Gopakumar–Vafa Type Invariants on Calabi–Yau 4-Folds via Descendent Insertions

Yalong Cao, Yukinobu Toda

Kavli Institute for the Physics and Mathematics of the Universe (WPI), The University of Tokyo Institutes for Advanced Study, The University of Tokyo, Kashiwa, Chiba 277-8583, Japan.
E-mail: yalong.cao@ipmu.jp; yukinobu.toda@ipmu.jp

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Abstract: The Gopakumar–Vafa type invariants on Calabi–Yau 4-folds (which are non-trivial only for genus zero and one) are defined by Klemm–Pandharipande from Gromov–Witten theory, and their integrality is conjectured. In a previous work of Cao–Maulik–Toda, DT$_4$ invariants with primary insertions on moduli spaces of one dimensional stable sheaves are used to give a sheaf theoretical interpretation of the genus zero GV type invariants. In this paper, we propose a sheaf theoretical interpretation of the genus one GV type invariants using descendent insertions on the above moduli spaces. The conjectural formula in particular implies nontrivial constraints on genus zero GV type (equivalently GW) invariants of CY 4-folds which can be proved by the WDVV equation.

0. Introduction

0.1. Background on GV invariants of CY 3-folds. Let $Y$ be a smooth projective Calabi–Yau 3-fold over $\mathbb{C}$. For each $g \in \mathbb{Z}_{\geq 0}$ and $\beta \in H_2(Y, \mathbb{Z})$, the Gromov–Witten invariant

$$GW_{g, \beta} = \int_{[\overline{M}_g(Y, \beta)]^{vir}} 1 \in \mathbb{Q}$$

enumerates stable maps $f : C \rightarrow Y$ with $f_*[C] = \beta$ and $g(C) = g$. Because of multiple cover phenomena, the invariants $GW_{g, \beta}$ are not necessarily integers, so may not be regarded as ideal curve counting. However if we write the generating series of GW invariants as

$$\sum_{\beta \geq 0, g \geq 0} GW_{g, \beta} \lambda^{2g-2} t^\beta = \sum_{\beta \geq 0, g \geq 0, k \geq 1} \frac{n_{g, \beta}}{k} \left( \frac{k \lambda}{2} \right)^{2g-2} t^{k \beta}, \quad (0.1)$$

for some $n_{g, \beta} \in \mathbb{Q}$ (called Gopakumar–Vafa invariants), then $n_{g, \beta}$ are expected to be integers. The above integrality conjecture for $n_{g, \beta}$ is proved by Ionel–Parkar [IP] via symplectic geometry.
Furthermore there should be a sheaf theoretical interpretation of the integrality of \( n_{g, \beta} \). From the perspective of Gopakumar–Vafa [GV] on type IIA and M-theory duality, the invariants \( n_{g, \beta} \) are expected to be related to characters of \( \mathfrak{sl}_2 \times \mathfrak{sl}_2 \)-actions on some cohomology theories of moduli spaces \( M_1(Y, \beta) \) of one dimensional stable sheaves \( F \) on \( Y \) with \([F] = \beta \) and \( \chi(F) = 1 \). For genus zero, Katz [Kat08] conjectured that the invariant \( n_{0, \beta} \) equals to the degree of the virtual fundamental class of \( M_1(Y, \beta) \), which is a special case of Donaldson–Thomas invariants. For higher genus, Maulik–Toda [MT18] conjectured the following identity (which itself modifies earlier approaches by Hosono–Saito–Takahashi [HST] and Kiem–Li [KL]):

\[
\sum_{i \in \mathbb{Z}} \chi(\mathcal{H}^i(\mathbf{R}\pi_\beta^*(\mathcal{M}))) y^i = \sum_{g \geq 0} n_{g, \beta}(y^{\frac{1}{2}} + y^{-\frac{1}{2}})^{2g}.
\]

Here \( \mathcal{M} \) is a perverse sheaf on \( M_1(Y, \beta) \) locally isomorphic to vanishing cycle sheaves of local Chern–Simons functions, \( \pi_\beta : M_1(Y, \beta) \to \text{Chow}_\beta(Y) \) is the Hilbert–Chow map, and \( \mathcal{H}^i(-) \) is the \( i \)th perverse cohomology. See [MT18, Section 7.1] for the relationship of the formula (0.1) with \( \mathfrak{sl}_2 \times \mathfrak{sl}_2 \)-action. The conjectural equality (0.2) is widely open (see [MT18, Section 9]).

0.2. GV type invariants of CY 4-folds. In what follows, we discuss similar stories for Calabi–Yau 4-folds. Let \( X \) be a smooth projective Calabi–Yau 4-fold\(^1\). As an analogy of GV invariants on Calabi–Yau 3-folds defined by (0.1), Klemm–Pandharipande [KP] defined Gopakumar–Vafa type invariants on \( X \) from corresponding primary GW invariants (see Sect. 1.1 for details). Here an insertion \( \gamma \in H^4(X, \mathbb{Z}) \) is needed in the definition of genus zero invariants, and the \( g \geq 2 \) invariants are defined to be zero because the virtual dimensions of corresponding GW moduli spaces are negative. The GV type invariants (0.3) are conjectured to be integers and verified in many examples from computations of GW invariants [KP]. The genus zero integrality conjecture has been proved by Ionel–Parker [IP, Theorem 9.2] using symplectic geometry.

Motivated by the conjectural identity (0.2) for CY 3-folds, it is desirable to give a sheaf theoretical interpretation of GV type invariants (0.3) for CY 4-folds in terms of moduli spaces \( M_1(X, \beta) \) of one dimensional stable sheaves \( F \) on \( X \) with \([F] = \beta \) and \( \chi(F) = 1 \).

For \( \gamma \in H^4(X, \mathbb{Z}) \), let \( \tau_0(\gamma) \) be the primary insertion:

\[
\tau_0(\gamma) := (\pi_M)_*(\pi_X^*\alpha \cup \text{ch}_3(\mathbb{F})) \in H^2(M_1(X, \beta), \mathbb{Z}).
\]

Here \( \mathbb{F} \) is a universal sheaf, and \( \pi_X, \pi_M \) are the projections from \( X \times M_1(X, \beta) \) onto corresponding factors. In [CMT18], Maulik and the authors proposed the following conjecture, which gives a sheaf theoretical interpretation of genus zero GV type invariants on CY 4-folds and could be regarded as an analogue of Katz conjecture [Kat08].

\(^1\) In this paper, a Calabi–Yau 4-fold is a complex smooth projective 4-fold \( X \) satisfying \( K_X \equiv \mathcal{O}_X \).
Conjecture 0.1 [CMT18, Conjecture 0.2]. For $\gamma \in H^4(X, \mathbb{Z})$, let $\langle \tau_0(\gamma) \rangle_\beta$ be the DT$_4$ invariant with primary insertions on $M_1(X, \beta)$, i.e.

$$\langle \tau_0(\gamma) \rangle_\beta := \int_{[M_1(X, \beta)]^{\text{vir}}} \tau_0(\gamma).$$

Then for a certain choice of an orientation on $M_1(X, \beta)$, we have

$$\langle \tau_0(\gamma) \rangle_\beta = n_{0, \beta}(\gamma). \quad (0.4)$$

Here $n_{0, \beta}(\gamma)$ is the $g=0$ Gopakumar–Vafa type invariant (0.3).

Here $[M_1(X, \beta)]^{\text{vir}} \in H_2(M_1(X, \beta), \mathbb{Z})$ is the DT$_4$-virtual class of $M_1(X, \beta)$, determined by a choice of an orientation [CGJ]. DT$_4$-virtual classes on moduli spaces of stable sheaves on CY 4-folds are constructed in general by Borisov–Joyce [BJ] via derived algebraic/C$^\infty$-geometry, and in some special cases by Cao–Leung [CL] via gauge theory and classical algebraic geometry (see Sect. 1.2 for a brief review). We refer to [CGJ, CK18, CK19, CKM19, CKM20, CL17, CMT19, CT19, CT20a, CT20b] for some recent developments of such theory. As we discuss below, the main purpose of this paper is to use DT$_4$-theory together with descendent insertions to give a conjectural sheaf theoretical interpretation of genus one GV type invariants.

0.3. Main conjecture. As we explained in Section 0.1, in order to give a sheaf theoretical interpretation of higher genus GV invariants on CY 3-folds, the theory of perverse sheaves of vanishing cycles plays a key role. However, such a theory is not feasible on CY 4-folds as moduli spaces of sheaves are no longer locally written as critical locus of functions on Zariski tangent spaces.

Our proposal is to use descendent DT$_4$ invariants to capture $g = 1$ GV type invariants on CY 4-folds. For $\alpha \in H^2(X, \mathbb{Z})$, the descendent insertion is defined by

$$\tau_1(\alpha) := (\pi_M)_*(\pi_X^* \alpha \cup \text{ch}_4(F)) \in H^2(M_1(X, \beta), \mathbb{Q}).$$

A subtlety here is that a universal sheaf $F$ is not unique, and $\text{ch}_4(F)$ may depend on its choice. We choose $F$ to be the normalized one (1.7) in the sense that the determinant of $R\pi_{M*}F$ is trivial. The descendent DT$_4$ invariants are defined by

$$\langle \tau_1(\alpha) \rangle_\beta := \int_{[M_1(X, \beta)]^{\text{vir}}} \tau_1(\alpha).$$

The following is our main conjecture of this paper:

Conjecture 0.2 (Conjecture 1.2). For a certain choice of an orientation on $M_1(X, \beta)$, there is an equality of functions of $\alpha \in H^2(X)$

$$\langle \tau_1(\alpha) \rangle_\beta = \frac{n_{0, \beta}(\alpha^2)}{2(\alpha \cdot \beta)} - \sum_{\beta_1 + \beta_2 = \beta} \frac{(\alpha \cdot \beta_1)(\alpha \cdot \beta_2)}{4(\alpha \cdot \beta)} m_{\beta_1, \beta_2} - \sum_{k \geq 1, k|\beta} \frac{(\alpha \cdot \beta)}{k} n_{1, \beta/k}. \quad (0.5)$$

Here $n_{0, \beta}(-)$ and $n_{1, \beta}$ are genus 0 and 1 GV type invariants (0.3) and $m_{\beta_1, \beta_2}$ are meeting invariants which can be inductively determined by genus 0 GV type invariants (see (1.4)).
Note that by the induction on the divisibility of $\beta$ together with the conjectural identity (0.4), the formula (0.5) in principle determines $n_{1,\beta}$ in terms of primary DT$_4$-invariants $\langle t_0(\gamma) \rangle_{\beta}$ and descendent DT$_4$-invariants $\langle t_1(\alpha) \rangle_{\beta}$. Therefore it gives a sheaf theoretical interpretation of $n_{1,\beta}$ from $M_1(X, \beta)$, in a different flavor from (0.2).

Also note that the LHS of the formula (0.5) is linear on $\alpha$, but the RHS is a priori rational function on $\alpha$. In order to make sense of the formula (0.5), we need to show the absence of pole in the RHS, which requires the following constraint

$$n_{0,\beta}(\alpha^2) = \frac{1}{2} \sum_{\beta_1 + \beta_2 = \beta \atop \beta_1, \beta_2 > 0} (\alpha \cdot \beta_1)(\alpha \cdot \beta_2) m_{\beta_1, \beta_2},$$

for any $\alpha \in H^2(X)$ with $\alpha \cdot \beta = 0$. This equality can be equivalently written using GW invariants (Proposition 1.4) and gives nontrivial constraints on GW invariants of Calabi–Yau 4-folds whose Picard numbers are bigger than one (Example 1.6). We prove that the above equality can be derived from the WDVV equation [W, DVV].

**Theorem 0.3** (Theorem 1.5). The identity (0.6) holds for any Calabi–Yau 4-fold. In particular, the formula in Conjecture 0.2 makes sense as an identity of linear functions on $\alpha \in H^2(X)$.

Our Conjecture 0.2 is written down using a heuristic argument in an ideal CY$_4$ geometry discussed in Sect. 1.5. At the moment, we are not able to fix the signs in front of genus one GV type invariants solely from the heuristic argument. The minus sign is in fact experimentally obtained by explicit computations of several examples in Sect. 2.

### 0.4. Verifications of Conjecture 0.2 in examples

We verify our main conjecture in several examples. The first example is an elliptic fibered CY 4-fold given by a Weierstrass model.

**Theorem 0.4** (Theorem 2.9). Let $\pi : X \rightarrow \mathbb{P}^3$ be an elliptic fibered Calabi–Yau 4-fold (2.1) given by a Weierstrass model, and $f = \pi^{-1}(p)$ be a generic fiber. Then Conjecture 0.2 holds for $\beta = r[f]$ with $r \geq 1$.

The proof of this case relies on a careful description of the normalized universal sheaf by Fourier–Mukai transforms (see Lemma 2.2–2.5).

The next example is the product of a CY 3-fold and an elliptic curve.

**Proposition 0.5** (Proposition 2.10). Let $X = Y \times E$ be the product of a smooth projective Calabi–Yau 3-fold $Y$ with an elliptic curve $E$. Then Conjecture 0.2 holds in the following cases

- $\beta = r[E]$ with $r \geq 1$,
- any $\beta \in H_2(Y, \mathbb{Z}) \subset H_2(X, \mathbb{Z})$.

The final examples are non-compact ones for small degree curve classes.

**Theorem 0.6** (Propositions 2.11, 2.12, 2.13). Conjecture 0.2 holds in the following cases:

- $X = \text{Tot}_{\mathbb{P}^2}(\mathcal{O}(-1) \oplus \mathcal{O}(-2))$ and $\beta = d[\mathbb{P}^1] \in H_2(X, \mathbb{Z})$ with $d \leq 3$.
- $X = \text{Tot}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}(-1, -1) \oplus \mathcal{O}(-1, -1))$ and $\beta = (d_1, d_2) \in H_2(X, \mathbb{Z})$ with $d_1, d_2 \leq 2$. 

• $X = K_{\mathbb{P}^3}$ and $\beta = d [\mathbb{P}^1] \in H_2(X, \mathbb{Z})$ with $d \leq 3$.

A common feature of the examples in Theorem 0.6 is that the CY 4-fold $X$ can be written as the total space of canonical bundle $K_Y$ of a (possibly non-compact) Fano 3-fold $Y$, and there is an isomorphism

$$i_* : M_1(Y, \beta) \cong M_1(X, \beta),$$

where $i : Y \hookrightarrow X$ is the zero section. Remarkably in this case, the DT$_4$ virtual class has a preferred choice of orientations:

$$[M_1(X, \beta)]^{vir} = (-1)^{c_1(Y) \cdot \beta - 1} \cdot [M_1(Y, \beta)]^{vir},$$

(0.7)

where the sign in the RHS is conjectured to give the correct choice of orientations which makes Conjecture 0.1 true (see [Cao, Conjecture 0.2]). We verify Conjecture 0.2 in the above examples using this particular choice of orientations.

Finally we remark that the proof of the $d = 3$ case for $X = K_{\mathbb{P}^3}$ requires an identification of the moduli space $M_1(X, 3)$ with the moduli space $P_1(X, 3)$ of PT stable pairs (see Proposition 2.13). By using the 4-fold stable pair vertex [CK19,CKM19], one can explicitly compute the invariant and obtain a matching. We thank Sergej Monavari for doing a computation for us using his Maple program. Other cases can be computed through explicit descriptions of moduli spaces and virtual classes, and are reduced to calculations using classical intersection theory.

1. Definitions and Conjectures

1.1. GV type invariants of CY 4-folds. Let $X$ be a Calabi–Yau 4-fold. The genus 0 Gromov–Witten invariants on $X$ are defined using insertions: for $\gamma \in H_4(X, \mathbb{Z})$, one defines

$$GW_{0,\beta}(\gamma) = \int_{[\overline{M}_{0,1}(X,\beta)]^{vir}} ev^*(\gamma) \in \mathbb{Q},$$

where $ev : \overline{M}_{0,1}(X, \beta) \to X$ is the evaluation map.

The genus 0 Gopakumar–Vafa type invariants

$$n_{0,\beta}(\gamma) \in \mathbb{Q}$$

(1.1)

are defined by Klemm–Pandharipande [KP] from the identity

$$\sum_{\beta > 0} GW_{0,\beta}(\gamma) q^{\beta} = \sum_{\beta > 0} n_{0,\beta}(\gamma) \sum_{d = 1}^{\infty} d^{-2} q^{d\beta}.$$ 

For the genus 1 case, virtual dimensions of moduli spaces of stable maps are zero, so Gromov–Witten invariants

$$GW_{1,\beta} = \int_{[\overline{M}_{1,0}(X,\beta)]^{vir}} 1 \in \mathbb{Q}$$

can be defined without insertions. The genus 1 Gopakumar–Vafa type invariants

$$n_{1,\beta} \in \mathbb{Q}$$

(1.2)
are defined in [KP] by the identity

\[
\sum_{\beta > 0} GW_{1,\beta} q^\beta = \sum_{\beta > 0} n_{1,\beta} \sum_{d=1}^\infty \frac{\sigma(d)}{d} q^{d\beta} + \frac{1}{24} \sum_{\beta > 0} n_{0,\beta} (c_2(X)) \log(1 - q^\beta) - \frac{1}{24} \sum_{\beta_1,\beta_2} m_{\beta_1,\beta_2} \log(1 - q^{\beta_1 + \beta_2}),
\]

where \(\sigma(d) = \sum_{i|d} i\) and \(m_{\beta_1,\beta_2} \in \mathbb{Z}\) are called meeting invariants defined as follows.

Take a basis \(S_1, \ldots, S_k\) of the free part of \(H^4(X, \mathbb{Z})\) and let

\[
\sum_{i,j} g^{ij} [S_i \otimes S_j] \in H^8(X \times X, \mathbb{Z})
\]

be the Künneth decomposition of the \((4, 4)\) component of the diagonal class (mod torsion). For \(\beta_1, \beta_2 \in H_2(X, \mathbb{Z})\), the meeting invariant \(^2\)

\[
m_{\beta_1,\beta_2} \in \mathbb{Z}
\]

is the virtual number of rational curves of class \(\beta_1\) meeting rational curves of class \(\beta_2\). They are uniquely determined by the following rules:

(i) The meeting invariants are symmetric, \(m_{\beta_1,\beta_2} = m_{\beta_2,\beta_1}\).
(ii) If either \(\deg(\beta_1) \leq 0\) or \(\deg(\beta_2) \leq 0\), we have \(m_{\beta_1,\beta_2} = 0\).
(iii) If \(\beta_1 \neq \beta_2\), then

\[
m_{\beta_1,\beta_2} = \sum_{i,j} n_{0,\beta_1}(S_i) g^{ij} n_{0,\beta_2}(S_j) + m_{\beta_1,\beta_2 - \beta_1} + m_{\beta_1 - \beta_2,\beta_2}.
\]

(iv) If \(\beta_1 = \beta_2 = \beta\), we have

\[
m_{\beta,\beta} = n_{0,\beta}(c_2(X)) + \sum_{i,j} n_{0,\beta}(S_i) g^{ij} n_{0,\beta}(S_j) - \sum_{\beta_1 + \beta_2 = \beta} m_{\beta_1,\beta_2}.
\]

In [KP], both of the invariants (1.1), (1.2) are conjectured to be integers, and Gromov–Witten invariants on \(X\) are computed to support the conjectures in many examples by localization technique or mirror symmetry. Note that the genus zero integrality conjecture has been proved by Ionel–Parker [IP, Theorem 9.2] using symplectic geometry.

1.2. Review of DT\(_4\) invariants. We fix an ample divisor \(\omega\) on \(X\) and take a cohomology class \(v \in H^*(X, \mathbb{Q})\). The coarse moduli space \(M_\omega(v)\) of \(\omega\)-Gieseker semistable sheaves \(E\) on \(X\) with \(\text{ch}(E) = v\) exists as a projective scheme. We always assume that \(M_\omega(v)\) is a fine moduli space, i.e. any point \([E] \in M_\omega(v)\) is stable and there is a universal family

\[
\mathcal{E} \in \text{Coh}(X \times M_\omega(v))
\]

flat over \(M_\omega(v)\). For instance, the moduli space of 1-dimensional stable sheaves \(E\) with \([E] = \beta, \chi(E) = 1\) and Hilbert schemes of closed subschemes satisfy this assumption [CK18,CMT18].

\(^2\) They are integers because of Ionel–Parker’s proof of genus zero integrality [IP, Theorem 9.2].
In [BJ,CL], under certain hypotheses, the authors construct a virtual class
\[ [M_{\omega}(v)]^{\text{vir}} \in H_{2-\chi(v,v)}(M_{\omega}(v), \mathbb{Z}), \tag{1.5} \]
where \( \chi(-,-) \) denotes the Euler pairing. Notice that this class could a priori be non-algebraic.

Roughly speaking, in order to construct such a class, one chooses at every point \([E] \in M_{\omega}(v)\), a half-dimensional real subspace
\[ \text{Ext}^2_+(E, E) \subseteq \text{Ext}^2(E, E) \]
of the usual obstruction space \( \text{Ext}^2(E, E) \), on which the quadratic form \( Q \) defined by Serre duality is real and positive definite. Then one glues local Kuranishi-type models of the form
\[ \kappa_+ = \pi_+ \circ \kappa : \text{Ext}^1(E, E) \to \text{Ext}^2_+(E, E), \]
where \( \kappa \) is the Kuranishi map for \( M_{\omega}(v) \) at \([E]\) and \( \pi_+ \) denotes projection on the first factor of the decomposition \( \text{Ext}^2(E, E) = \text{Ext}^2_+(E, E) \oplus \sqrt{-1} \cdot \text{Ext}^2_+(E, E) \).

In [CL], local models are glued in three special cases:

1. when \( M_{\omega}(v) \) consists of locally free sheaves only,
2. when \( M_{\omega}(v) \) is smooth,
3. when \( M_{\omega}(v) \) is a shifted cotangent bundle of a quasi-smooth derived scheme.

In each case, the corresponding virtual classes are constructed using either gauge theory or algebraic-geometric perfect obstruction theory.

The general gluing construction is due to Borisov–Joyce [BJ], based on Pantev–Toën–Vaquè–Vezzosi’s theory of shifted symplectic geometry [PTVV13] and Joyce’s theory of derived \( C^\infty \)-geometry. The corresponding virtual class is constructed using Joyce’s D-manifold theory (a machinery similar to Fukaya–Oh–Ohta–Ono’s theory of Kuranishi space structures used for defining Lagrangian Floer theory).

To construct the above virtual class (1.5) with coefficients in \( \mathbb{Z} \) (instead of \( \mathbb{Z}_2 \)), we need an orientability result for \( M_{\omega}(v) \), which can be stated as follows. Let
\[ \mathcal{L} := \det(\mathcal{R} \text{Hom}_{\pi_M}(\mathcal{E}, \mathcal{E})) \in \text{Pic}(M_{\omega}(v)), \quad \pi_M : X \times M_{\omega}(v) \to M_{\omega}(v) \]
be the determinant line bundle of \( M_{\omega}(v) \), which is equipped with the nondegenerate symmetric pairing \( Q \) induced by Serre duality. An orientation of \( (\mathcal{L}, Q) \) is a reduction of its structure group from \( \text{O}(1, \mathbb{C}) \) to \( \text{SO}(1, \mathbb{C}) = \{1\} \). In other words, we require a choice of square root of the isomorphism
\[ Q : \mathcal{L} \otimes \mathcal{L} \to \mathcal{O}_{M_{\omega}(v)}. \]
Existence of orientations is first proved when the Calabi–Yau 4-fold \( X \) satisfies \( \text{Hol}(X) = \text{SU}(4) \) and \( H^{\text{odd}}(X, \mathbb{Z}) = 0 \) [CL17], and is recently generalized to arbitrary Calabi–Yau 4-folds [CGJ]. Notice that the collection of orientations forms a torsor for \( H^0(M_{\omega}(v), \mathbb{Z}_2) \).

Examples computed in Sect. 2 only involve virtual class constructions in (2) and (3) mentioned above. We briefly review them (modulo discussions on choices of orientations) as follows:
When $M_\omega(v)$ is smooth, the obstruction sheaf $\text{Ob} \to M_\omega(v)$ is a vector bundle endowed with a quadratic form $Q$ via Serre duality. Then the virtual class is given by

$$[M_\omega(v)]^{\text{vir}} = \text{PD}(e(\text{Ob}, Q)).$$

Here $e(\text{Ob}, Q)$ is the half-Euler class of $(\text{Ob}, Q)$ (i.e. the Euler class of its real form $\text{Ob}_+$, the orientation of $(L, Q)$ in this case is equivalent to the orientation of $\text{Ob}_+$), and $\text{PD}(\cdot)$ denotes Poincaré dual. Note that the half-Euler class satisfies

$$e(\text{Ob}, Q)^2 = (-1)^{\frac{\text{rk}(\text{Ob})}{2}} e(\text{Ob}), \quad \text{if } \text{rk}(\text{Ob}) \text{ is even},$$

$$e(\text{Ob}, Q) = 0, \quad \text{if } \text{rk}(\text{Ob}) \text{ is odd}.$$  

Suppose $M_\omega(v)$ is the classical truncation of the shifted cotangent bundle of a quasi-smooth derived scheme. Roughly speaking, this means that at any closed point $[E] \in M_\omega(v)$, we have a Kuranishi map of the form

$$\kappa : \text{Ext}^1(E, E) \to \text{Ext}^2(E, E) = V_E \oplus V_E^*,$$

where $\kappa$ factors through a maximal isotropic subspace $V_E$ of $(\text{Ext}^2(E, E), Q)$. Then the virtual class of $M_\omega(v)$ is, roughly speaking, the virtual class of the perfect obstruction theory formed by $\{V_E\}_{E \in M_\omega(v)}$. When $M_\omega(v)$ is furthermore smooth as a scheme, then it is simply the Euler class of the vector bundle $\{V_E\}_{E \in M_\omega(v)}$ over $M_\omega(v)$.  

1.3. Sheaf theoretical interpretation with primary insertions. In [CMT18], a sheaf theoretical interpretation of genus zero GV type invariants (1.1) is proposed using DT$_4$ invariants.

To be precise, let $M_1(X, \beta)$ be the moduli scheme of one dimensional stable sheaves $F$ on $X$ with $[F] = \beta \in H_2(X, \mathbb{Z})$ and $\chi(F) = 1$. There exists a virtual class (see [CT19, Section 3.2]):

$$[M_1(X, \beta)]^{\text{vir}} \in H_2(M_1(X, \beta), \mathbb{Z}),$$

in the sense of Borisov–Joyce [BJ], which depends on the choice of an orientation [CGJ]. For integral classes $\gamma \in H^4(X, \mathbb{Z})$, consider insertions:

$$\tau_0 : H^4(X, \mathbb{Z}) \to H^2(M_1(X, \beta), \mathbb{Z}), \quad \tau_0(\gamma) := (\pi_M)_*(\pi_X^*\gamma \cup \text{ch}_3(F)),$$

(1.6)

where $\pi_X$, $\pi_M$ are projections from $X \times M_1(X, \beta)$ to corresponding factors, $F$ is a universal sheaf unique up to a twist of a line bundle from $M_1(X, \beta)$.

Note that $\text{ch}_3(F)$ is the Poincaré dual to the fundamental cycle of $F$ which is independent of the choice of $F$. We can define DT$_4$ invariants with primary insertions on $M_1(X, \beta)$:

$$\langle \tau_0(\gamma) \rangle_{\beta} := \int_{[M_1(X, \beta)]^{\text{vir}}} \tau_0(\gamma).$$

In [CMT18], this invariant is proposed to recover genus zero GV type invariants.
**Conjecture 1.1 ([CMT18, Conjecture 0.2]).** For a certain choice of an orientation on \( M_1(X, \beta) \), we have

\[
\langle \tau_0(\gamma) \rangle_\beta = n_{0, \beta}(\gamma),
\]

where \( n_{0, \beta}(\gamma) \) is the \( g=0 \) Gopakumar–Vafa type invariant (1.1).

In [CMT19], the authors also proposed how to recover all genus GV type invariants via stable pair theory which is a CY 4-fold analogy of Pandharipande–Thomas’ work [PT09] on CY 3-folds. The motivation of this work is to give a sheaf theoretical interpretation of genus one invariants via moduli spaces of one dimensional stable sheaves, which is in spirit an analogy of the work of Gopakumar–Vafa [GV] and Maulik–Toda [MT18].

1.4. \( DT_4 \) descendent invariants and conjecture. Of course, the theory of sheaves of vanishing cycles is infeasible on Calabi–Yau 4-folds. To capture genus one GV type invariants, we propose to use the insertion (1.6) involving higher Chern characters (\( ch_4 \)).

To fix the ambiguity in choosing a universal sheaf, we define

\[
F_{\text{norm}} := F \otimes \pi_M^*(\text{det}(\pi_M F))^{-1}.
\] (1.7)

It is independent of the choice of \( F \) and its push-forward to \( M_1(X, \beta) \) has trivial determinant. For \( \alpha \in H^2(X, \mathbb{Z}) \), we define (descendent) insertions

\[
\tau_1 : H^2(X, \mathbb{Z}) \to H^2(M_1(X, \beta), \mathbb{Q}), \quad \tau_1(\alpha) := (\pi_M)^*(\pi_X^*\alpha \cup ch_4(F_{\text{norm}})),
\]

and \( DT_4 \) invariants with descendent insertions:

\[
\langle \tau_1(\alpha) \rangle_\beta := \int_{[M_1(X, \beta)]^{\text{vir}}} \tau_1(\alpha).
\] (1.8)

**Conjecture 1.2.** For a certain choice of an orientation on \( M_1(X, \beta) \), there is an equality of functions of \( \alpha \in H^2(X) \)

\[
\langle \tau_1(\alpha) \rangle_\beta = \frac{n_{0, \beta}(\alpha^2)}{2(\alpha \cdot \beta)} - \sum_{\beta_1 + \beta_2 = \beta} \frac{(\alpha \cdot \beta_1)(\alpha \cdot \beta_2)}{4(\alpha \cdot \beta)} m_{\beta_1, \beta_2} - \sum_{k \geq 1, k|\beta} \frac{(\alpha \cdot \beta)}{k} n_{1, \beta/k}.
\]

Here \( n_{0, \beta}(\cdot) \) and \( n_{1, \beta} \) are genus 0 and 1 GV type invariants (1.1), (1.2) and \( m_{\beta_1, \beta_2} \) are meeting invariants (1.4) which can be inductively determined by genus 0 GV type invariants.

We believe the choices of orientations in Conjecture 1.1 and Conjecture 1.2 are the same when moduli spaces are connected. This will be checked in examples considered in Sect. 2.

Combining Conjectures 1.1 and 1.2, all genus GV type invariants can be interpreted in terms of \( DT_4 \) counting invariants of one dimensional stable sheaves.
1.5. Heuristic argument. In this section, we justify Conjecture 1.2 in an ideal situation:

let $X$ be an ‘ideal’ CY$_4$ in the sense that all curves of $X$ deform in families of expected dimensions, and have expected generic properties, i.e.

1. any rational curve in $X$ is a chain of smooth $\mathbb{P}^1$’s with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1, -1, 0)$, and moves in a compact 1-dimensional smooth family of embedded rational curves, whose general member is smooth with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1, -1, 0)$.

2. any elliptic curve $E$ in $X$ is smooth, super-rigid, i.e. the normal bundle is $L_1 \oplus L_2 \oplus L_3$ for general degree zero line bundle $L_i$ on $E$ satisfying $L_1 \otimes L_2 \otimes L_3 = \mathcal{O}_E$.

Furthermore any two elliptic curves are disjoint and disjoint from all families of rational curves on $X$.

3. there is no curve in $X$ with genus $g \geq 2$.

For a fixed curve class $\beta$, let $D_\beta$ be the total space of one dimensional family of rational curves in class $\beta$. It has a fibration

$$\pi: D_\beta \to M_\beta$$

to the (one dimensional) moduli space $M_\beta$ of rational curves in class $\beta$. In the ideal situation, most fibers of $\pi$ are smooth $\mathbb{P}^1$ with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1, -1, 0)$ except finitely many reducible fibers. As in [KP, Sect. 2.3], we may assume $D_\beta$ is modelled on the blow-up of a $\mathbb{P}^1$-bundle over finitely many general points. We first determine its contribution to our DT$_4$ descendent invariant. The universal family of $M_1(X, \beta)$ is $j_\ast \mathcal{O}_{D_\beta}$ for the embedding

$$j: D_\beta \hookrightarrow X \times M_\beta.$$

It is obvious that its push-forward to the moduli space has trivial determinant. By the Grothendieck–Riemann–Roch formula, we have

$$\text{ch}_4(j_\ast \mathcal{O}_{D_\beta}) = -\frac{1}{2} c_1(\omega_\pi).$$

Therefore for the family of rational curves (with respect to the orientation in [CMT18], i.e. taking the usual fundamental class of the moduli space), we have

$$\int_{M_\beta} \tau_1(\alpha) = -\frac{1}{2} \int_{D_\beta} c_1(\omega_\pi) \cdot \alpha|_{D_\beta}.$$ 

Below we explicitly compute this intersection number. Suppose that $D_\beta$ is the blow-up of general $k$-points of a $\mathbb{P}^1$-bundle. Then

$$H^2(D_\beta, \mathbb{Q}) = \mathbb{Q}\langle \beta, \psi, \beta_1^{(1)}, \ldots, \beta_1^{(k)} \rangle$$

is generated by $k+2$ elements, where $\beta$ is the fiber class, $\beta_1^{(i)}$ are the classes of exceptional curves, $\psi = c_1(\omega_\pi)$ is the first Chern class of the relative cotangent bundle of $\pi$.

We have the following intersection numbers

$$\beta \cdot \beta = \beta_1^{(i)} \cdot \beta = 0, \quad \beta \cdot \psi = -2, \quad \beta_1^{(i)} \cdot \psi = (\beta_1^{(i)})^2 = -1.$$ 

The restriction of $\alpha$ to $D_\beta$ satisfies

$$\alpha|_{D_\beta} = a\beta + b\psi + \sum_{i=1}^{k} d_i \beta_1^{(i)}.$$
for some $a, b, d_i \in \mathbb{Q}$. We have the following equations

$$(\alpha|_{D_{\beta}})^2 = b^2\psi^2 - \sum_{i=1}^{k} d_i^2 - 4ab - 2b \sum_{i=1}^{k} d_i,$$  \hspace{1cm} (1.9)

$$- \frac{1}{2} \psi \cdot \alpha|_{D_{\beta}} = a - \frac{b}{2} \psi^2 + \frac{1}{2} \sum_{i=1}^{k} d_i,$$  \hspace{1cm} (1.10)

$$\alpha \cdot \beta = -2b, \quad \alpha \cdot \beta_1^{(i)} = -b - d_i.$$  \hspace{1cm} (1.11)

We set $\beta_2^{(i)} = \beta - \beta_1^{(i)}$. From equation (1.11), we can solve $b, d_i$ and obtain

$$b = -\frac{1}{2} \alpha \cdot \beta, \quad d_i = \frac{1}{2} \alpha \cdot (\beta_2^{(i)} - \beta_1^{(i)}).$$

By substituting into (1.10), we can solve $a$ and obtain

$$a = -\frac{1}{2} \psi \cdot \alpha|_{D_{\beta}} - \frac{\psi^2}{4} (\alpha \cdot \beta) + \frac{1}{4} \sum_{i=1}^{k} \alpha \cdot (\beta_2^{(i)} - \beta_1^{(i)}).$$

By substituting these into (1.9), we obtain

$$- \frac{1}{2} \psi \cdot \alpha|_{D_{\beta}} = \frac{(\alpha|_{D_{\beta}})^2}{2(\alpha \cdot \beta)} + \frac{(\alpha \cdot \beta)}{8} \psi^2 + \sum_{i=1}^{k} \frac{(\alpha \cdot \beta_1^{(i)} - \alpha \cdot \beta_2^{(i)})^2}{8(\alpha \cdot \beta)} \cdot m_{\beta_1, \beta_2}.$$  \hspace{1cm} (1.12)

However some $\beta_1^{(i)}$ may be equal to $\beta_2^{(j)}$ in the ambient space $X$, and the number of times this could happen is given by the meeting invariant $m_{\beta_1, \beta_2}$ for such $\beta_1 = \beta_1^{(i)}$ and $\beta_2 = \beta_2^{(j)}$ (ref. [KP, pp. 10]). We also need to divide it by two in order to modulo the symmetry between $\beta_1$ and $\beta_2 = \beta - \beta_1$. Therefore, the above formula should be adjusted to be

$$- \frac{1}{2} \psi \cdot \alpha|_{D_{\beta}} = \frac{(\alpha|_{D_{\beta}})^2}{2(\alpha \cdot \beta)} + \frac{(\alpha \cdot \beta)}{8} \psi^2 + \sum_{\beta_1 + \beta_2 = \beta} \frac{(\alpha \cdot \beta_1 - \alpha \cdot \beta_2)^2}{8(\alpha \cdot \beta)} \cdot m_{\beta_1, \beta_2}.$$  \hspace{1cm} (1.13)

From [KP, pp. 11], we have

$$\psi^2 = -\frac{1}{2} \sum_{\beta_1 + \beta_2 = \beta} m_{\beta_1, \beta_2}.$$  \hspace{1cm} (1.14)

Therefore we obtain

$$- \frac{1}{2} \psi \cdot \alpha|_{D_{\beta}} = \frac{(\alpha|_{D_{\beta}})^2}{2(\alpha \cdot \beta)} - \sum_{\beta_1 + \beta_2 = \beta} \frac{(\alpha \cdot \beta_1)(\alpha \cdot \beta_2)}{4(\alpha \cdot \beta)} \cdot m_{\beta_1, \beta_2}.$$  \hspace{1cm} (1.15)

Thus considering only contribution from the rational curves family, we have

$$\langle \tau_1(\alpha) \rangle_{\beta} = \frac{n_{0, \beta}(\alpha^2)}{2(\alpha \cdot \beta)} - \sum_{\beta_1 + \beta_2 = \beta} \frac{(\alpha \cdot \beta_1)(\alpha \cdot \beta_2)}{4(\alpha \cdot \beta)} \cdot m_{\beta_1, \beta_2}.$$
Next, we determine the contribution from a super-rigid elliptic curve $E$. The moduli space $M_1(X, \beta)$ of one dimensional stable sheaves which are supported on $E$ with $[E] = \beta$ is $\text{Pic}^1(E)$, which is identified with $E$ by the map $x \mapsto O_E(x)$ for $x \in E$. It has a normalized universal sheaf

$$F = j_* \mathcal{O}_{E \times E}(\Delta) \in \text{Coh}(E \times E),$$

where $\Delta \subset E \times E$ is the diagonal, $j : E \times E \to X \times E$ is the product of the natural inclusion $E \hookrightarrow X$ and the identity map on $E$. For $\alpha \in H^2(X, \mathbb{Z})$, a direct calculation gives

$$\int_{\text{Pic}^1(E)} \tau_1(\alpha) = \alpha \cdot [E] = \alpha \cdot \beta.$$

As for $r \geq 1$ and $r|\beta$ (i.e. $\beta/r \in H_2(X, \mathbb{Z})$) and a super-rigid elliptic curve $E$ with $[E] = \beta/r$, the moduli space of one dimensional stable sheaves supported on $E$ is the moduli space $M_E(r, 1)$ of rank $r$ degree one stable bundles on $E$. The determinant map defines an isomorphism

$$\det : M_E(r, 1) \sim \to M_E(1, 1) = \text{Pic}^1(E) \sim \to E.$$

Under the above isomorphism, the normalized universal sheaf $F$ has the following $K$-theory class

$$[F] = [j_* \mathcal{O}_{E \times E}(\Delta)] + (r-1)[j_* \mathcal{O}_{E \times E}].$$

For example, the above identity follows from a simpler version of the argument in Lemma 2.6. Therefore we similarly get

$$\int_{M_E(r, 1)} \tau_1(\alpha) = \frac{\alpha \cdot \beta}{r}.$$

Therefore all super-rigid elliptic curves contribute

$$\sum_{r|\beta} \frac{(\alpha \cdot \beta)}{r} \cdot n_{1, \beta/r}$$

to our descendant invariant $\langle \tau_1(\alpha) \rangle_{\beta}$ for some choice of orientations. Here we did not explain why minus sign in front of genus one GV type invariants comes as the right choice of orientations. In fact, this choice emerges from the experimental study of all examples in Sect. 2. It will be an interesting question to give an explanation of this choice in the above ideal geometry.

1.6. Constraints on genus zero GV type invariants. In order to make sense of the formula in Conjecture 1.2, the right hand side has to be a linear function on $\alpha$. A priori, the right hand side is a rational function with simple pole along the hyperplane $\alpha \cdot \beta = 0$, so we need to show the absence of pole of this rational function. It requires the following constraint

$$n_{0, \beta}(\alpha^2) = \frac{1}{2} \sum_{\beta_1 + \beta_2 = \beta \atop \beta_1, \beta_2 > 0} (\alpha \cdot \beta_1)(\alpha \cdot \beta_2) m_{\beta_1, \beta_2}, \quad (1.12)$$

for any $\alpha \in H^2(X)$ with $\alpha \cdot \beta = 0$. Below we show that the above formula can be derived from the WDVV equation. We first solve the recursion for the meeting numbers and express the identity (1.12) in terms of genus zero GV type invariants.
Lemma 1.3. The identity (1.12) is equivalent to the identity
\[
n_{0,\beta}(\alpha^2) = \frac{1}{2} \sum_{k_1, k_2 \in \mathbb{Z}_{\geq 0}} \sum_{\substack{g.c.d.(k_1, k_2) = 1 \ k_1 | \beta_1, k_2 | \beta_2}} \frac{(\alpha \cdot \beta_1)(\alpha \cdot \beta_2)}{k_1^2 k_2^2} n_{0,\beta_1/k_1}(S_a) g^{ab} n_{0,\beta_2/k_2}(S_b).
\]
\[
(1.13)
\]
Here \(g.c.d.(k_1, k_2)\) denotes the greatest common divisor of \(k_1\) and \(k_2\). \(\{S_a\}\) is a basis of the free part of \(H^4(X, \mathbb{Z})\) and \((g^{ab})\) is the inverse matrix of \((g_{ab})\) given in (1.3).

Proof. For each \(k_1, \ k_2\) with \(g.c.d.(k_1, k_2) = 1\), we set
\[
a_{k_1, k_2} := \sum_{\substack{\beta_1 + \beta_2 = \beta \ \ k_1 | \beta_1, k_2 | \beta_2}} \frac{1}{k_1^2 k_2^2} (\alpha \cdot \beta_1)(\alpha \cdot \beta_2) m_{\beta_1/k_1, \beta_2/k_2},
\]
\[
b_{k_1, k_2} := \sum_{\substack{\beta_1 + \beta_2 = \beta \ \ k_1 | \beta_1, k_2 | \beta_2}} \sum_{a, b} \frac{1}{k_1^2 k_2^2} (\alpha \cdot \beta_1)(\alpha \cdot \beta_2) n_{0,\beta_1/k_1}(S_a) g^{ab} n_{0,\beta_2/k_2}(S_b).
\]
Since \(\alpha \cdot \beta = 0\), the only terms of \(\beta_1, \beta_2\) which are non-proportional contribute to the above sums. Therefore by the recursion for the meeting numbers (1.4), we obtain
\[
a_{k_1, k_2} - b_{k_1, k_2} = \sum_{\beta_1 + \beta_2 = \beta \ \ k_1 | \beta_1, k_2 | \beta_2} \frac{1}{k_1^2 k_2^2} (\alpha \cdot \beta_1)(\alpha \cdot \beta_2) m_{\beta_1/k_1 - \beta_2/k_2, \beta_2/k_2} + \sum_{\substack{\beta_1 + \beta_2 = \beta \ \ k_1 | \beta_1, k_2 | \beta_2}} \frac{1}{k_1^2 k_2^2} (\alpha \cdot \beta_1)(\alpha \cdot \beta_2) m_{\beta_1/k_1, \beta_2/k_2 - \beta_1/k_1}.
\]
\[
(1.14)
\]
In the above first sum, we set
\[
\beta_1' := k_1 \left( \frac{\beta_1}{k_1} - \frac{\beta_2}{k_2} \right), \quad \beta_2' := (k_1 + k_2)^2 \frac{\beta_2}{k_2}.
\]
Then we have \(k_1 | \beta_1', (k_1 + k_2) | \beta_2',\) and
\[
\beta_1' + \beta_2' = \beta, \quad m_{\beta_1/k_1 - \beta_2/k_2, \beta_2/k_2} = m_{\beta_1'/k_1, \beta_2'/k_1 + k_2}.
\]
Conversely, suppose that \(\beta_1' + \beta_2' = \beta\) is given such that \(k_1 | \beta_1'\) and \((k_1 + k_2) | \beta_2'.\) Then we have \(\beta_1 + \beta_2 = \beta\), where \(\beta_1, \beta_2\) are given by
\[
\beta_1 = k_1 \left( \frac{\beta_1'}{k_1} + \frac{\beta_2'}{k_1 + k_2} \right), \quad \beta_2 = k_2 \frac{\beta_2'}{k_1 + k_2},
\]
\[
(1.15)
\]
which satisfy \(k_1 | \beta_1, k_2 | \beta_2\). Therefore we can write the first sum in (1.14) as a sum over \(\beta_1' + \beta_2' = \beta\) with \(k_1 | \beta_1'\) and \((k_1 + k_2) | \beta_2'.\) Moreover using the relation \(\alpha \cdot \beta = 0\) and (1.15), we have
\[
\frac{1}{k_1^2 k_2^2} (\alpha \cdot \beta_1)(\alpha \cdot \beta_2) = \frac{1}{k_1^2 (k_1 + k_2)^2} (\alpha \cdot \beta_1')(\alpha \cdot \beta_2').
\]
Therefore the first sum of (1.14) is
\[
\sum_{\beta'_1 + \beta'_2 = \beta \atop k_1 | \beta'_1, (k_1 + k_2) | \beta'_2} \frac{1}{k_1^2 (k_1 + k_2)^2} (\alpha \cdot \beta'_1) (\alpha \cdot \beta'_2) m_{\beta'_1/k_1, \beta'_2/(k_1 + k_2)} = a_{k_1, k_1 + k_2}.
\]
Similarly the second sum in (1.14) equals to \(a_{k_1 + k_2, k_2}\). Therefore we obtain the relation
\[
a_{k_1, k_2} = b_{k_1, k_2} + a_{k_1, k_1 + k_2} + a_{k_1 + k_2, k_2}.
\]
Since the right hand side of (1.12) is \(a_{1,1}/2\), the identity (1.12) is equivalent to
\[
n_{0, \beta}(\alpha^2) = \frac{1}{2} \sum_{k_1, k_2 \in \mathbb{Z}_{>0}} b_{k_1, k_2}.
\]
Here in the above sum, \((k_1, k_2)\) are those which appear by iterated operations of either \((k_1, k_2) \mapsto (k_1, k_1 + k_2)\) or \((k_1, k_2) \mapsto (k_1 + k_2, k_2)\). By an elementary Euclidean algorithm argument, every coprime pair \((k_1, k_2)\) appears exactly once by the above operations. Therefore the identity (1.12) is equivalent to (1.13).

By the above lemma, we can give an equivalent form of (1.12) in terms of genus zero Gromov–Witten invariants.

**Proposition 1.4.** The identity (1.12) for all \((\alpha, \beta)\) with \(\alpha \cdot \beta = 0\) is equivalent to the following identity for all \((\alpha, \beta)\) with \(\alpha \cdot \beta = 0\),
\[
GW_{0, \beta}(\alpha^2) = \frac{1}{2} \sum_{\beta'_1 + \beta'_2 = \beta} \sum_{a, b} (\alpha \cdot \beta_1)(\alpha \cdot \beta_2) GW_{0, \beta_1}(S_a) g^{ab} GW_{0, \beta_2}(S_b),
\]
(1.16)
where \(\{S_a\}\) is a basis of the free part of \(H^4(X, \mathbb{Z})\).

**Proof.** Suppose that (1.12) holds for all \((\alpha, \beta)\) with \(\alpha \cdot \beta = 0\). Then by Lemma 1.3, we have
\[
GW_{0, \beta}(\alpha^2) = \sum_{k \geq 1, k \mid \beta} \frac{1}{k^2} n_{0, \beta/k}(\alpha^2)
\]
\[
= \frac{1}{2} \sum_{k \geq 1, k \mid \beta} \sum_{k_1 \in \mathbb{Z}_{>0}, k_2 \in \mathbb{Z}_{>0}} \sum_{\text{g.c.d.}(k_1, k_2) = 1} \sum_{a, b} \frac{1}{k_1^2 k_2^2}
\]
\[
(\alpha \cdot \beta_1)(\alpha \cdot \beta_2) n_{0, \beta_1/k_1}(S_a) g^{ab} n_{0, \beta_2/k_2}(S_b)
\]
\[
= \frac{1}{2} \sum_{k'_1 \in \mathbb{Z}_{>0}, k'_2 \in \mathbb{Z}_{>0}} \sum_{\text{g.c.d.}(k'_1, k'_2) = 1} \sum_{a, b} \frac{1}{k'^2_1 k'^2_2}
\]
\[
(\alpha \cdot \beta'_1)(\alpha \cdot \beta'_2) n_{0, \beta'_1/k'_1}(S_a) g^{ab} n_{0, \beta'_2/k'_2}(S_b).
\]
Here we have set \(\beta'_1 = k \beta_1, \beta'_2 = k \beta_2\) and \(k'_1 = k k_1, k'_2 = k k_2\) in the third identity, where \(\text{g.c.d.}(k'_1, k'_2) = k\) so that there is no constraint on \(\text{g.c.d.}(k'_1, k'_2)\) in the third identity. By the multiple cover formula, the last identity gives the desired identity (1.16).

Conversely suppose (1.16) holds for all \((\alpha, \beta)\) with \(\alpha \cdot \beta = 0\). Then by applying the above computation backwards, we can conclude (1.12) by an induction on the divisibility of \(\beta\). □
Theorem 1.5. The identity (1.12) holds for any Calabi–Yau 4-fold, i.e. the formula in Conjecture 1.2 is an identity of linear functions on $\alpha \in H^2(X)$.

Proof. We assume $H^{\text{odd}}(X, \mathbb{Q}) = 0$ for simplicity and the same argument works in the general case. Let $S_0 = 1, S_1, \ldots, S_m = \text{[pt]}$ be a basis of $H^2(X, \mathbb{Q})$, where $S_1, \ldots, S_m \in H^2(X, \mathbb{Q})$, $S_{i+1}, \ldots, S_{m+1} \in H^2(X, \mathbb{Q})$ and $S_{i+1}, \ldots, S_{m+1} \in H^2(X, \mathbb{Q})$. Write $\gamma := \sum_{i=0}^{m} t_i S_i$ where $t_i$ are formal variables. The Gromov–Witten potential (ref. [CoxKatz, (8.29)]) is

$$\Phi := \sum_{n=0}^{\infty} \sum_{\beta \in H_2(X, \mathbb{Z})} \frac{1}{n!} GW_{0, \beta}(\gamma^{\otimes n}) q^\beta$$

$$= \frac{1}{6} \int_X \gamma^3 + \sum_{b > 0, 0 \neq \beta \in H_2(X, \mathbb{Z})} \frac{1}{b!} GW_{0, \beta} \left( \left( \sum_{i=r+1}^{m} t_i S_i \right)^{\otimes b} \right) e^{\int_\beta (\sum_{i=r+1}^{m} t_i S_i)} q^\beta$$

$$= \frac{1}{6} \int_X \gamma^3 + \sum_{i=r+1}^{m} t_i \sum_{\beta \in H_2(X, \mathbb{Z})} GW_{0, \beta}(S_i) e^{\int_\beta (\sum_{i=r+1}^{m} t_i S_i)} q^\beta,$$

where we use CY4 condition in the last equality.

The WDVV equation (ref. [W, DVV], [CoxKatz, Theorem 8.2.4]) is

$$\sum_{a,b} \Phi_{ij \alpha} g^{ab} \Phi_{kl} = \sum_{a,b} \Phi_{jk \alpha} g^{ab} \Phi_{bi},$$

where $\Phi_{abc} = \frac{\partial^3 \Phi}{\partial t_a \partial t_b \partial t_c}$ denotes the third partial derivative of $\Phi$ with respect to $t_a, t_b, t_c$.

Note that

$$\Phi_{11a} = \int_X S_1^2 S_a + \sum_{0 \neq \beta \in H_2(X, \mathbb{Z})} GW_{0, \beta}(S_a)(S_1 \cdot \beta)^2 e^{\int_\beta (\sum_{i=r+1}^{m} t_i S_i)} q^\beta,$$

$$\Phi_{22b} = \int_X S_2^2 S_b + \sum_{0 \neq \beta \in H_2(X, \mathbb{Z})} GW_{0, \beta}(S_b)(S_2 \cdot \beta)^2 e^{\int_\beta (\sum_{i=r+1}^{m} t_i S_i)} q^\beta,$$

$$\Phi_{12a} = \int_X S_1 S_2 S_a + \sum_{0 \neq \beta \in H_2(X, \mathbb{Z})} GW_{0, \beta}(S_a)(S_1 \cdot \beta)(S_2 \cdot \beta) e^{\int_\beta (\sum_{i=r+1}^{m} t_i S_i)} q^\beta.$$

By taking $i = j = 1, k = l = 2$ in the WDVV equation, we obtain

$$GW_{0, \beta} \left( ((S_1 \cdot \beta) S_2 - (S_2 \cdot \beta) S_1)^2 \right)$$

$$= \sum_{r+1 \leq a, b \leq l} GW_{0, \beta_1}(S_a) g^{ab} GW_{0, \beta_2}(S_b) (S_1 \cdot \beta_1)(S_2 \cdot \beta_1)(S_1 \cdot \beta_2)(S_2 \cdot \beta_2)$$

$$- \sum_{r+1 \leq a, b \leq l} GW_{0, \beta_1}(S_a) g^{ab} GW_{0, \beta_2}(S_b) (S_1 \cdot \beta_1)^2 (S_2 \cdot \beta_2)^2$$

$$= \sum_{r+1 \leq a, b \leq l} GW_{0, \beta_1}(S_a) g^{ab} GW_{0, \beta_2}(S_b) (S_1 \cdot \beta_1)(S_2 \cdot \beta_1)(S_1 \cdot \beta_2)(S_2 \cdot \beta_2)$$

$$- \sum_{r+1 \leq a, b \leq l} GW_{0, \beta_1}(S_a) g^{ab} GW_{0, \beta_2}(S_b) (S_1 \cdot \beta_1)^2 (S_2 \cdot \beta_1)^2.$$
Let $\alpha := (S_1 \cdot \beta)S_2 - (S_2 \cdot \beta)S_1 \in H^2(X)$. By taking sum of the above two equations, we obtain

$$\text{GW}_{0, \beta}(\alpha^2) = \frac{1}{2} \sum_{\beta_1 + \beta_2 = \beta} \sum_{a, b} (\alpha \cdot \beta_1)(\alpha \cdot \beta_2)\text{GW}_{0, \beta_1}(S_a)g^{ab}\text{GW}_{0, \beta_2}(S_b).$$

By considering other $i, j, k = l \in \{1, 2, \ldots, r\}$, we can conclude for any $\alpha \in H^2(X)$ with $\alpha \cdot \beta = 0$, the above equality holds. Combining with Proposition 1.4, we are done. 

$\square$

The following example shows that (1.12) gives nontrivial constraints on GW invariants of Calabi–Yau 4-folds.

**Example 1.6.** When $X = \text{Tot}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}(-1, -1) \oplus \mathcal{O}(-1, -1))$. The formula (1.12) (equivalently (1.16) by Proposition 1.4) gives a recursion formula for genus zero GW invariants of $X$:

$$\text{GW}_{0,(d_1,d_2)}([pt]) = \frac{1}{2} \sum_{d_1',d_1''=d_1 \atop d_2',d_2''=d_2} \frac{(d_1''d_1'' - d_1'd_1'')^2}{d_1d_2} \text{GW}_{0,(d_1',d_2')([pt])} \cdot \text{GW}_{0,(d_1'',d_2'')([pt])}.$$ 

In [KP, Proposition 3, pp. 23], GW invariants are explicitly computed by torus localization:

$$\text{GW}_{0,(d_1,d_2)}([pt]) = \frac{1}{(d_1 + d_2)^2} \left( \frac{d_1}{d_1} + \frac{d_2}{d_2} \right)^2.$$ 

Combining them, we obtain a nontrivial identity of binomial coefficients for any $d_1, d_2 \in \mathbb{Z}_{\geq 0}$.

$$\frac{2d_1d_2}{(d_1 + d_2)^2} \left( \frac{d_1}{d_1} + \frac{d_2}{d_2} \right)^2 = \sum_{d_1',d_1''=d_1 \atop d_2',d_2''=d_2} \frac{(d_1''d_1'' - d_1'd_1'')^2}{d_1'd_1''d_1''d_1''} \left( \frac{d_1'}{d_1'} + \frac{d_1''}{d_1''} \right) \left( \frac{d_2'}{d_2'} + \frac{d_2''}{d_2''} \right).$$

### 1.7. Other DT\textsubscript{4} descendent invariants.

In Conjecture 1.1, 1.2, we use DT\textsubscript{4} invariants with $\text{ch}_3$ and $\text{ch}_4$ insertions to give sheaf theoretical interpretations of all genus GV type invariants of Calabi–Yau 4-folds. One can also consider other descendent invariants, though they do not seem to contain more information.

For instance, with the normalized universal sheaf $\mathbb{F}_{\text{norm}}$ (1.7), we can consider the $\text{ch}_5$-descendent insertions

$$\tau_2 : H^0(X, \mathbb{Z}) \to H^2(M_1(X, \beta), \mathbb{Q}), \quad \tau_2(r) := (\pi_M)_*(\pi_X^*r \cup \text{ch}_5(\mathbb{F}_{\text{norm}})),$$

and their DT\textsubscript{4} invariants

$$(\tau_2(r))_{\beta} := \int_{[M_1(X, \beta)]^{vir}} \tau_2(r).$$

Using an argument in the ideal geometry (as in Sect. 1.5), we obtain the following conjectural formula

$$(\tau_2(1))_{\beta} = -\frac{1}{12}n_{0,\beta}(c_2(X)),$$
for the same choice of orientations as in Conjecture 1.1. One can easily check the formula holds in examples considered in Sect. 2 and the corresponding choice of orientations is unique and coincides with those in Conjectures 1.1 and 1.2.

### 2. Computations of Examples

#### 2.1. Elliptic fibrations.

For $Y = \mathbb{P}^3$, we take general elements

$$u \in H^0(Y, \mathcal{O}_Y(-4K_Y)), \quad v \in H^0(Y, \mathcal{O}_Y(-6K_Y)).$$

Let $X$ be a Calabi–Yau 4-fold with an elliptic fibration

$$\pi : X \to Y$$

given by the equation

$$zy^2 = x^3 + uxz^2 + vz^3$$

in the $\mathbb{P}^2$-bundle

$$\mathbb{P}(\mathcal{O}_Y(-2K_Y) \oplus \mathcal{O}_Y(-3K_Y) \oplus \mathcal{O}_Y) \to Y,$$

where $[x : y : z]$ are homogeneous coordinates for the above projective bundle. A general fiber of $\pi$ is a smooth elliptic curve, and any singular fiber is either a nodal or cuspidal plane curve. Moreover, $\pi$ admits a section $\iota$ whose image corresponds to the fiber point $[0 : 1 : 0]$.

Let $h$ be a hyperplane in $\mathbb{P}^3$, $f$ be a general fiber of $\pi : X \to Y$ and set

$$B =: \pi^*h, \quad D := \iota(\mathbb{P}^3) \in H_6(X, \mathbb{Z}).$$

We consider moduli spaces $M_1(X, r[f])$ of one dimensional stable sheaves on $X$ in the multiple fiber classes $r[f]$ ($r \geq 1$).

**Lemma 2.1** [CMT18, Lemma 2.1]. For any $r \in \mathbb{Z}_{\geq 1}$, there is an isomorphism

$$M_1(X, r[f]) \cong X,$$

under which the virtual class of $M_1(X, r[f])$ is given by

$$[M_1(X, r[f])]^\text{vir} = \pm \text{PD}(c_3(X)) \in H_2(X, \mathbb{Z}),$$

where the sign corresponds to a choice of orientations in defining the LHS.

By the isomorphism in Lemma 2.1, we have the equivalence

$$\text{Coh}(X \times M_1(X, r[f])) \sim \text{Coh}(X \times X).$$

Under the above equivalence, the normalized universal sheaf $\mathbb{F}^\text{norm}$ is a coherent sheaf on $X \times X$. In order to compute $\tau_1(\alpha)_{r[f]}$, we need to give an explicit description of the $K$-theory class of $\mathbb{F}^\text{norm}$. Let $\Phi$ be the functor defined by

$$\Phi(-) = \text{Cone}(\pi^*R\pi_*(-) \to (-)) : D^b(\text{Coh}(X)) \to D^b(\text{Coh}(X)).$$
Lemma 2.2. The functor $\Phi$ is an equivalence such that, for any point $y \in Y$, the following diagram commutes

$$
\begin{array}{ccc}
D^b(\text{Coh}(f_y)) & \xrightarrow{i_*} & D^b(\text{Coh}(X)) \\
\Phi_{f_y} & \downarrow & \Phi \\
D^b(\text{Coh}(f_y)) & \xrightarrow{i_*} & D^b(\text{Coh}(X)).
\end{array}
$$

(2.4)

Here $f_y = \pi^{-1}(y)$, $i : f_y \hookrightarrow X$ is the inclusion and $\Phi_{f_y}$ is the spherical twist with respect to $O_{f_y}$:

$$
\Phi_{f_y}(-) = \text{Cone}(R\Gamma(-) \otimes O_{f_y} \to (-)).
$$

Proof. Since we have the decomposition

$$
R\pi_*O_X = O_Y \oplus \omega_Y[-1],
$$

(2.5)

the object $O_X$ is EZ-spherical in the sense of [Horja, Definition 2.1]. Therefore the equivalence of $\Phi$ follows from [Horja, Theorem 2.11]. The commutative diagram (2.4) follows from the definition of $\Phi$. □

We denote by $\Phi^{\times r}$ the $r$-times composition of $\Phi$. We define the following object

$$
(\Phi^{\times r} \boxtimes \text{id}_X)(O_{\Delta_X})[-1] \in D^b(\text{Coh}(X \times X)),
$$

(2.6)

and regard it as a family of objects over the second factor of $X \times X$.

Lemma 2.3. The object (2.6) is a flat family of stable sheaves $F$ on $X$ with $[F] = r[f]$ and $\chi(F) = -1$.

Proof. Let us take $x \in X$ and set $y = f(x)$. By the diagram (2.4), we have

$$
(\Phi^{\times r} \boxtimes \text{id}_X)(O_{\Delta_X})|_{X \times \{x\}} = i_*\Phi_{f_y}^{\times r}(O_x).
$$

It is enough to show that $\Phi_{f_y}^{\times r}(O_x)[-1]$ is a stable sheaf on $f_y$ with rank $r$ and Euler characteristic $-1$. We prove the claim by an induction on $r$. For $r = 1$, we have

$$
\Phi_{f_y}(O_x) = O_{f_y}(-x)[1],
$$

so the claim is obvious. Suppose that the claim holds for $r$. By the induction hypothesis,

$$
R\Gamma(\Phi_{f_y}^{\times r}(O_x)[-1]) = \mathbb{C}[-1].
$$

So there is a non-split extension of sheaves on $f_y$,

$$
0 \to \Phi_{f_y}^{\times r}(O_x)[-1] \to \Phi_{f_y}^{\times r+1}(O_x)[-1] \to O_{f_y} \to 0.
$$

Since $\Phi_{f_y}^{\times r}(O_x)[-1]$ is stable and the above sequence is non-split, it is straightforward to check that $\Phi_{f_y}^{\times r+1}(O_x)[-1]$ is also stable. □
We take the shifted derived dual of the object (2.6):

$$\mathbb{F}^{(r)} := R\mathcal{H}om_{X \times X}((\Phi^{\times r} \otimes \text{id}_X)(\mathcal{O}_{\Delta_X}), \mathcal{O}_{X \times X})[4] \in D^b(\text{Coh}(X \times X)).$$

Again we regard it as a family of objects over the second factor of $X \times X$.

**Lemma 2.4.** The object $\mathbb{F}^{(r)}$ is a (non-normalized) universal family of one dimensional stable sheaves $E$ on $X$ with $[E] = r[f]$ and $\chi(E) = 1$.

**Proof.** By Lemma 2.3, the object $\mathbb{F}^{(r)}$ is a flat family of one dimensional stable sheaves $E$ on $X$ with $[E] = r[f]$ and $\chi(E) = 1$. Therefore it induces the morphism $X \to M_{r[f],1}$. Since $\Phi$ is an equivalence, the above map is injective on closed points and isomorphisms on tangent spaces. Since $M_{r[f],1} \cong X$ (Lemma 2.1), the above morphism must be an isomorphism. Therefore $\mathbb{F}^{(r)}$ is a universal object. $\Box$

We next determine the $K$-theory class of $\mathbb{F}^{(r)}$. We use the following diagram, where the middle square is a derived Cartesian

Here $\iota$ is the diagonal embedding and $p$ is the second projection.

**Lemma 2.5.** We have the following identity in the $K$-theory of $X \times X$

$$[\mathbb{F}^{(r)}] = -\left( \sum_{j=0}^{r-1} i_* \eta^* \omega_Y^\otimes j - \mathcal{O}_{\Delta_X} \right)^\vee.$$

**Proof.** By the definition of $\mathbb{F}^{(r)}$, it is enough to show the identity in the $K$-theory

$$(\Phi^{\times r} \otimes \text{id}_X)(\mathcal{O}_{\Delta_X}) = -\sum_{j=0}^{r-1} i_* \eta^* \omega_Y^\otimes j + \mathcal{O}_{\Delta_X}.$$

(2.7)

We prove (2.7) by the induction on $r$. For $r = 1$, since

$$(\pi \times \text{id}_X)^*(\pi \times \text{id}_X)_* \mathcal{O}_{\Delta_X} \cong (\pi \times \text{id}_X)^*(\pi \times \text{id}_X)_* p_* i_* \mathcal{O}_X$$

$$\cong i_* \mathcal{O}_{X \times Y},$$

thus we have

$$(\Phi \otimes \text{id}_X)(\mathcal{O}_{\Delta_X}) = i_* \mathcal{O}_{X \times Y}(-\Delta_X)[1],$$

where $\mathcal{O}_{X \times Y}(-\Delta_X) \subset \mathcal{O}_{X \times Y}$ is the ideal sheaf of the diagonal. So (2.7) holds for $r = 1$. 

Suppose that (2.7) holds for $r$. We have isomorphisms

$$(\pi \times \text{id}_X)^* (\pi \times \text{id}_X)_* i_* \eta^* \omega^j_Y \cong (\pi \times \text{id}_X)^* (\pi, \text{id}_X)_* R* \eta^* \omega^j_Y$$

$$\cong (\pi \times \text{id}_X)^* (\pi^* \omega^j_Y \oplus \pi^* \omega^{j+1}_Y[-1])$$

$$\cong i_* p^* (\pi^* \omega^j_Y \oplus \pi^* \omega^{j+1}_Y[-1]).$$

Here we have used (2.5) for the second isomorphism. It follows that the object $(\Phi \Box r+1 \times \text{id}_X)(\mathcal{O}_{\Delta_X}) = -r-1 \sum_{j=0}^{r-1} i_* \eta^* \omega^{j+1}_Y - i_* \mathcal{O}_X \times \mathcal{O}_{\Delta_X} \text{,}$

which implies (2.7) holds for $r+1$. □

We define $\mathbb{P}^{(r)}_{\text{norm}}$ to be

$$\mathbb{P}^{(r)}_{\text{norm}} := \mathbb{P}^{(r)} \otimes p^*_{M*} \pi^* \omega^r_Y \in \text{Coh}(X \times X).$$

Here $p_M : X \times X \to X$ is the second projection.

**Lemma 2.6.** The object $\mathbb{P}^{(r)}_{\text{norm}}$ is the normalized universal sheaf of one dimensional stable sheaves $E$ on $X$ with $[E] = r[r]$ and $\chi(E) = 1$.

**Proof.** By a similar computation in Lemma 2.5 and the Grothendieck duality, we have

$$\mathbb{R}p_{M*}(i_* \eta^* \omega^j_Y) \cong \pi^* \omega^{-j}_Y \oplus \pi^* \omega^{-j-1}_Y[1].$$

Therefore by Lemma 2.5 we have an identity in the $K$-theory:

$$\mathbb{R}p_{M*} \mathbb{P}^{(r)} = -\sum_{j=0}^{r-1} (\pi^* \omega^{-j}_Y - \pi^* \omega^{-j-1}_Y) + \mathcal{O}_X$$

$$\cong \pi^* \omega^{-r}_Y.$$  

Therefore the lemma holds from the definition of normalized universal sheaf. □

Using above lemmas, we can compute $\langle \tau_1(\alpha) \rangle_{r[1]}$ in the following.

**Proposition 2.7.** Let $B, D \in H^2(X, \mathbb{Z})$ be divisors in (2.2). For $\alpha = \alpha_1[D] + \alpha_2[B]$, we have

$$\langle \tau_1(\alpha) \rangle_{r[1]} = (20 - 1920r^2)\alpha_1 + 960\alpha_2, \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}.$$  

Here we have chosen the minus sign for the virtual class (2.3).

**Proof.** Using Lemma 2.5 and Grothendieck Riemann–Roch theorem, we can easily calculate the Chern character of $\mathbb{P}^{(r)}_{\text{norm}}$ as follows (we only describe the terms with $\text{ch}_3$ and $\text{ch}_4$)

$$\text{ch}(\mathbb{P}^{(r)}_{\text{norm}}) = -\left( \sum_{j=0}^{r-1} \text{ch}(i_* \eta^* \omega^j_Y) - \text{ch}(\mathcal{O}_{\Delta_X}) \right) \vee e^{-4rp^*_{M*} \pi^* h}.$$
Therefore, we have

\[ \text{Lemma 2.8.} \]

\[ \begin{aligned}
\gamma_r = \left( \sum_{j=0}^{r-1} i_*(e^{-4ij\eta^* h} \cdot \eta^* \tau_{X/Y}^{-1}) - \text{ch}(O_{\Delta X}) \right)^{\vee} \cdot e^{-4rp_M^* \pi^* h} \\
= - \left\{ \sum_{j=0}^{r-1} i_*(1 - 4j\eta^* h - 2\eta^* h + \ldots) \right\}^{\vee} \cdot e^{-4rp_M^* \pi^* h} \\
= \left( r[X \times Y] + 2r^2i_*\eta^* h + \Delta X + \ldots \right) \cdot e^{-4rp_M^* \pi^* h} \\
= r[X \times Y] - 2r^2i_*\eta^* h + \Delta X + \ldots.
\end{aligned} \]

Therefore we have

\[ p_M*(\text{ch}_4^{(r)}(\tau_{\text{norm}}) \cdot p_X^* \alpha) = \alpha_1(D - 2r^2B) + \alpha_2B. \]

The proposition follows from \( D \cdot c_3(X) = -20 \) and \( B \cdot c_3(X) = -960 \). Here we note that the divisor \( E \) in [KP, pp. 38–39] satisfies \( E = D + 4B \), then \( D \cdot c_3(X) = -20 \) follows from \( E \cdot c_3(X) = -3860 \) and \( B \cdot c_3(X) = -960. \)

We next compute the meeting numbers.

**Lemma 2.8.** For any \( r_1 > 0, r_2 > 0 \), we have

\[ m_{r_1[f], r_2[f]} = 46080. \quad (2.8) \]

**Proof.** We set \( \gamma_1 = D \cdot B \) and \( \gamma_2 = B^2 \). By [KP, Section 6.3], we have

\[ n_{0,r[f]}(\gamma_1) = 960r, \quad n_{0,r[f]}(\gamma_2) = 0. \]

As in [KP, Section 6.3], to compute \( n_{0, p}(\gamma) \) for some \( \gamma \in H^4(X) \), it is enough to take \( \gamma_1, \gamma_2 \) as a basis and consider projections of \( \gamma \) to them. Therefore, we write

\[ \Delta = \gamma_1 \otimes \gamma_2 + \gamma_2 \otimes \gamma_1 - 4\gamma_2 \otimes \gamma_2, \quad c_2(X) = 48\gamma_1 + 182\gamma_2. \quad (2.9) \]

We prove (2.8) by an induction on \( r_1 + r_2 \). If \( r_1 = r_2 = 1 \), we have

\[ m_{1[f], 1[f]} = n_{0,1[f]}(c_2(X)) = 48 \cdot 960 = 46080. \]

For general \((r_1, r_2)\), suppose that \( r_1 > r_2 \). Then because of \( n_{0, r_1[f]}(\gamma_2) = 0 \) and the description of the diagonal (2.9), we have

\[ m_{r_1[f], r_2[f]} = m_{(r_1 - r_2)[f], r_2[f]} = 46080, \]

by the induction hypothesis. The same applies to the case of \( r_1 < r_2 \). Finally for \( r_1 = r_2 = r \), using the induction hypothesis we obtain

\[ m_{r[f], r[f]} = n_{0, r[f]}(c_2(X)) - \sum_{r' > 0, r'' > 0} m_{r'[f], r''[f]} \]

\[ = 46080r - 46080(r - 1) \]

\[ = 46080. \]

Therefore the identity (2.8) holds. \( \square \)
**Theorem 2.9.** Conjecture 1.2 holds for $\beta = r[f]$ with $r \geq 1$.

**Proof.** For $\alpha = \alpha_1[D] + \alpha_2[B]$ with $\alpha_1, \alpha_2 \in \mathbb{R}$, we have
\[
\frac{n_{0,r[f]}(\beta^2)}{2(\alpha \cdot r[f])} - \frac{1}{4} \sum_{r_1+r_2=r, r_i \geq 2} \frac{(\alpha \cdot r_1[f])(\alpha \cdot r_2[f])}{(\alpha \cdot r[f])} m_{r_1[f],r_2[f]} - \sum_{k|r} (\alpha \cdot r[f]/k)n_{1,r[f]/k} = 960(-2\alpha_1 + \alpha_2) - \frac{1}{4} \sum_{r_1+r_2=r, r_i \geq 2} \frac{r_1\alpha_1 \cdot r_2\alpha_1}{r\alpha_1} \cdot 46080 + 20\alpha_1
\]
\[
= 960(-2\alpha_1 + \alpha_2) - \frac{1}{4} \cdot \frac{1}{6}(r^2 - 1)\alpha_1 \cdot 46080 + 20\alpha_1
\]
\[
= (20 - 1920r^2)\alpha_1 + 960\alpha_2.
\]
Here we have used $n_{1,[f]} = -20$ and $n_{1,r[f]} = 0$ for $r > 1$ from [KP, Table 7]. By comparing with Proposition 2.7, we obtain the theorem. \qed

2.2. Product of CY 3-fold with elliptic curve. We next consider the case of a CY 4-fold given by the product of a smooth projective CY 3-fold and an elliptic curve.

**Proposition 2.10.** Let $X = Y \times E$ be the product of a smooth projective Calabi–Yau 3-fold $Y$ with an elliptic curve $E$. Then Conjecture 1.2 holds in the following cases:

1. $\beta = r[E]$ with $r \geq 1$,
2. any $\beta \in H_2(Y, \mathbb{Z}) \subset H_2(X, \mathbb{Z})$.

**Proof.** (1) By [CMT19, Lemma 2.16], genus zero GW invariants vanish and
\[
n_{1,[E]} = \chi(Y), \quad n_{1,r[E]} = 0, \quad \text{if } r > 1.
\]
We have isomorphisms
\[
M_1(X, [E]) \cong Y \times \text{Pic}^1(E), \quad M_1(X, r[E]) = \emptyset, \quad \text{if } r > 1,
\]
whose virtual class satisfies
\[
[M_1(X, [E])]^{vir} = -\chi(Y) \cdot [\text{Pic}^1(E)], \quad [M_1(X, r[E])]^{vir} = 0, \quad \text{if } r > 1,
\]
for certain choice of orientations. Then Conjecture 1.2 obviously holds when $r > 1$.

As for $r = 1$ case, let $\Delta_Y \hookrightarrow Y \times Y$ and $\Delta_E \hookrightarrow E \times E$ be the diagonals. Then
\[
F = \mathcal{O}_{\Delta_Y} \boxtimes \mathcal{O}_{E \times E}(\Delta_E)
\]
is the normalized universal sheaf of $M_1(X, [E])$. By the Grothendieck–Riemann–Roch formula,
\[
\text{ch}_4(F) = [\Delta_Y \times \Delta_E].
\]
For $\alpha = \alpha_1 \oplus \alpha_2 \in H^2(Y) \oplus H^2(E)$, we have $\tau_1(\alpha) = \alpha_1 \oplus \alpha_2$. Therefore
\[
\int_{[M_1(X,[E])]^{vir}} \tau_1(\alpha) = -\chi(Y) \cdot \int_Y \alpha_1 = -(\alpha \cdot [E])n_{1,[E]}.
\]
By [CMT19, Lemma 2.16], \( n_{1,\beta} = 0 \). By [CMT18, Lemma 2.6], we have an isomorphism

\[ M_1(X, \beta) \cong M_1(Y, \beta) \times E, \]

under which the virtual class satisfies

\[ [M_1(X, \beta)]^{\text{vir}} = [M_1(Y, \beta)]^{\text{vir}} \otimes [E], \]

for certain choice of orientations.

Let \( \mathcal{F}_Y \) be the normalized universal sheaf of \( M_1(Y, \beta) \) and \( \Delta_E \hookrightarrow E \times E \) be the diagonal embedding. Then

\[ \mathcal{F} = \mathcal{F}_Y \boxtimes \mathcal{O}_{\Delta_E} \]

is the normalized universal sheaf of \( M_1(X, \beta) \) whose Chern character satisfies

\[ \text{ch}_3(\mathcal{F}) = \text{ch}_2(\mathcal{F}_Y) \cdot [\Delta_E], \quad \text{ch}_4(\mathcal{F}) = \text{ch}_3(\mathcal{F}_Y) \cdot [\Delta_E]. \]

Let \( \pi_{M_Y} : M_1(Y, \beta) \times Y \to M_1(Y, \beta) \) be the projection. Then

\[ \int_{[M_1(X, \beta)]^{\text{vir}}} \tau_1(\alpha) = \int_E \alpha_2 \cdot \int_{[M_1(Y, \beta)]^{\text{vir}}} (\pi_{M_Y})_* \text{ch}_3(\mathcal{F}_Y) \]

\[ = \int_E \alpha_2 \cdot \int_{[M_1(Y, \beta)]^{\text{vir}}} \text{ch}_0((\pi_{M_Y})_* \mathcal{F}_Y) \]

\[ = \left( \int_E \alpha_2 \right) \cdot \deg[M_1(Y, \beta)]^{\text{vir}}, \]

where the second equality is by the Grothendieck–Riemann–Roch formula.

By [CMT18, Corollary 2.7], for certain choice of orientations, we have

\[ \int_{[M_1(X, \beta)]^{\text{vir}}} \tau_0(\alpha^2) = 2 \left( \int_{\beta} \alpha \right) \cdot \left( \int_{E} \alpha_2 \right) \cdot \deg[M_1(Y, \beta)]^{\text{vir}}. \]

Note that meeting invariants vanish obviously in this product case, therefore Conjecture 1.2 holds in this case. \( \square \)

2.3. Two local surfaces. In this section, we consider local surfaces

\[ X = \text{Tot}_{\mathbb{P}^2}(\mathcal{O}(-1) \oplus \mathcal{O}(-2)), \quad \text{Tot}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}(-1, -1) \oplus \mathcal{O}(-1, -1)). \]

Although they are non-compact, the moduli spaces \( M_1(X, \beta) \) of one dimensional stable sheaves are smooth projective varieties (ref. [CMT18, Section 3.1]). So it makes sense to study Conjecture 1.2 for these examples.

**Proposition 2.11.** Let \( X = \text{Tot}_{\mathbb{P}^2}(\mathcal{O}(-1) \oplus \mathcal{O}(-2)) \) and \( \beta = d [\mathbb{P}^1] \in H_2(X, \mathbb{Z}). \) Then Conjecture 1.2 holds for \( d \leq 3. \)
Proof. We present the proof for $d = 3$ case (the $d = 1, 2$ cases are easier versions of the same approach). We first compute the meeting invariants, consider a compactification of $X$:

$$\overline{X} = \mathbb{P}(\mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus \mathcal{O})$$

with the projection $\pi : \overline{X} \to \mathbb{P}^2$ and a section $\iota : \mathbb{P}^2 \to \overline{X}$. Then

$$H^4(\overline{X}, \mathbb{Z}) = \mathbb{Z}(T_1, T_2, T_3),$$

where $T_1 = [\iota(\mathbb{P}^2)]$, $T_2 = [\pi^{-1}(\text{pt})]$ and $T_3$ is the class of a surface in the infinite divisor. Their intersection matrix is

$$g = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & * \end{pmatrix} \quad \text{with} \quad g^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & (*)^{-1} \end{pmatrix}.$$ 

For any $\beta \in H_2(X) \subset H_2(\overline{X})$, we have

$$n_{0, \beta}(T_1) = (T_1 \cdot T_1) \cdot n_{0, \beta}([\text{pt}]), \quad n_{0, \beta}(T_2) = n_{0, \beta}([\text{pt}]), \quad n_{0, \beta}(T_3) = 0.$$ 

Then

$$m_{1,1} = n_{0,1}(c_2(X)) + \sum n_{0,1}(T_i) g^{ij} n_{0,1}(T_j) = 6,$$

$$m_{1,2} = m_{1,1} + \sum n_{0,1}(T_i) g^{ij} n_{0,2}(T_j) = 4.$$ 

So for $\alpha = [\mathbb{P}^1] \in H_2(X)$, the RHS of Conjecture 1.2 is

$$\frac{n_{0,3}([\text{pt}])}{6} - \frac{1}{3} m_{1,2} - 3 n_{1,3} = \frac{3}{2},$$

where we use $n_{0,3}([\text{pt}]) = n_{1,3} = -1$ (ref. [KP, pp. 22]).

Next we compute the DT$_4$ descendent invariant. By [CMT18, Prop. 3.1], there is an isomorphism

$$M_1(\mathbb{P}^2, d) \xrightarrow{\cong} M_1(X, d) \quad (2.10)$$

given by push-forward along the zero section $\mathbb{P}^2 \hookrightarrow X$. Under the above isomorphism, the virtual class can be computed as the Euler class of a certain obstruction bundle on $M_1(\mathbb{P}^2, d)$.

More specifically, there is a natural support morphism to the linear system of degree 3 curves

$$M_1(\mathbb{P}^2, 3) \to |\mathcal{O}(3)| = \mathbb{P}^9.$$ 

Moreover, if we denote

$$\mathcal{C} \hookrightarrow \mathbb{P}^9 \times \mathbb{P}^2$$

to be the universal curve over this linear system, there is an isomorphism $\mathcal{C} \cong M_1(\mathbb{P}^2, 3)$ which sends the pair $(\mathcal{C}, p)$ to the dual (on $\mathcal{C}$) of the ideal sheaf $\mathcal{I}_{\mathcal{C}, p}$. 


Let $\mathcal{V}$ denote the vector bundle on $M_1(\mathbb{P}^2, 3)$ whose fiber at a point $[E]$ is

$$\text{Ext}^1_{\mathbb{P}^2}(E, E(-1)) = \mathbb{R}\text{Hom}_{\mathbb{P}^2}(I_{C, p}, I_{C, p}(-1))[1].$$

Its top Chern class can be computed via the diagram:

$$\begin{array}{ccc}
\mathcal{C} \times \mathbb{P}^2 & \xrightarrow{j} & \mathbb{P}^9 \times \mathbb{P}^2 \\
\pi_C & \square & \pi_{1, 2} \\
\mathcal{C} & \xrightarrow{j} & \mathbb{P}^9 \times \mathbb{P}^2.
\end{array}$$

By Beilinson’s resolution of diagonal, the $K$–theory class of the universal ideal sheaf $\mathcal{I}$ on $\mathcal{C} \times \mathbb{P}^2$ is given by the pullback $[\mathcal{I}] = j^*[\mathcal{F}]$, where $[\mathcal{F}]$ is the $K$-theory class of $\mathbb{P}^9 \times \mathbb{P}^2$ given by

$$[\mathcal{F}] = \pi^*_{1, 3}[\mathcal{O}_C] - \pi^*_{2, 3}[\mathcal{O}_\Delta] = 3[\mathcal{O}(0, -1, 0)] - [\mathcal{O}(-1, 0, -3)] - [\mathcal{O}(0, -1, 1)] - [\mathcal{O}(0, -2, -1)].$$

The $K$-theory class of the corresponding universal sheaf of $M_1(\mathbb{P}^2, 3)$ is the pull-back of $[\mathcal{F}^\vee \otimes \mathcal{O}(0, 0, -3)]$. Its push-forward to the moduli space $M_1(\mathbb{P}^2, 3)$ has determinant $\mathcal{O}(1, 0)$. Therefore we define normalized universal sheaf (as an element in $K$-theory):

$$[\mathcal{E}] = -[\mathcal{F}^\vee \otimes \mathcal{O}(0, 0, -3) \otimes \mathcal{O}(-1, 0, 0)]$$

$$= -3[\mathcal{O}(-1, 1, -3)] + [\mathcal{O}(0, 0, 0)] + [\mathcal{O}(-1, 1, -4)] + [\mathcal{O}(-1, 2, -2)].$$

From this, we obtain

$$[\mathcal{V}] = j^*\pi_{1, 2, *}(-[\mathcal{E}]^\vee \otimes [\mathcal{E}] \otimes \mathcal{O}(0, 0, -1)) = (H_1^7 \cdot H_2^2)|_C.$$ 

Moreover $\text{ch}_4(\mathcal{E})$ is written as

$$\text{ch}_4(\mathcal{E}) = -\frac{3}{2} H_1 \cdot H_3 + \text{other terms},$$

where the other terms do not contribute to the descendent DT$_4$-invariant. Therefore

$$\int_{[M_1(X, 3)]^{vir}} \tau_1([\mathbb{P}^1]) = (-1) \cdot \int_{\mathbb{P}^9 \times \mathbb{P}^2} (H_1^7 \cdot H_2^2)(-\frac{3}{2} H_1 \cdot H_3)(H_1 + 3H_2) = \frac{3}{2},$$

where the minus sign $(-1)^{c_1(Y) \cdot \beta - 1} = -1$ (here $Y = \text{Tot}_{\mathbb{P}^2}(\mathcal{O}(-1))$) is the preferred choice of orientations (0.7). Therefore Conjecture 1.2 holds in this case. \hfill $\square$

**Proposition 2.12.** Let $X = \text{Tot}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}(-1, 1) \oplus \mathcal{O}(-1, -1))$ and $\beta = (d_1, d_2) \in H_2(X, \mathbb{Z})$. Then Conjecture 1.2 holds for $d_1, d_2 \leq 2$.

**Proof.** Similar to the proof of Proposition 2.11, we present the proof for $(2, 2)$ class here. Consider a compactification of $X$:

$$\overline{X} = \mathbb{P}(\mathcal{O}(-1, -1) \oplus \mathcal{O}(-1, -1) \oplus \mathcal{O}),$$

with the projection $\pi: \overline{X} \to \mathbb{P}^1 \times \mathbb{P}^1$ and a section $i: \mathbb{P}^1 \times \mathbb{P}^1 \to \overline{X}$. Then

$$H^4(\overline{X}, \mathbb{Z}) = \mathbb{Z}(T_1, T_2, T_3).$$
where \( T_1 = [t(\mathbb{P}^1 \times \mathbb{P}^1)] \), \( T_2 = [\pi^{-1}(\text{pt})] \) and \( T_3 \) is the class of a surface in the infinite divisor. Their intersection matrix is

\[
g = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & * \end{pmatrix}
\]

with \( g^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & (\ast)^{-1} \end{pmatrix} \).

For any \( \beta \in H_2(X) \subset H_2(\overline{X}) \), we have

\[
n_{0, \beta}(T_1) = (T_1 \cdot T_1) \cdot n_{0, \beta}(\text{pt}), \quad n_{0, \beta}(T_2) = n_{0, \beta}(\text{pt}), \quad n_{0, \beta}(T_3) = 0.
\]

By the definition of meeting invariants, we have

\[
m_{(0,1),(2,1)} = 2n_{0,(0,1)}([\text{pt}])n_{0,(2,1)}([\text{pt}]) + 2n_{0,(0,1)}([\text{pt}])n_{0,(2,0)}([\text{pt}]) = 2,
\]

\[
m_{(0,2),(2,0)} = 2n_{0,(0,2)}([\text{pt}])^2 = 0,
\]

\[
m_{(1,0),(1,2)} = 2n_{0,(1,0)}([\text{pt}])n_{0,(1,2)}([\text{pt}]) + 2n_{0,(1,0)}([\text{pt}])n_{0,(0,2)}([\text{pt}]) = 2,
\]

\[
m_{(1,1),(1,1)} = -2n_{0,(1,1)}([\text{pt}]) + 2n_{0,(1,1)}([\text{pt}])^2 - 4n_{0,(1,0)}([\text{pt}])^2 = -4.
\]

For \( \alpha = (a, b) \in H^2(X) \), the RHS of Conjecture 1.2 is

\[
\frac{2ab \cdot n_{0,(2,2)}([\text{pt}])}{4(a + b)} - \frac{1}{8(a + b)} \sum_{\beta_1 + \beta_2 = (2,2)} (\alpha \cdot \beta_1)(\alpha \cdot \beta_2)m_{\beta_1, \beta_2} - 2(a + b)n_{1,(2,2)} = -2(a + b),
\]

where we use \( n_{0,(2,2)}([\text{pt}]) = 2, n_{1,(2,2)} = 1 \) (ref. [KP, pp. 24]).

Next, we compute the DT_4 descendent invariant. The method is similar to the proof of Proposition 2.11. Similarly to (2.10), we have an isomorphism

\[
M_1(\mathbb{P}^1 \times \mathbb{P}^1, \beta) \xrightarrow{\cong} M_1(X, \beta).
\]

For \( \beta = (2, 2) \), it is isomorphic to the (1, 2, 2) divisor

\[
C \hookrightarrow \mathbb{P}^8 \times \mathbb{P}^1 \times \mathbb{P}^1.
\]

We have a commutative diagram

\[
\begin{array}{ccc}
\mathbb{C} \times (\mathbb{P}^1 \times \mathbb{P}^1) & \xrightarrow{j} & \mathbb{P}^8 \times (\mathbb{P}^1 \times \mathbb{P}^1) \\
\pi_C & & \square & \pi_{1,2} \\
C & \xrightarrow{j} & \mathbb{P}^8 \times (\mathbb{P}^1 \times \mathbb{P}^1).
\end{array}
\]

The \( K \)-theory class of the normalized universal sheaf of \( M_1(X, \beta) \) is the pullback of the following \( K \)-theory class of \( \mathbb{P}^8 \times (\mathbb{P}^1 \times \mathbb{P}^1) \times (\mathbb{P}^1 \times \mathbb{P}^1) \),

\[
[\mathcal{E}] = [\mathcal{O}] - [\mathcal{O}(-1, 1, 0, -1, -2)] - [\mathcal{O}(-1, 0, 1, -2, -1)] + [\mathcal{O}(-1, 1, 1, -1, -1)].
\]

From this, we can compute the virtual class

\[
[M_1(X, (2, 2))]^{\text{vir}} = -2H_1^5H_2H_3|_C.
\]
for certain choice of orientations and
\[
\int_{[M_1(X, (2, 2))]^{vir}} \tau_1(\alpha) = (-1) \int_{P^8 \times P^1 \times P^1} (-2) H_1^6 H_2 H_3 (H_1 + 2 H_2 + 2 H_3) (a + b) (-H_1) = -2(a + b),
\]
where the minus sign \((-1)^{c_1(Y) \cdot \beta - 1} = -1\) (here \(Y = \text{Tot}_{P^1 \times P^1}(O(-1, -1))\) is the preferred choice of orientations \(0.7\)). Therefore Conjecture 1.2 holds in this case. \(\square\)

### 2.4. Local Fano 3-fold.

**Proposition 2.13.** Let \(X = K_{P^3}\) and \(\beta = d [\mathbb{P}^1] \in H_2(X, \mathbb{Z})\). Then Conjecture 1.2 holds if \(d \leq 3\).

**Proof.** We first consider \(d = 2\) case (same method applies to \(d = 1\) case). Consider a compactification of \(X\):
\[
\overline{X} = P(\mathcal{O}_{P^3}(-4) \oplus \mathcal{O}_{P^3})
\]
with the projection \(\pi: \overline{X} \rightarrow \mathbb{P}^3\) and a section \(\iota: \mathbb{P}^3 \rightarrow \overline{X}\). Then
\[
H^4(\overline{X}, \mathbb{Z}) = \mathbb{Z}\langle T_1, T_2 \rangle,
\]
where \(T_1 = [\iota(\mathbb{P}^2)], T_2 = [\pi^{-1}(\mathbb{P}^1)]\). Their intersection matrix is
\[
g = \begin{pmatrix} -4 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{with} \quad g^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix}.
\]
For any \(\beta \in H_2(X) \subset H_2(\overline{X})\), we have
\[
n_{0, \beta}(T_1) = (T_1 \cdot T_1) n_{0, \beta}(\mathbb{P}^1), \quad n_{0, \beta}(T_2) = n_{0, \beta}(\mathbb{P}^1).
\]
By a direct calculation, we have
\[
c_2(X) = -10 \in \mathbb{Z} \cong H^4(X, \mathbb{Z}).
\]
Then meeting invariants can be calculated as
\[
m_{1, 1} = n_{0, 1}(c_2(X)) + \sum n_{0, 1}(T_i) g^{ij} n_{0, 1}(T_j) = -1400,
\]
\[
m_{1, 2} = m_{1, 1} + \sum n_{0, 1}(T_i) g^{ij} n_{0, 2}(T_j) = -67000,
\]
where we use (ref. [KP, Table 1, pp. 31]):
\[
n_{0, 1}(\mathbb{P}^1) = -20, \quad n_{0, 2}(\mathbb{P}^1) = -820, \quad n_{0, 3}(\mathbb{P}^1) = -68060.
\]
Hence for \(\alpha = [\mathbb{P}^2] \in H^2(X)\), the RHS of Conjecture 1.2 in this case is
\[
\frac{1}{4} n_{0, 2}(\mathbb{P}^1) - \frac{1}{8} m_{1, 1} = -30. \tag{2.11}
\]
Next we compute the DT_4 descendent invariant. It is easy to see that for any one dimensional stable sheaf \(F\) on \(\mathbb{P}^3\) with \([F] = 2[\mathbb{P}^1]\) and \(\chi(F) = 1\), there is an unique plane
$H \subset \mathbb{P}^3$ and a conic $C \subset H$ such that $F \cong \mathcal{O}_C$. Therefore $M_1(\mathbb{P}^3, 2)$ is a projective bundle over $|\mathcal{O}_{\mathbb{P}^3}(1)| \cong \mathbb{P}^3$ and we denote the projection by

$$g : M_1(\mathbb{P}^3, 2) \cong \mathbb{P}|\mathcal{O}_{\mathbb{P}^3}(1)|(\mathcal{E}) \to |\mathcal{O}_{\mathbb{P}^3}(1)|.$$  

Here $\mathcal{E} = f_*\mathcal{O}_H(2)$ and $H$ is the universal hyperplanes in the following diagram

$$
\begin{array}{c}
\mathcal{H} \\
\downarrow f \\
|\mathcal{O}_{\mathbb{P}^3}(1)|
\end{array}
\xrightarrow{\quad\quad\quad}\begin{array}{c}
\mathbb{P}^3 \times |\mathcal{O}_{\mathbb{P}^3}(1)|
\end{array}
$$

Let $\mathcal{C}$ be the relative universal conics

$$i : \mathcal{C} \hookrightarrow \mathcal{H} \times |\mathcal{O}_{\mathbb{P}^3}(1)| \mathbb{P}|\mathcal{O}_{\mathbb{P}^3}(1)|(\mathcal{E}) \hookrightarrow \mathbb{P}^3 \times \mathbb{P}|\mathcal{O}_{\mathbb{P}^3}(1)|(\mathcal{E}).$$

Then the normalized universal sheaf $\mathbb{F}_{\text{norm}}$ is given by $i_*\mathcal{O}_\mathcal{C}$, which has the following $K$-theory class

$$i_*\mathcal{O}_\mathcal{C} = [\mathcal{O}] - [\mathcal{O}(-1, -h_2)] - [\mathcal{O}(-2, -h_1)] + [\mathcal{O}(-3, -h_1 - h_2)],$$

(2.12)

where $h_1 = c_1(\mathcal{O}_g(1))$ and $h_2 = g^*\mathcal{O}(1)$. A direct calculation shows the identity in $K$-theory

$$R\pi_{M*}R\mathcal{H}om_{\mathbb{P}^3 \times \mathbb{P}|\mathcal{O}_{\mathbb{P}^3}(1)|(\mathcal{E})}(i_*\mathcal{O}_\mathcal{C}, i_*\mathcal{O}_\mathcal{C})[1]$$

$$= -4[\mathcal{O}] + 8[\mathcal{O}(h_2)] + 20[\mathcal{O}(h_1)] - 4[\mathcal{O}(h_1 - h_2)] - 20[\mathcal{O}(h_1 + h_2)].$$

Since $\mathbb{P}|\mathcal{O}_{\mathbb{P}^3}(1)|(\mathcal{E})$ is smooth whose tangent bundle has the $K$-theory class:

$$T_{\mathbb{P}|\mathcal{O}_{\mathbb{P}^3}(1)|(\mathcal{E})} = 4[\mathcal{O}(h_2)] + 10[\mathcal{O}(h_1)] - 4[\mathcal{O}(h_1 - h_2)] - 2[\mathcal{O}],$$

the obstruction bundle $\text{Obs}$ for $\text{DT}_3$ virtual class on $M_1(\mathbb{P}^3, 2)$ has the following $K$-theory class

$$\text{Obs} = T_{\mathbb{P}|\mathcal{O}_{\mathbb{P}^3}(1)|} + R\pi_{M*}R\mathcal{H}om_{\mathbb{P}^3 \times \mathbb{P}|\mathcal{O}_{\mathbb{P}^3}(1)|(\mathcal{E})}(i_*\mathcal{O}_\mathcal{C}, i_*\mathcal{O}_\mathcal{C}) - [\mathcal{O}]$$

$$= 20[\mathcal{O}(h_1 + h_2)] + [\mathcal{O}] - 10[\mathcal{O}(h_1)] - 4[\mathcal{O}(h_2)].$$

By taking the Euler class, the $\text{DT}_3$ virtual class on $M_1(\mathbb{P}^3, 2)$ is given by

$$[M_1(\mathbb{P}^3, 2)]^{\text{vir}} = 120h_1^7 + 840h_1^5h_2 + 3080h_1^3h_2^2 + 7700h_1h_2^3.$$  

On the other hand for $\alpha = [\mathbb{P}^2] \in H^2(X)$, using (2.12), we easily compute

$$\pi_{M*}(\alpha \boxtimes \text{ch}_4(i_*\mathcal{O}_\mathcal{C})) = -\frac{1}{2}h_1.$$  

Therefore

$$\int_{[M_1(\mathbb{P}^3, 2)]^{\text{vir}}} \tau_1(\alpha) = \left(120h_1^7 + 840h_1^5h_2 + 3080h_1^3h_2^2 + 7700h_1h_2^3\right) \cdot \left(-\frac{1}{2}h_1\right)$$

$$= -60h_1^8 - 420h_1^6h_2 - 1540h_1^4h_2^2 - 3850h_1^2h_2^3.$$
Here we have used equalities $h_1^8 = -4$, $h_1^7 h_2 = 6$, $h_1^6 h_2^2 = -4$ and $h_1^5 h_2^3 = 1$. By taking the sign $(-1)^{c_1(\mathbb{P}^3) \cdot \beta - 1} = -1$ as the preferred choice of orientations (0.7), we obtain a matching with (2.11).

Finally when $d = 3$, we consider the PT moduli space $P_1(X, 3)$ of stable pairs $(F, s)$ on $X$, where $F$ is a pure 1-dimensional sheaf with $[F] = 3[\mathbb{P}^1]$ and $\chi(F) = 1$, $s \in H^0(F)$ is a section with at most 0-dimensional cokernel. If $F$ is supported on a twisted cubic $C$, then $F = \mathcal{O}_C$ is a stable sheaf [FT, Lemma 3.1]. If $F$ is supported on a plane cubic $C$, then $F$ is a non-split extension of $\mathcal{O}_C$ with the structure sheaf of one point, hence also stable.

Therefore there is a well-defined forgetful map

$$f : P_1(X, \beta) \to M_1(X, \beta), \quad (s : \mathcal{O}_X \to F) \mapsto F,$$

which is in fact an isomorphism (see e.g. [FT, Lemma 3.2]). Therefore our descendant DT$_4$ invariants (1.8) can be calculated by similar descendant invariants on stable pair moduli spaces. In this case all stable pairs are scheme theoretically supported on the zero section $\mathbb{P}^3 \subset X$ and the moduli space $P_1(X, 3)$ has a natural orientation (up to an overall global sign) coming from $P_1(\mathbb{P}^3, 3)$ (see e.g. [CMT19, Proposition 3.3]). Using the 4-fold stable pair vertex [CK19, CKM19], one can explicitly compute the invariant and obtain $3 \langle \tau_1([\mathbb{P}^2]) \rangle_3 = -22610$.

In this case, the RHS of Conjecture 1.2 is

$$\frac{1}{6} n_{0,3}([\mathbb{P}^1]) - \frac{1}{3} m_{1,2} - 3 n_{1,3} = \frac{-68060}{6} - \frac{1}{3} \times (-67000) - 3 \times 11200 = -22610,$$

which matches with our descendant invariant. Here we have used $n_{1,3} = 11200$ from [KP, Table 1]. □

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