Anisotropic subloading surface Cam-clay plasticity model with rotational hardening: Deformation gradient-based formulation for finite strain

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Abstract
This study is aimed at developing an anisotropic elastoplastic constitutive model for geomaterials at finite strain and its stress calculation algorithm based on the fully implicit return-mapping scheme. The Cam-clay plasticity model is adopted as a specific prototype model of geomaterials. As a pertinent representation of deformation-induced anisotropy in geomaterials, nonlinear rotational hardening is incorporated into the model in a theoretically reasonable manner by introducing the dual multiplicative decompositions of the deformation gradient tensor. In addition to the usual decomposition into elastic and plastic parts, the plastic part is decomposed further into a part contributing to the rotational hardening and a remainder part. The former part leads to a back stress ratio tensor related to the rotational hardening via a hyperelastic-type hardening rule. The constitutive theory is thereby formulated on proper intermediate configurations entirely in terms of deformation-like tensorial variables possessing invariance property, without resort to any objective rates of stress or stress-like variables. Combining the Cam-clay plasticity with the concept of subloading surface, a class of unconventional plasticity, enables the model to be capable of reproducing complex hardening/softening accompanied by volumetric contractive/dilative responses. Basic characteristics and predictive capability of the proposed model, as well as the accuracy of the developed numerical scheme, are verified through several numerical examples including monotonic and cyclic loadings.

KEYWORDS
anisotropy, cam-clay plasticity, finite strain, return-mapping algorithm, rotational hardening, subloading surface concept
1 | INTRODUCTION

The general framework of computational plasticity has been established by extensive research efforts since the pioneering research works from the mid 1980s to the early 1990s by Simo, Simo and Taylor, Simo and Ortiz, Ortiz and Simo. We refer to the comprehensive monographs by Simo, Simo and Hughes, Simo and Ortiz, Ortiz and Simo for thorough description. Much progress in this field has been made to develop theoretically sound formulation for constitutive modeling as well as to devise efficient implementation into numerical methods for engineering analysis, especially in finite strain elastoplasticity. In this context, it would be no exaggeration to say that the theory of finite strain elastoplasticity based on the multiplicative decomposition of deformation gradient tensor with the use of hyperelastic stress–strain relation is the achievement of utmost importance in this research area, although the notion of multiplicative decomposition itself goes back to early works by Kröner, Lee, and Mandel. In view of the historical progress, prior to the emergence of the multiplicative framework, the hyperelastic-based plasticity framework were widely employed, in which an additive decomposition of the rate of deformation tensor into elastic and plastic parts is assumed, and the hyperelastic equation in rate form with some proper objective or corotational stress rate tensor is adopted for elastic stress–strain relation. It has been pointed out, however, that the hyperelastic-based framework has several serious drawbacks in theoretical and numerical aspects, and thereby associated controversial issues have been arisen, namely, the arbitrariness or non-uniqueness regarding the choice of an objective rate tensor, pathological model responses such as the oscillatory or unstable behavior of shear stress and/or back stress related to kinematic hardening under large simple shear deformation, energy dissipation and its accumulation even within purely elastic range during cyclic loadings, and the lack of “incremental objectivity” when using improper time-integration schemes. Due to the last mentioned property, a careful treatment is required in the numerical implementation of hyperelastic-based plasticity models to ensure the incremental objectivity.

In contrast, the hyperelastic-based plasticity relying upon the multiplicative decomposition completely bypasses the above mentioned problems encountered in the hyperelastic-based approach. The fundamental requirement of material frame-indifference and the energy conserving property in the elastic range are fulfilled automatically by the intrinsic natures of the multiplicative framework and the hyperelasticity. Nowadays, the hyperelastic-based approach has won the widespread acceptance, and has been adopted for modeling of metallic materials; see for example, the coupling with continuum damage. With regard to incorporation of plastic anisotropy into a model, there are mainly two different approaches to formulate the kinematic hardening at finite strain: One approach is a direct extension of the evolution law of the back stress tensor with the use of an objective rate tensor; see Tsakmakis and Svendsen et al. The other approach is based on a further decomposition of the plastic part of deformation gradient tensor; see for example, Lion, Wallin and Ristinmaa, Shutov and Kreißig, Henann and Anand, and Vladimirov et al., in which the kinematic hardening is described in terms of tensorial internal variables of strain type, and the back stress tensor is derived via a hyperelastic-type relation. In this way, one obtains a thermodynamically consistent formulation in a physically well-based manner as a multidimensional extension of one-dimensional rheological model. Dettmer and Reese presented a numerical comparison of the above-mentioned different approaches for nonlinear kinematic hardening at finite strain. In the context described above, Brepols et al. performed a numerical comparison of the inherently different formulations of finite strain elastoplasticity based on the hyperelastic- and hyperelastic-based approaches. In their article, they also presented a detailed review and numerical assessment of the objective integration algorithm for the hyperelastic-based plasticity, as well as of the return-mapping algorithm for the hyperelastic-based plasticity. We also refer to comparative studies on the different finite strain frameworks by Idesman, and Shutov and Ihlemann. As reviewed above, extensive research works on this subject have been done in both theoretical and numerical aspects, and thereby the constitutive framework for finite strain elastoplasticity seems to have already been well established, especially for metallic materials. As for geomaterials, indeed the multiplicative framework has been adopted by many researchers, for example, Simo and Meschke, Borja and Tamagnini, Callari et al., Meschke and Liu, Borja, Ortiz and Pandolfi, Andrade and Borja, Borja and Andrade, and Yamakawa et al., but most of their studies are limited to isotropic elastoplasticity, due to the fact that their formulations are mainly based on principal logarithmic strains or two invariants of logarithmic volumetric and deviatoric strains. As far as the authors’ knowledge, there are still not many studies that have addressed the constitutive formulation for geomaterials at finite strain within the multiplicative framework amenable to elastic and/or plastic anisotropy involving arbitrary large deformations. There are very early studies on the relevant topic by Jeremić et al. and Rouainia and Muir Wood. Their works, however, dealt with only isotropic models despite that their formulations are applicable to anisotropy. Thereafter, Mosler and Bruhns presented a variational framework for non-associative Drucker–Prager plasticity combined with Hill-type anisotropy, together with...
volumetric–deviatoric split of elastic and plastic deformations, in which they identified the Mandel stress tensor as a pertinent stress measure for thermodynamically consistent formulation. Gajo\textsuperscript{50} presented a finite strain model of saturated porous media with compressible solid and fluid constituents in a Lagrangian description for a volume element moving with the solid skeleton. Recently, Bennett et al.\textsuperscript{60} developed a thermodynamically consistent framework to formulate the modified Cam-clay and Drucker–Prager cap plasticity models. During the course of the model development, they also discussed the choice of suitable stress measure in the constitutive modeling of materials exhibiting large plastic volume changes. We also refer to a more recent study by Bennett et al.\textsuperscript{61} addressing the extension to include both elastic and plastic anisotropies. They dealt with only anisotropy represented in terms of structure (fabric) tensors, but not plastically induced anisotropy, although they stated in their paper that the provided framework includes the possibility of developing material specific constitutive equations inclusive of induced anisotropy.

Constitutive modeling with anisotropic plasticity for geomaterials has been and is still a very active research area. We refer to several recent research works: anisotropic viscoplastic Cam-clay model for shale by Borja et al.,\textsuperscript{62} adaptive constitutive modeling with rotational hardening by Dejaloud and Rezania,\textsuperscript{63} anisotropic model considering fabric evolution with the aid of DEM numerical tests by Wang et al.,\textsuperscript{64} and anisotropic model applicable to change in principal stresses and their axes rotation by Xue et al.\textsuperscript{65} These studies are, however, limited to the small strain framework.

Motivated by the status of past related researches mentioned above, the objective of this study is aimed at developing a novel Cam-clay plasticity model combined with an induced anisotropy at finite strain within the framework of multiplicative elastoplasticity. As a pertinent expression of the plastically induced anisotropy in geomaterials,\textsuperscript{66} the rotational hardening\textsuperscript{67–74} is incorporated into the model. To this end, according to the proposal by Lion,\textsuperscript{36} we adopt the dual multiplicative decompositions of the deformation gradient tensor; the rotational hardening is thereby described in terms of a strain-type tensorial internal variable, and then the back stress ratio tensor is given via a hyperelastic-type hardening rule defined on a proper intermediate configuration. In this way, the rotational hardening can be incorporated into the model in a well-grounded manner ensuring the frame-indifference property without resort to any objective rates of the back stress ratio tensor.

In order to further improve the capability of the model, particularly for cyclic loadings, we introduce the concept of subloading surface, a class of unconventional plasticity,\textsuperscript{75} originally proposed by Hashiguchi and Ueno,\textsuperscript{76} and subsequently improved and extended by Hashiguchi.\textsuperscript{77} We refer to the comprehensive monographs by Hashiguchi.\textsuperscript{78–80} The essence of the subloading surface concept is the hypothesis that plastic deformations can occur even at a state of stress below the yield stress (in other words, at a stress point inside the yield surface). Combining the Cam-clay plasticity with the subloading surface concept enables the model to be applicable to a broad class of geomaterials, such as normally and over-consolidated clays, as well as loose and dense sands. A notable feature is that the Cam-clay plasticity equipped with the subloading surface is capable of simulating a smooth transition from initial hardening to softening accompanied by volumetric contractive/dilative responses, typically observed in over-consolidated clays and dense sands; see the references\textsuperscript{81,82} for the assessment of predictive capability of the subloading surface model.

The organization of this paper is as follows: Section 2 describes two simple small strain examples. One is a one-dimensional elastoplastic model with the nonlinear kinematic hardening based on a rheological model. The other is a formulation of the rotational hardening under axisymmetric condition. Analogy of evolution equations appearing in these two examples motivates the development in the subsequent section. Section 3 delineates the entire formulation of general three-dimensional constitutive model. Section 4 presents the implicit stress calculation algorithm using return-mapping for the proposed model. Section 5 presents numerical examples to demonstrate the basic characteristics of the proposed model and to verify the performance of the developed stress calculation algorithm. Finally, Section 6 summarizes the achievement and remaining issues in this study, and mentions potential future developments.

We list notations and symbols used throughout the paper. Bold italic letters denote vectors or second-order tensors. Bold sans-serif letters denote fourth-order tensors. The summation convention is applied over repeated indices. The dot symbol “.” denotes the scalar product of two vectors, for example, \( \mathbf{a} \cdot \mathbf{b} = a_i b_i \). A single contraction of adjacent indices of any two second-order tensors is denoted by for example, \((\mathbf{cd})_{ij} = c_{ik} d_{kj}\). The colon symbol “:” denotes the scalar product of two second-order tensors, for example, \( \mathbf{c} : \mathbf{d} = c_{ij} d_{ij} \), or the double contraction of two adjacent indices of second- or higher-order tensors, for example, \((\mathbf{a} : \mathbf{d})_{ij} = a_{ijkl} d_{ijkl}\). The symbol “\( \otimes \)” denotes the dyadic product, for example, \((\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j\) for any two vectors, and \((\mathbf{c} \otimes \mathbf{d})_{ijkl} = c_{ik} d_{kl}\) for any two second-order tensors. The symbols “\( \boxtimes \)” and “\( \boxtimes \)” denote the tensor products of two second-order tensors, for example, \((\mathbf{c} \boxtimes \mathbf{d})_{ijkl} = c_{ik} d_{jl}\) and \((\mathbf{c} \boxtimes \mathbf{d})_{jkl} = c_{lj} d_{jk}\). The second-order identity tensor is denoted by \( \mathbf{I} \), whose Cartesian components are given by the Kronecker delta, that is, \((\mathbf{I})_{ij} = \delta_{ij}\). The fourth-order identity tensor \( \mathbf{I} \), the transposing tensor \( \mathbf{I} \), and the tracing identity tensor \( \mathbf{I} \) are defined by \( \mathbf{I} = I \otimes \mathbf{I} \), \( \mathbf{I} = \mathbf{I} \otimes \mathbf{I} \), and \( \mathbf{I} = (\text{tr} \mathbf{c}) \mathbf{I} \) for any second-order tensor \( \mathbf{c} \).
where the superscript “T” means the transpose, for example, \((c^T)_{ij} = c_{ji}\), and \(\text{tr } c = c : I = c_{kk}\) is the trace of a second-order tensor \(c\). The symbol \(\| \cdot \|\) denotes the magnitude of a vector or a tensor, for example, \(\|a\| = (a \cdot a)^{1/2}\) for a vector \(a\), and \(\|c\| = (c : c)^{1/2}\) for a second-order tensor \(c\). The superposed dot designates the material time derivative. The subscript “dev” means the deviatoric part of a second-order tensor, for example, \(c_{\text{dev}} = c - (\text{tr } c)I/3\). The superscripts “e” and “p” denote an elastic part and a plastic part, respectively. For simple notation we simply write the inverse and/or transpose of a tensor with the superscript “e” or “p” as for example, \((F^{-1})_e = F_e^{-1}\), \((F_e)^T = F_e^T\), and \(((F_p)^{-1})^T = ((F_p)^T)^{-1} = F_p^{-T}\), where the parentheses are omitted.

Remark 1. In the definition of stress and deformation variables, tension is positive and compression is negative as is customary in the continuum mechanics, although the subject of this study is constitutive modeling for geomaterials. In this study on a one-phase medium of the assemblage of soil particles, we shall hereinafter consider the effective (intergranular) stress acting on a soil skeleton.

2 | ROTATIONAL HARDENING UNDER AXISYMMETRIC CONDITION: SMALL STRAIN FORMULATION

This section firstly presents a brief introduction of one-dimensional small-strain elastoplastic model with the Armstrong–Frederick type nonlinear kinematic hardening in metal plasticity. The constitutive formulation herein is based on a rheological model, which provides a simple but clear interpretation for the mechanical basis of the nonlinear kinematic hardening. A key idea is the decomposition of plastic strain into a part related to kinematic hardening and a remainder dissipative part.

The latter part of this section presents a re-formulation of the rotational hardening rule for geomaterials in the small strain framework. The resulting equation is compared with the rotational hardening rule proposed by Hashiguchi and Chen. A limited case of axisymmetric condition is considered herein, while this is a preparation to develop the general three-dimensional rotational hardening model for finite strains in the subsequent sections.

2.1 | Motivation: One-dimensional rheological model of nonlinear kinematic hardening

We begin with a one-dimensional constitutive model for small strains with the Armstrong–Frederick type nonlinear kinematic hardening based on the rheological model shown in Figure 1. This type of kinematic hardening, together with its interpretation based on the rheological model presented herein, is widely accepted as a suitable plasticity model to capture the characteristic behavior of anisotropic hardening in metallic materials, namely, the Bauschinger effect. Our discussion here is limited to the rate-independent plasticity, although the rate-dependent viscoplasticity with nonlinear kinematic hardening can also be discussed based on the similar rheological model. We refer to Lion for the development of one-dimensional thermo-viscoplasticity based on a rheological model.

The rheological model is constituted by mechanical devices arranged as shown in Figure 1. The whole system of the model has unit length and unit cross-sectional area. The system is a series connection of an elastic spring with stiffness \(E\) and an inelastic device. The inelastic device consists of a frictional slider with maximum resistance \(\sigma_y\) (\(\geq 0\)), and

\[
\begin{align*}
\varepsilon_p & \quad \varepsilon_e \\
\sigma & \quad \sigma - \alpha
\end{align*}
\]
components for plastic hardening, which are connected in parallel. The subscript “y” in \( \sigma_y \) implies the yield stress. In the components for plastic hardening, a dashpot with pseudo-viscosity \( c/(\dot{\lambda}b) \) and a spring with stiffness \( c \) are connected serially. This spring is related to kinematic hardening, and so it is called "hardening spring." \( b \) is a dimensionless constant, and \( \dot{\lambda} (\geq 0) \) denotes the magnitude of a slip rate, usually termed "plastic multiplier" in the plasticity theory.

Let a stress \( \sigma \) be applied on the device, and let \( \varepsilon \) be a change in the length of device (i.e., strain). The commonly accepted kinematic assumption of the additive decomposition of the strain into elastic and plastic parts is adopted.

\[
\varepsilon = \varepsilon^e + \varepsilon^p, \tag{1}
\]

where \( \varepsilon, \varepsilon^e, \) and \( \varepsilon^p \) are the total strain and its elastic and plastic parts, respectively. The plastic strain is further decomposed additively as

\[
\varepsilon^p = \varepsilon^{ps} + \varepsilon^{pd}, \tag{2}
\]

where \( \varepsilon^{ps} \) is a plastic energy storage part contributing to the kinematic hardening, and \( \varepsilon^{pd} \) is a dissipative part. The quantity \( \varepsilon^{ps} \) plays a role of internal variable in the model. Differentiating Equations (1) and (2) with respect to time gives the additive decomposition of strain rates

\[
\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p, \quad \dot{\varepsilon}^p = \dot{\varepsilon}^{ps} + \dot{\varepsilon}^{pd}. \tag{3}
\]

The approach based on the rheological model with the idea of the decomposition of plastic strain gives a clear mechanical interpretation of the kinematic hardening, and enables the incorporation of nonlinear kinematic hardening into an elastoplastic model in a physically well-grounded manner without direct use of a rate evolution law for the back-stress as an internal variable.

As observed in Figure 1, the stress \( \sigma \) in the elastic spring and the stress \( \alpha \) in the hardening spring are given by the following two constitutive equations

\[
\sigma = E\varepsilon^e = E(\varepsilon - \varepsilon^p), \tag{4}
\]

and

\[
\alpha = c\varepsilon^{ps} = c(\varepsilon^p - \varepsilon^{pd}). \tag{5}
\]

where \( \alpha \) plays as a role of the stress-ordered variable for kinematic hardening, usually termed "back-stress", in the context of continuum plasticity. In order to define a criterion for activation of the frictional slider, we introduce the yield function

\[
f(\sigma, \alpha, \sigma_y) = |\sigma - \alpha| - \sigma_y \leq 0. \tag{6}
\]

This yield function means that the absolute value of stress \( |\sigma - \alpha| \) in the frictional slider must be less than or equal to \( \sigma_y \), and defines the admissible stresses, \( \sigma \) and \( \alpha \). History-dependent changes in \( \sigma_y \), which corresponds to isotropic hardening in the context of continuum plasticity, can be incorporated into the model, while we here suppose that \( \sigma_y \) is constant for simplicity.

We introduce the evolution equations for \( \varepsilon^p \) and \( \varepsilon^{pd} \):

\[
\dot{\varepsilon}^p = \dot{\lambda} n, \quad n := \frac{\partial f}{\partial \sigma} \left/ \left| \frac{\partial f}{\partial \sigma} \right| \right. = \text{sign}(\sigma - \alpha) \tag{7}
\]

and

\[
\dot{\varepsilon}^{pd} = \frac{b}{c} \alpha |\dot{\varepsilon}^p| = \dot{\lambda} \frac{b}{c} \alpha, \tag{8}
\]

where \( n \) having unit magnitude specifies the direction of plastic flow (extension or compression in one-dimensional case), and \( \text{sign}(\cdot) \) is the signum function. In the above derivation, we have used \( \partial f/\partial \sigma = \text{sign}(\sigma - \alpha), |\partial f/\partial \sigma| = 1 \), and \( |\dot{\varepsilon}^p| = \dot{\lambda} \).
Differentiating Equation (5) with respect to time, and then substituting Equations (7) and (8) lead to

\[ \dot{\alpha} = c \dot{\varepsilon}_p = c (\dot{\varepsilon}_p - \dot{\varepsilon}_{pd}) = c \left( n - \frac{b}{c} \alpha \right) \dot{\lambda} = (c \mp b \alpha) \dot{\varepsilon}_p, \]

(9)

where the upper sign corresponds to a positive (tensile) plastic strain rate \((\dot{\varepsilon}_p > 0)\), and the lower sign to a negative (compressive) one \((\dot{\varepsilon}_p < 0)\). Equation (9) can be rewritten as

\[ \dot{\alpha} = c \left( \frac{\dot{\varepsilon}_p - b}{c} |\dot{\varepsilon}_p| \right), \]

(10)

which is none other than the one-dimensional expression of the well-known evolution equation for the back-stress in the Armstrong–Frederick type nonlinear kinematic hardening model.

For further understanding of the hardening characteristic of this model, we consider a monotonic loading process within a time interval \([t_0, t]\). Integrating the rate equation (9) over the time interval with an initial condition \(\varepsilon_p = \varepsilon^p_0\) and \(\alpha = \alpha_0\) at \(t = t_0\) leads to

\[ \alpha(\varepsilon_p) = \pm \frac{1}{b} \left[ c - (c \mp b \alpha_0) \exp \left\{ \mp b (\varepsilon_p - \varepsilon^p_0) \right\} \right] \]

(11)

with its derivative

\[ \alpha'(\varepsilon_p) = c \mp b \alpha, \]

(12)

where we should note that double-sign corresponds in the same order. From Equation (11), the saturation values of the back-stress are

\[ \lim_{\varepsilon^p \to +\infty} \alpha(\varepsilon^p) = \frac{c}{b}, \quad \lim_{\varepsilon^p \to -\infty} \alpha(\varepsilon^p) = -\frac{c}{b}. \]

(13)

The behavior of nonlinear kinematic hardening presented above is illustrated in Figure 2. It should also be noted that, when \(b = 0\), the model reduces to Prager’s linear kinematic hardening.

The notion of the decomposition of inelastic strains facilitates the general three-dimensional constitutive formulation for finite strains combined with kinematic hardening and/or other inelastic mechanisms; it is extended to the multiplicative decomposition of inelastic parts of the deformation gradient tensor. Such approaches have widely been adopted for finite strain formulations of thermo-viscoplastic model, \(^{36}\) rate-dependent/independent elastoplastic models with nonlinear kinematic hardening, \(^{25,39–43}\) as well as elastoplastic models coupled to continuum damage. \(^{32,33}\)
2.2 Rotational hardening under axisymmetric condition: Small strain case

The rotational hardening has widely been accepted as a pertinent plastic hardening model to describe induced anisotropy observed in geomaterials.\(^\text{66–74}\) Figure 3 illustrates a conceptual diagram of the rotational hardening, in which the modified stress tensor

\[ \sigma^\text{mod} = \sigma - (-p)\beta, \]

with the back stress ratio tensor \(\beta\) represented by a deviatoric tensor (i.e., \(\text{tr} \beta = 0\)), is typically used. We should note here that the stress tensor \(\sigma\), as well as its mean normal stress \(p\), is defined as positive in tension. Hashiguchi\(^\text{79}\) proposed the evolution rule of \(\beta\) in the form applicable to finite strains:

\[ \dot{\beta} = \dot{\beta}_r \left( d_{\text{dev}}^p - \frac{1}{M_r} \beta \left\| d_{\text{dev}}^p \right\| \right), \]

where \(\dot{\beta}\) is some proper objective rate of \(\beta\), \(d_{\text{dev}}^p\) is the deviatoric part of the plastic deformation rate tensor, \(\dot{\beta}_r\) is the evolution coefficient, and \(M_r\) is the constant specifying the stress ratio at a rotation limit. Equation (15) has a form different from the one originally proposed in the earlier studies by him and his co-workers,\(^\text{68,69,72,73}\) but it is an improved version consistent with the development presented herein. The small strain counterpart of Equation (15) can be expressed as

\[ \dot{\beta} = \dot{\beta}_r \left( \varepsilon_{\text{dev}}^p - \frac{1}{M_r} \beta \left\| \varepsilon_{\text{dev}}^p \right\| \right). \]

To simplify the following discussion, we shall consider an axisymmetric condition, in which the matrix representations of stress and strain tensors in terms of Cartesian components are given, respectively, by

\[ \sigma = \begin{pmatrix} \sigma_a & 0 & 0 \\ 0 & \sigma_\ell & 0 \\ 0 & 0 & \sigma_\ell \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} \varepsilon_a & 0 & 0 \\ 0 & \varepsilon_\ell & 0 \\ 0 & 0 & \varepsilon_\ell \end{pmatrix}, \]

where the subscripts ‘\(a\)’ and ‘\(\ell\)’ mean the axial and lateral components, respectively. Here, we assume the axial compression or extension in \(x_1\)-direction under axisymmetric stress state with equal lateral confining stresses in \(x_2\)- and \(x_3\)-directions. In this case, the mean normal stress \(p\) and the deviatoric stress invariant \(q\) are given by

\[ p := \frac{1}{3} \text{tr} \sigma = \frac{1}{3} (\sigma_a + 2\sigma_\ell), \quad q := \sqrt{\frac{3}{2} \left\| \sigma_{\text{dev}} \right\|} = |\sigma_a - \sigma_\ell|. \]
The matrix representation of the deviatoric stress tensor $\sigma_{\text{dev}}$ and its axial and lateral components are

$$
\sigma_{\text{dev}} = \begin{pmatrix}
\sigma_a - p & 0 & 0 \\
0 & \sigma_\ell - p & 0 \\
0 & 0 & \sigma_\ell - p
\end{pmatrix} \implies \sigma_{\text{dev},a} = \sigma_a - p = \frac{2}{3}(\sigma_a - \sigma_\ell), \quad \sigma_{\text{dev},\ell} = \sigma_\ell - p = -\frac{1}{3}(\sigma_a - \sigma_\ell).
$$

(19)

Similar relations to Equations (18) and (19) hold for the modified stress tensor $\sigma_{\text{mod}}$ defined in Equation (14). The volumetric strain $\varepsilon_v$ and the deviatoric strain invariant $\varepsilon_s$ are given by

$$
\varepsilon_v := \text{tr} \varepsilon = \varepsilon_a + 2\varepsilon_\ell, \quad \varepsilon_s := \sqrt{\frac{2}{3} \|\varepsilon_{\text{dev}}\|} = \frac{2}{3} |\varepsilon_a - \varepsilon_\ell|.
$$

(20)

The matrix representation of the deviatoric strain tensor $\varepsilon_{\text{dev}}$ and its axial and lateral components are

$$
\varepsilon_{\text{dev}} = \begin{pmatrix}
\varepsilon_a - \varepsilon_v/3 & 0 & 0 \\
0 & \varepsilon_\ell - \varepsilon_v/3 & 0 \\
0 & 0 & \varepsilon_\ell - \varepsilon_v/3
\end{pmatrix} \implies \varepsilon_{\text{dev},a} = \varepsilon_a - \frac{1}{3}\varepsilon_v = \frac{2}{3}(\varepsilon_a - \varepsilon_\ell), \quad \varepsilon_{\text{dev},\ell} = \varepsilon_\ell - \frac{1}{3}\varepsilon_v = -\frac{1}{3}(\varepsilon_a - \varepsilon_\ell).
$$

(21)

Similar to Equation (1), we assume the additive decomposition of the strain tensor into elastic and plastic parts. Based on this assumption, similar relations to Equations (20) and (21) hold for the elastic strain tensor $\varepsilon^e$ and the plastic strain tensor $\varepsilon^p$, as well as their rates, $\dot{\varepsilon}^e$ and $\dot{\varepsilon}^p$.

Using the equations presented above, we have the following relation for the rate of plastic deviatoric strain tensor:

$$
\|\dot{\varepsilon}^p_{\text{dev}}\| = \sqrt{\frac{2}{3} \|\dot{\varepsilon}^p_{\text{dev},a} - \dot{\varepsilon}^p_\ell\|} = \sqrt{\frac{2}{3} \|\dot{\varepsilon}^p_{\text{dev},a}\|}.
$$

(22)

Substitution of Equation (22) into Equation (16) gives the rotational hardening rule for the axial component under axisymmetric condition:

$$
\dot{\beta}_a = b_t \left( \dot{\varepsilon}^p_{\text{dev},a} - \sqrt{\frac{3}{2} M_t} \beta_a \right)
$$

(23)

with its derivative

$$
\frac{d\beta_a}{d\dot{\varepsilon}^p_{\text{dev},a}} = b_t \left( 1 \mp \sqrt{\frac{3}{2} M_t} \beta_a \right).
$$

(24)

which implies that $d\beta_a/d\dot{\varepsilon}^p_{\text{dev},a} = 0$ when $\beta_a = \pm \sqrt{2/3} M_t$, leading to a saturation of rotational hardening.

Transforming Equation (23) readily gives

$$
\dot{\varepsilon}^p_{\text{dev},a} = \frac{1}{b_t} \dot{\beta}_a + \sqrt{\frac{3}{2} M_t} \beta_a \dot{\varepsilon}^p_{\text{dev},a}.
$$

(25)

Putting the first and second term in the right-hand side of Equation (25) to $\dot{\varepsilon}^p_{\text{dev},a}$ and $\dot{\varepsilon}^p_{\text{dev},a}$, respectively, we then have

$$
\dot{\varepsilon}^p_{\text{dev},a} = \dot{\varepsilon}^p_{\text{dev},a} + \varepsilon_{\text{dev,}a} \quad \text{with} \quad \varepsilon_{\text{dev,}}^{p} + \frac{1}{b_t} \dot{\beta}_a, \quad \varepsilon_{\text{dev,}a} = \sqrt{\frac{3}{2} M_t} \beta_a \dot{\varepsilon}^p_{\text{dev},a}.
$$

(26)
### Table 1: Analogy of nonlinear kinematic hardening and rotational hardening

| Hardening rule and its rate form | Evolution eq. of strain-type variable |
|---------------------------------|--------------------------------------|
| **Kinematic hardening (One-dimensional)** | Equation (5): $\dot{\alpha} = c (\varepsilon^p - \varepsilon^p_\text{ps})$ |
| | Equation (9): $\dot{\alpha} = c \left( \dot{\varepsilon}^p - \frac{b}{c} \dot{\alpha} \left| \dot{\varepsilon}^p \right| \right)$ |
| **Rotational hardening (Axial component in axisymmetric condition)** | Equation (29): $\dot{\beta}_a = b_r \dot{\varepsilon}^p_\text{dev,ax} = b_t \left( \dot{\varepsilon}^p_\text{dev,ax} - \dot{\varepsilon}^p_\text{dev,ax} \right)$ |
| | Equation (28): $\dot{\beta}_a = b_t \left( \dot{\varepsilon}^p_\text{dev,ax} - \sqrt{\frac{1}{3}} \frac{1}{M_t} \beta_a \left| \dot{\varepsilon}^p_\text{dev,ax} \right| \right)$ |

The time-integration of the first equation in Equation (26) leads to the additive decomposition of the axial component of plastic deviatoric strain:

$$\varepsilon^p_{\text{dev,ax}} = \varepsilon^p_{\text{dev,ax}} + \varepsilon^p_{\text{dev,ax}}.$$  \hspace{2cm} (27)

From the second equation in Equation (26), we have

$$\dot{\beta}_a = b_t \dot{\varepsilon}^p_{\text{dev,ax}} = b_t \left( \dot{\varepsilon}^p_{\text{dev,ax}} - \dot{\varepsilon}^p_{\text{dev,ax}} \right),$$  \hspace{2cm} (28)

the time-integration of which gives the constitutive equation of rotational hardening for axial component:

$$\beta_a = b_t \varepsilon^p_{\text{dev,ax}} = b_t \left( \varepsilon^p_{\text{dev,ax}} - \varepsilon^p_{\text{dev,ax}} \right).$$  \hspace{2cm} (29)

We should also note that the last equation in Equation (26) defines the evolution rule of $\varepsilon^p_{\text{dev,ax}}$.

The key equations of the nonlinear kinematic hardening and the rotational hardening described above are summarized in Table 1, from which one can observe a similarity in the mathematical structure between these two hardening models. This similarity was discussed also in Hashiguchi based on the rate equations (10) and (15). Although the kinematic hardening model is applied to metallic materials in general, this formal similarity motivates to develop a re-formulation of the rotational hardening in an analogous manner.

### 3 | CAM-CLAY PLASTICITY COMBINED WITH ROTATIONAL HARDENING AND CONCEPT OF SUBLOADING SURFACE

This section presents a general three-dimensional constitutive formulation of the Cam-clay plasticity model at finite strain combined with the rotational hardening and the subloading surface concept. The present formulation is entirely based on the multiplicative framework. We employ a representation of the plasticity model in terms of the three stress invariants in order to take account of the Lode angle dependencies of both yielding and plastic flow, that is, the effect of intermediate principal stress, commonly observed in geomaterials.

#### 3.1 | Basic kinematics and deformation and stress tensors in finite strain elastoplasticity

This section briefly summarizes the basic framework of the finite strain elastoplasticity as a preliminary for the constitutive formulation in the subsequent sections. The basic kinematic setting and the definitions of deformation and stress tensors used in the formulation of the present constitutive model are described in this section. A comprehensive description of the deformation and stress variables used in the finite strain elastoplasticity is referred to Hashiguchi and Yamakawa.

We employ the widely-accepted kinematic assumption for the finite strain elastoplasticity based on the multiplicative elastic–plastic decomposition of the deformation gradient tensor,$^{12,13}$ namely,

$$F = F^e F^p.$$  \hspace{2cm} (30)

Figure 4 shows a schematic view of configurations introduced to establish the basic kinematic setting for finite strain elastoplasticity, wherein $F^p$ describes the plastic deformation at each material point from the reference (undeformed)
configuration $\mathcal{K}_0$ to a local, stress-free, intermediate configuration, denoted by $\tilde{\mathcal{K}}$, and $\mathcal{F}^e$ describes the elastic deformation from $\tilde{\mathcal{K}}$ to the current (deformed) configuration $\mathcal{K}$. The notion of multiplicative decomposition is fundamentally based on an assumption of existence of the intermediate configuration $\tilde{\mathcal{K}}$ such that it can be obtained at each material point from $\mathcal{K}$ by releasing the stress and the elastic strain in a neighborhood of the material point, that is, by a purely elastic unloading prescribed by $\mathcal{F}^e = \mathcal{F}^e \mathcal{F}^{-1}$. The intermediate configuration $\tilde{\mathcal{K}}$ at each material point is thereby specified by $\mathcal{F}^p = \mathcal{F}^e \mathcal{F}^{-1} \mathcal{F}$.

Following the proposal by Lion, we introduce an additional kinematic assumption that the plastic part $\mathcal{F}^p$ may further be decomposed multiplicatively as

$$\mathcal{F}^p = \mathcal{F}^p \mathcal{F}^{p\omega},$$

where $\mathcal{F}^{p\omega}$ denotes a part contributing to the rotational hardening and $\mathcal{F}^{p\delta}$ denotes a remainder part. According to the decomposition of Equation (31), a local intermediate configuration $\tilde{\mathcal{K}}$ at each material point is obtained by releasing the rotational hardening; namely, $\mathcal{F}^{p\delta} = \mathcal{F}^{p\omega} \mathcal{F}^{-1} \mathcal{F}^p$. Since $\mathcal{F}^e = \mathcal{F}^e \mathcal{F}^{p\omega} \mathcal{F}^{-1} \mathcal{F}^p = \mathcal{F}^p \mathcal{F}^{p\delta}$, the changes in the stress and a stress ratio variable for rotational hardening in a deformation-driven problem are governed by evolution equations of $\mathcal{F}^p$ and $\mathcal{F}^{p\delta}$.

A notable feature of the above-described theory for finite strains based on the multiplicative factorization is that, $\mathcal{F}^p$, and hence, its factors $\mathcal{F}^{p\omega}$ and $\mathcal{F}^{p\delta}$, are invariant under a change in reference frame, as expatiated by Henann and Anand. Therefore, the fundamental requirement of frame-indifference property of the evolution equations for $\mathcal{F}^p$ and $\mathcal{F}^{p\delta}$, and also the constitutive relation for the rotational hardening variable formulated in terms of $\mathcal{F}^{p\omega}$, is automatically satisfied. There has been a long-debated issue concerning the indeterminacy in choosing the intermediate configuration in the multiplicative decomposition (see the references and page 334, Remark 34.2 in the reference). This arbitrariness, or lack of uniqueness, of the intermediate configuration raises an issue on the invariance requirement for constitutive relations under rigid body motions superimposed on the intermediate configuration. If one assumes the intermediate configuration to be isoclinic, however, the arbitrariness issue of the intermediate configuration and the invariance questions can be eliminated.

Remark 2. The nomenclature “rotational hardening” is not attributed to any physical origin of deformation-induced anisotropy but to a constitutive modeling of anisotropy in frictional materials in such a way that the yield surface rotates around a specific point in the stress space. This is in contrast to the kinematic hardening for metallic materials, in which translation of the yield surface is considered. In cohesionless (purely frictional) materials, the rotation center is usually taken as the origin of the stress space.

3.1.1 Deformation and velocity gradient tensors

In the constitutive formulation in this study, tensorial variables defined on the configurations $\mathcal{K}_0$, $\tilde{\mathcal{K}}$, $\tilde{\mathcal{K}}$ and $\mathcal{K}$ are denoted, respectively, by uppercase letters, overtilded uppercase letters, overlined uppercase letters, and lowercase letters, unless otherwise stated.
The right Cauchy–Green deformation tensor and its plastic and elastic counterparts are given, respectively, by
\[ \mathbf{C} := \mathbf{F}^T \mathbf{F}, \quad \mathbf{C}^p := \mathbf{F}^{pT} \mathbf{F}^p, \quad \mathbf{C}^e := \mathbf{F}^{eT} \mathbf{F}^e, \]
(32)
where \( \mathbf{C} \) and \( \mathbf{C}^p \) are defined on the reference configuration \( \mathcal{K}_0 \), while \( \mathbf{C}^e \) on the intermediate configuration \( \mathcal{K} \). In a similar manner, the right Cauchy–Green-type deformation tensors related to \( \mathbf{F}^{pd} \) and \( \mathbf{F}^{ps} \) are defined, respectively, by
\[ \mathbf{C}^{pd} := \mathbf{F}^{pdT} \mathbf{F}^{pd}, \quad \mathbf{C}^{ps} := \mathbf{F}^{psT} \mathbf{F}^{ps}, \]
(33)
where \( \mathbf{C}^{pd} \) and \( \mathbf{C}^{ps} \) are defined on the configurations \( \mathcal{K}_0 \) and \( \mathcal{K} \), respectively.

Taking time-derivative of the both sides of Equation (30), and then multiplying by \( \mathbf{F}^{-1} \) from the right, we obtain the additive decomposition of the spatial velocity gradient tensor into two terms, namely,
\[ \mathbf{I} = \mathbf{I}^e + \mathbf{F}^e \mathbf{L}^p \mathbf{F}^e \mathbf{F}^{-1}, \]
(34)
in which the velocity gradient tensor \( \mathbf{I} \) and its elastic part \( \mathbf{I}^e \) defined on \( \mathcal{K} \) are given by
\[ \mathbf{I} := \dot{\mathbf{F}} \mathbf{F}^{-1}, \quad \mathbf{I}^e := \dot{\mathbf{F}}^e \mathbf{F}^{-1}, \]
(35)
while we define the plastic velocity gradient tensor on \( \mathcal{K} \) as
\[ \mathbf{L}^p := \mathbf{F}^{pT} \mathbf{F}^p \mathbf{F}^{-1}. \]
(36)
The second term in the right-hand side of Equation (34) signifies the push-forward operation of \( \mathbf{L}^p \) from \( \mathcal{K} \) toward \( \mathcal{K} \).

In a similar way, the plastic velocity gradient tensor on \( \mathcal{K} \) related to \( \mathbf{F}^{pd} \) is defined by
\[ \mathbf{L}^{pd} := \dot{\mathbf{F}}^{pd} \mathbf{F}^{pd} \mathbf{F}^{-1}. \]
(37)

### 3.1.2 Stress tensors and their invariants

The pull-back operation for the spatial Kirchhoff stress tensor \( \boldsymbol{\tau} \) defined on \( \mathcal{K} \) using \( \mathbf{F}^e \) derives the stress tensors defined on \( \mathcal{K} \), namely,
\[ \mathbf{S} := \mathbf{F}^{e-1} \mathbf{F}^{e-1} \mathbf{\tau}, \quad \mathbf{M} := \mathbf{F}^{eT} \mathbf{\tau} \mathbf{F}^{-T} = \mathbf{C}^e \mathbf{S}, \]
(38)
where \( \mathbf{S} \) is the second Piola–Kirchhoff-type stress tensor, and \( \mathbf{M} \) is the Mandel stress tensor; they are both defined on \( \mathcal{K} \). The Mandel stress tensor \( \mathbf{M} \) is work-conjugate to the plastic velocity gradient tensor \( \mathbf{L}^p \) with respect to the plastic work rate; hence, in this study these measures of stress and plastic strain rate are used in the formulation of plasticity based on the description on the intermediate configuration \(^1\). Note that \( \mathbf{S} \) is symmetric by definition, whereas \( \mathbf{M} \) is not symmetric in general except the case of elastic isotropy.

In addition to the standard stress tensors defined above, we originally introduce the second Piola–Kirchhoff-type (dimensionless) back stress ratio tensor \( \mathbf{S}^{\text{rot}} \) defined on \( \mathcal{K} \) related to the rotational hardening. Similar to Equation (38)\(^2\), the Mandel-type back stress ratio tensor is given by
\[ \mathbf{M}^{\text{rot}} := \mathbf{C}^{pT} \mathbf{S}^{\text{rot}}. \]
(39)

\(^1\) This approach is adopted in numerous studies on constitutive formulation of finite strain elastoplasticity. In contrast to this, there is another approach based on the work-conjugacy of different measures of stress and plastic strain rate, \( \mathbf{S} \) and \( \text{sym}(\mathbf{C}^e \mathbf{L}^p) \). See Chapter III, Sections 35 and 36 in the monograph by Simo\(^7\) for a concise review on this issue.
This dimensionless tensor $\tilde{M}^{\text{rot}}$ is a finite strain counterpart of the back stress ratio tensor $\beta$ appearing in Equation (14) in the small strain theory, and will be used in Section 3.1.3 to define the modified stress tensor for rotational hardening via the push-forward operation by Equation (40).

A specific function relating $\tilde{C}^{\text{ps}}$ and $\tilde{S}^{\text{rot}}$, as well as $\tilde{M}^{\text{rot}}$, which characterizes the rotational hardening, will be presented in Section 3.4. Note that $\tilde{S}^{\text{rot}}$ is intrinsically symmetric by definition, whereas $\tilde{M}^{\text{rot}}$ is not symmetric except the limited case when the two symmetric tensors $\tilde{C}^{\text{ps}}$ and $\tilde{S}^{\text{rot}}$ commute. To be more specific, this is the case when $\tilde{S}^{\text{rot}}$ is given via an isotropic function of $\tilde{C}^{\text{ps}}$. In this case, $\tilde{S}^{\text{rot}}$ and $\tilde{C}^{\text{ps}}$ are coaxial, and thus commute. This is the case in the rotational hardening rule described in Section 3.4.

The push-forward operations with the use of $F^{\text{ps}}$ converting $\tilde{S}^{\text{rot}}$ and $\tilde{M}^{\text{rot}}$ from $\tilde{\mathcal{K}}$ to $\tilde{\mathcal{K}}$ derive the back stress ratio tensors defined on $\tilde{\mathcal{K}}$, namely,

$$\tilde{S}^{\text{rot}} := F^{\text{ps}} \tilde{S}^{\text{rot}} F^{\text{ps}}^T, \quad \tilde{M}^{\text{rot}} := F^{\text{ps}}^{-T} \tilde{M}^{\text{rot}} F^{\text{ps}}^T = \tilde{C}^{\text{ps}} \tilde{S}^{\text{rot}} = \tilde{S}^{\text{rot}},$$

in which we have used the relation $\tilde{C}^{\text{ps}} := F^{\text{ps}}^{-T} \tilde{C}^{\text{ps}} F^{\text{ps}}^{-1} = I$.

The pull-back operations for the above-defined stress tensors and the back stress ratio tensors derive the ones defined on $\mathcal{K}_0$, namely,

$$S := F^{\text{p}^{-1}} \tilde{S} F^{\text{p}^{-T}}, \quad M := F^{\text{p}^{-T}} \tilde{M} F^{\text{p}^{-T}} = CS$$

and

$$\tilde{S}^{\text{rot}} := F^{\text{p}^{-1}} \tilde{S}^{\text{rot}} F^{\text{p}^{-T}}, \quad \tilde{M}^{\text{rot}} := F^{\text{p}^{-T}} \tilde{M}^{\text{rot}} F^{\text{p}^{-T}} = C^{\text{ps}} S^{\text{rot}},$$

where $S$ is the second Piola–Kirchhoff stress tensor, and $M$ is the (generally non-symmetric) Mandel-type material stress tensor; $S^{\text{rot}}$ and $M^{\text{rot}}$ are the back stress ratio tensors in the material description.

Let $I_A$, $II_A$, and $III_A$ denote the first, second, and third principal invariants of an arbitrary (generally non-symmetric) second-order tensor $A$, respectively, and let them be defined as

$$I_A := \text{tr} A,$$

$$II_A := \frac{1}{2} \left\{ (\text{tr} A)^2 - \text{tr} A^2 \right\} = \text{tr}(\text{cof} A),$$

$$III_A := \frac{1}{3} \left\{ \text{tr} A^3 - \frac{3}{2} \text{tr} A^2 \text{tr} A + \frac{1}{2} (\text{tr} A)^3 \right\} = \det A,$$

where $\text{tr} A$, $\text{cof} A$, and $\det A$ denote the trace, cofactor, and determinant of $A$, respectively. The principal invariants of the deviatoric part of $A$, defined by $A^{\text{dev}} := A - (1/3)(\text{tr} A)I$, are thus given by

$$I_{A^{\text{dev}}} = \text{tr} A^{\text{dev}} \equiv 0,$$

$$II_{A^{\text{dev}}} = -\frac{1}{2}(\text{tr}(A^{\text{dev}})^2) = -\frac{1}{2} A^{\text{dev}} : A^{\text{dev}}^T,$$

$$III_{A^{\text{dev}}} = \frac{1}{3} \text{tr}(A^{\text{dev}})^3 = \det A^{\text{dev}}.$$

Based on the above definitions, we introduce the three stress invariants of the Mandel stress tensor $\tilde{M}$ defined on $\tilde{\mathcal{K}}$:

$$\tilde{P} := \frac{1}{3} I_M = \frac{1}{3} \text{tr} \tilde{M},$$
\[ Q := (-3 \Pi_{\text{dev}})^{1/2} = \sqrt{\frac{3}{2}} \left( \text{tr}(\mathbf{M}_{\text{dev}})^2 \right)^{1/2} = \sqrt{\frac{3}{2}} \left( \mathbf{M}_{\text{dev}} : \mathbf{M}_{\text{dev}}^{T} \right)^{1/2}, \]  
\[ \Theta := \frac{1}{3} \cos^{-1} \left[ \frac{27 \Pi_{\text{dev}}}{2(-3 \Pi_{\text{dev}})^{3/2}} \right] = \frac{1}{3} \cos^{-1} \left[ \frac{\sqrt{6} \text{tr}(\mathbf{M}_{\text{dev}})^3}{\left( \text{tr}(\mathbf{M}_{\text{dev}})^2 \right)^{3/2}} \right] \text{ for } \Pi_{\text{dev}} \neq 0, \]  

with \( \mathbf{M}_{\text{dev}} := \mathbf{M} - \mathbf{P} \mathbf{I} \) denoting the deviatoric part. In Equations (49)–(51), \( \mathbf{P} \) is the mean normal stress, \( Q \) is the deviatoric invariant, and \( \Theta \) is the Lode angle with respect to \( \mathbf{M} \). In Equation (51) we take a principal branch of the arcsine function, so that \( \Theta \) takes values in the range of \( 0 \leq \Theta \leq \pi/3 \). The values of \( \Theta \), \( 0 \) and \( \pi/3 \), correspond to the stress states of axisymmetric (triaxial) extension and compression, respectively. In this study, the Lode angle dependencies in plastic yielding and flow property are introduced to describe their differences in the extension and compression sides typically observed in geomaterials.

The first and second derivatives of \( \mathbf{P}, \mathbf{Q}, \) and \( \Theta \) with respect to \( \mathbf{M} \) are required in the following constitutive formulation as well as the numerical algorithm based on the return mapping scheme. The explicit expression of these derivatives is given in Appendix A.

Remark 3. From the standpoint of plasticity modeling at finite strain, it is worth noting that the three stress invariants \( \mathbf{P}, \mathbf{Q}, \) and \( \Theta \) are identical to those of \( \mathbf{r} \), as well as to those of \( \mathbf{M} \). Let \( p, q, \) and \( \vartheta \) denote the three stress invariants of \( \mathbf{r} \), and be defined by the form similar to Equations (49)–(51) as follows:

\[ p := \frac{1}{3} r, \quad q := (-3 \Pi_{\mathbf{r}_{\text{dev}}})^{1/2}, \quad \vartheta := \frac{1}{3} \cos^{-1} \left[ \frac{27 \Pi_{\mathbf{r}_{\text{dev}}}}{2(-3 \Pi_{\mathbf{r}_{\text{dev}}})^{3/2}} \right] \text{ for } \Pi_{\mathbf{r}_{\text{dev}}} \neq 0, \]  

with \( \mathbf{r}_{\text{dev}} = \mathbf{r} - p \mathbf{I} \) denoting the deviatoric part of \( \mathbf{r} \). In a similar way, we define the three stress invariants of \( \mathbf{M} \) as

\[ P := \frac{1}{3} l_{\mathbf{M}}, \quad Q := (-3 \Pi_{\mathbf{M}_{\text{dev}}})^{1/2}, \quad \Theta := \frac{1}{3} \cos^{-1} \left[ \frac{27 \Pi_{\mathbf{M}_{\text{dev}}}}{2(-3 \Pi_{\mathbf{M}_{\text{dev}}})^{3/2}} \right] \text{ for } \Pi_{\mathbf{M}_{\text{dev}}} \neq 0, \]  

with \( \mathbf{M}_{\text{dev}} = \mathbf{M} - \mathbf{P} \mathbf{I} \) denoting the deviatoric part of \( \mathbf{M} \). Noting the identity in regard to the traces, \( \text{tr} \mathbf{M}^n = \text{tr} \mathbf{M}_{\text{dev}}^n = \text{tr} \mathbf{r}^n \) \( (n = 1, 2, 3) \) according to Equations (38)_2 and (41)_2, we can readily find the identities among the principal invariants, \( l_{\mathbf{M}} = l_{\mathbf{r}}, \Pi_{\mathbf{M}} = \Pi_{\mathbf{r}}, \) and \( \Pi_{\mathbf{M}} = \Pi_{\mathbf{r}} = \Pi_{\mathbf{r}_{\text{dev}}} = \Pi_{\mathbf{r}_{\text{dev}}} \), and thereby the identities of the three stress invariants of \( \mathbf{M}, \mathbf{M}, \) and \( \mathbf{r} \) hold, that is,

\[ P = P = p, \quad Q = Q = q, \quad \Theta = \Theta = \vartheta. \]  

It should be noted, however, that, unlike Equation (54), the identity of the stress invariants does not hold among \( \mathbf{S}, \mathbf{S}, \) and \( \mathbf{r} \).

3.1.3 Modified stress tensor for rotational hardening

The schematic diagram of the rotational hardening in the small strain case was shown in Figure 3. In order to introduce the concept of rotational hardening into the finite strain plasticity, we define the Mandel-type modified stress tensor for rotational hardening based on the intermediate configuration \( \mathbf{K} \) as

\[ \mathbf{M}^{\text{mod}} := \mathbf{M} - (-\mathbf{P} + \mathbf{P}_1) \mathbf{M}^{\text{rot}}_{\text{dev}}, \]  

where the deviatoric part of the (dimensionless) back stress ratio tensor, \( \mathbf{M}^{\text{rot}}_{\text{dev}} \), defined in Equation (40)_2, is used. \( \mathbf{P}_1 \) is the plastic tensile limit hydrostatic stress \( (0 \leq P_1 \leq P_i) \); \( P_i \) is the elastic tensile limit hydrostatic stress, which will be used in

\footnotetext{2}The square root of \( (-3 \Pi_{\mathbf{M}_{\text{dev}}}) \) contained in Equation (51) takes a real value, since, as will be seen in Remark 3, \( (-3 \Pi_{\mathbf{M}_{\text{dev}}}) = (-3 \Pi_{\mathbf{r}_{\text{dev}}}) = (3/2) \text{tr}(\mathbf{r}_{\text{dev}})^2 = (3/2) \| \mathbf{r}_{\text{dev}} \|^2 \geq 0 \), in which the equal sign applies only when \( \mathbf{r}_{\text{dev}} = \mathbf{M}_{\text{dev}} = \mathbf{O} \), that is, hydrostatic stress states.
FIGURE 5  Normal-yield and subloading surfaces, as well as the elastic limit boundary surface, of the Cam-clay plasticity with rotational hardening on the $P$–$Q$ stress plane

the formulation of hyperelasticity in Section 3.3. Note that both $P_1$ and $P_t$ are assumed to be constant in this study, whereas $P_1$ may depend on the plastic deformation inducing degradation or development of bonding between granular particles.\(^72\)

The second term in the right-hand side of Equation 55 consists of the two factors, namely, the pressure relative to $P_1$ and the deviatoric part of the back stress ratio tensor $M_{\text{dev}}^{\text{rot}}$. This term plays a role like the back stress tensor in the kinematic hardening for metal plasticity, while in the rotational hardening it is proportional to the relative pressure. This thus means that the center of rotation in the stress space is at the plastic tensile limit hydrostatic stress $P = P_t (\geq 0)$, and the effect of rotational hardening vanishes at this stress state. When $P_t = 0$, the center of rotation is specified to be at the origin of the stress space, similar to the rotational hardening models in past studies.\(^67,68,70,72–74\)

The pull-back operation for $\bar{M}_{\text{mod}}$ leads to the modified stress tensor based on the reference configuration $\mathcal{K}_0$, namely,

$$M_{\text{mod}} = F^T \bar{M}_{\text{mod}} F = M - (P - P_t)M_{\text{rot}}^{\text{dev}},$$  \hspace{1cm} (56)

where $M_{\text{rot}}$ based on $\mathcal{K}_0$ is the counterpart of $\bar{M}_{\text{rot}}^{\text{rot}}$ in Equation (55), and it is given by the constitutive equation (107).

The three stress invariants of $M_{\text{mod}}$, denoted by $\bar{P}_{\text{mod}}$, $\bar{Q}_{\text{mod}}$, and $\Theta_{\text{mod}}$, as well as those of $M_{\text{mod}}$, denoted by $P_{\text{mod}}$, $Q_{\text{mod}}$, and $\Theta_{\text{mod}}$, are defined in similar formats to Equations (49)–(51) and Equation (53), respectively. Similar to Equation (54), the identities of the three stress invariants of $M_{\text{mod}}$ and $M_{\text{mod}}$ hold, that is,

$$\bar{P} = P_{\text{mod}} = P_{\text{mod}} = P, \quad \bar{Q} = Q_{\text{mod}} = Q_{\text{mod}}, \quad \bar{\Theta} = \Theta_{\text{mod}} = \Theta_{\text{mod}},$$  \hspace{1cm} (57)

where we note the vanishing trace of the deviatoric tensor, $\text{tr} \bar{M}_{\text{dev}}^{\text{rot}} = \text{tr} M_{\text{dev}}^{\text{rot}} = 0$, in the first equation.

3.2  Cam-clay yield and plastic potential functions combined with subloading surface

3.2.1  Concept of subloading surface

We introduce the subloading surface\(^76,78–80\) inside the yield surface; see Figure 5. Hereafter, the term “yield surface” commonly used in the conventional plasticity is rephrased as the “normal-yield surface” to clarify the distinction from the subloading surface. The following fundamental properties of the subloading surface are postulated:

1. The subloading surface has a shape and orientation similar to the normal-yield surface. In the present model, the center of their similarity is fixed to the stress point $P = P_t$ on the hydrostatic axis.\(^3\)

\(^3\) An extended version of the subloading surface model has been proposed,\(^77\) which is called the “extended subloading surface model.” In this extended model, the similarity center is adopted as an internal variable, and thus moves inside the normal-yield surface depending on the evolution of plastic...
2. The ratio of the size of the subloading surface to the size of the normal-yield surface is represented by a scalar variable denoted by $R$ ($0 \leq R \leq 1$). We call it the “normal-yield ratio.”

3. A current stress point always lies on the subloading surface, even in elastic unloading processes. In other words, the subloading surface always passes through the current stress point. This is the crucial difference from the conventional plasticity model. Being endowed with this property, the subloading surface model is capable of describing the evolution of plastic strain even when a current stress point is inside the normal-yield surface.

4. The state $0 \leq R \leq R_{e}$ is the “quasi-elastic state,” where the model behaves elastically with no evolution of plastic strain. The material constant $R_{e}$ ($0 \leq R_{e} < 1$) defines the value of $R$ at the elastic limit. When $R = 0$, the subloading surface degenerates to the stress point of similarity center. The state $R_{e} < R < 1$ is referred to as the “sub-yield state,” where the subloading surface resides within the normal-yield surface. The state $R = 1$ is called the “normal-yield state,” where the subloading surface coincides with the normal-yield surface, and thus the model behaves the same as the conventional plasticity model during plastic loading.

5. The value of $R$ increases in the sub-yield state as the plastic strain evolves, approaching the unity corresponding to the normal-yield state. During this process, the subloading surface expands toward the normal-yield surface. The degree of increase in $R$ during plastic loading obeys an evolution rule, which will be described in Section 3.7. Once the value of $R$ reaches the unity, the evolution rule of $R$ becomes inactive, and thereby $R$ ceases to evolve, keeping its unity value during plastic loading.

6. The value of $R$ is decreased by elastic unloading so that the subloading surface passes through the stress point that is determined by the elastic law. In this case the subloading surface shrinks, while the normal-yield surface maintains its size and shape, since elastic unloading does not induce any change in the variables associated to plasticity.

Remark 4. The state $R > 1$, which could be called a “super-yield state,” is the one in which the subloading surface is larger than the normal-yield surface, and a current stress point is outside the normal-yield surface. This is an inadmissible state in the concept of subloading surface for rate-independent plasticity, except when viscoplasticity involving over-stress state is considered.

3.2.2 Yield function, subloading function, and plastic potential function

We adopt the subloading function $f$ and the plastic potential function $g$ expressed in terms of the three invariants of the modified stress tensor $\mathbf{M}^{\text{mod}}$, involving anisotropy by the rotational hardening, as

$$f(\mathbf{M}, \mathbf{M}^{\text{rot}}, P_{c}, R) := \frac{(\nabla^{\text{mod}})^{2}}{M(\Theta^{\text{mod}})^{2}} - \frac{(\eta^{\text{rot}})^{2}}{2} + (P^{\text{mod}} - P_{t}) \left[ \frac{P_{\text{mod}} - \{ P_{1} + R(P_{c} - P_{t}) \}}{2} \right]$$

and

$$g(\mathbf{M}, \mathbf{M}^{\text{rot}}, P_{c}, R) := \frac{(\nabla^{\text{mod}})^{2}}{N(\Theta^{\text{mod}})^{2}} - \frac{(\eta^{\text{rot}})^{2}}{2} + (P^{\text{mod}} - P_{t}) \left[ \frac{P_{\text{mod}} - \{ P_{1} + R(P_{c} - P_{t}) \}}{2} \right],$$

where $P_{c}$ denotes the compression yield stress defined by the isotropic hardening rule, which will be described in Section 3.5. The ellipsoidal subloading surface and plastic potential surface are represented by $f = 0$ and $g = 0$, respectively. The normal-yield function $f_{\text{yield}}$ is given by putting $R = 1$ in Equation (58), and the elastic limit boundary surface function $f_{\text{elast}}$ is given by putting $R = R_{e}$, namely,

$$f_{\text{yield}}(\mathbf{M}, \mathbf{M}^{\text{rot}}, P_{c}) := f(\mathbf{M}, \mathbf{M}^{\text{rot}}, P_{c}, 1), \quad f_{\text{elast}}(\mathbf{M}, \mathbf{M}^{\text{rot}}, P_{c}, R_{e}) := f(\mathbf{M}, \mathbf{M}^{\text{rot}}, P_{c}, R_{e}).$$

The normal-yield surface in the stress space is represented by $f_{\text{yield}} = 0$, and the elastic limit boundary surface by $f_{\text{elast}} = 0$. $M(\Theta^{\text{mod}})$ and $N(\Theta^{\text{mod}})$ control the aspect ratio of the ellipsoids, while their mechanical meanings are the critical stress ratio and its counterpart in the plastic potential, respectively.
The dependence of \( f \) and \( g \) on the Lode angle is taken into account by \( M(\tilde{\Theta}^\text{mod}) := M_{TC}/\zeta_f(\tilde{\Theta}) \) and \( N(\tilde{\Theta}^\text{mod}) := N_{TC}/\zeta_g(\tilde{\Theta}) \) with their values \( M_{TC} \) and \( N_{TC} \) at the axisymmetric compression state \( (\tilde{\Theta}^\text{mod} = \pi/3) \). The functions representing the Lode angle dependency in \( f \) and \( g \) are given, respectively, by \( \zeta_f(\tilde{\Theta}) := \zeta(\tilde{\Theta}^\text{mod}; \rho_f) \) and \( \zeta_g(\tilde{\Theta}) := \zeta(\tilde{\Theta}^\text{mod}; \rho_g) \) with parameters \( \rho_f \) and \( \rho_g \). The specific form of the function \( \zeta(\tilde{\Theta}^\text{mod}; \rho) \) is given in Appendix B. Usually, \( M_{TC} \geq N_{TC} \) and \( \rho_f \leq \rho_g \) are taken. When \( M_{TC} = N_{TC} \) and \( \rho_f = \rho_g \), the functions \( f \) and \( g \) are identical, and thus the plastic flow rule is associative, whereas the non-associative plastic flow rule is often adopted for geomaterials.\(^4\)

It should also be noted, in the denominator of the first term in Equations (58) and (59), that the square of \( M(\tilde{\Theta}) \) and \( N(\tilde{\Theta}) \) is subtracted by \( \xi(\eta^\text{rot})^2 \), where \( \eta^\text{rot} \) is a measure of the degree of rotational hardening defined by the deviatoric invariant of \( \tilde{M}^\text{rot} \), that is,

\[
\eta^\text{rot} := \left( -3\mathbf{I} \tilde{M}^\text{rot} \right)^{1/2} = \sqrt{\frac{3}{2} \left( \text{tr} \left( \tilde{M}^\text{rot} \right)^2 \right)}^{1/2},
\]

and \( 0 \leq \xi \leq 1 \) is the coefficient. This subtraction induces distortion, or flattening, of the ellipsoids accompanying their rotation. Figure 6 illustrates the rotation of the ellipsoidal normal-yield and subloading surfaces accompanied by the distortion (flattening). The slope of the rotation axis is represented by \( \eta^\text{rot} \). When \( \eta^\text{rot} = 0 \) and/or \( \xi = 0 \), the shape of the subloading surface is identical to the standard modified Cam-clay ellipsoid without rotation or distortion. When the rotation axis coincides with the critical state line, the ellipsoid flattens completely, and the subloading surface eventually degenerates to a line segment having the same slope as the critical state line.

This form of yield function, involving anisotropy pertinent to geomaterials, was originally proposed by Dafalias.\(^5\) He derived the anisotropic yield function in a theoretically natural manner based on the rate equation of plastic work dissipation, which is consistent with the methodology in the critical state soil mechanics. The resulting yield function was a rotated and distorted ellipse representing anisotropy induced by deviatoric plastic deformations.\(^5\)

Remark 5. Another expression of Equations (58) and (59) can be obtained by replacing \( \tilde{p}^\text{mod}, \tilde{q}^\text{mod}, \) and \( \tilde{\Theta}^\text{mod} \) directly with \( p^\text{mod}, q^\text{mod}, \) and \( \Theta^\text{mod} \), respectively. Owing to Equation (57), this alternative expression in terms of the invariants of \( M^\text{mod} \) based on \( K_0 \) is physically equivalent to Equations (58) and (59).

\(^4\)The non-associativity of the plastic flow rule is sometimes classified into two types.\(^5\) In the present model, the difference between \( M_{TC} \) and \( N_{TC} \) concerns the volumetric non-associativity, while the difference between \( \rho_f \) and \( \rho_g \) the deviatoric non-associativity.

\(^5\)Dafalias et al.\(^7\) subsequently extended this form of function to a non-associative case. Then, Dafalias and Taleb\(^4\) generalized this model to the multiaxial stress space and applied to the simulation of experiments in various conditions. It is worth mentioning that they adopted the non-associative form with \( M_{TC} > N_{TC} \), that is, volumetric non-associativity, to obtain a better fit to experimental data. We should also refer to Zhang et al.,\(^7\) in which they independently developed a similar functional form, and adopted it for their successful simulation of cyclic mobility.
Remark 6. Equations (58) and (59) take the form of quadratic function of the modified stress tensor via its invariants to represent the ellipsoidal shape in the stress space. They can be rewritten in a physically equivalent form having the order of stress. The subloading function (58) is rewritten as

\[ f(\mathbf{M}, \mathbf{M}^{\text{rot}}, P_c, R) := \sqrt{\frac{(Q_{\text{mod}})^2}{M(\mathbf{G}_{\text{mod}})}} + \left[ \rho_{\text{mod}} - \left\{ \left( 1 - \frac{R}{2} \right) P_t + \frac{R}{2} P_c \right\} \right]^2 - \frac{R}{2} (P_t - P_c). \]  

(62)

The plastic potential function (59) is also rewritten in the same way. Other alternative expressions having mathematically different but physically equivalent form can of course be available. However, the authors’ numerical experience has shown that the above form provides a better convergent property in the iterative solution process for the return-mapping equation.

### 3.3 Hyperelastic model with pressure-dependent bulk and shear moduli

In this study, we propose a new hyperelastic model for geomaterials with variable elastic bulk and shear moduli in the description on the intermediate configuration. Hyperelastic models expressed in terms of variables defined on the intermediate configuration satisfy the principle of material frame-indifference.

The pioneering study on hyperelasticity for geomaterials is the one by Houlsby.\(^8^9\) He pointed out that simplistic elasticity models with variable tangent moduli dependent on stress can result in non-conservative elastic response, and proposed an energy-conserving nonlinear elasticity model for clays with pressure-dependent bulk and shear moduli based on a two-invariant elastic potential function within the small strain framework. Borja et al.\(^9^0\) assessed the predictive capability of the two-invariant nonlinear elasticity model by comparing with measured data in clays. Borja and Tamagnini,\(^4^7\) and Callari et al.,\(^4^8\) independently extended Houlsby’s small strain nonlinear elasticity model to include finite strains using the principal logarithmic strains. Extensions to make this class of elasticity model to be applicable in tensile stress range were proposed by Tamagnini et al.\(^9^1\) within the small strain theory, and by Yamakawa et al.\(^5^4,5^5\) within the finite strain theory. Niemunis and Cudny\(^9^2\) performed a comparative study on several elasticity models for clays in view of stress and energy recoverable properties during cyclic deformations, as well as homogeneity degrees of the constitutive functions. Later, Houlsby et al.\(^9^3\) proposed an improved version of his original hyperelastic model\(^8^9\) also within the small strain framework, in which an elastic stress–strain relation, together with variable tangent moduli as power functions of the mean normal stress, is derived from an elastic potential function in terms of two invariants of the elastic strain tensor, or alternatively, from a complementary energy function in terms of two invariants of the stress tensor.

The hyperelastic model by Houlsby et al.\(^9^3\) underlies the theoretical basis of the model proposed herein. We extend their small strain nonlinear elasticity model to include finite strains. Although hyperelastic models taking account of anisotropy have been developed by several recent research works,\(^9^4 - 9^7\) we shall limit the present formulation to isotropic hyperelasticity. Limiting to isotropic elasticity, the model formulation is realized by a two-invariant expression in terms of the elastic logarithmic strain tensor and the appropriate conjugate stress tensor, and thus it is entirely based on the description on the intermediate configuration \(\mathbf{K}\). Therefore, the proposed hyperelastic model can readily be combined with plasticity models for finite strains based on the multiplicative framework. In addition, a further extension is introduced for the model to be applicable in tensile stress range.

We use the logarithmic elastic strain tensor defined on \(\mathbf{K}\):

\[ \mathbf{E}^e_{\log} := \log \mathbf{U}^e, \]  

(63)

where the elastic stretch tensor \(\mathbf{U}^e\) is given by \(\mathbf{U}^e = (\mathbf{C}^e)^{1/2}\). It can also be obtained via the right polar decomposition of \(\mathbf{F}^e\), namely, \(\mathbf{F}^e = \mathbf{R}^e \mathbf{U}^e\), where \(\mathbf{R}^e\) is the elastic rotation tensor. The volumetric and deviatoric invariants of \(\mathbf{E}^e_{\log}\) are defined by

\[ \varepsilon^e_v := \text{tr} \mathbf{E}^e_{\log} = \log(\text{det} \mathbf{U}^e) = \log J^e, \]  

(64)
\[ \epsilon_e := \sqrt{\frac{2}{3} \left\{ \text{tr} \left( \left( F_{\log}^e \right)_{\text{dev}} \right) \right\}^2} = \sqrt{\frac{2}{3} \left\| (E_{\log}^e)_{\text{dev}} \right\|} = \sqrt{\frac{2}{3} \left\| \log U_e^e \right\|}, \]  

(65)

where \( J^e := \det U^e = \det F^e \) is the elastic volume change, and \( U_{\text{dis}}^e := (J^e)^{-1/3} U^e \) is the isochoric (or purely distortional) part of \( U^e \), which is a unimodular tensor having unit determinant, \( \det U_{\text{dis}}^e = 1 \).

We define the following form of hyperelastic potential function expressed in terms of the two strain invariants:

\[
W^e(\epsilon_v^e, \epsilon_s^e) := \begin{cases} 
P_i(\epsilon_v^e - \epsilon_v^{e_0}) + \frac{\kappa^e(P_0 - P_i)}{2 - \alpha} \left\{ 1 - A \frac{(2 - \alpha)/(2 - 2\alpha)}{A} \right\} & \text{for } 0 \leq \alpha < 1, \\
\frac{P_i}{A} (\epsilon_v^e - \epsilon_v^{e_0}) + \kappa^e(P_0 - P_i)(1 - \exp B) & \text{for } \alpha = 1,
\end{cases}
\]

(66)

where

\[
A := \left\{ 1 + (1 - \alpha) \left( -\frac{\epsilon_v^e - \epsilon_v^{e_0}}{\kappa^e} \right) \right\}^2 - \frac{3\mu_0^e(1 - \alpha)}{\kappa^e(P_0 - P_i)} \left( \epsilon_s^e \right)^2 & \text{for } 0 \leq \alpha < 1, \\
B := -\frac{\epsilon_v^e - \epsilon_v^{e_0}}{\kappa^e} - \frac{3\mu_0^e}{2\kappa^e(P_0 - P_i)} \left( \epsilon_s^e \right)^2 & \text{for } \alpha = 1.
\]

(67)

The material constants in Equation (66) are the elastic compressibility index \( \kappa^e > 0 \), the reference elastic shear modulus \( \mu_0^e > 0 \), the elastic tensile limit hydrostatic stress \( P_i \geq P_t \geq 0 \), and the parameter \( \alpha \) (\( 0 \leq \alpha \leq 1 \)) controlling the pressure dependency of elastic shear modulus. The reference mean normal stress at a reference elastic volumetric strain \( \epsilon_v^{e_0} \) is denoted by \( P_0 \) (< \( P_i \)).

The second equation in Equation (66) for \( \alpha = 1 \) is derived by considering the limit \( A \frac{(2 - \alpha)/(2 - 2\alpha)}{A} \rightarrow \exp B \) as \( \alpha \rightarrow 1^- \).

Note that \( A = 1 \), \( B = 0 \), and thus \( W^e(\epsilon_v^e, \epsilon_s^e, 0) = 0 \) hold at the reference state \( (\epsilon_v^e, \epsilon_s^e, 0) = (\epsilon_v^{e_0}, 0) \). When \( 0 \leq \alpha < 1 \), \( A \geq 0 \) holds for any values of \( \epsilon_v^e \in (-\infty, +\infty) \) and \( \epsilon_s^e \in [0, +\infty) \); \( A \rightarrow +\infty \) as \( \epsilon_v^e \rightarrow +\infty \) and/or \( \epsilon_s^e \rightarrow +\infty \), since \( P_0 < P_i \). When \( \alpha = 1 \), \( B \rightarrow -\infty \) as \( \epsilon_v^e \rightarrow +\infty \) and/or \( \epsilon_s^e \rightarrow +\infty \); \( B \rightarrow +\infty \) as \( \epsilon_v^e \rightarrow -\infty \) and/or \( \epsilon_s^e \rightarrow +\infty \).

The first derivatives of Equation (66) with respect to \( \epsilon_v^e \) and \( \epsilon_s^e \) respectively give the two stress invariants, \( p \) and \( q \), defined in Equation (52)\textsubscript{1,2}, namely,

\[
p = \frac{\partial W^e(\epsilon_v^e, \epsilon_s^e)}{\partial \epsilon_v^e} = \begin{cases} 
P_i + (P_0 - P_i) \left\{ 1 + (1 - \alpha) \left( -\frac{\epsilon_v^e - \epsilon_v^{e_0}}{\kappa^e} \right) \right\} A \frac{(2 - \alpha)/(2 - 2\alpha)}{A} & \text{for } 0 \leq \alpha < 1, \\
P_i + (P_0 - P_i) \exp B & \text{for } \alpha = 1;
\end{cases}
\]

(68)

and

\[
q = \frac{\partial W^e(\epsilon_v^e, \epsilon_s^e)}{\partial \epsilon_s^e} = \begin{cases} 
3\mu_0^e \epsilon_s^e A \frac{(2 - \alpha)/(2 - 2\alpha)}{A} & \text{for } 0 \leq \alpha < 1, \\
3\mu_0^e \exp B = 3\mu_0^e \left( \frac{P_i}{P_0 - P_i} \right) \epsilon_s^e & \text{for } \alpha = 1.
\end{cases}
\]

(69)

In Equations (68) and (69), one can confirm \( p = P_0 \) and \( q = 0 \) at the reference state \( (\epsilon_v^e, \epsilon_s^e, 0) = (\epsilon_v^{e_0}, 0) \). The second derivatives of Equation (66) give the hyperelastic tangent moduli in the two-invariant expression. For \( 0 \leq \alpha < 1 \), we

\footnote{In the small strain theory, the elastic compressibility index (also commonly called the swelling index) usually means the slope of the unloading/reloading line, or the swelling/recompression line, on the semi-logarithmic \( e - \log(-p) \) plane or the \( v - \log(-p) \) plane, where \( e \) and \( v \) denote the void ratio and the specific volume, respectively (\( v = 1 + e \)). Note, in contrast, that the elastic compressibility index \( \kappa^e \) in the present formulation for finite strains means the slope of the linear double-logarithmic \( \log v - \log(-p) \) relation. It is thus different from the one in the small strain theory. Note further that \( p \) in the present formulation denotes the mean normal Kirchhoff stress, not the mean normal Cauchy stress used in the small strain theory. This issue was mentioned in the references.}
have

\[
\begin{align*}
\varepsilon^{e}_{vv} &= \frac{3\mu_0^e}{\kappa^*} \left( 1 + (1 - \alpha) \left( -\frac{\varepsilon^e_v - \varepsilon^0_v}{\kappa^*} \right) \right) - \frac{3\mu_0^e}{\kappa^*} \left( \varepsilon^e_v \right)^2 \left( P_0 - P_i \right) - \frac{3\mu_0^e}{\kappa^*} \left( \varepsilon^e_v \right)^2 A^{(3\alpha - 2)/(2 - 2\alpha)}, \\
\varepsilon^{e}_{vs} &= \frac{3\mu_0^e}{\kappa^*} \left( 1 + (1 - \alpha) \left( -\frac{\varepsilon^e_v - \varepsilon^0_v}{\kappa^*} \right) \right) \exp \frac{B}{\kappa^*} \left( P_0 - P_i \right) - \frac{3\mu_0^e}{\kappa^*} \left( \varepsilon^e_v \right)^2 \left( P_0 - P_i \right) - \frac{3\mu_0^e}{\kappa^*} \left( \varepsilon^e_v \right)^2 A^{(3\alpha - 2)/(2 - 2\alpha)}, \\
\varepsilon^{e}_{sv} &= \frac{3\mu_0^e}{\kappa^*} \left( 1 + \left( -\frac{\varepsilon^0_v}{\kappa^*} \right) \right) \exp \frac{B}{\kappa^*} \left( P_0 - P_i \right) - \frac{3\mu_0^e}{\kappa^*} \left( \varepsilon^e_v \right)^2 \left( P_0 - P_i \right) - \frac{3\mu_0^e}{\kappa^*} \left( \varepsilon^e_v \right)^2 A^{(3\alpha - 2)/(2 - 2\alpha)}.
\end{align*}
\]  

(70)

For \( \alpha = 1 \), we have

\[
\begin{align*}
\varepsilon^{e}_{vv} &= -\frac{P_0 - P_i}{\kappa^*} \exp B = -\frac{p - P_i}{\kappa^*}, \\
\varepsilon^{e}_{vs} &= \varepsilon^{e}_{sv} = -\frac{3\mu_0^e}{\kappa^*} \exp B = -\frac{3\mu_0^e}{\kappa^*} \left( \frac{p - P_i}{P_0 - P_i} \right) \varepsilon^e_s = -q, \\
\varepsilon^{e}_{ss} &= 3\mu_0^e \left( 1 - \frac{3\mu_0^e}{\kappa^*} \left( \frac{P_0 - P_i}{P_0 - P_i} \right) \varepsilon^e_s \right)^2 \exp B = 3\mu_0^e \left( \frac{p - P_i}{P_0 - P_i} \right) - \frac{q^2}{(p - P_i)} \varepsilon^e_s.
\end{align*}
\]  

(71)

Using the hyperelastic tangent moduli given by Equations (70) and (71), we have the rate equation in the two-invariant expression:

\[
\begin{align*}
\begin{bmatrix}
\dot{p} \\
\dot{q}
\end{bmatrix} =
\begin{bmatrix}
\varepsilon^{e}_{vv} & \varepsilon^{e}_{vs} \\
\varepsilon^{e}_{sv} & \varepsilon^{e}_{ss}
\end{bmatrix}
\begin{bmatrix}
\dot{\varepsilon}^e_v \\
\dot{\varepsilon}^e_s
\end{bmatrix}.
\end{align*}
\]  

(72)

Let us consider the behavior of this hyperelastic model in some typical cases.

First, we consider the case of \( \alpha = 0 \). In this case, Equation (66) takes the form

\[
W^e(\varepsilon^e_v, \varepsilon^e_s) \mid_{\alpha=0} = \frac{\kappa^*(P_0 - P_i)}{2} + \left( P_1 \varepsilon^e_v - \varepsilon^0_v \right) - \frac{\kappa^*(P_0 - P_i)}{2} \left( 1 - \frac{\varepsilon^0_v}{\kappa^*} \right) - \frac{3\mu_0^e}{\kappa^*} \left( \varepsilon^e_v \right)^2 \left( P_0 - P_i \right) - \frac{3\mu_0^e}{\kappa^*} \left( \varepsilon^e_v \right)^2 A^{(3\alpha - 2)/(2 - 2\alpha)},
\]  

(73)

from which we have the two stress invariants

\[
p = P_i + (P_0 - P_i) \left( 1 - \frac{\varepsilon^0_v}{\kappa^*} \right),
\]

\[
q = 3\mu_0^e \varepsilon^e_s
\]

(74)

and the hyperelastic tangent moduli

\[
\varepsilon^{e}_{vv} = -\frac{P_0 - P_i}{\kappa^*}, \quad \varepsilon^{e}_{sv} = \varepsilon^{e}_{sv} = 0, \quad \varepsilon^{e}_{ss} = 3\mu_0^e.
\]  

(75)

One can observe in Equations (74) and (75) that, when \( \alpha = 0 \), \( p \) is given only by \( \varepsilon^0_v \), and \( q \) only by \( \varepsilon^e_s \), leading to vanishing off-diagonal components of the hyperelastic tangent moduli matrix in Equation (72). This means no volumetric–deviatoric coupling. Further, the stress–strain relation is linear with constant bulk and shear moduli. Thus, \( p \to \pm \infty \) for infinite elastic volumetric expansion/contraction \( \varepsilon^0_v \to \pm \infty \) (double-sign corresponds), irrespective of the value of \( P_i \).

Next, we consider the case of \( \alpha = 1 \). In this case, the volumetric and deviatoric responses are fully coupled. From Equations (68) and (69) for \( \alpha = 1 \), one can observe that an infinite elastic volumetric contraction \( \varepsilon^0_v \to -\infty \) leads to \( p \to -\infty \), while expansion \( \varepsilon^0_v \to +\infty \) with \( \varepsilon^e_s \to +\infty \) leads to \( p \to P_i^\cdot \). An infinite elastic deviatoric deformation \( \varepsilon^e_s \to +\infty \) with \( \varepsilon^0_v \to +\infty \) leads to \( p \to -\infty \) and \( q \to +\infty \).
Finally, we consider purely volumetric elastic deformations with \( \varepsilon^v_s = 0 \). In this case, no deviatoric stress is induced \((q = 0)\), while the mean normal stress is given by

\[
p = \begin{cases} 
  P_1 + (P_0 - P_i) \left( 1 + (1 - \alpha) \left( -\frac{\varepsilon^v - \varepsilon^v_0}{\kappa^*} \right) \right)^{1/(1-\alpha)} & \text{for } 0 \leq \alpha < 1, \\
  P_1 + (P_0 - P_i) \exp \left( -\frac{\varepsilon^v - \varepsilon^v_0}{\kappa^*} \right) & \text{for } \alpha = 1.
\end{cases}
\]  

(76)

The hyperelastic tangent moduli for \( 0 \leq \alpha < 1 \) are

\[
\begin{align*}
  c^e_{vv} &= -\frac{P_0 - P_i}{\kappa^*} \left( 1 + (1 - \alpha) \left( -\frac{\varepsilon^v - \varepsilon^v_0}{\kappa^*} \right) \right)^{(\alpha/(1-\alpha))} = -\frac{P_0 - P_i}{\kappa^*} \left( \frac{P - P_i}{P_0 - P_i} \right)^\alpha, \\
  c^e_{sv} &= c^e_{sv} = 0, \\
  c^e_{ss} &= 3\mu_0 \left( 1 + (1 - \alpha) \left( -\frac{\varepsilon^v - \varepsilon^v_0}{\kappa^*} \right) \right)^{(\alpha/(1-\alpha))} = 3\mu_0 \left( \frac{P - P_i}{P_0 - P_i} \right)^\alpha,
\end{align*}
\]

(77)

from which, taking a limit \( \alpha \to 1^- \), we have for \( \alpha = 1 \):

\[
\begin{align*}
  c^e_{vv} &= -\frac{P_0 - P_i}{\kappa^*} \exp \left( -\frac{\varepsilon^v - \varepsilon^v_0}{\kappa^*} \right) = -\frac{P_i}{\kappa^*}, \\
  c^e_{sv} &= c^e_{sv} = 0, \\
  c^e_{ss} &= 3\mu_0 \exp \left( -\frac{\varepsilon^v - \varepsilon^v_0}{\kappa^*} \right) = 3\mu_0 \left( \frac{P_i}{P_0 - P_i} \right).
\end{align*}
\]

(78)

One can observe in Equations (77) and (78) that, during purely volumetric processes, the hyperelastic bulk and shear moduli are both proportional to the \( \alpha \)-th power of the normalized pressure, with their reference values \( c^e_{vv} = -(P_0 - P_i)/\kappa^* \) and \( c^e_{ss} = 3\mu_0 \) at \( p = P_0 \). As the stress approaches the elastic tensile limit, namely when \( p \to P_i^- \), the moduli approach null values, i.e. \( c^e_{vv} \to 0^+ \) and \( c^e_{ss} \to 0^+ \). It is also worth noting that Equation (76) is consistent with Equation (124) representing the unloading/reloading response in purely volumetric processes. This point will be discussed again later in Section 3.5.1.

According to the standard expression, the hyperelastic constitutive equation in terms of the stress and elastic deformation tensors defined on \( \tilde{\mathbf{C}} \) is given by

\[
\dot{\mathbf{S}} = 2\frac{\partial W^e}{\partial \mathbf{C}^e} = 2 \left( \frac{\partial W^e}{\partial \varepsilon^v} \frac{\partial \varepsilon^v}{\partial \mathbf{C}^e} + \frac{\partial W^e}{\partial \varepsilon^s} \frac{\partial \varepsilon^s}{\partial \mathbf{C}^e} \right) = P \tilde{\mathbf{C}}^e - 1 + \frac{Q}{3\varepsilon_s^0} (\log \tilde{\mathbf{C}}^e)_{\text{dev}} \tilde{\mathbf{C}}^{e-1},
\]

(79)

where we have used Equations (68) and (69), as well as the identity of the stress invariants argued in Equation (54). Also used in the above are the first derivatives of \( \varepsilon^v_s \) and \( \varepsilon^s_s \) with respect to \( \mathbf{C}^e \) given by

\[
\frac{\partial \varepsilon^v_s}{\partial \mathbf{C}^e} = \frac{1}{2} \tilde{\mathbf{C}}^{e-1},
\]

(80)

\[
\frac{\partial \varepsilon^s_s}{\partial \mathbf{C}^e} = \frac{1}{6\varepsilon_s^0} (\log \tilde{\mathbf{C}}^e)_{\text{dev}} \tilde{\mathbf{C}}^{e-1} \quad \text{for } \varepsilon^s_s \neq 0,
\]

(81)

while the derivation is omitted here. Since the logarithm of a second-order tensor is an isotropic function of the argument tensor, \((\log \mathbf{C}^e)_{\text{dev}} \) and \( \mathbf{C}^{e-1} \) are coaxial and both symmetric, and thus they commute. Hence, \( \frac{\partial \varepsilon^e_s}{\partial \mathbf{C}^e} \) is symmetric. Using Equation (38)\_2, we can convert Equation (79) to the expression in terms of the Mandel stress tensor defined on \( \mathbf{C}^e \):

\[
\mathbf{M} := \tilde{\mathbf{C}}^e \mathbf{S} = P I + \frac{Q}{3\varepsilon_s^0} (\log \tilde{\mathbf{C}}^e)_{\text{dev}}.
\]

(82)
One can note in Equation (82) that \( \tilde{M} \) is symmetric, since the present hyperelastic model is isotropic. Under purely volumetric elastic processes \( (\varepsilon_e^s = 0 \text{ and } \tilde{Q} = 0) \), the deviatoric part represented by the second term in the right-hand side of Equation (82) vanishes, and the stress tensor is composed only of the spherical part.

Taking material-time derivative of Equation (79) leads to the rate form

\[
\dot{S} = \mathbf{C}^e : \frac{1}{2} \dot{\mathbf{C}}^e
\]

with the fourth-order hyperelastic tangent moduli tensor

\[
\mathbf{C}^e := 2 \frac{\partial \mathbf{S}}{\partial \mathbf{C}^e} = 4 \frac{\partial^3 \mathcal{W}^e}{\partial \mathbf{C}^e \otimes \partial \mathbf{C}^e} = 4 \left\{ c_{uv}^e \frac{\partial \varepsilon_v^e}{\partial \mathbf{C}^e} \otimes \frac{\partial \varepsilon_u^e}{\partial \mathbf{C}^e} + c_{us}^e \frac{\partial \varepsilon_s^e}{\partial \mathbf{C}^e} \otimes \frac{\partial \varepsilon_u^e}{\partial \mathbf{C}^e} + c_{sv}^e \frac{\partial \varepsilon_s^e}{\partial \mathbf{C}^e} \otimes \frac{\partial \varepsilon_v^e}{\partial \mathbf{C}^e} + c_{ss}^e \frac{\partial \varepsilon_s^e}{\partial \mathbf{C}^e} \otimes \frac{\partial \varepsilon_s^e}{\partial \mathbf{C}^e} + P \frac{\partial^2 \varepsilon_{ss}^e}{\partial \mathbf{C}^e \otimes \partial \mathbf{C}^e} + Q \frac{\partial^2 \varepsilon_{ss}^e}{\partial \mathbf{C}^e \otimes \partial \mathbf{C}^e} \right\}. \tag{84}
\]

The second derivatives of \( \varepsilon_v^e \) and \( \varepsilon_s^e \) with respect to \( \mathbf{C}^e \) appearing in the above expression are given by

\[
\frac{\partial^2 \varepsilon_v^e}{\partial \mathbf{C}_{ij}^e \partial \mathbf{C}_{KL}^e} = -\frac{1}{4} \left( c_{ij}^e \mathbf{C}_{LK}^e - c_{ij}^e \mathbf{C}_{LK}^e - c_{ij}^e \mathbf{C}_{LK}^e - c_{ij}^e \mathbf{C}_{LK}^e \right), \tag{85}
\]

\[
\frac{\partial^2 \varepsilon_s^e}{\partial \mathbf{C}_{ij}^e \partial \mathbf{C}_{KL}^e} = -\frac{1}{4} \left( c_{ij}^e \varepsilon_{KL}^e - c_{ij}^e \varepsilon_{KL}^e - c_{ij}^e \varepsilon_{KL}^e - c_{ij}^e \varepsilon_{KL}^e \right) \tag{86}
\]

in which we have used the relation \( \text{tr}(\mathbf{log} \mathbf{C}^e) = \log(\det \mathbf{C}^e) \) in the derivation. The fourth-order tensor \( \mathbf{C}^e \) possesses the major and minor symmetry properties, namely, \( \mathbf{C}_{KL}^e = \mathbf{C}_{KL}^e \) and \( \mathbf{C}_{IJ}^e = \mathbf{C}_{IJ}^e \). Detailed description on the numerical computation of a tensor logarithm and its derivative can be referred to the references.103–106 For \( \varepsilon_s^e = 0 \), an approximate computation using numerical differentiation technique is utilized in this study, instead of the analytical formulae (81) and (86).

The pull-back operation for Equation (82) using Equation (41)2, with the help of Equation (54), leads to the expression on \( \mathbf{K}_0 \):

\[
\mathbf{M} = \mathbf{P}^I + Q \frac{3}{\varepsilon_s^e} \left( \log(\mathbf{C}^{C^p(-1)}) \right)_{\text{dev}}, \tag{87}
\]

where we should note that \( \mathbf{C} \) and \( \mathbf{C}^p \) are not necessarily coaxial and thus do not commute. Therefore, \( \mathbf{M} \) is not symmetric in general, even if \( \mathbf{M} \) in Equation (82) is symmetric. The derivative of \( \mathbf{M} \) with respect to \( \mathbf{C}^p \) is given by

\[
\frac{\partial \mathbf{M}}{\partial \mathbf{C}^p} = \mathbf{I} \otimes \left( c_{uv}^e \frac{\partial \varepsilon_v^e}{\partial \mathbf{C}^p} + c_{us}^e \frac{\partial \varepsilon_s^e}{\partial \mathbf{C}^p} + \frac{1}{3\varepsilon_s^e} \left( \log(\mathbf{C}^{C^p(-1)}) \right)_{\text{dev}} \otimes \left( c_{uv}^e \frac{\partial \varepsilon_v^e}{\partial \mathbf{C}^p} + c_{us}^e \frac{\partial \varepsilon_s^e}{\partial \mathbf{C}^p} \right) \right)
\]

\[
- \frac{Q}{3(\varepsilon_s^e)^2} \left( \log(\mathbf{C}^{C^p(-1)}) \right)_{\text{dev}} \otimes \frac{\partial \varepsilon_s^e}{\partial \mathbf{C}^p} + \frac{Q}{3\varepsilon_s^e} \frac{\partial \left( \log(\mathbf{C}^{C^p(-1)}) \right)_{\text{dev}}}{\partial \mathbf{C}^p}, \tag{88}
\]

where the derivative of tensor logarithm appearing in the last term is calculated as

\[
\left\{ \frac{\partial \left( \log(\mathbf{C}^{C^p(-1)}) \right)_{\text{dev}}}{\partial \mathbf{C}^p} \right\}_{IJKL} = -\frac{\delta \left( \log(\mathbf{C}^{C^p(-1)}) \right)_{IJ}}{\partial \left( \mathbf{C}^{C^p(-1)} \right)_{PQ}} \mathbf{C}_{PK} \mathbf{C}^{p-1}_{KL} \mathbf{C}^{p-1}_{Q} + \frac{1}{3} \delta_{IJ} \frac{\delta \left( \log(\mathbf{C}^{C^p(-1)}) \right)_{PQ}}{\partial \left( \mathbf{C}^{C^p(-1)} \right)_{RS}} \mathbf{C}_{RK} \mathbf{C}^{p-1}_{LS}. \tag{89}
\]
The two strain invariants $\varepsilon_v^e$ and $\varepsilon_s^e$, as well as their derivatives, can be expressed in terms of $C$ and $C^{p-1}$ as follows:

$$\varepsilon_v^e = \log \left( \frac{\det C}{\det C^p} \right)^{1/2} = \log \left\{ \det(CC^{p-1}) \right\}^{1/2} = \frac{1}{2} \text{tr} \left\{ \log(CC^{p-1}) \right\}. \quad (90)$$

$$\varepsilon_s^e = \sqrt{\frac{2}{3}} \left\{ \text{tr} \left\{ (\log(CC^{p-1}))_{\text{dev}} \right\} \right\}^{1/2}; \quad (91)$$

and

$$\frac{\partial \varepsilon_v^e}{\partial C^p_{IJ}} = - \frac{1}{2} (CC^{p-1})^{-1} QP (CC^{p-1})_{PQ} C_{JQ}^{p-1} = - \frac{1}{2} C^{p-1}_{JI}, \quad (92)$$

$$\frac{\partial \varepsilon_s^e}{\partial C^p_{IJ}} = - \frac{1}{6} \varepsilon_s^e \left\{ (\log(CC^{p-1}))_{\text{dev}} \right\}_{QP} \frac{\partial (\log(CC^{p-1}))_{RS}}{\partial (C^{p-1})_{RT}} C_{RT}^{p-1} C_{JI}^{p-1} C_{JS}^{p-1}. \quad (93)$$

### 3.4 Rotational hardening rule

Extending the basic idea described in Section 2.2, we define the constitutive equation for rotational hardening as a function of $\tilde{C}^p$ that gives the back stress ratio tensor $\tilde{\mathcal{S}}^{\text{rot}}$. The formulation presented herein is made on the configuration $\tilde{\mathcal{C}}$ in a hyperelasticity-like format, while the essential assumption is that only plastic deviatoric deformations contribute to the rotational hardening. In other words, we assume that the rotational hardening is not induced by plastic volumetric deformations nor any elastic deformations.

We use the logarithmic strain measure defined on $\tilde{\mathcal{C}}$:

$$\tilde{E}_{\log}^{p} := \log \tilde{\mathcal{U}}^p,$$  \hspace{1cm} (94)

where the stretch tensor related to rotational hardening, $\tilde{\mathcal{U}}^p$, is given by $\tilde{\mathcal{U}}^p = (\tilde{C}^p)^{1/2}$, or the right polar decomposition $F^p = R^p \tilde{U}^p$. Similar to Equations (64) and (65), the volumetric and deviatoric invariants of $\tilde{E}_{\log}^{p}$ are defined by

$$\varepsilon_v^{p} := \text{tr} \tilde{E}_{\log}^{p} = \log(\det \tilde{U}^p) = \log J^p,$$  \hspace{1cm} (95)

$$\varepsilon_s^{p} := \sqrt{\frac{2}{3}} \left\{ \text{tr} \left\{ (\tilde{E}_{\log}^{p})_{\text{dev}} \right\} \right\}^{1/2} = \sqrt{\frac{2}{3}} \left\| (\tilde{E}_{\log}^{p})_{\text{dev}} \right\| = \sqrt{\frac{2}{3}} \left\| \log \tilde{U}^p_{\text{dis}} \right\|; \quad (96)$$

where $J^p := \det \tilde{\mathcal{U}}^p = \det F^p$ is the volume change related to $F^p$, and $\tilde{U}^p_{\text{dis}} := (J^p)^{-1/3} \mathcal{U}^p$ is the isochoric part of $\mathcal{U}^p$. As mentioned above, however, the volumetric invariant $\varepsilon_v^{p}$ does not contribute to the rotational hardening. It is thus excluded in the following formulation of rotational hardening, even if $F^p$ contains volumetric component.

We define the potential function for the rotational hardening in terms of $\varepsilon_s^{p}$:

$$W^{\text{rot}}(\varepsilon_s^{p}) := \begin{cases} \frac{1}{2 - \beta} \left( 1 + 2(1 - \beta) \frac{3b_r}{2} \varepsilon_s^{p} \right)^{(2-\beta)/(2-2\beta)} & \text{for } 0 \leq \beta < 1, \\ \exp \left[ \frac{3b_r}{2} (\varepsilon_s^{p})^2 \right] - 1 & \text{for } \beta = 1, \end{cases} \quad (97)$$

where $b_r (\geq 0)$ is the dimensionless coefficient of rotational hardening, and the parameter $\beta$ ($0 \leq \beta \leq 1$) controls the nonlinearity of rotational hardening. The second equation in Equation (97) for $\beta = 1$ is derived by taking the limit of the first equation as $\beta \to 1^−$. Equation (97) satisfies $W^{\text{rot}}(0) = 0$ at the reference state $\varepsilon_s^{p} = 0$. When $b_r = 0$, $W^{\text{rot}}(\varepsilon_s^{p}) = 0$ holds, and thereby no rotational hardening evolves.
The first and second derivatives of Equation (97) provide, respectively, the deviatoric invariant of the back stress ratio tensor and the tangent modulus of the rotational hardening, namely,

\[
q_{\text{rot}} := \frac{dW_{\text{rot}}(\varepsilon^p_s)}{d\varepsilon^p_s} = \begin{cases} 
3b_t \varepsilon^p_s \left( 1 + 3b_t (1 - \beta) (\varepsilon^p_s)^2 \right)^{\beta/(2-2\beta)} & \text{for } 0 \leq \beta < 1, \\
3b_t \varepsilon^p_s \exp \left[ \frac{3b_t}{2} (\varepsilon^p_s)^2 \right] & \text{for } \beta = 1;
\end{cases}
\] (98)

and

\[
c_{\text{rot}}^{ss} := \frac{d^2W_{\text{rot}}(\varepsilon^p_s)}{d\varepsilon^p_s d\varepsilon^p_s} = \begin{cases} 
3b_t \left( 1 + 3b_t (\varepsilon^p_s)^2 \right) \left( 1 + 3b_t (1 - \beta) (\varepsilon^p_s)^2 \right)^{(3-2\beta)/(2-2\beta)} & \text{for } 0 \leq \beta < 1, \\
3b_t \left[ 1 + 3b_t (\varepsilon^p_s)^2 \right] \exp \left[ \frac{3b_t}{2} (\varepsilon^p_s)^2 \right] & \text{for } \beta = 1.
\end{cases}
\] (99)

The rate equation of the rotational hardening is expressed as

\[
\dot{q}_{\text{rot}} = c_{\text{rot}}^{ss} \dot{\varepsilon}^p_s. \tag{100}
\]

At the reference state \(\varepsilon^p_s = 0\), Equations (98) and (99) take the values \(q_{\text{rot}} = 0\) and \(c_{\text{rot}}^{ss} = 3b_t\), respectively, irrespective of the value of \(\beta\). When \(\beta = 0\), Equation (98) reduces to the linear relation \(q_{\text{rot}} = 3b_t \varepsilon^p_s\) with the constant tangent modulus \(c_{\text{rot}}^{ss} = 3b_t\).

In an analogous manner to the hyperelastic equation, the constitutive equation for rotational hardening giving the second Piola–Kirchhoff-type back stress ratio tensor \(\tilde{S}_{\text{rot}}\) defined on \(\tilde{\mathbf{C}}\) is expressed as

\[
\tilde{S}_{\text{rot}} = 2 \frac{\partial W_{\text{rot}}(\varepsilon^p_s)}{\partial \tilde{\mathbf{C}}^p_s} = 2 \frac{dW_{\text{rot}}(\varepsilon^p_s)}{d\varepsilon^p_s} \frac{\partial \varepsilon^p_s}{\partial \tilde{\mathbf{C}}^p_s} = \frac{Q_{\text{rot}}^{\text{tot}}}{3\varepsilon^p_s} (\log \tilde{\mathbf{C}}^p_s)_{\text{dev}} \tilde{\mathbf{C}}^{p_s-1}, \tag{101}
\]

in which we have used the identity of the invariants, \(q_{\text{rot}}^{\text{tot}} = Q_{\text{rot}}^{\text{tot}}\), as well as the following derivative having a similar form to Equation (81):

\[
\frac{\partial \varepsilon^p_s}{\partial \tilde{\mathbf{C}}^p_s} = \frac{1}{6\varepsilon^p_s} (\log \tilde{\mathbf{C}}^p_s)_{\text{dev}} \tilde{\mathbf{C}}^{p_s-1} \quad \text{for } \varepsilon^p_s \neq 0. \tag{102}
\]

The tensorial factors in Equation (102), \((\log \tilde{\mathbf{C}}^p_s)_{\text{dev}}\) and \(\tilde{\mathbf{C}}^{p_s-1}\), are coaxial and both symmetric, and thus they commute. Hence, \(\partial \varepsilon^p_s / \tilde{\mathbf{C}}^{p_s}\) is symmetric, and thereby one can confirm that \(\tilde{S}_{\text{rot}}\) in Equation (101) is symmetric.

Applying the conversion (39) to Equation (101) leads to the expression in terms of the Mandel-type back stress ratio tensor defined on \(\hat{\mathbf{G}}\):

\[
\hat{\mathbf{M}}_{\text{rot}} = \frac{Q_{\text{rot}}^{\text{tot}}}{3\varepsilon^p_s} (\log \tilde{\mathbf{C}}^p_s)_{\text{dev}}. \tag{103}
\]

One can observe that \(\hat{\mathbf{M}}_{\text{rot}}\) is symmetric in the present isotropic model. Further, \(\hat{\mathbf{M}}_{\text{rot}}\) is deviatoric; this is consistent with the fundamental assumption for the rotational hardening stated at the beginning of this section. Remember that Equation (101) as well as Equation (103) is the extended counterpart of Equation (29) in the axisymmetric description within the small strain theory as the multidimensional rotational hardening rule including finite strains.

Taking material-time derivative of Equation (101) leads to the rate form:

\[
\dot{\tilde{S}}_{\text{rot}} = \tilde{\mathbf{C}}^{p_s} : \frac{1}{2} \dot{\tilde{\mathbf{C}}^p_s}, \tag{104}
\]
with the fourth-order tangent moduli tensor

$$\hat{C}^\text{rot} := 2\frac{\partial^2 \hat{S}^\text{rot}}{\partial \hat{C}^\text{rot} \partial \hat{C}^\text{rot}^T} = 4\frac{\partial^2 W^\text{rot}}{\partial \hat{C}^\text{rot} \partial \hat{C}^\text{rot}^T} = 4 \left( c_{\text{rot}}^\text{ss} \frac{\partial \hat{e}_{\text{ps}}}{\partial \hat{C}^\text{rot}} \otimes \frac{\partial \hat{e}_{\text{ps}}}{\partial \hat{C}^\text{rot}} + \hat{Q}^\text{rot} \frac{\partial^2 \hat{e}_{\text{ps}}}{\partial \hat{C}^\text{rot} \partial \hat{C}^\text{rot}^T} \right),$$  

(105)

where the second derivative of $\varepsilon_{ps}^s$ with respect to $\hat{C}^\text{rot}$ is given by a similar form to Equation (86), namely,

$$\frac{\partial^2 \varepsilon_{ps}^s}{\partial C^p_{IJKL} \partial C^p_{KL}} = -\frac{1}{\varepsilon_{ps}^s} \frac{\partial \varepsilon_{ps}^s}{\partial C^p_{IJKL}} \frac{\partial \varepsilon_{ps}^s}{\partial C^p_{KL}} + \frac{1}{6 \varepsilon_{ps}^s} \left\{ \frac{\delta}{\partial C^p_{IJKL}} \frac{\partial (\log C^p_{IJKL})}{\partial C^p_{KL}} C^p_{IJ} C^p_{KL} - \frac{1}{3} \frac{\partial C^p_{IJKL}}{\partial C^p_{KJ}} \right\} \frac{\partial \varepsilon_{ps}^s}{\partial C^p_{KL}}$$

for $\varepsilon_{ps}^s \neq 0$.  

(106)

The pull-back operation for Equation (103) using Equation (42) leads to the expression on $\mathcal{K}_0$:

$$M^\text{rot} = \frac{Q^\text{rot}}{3 \varepsilon_{ps}^s} \left( \log(C^p C^p_{pq-1}) \right)_{\text{dev}},$$  

(107)

in which the identity of invariants, $\hat{Q}^\text{rot} = Q^\text{rot}$, has been used. Note that $C^p$ and $C^p_{pq}$ are not necessarily coaxial, and thus do not necessarily commute. Hence, $M^\text{rot}$ is not symmetric in general, even if $\hat{M}^\text{rot}$ in Equation (103) is symmetric. The derivatives of $M^\text{rot}$ with respect to $C^p$ and $C^p_{pq}$ are given as follows:

$$\frac{\partial M^\text{rot}}{\partial C^p} = \frac{1}{3 \varepsilon_{ps}^s} \left( \log \left( C^p C^p_{pq-1} \right) \right)_{\text{dev}} \otimes \left( c_{\text{rot}}^\text{ss} \frac{\partial \hat{e}_{ps}^s}{\partial C^p} \right) - \frac{Q^\text{rot}}{3 \varepsilon_{ps}^s} \left( \log \left( C^p C^p_{pq-1} \right) \right)_{\text{dev}} \otimes \frac{\partial \varepsilon_{ps}^s}{\partial C^p}$$

$$+ \frac{Q^\text{rot}}{3 \varepsilon_{ps}^s} \frac{\partial (\log(C^p C^p_{pq-1}))}{\partial C^p} \frac{\partial \varepsilon_{ps}^s}{\partial C^p} + \frac{Q^\text{rot}}{3 \varepsilon_{ps}^s} \frac{\partial (\log(C^p C^p_{pq-1}))}{\partial C^p} \frac{\partial \varepsilon_{ps}^s}{\partial C^p},$$  

(108)

and

$$\frac{\partial M^\text{rot}}{\partial C^p_{pq}} = \frac{1}{3 \varepsilon_{ps}^s} \left( \log \left( C^p C^p_{pq-1} \right) \right)_{\text{dev}} \otimes \left( c_{\text{rot}}^\text{ss} \frac{\partial \hat{e}_{ps}^s}{\partial C^p_{pq}} \right) - \frac{Q^\text{rot}}{3 \varepsilon_{ps}^s} \left( \log \left( C^p C^p_{pq-1} \right) \right)_{\text{dev}} \otimes \frac{\partial \varepsilon_{ps}^s}{\partial C^p_{pq}}$$

$$+ \frac{Q^\text{rot}}{3 \varepsilon_{ps}^s} \frac{\partial (\log(C^p C^p_{pq-1}))}{\partial C^p_{pq}} \frac{\partial \varepsilon_{ps}^s}{\partial C^p_{pq}} + \frac{Q^\text{rot}}{3 \varepsilon_{ps}^s} \frac{\partial (\log(C^p C^p_{pq-1}))}{\partial C^p_{pq}} \frac{\partial \varepsilon_{ps}^s}{\partial C^p_{pq}},$$  

(109)

where the derivative of tensor logarithm appearing in the last term in each of the above two equations are calculated as

$$\left\{ \frac{\delta (\log \left( C^p C^p_{pq-1} \right))_{\text{dev}}}{\partial C^p} \right\}_{IJKL} = \frac{\delta (\log \left( C^p C^p_{pq-1} \right))_{IJK} C^p_{KL}}{\partial C^p_{KL}} - \frac{1}{3} \delta_{IJK} \frac{\delta (\log \left( C^p C^p_{pq-1} \right))_{KL}}{\partial C^p_{KL}}$$

(110)

and

$$\left\{ \frac{\delta (\log \left( C^p C^p_{pq-1} \right))_{\text{dev}}}{\partial C^p_{pq}} \right\}_{IJKL} = -\frac{\delta (\log \left( C^p C^p_{pq-1} \right))_{IJK} C^p_{KL}}{\partial C^p_{KL}} - \frac{1}{3} \delta_{IJK} \frac{\delta (\log \left( C^p C^p_{pq-1} \right))_{KL}}{\partial C^p_{KL}} C^p_{IJK} C^p_{KL} + \frac{1}{3} \delta_{IJK} \frac{\delta (\log \left( C^p C^p_{pq-1} \right))_{KL}}{\partial C^p_{KL}} C^p_{IJK} C^p_{KL}. $$

(111)

The two strain invariants $\varepsilon_{ps}^p$ and $\varepsilon_{ps}^s$ can be expressed in terms of $C^p$ and $C^p_{pq}$ as follows:

$$\varepsilon_{ps}^p = \log \left( \frac{\det C^p}{\det C^p_{pq}} \right)^{1/2} = \log \left\{ \det (C^p C^p_{pq-1}) \right\}^{1/2} = \frac{1}{2} \text{tr} \left\{ \log \left( C^p C^p_{pq-1} \right) \right\},$$

(112)

$$\varepsilon_{ps}^s = \sqrt{\frac{2}{3}} \left[ \text{tr} \left\{ \left( \log \left( C^p C^p_{pq-1} \right) \right)_{\text{dev}} \right\} \right]^{1/2}. $$

(113)
Their derivatives with respect to $C^p$ and $C^{pd}$ are given by

$$
\frac{\partial \varepsilon_{ps}}{\partial C^p_{IJ}} = \frac{1}{2} \left( C^p C^{pd} - 1 \right)_{QP} \delta_{RJ} C^{pd - 1}_{RQ} = \frac{1}{2} C^{p - 1}_{JI},
$$

(114)

$$
\frac{\partial \varepsilon_{ps}}{\partial C^{pd}_{IJ}} = \frac{1}{6 \varepsilon_s} \left\{ \log \left( C^p C^{pd - 1} \right)_{dev} \right\}_{QP} \frac{\partial \left( \log \left( C^p C^{pd - 1} \right) \right)_{JS}}{\partial \left( C^p C^{pd - 1} \right)_{JS}} C^{pd - 1}_{JS}^{-1},
$$

(115)

$$
\frac{\partial \varepsilon_{ps}}{\partial C^p_{IJ}} = \frac{1}{2} \left( C^p C^{pd - 1} \right)_{QP} \left( C^p C^{pd - 1} \right)_{PQ} C^{pd - 1}_{JI} = \frac{1}{2} C^{p - 1}_{JI},
$$

(116)

$$
\frac{\partial \varepsilon_{ps}}{\partial C^{pd}_{IJ}} = \frac{1}{6 \varepsilon_s} \left\{ \log \left( C^p C^{pd - 1} \right)_{dev} \right\}_{QP} \frac{\partial \left( \log \left( C^p C^{pd - 1} \right) \right)_{RS}}{\partial \left( C^p C^{pd - 1} \right)_{RS}} C^p_{RT} C^{pd - 1}_{IT} C^{pd - 1}_{JS}.
$$

(117)

These derivatives will be used in the Jacobi matrix of the return-mapping equation expressed in terms of $C^p$ and $C^{pd}$, which will be presented in Section 4.5.

### 3.5 Isotropic hardening rule

#### 3.5.1 Isotropic hardening rule based on plastic volumetric strain

The isotropic hardening rule in terms of the plastic volumetric strain and the pre-consolidation pressure is one of the essential ingredients of the Cam-clay plasticity model. This isotropic hardening rule includes softening, that is, the decrease in pre-consolidation pressure, induced by expansive plastic volumetric strain. Another feature to be mentioned is that volumetric expansion/compression due to isotropic unloading/reloading processes is modeled as a nonlinear elasticity with pressure-dependent bulk modulus. In this study, the isotropic hardening rule is formulated as the linear double-logarithmic $\log v - \log (-p)$ relation.98–102

In accordance with the multiplicative decomposition in Equation (30), the volume change $J := \det F$ is factorized to

$$
J = \det \left( F^e F^p \right) = \left( \det F^e \right) \left( \det F^p \right) = J^e J^p,
$$

(118)

where the elastic and plastic parts are defined by $J^e := \det F^e = (\det \hat{C}^e)^{1/2}$ and $J^p := \det F^p = (\det C^p)^{1/2}$, respectively. Taking logarithm of both sides of Equation (118) leads to the additive decomposition of the logarithmic volumetric strain with its rate form:

$$
\varepsilon_v = \varepsilon^e_v + \varepsilon^p_v, \quad \dot{\varepsilon}_v = \dot{\varepsilon}^e_v + \dot{\varepsilon}^p_v
$$

(119)

where the logarithmic volumetric strain and its plastic part is defined by $\varepsilon_v := \log J$ and $\varepsilon^p_v := \log J^p$, respectively, while the elastic part has already been defined in Equation (64) and used in the formulation of hyperelasticity. Their rates are given by $\dot{\varepsilon}_v = J / J, \dot{\varepsilon}^e_v = J^e / J^e$, and $\dot{\varepsilon}^p_v = J^p / J^p$. In the following, we develop the double-logarithmic relation between the specific volume $v$ and the mean normal stress $P$. Referring to Figure 7 and noting $J = v/v_0, J^e = v^e/v_0, J^p = v^p/v_0, I^e_0 = v^e_0/v_0$, and $J^p = v/v^p$, we have

$$
\varepsilon_v = \log \frac{v}{v_0}, \quad \varepsilon^e_v - \varepsilon^e_{v0} = \log \frac{v^e}{v_0} - \log \frac{v^e_0}{v_0} = \log \frac{v^e}{v^e_0}, \quad \varepsilon^p_v = \log \frac{v}{v^p} = \log \frac{v^p}{v^p_0}.
$$

(120)

We define the following form of isotropic hardening rule expressing the compression yield stress $P_c$ (tension is positive) as a function of $\varepsilon^p_v$:

$$
P_c = P_{c+} + (P_{c0} - P_{c+}) \exp \left( -\frac{1}{\gamma_v} \varepsilon^p_v \right),
$$

(121)
where $\gamma^*$ ($>0$) is the plastic compressibility index, and $P_{c0}$ is the reference value of $P_c$ at $\varepsilon^p_v = 0 \ (J^p = 1)$. The limit of $P_c$ at infinite plastic volumetric expansion $\varepsilon^p_v \to +\infty \ (J^p \to +\infty)$ is designated by the constant $P_{c+} \ (P_{c0} < P_{c+} \leq 0)$, while the limit in compression side is $P_c \to -\infty \ as \ \varepsilon^p_v \to -\infty \ (J^p \to 0^+)$. Therefore, $P_c$ takes values in the range $-\infty < P_c < P_{c+}$.

Taking time-derivative of Equation (121) leads to the rate form of the isotropic hardening rule:

$$\dot{P}_c = -\frac{1}{\gamma^*}(P_c - P_{c+}) \dot{\varepsilon}^p_v \iff \frac{(P_c - P_{c+})^*}{P_c - P_{c+}} = -\frac{1}{\gamma^*} \dot{\varepsilon}^p_v,$$

in which $-(P_c - P_{c+})/\gamma^*$ ($>0$) can be regarded as the instantaneous modulus of isotropic hardening. Using Equation (120)$_2$, we can rewrite Equation (121) as the double-logarithmic relation

$$(\varepsilon^p_v) = \log \frac{v^p}{v^0} = -\gamma^* \log \frac{P_c - P_{c+}}{P_{c0} - P_{c+}},$$

from which we note that this isotropic hardening rule represents a linear relation between $v^p$ versus $(P_c - P_{c+})$ on the double-logarithmic scale.

Next, we consider elastic response under purely volumetric deformation. Recalling Equation (76), we have its inverse relation

$$\left( \log \frac{v^e}{v^e_0} \right) = \varepsilon^e_v - \varepsilon^e_{v0} = \begin{cases} \frac{\kappa^*}{1 - \alpha} \left\{ \left( \frac{P - P_i}{P_0 - P_i} \right)^{1-\alpha} - 1 \right\} & \text{for } 0 \leq \alpha < 1, \\
-\kappa^* \log \frac{P - P_i}{P_0 - P_i} & \text{for } \alpha = 1, \end{cases}$$

in which $(\varepsilon^e_v - \varepsilon^e_{v0})$ has been rewritten in terms of the specific volume with the use of Equation (120)$_2$. We then note that the second equation for $\alpha = 1$ in Equation (124) represents a linear swelling/recompression line (elastic compression/expansion line) as the double-logarithmic $v^e$ versus $(P - P_i)$ relation. It should also be noted, on the other hand, that the first equation for $0 \leq \alpha < 1$ represents a nonlinear relation on the double-logarithmic scale.
Taking the sum of Equations (123) and (124) gives the total logarithmic volumetric strain as the function of \( P \) and \( P_c \), namely,

\[
\left( \log \frac{v}{v_0^c} = \right) \varepsilon_v - \varepsilon_v^0 = (\varepsilon_v^e - \varepsilon_v^{e0}) + \varepsilon_v^p = \begin{cases} 
\frac{\kappa^*}{1 - \alpha} \left\{ \left( \frac{P - P_i}{P_0 - P_i} \right)^{1 - \alpha} - 1 \right\} - \gamma^* \log \frac{P_c - P_{c+}}{P_{c0} - P_{c+}} & \text{for } 0 \leq \alpha < 1, \\
-\kappa^* \log \frac{P - P_i}{P_0 - P_i} - \gamma^* \log \frac{P_c - P_{c+}}{P_{c0} - P_{c+}} & \text{for } \alpha = 1,
\end{cases}
\]

(125)
in which the first and second terms in each of the right-hand sides correspond to the elastic and plastic parts, respectively.

Let us consider the elastoplastic compression process under normal compression condition. In this case, we can put \( v^0_c = v_{c0}, P = P_c, P_0 = P_{c0} \) in Equation (125). For simplicity of the discussion, we shall consider a specific case of \( \alpha = 1 \). The left-hand side of Equation (125) can be rewritten in terms of the specific volume, and we then have for \( \alpha = 1 \):

\[
\left( \log \frac{v_c}{v_{c0}} = \right) \log \frac{v_c}{v_{c0}} + \log \frac{P_c}{P_{c0}} = -\kappa^* \log \frac{P_c - P_i}{P_{c0} - P_i} - \gamma^* \log \frac{P_c - P_{c+}}{P_{c0} - P_{c+}},
\]

(126)

where the first and second terms in both sides correspond to the elastic and plastic parts, respectively. The above equation implies that the normal compression curve as the \( v_c \) versus \( P_c \) relation on the double-logarithmic scale is nonlinear in general even when \( \alpha = 1 \), while it is linear only when \( P_1 = P_{c+} = 0 \) (Note that \( P_1 \geq 0 \) and \( P_{c+} \leq 0 \) by their definitions). In this special case, the plastic part of Equation (126) reduces to the linear double-logarithmic relation

\[
(\varepsilon_v^p =) \log \frac{v_c}{v_{c0}} = -(\lambda^* - \kappa^*) \log \frac{P_c}{P_{c0}},
\]

(127)

where we have defined the elastoplastic compressibility index \( \lambda^* := \kappa^* + \gamma^* > 0 \) which signifies the slope of the normal compression line (see Figure 7), satisfying \( \lambda^* > \kappa^* \), since \( \gamma^* > 0 \). The inverse relation of Equation (127) results in the well-known form of the isotropic hardening rule with its rate form

\[
P_c = P_{c0} \exp \left( -\frac{1}{\lambda^* - \kappa^*} \varepsilon_v^p \right), \quad \dot{P}_c = -\frac{1}{\lambda^* - \kappa^*} P_c \dot{\varepsilon}_v^p.
\]

(128)

Figure 7 illustrates the normal compression line and the swelling/recompression line in the case of \( P_1 = P_{c+} = 0 \) on the double-logarithmic scale.

### 3.5.2  Extended isotropic hardening rule dependent on deviatoric plastic strain and stress ratio

The isotropic hardening rule described in the previous section is dependent only on the plastic volumetric strain \( \varepsilon_v^p \). In this section, we extend the isotropic hardening rule by incorporating the influence of the plastic deviatoric strain \( \varepsilon_s^p \) in order to improve the capability of the model. Here, we call it “deviatoric hardening.”

Let Equation (122) be extended to

\[
P_c = -\frac{1}{\gamma^*} (P_c - P_{c+}) \varepsilon_v^p + (P_c - P_{c+}) h_s \varepsilon_s^p \quad \leftrightarrow \quad \frac{(P_c - P_{c+})}{P_c - P_{c+}} = -\frac{1}{\gamma^*} \varepsilon_v^p + h_s \varepsilon_s^p,
\]

(129)

where the second term in the right-hand side has been added so as to include the effect of the plastic deviatoric strain \( \varepsilon_s^p \). The dimensionless coefficient \( h_s \) denotes the deviatoric hardening modulus, whose definition is

\[
h_s(\chi, P, \varepsilon_s^p) := \frac{1}{\gamma^*} h_{s1} \frac{\chi^a - 1}{\chi^a - 1 + b \left( \frac{P - P_i}{P_0 - P_i} \right)^c} \exp(-s \varepsilon_s^p),
\]

(130)

where \( h_{s1} \geq 0, a \geq 0, b > 1, c (-\infty < c < +\infty), \) and \( s \geq 0 \) are material constants. Equation (130) takes account of the effects of pressure and stress ratio, as well as the saturation of deviatoric hardening when \( \varepsilon_s^p \to +\infty \). In order to introduce
Deviatoric hardening: $h_n > 0$ for $\chi > 1$

Deviatoric softening: $h_n < 0$ for $0 \leq \chi < 1$

Critical state line $h_n = 0$ for $\chi = 1$

$Q = M_d(\Theta)(-P + P_t)$

$\chi(\Psi, Q, \Theta) := \frac{Q}{M_d(\Theta)(-P + P_t)}$.

The above definition is applicable to the range $-\infty < P < P_t$ including tensile stresses; $\chi$ takes a value within the range $0 \leq \chi < +\infty$. $M_d(\Theta)$ denotes the boundary stress ratio of the deviatoric hardening and softening; the Lode angle dependency is taken into account by $M_d(\Theta) := M_{d,TC}/\zeta_d(\Theta)$ with the boundary stress ratio in the axisymmetric (triaxial) compression side, $M_{d,TC}$, and the function $\zeta_d(\Theta) := \zeta(\Theta; \rho_d)$ describing the Lode angle dependency with a parameter $\rho_d$. The specific form of the function $\zeta(\Theta; \rho_d)$ will be described later.

Here, we assume that the model exhibits deviatoric hardening when the stress ratio $Q/(-P + P_t)$ is higher than $M_d(\Theta)$, that is, $\chi > 1$, while deviatoric softening when the stress ratio $Q/(-P + P_t)$ is lower than $M_d(\Theta)$, that is, $\chi < 1$. No deviatoric hardening or softening occurs on the boundary, namely, $h_n = 0$, when $Q/(-P + P_t) = M_d(\Theta)$, that is, $\chi = 1$. This idea is illustrated on the $P$--$Q$ stress plane in Figure 8 (a). In the principal stress space, the boundary of the deviatoric hardening/softening takes a form of conical surface having its apex at $P = P_t$ on the hydrostatic axis, while its cross-sectional shape on the deviatoric stress plane is a distorted circle due to the Lode angle dependency.

The deviatoric hardening/softening property dependent on the stress ratio assumed above is modeled by the factor containing $\chi$ in Equation (130). Here, a hyperbolic form is employed to model nonlinear hardening/softening property. Figure 8 (B) shows several plots of the deviatoric hardening modulus $h_n$ as a function of $\chi$ for different values of the index $a$. When $h_{s1} > 0$, the sign of $h_n$ varies depending on the value of $\chi$ as follows:

$$h_n(\chi, P, \varepsilon_p^b) = \begin{cases} 
\frac{1}{\gamma^s} h_{s1} \left( \frac{P - P_t}{P_0 - P_t} \right)^c \exp(-s \varepsilon_p^b) < 0 & (\chi = 0), \\
0 & (0 < \chi < 1), \\
>0 & (\chi = 1), \\
\frac{1}{\gamma^s} h_{s1} \left( \frac{P - P_t}{P_0 - P_t} \right)^c \exp(-s \varepsilon_p^b) > 0 & (\chi \to +\infty).
\end{cases}$$

In addition, the influence of $P$ and $\varepsilon_p^b$ are considered in Equation (130). When $h_{s1} > 0$, $c > 0$, and $s > 0$, the limits of $h_n$ are as follows: $h_n \to +\infty$ as $P \to -\infty$; $h_n \to 0$ as $P \to P_t^-$ and/or $\varepsilon_p^b \to +\infty$.

It should be mentioned here that, unlike Equation (122) leading to Equation (121), the extended version of the isotropic hardening rule is given by the rate form (129), and it is not analytically integrable except when $h_n$ is constant. Hence, a numerical time-integration scheme is required to compute an updated value of $P_c$. This will be described in Section 4.2.
3.6 Plastic evolution rule

The plastic evolution rule in the present model is composed of the plastic flow rule and the evolution rule for rotational hardening variable. This section describes these two evolution rules. The former specifies the evolution of $F_p$, which directly determines the plastic deformation. The latter specifies the evolution of $F_{pd}$; then, $F_{ps}$ is determined by $F_{ps} = F_p F_{pd}^{-1}$, which is used to calculate the back stress ratio tensor $\mathbf{M}_{\text{rot}}$ via the rotational hardening rule described in Section 3.4.

3.6.1 Plastic flow rule

According to Equation (36), the evolution equation for the plastic deformation gradient tensor $F_p$ is given by

$$\dot{F}_p = \bar{L}_p F_p,$$

in which $\bar{L}_p$ prescribes the evolution of $F_p$.

The plastic part of the velocity gradient tensor $\bar{L}_p$ based on the intermediate configuration $\bar{\mathcal{C}}$ can be decomposed additively as follows:

$$L_p = D_p + W_p, \quad D_p := \text{sym } L_p, \quad W_p := \text{skew } L_p,$$

where $D_p$ denotes the symmetric part of $L_p$, called the plastic rate of deformation tensor on $\bar{\mathcal{C}}$, and $W_p$ the skew-symmetric part, named the plastic material spin tensor on $\bar{\mathcal{C}}$ after Dafalias.88

We define the plastic flow rule for the symmetric part $D_p$ in the non-associative form using the function $g$, namely,

$$D_p = \dot{\lambda} N^p, \quad N^p := \text{sym } \frac{\partial g}{\partial \mathbf{M}} / \left\| \text{sym } \frac{\partial g}{\partial \mathbf{M}} \right\|,$$

where $\dot{\lambda} \geq 0$ is the plastic multiplier. The direction of plastic flow is given by $N^p$, denoting the symmetric part of the outward normal to the plastic potential surface at a current stress point. The symmetric plastic flow direction tensor $N^p$ is normalized so that it has a unit magnitude satisfying $\|N^p\| = 1$; hence, the magnitude of plastic flow is given by $\|D_p\| = \dot{\lambda}$. Figure 9 illustrates the flow direction at typical stress points on the rotated plastic potential surface. As for the determination of the intermediate configuration $\bar{\mathcal{C}}$, we opt for the simplest choice of a spinless intermediate configuration.
by specifying the skew-symmetric part as
\[ \mathbf{W}^p = \mathbf{O}. \] (136)

The pull-back operation for Equation (135), with the help of the formula \( \dot{\mathbf{C}}^p = 2 \mathbf{F}^p \mathbf{D}^p \mathbf{F}^p \), leads to the expression on the reference configuration \( \mathcal{K}_0 \):
\[ \dot{\mathbf{C}}^p = \frac{2 \lambda}{\sqrt{\frac{1}{2} \text{tr} \left\{ \mathbf{C}^p \frac{\partial g}{\partial \mathbf{M}} \mathbf{C}^{-1} \left( \frac{\partial g}{\partial \mathbf{M}} \right)^T + \left( \frac{\partial g}{\partial \mathbf{M}} \right)^T \mathbf{C}^p \right\}}} \text{sym} \left\{ \left( \frac{\partial g}{\partial \mathbf{M}} \right)^T \mathbf{C}^p \right\}. \] (137)

This form will be used in the stress calculation algorithm described in Section 4.

Remark 7. In the case of isotropic hyperelasticity, \( \dot{\mathbf{S}} \) and \( \dot{\mathbf{C}}^o \) are coaxial, and thus \( \mathbf{M} \) is symmetric. It follows that \( \partial g / \partial \mathbf{M} \) in Equation (135) is symmetric. In this case, the following equality holds (the proof is omitted but straightforward using a pull-back operation):
\[ \left( \frac{\partial g}{\partial \mathbf{M}} \right)^T \mathbf{C}^p = \mathbf{C}^p \left( \frac{\partial g}{\partial \mathbf{M}} \right), \] (138)
which implies that both \( \left( \frac{\partial g}{\partial \mathbf{M}} \right)^T \mathbf{C}^p \) and \( \mathbf{C}^p \left( \frac{\partial g}{\partial \mathbf{M}} \right) \), appearing in Equation (137) as the argument of symmetrization operator, are symmetric, although \( \partial g / \partial \mathbf{M} \) is not symmetric or coaxial with \( \mathbf{C}^p \), and therefore they do not commute in general.

3.6.2 Evolution rule for rotational hardening variable

According to Equation (37), the evolution equation for \( \mathbf{F}^{pd} \) is given by
\[ \dot{\mathbf{F}}^{pd} = \mathbf{L}^{pd} \mathbf{F}^{pd}. \] (139)

The velocity gradient tensor \( \mathbf{L}^{pd} \) on \( \dot{\mathbf{K}} \) related to the rotational hardening can be decomposed additively into the symmetric and skew-symmetric parts, namely,
\[ \mathbf{L}^{pd} = \mathbf{D}^{pd} + \mathbf{W}^{pd}, \quad \mathbf{D}^{pd} := \text{sym} \mathbf{L}^{pd}, \quad \mathbf{W}^{pd} := \text{skew} \mathbf{L}^{pd}. \] (140)

Referring to Equation (26) for small strain axisymmetric case, we define the evolution rule for \( \mathbf{D}^{pd} \) and \( \mathbf{W}^{pd} \) as follows:
\[ \mathbf{D}^{pd} = \lambda \left[ \frac{3}{2 M_r(\dot{\Theta})} \right] \text{sym} \mathbf{M}^{\text{rot}}_{\text{dev}}, \quad \mathbf{W}^{pd} = \mathbf{O}, \] (141)
where the latter equation specifies that the configuration \( \dot{\mathbf{K}} \) is spinless, similar to Equation (136) for the intermediate configuration \( \dot{\mathbf{K}} \). The back stress ratio tensor \( \mathbf{M}^{\text{rot}}_{\text{dev}} \) on \( \dot{\mathbf{K}} \) is given by the hyperelastic-type constitutive equation (103), which intrinsically provides a deviatoric tensor. The limit of rotational hardening is specified by the stress ratio at a rotation limit, defined by \( M_r(\dot{\Theta}) := M_{r,TC} / \zeta_r(\dot{\Theta}) = M_{r,TC} / \zeta_r(\dot{\Theta}; \rho_r) \) with its value at the axisymmetric (triaxial) compression state, \( M_{r,TC} \), and the parameter representing the Lode angle dependency, \( \rho_r \). Note that the identity of the Lode angle, like Equation (54), also applies to \( \dot{\Theta} \) on \( \dot{\mathbf{K}} \).
The pull-back operation of Equation (141), with the help of the formula $C^{pd} = 2F^{pd} \dot{D}^{pd} F^{pd}$, leads to the expression on the reference configuration $\mathcal{K}_0$:

$$\dot{C}^{pd} = 2\dot{\lambda} \sqrt{\frac{3}{2}} \frac{1}{M_r(\Theta)} \text{sym} (M^{rot}_{dev} C^{pd}).$$

(142)

**Remark 8.** As is the case mentioned in Remark 7, if the constitutive equation for rotational hardening described in Section 3.4 is isotropic, $M^{rot}_{dev}$ is symmetric (this is the case in the specific form of Equation (103)). In this case, the product $M^{rot}_{dev} C^{pd}$ appearing in Equation (142) is symmetric (it can readily be proved via the pull-back of $M^{rot}_{dev}$, while omitted here). This symmetricity holds even though $M^{rot}_{dev}$ is not symmetric or coaxial with $C^{pd}$, and thus they do not commute in general.

### 3.7 Evolution rule of normal-yield ratio $R$ for subloading surface

The normal-yield ratio $R$ plays a crucial role in the subloading surface concept; hence, the evolution rule of $R$ is of the utmost importance in the constitutive formulation incorporating the subloading surface. Based on the fundamental assumption that the normal-yield ratio $R$ evolves along with the progress of plastic strain, we define the evolution rule of $R$ in the following form $^{78-80,85}$:

$$\dot{R} = \dot{\lambda} U(R),$$

(143)

where the plastic multiplier $\dot{\lambda}$ represents the magnitude of plastic strain rate, and $U(R)$ is the function that controls the degree of evolution satisfying

$$\begin{align*}
U(R) &\to +\infty & (0 \leq R \leq R_c): \text{quasi-elastic state}; \\
U(R) &> 0, \ U'(R) < 0 & (R_c < R < 1): \text{sub-yield state}; \\
U(R) &\to 0^+ & (R \to 1^-): \text{approaching to normal-yield state}; \text{ and} \\
U(1) & = 0 & (R = 1): \text{normal-yield state}.
\end{align*}$$

(144)

The material constant $R_c (0 \leq R_c < 1)$ denotes the value of normal-yield ratio at an upper limit of quasi-elastic state. Figure 10 illustrates a typical behavior of the function $U(R)$. When the material is in a state within the range $0 \leq R \leq R_c$, the right-hand side of Equation (143) must take an indeterminate form with $U(R) \to +\infty$ and $\dot{\lambda} = 0$ so that $\dot{R}$ in the left-hand side takes a finite value. This means that the material exhibits purely elastic response when $0 \leq R \leq R_c$. When $R \to 1^-$, we have $U(R) \to 0^+$ implying the convergence of $\dot{R}$, which means that the material approaches a normal-yield
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state taking $R = 1$. We here employ the following specific form of $U(R)$ which satisfies the above-stated function property:

$$U(R) := u \cot \left( \frac{\pi}{2} \langle R - R_e \rangle \right)$$  \hspace{1cm} (145)

with two material constants $u (> 0)$ and $R_e$. The evolution coefficient $u$ regulates the degree of evolution of $R$ with plastic strain rate, and thereby regulates the smoothness of the transition from elastic to plastic state. The angle brackets $\langle \cdot \rangle$ denote the ramp function of a scalar argument $a$, namely,

$$\langle a \rangle := \max(a, 0) = \begin{cases} a & \text{for } a \geq 0, \\ 0 & \text{for } a < 0, \end{cases}$$  \hspace{1cm} (146)

which is also often denoted by the Macaulay brackets $\{a\}$.

According to Hashiguchi and Mase\textsuperscript{73} and Zhang et al.\textsuperscript{71} we extend the evolution coefficient $u$ in Equation (145) by introducing the functional dependence of $u$ on the mean normal stress $P$, the Lode angle $\Theta^\text{mod}$, and the accumulated logarithmic plastic deviatoric strain $\varepsilon^p_{s,\text{accum}}$, in order to improve the model capability to describe cyclic behavior. This extension leads to\textsuperscript{7}

$$u(P, \Theta^\text{mod}, \varepsilon^p_{s,\text{accum}}) = u_\infty + (u_0 - u_\infty) \left( \frac{P - P_1}{P_0 - P_1} \right)^m \{M(\Theta^\text{mod})\}^{-\nu} \exp \left( -\zeta \varepsilon^p_{s,\text{accum}} \right),$$  \hspace{1cm} (147)

in which we define

$$\varepsilon^p_{s,\text{accum}} := \int_0^t |\dot{\varepsilon}^p_s| \, dt.$$  \hspace{1cm} (148)

$u_0 (> 0)$ is the reference evolution coefficient; $u_\infty$ ($0 \leq u_\infty < u_0$) regulates the lower limit of $u$; and $m$, $\nu$, and $\zeta$ are non-negative material constants denoting the effects of $P$, $\Theta^\text{mod}$, and $\varepsilon^p_{s,\text{accum}}$ on the evolution of $R$, respectively. Equation (147) reduces to $u = u_0$ (constant) when $m = \nu = \zeta = 0$; otherwise the function $u$ behaves as:

$$\begin{cases} u \to u^+_0 & (m > 0, \ P \to P_1^-); \text{ and} \\
 u \to u^+_\infty & (\zeta > 0, \ \varepsilon^p_{s,\text{accum}} \to +\infty), \end{cases}$$  \hspace{1cm} (149)

which mean that the value of function $u$ becomes smaller as the pressure becomes smaller and/or the deviatoric plastic strain accumulates, leading to the stagnation of evolution of $R$. In addition, the dependence of $u$ on the Lode angle $\Theta^\text{mod}$ describes the difference in the evolution of $R$ in the axisymmetric (triaxial) compression and extension sides. More specifically, the critical stress ratio $M(\Theta^\text{mod})$ is generally larger in the compression side than in the extension side. Therefore, when $\nu > 0$, the function $u$ takes smaller values in the compression side than in the extension side, leading to slower evolution of $R$ and inducing larger plastic strains due to plastic loading in the compression side. It should also be noted in Equation (148) that the equality $\varepsilon^p_{s,\text{accum}} = \varepsilon^p_s$ does not necessarily hold, since $\varepsilon^p_s$ may take negative values in, for instance, reversal plastic loadings.

By applying the extended function $u$ of Equation (147) to Equation (145), we have the extended version of Equation (143), namely,

$$\dot{R} = \lambda U(R, P, \Theta^\text{mod}, \varepsilon^p_{s,\text{accum}}) \quad \text{with} \quad U(R, P, \Theta^\text{mod}, \varepsilon^p_{s,\text{accum}}) := u(P, \Theta^\text{mod}, \varepsilon^p_{s,\text{accum}}) \cot \left( \frac{\pi}{2} \langle R - R_e \rangle \right).$$  \hspace{1cm} (150)

The numerical integration scheme for the evolution rule (150) to compute an updated value of $R$ will be described in Section 4.3.

\textsuperscript{7} Zhang et al.\textsuperscript{71} adopted a quadratic dependence on the pressure by setting $m = 2$. 

4 IMPLICIT STRESS CALCULATION ALGORITHM USING RETURN-MAPPING

4.1 Incremental form of plastic evolution rules

The incremental form of the plastic evolution rules is derived in this section by integrating the rate equations, presented in Section 3.6, over a time interval \([t_n, t_{n+1}]\) corresponding to an incremental step in the stress calculation. This provides the update formulas for deformation variables associated to plasticity.

The plastic evolution rules (137) and (142) can be expressed symbolically as

\[
\dot{A}(t) = \text{sym}\{\mathcal{Z}(t)A(t)\}, \tag{151}
\]

where \(A(t)\) denotes a symmetric tensor symbolizing \(C^p\) in Equation (137), and \(C^{p_d}\) in Equation (142), while \(\mathcal{Z}(t)\) denotes a generally non-symmetric tensor representing their evolution coefficient tensors. When \(\mathcal{Z}\) is a constant tensor, that is, \(\mathcal{Z}(t) \equiv \mathcal{Z}\), Equation (151) is analytically integrable; integration over the time interval \([t_n, t_{n+1}]\) exactly gives

\[
A_{n+1} = QA_nQ^\top, \quad Q := \exp\left(\frac{1}{2}Z\Delta t\right), \tag{152}
\]

in which we have put \(A_n := A(t_n), A_{n+1} := A(t_{n+1})\), and \(\Delta t := |t_{n+1} - t_n|\). The tensor exponential function, denoted by \(Q\), plays a role as the integration operator. Numerical computation of the exponential function of a tensor, as well as its derivative, can be referred to the references.\(^{103-105,107}\) In general, \(\mathcal{Z}(t)\) may vary depending on stress states. We thereupon employ an approximate integration like the backward difference scheme with the use of \(Z_{n+1} := Z(t_{n+1})\), which leads to the implicit update formula\(^8\)

\[
A_{n+1} = Q_{n+1}A_nQ_{n+1}^\top, \quad Q_{n+1} := \exp\left(\frac{1}{2}Z_{n+1}\Delta t\right). \tag{153}
\]

Based on the above formula, we obtain for \(C^p\) and \(C^{p_d}\):

\[
C_{n+1}^p = Q_{n+1}^pC_n^pQ_{n+1}^p\top, \quad C_{n+1}^{p_d} = Q_{n+1}^{rot}C_n^{p_d}Q_{n+1}^{rot}\top \tag{154}
\]

with the exponential integration operators

\[
Q_{n+1}^p := \exp\left(\frac{1}{2}Z_{n+1}\Delta t\right), \quad \frac{1}{2}Z_{n+1}\Delta t := \frac{\Delta \lambda}{\sqrt{\frac{1}{2} \text{tr} \left\{ C_n^p \left( \frac{\partial g}{\partial M} \right)_{n+1} - C_{n+1}^{p-1} \left( \frac{\partial g}{\partial M} \right)_{n+1} + \left( \frac{\partial g}{\partial M} \right)_{n+1}^2 \right\} + \left( \frac{\partial g}{\partial M} \right)_{n+1}^2}} \left( \frac{\partial g}{\partial M} \right)_{n+1}^\top \right. \tag{155}
\]

and

\[
Q_{n+1}^{rot} := \exp\left(\frac{1}{2}Z_{n+1}\Delta t\right), \quad \frac{1}{2}Z_{n+1}\Delta t := \Delta \lambda \sqrt{\frac{3}{2} M_{i(\Theta_{n+1})} M_{i,\text{dev},n+1}}, \tag{156}
\]

where we have put the incremental plastic multiplier \(\Delta \lambda := \dot{\lambda}\Delta t\).

4.2 Integration of isotropic hardening rule

As was previously mentioned in Section 3.5.2, the extended version of the isotropic hardening rule, given by the rate form (129), is not analytically integrable except when \(h_s\) is constant. Hence, an approximate time-integration scheme is required

\(^8\) Alternative schemes, such as the forward difference and the generalized mid-point rule, can of course be employed. If the forward difference scheme is adopted, we obtain the explicit update formula.
to compute an updated value of the compression yield stress $P_c$ at the time $t_{n+1}$. Integrating Equation (129) over the time interval $[t_n, t_{n+1}]$ leads to

$$
\int_{t_n}^{t_{n+1}} \frac{(P_c - P_{c+})}{P_c - P_{c+}} \, dt = -\frac{1}{\gamma^*} \int_{t_n}^{t_{n+1}} \epsilon_v^p \, dt + \int_{t_n}^{t_{n+1}} h_s(\chi, P, \epsilon_v^p) \epsilon_v^p \, dt,
$$

(157)

from which we obtain the update formula for $P_c$:

$$
P_{c,n+1} = P_{c+} + (P_{c,n} - P_{c+}) \exp \left\{ -\frac{1}{\gamma^*} (\epsilon_v^{p,n} - \epsilon_v^{p,n+1}) + h_{s,n+1}(\epsilon_v^{p,n+1} - \epsilon_v^{p,n}) \right\}.
$$

(158)

Since, in fact, the deviatoric hardening modulus $h_s(\chi, P, \epsilon_v^{p,n})$ varies within the time interval, the above result is not an exact integration but an implicit approximation using $h_{s,n+1} := h_s(\chi_{n+1}, P_{n+1}, \epsilon_v^{p,n+1})$.

### 4.3 Integration of evolution rule of normal-yield ratio $R$

When the evolution coefficient $u$ in Equation (145) is constant, the evolution rule of $R$ given by (143) is analytically integrable in the following way. Consider again the time interval $[t_n, t_{n+1}]$ on which an incremental deformation is given. Suppose that the model exhibits elastic response within a subinterval $[t_n, t_{n+\vartheta}]$ ($0 \leq \vartheta \leq 1$), and plastic loading in the remainder $[t_{n+\vartheta}, t_{n+1}]$; that is, the model transits from elastic to plastic state at $t_{n+\vartheta}$. In this case, the inequality $R_{n} \leq R_{n+\vartheta} = R_e \leq R_{n+1}$ holds, and we can put $\Delta \lambda := \int_{t_{n+\vartheta}}^{t_{n+1}} \dot{\lambda} \, dt$. Equation (143), with the use of Equation (145), can be converted to a form in which the two variables $R$ and $\lambda$ are separated as follows:

$$
\dot{R} = \lambda u \cot \left( \frac{\pi}{2} \frac{\langle R - R_e \rangle}{R_e - 1} \right) \cot \left( \frac{\pi}{2} \frac{\langle R - R_e \rangle}{R_e - 1} \right) \left( \frac{R - R_e}{R_e - 1} \right) = \frac{u}{R_e - 1} \Delta \lambda,
$$

(159)

which is integrated analytically by

$$
\int_{t_{n+\vartheta}}^{t_{n+1}} \left\{ \cos \left( \frac{\pi}{2} \frac{\langle R - R_e \rangle}{R_e - 1} \right) \right\} \left( \frac{R - R_e}{R_e - 1} \right) \, dR = -\frac{u}{R_e - 1} \int_{t_{n+\vartheta}}^{t_{n+1}} \cos \left( \frac{\pi}{2} \frac{\langle R_{n+\vartheta} - R_e \rangle}{R_e - 1} \right) \, d\lambda \rightarrow \log \left\{ \cos \left( \frac{\pi}{2} \frac{\langle R_{n+1} - R_e \rangle}{R_e - 1} \right) \right\} = -\frac{u}{R_e - 1} \Delta \lambda,
$$

(160)

leading to the update formula for the normal-yield ratio $R$:

$$
R_{n+1} = R_e + \frac{2}{\pi} (1 - R_e) \cos^{-1} \left\{ \cos \left( \frac{\pi}{2} \frac{\langle R_{n+\vartheta} - R_e \rangle}{R_e - 1} \right) \exp \left( -\frac{u}{R_e - 1} \Delta \lambda \right) \right\},
$$

(161)

Note again that $R_{n+\vartheta} = R_e$ at the transition from elastic to plastic state, and thereby the argument of ramp function $\langle \cdot \rangle$ in Equation (161) vanishes. Otherwise, when the model exhibits plastic loading over the whole interval $[t_n, t_{n+1}]$, the inequality $R_e \leq R_n < R_{n+1}$ holds, and thus we can put $\vartheta = 0$, $R_{n+\vartheta} = R_n$, and $\Delta \lambda := \int_{t_{n+1}}^{t_{n+1}} \dot{\lambda} \, dt$ in Equation (161).

The extended version (150) with $u$, defined by Equation (147) taking account of the dependence on stress and plastic strain, is not integrable in analytical ways. In this case, we can integrate Equation (159) in a semi-analytical manner by replacing the constant coefficient $u$ in Equation (161) with $u_{n+1} := u(P_{n+1}, \Theta_{mod,n+1}, \epsilon_v^{p,n+1}, \epsilon_s^{p,accum,n+1})$. The updated value $R_{n+1}$ is computed directly by the update formula at every iteration during the solution process for the return-mapping equation.

The accumulated logarithmic plastic deviatoric strain $\epsilon_s^{p,accum}$, included in the function $u$, is updated according to Equation (148), namely,

$$
\epsilon_s^{p,accum,n+1} = \epsilon_s^{p,accum,n} + \left| \epsilon_s^{p,n+1} - \epsilon_s^{p,n} \right|.
$$

(162)

Another way to integrate Equation (150) would be the use of forward or backward Euler difference method. However, the authors’ numerical experience has shown that the semi-analytical integration presented herein provides a better accuracy.
4.4 Elastic predictor and loading/unloading judgment

Let the stress, strain, and all internal variables associated to plasticity be already computed and completely known at the time \( t_n \). Consider a time interval \([t_n, t_{n+1}]\) where an incremental deformation is imposed onto the known state at \( t_n \) by the relative deformation gradient tensor \( f_{[n,n+1]} \). The deformation gradient tensor is thereby updated by

\[
F_{n+1} = f_{[n,n+1]} F_n,
\]

where \( f_{[n,n+1]} \) corresponding to the incremental displacement \( \Delta u \) at a spatial point \( x_n \) is given by

\[
f_{[n,n+1]} := \frac{\partial x_{n+1}}{\partial x_n} = \frac{\partial (x_n + \Delta u)}{\partial x_n} = I + \Delta u \otimes \nabla x_n \quad (= F_{n+1} F_n^{-1}).
\]

The right Cauchy–Green deformation tensor \( C_{n+1} \) is then computed by

\[
C_{n+1} := F_{n+1}^T F_{n+1}.
\]

The elastic trial state is determined under the assumption that this incremental process is purely elastic without any progress of plastic deformations. This can be done by freezing plastic flow, so that we put

\[
C^p_{n+1} := C^p_n, \quad C^{p,\text{tr}}_{n+1} := C^{p,\text{tr}}_n, \quad C^d_{n+1} := C^d_n, \quad R^\text{tr}_{n+1} := R_n,
\]

where the superscript "tr" means the trial value. The elastic trial stresses \( M_{n+1}^\text{tr} \) and \( M_{n+1}^{\text{rot, tr}} \) are computed by substituting \( C^p_{n+1}, C^{p,\text{tr}}_{n+1}, \) and \( C^d_{n+1} \) into the hyperelastic equation (87) and the rotational hardening rule (107), respectively, and then converting \( S_{n+1}^\text{tr} \) and \( S_{n+1}^{\text{rot, tr}} \) by Equations (41)\_2 and (42)\_2. These elastic trial stresses, together with \( P^\text{tr}_{c,n+1} \) and \( R^\text{tr}_{n+1} \) given by Equation (166)\_3,\_4, determine the elastic trial state.

Once the elastic trial state is determined, we can make a judgment as to whether the incremental process is elastic or plastic according to the loading/unloading criterion \(^{10}\):

\[
\begin{cases}
  f_{n+1}^\text{tr} \leq 0 \quad \text{or} \quad f_{\text{elast},n+1}^\text{tr} \leq 0 & \quad \text{Elastic step}, \\
  f_{n+1}^\text{tr} > 0 \quad \text{and} \quad f_{\text{elast},n+1}^\text{tr} > 0 & \quad \text{Plastic step}.
\end{cases}
\]

Here we use the values of the subloading function (58) and the elastic limit boundary surface function (60)\_2 at the elastic trial state

\[
f_{n+1}^\text{tr} := f(M_{n+1}^\text{tr}, M_{n+1}^{\text{rot, tr}}, p^\text{tr}_{c,n+1}, R_{n+1}^\text{tr}), \quad f_{\text{elast},n+1}^\text{tr} := f(M_{n+1}^\text{tr}, M_{n+1}^{\text{rot, tr}}, p_{c,n+1}^{\text{tr}}, R_{n+1}).
\]

If the incremental step is judged to be elastic, the variables at the elastic trial state is the actual solution for the updated state, while the value of \( R \) needs to be updated so that the subloading surface passes through the updated stress point. This can be accomplished by solving the equation \( f_{n+1} = 0 \) for \( R_{n+1} \). Otherwise, if the step is plastic, the return-mapping described in the subsequent section is performed to compute the variables at the updated state.

\(^{10}\) Stress increments pointing inward of the subloading surface do not always lead to purely elastic unloading. In other words, the subloading surface model may exhibit a reverse plastic loading after an elastic unloading when a large stress increment in such direction is imposed. In such cases, the loading/unloading criterion (167), merely based on the elastic trial state, may make an incorrect judgment. To overcome this drawback, improved methods of loading/unloading judgment have been developed,\(^{108,109}\) while not yet implemented to the analysis code used in this study.
in the conventional plasticity, but also the elastic limit boundary surface function \( f_{\text{elast}} \). This is due to the essential nature of the subloading surface plasticity stated in Section 3.2.1 (particularly, in the items 3, 4, and 6, therein).

### 4.5 Return-mapping

The return-mapping equation is comprised of the incremental form of the plastic evolution rules (154) and the subloading surface function (58), or its variant form (62), namely,

\[
\begin{align*}
Y_{Cp}(C_{n+1}, C_{n+1}^d, \Delta \lambda) & := C_{n+1}^p - Q_{n+1}^{p} C_{n+1}^{pT} = 0, \\
Y_{Cpd}(C_{n+1}^{p}, C_{n+1}^{pd}, \Delta \lambda) & := C_{n+1}^{pd} - Q_{n+1}^{pd} C_{n}^{pdT} = 0, \\
Y_{f}(C_{n+1}^{p}, C_{n+1}^{pd}, \Delta \lambda) & := f(M_{n+1}, M_{n+1}^\text{rot}, P_{c,n+1}, R_{n+1}) = 0,
\end{align*}
\]

which is complemented by Equations (87), (107), (158), and (161). Given that \( C^p \) and \( C^{pd} \) are symmetric, the set of simultaneous equations (169a)–(169c) has totally 13 unknowns, that is, six independent components of each of the two tensors \( C_{n+1}^p \) and \( C_{n+1}^{pd} \), and the scalar \( \Delta \lambda \).

This system of nonlinear equations can be solved by means of the Newton–Raphson iterative method. Let \( X \) and \( Y \) denote symbolically the unknown vector and the residual vector, respectively. We then have a symbolic expression of the return-mapping equation:

\[
Y(X) = 0, \quad Y := \begin{bmatrix} Y_{Cp} \\ Y_{Cpd} \\ Y_f \end{bmatrix}, \quad X := \begin{bmatrix} C_{n+1}^p \\ C_{n+1}^{pd} \\ \Delta \lambda \end{bmatrix}.
\]

Further, let \( \delta X \) denote the corrector of \( X \) in each iteration. Linearization of \( Y(X) \) with respect to \( \delta X \), and then equating its result to zero vector lead to

\[
Y(X^{(k)}) + \frac{\partial Y}{\partial X} \bigg|_{X^{(k)}} \delta X = 0,
\]

where an entity affixed with the superscript “\((k)\)” signifies the one at \( k \)-th iteration. \( \frac{\partial Y}{\partial X} \) is the \( 13 \times 13 \) Jacobi matrix consisting of the derivatives of Equations (169a)–(169c) with respect to the unknown variables, \( C_{n+1}^p \), \( C_{n+1}^{pd} \), and \( \Delta \lambda \). Its analytical derivation involves cumbersome manipulation of differentiation, or, alternatively, numerical differentiation using a perturbation technique based on a forward difference approximation can be employed. The procedure to update the unknowns in an iteration is as follows:

\[
\delta X = - \left( \frac{\partial Y}{\partial X} \bigg|_{X^{(k)}} \right)^{-1} Y(X^{(k)}) \implies X^{(k+1)} := X^{(k)} + \delta X,
\]

where the convergence criterion is

\[
\|Y(X^{(k+1)})\| < \text{TOL},
\]

with “TOL” being the tolerance. In the numerical examples presented in Section 5, we set \( \text{TOL} = 1.0 \times 10^{-12} \).

### 5 NUMERICAL EXAMPLES

This section presents several numerical examples to demonstrate the verification of the constitutive model and the stress calculation algorithm developed in this study. The material constants and initial values required in this model are listed in Table 2; their values used in the analysis are given in the succeeding texts.
TABLE 2  Material constants and initial values used in numerical analysis

| Material constants and symbols | Material constants and symbols |
|--------------------------------|--------------------------------|
| Hyperelasticity:              | Evolution of normal-yield ratio: |
| Elastic compressibility index, \( \kappa^* \) | Elastic limit normal-yield ratio, \( R_e \) |
| Reference elastic shear modulus, \( \mu_0^* \) | Reference evolution coefficient, \( u_0 \) |
| Pressure dependency parameter, \( \alpha \) | Minimum limit of evolution coefficient, \( u_* \) |
| Elastic tensile limit hydrostatic stress, \( P_i \) | Parameters for evolution coefficient, \( m, u, \zeta \) |
| Rotational hardening rule:     | Lode angle dependency parameters for |
| Rotational limit stress ratio, \( M_{r,TC} \) | normal-yield and subloading surfaces, \( \rho_f \) |
| Rotational hardening coefficient, \( b_r \) | plastic potential surface, \( \rho_g \) |
| Nonlinearity parameter, \( \beta \) | rotational limit stress ratio, \( \rho_r \) |
| Deviatoric hardening/softening boundary stress ratio, \( \rho_d \) | deviatoric hardening/softening boundary stress ratio, \( \rho_d \) |
| Isotropic hardening rule:      | Initial values: |
| Plastic compressibility index, \( \gamma^* \) | Reference mean normal stress, \( P_0 \) |
| Absolute minimum of compression yield stress, \( P_c+ \) | Reference elastic volumetric strain, \( \epsilon_v^0 \) |
| Deviatoric hardening/softening boundary stress ratio, \( M_{d,TC} \) | Initial value of compression yield stress, \( P_{c0} \) |
| Deviatoric hardening/softening modulus, \( h_{s1} \) | Initial value of normal-yield ratio, \( R_0 \) |
| Deviatoric hardening/softening parameters, \( a, b, c, s \) | |
| Normal-yield, subloading, and plastic potential surfaces: | |
| Critical stress ratio, \( M_{TC} \) | |
| Aspect ratio of ellipsoidal plastic potential surface, \( N_{TC} \) | |
| Plastic tensile limit hydrostatic stress, \( P_i \) | |
| Flattening coefficient of ellipsoidal yield surface, \( \xi \) | |

5.1  Basic characteristics of rotational hardening: Effect of parameters \( M_{r,TC} \) and \( b_r \)

This numerical example demonstrates the basic characteristics of the rotational hardening. Another feature of the present model, namely, the subloading surface, was examined in our past study.\(^5\)\(^4\)\(^,\)\(^5\)\(^1\)\(^,\)\(^6\)\(^0\) as well as in relevant studies by other researchers,\(^7\)\(^1\) so that we here focus on the rotational hardening. To this end, a series of analyses was performed for various values of the two parameters, \( M_{r,TC} \) and \( b_r \), related to the rotational hardening. We analyzed axisymmetric (triaxial) compression, followed by extension, under a constant lateral confining stress \( \sigma_l = -100 \) kPa in \( x_2 \) and \( x_3 \)-directions. The axial stress \( \sigma_a \) in \( x_1 \)-direction was computed by controlling \((1,1)\)-component of the deformation gradient tensor, \( F_{11} \), to impose axial compression and then extension. The analysis was started from an initially isotropic stress state of \(-100 \) kPa.

The values of material constants and initial values used in the analysis were as follows: \( \kappa^* = 0.0065 \), \( \mu_0^* = 20 \) MPa, \( \alpha = 0.60 \), \( P_l = 0.10 \) kPa, \( \beta = 0.60 \), \( \gamma^* = 0.0185 \), \( P_{c+} = 0 \) kPa, \( M_{TC} = N_{TC} = 1.2 \), \( P_l = 0.0010 \) kPa, \( R_e = 0.010 \), \( u_0 = 10 \), \( u_* = 0 \), \( m = v = \zeta = 0 \), \( P_0 = -100 \) kPa, \( \epsilon_v^0 = 0 \), \( P_{c0} = -100 \) kPa, and \( R_0 = 1.0 \) (normally consolidated). The deviatoric hardening/softening was not used, that is, \( h_{s1} = 0 \). The Lode angle dependencies of the subloading and plastic potential functions, as well as of the rotational limit stress ratio \( M_r(\bar{\Theta}) \), were taken into account using Equation \( (B7) \). Initial isotropy without rotational hardening \( (\bar{\eta}^{rot} = 0) \) was assumed.

Figure 11 demonstrates the effect of \( M_{r,TC} \). The figures (A) and (B) compare the results obtained for \( \xi = 1 \) and 0 to examine the effect of distortion (flattening) of the subloading surface associated with the rotational hardening. Each of the figures (A) and (B) shows three graphs of the deviatoric stress (the difference between axial and lateral stresses), the volume change, and the degree of rotational hardening versus the axial compressive deformation, in which the analysis results for \( M_{r,TC} = 0.2, 0.6, \) and 1.0, with a fixed value of \( b_r = 10 \), are plotted. As a reference for comparison, the case of \( M_{r,TC} = 0.0 \) excluding the rotational hardening is also plotted. In (A), a gentle peak of the deviatoric stress is observed in \( M_{r,TC} = 1.0 \) and 0.6, which can be associated with slight dilative deformation around and after the peak stress. The larger the specified value of \( M_{r,TC} \) is, the higher the peak stress is observed, while the difference in the peak stress is not very significant, and the stresses after reaching the critical state are almost identical regardless of the value of \( M_{r,TC} \). In (B), in contrast, significant difference in the stress at the critical state depending on \( M_{r,TC} \) is observed. These results are consistent with the interpretation of the model property based on the schematic diagram in Figure 6.
With regard to the plots of $\overline{\eta}^{\text{rot}}$ in Figure 11 (A) and (B), one can observe, in the compression process, the evolution of $\overline{\eta}^{\text{rot}}$ toward the specified limit $M_r,TC$ and its eventual saturation. In the extension process, $\overline{\eta}^{\text{rot}}$ saturates at a value smaller than $M_r,TC$ due to the Lode angle dependency, namely, $M_r,TC \geq M_r,TE = M_r,TC / \zeta_r(0)$, where $\zeta_r(0) = 9/7$ in this case using Equation (B7).

Figure 12 shows the analysis results for $b_r = 0$, 5, 10, and 20, with a fixed value of $M_r,TC = 1.0$. The case of $b_r = 0$ is the analysis without rotational hardening. One can observe commonly in the figures (A) and (B) that the stress increases the more rapidly the larger the value of $b_r$ is, while the stresses after reaching the critical state are almost identical except the case of $b_r = 0$ in (B). These material responses are attributed to the fact that, as can be seen in the plots of $\overline{\eta}^{\text{rot}}$, the rotational hardening evolves the more rapidly the larger the value of $b_r$ is, and it eventually saturates at the specified rotation limit regardless of the value of $b_r$.

5.2 Accuracy assessment: Deformation-controlled analysis with large incremental steps

A series of deformation-controlled analyzes was conducted to assess the accuracy of the developed stress calculation algorithm in various deformation conditions. We imposed volumetric–shearing mixed deformation to the model by specifying the deformation gradient tensor,\textsuperscript{54} whose matrix representation in terms of Cartesian components is given by

$$F = F_s F_0 = f_s \cdots f_s F_0,$$

where $F_0$ defines an initial state, and $F_s$ prescribes a deformation imposed on the initial state. Two control parameters, $H_v$ and $H_s$, specify the volumetric and shear deformations, respectively. $F_s$ can be divided into $N_{\text{step}}$ equal increments

$$(174)$$

where $F_0$ defines an initial state, and $F_s$ prescribes a deformation imposed on the initial state. Two control parameters, $H_v$ and $H_s$, specify the volumetric and shear deformations, respectively. $F_s$ can be divided into $N_{\text{step}}$ equal increments
defined by the relative deformation gradient tensor with two control parameters of increment, $h_v$ and $h_s$, namely,

$$ f_* := F_*^{1/N_{\text{step}}} = \begin{pmatrix} 1 + h_v & h_s & 0 \\ 0 & 1 + h_v & 0 \\ 0 & 0 & 1 + h_v \end{pmatrix}, \quad h_v = (1 + H_v)^{1/N_{\text{step}} - 1}, \quad h_s = \frac{1}{(1 + H_v)^{(N_{\text{step}} - 1)/N_{\text{step}}}} H_s. $$

The accuracy of a computed stress was assessed by the relative error with respect to the Kirchhoff stress tensor:

$$ \text{Error} := \frac{\|\tau - \tau_{\text{ref}}\|}{\|\tau_{\text{ref}}\|}, $$

where $\tau$ denotes a solution computed by a prescribed number of increments, $N_{\text{step}}$, and $\tau_{\text{ref}}$ is a reference solution. We used the value of stress computed by sufficiently small increments specified by $N_{\text{step}} = 1000$ as a substitute for the reference solution, since exact solution for the present complex constitutive model cannot be available in analytical way.

The values of material constants and initial values used in the analysis were as follows: $\epsilon^e = 0.050$, $\mu^e = 2.0$ MPa, $\alpha = 1.0$, $P_c = 1.0$ kPa, $M_{d TC} = 1.2$, $b_1 = 1.0$, $\beta = 1.0$, $\gamma^* = 0.040$, $P_{c0} = 0$ kPa, $M_{d, TC} = 0.20$, $h_1 = 0.010$, $a = 5.0$, $b = 1.2$, $c = 0$, $s = 0$, $M_{TC} = N_{TC} = 1.2$, $P_i = 0$ kPa, $\xi = 1$, $R_0 = 1.0 \times 10^{-6}$, $m = 100$, $u_0 = 100$, $u_\infty = 1.0 \times 10^{-6}$, $m = 0$, $v = 0.010$, $\zeta = 0.20$, $P_0 = -100$ kPa, and $\epsilon^{e}_0 = 0$. The Lode angle dependency was taken into account using Equation (B8) with $\rho_f = 0.75$, $\rho_g = 0.90$, $\rho_l = 0.75$, and $\rho_d = 0.80$. Initially isotropic stress state of $\sigma_{11} = \sigma_{22} = \sigma_{33} = -100$ kPa without rotational hardening ($\dot{\eta}_{\text{rot}} = 0$) was assumed. The following two cases were analyzed:

- Case A: Normally consolidated soil ($OCR = P_{c0}/P_0 = 1.0$, $P_{c0} = -100$ kPa, $R_0 = 1/OCR = 1.0$);
- Case B: Over-consolidated soil ($OCR = P_{c0}/P_0 = 5.0$, $P_{c0} = -500$ kPa, $R_0 = 1/OCR = 0.20$).

Various deformation modes were imposed to the model by controlling the parameters $H_v \in [-0.02, +0.02]$ and $H_s \in [0, 0.2]$ with the number of incremental steps, $N_{\text{step}} = 10$, to compute stresses, and their error relative to the reference solution were evaluated by Equation (176). Figures 13 (A) and (B) depict the stress versus strain curves, the stress paths
FIGURE 13  Accuracy assessment of stress calculation with large strain increments. Deformation-controlled analysis imposing volumetric–shearing mixed modes

on the $p-q$ stress plane, and the error maps obtained in Cases A and B, respectively. The graphs on the left and center compare the stresses computed by $N_{\text{step}} = 10$ (plotted as markers) with the corresponding reference solution (solid lines). While slight deviations of the computed stress points from the reference solution are observed in some cases with very large strain increments, the error is not significant as a whole and the analysis results demonstrate enough accuracy and robustness for practical use. It is also worthy to note here that, in Case B for OCR = 5.0, a smooth transition from elastic to plastic responses around the peak stress is well simulated by the subloading surface model. The error maps on the right represent the relative error of the stresses computed at the final incremental step (i.e., 10th step) corresponding to $H_v$ and $H_s$, which also convince us of high accuracy in the present stress calculation even with large strain increments.

5.3 | Simulation of cyclic mobility observed in experiments

Cyclic mobility is typically observed in sands during undrained cyclic loading. Its accurate prediction by constitutive description necessitates highly sophisticated constitutive models with careful parameter determination.71,73,111–117 This phenomenon leading toward eventual flow liquefaction is essentially concerned with instability problem.118–122 Setting aside the underlying theoretical basis on this aspect, we shall here focus on constitutive simulation of cyclic mobility observed in experiment in a phenomenological level to calibrate the proposed model.

The experimental data simulated in the present analysis were obtained by two different testing methods, namely, cyclic torsional shear test on hollow cylindrical specimen and cyclic triaxial test on circular column specimen of a sand sample under undrained condition (CU test). The testing conditions in total three tests are listed in Table 3. The sand sample in all the tests was Gifu silica sand No. 8 with physical properties: the soil particle density $\rho_s = 2.623 \text{ g/cm}^3$, the minimum dry density $\rho_{d,\text{min}} = 1.125 \text{ g/cm}^3$, and the maximum dry density $\rho_{d,\text{max}} = 1.564 \text{ g/cm}^3$ (the corresponding maximum and minimum void ratios were $e_{\text{max}} = 1.332$ and $e_{\text{min}} = 0.677$). The cyclic shear loads applied to the specimens were sinusoidal wave with a frequency of 0.1 Hz.
TABLE 3  Experiment for silica sand sample. Testing methods and conditions

| Testing method | Test 1 | Test 2 | Test 3 |
|----------------|--------|--------|--------|
| Relative density, $D_r$ | 87.7% | 88.6% | 87.4% |
| Initial void ratio, $e_{init}$ | 0.749 | 0.743 | 0.751 |
| Effective confining pressure, $\sigma'_0$ | 100 kN/m² | 100 kN/m² | 100 kN/m² |
| Amplitude of cyclic shear stress ratio | $\tau/\sigma'_0 = 0.284$ | $\tau/\sigma'_0 = 0.303$ | $\sigma_d/(2\sigma'_0) = 0.305$ |

The simulation results are compared with the experimental data in Figure 14, in which the time histories of the shear strain and the excess pore water pressure ratio, the shear stress versus shear strain curves, and the stress paths are shown. The constitutive simulation successfully reproduces the cyclic mobility observed in the experiments, even though the same values of material constants were used throughout to simulate the experimental data measured in three different testing methods and conditions. This realistic simulation demonstrates the predictive capability of the proposed constitutive model, while some deviations from the experimental data are observed, particularly in Test 1 in that the overestimation of
the decrease in effective mean normal stress toward the zero pressure during several initial cycles. This can be associated with the overestimation of the buildup of excess pore water pressure due to excessive volume contraction in initial cycles. One can also observe in Test 3 that the different responses in compression side and extension side are fairly well simulated by the Lode angle dependency, while deviations from the experiment are still seen especially in the strain evolution in the extension side.

5.4 Application to finite element analysis

This example verifies the applicability to finite element (FE) analysis. A plane strain compression test for a rectangular specimen was simulated by FE analysis. The material subroutine developed in this study was implemented into a research numerical code by the authors for nonlinear FE analysis based on the updated Lagrangian formulation. The algorithmic (consistent) tangent moduli tensor was computed by numerical differentiation using a perturbation technique based on the forward Euler difference scheme\textsuperscript{125} to construct the global tangent stiffness matrix. The efficiency of this numerical technique has been reported also for constitutive model of geomaterials.\textsuperscript{126–128}

The analysis model is shown in Figure 15 (A). The specimen with a size of width \times height = 10 \text{cm} \times 20 \text{cm} was divided into FE mesh consisting of 10 \times 20 eight-node quadratic elements using 2 \times 2 reduced Gaussian quadrature points. Initially isotropic stress state of \( \sigma_{11} = \sigma_{22} = \sigma_{33} = -300 \text{ kPa} \) was assumed. The nodal displacements in \( x_1 \)-direction on the top and bottom faces were fully fixed to simulate frictional end boundaries. The specimen was compressed by imposing a downward displacement \( u \) in \( x_2 \)-direction uniformly on the top face under a constant lateral confining pressure of 300 kPa, and the reaction force \( F \) on the top face was computed. The gravity effect was ignored. The values of material constants and initial values used in the analysis were as follows: \( \kappa^* = 0.0025, \mu_0^* = 46 \text{ MPa}, \alpha = 1.0, P_l = 0 \text{ kPa}, M_{T,TC} = 0.50, b_t = 20, \gamma = 1.0, \gamma^* = 0.0225, P_{c+} = 0 \text{ kPa}, M_{TC} = N_{TC} = 0.98, P_t = 0 \text{ kPa}, \xi = 0, R_e = 1.0 \times 10^{-4}, u_0 = 5.5, u_{\infty} = 0, m = n = \varsigma = 0, P_0 = -300 \text{ kPa}, \) and \( \varepsilon_{v0}^* = 0 \). The deviatoric hardening/softening was not used, that is, \( h_{s1} = 0 \).

The Lode angle dependency was taken into account only for the normal-yield and subloading functions using Equation (B8) with \( \rho_f = 0.753, \rho_g = 1.0, \rho_t = 1.0, \) and \( \rho_d = 1.0 \). The following two cases were analyzed to examine the effect of difference in initial state of material:

- Case A: Normally consolidated soil (OCR = \( P_{c0}/P_0 = 1.0, P_{c0} = -300 \text{ kPa}, R_0 = 1/\text{OCR} = 1.0 \));
- Case B: Over-consolidated soil (OCR = \( P_{c0}/P_0 = 10.0, P_{c0} = -3000 \text{ kPa}, R_0 = 1/\text{OCR} = 0.10 \)).

Figures 15 (B) and (C) plot, respectively, the macroscopic response of nominal axial stress and volumetric strain versus the nominal axial strain. Figure 16 shows the contour of shear strain and the distribution of plastic loading/elastic unloading region at an macroscopic axial strain \( u/H = 0.1 \). The analysis result of Case A exhibits a gentle increase in the axial stress along with the contractive volumetric deformation, while a slight decrease in the axial stress is observed after a peak stress. In contrast, Case B exhibits a steep increase in the axial stress, followed by a remarkable decrease after a sharp peak of stress accompanied by the expansive deformation, which can be associated with the localized deformation observed in
Figure 16 (B). Note that, in Figure 16, the X-shaped localized shear zone is more remarkable in Case B than in Case A, and quite large shear strain to reach around 0.3–0.5 is observed.

Figure 17 (A) and (B) depict the convergence profile of the global Newton–Raphson iteration at typical three loading steps before, at, and after the peak stress (marked on the curves in Figure 15 (B) and (C)) in the analyses for Cases A and B. In both cases, nicely convergent profiles with quadratic or superlinear rate are observed even in the loading steps involving unstable macroscopic responses due to material softening and localized deformations. This result assures the performance of the stress calculation algorithm developed in this study.

6 CONCLUDING REMARKS

In this article, we have presented a novel formulation of anisotropic Cam-clay plasticity model at finite strains based on the multiplicative decomposition of the deformation gradient tensor. The rotational hardening has been incorporated into the model as a pertinent expression of the anisotropy induced by plastic deformation in geomaterials in a reasonable manner consistent with the theoretically sound structure of finite strain elastoplasticity. Furthermore, being combined with the subloading surface concept, the model is endowed with the capability of reproducing complex mechanical behavior of geomaterials, such as hardening/softening accompanied by contractive/dilative responses.

The basic characteristics and the predictive capability of the proposed model have been demonstrated through numerical examples involving large deformations and cyclic loadings. Critical assessments with regard to the numerical aspect
have ensured the accuracy of the stress calculation, as well as the applicability to boundary-value problem analysis using FEM.

The proposed constitutive model requires not a few material constants in return for its high capability. Therefore, developing a systematic parameter determination technique with a guidance for practical engineering use should be our future task.

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AUTHOR CONTRIBUTIONS
Y. Yamakawa conceptualized and directed this research work, and developed constitutive formulation with its numerical implementation. K. Hashiguchi contributed to the essential idea and the development of constitutive model. T. Sasaki implemented the material subroutine into finite element code, and guided the direction of numerical analysis. M. Higuchi, K. Sato, and T. Kawai performed experimental tests and numerical simulation presented in Section 5.3. T. Machishima and T. Iguchi carried out numerical analysis presented in Section 5. All authors have contributed to the original draft preparation, and they have read and agreed to the submission of the manuscript.

DATA AVAILABILITY STATEMENT
The data that support the findings of this study are available from the corresponding author upon reasonable request.

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APPENDIX A: FIRST AND SECOND DERIVATIVES OF STRESS INVARIANTS
The first derivatives of the three stress invariants, \( P, Q, \) and \( \Theta \), defined by Equation (53), with respect to the stress tensor are given as follows:

\[
\frac{\partial P}{\partial M} = \frac{1}{3} I, \quad \frac{\partial Q}{\partial M_{\text{dev}}} = \sqrt{\frac{3}{2}} \left( \frac{M_{\text{dev}}^T}{\text{tr}(M_{\text{dev}})^2} \right)^{1/2}, \quad \frac{\partial \Theta}{\partial M_{\text{dev}}} = -\frac{2}{\sqrt{6}} \frac{1}{\sin(3\Theta)} \frac{\partial y}{\partial M_{\text{dev}}},
\]

(A1)

in which we have used \( y \) with its derivative given by

\[
y := \frac{1}{\sqrt{6}} \cos(3\Theta), \quad \frac{\partial y}{\partial M_{\text{dev}}} = \frac{3}{\text{tr}(M_{\text{dev}})^2} \left( \frac{M_{\text{dev}}^T}{\text{tr}(M_{\text{dev}})^2} \right)^{1/2} \left( \frac{(M_{\text{dev}}^T}_{\text{dev}})^2 - \frac{\text{tr}(M_{\text{dev}})^3}{\text{tr}(M_{\text{dev}})^2} M_{\text{dev}}^T \right),
\]

(A2)
The second derivatives are given as follows:

\[
\frac{\partial^2 P}{\partial M \otimes \partial M} = O, \tag{A3}
\]

\[
\frac{\partial^2 Q}{\partial M_{\text{dev}} \otimes \partial M_{\text{dev}}} = \sqrt{3} \left( I \otimes I - \frac{M_{\text{dev}}^T}{(\text{tr}(M_{\text{dev}})^2)^{1/2}} \otimes \frac{M_{\text{dev}}^T}{(\text{tr}(M_{\text{dev}})^2)^{1/2}} \right), \tag{A4}
\]

\[
\frac{\partial^2 \Theta}{\partial M_{\text{dev}} \otimes \partial M_{\text{dev}}} = -\frac{3}{\tan(3\Theta)} \frac{\partial \Theta}{\partial M_{\text{dev}}} \otimes \frac{\partial \Theta}{\partial M_{\text{dev}}} - \frac{2}{\sqrt{6}} \frac{1}{\sin(3\Theta)} \frac{\partial^2 y}{\partial M_{\text{dev}} \otimes \partial M_{\text{dev}}}, \tag{A5}
\]

in which \( O \) denotes the fourth-order zero tensor, and the second derivative of \( y \) is given by

\[
\frac{\partial^2 y}{\partial M_{\text{dev}} \otimes \partial M_{\text{dev}}} = \frac{3}{(\text{tr}(M_{\text{dev}})^2)^{3/2}} \left( I \otimes M_{\text{dev}} + M_{\text{dev}}^T \otimes I \right)
- \frac{3}{(\text{tr}(M_{\text{dev}})^2)^{5/2}} \left( I \otimes I + 3M_{\text{dev}}^T \otimes (M_{\text{dev}})^2 + 3(M_{\text{dev}}^T)^2 \otimes M_{\text{dev}}^T \right)
+ \frac{15}{(\text{tr}(M_{\text{dev}})^2)^{7/2}} M_{\text{dev}}^T \otimes M_{\text{dev}}^T. \tag{A6}
\]

Among the above equations, those that contain \( \text{tr}(M_{\text{dev}})^2 \) in denominator are defined only when \( \text{tr}(M_{\text{dev}})^2 \neq 0 \). Note also that, when computing Equations (A1) and (A5) for \( \theta = 0 \) and \( \pi/3 \), a perturbation would be useful to avoid null values in their denominators.

The first and second derivatives of \( P, Q, \) and \( \Theta \), defined by Equations (49)–(51), and those of \( p, q, \) and \( \theta \), defined by Equation (52), are also given similarly.

**APPENDIX B: FUNCTION DESCRIBING LODGE ANGLE DEPENDENCY**

There are many forms of the function \( \zeta(\theta; \rho) \) to describe the Lode angle dependency; see, for example, the references \(^{52,129-134} \) for their developments and applications, and the historical surveys and comparisons.\(^ {135,136} \) The functional property of \( \zeta(\theta; \rho) \) that ensures the convexity of a yield surface incorporating the Lode angle dependency was examined in a strict mathematical manner by Hashiguchi.\(^ {137} \)

Two specific forms employed in this study are described below. Either form takes a value of \( \zeta = 1 \) at axisymmetric (triaxial) compression states, that is, \( \theta = \pi/3 \). The first and second derivatives of \( \zeta(\theta; \rho) \) are required in computation, but they are omitted here.

Hashiguchi\(^ {137} \) proposed the following simple function:

\[
\zeta(\theta) = \frac{8 + \cos 3\theta}{7}, \tag{B7}
\]

which always satisfies the convexity and smoothness requirement, not requiring a parameter \( \rho \). Equation (B7) gives \( \zeta(0) = 9/7 \) at axisymmetric (triaxial) extension states.

Based on the Lade–Duncan\(^ {138,139} \) and Matsuoka–Nakai\(^ {40} \) yield surfaces, Barde\(^ {135} \) proposed the following form, which he called “LMN dependence”:

\[
\zeta(\theta; \rho) = \frac{2\sqrt{\rho^2 - \rho + 1}}{\sqrt{3}\rho} \cos \alpha(\theta; \rho), \tag{B8}
\]
where $\alpha(\theta; \rho)$ is given by

\[
\alpha(\theta; \rho) = \begin{cases} 
\frac{1}{6} \cos^{-1} \left[ -1 + \frac{27}{2} \frac{\rho(1 - \rho)^2}{(\rho^2 - \rho + 1)^3} \cos^2 3\theta \right] & \text{for } 0 \leq \theta \leq \frac{\pi}{6}, \\
\frac{\pi}{3} - \frac{1}{6} \cos^{-1} \left[ -1 + \frac{27}{2} \frac{\rho(1 - \rho)^2}{(\rho^2 - \rho + 1)^3} \cos^2 3\theta \right] & \text{for } \frac{\pi}{6} < \theta \leq \frac{\pi}{3}.
\end{cases}
\] 

This form satisfies the convexity and smoothness requirement for $\rho$ in the range $1/2 < \rho \leq 1$. Equation (B8) gives $\zeta(0) = 1/\rho$ at axisymmetric (triaxial) extension states. Setting $\rho = 1$ leads to $\zeta(\theta; 1) = 1$ for any $\theta$, which means the independence from Lode angle.