Lower bound for the blow-up time for a general nonlinear nonlocal porous medium equation under nonlinear boundary condition

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Abstract

In this paper, we study the blow-up phenomenon for a general nonlinear nonlocal porous medium equation in a bounded convex domain (Ω ∈ ℝⁿ, n ≥ 3) with smooth boundary. Using the technique of a differential inequality and a Sobolev inequality, we derive the lower bound for the blow-up time under the nonlinear boundary condition if blow-up does really occur.

Keywords: Lower bound; Blow-up time; Robin boundary condition; Nonlocal porous medium equation

1 Introduction

Liu in paper [1] studied the blow-up phenomena for the solution of the following problems:

\[ \frac{\partial u}{\partial t} = \Delta u^m + u^p \int_{\Omega} u^q \, dx, \quad (x, t) \in \Omega \times (0, t^*), \]

\[ u(x, 0) = f(x) \geq 0, \quad x \in \Omega, \]

under the Robin boundary condition

\[ \frac{\partial u}{\partial v} + ku = 0, \quad (x, t) \in \Omega \times (0, t^*). \]

He obtained a lower bound for the blow-up time of the system when the solution blows up.

In paper [2], the authors also studied equations (1.1) and (1.2) subject to either homogeneous Dirichlet boundary condition or homogeneous Neumann boundary condition. The lower bounds for the blow-up time under the above two boundary conditions were obtained. Equation (1.1) is used in the study of population dynamics (see [3]). For other
systems in porous medium, one could see [4]. There have been a lot of papers in the literature on studying the question of blow-up for the solution of parabolic problems under a homogeneous Dirichlet boundary condition and Neumann boundary condition (one can see [5–12]). Some authors have started to consider the blow-up of these problems under Robin boundary conditions (see [13–17]). In papers [18–21], the authors studied the blow-up phenomena for the heat equation under nonlinear boundary conditions. Some new results about the nonlinear evolution equations may be founded in [22–24]. These papers have mainly focused on the bounded convex domain in \( \mathbb{R}^3 \). Recently, there have been some papers starting to study the blow-up problems in \( \mathbb{R}^n \) ( \( n \geq 3 \)) (see [25–29]). We continue the work of [2] for a more general equation. Until now, the authors have not found any paper dealing with lower bound for the blow-up time of a nonlinear nonlocal porous medium equation under nonlinear boundary condition in \( \mathbb{R}^n \) ( \( n \geq 3 \)). In this sense, the result obtained in this paper is new and interesting.

In this paper, we consider the blow-up phenomena of the solution for the following equation:

\[
(h(u))_t = \Delta u^m + k_1(t)u^p \int_\Omega u^q \, dx, \quad (x,t) \in \Omega \times (0,t^*),
\]

(1.4)

with the following boundary initial conditions:

\[
u(x,0) = f(x) \geq 0, \quad x \in \Omega, \]

(1.5)

\[
\frac{\partial u}{\partial \nu} = k_2(t) \int_{\partial \Omega} g(u) \, dx, \quad (x,t) \in \partial \Omega \times (0,t^*),
\]

(1.6)

where \( \Omega \) is a bounded convex domain in \( \mathbb{R}^n \), \( n \geq 3 \), with sufficiently smooth boundary, \( \Delta \) is the Laplace operator, \( \partial \Omega \) is the boundary of \( \Omega \), and \( t^* \) is the possible blow-up time, \( \frac{\partial u}{\partial \nu} \) is the outward normal derivative of \( u \). We assume \( \frac{k_1(t)}{k_1(0)} \leq \alpha \) and \( \frac{d\sigma}{dt} \geq M > 0 \).

The function \( g(\xi) \) satisfies

\[
0 \leq g(\xi) \leq \xi^s, \quad \forall \xi > 0,
\]

(1.7)

where \( s > \max\{\frac{2n}{2n-1}, p + q + 1 - m\} \).

2 Some useful inequalities

We will use the following useful inequalities later in the proof.

**Lemma 2.1** We suppose that \( u \) is a nonnegative function and \( \sigma, m \) are positive constants, then we have the result as follows:

\[
\int_{\partial \Omega} u^{\sigma+m-2} \, dA \leq \frac{H}{\rho_0} \int_{\Omega} u^{\sigma+m-2} \, dx + \frac{(\sigma + m - 2)d}{\rho_0} \int_{\Omega} u^{\sigma+m-3} |\nabla u| \, dx,
\]

(2.1)

where \( \rho_0 := \min_{\partial \Omega} |x \cdot \vec{v}|, \vec{v} \) is the outward normal vector of \( \partial \Omega \) and \( d := \max_{\partial \Omega} |x| \).

**Proof** Applying the divergence definition, we have

\[
\text{div}(u^{\sigma+m-2} \nabla u) = nu^{\sigma+m-2} + (\sigma + m - 2)u^{\sigma+m-3}(x \cdot \nabla u).
\]

(2.2)
Integrating (2.2), we deduce
\[
\int_\Omega \text{div}(u^{\sigma + m - 2}x) \, dx \leq n \int_\Omega u^{\sigma + m - 2} \, dx + (\sigma + m - 2) \int_\Omega u^{\sigma + m - 3} |x \cdot \nabla u| \, dx.
\]

Applying the divergence theorem, we obtain
\[
\int_{\partial \Omega} u^{\sigma + m - 2} x \cdot \nu \, dA = n \int_\Omega u^{\sigma + m - 2} \, dx + (\sigma + m - 2) \int_\Omega u^{\sigma + m - 3} |x \cdot \nabla u| \, dx.
\]

Because \( \Omega \) is a convex domain, we have \( \rho_0 := \min_{\partial \Omega} |x \cdot \nu| > 0 \). Then we derive
\[
\int_{\partial \Omega} u^{\sigma + m - 2} \, dA \leq \frac{n}{\rho_0} \int_\Omega u^{\sigma + m - 2} \, dx + (\sigma + m - 2) \int_\Omega u^{\sigma + m - 3} |x \cdot \nabla u| \, dx.
\]

**Lemma 2.2** Supposing that \( u \in W^{1,2}(\Omega) \) and \( n \geq 3 \), we have
\[
\int_\Omega u^{\sigma + m - 1} \, dx \leq C \left( \int_\Omega |u|^{\frac{n}{n-2}} \, dx \right)^{\frac{n}{n-2}} + \left( \int_\Omega |\nabla u|^{\frac{n}{n-2}} \, dx \right)^{\frac{n}{n-2}},
\]
where \( C = C(n, \Omega) \) is a Sobolev embedding constant depending on \( n \) and \( \Omega \).

**Proof** In paper [30], we have \( W^{1,2}(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega), \ n \geq 3 \). Then we deduce the Sobolev inequality as follows:
\[
\left( \int_\Omega |w|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{n}} \leq C \left( \int_\Omega |w|^{\frac{n}{n-2}} \, dx \right)^{\frac{n}{n-2}} \left( \int_\Omega |\nabla w|^{\frac{n}{n-2}} \, dx \right)^{\frac{n}{n-2}},
\]
that is,
\[
\left( \int_\Omega u^{\frac{\sigma + m - 1}{2}} \, dx \right)^{\frac{2n}{n-2}} \leq C \left( \int_\Omega u^{\frac{\sigma + m - 1}{2}} \, dx + \int_\Omega |\nabla u|^{\frac{\sigma + m - 1}{2}} \, dx \right)^{\frac{n}{n-2}}.
\]

We can get
\[
\int_\Omega u^{\frac{\sigma + m - 1}{n-1}} \, dx \leq C^{\frac{2n}{n-2}} \left( \int_\Omega u^{\frac{\sigma + m - 1}{2}} \, dx + \int_\Omega |\nabla u|^{\frac{\sigma + m - 1}{2}} \, dx \right)^{\frac{n}{n-2}}
\]
\[
\leq C^{\frac{2n}{n-2}} \left( \int_\Omega u^{\sigma + m - 1} \, dx \right)^{\frac{n}{n-2}} \left( \int_\Omega |\nabla u|^{\frac{\sigma + m - 1}{2}} \, dx \right)^{\frac{n}{n-2}}.
\]

**Remark 2.1** For any nonnegative function \( u \), the following Hölder inequality holds:
\[
\int_\Omega u^{n_1 + n_2} \, dx \leq \left( \int_\Omega u^{n_1} \, dx \right)^{x_1} \left( \int_\Omega u^{n_2} \, dx \right)^{x_2},
\]
where \( n_1, n_2, x_1, x_2 \) are positive constants and \( x_1, x_2 \) satisfy \( x_1 + x_2 = 1 \).

**Remark 2.2** The fundamental inequality
\[
(a + b)^l \leq a^l + b^l,
\]
where \( a, b \geq 0 \) and \( 0 < l \leq 1 \), holds.
3 Lower bound for the blow-up time

In this section it is useful in the sequel to define an auxiliary function of the following form:

\[ \phi(t) = k_1^n(t) \int_{\Omega} u^{2n(s-1)} \, dx = k_1^n(t) \int_{\Omega} u^\sigma \, dx, \quad 0 \leq t < t^*. \quad (3.1) \]

We will derive a differential inequality for \( \phi(t) \). From the inequality, we can establish the following theorem.

**Theorem 3.1** Let \( u(x,t) \) be the classical nonnegative solution of problem (1.4)–(1.7) in a bounded convex domain \( \Omega \) (\( \Omega \subset \mathbb{R}^n \) (\( n \geq 3 \))). We assume that \( m + s > p + q + 1 > 2, \, m > 3, \, p > 0, \, q > 0 \). Then the quantity \( \phi(t) \) defined in (3.1) satisfies the differential inequality

\[ \phi'(t) \phi^{-5}(t) \leq a(t) \phi^{-4}(t) + b(t), \quad (3.2) \]

from which it follows that the blow-up time \( t^* \) is bounded below. We have

\[ t^* \geq \Theta^{-1} \left( \frac{1}{4\phi^4(0)} \right), \quad (3.3) \]

where \( \Theta^{-1} \) is the inverse function of \( \Theta \), and \( a(t), \, b(t) \) are defined in (3.21), (3.22) respectively.

**Proof** Now we prove Theorem 3.1. For convenience, we set \( \phi(t) = \phi, \, k_1(t) = k_1, \, k_2(t) = k_2 \).

First we compute

\[
\phi'(t) = nk_1^{n-1}k_1' \int_{\Omega} u^\sigma \, dx + k_1^n \int_{\Omega} u^{\sigma-1} u_t \, dx
\]

\[= nk_1^{n-1}k_1' \int_{\Omega} u^\sigma \, dx + k_1^n \int_{\Omega} \frac{1}{h(u)} \left[ \Delta u^m + k_1 u^p \int_{\Omega} u^q \, dx \right] \, dx
\]

\[\leq n\alpha \phi + \frac{k_1^n \sigma}{M} \int_{\Omega} u^{\sigma-1} \left[ \Delta u^m + k_1 u^p \int_{\Omega} u^q \, dx \right] \, dx.
\]

Integrating by parts, we have

\[
\phi'(t) \leq n\alpha \phi + \frac{k_1^n \sigma}{M} \left[ \frac{m}{\phi} \int_{\Omega} u^{\sigma+m-2} \frac{\partial u}{\partial v} \, dA - m(\sigma - 1) \int_{\Omega} u^{\sigma+m-3} \, dx \right]
\]

\[+ \frac{k_1^n \sigma |\Omega|}{M} \int_{\Omega} u^{\sigma+p-1} \, dx
\]

\[\leq n\alpha \phi + \frac{\sigma m k_1^n k_2}{M} \int_{\Omega} u^{\sigma+m-2} \, dA \int_{\Omega} u^p \, dx - \frac{\sigma m(\sigma - 1) k_1^n}{M} \int_{\Omega} u^{\sigma+m-3} \, dx
\]

\[+ \frac{k_1^n \sigma |\Omega|}{M} \int_{\Omega} u^{\sigma+p-1} \, dx.
\]
Using the result of Lemma 2.1, we obtain

\[ \phi'(t) \leq n a \phi + \frac{\sigma m k_1^n k_2}{M} \int_\Omega u^{\sigma+m-2} \, dx \int_\Omega u^r \, dx \\
+ \frac{\sigma m k_1^n k_2}{M} \frac{(\sigma + m - 2)d}{\rho_0} \int_\Omega u^{\sigma+m-3} |\nabla u| \, dx \int_\Omega u^r \, dx \\
- \frac{\sigma m (\sigma - 1) k_1^n}{M} \frac{4}{(\sigma + m - 1)^2} \int_\Omega |\nabla u|^{\frac{\sigma+m-1}{2}} \, dx + \frac{k_{n+1}^\sigma |\Omega|}{M} \int_\Omega u^{\sigma+p+q-1} \, dx \]

where \( r_1 = \frac{\sigma m n |\Omega|}{M}, r_2 = \frac{\sigma m (\sigma+m-2)d}{\rho_0}, r_3 = \frac{\sigma m (\sigma-1)}{M}, r_4 = \frac{|\Omega|}{M}. \)

Now we estimate the third term of the right-hand side of (3.4). Using Hölder’s inequality, we have

\[ \int_\Omega u^r \, dx \leq \left( \int_\Omega u^\frac{r}{2} \, dx \right)^{\frac{2}{r}} |\Omega|^{\frac{r}{2}} = k_1^\frac{r}{2} \phi^\frac{1}{2} |\Omega|^{\frac{r}{2}}. \]

Then we obtain

\[ k_1^n \int_\Omega u^{\sigma+m-3} |\nabla u| \, dx \int_\Omega u^r \, dx \]

\[ \leq k_1^n \int_\Omega u^{\sigma+m-3} |\nabla u| \, dx k_1^\frac{2}{\sigma \pm m - 1} |\nabla u|^{\frac{\sigma+m-1}{2}} \]

\[ = k_1^\frac{2}{\sigma \pm m - 1} |\nabla u|^{\frac{\sigma+m-1}{2}} \frac{2}{\sigma + m - 1} \phi^\frac{-1}{2} k_1^n \int_\Omega u^{\frac{\sigma+m-3}{2}} |\nabla u|^{\frac{\sigma+m-1}{2}} \, dx \]

\[ \leq \left( \varepsilon_1^{-1} r_3 k_1^n \phi^\frac{-1}{2} \int_\Omega u^\frac{\sigma+m-3}{2} \, dx \right)^{\frac{1}{2}} \left( k_1^n \int_\Omega |\nabla u|^{\frac{\sigma+m-1}{2}} \, dx \right)^{\frac{1}{2}} \]

\[ \leq \frac{1}{2} \varepsilon_1^{-1} r_3 k_1^n \phi^\frac{-1}{2} \int_\Omega u^{\sigma+m-3} \, dx + \frac{1}{2} \varepsilon_1 \int_\Omega |\nabla u|^{\frac{\sigma+m-1}{2}} \, dx, \]

where \( r_5 = (k_1^\frac{2}{\sigma \pm m - 1} |\nabla u|^{\frac{\sigma+m-1}{2}}) \), \( \varepsilon_1 \) is a positive constant which will be defined later.

From the above deductions, we get

\[ r_2 k_2 k_1^n \int_\Omega u^{\sigma+m-3} |\nabla u| \, dx \int_\Omega u^r \, dx \]

\[ \leq \frac{1}{2} r_2 k_2 \varepsilon_1^{-1} r_5 k_1^n \phi^\frac{-1}{2} \int_\Omega u^{\sigma+m-3} \, dx + \frac{1}{2} r_2 k_2 \varepsilon_1 k_1^n \int_\Omega |\nabla u|^{\frac{\sigma+m-1}{2}} \, dx. \]  

Combining (3.4) and (3.5), we obtain

\[ \phi'(t) \leq n a \phi + r_1 k_1^n k_2 \int_\Omega u^{\sigma+m+1-2} \, dx \int_\Omega u^r \, dx + \frac{1}{2} r_2 k_2 \varepsilon_1^{-1} r_5 k_1^n \phi^\frac{-1}{2} \int_\Omega u^{\sigma+m-3} \, dx \\
+ r_4 k_1^{n+1} \int_\Omega u^{\sigma+p+q-1} \, dx + \left( \frac{1}{2} r_2 k_2 \varepsilon_1 - r_5 \right) k_1^n \int_\Omega |\nabla u|^{\frac{\sigma+m-1}{2}} \, dx. \]
Using (2.3), (2.4), and (2.5), we obtain

\[
\begin{align*}
\int_{\Omega} u^{\sigma + m - s - 2} \, dx & \leq \left( \int_{\Omega} u^{\frac{(\sigma + m - 1)n}{n+2}} \, dx \right)^{x_1} \left( \int_{\Omega} u^\omega \, dx \right)^{x_2} \\
& \leq \left( C \frac{2n}{n+2} \right)^{\frac{x_1}{x_2}} \left( \int_{\Omega} u^{\sigma + m - 1} \, dx \right)^{\frac{x_1}{x_2}} \\
& \quad + \left( \int_{\Omega} \left| \nabla u^{\frac{\sigma + m - 1}{2}} \right|^2 \, dx \right)^{\frac{x_1}{x_2}} \left( \int_{\Omega} u^\omega \, dx \right)^{x_2} \\
& = r_6 \left( \int_{\Omega} u^{\sigma + m - 1} \, dx \right)^{\frac{x_1}{x_2}} \left( \int_{\Omega} u^\omega \, dx \right)^{x_2} \\
& \quad + r_6 \left( \int_{\Omega} \left| \nabla u^{\frac{\sigma + m - 1}{2}} \right|^2 \, dx \right)^{\frac{x_1}{x_2}} \left( \int_{\Omega} u^\omega \, dx \right)^{x_2},
\end{align*}
\]

(3.7)

where

\[
\begin{align*}
x_1 & = \frac{(m + s - 2)(n - 2)}{(m - 1)n + 2\sigma}, \quad x_2 = \frac{(m - 1)n + 2\sigma + (2 - m - s)(n - 2)}{(m - 1)n + 2\sigma}, \\
r_6 & = \left( C \frac{2n}{n+2} \right)^{\frac{x_1}{x_2}}.
\end{align*}
\]

Using H"{o}lder's and Young's inequalities, we have

\[
\begin{align*}
& \quad r_6 \left( \int_{\Omega} u^{\sigma + m - 1} \, dx \right)^{\frac{x_1}{x_2}} \left( \int_{\Omega} u^\omega \, dx \right)^{x_2} \\
& \leq \left( \int_{\Omega} u^{\sigma + m - 1} \, dx \right)^{\frac{x_1}{x_2}} \left( \int_{\Omega} u^\omega \, dx \right)^{x_2} \\
& \quad + r_7 \left( \int_{\Omega} u^\omega \, dx \right)^{\frac{x_1}{x_2}} \left( \int_{\Omega} \left| \nabla u^{\frac{\sigma + m - 1}{2}} \right|^2 \, dx \right)^{\frac{x_1}{x_2}} \left( \int_{\Omega} u^\omega \, dx \right)^{x_2},
\end{align*}
\]

(3.8)

where \( r_7 = \left( \frac{n - 2 - x_1 n}{n - 2} \right) \left( \frac{n - 2}{n - 2 - x_1 n} \right)^{\frac{x_1}{x_2}} r_6^{\frac{n - 2}{n - 2 - x_1 n}}. \)

By H"{o}lder's and Young's inequalities, we get

\[
\begin{align*}
\int_{\Omega} u^{\sigma + m - 1} \, dx & \leq \left( \varepsilon_2 \int_{\Omega} u^{\sigma + m s - 2} \, dx \right)^{\frac{x_{10}}{\varepsilon_2}} \left( \int_{\Omega} u^\omega \, dx \right)^{\frac{x_{20}}{\varepsilon_2}} \\
& \leq x_{10} \varepsilon_2 \int_{\Omega} u^{\sigma + m s - 2} \, dx + x_{20} \varepsilon_2 \int_{\Omega} u^\omega \, dx,
\end{align*}
\]

where \( x_{10} = \frac{m - 1}{m + s}, \quad n_{10} = \frac{(\sigma + m s - 2)(m - 1)}{m + s}, \quad x_{20} = \frac{s - 1}{m + s - 2}, \quad n_{20} = \frac{(s - 1)n}{m + s - 2}. \)

If we choose \( \varepsilon_2 \) such that \( x_{10} \varepsilon_2 = \frac{1}{2} \), we have

\[
\int_{\Omega} u^{\sigma + m - 1} \, dx \leq \frac{1}{2} \int_{\Omega} u^{\sigma + m s - 2} \, dx + x_{20} \varepsilon_2 \int_{\Omega} u^\omega \, dx.
\]

(3.9)
Combining (3.7)–(3.9), we obtain
\[
\int_{\Omega} u^{\sigma + m s - 2} \, dx \leq 2x_{20} \varepsilon_{2} \int_{\Omega} u^{\sigma} \, dx + 2r_{\varepsilon}^{2} \left( \int_{\Omega} u^{\sigma} \, dx \right)^{2} + 2r_{\varepsilon} \left( \int_{\Omega} \left| \nabla u^\frac{\sigma + m - 1}{2} \right| \, dx \right)^{\frac{\sigma + m - 1}{2}} \left( \int_{\Omega} u^{\sigma} \, dx \right)^{\frac{\sigma - 1}{2}}. 
\]  
(3.10)

Then we can deduce
\[
k_{1}^{\sigma} \int_{\Omega} u^{\sigma + m s - 2} \, dx 
\leq 2x_{20} \varepsilon_{2} \int_{\Omega} u^{\sigma} \, dx + 2r_{\varepsilon}^{2} k_{1}^{\sigma} \left( \int_{\Omega} u^{\sigma} \, dx \right)^{\frac{\sigma + m s - 2}{\sigma}} + 2r_{\varepsilon} \left( \int_{\Omega} \left| \nabla u^\frac{\sigma + m - 1}{2} \right| \, dx \right)^{\frac{\sigma + m - 1}{2}} \left( \int_{\Omega} u^{\sigma} \, dx \right)^{\frac{\sigma - 1}{2}}. 
\]

(3.11)

where \( \varepsilon_{3} \) is a positive constant which will be defined later.

If we choose \( x_{11} = \frac{m - 3}{m + s - 2}, n_{11} = \frac{\sigma + m s - 2(m - 3)}{m + s - 2}, x_{21} = \frac{s + 1}{m + s - 2}, n_{21} = \frac{s + 1}{m + s - 2} \), using (2.4), we get
\[
\int_{\Omega} u^{\sigma + m s - 3} \, dx \leq \left( \int_{\Omega} u^{\sigma + m s - 2} \, dx \right)^{x_{11}} \left( \int_{\Omega} u^{\sigma} \, dx \right)^{x_{21}} \leq x_{11} \int_{\Omega} u^{\sigma + m s - 2} \, dx + x_{21} \int_{\Omega} u^{\sigma} \, dx.
\]

Then we obtain
\[
k_{1}^{\sigma} \phi^{\frac{2}{\sigma}} \int_{\Omega} u^{\sigma + m s - 2} \, dx \leq x_{11} \phi^{\frac{2}{\sigma}} k_{1}^{\sigma} \int_{\Omega} u^{\sigma + m s - 2} \, dx + x_{21} \phi^{\frac{2}{\sigma} + 1}. 
\]  
(3.12)

Combining (3.10) and (3.12), we have
\[
k_{1}^{\sigma} \phi^{\frac{2}{\sigma}} \int_{\Omega} u^{\sigma + m s - 2} \, dx \leq \left( 2x_{20} \varepsilon_{2} \int_{\Omega} u^{\sigma} \, dx + x_{21} \right) \phi^{\frac{2}{\sigma} + 1} + x_{11} k_{1}^{\sigma} \phi^{\frac{2}{\sigma}} \left( \frac{\sigma + m s - 2}{m + s - 2} \right) + 2r_{\varepsilon} \left( \int_{\Omega} \left| \nabla u^\frac{\sigma + m - 1}{2} \right| \, dx \right)^{\frac{\sigma + m - 1}{2}} \phi^{\frac{2}{\sigma} + x_{21}}.
\]
\[ k^{n+1}_1 \int_{\Omega} u^{\alpha+\beta+\gamma-1} dx \]
\[ \leq x_{12} k^{n+1}_1 \int_{\Omega} u^{\alpha+m+s-2} dx + x_{22} k^{n+1}_1 \int_{\Omega} u^{\gamma} dx \]
\[ \leq (2x_{20} \varepsilon_2^{-1} + x_{12} x_{22} k_1) \phi \]
\[ + \left( 2r_2 x_{12} k_1^{n+1} \frac{\varepsilon_2^{-1}}{n-2} + 2r_4 x_{12} k_1^{n+1} \frac{\varepsilon_2^{-1}}{n-2} \int_{\Omega} u^{\alpha+\beta+\gamma-2} \frac{x_{11} n \varepsilon_4^{-1}}{n-2} \right) \phi \]
\[ + \left( 2r_2 x_{12} k_1^{n+1} \frac{\varepsilon_2^{-1}}{n-2} + 2r_4 x_{12} k_1^{n+1} \frac{\varepsilon_2^{-1}}{n-2} \int_{\Omega} u^{\alpha+\beta+\gamma-2} \frac{x_{11} n \varepsilon_4^{-1}}{n-2} \right) \phi \]
\[ \leq x_{12} \int_{\Omega} u^{\alpha+\beta+\gamma-2} dx + x_{22} \int_{\Omega} u^{\gamma} dx. \]

Combining (3.10) and (3.14), we obtain

\[ x_{12} k^{n+1}_1 \int_{\Omega} u^{\alpha+\beta+\gamma-1} dx \]
\[ \leq x_{12} k^{n+1}_1 \int_{\Omega} u^{\alpha+m+s-2} dx + x_{22} k^{n+1}_1 \int_{\Omega} u^{\gamma} dx \]
\[ \leq (2x_{20} \varepsilon_2^{-1} + x_{12} x_{22} k_1) \phi \]
\[ + \left( 2r_2 x_{12} k_1^{n+1} \frac{\varepsilon_2^{-1}}{n-2} + 2r_4 x_{12} k_1^{n+1} \frac{\varepsilon_2^{-1}}{n-2} \int_{\Omega} u^{\alpha+\beta+\gamma-2} \frac{x_{11} n \varepsilon_4^{-1}}{n-2} \right) \phi \]
\[ + \left( 2r_2 x_{12} k_1^{n+1} \frac{\varepsilon_2^{-1}}{n-2} + 2r_4 x_{12} k_1^{n+1} \frac{\varepsilon_2^{-1}}{n-2} \int_{\Omega} u^{\alpha+\beta+\gamma-2} \frac{x_{11} n \varepsilon_4^{-1}}{n-2} \right) \phi \]
\[ \leq x_{12} \int_{\Omega} u^{\alpha+\beta+\gamma-2} dx + x_{22} \int_{\Omega} u^{\gamma} dx. \]

where \( \varepsilon_5 \) is a positive constant which will be defined later.

Combining (3.6), (3.11), (3.13), and (3.15), we have

\[ \phi'(t) \leq (na + 2r_1 k_2 x_{20} \varepsilon_2^{-1} + 2r_4 x_{20} \varepsilon_2^{-1} - x_{12} k_1 + r_4 x_{22} k_1) \phi \]
\[ + \left( 2r_1 k_2^{-1} x_{12} k_1^{n+1} \frac{\varepsilon_2^{-1}}{n-2} + 2r_4 x_{12} k_1^{n+1} \frac{\varepsilon_2^{-1}}{n-2} \int_{\Omega} u^{\alpha+\beta+\gamma-2} \frac{x_{11} n \varepsilon_4^{-1}}{n-2} \right) \phi \]
\[ + \left( 2r_1 k_2^{-1} x_{12} k_1^{n+1} \frac{\varepsilon_2^{-1}}{n-2} + 2r_4 x_{12} k_1^{n+1} \frac{\varepsilon_2^{-1}}{n-2} \int_{\Omega} u^{\alpha+\beta+\gamma-2} \frac{x_{11} n \varepsilon_4^{-1}}{n-2} \right) \phi \]
\[ \leq \phi. \]

}\]
If we choose suitable $\varepsilon_1, \varepsilon_3, \varepsilon_4, \varepsilon_5$ such that

\[
2r_1k_2r_6k_1^{n+1-\frac{x_1^2}{n}}x_1H - \frac{n-2}{n-2} \varepsilon_3 + \frac{1}{2}r_2k_2\varepsilon_1 + r_2k_2^{n+1}x_1k_1^{\frac{x_1^2}{n}} - \frac{n-2}{n-2} \varepsilon_4
\]

\[+ 2r_4r_6x_1k_1^{n+1-\frac{x_1^2}{n}}x_1H - \frac{n-2}{n-2} \varepsilon_5 - r_3 = 0. \tag{3.17}
\]

Substituting (3.17) into (3.16), we derive

\[
\phi'(t) \leq (n\alpha + 2r_1k_3x_2t_2\varepsilon_2 - \frac{s_1\varepsilon_2}{20} + 2r_4x_2\varepsilon_2 - \frac{s_1\varepsilon_2}{20} x_12k_1 + r_4x_22k_1) \phi
\]

\[+ \left( 2r_1k_2r_6k_1^{n+1-\frac{x_1^2}{n}}x_1H - \frac{n-2}{n-2} \varepsilon_3 + \frac{1}{2}r_2k_2\varepsilon_1 + r_2k_2^{n+1}x_1k_1^{\frac{x_1^2}{n}} - \frac{n-2}{n-2} \varepsilon_4 \right) \phi^{1+\frac{2k_1}{n-2-x_1n}}
\]

\[+ \frac{1}{2}r_2k_2^{n+1}x_1k_1^{\frac{x_1^2}{n}}x_1k_1^{\frac{x_1^2}{n}} - \frac{n-2}{n-2} \varepsilon_3 + r_4r_6x_1k_1^{n+1-\frac{x_1^2}{n}}x_1H - \frac{n-2}{n-2} \varepsilon_5 \phi^{1+\frac{2k_1}{n-2-x_1n}}).
\tag{3.18}
\]

Using Hölder’s and Young’s inequalities, we have

\[
\phi^{1+\gamma} \leq \left( 1 - \frac{\gamma}{4} \right) \phi + \frac{\gamma}{4} \phi^5. \tag{3.19}
\]

Applying (3.19) to $\phi^{1+\frac{2k_1}{n-2-x_1n}}, \phi^{1+\frac{2k_1}{n-2-x_1n}}, \phi^{1+\frac{2k_1}{n-2-x_1n}}$ in (3.18), respectively, we obtain

\[
\phi'(t) \leq a(t)\phi(t) + b(t)\phi^5(t),
\tag{3.20}
\]

where

\[
a(t) = \left( n\alpha + 2r_1k_3x_2t_2\varepsilon_2 - \frac{s_1\varepsilon_2}{20} + 2r_4x_2\varepsilon_2 - \frac{s_1\varepsilon_2}{20} x_12k_1 + r_4x_22k_1 \right)
\]

\[+ \left( 2r_1k_2r_6k_1^{n+1-\frac{x_1^2}{n}}x_1H - \frac{n-2}{n-2} \varepsilon_3 + \frac{1}{2}r_2k_2\varepsilon_1 + r_2k_2^{n+1}x_1k_1^{\frac{x_1^2}{n}} - \frac{n-2}{n-2} \varepsilon_4 \right) \phi^{1+\frac{2k_1}{n-2-x_1n}}
\]

\[+ \frac{1}{2}r_2k_2^{n+1}x_1k_1^{\frac{x_1^2}{n}}x_1k_1^{\frac{x_1^2}{n}} - \frac{n-2}{n-2} \varepsilon_3 + r_4r_6x_1k_1^{n+1-\frac{x_1^2}{n}}x_1H - \frac{n-2}{n-2} \varepsilon_5 \phi^{1+\frac{2k_1}{n-2-x_1n}}.
\tag{3.21}
\]
and
\[
b(t) = \left( 2r_1 k_2 r_7 k_1^{n-\frac{x_1}{n-2} \frac{2}{n-3} \frac{n}{n-1} m} + 2r_1 k_2 r_6 k_1^{n-\frac{x_1}{n-2} \frac{2}{n-3} \frac{n}{n-1} m} \right) \frac{n-2 - x_1 n - \frac{x_1}{n-2} \frac{2}{n-3} \frac{n}{n-1} m}{e_3} + 2r_4 \epsilon_1 x_1 + \frac{1}{2} r_3 k_2 x_1^{-1} r_5 \left( 2x_2 \epsilon_2 - \frac{x_1}{n} + x_1 k_1 \right) k n \left( \frac{x_1}{n-2} \frac{2}{n-3} \frac{n}{n-1} m \right) \frac{2}{n-2 - x_1 n} \left( n - 2 - x_1 n \right) \right) \frac{x_1}{2(n-2 - x_1 n)}.
\] (3.22)

Multiplying both sides of (3.20) by \( \phi^{-5}(t) \), we obtain
\[
\phi'(t) \phi^{-5}(t) \leq a(t) \phi^{-4}(t) + b(t). \quad (3.23)
\]
That is,
\[
- \left( \phi^{-4}(t) \right)' \leq 4a(t) \phi^{-4}(t) + 4b(t). \quad (3.24)
\]
Setting \( H(t) = \int_0^t a(\tau) d\tau \), (3.24) can be rewritten as
\[
\left( \phi^{-4}(t) e^{4H(t)} \right)' \geq -4b(t) e^{4H(t)}. \quad (3.25)
\]
Integrating (3.25) from 0 to \( t \), we have
\[
\phi^{-4}(t) e^{4H(t)} - \phi^{-4}(0) \geq -4 \int_0^t b(\tau) e^{4H(\tau)} d\tau. \quad (3.26)
\]
That is to say,
\[
\frac{e^{4H(\tau)}}{\phi^4(t)} - \frac{1}{\phi^4(0)} \geq -4 \Theta(t), \quad (3.27)
\]
where \( \Theta(t) = \int_0^t b(\tau) e^{4H(\tau)} d\tau \).

Taking the limit to (3.27) as \( t \to t^* \), we get
\[
\Theta(t^*) \geq \frac{1}{4 \phi^4(0)}.
\]
From the definition of \( \Theta(t) \), we have \( \frac{d\Theta(t)}{dt} = b(t) e^{4H(t)} > 0 \). We get \( \Theta(t) \) is a strictly increasing function. So we can get
\[
t^* \geq \Theta^{-1} \left( \frac{1}{4 \phi^4(0)} \right),
\]
from which we complete the proof of Theorem 3.1. \( \square \)
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