SURFACES AS GRAPHS OF FINITE TYPE IN $\mathbb{H}^2 \times \mathbb{R}$

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ABSTRACT. In this paper, we prove that $\Delta X = 2H$ where $\Delta$ is the Laplacian operator, $r = (x, y, z)$ the position vector field and $H$ is the mean curvature vector field of a surface $S$ in $\mathbb{H}^2 \times \mathbb{R}$ and we study surfaces as graphs in $\mathbb{H}^2 \times \mathbb{R}$ which has finite type immersion.

1. Introduction

The $\mathbb{H}^2 \times \mathbb{R}$ geometry is one of eight homogeneous Thurston 3-geometries

$$E^3, S^3, H^3, S^2 \times \mathbb{R}, H^2 \times \mathbb{R}, \widetilde{SL(2, \mathbb{R})}, Nil, Sol.$$ 

The Riemannian manifold $(M,g)$ is called homogeneous if for any $x, y \in M$ there exists an isometry $\phi : M \to M$ such that $y = \phi(x)$. The two and three-dimensional homogeneous geometries are discussed in detail in [6].

A Euclidean submanifold is said to be of finite Chen-type if its coordinate functions are a finite sum of eigenfunctions of its Laplacian [3]. B. Y. Chen posed the problem of classifying the finite type surfaces in the...
3-dimensional Euclidean space \( \mathbb{E}^3 \). Further, the notion of finite type can be extended to any smooth function on a submanifold of a Euclidean space or a pseudo-Euclidean space.

Let \( S \) be a 2-dimensional surface of the Euclidean 3-space \( \mathbb{E}^3 \). If we denote by \( r, H \) and \( \Delta \) the position vector field, the mean curvature vector field and the Laplace operator of \( S \) respectively, then it is well-known that [3]

\[
\Delta r = -2H.
\]

(1.1)

A well-known result due to Takahashi states that minimal surfaces and spheres are the only surfaces in \( \mathbb{E}^3 \) satisfying the condition \( \Delta r = \lambda r \) for a real constant \( \lambda \). From (1.1), we know that minimal surfaces and spheres also verify the condition

\[
\Delta H = \lambda H, \quad \lambda \in \mathbb{R}.
\]

(1.2)

Equation (1.1) shows that \( S \) is a minimal surface of \( \mathbb{E}^3 \) if and only if its coordinate functions are harmonic. In [9], D. W. Yoon studied surfaces invariant under the 1-parameter subgroup in \( Sol_3 \).

In 2012, M. Bekkar and B. Senoussi [1] studied the translation surfaces in the 3-dimensional Euclidean and Lorentz-Minkowski space under the condition

\[
\Delta^{III} r_i = \mu_i r_i, \quad \mu_i \in \mathbb{R},
\]

where \( \Delta^{III} \) denotes the Laplacian of the surface with respect to the third fundamental form \( III \).

A surface \( S \) in the Euclidean 3-space \( \mathbb{E}^3 \) is called minimal when locally each point on the surface has a neighborhood which is the surface of least area with respect to its boundary [5]. In 1775, J. B. Meusnier showed that the condition of minimality of a surface in \( \mathbb{E}^3 \) is equivalent with the vanishing of its mean curvature function, \( H = 0 \).

Let \( z = f(x,y) \) define a graph \( S \) in the Euclidean 3-space \( \mathbb{E}^3 \). If \( S \) is minimal, the function \( f \) satisfies

\[
(1 + (f_y)^2) f_{xx} - 2f_{xy}f_x f_y + (1 + (f_x)^2) f_{yy} = 0,
\]

which was obtained by J. L. Lagrange in 1760.

In 1835, H. F. Scherk studied translation surfaces in \( \mathbb{E}^3 \) and proved that, besides the planes, the only minimal translation surfaces are given by

\[
z(x, y) = \frac{1}{\lambda} \log |\cos(\lambda x)| - \frac{1}{\lambda} \log |\cos(\lambda y)|,
\]

where \( \lambda \) is a non-zero constant. In 1991, F. Dillen, L. Verstraelen and G. Zafindratafa [4] generalized this result to higher-dimensional Euclidean space.
In 2015, D. W. Yoon [8] studied translation surfaces in the product space \( \mathbb{H}^2 \times \mathbb{R} \) and classified translation surfaces with zero Gaussian curvature in \( \mathbb{H}^2 \times \mathbb{R} \).

In 2019, B. Senoussi, M. Bekkar [7] studied translation surfaces of finite type in \( H_3 \) and \( Sol_3 \) and the authors gave some theorems.

A surface \( S(\gamma_1, \gamma_2) \) in \( \mathbb{H}^2 \times \mathbb{R} \) is a surface parametrized by
\[
S : \Omega \subseteq \mathbb{R}^2 \rightarrow \mathbb{H}^2 \times \mathbb{R}, \quad X(s, t) = \gamma_1(s) \ast \gamma_2(t) \text{ or } X(s, t) = \gamma_2(t) \ast \gamma_1(s),
\]
where \( \gamma_1 \) and \( \gamma_2 \) are any generating curves in \( \mathbb{R}^3 \). Since the multiplication \( \ast \) is not commutative.

In this work we study the surfaces as graphs of functions \( \varphi = f(s, t) \) in \( \mathbb{H}^2 \times \mathbb{R} \) satisfy the condition
\[
(1.3) \quad \Delta x_i = \lambda_i x_i, \quad \lambda_i \in \mathbb{R}.
\]

2. Preliminaries

Let \( \mathbb{H}^2 \) be represented by the upper half-plane model \( \{(x, y) \in \mathbb{R} \mid y > 0\} \) equipped with the metric
\[
g_{\mathbb{H}} = \frac{(dx^2 + dy^2)}{y^2}.
\]
The space \( \mathbb{H}^2 \), with the group structure derived by the composition of proper affine maps, is a Lie group and the metric \( g_{\mathbb{H}} \) is left invariant.

Therefore, the product space \( \mathbb{H}^2 \times \mathbb{R} \) is a Lie group with the left invariant product metric
\[
g = \frac{(dx^2 + dy^2)}{y^2} + dz^2,
\]
we can define the multiplication law on \( \mathbb{H}^2 \times \mathbb{R} \) as follows
\[
(x, y, z) \ast (\bar{x}, \bar{y}, \bar{z}) = (y\bar{x} + x, y\bar{y} + y, z + \bar{z}).
\]
The left identity is \((0, 1, 0)\) and the inverse of \((x, y, z)\) is \((-\frac{x}{y}, \frac{1}{y}, -z)\), on \( \mathbb{H}^2 \times \mathbb{R} \) a left-invariant metric
\[
ds^2 = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2,
\]
where
\[
\omega^1 = \frac{dx}{y}, \quad \omega^2 = \frac{dy}{y}, \quad \omega^3 = dz,
\]
is the orthonormal coframe associated with the orthonormal frame
\[
e_1 = y \frac{\partial}{\partial x}, \quad e_2 = y \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z},
\]
The corresponding Lie brackets are
\[
[e_1, e_2] = -e_1, \quad [e_i, e_i] = [e_3, e_1] = [e_2, e_3] = 0, \forall i = 1, 2, 3.
\]
The Levi-Civita connection $\nabla$ of $H^2 \times \mathbb{R}$ is given by
\[
\begin{pmatrix}
\nabla_{e_1} e_1 \\
\nabla_{e_2} e_2 \\
\nabla_{e_3} e_3
\end{pmatrix} =
\begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix},
\nabla_{e_2} e_i = \nabla_{e_3} e_i = 0, \ \forall i = 1, 2, 3.
\]

Let $S$ be an immersed surface in $H^2 \times \mathbb{R}$ given as the graph of the function $z = f(x, y)$. Hence, the position vector is described by $r(x, y) = (x, y, f(x, y))$ and the tangent vectors $r_x = \frac{\partial r}{\partial x}$ and $r_y = \frac{\partial r}{\partial y}$ in terms of the orthonormal frame $(e_1, e_2, e_3)$ are described by
\[
\begin{align*}
    r_x &= \frac{\partial}{\partial x} + f_r \frac{\partial}{\partial z} = \frac{1}{y} e_1 + f_x e_3, \\
    r_y &= \frac{\partial}{\partial y} + f_y \frac{\partial}{\partial z} = \frac{1}{y} e_2 + f_y e_3.
\end{align*}
\]

**Definition 2.1.** [3] The immersion $(S, r)$ is said to be of finite Chen-type $k$ if the position vector $X$ admits the following spectral decomposition
\[
    r = r_0 + \sum_{i=1}^{k} r_i,
\]
where $r_i$ are $\mathbb{E}^3$-valued eigenfunctions of the Laplacian of $(S, r) : \Delta r_i = \lambda_i r_i, \ \lambda_i \in \mathbb{R}, i = 1, 2, \ldots, k$. If $\lambda_i$ are different, then $S$ is said to be of $k$-type.

For the matrix $G = (g_{ij})$ consisting of the components of the induced metric on $S$, we denote by $G^{-1} = (g^{ij})$ the inverse matrix of the determinant $D = \det(g_{ij})$ of the matrix $(g_{ij})$. The Laplacian $\Delta$ on $S$ is, in turn, given by
\[
\Delta = \frac{-1}{\sqrt{|D|}} \sum_{ij} \frac{\partial}{\partial r^i} (\sqrt{|D|} g^{ij} \frac{\partial}{\partial r^j}).
\]

If $r = r(x, y) = (r_1 = r_1(x, y), r_2 = r_2(x, y), r_3 = r_3(x, y))$ is a function of class $C^2$ then we set
\[
\Delta r = (\Delta r_1, \Delta r_2, \Delta r_3).
\]

**3. Surfaces as graphs of finite type in $H^2 \times \mathbb{R}$**

Let $S$ be a graph of a smooth function
\[
f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}.
\]

We consider the following parametrization of $S$
\[
r(x, y) = (x, y, f(x, y)), \ (x, y) \in \Omega.
\]
Theorem 3.1. A Beltrami formula in $\mathbb{H}^2 \times \mathbb{R}$ is given by the following:

$$\Delta r = 2H,$$

where $\Delta$ is the Laplacian of the surface and $H$ is the mean curvature vector field of $S$.

Proof. A basis of the tangent space $T_pS$ associated to this parametrization is given by

$$r_x = \frac{\partial}{\partial x} + f_x \frac{\partial}{\partial z} = \frac{1}{y} e_1 + f_x e_3,$$

$$r_y = \frac{\partial}{\partial y} + f_y \frac{\partial}{\partial z} = \frac{1}{y} e_2 + f_y e_3,$$

The coefficients of the first fundamental form of $S$ are given by

$$E = g(r_x, r_x) = \frac{1}{y^2} + f_x^2,$$

$$F = g(r_x, r_y) = f_x f_y,$$

$$G = g(r_y, r_y) = \frac{1}{y^2} + f_y^2.$$

The unit normal vector field $N$ on $S$ is given by

$$N = \frac{1}{W} \left( -\frac{1}{y} f_x e_1 - \frac{1}{y} f_y e_2 + \frac{1}{y^2} e_3 \right),$$

where $W = \sqrt{\frac{1}{y^4} + \frac{1}{y^2} f_x^2 + \frac{1}{y^2} f_y^2}$.

To compute the second fundamental form of $S$, we have to calculate the following

$$r_{xx} = \nabla_{r_x} r_x = \frac{1}{y^2} e_2 + f_{xx} e_3,$$

$$r_{xy} = \nabla_{r_x} r_y = \nabla_{r_y} r_x = -\frac{1}{y^2} e_1 + f_{xy} e_3,$$

$$r_{yy} = \nabla_{r_y} r_y = -\frac{1}{y^2} e_2 + f_{yy} e_3.$$

So, the coefficients of the second fundamental form of $S$ are given by

$$L = g(\nabla_{r_x} r_x, N) = \frac{1}{W y^2} \left( f_{xx} - \frac{1}{y} f_y \right),$$

$$M = g(\nabla_{r_x} r_y, N) = \frac{1}{W y^2} \left( f_{xy} + \frac{1}{y} f_x \right),$$

$$N = g(\nabla_{r_y} r_y, N) = \frac{1}{W y^2} \left( f_{yy} + \frac{1}{y} f_y \right),$$

where $W = \sqrt{\frac{1}{y^4} + \frac{1}{y^2} f_x^2 + \frac{1}{y^2} f_y^2}$.

Thus, the mean curvature $H$ of $S$ is given by

$$H = \frac{EN - 2FM + GL}{2W^2}.$$
By (2.3), the Laplacian operator $\Delta$ of $r$ can be expressed as

$$
\Delta = -\frac{1}{W^4} \left[ W^2 \left( G \frac{\partial^2}{\partial x^2} - 2F \frac{\partial^2}{\partial x \partial y} + E \frac{\partial^2}{\partial y^2} \right) + \Delta_1 \frac{\partial}{\partial x} + \Delta_2 \frac{\partial}{\partial y} \right],
$$

where

$$
\Delta_1 = \frac{2}{y^2} f_y f_x^2 f_{xy} - \frac{1}{y^4} f_x f_{xx} - \frac{1}{y^2} f_x f_y^2 f_{xy} - \frac{1}{y^4} f_x f_{yy} - \frac{1}{y^2} f_x f_{y}^3 f_{yy},
$$

and

$$
\Delta_2 = \frac{2}{y^2} f_y f_x^2 f_{xy} - \frac{1}{y^4} f_y f_{yy} - \frac{1}{y^2} f_y f_x f_{yy} - \frac{1}{y^4} f_y f_{xx} - \frac{1}{y^2} f_y f^3 f_{xx}
$$

$$
- \frac{1}{y^6} f_x^2 + \frac{1}{y^4} f_x^2 + \frac{1}{y^3} f_x^2 f_y^2.
$$

By a straightforward computation, the Laplacian operator $\Delta$ of $r$ with the help of (3.1) and (3.2) turns out to be

$$
\Delta r = -\frac{1}{W^4} \left[ \left( \frac{2}{y^3} f_x^2 f_y f_{xy} - \frac{1}{y^2} f_x f_{xx} - \frac{1}{y^2} f_x f_y f_{xy} - \frac{1}{y^4} f_x f_{yy} - \frac{1}{y^2} f_x f_{y}^3 f_{yy} \right) e_1 \right]
$$

$$
+ \left( \frac{2}{y^3} f_x f_y f_{xy} - \frac{1}{y^2} f_y f_{yy} - \frac{1}{y^2} f_x f_{yy} - \frac{1}{y^4} f_x f_{xx} + \frac{1}{y^2} f_x f_y^2 f_{yy} \right) e_2
$$

$$
+ \left( \frac{1}{W^2} \frac{f_x}{W y} \frac{1}{W^2} \frac{f_y}{W y} \frac{1}{y^2} (f_{xx} + f_{yy}) + \frac{f_x^2 f_{yy} + f_y^2 f_{xx}}{y^2} - \frac{1}{y} (f_x f_y^2 + f_y^3) - 2 f_x f_y f_{xy} \right) e_3
$$

$$
\Delta r = \frac{1}{W^3 y^2} \left[ \frac{1}{y^2} (f_{xx} + f_{yy}) + \frac{f_x^2 f_{yy} + f_y^2 f_{xx}}{y^2} - \frac{1}{y} (f_x f_y^2 + f_y^3) - 2 f_x f_y f_{xy} \right] e_1
$$

$$
+ \frac{1}{W y^2} \left[ \frac{1}{y^2} (f_{xx} + f_{yy}) + \frac{f_x^2 f_{yy} + f_y^2 f_{xx}}{y^2} - \frac{1}{y} (f_x f_y^2 + f_y^3) - 2 f_x f_y f_{xy} \right] e_2
$$

$$
+ \frac{1}{W^2 y^2} \left[ \frac{1}{y^2} (f_{xx} + f_{yy}) + \frac{f_x^2 f_{yy} + f_y^2 f_{xx}}{y^2} - \frac{1}{y} (f_x f_y^2 + f_y^3) - 2 f_x f_y f_{xy} \right] e_3
$$

thus we get

$$
\Delta r = 2HN,
$$

$$
= 2H,
$$
where $H$ is the mean curvature vector field of $S$.

$S$ is a minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ if and only if its coordinate functions are harmonic.

4. Surfaces as graphs in $\mathbb{H}^2 \times \mathbb{R}$ satisfying $\triangle x_i = \lambda_i x_i$

Let $S$ be an immersed surface in $\mathbb{H}^2 \times \mathbb{R}$ given as the graph of function $z = f(x,y)$. Hence, the vector position is described by $r(x,y) = (x,y,f(x,y))$.

We have

$$r_x = \frac{1}{y} e_1 + f_x e_3, \quad r_y = \frac{1}{y} e_2 + f_y e_3,$$

where $r_x = \frac{\partial r}{\partial x}$, $r_y = \frac{\partial r}{\partial y}$, and $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$.

From an earlier results the mean curvature $H$ of $S$ and the unit normal vector field $N$ on $S$ are given by

$$H = \frac{1}{2W^3y^2} \left[ \frac{1}{y^2} (f_{xx} + f_{yy}) + (f_x^2 f_{yy} + f_y^2 f_{xx}) - \frac{1}{y} (f_x^2 f_y + f_y^2 f_x) - 2f_x f_y f_{xy} \right],$$

and

$$N = \frac{1}{W} \left( -\frac{1}{y} f_x e_1 - \frac{1}{y} f_y e_2 + \frac{1}{y^2} e_3 \right),$$

where $W = \sqrt{\frac{1}{y^4} + \frac{1}{y^2 f_x^2} + \frac{1}{y^2 f_y^2}}$.

If the vector position on the tangent space $T_p S$ is described by $r(x,y)$

$$r(x,y) = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + f(x,y) \frac{\partial}{\partial z},$$

then

$$r(x,y) = \frac{x}{y} e_1 + e_2 + f(x,y) e_3.$$

The equation (1.3) by means of (3.3), (4.1) and (4.2) gives rise to the following system of ordinary differential equations

$$\left( \frac{2H}{W} \right) f_x = -\lambda_1 x,$$

$$\left( \frac{2H}{W} \right) f_y = -\lambda_2 y,$$

$$\frac{2H}{W} = \lambda_3 y^2 f.$$
Therefore, the problem of classifying the surfaces $S$ of (1.3) is reduced to the integration of this system of ordinary differential equations.

Next we study it according to the constants $\lambda_1$, $\lambda_2$ and $\lambda_3$.

**Case 1.** Let $\lambda_3 = 0$. In this case the system (4.3), (4.4) and (4.5) is reduced equivalently to

\[(4.6) \quad \left(\frac{2H}{W}\right) f_x = -\lambda_1 x,\]
\[(4.7) \quad \left(\frac{2H}{W}\right) f_y = -\lambda_2 y,\]
\[(4.8) \quad \frac{2H}{W} = 0.\]

The equation (4.8) implies that the mean curvature $H$ is identically zero. Thus, the surface $S$ is minimal; and we get also $\lambda_1 = \lambda_2 = 0$.

**Case 2.** Let $\lambda_3 \neq 0$. In this case we study the general system (4.3), (4.4) and (4.5).

2-i): If $\lambda_1 = \lambda_2 = 0$, then $H = 0$. From (4.5) we obtain $\lambda_3 = 0$, so we get a contradiction.

2-ii): If $\lambda_1 = 0$ and $\lambda_2 \neq 0$, from (4.3) we obtain $H f_x = 0$.

2-ii-a: If $H = 0$ (4.4), (4.5) implies that $\lambda_2 = \lambda_3 = 0$. So we get a contradiction.

2-ii-b: if $f_x = 0$, then $f(x, y) = \varphi(y)$, where $\varphi$ is smooth function of $y$.

The mean curvature $H$ turns to

\[(4.9) \quad H = \frac{1}{2W y^3} \left(\frac{1}{y} \varphi'' - \varphi'^2\right),\]

where $\varphi' = \frac{d\varphi}{dy}$.

Using (4.4) and (4.5) we obtain

$\varphi' = \frac{-\lambda_2}{\lambda_3 y \varphi},$

which leads to,

$\lambda_3 \varphi' \varphi = \frac{-\lambda_2}{y}.$

After integrating with respect to $y$, we obtain
\[ \frac{\lambda_3}{2} \varphi^2(y) = -\lambda_2 \ln y + \phi(x); \quad y > 0, \]

where \( \phi \) is smooth function of \( x \), and hence

\[ f(x, y) = \varphi(y) = \pm \sqrt{\frac{\lambda_2}{\lambda_3} \ln \frac{1}{y^2}} + \phi(x). \]

Using the condition \( f_x = 0 \) we get \( \phi(x) = a, \quad a \in \mathbb{R} \).

Thus,

\[ f(x, y) = \varphi(y) = \pm \sqrt{\frac{\lambda_2}{\lambda_3} \ln \frac{1}{y^2} + c}; \quad c = \frac{2}{\lambda_3} a, \]

in this subcase, the surfaces \( S \) are given by

\[ r(x, y) = \left( x, y, \pm \sqrt{\frac{\lambda_2}{\lambda_3} \ln \frac{1}{y^2} + c} \right); \quad \lambda_2 \neq 0, \lambda_3 \neq 0, \quad c \in \mathbb{R}. \]

2-iii): If \( \lambda_1 \neq 0 \) and \( \lambda_2 = 0 \), from (4.4) we obtain \( H f_y = 0 \).

2-iii-a: If \( H = 0 \), (4.3) and (4.5) implies that \( \lambda_2 = \lambda_3 = 0 \). So we get a contradiction.

2-iii-b: If \( f_y = 0 \), then \( f(x, y) = \psi(x) \), where \( \psi \) is smooth function of \( x \).

The mean curvature \( H \) turns to

\[ (4.10) \quad H = \frac{1}{2W y^4} \psi'', \]

where \( \psi' = \frac{d\psi}{dx} \).

Using (4.3) and (4.5) we get

\[ \psi' = -\frac{\lambda_1 x}{\lambda_3 y^2 \psi}, \]

so we can write

\[ (4.11) \quad \lambda_3 y^2 + \lambda_1 \frac{x}{\psi \psi'} = 0, \]

A differentiation with respect to \( y \) gives

\[ \lambda_3 y = 0, \]

this implies that \( \lambda_3 = 0 \) and from (4.8) we get the mean curvature \( H \) is identically zero. From (4.6) and (4.7) we obtain \( \lambda_1 = \lambda_2 = 0 \), which leads to a contradiction.
2-iv): If $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ From (4.3), we have

\begin{equation}
(4.12) \quad \frac{2H}{W} = -\frac{\lambda_1 x}{\psi'}. \tag{4.12}
\end{equation}

Substituting (4.12) into (4.5), we get

\[-\frac{\lambda_1 x}{\psi'} = \lambda_3 y^2 \psi',\]

A differentiation with respect to $x$ gives

\[-\lambda_1 \left( \frac{\psi - x\psi''}{\psi'^2} \right) = \lambda_3 y^2 \psi',\]

this equation gives

\begin{equation}
(4.13) \quad \lambda_1 \left( \frac{\psi' - x\psi''}{\psi'^3} \right) + \lambda_3 y^2 = 0. \tag{4.13}
\end{equation}

A differentiation with respect to $y$ gives

\[\lambda_3 y = 0,\]

this implies that $\lambda_3 = 0$ and from (4.8) we get the mean curvature $H$ is identically zero. From (4.6) and (4.7) we obtain $\lambda_1 = \lambda_2 = 0$, which leads to a contradiction.

Therefore, we have the following theorem,

**Theorem 4.1.** Let $S$ be a surface as graph of function parametrized by $r(x, y) = (x, y, f(x, y))$ in $\mathbb{H}^2 \times \mathbb{R}$. Then, $S$ satisfies the equation $\Delta r_i = \lambda_i r_i$, $\lambda_i \in \mathbb{R}$ if and only if $S$ is minimal surfaces or parametrized as

\[S : r(x, y) = \left( x, y, \pm \sqrt{\frac{\lambda_2}{\lambda_3}} \ln \frac{1}{y^2} + c \right); \quad \lambda_2 \neq 0, \lambda_3 \neq 0, c \in \mathbb{R}.\]

**Conflicts of Interest:** The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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