A characterization of signed discrete infinitely divisible distributions

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Abstract

In this article, we give some reviews concerning negative probabilities model and quasi-infinitely divisible at the beginning. We next extend Feller’s characterization of discrete infinitely divisible distributions to signed discrete infinitely divisible distributions, which are discrete pseudo compound Poisson (DPCP) distributions with connections to the Lévy-Wiener theorem. This is a special case of an open problem which is proposed by Sato (2014), Chaumont and Yor (2012). An analogous result involving characteristic functions is shown for signed integer-valued infinitely divisible distributions. We show that many distributions are DPCP by the non-zero p.g.f. property, such as the mixed Poisson distribution and fractional Poisson process. DPCP has some bizarre properties, and one is that the parameter $\lambda$ in the DPCP class cannot be arbitrarily small.

Keywords: discrete distribution, quasi infinitely divisible, pseudo compound Poisson, signed measure, negative probability, absolutely convergent Fourier series, Jørgensen set.

MSC2010: 60E07 60E10 28A20 42A32

1. Convolutions of signed measure model and quasi-infinitely divisible

\textsuperscript{Székely} (2005) spoke of flipping two “half-coins” (which have infinitely many sides numbered 0, 1, 2, \ldots whose even values are assigned negative probabilities) to obtain a fair coin with outcomes 0 or 1 with probability 1/2 each. The negative probabilities arise because his probability generating function (p.g.f.) $G(z) = \sqrt{0.5 + 0.5z}$ has negative coefficients. He went on to consider the general $n$-th root of a p.g.f. as a generating function with negative coefficients. In this work we continue along the same lines. In short, the aim of this paper is to determine necessary and sufficient conditions on a discrete distribution such that $[G(z)]^{\frac{1}{n}}$ (or $[\varphi(\theta)]^{\frac{1}{n}}$) is also the p.g.f. (or characteristic function) of a signed measure with bounded total variation. Székely used the word “signed infinitely divisible” to describe a phenomenon that writing a discrete probability mass as a convolution of signed point measures.

Notice that central to the inversion problem of the Central Limit Theorem is the search for some characteristic function $\varphi(\theta)$ such that $[\varphi(\theta)]^{\frac{1}{n}}$ is also a characteristic function. This is indeed the very definition of an infinitely divisible distribution, or the deconvolution problem. In the foundation of this body of work is the fundamental Lévy-Khinchine result: “A distribution is infinitely divisible if and only if it has a Lévy-Khinchine representation.”

Continuing along these lines, \textsuperscript{Sato} (2014), \textsuperscript{Nakamura} (2013) proposed the following definition of quasi infinite divisibility for distributions having a Lévy-Khinchine representation but with signed Lévy measure. By constructing a complete Riemann zeta distribution corresponding to Riemann hypothesis, \textsuperscript{Nakamura} (2015) showed that a complete Riemann zeta distribution is quasi-infinitely divisible for some conditions. Based on quantum physics, \textsuperscript{Denni and Mouayn} (2015) constructed a generalized Poisson distribution and derived a Lévy-Khintchine-type representation of its characteristic function with signed Lévy measure.

\textsuperscript{1}Accepted for publication in Studia Scientiarum Mathematicarum, 10 October 2016.

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A random variable $X$ on $\mathbb{R}$ is called quasi-infinitely divisible if the characteristic function of $X$ has the following form.

**Definition 1.1** (Quasi-infinitely divisible). A distribution $\mu$ on $\mathbb{R}$ is said to be quasi-infinitely divisible if its characteristic function has the form

$$\mathbb{E} e^{itX} = \exp \left\{ \alpha i\theta - \frac{1}{2} \sigma^2 \theta^2 + \int_{\mathbb{R}} \left( e^{ix\theta} - 1 - \frac{i\theta x}{1 + x^2} \right) \nu(dx) \right\},$$

with $\alpha \in \mathbb{R}$, $\sigma \geq 0$, and corresponding measure $\nu$ a bounded signed measure (that is, a quasi-Lévy measure) on $\mathbb{R}$ with total variation measure $|\nu|$ satisfying $\int_{\mathbb{R}} \min(x^2,1) |\nu|(dx) < \infty$.

The above definition also appears in Exercise 12.3 of [Sato (2013), p. 66](https://www.notationproject.org), where it is shown that $\mathbb{X}$ is not infinitely divisible (in the classical sense) if $\nu$ is a signed measure. The Lévy-Khinchine representation with signed Lévy measure is unique; see Exercise 12.3 in [Sato (2013)](https://www.notationproject.org).

**Problem 1.1.** Find a necessary and sufficient condition for the Lévy-Khinchine representation with signed Lévy measure to hold.

This is an open problem posed by Professor Ken-iti Sato, see p29 of [Sato (2014)](https://www.notationproject.org). When $X$ in (1) is non-negative (a “subordinator” version of (1)), it is given as an unsolved problem in Exercise 4.15(6) of [Chaumont and Yor (2012)](https://www.notationproject.org).

Denote the nonnegative integers by $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let $\nu$ be a signed point measure on $\mathbb{N}$ and remove the normal component in (1). Then the characteristic function of the DPCP distribution (see Definition 2.1 below) is a discrete version of the Lévy-Khinchine representation with signed Lévy measure.

[Baishanski (1998)](https://www.notationproject.org) considered a complex-valued (which includes negative-valued) probability model for $n$-fold convolutions of i.i.d. integer-valued “random variables” $(X_1, \ldots, X_n)$ with complex-valued probabilities $P(X_1 = v) = a_v$. The characteristic function is $\varphi(\theta) = \sum a_v e^{i\theta v}$, with $\sum a_v = 1$, and $P(X_1 + \cdots + X_n = v) = a_{nv}$. He charted this territory perhaps not for the sake of statisticians or probabilists, but certainly to the benefit of analysts. His work stemmed from an open problem related to the complex-valued probability model which was first posed by [Beurling (1938)](https://www.notationproject.org) and later quoted by [Beurling and Helson (1953)](https://www.notationproject.org). It is called the problem of “Norms of powers of absolutely convergent Fourier series” and has been investigated at length in many, many papers; see [Baishanski (1998)](https://www.notationproject.org) for a review. We only present the problem here and show its relationship to the DPCP distribution.

**Problem 1.2.** Let $f(\theta) = \sum_{j=0}^{\infty} a_j e^{i\theta j}$ and $\|f\| = \sum_{j=0}^{\infty} |a_j| < \infty$. Under what conditions on $f$ are the $\|f^n\|$ bounded? Discuss the asymptotic behavior of $\|f^n\|$ as $n \to \infty$.

When $a_v \geq 0$, the behavior of $a_{nv}$ has been firmly established via the Central Limit Theorem and gives rise to the normal law in particular, and stable laws in general. In the case of complex-valued probability, [Baishanski (1998)](https://www.notationproject.org) asks,

**Problem 1.3.** What is the Central Limit Theorem for complex-valued probabilities?

For some results in this direction, see [Baez-Duarte (1993)](https://www.notationproject.org) and [Baishanski (1998)](https://www.notationproject.org).

Here, we consider the case of real-valued probabilities. [van Haagen (1981)](https://www.notationproject.org) proved Kolmogorov’s Extension Theorem for finite signed measures which guarantees that a suitably “consistent” collection of finite-dimensional distributions will define a signed r.v.. [Kemp (1979)](https://www.notationproject.org) studied the circumstances under which convolutions of binomial variables and binomial pseudo-variables (with negative probability density) lead to valid distributions (with positive probability density). [Karymova (2005)](https://www.notationproject.org) obtained asymptotic decompositions into convolutions of Poisson signed measure that is appropriate for a broad range of lattice distributions.

Let $(\Omega, \mathcal{F}, \mu)$ be a signed measure space with $\mu(\Omega) = 1$ and $(S, \mathcal{B})$ be a measurable space. A mapping $X : \Omega \to \mathbb{R}$ is a signed random variable if it is a measurable function from $(\Omega, \mathcal{F})$ to $(S, \mathcal{B})$. Every signed random variable $X$ has an associated signed measure. The signed measure is $\mu(B) = P(X \in B) = P(X^{-1}B)$ for each $B \in \mathcal{B}$. 

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Remark 1: In this paper, we write “r.v.” (or “distribution”) for a random variable with ordinary (not negative) probabilities, and we write “signed r.v.” (or “signed distribution”) for a random variable that permits negative probabilities.

Definition 1.2 (Signed probability density). A signed random variable $X$ has signed probability distribution $\mu(dx) \in \mathbb{R}$ satisfying $\int_{\Omega} |\mu|(dx) < \infty$. The signed discrete distribution is well-defined provided the total variation $\int_{\Omega} |\mu|(dx)$ is finite. Taking a signed discrete distribution as an example, the absolute convergence of $\sum_{k=1}^{\infty} a_k$ guarantees that all rearrangements of the series are convergent to the same value. Signed random variables defined this way allow for treatment analogous to the classical case with the concepts of independence, expectation, variance, $r$th moments, characteristic functions, etc. operating in the natural way.

Without the condition of absolute convergence the negativity of $\alpha_k$ would cause $\exp \left\{ \sum_{i=1}^{\infty} \alpha_i (z^i - 1) \right\}$ to be undefinable, since we know from the Riemann series theorem that a conditionally convergent series can be rearranged to converge to any desired value. The next example drives home this point.

Example 1.1 (Discrete uniform distribution). Let $X$ be Bernoulli r.v. with probability of success $p = 0.5$. The logarithm of the p.g.f. of $X$ is

$$\ln \left( \frac{1}{2} + \frac{1}{2} z \right) = \ln \frac{1}{2} + \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} z^i = \ln 2 \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i \ln 2} (z^i - 1).$$

However, $\sum_{i=1}^{\infty} |a_i| = \sum_{i=1}^{\infty} \frac{1}{i \ln 2} = \infty$.

The present paper has concerned itself with the study of certain classes of discrete random variables, in particular, those of the infinitely divisible variety. We started from William Feller’s famous characterization that all discrete infinitely divisible distributions are compound Poisson distributed. Now, not all discrete distributions are infinitely divisible (not, at least, in the classical sense). But following the idea of Széky and others we extend our notion of infinitely divisible to include those distributions whose $n$th root of their p.g.f. is also a p.g.f. in a generalized sense. Székely’s “signed” ID is based on convolution of signed measures and Sato defines his “quasi” ID from Lévy-Khinchine representation with signed Lévy measure. The goal of this paper is to find a connection with these two kinds of generalised ID under the discrete r.v..

The paper is structured as follows: In Section 2, after giving the definition of signed probability model, we present several conditions guaranteeing that a discrete distribution is a signed discrete infinitely divisible distribution (or discrete pseudo compound Poisson). Also, we exemplify some famous discrete distributions which belong to the discrete quasi infinitely divisible distributions, such as the mixed Poisson distribution and the fractional Poisson process. In the same way, in Section 3, we conclude that a distribution is signed integer-valued ID if and only if it is integer-valued pseudo compound Poisson. In Section 4, some bizarre properties of signed discrete infinitely divisible distributions are discussed, and we mention a research problem of finding characteristic function's Jørgensen set.

2. Feller’s characterization and extension of signed discrete ID

2.1. Discrete pseudo compound Poisson distribution

Feller’s characterization of the compound Poisson distribution states that a non-negative integer valued r.v. $X$ is infinitely divisible (ID) if and only if its distribution is a discrete compound Poisson distribution. Taking $N$ and $Y_i$’s to be independent, $X$ can be written as

$$X = Y_1 + Y_2 + \cdots + Y_N,$$ (2)
where $N$ is Poisson distributed with parameter $\lambda$ and the $Y_i$’s are independently and identically distributed (i.i.d.) discrete r.v.’s with $\Pr(Y_1 = k) = \alpha_k$. Hence the p.g.f. of the compound Poisson distribution can be written as

$$P(z) = \sum_{k=0}^{\infty} P_k z^k = \exp \left\{ \lambda \sum_{k=1}^{\infty} \alpha_k (z^k - 1) \right\}, \quad |z| \leq 1, \quad \lambda > 0, \quad \sum_{k=1}^{\infty} \alpha_k = 1, \quad \alpha_k \geq 0,$$

(3)

where $z$ is a real number. (In this paper, all the arguments $z$ of a p.g.f. are taken to be real numbers $z \in [-1,1]$.) For more properties and characterizations of discrete compound Poisson, see Jánossy et al. (1950), Section 12.2 of Feller (1968), Section 9.3 of Johnson et al. (2005), and Zhang and Li (2016).

However, Feller neither shows nor claims that the sum of the coefficients in a $\sqrt{P(z) - 1}$ is bounded for any $n$, that is, it leaves the question of whether

$$\ln P(z) = \lim_{n \to \infty} \sum_{k=1}^{\infty} n a_{nk} (z^k - 1) = \sum_{k=1}^{\infty} b_k (z^k - 1), \quad \text{where} \sum_{k=1}^{\infty} b_k < +\infty.$$

This result of Feller’s can be viewed as a discrete analogue of the derivation of Lévy-Khinchine’s formula; see Itô (2004), Sato (2013), Meerschaert and Scheffler (2001) for the general case. When some $\alpha_k$ are negative, it is necessary to find a new method which guarantees that the extension of Feller’s characterization relating to the $n$th convolution of a signed measure is valid.

If it turns out that some $\alpha_i$ are negative in the p.g.f. of (3), then the $Y_i$ in (2) will have negative probability and a fortiori will rule out any chance for $\varphi(z) = \exp \left\{ \sum_{k=1}^{\infty} \alpha_k \lambda (z^k - 1) \right\}$ to be an infinitely divisible p.d.f. For instance:

**Example 2.1.** From the Taylor series expansion it follows that

$$\frac{2}{3} + \frac{1}{3} z = \exp \left\{ \ln \frac{2}{3} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2 z^k} \right\} = \exp \left\{ \ln \frac{2}{3} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2 \ln \frac{2}{3}} (z - 1) \right\}.$$

Note that $\alpha_k = \frac{(-1)^{k-1}}{k^2 \ln \frac{2}{3}}$ is negative whenever $k$ is even.

**Definition 2.1** (Discrete pseudo compound Poisson distribution, DPCP). If a discrete r.v. $X$ with $\Pr(X = i) = P_i$, $i \in \mathbb{N}$, has a p.g.f. of the form

$$P(z) = \sum_{i=0}^{\infty} P_i z^i = \exp \left\{ \sum_{k=1}^{\infty} \alpha_k \lambda (z^k - 1) \right\},$$

(4)

where $\sum_{i=1}^{\infty} \alpha_i = 1$, $\sum_{i=1}^{\infty} |\alpha_i| < \infty$, $\alpha_i \in \mathbb{R}$, and $\lambda > 0$, then $X$ is said to have a discrete pseudo compound Poisson distribution, abbreviated DPCP. We denote it as $X \sim \text{DPCP}(\alpha_1, \alpha_2, \cdots, \alpha_r, \lambda)$. The r-parameter case: If $X \sim \text{DPCP}(\alpha_1, \alpha_2, \cdots, \alpha_r, \lambda)$, we say $X$ has a DPCP distribution of order $r$.

Here is an explicit expression for the probability mass function of the DPCP distribution:

$$P_0 = e^{-\lambda}, \quad P_1 = \alpha_1 \lambda e^{-\lambda}, \quad P_2 = \left( \alpha_2 \lambda + \frac{1}{2!} \alpha_1^2 \lambda^2 \right) e^{-\lambda}, \quad P_3 = \left( \alpha_3 \lambda + \alpha_1 \alpha_2 \lambda^2 + \frac{1}{3!} \alpha_1^3 \lambda^3 \right) e^{-\lambda}, \cdots$$

$$P_n = \left( \alpha_n \lambda + \cdots + \sum_{k_1 + \cdots + k_n = i, k_1, \cdots, k_n \in \mathbb{N}} \frac{\alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_n^{k_n} \lambda^i}{k_1! k_2! \cdots k_n! n!} \right) e^{-\lambda},$$

(5)

see Jánossy et al. (1950), Johnson et al. (2005) for the discrete compound Poisson case (all $\alpha_i$ are non-negative).
2.2. Characterizations

It turns out that there already exist a few characterizations of the DPCP distribution in the literature. Indeed, [Lévy (1935)] derived the recurrence relation

$$P_{j+1} = \frac{\lambda}{j+1} [\alpha_1 P_j + 2\alpha_2 P_{j-1} + \cdots + (j+1)\alpha_{j+1} P_0], \quad P_0 = e^{-\lambda}, \; j = 1, 2, \ldots$$

(6)

of the density of the DPCP distribution when \( \alpha_i \) might be negative-valued. If we take \( P_j \) to be the empirical probability mass function, then the recursive formula in (6) can be used to estimate the parameters \( \alpha_j \), for \( j = 1, 2, \ldots \) (see [Buchmann and Grübel (2003), p. 1059]).

The name “pseudo compound Poisson” was introduced by H¨urlimann (1990). For the general situation, the following L´evy-Wiener theorem provides us a shortcut on necessary and sufficient conditions for a distribution to be DPCP. The proof is non-trivial; see [Zygmund (2002)] or [Lévy (1935)]. The simple case \( H(t) = t^{-1} \) is due to Wiener.

**Lemma 2.1** (Lévy-Wiener theorem). Let \( F(\theta) = \sum_{k=-\infty}^{\infty} c_k e^{ik\theta} \), for \( \theta \in [0, 2\pi] \), be an absolutely convergent Fourier series with \( \|F\| = \sum_{k=-\infty}^{\infty} |c_k| < \infty \). The values of \( F(\theta) \) lie on a curve \( C \). Let \( H(t) \) be a holomorphic function (analytic function) of a complex variable \( t \) which is regular at every point of \( C \). Then \( H[F(\theta)] \) has an absolutely convergent Fourier series.

The next two corollaries are direct consequences of the Lévy-Wiener theorem.

**Corollary 2.1.** For each \( k \in \mathbb{N} \), the \( t^k \) power of the p.g.f. \( G(z) = \sum_{i=0}^{\infty} p_i z^i \) is

$$\sqrt[k]{G(z)} = \sum_{i=0}^{\infty} q^{(k)}_i z^i, \quad |z| \leq 1.$$ 

If \( G(z) \) has no zero, then \( \sqrt[k]{G(z)} \) is absolutely convergent, namely, \( \sum_{i=0}^{\infty} |q^{(k)}_i| < \infty \).

When \( p_0 \geq p_1 \geq p_2 \geq \ldots \geq 0 \) as in Corollary 2.1, this is Székely’s Discrete Convex Theorem (see Theorem 2.3.1 of [Kerns (2004), p. 29]).

**Corollary 2.2.** Let \( f(\theta) = \sum_{j=0}^{\infty} a_j e^{ij\theta} \) and \( \|f\| = \sum_{j=0}^{\infty} |a_j| < \infty \). If \( f(\theta) \) has no zero, then the \( \|f^n\| \) are bounded.

Next, we give a lemma about the non-vanishing p.g.f. characterization of DPCP, see [Zhang et al. (2014)]. We restate the proof here.

**Lemma 2.2** (Non-zero p.g.f. of DPCP). For any discrete r.v. \( X \), its p.g.f. \( G(z) \) has no zeros if and only if \( X \) is DPCP distributed.

**Proof.** It is easy to see that the p.g.f. of a DPCP distribution has no zero. On the other hand, if \( G(z) \) has no zeros, taking \( z \) as a complex number, let \( z = re^{i\theta} \), for \( 1 \geq r \geq 0 \). We have \( G(re^{i\theta}) = \sum_{k=0}^{\infty} p_k r^k e^{ik\theta} \), thus

$$\sum_{k=0}^{\infty} |p_k r^k| = \sum_{k=0}^{\infty} |p_k| = 1.$$ 

By applying the Lévy-Wiener theorem for all \( r \in [0, 1] \), \( \ln G(e^{i\theta}) \) has an absolutely convergent Fourier series. Therefore, \( X \) is DPCP distributed from Definition 2.1.

For instance, \( P(z) = \frac{1}{3} + \frac{2}{3}z \) on \( |z| \leq 1 \) has no pseudo compound Poisson representation since \( \frac{1}{3} + \frac{2}{3}z = 0 \) for \( z = -\frac{1}{2} \).

Next, we define signed discrete infinite divisibility (ID) as an extension of discrete infinite divisibility. Firstly, we show that p.g.f. of a signed discrete infinitely divisible distribution never vanishes. Secondly, we obtain an extension of Feller’s characterization by employing the Lévy-Wiener theorem.
Definition 2.2. A p.g.f. is said to be signed discrete infinitely divisible if for every \( n \in \mathbb{N} \), \( G(z) \) is the \( n \)-th-power of some p.g.f. with signed probability density, namely

\[
G(z) = [G_n(z)]^n = \left( \sum_{i=0}^{\infty} p_i^{(n)} z^i \right)^n, \quad \sum_{i=0}^{\infty} p_i^{(n)} = 1, \sum_{i=0}^{\infty} |p_i^{(n)}| < \infty, p_i^{(n)} \in \mathbb{R}.
\]

The notion of signed discrete infinite divisibility first appeared in Székely (2005) where he discusses the conditions under which \( \sqrt[n]{G(z)} \) is absolutely convergent in the special case that \( G(z) \) is the p.g.f. of a Bernoulli distribution.

To get a characterization for signed discrete ID distributions, we need Prohorov’s theorem for signed measures, see p. 202 of Bogachev (2007). Applying Prohorov’s theorem for bounded and uniformly tight measures generalizes the continuity theorem for signed p.g.f.'s.

Lemma 2.3 (Prohorov’s theorem for signed measures, Bogachev (2007)). Let \( (E, \tau) \) be a complete separable metric space and let \( \mathcal{M} \) be a family of signed Borel measures on \( E \). Then the following conditions are equivalent:

(i) Every sequence \( \mu_n \in \mathcal{M} \) contains a weakly convergent subsequence.

(ii) The family \( \mathcal{M} \) is uniformly tight and bounded in the variation norm (a signed measure \( \mu \) in a topological space \( (E, \tau) \) is called uniformly tight if for every \( \varepsilon > 0 \) there exists a compact set \( K_\varepsilon \) such that \( |\mu|(E \setminus K_\varepsilon) < \varepsilon \) for all \( \mu \in \mathcal{M} \)).

Let \( G \) be as in Definition 2.2 and take \( E = \mathbb{N} \) and \( K_{\varepsilon M} = \{0, 1, \ldots, M - 1, M\} \) in Lemma 2.3. For every \( n \in \mathbb{N} \), \( \sum_{i=0}^{\infty} |p_i^{(n)}| < \infty \), there exists a compact set \( K_{\varepsilon M} \) such that \( \sum_{i=M+1}^{\infty} |p_i^{(n)}| < \varepsilon \) for every \( \varepsilon > 0 \), namely \( \limsup_{M \to \infty} \sum_{i=M+1}^{\infty} |p_i^{(n)}| = 0 \) (see p. 42 in Sato (2013) for the case of positive measure). So \( \{p_i^{(n)}\}_{i=0}^{\infty} \) is a sequence of uniformly tight signed bounded point measures.

The next lemma is an extension of the continuity theorem for p.g.f.’s. With slight modifications, the necessity part directly follows the proof of the p.g.f. case; see Feller (1968) p. 280.

Lemma 2.4 (Continuity theorem for signed p.g.f.’s). Let \( G_n(z) = \sum_{k=0}^{\infty} p_k^{(n)} z^k \) on \( |z| \leq 1 \) be p.g.f.’s for a sequence of bounded signed point measures \( \{p_k^{(n)}\}_{k=0}^{\infty} \) with \( \sum_{k=0}^{\infty} |p_k^{(n)}| < \infty \). Then there exists a sequence \( p_k \) such that

\[
\lim_{n \to \infty} p_k^{(n)} = p_k, \quad \text{for } k = 0, 1, \ldots, \text{if and only if the limit}
\]

\[
\lim_{n \to \infty} G_n(z) = \lim_{n \to \infty} \sum_{k=0}^{\infty} p_k^{(n)} z^k = G(z)
\]

exists for each \( z \) in the open interval \( (0, 1) \). Furthermore, let \( G(z) = \sum_{k=0}^{\infty} p_k z^k \). Then \( \sum_{k=0}^{\infty} p_k = 1 \) if and only if \( \lim G(z) = 1 \).

Proof. Necessity: We suppose (7) holds and define \( G(z) \) by (8). If \( p_k^{(n)} \) is a bounded and tight signed measure, then there exists an \( M \) such that \( \max \{ |p_k|, |p_k^{(n)}| \} \leq M \) for \( k = K + 1, K + 2, \ldots \). When \( 0 < z < 1 \), it follows that

\[
|G_n(z) - G(z)| = \left| \sum_{k=0}^{\infty} (p_k^{(n)} - p_k) z^k \right| \leq \sum_{k=0}^{K} |p_k^{(n)} - p_k| + 2M \sum_{k=K+1}^{\infty} z^k.
\]
Set $\varepsilon > 0$. Fix $K$ such that \[ \left| \sum_{k=K+1}^{\infty} z^k \right| = \frac{z^K}{1-z} \leq \varepsilon/2, \]
and choose $N$ sufficiently large such that
\[ \left| \sum_{k=0}^{K} (p_k^{(n)} - p_k) \right| \leq \varepsilon/2 \text{ for all } n \geq N. \]
Then $|G_n(z) - G(z)| \leq \varepsilon$ for $n \geq N$. Hence (8) is true.

Sufficiency: Assuming (8) is true for $0 < z < 1$, then there is a subsequence $\{n(1)\}$ such that $\lim_{n(1) \to \infty} G_{n(1)}(z) = G(z)$. Note that $\{p_k^{(n(1))}\}$ is bounded and uniformly tight. From Prohorov’s theorem for signed measures, we get a sub-subsequence $\{n(2)\}$, such that $\lim_{n(2) \to \infty} p_k^{(n(2))} = p_k$, thus $\lim_{n(2) \to \infty} G_{n(2)}(z) = G(z)$. The coefficients of the signed p.g.f. (identity theorem) do not depend on the choice of $\{n(1)\}$ and $\{n(2)\}$. Thus we have (7). If every convergent subsequence $p_k^{(n(1))}$ did not have the same limit $p_k$, then neither would $p_k^{(n(2))}$. This contradiction leads to the truth of (7).

Moreover, the final result is deduced from following equalities:
\[ 1 = \lim_{z \to 1} \lim_{n \to \infty} G_n(z) = \lim_{n \to \infty} \lim_{z \to 1} G_n(z) = \lim_{n \to \infty} \sum_{k=0}^{\infty} p_k^{(n)} z^k = \sum_{k=0}^{\infty} p_k, \]
for $0 < z < 1$. □

Lemma 2.5. Let $G(z)$ be a p.g.f. of signed discrete infinitely divisible r.v.. Then $G(z) \neq 0$ for all $|z| \leq 1$.

Proof. Given $n \in \mathbb{N}$, if $X$ is signed discrete infinitely divisible, then $G(z) = [G_n(z)]^{\frac{1}{z}}$ where $G_n(z)$ is a signed p.g.f. Let
\[ P(z) = \lim_{n \to \infty} G_n(z) = \begin{cases} 1, & \text{if } G(z) \neq 0, \\ 0, & \text{if } G(z) = 0. \end{cases} \]
The function $G(z)$ is continuous in $z$ for $|z| \leq 1$ and $G(1) = 1$, likewise so is $G_n(z) = [G(z)]^{\frac{1}{z}}$. By the continuity theorem for signed p.g.f.’s, $P(z)$ is a signed p.g.f. and $P(1) = 1$. Since $P(z)$ is continuous for $|z| \leq 1$, hence $P(z) = 1$ for all $|z| \leq 1$. The above statements show that $G(z) \neq 0$ for every $|z| \leq 1$. □

The continuity theorem for the p.g.f.’s of signed r.v.’s will be applied in the derivation of the general form for discrete quasi ID distributions. Putting all the above together we get a generalisation of the discrete compound Poisson distribution. Now we state and prove our characterization of discrete quasi infinitely divisible distributions.

Theorem 2.1 (Characterization of signed discrete ID distributions). A discrete distribution is signed discrete infinitely divisible if and only if it is a discrete pseudo compound Poisson distribution.

Proof. Sufficiency: Given $n \in \mathbb{N}$ and $P_X(z)$, if $X$ is DPCP distributed then
\[ [P_X(z)]^{\frac{1}{z}} = \exp \left( \sum_{i=1}^{\infty} \frac{1}{n_i} \alpha_i (z^i - 1) \right) = \sum_{k=0}^{\infty} p_k^{(n)} z^k. \]
By using the Lévy-Wiener theorem, we have $\sum_{k=0}^{\infty} |p_k^{(n)}| < \infty$. So $X$ is signed discrete ID distributed.

Necessity: Lemma 2.5 and Lemma 2.2 say respectively that the p.g.f. of a signed discrete ID r.v. $X$ has no zeros and any p.g.f. that has no zeros is the p.g.f. of a DPCP distribution. Consequently, $X$ is DPCP distributed.

Similar to the criterion for discrete infinite divisibility from Feller’s characterization: a function $h$ is an infinitely divisible p.g.f. if and only if $h(1) = 1$ and
\[ \ln h(z) - \ln h(0) = \sum_{k=1}^{\infty} a_k z^k, \]
Thus we have a Corollary 2.4. For any discrete r.v.

where \( a_k \geq 0 \) and \( \sum_{k=1}^{\infty} a_k = \lambda < \infty \) (see p. 290 of Feller [1968]). The related open problem was first studied by Lévy [1937b].

**Problem 2.1.** If some \( a_i \) are negative, under what necessary and sufficient conditions on \( a_i \) is

\[
\exp \left\{ \sum_{i=0}^{\infty} a_i (z^i - 1) \right\}
\]

a p.g.f.?\[\text{[\text{Lévy} [1937b]}\]

proved that \( P(z) = \exp \left\{ \sum_{i=0}^{m} a_i (z^i - 1) \right\} \) is not a p.g.f. unless a term with a sufficiently small negative coefficient is preceded by one term with a positive coefficient and followed by at least two terms with positive coefficients as well (see Johnson et al. [2005], pp. 393–394), namely, the conditions are \( a_1 > 0, a_{m-1} > 0, \) and \( a_m > 0 \). Milne and Westcott [1993] considered the multivariate form of (6) and gave some conditions under which the exponential of a multivariate polynomial is a p.g.f. For \( m = 4 \), van Harn [1978] gave four inequalities to ensure

\[
A(z) = e^{a(z-1) - b(z^2 - 1) + c(z^3 - 1) + d(z^4 - 1)}
\]

is a p.g.f.; the restrictions are \( a, b, c, d > 0 \) and \( b \leq \min \{ \frac{a^2}{3}, \frac{c}{2}, \frac{d^2}{3} \} \).

Next we give a few examples of DPCP distributions and the Bernoulli distribution.

**Example 2.2.** The p.g.f. \( P(z) = p + (1 - p)z \) on \( |z| \leq 1 \) has the pseudo compound Poisson representation

\[
\ln[p + (1 - p)z] = \ln p - \ln p \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{-i \ln p} \left( \frac{1 - p}{p} \right)^i z^i.
\]

Thus we have \( a_i = \frac{(-1)^{i-1}}{-i \ln p} \left( \frac{1 - p}{p} \right)^i \).

(i) If \( p > 0.5 \) then \( X \) is DPCP distributed since \( P(z) \) has no zeros.

(ii) If \( p = 0.5 \) then \( \sum_{i=1}^{\infty} a_i \) is conditionally convergent and \( P(z) \) has the zero \( z = -1 \).

(iii) If \( p < 0.5 \), then \( \sum_{i=1}^{\infty} a_i \) is divergent and \( P(z) \) has zero \( z = \frac{p}{p - 1} \).

A more general example comes from the following corollary, see also Zhang et al. [2014].

**Corollary 2.3.** For any discrete r.v. \( X \) with p.g.f. \( G(z) \), if \( p_0 > p_1 > p_2 > \cdots \), then \( X \) is DPCP distributed.

**Proof.** It can be shown that \( G(z) \) has no zeros in \( |z| < 1 \), since \( |(1 - z)G(z)| > 0 \) for \( |z| < 1 \), see Theorem 2.3.1 of Kerns [2004], p. 29. And \( z = \pm 1 \) are not zeros due to the facts that \( G(1) = 1 \) and \( G(-1) = p_0 - p_1 + p_2 - p_3 + \cdots > 0 \).

The next corollary will be useful in fitting zero-inflated discrete distribution.

**Corollary 2.4.** For any discrete r.v. \( X \), if \( P(X = 0) > 0.5 \) then \( X \) is DPCP distributed.

**Proof.** If \( p_0 = P(X = 0) > 0.5 \), then \( |G(z)| \geq p_0 - \sum_{i=1}^{\infty} p_i = 2p_0 - 1 > 0 \).

Based on the physical background of the fractal calculus, Laskin [2003] generalised \( e^x \) to Mittag-Leffler functions:

\[
E_\mu(x) = \sum_{m=0}^{\infty} \frac{x^m}{\Gamma(\mu m + 1)}, \quad (0 < \mu \leq 1) \tag{9}
\]
Then, the fractional Poisson process $N^\nu_\lambda(t)$ has p.g.f.
\[ E^\nu_\lambda[\lambda^\nu(z - 1)], (|z| \leq 1, \lambda > 0, 0 < \nu \leq 1). \]

The paper [Beghin and Macci, 2014] extended the fractional Poisson process to the discrete compound fractional Poisson process $M(t) = \sum_{k=1}^{\infty} X_k$. The p.g.f. of $M(t)$ is
\[ E^\nu_\lambda(t^\nu \lambda \sum_{i=1}^{\infty} \alpha_i(z^i - 1)), \]
where $|z| \leq 1, \alpha_i \geq 0, 0 < \nu \leq 1$, and $\{X_n\}_{n \geq 1}$ is a sequence of i.i.d. discrete r.v.’s independent of the fractional Poisson process $N^\nu_\lambda(t)$.

The Proposition 3.1 of Vellaisamy and Maheshwari (2016) shows that the one-dimensional distributions of the fractional Poisson process $N^\nu_\lambda(t), 0 < \nu < 1,$ are not infinitely divisible. Fortunately for us, it is quasi ID distributed because Mittag-Leffler functions have no real zero as $0 < \nu < 1$. It remains to use the following lemma, then Corollary 2.5 follows.

**Lemma 2.6.** For the single parameter Mittag-Leffler function (10), if $0 < \nu \leq 1$, then $E^\nu_\lambda(x)$ has no real zeros.

**Proof.** It can be shown that $E_1(z) = e^z$ has no zeros for all non-negative $z$. Just considering negative $z$ in the case $0 < \mu < 1$, the proof can be found in Theorem 4.1.1 of Popov and Sedletskii (2013) which states: “The Mittag-Leffler function $E^\rho_\lambda(z; a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)^\rho a^n}$, where $\rho > 0, a \in \mathbb{C}$, and if $a \in \left( \bigcup_{n=0}^{\infty} \bigcup_{n=0}^{\infty} \left[ -n + \frac{1}{\rho}, -n + 1 \right] \right) \cup \left[ 1, +\infty \right)$, then $E^\rho_\lambda(z; a)$ has no negative roots.”

**Remark 2:** The other proof can be found in p. 453 of Feller (1971). The Mittag-Leffler function $E^\rho_\lambda(x)$ can be written as a moment generating function $E^\rho_\lambda(t) = E^{\rho X} > 0$, where $X$ is the transformation of a positive $\alpha$ stable distributed r.v. $Y = X^{-\frac{1}{\alpha}}$ with moment generating function $E^{-\gamma Y} = e^{-c\gamma}$.

**Corollary 2.5.** The discrete compound fractional Poisson process $M(t)$ is DPCP distributed; so too is the fractional Poisson process.

As another special case, we have the following result. Another generalization of Poisson distribution, the mixed Poisson distribution, is also DPCP.

**Corollary 2.6.** Let $X$ be a mixed Poisson r.v. with p.m.f.
\[ P(X = n) = \int_0^{+\infty} \frac{\lambda^n}{n!} e^{-\lambda} dF(\lambda), (n = 0, 1, 2, \ldots), \]
where $F(\lambda)$ is a distribution function, then $X$ is DPCP distributed.

**Proof.** To prove this corollary we need to show that the p.g.f. of the mixed Poisson distribution has no zeros. Obviously,
\[ G(z) = \int_0^{+\infty} e^{\lambda(z-1)} dF(\lambda) > 0. \]

Notice that if $F(\lambda)$ is an infinitely divisible distribution, then $X$ is discrete compound Poisson distributed, see [Maceda, 1948]. This is the well-known Maceda’s mixed Poisson with infinitely divisible mixing law. The mixed Poisson distribution is widely applicable in the non-life insurance science.
3. Signed integer-valued ID

In this section we define a class of signed integer-valued infinitely divisible distributions which is wider than the class of integer-valued infinitely divisible distributions. Next, we show that a signed integer-valued infinitely divisible characteristic function never vanishes. Then we show that a distribution is signed integer-valued infinitely divisible if and only if it is an integer-valued pseudo compound Poisson distribution.

Definition 3.1 (Integer-valued pseudo compound Poisson, IPCP). Let $X$ be an integer-valued random variable with $P(X = k) = P_k$, $k \in \mathbb{Z}$. We say that $X$ has an integer-valued pseudo compound Poisson distribution if its characteristic function has the form

$$
\varphi(\theta) = \sum_{k=-\infty}^{\infty} P_k e^{i k \theta} = \exp \left\{ \sum_{k=-\infty}^{\infty} \alpha_k \lambda (e^{i k \theta} - 1) \right\},
$$

(10)

where $\alpha_0 = 0$, $\sum_{k=-\infty}^{\infty} \alpha_k = 1$, $\sum_{k=-\infty}^{\infty} |\alpha_k| < \infty$, $\alpha_k \in \mathbb{R}$, and $\lambda > 0$.

The early documental record of IPCP is in Paul Lévy’s monumental monograph of modern probability theory, see page 191 of Lévy (1937a). Recent research about IPCP can be found in Karymov (2005).

As an example, consider $\varphi(\theta) = \frac{1}{3} + \frac{2}{3} e^{i \theta}$. We would like to write $\varphi(\theta)$ as an exponential function. On our first attempt we might try the Taylor series expansion for $\ln \left( \frac{1}{3} + \frac{2}{3} e^{i \theta} \right)$, but $\frac{1}{3} + \frac{2}{3} e^{i \theta}$ vanishes at $z = -1/2$, so this method cannot be employed. On our second attempt we might look to the Fourier inversion formula for $\ln \left( \frac{1}{3} + \frac{2}{3} e^{i \theta} \right)$ since $\frac{1}{3} + \frac{2}{3} e^{i \theta}$ has no zeros. Let $f(\theta)$ be an integrable function on $[0, 2\pi]$. The Fourier coefficients $c_n$ for $n \in \mathbb{Z}$ of $f(\theta)$ are defined by

$$
c_n = \frac{1}{2\pi} \int_{0}^{2\pi} f(\theta) e^{-i n \theta} d\theta, \quad n \in \mathbb{Z}.
$$

In this example the Fourier coefficients of $\varphi(\theta)$ are $c_n = \frac{1}{2\pi} \int_{0}^{2\pi} \ln \left( \frac{1}{3} + \frac{2}{3} e^{i \theta} \right) e^{-i n \theta} d\theta$, for $n \in \mathbb{Z}$, and

$$
\lim_{M \to \infty} \sum_{k=-M}^{M} c_k e^{i k \theta} = f(\theta) = \ln \left( \frac{1}{3} + \frac{2}{3} e^{i \theta} \right).
$$

Remark 3 This example illustrates that a discrete r.v. may be signed integer-valued ID r.v. but not be signed discrete ID r.v.!

We enlisted Maple® to compute the $c_n$’s and graphed them in Figure 1. Here, $c_0 = \lambda$ and $c_n = \lambda \alpha_n$ for $n \neq 0$. The Lévy-Wiener theorem guarantees that $\sum_{n=-\infty}^{\infty} |\alpha_n| < \infty$.

In the following, we list three equivalent characterizations of an IPCP distributed r.v. with characteristic function $\exp \left\{ \sum_{k=-\infty}^{\infty} \alpha_k \lambda (e^{i k \theta} - 1) \right\}$ in (10). The axiomatic derivations of IPCP distributions can be obtained from any one of these characterizations.

Corollary 3.1. (1°) Signed compound Poisson: Assume that $N$ is Poisson distributed with $P(N = i) = \frac{\lambda^i}{i!} e^{-\lambda}, (\lambda \in \mathbb{R})$, denoted by $N \sim Po(\lambda)$. Then, $X$ can be decomposed as

$$
X = Y_1 + Y_2 + \cdots + Y_N,
$$

where the $Y_i$ are i.i.d. integer-valued signed r.v.’s with signed probability density $P(Y_1 = k) = \alpha_k$, with $\sum_{k=-\infty}^{\infty} \alpha_k = 1$, $\sum_{k=-\infty}^{\infty} |\alpha_k| < \infty$ and $N$ independent of $Y_i$. 

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Figure 1: Fourier coefficients for $\ln \left( \frac{1}{3} + \frac{2}{3} e^{i\theta} \right) = \sum c_k e^{i k \theta}$.

(2°) Sum of weighted signed Poisson: $X$ can be decomposed as

$$X = \sum_{k=-\infty}^{\infty} k N_k,$$

where $N_k$ for $k \in \mathbb{Z}$ are independently signed Poisson distributed $N_k \sim Po(\alpha_k \lambda)$ with signed probability density $P(N_k = n) = \frac{(\lambda \alpha_k)^n}{n!} e^{-\alpha_k \lambda}$, where $\alpha_k$ and $\lambda$ are defined in 1°.

(3°) Difference of discrete pseudo compound Poisson: Let $\exp \left\{ \sum_{i=1}^{\infty} \alpha_i^+ \lambda_+ (z^i - 1) \right\}$ and $\exp \left\{ \sum_{i=1}^{\infty} \alpha_i^- \lambda_- (z^i - 1) \right\}$ be the p.g.f.’s of discrete r.v.’s $X_+$ and $X_-$, respectively. $\alpha_k$ and $\lambda$ are defined in 1°. Then $X$ can be seen as a difference of two independent r.v.’s

$$X = X_+ - X_-,$$

where $\lambda_+ = \sum_{i=1}^{\infty} \alpha_i$ and $\alpha_i^+ = \alpha_i / \lambda_+$; also $\lambda_- = \sum_{i=1}^{\infty} \alpha_i^-$ and $\alpha_i^- = \alpha_i / \lambda_-$. 

Proof. It is easy to check (1°) – (3°) by examining the characteristic function. □

Next, we discuss the if-and-only-if relationship between the signed integer-valued ID and integer-valued pseudo compound Poisson distributions. This equivalence also holds for integer-valued infinitely divisible and integer-valued compound Poisson distributions.

Definition 3.2. A characteristic function (or the integer-valued r.v. $X$) is said to be signed integer-valued infinitely divisible if for every $n \in \mathbb{N}$, $\varphi_X(\theta)$ is the $n$-power of some characteristic function with signed probability density, namely,

$$\varphi_X(\theta) = [\varphi_{X_n}(\theta)]^n = \left( \sum_{k=-\infty}^{\infty} p_k^{(n)} e^{i k \theta} \right)^n,$$

where

$$\sum_{k=-\infty}^{\infty} p_k^{(n)} = 1, \quad \sum_{k=-\infty}^{\infty} |p_k^{(n)}| < \infty, \quad p_k^{(n)} \in \mathbb{R}.$$
To obtain our characterization for signed integer-valued ID distributions we need Prohorov’s theorem for signed measures, the proof of which being found in [Bogachev (2007)]. Applying Prohorov’s theorem, we obtain a continuity theorem for characteristic functions with signed probability densities. For more reading on the application of Prohorov’s theorem to signed measures, see Theorem 2.2 and Theorem 2.3 of [Baez-Duarte (1993)].

**Lemma 3.1** (Continuity theorem for signed characteristic functions, [Baez-Duarte (1993)].) (i) \( \mu_n \Rightarrow \mu \) if and only if \( \hat{\mu}_n \Rightarrow \hat{\mu} \) a.e. and \( \{ \mu_n \} \) is bounded and tight; (ii) Let \( \hat{\mu}_n \) be a complex measure and \( \hat{\mu}_n \) be the characteristic function of \( \mu_n \). If \( \hat{\mu}_n \rightarrow \hat{\mu} \) a.e. and \( \{ \mu_n \} \) is bounded and tight, then there exists a signed measure such that \( \hat{\mu} = g \) and \( \mu_n \Rightarrow g \).

**Lemma 3.2.** Let \( \varphi_X(\theta) \) be a signed integer-valued infinitely divisible characteristic function. Then \( \varphi_X(\theta) \neq 0 \) for all \( \theta \).

*Proof.* Given \( n \in \mathbb{N} \) and \( \varphi_X(\theta) \), if \( X \) is signed integer-valued infinitely divisible then \( \varphi_X(\theta) = \left( |\varphi_X(\theta)|^2 \right)^{\frac{1}{2}} \) where \( \varphi_X(\theta) \) is a characteristic function with signed probability. Note that \( |\varphi_X(\theta)|^2 = \varphi_X(\theta)\varphi_{X_{\text{op}}}(\theta) = \mathbb{E}e^{i\theta X_n}\mathbb{E}e^{-i\theta Y_n} = \mathbb{E}e^{i\theta(Y_n-X_n)} \),

where \( X_n \overset{d}{=} Y_n \), is also a characteristic function with signed probability. Let

\[
\psi(\theta) = \lim_{n \to \infty} |\varphi_X(\theta)|^2 = \begin{cases} 1, & \text{if } \varphi_X(\theta) \neq 0, \\ 0, & \text{if } \varphi_X(\theta) = 0. \end{cases}
\]

The function \( \varphi_X(\theta) \) is continuous in \( \theta \) for all \( \theta \) and \( \varphi_X(0) = 1 \), so too is \( |\varphi_X(\theta)|^2 = [\varphi_X(\theta)]^{2n} \). By the continuity theorem for signed characteristic functions, \( \psi(\theta) \) is a signed characteristic function and \( \psi(0) = 1 \). Since \( \varphi_X(\theta) \) is continuous for all \( \theta \), so is \( \psi(\theta) \). Hence, \( \psi(\theta) = 1 \) for all \( \theta \). The above statements show that \( \varphi_X(\theta) \neq 0 \) for every \( \theta \in \mathbb{R} \).

The continuity theorem for signed characteristic functions can be used to deduce the general theorem for signed integer-valued ID.

**Theorem 3.1** (Characterization for signed integer-valued ID). An integer-valued distribution is signed integer-valued infinitely divisible if and only if it is an integer-valued pseudo compound Poisson distribution.

*Proof.* Sufficiency: For every \( n \in \mathbb{N} \), \( \varphi_X(\theta) \), if \( X \) is IPCP distributed, then

\[
|\varphi_X(\theta)|^2 = \exp \left\{ \sum_{k=-\infty}^{\infty} \frac{1}{n} \alpha_k \lambda(e^{i\theta} - 1) \right\} \overset{\lambda \geq 1}{=} \sum_{k=-\infty}^{\infty} p_k^{(n)} e^{ik\theta}.
\]

By the Lévy-Wiener theorem we have \( \sum_{k=-\infty}^{\infty} |p_k^{(n)}| < \infty \). Hence, \( X \) is signed integer-valued ID.

Necessity: Lemma 3.2 says that the characteristic function of a signed integer-valued ID r.v. \( X \) never vanishes. Judging from the Lévy-Wiener theorem, any characteristic function with no zeros is the characteristic function of an IPCP distribution. Therefore \( X \) is IPCP distributed.

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4. **Some bizarre properties of DCP related to signed r.v.**

[Ruzsa and Székely (1983)] proves a theorem related to signed random variables and [Székely (2005)] gives the following result.

**Proposition 4.1** (Construction of signed random variables). If \( f \in L^1 \) and \( f \ d\omega > 0 \), then one can find a \( g \in L^1_{\omega} \) with \( g \neq 0 \) such that \( f \ast g \in L^1_{\omega} \), where \( L^1_{\omega} \) is the set of all non-negative integrable functions. Moreover, we can choose \( g \) such that its Fourier transform is always positive.
**Proposition 4.2** (Fundamental theorem of negative probability). If a random variable $R$ has a negative probability density, then there exist two other independent random variables $Y,Z$ with ordinary (not negative) distributions such that $R+Y = Z$ in distribution, where the operation $+$ is under ‘convolutional plus’. Thus $R$ can be seen as the ‘difference’ of two ordinary random variables.

This fact gives a new explanation for why some $\alpha_k$ may be negative by Theorem 2.1. Let $\varphi_R(\theta)$, $\varphi_Y(\theta)$, and $\varphi_Z(\theta)$ be the corresponding characteristic functions in Proposition 4.2.

Then we have $\varphi_R(\theta)\varphi_Y(\theta) = \varphi_Z(\theta)$. If $R$ is integer-valued quasi ID, then $\varphi_R(\theta) = [\varphi_R^{(n)}(\theta)]^n$ and $\varphi_R^{(n)}(\theta) \neq 0$, so we have

$$\varphi_R^{(n)}(\theta) \sum_{k=1}^{\infty} b_{nk} e^{ik\theta} = \sum_{k=1}^{\infty} a_{nk} e^{ik\theta} \text{ and } \sum_{k=1}^{\infty} b_{nk} e^{ik\theta} \neq 0,$$

from Proposition 4.1. Hence,

$$\varphi_R(\theta) = \left[\varphi_R^{(n)}(\theta)\right]^n = \left(\frac{\sum_{k=1}^{\infty} a_{nk} e^{ik\theta}}{\sum_{k=1}^{\infty} b_{nk} e^{ik\theta}}\right)^n = \exp\left(\sum_{k=1}^{\infty} \frac{v'_{nk} - v''_{nk}}{b_{nk}} e^{ik\theta}\right),$$

where $v'_{nk}$ and $v''_{nk}$ are non-negative. They satisfy $\sum_{k=1}^{\infty} |v'_{nk} - v''_{nk}| < \infty$ because $\sum_{k=1}^{\infty} a_{nk} e^{ik\theta}$ and $\sum_{k=1}^{\infty} b_{nk} e^{ik\theta}$ both have no zero, and the Lévy-Wiener theorem makes sense.

For those r.v.’s $X$ whose characteristic function can be written as $\varphi(\theta) = e^{\theta\eta(\theta)}$, the Jørgensen set $\Lambda(X)$ of positive reals (see Jørgensen (1997)) is defined by

$$\Lambda(X) = \{\lambda > 0 : e^{\lambda\eta(\theta)} \text{ is a characteristic function}\}.$$

Albeverio et al. (1998) deduces the following two Propositions.

**Proposition 4.3.** $\Lambda(X)$ is a semigroup containing 1. Furthermore, $\Lambda(X)$ is closed in $(0,\infty)$.

**Proposition 4.4.** Let $S \subset (0,\infty)$ be a closed semigroup with $1 \in S$. Then,

1. Either $S = (0,\infty)$ or $S \subset [\lambda,\infty)$ for some $\lambda > 0$.
2. If the interior of $S$ is non-empty, then $S \supset [\lambda,\infty)$ for some $\lambda$, $0 < \lambda < \infty$.

Here we show an example. Suppose there exists a negative $a_k$ in

$$\varphi(\theta) = \exp\left\{\lambda \sum_{k=1}^{\infty} a_k (e^{ik\theta} - 1)\right\}.$$

According to Corollary 4.2 below, for $\lambda > 0$ sufficiently small, $\varphi(\theta)$ cannot be a characteristic function. It is evident that $[\varphi(\theta)]^n$ is still a characteristic function since $\lambda \in \mathbb{N} \setminus \{0\}$ implies $\mathbb{N} \setminus \{0\} \subset \Lambda(X)$. Given the independent convolution of a Bernoulli r.v. and a Gamma distributed r.v. $X + Y$, where $X,Y$ are non-degenerate, [Nahla and Afif (2011)] find the set $\Lambda(X + Y)$. If $X$ and $Y$ are non-degenerate independent r.v.’s which have respectively Bernoulli and negative binomial distributions, [Letac et al. (2002)] find the set $\Lambda(X + Y)$. [Nakamura (2013)] gives a signed discrete infinitely divisible characteristic function $\varphi(\theta)$ such that $[\varphi(\theta)]^u$, $u \in \mathbb{R}$, is not a characteristic function except for the non-negative integers. To find the Jørgensen set of a signed discrete ID characteristic function or a quasi ID distribution defined by Lévy-Khintchine representation with signed Lévy measure $\mu$ is a research problem.

Here we show that the Jørgensen set of signed discrete infinitely divisible Bernoulli r.v.’s $X$ is $\mathbb{N}$.

**Corollary 4.1.** Let $G_X(z) = p + (1-p)z$ be the p.g.f., where $p > 0.5$. Then the Jørgensen set is $\mathbb{N}$. 

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Proof. Trivially, \( \mathbb{N} \) belongs to the Jørgensen set of \( X \). It remains to show that \( r \in (0, \infty) \setminus \mathbb{N} \) cannot be in the Jørgensen set. From Taylor’s formula, we have

\[
[p + (1 - p)z]^r = p^r \left[ 1 + \sum_{i=1}^{\infty} r(r - 1) \cdots (r - i + 1) \cdot \frac{1}{i!} \left( \frac{1 - p}{p} \right)^i z^i \right] \approx \sum_{i=0}^{\infty} c_iz^i.
\]

For each \( r < i - 1, i = 2, 3, \ldots \), we consider two cases: (i) if there exists an \( i \) such that \( c_i < 0 \), then the result follows; (ii) if there exists an \( i \) such that \( c_i > 0 \), note that

\[
c_i = r(r - 1) \cdots (r - i + 1) \cdot \frac{1}{i!} \left( \frac{1 - p}{p} \right)^i \quad \text{and} \quad c_{i+1} = r(r - 1) \cdots (r - i) \cdot \frac{1}{(i+1)!} \left( \frac{1 - p}{p} \right)^{i+1}.
\]

Then \( c_{i+1} = \frac{c_i}{(r+1)(r-i+1)} \left( \frac{1-p}{p} \right) < 0 \) since \( r - i + 1 < 0 \). \(\square\)

**Corollary 4.2.** If there exists a negative \( \alpha_k \) in \( [0, \infty) \), then \( \varphi_X(\theta) = \exp \left\{ \sum_{k=1}^{\infty} \alpha_k \lambda (e^{i k \theta} - 1) \right\} \) is not a characteristic function for \( \lambda > 0 \) sufficiently small.

**Proof.** The p.m.f. of \( X \) is \([5]\). For \( m \geq 1 \), we have \( \lim_{\lambda \to 0} \frac{P_m}{\lambda} = \alpha_m < 0 \). This is in contradiction of the fact that \( P_m/\lambda > 0 \). \(\square\)

**Remark 4** This theorem shows that we should not replace \( X \) by \( X(t) \) for \( t \in [0, t) \) (namely, replace \( \lambda \) by \( \lambda t \)) in (4) to have the DPCP process. The DPCP processes have a strange property that the \( t \) can only belong to the set \( \Lambda(X) \) which have semigroup properties.

It is well-known that Poisson processes are characterized by stationary and independent increments.

**Proposition 4.5 (Axioms for the Poisson process).** If a nonnegative integer-valued process \( \{X(t), t \geq 0\} \) satisfies the following conditions:

(i) Initial condition \( X(0) = 0 \) and \( 0 < P(X(t) > 0) < 1 \).

(ii) \( X(t) \) has stationary increments.

(iii) \( X(t) \) has independent increments.

(iv) Let discrete probability mass be given by \( P_i(t) = P(X(t) = i | X(0) = 0) \), with the probability of 1 and \( i \geq 2 \) events taking place in \( [t, \Delta t + t) \) given by \( P_1(\Delta t) = \lambda \Delta t + o(\Delta t) \) and \( \sum_{i=2} P_i(\Delta t) = o(\Delta t) \), respectively.

Then \( \{X(t), t \geq 0\} \) is a Poisson process.

**Proposition 4.6 (Axioms for the discrete compound Poisson process).** If a nonnegative integer-valued process \( \{X(t), t \geq 0\} \) satisfies each of (i)–(iii) in Proposition 4.5 and in addition satisfies

(iv) \( P_k(\Delta t) = \alpha_k \lambda \Delta t + o(\Delta t) \) where \( \sum_{k=1}^{\infty} \alpha_k = 1 \) and \( 0 \leq \alpha_k \leq 1 \),

then \( \{X(t), t \geq 0\} \) is a discrete compound Poisson process.

**Proposition 4.5** is called the Bateman theorem, see Section 2.3 of [Haight] (1967). The processes in Proposition 4.6 can be seen in Section 3.8 of [Haight] (1967), where they are known as stuttering Poisson processes. For the case discrete r.v., [Jánossy et al.] (1950) extended axioms for the Poisson process to Proposition 4.6. We may ask if there exist similar extensions of Proposition 4.6 to DPCP processes. As the contradiction in the proof of Corollary 4.2 shows, if some \( \alpha_k \) in (3) are negative, this will yield \( P_k(\Delta t) = \).
\( \alpha_k \lambda \Delta t + o(\Delta t) < 0. \) However, \( P_k(\Delta t) \) must be non-negative. So there are not DPCP processes \( X(t) \) on \( t \in [0, +\infty) \) as some \( a_k \) in (3) are negative.

To avoid the above contradictions, we highlight a strange property of DPCP processes on the semigroup \( \Lambda(X) \).

The properties of discrete pseudo compound Poisson: The DPCP process \( \{X(t), t \in \Lambda(X)\} \) satisfies the following conditions:

(i) Initial condition \( X(t_0) = 0 \), where \( t_0 = \inf \{\lambda > 0 : [\lambda, \infty) \subset \Lambda(X)\} \) and \( 0 < \mathbb{P}(X(t) > 0) < 1. \)

(ii) \( X(t) \) has stationary increments.

(iii) \( X(t) \) has independent increments.

5. Conclusion and comment

It was the great mathematician Leopold Kronecker who once said, “God made the integers; all else is the work of man.” It is in the spirit of that proverb that the present work deals with discrete(integer-valued) r.v.

Feller’s characterization of discrete ID (namely, non-negative integer-valued infinitely divisible distributions) says that a distribution is discrete ID if and only if it is discrete compound Poisson. Further, we define the discrete pseudo compound Poisson (DPCP) distribution whose p.g.f. has the exponential polynomial form \( e^{P(z)} \), where \( P(z) \) may contain negative coefficients (except coefficient \( P(0) \)). Using the definition of generalized infinitely divisible, then, we have an extension of Feller’s characterization which is that a distribution is discrete quasi-ID if and only if it is discrete pseudo compound Poisson, which is made possible by Prohorov’s theorem for bounded and tight signed measures. This theorem could be applied to the continuity theorem for p.g.f.’s with negative coefficients, or the continuity theorem for characteristic functions with signed measure.

If \( \exp \left\{ \lambda \sum_{k=1}^{\infty} \alpha_k (e^{i k \theta} - 1) \right\} \) is characteristic function, the parameter \( \lambda \) can not tend to 0 when some \( \alpha_k \) take negative values. This property is related to a characteristic function’s Jørgensen set, which is the set such that the positive real power of the characteristic function maintains as a characteristic function. It is easy to see that the Jørgensen set of an infinitely divisible characteristic function is \( \mathbb{R}^+ \). To find the Jørgensen set of a quasi-infinitely divisible characteristic function is an open problem; there are only some special cases of Jørgensen sets in the literature, see [Nakamura (2013), Nahla and Aifi (2011), Letac et al. (2002), Albeverio et al. (1998)].

6. Acknowledgments

The authors want to thank Prof. Székely, G. J. for his helpful suggestion. This work was partly supported by the National Natural Science Foundation of China (No.11201165). After Zhang and Li (2016) was accepted in Feb 2014, the first edition manuscript of this work was originally written by studying some concepts and references in Sato (2014). More theoretical results of quasi-infinitely divisible distributions can be found in recent manuscript Lindner, Pan and Sato (2017).

7. References

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