Abstract: We extend Panella and Roy’s work for massless Dirac particles with position-dependent (PD) velocity. We consider Dirac particles where the mass and velocity are both position-dependent. Bound states in the continuum (BIC)-like and discrete bound state solutions are reported. It is observed that BIC-like solutions are not only feasible for the ultra-relativistic (massless) Dirac particles but also for Dirac particles with PD-mass and PD-velocity that satisfy the condition \( m(x)v_F^2(x) = A \), where \( A \geq 0 \) is constant. A Dirac Pöschl-Teller and a harmonic oscillator models are also reported.

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I. INTRODUCTION

In heterostructure physics, it was believed that electrons are effectively described by the position-dependent mass (PDM) Schrödinger Hamiltonian (i.e., von Roos Hamiltonian, e.g., [1–3]). Using the Pauli spin matrices in the Schrödinger Hamiltonian, the Dirac Hamiltonian was ignored. However, this perspective has drastically changed since the discovery of graphene [4, 5]. Many studies on the applicability of Dirac Hamiltonian in condensed matter were carried out (cf., e.g., [4–13] and related references cited therein). It is found that the effective low-energy model for the quasi-particles is ultrarelativistic (i.e., massless) and is described by the Hamiltonian

\[ H = v_F \sigma \cdot p. \] (1)

Which is in fact the Dirac Hamiltonian for massless particles with an effective Fermi velocity \( v_F \) (where \( v_F = c/300 \), \( c \) is the speed of light, \( \sigma \) is a vector using Pauli matrices and \( p = -i\nabla \), with \( \hbar = 1 \)). However, the information on the material properties may be encoded in the Fermi velocity of the Dirac particles [6, 7]. In this case, the Dirac Hamiltonian (1) takes the form

\[ H = \sqrt{v_F(x)} \sigma_x p_x \sqrt{v_F(x)}. \] (2)

Hereby, one should notice that the replacement of the constant velocity, \( v_F \), by the position-dependent one, \( v_F(x) \), would render Hamiltonian (1) non-Hermitian. Whereas the form of Hamiltonian (2) preserves Hermiticity and recovers the constant \( v_F \) setting. Panella and Roy [13], for example, have used Hamiltonian (2) to study bound states in the continuum (BIC) (cf., e.g., [13] and related references therein) and discrete energy states for massless Dirac particle. Throughout this paper, we shall refer to their study as Panella-Roy’s model (namely, their model with \( m(x) = 0 \) and \( v_F(x) = v_0 \cosh^2 ax \)). They have found that with proper PD-Fermi velocity profile it is possible to create BIC-like and discrete bound-state solutions.

In this paper, motivated by theoretical curiosity and/or possible practical applicability, we propose that the information on the material properties is not only encoded in the Fermi velocity but also encoded in the mass of the Dirac particles. We therefore extend Panella and Roy’s work and consider the Dirac-Hamiltonian where the Fermi velocity and the mass are both position dependent. That is, we shall work with the Hamiltonian

\[ H = \sqrt{v_F(x)} \sigma_x p_x \sqrt{v_F(x)} + \beta m(x) v_F(x)^2, \] (3)

where \( \sigma_x \) and \( \beta \) are the usual Pauli matrices [6, 7]. Moreover, it is obvious that the second term in (3) is analogous PDM Dirac particle in a Lorentz scalar potential (cf., e.g., [14] and related references therein). The addition of such term leaves the corresponding Dirac Hamiltonian invariant under Lorentz transformation. The organization of this paper is in order.

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We discuss Hamiltonian (3) and give our methodical proposal in section II. We provide illustrative examples, including ultra-relativistic Dirac quasi-particles (i.e., particles with \( m(x) = 0 \)), in section III. In the same section, we show that similar scenarios (as those in the Panella-Roy’s model \[13\] for BIC-like and for discrete bound-states solutions) are observed for a wider class of \( m(x) \) and \( v_F(x) \) (i.e., for \( m(x) v_F^2(x) = A \), where \( A \geq 0 \) is constant). For such mass and Fermi velocity settings, a shift in the energy levels is obtained. In section IV, we show that Dirac particles may be trapped in an effective Pöschl-Teller potential \[15\] produced by both their PD-mass and PD-Fermi velocity. Moreover, we show, in section V, that the (1+1)-Dirac oscillator may just be a consequence of a linear PD-Fermi velocity and a singular PD-mass (i.e., \( v_F(x) = v_0 x \) and \( m(x) = A/x \)). Our concluding remarks are given in section VI.

II. (1+1)-DIRAC PARTICLES WITH POSITION-DEPENDENT VELOCITY AND MASS

With the usual textbook Pauli matrices and Dirac spinors, the (1+1)-Dirac equation \( H \psi(x) = E \psi(x) \), for \( H \) in (3), would decouple into

\[
- i F(x) \partial_x [F(x) \psi_2(x)] = \zeta_1(x) \psi_1(x) ; \quad \zeta_1(x) = E - m(x) F(x)^4, \\
- i F(x) \partial_x [F(x) \psi_1(x)] = \zeta_2(x) \psi_2(x) ; \quad \zeta_2(x) = E + m(x) F(x)^4,
\]

where \( F(x) = \sqrt{v_F(x)} \) and

\[
\psi(x) = N \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix},
\]

(with \( N \) as the normalization constant) are used. Now let us multiply (4) and (5), from the left, by \( F(x) \) and use the substitutions \( \tilde{\psi}_{1,2} (x) = F(x) \psi_{1,2} (x) \) to imply

\[
\tilde{\psi}_2(x) = -i \left( \frac{F(x)^2}{\zeta_2(x)} \right) \partial_x \tilde{\psi}_1(x).
\]

Which when substituted in (4) yields

\[
- \partial_x^2 \tilde{\psi}_1(x) + \left[ \frac{\zeta_2(x)}{\zeta_2(x)} - 2 \left( \frac{F'(x)}{F(x)} \right) \right] \partial_x \tilde{\psi}_1(x) = \left( \frac{\zeta_1(x) \zeta_2(x)}{F(x)^4} \right) \tilde{\psi}_1(x),
\]

where primes denote derivatives with respect to \( x \). To get rid of the first order derivative and bring (8) into the one-dimensional form of Schrödinger equation we use

\[
\tilde{\psi}_1(x) = \zeta_2(x)^\nu \phi_1(q) ; \quad q \equiv q(x).
\]

This would suggest that

\[
q'(x) = \zeta_2(x)^{1-2\nu} F(x)^{-1/2}.
\]

However, one also needs to avoid position-dependent energies and choose \( \nu = 1/2 \) to imply

\[
- \partial_q^2 \phi_1(q) + v_F(x)^2 \left[ \frac{3}{4} \left( \frac{\zeta_2'(x)}{\zeta_2(x)} \right)^2 - \frac{1}{2} \frac{\zeta_2'(x)}{\zeta_2(x)} - \frac{1}{2} \left( \frac{v_F'(x)}{v_F(x)} \right) \frac{\zeta_2'(x)}{\zeta_2(x)} \right] \phi_1(q) = \left[ E^2 - m(x)^2 v_F(x)^4 \right] \phi_1(q),
\]

with

\[
q(x) = \int^x \frac{1}{v_F(y)} \, dy,
\]

where \( q(x) \) represents a point canonical transformation. It is obvious that for massless particles equation (11) collapses into its most simplistic form

\[
- \partial_q^2 \phi_1(q) = E^2 \phi_1(q),
\]
that looks very much like the one-dimensional Schrödinger equation for free particles. However, the form of \(v_F(x)\) would determine the domain of \(q(x)\) in (12) and, therefore, has its say in the process. This is to be clarified in the illustrative examples below.

Nevertheless, one may use the non-relativistic limit \(m(x)v_F(x)^2 >> E_{\text{binding}} \equiv E_{\text{bind}}\), where \(E_{\text{bind}} = E - m(x)v_F(x)^2\) (analogous to the textbook non-relativistic limit for Dirac particles with rest mass energy \(m_0c^2 >> E_{\text{bind}}\)). One would, in this way, recover the constant non-zero mass and constant velocity settings as well as accommodate illustrative examples below. Consequently, one may recast (11) as

\[
\frac{1}{\zeta_2(x)} = \frac{1}{E + m(x)v_F(x)^2} \approx \frac{1}{E_{\text{bind}} + 2m(x)v_F(x)^2} \approx \frac{1}{2m(x)v_F(x)^2}.
\]

Consequently, one may recast (11) as

\[
-\partial_q^2 \phi_1(q) + V_{\text{eff}}(q) \phi_1(q) = E^2 \phi_1(q)
\]

where

\[
V_{\text{eff}}(q) = \frac{3}{16} \left[ \frac{m(x)v_F(x)^2}{m(x)^2 v_F(x)^2} \right]^2 - \frac{1}{4} \left[ \frac{(m(x)v_F(x)^2)''}{m(x)} + \frac{v_F'(x)}{v_F(x)} \left( \frac{m(x)v_F(x)^2)'}{m(x)} \right) \right] + m(x)^2 v_F(x)^4.
\]

Obviously, this approximation may only be used for non-zero constant mass and not for massless Dirac particles. To illustrate our methodical proposal above we discuss the following illustrative examples.

### III. BIC-LIKE AND DISCRETE BOUND-STATES SOLUTIONS: PARALLEL AND COMPLEMENTARY TO PANELLA-ROY’S MODEL

One considers the class of PD-mass and PD-Fermi velocity satisfying the condition \(m(x)v_F(x)^2 = A\), where \(A \geq 0\) is a constant. Under such assumptions, equation (11) would read

\[
-\partial_q^2 \phi_1(q) = \lambda^2 \phi_1(q),
\]

where \(\lambda^2 = E^2 - A^2\). Yet, one may notice that \(m(x) = 0\) is just a special case of the current more general proposal than that used in Panella-Roy’s model [13]. Although this equation looks like Schrödinger equation for free particle, the domain of \(q(x)\) in (12) would determine the boundary conditions on the related state functions. This is to be clarified in the following two examples. The first of which is discussed here as a complementary model that reports on the consequences of using \(m(x) = m_0/\cosh^{4} \alpha x\), and \(v_F(x) = v_0 \cosh^{2} \alpha x\) settings on Panella-Roy’s model. The second example considers \(v_F(x) = v_0 (1 + \alpha^2 x^2)\), \(m(x) = m_0/(1 + \alpha^2 x^2)^2\) and shares similar scenario on the BIC-like and the discrete bound-states solutions as that reported in Panella-Roy’s model [13].

#### A. Consequences of \(m(x)v_F(x)^2 = A\) on Panella-Roy’s model: complementary

Let the PD-mass and the PD-Fermi velocity take the following forms

\[
m(x) = \frac{m_0}{\cosh^{4} \alpha x}, \quad v_F(x) = v_0 \cosh^{2} \alpha x
\]

respectively, with the constants \(m_0, v_0 \geq 0\) and \(A = m_0v_0^2\). This would, in turn, imply

\[
q(x) = \int_{-1}^{x} \frac{1}{v_F(y)} dy = \frac{1}{\alpha v_0} \tanh \alpha x,
\]

and suggest that \(q(x) \in (-1/\alpha v_0, 1/\alpha v_0)\). Therefore, our particle under consideration is not free but rather quasi-free (i.e., trapped in a force field produced by its own PD-mass and PD-Fermi velocity in (17)) and is confined to move between \(-1/\alpha v_0\) and \(+1/\alpha v_0\).

Whilst the solution of (16) is straightforward and takes the form

\[
\phi_1(q) = \sin \lambda q,
\]
it is rather unphysical (i.e., it does not satisfy the boundary conditions imposed by the range of \( q(x) \)). Nevertheless, one may use this unphysical solution to obtain the related un-normalized wave function components through (9) and (7) as

\[
\psi_1(x) = \frac{\hat{\psi}_1(x)}{\sqrt{v_F(x)}} = \sqrt{\frac{\zeta_2}{v_0}} \sech(\alpha x) \sin \left[ \frac{\tilde{\lambda} \tanh \alpha x}{\alpha v_0} \right]; \quad \zeta_2 = E + m_\alpha v_0^2,
\]

and

\[
\psi_2(x) = \frac{\hat{\psi}_2(x)}{\sqrt{v_F(x)}} = -i \sqrt{\frac{\zeta_1}{v_0}} \sech(\alpha x) \cos \left[ \frac{\tilde{\lambda} \tanh \alpha x}{\alpha v_0} \right]; \quad \zeta_1 = E - m_\alpha v_0^2.
\]

Under such settings, the probability density \( \rho(x) \) is given by

\[
\rho(x) = |\psi_1(x)|^2 + |\psi_2(x)|^2 = N^2 \left( \frac{\zeta_1}{v_0} \sech^2(\alpha x) + 2m_\alpha v_0^2 \sech(\alpha x) \cos^2 \left[ \frac{\tilde{\lambda} \tanh \alpha x}{\alpha v_0} \right] \right),
\]

and the normalization constant \( N \) is obtained through

\[
\int_{-\infty}^{\infty} \rho(x) \, dx = 2N^2 \left( \frac{\zeta_1}{v_0} + \frac{2m_\alpha v_0^4}{\lambda} \right) = 1 \implies N = \sqrt{\frac{\alpha v_0 \lambda}{2(\tilde{\lambda} \zeta_1 + 2\alpha m_\alpha v_0^3)}}.
\]

Moreover, the probability current density

\[
j_x = \sqrt{v_F(x)} \psi_1^*(x) \sqrt{v_F(x)} \psi_2(x) + c.c.
\]

\[
= N^2 \tilde{\lambda} \left[ i - i \sin \left( \frac{\tilde{\lambda} \tanh \alpha x}{\alpha v_0} \right) \cos \left( \frac{\tilde{\lambda} \tanh \alpha x}{\alpha v_0} \right) \right]
\]

\[
= 0,
\]

which is expected to vanish for bound states. As a result, the Dirac spinor in (6) and the related two components \( \psi_1(x) \) and \( \psi_2(x) \) represent BIC-like solutions.

However, to make the unphysical solution in (19) satisfy the related boundary conditions \( \phi_1(q) = 0 \) at \( q(x) = \pm 1/\alpha v_0 \) (hence becomes a physically admissible solution) one may shift \( q(x) \rightarrow q(x) + 1/\alpha v_0 \) and recast the solution as

\[
\phi_1(q) = \sin \left[ \frac{\tilde{\lambda}}{\alpha v_0} \left( q + \frac{1}{\alpha v_0} \right) \right].
\]

This would immediately vanish at \( q = -1/\alpha v_0 \), and yield

\[
\tilde{\lambda}_n = \frac{n\pi \alpha v_0}{2} \implies E_n = \pm \sqrt{\left( \frac{n\pi \alpha v_0}{2} \right)^2 + m_\alpha^2 v_0^4}; \quad n = 1, 2, 3, \ldots
\]

for \( q = +1/\alpha v_0 \). In this case, one obtains the un-normalized components as

\[
\psi_1(x) = \sqrt{\frac{\zeta_2}{v_0}} \sech(\alpha x) \sin \left[ \frac{\tilde{\lambda}_n}{\alpha v_0} (\tanh \alpha x + 1) \right],
\]

\[
\psi_2(x) = -i \sqrt{\frac{\zeta_1}{v_0}} \sech(\alpha x) \cos \left[ \frac{\tilde{\lambda}_n}{\alpha v_0} (\tanh \alpha x + 1) \right],
\]

and the Dirac spinor would consequently read

\[
\psi_n(x) = N_n \sech(\alpha x) \begin{pmatrix} \sqrt{\frac{\zeta_2}{v_0}} \sin \left[ \frac{\tilde{\lambda}_n}{2} (\tanh \alpha x + 1) \right] \\ -i \sqrt{\frac{\zeta_1}{v_0}} \cos \left[ \frac{\tilde{\lambda}_n}{2} (\tanh \alpha x + 1) \right] \end{pmatrix},
\]
Where $N_n$ is given in (23). Yet, it should be noticed here that for $m_o = 0$ one may recover the final results of Panella-Roy’s model [13] to obtain $E_n = ±n\pi\alpha v_0/2$, $N_n = \sqrt{\alpha v_0/2E_n}$, and

$$\psi_n (x) = \sqrt{\frac{\alpha}{2}} \text{sech} (\alpha x) \left( \sin \left[ \frac{\alpha}{2} \left( \tanh \alpha x + 1 \right) \right] - i \cos \left[ \frac{\alpha}{2} \left( \tanh \alpha x + 1 \right) \right] \right) \quad (30)$$

As such, what may look like as a BIC-solution (documented in (19) and (24)) may turn out to be a bound state solution with discrete energy levels, if the proper physical boundary conditions are invested in the process (documented in (25) and (26)). Moreover, one observes a shift-up of order $m_0^2 v_0^2$ in the total energy squared, $E_n^2$, and some scaling factors in the components of the Dirac spinor (i.e., $\sqrt{\zeta_2/v_0}$ for $\psi_1 (x)$ and $\sqrt{\zeta_1/v_0}$ for $\psi_2 (x)$), as discrepancies between our current model and Panella-Roy’s model [13]. Obviously, should our $m (x)$ (i.e., the rest mass) and $v_F (x) = c$ (i.e., speed of light), then our $q (x) = x/c$ and equation (11) would collapse into the regular textbook Dirac equation for a free particle where the total energy reads $E = ±m_o c^2$.

B. Parallel to Panella-Roy’s model: $v_F (x) = v_0 (1 + \alpha^2 x^2)$

We now consider that the PD-mass as

$$m (x) = \frac{m_o}{(1 + \alpha^2 x^2)^2}$$

and the PD-Fermi velocity as

$$v_F (x) = v_0 (1 + \alpha^2 x^2) \implies q (x) = \frac{1}{\alpha v_0} \arctan \alpha x. \quad (31)$$

It is easy to observe similar scenario as that associated with $\phi_1 (q)$ of (19), where in the current case the particle described in (16) is now confined to move within $-\pi/2\alpha v_0 \leq q (x) \leq \pi/2\alpha v_0$. The unphysical solution then reads

$$\phi_1 (q) = \sin \tilde{\lambda} q = \sin \left( \frac{\tilde{\lambda}}{\alpha v_0} \arctan \alpha x \right). \quad (32)$$

This would, in turn, imply that

$$\psi_1 (x) = \frac{\tilde{\psi}_1 (x)}{\sqrt{v_F (x)}} = \sqrt{\frac{\zeta_2}{v_0}} \frac{1}{\sqrt{1 + \alpha^2 x^2}} \sin \left[ \frac{\tilde{\lambda}}{\alpha v_0} \arctan \alpha x \right] \quad (33)$$

and

$$\psi_2 (x) = \frac{\tilde{\psi}_2 (x)}{\sqrt{v_F (x)}} = -i \sqrt{\frac{\zeta_1}{v_0}} \frac{1}{\sqrt{1 + \alpha^2 x^2}} \cos \left[ \frac{\tilde{\lambda}}{\alpha v_0} \arctan \alpha x \right]. \quad (34)$$

Therefore,

$$\int_{-\infty}^\infty \rho (x) dx = \int_{-\infty}^\infty \frac{N^2}{v_0 (1 + \alpha^2 x^2)} \left[ \zeta_1 + 2m_o v_0^2 \sin^2 \left( \frac{\tilde{\lambda}}{\alpha v_0} \arctan \alpha x \right) \right] dx = 1 \implies N = \sqrt{\frac{\alpha v_0}{\pi \zeta_1 + \pi m_o v_0^2}}, \quad (35)$$

and

$$j_x = N^2 \tilde{\lambda} \left[ (i - i) \sin \left( \frac{\tilde{\lambda}}{\alpha v_0} \arctan \alpha x \right) \cos \left( \frac{\tilde{\lambda}}{\alpha v_0} \arctan \alpha x \right) \right] = 0, \quad (36)$$

indicating the existence of bound states. As such, the Dirac spinor in (6) and the related components $\psi_1 (x)$ and $\psi_2 (x)$ represent a BIC-like solution.

However, the physically admissible solution would be achieved through a shift in $q (x) \longrightarrow q (x) + \pi/2\alpha v_0$ to read

$$\phi_1 (q) = \sin \left[ \tilde{\lambda} \left( q + \frac{\pi}{2\alpha v_0} \right) \right]. \quad (37)$$
and yields

\[ \hat{\lambda}_n = n \alpha \nu_0 \implies E_n = \pm \sqrt{(n \alpha \nu_0)^2 + m^2 \nu_0^4}; \ n = 1, 2, 3, \ldots . \] (38)

Therefore,

\[ \psi_1 (x) = \frac{\psi_1 (x)}{\sqrt{v_F (x)}} = \sqrt{\frac{\xi_2}{\nu_0}} \frac{1}{\sqrt{1 + \alpha^2 x^2}} \sin \left[ n \left( \arctan \alpha x + \frac{\pi}{2} \right) \right] , \] (39)

and

\[ \psi_2 (x) = \frac{\psi_2 (x)}{\sqrt{v_F (x)}} = \frac{1}{\sqrt{\frac{\xi_1}{\nu_0}}} \frac{1}{\sqrt{1 + \alpha^2 x^2}} \cos \left[ n \left( \arctan \alpha x + \frac{\pi}{2} \right) \right] . \] (40)

Consequently, the Dirac spinor would read

\[ \psi_n (x) = N_n \frac{1}{\sqrt{1 + \alpha^2 x^2}} \left( \sqrt{\frac{\xi_2}{\nu_0}} \sin \left[ n \left( \arctan \alpha x + \frac{\pi}{2} \right) \right] \right) + \frac{1}{\sqrt{\frac{\xi_1}{\nu_0}}} \frac{1}{\sqrt{1 + \alpha^2 x^2}} \left( \sqrt{\frac{\xi_0}{\nu_0}} \cos \left[ n \left( \arctan \alpha x + \frac{\pi}{2} \right) \right] \right) ; \ n = 1, 2, 3, \ldots \] (41)

Moreover, for the case when \( m (x) = 0 \) one may obtain \( E_n = \pm \sqrt{(n \alpha \nu_0)^2} \) and

\[ \psi_n (x) = N_n \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{1 + \alpha^2 x^2}} \left( \sqrt{\frac{\xi_2}{\nu_0}} \sin \left[ n \left( \arctan \alpha x + \frac{\pi}{2} \right) \right] \right) + \frac{1}{\sqrt{\frac{\xi_1}{\nu_0}}} \frac{1}{\sqrt{1 + \alpha^2 x^2}} \left( \sqrt{\frac{\xi_0}{\nu_0}} \cos \left[ n \left( \arctan \alpha x + \frac{\pi}{2} \right) \right] \right) . \] (42)

Again one observes similar effects of the \( m (x) v_F (x)^2 \) setting on the total energy and on the components of the Dirac spinor as those mentioned in the above example.

**IV. (1+1)-DIRAC PÖSCHL-TELLER HOLES FOR \( v_F (x) = v_0 \) AND A PDM \( m(x) \neq 0 \)**

Let us now consider the case where \( v_F (x) = v_0 \implies q (x) = x/v_0 \) and

\[ m (x) = \frac{m_o}{\sin \alpha x}. \] (43)

Under such settings, the effective potential in (15) would read

\[ V_{eff} (q) = \frac{\alpha^2 v_0^2}{16} + \frac{5 \alpha^4 v_0^4}{16 \sin^2 (\alpha v_0 q)} . \] (44)

Which is obviously a shifted Pöschl-Teller type periodical potential (cf., e.g., [15]). In this case, one may rewrite (14) as

\[ - \partial_q^2 \phi_1 (q) + \frac{\tilde{V}_o}{2} \left[ \frac{s(s - 1)}{\sin^2 (\alpha v_0 q)} \right] \phi_1 (q) = \left( E^2 - \frac{\alpha^2 v_0^2}{16} \right) \phi_1 (q) , \] (45)

where \( \tilde{V}_o = 2 \alpha^2 v_0^2 \) and \( s(s - 1) = m^2 / \alpha^2 - 5/16 \). Such periodical potential setting imposes infinite impenetrable barriers manifested by the singularities between the holes (i.e., at \( q = 0, \pi/v_0 \alpha, 2\pi/v_0 \alpha, \ldots \) or equivalently at \( x = 0, \pi/\alpha, 2\pi/\alpha, \ldots \)). Here we pick up the hole within \( 0 \leq x = v_0 q \leq \pi/\alpha \) to obtain

\[ \phi_1 (q) = \sin^4 (\alpha v_0 q) \left( -n, s + n, s + \frac{1}{2}; \sin^2 (\alpha v_0 q) \right) \] (46)

\[ \phi_1 (q) = \sin^4 (\alpha v_0 q) \left( -n, s + n, s + \frac{1}{2}; \sin^2 (\alpha v_0 q) \right) . \]
and
\[ E_n = \pm \frac{\alpha v_0}{4} \sqrt{1 + 16(s + 2n)^2} ; \, n = 0, 1, 2, \ldots . \] (47)

Where
\[ s = \frac{1}{2} + \sqrt{m_0^2 \alpha^2 - \frac{1}{16}} > 1 \] (48)

Then, one would, in a straightforward manner, cast
\[ \psi_1 (x) = N_n \sqrt{\frac{E_n}{v_0}} + \frac{m_0 \sin \alpha x}{\sin \alpha x} \sin^s (\alpha x) \, _2F_1 \left( -n, s + n; s + \frac{1}{2}; \sin^2 (\alpha x) \right) \] (49)

and find \( \psi_2 (x) \), using (7), to construct the Dirac spinor of (5). Obviously, BIC-like bound states are not feasible here and only discrete bound state solutions are obtained.

V. (1+1)-DIRAC EFFECTIVE HARMONIC OSCILLATOR TOY: A BY-PRODUCT OF \( v_F (x) = v_0 x \) AND A \( m(x) = A/x \)

Consider a singular position-dependent mass along with a linear position-dependent Fermi velocity of the forms
\[ m(x) = A/x; \, v_F (x) = v_0 x ; \, x \in (0, \infty) \implies x = e^{v_0 q} \implies q (x) = \ln x^{1/v_0} ; \, q \in (-\infty, \infty) . \] (50)

In this case, equation (14) along with (15) would yield
\[ -\partial_q^2 \phi_1 (q) + \frac{1}{4} \omega^2 q^2 \phi_1 (q) = \left( E^2 + \frac{v_0^2}{16} \right) \phi_1 (q) , \] (51)

with \( \omega = 2A v_0^3 \). Obviously, \( \phi_1 (q = \pm \infty) = 0 \) represent the boundary conditions for the current Dirac harmonic oscillator at hand. In a straightforward manner, one would use the traditional textbook procedure and find that
\[ E^2 + \frac{v_0^2}{16} = \omega \left( n + \frac{1}{2} \right) \implies E_n = \pm \sqrt{Av_0^3 (2n + 1) - \frac{v_0^2}{16}} . \] (52)

and
\[ \phi_{1,n} (q) = e^{-Av_0^3 q^2/2} H_n \left( \sqrt{Av_0^3 q} \right) , \] (53)

where \( H_n \left( \sqrt{Av_0^3 q} \right) \) are the Hermite polynomials. Then we may obtain
\[ \psi_{1,n} (x) = N_n \sqrt{\frac{E_n}{v_0 x}} + Av_0 e^{-\frac{A}{A^{v_0^3}} \ln^2 x} H_n \left( \sqrt{Av_0 \ln x} \right) , \] (54)

and find \( \psi_2 (x) \) using (7) to construct the Dirac spinor of (5). Only discrete bound state solutions are observed here.

VI. CONCLUDING REMARKS

In this work, we have considered the (1+1)-Dirac particles where the mass and the Fermi velocity are both position-dependent. An alternative methodical proposal is proposed in such a way that the Panella-Roy’s model [13] becomes a special case. The set of \( m (x) \) and \( v_F (x) \) that satisfies \( m (x) v_F (x)^2 = A \) is a wider set than that used by Panella and Roy who have used massless Dirac particles. Moreover, analogous to the well known textbook non-relativistic limit for Dirac particles (i.e., rest mass energy \( m_0 c^2 \gg E_{\text{bind}} \), where \( E_{\text{bind}} = E - m_0 c^2 \)), we have used the limit where \( m (x) v_F (x)^2 \gg E_{\text{bind}} \) for non-zero PD-masses. To the best of our knowledge such methodical proposal has not been reported elsewhere.
For Dirac particles with $m(x)$ and $v_F(x)$ satisfying \( m(x)v_F(x)^2 = A \), we have reported feasible BIC-like and discrete bound-states solutions (documented in section III). They are in an almost exact accord with the scenario reported in the Panella-Roy’s model. However, we have also observed a shift-up of order \( m_0^2v_0^4 \) in the total energy squared, \( E_n^2 \), and some scaling factors in the components of the Dirac spinor (i.e., \( \sqrt{\xi_2/v_0} \) for \( \psi_1(x) \) and \( \sqrt{\xi_1/v_0} \) for \( \psi_2(x) \)). Moreover, the results of our methodical proposal collapse into those of Panella and Roy in [13] for \( m_0 = 0 \). Yet, should one use \( m(x) = m_0 \) (i.e., the rest mass) and \( v_F(x) = c \) (i.e., speed of light), then \( q(x) = x/c \) and equation (11) would collapse into the regular textbook Dirac equation for free particle, where the total energy is \( E = \pm m_0c^2 \).

Finally, for the case where \( m(x)v_F(x)^2 \neq A \), we have shown that Dirac particles may be trapped in an effective force fields produced by both their PD-mass and PD-Fermi velocity. This is documented in the effective Pöschl-Teller and the effective harmonic oscillator models discussed in sections IV and V, respectively. No BIC-like bound state solutions are observed for these models.

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