Finite Disks with Power-Law Potentials

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ABSTRACT

We describe a family of circular, and elliptical, finite disks with a disk potential that is a power of the radius. These are all flattened ellipsoids, obtained by squashing finite spheres with a power-law density distribution, cutoff at some radius $R_o$. First we discuss circular disks whose circular rotation speed $v \propto r^{\alpha}$, with any $\alpha > -1/2$. The surface-density of the disks is expressed in terms of hypergeometric functions of the radius: $\Sigma(\alpha)(r) \propto (r/R_o)^{2(\alpha - 1)}[1 - (r/R_o)^2]^{1/2} F_1(1 - \alpha, 1/2; 3/2; 1 - R_o^2/r^2)$. The disk then has the same rotation curve as the sphere within the cutoff radius (up to a constant factor). We give closed expressions for the full 3-D potentials in terms of hypergeometric functions of two variables. We express the potential and acceleration in the plane at $r > R_o$, and along the rotation axis, in terms of simple hypergeometric functions. All the multipoles of the disk are given. We then generalize to non-axisymmetric disks with midplane axis ratio $p < 1$. The potential in the midplane is given by $\varphi(r, \phi) \propto r^{2\alpha} F_1(\alpha + 1/2, -\alpha, \alpha + 1/2, \alpha + 1; \tau \sin^2 \phi; \tau) + c$, where $F_1$ is the hypergeometric function of two variables, and $\tau = 1 - p^2$. For integer values of $2\alpha$ the above quantities are given in more elementary terms. All these results follow straightforwardly from formulae we derive for the general, cutoff, power-law, triaxial ellipsoid.

Subject headings: Disk galaxies, potentials

1. Introduction

Models of disk galaxies with explicit expressions for the disk potential and for the surface density are quite useful in studies of galactic structure, for example as initial states for studies of stability, spiral-arm formation etc.. Infinite-extent, power-law disk models—with power-law rotation curves: $v(r) \propto r^{\alpha}$, and power-law surface densities: $\Sigma(r) \propto r^{2\alpha-1}$—have their uses in this context (e.g Binney and Tremaine 1987, Evans and de Zeeuw 1992, Evans 1994). They are, however, rather unrealistic. Qian 1992 describes an even wider class of disks with known surface densities and potentials in the plane of the disk; these too are of infinite extent. We hope to contribute to the subject by describing a family of finite disks with power-law potentials (within
the material disk). These are produced by squashing spheres with a power-law density profile, cutoff at some radius $R_o$, into ellipsoids (disks in the limiting case), the rotation curve in the equatorial plane, within $R_o$, is then the same as that of the sphere (up to a constant factor).

Milgrom 1989 used this method to determine the surface density of a constant-acceleration (isodynamic) disk with $v \propto r^{1/2}$. We describe here the general case since, as far as we know, this useful class of models does not seem to be part of the known lore. In the Appendix we derive expressions for the gravitational field of a cutoff, power-law, triaxial ellipsoid. From these follow, after some straightforward manipulations, all the results of the paper. In section 2 we discuss thin, and thick axisymmetric disks; in section 3 we derive their full 3-D potentials; and in section 4 we describe the generalization to non-axisymmetric disks, which may also model galactic bars.

2. Axisymmetric disk models

Start with a spherical mass distribution, with density

$$\rho = AR^\gamma,$$

(1)
cutoff at $R = R_o$. Now squash the configuration in one direction, by a constant factor $q < 1$, to obtain an ellipsoidal mass distribution. The gravitational acceleration, $g(r)$, at a point in the midplane, a distance $r \leq R_o$ from the centre, in the equatorial plane, is determined only by the mass interior to the ellipsoid going through $r$ (by “Newton’s third theorem”; see e.g. Binney and Tremaine 1987). In particular, $g(r)$ is independent of the cutoff radius $R_o$ (provided $R_o \geq r$). The only dimensional parameter characterizing the system, on which $g(r)$ may depend is then $A$, to which, in fact, $g$ must be proportional. On dimensional grounds, the only form that $g(r)$ can have is then

$$g(r) = -QAGr^{\gamma+1},$$

(2)
where $Q$ is a dimensionless constant that depends on the axis ratio and on $\gamma$. This argument generalizes to non-axisymmetric disks, and indeed to the interior field of a cutoff, power-law, triaxial ellipsoid (see section 4 and the Appendix). We see that the equatorial rotation curves of this family of ellipsoidal mass distributions is a power-law within the cutoff radius, with $v(r) = (QAG)^{1/2}r^\alpha$, $\alpha \equiv 1 + \gamma/2$. Beyond $R_o$, $g(r)$ does depend on $R_o$, and the above dimensional argument breaks down, as the expression for $g(r)$ may now contain a factor $f(r/R_o)$.

The surface density is just that of the spherical distribution

$$\Sigma^\alpha (r) = 2A \int_0^{(R_o^2 - r^2)^{1/2}} (r^2 + x^2)^{\gamma/2} dx.$$

(3)
Defining $u \equiv r/R_o$, $s \equiv [(R_o/r)^2 - 1]^{1/2} = (u^{-2} - 1)^{1/2}$, and $\tilde{\Sigma} \equiv 2AR_o^{\gamma+1}$, we can write after a change of variable

$$\Sigma^\alpha (r) = \frac{1}{2} \tilde{\Sigma} u^{\gamma+1} s \int_0^1 \lambda^{-1/2}(1 + s^2\lambda)^{\gamma/2} d\lambda.$$

(4)
This, in turn, can be written in terms of the hypergeometric function
\[ \Sigma^\alpha(r) = \hat{\Sigma} u^{2(\alpha-1)}(1 - u^2)^{1/2} _2F_1(1 - \alpha, 1/2; 3/2; -s^2), \] (5)
(see Gradshteyn and Ryzhik 1980 3.197.3). We see that near the disk’s edge, where \( s \approx 0 \), and \( _2F_1 \approx 1 \), \( \Sigma^\alpha(r) \approx \hat{\Sigma}(1 - u^2)^{1/2} \), for all the disks.

For integer values of \( 2\alpha \) the integrals in eqs.(3)(4) can be expressed in terms of elementary functions. Some interesting special cases are: \( \alpha = 1 \)–which in the thin-disk limit gives the Kalnajs disk (Kalnajs 1972)–has
\[ \Sigma(r) = \hat{\Sigma}(1 - u^2)^{1/2}; \] (6)
\( \alpha = 0 \)–which is the Mestel disk (Mestel 1963) in the same limit–has
\[ \Sigma(r) = \hat{\Sigma} u^{-1} \tan^{-1}(u^2 - 1)^{1/2}; \] (7)
and the isodynamic disk (Milgrom 1989) with constant acceleration (\( \alpha = 1/2 \)) has
\[ \Sigma(r) = \hat{\Sigma} \sinh^{-1}(u^2 - 1)^{1/2}. \] (8)
The limit \( \alpha \to -1/2 \) corresponds to a point mass at the origin (a Keplerian rotation curve), and smaller values of \( \alpha \) are excluded as they correspond to an infinite total mass.

For \( \alpha \) a positive integer, \( _2F_1 \) in eq.(3) is a polynomial of order \( \alpha - 1 \) in \( s^2 \), and so, \( \Sigma(r) \) is, up to the factor \( (1 - u^2)^{1/2} \), a polynomial of (even) order \( 2(\alpha - 1) \) in \( r \).

Values of \( \alpha > 1 \) correspond to disks with non-monotonic surface densities: for \( 1 < \alpha < 3/2 \) \( \Sigma(r) - \Sigma(0) \propto r^{2(\alpha-1)} \) near the origin; for \( \alpha = 3/2 \) \( \Sigma(r) - \Sigma(0) \propto -r^2 \ln r \) there; and for \( \alpha > 3/2 \) \( \Sigma(r) - \Sigma(0) \propto r^2 \). For all these cases \( \Sigma \) is then increasing near the origin, but decreases further out.

The proportionality constant, \( Q \), in expression(3) for \( g \) may be obtained by specializing results from Appendix A.
\[ Q(\kappa, \alpha) = \frac{4\pi}{2\alpha + 1} _2F_1(1/2, \alpha + 1/2; \alpha + 3/2; \kappa), \] (9)
where \( \kappa \equiv 1 - q^2 \).

The interior potential for \( R \leq R_o[1 + (\kappa/q^2) \cos^2 \theta]^{-1/2} \) (for \( \gamma \neq -2 \)) as a function of the polar coordinates \( R \) and \( \theta \) is
\[ \varphi(R, \theta) = \frac{2\pi AG}{\alpha(2\alpha + 1)} R^{2\alpha+1} F_1(\alpha + 1/2, -\alpha, \alpha + 1/2; \alpha + 3/2; \kappa \sin^2 \theta, \kappa) + c, \] (10)
Where \( F_1 \) is the first Appell function, or the first hypergeometric function of two variables (see Gradshteyn and Ryzhik 1980 3.211), and \( c = -2\pi \alpha^{-1} AG R_o^{2\alpha} \kappa^{-1/2} \sin^{-1} \kappa^{1/2} \).
Hereafter we concentrate on infinitely thin disks, for which the results may be put in simpler terms. For these

\[ Q = Q_0(\alpha) = 2\pi^{3/2}\Gamma(\alpha + 1/2)/\Gamma(\alpha + 1). \]  

(11)

The potential in the disk is given (except for the Mestel case) by

\[ \varphi(r) = \frac{Q_0 AGr^{2\alpha}}{2\alpha} + c. \]  

(12)

The constant \( c \) can be determined from the potential outside the disk treated in section 3:

\[ c = \frac{\pi^2 AGR_o^{2\alpha}}{\alpha} = -\frac{MG \pi(\alpha + 1/2)}{2\alpha}, \]  

(13)

where

\[ M = 4\pi AR_o^{(2\alpha+1)/(2\alpha + 1)} \]  

is the total mass of the disk. Note that \( \varphi \), unlike \( g \), does depend on \( R_o \), in general.

The total energy on a circular orbit at radius \( r \leq R_o \) may now be written

\[ E(r) = -\frac{MG \pi(\alpha + 1/2)}{2\alpha} \left[ 1 - \left( \frac{r}{R_o} \right)^{2\alpha} (\alpha + 1/2) \right]. \]  

(15)

This energy is negative everywhere in the disk for \( \alpha < 1 \), vanishes just at the edge, for the Kalnajs case, and is positive in some region near the edge for \( \alpha > 1 \), viz. for \( r > \eta(\alpha)R_o \), where \( \eta(\alpha) = [(\alpha + 1)\Gamma(\alpha + 1/2)/\pi^{1/2}\Gamma(\alpha + 1)]^{-1/2\alpha} \). It can be shown that \( \eta < 1 \) for all \( \alpha > 1 \), and we find numerically that it has a minimum of \( \approx 0.96 \), at \( \alpha \approx 4.37 \); it is thus only in a narrow region near the edge that \( E(r) > 0 \). We have seen that \( \alpha > 1 \) disks have nonmonotonic surface densities. Such disks may not be so realistic models by themselves; they are useful as components when we expand the potential of more general disks. We may find advantage, e.g., in disks whose potential is a finite sum of power laws, some possibly with \( \alpha > 1 \).

As an example, consider two of our disks with \( \alpha_1 < \alpha_2 \), but with the same \( R_o \), and the same \( \Sigma = 2AR_o^{2\alpha-1} \) (the values of \( A \) are different). The corresponding, unsquashed spheres then have the same density at \( R_o \). We see from expression (3) for the surface density that, for a fixed \( \Sigma \), \( \Sigma^\alpha(r) \) is a monotonically decreasing function of \( \alpha \) for all \( r < R_o \). Thus

\[ \Sigma^{\alpha_1\alpha_2} = \Sigma^{\alpha_1} - \Sigma^{\alpha_2} \]  

(16)

is positive everywhere. The acceleration in such a disk is

\[ g = -\Sigma G\pi^{3/2} \left[ \frac{\Gamma(\alpha_1 + 1/2)}{\Gamma(\alpha_1 + 1)} u^{2\alpha_1-1} - \frac{\Gamma(\alpha_2 + 1/2)}{\Gamma(\alpha_2 + 1)} u^{2\alpha_2-1} \right]. \]  

(17)

\((u = r/R_o)\), and is directed inward everywhere, because \( u^{2\alpha-1}\Gamma(\alpha + 1/2)/\Gamma(\alpha + 1) \) is decreasing with \( \alpha \). Furthermore, \( \Sigma^{\alpha_1\alpha_2}(r) \) vanishes at the edge faster than \((1 - u^2)^{1/2} \); in fact, there

\[ \Sigma^{\alpha_1\alpha_2}(r) \approx \frac{2}{3} \Sigma(\alpha_2 - \alpha_1)(1 - u^2)^{3/2}, \]  

(18)
which can be seen by taking \( _2F_1 \) to first order in \( s^2 \) in eq.(5). We can get disks that vanish even faster at the edge by taking the difference of two such disks with different pairs of \( \alpha \)'s with the same difference. In the limit \( \alpha_2 \to \alpha_1 \) we get a disk whose surface density is the derivative of \( \Sigma^\alpha(r) \) with respect to \( \alpha \) at a fixed \( \hat{\Sigma} \).

3. The field outside the axisymmetric thin disk

Building on formulae in Appendix A we get for the potential at a general position outside the disk (for \( \gamma \neq -2 \) the Mestel disk requires separate treatment)

\[
\varphi(r, z) = -\frac{MG}{R_o} \hat{\varphi}(r/R_o, z/R_o),
\]

where the dimensionless potential \( \hat{\varphi} \), as a function of the dimensionless variables \( \varrho \equiv r/R_o, \zeta \equiv z/R_o \), is

\[
\hat{\varphi}(\varrho, \zeta) = \frac{\alpha + 1/2}{\alpha} \left[ \sin^{-1}(1 + L)^{-1/2} - \frac{1}{2} \int_0^W \frac{t^{-1/2}(\Delta^2 - \zeta^2 t)^\alpha}{(1 - t)^{\alpha-1/2}} \, dt \right].
\]

Here, \( \Delta \) is the dimensionless polar radius \( \Delta^2 \equiv \zeta^2 + \varrho^2 \),

\[
L = \frac{1}{2} \{ \Delta^2 - 1 + [(\Delta^2 - 1)^2 + 4\zeta^2]^{1/2} \},
\]

(\( L \) is \( \omega(1, \vec{R}) \) of appendix A, with \( Y = R_o \), and \( W \equiv 1/(1 + L) \). The integral in eq.(24) may be identified, after some manipulations, as an integral representation of the first Appell function. We thus get finally

\[
\hat{\varphi}(\varrho, \zeta) = \frac{\alpha + 1/2}{\alpha} \sin^{-1}(1 + L)^{-1/2} - \frac{1}{2\alpha} \Delta^{2\alpha} W^{\alpha+1/2} F_1(\alpha + 1/2, -\alpha, \alpha + 1/2, \alpha + 3/2; U, W),
\]

where

\[
U \equiv 1 - L/\Delta^2.
\]

For all non-zero integer values of \( 2\alpha \) the integral in eq.(24) can be performed and expressed in terms of elementary functions. One case in point is the Kalnajs disk which is just an homogeneous ellipsoid, for which the potential is given in McMillan 1958 or in Binney and Tremaine 1987.

Another instance is that of the constant-acceleration disk where, integrating explicitly, we get for the dimensionless 3-D potential

\[
\hat{\varphi}(\varrho, \zeta) = 2L^{1/2} - 2\Delta + 2 \sin^{-1}(1 + L)^{-1/2} + |\zeta| \ln \left\{ \frac{(L^{1/2} - |\zeta|)(\Delta + |\zeta|)}{(L^{1/2} + |\zeta|)(\Delta - |\zeta|)} \right\}.
\]
For the Mestel case with $\alpha = 0$, eqs.\((20)\) and \((22)\) give 0/0, and it requires special treatment, which does not seem to yield a simple closed expression. In general, all expressions for the accelerations hold for the $\alpha = 0$ case; those for potentials and energies require either a separate derivation, or the application of the L’Hospital rule. For example, we can use this rule with eq.\((22)\) to obtain for the Mestel case

$$
\hat{\phi}(\hat{\varrho}, \zeta) = \frac{-1}{2}(\ln \Delta^2 + \ln W)\sin^{-1}(1 + L)^{-1/2} - \frac{1}{2}W^{1/2}\frac{\partial}{\partial\alpha}F_1(\alpha + 1/2, -\alpha, \alpha + 1/2, \alpha + 3/2; U, W)|_{\alpha=0}.
$$

(25)

The integral in eq.\((20)\), for $\alpha \neq 0$, can also be expressed in terms of simpler functions when we confine ourselves to the plane ($z = 0$, $r > R_o$), or to the symmetry axis ($r = 0$). We then get for the dimensionless potential for the former case

$$
\hat{\phi}(\hat{\varrho} > 1, 0) = \frac{2\alpha + 1}{2\alpha} \sin^{-1} \frac{\varrho}{\hat{\varrho}} - \frac{1}{2\alpha\hat{\varrho}} \ _2F_1\left(1/2, \alpha + 1/2; \alpha + 3/2; \varrho^{-2}\right);
$$

(26)

along the axis we have

$$
\hat{\phi}(0, \zeta) = \frac{2\alpha + 1}{2\alpha} \sin^{-1}(1 + \zeta^2)^{-1/2} - \frac{1}{2\alpha|\zeta|} \ _2F_1\left(1, \alpha + 1/2; \alpha + 3/2; -\zeta^{-2}\right).
$$

(27)

[Equations\((26),(27)\) can also be obtained as special cases of eq.\((22)\); for $\zeta = 0$ we have $U = W$, and for $\varrho = 0$ we have $U = 0$; in both cases $F_1$ can be expressed as a hypergeometric function of one variable.]

The acceleration in the plane can be brought to the form (valid also for $\alpha = 0$)

$$
g(r \geq R_o, z = 0) = -\frac{MG}{r^2} \ _2F_1\left(1/2, \alpha + 1/2; \alpha + 3/2; R_o^2/r^2\right). \tag{28}
$$

For $r \to \infty$ $\ _2F_1 \to 1$, and we get the Keplerian expression. At the disk’s edge $\ _2F_1(1/2, \alpha + 1/2; \alpha + 3/2; 1) = \pi^{1/2}(\alpha + 1/2)\Gamma(\alpha + 1/2)/\Gamma(\alpha + 1)$ is the ratio of the disk’s field to that of a sphere with the same mass

The acceleration along the symmetry axis can be written as

$$
g(r = 0, z) = -\frac{MG}{z^2} \ _2F_1\left(1, \alpha + 1/2; \alpha + 3/2; -R_o^2/z^2\right). \tag{29}
$$

Using relations found in Magnus and Oberhettinger 1949 to get explicit expressions for $\ _2F_1$ in special cases we obtain from eqs.\((28)\) and \((29)\) (with $t \equiv \varrho^{-1} = R_o/r$, and $t \equiv \zeta^{-1} = R_o/z$): for the Kalnajs disk

$$
g(r \geq R_o, z = 0) = -\frac{MG}{r^2} \frac{3[t \sin^{-1} \hat{t} \ - \hat{t}^2(1 - \hat{t}^2)^{1/2}]}{2}\hat{t}^3. \tag{30}
$$
(reproducing a result of McMillan 1958), and
\[ g(r = 0, z) = -\frac{MG}{z^2} \frac{3t^2}{4} \left( 1 - \frac{\tan^{-1}t}{t} \right). \] (31)

For the isodynamic disk
\[ g(r \geq R_o, z = 0) = -\frac{MG}{r^2} \frac{2}{1 + (1 - t^2)^{1/2}}. \] (32)

and
\[ g(r = 0, z) = -\frac{MG}{z^2} \frac{\ln(1 + t^2)}{t^2}. \] (33)

For the Mestel disk we get the Mestel 1963 result for the plane
\[ g(r \geq R_o, z = 0) = -\frac{MG}{r^2} \hat{t}^{-1} \sin^{-1} \hat{t}, \] (34)

and
\[ g(r = 0, z) = -\frac{MG}{z^2} \frac{\tan^{-1}t}{t}. \] (35)

along the axis.

Because \( _2F_1 \) is a power series with simple expressions for the coefficients, any of eqs. (26),(27),(28),(29) gives straightforwardly the multipoles of the disk: Write the potential (using polar coordinates) as
\[ \varphi(R, \theta, \phi) = \sum_{\ell=0}^{\infty} \mu_\ell R^{-\ell+1} P_\ell(\cos \theta) \] (36)

(only even \( \ell \)'s appear). Use, e.g., eq. (28); the hypergeometric function appearing in it is a power series
\[ _2F_1(1/2, \alpha + 1/2; \alpha + 3/2; R_o^2/r^2) = \sum_{k=0}^{\infty} f_k R_o^{2k} r^{-2k}, \] (37)

with
\[ f_k = \frac{\Gamma(k + 1/2)(\alpha + 1/2)}{\pi^{1/2}k!(\alpha + k + 1/2)}. \] (38)

The radius of convergence of this series in \( R_o/r \) is 1 (i.e. the expansion is valid for \( r > R_o \)). Comparing with the \( R \) derivative of expression (36), putting \( P_{2k}(0) = (-1)^k (2k)! 2^{-2k} (k!)^{-2} \), and using some identities for the \( \Gamma \) functions, we get for the multipole coefficients of the disk
\[ \mu_{2k} = (-1)^k GMR_o^{2k} \frac{\alpha + 1/2}{(2k + 1)(\alpha + k + 1/2)}. \] (39)

We see that, up to a numerical factor, \( \mu_{2k} \) is given by \( GMR_o^{2k} \), which is to be expected on dimensional grounds as, in constructing \( \mu \), we can only avail ourselves of the dimensional quantities \( M \) and \( R_o \) (or \( A \) and \( R_o \)). For example, the quadrupole moment is \( \mu_2 = -GMR_o^2(\alpha + 1/2)/3(\alpha + 3/2) \). We reiterate that while expressions (20),(22) for the potential are valid everywhere outside the disk, the multipole expansion is valid only outside the sphere of radius \( R_o \).
4. Non-axisymmetric thin disk models

Our disks may be generalized to non-axisymmetric ones, which may serve as models for non-axisymmetric galactic disks or for bars. Start again with our cut-off-power-law sphere and squash it by a factor $p$ along the $x$ axis and into a thin disk along the $z$ axis, to obtain a finite, elliptical disk of axis ratio $p$. The field vanishes within a triaxial homoeoid obtained by squashing a spherical shell of constant surface density by different factors along different axes. Thus, our dimensional argument of section 1 still holds, and the acceleration within the disk itself must still be proportional to $r^{\gamma+1}$, and oblivious to the cutoff. Thus

$$g_r(r, \phi) = -A Gr^{\gamma+1} P_r(\tau, \gamma, \sin^2 \phi), \quad (40)$$

$$g_\phi(r, \phi) = -A Gr^{\gamma+1} P_\phi(\tau, \gamma, \sin^2 \phi), \quad (41)$$

where $r, \phi$ are polar coordinates, and $\tau \equiv 1 - p^2$ is a measure of the departure from axisymmetry. When $\gamma \neq -2$ eq.(40) tells us that the midplane potential inside the disk must be of the form

$$\varphi(r, \phi) = \frac{AGr^{\gamma+2}}{\gamma + 2} P_r(\tau, \gamma, \sin^2 \phi) + C(\phi, R_o) \quad (42)$$

(note that the potential, unlike the acceleration, does depend on $R_o$). Deriving $g_\phi = -r^{-1} \frac{\partial \varphi}{\partial \phi}$ from eq.(42) and comparing with eq.(41) we see that $C$ may not depend on $\phi$ and that

$$P_\phi = \frac{1}{\gamma + 2} \frac{\partial P_r}{\partial \phi}. \quad (43)$$

All that remains is to derive one angular factor $P_r$. For the special case $\gamma = -2$ eq.(40) tells us that the potential must be of the form

$$\varphi(r, \phi) = AG \ln r P_r(\tau, \gamma, \sin^2 \phi) + C(\phi, R_o), \quad (44)$$

but now for eq.(41) to hold we must have $\frac{\partial P_r}{\partial \phi} = 0$, $C = AG \dot{C}(\phi) + c(R_o)$, and $P_\phi = \frac{\partial C}{\partial \phi}$. We learn then that for this case the radial acceleration is $\phi$-independent.

The surface-density of the non-axisymmetric disk is obtained simply from expression(3) for the axisymmetric case by multiplying the latter by a factor $1/p$–to account for the squashing along $x$–and by replacing the variable $r$ in eq.(3) by $r[1 + (\tau/p^2) \cos^2 \phi]^{1/2}$; the boundary is at $r = R_o/[1 + (\tau/p^2) \cos^2 \phi]^{1/2}$.

We have derived the field inside the disk by integrating the contributions of thin triaxial homoeoids to the exterior potential (taken from table 2-1 of Binney and Tremaine 1987) weighted by the power-law density profile. The derivation is given in Appendix A. The components of the acceleration may be written as

$$g_r = -2\pi AGr^{2\alpha - 1} \int_0^1 \frac{t^{\alpha - 1/2}(1 - \tau \sin^2 \phi t)^\alpha}{(1 - t)^{1/2}(1 - \tau t)^{\alpha + 1/2}} dt, \quad (45)$$
\[ g_\phi = 2\pi AG r^{2\alpha-1} \tau \sin \phi \cos \phi \int_0^1 \frac{t^{\alpha+1/2}(1-\tau \sin^2 \phi t)^{\alpha-1}}{(1-t)^{1/2}(1-\tau t)^{\alpha+1/2}} \, dt, \]

where \( \alpha = 1 + \gamma/2 \). The integrals in eqs. (45)(46) may be written in terms of the first Appell function

\[ g_r = -Q_o AG r^{2\alpha-1} F_{1}(\alpha + 1/2, -\alpha, \alpha + 1/2, \alpha + 1; \tau \sin^2 \phi; \tau), \]

\[ g_\phi = Q_o AG r^{2\alpha-1} \frac{(\alpha + 1/2)}{(\alpha + 1)} \tau \sin \phi \cos \phi \times \]

\[ \times F_{1}(\alpha + 3/2, -\alpha + 1, \alpha + 1/2, \alpha + 2; \tau \sin^2 \phi; \tau). \]

These are valid for all values of \( \gamma > -3 \), including \( \gamma = -2 \ (\alpha = 0) \). In the last case we see from eq.(45) that indeed \( g_r \) does not depend on \( \phi \).

When \( \gamma \neq -2 \), the potential in the disk is given by

\[ \varphi(r, \phi) = \frac{Q_o AG r^{2\alpha}}{2\alpha} F_{1}(\alpha + 1/2, -\alpha, \alpha + 1/2, \alpha + 1; \tau \sin^2 \phi; \tau) + c, \]

where

\[ c = -\frac{\pi^2 AGR_0^{2\alpha}}{\alpha} 2F_{1}(1/2, 1/2; 1; \tau) \]

is calculated in Appendix A. In the axisymmetric case \( 2F_{1} = F_{1} = 1 \) as the arguments vanish and we get the results of section 2.

For \( \gamma = -2 \) the interior potential is

\[ \varphi(r, \phi) = 2\pi^2 AG \int_0^1 \ln[(1 - \tau \sin^2 \phi t)] dt - \]

\[ -\pi AG \left[ \int_0^1 \ln[(1 - \tau t)] dt \right] \]

\[ + \pi AG \int_0^1 \ln[(1 - \tau t^2)] dt - \]

\[ -\pi AG \int_0^1 \ln[(1 - \tau t^2)] dt, \]

(51)

to be compared with eq.(44). The first term gives \( g_r \), the second \( g_\phi \), and the third contributes to the constant that corresponds to \( \varphi = 0 \) at infinity.

We next discuss various special cases. As in the axisymmetric case non-zero integer values of \( 2\alpha \) afford simplifications. We get for \( \alpha = 1 \):

\[ \varphi(r, \phi) = \frac{1}{2} \pi^2 AG r^2 [ 2F_{1}(3/2, 3/2; 2; \tau) - \frac{3\tau \sin^2 \phi}{4} 2F_{1}(3/2, 5/2; 3; \tau) ] + c, \]

(52)

which is simply an anisotropic, harmonic potential (known to be the case inside an homogeneous ellipsoid—it is the basis, e.g., of the Freeman bar model described by Binney and Tremaine 1987).

For the \( \alpha = 1/2 \) disk we have

\[ \varphi(r, \phi) = 4\pi AG r B(\tau, \sin^2 \phi) + c, \]

(53)
where $c$ is given by eq.(50), and
\[
B = \frac{|\cos \phi|}{(1 - \tau)^{1/2} \tau^{1/2}} \tan^{-1} \left( \frac{\tau \cos^2 \phi}{1 - \tau} \right)^{1/2} + \frac{|\sin \phi|}{2 \tau^{1/2}} \ln \frac{1 + \tau^{1/2} |\sin \phi|}{1 - \tau^{1/2} |\sin \phi|}. \tag{54}
\]

The weak-asymmetry limit, $\tau \ll 1$, is readily obtained for the a general $\alpha$, as $F_1$ is a known power expansion in its two arguments. To first order in $\tau$ we have
\[
\varphi(r, \phi) = \frac{Q_0 AG r^{2\alpha}}{2\alpha} \left[ 1 + \frac{\tau(\alpha + 1/2)}{2(\alpha + 1)} (1 + 2\alpha \cos^2 \phi) \right] - \frac{\pi^2 AG r^{2\alpha}}{\alpha} \left[ 1 + \frac{\tau}{4} \right], \tag{55}
\]
\[
g_r = -Q_0 AG r^{2\alpha - 1} \left[ 1 + \frac{\tau(\alpha + 1/2)}{2(\alpha + 1)} (1 + 2\alpha \cos^2 \phi) \right], \tag{56}
\]
and
\[
g_\phi = Q_0 AG r^{2\alpha - 1} \frac{(\alpha + 1/2)}{2(\alpha + 1)} \tau \sin \phi \cos \phi. \tag{57}
\]

The disk potential and (radial) acceleration along the principal axes may also be written in simpler terms for general $\alpha$. Along the (shorter) $x$-axis, where $\phi = 0$
\[
g_r = -Q_0 AG r^{2\alpha - 1} \left. F_1(\alpha + 1/2, \alpha + 1/2; \alpha + 1; \tau) \right., \tag{58}
\]
and along the longer axis
\[
g_r = -Q_0 AG r^{2\alpha - 1} \left. F_1(1/2, \alpha + 1/2; \alpha + 1; \tau) \right.. \tag{59}
\]

The most general case pertains to the external field of a thick disk. This is given in Appendix A in terms of a Lauricella function of four variables, $F_1^{(4)}$, which is a generalization of the hypergeometric function (see Exton 1976). In any of the three symmetry planes of the ellipsoid this reduces to a Lauricella function of three variables $F_1^{(3)}$ ($F_0^{(2)}$ is the same as $F_1$). It is straightforward to obtain all these from formulae in Appendix A. All these functions are single integrals between 0 and 1 of simple algebraic functions. Exton 1976 gives a short FORTRAN program to evaluate these functions.

We thank David Earn for useful comments.

A. Formulae for the field of a cutoff, power-law ellipsoid

We start with the expression, given in Table 2-1 of Binney and Tremaine 1987 for the potential at position $\vec{R}$ outside a thin, triaxial homoeoid of axes $a_i$, $i = 1, 2, 3$
\[
\varphi(\vec{R}) = -\frac{G}{2} M_{\text{shel}} K(\vec{b}), \tag{A1}
\]
where \( b_i \) are the axes of the ellipsoid that is confocal with the shell and passes through \( \vec{R} \):
\[
b_i^2 = a_i^2 + \lambda(\vec{a}, \vec{R}),
\]
and \( \lambda \) is defined by
\[
\sum R_i^2/(a_i^2 + \lambda) = 1. \tag{A2}
\]
The quantity \( K \) is given by
\[
K(\vec{b}) = \int_0^\infty \frac{ds}{[(b_1^2 + s)(b_2^2 + s)(b_3^2 + s)]^{1/2}}. \tag{A3}
\]
We now calculate the potential outside a cutoff, power-law ellipsoid, whose external axes are \( pY \), \( Y \), and \( qY \), integrating the contributions of thin-shells of axes \( kpY \), \( kY \), \( kqY \), with \( 0 \leq k \leq 1 \). The mass of the shell between \( k \) and \( k + dk \) is that of the original spherical shell
\[
dM = 4\pi AY^{\gamma+3}k^{\gamma+2} dk,
\]
so that
\[
\varphi(\vec{R}) = -2\pi AGY^{\gamma+2} J, \tag{A4}
\]
with
\[
J = Y \int_0^1 k^{\gamma+2} dk \int_0^\infty \frac{ds}{[[pk^2Y + \lambda + s][k^2Y + \lambda + s][qk^2Y + \lambda + s]]^{1/2}}. \tag{A5}
\]
Changing variables to \( t \equiv (\lambda + s)/(kY)^2 \) we have
\[
J = \int_0^1 k^{\gamma+1} dk \int_{\omega(\vec{R})}^\infty f(t) \ dt, \tag{A6}
\]
with
\[
f(t) \equiv [(p^2 + t)(1 + t)(q^2 + t)]^{-1/2} \tag{A7}
\]
that is independent of \( \vec{R}, k, \) or \( Y \), the dependence on which enters through the lower integration boundary \( \omega \equiv \lambda(k, \vec{R})/(kY)^2 \). We now integrate by parts over \( k \) (the case \( \gamma = -2 \) requires a special treatment) to get
\[
J = \frac{1}{\gamma + 2} \int_{\omega(\vec{R})}^\infty f(t) \ dt + \frac{1}{\gamma + 2} \int_0^1 dk k^{\gamma+2} \frac{d\omega}{dk} f(\omega). \tag{A8}
\]
In the second integral—call it \( \dot{J} \)—we change variables to \( \omega \), noting that, from the definition of \( \lambda \) in eq.(A2) we have
\[
k^2 = \frac{\xi^2}{p^2 + \omega} + \frac{\nu^2}{q^2 + \omega} + \frac{\zeta^2}{1 + \omega}. \tag{A9}
\]
Here \( \xi \equiv x/Y, \nu \equiv y/Y, \zeta \equiv z/Y \). At \( k = 0 \) \( \omega(0, \vec{R}) = \infty \). We then get
\[
\dot{J} = -\int_{\omega(\vec{R})}^\infty d\omega \ f(\omega) \left( \frac{\xi^2}{p^2 + \omega} + \frac{\nu^2}{q^2 + \omega} + \frac{\zeta^2}{1 + \omega} \right)^{\gamma+2} \tag{A10}
\]
It is useful to change variables to \( t \equiv 1/(1 + \omega) \), and defining \( \tau \equiv 1 - p^2, \kappa \equiv 1 - q^2 \) we can write
\[
\dot{J} = -\int_0^W \frac{d^{(\kappa+1)/2}\left[\xi^2(1 - \kappa t) + \nu^2(1 - \kappa t)(1 - \tau t) + \zeta^2(1 - \tau t)\right]^{(\gamma+2)/2}}{[(1 - \kappa t)(1 - \tau t)]^{(\gamma+3)/2}} \ dt, \tag{A11}
\]
where \( W \equiv 1/\left[1 + \omega(1, \vec{R})\right] \). For \( \vec{R} \) on the surface of the cutoff ellipsoid \( \omega(1, \vec{R}) \) vanishes, so the upper integration boundary is 1.

To obtain the acceleration at position \( \vec{R} \) inside the ellipsoid we make use of the fact that the layers outside \( \vec{R} \) do not contribute. Thus we can use the above expression for the potential outside an ellipsoid that is cutoff beyond that through \( \vec{R} \), then take the \( \vec{R} \) gradient, and calculate the integrals at the boundary. When taking the \( \vec{R} \) gradient, we note that the two contributions from differentiating the boundaries of the integrals cancel, and we are left with the following expression for the acceleration inside the cutoff ellipsoid

\[
\vec{g} = -\frac{2\pi AG}{\gamma + 2} \sqrt{R^2 I(\hat{x}, \hat{y}, \hat{z})},
\]

(A12)

with \( I \) a function of the polar angles, but not of \( R \).

By comparison with the exterior potential at the boundary, where \( W = 1 \) we get

\[
c = -\frac{2\pi AG R_o^{\gamma + 2}}{\gamma + 2} \int_0^\infty f(t) \, dt = -\frac{4\pi AG R_o^{\gamma + 2}}{\gamma + 2} F_1(1/2, 1/2, 1/2, 3/2; \kappa, \tau).
\]

(A15)

To get our results in the midplane of a disk we take \( \theta = \pi/2 \), and go to the limit \( q = 0 \) (\( \kappa = 1 \)). All our expressions for the interior acceleration then follow. Our expression for the potential outside an axisymmetric disk eqs.(A4)(A8)(A11), and then writing the resulting integrals in terms of hypergeometric functions.

Regarding the field at an arbitrary position outside the ellipsoid, we note that in eq.(A11) the term in square brackets in the numerator of the integrand may be written in the form \( (R/R_o)^2(1 - b t)(1 - c t) \). Thus, by changing variables to \( t = s W \) we can write

\[
\hat{J} = -B \int_0^1 s^{(\gamma + 1)/2}[(1 - b W s)(1 - c W s)]^{(\gamma + 2)/2}[(1 - \kappa W s)(1 - \tau W s)]^{-(\gamma + 3)/2} \, ds,
\]

(A16)

with \( B = (R/R_o)^{2\alpha} W^{\alpha + 1/2} \). This can be written in terms of the Lauricella function of type \( D \), which is a generalization of the hypergeometric function to many variables (see Exton 1976 2.3.6). The first integral in eq.(A8) can also be brought to a simpler form, and we finally get for the
potential outside the cutoff, power-law ellipsoid

\[
\varphi(\vec{R}) = -\frac{MG}{R_o} \frac{2\alpha + 1}{4\alpha} \left\{ 2W^{1/2} F_1(1/2,1/2,1/2,3/2;\kappa W,\tau W) - \right.
\]
\[
- \left( \frac{R}{R_o} \right)^{2\alpha} W^{\alpha+1/2}(\alpha + 1/2)^{-1} \times 
\]
\[
\times F^{(i)}_4(\alpha + 1/2, -\alpha, -\alpha, \alpha + 1/2, \alpha + 1/2; \alpha + 3/2; bW, eW, \kappa W, \tau W) \right\}. \tag{A17}
\]

We remind the reader that \( W = (1 + \omega)^{-1} \), where \( \omega \) is the positive solution of
\[
(x/R_o)^2/(p^2 + \omega) + (y/R_o)^2/(1 + \omega) + (z/R_o)^2/(q^2 + \omega) = 1.
\]

The exterior potential may be written in terms of hypergeometric functions of three variables when we are in any of the symmetry planes of the ellipsoid.

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