\textbf{$L^2$-blowup estimates of the plate equation}

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Abstract

We consider the Cauchy problems in $\mathbb{R}^n$ for the plate equation with a weighted $L^1$-initial data. We derive optimal estimates of the $L^2$-norm of solutions for $n = 1, 2, 3, 4$. In particular, such obtained results express infinite time blowup property in the case when the 0-th moment of the initial velocity does not vanish. The idea to derive them is strongly inspired from an already developed technique \cite{6, 11, 7}.

1 Introduction

This is one more application to the plate and/or beam equation of the method recently applied in \cite{7} to the wave equation. A similar argument to these wave-like estimates to capture a singularity is introduced in \cite{6, 11}, and their related papers are published by the author’s collaborative works (see \cite{3} and the references therein).

Now, we consider the Cauchy problem of the plate equation:

$$
\begin{align*}
&u_{tt} + \Delta^2 u = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \\
&u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n.
\end{align*}
$$

Here, we assume, for the moment, $[u_0, u_1] \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

Concerning the existence of a unique energy solution to problem (1.1)-(1.2), by the standard semi-group theory one can find that the problem (1.1)-(1.2) has a unique weak solution

$$
\begin{align*}
u \in C((0, \infty); H^2(\mathbb{R}^n)) \cap C^1([0, \infty); L^2(\mathbb{R}^n))
\end{align*}
\]satisfying the energy conservation law such that

$$
E(t) = E(0), \quad t \geq 0,
$$

where the total energy $E(t)$ for the solution to problem (1.1)-(1.2) can be defined by

$$
E(t) := \frac{1}{2} \left( \|u_t(t, \cdot)\|^2 + \|\Delta u(t, \cdot)\|^2 \right).
$$

Here, $\|u\|$ denotes the usual $L^2$-norm of $u \in L^2(\mathbb{R}^n)$.

As is frequently appeared in several research papers on the plate and/or beam models, the assumption $n \geq 5$ is sometimes imposed to treat the problems in unbounded domains, and for this observation one can cite several typical papers such as \cite{1}, \cite{2}, \cite{12}, \cite{13}, \cite{15}, \cite{16} and \cite{14}, however, it seems that there are no any related papers to investigate actively a reason why $n \geq 5$ (see also \cite{18} for a topic on the $L^p$-regularity condition). The purpose of this paper is to give a partial answer on this kind of problem.

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Theorem 1.1
Let $n$.

The following two theorems give a hint about a question why

where $u$

$\|s\|_{small\ frequency\ region}$, so we are particular about dealing with the asymptotic behavior of the quantity $\|u(t,\cdot)\|$. By the way, it should be mentioned that the equation considered in [13] does not have any singularity near 0 frequency region, so a topic taken up in [13] seems to be a little different from ours.

Before going to introduce our Theorems, we set

$$I_{0,n} := \|u_0\| + \|u_0\|_{L^1(R^n)} + \|u_1\| + \|u_1\|_{L^1(R^n)}.$$ 

The following two theorems give a hint about a question why $n \geq 5$ in the plate equation.

**Theorem 1.1** Let $n = 1, 2, 3$. Let $[u_0, u_1] \in H^2(R^n) \times L^2(R^n)$. Then, the solution $u(t,x)$ to problem (1.1) satisfies the following properties under the additional regularity on the initial data:

$$[u_0, u_1] \in L^1(R^n) \times L^1(R^n) \Rightarrow \|u(t,\cdot)\| \leq C_{1} I_{0,n} t^{1-\frac{\gamma}{2}},$$

$$[u_0, u_1] \in L^1(R^n) \times L^{1,1}(R^n) \Rightarrow C_2 \left| \int_{R^n} u_1(x) dx \right| t^{1-\frac{\gamma}{2}} \leq \|u(t,\cdot)\|$$

for $t \gg 1$, where $C_j > 0 (j = 1, 2)$ are constants depending only on $n$.

Our next result is the case of $n = 4$.

**Theorem 1.2** Let $n = 4$. Let $[u_0, u_1] \in H^2(R^4) \times L^2(R^4)$. Then, the solution $u(t,x)$ to problem (1.1) satisfies the following properties under the additional regularity on the initial data:

$$[u_0, u_1] \in L^1(R^4) \times L^1(R^4) \Rightarrow \|u(t,\cdot)\| \leq C_{1} I_{0,4} \sqrt{\log t},$$

$$[u_0, u_1] \in L^1(R^4) \times L^{1,1}(R^4) \Rightarrow C_2 \left| \int_{R^4} u_1(x) dx \right| \sqrt{\log t} \leq \|u(t,\cdot)\|$$

for $t \gg 1$, where $C_j > 0 (j = 1, 2)$ are constants depending only on the space dimension $n = 4$.

**Remark 1.1** One of our advantages in the results is that we never use the compactness argument as is frequently developed in the wave equation case because of the non-Kowalewskian property of the equation (1.1). The method will be widely applicable to the other types of evolution equations. Indeed, it will be possible to generalize our results to the more general $\sigma$-evolution equations:

$$u_{tt} + (-\Delta)^\sigma u = 0,$$

where $((-\Delta)^\sigma f)(x) := \mathcal{F}_{\xi \rightarrow x}^{-1}(|\xi|^{2\sigma} \hat{f}(\xi))(x)$ and $\sigma > 0$. A threshold number $n^*$ on the dimension $n$ to divide whether the blow up phenomenon occurs or not can be defined by $n^* = 2\sigma$.

**Remark 1.2** In both Theorems, the so-called infinite time blowup property can be observed in the case when $n < 5$. One has to treat more delicately when we study decay estimates, asymptotic profiles, nonlinear problems and so on, of the plate equation considered in unbounded domains, and in particular, in $R^n$.

Contrary to infinite time blowup results above, in the case when $\int_{R^n} u_1(x) dx = 0$, one has the $L^2$-boundedness property (at least) for $n = 3, 4$.

**Theorem 1.3** Let $n = 3, 4$, and $[u_0, u_1] \in H^2(R^n) \times L^2(R^n)$. Assume further that $u_1 \in L^{1,\gamma}(R^n)$ with

$$n = 4 \Rightarrow \gamma \in (0, 1],$$

$$n = 3 \Rightarrow \gamma \in \left(\frac{1}{2}, 1\right].$$

If

$$\int_{R^n} u_1(x) dx = 0,$$

then the solution $u(t,x)$ to problem (1.1) satisfies

$$\|u(t,\cdot)\| \leq C(\|u_0\| + \|u_1\| + \|u_1\|_{1,\gamma}),$$

where $C > 0$ is a constant depending on the space dimension $n$ and $\gamma$. 

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In the case of $n = 1, 2$, if one assumes an additional vanishing condition of the 1th-order moment of the initial velocity $u_1$ one can state the following $L^2$-boundedness result.

**Theorem 1.4** Let $n = 1, 2$, and $[u_0, u_1] \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Assume further that $u_1 \in L^{1, \gamma}(\mathbb{R}^n)$ with

$$n = 2 \implies \gamma \in (1, 2],$$

$$n = 1 \implies \gamma \in \left(\frac{3}{2}, 2\right].$$

Under the following two conditions,

$$\int_{\mathbb{R}^n} u_1(x)dx = 0, \quad \int_{\mathbb{R}^n} x_j u_1(x)dx = 0 \quad (j = 1, 2, \cdots, n),$$

the solution $u(t, x)$ to problem (1.4) satisfies

$$\|u(t, \cdot)\| \leq C(\|u_0\| + \|u_1\| + \|u_1\|_{1, \gamma}),$$

where $C > 0$ is a constant depending on the space dimension $n$ and $\gamma$.

**Example.** Let $n = 1$, and let $g \in C^\infty_0(\mathbb{R})$ be an odd function, and choose $u_1(x) := g'(x) \in C^\infty_0(\mathbb{R})$. Then, one has

$$\int_{\mathbb{R}} u_1(x)dx = 0, \quad \int_{\mathbb{R}} x u_1(x)dx = \int_{\mathbb{R}} x g'(x)dx = -\int_{\mathbb{R}} g(x)dx = 0.$$ 

This expresses an example of the initial velocity satisfying (1.4) for $n = 1$.

Similarly, in the case when $n = 2$, one can construct an example satisfying (1.4) by

$$u_1(x_1, x_2) := g'(x_1)h'(x_2),$$

where $g, h \in C^\infty_0(\mathbb{R})$.

As a counter part of Theorem 1.4 one can get the infinite time blowup result in the case when $n = 1, 2$.

For simplicity, we mention only the case of $n = 2$ in order to compare it with Theorem 1.4.

**Theorem 1.5** (1) Let $n = 1$, and $[u_0, u_1] \in H^2(\mathbb{R}) \times L^2(\mathbb{R})$. Assume further that $u_1 \in L^{1, 2}(\mathbb{R})$ satisfies

$$\left| \int_{\mathbb{R}} u_1(x)dx \right| + \left| \int_{\mathbb{R}} x u_1(x)dx \right| > 0. \quad (1.5)$$

Then the solution $u(t, x)$ to problem (1.1) - (1.2) satisfies

$$C \left( \left| \int_{\mathbb{R}} x u_1(x)dx \right|^\frac{4}{3} + \left| \int_{\mathbb{R}} u_1(x)dx \right|^\frac{4}{3} \right) \leq \|u(t, \cdot)\|, \quad t \gg 1,$$

where $C > 0$ is a constant.

(2) Let $n = 2$, and $[u_0, u_1] \in H^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$. Assume further that $u_1 \in L^{1, 2}(\mathbb{R}^2)$ satisfies

$$\left| \int_{\mathbb{R}^2} u_1(x)dx \right| + \left| \int_{\mathbb{R}^2} x_1 u_1(x)dx \right| + \left| \int_{\mathbb{R}^2} x_2 u_1(x)dx \right| > 0. \quad (1.6)$$

Then the solution $u(t, x)$ to problem (1.1) - (1.2) satisfies

$$C \left( \left| \int_{\mathbb{R}^2} x_1 u_1(x)dx \right| + \left| \int_{\mathbb{R}^2} x_2 u_1(x)dx \right| \right)^\sqrt{\log t} + C \left| \int_{\mathbb{R}^2} u_1(x)dx \right| \sqrt{t} \leq \|u(t, \cdot)\|, \quad t \gg 1,$$

where $C > 0$ is a constant.

**Remark 1.3** As a result of Theorem 1.5, (for example) in case of $n = 1$, even if $\int_{\mathbb{R}} u_1(x)dx = 0$, if $\int_{\mathbb{R}} x u_1(x)dx \neq 0$, then one has the infinite time blowup result with its rate $t^{1/4}$, however one can know nothing about the optimality of the blowup rate $t^{1/4}$. A corresponding result for $n = 2$ is true similarly. Of course, one should investigate (optimal) upper bound estimates of Theorem 1.5, however, this study will be left to the reader’s interest.
Notation. Throughout this paper, \( \| \cdot \|_q \) stands for the usual \( L^q(\mathbb{R}^n) \)-norm. For simplicity of notation, in particular, we use \( \| \cdot \| \) instead of \( \| \cdot \|_2 \). We also introduce the following weighted functional spaces.

\[
L^{1,\gamma}(\mathbb{R}^n) := \left\{ f \in L^1(\mathbb{R}^n) \mid \| f \|_{1,\gamma} := \int_{\mathbb{R}^n} (1 + |x|^\gamma)|f(x)|dx < +\infty \right\}.
\]

One denotes the Fourier transform \( \mathcal{F}_{x \to \xi}(f)(\xi) \) of \( f(x) \) by

\[
\mathcal{F}_{x \to \xi}(f)(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x)dx, \quad \xi \in \mathbb{R}^n,
\]

as usual with \( i := \sqrt{-1} \), and \( \mathcal{F}_{\xi \to x}^{-1} \) expresses its inverse Fourier transform. Finally, we denote the surface area of the \( n \)-dimensional unit ball by \( \omega_n := \int_{|\xi| = 1} d\omega, \) and we set \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \).

The paper is organized as follows. In Section 2 we try to get \( L^2 \)-bound of solutions via known method due to [9]. In Section 3 we derive the lower bound estimates of the \( L^2 \)-norm of solutions, and in Section 4 we obtain the upper bound estimates of the \( L^2 \)-norm of solutions, and by combining the results obtained in Sections 3 and 4 one can prove Theorems 1.1 and 1.2. Finally, we prove Theorems 1.3, 1.4 and 1.5 in Section 5.

2 \( L^2 \)-upper bound estimates: higher dimensional case

In this section, we introduce a device to derive \( L^2 \)-estimates of solutions to problem (1.1)-(1.2) by relying on the modified Morawetz method developed newly in [9]. For this we prepare the following Hardy type inequality (see Davies-Hinz [4]). The basic concept in [9] is that the Hardy type inequality implies the \( L^2 \)-upper bound estimate of the solution in several types of linear evolution equations.

Lemma 2.1 Let \( n \geq 5 \). Then there exists a constant \( C^* > 0 \) such that

\[
\int_{\mathbb{R}^n} \frac{|w(x)|^2}{|x|^4}dx \leq C^* \int_{\mathbb{R}^n} |\Delta w(x)|^2 dx
\]

for all \( w \in H^2(\mathbb{R}^n) \).

As in the idea [9], for the solution \( u(t, x) \) to problem (1.1)-(1.2) we set

\[
v(t, x) := \int_0^t u(s, x)ds.
\]

Then, the function \( v(t, x) \) satisfies the following equation and initial data.

\[
\begin{align*}
v_{tt} + \Delta^2 v &= u_1, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \\
v(0, x) &= 0, \quad v_t(0, x) = u_0(x), \quad x \in \mathbb{R}^n.
\end{align*}
\]

(2.1) \hspace{1cm} (2.2)

Multiplying the both sides of (2.1)-(2.2) by \( v_1 \), and integrating it over \( [0, t] \times \mathbb{R}^n \) one can get the following energy equality such that

\[
\frac{1}{2} \|v(t, \cdot)\|^2 + \frac{1}{2} \|\Delta v(t, \cdot)\|^2 = \frac{1}{2} \|v_0\|^2 + \int_{\mathbb{R}^n} u_1(x)v(t, x)dx.
\]

(2.3)

Now we are in a position to use Lemma 2.1 to proceed the following computations based on the Schwarz inequality:

\[
\left| \int_{\mathbb{R}^n} u_1(x)v(t, x)dx \right| \leq \int_{\mathbb{R}^n} |u_1(x)||v(t, x)|dx = \int_{\mathbb{R}^n} \left( |x|^4 |u_1(x)| \right) \left( \frac{|v(t, x)|}{|x|^2} \right) dx
\]

\[
\leq \left( \int_{\mathbb{R}^n} |x|^4 |u_1(x)|^2 dx \right)^{1/2} \left( \int_{\mathbb{R}^n} \frac{|v(t, x)|^2}{|x|^4} dx \right)^{1/2}.
\]
\[ \leq C_\varepsilon \int_{\mathbb{R}^n} |x|^4 |u_1(x)|^2 \, dx + \varepsilon \int_{\mathbb{R}^n} \frac{v(t,x)^2}{|x|^4} \, dx \]

\[ \leq C_\varepsilon \int_{\mathbb{R}^n} |x|^4 |u_1(x)|^2 \, dx + \varepsilon C^* \int_{\mathbb{R}^n} |\Delta v(t,x)|^2 \, dx \]  

(2.4)

with some constant \(C_\varepsilon > 0\), which depends on each \(\varepsilon > 0\). Thus, from (2.3) and (2.4) it follows that

\[ \frac{1}{2} \|v_t(t,\cdot)\|^2 + \frac{1}{2} \|\Delta v(t,\cdot)\|^2 \leq \frac{1}{2} \|u_0\|^2 + C_\varepsilon \int_{\mathbb{R}^n} |x|^4 |u_1(x)|^2 \, dx + \varepsilon C^* \int_{\mathbb{R}^n} |\Delta v(t,x)|^2 \, dx, \]

which implies

\[ \frac{1}{2} \|v_t(t,\cdot)\|^2 + \left( \frac{1}{2} - \varepsilon C^* \right) \|\Delta v(t,\cdot)\|^2 \leq \frac{1}{2} \|u_0\|^2 + C_\varepsilon \int_{\mathbb{R}^n} |x|^4 |u_1(x)|^2 \, dx \]

provided that

\[ \int_{\mathbb{R}^n} |x|^4 |u_1(x)|^2 \, dx < + \infty. \]

By choosing \(\varepsilon > 0\) small enough one can arrive at the crucial \(L^2\)-estimate by means of the multiplier method only because of \(v_t(t,x) = u(t,x)\). The result below compensates the unknown information on \(n \geq 5\).

**Proposition 2.1** Let \(n \geq 5\), and \([u_0,u_1] \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\). Then, there exists a constant \(C > 0\) such that the solution \(u(t,x)\) to problem (1.1)-(1.2) satisfies

\[ \|u(t,\cdot)\|^2 \leq C \left( \|u_0\|^2 + \int_{\mathbb{R}^n} |x|^4 |u_1(x)|^2 \, dx \right) \quad t \geq 0, \]

provided additionally that

\[ \int_{\mathbb{R}^n} |x|^4 |u_1(x)|^2 \, dx < + \infty. \]  

(2.5)

** Remark 2.1** An essential part of this argument above has already been developed previously in [2] for the plate equation with damping and the rotational inertia term by employing the idea due to [9]. This part is just a trial to observe the case of \(n \geq 5\).

**Remark 2.2** Note that in the case of \(4 \leq n\), the condition [25] together with \(u_1 \in L^2(\mathbb{R}^n)\) does not necessarily imply \(u_1 \in L^1(\mathbb{R}^n)\).

So, one can apply the method due to [9] to get the \(L^2\)-bound in the case when \(n \geq 5\), however, this method decisively depends on the existence of the Hardy type inequality. So, the results for \(n = 1, 2, 3, 4\) of Theorems 1.1 and 1.2 are quite important since those are independent of any such inequalities. We have to derive \(L^2\)-bound estimates from the equation itself.

One can proceed the similar argument by using the following inequality in place of Lemma 2.1 (see [2] for its detail).

**Lemma 2.2** Let \(n \geq 5\). Then there exists a constant \(C^* > 0\) such that

\[ \|w\|_{L^{\frac{2n}{n-2}}} \leq C^* \|\Delta w\| \]

for all \(w \in H^2(\mathbb{R}^n)\).

### 3 \(L^2\)-lower bound estimates

In this section, we derive the lower bound estimates of the \(t\)-function \(\|u(t,\cdot)\|\) based on the Plancherel Theorem combined with low frequency estimates (cf. [9]). For the proof, it suffices to assume \([u_0,u_1] \in C^0_0(\mathbb{R}^n) \times C^0_0(\mathbb{R}^n)\) to have sufficient regularity of the solution \(u(t,x)\) because the density argument can
be applied in the final estimates. We first set
\[ L := \sup_{\theta \neq 0} \left| \frac{\sin \theta}{\theta} \right| < +\infty. \] (3.1)

On the other hand, because of
\[ \lim_{\theta \to +0} \frac{\sin \theta}{\theta} = 1, \]
there exists a real number \( \delta_0 \in (0, 1) \) such that
\[ \left| \frac{\sin \theta}{\theta} \right| \geq \frac{1}{2} \] (3.2)
for all \( \theta \in (0, \delta_0^2] \). The following fundamental inequality will be useful, too:
\[ |a + b|^2 \geq \frac{1}{2} |a|^2 - |b|^2 \] (3.3)
for all \( a, b \in \mathbb{C} \).

In order to get the lower bound estimate for the quantity \( \|w(t, \cdot)\| \), it suffices to treat \( \|w(t, \cdot)\| \) with \( w(t, \xi) := F_{x ightarrow \xi}(u(t, \cdot))(\xi) = \tilde{u}(t, \xi) \) because of the Plancherel Theorem.

Now we decompose the quantity \( \|w(t, \cdot)\| \) as follows: for each \( n \geq 1 \)
\[ \|w(t, \cdot)\|^2 = \left( \int_{|\xi| \leq \frac{\delta_0}{\sqrt{t}}} + \int_{|\xi| \geq \frac{\delta_0}{\sqrt{t}}} \right) |w(t, \xi)|^2 d\xi = I_{low}^{(n)}(t) + I_{high}^{(n)}(t). \] (3.4)

Here we have just chosen \( t > 0 \) large enough such that
\[ \frac{\delta_0}{\sqrt{t}} \leq 1. \]

By the way, in the Fourier space \( \mathbb{R}^n_\xi \) the problem (1.1)-(1.2) and its solution \( u(t, x) \) can be transformed into the following ODE with parameter \( \xi \in \mathbb{R}^n_\xi \):
\[ w_{tt} + |\xi|^4 w = 0, \quad t > 0, \quad \xi \in \mathbb{R}^n_\xi, \]
\[ w(0, \xi) = w_0(\xi), \quad w_t(0, \xi) = w_1(\xi), \quad \xi \in \mathbb{R}^n_\xi, \] (3.5) (3.6)
where \( w_0(\xi) := \tilde{u}_0(\xi) \) and \( w_1(\xi) := \tilde{u}_1(\xi) \). In addition, one can solve the problem (3.5) as follows:
\[ w(t, \xi) = \frac{\sin(t|\xi|^2)}{|\xi|^2} w_1(\xi) + \cos(t|\xi|^2)w_0(\xi). \] (3.7)

Now, let us derive the lower bound estimates for \( I_{low}^{(n)}(t) \) because one has \( \|w(t, \cdot)\|^2 \geq I_{low}^{(n)}(t) \). One relies on a device coming from an idea in [6]. Indeed, it follows from (3.7) and (3.3) that
\[ I_{low}^{(n)}(t) = \int_{|\xi| \leq \frac{\delta_0}{\sqrt{t}}} \left| \frac{\sin(t|\xi|^2)}{|\xi|^2} w_1(\xi) + \cos(t|\xi|^2)w_0(\xi) \right|^2 d\xi \geq \frac{1}{2} \left( \int_{|\xi| \leq \frac{\delta_0}{\sqrt{t}}} \frac{\sin^2(t|\xi|^2)}{|\xi|^2} |w_1(\xi)|^2 d\xi - \int_{|\xi| \leq \frac{\delta_0}{\sqrt{t}}} \cos^2(t|\xi|^2) |w_0(\xi)|^2 d\xi \right) =: \frac{1}{2} J_1(t) - J_2(t). \] (3.8)

Let us first estimate \( J_1(t) \) by using the decomposition of the initial data \( w_1(\xi) \) in the Fourier space:
\[ w_1(\xi) = P + (A(\xi) - iB(\xi)), \quad \xi \in \mathbb{R}^n_\xi, \]
where
\[ P := \int_{\mathbb{R}^n} u_1(x)dx, \]
\[ A(\xi) := \int_{\mathbb{R}^n} (\cos(x\xi) - 1)u_1(x)dx, \quad B(\xi) := \int_{\mathbb{R}^n} \sin(x\xi)u_1(x)dx. \]

It is known (see [5]) that there is a constant \( M > 0 \) such that
\[ |A(\xi) - iB(\xi)| \leq M|\xi||u_1||_{L^{1,1}}, \quad \xi \in \mathbb{R}_\xi^n, \tag{3.9} \]
in case of \( u_1 \in L^{1,1}(\mathbb{R}^n) \). Then, from (3.9) we see that
\[
J_1(t) = \int_{|\xi| \leq \frac{\delta_0}{\sqrt{t}}} \frac{\sin^2(t|\xi|^2)}{|\xi|^4} |w_1(\xi)|^2 d\xi \\
\geq \frac{P^2}{2} \int_{|\xi| \leq \frac{\delta_0}{\sqrt{2}}} \frac{\sin^2(t|\xi|^2)}{|\xi|^4} d\xi - \int_{|\xi| \leq \frac{\delta_0}{\sqrt{2}}} |A(\xi) - iB(\xi)|^2 \frac{\sin^2(t|\xi|^2)}{|\xi|^4} d\xi \\
= \frac{P^2}{2} K_1(t) - K_2(t). \tag{3.10} \]

\( K_2(t) \) can be estimated from above by using and [5.1] and [5.2]:
\[
K_2(t) \leq M^2 \|u_1\|_{L^{1,1}}^2 t \int_{0}^{\frac{\delta_0}{\sqrt{2}}} t^2 \left( \frac{\sin(t|\xi|^2)}{t|\xi|^2} \right)^2 |\xi|^2 d\xi \\
\leq M^2 \|u_1\|_{L^{1,1}}^2 t \int_{0}^{\frac{\delta_0}{\sqrt{2}}} t^2 |\xi|^2 d\xi = M^2 \|u_1\|_{L^{1,1}}^2 t \int_{0}^{\frac{\delta_0}{\sqrt{2}}} \frac{t^2}{n+2} r^{-n+2} dr \\
= \frac{t^2}{n+2} \frac{M^2 w_1 \delta_0^{n+2} \|u_1\|_{L^{1,1}}}{t^{1-\frac{n}{2}}, \quad t \gg 1. \tag{3.11} \]

On the other hand, one can obtain the lower bound estimate for \( K_1(t) \) because of (3.2):
\[
K_1(t) = \frac{t^2}{4n} \int_{|\xi| \leq \frac{\delta_0}{\sqrt{2}}} \frac{\sin^2(t|\xi|^2)}{|\xi|^2 t^2} d\xi \\
= \frac{\omega_n \delta_0^n}{4n} t^{2-\frac{n}{2}}, \quad t \gg 1. \tag{3.12} \]

Therefore from (3.10), (3.11) and (3.12) one can get the estimate from below for \( J_1(t) \):
\[
J_1(t) \geq \frac{P^2 \omega_n \delta_0^n}{4n} t^{2-\frac{n}{2}} - \frac{L^2}{n+2} M^2 \omega_n \delta_0^{n+2} \|u_1\|_{L^{1,1}}^2 t^{1-\frac{n}{2}}, \quad t \gg 1. \tag{3.13} \]

Since the upper bound estimate of \( J_2(t) \) can be easily obtained as follows:
\[
J_2(t) \leq \int_{|\xi| \leq \frac{\delta_0}{\sqrt{2}}} |w_0(\xi)|^2 d\xi \leq \|w_0\|_{L^{1,1}}^2 \omega_n \int_0^{\frac{\delta_0}{\sqrt{2}}} r^{-n-1} dr = \frac{\omega_n \delta_0^n}{n} \|w_0\|_{L^{1,1}}^2 t^{-\frac{n}{2}},
\]
because of (3.13), (3.4) and (3.5) one has just arrived at the following lower bound estimate for \( \|w(t, \cdot)\| \):
\[
\|w(t, \cdot)\|^2 \geq I_{w_0}(t) \geq \frac{P^2 \omega_n \delta_0^n}{4n} t^{2-\frac{n}{2}} - \frac{M^2 \omega_n \delta_0^{n+2} \|u_1\|_{L^{1,1}}^2 t^{1-\frac{n}{2}} - \frac{\omega_n \delta_0^{n+2}}{n} \|u_0\|_{L^{1,1}}^2 t^{-\frac{n}{2}}, \quad t \gg 1. \tag{3.14} \]

Therefore, there is a positive real number \( t_0 \) such that
\[
\|w(t, \cdot)\|^2 \geq \frac{P^2 \omega_n \delta_0^n}{32n} t^{2-\frac{n}{2}}, \tag{3.15} \]
for all \( t \geq t_0 \) and all \( n \in \mathbb{N} \). It should be mentioned that \( t_0 > 0 \) depends on \( n \) and the quantities \( \|u_1\|_{L^{1,1}} \) and \( \|u_0\|_{L^{1,1}} \). From (3.15) with \( n \geq 1 \) one has the following lemma.
Lemma 3.1 Let $n \geq 1$, and $[u_0, u_1] \in L^1(\mathbb{R}^n) \times L^{1,1}(\mathbb{R}^n)$. Then, it holds that

$$\|w(t, \cdot)\|^2 \geq CP^2 t^{2-\frac{2\pi}{\sqrt{n}}}, \quad t \gg 1.$$ 

Note that the results for $n = 4$ in Lemma 3.1 seems to be weak because it shows just the non-decay property. We expect that $t$-blowup as in wave equation case for $n = 2$ (17). For this purpose we improve the statement for $n = 4$ by relying on a similar argument as in (7) with a trick function $e^{-r^2}$. Indeed, it follows from (3.3) and a similar argument to the previous one that

$$\|w(t, \cdot)\|^2 \geq \frac{1}{2} \int_{\mathbb{R}^4} \frac{\sin^2(tr^2)}{r^4} |w_1(\xi)|^2 d\xi - \int_{\mathbb{R}^4} \frac{\cos^2(tr^2)}{r^4} |w_0(\xi)|^2 d\xi$$

$$\geq \frac{1}{2} \int_{\mathbb{R}^4} e^{-r^2} \frac{\sin^2(tr^2)}{r^4} |P + (A(\xi) - iB(\xi))|^2 d\xi - \|u_0\|^2$$

$$\geq \frac{1}{4} P^2 \int_{\mathbb{R}^4} e^{-r^2} \frac{\sin^2(tr^2)}{r^4} d\xi - \frac{M^2}{2} \|u_1\|_{L^{1,1}}^2 \int_{\mathbb{R}^4} e^{-r^2} \frac{1}{r^2} d\xi - \|u_0\|^2$$

$$=: \frac{1}{4} P^2 U(t) - \frac{1}{4} M^2 \|u_1\|_{L^{1,1}}^2 \omega_4 - \|u_0\|^2. \quad (3.16)$$

Now, we apply an useful idea coming from (11). For this purpose, we set

$$\theta_j := \sqrt{(\frac{1}{4} + j) \frac{\pi}{t}}, \quad \tau_j := \sqrt{(\frac{3}{4} + j) \frac{\pi}{t}} \quad (j = 0, 1, 2, \cdots),$$

and choose $t > 1$ large enough such that $\theta_0 = \sqrt{\frac{\pi}{4t}} < 1$. Then, because of the fact that

$$|\sin(tr^2)| \geq \frac{1}{\sqrt{2}}$$

for $r \in [\theta_j, \tau_j]$ and $j = 0, 1, 2, \cdots$, one can proceed to estimate:

$$U(t) = \int_{\mathbb{R}^4} e^{-r^2} \frac{\sin^2(tr^2)}{r^4} d\xi \geq \frac{1}{2} \sum_{j=0}^\infty \int_{\theta_j \leq r \leq \tau_j} e^{-r^2} \frac{1}{r^4} d\xi = \frac{\omega_4}{2} \left( \sum_{j=0}^\infty \frac{\int_{\theta_j}^{\tau_j} e^{-r^2} dr}{r^4} \right) \quad (3.17)$$

$$\geq \frac{\omega_4}{2} \left( \frac{1}{2} \int_{\theta_0}^{\infty} e^{-r^2} r^2 dr \right) \geq \frac{\omega_4}{4} \int_{\theta_0}^{1} e^{-r^2} r^2 dr \quad (3.18)$$

$$\geq \omega_4 e^{-1} \frac{1}{4} \int_{\theta_0}^{1} \frac{1}{r^4} dr = \omega_4 e^{-1} \frac{1}{8} (\log t + \log 4 - \log \pi), \quad (3.19)$$

where in the inequality from (3.17) to (3.18) one has just used the monotone decreasing property of the function $r \mapsto e^{-r^2}$, and the fact that the length

$$\tau_j - \theta_j = \frac{\pi}{2t} \frac{1}{\sqrt{(\frac{1}{4} + j) \frac{\pi}{t}} + \sqrt{(\frac{3}{4} + j) \frac{\pi}{t}}}$$

is decreasing to 0 as $j \to \infty$ for each fixed $t > 1$. Therefore, by (3.15) and (3.16) one has the following estimates for $n = 4$ for large $t > 1$.

**Lemma 3.2** Let $n = 4$, and $[u_0, u_1] \in L^1(\mathbb{R}^n) \times L^{1,1}(\mathbb{R}^n)$. Then, it holds that

$$\|w(t, \cdot)\|^2 \geq CP^2 \log t, \quad t \gg 1.$$ 

Proofs of the lower bound estimates of Theorems 1.1 and 1.2 are direct consequence of Lemmas 3.1 and 3.2.
4 $L^2$-upper bound estimates of the solution

In this section, let us derive upper bound estimates of $\|u(t, \cdot)\|$ as $t \to \infty$ by treating the function $w(t, \xi)$ in both high and low frequency region.

As in Section 3, one again assumes $[u_0, u_1] \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$ to proceed the proof.

From (3.4) one first derives the upper bound estimate for $I_{low}^{(n)}(t)$ for all $n \geq 1$. Indeed, by (3.8) one has
\[
I_{low}^{(n)}(t) \leq \int_{|\xi| \leq \frac{2}{n}} \frac{\sin^2(t|\xi|^2)}{|\xi|^4}|w_1(\xi) + \cos(t|\xi|^2)w_0(\xi)|^2 d\xi
\leq 2 \int_{|\xi| \leq \frac{2}{n}} \frac{\sin^2(t|\xi|^2)}{|\xi|^4}|w_1(\xi)|^2 d\xi + 2 \int_{|\xi| \leq \frac{2}{n}} \cos^2(t|\xi|^2)|w_0(\xi)|^2 d\xi
= 2L_1(t) + 2L_2(t).
\]

It is easy to obtain the estimate for $L_2(t)$ as in the estimate for $J_2(t)$ in Section 2:
\[
L_2(t) \leq \int_{|\xi| \leq \frac{2}{n}} |w_0(\xi)|^2 d\xi \leq \|u_0\|^2_{L^1} \omega_n \int_0^{\frac{2}{n}} r^{n-1} dr = \frac{\omega_n \delta_0^n}{n} \|u_0\|_{L^1}^2 t^{-\frac{n}{2}}, \quad t \gg 1.
\]

where one has just used the fat that
\[
\int_{|\xi| \leq \frac{2}{n}} d\xi = \frac{\omega_n \delta_0^n}{n} t^{-\frac{n}{2}}, \quad t \gg 1.
\]

Let us estimate $L_1(t)$. Indeed, from (3.1) and (4.3) one has
\[
L_1(t) = \int_{|\xi| \leq \frac{2}{n}} \frac{\sin^2(t|\xi|^2)}{|\xi|^4}|w_1(\xi)|^2 d\xi \leq \|u_1\|^2_{L^1} t^2 \int_{|\xi| \leq \frac{2}{n}} \frac{\sin^2(t|\xi|^2)}{|(t|\xi|^2)|^2} d\xi
\leq \|u_1\|^2_{L^1} L^2 t^2 \frac{\omega_n \delta_0^n}{n} t^{-\frac{n}{2}} = \|u_1\|^2_{L^1} L^2 \frac{\omega_n \delta_0^n}{n} t^{-\frac{n}{2}}, \quad t \gg 1.
\]

Thus, from (3.1), (4.2) and (4.4) one can obtain the low-frequency estimate
\[
I_{low}^{(n)}(t) \leq C \left( \|u_0\|^2_{L^1} t^{-\frac{n}{2}} + \|u_1\|^2_{L^1} t^{-\frac{n}{2}} \right), \quad t \gg 1,
\]

with some constant $C > 0$.

Next, let us treat $I_{high}^{(n)}(t)$ to get the upper bound estimate. Indeed, similar to the above estimate one stars with the following inequalities.
\[
I_{high}^{(n)}(t) \leq \int_{|\xi| \geq \frac{2}{n}} \frac{\sin(t|\xi|^2)}{|\xi|^2}|w_1(\xi) + \cos(t|\xi|^2)w_0(\xi)|^2 d\xi
\leq 2 \int_{|\xi| \geq \frac{2}{n}} \frac{\sin^2(t|\xi|^2)}{|\xi|^4}|w_1(\xi)|^2 d\xi + 2 \int_{|\xi| \geq \frac{2}{n}} \cos^2(t|\xi|^2)|w_0(\xi)|^2 d\xi
= 2N_1^{(n)}(t) + 2N_2^{(n)}(t).
\]

It is easy to treat $N_2^{(n)}(t)$ as follows. This can be derived for all $n \geq 1$:
\[
N_2^{(n)}(t) \leq \int_{|\xi| \geq \frac{2}{n}} |w_0(\xi)|^2 d\xi \leq \|u_0\|^2.
\]
Let us estimate $N_1^{(n)}(t)$. First, for all $n \geq 1$ one has
\[
N_1^{(n)}(t) = \int_{|\xi| \geq \frac{40}{(\log t)^{1/2}}} \frac{\sin^2(t|\xi|^2)}{|\xi|^4} |w_1(\xi)|^2 d\xi + \int_{\frac{40}{(\log t)^{1/2}}}^{40} \frac{\sin^2(t|\xi|^2)}{|\xi|^4} |w_1(\xi)|^2 d\xi
\]
\[
=: R_1^{(n)}(t) + R_2^{(n)}(t). \quad (4.8)
\]
To begin with, we treat $R_1^{(n)}(t)$ to get the estimate
\[
R_1^{(n)}(t) \leq \frac{t^{1/2}}{2^n} \int_{|\xi| \geq \frac{40}{(\log t)^{1/2}}} \sin^2(t|\xi|)|w_1(\xi)|^2 d\xi
\]
\[
\leq \frac{t^{1/2}}{2^n} \int_{|\xi| \geq \frac{40}{(\log t)^{1/2}}} |w_1(\xi)|^2 d\xi \leq \frac{\sqrt{t}}{2^n} \|u_1\|^2, \quad t \gg 1. \quad (4.9)
\]
On the other hand, in the case of $n \neq 4$, $R_2^{(n)}(t)$ can be estimated as follows:
\[
R_2^{(n)}(t) \leq \|u_1\|^2 \int_{\frac{40}{(\log t)^{1/2}}}^{40} \frac{1}{|\xi|^2} d\xi
\]
\[
= \|u_1\|^2 \omega_n \int_{\frac{40}{(\log t)^{1/2}}}^{40} r^{-n-5} dr = \frac{\omega_n}{(4 - n)2^n} \|u_1\|^2 t^{2 - \frac{2}{n} - t^{\frac{1}{2} - \frac{1}{4}}}, \quad t \gg 1. \quad (4.10)
\]
Thus, from (4.6), (4.7), (4.8), (4.9) and (4.10) one has the infinite time blowup estimate for $I_{\text{high}}^{(n)}(t)$ in the case of $n = 1, 2, 3$:
\[
I_{\text{high}}^{(n)}(t) = C \left( \|u_0\|^2 + \|u_1\|^2 t^{2 - \frac{2}{n} - t^{\frac{1}{2} - \frac{1}{4}}} \right), \quad t \gg 1.
\]
\[
\leq C \left( \|u_0\|^2 + \|u_1\|^2 + \|u_1\|^2 t^{2 - \frac{2}{n}} \right), \quad t \gg 1. \quad (4.11)
\]
Next, let us give sharp estimates for $N_1^{(4)}(t)$ in the case of $n = 4$ by using a more delicate computation. For this purpose, by choosing $t > 1$ sufficiently large to get the relation $1 \leq \log t \leq t \leq t^2$ one has a decomposition of the integrand:
\[
N_1^{(4)}(t) = \int_{|\xi| \geq \frac{40}{(\log t)^{1/2}}} \frac{\sin^2(t|\xi|^2)}{|\xi|^4} |w_1(\xi)|^2 d\xi
\]
\[
+ \int_{\frac{40}{(\log t)^{1/2}}}^{40} \frac{\sin^2(t|\xi|^2)}{|\xi|^4} |w_1(\xi)|^2 d\xi + \int_{\frac{40}{(\log t)^{1/2}}}^{40} \frac{\sin^2(t|\xi|^2)}{|\xi|^4} |w_1(\xi)|^2 d\xi
\]
\[
=: S_1(t) + S_2(t) + S_3(t). \quad (4.12)
\]
Let us estimate them in order.
First, one has
\[
S_1(t) \leq \frac{\log t}{2^n} \int_{|\xi| \geq \frac{40}{(\log t)^{1/2}}} |w_1(\xi)|^2 d\xi
\]
\[
\leq \frac{\log t}{2^n} \|u_1\|^2, \quad t \gg 1. \quad (4.13)
\]
For $S_2(t)$ we see that
\[
S_2(t) \leq \|u_1\|_{L^1}^2 \omega_4 \int_{\frac{40}{(\log t)^{1/2}}}^{40} \frac{1}{r} dr
\]
\[
\leq \omega_4 \|u_1\|_{L^2}^2 t^{4-1} (\log t - \log(\log t)), \quad t \gg 1. \quad (4.14)
\]
Finally, let us treat \( S_3(t) \) to obtain the following estimate:

\[
S_3(t) \leq \|u_1\|_{L^1}^2 \int_{\frac{n_0}{r_1} \leq |\xi| \leq \frac{n_0}{r_1 t}} \frac{1}{r_1^4} d\xi
\]

\[
\leq \omega_n \|u_1\|_{L^1}^2 t^{-4 \log t}, \quad t \gg 1.
\]  (4.15)

Thus, it follows from (4.12), (4.13), (4.14) and (4.15) that for large \( t \gg 1 \)

\[
N_1^{(4)}(t) \leq C \left( \|u_1\|_{L^1}^2 \log t + \|u_1\|^2 \log t + \|u_1\|_{L^1}^2 \log(t - \log(t)) \right), \quad t \gg 1
\]  (4.16)

with some constant \( C > 0 \). Therefore, by combining (4.6), (4.7) and (4.16) one can get the high-frequency estimate for \( n = 4 \):

\[
I_{\text{high}}^{(4)}(t) \leq C \left( \|u_0\|^2 + \|u_1\|^2 + \|u_1\|_{L^1}^2 \right) \log t, \quad t \gg 1.
\]  (4.17)

Finally, it follows from (3.4), (3.5), (4.11) and (4.17) one can get the crucial upper bound estimates for each \( n = 1, 2, 3, 4 \).

**Lemma 4.1** Let \( n = 1, 2, 3, \) and \([u_0, u_1] \in (L^1(R^n) \cap L^2(R^n)) \times (L^1(R^n) \cap L^2(R^n)) \). Then, it holds that

\[
\|w(t, \cdot)\|^2 \leq C(\|u_0\|^2 + \|u_1\|^2 + \|u_1\|_{L^1}^2) + \|u_1\|_{L^1}^2, \quad t \gg 1.
\]

**Lemma 4.2** Let \( n = 4 \), and \([u_0, u_1] \in (L^1(R^n) \cap L^2(R^n)) \times (L^1(R^n) \cap L^2(R^n)) \). Then, it holds that

\[
\|w(t, \cdot)\|^2 \leq C(\|u_0\|^2 + \|u_1\|^2 + \|u_1\|_{L^1}^2) \log t, \quad t \gg 1.
\]

Finally, proofs of the upper bound estimates part of Theorems 1.1 and 1.2 are direct consequence of Lemmas 4.1 and 4.2 and the Plancherel Theorem.

In connection with these estimates above, one can show the \( L^2 \)-upper bound estimate of the solution itself independently from the Hardy type inequality (see Proposition 2.1 and Remark 2.2).

**Proposition 4.1** Let \( n \geq 5 \), and \([u_0, u_1] \in H^2(R^n) \times (L^1(R^n) \cap L^1(R^n)) \). Then, there exists a constant \( C > 0 \) such that the solution \( u(t, x) \) to problem (1.1)-(1.2) satisfies

\[
\|u(t, \cdot)\| \leq C \left( \|u_0\| + \|u_1\| + \|u_1\|_{L^1} \right) \quad (t \geq 0).
\]

**Remark 4.1** A similar \( L^2 \)-upper bound estimates can be derived to the solution itself of the (free) wave equation in the Euclidean space \( R^n \) in the case of \( n \geq 3 \), however, the method developed in Proposition 2.1 is widely applicable to the exterior problem and variable coefficient cases.

**Proof of Proposition 4.1** It suffices to assume the initial data \( u_j \) \((j = 0, 1) \) belongs to \( C_0^\infty(R^n) \). Then, one can proceed to estimate as follows:

\[
C_n \|u(t, \cdot)\|^2 = \|w(t, \cdot)\|^2 = \int_{R^n} \left| \frac{\sin(t|\xi|^2)}{|\xi|^2} w_1(\xi) + \cos(t|\xi|^2) w_0(\xi) \right|^2 d\xi
\]

\[
\leq 2 \int_{R^n} \frac{\sin^2(t|\xi|^2)}{|\xi|^4} |w_1(\xi)|^2 d\xi + 2 \int_{R^n} \frac{\cos^2(t|\xi|^2) |w_0(\xi)|^2}{|\xi|^2} d\xi
\]

\[
\leq 2 \int_{|\xi| \leq 1} \frac{\sin^2(t|\xi|^2)}{|\xi|^4} |w_1(\xi)|^2 d\xi + 2 \int_{|\xi| \geq 1} \frac{\sin^2(t|\xi|^2)}{|\xi|^4} |w_1(\xi)|^2 d\xi + \|u_0\|^2
\]

\[
= 2T_1(t) + 2T_2(t) + \|u_0\|^2.
\]  (4.18)

Now, about \( T_1(t) \) one can get

\[
T_1(t) \leq \|u_1\|_{L^1}^2 \int_{|\xi| \leq 1} \frac{1}{|\xi|^4} d\xi \leq \omega_n \|u_1\|_{L^1}^2 \int_0^1 r^{n-5} dr
\]

\[
= \frac{\omega_n}{n-4} \|u_1\|_{L^1}^2, \quad (t \geq 0).
\]  (4.19)
For the estimate of $T_2(t)$ one can obtain

$$T_2(t) \leq \int_{|\xi| \geq 1} \sin^2(t|\xi|^2)|w_1(\xi)|^2 d\xi \leq \int_{|\xi| \geq 1} |w_1(\xi)|^2 d\xi$$

$$\leq \|w_1\|^2 = \|u_1\|^2, \quad (t \geq 0).$$

(4.20)

The statement of Proposition 1.1 can be derived because of (4.18)-(4.20).

\[\square\]

5 Proof of Theorems 1.3, 1.4 and 1.5.

In this section, let us prove Theorems 1.3, 1.4 and 1.5. The first part is devoted to the proof of Theorem 1.3.

Proof of Theorem 1.3. The whole idea comes from [8]. By applying the spatial Fourier transform to the both sides of (1.1), again the problem (1.1)-(1.2) can be reduced to ODE with parameter $\xi$:

$$\hat{u}_{tt}(t, \xi) + |\xi|^4 \hat{u}(t, \xi) = 0, \quad (t, \xi) \in (0, \infty) \times \mathbb{R}_\xi^n;$$

(5.1)

$$\hat{u}(0, \xi) = \hat{u}_0(\xi), \quad \hat{u}_t(0, \xi) = \hat{u}_1(\xi), \quad \xi \in \mathbb{R}_\xi^n.$$  \hspace{1cm} (5.2)

For the solution $\hat{u}(t, \xi)$ to problem (5.1)-(5.2), one introduces an auxiliary function

$$\hat{w}(t, \xi) := \int_0^t \hat{u}(s, \xi) ds.$$  \hspace{1cm} (5.3)

Then $\hat{w}(t, \xi)$ satisfies

$$\hat{w}_{tt}(t, \xi) + |\xi|^4 \hat{w}(t, \xi) = \hat{u}_1(\xi), \quad (t, \xi) \in (0, \infty) \times \mathbb{R}_\xi^n;$$

(5.4)

$$\hat{w}(0, \xi) = 0, \quad \hat{w}_t(0, \xi) = \hat{u}_0(\xi), \quad \xi \in \mathbb{R}_\xi^n.$$  \hspace{1cm} (5.5)

Now, we introduce one more auxiliary function $\hat{v}(t, \xi)$ defined on $\mathbb{R}_\xi^n \setminus \{0\}$ as follows:

$$\hat{v}(t, \xi) = \hat{w}(t, \xi) - \frac{\hat{u}_1(\xi)}{|\xi|^4}, \quad \xi \in \mathbb{R}_\xi^n \setminus \{0\}.$$  \hspace{1cm} (5.6)

Then, the function $\hat{v}(t, \xi)$ satisfies

$$\hat{v}_{tt}(t, \xi) + |\xi|^4 \hat{v}(t, \xi) = 0, \quad t > 0, \quad \xi \in \mathbb{R}_\xi^n \setminus \{0\},$$

(5.7)

$$\hat{v}(0, \xi) = -\frac{\hat{u}_1(\xi)}{|\xi|^4}, \quad \hat{v}_t(0, \xi) = \hat{u}_0(\xi), \quad \xi \in \mathbb{R}_\xi^n \setminus \{0\}.$$  \hspace{1cm} (5.8)

By multiplying both sided of (5.6) by $\overline{\hat{v}_t(t, \xi)}$, integrating it over $\{|\xi| \geq \delta\}$ with small $\delta > 0$, and taking real parts of the resulted equality one can gets

$$\frac{d}{dt} \int_{|\xi| \geq \delta} (|\hat{v}_t(t, \xi)|^2 + |\xi|^4|\hat{v}(t, \xi)|^2) d\xi = 0,$$

so that by integrating it over $[0, t]$ one has

$$\int_{|\xi| \geq \delta} (|\hat{v}_t(t, \xi)|^2 + |\xi|^4|\hat{v}(t, \xi)|^2) d\xi = \int_{|\xi| \geq \delta} (|\hat{v}_0(\xi)|^2 + \frac{|\hat{u}_1(\xi)|^2}{|\xi|^4}) d\xi,$$

where we have just used (5.7). Since $\hat{v}_t(t, \xi) = \hat{w}_t(t, \xi) = \hat{u}(t, \xi)$, one can arrive at

$$\int_{|\xi| \geq \delta} |\hat{u}(t, \xi)|^2 d\xi \leq \int_{\mathbb{R}_\xi^n} |\hat{u}_0(\xi)|^2 d\xi + \int_{|\xi| \geq \delta} \frac{|\hat{u}_1(\xi)|^2}{|\xi|^4} d\xi$$
\[ \|u_0\|^2 + \int_{1 \leq |\xi| \leq \delta} \frac{|\hat{u}_1(\xi)|^2}{|\xi|^4} d\xi + \int_{|\xi| \leq 1} \frac{|\hat{u}_1(\xi)|^2}{|\xi|^4} d\xi \leq \|u_0\|^2 + \int_{1 \leq |\xi| \leq \delta} \frac{|\hat{u}_1(\xi)|^2}{|\xi|^4} d\xi + \|u_1\|^2. \]  

(5.8)

By applying Lemma 3.1 of [5, page 879] to \( \hat{u}_1(\xi) \) one has

\[ |\hat{u}_1(\xi)| \leq C |\xi|^\gamma \|u_1\|_{1,\gamma}, \]  

(5.9)

because of the assumption \( P_1 = \int_{\mathbb{R}^n} u_1(x) dx = 0 \), where \( \gamma \in [0, 1] \). By substituting (5.9) into (5.8) one has arrived at the estimate

\[ \int_{|\xi| \geq \delta} |\hat{u}(t, \xi)|^2 \leq \|u_0\|^2 + \|u_1\|^2 + C \|u_1\|_{1,\gamma}^2 \int_{1 \geq |\xi| \geq \delta} |\xi|^{2\gamma - 4} d\xi \]

\[ \leq \|u_0\|^2 + \|u_1\|^2 + C \|u_1\|_{1,\gamma}^2 \cdot \frac{\omega_n}{2\gamma + n - 4}, \]  

(5.10)

provided that \( 2\gamma + n - 4 > 0 \) for \( \gamma \in (0, 1] \) in the case of \( n = 4 \), and for \( \gamma \in (\frac{4}{7}, 1] \) in the case of \( n = 3 \). Note that the constant \( C > 0 \) does not depend on any small \( \delta > 0 \). By letting \( \delta \to +0 \) in (5.10), one has the desired estimates via the Plancherel Theorem.

In order to prove Theorem 1.4, we use the following lemma, which is a direct consequence of Lemma 5.1 with \( \gamma \in (1, 2] \) of [10].

**Lemma 5.1** Let \( n \geq 1 \), and for \( \gamma \in (1, 2] \), suppose that \( f \in L^{1,\gamma}(\mathbb{R}^n) \) satisfies

\[ \int_{\mathbb{R}^n} f(x) dx = 0, \quad \int_{\mathbb{R}^n} x_j f(x) dx = 0 \quad (j = 1, 2, \ldots, n). \]

Then, it holds that

\[ |\hat{f}(\xi)| \leq C |\xi|^\gamma \|f\|_{1,\gamma}, \quad \xi \in \mathbb{R}^n \]

with some constant \( C = C_\gamma > 0 \).

Based on Lemma 5.1 let us prove Theorem 1.4.

**Proof of Theorem 1.4.** The proof is just a slight modification of those for Theorem 1.3. Indeed, we use Lemma 5.1 with \( f(x) := u_1(x) \) and \( \gamma \in (1, 2] \) in place of (5.10) with \( \gamma \in [0, 1] \). Then, (5.10) above can be interpreted as follows: the positivity \( 2\gamma + n - 4 > 0 \) holds for \( \gamma \in (1, 2] \) in the case of \( n = 2 \), and for \( \gamma \in (\frac{4}{7}, 2] \) in the case of \( n = 1 \). This implies the desired results.

Finally, let us prove Theorem 1.5 basing on the following equality which is a direct consequence of [10] Lemma 5.1 with \( f(x) := u_1(x) \):

\[ w_1(\xi) = P + i\xi P_1 + E(\xi), \]  

(5.11)

where

\[ P := \int_{\mathbb{R}} u_1(x) dx, \quad P_1 := \int_{\mathbb{R}} x u_1(x) dx, \]

and the error term \( E(\xi) \) satisfies

\[ |E(\xi)| \leq C |\xi|^2 \|u_1\|_{1,2}, \quad \xi \in \mathbb{R}. \]  

(5.12)

**Proof of (1) of Theorem 1.5.** For our purpose it suffices to obtain the lower bound estimate on \( J_1(t) \) in (3.8). Indeed, from (5.10) and (5.3) one has

\[ J_1(t) = \int_{|\xi| \leq \frac{2\pi}{\gamma}} \frac{\sin^2(t|\xi|^2)}{|\xi|^4} |P + i\xi P_1 + E(\xi)|^2 d\xi \]
Similarly to the estimates developed in (3.16), by using (5.17) one can proceed the computation as follows:

\[
\begin{align*}
\geq & \frac{1}{2} \int_{|\xi| \leq \frac{4}{\delta_0}} \frac{\sin^2(t|\xi|^2)}{|\xi|^4} (P^2 + |\xi|^2 P_1^2) d\xi - \int_{|\xi| \leq \frac{4}{\delta_0}} \frac{\sin^2(t|\xi|^2)}{|\xi|^4} |E(\xi)|^2 d\xi \\
\geq & \frac{P^2}{2} \int_{|\xi| \leq \frac{4}{\delta_0}} \frac{\sin^2(t|\xi|^2)}{|\xi|^4} d\xi + \frac{P^2}{2} \int_{|\xi| \leq \frac{4}{\delta_0}} \frac{\sin^2(t|\xi|^2)}{|\xi|^4} d\xi - \int_{|\xi| \leq \frac{4}{\delta_0}} \frac{\sin^2(t|\xi|^2)}{|\xi|^4} |E(\xi)|^2 d\xi \\
= & \frac{P^2}{2} K_1(t) + \frac{P^2}{2} K_3(t) - K_4(t),
\end{align*}
\]

where \(K_1(t)\) is the same function defined in (3.10), and

\[
\begin{align*}
K_3(t) := & \int_{|\xi| \leq \frac{4}{\delta_0}} \frac{\sin^2(t|\xi|^2)}{|\xi|^4} d\xi, \\
K_4(t) := & \int_{|\xi| \leq \frac{4}{\delta_0}} \frac{\sin^2(t|\xi|^2)}{|\xi|^4} |E(\xi)|^2 d\xi.
\end{align*}
\]

First, it is easy to see by using (5.17)

\[
K_4(t) \leq C \|u_1\|_{L^2}^2 t^{-\frac{n}{2}}, \quad (n = 1)
\]

with some constant \(C > 0\). While,

\[
\begin{align*}
K_3(t) & \geq \frac{t}{\delta_0} \omega_1 \int_{0}^{\frac{4}{\delta_0}} \sin^2(tr^2) dr \\
& = \left( \frac{\omega_1}{\delta_0} \int_{0}^{\frac{4}{\delta_0}} \sin^2(a^2) da \right) t^{\frac{1}{2}} \quad (t \gg 1).
\end{align*}
\]

On the lower bound of \(K_1(t)\) one has already derived it in (3.12) with \(n = 1\). Thus, by combining it with (5.18) one has the desired estimates.

\[\square\]

**Proof of (2) of Theorem 1.5.** Let \(n = 2\). The following result, once more, is a direct consequence of Lemma 5.1 with \(f(x) := u_1(x)\):

\[
w_1(\xi) = P + i\xi \cdot P_1 + E(\xi),
\]

where

\[
P := \int_{\mathbb{R}^2} u_1(x) dx, \quad P_1 := (p_1, p_2),
\]

\[
p_1 := \int_{\mathbb{R}^2} x_1 u_1(x) dx, \quad p_2 := \int_{\mathbb{R}^2} x_2 u_1(x) dx,
\]

and the error term \(E(\xi)\) satisfies

\[
|E(\xi)| \leq C |\xi|^2 \|u_1\|_{L^2}, \quad \xi \in \mathbb{R}^2.
\]

(5.17)

Similarly to the estimates developed in (3.16), by using (5.17) one can proceed the computation as follows:

\[
\begin{align*}
\|w(t, \cdot)\|^2 \geq & \frac{1}{2} \int_{\mathbb{R}^2} \sin^2(tr^2) |w_1(\xi)|^2 d\xi - \int_{\mathbb{R}^2} \cos^2(tr^2) |w_0(\xi)|^2 d\xi \\
\geq & \frac{1}{2} \int_{\mathbb{R}^2} e^{-r^2 \sin^2(2\theta)} t^2 |P + i\xi \cdot P_1 + E(\xi)|^2 d\xi - \|u_0\|^2 \\
\geq & \frac{1}{4} P^2 \int_{\mathbb{R}^2} e^{-r^2 \sin^2(2\theta)} d\xi + \frac{1}{4} K_5(t) - C \|u_1\|_{L^2}^2 \int_{\mathbb{R}^2} e^{-r^2 \sin^2(2\theta)} r^4 d\xi - \|u_0\|^2 \\
= & \frac{1}{4} P^2 U(t) + \frac{1}{4} K_5(t) - C \|u_1\|_{L^2}^2 \omega_2 - \|u_0\|^2,
\end{align*}
\]

\[(5.18)\]
where
\[
K_5(t) := \int_{\mathbb{R}^2} e^{-r^2 \sin^2(tr^2) r^4} |\xi \cdot P_1|^2 d\xi.
\]

First, by similar computations as in (3.17), (3.18) and (3.19) one can get
\[
U(t) \geq Ct, \quad t \gg 1,
\]
with some constant \(C > 0\). So, it suffices to estimate \(K_5(t)\). In order to estimate \(K_5(t)\), we employ an idea coming from [11]. For this, we set
\[
K := \left\{ \xi \in \mathbb{R}^2 : \left| \frac{\xi}{|\xi|} \right| \frac{P_1}{|P_1|} \geq \frac{1}{2} \right\}.
\]

Then, one can estimate as follows:
\[
K_5(t) := \int_{\mathbb{R}^2} e^{-r^2 \sin^2(tr^2) r^4} |\xi \cdot P_1|^2 d\xi
\geq |P_1|^2 \int_K e^{-r^2 \sin^2(tr^2) r^4} \frac{\xi}{|\xi|} \frac{P_1}{|P_1|}^2 d\xi \geq \frac{|P_1|^2}{4} \int_K e^{-r^2 \sin^2(tr^2) r^4} d\xi
= C\pi \frac{|P_1|^2}{4} \int_0^\infty e^{-r^2 \sin^2(tr^2) r} dr := C\pi \frac{|P_1|^2}{4} K_6(t) .
\]

Once more, as in (3.17), (3.18) and (3.19) one can estimate \(K_6(t)\):
\[
K_6(t) \geq \sum_{j=0}^{\infty} \int_{\theta_j}^{\tau_j} e^{-r^2 \sin^2(tr^2) r} r \geq \frac{1}{2} \sum_{j=0}^{\infty} \int_{\theta_j}^{\tau_j} e^{-r^2} r^{-1} dr
\geq \frac{1}{4} \int_{\theta_0}^{\infty} e^{-r^2} r^{-1} dr \geq \frac{e^{-1}}{4} \int_{\theta_0}^{1} r^{-1} dr
\geq C \log t, \quad t \gg 1.
\]

The desired estimate can be established from (5.18)–(5.21).

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