On Double-Elliptic Integrable Systems
1. A Duality Argument for the case of SU(2)

H.W.Braden†, A.Marshakov‡§, A.Mironov¶∥, A.Morozov∗∗

We construct a two parameter family of 2-particle Hamiltonians closed under the duality operation of interchanging the (relative) momentum and coordinate. Both coordinate and momentum dependence are elliptic, and the modulus of the momentum torus is a non-trivial function of the coordinate. This model contains as limiting cases the standard Ruijsenaars-Calogero and Toda family of Hamiltonians, which are at most elliptic in the coordinates, but not in the momenta.

1 Introduction

The theory of (classical) integrable systems has been the subject of renewed interest following the realisation that integrability is a crucial and characteristic property of non-perturbative effective actions (see, for example, [1]). From this point of view the low-energy effective actions of Yang-Mills theories [2, 3] (they are non-trivial, for example, in the presence of $N = 2$ supersymmetry) belong to universality classes represented by the simplest finite-dimensional integrable models [1, 3, 4]. At the same time these Yang-Mills theories may be associated with D-branes [3], which can be embedded in various target spaces. Unfortunately the set of known integrable models (the Calogero-Ruijsenaars and Toda family) does not include all of the universality classes arising from the various brane constructions. The main gap is a putative “double-elliptic” integrable system, where both coordinates and momenta take values in elliptic curves (complex tori), which should play a role in the description of toric, K3 and Calabi-Yau target spaces. These will be associated with (compactified) six-dimensional Yang-Mills theories.

It is the task of this paper to suggest what such a double-elliptic system can look like. We will discuss the most straightforward construction based on a duality argument for the case of Yang-Mills gauge group SU(2) (2-particle integrable system). By itself our argument is not conclusive, for in this situation there is only one

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1Department of Mathematics and Statistics, University of Edinburgh, Edinburgh EH9 3JZ Scotland; e-mail address: hwb@ed.ac.uk
†Theory Department, Lebedev Physics Institute, Moscow 117294, Russia; e-mail address: mars@lpi.ac.ru
‡ITEP, Moscow 117259, Russia; e-mail address: andrei@heron.itep.ru
§ITEP, Moscow 117259, Russia; e-mail address: mironov@lpi.ac.ru
¶ITEP, Moscow 117259, Russia; e-mail address: mironov@itep.ru
∥ITEP, Moscow 117259, Russia; e-mail address: morozov@vx.itep.ru
∗∗ITEP, Moscow 117259, Russia; e-mail address: morozov@itep.ru

1 The (second) Hamiltonian of the Calogero system [8, 9] is

$$H_{Cal}^2 = \sum_i p_i^2 + \sum_{i<j} V(q_{ij})$$

($q_{ij} = q_i - q_j$ and in the centre-of-mass $H_{Cal}^1 = \sum_i p_i = 0$), with $V(q)$ a rational, trigonometric or elliptic function of the coordinates with second order pole. The conserved Hamiltonians of the Calogero family may be written so as to exhibit a rational (polynomial) dependence on the momenta.

The (first) Ruijsenaars Hamiltonian [10] is

$$H_{Ru}^1 = \sum_i \cosh p_i \prod_{k \neq i} F(q_{ik})$$

where $F(q)$ is a rational, trigonometric or an elliptic function. The momentum dependence of the conserved Hamiltonians is now trigonometric, with the Calogero family arising as a limit of the Ruijsenaars one.

The Toda-chain family [11] (which warrants a special mention because it is associated with pure gauge $N = 2$ SUSY Yang-Mills models in 4 dimensions [12, 13, 14]) is a special double-scaling limit of the elliptic Calogero model [14].

Each of these models, while being various limits of the elliptic Ruijsenaars system, may further be embedded into the double-elliptic system introduced in the present paper.
non-trivial Hamiltonian and the multiparticle generalisation is needed. Nonetheless, it provides an important insight.

Our paper first reviews duality in the context of integrable systems. Using this, section three constructs a Hamiltonian dual to the (elliptic) Calogero model that is elliptic in the momentum. The resulting “rational-elliptic” model is investigated with comparison made both to a direct solution of the elliptic Calogero model and that resulting from use of the “projection” method of Olshanetsky and Perelomov. An interesting feature of our model is the appearance of “dressed” elliptic curves. Section four simply states the result for the dual of the (elliptic) Ruijsenaars model, our “trigonometric-elliptic” model, while section five details the construction of the wholly new “elliptic-elliptic” model. We show here how the Ruijsenaars-Calogero and Toda families arise as limits of our model. Some comments are also made on the ansatz involved in the construction of our model. We end with a brief conclusion.

## 2 Duality of integrable systems

Integrable systems may be introduced and solved in a variety of ways. Some of these include

1. The projection method \[15\], where solvable and often trivial dynamics on a given space look non-trivial after dimensional reduction or projection to a lower dimensional space.

2. A full list of Hamiltonians in involution may be given.

3. The system is exhibited in the Lax form (possibly with spectral parameter) and R-matrices given \[16\], showing the Poisson commutativity of the powers of the traces. In field theory applications the Lax representation may be deduced either from (possibly quantum) group theory, or from the dynamics of scalars – describing the shape of branes embedding \[17, 18\] – in higher-dimensional SUSY Yang-Mills models. The latter is essentially the DKN-Hitchin approach \[17, 18\].

4. Generalized WDVV equations \[19, 20, 21, 22\]. In the simplest cases they are related to Hodge theory \[20\] and, in more interesting situations, to non-trivial algebras of forms \[21\].

5. Coordinate-momentum duality.

In what follows we shall introduce and exploit the last approach, which is quite a constructive procedure in the case of \(SU(2)\). Connection will also be made with the projection method and, simply because of dimensionality, the Hamiltonian we construct together with the centre-of-mass fully describes the system. In this section we will first explain the general idea behind “duality” and then apply it in the two-particle context. We conclude the section by relating this duality to an underlying DKN-Hitchin-Seiberg-Witten structure.

### 2.1 Duality: the general idea

The idea of duality here expresses a relationship between two completely integrable systems \(S_1, S_2\) on a fixed symplectic manifold with given symplectic structure \((M, \omega)\) and goes back to \[24, 25\]. We say the Hamiltonian systems are dual when the conserved quantities of \(S_1\) and \(S_2\) together form a coordinate system for \(M\). Consider for example free particles, \(H^{(1)}_k = \sum_i p_i^k/k\). For this system the free particles momenta are identical to the conserved quantities or action variables. Now consider the Hamiltonian \(H^{(2)}_k = \sum_i q_i^k/k\) with conserved quantities \(q_i\). Together \(\{p_i, q_i\}\) form a coordinate system for phase space, and so the two sets of Hamiltonians are dual. Duality then in this simplest example is a transformation which interchanges momenta and coordinates. For more complicated interacting integrable systems finding dual Hamiltonians is a nontrivial exercise. Note that this whole construction manifestly depends on the particular choice of conserved quantities. A clever choice may result in the dual system arising by simply interchanging the momentum and coordinate dependence, as in the free system.

Some years ago Ruijsenaars \[24\] observed such dualities between various members of the Calogero-Moser and Ruijsenaars families: the rational Calogero and trigonometric Ruijsenaars models were dual to themselves while trigonometric Calogero model was dual with the rational Ruijsenaars system (see \[23\] for more examples). These dualities were shown by starting with a Lax pair \(L = L(p, q)\) and an auxiliary diagonal matrix \(A = A(q)\). When \(L\) was diagonalized the matrix \(A\) became the Lax matrix for the dual Hamiltonian, while \(L\) was a function of the coordinates of the dual system. Dual systems for a model possessing a Lax representation are then related to the eigenvalue motion of the Lax matrix.
Our approach to finding a dual system \([25]\) is to make a canonical transformation which substitutes the original set of Poisson-commuting coordinates \([q_i, q_j] = 0\), by another obvious set of the Poisson-commuting variables: the Hamiltonians \(h_i(\vec{p}, \vec{q})\) or, better, the action variables \(a_i(\vec{h}) = a_i(\vec{p}, \vec{q})\). It will be clear below that in practice really interesting transformations are a little more sophisticated: \(h_i\) are identified with certain functions of the new coordinates (these functions determine the Ruijsenaars matrix \(A(q)\)), which – in the most interesting cases – are just the same Hamiltonians with the interactions switched-off. Such free Hamiltonians are functions of momenta alone, the and dual coordinates substitute these momenta, just as one had for the system of free particles.

The most interesting question for our purposes is: what are the duals of the elliptic Calogero and Ruijsenaars systems \(?\) Since the elliptic Calogero (Ruijsenaars) is rational (trigonometric) in momenta and elliptic in the coordinates, the dual will be elliptic in momenta and rational (trigonometric) in coordinates. Having found such a model the final elliptization of the coordinate dependence is straightforward, providing us with the wanted double-elliptic systems.

2.2 The 2-particle (\(SU(2)\)) case

The calculations are especially simple in the case of \(SU(2)\) which, in the center-of-mass frame, has only one coordinate and one momentum. In this case the duality transformation can be described explicitly since the equations of motion can be integrated in a straightforward way. Technically, given two Hamiltonian systems, one with the momentum \(p\), coordinate \(q\) and Hamiltonian \(h(p, q)\) and another with the momentum \(P\), coordinate \(Q\) and Hamiltonian \(H(P, Q)\) we may describe duality by the relation

\[
\begin{align*}
    h(p, q) &= f(Q), \\
    H(P, Q) &= F(q).
\end{align*}
\]

Here the functions \(f(Q)\) and \(F(q)\) are such that

\[
dP \wedge dQ = -dp \wedge dq,
\]

which expresses the fact we have a canonical transformation. This relation entails that

\[
F'(q) \frac{\partial h(p, q)}{\partial p} = f'(Q) \frac{\partial H(P, Q)}{\partial P}.
\]

At this stage the functions \(f(Q)\) and \(F(q)\) are arbitrary. However, when the Hamiltonians depend on a coupling constant \(g^2\) and are such that their “free” part can be separated and depends only on the momenta,\(^2\) the free Hamiltonians provide a natural choice for these functions: \(F(q) = h_0(q)\) and \(f(Q) = H_0(Q)\) where

\[
\begin{align*}
    h(p, q)|_{g^2=0} &= h_0(p), \\
    H(P, Q)|_{g^2=0} &= H_0(P).
\end{align*}
\]

With such choice the duality equations become

\[
\begin{align*}
    h_0(Q) &= h(p, q), \\
    H_0(q) &= H(P, Q), \\
    \frac{\partial h(p, q)}{\partial p} H_0'(q) &= h_0'(Q) \frac{\partial H(P, Q)}{\partial P}.
\end{align*}
\]

Free rational, trigonometric and elliptic Hamiltonians are \(h_0(p) = \frac{p^2}{2}\), \(h_0(p) = \cosh p\) and \(h_0(p) = \text{cn}(p|k)\) respectively.

\(^2\) Note, that this kind of duality relates the weak coupling regime for \(h(p, q)\) to the weak coupling regime for \(H(P, Q)\). For example, in the rational Calogero case

\[
\begin{align*}
    h(p, q) &= \frac{p^2}{2} + \frac{g^2}{4q^2} = \frac{Q^2}{2}, \\
    H(P, Q) &= \frac{P^2}{2} + \frac{g^2}{4Q^2} = \frac{q^2}{2}.
\end{align*}
\]

We recall that in the brane picture the coupling constant \(g\) is related to the mass of adjoint hypermultiplet and thus remains unchanged under \(T\)-duality transformations.
2.3 The DKN-Hitchin-Seiberg-Witten structure

The main duality relation,
\[ H_0(q) = H(P, Q) \]  \hspace{1cm} (6)
can be considered as defining a family of spectral curves \( q(P) \), parameterised by a parameter (modulus) \( Q \). The symplectic structure \( dP \wedge dQ = -dp \wedge dq \), used in the formulation of duality, is then related to the generating “Seiberg-Witten” 1-form \( dS = pdq \).

From (6) it follows that
\[ \frac{\partial q}{\partial P} \bigg|_{Q} = \frac{1}{H'_0(q)} \frac{\partial H(P, Q)}{\partial P} \]  \hspace{1cm} (7)
which together with (6) implies:
\[ \frac{\partial q}{\partial P} \bigg|_{Q} = \frac{1}{h'_0(Q)} \frac{\partial h(p, q)}{\partial p} \]  \hspace{1cm} (8)
When compared with the Hamiltonian equation for the original system,
\[ \frac{\partial q}{\partial t} = \frac{\partial h(p, q)}{\partial p} \],  \hspace{1cm} (9)
we see that \( P = h'_0(Q)t \) is proportional to the ordinary time-variable \( t \). This is a usual feature of classical integrable systems, exploited in Seiberg-Witten theory [4]: in the \( SU(2) \) case the spectral curve \( q(t) \) can be described by
\[ h \left( p \left( \frac{\partial q}{\partial t}, q \right), q \right) = E. \]  \hspace{1cm} (10)
where \( p \) is expressed through \( \partial q/\partial t \) and \( q \) from the Hamiltonian equation \( \partial q/\partial t = \partial H/\partial p \). In other words, the spectral curve is essentially the solution of the equation of motion of integrable system, where the time \( t \) plays the role of the spectral parameter and the energy \( E \) that of the modulus.

3 Elliptic Calogero model and its dual

Here we begin with the elliptic Calogero Hamiltonian
\[ h(p, q) = \frac{p^2}{2} + \frac{g^2}{sn^2(q|k)}, \]  \hspace{1cm} (11)
and seek a dual Hamiltonian elliptic in the momentum. Thus \( h_0(p) = \frac{k^2}{2} \) and we seek \( H(P, Q) = H_0(q) \) such that \( H_0(q) = cn(q|k) \). Eqs.(6) become
\[ \frac{Q^2}{2} = \frac{p^2}{2} + \frac{g^2}{sn^2(q|k)}, \]
\[ cn(q|k) = H(P, Q), \]  \hspace{1cm} (12)
\[ p \cdot cn'(q|k) = Q \frac{\partial H(P, Q)}{\partial P}. \]

3 In the general case of a \( g \)-parameter family of complex curves (Riemann surfaces) of genus \( g \), the Seiberg-Witten differential \( dS \) is characterised by the property \( \delta dS = \sum_{\lambda=1}^{g} d\omega_\lambda \wedge dv_\lambda \), where \( dv_\lambda(z) \) are the \( g \) holomorphic 1-differentials on the curves (on the fibers), while \( d\omega_\lambda \) are the variations of \( g \) moduli (along the base). The associated integrable system has \( u_i \) as coordinates and \( \pi_i \) – some \( g \) points on the curve – as the momenta. The symplectic structure is
\[ \sum_{i=1}^{g} d\omega_\lambda \wedge dp_\lambda = \sum_{i,k=1}^{g} d\omega_\lambda \wedge dv_\lambda(\pi_k) \]
The vector \( p_\lambda = \sum_{k=1}^{g} \int A_k d\omega_\lambda \) is a point of the Jacobian, and the Jacobi map identifies this with the \( g \)-th power of the curve, \( Jac \cong \mathbb{C}^g \). Here \( d\omega_\lambda \) are canonical holomorphic differentials, \( dv_\lambda = \sum_{j=1}^{g} \int_{A_j} d\omega_\lambda \). The Seiberg-Witten integrals
\[ a_i(u) = \int_{A_i} dS, \]
which satisfy
\[ \frac{\partial a_i}{\partial u_j} = \int_{A_j} dv_\lambda, \]
define a flat structure on the moduli space. The generalized WDVV equations are written in terms of these coordinates [19, 20].
Upon substituting
\[ \text{cn}'(q|k) = -\text{sn}(q|k)\text{dn}(q|k) = -\sqrt{(1 - H^2)(k'^2 + k^2 H^2)}, \]
\[ (\text{this is because } \text{sn}^2 q = 1 - \text{cn}^2 q, \text{dn}^2 q = k'^2 + k^2 \text{cn}^2 q, k'^2 + k^2 = 1 \text{ and } \text{cn} q = H) \]
\[ \text{we get for } (12): \]
\[ \left( \frac{\partial H}{\partial P} \right)^2 = \frac{p^2}{Q^2} (1 - H^2) (k'^2 + k^2 H^2). \]  
\[ \]  
Now from the first eqn. (12), \( p^2 \) can be expressed through \( Q \) and \( \text{sn}^2(q|k) = 1 - \text{cn}^2(q|k) = 1 - H^2 \) as
\[ \frac{p^2}{Q^2} = 1 - \frac{2g^2}{Q^2(1 - H^2)}. \]  
so that
\[ \left( \frac{\partial H}{\partial P} \right)^2 = \left( 1 - \frac{2g^2}{Q^2} - H^2 \right) (k'^2 + k^2 H^2). \]  
Therefore \( H \) is an elliptic function of \( P \), namely
\[ H(P, Q) = \text{cn}(q|k) = \alpha(Q) \cdot \text{cn} \left( P \sqrt{k'^2 + k^2 \alpha^2(Q)} \left| \frac{k \alpha(Q)}{\sqrt{k'^2 + k^2 \alpha^2(Q)}} \right. \right) \]  
with
\[ \alpha^2(Q) = \alpha_{rat}^2(Q) = 1 - \frac{2g^2}{Q^2}. \]  

In the limit \( g^2 = 0 \), when the interaction is switched off, \( \alpha(q) = 1 \) and \( H(P, Q) \) reduces to \( H_0(P) = \text{cn}(P|k) \), as assumed in (12).

We have therefore obtained a dual formulation of the elliptic Calogero model (in the simplest SU(2) case). At first glance our dual Hamiltonian looks somewhat unusual. In particular, the relevant elliptic curve is “dressed”: it is described by an effective modulus
\[ k_{eff} = \frac{k \alpha(Q)}{\sqrt{k'^2 + k^2 \alpha^2(Q)}} = \frac{k \alpha(Q)}{\sqrt{1 - k^2(1 - \alpha^2(Q))}} \]  
which differs from the “bare” one \( k \) in a \( Q \)-dependent way. In fact \( k_{eff} \) is nothing but the modulus of the “reduced” Calogero spectral curve \( \beta(\theta) \), see eq. (8) below.

Let us rewrite (17) in several equivalent forms. First, we may solve for \( \alpha(Q) \) in terms of \( k \) and \( k_{eff} \),
\[ \alpha(Q) = \frac{k' k_{eff}}{k k_{eff}} \quad \text{and} \quad \beta(Q) \equiv \sqrt{k'^2 + k^2 \alpha^2(Q)} = \frac{k'}{k_{eff}}. \]  
Thus (17) may be expressed as
\[ H(P, Q) = \text{cn}(q|k) = \frac{k' k_{eff}}{k k_{eff}} \text{cn} \left( P \frac{k'}{k_{eff}} \left| k_{eff} \right. \right), \]  
from which it follows
\[ \text{dn}(q|k) = \sqrt{k'^2 + k^2 \text{cn}^2(q|k)} = \frac{k'}{k_{eff}} \text{dn} \left( P \frac{k'}{k_{eff}} \left| k_{eff} \right. \right). \]  

Interesting expressions arise when we express our results in terms of of theta-functions. Recall the standard relations:
\[ \text{sn}(q) = \sqrt{\frac{\epsilon_1}{\psi(q) - e_3}} = \frac{1}{k} \frac{\theta_1(q)}{\theta_4(q)} = \frac{1}{k} \frac{\theta_{11}(q)}{\theta_{01}(q)}, \]  
\[ \text{cn}(q) = \sqrt{\frac{\varphi(q) - e_1}{\varphi(q) - e_3}} = \sqrt{\frac{k'}{k}} \frac{\theta_2(q)}{\theta_4(q)} = \sqrt{\frac{k'}{k}} \frac{\theta_{10}(q)}{\theta_{01}(q)}, \]  
\[ \text{dn}(q) = \sqrt{\frac{\varphi(q) - e_2}{\varphi(q) - e}} = \sqrt{\frac{k'}{k}} \frac{\theta_3(q)}{\theta_4(q)} = \sqrt{\frac{k'}{k}} \frac{\theta_{00}(q)}{\theta_{01}(q)}. \]  

\[ \]
Here the Jacobi moduli $k^2$ and $k'^2 = 1 - k^2$ are the cross-ratios of the ramification points of the (hyper-) elliptic representation of the torus,

$$y^2 = \prod_{a=1}^{3} (x - e_a(r)), \quad \sum_{a=1}^{3} e_a = 0, \quad x = \varphi(q), \quad y = \frac{1}{2} \varphi'(q). \quad (24)$$

Then

$$k^2 = \frac{e_{23}}{e_{13}} = \frac{\theta_1'(0)}{\theta_0'(0)} = k^2 = 1 - k^2 = \frac{e_{12}}{e_{13}} = \frac{\theta_3'(0)}{\theta_0'(0)} \quad (25)$$

and

$$e_{ij} = e_i - e_j, \quad q = 2K \hat{q}, \quad \hat{q} = 2\omega \hat{q}, \quad e_{13} = \frac{K^2}{\omega^2} \quad (26)$$

Similarly,

$$P = 2K_{eff} \hat{P}, \quad E_{13} = \frac{K_{eff}^2}{\omega_{eff}^2} \quad (27)$$

where $E_1, E_2, E_3$ are the ramification points of the “hyperelliptic” representation of the “dressed” torus with modulus $\tau_{eff}$. This has two equivalent “hyperelliptic” representations \[26\]:

$$Y^2 = \prod_{a=1}^{3} (X - E_a) \quad \text{and} \quad \hat{Y}^2 = (x - u) \prod_{a=1}^{3} (x - e_a). \quad (28)$$

The equivalence of these representations follows from the rational map

$$\frac{u - e_3 x - e_1}{u - e_1 x - e_3} = \frac{X - E_3}{X - E_3}. \quad (29)$$

This allows another interpretation of formula \[17\]: here $x$ is the Weierstrass function related to the elliptic cosine in the left hand side of \[17\], while $X$ is that related to the elliptic cosine in the right hand side of \[17\] ($Y', \ Y$ are the first derivatives of the corresponding Weierstrass functions). Note that $u$ in \[28\] is related to the standard Seiberg-Witten modulus by a factor of $2g^2$: $u = u_{SW}/2g^2$ and to the energy parameter $E$ by $E = u_{SW} - 2g^2\epsilon_a$. This rescaling factor is responsible for the unusual coefficient $\sqrt{-2g^2}$ in our definition of the modulus $\alpha$ below (in \[35\]).

The relation between ramification points in the different representations is easily obtained, since the rational equivalence between the two sets of points, $e_1, e_2, e_3, u, x, x'$ and $E_1, E_2, E_3, \infty, X, X'$ implies the following cross-ratio identities for quadruples:

$$\frac{E_{ac}}{E_{bc}} = \frac{e_{ac}}{e_{bc}}, \quad \frac{X - E_c}{E_{ac}} = \frac{x - e_c}{x - u} \frac{e_{ac}}{e_a - u}, \quad \frac{X - X'}{E_{ac}} = \frac{x - x'}{x - e_c} \frac{e_{ac}}{e_a - x'},$$

i.e.

$$\frac{E_{ac}dx}{(X - E_a)(X - E_c)} = \frac{e_{ac}dx}{(x - e_a)(x - e_c)}. \quad (30)$$

It then follows that

$$\frac{dx}{y \sqrt{x - u}} = \frac{dX}{Y} \sqrt{\frac{E_{12}/e_{12}}{e_3 - u}} = \frac{dX}{Y} \sqrt{\frac{E_{23}/e_{23}}{e_1 - u}} = \frac{dX}{Y} \sqrt{\frac{E_{31}/e_{31}}{e_2 - u}}. \quad (31)$$

Note that with the above definitions, we find that \[22\] takes the form

$$\frac{\theta_{00}(q | \tau)}{\theta_{01}(q | \tau)} = \sqrt{\frac{k'}{k_{eff}^2}} \frac{\theta_{00}(P k'/k_{eff}^2 | \tau_{eff})}{\theta_{01}(P k'/k_{eff}^2 | \tau_{eff})} \quad (32)$$

or, more symmetrically,

$$\frac{\theta_{00}(q | \tau)\theta_{00}(0 | \tau)}{\theta_{01}(q | \tau)\theta_{01}(0 | \tau)} = \frac{\theta_{00}(P k'/k_{eff}^2 | \tau_{eff})\theta_{00}(0 | \tau_{eff})}{\theta_{01}(P k'/k_{eff}^2 | \tau_{eff})\theta_{01}(0 | \tau_{eff})}. \quad (33)$$
In fact, after a Landen transformation, we find that
\[
\frac{\theta_{00}(\frac{1}{2}, \frac{1}{2})}{\theta_{01}(\frac{1}{2}, \frac{1}{2})} = \pm \left(\frac{\theta_{00}\left(\frac{\tilde{p}_{k'}}{2k_{eff}}, \frac{\tau_{eff}}{2}\right)}{\theta_{01}\left(\frac{\tilde{p}_{k'}}{2k_{eff}}, \frac{\tau_{eff}}{2}\right)}\right)^{\pm 1}.
\] (34)

In terms of \(P\) and \(k_{eff}\) or \(\tau_{eff}\) the symplectic structure \(dP \wedge dQ\) looks somewhat more complicated, but these alternate representations can be useful for other purposes, including discussion of the algebraic geometry of the spectral curves.

At this stage we have everything to relate the symplectic structure \(dP \wedge dQ\) to the “canonical” one, \(dp^{Jac} \wedge da\) (see footnote 3). First of all, the variation of the flat modulus \(a = \oint_A dS\) is
\[
da = \sqrt{-2g^2}d\left(\int_A \frac{\sqrt{x-u}}{2y}dx\right) = -\frac{\sqrt{-2g^2}}{4} \left(\oint_A \frac{dx}{y\sqrt{x-u}}\right)du = -\frac{\sqrt{-2g^2}}{4} \sqrt{\frac{E_{13}}{e_{13}}} \frac{du}{\sqrt{c_2 - u}} \oint_A \frac{dX}{Y}.\] (35)

Now on the one hand we have
\[k_{eff}^2 = \frac{E_{23}}{E_{13}} = \frac{e_{23}}{e_{13}} \frac{e_1 - u}{e_2 - u} = k^2 \frac{e_1 - u}{e_2 - u} = k^2 \frac{E - 2g^2}{E - 2g^2k^2} \] (36)
while on the other hand we have
\[k_{eff}^2 = \frac{k^2\alpha^2}{\beta^2}, \] (37)
where
\[\alpha^2 = 1 - \frac{2g^2}{Q^2} \quad \text{and} \quad \beta^2 = k^2 + k^2 \alpha^2 = 1 - \frac{2g^2k^2}{Q^2}. \] (38)

Thus
\[\frac{e_1 - u}{e_2 - u} = 1 + \frac{e_{12}}{e_2 - u} = \frac{\alpha^2}{\beta^2} = 1 - \frac{2g^2k^2}{\beta^2Q^2} \] (39)
with \(k^2 = e_{12}/e_{13}\), and so
\[\frac{e_{13}}{e_2 - u} = -\frac{2g^2}{\beta^2Q^2}, \quad du = \frac{e_{13}}{g^2}QdQ. \] (40)

Utilizing these gives
\[\oint_A \frac{dX}{Y} = \frac{\sqrt{E_{13}}}{\sqrt{e_2 - u}} \oint_A \frac{du}{\sqrt{e_{13}}} \oint_A \frac{dX}{Y} = -\frac{1}{2\beta} \oint_A \frac{dQ}{\left(\sqrt{E_{13}} \oint_A \frac{dX}{Y}\right)} \] (41)

Combining these expressions then yields
\[dP \wedge dQ = 2d(\beta P) \wedge da \left(\sqrt{E_{13}} \oint_A \frac{dX}{Y}\right)^{-1}. \] (42)

Since the coordinate on the Jacobian differs from the argument of the Jacobi function by a factor \(2\omega_{eff}\sqrt{E_{13}}\),
\[P\beta = 2\omega_{eff}\sqrt{E_{13}} \cdot p^{Jac}, \] (43)
and \(\oint_A \frac{dX}{Y} = 4\omega_{eff}\), we finally have
\[dP \wedge dQ = dp^{Jac} \wedge da. \] (44)

Thus our symplectic form is the canonical one.

### 3.1 Comment 1. Elliptic solution of Calogero model

According to the argument of §2.3 our Hamiltonian \(H_{Cal}\) should be simply related to the solution \(q(t)\) of the equations of motion of the Calogero Hamiltonian, which in the case of \(SU(2)\) are immediately integrated to give
\[H_{Cal} \left(\frac{\partial q}{\partial P}, q\right) = E \] (45)
More explicitly, the equation

\[ \frac{dq}{dt} = \sqrt{E - \frac{2g^2}{\text{sn}^2(q|k)}}, \] (46)

has a solution \([27, 28]\):

\[ \text{cn}(q|k) = \sqrt{1 - \frac{2g^2}{E}} \cdot \text{cn} \left( t\sqrt{E - 2g^2k^2} \right), \] (47)

This may be derived straightforwardly by differentiating both sides and applying (13). Note that the Calogero equation (46) and the family of Calogero spectral curves are essentially independent of the value of coupling constant \(g^2\): it can be absorbed into rescaling of moduli (like \(E\)) and the time-variables (like \(t\)).

In order to see that (47) is identical to (17) one needs to put \(E = Q^2\) and make the rescaling \(P = h'_0(Q)t = Qt\). With these substitutions we find that

\[ \sqrt{1 - \frac{2g^2}{E}} = \sqrt{1 - \frac{2g^2}{Q^2}} = \alpha_{\text{rat}}(Q), \] (48)

and

\[ t\sqrt{E - 2g^2k^2} = P\sqrt{1 - \frac{2g^2k^2}{Q^2}} = P\sqrt{k'^2 + k^2\alpha_{\text{rat}}^2(Q)}, \]
\[ k\sqrt{\frac{E - 2g^2}{E - 2g^2k^2}} = k\sqrt{\frac{1 - 2g^2/Q^2}{1 - 2g^2k^2/Q^2}} = \frac{k\alpha_{\text{rat}}(Q)}{\sqrt{k'^2 + k^2\alpha_{\text{rat}}^2(Q)}}. \] (49)

We then see that (17) is identical to (17).

We remark that the relevant symplectic structure here is

\[ dE \wedge dt = 2QdQ \wedge dt = -2dP \wedge dQ. \] (50)

### 3.2 Comment 2. Projection method

Another important remark about the Calogero model is that its elliptic solution – and thus the \(SU(2)\) dual Hamiltonian (17) – can be obtained by the projection method: the spectral curve \(q(P)\) is embedded into its Jacobian (an abelian variety, i.e. a torus of complex dimension \(N = 2\)) by a simple algebraic equation

\[ \Theta(\hat{q}, \hat{P}) \equiv \Theta(\hat{q} + \hat{P}, \hat{q} - \hat{P}) = 0 \] (51)

where \(\hat{q}\) and \(\hat{P}\) are just the two coordinates on the Jacobian (hats appear because of the difference in normalization of arguments of theta and Jacobi elliptic functions, see (23)). Now the Calogero spectral curve – and consequently the relevant genus-two theta-function – has a very particular period matrix: the sum of all the elements in every row is the same (independent of the number of a row),

\[ \sum_{j=1}^{N} T_{ij} = \text{const} \] (52)

\[ ^4 \text{In the following sections we prove that not only the elliptic-rational (the dual of the elliptic Calogero model) but also the elliptic-trigonometric (the dual of the elliptic Ruijsenaars model) and the elliptic-elliptic (our new double-elliptic) Hamiltonians have the same form (47), but the latter with the identifications } E = \sin^2 Q \text{ and } E = \text{sn}^2(Q|k). \text{ Thus they are also related to Calogero equation (44). However, the relevant symplectic structures – which are always given by } dP \wedge dQ = h'_0(Q)dt \wedge dQ = dh_0(Q) \wedge dt – \text{ are no longer equivalent to } dE \wedge dt \text{ (since } E \neq h_0(Q), \text{ i.e. } E \text{ is no longer associated with the proper Hamiltonian).} \]
In the case of $N = 2$ this means that the period matrix has $T_{11} = T_{22}$. The corresponding theta-functions are then easily represented in terms of genus-one theta-functions. For example:

$$\hat{\Theta}(\xi_+, \xi_-) = \Theta$$

$$= \sum_{m,n \in \mathbb{Z}} \exp i\pi \left( (m + 1/2)\tau_r + 2(m + 1/2)(n + 1/2)s + (n + 1/2)^2 \tau_{\text{eff}} + 2(m + 1/2)(\xi_+ + 1/2) + 2(n + 1/2)(\xi_- + 1/2) \right)$$

$$= \left( \sum_{k \in \mathbb{Z}, l \in \mathbb{Z}} - \sum_{k \in \mathbb{Z}, l \in \mathbb{Z}} \right) \exp i\pi \left( 2k^2\tau + 2\tau_{\text{eff}} + 4k\xi_+ + 4l\xi_- \right)$$

$$= \theta_{00}(2\xi_+ | 2\tau) \theta_{10}(2\xi_- | 2\tau_{\text{eff}}) - \theta_{10}(2\xi_+ | 2\tau) \theta_{00}(2\xi_- | 2\tau_{\text{eff}})$$

$$= \frac{1}{2} \left( \theta_{01}(\xi_+ | \tau) \theta_{00}(\xi_- | \tau_{\text{eff}}) - \theta_{00}(\xi_+ | \tau) \theta_{01}(\xi_- | \tau_{\text{eff}}) \right)$$

(53)

where $\tau = r + s$ and $\tau_{\text{eff}} = r - s$ and $\xi_\pm = \frac{1}{2}(\xi_1 \pm \xi_2)$.

We see that (after the appropriate $\tau$ and $\tau_{\text{eff}}$-dependent rescaling of $\hat{P}$) the equation $\hat{\Theta}(\hat{q}, \hat{P}) = 0$ has (35) and thus (17) as a solution. The different choices of sign in (34) correspond to different choices of theta-characteristics in (25) and (53), and these are related by modular transformations. This result follows from the work of [1]; see also [2, 28, 29].

The projection method provides the most direct generalisation from $SU(2)$ to $SU(N)$, i.e. to the $N$-particle systems, described (in the center of mass) by $g = N - 1$ independent coordinates and momenta. Because of (22) the genus-$N$ theta function on the Jacobian of the Calogero spectral curve always decomposes into bilinear combinations of genus-one and genus-$g$ theta functions:

$$\hat{\Theta}^{(N)}(\hat{q}, \hat{P}) = \frac{1}{N} \sum_{i=1}^{N} \theta_i \left( \hat{q} + i \frac{\tau}{N} \right) \Theta^{(g)}_{e_i} \left( \hat{P} | \frac{T^{(g)}_{\text{eff}}}{N} \right)$$

(54)

The equation

$$\hat{\Theta}^{(N)}(\hat{q}, \hat{P}) = 0$$

(55)

then defines $N$ branches of the solution $q_i(\hat{P})$, which generalises (17). As usual in Seiberg-Witten theory (see eq. (10)) the dual momenta $\hat{P}$ may be associated with the first $g$ time-variables while the dual coordinates are the moduli, parameterising the period matrix $T^{(g)}$ which characterises the covering of the bare elliptic curve.

Alternatively $\hat{\Theta}^{(N)}(\hat{q}, \hat{P})$ may be considered as a generating function for the dual Hamiltonians, with the original coordinates $q_i$ playing the role of the spectral parameter (which carries an index $i$, labeling the sheet of the $N$-sheet covering).

The Hamiltonians themselves are made from the genus-$g$ theta functions $\Theta^{(g)}_{e_i} \left( \hat{P} | \frac{T^{(g)}_{\text{eff}}}{N} \right)$ with $N$ different theta-characteristics $e_i$ (as in (17) in the case of $SU(2)$ is made from two genus-one theta-functions with half-integer characteristics – which form an elliptic cosine). The Seiberg-Witten symplectic structure defines the

5 In [28] the period matrix is taken to be [28]: $\left( \frac{\tau/2}{1/2} \frac{1/2}{-\tau_{\text{eff}}/2} \right)$, which is modular equivalent, but different from our choice in [28], $\left( \frac{r}{s} \frac{s}{r} \right)$. The obvious choice of cycles and holomorphic differentials in the case of Calogero curve, which is the double-covering of the bare torus $(0, \tau)$, is:

$$\oint_{A_1} d\omega_+ = \oint_{A_2} d\omega_- = 1$$

and

$$\oint_{B_2} d\omega_+ = -\oint_{B_1} d\omega_- = \tau_{\text{eff}}$$

Canonical differentials for such cycles are $d\omega_1 = \frac{d\omega_+ + d\omega_-}{2}$ and $d\omega_2 = \frac{d\omega_+ - d\omega_-}{2}$ and the period matrix is our $\left( \frac{r}{s} \frac{s}{r} \right)$. For another choice of cycles $\left( \frac{A_1}{A_2} \right)$, $A_1 = A_1 + A_2$, $A_2 = B_1 - B_2$, $B_1 = B_1$, $B_2 = A_2$ the period matrix (defined from the relations $\oint_{B_i} d\omega = T_{ij} \oint_{A_j} d\omega$ for arbitrary holomorphic $d\omega$) is $\left( \frac{\tau/2}{1/2} \frac{1/2}{-1/2\tau_{\text{eff}}} \right)$. For
Poisson bracket between $\vec{P}$ and $T_{e ff}^{(g)}$ such that the Hamiltonians are Poisson-commuting. Commutativity is implied by the claim [23, 24, 28] that for any $N$ (and, at least, at a special value of the coupling constant $g^2$) in addition to the explicit decomposition (54) there is also an implicit (at today's level of knowledge) one into elliptic (genus-one) theta (sigma)-functions:

$$\Theta^{(N)}(\hat{q},\hat{P}) \sim \prod_{i=1}^{N} \theta_{*} \left( \hat{q} - \hat{q}_{i}(\hat{P}) \mid \tau \right).$$

Here $q_{i}(\hat{P} \mid \tau, T_{e ff}^{(g)})$ are coordinates of the $SU(N)$ Calogero equations and so, for a given set of $g$ times $\vec{P}$, these do Poisson-commute. Note that the non-trivial coefficient of proportionality in (56) means the $\tau$-functions $\Theta^{(N)}(\hat{q},\hat{P})$ need not commute at different values of the spectral parameter $q$. In particular, the individual coefficients $\theta_{*}^{(g)} \left( \hat{P} \mid \tau_{0}, \tau \right)$ also need not commute and only their ratios will. These ratios form the Poisson-commuting Hamiltonians.

In order to get a double-elliptic system one needs to change the parameterisation of $T^{(g)}$ from rational to elliptic, or, equivalently, to adequately deform the Seiberg-Witten symplectic structure. In section 5 below we present such a deformation for the $SU(2)$ case.

4 Elliptic Ruijsenaars System

All of the above formulae are straightforwardly generalised from the Calogero (rational-elliptic) system to the Ruijsenaars (trigonometric-elliptic) system. The only difference ensuing is that the $q$-dependence of the dual (elliptic-trigonometric) Hamiltonian is now trigonometric rather than rational:

$$\alpha^{2}(q) = \alpha^{2}_{trig}(q) = 1 - \frac{2g^2}{\sinh^2 q}.$$  

For details on the geometry of the Ruijsenaars spectral curves see [31]. Rather than giving further details we will proceed directly to a consideration of the double-elliptic model.

5 The double-elliptic system

5.1 Solution of duality equations

In order to get a double-elliptic system one needs to exchange the rational $Q$-dependence in (12) for elliptic an one, and so we substitute $\alpha^{2}_{rat}(Q)$ by the obvious elliptic analogue $\alpha^{2}_{ell}(Q) = 1 - \frac{2g^2}{\sinh^2 Q}$. Moreover, now the elliptic curves for $q$ and $Q$ need not in general be the same, i.e. $\tilde{k} \neq k$.

Instead of (12) the duality equations now become

$$\begin{align*}
\text{cn}(q|k) &= H(P,Q|k,\tilde{k}), \\
\text{cn}(Q|\tilde{k}) &= H(p,q|\tilde{k},k), \\
\text{cn}'(Q|\tilde{k}) \frac{\partial H(P,Q|k,\tilde{k})}{\partial P} &= \text{cn}'(q|k) \frac{\partial H(p,q|\tilde{k},k)}{\partial p},
\end{align*}$$

and the natural ansatz for the Hamiltonian (suggested by (17)) is

$$H(p,q|\tilde{k},k) = \alpha(\tilde{k},k) \cdot \text{cn} \left( p \beta(q|\tilde{k},k) \mid \gamma(q|\tilde{k},k) \right)$$

$$H(P,Q|k,\tilde{k}) = \alpha(Q|\tilde{k},k) \cdot \text{cn} \left( \tilde{P} \beta(Q|k,\tilde{k}) \mid \gamma(Q|k,\tilde{k}) \right) = \hat{\alpha}_N(\tilde{P} \beta | \gamma).$$

---

6 For example, in the case of $N = 2$,

$$\begin{align*}
\left\{ \theta_{00}(\vec{P} \mid \frac{T}{2}), \theta_{01}(\vec{P} \mid \frac{T}{2}) \right\} &= \frac{i}{4\pi} \left\{ \vec{F} \mid \frac{T}{2} \right\} \cdot (\theta_{00}' - \theta_{01}' \theta_{01}' - \theta_{00}' \theta_{01}')(\vec{P} \mid \frac{T}{2}) \neq 0
\end{align*}$$
For ease of expression we will suppress the dependence of $\alpha, \beta, \gamma$ on $k$ and $\tilde{k}$ in what follows using $\alpha(q)$ for $\alpha(q|k, \tilde{k})$ and $\tilde{\alpha}(Q)$ for $\alpha(Q|k, \tilde{k})$ etc.

Substituting these ansatz into (65) and making use of (13), the square of the final eqn. (68) becomes

$$
\left(1 - \text{cn}^2(Q|\tilde{k})\right) \left(\tilde{k}'^2 + \tilde{k}^2 \text{cn}^2(Q|\tilde{k})\right) \left(\tilde{\alpha}'^2(Q) - \text{cn}^2(q|\tilde{k})\right) \left(\tilde{\beta}'^2(Q) - \text{cn}^2(p|\tilde{k})\right)
= \left(1 - \text{cn}^2(q|k)\right) \left(k'^2 + k^2 \text{cn}^2(q|k)\right) \left(\alpha'^2(q) - \text{cn}^2(q|k)\right) \left(\beta'^2(q) - \text{cn}^2(p|k)\right)
$$

The first two eqs.(65) together with (69) allow this to be simplified yielding

$$
\beta^2(Q) \left(1 - \text{cn}^2(Q|\tilde{k})\right) \left(\tilde{k}'^2 + \tilde{k}^2 \text{cn}^2(Q|\tilde{k})\right) \left(\tilde{\alpha}'^2(Q) - \text{cn}^2(q|\tilde{k})\right) \left(\gamma'^2 + \gamma^2 \text{cn}^2\right) = \beta^2(q) \left(1 - \text{cn}^2(q|k)\right) \left(k'^2 + k^2 \text{cn}^2(q|k)\right) \left(\alpha'^2(q) - \text{cn}^2(q|k)\right) \left(\gamma'^2 + \gamma^2 \text{cn}^2\right)
$$

Now there is cancellation between the third and fifth terms of the left and right hand sides provided

$$
\frac{\tilde{k}'^2}{k'^2} = \frac{\alpha'^2(q)}{\gamma^2}, \\
\frac{k'^2}{\tilde{k}'^2} = \frac{\tilde{\alpha}'^2(Q)}{\gamma^2}.
$$

Then, since $\gamma'^2 = 1 - \gamma^2$, these may be reexpressed as

$$
\gamma^2(q) = \frac{k^2 \alpha^2(q)}{k'^2 + k^2 \alpha^2(q)}, \\
\tilde{\gamma}^2(Q) = \frac{k^2 \tilde{\alpha}^2(Q)}{k'^2 + k^2 \tilde{\alpha}^2(Q)}
$$

With these identifications we now obtain from (61) that

$$
\frac{\beta^2(Q)}{k'^2 + k^2 \alpha^2(Q)} \left(1 - \text{cn}^2(Q|\tilde{k})\right) \left(\tilde{\alpha}'^2(Q) - \text{cn}^2(q|\tilde{k})\right) = \frac{\beta^2(q)}{k'^2 + k^2 \alpha^2(q)} \left(1 - \text{cn}^2(q|k)\right) \left(\alpha'^2(q) - \text{cn}^2(q|k)\right).
$$

This relation should hold for all values of the two independent variables $q$ and $Q$. These variables can be separated provided the terms $\text{cn}^2(q|k) \cdot \text{cn}^2(Q|\tilde{k})$ on both sides cancel each other. This implies that

$$
\beta^2(q) = \tilde{k}'^2 + \tilde{k}^2 \alpha^2(q), \\
\tilde{\beta}^2(Q) = k'^2 + k^2 \tilde{\alpha}^2(Q),
$$

and

$$
\left(1 - \text{cn}^2(Q|\tilde{k})\right) \left(\tilde{\alpha}'^2(Q) - \text{cn}^2(q|\tilde{k})\right) = \left(1 - \text{cn}^2(q|k)\right) \left(\alpha'^2(q) - \text{cn}^2(q|k)\right),
$$

i.e.

$$
\alpha^2(q) \text{sn}^2(q|k) + \text{cn}^2(q|k) = \tilde{\alpha}^2(Q) \text{sn}^2(Q|\tilde{k}) + \text{cn}^2(Q|\tilde{k}) = 1 - 2g^2 = \text{const.}
$$

Here we have represented the $q$ and $Q$-independent constant as $1 - 2g^2$ to introduce the coupling constant $g^2$ in the conventional manner. Thus we arrive at

$$
\alpha^2(q|\tilde{k}, k) = \alpha^2(q|k) = 1 - \frac{2g^2}{\text{sn}^2(q|k)}, \\
\beta^2(q|\tilde{k}, k) = \tilde{k}'^2 + \tilde{k}^2 \alpha^2(q|k), \\
\gamma^2(q|\tilde{k}, k) = \frac{\tilde{k}'^2 \alpha^2(q|k)}{k'^2 + k^2 \alpha^2(q|k)}.
$$
and finally the double-elliptic duality becomes

$$H(P, Q|k, \tilde{k}) = \text{cn}(q|k) = \alpha(Q|\tilde{k})\text{cn}\left(\sqrt{\frac{k\alpha(Q|\tilde{k}}{\sqrt{k^2 + k^2\alpha^2(Q|\tilde{k})}}\right),$$  \hspace{1cm} (69)

$$H(p, q|\tilde{k}, k) = \text{cn}(Q|\tilde{k}) = \alpha(q|k)\text{cn}\left(\sqrt{\frac{\tilde{k}\alpha(q|k)}{\sqrt{\tilde{k}^2 + \tilde{k}^2\alpha^2(q|k)}}} \right).$$  \hspace{1cm} (70)

These double-elliptic Hamiltonians are our main new result. We shall now consider various limiting cases arising from these, and discuss various other choices that can be made as an ansatz.

### 5.2 Limiting cases

We now show that the double-elliptic Hamiltonian

$$H_{\text{dell}}^\alpha(p, q|\tilde{k}, k) \equiv \alpha(q|k) \cdot \text{cn}\left(\sqrt{\frac{\tilde{k}^2 + \tilde{k}^2\alpha^2(q|k)}{\sqrt{\tilde{k}^2 + \tilde{k}^2\alpha^2(q|k)}}} \right), \hspace{1cm} (71)$$

contains the entire Ruijsenaars-Calogero and Toda family as its limiting cases, as desired. (Of course we have restricted ourselves to the $SU(2)$ members of this family in this paper.)

In order to convert the elliptic dependence of the momentum $p$ into the trigonometric one, the corresponding “bare” modulus $\tilde{k}$ should vanish: $\tilde{k} \to 0$, $\tilde{k}^2 = 1 - \bar{k}^2 \to 1$ (while $k$ can be kept finite). Then, since $\text{cn}(x|\tilde{k} = 0) = \cosh x$,

$$H_{\text{dell}}^\alpha(p, q) \to \alpha(q) \cosh p = H_{\text{Ru}}(p, q)$$  \hspace{1cm} (72)

with the same

$$\alpha^2(q|k) = 1 - \frac{2g^2}{\text{sn}^2(q|k)\text{cn}^2(q|k)}.$$  \hspace{1cm} (73)

Thus we obtain the $SU(2)$ elliptic Ruijsenaars Hamiltonian\(^7\) The trigonometric and rational Ruijsenaars as well as all of the Calogero and Toda systems are obtained through further limiting procedures in the standard way.

The other limit $k \to 0$ (with $\tilde{k}$ finite) gives $\alpha(q|k) \to \alpha_{\text{trig}}(q) = 1 - \frac{2g^2}{\cosh q}$ and

$$H_{\text{dell}}^\alpha(p, q) \to \alpha_{\text{trig}}(q) \cdot \text{cn}\left(\sqrt{\frac{\tilde{k}^2 + \tilde{k}^2\alpha^2_{\text{trig}}(q)}{\sqrt{\tilde{k}^2 + \tilde{k}^2\alpha^2_{\text{trig}}(q)}}} \right) = \tilde{H}_{\text{Ru}}(p, q).$$  \hspace{1cm} (74)

This is the elliptic-trigonometric model, dual to the conventional elliptic Ruijsenaars (i.e. the trigonometric-elliptic) system. In the further limit of small $q$ this degenerates into the elliptic-rational model with $\alpha_{\text{trig}}(q) \to \alpha_{\text{rat}}(q) = 1 - \frac{2g^2}{q}$, which is dual to the conventional elliptic Calogero (i.e. the rational-elliptic) system, analysed in some detail in section three above.

### 5.3 Other double-elliptic ansatze

Our approach has been based on choosing appropriate functions $f(q)$ and $F(Q)$ and implementing duality. Other choices of functions associated with alternative free Hamiltonians may be possible. Instead of the duality relations (58) one could consider those based on $h_0(p) = \text{sn}(p|\tilde{k})$ instead of $\text{cn}(p|\tilde{k})$. These give

$$\text{sn}(q|k) = H_\alpha(P, Q|k, \tilde{k}),$$

$$\text{sn}(Q|\tilde{k}) = H_\alpha(p, q|\tilde{k}, k),$$

$$\text{sn}'(Q|\tilde{k}) \frac{\partial H_\alpha(P, Q|k, \tilde{k})}{\partial P} = \text{sn}'(q|k) \frac{\partial H_\alpha(p, q|\tilde{k}, k)}{\partial p}.$$  \hspace{1cm} (75)

\(^7\) Indeed,

$$F^2(q) = \frac{\epsilon^2(\epsilon|k)(\mu(\epsilon) - \nu(q))}{\text{sn}^2(\sqrt{13}q|k)} = \frac{\epsilon^2(\epsilon|k)}{\text{sn}^2(\sqrt{13}q|k)} - \frac{\epsilon_{13}^2}{\text{sn}^2(\sqrt{13}q|k)} = \frac{\epsilon^2(\epsilon|k)e_{13}}{\text{sn}^2(\epsilon|k)} \left(1 - \frac{\text{sn}^2(\epsilon|k)}{\text{sn}^2(q|k)}\right)$$

where $q = 2\omega\sqrt{13}$ and $2g^2 = \text{sn}^2(\epsilon|k)$. 

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and now the natural ansatz is

\[ H_s(p, q | \tilde{k}, k) = \alpha_s(q | k) \cdot \text{sn} \left( p \beta_s(q | k) \mid \gamma_s(q | k) \right). \] (76)

All of our calculations above may be repeated with little difference, the only significant one being that instead of (13) one now uses

\[ \text{sn}'(q | k) = \sqrt{(1 - \text{sn}^2(q | k))(1 - k^2 \text{sn}^2(q | k))}. \] (77)

With this choice one gets somewhat simpler expressions for \( \beta_s \) and \( \gamma_s \):

\[ \beta_s = 1, \]
\[ \gamma_s(q | \tilde{k}, k) = \tilde{k} \alpha_s(q | k), \] (78)
\[ \alpha_s(q | k) = 1 - \frac{2g^2}{\text{cn}^2(q | k)} \]

and the final Hamiltonian is now

\[ H_s(p, q | \tilde{k}, k) = \alpha_s(q | k) \cdot \text{sn}(p \tilde{k} \alpha_s(q | k)). \] (79)

Although this Hamiltonian is somewhat simpler than our earlier choice, the limits involved in obtaining the Ruijsenaars-Calogero-Toda reductions are somewhat more involved, and that is why we chose to present the Hamiltonian (71) first.

One might further try other elliptic functions for \( h_0(p) \). Every solution we have obtained by making a different ansatz has been related to our solution (71) via modular transformations of the four moduli \( \tilde{k}, k, \tilde{k}_{\text{eff}} = \tilde{\gamma}, \text{and } k_{\text{eff}} = \gamma \).

6 Conclusion

In this paper we suggest that the 2-particle \( (SU(2)) \) Hamiltonian of the double-elliptic system is given by:

\[ H(p, q | \tilde{k}, k) = \alpha(q | k) \text{cn} \left( p \sqrt{\tilde{k}^2 + \tilde{k}^2 \alpha^2(q | k)} \mid \frac{\tilde{k} \alpha(q | k)}{\sqrt{\tilde{k}^2 + \tilde{k}^2 \alpha^2(q | k)}} \right), \] (80)
\[ \alpha^2(q | k) = 1 - \frac{2g^2}{\text{sn}^2(q | k)} \] (81)

As particular limits this model provides the elliptic Ruijsenaars system (\( \tilde{k} \to 0 \)) and its dual (\( k \to 0 \)).

A non-trivial feature of our double-elliptic model is the “dressing” of the bare elliptic moduli \( \tilde{k} \) and \( k \) which characterise the momentum and coordinate tori respectively. In general the geometry of the double-elliptic system involves two elliptic curves and two “dressed” Jacobians \( (N - 1) \)-dimensional algebraic varieties) with the period matrices \( \tilde{\tau}, \tau, \tilde{T}_{\text{eff}} \) and \( T_{\text{eff}} \). The spectral parameters can be associated with the center-of-mass momentum and coordinate or, equivalently, with theta-characteristics, as implicitly explained in [31]. The multi-particle generalisation, introduction of spectral parameters, and the relationship between these models and Yang-Mills in 6d, to double-loop algebras and conformal models, and to \( \tau \)-functions and their fermionic representations will be discussed elsewhere.

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