \in M_2(\mathbb{F}_q)} \chi(-x \odot m) f(m). \]

For instance, for \( i \in \mathbb{F}_q^* \) and \( y \in M_2(\mathbb{F}_q) \), we have
\[ \tilde{D}_i(y) = \sum_{x \in D_i} \chi(-x \odot y). \]

**Lemma 2.5.** For \( i \in \mathbb{F}_q^* \) and \( y \in M_2(\mathbb{F}_q) \), \( \tilde{D}_i(y) \) is expressed as
\[ \tilde{D}_i(y) = q^3 \delta_0(y) + q \sum_{r \in \mathbb{F}_q^*} \chi \left( -ir - \frac{\|y\|_*}{r} \right), \]
where \( \delta_0(y) = 1 \) if \( y = 0 \), and 0 otherwise.

**Proof.** By the orthogonality of \( \chi \), we can write that
\[ \tilde{D}_i(y) = q^{-1} \sum_{x \in M_2(\mathbb{F}_q)} \sum_{r \in \mathbb{F}_q} \chi(r(\|x\|_* - i)) \chi(-x \odot y) \]
Then the lemma follows from a calculation of the sums over \(x_1, x_2 \in \mathbb{F}_q\) using the orthogonality of \(\chi\).

\[ q^3 \delta_0(y) + q^{-1} \sum_{x \in M_2(\mathbb{F}_q)} \sum_{r \neq 0} \chi(r(\|x\|_* - i)) \chi(-x \odot y) \]

Then, for every \(\chi\) \(|(3.2)|

**3. The Key Lemma and Proof of Theorem 1.4**

This section is dedicated to proving Theorem 1.4. We begin by introducing notations for our interested quantities.

**Notation 3.1.** Let \(E, F\) be sets in \(M_2(\mathbb{F}_q)\).

1. For \(t \in \mathbb{F}_q\), we denote by \(N_t(E, F)\) the number of pairs \((x, y) \in E \times F\) such that \(\det(x+y) = t\).
2. For \(\ell \in \mathbb{F}_q\), we let \(W_\ell(E, F)\) denote the number of pairs \((x, y) \in E \times F\) such that \(x \odot y = \ell\).
3. For \(\ell \in \mathbb{F}_q\), we write \(R_\ell(E, F)\) for \(W_\ell(E, F) - |E||F|/q\). Namely,

\[ R_\ell(E, F) := W_\ell(E, F) - \frac{|E||F|}{q}. \]

4. We denote by \(M(E, F)\) the maximum value of the set \(\{W_\ell(E, F) : \ell \in \mathbb{F}_q\}\). Namely,

\[ M(E, F) := \max_{\ell \in \mathbb{F}_q} \sum_{x \in E, y \in F : x \odot y = \ell} 1. \]

A bound on \(N_t(E, F)\) plays an essential role in proving Theorem 1.4, as well as it is interesting on its own right. To obtain an upper bound for \(N_t(E, F)\), we need a couple of technical lemmas.

**Lemma 3.2.** For \(i, j \in \mathbb{F}_q^*\), let \(E \) and \(F\) be subsets of \(D_i\) and \(D_j\), respectively. Suppose that for all \(\ell \in \mathbb{F}_q\), the following two inequalities hold:

\[ |R_\ell(E, F)|^2 \leq 2q^2|E||F| + 7|E||F|^2 + 2q|E|M(E, F) \]

and

\[ |R_\ell(F, F)|^2 \leq 2q^2|F|^2 + 7|F|^3 + 2q|F|M(F, F). \]

Then, for every \(t \in \mathbb{F}_q\), we have

\[ \left| N_t(E, F) - \frac{|E||F|}{q} \right| \leq \sqrt{18q^2|E||F| + 11|E||F|^2 + 4\sqrt{7}q|E||F|^{3/2}}. \]

**Proof.** By definition, we can write

\[ N_t(E, F) = \sum_{x \in E, y \in F : \|x+y\|_* = t} 1. \]

Since \(E \subseteq D_i, F \subseteq D_j\), by Lemma 2.4, this can be written as

\[ N_t(E, F) = \sum_{x \in E, y \in F : x \odot y = t - i - j} 1. \]

Letting \(\ell = t - i - j\), we see that \(N_t(E, F) = W_\ell(E, F)\). Hence, to prove the lemma, it suffices to show that for all \(\ell \in \mathbb{F}_q\), we have

\[ |R_\ell(E, F)|^2 \leq 18q^2|E||F| + 11|E||F|^2 + 4\sqrt{7}q|E||F|^{3/2}. \]
Notice from the assumption \((3.1)\) that to prove the above inequality it is enough to show that 
\[ M(F, F) \leq \frac{2|F|^2}{q} + 8q|F| + 2\sqrt{7}|F|^{3/2}. \]

Since \(W_{\ell}(F, F) = |F|^2/q + R_{\ell}(F, F)\) by definition, it is clear that 
\[ M(F, F) = \frac{|F|^2}{q} + \max_{\ell \in F_q} R_{\ell}(F, F), \]
and the assumption \((3.2)\) implies that 
\[ \max_{\ell \in F_q} R_{\ell}(F, F) \leq \sqrt{2q}|F| + \sqrt{7}|F|^{3/2} + \sqrt{2q^{1/2}|F|^{1/2}} M(F, F)^{1/2}. \]

Therefore, we have 
\[
M(F, F) \leq \frac{|F|^2}{q} + 2\sqrt{2q}|F| + \sqrt{7}|F|^{3/2} + \sqrt{2q^{1/2}|F|^{1/2}} M(F, F)^{1/2}
\leq 2 \max \left\{ \frac{|F|^2}{q} + \sqrt{2q}|F| + \sqrt{7}|F|^{3/2}, \sqrt{2q^{1/2}|F|^{1/2}} M(F, F)^{1/2} \right\}.
\]

From this estimate, we obtain the inequality \((3.3)\) as follows:
\[
M(F, F) \leq \max \left\{ \frac{2|F|^2}{q} + 2\sqrt{2q}|F| + 2\sqrt{7}|F|^{3/2}, 8q|F| \right\}
\leq \frac{2|F|^2}{q} + 8q|F| + 2\sqrt{7}|F|^{3/2}.
\]

□

As we will see, Proposition 3.7 given in the last part of this section plays a key role in proving Theorem 1.4. Notice that the proof of Proposition 3.7 uses bounds of several summations. To make the exposition better, we separately treat these summations in several lemmas.

**Lemma 3.3.** Let \(F\) be a subset of \(D_j\) with \(j \in \mathbb{F}_q^*\). Then, for every \(\ell \in \mathbb{F}_q\), we have 
\[
\mathcal{I}(\ell) := \sum_{\substack{y, y' \in F \\colon s, s' \in \mathbb{F}_q^* \colon s'y' = sy}} \delta_0(s' y' - s y) \chi(\ell(s' - s)) \leq 2q|F|.
\]

**Proof.** The value \(\mathcal{I}(\ell)\) can be written as 
\[
\mathcal{I}(\ell) = \sum_{\substack{y, y' \in F \\colon s, s' \in \mathbb{F}_q^* \colon s'y' = sy}} \chi(\ell(s' - s)).
\]

It is clear that the sum over pairs \((s, s')\) with \(s = s'\) is \((q - 1)|F|\), and the sum over pairs \((s, s')\) with \(s \neq s'\) is 
\[
\sum_{\substack{y, y' \in F \\colon s, s' \in \mathbb{F}_q^* \colon s'y' = sy, s \neq s'}} \chi(\ell(s' - s)) = \sum_{\substack{y, y' \in F \\colon a \neq 0, b \neq 0, 1y' = by}} \chi(\ell a(1 - b)),
\]
where we use a change of variables by letting \(a = s', b = s/s'\).
If $\ell \neq 0$, then this value is less than or equal to zero, because the sum over $a \neq 0$ is $-1$ by the orthogonality of $\chi$. If $\ell = 0$, then the value above is given by

$$\sum_{y, y' \in F, a \neq 0, b \neq 0, 1} 1 = (q - 1) \sum_{y, y' \in F, b \neq 0, 1} 1.$$  

Observe that if $b \neq 1$ and $y, y' \in D_j$ with $j \neq 0$, then $y' = by$ only if $b = -1$. Thus, the value above is at most $(q - 1)|F|$. In summary, we have proved that for any $\ell \in \mathbb{F}_q$,

$$\mathcal{I}(\ell) \leq 2(q - 1)|F| \leq 2q|F|,$$

as desired.

Lemma 3.4. Let $i \in \mathbb{F}_q^*$ and $F$ be a subset of $D_j$ with $j \in \mathbb{F}_q^*$. Then, for all $\ell \in \mathbb{F}_q$, we have

$$\mathcal{A}(\ell) := \sum_{y, y' \in F, r, s, s' \in \mathbb{F}_q^*, \|s'y' - sy\|_* = 0} \chi(-ir) \chi(\ell(s' - s)) \leq q|F|^2.$$

Proof. Since $i \neq 0$, the sum over $r \in \mathbb{F}_q^*$ of $\mathcal{A}(\ell)$ is $-1$. Thus, we have

$$\mathcal{A}(\ell) = \sum_{y, y' \in F, s, s' \in \mathbb{F}_q^*, \|s'y' - sy\|_* = 0} -\chi(\ell(s' - s)).$$

Notice that $\mathcal{A}(\ell)$ is a real number since $\mathcal{A}(\ell) = \overline{\mathcal{A}(\ell)}$. It is clear that the contribution of the case $s = s'$ to $\mathcal{A}(\ell)$ is negative. Hence,

$$\mathcal{A}(\ell) \leq \sum_{y, y' \in F, s, s' \in \mathbb{F}_q^*, \|s'y' - sy\|_* = 0, s \neq s'} -\chi(\ell(s' - s)).$$

Since $F \subseteq D_j$, the condition $\|s'y' - sy\|_* = 0$ is equivalent to $js'^2 + js^2 - s|y' \oplus y| = 0$. Using a change of variables by letting $a = s'$, $b = s/s'$, we have

$$\mathcal{A}(\ell) \leq \sum_{y, y' \in F, a \neq 0, b \neq 0, 1, j \neq b^2 \oplus (y' \ominus y) = 0} -\chi(\ell a(1 - b)).$$

If $\ell = 0$, then this value is obviously a non-positive real number. If $\ell \neq 0$, then the sum over $a \neq 0$ is $-1$. Hence,

$$\mathcal{A}(\ell) \leq \sum_{y, y' \in F, b \neq 0, 1, j \neq b^2 \oplus (y' \ominus y) = 0} 1 \leq q|F|^2,$$

as required.

Lemma 3.5. Let $F$ be a subset of $M_2(\mathbb{F}_q)$. Then, for every $i \in \mathbb{F}_q^*$, we have

$$\mathcal{B}(i) := \sum_{y, y' \in F, r, s \in \mathbb{F}_q^*, \|y' - y\|_* \neq 0} \chi \left( -ir - \frac{s^2 \|y' - y\|_*}{r} \right) \leq 2q|F|^2.$$
Proof. We apply Lemma 2.2 with \( b = 0 \) to get the following:

\[
\mathcal{B}(i) = G_1 \sum_{y,y' \in F, r \in \mathbb{F}_q^* : \|y - y\|_* \neq 0} \chi(-ir) \eta \left(-\frac{\|y' - y\|_*}{r}\right) - \sum_{y,y' \in F, r \in \mathbb{F}_q^* : \|y' - y\|_* \neq 0} \chi(-ir)
\]

\[
= G_1 \sum_{y,y' \in F, r \in \mathbb{F}_q^* : \|y - y\|_* \neq 0} \chi(-ir) \eta \left(-\frac{\|y' - y\|_*}{r}\right) + \sum_{y,y' \in F, r \in \mathbb{F}_q^* : \|y' - y\|_* \neq 0} 1.
\]

Since the sum over \( r \in \mathbb{F}_q^* \) of the first term above is a Gauss sum, it is easy to see that

\[
\mathcal{B}(i) \leq q|F|^2 + |F|^2 \leq 2q|F|^2.
\]

\[\square\]

**Lemma 3.6.** Let \( i \in \mathbb{F}_q^* \) and \( F \) be a subset of \( D_j \) with \( j \in \mathbb{F}_q^* \). Then for all \( \ell \in \mathbb{F}_q \), we have

\[
\mathcal{C}(\ell) := \sum_{y,y' \in F, r,s,s' \in \mathbb{F}_q^* : \|y' - y - sy\|_* \neq 0, s \neq s'} \chi \left(-ir - \frac{\|s'y' - sy\|_*}{r}\right) \chi \left((s' - s)\ell\right).
\]

\[
\leq 2q|F|^2 + G_1 \sum_{y,y' \in F, r \neq 0, b \neq 0} \eta(-r) \chi(r(\ell^2 - 4ij)) \chi\left(r(\ell^2 - 4ij)b^2 + 2r(2i(y' \odot y) - \ell^2)b\right).
\]

**Proof.** The value \( \mathcal{C}(\ell) \) is rewritten as follows:

\[
\mathcal{C}(\ell) = \sum_{y,y' \in F, r,s,s' \in \mathbb{F}_q^* : \|y' - (s/s')y\|_* \neq 0, s \neq s'} \chi \left(-ir - \frac{s^2\|y' - (s/s')y\|_*}{r}\right) \chi \left(\ell s'(1 - s/s')\right).
\]

By a change of variables with \( a = s', b = s/s' \), we have

\[
\mathcal{C}(\ell) = \sum_{y,y' \in F, r,a \neq 0, b \neq 0, 1: \|y' - by\|_* \neq 0} \chi \left(-ir - \frac{a^2\|y' - by\|_*}{r}\right) \chi \left(\ell a(1 - b)\right).
\]

Computing the sum over \( a \in \mathbb{F}_q^* \) by Lemma 2.2, we have

\[
\mathcal{C}(\ell) = G_1 \sum_{y,y' \in F, r \neq 0, b \neq 0, 1: \|y' - by\|_* \neq 0} \eta\left(\frac{\|y' - by\|_*}{r}\right) \chi(-ir) \chi\left(\frac{r\ell^2(b - 1)^2}{4\|y' - by\|_*}\right)
\]

\[+ \sum_{y,y' \in F, r \neq 0, b \neq 0, 1: \|y' - by\|_* \neq 0} -\chi(-ir).
\]

In the first term we use a change of variables by replacing \( r/(4\|y' - by\|_*) \) by \( r \) and in the second term we compute the sum over \( r \neq 0 \). Then we see that

\[
\mathcal{C}(\ell) = G_1 \sum_{y,y' \in F, r \neq 0, b \neq 0, 1: \|y' - by\|_* \neq 0} \eta\left(\frac{1}{-4r}\right) \chi(-4ir\|y' - by\|_*) \chi\left(r\ell^2(b - 1)^2\right)
\]

\[= G_1 \sum_{y,y' \in F, r \neq 0, b \neq 0, 1: \|y' - by\|_* \neq 0} \eta\left(\frac{1}{-4r}\right) \chi(-4ir\|y' - by\|_*) \chi\left(r\ell^2(b - 1)^2\right).
\]

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\[ + \sum_{y, y' \in F \atop r \neq 0, b \neq 0; \|y' - by\|_\ast \neq 0} 1. \]

Since the second term above is less than \( q|F|^2 \), it follows that

\[ C(\ell) \leq q|F|^2 + G_1 \sum_{y, y' \in F \atop r \neq 0, b \neq 0; \|y' - by\|_\ast \neq 0} \eta \left( \frac{1}{-4r} \right) \chi(-4ir\|y' - by\|_\ast) \chi(r\ell^2(b - 1)^2) \]

\[ = q|F|^2 + G_1 \sum_{y, y' \in F \atop r \neq 0, b \neq 0; \|y' - by\|_\ast \neq 0} \eta \left( \frac{1}{-4r} \right) \chi(-4ir\|y' - by\|_\ast) \chi(r\ell^2(b - 1)^2) \]

\[-G_1 \sum_{y, y' \in F \atop r \neq 0; \|y' - by\|_\ast = 0} \eta \left( \frac{1}{-4r} \right) \chi(r\ell^2(b - 1)^2). \]

Using the formula (2.3) and the fact that \( G_1^2 \) is a real number with \( G_1^2 = \pm q \), we see that the third term above is a real number which is less than or equal to \( q|F|^2 \). Hence, \[ C(\ell) \leq 2q|F|^2 + G_1 \sum_{y, y' \in F \atop r \neq 0, b \neq 0; \|y' - by\|_\ast = 0} \eta \left( \frac{1}{-4r} \right) \chi(-4ir\|y' - by\|_\ast) \chi(r\ell^2(b - 1)^2) \]

\[ = 2q|F|^2 + G_1 \sum_{y, y' \in F \atop r \neq 0, b \neq 0} \eta \left( \frac{1}{-4r} \right) \chi(-4ir\|y' - by\|_\ast) \chi(r\ell^2(b - 1)^2) \]

\[-G_1 \sum_{y, y' \in F \atop r \neq 0; \|y' - by\|_\ast = 0} \eta \left( \frac{1}{-4r} \right) \chi(r\ell^2(b - 1)^2). \]

By the orthogonality of \( \eta \), we see that if \( \ell = 0 \) or \( b = 1 \), then the last term above is zero. On the other hand, if \( \ell \neq 0 \) and \( b \neq 1 \), then it follows from the formula (2.3) that the last term above is \[-\eta(-1)G_1^2 \sum_{y, y' \in F \atop b \neq 0; \|y' - by\|_\ast = 0} 1. \]

This value is a negative real number since \( \eta(-1)G_1^2 = q \) (see (2.1)). Hence, \[ C(\ell) \leq 2q|F|^2 + G_1 \sum_{y, y' \in F \atop r \neq 0, b \neq 0} \eta \left( \frac{1}{-4r} \right) \chi(-4ir\|y' - by\|_\ast) \chi(r\ell^2(b - 1)^2) \]

Since \( \|y' - by\|_\ast = j + jb^2 - b(y' \odot y) \) for \( b \in F_q, y, y' \in F \subseteq D_j \), it follows that \[ C(\ell) \leq 2q|F|^2 + G_1 \sum_{y, y' \in F \atop r \neq 0, b \neq 0} \eta(-r)\chi(r(\ell^2 - 4ij)) \chi(r(\ell^2 - 4ij)b^2 + 2r(2i(y' \odot y) - \ell^2)b) \]

where we also used the fact that \( \eta(1/(-4r)) = \eta( -r) \). Thus, the proof is complete. \( \square \)

Based on the previous lemmas, we can deduce the following result.
Proposition 3.7. Let $i, j$ be elements in $\mathbb{F}_q^*$, and $E$ and $F$ be subsets of $D_i$ and $D_j$, respectively. For each $t \in \mathbb{F}_q$, we have

$$|N_t(E, F) - \frac{|E||F|}{q}| \leq \sqrt{18q^2|E||F| + 11|E||F|^2 + 4\sqrt{7}q|E||F|^2}.$$ 

Proof. To prove the proposition, we invoke Lemma 3.2, i.e., show that the conditions (3.1) and (3.2) in Lemma 3.2 are satisfied. Note that if we prove the condition (3.1), then we easily see that the condition (3.2) would be automatic by considering the case $E = F$. Thus it is enough to prove the condition (3.1); for each $\ell \in \mathbb{F}_q$,

$$(3.4) \hspace{1cm} |R_\ell(E, F)|^2 \leq 2q^2|E||F| + 7|E||F|^2 + 2q|E|M(F, F).$$

To prove the above inequality, we first notice by the orthogonality of $\chi$ that

$$W_\ell(E, F) = q^{-1} \sum_{x \in E, y \in F} \sum_{s \in \mathbb{F}_q} \chi(s(x \ominus y - \ell)) = \frac{|E||F|}{q} + q^{-1} \sum_{x \in E, y \in F} \sum_{s \neq 0} \chi(sx \ominus y)\chi(-s\ell).$$

From this equality, we see that

$$R_\ell(E, F) = q^{-1} \sum_{x \in E, y \in F} \sum_{s \neq 0} \chi(sx \ominus y)\chi(-s\ell).$$

By the Cauchy-Schwarz inequality w.r.t $x \in E$, we have

$$|R_\ell(E, F)|^2 \leq q^{-2}|E| \left| \sum_{y \in F, s \in \mathbb{F}_q} \chi(sx \ominus y)\chi(-s\ell) \right|^2.$$

Since $E \subseteq D_i$, we have

$$|R_\ell(E, F)|^2 \leq q^{-2}|E| \sum_{x \in D_i} \sum_{y, y' \in F, s, s' \in \mathbb{F}_q^*} \chi(-x \ominus (s'y' - sy)) \chi(\ell(s' - s)) = q^{-2}|E| \sum_{y, y' \in F, s, s' \in \mathbb{F}_q^*} \tilde{D}_i(s'y' - sy)\chi(\ell(s' - s)).$$

Using Lemma 2.5 and Lemma 3.3,

$$|R_\ell(E, F)|^2 \leq 2q^2|E||F| + q^{-1}|E| \sum_{y, y' \in F, r \in \mathbb{F}_q^*, s, s' \in \mathbb{F}_q^*} \chi(-ir - \frac{||s'y' - sy||}{r}) \chi(\ell(s' - s)).$$

Let $A(\ell)$ denote the second term of the RHS of the above inequality. Then, to prove the inequality (3.4), it is enough to show that

$$A(\ell) \leq 7|E||F|^2 + 2q|E|M(F, F).$$

To prove this inequality, we split up the sum $A(\ell)$ into two summands as follows:

$$A(\ell) = q^{-1}|E| \sum_{y, y' \in F, r, s, s' \in \mathbb{F}_q^*: ||s'y' - sy|| = 0} \chi(-ir) \chi(\ell(s' - s)).$$
\[ + q^{-1} |E| \sum_{r, s, s' \in F_q^* : \|y' - sy\|_r \neq 0} \chi \left( -ir - \frac{\|s'y' - sy\|_r}{r} \right) \chi \left( \ell(s' - s) \right) . \]

From Lemma 3.4, it is clear that the first term of the RHS of the above equality is \( \leq |E||F|^2 \). Hence, letting \( B(\ell) \) denote the second term of the RHS of the above equality, we only need to show that
\[ B(\ell) \leq 6|E||F|^2 + 2q|E|M(F, F). \]

To estimate \( B(\ell) \), we consider two cases that \( s = s' \) and \( s \neq s' \). It follows that
\[ B(\ell) = q^{-1}|E| \sum_{r, s \in F_q^* : \|y - sy\|_r \neq 0} \chi \left( -ir - \frac{s^2\|y - y\|_r}{r} \right) \]
\[ + q^{-1}|E| \sum_{r, s, s' \in F_q^* : \|s'y' - sy\|_r \neq 0, s \neq s'} \chi \left( -ir - \frac{\|s'y' - sy\|_r}{r} \right) \chi \left( \ell(s' - s) \right) . \]

It is obvious from Lemma 3.5 that the first term of the RHS of the above equality is \( \leq 2|E||F|^2 \). Therefore, letting \( C(\ell) \) be the second term of the RHS of the above equality, our problem is reduced to showing that
\[ C(\ell) \leq 4|E||F|^2 + 2q|E|M(F, F). \]

Using Lemma 3.6, it follows that
\[ C(\ell) \leq 2|E||F|^2 + G_1 q^{-1}|E| \sum_{y, y' \in F_q^* : r \neq 0, b \neq 0} \eta(-r) \chi(r(\ell^2 - 4ij)) \chi(r(\ell^2 - 4ij)b^2 + 2r(2i(y' \circ y) - \ell^2)b) . \]

Letting \( D(\ell) \) denote the second term of the RHS of the above inequality, it is enough to prove that
\[ (3.5) \quad D(\ell) \leq 2|E||F|^2 + 2q|E|M(F, F). \]

When \( \ell^2 - 4ij = 0 \), it is not hard to see that \( D(\ell) = 0 \). Thus, assuming that \( \ell^2 - 4ij \neq 0 \), we will prove the inequality (3.5). Computing the sum over \( b \neq 0 \) of the term \( D(\ell) \) by using Lemma 2.2, we have
\[ (3.6) \quad D(\ell) = q^{-1}|E| G_1^2 \sum_{y, y' \in F_q^* : r \neq 0} \eta(4ij - \ell^2) \chi \left( \left( \frac{(\ell^2 - 4ij)^2 - (2i(y' \circ y) - \ell^2)^2}{\ell^2 - 4ij} \right) r \right) \]
\[ - q^{-1}|E| G_1 \sum_{y, y' \in F_q^* : r \neq 0} \eta(-r) \chi(r(\ell^2 - 4ij)) . \]

The last value above is the same as
\[ - q^{-1}|E| G_1^2 \sum_{y, y' \in F_q^*} \eta(4ij - \ell^2) \]
which is clearly \( \leq |E||F|^2 \). Hence, letting \( F(\ell) \) be the first term of the RHS of the above equality (3.6), our final task is to show that
\[ (3.7) \quad F(\ell) \leq |E||F|^2 + 2q|E|M(F, F). \]
Notice that the value in the bracket \([\quad]\) in (3.6) is zero if and only if \(y' \cap y = 2j\) or \((\ell^2 - 2ij)/i\). Hence, in the case of \(y' \cap y \neq 2j, (\ell^2 - 2ij)/i\), the contribution to \(F(\ell)\) is at most \(|E||F|^2\), because the sum over \(r \neq 0\) is \(-1, G_1^2 = \pm q,\) and \(\eta\) takes \(\pm 1\). On the other hand, in the case of \(y' \cap y = 2j\) or \((\ell^2 - 2ij)/i\), the contribution to \(F(\ell)\) is clearly dominated by
\[
2q|E|\max_{k \in \mathbb{F}_q} \sum_{y, y' : E : y' \cap y = k} 1.
\]
Thus, the inequality (3.7) holds and the proof of the proposition is complete. \(\square\)

3.1. **Proof of Theorem 1.4.** In this subsection, we give a proof of Theorem 1.4 for which we heavily use Proposition 3.7.

**Proof.** Note that the hypothesis \(|E||F| \geq 15^2q^4\) implies that \(|E| \geq 15q^2\) or \(|F| \geq 15q^2\), say that \(|E| \geq 15q^2\). Then we see that \(|E|^{1/2}|F|^{1/2} \geq 15q^2\) and \(|E|^{1/2} \geq \sqrt{15}q\). This clearly implies that

\[(3.8) \quad |E||F|^{1/2} \geq 15\sqrt{15}\ q^3.\]

In view of Proposition 3.7, it suffices to prove that if \(|E||F| \geq 15^2q^4\) and \(|E| \geq 15q^2\), then

\[(3.9) \quad \frac{|E||F|}{q} > \sqrt{18q^2|E||F| + 11|E||F|^2 + 4\sqrt{7}q|E||F|^{3/2}}.\]

By squaring both sides of (3.9) and simplifying it, we see that, to obtain the inequality (3.9), it is enough to show that

\[(3.10) \quad |E||F| > 18q^4 + 11q^2|F| + 4\sqrt{7}q^3|F|^{3/2}.\]

Since \(|E| \geq 15q^2\), and hence \(|E||F| = \frac{11}{15}|E||F| + \frac{4}{15}|E||F|\) and \(\frac{11}{15}|E||F| \geq 11q^2|F|\), for the inequality (3.10) it is enough to prove that

\[(3.11) \quad \frac{4}{15}|E||F| > 18q^4 + 4\sqrt{7}q^3|F|^{3/2}.\]

Write

\[
\frac{4}{15}|E||F| = \frac{2}{25}|E||F| + \frac{14}{75}|E||F|.
\]

Then the inequality (3.11) would follow if we show two inequalities;

\[(3.12) \quad \frac{2}{25}|E||F| \geq 18q^4\]

and

\[(3.13) \quad \frac{14}{75}|E||F| > 4\sqrt{7}q^3|F|^{3/2}.\]

The inequality (3.12) follows immediately from our assumption that \(|E||F| \geq 15^2q^4\). The inequality (3.13) is equivalent to

\[
|E||F|^{1/2} > \frac{150\sqrt{7}}{7} q^3,
\]

which is immediate from (3.8). This proves the theorem. \(\square\)

4. **Sharpness of Theorem 1.4**

In this section, we will show that by giving a concrete example, Theorem 1.4 can not be improved in general. Let \(H\) be a subvariety of \(M_2(\mathbb{F}_q)\) defined by the equation \(x_2 + x_3 = 0\), and \(H_i := H \cap D_i\),
so that we have
\[(4.1) \quad H_i = \left\{ \left( \begin{array}{cc} x_1 & x_2 \\ -x_2 & x_4 \end{array} \right) \in M_2(\mathbb{F}_q) : x_1x_4 + x_2^2 = i \right\}. \]

Then it is clear that \(|H_i| \sim q^2\), and \(-x \in H_i\) if and only if \(x \in H_i\). Let \(E\) be a maximal subset of \(H_i\) such that \(E \cap (-E) = \phi\). Then it is obvious that
\[(4.2) \quad |E| \sim q^2. \]

**Proposition 4.1.** Let \(i\) be a non-square number in \(\mathbb{F}_q^*\), and let \(E\) be a subset of \(H_i\) given as in the above. Fix \(y \in E\). Then the equation for \(x; x \cap y = -2i\) has a unique solution \(x = -y\) in \(E\).

A proof of Proposition 4.1 will be given shortly after a proof of Corollary 4.2 below. The following indicates that Theorem 1.4 is sharp in general.

**Corollary 4.2.** Let \(i\) and \(E\) be given as in Proposition 4.1. Then we have
\[0 \notin \text{det}(E + E).\]

**Proof.** Since \(\text{det}(x + y) = \|x + y\|_s = 2i + x \cap y\) for \(x, y \in E \subset D_i\), it suffices to show that
\[x \cap y \neq -2i \quad \text{for any } x, y \in E.\]

Let us assume that \(x \cap y = -2i\) for some \(x, y \in E\). Then by Proposition 4.1, we have the relation \(y = -x\), so \(-E \cap E\) is not empty. However, this is impossible by the condition on \(E\). This proves the corollary.

4.1. **Proof of Proposition 4.1.**

**Proof of Proposition 4.1.** It is obvious that \(-y \cap y = -2i\), so \(x = -y\) is a solution to \(x \cap y = -2i\). Let us show the uniqueness. The conditions \(x \in E, y \in E, x \cap y = -2i\), respectively, turn into
\[x_1x_4 + x_2^2 = i,\]
\[y_1y_4 + y_2^2 = i,\]
\[(x_1, x_2, -x_2, x_4) \cap (y_1, y_2, -y_2, y_4) = -2i.\]

Let \(N\) denote the number of solutions to the above equations for \(x_1, x_2, x_4\). We aim to prove that \(N = 1\). Since \((x_1, x_2, -x_2, x_4) \cap (y_1, y_2, -y_2, y_4) = y_4x_1 + 2y_2x_2 + y_1x_4\), we can write
\[N = \sum_{x_1,x_2,x_4 \in \mathbb{F}_q; \quad y_4x_1+2y_2x_2+y_1x_4=-2i, \quad x_1x_4+x_2^2=i} 1.\]

By the orthogonality of \(\chi\), we have
\[N = q^{-2} \sum_{x_1,x_2,x_4 \in \mathbb{F}_q} \sum_{s,r \in \mathbb{F}_q} \chi(s(y_4x_1 + 2y_2x_2 + y_1x_4 + 2i))\chi(r(x_1x_4 + x_2^2 - i)).\]

Decomposing the ‘internal’ sum \(\sum_{s,r \in \mathbb{F}_q}\) into four summands
\[\sum_{s,r \in \mathbb{F}_q} = \sum_{s=0,r=0} + \sum_{s=0,r\neq 0} + \sum_{s\neq 0,r=0} + \sum_{s\neq 0,r\neq 0},\]
we obtain four corresponding summands of \(N\) (in order)
\[N = N_1 + N_2 + N_3 + N_4.\]
Now we calculate $N_s$. First of all, $N_1$ is computed:

\[(4.3) \quad N_1 = q^{-2} \sum_{x_1, x_2, x_4 \in \mathbb{F}_q} 1 = q.\]

Secondly, $N_2$ is given as follows:

\[
N_2 = q^{-2} \sum_{x_1, x_2, x_4 \in \mathbb{F}_q} \sum_{r \neq 0} \chi(r(x_1 x_4 + x_2^2 - i))
\]

\[(4.4) \quad = q^{-2} \sum_{r \neq 0} \chi(-ir) \left( \sum_{x_2 \in \mathbb{F}_q} \chi(rx_2^2) \right) \left( \sum_{x_1, x_4 \in \mathbb{F}_q} \chi(rx_1 x_4) \right).\]

In (4.4), the sum over $x_2 \in \mathbb{F}_q$ is equal to $\eta(r)G_1$ by Lemma 2.2, and the one over $x_1, x_4 \in \mathbb{F}_q$ is equal to $q$ by the orthogonality of $\chi$. Therefore, we see that

\[
N_2 = q^{-1}G_1 \sum_{r \neq 0} \eta(r)\chi(-ri).
\]

Now, by the formula in (2.3), we have

\[(4.5) \quad N_2 = q^{-1}G_1^2 \eta(-i).\]

Thirdly, the term $N_3$ is given as follows:

\[
N_3 = q^{-2} \sum_{x_1, x_2, x_4 \in \mathbb{F}_q} \sum_{s \neq 0, r \neq 0} \chi(s(y_4 x_1 + 2y_2 x_2 + y_1 x_4 + 2i)).
\]

Since $y_1y_4 + y_2^2 = i \neq 0$, one of $y_i, i = 1, 2, 3,$ is not a zero. Then the orthogonality of $\chi$ yields that

\[(4.6) \quad N_3 = 0.\]

Lastly, the term $N_4$ is written as follows:

\[
N_4 = q^{-2} \sum_{x_1, x_2, x_4 \in \mathbb{F}_q} \sum_{s \neq 0, r \neq 0} \chi(s(y_4 x_1 + 2y_2 x_2 + y_1 x_4 + 2i))\chi(r(x_1 x_4 + x_2^2 - i))
\]

\[(4.7) \quad = q^{-2} \sum_{s \neq 0, r \neq 0} \chi(2is)\chi(-ir) \sum_{x_4 \in \mathbb{F}_q} \chi(sy_1 x_4) \left( \sum_{x_1 \in \mathbb{F}_q} \chi((sy_4 + rx_4)x_1) \right) \left( \sum_{x_2 \in \mathbb{F}_q} \chi(rx_2^2 + 2sy_2 x_2) \right).\]

In the term (4.7), by the orthogonality of $\chi$, the sum over $x_1 \in \mathbb{F}_q$ is equal to $q$ if $x_4 = -sy_4/r$, and $0$ otherwise. By the formula (2.2), the sum over $x_2 \in \mathbb{F}_q$ is equal to

\[
\eta(r)G_1 \chi \left( \frac{s^2y_2^2}{-r} \right).
\]

It follows that

\[
N_4 = q^{-1}G_1 \sum_{s \neq 0, r \neq 0} \eta(r)\chi(-ir)\chi(2is)\chi \left( \frac{(y_1y_4 + y_2^2)s^2}{-r} \right).
\]

Since $y_1y_4 + y_2^2 = i$, $N_4$ is written as

\[(4.8) \quad N_4 = q^{-1}G_1 \sum_{r \neq 0} \eta(r)\chi(-ir) \left( \sum_{s \neq 0} \chi \left( \frac{is^2}{-r} + 2is \right) \right).\]
Using Lemma 2.2 to compute the sum over \( s \neq 0 \) in (4.8), we obtain that
\[
N_4 = q^{-1}G_1^2 \sum_{r \neq 0} \eta(-i) - q^{-1}G_1 \sum_{r \neq 0} \eta(r) \chi(-ir)
\]
\[
= q^{-1}G_1^2 \eta(-i)(q - 1) - q^{-1}G_1^2 \eta(-i) = G_1^2 \eta(-i) - 2q^{-1}G_1^2 \eta(-i).
\]
Adding all \( N_i \) with \( 1 \leq i \leq 4 \), we obtain
\[
N = N_1 + N_2 + N_3 + N_4 = q + \eta(i)G_1^2 \eta(-1) - \eta(i)q^{-1}G_1^2 \eta(-1).
\]
Since \( i \) is a non-square number, \( \eta(i) = -1 \). Recall from (2.1) that \( G_1^2 \eta(-1) = q \). Thus \( N = 1 \), as required. This completes the proof of Lemma 4.2.

\[\Box\]

5. Proof of Theorem 1.6

In this section we prove Theorem 1.6 by using Theorem 1.4 and a result on the size of the intersection of a product type subset \( S \) and \( D_i \) with \( i \neq 0 \). For the latter result, we estimate \( |S \cap D_i| \) by adapting the method which Hart and Iosevich [20] used in studying the size of the dot-product set determined by a set in \( \mathbb{F}_q^d \).

**Lemma 5.1.** Let \( S \subseteq M_2(\mathbb{F}_q) \) be of product type. Then, for each \( i \in \mathbb{F}_q^* \), we have
\[
\left| |S \cap D_i| - \frac{|S|}{q} \right| \leq q^{1/2}|S|^{1/2}.
\]

**Proof.** Let \( S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} \) for some \( S_1, S_2 \subseteq \mathbb{F}_q^2 \). It is clear that \( |S| = |S_1||S_2| \). It follows that
\[
|S \cap D_i| = \sum_{\alpha \in S_1, \beta \in S_2; \det(\alpha, \beta) = i} 1,
\]
where \( \det(\alpha, \beta) := \alpha_1 \beta_2 - \alpha_2 \beta_1 \) for \( \alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathbb{F}_q^2 \). By the orthogonality of \( \chi \), we have
\[
|S \cap D_i| = q^{-1} \sum_{\alpha \in S_1, \beta \in S_2} \sum_{r \in \mathbb{F}_q} \chi(r(\det(\alpha, \beta) - i))
\]
\[
= \frac{|S_1||S_2|}{q} + q^{-1} \sum_{\alpha \in S_1, \beta \in S_2} \sum_{r \neq 0} \chi(r(\det(\alpha, \beta) - i))
\]
\[
:= \frac{|S|}{q} + R(i).
\]
Hence, in order to prove the lemma, it will be enough to show that
\[
|R(i)|^2 \leq q|S|.
\]
Now, applying the Cauchy-Schwarz inequality to \( |R(i)|^2 \) w.r.t \( \alpha \in S_1 \), and then replacing the index set “\( \alpha \in S_1 \)” by “\( \alpha \in \mathbb{F}_q^2 \), we see
\[
|R(i)|^2 \leq q^{-2} \left( \sum_{\alpha \in S_1} \left| \sum_{\beta \in S_2, r \neq 0} \chi(r(\det(\alpha, \beta) - i)) \right| \right)^2 \leq q^{-2} |S_1| \sum_{\alpha \in \mathbb{F}_q^2} \left| \sum_{\beta \in S_2, r \neq 0} \chi(r(\det(\alpha, \beta) - i)) \right|^2.
\]
Note that the rightmost term of this inequality is in turn equal to
\[ q^{-2}|S_1| \sum_{\alpha \in \mathbb{F}_q^2} \sum_{\beta, \beta' \in S_2, r, r' \neq 0} \chi(i(r' - r)) \chi(r \det(\alpha, \beta) - r' \det(\alpha, \beta')). \]

Next, we compute the sum over \( \alpha \in \mathbb{F}_q^2 \) by using the orthogonality of \( \chi \) and obtain
\[ |R(i)|^2 \leq |S_1| \sum_{\beta \in \mathbb{F}_q^2, r \neq 0} 1 + |S_1| \sum_{\beta, \beta' \in S_2, r, r' \neq 0: r \neq r', r \beta = r' \beta'} \chi(i(r' - r)). \]

Considering the cases that \( r = r' \) and \( r \neq r' \), we have
\[ |R(i)|^2 \leq q|S_1||S_2| + |S_1| \sum_{\beta, \beta' \in S_2, \alpha \neq 0, b \neq 0, 1: r / r' \neq 1, (r / r') \beta = \beta'} \chi(i r (1 - r/r')). \]

By a change of variables with \( a = r', b = r/r' \),
\[ |R(i)|^2 \leq q|S_1| + |S_1| \sum_{\beta, \beta' \in S_2, \alpha \neq 0, b \neq 0, 1} \chi(ia(1 - b)). \]

The second term in RHS of the inequality (5.1) is non-positive, because the sum over \( a \neq 0 \) is -1 by the orthogonality of \( \chi \). Hence, we obtain \( |R(i)|^2 \leq q|S| \), as required. □

5.1. Proof of Theorem 1.6.

Proof of Theorem 1.6. Since \( |E|, |F| \geq Cq^3 \), we see from Lemma 5.1 that \( |E \cap D_i| \sim |E|/q \) and \( |F \cap D_j| \sim |F|/q \). Since \( (E \cap D_i) \subseteq D_i \), and \( (F \cap D_j) \subseteq D_j \), the theorem follows from Theorem 1.4. □

6. Sum of two matrix sets

For \( E, F \subseteq M_2(\mathbb{F}_q) \), the sum set \( E + F \) is defined by
\[ E + F := \{ x + y \in M_2(\mathbb{F}_q) : x \in E, y \in F \}. \]

In this section, we shall give a ‘general’ lower bound for sizes of sets \( E + F \) when \( E \) and \( F \) are subsets \( D_i \) and \( D_j \) for nonzero \( i, j \in \mathbb{F}_q \). This result is one of main ingredients of the proof of Theorem 1.7 given in the next section.

Recall that \( N_t(E, F) \) denotes the number of pairs \( (x, y) \in E \times F \) such that \( \det(x + y) = t \).

Lemma 6.1. If \( E \subseteq D_i, F \subseteq D_j \) for nonzero \( i, j \in \mathbb{F}_q \), then we have
\[ \max_{t \in \mathbb{F}_q} N_t(E, F) \leq \frac{|E||F|}{q} + q|E|^{1/2}|F|^{1/2}. \]

Proof. From Proposition 3.7, we have
\[ \max_{t \in \mathbb{F}_q} N_t(E, F) \leq \frac{|E||F|}{q} + \sqrt{18q^2|E||F|} + 11|E||F|^2 + 4\sqrt{q}|E||F|^{3/2}. \]
Using the basic fact that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$, we obtain the estimate:

\begin{equation}
(6.1) \quad \max_{t \in \mathbb{F}_q} N_t(E, F) \ll \frac{|E||F|}{q} + q|E|^{\frac{1}{2}}|F|^{\frac{1}{2}} + |E|^{\frac{1}{2}}|F| + q^{\frac{3}{2}}|E|^{\frac{3}{2}}|F|^{\frac{1}{2}}.
\end{equation}

Switching roles of $E$ and $F$ in (6.1), we also obtain

\begin{equation}
(6.2) \quad \max_{t \in \mathbb{F}_q} N_t(E, F) \ll \frac{|E||F|}{q} + q|E|^{\frac{1}{2}}|F|^{\frac{1}{2}} + |E|^{\frac{1}{2}}|F| + q^{\frac{3}{2}}|E|^{\frac{3}{2}}|F|^{\frac{1}{2}}.
\end{equation}

For $1 \leq r \leq 4$, let $a_r$ be the $r$-th term in RHS of (6.1), and for $r = 3, 4$, $a'_r$ the $r$-th term in RHS of (6.2):

\begin{align*}
a_1 + a_2 + a_3 + a_4 &= \frac{|E||F|}{q} + q|E|^{\frac{1}{2}}|F|^{\frac{1}{2}} + |E|^{\frac{1}{2}}|F| + q^{\frac{3}{2}}|E|^{\frac{3}{2}}|F|^{\frac{1}{2}} \\
a_1 + a_2 + a'_3 + a'_4 &= \frac{|E||F|}{q} + q|E|^{\frac{1}{2}}|F|^{\frac{1}{2}} + |E|^{\frac{1}{2}}|F| + q^{\frac{3}{2}}|E|^{\frac{3}{2}}|F|^{\frac{1}{2}}.
\end{align*}

To prove the lemma, we consider two cases.

**Case 1:** Assume that $|E| \leq q^2$ or $|F| \leq q^2$. Indeed, if $|E| \leq q^2$, then it follows from (6.2) that $a_2 \geq a'_3$ and $a_2 \geq a'_4$. If $|F| \leq q^2$, then we see from (6.1) that $a_2 \geq a_3$ and $a_2 \geq a_4$. Thus, in this case, we have

\[ \max_{t \in \mathbb{F}_q} N_t(E, F) \ll a_1 + a_2. \]

**Case 2:** Assume that $|E| > q^2$ and $|F| > q^2$. It follows from (6.1) that $a_1 > a_3$ and $a_1 > a_4$. Hence, in this case we also have

\[ \max_{t \in \mathbb{F}_q} N_t(E, F) \ll a_1 + a_2. \]

This completes the proof. \hfill \Box

Recall that $W_t(E, F)$ denotes the number of pairs $(x, y) \in E \times F$ such that $x \odot y = t$. Note that if $E \subseteq D_i$ and $F \subseteq D_j$ for some $i, j \in \mathbb{F}_q$, then we have $W_t(E, F) = N_t(E, F)$, where $t = \ell + i + j$. Thus Lemma 6.1 can be restated as follows.

**Corollary 6.2.** Let $E, F$ be the sets given in Lemma 6.1. Then we have

\[ \max_{t \in \mathbb{F}_q} W_t(E, F) \ll \frac{|E||F|}{q} + q|E|^{\frac{1}{2}}|F|^{\frac{1}{2}}. \]

For any $E, F \subseteq M_2(\mathbb{F}_q)$, not necessarily contained in $D_i$ for some $i$, we produce an upper bound of $W_0(E, F)$, which will be also used in proving the main result of this section.

**Lemma 6.3.** Let $E, F \subseteq M_2(\mathbb{F}_q)$. Then we have

\begin{equation}
(6.3) \quad W_0(E, F) \leq \frac{|E||F|}{q} + \sqrt{2}q^2|E|^{\frac{1}{2}}|F|^{\frac{1}{2}}.
\end{equation}

**Proof.** We proceed as in the proof of Lemma 5.1. By the orthogonality of $\chi$, we can write

\[ W_0(E, F) = \sum_{x \in E, y \in F : x \odot y = 0} 1 = \frac{|E||F|}{q} + q^{-1} \sum_{x \in E, y \in F : s \neq 0} \chi(s(x \odot y)). \]

Let

\[ \Omega := q^{-1} \sum_{x \in E, y \in F : s \neq 0} \chi(s(x \odot y)). \]
Notice that to complete the proof of the lemma, it suffices to prove that
\[ |\Omega|^2 \leq 2q^4 |E||F|.\]
Let us bound $|\Omega|^2$. First, applying the Cauchy-Schwarz inequality to $|\Omega|^2$ w.r.t $x \in E$, and next replacing the index set “$x \in E$” by “$x \in M_2(\mathbb{F}_q)$”, we obtain
\[ |\Omega|^2 \leq q^{-2} |E| \sum_{x \in M_2(\mathbb{F}_q)} \left| \sum_{y \in F, y \neq y^*} \chi(x \odot (s \odot y)) \right|^2. \]
Note that the term of the RHS of this inequality is in turn equal to
\[ q^{-2} |E| \sum_{x \in M_2(\mathbb{F}_q)} \sum_{y, y' \in F, y \neq y'} \chi(x \odot (s \odot y' - s \odot y)). \]
Using the orthogonality of $\chi$ for the Odot-product to compute the sum over $x \in M_2(\mathbb{F}_q)$, we obtain
\[ |\Omega|^2 \leq q^2 |E| \sum_{y, y' \in F, s \neq s', sy = sy'} 1. \]
Considering the cases that $s = s'$ and $s \neq s'$, we have
\[ |\Omega|^2 \leq q^2 |E| \sum_{y, y' \in F, s \neq s', sy = sy'} 1 \leq q^3 |E||F| + q^2 |E| \sum_{y, y' \in F, s \neq s', sy = sy'} 1. \]
Whenever we fix $y \in F, s, s' \neq 0$, there is at most one $y' \in F$ such that $sy = s'y'$. Therefore,
\[ |\Omega|^2 \leq q^3 |E||F| + q^1 |E||F| \leq 2q^4 |E||F|, \]
as desired. \(\square\)

For two subsets $E, F$ of $M_2(\mathbb{F}_q)$, we denote by $\Lambda(E, F)$ the additive energy defined by
\[ \Lambda(E, F) := \left| \{(x, y, z, w) \in E \times F \times E \times F : x + y = z + w\} \right|. \]
The following proposition, whose proof will be given at the end of this section, plays a key role in the proof of Theorem 6.5 below.

**Proposition 6.4.** Assume that $E \subseteq D_i$ and $F \subseteq D_j$ for $i, j \neq 0$. Then we have
\[ \Lambda(E, F) \ll q^{-1} |E|^2 |F| + q |E||F| + q |E|^3/2 |F|^{1/2}. \]

The following is a main result of ours for the sum of two sets, whose proof heavily depends on Proposition 6.4

**Theorem 6.5.** Assume that $E \subseteq D_i$ and $F \subseteq D_j$ for $i, j \neq 0$. Then we have
\[ |E + F| \gg \min \left\{ q |F|, \left( \frac{|E||F|}{q} \right), \frac{|E|^{1/2} |F|^{3/2}}{q} \right\}. \]

**Proof.** From the Cauchy-Schwarz inequality, it follows that
(6.4) \[ |E + F| \geq \frac{|E|^2 |F|^2}{\Lambda(E, F)}. \]
By Proposition 6.4, we have
\[ |E + F| \gg \frac{|E|^2 |F|^2}{q^{-1} |E|^2 |F| + q |E||F| + q |E|^{3/2} |F|^{1/2}}. \]
Then from this inequality, the proposition is immediate. \(\square\)

In fact, in Theorem 6.5, if we know which one of \(E\) and \(F\) is larger than the other, then we can give a simpler statement.

**Corollary 6.6.** For \(i, j \in \mathbb{F}_q^*\), let \(E \subseteq D_i\) and \(F \subseteq D_j\). Suppose, say, \(|F| \geq |E|\). Then, we have

\[
|E + F| \gg \min \left\{ q|F|, \frac{|E||F|}{q} \right\}.
\]

**Proof.** Since \(|F| \geq |E|\), we see that \(|E|/q \leq |E|^{1/2}|F|^{3/2}/q\). Hence, the corollary follows immediately from Theorem 6.5. \(\square\)

6.1. **Proof of Proposition 6.4.** Here we give a proof of Proposition 6.4. We begin by giving a simple lemma.

**Lemma 6.7.** Let \(X\) be a finite set, and \(X = \bigcup_{k=1}^m X_k\) be a partition on \(X\) with \(a := |X_k| = |X_\ell|\) for all \(k, \ell\). If \(Y\) is a subset of \(X\) such that \(Y \cap X_k \neq \emptyset\) for any \(k = 1, \ldots, m\), then the cardinality of \(Y\) is bounded by

\[
\frac{|X|}{a} \leq |Y| \leq \frac{b|X|}{a},
\]

where \(b := \max_{1 \leq k \leq m} \{|Y \cap X_k|\}\).

**Proof.** Notice that \(m = |X|/a\) is the number of members of the partition. Since \(Y \cap X_k \neq \emptyset\) for any \(k = 1, \ldots, m\), we have \(m \leq |Y|\). Since \(|Y \cap X_k| \leq b\) for all \(k = 1, \ldots, m\) and \(|Y| = \sum_{1 \leq k \leq m} |Y \cap X_k|\), we have \(|Y| \leq mb\). This proves the lemma. \(\square\)

Lemma 6.7 is useful when we want to obtain a bound on the cardinality of a set \(Y\) in question. It is enough to find a larger set \(X\) allowing an embedding \(Y \hookrightarrow X\) of sets satisfying the conditions in the lemma. Indeed, we will use this lemma at the last moment to complete the proof of Proposition 6.4 below.

**Proof of Proposition 6.4.** Since \(F \subseteq D_j\), we can write

\[
\Lambda(E, F) \leq \sum_{x, z \in E, y \in F : \det(x+y-z) = j} 1 = \sum_{x, z \in E, y \in F : (x+y) \odot (x-z) = 0} 1.
\]

Here the equality in (6.5) follows from the equivalence of two conditions: for \(x, z \in E \subseteq D_i, y \in F \subseteq D_j\),

\[
\det(x + y - z) = j \Leftrightarrow (x + y) \odot (x - z) = 0.
\]

To make the computation easy, we split the RHS of (6.5) into two summands:

\[
\sum_{x, z \in E, y \in F : (x+y) \odot (x-z) = 0} 1 = I + II,
\]

where \(I\) denotes the sum over \(x, y, z\) with \(\det(x + y) = 0\) or \(\det(x - z) = 0\), and \(II\) the sum over \(x, y, z\) with \(\det(x + y) \neq 0\) and \(\det(x - z) \neq 0\). Let us bound \(I\) and \(II\) separately.

For \(I\), the following is obvious.

\[
I \leq \sum_{x, z \in E, y \in F : \det(x+y) = 0} 1 + \sum_{x, z \in E, y \in F : \det(x-z) = 0} 1
\]
Lemma 6.1 directly gives a bound on the first sum in (6.6). To bound the second sum, notice that since \( E \) is a subset of \( D_i \), \(-E\) is also contained in \( D_i \) and \(|E| = |-E|\). Thus Lemma 6.1 is also applicable to the second sum. Therefore, we have obtained

\[
I \ll |E|(q^{-1}|E||F| + |q|E|^{1/2}|F|^{1/2}) + |F|(q^{-1}|E|^2 + |q||E|)
\]

\[
\ll q^{-1}|E|^{3/2}|F|^{1/2}.
\]

Next, we bound \( II \). Recall that

\[
II = \sum_{x,z \in E, y \in F: (x+y) \cap (x-z) = 0} 1 = \sum_{x \in E} \left( \sum_{z \in E, y \in F: (x+y) \cap (x-z) = 0} 1 \right).
\]

Fix \( x \in E \), and let \( \beta = x - y \) and \( \alpha = x - z \). Then we see that

\[
II = \sum_{x \in E} \left( \sum_{\alpha \in (-E+x), \beta \in (F+x): \alpha \odot \beta = 0, \det(\alpha) \neq 0 \neq \det(\beta)} 1 \right),
\]

where \(-E + x := \{ -e + x : e \in E \}\) and \( F + x := \{ f + x : f \in F \}\). Let \( II(x) \) be the sum in the bracket in (6.7); namely,

\[
II(x) := \sum_{\alpha \in (-E + x), \beta \in (F + x): \alpha \odot \beta = 0, \det(\alpha) \neq 0 \neq \det(\beta)} 1.
\]

Now for each \( x \in E \) we bound \( II(x) \).

For a nonzero vector \( \gamma \in M_2(F_q) \), let \( [\gamma] \) be the one dimensional subspace (i.e., the line) in \( M_2(F_q) \) generated by \( \gamma \) and \( [\gamma]^* := [\gamma] \setminus \{0\} \). For \( H \subseteq M_2(F_q) \), let

\[ H_x := \{ s \alpha : s \in F_q^*, \alpha \in H + x, \det(\alpha) \neq 0 \}. \]

In other words, \( H_x \) is the union of all \([\gamma]^*\) with \( \gamma \in H + x, \det(\gamma) \neq 0\). Notice that for \((\alpha, \beta) \in (-E + x) \times (F + x)\), \( \alpha \odot \beta = 0 \) iff \((s \alpha) \odot (t \beta) = 0\) for all \( s, t \in F_q^*\), and for any \( \gamma \in M_2(F_q) \), \( \det(\gamma) \neq 0 \) iff \( \det(s \gamma) \neq 0 \) for any nonzero \( s \in F_q \). We claim that for every \( x \in E \), we have

\[
II(x) = \sum_{\alpha \in (-E+x), \beta \in (F+x): \alpha \odot \beta = 0, \det(\alpha) \neq 0 \neq \det(\beta)} 1 \sim q^{-2} \sum_{u \in (-E)_x, v \in F_q : u \odot v = 0} 1.
\]

To prove the claim, we use Lemma 6.7. Let \( Y \) be the index set of the first summation in (6.8) which we want to count, and \( X \) the index set of the second summation, i.e.,

\[ X = \bigcup_{(\alpha, \beta) \in Y} [\alpha]^* \times [\beta]^*, \]

where we take the ordinary (not necessarily disjoint) union of sets. Obviously we have a natural embedding \( Y \hookrightarrow X \), \((\alpha, \beta) \mapsto (\alpha, \beta)\). For the remaining conditions in Lemma 6.7, it is enough to show that for any \((\alpha, \beta) \in Y\),

\[ 1 \leq Y \cap ([\alpha]^* \times [\beta]^*) \leq 4. \]
The inequality $1 \leq Y \cap ([\alpha]^* \times [\beta]^*)$ is trivially true. For the other inequality, it is enough to show two inequalities

\begin{equation}
(6.9) \quad ||\alpha|^* \cap (-E + x)| \leq 2, \quad ||\beta|^* \cap (F + x)| \leq 2.
\end{equation}

We only prove the first one in (6.9) (in fact, the proof below works for the second inequality.) First, notice that $-E \subseteq D_i$ since $E \subseteq D_i$, and so $-E + x \subseteq D_i + x$. Since $||\alpha|^* \cap (-E + x)| \leq ||\alpha|^* \cap (D_i + x)|$, it suffices to show

$$||\alpha|^* \cap (D_i + x)| \leq 2.$$ 

Note that for a (fixed) $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in E$, the variety $D_i + x$ is defined by the equation

$$(z_1 - x_1)(z_4 - x_4) - (z_2 - x_2)(z_3 - x_3) = i,$$

where $z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \in M_2(\mathbb{F}_q)$.

Therefore, for $\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$, an element $s\alpha \in [\alpha]^*$ for $s \in \mathbb{F}_q^*$ lies in the variety $[\alpha]^* \cap (D_i + x)$ if and only if $s$ satisfies

$$(s\alpha_1 - x_1)(s\alpha_4 - x_4) - (s\alpha_2 - x_2)(s\alpha_3 - x_3) = i;$$

equivalently,

\begin{equation}
(6.10) \quad (\alpha_1\alpha_4 - \alpha_2\alpha_3)s^2 + (\alpha_2x_3 + \alpha_3x_2 - \alpha_1x_4 - \alpha_4x_1)s + x_1x_4 - x_2x_3 - i = 0.
\end{equation}

In other words, the number $||[\alpha]^* \cap (D_i + x)|$ is equal to the number of solutions to this equation (6.10) for $s$. Since $\det(\alpha) \neq 0$, this equation is quadratic, and so it has at most two solutions. Thus we have

$$||[\alpha]^* \cap (D_i + x)| \leq 2,$$

as desired. Hence, the inequality (6.9) holds. Note that the number of the above partitions on $X$ is equal to $\frac{X}{(q - 1)^2}$. From Lemma 6.7, it follows that

$$\frac{|X|}{(q - 1)^2} \leq II(x) \leq 4 \frac{|X|}{(q - 1)^2}.$$ 

This proves the claim (6.8).

Now we are ready to bound $II(x)$ in (6.8). It is clear that

$$|(-E)_x| \sim q|(-E + x)|, \quad |F_x| \sim q|(F + x)|$$

for all $x \in E$.

Applying the inequality (6.3) in Lemma 6.3, we see that for every $x \in E$,

$$II(x) \ll q^{-2}\left(\frac{|(-E)_x||F_x|}{q} + q^2|(-E)_x|^{1/2}|F_x|^{1/2}\right)$$

$$\sim \frac{|E||F|}{q} + q|E|^{1/2}|F|^{1/2}.$$ 

Summing over $x \in E$,

$$II \ll \frac{|E|^2|F|}{q} + q|E|^{3/2}|F|^{1/2}.$$ 

To conclude,

$$\Lambda(E, F) \leq I + II \ll (q^{-1}|E|^2|F| + q|E||F| + q|E|^3|F|^{1/2}) + (q^{-1}|E|^2|F| + q|E|^3|F|^{1/2})$$
\[ q^{-1}|E|^2|F| + q|E||F| + q|E|^{3/2}|F|^{1/2}, \]
as desired. \( \square \)

7. Determinants of finitely iterated sum sets (Proof of Theorem 1.7)

As we will see, the proof of Theorem 1.7 uses some other results as well as Corollary 6.6. We will list them below. The following result was given by Li and Su [31] by using Fourier techniques. A graph theoretic proof was recently given by Demirogly Karabulut [5]. For the sake of completeness, we will include a short proof in Appendix.

Proposition 7.1 ([31, 5]). Let \( E, F \subseteq M_2(\mathbb{F}_q) \). If \(|E||F| > 4q^5\), then we have
\[ \det(E + F) \supseteq \mathbb{F}_q^* \]

The following result is an immediate consequence from Corollary 6.6 for the balance case.

Lemma 7.2. For \( i \in \mathbb{F}_q^* \), let \( E \subseteq D_i \). Then we have
\[ |E + E| \gg \min \left\{ q|E|, \frac{|E|^2}{q} \right\}. \]

We note that Corollary 6.6 only gives us the lower bound when \( F \) is a set in \( D_j \) for some \( j \in \mathbb{F}_q^* \). To make the inductive argument in the proof of Theorem 1.7 below work, we also need the following result from [8] in the case when \( F \) is an arbitrary set in \( M_2(\mathbb{F}_q) \). We refer readers to [8] for a detailed proof using spectrum of the unit-special Cayley graph.

Lemma 7.3 (Proof of Corollary 1.7, [8]). For \( i \in \mathbb{F}_q^* \), let \( E \) be a set in \( D_i \), and \( F \) be a set in \( M_2(\mathbb{F}_q) \). Then we have
\[ |E + F| \gg \min \left\{ q|E|, \frac{|E|^2|F|}{q^2} \right\}. \]

It is worth noting that the bound in Corollary 6.6 is stronger than that of Lemma 7.3 whenever \(|E| \leq q^2\). Another key ingredient in proving Theorem 1.7 is the following lemma whose proof is based on an induction argument with Lemma 7.2 and Lemma 7.3.

Lemma 7.4. Let \( k \geq 2 \) be an integer and \( i \) be an element in \( \mathbb{F}_q^* \). Let \( E \) be a set in \( D_i \) with \(|E| \geq Cq^2\) for a sufficiently large constant \( C \). We have
\[ |kE| \gg \min \left\{ q|E|, \frac{|E|^{2k-2}}{q^{3k-5}} \right\}. \]

Proof. The proof proceeds by induction on \( k \). Suppose \( k = 2 \). Then Lemma 7.2 gives us
\[ |E + E| \gg \min \left\{ q|E|, \frac{|E|^2}{q} \right\}. \]
Thus the base case follows. Suppose that the theorem holds for any \( k - 1 \geq 2 \). We now show that it also holds for \( k \). Indeed, by inductive hypothesis, we have
\[ |(k - 1)E| \gg \min \left\{ q|E|, \frac{|E|^{2k-4}}{q^{3k-8}} \right\}. \]
Applying Lemma 7.3 with \( F = (k - 1)E \), we have
\[
|kE| \gg \min \left\{ \frac{|E|}{q}, \frac{|E|^3}{q^2}, \frac{|E|^{2k - 2}}{q^{3k - 5}} \right\} \gg \min \left\{ \frac{|E|}{q}, \frac{|E|^{2k - 2}}{q^{3k - 5}} \right\},
\]
since \( |E| \geq C q^2 \). This concludes the proof of Lemma 7.4. \( \square \)

**Proof of Theorem 1.7**: From Lemma 7.4, we see that one of the following cases happens.

**Case 1**: If \( |kE| \gg q|E| \), then by applying Proposition 7.1 for the set \( kE \), we have
\[
\det(2kE) \supseteq \mathbb{F}_q^*,
\]
whenever \( |E| \geq C q^\frac{3}{2} \).

**Case 2**: If \( |kE| \gg \frac{|E|^{2k - 2}}{q^{3k - 5}} \), then we apply Proposition 7.1 again to obtain
\[
\det(2kE) \supseteq \mathbb{F}_q^*,
\]
whenever \( |E| \geq C q^{\frac{6k - 5}{4k - 4}} \).

This completes the proof of the theorem. \( \square \)

8. **Appendix**

In this appendix, we give an alternative proof of Proposition 7.1. We begin by proving a preliminary lemma below.

**Lemma 8.1**. For \( t \in \mathbb{F}_q, m \in M_2(\mathbb{F}_q) \), we have
\[
\hat{D}_t(m) = \frac{\delta_0(m)}{q} + \frac{1}{q^3} \sum_{s \neq 0} \chi \left( -st - \frac{||m||_*}{s} \right),
\]
where \( \delta_0(m) = 1 \) if \( m = 0 \), and 0 otherwise.

**Proof**. By the orthogonality of \( \chi \), we have
\[
\hat{D}_t(m) = q^{-4} \sum_{x \in M_2(\mathbb{F}_q): ||x||_s = t} \chi(-m \cdot x) = q^{-5} \sum_{x \in M_2(\mathbb{F}_q)} \sum_{s \in \mathbb{F}_q} \chi(s(||x||_* - t))\chi(-m \cdot x)
\]
\[
= \frac{\delta_0(m)}{q} + q^{-5} \sum_{s \neq 0} \sum_{x \in M_2(\mathbb{F}_q)} \chi(s(||x||_* - t))\chi(-m \cdot x)
\]
\[
= \frac{\delta_0(m)}{q} + q^{-5} \sum_{s \neq 0} \chi(-st) \left( \sum_{x_1, x_4 \in \mathbb{F}_q} \chi((sx_4 - m_1)x_1 - m_4x_4) \right) \left( \sum_{x_2, x_3 \in \mathbb{F}_q} \chi((-sx_3 - m_2)x_2 - m_3x_3) \right)
\]

Using the orthogonality of \( \chi \) again, we compute the sums over \( x_1, x_3 \in \mathbb{F}_q \). Then we see that
\[
\hat{D}_t(m) = \frac{\delta_0(m)}{q} + q^{-3} \sum_{s \neq 0} \chi(-st)\chi \left( -\frac{||m||_*}{s} \right),
\]
which completes the proof. \( \square \)

**Proof of Proposition 7.1**. To complete the proof, it will be enough to show that if \( |E||F| > 4q^5 \), then \( N_t(E, F) > 0 \) for all \( t \in \mathbb{F}_q^* \). We proceed as in [25]. By definition,
\[
N_t(E, F) = \sum_{x \in E, y \in F: \det(x + y) = t} 1 = \sum_{x \in E, y \in F} D_t(x + y).
\]
Applying the Fourier inversion theorem to the function \( D_t(x+y) \) and using the definition of the Fourier transform, we see that

\[
N_t(E, F) = q^{\frac{1}{2}} \sum_{m \in M_2(\mathbb{F}_q)} \hat{D}_t(m) \hat{E}(m) \hat{F}(m). \tag{8.2}
\]

Combining (8.2) with (8.1), we get

\[
N_t(E, F) = \frac{|E||F|}{q} + R(t),
\]

where \( R(t) \) is given by

\[
R(t) = q^{\frac{3}{2}} \sum_{m \in M_2(\mathbb{F}_q)} \left( \sum_{s \neq 0} \chi\left( -st - \frac{\|m\|}{s} \right) \right) \hat{E}(m) \hat{F}(m).
\]

For \( t \neq 0 \), the sum over \( s \neq 0 \) is the Kloosterman sum whose absolute value is less than or equal to \( 2\sqrt{q} \). Thus we have

\[
N_t(E, F) \geq \frac{|E||F|}{q} - |R(t)| \geq \frac{|E||F|}{q} - 2q^{3/2} \sum_{m \in \mathbb{F}_q^3} |\hat{E}(m)||\hat{F}(m)|
\]

\[
\geq \frac{|E||F|}{q} - 2q^{3/2}|E|^{1/2}|F|^{1/2},
\]

where the last inequality follows from the Cauchy-Schwarz inequality and the Plancherel theorem. Thus \( N_t(E, F) > 0 \), provided that \( |E||F| > 4q^5 \). This completes the proof. \( \square \)

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