ON REPRODUCING KERNELS, AND ANALYSIS OF MEASURES

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To the memory of Jørgen Hoffmann-Jørgensen.

Abstract. Starting with the correspondence between positive definite kernels on the one hand and reproducing kernel Hilbert spaces (RKHSs) on the other, we turn to a detailed analysis of associated measures and Gaussian processes. Point of departure: Every positive definite kernel is also the covariance kernel of a Gaussian process.

Given a fixed sigma-finite measure \( \mu \), we consider positive definite kernels defined on the subset of the sigma algebra having finite \( \mu \) measure. We show that then the corresponding Hilbert factorizations consist of signed measures, finitely additive, but not automatically sigma-additive. We give a necessary and sufficient condition for when the measures in the RKHS, and the Hilbert factorizations, are sigma-additive. Our emphasis is the case when \( \mu \) is assumed non-atomic. By contrast, when \( \mu \) is known to be atomic, our setting is shown to generalize that of Shannon-interpolation. Our RKHS-approach further leads to new insight into the associated Gaussian processes, their Ito calculus and diffusion. Examples include fractional Brownian motion, and time-change processes.

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1. Introduction

A reproducing kernel Hilbert space (RKHS) is a Hilbert space $\mathcal{H}$ of functions (defined on a prescribed set) in which point-evaluation is a continuous linear functional; so continuity is required to hold with respect to the norm in $\mathcal{H}$. These Hilbert spaces (RKHS) have a host of applications, including to complex analysis, to harmonic analysis, and to quantum mechanics.

A fundamental theorem of Aronszajn yields an explicit correspondence between positive definite kernels on the one hand and RKHSs on the other. Now every positive definite kernel is also the covariance kernel of a Gaussian process; a fact which is a point of departure in our present analysis: Given a positive definite kernel, we shall explore its use in the analysis of the associated Gaussian process; and vice versa.

This point of view is especially fruitful when one is dealing with problems from stochastic analysis. Even restricting to stochastic analysis, we have the exciting area of applications to statistical learning theory [SZ07, Wes13]. The RKHSs are useful in statistical learning theory on account of a powerful representer theorem: It states that every function in an RKHS that minimizes an associated empirical risk-function can be written as a generalized linear combination of samplings of the kernel function; i.e., samples evaluated at prescribed training points. Hence, it is a popular tool for empirical risk minimization problems, as it adapts perfectly to a host of infinite dimensional optimization problems.

Recall that a reproducing kernel Hilbert space (RKHS) is a Hilbert space $\mathcal{H}$ of functions, say $f$, on a fixed set $X$ such that every linear functional (induced by $x \in X$), so when $x$ is fixed, set

$$E_x (f) = f (x), \quad f \in \mathcal{H}. \quad (1.1)$$

We require that $E_x$ is continuous in the norm of $\mathcal{H}$.

Hence, by Riesz’ representation theorem, there is a corresponding $h_x \in \mathcal{H}$ such that

$$E_x f = \langle f, h_x \rangle_{\mathcal{H}} \quad (1.2)$$
where $(\cdot,\cdot)_{\mathcal{H}}$ denotes the inner product in $\mathcal{H}$. Setting
\[ K(x,y) = \langle h_y, h_x \rangle_{\mathcal{H}}, \quad (x,y) \in X \times X \]
we get a positive definite (p.d.) kernel, i.e., for $\forall n \in \mathbb{N}, \forall \{\alpha_i\}_1^n, \forall \{x_i\}_1^n, \alpha_i \in \mathbb{C}, x_i \in X$, we have
\[ \sum_i \sum_j \alpha_i \bar{\alpha}_j K(x_i, x_j) \geq 0. \tag{1.3} \]

Conversely, if $K$ is given p.d., i.e., satisfying (1.3), then by [Aro50], there is a RKHS such that (1.2) holds.

Given $K$ p.d., we may take $\mathcal{H}(K)$ to be the completion of
\[ \varphi = \sum_i \alpha_i K(\cdot,x_i) \tag{1.4} \]
in the norm
\[ \|\varphi\|_{\mathcal{H}(K)}^2 = \sum_i \sum_j \alpha_i \bar{\alpha}_j K(x_i, x_j), \tag{1.5} \]
but quotiented out by those functions $\varphi$ in (1.4) with $\|\varphi\|_{\mathcal{H}(K)}^2 = 0$.

A key fact which we shall be using throughout the paper is the following:

**Lemma 1.1.** Let $K$ be a positive definite kernel on $X \times X$, and let $\mathcal{H}(K)$ be the corresponding RKHS.

Then a function $f$ on $X$ is in $\mathcal{H}(K)$ iff there is a finite constant $C = C_f$, depending on $f$, such that for $\forall n \in \mathbb{N}, \forall \{x_i\}_1^n, \{\alpha_i\}_1^n, x_i \in X, \alpha_i \in \mathbb{C}$, we have:
\[ \left| \sum_{i=1}^n \alpha_i f(x_i) \right|^2 \leq C \sum_i \sum_j \alpha_i \bar{\alpha}_j K(x_i, x_j). \tag{1.6} \]

**Remark 1.2.** Our present focus is on the case when the prescribed $\sigma$-finite measure $\mu$ is non-atomic. But the atomic case is also important, for example in interpolation theory in the form of Shannon, see e.g., [DM72].

Consider, for example, the case $X = \mathbb{R}$, and
\[ K(x,y) = \frac{\sin \pi (x-y)}{\pi (x-y)}, \tag{1.7} \]
defined for $(x,y) \in \mathbb{R} \times \mathbb{R}$. In this case, the RKHS $\mathcal{H}(K)$ is familiar: It may be realized as functions $f$ on $\mathbb{R}$, such that the Fourier transform
\[ \hat{f}(\xi) = \int_{\mathbb{R}} e^{-i2\pi \xi} f(x) dx \tag{1.8} \]
is well defined, and supported in the compact interval $[-\frac{1}{2}, \frac{1}{2}]$, frequency band, with $\|f\|^2_{\mathcal{H}(K)} = \int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{f}(\xi)|^2 d\xi$.

Set $\mu = \sum_{n \in \mathbb{Z}} \delta_n$ (the Dirac-comb). Then Shannon’s theorem states that

$$\ell^2(\mathbb{Z}) \ni (\alpha_n)_{n \in \mathbb{Z}} \xrightarrow{T} \mathcal{H}(K),$$

given by

$$(T((\alpha_n)))(x) = \sum_{n \in \mathbb{Z}} \alpha_n \frac{\sin \pi (x-n)}{\pi (x-n)}$$

is isometric, mapping $\ell^2$ onto $\mathcal{H}(K)$. Its adjoint operator

$$T^*: \mathcal{H}(K) \rightarrow \ell^2(\mathbb{Z})$$

is

$$(T^*f)_n = f(n), \quad n \in \mathbb{Z}. \quad (1.11)$$

Compare (1.11) with (3.6) below in a much wider context.

The RKHS for the kernel (1.7) $\mathcal{H}(K)$ is called the Paley-Wiener space. Functions in $\mathcal{H}(K)$ also go by the name, band-limited signals. We refer to (1.11) as (Shannon) sampling. It states that functions (continuous time-signals) $f$ from $\mathcal{H}(K)$ may be reconstructed “perfectly” from their discrete $\mathbb{Z}$ samples.

2. Sigma-algebras and RKHSs of signed measures

Now our present focus will be a class of p.d. kernels, defined on subsets of a fixed $\sigma$-algebra. Specifically, if $(M, \mathcal{B}, \mu)$ is a $\sigma$-finite measure space, we set $X = X(\mu) = \mathcal{B}_{\text{fin}}$; see (2.1) below.

**Definition 2.1.** Consider a measure space $(M, \mathcal{B}, \mu)$ where $\mathcal{B}$ is a $\sigma$-algebra of subsets in $M$, and $\mu$ is a $\sigma$-finite measure on $\mathcal{B}$. Set

$$\mathcal{B}_{\text{fin}} = \mathcal{B}_{\text{fin}}(\mu) = \{A \in \mathcal{B} \mid \mu(A) < \infty\}. \quad (2.1)$$

Let $\mathcal{H}$ be a Hilbert space having the following property:

$$\{\chi_A \mid A \in \mathcal{B}_{\text{fin}}\} \subset \mathcal{H}, \quad (2.2)$$

where $\chi_A$ denotes the indicator function for the set $A$.

We shall restrict the discussion to real valued functions.

**Theorem 2.2.** Let $\beta$ be a function, $\mathcal{B}_{\text{fin}} \times \mathcal{B}_{\text{fin}} \rightarrow \mathbb{R}$. Then TFAE:

(i) $\beta$ is positive definite, i.e., for $\forall n \in \mathbb{N}$, $\forall \{\alpha_i\}_{1}^{n}$, $\forall \{A_i\}_{1}^{n}$, $\alpha_i \in \mathbb{R}$, $A_i \in \mathcal{B}_{\text{fin}}$, we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \beta(A_i, A_j) \geq 0. \quad (2.3)$$
(ii) There is a Hilbert space $H$ which satisfies (2.2); and also
\[ \beta(A, B) = \langle \chi_A, \chi_B \rangle_H, \quad \forall (A, B) \in B_{\text{fin}} \times B_{\text{fin}}. \] (2.4)

(iii) There is a Hilbert space $H$ which satisfies (2.2); and also a linear mapping:
\[ H \ni f \mapsto \mu_f \in \left( \text{signed finitely additive measures on } (M, \mathcal{B}) \right) \] (2.5)
with
\[ \mu_f(A) = \langle \chi_A, f \rangle_H, \quad \forall A \in B_{\text{fin}}. \] (2.6)

Proof. We shall divide up the reasoning in the implications: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

Case (i) $\Rightarrow$ (ii). Given a function $\beta$ as in (i), we know that, by [Aro50], there is an associated reproducing kernel Hilbert space $H(\beta)$. The vectors in $H(\beta)$ are obtained by the quotient and completion procedures applied to the functions
\[ B_{\text{fin}} \ni B \mapsto \beta(A, B) \in \mathbb{R} \] (2.7)
defined for every $A \in B_{\text{fin}}$. Moreover, the inner product in $H(\beta)$, satisfies
\[ \langle \beta(A_1, \cdot), \beta(A_2, \cdot) \rangle_{H(\beta)} = \beta(A_1, A_2). \] (2.8)

Now let
\[ H = (\text{span} \{ \chi_A \mid A \in B_{\text{fin}} \})^\sim \] (2.9)
with $(\cdot \cdot)^\sim$ denoting the Hilbert completion:
\[
\left\| \sum_i \alpha_i \chi_{A_i} \right\|_{H}^2 = \sum_i \sum_j \alpha_i \alpha_j \beta(A_i, A_j)
= \sum_i \sum_j \alpha_i \beta(A_i, A_j)^2 \quad \text{see (2.7)}
\]
It is then immediate from this that the Hilbert space $H$ satisfies the conditions stated in (ii) of the theorem.

Case (ii) $\Rightarrow$ (iii). Let $H$ satisfy the conditions in (ii); and for $f \in H$, let $\mu_f$ be as in (2.5). We must show that if $n \in \mathbb{N}$, $\{A_i\}_1^n$, $A_i \in B_{\text{fin}}$, satisfy $A_i \cap A_j = \emptyset$, $i \neq j$, then
\[ \mu_f(\cup_{1}^{n}A_i) = \sum_{1}^{n} \mu_f(A_i). \] (2.10)
But
\[
\text{LHS}_{(2.10)} = \langle \chi_{\cup_{i=1}^{n}A_i}, f \rangle_H
= \sum_i \langle \chi_{A_i}, f \rangle_H = \sum_i \mu_f(A_i) = \text{RHS}_{(2.10)}.\]
The remaining assertions in (iii) are clear.

Case (iii) ⇒ (i). This step is immediate from (2.4). □

**Proposition 2.3.** Let \((M, \mathcal{B}, \mu)\) be a \(\sigma\)-finite measure. As in Theorem 2.2, we specify a pair \((\beta, \mathcal{H})\) where \(\beta\) is defined on \(\mathcal{B}_{\text{fin}} \times \mathcal{B}_{\text{fin}}\), and \(\mathcal{H}\) is a Hilbert space subject to condition (2.2). For \(f \in \mathcal{H}\), set

\[
\mu_f(A) = \langle \chi_A, f \rangle_{\mathcal{H}}, \quad A \in \mathcal{B}_{\text{fin}}.
\]

Then \(\mu_f \in \mathcal{H}(\beta) (= \text{the RKHS of } \beta)\). Moreover,

\[
\|\mu_f\|_{\mathcal{H}(\beta)} \leq \|f\|_{\mathcal{H}}.
\]

**Proof.** This will be a direct application of Lemma 1.1, but now applied to \(X = \mathcal{B}_{\text{fin}}\). Hence we must show that, for \(\forall n \in \mathbb{N}\), \(\{A_i\}_{i=1}^n\), \(\{\alpha_i\}_{i=1}^n\), \(A_i \in \mathcal{B}_{\text{fin}}\), \(\alpha_i \in \mathbb{R}\), the estimate (1.6) holds, and with a finite constant \(C_f\).

In fact, we may take \(C_f = \|f\|^2_{\mathcal{H}}\), so \(\|\mu_f\|_{\mathcal{H}(\beta)} \leq \|f\|_{\mathcal{H}}\) as claimed. Specifically,

\[
\sum_{i=1}^n \alpha_i \mu_f(A_i) = \sum_{i=1}^n \alpha_i \langle \chi_{A_i}, f \rangle_{\mathcal{H}}
\]

\[
\leq \sum_{i=1}^n \alpha_i \|\chi_{A_i}\|_{\mathcal{H}} \|f\|_{\mathcal{H}}
\]

\[
\leq \|f\|^2_{\mathcal{H}} \sum_i \sum_j \alpha_i \alpha_j \beta(A_i, A_j),
\]

which is the desired conclusion. □

3. **The sigma-additive property**

The sigma-additive property alluded to here is not a minor technical point. Indeed, one of the basic problems related to the propositional calculus and the foundations of quantum mechanics is the description of probability measures (called states in quantum physical terminology) on the set of experimentally verifiable propositions. In the quantum setting, the set of propositions is then realized as an orthomodular partially ordered set, where the order is induced by a relation of implication, called a quantum logic. Now quantum-observables are generally non-commuting, and the precise question is in fact formulated for states (measures) on \(C^*\)-algebras; i.e., normalized positive linear functionals (see e.g., [JT17b]).

The classical Gleason theorem is the assertion that a state on the \(C^*\)-algebra \(\mathcal{B}(\mathcal{H})\) of all bounded operators on a Hilbert space is uniquely described by the values it takes on orthogonal projections, assuming the
dimension of the Hilbert space $\mathcal{H}$ is not 2. The precise result entails extension of finitely additive measures to sigma-additive counterparts, i.e., when we have additivity on countable unions of disjoint sets from the underlying sigma-algebra.

We now turn to the question of when the finitely additive measures $\mu_f$ are in fact $\sigma$-additive. (See Theorem 2.2, part (iii).)

Given $(M, \mathcal{B}, \mu)$ as above, we shall set
\[
\mathcal{D}_{\text{fin}}(\mu) = \text{span} \{ \chi_A \mid A \in \mathcal{B}_{\text{fin}} \}.
\] (3.1)

Recall that $\mathcal{D}_{\text{fin}}(\mu)$ is automatically a dense subspace in $L^2(\mu)$.

**Theorem 3.1.** Let $\mathcal{B}_{\text{fin}} = \mathcal{B}_{\text{fin}}(\mu)$ be as specified in (2.1) with a fixed $(M, \mathcal{B}, \mu)$ $\sigma$-finite measure. Let $\beta$ be given, assumed positive definite on $\mathcal{B}_{\text{fin}} \times \mathcal{B}_{\text{fin}}$, and let $\mathcal{H}$ be a Hilbert space which satisfies conditions (2.2) and (2.4).

Then there is a dense subspace $\mathcal{H}_\mu \subset \mathcal{H}$ such that the signed measures
\[
\{ \mu_f \mid f \in \mathcal{H}_\mu \}
\] (3.2)
are $\sigma$-additive if and only if the following implication holds:
\[
\begin{align*}
(\alpha) & \quad \{ \varphi_n \}_{n \in \mathbb{N}} : \varphi_n \in \mathcal{D}_{\text{fin}}(\mu), \|\varphi_n\|_{L^2(\mu)} \xrightarrow{n \to \infty} 0 \\
(\beta) & \quad f \in \mathcal{H}, \|\varphi_n - f\|_{\mathcal{H}} \xrightarrow{n \to \infty} 0 \\
\end{align*}
\Rightarrow f = 0,
\]
i.e., if a vector $f \in \mathcal{H}$ satisfies $(\alpha)$ and $(\beta)$, it must be the zero vector in $\mathcal{H}$.

**Proof.** Note that, because of assumptions (2.2) and (2.4), we get a natural inclusion mapping, denoted $T$,
\[
L^2(\mu) \xrightarrow{T} \mathcal{H}
\] (3.3)
with dense domain $\mathcal{D}_{\text{fin}}(\mu)$ in $L^2(\mu)$. Recall, if $A \in \mathcal{B}_{\text{fin}}$, then the indicator function $\chi_A$ is assumed to be in $\mathcal{H}$.

With these assumptions, we see that the implication in the statement of the theorem simply states that $T$ is closable when viewed as a densely defined operator as in (3.3).

By a general theorem (see e.g., [JT17b]), $T$ is closable if and only if the domain $\text{dom}(T^*)$ of its adjoint $T^*$ is dense in $\mathcal{H}$.

We have that a vector $f \in \mathcal{H}$ is in $\text{dom}(T^*)$ if and only if $\exists C_f < \infty$ such that
\[
|\langle T\varphi, f \rangle_{\mathcal{H}}| \leq C_f \|\varphi\|_{L^2(\mu)}
\] (3.4)
holds for $\forall \varphi \in \mathcal{D}_{\text{fin}}(\mu)$. Also note that, if $\varphi = \chi_A, A \in \mathcal{B}_{\text{fin}}$, then
\[
\langle T\varphi, f \rangle_{\mathcal{H}} = \mu_f(A);
\] (3.5)
and so if $f \in \text{dom } (T^*)$, then

$$
\mu_f (A) = \left< \chi_A, T^* f \right>_{L^2(\mu)} = \int_A (T^* f) \, d\mu, \quad \forall A \in \mathcal{B}_{\text{fin}}. \tag{3.6}
$$

Note, by definition, $T^* f \in L^2(\mu)$. Indeed, the converse holds as well. Since the right-hand side in (3.6) is clearly $\sigma$-additive, one implication holds. Moreover, the other implication follows from general facts about $L^2(M, \mathcal{B}, \mu)$ valid for any $\sigma$-finite measure $\mu$ on $(M, \mathcal{B})$. □

**Corollary 3.2.** Let $(\beta, \mathcal{H})$ be as in the statement of Theorem 3.1, and let $T$ be the closable inclusion $L^2 (\mu) \xrightarrow{T} \mathcal{H}$. Then for $f \in \text{dom } (T^*)$, dense in $\mathcal{H}$, the corresponding signed measure $\mu_f$ is absolutely continuous w.r.t. $\mu$ with Radon-Nikodym derivative

$$
\frac{d\mu_f}{d\mu} = T^* f. \tag{3.7}
$$

**Example 3.3.** Let $(M, \mathcal{B}, \mu)$ be a $\sigma$-finite measure space, and on $\mathcal{B}_{\text{fin}} \times \mathcal{B}_{\text{fin}}$ define

$$
\beta_\mu (A, B) := \mu (A \cap B), \quad \forall A, B \in \mathcal{B}_{\text{fin}}. \tag{3.8}
$$

Let $\mathcal{H}(\beta_\mu) = \text{RKHS}(\beta_\mu)$, i.e., the reproducing kernel Hilbert space associated with the p.d. function $\beta_\mu$. Then $\mathcal{H}(\beta_\mu)$ consists of all signed measures $m$ of the form

$$
m (A) = \int_A \varphi \, d\mu, \quad \varphi \in L^2 (\mu); \tag{3.9}
$$

and when (3.9) holds,

$$
\|m\|_{\mathcal{H}(\beta_\mu)}^2 = \int_M |\varphi|^2 \, d\mu. \tag{3.10}
$$

**Proof.** When $\beta_\mu$ is specified as in (3.8), then one checks immediately that the inclusion operator $T : L^2 (\mu) \rightarrow \mathcal{H}(\beta_\mu)$ is isometric, and maps onto $\mathcal{H}(\beta_\mu)$. Indeed, for finite linear combinations $\sum_{i=1}^n \alpha_i \chi_{A_i}$ as above, we have

$$
\left\| \sum_{i} \alpha_i \chi_{A_i} \right\|_{L^2(\mu)}^2 = \sum_{i} \sum_{j} \alpha_i \alpha_j \mu (A_i \cap A_j) = \left\| \sum_{i} \alpha_i \beta_\mu (A_i, \cdot) \right\|_{\mathcal{H}(\beta_\mu)}^2,
$$

so $T$ is isometric and onto. □
4. Gaussian Fields

Let \((M, \mathcal{B}, \mu)\) be a \(\sigma\)-finite measure space. By a Gaussian field based on \((M, \mathcal{B}, \mu)\), we mean a probability space \((\Omega, \mathcal{C}, P^\mu)\), depending on \(\mu\), such that \(\mathcal{C}\) is a \(\sigma\)-algebra of subsets of \(\Omega\), and \(P^\mu\) is a probability measure on \((\Omega, \mathcal{C})\). For every \(A \in \mathcal{B}_{\text{fin}}(\mu)\), it is assumed that \(X_A^{(\mu)}\) is in \(L^2(\Omega, \mathcal{C}, P^\mu)\); and in addition,

\[
X_A^{(\mu)} \sim N(0, \mu(A)),
\]

i.e., the distribution of \(X_A^{(\mu)}\), computed for \(P^\mu\) is the standard Gaussian with variance \(\mu(A)\).

Finally, set \(E^\mu(\cdot) = \int_\Omega (\cdot) dP^\mu\); then it is required that

\[
E^\mu(X_A^{(\mu)} X_B^{(\mu)}) = \mu(A \cap B), \quad \forall A, B \in \mathcal{B}_{\text{fin}}.
\] (4.2)

For a background reference on probability spaces, see e.g., \([HJr94]\).

**Proposition 4.1.** Given \((M, \mathcal{B}, \mu)\), \(\sigma\)-finite, then a Gaussian field \(\{X_A^{(\mu)}\}_{A \in \mathcal{B}_{\text{fin}}}\) may be constructed as follows:

For \(\forall n \in \mathbb{N}\), \(\{A_i\}_1^n, A_i \in \mathcal{B}_{\text{fin}}, \) let \(g^{(A_i)}\) be the Gaussian distribution on \(\mathbb{R}^n\), with mean zero, and covariance matrix

\[
[\mu(A_i \cap A_j)]_{i,j=1}^n.
\] (4.3)

**Proof.** By Kolmogorov’s theorem \([Kol50, Kol62, SSBR71, Hid80, Moh14, JT17b]\), there is a unique probability measure \(P^\mu\) on the infinite Cartesian product

\[
\Omega = \mathbb{R}^{\mathcal{B}_{\text{fin}}}
\] (4.4)

such that

\[
E^\mu(\cdot | \{A_1, \cdots, A_n\}) = g^{(A_i)}.
\] (4.5)

For \(\omega \in \Omega = \mathbb{R}^{\mathcal{B}_{\text{fin}}},\) set

\[
X_A^{(\mu)}(\omega) = \omega(A), \quad A \in \mathcal{B}_{\text{fin}}.
\] (4.6)

For the \(\sigma\)-algebra \(\mathcal{C}\) of subsets in \(\Omega\), we take the cylinder \(\sigma\)-algebra, i.e., the subsets of \(\Omega = \mathbb{R}^{\mathcal{B}_{\text{fin}}}\) generated by \(\{A_i\}_1^n, (a_i, b_i)\),

\[
\{\omega \in \Omega | a_i < \omega(A_i) < b_i\},
\] (4.7)

see Figure 4.1 on page 10.

**Corollary 4.2.** Let \((M, \mathcal{B}, \mu)\) be given, \(\sigma\)-finite, and let \(X^{(\mu)}\) be an associated Gaussian field; see Proposition 4.1, and (4.2).
Let $\mathcal{D}_{\text{fin}}(\mu) = \text{span} \{ \chi_A \mid A \in \mathcal{B}_{\text{fin}} \}$; then
\[
\mathcal{D}_{\text{fin}}(\mu) \ni \sum_i \alpha_i \chi_{A_i} \mapsto \sum_i \alpha_i X_{A_i}^{(\mu)} \quad (4.8)
\]
extends by closure to an isometry of $L^2(\mu)$ into $L^2(\Omega, \mathbb{P}(\mu))$, called the generalized Ito-Wiener integral.

Proof. We have for all linear combinations as above,
\[
\left\| \sum_i \alpha_i X_{A_i}^{(\mu)} \right\|_{L^2(\Omega, \mathbb{P}(\mu))}^2 = \sum_i \sum_j \alpha_i \alpha_j \mathbb{E}_\mu(X_{A_i}^{(\mu)} X_{A_j}^{(\mu)}) = \sum_i \sum_j \alpha_i \alpha_j \mu(A_i \cap A_j) = \left\| \sum_i \alpha_i \chi_{A_i} \right\|_{L^2(\mu)}^2
\]
which is the desired isometry. Hence
\[
T_\mu : \sum_i \alpha_i \chi_{A_i} \mapsto \sum_i \alpha_i X_{A_i}^{(\mu)} \quad (4.9)
\]
extends by closure to an isometry
\[
T_\mu(\varphi) := X_{\varphi}^{(\mu)},
\]
\[\text{i.e.,}
\mathbb{E}_\mu\left(|X_{\varphi}^{(\mu)}|^2\right) = \int_M |\varphi|^2 \, d\mu, \quad \text{and} \quad \mathbb{E}_\mu(X_{\varphi_1}^{(\mu)} X_{\varphi_2}^{(\mu)}) = \int_M \varphi_1 \varphi_2 \, d\mu
\]
hold for all $\varphi_1, \varphi_2 \in L^2(\mu)$. Moreover, $X_{\varphi}^{(\mu)} \sim N(0, \|\varphi\|^2_{L^2(\mu)})$ as stated. \hfill \square
Corollary 4.3. Let \((M, \mathcal{B}, \mu)\) be as above, i.e., \(\mu\) is assumed \(\sigma\)-finite. Suppose, in addition, that \(\mu\) is non-atomic; then the quadratic variation of the Gaussian process \(X(\mu)\) coincides with the measure \(\mu\) itself.

Proof. Consider \(B \in \mathcal{B}_{\text{fin}}\), and consider all partitions \(\text{PAR}(B)\) of the set \(B\), i.e.,
\[
\pi = \{(A_i)\}\quad(4.11)
\]
specified as follows: \(A_i \in \mathcal{B}_{\text{fin}}, A_i \cap A_j = \emptyset\) if \(i \neq j\), and \(\cup_i A_i = B\).

We consider the limit over the net of such partitions, i.e., \(\pi \to 0\) means \(\max_i \mu(A_i) \to 0\). We show that
\[
\mathbb{E}_\mu \left( \left| \mu(B) - \sum_i (X_{A_i}^{(\mu)})^2 \right|^2 \right) \to 0 \quad(4.12)
\]
as \(\pi \to 0\), i.e., \(\max_i \mu(A_i) \to 0\), \(\pi = (A_i) \in \text{PAR}(B)\).

Since, for \(\pi = (A_i) \in \text{PAR}(B)\), we have \(\sum_i \mu(A_i) = \mu(B)\), to prove (4.12), we need only consider the individual terms; \(i\) fixed:
\[
\mathbb{E}_\mu \left( \left| \mu(A_i) - (X_{A_i}^{(\mu)})^2 \right|^2 \right) = \mu(A_i)^2 - 2\mu(A_i) \mathbb{E}_\mu \left( (X_{A_i}^{(\mu)})^2 \right) + \mathbb{E}_\mu \left( (X_{A_i}^{(\mu)})^4 \right).
\]

But
\[
\mathbb{E}_\mu \left( (X_{A_i}^{(\mu)})^2 \right) = \mu(A_i), \quad \text{and} \quad \mathbb{E}_\mu \left( (X_{A_i}^{(\mu)})^4 \right) = 3\mu(A_i)^2;
\]
and so
\[
\mathbb{E}_\mu \left( \left| \mu(A_i) - (X_{A_i}^{(\mu)})^2 \right|^2 \right) = 2\mu(A_i)^2.
\]

Now, for \(\pi = (A_i) \in \text{PAR}(B)\), we have:
\[
\sum_i \mu(A_i)^2 \leq \left( \max_i \mu(A_i) \right)^2 \mu(B) \quad \text{as} \quad \pi \to 0;
\]
and the desired conclusion (4.12) follows. \(\square\)

Corollary 4.4. Let \(\mu\) and \(\nu\) be two positive \(\sigma\)-finite measures on a fixed measure space \((M, \mathcal{B})\); see Corollary 4.3 for the detailed setting. Let \(X^{(\mu)}\) and \(X^{(\nu)}\) be the corresponding Gaussian fields. Consider nets of partitions \(\pi = \{(A_i)\}\) from \((M, \mathcal{B})\).

(i) If \(B \in \mathcal{B}\), then the limit
\[
\lim_{\pi \to 0} \sum_{\pi \in \text{PAR}(B)} X_{A_i}^{(\mu)} X_{A_i}^{(\nu)}
\]

(4.13)
exists; and it defines a signed measure, denoted \( \langle X^{(\mu)}, X^{(\nu)} \rangle \), satisfying
\[
\langle X^{(\mu)}, X^{(\nu)} \rangle = \frac{1}{2} \left( \langle X^{(\mu)} \rangle + \langle X^{(\nu)} \rangle - \langle X^{(\mu)} - X^{(\nu)} \rangle \right).
\]

(ii) If \( \lambda \) is a positive measure on \((M, \mathcal{B})\) satisfying \( \mu \ll \lambda \), and \( \nu \ll \lambda \), with respective Radon-Nikodym derivatives \( d\mu/d\lambda \) and \( d\nu/d\lambda \), then
\[
\langle X^{(\mu)}, X^{(\nu)} \rangle = \sqrt{\frac{d\mu}{d\lambda} \frac{d\nu}{d\lambda}} d\lambda,
\]
where the representation in (4.15) is independent of the choice of measures \( \lambda \) subject to: \( \mu \ll \lambda \), \( \nu \ll \lambda \).

Proof. The details follow those in the proof of Corollary 4.3 above; and we also make use of the theory of sigma-Hilbert spaces (universal Hilbert spaces); see e.g., [Nel69, BJ18, JT18]. \( \square \)

Corollary 4.5. Let \((M, \mathcal{B}, \mu), X^{(\mu)}, \) and \( T_{\mu} : L^2(\mu) \rightarrow L^2(\mathbb{P}^{(\mu)}) \) be as in Proposition 4.1, then the adjoint
\[
T_{\mu}^* : L^2(\Omega, \mathbb{P}^{(\mu)}) \rightarrow L^2(M, \mu)
\]
is specified as follows:

Let \( n \in \mathbb{N} \), and let \( p(x_1, x_2, \ldots, x_n) \) be a polynomial on \( \mathbb{R}^n \). For
\[
F := p \left( X^{(\mu)}_{\varphi_1}, \cdots, X^{(\mu)}_{\varphi_n} \right), \quad \{\varphi_i\}_{i=1}^n, \varphi_i \in L^2(\mu);
\]
set
\[
D(F) := \sum_{i=1}^n \frac{\partial p}{\partial x_i} \left( X^{(\mu)}_{\varphi_1}, \cdots, X^{(\mu)}_{\varphi_n} \right) \otimes \varphi_i.
\]

Then we get the adjoint \( T_{\mu}^* \) of the isometry \( T_{\mu} \) expressed as:
\[
T_{\mu}^*(F) = \sum_{i=1}^n \mathbb{E}_{\mu} \left( \frac{\partial p}{\partial x_i} \left( X^{(\mu)}_{\varphi_1}, \cdots, X^{(\mu)}_{\varphi_n} \right) \right) \varphi_i.
\]

(Note that the right-hand side in (4.18) is in \( L^2(\mu) \).)

Proof sketch. Recall that
\[
T_{\mu}\psi := X^{(\mu)}_{\psi} : L^2(\mu) \rightarrow L^2(\Omega, \mathbb{P}^{(\mu)})
\]
as in (4.10), and
\[
X^{(\mu)}_{\psi} = \int_M \psi \, dX^{(\mu)}
\]
is the stochastic integral, where \( dX^{(\mu)} \) denotes the Ito-Wiener integral.
The arguments combine the results in the present section, and standard facts regarding the Malliavin derivative. (See, e.g., [JP17, Kul02, Ewa08, DMOkRs16].) Recall that the operator
\[ D : L^2(\Omega, \mathbb{P}^{(\mu)}) \rightarrow L^2(\Omega, \mathbb{P}^{(\mu)}) \otimes L^2(\mu) \] from (4.17) is the Malliavin derivative corresponding to the Gaussian field (4.19); see also Corollary 4.2.

In the arguments below, we restrict consideration to the case of real valued functions. We shall also make use of the known fact that the space of functions \( F \) in (4.16) is dense in \( L^2(\Omega, \mathbb{P}^{(\mu)}) \) as \( n \in \mathbb{N} \), polynomials \( p(x_1, \ldots, x_n) \), and \( \{\varphi_i\}_{i=1}^n \) vary, \( \varphi_i \in L^2(\mu) \).

The key step in the verification of the formula (4.18) for \( T^* \), from \( L^2(\Omega, \mathbb{P}^{(\mu)}) \) onto \( L^2(\mu) \), is the following assertion: Let \( F \) and \( X_\psi^{(\mu)} \), \( \psi \in L^2(\mu) \), be as stated; then
\[
\langle F, X_\psi^{(\mu)} \rangle_{L^2(\Omega, \mathbb{P}^{(\mu)})} = \mathbb{E}_\mu \left( FX_\psi^{(\mu)} \right) = \sum_{i=1}^n \mathbb{E}_\mu \left( \frac{\partial p}{\partial x_i}(X_\varphi^{(\mu)}, \ldots, X_\psi^{(\mu)}) \langle \varphi_i, \psi \rangle_{L^2(\mu)} \right).
\] (4.21)

But (4.21) in turn follows from the basic formula for the finite-dimensional Gaussian distributions \( g^{(n)}(x) \) in Proposition 4.1 above. We have:
\[
\int_{\mathbb{R}^n} \frac{\partial p}{\partial x_i}(x_1, \ldots, x_n) g^{(n)}(x_1, \ldots, x_n) d^{(n)}x = \int_{\mathbb{R}^n} x_i p(x_1, \ldots, x_n) g^{(n)}(x_1, \ldots, x_n) d^{(n)}x
\]
where \( d^{(n)}x = dx_1dx_2\ldots dx_n \) is the standard Lebesgue measure on \( \mathbb{R}^n \).

The general case is as follows: Set \( C = [\mu(A_i \cap A_j)]_{i,j} \), the covariance matrix from (4.3), and
\[
g(x) := g^{(A_i)}(x) = (\det C)^{-n/2} e^{-\frac{1}{2} \langle x, C^{-1} x \rangle_{\mathbb{R}^n}};
\]
then
\[
\mathbb{E}_\mu \left( \sum_i \frac{\partial p}{\partial x_i} \langle \varphi_i, \psi \rangle_{L^2(\mu)} \right) = \int_{\mathbb{R}^n} \sum_i \frac{\partial p}{\partial x_i}(x) g(x) \langle \varphi_i, \psi \rangle_{L^2(\mu)} d^{(n)}x
\]
\[
= \int_{\mathbb{R}^n} p(x) \left( \sum_{i,j} C^{-1}_{ij} x_j \right) g(x) \langle \varphi_i, \psi \rangle_{L^2(\mu)} d^{(n)}x.
\]
\[
\mathbb{E}_\mu \left( p T_\mu \left( \sum_{i,j} C_{ij}^{-1} \varphi_j \langle \varphi_i, \psi \rangle_{L^2(\mu)} \right) \right),
\]
where
\[
\psi \mapsto \sum_{i,j} C_{ij}^{-1} \varphi_j \langle \varphi_i, \psi \rangle_{L^2(\mu)}
\]
is the projection from \(\psi\) onto \(\text{span}\ \{\varphi_i\}\).

Recall the correspondence \((p, \varphi_1, \ldots, \varphi_n) \mapsto F\) in (4.16), where \(p = p(x_1, \ldots, x_n)\), \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n\). The random variable \(F\) has the Wiener-chaos representation in (4.16).

**Corollary 4.6.** Let \((M, \mathcal{B}, \mu)\) be a \(\sigma\)-finite measure, and let \(\{X_\varphi^{(\mu)} | \varphi \in L^2(\mu)\}\) be the corresponding Gaussian field. We then have the following covariance relations for \((X_\varphi^{(\mu)})^m\) corresponding to the even and odd values of \(m \in \mathbb{N}\):

\[
\mathbb{E}_\mu \left( (X_\varphi^{(\mu)})^{2n} X_\psi^{(\mu)} \right) = 0, \quad \forall \varphi, \psi \in L^2(\mu); \quad \text{and}
\]
\[
\mathbb{E}_\mu \left( (X_\varphi^{(\mu)})^{2n+1} X_\psi^{(\mu)} \right) = \langle \varphi, \psi \rangle_{L^2(\mu)} \|\varphi\|_{L^2(\mu)}^{2n} (2n+1)!!
\]

where
\[
(2n+1)!! = (2n+1)(2n-1)\cdots5 \cdot 3 = \frac{(2(n+1))!}{2^{n+1}(n+1)!}.
\]

**Proof.** This is immediate from (4.21), and an induction argument. Take \(n = 1\), and \(p(x) = x^m\); starting with
\[
\mathbb{E}_\mu \left( (X_\varphi^{(\mu)})^2 X_\psi^{(\mu)} \right) = 2\mathbb{E}_\mu \left( X_\varphi^{(\mu)} \langle \varphi, \psi \rangle_{L^2(\mu)} \right) = 0
\]
and
\[
\mathbb{E}_\mu \left( (X_\varphi^{(\mu)})^3 X_\psi^{(\mu)} \right) = 3\mathbb{E}_\mu \left( (X_\varphi^{(\mu)})^2 \langle \varphi, \psi \rangle_{L^2(\mu)} \right)
\]
\[
\|\varphi\|_{L^2(\mu)}^2
\]

\(\square\)

### 4.1. Ito calculus.

In this section we discuss properties of the Gaussian process corresponding to the Hilbert space factorizations from the setting in Theorem 3.1.

The initial setting is a fixed \(\sigma\)-finite measure \((M, \mathcal{B}, \mu)\) with corresponding
\[
\mathcal{B}_{fin} = \mathcal{B}_{fin}(\mu) = \{ A \in \mathcal{B} | \mu(A) < \infty \}.
\]

As in Section 3, we shall study positive functions \(\beta\)
\[
\mathcal{B}_{fin} \times \mathcal{B}_{fin} \xrightarrow{\beta} \mathbb{R};
\]

\(\square\)
i.e., it is assumed that for $\forall n \in \mathbb{N}$, $\forall \{c_i\}_1^n$, $\{A_i\}_1^n$, $c_i \in \mathbb{R}$, $A_i \in \mathcal{B}_{\text{fin}}$, we have

$$
\sum_i \sum_j c_i c_j \beta(A_i, A_j) \geq 0. \tag{4.24}
$$

Then let $X = X^{(\beta)}$ be the Gaussian process with

$$
\begin{cases}
\mathbb{E}(X_A) = 0, \\
\mathbb{E}(X_A X_B) = \beta(A, B), \quad \forall A, B \in \mathcal{B}_{\text{fin}}.
\end{cases} \tag{4.25}
$$

Theorem 4.7. Let $(M, \mathcal{B}, \mu)$ be as above, and let $\beta$ be a corresponding p.d. function, i.e., we have (4.22)–(4.24) satisfied.

Now suppose there is a Hilbert space $\mathcal{H}$ such that the conditions in Theorem 3.1 are satisfied.

Then the Gaussian process $X = X^{(\beta)}$ admits an Ito-integral representation: Let $X^{(\mu)}$ denote the Gaussian field from Proposition 4.1 and Corollary 4.2. Then there is a function $l$, as follows:

$$
\begin{array}{c}
\mathcal{B}_{\text{fin}} \xrightarrow{l} L^2(M, \mu) \\
\psi \xrightarrow{l} l_A
\end{array} \tag{4.26}
$$

such that

$$
X_A = \int_M l_A(x) dX^{(\mu)}(x), \quad \forall A \in \mathcal{B}_{\text{fin}}; \tag{4.27}
$$

where (4.27) is the Ito-integral from Corollary 4.2.

We shall first need a lemma which may be of independent interest.

Lemma 4.8. With the conditions on $(\beta, \mu)$ as in the statement of Theorem 3.1 and Theorem 4.7, we get existence of an $L^2(\mu)$-factorization for the initially given p.d. function $\beta$ (see (4.22)–(4.24)). Specifically, $\beta$ admits a representation:

$$
\beta(A, B) = \int_M l_A(x) l_B(x) d\mu(x), \quad \forall A, B \in \mathcal{B}_{\text{fin}} \tag{4.28}
$$

with $l_A \in L^2(\mu), \forall A \in \mathcal{B}_{\text{fin}}$.

Proof of the lemma. An application of Theorem 3.1 yields a closed linear operator $T$ from $L^2(\mu)$ into $\mathcal{H}$, having $\mathcal{D}_{\text{fin}}(\mu) \subset L^2(\mu)$ as dense domain. Moreover, we have:

$$
\begin{align*}
\beta(A, B) & = \langle \chi_A, \chi_B \rangle_{\mathcal{H}} \\
& = \langle T(\chi_A), T(\chi_B) \rangle_{\mathcal{H}} \\
& = \langle T^* T \chi_A, \chi_B \rangle_{L^2(\mu)}
\end{align*}
$$
since $T^*T$ is selfadjoint

$$\left\langle \left( (T^*T)^{\frac{1}{2}} \right)^2 \chi_A, \chi_B \right\rangle_{L^2(\mu)}$$

= $$\left\langle (T^*T)^{\frac{1}{2}} \chi_A, (T^*T)^{\frac{1}{2}} \chi_B \right\rangle_{L^2(\mu)}.$$

Now setting,

$$l_A := (T^*T)^{\frac{1}{2}} \chi_A, \quad A \in \mathcal{B}_{fin}, \quad (4.29)$$

the desired conclusion (4.28) follows. □

**Proof of Theorem 4.7.** Let $(\beta, \mu)$ be as in the statement of Theorem 4.7, and let $\{l_A\}_{A \in \mathcal{B}_{fin}}$ be the $L^2(\mu)$-function in (4.29). We see that the factorization (4.28) is valid.

Hence, by Corollary 4.2, the corresponding Ito-integral (4.27) is well defined; and the resulting Gaussian process $X_A := \int_M l_A(x) dX^{(\mu)}_x$ is a Gaussian field with $\mathbb{E}(X_A) = 0$. Hence we only need to verify the covariance condition in (4.25) above:

Let $A, B \in \mathcal{B}_{fin}$, and compute:

$$\mathbb{E}(X_A X_B) = \mathbb{E} \left[ \left( \int_M l_A(x) dX^{(\mu)}_x \right) \left( \int_M l_B(x) dX^{(\mu)}_x \right) \right]$$

= by Cor. 4.3, the Ito-isometry

$$\int_M l_A(x) l_B(x) d\mu(x) \quad (\mu = QV(X^{(\mu)}))$$

= by Lem. 4.8, see (4.28)

$$\beta(A, B);$$

and the proof is completed. □

**Remark 4.9** (fractional Brownian motion). As an application of Theorem 4.7, consider the case of $(\mathbb{R}, \mathcal{B}, \lambda_1)$ (so $\mu = \lambda_1$), i.e., standard Lebesgue measure on $\mathbb{R}$, with $\mathcal{B}$ denoting the standard Borel-sigma-algebra. We shall discuss fractional Brownian motion with Hurst parameter $H$ (see [MVN68, Man82, DvZ05, DvZZ05, AJL11, AJ12]).

Recall, on $[0, \infty)$, fractional Brownian motion $\{X^{(H)}_t\}_{t \in [0, \infty)}$, $0 < H < 1$, fixed, may be normalized as follows: $X^{(H)}_0 = 0$,

$$\mathbb{E}(X^{(H)}_t) = 0, \quad \text{and} \quad (4.30)$$

$$\mathbb{E}(X^{(H)}_s X^{(H)}_t) = \frac{1}{2} \left( s^{2H} + t^{2H} - |s - t|^{2H} \right), \quad \forall s, t \in [0, \infty). \quad (4.31)$$

The corresponding process induced by $\mathcal{B}_{fin}$ is

$$X^{(H)}_{[0,t]} := X^{(H)}_t; \quad (4.32)$$
and we shall adapt (4.32) as an identification. The following spectral representation is known: Set, for \( \lambda \in \mathbb{R} \),
\[
d\mu^{(H)}(\lambda) = \frac{\sin(\pi H) \Gamma(1 + 2H)}{2\pi} |\lambda|^{1-2H} d\lambda; \tag{4.33}
\]
then
\[
\mathbb{E} \left( X_s^{(H)} X_t^{(H)} \right) = \int_{\mathbb{R}} \frac{e^{i\lambda s} - 1}{\lambda^2} \frac{e^{-i\lambda t} - 1}{\lambda^2} d\mu^{(H)}(\lambda). \tag{4.34}
\]

A choice of factorization for the kernel \( K^{(H)}(s, t) \) in (4.31) is then as follows:
\[
K^{(H)}(s, t) = \frac{1}{2} \left( s^{2H} + t^{2H} - |s - t|^{2H} \right)
= \int_{\mathbb{R}} l_s(x) l_t(x) \, dx \quad (s, t \in [0, \infty))
\]
with
\[
l_t(x) = \frac{1}{\Gamma(H + \frac{1}{2})} \left( \chi_{(-\infty, 0]}(x) \left( (t - x)^{H - \frac{1}{2}} - (-x)^{H - \frac{1}{2}} \right) + \chi_{[0, t]}(x) (t - x)^{H - \frac{1}{2}} \right), \quad x \in \mathbb{R}, \ t \in [0, \infty). \tag{4.35}
\]

**4.2. Application to fractional Brownian motion.**
Fix \( H, 0 < H < 1 \), the Hurst parameter, and let \( \{X_t^{(H)}\}_{t \in [0, \infty)} \) be fractional Brownian motion (fBM), see Remark 4.9. Then the special case \( H = \frac{1}{2} \) corresponds to standard Brownian motion (BM). We shall write \( X_t^{(H)} = W_t \); where “\( W \)” is for Wiener. Now \( \{W_t\}_{t \in [0, \infty)} \) is a martingale; and standard Brownian motion has independent increments, by contrast to the case when \( H \neq \frac{1}{2} \), i.e., fBM.

(i) Ito-integral representation for \( X_t^{(H)} \) when \( H \neq \frac{1}{2} \).

We now combine Theorem 4.7, (4.35) and (4.27) to conclude that \( X_t^{(H)} \) has the following Ito-integral representation:

Let \( \{l_t^{(H)}\}_{t \in [0, \infty)} \) be the integral kernel from (4.35). Note, it depends on the value of \( H \), but we shall fix \( H, H \neq \frac{1}{2} \). Then
\[
X_t^{(H)} = \int_{\mathbb{R}} l_t^{(H)}(x) \, dW_x; \tag{4.36}
\]
where \( \text{RHS}(4.36) \) is the Ito-integral introduced in Corollary 4.2 in the more general setting of \( X^{(\mu)} \). Here, \( \mu = \lambda_1 = dx \) is standard Lebesgue measure; and \( QV(W_x) = dx \); see Corollary 4.3.

(ii) Filtrations.
Returning to the probability space \((\Omega, \mathcal{C})\) for \(\{W_t\}_{t \in [0, \infty)}\); see Proposition 4.1, and let \(\mathcal{B}\) be the standard Borel \(\sigma\)-algebra of subsets of \(\mathbb{R}\). For \(A \in \mathcal{B}\), we denote by \(\mathcal{F}(A) := \) the sub \(\sigma\)-algebra of the cylinder \(\sigma\)-algebra in \(\Omega\) (see (4.4)) generated by the random variables \(W_B\), as \(B \in \mathcal{B}\) varies over subsets \(B \subseteq A\). Let \(l_t^{(\pm)}(x)\) denote the two separate terms on RHS (4.35), i.e.,

\[
l_t^{(-)}(x) = \chi_{(-\infty, 0]}(x) \left( (t - x)^{H - \frac{1}{2}} - (-x)^{H - \frac{1}{2}} \right) / \Gamma \left( H + \frac{1}{2} \right)
\]

and

\[
l_t^{(+)}(x) = \chi_{[0, t]}(x) (t - x)^{H - \frac{1}{2}} / \Gamma \left( H + \frac{1}{2} \right).
\]

Then there are two components (of fractional Brownian motion):

\[
X_t^{(-)} = \int_{-\infty}^0 l_t^{(-)}(x) \, dW_x,
\]

and

\[
X_t^{(+)} = \int_0^t l_t^{(+)}(x) \, dW_x;
\]

where \(H \neq \frac{1}{2}\) is fixed; (supposed in the notation.)

The two processes \((X_t^{(-)})\) and \((X_t^{(+)})\) are independent, and

\[
X_t = X_t^{(H)} = X_t^{(-)} \oplus X_t^{(+)}
\]

with

\[
\mathbb{E}(X_t X_t) = \mathbb{E}(X_t^{(-)} X_t^{(-)}) + \mathbb{E}(X_t^{(+)} X_t^{(+)}), \quad \forall s, t \in [0, \infty). \tag{4.38}
\]

These processes \((X_t^{(\pm)})\) result from the initial fBM \(X_t^{(H)}\) itself, as conditional Gaussian processes as follows:

\[
\mathbb{E}\left( X_t^{(+)} | \mathcal{F}((-\infty, 0]) \right) = 0; \tag{4.39}
\]

and

\[
\mathbb{E}(X_t | \mathcal{F}((-\infty, 0])) = X_t^{(-)} \tag{4.40}
\]

and

\[
\mathbb{E}(X_t | \mathcal{F}([0, t])) = X_t^{(+)} \tag{4.41}
\]

So \(X_t^{(-)}\) in (4.40) is the backward process, while \(X_t^{(+)}\) is the corresponding forward process.

**Corollary 4.10.** Fix \(H\) (Hurst parameter) as above, and consider the fractional Brownian motion \(X_t^{(H)}\), and its forward part \(X_t^{(+)} := (X_t^{(H)})^+\) given in (4.41). Then \((X_t^{(H)})^+\) is a semimartingale, i.e., if \(0 < s < t\), then

\[
\mathbb{E}\left( X_t^{(+)} | \mathcal{F}([0, s]) \right) = X_s^{(+)}. \tag{4.42}
\]
Proof.

\[
\text{LHS}_{(4.42)} \overset{(4.41)}{=} E \left( E \left( X_t^{(H)} \mid \mathcal{F}([0,t]) \right) \mid \mathcal{F}([0,s]) \right) \\
= E \left( X_t^{(H)} \mid \mathcal{F}([0,s]) \right) \\
= E \left( \left( X_t^{(H)} - X_s^{(H)} \right) + X_s^{(H)} \mid \mathcal{F}([0,s]) \right) \\
\overset{(4.39)}{=} E \left( X_s^{(H)} \mid \mathcal{F}([0,s]) \right) \overset{(4.41)}{=} X_s^{(+)}.
\]

\[\Box\]

We stress that the proofs of these properties of fBM, (with \(H \neq \frac{1}{2}\)) follow essentially from our conclusions in Remark 4.9, as well as Corollaries 4.2 and 4.3.

The spectral representation.

The formula (4.34) is a spectral representation in following sense: The choice of \(d\mu^{(H)}\) in (4.33) yields the following generalized Paley-Wiener space (compare (1.7)–(1.8) above):

Let \(\mathcal{H}(\mu^{(H)})\) denote the Hilbert space of functions \(f\) on \(\mathbb{R}\) such that the Fourier transform \(\widehat{f}\) is well defined and is in \(L^2(\mu^{(H)})\). Then set

\[
\|f\|_{\mathcal{H}(\mu^{(H)})}^2 = \|\widehat{f}\|_{L^2(\mu^{(H)})}^2 = \int_{\mathbb{R}} |\widehat{f}(\lambda)|^2 d\mu^{(H)}(\lambda). \tag{4.43}
\]

For \(f \in \mathcal{H}(\mu^{(H)})\), consider the Ito-integral,

\[
X^{(H)}(f) := \int f(t) dX_t^{(H)}. \tag{4.44}
\]

Then it follows from (4.34), and Theorems 3.1 and 4.7 that

\[
E \left( |X^{(H)}(f)|^2 \right) = \|f\|_{\mathcal{H}(\mu^{(H)})}^2. \tag{4.45}
\]

In particular,

\[
E \left( \left| X^{(H)}(f(t + \cdot)) \right|^2 \right) = E \left( \left| X^{(H)}(f) \right|^2 \right). \tag{4.46}
\]

This follows since the RHS in (4.43) is translation invariant, i.e., we have:

\[
f(t + \cdot)(\lambda) = e^{it\lambda} \widehat{f}(\lambda). \tag{4.47}
\]
4.3. A Karhunen-Loève representation.
The Karhunen-Loève (KL) theorem is usually stated for the special case of positive definite kernels \( K \) which are also continuous (typically on a bounded interval), so called Mercer-kernels. The starting point is then an application of the spectral theorem to the corresponding self-adjoint integral operators, \( T_K \) in \( L^2 \) of the interval. Mercer’s theorem states that if \( K \) is Mercer, then the integral operator \( T_K \) is trace-class. A Karhunen-Loève representation for a stochastic process (with specified covariance kernel \( K \)) is a generalized infinite linear combination, or orthogonal expansion, for the random process, analogous to a Fourier series representation for (deterministic) functions on a bounded interval; see e.g., [FR42, BS06]. The KL representation we give below is much more general, and it applies to the most general positive definite kernel, and makes essential use of our RKHS theorem (Corollary 4.11 below). In our KL-theorem, we also make precise the random i.i.d \( \mathcal{N}(0, 1) \)-terms inside the KL-expansion; see (4.48).

**Corollary 4.11.** Let \((M, \mathcal{B}, \mu)\) be a \(\sigma\)-finite measure space, and let \(\{X_A^{(\mu)}\}_{A \in \mathcal{B}_{fin}}\) be the associated Gaussian field (see Proposition 4.1 and Corollary 4.2.) Let \(\{\varphi_k\}_{k \in \mathbb{N}}\) be an orthonormal basis (ONB) in \(L^2(\mu)\), and set
\[
Z_k := X_{\varphi_k}^{(\mu)} = \int_M \varphi_k \ dX^{(\mu)}. \tag{4.48}
\]

(i) Then \(\{Z_k\}_{k \in \mathbb{N}}\) is an i.i.d. \(\mathcal{N}(0, 1)\)-system (i.e., a system of independent, identically distributed, \(Z_k \sim \mathcal{N}(0, 1)\) standard Gaussians.)

(ii) Moreover, \(X^{(\mu)}\) admits the following Karhunen-Loève representation \((A \in \mathcal{B}_{fin})\):
\[
X_A^{(\mu)}(\cdot) = \sum_{k \in \mathbb{N}} \left( \int_A \varphi_k \ d\mu \right) Z_k(\cdot), \tag{4.49}
\]
and, more generally, for \(\psi \in L^2(\mu)\),
\[
X_{\psi}^{(\mu)}(\cdot) = \sum_{k \in \mathbb{N}} \langle \psi, \varphi_k \rangle_{L^2(\mu)} Z_k(\cdot). \tag{4.50}
\]

(iii) In particular, \(X^{(\mu)}\) admits a realization on the infinite product space \(\Omega = \mathbb{R}^\mathbb{N}\), equipped with the usual cylinder-set \(\sigma\)-algebra, and the infinite-product measure
\[
\mathbb{P} := X_{\mathbb{N}}g_1 = g_1 \times g_1 \times \cdots, \tag{4.51}
\]
where \( g_1(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \) is the \( N(0, 1) \)-distribution. (We compute the expectation \( \mathbb{E} \) with respect to \( \mathbb{P} \), the infinite product measure \( \mathbb{P} \) in (4.51))

**Proof sketch.** When the system \( \{Z_k\}_{k \in \mathbb{N}} \) is specified as in (4.48), it follows from standard Gaussian theory (see e.g., [JS07, GDV07, DP10, AJL11, AJ12, AJ15, AJL17] and the papers cited there) that it is an i.i.d. \( N(0, 1) \)-system.

For \( A, B \in \mathcal{B}_{fin} \),

\[
\mathbb{E} \left( X_A^{(\mu)} X_B^{(\mu)} \right) = \sum_{k \in \mathbb{N}} \sum_{l \in \mathbb{N}} \int_A \varphi_k d\mu \int_B \varphi_l d\mu \mathbb{E} (Z_k Z_l) = \sum_{k \in \mathbb{N}} \int_A \varphi_k d\mu \int_B \varphi_k d\mu = \langle \chi_A, \chi_B \rangle_{L^2(\mu)} = \mu(A \cap B).
\]

Since the representation in (4.49) yields a Gaussian process with mean zero, it is determined by its covariance kernel, and the result follows.

**Remark 4.12.** In this section, we have addressed some questions that are naturally implied by our present setting, but we wish to stress that there is a vast literature in the general area of the subject, and dealing with a variety of different important issues for Gaussian fields. Below we cite a few papers, and readers may also want to consult papers cited there: [Min10, K11, PR14, DPLT18, Kul02, Ewa08, DMOkRs16, She07].

5. Measures on \((I, \mathcal{B})\) When \( I \) Is An Interval

We consider the spaces consisting of the measure spaces when \( I \) is an interval (taking \( I = [0, 1] \) for specificity); and where \( \mathcal{B} \) is the standard Borel \( \sigma \)-algebra of subsets in \( I \).

In this case, our results above, especially Corollary 3.2, take the following form:

**Theorem 5.1.** Let \( \mu \) be a \( \sigma \)-finite measure on \((M, \mathcal{B})\), and \( \beta_\mu (A, B) = \mu(A \cap B) \) the p.d. function from (3.8). Let \( \mathcal{H}(\beta_\mu) \) be the corresponding RKHS.

(i) Then \( \mathcal{H}(\beta_\mu) \) consists of all functions \( F \) on \([0, 1] \), such that \( F(0) = 0 \), and

\[
\sup_{0 \leq a < b \leq 1} \frac{|F(b) - F(a)|}{\mu([a, b])} < \infty,
\]

\[(5.1)\]
supremum over all intervals contained in \([0,1]\).

(ii) If \(dF/d\mu\) denotes the Radon-Nikodym derivative corresponding to (5.1), then the \(H(\beta_\mu)\)-norm is as follows:

First, \(dF/d\mu \in L^2(\mu)\), and

\[
\|F\|_{H(\beta_\mu)}^2 = \int_0^1 \left| \frac{dF}{d\mu} \right|^2 d\mu. \tag{5.2}
\]

Proof sketch. The idea is essentially contained in the considerations above from Section 3. Indeed, if \(F\) is as specified in (5.1) & (5.2), set for all \(A \in \mathcal{B}\),

\[
\mu_F(A) = \int_A \frac{dF}{d\mu} d\mu. \tag{5.3}
\]

Then the Radon-Nikodym derivative \(d\mu_F/d\mu\) in (3.7) satisfies \(d\mu_F/d\mu = dF/d\mu\) (see (5.2)–(5.3)).

Moreover,

\[
L^2(\mu) \ni \varphi \mapsto \int_A \varphi d\mu = F_\varphi(A) \quad (5.4)
\]
defines an isometry, mapping onto \(H(\beta_\mu)\).

\[\square\]

Example 5.2 (Cantor measures). If, for example, \(\mu = \mu_3\) is the middle-third Cantor measure, then the Devil’s Staircase function (see Figure 5.2 on page 23) is

\[
F(x) = \mu_3([0,x]). \tag{5.5}
\]

It is in \(H(\beta_\mu)\), and

\[
\frac{dF}{d\mu_3} = \chi_{[0,1]}. \tag{5.6}
\]

Note that it is important that the Radon-Nikodym derivative in (5.6) is with respect to the Cantor measure \(\mu_3\). If, for example, \(\lambda\) denotes the Lebesgue measure on \([0,1]\), then \(dF/d\lambda = 0\).

For graphical illustration of these functions, see Figures 5.1–5.2 below.

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5.1. **Time-change.** While there is earlier work in the literature, dealing with time-change in Gaussian processes, see e.g., [BNS08, BNS15]; our aim here is to illustrate the use of our results in Sections 3 and 4 as they apply to the change of the time-variable in a Gaussian process. To make our point, we have found it sufficient to derive the relevant properties for time-change for time in a half-line.

![Cumulative Distributions](image)

**Figure 5.2.** The two cumulative distributions, with support sets $[0, 1]$ and $C_{1/3}$.

**Proposition 5.3.** Let $J = [0, \infty)$ denote the positive half-line, and let $\{B_t\}_{t \in J}$ be the standard Brownian motion, i.e., $B_t \sim N(0, t)$, and

$$E(B_sB_t) = s \land t, \quad \forall s, t \in J \tag{5.7}$$

where $s \land t = \min(s, t)$. Let $h : J \to J$ be a monotone (increasing) function such that $h(0) = 0$, and set $X = X^{(h)}$ given by

$$X_t := B_{h(t)}, \quad t \in J. \tag{5.8}$$

(i) Then $X_t$ is the Gaussian process determined by the following induced covariance kernel:

$$E(X_sX_t) = h(s \land t) \tag{5.9}$$

(ii) The quadratic variation measure for $\{X_t^{(h)}\}_{t \in J}$ is

$$d\mu(t) = h'(t) \, dt, \tag{5.10}$$

where $dt$ is the usual Lebesgue measure on $J$.

(Recall that, since $h(s) \leq h(t)$ for all $s, t, s \leq t$; it follows, by Lebesgue’s theorem, that $h$ is differentiable almost everywhere on $J$ with respect to $dt$.)
Remark 5.4. Note that, if \( h(t) = t^2 \), then \( \mathbb{E}(X_sX_t) = (s \wedge t)^2 \); see Figure 5.3 on page 24 for an illustration.

Proof. Since \( h \) is monotone (increasing) and \( h(0) = 0 \), we get
\[
    h(s) \wedge h(t) = h(s \wedge t), \quad \forall s, t \in J
\]
and so the covariance kernel satisfies:
\[
    \mathbb{E}(X_sX_t) = \mathbb{E}(B_{h(s)}B_{h(t)}) \\
    = \text{by (5.7)} \quad h(s) \wedge h(t) \quad \text{by (5.11)} \quad h(s \wedge t) \\
    = \int_0^{s \wedge t} h'(x) \, dx = \mu(s \wedge t) \\
    = \mu([0, s] \cap [0, t])
\]
where \( \mu \) is the measure given in (5.10).

It now follows from Corollary 4.3 that then \( \mu \) is indeed the quadratic variation measure for \( \{X_t^{(h)}\}_{t \in J} \), as asserted. \( \square \)

![Figure 5.3. Time-change of Brownian motion](image)

Corollary 5.5. Let \( h : J \to J \) be as in Proposition 5.3, i.e., \( h(0) = 0 \), \( h(s) \leq h(t) \), for \( s \leq t \); and, as in (5.8), consider:
\[
    X_t = B_{h(t)}, \quad t \in J.
\]
Let \( f : \mathbb{R} \to \mathbb{R} \) be given, assumed twice differentiable. Then the Itô-integral formula for \( f(X_t) \) is as follows: For \( t > 0 \), we have:
\[
    f(X_t) = \int_0^t f'(X_s) \, dX_s + \frac{1}{2} \int_0^t f''(X_s) h'(s) \, ds.
\]

Proof. The result is immediate from Itô’s lemma applied to the quadratic variation term on the right-hand side in (5.13).

Recall, we proved in Proposition 5.3 (ii), eq. (5.10) that the quadratic variation of a Gaussian process with covariance measure \( \mu \) is \( \mu \) itself.
Hence (5.13) follows from a direct application to \( d\mu(s) = h'(s)\,ds \), where \( ds \) is standard Lebesgue measure on the interval \( J \).

**Corollary 5.6.** Let \( h : J \to J \), \( h(0) = 0 \), \( h \) monotone be as specified as Corollary 5.5, and let \( X_t = B_{h(t)} \) be the corresponding time-change process.

Set \( d\mu = dh = (\text{the Stieltjes measure}) = h'(t)\,dt \); see (5.13). For \((t,x) \in J \times J \), and \( f \in L^2(\mu) \), let

\[
u(t,x) = \mathbb{E}_{X_0=x} (f(X_t)).
\]

(5.14)

Then \( u \) satisfies the following diffusion equation

\[
\frac{\partial}{\partial t} u(t,x) = \frac{1}{2} h'(t) \frac{\partial^2}{\partial x^2} u(t,x),
\]

(5.15)

with boundary condition

\[
u(t,\cdot)|_{t=0} = f(\cdot).
\]

**Proof.** The assertion follows from an application of the conditional expectation \( \mathbb{E}_{X_0=x} \) to both sides in (5.13). Since the expectation of the first of the two terms on the right-hand side in (5.13) vanishes, we get from the definition (5.14) that:

\[
u(t,x) = \frac{1}{2} \int_0^t \mathbb{E}_{X_0=x} (f''(X_s)) h'(s)\,ds,
\]

and so

\[
\frac{\partial}{\partial t} u(t,x) = \frac{1}{2} h'(t) \frac{\partial^2}{\partial x^2} u(t,x)
\]

as claimed in (5.15). The remaining conclusions in the corollary are immediate. \(\square\)

**Remark 5.7.** Let \( 0 < H < 1 \) be fixed, and set

\[
h(t) := t^{2H}, \quad t \in J.
\]

(5.16)

Then the corresponding process

\[
X_t^{(H)} := B_{t^{2H}}, \quad t \in J,
\]

is a time-changed process, as discussed in Proposition 5.3. We have

\[
\mathbb{E} \left( (X_t^{(H)})^2 \right) = t^{2H}.
\]

(5.17)

Now this is the same variance as the fractional Brownian motion \( Y_t^{(H)} \); but we stress that (when \( H \) is fixed, \( H \neq 1/2 \)), then the two Gaussian processes \( X_t^{(H)} \) (time-change), and \( Y_t^{(H)} \) (fractional Brownian motion with Hurst parameter \( H \)), are different. (See Figure 5.4 on page 26.)
Figure 5.4. Fractional Brownian motion. The top two figures are sample paths of fractional Brownian motion, while the bottom two are the corresponding processes resulting from time change in the standard Brownian motion.

The reason is that the two covariance kernels are different. Indeed, when $H \neq 1/2$,

$$
\mathbb{E}(x_t^{(H)} x_t^{(H)}) 
eq \frac{1}{2} \left( s^{2H} + t^{2H} - |s-t|^{2H} \right);
$$

i.e., the two functions from (5.18) are different on $J \times J$.

Remark 5.8. For general facts on fractional Brownian motion, and Hurst parameter, see e.g., [AJL11] and [HOk02].

6. Laplacians

The purpose of the present section is to show that there is an important class of Laplace operators, and associated energy Hilbert spaces $\mathcal{H}$, which satisfies the conditions in our results from Sections 2 and 3 above. Starting with a fixed sigma-finite measure $\mu$, the setting from
sect 3 entails pairs \((\beta, H)\), subject to conditions (2.2) and (2.4), which admit a certain spectral theory. With the condition in Theorem 3.1, we showed that there are then induced sigma-finite measures \(\mu_f\), indexed by \(f\) in a dense subspace in \(H\). The key consideration implied by this is a closable, densely defined, operator \(T\) from \(L^2(\mu)\) into \(H\). The induced measures \(\mu_f\) are then indexed by \(f\) in a dense subspace in \(H\). The key consideration implied by this is a closable, densely defined, operator \(T\) from \(L^2(\mu)\) into \(H\). The key consideration implied by this is a closable, densely defined, operator \(T\) from \(L^2(\mu)\) into \(H\). The in-

duced measures \(\mu_f\) are then indexed by \(f\) in \(\text{dom}(T^*)\), the dense domain of the adjoint operator \(T^*\). If \(H\) is one of the energy Hilbert spaces, then \(T^*\) will be an associated Laplacian; see details in Proposition 6.4.

Now the Laplacians we introduce include variants from both discrete network analysis, and more classical Laplacians from harmonic analysis. As well as more abstract Laplacians arising in potential theory. There is a third reason for the relevance of such new classes of Laplace-operators: Each one of these Laplacians corresponds to a reversible Markov process (and vice versa.) The latter interconnection will be addressed at the end of section, but the more detailed implications, following from it, will be postponed to future papers. As for the research literature, it is fair to say that papers on reversible Markov processed far outnumber those dealing with generalized Laplacians.

Let \((M, \mathcal{B}, \mu)\) be a fixed \(\sigma\)-finite positive measure, and let \(\rho\) be a symmetric positive measure on the product space \((M \times M, \mathcal{B}_2)\) where \(\mathcal{B}_2\) denotes the product \(\sigma\)-algebra on \(M \times M\), i.e., the \(\sigma\)-algebra of subsets of \(M \times M\) generated by the cylinder sets

\[
\{ A \times B \mid A, B \in \mathcal{B} \}.
\]

(6.1)

We assume that \(\rho\) admits a disintegration with \(\mu\) as marginal measure:

\[
d\rho(x, y) = \rho^{(x)}(dy)\,d\mu(x);
\]

(6.2)
equivalently,

\[
\rho(A \times B) = \int_A \rho^{(x)}(B)\,d\mu(x),
\]

(6.3)
for \(\forall A, B \in \mathcal{B}\). Note that since \(\rho\) is symmetric, we also have a field of measures \(\rho^{(y)}(dx)\) such that

\[
\rho(A \times B) = \int_B \rho^{(y)}(A)\,d\mu(y).
\]

(6.4)
For the theory of disintegration of measures, we refer to [BJ15, BJ18] and the papers cited there.

Let \(\pi_i, i = 1, 2\), denote the coordinate projections

\[
\pi_1(x, y) = x, \quad \text{and} \quad \pi_2(x, y) = y,
\]

for \((x, y) \in M \times M\). Then from the assumptions above, we get

\[
\mu = \rho \circ \pi_1^{-1} = \rho \circ \pi_2^{-1}.
\]
We shall finally assume that
\[ \rho (A \times M) < \infty, \quad \forall A \in \mathcal{B}_{\text{fin}} (\mu) \] (6.5)
where \( \mathcal{B}_{\text{fin}} (\mu) = \{ A \in \mathcal{B} | \mu (A) < \infty \} \); and we set
\[ c(x) = \rho^{(x)} (M), \quad x \in M, \] (6.6)
where the measures \( \rho^{(x)} \) are the slice measures from the disintegration formula (6.2), or equivalently (6.3). We note that assumption (6.5) may be relaxed. For the results proved below, it will be enough to assume only that the function \( c(x) \) defined by the RHS in (6.6) be finite for almost all \( x \), so for a.a. \( x \) with respect to the measure \( \mu \). See (6.3).

We shall need the measure \( \nu \), given by
\[ d\nu (x) = c(x) \, d\mu (x). \] (6.7)

Given a pair \((\mu, \rho)\), as above, set
\[ (Rf) (x) = \int_M f(y) \rho^{(x)} (dy), \] (6.8)
defined on all measurable functions \( f \) on \((M, \mathcal{B})\).

The associated Laplacian (Laplace operator) is as follows:
\[ (\Delta f) (x) = \int_M (f(x) - f(y)) \rho^{(x)} (dy) \]
\[ = c(x) f(x) - (Rf) (x). \] (6.9)

**Definition 6.1.** Let \((\mu, \rho)\) be as above, and let \( \mathcal{E} \) be the associated energy Hilbert space consisting of measurable functions \( f \) on \((M, \mathcal{B})\) such that
\[ \|f\|^2_{\mathcal{E}} = \frac{1}{2} \iint_{M \times M} |f(x) - f(y)|^2 \, d\rho (x,y) < \infty; \] (6.10)
modulo functions \( f \) s.t. RHS_{(6.10)} = 0.

**Lemma 6.2.** Let a fixed pair \((\mu, \rho)\) be as above; and let \( \nu \) be the induced measure on \((M, \mathcal{B})\) given by (6.7).

(i) Then condition (2.2) is satisfied for \( \mathcal{H} = \mathcal{E} \) (the energy Hilbert space), and with
\[ \langle f, g \rangle_{\mathcal{E}} = \frac{1}{2} \iint_{M \times M} (f(x) - f(y)) (g(x) - g(y)) \, d\rho (x,y), \] (6.11)
we have, for \( A, B \in \mathcal{B}_{\text{fin}} \):
\[ \langle \chi_A, \chi_B \rangle_{\mathcal{E}} = \nu (A \cap B) - \rho (A \times B), \] (6.12)
and for \( A = B \),
\[ \|\chi_A\|^2_{\mathcal{E}} = \nu (A) - \rho (A \times A). \] (6.13)
(ii) If \( \varphi \in \mathcal{D}_{\text{fin}}(\mu) \), and \( f \in \mathcal{E} \), then
\[
\langle \varphi, f \rangle_{\mathcal{E}} = \int_{M} \varphi(x)(\Delta f)(x) \, d\mu(x) .
\] (6.14)

Proof sketch. Most of the assertions follow by direct computation, using the results in Sections 2–3 above; see also [BDM05, BJ15, JT17a, JP17, AJL18], and the papers cited there. \( \square \)

Corollary 6.3. Let the pair \((\mu, \rho)\) be as stated in Lemma 6.2, and let \( \nu \) be the measure \( d\nu(x) = c(x) \, d\mu(x) \) where \( c(x) = \rho^{(x)}(M) \) as in (6.7). On \( \mathcal{B}_{\text{fin}} \times \mathcal{B}_{\text{fin}} \), set
\[
\beta(A, B) = \nu(A \cap B) - \rho(A \times B) .
\] (6.15)

Then \( \beta \) is positive definite, and the corresponding RKHS \( \mathcal{H}(\beta) \) naturally and isometrically, embeds as a closed subspace in the energy Hilbert space \( \mathcal{E} \) from (6.11).

Proposition 6.4. Let \((\mu, \rho)\) be as above, we denote by \( T \) the inclusion identification
\[
\mathcal{D}_{\text{fin}}(\mu) \subset L^2(\mu) \underset{T}{\longrightarrow} \mathcal{E} \quad (\varphi \in L^2(\mu)) \longmapsto (\varphi \in \mathcal{E}) .
\] (6.16)

(i) Then \( T \) is closable with respect to the respective inner products in \( L^2(\mu) \) and \( \mathcal{E} \); see (6.11).
Moreover, for \( f \in \text{dom}(T^*) \subseteq_{\text{dense}} \mathcal{E} \) we have
\[
T^* f = \Delta f
\] (6.17)
where \( \Delta \) is the Laplacian in (6.9).

(ii) For \( f \in \text{dom}(T^*) \), the induced measure \( \mu_f \) from Section 3, satisfies
\[
\mu_f(A) = \int_{A} (\Delta f) \, d\mu, \quad \forall A \in \mathcal{B}_{\text{fin}}.
\] (6.18)

Proof. The details are essentially contained in the above. It is convenient to derive the closability of \( T \) as a consequence of the following symmetry property:

For operators \( T \) and \( T^* \), \( L^2(\mu) \underset{T}{\longrightarrow} \mathcal{E} \), and \( \mathcal{E} \underset{T^*}{\longrightarrow} L^2(\mu) \), we consider the following dense subspaces, respectively:
\[
\mathcal{D}_{\text{fin}}(\mu) \subset L^2(\mu) \text{, dense w.r.t. the } L^2(\mu)-\text{norm}; \quad \{ f \in \mathcal{E} \mid \Delta f \in L^2(\mu) \} \subset \mathcal{E} \text{, dense w.r.t. the } \mathcal{E}\text{-norm (6.10).}
\] (6.19) (6.20)
Then a direct verification, using Lemma 6.2, \((6.10)-(6.14)\), yields:

\[
\langle T\varphi, f \rangle_{\mathcal{E}} = \langle \varphi, \Delta f \rangle_{L^2(\mu)}
\]

for all \(\varphi \in \text{dom}(T)\) (see \((6.19)\)), and all \(f \in \text{dom}(T^*)\) (see \((6.20)\)).

Equivalently,

\[
\langle \varphi, f \rangle_{\mathcal{E}} = \int_{M} \varphi(\Delta f) \, d\mu,
\]

for functions \(\varphi\) and \(f\) in the respective domains. But we already established \((6.22)\) in Lemma 6.2 above; see \((6.14)\). Now the conclusions in the Proposition follow. \(\square\)

6.1. **Discrete time reversible Markov processes.** Let \((M, \mathcal{B})\) be a measure space. A Markov process with state space \(M\) is a stochastic process \(\{X_n\}_{n \in \mathbb{N}_0}\) having the property that, for all \(n, k \in \mathbb{N}_0\),

\[
\text{Prob}(X_{n+k} \in A \mid X_1, \ldots, X_n) = \text{Prob}(X_{n+k} \mid X_n)
\]

holds for all \(A \in \mathcal{B}\). A Markov process is determined by its transition probabilities

\[
P_n(x, A) = \text{Prob}(X_n \in A \mid X_0 = x),
\]

indexed by \(x \in M\), and \(A \in \mathcal{B}\).

It is known and easy to see that, if \(\{X_n\}_{n \in \mathbb{N}_0}\) is a Markov process, then

\[
P_{n+k}(x, A) = \int_M P_n(x, dy) P_k(y, A);
\]

and so, in particular, we have:

\[
P_n(x, A) = \int_{y_1} \int_{y_2} \cdots \int_{y_{n-1}} P(x, dy_1) P(y_1, dy_2) \cdots P(y_{n-1}, A),
\]

for \(x \in M, A \in \mathcal{B}\).

**Definition 6.5.** Let \(\mu\) be a \(\sigma\)-finite measure on \((M, \mathcal{B})\). We say that a Markov process is reversible iff there is a positive measurable function \(c\) on \(M\) such that, for all \(A, B \in \mathcal{B}\), we have:

\[
\int_A c(x) P(x, B) \, d\mu(x) = \int_B c(y) P(y, A) \, d\mu(y).
\]

**Proposition 6.6.** Let \((M, \mathcal{B}, \mu)\) be as usual, and let \((P(x, \cdot))\) be the generating transition system for a Markov process. Then this Markov
process $\{X_n\}_{n \in \mathbb{N}_0}$ is reversible if and only if there is a positive measurable function $c$ on $M$ such that the assignment $\rho$:

$$\rho(A \times B) = \int_A c(x) P(x, B) d\mu(x), \quad A, B \in \mathcal{B}, \quad (6.28)$$

extends to a symmetric sigma-additive positive measure on the product $\sigma$-algebra $\mathcal{B}_2$, i.e., the $\sigma$-algebra on $M \times M$ generated by product sets $\{A \times B \mid A, B \in \mathcal{B}\}$.

Proof. The conclusion follows from the considerations above, and the remaining details are left to the reader. □

Corollary 6.7. Let $(\mu, \rho)$ be a pair of measures, $\mu$ on $(M, \mathcal{B})$, $\rho$ on $(M \times M, \mathcal{B}_2)$ satisfying the conditions in (6.3)-(6.4), and let $c$ be the function from (6.6), then

$$P(x, A) := \frac{1}{c(x)} \rho^{(x)}(A) \quad (6.29)$$

defines a reversible Markov process.

Proof. For measurable function $f$ on $(M, \mathcal{B})$, i.e., $f : M \to \mathbb{R}$, set

$$(Pf)(x) = \int_M f(y) P(x, dy).$$

Then the path space measure for the associated Markov-process $\{X_n\}_{n \in \mathbb{N}_0}$ is determined by its conditional expectations evaluated on cylinder functions:

$$\mathbb{E}_{X_0=x} [f_0(X_0) f_1(X_1) f_2(X_2) \cdots f_n(X_n)] = f_0(x) P (f_1 P (f_2 \cdots P (f_{n-1} P (f_n)))) \cdots (x).$$

The result is now immediate from Definition 6.5. □

Corollary 6.8. Let the pair $(\mu, \rho)$ be as above, and as in Lemma 6.2. Let $\{X_n\}_{n \in \mathbb{N}_0}$ be the corresponding reversible Markov process; see Corollary 6.7.

(i) Then, for measurable functions $f$ on $(M, \mathcal{B})$, we have the following variance formula:

$$\text{VAR}_{X_0=x} (f(X_1)) = \int_M |f(y) - P(f)(x)|^2 P(x, dy)$$

(ii) Set $d\nu = c(x) d\mu(x)$, and let $\mathcal{E}$ denote the energy Hilbert space from Definition 6.1. Then a measurable function $f$ on $(M, \mathcal{B})$
is in $\mathcal{E}$ iff $f - P(f) \in L^2(\nu)$, and $\text{VAR}_x(f(X_1)) \in L^1(\nu)$. In this case,
\[
\|f\|_\mathcal{E}^2 = \frac{1}{2} \left[ \int_M |f - Pf|^2 \, d\nu + \int_M \text{VAR}_x(f(X_1)) \, d\nu(x) \right].
\]

**Proof.** Immediate from the details in Proposition 6.6 and Corollary 6.7.

**Remark 6.9.** In the last section we pointed out the connection between reversible Markov processes, and the Laplace operators, the energy Hilbert space, and our results in Sections 2 and 3. However we have postponed applications to reversible Markov processes to future papers. For earlier papers regarding Laplace operators and associated energy Hilbert space, see eg., [JP13]. The literature on reversible Markov processes is vast; see e.g., [CSC10, Lon17, BJ15, ABOPS16].

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