Three Introductory Lectures in Helsinki on Matrix Models of Superstrings

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Abstract

These are short notes of three introductory lectures on recently proposed matrix models of Superstrings and M theory given at 5th Nordic Meeting on Supersymmetric Field and String Theories in Helsinki (March 10–12, 1997).

Contents:

1. M(atrix) theory of BFSS,
2. From IIA to IIB with IKKT,
3. The NBI matrix model.

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Preface

These lecture notes are based on three lectures at 5th Nordic Meeting on Supersymmetric Field and String Theories in Helsinki (March 10–12, 1997).

The request from the Organizers was to make the lectures understandable for graduate students. For this reason the literal title of the transparencies was

A guide (for graduate students)
on how to read (and write)papers on hep-th onMatrix Models of Superstrings

and the presentation was along this line.

The main goal of the lectures was to introduce the audience into a fast developing subject of application of matrix models in Superstring Theory. The knowledge of superstrings is assumed at the level of about first nine chapters of the book by Green, Schwartz and Witten. A fascinating subject of string dualities and, correspondingly, applications to M theory is practically left outside for this reason. No preliminary knowledge of matrix models is assumed. The proper terminology, which is clarified in the lectures, is listed in the next page.

Each of the three lectures is mostly concentrated around one of the three selected papers [1, 2, 3]. The references in the text are only to the results quoted. More complete list of references can be found clicking a mouse on the number of citations to the pioneering paper of Banks, Fischler, Shenker and Susskind [4] in HEP database at SLAC.
Dictionary of the Language

BPS, 6, 18
branes
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  IIB, 12
supersymmetry
  N=2, 13, 14
target-space, 4
super Yang-Mills, 12, 14
ten-dimensional, 5
Witten, 7
Introduction

The standard non-perturbative approach to bosonic (Polyakov) string, which is based on discretized random surfaces and matrix models, exists since the middle of the eighties \cite{4,5,6}. The main result of the investigations (both analytical and numerical) within this approach is that the bosonic string is not in the stringy phase but rather in a branched polymer phase when the dimension $D$ of the embedding space is larger than one\cite{1}. This is the way how the tachyonic problem is resolved for $D > 1$. In other words the perturbative vacuum with the tachyon is unstable and the system chooses a stable vacuum which is not associated with strings.

A question immediately arises what about superstrings where the GSO-projection kills tachyons (at least perturbatively). This is a strong argument supporting the expectation for superstrings to live in a stringy phase, which agrees with the common belief that fermions smooth out the dynamics.

The attempts (not quite successful until very recent time) of discretizing superstrings are performed starting from \cite{8}. The problem resides, roughly speaking, in the fact that is not easy to discretize the target-space supersymmetry (SUSY). A progress had been achieved only for the simplest case of pure two dimensional supergravity which can be associated with a supereigenvalue model \cite{9}. For a more detail review, see Ref. \cite{7}.

The dramatic recent progress in a non-perturbative formulation of superstrings by supersymmetric matrix models, which has occurred during last few months, is the subject of these lecture notes. I shall mostly concentrate on ten dimensional superstrings practically leaving outside presumably most interesting question of constructing the fundamental Lagrangian of eleven dimensional M theory in the language of matrix models.

\footnote{For a review see Ref. \cite{7}.}
1 M(atrix) theory of BFSS [1]

Eleven dimensional M theory combines different ten dimensional superstring theories (IIA, IIB, . . . ), which are in fact related by duality transformations, into a single fundamental theory. BFSS proposed [1] to describe it by a supersymmetric matrix quantum mechanics in the limit of infinite matrices. This construction is called M(atrix) theory.

1.1 The set up

The point of interest of Ref. [1] is $D = 10 + 1$ dimensional M theory (characterized by its Planck’s length, $l_p$). The eleven coordinates

$$x^\mu = \left( t, x^i, x^{11} \right) \quad (i = 1, \ldots, 9)$$

are split into time, $t$, the nine transverse ones, $x^i$ or $x^\perp$, and the longitudinal one, called $x^{11}$, which is compactified:

$$x^{11} = x^{11} + 2\pi R.$$

The radius of compactification $R$ plays the role of an infrared cutoff in the theory.

The system is considered in the Infinite Momentum Frame (IMF), which is the same as the light cone frame, boosting along the longitudinal axis. The same notations $t$ and $x^{11}$ are used for $t \pm x^{11}$. The advantage of using IMF is that only positive momenta $p_{11}$ are essential while systems with zero or negative $p_{11}$ do not appear as independent dynamical degrees of freedom. A price for this is the absence of manifest Lorentz invariance.

Due to compactness all systems have (positive) longitudinal momentum

$$p_{11} = \frac{N}{R} \quad (N > 0),$$

where $N > 0$ is integer. At the end of calculations $R$ and $N/R$ should tend to infinity,

$$R \to \infty, \quad \frac{N}{R} \to \infty,$$

(1.2)

to get uncompactified infinite momentum limit of 11D theory. $N$ will be identified in what follows with the Ramond-Ramond (RR) charge of the system.

1.2 10D versus 11D language

M theory with compactified $x^{11}$ is by construction type IIA superstring in $D = 9 + 1$ dimensions. The parameters $R$ and $l_p$ of eleven dimensional M theory and those $g_s$ and $l_s$ of the ten dimensional superstring are related by

$$R = g_s^{2/3} l_p, \quad l_s = g_s^{-1/3} l_p,$$

(1.3)
where $g_s$ is the *string coupling constant* and $l_s \equiv \sqrt{\alpha'}$ is the string length scale related to the *string tension* $T$ by

$$T = \frac{1}{2\pi \alpha'}.$$  \hfill (1.4)

The 11D M theory is in turn a strong coupling limit of 10D IIA superstring, since (1.2) is guaranteed as $g_s \rightarrow \infty$.

No perturbative string states carry RR charge $\Rightarrow$ they are associated with vanishing momentum $p_{11}$. 1 unit of RR charge is carried by D0-brane of Polchinski \cite{10} for which

$$p_{11} = \frac{1}{R},$$  \hfill (1.5)

in accord with Eq. (1.1) at $N = 1$.

The *low-energy* limit of M theory is 11D supergravity having 256 massless states: 44 gravitons, 84 three-forms and 128 gravitinos. These 256 states are referred to as supergravitons which are massless as 11D objects $\Rightarrow$ they are Bogomolny–Prasad–Sommerfield (BPS) saturated states in 10D theory. Their 10D mass $\sim 1/R$.

States with $N \neq 1$ are not associated with elementary D0-branes. The states with $N > 1$ are bound composites of $N$ D0-branes as is discussed in the next subsection.

### 1.3 The appearance of matrices

The world-volume of a $p$-brane is parametrized by $p + 1$ coordinates $\xi_0, \ldots, \xi_p$. The $p$-branes emerge as classical solutions in 10D supergravities\cite{4} which describe low-energy limits of 10D superstrings. They possess an intrinsic abelian gauge field $A_\alpha(\xi)$ ($\alpha = 0, \ldots, p$) which can be viewed as tangent (to $p$-brane) components of 10D abelian gauge field reduced to $p$-brane. Otherwise, the remaining $9 - p$ components of the 10D abelian gauge field, which are orthogonal to the $p$-brane, are associated with its coordinates \cite{4}

$$X_i(\xi) = 2\pi \alpha' A_i(\xi) \quad (i = p + 1, \ldots, 9).$$  \hfill (1.6)

A D(irichlet) $p$-brane can emit a fundamental open string which has the Dirichlet boundary condition on a $p + 1$ dimensional hyperplane and the Neumann boundary condition in the $9 - p$ dimensional bulk of space. This string can end either on the same D$p$-brane or on another one as is illustrated by Fig. 1.

If one has $N$ parallel D$p$-branes separated by some distances in the $9 - p$ dimensional space, then massless vector states emerge only when the string begins and ends at the same brane, so the gauge group $U(1)^N$ appears in a natural way. Since the energy of strings stretched between different D-branes is

$$M \sim T |X^i - X^j|,$$  \hfill (1.7)

\footnote{For a review, see Ref. \cite{11}.}
more massless vector states appear when the branes are practically on the top of each other. Since the string is oriented, all possible massless states when the string begins and ends either on same or different Dp-branes form a U(\(N\)) multiplet when strings are very short. The example of \(N = 2\) is illustrated by Fig. 2. This is how hermitian \(N \times N\) matrices appear in the description of bound composites of \(N\) Dp branes according to Witten [12].

For our case of \(N\) D0-branes, their coordinates \(X_i(t)\) become 9 Hermitean \(N \times N\) matrices \(X_i^{ab}(t)\) (accompanied by the fermionic superpartners \(\theta_{\alpha}^{ab}(t)\) which are 16 component nine dimensional spinors). They can be thought as spatial components of the vector field in ten dimensional super Yang–Mills theory after reduction to zero space dimension (same for the superpartners). \(N\) is associated with the value of the RR charge of these states.

1.4 The fundamental Lagrangian

The possibility of formulating the fundamental Lagrangian of M theory as a matrix model is formulated in Ref. [1] as the

**Conjecture:** M theory in IMF is a theory with the only dynamical degrees of freedom of D0-branes.

In other words *all* systems are composed of D0-branes. Therefore, the fundamental Lagrangian of M theory is completely expressed via the hermitian \(N \times N\) matrices \(X_i^{ab}(t)\) describing coordinates of D0-branes (and their fermionic superpartners \(\theta_{\alpha}^{ab}(t)\)), so that

\[
M \text{ theory} = M(\text{atrix}) \text{ theory}
\]
Figure 2: Appearance of matrices in the example of bound states of two parallel D-branes ($N = 2$). The fundamental string can begin and end either at the same or different D-branes. Since the string is oriented, there are four massless vector states when the branes are practically on the top of each other. They form a representation of U(2).

M(atrix) theory is described (in units of $l_s = 1$) by the Lagrangian

$$L = \frac{1}{2g_s} \text{tr} \left( \dot{X}^i \dot{X}^i + 2\theta^T \dot{\theta} - \frac{1}{2} [X^i, X^j]^2 - 2\theta^T \gamma_i [\theta, X^i] \right).$$

(1.8)

Here $N \to \infty$ in order to satisfy (1.2).

Changing the units to those where eleven dimensional $l_p = 1$ and introducing

$$Y = X/g_s^{1/3},$$

Eq. (1.8) can be rewritten as

$$L = \text{tr} \left( \frac{1}{2R} D_t Y^i D_t Y^i - \frac{1}{4} R [Y^i, Y^j]^2 - \theta^T D_t \theta - R \theta^T \gamma_i [\theta, Y^i] \right),$$

(1.9)

where

$$D_t = \partial_t + iA_0$$

(1.10)

is the covariant derivative with respect to the $A_0$ field. Equation (1.8) is written in the $A_0 = 0$ gauge.

The Lagrangian (1.9) is invariant under two SUSY transformations

$$\delta_{\text{SUSY}} X^i = -2\epsilon^T \gamma^i \theta,$$

(1.11)
\[ \delta_{\text{SUSY}} \theta = \frac{1}{2} \left( D_t X^i \gamma_i + \gamma_- + \frac{1}{2} [X^i, X^j] \gamma_{ij} \right) \epsilon + \epsilon', \]  
\[ \delta A_0 = -2 \epsilon^T \theta, \]  
(1.12)  
(1.13)

where \( \epsilon \) and \( \epsilon' \) are two independent 16 component (\( t \)-independent) parameters. It is seen from this formula that \( A_0 \) is needed to close the SUSY algebra.

### 1.5 Matrix quantum mechanics

The *Hamiltonian* which is associated with the Lagrangian (1.9) reads

\[ H = R \text{tr} \left\{ \frac{\Pi_i \Pi_i}{2} + \frac{1}{4} [Y^i, Y^j]^2 + \theta^T \gamma_i [\theta, Y^i] \right\}, \]  
(1.14)

where \( \Pi_i \) is the canonical conjugate to \( Y^i \). As is usual for fermions, a half of \( \theta^a_{ab} \) plays the role of coordinates and the other half plays the role of canonical conjugate momenta in the language of 1st quantization.

All finite energy states of the 10D Hamiltonian (1.14) acquire infinite energy as \( R \to \infty \), *i.e.* in the uncompactified 11D limit. Only the states whose energy \( \sim 1/N \) as \( N \to \infty \) yield

\[ H \sim \frac{R}{N} = \frac{1}{p_{11}}, \]  
(1.15)

as is expected since \( p^2 = 2 E p_{11} - p_{\perp}^2 \) in 11D IMF, so that

\[ p^2 = 0 \text{ (in 11D)} \implies E = \frac{p_{\perp}^2}{2p_{11}} \text{ (in 10D)} \]  
(1.16)

in 10D.

The simplest states of the Hamiltonian (1.14) is when the matrices \( Y^i \) are diagonal with only one nonvanishing diagonal component and all \( \theta \)'s equal zero. For nonvanishing \( p_{\perp} \) — the eigenvalue of \( \Pi_{\perp} \) — Eq. (1.14) yields

\[ E = \frac{R}{2} p_{\perp}^2 = \frac{p_{\perp}^2}{2p_{11}} \]  
(1.17)

since the commutators vanish. Thus we get Eq. (1.3) with \( N = 1 \) and this state corresponds to a single D0-brane in 10D language.

Each of these states is accompanied by the fermionic superpartners and they form a representation of the algebra of 16 \( \theta \)'s with

\[ 2^{16/2} = 2^8 = 256 \]

components. They are exactly 256 states of supergraviton in 11D. In the 10D language these are BPS states of the mass \( \sim 1/R \) which become massless in the uncompactified limit \( R \to \infty \).
A more general eigenstate of the Hamiltonian (1.14) has a form of the diagonal $N \times N$ matrix

$$Y_i = \begin{pmatrix}
Y_i^{(1)} \\
\vdots \\
Y_i^{(N)}
\end{pmatrix}.$$  

(1.18)

The commutator obviously vanishes in this case.

It is convenient to split the $U(N)$ group as $U(1) \otimes SU(N)$ and to associate the $U(1)$ part with the center mass coordinate

$$Y_i(cm) = \frac{1}{N} \text{tr} Y_i.$$  

(1.19)

Then

$$p_i(cm) = \text{tr} \Pi_i = \frac{N}{R} \dot{Y}_i(cm),$$  

(1.20)

and using $p_{11} = N/R$ we get the usual relation

$$\frac{1}{p_{11}} p_i(cm) = \dot{Y}_i(cm)$$  

(1.21)

between transverse velocity and momentum.

Interaction states are described in this construction by non-diagonal matrices. They correspond to scattering states of supergravitons in 11D. The interaction of supergravitons at the tree level is correctly reproduced within M(atrix) theory.

### 1.6 The relation to membranes

The Hamiltonian (1.14) of the $N \rightarrow \infty$ supersymmetric quantum mechanics looks pretty much like the one [13] for a 11D supermembrane in IMF. While there are no truly stable finite energy membranes in the decompactified limit, there exist very long lived classical membranes.

The membrane action can be derived in the Weyl basis on $\text{gl}(N)$, which is given by two unitary $N \times N$ matrices $g$ and $h$ (clock and shift operators) obeying

$$hg = \omega gh, \quad \omega = e^{\frac{2\pi i}{N}},$$  

$$h^N = 1 = g^N.$$  

(1.22) 

(1.23)

Any hermitian $N \times N$ matrix $Z$ can be expanded in this basis as

$$Z = \sum_{n,m=1}^{N} Z_{n,m} g^n h^m.$$  

(1.24)

As $N \rightarrow \infty$, we can introduce a pair of canonical variables $q$ and $p$, so that

$$g = e^{ip}, \quad h = e^{iq},$$  

$$[q, p] = \frac{2\pi i}{N}.$$  

(1.25) 

(1.26)
As usual in quantum mechanics, the last equality is possible only as $N \to \infty$. Then, we have

$$\text{tr } Z \Rightarrow N \int dp dq Z(p, q), \quad (1.27)$$

$$[X, Y] \Rightarrow \frac{i}{N} \{ \partial_q X \partial_p Y - \partial_p X \partial_q Y \}, \quad (1.28)$$

for the trace and the commutator, and finally

$$\boxed{\text{M(atrix) action} \implies \text{Supermembrane action}}$$

as $N \to \infty$.

A special comment is needed concerning the continuum spectrum of the supermembrane \[14\]. From the point of view of the M(atrix) theory, it is as a doctor ordered for describing the supergraviton scattering states. The conjecture of M(atrix) theory is that there exists a normalizable bound state at the beginning of the continuum spectrum at $p^2 = 0$.

The emergence of membranes in M(atrix) theory can be seen from the classical equations of motion

$$\left[ Y^i, \left[ Y^j, Y^i \right] \right] = 0, \quad \left[ Y^i, (\gamma_i \theta)_{\alpha} \right] = 0, \quad (1.29)$$

which are satisfied by static configurations.

An infinite membrane stretched out in the 8,9 plane is given by \[1\]

$$Y^8 = R_8 \sqrt{N} p, \quad Y^9 = R_9 \sqrt{N} q, \quad \text{all other } Y\text{'s and } \theta\text{'s} = 0, \quad (1.30)$$

where $p$ and $q$ are $N = \infty$ matrices (operators), and $R_8$ and $R_9$ are (large enough) compactification radii. Equations (1.29) are satisfied by (1.30) because

$$\left[ Y^8, Y^9 \right] = \text{c-number}, \quad (1.31)$$

The membrane in this picture is built out of infinitely many D0-branes.

The interaction between these membrane configurations has been studied \[15, 16, 17\] and compared with the superstring results.
2 From IIA to IIB with IKKT [2]

M(atrix) theory naturally describes ten dimensional IIA superstring. IKKT proposed [2] another matrix model associated with IIB superstring, which is in spirit of the Eguchi–Kawai large-N reduced ten dimensional super Yang–Mills theory. This non-perturbative formulation of IIB superstring is called the IKKT matrix model.

2.1 Preliminaries

IIB superstring differs from IIA superstring by chiralities of the fermionic superpartners. They are opposite for IIA superstring and same for IIB superstring.

As a consequence of this, Dp-branes of even \( p \) \( (p = 0, 2, 4, \ldots) \) are consistently incorporated by type IIA superstring theory while type IIB superstring is associated with Dp-branes of odd \( p \) \( (p = -1, 1, 3, 5, \ldots) \) [10]. This is due to the rank of the antisymmetric field which is odd for IIA superstring and even for IIB superstring. Correspondingly, the analog of D0-brane (associated with \( p = 0 \) in the IIA case) is D-instanton (associated with \( p = -1 \) in the IIB case) and the analog of D-membrane (associated with \( p = 2 \) in the IIA case) is D-string (associated with \( p = 1 \) in the IIB case).

In analogy with Ref. [1] where the fundamental Lagrangian is expressed in terms of D0-branes, one might expect that IIB superstring is described in terms of D-instanton variables, \( i.e. \) by the ten dimensional super Yang–Mills dimensionally reduced to a point [12].

2.2 Schild formulation of IIB superstring

The starting point in the IKKT approach is the Green–Schwartz action of type IIB superstring theory with fixed \( \kappa \)-symmetry:

\[
S_{GS} = -T \int d^2 \sigma \left\{ \sqrt{-\sigma^2} + 2i \varepsilon^{ab} \partial_a X^\mu \bar{\Psi} \gamma_\mu \partial_b \Psi \right\},
\]

(2.1)

where

\[
\sigma^{\mu\nu} = \varepsilon^{ab} \partial_a X^\mu \partial_b X^\nu,
\]

(2.2)

the vector index \( \mu \) of \( X^\mu (\sigma_1, \sigma_2) \) runs from 0 to 9 and the spinor index \( \alpha \) of \( \Psi_\alpha (\sigma_1, \sigma_2) \) runs from 1 to 32. The fermion \( \Psi \) is a Majorana–Weyl spinor in 10D which satisfies the condition \( \gamma_{11} \Psi = \Psi \), so that only 16 components effectively remain.

The action (2.1) is invariant under the \( \mathcal{N} = 2 \) SUSY transformation:

\[
\delta_{\text{SUSY}} \Psi_\alpha = \frac{1}{2 \sqrt{-\frac{2}{3} \sigma^2}} \sigma^{\mu\nu} (\gamma_{\mu\nu} \epsilon)_\alpha + \xi_\alpha,
\]

\[
\delta_{\text{SUSY}} X^\mu = 4i \bar{\epsilon} \gamma^\mu \Psi,
\]

(2.3)

whose parameters \( \epsilon \) and \( \xi \) do not depend on \( \sigma_1 \) and \( \sigma_2 \).
The action (2.1) can be rewritten in the Schild form

\[ S_{\text{Schild}} = \int d^2 \sigma \left\{ \sqrt{g} \alpha \left( \frac{1}{4} \left\{ X^\mu, X^\nu \right\}^2 - \frac{i}{2} \bar{\Psi} \gamma^\mu \left\{ X_\mu, \Psi \right\} \right) + \beta \sqrt{g} \right\}, \]  

(2.4)

where \( \sqrt{g} (\sigma_1, \sigma_2) \) is positive definite scalar density (which is considered as an independent dynamical variable) and the Poisson bracket is defined by

\[ \left\{ X, Y \right\} \equiv \frac{1}{\sqrt{g}} \varepsilon^{ab} \partial_a X \partial_b Y. \]  

(2.5)

Note that \( \sqrt{g} \) cancels in the fermionic term in the action.

The equivalence of (2.1) and (2.4) at the classical level can be proven by using the classical equation of motion for \( \sqrt{g} \). Varying the Schild action (2.4) with respect to \( \sqrt{g} \), we get

\[ -\frac{1}{4} \frac{1}{\sqrt{g}} \left( \varepsilon^{ab} \partial_a X^\mu \partial_b X^\nu \right)^2 + \beta = 0. \]  

(2.6)

Substitution of the solution

\[ \sqrt{g} = \frac{1}{2} \sqrt{\frac{\alpha}{\beta}} \sqrt{\left( \varepsilon^{ab} \partial_a X^\mu \partial_b X^\nu \right)^2} \]  

(2.7)

into (2.4) restores the Nambu–Goto form (2.1) of the Green–Schwartz action:

\[ S_{NG} = T \int d^2 \sigma \left\{ \sqrt{\alpha \beta} \sqrt{\left( \varepsilon^{ab} \partial_a X^\mu \partial_b X^\nu \right)^2} - \frac{i}{2} \alpha \varepsilon^{ab} \partial_a X^\mu \bar{\Psi} \gamma^\mu \partial_b \Psi \right\}. \]  

(2.8)

The action (2.8) is invariant under the \( \mathcal{N} = 2 \) SUSY transformation

\[ \delta_{\text{SUSY}} \Psi_\alpha = -\frac{1}{2} \sqrt{g} \{ X_\mu, X_\nu \} (\gamma^\mu \epsilon)_\alpha + \xi_\alpha, \]
\[ \delta_{\text{SUSY}} X^\mu = i \bar{\epsilon} \gamma^\mu \Psi, \]  

(2.9)

where the parameters \( \epsilon \) and \( \xi \) do not depend again on \( \sigma_1 \) and \( \sigma_2 \).

Finally the partition function in the Schild formulation of IIB superstring is defined by the path integral over the positive definite function \( \sqrt{g} \), and over \( X^\mu \) and \( \Psi_\alpha \):

\[ Z_{\text{Schild}} = \int D\sqrt{g} DX^\mu D\Psi_\alpha e^{-S_{\text{Schild}}}. \]  

(2.10)

It is invariant under the SUSY transformation (2.9) since both the action (2.4) and the measure \( DX^\mu D\Psi_\alpha \) are invariant.

Equations (2.4) and (2.10) represent IIB superstring in the Schild formalism with fixed \( \kappa \)-symmetry [3].

In addition to the \( \mathcal{N} = 2 \) SUSY transformation (2.9), the partition function is invariant at fixed \( \sqrt{g} \) under area-preserving or symplectic \( \text{diffomorphisms} \)

\[ \delta_{\text{diff}} X^\mu = \left\{ X^\mu, \Omega \right\}, \quad \delta_{\text{diff}} \Psi_\alpha = \left\{ \Psi_\alpha, \Omega \right\} \]  

(2.11)
which is only a part of the whole reparametrization (or diffeomorphism) transformations. The invariance of the string theory under the whole group of reparametrizations is restored when \( \sqrt{g} \) is transformed. The symmetry (2.11) reminds the non-abelian gauge symmetry in Yang–Mills theory and is to be fixed for doing perturbative calculations.

2.3 The IKKT matrix model

The IKKT matrix model can be obtained from the representation (2.10) of IIB superstring in the Schild formalism by replacing

\[
X_\mu (\sigma_1, \sigma_2) \quad \mapsto \quad A^{ab}_\mu, \\
\Psi_\alpha (\sigma_1, \sigma_2) \quad \mapsto \quad \psi^{ab}_\alpha,
\]

(2.12)
(2.13)

where \( A^{ab}_\mu \) and \( \psi^{ab}_\alpha \) are hermitian \( n \times n \) bosonic and fermionic matrices, respectively.

The IKKT matrix model is defined by the partition function

\[
Z = \sum_{n=1}^{\infty} \int dA_\mu d\psi_\alpha e^{-S},
\]

(2.14)

which is of the type of 2nd quantized (euclidean) field theory, with the action

\[
S = \alpha \left( -\frac{1}{4} \text{tr} [A_\mu, A_\nu]^2 - \frac{1}{2} \text{tr} (\bar{\psi} \gamma^\mu [A_\mu, \psi]) \right) + \beta n.
\]

(2.15)

The summation over the matrix size \( n \) in Eq. (2.14) implies that \( n \) is a dynamical variable (an analog of \( \sqrt{g} \) in Eq. (2.10)).

The action (2.15) and the measure \( dA_\mu d\psi_\alpha \) in (2.14) are invariant under the \( \mathcal{N} = 2 \) SUSY transformation

\[
\delta_{\text{SUSY}} \psi^{ab}_\alpha = \frac{i}{2} [A_\mu, A_\nu]^{ab}(\gamma^{\mu\nu} \epsilon)_\alpha + \xi_\alpha \delta^{ab}, \\
\delta_{\text{SUSY}} A^{ab}_\mu = i\bar{\epsilon} \gamma_\mu \psi^{ab},
\]

(2.16)

where the parameters \( \epsilon \) and \( \xi \) are numbers rather than matrices, as well as under the SU(\( n \)) gauge transformation

\[
\delta_{\text{gauge}} A_\mu = i [A_\mu, \omega], \\
\delta_{\text{gauge}} \psi_\alpha = i [\psi_\alpha, \omega].
\]

(2.17)

The formulas (2.16) and (2.17) look like as if ten dimensional super Yang–Mills theory is reduced to a point. For instance only the commutator is left in the non-abelian field strength

\[
f_{\mu\nu} = i [A_\mu, A_\nu]
\]

(2.18)

and there are no space-time derivatives. However, the action (2.13) coincides with the one of 10D super Yang–Mills dimensionally reduced to zero dimensions only if \( \beta = 0 \) and \( n \) is fixed. This differs the IKKT matrix model from a pure D-instanton matrix model.
As was argued in [2], if large values of \( n \) and smooth matrices \( A_{\mu}^{ab} \) and \( \psi_\alpha^{ab} \) dominate in (2.14), one substitutes

\[
[\cdot, \cdot] \quad \Rightarrow \quad i\{\cdot, \cdot\} \quad \text{(2.19)}
\]
\[
\text{tr} \ldots \quad \Rightarrow \quad \int d^2 \sigma \sqrt{g} \ldots \quad \text{(2.20)}
\]
similarly to what is discussed in Subsect. 1.6. Then the formulas (2.14) to (2.17) for the IKKT matrix models reproduce the ones (2.4) to (2.11) for the Schild formulation of IIB superstring.

This passage from the IKKT matrix model to the Schild formulation of IIB superstring can be formalized introducing the matrix function

\[
L(\sigma_1, \sigma_2)^{ab} = \sum_{m_1, m_2} j_{m_1, m_2}(\sigma_1, \sigma_2) J_{m_1, m_2}^{ab}, \quad \text{(2.21)}
\]

where \( J_{m_1, m_2}^{ab} \) form a basis for \( \text{gl}(\infty) \) and \( j_{m_1, m_2}(\sigma_1, \sigma_2) \) form a basis in the space of functions of \( \sigma_1 \) and \( \sigma_2 \). An explicit form of \( j \)'s depends on the topology of the \( \sigma \)-space. Explicit formulas are available for a sphere and a torus.

With the aid of (2.21) we can relate matrices with functions of \( \sigma_1 \) and \( \sigma_2 \) by

\[
A_{\mu} = \int d^2 \sigma \sqrt{g} X_{\mu} L, \quad \text{(2.22)}
\]
\[
X_{\mu} = \text{tr} A_{\mu} L. \quad \text{(2.23)}
\]

These formulas result for smooth configurations in Eqs. (2.19) and (2.20). The word “smooth” means that configurations can be reduced by a gauge transformation to the form where high modes are not essential in the expansions (2.22) or (2.23).

The commutators of \( J \)'s coincide with the Poisson brackets of \( j \)'s as \( n \to \infty \). This demonstrates the equivalence between the group of symplectic diffeomorphisms and the gauge group \( \text{SU}(\infty) \) for smooth configurations.

### 2.4 D-strings as classical solutions

The classical equations of motion for the Schild action (2.4) read

\[
\{X^\mu, \{X_\mu, X_\nu\}\} = 0, \quad \{X^\mu, (\gamma_\mu \Psi)_\alpha\} = 0. \quad \text{(2.24)}
\]

Their matrix model counterparts are

\[
[A^\mu, [A_\mu, A_\nu]] = 0, \quad [A^\mu, (\gamma_\mu \psi)_\alpha] = 0, \quad \text{(2.25)}
\]

which are to be solved for \( n \times n \) matrices \( A_{\mu} \) at infinite \( n \).

Since Eqs. (2.25) look like Eq. (1.29) for M(atrix) theory, they possess operator-like solutions of the form (1.30), which are now associated with D-strings [2]. The solution associated with static D-string along 1st axis reads

\[
A_{\mu}^{\text{cl}} = \left( \frac{T}{2\pi} q, \frac{L}{2\pi} p, 0, \ldots, 0 \right), \quad \psi_{\alpha}^{\text{cl}} = 0, \quad \text{(2.26)}
\]
where the (infinite) $n \times n$ matrices $p$ and $q$ obey the canonical commutation relation (1.26), while $T/2\pi$ and $L/2\pi$ are (large enough) compactification radii.

The arguments in favor of identification of the classical solution (2.26) with static D-string are

- It is one dimension less than D-membrane of [1];
- Interaction between the two D-strings is reproduced at large distances [2];
- It is a BPS state (a proper central charge of SUSY algebra exists [18, 19]);
- It can be extended to $p = 3, 5$ [19, 20].

### 2.5 Zoo of D$p$-branes

A solution associated with two D-strings has a block-diagonal form and is built out of the ones given by Eq. (2.26) for single D-strings.

The solution for two \textit{parallel} static D-strings separated by the distance $b$ along 2nd axis reads [2]

\[
A^{\text{cl}}_0 = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}, \quad A^{\text{cl}}_1 = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}, \quad A^{\text{cl}}_2 = \begin{pmatrix} b/2 & 0 \\ 0 & -b/2 \end{pmatrix}, \quad A^{\text{cl}}_3 = \ldots = A^{\text{cl}}_9 = 0, \quad (2.27)
\]

where we have denoted

\[
Q \equiv \frac{T}{2\pi} q, \quad P \equiv \frac{L}{2\pi} p. \quad (2.28)
\]

The solution associated with two \textit{anti-parallel} static D-strings separated by the distance $b$ along 2nd axis is

\[
A^{\text{cl}}_0 = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}, \quad A^{\text{cl}}_1 = \begin{pmatrix} P & 0 \\ 0 & -P \end{pmatrix}, \quad A^{\text{cl}}_2 = \begin{pmatrix} b/2 & 0 \\ 0 & -b/2 \end{pmatrix}, \quad A^{\text{cl}}_3 = \ldots = A^{\text{cl}}_9 = 0. \quad (2.29)
\]

The solution associated with two static D-strings \textit{rotated} through the angle $\theta$ in the 1,2 plane and separated by the distance $b$ along 3rd axis is

\[
A^{\text{cl}}_0 = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}, \quad A^{\text{cl}}_1 = \begin{pmatrix} P & 0 \\ 0 & P \cos\theta \end{pmatrix}, \quad A^{\text{cl}}_2 = \begin{pmatrix} P & 0 \\ 0 & P \sin\theta \end{pmatrix}, \quad A^{\text{cl}}_3 = \begin{pmatrix} b/2 & 0 \\ 0 & -b/2 \end{pmatrix}, \quad A^{\text{cl}}_4 = \ldots = A^{\text{cl}}_9 = 0. \quad (2.30)
\]

The solution associated with one D$p$-brane, which extends (2.28) to $p > 1$, is given by

\[
A^{\text{cl}}_\mu = \left( P_1, Q_1, \ldots, P_{p+1}, Q_{p+1}, 0, \ldots, 0 \right), \quad \psi^{\text{cl}}_\alpha = 0, \quad (2.31)
\]
where \(P\)'s and \(Q\)'s form \((p+1)/2\) pairs of operators (infinite matrices) as in Eq. (2.28) obeying canonical commutation relation on a torus associated with compactification (of large enough radii \(L_a/2\pi\)) along the axes 0, \ldots, \(p\), so that

\[
\omega_k = \frac{L_{2k-2}L_{2k-1}}{2\pi n^{p+1}} \quad \left( k = 1, \ldots, \frac{p+1}{2} \right)
\]  

(2.32)
is fixed as \(n \to \infty\). This is because of the fact that the full Hilbert space of the dimension \(n\) is represented as the tensor product of \((p+1)/2\) Hilbert spaces of the dimension \(n^{p+1}\), each [13]. The value of \(n\) is related to the \(p+1\) dimensional volume

\[
V_{p+1} \equiv L_0 L_1 \cdots L_p
\]  

(2.33)
of the \(p\)-brane by

\[
n = V_{p+1} \prod_{i=1}^{p+1} (2\pi \omega_i)^{-1}.
\]  

(2.34)

These formulas allows one to extract world-volume characteristics of \(Dp\)-branes from the matrix model.

A general multi-brane solution has a block-diagonal form and is built out of single \(p\)-brane solutions \(\text{(2.31)}\) quite similar to \(\text{(2.27)--(2.30)}\).

### 2.6 One-loop effective action

The calculation of the one-loop effective action in the IKKT matrix model at fixed \(n\) can be performed for an arbitrary background, \(A_{\mu}^{\text{cl}}\) and \(\psi_{\alpha}^{\text{cl}} = 0\), obeying the classical equations of motion \(\text{(2.25)}\). The calculation is quite similar to the one in the Eguchi–Kawai reduced model.

Expanding around the classical solution

\[
A_{\mu} = A_{\mu}^{\text{cl}} + a_{\mu}
\]  

(2.35)
and adding the gauge fixing and ghost terms to the action \(\text{(2.15)}\):

\[
S_{\text{g.f.t.}} = - \text{tr} \left( \frac{1}{2} \left[ A_{\mu}^{\text{cl}}, a_{\mu} \right]^2 + \left[ A_{\mu}^{\text{cl}}, b \right] \left[ A_{\nu}^{\text{cl}}, c \right] \right),
\]  

(2.36)
where the matrices \(b\) and \(c\) represent ghosts, we get [2]

\[
W = \frac{1}{2} \text{Tr} \ln(P^2\delta_{\mu\nu} - 2iF_{\mu\nu}) - \frac{1}{4} \text{Tr} \ln \left( P^2 + i2 F_{\mu\nu} \gamma^{\mu\nu} \left( \frac{1+\gamma_{11}}{2} \right) \right) - \text{Tr} \ln P^2.
\]  

(2.37)

Here the \textit{adjoint} operators \(P_\mu\) and \(F_{\mu\nu}\) are defined on the space of matrices by

\[
P_\mu = \left[ A_{\mu}^{\text{cl}}, \cdot \right], \quad F_{\mu\nu} = \left[ f_{\mu\nu}^{\text{cl}}, \cdot \right] = i \left[ \left[ A_{\mu}^{\text{cl}}, A_{\nu}^{\text{cl}} \right], \cdot \right].
\]  

(2.38)
For the solution (2.31), Im $W$ vanishes for $p = 1, 3, 5, 7$ since $P_\mu = 0$ at least in one direction.

The first term on the right hand side of Eq. (2.37) comes from the quantum fluctuations of $A_\mu$, the second and third terms which come from fermions and ghosts have the minus sign for this reason. The extra factor $1/2$ in the first and second terms is because the matrices $A$ and $\psi$ are hermitian.

If $A_\mu^{cl}$ is diagonal
\begin{equation}
A_\mu^{cl} = \text{diag} \left( p_\mu^{(1)}, \ldots, p_\mu^{(n)} \right), \quad \psi_\alpha^{cl} = 0, \quad (2.39)
\end{equation}
which is a solution of Eq. (2.23) associated with the flat space-time, then $F_{\mu\nu} = 0$ and
\begin{equation}
W = \left( \frac{1}{2} \cdot 10 - \frac{1}{4} \cdot 16 - 1 \right) \text{Tr} \ln P^2 = 0. \quad (2.40)
\end{equation}
The plane vacuum is a BPS state.

The same is true (to all loops) for any $A_\mu^{cl}$ whose commutator is diagonal:
\begin{equation}
\left[ A_\mu^{cl}, A_\nu^{cl} \right] = c_{\mu\nu} 1_n, \quad (2.41)
\end{equation}
where $c_{\mu\nu}$ are c-numbers rather than matrices. Such solutions preserve a half of SUSY and are BPS states. The solution (2.27) associated with parallel D-strings is an example of such a BPS state.

For a general background $A_\mu^{cl}$, the matrix $F_{\mu\nu}$ can always be represented in the canonical (Jordan) form
\begin{equation}
F_{\mu\nu} = \begin{pmatrix}
0 & -\omega_1 \\
\omega_1 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & -\omega_5 \\
\omega_5 & 0
\end{pmatrix}, \quad (2.42)
\end{equation}
so that
\begin{equation}
\text{Tr} \ln (P^2 \delta_{\mu\nu} - 2i F_{\mu\nu}) = \sum_{i=1}^{5} \text{Tr} \ln ((P^2)^2 - 4\omega_i^2) \quad (2.43)
\end{equation}
and
\begin{equation}
\text{Tr} \ln \left( \left( P^2 + \frac{i}{2} F_{\mu\nu} \gamma^{\mu\nu} \right) \left( 1 + \frac{\gamma_{11}}{2} \right) \right) = \sum_{s_1, \ldots, s_5 = \pm 1} \text{Tr} \ln \left( P^2 - \sum_{i} \omega_i s_i \right). \quad (2.44)
\end{equation}
There are 16 terms on the right hand side of Eq. (2.44) representing the trace over $\gamma$-matrices. Equations (2.43) and (2.44) are most useful in practical calculations for the background of Dp-brane given by (2.31), when only $(p+1)/2$ of 5 omegas are nonvanishing.
2.7 Brane-brane interaction

The interaction between two Dp-branes is calculated by substituting the proper classical solutions into (2.37) and using Eqs. (2.43), (2.44).

For parallel Dp-branes, \( W = 0 \) is accordance with the general arguments of the previous subsection.

For anti-parallel Dp-branes, we get (\[P\] for \( p = 1 \), \[19\] for \( p \geq 3 \))

\[
W = -2n \int_0^\infty \frac{ds}{s} e^{-b^2s} \left[ \frac{\sum_{i=1}^{p+1} (\cosh 4\omega_i s - 1)}{2 \prod_{i=1}^{p+1} \cosh 2\omega_i s} - 4 \prod_{i=1}^{p+1} \frac{1}{2 \sinh 2\omega_i s} \right].
\]

(2.45)

The asymptotics of this formula at large \( b \):

\[
W = -\frac{1}{16} n \left( \frac{5-p}{2} \right)! \left[ 2 \sum_{i=1}^{p+1} \omega_i^4 - \left( \sum_{i=1}^{p+1} \omega_i^2 \right)^2 \right] \prod_{i=1}^{p+1} \frac{1}{\omega_i} \left( \frac{2}{b} \right)^{7-p} + O \left( \frac{1}{b^{9-p}} \right),
\]

(2.46)

agrees with the superstring calculation at large distances.

However, the superstring result \[10, 21, 22\] for the interaction between two anti-parallel Dp-branes at arbitrary distances \( b \), which is given by the annulus diagram in the open-string language or by the cylinder diagram in the closed-string language,

\[
W = -V_{p+1} \int_0^\infty \frac{dt}{t} \left( \frac{1}{8\pi^2 \alpha' t} \right)^{p+1} \frac{1}{q-1} \prod_{n=1}^{\infty} \frac{(1 - q^{2n-1})^8}{\prod_{n=1}^{\infty} (1 - q^{2n})^8}.
\]

(2.47)

with \( q = e^{-\pi t} \), does not coincide with the matrix-model result (2.45). There is no agreement even if one truncates to the lightest open string modes.

A way out could be to interpret \[23\] the classical solutions in the IKKT matrix model as D-branes with magnetic field, in analogy with previous work \[16\] on M(atrix) theory \[1\]. An alternative possibility is to modify the IKKT matrix model to better reproduce the superstring calculation.
3 The NBI matrix model [3]

Calculations of the brane-brane interaction in the matrix model can be extended to the case of moving and rotated static Dp-branes. The results agree with the superstring calculations for empty branes only at large distances between them. This was one of the motivations of Ref. [3] to modify the IKKT matrix model introducing (instead of \( n \)) an additional dynamical variable — a positive definite hermitian matrix \( Y^{ab} \) — which is the direct analog of \( \sqrt{g} \) in the Schild formulation of IIB superstring. Integration over \( Y^{ab} \) results in the Non-abelian Born–Infeld (NBI) action which reproduces the Nambu–Goto version of the Green–Schwarz action of IIB superstring.

3.1 Parallel moving branes

The operator-like solution to Eqs. (2.25), which is associated with two parallel branes separated by the distance \( b \) along the \((p+2)\)-th axis and moving with velocities \( v \) and \(-v\) along the \((p+1)\)-th axis, can be obtained by boosting the one for parallel branes (see (2.27)) along the \((p+1)\)-th axis:

\[
A^{cl}_0 = \begin{pmatrix} B_0 \cosh \epsilon & 0 \\ 0 & B_0 \cosh \epsilon \end{pmatrix},
\]

\[
A^{cl}_a = \begin{pmatrix} B_a & 0 \\ 0 & B_a \end{pmatrix}, \quad a = 1, \ldots p,
\]

\[
A^{cl}_{p+1} = \begin{pmatrix} B_0 \sinh \epsilon & 0 \\ 0 & -B_0 \sinh \epsilon \end{pmatrix},
\]

\[
A^{cl}_{p+2} = \begin{pmatrix} b \tfrac{1}{2} & 0 \\ 0 & -\tfrac{b}{2} \end{pmatrix},
\]

\[
A^{cl}_i = 0, \quad i = p + 3, \ldots 9.
\]

(3.1)

Here

\[
v = \tanh \epsilon,
\]

(3.2)

and we have denoted

\[
B_0 \equiv Q_1, \quad B_1 \equiv P_1, \ldots \quad B_{p-1} \equiv Q_{\frac{p-1}{2}}, \quad B_p \equiv P_{\frac{p-1}{2}}.
\]

(3.3)

The substitution of (3.1) into the one-loop (euclidean) effective action (2.37) yields [3]

\[
W = -n \frac{2\pi}{p+1} \prod_{a \neq 1} L_a^{-1} \int_0^\infty \frac{ds}{s} \left( \frac{\pi}{2s} \right)^\frac{p}{2} e^{-b^2 s} \frac{(\cosh(4\omega_1 s \sinh \epsilon) - 4 \cosh(2\omega_1 s \sinh \epsilon) + 3)}{\cosh \epsilon \sinh(2\omega_1 s \sinh \epsilon)}.
\]

(3.4)

Using Eqs. (2.32), (2.33) and Wick rotating back to Minkowski space-time, we get for the phase shift

\[
\delta = -\frac{V_p}{(2\pi)^p} \omega_1 \prod_{i=1}^l \frac{1}{\omega_i^2} \int_0^\infty \frac{ds}{s} \left( \frac{\pi}{2s} \right)^\frac{p}{2} e^{-b^2 s} \frac{(\cos(4\omega_1 s \sinh \epsilon) - 4 \cos(2\omega_1 s \sinh \epsilon) + 3)}{\cosh \epsilon \sin(2\omega_1 s \sinh \epsilon)}
\]

(3.5)
where
\[ V_p = \prod_{a=1}^{p} L_a. \] (3.6)

This result was shown [3] to agree with the superstring calculation of Bachas [24] at large \( b \) for the real part of \( \delta \). Analogously, the imaginary part of (3.5) which comes from the poles at zeros of the denominator agrees at small \( v \) providing \( \omega_i = 2\pi\alpha' \).

### 3.2 Rotated branes

Taking the configuration of two parallel Dp-branes separated by the distance \( b \) along the \((p+2)\)-th axis and rotating them in the opposite directions in the \((p,p+1)\) plane through the angle \( \theta/2 \), one obtains the following solution to Eq. (2.25)

\[
A_{cl}^a = \begin{pmatrix}
B_a & 0 \\
0 & B_a
\end{pmatrix}, \quad a = 0, \ldots, p-1,
\]

\[
A_{cl}^p = \begin{pmatrix}
B_p \cos \frac{\theta}{2} & 0 \\
0 & B_p \cos \frac{\theta}{2}
\end{pmatrix},
\]

\[
A_{cl}^{p+1} = \begin{pmatrix}
B_p \sin \frac{\theta}{2} & 0 \\
0 & -B_p \sin \frac{\theta}{2}
\end{pmatrix},
\]

\[
A_{cl}^{p+2} = \begin{pmatrix}
\frac{b}{2} & 0 \\
0 & -\frac{b}{2}
\end{pmatrix},
\]

\[
A_{cl}^i = 0, \quad i = p+3, \ldots, 9,
\] (3.7)

which extends (2.30) to \( p > 1 \). This looks pretty much like an analytic continuation of Eq. (3.1) \( (\epsilon \to i\theta/2) \).

The interaction between two rotated Dp-branes is given by [3]

\[
W = -4n \frac{2^{2p}}{\pi^p} \frac{1}{\cos \frac{\theta}{2}} \prod_{a \neq p-1} L_a^{-1}
\]

\[
\times \int_0^\infty \frac{ds}{s} \left( \frac{\pi}{2s} \right)^{\frac{3}{2}} e^{-b^2 s} \operatorname{tanh} \left( \frac{\omega_{p+1}}{s} \sin \frac{\theta}{2} \right) \sinh^2 \left( \frac{\omega_{p+1}}{s} \sin \frac{\theta}{2} \right).
\] (3.8)

It can be obtained from (3.4) substituting \( \epsilon = i\theta/2 \).

Expanding in \( 1/b^2 \) and using Eq. (2.32), one gets

\[
W = -\frac{1}{16} n \left( \frac{4-p}{2} \right)! \frac{L_{p-1}}{\sqrt{2\pi n^{p+1}}} \omega_{p+1}^{-1} \prod_i \omega_i^{-1} \operatorname{tan} \frac{\theta}{2} \sin^2 \frac{\theta}{2} \frac{1}{b^{p-\theta}} + O \left( \frac{1}{b^{8-p}} \right) \] (3.9)

for large distances, which agrees with the supergravity result. For \( p = 1 \) this is first shown in [3].
3.3 The NBI action

In the IKKT model the matrix size $n$ is considered as a dynamical variable, so the partition function (2.14) includes the summation over $n$. This sum is expected to recover the integration over $\sqrt{g}$ in (2.10) while the proof is missing. Even at the classical level, the minimization of Eq. (2.15) with respect to $n$ does not result in a nice matrix-model action which could be associated with the Nambu–Goto action (2.8).

These problems can be easily resolved by a slight modification of the IKKT matrix model. Let us introduce a positive definite $N \times N$ hermitian matrix $Y^{ab}$ which would play the role of a dynamical variable instead of $n$. In other words, the matrix size $N$ is fixed (to be distinguished from fluctuating $n$) while the elements of $Y$ fluctuate.

The classical action has the form

$$S_{cl} = -\alpha \left(\frac{1}{4} \text{tr} \ Y^{-1} [A_\mu, A_\nu]^2 + \frac{1}{2} \text{tr} \ (\bar{\psi} \gamma^\mu [A_\mu, \psi])\right) + \beta \text{tr} \ Y, \quad (3.10)$$

which yields the following classical equation of motion for the $Y$-field:

$$\frac{\alpha}{4} \left( Y^{-1} [A_\mu, A_\nu]^2 Y^{-1} \right)_{ij} + \beta \delta_{ij} = 0. \quad (3.11)$$

The solution to Eq. (3.11) reads

$$Y = \frac{1}{2} \sqrt{\frac{\alpha}{\beta}} \sqrt{\ - [A_\mu, A_\nu]^2}. \quad (3.12)$$

Here $-[A_\mu, A_\nu]^2$ is positive definite, since the commutator is anti-hermitian (cf. Eq. (2.18)). The square root in (3.12) is unique, provided $Y$ is positive definite which is the case. After the substitution of (3.12), the classical action (3.10) reduces to

$$S_{\text{NBI}}^{cl} = \sqrt{\alpha \beta} \text{tr} \sqrt{\ - [A_\mu, A_\nu]^2} - \frac{\alpha}{2} \text{tr} \ (\bar{\psi} \gamma^\mu [A_\mu, \psi]). \quad (3.13)$$

The bosonic part of (3.13) coincides with the strong field limit of the Non-abelian Born–Infeld (NBI) action. The action (3.13) is called for this reason the NBI action. Notice that it is field-theoretic rather than widely discussed stringy NBI action which has a different structure [23].

The formulas above in this subsection are very similar to the ones of Subsect. 2.2 for the Schild formulation. Thus the hermitian matrix $Y^{ab}$ with positive definite eigenvalues is the direct analog of $\sqrt{g(\sigma_1, \sigma_2)}$ so that

$$Y^{ab} \Rightarrow \sqrt{g(\sigma_1, \sigma_2)} \quad (3.14)$$

in the same sense as in (2.12), (2.13). In the next subsection we discuss that it is possible to choose such a measure of integration over $Y$ which reproduces the Nambu–Goto version of the Green–Schwarz action even at the quantum level.
3.4 The NBI model of IIB superstring

The NBI matrix model is defined by the action

\[
S_{\text{NBI}} = -\alpha \left( \frac{1}{4} \text{tr} Y^{-1}[A_\mu, A_\nu]^2 + \frac{1}{2} \text{tr} \left( \bar{\psi} \gamma^\mu [A_\mu, \psi] \right) \right) + V(Y),
\]

where \( Y \) is a hermitian \( N \times N \) matrix with positive eigenvalues. The potential is

\[
V(Y) = \beta \text{Tr} Y + \gamma \text{Tr} \ln Y,
\]

where

\[
\gamma = N - \frac{1}{2}.
\]

The partition function is then given by the matrix integral

\[
Z_{\text{NBI}} = \int dA_\mu d\psi_\alpha dY e^{-S_{\text{NBI}}}.
\]

The action (3.15) is invariant under the SUSY transformation

\[
\delta_{\text{SUSY}} \psi = \frac{i}{4} \left[ Y^{-1}, [A_\mu, A_\nu] \right]_+ \gamma^{\mu\nu} \epsilon + \xi
\]

\[
\delta_{\text{SUSY}} A_\mu = i \bar{\epsilon} \gamma_\mu \psi,
\]

in the limit \( N \to \infty \), where \([\cdot, \cdot]_+\) stands for the anticommutator. \( Y \) is not changed under this transformation.

The action (3.15) differs from its classical counterpart (3.10) by the second term on the right-hand side of Eq. (3.16). It is associated with the measure of integration over \( Y \) rather than with the classical action. The classical action (3.10) can be obtained from (3.15) in the limit \( \alpha \sim \beta \to \infty, \alpha/\beta \sim 1 \sim \gamma/N \). This corresponds to the usual classical limit in string theory since \( \alpha \sim \beta \sim g_s^{-1} \).

The matrix \( Y \) can be always brought to the diagonal form

\[
Y = \Omega^\dagger \text{diag} (y_1, \ldots, y_N) \Omega \quad (y_1, \ldots, y_N \geq 0),
\]

where \( \Omega \) is unitary. The measure for integration over \( Y \) reads explicitly

\[
\int dY \ldots = \int_0^\infty \prod_{i=1}^N dy_i \Delta^2[y] \cdot d\Omega \ldots
\]

with

\[
\Delta[y] = \prod_{i>j} (y_i - y_j)
\]

being the Vandermonde determinant.

The integral over \( Y \) in (3.18) can be done. Let us mention that the fermionic term in (3.15) is \( Y \)-independent and denote

\[
\mathcal{F}(z) = \int dY \ e^{-\alpha \text{tr} Y^{-1}z^2/4 - \beta \text{tr} Y - \gamma \text{tr} \ln Y},
\]

23
where \( z^2 = -[A_\mu, A_\nu]_+^2 \). This matrix integral looks like an external field problem for the Penner matrix model.

Doing the Itzykson–Zuber integral over the “angular” variable \( \Omega \), (3.23) takes the form

\[
\mathcal{F}(z) \propto \int_0^\infty \prod_{i=1}^N dy_i \Delta^2[y] \frac{e^{-\alpha \sum_i y_i^{-1} z_i^2/4 - \beta \sum_i y_i - \gamma \sum_i \log y_i}}{\Delta[z]} e^{-\sqrt{\alpha \beta} \sum_i z_i},
\]

where \( z_i^2 \) stand for the eigenvalues of \( z^2 \).

Hence, it is shown that

\[
\int dA_\mu d\psi_\alpha dY e^{-S_{NBI}} = \int \prod_{i>j} \frac{dA_\mu d\psi_\alpha}{(z_i + z_j)} e^{-S_{NBI}^\text{cl}}.
\]

(3.25)

Thus the NBI action \( S_{NBI}^\text{cl} \) defined by Eq. (3.13) is reproduced modulo the change of the measure for integration over \( A_\mu \).

The significance of this result is that it can be explicitly shown that

\[
S_{NBI}^\text{cl} \Rightarrow S_{NG}
\]

given by Eq. (2.8), where the arrow is in the same sense as in (2.12), (2.13) and (3.14).

Analogously, the Schild action (2.4) can be reproduced from the model (3.18) with the additional integration over \( Y \) (without explicitly doing it).

A proposal of Ref. [3] is to modify the measure for the integration over \( A_\mu \) from the outset to get

\[
\int dA_\mu d\psi_\alpha dY \prod_{i>j} (z_i + z_j) e^{-S_{NBI}} = \int dA_\mu d\psi_\alpha e^{-S_{NBI}^\text{cl}}
\]

\[
N=\infty \int DX^\mu D\Psi_\alpha e^{-S_{NG}}.
\]

(3.26)

Then the Nambu–Goto version of the Green–Schwartz action of IIB superstring is exactly reproduced by the NBI matrix model.

### 3.5 Remark on D-brane solutions in the NBI model

The classical solutions (2.31) associated with D-brane configurations are also classical solutions to the NBI matrix model whose classical equations of motion, which result from the variation of the action (3.13) with respect to \( A_\mu \) and \( \psi_\alpha \), read

\[
\left[ A_\mu, \left[ Y^{-1}, [A_\mu, A_\nu]_+ \right]_+ \right] = 0, \quad [A_\mu, (\gamma^\nu \psi)_\alpha] = 0.
\]

(3.27)

The reason is that these classical solutions are BPS states and the commutator \([A_\mu, A_\nu]_+\) is proportional to the unit matrix (see Eq. (2.41)).
A more general property holds in the large–$N$ limit when any classical solution of the IKKT matrix model is simultaneously a solution of the classical equations of motion of the NBI model. However, the structure of the classical equations (3.11) and (3.27) in the NBI matrix model is, generally speaking, richer than Eq. (2.25) in the IKKT model, since $Y^{\text{cl}}$ may have some nontrivial distribution of eigenvalues (typical for the large–$N$ saddle points).

One of most urgent checks of the NBI model would be to perform the calculation of the brane-brane interaction to compare with the superstring result. This calculation will take into account the fact that $Y$ is a dynamical field while the ones described above for the IKKT matrix model are done at fixed $n$, i.e. without considering $n$ as a dynamical variable.
Conclusion

It is now too early to make any definite conclusions since it is not yet clear whether or not this formulation of superstrings, which is based on the supersymmetric matrix models, would survive. Nevertheless, such an approach to M theory looks most promising among those proposed so far.

This situation reminds me somewhat of the one with QCD in the very beginning of the seventies about the time when the QCD Lagrangian was introduced. Before that there existed the approach to the theory of strong interaction based on strings and dual resonance models, while the new theory looked quite different and was most convenient to study strong interaction at small distances. Once again, it is now too early to predict whether the same could happen with superstrings in the nearest future, but this option should not be immediately excluded.

One of the simplest checks of the matrix models of superstrings is the study of the interaction between D-branes. It should answer, in particular, the question whether the classical operator-like solutions of the matrix models are associated with empty D-branes or D-branes carrying magnetic field.

A more serious problem is to show how string perturbation theory emerges from the matrix models. The NBI matrix model is very promising from this point of view since it reproduces the Nambu–Goto version of the Green–Schwartz action.

While the proposed matrix models of IIB superstring are of the type of reduced ten-dimensional super Yang–Mills, they have additional degrees of freedom which are essential to have strings. This differs the situation from the one in large–$N$ QCD where the fundamental Lagrangian is fixed, and the problem to obtain strings in the Eguchi–Kawai reduced model is almost as difficult as in whole QCD. Now, for the matrix models of superstrings, the true model is not know from the outset. The reader is still free to introduce his/her own model to describe superstrings in the best way.

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