A simple family of nonadditive quantum codes

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Most known quantum codes are additive, meaning the codespace can be described as the simultaneous eigenspace of an abelian subgroup of the Pauli group. While in some scenarios such codes are strictly suboptimal, very little is understood about how to construct nonadditive codes with good performance. Here we present a family of nonadditive quantum codes for all odd blocklengths n, that has a particularly simple form. Our codes correct single qubit erasures while encoding a higher dimensional space than is possible with an additive code or, for n ≥ 11, any previous codes.

Quantum error correcting codes will be essential if large-scale quantum computers are ever to be built. Nearly all known quantum codes are stabilizer (or additive) codes, in the framework of [1], but it has been known for some time that nonadditive codes can perform better [2]. So far this has only been shown for distance 2 quantum codes. A distance 2 code corrects any single qubit error with unknown location. Particularly in this second capacity, such codes may be quite important in the ancilla preparation phase of fault-tolerant quantum computing [3]. Furthermore, distance 2 codes promise to shed light on the structure of general quantum codes due to their simplicity.

It is shown in [4] that the best additive distance-2 codes are [n, n − 2, 2] for n even and [n, n − 3, 2] for n odd, where we have used the notation [n, k, d] to indicate an n qubit additive code with distance d and k encoded qubits. In [5] it was shown that these codes are optimal for even n, For n = 5, a ((5, 6, 2)) nonadditive code was found in [2], where we have used the notation ((n, K, d)) to indicate a distance d code of size n, protecting a K dimensional code space, in analogy with the standard classical notation of (n, K, d) denoting an n bit code with K codewords and distance d. This code, together with the family of ((n, 3 · 2n−3, 2)) codes it generates [2], gives the only performance improvement from nonadditive codes known for qubits. Here, we present a family of new nonadditive codes that improve on all known constructions. Our codes correct single qubit erasures while encoding a higher dimensional space than is possible with any additive code or, for n ≥ 11, any previous codes.

A quantum code must detect more than simply bitflips. It must detect the errors whose ith bits are given by

\[ x_i^{(j)} = \delta_{ij} \quad 0 \leq i < 5 \]

These, together with their bitwise complements \( \bar{x}^{(j)} \), form a classical (5, 10, 2) code. This can be seen easily as every codeword is either weight 1 or weight 4, and single bit errors necessarily change the weight of a codeword by 1, taking it out of the codespace. In the following, we will call a code self-complementary if the complement of each codeword is also in the code.

A quantum code must detect more than simply bitflips. It must detect the errors X = \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) (“amplitude” errors, analogous to the classical bitflip), Z = \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) (errors in phase) and Y = XZ (when both errors occur) [6]. It will be sufficient to show for our code that any of these errors map code vectors to some vector orthogonal to the codespace. The full necessary and sufficient error-correction conditions were derived in [7].

We now show how to turn the classical code above into a quantum code. The five basis vectors of our quantum code are related to the classical code by taking superpositions of codewords with their complements:

\[ v^{(j)} = |x^{(j)}\rangle + |\bar{x}^{(j)}\rangle \quad 0 \leq j < 5 \]

While we would like to approach this bound, at least for large n, we achieve a more modest goal. Our code is asymptotically almost optimal, but the rate of convergence is weaker due to the square root in \( M_{k,l} \).

**THE CODE FOR n = 5**

We first describe our construction for n = 5 and then show how to extend it to general odd n. We obtain our quantum codes from specific classical codes. Consider the following five five-bit strings:

\[ x^{(j)} \quad 0 \leq j < 5 \]

whose ith bits are given by

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\[ v^{(j)} = |x^{(j)}\rangle + |\bar{x}^{(j)}\rangle \quad 0 \leq j < 5 \]

\[ v^{(0)} = |00000\rangle + |01111\rangle \]
\[ v^{(1)} = |01000\rangle + |10111\rangle \]
\[ v^{(2)} = |00100\rangle + |11011\rangle \]
\[ v^{(3)} = |00010\rangle + |11101\rangle \]
\[ v^{(4)} = |00001\rangle + |11110\rangle . \]
What is the effect of single qubit errors? An $X$ error on the $i$th qubit acts on the codewords as

$$X_i v^{(j)} = |x^{(i)} \oplus x^{(j)}| + |x^{(i)} \oplus \bar{x}^{(j)}|,$$

so that for $i \neq j$ the $X_i v^{(j)}$'s are all the + signs superpositions of weight two vectors with their weight three complements, while for $i = j$ we always have the state $(00000) + (11111)$, the superposition of the weight zero vector with its weight five complement. These are all orthogonal to the codespace since they are constructed entirely of superpositions of weights not in the codewords, which is a direct result of having started with a classical distance $d = 2$ code.

Turning our attention to the $Z$ errors, we find

$$Z_i v^{(j)} = (-1)^{\delta_{ij}} (|x^{(j)}| - |\bar{x}^{(j)}|).$$

These five vectors are the original code vectors but with a + sign as relative phase and sometimes an overall phase. They are all orthogonal to the code vectors and to one another, as well as to the $X$ error states of Eq. (2).

Finally, $Y$ errors act on the codewords according to

$$Y_i v^{(j)} = (-1)^{\delta_{ij}} (|x^{(i)} \oplus x^{(j)}| - |x^{(i)} \oplus \bar{x}^{(j)}|).$$

These are again orthogonal to the codespace as they consist of superpositions of weights not included in the original code, and also to the image of the code under $X$ and $Z$ errors (the $X$'s due to the relative phase, and the $Z$'s due to being made of the wrong weight strings).

We emphasize that this code is not a subcode of the $((5, 2))$ code in $[2]$. Indeed, it is not a subcode of any $((5, 2))$ code, since there are no vectors orthogonal to the codespace with distance 2 to all five vectors. To see this, observe that the set of vectors obtained by single qubit errors on the code vectors spans the entire space orthogonal to the codespace. Since every vector outside the codespace might be mistaken for a single error on some state in the code, no such vector can be added to our code while maintaining a minimum distance of 2.

### The Code for Odd $n$

We now show how to extend our construction to any odd $n$. The method we have used is quite general, and is summarized in the following lemma, whose proof follows immediately from the reasoning above:

**Lemma 1**: Any self-complementary $(n, K, d > 1)$ classical code leads to a $(n, K/2, 2)$ quantum code by pairing up codewords with their complements in superposition.

Our construction again starts from a classical code, which is obtained as follows: Choose an ordering of the weight $i$ bitstrings of length $n$ and let $w^{(i, j, n)}$ be the $j$th such string, where $0 \leq j < \binom{n}{i}$. Letting $n = 4k + 2l + 3$, we consider the classical distance-2 codes indexed by $(k, l)$ whose codewords (indexed by $(i, j)$) are

$$v^{(i, j)}_{(k, l)} = w^{(2i + 1, j, 4k + 2l + 3)}$$

for

$$0 \leq i \leq k,$$

$$0 \leq j < \left(\frac{4k + 2l + 3}{2i + 1}\right).$$

Together with their complements $\bar{v}^{(i, j)}_{(k, l)}$. Note that our $(5, 10, 2)$ code is the special case of $[3]$ with $k = 0, l = 1$.

Using Lemma 1, we now turn this code into a $(n = 4k + 2l + 3, M_{(k,l)})$ quantum code, spanned by

$$v^{(i, j)}_{(k, l)} = |v^{(i, j)}_{(k, l)}\rangle + |\bar{v}^{(i, j)}_{(k, l)}\rangle$$

$$l = 0, 1,$$

$$0 \leq i \leq k,$$

$$0 \leq j < \left(\frac{4k + 2l + 3}{2i + 1}\right).$$

Let $C_n$ denote our code.

It remains for us to count $M_{(k,l)}$, the total number of code vectors. We have for $l \in \{0, 1\}$ that

$$M_{(k,l)} = \sum_{i=0}^{k} \left(\frac{4k + 2l + 3}{2i + 1}\right) = 2^{k + 2l + 3} - \frac{1}{2} \left(\frac{4k + 2l + 2}{2k + l + 1}\right),$$

where we have evaluated the sum using Pascal's first identity $(\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m})$.

As mentioned above, $(n, 3 \cdot 2^{n-4}, 2)$ codes were constructed in $[3]$. For $n \leq 9$, these codes encode more elements than ours, while for $n \geq 11$ our codes have a larger codespace. Evaluating $M_{(k,l)}$ as $n \to \infty$ gives

$$M_{(k,l)} \approx 2^{n-2} \left(1 - \frac{\binom{n-1}{2}}{2^{n-1}}\right) = 2^{n-2} \left(1 - \sqrt{n} \sqrt{\pi(n-1)}\right),$$

allowing us to asymptotically encode $n - 2 - \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{n(n-1)}}$ qubits. While the rate of convergence may be suboptimal ($O(\frac{1}{\sqrt{n}})$ vs. $O(\frac{1}{\sqrt{n}})$), in light of Eq. (1) the resulting limit of $n - 2$ encoded qubits cannot be surpassed and our code is asymptotically close to optimal.

### The Automorphism Group

The automorphism group of a code is the group of unitaries that map the codespace to itself and consist of the composition of local unitaries on each qubit and a permutation of qubits. The automorphisms of a code characterize the code's symmetries. Since they correspond to the logical operations that can be applied transversally, they are relevant for fault tolerant quantum computation. The following lemma gives the automorphism group of our code.
Lemma 2: The automorphisms of $C_n$ are exactly

$$ (X^\otimes n)^b Z^f \circ \pi_n \equiv (X^\otimes n)^b (\bigotimes_{i=1}^n Z_i^{f_i}) \circ \pi_n, \quad (4) $$

with $|f|$ even, $b \in \{0, 1\}$ and $\pi_n \in S_n$ is a permutation.

This is most easily proved as a consequence of the following characterization of the projector onto $C_n$.

Lemma 3: The projector onto $C_n$ is given by

$$ P_{C_n} = \frac{1}{2^n} \left( I^\otimes n + X^\otimes n \right) \sum_{s=0}^{n-1} K^{(2s)} \left( \sum_{|x|=2s} Z^x \right) \quad (5) $$

where we have let $K^{(2s)} = \sum_{i=0}^k K^{(2s)}_{2i+l}$, as well as $K^{(2s)}_{2i+l} = 2 \sum_{t=0}^{2i+l} \binom{2s}{s-t}$ for $s > 0$ and $K^{(2s)}_{2i+l} = \binom{n}{2i+l}$.

Proof: Letting $P_{2i+l} = \sum_j \langle \psi^{(j)}_{k,l} | \psi^{(j)}_{k,l} \rangle$, we have

$$ P_{2i+l} = \frac{1}{2} \sum_{|w|=2i+l} \sum_{b_1 b_2=0} \binom{2s}{s-t} \left( (X^\otimes n)^{b_1} |w\rangle \langle w| (X^\otimes n)^{b_2} \right), \quad (6) $$

which, using the fact that $|w\rangle \langle w| = \bigotimes_{i=1}^n \left( \frac{I-(1+(-1)^{|w|} Z_i)}{2} \right)$, is equal to

$$ \frac{1}{2^n} \sum_{|x| \text{ even}} \left( \sum_{|w|=2i+l} (-1)^{|w|} \right) \left( I^\otimes n + X^\otimes n \right) Z^x $$

$$ = \frac{1}{2^n} \sum_{s=0}^{n-1} K^{(2s)}_{2i+l} \left( I^\otimes n + X^\otimes n \right) \left( \sum_{|x|=2s} Z^x \right) \quad (7) $$

Summing over $i = 0 \ldots k$ proves the claim. □

Proof (of Lemma 2): Since our code is permutation invariant, we only need to show that the unitaries of the form $\bigotimes_{i=1}^n U_i$ leaving $C_n$ invariant are exactly of the form $Z^f$ or $X^\otimes n Z^f$ with $|f|$ even. To see this, note that Eq. 5 has a terms of two types: those of weight (the number of non-identity Paulis) less than $n$, namely

$$ \frac{1}{2^n} \sum_{s=0}^{n-1} K^{(2s)} \left( \sum_{|x|=2s} Z^x \right), \quad (8) $$

and those of weight $n$. Since conjugation by local unitaries does not change the weight of a Pauli operator and indeed, acts trivially on identity factors, if we wish $P_{C_n}$ to be invariant under conjugation by $\bigotimes_{i=1}^n U_i$, we must have

$$ \left( \bigotimes_{i=1}^n U_i \right) Z^x \left( \bigotimes_{i=1}^n U_i \right)^* = Z^x \quad (9) $$

whenever $|x|$ is even. In other words, $\bigotimes_{i=1}^n U_i$ must commute with $Z^x$ for all even weight $x$. This implies $\bigotimes_{i=1}^n U_i = (X^\otimes n)^b Z^f$ for $b \in \{0, 1\}$. To see that we must have $|f|$ even, note that $(X^\otimes n)^b Z^f$ commutes with

$$ \frac{1}{2^n} \sum_{s=0}^{n-1} K^{(2s)} \left( \sum_{|x|=2s} Z^x \right), $$

so that to commute with $P_{C_n}$ it must also commute with $X^\otimes n$. □

CONCLUSION

Our family of codes can be extended to Heisenberg-Weyl type errors on higher dimensions with exactly the same counting as before. Instead of using classical codewords paired with their complements, one uses codewords of the form

$$ |v\rangle + X^\otimes n |v\rangle + (X^\otimes n)^2 |v\rangle \ldots (X^\otimes n)^{D-1} |v\rangle \quad (10) $$

in $D$ dimensions, where $X|i\rangle = |(i+1) \mod D\rangle$ defines the $D$-dimensional $X$ operator. The same classical codewords $v$ of Eq. 4 generate quantum $((4k+2l+3, M(k,l)2))$-codes with $M(k,l)$ as before.

So far we have not been able to find similar constructions for higher distances as Lemma 1 is specific to distance 2, but we are hopeful our work will inspire new thinking on nonadditive codes.

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