The Eigenvalue Spectrum of the Inertia Operator

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Abstract

We view the inertia construction of algebraic stacks as an operator on the Grothendieck groups of various categories of algebraic stacks. We show that the inertia operator is locally finite and diagonalizable. This is proved for the Grothendieck group of Deligne-Mumford stacks over a base scheme and the category of quasi-split Artin stacks defined over a field of characteristic zero. Motivated by the quasi-splitness condition, in [2] we consider the inertia operator of the Hall algebra of algebroids, and applications of it in generalized Donaldson-Thomas theory.

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1 Introduction

Let $B$ be a unital noetherian commutative ring and $\text{Sch}/B$, the big étale site of schemes of finite type over $B$. $\text{St}/B$ denotes the category of algebraic stacks of finite type over $\text{Sch}/B$ with affine diagonals. This in particular means that for every algebraic stack $\mathcal{X}$ and an $S$-point $s : S \to \mathcal{X}$, the sheaf of automorphisms $\text{Aut}(s) \to S$ is an affine $S$-group scheme.

The Grothendieck group of $\text{Sch}/B$, denoted by $K(\text{Sch}/B)$ is the free abelian group of isomorphism classes of such schemes, modulo the scissor relations, $[X] = [Z] + [X \setminus Z]$, for $Z \subset X$ a closed subscheme and equipped with structure of a commutative unital ring according to the fibre product in $\text{Sch}/B$, $[X] \cdot [Y] = [X \times Y]$. We will always tensor with $\mathbb{Q}$.

The Grothendieck ring of the category $\text{St}/B$ of algebraic stacks over $\text{Sch}/B$, is defined similar to above: $K(\text{St}/B)$ is the $\mathbb{Q}$-vector space generated by isomorphism classes of algebraic stacks modulo similar relations; i.e. for any closed immersion $\mathfrak{Z} \to \mathcal{X}$ of algebraic stacks we have $[\mathcal{X}] = [\mathfrak{Z}] + [\mathcal{X} \setminus \mathfrak{Z}]$. And the fiber product over the base category turns $K(\text{St}/B)$ into a commutative ring. Hence for any algebraic stack $\mathfrak{Y}$, isomorphic to a fiber product $\mathcal{X} \times \mathfrak{Z}$, we have $[\mathfrak{Y}] = [\mathcal{X}][\mathfrak{Z}]$. Moreover, $K(\text{St}/B)$ is a unital associative $K(\text{Sch}/B)$-algebra in the obvious way.

For any algebraic stack, $\mathcal{X}$, the inertia stack $\mathcal{I}\mathcal{X}$ is the fiber product

\[
\begin{array}{ccc}
\mathcal{I}\mathcal{X} & \square & \mathcal{X} \\
\downarrow & & \downarrow \\
\mathcal{X} & \Delta & \mathcal{X} \times \mathcal{X}
\end{array}
\]

where $\Delta$ is the diagonal morphism. $\mathcal{I}\mathcal{X}$ is isomorphic to the stack of objects $(x, f)$, where $x$ is an object of $\mathcal{X}$ and $f : x \to x$ is an automorphism of it. Here a morphism $h : (x, f) \to (y, g)$ is an arrow $h : x \to y$ in $\mathcal{X}$ satisfying $goh = hof$. 

This construction respects the equivalence and scissor relations and is $K(\text{Sch}/B)$-linear, hence inducing a well-defined inertia operator on $K(\text{St}/B)$ as a $K(\text{Sch}/B)$-algebra, and in particular an inertia endomorphism on the $K(\text{Sch}/B)$-module $K(\text{St}/B)$.

Other important variants of the inertia operator are the semisimple and unipotent inertia operators, respectively $I^{ss}$ and $I^{u}$, which are defined in §5.4.

We always assume that $K$ is tensored with $\mathbb{Q}$. Our main results on local finiteness and diagonalization of the inertia endomorphism are as follows. Here $q = [\mathbb{A}^1]$ is our notation for the class of the affine line.

1. Corollary 4.9. In the case of Deligne-Mumford stacks the operator $I$ is diagonalizable as a $K(\text{Sch}/B)$-linear endomorphism, and the eigenvalue spectrum of it is equal to $\mathbb{N}$, the set of positive integers.

2. Corollary 5.13. In the case of Artin stacks, the unipotent inertia $I^{u}$ is diagonalizable on $K(\text{St})[q^{-1}, \{(q^k - 1)^{-1} : k \geq 1\}]$ and the eigenvalue spectrum of it is the set $\{q^k : k \geq 0\}$ of all power of $q$.

3. Theorem 6.6. In the case of quasi-split stacks, the operator $I$ is diagonalizable as a $\mathbb{Q}(q)$-linear endomorphism and the eigenvalue spectrum of it is the set of all polynomials of the form

$$n q^k \prod_{i=1}^{k} (q^{r_i} - 1).$$

4. Theorem 6.10. In the case of quasi-split stacks the endormorphism $I^{ss}$ is diagonalizable as a $\mathbb{Q}(q)$-linear operator and the eigenvalue spectrum of it is the set of all polynomials of the form

$$n \prod_{i=1}^{k} (q^{r_i} - 1).$$

This paper is organized as follows. §2 contains several results needed for stratification of group schemes in the rest of the paper. The reader may skip this section and refer to it as needed in the proofs of the results in other sections. In §3 we consider the simplest case; i.e. the category of groupoids. The arguments in this section illustrate the main ideas behind the results of this paper. In §4 we consider the full subcategory $\text{DM}$ of $\text{St}$, of Deligne-Mumford stacks. For the cases of Artin stacks in §5 and §6 the base category would be the category of varieties over the spectrum of an algebraically closed field in characteristic zero.

2 Stratification of group schemes

2.1 Connected component of unity

By a group space $G \rightarrow X$, we mean a group object in the category $\text{St}/B$. In this section we will see that we can always stratify such objects by nicely-behaved group schemes. The results of this section will be used in §5 and §6.
Let $G$ be a finitely presented group scheme over a base scheme $X$. Recall that the connected component of unity, $G^0$, is defined as the functor that assigns to any morphism $S \to X$, the set

$$G^0(S) = \{ u \in G(S) : \forall x \in X, u_x(S_x) \subset G^0_x \}.$$ 

Here $G^0_x$ is the connected component of unity of the algebraic group $G_x = G \otimes_X \kappa(x)$. By [1] Exp. VI(B), Thm. 3.10], if $G$ is smooth over $X$, this functor is representable by a unique open subgroup scheme of $G$. Also note that in this case, $G^0$ is smooth and finitely presented and is preserved by base change [1] Exp. VI(B), Prop. 3.3.

When $G$ is finitely presented and smooth, the quotient space $G/G^0$ exists as a finitely presented and étale algebraic space over $X$. For sheaf theoretic reasons, the formation of this quotient is also preserved by base change. $G^0$ is not closed in general (interesting examples can be found in [1] §7.3 (iii) and [1] Exp. XIX, §5]), however we have the following

**Lemma 2.1.** Let $G \to X$ be a smooth group scheme and assume that $\overline{G} = G/G^0$ is finite and étale. Then $G^0$ is a closed subscheme of $G$.

**Proof.** $\overline{G} \to X$ is finite, hence proper and consequently universally closed. Thus in the cartesian diagram

$$
\begin{array}{ccc}
G^0 \times_X G & \longrightarrow & G \times_X G \\
\downarrow & & \downarrow \\
\overline{G} & \longrightarrow & G \times_X \overline{G}
\end{array}
$$

the morphism $\varphi : (h, g) \mapsto (hg, g)$ is a closed immersion. The property of being a closed immersion is local in the fppf topology and thus by the cartesian diagram

$$
\begin{array}{ccc}
G^0 \times_X G & \longrightarrow & G \times_X G \\
\downarrow & & \downarrow \\
G^0 & \longrightarrow & G
\end{array}
$$

the embedding of $G^0$ in $G$ is also a closed immersion. The vertical right hand arrow is described by $(g, h) \mapsto gh^{-1}$. □

Now from Zariski’s main theorem (in particular [17] Lem. 03I1)) it follows that:

**Corollary 2.2.** Let $G$ be a smooth finitely presented group scheme over $X$. There exists a nonempty open scheme $U$ of $X$, such that $G^0_U$ is closed and $G_U/G^0_U$ is finite and étale over $U$. 

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2.2 Centre and centralizers

We refer the reader to [1, §2.2] for a definition of functorial centralizer, $Z_G(Y)$, for a closed subscheme $Y$ of a group scheme $G \to X$. It is not generally true that these functors are representable by schemes. However:

**Corollary 2.3.** Let $G$ be a smooth group scheme over an integral base scheme $X$. There exists a nonempty Zariski open $U$ in $X$, such that $G_U$ has a closed subscheme representing its centre.

**Proof.** By Corollary 2.2 we may assume that $G^0$ is a closed and open connected subscheme and $G = G/G^0$ is a finite étale group scheme over $X$. Let $\bar{X} \to X$ be a finite étale cover such that $G|_{\bar{X}}$ is a constant finite group over $\bar{X}$; i.e. a union of connected components all of which are isomorphically mapped to $\bar{X}$. Since $G \to \bar{G}$ is a torsor for a connected group, so is $G|_{\bar{X}} \to \bar{X}$. Therefore the connected components of the source and target correspond bijectively. But every connected component of $G|_{\bar{X}}$ maps isomorphically to $\bar{X}$, thus each connected component of $G|_{\bar{X}}$ is isomorphic to $G^0$. Now by [1, Lem. 2.2.4], the centralizer of each connected component exists over $\bar{X}$ and their intersection is the centre of $G|_{\bar{X}}$. Finally $\bar{X} \to X$ is étale and in particular an fpqc covering and the centre of $G|_{\bar{X}}$ is affine over $\bar{X}$, so affine descent finishes the proof. □

2.3 Groups of multiplicative type

**Quasi-split tori**

A commutative group scheme $T \to X$ is said to be of multiplicative type if it is locally diagonalizable over $X$ in the fppf topology (and therefore in the étale topology [1, Exp. X, Cor. 4.5]). For the general theory of group schemes of multiplicative type we refer the reader to [1, Ch. IV–X], however we recall a few preliminary facts here. Associated to $T$ there exists [1, Exp. X, Prop. 1.1] a locally constant étale abelian sheaf

$$T \mapsto M = \text{Hom}_{X-\text{gp}}(T, \mathbb{G}_m),$$

and $T$ is the scheme representing the sheaf $\text{Hom}_{X-\text{gp}}(M, \mathbb{G}_m)$. This is an anti-equivalence of the categories of $X$-group schemes of multiplicative type and locally constant étale sheaves on $X$ whose geometric fibers are finitely generated abelian groups.

Since every étale morphism is Zariski locally finite [17, Lem. 03H1], by shrinking $X$ to a dense open subset of it, we may assume that $T$ is *isotrivial*; i.e. isomorphic to $H \times \mathbb{G}_m^n$ after base change along a finite étale cover of $X' \to X$. Here $H$ is a constant finite commutative group. Equivalently, $T_{X'}$ is isomorphic to $\text{Hom}_{X-\text{gp}}(M_{X'}, \mathbb{G}_m, X')$ where $M$ is a finitely generated abelian group.

We may also assume that $X' \to X$ is connected and Galois by [13, Proposition 6.18]). Let $\Gamma$ be the associated Galois group. The action of $\Gamma$ on $X'$ induces an action of it on $M_{X'}$. 

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An $X$-torus, $T$, is an $X$-group scheme which is fppf locally isomorphic to $\mathbb{G}_{m,X}$. This is equivalent to asking for $M$ to be torsion-free. We say $T$ splits over $X'$ if $T_{X'} = \mathbb{G}_{m,X'}$. Our anti-equivalence of categories is now between isotrivial $X$-tori that split over $X'$ and $\Gamma$-lattices (i.e. a finitely generated torsion free abelian groups equipped with the structure of some $\Gamma$-module of finite type) given by 

$$T \mapsto H^0(X', \text{Hom}_{X'}(X' \times X T, \mathbb{G}_m))$$

and by 

$$A \mapsto \text{Hom}_{X}(X' \times A/\Gamma, T)$$

in the reserve direction where $A$ is a $\Gamma$-lattice. In this case, $\chi_T = M_{X'}$ is called the character lattice of $T$. $T$ is called a quasi-split torus if $\chi_T$ is a permutation $\Gamma$-lattice (i.e. the action of $\Gamma$ on $\chi_T$ is by permutation of the elements of a $\mathbb{Z}$-basis).

**Maximal tori of group schemes**

We recall [1, Exposé IXX, Def. 1.5] that the reductive rank of algebraic $k$-group $G$, is the rank of a maximal torus $T$ of $G$ where $k$ is the algebraic closure of $k$: $\rho_r(G) = \dim_k T$. Likewise, the unipotent rank of $G$ is the dimension of the unipotent radical $U$ of $G$ and denoted by $\rho_u(G) = \dim_k U$.

For an affine smooth $X$-group scheme $G$, the above integers can be more generally considered as functions on $X$ that assign to every point $x \in X$, the corresponding $\rho_r(x) = \rho_r(G_x)$, and $\rho_u(x) = \rho_u(G_x)$.

The function $\rho_r$ is lower semi-continuous in the Zariski topology [1, Exposé XII, Thm. 1.7]. Moreover, the condition of $\rho_r$ being a locally constant function in the Zariski topology, is equivalent to existence of a global maximal $X$-torus for $G$ in the étale topology by [1, Exposé XII, Thm. 1.7]. If $G$ is commutative, then this is furthermore equivalent to existence of a global maximal $X$-torus for $G$ in the Zariski topology [1, Exposé XII, Cor. 1.15]. This immediately implies the following

**Proposition 2.4.** Let $G$ be an affine smooth group scheme over a noetherian base scheme $X$. Then there exists a stratification of $X$ by finitely many locally closed Zariski subschemes $\{X_i\}$ such that each group scheme $G|_{X_i}$ admits an isotrivial maximal torus. If $G$ is moreover commutative, $G|_{X_i}$ admits a maximal torus in Zariski topology.

**Proof.** Since $\rho_r$ is lower semi-continuous and integer valued there exists a stratification by locally closed subspaces on which $\rho_r$ is constant. By our earlier discussion, we may further assume each global torus is isotrivial. □
2.4 Commutative group schemes

An affine group scheme $U \to X$ is said to be unipotent if it is unipotent over each geometric fibre. The goal of this section is to prove the following structure decomposition for a commutative group scheme.

**Proposition 2.5.** Let $X$ be an integral scheme and $G$ a finitely presented smooth affine commutative group scheme over $X$. Let $\eta$ be the generic point of $X$. Then the decomposition of $G_{\eta}$ as $T_{\eta} \times U_{\eta}$ to a maximal torus and the unipotent radical spreads out; i.e. there exists a non-empty Zariski open $V \subseteq X$ such that $G_{V}$ is isomorphic to $T_{V} \times U_{V}$ where $T_{V}$ is a maximal torus with $T_{\eta}$ as generic fiber and $U_{V}$ is a unipotent $V$-group scheme with $U_{\eta}$ as generic fiber. Moreover, if $T_{\eta}$ is quasi-split we may assume $T_{V}$ is also quasi-split.

We say a group scheme $H_{\eta} \to X$ (or a property of it with respect to another group scheme $G_{\eta}$) spreads out to a neighborhood of the generic point $\eta \in X$ if there exists a dense open subset $U \subseteq X$ over which there is a $U$-group scheme $H_{U}$ pulling back to the prior one (and satisfying the same property with respect to a spreading out $G_{U}$ of $G_{\eta}$).

**Lemma 2.6.** Let $X$ be an integral scheme and $G$ a finitely presented affine group scheme over $X$. Then closed subgroups of the generic fiber spread out: i.e. let $\eta$ be the generic point of $X$ and $H_{\eta}$ a closed subgroup of $G_{\eta}$. Then there exists a non-empty open set $U \subseteq X$ such that $G_{U}$ contains a subgroup scheme $H_{U}$ fitting in the commutative diagram

$$
\begin{array}{ccc}
H_{\eta} & \longrightarrow & H_{U} \\
\downarrow & & \downarrow \\
G_{\eta} & \longrightarrow & G_{U}
\end{array}
$$

where the horizontal arrows are pull-back morphisms and the vertical arrows are monomorphisms of group schemes.

**Proof.** It suffices to consider the case where $X$ is an affine scheme $X = \text{Spec } R$. Let $K$ be the function field of $R$ with the canonical homomorphism $R \to K$ corresponding to the inclusion of the generic point $\eta \to X$. Then $G = \text{Spec } S$, where $S = R[x_{1}, \ldots, x_{k}] / \mathfrak{I}$ for some finitely generated ideal $\mathfrak{I} = \langle p_{1}, \ldots, p_{\ell} \rangle \subseteq R[x_{1}, \ldots, x_{k}]$ and therefore $G_{\eta} = \text{Spec } K \otimes_{R} S$. With this notation, $H_{\eta}$ is cut out as a subscheme by a finitely generated ideal $\mathfrak{p} \subseteq K \otimes_{R} S = K[x_{1}, \ldots, x_{k}] / \mathfrak{I} K$. Thus each generator of $\mathfrak{p}$ can be considered as a polynomial with coefficients in $K$. Since $K$ is the inverse limit of localizations of $R$ in its elements, there exists $f \in R$ such that all elements of $\mathfrak{p}$ are defined with coefficients in $R_{f}$. This defines a subscheme $H_{U}$ of $G_{U}$ satisfying the commutativity of the above diagram, if we set $U = \text{Spec } R_{f}$.

Now we put a group scheme structure on $H_{U}$ by shrinking $U$ further. Let $i : G \to G$ and $m : G \times_{X} G \to G$, respectively be the inversion and multiplication morphisms on $G$. Considering the inversion morphism, existence of
group structure on $H_\eta$ means that in level of coordinate rings, we are given a commutative diagram

$$
\begin{array}{ccc}
R_f[x_1, \ldots, x_k]/\mathcal{J}R_f & \xrightarrow{i^\#} & R_f[x_1, \ldots, x_k]/\mathcal{J}R_f \\
\downarrow & & \downarrow \\
K[x_1, \ldots, x_k]/\mathcal{J}K & \xrightarrow{q} & K[x_1, \ldots, x_k]/\mathcal{P}K
\end{array}
$$

and the composition of the induced morphism $i^\#$ and the quotient map $q$ has precisely $\mathcal{P}K$ as its kernel. Hence by a similar argument, there exists some $g \in R_f$ lifting this composition as in the cartesian diagram

$$
\begin{array}{ccc}
R_{fg}[x_1, \ldots, x_k]/\mathcal{P}R_{fg} & \xrightarrow{\varphi} & R_{fg}[x_1, \ldots, x_k]/\mathcal{P}R_{fg} \\
\downarrow & & \downarrow \\
K[x_1, \ldots, x_k]/\mathcal{P}K & \xrightarrow{\varphi_{|\varphi}} & K[x_1, \ldots, x_k]/\mathcal{P}K.
\end{array}
$$

The case of multiplication morphism is similar. So by shrinking further we may assume that $H_U$ is a $U$-group scheme. Commutativity of the diagram

$$
\begin{array}{ccc}
H_U \times H_U & \longrightarrow & H'_U \\
\downarrow & & \downarrow \\
G_U \times G_U & \longrightarrow & G'_U
\end{array}
$$

where the horizontal arrows are morphisms $(x,y) \mapsto xy^{-1}$ is now obvious. Thus $H_U$ is the desired subgroup scheme of $G_U$. □

**Lemma 2.7.** Group homomorphisms (respectively isomorphisms) spread out. Let $G \rightarrow X$ and $\eta \in X$ be as in previous lemma. If $G' \rightarrow X$ is another group scheme and $\varphi: G_\eta \rightarrow G'_\eta$ is a group scheme homomorphism (resp. isomorphism), then there exists a non-empty $U \subseteq X$ and a homomorphism (resp. isomorphism) $\varphi: G'_U \rightarrow G_U$ such that $\varphi|_U = \varphi_\eta$.

**Proof.** The proof is by similar arguments as in previous lemma. □

**Lemma 2.8.** The property of being a quasi-split torus spreads out. Let $G \rightarrow X$ and $\eta \in X$ be as in the previous lemmas. If $G_\eta$ is a quasi-split torus, then there exists a non-empty $U \subseteq X$ such that $G'_U$ is a quasi-split torus.

**Proof.** We may assume that $G$ is also isotrivial. We recall that the character lattice of a torus is expressed in terms of the (étale locally constant) sheaf
\( \chi(G) = \text{Hom}_{U - \text{gp}}(G, \mathbb{G}_{m,U}) \). Restriction from \( U \) to \( \eta \) induces a homomorphism of finitely generated \( \mathbb{Z} \)-modules

\[ \text{Hom}_{U - \text{gp}}(G, \mathbb{G}_{m,U}) \to \text{Hom}_{K - \text{gp}}(G_{\eta}, \mathbb{G}_{m,\eta}) \]

and by Lemma \([2.7]\) we may assume that this is an isomorphism of \( \mathbb{Z} \)-modules. We also note that if \( \tilde{U} \to U \) is a finite \`e\tel\-etale covering that trivializes \( G_U \) and restricts to the finite separable extension \( L/\kappa(\eta) \), then \( \Gamma = \text{Gal}(L/K) \) is at the same time the fundamental group of this covering and the associated action of \( \Gamma \) on \( \chi(G_{\eta}) \) induces same action of this group on \( \chi(G) \). □

**Lemma 2.9.** Let \( G \) be unipotent group scheme over an integral base scheme \( X \). Then there exists a non-empty Zariski open set \( U \subset X \) such that \( G_U \) has a filtration in subgroups \( 1 \subset G_1 \subset \ldots \subset G_{r-1} \subset G_r = G \) with all factors \( G_\ell/G_{\ell-1} \) isomorphic to the constant \( U \)-group scheme \( \mathbb{G}_a,U \).

**Proof.** Let \( H_\eta \) be a subgroup of the generic fiber \( G_{\eta} \). By Lemma \([2.6]\) the property of being a subgroup spreads out to a non-empty open in \( X \). We also observe that the property of being isomorphic to \( \mathbb{G}_a \) spreads out. That is, if \( H_\eta \) is isomorphic to \( \mathbb{G}_{a,\eta} \), then there exists a non-empty open \( U \subset X \) such that \( H_\eta \) spreads out over it to the constant group scheme \( \mathbb{G}_{a,U} \). This is another straightforward spreading out argument: let \( K \) be the field of fraction of an integral domain \( R \). Let \( S \) be a finitely presented \( R \)-algebra generated by \( x_1, \ldots, x_\ell \) and that there exists an \( R \)-algebra isomorphism \( \varphi : K \otimes_R S \to K[t] \). It is easy to check that there exists a localization \( R_I \) of \( R \) that extends \( \varphi \) to an isomorphism \( \tilde{\varphi} : R_I \otimes_R S \to R_I[t] \). The claim now follows by induction on the quotient scheme \( G_U/\mathbb{G}_{a,U} \). □

The proof of Proposition \([2.5]\) is now immediate by spreading the maximal torus of the generic fibre out by Proposition \([2.4]\) and spreading the unipotent radical out by Lemma \([2.9]\) and observing that the group structure of \( T_\eta \times U_\eta \) also spreads out by Lemma \([2.7]\).

### 3 Inertia of groupoids

Let \( K(\text{gpd}) \) be the \( \mathbb{Q} \)-vector space generated by finite groupoids, modulo equivalence and scissor relations. It is easy to verify that the vector space \( K(\text{gpd}) \) is generated by \([BG]\), for finite groups \( G \).

Denote by \( I : K(\text{gpd}) \to K(\text{gpd}) \) the endomorphism sending \([X]\) to \([IX]\), where \( IX = X \times_{X \times X} X \) is the inertia groupoid of \( X \). Note that inertia is compatible with equivalence and scissor relations, so that \( I \) is well-defined.

\(^1\)A relevant note is that \( \mathbb{A}_1 \)-fibrations are always Zariski locally trivial (cf. \( \[8\] \)).
3.1 Filtration by central order

The vector space $K(\text{gpd})$ has two natural gradings: by order of the automorphism group, and by order of the centre of the automorphism group. Let $K^n(\text{gpd})$ be the subspace of $K(\text{gpd})$ generated by those $[BG]$ such that $\#G = n$. Let $K_i(\text{gpd})$ be the subspace generated by $[BG]$ such that $\#Z(G) = i$. Finally $K^n_i(\text{gpd})$ is generated by those $[BG]$ such that $\#G = n$, and $\#Z(G) = i$. We have

$$K(\text{gpd}) = \bigoplus_{n=1}^{\infty} \bigoplus_{i=1}^{\infty} K^n_i(\text{gpd}).$$

Clearly, $K^n(\text{gpd})$ is finite-dimensional for every $n$, but $K_i(\text{gpd})$ is infinite-dimensional, for all $i$. This grading defines an ascending filtration $K^{\leq n}(\text{gpd})$ and a descending filtration $K_{\geq i}(\text{gpd})$.

**Lemma 3.1.** The endomorphism $I$ preserves $K^{\leq n}(\text{gpd})$ and $K_{\geq i}(\text{gpd})$ and on the associated graded, $K_{\geq i}/K_{>i}(\text{gpd})$, $I$ is multiplication by $i$.

**Proof.** For any finite group $G$,

$$I[BG] \cong \bigcup_{g \in C(G)} BZ_G(g),$$

where $C(G)$ denotes the set of conjugacy classes of $G$, and $Z_G(g)$ is the centralizer of $g$ in $G$. Thus,

$$I[BG] = \sum_{g \in C(G)} [BZ_G(g)] = \#Z(G)[BG] + \sum_{g \in C(G)^*} [BZ_G(g)],$$

where $C(G)^*$ denotes the set of non-central conjugacy classes. Now we note that for non-central $g$ we have strict inequalities

$$\#Z(G) < \#Z_G(g) < \#G.$$

This is enough to prove the claim. □

3.2 Local finiteness and diagonalization

**Proposition 3.2.** The endomorphism $I : K(\text{gpd}) \to K(\text{gpd})$ is diagonalizable, with spectrum of eigenvalues equal to the positive integers.

**Proof.** Every subspace $K^{\leq n}(\text{gpd})$ is finite dimensional, and preserved by $I$. On this finite dimensional subspace, $I$ is triangular, with respect to the grading with central order, and with distinct eigenvalues on the diagonal. This proves that $I$ is diagonalizable when restricted to $K^{\leq n}(\text{gpd})$ for all $n$. □
Thus $K(gpd)$ has another natural grading, namely the grading induced by the direct sum decomposition into eigenspaces under $I$. This can be interpreted as a grading by the virtual order of the centre. Denote the corresponding projection operators by $\pi_n$.

For instance, if $A$ is a finite abelian group, then $[BA]$ is an eigenvector for $I$, with eigenvalue $\#A$. Thus $\pi_n[BA] = [BA]$, if $A$ had $n$ elements, and $\pi_n[BA] = 0$, otherwise.

Example 3.3. For the dihedral group $D_4$ with eight elements, we have

$$ I[BD_4] = 2[BD_4] + [BZ_4] + 2[BD_2]. $$

Hence

$$ [BD_4] - \frac{1}{2}[BZ_4] - [BD_2] $$

is an eigenvalue of $I$ with eigenvalue 2. It follows that

$$ \pi_n[BD_4] = \begin{cases} [BD_4] - \frac{1}{2}[BZ_4] - [BD_2] & \text{if } n = 2 \\ \frac{1}{2}[BZ_4] + [BD_2] & \text{if } n = 4 \\ 0 & \text{otherwise} \end{cases}. $$

3.3 The operators $I_r$ and eigenprojections

Let $I_r BG$ be the stack of tuples $(s_1, \ldots, s_r)$ where $s_i$ are $r$ distinct pairwise commuting elements of $G$:

$$ I_r(BG) = [(G^*r)^* / G], $$

where the brackets are used as the notation for quotient algebroids. In $K(gpd)$ we write

$$ I_r[BG] = [(G^*r)^* / G], $$

where bracket stands for the element in the $K$-group and the quotient notation is omitted.

This defines another family of operators on $K(gpd)$. For $r = 0$, $I_0$ is identity on all $BG$ and $I_1$ is the usual inertia operator.

**Proposition 3.4.** The operators $I_r$ for $r \geq 0$, preserve the filtration $K_{\geq k}(gpd)$ and act as multiplication by $r! \binom{k}{r}$ on the quotient $K_{\geq k}(gpd)/K_{> k}(gpd)$.

**Proof.** Let $n$ be the size of the group $G$ and $k$ the size of its centre. Notice that there are $r! \binom{k}{r}$ ways of choosing a labelled set of $r$ elements from the centre and therefore,

$$ I_r[BG] = r! \binom{k}{r} [BG] + \sum_{S \in (G^*r)^*} [BZ_G(S)]. $$

\(\square\)
Corollary 3.5. The operators $I_r$, for $r \geq 0$ are simultaneously diagonalizable. The common eigenspaces form a family $\Pi_k(\text{gpd})$ of subspaces of $K(\text{gpd})$ indexed by positive integers $k \geq 0$, and

$$K(\text{gpd}) = \bigoplus_{k \geq 0} \Pi_k(\text{gpd}).$$

Let $\pi_k$ denote the projection onto $\Pi_k(\text{gpd})$. We have

$$I_r \pi_k = r! \binom{k}{r} \pi_k,$$

for all $r, k \geq 0$.

Corollary 3.6. For $r \geq 0$, we have

$$\ker I_r = \bigoplus_{k < r} \Pi_k(\text{gpd}).$$

Corollary 3.7. For every $k \geq 0$, we have

$$\pi_k = \sum_{r=k}^{\infty} \frac{(-1)^{r+k} r!}{r!} \binom{k}{r} I_r.$$

In particular, $\pi_0 = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} I_r$, and $\pi_1 = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{(r-1)!} I_r$.

PROOF. We use the “beautiful identity” [16]

$$\sum_r (-1)^{r+k} \binom{r}{k} \binom{k}{r} = \delta_{l,k}$$

to find the projections. We have $\text{id} = \sum_{\ell \geq 0} \pi_\ell$, and hence

$$I_r = \sum_{\ell \geq 0} I_r \pi_\ell = \sum_{\ell \geq 0} r! \binom{\ell}{r} \pi_\ell,$$

and therefore

$$\sum_{r \geq 0} \frac{(-1)^{r+k}}{r!} \binom{k}{r} I_r = \sum_{r \geq 0} \frac{(-1)^{r+k}}{r!} \binom{k}{r} \left( \sum_{\ell \geq 0} r! \binom{\ell}{r} \pi_\ell \right)$$

$$= \sum_{\ell \geq 0} \left( \sum_{r \geq 0} (-1)^{r+k} \binom{r}{k} \binom{\ell}{r} \right) \pi_\ell = \sum_{\ell \geq 0} \delta_{\ell,k} \pi_\ell = \pi_k.$$

$\square$
Example 3.8. For the group of permutations of 3 letters, $S_3$ we have

$$I_r[B_{S_3}] = \begin{cases} [B_{S_3}] + [B_{Z_2}] + [B_{Z_3}] & \text{if } r = 1 \\ 2[B_{Z_2}] + 3[B_{Z_3}] & \text{if } r = 2 \\ 3[B_{Z_3}] & \text{if } r = 3 \\ 0 & \text{otherwise} \end{cases}$$

This gives us a way of computing

$$\pi_n[B_{S_3}] = \begin{cases} I_1 - I_2 + \frac{1}{2}I_3 = [B_{S_3}] - [B_{Z_2}] - \frac{1}{2}[B_{Z_3}] & \text{if } n = 1 \\ \frac{1}{2}I_2 - \frac{1}{3}I_4 = [B_{Z_2}] & \text{if } n = 2 \\ \frac{3}{2}I_3 = \frac{1}{2}[B_{Z_3}] & \text{if } n = 3 \\ 0 & \text{otherwise} \end{cases}$$

4 Inertia operator on Deligne-Mumford stacks

Let $DM$ be the full subcategory of $St/B$ of all Deligne-Mumford stacks over $B$. The inertia of a Deligne-Mumford stack is another Deligne-Mumford stack. The Grothendieck ring, $K(DM)$, has the structure of a $K(Sch)$-algebra. The inertia endomorphism, respects the scissor relations and is linear with respect to multiplication by schemes. Therefore we have an induced inertia endomorphism on $K(DM)$.

4.1 Irreducible gerbes

An irreducible gerbe is a connected Deligne-Mumford stack, $X$, with finite étale inertia, $I_X \to X$.

Proposition 4.1. Every noetherian Deligne-Mumford stack can be stratified into finitely many locally closed irreducible gerbes.

Proof. It suffices to show that restricting $I_X \to X$ to some nonempty open substack of $X$ is a finite étale morphism. Using [15 Prop. 5.7.6] we may assume that $X$ is an integral Deligne-Mumford stack. Flatness is an fpqc-local property hence by generic flatness [7 Thm. 6.9.1] there exists an open substack of $X$ such that $I_X \to X$ is flat. Since $I_X \to X$ is unramified, it is étale as well. A quasi-finite morphism of schemes is generically finite [17 Lem. 03I1]. Therefore by fpqc-descent of finiteness on base [17 Lem. 02LA], $I_X \to X$ is finite on an open substack of $X$. □

Central inertia

Let $ZI_X$ be the full subcategory of $I_X$, consisting of tuples $(x, \varphi)$ of objects $x$ of $X$ and central automorphisms $\varphi \in Z(Aut(x))$. Let $U \to \mathfrak{X}$ be an étale cover of $X$ by a scheme $U$ such that $\mathfrak{X}_U$ is the neutral gerbe $B_U G$ for an constant
finite $U$-group scheme $G$. Then $ZI\mathcal{X}|_{U'}$ is isomorphic to the gerbe $B_{I';Z(G)}$. We conclude that $ZI\mathcal{X}$ is a Deligne-Mumford stack and $ZI\mathcal{X} \to I\mathcal{X}$ is a closed immersion. In particular, if $\mathcal{X}$ is an irreducible gerbe, then $ZI\mathcal{X} \to \mathcal{X}$ is also a finite étale cover.

**Remark 4.2.** If $\mathcal{X}$ is an irreducible gerbe, then each connected component $Y$ of $I\mathcal{X}$ is an irreducible gerbe. This is clear since $I\mathcal{Y} \to I\mathcal{X}$ is finite étale and therefore so is $I\mathcal{Y} \to I\mathcal{X}$ by definition of the inertia stack. In other words, an inertia stack of an irreducible gerbe has a canonical stratification into irreducible gerbes by its connected components.

### 4.2 Filtration by split central order

Recall [7, Cor. 17.9.3] that if $\varphi : Y \to X$ is a separated étale morphism over a connected base scheme $X$, there is a one-to-one correspondence between the sections of $\varphi$ and the number of connected components of $Y$ isomorphic to $X$. Thus for a finite étale covering, the number of such sections is an indication of how close $Y$ is to being a trivial degree $n$ covering, $\bigsqcup_n X \to X$.

**Definition 4.3.** For an irreducible gerbe $\mathcal{X}$ we define the *split central order*, to be the number of sections of $ZI\mathcal{X} \to \mathcal{X}$.

We define an ascending filtration of $K(\text{DM})$ by declaring $[\mathcal{X}]$, for an irreducible gerbe $\mathcal{X}$, to belong to $K_{\geq n}(\text{DM})$ if its split central order is at least $n$.

**Proposition 4.4.** The inertia endomorphism on $K(\text{DM})$ preserves the filtration by split central order. Furthermore, on the associated graded piece $K_{\geq n}(\text{DM})/K_{>n}(\text{DM})$, the inertia endomorphism operates by multiplication by the integer $n$.

**Proof.** Consider an irreducible gerbe $\mathcal{X}$, with split central order $n$ and $I\mathcal{X} = \bigsqcup Y_n$ be the stratification of $I\mathcal{X}$ by connected components (hence irreducible gerbes). There are precisely $n$ of the $Y_n$ which are contained in $ZI\mathcal{X}$. Let $\mathcal{Y}$ be the associated graded piece of $K(\text{DM})$, and let $\mathcal{Z}$ be the inertia stack of $\mathcal{X}$. It suffices to show that any other strata $Y$ has split central number strictly larger than $n$.

There exists a diagram

$$
\begin{array}{ccc}
ZI(I\mathcal{X}) & \longrightarrow & I\mathcal{X} \times_{\mathcal{X}} ZI\mathcal{X} \\
\downarrow j & & \downarrow \pi_2 \\
I\mathcal{X} & \longrightarrow & ZI\mathcal{X}
\end{array}
$$

where the square is cartesian. For any object $x$ of $\mathcal{X}$, elements of $I\mathcal{X}$ over $x$ are pairs $(x, \varphi)$ such that $\varphi \in \text{Aut}(x)$ and objects of $ZI\mathcal{X}$ over $x$ are pairs $(x, \psi)$ where $\psi \in Z(\text{Aut}(x))$. The fibered product $I\mathcal{X} \times_{\mathcal{X}} ZI\mathcal{X}$ is hence the stack of triples $(x, \varphi, \psi)$ with $x, \varphi$ and $\psi$ as above. On the other hand, $ZI(I\mathcal{X})$ is the stack
of the objects \((x, \varphi, \psi)\) such that \(\varphi \in \text{Aut}(x), \psi \in Z(Z_{\text{Aut}(x)}(\varphi))\). Hence there is an embedding of the fibered product into \(\mathcal{Z}(I\mathcal{X})\). Restricting to a substack \(\mathcal{Y} \subset I\mathcal{X}\) we get the following diagram.

\[
\begin{array}{ccc}
\mathcal{Z}\mathcal{Y} & \xrightarrow{j} & \mathcal{Y} \times_{\mathcal{X}} \mathcal{Z}\mathcal{X} \\
\pi_3 & \rightarrow & \pi_2 \\
\mathcal{Y} & \xrightarrow{j} & I\mathcal{X}
\end{array}
\]

In this diagram the embedding \(j\) is necessarily a union of connected components, because all vertical and diagonal maps in the diagram are representable finite étale covering maps. Note also that there is a canonical section, \(\delta\), to \(\pi_3\): \(I(Y) \to Y\) via the diagonal \(Y \to Y \times Y\) since any automorphism of an object \(x\) in \(\mathcal{X}\) is in its own centralizer. It is obvious that any section of \(\pi_1\) pullback to a (distinct) section of \(\pi_2\) and gives a (distinct) section of \(\pi_3\). This shows that inertia endomorphism preserves \(K_{\geq n}\).

For the action of inertia on the graded piece \(K_{\geq n}/K_{>n}\) we show that if \(\mathcal{Y}\) is a component of \(I\mathcal{X}\) which is not a section of \(\pi_1\), then the associated section \(\delta\) is not induced by pulling back sections of \(\pi_1\). In fact, if \(\mathcal{Y}\) is not contained in \(\mathcal{Z}\mathcal{X}\), \(\delta\) does not lift to \(\pi_2\) and we are done. Otherwise, (when \(\mathcal{Y}\) is completely contained in \(\mathcal{Z}\mathcal{X}\)), \(\delta\) lifts to a section of \(\pi_2\) but the image of this section in \(\mathcal{Z}\mathcal{X}\) is \(Y\) itself, which is not a degree one cover of \(\mathcal{X}\). □

4.3 Local finiteness and diagonalization

In this section we use the notation \(\prod_{\mathcal{X}}^k \mathcal{Y}\) to denote the \(k\)-fold fiber product of a stack \(\mathcal{Y}\) by itself over \(\mathcal{X}\). We use the notation \(I^{(k)}\mathcal{X}\) for \(k\)-times application of the inertia construction on the stack \(\mathcal{X}\). We may think of the objects of \(I^{(k)}\mathcal{X}\) as tuples \((x, f_1, \ldots, f_k)\) of an object \(x\) in \(\mathcal{X}\) and pairwise commuting automorphisms \(f_1, \ldots, f_k\). A morphism \((x, f_1, \ldots, f_k) \to (y, g_1, \ldots, g_k)\) is an arrow \(h: x \to y\) of \(\mathcal{X}\) satisfying \(h \circ f_i = g_i \circ h\) for all \(i = 1, \ldots, k\).

**Lemma 4.5.** Let \(\mathcal{Y} \to \mathcal{X}\) be a finite étale representable morphism of algebraic stacks. Then the following family is a finite set up to isomorphism of stacks.

\[C(\mathcal{Y} \to \mathcal{X}) = \{\mathcal{W} : \mathcal{W}\ is\ a\ connected\ component\ of\ \prod_{\mathcal{X}}^k \mathcal{Y}\ for\ some\ k \geq 0\}\]

**Proof.** This is trivial since the Galois closure of \(\mathcal{Y}\) with respect to \(\mathcal{X}\) is a finite étale \(\mathcal{X}\)-stack \(\overline{\mathcal{Y}} \to \mathcal{X}\). And every element in the above family is isomorphic to an intermediate cover, in between \(\overline{\mathcal{Y}}\) and \(\mathcal{Y}\). □

**Corollary 4.6.** Let \(\mathcal{Y}_1, \ldots, \mathcal{Y}_s\) be finitely many algebraic stacks, finite étale over \(\mathcal{X}\). Then the following family is finite up to isomorphism.

\[\{\mathcal{W} : \mathcal{W}\ is\ connected\ component\ of\ \prod_{\mathcal{X}}^{k_1} \mathcal{Y}_1 \times \cdots \times \prod_{\mathcal{X}}^{k_s} \mathcal{Y}_s, for\ k_1, \ldots, k_s \geq 0\}\]
Proof. There are $s$ projection maps

$$p_\ell : \prod_i \mathcal{Y}_1 \times_x \cdots \times \prod_i \mathcal{Y}_s \to \prod_i \mathcal{Y}_\ell, \quad \ell = 1, \ldots, s$$

which are all finite étale and in particular closed and open. The immersion

$$i : \prod_i \mathcal{Y}_1 \times_x \cdots \times \prod_i \mathcal{Y}_s \to \prod_i \mathcal{Y}_1 \times \cdots \times \prod_i \mathcal{Y}_s$$

is similarly closed and open. Hence $\mathcal{W}$ is isomorphic to its image $i(\mathcal{W})$ which is a connected component of $\pi_1(\mathcal{W}) \times \cdots \times \pi_s(\mathcal{W})$. However any fiber product $\mathcal{W}_1 \times \cdots \times \mathcal{W}_s$ where $\mathcal{W}_i \in C(\mathcal{Y}_i \to \mathcal{X})$ has finitely many connected components and by Lemma 4.5 there are only finitely many such fiber products. □

Corollary 4.7. Let $\mathcal{X}$ be an irreducible gerbe. Then the following family is finite up to isomorphism.

$$\{ \mathcal{W} : \mathcal{W} \text{ is a connected component of } I^{(m)} \mathcal{X} \text{ for some } m \geq 0 \}$$

Proof. For an irreducible gerbe $I \mathcal{X} \to \mathcal{X}$ is finite étale, hence closed and open and therefore the inertia stratifies to finitely many connected components $\mathcal{Y}_1, \ldots, \mathcal{Y}_s$ which are finite étale over $\mathcal{X}$. In the commutative diagram

$$\begin{align*}
I^{(m)} \mathcal{X} & \xrightarrow{j} \prod_i^{m} I \mathcal{X} \\
& \searrow \downarrow \\
& \mathcal{I} \mathcal{X}
\end{align*}$$

the downward arrows are finite étale and hence so is the inclusion $j$. Consequently $j$ is open and closed, and therefore any connected component of $I^{(m)} \mathcal{X}$ is a stratum of some substack

$$\mathcal{Y}_{i_1} \times_x \cdots \times \mathcal{Y}_{i_m} \subset \prod_i^{m} I \mathcal{X}$$

for a choice of $i_1, \ldots, i_m \in \{1, \ldots, s\}$. The claim now follows from Corollary 4.6. □

This completes the proof of our main results for Deligne-Mumford stacks:

Theorem 4.8 (Local finiteness). Let $\mathcal{X}$ be a noetherian Deligne-Mumford $B$-stack and $\{ \mathcal{X}_i \}_{i \in A}$, the stratification of it by irreducible gerbes. Then the $K(Sch)$-submodule of $K(DM)$ generated by the set of motivic classes of all $\mathcal{X}_i$ and all intermediate Galois covers between $I \mathcal{X}_i \to I \mathcal{X}_j$ is finitely generated, invariant under inertia endomorphism, and contains $[\mathcal{X}]$.

Corollary 4.9 (Diagonalization). The endomorphism $I : K(DM) \to K(DM)$ is diagonalizable, with eigenvalue spectrum equal to $\mathbb{N}$, the set of positive integers.
4.4 The operators $I_r$ and eigenprojections

Let $I_r \mathcal{X}$ be the stack of tuples $(x, s_1, \ldots, s_r)$ where $s_i$ are distinct pairwise commuting automorphisms of $x$. By this we mean that of $x : X \to \mathcal{X}$ is an $X$-point of $\mathcal{X}$, and $G = \text{Aut}(x)$ is the $X$-group scheme of automorphisms of $x$, then $s_i$ are sections of $G \to X$ and not any two of them are identical sections. This definition applies also to $r = 0$. The stack $I_0 \mathcal{X}$ is just $\mathcal{X}$. For $r = 1$, $I_1 \mathcal{X}$ is the usual inertia. Hence $I_1$ is diagonalizable with integer eigenvalues.

Note that $I_r$ is closely related to the $k$-fold inertia operators $I^{(k)}$. It is easy to see that by an inclusion-exclusion argument that they satisfy the following identity,

$$I_r = \sum_{k=1}^{r} s(r, k) I^{(k)},$$

where $s(r, k)$ are the signed Stirling number of the first kind.

We use the notation $ZI_r \mathcal{X}$ for the substack of $I_r \mathcal{X}$ consisting of objects $(x, s_1, \ldots, s_r)$ such that all $s_i$ are in the centre of $\text{Aut}(x)$. The complement locus will be denoted by $NZI_r \mathcal{X}$.

Let $\mathcal{X}$ be an irreducible gerbe with $I \mathcal{X} \to \mathcal{X}$ an étale morphism of degree $n$. Let $\mathcal{X}$ be a Galois covering $\tilde{\mathcal{X}} \to \mathcal{X}$ of $\mathcal{X}$ such that $ZI \mathcal{X} \mid \tilde{\mathcal{X}}$ is a disjoint union of $n$ copies of $\tilde{\mathcal{X}}$. So we have

$$[ZI \mathcal{X} \mid \tilde{\mathcal{X}}] = r! \binom{n}{r} [\tilde{\mathcal{X}}].$$

We use the notation $\text{Inj}(r, n)$ for the set of injections from a set of cardinality $r$ to a set of of cardinality $n$. So

$$\# \text{Inj}(r, n) = r! \binom{n}{r}.$$
morphism. Since \( \bar{X}/\Gamma \cong \bar{X} \), the above happens precisely when \( \mathcal{Y} \) is isomorphic to a copy of \( \bar{X} \) by the vertical morphism. By Proposition 4.4 this only is the case if \( \mathcal{Y} \) is one of the \( k \) copies of \( \bar{X} \) contributing to the split central number of \( \bar{X} \). Hence the set of fixed points of \( \Gamma \) is of size \( k \). Also, 

\[
\bar{X} \times \Inj(r, n) \cong \ZI_r \, \bar{X}]
\]

and the action of \( \Gamma \) on \( \mathcal{Y} \) induces an action of it on \( \Inj(r, n) \). A morphism \( \varphi : \mathcal{Y} \to \mathcal{Y} \) is invariant under this action if every element in the image of \( \varphi \) is so. Therefore the number of fixed points of \( \Inj(r, n) \) is \( r! \binom{k}{r} \). We may hence calculate as follows:

\[
ZI_r[\bar{X}] = \left[ \bar{X} \times \Inj(r, n) \right] = \sum_{\varphi \in \Inj(r, n)} \left[ \bar{X} / \Stab_{\Gamma} \varphi \right] = \sum_{\varphi \in \Inj(r, n) / \Gamma} \left[ \bar{X} \right] + \sum_{\varphi \in \Inj(r, n) / \Gamma, \Stab_{\Gamma} \varphi \neq \Gamma} \left[ \bar{X} / \Stab_{\Gamma} \varphi \right]
\]

Thus, we conclude,

\[
ZI_r[\bar{X}] = r! \binom{k}{r} [\bar{X}] + \sum_{\varphi \in \Inj(r, n) / \Gamma, \Stab_{\Gamma} \varphi \neq \Gamma} \left[ \bar{X} / \Stab_{\Gamma} \varphi \right].
\]

Finally note that each intermediate cover \( \mathcal{Y} = \bar{X} / \Stab_{\Gamma} \varphi \) has a strictly larger split central number \( k \). In fact, \( ZI \mathcal{Y} = ZI \bar{X} |_{\mathcal{Y}} \) so every section of \( ZI \bar{X} \to \bar{X} \) pulls back to a section of \( ZI \mathcal{Y} \to \mathcal{Y} \) but also \( ZI \mathcal{Y} \to \mathcal{Y} \) has sections induced by \( \varphi \) that do not descend to \( ZI \bar{X} \to \bar{X} \).

Finally for every irreducible gerbe \( \mathcal{Y} \subseteq \NZI_r \bar{X} \), the split central rank is strictly larger than \( n \), because at least one of the sections \( s_i \) is noncentral. □

**Corollary 4.11.** The operators \( I_r \), for \( r \geq 0 \) are simultaneously diagonalizable. The common eigenspaces form a family \( \Pi_k(\DM) \) of subspaces of \( K(\DM) \) indexed by non-negative integers \( k \geq 0 \), and

\[
K(\DM) = \bigoplus_{k \geq 0} \Pi_k(\DM).
\]

Let \( \pi_k \) denote the projection onto \( K^k(\DM) \). We have

\[
I_r \pi_k = r! \binom{k}{r} \pi_k,
\]

for all \( r \geq 0, k \geq 0 \).
Corollary 4.12. For \( r \geq 1 \), we have
\[
\ker I_r = \bigoplus_{k<r} \Pi_k(\text{DM}) .
\]

Corollary 4.13. For every \( k \geq 0 \), we have
\[
\pi_k = \sum_{r=k}^{\infty} \frac{(-1)^{r+k}}{r!} \binom{r}{k} I_r .
\]
In particular, \( \pi_0 = \sum_{r=0}^{\infty} \frac{(-1)^{r}}{r!} I_r \), and \( \pi_1 = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{(r-1)!} I_r \).

The proof is similar to that of Corollary 3.7.

5 Inertia endomorphism of algebraic stacks

In the rest of this paper, we need to work over a field of characteristic zero. So we let \( k = \mathbb{C} \), we shorten our notation for the category of algebraic stacks \( \text{St} / \mathbb{C} \) to \( \text{St} \), and let the base category be that of the \( \mathbb{C} \)-varieties, denoted by \( \text{Var} \).

5.1 Central band of a gerbe

We recall that to any algebraic stack \( \mathcal{X} \) we can associate an fppf coarse moduli sheaf \( \mathcal{X} \) of isomorphism classes of objects of \( \mathcal{X} \) [9, Rmk. 3.19]. The morphism of stacks \( \mathcal{X} \to X \) is always an fppf (in particular, étale) gerbe.

Proposition 5.1. Let \( \mathcal{X} \to X \) be an étale gerbe. Then there exists a sheaf of abelian groups \( Z \to X \) and a morphism of sheaves of groups \( \varphi : Z \times_X \mathcal{X} \to I \mathcal{X} \) such that

1. For every \( s : S \to \mathcal{X} \), the induced morphism of sheaves of groups \( s^* \varphi : Z|_S \to \text{Aut}(s) \) identifies \( Z|_S \) with the centre of the sheaf of groups \( \text{Aut}(s) \); and,

2. The pair \( (Z, \varphi) \) is unique, up to isomorphism of sheaves of groups over \( X \).

Proof. This is explained in [6, Ch. IV, §1.5.3.2] for existence of the sheaf and to [6, Ch. IV, Cor. 1.5.5] for the properties of it. \( \square \)

In the above setting, \( Z \) is called the central band associated to \( \mathcal{X} \) and if it is a scheme we call it the central group scheme. The central inertia of \( \mathcal{X} \) is defined to be the fiber product

\[
\begin{array}{ccc}
ZI \mathcal{X} & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
Z & \longrightarrow & X.
\end{array}
\]

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Discrete central inertia

Suppose we are in the case that $Z \to X$ is a group scheme and consider the open subgroup scheme $Z^0$ and the quotient algebraic space $Z/Z^0$ over $X$. Pulling back to $X$, we define the connected component of the central inertia, $Z^0I_X$ as the subgroup space, and the discrete central inertia as the quotient group space, which are respectively given by the following fiber products

$$
\begin{array}{ccc}
Z^0I_X & \longrightarrow & X \\
\downarrow & & \downarrow \\
Z^0 & \longrightarrow & X \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
DZI_X & \longrightarrow & X \\
\downarrow & & \downarrow \\
Z/Z^0 & \longrightarrow & X \\
\end{array}
$$

Here $Z/Z^0 \to X$ is called the discrete central band of $X \to X$.

5.2 Clear gerbes

**Definition 5.2.** Let $X$ be an algebraic stack of finite type over the associated coarse moduli space $X$. We say $X$ is a clear gerbe over the algebraic space $X$, if all the following conditions are satisfied.

1. $X \to X$ is étale gerbe with faithfully flat structure morphism of finite type;
2. the projection $I_X \to X$ is a representable smooth morphism of finite type;
3. $X$ is a $k$-variety (i.e. a reduced, separated, $k$-scheme of finite type);
4. the discrete inertia $D_X \to X$ is finite étale morphism;
5. the central band is a smooth commutative $X$-group scheme;
6. the central inertia is a closed substack of the inertia stack;
7. the discrete central band is an étale finite $X$-group scheme;
8. the central band admits a maximal torus.

We start with a modification of the stratification in [9, Thm. 11.5].

**Theorem 5.3.** Every algebraic stack of finite type with affine diagonal has a stratification into a disjoint union of finitely many locally closed clear gerbes.

**Proof.** By replacing generic smoothness instead of generic flatness in the proof of [9 Thm. 11.5] we may assume (C1) and (C2) are already satisfied and the coarse moduli sheaf $X$, is a noetherian algebraic space of finite type and in particular quasi-compact and quasi-separated. Now we stratify $X$ into $k$-varieties by [5 Thm. 3.1.1]. It now suffices to show that there exists a Zariski open $U \subseteq X$ such that $X|_U$ satisfies conditions (4)--(8).

Let $\tilde{X} \to X$ be an étale cover of $X$ trivializing $X$ to $B_X G$ for a $\tilde{X}$-group scheme $G$. By generic smoothness we may shrink $X$ such that $G$ is smooth over $\tilde{X}$ and consequently ([7 Cor. 17.7.3]), over $X$. By Corollary 2.2 we may now assume that (4) is satisfied.
Let $Z$ be the central band of $\mathcal{X} \to X$. It is easy to check that the diagonal morphism $Z \to Z \times_X Z$ is representable. By Corollary 2.3 we may further assume that $G$ admits its centre $Z(G)$ as a closed subscheme. Then $Z(G)$ is an étale cover of the central band and therefore the latter is an algebraic space. Since all stabilizers are affine, we conclude that $Z$ is a scheme. This takes care of (5). By descent of closed immersions (6) is also satisfied. Applying Corollary 2.2, this time to $Z \to X$, will assure (7).

Recall that a smooth commutative algebraic group over a perfect field has a decomposition $T \times U$ where $U$ is the unipotent radical of the algebraic group and $T$ is a group of multiplicative type. If the group is connected then so is $T$, in which case $T$ is a torus [12, XIV, Theorem 2.6]. Let $\eta$ be a generic point of $X$. Then the mentioned decomposition holds for the generic fiber $Z_\eta$ of central band over $X$. We now use the machinery of §2.3 on how the structure of a commutative algebraic groups spread out by Corollary 2.5. □

**Definition 5.4.** Let $\mathcal{X} \to X$ be a clear gerbe. The relative dimensions $\dim_\mathcal{X}(I\mathcal{X})$ and $\dim_\mathcal{X}(ZI\mathcal{X})$ are well-defined. The former is called the total rank. The latter is called the central rank of $\mathcal{X}$, $\rho(\mathcal{X})$, and is bounded above by the former. It is also the rank of the associated commutative group scheme $Z^0 \to X$.

The split central degree is defined to be the number of sections of $\text{D}ZI\mathcal{X} \to \mathcal{X}$ and is denoted by $\nu(\mathcal{X})$. This is bounded above by the total degree, defined as the degree of finite étale map $\text{D}\mathcal{X} \to \mathcal{X}$.

The unipotent and reductive ranks of the central band are constant over the coarse moduli space. We will denote them respectively by $\rho_u(\mathcal{X})$ and $\rho_r(\mathcal{X})$.

### 5.3 Filtration by central rank and split central degree

Let $K_{\geq (r,n)}$ be the subspace of $K(\text{St})$ spanned by clear gerbes $\mathcal{X} \to X$, for which the central rank, is at least $r$ and if this rank is exactly $r$ then the split central order is at least $n$. We will now show that the inertia preserves this filtration. First we need a lemma. If $u : G \to H$ is any homomorphism of group schemes, it follows from the definition of the functor of connectedness component of identity [1, Exp. VI, Def. 3.1] that $G^0$ maps to $H^0$. Moreover we have:

**Lemma 5.5.** Let $u : G \to H$ be a closed immersion of smooth group schemes of same dimensions over a field $k$, then $u^0 : G^0 \to H^0$ is an isomorphism of connected group schemes and the induced quotient map $G/G^0 \to H/H^0$ is a closed immersion of finite group schemes.

**Proof.** From the fact that $G^0$ and $H^0$ are irreducible ([17, Tag 0B7Q]) we conclude that $G^0 \to H^0$ is a surjective closed immersion. Being a surjective closed immersions is equivalent to being a thickening, but both $G^0$ and $H^0$ are reduced. Therefore $G^0 \to H^0$ is an isomorphism. □

Let $\mathcal{X}$ is a clear gerbe. We fix a stratification of $I\mathcal{X}$ by clear gerbes and let $\mathcal{Q}$ be one such a stratum. We prove that if $\mathcal{Q}$ is not contained in $ZI\mathcal{X}$ then either
\[ \rho(\mathcal{Y}) > \rho(\mathcal{X}) \text{ or } \nu(\mathcal{Y}) > \nu(\mathcal{X}). \] And if \( \mathcal{Y} \) is contained in \( \text{ZI} \mathcal{X} \), and maps to a component of \( \text{DZI} \mathcal{X} \), not of degree 1 over \( \mathcal{X} \), then \( \rho(\mathcal{X}) = \rho(\mathcal{Y}) \) but \( \nu(\mathcal{Y}) > \nu(\mathcal{X}) \).

**Proposition 5.6.** Let \( \mathcal{Y} \) be a stratum of \( \mathcal{X} \) not (completely) contained in \( \text{ZI} \mathcal{X} \). Then \( \rho(\mathcal{Y}) \geq \rho(\mathcal{X}) \) and if \( \rho(\mathcal{Y}) = \rho(\mathcal{X}) \) then \( \nu(\mathcal{Y}) > \nu(\mathcal{X}) \).

**Proof.** Consider the diagram below, where over every object \((x, \varphi) \in \text{Aut}(x)\) in \( \mathcal{Y} \), \( j \) maps \( Z(\text{Aut}(x)) \) to \( Z(\text{ZAut}(x)(\varphi)) \). By [4, Lem. 2.2.4], \( j \) is a closed immersion of group objects on \( \mathcal{Y} \).

\[
\begin{array}{ccc}
\text{ZI} \mathcal{Y} & \xrightarrow{j} & \mathcal{Y} \\
\downarrow{\pi_3} & & \downarrow{\pi_2} \\
\mathcal{Y} & \xrightarrow{\pi_1} & \mathcal{X}
\end{array}
\]

From this diagram, it is obvious that \( \rho(\mathcal{Y}) \geq \rho(\mathcal{X}) \).

If \( \rho(\mathcal{Y}) = \rho(\mathcal{X}) \), by Lemma 5.5 and the fact that the formation of the connectedness component of identity and the quotient by it, are preserved by base change, we have another commutative diagram:

\[
\begin{array}{ccc}
\text{DZI} \mathcal{Y} & \xrightarrow{j} & \text{DZI} \mathcal{X} \\
\downarrow{\pi_3} & & \downarrow{\pi_1} \\
\mathcal{Y} & \xrightarrow{\pi_1} & \mathcal{X}
\end{array}
\]

The morphism \( \pi_3 \) has a canonical section induced by the diagonal morphism \( \mathcal{Y} \to \mathcal{Y} \times_X \mathcal{Y} \) (explicitly \((x, \varphi) \mapsto (x, \varphi, [\varphi])\), where \( x \) is an object of \( \mathcal{X} \), \( \varphi \in \text{Aut}(x) \) and \([\varphi]\) is the orbit of \( \varphi \) by the action of connectedness component of unity). Since \( \mathcal{Y} \) is not contained in \( \text{ZI} \mathcal{X} \), this section is not in the image of \( j \). Therefore \( \nu(\mathcal{Y}) > \nu(\mathcal{X}) \). \( \square \)

When \( \mathcal{X} \) is a clear gerbe, the connectedness components of the central inertia \( \text{ZI} \mathcal{X} \to \mathcal{X} \) are also clear gerbes; this yields a canonical stratification of the central inertia by clear gerbes. The next two propositions pertain to these strata.

**Proposition 5.7.** The connectedness components of the central inertia \( \text{ZI} \mathcal{X} \to \mathcal{X} \) satisfy the following properties: (1) they all have the same central rank as that of \( \mathcal{X} \); and (2) there is a one-to-one correspondence between connectedness components of \( \mathcal{X} \) and that of \( \text{DZI} \mathcal{X} \) and each component \( \mathcal{Y} \subseteq \text{ZI} \mathcal{X} \) is a (\( Z^0 \mathcal{X} \))-torsor over its associated connected component \( \mathcal{Y}' \subseteq \text{DZI} \mathcal{X} \).

**Proof.** It is easy to verify the isomorphism \( \text{ZI}(\text{ZI} \mathcal{X}) \cong \text{ZI} \mathcal{X} \times_{\mathcal{X}} \text{ZI} \mathcal{X} \) of stacks which also fit in the commutative diagram

\[
\begin{array}{ccc}
\text{ZI}(\text{ZI} \mathcal{X}) & \xrightarrow{j} & \text{ZI} \mathcal{X} \\
\downarrow{\square} & & \downarrow{\square} \\
\mathcal{X} \times_{\mathcal{X}} \mathcal{X} & \xrightarrow{\pi} & \mathcal{X}
\end{array}
\]
Thus for any locally closed substack \( \mathcal{Y} \) is \( ZI \mathcal{X} \) which descends to a locally closed subspace \( Y \) of \( Z \), we have

\[
\begin{array}{ccc}
ZI(\mathcal{Y}) & \to & \mathcal{Y} \\
\downarrow & & \downarrow \\
Z \times_X Y & \to & Y.
\end{array}
\]

(3)

In particular the central \( Y \)-group scheme associated to \( \mathcal{Y} \) is the pull-back \( Z|_Y \). (1) is now obvious form diagram (3). For (2) notice that the morphism \( ZI \mathcal{X} \to DZI \mathcal{X} \) is a principal bundle for the connected group scheme \( Z^0I \mathcal{X} \) over \( \mathcal{X} \), and therefore there is a bijection between connectedness components of the source and the target of this morphism. Passing to a component gives us the cartesian diagram

\[
\begin{array}{ccc}
\mathcal{Y} & \to & \mathcal{Y}' \\
\downarrow & & \downarrow \\
ZI \mathcal{X} & \to & DZI \mathcal{X}
\end{array}
\]

together with a finite étale mapping \( \mathcal{Y}' \to \mathcal{X} \) proving the lemma. \( \square \)

**Proposition 5.8.** Let \( \mathcal{Y} \) be a connected component of \( ZI \mathcal{X} \). We always have \( \nu(\mathcal{Y}) \geq \nu(\mathcal{X}) \), with equality happening if and only if the image of \( \mathcal{Y} \) in \( DZI \mathcal{X} \) maps down isomorphically to \( \mathcal{X} \).

**Proof.** By last lemma, \( \mathcal{Y} \) sits over a connectedness component \( \mathcal{Y}' \subseteq DZI \mathcal{X} \). The homomorphism of commutative group schemes \( (ZI \mathcal{X})_{\mathcal{Y}} \to ZI \mathcal{Y} \) over \( \mathcal{Y} \) is an isomorphism giving the left cartesian square of homomorphisms of commutative group schemes and inducing the right hand one:

\[
\begin{array}{ccc}
ZI \mathcal{Y} & \to & ZI \mathcal{X} \\
\downarrow \pi_2 & & \downarrow \pi_1 \\
\mathcal{Y} & \to & \mathcal{X}
\end{array}
\quad \quad \quad
\begin{array}{ccc}
DZI \mathcal{Y} & \to & DZI \mathcal{X} \\
\downarrow \pi_2 & & \downarrow \pi_1 \\
\mathcal{Y} & \to & \mathcal{X}
\end{array}
\]

(4)

So distinct sections of \( \pi_1 \) pull back to distinct sections of \( \pi_2 \), therefore \( \nu(\mathcal{Y}) \geq \nu(\mathcal{X}) \).

Now suppose \( \mathcal{Y}' \) is not isomorphic to \( \mathcal{X} \). Then, as the structure map \( \mathcal{Y} \to \mathcal{X} \) factors through \( \mathcal{Y}' \), and yields a section of \( \pi_2 \) that is not induced by \( \pi_1 \). In this case we have \( \nu(\mathcal{Y}) > \nu(\mathcal{X}) \). If \( \mathcal{Y}' \) maps isomorphically to \( \mathcal{X} \), the structure map \( \mathcal{Y} \to \mathcal{X} \) is an \( Z^0I \mathcal{X} \)-torsor over \( \mathcal{X} \). Hence, the upper horizontal map in the left hand diagram of (4) is a torsor for a connected group scheme, and therefore we can push the sections forward. In this case, \( \nu(\mathcal{Y}) = \nu(\mathcal{X}) \). \( \square \)

**Remark 5.9.** The \( Z^0I \mathcal{X} \)-torsors \( \mathcal{Y} \) in Proposition 5.7 and 5.8 all come from scheme torsors. In fact, the \( Z^0I \mathcal{X} \)-principal bundle \( ZI \mathcal{X} \to DZI \mathcal{X} \) is the pull-back of the \( Z^0 \)-principal bundle \( Z \to Z/Z^0 \). Passing to a strata \( \mathcal{Y} \), we likewise observe that \( \mathcal{Y} \to \mathcal{Y}' \) is the pull back of a \( Z^0 \)-torsor.
Filtration by total rank and total degree

Let $K \leq d$ be the span of clear gerbes $X \to X$, for which the total rank is at most $d$, and $K \leq (d, \delta)$ be the span of clear gerbes $X \to X$, for which the total rank is at most $d$ and if it is exactly $d$ then the total degree is at most $\delta$.

**Proposition 5.10.** The inertia operator $I$ preserves $K \leq d$ and $K \leq (d, \delta)$.

**Proof.** Let $\mathcal{Y}$ be a clear gerbe which is a locally closed substack of $I \mathcal{X}$. Consider the commutative diagram

$$
\begin{array}{ccc}
I \mathcal{Y} & \xrightarrow{i} & I \mathcal{X} |_{\mathcal{Y}} \\
\downarrow & & \downarrow \\
\mathcal{Y} & \xrightarrow{i} & \mathcal{X}
\end{array}
$$

(5)

where $i$ is given over every object $(x, \varphi \in Aut(x))$ by the closed immersion $Z_{Aut(x)}(\varphi) \to Aut(x)$. This shows that $\mathcal{Y}$ does not have a total rank larger than that of $\mathcal{X}$. If the two total ranks are the same, then by Lemma 5.5 we have another commutative diagram

$$
\begin{array}{ccc}
D \mathcal{Y} & \xrightarrow{i} & D \mathcal{X} |_{\mathcal{Y}} \\
\downarrow & & \downarrow \\
\mathcal{Y} & \xrightarrow{i} & \mathcal{X}
\end{array}
$$

(6)

which finished the proof. □

5.4 Semisimple and unipotent inertia

If $G$ is an affine group scheme of finite type on base scheme $S$, an element $g \in G(S)$ is defined to be semisimple (unipotent) if for all scheme points $s \in S$ given as spectrum of a field, $g_s$ is semisimple (unipotent) in $G_s$.

**Definition 5.11.** We define the semisimple (unipotent) inertia of an algebraic stack $\mathcal{X}$ to be the strictly full subcategory $I^{ss} \mathcal{X}$ (or $I^u \mathcal{X}$) of the inertia stack $I \mathcal{X}$ consisting over a base $S$ of those objects $(x, \varphi)$ such that $\varphi \in Aut(x)$ is a semisimple (unipotent) element of the $S$-group scheme $Aut(x)$.

It is easy to check that $I^u \mathcal{X} \subset I \mathcal{X}$ is a closed substack of the inertia. According to [9, 3.5.1] in order to check that $I^{ss} \mathcal{X}$ is a substack of $I \mathcal{X}$ we only need to observe that if $f : U \to S$ is an étale surjection and $\varphi \in Aut(x)$ is an automorphism of an object $x$ over $S$ where $f^*\varphi \in Aut(f^*x)$ is semisimple then $f$ is also semisimple. But this follows from the above definition and the fact that being semisimple is preserved along field extensions. That is, given a group scheme $G \to S$, if $g$ is a $k$-valued point of $G_k$, and $K/k$ is an algebraically closed extension, and $g'$ the
$K$-valued point of $G_K$ obtained by pullback, then $g$ is semisimple if and only if $g'$ is [1, Exposé XII].

Let $u : U \to \mathfrak{X}$ be an étale covering of $\mathfrak{X}$ by an algebraic space. We note that $\text{Aut}^{ss}(u)$ may fail to be an algebraic space, however it is a locally constructible space by [1, Exposé XII, Proposition 8.1]. On the other hand, the diagonal of $\mathfrak{I}^{ss}$ is easily seen to be representable, separated and quasi-compact. Therefore $\mathfrak{I}^{ss}$ can be written as a well-defined element in $K(\text{St})$ even though it is not necessarily an algebraic stack.

A smooth commutative algebraic group $Z$ over a perfect field has a decomposition $Z = Z^{ss} \times U$ where $U$ is the unipotent radical of $Z$ and $Z^{ss}$ is a group of multiplicative type. If $Z$ is connected then so is $Z^{ss}$, in which case the latter is a torus [12, XIV]. The locus determined by the semisimple central automorphisms of objects of $\mathfrak{X}$ is denoted by $Z^{ss}I X$ and fits in the cartesian diagram

$$
\begin{array}{c}
Z_u I X \longrightarrow \mathfrak{X} \\
\downarrow \downarrow \\
Z^{ss} = Z/U \longrightarrow X
\end{array}
$$

and hence has the structure of a group of multiplicative type over $\mathfrak{X}$. We similarly define $Z_u I \mathfrak{X}$ as the pull-back of $U$ along $\mathfrak{X} \to X$. It is easy to check that

$$
Z_u I \mathfrak{X} = ZI \mathfrak{X} \cap I^{u} \mathfrak{X}, \quad \text{and} \quad Z_{ss} I \mathfrak{X} = ZI \mathfrak{X} \cap I^{ss} \mathfrak{X}.
$$

5.5 Spectrum of the unipotent inertia

Let $K_{\geq u}$ be the sub-vector space of $K(\text{St})$ spanned by clear gerbes $\mathfrak{X} \to X$, for which the unipotent central rank, is at least $u$. We use the notation $K_{\geq u}^{\leq d} = K_{\geq u} \cap K^{\leq d}$.

**Proposition 5.12.** The operator $I^u$ preserves $K_{\geq u}^{\leq d}$ and on the graded piece $K_{\geq u}^{\leq d}/K_{> u}^{\leq d}$ operates by multiplication by $q^u$.

**Proof.** Let $\mathfrak{Y}$ be a clear gerbe which is a locally closed substack of the clear gerbe $\mathfrak{X}$ with total rank at most $d$ and unipotent central rank at least $u$. Diagram [1] is enough to conclude that $K_{\geq u}^{\leq d}$ is preserved by the unipotent inertia. Also in diagram [1] all unipotent automorphisms are mapped to unipotent automorphisms therefore inducing another commutative diagram

$$
\begin{array}{c}
Z_u I \mathfrak{Y} \longrightarrow 2 \longrightarrow \mathfrak{Y} \times_\mathfrak{X} Z_u I \mathfrak{X} \longrightarrow Z_u I \mathfrak{X}, \\
\pi_3 \downarrow \downarrow \downarrow \pi_2 \quad \square \quad \pi_1 \\
\mathfrak{Y} \longrightarrow \mathfrak{X}
\end{array}
$$

In the level of group schemes, the unipotent radical of the central group scheme of $\mathfrak{X}$, pulls back to a subgroup of the unipotent radical of the central group.
scheme of \( \mathcal{Y} \). This shows that \( K \geq u \) is preserved under \( I^u \). However if \( \mathcal{Y} \) is not contained in \( ZI^X \), \( \pi_3 \) has a trivial section that does not lift to \( \pi_2 \), this means that \( \mathcal{Y} \) will have a strictly higher rank unipotent central group scheme. We conclude that

\[
[I^u \mathcal{X}] = [Z_u I^X] \mod K^{<d}_{\geq u} = q^u [\mathcal{X}] \mod K^{<d}_{\geq u}
\]

The second identity follows from the fact that \( U \times_X \mathcal{X} \to \mathcal{X} \) is a Zariski locally trivial fibration by Hilbert’s Theorem 90. □

In immediately follows that,

**Corollary 5.13.** The endomorphism \( I^u : K(\text{St}) \to K(\text{St}) \) is locally finite and diagonalizable \( \mathbb{Z}[q] \)-module operator on \( K(\text{St})[q^{-1}, \{q^k - 1 : k \geq 1\}] \) with eigenvalue spectrum of it consisting of all monomials \( q^u \) for \( u \geq 0 \).

**Proof.** Local finiteness follows inductively from

\[
I^u K^{<d}_{\geq u} = q^u K^{<d}_{\geq u} + K^{<d}_{\geq u+1}
\]

and the fact that \( K^{<d}_{\geq u} \) is empty for all \( u > d \). Diagonalizability is clear from the previous Proposition. □

### 6 Quasi-split stacks

In this section we present a criteria that if satisfied guarantees the inertia endomorphism is locally finite and diagonalizable. For some preliminaries on group schemes of multiplicative type and unipotent group schemes we refer the reader to §2.3.

**Definition 6.1.** A clear gerbe \( \mathcal{X} \to X \) is called quasi-split if the maximal torus of the central group scheme of it is a quasi-split torus. A full subcategory \( \text{QS} \) of \( \text{St} \) is called a quasi-split category if it is closed under inertia, scissor relations and fibre products and every object of it has a stratification by quasi-split gerbes.

So there is a well-defined induced \( K(\text{Var}) \)-linear inertia endomorphism on the algebra \( K(\text{QS}) \).

#### 6.1 Motivic classes of quasi-split tori

Let \( \Gamma \) be a finite group acting on the finite set \( \mathcal{Z} = \{1, 2, \ldots, r\} \). We consider an integer partition \( \lambda \vdash r \) with declaring \( \lambda_i \) to be the number of orbits of size \( i \) and define a polynomial

\[
\varphi_\lambda = \prod_{i=1}^r (q^i - 1)^{\lambda_i}.
\]
Proposition 6.2. Let \( T \) be an isotrivial quasi-split torus over the integral scheme \( X \) and \( \overline{X} \) a splitting cover of it, with Galois group \( \Gamma \) acting by permutation of a basis of \( \chi(T) \). The motivic class of \( T \) is given by
\[
[T] = 2\lambda(q)[X] + \sum_{I* \in F(r)/\Gamma \text{Stab}(I) \subseteq \Gamma} (-1)^{\ell(I*)} q^{\ell(I*)} [\overline{X}/\text{Stab}_{\Gamma}(I)].
\] (7)
where \( \lambda \vdash r \) is the partition associated to the action of \( \Gamma \) on a basis of \( \chi(T) \).

Proof. For a subset \( I \subseteq r \), we denote by \( \mathbb{A}^I \subseteq \mathbb{A}^r \) the subset of all \((x_1, \ldots, x_r)\) such that \( x_i = 0 \), for \( i \notin I \), and by \( G_m^I \subseteq \mathbb{A}^I \), the set of all \((x_1, \ldots, x_r) \in \mathbb{A}^I \), such that \( x_i \neq 0 \), for all \( i \in I \). For every \( I \subseteq r \), we have a \( \Gamma \)-equivariant stratification
\[
\mathbb{A}^I = \bigcup_{J \subseteq I} G_m^J,
\]
and hence,
\[
[G_m^I] = [\mathbb{A}^I] - \sum_{J \subset I} [G_m^J].
\]
By induction, we get an equivariant inclusion-exclusion principle
\[
[G_m^r] = \sum_{k \geq 0} (-1)^k \sum_{I* \in F^k(r)} [\mathbb{A}^I].
\]
Here \( F^k(r) \) is the set of all flags \( I* = (I_k \subseteq \cdots \subseteq I_1 \subseteq I_0 = r) \). We denote the length of a flag \( I* \) by \( k = \ell(I*) \), the maximal index by \( k = \max \), and the set of all flags, regardless of their lengths, by \( F(r) \).

\( \Gamma \) acts on the split torus \( T_X = \text{Spec}(\mathcal{O}_X(\chi_T)) \) and by [10 Prop. 5.21] we can revive \( T \) from this pull-back as \( T \cong (\overline{X} \times_X T)/\Gamma \). Then the surjection of \( \mathbb{Z} \)-module, \( \bigoplus_{b \in \mathbb{Z}} h_b \mathcal{O}_X \to \chi_T \), induces a sheaf homomorphism \( \mathcal{O}_\overline{X}(b_1, \cdots, b_r) \to \mathcal{O}_\overline{X}(\chi_T) \) and consequently an open immersion \( G_m^r, \overline{X} \cong T_X \to \mathbb{A}^r, \overline{X} \) which is equivariant for the \( \Gamma \)-action. Hence we may pass to quotient schemes and get
\[
T = \overline{X} \times_{\Gamma} G_m^r \hookrightarrow \overline{X} \times_{\Gamma} \mathbb{A}^r.
\]
We have

\[ [T] = [\overline{X} \times_{\Gamma} \mathbb{G}_m^r] \]

\[ = \sum_{k \geq 0} (-1)^k [\overline{X} \times_{\Gamma} \bigsqcup_{\lambda \in F^k(\mathbb{Z})} \mathbb{A}^{\lambda_k}] \]

\[ = \sum_{k \geq 0} (-1)^k \sum_{\lambda \in F^k(\mathbb{Z})/\Gamma} [\overline{X} \times_{\text{Stab}_\Gamma(I_\lambda)} \mathbb{A}^{\lambda_k}] \]

\[ = \sum_{\lambda \in F(\mathbb{Q})/\Gamma} (-1)^{\ell(I_\lambda)} q^{I_{\max}} [\overline{X} / \text{Stab}_\Gamma(I_\lambda)] \]

\[ + \sum_{\lambda \in F(\mathbb{Q})/\Gamma} (-1)^{\ell(I_\lambda)} q^{I_{\max}} [\overline{X} / \text{Stab}_\Gamma(I)] \]

\[ = 2\Omega(q)[X] + \sum_{\lambda \in F(\mathbb{Q})/\Gamma} (-1)^{\ell(I_\lambda)} q^{I_{\max}} [\overline{X} / \text{Stab}_\Gamma(I)] . \]

Note that all forms of affine spaces occurring in this computation are vector bundles over their base by Hilbert’s Theorem 90. This is the reason for the appearance of the terms \( q^{I_{\lambda}} \) in the calculation. \( \square \)

### 6.2 Spectrum of the inertia

Let \( \mathfrak{X} \to X \) be a quasi-split clear gerbe with a central \( X \)-group scheme \( Z \) of reductive rank \( t \). The Galois group \( \Gamma = \pi_1(\mathfrak{X}/X) \) of the minimal splitting Galois cover is then a subgroup of \( S_t \), the group of permutations of \( t \) letters. This action induces an integer partition \( \lambda \vdash t \) as explained in §6.1.

**Definition 6.3.** For a quasi-split clear gerbe \( \mathfrak{X} \), the partition \( \lambda \) constructed as above is called the *twist type* of it.

We impose a well-ordering on the set of all integer partitions:

1. if \( \lambda \vdash t \) and \( \mu \vdash s \) and \( t < s \) then \( \lambda < \mu \); and
2. if \( \lambda, \mu \vdash t \) and \( b(\lambda) < b(\mu) \) then \( \lambda < \mu \).

Here \( b(\lambda) = \sum_i \lambda_i \) is the *number of blocks* of the partition \( \lambda \). If \( s = t \) and \( b(\lambda) = b(\mu) \) the ordering between \( \lambda \) and \( \mu \) is not important. We may consider the lexicographic ordering for this case.

The double-filtration in section 5.3 is similarly preserved in \( K(\mathbb{Q}s) \). We now introduce a refinement, \( K_{\geq(r,n,\lambda)} \), of this filtration by declaring \( K_{\geq(r,n,\lambda)} \) to be generated by those quasi-split clear gerbes that have central rank at least \( r \), and if they have central rank exactly \( r \), then their split central degree is at least \( n \), and if it is exactly \( n \), then their twist type is at least \( \lambda \).
Lemma 6.4. The inertia endomorphism of $K(QS)$ respects the filtration $K_{\geq (r,n)}$ and on each graded piece $K_{(r,n,\gamma)}/K_{(r,n,\gamma)}$ operates by multiplication by the polynomial $q^{r-t}G_{\lambda+t}(q)$.

PROOF. In view of results of section 5.3 it suffices to consider a tuple $(r,n,\lambda)$, and a quasi-split clear gerbe $X \to X$ with central group scheme $Z \to X$ of rank $r$, split central degree $n$, and twist type $\lambda \vdash t$ where $t$ is the rank of the maximal torus $T \subseteq Z$. Also from Propositions 5.4 and 5.7 we only need to consider a central stratum $\mathfrak{G}$ of $1X$ which is an $Z^01X$-torsor over $X$ and descends over $X$ to a $Z^0$-torsor $Y$. There are $n$ such strata.

Let $\eta \in X$ be the generic point with residue field $K$. Over the algebraic closure, we have a decomposition $Z_{\mathfrak{G}} = T_{\mathfrak{G}} \times U_{\mathfrak{G}}$ where $U_{\mathfrak{G}}$ is the unipotent radical. By [11, Cor. 15.10], this direct product decomposition descends to $K$ and by Lemma 2.3 and Corollary 5.3 spreads out to a non-empty open of $X$ which we may without loss of generality assume is $X$ itself. Both $U_X$ and $T_X$ are special ([3, Prop 2.2]) and therefore so is $Z$. By Proposition 6.2

$$[Y] = q^{r-t}G_{\lambda+t}(q)[X] + \sum_{I \in F(\mathfrak{G})/\Gamma} (-1)^{\ell(I)} q^{r-t+|I_{\max}|} [\mathfrak{X}/\text{Stab}_I(I)],$$

for a finite étale covering $\mathfrak{X} \to X$. And pulling back along $X \to X$,

$$[\mathfrak{G}] = q^{r-t}G_{\lambda+t}(q)[\mathfrak{X}] + \sum_{I \in F(\mathfrak{G})/\Gamma} (-1)^{\ell(I)} q^{r-t+|I_{\max}|} [\mathfrak{X}/\text{Stab}_I(I)], \quad (8)$$

where $\mathfrak{X} = [X/\text{Stab} I]$. Note that each of these stacks is a clear gerbe over the intermediate cover $X/\text{Stab} I$ and the maximal torus of $X$ pulls back to the maximal torus $T|X/\text{Stab} I$ of the central band of $\mathfrak{X}/\text{Stab}_I(I)$. Since $I' = \text{Stab} I$ is a proper subgroup of $\Gamma$, the orbit space $\mathfrak{X}/\Gamma'$ has strictly more elements than $\mathfrak{X}/\Gamma$. Therefore the twist type of $\mathfrak{X}/\text{Stab}_I(I)$ is strictly larger than that of $\mathfrak{X}$. □

Proposition 6.5. The endomorphism $I : K(QS) \to K(QS)$ is locally finite. For every element $\mathfrak{x}$ in $QS$ there exists a finite-dimensional $Z[\eta]$-module of $K(QS)$ that is invariant under inertia and contains the motivic class $[\mathfrak{x}]$.

PROOF. Recall the ascending filtration $K^{\leq (d,\delta)}$ of $\text{St}$ by total rank and total degree. Similar filtration is well-defined on $QS$. There are finitely many possible twist types for the maximal tori of clear gerbes in $K^{\leq (d,\delta)}$. The central rank is bounded above by $d$, and the finest twist type is the partition of $d$ by $d$ blocks, which we denote by $1_d \vdash d$ and we may write

$$K^{\leq (d,\delta)} = K^{\leq (d,\delta,1_d)}, \quad K^{\leq (r,n,t)}_{\geq (d,\delta,1_d)} = K^{\leq (d,\delta,1_d)} \cap K^{\geq (r,n,t)}.$$

By a similar argument as of Proposition 5.10 $K^{\leq (d,\delta,1_d)}$ is preserved by the inertia. By the previous proposition we have

$$\mathfrak{I}K^{\leq (d,\delta,1_d)}_{\geq (r,n,\lambda)} = nq^{r-t}G_{\lambda+t}(q) \cdot K^{\leq (d,\delta,1_d)}_{\geq (r,n,\lambda)} + K^{\leq (d,\delta,1_d)}_{\geq (r,n,\lambda)}$$

29
and the claim follows by induction. For the base case of this induction we only need to observe that $K \leq (d, \delta, 1)_{d, n, \lambda}$ is empty if either $r > d$ or $n > d$ and a clear gerbe in $K \leq (d, \delta, 1)$ consists of $\delta$ strata each of which is a torsor for the split torus $G_{d, n}$. Therefore the inertia operator acts on this filtered piece by multiplication by $\delta(q - 1)^d$. □

It immediately follows that,

**Theorem 6.6.** The operator $I$ is diagonalizable on $K(\mathbb{Q})[q] = \mathbb{Q}(q) \otimes_{\mathbb{Q}[q]} K(\mathbb{Q})$ as a linear endomorphism of a $\mathbb{Q}(q)$-vector space. The eigenvalue spectrum of it is the set of all polynomials of the form

$$np^{d} \prod_{i=1}^{k} (q^r - 1).$$

**Remark 6.7.** In fact $I$ is diagonalizable as an endomorphism of the $\mathbb{Z}[q]$-module $K(\mathbb{Q})[q^{-1}, \{ (\mathcal{Q}_\lambda - \mathcal{Q}_\mu)^{-1} : \forall \lambda \vdash t, \mu \vdash s \}]$.

### 6.3 Spectrum of the semisimple inertia

We can now prove a semisimple version of Proposition 6.6 in terms of the reductive ranks of the quasi-split clear gerbes rather than their central ranks. Note that the finite group scheme $Z/Z^0$ is semisimple and product of semisimple commuting elements is semisimple. Therefore there are $\nu(X)$ connected components of $DZ_{ss}^0 X$ that map isomorphically to $X$. In fact, the morphism $Z^0 ss / Z^0 ss, 0 \to Z^0 / Z^0$ is an isomorphism of finite $X$-group schemes.

**Proposition 6.8.** Let $\mathcal{Y}$ be a stratum of $I^{ss} X$ not completely contained in $Z_{ss} I^X$. Then $\rho_r(\mathcal{Y}) \geq \rho_r(X)$ and if $\rho_r(\mathcal{Y}) = \rho_r(X)$ then $\nu(\mathcal{Y}) > \nu(X)$.

**Proof.** Let $\mathcal{Y} \subseteq I^{ss} X$ be a strata of the semisimple inertia, in particular a locally closed substack of $I^X$. Diagram (9) has a semisimple version. The downward arrows $\pi_1$, $\pi_2$ and $\pi_3$ are all structure morphisms of relative commutative group schemes. Since unipotence is preserved under the group homomorphisms we may divide each of these commutative group schemes with their unipotent radical.

\[ Z_{ss} I \mathcal{Y} \xleftarrow{\pi_3} Z_{ss} I X \xrightarrow{\pi_2} Z_{ss} I \mathcal{Y} \]

\[ \mathcal{Y} \]

It now follows that $\rho_r(\mathcal{Y}) \geq \rho_r(X)$. If $\pi_2$ and $\pi_3$ are of identical relative dimensions, then we may pass to the discrete central semisimple inertia similar to the
argument of Proposition 5.6

\[\begin{array}{c}
\text{DZ}_{ss}I\mathcal{Y} \xrightarrow{j} (\text{DZ}_{ss}I\mathfrak{X})_{\mathcal{Y}} \xrightarrow{\pi_3} \text{DZ}_{ss}I\mathfrak{X} \\
\mathfrak{Y} \xrightarrow{\pi_2} \xrightarrow{\pi_1} \mathfrak{X}
\end{array}\] (10)

Same analysis as in case of Proposition 5.6 shows that \(\pi_3\) has strictly more sections than \(\pi_1\). \(\square\)

When \(\mathfrak{X}\) is a quasi-split clear gerbe, the connectedness components of the semisimple central inertia \(\text{Z}_{ss}I\mathfrak{X} \rightarrow \mathfrak{X}\) are also quasi-split clear gerbes; this yields a canonical stratification of \(\text{Z}_{ss}I\mathfrak{X}\). The analogue of Propositions 5.7 and 5.8 is stated below:

**Proposition 6.9.** Let \(\mathcal{Y}\) be a connectedness component of \(\text{Z}_{ss}I\mathfrak{X} \rightarrow \mathfrak{X}\). Then \(\rho_{\mathcal{Y}}(\mathcal{Y}) = \rho_{\mathfrak{X}}(\mathfrak{X})\). We always have \(\nu(\mathcal{Y}) \geq \nu(\mathfrak{X})\), with equality happening if and only if the image of \(\mathcal{Y}\) in \(\text{DZ}_{ss}I\mathfrak{X}\) maps down isomorphically to \(\mathfrak{X}\).

**Proof.** The proof is similar to that of Propositions 5.7 and 5.8 by considering the commutative diagram

\[\begin{array}{ccc}
\text{Z}_{ss}I(\text{Z}_{ss}I\mathfrak{X}) & \longrightarrow & \text{Z}_{ss}I\mathfrak{X} \\
\downarrow & & \downarrow \\
\text{Z}^{ss} \times_{\mathfrak{X}} \text{Z}^{ss} & \longrightarrow & \text{Z}^{ss} \longrightarrow \mathfrak{X}
\end{array}\]

for the first claim where by restricting to \(\mathcal{Y}\) we get

\[\begin{array}{ccc}
\text{Z}_{ss}I(\mathcal{Y}) & \longrightarrow & \mathcal{Y} \\
\downarrow & & \downarrow \\
\text{Z}^{ss} \times_{\mathfrak{X}} Y & \longrightarrow & Y.
\end{array}\]

Note that \(\text{Z}_{ss}I\mathfrak{X} \longrightarrow \text{DZ}_{ss}I\mathfrak{X}\) is a principal bundle for the connected group scheme \(\text{Z}^{ss}_{ss}I\mathfrak{X}\) over \(\mathfrak{X}\), and therefore there is a bijection between connectedness components of the source and the target of this morphism. So \(\mathcal{Y}\) sits over a connectedness component \(\mathcal{Y}' \subseteq \text{DZ}_{ss}I\mathfrak{X}\). Similar to the case in Proposition 5.8 we now can form the following cartesian diagram

\[\begin{array}{ccc}
\text{DZ}_{ss}I\mathcal{Y} & \longrightarrow & \text{DZ}_{ss}I\mathfrak{X} \\
\downarrow & & \downarrow \\
\mathcal{Y} & \longrightarrow & \mathfrak{X}
\end{array}\]

and the rest of the proof is now similar to Proposition 5.8. \(\square\)
The proofs of local finiteness and diagonalization of $I^s$ now follow similar to the case of the full inertia operator. The filtration to consider in this case is $K\leq(d,\delta,1_d)$ where $K\geq(r,n,\lambda)$ is generated by those clear gerbes for which the reductive rank is at least $t$, and if it is exactly $t$ then the discrete central order is at least $n$ and if it is exactly $n$, then the twist type of the maximal torus is at least $\lambda$. The semisimple inertia acts on the filtered pieces via

$$I^sK\leq(d,\delta,1_d) = n\prod_{i=1}^k(q^{r_i}-1),$$

We conclude that,

**Theorem 6.10.** The endomorphism $I^s : K(QS) \to K(QS)$ is locally finite and triangularizable and the eigenvalue spectrum of it is the set of all polynomials of the form

$$n \prod_{i=1}^k(q^{r_i}-1).$$

Moreover $I^s$ is a diagonalizable $\mathbb{Q}(q)$-linear endomorphism of the vector space $K(QS)(q)$.

**7 Examples**

**Example 7.1.** A first simple example is the case of $[BGL_2]$. Here, and in following examples we are suppressing the notation $[]$ for quotient stacks; thus unless mentioned otherwise, all quotients (of schemes) are stack quotients. Note that we have

$$I_{BGL_2} = GL_2 / GL_2 = (GL_2)^{ss, eq} / (GL_2)^{dist} / GL_2 \cup (GL_2)^{ss} / GL_2.$$  

The first stratum contains diagonalizable matrices with one eigenvalue, the second stratum diagonalizable matrices with distinct eigenvalues, and the third stratum the non-semisimple matrices. We study these three strata and their inertia:

**First stratum:** Consider the mapping $\mathbb{G}_m \to GL_2$ via $x \mapsto \left( \begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix} \right)$. This is equivariant with respect to the natural $GL_2$-action, so we get an induced morphism of stacks

$$\mathbb{G}_m \times BGL_2 \to GL_2 / GL_2$$

which is easily seen to be an isomorphism onto the first stratum.

**Second stratum:** Let $T$ be the standard maximal torus of $GL_2$. Let $\Delta$ be the centre of $GL_2$, which is the diagonal subtorus of $T$. Let $N$ be the normalizer of $T$. We have a short exact sequence

$$0 \to T \to N \to \mathbb{Z}_2 \to 0$$

where $\mathbb{Z}_2$ is the Weyl group of $GL_2$. Note that $N = \mathbb{G}_m^2 \times \mathbb{Z}_2$ is in fact a semi-direct product, by taking $\left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$ as the nontrivial element of $\mathbb{Z}_2 \subset N$. The
induced action of the Weyl group $\mathbb{Z}_2$ on $T$ is by swapping the two entries. The natural inclusion map $T \setminus \Delta \to \text{GL}_2$ is equivariant for the inclusion $N \subset \text{GL}_2$, so we get an induced morphism of stacks

$$(T \setminus \Delta)/N \to \text{GL}_2/\text{GL}_2$$

which is an isomorphism onto the second stratum. We will abbreviate this as $\mathfrak{X} = (T \setminus \Delta)/N$.

**Third stratum:** Let $H$ be the (commutative) subgroup of all matrices of the form $\begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix}$. Note that $H$ is the centralizer of every matrix of the form $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$, with $a \neq 0$. Thus we see that the third stratum is isomorphic to $\mathbb{G}_m \times B H$.

We conclude that in the level of motivic classes the inertia of the class $[B \text{GL}_2]$ is given by

$$I[B \text{GL}_2] = (q - 1)[B \text{GL}_2] + [\mathfrak{X}] + (q - 1)[B H].$$

Since $H$ is commutative, we also have $I[B H] = q(q - 1)[B H]$. We will now find the inertia of the second stratum $\mathfrak{X}$. Note that the coarse moduli space of $\mathfrak{X}$ is the smooth variety $X = T \setminus \Delta/\mathbb{Z}_2$. Note also that $I \mathfrak{X} = Z I \mathfrak{X}$ as the stabilizer of any point in $T \setminus \Delta$ is commutative. We will write $\widetilde{X} = T \setminus \Delta$ to emphasize the fact that $T \setminus \Delta$ is a degree 2 cover of $X$.

Associated to the $\mathbb{Z}_2$-action on the group $T$, there exists a commutative $X$-group scheme

$$T' = \widetilde{X} \times_{\mathbb{Z}_2} T,$$

with fibre $T$. By Lemma 7.2, $\mathfrak{X}$ is the neutral gerbe, $\mathfrak{X} = B_X T'$. And $Z I \mathfrak{X} = I \mathfrak{X}$ fits in the cartesian diagram

\[
\begin{array}{ccc}
I \mathfrak{X} & \longrightarrow & \mathfrak{X} \\
\downarrow & & \downarrow \\
T' & \longrightarrow & X.
\end{array}
\]

The representation of $\mathbb{Z}_2$ on $\mathbb{A}^2$ given by swapping entries, yields a canonical closed embedding $T' \subset V$, into a rank 2 vector bundle over $X$. As in Proposition 6.2, this leads to

$$I[\mathfrak{X}] = (q^2 - 1)[\mathfrak{X}] - (q - 1)^2(q - 2)[B \mathbb{G}_m^2].$$

Thus, the 4-dimensional $K(\text{Var})$-module, $L$, of motives generated by the 4 motives

$$[B \text{GL}_2], [B H], [\mathfrak{X}], \text{ and } [B \mathbb{G}_m^2]$$

is preserved by the inertia endomorphism $I$. The first element in this set is of central rank 1 and the other three are of central rank 2. $B H$ has reductive rank 1, $\mathfrak{X}$ has reductive rank 2 with the nontrivial partition of 2 associated to it, and $B \mathbb{G}_m^2$ has a rank 2 torus with the trivial partition of 2 associated to
it. The eigenvalue spectrum is hence \( \{ q - 1, q(q - 1), q^2 - 1, (q - 1)^2 \} \). Inertia endomorphism is lower triangularizable on \( L \) and we have

\[
I = \begin{pmatrix}
q - 1 & 0 & 0 & 0 \\
q - 1 & q(q - 1) & 0 & 0 \\
1 & 0 & q^2 - 1 & 0 \\
0 & 0 & -(q - 1)^2(q - 2) & (q - 1)^2
\end{pmatrix}
\]

with a set of eigenvectors

| Eigenvalues | Eigenvectors |
|-------------|--------------|
| \( q - 1 \) | \(-q(q - 1)[\text{BGL}_2] + q[\text{B} \text{H}] + [\mathfrak{X}] + (q - 1)[\text{B} \mathbb{G}^2_m] \) |
| \( q(q - 1) \) | \([\text{B} \text{H}]\) |
| \( q^2 - 1 \) | \([\mathfrak{X}] - \frac{(q - 1)(q - 2)}{2}[\text{B} \mathbb{G}^2_m] \) |
| \( (q - 1)^2 \) | \([\text{B} \mathbb{G}^2_m]\) |

Table 1: Spectrum of the inertia endomorphism on a 4-dimensional \( K(\text{Var}) \)-submodule of \( K(\text{St}) \) containing \([\text{BGL}_2]\)

Also \( I \) is diagonalizable on \( L[q^{-1}, (q - 1)^{-1}] \) and the eigenprojections of \([\text{BGL}_2]\) are

| Eigenvalues | Eigenvectors |
|-------------|--------------|
| \( \Pi_{q - 1} \) | \([\text{BGL}_2] - \frac{q}{q - 1}[\text{B} \text{H}] - \frac{1}{q(q - 1)}[\mathfrak{X}] - \frac{1}{4q}[\text{B} \mathbb{G}^2_m] \) |
| \( \Pi_{q(q - 1)} \) | \( \frac{q - 1}{q}[\text{B} \text{H}] \) |
| \( \Pi_{q^2 - 1} \) | \( \frac{1}{q(q - 1)}[\mathfrak{X}] - \frac{(q - 2)}{2qq}[\text{B} \mathbb{G}^2_m] \) |
| \( \Pi_{(q - 1)^2} \) | \( \frac{1}{4q}[\text{B} \mathbb{G}^2_m] \) |

Table 2: Eigenprojections of \([\text{BGL}_2]\)

**Lemma 7.2.** Let \( \Gamma \) be a group acting on the group \( T \) by automorphisms, \( \Gamma \to \text{Aut}(T) \), and let \( G = N \rtimes H \) be the associated semi-direct product of groups. Let \( X \) be a variety, \( \tilde{X} \to X \) a principal \( \Gamma \)-bundle, and \( T' \to X \) is the associated form of \( T \) over \( X \). The \( B_X T' = \tilde{X}/G \).

**Proof.** Consider the diagram

\[
\begin{array}{ccc}
\tilde{X} \times T & \longrightarrow & \tilde{X} \times T \\
\downarrow & & \downarrow \\
\tilde{X} & \longrightarrow & \tilde{X}
\end{array}
\]
where $\Gamma$ acts on the first column and $G$ on the second column in the obvious way. Then the horizontal arrows are a morphism of $T$-bundles which is $\Gamma \to G$ equivariant. Thus we get an induced cartesian diagram of stacks

\[
\begin{array}{ccc}
(\tilde{X} \times T)/\Gamma & \longrightarrow & (\tilde{X} \times T)/G \\
\downarrow & & \downarrow \\
\tilde{X}/H & \longrightarrow & \tilde{X}/G
\end{array}
\]

which we may rewrite as

\[
\begin{array}{ccc}
T' & \longrightarrow & X \\
\downarrow & & \downarrow \\
\tilde{X}/H & \longrightarrow & \tilde{X}/G
\end{array}
\]

Then the latter diagram induces a morphism $\mathfrak{x} \to \tilde{X}/G$, which is then obviously an isomorphism. □

**Example 7.3.** In the previous example the central group schemes were always connected. We will now present an example that demonstrates how non-connected central group schemes contribute to non-monic eigenvalues. Let $N = G_m^2 \ltimes \mathbb{Z}_2$ be the group scheme introduced in previous example. In this example we study $B.N$. The inertia of $B.N$ has two obvious connectedness components:

\[
I_{B.N} = N/N = G_m^2/N \sqcup G_m^2 \times \{ \sigma \}/N
\]

where $\sigma$ is the nontrivial element of $\mathbb{Z}_2$.

**First stratum:** This stratum is not already a gerbe (as the stabilizer of points on diagonal $\Delta \subset G_m^2$ is not isomorphic to the stabilizer of other points). However the following is a stratification of it into clear gerbes:

\[
G_m^2/N = \Delta/N \sqcup \mathfrak{X},
\]

where $\mathfrak{X} = G_m^2 \backslash \Delta/N$ is the same quotient stack that appeared in previous example. The action of $N$ on $\Delta$ is trivial so we have

\[
[G_m^2/N] = (q - 1)[B.N] + [\mathfrak{X}].
\]

**Second stratum:** This stratum is already a clear gerbe and we will denote it as $\mathfrak{Y}$. Any point $((\mu, \gamma), \sigma)$ of $G_m^2 \times \{ \sigma \}$ is conjugate to $((\mu \gamma, 1), \sigma)$ which is canonical for the orbit. Thus the subscheme $Y$ representing the points,

\[
\{(x, 1), \sigma\} \subset G_m^2 \times \{ \sigma \},
\]

is a coarse moduli space for this gerbe. This is isomorphic to $\mathbb{A}^1 \setminus \{0\}$. The stabilizer of this subscheme is a subgroup scheme $N' \subseteq N$, the fiber of which over a geometric point $x$ of $Y$ is the $\kappa(x)$-algebraic group

\[
N' = \{(t, t), 1 \} \cup \{(xt, t), \sigma \} : t \in \kappa(x)^* \}
\]

We notice that the mapping $Y/N' \to \mathfrak{Y}$ is an isomorphism of stacks and also that $N'$ is a commutative $Y$-group scheme, acting trivially on $Y$. Therefore

\[
[\mathfrak{Y}] = [Y][B.N'] = (q - 1)[B.N']
\]

\[
I[B.N'] = [N'/N'] = [N'][B.N']
\]

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Finally $N'(N')^0 \to Y$ is a degree two covering of $X$. The image of $(N')^0$ in $N'/(N')^0$ is isomorphic to $Y$ and therefore so is the image of the other connected component. So $N'$ is Zariski locally the union of two $\mathbb{G}_m$-torsors over $Y$. Pulling back along $\mathcal{Y} \to Y$ we have

$$I[B N'] = 2(q - 1)[B N'].$$

We conclude that the $K(\text{Var})$-submodule of $K(\text{St})$ generated by $[B N], [B N'], [X]$, and $[B \mathbb{G}_m^2]$ is invariant under inertia endomorphism. The first two generators have central rank one, and $[B N]$ has split central number one whereas $[B N']$ has split central number two. The spectrum of $I$ restricted to this submodule is the set

$$\{(q - 1), 2(q - 1), q^2 - 1, (q - 1)^2\}$$

as expected.

**Example 7.4.** Another simple example that shows many features of this theory is the stack $B\text{GL}_3$. As before, the inertia stack is isomorphic to the quotient stack $[\text{GL}_3 / \text{GL}_3]$ via conjugation action of $\text{GL}_3$ on itself. We first stratify this quotient according to Jordan canonical forms: let $J^k_\lambda$ be the subscheme of all general linear matrices with $k$-distinct eigenvalues and $\lambda \vdash 3$ is a partition of 3 indicating format of the Jordan blocks and $R^k_\lambda \Rightarrow J^k_\lambda$ is the groupoid representation of restriction of $[\text{GL}_3 / \text{GL}_3]$ to $J^k_\lambda$. Then we have a stratification

$$[\text{GL}_3 / \text{GL}_3] = [J^1_3/R^1_3] \sqcup [J^1_{(2,1)}/R^1_{(2,1)}] \sqcup [J^1_{(1,1,1)}/R^1_{(1,1,1)}]$$

$$\sqcup [J^2_{(2,1)}/R^2_{(2,1)}] \sqcup [J^2_{(1,1,1)}/R^2_{(1,1,1)}]$$

$$\sqcup [J^3_{(1,1,1)}/R^3_{(1,1,1)}]$$

The action of $R^k_\lambda$ on $J^k_\lambda$ by conjugation is always trivial unless in presence of Jordan blocks of same dimension with distinct eigenvalues (which can then be permuted). Thus

$$[\text{GL}_3 / \text{GL}_3] = J^1_{(3)} \times B J^1_{(2,1) \times B R^1_{(2,1)} \times B J^1_{(1,1,1) \times B R^1_{(1,1,1)}}$$

$$\sqcup J^2_{(2,1) \times B R^2_{(2,1)} \times B J^2_{(1,1,1) \times B R^2_{(1,1,1)}}$$

$$\sqcup J^3_{(1,1,1) \times B R^3_{(1,1,1)}}$$

We recall the notation of Example 7.1 for the subgroup of upper-triangular $2 \times 2$ matrices with a single eigenvalue of multiplicity two:

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{G}_m \right\}.$$ 

This represents a commutative group scheme. Now, easy computations show that all $R^k_\lambda$’s are subgroup schemes of $\text{GL}_3$ and in fact
| Groupoid   | Group scheme structure                                                                 | Commutative? |
|------------|----------------------------------------------------------------------------------------|--------------|
| \( R^1_{(3)} \) | \[
\begin{pmatrix}
\begin{array}{ccc}
a & b & c \\
0 & a & b \\
0 & 0 & a
\end{array}
\end{pmatrix} : a \in G_m, b, c \in \mathbb{A}^1
\] | Yes          |
| \( R^1_{(2,1)} \) | \[
\begin{pmatrix}
\begin{array}{ccc}
a & b & c \\
0 & a & 0 \\
0 & d & e
\end{array}
\end{pmatrix} : a, e \in G_m, b, c, d \in \mathbb{A}^1
\] | No           |
| \( R^1_{(1,1,1)} \) | \( GL_3 \)                                                                                   | No           |
| \( R^2_{(2,1)} \) | \( H \times G_m \)                                                                            | Yes          |
| \( R^2_{(1,1,1)} \) | \( GL_2 \times G_m \)                                                                        | No           |
| \( R^3_{(1,1,1)} \) | \( G_m^3 \rtimes S_3 \)                                                                      | Yes          |

Table 3: Stratification of GL\(_3\)

\[
[GL_3 \, / \, GL_3] = (q - 1)[B \, R^1_{(3)}] + (q - 1)[B \, R^1_{(2,1)}] + (q - 1)[B \, GL_3] \\
\quad + (q - 1)(q - 2)[B \, H \rtimes G_m] + (q - 1)(q - 2)[B \, GL_2 \rtimes G_m] \\
\quad + [G_m^3 \, / \, G_m^3 \rtimes S_3]
\]

Since inertia respects the commutative algebra structure of \( K(\text{St}) \) we may use the previous example to compute the effect of inertia on terms of the second line above. Since \( R^1_{(3)} \) is commutative we also have

\[
I[B \, R^1_{(3)}] = [R^1_{(3)}][B \, R^1_{(3)}] = q^2(q - 1)[B \, R^1_{(3)}].
\]

The case of \( \mathcal{G} = [G_m^3 \, / \, G_m^3 \rtimes S_3] \) is similar to that of \( [G_m^2 \, / \, G_m^2 \rtimes \mathbb{Z}_2] \). It remains to analyze the action of \( G = R^1_{(2,1)} \) on itself. We need to stratify \( G/G \) to several substacks which is carried out in Table 4. It follows that

\[
I[B \, G] = q(q - 1)[B \, G] + q^3(q - 1)(q - 2)[B \, G_1] \\
\quad + q(q - 1)^3[B \, G_2] + q(q - 1)^2[B \, G_3] + q(q - 1)^2[B \, G_4].
\]

We conclude that \( [B \, GL_3] \) is contained in a 9-dimensional \( K(\text{Var}) \)-submodule of \( K(\text{St}) \) which is diagonalizable (Table 5).

References

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\[
\begin{bmatrix}
  a & b & c \\
  0 & a & 0 \\
  0 & d & e \\
\end{bmatrix}
\] : \(a \neq e\)

\[
\begin{bmatrix}
  a & b + cd/(a - e) & 0 \\
  0 & a & 0 \\
  0 & 0 & e \\
\end{bmatrix}
\]

\[
G_1 = \begin{cases}
  \begin{bmatrix}
    x & y & 0 \\
    0 & x & 0 \\
    0 & 0 & z \\
  \end{bmatrix} : x, z \neq 0 \\
\end{cases}
\]

\[
\begin{bmatrix}
  a & 0 & 1 \\
  0 & a & 0 \\
  0 & 0 & a \\
\end{bmatrix}
\] itself

\[
G
\]

| Strata | Canonical form for an orbit | Centralizer of the canonical form |
|--------|-----------------------------|----------------------------------|
| \(a b c\) : \(a \neq e\) | \(a b \) + \(cd/(a - e)\) 0 | \[
\begin{bmatrix}
  x & y & 0 \\
  0 & x & 0 \\
  0 & 0 & z \\
\end{bmatrix}
\] : x, z \(\neq 0\) |
| \(0 a 0\) : c, \(d \neq 0\) | \[
\begin{bmatrix}
  a & 0 & 1 \\
  0 & a & 0 \\
  0 & d & a \\
\end{bmatrix}
\] | \[
G_2 = \begin{cases}
  \begin{bmatrix}
    x & y & z \\
    0 & w & x \\
    \end{bmatrix} \neq 0 \\
\end{cases}
\]
| \(0 d a\) : \(d \neq 0\) | \[
\begin{bmatrix}
  0 & a & 0 \\
  0 & a & 0 \\
  0 & 0 & a \\
\end{bmatrix}
\] | \[
G_3 = \begin{cases}
  \begin{bmatrix}
    x & 0 & 0 \\
    0 & z & x \\
    \end{bmatrix} \neq 0 \\
\end{cases}
\]
| \(a b c\) : \(c \neq 0\) | \[
\begin{bmatrix}
  a & 0 & 1 \\
  0 & a & 0 \\
  0 & 0 & a \\
\end{bmatrix}
\] | \[
G_4 = \begin{cases}
  \begin{bmatrix}
    x & y & z \\
    0 & x & 0 \\
    \end{bmatrix} \neq 0 \\
\end{cases}
\]

Table 4: Stratification of \(R_{(2,1)}^1\)

| Central rank | Reductive rank | Twist type | Pivot elements | Eigenvalue |
|--------------|---------------|------------|----------------|------------|
| 1            | 1             | (1) [BGL_3]| \(q - 1\)      |            |
| 2            | 2             | (2,0) [BGL_2][BG_m]| \((q - 1)^2\) |            |
| 3            | 3             | (0,0,1) [\mathbb{Q}] | \(q^3 - 1\) |            |
|              |               | (1,1,0) [\mathbb{Q}][BG_m]| \((q^2 - 1)(q - 1)\) |            |
|              |               | (3,0,0) [BG_m]| \((q - 1)^3\) |            |
| 4            | 1             | (1) [BH][BG_m],[BG_1]| \(q(q - 1)^2\) |            |
|              |               | (1) [BH_1][BG_2],[BG_3],[BG_4]| \(q^4(q - 1)\) |            |

Table 5: Spectrum of the inertia endomorphism of a 9-dimensional K(Var)-submodule of K(St) containing [BGL_3]

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