Hamiltonian Formulation of Two Body Problem in Wheeler–Feynman electrodynamics

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Summary. – A problem of electromagnetic interaction of two charged relativistic particles is considered in Wheeler–Feynman approach to the classical electrodynamics. We formulate this theory in such a way that its Hamiltonian description becomes available. This description is a kind of constrained Hamiltonian mechanics. Hamiltonian equations obtained have simpler form than Lagrangian ones.

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A conventional framework for the description of electromagnetic interaction is a field theory. There is less a known approach to electrodynamics as the theory with action at a distance in space-time. This approach was proposed by Schwarzschild [1] in 1903, was considered by Tetrode [2] and Fokker [3] in the 20th and got its net formulation in papers of Wheeler and Feynman [4] in 1945 and 1949. In this theory electromagnetic field is expressed in
terms of coordinates and velocities of charged particles using Maxwell equations. The field obtained is substituted into equations of particles motion. An action in terms of the world lines of particles is offered which reproduces these equations.

A general solution of these equations is unknown. There is no significant progress in the investigation of this area. The main obstacle is the following. The action of electrodynamics contains advanced and retarded terms, hence the equations of motion relate coordinates and velocities at different instants of time. They are differential equations with a deviating argument. A theory of such equations is still being developed [4]. As the equations of electrodynamics are concerned, it is not clear even whether they have unique solution. Only some particular solutions have been studied [6].

The presence of advanced and retarded terms in action hinders Hamiltonian mechanics construction. Hamiltonian description is available only for such systems, whose Lagrangian depends on coordinates and velocities at one value of the evolution parameter. Hamiltonian formalism is an effective tool for the solution of Lagrangian equations of motion, and it is also a basis for canonical quantization. Failure in constructing Hamiltonian description of action-at-a-distance electrodynamics stimulated a search for an alternative quantization procedure. Quantum theory with path integrals was proposed by Feynman in application to the problem involved [7].

In this article the Hamiltonian formulation for the problem of motion of two charged particles in the space free from other charges is presented. Parametrical invariance of the action is used. This invariance allows ones to choose any convenient parametrization of the world lines. In Section 1 such parametrization is chosen, that the argument deviation in advanced and retarded terms of the action will be constant. Lagrangian equations are obtained for this the action. In Section 2 boundary conditions are chosen for the equations. In Section 3 the action is rewritten in the form of integrated Lagrangian depending on the coordinates and velocities at one value of the integration parameter. The Hamiltonian formulation is found for this mechanics.

Only one-dimensional motion of charges will be considered in this article. Generalization to greater number of dimensions adds complexity to the algebra, but the construction scheme might be preserved.
1 Equations of motion

1. The motion of charged particles $x$ and $y$, interacting via electromagnetic field, can be described by stationary principle for the action $S$:

$$S = \int d\tau \left( -m_1 \sqrt{\dot{x}^2} - m_2 \sqrt{\dot{y}^2} \right) - e_1 e_2 \int d\tau_1 d\tau_2 \frac{dx^\mu}{d\tau_1} \frac{dy^\mu}{d\tau_2} \delta \left( ((x(\tau_1) - y(\tau_2))^2) \right) \quad (1)$$

(metric $g^{\mu\nu} = \text{diag} (+1, -1, -1, -1)$ is used)

Performing one integration in (1), one can cancel $\delta$-function:

$$S = \int d\tau \left( -m_1 \sqrt{\dot{x}^2} - m_2 \sqrt{\dot{y}^2} \right) - \frac{e_1 e_2}{2} \left( \dot{x}(\tau) y(\tau^+(\tau)) + \dot{x}(\tau) y(\tau^-(\tau)) \right) \quad (2)$$

where $(x(\tau) - y(\tau^\pm(\tau)))^2 = 0$, $\tau^+ > \tau^-$. Dot denotes differentiation with respect to $\tau$. The action is parametrically invariant.

Functions $\tau^\pm(\tau)$ mark on the world line of the particle $y$ the points of its intersection by a light cone with an origin placed in the point $x(\tau)$. They are moments when retarded and advanced fields from the point $x(\tau)$ reach the world line $y$. The $\tau^\pm(\tau)$ are monotonically increasing functions of $\tau$.

2. There is a parametrization of the world lines satisfying the following condition:

$$\tau^+(\tau) = \tau, \quad \tau^-(\tau) = \tau - 1$$

Let the world lines be given. Let us consider a light ray, emitted into the future from an arbitrary point on the world line $x$, until its intersection with the world line $y$. From the point of intersection we draw a ray to the intersection with the line $x$. Let us continue this light stairway (fig.1). We will mark the points of reflection of the light ray from the world lines by the values of parameter $\tau$ with a unit step. We specify an arbitrary parametrization $\tau \in [0, 1]$ on the segment of the world line $x$ between two sequential reflections, then define parametrization of the whole world line using process described above (shifting the stairway along the segment with initial parametrization). We note that parametrization constructed is continuous. We can choose
parametrization on the segment [0, 1] so that parametrization will be smooth, double differentiable and so on. Continuous parametrization is sufficient for our purpose.

Let us consider one-dimensional motion caused by restricting the initial data to a straight line. We introduce light coordinates \( x^\pm = x_0 \pm x_1 \) in a space-time plane containing the world lines. In the parametrization constructed the following relations hold:

\[
y^+(\tau) = x^+(\tau), \quad y^-(\tau) = x^-(\tau + 1) \tag{3}
\]

- position \( y \) is determined by parametrization of the world line \( x \).

\( x^\pm, y^\pm \) are monotonically increasing functions of \( \tau \):

\[
\dot{x}^+ \dot{x}^- = \dot{x}^2 > 0, \quad \frac{1}{2}(\dot{x}^+ + \dot{x}^-) = \dot{x}_0 > 0 \implies \dot{x}^\pm > 0
\]

3. Let us substitute expressions (3) into the action (2):

\[
S = \int d\tau \left( -m_1 \sqrt{\dot{x}^+ \dot{x}^-} - m_2 \sqrt{\dot{x}^+ \dot{x}^-} - e_1 e_2 \left( \frac{\dot{x}^-}{x^-_a - x^-} + \frac{\dot{x}^+}{x^+_a - x^+} \right) \right) \tag{4}
\]

We denote \( x_a(\tau) = x(\tau + 1), \quad x_r(\tau) = x(\tau - 1) \).

The action (4) contains advanced terms only (shifting the parameter \( \tau \to \tau - 1 \) in (4), one can obtain the action containing retarded terms only). The stationary principle for this action leads to equations:

\[
\begin{align*}
\left( m_1 \sqrt{\frac{\dot{x}^-}{\dot{x}^+} + \frac{\dot{x}^+}{\dot{x}^-}} \right)' &= 2e_1 e_2 \left( \frac{\dot{x}^+_a}{(x^+_a - x^+_)^2} - \frac{\dot{x}^-_r}{(x^-_r - x^-)^2} \right) \tag{5} \\
\left( m_1 \sqrt{\frac{\dot{x}^-}{\dot{x}^+} + m_2 \sqrt{\frac{\dot{x}^-}{\dot{x}^+}} \right)' &= 2e_1 e_2 \left( \frac{\dot{x}^+_a}{(x^+_a - x^+_)^2} - \frac{\dot{x}^-_r}{(x^-_r - x^-)^2} \right)
\end{align*}
\]

The equations contain both retarded and advanced terms. Under the action variation the terms of the form \( F \delta x_a \) are transformed into the form \( F \delta x \) via the shift of integration parameter. In this way retarded terms appear in equations of motion.

Equations (3) are still complicated to solve them immediately.
2 Boundary conditions

In this Section we show that equations of motion (5) have a more wide class of solutions than equations (6), obtained from the same action in parametrization \(\tau = \text{Minkowski time}\). Additional solutions are excluded by imposing proper boundary conditions.

For additional solutions an extra force appears in the right hand side of the equations of motion. We show that this force is determined by the type of boundary conditions rather than by the stationary action principle. For the problem in question the extra force is an artifact of parametrization used, it looks like physical phenomenon. We consider in detail how this force appears.

The stationary principle for the action (1) in parametrization \(\tau = x_0\) leads to equations (6):

\[
\frac{d}{dx_0} \frac{v_x}{\sqrt{1 - v_x^2}} = e_1 E_x, \quad \frac{d}{dx_0} \frac{v_y}{\sqrt{1 - v_y^2}} = -e_2 E_y
\]

\[
E_x = \frac{e_2}{2} \frac{1 + v_y^r}{1 - v_y^r} \left( \frac{1}{(x_1 - y_1^r)^2} \right) + \frac{1}{1 + v_y^r (x_1 - y_1^r)^2}
\]

\[
E_y = \frac{e_1}{2} \frac{1 - v_x^r}{1 + v_x^r} \left( \frac{1}{(y_1 - x_1^r)^2} \right) + \frac{1}{1 - v_x^r (y_1 - x_1^r)^2}
\]

Let us transform these equations in the stairway parametrization:

\[
m_1 \left( \sqrt{\frac{\dot{x}^+}{\dot{\alpha}^-}} \right) = 2 e_1 e_2 \left( \frac{\dot{x}^+ \dot{\alpha}^-}{\dot{\alpha}^- (x^+ - x_1^r)^2} + \frac{\dot{x}^-}{(x^- - x_1^-)^2} \right)
\]

\[
m_2 \left( \sqrt{\frac{\dot{\alpha}^-}{\dot{x}^+}} \right) = 2 e_1 e_2 \left( \frac{\dot{\alpha}^- \dot{x}^+}{\dot{x}^+ (x^- - x_1^-)^2} + \frac{\dot{\alpha}^+}{(x^+ - x_1^+)^2} \right)
\]

It is easy to check by substitution that equations (5) follow from these equations: any solution of (7) is a solution of (5).

The converse is incorrect, (3) are equivalent to the equations of a more general form:

\[
L_1 = R_1 + \frac{H}{\dot{x}^-}, \quad L_2 = R_2 + \frac{H}{\dot{x}^+}
\]
We denote the left (right) hand side of equations (7) by \( L_{1,2} \) (\( R_{1,2} \)). \( H(\tau) \) is an arbitrary function with a period 1.

Proof.

\[ \begin{align*}
\{ & -\frac{\dot{x}}{x}L_1 + L_2 = R^+ \quad (R^+ - \text{the right hand sides of equations (5)}) \\
& L_1 - \left( \frac{\dot{x}^+}{\dot{x}} L_2 \right)_r = R^- \\
\end{align*} \]

\[ \begin{align*}
\begin{bmatrix} \\
\dot{x}^+ L_2 - (\dot{x}^+ L_2)_r = \dot{x}^+ R^+ + \dot{x}^- R^- \\
L_1 = \frac{\dot{x}^+}{\dot{x}} (L_2 - R^+) \\
\end{bmatrix} \quad (9)
\end{align*} \]

The first equation in (9) is a linear difference equation for \( \dot{x}^+ L_2 \). A particular solution of this equation is 

\[ L_2 = R_2. \]

The general solution of this equation has the form:

\[ \dot{x}^+ L_2 = \dot{x}^+ R_2 + H, \]

where \( H \) is a general solution of the homogeneous equation \( H - H_r = 0 \), so \( H \) is a periodical function with period 1. Substituting this solution into second equation (8), we have (8).

Let us transform equations (8) into parametrization \( \tau = x_0 \):

\[ \begin{align*}
m_1 \frac{d}{dx_0} \sqrt{1 - v_x^2} x &= e_1 E_x + \left( \frac{d\tau}{ds_x} \right)^2 H(\tau) \quad (10) \\
m_2 \frac{d}{dx_0} \sqrt{1 - v_y^2} \sqrt{2} y &= -e_2 E_y - \left( \frac{d\tau}{ds_y} \right)^2 H(\tau) \\
( \quad ds_x^2 = dx_0^2 - dx_1^2, \quad ds_y^2 = dy_0^2 - dy_1^2 \quad )
\end{align*} \]

A new force appears in the right hand side of equations of motion. The force acts on segments of the world lines intercepted by two rays of the light stairway with the initial values of the parameter \( \tau \) and \( \tau + d\tau \) (fig.2).
The force is inversely proportional to the square of the segment interval, the proportionality coefficient \( d\tau^2 H(\tau) \) is common for all segments.

The force of an identical form appears in the other problem. The same force acts between two mirrors, moving along a straight line, when radiation is placed between them.

Let us consider a light impulse of energy \( k \) and duration \( \Delta t \), reflected from a moving mirror. From geometrical reasoning (fig.3) we have:

\[
\begin{align*}
\Delta t' &= \Delta t \frac{1 - v}{1 + v} \\
\Delta x_0 &= \frac{\Delta t}{1 + v}, \quad \Delta x_1 = \frac{\Delta t v}{1 + v} \\
\Delta s^2 &= \Delta t^2 \frac{1 - v}{1 + v}
\end{align*}
\]

Conservation of momentum leads to relations:

\[
k' = k \frac{1 + v}{1 - v}, \quad \Delta p = |p' - p| = \frac{2k}{1 - v} \quad (\text{for small } k)
\]

Therefore,

\[
k' \Delta t' = k \Delta t, \quad \frac{\Delta p}{\Delta x_0} = \frac{2k\Delta t}{\Delta s^2} \quad (11)
\]

The force (11) has the same form as the extra term in (10). The coefficient \( k \Delta t \), analogous to \( d\tau^2 H(\tau) \) in (10), conserves in reflections. The analogy is possible for \( H(\tau) > 0 \), in the case of repulsion.

The extra force essentially affects the structure of the solution of equations of motion. The arbitrary function \( H(\tau) \) determining the energy distribution for the radiation between the mirrors brings about the functional ambiguity in the solution. This ambiguity remains both in the non-relativistic limit and the limit \( e \to 0 \). (The mirrors repel even though they move slowly or they are not charged.)

The radiation between the mirrors vanishes at \( t \to \infty \). In reflections from the mirrors, moving in opposite directions with asymptotically constant velocity, the duration of light impulse exponentially increases: \( \Delta t_n \sim \Delta t_0 \left(\frac{1 + v}{1 - v}\right)^n \to \infty \) (see fig.2), hence \( k \to 0 \). The radiation vanishes at \( t \to -\infty \) also, owing to the time symmetry of the problem. In other words, the initial impulse, as small as one likes is amplified in reflections, causes the finite effect of repulsion, and then it is attenuated down to zero.
Nevertheless, in Wheeler-Feynman electrodynamics the whole field of radiation is expressed in terms of the sources from the Maxwell equations using conditions of absence of radiation at infinity, and it is already presented by retarded and advanced terms in equation (1). The new force has no relation to electrodynamics.

We ask, how the replacement of the parameter in parametrically invariant action changes the equations of motion. Actions calculated in different parametrizations coincide. Certainly, the calculation must be performed for one and the same segment of the trajectory. When parametrization changes, one must take care that the initial and final points of the trajectory do not change. The values of the parameter, marking boundary points, are obtained from an equation \( x(\tau^*) = x^* \), and they can become variables depending on the trajectory. Simple example: \( S = \int_0^{\tau_f} d\tau \sqrt{\dot{x}^2} = \int_0^{s^*} ds \), \( s^* \) must be a dynamical variable for substantial equations to be obtained.

In the action variation the initial and final points of the trajectories are fixed: \( \delta x(\tau_i) = \delta x(\tau_f) = \delta y(\tau_i) = \delta y(\tau_f) = 0 \). In the stairway parametrization dynamical variables are the coordinates of particle \( x \), the information about the trajectory \( y \) is “encoded” in the parametrization of the trajectory \( x \). When the trajectory \( x \) is varied so that the initial and final positions \( x \) are fixed than the boundary points of \( y \) change freely (fig.4). Variations of \( x \) should be restricted to such class, that boundary points of \( y \) would be fixed. When minimum of the action is sought in this class, the extra force is excluded.

We describe this calculation, omitting details. The initial and final values of parameter \( \tau_i \) and \( \tau_f \) become dynamical variables. Conditions, fixing the positions of the boundary points, are included in the action with Lagrangian multipliers. Minimum of such action is found with respect to all dynamical variables. All off-integral terms appearing in calculation should be taken into account.

Different terms in the action are turned on at different instants of time (see fig.4). In interval \( \tau \in (\tau_f - 1, \tau_f] \) only the term representing length of the world line \( x \) is active. The stationary action principle applied to this interval leads to equations of motion without additional \( H \)-term. Periodical condition on function \( H \) excludes this term for all instants of time.

Extra force is determined by boundary conditions. If boundary conditions different from fixation of initial and final coordinates are imposed, distinct
equations of motion would be obtained. When boundary conditions have the form:

“light emitted from the initial point \( x_i \), after integer number of reflections from the world lines arrives at the final point \( x_f \)” \( (12) \)

force \( (11) \) appears in equations. One can derive this force in any parametrization.

Let us consider infinite trajectories. In variation of the action off-integral term \( \frac{\delta S}{\delta x_\mu} \delta x_\mu \bigg|_{-\infty}^{+\infty} \) appears. One excludes this term by requiring \( \delta x(\pm\infty) = 0 \). In commonly used parametrizations this requirement can be satisfied, because variation \( \delta x \) can always be made local. In stairway parametrization local disturbance affects the parametrization of the whole trajectory. Size of response to disturbance exponentially increases when \( t \to \infty \) (fig. 2). Requirement \( \delta x(\pm\infty) = 0 \) can be satisfied, if only those correlated variations of the trajectories are taken, for which responses induced are cancelled at infinity (fig. 5). Therefore, equations (5) give only conditional minimum of the action in the class of trajectories variations, where calculations in the stairway parametrization are well defined.

An exact minimum is defined as follows

\[
F_x = \frac{\delta S}{\delta x} \bigg|_{\text{in cond.min.}} = 0
\]

Variation is performed with respect to all other changes of trajectories.

In calculation of variational derivative the variables \( x \) and \( \dot{x} \) are not independent:

\[
\frac{\delta \dot{x}(\tau_1)}{\delta x(\tau_2)} = \delta'(\tau_1 - \tau_2)
\]

In variation of the action all terms of form \( F\delta \dot{x} \) can be transformed into the form \(-F\dot{\delta}x\) using integration in parts. In this the coefficient at \( \delta x \) in \( \delta S \) is the required derivative \( \frac{\delta S}{\delta x} \).

This calculation should be done in some other parametrization, where it is defined. As a result we have

\[
F_x = \left( m_1 \frac{d}{dx_0} \frac{v_x}{\sqrt{1 - v_x^2}} - e_1 E_x \right) \bigg|_{\text{in cond.min.}} = \left( \frac{d\tau}{ds_x} \right)^2 H = 0,
\]
hence, the extra force is excluded.

A similar mechanism determines an additional force in the problem of free motion of mass on a surface. Here equations of motion define conditional minimum of the action in a class of trajectories lying on the surface. The surface acts on a mass through a reaction force. This force is equal to a derivative of an action \( \frac{\delta S}{\delta x^\perp} \) with respect to variations of coordinates leading away the mass from the surface. This force can not be obtained from the stationary principle of some action formulated in terms of surface coordinates only. This analogy is correct with the following refinement: the stairway parametrization constrains not the paths themselves but their variations in the vicinity of an arbitrary path (fig.6).

Let us consider the finite trajectories again. The requirement that \( \tau_i \) and \( \tau_f \) be dynamical variables is not necessary. They can be constant, but their difference should not be integer. Only when \( \tau_f - \tau_i \in \mathbb{Z} \) a restriction of class of trajectories occurs — for such trajectories the light stairway has integer number of steps (12). When \( \tau_f - \tau_i \notin \mathbb{Z} \), positions of boundary points are not related because of sufficient freedom in choice of parametrization of trajectories in interval \( \tau \in [0, 1] \). After any sufficiently small variation of the trajectories with \( \tau_f - \tau_i \notin \mathbb{Z} \), light stairway parametrization can be chosen on them with the same \( \tau_i \) and \( \tau_f \).

Further we restrict ourselves just to the trajectories with \( \tau_f - \tau_i \in \mathbb{Z} \). The ends of the trajectories will be fixed. In this case the equations of motion determine the conditional minimum of the action in class of trajectories satisfying condition (12). Light stairway with the initial value of parameter \( \tau_i \) divides the trajectories into \( N \) segments. The requirement \( \frac{\delta S}{\delta x} \bigg|_{\text{in cond.min.}} = 0 \) for inner points of the segments is automatically fulfilled: even though the positions of the boundary points of the segments can be related, there is nothing to constrain the position of inner points. For “free” motion ( in the limit \( \epsilon \to 0 \) ) the action reaches the minimum on straight-line segments (fig.7). If condition (12) is not fulfilled for straight lines connecting the ends of the trajectories, then the straight trajectories do not belong to the class, in which the variation is performed. The conditional minimum of the action is reached on a polygonal line.

The momentum of a particle is proportional to unit tangent vector to the world line \( n^\mu_x = \dot{x}^\mu / \sqrt{\dot{x}^2} \). On the polygonal world lines the momentum of each particle changes. From the relations of the next Section one can show
that for such trajectories the total momentum conserves (see (13)):

\[-m_1 \Delta n_x^+(\tau_i + n) = m_2 \Delta n_y^+(\tau_i + n - 1) = k_n^+\]
\[m_1 \Delta n_x^-(\tau_i + n) = -m_2 \Delta n_y^-(\tau_i + n) = k_n^-\]

where \(\Delta n^\mu(\tau) = n^\mu(\tau + o) - n^\mu(\tau - o)\) is discontinuity of the unit tangent vector. The force acting on a particle is singular. It has form (11).

The requirement \(\frac{\delta S}{\delta x} \bigr|_{\text{in cond.min.}} = 0\) for the vertices of the polygonal line represents a condition of smoothness of trajectories:

\[S = -m_1 \sum_{i=1}^{N} \sqrt{(x_{i+1} - x_i)^2} - m_2 \sum_{i=1}^{N-1} \sqrt{(y_{i+1} - y_i)^2} \]
\[\frac{\partial S}{\partial x_i} = m_1 (\Delta n_x)_i = 0\]

This condition can be satisfied, only if the exact minimum of the action belongs to the class of the allowed variations. It implies that the position of the ends of the trajectories (fixed in variations) is chosen so that the solution obeys condition (12).

By this means, the condition of smoothness of trajectories selects physical solutions among the trajectories with \(\tau_f - \tau_i \in \mathbb{Z}\), minimizing an action. For continuity of the unit tangential vector in 2-dimensional space it is sufficient to require the continuity for one of its components. By virtue of (13), it is sufficient to require continuous linking of tangent in single point \(\tau_i + n\).

### 3 Hamiltonian formulation

1. Let us break up the axis of parameter \(\tau\) into unit intervals. We denote the values of the coordinates of particle \(x\) in each interval as \(x_n\):

\[x_n^\pm(\tau) = x^\pm(\tau + n), \quad \tau \in [0, 1], \quad n \in \mathbb{Z}\]

In view of continuity of function \(x(\tau)\), a condition is imposed on the coordinates \(x_n\):

\[x_n^\pm(1) = x_{n+1}^\pm(0)\]
Let us rewrite action (4) into a form

$$ S = \int_0^1 d\tau \sum_n \left( -m_1 \sqrt{x_n^- x_{n+1}^-} - m_2 \sqrt{x_n^+ x_{n+1}^+} ight) - e_1 e_2 \left( \frac{x_n^-}{x_{n+1}^- - x_n^+} + \frac{x_n^+}{x_{n+1}^+ - x_n^-} \right) $$

(15)

To avoid divergence we will consider finite trajectories: summation in (15) is performed from $n = 1$ to $n = N$. When $n = N$ only the first term $-m_1 \sqrt{x_N^+ x_N^-}$ is present in the action.

The action (15) is given in the form of integral $S = \int d\tau \, L(x_n, \dot{x}_n)$ of the function depending on the coordinates and their first derivatives at one value of parameter $\tau$. Hamiltonian description is available for this mechanics.

2. Let us define momenta $p_n^\mu = \frac{\partial L}{\partial \dot{x}_n^\mu}$, conjugated to the coordinates $x_n^\mu$:

$$ p_n^+ = -\frac{m_1}{2} \sqrt{\dot{x}_n^-} - \frac{m_2}{2} \sqrt{\dot{x}_{n+1}^+} - \frac{e_1 e_2}{x_{n+1}^- - x_n^+} \quad n = 2..N $$

(16)

$$ p_n^- = -\frac{m_1}{2} \sqrt{\dot{x}_n^-} - \frac{m_2}{2} \sqrt{\dot{x}_{n+1}^-} - \frac{e_1 e_2}{x_{n+1}^- - x_n^+} \quad n = 2..N $$

$$ p_1^+ = -\frac{m_1}{2} \sqrt{\dot{x}_1^-} - \frac{m_2}{2} \sqrt{\dot{x}_2^-} - \frac{e_1 e_2}{x_2^- - x_1^-} \quad p_N^+ = -\frac{m_1}{2} \sqrt{\dot{x}_N^-} $$

$$ p_1^- = -\frac{m_1}{2} \sqrt{\dot{x}_1^-} - \frac{e_1 e_2}{x_2^- - x_1^-} \quad p_N^- = -\frac{m_1}{2} \sqrt{\dot{x}_N^-} - \frac{m_2}{2} \sqrt{\dot{x}_{N-1}^-} $$

The stationary principle of action (15) has a form

$$ \delta S = \int_0^1 d\tau \sum_n \left( -\dot{p}_n^\mu + \frac{\partial L}{\partial x_n^\mu} \right) \delta x_n^\mu + \sum_n p_n^\mu \delta x_n^\mu \bigg|_0^1 = 0 $$

(17)

An equality to zero of the first item manifests a minimum of the action with respect to those variations of the trajectories, when points $x_n(0), x_n(1)$ are fixed. This condition leads to equation (5). The second item in (15)
represents off-integral terms extracted in variation of the action (15). An equality to zero of the second item is a condition of the action minimum with respect to variations of positions \( x_n(0), \ x_n(1) \). The boundary points \( x_1(0), \ y_1(0) \) and \( x_N(1), \ y_{N-1}(1) \) are fixed in the variations:

\[
\begin{align*}
\delta x_1^+(0) &= \delta y_1^+(0) = 0 \quad \delta x_1^-(0) = \delta y_1^-(0) = \delta x_2^- (0) = 0 \\
\delta x_N^-(1) &= \delta y_{N-1}^-(1) = 0 \quad \delta x_N^+(0) = \delta y_{N-1}^+(1) = \delta x_{N-1}^+ (1) = 0
\end{align*}
\] (18)

With regard to these conditions and the condition of continuous linking of coordinates (14) the requirement of the equality to zero of the second item in (17) is written as condition of continuous linking of the momenta

\[
p_n^\mu(1) = p_{n+1}^\mu(0) \quad n = 2..N - 2 \quad \mu = +, - ; \quad n = 1 \quad \mu = + ; \quad n = N - 1 \quad \mu = -
\] (19)

3. The action (15) is parametrically invariant, hence Legendre transformation (16) is degenerate: momenta \( p_n^\mu \) do not change in replacing \( \dot{x}_n^\mu \rightarrow \lambda \dot{x}_n^\mu \). This implies that the momenta can not be independent, there is a relation among them. This relation (Hamiltonian constraint) has a sense of compatibility condition of system (16) for the velocities \( \dot{x}_n \). Let us obtain this condition.

We introduce notations

\[
r_n^\pm = \frac{2}{m_1} \left( p_n^\pm + \frac{e_1 e_2}{x_{n+1}^\pm - x_n^\pm} \right) \quad n = 1..N - 1 , \quad r_N^\pm = \frac{2}{m_1} p_N^\pm
\] (20)

\[
\beta = \frac{m_1}{m_2}, \quad v_n = \frac{\dot{x}_n^-}{\dot{x}_n^+} \quad n = 1..N , \quad u_n = \frac{\dot{x}_{n+1}^-}{\dot{x}_n^+} \quad n = 1..N - 1
\]

Variables \( v_n, u_n \) are sufficient to determine the velocities:

\[
\sqrt{\dot{x}_n^-} = v_n \sqrt{\dot{x}_n^+} \quad \sqrt{\dot{x}_n^+} = \frac{u_{n-1}}{v_n} \sqrt{\dot{x}_{n-1}^+} = \prod_{k=2}^{n} \frac{u_k-1}{v_k} \cdot \sqrt{\dot{x}_1^+}
\]

value \( \dot{x}_1^+ \) is arbitrary.

Let us rewrite equations (16):

\[
\begin{align*}
r_n^+ &= -v_n - u_n / \beta \quad n = 1..N - 1 \\
r_n^- &= -\frac{1}{v_n} - \frac{1}{\beta u_n} \quad n = 2..N
\end{align*}
\] (21)

13
\[ r_N^+ = -v_N \quad r_1^- = -\frac{1}{v_1} \]

(2N equations on 2N - 1 variables \( v_n, u_n \))

\[ v_n = -u_n/\beta - r_n^+ \quad n = 2..N - 1 \quad (22) \]

\[ u_n = \frac{\beta(r_n^+r_n^- - 1) \cdot u_{n-1} + r_n^+}{-r_n^- \cdot u_{n-1} - 1/\beta} \quad n = 2..N - 1 \quad (23) \]

\[ v_1 = -\frac{1}{r_1} \quad u_1 = \frac{\beta(r_1^+r_1^- - 1)}{-r_1} \quad (24) \]

\[ v_N = -r_N^+ \quad u_{N-1} = \frac{-r_N^+}{\beta(r_N^+r_N^- - 1)} \quad (25) \]

Relation (22) expresses \( v_n \) in terms of \( u_n \). Sequential values \( u_n \) are related via linear-fractional transformation (23). Let us represent this transformation in a matrix form:

\[ u_n = \frac{\Psi^1_n}{\Psi^2_n}, \quad \left( \begin{array}{l} \Psi^1_n \\ \Psi^2_n \end{array} \right) = g_n \left( \begin{array}{l} \Psi^1_{n-1} \\ \Psi^2_{n-1} \end{array} \right), \quad g_n = \left( \begin{array}{cc} \beta(r_n^+r_n^- - 1) & r_n^+ \\ -r_n^- & -1/\beta \end{array} \right) \quad (26) \]

The initial value \( u_1 \) for recurrent formula (23) is given by expression (24). In a matrix form:

\[ u_1 = \frac{\Psi^1_1}{\Psi^2_1} \quad \Psi_1 = g_1 \left( \begin{array}{l} 1 \\ 0 \end{array} \right) \implies \Psi_n = g_n...g_1 \left( \begin{array}{l} 1 \\ 0 \end{array} \right) \]

Condition (25) for value \( u_{N-1} \) can be written as

\[ \beta(r_N^+r_N^- - 1)\Psi^1_{N-1} + r_N^+\Psi^2_{N-1} = 0 \iff (1 0)g_N\Psi_{N-1} = 0 \]

or

\[ \Delta = (1 0) \quad g_N...g_1 \left( \begin{array}{l} 1 \\ 0 \end{array} \right) = 0 \quad (27) \]

Equation (27) is a desired condition on \( (x_n, p_n) \), when system (13) is compatible.
Let us consider a limit of infinite trajectories. We change the numeration of the variables: let $n$ changes from $-N$ to $N$, in the limit $N \to \infty$ the initial and final points $x_{-N}, x_N$ tend to infinity. The constraint is

$$
\Delta = (1 \ 0) \ g_N \ldots g_{-N} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0
$$

(28)

Variables $v_n, u_n$ determine the slopes of tangents to the world lines $x$ and $y$ (see (20)). They have definite limits at $n \to \infty$:

$$
\lim_{n \to \pm\infty} v_n = v_{\pm\infty}, \quad \lim_{n \to \pm\infty} u_n = u_{\pm\infty}
$$

Variables $r_n^\mu$ have definite limits also. In view of (21), values $r_n^\mu$ at the initial and final segments of trajectories are defined in a special way to compared with other $r_n^\mu$. The limiting values

$$
r_{-N}^- \to -\frac{1}{v_{-\infty}}, \quad r_{+N}^+ \to v_{+\infty}, \quad N \to \infty
$$

(29)

$$
r_{-N}^- \to -\frac{1}{v_{+\infty}} - \frac{1}{\beta u_{+\infty}}, \quad r_{+N}^+ \to -v_{-\infty} - u_{-\infty}/\beta
$$

differ from those for inner segments

$$
r_{-n}^- \to -\frac{1}{v_{\pm\infty}} - \frac{1}{\beta u_{\pm\infty}}, \quad r_{+n}^+ \to -v_{\pm\infty} - u_{\pm\infty}/\beta
$$

(30)

$$
|n| < N, \quad n \to \pm\infty, \quad N \to \infty,
$$

the associated matrices are different too:

$$
g_{\pm\infty} = \lim_{N \to \infty} g_{\pm(N-1)} = g^{(0)}_{\pm\infty} = \lim_{N \to \infty} g_{\pm N}.
$$

Asymptotic values $u_{\pm\infty}$ are stationary points of transformation (23) at $n \to \infty$. This means that $\Psi_{\pm\infty} = \begin{pmatrix} u_{\pm\infty} \\ 1 \end{pmatrix}$ are eigen vectors of matrices $g_{\pm\infty}$:

$$
g_{\pm\infty} \Psi_{\pm\infty} \sim \Psi_{\pm\infty}.
$$

(31)

One can verify this directly, substituting (30) into (20):

$$
\begin{pmatrix}
\frac{1}{\beta} + \frac{u_{\pm\infty}}{v_{\pm\infty}} + \frac{v_{\pm\infty}}{u_{\pm\infty}}, \quad -\frac{1}{\beta} u_{\pm\infty} - v_{\pm\infty}, \quad \frac{1}{v_{\pm\infty}} - \frac{1}{\beta}
\end{pmatrix}
\begin{pmatrix}
u_{\pm\infty} \\ u_{\pm\infty} \\ 1
\end{pmatrix} =
\begin{pmatrix}
u_{\pm\infty} \\ u_{\pm\infty} \\ 1
\end{pmatrix}
$$
Condition (31) can also be written in a form
\[ \tilde{\Psi}_{\pm\infty} g_{\pm\infty} \sim \tilde{\Psi}_{\pm\infty}, \quad \tilde{\Psi}_{\pm\infty} = (1 - u_{\pm\infty}) \]

Proof.

It follows from \( g\Psi \sim \Psi \) that \( \tilde{\Psi} g\Psi = 0 \), where \( \tilde{\Psi} = \Psi^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \).

On the other hand, a general solution of a linear equation on \( V \):
\( V\Psi = 0 \) in 2-dimensional space is \( V \sim \tilde{\Psi} \). Hence \( \tilde{\Psi} g \sim \tilde{\Psi} \).

Substituting (29) into (26), one can easily show, that
\[ g^{(0)}_{-\infty} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{v_{-\infty}} \begin{pmatrix} u_{-\infty} \\ 1 \end{pmatrix} \sim \Psi_{-\infty} \]
\[ (1 \ 0) \ g^{(0)}_{+\infty} = \frac{v_{+\infty}}{u_{+\infty}} (1 - u_{+\infty}) \sim \tilde{\Psi}_{+\infty} \]

Therefore, in the limiting expression of constraint (28):
\[ \Delta = (1 \ 0) \ g^{(0)}_{+\infty} g_{+\infty} \cdots g_1 \ g_0 \ g_{-1} \cdots g_{-\infty} \ g^{(0)}_{-\infty} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \] (32)

the facings \( 1 \ 0 \) \( g^{(0)}_{+\infty} \) and \( g^{(0)}_{-\infty} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) are eigen vectors of the asymptotic matrices \( g_{+\infty} \) and \( g_{-\infty} \), respectively. Hence one can insert any number of asymptotic matrices after left facing and before right facing, and equation (32) holds true.

The constraint is just a consequence of momenta definition (16), it is fulfilled for any world line. We will draw an analogy to mechanics of free relativistic particle \( L = m\sqrt{x^2} \): here the constraint \( p^2 - m^2 = 0 \) on momentum \( p^\mu = m\dot{x}^\mu / \sqrt{x^2} \) is fulfilled for any world line. Equations of motion have not been taken into account yet.

4. Canonical Hamiltonian for the action (15) vanishes:
\[ H_c = \sum_n p_n^\mu \dot{x}_n^\mu - L = 0 \]
According to Dirac’s description of constrained Hamiltonian systems, the Hamiltonian is constraint (27):

\[ H = \lambda \Delta \approx 0 , \]

\( \lambda \) is Lagrangian multiplier (an arbitrary function of \( \tau \)). Weak equality symbol implies that condition \( \Delta = 0 \) must be considered after Poisson brackets calculation.

Hamiltonian equations of motion are

\[ \dot{x}_n^\pm = \lambda \{ x_n^\pm, \Delta \} = \lambda \frac{\partial \Delta}{\partial p_n^\pm} , \quad \dot{p}_n^\pm = \lambda \{ p_n^\pm, \Delta \} = -\lambda \frac{\partial \Delta}{\partial x_n^\pm} \]

In view of (20),

\[ \frac{\partial}{\partial p_n^\pm} \Delta(r) = \frac{2}{m_1} \frac{\partial}{\partial r_n^\pm} \Delta(r) \]

\[ \frac{\partial}{\partial x_n^\pm} \Delta(r) = \frac{2 e_1 e_2}{m_1} \left( \frac{1}{(x_{n+1}^\pm - x_n^\pm)^2} \frac{\partial}{\partial r_n^\pm} - \frac{1}{(x_n^\pm - x_{n-1}^\pm)^2} \frac{\partial}{\partial r_{n-1}^\pm} \right) \Delta(r) \quad \text{see}^1 \]

Hence

\[ \dot{r}_n^\pm = \frac{2}{m_1} \left( \dot{p}_n^\pm - e_1 e_2 \frac{\dot{x}_n^\pm - \dot{x}_n^\pm}{(x_{n+1}^\pm - x_n^\pm)^2} \right) \quad \text{see}^2 \]

\[ = -\frac{4 e_1 e_2}{m_1^2} \left( \frac{1}{(x_{n+1}^\pm - x_n^\pm)^2} \frac{\partial}{\partial r_{n+1}^\pm} - \frac{1}{(x_n^\pm - x_{n-1}^\pm)^2} \frac{\partial}{\partial r_{n-1}^\pm} \right) \Delta(r) \quad \text{see}^3 \]

Let us calculate the derivatives \( \frac{\partial \Delta}{\partial \gamma} \)

\[ \frac{\partial g_n}{\partial r_n^\pm} = \left( \begin{array}{c} \beta r_n^- \ 1 \\ 0 \ 0 \end{array} \right) = -\beta \left( \begin{array}{c} 1 \\ 0 \end{array} \right) g_n \]

\[ \frac{\partial g_n}{\partial r_n^-} = \left( \begin{array}{c} \beta r_n^+ \ 0 \\ -1 \ 0 \end{array} \right) = \beta g_n \left( \begin{array}{c} 0 \\ 1 \end{array} \right) (1 0) \quad \frac{\partial g_k}{\partial r_n^\pm} = 0 , \ k \neq n \]

\[ \Delta_n^\pm = \frac{\partial \Delta}{\partial r_n^\pm} = -\beta \cdot (1 0) g_N g_{n+1} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \cdot (0 1) g_n \cdots g_1 \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \]

---

1When \( n = N \), the first item is absent inside brackets, when \( n = 1 \), the second item is absent

2When \( n = N \), the second item is absent
\[
\Delta_n^- = \frac{\partial \Delta}{\partial r_n^-} = \beta \cdot (1 0) g_N \cdots g_n \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \cdot (1 0) \ g_{n-1} \cdots g_1 \left( \begin{array}{c} 1 \\ 0 \end{array} \right)
\]
\[
\Delta_n^+ = -\beta \tilde{\Psi}_n^1 \Psi_n^2
\]
\[
\Delta_n^- = \beta \tilde{\Psi}_{n-1}^2 \Psi_{n-1}^1
\]

where
\[
\Psi_n = g_n \cdots g_1 \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \sim \left( \begin{array}{c} u_n \\ 1 \end{array} \right),
\]
\[
\tilde{\Psi}_n = (1 0) g_N \cdots g_{n+1} \sim (1 - u_n)
\]
\[
\Delta = \tilde{\Psi}_n^1 \Psi_n^1 + \tilde{\Psi}_n^2 \Psi_n^2 = 0
\]

Hamiltonian equations are
\[\begin{aligned}
\dot{x}_n^+ &= \frac{2\lambda}{m_1} \Delta_n^+(r) \\
\dot{r}_n^+ &= -\frac{4\lambda e_1 e_2}{m_1} \left( \frac{\Delta_n^+(r)}{x_n^+ - x_n^+ - 1} \right) - \frac{\Delta_n^-(r)}{(x_n^+ - x_n^+ - 1)^2}
\end{aligned}\]

Let us derive the ratios
\[\frac{\dot{x}_{n+1}^-}{\dot{x}_n^+} = \frac{\Delta_{n+1}^-}{\Delta_n^+} \frac{\Delta_n^2}{\Delta_n^1} \frac{\Psi_n^1}{\Psi_n^2} \left( \frac{\Psi_n^1}{\Psi_n^2} \right)^2 \left( \frac{\Psi_n^1}{\Psi_n^2} \right)^2 \left( u_n \right)^2\]
\[\frac{\dot{x}_n^-}{\dot{x}_n^+} = \frac{\Delta_n^-}{\Delta_n^+} \frac{\Psi_n^1}{\Psi_n^2} \left( \frac{\Psi_n^1}{\Psi_n^2} \right)^2 \frac{\Psi_n^1}{\Psi_n^2} = (v_n)^2\]

In the proof of the last equality the following identities were used
\[\psi_n = -u_n / \beta - r_n^+ (1 - u_n) g_n \left( \begin{array}{c} 0 \\ 1 \end{array} \right)\]
\[= -\frac{1}{\Psi_n^1} \cdot \tilde{\Psi}_n g_n \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \frac{\Psi_n^1}{\Psi_n^2} = \frac{\Psi_n^1}{\Psi_n^2} \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \left( \frac{-1 / \beta}{r_n^+} \left( r_n^+ r_n^- - 1 \right) \right) \Psi_n = -u_n / \beta - r_n^+ = v_n\]

The first equation in (36) is fulfilled for any world line when momenta definitions in terms of velocities are substituted and a proper Lagrangian multiplier
\[ \lambda = m_1 \dot{x}_1^+ / \left( 2 \Delta^+_1 (r(\dot{x})) \right) \] is chosen. Analogously to mechanics of relativistic particle, where an equation \( \dot{x}^\mu = \lambda \{ x^\mu, p^2 - m^2 \} = 2 \lambda p^\mu \) after substitution \( p^\mu = m \frac{x^\mu}{\sqrt{x^2}} \) and choice \( \lambda = \frac{\sqrt{x^2}}{2m} \) is satisfied identically on any world line. The shape of the world line is determined by the second equation in (36).

Equations (36) conserve the constraint condition: \( \Delta = \lambda \{ \Delta, \Delta \} = 0 \). Let us verify this directly:

\[
\dot{\Delta} = \sum_{n=1}^{N} \dot{r}^\mu_n \frac{\partial \Delta}{\partial r^\mu_n} = - \frac{4 \lambda e_1 e_2}{m_1^2} \sum_{\mu} \left( \sum_{n=1}^{N-1} \left( \Delta^\mu_{n+1} \Delta^\mu_n \right) \frac{\Delta^\mu_n}{(x^\mu_{n+1} - x^\mu_n)^2} - \sum_{n=2}^{N} \left( \Delta^\mu_{n-1} \Delta^\mu_n \right) \frac{\Delta^\mu_n}{(x^\mu_n - x^\mu_{n-1})^2} \right) = 0
\]

Hamiltonian equations (36) specify a phase flow on a surface of constraint (27), which stratifies this surface into non-intersecting phase trajectories. The projection of the phase trajectory into configuration space \( x_n \) gives the solution of Lagrangian equations for the action (15). It is necessary to select from all phase trajectories those, that match continuous linking conditions.

5. Lagrangian multiplier in (36) affects only the parametrization of phase trajectories. It can be excluded by the parameter replacing:

\[ \tau \rightarrow \tilde{\tau}(\tau) = \int_{0}^{\tau} \lambda(\tau') \, d\tau' \]

Value \( \tilde{\tau}(1) = T \) enters in the linking conditions. In new parametrization these conditions have the form:

\[
\begin{align*}
\hat{x}^\mu_n(T) &= x^\mu_{n+1}(0) \quad n = 1..N - 1, \quad \hat{x}^\mu_N(T) = x^\mu_f \\
\hat{p}^\mu_n(T) &= p^\mu_{n+1}(0) \quad n = 2..N - 2 \quad \mu = +, -; \\
& \quad n = 1 \quad \mu = +; \quad n = N - 1 \quad \mu = -
\end{align*}
\]

Values \( \left( x^\mu_n, p^\mu_n \right)_T \) are determined in the solution of differential equations (36), they are functions of initial data \( \left( x^\mu_n, p^\mu_n \right)_0 \) and “time” \( T \). Conditions (37) are the system of non-linear equations on initial data and \( T \). Counting shows that the number of equations equals the number of independent variables.
(see fig.8):

\[ 2N \text{ conditions on coordinates} + 2(N - 2) \text{ conditions on momenta} + \]
\[ + 1 \text{ constraint (27)} = (2N - 4) \text{ coordinates} + \]
\[ + 2N \text{ momenta} + 1 \text{ Lagrangian multiplier ( or “time”) } \]

In the previous Section it was shown that excluding of non-physical solutions requires continuous linking of tangent in one point of the trajectory

\[ u_n(T) = u_{n+1}(0) \] (38)

To satisfy this condition one should carry some fixed boundary value ( e.g., \( x_N(0) \) ) into the set of independent variables.

In the limit \( N \to \infty \) \((n = -N..N)\) the sense of the linking conditions becomes more clear:

\[ (x, p)_T = S (x, p)_0 \] (39)

where a transformation \( S (x_n, p_n) = (x_{n+1}, p_{n+1}) \) is a shift of index \( n \).

The index shift does not change constraint condition (32):

\[ \Delta = \tilde{\Psi} + \cdots + g_{n+1} g_n g_{n-1} \cdots \Psi_{-\infty} = 0 \xrightarrow{S} \Delta = \tilde{\Psi} + \cdots + g_{n+2} g_{n+1} g_n \cdots \Psi_{-\infty} = 0 \]

and equations of motion (36) \( \because \Delta_n^{\pm} (Sr) = \Delta_{n+1}^{\pm} (r) \). This means that transformation \( S \) transfers the phase trajectories into the phase trajectories. Requirement (39) implies that phase trajectories \( (x, p)_T \) and \( S (x, p)_T \) should have a common point. This is possible only when they coincide \((\text{up to reparametrizations})\). Therefore, condition (39) selects those phase trajectories, that transform into themselves by transformation \( S \).

Let point \( (x, p) \) belong to a plane \( x_0^+ = C \). Let us apply to it the transformation \( S \), then bring back it onto the plane \( x_0^+ = C \) by movement \( R \) along the phase trajectory. The phase trajectory involved transforms into itself by transformation \( S \) only if

\[ RS (x, p) = (x, p) \]

Therefore, the problem reduces to a search for stationary points of discrete transformation \( RS \).
Continuous linking conditions (37), (38) are non-Hamiltonian. They relate values of dynamical variables at distinct evolution parameters \( \tau \). Mechanics considered has yet another element non-traditional for Hamiltonian description: parameter \( \tau \), used in this Section, is not time in any reference frame. Coordinates \( x_n(\tau) \) at one \( \tau \) and different \( n \) are separated by time-like interval. Usually the parametrization of the world line is introduced with the aid of its sectioning by a specified set of space-like surfaces [8]. This construction is convenient to achieve the monotonic parametrization. However, it is not necessary, its rejection does not lead to contradictions. Let us remember that Hamiltonian description is applied just to those variational problems, whose formulation does not mention time. So, Hamiltonian formalism enables one to obtain the shape of minimal surfaces both in the Minkowski space (world sheets of relativistic string) and Euclidean space (soap films).

Nevertheless, just the absence of time interpretation for parameter \( \tau \) in our problem allows one to bypass “no-interaction” theorem [9]. According to this theorem, the following requirements

1. single-time Hamiltonian description is applied (i.e. one-parametrical Hamiltonian description, in which the parameter is time in some reference frame)

2. coordinates of particles in the Minkowski space are canonical coordinates of Hamiltonian mechanics

3. world lines of particles are curves in the Minkowski space, on which Lorentz group acts by rotations and reparametrizations

are compatible only in mechanics of non-interacting particles. For the mechanics under discussion requirements 2 and 3 are satisfied. Requirement 1 is violated: canonical coordinates are the set \( x_n^\mu \) of coordinates of one particle in different moments of the Minkowski time.

**Results**

Hamiltonian mechanics of one-dimensional two body problem in Wheeler-Feynman electrodynamics is constructed as follows. In phase space \((x_n^\mu, p_n^\mu)\)
\( n = 1..N \) constraint (27) is given. Hamiltonian equations (36) specify the phase flow on the surface of the constraint. Conditions (37),(38) select the physical solutions among phase trajectories.

So, the structure of solutions of Lagrangian equations – differential-difference equations (5) – is determined by the system of ordinary differential equations (36) and the system of non-linear equations (37),(38). A separate paper will be devoted to the examination of this structure.

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Captions to figures

fig.1 Stairway parametrization.
fig.2 Extra force acts on the marked segments of the world lines.
fig.3 Light impulse reflects from the moving mirror.
fig.4 When the boundary points of $x$ are fixed, the boundary points of $y$
change freely.
fig.5 Under the allowed variations the shifts of parametrization are can-
celled at infinity.
fig.6 Action minima.
fig.7 If the light stairway is forced to have an integer number of steps, the
minimum of the action is reached on a polygonal line.
fig.8 Number of initial data is equal to the number of linking conditions.
fig. 1
fig. 2
\[ \tan \alpha = v \]

\[ \Delta x_0 \]

\[ \Delta x_1 \]

\[ \Delta t \]

\[ \Delta t' \]

\[ \Delta t' \]

\[ \Delta t \]

fig. 3
fig. 4
fig. 6
\( \tau_{z+n} \)

\( \tau_{z+n-1} \)

\( k_n^- \)

\( k_n^+ \)

fig. 7
fixed values
● initial data 4
× final data 2

variables
○ initial data $2N-4 \ (x) + 2N \ (p)$

conditions $2N \ (x) + 2(N-2) \ (p)$

fig. 8