Martingale Wasserstein inequality for probability measures in the convex order

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It was shown by the authors that two one-dimensional probability measures in the convex order admit a martingale coupling with respect to which the integral of $|x - y|$ is smaller than twice their $W_1$-distance (Wasserstein distance with index 1). We showed that replacing $|x - y|$ and $W_1$ respectively with $|x - y|$\rho and $W_\rho$ does not lead to a finite multiplicative constant. We show here that a finite constant is recovered when replacing $W_\rho$ with the product of $W_\rho$ times the centred $\rho$-th moment of the second marginal to the power $\rho - 1$. Then we study the generalisation of this new martingale Wasserstein inequality to higher dimension.

Keywords: Convex order; Martingale Optimal Transport; Wasserstein distance; Martingale couplings

1. Introduction

For all $d \in \mathbb{N}^*$, let $\mathcal{P}(\mathbb{R}^d)$ denote the set of probability measures on $\mathbb{R}^d$ and for $\rho \geq 1$, let $\mathcal{P}_\rho(\mathbb{R}^d)$ denote the subset of probability measures with finite $\rho$-th moment. For $\mu, \nu \in \mathcal{P}_\rho(\mathbb{R}^d)$, we define the Wasserstein distance with index $\rho$ by

$$W_\rho(\mu, \nu) = \left( \inf_{P \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^\rho P(dx, dy) \right)^{1/\rho},$$

where $\Pi(\mu, \nu)$ denotes the set of couplings between $\mu$ and $\nu$, that is

$$\Pi(\mu, \nu) = \{ P \in \mathcal{P}_\rho(\mathbb{R}^d \times \mathbb{R}^d) \mid \forall A \in \mathcal{B}(\mathbb{R}^d), P(A \times \mathbb{R}^d) = \mu(A) \text{ and } P(\mathbb{R}^d \times A) = \nu(A) \}.$$

Let $\Pi^M(\mu, \nu)$ be the set of martingale couplings between $\mu$ and $\nu$, that is

$$\Pi^M(\mu, \nu) = \left\{ M \in \Pi(\mu, \nu) \mid \mu(dx)\text{-a.e., } \int_{\mathbb{R}^d} y m(x, dy) = x \right\},$$

where for all $M \in \Pi(\mu, \nu)$, $(m(x, dy))_{x \in \mathbb{R}}$ denotes a regular conditional probability distribution of $M$ with respect to $\mu$. The celebrated Strassen theorem [24] ensures that if $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$, then $\Pi^M(\mu, \nu) \neq \emptyset$ iff $\mu$ and $\nu$ are in the convex order. We recall that two probability measures $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$ are in the convex order, and denote $\mu \leq_{cx} \nu$, if

$$\int_{\mathbb{R}^d} f(x) \mu(dx) \leq \int_{\mathbb{R}^d} f(y) \nu(dy),$$

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for any convex function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \). For all \( \rho \geq 1 \) and \( \mu, \nu \in \mathcal{P}_\rho(\mathbb{R}^d) \), we define \( \mathcal{M}_\rho(\mu, \nu) \) by

\[
\mathcal{M}_\rho(\mu, \nu) = \left( \inf_{M \in \Pi^\rho(\mu, \nu)} \int_{\mathbb{R}^d} |x-y|^\rho M(dx, dy) \right)^{1/\rho}.
\]

Notice that when \( \mathbb{R}^d \) is endowed with the Euclidean norm, the martingale property \( \int_{\mathbb{R}^d} |x|^2 \mu(dx, dy) = \int_{\mathbb{R}^d} |x|^2 \mu(dx) \) valid for any martingale coupling \( M \in \Pi^\rho(\mu, \nu) \) yields the remarkable property that \( \mathcal{M}_2(\mu, \nu) \) depends only on the marginals, namely

\[
\mathcal{M}_2^2(\mu, \nu) = \int_{\mathbb{R}^d} |y|^2 \nu(dy) - \int_{\mathbb{R}^d} |y|^2 \mu(dx) = \int_{\mathbb{R}^d} |y-c|^2 \nu(dy) - \int_{\mathbb{R}^d} |x-c|^2 \mu(dx),
\]

for each \( c \in \mathbb{R}^d \).

It was shown in [21] that if \( \mu \) and \( \nu \) are in the convex order and close to each other, then there exists a martingale coupling which expresses this proximity:

\[
\forall \mu, \nu \in \mathcal{P}_1(\mathbb{R}) \text{ such that } \mu \leq_{\text{conv}} \nu, \quad \mathcal{M}_1(\mu, \nu) \leq 2W_1(\mu, \nu),
\]

(3)

where the constant 2 is sharp. We call this inequality which measures the impact of the restriction to martingale couplings in the minimisation problem defining the Wasserstein distance a martingale Wasserstein inequality. It was proved by exhibiting for all \( \mu, \nu \in \mathcal{P}_1(\mathbb{R}) \) in the convex order a subset \( Q \) of two dimensional probability measures on the unit square and a family \( (M^Q)_{Q \in \mathcal{Q}} \) of martingale couplings between \( \mu \) and \( \nu \) such that for all \( Q \in \mathcal{Q} \), \( \int_{\mathbb{R}^d} |y-x|^2 \mu(dx, dy) \leq 2W_1(\mu, \nu) \). A particular martingale coupling stands out from the latter family: the so called inverse transform martingale coupling. This coupling is explicit in terms of the cumulative distribution functions of the marginal distributions and their left-continuous generalised inverses. It is more explicit than the left-curtain (and right-curtain) coupling introduced by Beiglböck and Juillet [8] and which under the condition that \( \nu \) has no atoms and the set of local maximal values of \( F_\nu - F_\mu \) is finite can be explicited according to Henry-Labordère and Touzi [20] by solving two coupled ordinary differential equations starting from each right-most local maximiser. Many properties of the inverse transform martingale coupling and the family from which it derives are discussed in [21]. In this paper, we prove a more general martingale Wasserstein inequality:

\[
\forall \rho \geq 1, \quad \exists C \in \mathbb{R}^*_+, \quad \forall \mu, \nu \in \mathcal{P}_\rho(\mathbb{R}) \text{ such that } \mu \leq_{\text{conv}} \nu, \quad \mathcal{M}_\rho^\rho(\mu, \nu) \leq CW_\rho(\mu, \nu)\sigma_\rho^{\rho-1}(\nu),
\]

(4)

where the centred moment \( \sigma_\rho(\eta) \) of order \( \rho \) of \( \eta \in \mathcal{P}_\rho(\mathbb{R}^d) \) is defined by

\[
\sigma_\rho(\eta) = \min_{c \in \mathbb{R}^d} \left( \int_{\mathbb{R}^d} |y-c|^\rho \eta(dy) \right)^{1/\rho}.
\]

For all \( \rho \geq 1 \), let \( C_\rho \) denote the optimal constant \( C \) in (4), that is

\[
C_\rho = \inf \left\{ C > 0 \mid \forall \mu, \nu \in \mathcal{P}_\rho(\mathbb{R}) \text{ such that } \mu \leq_{\text{conv}} \nu, \quad \mathcal{M}_\rho^\rho(\mu, \nu) \leq CW_\rho(\mu, \nu)\sigma_\rho^{\rho-1}(\nu) \right\}.
\]

(5)

One readily notices that (3) is a particular case of (4) for \( \rho = 1 \) and \( C = 2 \). Moreover, since 2 is sharp for (3), we have \( C_1 = 2 \). One can also obtain that \( C_2 = 2 \) when \( \mathbb{R}^d \) is endowed with the Euclidean norm with simple arguments which hold in general dimension and actually lead us to generalise (3) into (4).
Martingale Wasserstein inequality

Indeed, let \( \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d) \) be such that \( \mu \leq_{ct} \nu \) and \( \pi \in \Pi(\mu, \nu) \) be optimal for \( W_2(\mu, \nu) \). Then by (2), the martingale property and the Cauchy-Schwarz inequality, we have

\[
\mathcal{M}^2_2(\mu, \nu) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( |y - c|^2 - |x - c|^2 \right) \pi(dx, dy)
\]

\[
= \int_{\mathbb{R}^d \times \mathbb{R}^d} (y - x, y - c + x - c) \pi(dx, dy) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x||y - c + x - c| \pi(dx, dy)
\]

\[
\leq W_2(\mu, \nu) \sqrt{\int_{\mathbb{R}^d \times \mathbb{R}^d} |y - c + x - c|^2 \pi(dx, dy)}.
\]

By Jensen’s inequality and the definition of the convex order, the integral in the square root is bounded from above by

\[
2 \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( |y - c|^2 + |x - c|^2 \right) \pi(dx, dy) \leq 4 \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - c|^2 \nu(dy),
\]

where the right-hand side is minimal and equal to \( 4\sigma^2_2(\nu) \) for \( c \) equal to the common mean of \( \mu \) and \( \nu \) so that \( \mathcal{M}^2_2(\mu, \nu) \leq 2W_2(\mu, \nu)\sigma_2(\nu) \) and \( C_2 \leq 2 \). Note that this inequality is preferred to the also derived, sharper but more complex \( \mathcal{M}^2_2(\mu, \nu) \leq W_2(\mu, \nu)\sqrt{2(\sigma^2_2(\nu) + \sigma^2_2(\nu))} \), since in the limit \( W_2(\mu, \nu) \to 0 \) where the martingale Wasserstein inequality is particularly interesting, \( \sigma^2_2(\mu) \) goes to \( \sigma^2_2(\nu) \).

On the other hand, for all \( n \in \mathbb{N}^* \), let \( \mu_n \) be the centred Gaussian distribution with variance \( n^2 \). Then we get that \( \mathcal{M}^2_2(\mu_n, \mu_{n+1}) = \sqrt{2n+1} \). It is well known (see for instance Remark 2.19 (ii) Chapter 2 [25]) that for all \( \rho \geq 1 \) and \( \mu, \nu \in \mathcal{P}_\rho(\mathbb{R}) \),

\[
W_\rho(\mu, \nu) = \left( \int_0^1 \left| F^{-1}_\mu(u) - F^{-1}_\nu(u) \right|^\rho \left( du \right) \right)^{1/\rho}, \tag{6}
\]

where we denote by \( F_\eta(x) = \eta((-\infty, x]) \), \( x \in \mathbb{R} \) and \( F^{-1}_\eta(u) = \inf \{ x \in \mathbb{R} \mid F_\eta(x) \geq u \}, u \in (0, 1) \), the cumulative distribution function and the quantile function of a probability measure \( \eta \) on \( \mathbb{R} \). Therefore, for \( G \sim \mathcal{N}_{1}(0, 1) \), \( W_2(\mu_n, \mu_{n+1}) = \left( \int_0^1 |nF^{-1}_\mu(u) - (n+1)F^{-1}_\nu(u)|^2 du \right)^{1/2} = E[|G|^2]^{1/2} = 1 \). We deduce that for all \( n \in \mathbb{N}^* \), \( 2n+1 \leq C_2 \sqrt{(n+1)^2} \), which implies for \( n \to +\infty \) that \( C_2 \geq 2 \). Hence \( C_2 = 2 \).

The generalisation of the martingale Wasserstein inequality (3) is motivated by the resolution of the Martingale Optimal Transport (MOT) problem introduced by Beiglböck, Henry-Labordère and Penkner [7] in a discrete time setting, and Galichon, Henry-Labordère and Touzi [16] in a continuous time setting. For adaptations of celebrated results on classical optimal transport theory to the MOT problem, we refer to Beiglböck and Juillet [8], Henry-Labordère, Tan and Touzi [19] and Henry-Labordère and Touzi [16]. On duality, we refer to Beiglböck, Nutz and Touzi [10], Beiglböck, Lim and Oblój [9] and De March [12]. We also refer to De March [13] and De March and Touzi [15] for the multi-dimensional case.

About the numerical resolution of the MOT problem, one can look at Alfonsi, Corbetta and Jourdain [1, 2], De March [14], Guo and Oblój [17] and Henry-Labordère [18]. When \( \mu \) and \( \nu \) are finitely supported, then the MOT problem amounts to linear programming. In the general case, once the MOT problem is discretised by approximating \( \mu \) and \( \nu \) by probability measures with finite support and in the convex order, Alfonsi, Corbetta and Jourdain raised the question of the convergence of the discrete optimal cost towards the continuous one. Partial results were first brought by Guo and Oblój [17] and
the stability of left-curtain couplings obtained by Juillet [22]. Backhoff-Veraguas and Pammer [6] and Wiesel [27] independently proved stability of the Martingale Optimal Transport value with respect to the marginal distributions in dimension one under mild regularity assumption on the cost function. Very recently, Brückerhoff and Juillet [11] proved that in dimension \( d \geq 2 \), stability fails and (4) does not generalise. Since, on the contrary, for \( \rho = 2 \), the generalisation to any dimension is possible, we may wonder for which values of \( \rho \) it is also the case. Note that such a generalization would imply stability of the Martingale Optimal Transport problem for continuous costs which satisfy a growth constraint related to \( \rho \) restricted to the case when the second marginal is increased in the convex order.

More precisely, let \( \rho > 1 \), \( \mu, \nu \in \mathcal{P}_\rho(\mathbb{R}^d) \) be such that \( \mu \leq_{cx} \nu \) and \( (\nu_n)_{n \in \mathbb{N}} \in \mathcal{P}_\rho(\mathbb{R}^d)^\mathbb{N} \) be such that \( \nu \leq_{cx} \nu_n \) for all \( n \in \mathbb{N} \) and \( \nu_n \) converges to \( \nu \) in \( W_\rho \) as \( n \to +\infty \). Let \( c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) be continuous and growing at most as the \( \rho \)-th power of its variables, i.e. \( |c(x, y)| \leq K(1 + |x|^{\rho} + |y|^{\rho}) \) for all \( (x, y) \in \mathbb{R}^d \times \mathbb{R}^d \) and a certain \( K \in \mathbb{R}_+ \). It is well known that any sequence \( (\pi_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} \Pi^M(\mu, \nu_n) \) is tight and has all its accumulation points with respect to the weak convergence topology in \( \Pi^M(\mu, \nu) \).

Then one can readily derive the first inequality

\[
V(\mu, \nu) := \inf_{\pi \in \Pi^M(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y) \pi(dx, dy) \leq \liminf_{n \to +\infty} V(\mu, \nu_n).
\]

On the other hand, for any \( (\pi_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} \Pi^M(\mu, \nu_n) \), we have

\[
\limsup_{n \to +\infty} V(\mu, \nu_n) \leq \limsup_{n \to +\infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y) \pi_n(dx, dy).
\]

Recall that a sequence \( (\pi_n)_{n \in \mathbb{N}} \) of probability measures on \( \mathbb{R}^d \times \mathbb{R}^d \) converges to \( \tau \) in \( W_\rho \) iff the sequence \( (\int_{\mathbb{R}^d \times \mathbb{R}^d} f(x, y) \pi_n(dx, dy))_{n \in \mathbb{N}} \) converges to \( \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x, y) \tau(dx, dy) \) for any real-valued continuous map \( f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) which grows at most as the \( \rho \)-th absolute power of its variables. Hence it suffices to find \( (\pi_n)_{n \in \mathbb{N}} \) converging in \( W_\rho \) to some optimal coupling \( \pi \in \Pi^M(\mu, \nu) \) for \( V(\mu, \nu) \), which exists by Lemma 3 below. This lemma also ensures that there exist for all \( n \in \mathbb{N} \) a martingale coupling \( M_n \in \Pi^M(\nu, \nu_n) \) optimal for \( \mathcal{M}_\rho(\nu, \nu_n) \). Let \( (m_n(y, dy'))_{y \in \mathbb{R}^d} \) be a regular conditional probability distribution of \( M_n \) with respect to \( \nu \) and \( \pi_n(dx', dy') = \int_{\mathbb{R}^d} m_n(y, dy') \pi(dx', dy) \in \Pi^M(\mu, \nu_n) \).

Since \( \pi(dx', dy') |\sigma(x)| \) \( M_n(dy, dy') \) is a coupling between \( \pi \) and \( \pi_n \), we have

\[
\mathcal{W}_\rho^\rho(\pi, \pi_n) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - y'|^\rho \pi(dx, dy) m_n(y, dy') = \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - y'|^\rho \mathcal{M}_\rho(dy, dy').
\]

If \( C_\rho \) is finite, then by convergence of \( (\nu_n)_{n \in \mathbb{N}} \) in \( W_\rho \), the sequences \( (\mathcal{W}_\rho(\nu, \nu_n))_{n \in \mathbb{N}} \) and \( (\sigma_\rho^{-1}(\nu_n))_{n \in \mathbb{N}} \) are bounded, hence \( (\pi_n)_{n \in \mathbb{N}} \) converges to \( \tau \) in \( W_\rho \). For the above mentioned numerical motivation, this covers the case when the support of \( \nu \) is bounded and this measure is approximated by dual quantization.

We present our main result in Section 2, namely the new one-dimensional martingale Wasserstein inequality which extends the previous one, see [21], to any index \( \rho \geq 1 \). Then Section 3 addresses the extension of this inequality to higher dimension. It turns out that the nice example given in [11] to prove that the generalisation fails for \( \rho = 1 \) also prevents it for \( \rho < \frac{1 + \sqrt{5}}{2} \). For \( \rho \) larger than this threshold, apart in the particular case \( \rho = 2 \) addressed above, we were not able to prove the generalisation. But we exhibit restricted classes of couples \( (\mu, \nu) \in \mathcal{P}_\rho(\mathbb{R}^d) \times \mathcal{P}_\rho(\mathbb{R}^d) \) with \( \mu \leq_{cx} \nu \) such that the inequality holds with a finite constant \( C \) (possibly equal to the one-dimensional constant \( C_\rho \)) uniform over the
class whatever \( \rho \geq 1 \). In subsection 3.2, we exhibit three classes such that \( C \) is equal to the one-dimensional constant \( C_\rho \). In subsection 3.3, we deal with the scaling case where \( C = 3 \times 2^{\rho-1} \).

Finally Section 4 is devoted to the proof of some technical lemmas.

2. A new martingale Wasserstein inequality in dimension one

We come back a moment on the family \((M^Q)_{Q \in \mathcal{Q}}\) parametrised by \( \mathcal{Q} \) mentioned in the introduction since it will have particular significance in the present section. We briefly recall the construction and main properties, see [21] for an extensive study. Let \( \mu, \nu \in \mathcal{P}_1(\mathbb{R}) \) be such that \( \mu \preceq_{ex} \nu \) and \( \mu \neq \nu \). For \( u \in [0, 1] \) we define

\[
\Psi_+(u) = \int_0^u (F^{-1}_{\mu} - F^{-1}_{\nu})^+(v) \, dv \quad \text{and} \quad \Psi_-(u) = \int_0^u (F^{-1}_{\mu} - F^{-1}_{\nu})^-(v) \, dv,
\]

with respective left continuous generalised inverses \( \Psi_+^{-1} \) and \( \Psi_-^{-1} \). We then define \( \mathcal{Q} \) as the set of probability measures on \((0, 1)^2\) with first marginal \( \frac{1}{\Psi_+(1)} d\Psi_+ \), second marginal \( \frac{1}{\Psi_-(1)} d\Psi_- \) and such that \( u < v \) for \( Q(du, dv) \)-almost every \((u, v) \in (0, 1)^2\). Since \( d\Psi_+ \) and \( d\Psi_- \) are concentrated on two disjoint Borel sets, there exists for each \( Q \in \mathcal{Q} \) a probability kernel \((\pi^Q(u, dv))_{u \in (0,1)}\) such that

\[
Q(du, dv) = \frac{1}{\Psi_+(1)} d\Psi_+(u) \pi^Q(u, dv) = \frac{1}{\Psi_-(1)} d\Psi_-(v) \pi^Q(v, du),
\]

\[
|F^{-1}_{\mu} - F^{-1}_{\nu}|(u) du \pi^Q(u, dv) \text{ a.e. } |F^{-1}_{\mu}(v) - F^{-1}_{\nu}(u)| + |F^{-1}_{\mu}(u) - F^{-1}_{\nu}(u)| = |F^{-1}_{\nu}(v) - F^{-1}_{\nu}(u)|.
\]

We define a probability kernel \((\tilde{m}^Q(u, dy))_{u \in (0,1)}\) which satisfies for \( du \)-almost all \( u \in (0, 1) \) such that \( F^{-1}_{\mu}(u) \neq F^{-1}_{\nu}(u) \)

\[
\tilde{m}^Q(u, dy) = \int_{v \in (0,1)} \left( \frac{F^{-1}_{\mu}(u) - F^{-1}_{\nu}(u)}{F^{-1}_{\nu}(v) - F^{-1}_{\nu}(u)} \delta_{F^{-1}_{\mu}(v)}(dy) + \frac{F^{-1}_{\mu}(v) - F^{-1}_{\nu}(u)}{F^{-1}_{\nu}(v) - F^{-1}_{\nu}(u)} \delta_{F^{-1}_{\nu}(v)}(dy) \right) \pi^Q(u, dv),
\]

and \( \tilde{m}^Q(u, dy) = \delta_{F^{-1}_{\nu}(u)}(dy) \) for all \( u \in (0, 1) \) such that \( F^{-1}_{\mu}(u) = F^{-1}_{\nu}(u) \). Then for \( du \)-almost all \( u \in (0, 1) \),

\[
\int_{\mathbb{R}} |y - F^{-1}_{\nu}(u)| \tilde{m}^Q(u, dy) = |F^{-1}_{\mu}(u) - F^{-1}_{\nu}(u)|.
\]

The measure

\[
M^Q(dx, dy) = \int_0^1 \delta_{F^{-1}_{\mu}(u)}(dx) \tilde{m}^Q(u, dy) \, du
\]

is a martingale coupling between \( \mu \) and \( \nu \) which satisfies

\[
\int_{\mathbb{R} \times \mathbb{R}} |y - x| \, M^Q(dx, dy) \leq \int_0^1 \int_{\mathbb{R}} |F^{-1}_{\mu}(u) - F^{-1}_{\nu}(u)| + |F^{-1}_{\nu}(u) - y| \, \tilde{m}^Q(u, dy) \, du = 2W_1(\mu, \nu).
\]
We also recall some standard results about cumulative distribution functions and quantile functions since they will prove very handy one-dimensional tools. Proofs can be found for instance in [21, Appendix A]. For any probability measure $\eta$ on $\mathbb{R}$:

1. $F_\eta$, resp. $F_\eta^{-1}$, is right continuous, resp. left continuous, and nondecreasing;
2. For all $(x, u) \in \mathbb{R} \times (0, 1)$,
   \[ F_\eta^{-1}(u) \leq x \iff u \leq F_\eta(x), \tag{14} \]
   which implies
   \[ F_\eta(x-) \leq u \leq F_\eta(x) \implies x = F_\eta^{-1}(u), \tag{15} \]
   and
   \[ F_\eta(F_\eta^{-1}(u)-) \leq u \leq F_\eta(F_\eta^{-1}(u)); \tag{16} \]
3. For $\mu(dx)$-almost every $x \in \mathbb{R}$,
   \[ 0 < F_\eta(x), \quad F_\eta(x-) < 1 \quad \text{and} \quad F_\eta^{-1}(F_\eta(x)) = x; \tag{17} \]
4. Denoting by $\lambda_{(0,1)}$, resp. $\lambda_{(0,1)^2}$, the Lebesgue measure on $(0, 1)$, resp. $(0, 1)^2$, we have
   \[ \left( (u, v) \mapsto F_{\mu}(F_{\mu}^{-1}(u)-) + v\mu\{F_{\mu}^{-1}(u)\} \right)_\sharp \lambda_{(0,1)^2} = \lambda_{(0,1)}, \tag{18} \]
where $\sharp$ denotes the pushforward operation.
5. The image of the Lebesgue measure on $(0, 1)$ by $F_\eta^{-1}$ is $\eta$.

The property 5 is referred to as inverse transform sampling.

We can now state and prove our main result. For all $\rho \geq 1$ and $\mu, \nu \in \mathcal{P}_\rho(\mathbb{R})$ in the convex order, we provide an estimate of the martingale Wasserstein function $\mathcal{M}_\rho(\mu, \nu)$ in terms of the Wasserstein distance $\mathcal{W}_\rho(\mu, \nu)$ and the centred $\rho$-th moment of $\nu$.

**Proposition 1.** Let $\rho \geq 1$ and $\mu, \nu \in \mathcal{P}_\rho(\mathbb{R})$ be such that $\mu \leq_{ex} \nu$. Then

(i) For all $Q \in \mathcal{Q}$, the martingale coupling $M^Q \in \Pi^M(\mu, \nu)$ defined by (13) satisfies
   \[ \mathcal{M}_\rho^\rho(\mu, \nu) \leq \int_{\mathbb{R} \times \mathbb{R}} |x - y|^\rho M^Q(dx, dy) \leq K_{\rho} \mathcal{W}_\rho(\mu, \nu) \sigma_{\rho}^{\rho-1}(\nu), \tag{19} \]
   where
   \[ K_{\rho} = \inf \left\{ 2^{\rho-1} \gamma_1 + 2(2^{\rho-2} \vee 1) \gamma_2 \mid (\gamma_1, \gamma_2) \in \mathbb{R}^2_+ \text{ and } \forall x \in \mathbb{R}_+, \frac{x + x^\rho}{1 + x} \leq \gamma_1 + \gamma_2 (1 + x)^{\rho-1} \right\}. \tag{20} \]

(ii) The constant $C_{\rho}$ defined by (5) satisfies $C_1 = K_1 = 2$, $C_\rho = K_\rho = 2^{\rho-1}$ when $\rho \geq 2$ and, for $1 < \rho < 2$,
   \[ 2^{\rho-1} \sup_{x \in (1, +\infty)} \frac{x + x^\rho}{(1 + x)^\rho} \leq C_{\rho} \leq K_{\rho} \leq \left( 2^{\rho-1} + 2 \right) \wedge \left( 2 \sup_{x \in (1, +\infty)} \frac{x + x^\rho}{(1 + x)^\rho} \right). \tag{21} \]

(iii) $\mathcal{W}_{\rho}(\mu, \nu)$ and $\sigma_{\rho}(\nu)$ have the right exponent in (4) in the following sense:
   \[ \forall \rho > 1, \quad \forall s \in (1, \rho], \quad \sup_{\mu, \nu \in \mathcal{P}_\rho(\mathbb{R})} \frac{\mathcal{M}_s^\rho(\mu, \nu)}{\mathcal{W}_s^\rho(\mu, \nu) \sigma_{\rho}^{\rho-s}(\nu)} = +\infty. \tag{22} \]
Remark 1. (i) Let $\rho \in (1, 2)$. For $x \geq 0$, $(1 + x)^\rho = x^\rho + \int_x^{1+x} py^{\rho-1} dy \leq x^\rho + \rho(1 + x)^{\rho-1}$. Hence for $x > \rho(1 + x)^{\rho-1}$, $x^\rho + \rho(1 + x)^{\rho-1} < x^\rho + x$ so that $\sup_{x \in (1, +\infty)} \frac{x + x^\rho}{(1+x)^\rho} > 1$ Moreover, $\gamma_2 \geq 1$ is necessary for $(\gamma_1, \gamma_2)$ to belong in the set which appears in the definition (20) of $K_\rho$ and when $\gamma_2 = 1$, then, by a easy generalization of the previous reasoning, $\gamma_1 \geq 1$ is necessary. In the last step of the proof of Proposition 1, we check that both $(\gamma_1, \gamma_2) = (1, 1)$ and $(\gamma_1, \gamma_2) = \left(0, \sup_{x \in (1, +\infty)} \frac{x + x^\rho}{(1+x)^\rho}\right)$ are admissible. The upper bound of $K_\rho$ in (21) is the minimum of $2\rho^{-1}\gamma_1 + 2(2\rho^{-2} \vee 1)\gamma_2 = 2\rho^{-1}\gamma_1 + 2\gamma_2$ over these two couples. In Figure 1, we plot for $\rho \in (1, 2]$, the lower and upper bounds in (21) together with some numerical estimation of $K_\rho$. The supremum of $x \mapsto f_\rho(x) := \frac{x + x^\rho}{(1+x)^\rho}$ over $(1, +\infty)$ is computed numerically by iterating the function $x \mapsto \frac{x + x^\rho}{(1+x)^\rho}$ in order to find the unique root of the derivative $f'_\rho(x) = \frac{\rho x^{\rho-1} + 1 - (\rho-1)x}{(1+x)^{\rho+1}}$ and $K_\rho$ is obtained by minimising $2\rho^{-1}\gamma_1 + 2 \sup_{x \in \mathbb{R}_+} \frac{x^\rho + (1-\gamma_1)x - \gamma_1}{(1+x)^\rho}$ over $\gamma_1$ in a grid of the interval $[0, 1]$ with the supremum computed in the same way as for $\gamma_1 = 0$. It turns out that away from the neighbourhood $[1.1, 1.2]$ of the point where the two functions of $\rho$ involved in the minimum in the right-hand side of (21) intersect, this right-hand side coincides with $K_\rho$. 

Figure 1. Plot of $(1, 2] \ni \rho \mapsto K_\rho$ with the lower and upper bounds in (21).
(ii) When \( \rho = 1 \), the condition \( \gamma_2 \geq 1 \) is no longer necessary for \((\gamma_1, \gamma_2)\) to belong in the set which appears in the definition (20) of \( K_\rho \) and \((\gamma_1, \gamma_2) = (2, 0) \) is admissible so that \( K_1 \leq 2 \), while \( \lim_{\rho \to 1^+} K_\rho \) appears to be equal to 3 according to Figure 1.

(iii) When \( \rho \geq 2 \), in the last step of the proof of the proposition, we obtain that \( K_\rho \leq 2^{\rho - 1} \) from the admissibility of \((\gamma_1, \gamma_2) = (0, 1)\).

(iv) We show (see (29) below) that \( \mathcal{W}_\rho(\mu, \nu) \leq 2\sigma_\rho(\nu) \), so that for \( s \in [0, 1] \),

\[
\mathcal{M}_\rho^s(\mu, \nu) \leq 2^{1-s} C_\rho \mathcal{W}_\rho^s(\mu, \nu) \sigma_\rho^{-s}(\nu).
\]

**Proof of Proposition 1.** Let us prove iii first. One can readily show (see for instance [21, (2.24)]) that for \( a, b \in \mathbb{R} \) such that \( 0 < a < b \),

\[
H = \frac{(b + a)}{4b} \delta_{(-a, -b)} + \frac{(b - a)}{4b} \delta_{(-a, b)} + \frac{(b + a)}{4b} \delta_{(a, b)} + \frac{(b - a)}{4b} \delta_{(a, -b)}
\]

is the only martingale coupling between \( \mu = \frac{1}{2} \delta_{-a} + \frac{1}{2} \delta_a \) and \( \nu = \frac{1}{2} \delta_{-b} + \frac{1}{2} \delta_b \). Consequently, for \( \rho \geq 1 \) we trivially have

\[
\mathcal{M}_\rho^0(\mu, \nu) = \int_{\mathbb{R}^2} |x - y|^\rho H(dx, dy) = \frac{1}{2b} \left( (a + b)(b - a)^\rho + (b - a)(a + b)^\rho \right).
\]

On the other hand, since \( \mathcal{W}_\rho(\mu, \nu) = \left( \int_0^1 |F_{\mu}^{-1}(u) - F_{\nu}^{-1}(u)| \, du \right)^{1/\rho} \) (see for instance Remark 2.19 (ii) Chapter 2 [25]),

\[
\mathcal{W}_\rho(\mu, \nu) = \left( \int_0^{1/2} |x - (x - b)|^\rho \, dx + \int_0^{1/2} |a - (x - b)|^\rho \, dx \right)^{1/\rho} = b - a.
\]

Moreover, for all \( c \in \mathbb{R} \), \( \int_{\mathbb{R}} |y - c|^\rho \, \nu(dy) = \frac{1}{2} (|b - c|^\rho + |b + c|^\rho) \), which attains its infimum for \( c = 0 \), hence \( \sigma_\rho(\nu) = b \). So for all \( s \in [1, \rho] \), we have

\[
\frac{\mathcal{M}_\rho^s(\mu, \nu)}{\mathcal{W}_\rho(\mu, \nu) \sigma_\rho(\nu)^{\rho - s}} = \frac{1}{2b^{\rho + 1 - s}} \left( (a + b)(b - a)^{\rho - s} + (a + b)^\rho(b - a)^{1-s} \right) \geq \frac{(a + b)^\rho(b - a)^{1-s}}{2b^{\rho + 1 - s}},
\]

which tends to \( +\infty \) as \( b \) tends to \( a \) as soon as \( \rho > 1 \) and \( s \in (1, \rho] \), which proves iii. Furthermore, (24) applied with \( s = 1, a = 1 \) and \( b > 1 \) yields

\[
\frac{\mathcal{M}_\rho^0(\mu, \nu)}{\mathcal{W}_\rho(\mu, \nu) \sigma_\rho(\nu)^{\rho - 1}} = \frac{(1 + b)(b - 1)^{\rho - 1} + (1 + b)^\rho}{2b^\rho},
\]

In particular for \( b = \frac{x + 1}{x} \) where \( x \) denotes any real number in \((1, +\infty)\), the latter equality writes

\[
\frac{\mathcal{M}_\rho^0(\mu, \nu)}{\mathcal{W}_\rho(\mu, \nu) \sigma_\rho(\nu)^{\rho - 1}} = 2^{\rho - 1} \frac{x + x^\rho}{(1 + x)^\rho},
\]

which proves the lower bound in (21):

\[
C_\rho \geq 2^{\rho - 1} \sup_{x \in [1, +\infty)} \frac{x + x^\rho}{(1 + x)^\rho}.
\]
Note that considering more general measures $\mu$ and $\nu$ in the convex order, each concentrated on two atoms, does not yield a greater lower bound.

We now show i. Let $Q \in \mathcal{Q}$. Since the probability measure $M^Q$ defined by (13) belongs to $\Pi^M(\mu, \nu)$, we have by definition of $\mathcal{M}_\rho(\mu, \nu)$ and the definition (11) of $\tilde{m}^Q$ that

$$
\mathcal{M}_\rho^Q(\mu, \nu) \leq \int_{\mathbb{R} \times \mathbb{R}} |y - x|^\rho \pi^Q(dx, dy) = \int_{(0,1) \times \mathbb{R}} |y - F_u^{-1}(u)|^\rho \, du \pi^Q(u, dy)
$$

$$
= \int_{(0,1)^2} |F_{v}^{-1}(v) - F_{\mu}^{-1}(u)|^\rho \frac{|F_{\mu}^{-1}(u) - F_{\nu}^{-1}(u)|}{|F_{v}^{-1}(v) - F_{\mu}^{-1}(u)|} \mathbf{1}_{\{|F_{\mu}^{-1}(u) - F_{\nu}^{-1}(u)| > 0\}} \, du \pi^Q(u, dv)
$$

$$
+ \int_{(0,1)^2} |F_{v}^{-1}(v) - F_{\mu}^{-1}(u)|^\rho \frac{|F_{\mu}^{-1}(u) - F_{v}^{-1}(u)|}{|F_{v}^{-1}(v) - F_{\mu}^{-1}(u)|} \mathbf{1}_{\{|F_{\mu}^{-1}(u) - F_{\nu}^{-1}(u)| > 0\}} \, du \pi^Q(u, dv).
$$

(26)

Let us recall (10):

$$
|F_{\mu}^{-1}(u) - F_{\nu}^{-1}(u)| \, du \pi^Q(u, dv) \text{ a.e. } |F_{\mu}^{-1}(v) - F_{\mu}^{-1}(u)| + |F_{\mu}^{-1}(u) - F_{\nu}^{-1}(u)| = |F_{\nu}^{-1}(v) - F_{\nu}^{-1}(u)|.
$$

Let $\gamma_1, \gamma_2 \geq 0$ be such that for all $x \in \mathbb{R}_+$, $\frac{x + x^\rho}{1 + x} \leq \gamma_1 + \gamma_2 (1 + x)^{\rho - 1}$. Therefore, for all $(a, b) \in \mathbb{R}_+ \times \mathbb{R}_+$ such that $a + b > 0$, we have

$$
\frac{a\rho b + ab^\rho}{a + b} \leq \gamma_1 a^\rho + \gamma_2 a(b + b)^{\rho - 1}.
$$

(27)

By (26), (10) and (27) with $(a, b) = (|F_{\nu}^{-1}(u) - F_{\mu}^{-1}(u)|, |F_{\nu}^{-1}(v) - F_{\nu}^{-1}(u)|)$, we get

$$
\mathcal{M}_\rho^Q(\mu, \nu) \leq \gamma_1 \int_{(0,1)} |F_{\nu}^{-1}(v) - F_{\mu}^{-1}(u)|^\rho \, du
$$

$$
+ \gamma_2 \int_{(0,1)} |F_{\nu}^{-1}(v) - F_{\mu}^{-1}(u)|^\rho \frac{|F_{\mu}^{-1}(u) - F_{\nu}^{-1}(u)|}{|F_{\nu}^{-1}(v) - F_{\mu}^{-1}(u)|} \, du \pi^Q(u, dv)
$$

$$
= \gamma_1 W_\rho^Q(\mu, \nu) + \gamma_2 \int_{(0,1)} |F_{\nu}^{-1}(v) - F_{\mu}^{-1}(u)|^\rho \frac{|F_{\mu}^{-1}(u) - F_{\nu}^{-1}(u)|}{|F_{\nu}^{-1}(v) - F_{\mu}^{-1}(u)|} \, du \pi^Q(u, dv).
$$

(28)

Let $c \in \mathbb{R}$. On the one hand, the inverse transform sampling and the definition (1) of the convex order applied with $x \mapsto |x - c|^\rho$ yield

$$
W_\rho^Q(\mu, \nu)
$$

$$
= \int_{(0,1)} |F_{\nu}^{-1}(u) - F_{\mu}^{-1}(u)|^\rho \, du \leq 2^{\rho - 1} \left( \int_{(0,1)} |F_{\nu}^{-1}(u) - c|^\rho \, du + \int_{(0,1)} |F_{\mu}^{-1}(u) - c|^\rho \, du \right)
$$

$$
= 2^{\rho - 1} \left( \int_{\mathbb{R}} |y - c|^\rho \nu(dy) + \int_{\mathbb{R}} |x - c|^\rho \mu(dx) \right) \leq 2^\rho \int_{\mathbb{R}} |y - c|^\rho \nu(dy).
$$

(29)
We deduce that
\[ W^p_\rho(\mu, \nu) = W_\rho(\mu, \nu) W^{p-1}_\rho(\mu, \nu) \leq W_\rho(\mu, \nu) \times 2^{p-1} \left( \int_{\mathbb{R}} |y - c|^p \nu(dy) \right)^{(p-1)/p}. \tag{30} \]

On the other hand we have
\[
\int_{(0,1)} |F_\nu^1(v) - F_\nu^1(u)|^{\rho-1} |F_\mu^1(u) - F_\nu^1(u)| \, du \, \pi^Q(u, dv) \\
= \int_{(0,1)}^2 |F_\nu^1(v) - F_\nu^1(u)|^{\rho-1} (F_\mu^1(u) - F_\nu^1(u)) \, du \, \pi^Q(u, dv) \\
+ \int_{(0,1)} |F_\nu^1(v) - F_\nu^1(u)|^{\rho-1} (F_\mu^1(u) - F_\nu^1(u)) \, du \, \pi^Q(u, dv). \tag{31}
\]

Using the inequality \(|x - y|^{\rho-1} \leq (2^{\rho-2} \vee 1)(|x|^\rho + |y|^{\rho-1})\) valid for all \((x, y) \in \mathbb{R}\) and the fact that \((F_\mu^1 - F_\nu^1)^+(u) \, du \, \pi^Q(\nu, du) = Q(du, dv) = (F_\mu^1 - F_\nu^1)^-(v) \, dv \, \pi^Q(v, du)\) according to (9), we get
\[
\int_{(0,1)}^2 |F_\nu^1(v) - F_\nu^1(u)|^{\rho-1} (F_\mu^1(u) - F_\nu^1(u)) \, du \, \pi^Q(u, dv) \\
\leq (2^{\rho-2} \vee 1) \left( \int_{(0,1)}^2 |F_\nu^1(v) - c|^{\rho-1} (F_\mu^1(u) - F_\nu^1(u)) \, du \, \pi^Q(u, dv) \\
+ \int_{(0,1)}^2 |F_\nu^1(u) - c|^{\rho-1} (F_\mu^1(u) - F_\nu^1(u)) \, du \, \pi^Q(u, dv) \right) \tag{32}
\]
\[
= (2^{\rho-2} \vee 1) \left( \int_{(0,1)}^2 |F_\nu^1(u) - c|^{\rho-1} (F_\mu^1(u) - F_\nu^1(u)) \, du \, \pi^Q(u, dv) \\
+ \int_{(0,1)}^2 |F_\nu^1(u) - c|^{\rho-1} (F_\mu^1(u) - F_\nu^1(u)) \, du \, \pi^Q(u, dv) \right)
\]
\[
= (2^{\rho-2} \vee 1) \int_{(0,1)} |F_\nu^1(u) - c|^{\rho-1} |F_\mu^1(u) - F_\nu^1(u)| \, du.\]

Similarly, we have
\[
\int_{(0,1)}^2 |F_\nu^1(v) - F_\nu^1(u)|^{\rho-1} (F_\mu^1(u) - F_\nu^1(u)) \, du \, \pi^Q(u, dv) \leq (2^{\rho-2} \vee 1) \int_{(0,1)} |F_\nu^1(u) - c|^{\rho-1} |F_\mu^1(u) - F_\nu^1(u)| \, du. \tag{33}
\]

Plugging (32) and (33) in (31) for the first inequality, using Hölder’s inequality for the second inequality and the inverse transform sampling for the equality, we have
\[
\int_{(0,1)} |F_\nu^1(v) - F_\nu^1(u)|^{\rho-1} |F_\mu^1(u) - F_\nu^1(u)| \, du \, \pi^Q(u, dv)
\]
Martingale Wasserstein inequality

\[
\leq 2(2^{\rho-2} + 1) \int_{(0,1)} |F^{-1}_\nu(u) - c|^{\rho-1}|F^{-1}_\mu(u) - F^{-1}_\nu(u)| \, du
\]
\[
\leq 2(2^{\rho-2} + 1) \left( \int_{(0,1)} |F^{-1}_\mu(u) - F^{-1}_\nu(u)|^\rho \, du \right)^{1/\rho} \left( \int_{(0,1)} |F^{-1}_\nu(u) - c|^\rho \, du \right)^{(\rho-1)/\rho}
\]
\[
= 2(2^{\rho-2} + 1) W_\rho(\mu, \nu) \left( \int_\mathbb{R} |y - c|^\rho \, \nu(dy) \right)^{(\rho-1)/\rho}.
\]

The latter inequality and (30) plugged in (28) then yields
\[
\mathcal{M}_\rho^\rho(\mu, \nu) \leq (2^{\rho-1} \gamma_1 + 2(2^{\rho-2} \lor 1) \gamma_2) W_\rho(\mu, \nu) \left( \int_\mathbb{R} |y - c|^\rho \, \nu(dy) \right)^{(\rho-1)/\rho}.
\]

By taking in the right-hand side the infimum over all \((\gamma_1, \gamma_2) \in \mathbb{R}_+ \times \mathbb{R}_+\) such that for all \(x \in \mathbb{R}_+, \frac{x + x^\rho}{1 + x} \leq \gamma_1 + \gamma_2 x^{\rho-1}\) and over all \(c \in \mathbb{R}\), we deduce that
\[
\mathcal{M}_\rho^\rho(\mu, \nu) \leq K_\rho W_\rho(\mu, \nu) \sigma_\rho^{\rho-1}(\nu).
\]

To complete the proof, it remains to prove ii. The definition (5) of \(C_\rho\) implies that \(C_\rho \leq K_\rho\).

We have \(\sup_{x \in \mathbb{R}_+} \frac{x + x^\rho}{1 + x} = 2\). Hence \(K_1 \leq 2\) and, by (25), \(2 \leq C_1\). Therefore \(C_1 = K_1 = 2\).

Let us next suppose that \(\rho \geq 2\). Then \(\mathbb{R}_+ \ni x \mapsto (1 + x)^\rho - x^\rho\) is a convex function above its tangent at the origin which writes \((1 + x)^\rho - x^\rho \geq 1 + \rho x\) so that \(\frac{x + x^\rho}{1 + x} \leq (1 + x)^{\rho-1}\) for all \(x \in \mathbb{R}_+\) and \(K_\rho \leq 2^{\rho-1} \times 2(2^{\rho-2} \lor 1) = 2^{\rho-1}\). Since the lower bound in (25) is equal to \(2^{\rho-1}\), we deduce that \(C_\rho = K_\rho = 2^{\rho-1}\).

Let us finally suppose that \(\rho \in (1, 2)\). For all \(x \in \mathbb{R}_+, \frac{x + x^\rho}{1 + x} \leq 1 + x^{\rho-1} \leq 1 + (1 + x)^{\rho-1}\), hence \(K_\rho \leq 2^{\rho-1} + 2\). Moreover, let \(\gamma_2 = \sup_{x \in (1, +\infty)} \frac{x + x^\rho}{(1 + x)^\rho}\). By Remark 1, \(\gamma_2 > 1\). Since \(\mathbb{R}_+ \ni x \mapsto (1 + x)^\rho - x^\rho\) is non-decreasing, for \(x \in [0, 1]\), \((1 + x)^\rho \geq x^\rho + 1 \geq x^\rho + x\). Hence \(\forall x \in \mathbb{R}_+, \frac{x + x^\rho}{1 + x} \leq \gamma_2(1 + x)^{\rho-1}\) and \(K_\rho \leq 2\gamma_2 = 2 \sup_{x \in (1, +\infty)} \frac{x + x^\rho}{(1 + x)^\rho}\). We conclude that
\[
K_\rho \leq \left(2^{\rho-1} + 2\right) \land \left(2 \sup_{x \in (1, +\infty)} \frac{x + x^\rho}{(1 + x)^\rho}\right).
\]

\[\square\]

**Remark 2.** For \(\rho = 2\), by (2) for the first inequality and the fact that \(\sigma_2(\nu)\) is the standard deviation of \(\nu\), consequence of the bias-variance decomposition, for the last equality, we have
\[
\mathcal{W}_2^2(\mu, \nu) \leq \mathcal{M}_2^2(\mu, \nu)
\]
\[
= \int_\mathbb{R} y^2 \, \nu(dy) - \int_\mathbb{R} x^2 \, \mu(dx)
\]
\[
\leq \int_\mathbb{R} y^2 \, \nu(dy) - \left( \int_\mathbb{R} x \, \mu(dx) \right)^2 = \int_\mathbb{R} y^2 \, \nu(dy) - \left( \int_\mathbb{R} y \, \nu(dy) \right)^2
\]
\[
= \sigma_2(\nu),
\]
where the inequalities are equalities as soon as \( \mu \) is reduced to an atom. Therefore we can improve the constant \( 2^\rho \) in (29) at least in the case \( \rho = 2 \). We can then naturally wonder whether we can also improve this constant for any \( \rho > 1 \). The constant

\[
C'_\rho = \sup_{\nu \in \mathcal{P}_\rho(\mathbb{R})} \frac{\int_\mathbb{R} |y - \int_\mathbb{R} z \nu(dz)|^\rho \nu(dy)}{\int_\mathbb{R} |y|^\rho \nu(dy)}
\]

is studied in [23]. For all \( \nu \in \mathcal{P}_\rho(\mathbb{R}) \), let \( c \in \mathbb{R} \) be such that \( \sigma^\rho_\rho(\nu) = \int_\mathbb{R} |y - c|^\rho \nu(dy) \) and \( \nu_c \) be the image of \( \nu \) by \( y \mapsto y - c \). Then we have

\[
\int_\mathbb{R} |z|^\rho \nu_c(dz) = \sigma^\rho_\rho(\nu), \quad \int_\mathbb{R} \left| y - \int_\mathbb{R} z \nu_c(dz) \right|^\rho \nu_c(dy) = \int_\mathbb{R} \left| y - \int_\mathbb{R} z \nu(dz) \right|^\rho \nu(dy) = \mathcal{W}_\rho^\rho(\mu_\nu, \nu),
\]

where we denote \( \mu_\nu = \delta_{\int_\mathbb{R} y \nu(dy)} \), which is dominated by \( \nu \) in the convex order. We deduce that

\[
C'_\rho = \sup_{\nu \in \mathcal{P}_\rho(\mathbb{R})} \frac{\mathcal{W}_\rho^\rho(\mu_\nu, \nu)}{\sigma^\rho_\rho(\nu)}.
\]

Yet by [23, Theorem 2.3] we have \( C'_\rho \sim \rho \to +\infty \frac{\rho - 1}{\sqrt{2\rho}} \), which shows that we cannot lower the constant \( 2^\rho \) in (29) by a factor more than \( 2\sqrt{2\rho} \) asymptotically for \( \rho \to +\infty \).

3. On multidimensional generalisations

One may legitimately wonder whether the new martingale Wasserstein inequality (4) holds in higher dimension \( d \in \mathbb{N}^* \), that is if for all \( \rho \geq 1 \), there exists \( C \in \mathbb{R}_+^* \) such that for all \( \mu, \nu \in \mathcal{P}_\rho(\mathbb{R}^d) \) satisfying \( \mu \leq_{cx} \nu \),

\[
\mathcal{M}_\rho^\rho(\mu, \nu) \leq C \mathcal{W}_\rho(\mu, \nu)\sigma^\rho_{\rho-1}(\nu). \tag{34}
\]

For all \( d \in \mathbb{N}^* \) and \( \rho \geq 1 \), we define \( C_{\rho,d} \) by

\[
C_{\rho,d} = \inf \left\{ C > 0 \mid \forall \mu, \nu \in \mathcal{P}_\rho(\mathbb{R}^d) \text{ such that } \mu \leq_{cx} \nu, \mathcal{M}_\rho^\rho(\mu, \nu) \leq C \mathcal{W}_\rho(\mu, \nu)\sigma^\rho_{\rho-1}(\nu) \right\}. \tag{35}
\]

The constant \( C_{\rho,d} \) is well defined but is potentially infinite. Of course, for \( d = 1 \), we get \( C_{\rho,d} = C_\rho \). Moreover, \( C_{\rho,d} \) depends a priori on the choice of the norm in \( \mathbb{R}^d \), but since all norms on \( \mathbb{R}^d \) are equivalent, \( C_{\rho,d} \) is finite for one specific norm if it is finite for any norm. In the next subsection, we give lower-bounds of the constant \( C_{\rho,d} \) depending on \( \rho \geq 1 \) but neither on \( d \geq 2 \) nor on the norm \( \mathbb{R}^d \) is endowed with. In particular, by investigating the consequences for \( \rho \geq 1 \) of the very nice example introduced by Brückerhoff and Juillet [11] to show that \( C_{1,d} = +\infty \) for \( d \geq 2 \), we extend this equality to \( \rho \in \left[ 1, \frac{1 + \sqrt{5}}{2} \right) \). Unfortunately, apart in the particular case \( \rho = 2 \) already addressed in the introduction, we were not able to prove the finiteness of \( C_{\rho,d} \) for some \( \rho \) in the complement interval \( \left[ \frac{1 + \sqrt{5}}{2}, +\infty \right) \).

That is the reason why, we investigated restricted classes of couples \( (\mu, \nu) \in \mathcal{P}_\rho(\mathbb{R}^d) \times \mathcal{P}_\rho(\mathbb{R}^d) \) with \( \mu \leq_{cx} \nu \) such that (34) holds with a finite constant \( C \) uniform over the class. In subsection 3.2, we exhibit three classes such that \( C \) is equal to the one-dimensional constant \( C_\rho \). In subsection 3.3, we deal with the scaling case where \( C = 3 \times 2^{\rho-1} \).
3.1. Lower-bounds on the constant $C_{\rho,d}$

**Proposition 2.** Let $d \in \mathbb{N}^* \setminus \{1\}$ and $\rho \geq 1$. Regardless of the norm $\mathbb{R}^d$ is endowed with, we have

$$C_{\rho,d} = +\infty \text{ for } \rho \in \left[1, \frac{1 + \sqrt{5}}{2}\right), \quad C_{\frac{1 + \sqrt{5}}{2},d} \geq 2 \cdot \frac{\sqrt{5} - 1}{2} \left( \frac{3 + \sqrt{5}}{2}\right)^{\frac{3 - \sqrt{5}}{2}} \approx 2.217, \quad (36)$$

$$C_{\rho,d} \geq 2 \text{ for } \rho \in \left(\frac{1 + \sqrt{5}}{2},2\right) \text{ and } C_{\rho,d} \geq 2^{\rho - 1} \text{ for } \rho \geq 2. \quad (37)$$

The equality $C_{1,d} = +\infty$ for $d \geq 2$ was recently obtained by Brückerhoff and Juillet [11] by exhibiting the very nice Example 1 in dimension $d = 2$. We derive (36) for $d = 2$ by investigating the consequences of this example for $\rho \geq 1$. The extension to $d \geq 2$ then follows from the next lemma and the fact that $\mathbb{R}^2 \ni (x_1, x_2) \mapsto |(x_1, x_2, 0, \ldots, 0)|$ is a norm whatever the norm $| \cdot | : \mathbb{R}^d$ is endowed with. The case $\rho \in \left(\frac{1 + \sqrt{5}}{2},2\right)$ is deduced in the same way from Example 2 in subsection 3.3 while the case $\rho \geq 2$ follows from the next lemma combined with the equality $C_{\rho} = 2^{\rho - 1}$ established in Proposition 1.

**Lemma 1.** Let $d, d' \in \mathbb{N}^*$ be such that $d' < d$, $\rho \geq 1$, $| \cdot |$ be a norm on $\mathbb{R}^d$ and $| \cdot |'$ be a norm on $\mathbb{R}^{d'}$ satisfying the following consistency condition:

$$\exists \lambda > 0, \forall x_1, \ldots, x_{d'} \in \mathbb{R}, \quad |(x_1, \ldots, x_{d'})|' = \lambda |(x_1, \ldots, x_{d'}, 0, \ldots, 0)|. \quad (38)$$

Then $C_{\rho,d'} \leq C_{\rho,d}$ for $\mathbb{R}^d$ and $\mathbb{R}^{d'}$ respectively endowed with $| \cdot |$ and $| \cdot |'$. In particular, $C_\rho \leq C_{\rho,d}$ regardless of the norm $\mathbb{R}^d$ is endowed with.

**Proof of Lemma 1.** Let $\mu', \nu' \in \mathcal{P}_\rho(\mathbb{R}^{d'})$ be such that $\mu' \leq_{\text{ex}} \nu'$. Let $\mu$ and $\nu$ be the respective images of $\mu'$ and $\nu'$ by the map $\mathbb{R}^{d'} \ni (x_1, \ldots, x_{d'}) \mapsto (x_1, \ldots, x_{d'}, 0, \ldots, 0) \in \mathbb{R}^d$. Let $c' = (c'_1, \ldots, c'_{d'}) \in \mathbb{R}^{d'}$ and $c = (c'_1, \ldots, c'_{d'}, 0, \ldots, 0) \in \mathbb{R}^d$. By (38) and the definition of $C_{\rho,d'}$, we have

$$M^\rho_{\rho} (\mu', \nu') = \lambda^\rho M^\rho_{\rho} (\mu, \nu) \leq C_{\rho,d} \lambda W_{\rho} (\mu, \nu) \left( \lambda^\rho \int_{\mathbb{R}^d} |y - c'|^\rho \nu'(dy') \right)^{(\rho - 1)/\rho}$$

$$= C_{\rho,d} W_{\rho} (\mu', \nu') \left( \int_{\mathbb{R}^d} |y' - c'|^\rho \nu'(dy') \right)^{(\rho - 1)/\rho}.$$

By taking the infimum over all $c' \in \mathbb{R}^{d'}$, we get $M^\rho_{\rho} (\mu', \nu') \leq C_{\rho,d} W_{\rho} (\mu', \nu') \sigma_{\rho}^{-1} (\nu')$, hence $C_{\rho,d'} \leq C_{\rho,d}$.

In the particular case $d' = 1$, since the absolute value on $\mathbb{R}$ is consistent with $| \cdot |$ for the coefficient $\lambda = \frac{1}{\|1,0,\ldots,0\|}$, we obtain that $C_\rho \leq C_{\rho,d}$. \hfill \(\square\)

**Example 1 (taken from [11]).** Let $\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{(i,0)}$ and $\mu_n P_\theta(dy) = \int_{\mathbb{R}^2} P_\theta(x, dy) \mu_n(dx)$ for $\theta \in [0, \pi)$ where $P_\theta$ denotes the two dimensional Markov kernel defined by

$$P_\theta(x, dy) = \frac{1}{2} \left( \delta_{x-(\cos \theta, \sin \theta)}(dy) + \delta_{x+(\cos \theta, \sin \theta)}(dy) \right).$$
Since $\mu_n(dx)P_0(x,dy)P_0(x,dz)$ is a coupling between $\mu_nP_0(dy)$ and $\mu_nP_0(dz)$, one has

$$W_p(\mu_n, \mu_nP_0) \leq |(1 - \cos \theta, -\sin \theta)|^{\frac{\theta - 1}{p}} 0.$$  

As a consequence $\lim_{\theta \to 0} W_p(\mu_n, \mu_nP_0) = W_p(\mu_n, \mu_nP_0)$ and $\lim_{\theta \to 0} \sigma_p(\mu_nP_0) = \sigma_p(\mu_nP_0)$. Let us now compute those two limits. Since $\mu_n$ and $\mu_nP_0 = \frac{1}{2n} \left( \delta_{(0,0)} + \delta_{(1,0)} + \delta_{(n,0)} + \delta_{(n+1,0)} \right) + \frac{1}{n} \sum_{i=2}^{n-1} \delta_{(i,0)}$ are both supported on the abscissa axis, using the comonoton coupling, one computes $W_p(\mu_n, \mu_nP_0) = n^{-\frac{1}{p}} |(1,0)|$.

Let $c_* = (\frac{a+1}{2}, 0)$. By invariance of $\mu_nP_0$ by $x \mapsto 2c_* - x$ and convexity of the norm, for each $c \in \mathbb{R}^2$,

$$\int_{\mathbb{R}^2} \langle x - c \rangle^p \mu_nP_0(dx) = \int_{\mathbb{R}^2} \frac{1}{2} (|x - c| + |2c_* - x - c|) \mu_nP_0(dx) \geq \int_{\mathbb{R}^2} \frac{1}{2} (|x - c_*|^p + |2c_* - x - c_*|) \mu_nP_0(dx) = \int_{\mathbb{R}^2} |x - c_*|^p \mu_nP_0(dx).$$

Hence $\sigma_p^0(\mu_nP_0) = \int_{\mathbb{R}^2} |x - c_*|^p \mu_nP_0(dx)$ and

$$\sigma_p^0(\mu_nP_0) = \frac{1}{2pn} \left( (n+1)^p + (n-1)^p + 2 \sum_{i=2}^{n+1} (n+1-2i)^p \right) \sim_n \infty \frac{n^p}{2^p(\rho+1)}.$$  

According to Lemma 1.1 [11], for $\theta \in (0,\pi)$, $P_0$ is the only martingale coupling between $\mu_n$ and $\mu_nP_0(dy)$ so that $\mathcal{M}_p(\mu_n, \mu_nP_0) = |(\cos \theta, \sin \theta)|$ and $\lim_{\theta \to 0} \mathcal{M}_p(\mu_n, \mu_nP_0) = |(1,0)|$. Note that this limit is not equal to $\mathcal{M}_p(\mu_n, \mu_nP_0)$ when $\rho < 2$. Indeed,

$$\frac{1}{n} \sum_{i=2}^{n-1} \delta_{(i,0),(i,0)} + \frac{1}{2n} \left( \frac{n}{n+1} \delta_{((1,0),(0,0))} + \delta_{((1,0),(1,0))} \right) + \frac{1}{n} \delta_{((1,0),(n+1,0))}$$

$$+ \frac{1}{n+1} \delta_{((n+1,0),(0,0))} + \delta((n,0),(n,0)) + \frac{n}{n+1} \delta((n,0),(n+1,0))$$

is a martingale coupling between $\mu_n$ and $\mu_nP_0$ (in fact, one may easily check that its image by the projection on the first and third coordinates is the only martingale coupling between the first marginals of $\mu_n$ and $\mu_nP_0$ in the family $(\mathcal{M}^Q)_{Q \in \mathcal{Q}}$) so that $\mathcal{M}_p(\mu_n, \mu_nP_0) \leq \left( \frac{n^p}{n^p+n^p} \right)^\frac{1}{p} |(1,0)|$.

Hence

$$\lim_{\theta \to 0} \frac{\mathcal{M}_p(\mu_n, \mu_nP_0)}{W_p(\mu_n, \mu_nP_0)\sigma_p^{-1}(\mu_nP_0)} = \frac{|(1,0)|}{W_p(\mu_n, \mu_nP_0)\sigma_p^{-1}(\mu_nP_0)} \sim_n \infty 2^p-1(p+1) \frac{n^p}{n^p-n^p+1}$$

One has $\frac{p}{\rho} - \rho + 1 > 0$ for $p \in \left[ 1, \frac{1+\sqrt{5}}{2} \right)$ and $\frac{p}{\rho} - \rho + 1 = 0$ for $p = \frac{1+\sqrt{5}}{2}$. Since $C_{\rho,2} \leq \lim_{n \to \infty} \lim_{\theta \to 0} \frac{\mathcal{M}_p(\mu_n, \mu_nP_0)}{W_p(\mu_n, \mu_nP_0)\sigma_p^{-1}(\mu_nP_0)}$, we conclude that $C_{\rho,2} = +\infty$ when $p \in \left[ 1, \frac{1+\sqrt{5}}{2} \right)$ and $C_{\rho,2} \geq 2^p-1(p+1)^\frac{1}{p}$ when $p = \frac{1+\sqrt{5}}{2}$, whatever the norm $\mathbb{R}^2$ is endowed with.
3.2. Extensions of the one dimensional inequality

We first look in Propositions 3, 4 and 5 at extensions of the one dimensional inequality which give the same optimal constant. We begin with the fact that the martingale Wasserstein inequality (4) can be tensorised: it holds in greater dimension when the marginals are independent.

**Proposition 3.** Let $d \in \mathbb{N}^*$ and $\mu_1, \nu_1 \ldots, \mu_d, \nu_d \in \mathcal{P}_\rho(\mathbb{R})$ be such that for all $1 \leq i \leq d$, $\mu_i \leq_{cx} \nu_i$. Let $\mu = \mu_1 \otimes \ldots \otimes \mu_d$ and $\nu = \nu_1 \otimes \ldots \otimes \nu_d$. Then $\mu \leq_{cx} \nu$ and

$$\mathcal{M}_p^\rho(\mu, \nu) \leq C_\rho \mathcal{W}_\rho(\mu, \nu) \sigma_p^{\rho-1}(\nu),$$

where $\mathbb{R}^d$ is endowed with the $L^\rho$-norm.

**Proof.** For all $1 \leq i \leq d$, there exists by Lemma 3 below a martingale coupling $M_i \in \Pi^\mathcal{M}(\mu_i, \nu_i)$ between $\mu_i$ and $\nu_i$, optimal for $\mathcal{M}_\rho(\mu_i, \nu_i)$. Let then $M$ be the probability measure on $\mathbb{R}^d \times \mathbb{R}^d$ defined by

$$M(dx, dy) = \mu(dx) m_1(x_1, dy_1) \ldots m_d(x_d, dy_d) = M_1(dx_1, dy_1) \otimes \ldots \otimes M_d(dx_d, dy_d).$$

It is clear that $M$ is a martingale coupling between $\mu$ and $\nu$, which shows that $\mu \leq_{cx} \nu$, and

$$\mathcal{M}_p^\rho(\mu, \nu) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^\rho M(dx, dy) = \sum_{i=1}^d \int_{\mathbb{R}^d \times \mathbb{R}^d} |x_i-y_i|^\rho M_i(dx_i, dy_i).$$

Then for all $c = (c_1, \ldots, c_d) \in \mathbb{R}^d$ we have

$$\mathcal{M}_p^\rho(\mu, \nu) \leq \sum_{i=1}^d \int_{\mathbb{R} \times \mathbb{R}} |x_i-y_i|^\rho M_i(dx_i, dy_i) = \sum_{i=1}^d \mathcal{M}_p^\rho(\mu_i, \nu_i)$$

$$\leq C_\rho \sum_{i=1}^d \mathcal{W}_\rho(\mu_i, \nu_i) \left( \int_{\mathbb{R}} |y_i-c_i|^\rho \nu_i(dy_i) \right)^{(\rho-1)/\rho}$$

$$\leq C_\rho \left( \sum_{i=1}^d \mathcal{W}_\rho^p(\mu_i, \nu_i) \right)^{1/\rho} \left( \sum_{i=1}^d \int_{\mathbb{R}} |y_i-c_i|^\rho \nu_i(dy_i) \right)^{(\rho-1)/\rho},$$

where for the last inequality we applied Hölder’s inequality to the sum over $i$. Let $P \in \Pi(\mu, \nu)$ be a coupling between $\mu$ and $\nu$. For $1 \leq i \leq d$, let $P_i$ be the marginals of $P$ with respect to the coordinates $i$ and $i+d$, so that $P_i$ is a coupling between $\mu_i$ and $\nu_i$. Then

$$\sum_{i=1}^d \mathcal{W}_\rho^p(\mu_i, \nu_i) \leq \sum_{i=1}^d \int_{\mathbb{R} \times \mathbb{R}} |x_i-y_i|^\rho P_i(dx_i, dy_i) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \sum_{i=1}^d |x_i-y_i|^\rho P(dx, dy)$$

$$= \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^\rho P(dx, dy).$$
Since the inequality above is true for any coupling $P$ between $\mu$ and $\nu$, we get
\[
\sum_{i=1}^{d} W^p_\rho(\mu_i, \nu_i) \leq W^p_\rho(\mu, \nu),
\]
which is in fact even an equality according to [3, Proposition 1.1]. We then deduce from (40) and (41) that
\[
\mathcal{M}^p_\rho(\mu, \nu) \leq C_\rho W^p_\rho(\mu, \nu) \left( \sum_{i=1}^{d} \int_{\mathbb{R}} |y_i - c_i|^p \nu(dy_i) \right)^{(\rho-1)/\rho}
\]
\[
= C_\rho W^p_\rho(\mu, \nu) \left( \int_{\mathbb{R}^d} \sum_{i=1}^{d} |y_i - c_i|^p \nu(dy) \right)^{(\rho-1)/\rho}
\]
\[
= C_\rho W^p_\rho(\mu, \nu) \left( \int_{\mathbb{R}^d} |y - c|^p \nu(dy) \right)^{(\rho-1)/\rho}.
\]
By taking the infimum over all $c \in \mathbb{R}^d$, we get (39).

We next turn to the case when, for some $\alpha \in \mathbb{R}^d$, the images of $\mu$ and $\nu$ by
\[
\mathbb{R}^d \ni x \mapsto (|x - \alpha|, 1_{\{|x-\alpha|>0\}}(x - \alpha))
\]
are product measures sharing the same second marginal. This in particular covers the case of radially symmetric measures $\mu$ and $\nu$.

**Proposition 4.** Let $\mathbb{R}^d$ be endowed with any norm and $\mu, \nu \in \mathcal{P}_\rho(\mathbb{R}^d)$ be the respective images of $\tilde{\mu}(dr)\eta(d\theta)$ and $\tilde{\nu}(dr)\eta(d\theta)$ by $(r, \theta) \mapsto \alpha + r\theta$ where $\alpha \in \mathbb{R}^d$, $\tilde{\mu}, \tilde{\nu} \in \mathcal{P}_\rho(\mathbb{R})$ are such that $\tilde{\mu}(\mathbb{R}_+) = \tilde{\nu}(\mathbb{R}_+) = 1$ and $\eta \in \mathcal{P}(\mathbb{R})$ is such that $\eta(S_{d-1}) = 1$ for $S_{d-1} = \{\theta \in \mathbb{R}^d : |\theta| = 1\}$ and $\eta$ is invariant by $x \mapsto -x$. If $\mu \leq_{Cx} \nu$, then $\mathcal{M}^\rho_\rho(\mu, \nu) \leq C_\rho W^\rho_\rho(\mu, \nu)\sigma^{\rho+1}_\rho(\nu)$.

**Proof.** Since $\mathcal{M}_\rho$ and $W_\rho$ (resp. $\sigma_\rho$) are (resp. is) preserved by taking the image of its two arguments (resp. its argument) by the translation vector $\alpha$, we suppose without loss of generality that $\alpha = 0$. Let $\tilde{\mu}(dt), \tilde{\nu}(du) \in \mathcal{P}_\rho(\mathbb{R})$ be the respective images of $\mu(dt)\eta(\theta)$ and $\nu(du)\eta(\theta)$ by $(r, s, \theta) \mapsto s\theta$. Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be a convex function. The function $\mathbb{R}_+ \ni r \mapsto \varphi(-r) + \varphi(r)$ is non-decreasing and convex so that the function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ defined by $f(x) = \frac{\varphi(-|x|) + \varphi(|x|)}{2}$ is convex. We have
\[
\int_{\mathbb{R}^d} f(x)\mu(dx) = \int_{\mathbb{R}^d} \frac{\varphi(-|t\theta|) + \varphi(|t\theta|)}{2} \tilde{\mu}(dt)\eta(d\theta) = \int_{\mathbb{R}} \varphi(t)\tilde{\mu}(dt)
\]
and, in the same way,
\[
\int_{\mathbb{R}^d} f(y)\nu(dy) = \int_{\mathbb{R}} \varphi(u)\tilde{\nu}(du)
\]so that $\mu \leq_{Cx} \nu$ implies that $\tilde{\mu} \leq_{Cx} \tilde{\nu}$. Let $\tilde{M} \in \Pi^\rho(\tilde{\mu}, \tilde{\nu})$ be optimal for $\mathcal{M}_\rho(\mu, \nu)$ and $M(dt, du)$ denote the image of $\tilde{M}(dt, du)\eta(d\theta)$ by $(t, u, \theta) \mapsto (\theta, ut\theta)$. The marginals of $M$ are the respective images of $\tilde{\mu}(dr)\eta(\theta)$ and $\tilde{\nu}(dr)\eta(\theta)$ by $(s, \theta) \mapsto rs\theta$. As the image of $\eta(S_{d-1}) \eta(d\theta)$ by $(s, \theta) \mapsto s\theta$ is equal
to \( \eta \), they are equal to \( \mu(dx) \) and \( \nu(dy) \). Moreover, for \( \psi : \mathbb{R}^d \to \mathbb{R} \) measurable and bounded, using the martingale property of \( \bar{M} \) for the second equality, we obtain

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x) y M(dx, dy) = \int_{\mathbb{R}^d \times \mathbb{R}} \psi(t \theta) u \bar{M}(dt, du) \eta(d\theta)
\]

so that \( M \in \Pi^M(\mu, \nu) \). As a consequence,

\[
\mathcal{M}^\rho_p(\mu, \nu) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^\rho M(dx, dy) = \int_{\mathbb{R}^d \times \mathbb{R} \times S_{d-1}} |t \theta - u \bar{M}(dt, du)| \eta(d\theta)
\]

so that \( \mathcal{M}^\rho_p(\mu, \nu) \leq C_p \mathcal{W}_p(\bar{\mu}, \bar{\nu}) \sigma^\rho_{\rho-1}(\bar{\nu}). \tag{42} \)

For \( y, c \in \mathbb{R}^d \), by the triangle inequality \( |y| \leq \frac{|y-c| + |y|}{2} \) so that by Jensen’s inequality, \( |y|^\rho \leq \frac{|y-c|^\rho + |y|^\rho}{2} \) so that

\[
\int_{\mathbb{R}^d} |y|^\rho \nu(dy) = \int_{\mathbb{R}^d} |rs \theta|^\rho \bar{\nu}(dr) \frac{\delta_{-1} + \delta_1}{2}(d\theta) ≤ \int_{\mathbb{R}^d} |rs \theta - c|^\rho \bar{\nu}(dr) \frac{\delta_{-1} + \delta_1}{2}(d\theta)
\]

As a consequence,

\[
\sigma^\rho(\nu) = \int_{\mathbb{R}^d} |y|^\rho \nu(dy) = \int_{\mathbb{R}^d} |u|^\rho \bar{\nu}(du) \eta(d\theta) = \int_{\mathbb{R}^d} |u|^\rho \bar{\nu}(du) ≥ \sigma^\rho(\nu), \quad \tag{43}
\]

where the last inequality is in fact an equality by a reasoning similar to the one which just lead to the first equality.

The image of \( P \in \Pi(\mu, \nu) \) optimal for \( \mathcal{W}_p(\mu, \nu) \) by \( \mathbb{R}^d \times \mathbb{R}^d \ni (x, y) \mapsto (|x|, |y|) \) belongs to \( \Pi(\bar{\mu}, \bar{\nu}) \) so that, with the inequality \( |x - y| \geq ||x| - |y|| \) deduced from the triangle inequality,

\[
\mathcal{W}_p^\rho(\mu, \nu) = \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^\rho P(dx, dy) ≥ \int_{\mathbb{R}^d \times \mathbb{R}^d} ||x| - |y||^\rho P(dx, dy) ≥ \mathcal{W}_p^\rho(\bar{\mu}, \bar{\nu}).
\]

On the other hand, for \( \bar{P} \in \Pi(\bar{\mu}, \bar{\nu}) \) optimal for \( \mathcal{W}_p(\bar{\mu}, \bar{\nu}) \), the image of \( \bar{P}(dr, dv)\frac{\delta_{-1} + \delta_1}{2}(ds) \) by \( (r, v, s) \mapsto (rs, vs) \) belongs to \( \Pi(\bar{\mu}, \bar{\nu}) \) so that

\[
\mathcal{W}_p^\rho(\bar{\mu}, \bar{\nu}) = \int_{\mathbb{R}^d \times \mathbb{R}^d} |rs - vs|^\rho \bar{P}(dr, dv) \frac{\delta_{-1} + \delta_1}{2}(ds) ≥ \mathcal{W}_p^\rho(\bar{\mu}, \bar{\nu}).
\]

Hence \( \mathcal{W}_p(\mu, \nu) ≥ \mathcal{W}_p(\bar{\mu}, \bar{\nu}) ≥ \mathcal{W}_p(\bar{\mu}, \bar{\nu}) \) where the first (resp. second) inequality can be proved to be an equality by an adaptation of the reasoning leading to the second (resp. first) one. Plugging \( \mathcal{W}_p(\mu, \nu) ≥ \mathcal{W}_p(\bar{\mu}, \bar{\nu}) \) together with (43) into (42), we conclude that \( \mathcal{M}^\rho_p(\mu, \nu) \leq C_p \mathcal{W}_p(\mu, \nu) \sigma^\rho_{\rho-1}(\bar{\nu}). \)
We now look at two measures $\mu$ and $\nu$ such that for $X$ distributed according to $\mu$, there exists $\lambda \geq 0$ such that $\nu$ is the probability distribution of $X + \lambda(X - \mathbb{E}[X])$ and the conditional probability distribution of $X$ given the direction of $X - \mathbb{E}[X]$ has mean $\mathbb{E}[X]$. In order to transcribe formally the latter condition, we give the following definition.

**Definition.** Let $d \in \mathbb{N}^* \setminus \{1\}$ and $H : \mathbb{R}^d \to \mathbb{R}^d$ be a measurable map such that $H(\mathbb{R}^d)$ is a Borel subset of $\mathbb{R}^d$, We say that $H$ is direction-dependent iff $|H(x)| = 1$ for all $x \in \mathbb{R}^d$ and

$$
\forall x, y \in \mathbb{R}^d \setminus \{0\}, \quad H(x) = H(y) \iff y \in \text{Span}(x).
$$

In dimension $d \in \mathbb{N}^* \setminus \{1\}$, a natural example of a direction-dependent map $H : \mathbb{R}^d \to \mathbb{R}^d$ is given by the one defined for all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d \setminus \{0\}$ by

$$
\begin{align*}
H(x) &= \frac{x}{|x|} \text{ if } x_1 > 0 \text{ or there exists } i \in \{1, \ldots, d - 1\} \text{ such that } x_1 = \cdots = x_i = 0 \text{ and } x_{i+1} > 0; \\
H(x) &= -\frac{x}{|x|} \text{ otherwise},
\end{align*}
$$

and $H(0)$ is any vector with norm 1.

**Proposition 5.** Let $d \in \mathbb{N}^* \setminus \{1\}$, $r \in [1, +\infty]$ and $\mathbb{R}^d$ be endowed with the $L^r$-norm. Let $\rho \geq 1$, $\mu \in \mathcal{P}_\rho(\mathbb{R}^d)$ be with mean $\alpha \in \mathbb{R}^d$, $\lambda : \mathbb{R}^d \to \mathbb{R}_+$, $H : \mathbb{R}^d \to \mathbb{R}^d$ be a direction-dependent measurable map in the sense of Definition 3.2 and $\nu$ be the image of $\mu$ by the map $x \mapsto x + \lambda(H(x - \alpha))(x - \alpha) = \alpha + (1 + \lambda(H(x - \alpha)))(x - \alpha)$.

If $\mathbb{E}[\lambda(H(X - \alpha))(X - \alpha)|\rho] < +\infty$ and $\mathbb{E}[X|H(X - \alpha)] = \alpha$ almost surely for $X$ distributed according to $\mu$, then $\mu \leq_{cx} \nu$. If moreover $\lambda$ is constant, then

$$
\mathcal{M}_\rho^\mu(\mu, \nu) \leq C_\rho \mathcal{W}_\rho(\mu, \nu) \sigma_\rho^{\rho - 1}(\nu).
$$

**Remark 3.** Suppose that $\mu \in \mathcal{P}_\rho(\mathbb{R}^d)$ is symmetric with mean $\alpha \in \mathbb{R}^d$, that is $(x - \alpha)_\sharp \mu(dx) = (\alpha - x)_\sharp \mu(dx)$. Let $H$ be defined by (44) and $X$ be distributed according to $\mu$. Then $(X - \alpha, H(X - \alpha)) = (\alpha - X, H(\alpha - X)) = (\alpha - X, H(X - \alpha)) = \mathbb{E}[X - \alpha|H(X - \alpha)] = \mathbb{E}[\alpha - X|H(X - \alpha)]$ a.s., hence $\mathbb{E}[X|H(X - \alpha)] = \alpha$ a.s.

The proof of Proposition 5 relies on the following lemma, whose proof is deferred to Section 4, which explains why we can endow $\mathbb{R}^d$ with the $L^r$-norm for $r \in [1, \infty]$. In the case $r = 2$ of the Euclidean norm, the result is a simple property of the orthogonal projection.

**Lemma 2.** Let $d \in \mathbb{N}^* \setminus \{1\}$, $r \in [1, +\infty]$, $\mathbb{R}^d$ be endowed with the $L^r$-norm, $\mathbb{S}^{d-1} = \{ a \in \mathbb{R}^d \mid |a| = 1 \}$ and $\text{sgn} : \mathbb{R} \to \mathbb{R}, x \mapsto 1_{\{x \geq 0\}} - 1_{\{x < 0\}}$. For all $a = (a_1, \ldots, a_d) \in \mathbb{S}^{d-1}$ and $c = (c_1, \ldots, c_d) \in \mathbb{R}^d$, let $c_a$ be defined by

$$
c_a = \begin{cases} 
\frac{\sum_{i=1}^d c_i \text{sgn}(a_i)|a_i|r^{-1}}{a} & \text{if } r < +\infty \\
\frac{c_i \text{sgn}(a_i)a}{r} & \text{if } r = +\infty, \text{ where } i = \min\{j \in \{1, \ldots, d\} \mid |a_j| = 1\}.
\end{cases}
$$

Then

$$
\forall a \in \mathbb{S}^{d-1}, \quad \forall c \in \mathbb{R}^d, \quad \forall y \in \text{Span}(a), \quad |y - c_a| \leq |y - c|.
$$
Proof of Proposition 5. Up to replacing $\mu$ and $\nu$ by their respective images by the map $x \mapsto x - \alpha$, we may suppose without loss of generality that $\alpha = 0$.

Let $(p(a, dx))_{a \in H(\mathbb{R}^d)}$ be a probability kernel such that $(H_{2\mu})(da) p(a, dx)$ is the image of $\mu$ by the map $x \mapsto (H(x), x)$. For all $a \in H(\mathbb{R}^d)$, let $\tilde{\rho}(a, dy)$ be the image of $p(a, dx)$ by the map $x \mapsto (1 + \lambda(H(x)))x$ for all $a \in H(\mathbb{R}^d)$, let $q(a, \cdot)$, resp. $\tilde{q}(a, \cdot)$, be the image of $p(a, \cdot)$, resp. $\tilde{\rho}(a, \cdot)$, by the map $y \mapsto \langle y, a \rangle$. We have

\[
\int_{H(\mathbb{R}^d)} \left( \int_{\mathbb{R}^d} |x|^\rho \, p(a, dx) \right) (H_{2\mu})(da) = \int_{\mathbb{R}^d} |x|^\rho \, \mu(dx) < +\infty,
\]

and

\[
\int_{H(\mathbb{R}^d)} \left( \int_{\mathbb{R}^d} |y|^\rho \, \tilde{\rho}(a, dy) \right) (H_{2\mu})(da) = \int_{\mathbb{R}^d} |(1 + \lambda(H(x)))x|^\rho \, \mu(dx) < +\infty,
\]

so $H_{2\mu}(da)$-almost everywhere, $p(a, \cdot)$ and $q(a, \cdot)$, and therefore $\tilde{\rho}(a, \cdot)$ and $\tilde{q}(a, \cdot)$, belong to $\mathcal{P}_p(\mathbb{R}^d)$. Moreover we see by definition of $H$ that for $H_{2\mu}(da)$-almost all $a \in H(\mathbb{R}^d)$, $p(a, \text{Span}(a)) = 1$, so

\[
p(a, dx) = \int_{\mathbb{R}} \delta_{ta}(dx) q(a, dt) \quad \text{and} \quad \tilde{\rho}(a, dy) = \int_{\mathbb{R}} \delta_{sa}(dy) \tilde{q}(a, ds).
\]

By assumption, we have $H_{2\mu}(da)$-almost everywhere

\[
a \int_{\mathbb{R}} t \, q(a, dt) = \int_{\mathbb{R}^d} x \, p(a, dx) = 0.
\]

Since $\tilde{q}(a, \cdot)$ is the image of $q(a, \cdot)$ by the map $y \mapsto (1 + \lambda(H(y)))y$, or equivalently by the map $y \mapsto (1 + \lambda(a))y$, by Lemma 7 below, for $H_{2\mu}(da)$-almost all $a \in H(\mathbb{R}^d)$, $q(a, \cdot) \leq_{cx} \tilde{q}(a, \cdot)$. Up to replacing $q(a, \cdot)$ and therefore $\tilde{q}(a, \cdot)$ by $\delta_0$ on a $H_{2\mu}$-null set, we may suppose without loss of generality that $q(a, \cdot), \tilde{q}(a, \cdot) \in \mathcal{P}_p(\mathbb{R})$ and $q(a, \cdot) \leq_{cx} \tilde{q}(a, \cdot)$ for all $a \in H(\mathbb{R}^d)$. By [4, Theorem 19.12] and since $\mathcal{P}_p(\mathbb{R}^d)$ is a closed subset of $\mathcal{P}(\mathbb{R}^d)$ endowed with the weak convergence topology, the map $H(\mathbb{R}^d) \ni a \mapsto (q(a, \cdot), \tilde{q}(a, \cdot)) \in \mathcal{P}_p(\mathbb{R}) \times \mathcal{P}_p(\mathbb{R})$ is measurable when the codomain is endowed with the product $\mathcal{B} \otimes \mathcal{B}$ of the Borel $\sigma$-algebra $\mathcal{B}$ on $\mathcal{P}_p(\mathbb{R})$ associated with the weak convergence topology. With Lemma 4 below, we deduce that there exists a measurable map $H(\mathbb{R}^d) \ni a \mapsto M^a \in \mathcal{P}(\mathbb{R} \times \mathbb{R})$ endowed with the $\sigma$-field generated by the weak convergence topology such that for each $a \in H(\mathbb{R}^d)$, $M^a$ belongs to $\Pi^M(q(a, \cdot), \tilde{q}(a, \cdot))$ and is optimal for $\mathcal{M}_p(q(a, \cdot), \tilde{q}(a, \cdot))$. Let $\tilde{M}^a$ be the image of $M^a$ by the map $(t, s) \mapsto (ta, sa)$. Then the map $a \mapsto \tilde{M}^a$ is also measurable, which is equivalent to say (see again [4, Theorem 19.12]) that $(\tilde{M}^a)_{a \in H(\mathbb{R}^d)}$ is a probability kernel from $H(\mathbb{R}^d)$ to $\mathbb{R} \times \mathbb{R}$. Hence we can define

\[
\tilde{M}(dx, dy) = \int_{a \in H(\mathbb{R}^d)} \tilde{M}^a(dx, dy) (H_{2\mu})(da).
\]

For all $a \in H(\mathbb{R}^d)$, $M^a$ is a martingale coupling between $q(a, \cdot)$ and $\tilde{q}(a, \cdot)$, hence we easily see that $\tilde{M}^a$ is a martingale coupling between the respective images of $q(a, \cdot)$ and $\tilde{q}(a, \cdot)$ by the map $t \mapsto at$, namely $p(a, \cdot)$ and $\tilde{\rho}(a, \cdot)$. Therefore one can also readily show that $\tilde{M}$ is a martingale coupling between
\[
\int_{a \in H(\mathbb{R}^d)} p(a, dx) (H_2 \mu)(da) = \mu(dx) \quad \text{and} \quad \int_{a \in H(\mathbb{R}^d)} \tilde{p}(a, dy) (H_2 \mu)(da) = \nu(dy). \]
Consequently,
\[
\mathcal{M}_p^\rho(\mu, \nu) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x|^{\rho} \, \mathcal{M}(dx, dy) = \int_{H(\mathbb{R}^d)} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x|^{\rho} \, \mathcal{M}(dx, dy) \right) (H_2 \mu)(da)
\]
\[
= \int_{H(\mathbb{R}^d)} \left( \int_{\mathbb{R} \times \mathbb{R}} |s - t|^{\rho} \, M^a(dt, ds) \right) (H_2 \mu)(da) = \int_{H(\mathbb{R}^d)} \mathcal{M}_p^\rho(q(a, \cdot), \tilde{q}(a, \cdot)) (H_2 \mu)(da).
\]  
(47)

Let \( c \in \mathbb{R}^d \). For all \( a \in H(\mathbb{R}^d) \), let \( c_a \) be defined by (46) and \( s_a \in \mathbb{R} \) be such that \( c_a = s_a a \). If the map \( \lambda \) is constant equal to some \( \lambda \in \mathbb{R}_+ \) (with a slight abuse of notation), then using the definition of \( C_p \) for the first inequality, Lemma 6 below for the first equality, Lemma 2 for the second inequality, Hölder’s inequality for the third inequality and Lemma 6 again for the last equality (there the constancy of \( \lambda \) plays a crucial role), we deduce that
\[
\mathcal{M}_p^\rho(\mu, \nu) \leq \int_{H(\mathbb{R}^d)} C_p \mathcal{W}_p \left( q(a, \cdot), \tilde{q}(a, \cdot) \right) \left( \int_{\mathbb{R}} |s - s_a|^{\rho} \tilde{q}(a, ds) \right)^{\rho / p} (H_2 \mu)(da)
\]
\[
= C_p \lambda \int_{H(\mathbb{R}^d)} \left( \int_{\mathbb{R}} |t|^{\rho} q(a, dt) \right)^{1 / \rho} \left( \int_{\mathbb{R}} |sa - s_a a|^{\rho} \tilde{q}(a, ds) \right)^{(p - 1) / p} (H_2 \mu)(da)
\]
\[
= C_p \lambda \int_{H(\mathbb{R}^d)} \left( \int_{\mathbb{R}^d} |x|^{\rho} p(a, dx) \right)^{1 / \rho} \left( \int_{\mathbb{R}} |y - c_a^{\rho} \tilde{p}(a, dy) \right)^{(p - 1) / p} (H_2 \mu)(da)
\]
\[
\leq C_p \lambda \int_{H(\mathbb{R}^d)} \left( \int_{\mathbb{R}^d} |x|^{\rho} p(a, dx) \right)^{1 / \rho} \left( \int_{\mathbb{R}^d} |y - c|^{\rho} \tilde{p}(a, dy) \right)^{(p - 1) / p} (H_2 \mu)(da)
\]
\[
\leq C_p \lambda \left( \int_{H(\mathbb{R}^d)} \int_{\mathbb{R}^d} |x|^{\rho} p(a, dx) (H_2 \mu)(da) \right)^{1 / \rho}
\]
\[
\times \left( \int_{H(\mathbb{R}^d)} \int_{\mathbb{R}^d} |y - c|^{\rho} \tilde{p}(a, dy) (H_2 \mu)(da) \right)^{(p - 1) / p}
\]
\[
\leq C_p \lambda \left( \int_{\mathbb{R}^d} |x|^{\rho} \mu(dx) \right)^{1 / \rho} \left( \int_{\mathbb{R}^d} (1 + \lambda)x - c|^{\rho} \mu(dx) \right)^{(p - 1) / p}
\]
\[
= C_p \mathcal{W}_p(\mu, \nu) \left( \int_{\mathbb{R}^d} |y - c|^{\rho} \nu(dy) \right)^{(p - 1) / p}.
\]  
(48)

By taking the infimum over all \( c \in \mathbb{R}^d \), we get (45).

3.3. The scaling case

We call scaling case the situation in which two measures \( \mu \) and \( \nu \) are such that for \( X \) distributed according to \( \mu \), there exists \( \lambda \geq 0 \) such that \( \nu \) is the probability distribution of \( X + \lambda(X - \mathbb{E}[X]) = \mathbb{E}[X] + (1 + \lambda)(X - \mathbb{E}[X]) \). In the previous section we already considered this case under an additional
assumption on the conditional probability distribution of $X$, see Proposition 5. We release here the latter constraint and study the impact on the constant $C$ in (34).

**Proposition 6.** Let $d \in \mathbb{N}^*$ and $\mathbb{R}^d$ be endowed with any norm. Let $\rho \geq 1$, $\lambda \geq 0$ and $\mu \in \mathcal{P}_\rho(\mathbb{R}^d)$ be with mean $\alpha \in \mathbb{R}^d$. Let $\nu$ be the image of $\mu$ by the map $x \mapsto x + \lambda(x - \alpha)$. Then

$$
\mathcal{M}_\rho^p(\mu, \nu) \leq 2^{\rho - 1} \frac{3 + \lambda}{1 + \lambda} W^p_\rho(\mu, \nu) \sigma^{p - 1}_\rho(\nu). 
$$

(49)

**Remark 4.** Suppose that there exists a direction-dependent measurable map $H : \mathbb{R}^d \to \mathbb{R}^d$ in the sense of Definition 3.2 such that for $X$ distributed according to $\mu$, $\mathbb{E}[X | H(X - \alpha)] = \alpha$ almost surely. Then by Proposition 5, we see that $2^{\rho - 1} \frac{3 + \lambda}{1 + \lambda}$ could be replaced in (49) with $C_\rho$. In view of (21) and Remark 1, for $\rho \in (1, 2)$, $C_\rho > 2^{\rho - 1}$ so $2^{\rho - 1} \frac{3 + \lambda}{1 + \lambda}$ is sharper for $\lambda$ in a neighbourhood of $+\infty$. However, the smallest constant independent of $\lambda$ induced by (49) is $3 \times 2^{\rho - 1}$, which is greater than $C_\rho$ by Proposition 1 ii (using $2 \leq 2^\rho = 2 \times 2^{\rho - 1}$ when $\rho \in (1, 2)$).

**Proof.** For all $x \in \mathbb{R}^d$, let $m(x, dy)$ be the probability kernel defined by

$$
m(x, dy) = \frac{1}{1 + \lambda} \delta_{x + \lambda(x - \alpha)}(dy) + \frac{\lambda}{1 + \lambda} \nu(dy). 
$$

(50)

For all measurable and bounded map $h : \mathbb{R}^d \to \mathbb{R}$, we have

$$
\int_{\mathbb{R}^d \times \mathbb{R}^d} h(y) \mu(dx) \, m(x, dy) = \frac{1}{1 + \lambda} \int_{\mathbb{R}^d} h(x + \lambda(x - \alpha)) \mu(dx) + \frac{\lambda}{1 + \lambda} \int_{\mathbb{R}^d} h(y) \nu(dy)
$$

$$
= \int_{\mathbb{R}^d} h(y) \nu(dy).
$$

Moreover, for all $x \in \mathbb{R}^d$,

$$
\int_{\mathbb{R}^d} y \, m(x, dy) = \frac{1}{1 + \lambda} (x + \lambda(x - \alpha)) + \frac{\lambda}{1 + \lambda} \int_{\mathbb{R}^d} (x' + \lambda(x' - \alpha)) \mu(dx')
$$

$$
= \frac{1}{1 + \lambda} (x + \lambda(x - \alpha)) + \frac{\lambda}{1 + \lambda} \alpha = x.
$$

So $\mu(dx) \, m(x, dy)$ is a martingale coupling between $\mu$ and $\nu$, and

$$
\mathcal{M}_\rho^p(\mu, \nu) \leq \int_{\mathbb{R} \times \mathbb{R}} |y - x|^{\rho} \mu(dx) \, m(x, dy)
$$

$$
= \frac{1}{1 + \lambda} \int_{\mathbb{R}^d} \lambda^\rho |x - \alpha|^\rho \mu(dx) + \frac{\lambda}{1 + \lambda} \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x|^{\rho} \mu(dx) \nu(dy).
$$

On the one hand, using Lemma 6 below and the fact that $\mu(dx) \, \nu(dy)$ is a coupling between $\mu$ and $\nu$, we have

$$
\int_{\mathbb{R}^d} |x - \alpha|^\rho \mu(dx) = \frac{1}{\lambda^\rho} W^\rho_\rho(\mu, \nu) = \frac{1}{\lambda^\rho} W^p_\rho(\mu, \nu) W^{p - 1}_\rho(\mu, \nu)
$$

$$
\leq \frac{1}{\lambda^\rho} W^p_\rho(\mu, \nu) \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x|^\rho \mu(dx) \nu(dy) \right)^{(\rho - 1)/\rho}.
$$
On the other hand, Minkowski’s inequality and Lemma 6 below yield
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x|^\rho \mu(dx) \nu(dy)
\]
\[
= \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x|^\rho \mu(dx) \nu(dy) \right)^{\frac{1}{\rho}} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x|^\rho \mu(dx) \nu(dy) \right)^{\frac{\rho - 1}{\rho}}
\]
\[
\leq \left( \left( \int_{\mathbb{R}^d} |x - \alpha|^\rho \mu(dx) \right)^{1/\rho} + \left( \int_{\mathbb{R}^d} |y - \alpha|^\rho \nu(dy) \right)^{1/\rho} \right)^{\rho - 1}
\]
\[
\leq 2^{\rho - 1} \left( \int_{\mathbb{R}^d} |y - \alpha|^\rho \nu(dy) \right)^{(\rho - 1)/\rho}.
\]
We deduce that
\[
\mathcal{M}_\rho^\rho(\mu, \nu) \leq \frac{3 + \lambda}{1 + \lambda} W_\rho(\mu, \nu) \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x|^\rho \mu(dx) \nu(dy) \right)^{(\rho - 1)/\rho}.
\]

Using Minkowski’s inequality and the definition of convex order, for all \( c \in \mathbb{R} \) we get
\[
\left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x|^\rho \mu(dx) \nu(dy) \right)^{(\rho - 1)/\rho}
\]
\[
\leq \left( \left( \int_{\mathbb{R}^d} |x - c|^\rho \mu(dx) \right)^{1/\rho} + \left( \int_{\mathbb{R}^d} |y - c|^\rho \nu(dy) \right)^{1/\rho} \right)^{\rho - 1}
\]
\[
\leq 2^{\rho - 1} \left( \int_{\mathbb{R}^d} |y - c|^\rho \nu(dy) \right)^{(\rho - 1)/\rho}.
\]
By taking the infimum over all \( c \in \mathbb{R} \), we get
\[
\mathcal{M}_\rho^\rho(\mu, \nu) \leq 2^{\rho - 1} \frac{3 + \lambda}{1 + \lambda} W_\rho(\mu, \nu) \sigma_\rho^{\rho - 1}(\nu).
\]

As already seen in the previous proof, in the scaling case, by Lemma 6 below, \( W_\rho^\rho(\mu, \nu) = \lambda^\rho \int_{\mathbb{R}^d} |x - \alpha|^\rho \mu(dx) \). On the other hand,
\[
\sigma_\rho^\rho(\nu) \leq \int_{\mathbb{R}^d} |y - \alpha|^\rho \nu(dy) = (1 + \lambda)^\rho \int_{\mathbb{R}^d} |x - \alpha|^\rho \mu(dx)
\]
so that
\[
C_{\rho,d} \geq \frac{\mathcal{M}_\rho^\rho(\mu, \nu)}{\lambda (1 + \lambda)^{\rho - 1} \int_{\mathbb{R}^d} |x - \alpha|^\rho \mu(dx)}.
\]
When, moreover, \( \nu \) is supported on an affine basis of \( \mathbb{R}^d \), there is a single martingale coupling between \( \mu \) and \( \nu \) (thus given by \( \mu(dx)\mu(dy) \) with the kernel \( m \) defined in (50)) and we are in a good position to derive a lower-bound for the constant \( C_{\rho,d} \) defined in (35). In the next example, we exploit this idea in dimension \( d = 2 \).

**Example 2.** Let \( n \in \mathbb{N}^* \), \( i = (0, 0) \), \( j = (1, 0) \), \( k = (1, \frac{1}{n}) \), \( p = \frac{1}{2m} \), \( q = \frac{1}{2n} \), \( r = 1 - \frac{1}{n} \) and \( \alpha = pi + qj + rk = (\frac{1}{m} - \frac{1}{n}) \). We set \( \mu = p\delta_i + q\delta_j + r\delta_k \) and define \( \nu \) as the image of \( \mu \) by \( x \mapsto x + \lambda(x - \alpha) \) with \( \lambda > 0 \). By the definition (50) of the kernel \( m \), we have

\[
M^\rho_{\alpha}(\mu, \nu) = \frac{\lambda \rho}{1 + \lambda} \int_{\mathbb{R}^2} |x - \alpha|^\rho \mu(dx) + \frac{\lambda}{1 + \lambda} \int_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y|^\rho \mu(dx)\nu(dy)
\]

where \( \int_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y|^\rho \mu(dx)\nu(dy) \) goes to \( \int_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y|^\rho \mu(dx)\mu(dy) \) as \( \lambda \to 0^+ \). Taking the limit \( \lambda \to 0^+ \) in (51), we deduce that

\[
C_{\rho,2} \geq \mathbf{1}_{\{\rho = 1\}} + \frac{\int_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y|^\rho \mu(dx)\mu(dy)}{\int_{\mathbb{R}^2} |x - \alpha|^\rho \mu(dx)}.
\]

As \( n \to \infty \), whatever the norm \( |\cdot| \) on \( \mathbb{R}^2 \), \( |i - j| \), \( |i - k| \), \( |i - \alpha| \) and \( |j - \alpha| \) converge to \( |i - j|/2 \) and \( |k - \alpha| = \frac{1}{\sqrt{n}} \) of \( (0, 1) \), which implies that

\[
\int_{\mathbb{R}^2} |x - \alpha|^\rho \mu(dx) = p|i - \alpha|^\rho + q|j - \alpha|^\rho + r|k - \alpha|^\rho \sim \frac{|i - j|^\rho}{2^\rho n}
\]

\[
\int_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y|^\rho \mu(dx)\mu(dy) = 2 (pq |j - i|^\rho + qr |k - j|^\rho + rp |i - k|^\rho) \sim \frac{|i - j|^\rho}{2^{\rho - 1} n}.
\]

By taking the limit \( n \to \infty \) in the last inequality, we conclude that \( C_{\rho,2} \geq \mathbf{1}_{\{\rho = 1\}} + 2 \), whatever the norm \( \mathbb{R}^2 \) is endowed with.

## 4. Technical lemmas

This section is devoted to the statements and proofs of technical lemmas needed earlier in the paper.

**Lemma 3.** Let \( d \in \mathbb{N}^* \), \( \rho \geq 1 \) and \( \mu, \nu \in P_\rho(\mathbb{R}^d) \) be such that \( \mu \leq c_x \nu \). Let \( c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+ \) be lower semicontinuous. Then the infimum

\[
C(\mu, \nu) = \inf_{M \in \Pi^M(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y) M(dx, dy)
\]

is attained, i.e. there exists \( M \in \Pi^M(\mu, \nu) \) such that \( C(\mu, \nu) = \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y) M(dx, dy) \).

**Proof.** Let \( M_n \in \Pi^{M}(\mu, \nu) \), \( n \in \mathbb{N} \) be a sequence of martingale couplings between \( \mu \) and \( \nu \) such that

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y) M_n(dx, dy) \to_{n \to +\infty} C(\mu, \nu).
\]
The probability measures \( \mu \) and \( \nu \) are tight: for all \( \varepsilon > 0 \) there exists a compact subset \( K \subset \mathbb{R}^d \) such that \( \mu(K) \geq 1 - \varepsilon \) and \( \nu(K) \geq 1 - \varepsilon \). Therefore, for all \( n \in \mathbb{N} \),

\[
M_n((K \times K)^C) \leq M_n((K^C \times \mathbb{R}^d) \cup (\mathbb{R}^d \times K^C)) \leq \mu(K^C) + \nu(K^C) \leq 2\varepsilon.
\]

We deduce that \( (M_n)_{n \in \mathbb{N}} \) is tight. By Prokhorov's theorem, there exists an increasing map \( \varphi : \mathbb{N} \to \mathbb{N} \) such that \( (M_{\varphi(n)})_{n \in \mathbb{N}} \) converges weakly towards \( M \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \). Since the projections maps \((x, y) \mapsto x \) and \((x, y) \mapsto y \) are continuous, the respective marginals of \( M_{\varphi(n)} \) converge to the respective marginals of \( M \). Since for all \( n \in \mathbb{N} \), \( M_{\varphi(n)} \) has marginals \( \mu \) and \( \nu \), so does \( M \), hence \( M \in \Pi(\mu, \nu) \). Moreover, for all \( n \in \mathbb{N} \) let \((X_n, Y_n)\) be a bivariate random variable distributed according to \( M_n \) and \((X, Y)\) be distributed according to \( M \). Since \( \mu \) and \( \nu \) belong to \( \mathcal{P}_1(\mathbb{R}^d) \), \((X_n, Y_n)_{n \in \mathbb{N}} \) is uniformly integrable. Let \( f : \mathbb{R}^d \to \mathbb{R} \) be a continuous and bounded map. Since \((X_{\varphi(n)}, Y_{\varphi(n)})_{n \in \mathbb{N}} \) converges in distribution to \((X, Y)\), is uniformly integrable and \((x, y) \mapsto f(x)(y - x)\) is continuous with at most linear growth, we have

\[
0 = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x)(y - x) \, M_{\varphi(n)}(dx, dy) = \mathbb{E}[f(X_{\varphi(n)})(Y_{\varphi(n)} - X_{\varphi(n)})]
\]

\[
\lim_{n \to +\infty} \mathbb{E}[f(X)(Y - X)] = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x)(y - x) \, M(dx, dy).
\]

We deduce that \( M \in \Pi^M(\mu, \nu) \). Then by the Portmanteau theorem, we get

\[
C(\mu, \nu) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y) \, M(dx, dy) \leq \liminf_{n \to +\infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y) \, M_{\varphi(n)}(dx, dy) = C(\mu, \nu),
\]

so \( M \) is optimal for \( C(\mu, \nu) \). \( \square \)

**Lemma 4.** Let \( \rho \geq 1 \) and \( \bar{\Pi}_\rho = \{(q, q') \in \mathcal{P}_\rho(\mathbb{R}) \times \mathcal{P}_\rho(\mathbb{R}) \mid q \leq_{\text{ex}} q'\} \). There exists a measurable map \( M_* : \bar{\Pi}_\rho \to \mathcal{P}(\mathbb{R} \times \mathbb{R}) \) such that for each \( (\mu, \nu) \in \bar{\Pi}_\rho \), \( M_* (\mu, \nu) \in \text{Opt}_\rho(\mu, \nu) := \{ M \in \Pi^M(\mu, \nu) : M^\rho_q(\mu, \nu) = \int_{\mathbb{R} \times \mathbb{R}} |x - y|^\rho M(dx, dy) \} \), where \( \bar{\Pi}_\rho \) is endowed with the trace of \( \mathcal{B} \otimes \mathcal{B} \) with \( \mathcal{B} \) denoting the Borel \( \sigma \)-algebra of \( \mathcal{P}_\rho(\mathbb{R}) \) endowed with the weak convergence topology and \( \mathcal{P}(\mathbb{R} \times \mathbb{R}) \) with the Borel \( \sigma \)-algebra associated with the weak convergence topology.

The reason why this statement is restricted to \( d = 1 \) is that the proof relies on the continuity of \( M_\rho \) with respect to its marginals which fails in higher dimension for \( \rho \in [1, 2) \) according to Example 1 taken from [11].

**Proof.** The reasoning is inspired from the proof of Corollary 5.22 [26]. The set \( \bar{\Pi}_\rho \) is a closed subset of the Polish space \( \mathcal{P}_\rho(\mathbb{R}) \times \mathcal{P}_\rho(\mathbb{R}) \) endowed with the product of the \( \mathcal{W}_\rho \) topology. Therefore it is Polish. The set \( \bigcup_{(q, q') \in \bar{\Pi}_\rho} \Pi^M_q \) is a closed subset of the set \( \mathcal{P}_\rho(\mathbb{R} \times \mathbb{R}) \) endowed with \( \mathcal{W}_\rho \) where the map \( M \mapsto \int_{\mathbb{R} \times \mathbb{R}} |x - y|^\rho M(dx, dy) \) is continuous. Since \( \bar{\Pi}_\rho \ni (\mu, \nu) \mapsto M^\rho_q(\mu, \nu) \) is continuous according to Corollary 1.2 [6], we deduce that the set \( \bigcup_{(q, q') \in \bar{\Pi}_\rho} \text{Opt}_\rho(q, q') \) is Polish as a closed subset of the Polish space \( \mathcal{P}_\rho(\mathbb{R} \times \mathbb{R}) \) endowed with \( \mathcal{W}_\rho \). For each \( (\mu, \nu) \in \bar{\Pi}_\rho \), \( \text{Opt}_\rho(\mu, \nu) \) is non-empty and compact for the \( \mathcal{W}_\rho \) topology. The map

\[
\bigcup_{(q, q') \in \bar{\Pi}_\rho} \text{Opt}_\rho(q, q') \ni (M(dx, dy)) \mapsto (M(dx, \mathbb{R}^d), M(\mathbb{R}^d, dy)) \in \bar{\Pi}_\rho
\]
is onto and continuous (and therefore measurable). The measurable selection theorem implies that it admits a measurable right-inverse \( \tilde{\Pi}_\rho \ni (\mu, \nu) \mapsto M_\ast(\mu, \nu) \in \bigcup_{(q, q') \in \widetilde{\Pi}_\rho} \text{Opt}_\rho(q, q') \) such that for each \( (\mu, \nu) \in \widetilde{\Pi}_\rho \), \( M_\ast(\mu, \nu) \in \text{Opt}_\rho(\mu, \nu) \). Since \( \mathcal{P}_\rho(\mathbb{R} \times \mathbb{R}) \) is a closed subset of \( \mathcal{P}(\mathbb{R} \times \mathbb{R}) \) for the weak convergence topology and the Borel \( \sigma \)-algebras on \( \mathcal{P}_\rho(\mathbb{R} \times \mathbb{R}) \) associated to the \( \mathcal{W}_\rho \) and the weak convergence topologies coincide according to the next lemma, the map is still measurable if we consider \( \mathcal{P}_\rho(\mathbb{R} \times \mathbb{R}) \) endowed with the Borel \( \sigma \)-algebra associated with the weak convergence topology as the codomain. The Borel \( \sigma \)-algebra on \( \mathcal{P}_\rho(\mathbb{R}) \) endowed with \( \mathcal{W}_\rho \) coincides with \( \mathcal{B} \) according to the next lemma. Since \( \mathcal{P}_\rho(\mathbb{R}) \) endowed with \( \mathcal{W}_\rho \) is Polish, the Borel \( \sigma \)-algebra on the product space \( \mathcal{P}_\rho(\mathbb{R}) \times \mathcal{P}_\rho(\mathbb{R}) \) thus coincides with \( \mathcal{B} \), which concludes the proof.

\[ \square \]

**Lemma 5.** Let \((E, d_E)\) be a Polish space, \(\rho \geq 1\) and \(\mathcal{P}_\rho(E)\) be the set of probability measures on \(E\) with finite \(\rho\)-th moment. Let \(\mathcal{B}, \text{ resp. } \mathcal{B}_\rho\) be the Borel \(\sigma\)-algebra on \(\mathcal{P}_\rho(E)\) with respect to the weak convergence topology, resp. the \(\mathcal{W}_\rho\)-distance topology. Then \(\mathcal{B} = \mathcal{B}_\rho\).

**Proof.** Since the \(\mathcal{W}_\rho\)-distance topology is finer than the weak convergence topology, we clearly have \(\mathcal{B} \subset \mathcal{B}_\rho\). Therefore it remains to prove that \(\mathcal{B}_\rho \subset \mathcal{B}\).

Let \(x_0 \in E\) and \(\Phi_\rho(E)\) be the set of all real-valued continuous functions \(f\) on \(E\) which satisfy the growth constraint

\[ \exists \alpha > 0, \quad \forall x \in E, \quad |f(x)| \leq \alpha(1 + d_E(x, x_0)). \]

For all \(f \in \Phi_\rho(E)\), let \(\tilde{f} : \mathcal{P}_\rho(E) \to \mathbb{R}\) be the map defined for all \(p \in \mathcal{P}_\rho(E)\) by \(\tilde{f}(p) = \int_E f(x) p(dx)\). The \(\mathcal{W}_\rho\)-distance topology is then the weak topology on \(\mathcal{P}_\rho(E)\) induced by the family \((\tilde{f})_{f \in \Phi_\rho(E)}\), that is the coarsest topology on \(\mathcal{P}_\rho(E)\) for which \(\tilde{f}\) is continuous for all \(f \in \Phi_\rho(E)\). Any open set for this topology is a union of finitely many intersections of sets of the form \(\tilde{f}^{-1}(U)\) where \(f \in \Phi_\rho(E)\) and \(U\) is an open subset of \(\mathbb{R}\). On the one hand, \((\mathcal{P}_\rho(E), \mathcal{W}_\rho)\) is Polish [26, Theorem 6.18] and therefore strongly Lindelöf, hence the latter union can be assumed at most countable. On the other hand, any open subset of \(\mathbb{R}\) is an at most countable union of open intervals of \(\mathbb{R}\). We deduce that any open set for the \(\mathcal{W}_\rho\)-distance topology is an at most countable union of finitely many intersections of at most countable unions of sets of the form \(\tilde{f}^{-1}((a, b))\) where \(f \in \Phi_\rho(E)\) and \((a, b) \subset \mathbb{R}\). Since \(\mathcal{B}\) is closed under countable unions and intersections, it suffices to show that every set of the form \(\tilde{f}^{-1}((a, b))\) belongs to \(\mathcal{B}\) to conclude that any open set of the \(\mathcal{W}_\rho\)-distance topology belongs to \(\mathcal{B}\) and therefore \(\mathcal{B}_\rho \subset \mathcal{B}\).

Let then \(f \in \Phi_\rho(E)\) and \(a, b \in \mathbb{R}\) be such that \(a < b\) and let us show that \(\tilde{f}^{-1}((a, b)) \in \mathcal{B}\), which will end the proof. For all \(n \in \mathbb{N}\), let

\[ f_n : x \mapsto (f(x) \lor (-n)) \land n, \]

which is clearly continuous and bounded. Then for all \(n \in \mathbb{N}\) and \(p \in \mathcal{P}_\rho(E)\),

\[ \tilde{f}_n(p) = \int_X ((f(x) \lor (-n)) \land n) p(dx), \]

which by the dominated convergence theorem converges to \(\tilde{f}(p)\) as \(n \to +\infty\), hence

\[ \tilde{f}^{-1}((a, b)) = \bigcup_{k \in \mathbb{N}} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \tilde{f}_n^{-1}
\left(\left(\left\{ a + \frac{1}{k} - \frac{1}{k}\right\} \right)\right). \]
Since the weak convergence topology is induced by the family of $\tilde{g}$ for $g$ continuous and bounded, we have that $f^{-1}_n((a, b)) \in \mathcal{B}$ for all $n \in \mathbb{N}$, hence $f^{-1}((a, b)) \in \mathcal{B}$.

**Proof of Lemma 2.** Let $a = (a_1, \cdots, a_d) \in \mathbb{R}^{d-1}$, $c = (c_1, \cdots, c_d) \in \mathbb{R}^d$, $y \in \text{Span}(a)$ and $t \in \mathbb{R}$ be such that $y = ta$. Suppose first that $r = +\infty$. Then

$$|y - c_a| = |ta - c_0 a| = |t - c_0 a| = |t| a_i - c_i sgn(a_i)| = |(ta_i - c_i) sgn(a_i)|$$

$$= |ta_i - c_i| \leq |ta - c| = |y - c|.$$

Suppose now that $r < +\infty$. Using the fact that $|a| = 1$ for the second and third equalities, Hölder’s inequality for the second inequality and the fact that $|sgn(x)| = 1$ for all $x \in \mathbb{R}$ for the last but one equality, we get

$$|y - c_a| = |ta - \left(\sum_{i=1}^{d} c_i sgn(a_i)|a_i|^{r-1}\right) a|$$

$$= |t\sum_{i=1}^{d} |a_i|^{r} - \sum_{i=1}^{d} c_i sgn(a_i)|a_i|^{r-1}|$$

$$\leq \sum_{i=1}^{d} |t|a_i| - c_i sgn(a_i)|a_i|^{r-1}|$$

$$\leq \left(\sum_{i=1}^{d} |t|a_i| - c_i sgn(a_i)|^{r}\right)^{1/r} \left(\sum_{i=1}^{d} |a_i|^{r}\right)^{(r-1)/r}$$

$$= \left(\sum_{i=1}^{d} |(ta_i - c_i) sgn(a_i)|^{r}\right)^{1/r}$$

$$= |ta - c| = |y - c|.$$

**Lemma 6.** Let $d \in \mathbb{N}^*$, $\mathbb{R}^d$ be endowed with any norm, $\rho \geq 1$, $\lambda \geq 0$, $\mu \in \mathcal{P}_\rho(\mathbb{R}^d)$ and $\alpha \in \mathbb{R}^d$. Let $\nu$ be the image of $\mu$ by the map $x \mapsto x + \lambda(x - \alpha)$. Then

$$\mathcal{W}_\rho(\mu, \nu) = \lambda \left(\int_{\mathbb{R}^d} |x - \alpha|^\rho \mu(dx)\right)^{1/\rho}.$$

**Remark 5.** Let $\eta_0, \eta_1 \in \mathcal{P}_\rho(\mathbb{R}^d)$ and $\gamma \in \Pi(\eta_0, \eta_1)$ be optimal for $\mathcal{W}_\rho(\eta_0, \eta_1)$. For all $t \in [0, 1]$, let $\eta_t$ be the image of $\gamma$ by $(x, y) \mapsto (1 - t)x + ty$. It is well known that the curve $[0, 1] \ni t \mapsto \eta_t$ is a constant speed geodesic in $(\mathcal{P}_\rho(\mathbb{R}^d), \mathcal{W}_\rho)$ connecting $\eta_0$ to $\eta_1$ [5, Theorem 7.2.2]. Moreover, for all $0 \leq s \leq t \leq 1$, the image of $\gamma$ by $((1 - s)x + sy, (1 - t)x + ty)$ is an optimal transport plan between $\gamma_s$ and $\eta_t$ for the $\mathcal{W}_\rho$-distance.

In particular for $\eta_0 = \delta_\alpha$ and $\eta_1 = \nu$, the unique coupling $\gamma(dx, dy) = \delta_\alpha(dx) \nu(dy)$ is optimal for $\mathcal{W}_\rho(\eta_0, \eta_1)$, and for $t = 1/(1 + \lambda)$, $\eta_t = \mu$. Therefore, the image of $\gamma$ by $(x, y) \mapsto ((1 - t)x + ty, y)$, that is the image of $\mu$ by $x \mapsto (x, x + \lambda(x - \alpha))$, is an optimal transport plan between $\mu$ and $\nu$ for the $\mathcal{W}_\rho$-distance, which implies (52).
We add here a quick proof with the central elements of Remark 5.

**Proof of Lemma 6.** We have, by the triangle inequality for the metric $W_\rho$,

$$
\left( \int_{\mathbb{R}^d} |y - \alpha|^\rho \nu(dy) \right)^{1/\rho} = W_\rho(\delta_\alpha, \nu) \leq W_\rho(\delta_\alpha, \mu) + W_\rho(\mu, \nu)
$$

so

$$
W_\rho(\mu, \nu) \geq \left( \int_{\mathbb{R}^d} |y - \alpha|^\rho \nu(dy) \right)^{1/\rho} - \left( \int_{\mathbb{R}^d} |x - \alpha|^\rho \mu(dx) \right)^{1/\rho} = \lambda \left( \int_{\mathbb{R}^d} |x - \alpha|^\rho \mu(dx) \right)^{1/\rho}.
$$

Since $\mu(dx) \delta_{x+\lambda(x-\alpha)}(dy)$ is a coupling between $\mu$ and $\nu$, we also have

$$
W_\rho(\mu, \nu) \leq \lambda \left( \int_{\mathbb{R}^d} |x - \alpha|^\rho \mu(dx) \right)^{1/\rho},
$$

hence $W_\rho(\mu, \nu) = \lambda \left( \int_{\mathbb{R}^d} |x - \alpha|^\rho \mu(dx) \right)^{1/\rho}$.

**Lemma 7.** Let $d \in \mathbb{N}^*$, $\lambda > 0$, $\mu \in P_1(\mathbb{R}^d)$ and $\alpha \in \mathbb{R}^d$. Let $\nu$ be the image of $\mu$ by the map $x \mapsto x + \lambda(x-\alpha)$. Then $\mu \preceq_{ex} \nu$ iff $\alpha$ is the mean of $\mu$.

**Proof.** If $\mu \preceq_{ex} \nu$, then $\mu$ and $\nu$ have the same mean, so

$$
\int_{\mathbb{R}^d} x \mu(dx) = \int_{\mathbb{R}^d} y \nu(dy) = \int_{\mathbb{R}^d} x \mu(dx) + \lambda \left( \int_{\mathbb{R}^d} x \mu(dx) - \alpha \right),
$$

which implies that $\alpha = \int_{\mathbb{R}^d} x \mu(dx)$.

Conversely, suppose that $\alpha = \int_{\mathbb{R}^d} x \mu(dx)$. Then $\alpha = \int_{\mathbb{R}^d} y \nu(dy)$ and for all convex function $f : \mathbb{R}^d \to \mathbb{R}$,

$$
\int_{\mathbb{R}^d} f(x) \mu(dx) = \int_{\mathbb{R}^d} f \left( \frac{\lambda}{1 + \lambda} x + \frac{1}{1 + \lambda} (x + \lambda(x-\alpha)) \right) \mu(dx)
$$

$$
\leq \int_{\mathbb{R}^d} \left( \frac{\lambda}{1 + \lambda} f(\alpha) + \frac{1}{1 + \lambda} f(x + \lambda(x-\alpha)) \right) \mu(dx)
$$

$$
= \frac{\lambda}{1 + \lambda} \int_{\mathbb{R}^d} y \nu(dy) + \frac{1}{1 + \lambda} \int_{\mathbb{R}^d} f(y) \nu(dy)
$$

$$
\leq \frac{\lambda}{1 + \lambda} \int_{\mathbb{R}^d} f(y) \nu(dy) + \frac{1}{1 + \lambda} \int_{\mathbb{R}^d} f(y) \nu(dy)
$$

$$
= \int_{\mathbb{R}^d} f(y) \nu(dy),
$$

where we used Jensen’s inequality in the last inequality. We deduce that $\mu \preceq_{ex} \nu$. 

\qed
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