Witnessing entanglement via the geometric phase in a impurity-doped Bose-Einstein condensate

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We propose a theoretical scheme to witness quantum entanglement via the geometric phase in an impurity-doped Bose-Einstein condensate (BEC), which is a micro-macro quantum system consisting of two Rydberg impurity qubits and the BEC. We calculate the geometric phase of the impurity qubits in the presence of the initial micro-micro and micro-macro entanglement, respectively. It is demonstrated that the geometric phase of the impurity qubits can witness not only inter-qubit micro-micro entanglement, but also qubit-BEC micro-macro entanglement. Our work provide a new insight to witness micro-micro and micro-macro entanglement in a impurity-doped BEC.

I. INTRODUCTION

Quantum entanglement lies at the heart of quantum physics and the emerging second quantum revolution [1, 2], and is a key resource for quantum information science and technology [3, 4]. Its detection is thus of crucial importance and has been studied extensively, notably with so-called entanglement witnesses [4]. The fact that there exist entanglement witness for every entangled state [5] has raised the importance on the theoretical point of view even further [6, 7]. In particular, it is one of fascinating problems in the field of quantum physics and quantum information to detect quantum entanglement of a micro-macro system [8–21]. The difficulties inherent in such a question are manifolds, and they are related not only to quantum decoherence induced by the surrounding environment [22–32], but also to a measurement precision sufficient to observe quantum effects at such macroscales.

A Bose-Einstein condensate (BEC) doped with impurities [33–39] provides an ideal platform for the study of micro-micro and micro-macro entanglement where microscopic impurities meets a macroscopic matter, the BEC. As the interaction among Rydberg impurity atoms [40–45] can be tailored by electric fields and microwave fields [46, 47] while the BEC allows for an extremely precise control of interatomic interactions by manipulating s-wave scattering length [48, 49], they can build a precisely controllable micro-macro quantum systems [50].

Since Berry demonstrated that a geometric phase can be created in a quantum system undergoing a cyclic adiabatic evolution [51], much attention has been paid to the geometric phase [52–54]. Geometric phases of pure states have been generalized to mixed states [55–57] and open quantum systems [58]. Important experimental progresses have been made on the geometric phase [59–74]. In particular, the rise of quantum information science has opened up a new direction for applications of the geometric phase as well as triggered new insights into its physical nature and applications, such as geometric quantum computation [75–78] and a new type of topological phases for entangled quantum systems [79, 80].

In the present paper, we want to propose a theoretical scheme to witness micro-micro and micro-macro entanglement via the geometric phase of the impurity qubits in an impurity-doped BEC, which consists of the BEC and two Rydberg impurity atoms treated as qubits. We calculate the geometric phase of the impurity qubits and obtain direct relations between the geometric phase and quantum entanglement. It is shown that the geometric phase of the impurity qubits can witness not only the micro-micro entanglement between two impurity qubits, but also micro-macro entanglement between the impurity qubits and the BEC.

The remainder of this paper is organized as follows. In Sec. II, we introduce the impurities-doped BEC model consisting of the BEC and two Rydberg impurities. We present an analytical solution of the impurities-doped BEC model for a general initial state of the model. In Sec. III, we calculate the geometric phase of the impurity qubits when the two qubits are initially in an entangled state while qubits and the BEC are initially unentangled. It is indicated that the geometric phase can witness micro-micro entanglement between two Rydberg impurities. In Sec. IV, we investigate the geometric phase of the impurity qubits when the qubits and the BEC are initially in entangled states. We find a direct relation between micro-macro entanglement and the geometric phase of the impurity qubits, and demonstrate that the geometric phase can witness micro-micro entanglement. Finally, Sec. V is devoted to some concluding remarks.

II. THE IMPURITY-DOPED BEC MODEL

The impurity-doped BEC system under our consideration consists of a BEC and two localized Rydberg impurity atoms immersed in the BEC [81]. The two separated Rydberg impurities are frozen in place and they interact with each other via a repulsive van der waals interaction [46, 47]. The relevant internal level structure for each Rydberg atom is given by the atomic ground state |0⟩ and the excited Rydberg state |1⟩, which form an effective two-level system, i.e., an impurity qubit. The Hamiltonian of two Rydberg impurities in the absence of the external laser field [47] is given by

$$H_R = \frac{\omega}{2}(\sigma_z^1 + \sigma_z^2) + J\sigma_z^1\sigma_z^2,$$

(1)
where $\omega$ is the transition frequency between two internal states of each Rydberg impurity atom, the second term accounts for the van der Waals interaction between the Rydberg impurities with the coupling strength $J = C_6 / R^6$ where $R$ is the distance between two localized Rydberg impurities, $C_6 \propto n^{11}$ with $n$ being the principal quantum number of the Rydberg excitation. We have set $\hbar = 1$ in Hamiltonian (1) and through out the paper.

**FIG. 1.** Schematic diagram of two Rydberg impurities immersed in a Bose-Einstein condensate. Here $J$ denotes the van der Waals interaction between two Rydberg impurities.

Under the single-mode approximation, the Hamiltonian of a BEC confined in a trapping potential has a Kerr-interaction form given by

$$H_B = \omega_b a^\dagger a + \chi a^\dagger a^\dagger a a,$$

(2)

where $\omega_b$ is the mode frequency of the BEC which depends on the BEC mode function and the trapping potential, the nonlinear coupling constant describes the inter-atomic $s$-wave scattering interaction in the BEC.

The two Rydberg impurities interact with the BEC via coherent collisions. The impurity-BEC interaction Hamiltonian can be given by

$$H_I = \frac{\lambda}{2} (\sigma_z^1 + \sigma_z^2) a^\dagger a,$$

(3)

and $\lambda$ is the interaction strength.

Hence, The Hamiltonian of the total system including the two Rydberg impurities and the BEC is given by

$$H = H_R + H_B + H_I,$$

(4)

which is a diagonal Hamiltonian with the following eigenvalues and eigenstates

$$E_{ijn} = \frac{1}{2} \omega \left( (-1)^i + (-1)^j \right) + \omega_b n + (-1)^{i+j} J$$

$$+ \frac{1}{2} \lambda \left( (-1)^i + (-1)^j \right) n + \chi n(n-1),$$

(5)

$$|\psi\rangle_{ijn} = |ijn\rangle,$$

(6)

where $|ijn\rangle = |i\rangle \otimes |j\rangle \otimes |n\rangle$ with $|i\rangle(|j\rangle)$ is a eigenstate of $\sigma_z^i (\sigma_z^j)$ with $i = 0, 1 (j = 0, 1)$, and $|n\rangle$ is a Fock state with $n = 0, 1, 2, \cdots$. 

Assume that the two impurity qubits and the BEC are initially in the following state

$$|\Psi(0)\rangle = (c_0 |00\rangle + c_1 |11\rangle + c_2 |01\rangle + c_3 |10\rangle) \otimes |\alpha\rangle,$$

(7)

where $|\alpha\rangle$ is the Glauber coherent state defined by

$$|\alpha\rangle = e^{-\frac{\alpha^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle,$$

(8)

and the superposition coefficients satisfy the normalization condition

$$|c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2 = 1.$$

(9)

It is straightforward to obtain the wavefunction of the impurity-doped BEC at an arbitrary time with the following expression

$$|\Psi(t)\rangle = (c_0 |00\rangle \otimes |\varphi_0(t)\rangle + c_1 |11\rangle \otimes |\varphi_1(t)\rangle$$

$$+ c_2 |01\rangle \otimes |\varphi_2(t)\rangle + c_3 |10\rangle \otimes |\varphi_3(t)\rangle),$$

(10)

where $|\varphi_i(t)\rangle (i = 0, 1, 2, 3)$ are four generalized coherent states given by

$$|\varphi_i(t)\rangle = e^{-\frac{\alpha^2}{2}} \sum_{n=0}^{\infty} e^{-\theta_i(n)} \frac{\alpha^n}{\sqrt{n!}} |n\rangle,$$

(11)

where the four running frequencies are given by

$$\theta_0(n) = -\omega + J - (\lambda - \omega_b)n + \chi n(n-1),$$

$$\theta_1(n) = \omega + J + (\lambda + \omega_b)n + \chi n(n-1),$$

$$\theta_2(n) = \theta_3(n) = -J + \omega_n + \chi n(n-1).$$

(12)

From Eq. (10) we can obtain the reduced density operator of the two impurity qubits with the following form

$$\rho(t) = \begin{pmatrix}
\rho_{00}(t) & \rho_{01}(t) & \rho_{02}(t) & \rho_{03}(t) \\
\rho_{10}(t) & \rho_{11}(t) & \rho_{12}(t) & \rho_{13}(t) \\
\rho_{20}(t) & \rho_{21}(t) & \rho_{22}(t) & \rho_{23}(t) \\
\rho_{30}(t) & \rho_{31}(t) & \rho_{32}(t) & \rho_{33}(t)
\end{pmatrix},$$

(13)

where the diagonal elements of the reduced density operator are given by

$$\rho_{ii}(t) = |c_i|^2 \sum_{n=0}^{\infty} |\langle n| \varphi_i(t)\rangle|^2 \quad (i = 0, 1, 2, 3),$$

(14)

and the off-diagonal elements of the reduced density operator are given by

$$\rho_{ij}(t) = \rho_{ji}^*(t)$$

$$= c_i c_j^* \sum_{n=0}^{\infty} \langle n| \varphi_i(t)\rangle \langle \varphi_j(t)| n\rangle, \quad (i \neq j).$$

(15)
III. MICRO-MICRO ENTANGLEMENT WITNESSS VIA THE GEOMETRIC PHASE

In this section we calculate the geometric phase of the two Rydberg impurities in the BEC when the two impurity qubits are initially in an entangled state while the BEC is in a coherent state. We will obtain a direct relation between the inter-qubit entanglement measured by the concurrence and the geometric phase of the two qubits, and indicate that the geometric phase can witness the inter-qubit entanglement.

We assume that two impurity qubits are initially in a Bell-type state while the BEC is initially in a coherent state given by Eq. (8), then the initial state of the impurity-doped BEC is

$$|\Psi(0)\rangle = (\cos \eta_0 |00\rangle + \sin \eta_0 |11\rangle) \otimes |\alpha\rangle.$$  \hspace{1cm} (16)

Making Eqs. (10) and (13), we can find the reduced density operator of the two qubits in the subspace \{00, 11\} at an arbitrary time

$$\rho(t) = \cos^2 \eta_0 |00\rangle \langle 00| + \sin^2 \eta_0 |11\rangle \langle 11| + \rho_{01}(t) |00\rangle \langle 11| + \rho_{10}(t) |11\rangle \langle 00|,$$  \hspace{1cm} (17)

where the off-diagonal elements are given by

$$\rho_{01}(t) = \rho_{10}^*(t) = \frac{1}{2} \sin(2\eta_0) \sum_{n=0}^{\infty} \langle n | \varphi_0(t) \rangle \langle \varphi_1(t) | n \rangle,$$ \hspace{1cm} (18)

where $|\varphi_0(t)\rangle$ and $|\varphi_1(t)\rangle$ are given by Eq. (11).

The off-diagonal element (18) can be simply expressed as

$$\rho_{01}(t) = \frac{1}{2} \sin(2\eta_0) e^{iA_1(t) - \Gamma_1(t)},$$ \hspace{1cm} (19)

where we have introduced the following functions

$$A_1(t) = 2\omega t + |\alpha|^2 \sin(2\lambda t),$$ \hspace{1cm} (20)

$$\Gamma_1(t) = 2 |\alpha|^2 \sin^2(\lambda t).$$ \hspace{1cm} (21)

Then the reduced density operator of the qubits (17) can be rewritten as

$$\rho(t) = \left( \frac{1}{2} \sin(2\eta_0) e^{-iA_1(t) - \Gamma_1(t)} \right)^2 \sin^2 \eta_0,$$ \hspace{1cm} (22)

which has the eigenvalue equation

$$\rho(t) |\varepsilon_i(t)\rangle = \varepsilon_i(t) |\varepsilon_i(t)\rangle \quad (i = 1, 2),$$ \hspace{1cm} (23)

where the eigenvalues are given by

$$\varepsilon_{1,2} = \frac{1}{2} \left[ 1 \pm E(t) \right],$$ \hspace{1cm} (24)

where $E(t)$ is given by

$$E(t) = \sqrt{1 + \sin^2(2\eta_0) \left[ e^{-2\Gamma_1(t)} - 1 \right]}.$$ \hspace{1cm} (25)

The corresponding eigenstates are given by

$$|\varepsilon_1(t)\rangle = \cos \theta(t) |00\rangle + \sin \theta(t) e^{-iA_1(t)} |11\rangle,$$ \hspace{1cm} (26)

$$|\varepsilon_2(t)\rangle = \sin \theta(t) |00\rangle - \cos \theta(t) e^{-iA_1(t)} |11\rangle.$$ \hspace{1cm} (27)

where the mixing angle in the eigenstates is defined by

$$\sin \theta(t) = \sqrt{\frac{E(t) - \cos(2\eta_0)}{2E(t)}},$$ \hspace{1cm} (28)

$$\cos \theta(t) = \sqrt{\frac{E(t) + \cos(2\eta_0)}{2E(t)}},$$ \hspace{1cm} (29)

According to the kinematic approach [58], the geometric phase of two qubits with the reduced density operator of the qubits (22) is given by the following expression

$$\Phi_G = \arg \left\{ \sum_{i=1}^{2} \sqrt{\varepsilon_i(0) \varepsilon_i(\tau) \langle \varepsilon_i(0) | \varepsilon_i(\tau) \rangle} e^{-\int_0^\tau dt \langle \varepsilon_i(t) | \dot{\varepsilon}_i(t) \rangle} \right\},$$ \hspace{1cm} (30)

where the dot denotes the time derivative, $\tau$ is the evolution time of the qubits along a quasicyclic path of the impurity qubits, it is determined by the characteristic frequency of the qubits

$$\tau = \frac{2\pi}{\omega}.$$ \hspace{1cm} (31)

From Eqs. (21), (24) and (25) we can obtain the initial eigenvalues of the the impurity qubits

$$\varepsilon_1(0) = 1, \quad \varepsilon_2(0) = 0.$$ \hspace{1cm} (32)

Then the geometric phase (30) becomes

$$\Phi_G = \arg \left\{ \varepsilon_1(0) \varepsilon_1(\tau) \langle \varepsilon_1(0) | \varepsilon_1(\tau) \rangle e^{-\int_0^\tau dt \langle \varepsilon_1(t) | \dot{\varepsilon}_1(t) \rangle} \right\},$$ \hspace{1cm} (33)

where the two inner products are given by

$$\langle \varepsilon_1(0) | \varepsilon_1(\tau) \rangle = \cos \eta_0 \cos \theta(\tau) + e^{-iA_1(\tau)} \sin \eta_0 \sin \theta(\tau),$$

$$\langle \varepsilon_1(t) | \dot{\varepsilon}_1(t) \rangle = -i\dot{A}_1(t) \sin^2 \theta(t).$$ \hspace{1cm} (34)

Substituting Eq. (34) into (33) we can express the geometric phase

$$\Phi_G = \arg \left\{ \cos \eta_0 \cos \theta(\tau) + e^{-iA_1(\tau)} \sin \eta_0 \sin \theta(\tau) \right\}$$

$$+ \int_0^\tau dt \dot{A}_1(t) \sin^2 \theta(t),$$ \hspace{1cm} (35)

where $A_1(\tau)$ is given by Eq. (20).

In the following we show the geometric phase given by Eq. (35) can be used to witness the initial entanglement of the impurity qubits. We can use quantum concurrence to measure the amount of entanglement for an arbitrary quantum state of the two Rydberg impurities. The concurrence of an arbitrary quantum state of two qubits with a density operator $\rho_R(t)$ [83] is given by

$$C = \max \{ 0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 \},$$ \hspace{1cm} (36)

where the $\lambda_i$ ($i = 1, 2, 3, 4$) are the square roots of the eigenvalues in descending order of the operator $R = \rho_R(t) (\sigma^y \otimes \sigma^y)$. 

\[ \]
\( \sigma_y^2 \rho_R(t)(\sigma_y^\dagger \otimes \sigma_y^\dagger) \) with \( \sigma_y \) being the Pauli operator in the computational basis. It ranges from \( C_1 = 0 \) for a separable state to \( C_1 = 1 \) for a maximally entangled state.

If the time-dependent density matrix of a two qubit system can be expressed as

\[
\rho(t) = \begin{pmatrix}
    u(t) & 0 & 0 & z(t) \\
    0 & x(t) & 0 & 0 \\
    0 & 0 & x(t) & 0 \\
    z^*(t) & 0 & 0 & y(t)
\end{pmatrix},
\]

one finds that the concurrence corresponding to this state is given by [84–86]

\[
C(t) = \max \{0, 2|z(t)| - 2x(t)\}.
\]

For the initial state (16) of two impurity qubits, we can obtain the quantum concurrence with the following expression

\[
C(0) = |\sin (2\eta_0)|.
\]

Then we can find the relation between the mixing angle and the initial-state entanglement

\[
\cos \theta(t) = \frac{E(t) + \sqrt{1 - C^2}}{E(t)},
\]

\[
\sin \theta(t) = \frac{E(t) - \sqrt{1 - C^2}}{E(t)},
\]

where \( E(t) \) can be expressed as

\[
E(t) = \sqrt{1 + C^2 \left[ e^{-2\Gamma_1(t)} - 1 \right]}.
\]

which leads to

\[
\cos \theta(t) = \sqrt{1 + \sqrt{1 - C^2}}, \quad \sin \theta(t) = \sqrt{1 - \sqrt{1 - C^2}}.
\]

From Eq. (35) we find that the geometric phase in the weak coupling regime takes the following simple form

\[
\Phi_G = \frac{4\pi\lambda}{\omega}|\alpha|^2 \left(1 - \sqrt{1 - C^2}\right),
\]

which indicates that the geometric phase of the qubits can be manipulated through the coupling strength between the impurity qubits and the BEC or/and the initial-state parameters. In Fig. 2, we have plotted the geometric phase of the impurity qubits with respect to the initial inter-qubit entanglement. Fig. 2 indicates that the geometric phase of the qubits increases with increasing the initial-state entanglement of the two impurity qubits.

It is interesting to note that we can obtain the direct expression of the initial entanglement of impurity qubits in terms of the geometric phase

\[
C = 1 - \left(1 - \frac{\omega\Phi_G}{4\pi\lambda|\alpha|^2}\right)^2,
\]

which indicates that we can see that the accumulated geometric phase of two impurity qubits during the quasicyclic evolution time \( \tau = 2\pi/\omega \) can witness the initial entanglement of impurity qubits.

\textbf{IV. MICRO-MACRO ENTANGLEMENT WITNESSS VIA THE GEOMETRIC PHASE}

In this section we investigate the geometric phase of the two impurity qubits in the BEC when the two impurity qubits are initially entangled with the BEC. We will obtain a direct relation between the initial qubit-BEC entanglement measured by the concurrence and the geometric phase of the two qubits, and indicate that the geometric phase can witness the hybrid entanglement between the impurity qubits and the BEC.

We assume that two impurity qubits and the BEC are initially in the following hybrid entangled state

\[
|\Psi(0)\rangle = \cos \eta_0 |00\rangle \otimes |\alpha\rangle + \sin \eta_0 |11\rangle \otimes |-\alpha\rangle,
\]

where \(|-\alpha\rangle\) and \(|\alpha\rangle\) are two anti-phase coherent states defined by Eq. (8).

At an arbitrary time \( t \), the state of the system under our consideration is a two-component wavefunction

\[
|\Psi(t)\rangle = \cos \eta_0 |00\rangle \otimes \varphi_0'(t) + \sin \eta_0 |11\rangle \otimes \varphi_1'(t),
\]

where the two component wavefunctions of the BEC are given by

\[
|\varphi_0'(t)\rangle = e^{-i|\alpha|^2} \sum_{n=0}^{\infty} e^{-iE_{n,0}t} \frac{\alpha^n}{\sqrt{n!}} |n\rangle,
\]

\[
|\varphi_1'(t)\rangle = e^{-i|\alpha|^2} \sum_{n=0}^{\infty} e^{-iE_{1,1}t} \frac{(-\alpha)^n}{\sqrt{n!}} |n\rangle.
\]
Above two component wavefunctions of the BEC are generally nonorthogonal with the following inner product
\[
\langle \varphi'_0(t)|\varphi'_1(t) \rangle = e^{-i[2\omega t + |\alpha|^2 \sin(2\lambda t)]} e^{i\alpha^2 \cos^2(\lambda t)},
\]
(50)

It is easy to obtain the reduced density operator of the two qubits in the two-state space \{|00\}, |11\} to be
\[
\rho(t) = \left( \frac{1}{2} \sin^2(2\eta_0) e^{-i\Lambda_2(t)-\Gamma_2(t)} \right) \frac{1}{2} \sin^2(\eta_0),
\]
(51)
where \(\Lambda_2(t)\) and \(\Gamma_2(t)\) are given by
\[
\Lambda_2(t) = 2\omega t - |\alpha|^2 \sin(2\lambda t),
\]
(52)
\[
\Gamma_2(t) = 2 |\alpha|^2 \cos^2(\lambda t).
\]
(53)
The reduced density operator of the two qubits (51) has the following eigenvalues
\[
\tilde{\varepsilon}_{1,2}(t) = \frac{1}{2} \left[ 1 \pm \tilde{E}(t) \right],
\]
(54)
where the function \(\tilde{E}(t)\) is defined by
\[
\tilde{E}(t) = \sqrt{1 + \sin^2(2\eta_0) \left[ e^{-2\Gamma_2(t)} - 1 \right]}.
\]
(55)
The corresponding eigenstates are given by
\[
|\tilde{\varepsilon}_1(t)\rangle = \cos \tilde{\theta}(t) |00\rangle + \sin \tilde{\theta}(t) e^{-i\Lambda_2(t)} |11\rangle,
\]
(56)
\[
|\tilde{\varepsilon}_2(t)\rangle = \sin \tilde{\theta}(t) |00\rangle - \cos \tilde{\theta}(t) e^{-i\Lambda_2(t)} |11\rangle,
\]
(57)
where the mixing angle functions are defined by
\[
\cos \tilde{\theta}(t) = \sqrt{\frac{\tilde{E}(t) + \cos(2\eta_0)}{2\tilde{E}(t)}},
\]
(58)
\[
\sin \tilde{\theta}(t) = \sqrt{\frac{\tilde{E}(t) - \cos(2\eta_0)}{2\tilde{E}(t)}}.
\]
(59)

Through a calculation similar to the previous section, we can obtain the geometric phase of the two impurity qubits with the following form
\[
\tilde{\Phi}_G = \tilde{\Phi}_1 + \tilde{\Phi}_2,
\]
(60)
where the first part of the geometric phase is given by
\[
\tilde{\Phi}_1 = \arg \langle \tilde{\varepsilon}_1(0)|\tilde{\varepsilon}_1(\tau) \rangle + \int_0^\tau d\tilde{\Lambda}_2(t) \sin^2 \tilde{\theta}(t),
\]
(61)
where the inner product \(\langle \tilde{\varepsilon}_1(0)|\tilde{\varepsilon}_1(\tau) \rangle\) is given by
\[
\langle \tilde{\varepsilon}_1(0)|\tilde{\varepsilon}_1(\tau) \rangle = \cos \tilde{\theta}(0) \cos \tilde{\theta}(\tau) + \sin \tilde{\theta}(0) \sin \tilde{\theta}(\tau) e^{i(\Lambda_2(0) - \Lambda_2(\tau))}.
\]
(62)
The second part of the geometric phase can be expressed as
\[
\tilde{\Phi}_2 = \arg \left\{ 1 + \tilde{F}_1(\tau) \tilde{F}_2(\tau) \tilde{F}_3(\tau) \right\},
\]
(63)
where the three factorization functions are defined by
\[
\tilde{F}_1(\tau) = \frac{\tilde{\varepsilon}_2(0) \tilde{\varepsilon}_2(\tau)}{\tilde{\varepsilon}_1(0) \tilde{\varepsilon}_1(\tau)},
\]
(64)
\[
\tilde{F}_2(\tau) = \frac{\langle \tilde{\varepsilon}_2(0)|\tilde{\varepsilon}_2(\tau) \rangle}{\langle \tilde{\varepsilon}_1(0)|\tilde{\varepsilon}_1(\tau) \rangle},
\]
(65)
\[
\tilde{F}_3(\tau) = \exp \left\{ -\int_0^\tau d\tau \left[ -\langle \tilde{\varepsilon}_2(\tau)|\tilde{\varepsilon}_2(\tau) \rangle - \langle \tilde{\varepsilon}_1(\tau)|\tilde{\varepsilon}_1(\tau) \rangle \right] \right\}
\]
(66)

Substituting Eqs. (54)-(57) into Eqs. (64)-(66), we find that
\[
\tilde{F}_1(\tau) = \sqrt{\frac{1 - \tilde{E}(0)}{1 + \tilde{E}(0)}}.
\]
(67)
\[
\tilde{F}_2(\tau) = \frac{\cos \tilde{\theta}(0) \cos \tilde{\theta}(\tau) + \sin \tilde{\theta}(0) \sin \tilde{\theta}(\tau) e^{i(\Lambda_2(0) - \Lambda_2(\tau))}}{\sin \tilde{\theta}(0) \sin \tilde{\theta}(\tau) + \cos \tilde{\theta}(0) \cos \tilde{\theta}(\tau) e^{i(\Lambda_2(0) - \Lambda_2(\tau))}},
\]
\[
\tilde{F}_3(\tau) = \exp \left\{ i \int_0^\tau d\tau \tilde{\Lambda}_2(\tau) \cos \left[ 2\theta(\tau) \right] \right\}
\]
(68)

In order to observe properties of the geometric phase, we consider the specific case of \(\eta_0 = \frac{\pi}{4}\) and \(\lambda \tau = \frac{\pi}{4}\). In this case, we have
\[
\Lambda_2(0) = 0, \quad \Lambda_2(\tau) = 2\omega \tau - |\alpha|^2,
\]
(69)
\[
\cos \tilde{\theta}(\tau) = \sin \tilde{\theta}(\tau) = 1, \quad \tilde{E}(0) = e^{-2|\alpha|^2}, \quad \tilde{E}(\tau) = e^{-|\alpha|^2},
\]
(70)
\[
\langle \tilde{\varepsilon}_1(0)|\tilde{\varepsilon}_1(\tau) \rangle = \cos \left[ \frac{1}{2} \tilde{\Lambda}_2(\tau) \right] e^{-i\frac{1}{2} \tilde{\Lambda}_2(\tau)}
\]
(72)

And the three factorization functions are given by
\[
\tilde{F}_1(\tau) = \frac{1 - e^{-|\alpha|^2}}{\sqrt{1 + e^{-2|\alpha|^2}}},
\]
(73)
\[
\tilde{F}_2(\tau) = \tilde{F}_3(\tau) = 1.
\]
(74)

Making use of Eqs. (70)-(75), we find that two parts of the geometric phase of the two qubits are given by
\[
\tilde{\Phi}_1 = 2\pi \left( \frac{1}{4} + \frac{\lambda}{8} \right) \frac{|\alpha|^2}{\pi}, \quad \tilde{\Phi}_2 = 0.
\]
(75)

Therefore, the geometric phase of the two qubits takes the following form
\[
\tilde{\Phi}_G = 2\pi \left( \frac{1}{4} + \frac{\omega}{64} \right) \frac{|\alpha|^2}{\pi},
\]
(76)
where we have used the condition of \(\lambda \tau = \pi/4\).

In order to obtain the direct relation between the geometric phase of the two qubits and the initial entanglement between the two qubits and the BEC, we need to calculate the entanglement amount of the initial state (47). The initial state given by Eq. (47) is a two-component entangled state. Its entanglement amount can be measured by the quantum concurrence.
A calculation similar to the previous section gives the quantum concurrence of the initial state (47) with the following expression

\[
C(\eta_0, \alpha) = |\sin(2t\eta_0)| \sqrt{1 - e^{-2|\alpha|^2}}. 
\] (77)

Taking into account \(\eta_0 = \pi/2\), from Eqs. (76) and (77) we can obtain the direct relation between the geometric phase and the initial entanglement between the two qubits and the BEC.

\[
\tilde{\Phi}_G = -\frac{1}{64} (16 + \omega) \ln(1 - C^2). 
\] (78)

which indicates that the geometric phase of the qubits can be controlled through changing the initial-state entanglement between the impurity qubits and the BEC. In Fig. 2, we have plotted the geometric phase of the impurity qubits with respect to the initial inter-qubit entanglement. Fig. 2 indicates that the geometric phase of the qubits increases with increasing the initial-state entanglement between the impurity qubits and the BEC.

From Eq. (78) we can find that the initial entanglement between the impurity qubits and the BEC can be directly expressed in terms of the geometric phase as

\[
C = \sqrt{1 - \exp\left(-\frac{64}{16 + \omega} \tilde{\Phi}_G\right)}. 
\] (79)

which indicates that the accumulated geometric phase of two impurity qubits during the quasicyclic evolution time \(t = \tau = 2\pi/\omega\) can witness the initial entanglement between the impurity qubits and the BEC.

In this case, we find that at an arbitrary time \(t\) the wavefunction of the impurity-doped BEC is given by

\[
|\Psi(t)\rangle = \cos \eta_0 |00\rangle \otimes |\varphi_0'(t)\rangle + \sin \eta_0 |01\rangle \otimes |\varphi_{01}(t)\rangle, 
\] (81)

where \(|\varphi_0'(t)\rangle\) is given by Eq.(48) while \(|\varphi_{01}(t)\rangle\) is given by

\[
|\varphi_{01}(t)\rangle = e^{-\frac{\eta_0^2}{2}} \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{\sqrt{n!}} |n\rangle. 
\] (82)

The inner product the of two component wavefunctions of the BEC is given by

\[
\langle\varphi_0'(t)|\varphi_{01}(t)\rangle = \int \langle \omega - 4Jt + |\alpha|^2 \sin(2\lambda t) \rangle e^{-|\alpha|^2 \cos^2(\lambda t)} dt. 
\] (83)

Then we can obtain the reduced density operator of two impurity qubits

\[
\rho(t) = \left(\frac{1}{2} \sin(2t\eta_0) e^{-|\alpha|^2} \cos^2(\lambda t) - \frac{1}{2} \sin(2t\eta_0) e^{|\alpha|^2} \cos^2(\lambda t) \right), 
\] (84)

where the phase and the decaying factor are given by

\[
\Lambda_3(t) = (\omega - 4Jt) - |\alpha|^2 \sin(2\Lambda t), 
\] (85)

\[
\Gamma_3(t) = \Gamma_3(t) = 2 |\alpha|^2 \cos^2(\lambda t). 
\] (86)

Two eigenvalues of the reduced density operator (85) are

\[
\tilde{\epsilon}_{1,2} = \frac{1}{2} \left[1 \pm \tilde{E}(t)\right], 
\] (87)

where \(\tilde{E}(t)\) has been given by Eq.(55). The corresponding two eigenstates are

\[
|\tilde{\epsilon}_1(t)\rangle = \cos \tilde{\theta}(t) |00\rangle + \sin \tilde{\theta}(t) e^{-i\Lambda_3(t)} |01\rangle, 
\] (88)

\[
|\tilde{\epsilon}_2(t)\rangle = \sin \tilde{\theta}(t) |00\rangle - \cos \tilde{\theta}(t) e^{-i\Lambda_3(t)} |01\rangle, 
\] (89)

where the mixing angle functions and \(\Lambda_3(t)\) are given by Eqs. (58), (59) and (86), respectively.

We can obtain the geometric phase of the two impurity qubits with the following form

\[
\tilde{\Phi}_G = \tilde{\Phi}_1 + \tilde{\Phi}_2, 
\] (90)

where the first part of the geometric phase is given by

\[
\tilde{\Phi}_1 = \arg \left\langle \tilde{\epsilon}_1(0)|\tilde{\epsilon}_1(t)\rangle + \int_0^t dt \Lambda_3(t) \sin^2 \tilde{\theta}(t) \right\rangle. 
\] (91)

The second part of the geometric phase can be expressed as

\[
\tilde{\Phi}_2 = \arg \left\{1 + \tilde{F}_1(\tau) \tilde{F}_2(\tau) \tilde{F}_3(\tau) \right\}, 
\] (92)

where the three factorization functions are given by

\[
\tilde{F}_1(\tau) = \tilde{F}(\tau), 
\] (93)

\[
\tilde{F}_2(\tau) = \cos \tilde{\theta}(0) \cos \tilde{\theta}(\tau) + \sin \tilde{\theta}(0) \sin \tilde{\theta}(\tau) e^{i\Lambda_3(0) - \Lambda_3(\tau)}, 
\] (94)

\[
\tilde{F}_3(\tau) = \exp \left\{i \int_0^\tau dt \tilde{\Lambda}_3(t) \cos \left[2\tilde{\theta}(t)\right] \right\}. 
\] (95)

In what follows we consider another hybrid entangled state between the impurity qubits and the BEC

\[
|\Psi(0)\rangle = |0\rangle (\cos \eta_0 |0\rangle \otimes |\alpha\rangle + \sin \eta_0 |1\rangle \otimes |\alpha\rangle), 
\] (80)

where only the first qubit of the two impurity qubits is entangled with the BEC.

![FIG. 3. (Color online) The geometric phase of the impurity qubits with respect to the hybrid entanglement between the impurity qubits and the BEC. The geometric phase is scaled by \((16 + \omega)/64\).](image)
When we take $\eta_0 = \frac{\pi}{4}$ and $\lambda \tau = \frac{\pi}{4}$, we have

$$\tilde{F}_1'\left(\tau\right) = \frac{1 - e^{-|\alpha|^2}}{\sqrt{1 + e^{-2|\alpha|^2}}},$$

(96)

$$\tilde{F}_2'\left(\tau\right) = \tilde{F}_3'\left(\tau\right) = 1.$$  

(97)

In this case the geometric phase of the two qubits takes the following simple form

$$\tilde{\Phi}_G = -\pi \left(1 - \frac{4J}{\omega}\right) + \frac{1}{2} |\alpha|^2,$$  

(98)

where we have used $\tau = 2\pi/\omega$.

For the hybrid entangled state between the second qubit and the BEC, the quantum concurrence is given by

$$C(\eta_0, \alpha) = |\sin(2\eta_0)| \sqrt{1 - e^{-2|\alpha|^2}}.$$  

(99)

When $\eta_0 = \frac{\pi}{4}$, it becomes

$$C = \sqrt{1 - e^{-2|\alpha|^2}}.$$  

(100)

From Eqs. 99 and (101) we can obtain the direct relation between the geometric phase and the initial hybrid entanglement

$$\tilde{\Phi}_G' = -\pi \left(1 - \frac{4J}{\omega}\right) + \frac{1}{4} \ln \left(1 - C^2\right).$$  

(101)

Therefore, from Eq. (102) we can express the initial hybrid entanglement between the two qubits and the BEC in terms of the geometric phase of the two qubits

$$C = \sqrt{1 - \exp \left(4\tilde{\Phi}_G' - \frac{16J\pi}{\omega}\right)},$$  

(102)

which indicates that the accumulated geometric phase of two impurity qubits during the quasicyclic evolution time $\tau = 2\pi/\omega$ can witness the initial entanglement between the impurity qubits and the BEC for the qubit-BEC system under our consideration.

V. CONCLUDING REMARKS

We have presented a theoretical proposal to witness microscopic and macroscopic entanglement in terms of the geometric phase of the impurity qubits in the impurities-doped BEC system, which is a micro-macro quantum system consisting of microscopic impurity qubits and the macroscopic BEC. We have calculated the geometric phase of the impurity qubits when the two qubits are initially in an entangled state while qubits and the BEC are initially unentangled, and found that the initial micro-macro entanglement between two impurity qubits can be witnessed in terms of the accumulated geometric phase of two impurity qubits along the quasicyclic evolution path. When the qubits and the BEC are initially in entangled states, i.e., micro-macro entangled states, we have obtained the geometric phase of the qubits for two types of micro-macro entangled states. We have found a direct relation between the initial micro-macro entanglement and the geometric phase of the impurity qubits, and demonstrated that the geometric phase can witness micro-micro entanglement between the impurity qubits and the BEC. The ability of the geometric phase for the impurity qubits to witness micro-micro and micro-macro entanglement in the micro-macro hybrid quantum system provides a new insight for the entanglement detection in hybrid quantum systems which involve micro-macro quantum systems.

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