Abstract

Suppose $S$ is a surface of genus $\geq 2$, $f : S \to S$ is a surface homeomorphism isotopic to a pseudo-Anosov map $\alpha$ and suppose $\tilde{S}$ is the universal cover of $S$ and $F$ and $A$ are lifts of $f$ and $\alpha$ respectively. A result of A. Fathi shows there is a semiconjugacy $\Theta : \tilde{S} \to \bar{\mathcal{L}}^s \times \bar{\mathcal{L}}^u$ from $F$ to $\bar{A}$, where $\bar{\mathcal{L}}^s$ ($\bar{\mathcal{L}}^u$) is the completion of the $R$-tree of leaves of the stable (resp. unstable) foliation for $A$ and $\bar{A}$ is the map induced by $A$.

We generalize a result of Markovich and show that for any $g \in \text{Homeo}(S)$ that commutes with $f$ and is isotopic to the identity with identity lift $G$ and for any $(c, w)$ in the image of $\Theta$ each component of $\Theta^{-1}(c, w)$ is $G$-invariant.

1 Introduction

Suppose that $S$ is a closed surface and that $\alpha : S \to S$ is either an orientation preserving linear Anosov map of $T^2$ or an orientation preserving pseudo-Anosov homeomorphism of a higher genus surface. In the former case note that $\alpha$ fixes the point $e$ that is the image of $(0, 0)$ in the usual projection of $\mathbb{R}^2$ to $T^2$. The first author proved that if $f \in \text{Homeo}(T^2)$ is isotopic to $\alpha$ and fixes $e$ then there is a unique map $p : T^2 \to T^2$ that fixes $e$, is isotopic to the identity and that semi-conjugates $f$ to $\alpha$; i.e $pf = \alpha p$.

To describe this case further we work in the universal cover $\mathbb{R}^2$ of $T^2$. Let $A, F$ and $P$ be the lifts of $\alpha, f$ and $p$ respectively that fix $(0, 0)$ and note that $PF^k = A^k P$ for all $k \in \mathbb{Z}$ (because $PF^k((0, 0)) = A^k F((0, 0))$). We say that the $F$-orbit of $\tilde{y} \in \mathbb{R}^2$ shadows the $A$-orbit of $\tilde{x} \in \mathbb{R}^2$ if $\bar{d}(F^k(\tilde{y}), A^k(\tilde{x})) \leq C$ for some $C$ and all $k \in \mathbb{Z}$; we also say that the $f$-orbit of $y \in T^2$ globally shadows the $\alpha$-orbit of $x \in T^2$. Since $p$ is homotopic to the identity, there exists $C > 0$ so that $\text{dist}(P(\tilde{y}), \tilde{y}) < C$ for all $\tilde{y}$.

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\[ \tilde{y} \in \mathbb{R}^2. \] In particular, \( \text{dist}(F^k(\tilde{y}), A^k(P(\tilde{y}))) = \text{dist}(F^k(\tilde{y}), PF^k(\tilde{y})) < C \) for all \( k \) so the \( F \)-orbit of \( \tilde{y} \) shadows the \( A \)-orbit of \( P(\tilde{y}) \) for all \( \tilde{y} \in \mathbb{R}^2 \). It is well known that no two \( A \)-orbits shadow each other so \( P \) is completely determined by this shadowing property. The surjectivity of \( P \) reflects the fact that every \( A \)-orbit is shadowed by some \( F \)-orbit. The fact that \( P \) is defined on all of \( \mathbb{R}^2 \) reflects the fact that every \( F \)-orbit shadows some \( A \)-orbit.

Suppose now that \( \alpha \) is pseudo-Anosov, that \( f \) is isotopic to \( \alpha \) and that \( A : \hat{S} \to \hat{S} \) is a lift of \( \alpha \). The isotopy between \( \alpha \) and \( f \) lifts to an isotopy between \( A \) and a lift \( F : \hat{S} \to \hat{S} \) of \( f \). Equivalently, \( F \) is the unique lift of \( f \) that induces the same action on covering translations as \( A \). Let \( d(x, y) \) be any path metric on \( S \) and let \( \tilde{d}(\tilde{x}, \tilde{y}) \) be its lift to \( \hat{S} \). Shadowing in \( \hat{S} \) and global shadowing in \( S \) are defined as above using \( \tilde{d} \) in place of the Euclidean metric on \( \mathbb{R}^2 \). It is not hard to construct examples (c.f. Proposition 2.1 of [H2]) for which there are \( F \)-orbits that are not shadowed by any \( A \)-orbit. On the other hand, the second author proved [HI] that every \( A \)-orbit is shadowed by some \( F \)-orbit. More precisely, there exists a closed \( f \)-invariant set \( Y \subset S \) with full pre-image denoted \( \tilde{Y} \subset \hat{S} \) and a continuous equivariant surjection \( P : \hat{Y} \to \hat{S} \) such that \( PF = AP \).

The \( F \)-orbits of \( \hat{S} \setminus \hat{Y} \) do not shadow \( A \)-orbits. It is natural to ask if there is some larger context in which one can understand these orbits. Fatih [Fa] answered this by considering leaf spaces of the stable and unstable foliations.

In the Anosov case, the hyperbolic linear map \( A \) has stable and unstable invariant foliations \( \mathcal{F}^s \) and \( \mathcal{F}^u \) consisting of straight lines parallel to the eigenvectors of \( A \). Let \( \mathcal{L}^s \) and \( \mathcal{L}^u \) be the leaf spaces of \( \mathcal{F}^s \) and \( \mathcal{F}^u \) respectively. Then \( \mathcal{L}^s \) and \( \mathcal{L}^u \) can be identified with \( \mathbb{R} \) in such a way that \( A \) induces homotheties \( A^s : \mathcal{L}^s \to \mathcal{L}^s \) and \( A^u : \mathcal{L}^u \to \mathcal{L}^u \) defined respectively by \( x \to \lambda x \) and by \( x \to x/\lambda \) where \( \lambda > 1 \) is an eigenvalue of \( A \). Thus \( (A^s, A^u) : \mathcal{L}^s \times \mathcal{L}^u \to \mathcal{L}^s \times \mathcal{L}^u \) is naturally identified with \( A : \mathbb{R}^2 \to \mathbb{R}^2 \) and the shadowing argument that defines \( P \) can be done in this product of leaf spaces.

In the pseudo-Anosov case the stable and unstable foliations \( \mathcal{F}^s \) and \( \mathcal{F}^u \) have singularities and their leaf spaces \( \mathcal{L}^s \) and \( \mathcal{L}^u \) are more complicated. Namely, \( \mathcal{L}^s \) and \( \mathcal{L}^u \), and their metric completions \( \tilde{\mathcal{L}}^s \) and \( \tilde{\mathcal{L}}^u \) have the structure of \( \mathbb{R} \)-trees [MS]. As in the previous case \( A \) induces a homothety \( A^s : \tilde{\mathcal{L}}^s \to \tilde{\mathcal{L}}^s \) that uniformly expands distance by a factor \( \lambda > 1 \) and a homothety \( A^u : \tilde{\mathcal{L}}^u \to \tilde{\mathcal{L}}^u \) that uniformly contracts distance by the factor \( 1/\lambda \). Let \( \tilde{A} = (A^s, A^u) : \tilde{\mathcal{L}}^s \times \tilde{\mathcal{L}}^u \to \tilde{\mathcal{L}}^s \times \tilde{\mathcal{L}}^u \).

Let \( Q_s : \hat{S} \to \mathcal{L}^s \) be the natural map that sends a point in \( \hat{S} \) to the leaf of \( \mathcal{F}^s \) that contains it. Define \( Q_u \) similarly and let \( Q = Q_s \times Q_u : \hat{S} \to \mathcal{L}^s \times \mathcal{L}^u \). Denote the subset of \( \mathcal{L}^s \times \mathcal{L}^u \) consisting of pairs of leaves of \( \mathcal{F}^s \) and \( \mathcal{F}^u \) that have a point in common by \( \Delta \). Then \( Q(\hat{S}) = \Delta \), and \( Q : \hat{S} \to \Delta \) is a homeomorphism. Moreover,
the following diagram commutes.

\[
\begin{array}{c}
\tilde{Y} \\
\downarrow P \\
\tilde{S} \\
\downarrow A \\
\Delta \\
\end{array}
\quad F
\quad \begin{array}{c}
\tilde{Y} \\
\downarrow P \\
\tilde{S} \\
\downarrow A \\
\Delta \\
\end{array}
\quad \begin{array}{c}
\tilde{Y} \\
\downarrow P \\
\tilde{S} \\
\downarrow A \\
\Delta \\
\end{array}
\]

Fathi [Fa] extended \(QP\) to a non-surjective map \(\Theta : \tilde{S} \to \tilde{L}^s \times \tilde{L}^u\) that makes the following diagram commutes.

\[
\begin{array}{c}
\tilde{S} \\
\downarrow \Theta \\
\tilde{L}^s \times \tilde{L}^u \\
\downarrow A \\
\Delta \\
\end{array}
\quad F
\quad \begin{array}{c}
\tilde{S} \\
\downarrow \Theta \\
\tilde{L}^s \times \tilde{L}^u \\
\downarrow A \\
\Delta \\
\end{array}
\]

See also [RHU] for more details on the map \(\Theta\).

The maps \(p : T^2 \to T^2\) and \(\Theta : S \to \tilde{L}^s \times \tilde{L}^u\) depend canonically on \(f\) and so determine canonical decompositions \(\{p^{-1}(x)\}\) of \(T^2\) and \(\{\Theta^{-1}(c,w)\}\) of \(S\). If \(g\) commutes with \(f\) then one expects \(g\) to preserve this decomposition. If \(g\) is isotopic to the identity then one might even expect \(g\) to setwise preserve each element of the decomposition.

Implicit in [Mark] is an even stronger and more surprising fact for the Anosov case. Namely that if \(g\) commutes with \(f\) and is isotopic to the identity then \(g\) setwise preserves each component of the decomposition. It is not hard to see that this decomposition is upper semi-continuous and that each element is cellular.

**Theorem 1.1. (Markovich)** Suppose that \(\alpha : T^2 \to T^2\) is a linear Anosov map, that \(f \in \text{Homeo}(T^2)\) is isotopic to \(\alpha\) and fixes \(e\) and that \(p : T^2 \to T^2\) is the unique map that fixes \(e\) and satisfies \(pf = \alpha p\). If \(g \in \text{Homeo}(T^2)\) commutes with \(f\), is isotopic to the identity and setwise preserves \(p^{-1}(e)\) then each component of \(p^{-1}(x)\) is \(g\)-invariant for all \(x \in T^2\).

The main result of this paper is the following extension of Markovich’s result to the pseudo-Anosov case. If \(g\) is isotopic to the identity then the identity lift \(G\) of \(g\) is the unique lift that commutes with all covering translations.
Theorem 1.2. Suppose that \( \alpha : S \to S \) is pseudo-Anosov, that \( f \in \text{Homeo}(S) \) is isotopic to \( \alpha \) and that \( A, F : \tilde{S} \to \tilde{S} \) and \( \Theta : \tilde{S} \to \tilde{L}^s \times \tilde{L}^u \) are as above. If \( g \in \text{Homeo}(S) \) commutes with \( f \) and is isotopic to the identity then the identity lift \( \bar{G} : \tilde{S} \to \tilde{S} \) of \( g \) commutes with \( F \) and preserves each component of \( \Theta^{-1}(c, w) \) for all \( (c, w) \in \Theta(\bar{S}) \).

Our proof of Theorem 1.2 makes use of arguments from [Mark]. We give complete details for the reader’s convenience and because the arguments from [Mark] are not easily referenced. It is straightforward to modify our proof of Theorem 1.2 to obtain a proof of Theorem 1.1. We leave that to the interested reader.

2 Proof of Theorem 1.2

We assume throughout this section that \( \alpha : S \to S \) is pseudo-Anosov with expansion factor \( \lambda > 1 \), that \( f \in \text{Homeo}(S) \) is isotopic to \( \alpha \), and that \( A : \tilde{S} \to \tilde{S} \) and \( F : \tilde{S} \to \tilde{S} \) are lifts of \( \alpha \) and \( f \) that induce the same action on covering translations.

The transverse measures on the lifts \( \bar{\mathcal{F}}^s \) and \( \bar{\mathcal{F}}^u \) of the stable and unstable measured foliations \( \mathcal{F}^s \) and \( \mathcal{F}^u \) for \( \alpha \) determine pseudo-metrics \( d^u \) and \( d^s \) on \( \tilde{S} \) such that

\[
d^u(A(x), A(y)) = \lambda d^u(x, y) \quad \text{and} \quad d^s(A(x), A(y)) = \lambda^{-1} d^s(x, y),
\]

for all \( x, y \in \tilde{S} \). Moreover, \( d^u(x, y) = 0 \) [resp. \( d^s(x, y) = 0 \)] if and only if \( x \) and \( y \) belong to the same leaf of \( \mathcal{F}^s \) [resp. \( \mathcal{F}^u \)]. There is also [B] a singular Euclidean metric \( d \) on \( \bar{S} \) such that

\[
d(x, y) = \sqrt{d^s(x, y)^2 + d^u(x, y)^2}
\]
on each standard Euclidean chart (i.e. one without singularities). In particular, \( d(x, y) \leq 2 \max\{d^s(x, y), d^u(x, y)\} \).

For any \( x, y \in \bar{S} \) there is a unique (up to parametrization) path \( \rho \) with endpoints \( x \) and \( y \) and with length equal to \( d(x, y) \). Subdividing \( \rho \) at the singularities that it intersects decomposes \( \rho \) into a concatenation of linear subpaths. We will refer to \( \rho \) as the geodesic joining \( x \) to \( y \). If the intersection of two geodesics \( \gamma_1 \) and \( \gamma_2 \) is more than a single point, then it is a path whose endpoints are either singularities or endpoints of \( \gamma_1 \) or \( \gamma_2 \). Any leaf without singularities of either foliation is a geodesic, as is any embedded copy of \( \mathbb{R} \) in a leaf with singularities.

Remark 2.1. Since the geodesic \( \gamma \) joining \( x \) to \( y \) (minimizing length as measured by \( d \)) consists of a finite collection of segments whose interiors lie in Euclidean charts we can conclude \( \max\{d^s(x, y), d^u(x, y)\} \leq d(x, y) \leq 2 \max\{d^s(x, y), d^u(x, y)\} \).

We denote the leaf of \( \mathcal{F}^s \) that contains \( x \) by \( W^s(x) \), the leaf space of \( \mathcal{F}^s \) by \( \mathcal{L}^s \) and the image of \( W^s(x) \) in \( \mathcal{L}^s \) by \( \mathcal{Q}_s(x) \). Thus \( \mathcal{Q}_s^{-1}(\mathcal{Q}_s(x)) = W^s(x) \). Moreover, \( d^u \) induces a metric on \( \mathcal{L}^s \) (which we also denote \( d^u \)) by setting \( d^u(\mathcal{Q}_s(x_1), \mathcal{Q}_s(x_2)) = d^u(x_1, x_2) \). This is easily seen to be independent of the choice of \( x_1 \) and \( x_2 \). This
metric gives $\mathcal{L}^s$ the structure of an $\mathbb{R}$-tree $[MS]$. As illustrated in Remark $2.2$, $\mathcal{L}^s$ is not complete. We denote the metric completion by $(\bar{\mathcal{L}}^s, d^u)$

**Remark 2.2.** One can construct a non-converging Cauchy sequence $L_i = Q_s(x_i)$ in $\mathcal{L}^s$ as follows. Choose a singularity $\bar{x}_0 \in \bar{S}$, a stable ray $R_0$ initiating at $x_0$ and a sequence $\epsilon_i > 0$ whose sum is finite. Assuming inductively that $x_i$ and $R_i$ have been defined, choose a singularity $x_{i+1} \in \bar{S}$ such that $W^u(x_{i+1}) \cap R_i \neq \emptyset$ and such that $d^u(x_{i+1}, x_i) < \epsilon_{i+1}$. Let $R_{i+1}$ be a stable ray in $W^s(x_{i+1})$ that initiates at $x_{i+1}$ and whose interior is contained in a component of the complement of $W^u(x_{i+1})$ that is disjoint from $x_i$. By construction, $d^u(L_i, L_j) = \epsilon_{i+1} + \cdots + \epsilon_j$ for all $i \leq j$ and there are no accumulation points of the $W^s(x_i)$'s. The former shows that the $L_i$'s are a Cauchy sequence and the latter implies that this sequence is non-convergent.

Similarly $\mathcal{L}^u$, the space of unstable leaves in $\bar{S}$ is an $\mathbb{R}$-tree with metric $d^s$ and $Q_u : \bar{S} \to \mathcal{L}^u$ satisfies $Q_u^{-1}(Q_u(x)) = W^u(x)$. The metric completion of $\mathcal{L}^u$ is $(\bar{\mathcal{L}}^u, d^s)$ and we use the metric $d = \max\{d^s, d^s\}$ on $\bar{\mathcal{L}}^u \times \bar{\mathcal{L}}^u$. Define $Q : \bar{S} \to \bar{\mathcal{L}}^u \times \bar{\mathcal{L}}^u$ by $Q(x) = (Q^s(x), Q^u(x))$ and let $\Delta = Q(\bar{S})$ equipped with the subset topology. Note that $\Delta$ can be characterized as the subset of $\mathcal{L}^s \times \mathcal{L}^u$ consisting of pairs of leaves of $\mathcal{F}^s$ and $\mathcal{F}^u$ that have a point in common.

The pseudo-Anosov map $A$ on $\bar{S}$ induces a homeomorphism $A_s : \mathcal{L}^s \to \bar{\mathcal{L}}^s$. From the properties of $d^s$ (equation (1) above) it is clear that $A_s$ is an expanding homothety, i.e. for any $L_1, L_2 \in \mathcal{L}^s$

$$d^u(A_s(L_1), A_s(L_2)) = \lambda d^u(L_1, L_2). \quad (2)$$

The map $A_u : \mathcal{L}^u \to \mathcal{L}^u$ is defined analogously and has similar properties. Likewise if $T$ is a covering translation on $\bar{S}$ it induces an isometry $T_s : \mathcal{L}^s \to \mathcal{L}^s$. It is clear that $A_s$ and $A_u$ extend to expanding and contracting homotheties respectively of $\bar{\mathcal{L}}^s$ and $\bar{\mathcal{L}}^u$ which we also denote by $A_s$ and $A_u$. Also $T_s$ extends to an isometry of $\bar{\mathcal{L}}^s$. The map $T_u : \mathcal{L}^u \to \mathcal{L}^u$ is defined analogously and has similar properties. Let $\bar{A} : \bar{\mathcal{L}}^s \times \bar{\mathcal{L}}^u \to \bar{\mathcal{L}}^s \times \bar{\mathcal{L}}^u$ be the map $A_s \times A_u$.

As a consequence of Fathi’s theorem we have the following.

**Proposition 2.3.** There exists a unique continuous map $\theta_s : \bar{S} \to \bar{\mathcal{L}}^s$ satisfying

(i) There is a commutative diagram

![Diagram](image_url)
(ii) There exists $C_1 > 0$ such that $d^u(Q_s(x), \theta_s(x)) < C_1$ for all $x \in \tilde{S}$.

(iii) For all $x \in \tilde{S}$, $\theta_s(x)$ is the unique point in $\tilde{L}^u$ with the property that there exists $C_2 > 0$ such that $d^u(Q_s(F^k(x)), A^k_s \theta_s(x)) < C_2$ for all $k \geq 0$.

(iv) $\theta_s(T(x)) = T_1(\theta_s(x))$ for all $x \in \tilde{S}$ and each covering transformation $T$.

(v) The image of $\theta_s$ contains $\mathcal{L}^u$.

Proof. Let $\tilde{S} \rightarrow \tilde{L}^u$ be the composition of $\Theta$ with the projection of $\tilde{L}^s \times \tilde{L}^u$ onto its first factor. Then properties (i)-(v) follow from the corresponding properties of $\Theta$ as proved in [Fa].

Let $\tilde{S} \rightarrow \tilde{L}^u$ be the analogous semiconjugacy from $F$ to $A_u : \tilde{L}^u \rightarrow \tilde{L}^u$. Note then that

$$\Theta(x) = (\theta_u(x), \theta_s(x)).$$

**Remark 2.4.** Proposition 2.3 can be proved directly as follows. Given $x \in \tilde{S}$ let $z_k = A^{-k} F^k(x)$ and let $L_k = Q_s(z_k)$. Then it is straight forward to show that $\{L_k\}$ is a Cauchy sequence in the $d^u$ metric on $\tilde{L}^u$. It’s limit can be taken as the definition of $\theta_s(x)$. Property (i) is then immediate and it is not difficult to show $\theta_s$ is continuous and satisfies the other properties.

We denote open $\epsilon$-neighborhoods by $N_\epsilon(\cdot)$. By item (iii) of Proposition 2.3 applied with $k = 0$ there is a constant $C_2$ such that

$$d^u(Q_s(x), \theta_s(x)) < C_2$$

for all $x \in \tilde{S}$. Since $d^u(A(x), F(x))$ is bounded independently of $x$ and since $Q_s$ preserves $d^u$ we may also assume that

$$d^u(Q_s F(x), A_s Q_s(x)) = d^u(Q_s F(x), Q_s A(x)) < C_2.$$  \hspace{1cm} (4)

Choose $C > \max(C_2, C_2/(\lambda - 1))$ and note that

$$\lambda C - C_2 > C$$ \hspace{1cm} (5)

**Lemma 2.5.** $\Theta^{-1}(N_\epsilon(c) \times N_\epsilon(w))$ is a bounded subset of $\tilde{S}$ for all $\epsilon > 0$ and all $(c, w) \in \mathcal{L}^s \times \mathcal{L}^u$.

Proof. For $x \in Q_s^{-1}(c)$ and $z \in \Theta^{-1}(N_\epsilon(c) \times N_\epsilon(w))$ we have

$$d^u(x, z) = d^u(Q_s(x), Q_s(z))$$

$$\leq d^u(Q_s(x), \theta_s(z)) + d^u(\theta_s(z), Q_s(z))$$

$$\leq \epsilon + C_2.$$

Symmetrically, if $\theta_u(y) = w$ then

$$d^s(y, z) \leq \epsilon + C_2.$$

It follows that $d(z, z') \leq 2 \max\{d^u(z, z'), d^u(z, z')\} \leq 2(\epsilon + C_2)$ for all $z, z' \in \Theta^{-1}(N_\epsilon(c) \times N_\epsilon(w))$. \hfill \square
Lemma 2.6. If \( V(c, \epsilon) \) is defined to be \( \theta_s^{-1}(N_\epsilon(c)) \), then

(i) The set \( V(c, \epsilon) \) is an open, connected, simply connected, unbounded set for all \( c \in L^s \) and all \( \epsilon > 0 \).

(ii) If \( G : \hat{S} \to \hat{S} \) commutes with \( F \) and if there is a constant \( C_1 \) such that \( d(x, G(x)) < C_1 \) for all \( x \in \hat{S} \) then there is a ray \( R \) that is properly embedded in \( \hat{S} \) such that \( R, G(R) \subset V(c, \epsilon) \) and such that \( R \) is properly homotopic to \( G(R) \) in \( V(c, \epsilon) \); i.e. there is a one parameter family \( R_t \) of rays in \( V(c, \epsilon) \) such that \( R_0 = R \) and \( R_1 = G(R) \) and such that each \( R_t \) is properly embedded in \( \hat{S} \).

Proof. Define

\[ Y_k = \{ x \in \hat{S} : d^\mu(x, W^s(A^k(c))) < \lambda^k \epsilon - C \} \]

or equivalently

\[ Y_k = Q_s^{-1}(N_{\lambda^k \epsilon - C}(A^k_s(c))). \]

Then \( Y_k \) is an open convex subset of \( \hat{S} \) that is a union of leaves of \( F^\ast \). In particular, \( Y_k \) is an open, connected, simply connected, unbounded set. If \( R_1(t) \) and \( R_2(t) \) are any two rays in \( Y_k \) that are properly embedded in \( \hat{S} \) and if there is a constant \( C_0 \) such that \( d(R_1(t), R_2(t)) \leq C_0 \) for all \( t \), then these rays are properly homotopic in \( Y_k \) (by a homotopy along geodesics).

Define

\[ X_k = F^{-k}(Y_k). \]

Thus

\begin{align*}
X_k &= F^{-k} Q_s^{-1}(N_{\lambda^k \epsilon - C}(A^k_s(c))) \\
&= \{ x \in \hat{S} : d^\mu(Q_s F^k(x), A^k_s(c)) < \lambda^k \epsilon - C \}.
\end{align*}

Since \( F \) is a homeomorphism, each \( X_k \) is an open, connected, simply connected, unbounded set. Also if \( R_1(t) \) and \( R_2(t) \) are properly embedded rays in \( \hat{S} \), contained in \( X_k \), for which there is a constant \( C_0 \) such that \( d(R_1(t), R_2(t)) \leq C_0 \), then these rays are properly homotopic in \( X_k \).

If \( x \in X_k \) then Equation (3) implies that \( A^k_s \theta_s(x) = \theta_s F^k(x) \in N_{\lambda^k \epsilon}(A^k_s(c)) \) and so \( \theta_s(x) \in N_\epsilon(c) \) by Equation (2). This proves that \( X_k \subset V(c, \epsilon) \). Moreover, by the triangle inequality and Equations (1), (2) and (4) we have

\begin{align*}
d^\mu(Q_s F^{k+1}(x), A^{k+1}_s(c)) &= d^\mu(Q_s F F^k(x), A_s A^k_s(c)) \\
&\leq d^\mu(A_s Q_s F^k(x), A_s A^k_s(c)) + C_2 \\
&= \lambda d^\mu(Q_s F^k(x), A^k_s(c)) + C_2 \\
&\leq \lambda(\lambda^k \epsilon - C) + C_2 \\
&= \lambda^{k+1} \epsilon - (\lambda C - C_2) \\
&\leq \lambda^{k+1} \epsilon - C.
\end{align*}
which proves that \( X_k \subset X_{k+1} \).

If \( \theta_s(w) \in N_\delta(c) \) then we may choose \( \delta < \epsilon \) and \( k \geq 1 \) so that \( \theta_s(w) \in N_\delta(c) \) and so that \( \lambda^k \delta < \lambda^k \epsilon - C_2 - C \). Then

\[
\theta_s F^k(w) = A^k_s \theta_s(w) \in N_{\lambda^k \epsilon - C - C_2}(A^k_s(c))
\]

and Equation (3) implies that

\[
Q_s F^k(w) \in N_{\lambda^k \epsilon - C}(A^k_s(c))
\]

and hence that \( w \in X_k \). We have now shown that \( V(c, \epsilon) \) is the increasing union of open, connected, simply connected and unbounded sets \( X_k \) thereby completing the proof of (i).

Choose \( k \geq 1 \) so that \( C_1 < \lambda^k \epsilon - C \) and let \( R' \) be a ray in \( W^s(A^k_s(c)) \). Then \( R' \) and \( G(R') \) are properly embedded in \( \tilde{S} \) and are contained in \( Y_k \) and \( \bar{R} := F^{-k}(R') \) and \( G(R) = F^{-k}G(R') \) are properly embedded in \( \tilde{S} \) and are contained in \( X_k \). Choosing \( C_0 \) so that \( d(G(x), x) \leq C_0 \) for all \( x \), we conclude \( d(R(t), G(R(t))) \leq C_0 \) and as noted above, \( R \) and \( G(R) \) are properly homotopic in \( X_k \subset V(c, \epsilon) \). This proves (ii). \( \square \)

The following proof is an adaptation of one that appears in [Mark].

**Proof of Theorem 1.2** Let \( \varphi : \tilde{S} \to \tilde{S} \) be the unique lift of \( g \) that is equivariantly isotopic to the identity; equivalently \( \varphi \) is the lift that commutes with all covering translations \( T \) of \( \tilde{S} \). The commutator \([F, G]\), which must be a covering translation since it is a lift of the identity on \( S \), commutes with all covering translations and hence is the identity. It follows that \( F \) and \( G \) commute.

Since \( \tilde{S} \) is compact and \( \bar{d}(G(Ty)), Q(Ty)) = \bar{d}(Q(G(y)), Q(y)) \) for all \( y \in \tilde{S} \) and all covering translations \( T \), there is a constant \( C' \) such that \( \bar{d}(Q(G(y)), Q(y)) < C' \) for all \( y \in \tilde{S} \). It follows that

\[
\bar{d}(QF^k(x), \tilde{A}^k(\Theta(G(x)))) \leq \bar{d}(QF^k(x), QF^k(G(x)) + \bar{d}(QF^k(G(x)), \tilde{A}^k \Theta(G(x)))
\]

\[
= \bar{d}(QF^k(x), G(F^k(x)) + \bar{d}(QF^k(G(x)), \tilde{A}^k \Theta(G(x)))
\]

\[
\leq C' + \bar{d}(QF^k(G(x)), \tilde{A}^k \Theta(G(x)))
\]

\[
\leq C' + \bar{d}_2
\]

where \( \bar{d}_2 \) is the maximum of the constants produced by item (iii) of Proposition 2.3 applied to \( \theta_s \) and to \( \theta_u \). The uniqueness part of Proposition 2.3(iii) therefore implies that \( \Theta G = \Theta \). In particular, \( \Theta^{-1}(c, w) \) is \( G \)-invariant. It suffices to show that each component of \( \Theta^{-1}(c, w) \) is \( G \)-invariant.

Choose \( \epsilon_n \to 0 \). Let \( V_n = V(c, \epsilon_n) \) be as in Lemma 2.6 and let \( H_n = H(w, \epsilon_n) \) be the open set obtained by applying Lemma 2.6 with \( \mathcal{L}^s \) replaced by \( \mathcal{L}^u \) and \( \theta_s \) replaced by \( \theta_u \). Then \( V_n \cap H_n = \Theta^{-1}(N_\epsilon(c) \times N_\epsilon(w)) \) is bounded by Lemma 2.5 and

\[
\Theta^{-1}(c, w) = \cap_{n=1}^{\infty}(V_n \cap H_n).
\]
Moreover if Λ is a component of Θ^{-1}(c, w) and we let \( K_n \) be the component of \( V_n \cap H_n \) containing Λ then \( \overline{K_{n+1}} \subset K_n \) for all \( n \) where \( K_{n+1} \) denotes the closure of \( K_{n+1} \). In particular,

\[
\bigcap_{n=1}^{\infty} K_n = \bigcap_{n=1}^{\infty} \overline{K_{n+1}}.
\]

Since each \( \overline{K_{n+1}} \) is compact, \( \bigcap_{n=1}^{\infty} K_n \) is non-empty and connected and hence equal to \( Λ \). It therefore suffices to show that each \( K_n \) is \( G \)-invariant. In fact it is enough to prove that \( G(\overline{K_n}) \cap \overline{K_n} \neq \emptyset \), because then \( G(\overline{K_n}) \) and \( \overline{K_n} \) are both subsets of \( K_{n-1} \) and hence \( G(K_{n-1}) = K_{n-1} \).

If there were infinitely many components of \( V_n \cap H_n \) that contain an element of \( \Theta^{-1}(c, w) \) then there would be a sequence \( \{x_i\} \subset \Theta^{-1}(c, w) \) converging to some \( x \in \Theta^{-1}(c, w) \) and with each \( x_i \) in a different component of \( V_n \cap H_n \). Since these components are disjoint, \( x \) is also an accumulation point of the frontiers of these components. Hence, since the frontier of a component of \( V_n \cap H_n \) is contained in the union of the frontier of \( V_n \) and the frontier of \( H_n \) there is a sequence \( \{y_i\} \) converging to \( x \) with each \( y_i \) in the frontier of either \( V_n \) or \( H_n \). This contradicts the continuity of \( Θ \) and the fact that \( d(Θ(y_i), Θ(x_i)) = \epsilon_n > 0 \) for all \( i \). We conclude that there are only finitely many components of \( V_n \cap H_n \) that contains an element of \( Θ^{-1}(c, w) \). Since \( G \) commutes with \( Θ \), \( G^{m}(\overline{K_n}) \) is a such a component for all \( i \geq 0 \). Thus \( G^{m}(\overline{K_n}) = \overline{K_n} \) for some smallest \( m \geq 1 \).

Let \( S^2 \) denote the one point compactification of \( \hat{S} \) obtained by adding a point \( \infty \). The set \( V_n \subset \hat{S} \) can be thought of as a subset of \( S^2 \) and when we do so we refer to it simply as \( V \). By Lemma 2.6, \( V \) is open, connected and simply connected and so has a prime end compactification \( \hat{V} \). For our purposes the key properties are:

(i) \( \hat{V} \) is topologically a disk \( D \) whose interior is identified with \( V \).

(ii) The function \( G|_V \) extends continuously to a homeomorphism \( \hat{G} : D \to D \).

(iii) For each continuous arc \( γ : [0, 1] \to S^2 \) with \( γ([0,1)) \subset V \) and \( γ(1) \) in the frontier of \( V \) there is a continuous arc \( \hat{γ} : [0, 1] \to D \) with \( \hat{γ}(t) = γ(t) \) for \( t \in [0, 1] \). The point \( γ(1) \) is called an accessible point of the frontier of \( V \) and \( \hat{γ}(1) \) is a prime end corresponding to it (there may be more than one prime end corresponding to an accessible point).

(iv) If \( γ_t \) is a continuous one parameter family of arcs in \( S^2 \) as in (iii) and if \( γ(1) \) is independent of \( t \) then \( \hat{γ}(1) \) is also independent of \( t \).

Properties (i)-(iii) go back to Caratheodory. An excellent modern exposition can be found in Mather’s paper [M]. In particular see §17 of [M] for a discussion of accessible points and §18 for property (iv).

A properly embedded ray \( R \) in \( \hat{S} \) that is contained in \( V_n \) determines an arc \( γ \) as in (iii) with \( γ(1) = \infty \). Considering the rays \( R \) and \( G(R) \) and applying item (ii) of
Lemma 2.6 we obtain \( \gamma \) and \( \hat{G}(\gamma) \) to which Property (iv) applies. This implies that \( \hat{\gamma} \) and \( \hat{G}(\hat{\gamma}) \) converge to the same prime end \( \hat{\infty} \) which is evidently fixed by \( \hat{G} \).

Now choose a properly embedded ray \( R_1 \) in \( H_{n+1} \) with initial endpoint in \( K_n \) and note that \( R_1 \) is disjoint from the frontier of \( H_n \). Since \( K_n \) is bounded, \( R_1 \) intersects the frontier of \( K_n \) in some first point \( z \) which is necessarily in the frontier of \( V_n \). The initial segment \( \gamma_1 \) of \( R_1 \) that terminates at \( z \) satisfies the hypotheses of (iii) with \( \gamma(1) = z \). Let \( \hat{\gamma} \) be the associated path in \( D \) and let \( \hat{z} \in \partial D \) be the prime end \( \hat{\gamma}(1) \).

Our proof is by contradiction. Let \( m \) be the smallest natural number with \( G^m(K_n) = K_n \) and assume \( m \neq 1 \) and \( G(K_n) \cap \overline{K}_n = \emptyset \). Then \( \hat{z}, \hat{G}(\hat{z}), \hat{G}^m(\hat{z}) \) and \( \hat{G}^{m+1}(\hat{z}) \) are distinct and in this order on \( \partial D \setminus \{ \hat{\infty} \} \) (oriented so that \( \hat{z} < \hat{G}(\hat{z}) \)). Since \( G^m(K_n) = K_n \) the initial endpoints of the \( \hat{\gamma}_1 \) and \( \hat{G}^m(\hat{\gamma}_1) \) can be joined in \( K_n \) to form an arc \( \hat{\beta} \) with interior in \( K_n \) and endpoints \( \hat{z} \) and \( \hat{G}^m(\hat{z}) \). Therefore \( \hat{G}(\hat{\beta}) \) has one endpoint \( \hat{G}(\hat{z}) \) and the other \( \hat{G}^{m+1}(\hat{z}) \). It follows that \( \hat{G}(\hat{\beta}) \cap \hat{\beta} \neq \emptyset \) and indeed the points of intersection must lie in the interior of these arcs in contradiction to the assumption that \( G(K_n) \cap \overline{K}_n = \emptyset \). We conclude that \( m = 1 \) and hence that \( G(K_n) = K_n \). \( \square \)

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