On the Extension of B. Sz.-Nagy’s Dilation Theorem to Linear Pencils of Operators

Dmitriy S. Kalyuzhniy

The explicit constructions of minimal isometric, and minimal unitary dilations of an arbitrary linear pencil of operators $T(\lambda) = T_0 + \lambda T_1$ consisting of contractions on a separable Hilbert space for $|\lambda| = 1$, which generalize the classical constructions (the case $T_1 = 0$), are presented. In contrast to the classical case these dilations are essentially non-unique.

1 Introduction

The classical Sz.-Nagy dilation theorem [14] asserts that any contractive linear operator $T$ on a Hilbert space $\mathcal{H}$ has a unitary dilation, i.e. a unitary operator $U$ on some Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that

$$\forall n \in \mathbb{Z}_+ \quad T^n = P_\mathcal{H} U^n |\mathcal{H}$$

(here $P_\mathcal{H}$ denotes the orthogonal projector onto $\mathcal{H}$, $A|\mathcal{H}$ denotes the restriction of an operator $A$ onto $\mathcal{H}$); moreover, this unitary dilation $U$ can be chosen minimal (in the sense of natural partial order in the set of all unitary dilations of $T$), that is equivalent to the following:

$$\mathcal{K} = \bigvee_{n=-\infty}^{\infty} U^n \mathcal{H}$$

(here $\bigvee_n \mathcal{L}_n$ denotes the closure of the linear span of subsets $\mathcal{L}_n$ in $\mathcal{K}$); the minimal unitary dilation $U$ of a contraction $T$ is unique up to unitary equivalence.

There is a quantity of generalizations of this theorem to commutative families of contractions (see [13], [11], [5], [4] for the bibliography), and noncommutative families of contractions (e.g. [3], [6], [7], [12]). In the present paper we obtain the extension of the Sz.-Nagy dilation theorem (in the existence part) to linear pencils of operators.

A linear pencil $\sum_{k=1}^{N} z_k \tilde{T}_k$ of bounded linear operators on a Hilbert space $\tilde{\mathcal{H}}$ is called a dilation of a linear pencil $\sum_{k=1}^{N} z_k T_k$ of bounded linear operators on a Hilbert space $\mathcal{H}$ if $\tilde{\mathcal{H}} \supset \mathcal{H}$, and each of the following three equivalent conditions is fulfilled:

(i) $\forall z = (z_1, \ldots, z_N) \in \mathbb{C}_N, \forall n \in \mathbb{Z}_+$ \quad $\left(\sum_{k=1}^{N} z_k T_k\right)^n = P_\mathcal{H} \left(\sum_{k=1}^{N} z_k \tilde{T}_k\right)^n |\mathcal{H}$,

(ii) $\forall \zeta = (\zeta_1, \ldots, \zeta_N) \in \mathbb{T}_N, \forall n \in \mathbb{Z}_+$ \quad $\left(\sum_{k=1}^{N} \zeta_k T_k\right)^n = P_\mathcal{H} \left(\sum_{k=1}^{N} \zeta_k \tilde{T}_k\right)^n |\mathcal{H}$,

(iii) $\forall t \in \mathbb{Z}_+^N \quad T^t = P_\mathcal{H} \tilde{T}^t |\mathcal{H}$

is fulfilled; here $\mathbb{T}_N := \{ \zeta = (\zeta_1, \ldots, \zeta_N) \in \mathbb{C}_N : |\zeta_k| = 1, k = 1, \ldots, N \}$ is the unit $N$-fold torus, $\mathbb{Z}_+^N := \{ t = (t_1, \ldots, t_N) \in \mathbb{Z}_N : t_k \geq 0, k = 1, \ldots, N \}$ is the discrete positive octant, $\forall t \in \mathbb{Z}_+^N$ \quad $T^t$ is the $t$-th symmetrized multipower of the $N$-tuple $T = (T_1, \ldots, T_N)$.
of operators, e.g. for \( t = (1, 2, 0, \ldots, 0) \) \( T^t = (T^1 T^2_2 + T^2 T^1 T^2 + T^2_2 T^1_1)/3 \) (obviously, for a commutative \( N \)-tuple \( T \) this is a usual multipower: \( T^t = \prod_{k=1}^N T^t_k \)). If, moreover, for any \( \{j_k\}_1^n \subset \{1, \ldots, N\} \)
\[
T_{j_1} \cdots T_{j_n} = P_\delta \tilde{T}_{j_1} \cdots \tilde{T}_{j_n} |\mathcal{F} \tag{1.1}
\]
holds then a pencil \( \sum_{k=1}^N \zeta_k T_k \) is said to be a uniform dilation of a pencil \( \sum_{k=1}^N \zeta_k T_k \). If for each \( \zeta \in \mathbb{T}^N \) the operator \( \sum_{k=1}^N \zeta_k T_k \) is contractive (resp., isometric, unitary) then we shall refer to the set of operators \( \sum_{k=1}^N \zeta_k T_k \), \( \zeta \in \mathbb{T}^N \), as a contractive (resp., isometric, unitary) linear pencil. In case when a pencil \( \sum_{k=1}^N \zeta_k T_k \) is contractive, and its dilation \( \sum_{k=1}^N \zeta_k \tilde{T}_k \) is an isometric (resp., unitary) pencil, the latter is said to be an isometric (resp., unitary) dilation of \( \sum_{k=1}^N \zeta_k T_k \). Contractive linear pencils appear as pencils of main operators of multiparametric dissipative linear stationary dynamical scattering systems (see [1], [10]). It was proved in [10] that a contractive linear pencil \( \sum_{k=1}^N \zeta_k T_k \) on a separable Hilbert space allows a unitary dilation if and only if for any \( N \)-tuple \( C = (C_1, \ldots, C_N) \) of commuting contractions on a common separable Hilbert space
\[
\| \sum_{k=1}^N C_k \otimes T_k \| \leq 1.
\]
For \( N = 1 \) this condition is, obviously, always fulfilled. For \( N = 2 \) it is also always fulfilled (this follows from [1]). For \( N \geq 3 \) this condition, in general, fails [8]. Thus, in the cases \( N = 1 \) and \( N = 2 \) a unitary dilation of a given contractive linear pencil is always exists. Since in the case \( N = 1 \) the structure of minimal unitary dilation is well known (the Sz.-Nagy dilation theorem) we shall concentrate our attention on the case \( N = 2 \).

It is convenient for the sequel to consider nonhomogeneous linear pencils of operators \( T(\lambda) := T_0 + \lambda T_1 \), \( \lambda \in \mathbb{T} \), instead of homogeneous ones \( T_\zeta := \zeta_0 T_0 + \zeta_1 T_1 \), \( \zeta = (\zeta_0, \zeta_1) \in \mathbb{T}^2 \). It is clear that \( T(\lambda) \), \( \lambda \in \mathbb{T} \), is a contractive (resp., isometric, unitary) pencil if and only if \( T_\zeta \), \( \zeta \in \mathbb{T}^2 \), is a contractive (resp., isometric, unitary) pencil. The definition of dilation is reformulated as follows. A linear pencil \( \tilde{T}(\lambda) \) of operators on a Hilbert space \( \tilde{\mathcal{F}} \) is said to be a dilation of a linear pencil \( T(\lambda) \) of operators on a Hilbert space \( \mathcal{F} \) if \( \tilde{\mathcal{F}} \supset \mathcal{F} \), and
\[
\forall \lambda \in \mathbb{T}, \forall n \in \mathbb{Z}_+ \quad T(\lambda)^n = P_\delta \tilde{T}(\lambda)^n |\mathcal{F}. \tag{1.2}
\]
Note that the dilation \( \tilde{T}(\lambda) \) of a pencil \( T(\lambda) \) is called uniform if \((1.1)\) holds, and this is equivalent to the condition
\[
\forall n \in \mathbb{N}, \forall \{\lambda_j\}_1^n \subset \mathbb{T} \quad T(\lambda_1) \cdots T(\lambda_n) = P_\delta \tilde{T}(\lambda_1) \cdots \tilde{T}(\lambda_n) |\mathcal{F}. \tag{1.3}
\]
We shall use the term “minimal” for minimal isometric dilations (resp., minimal unitary extensions, minimal unitary dilations, minimal uniform isometric dilations, minimal uniform unitary dilations) in the sense of natural partial order in the set of all isometric dilations (resp., all unitary extensions, all unitary dilations, all uniform isometric dilations, all uniform unitary dilations) of a given contractive linear pencil \( T(\lambda) \).

In Section 2 we construct a minimal isometric dilation of an arbitrary contractive linear pencil \( T(\lambda) \). This dilation is turned out to be uniform. We also give an example of non-uniform minimal isometric dilation, and show that both a minimal isometric dilation and a minimal uniform isometric dilation of a contractive linear pencil are essentially
non-unique. In Section 3 we construct a minimal unitary extension of an arbitrary isometric linear pencil. Together with the construction of a minimal isometric dilation this gives us the construction of a minimal unitary dilation of an arbitrary contractive linear pencil. This dilation is also turned out to be uniform. We give also an example of non-uniform minimal unitary dilation, and show that both a minimal unitary dilation and a minimal uniform unitary dilation of a contractive linear pencil are essentially non-unique. The question on the description of all minimal isometric (resp., unitary) dilations of a contractive linear pencil is still open.

2 Minimal isometric dilations of contractive linear pencils

Let \( X \) be a separable Hilbert space, \( \mathbb{D} \) denote the unit disk, \( H^2_{\mathbb{X}}(\mathbb{D}) \) denote the Hardy space of holomorphic \( X \)-valued functions \( x \) on \( \mathbb{D} \) such that
\[
\|x\|^2 = (2\pi)^{-1} \sup_{0<r<1} \int_0^{2\pi} \|x(re^{i\theta})\|^2 \, d\theta < \infty,
\]
\([\mathbb{X}, \mathbb{X}_*]\) denote the Banach space of all bounded linear operators from a separable Hilbert space \( \mathbb{X} \) into a separable Hilbert space \( \mathbb{X}_* \). Recall (see [15]) that a contractive holomorphic function \( \theta : \mathbb{D} \to [\mathbb{X}, \mathbb{X}_*] \) is called outer if \( \theta H^2_{\mathbb{X}}(\mathbb{D}) = H^2_{\mathbb{X}_*}(\mathbb{D}) \) (the closure is taken in the norm of the Hilbert space \( H^2_{\mathbb{X}_*}(\mathbb{D}) \)).

Let \( T(\lambda) = T_0 + \lambda T_1 \) be a linear pencil of contractions on a separable Hilbert space \( \mathcal{H} \), i.e.
\[
\forall \lambda \in \mathbb{T} \quad T(\lambda)^* T(\lambda) \leq I_{\mathcal{H}}
\]
where \( I_{\mathcal{H}} \) is the identity operator on \( \mathcal{H} \). Then by the operator Fejér–Riesz theorem (see [13]) there exist a separable Hilbert space \( \mathcal{Y} \) and a linear outer \([\mathcal{H}, \mathcal{Y}]\)-valued function \( F(z) = F_0 + z F_1 \) such that for boundary values \( F(\lambda) = F_0 + \lambda F_1 \) we have:
\[
\forall \lambda \in \mathbb{T} \quad F(\lambda)^*F(\lambda) = I_{\mathcal{H}} - T(\lambda)^* T(\lambda).
\]
This function \( F(z) \) is determined by pencil \( T(\lambda) \) uniquely, up to unitary operator factor from the left. Set
\[
\mathbb{K} := \left( \bigoplus_{-\infty}^{-1} \mathcal{Y} \right) \oplus \mathcal{H},
\]
and define the operators
\[
V(\lambda) := \begin{bmatrix} \ddots & & \\
& I_{\mathcal{H}} & \\
& I_{\mathcal{Y}} & \\
& \vdots & \\
& F(\lambda) & \\
& T(\lambda) &
\end{bmatrix} : \mathbb{K} \to \mathbb{K} \quad (\lambda \in \mathbb{T})
\]
(here and in the sequel empty places of matrices mean zeros). It follows from (2.2) that \( V(\lambda) \) is an isometric linear pencil. Let us show that \( V(\lambda) \) is a uniform dilation.
of $T(\lambda)$. Indeed, for any $\{\lambda_j\}_1^n \subset \mathbb{T}$ and $h \in \mathcal{H}$ (we identify such a vector $h$ with $\text{col}(\ldots, 0, 0, h) \in \mathfrak{H}_+)$ we have

$$V(\lambda_1) \cdots V(\lambda_n)h = \text{col}(\ldots, 0, F(\lambda_n)h, F(\lambda_{n-1})T(\lambda_n)h, \ldots, F(\lambda_1)T(\lambda_2) \cdots T(\lambda_n)h),$$

and therefore we obtain

$$P_\mathcal{H}V(\lambda_1) \cdots V(\lambda_n)|\mathcal{H} = T(\lambda_1) \cdots T(\lambda_n),$$

that agrees with (1.3) for $\tilde{T}(\lambda) = V(\lambda)$, $\lambda \in \mathbb{T}$. Thus, $V(\lambda)$ is a uniform isometric dilation of $T(\lambda)$. 

**Proposition 2.1** The isometric dilation $\tilde{T}(\lambda) \in [\tilde{\mathcal{H}}] := [\mathcal{H}, \tilde{\mathcal{H}}]$, $\lambda \in \mathbb{T}$, of the linear pencil of contractions $T(\lambda) \in [\mathcal{H}]$, $\lambda \in \mathbb{T}$, is a minimal isometric dilation of $T(\lambda)$ if and only if

$$\tilde{\mathcal{H}} = \bigvee_{n \in \mathbb{Z}_+, \{\lambda_j\}_1^n \subset \mathbb{T}} \tilde{T}(\lambda_1) \cdots \tilde{T}(\lambda_n)\mathcal{H}$$

(here for $n = 0$ the corresponding term is $\mathcal{H}$). If $\tilde{T}(\lambda)$ is a minimal uniform isometric dilation of $T(\lambda)$ then $\tilde{T}(\lambda)$ is a minimal isometric dilation of $T(\lambda)$.

**Proof.** Let $\tilde{T}(\lambda)$ be an isometric dilation of $T(\lambda)$, and (2.6) hold. Suppose that $\mathcal{H}'$ is a subspace of $\tilde{\mathcal{H}}$, $\mathcal{H}' \supset \mathcal{H}$, and $T'(\lambda) := P_{\mathcal{H}'}\tilde{T}(\lambda)|\mathcal{H}'$, $\lambda \in \mathbb{T}$, is an isometric dilation of $T(\lambda)$, $\lambda \in \mathbb{T}$. Since for any $h' \in \mathcal{H}'$ and $\lambda \in \mathbb{T}$

$$\|\tilde{T}(\lambda)h'\| = \|h'\| = \|T'(\lambda)h'\| = \|P_{\mathcal{H}'}\tilde{T}(\lambda)h'\|,$$

we have $\tilde{T}(\lambda)h' \in \mathcal{H}'$, and $\mathcal{H}'$ is invariant under $\tilde{T}(\lambda)$. Therefore,

$$\tilde{\mathcal{H}} = \bigvee_{n, \{\lambda_j\}_1^n} \tilde{T}(\lambda_1) \cdots \tilde{T}(\lambda_n)\mathcal{H} \subset \bigvee_{n, \{\lambda_j\}_1^n} \tilde{T}(\lambda_1) \cdots \tilde{T}(\lambda_n)\mathcal{H}' \subset \mathcal{H}' \subset \tilde{\mathcal{H}},$$

and $\mathcal{H}' = \tilde{\mathcal{H}}$. Thus, $\tilde{T}(\lambda)$ is a minimal isometric dilation of $T(\lambda)$. For the rest of this Proposition it is sufficient to prove that if $\tilde{T}(\lambda)$ is a minimal uniform isometric dilation of $T(\lambda)$ then (2.6) is true. The right-hand side of the equality in (2.6) (denote it by $\mathcal{H}''$) is an invariant subspace in $\tilde{\mathcal{H}}$ under operators $\tilde{T}(\lambda)$ for all $\lambda \in \mathbb{T}$. If $\tilde{T}(\lambda)$ is a uniform isometric dilation of $T(\lambda)$ then

$$T''(\lambda) := \tilde{T}(\lambda)|\mathcal{H}'', \quad \lambda \in \mathbb{T},$$

is also a uniform isometric dilation of $T(\lambda)$. Indeed, for any $n \in \mathbb{N}$ and $\{\lambda_j\}_1^n \subset \mathbb{T}$ we have

$$P_\mathcal{H}T''(\lambda_1) \cdots T''(\lambda_n)|\mathcal{H} = P_\mathcal{H}\tilde{T}(\lambda_1) \cdots \tilde{T}(\lambda_n)|\mathcal{H} = T(\lambda_1) \cdots T(\lambda_n).$$

Besides, (2.7) implies that $\tilde{T}(\lambda)$ is a uniform isometric dilation of $T''(\lambda)$. If $\tilde{T}(\lambda)$ is a minimal uniform isometric dilation of $T(\lambda)$ then $\mathcal{H}'' = \tilde{\mathcal{H}}$, $T''(\lambda) = \tilde{T}(\lambda)$ for all $\lambda \in \mathbb{T}$, and the proof is complete. $\square$
Now let us show that for $T(\lambda) = V(\lambda)$, $\tilde{\mathcal{H}} = \mathfrak{K}_+$, where $V(\lambda)$ is defined by (2.4), and $\mathfrak{K}_+$ is defined by (2.3), the equality in (2.6) is true. From (2.5) we get for any $n \in \mathbb{N}$, $\{\lambda_j\}_1^n \subset \mathbb{T}$, and $h \in \mathcal{H}$

$$V(\lambda_1) \cdots V(\lambda_n)h - V(\lambda_1) \cdots V(\lambda_{n-1})T(\lambda_n)h = \text{col}(\ldots, 0, 0, F(\lambda_n)h, 0, \ldots, 0),$$

(2.8)

with only non-zero entry $F(\lambda_n)h$ of this column vector in the $(-n)$-th $\mathcal{Y}$'s component of $\mathfrak{K}_+ = (\bigoplus_{-\infty}^{-1} \mathcal{Y}) \oplus \mathfrak{H}$. Since $F(z)$ is a linear outer function, it follows from Proposition V.2.4 in [15] that for any $\lambda \in \mathbb{T}$ the lineal $F(\lambda)\mathfrak{H}$ is dense in $\mathcal{Y}$. Hence vectors of the form (2.8) together with vectors from $\mathfrak{H}$ are dense in $\mathfrak{K}_+$, and the desired equality

$$\mathfrak{K}_+ = \bigvee_{n \in \mathbb{Z}_+, \{\lambda_j\}_1^n \subset \mathbb{T}} V(\lambda_1) \cdots V(\lambda_n)\mathfrak{H}$$

(2.9)

is true.

Summing up all that was said about $V(\lambda)$ in this Section, we obtain the following.

**Theorem 2.2** Formulas (2.3)–(2.4) define the minimal isometric dilation $V(\lambda)$ of a given contractive linear pencil $T(\lambda)$. Moreover, $V(\lambda)$ is a minimal uniform isometric dilation of $T(\lambda)$.

**Remark 2.3** It is clear that in the particular case $T_1 = 0$ the described construction of minimal isometric dilation coincides with the classical one for a contraction $T_0$ (see Section I.4 of [15]). In this case $\mathcal{Y} = D_{T_0}\mathfrak{H}$ where $D_{T_0} := (I_\mathfrak{H} - T_0^*T_0)^{1/2}$, $F(\lambda) = F_0 = D_{T_0} \in [\mathfrak{H}, \mathcal{Y}]$ and $V(\lambda) = V_0$ for all $\lambda \in \mathbb{T}$.

Two dilations $T'(\lambda) \in [\mathfrak{H}']$, $\lambda \in \mathbb{T}$, and $T''(\lambda) \in [\mathfrak{H}''], \lambda \in \mathbb{T}$, of a linear pencil $T(\lambda) \in [\mathfrak{H}], \lambda \in \mathbb{T}$, are said to be unitarily equivalent if there is a unitary operator $W : \mathfrak{H}' \to \mathfrak{H}''$ such that

1) $Wh = h$ for all $h \in \mathfrak{H}$;
2) $\forall \lambda \in \mathbb{T}$, $T''(\lambda) = WT'(\lambda)W^{-1}$.

The following example shows that minimal isometric dilations of a contractive linear pencil are essentially non-unique and not necessarily uniform.

**Example 2.4** Consider the trivial linear pencil $T(\lambda) = 0$ in $\mathfrak{H} = \mathbb{C}$. Then (see Remark 2.3) $\mathcal{Y} = \mathbb{C}$, $F(\lambda) = 1$ for all $\lambda \in \mathbb{T}$, $\mathfrak{K}_+ = (\bigoplus_{-\infty}^{-1} \mathbb{C}) \oplus \mathbb{C} = \bigoplus_{-\infty}^0 \mathbb{C}$, and we obtain the following minimal uniform isometric dilation of $T(\lambda) = 0$:

$$V(\lambda) = \begin{bmatrix}
\vdots & 0 \\
1 & 1 \\
0 & 1 \\
\end{bmatrix} : \bigoplus_{-\infty}^0 \mathbb{C} \to \bigoplus_{-\infty}^0 \mathbb{C}, \quad (\lambda \in \mathbb{T})$$

(2.10)

which coincides identically with the multiplicity one forward shift operator $S$. However, the linear pencil $V'(\lambda) := \lambda S$, $\lambda \in \mathbb{T}$, is also a minimal uniform isometric dilation of
Thus, the linear pencil \( \tilde{V}(\lambda) = 0 \) which is not unitarily equivalent to the linear pencil \( V(\lambda) = S \).

Now set

\[
\tilde{V}(\lambda) := \begin{bmatrix}
\ddots & \cdot \\
& 1 \\
\frac{1}{\sqrt{2}} & \lambda/\sqrt{2} \\
\lambda/\sqrt{2} & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} & \lambda/\sqrt{2} \\
0 & 0 \\
\end{bmatrix} : \bigoplus_{-\infty}^{0} \mathbb{C} \to \bigoplus_{-\infty}^{0} \mathbb{C}. \quad (\lambda \in \mathbb{T}) \quad (2.11)
\]

It is verified directly that \( \tilde{V}(\lambda) \) is an isometric linear pencil. Let us show that \( \tilde{V}(\lambda) \) is a minimal isometric dilation of the trivial linear pencil \( T(\lambda) = 0 \), however is not uniform. For any \( \lambda \in \mathbb{T} \) and \( h \in \mathcal{H} = \mathbb{C} \) (identified with \( \text{col}(\ldots, 0, 0, h) \in \bigoplus_{0}^{0} \mathbb{C} \)) we have

\[
\tilde{V}(\lambda) h = \text{col}(\ldots, 0, 0, \lambda h/\sqrt{2}, h/\sqrt{2}, 0), \\
\tilde{V}(\lambda)^2 h = \text{col}(\ldots, 0, 0, \lambda h, 0, 0), \\
\ldots \\
\tilde{V}(\lambda)^n h = \text{col}(\ldots, 0, 0, 0, \ldots, 0) \\
\ldots \\
\]

(2.12)

Therefore, \( P_{\mathcal{H}} \tilde{V}(\lambda)^n h = 0 = T(\lambda)^n h \) for any \( \lambda \in \mathbb{T}, \ h \in \mathcal{H}, \ n \in \mathbb{N} \), i.e. \( (1.2) \) holds with \( \tilde{T}(\lambda) = \tilde{V}(\lambda) \), and

\[
\bigoplus_{n=0}^{0} \mathbb{C} = \biggl( \bigoplus_{n=0}^{0} \tilde{V}(1)^n \mathcal{H} \biggr) \oplus \tilde{V}(-1) \mathcal{H} \subset \bigvee_{n \in \mathbb{Z}_{+}, \{\lambda_j\}^n_{j \in \mathbb{T}}} \tilde{V}(\lambda_1) \cdots \tilde{V}(\lambda_n) \mathcal{H} \subset \bigoplus_{n=0}^{0} \mathbb{C}.
\]

Thus, the linear pencil \( \tilde{V}(\lambda) \) is a minimal isometric dilation of the linear pencil \( T(\lambda) = 0 \). Since for any non-zero \( h \in \mathcal{H} = \mathbb{C} \) by virtue of \( (2.11) \) and \( (2.12) \) we have

\[
\tilde{V}(-1) \tilde{V}(1) h = \tilde{V}(-1) \text{col}(\ldots, 0, 0, h/\sqrt{2}, h/\sqrt{2}, 0) = \text{col}(\ldots, 0, 0, -h),
\]

we get \( P_{\mathcal{H}} \tilde{V}(-1) \tilde{V}(1) h = -h \neq 0 = T(-1)T(1) h \), and this dilation is not uniform. It is clear that a pencil \( \tilde{V}(\lambda) \) is not unitarily equivalent both to \( V(\lambda) \) and \( V'(\lambda) \) since the uniformity property of a dilation remains under unitary equivalence transformations.

### 3 Minimal unitary dilations of contractive linear pencils

First we shall construct a minimal unitary extension of an arbitrary isometric linear pencil. A unitary linear pencil \( U(\lambda) = U_0 + \lambda U_1 \in [\mathcal{H}] \), \( \lambda \in \mathbb{T} \), is said to be a unitary extension of an isometric linear pencil \( V(\lambda) = V_0 + \lambda V_1 \in [\mathcal{H}] \), \( \lambda \in \mathbb{T} \), if \( \mathcal{H}_+ \) is an invariant subspace in \( \mathcal{H} \) under all operators \( U(\lambda) \), \( \lambda \in \mathbb{T} \), and

\[
\forall \lambda \in \mathbb{T} \quad V(\lambda) = U(\lambda)|\mathcal{H}_+.
\quad (3.1)
\]
It is easy to see that \( U(\lambda) \) is a unitary extension of an isometric linear pencil \( V(\lambda) \) if and only if \( U(\lambda) \) is a uniform unitary dilation of this pencil.

Let \( V(\lambda) \in [\mathcal{K}_+] \), \( \lambda \in \mathbb{T} \), be an isometric linear pencil, i.e.

\[
\forall \lambda \in \mathbb{T} \quad (V_0 + \lambda V_1)^*(V_0 + \lambda V_1) = I_{\mathcal{K}_+}.
\]

Then \( V_1^*V_0 = 0 \), i.e. \( \overline{V_0\mathcal{K}_+} \perp V_1\mathcal{K}_+ \). Let us show that

\[
\mathcal{U} := \bigvee_{\lambda \in \mathbb{T}} V(\lambda)\mathcal{K}_+ = \overline{V_0\mathcal{K}_+} \oplus V_1\mathcal{K}_+.
\]

(3.2)

Indeed, the inclusion \( \subset \) is obvious. The inclusion \( \supset \) follows from the Fourier representations

\[
\forall k_+ \in \mathcal{K}_+ \quad V_jk_+ = (2\pi)^{-1} \int_0^{2\pi} e^{-ij\lambda}V(e^{i\lambda})k_+ \, ds. \quad (j = 0, 1)
\]

Set

\[
\mathcal{L} := \mathcal{K}_+ \oplus \mathcal{U}, \quad \mathcal{K}_+ := \mathcal{U} \oplus V(\lambda)\mathcal{K}_+, \quad (\lambda \in \mathbb{T})
\]

and define the unitary linear pencil

\[
P(\lambda) := P_{\overline{V_0\mathcal{K}_+}} + \lambda P_{\overline{V_1\mathcal{K}_+}} \in [\mathcal{U}], \quad \lambda \in \mathbb{T}.
\]

(3.4)

Then \( P(\lambda)V(1)\mathcal{K}_+ = V(\lambda)\mathcal{K}_+ \) for any \( \lambda \in \mathbb{T} \), and since \( P(\lambda) \) is unitary in \( \mathcal{U} \), we have \( P(\lambda)\mathcal{K}_1 = \mathcal{K}_\lambda \) for any \( \lambda \in \mathbb{T} \). Set

\[
\mathcal{U} := \mathcal{K}_1 \oplus \mathcal{L},
\]

(3.5)

\[
Q(\lambda) := \begin{bmatrix} P(\lambda)|\mathcal{K}_1 & 0 \\ 0 & I_{\mathcal{L}} \end{bmatrix} : \mathcal{U} = \mathcal{K}_1 \oplus \mathcal{L} \to \mathcal{K}_+ \quad (\lambda \in \mathbb{T})
\]

(3.6)

Then \( Q(\lambda) \) is an isometric linear pencil, and equalities

\[
I_{\mathcal{K}_+} - V(\lambda)V(\lambda)^* = Q(\lambda)Q(\lambda)^*, \quad (\lambda \in \mathbb{T})
\]

(3.7)

\[
V(\lambda)^*Q(\lambda) = 0 \quad (\lambda \in \mathbb{T})
\]

(3.8)

hold. Indeed, for any fixed \( \lambda \in \mathbb{T} \) the operator \( V(\lambda) \) is isometric, hence \( V(\lambda)V(\lambda)^* = P_{V(\lambda)\mathcal{K}_+} \), and by (3.3) \( I_{\mathcal{K}_+} - V(\lambda)V(\lambda)^* = P_{\mathcal{K}_+ \oplus \mathcal{L}} \); since \( Q(\lambda) \) is also an isometry and \( Q(\lambda)\mathcal{U} = \mathcal{K}_\lambda \oplus \mathcal{L}, \quad Q(\lambda)Q(\lambda)^* = P_{\mathcal{K}_\lambda \oplus \mathcal{L}}, \) and (3.7) holds. Since \( Q(\lambda)\mathcal{U} = \mathcal{K}_\lambda \oplus \mathcal{L} = \mathcal{K}_+ \oplus V(\lambda)\mathcal{K}_+ \), we have \( Q(\lambda)\mathcal{U} \perp V(\lambda)\mathcal{K}_+ \), and (3.8) follows. Define

\[
\mathcal{K} := \mathcal{K}_+ \oplus \left( \bigoplus_{n=1}^{\infty} \mathcal{U} \right),
\]

(3.9)

\[
U(\lambda) := \begin{bmatrix} V(\lambda) & Q(\lambda) \\ I_{\mathcal{U}} & I_{\mathcal{U}} \end{bmatrix} \quad : \mathcal{K} \to \mathcal{K} \quad (\lambda \in \mathbb{T})
\]

(3.10)

Since \( V(\lambda) \) and \( Q(\lambda) \) are isometric linear pencils and due to (3.7) and (3.8) \( U(\lambda) \) is a unitary linear pencil. By (3.10) the subspace \( \mathcal{K}_+ \) is invariant under \( U(\lambda), \lambda \in \mathbb{T} \), and (3.1) holds, i.e. \( U(\lambda) \) is a unitary extension of the isometric linear pencil \( V(\lambda) \).
Proposition 3.1 The unitary dilation \( \tilde{T}(\lambda) \in [\tilde{H}] \), \( \lambda \in \mathbb{T} \), of a linear pencil of contractions \( T(\lambda) \in [\hat{H}] \), \( \lambda \in \mathbb{T} \), is a minimal unitary dilation of this pencil if and only if
\[
\tilde{H} = \bigvee_{n \in \mathbb{Z}_+, \{\lambda_j\}_1^n \subset \mathbb{T}, \{k_j\}_1^n \subset \{-1,1\}} \tilde{T}(\lambda_1)^{k_1} \cdots \tilde{T}(\lambda_n)^{k_n} \tilde{H} \tag{3.11}
\]
(\text{here for } n = 0 \text{ the corresponding term is } \tilde{H}). If \( \tilde{T}(\lambda) \) is a minimal uniform unitary dilation of \( T(\lambda) \) then \( \tilde{T}(\lambda) \) is its minimal unitary dilation.

**Proof.** Let \( \tilde{T}(\lambda) \) be a unitary dilation of a linear pencil \( T(\lambda) \), and (3.11) hold. Suppose that \( \tilde{H}' \) is a subspace of \( \tilde{H} \), \( \tilde{H}' \supset \tilde{H} \), and \( T'(\lambda) := P_{\tilde{H}'} \tilde{T}(\lambda)|_{\tilde{H}'} \), \( \lambda \in \mathbb{T} \), is a unitary dilation of a linear pencil \( T(\lambda) \). In the same way as in Proposition 2.1 we show that \( \tilde{H}' \) is invariant under \( \tilde{T}(\lambda) \) and \( \tilde{T}(\lambda)^* \) for all \( \lambda \in \mathbb{T} \). Therefore,
\[
\tilde{H} = \bigvee_{n \in \mathbb{Z}_+, \{\lambda_j\}_1^n \subset \mathbb{T}, \{k_j\}_1^n \subset \{-1,1\}} \tilde{T}(\lambda_1)^{k_1} \cdots \tilde{T}(\lambda_n)^{k_n} \tilde{H} \subset \bigvee_{n \in \mathbb{Z}_+, \{\lambda_j\}_1^n \subset \mathbb{T}, \{k_j\}_1^n \subset \{-1,1\}} \tilde{T}(\lambda_1)^{k_1} \cdots \tilde{T}(\lambda_n)^{k_n} \tilde{H}' \subset \tilde{H}',
\]
and \( \tilde{H}' = \tilde{H} \). Thus, \( \tilde{T}(\lambda) \) is a minimal unitary dilation of \( T(\lambda) \). For the rest of this Proposition it is sufficient to prove that if \( \tilde{T}(\lambda) \) is a minimal uniform unitary dilation of \( T(\lambda) \) then (3.11) is true. The right-hand side in (3.11) (denote it by \( \tilde{H}' \)) is a reducing subspace in \( \tilde{H} \) for \( \tilde{T}(\lambda) \), \( \lambda \in \mathbb{T} \). In the same way as in Proposition 2.1 we can show that \( T''(\lambda) := \tilde{T}(\lambda)|_{\tilde{H}'} \), \( \lambda \in \mathbb{T} \), is a uniform unitary dilation of \( T(\lambda) \). If \( \tilde{T}(\lambda) \) is a minimal uniform unitary dilation of \( T(\lambda) \) then \( \tilde{H}' = \tilde{H} \), \( T''(\lambda) = \tilde{T}(\lambda) \) for all \( \lambda \in \mathbb{T} \), and the proof is complete. \( \square \)

Now let us show that for \( T(\lambda) = V(\lambda) \), \( \tilde{H} = \mathfrak{H}_+ \), \( \tilde{T}(\lambda) = U(\lambda) \), and \( \tilde{H} = \mathfrak{H} \), where \( U(\lambda) \) and \( \mathfrak{H} \) are defined by (3.10) and (3.3) respectively, (3.11) is true. Let us identify vectors \( k_+ \in \mathfrak{H}_+ \) with \( \text{col}(k_+, 0, 0, \ldots) \in \mathfrak{H} \). Then for any \( \{\lambda_j\}_1^n \subset \mathbb{T} \) and \( k_+ \in \mathfrak{H}_+ \) we have
\[
U(\lambda_1)^{-1} \cdots U(\lambda_n)^{-1} k_+ = \text{col}(V(\lambda_1)^* \cdots V(\lambda_n)^* k_+, Q(\lambda_1)^* V(\lambda_2)^* \cdots V(\lambda_n)^* k_+, \ldots, Q(\lambda_{n-1})^* V(\lambda_n)^* k_+, Q(\lambda_n)^* k_+, 0, 0, \ldots);
\]
and
\[
U(\lambda_1)^{-1} \cdots U(\lambda_n)^{-1} k_+ - U(\lambda_1)^{-1} \cdots U(\lambda_{n-1})^{-1} V(\lambda_n)^* k_+ = \text{col}(0, \ldots, 0, Q(\lambda_n)^* k_+, 0, 0, \ldots). \tag{3.12}
\]
Since by (3.6) for any \( \lambda \in \mathbb{T} \) we have \( Q(\lambda)^* \mathfrak{H}_+ = \mathfrak{H} \), vectors of the form (3.12) together with vectors from \( \mathfrak{H}_+ \) fill \( \mathfrak{H} \), and the desired equality
\[
\mathfrak{H} = \bigvee_{n \in \mathbb{Z}_+ \cup \{0\}, \{\lambda_j\}_1^n \subset \mathbb{T}, \{k_j\}_1^n \subset \{-1,1\}} U(\lambda_1)^{k_1} \cdots U(\lambda_n)^{k_n} \mathfrak{H}_+ \tag{3.13}
\]
is valid. Summing up all that was said about \( U(\lambda) \) in this Section, we obtain the following.
Theorem 3.2 Formulas (3.2), (3.3), and (3.10) define the minimal unitary extension $U(\lambda)$ of a given isometric linear pencil $V(\lambda)$.

Remark 3.3 It is clear that in the particular case $V_1 = 0$ the described construction of minimal unitary extension coincides with the classical one for an isometry $V_0$ (see Section I.2 of [15]). In this case (3.2) turns into $\mathfrak{V} = V_0 \mathfrak{R}_+$, (3.3) turns into $\mathfrak{L} = \mathfrak{R}_+ \ominus V_0 \mathfrak{R}_+$, $\mathfrak{R}_+ = \{0\} (\lambda \in \mathbb{T})$, (3.5) turns into $\mathfrak{L} = \mathfrak{L}$, and (3.6) turns into $Q(\lambda) = I_\mathfrak{L} : \mathfrak{L} \to \mathfrak{R}_+ (\lambda \in \mathbb{T})$. Thus, $U$ coincides with the wandering generating subspace $\mathfrak{L}$ of the forward shift part of $V_0$, and $U(\lambda) = U_0$, $\lambda \in \mathbb{T}$, where

$$U_0 = \begin{bmatrix} V_0 & I_\mathfrak{L} \\ I_\mathfrak{L} & I_\mathfrak{L} \\ \vdots & \ddots \end{bmatrix} : \mathfrak{R} \to \mathfrak{R}$$

(3.14)

is the classical minimal unitary extension of $V_0$.

The following example shows that minimal unitary extensions of an isometric linear pencil are essentially non-unique.

Example 3.4 Let $V(\lambda) = S$ be a forward shift operator in $\mathfrak{R}_+ = \bigoplus_{-\infty}^{0} \mathbb{C}$ for all $\lambda \in \mathbb{T}$ (see (2.10)). Then the construction of minimal unitary extension gives (see Remark 3.3) $U(\lambda) = U_0$, where $U_0$ is defined in (3.14) with $V_0 = S$, $\mathfrak{L} = \mathbb{C}$, $\mathfrak{R} = \bigoplus_{-\infty}^{\infty} \mathbb{C}$, i.e.

$$U(\lambda) = U_0 = \begin{bmatrix} \ddots & 1 \\ \vdots & 0 & 1 \\ \cdot & \cdot & \lambda \end{bmatrix} : \bigoplus_{-\infty}^{\infty} \mathbb{C} \to \bigoplus_{-\infty}^{\infty} \mathbb{C} \quad (\lambda \in \mathbb{T})$$

(3.15)

(Here and in the sequel the frame distinguishes the $(0, 0)$-th element of an infinite matrix). However,

$$U'(\lambda) := \begin{bmatrix} \ddots & 1 \\ \vdots & 0 & \lambda \\ \cdot & \cdot & \lambda \end{bmatrix} : \bigoplus_{-\infty}^{\infty} \mathbb{C} \to \bigoplus_{-\infty}^{\infty} \mathbb{C} \quad (\lambda \in \mathbb{T})$$

(3.16)

is also a minimal unitary extension of the isometric linear pencil $V(\lambda) = S$, which is not unitarily equivalent to the linear pencil $U(\lambda)$.

Proposition 3.5 If $V(\lambda) \in [\mathfrak{R}_+]$, $\lambda \in \mathbb{T}$, is a minimal isometric dilation of a contractive linear pencil $T(\lambda) \in \mathfrak{S}^T$, $\lambda \in \mathbb{T}$, and $U(\lambda) \in [\mathfrak{R}]$, $\lambda \in \mathbb{T}$, is a minimal unitary extension of $V(\lambda)$, then $U(\lambda)$ is a minimal unitary dilation of a pencil $T(\lambda)$. For that, $U(\lambda)$ is a minimal uniform unitary dilation of $T(\lambda)$ if and only if $V(\lambda)$ is a minimal uniform isometric dilation of $T(\lambda)$.
The following example shows that minimal unitary dilations of a contractive linear pencil over \( U \) are essentially non-unique and not necessarily uniform. Combining Examples 2.4 and 3.4 we see that our construction of minimal unitary dilation for the trivial linear pencil \( T(\lambda) = 0 \in [\mathbb{C}] \), \( \lambda \in \mathbb{T} \), coincides with the classical one (see Remarks 2.3 and 3.3) and gives identically (i.e. for all \( \lambda \in \mathbb{T} \)) the

\[
\mathcal{F} = \bigvee_{n, \{\lambda_j\}_1^n, \{k_j\}_1^n} U(\lambda_1)^{k_1} \cdots U(\lambda_n)^{k_n} \mathcal{F}_+ = \bigvee_{n, \{\lambda_j\}_1^n, \{k_j\}_1^n} U(\lambda_1)^{k_1} \cdots U(\lambda_n)^{k_n} \left( \bigvee_{m, \{\lambda_j\}_{n+m}^n} V(\lambda_{n+1}) \cdots V(\lambda_{n+m}) \mathcal{F} \right) = \bigvee_{n, \{\lambda_j\}_1^n, \{k_j\}_1^n, \{\lambda_j\}_{n+m}^{n+m}, \{k_j\}_1^n} U(\lambda_1)^{k_1} \cdots U(\lambda_n)^{k_n} \mathcal{F} \subset \mathcal{F},
\]

and by Proposition 3.1 \( U(\lambda) \) is a minimal unitary dilation of a pencil \( T(\lambda) \) (of course, a dilation of a dilation of \( T(\lambda) \) is again a dilation of \( T(\lambda) \)). Since for any \( \{\lambda_j\}_1^n \subset \mathbb{T} \)

\[
P_{\mathcal{F}}U(\lambda_1) \cdots U(\lambda_n)|\mathcal{F} = P_{\mathcal{F}}V(\lambda_1) \cdots V(\lambda_n)|\mathcal{F},
\]

\( U(\lambda) \) is a uniform dilation of \( T(\lambda) \) if and only if \( V(\lambda) \) is a uniform dilation of \( T(\lambda) \), and the proof is complete. \( \square \)

Now from Theorems 2.2 and 3.2, and Proposition 3.3 we obtain the following.

**Theorem 3.6** Formulas (2.3)–(2.4), (3.2)–(3.6), (3.9) and

\[
U(\lambda) := \begin{bmatrix}
\ddots & & & & \\
& I_{\mathcal{F}} & & & \\
& I_{\mathcal{F}_+} & F(\lambda) & & \\
& \bar{T}(\lambda) & P_{\mathcal{F}}Q(\lambda) & P_{\mathcal{F}}Q(\lambda) & \\
& I_{\mathcal{F}} & & & I_{\mathcal{F}} \\
& & & \ddots & \\
& & & \begin{pmatrix} -1 \\ \cdots \end{pmatrix} & + \mathcal{F} & + \begin{pmatrix} \cdots \\ 1 \end{pmatrix} & \rightarrow & \begin{pmatrix} -1 \\ \cdots \end{pmatrix} & + \mathcal{F} & + \begin{pmatrix} \cdots \\ 1 \end{pmatrix} & (= \mathcal{F}) & (\lambda \in \mathbb{T})
\end{bmatrix}
\]

define the minimal unitary dilation \( U(\lambda) \) of a given contractive linear pencil \( T(\lambda) \). Moreover, \( U(\lambda) \) is a minimal uniform unitary dilation of \( T(\lambda) \).

The following example shows that minimal unitary dilations of a contractive linear pencil are essentially non-unique and not necessarily uniform.

**Example 3.7** Combining Examples 2.4 and 3.4 we see that our construction of minimal unitary dilation for the trivial linear pencil \( T(\lambda) = 0 \in [\mathbb{C}] \), \( \lambda \in \mathbb{T} \), coincides with the classical one (see Remarks 2.3 and 3.3) and gives identically (i.e. for all \( \lambda \in \mathbb{T} \)) the
multiplicity one two-sided shift operator $U(\lambda) = U_0$ from (3.15), and another minimal unitary dilation of $T(\lambda) = 0$ is $U'(\lambda)$ from (3.16). These two minimal unitary dilations of $T(\lambda) = 0$ are uniform and not unitarily equivalent. Applying our construction of minimal unitary extension to the non-uniform minimal isometric dilation $\tilde{V}(\lambda)$ of $T(\lambda) = 0$, from (2.11), we obtain according to Proposition 3.5 the following non-uniform minimal unitary dilation of $T(\lambda) = 0$:

$$
\tilde{U}(\lambda) := \begin{pmatrix}
\cdots & 1 \\
1 & 1/\sqrt{2} & \lambda/\sqrt{2} \\
1/\sqrt{2} & \lambda/\sqrt{2} & \lambda/\sqrt{2} & 1/\sqrt{2} \\
-1/\sqrt{2} & \lambda/\sqrt{2} & 1 & \cdots \\
\end{pmatrix}
$$

$$
: \bigoplus_{-\infty}^{\infty} \mathbb{C} \to \bigoplus_{-\infty}^{\infty} \mathbb{C}. \quad (\lambda \in \mathbb{T})
$$

It is clear that the linear pencil $\tilde{U}(\lambda)$ is not unitarily equivalent both to $U(\lambda)$ and $U'(\lambda)$ since the uniformity property of a dilation remains under unitary equivalence transformations.

**Remark 3.8** In our construction of a minimal uniform unitary dilation of a contractive linear pencil $T(\lambda)$ (see (3.17)) we obtain a linear function

$$\theta(z) = \begin{bmatrix} \theta_{11}(z) & \theta_{12}(z) \\ \theta_{21}(z) & \theta_{22}(z) \end{bmatrix} := \begin{bmatrix} F(z) & P_\mathfrak{g}Q(z) \\ T(z) & P_\mathfrak{g}Q(z) \end{bmatrix}$$

taking values from $[\mathfrak{b} \oplus \mathfrak{u}, \mathfrak{b} \oplus \mathfrak{u}]$ which are contractions for all $z \in \mathbb{D}$, and unitary operators for all $z \in \mathbb{T}$ (i.e., a linear biinner function). Moreover, this function satisfies the condition

$$\theta_{11}L^2_{\mathfrak{X}}(\mathbb{T}) = L^2_{\mathfrak{g}}(\mathbb{T}), \quad \theta_{22}L^2_{\mathfrak{X}}(\mathbb{T}) = L^2_{\mathfrak{u}}(\mathbb{T}), \quad (3.18)$$

where $L^2_{\mathfrak{X}}(\mathbb{T})$ denotes the Lebesgue space of $\mathfrak{X}$-valued square integrable functions on $\mathbb{T}$, $\theta_{11}L^2_{\mathfrak{g}}(\mathbb{T})$ (resp. $\theta_{22}L^2_{\mathfrak{u}}(\mathbb{T})$) is the closure of the image of $L^2_{\mathfrak{g}}(\mathbb{T})$ under the operator of multiplication by the boundary function $\theta_{11}(\lambda)$ (resp. $\theta_{22}(\lambda)^*$). The representation of an arbitrary holomorphic contractive operator-valued function on $\mathbb{D}$ as the block $\theta_{21}(z)$ of a biinner function $\theta(z)$ is called Darlington’s (a $\mathcal{D}$-representation). Let us remark that, in general, for construction of a minimal uniform unitary dilation of a contractive linear pencil $T(\lambda)$ it suffices to find a $\mathcal{D}$-representation of a function $T(z)$ with additional requirements of linearity and fulfillment of condition (3.18) on the corresponding biinner function $\theta(z)$. One can show that Arov’s general method of construction of so-called minimal $\mathcal{D}$-representations which satisfy (3.18) (see [2]), applied to a linear operator-valued function $T(z)$ which is contractive on $\mathbb{D}$, gives a linear biinner function $\theta(z)$. Thus, minimal uniform unitary dilations of $T(\lambda)$ which are obtained in such a way deserve a special consideration.
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Department of Higher Mathematics
Odessa State Academy of Civil Engineering and Architecture
65029, Didrihson str. 4, Odessa, Ukraine