An anisotropic regularity condition for the 3D incompressible Navier-Stokes equations for the entire exponent range

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Abstract

We show that a suitable weak solution to the incompressible Navier-Stokes equations on $\mathbb{R}^3 \times (-1,1)$ is regular on $\mathbb{R}^3 \times (0,1]$ if $\partial_3 u$ belongs to $M^{2p/(2p-3),\alpha}((-1,0);L^p(\mathbb{R}^3))$ for any $\alpha > 1$ and $p \in (3/2,\infty)$, which is a logarithmic-type variation of a Morrey space in time. For each $\alpha > 1$ this space is, up to a logarithm, critical with respect to the scaling of the equations, and contains all spaces $L^q((-1,0);L^p(\mathbb{R}^3))$ that are subcritical, that is for which $2/q + 3/p < 2$.

1 Introduction

We address conditional regularity of suitable Leray-Hopf weak solutions to the incompressible Navier-Stokes equations (NSE),

$$u_t - \Delta u + u \cdot \nabla u + \nabla \pi = 0,$$

$$\text{div} \ u = 0,$$

in $\mathbb{R}^3 \times (0,T)$. Our main result is the following.

Theorem 1. Suppose that $(u, \pi)$ is a suitable Leray-Hopf weak solution to the Navier-Stokes equations on $\mathbb{R}^3 \times (-1,1)$ such that for some $\alpha > 1$ and $p \in (3/2,\infty)$ we have

$$\|\partial_3 u\|_{L^{2p/(2p-3),\alpha}((-1,0);L^p(\mathbb{R}^3))} \leq C_{p,\alpha} \left( \frac{-1}{\log |I|} \right)^{\alpha}$$

for every $I \subset (-1,0)$ with $|I| < \frac{1}{2}$. Then $u$ is regular on $\mathbb{R}^3 \times (-1,0]$.

Here we write $L^p \equiv L^p(\mathbb{R}^3)$, for brevity.

In order to put this result in a context, we note that the study of conditional regularity of the NSE goes back to Serrin, Ladyzhenskaya, and Prodi ([S, L, P]), who proved that if $u \in L^q_t L^p_x$ holds with $2/q + 3/p \leq 1$, where $p \in (3,\infty)$ then the solution is regular. On the other hand, Beirão da Veiga showed in [B] that the regularity holds if $\nabla u \in L^q_t L^p_x$ with $2/q + 3/p \leq 2$ and $p \in (3/2,\infty)$.

In [NP1], Neustupa and Penel proved that boundedness of only one component of the velocity (say $u_3$) implies regularity, with the approach based on the evolution equation for $\omega_3$ (cf. also [NP2]). Afterwards, there have been many results ([CC, H, NNP, P, PP1, SK], which approached the Serrin’s scale invariant condition in terms of one velocity component, until a recent breakthrough paper [CW], which achieved the range of exponents with strict inequality $2/q + 3/p < 1$. A

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subsequent paper [WWZ] has improved it up to the equality, but with the Lorenz spaces replacing Lebesgue spaces for integrability in time.

As for regularity conditions in terms of $\partial_3 u$, Penel and Pokorný proved in [PPT] regularity under the condition that $\partial_3 u$ belongs to $L^q_t L^p_x$ where $2/q + 3/p \leq 3/2$ and $2 \leq p \leq \infty$. The result in [KZ] then provided a scale invariant regularity criterion $2/q + 3/p \leq 2$, with a restricted exponent range $9/4 \leq p \leq 3$. The method in [KZ] was based on testing the equations for $(u_1, u_2)$ with $-\Delta_2 u_{1,2}$, and an identity for $\sum_{i,j=1}^{2} \int u_i \partial_i u_j$ in which every term contains $\partial_3 u$. The partial regularity methods [CKN, V, O, W1] allowed localization of this condition in [KRZ]. There have been several improvements on the criteria since then; cf. [BG, CZ, PP1, Sk1, Sk2] for a partial list of references. In particular, in [Sk2], Skaláč extended the range for $\partial_3 u$ to $3/2 < p \leq 3$ using sharp anisotropic inequalities, and, very recently, this range has been extended to $3/2 < p \leq 6$ in [CFZ].

In this context Theorem 1 provides the first conditional regularity criterion in terms of $\partial_3 u$ covering the full range of Lebesgue exponents $3/2 < p < \infty$ as well as all Lebesgue spaces $L^q_t L^p_x$ with sharp inequality $2/q + 2/p < 2$. To be more precise, letting (2) be the definition of a Morrey-type space $M^{2p, \alpha}_{2p-3}((-1,0); L^p)$, we immediately see that such Morrey space contains $L^q((-1,0); L^p)$ for every $q > \frac{2p}{2p-3}$ (that is such that $2/q + 3/p < 2$), since Hölder’s inequality implies

$$\|\partial_3 u\|_{L^q_t L^p_x(I; L^p)} \leq \left\| \partial_3 u \right\|_{L^q(I; L^p; L^p)} \leq C_{p,q,\alpha} \left( -\frac{1}{\log |I|} \right)^\alpha$$

for any $I \subset (-1,0)$, $|I| \leq \frac{1}{2}$. Furthermore, $M^{2p, \alpha}_{2p-3}((-1,0); L^p)$ is, up to a logarithm, critical with respect to the scaling of the equations; namely letting $u_\lambda := \lambda u(\lambda x, \lambda^2 t)$ we have

$$\|\partial_3 u_\lambda\|_{L^q_t L^p_x(I; L^p)} = \|\partial_3 u\|_{L^q_{2p-3}^{2p-3} (\lambda I; L^p)} \leq C_{p,\alpha} \left( -\frac{1}{\log |I|} \right)^\alpha \leq C_{p,\alpha} \left( -\frac{1}{\log |I|} \right)^\alpha \lambda^\alpha (\left( -\frac{1}{\log \lambda} \right)^\alpha)$$

as $\lambda \to 0^+$, for every $I \subset (-1,0)$ such that $|I| \leq \frac{1}{2}$.

The question whether the Morrey space $M^{2p, \alpha}_{2p-3}((-1,0); L^p)$ can be replaced by a critical Lebesgue-type space, $L^{2p, \alpha}_{2p-3}((-1,0); L^p)$, without any restriction on the range of $p$ as in Theorem 1 remains an open problem.

Our approach in proving Theorem 1 is inspired by the treatment of a related regularity condition in terms of one component of $u$ that was recently proved by Wang et al in [WWZ], which in turn drew from a recent result of Chae and Wolf [CW], which introduced a new approach based on partial regularity and testing the local energy equality with a one-dimensional backward heat kernel.

## 2 Proof of Theorem 1

Before proceeding to the proof of our main result, we recall that $(u, \pi)$ with

$$E := \|u\|_{L^\infty(-1,0); L^2}^2 + \|\nabla u\|_{L^2(\mathbb{R}^2 \times (-1,1))}^2 < \infty \quad (3)$$

is a suitable weak solution in $\mathbb{R}^3 \times (-1,1)$ if it satisfies the equation (1) in the distributional sense, if $\pi = (-\Delta)^{-1}(\partial_i u_j \partial_j u_i)$, the strong energy inequality

$$\int |u(t)|^2 + 2 \int \int |\nabla u|^2 \leq \int |u(s)|^2$$

holds for almost all $s \in (-1,1)$ and all $t \in (s,1)$, as well as the local energy inequality

$$\int |u|^2 \phi(t) + 2 \int |\nabla u|^2 \phi \leq \int |u|^2 (\partial_t \phi + \Delta \phi) + (\|u\|^2 + 2\pi)(u \cdot \nabla)$$
holds for all \( t \in (-1, 1) \) and \( \phi \in C_0^\infty(\mathbb{R}^3 \times (-1, 1); [0, 1]) \). We note that, due to the global integrability assumptions on \( u \), the local energy inequality can be extended to include the test functions \( \phi \in C^\infty(\mathbb{R}^3 \times (-1, 1); [0, 1]) \) that have compact support only in time and that have bounded derivatives. For \( n \in \mathbb{N}_0 \) we set \( r_n := 2^{-n} \) and
\[
U_n := \mathbb{R}^2 \times (-r_n, r_n), \quad Q_n := U_n \times (-r_n^2, 0).
\]
We also set
\[
E_n \equiv E(r_n) := \sup_{t \in (-r_n^2, 0)} \int_{U_n} |u(t)|^2 dx + \int_0^0 \int_{-r_n^2} |\nabla u|^2 dx ds
\]
and
\[
\Phi_n(x_3, t) := (4\pi(r_n^2 - t))^{-\frac{1}{2}} e^{-\frac{x_3^2}{4(r_n^2 - t)}}, \quad x_3 \in \mathbb{R}, \ t < 0.
\]
Note that
\[
r_k^{-1} \lesssim \Phi_n \lesssim r_k^{-1} \quad \text{on } Q_k, \quad k = 0, 1, \ldots, n.
\] (4)
Fix \( p \in (3/2, \infty) \) and set
\[
B_k := \||\partial_3 u||_{L_t^{\frac{2p}{p-1}} L^p(Q_k)}.
\]
Note that
\[
\sum_{k \geq 0} B_k \lesssim_{p, a} \sum_{k \geq 0} \left( \frac{-1}{\log |r_k^2|} \right)^\alpha \lesssim_\alpha \sum_{k \geq 0} k^{-\alpha} \lesssim_\alpha 1,
\] (5)
by the assumption \([2]\). We also set \( B := \||\partial_3 u||_{L_t^{\frac{2p}{p-1}}((-1,0); L^p)} \).

In order to prove the main result, Theorem 1, we need the following localization property.

**Proposition 2.** Let \( (u, \pi) \) be a suitable weak solution to the NSE on \( \mathbb{R}^3 \times (-1, 1) \) satisfying \([2]\). Then
\[
\frac{1}{r_n} E_n \lesssim E(1 + B) + \sum_{j,k=0}^{n-1} (r_k^{-1} E_k + \delta_j r_j^{-1} E_j + \delta_j r_j B_k)
\]
for all \( n \geq 0 \), where
\[
a_{jk} := \chi_{k \geq j} r_k^{-\frac{2}{3}} r_j^{-\frac{2}{3}} B_j + \delta_{jk} r_k^{-\frac{2}{3}} B + \delta_{jk} B_k
\]
for some \( a > 1 \).

Here \( \chi_{k \geq j} = 1 \) for \( k \geq j \) and 0 otherwise, and \( \delta_{kj} \) denotes the Kronecker delta.

**Proof of Proposition 2.** Let \( \eta(x_3, t) \) be such that \( \text{supp } \eta \subset (\frac{1}{3}, \frac{1}{3}) \times (-1, 0) \) and \( \eta = 1 \) on \( (-\frac{1}{3}, \frac{1}{3}) \times (-\frac{1}{16}, 0) \). The local energy inequality applied with \( \Phi_n \eta \), where \( n \in \mathbb{N} \) is fixed, gives
\[
\int_{U_0} |u(t)|^2 \Phi_n(t) \eta(t) dx + 2 \int_{-1}^t \int_{U_0} |\nabla u|^2 \Phi_n \eta
\leq \int_{-1}^t \int_{U_0} |u|^2 (\partial_t + \Delta)(\Phi_n \eta) + \int_{-1}^t \int_{U_0} (|u|^2 + 2\pi) u \cdot \nabla(\Phi_n \eta)
\] (6)
for almost every \( t \in (-1, 0) \). We show below that the right-hand side can be bounded from above by a constant multiple of \( E_0 + \sum_{k=0}^n r_k^{-1} E_k(r_k^2 B_0 + B_k) \), uniformly in \( t \). This and the bound \( \Phi_n \eta \lesssim r_n^{-1} \) on \( Q_n \) then give the claim.
For the first term on the right-hand side of (5), we have
\[ \int_{Q_0} |u|^2 |(\partial_k + \Delta)(\Phi_n \eta)| = \int_{Q_0} |u|^2 |2\partial_3 \Phi_n \partial_3 \eta + \Phi_n \partial_3 \eta| \lesssim \int_{Q_0} |u|^2 \lesssim E_0, \]
where we used that \( \Phi_n \) satisfies the one-dimensional heat equation in \( Q_0 \) in the first step, and the bounds \( |\nabla \Phi_n|, |\Phi_n| \lesssim 1 \) on \( \text{supp} \, \partial_3 \eta \cap Q_0 \) in the second step.

For the velocity component of the second term on the right-hand side of (6), we have
\[ -\sum_{k=0}^{n-1} \int_{Q_k \setminus Q_{k+1}} |\partial_3 u| |u|^2 \Phi_n \eta + \int_{Q_n} |\partial_3 u| |u|^2 \Phi_n \eta \lesssim \sum_{k=0}^{n-1} r_k^{-1} \int_{Q_k} |\partial_3 u| |u|^2 + r_n^{-1} \int_{Q_n} |\partial_3 u| |u|^2 \]
where we have estimated the last term, \( k = n \), using the term with \( k = n - 1 \), in the last step.

As for the term in (6) involving the pressure \( \pi = (-\Delta)^{-1}(\partial_3 u_3 \partial_3 u_i) \), for each \( k \in \mathbb{N}_0 \) we choose \( \chi_k(x_3, t) \in C^\infty([0, 1]) \) such that \( \chi_k = 1 \) on \( (-r_k, r_k) \times (0, 1] \) and set
\[ \phi_j := \begin{cases} 
\chi_j - \chi_{j+1}, & j = 0, \ldots, n - 1, \\
\chi_n, & j = n.
\end{cases} \]

Then we may write
\[ \frac{1}{2} \partial_3 \pi = (-\Delta)^{-1} \partial_i \partial_i (u_3 \partial_3 u_i) = (-\Delta)^{-1} \partial_i \partial_i (u_3 \partial_3 u_i \chi_0) + (-\Delta)^{-1} \partial_i \partial_i (u_3 \partial_3 u_i (1 - \chi_0)) \]
\[ = \sum_{j=0}^{n} (-\Delta)^{-1} \partial_i \partial_i (u_3 \partial_3 u_i \phi_j) + (-\Delta)^{-1} \partial_i \partial_i (u_3 \partial_3 u_i (1 - \chi_0)) =: \sum_{j=0}^{n} p_j + q, \]
and thus the pressure term may be decomposed as
\[ -\frac{1}{2} \int_{-1}^{t} \int_{U_{0}} \pi u \cdot \nabla (\Phi_n \eta) = \frac{1}{2} \int_{-1}^{t} \int_{U_{0}} \partial_3 \pi u_3 \Phi_n \eta + \frac{1}{2} \int_{-1}^{t} \int_{U_{0}} \pi \partial_3 u_3 \Phi_n \eta. \]

Using the notation in (11), we rewrite the first term as
\[ \frac{1}{2} \int_{-1}^{t} \int_{U_{0}} \partial_3 \pi u_3 \Phi_n \eta = \sum_{j=0}^{n} \int_{-1}^{t} \int_{U_{0}} p_j u_3 \Phi_n \eta + \int_{-1}^{t} \int_{U_{0}} q u_3 \Phi_n \eta \]
\[ = \sum_{k=0}^{n} \sum_{j=\max(0, k-3)}^{n} \int_{Q_k} p_j u_3 \Phi_n \phi_k \eta + \sum_{j=0}^{n} \sum_{k=j+4}^{n-4} \int_{Q_k} p_j u_3 \Phi_n \phi_k \eta + \int_{-1}^{t} \int_{U_{0}} q u_3 \Phi_n \eta \]
\[ =: I_1 + I_2 + I_3. \]

For \( I_1 \), we note that \( \sum_{j=k-3}^{n} p_j = (-\Delta)^{-1} \partial_i \partial_i (u_3 \partial_3 u_i \chi_{k-3}) \) for \( k \geq 3 \), which gives
\[ |I_1| \lesssim \sum_{k=0}^{n} r_k^{-1} \int_{Q_k} \left| \sum_{j=\max(0, k-3)}^{n} p_j u_3 \right| \lesssim \sum_{k=0}^{n} r_k^{-1} \left| \int_{Q_k} u \otimes \partial_3 u \right|_{L^4 L^{2p'}(Q_k)} \lesssim \sum_{k=0}^{n} r_k^{-1} \left| \int_{Q_k} u \right|_{L^4 L^{2p'}(Q_k)} \lesssim \sum_{k=0}^{n} r_k^{-1} \left| E_k B_k \right| \lesssim \sum_{k=0}^{n} r_k^{-1} \left| E_k B_k \right|. \]
as required. For $I_2$, we note that $p_j$ is harmonic with respect to the spatial variables in $Q_{j+2}$, and thus using the anisotropic interior estimates for harmonic functions (cf. [CW, Lemma A.2]) we obtain
\[ \|p_j\|_{L^m((\mathbb{R}^2 \times (-r_k, r_k)))} \lesssim r_k^{\frac{4}{p} - \frac{2}{q}} \|p_j\|_{L'(\mathbb{R}^2 \times (-r_{j+2}, r_{j+2}))} \tag{9} \]
for all $l \in [1, m]$. We fix any $a > \max\{2p/3, 4/3\}$ and then fix any $l \in (\max\{1, 2p/(p+2)\}, \min\{6p/(p+6), 6a/(3a+4)\})$, which is nonempty due to our choice of $a$, to obtain
\[ |I_2| \lesssim \sum_{j=0}^{n-4} \sum_{k=j+4}^{n} r_k^{-1} \int_{Q_k} |p_j u| \]
\[ \lesssim \sum_{j=0}^{n-4} \sum_{k=j+4}^{n} r_k^{-1} \|p_j\|_{L^p_{a/L} L_{2}^2(Q_k)} \|u\|_{L^p_{a/L} L_{2}^2(Q_k)} \]
\[ \lesssim \sum_{j=0}^{n-4} \sum_{k=j+4}^{n} r_k^{-\frac{1}{2}} r_j^{-\frac{1}{2}} r_j^{1-\frac{2}{p}} \|u\|_{L_{2}^{2a/3p-2p} L_{2}^{2a} (Q_k)} \|\partial_3 u\|_{L_{2}^{2a/3p-2p} L_{2}^{2a} (Q_k)} \|u\|_{L_{\infty}^{2} L_{2}^{2} (Q_k)} r_k^{\frac{2}{a}} \]
where we used the harmonic estimate (9) (note that $l \leq 2$, as required by (3)) and the fact that $l < p$ in the third inequality. Note also that our choice of $l$ gives that $lp/(p-1) \in (2, 6)$, which will allow us to estimate the term with the $L_x^{lp/(p-l)}$ norm using the energy $E$. We obtain
\[ |I_2| \lesssim \sum_{j=0}^{n-4} \sum_{k=j+4}^{n} r_k^{-\frac{1}{2}} r_j^{-\frac{1}{2}} r_j^{1-\frac{2}{p}} \|u\|_{L_{2}^{2a/3p-2p} L_{2}^{2a} (Q_k)} \|\partial_3 u\|_{L_{2}^{2a/3p-2p} L_{2}^{2a} (Q_k)} \|u\|_{L_{\infty}^{2} L_{2}^{2} (Q_k)} r_k^{\frac{2}{a}} \]
\[ \lesssim \sum_{j=0}^{n-4} \sum_{k=j+4}^{n} \left( r_j^{-1} E_j \right)^{\frac{1}{2}} \left( r_k^{-1} E_k \right)^{\frac{1}{2}} \|\partial_3 u\|_{L_{2}^{2a/3p-2p} L_{2}^{2a} (Q_k)} \|u\|_{L_{\infty}^{2} L_{2}^{2} (Q_k)} r_k^{\frac{2}{a}} r_j^{\frac{2}{a}} \]
\[ \lesssim \sum_{j=0}^{n-4} \sum_{k=j+4}^{n} \left( r_j^{-1} E_j \right)^{\frac{1}{2}} \left( r_k^{-1} E_k \right)^{\frac{1}{2}} \|\partial_3 u\|_{L_{2}^{2a/3p-2p} L_{2}^{2a} (Q_k)} \|u\|_{L_{\infty}^{2} L_{2}^{2} (Q_k)} r_k^{\frac{2}{a}} r_j^{\frac{2}{a}} B_j, \]
as required, where we used Hölder’s inequality in time in the first inequality (hence the upper bound $l < 6a/(3a+4)$ in our choice of $l$) in order to be able to bound all norms of $u$ by $E_j^{1/2}$ or $E_j^{1/2}$ in the second inequality (where we also moved $r_k$’s and $r_j$’s around), and we have estimated the case of $k = n$ in terms of the case $k = n - 1$ as well as used (2) in the last inequality.

The estimate on $I_3$ is analogous, but does not require summation in $j$. Indeed, recalling (7) we see that $q$ is harmonic in $(\text{supp } \eta) \cap (U_0 \times (-1, l))$, and so we perform the same estimate as in the first four inequalities in the estimate on $|I_2|$ above, but with $Q_j$ replaced by $\mathbb{R}^3 \times (-1, 0)$ and without the summation in $j$. We obtain
\[ |I_3| \lesssim \|u\|_{L_{\frac{4p}{mp-4p+mp}((-1,0);L_{\frac{4p}{mp}})} \|\partial_3 u\|_{L_{\frac{2p}{mp-4p+mp}((-1,0);L_{p})}} \sum_{k=0}^{\frac{n}{2}} r_k^{-\frac{1}{2} + \frac{2}{a}} E_k^{\frac{1}{2}} \]
\[ \lesssim E^{\frac{1}{2}} B \left( \sum_{k=0}^{n-1} r_k^{-1} E_k^{\frac{1}{2}} \right)^{\frac{1}{2}} \lesssim BE + B \sum_{k=0}^{n-1} r_k^{-1} E_k^{\frac{1}{2}} \]
as required.

It remains to estimate the second term in (8). For this we apply the splitting (7) to $\pi$ (rather than to $\partial_3 \pi$) to obtain
\[ \pi = \sum_{j=0}^{n} (-\Delta)^{-1} \partial_i \partial_i (u_i u_j \phi_j) + (-\Delta)^{-1} \partial_i \partial_i (u_i u_i (1 - \chi_0)) =: \sum_{j=0}^{n} \tilde{p}_j + \tilde{q}, \]
which allows us to estimate
\[
\int_{-1}^t \int_{U_0} \pi \partial_3 u_3 \Phi \eta 
\lesssim \sum_{k=0}^n r_k^{-1} \int_{Q_k} |(-\Delta)^{-1} \partial_t \partial_m (u_i u_m \chi_{\max\{0,k-3\}})| |\partial_3 u|
\]
\[+ \sum_{j=0}^{n-4} \sum_{k=j+4}^n r_k^{-1} \int_{Q_k} |\tilde{p}_j \partial_3 u| + \sum_{k=0}^n r_k^{-1} \int_{Q_k} |\tilde{q} \partial_3 u|\]
\[\lesssim \sum_{k=0}^n r_k^{-1} \|u\|_{L^\frac{q}{2} Q_k}^2 B_k
\]
\[+ \sum_{j=0}^{n-4} \sum_{k=j+4}^n r_k^{-1} \|\tilde{p}_j\|_{L^\frac{2}{q'} Q_{k+2}} \|\partial_3 u\|_{L^2 Q_k} + \sum_{k=0}^n r_k^{-1} \|\tilde{q}\|_{L^\frac{2}{q'} Q_k} \|\partial_3 u\|_{L^2 Q_k},\]
where \(a \in (1,p').\) Note such choice of \(a\) implies that \(a < 2p/(2p - 3).\) Therefore, choosing any \(l \in (1,3a/(a + 2))\) we obtain
\[
\int_{-1}^t \int_{U_0} \pi \partial_3 u_3 \Phi \eta 
\lesssim \sum_{k=0}^n r_k^{-1} \|u\|_{L^\frac{q}{2} Q_k}^2 B_k
\]
\[+ \sum_{j=0}^{n-4} \sum_{k=j+4}^n r_k^{-1} \|\tilde{p}_j\|_{L^\frac{2}{q'} Q_{k+2}} r_k \|\tilde{q}\|_{L^\frac{2}{q'} Q_k} + \sum_{k=0}^n r_k^{-1} \|\tilde{q}\|_{L^\frac{2}{q'} Q_k} \|\partial_3 u\|_{L^2 Q_k},\]
\[\lesssim \sum_{k=0}^n r_k^{-1} E_k B_k + \sum_{j=0}^{n-4} \sum_{k=j+4}^n r_k^{-1} \|u\|_{L^2(\rho k)^\frac{2}{q'}} \|\tilde{p}_j\|_{L^\frac{2}{q'} Q_{k+2}} r_k \|\tilde{q}\|_{L^\frac{2}{q'} Q_k} + \sum_{k=0}^n r_k^{-1} \|\tilde{q}\|_{L^\frac{2}{q'} Q_k} \|\partial_3 u\|_{L^2 Q_k},\]
\[\lesssim \sum_{k=0}^n r_k^{-1} E_k B_k + E B,
\]
as required, where, in the first inequality, we used the harmonic estimate \(\square\) (note that \(l < p'\), as required, thanks to our choice of \(a\) and \(l\)), as well as Hölder’s inequality \(\|f\|_{L^a(-r_k^2,0)} \lesssim \|f\|_{L^{\frac{q}{2}}(-r_k^2,0)} \|u\|_{L^\frac{q}{2} Q_k}\); in the second inequality we used the fact that \(p' \in (1,3)\), which allowed us to estimate \(\|u\|_{L^\frac{q}{2} Q_k}\); by \(E_k\), as well as the fact that \(l < p'\) to sum the last series, and the fact that \(B_j\)'s are nonincreasing, i.e., \(B_k \leq B_j \leq B\). In the last inequality, we used Hölder’s inequality \(\|f\|_{L^2(\rho k)^\frac{2}{q'}} \lesssim \|f\|_{L^{\frac{q}{2}}(r_k^2,0)}\); for \(2a' < 4l'/3\) by the choice of \(l\).

**Corollary 3.** Under the assumptions of Proposition \(\square\) we have \(r^{-1} E(r) \lesssim E(1 + B) e^{C \sum_{k,j \geq 0} a_{jk}} < \infty\) for all \(r \in (0,1)\).

**Proof.** Recall that if \(b_n, x_n \geq 0\) and \(C > 0\) are such that \(x_0 \leq C\) and \(x_n \leq C + \sum_{k=0}^{n-1} b_k x_k\) for all \(n \geq 1\), then
\[x_n \leq C e^{\sum_{k=0}^{n-1} b_k}\]
for all \(n \geq 1\), by the discrete Gronwall inequality (see Lemma A.1 in [WWZ] for a proof). Letting \(x_n := r_n^{-1} E_n\) and \(b_n := \sum_{k \geq 0} (a_{kn} + a_{nk})\) and using Young’s inequality \(ab \leq (a^2 + b^2)/2\), Proposition \(\square\) gives
\[x_n \lesssim E(1 + B) + \sum_{k,j=0}^{n-1} x_k a_{jk} \lesssim E(1 + B) + \sum_{k=0}^{n-1} x_k b_k\]
for each \(n \geq 1\). Since \(\square\) implies that \(\sum_{k \geq 0} b_k < \infty\), we obtain the claim for \(r = r_n\), where \(n \geq 0\). The claim for other \(r\) follows by approximating with a neighboring \(r_n\). \(\square\)
We note in passing that in the proof above we have in fact used a discrete Gronwall inequality of the form $x_n \leq Ce^{\sum_{i,j \geq 0} a_{ij}}$ whenever $(x_n)_{n \geq 0}$ is a nonnegative sequence such that $x_0 \leq C$ and $x_n \leq C + \sum_{i,j=0}^{n-1} a_{ij} x_i^{1/2} x_j^{1/2}$ for $n > 0$, and the coefficients $a_{ij} \geq 0$ are such that $\sum_{i,j} a_{ij} < \infty$.

We can now prove our main result.

**Proof of Theorem 1**. By the above corollary, $r^{-1}E(r) \leq C_{E,B}$ for all $r \in (0,1)$. Since

$$\|\partial_3 u\|_{L^{2n/(n-2)}((-r^2,0);L^p(B_r))} \to 0 \quad \text{as} \quad r \to 0,$$

the next lemma gives that $(0,0)$ is a regular point of $u$, in the sense that $u$ is essentially bounded in $(-p^2,0) \times B(0,p)$ for some $p$. Regularity at any other point in $\mathbb{R}^3$ at $t = 0$ follows analogously, by translating and rescaling $U_n$, $Q_n$, $A_n$, and $E_n$. We now show that this implies that $u$ is regular on $(-1,0]$. Indeed, note that, due to the existence of intervals of regularity of any Leray-Hopf weak solution (see Theorem 6.41 in [OP], for example) we can assume, by rescaling, that $(0,0)$ is a suitable weak solution in $\mathbb{R}^3$ at $t = 0$ shows that $\|u(t)\|_{L^\infty(B_R)}$ remains bounded as $t \to 0^-$ for each $R > 0$.

Due to the partial regularity theory of Caffarelli-Kohn-Nirenberg, there exists $\epsilon > 0$ such that if $\int_{B_1(x)} (|u|^3 + |p|^{3/2}) \leq \epsilon$ then $|u| \leq C(\epsilon)$ on $(-1/4,0) \times B_{1/2}(x)$, where $C(\epsilon) > 0$ is independent of $x$ (see Theorem 2.2 in [O], for example). Let $(B_1(x_n))_{n \geq 1}$ be a cover of $\mathbb{R}^3$ such that $x_n \in \mathbb{Z}^3/4$ for each $n \geq 1$. Since (3) together with interpolation and the Calderón-Zygmund inequality imply that $|u|^3 + |p|^{3/2} \in L^1((-1,0) \times \mathbb{R}^3)$, we see that $\int_{B_1(x_n)} (|u|^3 + |p|^{3/2}) > \epsilon$ for only finitely many $n$'s. Thus there exists $R > 0$ such that $|u| \leq C(\epsilon)$ on $(-1/2,0) \times \{|x| > R\}$. Hence $\|u(t)\|_{L^\infty(\mathbb{R}^3)}$ remains bounded as $t \to 0^-$, and so regularity of $u$ persist beyond $t = 0$, due to the classical Leray estimates (see Corollary 6.25 in [OP], for example). This concludes the proof of Theorem 1 once we establish the next lemma.

Let us introduce some notation. Given a suitable weak solution $(u, \pi)$ we denote by

$$Q_r := B_r \times (-r^2,0)$$

the finite cylinder of radius $r$ and set

$$P(\pi, r) := \frac{1}{r^2} \int_{Q_r} |\pi|^{3/2}, \quad C(u, r) := \frac{1}{r^2} \int_{Q_r} |u|^3, \quad A(u, r) := \frac{1}{r} \sup_{t \in (-r^2,0]} \int_{B_r} |u(t)|^2 \, dx, \quad E(u, r) := \frac{1}{r} \int_{Q_r} |\nabla u|^2.$$

We may now state the lemma that we used above.

**Lemma 4** (Conditional local regularity). *Given $M > 0$ there exists $\epsilon(M) > 0$ with the following property: If $(u, \pi)$ is a suitable weak solution in $Q_1$ such that

$$\sup_{r \in (0,1)} \frac{1}{r^2} \left( \sup_{t \in (-r^2,0]} \int_{B_r} |u(t)|^2 \, dx + \int_{Q_r} |\nabla u|^2 \right) \leq M < \infty$$

then $(0,0)$ is a regular point provided

$$r_0^{2-\frac{2}{a} - \frac{2}{b}} \|\partial_3 u\|_{L^a((-r_0^2,0);L^b(B_{r_0}))} \leq \epsilon(M)$$

for some $r_0 \in (0, C_0(u, \pi))$ and $a, b \geq 1$.**
Proof of Lemma 4. Note that, by interpolation (cf. \cite{O} Lemma 2.1, for example), the first assumption gives
\[ C(u, r) \lesssim M \]
for every \( r \in (0, 1) \). We shall show the claim with
\[ C_0(u, \pi) := \min \left\{ \frac{1}{2} \left( C(u, 1) + P(\pi, 1) \right)^{-2} \right\} . \]

Suppose that the claim does not hold. Then there exists a sequence \((u^k, \pi^k)\) and \( r_k \in (0, C_0(u^k, \pi^k)) \) such that
\[ C(u^k, r) \lesssim M \quad \text{and} \quad \int_{\Omega} |\nabla u^k|^2 = \frac{1}{k} \]
while \((0, 0)\) is a singular point of \( u^k \) for every \( k \). Using \cite{WZI} Lemma A.2, we obtain
\[ A(u^k, r) + E(u^k, r) + P(\pi^k, r) \leq C(M), \]
for all \( r \in (0, r_k) \). In order to relax the restriction on the range of \( r \) we apply the rescaling
\[ v^k(x, t) := r_k u^k(r_k x, r_k^2 t), \quad q^k(x, t) := r_k^2 \pi^k(r_k x, r_k^2 t) \]
to obtain
\[ A(v^k, r) + E(v^k, r) + P(q^k, r) + C(v^k, r) + k\|\partial_3 v^k\|_{L^p_t L^q_x(\Omega)} \leq C(M) \]
for all \( r \in (0, 1) \). This estimate on \((v^k, q^k)\) together with the Aubin-Lions Lemma (cf. \cite{RRS} Theorem 4.12, for example) is sufficient to extract a subsequence, which we relabel, such that
\[ v^k \to v \text{ in } L^3(\Omega_{1/2}), \quad q^k \to q \text{ in } L^\frac{3}{2}(\Omega_{1/2}), \quad \text{and} \quad \partial_3 v^k \to 0 \text{ in } L^3_t L^\frac{6}{5}_x(\Omega_1), \]
where \((v, q)\) is a suitable weak solution to the Navier-Stokes equations on \( \Omega_{1/2} \) such that \( \partial_3 v = 0 \). It follows that \( v \) and \( \nabla v \) are bounded functions in \( \Omega_{1/4} \) due to the localized regularity condition on \( \partial_3 v \) of \cite{KRZ}. Since also \( \int_{\Omega_{1/2}} |q|^{3/2} < \infty \), we get, using the elliptic regularity on the equation \(-\Delta q = \partial_i v_j \partial_j v_i \in L^\infty(\Omega_{1/4})\) at almost every time \( t \in (-1/4, 0) \), that \( \|q\|_{W^{2-p}(B_{1/4})} \lesssim \|\nabla v\|^2_{L^p_t(\Omega_{1/4})} \)
and \( \|q\|_{L^3(\Omega_{1/4})} \) for every \( p \in (1, \infty) \) and almost every \( t \in (-1/4, 0) \) \cite{GT} Theorem 9.11. Using this statement with \( p \) sufficiently large, we obtain \( q \in L^{3/2}_t L^\infty_x(\Omega_{1/8}) \). This immediately implies \( r^{-3} \int_{\Omega_r} |q|^{3/2} < \infty \) for \( r \in (0, 1/8) \), and consequently for every \( r \in (0, 1/16) \) we have
\[ \epsilon \leq \liminf_{k \to \infty} \frac{1}{r^2} \int_{\Omega_r} \left( |v^k|^3 + |q^k|^2 \right) = \frac{1}{r^2} \int_{\Omega_r} \left( |v|^3 + |q|^2 \right) \leq C_{v,q} (r^3 + r) \leq C_{v,q} r, \]
where \( \epsilon > 0 \) is given by the Caffarelli-Kohn-Nirenberg condition (cf. \cite{O} Theorem 2.2) and \( C \) depends on \( M \) and the uniform bound of \( v \) and \( \nabla v \) on \( \Omega_{1/4} \). The above inequality leads to a contradiction when we send \( r \to 0 \), concluding the proof. \( \square \)

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