New Integrable Family in the $n$-Dimensional Homogeneous Lotka–Volterra Systems with Abelian Lie Algebra

Kenji IMAI$^\dagger$ and Yoshihiro HIRATA$^\ddagger$

$^\dagger$School of Liberal Arts and Sciences, Daido Institute of Technology, Nagoya, 457-8530, Japan
kimai@daido-it.ac.jp

$^\ddagger$Graduate School of Human Informatics, Nagoya University, Nagoya, 464-8601, Japan
hirata@ncube.human.nagoya-u.ac.jp

Abstract. We present an $n$-dimensional integrable homogeneous Lotka–Volterra system, which has $(n^2 - 1)$-dimensional Lie symmetry algebra. Moreover a wider integrable family is derived from the structure of the Lie algebra.

The $n$-dimensional homogeneous Lotka–Volterra (HLV) system is given by

$$\frac{dx_i}{dt} = f_i(x) = x_i \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, \ldots, n,$$

where $a_{ij}, i, j = 1, \ldots, n$ are complex parameters. Integrability of the HLV system has been fairly investigated in some particular cases, e.g., soliton systems, the 2-dimensional systems, or the 3-dimensional ABC system. Almost all the above studies dealt with first integrals in the criterions of integrability. However it is known that a system of ordinary differential equations is also integrable if the system has enough Lie symmetry vector fields. In this letter we treat Lie symmetries and find a new integrable family in the $n$-dimensional HLV systems, named the ladder system or the generalized ladder system.

We rewrite the considered system into a vector field form as

$$X_f = \sum_{i=1}^{n} f_i(x) \frac{\partial}{\partial x_i}.$$ 

A Lie symmetry is a vector field which commutes the vector field $X_f$.

**Definition 1 (Lie symmetry).** A vector field $X$ is called a Lie symmetry with respect to the system (1) if $X$ commutes $X_f$, i.e.,

$$[X_f, X] := X_f X - XX_f = 0.$$
A set of all the Lie symmetries forms a Lie algebra, and if the algebra has an \((n - 1)\)-dimensional solvable sub-algebra, the system can be integrated by quadrature.

Let us consider a particular system in the HLV systems, which we call the \(n\)-dimensional ladder system. We introduced the 3-dimensional ladder system in Ref. [11]. The \(n\)-dimensional ladder system possesses polynomial Lie symmetries which constitute \((n - 1)\)-dimensional Abelian Lie algebra. Hence the \(n\)-dimensional ladder system can be integrated by using the Lie algebra.

**Definition 2 (The ladder system).** The \(n\)-dimensional HLV system \((1)\) with the coefficients

\[
A := (a_{ij})_{1 \leq i,j \leq n} = \begin{pmatrix}
1 & 0 & \cdots & -n + 2 \\
2 & 1 & \cdots & -n + 3 \\
\vdots & \vdots & \ddots & \vdots \\
n & n - 1 & \cdots & 1
\end{pmatrix}
\]  

(2)

is called the \(n\)-dimensional (homogeneous Lotka–Volterra) ladder system.

The ladder system has rational Lie symmetries as introduced below.

**Theorem 1.** The \(n\)-dimensional ladder system \((1)\) with eq. \((2)\) possesses the following Lie symmetries:

\[
Y^l_m = x_m u^{l-m-1} \left( D - u \frac{\partial}{\partial x_l} \right), \quad l, m = 1, \ldots, n,
\]

where

\[
u = \sum_{j=1}^{n} x_j,
\]

\[
D = \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}.
\]

A proof is straightforward and hence we omit it. Now we also present several basic properties for \(Y^l_m\) without proofs.

**Proposition 1.**

\[
[Y^l_m, Y^l'_{m'}] = \delta_{m',l} Y^l_{m'} - \delta_{m,l} Y^l_{m'},
\]

\[
\sum_{l=1}^{n} Y^l_i = 0.
\]

\[
Y^l_m(u) = 0,
\]

\[
[Y^l_m, D] = (l - m)Y^l_m,
\]

and especially,

\[
[Y^l_i, D] = 0.
\]

Now let us introduce the following linear space

\[
\mathcal{L} = \left\{ \sum_{l,m=1}^{n} a^m_l Y^l_m \mid a^m_l \in \mathbb{C} \right\}.
\]
Theorem 2. The linear space $\mathcal{L}$ forms $(n^2 - 1)$-dimensional Lie algebra.

Proof. It is evident that $\mathcal{L}$ forms a closed algebra from eq. (3). Hence we only show that $\mathcal{L}$ has $n^2 - 1$ dimensions.

$Y^l_n$ have homogeneous rational coefficients of order $l-m$. We therefore consider the dimension of each homogeneous subset. First, consider the linear combination of $Y^l_{l-k}$, $-n < k < n$, $k \neq 0$ as

$$\sum_{l=k+1}^{k+n} \alpha_l Y^l_{l-k} = 0,$$

where $\alpha_l \in \mathbb{C}$ and $\alpha_l = 0$ for $l \leq 0$, $n + 1 \leq l$. If $k > 0$, the coefficient of $\partial/\partial x_1$ is written as

$$u^{k-1} \sum_{l=k+1}^{k+n} \alpha_l x_{l-k} x_1.$$

Obviously $\forall l$, $\alpha_l = 0$. Hence when $k > 0$, $Y^l_{l-k}$ are linearly independent over $\mathbb{C}$. In the same way, if $k < 0$, by considering the coefficient of $\partial/\partial x_n$, one can show the linear independency.

Next we deal with the “diagonal” elements $Y^l_l$. The linear combination of $Y^l_l$, $1 \leq l \leq n$ is rewritten as

$$\sum_{l=1}^{n} \alpha_l Y^l_l = u^{-1} \left( \sum_{l=1}^{n} \alpha_l x_l \sum_{j=1}^{n} \frac{\partial}{\partial x_j} - \sum_{l=1}^{n} \alpha_l x_l \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_l} \right)$$

$$= u^{-1} \left( \sum_{j=1}^{n} \alpha_j x_j \sum_{l=1}^{n} x_l \frac{\partial}{\partial x_l} - \sum_{l=1}^{n} \alpha_l x_l \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_l} \right)$$

$$= u^{-1} \sum_{j=1}^{n} \sum_{l=1}^{n} (\alpha_j - \alpha_l) x_j x_l \frac{\partial}{\partial x_l}.$$

Hence $\sum \alpha_l Y^l_l = 0$ if and only if $\alpha_1 = \cdots = \alpha_n$. Thus the linear space

$$\mathcal{L}' = \left\{ \sum_{l=1}^{n} \alpha_l Y^l_l \bigg| \alpha_l \in \mathbb{C} \right\}$$

has $n-1$ dimensions. This completes a proof.

In particular $\mathcal{L}$ has $(n-1)$-dimensional Abelian sub-algebra constituted with polynomial Lie symmetries.

Corollary 1. The polynomial $(n-1)$-dimensional sub-algebra spanned by $\{Y^1_n, \ldots, Y^n_{n-1}\}$ is Abelian. Hence the ladder system is integrable.

The following statement, obtained as a by-product of the proof of Theorem 2, is essential for generalization of the ladder system.

Corollary 2. The $(n-1)$-dimensional linear space $\mathcal{L}'$ forms $(n-1)$-dimensional Abelian sub-algebra.
Now we generalize the ladder system by using the Abelian sub-algebra \( L' \). Since \([Y_l^l, Y_m^m] = 0\) and \([Y_l^l, D] = 0\), arbitrary linear combinations of \( Y_m^m, m = 1, \ldots, n \) and \( D \) over \( \mathbb{C} \) commute every \( Y_l^l \) as

\[
[Y_l^l, \alpha_0 D - \sum_{m=1}^{n} \alpha_m Y_m^m] = 0,
\]

where \( \alpha_m \in \mathbb{C}, m = 0, 1, \ldots, n \). Moreover from eq. (4), the following relation

\[
[Y_l^l, \alpha_0 u D - u \sum_{m=1}^{n} \alpha_m Y_m^m] = 0 \tag{5}
\]

holds.

On the other hand

\[
\alpha_0 u D - u \sum_{m=1}^{n} \alpha_m Y_m^m = \alpha_0 u \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} - \sum_{m=1}^{n} \alpha_m x_m \left( D - u \frac{\partial}{\partial x_m} \right)
\]

\[
= \alpha_0 \sum_{i,j=1}^{n} x_i x_j \frac{\partial}{\partial x_j} - \sum_{m=1}^{n} \alpha_m x_m \sum_{k=1}^{n} x_k \left( \frac{\partial}{\partial x_k} - \frac{\partial}{\partial x_m} \right)
\]

\[
= \alpha_0 \sum_{i,j=1}^{n} x_i x_j \frac{\partial}{\partial x_i} - \sum_{k,m=1}^{n} \alpha_m x_k x_m \frac{\partial}{\partial x_k} + \sum_{k,m=1}^{n} \alpha_m x_k x_m \frac{\partial}{\partial x_m}
\]

\[
= \sum_{i,j=1}^{n} (\alpha_0 + \alpha_i - \alpha_j)x_i x_j \frac{\partial}{\partial x_i}.
\]

Hence \( \alpha_0 u D - u \sum_{m=1}^{n} \alpha_m Y_m^m \) defines the HLV system with the coefficients \( a_{ij} = \alpha_0 + \alpha_i - \alpha_j \). It can be regarded as a generalization of the ladder system: if one puts \( \alpha_0 = 1, \alpha_j = j, j = 1, \ldots, n \), then the generalized ladder system becomes the ladder system.

**Definition 3 (The generalized ladder system).** The HLV system (1) with \( a_{ij} = \alpha_0 + \alpha_i - \alpha_j, \alpha_j \in \mathbb{C}, 0 \leq j \leq n \) is called the (homogeneous Lotka–Volterra) generalized ladder system.

Equation (5) implies the following statement.

**Theorem 3.** The generalized ladder system possesses the \((n-1)\)-dimensional Abelian Lie algebra \( L' \). Hence it is integrable.

The generalized ladder system also possesses the following Lie symmetries besides \( Y_l^l, l = 1, \ldots, n \).

**Proposition 2.** \( \tilde{Y}_m^l = x_m u^{\alpha_l - \alpha_m - 1}(D - u \partial / \partial x_l), l, m = 1, \ldots, n \) are Lie symmetries for the generalized ladder system.

A proof is straightforward. Note that the above Lie symmetries are logarithmic in general. Moreover the \((n^2 - 1)\)-dimensional linear space \( \tilde{\mathcal{L}} \) defined by

\[
\tilde{\mathcal{L}} = \left\{ \sum_{m,l=1}^{n} a_l^m \tilde{Y}_m^l \bigg| a_l^m \in \mathbb{C} \right\}
\]

has the same algebraic structure with \( \mathcal{L} \). One can easily show the following proposition.
Proposition 3. The commutators among $\tilde{Y}_m^l$, $l, m = 1, \ldots, n$ are given by

$$[\tilde{Y}_m^l, \tilde{Y}_{m'}^{l'}] = \delta_{m', l} \tilde{Y}_{m'}^{l'} - \delta_{m, l'} \tilde{Y}_m^{l'}.$$ 

Hence $\tilde{L} \cong L$.

Thus although we construct the generalized ladder system by extracting the Abelian sub-algebra $L'$, the same algebraic structure is restored.

Since the generalized ladder system is integrable, there exist $n - 1$ first integrals. However the first integrals are not always polynomial or rational functions. The first integrals can easily be composed by the Lie symmetries $Y_m^l$.

Proposition 4. $x_m^l x_m^{l'}$, $l, m = 1, \ldots, n$ are first integrals for the generalized ladder system.

Proof. If there exist two Lie symmetries $X$ and $X'$, and exists a function $F$ such that $X' = FX$, then $F$ is a first integrals. One can compose the first integrals by using the obtained Lie symmetries in Proposition 3. □

In this letter we have first introduced the $n$-dimensional ladder system (1) with eq. (2). The ladder system possesses the Lie symmetry algebra $L$, and it is integrable since $L$ has $(n-1)$-dimensional Abelian sub-algebra (Corollary 2). Moreover, by extracting the Abelian Lie sub-algebra $L' \subset L$, we have constructed the new integrable family in the HLV systems, the generalized ladder system. Although $L$ is not Lie symmetry algebra for the generalized ladder system anymore, the generalized ladder system possesses the Lie symmetry algebra $\tilde{L}$, which is isomorphic to $L$.

The first integrals given in this letter are rational for the ladder system and generally logarithmic for the generalized ladder system. Thus despite the first integrals which belong to different classes, the structure of the Lie symmetry algebra is just the same between the ladder and the generalized ladder systems.

We have presented the method to derive new integrable systems using Lie symmetry sub-algebra. The universality of the method is open.

References

[1] R. Hirota and J. Satsuma: Prog. Theor. Phys. Suppl. 59 (1976) 64.
[2] K. Narita: J. Phys. Soc. Jpn. 51 (1982) 1682.
[3] Y. Ito: Prog. Theor. Phys. 78 (1987) 507.
[4] M. R. Cairó, M. R. Feix and J. Llibre: J. Math. Phys. 40 (1999) 2074.
[5] L. Cairó and J. Llibre: J. Phys. A 33 (2000) 2407.
[6] M. A. Almeida, M. E. Magalhães and I. C. Moreira: J. Math. Phys. 36 (1995) 1854.
[7] S. Labrunie: J. Math. Phys. 37 (1996) 5539.
[8] J. Moulin-Ollagnier: Bull. Sci. Math. 121 (1997) 463.
[9] P. Gao: *Phys. Lett. A* **255** (1999) 253.

[10] L. Cairó and J. Llibre: *J. Phys. A* **33** (2000) 2395.

[11] Y. Hirata and K. Imai *J. Phys. Soc. Jpn.* **71** (2002) 2396.