Topological Order and Non-Abelian Statistics in Noncentrosymmetric $s$-Wave Superconductors

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We demonstrate that in two-dimensional noncentrosymmetric $s$-wave superconductors under applied magnetic fields for a particular electron density, topological order emerges, and there exists a zero energy Majorana fermion mode in a vortex core, which obeys non-Abelian statistics, in analogy with $p_x + ip_y$ superconductors, the Moore-Read Pfaffian quantum Hall state, and the gapped non-Abelian spin liquid phase of the Kitaev model.

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Recently, there is considerable interest in emergent topological states of many-body quantum systems characterized by a topologically nontrivial structure of the Hilbert space, i.e. “topological order” [1]. In certain classes of topological states in 2+1 dimension, quasiparticles are non-Abelian anyons [2, 3, 4, 5, 6, 7, 8, 9, 10]. The essential feature of the non-Abelian statistics is that the exchange of particles is described by unitary operations in the multi-dimensional space, which is spanned by the basis of the degenerate many-body ground state. Thus, the state depends on the order of the multiple exchange processes of particles. The possible realization of non-Abelian statistics in real systems has been extensively studied so far in connection with the ν = 5/2 fractional quantum Hall (FQH) state, and the vortex state of chiral $p_x + ip_y$ superconductors (superfluids) [2, 3, 4, 5, 6, 7, 8, 9, 10]. These topological states are of interest also in the context of quantum computing, since the non-Abelian anyon can be utilized as a decoherence-free qubit, and potentially applied to the construction of fault-tolerant topological quantum computers [8, 13, 14, 15]. In this paper, we present another candidate of a topological phase allowing the existence of non-Abelian anyons, which can be realized in strongly noncentrosymmetric (NC) s-wave superconductors. This topological phase belongs to the same class as those of the Moore-Read (MR) Pfaffian FQH state [2, $p_x + ip_y$ superconductors [2], and the gapped non-Abelian spin liquid phase of the Kitaev model [9, 10, 16]. In NC superconductors, the asymmetric spin-orbit (SO) interaction which breaks inversion symmetry plays important roles in various exotic superconducting properties [7, 13, 14, 16, 17, 18]. In our proposal, the asymmetric SO interaction combined with an external magnetic field yields the nontrivial topological state for a particular electron filling.

We consider type II NC $s$-wave superconductors with the Rashba type SO interaction in two dimension. We neglect the parity-mixing of triplet components of Cooper pairs due to the asymmetric SO interaction [17], because the inclusion of this effect does not change the essential part of our argument. For a concreteness, we define our model on the square lattice, though the following consideration do not reply on the particular choice of the crystal structure. Then the model Hamiltonian is

$$\mathcal{H} = \sum_{k,\sigma} \varepsilon_k c_{k\sigma}^\dagger c_{k\sigma} + \alpha \sum_{k,\sigma} \mathcal{L}_0(k) \cdot \sigma_{\sigma\sigma'} c_{k\sigma}^\dagger c_{k\sigma'}^\dagger$$

$$- \frac{1}{\hbar} \sum_k [\Delta_+ c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} + \text{h.c.}]$$

$$= \sum_{\nu=\pm} \sum_k \varepsilon_{k\nu} a_{k\nu}^\dagger a_{k\nu} - \sum_k \{\Delta_\nu(k) a_{k\uparrow}^\dagger a_{-k\downarrow}^\dagger + \text{h.c.}\} \tag{1}$$

Here, $c_{k\sigma}^\dagger$ ($c_{k\sigma}$) is a creation (an annihilation) operator for an electron with momentum $k$, spin $\sigma$. The energy band dispersion is $\varepsilon_k = -2t(\cos k_x + \cos k_y) - \mu$. The second term of eq. (1) is the Rashba SO interaction with $\mathcal{L}_0(k) = (\sin k_y, -\sin k_x, 0)$. Eq. (2) is expressed in terms of the chirality basis which diagonalize the SO term. The energy band is split into two parts by the SO interaction: $\varepsilon_{k\pm} = \varepsilon_k \pm \alpha |\mathcal{L}_0(k)|$. The gap function in this basis is odd parity, and possesses the momentum dependence $\Delta_\nu(k) = \Delta_\nu(k) + \text{h.c.}$ (with $\mu(k) = -\mathcal{L}_{0x} \pm i \mathcal{L}_{0y}/\sqrt{\mathcal{L}_{0x}^2 + \mathcal{L}_{0y}^2}$, which is, importantly, similar to that of $p_x + ip_y$ superconductors). Thus, for $\Delta \ll E_F$, we can exploit the same argument as that applied to $p$-wave superconductors [22, 23], and find that in the mixed state with vortices parallel to the $z$-axis, there is a zero-energy quasiparticle state of a vortex core which is described by a Majorana fermion. However, in this case, the existence of Majorana fermions in vortices does not directly lead to the non-Abelian statistics of them, because there are two bands ($\varepsilon_{k\mu}, \mu = \pm$), each of which contributes to a zero-energy mode with a different band index. The existence of two different species of Majorana fermions in a single vortex implies that the sign change of the fermion operators under the braiding of two vortices, which is a hallmark of the non-Abelian statistics [2, 4] (see below), is canceled. To eliminate this unwanted multiplicity of Majorana fermions, we tune the chemical potential as $\mu = -4t$ for which the Fermi level crosses the $\Gamma$ point in the Brillouin Zone (BZ). In this
situation, there are still two bands near the Fermi level in the model \( \text{(2)} \): one from \( \varepsilon_{k-} \) with a finite Fermi momentum, and the other in the vicinity of the \( \Gamma \) point which is given by the Dirac cone. To generate the mass gap in the Dirac cone, we introduce the Zeeman coupling \( \mu_B H_z \sum_k (c_{k\uparrow} c_{k\downarrow} - c_{k\downarrow} c_{k\uparrow}) \). The magnitude of the gap is of the order of \( \mu_B H_z \). Then, there is only a single energy band \( \varepsilon_{k-} \) which crosses the Fermi level. Let us assume that \( H_z \) is sufficiently smaller than the orbital depairing field \( H_{orb} \). The Pauli depairing effect due to \( H_z \) is negligible for \( \alpha \gg \mu_B H_z / \Delta \). Under these circumstances, we can integrate out contributions from quasiparticles with a gap \( \sim \mu_B H_z \) at the \( \Gamma \) point, and obtain the low-energy effective Hamiltonian \( H_{\text{eff}} \) for the single band superconductor. Taking account of the fact that the Zeeman field \( H_z \) induces the interband Cooper pairing between the \((+)\)-band and the \((-)\)-band, we obtain

\[
H_{\text{eff}} = \sum_k \varepsilon_{k-} a_{k-}^\dagger a_{k-} - \sum_k [\Delta_{k-}(k) a_{k-}^\dagger a_{k-} + \text{h.c.}]. \tag{3}
\]

Here the renormalized energy band is \( \varepsilon_{k-} = \varepsilon_{k-} + \varepsilon_0 \) with \( \varepsilon_0 = H_z^2 \Delta^2 / \alpha^2 |\mathbf{L}(k)|^2 m_0, |\mathbf{L}(k)| = \sqrt{L_{xy}^2 + \mu_B^2 H_z^2 / \alpha^2}, \) and \( m_0 = (4\alpha^2 |\mathbf{L}(k)|)^4 + \Delta^2 (L_{xy}^2 + L_{xy}^2) / 2\alpha |\mathbf{L}(k)|^3. \) \( m_0 \) is an energy gap of quasiparticles in the \((+)\)-band in the vicinity of the Fermi momentum of the \((-)\)-band. The superconducting gap function is \( \Delta_{k-}(k) = a \Delta \tilde{n}_-(k) \) where \( a = 1 + H_z^2 \Delta^2 / 2\alpha^3 |\mathbf{L}(k)|^2 m_0, \) and \( \tilde{n}_-(k) = (-L_{xy} + iL_{xy}) / |\mathbf{L}(k)| \). The above expression \( \text{(3)} \) is valid only in the vicinity of the Fermi momentum \( k_F \) defined by \( \varepsilon_{kF} = 0 \) (not \( \varepsilon_{kF} = 0 \)). In many type II superconductors, it is typical that for \( H_z < H_{orb} \), \( \mu_B H_z < \Delta \). This implies that the gap generated by the magnetic field at the \( \Gamma \) point might be smaller than the superconducting gap, which may invalidate the approximation used in the derivation of \( \text{(3)} \). However, our argument on the low-energy vortex core states which we are most concerned with is applied only to energy scale \( < \Delta^2 / v_F \). Therefore, the effective Hamiltonian \( \text{(3)} \) is applicable for our purpose, as long as the condition \( \Delta^2 / v_F < \mu_B H_z < H_{orb} \) is satisfied, which can be fulfilled in ordinary experimental situations.

The topological order of the model \( \text{(3)} \) clearly manifests in the Chern number which is, for the Hamiltonian of the form \( H_{\text{eff}} = \sum_{\mathbf{k} \in \text{BZ}} \sum_a (a_{k-}^\dagger a_{k-} - a_{k-} a_{k-}^\dagger) \sigma_i \mathbf{E}_i(k) (a_{k-}^\dagger a_{k-})^\dagger, \) defined as \( \text{(3)} \quad \text{[14]} \)

\[
\mathcal{N} = \int \frac{d^2 k}{8\pi} \varepsilon_{k-} \mathbf{E} \cdot \left( \frac{\partial \mathbf{E}}{\partial k_\gamma} \times \frac{\partial \mathbf{E}}{\partial k_\delta} \right) \tag{4}
\]

where \( \mathbf{E} = (E_x(k), E_y(k), E_z(k)) / |\mathbf{E}(k)|. \) The integral of \( \text{(3)} \) is taken over the whole BZ, while the expression of \( \text{(3)} \) is derived for \( k \) in the vicinity of \( k_F \). Nevertheless, we can consider the Chern number of the model \( \text{(3)} \) by re-interpreting eq. \( \text{(3)} \) as a lattice regularized version of the low-energy effective theory, and extending the \( k \)-space in which the model \( \text{(3)} \) is defined to the entire BZ. Then, the numerical evaluation of \( \mathcal{N} \) for the Hamiltonian \( \text{(3)} \) gives \( \mathcal{N} = 1 \). Therefore, the model \( \text{(3)} \) is classified as the same topological class as those of the MR state, spinless \( p_x + ip_y \) superconductors, and the gapped non-Abelian phase of the Kitaev model. The existence of the Zeeman field \( H_z \) in the model \( \text{(3)} \) is important for this topological characterization, because it does not only break time-reversal symmetry, but also ensures the differentiability of \( E_{x,y}(k) \) for Eq. \( \text{(3)} \) which is singular at \( k = 0 \) for \( H_z = 0 \).

The Chern number \( \mathcal{N} = 1 \) implies the existence of zero-energy Majorana fermion modes in vortices which obey the non-Abelian statistics, as in the case of \( p_x + ip_y \) superconductors \( \text{(3-16)} \). To demonstrate this, we proceed to solve the Bogoliubov-de-Gennes (BdG) equations for the model \( \text{(3)} \) with a single vortex inserted parallel to the \( z \)-axis. For simplicity, we switch to the continuum model replacing the energy band \( \varepsilon_k \) of eq. \( \text{(3)} \) with \( \varepsilon_z' = k^2 / 2m - \mu + \mathcal{L}_0 \) with \( \mathcal{L}_0 = (k_y - k_z, 0) \). Furthermore, we assume that the gap amplitude \( \Delta(r) \) vanishes inside of the vortex core, and is equal to a constant \( \Delta \) outside of the core, and \( \Delta \ll v_F \). Then, in the vicinity of the Fermi surface, the BdG equations corresponding to the model \( \text{(3)} \) with a single vortex are

\[
\left( \begin{array}{c}
-i v_F \cdot \nabla + \varepsilon_0 \\
\Delta_0 e^{-i\phi} \hat{P} e^{i\phi} \\
\Delta_0 e^{-i\phi} \hat{P} e^{-i\phi} - i v_F \cdot \nabla - \varepsilon_0
\end{array} \right) \Psi = \varepsilon \Psi \tag{5}
\]

where \( \Psi^i = (u(r), v(r)) \), \( \hat{P} = -(\partial_x + i\partial_y) \), \( \hat{P}^\dagger = -\hat{P}^* \), and \( \Delta_0 = a \Delta \mathcal{C}(r) / |\mathbf{E}(k)| \). The BdG equations \( \text{(5)} \) are equivalent to those of spinless \( p_x + ip_y \) superconductors except that there are the \( \varepsilon_0 \)-terms in the diagonal components, which can be formally absorbed into the shift of the Fermi momentum \( k_F \rightarrow k_F - \varepsilon_0 / v_F \). Thus, the solution of \( \text{(5)} \) is given by \( \Psi = e^{-i\mathbf{k}_0 / v_F} \Psi_{p_x + ip_y} \) with \( \Psi_{p_x + ip_y} \) the eigen function of the BdG equations for spinless \( p_x + ip_y \) superconductors, and there exists a zero energy mode inside the vortex core which is separated from the first excited state by a gap of energy size \( \Delta^2 / v_F \). \text{[22, 23]} The Bogoliubov quasiparticles for this zero energy state are described by a Majorana fermion field \( \mathcal{C} = \int d\mathbf{r} |u(r) a_{\downarrow}(r) + v(r) a_{\uparrow}(r)\rangle, \) since \( (v^*(r), u^*(r)) = (u(r), v(r)) \) for \( \varepsilon = 0 \). Here \( a_{\downarrow}^\dagger (r) = \sum_k a_{k-}^\dagger e^{-i k r} \).

To confirm the above prediction, we apply numerical analysis directly to the BdG equations for the tight-binding model \( \text{(1)} \) without referring to the low-energy effective theory \( \text{(3)} \). The energy spectrum and the eigen functions of the BdG equations were calculated for the model \( \text{(1)} \) with a vortex located at the center of the system on the square lattice with open boundaries. In this calculation, we assume that the GL parameter is so large that the Zeeman field \( H_z \) is approximated to be uniform, and the spatial dependence of the superconducting gap function due to the vortex is taken into account.
\[
\gamma_j(\theta_\sigma \exp(i\frac{\pi}{4}(2\psi^\dagger \psi - 1))) = \theta_{\sigma} \exp(i\pi \psi^\dagger \psi - 1)).
\]

Here \(\theta_\sigma\) is a phase factor called a topological spin. Under the exchange of two vortices, the Majorana fermion operators are transformed as,

\[
\gamma_i \rightarrow R_{\sigma\sigma} \gamma_i (R_{\sigma\sigma}^\dagger = \gamma_j, \gamma_i \rightarrow R_{\sigma\sigma} \gamma_j (R_{\sigma\sigma}^\dagger = -\gamma_i).
\]

The minus sign in the second transformation rule is essential for non-Abelian statistics. The topological spin \(\theta_\sigma\) in \(R_{\sigma\sigma}\) is associated with the conformal spin of the primary field for vortices which is the chiral Ising spin field, because of the bulk-edge correspondence of anyonic particles:

\[
\theta_\sigma = e^{i2\pi(h_0-\bar{h}_0)/\hbar} = e^{i\pi/8}.
\]

As a result, when two vortices are fused into \(\psi\), the braiding of them yields the phase factor \(R_{\psi\psi} = e^{\frac{3i}{8}\pi}\), while when the fusion gives the topological charge 1, the phase factor due to the braiding is \(R_{\sigma\sigma}^1 = e^{-i\pi/8}\). The braiding rules for the other particle types are \(R_{\sigma\psi} = -i\) and \(R_{\psi\psi} = -1\).

We, now, discuss the feasibility of the experimental detection of the non-Abelian statistics. One promising approach is to use the two-point-contact interferometer proposed in the context of the FQH state \([4, 11, 12]\). In the superconducting state considered here, this experiment is applicable only to the thermal transport. According to refs. \([11, 12]\), the interference term of the edge heat current \(J^\text{int}\) depends on the parity of the total number of vortices \(n\) in the bulk. \(J^\text{int}\) for odd \(n\) is much smaller than \(J^\text{int}\) for even \(n\), though both of them do not exhibit the dependence on a magnetic flux \(\Phi\), because the \(\sigma\)-particle is neutral. This parity dependence characterizes the non-Abelian statistics. Another possible experiment is a bit indirect but simpler. It uses a disk-shaped system with which two heat baths are attached at the boundary. (see FIG.2(a).) For this geometry, as in the case of \(px + i p_y\) superconductors \([24]\), the energy spectrum of the edge state depends on the parity of the total number of vortices \(n\) in the bulk. For even \(n\), the lowest energy state has a gap of the order \(\Delta/k_F L\) where \(L\) is the length of the boundary. Although the gap is small for a sufficiently large system size, it is nonzero, and thus the quasiparticle corresponding to this edge mode is a complex fermion interacting with \(\Phi\). This Bogoliubov quasiparticle is categorized as the same particle type as the \(\psi\)-fermion in the bulk, because, in the limit that the two vortices merge together at a position \(r\), the resulting \(\psi\)-particle is nothing but the Bogoliubov quasiparticle with a nonzero energy \([8]\).

For odd \(n\), the low-energy edge state has a gap of the order \(\Delta/k_F L\) which is smaller than \(J^\text{int}\) for even \(n\), though both of them do not exhibit the dependence on a magnetic flux \(\Phi\), because the \(\sigma\)-particle is neutral. This parity dependence characterizes the non-Abelian statistics. Another possible experiment is a bit indirect but simpler. It uses a disk-shaped system with which two heat baths are attached at the boundary. (see FIG.2(a).) For this geometry, as in the case of \(px + i p_y\) superconductors \([24]\), the energy spectrum of the edge state depends on the parity of the total number of vortices \(n\) in the bulk. For even \(n\), the lowest energy state has a gap of the order \(\Delta/k_F L\) where \(L\) is the length of the boundary. Although the gap is small for a sufficiently large system size, it is nonzero, and thus the quasiparticle corresponding to this edge mode is a complex fermion interacting with \(\Phi\). This Bogoliubov quasiparticle is categorized as the same particle type as the \(\psi\)-fermion in the bulk, because, in the limit that the two vortices merge together at a position \(r\), the resulting \(\psi\)-particle is nothing but the Bogoliubov quasiparticle with a nonzero energy \([8]\).

For odd \(n\), the low-energy edge state is a Majorana fermion mode, and can be fused with an unpaired Majorana fermion in the bulk resulting in the \(\psi\)-state or the 1-state. The phase accumulated by the current flow of the edge \(\sigma\)- or \(\psi\)-particles encircling the bulk \(n\) vortices is obtained from the square of the braiding oper-
FIG. 2: (a) Experimental setup for the thermal transport measurement of the edge state. A magnetic flux $\Phi$ penetrates in the bulk. Two heat baths are attached at the boundary so that the longer path between the two heat baths, $C_2$, encircles almost all vortices in the bulk. (b) Superconductor-metal-insulator junction.

The heat bath 2, the chiral edge heat current flows mainly in the boundary loop 4 times. Thus $J_{\text{int}}^{\text{odd}}$ is much suppressed. On the other hand, when $T_1 > T_2$, the edge current flows mainly in the path $C_1$, less affected by $\Phi$. In this case, the dependence of $J_{\text{int}}$ on the parity of $n$ is weaker than the case of $T_1 < T_2$. These observable effects can be utilized for the detection of the non-Abelian statistics.

An advantage of NC $s$-wave superconductors over $p_x + ip_y$-wave superconductors and the $\nu = 5/2$ FQH state is that the gap energy scale of the former can be typically much larger than those of the latter. Note that the superconductivity in NC systems needs not to be a bulk phenomenon. Let us consider the junction between an $s$-wave superconductor and a metallic thin film placed on an insulating substrate. (See FIG.2(b).) The thin film must be sufficiently clean so that the mean free path is larger than its thickness. In this system, inversion symmetry is broken, and an asymmetric potential gradient perpendicular to the interface is introduced. We can use a material with a high transition temperature such as MgB$_2$ ($T_c \sim 39$K) for the superconductor. Then, the proximity effect induces $s$-wave superconductivity in the 2D NC system realized in the thin film. If the Fermi energy of the film $E_F$ is much smaller than that of the bulk superconductor, the energy gap in the vortex core $\Delta^2 / E_F$ for the proximity-induced NC superconductor can be relatively large. The strength of the asymmetric SO interaction can be controlled by changing the substrate or applying a perpendicular voltage on the film. Although electrons should experience strong SO scatterings at the interface, the transition temperature and the gap of the $s$-wave pairing state are not affected by them. Also, note that the Majorana fermions in vortices of the NC superconductors do not require a half quantum vortex, i.e. a texture of the $d$-vector, because our system is essentially regarded as spinless. In this sense, zero energy Majorana states in NC $s$-wave superconductors are more realizable than in spinful $p$-wave superconductors.

In conclusion, NC $s$-wave superconductors under magnetic field have a topological order for a particular electron filling, and can be playgrounds for the non-Abelian anyons. Although we consider only the Rashba SO interaction here, our argument can be easily generalized to other asymmetric SO interactions.

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