Deformed “Commutative” Chern - Simons System

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Abstract

Noncommutative Chern - Simons’ system is non-perturbatively investigated at a full deformed level. A deformed “commutative” phase space is found by a non-canonical change between two sets of deformed variables of noncommutative space. It is explored that in the “commutative” phase space all calculations are similar to the case in commutative space. Spectra of the energy and angular momentum of the Chern - Simons’ system are obtained at the full deformed level. The noncommutative-commutative correspondence is clearly showed. Formalism for the general dynamical system is briefly presented. Some subtle points are clarified.
1 Introduction

Since the topological Chern-Simons (C-S) field theories proposed by Deser, Jackiw and Templeton [1], they have been extensively investigated in literature. Its interesting characteristic is that it provides topological mass terms for odd (space-time) dimensional gauge theories. There is correspondence between topologically massive electrodynamics in the Weyl gauge and a model in quantum mechanics, as explained by Dunne, Jackiw and Trugenberger [2], similar dynamical effects of C-S system at quantum mechanical level were also studied [2, 3]. The Chern-Simons interaction plays a crucial role in the quantum Hall effect, high $T_c$ superconductivity, cosmic string in planar gravity, etc. Their generalization to noncommutative space has been considered. The noncommutative extension of quantum field theory has attracted extensively attention in literature [4–12]. In order to get qualitative understanding of noncommutativity affecting quantum field theory one tries to understand these effects firstly in low energy sector at the level of noncommutative quantum mechanics (NCQM) [13–22]. In literature there has been a large number of papers dealing with noncommutative C-S gauge theory [23–31]. However, there has been relatively little work exploring the C-S system at the NCQM level. NCQM, as the one-particle sector of noncommutative quantum field theories, can be treated in a more or less self-contained way so that a more detailed study of noncommutative C-S quantum mechanics is useful. Noncommutative C-S system was solved at the NCQM level [22] through the undeformed variables of commutative space to represent the deformed variables of noncommutative space. It is interest to clarify whether such a noncommutative-commutative correspondence can be realized at a full deformed level through a non-canonical change among two sets of deformed variables of noncommutative space.

In this paper a noncommutative C-S quantum mechanical model, like the usual harmonic oscillator, serves as a typical example. A deformed “commutative” phase space is found by a non-canonical change between two sets of deformed variables of noncommutative space. It essentially defers from the treatment about the noncommutative-commutative correspondence in literature where a non-canonical change between a set of deformed variables of noncommutative space and a set of undeformed variables of commutative space was considered. The advantage of the “commutative” framework is that in which all calcu-
lations in noncommutative space are similar to ones in commutative space. In the deformed “commutative” phase space and deformed “commutative” Fock space the noncommutative C-S system is non-perturbatively solved at the full deformed level. Spectra of its energy and angular momentum are obtained. The noncommutative-commutative correspondence is clearly showed: results in commutative space smoothly emerge from ones in noncommutative space when the limit of vanishing noncommutative parameters is undertaken, even for cases of noncommutative versions with expressions where noncommutative parameters appear in the denominator. Formalism for the general noncommutative dynamical system is briefly presented. Finally, some subtle points are clarified.

2 The C-S Interactions

2.1 Realization of the C-S Process

At the quantum mechanical level, a C-S quantum mechanical model can be constructed as follows. A charged particle of mass $\mu$ and electric charge $q (> 0)$ moves in the following external crossed magnetic and electric fields. The electric field $\hat{E}$ acts radially in the $x-y$ plane, $\hat{E}_i = -\mathcal{E}\hat{x}_i$, $(i = 1, 2)$ providing a radial harmonic potential where $\mathcal{E}$ is a constant. The homogeneous magnetic field $\hat{B}$ aligned along the $z-$axis. The vector potential $\hat{A}_i$ of $\hat{B}$ is chosen as (Henceforth the summation convention is used) $\hat{A}_i = -B\epsilon_{ij}\hat{x}_j/2, (i, j = 1, 2)$, where $\epsilon_{ij}$ is a two-dimensional antisymmetric unit tensor, $\epsilon_{12} = -\epsilon_{21} = 1, \epsilon_{11} = \epsilon_{22} = 0$. The particle’s motion is confined to be planar and rotationally symmetric. The deformed C-S Hamiltonian is represented as

$$\hat{H} = \frac{1}{2\mu}(\hat{p}_i + \frac{1}{2}g\epsilon_{ij}\hat{x}_j)^2 + \frac{1}{2}\kappa\hat{x}_i^2 = \frac{1}{2\mu}\hat{p}_i^2 + \frac{1}{2}\mu g\epsilon_{ij}\hat{p}_i\hat{x}_j + \frac{1}{2}\mu\omega^2\hat{x}_i^2,$$

(1)

where the constant parameters $g = qB/c$ ($c$ is the speed of light) and $\kappa = q\mathcal{E}$. In Eq. (1) the term $g\epsilon_{ij}\hat{p}_i\hat{x}_j/2\mu$ plays a role of realizing analogs of the C-S theory $[2]$. The frequency $\omega = [g^2/4\mu^2 + \kappa/\mu]^{1/2}$, where the dispersive “mass” term $g/2\mu$ comes from the presence of the C-S term. If NCQM is a realistic physics, low energy quantum phenomena should be reformulated in this framework. In the above the noncommutative Hamiltonian (1) is obtained by reformulating the corresponding commutative one, $H = (p_i + g\epsilon_{ij}x_j/2)^2/2\mu + \kappa x_i^2/2$ where $x_i$ and $p_i$ are the undeformed canonical phase space variables in commutative
space, in terms of the deformed canonical phase space variables $\hat{x}_i$ and $\hat{p}_i$ in nocommutative space.

### 2.2 The Noncommutative Phase Space

The starting point is the deformed Heisenberg-Weyl algebra. We consider the case of both position-position noncommutativity (space-time noncommutativity is not considered) and momentum-momentum noncommutativity. In this case the consistent deformed Heisenberg-Weyl algebra is [22]:

\[
\begin{align*}
[\hat{x}_i, \hat{x}_j] &= i\xi^2\theta\epsilon_{ij}, \\
[\hat{p}_i, \hat{p}_j] &= i\xi^2\eta\epsilon_{ij}, \\
[\hat{x}_i, \hat{p}_j] &= i\hbar\delta_{ij}, (i, j = 1, 2),
\end{align*}
\] (2)

where $\theta$ and $\eta$ are constant parameters, independent of the position and momentum. Here the noncommutativity of canonical momenta $\hat{p}_i$ means the intrinsic noncommutativity. The scaling factor $\xi = (1 + \theta\eta/4\hbar^2)^{-1/2}$ is a dimensionless constant.

There are different ways to construct creation-annihilation operators. The deformed annihilation-creation operators ($\hat{a}_i, \hat{a}^\dagger_i$) ($i = 1, 2$) at the deformed level which are related to deformed variables ($\hat{x}_i, \hat{p}_i$) are:

\[
\begin{align*}
\hat{a}_i &= \sqrt{\frac{\mu\omega}{2\hbar}} (\hat{x}_i + \frac{i}{\mu\omega} \hat{p}_i), \\
\hat{a}^\dagger_i &= \sqrt{\frac{\mu\omega}{2\hbar}} (\hat{x}_i - \frac{i}{\mu\omega} \hat{p}_i).
\end{align*}
\] (3)

Equation (3) and the NCQM algebra (2) show that the operators $\hat{a}^\dagger_i$ and $\hat{a}^\dagger_j$ for the case $i \neq j$ do not commute. When the state vector space of identical bosons is constructed by generalizing one-particle quantum mechanics, because of such a noncommutativity the operators $\hat{a}_1^\dagger\hat{a}_2^\dagger$ and $\hat{a}_2^\dagger\hat{a}_1^\dagger$ applied successively to the vacuum state $|0, 0\rangle$ [where the definition of the vacuum state $|0, 0\rangle$ is $\hat{a}_i|0, 0\rangle = 0, (i = 1, 2)$] do not produce the same physical state, $\hat{a}_1^\dagger\hat{a}_2^\dagger|0, 0\rangle \neq \hat{a}_2^\dagger\hat{a}_1^\dagger|0, 0\rangle$. In order to maintain Bose-Einstein statistics at the non-perturbation level described by $\hat{a}_i^\dagger$ the basic assumption is that operators $\hat{a}_i^\dagger$ and $\hat{a}_j^\dagger$ should be commuting. This requirement leads to a condition between two noncommutative parameters $\eta$ and $\theta$: $\eta = \mu^2\omega^2\theta$. From Eqs. (2), (3) it follows that the commutation relations of $\hat{a}_i$ and $\hat{a}_j^\dagger$ read

\[
[\hat{a}_1, \hat{a}^\dagger_1] = [\hat{a}_2, \hat{a}^\dagger_2] = 1, [\hat{a}_1, \hat{a}_2] = 0; \quad [\hat{a}_1, \hat{a}^\dagger_2] = i\xi^2\mu\omega\theta/\hbar.
\] (4)

The first three equations in (4) are the same commutation relations as the one in commutative space.
The last equation in (4) codes effects of spatial noncommutativity. We emphasize that it is consistent with all principles of quantum mechanics and Bose - Einstein statistics.

The deformed variables \( \hat{x}_i, \hat{p}_i \) has different realizations by the undeformed variables \( x_i \) and \( p_i \) [18]. We consider the following consistent ansatz of expansions of \( \hat{x}_i \) and \( \hat{p}_i \) by \( x_i \) and \( p_i \):

\[
\hat{x}_i = \xi \left( x_i - \frac{\theta}{2\hbar} \epsilon_{ij} p_j \right), \quad \hat{p}_i = \xi \left( p_i + \frac{\eta}{2\hbar} \epsilon_{ij} x_j \right). \tag{5}
\]

where \( x_i \) and \( p_i \) satisfy the undeformed Heisenberg - Weyl algebra in commutative space, \([x_i, x_j] = [p_i, p_j] = 0, \ [x_i, p_j] = i\hbar \delta_{ij} \).

2.3 Investigation at the Undeformed Level

We briefly review the investigation of the deformed C-S Hamiltonian (1) at the undeformed level using undeformed phase space variables \( x_i \) and \( p_i \) [22]. Using (5) the C-S Hamiltonian (1) is represented by \( x_i \) and \( p_i \) as

\[
\hat{H} = \frac{1}{2M} \left( p_i + \frac{1}{2} G \epsilon_{ij} x_j \right)^2 + \frac{1}{2} K x_i^2 = \frac{1}{2M} p_i^2 + \frac{1}{2M} G \epsilon_{ij} p_i x_j + \frac{1}{2} M \Omega^2 x_i^2. \tag{6}
\]

The above effective parameters \( M, G, K \) and \( \Omega \) are defined as \( 1/2M \equiv \xi^2 \left[ c_1^2/2\mu + \kappa \theta^2/8\hbar^2 \right], \) \( G/2M \equiv \xi^2 (c_1 c_2/\mu + \kappa \theta/2\hbar), M\Omega^2/2 \equiv \xi^2 \left[ c_2^2/2\mu + \kappa/2 \right], \) \( K \equiv M\Omega^2 - G^2/4M, \) where \( c_1 = 1 + g\theta/4\hbar, \ c_2 = g/2 + \eta/2\hbar. \) Equation (6) is exactly solvable [32, 33]. We introduce new variables \( X_{\alpha} \) and \( P_{\alpha}, \)

\[
X_{\alpha} = \sqrt{\frac{M\Omega}{2\omega_{\alpha}}} x_1 - \sqrt{\frac{1}{2M\Omega \omega_{\alpha}}} p_2, \quad X_b = \sqrt{\frac{M\Omega}{2\omega_{b}}} x_1 + \sqrt{\frac{1}{2M\Omega \omega_{b}}} p_2, \\
P_{\alpha} = \sqrt{\frac{\omega_{\alpha}}{2M\Omega}} p_1 + \sqrt{\frac{M\Omega \omega_{\alpha}}{2}} x_2, \quad P_b = \sqrt{\frac{\omega_{b}}{2M\Omega}} p_1 - \sqrt{\frac{M\Omega \omega_{b}}{2}} x_2. \tag{7}
\]

where \( \omega_{\alpha} = \Omega + G/2M, \) \( \omega_{b} = \Omega - G/2M, \) and define the annihilation-creation operators

\[
A_{\alpha} = \sqrt{\omega_{\alpha}/2\hbar} X_{\alpha} + i\sqrt{\hbar/2\omega_{\alpha}} P_{\alpha}, \quad A_{\alpha}^\dagger = \sqrt{\omega_{\alpha}/2\hbar} X_{\alpha} - i\sqrt{\hbar/2\omega_{\alpha}} P_{\alpha}, \) \( (\alpha = a, b). \) Then the Hamiltonian (6) decomposes into two uncoupled harmonic oscillators of unit mass and frequencies \( \omega_{a} \) and \( \omega_{b}. \)

\[
\hat{H} = H_a + H_b, \ H_{a,b} = \hbar \omega_{a,b} (A_{a,b}^\dagger A_{a,b} + 1/2), \ E_{n_{a},n_{b}} = \hbar \omega_{a} \left( n_{a} + \frac{1}{2} \right) + \hbar \omega_{b} \left( n_{b} + \frac{1}{2} \right). \tag{8}
\]

3 The “commutative” phase space
3.1 Construction of Fock space

The number operators in the hat system are \( \hat{N}_1 = \hat{a}_\dagger_1 \hat{a}_1 \) and \( \hat{N}_2 = \hat{a}_\dagger_2 \hat{a}_2 \). Because the last equation in Eq. (4) correlates different degrees of freedom, \( \hat{N}_1 \) and \( \hat{N}_2 \) do not commute, \([\hat{N}_1, \hat{N}_2] \neq 0\). They have not common eigenstates. Thus it is impossible to construct Fock space in the hat system. In order to construct a Fock space we introduce the following auxiliary operators, the tilde annihilation operators \( \tilde{a}_1 \) and \( \tilde{a}_2 \) \( \big[22\big] \), and express \( \hat{a}_1 \) and \( \hat{a}_2 \) by \( \sqrt{2\alpha_1} \tilde{a}_1 \) and \( \sqrt{2\alpha_2} \tilde{a}_2 \) as follows

\[
\hat{a}_1 = \frac{1}{\sqrt{2}} \left( \sqrt{2\alpha_1} \tilde{a}_1 + \sqrt{2\alpha_2} \tilde{a}_2 \right), \quad \hat{a}_2 = -i \frac{\sqrt{2}}{\sqrt{2}} \left( \sqrt{2\alpha_1} \tilde{a}_1 - \sqrt{2\alpha_2} \tilde{a}_2 \right),
\]

(9)

where

\[
\alpha_{1,2} = 1 \pm \xi^2 \frac{\mu \omega \theta}{\hbar}.
\]

(10)

From Eqs. (4) and (9) it follows that the commutation relations of \( \tilde{a}_i \) and \( \tilde{a}_j^\dagger \) read

\[
[\tilde{a}_i, \tilde{a}_j^\dagger] = \delta_{ij}, \quad [\tilde{a}_i, \tilde{a}_j] = [\tilde{a}_i^\dagger, \tilde{a}_j^\dagger] = 0, \ (i, j = 1, 2).
\]

(11)

The algebra (11) is same as the bosonic one in commutative space. The operators \( \tilde{a}_i \) and \( \tilde{a}_i^\dagger \) are explained as the deformed annihilation and creation operators in the tilde system. The tilde number operators \( \tilde{N}_1 = \tilde{a}_\dagger_1 \tilde{a}_1 \) and \( \tilde{N}_2 = \tilde{a}_\dagger_2 \tilde{a}_2 \) commute each other, \([\tilde{N}_1, \tilde{N}_2] = 0\). The commutations between \( \tilde{a}_i \) and \( \tilde{N}_i \) are same as ones in commutative space. The eigenvalues of \( \tilde{N}_i \) are \( n_i = 0, 1, 2, \cdots \). A general tilde state is \( |\tilde{m}, n\rangle \equiv (m!n!)^{-1/2}(\tilde{a}_1^\dagger)^m(\tilde{a}_2^\dagger)^n|0, 0\rangle \), where the vacuum state \( |0, 0\rangle \) in the tilde system is defined as \( \tilde{a}_i|0, 0\rangle = 0 \), \((i = 1, 2)\). It is the common eigenstate of \( \tilde{N}_1 \) and \( \tilde{N}_2 \): \( \tilde{N}_1|\tilde{m}, n\rangle = m|\tilde{m}, n\rangle, \tilde{N}_2|\tilde{m}, n\rangle = n|\tilde{m}, n\rangle \), \((m, n = 0, 1, 2, \cdots)\), and satisfies \( \langle \tilde{m}', n'|\tilde{m}, n\rangle = \delta_{m'm}\delta_{n'n} \). The states \( \{|\tilde{m}, n\rangle\} \) constitute an orthogonal normalized complete basis of the tilde Fock space. In the tilde Fock space all calculations are the same as the case in commutative space, thus the concept of identical particles is maintained and the formalism of the deformed bosonic symmetry which restricts the states under permutations of identical particles in multi-boson systems can be similarly developed.

3.2 The “commutative” phase space

Starting from the “commutative” tilde annihilation-creation operators we introduce the “commutative” phase space variables as follows. The tilde annihilation-creation operators
\( \hat{a}_i \) and \( \hat{a}_i^\dagger \) \((i = 1, 2) \) and the tilde phase space variables \( \tilde{x}_i \) and \( \tilde{p}_i \) should satisfy the following relations:

\[
\tilde{a}_i = \sqrt{\frac{\mu \omega_i}{2\hbar}} \left( \tilde{x}_i + \frac{i}{\mu \omega_i} \tilde{p}_i \right), \quad \tilde{a}_i^\dagger = \sqrt{\frac{\mu \omega_i}{2\hbar}} \left( \tilde{x}_i - \frac{i}{\mu \omega_i} \tilde{p}_i \right), \quad \omega_i \equiv \alpha_i \omega. \tag{12}
\]

From Eqs. (9) and (12) it follows that \( \tilde{x}_i \) and \( \tilde{p}_i \) satisfy an algebra which is the same as the undeformed Heisenberg - Weyl algebra in commutative space:

\[
[\tilde{x}_i, \tilde{p}_j] = i\hbar \delta_{ij}, \quad [\tilde{x}_i, \tilde{x}_j] = 0, \quad [\tilde{p}_i, \tilde{p}_j] = 0, \quad (i, j = 1, 2). \tag{13}
\]

In view of Eqs. (13) the tilde phase space can be considered as “commutative” one. Calculations in the tilde phase space are the same as in the commutative phase space. Using Eqs. (8), (9) and (13) we obtain the following relations between two sets of deformed phase space variables \((\hat{x}_i, \hat{p}_i)\) and \((\tilde{x}_i, \tilde{p}_i)\)

\[
\hat{x}_1 = \frac{1}{\sqrt{2}} \left( \alpha_1 \tilde{x}_1 + \alpha_2 \tilde{x}_2 \right), \quad \hat{x}_2 = \frac{1}{\sqrt{2\mu \omega}} \left( \tilde{p}_1 - \tilde{p}_2 \right), \\
\hat{p}_1 = \frac{1}{\sqrt{2}} \left( \tilde{p}_1 + \tilde{p}_2 \right), \quad \hat{p}_2 = -\frac{\mu \omega}{\sqrt{2}} \left( \alpha_1 \tilde{x}_1 - \alpha_2 \tilde{x}_2 \right). \tag{14}
\]

One point that should be emphasized is that the two sets variables \((\hat{x}_i, \hat{p}_i)\) and \((\tilde{x}_i, \tilde{p}_i)\) are both the deformed phase space variables of noncommutative space. It is worthy noting that the tilde phase space variables \(\tilde{x}_i\) are always combined with a factor \(\alpha_i\), i.e. they are always represented in the form \(\alpha_1 \tilde{x}_1\) and \(\alpha_2 \tilde{x}_2\), and all effects of spatial noncommutativity are included in the parameters \(\alpha_i\). Thus results in commutative phase space smoothly emerge from ones in noncommutative phase space when the limit of vanishing noncommutative parameters is undertaken, even for cases of noncommutative versions with expressions where noncommutative parameters appear in the denominator (see below).

Like the hat variables \((\hat{x}_i, \hat{p}_i)\), the tilde variables \((\tilde{x}_i, \tilde{p}_i)\) can be expanded by undeformed variables \((x_i, p_i)\). From Eqs. (8) and (14) it follows that

\[
\tilde{x}_1 = \frac{\xi}{\sqrt{2\alpha_1}} \left[ \left( x_1 - \frac{1}{\mu \omega} p_2 \right) - \frac{\theta}{2\hbar} p_2 + \frac{\eta}{2\hbar \mu \omega} x_1 \right], \\
\tilde{x}_2 = \frac{\xi}{\sqrt{2\alpha_2}} \left[ \left( x_1 + \frac{1}{\mu \omega} p_2 \right) - \frac{\theta}{2\hbar} p_2 - \frac{\eta}{2\hbar \mu \omega} x_1 \right], \\
\tilde{p}_1 = \frac{\xi}{\sqrt{2}} \left[ (p_1 + \mu \omega x_2) + \frac{\eta}{2\hbar} x_2 + \frac{\theta \mu \omega}{2\hbar} p_1 \right], \\
\tilde{p}_2 = \frac{\xi}{\sqrt{2}} \left[ (p_1 - \mu \omega x_2) + \frac{\eta}{2\hbar} x_2 - \frac{\theta \mu \omega}{2\hbar} p_1 \right]. \tag{15}
\]
Comparing Eqs. (14) and (15), it clearly shows that Eq. (14) represents the non-canonical changes between two sets of deformed variables \((\hat{x}_i, \hat{p}_i)\) and \((\tilde{x}_i, \tilde{p}_i)\), both in noncommutative space.

3.3 Spectra of Energy and Angular Momentum

Using Eq. (14) the deformed C-S Hamiltonian (1) can be represented by \(\tilde{x}_i\) and \(\tilde{p}_i\) as

\[
\hat{H}(\hat{x}_i, \hat{p}_i) = \tilde{H}(\tilde{x}_i, \tilde{p}_i),
\]

Here

\[
\tilde{H}(\tilde{x}_i, \tilde{p}_i) = \left(\frac{1}{2\mu} \tilde{p}_1^2 + \frac{1}{2} \mu \omega^2 \alpha_1^2 \tilde{x}_1^2\right) + \left(\frac{1}{2\mu} \tilde{p}_2^2 + \frac{1}{2} \mu \omega^2 \alpha_2^2 \tilde{x}_2^2\right) + \\
+ \frac{g}{2\mu} \cdot \frac{1}{\omega} \left[\left(\frac{1}{2\mu} \tilde{p}_1^2 + \frac{1}{2} \mu \omega^2 \alpha_1^2 \tilde{x}_1^2\right) - \left(\frac{1}{2\mu} \tilde{p}_2^2 + \frac{1}{2} \mu \omega^2 \alpha_2^2 \tilde{x}_2^2\right)\right].
\]

(16)

\(\tilde{H}(\tilde{x}_i, \tilde{p}_i)\) decouples into two modes of harmonic oscillators with frequencies \(\omega \alpha_1\) and \(\omega \alpha_2\).

In Eq. (16) all effects of spatial noncommutativity are included in the parameters \(\alpha_1\) and \(\alpha_2\). Eigenvalues of \(\tilde{H}(\tilde{x}_i, \tilde{p}_i)\) can be directly read out from Eq. (16):

\[
\tilde{E}_{n_1,n_2} = \hbar \omega \left[\alpha_1(n_1 + \frac{1}{2}) + \alpha_2(n_2 + \frac{1}{2})\right] + \frac{\hbar g}{2\mu} \left[\alpha_1(n_1 + \frac{1}{2}) - \alpha_2(n_2 + \frac{1}{2})\right]
\]

(17)

Comparing with the results from the investigation at the undeformed level by using undeformed phase space variables \(x_i\) and \(p_i\) [22], calculations in the tilde system are simple. In the limit of vanishing \(\theta\) the parameters \(\alpha_i \to 1\) smoothly, Eq. (17) smoothly reduces to the spectrum in commutative space.

The C-S system (1) is rotationally symmetric, so the deformed angular momentum commutes with the Hamiltonian \(\hat{H}\) in Eq. (1). In noncommutative space there are different ways to define the deformed angular momentum. Here we consider the following definition [22]:

\[
\hat{J}_z = \frac{1}{1 - \xi^4 \theta \eta / \hbar^2} \left[\epsilon_{ij} \hat{x}_i \hat{p}_j + \frac{\xi^2}{2\hbar} (\eta \hat{x}_i \hat{x}_i + \theta \hat{p}_i \hat{p}_i)\right].
\]

(18)

This deformed angular momentum \(\hat{J}_z\) transforms \(\hat{x}_i\) and \(\hat{p}_j\) as two dimensional vectors:

\([\hat{J}_z, \hat{x}_i] = i \epsilon_{ij} \hat{x}_j, [\hat{J}_z, \hat{p}_j] = i \epsilon_{ij} \hat{p}_j\).

Thus it is a generator of rotations at the deformed level.

Using Eq. (14) the deformed angular momentum \(\hat{J}_z(\hat{x}_i, \hat{p}_i)\) can be reformulated by \(\tilde{x}_i\)
and $\tilde{p}_i$ as $\tilde{J}_z(\tilde{x}_i, \tilde{p}_i) = \tilde{J}(\tilde{x}_i, \tilde{p}_i)$, here

$$\tilde{J}(\tilde{x}_i, \tilde{p}_i) = \frac{1}{\alpha_1\alpha_2} \left[ \left( \frac{1}{2} \tilde{p}_2^2 + \frac{1}{2} \mu \omega^2 \alpha_2^2 \tilde{x}_2^2 \right) - \left( \frac{1}{2} \tilde{p}_1^2 + \frac{1}{2} \mu \omega^2 \alpha_1^2 \tilde{x}_1^2 \right) \right] + \frac{\xi^2 \theta \mu}{\alpha_1 \alpha_2 \hbar} \left[ \left( \frac{1}{2} \tilde{p}_2^2 + \frac{1}{2} \mu \omega^2 \alpha_2^2 \tilde{x}_2^2 \right) + \left( \frac{1}{2} \tilde{p}_1^2 + \frac{1}{2} \mu \omega^2 \alpha_1^2 \tilde{x}_1^2 \right) \right]$$

(19)

The first two terms in Eqs. (16) and (19) are the dominate ones. The behavior of $\tilde{J}(\tilde{x}_i, \tilde{p}_i)$ is different from $\tilde{H}(\tilde{x}_i, \tilde{p}_i)$: the dominate contribution of $\tilde{H}$ comes from mode 1 plus mode 2, but the dominate contribution of $\tilde{J}$ comes from mode 2 minus mode 1. Eigenvalues of $\tilde{J}(\tilde{x}_i, \tilde{p}_i)$ can be directly read out from Eq. (19):

$$\tilde{J}_{n_1, n_2} = \frac{\hbar}{\alpha_1 \alpha_2} (n_2 - n_1) + \frac{\xi^2 \theta \mu}{\alpha_1 \alpha_2} (n_1 + n_2 + 1)$$

(20)

In the limit of vanishing $\theta$ the parameters $\alpha_i \to 1$ smoothly. Though $\alpha_1$ and $\alpha_2$ appear in the denominator, Eqs. (19) and (20) smoothly reduces to the ones in commutative space.

The Hamiltonian and angular momentum of the C-S system can be also reformulated by the tilde annihilation-creation operators $\tilde{a}_i$ and $\tilde{a}^\dagger_i$. Using Eqs. (2) and (3), the Hamiltonian $\tilde{H}(\tilde{x}, \tilde{p})$ in Eq. (11) is first represented by the deformed annihilation-creation operators $\tilde{a}_i$ and $\tilde{a}^\dagger_i$ as

$$\tilde{H}(\tilde{x}, \tilde{p}) = \hbar \omega \left[ (\tilde{a}^\dagger_1 \tilde{a}_1 + \frac{1}{2}) + (\tilde{a}^\dagger_2 \tilde{a}_2 + \frac{1}{2}) \right] - i \frac{\hbar g}{2 \mu} \left( \tilde{a}^\dagger_2 \tilde{a}_1 - \tilde{a}^\dagger_1 \tilde{a}_2 + i \frac{\hbar}{\mu} \xi^2 \theta \mu \right).$$

(21)

Then using Eqs. (9) and (11), it can be further represented by $\tilde{a}_i$ and $\tilde{a}^\dagger_i$ as $\tilde{H}(\tilde{x}, \tilde{p}) = \tilde{H}(\tilde{a}, \tilde{a}^\dagger)$, where

$$\tilde{H}(\tilde{a}, \tilde{a}^\dagger) = \tilde{H} = \hbar \omega \left[ \alpha_1 (\tilde{a}^\dagger_1 \tilde{a}_1 + \frac{1}{2}) + \alpha_2 (\tilde{a}^\dagger_2 \tilde{a}_2 + \frac{1}{2}) \right] + \frac{\hbar g}{2 \mu} \left[ \alpha_1 (\tilde{a}^\dagger_1 \tilde{a}_1 + \frac{1}{2}) - \alpha_2 (\tilde{a}^\dagger_2 \tilde{a}_2 + \frac{1}{2}) \right]$$

(22)

Eigenvalues of $\tilde{H}(\tilde{a}, \tilde{a}^\dagger)$ can be directly read out from Eq. (22).

Similarly, the deformed angular momentum $\tilde{J}_z$ can be represented by $\tilde{a}_i$ and $\tilde{a}^\dagger_i$ as

$$\tilde{J}_z = \frac{1}{1 - \xi^4 \mu^2 \omega^2 \theta / \hbar^2} \left\{ i \hbar \left( \tilde{a}^\dagger_2 \tilde{a}_1 - \tilde{a}^\dagger_1 \tilde{a}_2 + \frac{i}{\hbar} \xi^2 \theta \mu \right) + \xi^2 \theta \mu \left[ (\tilde{a}^\dagger_1 \tilde{a}_1 + \frac{1}{2}) + (\tilde{a}^\dagger_2 \tilde{a}_2 + \frac{1}{2}) \right] \right\}.$$

(23)

Then it can be further represented by $\tilde{a}_i$ and $\tilde{a}^\dagger_i$ as $\tilde{J}_z(\tilde{a}_i, \tilde{a}^\dagger_i) = \tilde{J}(\tilde{a}_i, \tilde{a}^\dagger_i)$, here

$$\tilde{J}(\tilde{a}_i, \tilde{a}^\dagger_i) = \frac{\hbar}{\alpha_1 \alpha_2} \left[ \alpha_2 \left( \tilde{a}^\dagger_2 \tilde{a}_2 + \frac{1}{2} \right) - \alpha_1 \left( \tilde{a}^\dagger_1 \tilde{a}_1 + \frac{1}{2} \right) \right] + \frac{\xi^2 \theta \mu \omega}{\alpha_1 \alpha_2} \left[ \alpha_1 \left( \tilde{a}^\dagger_1 \tilde{a}_1 + \frac{1}{2} \right) + \alpha_2 \left( \tilde{a}^\dagger_2 \tilde{a}_2 + \frac{1}{2} \right) \right].$$

(24)
The eigenvalues of $\tilde{J}(\tilde{a}_i, \tilde{a}^\dagger_i)$ can be directly read out from Eq. (24).

4 The General Cases

We briefly express results for a general system. The general representations of the deformed annihilation-creation operators $\tilde{a}_i$ and $\tilde{a}^\dagger_i$ fixed by the deformed Heisenberg-Weyl algebra are [22]:

$$\tilde{a}_i = \sqrt{\frac{1}{2\hbar}} \sqrt{\frac{\eta}{\theta}} \left( \hat{x}_i + i \frac{\theta}{\eta} \hat{p}_i \right), \quad \tilde{a}^\dagger_i = \sqrt{\frac{1}{2\hbar}} \sqrt{\frac{\eta}{\theta}} \left( \hat{x}_i - i \frac{\theta}{\eta} \hat{p}_i \right).$$

(25)

In order to construct a Fock space we introduce the tilde annihilation and creation operators $\tilde{a}_i$ and $\tilde{a}^\dagger_i$ through Eq. (9), then introduce the tilde phase space variables $\tilde{x}_i$ and $\tilde{p}_i$ as the follows:

$$\tilde{a}_i = \sqrt{\frac{\alpha_i}{2\hbar}} \sqrt{\frac{\eta}{\theta}} \left( \tilde{x}_i + i \frac{\theta}{\alpha_i \eta} \tilde{p}_i \right), \quad \tilde{a}^\dagger_i = \sqrt{\frac{\alpha_i}{2\hbar}} \sqrt{\frac{\eta}{\theta}} \left( \tilde{x}_i - i \frac{\theta}{\alpha_i \eta} \tilde{p}_i \right),$$

(26)

where the parameters $\alpha_i$ read

$$\alpha_{1,2} = 1 \pm \xi^2 \sqrt{\theta \eta}/\hbar.$$  

(27)

Eq. (14) indicates that $\hat{x}_i$ and $\hat{p}_i$ can be expressed as functions of $\alpha_i \hat{x}_i$ and $\hat{p}_i$. From Eqs. (25), (9) and (26) it follows that $\tilde{H}(\tilde{x}_i, \tilde{p}_i)$ can be represented as a function of $\alpha_i \hat{x}_i$ and $\hat{p}_i$:

$$\tilde{H}(\tilde{x}_i, \tilde{p}_i) = \tilde{H}(\alpha_i \tilde{x}_i, \tilde{p}_i).$$

(28)

Eq. (9) indicates that $\tilde{a}_i$ and $\tilde{a}^\dagger_i$ can be expressed as functions of $\sqrt{\alpha_i} \hat{a}_i$ and $\sqrt{\alpha_i} \hat{a}^\dagger_i$. Using Eqs. (3) and (9), $\tilde{H}(\tilde{x}_i, \tilde{p}_i)$ can be represented as a function of $\sqrt{\alpha_i} \hat{a}_i$ and $\sqrt{\alpha_i} \hat{a}^\dagger_i$:

$$\tilde{H}(\tilde{x}_i, \tilde{p}_i) = \tilde{H}(\sqrt{\alpha_i} \hat{a}_i, \sqrt{\alpha_i} \hat{a}^\dagger_i).$$

(29)

In $\tilde{H}(\sqrt{\alpha_i} \hat{a}_i, \sqrt{\alpha_i} \hat{a}^\dagger_i)$ we combine $\sqrt{\alpha_i} \hat{a}_i$’s and $\sqrt{\alpha_i} \hat{a}^\dagger_i$’s into tilde number operators $\alpha_i \hat{N}_i$’s as possible. If $\tilde{H}$ only contains terms of $\alpha_i \hat{N}_i$’s, i.e., $\tilde{H}$ can be represented as $\tilde{H}(\alpha_i \hat{N}_i)$, eigenvalues of $\tilde{H}$ can be directly read out. In general cases, besides terms of $\alpha_i \hat{N}_i$’s, there are some terms with surplus $\tilde{a}_i$’s and/or $\tilde{a}^\dagger_i$’s, such as $\tilde{a}_i \tilde{N}_j$’s and/or $\tilde{a}^\dagger_i \tilde{N}_j$’s. In these cases the states $|\tilde{m}, \tilde{n}\rangle$ are not eigenstates of $\tilde{H}(\sqrt{\alpha_i} \hat{a}_i, \sqrt{\alpha_i} \hat{a}^\dagger_i)$. Expectations of these terms in the states $|\tilde{m}, \tilde{n}\rangle$ are zero, $\langle \tilde{m}, \tilde{n}|\tilde{a}_i \tilde{N}_j|\tilde{m}, \tilde{n}\rangle = \langle \tilde{m}, \tilde{n}|\tilde{a}^\dagger_i \tilde{N}_j|\tilde{m}, \tilde{n}\rangle = 0$, thus only terms of $\alpha_i \hat{N}_i$’s contribute to expectations of $\tilde{H}(\sqrt{\alpha_i} \hat{a}_i, \sqrt{\alpha_i} \hat{a}^\dagger_i)$. 
5 Concluding Remarks

In the tilde phase space ($\tilde{x}_i, \tilde{p}_i$) and tilde Fock space ($\tilde{a}_i, \tilde{a}^\dagger_i$) all calculations are the same as the case in commutative space. *All effects of spatial noncommutativity are included in the parameters* $\alpha_i$. The noncommutative-commutative correspondence is clearly showed: results in commutative space smoothly emerge from ones in noncommutative space when the commutative limit $\theta \rightarrow 0$ is undertaken. It is worth noting that even for cases of noncommutative versions with expressions where noncommutative parameters appear in the denominator, for example, $\tilde{J}(n_1, n_2)$ in Eq. (20), the commutative limit keeps smooth, because in the limit of vanishing $\theta$ the parameters $\alpha_i \rightarrow 1$ smoothly.

It is noticed that in many field theoretical problems, however, the passage from the noncommutative space to its commutative limit has not appeared to be smooth [34–37] because of noncommutative versions with expressions where noncommutative parameters appear in the denominator. Such behavior refers to the UV/IR mixing that arises in loop calculations in interacting quantum field theories. Quantum mechanics does not have any ultraviolet divergence [38] and, therefore, one does not expect any difficulty in taking the limit of vanishing $\theta$. It is noticed that there is a singularity of the coordinate transformation in Eq. (5).

The question is whether the limit for the vanishing noncommutative parameters can be smoothly undertaken. Eq. (5) shows that the determinant $R$ of the transformation matrix $R$ between $(\hat{x}_1, \hat{x}_2, \hat{p}_1, \hat{p}_2)$ and $(x_1, x_2, p_1, p_2)$ is $R = \xi^4(1 - \theta \eta / 4\hbar^2)^2$. When $\theta \eta = 4\hbar^2$, the matrix $R$ is singular. In this case the inverse of $R$ does not exist. But this singular condition does not prevent that the vanishing limit for noncommutative parameters can be smoothly undertaken. The point is that the noncommutativity of momenta in Eq. (2) means the intrinsic one. All the experiments show that corrections of spatial noncommutativity, if any, are extremely small. Therefore, the parameter $\eta$, like the parameter $\theta$, should be extremely small. But the singular condition gives $\eta = 4\hbar^2 / \theta$. Thus an extremely small $\theta$ corresponds to an extremely large $\eta$. It would lead to that corrections from the noncommutativity of the momenta would be extremely large which contradicts all the experiments. The above singular condition is un-physical. For the realistic noncommutative quantum theory $\theta \eta$ should be the order $o(\theta^2)$, $o(\eta^2)$ and $o(\theta \eta)$ which must be much less than $4\hbar^2$, thus there is
no problem to take the vanishing limit for noncommutative parameters smoothly. In fact, the existing upper bounds of $\theta$ and $\eta$ indicate that the singular condition cannot be met. The existing upper bounds of $\theta$ and $\eta$ are, respectively, $\theta/(\hbar c)^2 \leq (10 \ TeV)^{-2}$ and $|\sqrt{\eta}| \leq 1\mu eV/c$. From these upper bounds it follows that $\theta \eta = [\theta/(\hbar c)^2](\eta c^2)\hbar^2 \ll 4\hbar^2$. Therefore, in spite of the problems in quantum field theories because of loop effects, there is no problem in NCQM to have a smooth limit for the vanishing noncommutative parameters. This is guaranteed by the parameters $\alpha_i \to 1$ smoothly in the limit of vanishing $\theta$.

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