Addendum: Painlevé transcendents and \(\mathcal{PT}\)-symmetric Hamiltonians (2015 J. Phys. A: Math. Theor. 48 475202)

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Abstract

This paper is an Addendum to reference Bender and Komijani (2015 J. Phys. A: Math. Theor. 48 475202) (which stems from an earlier paper Bender et al (2014 J. Phys. A: Math. Theor. 47 235204)), where it was demonstrated that unstable separatrix solutions to the Painlevé equations I and II are determined by \(\mathcal{PT}\)-symmetric Hamiltonians. Here, unstable separatrix solutions of the fourth Painlevé transcendent are studied numerically and analytically. It is shown that for a fixed initial value such as \(y(0) = 1\) a discrete set of initial slopes \(y'(0) = b_n\) give rise to separatrix solutions. Similarly, for a fixed initial slope such as \(y'(0) = 0\) a discrete set of initial values \(y(0) = c_n\) give rise to separatrix solutions. For Painlevé IV the large-\(n\) asymptotic behavior of \(b_n\) is \(b_n \sim B_{IV} n^{3/4}\) and that of \(c_n\) is \(c_n \sim C_{IV} n^{1/2}\). The constants \(B_{IV}\) and \(C_{IV}\) are determined both numerically and analytically. The analytical values of these constants are found by reducing the nonlinear Painlevé IV equation to the linear eigenvalue equation for the sextic \(\mathcal{PT}\)-symmetric Hamiltonian \(H = \frac{1}{2}p^2 + \frac{1}{5}x^6\).

Keywords: PT-symmetric Hamiltonians, Painlevé transcendent, separatrix

1. Introduction

The six Painlevé transcendents satisfy nonlinear second-order differential equations having the property that their movable (spontaneous) singularities are poles (and not branch points or essential singularities). Many papers have been published on these equations (see, for example,
references [1–8] in [1] for background information and references [9–16] in [1] for applications in mathematical physics). This paper considers the fourth Painlevé transcendent, referred to here as P-IV. There are many studies of this equation; see, for example, references [3–11].

The initial-value problem for the P-IV differential equation examined here is

\[ y(t)y''(t) = \frac{1}{2}[y'(t)]^2 + 2t[y(t)]^2 + 4t[y(t)]^3 + \frac{3}{2}[y(t)]^4, \quad y(0) = c, \quad y'(0) = b. \]  

(1)

(For simplicity we have set the two arbitrary constants in P-IV, one of which is an additive constant, to 0. A complete asymptotic analysis of P-IV for nonzero values of these constants would be an interesting and nontrivial extension of this work.) There have been many asymptotic studies of the Painlevé transcendents, but here we present a numerical and asymptotic analysis that has not appeared in the literature. This analysis concerns the initial conditions that give rise to special unstable separatrix solutions of P-IV. These solutions are special critical cases of Clarkson–McLeod solutions discussed in references [3–6] and analyzed in depth by Its and Kapoiev in reference [7]. The asymptotic analysis here extends our earlier work on nonlinear differential-equation eigenvalue problems in [1, 12, 13].

The idea, originally proposed in reference [2], is that a nonlinear differential equation may have a discrete set of critical initial conditions that give rise to unstable separatrix solutions. These discrete initial conditions can be thought of as eigenvalues and the separatrices stemming from these initial conditions can be viewed as corresponding eigenfunctions. The objective in reference [2] is to find the large- \( n \) (semiclassical) asymptotic behavior of the \( n \)th eigenvalue. The analytical approach is to reduce the nonlinear differential-equation problem to a linear problem that could be solved to determine the asymptotic behavior of the eigenvalues as \( n \to \infty \).

A toy model used in reference [2] to explore the properties of nonlinear eigenvalue problems is the first-order differential-equation problem

\[ y'(t) = \cos[\pi t y(t)], \quad y(0) = a. \]  

(2)

The solutions to this initial-value problem exhibit \( n \) maxima before vanishing like \( 1/t \) as \( t \to \infty \). As the initial condition \( a \) increases past critical values \( a_n \), the number of maxima of \( y(t) \) jumps from \( n \) to \( n+1 \). At \( a_n \) the solution \( y(t) \) is an unstable separatrix: If \( y(0) \) is slightly below \( a_n \), \( y(x) \) merges with a bundle of stable solutions all having \( n \) maxima and when \( y(0) \) is slightly above \( a_n \), \( y(x) \) merges with a bundle of stable solutions all having \( n+1 \) maxima. We seek the asymptotic behavior of \( a_n \) for large \( n \), which is the analog of a high-energy semiclassical approximation in quantum mechanics. In reference [2] it is shown that for large \( n \) the nonlinear differential equation (2) reduces to a linear difference equation for a one-dimensional random walk. The difference equation is solved exactly, and it is shown that

\[ a_n \sim 2^{5/6} \sqrt{n} \quad (n \to \infty). \]  

(3)

Kerr subsequently found an alternative solution to this asymptotics problem and verified (3) [14].

The nonlinear eigenvalue problem described above is similar in many respects to the linear eigenvalue problem for the time-independent Schrödinger equation. For a potential \( V(x) \) that rises as \( x \to \pm \infty \), the eigenfunctions \( \psi(x) \) of the Schrödinger eigenvalue problem

\[ -\psi''(x) + V(x)\psi(x) = E\psi(x), \quad \psi(\pm \infty) = 0, \]  

(4)
are unstable with respect to small changes in the eigenvalue $E$; that is, if $E$ is increased or decreased slightly, $\psi(x)$ abruptly violates the boundary conditions (is not square integrable). Also, like the eigenfunctions (separatrix curves) of (2), the $n$th eigenfunction $\psi_n(x)$ has $n$ oscillations in the classically allowed region before decreasing monotonically to 0 in the classically forbidden region.

This paper considers two eigenvalue problems for P-IV. First, we find the large-$n$ behavior of the positive eigenvalues $b_n$ for the initial condition $y(0) = 1, y'(0) = b_n$ and also the large-$n$ behavior of the negative eigenvalues $c_n$ for the initial condition $y(0) = c_n, y'(0) = 0$. We show that

$$b_n \sim B_{IV} n^{3/4} \quad \text{and} \quad c_n \sim C_{IV} n^{1/2}. \quad (5)$$

In section 2 we compute the constants $B_{IV}$ and $C_{IV}$ numerically and in section 3 we find them analytically by reducing the large-eigenvalue problem to the linear time-independent Schrödinger equation for the sextic $\mathcal{PT}$-symmetric Hamiltonian $H = \frac{1}{2} p^2 + x^6$. Section 4 gives brief concluding remarks.

2. Numerical analysis of the fourth Painlevé transcendent

There are three possible asymptotic behaviors of the solutions to the P-IV equation as $t \to -\infty$; $y(t)$ can approach the straight lines $y = -2t$, $y = -2t/3$, or $y = 0$. An elementary asymptotic analysis shows that if $y(t)$ approaches $y = -2t/3$, the solution oscillates stably about this line with slowly decreasing amplitude [15]. However, while $y = -2t$ and $y = 0$ are possible asymptotic behaviors, these behaviors are unstable and nearby solutions veer away from them. Here we consider the eigenfunction solutions to P-IV that approach $y = -2t$ as $t \to -\infty$. These separatrix solutions resemble quantum-mechanical eigenfunctions because they have $n$ oscillations before exhibiting this asymptotic behavior. Because the P-IV equation is nonlinear these oscillations are unbounded; the $n$th eigenfunction passes through $2[n/2]$ or $1 + 2[(n - 1)/2]$ simple poles before smoothly approaching $y = -2t$ for the two cases studied below.

We consider two different eigenvalue problems for P-IV that are related to the instability of the asymptotic behavior $y = -2t$: (i) we fix the initial value $y(0) = 1$ and seek the discrete values of the initial slopes $y'(0) = b$ that give solutions approaching $-2t$, and (ii) we fix the initial slope $y'(0) = 0$ and seek the discrete initial values of $y(0) = c$ for which $y(t)$ approaches $-2t$.

2.1. Initial-slope eigenvalues for Painlevé IV

Let us examine the solutions to the initial-value problem for P-IV in (1) for $t < 0$. As in reference [1], we find these solutions numerically by using Runge-Kutta to integrate down the negative-real axis. When we approach a simple pole, we integrate along a semicircle in the complex-$t$ plane around the pole and continue integrating down the negative axis. We choose the initial value $y(0) = 1$ and allow the initial slope $y'(0) = b$ to have increasingly positive values. (We only present results for positive initial slope; the P-IV equation is symmetric under $t \to -t$ and also under $y \to -y$.) Numerical study shows that the choice of $y(0)$ is not crucial if $y(0) \neq 0$; for any $y(0)$ the large-$n$ behavior of the initial-slope eigenvalues $b_n$ is the same.

Above the first eigenvalue $b_1 = 3.15837325$ there is a continuous interval of $b$ for which $y(t)$ has an infinite sequence of simple poles (figure 1, left panel). When $b$ increases above the next eigenvalue $b_2 = 6.18498704$, the character of the solutions changes abruptly
Figure 1. Behavior of solutions $y(t)$ to the P-IV equation (1) for initial conditions $y(0) = 1$ and $b = y'(0)$. Left panel: $b = 5.18498704$, which lies between the eigenvalues $b_1 = 3.15837325$ and $b_2 = 6.18498704$. Right panel: $b = 7.18498704$, which lies between $b_2 = 6.18498704$ and $b_3 = 8.79172082$. The upper dashed line (red) is $y = -2t$, which is unstable and the lower dashed line (stable) is $y = -2t/3$. In the left panel $y(t)$ has an infinite sequence of simple poles but in the right panel the poles abruptly end and solution then oscillates stably about $-2t/3$.

Figure 2. Solutions to the P-IV equation (1) for $y(0) = 1$ and $b = y'(0)$. Left panel: $b = 10.1720921$, which lies between the eigenvalues $b_3 = 8.79172082$ and $b_4 = 11.1720921$. Right panel: $b = 12.1720921$, which lies between the eigenvalues $b_4 = 11.1720921$ and $b_5 = 13.3990049$.

and after $y(t)$ passes through a finite number of simple poles it begins to oscillate stably about $-2t/3$ (figure 1, right panel). When $b$ exceeds the third eigenvalue $b_3 = 8.79172082$, the solutions again pass through an infinite sequence of poles (figure 2, left panel). When $b$ increases above $b_4 = 11.1720921$, the solutions again oscillate stably about $-2t/3$ (figure 2, right panel). Numerical study verifies that there is an infinite sequence of eigenvalues at which the solutions to P-IV alternate between infinite sequences of simple poles and stable oscillation about $-2t/3$.

When $y'(0)$ is an eigenvalue the solutions exhibit a completely different and unstable behavior from those in figures 1 and 2. These solutions pass through a finite number of simple poles (like the oscillations of quantum-mechanical eigenfunctions in a classically allowed region) and then have a turning-point-like transition in which the poles cease and $y(t)$ exponentially
Figure 3. First two separatrix (eigenfunction) solutions of P-IV with initial condition \( y(0) = 1 \). Left panel: \( y'(0) = b_1 = 3.15837325 \); right panel: \( y'(0) = b_2 = 6.18498704 \). The dashed lines are \( y = -2t \) and \( y = -2t/3 \).

Figure 4. Third and fourth eigenfunctions of P-IV with initial condition \( y(0) = 1 \). Left panel: \( y'(0) = b_3 = 8.79172082 \); right panel: \( y'(0) = b_4 = 11.1720921 \). The solutions approaching the line \( -2t \) are insensitive to the choice of \( y(0) \). The graphs in figures 3, 4, and 5 show the character of the solutions changes abruptly and the solutions exhibit the two possible generic behaviors shown in figures 1 and 2.

As in reference [1] for P-I and P-II, we have performed a numerical asymptotic study of the critical values \( b_n \) for \( n \gg 1 \) by using Richardson extrapolation [15]. (In this paper we have taken \( y(0) = 1 \) but we find that if \( y(0) \) is held fixed, the large-\( n \) behavior of the initial slope \( b_n \) is insensitive to the choice of \( y(0) \).) By applying fifth-order Richardson extrapolation to the first twelve eigenvalues, we find the value of \( B_{IV} \) accurate to one part in seven decimal places:

\[
B_{IV} = 4.2568433. \tag{6}
\]
Figure 5. Eleventh and twelth eigenfunctions of P-IV with initial condition \( y(0) = 1 \). Left panel: \( y'(0) = b_{11} = 24.9911479 \); right panel: \( y'(0) = b_{12} = 26.7370929 \). As \( n \) increases, the eigenfunctions pass through more and more simple poles before exhibiting a turning-point transition and approaching the limiting curve \( -2t \) exponentially rapidly. This behavior is analogous to that of the eigenfunctions of a time-independent Schrödinger equation for a particle in a potential well; the higher-energy eigenfunctions exhibit more and more oscillations in the classically allowed region before entering the classically forbidden region, where they decay to zero.

Figure 6. First two separatrix solutions (eigenfunctions) of Painlevé IV with fixed initial slope \( y'(0) = 0 \). Left panel: \( y(0) = c_1 = -1.98740393 \); right panel: \( y(0) = c_2 = -3.23535569 \). The dashed curves are \( y = -2t \) and \( t = -2t/3 \).

2.2. Initial-value eigenvalues for Painlevé IV

If we fix the initial slope at \( y'(0) = 0 \) and allow the initial value \( y(0) = c \) to become increasingly negative, we find a sequence of negative eigenvalues \( c_n \) for which the solutions behave like the separatrix (eigenfunction) solutions in figures 3–5. The first two eigenfunctions are plotted in figure 6, the next two in figure 7, and the eleventh and twelth in figure 8.

Applying fourth-order Richardson extrapolation to the first 15 eigenvalues, we find that for large \( n \) the sequence of initial-value eigenvalues \( c_n \) is asymptotic to \( C_{IV}n^{1/2} \), where

\[
C_{IV} = -2.626587.
\]
Figure 7. Third and fourth eigenfunctions of Painlevé IV with initial slope \(y'(0) = 0\). Left panel: \(y(0) = c_3 = -4.161\,6081\); right panel: \(y(0) = c_4 = -4.919\,08695\).

Figure 8. Eleventh and twelfth eigenfunctions of Painlevé IV with initial slope \(y'(0) = 0\). Left panel: \(y(0) = c_{11} = -8.512\,11189\); right panel: \(y(0) = c_{12} = -8.908\,05963\).

3. Asymptotic determination of \(B_{IV}\) and \(C_{IV}\)

In this section we present an asymptotic analysis that yields analytic formulas for \(B_{IV}\) and \(C_{IV}\) in (6) and (7). To begin, we rewrite the P-IV equation (1) as

\[
2[y(t)]^{3/2} \left[ \sqrt{y(t)} \right]' = 2t^2[y(t)]^2 + 4t[y(t)]^3 + \frac{3}{2}[y(t)]^4.
\]

This suggests the substitution \(u(t) = \sqrt{y(t)}\), which gives the equation

\[
u''(t) = t^2 u(t) + 2t[u(t)]^3 + \frac{3}{4}[u(t)]^5.
\]

Following reference [1] we multiply by \(u'(t)\) and integrate from \(t = 0\) to \(t = x\):

\[
H \equiv \frac{1}{2} [u'(x)]^2 + \frac{1}{8} [u(x)]^6 = \frac{1}{2} [u'(0)]^2 + \frac{1}{8} [u(0)]^6 - I(x),
\]

Equation (8)
Figure 9. Numerical evidence that $I(x)$ [where $I(x)$ is normalized by dividing by $H(0)$] is small for fixed $x$ as $n \to \infty$.

where $I(x) = \int_0^x dt \left( t^2 u(t)u'(t) + 2t[u(t)]^3 u'(t) \right)$. The path of $t$ integration used here is like that used to compute $y(t)$ numerically in section 2; the path follows a straight line until it approaches a pole, at which point it makes a semicircular detour in the complex-$t$ plane to avoid the pole.

If we evaluate $H(x)$ on the imaginary-$t$ axis we obtain the Hamiltonian

$$\hat{H} = \frac{1}{2} \hat{p}^2 + \frac{1}{8} \hat{x}^6. \quad (9)$$

This Hamiltonian can be interpreted in two possible ways, either as a Hermitian Hamiltonian for which the eigenfunctions vanish as $x \to \pm \infty$ or as a $\mathcal{P}\mathcal{T}$-symmetric Hamiltonian for which the eigenfunctions vanish as $|x| \to \infty$ with arg $x = -\frac{1}{4}\pi$ and $-\frac{3}{4}\pi$. To see which quantization scheme is correct we calculate $I(x)$ numerically (see figure 9).

We find numerically that on the lines arg $x = -\frac{1}{4}\pi$ and $-\frac{3}{4}\pi$ the function $I(x)$ becomes small compared with $H$ as $n \to \infty$ for fixed $x$. Thus, for an eigenfunction of P-IV we can interpret $H$ as a time-independent quantum-mechanical Hamiltonian. We conclude that the large-$n$ (semiclassical) behavior of the P-IV eigenvalues can be determined by solving the linear quantum-mechanical eigenvalue problem $\hat{H}\psi = E\psi$, where $\hat{H} = \frac{1}{2} \hat{p}^2 + \frac{1}{8} \hat{x}^6$. The large eigenvalues of this Hamiltonian can be found by using the complex WKB techniques discussed in detail in reference [16]. For the general class of $\mathcal{P}\mathcal{T}$-symmetric Hamiltonians $\hat{H} = \frac{1}{2} \hat{p}^2 + g \hat{x}^2 (\hat{x})^\varepsilon (\varepsilon \geq 0)$, the WKB approximation to the $n$th eigenvalue ($n \gg 1$) is
given by
\[ E_n \sim \frac{1}{2} (2g)^{2/(4+\varepsilon)} \left[ \frac{\Gamma\left(\frac{1}{2} + \frac{1}{4+\varepsilon}\right)}{\sin\left(\frac{\pi}{4+\varepsilon}\right) \Gamma\left(1 + \frac{1}{4+\varepsilon}\right)} \right]^{(2\varepsilon+4)/(\varepsilon+4)}. \] (10)

Thus, for \( H \) in (9) we take \( g = 1/8 \) and \( \varepsilon = 4 \) and obtain the asymptotic behavior
\[ E_n \sim \left[ \sqrt{\pi} \frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(\frac{7}{6}\right)} \right]^{3/2} (n \to \infty). \] (11)

Since \( \tilde{H} \) in (9) is time independent, we can evaluate \( H \) in (8) for fixed \( y(0) \) and large \( y'(0) = b_n \) and obtain the result that
\[ b_n \sim 4\sqrt{E_n/2} = B_{IV} n^{3/4} \quad (n \to \infty), \] (12)
which verifies (5). We then read off the analytic value of the constant \( B_{IV} \):
\[ B_{IV} = 2^{3/2} \left[ \sqrt{\pi} \frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(\frac{7}{6}\right)} \right]^{3/4}, \] (13)
which agrees with the numerical result in (6). Also, if we take the initial slope \( y'(0) \) to vanish and take the initial condition \( y(0) = c_n \) to be large, we obtain an analytic expression for \( C_{IV} \),
\[ C_{IV} = -2 \left[ \sqrt{\pi} \frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(\frac{7}{6}\right)} \right]^{1/2}, \] (14)
which agrees with the numerical result in (7).

4. Concluding remarks

In this paper we have shown that the fourth Painlevé equation \( P-IV \) exhibits instabilities that are associated with separatrix solutions. The initial conditions that give rise to these separatrix solutions are eigenvalues. We have calculated the semiclassical (large-eigenvalue) behavior of the eigenvalues in two ways, first by using numerical techniques and then by using asymptotic methods to reduce the initial-value problems for the nonlinear \( P-IV \) equation (1) to the linear eigenvalue problem associated with the time-independent Schrödinger equation for the \( PT \)-symmetric \( x^\varepsilon \) potential. The agreement between these two approaches is exact.

The obvious continuation of this work is to examine the three remaining Painlevé equations, \( P-III, P-V, \) and \( P-VI \), to see if there are instabilities, separatrices, and eigenvalues for these equations as well. It is quite surprising that \( P-I, P-II, \) and \( P-IV \) are associated with the \( PT \)-symmetric \( x^\varepsilon(ix)^\varepsilon \) for the values \( \varepsilon = 1, 2, \) and 4 and it will be interesting to see if these more complicated Painlevé equations have associated values of \( \varepsilon \) as well.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Painlevé transcendents and $\mathcal{PT}$-symmetric Hamiltonians

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Abstract

Unstable separatrix solutions for the first and second Painlevé transcendents are studied both numerically and analytically. For a fixed initial condition, say $y(0) = y_0$, there is a discrete set of initial slopes $y'(0) = b_n$ that give rise to separatrix solutions. Similarly, for a fixed initial slope, say $y'(0) = 0$, there is a discrete set of initial values $y(0) = c_n$ that give rise to separatrix solutions. For Painlevé I the large-$n$ asymptotic behavior of $b_n$ is $b_n \sim B_1 n^{1/3} (n \to \infty)$ and that of $c_n$ is $c_n \sim C_1 n^{2/3} (n \to \infty)$, and for Painlevé II the large-$n$ asymptotic behavior of $b_n$ is $b_n \sim B_II n^{2/3} (n \to \infty)$ and that of $c_n$ is $c_n \sim C_{II} n^{1/3} (n \to \infty)$. The constants $B_1$, $C_1$, $B_{II}$, and $C_{II}$, which are the coefficients in these asymptotic behaviors, are first determined numerically. Then, by using asymptotic methods, they are found analytically by reducing the nonlinear equations to the linear eigenvalue problems associated with the cubic and quartic $\mathcal{PT}$-symmetric Hamiltonians $H = \frac{1}{2} p^2 + 2ix^3$ and $H = \frac{1}{2} p^2 - \frac{1}{2} x^4$.

Keywords: semiclassical, WKB, asymptotic, eigenvalue, separatrix

1. Introduction

The famous Painlevé transcendents are six nonlinear second-order differential equations whose key features are that their movable (spontaneous) singularities are poles (and not, for example, branch points or essential singularities). There is a vast literature on these remarkable differential equations [1–8]. These equations have arisen many times in mathematical physics; for a small sample, see [9–16]. This paper considers the first and second Painlevé transcendents, referred to here as P-I and P-II. The initial-value problem (IVP) for the P-I differential equation is

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and the IVP for P-II (we have set an arbitrary additive constant to 0) is
\[ y''(t) = 2[y(t)]^3 + ty(t), \quad y(0) = c, \quad y'(0) = b. \]

Many asymptotic studies of the Painlevé transcendents have been published, but in this paper we present a simple numerical and asymptotic analysis that to our knowledge has not appeared in the literature. This analysis concerns the initial conditions that give rise to special unstable separatrix solutions of P-I and P-II. Our asymptotic analysis verifies the numerical results given in this paper for P-I and P-II as well as some preliminary numerical calculations that were presented in an earlier paper on nonlinear differential-equation eigenvalue problems [17].

The main idea, originally introduced in [17], is that a nonlinear differential equation may have a discrete set of critical initial conditions that give rise to unstable separatrix solutions. These discrete initial conditions can be thought of as eigenvalues and the separatrices that stem from these initial conditions can be viewed as the corresponding eigenfunctions. The objective in [17] was to find the large-\( n \) (semiclassical) asymptotic behavior of the \( n \)th eigenvalue. The general analytical approach that was proposed was to simplify the nonlinear differential problem to a linear problem that could be used to determine the leading asymptotic behavior of the eigenvalues as \( n \to \infty \).

A toy model was used in [17] to explain the concept of a nonlinear eigenvalue problem. This model makes use of the elementary first-order differential equation problem
\[ y'(t) = \cos[\pi t y(t)], \quad y(0) = a. \tag{3} \]

It was shown that the solutions to this IVP pass through \( n \) maxima before vanishing like \( 1/t \) as \( t \to \infty \). As the initial condition \( a = y(0) \) increases past special critical values \( a_n \), the number of maxima jumps from \( n \) to \( n+1 \). At these critical values the solution \( y(t) \) to (3) is an unstable separatrix curve in the following sense: at values of \( y(0) \) infinitesimally below \( a_n \) the solution merges with a bundle of stable solutions all having \( n \) maxima and when \( y(0) \) is infinitesimally above \( a_n \) the solution merges with a bundle of stable solutions all having \( n+1 \) maxima. The challenge is to determine the asymptotic behavior of the critical values \( a_n \) for large \( n \). (This generic problem is the analog of a semiclassical high-energy approximation in quantum mechanics.) To solve this problem it was shown that for large \( n \), the nonlinear differential equation problem (3) reduces to a linear one-dimensional random-walk problem. The random-walk problem was solved exactly, and it was shown analytically that
\[ a_n \sim \frac{2^{5/6}}{\sqrt{n}} (n \to \infty). \tag{4} \]

Kerr subsequently found an alternative solution to this asymptotics problem and verified (4) [18].

The nonlinear eigenvalue problem described above is similar in many respects to the linear eigenvalue problem for the time-independent Schrödinger equation. For a potential \( V(x) \) that rises as \( x \to \pm \infty \), the eigenfunctions \( \psi(x) \) of the Schrödinger eigenvalue problem
\[ -\psi''(x) + V(x)\psi(x) = E\psi(x), \quad \psi(\pm \infty) = 0, \] (5)

are unstable with respect to small changes in the eigenvalue \( E \); that is, if \( E \) is increased or decreased slightly, \( \psi(x) \) abruptly ceases to obey the boundary conditions (and thus is not normalizable (square integrable)). Furthermore, like the eigenfunctions (separatrix curves) of (3), the eigenfunction \( \psi_n(x) \) corresponding to the \( n \)th eigenvalue has \( n \) oscillations in the classically allowed region before decreasing monotonically to 0 in the classically forbidden region.
This paper considers four eigenvalue problems. First, for P-I we find the large-$n$ behavior of the positive eigenvalues $b_n$ for the initial condition $y(0) = 0, y'(0) = b_n$ and also the large-$n$ behavior of the negative eigenvalues $c_n$ for the initial condition $y(0) = c_n, y'(0) = 0$. We show that
\[ b_n \sim B_1 n^{3/5} \quad \text{and} \quad c_n \sim C_1 n^{2/5}. \]
Second, for P-II we show that for large $n$ the asymptotic behaviors of $b_n$ and $c_n$ are given by
\[ b_n \sim B_{II} n^{2/3} \quad \text{and} \quad c_n \sim C_{II} n^{1/3}. \]
We determine the constants $B_1$, $C_1$, $B_{II}$, and $C_{II}$ both numerically and analytically.

This paper is organized as follows. In section 2 we obtain the constants $B_1$ and $C_1$ by using numerical techniques and in section 3 we do so analytically by reducing the large-eigenvalue problem to the linear time-independent Schrödinger equation for the cubic $\mathcal{PT}$-symmetric Hamiltonian $H = \frac{1}{2}p^2 + ix^3$. Next, we study the eigenvalue problem for the second Painlevé transcendent. In section 4 we present a numerical determination of the large-$n$ behavior of the eigenvalues and in section 5 we verify the numerical results in section 4 by using asymptotic analysis to reduce the nonlinear large-eigenvalue problem for P-II to the linear Schrödinger equation for the quartic $\mathcal{PT}$-symmetric Hamiltonian $H = \frac{1}{2}p^2 - \frac{1}{2}x^4$. In section 6 we make some brief concluding remarks.

2. Numerical analysis of the first Painlevé transcendent

In [17] there is a brief numerical study of the IVP for the first Painlevé transcendent (1). It is easy to see that there are two possible asymptotic behaviors as $t \to -\infty$; the solutions to the P-I equation can approach either $+\sqrt{-t/6}$ or $-\sqrt{-t/6}$. An elementary asymptotic analysis shows that if the solution $y(t)$ approaches $-\sqrt{-t/6}$, the solution oscillates stably about this curve with slowly decreasing amplitude [19]. However, while the curve $+\sqrt{-t/6}$ is a possible asymptotic behavior, this behavior is unstable and nearby solutions tend to veer away from it. We define the eigenfunction solutions to the first Painlevé transcendent as those solutions that do approach $+\sqrt{-t/6}$ as $t \to -\infty$. These separatrix solutions resemble the eigenfunctions of conventional quantum mechanics in that they exhibit $n$ oscillations before settling down to this asymptotic behavior. However, because the P-I equation is nonlinear, these oscillations are violent; the $n$th eigenfunction passes through $[n/2]$ double poles where it blows up before it smoothly approaches the curve $+\sqrt{-t/6}$. (The symbol $[n/2]$ means greatest integer in $n/2$.)

One can specify two different kinds of eigenvalue problems for P-I, each of which is fundamentally related to the instability of the asymptotic behavior $+\sqrt{-t/6}$. One can (i) fix the initial value $y(0)$ and look for (discrete) values of the initial slopes $y'(0) = b$ that give rise to solutions approaching $+\sqrt{-t/6}$, or else (ii) one can fix the initial slope $y'(0)$ and look for the (discrete) initial values of $y(0) = c$ that give rise to solutions approaching $+\sqrt{-t/6}$.

2.1. Initial-slope eigenvalues for Painlevé I

Let us examine the numerical solutions to the IVP for the P-I equation (1) for $t < 0$. To find these solutions we use Runge–Kutta to integrate down the negative-real axis. When we approach a double pole and the solution becomes large and positive, we estimate the location of the pole and integrate along a semicircle in the complex-$t$ plane around the pole. We then continue integrating down the negative-real axis. We choose the fixed initial value $y(0) = 0$.
and allow the initial slope \( y'(0) = b \) to have increasingly positive values. (We only present results for positive initial slope; the behavior for negative initial slope is analogous and describing it would be repetitive.) Our numerical analysis shows that the particular choice of \( y(0) \) is not crucial; for any fixed \( y(0) \) the large-\( n \) asymptotic behavior of the initial-slope eigenvalues \( b_n \) is the same.

We find that above the critical value \( b_1 = 1.851 \ 854 \ 034 \) (the first eigenvalue) there is a continuous interval of \( b \) for which \( y(t) \) first has a minimum and then has an infinite sequence of double poles (see figure 1, left panel). However, if \( b \) increases past the next critical value \( b_2 = 3.004 \ 031 \ 103 \) (the second eigenvalue), the character of the solutions changes abruptly

![Figure 1.](image1.png)

**Figure 1.** Typical behavior of solutions to the first Painlevé transcendent \( y(t) \) for the initial conditions \( y(0) = 0 \) and \( b = y'(0) \). In the left panel \( b = 2.504 \ 031 \ 103 \), which lies between the eigenvalues \( b_1 = 1.851 \ 854 \ 034 \) and \( b_2 = 3.004 \ 031 \ 103 \). In the right panel \( b = 3.504 \ 031 \ 103 \), which lies between the eigenvalues \( b_2 = 3.004 \ 031 \ 103 \) and \( b_3 = 3.905 \ 175 \ 320 \). The dashed curves are \( y = \pm \sqrt{-t/6} \). In the left panel the solution \( y(t) \) has an infinite sequence of double poles and in the right panel the solution oscillates stably about \(-\sqrt{t/6}\).

![Figure 2.](image2.png)

**Figure 2.** Solutions to the P-I equation (1) for \( y(0) = 0 \) and \( b = y'(0) \). Left panel: \( b = 4.583 \ 412 \ 410 \), which lies between the eigenvalues \( b_3 = 3.905 \ 175 \ 320 \) and \( b_4 = 4.683 \ 412 \ 410 \). Right panel: \( b = 4.783 \ 412 \ 410 \), which lies between the eigenvalues \( b_4 = 4.683 \ 412 \ 410 \) and \( b_5 = 5.383 \ 086 \ 722 \).
and $y(t)$ oscillates stably about $-\sqrt{-t/6}$ (figure 1, right panel). When $b$ exceeds the critical value $b_3 = 3.905175320$ (the third eigenvalue), the solutions again exhibit an infinite sequence of poles (figure 2, left panel). When $b$ increases past the fourth critical value $b_4 = 4.683412410$ (fourth eigenvalue), the solutions once again oscillate stably about $-\sqrt{-t/6}$ (figure 2, right panel). Our numerical analysis indicates that there is an infinite sequence of critical points (eigenvalues) at which the P-I solutions alternate between infinite sequences of double poles and stable oscillation about $-\sqrt{-t/6}$.

The solutions that arise when $y'(0)$ is at an eigenvalue have a completely different (and unstable) character from those in figures 1 and 2. These special solutions pass through a finite number of double poles (analogous to the oscillatory behavior of quantum-mechanical bound-state eigenfunctions in the classically allowed region of a potential well) and then undergo a turning-point-like transition in which the poles cease and $y(t)$ exponentially approaches the limiting curve $+\sqrt{-t/6}$. The solutions arising from the first and second critical points $b_1$ and $b_2$ are shown in figure 3, those arising from the third and fourth critical points $b_3$ and $b_4$ are shown in figure 4, and those arising from the tenth and eleventh critical points $b_{10}$ and $b_{11}$ are

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**Figure 3.** First two separatrix solutions (eigenfunctions) of Painlevé I with initial condition $y(0) = 0$. Left panel: $y'(0) = b_1 = 1.851854034$; right panel: $y'(0) = b_2 = 3.004031103$. The dashed curves are $y = \pm \sqrt{-t/6}$.

**Figure 4.** Third and fourth eigenfunctions of Painlevé I with initial condition $y(0) = 0$. Left panel: $y'(0) = b_3 = 3.905175320$; right panel: $y'(0) = b_4 = 4.683412410$. 
shown in figure 5. The critical points are analogous to eigenvalues because they give rise to unstable separatrix solutions; if \( y'(0) \) changes by an infinitesimal amount above or below a critical value, the character of the solutions changes abruptly and the solutions exhibit the two possible generic behaviors shown in figures 1 and 2.

In [17] a numerical asymptotic study of the critical values \( b_n \) for \( n \gg 1 \) was performed by using Richardson extrapolation\(^4\). (In [17] the initial value was chosen to be \( y(0) = 1 \) rather than \( y(0) = 0 \) as in the current paper. However, as emphasized above, if \( y(0) \) is held fixed, we find that the large-\( n \) asymptotic behavior of the initial slope \( b_n \) is insensitive to the choice of \( y(0) \).) It was found in [17] that for large \( n \), the \( n \)th critical value had the asymptotic behavior

\[
y_n'(0) = b_n \sim B_1 n^{3/5} \quad (n \to \infty).
\]

In [17] the constant \( B_1 \) was determined numerically to an accuracy of about four or five decimal places. However, we have now performed a more accurate numerical determination of the constant \( B_1 \) by applying fifth-order Richardson extrapolation to the first eleven eigenvalues, and we have found the value of \( B_1 \) accurate to one part in nine decimal places:

\[
B_1 = 2.09214674.
\]

On the basis of our numerical analysis, we can say with confidence that the underlined digit lies in the range from 3 to 5, so our determination of \( B_1 \) is accurate to one part in \( 2 \times 10^8 \).

2.2. Initial-value eigenvalues for Painlevé I

If we hold the initial slope fixed at \( y'(0) = 0 \) and allow the initial value \( y(0) = c \) to become increasingly negative, we find that there is a sequence of negative eigenvalues \( c_n \) for which

\( \text{Figure 5. Tenth and eleventh eigenfunctions of Painlevé I with initial condition } y(0) = 0. \text{ Left panel: } y'(0) = b_{10} = 8.244932302; \text{ right panel: } y'(0) = b_{11} = 8.738330156. \text{ Note that as } n \text{ increases, the eigenfunctions pass through more and more double poles before exhibiting a turning-point-like transition and approaching the limiting curve } +\sqrt{-1/6} \text{ exponentially rapidly. This behavior is analogous to that of the eigenfunctions of a time-independent Schrödinger equation for a particle in a potential well: the higher-energy eigenfunctions exhibit more and more oscillations in the classically allowed region before entering the classically forbidden region, where they decay exponentially.} \)
the solutions behave like the eigenfunction separatrix solutions in figures 3–5. The first four eigenfunctions are plotted in figures 6 and 7.

Applying fourth-order Richardson extrapolation to the first 15 eigenvalues, we find that for large \( n \) the sequence of initial-value eigenvalues \( c_n \) is asymptotic to \( C_1 n^{3/2} \), where the numerical value of the constant \( C_1 \) is

\[
C_1 = -1.030\,4844.
\]

We are confident that the final digit is accurate to an error of \( \pm 1 \) and thus \( C_1 \) is determined to an accuracy of one part in \( 10^7 \).

3. Asymptotic calculation of \( B_1 \) and \( C_1 \)

In this section we use asymptotic techniques to obtain analytic expressions for the constants \( B_1 \) and \( C_1 \) in (7) and (8). To begin, we multiply the P-I differential equation in (1) by \( y'(t) \) and
integrate from \( t = 0 \) to \( t = x \). We get

\[
H = \frac{1}{2} \left[ y'(x)^2 - 2y(x)^2 \right] = \frac{1}{2} \left[ y'(0)^2 - 2y(0)^2 \right] + I(x),
\]

where \( I(x) = \int_0^x dy \, y'(t) \). Note that the path of integration is taken to be the same as that used to calculate \( y(t) \) numerically in section 2; the path follows the negative-real axis until it gets near a pole, at which point it makes a semicircular detour in the complex-\( t \) plane to avoid the pole.

If we evaluate \( I(x) \) for large-negative \( x \) in the classically allowed region (that is, before the poles abruptly cease at the turning point), we find that as \( n \to \infty \), \( I(x) \) fluctuates and becomes small compared with \( H \). This is not surprising because \( I(x) \) receives many positive and negative contributions from the poles. To be precise, we can see from the definition of \( I(x) \) that \( I'(x) \) vanishes when \( y'(x) \) vanishes. Near the points where \( y'(x) \) vanishes, we have verified numerically that \( I(x) \) is small compared with \(-2[y(x)]^2\). Far from these points \( y(x) \) becomes large and so does \( I(x) \). However, \(-2[y(x)]^2\) blows up like a sixth-order pole and \( I(x) \) blows up like a second-order pole. These asymptotic estimates are difficult to verify analytically, but careful numerical analysis confirms these results. We emphasize that these estimates are valid when \( x \) is large and negative but only in the classically allowed region and not as \( x \to -\infty \).

In fact, by calculating \( I(x) \) as \( x \to -\infty \), we can see a clear signal of an eigenvalue; as \( y'(0) = b \) passes an eigenvalue, \( I(x) \) goes from having positive to negative (or negative to positive) fluctuations but at an eigenvalue \( I(x) \) is smooth and not fluctuating. Thus, for large \( n \) we treat the fluctuating quantity \( I(x) \) as small, and if we do so we can interpret \( H \) as a time-independent quantum-mechanical Hamiltonian. (The isomonodromic properties of \( H \) when \( I(x) \) is not neglected were studied in [6].)

We conclude that the large-\( n \) (semiclassical) behavior of the eigenvalues (that is, the initial conditions in (1)) can be determined by solving the linear quantum-mechanical eigenvalue problem \( \hat{H}\psi = E\psi \), where \( \hat{H} = \frac{1}{2}\hat{p}^2 - \hat{x}^2 \). To find these eigenvalues we rotate \( \hat{H} \) into the complex plane [20] and obtain the well-studied \( \mathcal{PT} \)-symmetric Hamiltonian [21]

\[
\hat{H} = \frac{1}{2}\hat{p}^2 + 2i\hat{x}^3.
\]

The large eigenvalues of this Hamiltonian can be found by using the complex WKB techniques discussed in detail in [21]. For the general class of \( \mathcal{PT} \)-symmetric Hamiltonians \( \hat{H} = \frac{1}{2}\hat{p}^2 + g\hat{x}^2(\pm\hat{x}) \) \((\epsilon > 0)\), the WKB approximation to the \( n \)th eigenvalue \((n \gg 1)\) is given by

\[
E_n \sim \frac{1}{2} (2g)^{2/(4+\epsilon)} \left[ \frac{\Gamma \left( \frac{3}{2} + \frac{1}{\epsilon + 2} \right) \sqrt{\pi} n^{2(\epsilon+4)/(\epsilon+4)}}{\sin \left( \frac{\pi}{\epsilon + 2} \right) \Gamma \left( 1 + \frac{1}{\epsilon + 2} \right)} \right].
\]

Thus, for \( H \) in (10) we take \( g = 2 \) and \( \epsilon = 1 \) and obtain the asymptotic behavior

\[
E_n \sim 2 \left[ \sqrt{3}\pi \Gamma \left( \frac{11}{6} \right) n / \Gamma \left( \frac{1}{3} \right) \right]^{6/5} \quad (n \to \infty).
\]

Since \( \hat{H} \) in (10) is time independent, we can evaluate \( H \) in (9) for fixed \( y(0) \) and large \( y'(0) = b_n \) and obtain the result that
which verifies (6). We then read off the analytic value of the constant $B_1$:

$$B_1 = 2 \left[ \sqrt{\frac{3\pi}{2}} \Gamma(\frac{11}{6})/\Gamma(\frac{1}{3}) \right]^{3/5},$$

which agrees with the numerical result in (7). Also, if we take the initial slope $y'(0)$ to vanish and take the initial condition $y(0) = c_t$ to be large, we obtain an analytic expression for $C_t$,

$$C_t = - \left[ \sqrt{\frac{3\pi}{2}} \Gamma(\frac{11}{6})/\Gamma(\frac{1}{3}) \right]^{2/5},$$

which agrees with the numerical result in (8).

4. Numerical analysis of the second Painlevé transcendent

To understand the behavior of solutions to the IVP in (2) for Painlevé II, we follow the procedure used in section 2 to study P-I. An elementary asymptotic analysis shows that as $t \to -\infty$, there are three possible asymptotic behaviors for solutions $y(t)$. First, $y(t)$ can oscillate stably about the negative axis. Second, $y(t)$ can approach the curves $\pm \sqrt{-t/2}$; however, both of these asymptotic behaviors are unstable. If we numerically integrate (2), we observe that when $t$ becomes large and negative, a typical solution to the P-II IVP either oscillates about the negative axis or passes through an infinite sequence of simple poles. However, it is also possible to find special eigenfunction solutions that pass through only a finite number of poles and then approach either the positive or the negative branches of the square-root curves. These eigenfunctions obey the boundary conditions $y(0) = 0$ and $y'(0) = \pm b$. (Note that P-II is symmetric under $y \to -y$, so there are two sets of eigenfunctions, one for each sign of $y'(0)$.) We study these eigenfunctions numerically in section 4.1. The P-II equation is particularly interesting because as $t \to +\infty$, the behavior $y \to 0$ becomes unstable. Thus, it is possible to have new kinds of eigenfunctions for positive $t$ as well. We seek eigenfunctions that satisfy $y'(0) = 0$ and $y(0) = c$ and examine the positive-$c$ eigenfunctions numerically in section 4.2.

4.1. Initial-slope eigenvalues for Painlevé II

Similar to what we found in section 2, if we choose $y(0) = 0$, there are critical values $y'(0) = b_n$ at which the solutions $y(t)$ change their character. In figures 8 and 9 we plot the solutions to the P-II equation for the initial condition $y(0) = 0$ and $y'(0) = b$ for $b_1 < b < b_2$, $b_2 < b < b_3$, $b_3 < b < b_4$, and $b_4 < b < b_5$. Note that in these figures the character of the solution alternates between having an infinite sequence of simple poles and oscillating stably about $y(t) = 0$. However, when $y'(0) = b$ is at a critical value (eigenvalue) $b_n$, the solution $y(t)$ passes through a finite number $\lfloor n/2 \rfloor$ of simple poles and then approaches either $+\sqrt{-t/2}$ or $-\sqrt{-t/2}$. These eigenfunctions (separatrices) are plotted in figures 10–12 for $n = 1, 2, 3, 4$, and 20, 21.

Note that the eigenfunctions in figures 10–12 alternate between approaching the upper-unstable branch $+\sqrt{-t/2}$ or the lower-unstable branch $-\sqrt{-t/2}$, and thus there are actually two sequences of eigenvalues, one for even $n$ and one for odd $n$. Using Richardson extrapolation, we find that the sequences of eigenvalues $b_{2n}$ and $b_{2n+1}$ have the same asymptotic behavior.
Our numerical calculations give

\[ B_{\Pi} = 1.862\,4128. \]  

The numerical data for P-II are slightly more noisy than those for P-I, and fourth-order Richardson extrapolation only gives the underlined eighth digit as 8 ± 2.

Figure 8. Typical behavior of solutions to the second Painlevé transcendent for the initial conditions \( y(0) = 0 \) and \( b = y'(0) \). In the left panel \( b = 1.028\,605\,106 \), which lies between the eigenvalues \( b_1 = 0.595\,082\,5526 \) and \( b_2 = 1.528\,605\,106 \). In the right panel \( b = 2.028\,605\,106 \), which lies between the eigenvalues \( b_2 = 1.528\,605\,106 \) and \( b_3 = 2.155\,132\,869 \). In the left panel the solution \( y(t) \) has an infinite sequence of simple poles and in the right panel the solution oscillates stably about \(-\sqrt{t/6}\). The dashed curves are the functions \( \pm \sqrt{-t/2} \).

Figure 9. Solutions to the P-II equation (2) for \( y(0) = 0 \) and \( b = y'(0) \). Left panel: \( b = 2.600\,745\,985 \), which lies between the eigenvalues \( b_3 = 2.155\,132\,869 \) and \( b_4 = 2.700\,745\,985 \). Right panel: \( b = 2.800\,745\,985 \), which lies between the eigenvalues \( b_4 = 2.700\,745\,985 \) and \( b_5 = 3.195\,127\,590 \).

\[ b_{2n} \sim b_{2n+1} \sim B_{\Pi} n^{2/3} \quad (n \to \infty). \]  

(16)
Next, we plot the positive-$t$ solutions to P-II for vanishing initial slope and positive initial condition for $t_0$. As $t \to \infty$, the $n$th eigenfunction passes through $n$ simple poles before it approaches zero monotonically. In figures 13–15 we plot the six eigenfunctions corresponding to $n = (1, 2), (3, 4), \text{and} (13, 14)$. (Because of the symmetry of P-II, for every positive eigenvalue there is a corresponding negative eigenvalue. We do not plot the negative-eigenvalue solutions.)

Using fourth-order Richardson we determine that for large $n$, $c_n \sim C_n n^{1/3}$, where

$$C_\Pi = 1.215 \, 81165.$$  \hfill (18)

The last digit 5 has an uncertainty of $\pm 1$. 

Figure 10. First two separatrix solutions (eigenfunctions) of Painlevé II with initial condition $y(0) = 0$. Left panel: $y'(0) = b_1 = 0.595 \, 082 \, 5526$; right panel: $y'(0) = b_2 = 1.528 \, 605 \, 106$. The dashed curves are $\pm \sqrt{-t/2}$.

Figure 11. Third and fourth eigenfunctions of Painlevé II with initial condition $y(0) = 0$. Left panel: $y'(0) = b_3 = 2.155 \, 132 \, 869$; right panel: $y'(0) = b_4 = 2.700 \, 745 \, 985$. 

4.2. Initial-value eigenvalues for Painlevé II

The initial-value eigenvalues for Painlevé II are crucial in understanding the behavior of the solutions. As $t \to \infty$, the $n$th eigenfunction passes through $n$ simple poles before it approaches zero monotonically. The figures 13–15 illustrate the six eigenfunctions corresponding to $n = (1, 2), (3, 4), \text{and} (13, 14)$. The eigenvalues for these cases are determined using fourth-order Richardson's method, which yields $C_\Pi = 1.215 \, 81165$ with an uncertainty of $\pm 1$.
To obtain analytic expressions for $B_{II}$ in (17) and $C_{II}$ in (18), we follow the same procedure as in section 3 for P-I. We multiply the P-II differential equation in (2) by $y'(t)$ and integrate from $t = 0$ to $t = x$, where $x$ is in the turning-point region in which the simple poles stop. The result is

$$H \equiv \frac{1}{2} [y'(x)]^2 - \frac{1}{2} [y(x)]^4 = \frac{1}{2} [y'(0)]^2 - \frac{1}{2} [y(0)]^4 + I(x), \quad (19)$$

where $I(x) = \int_0^x dt \, t y(t) y'(t)$. The path of integration is the same as that used to calculate P-II numerically in section 4; it follows the negative-real axis until it gets near a simple pole, at which point it makes a semicircular detour in the complex-$t$ plane to avoid the pole. Again, as in section 3, we argue that along this path the integrand of $I(x)$ is oscillatory and because of cancellations we may neglect $I(x)$ when $n$ is large.

**Figure 12.** The twentieth and twenty-first eigenfunctions of Painlevé II with initial condition $y(0) = 0$. Left panel: $y'(0) = b_{20} = 8.499 476 190$; right panel: $y'(0) = b_{21} = 8.787 666 814$.

**Figure 13.** First two separatrix solutions (eigenfunctions) of Painlevé II with fixed initial slope $y'(0) = 0$. Left panel: $y(0) = c_1 = 1.222 873 339$; right panel: $y(0) = c_2 = 1.533 883 935$. 

5. Asymptotic calculation of $B_{II}$ and $C_{II}$

To obtain analytic expressions for $B_{II}$ in (17) and $C_{II}$ in (18), we follow the same procedure as in section 3 for P-I. We multiply the P-II differential equation in (2) by $y'(t)$ and integrate from $t = 0$ to $t = x$, where $x$ is in the turning-point region in which the simple poles stop. The result is

$$H \equiv \frac{1}{2} [y'(x)]^2 - \frac{1}{2} [y(x)]^4 = \frac{1}{2} [y'(0)]^2 - \frac{1}{2} [y(0)]^4 + I(x), \quad (19)$$

where $I(x) = \int_0^x dt \, t y(t) y'(t)$. The path of integration is the same as that used to calculate P-II numerically in section 4; it follows the negative-real axis until it gets near a simple pole, at which point it makes a semicircular detour in the complex-$t$ plane to avoid the pole. Again, as in section 3, we argue that along this path the integrand of $I(x)$ is oscillatory and because of cancellations we may neglect $I(x)$ when $n$ is large.
We treat $H$ as the $\mathcal{PT}$-symmetric quantum-mechanical Hamiltonian

$$\hat{H} = \frac{1}{2} \hat{p}^2 - \frac{1}{2} \hat{x}^4$$

and we use (11) with $g = 1/2$ and $\epsilon = 2$ to obtain the formula

$$E_n \sim \frac{1}{2} \left[ 3n \sqrt{2\pi} \Gamma\left(\frac{3}{4}\right)/\Gamma\left(\frac{1}{4}\right) \right]^{1/3}$$

for the large eigenvalues of $\hat{H}$. Finally, we calculate the eigenvalues $b_n$ by using

$$\sqrt{2E_n} \sim \left[ 3n \sqrt{2\pi} \Gamma\left(\frac{3}{4}\right)/\Gamma\left(\frac{1}{4}\right) \right]^{2/3} \quad (n \to \infty).$$

Figure 14. Third and fourth eigenfunctions of Painlevé II with initial slope $y'(0) = 0$. Left panel: $y(0) = c_1 = 1.754537281$; right panel: $y(0) = c_4 = 1.93061783$.  

Figure 15. Thirteenth and fourteenth separatrix solutions (eigenfunctions) of Painlevé II with fixed initial slope $y'(0) = 0$. Left panel: $y(0) = c_1 = 2.858869051$; right panel: $y(0) = c_2 = 2.930357651$.  

We treat $H$ as the $\mathcal{PT}$-symmetric quantum-mechanical Hamiltonian

$$\hat{H} = \frac{1}{2} \hat{p}^2 - \frac{1}{2} \hat{x}^4$$

and we use (11) with $g = 1/2$ and $\epsilon = 2$ to obtain the formula

$$E_n \sim \frac{1}{2} \left[ 3n \sqrt{2\pi} \Gamma\left(\frac{3}{4}\right)/\Gamma\left(\frac{1}{4}\right) \right]^{1/3}$$

for the large eigenvalues of $\hat{H}$. Finally, we calculate the eigenvalues $b_n$ by using

$$\sqrt{2E_n} \sim \left[ 3n \sqrt{2\pi} \Gamma\left(\frac{3}{4}\right)/\Gamma\left(\frac{1}{4}\right) \right]^{2/3} \quad (n \to \infty).$$
This result allows us to identify the value of $B_{II}$ in (17) as

$$B_{II} = \left[3\sqrt[3]{2\pi \Gamma\left(\frac{1}{4}\right)}/\Gamma\left(\frac{3}{4}\right)\right]^{4/3}. \quad (23)$$

This result agrees with the numerical determination in (17).

To calculate $C_{II}$ we observe from figures 13–15 that the initial value $y(0)$ is positive. However, if we neglect $H(x)$ and assume a vanishing initial slope, we see that the right side of (19) is negative. Thus, as we did for the cubic Hamiltonian $\frac{1}{2}\hat{p}^2 - 2\hat{x}^3$, we perform a complex rotation of the coupling constant to convert the quartic Hamiltonian to the form

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\hat{x}^4. \quad (24)$$

This is the conventional Hermitian quartic-anharmonic-oscillator Hamiltonian, and does not belong to the class of $PT$-symmetric Hamiltonians $\hat{H} = \frac{1}{2}\hat{p}^2 + g\hat{x}^2(i\hat{x})^\gamma$. A WKB calculation gives the large-eigenvalue approximation

$$E_n \sim \left[3n\sqrt{\pi} \Gamma\left(\frac{1}{4}\right)/\Gamma\left(\frac{3}{4}\right)\right]^{4/3} \quad (n \to \infty). \quad (25)$$

Thus, we read off the value of $C_{II}$:

$$C_{II} = \left[3\sqrt[3]{\pi \Gamma\left(\frac{1}{4}\right)}/\Gamma\left(\frac{3}{4}\right)\right]^{4/3}, \quad (26)$$

which agrees exactly with the numerical result in (18).

6. Brief concluding remarks

In this paper we have shown that the first two Painlevé equations, P-I and P-II, exhibit instabilities that are associated with separatrix solutions. The initial conditions that give rise to these separatrix solutions are eigenvalues. We have calculated the semiclassical (large-eigenvalue) behavior of the eigenvalues in two ways, first by using numerical techniques and then by using asymptotic methods to reduce the IVPs for the nonlinear P-I and P-II equations to linear eigenvalue problems associated with the time-independent Schrödinger equation. The agreement between these two approaches is exact.

The obvious continuation of this work is to examine the next four Painlevé equations, P-III–P-VI, to see if there are instabilities, separatrices, and eigenvalues for these equations as well. However, the techniques we have applied here may also be useful for other nonlinear differential equations such as the Thomas–Fermi equation $y''(x) = [y(x)]^{5/2}/\sqrt{x}$, which is posed as a boundary-value problem satisfying the boundary conditions $y(0) = 1$ and $y(\infty) = 0$. The solution to this problem is unstable with respect to small changes in the initial data; if the initial slope $y'(0)$ is varied by a small amount, the solution develops a spontaneous singularity at some positive value $a$. A leading-order local analysis suggests that this singularity is a fourth-order pole of the form $400(x - a)^{-4}$. However, this singularity is not a pole. Indeed, a higher-order local analysis indicates that there is a logarithmic-branch-point singularity at $x = a$ as well and thus the solutions to the Thomas–Fermi equation live on multsheeted Riemann surfaces. It would be interesting to see if our work on nonlinear eigenvalue problems extends beyond meromorphic functions.
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