Noise models for superoperators in the chord representation

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The precise manipulation of coherent quantum processes is the ultimate goal at the basis of quantum information theory. However experimental quantum systems cannot be completely isolated and therefore unwanted interactions pose strict limits on its practical factibility.

Therefore understanding the effects of certain kinds of noise on algorithms and how to correct the errors they produce is a major subject of interest.

The purpose of this paper is to show how some simple noise models of interest in quantum information theory can be formulated in a phase space language. In doing so we reinterpret the depolarizing and the phase damping channels on many qubits as a special kind of diffusion in phase space, study their spectral properties and display their effects on selected pure initial states.

The Weyl representation, and its associated Wigner function, have been successfully used to understand many aspects of quantum mechanics [1], especially its classical limit [2, 3]. In its discrete version, it has also been used to study the properties of quantum maps and, recently, to analyze many-qubit quantum algorithms [4]. In phase space the Wigner distribution of a state displays both its classical properties, in the smooth part of the distribution, as well as its quantum properties, in the form of highly correlated oscillatory structures with sub-Planck scale sizes [5]. To reveal these structures occurring at widely different scales it is then natural to consider the Fourier transform of the Wigner function. This procedure leads to a different, and alternative, representation of quantum mechanics in phase space. This representation, when applied to density matrices is generally called characteristic or generating function representation. Its properties as a representation for general operators have been extensively studied in [6] where due to its geometrical features it has been called the chord representation. We adhere to this nomenclature. In this paper we show the advantages and the simplicity of using this representation as a “noise” basis, where various models of noise are given a simple phase space interpretation.

The paper is organized as follows. In Sec. II A we briefly review some fundamental concepts about quantum open systems and propose a two-stage scheme for the noisy propagator of density operators. In Sec. II B we describe noise models in phase space. Particularly, noise superoperators which are diagonal in the chord or characteristic representation [6] or, as we show, whose Kraus operators are proportional to translation operators in phase space. In Sec. III we find generalizations of the depolarizing and phase damping channels [7, 8] for many qubits using the chord representation. Finally, in Sec. IV we describe the diffusive noise recently proposed in [9, 10] and show how it can be used to obtain the main part of the spectrum of the noisy propagator.

II. A MODEL OF NOISY EVOLUTION

A. Two-stage Superoperators

Quantum systems coupled to a Markovian environment evolve according to the master equation [11, 12]

\[
\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] + \frac{1}{\hbar} \sum \left( \hat{L}_j \hat{\rho} \hat{L}^\dagger_j - \frac{1}{2} \hat{L}^\dagger_j \hat{L}_j \hat{\rho} - \frac{1}{2} \hat{\rho} \hat{L}^\dagger_j \hat{L}_j \right),
\]

(1)

where \( \hat{L}_j \) are the Lindblad operators. The first term on the right gives the unitary evolution. The second term represents interaction with an environment. Equation (1) generates a solution in the form of a one parameter family (quantum dynamical semigroup [13]) of linear maps or superoperators \( \hat{L}_\tau \), such that the state at time \( \tau \) is given by

\[
\hat{\rho}_\tau = \hat{L}_\tau (\hat{\rho}_0).
\]

(2)

A general way of representing a superoperator is as a Kraus operator sum [14]. Given a superoperator \( \hat{L} \) there exist a set of (Kraus) operators \( \hat{M}_\mu \), such that

\[
\hat{L} = \sum \hat{M}_\mu \circ \hat{M}_\mu^\dagger,
\]

(3)
where, for \( L \) to be trace preserving, the identity \( \sum_p \hat{M}_p \hat{M}^\dagger \hat{M}_a = \hat{I} \) must hold. Throughout the paper the \( \odot \) symbol should be interpreted as

\[
\hat{M} \odot \hat{M}^\dagger (\hat{\rho}) \equiv \hat{M} \hat{\rho} \hat{M}^\dagger.
\]

(4)

Not only does the Kraus form ensure that \( L(\hat{\rho}) \) is positive, for any density operator \( \hat{\rho} \), but it is also completely positive, meaning that tracing over any environment on which \( L \) acts trivially yields again a density operator \([15]\).

In this paper instead of modeling environments and solving the master equation we focus on the properties of \( L \) for open quantum systems and thus propose models for noisy propagators. This shift of emphasis is analogous to the shift from quantum systems and thus propose models for noisy propagators. There are several situations where this scheme applies naturally. One example is a kicked map in which the noise only acts between kicks. A billiard inside a bath constitutes another example if we consider that the interaction with the walls is purely unitary while the free propagation is noisy. A quantum algorithm, supposed perfect, sent through a noisy channel is another example if we consider that the interaction with the walls is purely unitary while the free propagation is noisy.

We center our attention on discrete time systems

\[
\hat{\rho}_n = L(\hat{\rho}_{n-1}) = L^n(\hat{\rho}_0)
\]

(5)

and propose a two-stage propagator consisting on the composition of a unitary with a noisy evolution. After one unitary step a density matrix \( \hat{\rho} \) becomes

\[
\hat{U} \hat{\rho} \hat{U}^\dagger \equiv U(\hat{\rho}).
\]

(6)

The operator \( U \) is a unitary superoperator with trivial Kraus form. The interaction with a Markovian environment is modeled by a noise superoperator \( D_\sigma \) depending on a parameter that quantifies its strength. The noisy one-step propagator is thus defined as

\[
L(\hat{\rho}) = D_\sigma \circ U(\hat{\rho}).
\]

(7)

Similar types of two-stage schemes can be found in \([3, 10, 16, 17, 18, 19]\). There are several situations where this scheme appears naturally. One example is a kicked map in which the noise only acts between kicks. A billiard inside a bath constitutes another example if we consider that the interaction with the walls is purely unitary while the free propagation is noisy. A quantum algorithm, supposed perfect, sent through a noisy channel is another example. In the case of a quantum algorithm that needs to be iterated, like Grover’s search, the noisy part would be an effective interaction acting after each iteration. Finally a numerical solution of Lindblad’s equation proceeding in small time steps will naturally alternate between the unitary and the noisy propagation.

**B. Diagonal noise in the chord representation**

Systems of \( K \) qubits are usually treated in a tensor product basis, the computational basis, defined as the tensor product of the eigenstates of \( \sigma_z \), \( \{|1\}, \{0\} \). It is labeled by an integer \( |n\rangle \) ranging from \( n = 0, \ldots, N-1 \) where \( N = 2^K \) (throughout the paper, for many qubit systems, we call \( K \) the number of qubits and \( N \) the dimension of phase space). The coefficients of the binary expansion of \( n \) represents the state of each single qubit. As an operator basis, it is also customary to take the tensor product of Pauli operators, consisting of the \( N^2 = 4^K \) possible tensor products of \( 1, \sigma_x, \sigma_y, \sigma_z \).

An alternative approach, standard in the treatment of quantum maps \([3]\), and also recently utilized in the context of quantum information \([4, 24, 21]\) is to treat \( K \)-qubit systems using a phase space setting. In this approach the computational basis is assimilated to the position basis \( B_q = \{|n\}, n = 0, \ldots, N-1 \) (with periodic boundary conditions). Therefore the conjugate discrete momentum basis \( B_p = \{|k\}, k = 0, \ldots, N-1 \) is obtained by means of the discrete Fourier transform (DFT). The dimension \( N \) is then naturally related to an effective Planck constant by

\[
N = 1/2\pi\hbar.
\]

(8)

The natural operator basis in this context is constituted by the phase space translations \([4, 6, 22]\) \( \hat{T}_{(q, p)} \) defined as

\[
\hat{T}_{(q, p)} |k\rangle = \exp \left[ -(2\pi i/N)q (k+p/2) \right] |k+p\rangle
\]

(9)

\[
\hat{T}_{(q, p)} |n\rangle = \exp \left[ (2\pi i/N)p (n+q/2) \right] |n+q\rangle,
\]

(10)

with the group composition rule

\[
\hat{T}_{(q_1, p_1)} \hat{T}_{(q_2, p_2)} = \hat{T}_{(q_1+q_2, p_1+p_2)} e^{(\pi i/N)(p_1 q_2 - q_1 p_2)},
\]

(11)

where the phase in the exponential is the area of the triangle defined by the vertices \((0, 0), (q_1, p_1) \) and \((q_2, p_2)\). Translations can be written in terms of compositions of translations in position and momentum. From Eq. (11) it is easy to show that

\[
\hat{T}_{0}^\dagger \hat{T}_{0} = \hat{I}.
\]

(14)

These operators have matrix representation in \( \mathcal{H}_{N^2} \) or “Liouville” space, the space of \( N \times N \) complex matrices with the Hilbert-Schmidt inner product

\[
\langle \hat{A}, \hat{B} \rangle = \text{Tr}(\hat{A}^\dagger \hat{B}),
\]

(15)

where \( A, B \in \mathcal{H}_{N^2} \) (we use “matrix” or “operator” to refer to elements in \( \mathcal{H}_{N^2} \) indistinctly, except where an explicit distinction is required). There are \( N^2 \) operators \( \hat{T}_{(q, p)} \) in \( \mathcal{H}_{N^2} \) and they form a complete orthogonal set since

\[
\text{Tr}(\hat{T}_{\alpha}^\dagger \hat{T}_{\alpha'}) = N \delta_{\alpha\alpha'}.
\]

(16)
Therefore any operator $\hat{A}$ in $\mathcal{H}_{N^2}$ can be expanded in this unitary basis as

$$\hat{A} = \frac{1}{\sqrt{N}} \sum_\alpha a(\alpha)\hat{T}_\alpha. \quad (17)$$

The $c$-number function (or symbol)

$$a(\alpha) = \frac{1}{\sqrt{N}} \text{Tr}(\hat{A}\hat{T}_\alpha^\dagger) \quad (18)$$
defines the chord representation [6] of $\hat{A}$. The DFT of the translation operators (with the dimension extended to $2N$, see [23]) are the generalized phase space point operators. Operators written in this basis constitute the Weyl or center [6] representation. The symbol of a density operator in the center representation is the well known discrete Wigner function, while in the chord representation it is also known as generating or characteristic function. A description of the phase space point operators and the peculiar features of the discrete Wigner function can be found in [4].

A general superoperator $\mathbf{D}$ can be written in the basis of translations as

$$\mathbf{D} = \frac{1}{N} \sum_{\alpha,\beta} C(\alpha, \beta)\hat{T}_\alpha \otimes \hat{T}_\beta^\dagger. \quad (19)$$

where the only requirement is that the matrix coefficients $C(\alpha, \beta)$ be non-negative and $C$ has unit trace. In this general setting the Kraus operators are linear superpositions of translations obtained by diagonalizing $C(\alpha, \beta)$. In this paper we only consider superoperators where $C(\alpha, \beta) = \delta_{\alpha,\beta}C(\alpha)$ is already in diagonal form in such a way that the Kraus operators are simply proportional to the translations $\hat{T}_\alpha$. Thus, the noise superoperator that we consider is explicitly defined by giving the Kraus operator sum form

$$\mathbf{D}_\sigma = \frac{1}{N} \sum_\alpha C_\sigma(\alpha)\hat{T}_\alpha \otimes \hat{T}_\alpha^\dagger. \quad (20)$$

Trace preservation is achieved if

$$\sum_\alpha C_\sigma(\alpha)/N = 1. \quad (21)$$

From Eq. (20) we see that $\mathbf{D}_\sigma$ is a convex sum of unitaries yielding a contracting superoperator. Physically the action of $\mathbf{D}_\sigma$ can be simply interpreted as performing an incoherent sum of all the possible translations $\hat{T}_\alpha$ each with a probability $C_\sigma(\alpha)/N$. In this way the choice of $C_\sigma(\alpha)$ determines different types of noise. The parameter $\sigma$ is introduced to control the strength of the noise.

One can think of this as a noise channel where the “errors” are unitary translations in phase space where they occur with probability $C_\sigma(\alpha)/N$. In this sense it constitutes a “very nice” [24] error basis with properties that are different from the more usual ones given by tensor products of Pauli matrices.

One of the motivations for using noise in the form of Eq. (20) is that all the spectral properties are readily available. From Eq. (11) it is clear that

$$\mathbf{D}_\sigma(\hat{T}_\lambda) = \frac{1}{N} \sum_\alpha C_\sigma(\alpha)\hat{T}_\alpha \hat{T}_\lambda \hat{T}_\alpha^\dagger = \frac{1}{N} \sum_\alpha C_\sigma(\alpha)e^{-i(2\pi/N)\lambda\wedge\alpha} \hat{T}_\lambda = \tilde{C}_\sigma(\lambda) \hat{T}_\lambda \quad (22)$$

where $\wedge$ is the wedge product $\lambda \wedge \mu = \mu p - \nu q$, with $\lambda = (\mu, \nu)$ and $\alpha = (q, p)$. Then the eigenfunctions of $\mathbf{D}_\sigma$ are the translation operators $\hat{T}_\lambda$ and the corresponding eigenvalues are given by $\tilde{C}_\sigma(\lambda)$, the DFT of $C_\sigma(\alpha)$. As $\mathbf{D}_\sigma$ is diagonal in the chord representation [17] its action is quite simple. If $\hat{\rho}$ is expanded as

$$\hat{\rho} = \frac{1}{\sqrt{N}} \sum_\lambda \rho_\lambda \hat{T}_\lambda \quad (23)$$

then

$$\mathbf{D}_\sigma(\hat{\rho}) = \frac{1}{\sqrt{N}} \sum_\lambda \tilde{C}_\sigma(\lambda) \rho_\lambda \hat{T}_\lambda. \quad (24)$$

In the chord representation the action of $\mathbf{D}_\sigma$ is simply to modulate the elements $\rho_\lambda$ with $\tilde{C}_\sigma(\lambda)$.

It is also of considerable interest to determine the spectral properties of the combined action of $\mathbf{D}_\sigma$ with a unitary step $\mathbf{U}$ as in Eq. (7). In many instances (see sections III and IV) a significant portion of the noise spectrum $C_\sigma(\lambda)$ is zero or is contained within a small boundary of zero in the complex plane, with just a few isolated eigenvalues in the annular region between it and the unit circle. This quasi-null subspace reduces the effective rank of the combined superoperator $\mathbf{D}_\sigma \circ \mathbf{U}$ and allows a very efficient calculation of the leading spectrum with diagonalizations of relatively small size (see Sec. IV A).

### III. GENERALIZED NOISE CHANNELS IN PHASE SPACE

In this section we obtain many-qubit generalizations of the depolarizing and phase damping channels and write their superoperators in terms of translations in phase space. With a slightly different approach these generalizations have been studied in [23].

In Sec. IIIA we proposed a noise superoperator which is diagonal in the chord representation and whose Kraus operators are proportional to the translation operators. The main feature of this type of noise is the (diagonal) supermatrix of coefficients $C(\alpha)$ (or equivalently the spectrum which from Eq. (22) is $\tilde{C}(\beta)$, i.e. the DFT of $C(\alpha)$). We introduce a parameter $\epsilon$ which, if we think the noise is due to a coupling to an environment, it allows to continuously change from no coupling ($\epsilon = 0$) to full-strength coupling ($\epsilon = 1$). Thus Eq. (20) can be split as a convex sum of two superoperators

$$\mathbf{S}_\epsilon = (1 - \epsilon)\hat{T}_0 \otimes \hat{T}_0 + \frac{\epsilon}{N} \sum_{\alpha=0}^{N^2-1} C(\alpha)\hat{T}_\alpha \otimes \hat{T}_\alpha^\dagger. \quad (25)$$
Hence $S_x$ is also diagonal in the chord representation. Following Eq. (22), the spectrum of $S_x$ is given by

$$\Sigma(\beta) = (1 - \epsilon) + \epsilon \tilde{C}(\beta)$$

(with $\beta \equiv (q, p)$ and $q, p = 0, \ldots, N - 1$). This means that the spectrum takes the constant value $(1 - \epsilon)$ for all $\tilde{T}_\beta$ plus an additional $\epsilon \tilde{C}(\beta)$. Moreover, $\tilde{C}(0) = 1$ for $S_x$ to be unital.

### A. Depolarizing Channel

The depolarizing channel for a single qubit, leaves it unchanged with probability $(1 - \epsilon)$ and *depolarizes* it, which means that it leaves it in a completely mixed state, with probability $\epsilon$. The Kraus operators for one-qubit depolarizing channel [7, 8] are

$$\begin{align*}
M_0 &= \sqrt{(1 - \epsilon)} \hat{I}, \\
M_1 &= \sqrt{\frac{\epsilon}{2}} \hat{\sigma}_1, \\
M_2 &= \sqrt{\frac{\epsilon}{2}} \hat{\sigma}_2, \\
M_3 &= \sqrt{\frac{\epsilon}{2}} \hat{\sigma}_3.
\end{align*}$$

The normalized operators

$$\begin{align*}
\hat{M}_0 &= \frac{1}{\sqrt{N}} \hat{I}, \\
\hat{M}_1 &= \frac{\epsilon}{\sqrt{2N}} \hat{\sigma}_1, \\
\hat{M}_2 &= \frac{\epsilon}{\sqrt{2N}} \hat{\sigma}_2, \\
\hat{M}_3 &= \frac{\epsilon}{\sqrt{2N}} \hat{\sigma}_3.
\end{align*}$$

If not stated explicitly matrices are written in the computational basis $\{|0\rangle, |1\rangle\}$. Notice that each $\hat{M}_\mu$ is a constant multiplied by the corresponding Pauli matrix $\hat{\sigma}_\mu$ (with $\hat{\sigma}_0 \equiv \hat{I}$), which are the generators of the $SU(2)$ group and constitute an orthonormal basis of $\mathcal{H}_2$. For $K$ qubits a straightforward generalization of Eq. (27) can be constructed with the Kraus operators given by

$$\begin{align*}
\hat{M}_0 &= \sqrt{(1 - \epsilon)} \hat{I}, \\
\hat{M}_\mu &= \sqrt{\frac{N}{2(N^2 - 1)}} \hat{\sigma}_\mu,
\end{align*}$$

where $\gamma_\mu$ is the $\mu$th element of the set of $N^2 - 1$ generators of $SU(N)$. They are hermitian operators which satisfy

$$\begin{align*}
\text{Tr}(\gamma_\mu) &= 0 \quad (29a) \\
\text{Tr}(\gamma_\mu \gamma_\nu) &= 2\delta_{\mu\nu}. \quad (29b)
\end{align*}$$

The normalized operators

$$\begin{align*}
\hat{Q}_0 &= \frac{1}{\sqrt{N}} \hat{I} \\
\hat{Q}_\mu &= \frac{1}{\sqrt{2}} \gamma_\mu
\end{align*}$$

form a complete basis in $\mathcal{H}_{N^2}$. We propose for the generalized depolarizing channel the following expression

$$S^{DC}_\epsilon = (1 - \epsilon) \hat{I} \odot \hat{I} + \frac{\epsilon}{N} \sum_{\mu=0}^{N^2-1} \hat{Q}_\mu \odot \hat{Q}_\mu$$

$$\equiv (1 - \epsilon) \hat{I} \odot \hat{I} + \epsilon S_2$$

Using the fact that the set $\{\hat{Q}_\mu\}_{\mu \neq 0}$ spans the same subspace of traces of operators as $\{\hat{T}_{\alpha}/\sqrt{N}\}_{\alpha \neq 0}$, it can be seen that $S_2$ in Eq. (31) is trace preserving.

The computational basis, which we arbitrarily took to be the position basis in discrete phase space, defines a “canonical” orthonormal basis in Liouville space $\mathcal{H}_{N^2}$. The elements are the skew projectors (transition operators)

$$\hat{P}_{ij} = |i\rangle \langle j|$$

with $i, j = 0, \ldots, N - 1$. Using Eq. (10) it is easy to see that

$$\hat{P}_{ij} = \frac{1}{N} \sum_{q,p=0}^{N-1} \text{Tr} \left[ \hat{T}_{(q,p)} \hat{P}_{ij} \hat{T}_{(q,p)}^+ \right]$$

$$= \frac{1}{N} \sum_{p=0}^{N-1} e^{-i\hat{p}(i-j)} \hat{T}_{(i-j,p)}.$$

where $1/\sqrt{N}$ is added for normalization. For clarity, we used the two indices of $\hat{T}_{(q,p)}$ explicitly.

Now, the generators $\gamma_\mu$ can be written in the computational basis in terms of skew projectors as (see, for example, [26])

$$\gamma_\mu \rightarrow \{\hat{U}_{12}, \hat{U}_{13}, \hat{U}_{23}, \ldots, \hat{V}_{12}, \hat{V}_{13}, \hat{V}_{23},$$

$$\ldots, \hat{W}_1, \hat{W}_2, \ldots, \hat{W}_{N-1}\}$$

with

$$\begin{align*}
\hat{U}_{jk} &= \hat{P}_{jk} + \hat{P}_{kj}, \\
\hat{V}_{jk} &= i(\hat{P}_{jk} - \hat{P}_{kj}), \\
\hat{W}_l &= -\sqrt{\frac{2}{l(l+1)}} (\hat{P}_{11} + \hat{P}_{22} + \cdots + \hat{P}_{ll} - l\hat{P}_{l+1,l+1})
\end{align*}$$

where $1 \leq j < k \leq N, 1 \leq l \leq N - 1$. Inserting Eqs. (33) and (35) into Eq. (31) it can be transformed into

$$S^{DC}_\epsilon = (1 - \epsilon) \hat{T}_0 \odot \hat{T}_0^+ + \frac{\epsilon}{N^2} \sum_{\alpha=0}^{N^2-1} \hat{T}_\alpha \odot \hat{T}_\alpha^+.$$
The physical interpretation is quite simple from a phase space point of view. With probability $(1 - \epsilon)$ it leaves the state unchanged while with uniform probability $\epsilon / N^2$ it performs all the possible translations, thus averaging over all phase space with equal weight (except at the origin). In FIG. 1 the action of $S^{DC}$ on a superposition of two coherent states (represented by the Wigner function) is shown, with $\epsilon = 0.9$. The value of $\epsilon$ is taken purposely large to make the effects of the averaging become apparent. Eventually, further averaging over the whole phase space leads to a completely depolarized state, i.e., the uniform state $\hat{I}/N$.

Equation (26) gives the spectrum

$$\Sigma^{DC}(\beta) = (1 - \epsilon) + \epsilon \sum_{\alpha} \frac{1}{N^2} e^{i(2\pi/N)\beta \wedge \alpha}$$

$$= (1 - \epsilon) + \epsilon \delta_{\beta,0}. \quad (37)$$

The spectrum takes the constant value $(1 - \epsilon)$ for all $\hat{T}_{(q,p)}$ except for $\hat{T}_{(0,0)}$ where it is equal to 1 (FIG. 2 left). The $(N^2 - 1)$-degeneracy makes the composition of this channel with any other superoperator trivial. The effect is to contract all the spectrum, except for a single 1, by a factor $(1 - \epsilon)$.

When $\epsilon \sim 1$ the spectrum lies within a small boundary of the origin leaving 1 as the unique non-zero eigenvalue. Physically this means that after one step all the modes that are orthogonal to the identity decay. Thus the only preferred state in this limit case is the uniform density $\hat{I}/N$.

### B. Phase-Damping

The phase-damping channel is widely used in the contexts of quantum to classical correspondence and quantum information because it provides a simple picture of how decoherence acts picking out preferred sets of states [7]. The Kraus operators for the one-qubit phase damping channel are

$$M_0 = \sqrt{(1 - \epsilon)\hat{I}}, \quad M_1 = \sqrt{\epsilon}\hat{0}\hat{0}, \quad M_2 = \sqrt{\epsilon}\hat{1}\hat{1}. \quad (38)$$

It is instructive to see what happens after repeated action of the phase-damping channel. It is easy to check that after $n$ steps, the initial density matrix $\rho_0$ is

$$(S^{PDC}_n)^n(\rho_0) = \left( \begin{array}{cc} \rho_{00} & (1 - \epsilon)^n \rho_{01} \\ (1 - \epsilon)^n \rho_{10} & \rho_{11} \end{array} \right). \quad (39)$$

So if $\epsilon$ represents a decay rate, exponential decay of the non-diagonal terms occurs leaving the state in a completely mixed state. Therefore decoherence picks out the computational states as preferred basis. Unlike the depolarizing channel, phase damping only produces loss of coherence.

We propose a generalized $S^{PDC}$ for $K$ qubits in terms of the skew projectors $\hat{P}_{ij}$

$$S^{PDC}_\epsilon = (1 - \epsilon)\hat{I} \otimes \hat{I} + \epsilon \sum_{i,j=0}^{N-1} C_{ij}\hat{P}_{ij} \otimes \hat{P}_{ij}. \quad (40)$$

To identify $\sqrt{\epsilon C_{ij}\hat{P}_{ij}}$ with Kraus operators then the coefficients $C_{ij}$ must be real and positive. Moreover, in order to be trace preserving the identity $\sum_{i,j} C_{ij} = 1$ must hold. Using the same arguments of Sec. II A an expression in terms of translation operators can be obtained. For example, if $C_{ij} = \delta_{ij}$ we get

$$S^{PDC}_\epsilon = (1 - \epsilon)\hat{I} \otimes \hat{I} + \frac{\epsilon}{N} \sum_{p=0}^{N-1} \hat{T}(0,p) \otimes \hat{T}^\dagger(0,p). \quad (41)$$

The physical interpretation follows intuitively. With probability $(1 - \epsilon)$ it leaves the state unchanged, while it averages in a preferred direction (in this case the vertical lines $q = \text{const}$, $p \in [0, N - 1]$). The channel picks out as pointer basis the position projectors $\hat{P}_i$. This is not a surprise since we (arbitrarily) took as computational basis, the position eigenstates $|q\rangle$. Had we chosen the momentum basis instead, then the averaging over lines $p = \text{const}$ would lead to momentum as pointer basis. Following this train of thought we show that different pointer bases can be selected by “diffusing” along any line in the phase space grid. We define a line on the $N \times N$ grid $G_N$ of points $(q, p)$ in phase space as

$$L_{n_1,n_2,n_3} = \{(q, p) \in G_N; n_1 p = n_2 q + n_3, n_1, n_2, n_3 \in \mathbb{N}\}. \quad (42)$$

The generalized phase damping channel on the line $L_{n_1,n_2,n_3}$ is given by

$$S^{PDC}_{n_1,n_2,n_3} = (1 - \epsilon)\hat{T}(0,0) \otimes \hat{T}^\dagger(0,0)$$

$$+ \frac{\epsilon}{R} \sum_{(q,p) \in L_{n_1,n_2,n_3}} \hat{T}(q,p) \otimes \hat{T}^\dagger(q,p).$$

$$= (1 - \epsilon)\hat{I} \otimes \hat{I} + \epsilon D_{n_1,n_2,n_3}, \quad (43)$$

where $D_{n_1,n_2,n_3}$ is a superoperator like (33). The integer $R$ is the number of points contained in the line. If either $n_1$ or $n_2$ is odd then $R \equiv N$. On the other hand, it is an integer power of 2 times $N$ if both $n_1$ and $n_2$ are even $[6]$. The eigenvalues corresponding to each translation $\hat{T}(q,p)$ follow from the analysis in Sects. IIE and III

$$\Sigma^{PDC}(q,p) = 1 - \epsilon \left(1 - e^{-i(2\pi/N)q}n_3/n_2\delta_{n_2q,n_1p}\right) \quad (44)$$

for $n_1 \neq 0$, while if $n_1 = 0$ and $n_2 \neq 0$ it is

$$\Sigma^{PDC}(q,p) = 1 - \epsilon \left(1 - e^{-i(2\pi/N)p}n_3/n_2\delta_{n_2q,n_0}\right). \quad (45)$$

This spectrum is displayed in FIG. 2(b) and (c).

In FIG. 3 the effect of $S^{PDC}_{n_1,n_2,n_3}$ acting on a superposition of coherent states can be seen, for the case $n_1 = 1, n_2 = -1$ and $n_3 = 0$. The right panel shows how the line $(p = -q)$ that the channel picks, and averages over, appears. The only difference with a line with non-zero $n_3$ would be a vertical translation of the line represented. Moreover, there is a progressive erasure of quantum interferences between the two coherent states.

The action of this type of noise channel can be best understood if we delve deeper into the action of $D_{n_1,n_2,n_3}$. Using
Figure 2: Complex plane plot of (a) The spectrum of the generalized depolarizing channel. Notice the $N^2 - 1$ degeneracy on $(1 - \epsilon, 0)$; (b) The spectrum for the Phase damping channel on the line with $N = 32$, $n_1 = 1$, $n_2 = 2$, $n_3 = 2$. There are $N^2 - N$ eigenvalues on $(1 - \epsilon, 0)$ and $N$ (doubly degenerate) eigenvalues on the circle of radius $\epsilon$ centered at $(1 - \epsilon, 0)$; and (c) The spectrum for the Phase damping channel on the line with $N = 32$, $n_1 = 1$, $n_2 = 0$, $n_3 = 2$. There are $N^2 - N$ eigenvalues on $(1 - \epsilon, 0)$ and $N$ eigenvalues equal to 1.

Figure 3: Wigner function representation of the action after one step of the phase damping on the line $S_{-1/2}$, with $N = 64$ and $\epsilon = 0.85$. The initial state is the same as in FIG. 1. The averaging over the line $p = -q (\mathrm{mod} \ 64)$ can be appreciated on the right panel. (Notice that the short wavelength interferences are due to the torus periodicity.)

Eq. (12) it can be written as a composition of two superoperators

$$D_{n_1 n_2 n_3} = \begin{cases} D_{n_1 n_2 0} \circ V_{n_3 / n_1}, & n_1 \neq 0 \\ D_{0 n_2 0} \circ U_{-n_3 / n_2}, & n_1 = 0 \end{cases}, \quad (46)$$

where the divisions in the exponents should be understood as the product with the inverse of an integer, modulo $N$ (whenever such inverse exists). Therefore $V_{n_3 / n_1} = V_{n_3 / n_1} \circ (\tilde{V}^\dagger)_{n_3 / n_1}$ is a unitary translation of $n_3 / n_1$ in momentum and $U_{-n_3 / n_2} = U_{-n_3 / n_2} \circ (\tilde{U}^\dagger)$ is a unitary translation of $-n_3 / n_2$ in position. In other words, first apply a translation in position or momentum and then average on the line of slope $n_2 / n_1$ that contains the origin.

This superoperator has interesting features when $\epsilon \sim 1$. Equations (44) and (45) show that there is an $N^2 - R$ dimensional null subspace. The remaining non-zero eigenvalues are on the unit circle and, depending on the parity of $n_1$, $n_2$ and $n_3$, may exhibit double (or some integer power of 2) degeneracy. The effect is that the averaging superoperator $D_{n_1 n_2 0}$ truncates the unitray superoperator $V_{n_3 / n_1}$ to a smaller submatrix which is still unitary (the same applies to $D_{0 n_2 0}$ composed with $U_{-n_3 / n_2}$).

When this noise channel acts on a unitary map (with $\epsilon \sim 1$) the full propagator is truncated to an $R$-dimensional subspace spanned by translations on the chosen line.

IV. DIFFUSIVE NOISE

Although the effect of the previous two types of noise is clear in the context of many-qubit quantum algorithms, it is not intuitive to picture them as diffusion in phase space, because they average over large distances (namely the whole space). On the other hand, although a purely diffusive noise is difficult to “see” acting on a system of qubits, it has a natural interpretation in phase space. However, in spite of this differences, they can be treated as different types (depending on a function $C(\alpha)$) of the same class of noise given in Eq. (25). Following recent works [3, 10, 18] we define as diffusive noise,

$$D_\sigma = \sum_\alpha C_\sigma(\alpha) \hat{T}_\alpha \circ \hat{T}_\alpha^{\dagger} \quad (47)$$

which is a particular case of Eq. (25) with $\epsilon = 1$ and where $C_\sigma(\alpha)$ is a periodic function narrowly peaked around $\alpha = 0$ and of approximate width $\sigma$. For practical purposes we take $C_\sigma(\alpha)$ to be a periodized Gaussian of half-width $\sigma$ and thus $C_\sigma(\lambda)$ is also a Gaussian of half-width $1/(2\pi\sigma N)$ (FIG. 4) where $\lambda \equiv (\mu, \nu)$). This implements an incoherent sum of all the possible translations in phase space weighed by $C_\sigma(\alpha)$. To avoid a net drift $C_\sigma(\alpha)$ should satisfy

$$C_\sigma(\alpha) = C_\sigma(-\alpha) \quad (48)$$

which makes $D_\sigma$ a Hermitian operator.

In the context of quantum optics a similar type of noise for continuous phase space variables has been called Gaussian noise [27].
A. Spectral properties of the noisy propagator

In the examples of Sec. III the effect of noise acting on the unitary propagation was very simple. On the other hand, in the case of diffusive noise the numerical calculation is more involved and its nature depends strongly on the properties of the unitary propagator, for chaotic systems, has gained interest because in the limits of the spectrum of the noisy propagator, for chaotic systems, has the unitary map. In the context of quantum classical transition involved and its nature depends strongly on the properties of the case of diffusive noise the numerical calculation is more complex. On the other hand, in classically chaotic systems with noise. Experimentally the appearance of RP resonances in quantum systems has been shown, for example, in [19]. Clearly the part of the spectrum that can be related to long time decays is the largest in modulus (and smaller than 1).

The chord representation is specially suited for these calculations because since \( C_\sigma(\alpha) \) is a Gaussian of half-width \( \sigma \) then the spectrum \( C_\sigma(\lambda) \) is also a Gaussian of complementary half-width \( 1/(2\pi\sigma) \) which means that for some ranges of \( \sigma N \) there is a large part of the spectrum of \( \mathbf{D}_\sigma \) very close to zero.

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The large \( (N^2 \times N^2) \) supermatrix of the unitary step in the chord representation is

\[
\mathcal{U}_{\alpha,\alpha'} = \text{Tr} \left[ \hat{T}_\alpha \hat{U} \hat{T}_\alpha \hat{U}^\dagger \right]
\]

Since the noise superoperator is diagonal in this representation its application is particularly simple and is given by

\[
\mathcal{L}_{\lambda,\lambda'} = \tilde{C}_\sigma(\lambda') \mathcal{U}_{\lambda,\lambda'}
\]

Thus for \( \lambda' \gg 1/(\sigma N) \) the matrix elements of Eq. (50) are negligibly small. Therefore \( \tilde{C}_\sigma(\lambda) \) tends to kill the long chord components of the unitary step. A schematic representation of \( \tilde{C}_\sigma(\lambda) \) is found in FIG. 4. The grid represents points in phase space. The whiter region gives the idea of which matrix elements can be neglected. The number of points inside the darker region represents the rank of the noisy superoperator reduced by the diffusion.

In the numerical treatment we pick a threshold of values for \( \tilde{C}_\sigma(\lambda) \). To do so we determine a range which is a coefficient \( a \) times the width of \( C_\sigma(\lambda) \). Outside this range the quai-null spectrum is set exactly equal to zero. The resulting matrix is of a size of order \( 4(\alpha/(2\pi\sigma N))^2 \) (see FIG. 4). This procedure provides, up to a reasonable accuracy, the largest part of the spectrum of \( \mathbf{L} \). The calculation becomes more difficult as \( \sigma \to 0 \) and to reobtain the unitary spectrum the full dimension \( N^2 \times N^2 \) is needed. For chaotic maps the spectrum is composed of the eigenvalue 1, corresponding to \( \rho_0 = \mathbf{I}/N \), a small set of eigenvalues with modulus smaller than one, and many eigenvalues close to zero. The action of the noise is to set the latter ones exactly equal to zero.

In FIG. 5 we see computation of the spectrum for three different sizes of truncation for a quantum map with diffusion. The leading part of the spectrum is unchanged while their appear slight in the less significant part. The stability of the spectrum up to orders of \( 10^{-3} \) can be clearly observed.

Many different schemes have been used to calculate the leading spectrum of the noisy propagator. The advantage over these methods is that although they use smaller matrices, the number of eigenvalues obtained is limited by accuracy and difficult error estimation.

In [19] there is an analogous analysis for diffusive noise on a spherical phase space.

V. CONCLUSIONS

We provided a generalization of two well known noise channels in the context of phase space representations of quantum mechanics. Specifically we showed how the depolarizing and phase damping channels have superoperator expressions that are very simple to study using the chord representation. Moreover their spectral features, which can be crucial when composed with a map or an arbitrary algorithm are well determined. Some useful properties of these generalized noise channels composed with quantum maps and algorithms are currently being studied.

The noises where formulated as a one parameter (\( \epsilon \)) family and their spectral properties where studied. In particular when the spectrum of the noise has a large null eigensubspace the convolution with a unitary map results in a truncated matrix whose effective rank depends on the rank of the noise. Thus reducing the computational requirements for systems with a large Hilbert space. This was shown to be useful when computing the leading spectrum of coarse grained propagators corresponding to classically chaotic quantum maps. The coarse graining was obtained by means of a diffusive noise and we showed that the leading spectrum is independent of the truncation.
Figure 5: Leading spectrum of $L = D \sigma U$, where $D \sigma$ is the diffusive noise of Sec. IV (with $\sigma = 0.063$) and $U$ is the unitary superoperator for the quantum perturbed Arnold cat used in [9,10] (with $N = 100$, $k = 0.02$). The dimension of the propagator after the truncation is $\dim = 4(\frac{a}{(2\pi \sigma)})^2$. (a) $a = 2$, $\dim = 4(2/2\pi \sigma)^2 \approx 100$; (b) $a = 2.8$ and $\dim = 4(2.8/2\pi \sigma)^2 \approx 196$; (c) $a = 4.8$ and $\dim = 4(4.8/2\pi \sigma)^2 \approx 576$. (Note that $N^2 \times N^2 = 10000$.) If $\lambda_i$ is the $i$-th eigenvalue, then $\log \lambda_i = \log(r_i) + i\phi_i$ (where $r_i = |\lambda_i|$) and the coordinates in the plots are $(\phi, -\log(r))$.

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[1] L. Davidovich, in Latin American School of Physics XXXI ELAF, edited by Shahen Hacyan, Rocío Jáuregui, and Ramón López-Peña, AIP Conf. Proc. 464 (AIP, Woodbury, NY, 1999); Proceedings of the first PASI Conference on Chaos, Decoherence and Entanglement 2000, available at http://kaiken.df.uba.ar;
[2] See, for example, W. H. Zurek, Phys. Today 44 (10), 36 (1991); J. P. Paz, S. Habib, and W. H. Zurek, Phys. Rev. D 47, 488 (1992);
[3] J. P. Paz and W. H. Zurek, in Coherent Matter Waves, Proceedings of the Les Houches Session LXII, edited by R. Kaiser, C. Westbrook, and F. David (Springer Verlag, Berlin, 2001);
[4] C. Miquel, J. P. Paz and M. Saraceno, Phys. Rev. A, 65, 2309 (2002);
[5] W. H. Zurek, Nature 412, 712 (2001);
[6] A.M. Ozorio de Almeida Phys. Rep. 295 266 (1998); A. Rivas and A. M. Ozorio de Almeida, Ann. Phys. (N.Y.) 276, 223 (1999);
[7] J. Preskill, “1998 Lecture Notes for Physics 229: Quantum Information and Computation” available at http://www.theory.caltech.edu/people/preskill;
[8] I. Chuang and M. Nielsen, Quantum Information and Computation (Cambridge University Press, Cambridge, UK, 2001);
[9] I. García-Mata, M. Saraceno, and M. E. Spina, Phys. Rev. Lett. 91, 064101 (2003);
[10] I. García-Mata and M. Saraceno, 69, 056211 (2004);
[11] G. Lindblad, Commun. Math Phys. 48, 119 (1976);
[12] G. W. Gardiner and P. Zoller, “Quantum Noise: A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods with Applications to Quantum Optics”, Springer-Verlag, Berlin-Heidelberg 2000; B. Kümmerer. Quantum Markov Processes in “Coherent Evolution in Noisy Environments”, A. Buchleitner and K. Hornberger (Eds.), Springer-Verlag, Berlin-Heidelberg (2002).
[13] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, J. Math. Phys. 17, 821 (1976).
[14] K. Kraus, States, Effects and Operations, Springer-Verlag, Berlin, 1983.
[15] See T. F. Havel, J. Math. Phys. 44, 534 (2003), and references therein.
[16] G. Palla, G. Vattay and Andre Voros, Phys. Rev. E 64, 012104 (2001).
[17] D. Braun, CHAOS, 9,730 (1999) and D. Braun, Physica D, 131, 265 (1999); D. Braun, “Dissipative Quantum Chaos and Decoherence”, Springer-Verlag, Berlin-Heidelberg (2001).
[18] S. Nonnenmacher, Nonlinearity 16, pp.1685-1713 (2003).
[19] M. E. Spina and M. Saraceno, e-print[nlin.CD/0408001]
[20] C. Miquel, J.P. Paz, M. Saraceno, E. Knill, R. Laflamme, and C. Negrevergne, Nature 418, 59 (2002).
[21] J. P. Paz, A. J. Roncaglia, and M. Saraceno, Phys. Rev. A 69, 032312 (2004).
[22] J. Schwinger, Proc. Nat. Acad. Sci. 46, 570 (1960).
[23] J. H. Hannay and M. V. Berry, Physica 1D,267 (1980).
[24] E. Knill, Los Alamos National Laboratory Report LAUR-96-2717 and Los Alamos National Laboratory Report LAUR-96-2807
[25] G. G. Carlo, G. Benenti, G. Casati, and C. Mejía-Monasterio Phys. Rev. A 69, 062317 (2004).
[26] G. Mahler and V. A. Weberuß, *Quantum Networks: Dynamics of Open Nanostructures*. (Springer Verlag, Berlin, 1998).
[27] M. J. Hall, Phys. Rev. A 50, 3295 (1994).
[28] M. Blank, G. Keller, and C. Liverani, Nonlinearity 15, pp.1905-1973 (2002).
[29] D. Ruelle, Phys. Rev. Lett. 56, 405 (1986); D. Ruelle, J. Stat. Phys 44, 281 (1986).
[30] K. Pance, W. T. Lu and S. Sridhar, Phys. Rev. Lett. 85, 2737 (2000).
[31] G. Blum and O. Agam, Phys. Rev. E 62, 1977 (2000).
[32] R. Florido, J. M. Martín-González and J. M. Gómez Llorente, Phys. Rev. E 66, 046208 (2002).
[33] C. Manderfeld, J. Weber, and F. Haake, J. Phys. A 34, 9893 (2001).
[34] M. L. Aolita and M. Saraceno, in preparation.