On Khovanov Homology of Quasi-Alternating Links

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Abstract. We prove that the length of any gap in the differential grading of the Khovanov homology of any quasi-alternating link is one. As a consequence, we obtain that the length of any gap in the Jones polynomial of any such link is one. This establishes a weaker version of Conjecture 2.3 in (Topol Appl 264:1–11, 2019). Moreover, we obtain a lower bound for the determinant of any such link in terms of the breadth of its Jones polynomial. This establishes a weaker version of Conjecture 3.8 in (Algebr Geom Topol 15:1847–1862, 2015). The main tool in obtaining this result is establishing the Knight Move Conjecture [(Algebr Geom Topol 2:337-370, 2002), Conjecture 1] for the class of quasi-alternating links.

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1. Introduction

The class of alternating links has played an important role in the development of knot theory since its early age. In particular, the study of their Jones polynomials led to the solution of long-lasting conjectures in knot theory. Thistlewaite [4] proved that the Jones polynomial of any prime alternating link, which is not a $(2,p)$-torus link, is alternating and has no gaps. Moreover, the breadth of the Jones polynomial of any connected alternating link is equal to its crossing number. Alternating links are also known to have simple Khovanov and link Floer homologies. Indeed, the Khovanov homology of a given alternating link $L$ is entirely determined by its signature $\sigma_L$ and its Jones polynomial $V_L(t)$. Similarly, the link Floer homology of any alternating link $L$ is entirely determined by its signature $\sigma_L$ and its Alexander polynomial $\Delta_L(t)$. Furthermore, Ozsváth and Szabó [5] studied the Heegaard Floer Homology of the branched double-cover of alternating links and proved that this homology is determined by the determinant of the link, $\det(L)$. This homological property extends to a larger family of links called quasi-
alternating. Unlike alternating links which admit a simple diagrammatic definition, quasi-alternating links are defined recursively as follows:

**Definition 1.1.** The set $Q$ of quasi-alternating links is the smallest set satisfying the following properties:

- The unknot belongs to $Q$.
- If $L$ is a link with a diagram $D$ containing a crossing $c$ such that
  1. both smoothings of the diagram $D$ at the crossing $c$, $L_0$ and $L_1$ as in Fig. 1 belong to $Q$, and
  2. $\det(L_0), \det(L_1) \geq 1$,
  3. $\det(L) = \det(L_0) + \det(L_1)$; then $L$ is in $Q$ and in this case we say $L$ is quasi-alternating at the crossing $c$ with quasi-alternating diagram $D$.

It is impossible to use the above definition to show that a given link is not quasi-alternating. As an alternative, several obstructions for a link to be quasi-alternating have been introduced through the past two decades. Many of these have been established for alternating links first, then extended to this new class of links. Some of these main obstructions are listed here.

1. the branched double-cover of any quasi-alternating link is an $L$-space [5, Proposition. 3.3];
2. the space of branched double-cover of any quasi-alternating link bounds a negative definite 4-manifold $W$ with $H_1(W) = 0$ [5, Proof of Lemma. 3.6];
3. the $\mathbb{Z}/2\mathbb{Z}$ knot Floer homology group of any quasi-alternating link is $\sigma$-thin [6, Theorem. 2];
4. the reduced ordinary Khovanov homology group of any quasi-alternating link is $\sigma$-thin [6, Theorem. 1];
5. the reduced odd Khovanov homology group of any quasi-alternating link is $\sigma$-thin [7, Remark after Proposition. 5.2];
6. the determinant of any quasi-alternating link is bigger than the degree of its $Q$-polynomial [2, Theorem 2.2]. This inequality was sharpened later to the determinant minus one is bigger than or equal to the degree of the $Q$-polynomial with equality holds only for $(2, n)$-torus links [8, Theorem 1.1].

The main purpose of this paper is to introduce new obstruction for a link to be quasi-alternating in terms of Khovanov homology. More precisely, we prove that if the differential grading of the Khovanov homology of any quasi-alternating link has a gap, then the length of this gap is one. As an immediate application, we obtain that any gap in the Jones polynomial of any quasi-alternating link has length one. Moreover, a lower bound for the determinant of any such link is obtained in terms of the breadth of its Jones polynomial. The main result of this paper totally relies on establishing the Knight Move Conjecture [3, Conjecture 1] for the class of quasi-alternating links.

Consequently, we prove that certain links are not quasi-alternating as the differential grading of their Khovanov homology has a gap of length big-
ger than one. On the other hand, we show that some subclasses of quasi-alternating links have no gaps in the differential grading of their Khovanov homology. This leads us to suggest Conjecture 4.13 that implies both Conjecture 2.3 in [1] and Conjecture 3.8 in [2].

This paper is organized as follows. In Sect. 2, we introduce some background and notations needed for the rest of the paper. In Sect. 3, we give the proof of our main result. Finally, Sect. 4 will be devoted to give some applications and consequences of our main result.

2. Background and Notations

In this section, we introduce some notations and definitions, and we review some properties of Khovanov homology needed in this paper. Without loss of generality, we assume that the three unoriented links \( L, L_0 \) and \( L_1 \) are given according to the scheme in Fig. 1. The crossing \( c \) of the link \( L \) will be called of type I if it looks like the crossing in Fig. 1, otherwise it will be of type II.

For an oriented link \( L \), we always assume that we smooth a positive crossing according to the scheme in Fig. 2.

All the results of this paper discuss the case where the crossing \( c \) is of type I and oriented positively unless mentioned otherwise. Similar results can be obtained in the other cases by taking the mirror image of \( L \) if it is required.

2.1. The Jones Polynomial

The Jones polynomial \( V_L(t) \) is an invariant of oriented links. It is a Laurent polynomial with integral coefficients that can be defined in several ways. In this subsection, we shall briefly recall the definition of this polynomial in terms of the Kauffman bracket and review some of its properties needed in the sequel.

**Definition 2.1.** The Kauffman bracket polynomial is a function from the set of unoriented link diagrams in the oriented plane to the ring of Laurent polynomials with integer coefficients in an indeterminate \( A \). It maps a link \( L \) to \( \langle L \rangle \in \mathbb{Z}[A^{-1}, A] \) and it is defined by the following relations:

1. \( \langle \bigcirc \rangle = 1 \),
2. \( \langle \bigcirc \cup L \rangle = (-A^{-2} - A^2) \langle L \rangle \),
3. \( \langle L \rangle = A \langle L_0 \rangle + A^{-1} \langle L_1 \rangle \),

\[ \begin{align*}
\text{L} & \quad \text{L}_0 & \quad \text{L}_1
\end{align*} \]

Figure 1. The link diagram \( L \) at the crossing \( c \) and its smoothings \( L_0 \) and \( L_1 \) respectively
where $\bigcirc$, in relation 2 above, denotes a trivial circle disjoint from the rest of the link, and $L, L_0$, and $L_1$ represent three unoriented links which are identical everywhere except in a small region where they are as indicated in Fig. 1.

Given an oriented link diagram $L$, we let $x(L)$ to denote the number of negative crossings and $y(L)$ to denote the number of positive crossings in $L$ according to the scheme in Fig. 2. The writhe of link diagram $L$ is defined to be the integer $w(L) = y(L) - x(L)$.

Definition 2.2. The Jones polynomial $V_L(t)$ of an oriented link $L$ is the Laurent polynomial in $t^{\pm 1/2}$ with integer coefficients defined by

$$V_L(t) = ((-A)^{-3w(L)} \langle L \rangle)_{t^{1/2} = A^{-2}} \in \mathbb{Z}[t^{-1/2}, t^{1/2}],$$

where $\langle L \rangle$ is the Kauffman bracket of the unoriented link obtained from $L$ by ignoring the orientation.

2.2. Khovanov Invariant

This link invariant is a bigraded cohomology theory, with rational coefficients, that was first introduced by Khovanov in [9]. For an oriented link $L$, we denote the $i$-th homology group of the complex $\mathcal{C}(L)$ and $\mathcal{C}(L)$ by $\mathcal{H}^i(L)$ and $\mathcal{H}^j(L)$, respectively. Also, we denote the $j$-th graded component of $\mathcal{H}^i(L)$, and $\mathcal{H}^j(L)$ by $\mathcal{H}^{i,j}(L)$, and $\mathcal{H}^{i,j}(L)$, respectively. Therefore, we have

$$\mathcal{H}^i(L) = \oplus_{j \in \mathbb{Z}} \mathcal{H}^{i,j}(L), \text{ and } \mathcal{H}^j(L) = \oplus_{i \in \mathbb{Z}} \mathcal{H}^{i,j}(L).$$

The homology groups $\mathcal{H}(L)$ and $\mathcal{H}(L)$ are isomorphic up to some shifts. More precisely

$$\mathcal{H}^{i,j}(L) = \mathcal{H}^{i+x(L),j+2y(L)-y(L)}(L),$$

where $x(L)$ and $y(L)$ are as defined above. The graded Euler characteristic of this invariant is equal to the normalized Jones polynomial. More precisely,

$$\sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim \mathcal{H}^{i,j}(L) = (q^{-1} + q)V_L(t)_{t^{1/2} = -q}.$$

It is worth mentioning here that the graded Euler characteristic is indeed the evaluation, at $t = -1$, of the two-variable Khovanov polynomial invariant of the link defined first in [3] as follows:

$$\text{Kh}(L)(t, q) = \sum_{i,j \in \mathbb{Z}} t^i q^j \dim \mathcal{H}^{i,j}(L).$$

Figure 2. Negative and positive crossings respectively
2.3. Exact Sequence

It is clear that $C(L_0)$ and $C(L_1)[-1][-1]$ are subcomplexes of $C(L)$ and form a short exact sequence

$$0 \rightarrow C(L_1)[-1][-1] \rightarrow C(L) \rightarrow C(L_0) \rightarrow 0,$$

with degree preserving maps. This induces the following exact sequences:

$$\cdots \rightarrow H^{i-1,j-1}(L_0) \overset{\delta}{\rightarrow} H^{i,j}(L_1)[-1][-1] \rightarrow H^{i,j}(L) \rightarrow H^{i+1,j}(L_0) \rightarrow \cdots$$

Consequently, we obtain the following long exact sequences in homology as a result of the facts that $x(L_0) = x(L), y(L_0) = y(L) - 1, x(L_1) = x(L) + e$ and $y(L_1) = y(L) - e - 1$:

$$\cdots \rightarrow H^{i-1,j-1}(L_0) \overset{\delta}{\rightarrow} H^{i-e-1,j-3e-2}(L_1) \rightarrow H^{i,j}(L) \rightarrow H^{i,j-1}(L_0) \rightarrow \cdots,$$

where $e$ denotes the difference between the number of negative crossings in $L_1$ and the number of such crossings in $L$.

We recall from [10] that a link $L$ is said to be $H$-thin if its Khovanov homology $H(L)$ is supported on the two diagonals that satisfy $j - 2i = -\sigma(L) \pm 1$. It has been proved in [11] that all alternating links are $H$-thin. This fact has been later generalized to quasi-alternating links in [6, Theorem 1].

**Remark 2.3.** The result of [6, Theorem 1] shows that the reduced Khovanov homology of any quasi-alternating link is $\sigma$-thin over $\mathbb{Z}$. Indeed, this implies that the unreduced rational Khovanov homology that we consider in this paper is $H$-thin.

2.4. Lee Invariant

The Lee invariant is a variant of the rational Khovanov homology obtained from the same underlying complex with a differential $\Phi$ of degree difference $(1, 4)$. We let $H^i(L)$ to denote the $i$-th homology group of the complex $C(D)$ using Lee’s differential. We summarize all results about this invariant needed in the sequel in the following proposition:

**Proposition 2.4.** [11, Sect. 4] Let $L$ be an oriented link of $k$ components $L_1, L_2, \ldots, L_k$. Then

1. $\dim(H(L)) = 2^k$.
2. There is a generator of homology in degree $i$ such that

$$\dim H^i(L) = 2 \times \left| \left\{ E \subseteq \{2, 3, \ldots, n\} \mid \sum_{l \in E, m \notin E} 2\text{lk}(L_l, L_m) = i \right\} \right|$$

where the set $\{1, 2, \ldots, n\}$ indexes the set of components of the link $L$. The linking number $\text{lk}(L_l, L_m)$ is the linking number (for the original orientation) between the components $L_l$ and $L_m$. 
3. For any $H$-thin link $L$, we have a spectral sequence converging to $H(L)$ with $E_{1}^{s,t} = H^{s+t,s}(L)$ and $E_{2} = H(L)$. In other words,

$$H(L) \cong \frac{\text{Ker}(\Phi : H(L) \to H(L))}{\text{Image}(\Phi : H(L) \to H(L))}.$$ 

Our main goal in this paper is to prove the following theorem and give some consequences in the last section.

**Theorem 2.5.** The length of any gap in the differential grading of the Khovanov homology of any quasi-alternating link is one.

3. **Proof of the Main Theorem**

The crucial step in the proof of Theorem 2.5 relies on showing that the Lee homology of any quasi-alternating link is bigraded and therefore the Khovanov homology of such a link satisfies the Knight Move Conjecture [3, Conjecture 1].

A simple way to describe the Khovanov homology of a given link is to locate a non negative integer $n$ at the point $(i, j)$, where $n$ represents the dimension of $H_{i,j}^{L}$ over $\mathbb{Q}$. An example is illustrated in Table 1. Based on this description of Khovanov homology, we introduce the following definition.

**Definition 3.1.** For a given link $L$,

1. We say that the differential grading of the Khovanov homology of $L$ has a gap of length $s$ if there are $s$ consecutive columns of the Khovanov homology of the link $L$ with zero entries and these columns correspond to vertical lines of $x$-intercepts $i, i+1, i+2, \ldots, i+s-1$ such that $i_{\text{min}} + 1 \leq i < i+1 < i+2 < \cdots < i+s-1 \leq i_{\text{max}} - 1$. This is equivalent to say that $H^{i-1}(L) \neq 0$ and $H^{i+s}(L) \neq 0$ while $H^{m}(L) = 0$ for $m = i, i+1, \ldots, i+s-1$.

2. We say that the quantum grading of the Khovanov homology of $L$ has a gap of length $s$ if there are $s$ consecutive rows of the Khovanov homology of the link $L$ with zero entries and these rows correspond to horizontal lines of $y$-intercepts $j, j+2, j+4, \ldots, j+2s-2$ such that $j_{\text{min}} + 2 \leq j < j+2 < j+4 < \cdots < j+2s-2 \leq j_{\text{max}} - 2$.

3. We say that a polynomial has a gap of length $s$ if this polynomial has two monomials of degrees $i$ and $i+s+1$ of nonzero coefficients and all the monomials of degree $m$ have zero coefficients for $m = i+1, i+2, \ldots, i+s$.

4. We say that there is a gap between the two polynomials $f(x)$ and $g(x)$ of length $s$ if either $M = \max \text{deg}(f(x)) < m = \min \text{deg}(g(x))$ or $M = \max \text{deg}(g(x)) < m = \min \text{deg}(f(x))$ and all the monomials $x^{M+1}, x^{M+2}, \ldots, x^{M+s}s = x^{m-1}$ have zero coefficients in the two polynomials.

To illustrate this definition, we consider the Khovanov homology of the trefoil knot $3_1$ given in Table 1. This homology has a gap of length one in the differential grading as $H^{-1}(3_1) = 0$ while both $H^{-2}(3_1)$ and $H^{0}(3_1)$ are nontrivial. Also, the Jone polynomial of the trefoil that is known to be
Table 1. The Khovanov homology of the trefoil knot

| j  | i |
|----|---|
| −3 | −3|
| −2 | −1|
| −1 | 0 |

−t^−4 + t^−3 + t^−1 has a gap of length one as the monomial t^−2 has zero coefficient while the monomials t^−1 and t^−3 have nonzero coefficients.

The signatures of the links L, L_0 and L_1 are related according to the following lemma if the link L is quasi-alternating at the crossing c.

**Lemma 3.2.** [6, Lemma 3] Let L be a quasi-alternating link and L_0 and L_1 be its smoothings at the crossing c where it is quasi-alternating. Then we have,

\[ \sigma(L_0) = \sigma(L) + 1 \]

and

\[ \sigma(L_1) = \sigma(L) + e, \]

where e is as defined in Sect. 2.

Unlike Khovanov homology which is bigraded as explained in Sect. 2, Lee homology theory is known to be filtered rather than bigraded. However, the Lee homology of a quasi-alternating link can be endowed with a bigrading structure as shown in the following theorem.

**Theorem 3.3.** If L is a quasi-alternating link, then the Lee homology H(L) is bigraded and \( H^{i,j}(L) \cong H^{i,j+2}(L) \cong \mathbb{Q}^n \) for some non-negative integer n and for any i and j such that \( j - 2i = -\sigma(L) - 1 \).

**Proof.** The link L is H-thin as it is quasi-alternating, [6, Theorem 1]. Consequently, the bigrading of H(L) induces a bigrading on H(L) as a result of Part 3 of Proposition 2.4. We use induction on the determinant of the link L to prove the second claim in the theorem. It is clear that the result holds when det(L) = 1 since the only quasi-alternating link of determinant 1 is the unknot [12, Prop. 3.2]. Now suppose that the result holds for all quasi-alternating links of determinant less than \( m = \det(L) \). In particular, it holds for the links L_0 and L_1. It is easy to see that if \( H^i(L) \neq 0 \) then \( H^{i\pm1}(L) = 0 \) as a result of Part 2 of Proposition 2.4. If we use this fact in the long exact sequence of the cohomology groups H(L), H(L_0) and H(L_1) that is mentioned for the first time in the proof of [11, Theorem 4.2], then we obtain

\[ 0 \rightarrow H^{i-1}(L_0) \rightarrow H^{i-1}(L_1) \rightarrow H^i(L) \rightarrow H^i(L_0) \rightarrow H^i(L_1) \rightarrow 0. \quad (6) \]

As a result of the facts that \( x(L_0) = x(L), y(L_0) = y(L) - 1, x(L_1) = x(L) + e \) and \( y(L_1) = y(L) - e - 1 \), we obtain the following exact sequences:
and
\[
0 \rightarrow H^{i-1}(L_0) \rightarrow H^{i-e-1}(L_1) \rightarrow H^i(L) \rightarrow H^i(L_0) \rightarrow H^{i-e}(L_1) \rightarrow 0. \quad (7)
\]

The above exact sequence still holds if we replace \( j \) by \( j + 2 \) as follows:
\[
0 \rightarrow H^{i-1,j-1}(L_0) \rightarrow H^{i-e-1,j-3e-2}(L_1) \rightarrow H^{i,j}(L) \rightarrow H^{i,j-1}(L_0) \rightarrow H^{i-e,j-3e-2}(L_1) \rightarrow 0 \quad (8)
\]

It is easy to see that \( H^{i-1,j+1}(L_0) = 0 = H^{i-e,j-3e-2}(L_1) \) since \( (j+1) - 2(i-1) = j - 2i + 3 = -\sigma(L) - 1 + 3 = -\sigma(L_0) + 3 \neq -\sigma(L_0) \pm 1 \) and \( (j-3e - 2) - 2(i-e) = j - 2i - e - 2 = -\sigma(L) - 1 - e - 2 = -\sigma(L_1) - 3 \neq -\sigma(L_1) \pm 1 \), respectively provided that \( j - 2i = -\sigma(L) - 1 \). Also as a result of the induction hypotheses on \( L_0 \) and \( L_1 \), we obtain \( H^{i-e-1,j-3e-2}(L_1) \cong H^{i-e-1,j-3e}(L_1) \) and \( H^{i,j-1}(L_0) \cong H^{i,j+1}(L_0) \) since \( (j - 3e) - 2(i - e - 1) = j - 2i - e + 2 = -\sigma(L) - 1 - e + 2 = -\sigma(L_1) + 1, (j - 3e - 2) - 2(i - e - 1) = j - 2i - e = -\sigma(L_1) - 1 - 1 = -\sigma(L_0) - 1, (j + 1) - 2i = j - 2i + 1 = -\sigma(L) - 1 + 1 = -\sigma(L_0) + 1 \). Now we have four cases to consider:

1. If \( H^{i-e-1,j-3e-2}(L_1) \cong H^{i-e-1,j-3e}(L_1) = 0 \) and \( H^{i,j-1}(L_0) \cong H^{i,j+1}(L_0) = 0 \), then \( H^{i,j+2}(L) = 0 \).

2. If \( H^{i-e-1,j-3e-2}(L_1) \cong H^{i-e-1,j-3e}(L_1) \neq 0 \) and \( H^{i,j-1}(L_0) \neq 0 \), then as a result of the fact that if \( H^i(L) \neq 0 \) then \( H^{i-1}(L_0) = 0 \) obtained from Part 2 of Proposition 2.4 we conclude that \( H^{i-1,j-1}(L_0) = 0 = H^{i-e,j-3e}(L_1) \). This implies that \( H^{i,j}(L) \cong H^{i,j-1}(L_0) \oplus H^{i-e-1,j-3e-2}(L_1) \cong H^{i,j+1}(L_0) \oplus H^{i-e-1,j-3e}(L_1) \cong H^{i,j+2}(L) \).

3. If \( H^{i-e-1,j-3e-2}(L_1) \cong H^{i-e-1,j-3e}(L_1) = 0 \) and \( H^{i,j-1}(L_0) \neq 0 \), then as a result of the fact that if \( H^i(L) = 0 \) then \( H^{i+1}(L) = 0 \) obtained from Part 2 of Proposition 2.4 we conclude that \( H^{i-1,j-1}(L_0) = 0 \). In particular, we obtain \( H^{i,j-1}(L_0) \cong H^{i,j}(L) \oplus H^{i-e,j-3e-2}(L_1) \cong H^{i,j}(L) \) and \( H^{i,j+1}(L_0) \cong H^{i,j+2}(L) \oplus H^{i-e,j-3e}(L_1) \). To prove the required result it is enough to prove that \( H^{i-e,j-3e}(L_1) = 0 \). We prove this by contradiction. If we assume that \( H^{i-e,j-3e}(L_1) \neq 0 \), then \( H^{i-e}(L_1) \neq 0 \) as a result of the fact that \( H^{i-e}(L_1) \cong H^{i-e,j-3e+2}(L_1) \oplus H^{i-e,j-3e}(L_1) \) obtained from Part 3 of Proposition 2.4 because \( L_1 \) is \( H \)-thin as it is quasi-alternating. Now according to the long exact sequence 7 and the assumption in this case we conclude that \( H^i(L_0) \cong H^i(0) \oplus H^{i-e}(L_1) \cong (H^{i,j}(L) \oplus H^{i,j+2}(L)) \oplus (H^{i-e,j-3e}(L_1) \oplus H^{i-e,j-3e+2}(L_1)) \). Thus we obtain \( 0 = H^{i-e,j-3e+2}(L_1) \cong H^{i-e,j-3e}(L_1) \), where the isomorphism follows by the induction hypothesis on the link \( L_1 \).


4. If \( H^{i-e-1,j-3e-2}(L_1) \cong H^{i-e-1,j-3e}(L_1) \neq 0 \) and \( H^{i,j-1}(L_0) \cong H^{i,j+1}(L_0) = 0 \). This case can be discussed using a similar argument as in the previous case.

\[ \square \]

Remark 3.4. In fact, if the \( E_2^{\cdot,\cdot} \)- and the \( E_{\infty}^{\cdot,\cdot} \)-pages of the spectral sequence of the double complex \((C(L),d(L),\Phi(L))\) are isomorphic, then the Lee homology \( H(L) \) is bigraded since these two pages are isomorphic to \( \ker(\Phi:H(L)\to H(L)) \) and \( H(L) \), respectively. As a result of this fact, we can define the Lee polynomial invariant of such a link as follows:

\[
Le(L)(t,q) = \sum_{i,j \in \mathbb{Z}} t^i q^j \dim H^{i,j}(L).
\]

(10)

As a consequence of Theorem 3.3 and Part 3 of Proposition 2.4, we establish the Knight Move Conjecture [3, Conjecture 1] for the class of quasi-alternating links. This generalizes [11, Theorem 4.5] to this class of links.

Corollary 3.5. Let \( L \) be an oriented quasi-alternating link of components \( L_1, L_2, \ldots, L_n \). Let \( l_{jk} \) denote the linking number of the components \( L_j \) and \( L_k \). Then, we have

\[
Le(L)(t,q) = q^{-\sigma(L)}(q^{-1} + q) \left( \sum_{E \subset \{2,3,\ldots,n\}} (tq^2)^{\sum_{j \in E, k \notin E} 2l_{jk}} \right).
\]

Therefore, we obtain

\[
Kh(L)(t,q) = q^{-\sigma(L)}((q^{-1} + q) \left( \sum_{E \subset \{2,3,\ldots,n\}} (tq^2)^{\sum_{j \in E, k \notin E} 2l_{jk}} \right)
+ (q^{-1} + tq^2)q) \text{Kh}'(L)(tq^2)),
\]

for some polynomial \( \text{Kh}'(L) \).

One can define two types of breadth for the Khovanov homology of any link \( L \) based on the differential and quantum gradings of Khovanov homology. These two types are denoted hereafter by \( \text{breadth}_i(\mathcal{H}(L)) \) and \( \text{breadth}_j(\mathcal{H}(L)) \) and are defined to be the difference between the maximum and minimum corresponding gradings. For instance, the \( \text{breadth}_i(\mathcal{H}(3_1)) = 3 \) while \( \text{breadth}_j(\mathcal{H}(3_1)) = 8 \) as one can see from Table 1. In the case of quasi-alternating links, these two types of breadth are related as follows.

Corollary 3.6. If \( L \) is quasi-alternating link, then \( \text{breadth}_j(\mathcal{H}(L)) = 2 \text{breadth}_i(\mathcal{H}(L)) + 2 \).

Proof. In the above description of Khovanov homology, we claim that the most left-bottom nonzero entry that comes right after any gap or at the beginning of the Khovanov homology of \( L \) will be on the lower diagonal with \( j - 2i = -\sigma(L) - 1 \) and the most right-top nonzero entry right before any gap or at the end of the Khovanov homology of \( L \) will be on the upper diagonal with \( j - 2i = -\sigma(L) + 1 \). Each one of these two entries is the only entry in that row.
We show the first case of the claim and the second case of the claim follows using a similar argument. Suppose that the most left-bottom entry shares the same row with another entry to the right. In this case, the leftmost entry will be on the upper diagonal and therefore it will be the only entry to survive in $H^i(L)$ in that column, where $i$ is the differential grading of the most left-bottom entry. This is impossible according to the result of Theorem 3.3.

Thus the minimum quantum grading $j_{\text{min}}$ is related to the minimum differential grading $i_{\text{min}}$ by $j_{\text{min}} - 2i_{\text{min}} = -\sigma(L) - 1$ and the maximum quantum grading $j_{\text{max}}$ is related to the maximum differential grading $i_{\text{max}}$ by $j_{\text{max}} - 2i_{\text{max}} = -\sigma(L) + 1$. Now the result follows because breadth$_j(H(L)) = j_{\text{max}} - j_{\text{min}} = (2i_{\text{max}} - \sigma(L) + 1) - (2i_{\text{min}} - \sigma(L) - 1) = 2(i_{\text{max}} - i_{\text{min}}) + 2 = 2$ breadth$_i(H(L)) + 2$. □

We can give an explicit formula for the Jones polynomial of any quasi-alternating link in terms of the integers that are given in the above description of its Khovanov homology. Indeed, these integers can be expressed in terms of the dimensions of the Khovanov and Lee homologies of the given link. More precisely, if $q_l$ is the $(i_{\text{min}} + l - 1, j_{\text{min}} + 2l - 2)$-entry of the lower diagonal in the above description of $H(L)$ after taking the quotient by $H(L)$. Then we have

\begin{align*}
\text{Lemma 3.7.} & \text{ If } L \text{ is a quasi-alternating link, then } \\
a_j & = \begin{cases} 
\dim H^{i_{\text{min}}}L - \dim H^{i_{\text{min}}}L, & \text{if } l = 1, \\
\dim H^{i_{\text{min}}}L - \dim H^{i_{\text{min}}}L - a_{l-1}, & \text{if } 1 < l < i_{\text{max}} - i_{\text{min}}, \\
\dim H^{i_{\text{max}}}L - \dim H^{i_{\text{min}}}L, & \text{if } i = i_{\text{max}} - i_{\text{min}}.
\end{cases}
\end{align*}

\text{Proof.} The result is a straightforward consequence of the fact that $L$ is $H$-thin and Part 3 of Proposition 2.4. □

\text{Corollary 3.8.} If the link $L$ is quasi-alternating, then its Jones polynomial is given by

\begin{align*}
V_L(t) & = (-1)^{i_{\text{min}}}a_1q^{j_{\text{min}}+1} + (-1)^{i_{\text{max}}}a_{i_{\text{max}} - i_{\text{min}}}q^{j_{\text{max}}-1} \\
& + \sum_{j} n_j q^j + \sum_{l=1}^{j_{\text{max}} - j_{\text{min}} - 2} (-1)^{j_{\text{min}} + l}(a_l + a_{l+1})q^{j_{\text{min}}+2l+1}q^{\text{max}+2l+1}q=q^{-\sqrt{T}}, \quad (11)
\end{align*}

where the first sum is taken over all $j_{\text{min}} + 1 \leq j \leq j_{\text{max}} - 1$ such that $H^{i,j+1}(L) \cong H^{i,j-1}(L) \cong \mathbb{Q}^{n_j}$ for some positive integer $n_j$.

\text{Proof.} The entries at each column of $H(L)$ contribute $n_j(q^m + q^{m+2}) = n_j q^{m+1}(q^{-1} + q)$ to $\text{Kh}(-1, q)$ for some integer $m$. Now the first sum follows after we divide by $q^{-1} + q$ and go over all the columns of $H(L)$.

As a result of Part 3 of Proposition 2.4, we conclude that there is a pairing (Knight move) in the entries of $H(L)/H(L)$. In particular, the $(i_{\text{min}} + l - 1, j_{\text{min}} + 2l - 2)$-entry of the lower diagonal is equal to the $(i_{\text{min}} + l, j_{\text{min}} + 2l + 2)$-entry for $1 \leq l \leq i_{\text{max}} - i_{\text{min}}$. It is not too hard to see that the $(i_{\text{min}} + l - 1, j_{\text{min}} + 2l - 2)$-entry is simply $a_l$ as a result of Lemma 3.7. These two entries contribute $(-1)^{i_{\text{min}} + l - 1}a_l(q^{j_{\text{min}}+2l-2} - q^{j_{\text{min}}+2l+2}) = (-1)^{i_{\text{min}} + l - 1}a_l(q^{-1} + q)q^{j_{\text{min}}+2l-1} - q^{j_{\text{min}}+2l+1}$ to the polynomial $\text{Kh}(-1, q)$. The second sum follows after we divide by $q^{-1} + q$ and go over all such entries. □
**Proposition 3.11.** If there is a gap between the two polynomials $q_i$ in the above formula follows easily from the fact that $i \in 3.11$. We have three cases to consider according to the value of $l$. The coefficient $l$ is equal to zero. Let $H_{\min}(L)$ denote the differential supports of $H_{\min}(L_1)$ and the differential supports of $H_{\min}(L)$, if $j = j_{\max} - 1$, and hence the result of the first part follows and the second part follows easily from the fact that $i$ has to be even if $\dim H^i(L) \neq 0$. \hfill $\Box$

**Proposition 3.10.** Let $L$ be a quasi-alternating link. The differential grading of $H(L)$ has a gap of length $s$ if and only if $V_L(t)$ has a gap of length $s$.

**Proof.** The result can be proved easily from the following three equivalences which are direct consequences of Lemma 3.9

1. $\dim H_{\min}^{\min}(L) = a_1 + \dim H_{\min}^{\min}(L) \neq 0$ is equivalent to $a_1 + \frac{1}{2} \dim H_{\min}^{\min}(L) \neq 0$ which is also equivalent to $b_{j_{\min} + 1} = (-1)^{i_{\min} + 1} \dim H_{\min}^{\min}(L) \neq 0$.

2. $\dim H^i(L) = a_{i-i_{\min}} + a_{i-i_{\min}+1} + \frac{1}{2} \dim H^i(L) \neq 0$ is equivalent to $a_{i-i_{\min}} + \frac{1}{2} \dim H^i(L) \neq 0$ which is also equivalent to $b_j = (-1)^{i_{\min}}(a_{i-i_{\min}} + a_{i-i_{\min}+1}) + \frac{1}{2} \dim H^i_{\min}(L) \neq 0$ where $i_{\min} < i < i_{\max}$ and $j = 2i - \sigma(L)$.

3. $\dim H^i_{\max}(L) = a_{i_{\max} - i_{\min} + 1} + \frac{1}{2} \dim H^i_{\max}(L) \neq 0$ is equivalent to $a_{i_{\max} - i_{\min}} + \frac{1}{2} \dim H^i_{\max}(L) \neq 0$ which is also equivalent to $b_{j_{\max} - 1} = (-1)^{i_{\max}}a_{i_{\max} - i_{\min}} + \frac{1}{2} \dim H^i_{\max}(L) \neq 0$.

According to the long exact sequence in Equation 5, the differential support of $H(L)$ is included in the union of the differential support of $H(L_0)$ and the differential support of $H(L_1)$ after doing the appropriate shift. We denote the differential supports of $H(L_0)$ and $H(L_1)$ by $S_0$ and $S_1$, respectively. Also, we let $S \{ k \}$ to denote the set $S$ shifted to the left by $k$ that is $i \in S \{ k \}$ if $i + k \in S$.

**Proposition 3.11.** If there is a gap between the two polynomials $A(L_0)$ and $A^{-1}(L_1)$ of length bigger than three, then $|i_1 - i_0| > 1$ for any $i_0 \in S_0$ and $i_1 \in S_1 \{ e + 1 \}$. In particular, the sets $S_0$ and $S_1 \{ e + 1 \}$ are disjoint and this implies that

$$
H^i(L) \cong \begin{cases} 
H^i(L_0), & \text{if } i \in S_0, \\
H^{i-e-1}(L_1), & \text{if } i \in S_1 \{ e + 1 \}, \\
0, & \text{otherwise}.
\end{cases}
$$
Proof. If there is a gap between the two polynomials $A(L_0)$ and $A^{-1}(L_1)$ of length bigger than three, then either $M = \maxdeg(A(L_0)) < \mindeg(A^{-1}(L_1)) = m$ or $M = \maxdeg(A^{-1}(L_1)) < \mindeg(A(L_0)) = m$ such that $A^{M+1}, A^{M+2}, \ldots, A^{M+s}$ as monomials in both polynomials have zero coefficients with $s$ at least seven as the powers of the monomials of $\langle L \rangle$ differ by multiples of four. We discuss the first case and the second case follows using a similar argument. Notice that with the substitution $A^{-2} = t^{1/2} = -q$, we have

$$(-A)^{-3w(L)}(-A^{-2} - A^2) \langle L \rangle = (-t^{-1/2} - t^{1/2})V_L(t) = (q + q^{-1})V_L(t)$$

$$= \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim H^{i,j}(L).$$

In addition, since $x(L_0) = x(L)$ and $y(L_0) = y(L) - 1$, we have $w(L_0) = w(L) - 1$. Now with the same substitution as before, we obtain

$$A(-A)^{-3w(L)}(-A^{-2} - A^2) \langle L_0 \rangle = A(-A)^{-3(w(L_0)+1)}(-A^{-2} - A^2) \langle L_0 \rangle$$

$$= -t^{1/2}(-t^{-1/2} - t^{1/2})V_{L_0}(t)$$

$$= q(q + q^{-1})V_{L_0}(t)$$

$$= \sum_{i,j \in \mathbb{Z}} (-1)^i q^{i+1} \dim H^{i,j}(L_0)$$

$$= \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim H^{i,j-1}(L_0).$$

Moreover, from the facts that $x(L_1) = x(L) + e$ and $y(L_1) = y(L) - e - 1$, we have $w(L_1) = w(L) - 2e - 1$ where $e$ is as defined in Lemma 3.2. Now also with the same substitution as before, we obtain

$$A^{-1}(-A)^{-3w(L)}(-A^{-2} - A^2) \langle L_1 \rangle = A^{-1}(-A)^{-3(w(L_1)+2e+1)}(-A^{-2} - A^2) \langle L_1 \rangle$$

$$= -t^{\frac{3e+2}{2}}(-t^{-1/2} - t^{1/2})V_{L_1}(t)$$

$$= (-1)^{3e-1}q^{3e+2}(q + q^{-1})V_{L_1}(t)$$

$$= \sum_{i,j \in \mathbb{Z}} (-1)^{i-e} q^{j+3e+2} \dim H^{i,j}(L_1)$$

$$= \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim H^{i-e-1,j-3e-2}(L_1).$$

As a direct consequence of the previous argument and the fact that $\langle L \rangle = A(L_0) + A^{-1}(L_1)$, we obtain the following equality that can be also confirmed from the fact that the Euler characteristic of any long exact sequence vanishes

$$\sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim H^{i,j}(L) = \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim H^{i,j-1}(L_0) +$$

$$\sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim H^{i-e-1,j-3e-2}(L_1),$$

$$\sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim H^{i,j}(L).$$
Now, if \((i_{\max}, j_{\max})\) are the coordinates of the most right-top nonzero entry of \(\mathcal{H}(L_0)\) that is part of the upper diagonal as pointed out in the proof of Corollary 3.6, then we obtain \(\mathcal{H}^{i_{\max}+1,j_{\max}}_0(L) \cong \mathcal{H}^{i_{\max},j_{\max}}(L_0)\) from the long exact sequence in Eq. (5). The last result follows because \(\dim \mathcal{H}^{i_{\max}+1,j_{\max}+3e-1}(L_1) = 0 = \dim \mathcal{H}^{i_{\max},j_{\max}+3e-1}(L_1)\) as they are the coefficients of \(q^{j_{\max}+1}\) in the polynomial \(\sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim \mathcal{H}^{i,j-1}(L_0)\) with

\[
\min \deg \left( \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim \mathcal{H}^{i-1,j-3e-2}(L_1) \right) > \max \deg \left( \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim \mathcal{H}^{i,j-1}(L_0) \right) = j_{\max} + 1.
\]

As a result of the fact \(j_{\max} - 2i_{\max} = -\sigma(L_0) + 1\), we obtain \(j_{\max} + 1 - 2i_{\max} = j_{\max} - 2i_{\max} + 1 = -\sigma(L_0) + 1 + 1 = -(\sigma(L_0) - 1) + 1 = -\sigma(L) + 1\). In other words, the last statement is equivalent to say that the above nonzero entry of the Khovanov homology of \(L\) with coordinates \((i_{\max}, j_{\max} + 1)\) will also lie on the upper diagonal of the Khovanov homology of \(L\).

Similarly, if \((i_{\min}, j_{\min})\) are the coordinates of the most left-bottom nonzero entry of \(\mathcal{H}(L_1)\) that is part of the lower diagonal as pointed in the proof of Corollary 3.6, then we obtain \(\mathcal{H}^{i_{\min}+e+1,j_{\min}+3e+2}_0(L) \cong \mathcal{H}^{i_{\min},j_{\min}}(L_1)\) from the long exact sequence in Eq. (5). The last result follows because \(\dim \mathcal{H}^{i_{\min}+e+1,j_{\min}+3e+1}(L_0) = 0 = \dim \mathcal{H}^{i_{\min}+e+1,j_{\min}+3e+1}(L_0)\) as they are the coefficients of \(q^{j_{\min}+3e+2}\) in the polynomial \(\sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim \mathcal{H}^{i,j-1}(L_0)\) with

\[
\min \deg \left( \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim \mathcal{H}^{i-1,j-3e-2}(L_1) \right) = \min \deg \left( \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim \mathcal{H}^{i,j-1}(L_0) \right) + 3e + 2 = \min \deg \left( \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim \mathcal{H}^{i,j-1}(L_0) \right) + j_{\min} + 1 = \min \deg \left( \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim \mathcal{H}^{i,j-1}(L_0) \right) .
\]

Since \(j_{\min} - 2i_{\min} = -\sigma(L_1) + 1\), we obtain \(j_{\min} + 3e + 2 - 2(i_{\min} + e + 1) = j_{\min} - 2i_{\min} + e = -\sigma(L_1) + e - 1 = -(\sigma(L_1) - e) - 1 = -\sigma(L) - 1\). In other words, the last statement is equivalent to say that the above nonzero entry of the Khovanov homology of \(L\) with coordinates \((i_{\min} + e + 1, j_{\max} + 3e + 2)\) will also lie on the lower diagonal of the Khovanov homology of \(L\).

Thus, we conclude that \(2(i_{\min} + e + 1 - i_{\max}) = j_{\min} + 3e + 2 + \sigma(L) + 1 - j_{\max} - \sigma(L) + 1 = j_{\min} + 3e + 2 - j_{\max} + 2 + 2 = 4\), where the inequality follows from the fact that \(j_{\min} + 3e + 2 - j_{\max} + 1 > 0\) that is obtained from Eq. (12). The last inequality implies that \(i_{\min} + e + 1 \geq i_{\max} + 2\) and hence
we conclude that \(|i_1 - i_0| \geq |(i_{\text{min}} + e + 1) - i_{\text{max}}| \geq 2\) for any \(i_0 \in S_0\) and \(i_1 \in S_1\{e + 1\}\) as a result of \(i_0 \leq i_{\text{max}}\) and \(i_{\text{min}} + e + 1 \leq i_1\).

To compute the Khovanov homology of the link \(L\), we need to consider only \(i \in S_0 \cup S_1\{e + 1\}\) because otherwise we obtain \(0 = \mathcal{H}^{i-e-1}(L_1) \to \mathcal{H}^i(L) \to \mathcal{H}^i(L_0) = 0\) from the long exact sequence and this implies that \(\mathcal{H}^i(L) = 0\). The other two cases are either \(i \in S_0\) or \(i \in S_1\{e + 1\}\) but not both since the two sets \(S_0\) and \(S_1\{e + 1\}\) are disjoint. We discuss the first case and the other case follows using a similar argument. If \(i \in S_0\), then \(i + e + 1, i + e \notin S_1\) as a result of what we just proved. Hence the result follows directly from the long exact sequence \(0 = \mathcal{H}^{i-e-1}(L_1) \to \mathcal{H}^i(L) \to \mathcal{H}^i(L_0) \to \mathcal{H}^{i-e-1}(L_1) = 0\).

**Corollary 3.12.** If there is a gap between the two polynomials \(A(L_0)\) and \(A^{-1}(L_1)\) of length bigger than three, then we have

\[
\mathcal{H}^i(L) \cong \begin{cases} 
\mathcal{H}^i(L_0), & \text{if } i \in S_0, \\
\mathcal{H}^{i-e-1}(L_1), & \text{if } i \in S_1\{e + 1\}, \\
0, & \text{otherwise.}
\end{cases}
\]

**Proof.** The \(E_\infty\)-page of the spectral sequence of the double complex \((\mathcal{C}(L), d(L), \Phi(L))\) is simply the disjoint union of the \(E_\infty\)-pages of the spectral sequences for the double complexes for the links \(L_0\) and \(L_1\). Thus the Lee homology of the link \(L\) is simply the disjoint union of the Lee homologies of the links \(L_0\) and \(L_1\).

**Corollary 3.13.** If \(l_0 \neq l - 1\) or \(l_1 \neq l - 1\) where \(l, l_0\) and \(l_1\) are the number of components of \(L, L_0\) and \(L_1\) respectively and if there is a gap between the two polynomials \(A(L_0)\) and \(A^{-1}(L_1)\), then this gap has to be of length three.

**Proof.** If the gap is of length bigger than three, then the results of Corollary 3.12 and Part 1 of Proposition 2.4 yields \(2^l = \dim H(L) = \dim H(L_0) + \dim H(L_1) = 2^{l_0} + 2^{l_1}\). Obviously, this holds only if \(l_0 = l - 1 = l_1\).

The following lemma is well known about the Jones polynomial and it can be obtained by an argument similar to the proof of Proposition 3.11.

**Lemma 3.14.** The Jones polynomial of the link \(L\) at the crossing \(c\) satisfies one of the following skein relations:

1. If \(c\) is a positive crossing of type I, then \(V_L(t) = -t^{\frac{3}{2}} V_{L_0}(t) - t^{\frac{3}{2} + 1} V_{L_1}(t)\).
2. If \(c\) is a negative crossing and of type I, then \(V_L(t) = -t^{\frac{3}{2}} V_{L_0}(t) - t^{\frac{3}{2} - 1} V_{L_1}(t)\).
3. If \(c\) is a positive crossing and of type II, then \(V_L(t) = -t^{\frac{3}{2} + 1} V_{L_0}(t) - t^{\frac{3}{2}} V_{L_1}(t)\).
4. If \(c\) is a negative crossing of type II, then \(V_L(t) = -t^{-\frac{1}{2}} V_{L_0}(t) - t^{\frac{3}{2} - 1} V_{L_1}(t)\).

**Proposition 3.15.** Let \(L\) be a link and \(c\) be a crossing of this link consisting of two arcs of two different components. Then the gap between the polynomials \(A(L_0)\) and \(A^{-1}(L_1)\) is of length at most seven.
Proof. We assume that $c$ is a crossing between the two components $L_1$ and $L_2$ of the link $L$. We use induction on the length of the sequence of crossing switches required to turn the sublink of $L$ consisting of the components $L_1$ and $L_2$ into disjoint union of two links.

This sequence is a subset of the set of all crossings between the two components $L_1$ and $L_2$ of the link $L$. Each such crossing in this set consists of two arcs one coming from the component $L_1$ and the other one from $L_2$. Therefore, this set can be written as a disjoint union of two subsets one contains all crossings with the arc from $L_1$ is above the arc from $L_2$ and the other way around for the second one. A choice of the above sequence can be made to be either one of these two subsets among many other possible choices. Without loss of generality, we can assume that the crossing $c$ is the first element in this sequence and the link $K$ is obtained from $L$ by switching the crossing $c$. Therefore, the two links $L$ and $K$ are identical except at the crossing $c$.

From the induction hypothesis, the result holds for the link $K$ at the crossing $c$. In particular, the gap between the polynomials $A^{-1}(K_0)$ and $A(K_1)$ is of length at most seven. The result follows directly if this gap is of length less than or equal to three or if $\mindeg(A(K_1)) > \maxdeg(A^{-1}(K_0))$ noting that the links $K_0$ and $L_0$ are identical and the links $K_1$ and $L_1$ are also identical. Thus we can assume that this gap is of length exactly seven and $\mindeg(A^{-1}(K_0)) > \maxdeg(A(K_1))$.

The choice of the skein relation in Lemma 3.14 that we can apply at the crossing $c$ to evaluate $V_L(t)$ or $V_K(t)$ depends on the type of the crossing $c$ and the orientation of the two components $L_1$ and $L_2$ either in $K$ or $L$. The fact that the Jones polynomials of the same link with two different orientations are related by some phase gives us the freedom to choose the orientations on the two components $L_1$ and $L_2$ without affecting the length of the gap. Without loss of generality and choosing the appropriate orientations on the components $L_1$ and $L_2$, we can assume that the crossing $c$ is positive in the link $L$ and negative in the link $K$.

As a result of the assumption $\mindeg(A^{-1}(K_0)) > \maxdeg(A(K_1))$, we can conclude that $\mindeg(V_{L_0}(t)) > \maxdeg(t^{\frac{3}{2}}V_{L_1}(t))$. In this case, the gap between the polynomials $-t^{\frac{3}{2}}V_{L_0}(t)$ and $-t^{\frac{3}{2}}V_{L_1}(t)$ in the link $K$ is of length $(\mindeg(V_{L_0}(t)) - \frac{1}{2}) - (\maxdeg(V_{L_1}(t)) - 1) - 1 = \mindeg(V_{L_0}(t)) - \maxdeg(V_{L_1}(t)) - \frac{1}{2}$ and the gap between the polynomials $-t^{\frac{3}{2}}V_{L_0}(t)$ and $-t^{\frac{3}{2}}V_{L_1}(t)$ in the link $L$ is of length $(\mindeg(V_{L_0}(t)) + \frac{1}{2}) - (\maxdeg(V_{L_1}(t)) + 1) - 1 = \mindeg(V_{L_0}(t)) - \maxdeg(V_{L_1}(t)) - \frac{3}{2}$. Thus the result follows since the length of the gap in $L$ is smaller than the length of the gap in the link $K$. 

As a consequence of Corollary 3.13 and Proposition 3.15, we obtain the following Corollary:

**Corollary 3.16.** Let $L$ be a link. If there is a gap between the two polynomials $A(L_0)$ and $A^{-1}(L_1)$, then it has to be of length at most seven.
Proof of Theorem 2.5. We use induction on the determinant of the given quasi-alternating link to prove the statement of the theorem. If det(\(L\)) = 1, then the result holds since the only quasi-alternating link of determinant one is the unknot [12, Prop. 3.2]. Now, suppose that the result holds for all quasi-alternating links of determinant less than the determinant of the link \(L\). In particular, any gap in the differential gradings of \(H(L_0)\) and \(H(L_1)\) is of length one.

We have two cases to consider. The first case if there is a gap between the polynomials \(A(L_0)\) and \(A^{-1}(L_1)\) of length seven or less and this includes the case of having no gap. The second case corresponds to a gap of length bigger than seven. In any case, there is no cancellation between these two polynomials as a result of \(L\) being quasi-alternating. Thus any gap in the Jones polynomial of the link \(L\) is induced either by a gap in one of the two polynomials of \(L_0\) and \(L_1\) or by the gap between these two polynomials.

In the first case, it is easy to see that the Jones polynomial of \(L\) has no gap of length bigger than one as a result of the facts that the above polynomials have no gap of length bigger than seven from the induction hypothesis on the links \(L_0\) and \(L_1\) and that the gap between them is of length seven or less. Now the result follows from Proposition 3.10 since having a gap in the differential grading of \(H(L)\) of length bigger than one induces a gap of length bigger than one in the Jones polynomial of \(L\) and this contradicts what we know already about the Jones polynomial of \(L\).

In the second case, the gap between the two polynomials is of length bigger than seven. In this case and according to Corollary 3.13, we can assume that \(L\) is a link of more than one component that is quasi-alternating at a crossing consisting of two arcs of two different components with \(l_0 = l - 1 = l_1\). This is impossible according to Proposition 3.15 and hence the result follows.

\[\square\]

4. Consequences of the Main Theorem

In this section, we discuss some consequences and applications of Theorem 2.5. We start by stating two corollaries. The first of which establishes a weaker version of Conjecture 2.3 in [1] and the second one establishes a weaker version of Conjecture 3.8 in [2].

Corollary 4.1. If \(L\) is a quasi-alternating link, then the length of any gap in the Jones polynomial \(V_L(t)\) is one.

Proof. The result follows easily by combining Theorem 2.5 with Proposition 3.10.

\[\square\]

Corollary 4.2. If \(L\) is a quasi-alternating link, then \(\left\lceil\frac{\text{breadth } V_L(t)}{2}\right\rceil + 1 \leq \det(L)\).

Proof. The result follows from the fact that any gap in the Jones polynomial of the link has length one. This implies that the Jones polynomial has at least \(\left\lceil\frac{\text{breadth } V_L(t)}{2}\right\rceil + 1\) monomials each of which contributes an increment of
Corollary 4.3. Let $L$ be a quasi-alternating link. Then, the length of any gap in the quantum grading of $\mathcal{H}(L)$ is one.

Proof. Suppose there are two consecutive rows that correspond to horizontal lines of $y$-intercepts $j$ and $j + 2$ of $\mathcal{H}(L)$ with zero entries such that $j_{\min} + 2 \leq j < j + 2 \leq j_{\max} - 2$. In this case, this implies that $\mathcal{H}_i(L) = 0$ for some $i$ with $j - 2i = -\sigma(L) - 1$. From the assumption, we have $\mathcal{H}_i-j(L) \cong \mathcal{H}_i-j+2(L) \cong \mathcal{H}_i+1,j+2(L) = 0$. As a result of the knight move, we obtain $\mathcal{H}_{i-1},j-2(L) \cong \mathcal{H}_{i+1},j+4(L) = 0$ and $\mathcal{H}_{i+1},j+4(L) \cong 0$. As a result of the knight move, we obtain $j_{\min}+4 \leq j \leq j_{\max}-4$. Therefore, we conclude that $i_{\min} + 2 \leq i \leq i_{\max} - 2$. This contradicts the fact that the length of any gap in the differential grading of $\mathcal{H}(L)$ is one.

As another application, we obtain a necessary condition for a Kanenobu knot to be quasi-alternating.

Corollary 4.4. If $|p + q| > 6$, then the Kanenobu knot $K(p, q)$ is not quasi-alternating.

Proof. Any Kanenobu knot $K(p, q)$ with $|p + q| > 6$ has gap in the differential grading of $\mathcal{H}(K(p, q))$ of length bigger than one as it was proved in [13, Theorem 1.3].

Remark 4.5. The above result can be also obtained in terms of $V_L(t)$. In particular, it is proven in [14] that $V_{K(p,q)}(t) = (-t)^{p+q}(V_{K(0,0)}(t) - 1) + 1 = (-t)^{p+q}((t^{-2} - t^{-1} + 1 - t + t^2)^2 - 1) + 1$. It is easy to see that if $|p + q| > 6$, then the above polynomial has a gap of length bigger than one. Thus according to Corollary 4.1, the knot $K(p, q)$ is not quasi-alternating. It is known though that all these knots but finitely many of them are not quasi-alternating as a result of Corollary 3.3 in [2].

Now we investigate the implications of the main result to the class of alternating links being considered as a special class of quasi-alternating links. We know that any prime alternating link that is not a $(2,n)$-torus link has no gap in its Jones polynomial as a result of [4, Theorem 1(iv)]. In addition, it is known that all alternating links are $H$-thin [11, Theorem 1.2]. If we combine all these results, we obtain

Corollary 4.6. If $L$ is a prime alternating link that is not a $(2,n)$-torus link, then it has no gap in the differential grading of $\mathcal{H}(L)$.

Proof. If we assume that the differential grading of $\mathcal{H}(L)$ has a gap, then Proposition 3.10 implies that the Jones polynomial has a gap. This contradicts [4, Theorem 1(iv)].

Remark 4.7. It is easy to see that if $L$ is a $(2,n)$-torus link, then it has only one gap in the differential grading of $\mathcal{H}(L)$ according to the computations in [9].
Corollary 4.8. We have breadth($V_L(t)$) ≤ breadth($V_{L_0}(t)$) + breadth($V_{L_1}(t)$) + 2, for any link $L$. In the case if $L$ is connected and alternating, then we obtain $c(L) ≤ c(L_0) + c(L_1) + 2$, where $L_0$ and $L_1$ are the links obtained by smoothing any crossing of any reduced connected alternating diagram of the link $L$.

Proof. If we combine the results of Corollary 3.13 and Proposition 3.15, we obtain breadth($\langle L \rangle$) ≤ breadth($\langle L_0 \rangle$) + breadth($\langle L_1 \rangle$) + 8. Now the first result follows as a consequence of the fact that breadth($V_L(t)$) = breadth($\langle L \rangle$). For the second result, we assume that $L$ is a reduced connected alternating diagram of the link $L$. In this case, we obtain $c(L) = breadth(V_L(t)) ≤ breadth(V_{L_0}(t)) + breadth(V_{L_1}(t)) + 2 = c(L_0) + c(L_1) + 2$ where the equalities follow as a consequence of [15, Theorem 2.10] that implies breadth($V_L(t)$) = $c(L)$ for any connected reduced alternating link diagram and the fact that the diagrams $L_0$ and $L_1$ are connected and alternating if $L$ is a reduced connected alternating diagram of given link.

Many questions arise naturally about the inequality in Corollary 4.8. We would like just to post the following two questions:

Question 4.9. For what class of links does the second inequality in Corollary 4.8 hold?

Question 4.10. Is there a link $L$ for which the second inequality in Corollary 4.8 does not hold?

An easy technique to produce new examples of quasi-alternating links from old ones was introduced by Champanerkar and Kofman [16, Page 2452]. It basically replaces the crossing $c$ where the link $L$ is quasi-alternating by an alternating rational tangle of the same type. This technique has been later generalized not only to a single rational tangle but also to a product of rational tangles [17, Definition 2.5] and to non-algebraic alternating tangles [18]. The following proposition shows how this twisting technique affects the gaps in the differential grading.

Proposition 4.11. Let $L$ be a quasi-alternating link at some crossing $c$ and let $L^*$ be the link obtained by replacing the crossing $c$ in $L$ by a product of rational tangles that extends $c$. Then the number of gaps in the differential grading of $\mathcal{H}(L^*)$ is less than or equal the number of gaps in $\mathcal{H}(L)$. In particular, if there are no gaps in differential grading of $\mathcal{H}(L)$, then it is also the case for the differential grading of $\mathcal{H}(L^*)$.

Proof. For a positive integer $n$, we let $L^n$ denote the link with the crossing $c$ replaced by an alternating integer tangle of $n$ vertical or horizontal crossings of the same type. To prove our result, we first show that the result holds for $L^n$. By writing the Kauffman bracket skein relations, one can easily prove that:

$$\langle L^n \rangle = A^{n-1} (A\langle L_0 \rangle + A^{-1}\langle L_1 \rangle) + \left( \sum_{i=1}^{n-1} (-1)^i A^{n-4i-2} \right) \langle L_1 \rangle,$$

and
⟨L^n⟩ = A^{-n+1}(A^0 + A^{-1}⟨L_1⟩) + \left(\sum_{i=1}^{n-1} (-1)^i A^{-n+4i+2}\right)⟨L_1⟩,

for the case of vertical and horizontal tangles, respectively. Now, we discuss the first case. The second case can be treated in a similar manner. It is not too hard to see that any gap in ⟨L^n⟩ is induced basically by a gap in ⟨L⟩. Finally the result follows since any product of rational tangles can be obtained by a sequence of integer tangles and the result of Proposition 3.10.

Quasi-alternating links of braid index 3 and quasi-alternating Montesinos links have been classified in [19,20], respectively. Now we apply the above result to these two classes of quasi-alternating links to obtain:

**Corollary 4.12.** All quasi-alternating Montesinos links and quasi-alternating links of braid index 3 have no gap in the differential grading of their Khovanov homologies.

**Proof.** The result follows as a result of Corollary 4.11 and the fact that the Jones polynomial of all quasi-alternating Montesinos links and quasi-alternating links of braid index 3 have no gap (Theorem 3.10 and Theorem 4.3 in [1], respectively).

Finally, we enclose our discussion with the following conjecture that is supported by the above results and implies both Conjecture 2.3 in [1] and Conjecture 3.8 in [2]. In particular, it implies that the Jones polynomial of any prime quasi-alternating link that is not a (2,n)-torus link has no gap and the breadth of the Jones polynomial of such a link is a lower bound of the determinant of this link.

**Conjecture 4.13.** If L is a prime quasi-alternating link that is not a (2,n)-torus link, then the differential grading of H(L) has no gap.

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