STABLE SETS OF PLANAR HOMEOMORPHISMS WITH TRANSLATION PSEUDO-ARCS

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Abstract. For every \( n \in \mathbb{N} \) we construct orientation preserving planar homeomorphisms \( g_n \) such that \( \text{Fix}(g_n) = \{ 0 \} \), the fixed point index of \( g_n \) at 0, \( i_{\mathbb{R}^2}(g_n,0) \), is equal to \(-n\) and the stable (respectively unstable) sets of \( g_n \) at 0 decompose into exactly \( n + 1 \) connected branches \( \{ S_j \}_{j \in \{1,2,\ldots,n+1\}} \) (resp. \( \{ U_j \}_{j \in \{1,2,\ldots,n+1\}} \)) such that:

a) \( S_i \cap S_j = \{ 0 \} \) for any \( i, j \in \{1,2,\ldots,n+1\} \) with \( i \neq j \).

b) \( S_i \cap U_j = \{ 0 \} \) for any \( i, j \in \{1,2,\ldots,n+1\} \).

For every \( j \in \{1,2,\ldots,n+1\} \), \( S_j \setminus \{ 0 \} \) and \( U_j \setminus \{ 0 \} \) admit translation pseudo-arcs. This means that there exist pseudo-arcs \( K_j \subset S_j \) and points \( p_j, g_n(p_j) \in K_j \), such that \( g_n(K_j) \cap K_j = \{ g_n(p_j) \} \) and

\[
S_j \setminus \{ 0 \} = \bigcup_{m=-\infty}^{\infty} g_n^m(K_j)
\]

and analogously for \( U_j \).

We also study the closure of the class of above homeomorphisms in the (complete) metric space of planar orientation preserving homeomorphisms.

1. Introduction. The stable and unstable manifolds theorem for hyperbolic diffeomorphisms is a key result in differential dynamics. From the topological point of view, S. Baldwin and E.E. Slaminka dealt, in [1], with a stable/unstable sets theorem for area and orientation preserving homeomorphisms of orientable two manifolds having isolated fixed points of index less than 1. On the other hand, Le Roux, in [13], and Ruiz del Portal and Salazar, see [15], [16] and [17], gave, independently, stable/unstable sets theorems in more general settings. If \( f \) is a local homeomorphism of the plane and \( p \) is a fixed point that is an isolated invariant set, using the fixed point index of the iterations of \( f \) at \( p \) it is possible to prove the existence of canonical connected branches of the stable/unstable sets. The number of such branches is, at least, \( 1 - i_{\mathbb{R}^2}(f^r,p) \) where \( r \) is any natural number \( \geq 2 \) such that \( i_{\mathbb{R}^2}(f^r,p) \neq 1 \). Besides of the above references we recommend [11] and [14] for information about the fixed point index.

Given a homeomorphism \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) such that \( \{ 0 \} \) is the unique compact invariant set of \( f \), the stable (resp. unstable) sets of \( f \) at 0, \( S(f,0) \) (resp. \( U(f,0) \)), is defined as \( S(f,0) = \{ z \in \mathbb{R}^2 : \lim_{n \to \infty} f^n(z) = 0 \} \) (resp. \( U(f,0) = \{ z \in \mathbb{R}^2 : \lim_{n \to \infty} f^{-n}(z) = 0 \} \)).

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The aim of this paper is to construct, for any $n \in \mathbb{N}$, planar homeomorphisms $g_n : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\{0\}$ is the unique compact invariant set, $i_{g_n}(g_n^r, 0) = -n$ for every $r \in \mathbb{N}$ and the stable and unstable sets at the origin decompose into exactly $n + 1$ branches that admit translation pseudo-arcs (Theorem 1 and Corollary 1). Given a homeomorphism $g : \mathbb{R}^2 \to \mathbb{R}^2$ and a $g$-invariant connected set $U$, a translation pseudo-arc for $g$ in $U$ is a pseudo-arc $K \subset U$ such that there is $p_* \in K$ with $g(p_*) \in K$, $g(K) \cap K = \{g(p_*)\}$ and

$$S = \bigcup_{m=-\infty}^{\infty} g^m(K)$$

Consider the space $E$ of all orientation preserving planar homeomorphisms endowed with the (complete) metric $\text{dist}(f, g) = \sup_{x \in \mathbb{R}^2} \left\{ \frac{\|f(x) - g(x)\|}{1 + \|f(x) - g(x)\|} : \|f^{-1}(x) - g^{-1}(x)\| \right\}$ and let $H_0$ be the subset of $E$ of all planar homeomorphisms such that $\{0\}$ is the unique compact invariant set.

For every $n \in \mathbb{N}$ we will consider a canonical (up to conjugation) planar homeomorphism, $G_n \in H_0$, such that $i_{G_n}(G_n, 0) = -n$ (see Figure 5 below). Consider $H = \{f \in E :$ there is $n \in \mathbb{N}$ such that $f$ is conjugate to $G_n\}$. For every $f \in H$ there is $n \in \mathbb{N}$ such that $i_{G_n}(f, 0) = -n$ and $U(f, 0) \setminus \{0\}$ and $S(f, 0) \setminus \{0\}$ decompose into exactly $n + 1$ alternating topological manifolds homeomorphic to $\mathbb{R}$.

We will use the construction of Theorem 1 to present a second main result of the paper. We show that the space of all planar homeomorphisms of $H_0$ satisfying the properties of that theorem is a dense subset in $H_0 \cap cl_E(H)$ (Theorem 2).

One of the main tools we will need in our constructions is the concept of Handel’s circle, see [12], and how this notion is useful to determine specific crooked curves.

2. Free homeomorphisms of the plane. In this section we recall some definitions and results extracted from the works by Brown [5] and [6].

Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation preserving homeomorphism from the plane onto itself. An oriented arc $\alpha$ with endpoints $p_0$ and $h(p_0) = p_1$ is a translation arc for $h$ if $h(\alpha \setminus \{p_1\}) \cap (\alpha \setminus \{p_1\}) = \emptyset$. A homeomorphism $h$ is free if, given any continuum $K$ with $h(K) \cap K = \emptyset$, then $h^n(K) \cap K = \emptyset$ for every $|n| > 1$. The simplest examples of free homeomorphisms are those $h$ such that $Fix(h) = \emptyset$.

Using Brouwer translation theorem (see [4]) it can be shown that another class of free homeomorphisms is composed by those $h$ such that $Fix(h) = \{p\}$ is a singleton and the fixed point index $i_{g_n}(h, p) \neq 1$.

Free homeomorphisms have simple dynamics. Indeed, if the set of fixed points $Fix(h)$ is totally disconnected, then every positive orbit $\{h^n(p_0)\}_{n \geq 0}$ is either convergent to a fixed point or divergent to $\infty$ (see Propositions 2.3 and 2.5 of [8]).

3. Crooked curves with symmetries. Let us consider the cylinder $A = (\mathbb{R} / \mathbb{Z}) \times [-1, 1]$ and its universal covering $\hat{A} = \mathbb{R} \times [-1, 1]$, $\pi : \hat{A} \to A$ defined as $\pi(x, y) = (\hat{x}, y)$, $\hat{x} = x + \mathbb{Z}$.

Given $N \geq 4$ we decompose $A$ and $\hat{A}$ into cells $\hat{a}_j = [j - 1/N, j/N] \times [-1, 1]$, $j \in \mathbb{Z}$ and $a_j = \pi(\hat{a}_j)$, $j = 1, 2, \cdots, N$.

**Definition 3.1.** A Jordan curve $C \subset A$ is crookedly embedded in $A$, with respect to the decomposition $\{a_j\}_{j=1,\ldots,N}$, if $C$ is homotopic to the circle $y = \text{constant}$ and
for any arc \( \alpha \) of the lift of \( C, \tilde{C} \), joining cells \( \tilde{a}_i \) and \( \tilde{a}_j \) with \( 2 < i - j < N \), there exists a decomposition of \( \alpha \) into 3 subarcs \( \alpha = \alpha_3 \ast \alpha_2 \ast \alpha_1 \) where \( \alpha_1 \) joins \( \tilde{a}_i \) and \( \tilde{a}_{j+1} \), \( \alpha_2 \) runs from \( \tilde{a}_{j+1} \) to \( \tilde{a}_{i-1} \) and \( \alpha_3 \) joins \( \tilde{a}_{i-1} \) and \( \tilde{a}_j \). See figures 1 and 2.

**Definition 3.2.** An arc \( \alpha \) in \( \tilde{A} \) is transversal to a segment \( x = \text{constant} \) if \( \alpha \cap \{ x = \text{constant} \} \) is finite and for every \( p \in \alpha \cap \{ x = \text{constant} \} \) and every neighborhood \( V \) of \( p \), \( V \cap \alpha \) intersects both components of \( \tilde{A} \setminus \{ x = \text{constant} \} \). A Jordan curve is transversal to \( \tilde{x} = \text{constant} \) if its lift to \( \tilde{A} \) is transversal to \( x = \text{constant} \).

Let \( C \) be a Jordan curve in \( A \) which is homotopic to \( \{ y = \text{constant} \} \). Take a parameterization of the lift of \( C \) to \( \tilde{A} \)

\[
\tilde{C} : \begin{cases}
  x = \theta(t) \\
  y = \rho(t)
\end{cases}
\]

with \( \theta(t + 1) = \theta(t) + 1 \) and \( \rho(t + 1) = \rho(t) \) continuous maps, \( \theta(0) = 0 \), \( \theta(1) = 1 \) and \( t \in [0,1] \Rightarrow (\theta(t), \rho(t)) \in C \) is bijective.

Consider \( \Theta = \min\{\theta(t) : t \in [0,1]\} \) and \( \Theta = \max\{\theta(t) : t \in [0,1]\} \).

Given \( \omega \in \mathbb{R} \) consider the rotation \( \tilde{T}_\omega : A \to A \) defined as \( \tilde{T}_\omega(x, y) = (\tilde{x} + \omega, y) \).

The lift \( \tilde{T}_\omega : \tilde{A} \to \tilde{A} \), that we shall denote in the same way, is \( \omega \)-

Note that the cells decompositions \( \{a_j\}_{j=1,\ldots,N} \) or \( \{\tilde{a}_j\}_{j \in \mathbb{Z}} \) are invariant by \( T_{1/N} \).

The next result follows from the discussions of [12] pages 164 - 165. See in particular, figures 1 and 3.

**Proposition 1.** Given \( N \geq 4 \) and \( \epsilon > 0 \) there exists a Jordan curve \( C \subset A \) such that:

1) \( C \) is crookedly embedded in \( A \) with respect to \( \{a_j\}_{j=1,\ldots,N} \),

2) \( T_{1/N}(C) = C \),

3) \( C \) intersects the segments \( \{\tilde{x} = j/N\} \) transversely,

4) \( -\epsilon \leq \Theta \leq 0 \) and \( 1 + \epsilon \geq \Theta \geq 1 \).

5) Let \( \tilde{C} \) be the lift of \( C \) to \( \tilde{A} \). The arc \( \alpha \subset \tilde{C} \) joining the points \( (0,0) \) and \( (1,0) \) is a translation arc for \( T_1 \).

To understand the proof Figure 3 in [12] is useful. There, the annulus is transformed in a cylinder and, after cutting the line \( T \), we obtain the rectangle of figure 1.1 below. The continuous paths correspond to \( S_1 \) while the dotted lines correspond to \( S_2 \). By juxtaposing this rectangle infinitely many times we obtain the curve \( \tilde{C} \) with \( N = 5 \), of figure 1.2.

4. **Periodic crooked chains.** We start with the band \( \mathbb{B} = \mathbb{R} \times [-1,1] \) and employ the notations \( \mathbb{B}_+ = \mathbb{B} \cap \{(x,y) \in \mathbb{R}^2 : x \geq 0\} \), \( \mathbb{B}_- = \mathbb{B} \cap \{(x,y) \in \mathbb{R}^2 : x \leq 0\} \) and \( \mathbb{B}_0 = \mathbb{B}_+ \cap \mathbb{B}_- \).

We consider the family of homeomorphisms of the plane \( \mathcal{H} \). A homeomorphism \( h \) is in this family if it is orientation preserving and satisfies \( h(0,0) = (0,0) \) for every \( y \in \mathbb{R} \) and \( h \circ T_1 = T_1 \circ h \) outside a compact set.

We shall consider bands \( B \) which are the image of \( \mathbb{B} \) under \( h^{-1} \) with \( h \in \mathcal{H} \).

We observe that \( B_+ := h^{-1}(\mathbb{B}_+) = B \cap \{(x,y) \in \mathbb{R}^2 : x \geq 0\} \), \( B_- := h^{-1}(\mathbb{B}_-) = B \cap \{(x,y) \in \mathbb{R}^2 : x \leq 0\} \) and \( B_0 := h^{-1}(\mathbb{B}_0) = B_+ \cap B_- \).

We shall decompose the band \( \mathbb{B} \) in cells determined by the vertical segments \( x = \delta_n^+ \) and \( x = \delta_n^- \) respectively, where \( \{\delta_n^+\}_{n \in \mathbb{Z}} \) and \( \{\delta_n^-\}_{n \in \mathbb{Z}} \) are strictly monotone sequences satisfying \( \delta_n^- \to -\infty, \delta_n^+ \to 0 \) as \( n \to -\infty \) and \( \delta_n^- \to 0, \delta_n^+ \to +\infty \) as \( n \to +\infty \). This decomposition produces also cell divisions \( \Delta^+ = \{\Delta_n^+\}_{n \in \mathbb{Z}} \) and \( \Delta^- = \{\Delta_n^-\}_{n \in \mathbb{Z}} \) of \( B_+ \) and \( B_- \) respectively.
In addition we shall assume that these sequences are eventually periodic. This
means that there exists an integer $N > 0$ (the period) and some $v > 0$ such that
$\delta_{n+N}^+ = \delta_n^+ + 1$ if $n \geq v$ and $\delta_{n+N}^- = \delta_n^- + 1$ if $n \leq -v$.

We observe that $\Delta_{n+N}^+ = T_1(\Delta_n^+)$ if $n \geq v$ and $\Delta_{n+N}^- = T_1(\Delta_n^-)$ if $n \leq -v$. The diameter of $\Delta$ is defined as
$$diam(\Delta) = \sup_{n \in \mathbb{Z}} \{ diam(\Delta_{n+\alpha}), diam(\Delta_{n-\beta}) \}.$$ 

The periodicity of the cells implies that $diam(\Delta)$ is always finite.

From now on we employ the notation $\Delta_n$ instead of $\Delta_{n+\alpha}$ or $\Delta_{n-\beta}$. A point $p$ is interior to the chain $\Delta$ if it belongs to the interior of some cell $\Delta_n$. Given two interior points $p \neq q$ we define
$$\Delta(p,q) = \cup_{n=\alpha}^{\beta} \Delta_n$$
where one of the points belongs to $int(\Delta_\alpha)$ while the other belongs to $int(\Delta_\beta)$.

Notice that $\Delta(p,q) = \Delta(q,p)$.

Given bands $B$ and $B'$ with chains $\Delta$ and $\Delta'$, we say that $\Delta'$ is a sub-chain of $\Delta$ if $B' \subset B$, the period of $B'$ is a multiple of the period of $\Delta$ and every cell of $\Delta'$ is contained in a cell of $\Delta$.

Assume now that the points $p$ and $q$ are interior to $\Delta$ and also to the subchain $\Delta'$. We say that the chain $\Delta'$ is crooked in $\Delta$ from $p$ to $q$ if $\Delta'(p,q) \subset \Delta(p,q)$ and given integers $h, k, i, j$ with $\alpha \leq h, k \leq \beta, k - h \geq 2, \alpha' \leq i, j \leq \beta'$ with $\Delta'_i \subset \Delta_k$, $\Delta'_j \subset \Delta_h$, then there exist $r, s$ with $i > r > s > j$ or $i < r < s < j$ and such that $\Delta'_r \subset \Delta_{k-1}, \Delta'_s \subset \Delta_{h+1}$.

Consider a sequence of bands $\{B^\lambda\}_{\lambda \in \mathbb{N}}$ with associated decompositions $\Delta^\lambda$. Assume that
0) $p \neq q$ are two points interior to each $\Delta^\lambda$
1) $\lim_{\lambda \to \infty} diam(\Delta^\lambda) = 0$,
2) $\Delta^{\lambda+1}$ is a sub-chain of $\Delta^\lambda$
3) $\Delta^{\lambda+1}$ is crooked in $\Delta^\lambda$ from $p$ to $q$.

By construction the set $K = \bigcap_{\lambda} \Delta^\lambda(p,q)$ is a non-degenerate continuum which is chained and hereditarily indecomposable. This implies that $K$ is a pseudo-arc (see [2] for a proof).
5. The family $F$. Let us consider the differential equations
\begin{align*}
\dot{x} &= \Phi(x, y) \\
\dot{y} &= 0
\end{align*}
where $\Phi : \mathbb{R}^2 \to \mathbb{R}$ is a $C^\infty$ map such that $\Phi(x, y) \geq 0$ for every $(x, y) \in \mathbb{R}^2$, $\Phi(x, y) = 0$ iff $x = 0$ and $|y| \leq \lambda$ for some $\lambda > 0$ and $\Phi(x, y) = 1$ outside a compact set.

Let $\phi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ be the flow induced by the above equation and $f_* = \phi(\cdot, 1) : \mathbb{R}^2 \to \mathbb{R}^2$ be the time-1 homeomorphisms.

We shall employ the notation $f_* \in F_*$ for maps $f_*$ as above. The class $F$ is obtained by conjugation under $H$, that is
$$f \in F, f = h^{-1} \circ f_* \circ h \text{ with } h \in H \text{ and } f_* \in F_*.$$

We observe that if $f \in F$ then $\text{Fix}(f) = \text{Fix}(f_*) = \{0\} \times [-\lambda, \lambda]$. Also, $B = h^{-1}(\mathbb{R} \times [-\lambda, \lambda])$ is a band in the sense previously defined with $B_+ = h^{-1}((0, \infty) \times$
\([-\lambda, \lambda]\), \(B_+ = h^{-1}((-\infty, 0] \times [-\lambda, \lambda])\). The bands \(B_+\) and \(B_-\) are invariant under \(f\). We shall consider decompositions \(\Delta\) which are also invariant under \(f\). This means that \(f(\Delta_n) = \Delta_{n+N}\) for each \(n \in \mathbb{Z}\).

We are ready to define the main objects in this section. An element of \(\mathcal{F}\) is a triplet \((f_*, h, \Delta)\) where \(f_* \in \mathcal{F}_*, h \in \mathcal{H}\) and \(\Delta\) is an invariant decomposition in cells of the band \(B = h^{-1}(\mathbb{R} \times [-\lambda, \lambda])\). These triplets will generate, as a final product, a homeomorphism \(f = h^{-1} \circ f_* \circ h\) in the class \(\mathcal{F}\).

Next we observe a property which will be employed later. Assume first that \(\gamma_* \subset (0, \infty) \times [-\lambda, \lambda]\) is a translation arc for \(f_*\). The iterates \(f_*^n(\gamma_*)\) will accumulate, if \(n \to -\infty\), on a segment contained in \(\{0\} \times [-\lambda, \lambda]\). This segment will be non-degenerate excepting if \(\gamma_*\) is horizontal. In such a case \(f_*^n(\gamma_*)\) accumulates on a single point. Assume now that \(f \in \mathcal{F}, f = h^{-1} \circ f_* \circ h\) and \(\gamma \subset B_*\) is a translation arc with the property \(c((\Theta) \setminus \Theta = \{(0, 0)\})\) where \(\Theta = \bigcup_{n \in \mathbb{Z}} f^n(\gamma)\). Then \(h(\gamma) = \gamma_*\) has a similar property with respect to \(f_*\) and so \(h(\gamma)\) must be a horizontal segment.

In consequence \(h(\Theta) = (0, \infty) \times \{0\}\).

6. The iterative method and the main results. In this section we assume the claim below and prove the main theorems of the paper. The proof of the claim will be given in the last section.

Claim 1. Given \((f_*, h, \Delta) \in \mathcal{F}\) with \(\text{Fix}(f_*) = \{0\} \times [-\lambda, \lambda]\), a point \(p\) interior to \(\Delta\) and \(\epsilon > 0\), there exists \((\hat{f}_*, \hat{h}, \hat{\Delta}) \in \mathcal{F}\) with \(\text{Fix}(\hat{f}_*) = \{0\} \times [-\lambda, \hat{\lambda}]\) such that:

- (i) \(\hat{\lambda} < 1/2\lambda\),
- (ii) \(\hat{\Delta} \subset B\) and \(\hat{\Delta}\) is a subchain of \(\Delta\) which is crooked from \(p\) to \(q = f(p) = \hat{f}(p)\),
- (iii) \(\text{diam} (\hat{\Delta}) \leq 1/2 \text{diam} (\Delta)\),
- (iv) \(\text{dist}(\hat{f}, \hat{f}) < \epsilon\) and
- (v) \(f = \hat{f}\) and \(f^{-1} = \hat{f}^{-1}\) in \(\mathbb{R}^2 \setminus ([-\lambda, \lambda] \times [-\lambda, \lambda])\).

Now we are in a position to state the main results of the paper.

Theorem 6.1. There exists an orientation preserving planar homeomorphism \(g\) such that \(\text{Fix}(g) = \{0\}\), the fixed point index of \(g\) at \(0\), \(i_{\mathbb{R}^2}(g, 0) = 0\) and the stable and the unstable sets of \(g\) at \(0\) are connected subsets \(S\) and \(U\) and such that:

- (a) \(S \cap U = \{0\}\) and
- (b) \(S \setminus \{0\}\) and \(U \setminus \{0\}\) admit a translation pseudo-arc. This means that there exist a pseudo-arc \(K\) with \(p_*, g(p_*) \in K\), \(g(K) \cap K = \{g(p_*)\}\) and

\[
S \setminus \{0\} = \bigcup_{n=-\infty}^{\infty} g^n(K)
\]

and analogously for \(U\).

Proof. We start with a fixed triplet \(((f_*)_0, h_0, \Delta^{(n)}) \in \mathcal{F}\) and a point \(p_*\) which is interior to \(\Delta^{(0)}\). Next we consider a decreasing sequence of positive numbers \(\{a_n\}_{n \in \mathbb{N}}\) such that \(\sum_{n \in \mathbb{N}} a_n\) converges and define inductively a sequence of triplets \(((f_n)_n, h_n, \Delta^{(n)}) \in \mathcal{F}\). They are obtained by applying the previous claim with \(\epsilon = a_n\).

The point \(p_*\) is interior to all decompositions \(\Delta^{(n)}\). For each \(n \in \mathbb{N}\) one has that \(\text{Fix}(f_n) = \{0\} \times [-\lambda_n, \lambda_n]\) with \(\lambda_n \to 0\) as \(n \to \infty\) and \(\text{dist}(f_n, f_{n+1}) < a_n\). This implies that \(f_n\) converges uniformly on compact sets to a homeomorphism which will be denoted by \(g\). Next we discuss the properties of \(g\).
that must converge to a fixed point (the origin) and any unbounded orbit tends to infinity.

Fix is either 0 or 1 and the unstable sets of 0 and 1 are invariant. We deduce that the orbit starting at \( z \in \mathbb{R}^2 \) goes to infinity as \( m \to \infty \).

The discussions at the end of Section 3 imply that \( K \) is a pseudo-arc.

e) For every \( z \in \mathbb{R}^2 \) the limit of the sequence \( \{g^m(z)\} \) as \( m \to +\infty \) or \( m \to -\infty \) is either 0 or \( \infty \).

f) The sets \( U(g,0) \) and \( S(g,0) \) are \( g \)-invariant because \( B_+^n \) and \( B_-^n \) coincide for both homeomorphisms \( f_n \) and \( g \). Thus \( g^m(z) \) goes to infinity as \( |m| \to \infty \).

Corollary 1. For every \( n \in \mathbb{N} \) there exists a planar homeomorphism \( g_n \) such that \( \text{Fix}(g_n) = \{0\} \), the fixed point index of \( g_n \) at 0, \( i_{\mathbb{R}^2}(g_n,0) \), and the stable and the unstable sets of 0 decompose into exactly \( n + 1 \) connected branches \( \{S_i\}_{i \in \{1,2,\ldots,n+1\}} \) and \( \{U_j\}_{j \in \{1,2,\ldots,n+1\}} \) and such that:

\[ a) S_i \cap S_j = \{0\} = U_i \cap U_j \text{ for any } i, j \in \{1,2,\ldots,n+1\} \text{ with } i \neq j, \]

\[ b) S_i \cap U_j = \{0\} \text{ for any } i, j \in \{1,2,\ldots,n+1\}. \]

\[ c) For every } j \in \{1,2,\ldots,n+1\}, S_j \setminus \{0\} \text{ and } U_j \setminus \{0\} \text{ admit translation pseudo-arcs.} \]
This corollary can be proved using the same techniques of the above theorem or by pasting copies of the above homeomorphisms along the stable and unstable sets to obtain any finite amount of branches. Indeed, using the construction of the above theorem we have a homeomorphism \( g : S^2 \to S^2 \) such that \( C = U(g, 0) \cup S(g, 0) \cup \{ \infty \} \) is an invariant continuum having the shape of the circle \( S^1 \). Indeed, \( C \) is the inverse limit of a sequence of decreasing continua \( \{ C_n \} \) such that the corresponding inclusions \( C_{n+1} \to C_n \) are homotopy equivalences. Then, \( C \) decomposes the sphere into two components \( S^+ \) and \( S^- \) such that \( \partial(S^+) = \partial(S^-) = U(g, 0) \cup S(g, 0) \cup \{ \infty \} \) (see Borsuk’s book, [3]). To obtain the required homeomorphism \( g_n \), see the next figure.

Consider the space \( E \) of all orientation preserving planar homeomorphisms endowed with the (complete) metric

\[
\text{dist}(f, g) = \sup_{x \in \mathbb{R}^2} \left\{ \frac{\|f(x) - g(x)\|}{1 + \|f(x) - g(x)\|} \frac{\|f^{-1}(x) - g^{-1}(x)\|}{1 + \|f^{-1}(x) - g^{-1}(x)\|} \right\}
\]

and \( H_0 \) be the subset of \( E \) of all planar homeomorphisms such that \( \{0\} \) is the unique compact invariant set.

For every \( n \in \mathbb{N} \) there is a unique (up to conjugation) canonical planar homeomorphism, \( G_n \in H_0 \), such that \( i_{\mathbb{R}^2}(G_n, 0) = -n \) (see Figure 5).

Consider the spaces \( H_0 \) of all planar homeomorphisms \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) such that \( \{0\} \) is the unique compact invariant set,

\[
H = \{ f \in E : \text{ there is } n \in \mathbb{N} \text{ such that } f \text{ is conjugate to } G_n \}
\]

and \( \text{cl}_E(H) \).

Now, define \( H \) the set of homeomorphisms \( f \in H_0 \cap \text{cl}_E(H) \) such that \( 0 \) is an asymptotically stable attractor for \( f \) \((f^{-1}) \) in \( S(f, 0) \cup U(f, 0) \).

Finally, take \( H_0 \) the set of planar homeomorphisms \( f \in H \) such that and \( U(f, 0) \) and \( S(f, 0) \) decompose into a finite amount of branches that admit a translation pseudo-arc.

**Theorem 6.2.** The subset \( H_0 \) is a dense subset of \( H \).
Proof. We will give a constructive proof to check that $\mathbf{H}_0$ is dense in $\mathbf{H}$. Consider first the case of $f \in \mathbf{H}$ such that $U(f,0) \setminus \{0\}$ and $S(f,0) \setminus \{0\}$ are connected. The proof in the general case is similar using Corollary 1 instead of Theorem 1.

For every $\epsilon > 0$ there is a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\text{dist}(h^{-1} \circ f \circ h, f_\ast) < \epsilon$. Let $\eta > 0$ such that $\text{dist}(f_\ast, \phi) < \eta$ implies that $\text{dist}(h \circ \phi \circ h^{-1}, f) < \epsilon$. Now consider any decreasing sequence of positive numbers $\{a_n\}_{n \in \mathbb{N}}$ such that $\sum_{n \in \mathbb{N}} a_n < \eta$ and using the construction of Theorem 1 obtain a homeomorphism $g : \mathbb{R}^2 \to \mathbb{R}^2 \in \mathbf{H}_0$ such that $\text{dist}(g, f_\ast) < \eta$. Then, $h \circ g \circ h^{-1} \in \mathbf{H}_0$ and $\text{dist}(f, h \circ g \circ h^{-1}) < \epsilon$. \hfill \qed

7. Proof of Claim 1.

Lemma 7.1. Given $f \in \mathcal{F}$ with $\text{Fix}(f) = \{0\} \times [-\lambda, \lambda]$ and $U$ a neighborhood of $\text{Fix}(f)$, there exists $\delta_1 > 0$ such that for any $\gamma$ translation arc for $f$ contained in $h^{-1}((0, \infty) \times [-\delta', \delta'])$ $\delta' < \delta_1$, there exists $f' \in \mathcal{F}$ satisfying

a) $d(f, f') \leq \omega(\delta')$ with $\lim_{\delta' \to 0^+} \omega(\delta') = 0$,

b) $f = f'$ and $f^{-1} = f'^{-1}$ in $\mathbb{R}^2 \setminus U$,

c) $\text{Fix}(f') = \{0\} \times [-2\delta, 2\delta]$ and

d) there exist an arc $\alpha \subset U$ such that one of its endpoints is the origin and $\alpha \setminus \{0\} \subset \{(x, y) \in \mathbb{R}^2 : x > 0, |y| \leq \delta\}$ and an integer $k \in \mathbb{N}$ such that the set $C = \alpha \cup (\bigcup_{n \geq k} f^n(\gamma))$ is homeomorphic to $[0, \infty)$ and is $f'$-invariant.

Proof. We shall assume $f = f_\ast$ and $h = \text{Id}$. The general case can be treated by conjugation.

As a first step we modify slightly the function $\Phi$ defining the flow. The new function $\Phi_1$ vanishes only if $|y| \leq \delta$ and $x = 0$ and $\Phi = \Phi_1$ outside a small neighborhood of $\{0\} \times [2\delta, \lambda]$ and $\{0\} \times [-\lambda, -2\delta]$. The new homeomorphism $f_1 \in \mathcal{F}_\ast$ associated to $\Phi_1$ satisfies (after choosing appropriate neighborhoods) $\text{Fix}(f_1) = \{0\} \times [-2\delta, 2\delta]$, $d(f, f_1) < \delta$ and $f = f_1$ in $\mathbb{R} \times [-\delta, \delta]$.

The arc $\gamma$ is still a translation arc for $f_1$ and the set $\Omega = \bigcup_{n \in \mathbb{Z}} f^n(\gamma)$ is contained in the band $[0, \infty) \times [-\delta, \delta]$. Here we are using that this strip is invariant under $f \in \mathcal{F}_\ast$. The set $\Omega$ will accumulate on a segment contained in $\{0\} \times [-\delta, \delta]$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure5.png}
\caption{Figure 5.}
\end{figure}
Our next task will be to construct an arc $\Gamma = \overline{q_0q_1}$ with endpoints $q_0, q_1$ lying in $y = \pm 2\delta$ and $y = -\pm 2\delta$ and such that $\Gamma \setminus \{q_0, q_1\} \subset \{(x, y) \in \mathbb{R}^2 : |y| < 2\delta\}$, $\Gamma \cap \Omega = \{p_0\}$ and $f_1(\Gamma) \cap \Gamma = \emptyset$. The point $p_0$ is the first endpoint of the arc $\gamma$, namely $\gamma = p_0p_\Gamma$, $f_1(p_0) = p_1$. To prove that $\Gamma$ exists we select two consecutive points in $\Omega$, say $p$ and $p_\ast$, such that the coordinate $y$ reaches its maximum and minimum value on them (with respect to $\Omega$). We can also assume that $p_0 \in \overline{p_\ast p^\ast}$. Next we connect the arc $\overline{p_\ast p^\ast}$ to the boundaries of the band $y \in [-2\delta, 2\delta]$ via two vertical segments. The resulting arc $\Gamma_\ast$ will satisfy that $f_1(\Gamma_\ast) \cap \Gamma = \emptyset$. Next we perturb $\Gamma_\ast$ a little in such a way that the new arc $\Gamma$ intersects $\Omega$ only at $p_0$ but still $f_1(\Gamma) \cap \Gamma = \emptyset$.

This perturbation argument can be justified with the Jordan-Schoenflies Theorem (see [9]).

Once we know that $\Gamma$ exists we find $\upsilon$ large enough so the backward iterates $\Gamma_n = f_1^n(\Gamma)$ $n \leq -\upsilon + 1$ are very close to $\{(x, y) \in \mathbb{R}^2 : x = 0\}$. For instance we can assume that $\Gamma_{-\upsilon + 1} \subset (\{0\} \times [-2\delta, 2\delta]) \cap U$.

Let $\Lambda_n$ denote the compact region determined by $\Gamma_n$, $\{0\} \times [-2\delta, 2\delta]$ and the appropriate segments inside the lines $|y| = 2\delta$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig6.png}
\caption{Figure 6.}
\end{figure}

The sets $\Gamma_{-\upsilon}$ and $\Lambda_{-\upsilon}$ are contained in $\Lambda_{-\upsilon+1}$ and we can construct an isotopy $\phi : (-\infty, 1] \times \Lambda_\upsilon \to \Lambda_{\upsilon+1}$ with the following properties:

i) The points in $\{0\} \times [-2\delta, 2\delta]$ are equilibria.

ii) The point in $\{0\} \times [-2\delta, 2\delta]$ are connected in a bijective way to the points in $\Gamma_\upsilon$ (from $x = 0$ at time $t = -\infty$ to $\Gamma_\upsilon$ and time $t = 0$).

iii) The points in $\Gamma_\upsilon$ are connected to the points in $\Gamma_{-\upsilon+1}$ (which are reached at time $t = 1$. That is, for $p \in \Gamma_{-\upsilon}$, $f_1(p) = \phi(1, p)$.

iv) The isotopy $\phi$ coincides with the flow initially associated to $f_1$ on the lines $|y| = 2\delta$.

v) The orbit travelling from $p_{-\upsilon}$ to $p_{-\upsilon+1}$ is precisely $f_1^{-\upsilon}(\gamma)$.

This construction is possible because we are dealing with orientation preserving homeomorphisms of the interval.

Define 
\[
 f'(p) = \begin{cases} 
 f_1(p) & \text{if } p \notin \Lambda_{-\upsilon} \\
 \phi(1, p) & \text{if } p \in \Lambda_{-\upsilon} 
\end{cases}
\]

It is easy to check that $f'$ satisfies all the requirements. Perhaps the most delicate point is to show that $f'$ is in the class $F$. To this end we observe that the region $\Omega$ determined by the arcs $\Gamma_{-\upsilon}$, $\Gamma_{-\upsilon+1}$ and the segments $|y| = 2\delta$ is a translation domain. This means that $\bigcup_{n \in \mathbb{N}} (f')^n(\Omega) = (0, \infty) \times [-2\delta, 2\delta]$. 

Moreover $\Omega_n = (f')^n(\Omega)$ satisfies $\Omega_{n+1} = T_1(\Omega_n)$ for large $n$. The set $\Omega$ is fibered by the orbits of the isotopy $\phi$ and each of these orbits converges to an equilibrium in $\{0\} \times [-2\delta, 2\delta]$. The homeomorphism $h'$ is first defined on the cell $\Omega$ and then extended periodically to the open band $(0, \infty) \times [-2\delta, 2\delta]$. The property of convergence of negative orbits guarantees that $h'$ has a continuous extension to the closed strip.

**Remark 2.** In the previous construction $f$ and $f'$ are close but $h'$ is not necessarily close to the identity. An important observation is that $(h')^{-1}(\gamma)$ is a horizontal segment and $(h')^{-1}(C) = (0, \infty) \times \{0\}$. Indeed, as we mentioned in Section 4, this is the only possibility if the backward iterates of $\gamma$ are going to accumulate on a point.

**Proof of Claim 1.** Take a narrow band $[0, \infty) \times [-\delta, \delta]$ with $\delta < \lambda/2$. In the region $x \geq \alpha$ where $f_\ast = T_1$ consider the arc $\gamma$ joining $p_0$ and $p_1 = T_1(p_0)$ inside a curve as given in Proposition 1 and contained in $|y| \leq \delta$. The condition 4 in Proposition 1 can be employed to make sure that $\gamma$ remains in $x \geq \alpha$. Next we take $C = \bigcup_{n \in \mathbb{N}} f_\ast^n(\gamma)$. This curve is crooked with respect to the decomposition in cells $\Delta$, with period $N$. We transport the curve $\gamma$ to $f$ via $h^{-1}$ and apply Lemma 2 to obtain $f' \in F$. Let $h' \in \mathcal{H}$ be such that $f' = (h')^{-1} \circ f'_\ast \circ h'$ then $h'(\gamma)$ must be a horizontal segment, indeed $h'(\alpha \cup (\bigcup_{n \in \mathbb{N}}(f')_\ast^n(\gamma))) = \{y = 0, x > 0\}$. We can now inflate this curve to produce the new band and partition. Here we only employ the eventual periodicity of partitions and the transversality of $\gamma$ with the vertical segments. □
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