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Abstract  
A new approach to solving linear ill-posed problems is proposed. The approach consists of solving a Cauchy problem for a linear operator equation and proving that this problem has a global solution whose limit at infinity solves the original linear equation.

1 Introduction  
Let $A$ be a linear, bounded, injective operator on a Hilbert space $H$, and assume that $A^{-1}$ is unbounded and that $\|A\| \leq \sqrt{m}$, where $m > 0$ is a constant. For example, $A$ may be a compact injective linear operator. Consider the equation,

$$Au = f. \quad (1.1)$$

Assume that (1.1) is solvable, so that $f = Ay$ for a unique $y \in H$. Problem (1.1) is ill-posed since $A^{-1}$ is unbounded. Equation (1.1) cannot be solvable for all $f \in H$ because if $A$ is injective, linear, closed and $R(A) = H$, then $A^{-1}$ must be bounded (by the Banach theorem). Let $f_\delta$ be given, such that

$$\|f_\delta - f\| \leq \delta \quad (1.2)$$

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Equation (1.1) with \( f_\delta \) in place of \( f \) may have no solution, and if it has a solution \( u_\delta \) then it may be that \( \| u - u_\delta \| \) is large, although \( \delta > 0 \) is small. There is a large literature on ill-posed problems since they are important in applications. (See e.g. [1], [3]). In this paper a new approach to solving linear ill-posed problems is proposed. This approach consists of the following steps:

**Step 1.** Solve the Cauchy problem:

\[
\dot{u} = -[Bu + \varepsilon(t)u - F_\delta], \quad u(0) = u_0,
\]

(1.3)

where

\[
\dot{u} := \frac{du}{dt}, \quad B := A^*A, \quad F_\delta := A^*f_\delta, \quad ||F_\delta - F|| \leq \delta \sqrt{m}, \quad F = By,
\]

and

\[
\varepsilon(t) \in C^1[0, \infty); \quad \varepsilon(t) > 0; \quad \varepsilon(t) \searrow 0 \text{ as } t \to \infty; \quad \frac{|\dot{\varepsilon}(t)|}{\varepsilon^{5/2}(t)} \to 0 \text{ as } t \to \infty.
\]

(1.4)

One has \( ||A^*(f_\delta - f)|| \leq \sqrt{m}\delta \), where we have used the estimate \( ||A|| = ||A^*|| \leq \sqrt{m} \).

Examples of functions \( \varepsilon(t) \) satisfying (1.4) can be constructed by the formula:

\[
\varepsilon(t) = [c + \int_0^t h(s)ds]^{-\frac{3}{4}},
\]

where \( c > 0 \) is a constant, \( h(s) > 0 \) is a continuous function defined for all \( s \geq 0 \), such that \( h(s) \to 0 \) as \( s \to \infty \) and \( \int_0^\infty h(s)ds = \infty \). One has \( \frac{|\dot{\varepsilon}(t)|}{\varepsilon^{5/2}(t)} = \frac{2h(t)}{3} \to 0 \) as \( t \to \infty \). For example, \( \varepsilon(t) = \frac{1}{\log(t+2)} \) satisfies (1.4). If \( h(t) = \frac{1}{(2 + t) \log(2 + t)} \), then

\[
\varepsilon(t) = \frac{1}{(1 + \log \log(2 + t))^{\frac{3}{4}}}.
\]
This $\varepsilon(t)$ yields nearly fastest decay of $h(t)$ allowed by the restriction $\int_0^\infty h(s)ds = \infty$.

**Step 2.** Calculate $u(t_\delta)$, where $t_\delta > 0$ is a number which is defined by formula (1.9) below.

Then $t_\delta \to \infty$ as $\delta \to 0$ and satisfies the inequality:

$$\|u(t_\delta) - y\| \leq \eta(\delta) \to 0 \text{ as } \delta \to 0,$$

(1.5)

for a certain function $\eta(\delta) > 0$. If $\delta = 0$, so that $F_\delta = By$, then Step 2 yields the relation

$$\lim_{t \to \infty} \|u(t) - y\| = 0.$$

(1.6)

The foregoing approach is justified in Section 2. Our basic results are formulated as follows.

**Theorem 1.1.** Assume that equation (1.1) is uniquely solvable, (1.4) holds, and $\delta = 0$. Then for any $u_0$, problem (1.3), with $F = By$ replacing $F_\delta$, has a unique global solution and (1.6) holds.

By global solution we mean the solution defined for all $t > 0$.

**Theorem 1.2.** Assume that equation (1.1) is uniquely solvable, (1.4) holds, and $\delta > 0$. Then for any $u_0$ problem (1.3) has a unique global solution $u(t)$ and there exists a $t_\delta \to \infty$ as $\delta \to 0$, such that $\|u(t_\delta) - y\| \to 0$ as $\delta \to 0$. The number $t_\delta$ is defined by formula (1.9).

Let $y$ solve (1.1). Then $By = F := A^* f$ and $\|B\| \leq m$. If

$$\phi(\beta) := \phi(\beta, y) := \beta \left\| \int_0^m \frac{dE_\lambda y}{\lambda + \beta} \right\|,$$

(1.7)

where $E_\lambda$ is the resolution of the identity of the selfadjoint operator $B$, $E_{\lambda-0} = E_\lambda$, $\beta(\delta)$ is the minimizer of the function

$$h(\beta, \delta) := \phi(\beta) + \frac{\delta}{2\beta^2}$$

(1.8)

on $(0, \infty)$, (see formula (2.20) and Remark 2.3 below), and

$$\eta(\delta) := h(\beta(\delta), \delta), \quad \varepsilon(\delta) = \beta(\delta),$$

(1.9)
then \( t_\delta \to \infty \) as \( \delta \to 0 \), \( \eta(\delta) \to 0 \) as \( \delta \to 0 \), and

\[
\lim_{\delta \to 0} \| u(t_\delta) - y \| = 0. \tag{1.10}
\]

Because \( B \) is injective, zero is not an eigenvalue of \( B \), so, for any \( y \in H \), one has \( \| \int_0^s dE_\lambda y \| \to 0 \) as \( s \to 0 \). Therefore \( \phi(\beta, y) \to 0 \) as \( \beta \to 0 \), for any fixed \( y \). From (2.15) (see below) one gets

\[
\| u(t_\delta) - y \| < \eta(\delta) + g_\delta(t_\delta) \to 0 \text{ as } \delta \to 0, \tag{1.11}
\]

where \( g_\delta(t) \) is given by the right-hand side of (2.12) with \( ||f_\delta|| \) replacing \( ||f|| \).

**Remark 1.1.** Theorem 1.2 shows that solving the Cauchy problem (1.3) and calculating its solution at a suitable time \( t_\delta \) yields a stable solution to ill-posed problem (1.1) and this stable approximate solution satisfies the error estimate (1.11).

For nonlinear ill-posed problems a similar approach is proposed in [1].

## 2 Proofs

### 2.1 Proof of Theorem 1.1

We start with a simple, known fact: if equation (1.1) is solvable, then it is equivalent to the equation

\[
Bu = A^* f = By \tag{2.1}
\]

Indeed, if \( Ay = f \), then apply \( A^* \) and get (2.1). Conversely, if (2.1) holds, then \( (B(u - y), u - y) = \| A(u - y) \|^2 = 0 \), thus \( Au = Ay \) and \( u = y \), so (1.1) is solvable and its solution is the solution to (2.1). Therefore we will study equation (2.1). The operator \( B = A^* A \) is selfadjoint and nonnegative, that is, \( (Bu, u) \geq 0 \). Let \( E_\lambda \) be its resolution of the identity.

We make another observation: If (1.4) holds, then

\[
\int_0^\infty \varepsilon(t) dt = \infty. \tag{2.2}
\]

Indeed, (1.4) implies

\[
-\frac{\ddot{\varepsilon}}{\varepsilon^2} \leq c,
\]
where \( c = \text{const} > 0 \), so
\[
\frac{d}{dt} \frac{1}{\varepsilon(t)} \leq c,
\]
\[
\frac{1}{\varepsilon(t)} - \frac{1}{\varepsilon(0)} \leq ct,
\]
\[
\frac{1}{\varepsilon(t)} \leq c_0 + ct,
\]
and
\[
\varepsilon(t) \geq \frac{1}{c_0 + ct}.
\]
Formula (2.2) follows from the foregoing inequality.

Consider the problem
\[
Bw + \varepsilon(t)w - F = 0, \quad F := A \ast f = By.
\]
(2.3)
Since \( B \geq 0 \) and \( \varepsilon(t) > 0 \), the solution \( w(t) \) of (2.3) exists, is unique and admits the estimate
\[
\|w\| \leq \|(B + \varepsilon(t))^{-1}F\| \leq \frac{\|F\|}{\varepsilon(t)}.
\]
(2.4)
If \( F = A \ast f \), then (see Remark 2.3 below) one gets:
\[
\|w\| \leq \|(B + \varepsilon(t))^{-1}F\| = \|(B + \varepsilon(t))^{-1}A \ast f\| \leq \frac{\|f\|}{2\varepsilon(t)}.
\]
(2.4’)
Differentiate (2.3) with respect to \( t \) (this is possible by the implicit function theorem) and get
\[
[B + \varepsilon(t)]\dot{w} = -\dot{\varepsilon}w, \quad \|
\dot{w}\| \leq \frac{|\dot{\varepsilon}|}{\varepsilon} \|w\| \leq \frac{|\dot{\varepsilon}(t)|}{\varepsilon(t)} \|F\|,
\]
(2.5)
where (2.4) was used.
Using (2.4) yields:
\[
\|
\dot{w}\| \leq \frac{|\dot{\varepsilon}|}{\varepsilon} \|w\| \leq \frac{|\dot{\varepsilon}(t)|}{2\varepsilon(t)} \|f\|.
\]
(2.5’)

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Denote
\[ z(t) := u(t) - w(t). \quad (2.6) \]

Subtract (2.3) from (1.3) (with \( F \) in place of \( F_\delta \)) and get
\[ \dot{z} = -\dot{w} - [B + \varepsilon(t)]z, \quad z(0) = u_0 - w(0). \quad (2.7) \]

Multiply (2.7) by \( z(t) \) and get
\[ (\dot{z}, z) = -(\dot{w}, z) - (Bz, z) - \varepsilon(t)(z, z). \quad (2.8) \]

Denote
\[ \|z(t)\| := g(t) \quad (2.9) \]

Then the inequality \((Bz, z) \geq 0\) and equation (2.8) imply:
\[ g\dot{g} \leq \|\dot{w}\|g - \varepsilon(t)g^2. \quad (2.10) \]

Because \( g \geq 0 \), it follows from (2.10) and (2.5') that
\[ \dot{g} \leq \|f\| \frac{|\dot{\varepsilon}(t)|}{2\varepsilon^2(t)} - \varepsilon(t)g(t), \quad g(0) = \|u_0 - w_0\|, \quad (2.11) \]
so
\[ g(t) \leq e^\int_0^t \frac{|\dot{\varepsilon}(s)|}{2\varepsilon^2(s)} ds \left[ g(0) + \int_0^t e^{\int_0^\tau \frac{|\dot{\varepsilon}(s)|}{2\varepsilon^2(s)} ds} d\tau \|f\| \right]. \quad (2.12) \]

Assumption (1.4) (the last one in (1.4)) and (2.12) imply (use L'Hospital's rule) that
\[ \|u(t) - w(t)\| := g(t) \to 0 \text{ as } t \to +\infty. \quad (2.13) \]

The existence of the global solution to (1.3) is obvious since equation (1.3) is linear and the operator \( B \) is bounded.

To prove (1.6) it is sufficient to prove that
\[ \|w(t) - y\| \to 0 \text{ as } t \to +\infty. \quad (2.14) \]
Indeed, if (2.14) holds then (2.13) and (2.14) imply:

\[ \| u(t) - y \| \leq \| u(t) - w(t) \| + \| w(t) - y \| \to 0 \text{ as } t \to \infty. \]  

(2.15)

We now prove (2.14). One has:

\[ \| w(t) - y \| = \left\| \int_0^m \frac{\lambda}{\lambda + \varepsilon(t)} dE_\lambda y - \int_0^m dE_\lambda y \right\| = \left\| \int_0^m \frac{\varepsilon(t)}{\lambda + \varepsilon(t)} dE_\lambda y \right\|. \]  

(2.16)

Thus

\[ \| w(t) - y \| = \phi(\varepsilon(t), y), \]  

(2.17)

where \( \phi(\varepsilon, y) := \phi(\varepsilon) \) is as defined in (1.7). Since \( B \) is injective, the point \( \lambda = 0 \) is not an eigenvalue of \( B \). Therefore

\[ \lim_{\varepsilon \to 0} \phi(\varepsilon) = 0, \]  

(2.18)

by the Lebesgue dominant convergence theorem.

Thus (2.14) follows and Theorem 1.1 is proved.

\[ \Box \]

\section{2.2 Proof of Theorem 1.2}

The proof is quite similar to the above, so we indicate only the new points. Equation (2.3) is now replaced by the equation

\[ Bw + \varepsilon(t)w - F_\delta = 0. \]  

(2.19)

Estimates (2.4), (2.4), (2.5), (2.5) and (2.13) hold with \( F_\delta \) and \( f_\delta \) in place of \( F \) and \( f \), respectively. The main new point is the estimate of \( w(t) - y \):

\[ \| w(t) - y \| = \left\| \int_0^m \frac{dE_\lambda F_\delta}{\lambda + \varepsilon(t)} - \int_0^m dE_\lambda y \right\| = \left\| \int_0^m \frac{dE_\lambda(F_\delta - F)}{\lambda + \varepsilon(t)} \right\| + \phi(\varepsilon(t)) \leq \phi(\varepsilon(t)) + \frac{\delta}{2\varepsilon^2(t)}, \]  

(2.20)

where \( \| f - f_\delta \| \leq \delta \) and estimate (2.4') was used.
If $\beta(\delta)$ is the minimizer of the function (1.8), then

$$h(\beta(\delta), \delta) := \eta(\delta) \to 0 \text{ as } \delta \to 0; \quad \beta(\delta) \to 0 \text{ as } \delta \to 0.$$  \hspace{1cm} (2.21)

The latter relation in (2.21) holds because $\phi(\beta) \to 0$ as $\beta \to 0$.

Since $\varepsilon(t) \searrow 0$ as $t \to \infty$, one can find the unique $t_\delta$ such that

$$\varepsilon(t_\delta) = \beta(\delta) \to 0 \text{ as } \delta \to 0.$$  \hspace{1cm} (2.22)

Thus

$$\|w(t_\delta) - y\| \leq \eta(\delta) \to 0 \text{ as } \delta \to 0.$$  \hspace{1cm} (2.23)

The function $\eta(\delta) = \eta(\delta, y)$ depends on $y$ because $\phi(\varepsilon) = \phi(\varepsilon, y)$ does (see formula (1.7)).

Combining (2.23), (2.13) and (2.15) one gets the conclusion of Theorem 1.2. \hfill \Box

**Remark 2.1.** We also give a proof of (2.14) which does not use the spectral theorem.

From (2.3) one gets

$$Bx + \varepsilon(t)x = -\varepsilon(t)y, \quad x(t) := w(t) - y.$$  \hspace{1cm} (2.24)

Thus $(Bx, x) + \varepsilon(x, x) = -\varepsilon(y, x)$. Since $(Bx, x) \geq 0$ and $\varepsilon > 0$, one gets

$$\langle x, x \rangle \leq |\langle y, x \rangle|, \quad \|x(t)\| \leq \|y\| = \text{const} < \infty.$$  \hspace{1cm} (2.25)

Bounded sets in $H$ are weakly compact. Therefore there exists a sequence $t_n \to \infty$ such that

$$x_n := x(t_n) \rightharpoonup x_\infty, \quad n \to \infty$$  \hspace{1cm} (2.26)

where $\rightharpoonup$ stands for the weak convergence. From (2.24) and (2.25) it follows that

$$Bx_n \to 0, \quad n \to \infty.$$  \hspace{1cm} (2.27)

A monotone hemicontinuous operator is weakly closed. This claim, which we prove below, implies that (2.26) and (2.27) yield $Bx_\infty = 0$. Because $B$ is injective, $x_\infty = 0$, that is, $x(t_n) \to 0$. From (2.25) it follows that $\|x(t_n)\| \to 0$...
as \( n \to \infty \), because \((y, x(t_n)) \to 0\) as \( n \to \infty \), due to \( x(t_n) \to 0 \). By the uniqueness of the limit, one concludes that \( \lim_{t \to \infty} \|x(t)\| = 0 \), which is (2.14).

Let us now prove the claim.

We wish to prove that \( x_n \rightharpoonup x \) and \( Bx_n \to f \) imply \( Bx = f \) provided that \( B \) is monotone and hemicontinuous. The monotonicity implies \((Bx_n - B(x - \varepsilon p), x_n - x + \varepsilon p) \geq 0\) for all \( \varepsilon > 0 \) and all \( p \in H \). Take \( \varepsilon \to 0 \) and use hemicontinuity of \( B \) to get \((f - Bx, p) \geq 0 \) \( \forall p \in H \). Take \( p = f - Bx \) to obtain \( Bx = f \), as claimed.

The above argument uses standard properties of monotone hemicontinuous operators [2].

**Remark 2.2.** In (2.23) \( \eta(\delta) = O(\delta^{\frac{2}{3}}) \) is independent of \( y \) if \( y \) runs through a set \( S_a := S_{a,R} := \{ y : y = B^a h, \|h\| \leq R \} \), where \( R > 0 \) is an arbitrary large fixed number and \( a \geq 1 \). If \( 0 < a < 1 \), then \( \eta(\delta) = O(\delta^{\frac{2a}{2a+1}}) \) as \( \delta \to 0 \), and this estimate is uniform with respect to \( y \in S_{a,R} \).

Consider, for example, the case \( a \geq 1 \). If \( y = B^a h \), then \( \phi(\varepsilon) \) in (2.20) can be chosen for all \( y \in S_{a,R} \) simultaneously. Using (1.7), one gets:

\[
\phi(\varepsilon) = \varepsilon \sup_{\|h\| \leq R} \left\| \int_0^m \frac{\lambda^a}{\lambda + \varepsilon} dE_\lambda h \right\| \leq \varepsilon m^{a-1} R,
\]

where \( a \geq 1 \) and \( \varepsilon \) is positive and small. For a fixed \( \delta > 0 \) one finds the minimizer \( \varepsilon(\delta) = O(\delta^{\frac{2}{3}}) \) of the function \( \frac{\delta}{2\varepsilon^{\frac{2}{3}}} + \varepsilon m^{a-1} R \) and the minimal value \( \eta(\delta) \) of this function is \( O(\delta^{\frac{2}{3}}) \).

If \( B \) is compact, then the condition \( y \in S_a \) means that \( y \) belongs to a compactum which is the image of a bounded set \( \|h\| \leq R \) under the map \( B^a \).

The case \( 0 < a < 1 \) is left to the reader. It can be treated by the method used above.

**Remark 2.3.** It can be checked easily that

\[
A(A^* A + \varepsilon I)^{-1} = (AA^* + \varepsilon I)^{-1} A.
\]

This implies

\[
\|(B + \varepsilon I)^{-1} A f\|^2 = ((b + \varepsilon I)^{-2} b f, f) := J,
\]
where \( B := A^*A \) and \( b := AA^* \geq 0 \).

Thus,
\[
J = \int_0^m s(s + \epsilon)^{-2}d(e_s f, f) \leq \frac{1}{4\epsilon}||f||^2,
\]
where \( e_s \) is the resolution of the identity corresponding to the selfadjoint operator \( b \). Therefore one gets the following estimate:
\[
||(B + \epsilon I)^{-1}A^*f|| \leq \frac{1}{2\sqrt{\epsilon}}||f||.
\]
This estimate was used to obtain estimates (2.4) and (2.5). For example, estimate (2.4) was replaced by the following one:
\[
||(B + \epsilon I)^{-1}F|| \leq \frac{1}{2\epsilon^{\frac{1}{2}}}||f||,
\]
and (2.5) can be replaced by the estimate:
\[
||\dot{w}|| \leq \frac{||\dot{\epsilon}||}{\epsilon}||w|| \leq \frac{|\dot{\epsilon}(t)|}{2\epsilon^{3/2}(t)}||f||.
\]

These estimates were used to improve the estimate for \( \eta(\delta) \) in the previous remark.

Estimate (2.4') was used by a suggestion of a referee. The author thanks the referee for the suggestion.

In fact, one can prove a stronger estimate than (2.4'), namely \( ||w|| \leq ||y|| \). Indeed, multiply (2.3) by \( w - y \), use the nonnegativity of \( B \) and positivity of \( \epsilon \) and get \( (w, w - y) \leq 0 \). Thus \( ||w||^2 \leq ||w||||y|| \), and the desired inequality \( ||w|| \leq ||y|| \) follows.

Appendix. Let us give an alternative proof of Theorem 1.2. Let \( u_\delta(t) \) solve (1.3), \( u(t) \) solve (1.3) with \( F_\delta \) replaced by \( F \), and \( u_\delta(t) \) and \( u(t) \) satisfy the same initial condition. Denote \( w_\delta := u_\delta(t) - u(t) \) and let \( ||w_\delta|| := g_\delta(t) \). One has:
\[
\dot{w}_\delta = -[Bw_\delta + \epsilon(t)w_\delta - h_\delta], \quad w_\delta(0) = 0,
\]
where \( h_\delta := F_\delta - F, ||h_\delta|| < \sqrt{m}\delta := c\delta \). Multiply the above equation by \( w_\delta \) in \( H \), use the inequality \( B \geq 0 \) and get
\[
\dot{g_\delta} \leq -\epsilon(t)g_\delta + c\delta.
\]
Since \( g_\delta(0) = 0 \), this implies:

\[
g_\delta(t) \leq c_\delta \exp\left[- \int_0^t \epsilon(s) ds \right] \int_0^p \epsilon(s) ds dp \leq c \frac{\delta}{\epsilon(t)}.
\]

Thus

\[
||u_\delta(t) - y|| \leq ||u_\delta(t) - u(t)|| + ||u(t) - y|| \leq c \frac{\delta}{\epsilon(t)} + a(t),
\]

where \( a(t) := ||u(t) - y|| \to 0 \) as \( t \to \infty \). Define \( t_\delta \) as the minimal minimizer of the following function of \( t \) for a fixed \( \delta > 0 \):

\[
c_\frac{\delta}{\epsilon(t)} + a(t) = \min := \mu(\delta).
\]

Since \( a(t) \to 0 \) and \( \epsilon(t) \to 0 \) as \( t \to \infty \), one concludes that the minimal minimizer \( t_\delta \to \infty \) as \( \delta \to 0 \) and \( \mu(\delta) \to 0 \) as \( \delta \to 0 \). Theorem 1.2 is proved.

\[\square\]

References

[1] Airapetyan, R., Ramm, A.G., Dynamical systems and discrete methods for solving nonlinear ill-posed problems, Appl.Math.Rev.,Vol.1, World Sci. Publishers, (ed. G.Anastassiou), (2000), pp.491-536.

[2] Deimling, K., Nonlinear functional analysis, Springer Verlag, Berlin, 1985

[3] Engl, W., Hanke, M., Neubauer, A., Regularization of ill-posed problems, Kluwer, Dordrecht, 1996.

[4] Groetsch, C., Inverse problems in mathematical sciences, Vieweg, Braunschweig, 1993.