Evolution Equations in Hilbert Spaces via the Lacunae Method

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Abstract

In this paper we consider evolution equations in the abstract Hilbert space under the special conditions imposed on the operator at the right-hand side of the equation. We establish the method that allows us to formulate the existence and uniqueness theorem and find a solution in the form of a series on the root vectors of the right-hand side. We consider fractional differential equations of various kinds as an application. Such operators as the Riemann-Liouville fractional differential operator, the Riesz potential, the difference operator have been involved.

Keywords: Evolution equation; Fractional differential equations; Strictly accretive operator; Abel-Lidskii basis property; Schatten-von Neumann class; convergence exponent.

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1 Introduction

In the paper [17] we obtained the clarification of the results by Lidskii V.B. [20] on the decomposition on the root vector system of the non-selfadjoint operator. We used a technique of the entire function theory and introduce a so-called Schatten-von Neumann class of the convergence exponent. Considering strictly accretive operators satisfying special conditions formulated in terms of the norm, we constructed a sequence of contours of the power type in the contrary to the results by Lidskii V.B. [20], where a sequence of contours of the exponential type was used. In this paper we produce the application of the mentioned method to evolution equations in the abstract Hilbert space with the right-hand side of the special type. Here, we should appeal to a plenty of applications to concrete differential equations connected with modeling various physical-chemical processes: filtration of liquid and gas in highly porous fractal medium; heat exchange processes in medium with fractal structure and memory; casual walks of a point particle that starts moving from the origin by self-similar fractal set; oscillator motion under the action of elastic forces which is characteristic for viscoelastic media, etc. In particular, we would like to study the existence and uniqueness theorems for evolution equations with the right-hand side –
a differential operator with a fractional derivative in final terms. In this connection such operators as a Riemann-Liouville fractional differential operator, Kipriyanov operator, Riesz potential, difference operator are involved. Note that analysis of the required conditions imposed upon the right-hand side of the studied class of evolution equations deserves to be mentioned. In this regard we should note a well-known fact (see for instance [33]) that a particular interest appears in the case when a senior term of the operator (see [15]) is not selfadjoint at least for in the contrary case there is a plenty of results devoted to the topic within the framework of which the following papers are well-known [10], [19], [22], [23], [33]. Indeed, most of them deal with a decomposition of the operator to a sum where the senior term must be either a selfadjoint or normal operator. In other cases the methods of the papers [14], [15] become relevant and allow us to study spectral properties of operators whether we have the mentioned above representation or not. Here, we ought to stress that the results of the papers [2], [22] can be also applied to study non-selfadjoint operators but based on the sufficiently strong assumption regarding the numerical range of values of the operator (the numerical range belongs to a parabolic domain). The methods of [15] that are applicable to study non-selfadjoint operators can be used in the natural way if we deal with a more abstract construction – the infinitesimal generator of a semigroup of contraction [16]. The central challenge of the latter paper is how to create a model representing a composition of fractional differential operators in terms of the semigroup theory. Here we should note that motivation arouse in connection with the fact that a second order differential operator can be presented as a some kind of a transform of the infinitesimal generator of a shift semigroup and stress that the eigenvalue problem for the operator was previously studied by methods of theory of functions [28], [3]. Having been inspired by novelty of the idea we generalize a differential operator with a fractional integro-differential composition in the final terms to some transform of the corresponding infinitesimal generator of the shift semigroup. By virtue of the methods obtained in the paper [15] we managed to study spectral properties of the infinitesimal generator transform and obtained an outstanding result – asymptotic equivalence between the real component of the resolvent and the resolvent of the real component of the operator. The relevance is based on the fact that the asymptotic formula for the operator real component can be established in most cases due to well-known asymptotic relations for the regular differential operators as well as for the singular ones [31]. Thus, we have theorems establishing spectral properties of some class of non-selfadjoint operators which allow us, jointly with the results [17], to study the Cauchy problem for the evolution equation by the functional analysis methods. Note that the abstract approach to the Cauchy problem for the fractional evolution equation was previously implemented in the papers [1], [5]. However, the main advantage of this paper is the obtained formula for the solution of the evolution equation with the relatively wide conditions imposed upon the right-hand side, wherein the derivative at the left-hand side is supposed to be of the real order. We consider the evolution equations with the right-hand side – an operator function of the power type. This problem appeals to many ones that lie in the framework of the theory of differential equations for instance in the paper [27] the solution of the evolution equation modeling the switching kinetics of ferroelectrics in the injection mode can be obtained in the analytical way if we impose the conditions upon the right-hand side. The following papers deal with equations which can be studied by the obtained in this paper abstract method [24], [26], [25], [30], [34]. Thus, we can claim that the offered approach is undoubtedly novel and relevant.
2 Preliminaries

Let $C, C_i, i \in \mathbb{N}_0$ be real positive constants. We assume that a value of $C$ can be different in various formulas and parts of formulas but values of $C_i$ are certain. Denote by $\text{Fr} M$ the set of boundary points of the set $M$. Everywhere further, if the contrary is not stated, we consider linear densely defined operators acting on a separable complex Hilbert space $\mathfrak{H}$. Denote by $\mathcal{B}(\mathfrak{H})$ the set of linear bounded operators on $\mathfrak{H}$. Denote by $\hat{L}$ the closure of an operator $L$. We establish the following agreement on using symbols $\tilde{\cdot}$ of an operator numerical range $\mathfrak{R}(\cdot)$.

Consider a pair of complex Hilbert spaces $\mathfrak{H}, \mathfrak{K}$, the notation $\mathfrak{H}_+ \subset \mathfrak{K}$ means that $\mathfrak{H}_+$ is dense in $\mathfrak{K}$ as a set of elements and we have a bounded embedding provided by the inequality $\|f\|_\mathfrak{H} \leq C_0 \|f\|_\mathfrak{K}, C_0 > 0, f \in \mathfrak{H}_+$, moreover any bounded set with respect to the norm $\mathfrak{H}_+$ is compact with respect to the norm $\mathfrak{K}$. Let $L$ be a closed operator, for any closable operator $S$ such that $\hat{S} = L$, its domain $\text{D}(S)$ will be called a core of $L$. Denote by $\text{D}_0(L)$ a core of a closable operator $L$. Let $\mathcal{P}(L)$ be the resolvent set of an operator $L$ and $\mathcal{R}_L(\zeta), \zeta \in \mathcal{P}(L), [\mathcal{R}_L := \mathcal{R}_L(0)]$ denotes the resolvent of an operator $L$. Denote by $\lambda_i(L), i \in \mathbb{N}$ the eigenvalues of an operator $L$. Suppose $L$ is a compact operator and $N := (L^* L)^{1/2}$, $r(N) := \dim \mathcal{R}(N)$; then the eigenvalues of the operator $N$ are called the singular numbers (s-numbers) of the operator $L$ and are denoted by $s_i(L), i = 1, 2, \ldots, r(N)$. If $r(N) < \infty$, then we put by definition $s_i = 0, i = r(N) + 1, 2, \ldots,$. Let $\nu(L)$ denotes the sum of all algebraic multiplicities of an operator $L$. Denote by $n(r)$ a function equals to the quantity of the elements of the sequence $\{a_n\}_1^\infty, |a_n| \uparrow \infty$ within the circle $|z| < r$. Let $A$ be a compact operator, denote by $n_A(r)$ counting function a function $n(r)$ corresponding to the sequence $\{s^{-1}_i(A)\}^\infty_1$. Let $\mathfrak{S}_p(\mathfrak{H}), 0 < p < \infty$ be a Schatten-von Neumann class and $\mathfrak{S}_\infty(\mathfrak{H})$ be the set of compact operators. Denote by $\mathfrak{S}_p(\mathfrak{H})$ the class of the operators such that $A \in \mathfrak{S}_p(\mathfrak{H}) \Rightarrow \{A \in \mathfrak{S}_{p+\varepsilon}, \forall \varepsilon > 0\}$. In accordance with [17] we will call it Schatten-von Neumann class of the convergence exponent. Suppose $L$ is an operator with a compact resolvent and $s_n(R_L) \leq C n^{-\mu}, n \in \mathbb{N}, 0 \leq \mu < \infty; \text{then we denote by } \mu(L) \text{ order of the operator } L \text{ (see [33]). Denote by } \mathfrak{R} \text{ by } \mathfrak{R}_L := (L + L^*)/2, \mathcal{M} \mathfrak{L} := (L - L^*)/2i \text{ the real and imaginary components of an operator } L \text{ respectively. In accordance with the terminology of the monograph [9] the set } \Theta(L) \equiv \{z \in \mathbb{C} : \zeta = (L f, f)_\mathfrak{H}, f \in \mathfrak{D}(L), \|f\|_\mathfrak{H} = 1\} \text{ is called the numerical range of an operator } L. \text{ An operator } L \text{ is called sectorial if its numerical range belongs to a closed sector } \mathfrak{L}_\theta(\theta) := \{\zeta : |\arg(\zeta - i)| \leq \theta < \pi/2\}, \theta \text{ is the semi-angle of the sector } \mathfrak{L}_\theta. \text{ If we want to stress the correspondence between } \nu \text{ and } \theta, \text{ then we will write } \theta_L. \text{ An operator } L \text{ is called bounded from below if the following relation holds } \mathfrak{R}_L(f, f)_\mathfrak{H} \geq \gamma_L \|f\|^2_\mathfrak{H}, f \in \mathfrak{D}(L), \gamma_L \in \mathbb{R}, \text{ where } \gamma_L \text{ is called a lower bound of } L. L \text{ is called accretive if } \gamma_L = 0. \text{ An operator } L \text{ is called strictly accretive if } \gamma_L > 0. \text{ An operator } L \text{ is called } m\text{-accretive if the next relation holds } (A + \zeta)^{-1} \in \mathcal{B} \mathfrak{R}(\mathfrak{H}), \| (A + \zeta)^{-1} \| \leq (\mathfrak{R} \zeta)^{-1}, \mathfrak{R} \zeta > 0. \text{ An operator } L \text{ is called symmetric if one is densely defined and the following equality holds } (L f, g)_\mathfrak{H} = (f, L g)_\mathfrak{H}, f, g \in \mathcal{D}(L). \text{ Consider a sesquilinear form } t[\cdot, \cdot] (\text{ see [9]} \text{ defined on a linear manifold of the Hilbert space } \mathfrak{H}. \text{ Let } \mathfrak{h} = (t + t^*)/2, \mathfrak{e} = (t - t^*)/2i \text{ be a real and imaginary component of the form } t \text{ respectively, where } t^*[u, v] = t[v, u], D(t^*) = D(t). \text{ Denote by } t[\cdot] \text{ the quadratic form corresponding to the sesquilinear form } t[\cdot, \cdot]. \text{ According to these definitions, we have } \mathfrak{h}[\cdot] = \mathfrak{R} t[\cdot], \mathfrak{e}[\cdot] = \mathfrak{I} m t[\cdot]. \text{ Denote by } \check{t} \text{ the closure of a form } t. \text{ The range of a quadratic form }
t[f], f ∈ D(t), ∥f∥₀ = 1 is called range of the sesquilinear form t and is denoted by Θ(t). A form t is called sectorial if its range belongs to a sector having a vertex υ situated at the real axis and a semi-angle 0 ≤ θ < π/2. Due to Theorem 2.7 [9, p.323] there exist unique m-sectorial operators $T_t, T_h$ associated with the closed sectorial forms $t, h$ respectively. The operator $T_h$ is called a real part of the operator $T_t$ and is denoted by Re $T_t$.

Everywhere further, unless otherwise stated, we use notations of the papers [8], [9], [11], [12], [32]. Consider the following hypotheses regarding an operator.

(H1) There exists a Hilbert space $H_+ ⊂⊂ H$ and a linear manifold $M$ that is dense in $H_+$. The operator $L$ is defined on $M$.

(H2) $|(Lf, g)_θ| ≤ C_1 ∥f∥_{θ_+} ∥g∥_{θ_+}, \ Re(Lf, f)_θ ≥ C_2 ∥f∥_{θ_+}^2$, $f, g ∈ M$, $C_1, C_2 > 0$.

Throughout the paper we consider a restriction $W$ of the operator $L$ on the set $M$. We also use the short-hand notations $A := R_{iW}$, $µ := µ(H)$, where $H := ReW$.

**Auxiliary propositions**

Firstly, we consider general statements proved in [17] with the made refinement related to the involved notion of the convergence exponent as well as newly constructed sequence of contours allowing to arrange the eigenvectors in the power type way, we used this expression following the literary style of the monograph [20]. We implement the approach that refers us to the notion – operator order, it gives us an opportunity to reformulate results of the spectral theory in the more convenient and applicable way. Recall that in the paper [20] there was considered a sequence of contours of the exponential type, the condition $α > ρ$ (here and further $ρ$ denotes the index of the Schatten-von Neumann class of the convergence exponent) is imposed (see [20], [17]). We improved this result in the paper [17] in the following sense, we produced a sequence of the power type contours what gives us the opportunity to obtain a solution of the problem in the case $α = ρ$. Moreover, we have omitted the conditions imposed on the semi-angle of the sector containing the numerical range of values of the involved operator. Such a significant achievement is obtained by virtue of the way of choosing a contour which we consider throughout the paper $γ := Fr \{ θ ∈ \mathbb{C} : |θ| < r, |argθ| ≤ θ_0 \} \cup M_r := \{ λ : |λ| < r, |argλ| ≤ θ_0 \}$, $r = C_2(1 - C_1ctgθ/C_2)$, where the semi-angle $θ$ related to the operator $W$ is sufficiently small (see reasonings of Theorem 2 [17]), $r$ is chosen so that the operator $(E−λA)^{−1}$ is regular within the corresponding circle, $ε > 0$ is sufficiently small. The auxiliary theorems given bellow (see [17]) give us a tool to study the existence and uniqueness theorems in the abstract Hilbert space. Moreover, we obtain a solution that can be presented by the series on the operator $A$ root vectors $e_{q+i}$ with the coefficients $c_{q+i}$, where the index $q$ relates to the eigenvalue, the index $ξ$ relates to the geometrical multiplicity, the index $i$ relates to the algebraic multiplicity, the convergence is understood in the Abel-Lidskii sense (see [20]). The idea of the proofs of the auxiliary theorems given bellow belongs to Lidskii V.B. However, we produce the proofs in [17], since the made refinement corresponding to the case, when $ρ$ does not equal the index of the Schatten-von Neumann class, deserves to be considered itself.
Theorem 1. Assume that hypothesis H1, H2 hold, \( A \in \tilde{S}_p \), \( p \leq \alpha \). Moreover in the case \( A \in \tilde{S}_p \setminus S_p \) the additional condition holds

\[
\frac{n_{A^m+1}(r_{m+1})}{r^p} \to 0, \quad m = [\rho].
\]

Then a sequence of natural numbers \( \{N_\nu\}_0^\infty \) can be chosen so that

\[
\frac{1}{2\pi i} \int \frac{e^{-\lambda^\alpha t}A(E - \lambda A)^{-1}}{\gamma} \, d\lambda = \sum_{\nu=0}^{\infty} \sum_{q=N_{\nu}+1}^{m(q)} \sum_{\xi=1}^{k(q)} \sum_{i=0}^{\nu_{\xi+1}} e_{q+i}c_{q+i}(t),
\]

moreover

\[
\sum_{\nu=0}^{\infty} \left\| \sum_{q=N_{\nu}+1}^{m(q)} \sum_{\xi=1}^{k(q)} \sum_{i=0}^{\nu_{\xi+1}} e_{q+i}c_{q+i}(t) \right\|_{L^\alpha} < \infty.
\]

Theorem 2. Assume that hypotheses H1, H2 hold, \( \alpha > 2/\mu, \mu \in (0, 1) \) and \( \alpha > 1, \mu \in (1, \infty) \). Then a sequence of the natural numbers \( \{N_\nu\}_0^\infty \) can be chosen so that

\[
\frac{1}{2\pi i} \int \frac{e^{-\lambda^\alpha t}A(E - \lambda A)^{-1}}{\gamma} \, d\lambda = \sum_{\nu=0}^{\infty} \sum_{q=N_{\nu}+1}^{m(q)} \sum_{\xi=1}^{k(q)} \sum_{i=0}^{\nu_{\xi+1}} e_{q+i}c_{q+i}(t),
\]

where

\[
\sum_{\nu=0}^{\infty} \left\| \sum_{q=N_{\nu}+1}^{m(q)} \sum_{\xi=1}^{k(q)} \sum_{i=0}^{\nu_{\xi+1}} e_{q+i}c_{q+i}(t) \right\|_{L^\alpha} < \infty,
\]

the following relation holds for the eigenvalues

\[
|\lambda_{N_\nu+k} - |\lambda_{N_\nu+k-1}| \leq C|\lambda_{N_\nu+k}|^{1-1/\tau}, \quad k = 2, 3, ..., N_{\nu+1} - N_\nu, \quad 0 < \tau < \mu.
\]

Theorem 3. Assume that a normal operator satisfies the hypotheses H1, H2, \( \alpha > 1 \), the condition \( (\ln^{1+1/\alpha} x)_{\lambda_{\nu}(H)} = o(i^{-1/\alpha}) \) holds. Then a sequence of the natural numbers \( \{N_\nu\}_0^\infty \) can be chosen so that

\[
\frac{1}{2\pi i} \int \frac{e^{-\lambda^\alpha t}A(E - \lambda A)^{-1}}{\gamma} \, d\lambda = \sum_{\nu=0}^{\infty} \sum_{q=N_{\nu}+1}^{m(q)} \sum_{\xi=1}^{k(q)} \sum_{i=0}^{\nu_{\xi+1}} e_{q+i}c_{q+i}(t),
\]

moreover

\[
\sum_{\nu=0}^{\infty} \left\| \sum_{q=N_{\nu}+1}^{m(q)} \sum_{\xi=1}^{k(q)} \sum_{i=0}^{\nu_{\xi+1}} e_{q+i}c_{q+i}(t) \right\|_{L^\alpha} < \infty,
\]

the following relation holds for the corresponding eigenvalues

\[
|\lambda_{N_\nu+k} - |\lambda_{N_\nu+k-1}| \leq C|\lambda_{N_\nu+k}|^{1-1/\tau}, \quad k = 2, 3, ..., N_{\nu+1} - N_\nu, \quad 0 < \tau < 1/\alpha.
\]
3 Main results

In this section we consider evolution equations in the abstract Hilbert space. Having used an abstract theorem formulated in terms of the operator order, we produce an example of the class of differential equations for which the made refinement regarding the convergence exponent is relevant. More precisely, under the assumption \( \rho = \alpha \) the sequence of contours may be chosen in a concrete – power type way, what provides a peculiar validity of the statement. We prove the existence and uniqueness theorem and supply it with a plenty of applications. We consider applications to the differential equations in the concrete Hilbert spaces and involve such operators as Riemann-Liouville operator, Kipriyanov operator, Riesz potential, difference operator. Moreover, we produce the artificially constructed normal operator for which the clarification of the Lidskii V.B. results relevantly works. Further, we will consider a Hilbert space \( \mathcal{H} \) which consists of element-functions \( u : \mathbb{R}_+ \to \mathcal{H}, u := u(t), t \geq 0 \) and we will assume that if \( u \) belongs to \( \mathcal{H} \) then the fact holds for all values of the variable \( t \). Notice that under such an assumption all standard topological properties as completeness, compactness etc remain correctly defined. We understand such operations as differentiation and integration in the generalized sense that is caused by the topology of the Hilbert space \( \mathcal{H} \), more detailed information can be found in the Chapter 4 \[18\]. Consider a Cauchy problem

\[
\frac{du}{dt} = -\tilde{W}^n u, \; u(0) = h \in D(\tilde{W}), \; n = 1, 2, \ldots
\]

(1)

In the case when \( \tilde{W}^n \) is accretive, we can suppose \( h \in \mathcal{H} \) (here we should note that the case \( n = 1 \) was considered by Lidskii V.B. \[20\]). Bellow, we formulate a result which follows from the auxiliary theorems given above.

Theorem 4. Assume that the conditions of one of the Theorems \[1 \] \[2 \] \[3 \] hold under the assumption \( \alpha = n \), then there exists a solution of the Cauchy problem (1) in the form

\[
u(t) = \frac{1}{2\pi i} \int_{\gamma} e^{-\lambda^nt} A(E - \lambda A)^{-1} h d\lambda = \sum_{\nu=0}^{\infty} \sum_{q=N_{\nu}+1}^{\infty} \sum_{\xi=0}^{N_{\nu}+1} e^{q_{\xi}+iC_{q_{\xi}+1}}(t),
\]

(2)

where

\[
\sum_{\nu=0}^{\infty} \left\| \sum_{q=N_{\nu}+1}^{\infty} \sum_{\xi=0}^{N_{\nu}+1} e^{q_{\xi}+iC_{q_{\xi}+1}}(t) \right\|_{\mathcal{H}} < \infty,
\]

a sequence of natural numbers \( \{N_{\nu}\}_0^{\infty} \) can be chosen in accordance with the claim of the corresponding theorem. Moreover, the existing solution is unique, if the operator \( \tilde{W}^n \) is accretive.

Proof. Let us find a solution of problem (1) in the form (2). We need prove that the following integral converges i.e.

\[
\frac{1}{2\pi i} \int_{\gamma} e^{-\lambda^nt}(E - \lambda A)^{-1} h d\lambda \in \mathcal{H}, \; h \in \mathcal{H}.
\]

(3)

For this purpose consider a contour \( \gamma_k := \{\lambda \in \gamma, |\lambda| < R_k, k \in \mathbb{N}\}, R_k \uparrow \infty \). Using simple estimating, then applying Lemma 6 \[17\], we get

\[
\left\| \int_{\gamma_k} e^{-\lambda^nt}(E - \lambda A)^{-1} h d\lambda \right\|_{\mathcal{H}} \leq \int_{\gamma_k} |e^{-\lambda^nt}| \cdot \|(E - \lambda A)^{-1} h\|_{\mathcal{H}} d\lambda \leq C\|h\|_{\mathcal{H}} \int_{\gamma_k} e^{-t\text{Re}\lambda^n} d\lambda.
\]
It is clear that
\[ \int_{\gamma_h} e^{-t\text{Re}\lambda^n}|d\lambda| \to C, k \to \infty. \]
The latter fact gives us the desired result. Since \( A \) is bounded, then we have
\[ \tilde{W}u(t) = \tilde{W} \left( \frac{1}{2\pi i} \int_{\gamma} e^{-\lambda^n t} A(E - \lambda A)^{-1} h d\lambda \right) = \tilde{W} A \left( \frac{1}{2\pi i} \int_{\gamma} e^{-\lambda^n t} (E - \lambda A)^{-1} h d\lambda \right) = \frac{1}{2\pi i} \int_{\gamma} e^{-\lambda^n t} (E - \lambda A)^{-1} h d\lambda. \]
Combining the latter relation with (3), we obtain \( u \in D(\tilde{W}_n) \).

Analogously to the above, using Lemma 6 [17], we can show that the following derivative exists i.e.
\[ \frac{du}{dt} = -\frac{1}{2\pi i} \int_{\gamma} e^{-\lambda^n t} \lambda^n A(E - \lambda A)^{-1} h d\lambda \in \mathcal{H}. \]
Notice that \( \lambda^n A^n (E - \lambda A)^{-1} = (E - \lambda A)^{-1} - (E + \lambda A + ... + \lambda^{n-1} A^{n-1}) \), substituting this relation to the above formula, we obtain
\[ A^{n-1} \frac{du}{dt} = -\frac{1}{2\pi i} \int_{\gamma} e^{-\lambda^n t} (E - \lambda A)^{-1} h d\lambda + \frac{1}{2\pi i} \int_{\gamma} e^{-\lambda^n t} \sum_{k=0}^{n-1} \lambda^k A^k h d\lambda. \]
The second integral equals zero by virtue of the fact that the function under the integral is analytical inside the intersection of the domain \( G \) with the circle of the arbitrary radius \( R \) and it decreases sufficiently fast on the arch of the radius \( R \), when \( R \to \infty \), here we denote by \( G \) the interior of the contour \( \gamma \). Thus, we have come to the relation
\[ A^{n-1} \frac{du}{dt} = -\frac{1}{2\pi i} \int_{\gamma} e^{-\lambda^n t} (E - \lambda A)^{-1} h d\lambda. \]

Since the left-hand side of the latter relation belongs to \( D(\tilde{W}^{n-1}) \), then we can claim that it is so for the right-hand side also. It follows that \( u \in D(\tilde{W}^n) \). Now, if we recall the expression for \( u \), we get \( A^{n-1} u' + \tilde{W} u = 0 \). Applying the operator \( \tilde{W}^{n-1} \) to both sides of the latter relation, we obtain the fact that \( u \) is a solution of equation (1). Let us show that the initial condition holds in the sense \( u(t) \to h, t \to +0 \). It becomes clear in the case \( h \in D(\tilde{W}) \) for in this case it suffices to apply Lemma 7 [17], what gives us the desired result i.e. we can put \( u(0) = h \). Consider a case when \( h \) is an arbitrary element of the Hilbert space \( \mathcal{H} \) and let us involve the accretive property of the operator \( \tilde{W}^n \). Consider an operator
\[ S_t h = \frac{1}{2\pi i} \int_{\gamma} e^{-\lambda^n t} A(E - \lambda A)^{-1} h d\lambda, t > 0. \]
In accordance with the above, it is clear that \( S_t : \mathcal{H} \to \mathcal{H} \). Let us prove that \( \| S_t \|_{\mathcal{H} \to \mathcal{H}} \leq 1, t > 0 \). Firstly, assume that \( h \in D(\tilde{W}) \). Let us multiply the both sides of relation (1) on \( u \) in the
Remark 1. Note that the assumption of the inner product, we get \((u', u)_\beta + (\bar{W}^\alpha u, u)_\beta = 0\). Consider a real part of the latter relation, we have \(\text{Re} (u', u)_\beta + \text{Re}(\bar{W}^\alpha u, u)_\beta = (u', u)_\beta / 2 + (u, u')_\beta / 2 + \text{Re}(\bar{W}^\alpha u, u)_\beta\). Therefore, \(||u(t)||_\beta^2 \leq -2\text{Re}(\bar{W}^\alpha u, u)_\beta \leq 0\). Integrating both sides, we get

\[
\|u(\tau)\|_\beta^2 - \|u(0)\|_\beta^2 = \int_0^\tau \frac{d}{dt}\|u(t)\|_\beta^2 dt \leq 0.
\]

The last relation can be rewritten in the form \(\|S_t h\|_\beta \leq \|h\|_\beta\), \(h \in D(\bar{W})\). Since \(D(\bar{W})\) is a dense set in \(\mathfrak{H}\), then we obviously obtain the desired result i.e. \(\|S_t\|_{\beta \to \beta} \leq 1\). Now, having assumed that \(h_n \overset{\beta}{\to} h, n \to \infty, \{h_n\} \subset D(\bar{W}), h \in \mathfrak{H}\), consider the following reasonings \(\|u(t) - h\|_\beta = \|S_t h - S_t h_n + S_t h_n - h_n + h_n - h\|_\beta \leq \|S_t\| \cdot \|h - h_n\|_\beta + \|S_t h_n - h_n\|_\beta + \|h_n - h\|_\beta\). Note that \(S_t h_n \overset{\beta}{\to} h_n, t \to +0\). It is clear that if we chose \(n\) so that \(\|h - h_n\|_\beta < \varepsilon / 3\) and after that chose \(t\) so that \(\|S_t h_n - h_n\|_\beta < \varepsilon / 3\), then we obtain \(\forall \varepsilon > 0, \exists \delta(\varepsilon) : \|u(t) - h\|_\beta < \varepsilon, t < \delta\). Thus, we can put \(u(0) = h\) and claim that the initial condition holds in the case \(h \in \mathfrak{H}\). The decomposition on the series of the root vectors \(\{\varphi\}_{\text{finite}}\) is given by virtue of Theorems \(2\) \(3\) respectively. The uniqueness follows easily from the fact that \(\bar{W}^\alpha\) is accretive. In this case, repeating the previous reasonings we come to

\[
\|\varphi(\tau)\|_\beta^2 - \|\varphi(0)\|_\beta^2 = \int_0^\tau \frac{d}{dt}\|\varphi(t)\|_\beta^2 dt \leq 0,
\]

where \(\varphi\) is a sum of solutions \(u_1\) and \(u_2\). Notice that by virtue of the initial conditions, we have \(\varphi(0) = 0\), thus relation \(1\) can hold only if \(\varphi = 0\). The proof is complete. \(\Box\)

**Remark 1.** Note that the assumption \(n > \rho\), that is additional to the ones of the above theorems, guaranties an opportunity to chose a sequence of numbers \(\{N_n\}_0^\infty\) so that in the used terms, we have

\[
|\lambda_{N_n+k} - |\lambda_{N_n+k-1}| \leq e^{-|\lambda_{N_n}|\tau}, \ 0 < \tau < n - \rho.
\]

This fact follows from the results by Lidskii V.B. [27, p.23].

**Evolution equations with the quasi-polynomial right-hand side**

Let \(I := (a, b) \subset \mathbb{R}, \Omega := [0, \infty), \) consider the functions \(u(t, x), t \in \Omega, x \in \Omega\). In accordance with the said above, we will consider functional spaces with respect to the variable \(x\) and we will assume that if \(u\) belongs to a functional space then the fact holds for all values of the variable \(t\). Consider a Cauchy problem

\[
\frac{du}{dt} = \sum_{k=1}^n C_k D^{\kappa_\theta}_{a^+} u =: P_{s, \vartheta} u, \ \vartheta > 0, \ u(0, x) = h(x) \in L_2(I),
\]

where at the right-hand side we have a linear combination of the Riemann-Liouville fractional differential operators acting in \(L_2(I)\) with respect to the variable \(x\). We will call (analogously to the theory of ordinary differential equations) the right-hand side of equation \(5\) *quasi-polynomial*. Consider a case when the right-hand side of equation \(5\) can be represented as follows

\[
P_{s, \vartheta} u = -(\eta D^2 + \xi D^{\mu}_{a^+})^n u, \ 0 < \beta < 1/n, \ \eta < 0, \ \xi > 0.
\]
Let us show that under such assumptions problem \( \mathbf{[5]} \) has a unique solution which can be found due to a certain formula. Consider the operator \( W := \eta D^2 + \xi D_{a+}^\beta, \ D(W) = C_0^\infty(I) \). Note that the hypotheses H1, H2 hold regarding the operator, if we assume that \( \mathcal{S} := L_2(I), \ \mathcal{S}_+ := H^1_0(I) \).

It follows from the strictly accretive property of the fractional differential operator (see \( \mathbf{[14]} \)) and the estimate
\[
\|D_{a+}^\beta f\|_{L^2} \leq C\|f\|_{H^1_0}, \ f \in C_0^\infty(I), \ \beta \in (0,1).
\]

Let us prove that
\[
-C(D^2 f, f)_{L^2} \leq \text{Re}(W f, f)_{L^2} \leq -C(D^2 f, f)_{L^2}, \ f \in C_0^\infty(I).
\]

Using relation \( \mathbf{[7]} \) and the Friedrichs inequality, we obtain
\[
\text{Re}(D_{a+}^\beta f, f)_{L^2} = \|f\|_{H^1_0}^2, \ f \in C_0^\infty(I), \text{ what gives us the upper estimate.}
\]

The lower estimate follows easily from the accretive property of the fractional differential operator of the order less than one. Using relation \( \mathbf{[8]} \), the corollary of the minimax principle, we get
\[
-C(t)^\lambda_j(H) \times \lambda_j(D^2),
\]

where \( H \) is a real part of the operator \( \tilde{W} \). Therefore, taking into account the well-known fact \( \lambda_j(D^2) = -\pi^2 j^2/(b-a)^2 \), we get \( \lambda_j(H) \propto j^2 \), it follows that \( \mu(H) > 1 \). Now we can study the Cauchy problem \( \mathbf{[5]} \) by restricting the one to the problem
\[
\frac{du}{dt} = -(\eta D^2 + \xi D_{a+}^\beta)n u, \ u(0) = h \in D(\tilde{W}).
\]

In accordance with Theorem \( \mathbf{[3]} \) we are able to present a solution of the problem \( \mathbf{[3]} \), with the restricted assumptions regarding \( h \), as follows
\[
u(t) = \frac{1}{2\pi i} \int e^{-\lambda^n t} A(E - \lambda A)^{-1} h d\lambda,
\]

where the used terms relate to the operator \( \tilde{W} \). Thus, we have in the reminder a question how to weaken conditions imposed upon the function \( h \) as well as wether the representation \( \mathbf{[6]} \) holds. To answer the questions consider the following reasonings. Further, for the sake of the simplicity, we consider a case when \( \eta = -1, \ \xi = 1 \). This assumption does not restrict generality of reasonings. Let us show that \( D(\tilde{W}) \subset H^3_0(I) \). Using H2, we have the implication
\[
 f_k \xrightarrow{W} f \implies f_k \xrightarrow{H^1_0} f, \ \{f_k\}_{\infty} \subset C_0^\infty(I).
\]

Applying \( \mathbf{[7]} \), we get \( D_{a+}^\beta f_k \xrightarrow{L^2} D_{a+}^\beta f \). The following fact can be obtained easily, we have omitted the proof
\[
\{f_k \xrightarrow{W} f, \ D_{a+}^\beta f_k \xrightarrow{L^2} D_{a+}^\beta f\} \implies D^2 f_k \xrightarrow{L^2} D^2 f.
\]

Combining the above implications we obtain the desired result i.e. \( D(\tilde{W}) \subset H^3_0(I) \). Consider a set \( H^s_{0+}(I) := \{f : f \in H^s(I), \ f^{(k)}(a) = 0, \ k = 0, 1, ..., s - 1\}, \ s \in \mathbb{N} \). It is clear that \( H^s_{0+}(I) \subset H^s_{0+}(I) \), thus we can define the operator \( W_{+} \) as the extension of the operator \( \tilde{W} \) on the set \( H^s_{0+}(I) \), we have \( \tilde{W} \subset W_{+} \). Let us show that \( D(W^n) = H^{2n}_{0+}(I) \). Assume that \( f \in H^{2n}_{0+}(I) \), then \( f \in \tilde{W} \), it can be verified directly. If \( f \in D(W^n) \), then in accordance with the definition, we have
\[
W^n_{+} f \in H^s_{0+}(I). \text{ It follows that } W_{+} g_1 \in H^s_{0+}(I), \text{ where } g_1 = W^n_{+} f \in H^s_{0+}(I). \text{ Hence}
\]
we obtain the desired result. Let us show that
\[ f D_{a+}^{2 \beta} g_1 \in H^2_{0+}(I). \]
Applying the operator \( I_{a+}^2 \) to the both sides of the last relation, we easily get
\[ g_1 + I_{a+}^{2 \beta} g_1 \in H^4_{0+}(I). \]
Using the constructive features of relation (10), we can conclude firstly \( g_1 \in H^3_{0+}(I) \) and due to the same reasonings establish the fact \( g_1 \in H^4_{0+}(I) \) secondly, what gives us \( W_{I_{a+}}^{n-2} f \in H^2_{0+}(I) \).
Using the absolutely analogous reasonings we prove that \( W_{I_{a+}}^{n-k} f \in H^{2k}_{0+}(I) \), \( k = 1, 2, \ldots, n \). Thus, we obtain the desired result. Let us show that
\[
W_{I_{a+}}^n f = \sum_{k=0}^n (-1)^{n-k} C_n^k D_{a+}^{\beta k + 2(n-k)} f, \quad f \in H^{2n}_{0+}(I). \tag{11}
\]
We need establish the formula \( D_{a+}^{\beta k} D_{a+}^{2(n-k)} f = D_{a+}^{2(n-k)} D_{a+}^{\beta k} f = D_{a+}^{\beta k + 2(n-k)} f \), \( f \in H^{2n}_{0+}(I) \), \( k = 1, 2, \ldots, n \) for this purpose, in accordance with Theorem 2.5 [32, p.46], we should prove that \( f \in I_{a+}^{\beta k + 2(n-k)}(L_1) \) or \( f = a.e. I_{a+}^{2(n-k)} \varphi, \quad \varphi \in I_{a+}^{\beta k}(L_1) \). We get \( f = a.e. I_{a+}^{2(n-k)} D_{a+}^{\beta k} f = I_{a+}^{2(n-k)} D_{a+}^{\beta k} f = f_{a+}^{2(n-k)} \varphi \), where \( \varphi := D_{a+}^{2(n-k)} f \). Note that the conditions of Theorem 13.2 [32, p.229] hold i.e. the Marchaud derivative of the function \( \varphi \) belongs to \( L_1(I) \). Hence \( \varphi \in I_{a+}^{\beta k}(L_1) \) and we obtain the required formula. Using the well-known formulas for linear operators \( (A + B)C \supseteq AC + BC, \quad C(A + B) = CA + CB \), applying the Leibniz formula, we obtain (11). Now, combining the obvious inclusion \( \tilde{W}^n \subset W_{I_{a+}}^n \) with (11), we get
\[
\tilde{W}^n \subset \sum_{k=0}^n (-1)^{n-k} C_n^k D_{a+}^{\beta k + 2(n-k)}.
\]
The next question is wether the operator \( \tilde{W}^n \) is accretive. By direct calculation, we have
\[
\Re(\tilde{W}^n f, f)_{L_2} = \sum_{k=0}^n C_n^k \Re \left( D_{a+}^{\beta k + 2(n-k)} f, D_{a+}^{n-k} f \right)_{L_2} = \sum_{k=0}^n C_n^k \Re \left( D_{a+}^{\beta k} g_k, g_k \right)_{L_2} \geq 0,
\]
where \( g_k := D_{a+}^{n-k} f, \quad f \in D(\tilde{W}^n) \). Note that the last inequality holds by virtue of the strictly accretive property of the fractional differential operator of the order less than one (see [14]). Thus, the uniqueness and the opportunity to weaken conditions imposed on \( h \) follow from Theorem 4. Here we should remark that the latter theorem gives us the fact that the existing solution is unique in the set \( D(\tilde{W}^n) \), but using the same method we can establish the uniqueness of the solution of problem [5]. Having known the root vectors of the operators \( A \), applying formula (2), we can represent the obtained solution as a series.

Kipriyanov operator

Using notations of the paper [11] we assume that \( \Omega \) is a convex domain of the \( m \)-dimensional Euclidean space \( \mathbb{E}^m \), \( P \) is a fixed point of the boundary \( \partial \Omega \), \( Q(\mathbf{r}, \mathbf{e}) \) is an arbitrary point of \( \Omega \); we denote by \( \mathbf{e} \) a unit vector having a direction from \( P \) to \( Q \), denote by \( r = |P - Q| \) the Euclidean distance between the points \( P, Q \), and use the shorthand notation \( T := P + \mathbf{e} t, \quad t \in \mathbb{R} \). We consider the Lebesgue classes \( L_p(\Omega) \), \( 1 \leq p < \infty \) of complex valued functions. For the function \( f \in L_p(\Omega) \), we have
\[
\int_{\Omega} |f(Q)|^p dQ = \int_{\omega} d\mathbf{e} \int_0^{d(e)} |f(Q)|^{p r^{m-1}} d\mathbf{e} < \infty, \tag{12}
\]
The properties of these operators are described in detail in the paper [13]. We suppose (see [13]). By definition, put also, we consider auxiliary operators, the so-called truncated directional fractional derivatives now, we can define the directional fractional derivatives as follows (see [32, p.175]) defined by the following formal construction

\[ (D^\beta_{0+,e} f)(Q) := \frac{\beta}{\Gamma(1 - \beta)} \int_0^r \frac{f(P + te)}{(r-t)^{1-\beta}} \frac{dt}{t^m} \text{, } (D^\beta_{d-} f)(Q) := \frac{d}{\Gamma(1 - \beta)} \int_{r+\varepsilon}^d \frac{f(P + te)}{(t-r)^{1-\beta}} \frac{dt}{t^m} \text{, } f \in L_p(\Omega) \text{, } 1 \leq p \leq \infty. \]

Also, we consider auxiliary operators, the so-called truncated directional fractional derivatives (see [13]). By definition, put

\[ (D^\beta_{0+,\varepsilon} f)(Q) := \frac{\beta}{\Gamma(1 - \beta)} \int_0^{r-\varepsilon} \frac{f(Q)r^{m-1} - f(P + et)_{m-1}}{(r-t)^{\beta+1}r^{m-1}} \frac{dt}{t^m} \text{, } (D^\beta_{d-\varepsilon} f)(Q) := \frac{\beta}{\Gamma(1 - \beta)} \int_{r+\varepsilon}^d \frac{f(Q) - f(P + et)_{m-1}}{(t-r)^{\beta+1}} \frac{dt}{t^m} \text{, } 0 \leq r \leq d - \varepsilon, \]

\[ (D^\beta_{0+,\varepsilon} f)(Q) := \frac{f(Q)}{\varepsilon^\beta} \text{, } 0 \leq r < \varepsilon; \]

\[ (D^\beta_{d-\varepsilon} f)(Q) := \frac{f(Q)}{\beta} \left( \frac{1}{\varepsilon^\beta} - \frac{1}{(d-r)^\beta} \right) \text{, } d - \varepsilon < r \leq d. \]

Now, we can define the directional fractional derivatives as follows

\[ D^\beta_{0+} f = \lim_{\varepsilon \to 0+} \frac{D^\beta_{0+,\varepsilon} f}{(L_p)} \text{, } D^\beta_{d-} f = \lim_{\varepsilon \to 0-} \frac{D^\beta_{d-\varepsilon} f}{(L_p)} \text{, } 1 \leq p \leq \infty. \]

The properties of these operators are described in detail in the paper [13]. We suppose \( J^0_{0+} = I \). Nevertheless, this fact can be easily proved by virtue of the reasonings corresponding to the one-dimensional case and given in [32]. We also consider integral operators with a weighted factor (see [32, p.175]) defined by the following formal construction

\[ (J^\beta_{0+} f)(Q) := \frac{1}{\Gamma(1 - \beta)} \int_0^r \frac{(\xi f)(P + te)}{(r-t)^{1-\beta}} \frac{(t)^{m-1}}{t^m} \frac{dt}{t^m}, \]
where $\xi$ is a real-valued function.

Consider a linear combination of the uniformly elliptic operator, which is written in the divergence form, and a composition of a fractional integro-differential operator, where the fractional differential operator is understood as the adjoint operator regarding the Kipriyanov operator (see [11], [12], [14])

$$L := -\mathcal{T} + \mathcal{A}_0 + \xi \mathcal{D}_d^\beta, \quad \sigma \in [0, 1),$$

$$\text{D}(L) = H^2(\Omega) \cap H^1_0(\Omega),$$

where $\mathcal{T} := D_j(a^{ij}D_i)$, $i, j = 1, 2, \ldots, m$, under the following assumptions regarding coefficients $a^{ij}(Q) \in C^2(\bar{\Omega})$, $\text{Re}^{ij}\xi_\xi_j \geq \gamma_a|\xi|^2$, $\gamma_a > 0$, $\text{Im}^{ij} = 0$ $(m \geq 2)$, $\xi \in L_\infty(\Omega)$.

Note that in the one-dimensional case the operator $\mathcal{A}_0 + \xi \mathcal{D}_d^\beta$ is reduced to a weighted fractional integro-differential operator composition, which was studied properly by many researchers [6], [7], [21], [29], more detailed historical review see in [32, p.175]. In accordance with Theorem 3 [16], we claim that the hypotheses H1, H2 are fulfilled if $\gamma_a$ is sufficiently large in comparison with $\|\xi\|_\infty$, where we put $\mathfrak{M} := C_0^\infty(\Omega)$. Note that the order $\mu$ of the operator $\mathcal{H}$ can be evaluated easily through the order of the regular differential operator and since the latter can be found by methods described in [31]. More precisely, we have

$$C(\mathfrak{M} f, f)_\mathcal{H} \leq (H f, f)_\mathcal{H} \leq C(\mathfrak{M} f f, f)_\mathcal{H}, \quad f \in C_0^\infty(\Omega).$$

Applying the minimax principle, we get $\lambda_\mu(H) \propto \lambda_\mu(\mathfrak{M} f f)$. Using the well-known formula for regular differential operators $\lambda_j(\mathfrak{M} f f) \propto j^{2/m}$ (see [31]), we get $\mu(H) = 2/m$. Therefore, if we assume that $2 \leq m < n$, then in accordance with Theorem 4, we can claim that there exists a solution of problem (11) where $W$ is a restriction of $L$ on the set $C_0^\infty(\Omega)$, the coefficients (13) are sufficiently smooth to guaranty the fact the right-hand side of (11) has a sense. Note that the solvability of the uniqueness problem as well as the opportunity to extend the initial condition depends on the accretive property of the operator $\mathcal{W}^m$. The latter problem can be studied by the methods similar to the ones used in the previous paragraph. Indeed, we have established the accretive property in the one-dimensional case.

### Riesz potential

Consider a space $L_2(\Omega)$, $\Omega := (-\infty, \infty)$. We denote by $H^{2, \gamma}_0(\Omega)$ the completion of the set $C_0^\infty(\Omega)$ with the norm

$$\|f\|_H^{2, \gamma}(\Omega) = \left\{\|f\|_{L_2(\Omega)}^2 + \|D^2 f\|_{L_2(\Omega, \omega)}^2\right\}^{1/2}, \quad \gamma \in \mathbb{R},$$

where $\omega(x) := (1 + |x|)$. Let us notice the following fact (see Theorem 1 [11]), if $\gamma > 4$, then $H^{2, \gamma}_0(\Omega) \subset \subset L_2(\Omega)$. Consider a Riesz potential

$$I^\beta f(x) = B_\beta \int_{-\infty}^{\infty} f(s)|s - x|^\beta - 1ds, \quad B_\beta = \frac{1}{2\Gamma(\beta) \cos \beta \pi/2}, \beta \in (0, 1),$$

where $f$ is in $L_\rho(\Omega)$, $1 \leq \rho < 1/\beta$. It is obvious that $I^\beta f = B_\beta \Gamma(\beta)(I^\beta f + I^\beta f)$, where

$$I^\pm f(x) = \frac{1}{\Gamma(\beta)} \int_{0}^{\infty} f(s \mp x)s^{\beta - 1}ds,$$
the last operators are known as fractional integrals on the whole real axis (see \[32\] p.94). Following the idea of the monograph \[32\] p.176 consider a sum of a differential operator and a composition of fractional integro-differential operators
\[
W := D^2aD^2 + I^{2(1-\beta)}D^2 + \delta, \quad D(W) = C_0^\infty(\Omega), \quad 3/4 < \beta < 1,
\]
where \(a(x) \in L_\infty(\Omega) \cap C^2(\Omega), \quad \text{Re}a(x) > \gamma_a(1 + |x|)^5, \quad \gamma_a, \delta > C_\beta\). Let \(\Omega' := [0, \infty), \) consider the functions \(u(t, x), \) \(t \in \Omega', x \in \Omega\). Similarly to the previous paragraph, we will consider functional spaces with respect to the variable \(x\) and we will assume that if \(u\) belongs to a functional space then this fact holds for all values of the variable \(t\), wherewith all standard topological properties of a space as completeness, compactness e.t.c. remain correctly defined. Consider a Cauchy problem \[1\] in the corresponding terms, under the additional assumptions \(a(x) \in C^{2n}(\Omega), \) \(a^{ij}(x) \in L_\infty(\Omega), \) \(i = 1, 2, ..., 2n\). Notice that in accordance with the results \[16\], we claim that the hypothesis \(H1, H2\) hold regarding: the operator \(\hat{W}\), the set \(C_0^\infty(\Omega), \) the spaces \(L_2(I), H_0^{2,5}(\Omega)\), more precisely we should put \(\mathcal{H} := L_2(\Omega), \) \(\mathcal{H}_+ := H_0^{2,5}(\Omega), \) \(\mathfrak{M} := C_0^\infty(\Omega)\). Thus, in accordance with the hypothesis \(H2\), we have
\[
(Hf, f)_{L_2} = \text{Re}(Wf, f)_{L_2} \geq C_\gamma \|f\|^2_{H_0^{2,5}} = C(D^2wD^2f, f)_{L_2} + C(f, f)_{L_2}, \quad f \in C_0^\infty(\Omega).
\]
where \(w(x) = (1 + |x|)^5\). Let us consider the operator \(B = D^2wD^2 + I, \quad D(B) = C_0^\infty(\Omega)\) it is clear that by virtue of the minimax principle, we can estimate the eigenvalues of the operator \(\hat{W}\) via estimating the eigenvalues of the operator \(\hat{B}\). Hence, we have come to the problem of estimating the eigenvalues of the singular operator. Here, we should point out that there exists the Fefferman concept that covers such a kind of problems. For instance, the Rozenblyum result is presented in the monograph \[31\] p.47], in accordance with which we can chose such an unbounded subset of \(\mathbb{R}\) that the relation \(\lambda_j(\hat{B}) \approx j^4\) holds. Thus, we left this question to the reader for a more detailed study and reasonably allow ourselves to assume that the condition \(\mu(H) = 4\) holds. In this case, in accordance with Theorem \[1\] we are able to present a solution of the problem \[1\] in the form
\[
u(t) = \frac{1}{2\pi i} \int e^{-\lambda t}A(E - \lambda A)^{-1}hd\lambda.
\]
Now assume additionally that \(\text{Im}a = 0\), then \(D^2aD^2\) is selfadjoint. It follows that \(\hat{W}\) is selfadjoint and we can easily prove that \(\text{Re}(\hat{W}^n f, f)_{B} \geq 0, \) \(n = 1, 2, ..., \) Therefore, applying Theorem \[1\] we can assume that \(h \in \mathcal{H}\) and claim that the existing solution is unique. Thus, we have established the existence and uniqueness of the solution of problem \[1\]. Having known the root vectors of the operator \(A\), applying formula \[2\], we can represent the obtained solution as a series.

**Difference operator**

The approach implemented in studying the difference operator is remarkable due to the appeared opportunity to set the problem within the framework of the created theory, having constructed a suitable perturbation of the operator composition. Consider a difference operator and its adjoint operator
\[
Jf(x) = c[f(x) - f(x - d)], \quad J^*f(x) = c[f(x) - f(x + d)], \quad f \in L_2(\Omega), \quad \Omega = (-\infty, \infty), \quad c, d > 0.
\]
Let us find a representation for fractional powers of the operator $A$. Using formula (45) [16], we get

$$J^\beta f = \sum_{k=0}^{\infty} C_k f(x - kd), \quad f \in L_2(\Omega), \quad C_k = -\frac{\beta \Gamma(k - \beta)}{k! \Gamma(1 - \beta)} c^\beta, \quad \beta \in (0, 1).$$

We need the following theorem (see Theorem 5 [16]).

**Theorem.** Assume that $Q$ is a closed operator acting in $L_2(\Omega)$, $Q^{-1} \in \mathcal{S}_\infty(L_2)$, the operator $N$ is strictly accretive, bounded, $R(Q) \subset D(N)$. Then a perturbation

$$L := J^\beta Q + b J^\beta + Q^* N Q, \quad a, b \in L_\infty(\Omega),$$

satisfies conditions H1–H2, if $\gamma_N > \sigma \|Q^{-1}\|^2$, where we put $M := D_0(Q)$,

$$\sigma = 4c \|a\|_{L_\infty} + \|b\|_{L_\infty} \frac{\beta c^\beta}{\Gamma(1 - \beta)} \sum_{k=0}^{\infty} \frac{\Gamma(k - \beta)}{k!}.$$

Observe that by virtue of the made assumptions regarding $Q$, we have $\mathcal{H}_Q \subset \subset L_2(\Omega)$. We have chosen the space $L_2(\Omega)$ as a space $\mathcal{H}_Q$ as a space $\mathcal{H} +$. Applying the condition H2, we get

$$C(Q^* N Q f, f)_{\mathcal{H}} \leq (H f, f)_{\mathcal{H}} \leq C(Q^* N Q f, f)_{\mathcal{H}}, \quad f \in D_0(Q),$$

where $H$ is a real part of $\tilde{W}$. Therefore, by virtue of the minimax principle, we get $\lambda_j(H) \asymp \lambda_j(Q^* N Q)$. Hence $\mu(H) = \mu(Q^* N Q)$. Thus, we have naturally come to the significance of the operator $Q$ and the remarkable fact that we can fulfill the conditions of Theorem 4 choosing the operator $Q$ in the artificial way. Applying Theorem 4, we can claim that there exists a solution of problem (1), where $W$ is a restriction of $L$ on the set $\mathfrak{M}$ (see introduction), functions $a, b$ are sufficiently smooth to guarantee the fact the right-hand side of (1) has a sense. The extension of the initial conditions on the whole space $\mathfrak{H}$, as well as solvability of the uniqueness problem can be implemented in the case when the operator $\tilde{W}$ is accretive. In its own turn, it is clear that the particular methods, to establish the accretive property, can differ and may depend on the concrete form of the operator $Q$.

**Artificially constructed normal operator**

In this paragraph we consider an operator class which cannot be completely studied by methods [20], at the same time Theorem 3 gives us a rather relevant result. Our aim is to construct a normal operator $N$ being satisfied the Theorem 3 conditions, such that $N \in \tilde{\mathcal{S}}_\alpha \setminus \mathcal{S}_\alpha$. Let us consider the following example as a perquisite for the further reasonings.

**Example 1.** Here we would like to produce an example of the sequence $\{\mu_n\}_{n=1}^{\infty}$ that satisfies the condition $(\ln^{\kappa+1} x)_{\mu_n} = o(n^{-\kappa})$, ($0 < \kappa < 1$), and at the same time

$$\sum_{n=1}^{\infty} \frac{1}{|\mu_n|^{1/\kappa}} = \infty.$$  

Consider a sequence $\mu_n = n^\kappa \ln^\kappa n \cdot \ln^\kappa \ln n$, then using the integral test for convergence we can easily see that the previous series diverges. At the same time substituting, we get

$$\frac{\ln^\kappa \mu_n}{\mu_n} \leq \frac{C}{n^\kappa \ln^\kappa \ln n}, \quad n = 1, 2, \ldots.$$
what gives us the fulfilment of the first condition.

Consider the abstract separable Hilbert space $\mathcal{H}$ and an operator $N$ acting in the space as follows

$$Nf = \sum_{n=1}^{\infty} \lambda_n f_n e_n, \quad f_n = (f, e_n)_\mathcal{H}, \quad \lambda_n = \mu_n + i\eta_n$$

where $\{e_n\}_{n=1}^{\infty} \subset \mathcal{H}$ is an orthonormal basis, the sequence $\{\mu_n\}_{n=1}^{\infty}$ is defined in Example 1, $|\eta_n| < M\mu_n$, $M > 0$, $n = 1, 2, \ldots$. Define the space $\mathcal{H}_+$ as follows

$$\mathcal{H}_+ := \left\{ f \in \mathcal{H} : \|f\|^2_{\mathcal{H}_+} := \sum_{n=1}^{\infty} |\lambda_n| |f_n|^2 < \infty \right\}.$$  

It is clear that $\mathcal{H}_+$ is dense in $\mathcal{H}$, since $\{e_n\}_{n=1}^{\infty} \subset \mathcal{H}_+$. Let us show that embedding of the spaces $\mathcal{H}_+ \subset \mathcal{H}$ is compact. Consider the operator $B : \mathcal{H} \rightarrow \mathcal{H}$ defined as follows

$$Bf = \sum_{n=1}^{\infty} |\lambda_n|^{-1/2} f_n e_n.$$  

Note that compactness of the operator $B$ can be proved easily due to the well-known criterion of compactness in the Banach space endowed with a basis (we left the prove to the reader). Notice that if $f \in \mathcal{H}_+$, then $g \in \mathcal{H}$, where $g$ is defined by its fourier coefficients $g_n = |\lambda_n|^{1/2} f_n$. By virtue of such a correspondence we can consider any bounded set in the space $\mathcal{H}_+$ as a bounded set in the space $\mathcal{H}$. Applying the operator $B$ to the element $g$, we get the element $f$. Due to the compactness of the operator $B$ we can conclude that the image of the bounded set in the sense of the norm $\mathcal{H}_+$ is a compact set in the sense of the norm $\mathcal{H}$. Define the set $\mathcal{M}$ as a linear manifold generated by the basis vectors. Thus, we have obtained the relation $\mathcal{H}_+ \subset \subset \mathcal{H}$ and established the fulfilment of the hypotheses H1. The first relation of the hypotheses H2 can be obtained easily due to the application of the Cauchy-Schwarz inequality. To obtain the second one consider

$$\text{Re}(Nf, f)_\mathcal{H} = \sum_{n=1}^{\infty} \text{Re} \lambda_n |f_n|^2 \geq (1 + M^2)^{-1/2} \sum_{n=1}^{\infty} |\lambda_n| |f_n|^2 = (1 + M^2)^{-1/2} \|f\|^2_{\mathcal{H}_+}.$$  

Now, we conclude that hypotheses H1, H2 hold. Consider a Cauchy problem

$$\frac{du}{dt} = N^{1/\kappa} u, \quad u(0) = h \in \text{D}(N), \quad \kappa = 1/2, 1/3, \ldots, \quad (14)$$

where $h$ is supposed to be an arbitrary element if the operator $N^{1/\kappa}$ is accretive. In accordance with Theorem 4, we conclude that there exists a solution of problem (14) presented by the series (2). Moreover, we claim that under the assumption $M \leq \tan\{\pi\kappa/2\}$, the existing solution is unique and we can extend the initial condition assuming that $h \in \mathcal{H}$. For this purpose, in accordance with Theorem 3 let us prove that $\text{Re}(N^{1/\kappa} f, f)_\mathcal{H} \geq 0$. The latter fact follows from the relation

$$\text{Re} \lambda_n^{1/\kappa} = |\lambda_n|^{1/\kappa} \cos \left( \frac{\arg \lambda_n}{\kappa} \right) \geq |\lambda|^{1/\kappa} \cos \frac{\pi}{2},$$

and the representation

$$\text{Re}(N^{1/\kappa} f, f)_\mathcal{H} = \sum_{n=1}^{\infty} \text{Re} \lambda_n^{1/\kappa} |f_n|^2.$$  

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Thus, we obtain the desired result. Note that the constructed normal operator indicates the significance of the made in Theorem 3 clarification of the results [20]. However, we produce one more relevant application of the mentioned theorem in the following paragraph.

**Evolution equations with the fractional derivative at the left-hand side**

In this paragraph, we still consider a Hilbert space $H$ consisting of element-functions $u : \mathbb{R}_+ \to \mathcal{H}$, $u := u(t)$, $t \geq 0$ assuming that if $u$ belongs to $H$ then the fact holds for all values of the variable $t$. We understand such operations as differentiation and integration in the generalized sense that is caused by the topology of the Hilbert space $H$. The derivative is understood as a limit

$$
\frac{u(t + \Delta t) - u(t)}{\Delta t} \to \frac{du}{dt}, \Delta t \to 0.
$$

Let $t \in I := [a, b]$, $0 < a < b < \infty$. The following integral is understood in the Riemann sense as a limit of partial sums

$$
\sum_{i=0}^{n} u(\xi_i)\Delta t_i \to \int_{I} u(t)dt, \zeta \to 0,
$$

where $(a = t_0 < t_1 < ... < t_n = b)$ is an arbitrary splitting of the segment $I$, $\zeta := \max(t_{i+1} - t_i)$, $\xi_i$ is an arbitrary point belonging to $[t_i, t_{i+1}]$. The sufficient condition of the last integral existence is a continuous property (see [18, p.248]) i.e. $u(t) \to u(t_0)$, $t \to t_0$, $\forall t_0 \in I$. The improper integral is understood as a limit

$$
\int_{a}^{b} u(t)dt \to \int_{a}^{c} u(t)dt, b \to c, c \in [0, \infty].
$$

Combining the operations we can consider a generalized fractional derivative in the Riemann-Liouville sense (see [32]), in the formal form, we have

$$
D_{1/\alpha}^{-} f(t) := - \frac{1}{\Gamma(1 - 1/\alpha)} \frac{d}{dt} \int_{0}^{t} f(t + x)x^{-1/\alpha}dx, \alpha > 1.
$$

Let us study a Cauchy problem

$$
D_{1/\alpha}^{-} u = \check{W}u, \quad u(0) = h \in \mathcal{D}(\check{W}), \quad (15)
$$

where in the case when the operator composition $D_{1/\alpha}^{-} \check{W}$ is accretive, we assume that $h \in \mathcal{H}$.

**Theorem 5.** Assume that the Theorem 3 conditions hold, then there exists a solution of the Cauchy problem (15) in the form

$$
u=0 \sum_{q=N_{\nu}+1} \sum_{\xi=1} \sum_{i=0} e_{q}\nu+c_{q+t}(t),
$$

(16)
where
\[ \sum_{\nu=0}^{\infty} \left| \sum_{q=N_{\nu}+1}^{m(q)} \sum_{\xi=1}^{k(q)} e_{q+i} c_{q+i}(t) \right|_{\mathcal{H}} < \infty, \]
a sequence of natural numbers \( \{N_{\nu}\}_{\nu=0}^{\infty} \) can be chosen in accordance with the claim of Theorem 3.

Moreover, the existing solution is unique if the operator composition \( D^{1-1/\alpha} \tilde{W} \) is accretive.

**Proof.** Let us find a solution of problem (15) in the form (16). Analogously to the reasonings of Theorem 4, using Lemma 6 [17], it is not hard to prove that the following integrals exist i.e.
\[ \frac{1}{2\pi i} \int_{\gamma} e^{-\lambda^\alpha t}(E - \lambda A)^{-1}h d\lambda \in \mathcal{H}; \quad \frac{du}{dt} = -\frac{1}{2\pi i} \int_{\gamma} e^{-\lambda^\alpha t}\lambda A(E - \lambda A)^{-1}h d\lambda \in \mathcal{H}. \quad (17) \]

Note that the first relation gives us the fact \( u(t) \in D(\tilde{W}) \). Applying the scheme of the proof corresponding to the ordinary integral calculus, using the contour \( \gamma_k \) (see reasonings of Theorem 4), applying Lemma 6 [17], we can establish a formula
\[ \int_{0}^{\infty} x^{-\sigma} dx \int_{\gamma} e^{-\lambda^\alpha(t+x)} \lambda^m A(E - \lambda A)^{-1}h d\lambda = \int_{\gamma} e^{-\lambda^\alpha t} \lambda^m A(E - \lambda A)^{-1}h d\lambda \int_{0}^{\infty} x^{-\sigma} e^{-\lambda^\alpha x} dx, \quad (18) \]
where \( \sigma \in (0, 1) \), \( \zeta > 0 \), \( m \in \mathbb{N}_0 \). In the same way, we get
\[ \frac{d}{dt} \int_{\gamma} \lambda^{1-\alpha} e^{-\lambda^\alpha t} A(E - \lambda A)^{-1}h d\lambda = -\int_{\gamma} \lambda e^{-\lambda^\alpha t} A(E - \lambda A)^{-1}h d\lambda. \]

Applying the obtained formulas, taking into account a relation
\[ \int_{0}^{\infty} x^{-\frac{1}{\alpha}} e^{-\lambda^\alpha x} dx = \Gamma(1 - 1/\alpha)\lambda^{1-\alpha}, \]
we get
\[ D^{1/\alpha} u = \frac{1}{2\pi i} \int_{\gamma} e^{-\lambda^\alpha t} \lambda A(E - \lambda A)^{-1}h d\lambda. \quad (19) \]

Making the substitution using the formula \( \lambda A(E - \lambda A)^{-1} = (E - \lambda A)^{-1} - E \), we obtain
\[ D^{1/\alpha} u = \frac{1}{2\pi i} \int_{\gamma} e^{-\lambda^\alpha t}(E - \lambda A)^{-1}h d\lambda - \frac{1}{2\pi i} \int_{\gamma} e^{-\lambda^\alpha t}h d\lambda = I_1 + I_2. \]

The second integral equals zero by virtue of the fact that the function under the integral is analytical inside the intersection of the domain \( G \) with the circle of an arbitrary radius \( R \) and it decreases sufficiently fast on the arch of the radius \( R \), when \( R \to \infty \), here we denote by \( G \) the interior of the contour \( \gamma \). Now, if we consider the expression for \( u \), we obtain the fact that \( u \) is a solution of the equation i.e. \( D^{1/\alpha} u = \tilde{W} u \). We obtain the decomposition on the series of the root vectors (16) due to Theorem 3. Let us show that the initial condition holds in the sense
Let us multiply the both sides of the latter relation on \( u \). Using formula (20), we get

\[
\int_0^\infty x^{1/\alpha-1} dx \int_\gamma e^{-\lambda\gamma(t+x)} \lambda A(E - \lambda A)^{-1} h d\lambda = \int_\gamma e^{-\lambda\gamma t} \lambda A(E - \lambda A)^{-1} h d\lambda \int_0^\infty x^{1/\alpha-1} e^{-\lambda\gamma x} dx = \]

\[
= \Gamma(1/\alpha) \int_\gamma e^{-\lambda\gamma t} A(E - \lambda A)^{-1} h d\lambda = 2\pi i \Gamma(1/\alpha) u(t).
\]

Substituting (17) to the first term of the latter relation, using the given above definition of the generalized fractional derivative, we obtain (20). Now, assume that \( h \in D(\tilde{W}) \). Applying the operator \( \mathcal{D}_{-1/\alpha} \) to both sides of relation (15), using formula (20), we get \( u' + \mathcal{D}_{-1/\alpha} \tilde{W} u = 0 \). Let us multiply the both sides of the latter relation on \( u \) in the sense of the inner product, we get \( (u', u)_\beta + (\mathcal{D}_{-1/\alpha} \tilde{W} u, u)_\beta = 0 \). Consider a real part of the latter relation, we have \( \text{Re} (u', u)_\beta + \text{Re} (\mathcal{D}_{-1/\alpha} \tilde{W} u, u)_\beta = (u', u)_\beta / 2 + (u, u')_\beta / 2 + \text{Re} (\mathcal{D}_{-1/\alpha} \tilde{W} u, u)_\beta \). Therefore \( \langle u(t) \rangle^2_t = -2 \text{Re} (\mathcal{D}_{-1/\alpha} \tilde{W} u, u)_\beta \leq 0 \). Integrating both sides, we get

\[
\|u(\tau)\|_\beta^2 - \|u(0)\|_\beta^2 = \int_0^\tau \frac{d}{dt} \|u(t)\|_\beta^2 dt \leq 0.
\]

The last relation can be rewritten in the form \( \|S_t h\|_\beta \leq \|h\|_\beta \), \( h \in D(\tilde{W}) \). Since \( D(\tilde{W}) \) is a dense set in \( \mathcal{H} \), then we obviously obtain the desired result i.e. \( \|S_t\|_{\mathcal{H} \to \mathcal{H}} \leq 1 \). The rest part of the proof is absolutely analogous to the proof of Theorem 4. The decomposition on the series of the root vectors (16) is given by virtue of Theorem 3. The uniqueness follows from the accretive property of the operator composition \( \mathcal{D}_{-1/\alpha} \tilde{W} \). \( \square \)

**Remark 2.** Eventually, we want to note that Theorem 2 is applicable to the operators studied in the previous paragraphs. The question that may appear is related to the description of the operator class for which the considered above operator composition is accretive.

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4 Conclusions

In this paper we studied the Cauchy problems for the evolution equation in the abstract Hilbert space. The made approach allows us to obtain a solution analytically for the right-hand side belonging to a sufficiently wide class of operators. In this regard such operators as the Riemann-Liouville fractional differential operator, the Kipriyanov operator, the Riesz potential, the difference operator have been involved. Moreover, we produced the artificially constructed normal operator for which the clarification of the Lidskii V.B. results relevantly works. It is remarkable that Theorem 5 covers many previously obtained results in the framework of the theory of fractional differential equations. However, the main advantage is the obtained abstract formula for the solution. Moreover, the norm-convergence of the series representing the solution allows us to apply the methods of the approximation theory. It is also worth noticing that Theorem 5 jointly with Theorem 3 give us the opportunity to minimize conditions imposed on the fractional order at the left-hand side of the equation and the artificially constructed normal operator provides the relevance and significance of such an achievement. Here, we should admit that the analogs of Theorem 5 can be obtained by Lidskii V.B. methods [20], but the used in the paper [17] variant of the natural lacunae method allows us to formulate the optimal conditions in comparison with the Lidskii V.B. results.

The interesting problem that may appear is how to consider the evolution equations of the arbitrary real order. We should note that the applied technique can be relevant in the case of the integer order higher than one for the operator function under the integral presenting the solution is sufficiently ”good” what allows us to differentiate it. The latter opportunity becomes more valuable due to the appeared formula connecting the derivative of the integer order and the corresponding power of the operator. This creates a prerequisite to consider a generalized Taylor series and raise a question on convergence at least and representing the solution at most. It is remarkable that the supposed correspondence between derivatives and the operator powers leads us to the study of series of unbounded operators. Apparently, the hypotheses on the representing the solution by the Taylor series is equivalent to existing the fixed point of the mentioned operator series. We hope that this idea creates a prerequisite for further study in the direction.

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