Superintegrable chiral Potts model: Proof of the conjecture for the coefficients of the generating function $G(t, u)$

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Abstract. In this paper, we prove the conjecture for the coefficients of the two variable generating function used in our previous paper. The conjecture was tested numerically before, but its proof was lacking up to now.

1. Introduction

In our previous paper [1] we made a conjecture for the coefficients of a two-variable generation function, backed by exact results for special cases and numerical evidence. This enabled us to derive a formula of Baxter [2, 3] for the order parameter of the chiral Potts model, from which an algebraic proof [3, 4, 5, 6] of the old conjecture of Albertini et al. [7] followed. Thus we were able to obtain the same results by a different method, namely using explicit expressions for the ground state eigenvectors.

More specifically, the coefficients of the generating function

\[ G(t, u) = \sum_{\{0 \leq n_j \leq N-1\}} \prod_{j=1}^{L} \frac{(1-t^N)^{L-1}(1-u^N)^{L-1}}{(1-t\omega^N_j)(1-u\omega^{-N_j})}, \quad N_j = \sum_{i<j} n_i, \quad (1.1) \]

are symmetric, i.e. $G_{\ell,k} = G_{k,\ell}$, and for $P \geq Q$ they have been conjectured to be

\[ G_{\ell N+Q,jN+P} = G_{jN+P,\ell N+Q} \]

\[ = \sum_{n=0}^{j} \left[ (j-n+1)\Lambda_n^Q \Lambda_{\ell+1+j-n}^P - (n-\ell)\Lambda_{\ell+1+j-n}^Q \Lambda_n^P \right], \quad (1.3) \]

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where $\Lambda_n^P = c_{n,N+P}$ are the coefficients of the polynomial

$$Q(t) = \frac{(1-t^N)^L}{(1-t)^L} = \sum_{m=0}^{L(N-1)} c_m t^m. \quad (1.4)$$

For $P = Q$ this formula has been proven in [9], using a transformation formula [8, chapter 10] for the hypergeometric series. For $P \neq Q$, we have shown in [11], that there exist messy extra terms which make it impossible to adopt the $P = Q$ proof of [9].

To calculate the pair correlation of the superintegrable chiral spin chain, we need to derive identities for a Fourier transform of the generating function in (1.1). For this reason, we feel it may be necessary to prove our conjecture using different approaches.

As can be seen from the definition of $N_j$ in (1.1), we have $N_1 = 0$. The condition $\sum n_i = N$ in (1.1) may be replaced by $0 \leq N_2 \leq N_3 \cdots \leq N_L \leq N$ by excluding the case $N_i = N$ for all $i = 1, \cdots, L$. We rewrite (1.1) as

$$G(t, u) = \frac{1}{(1-t^N)(1-u^N)} \left[ f_0 \sum_{0 \leq N_2 \leq N_3 \cdots \leq N_L \leq N} f_{N_2} f_{N_3} \cdots f_{N_L} - L f_0^L \right], \quad (1.5)$$

where $f_N = f_0$ and

$$f_{N_i} = \frac{(1-t^N)(1-u^N)}{(1-t^N_i)(1-u^N_i)} = \sum_{\mu_i = 0}^{N-1} \sum_{\nu_i = 0}^{N_i} t^{\mu_i} u^{\nu_i} \omega^{N_i(\mu_i - \nu_i)}. \quad (1.6)$$

Substituting the above equation into (1.5), we find

$$G(t, u) = \frac{1}{(1-t^N)(1-u^N)} \left[ F(t, u) - L f_0^L \right], \quad (1.7)$$

where $f_0^L = Q(t)Q(u)$, and

$$F(t, u) = \prod_{i=1}^{L} \sum_{\mu_i = 0}^{N-1} \sum_{\nu_i = 0}^{N_i} t^{\mu_i} u^{\nu_i} \mathcal{V}(\{\mu_i\}, \{\nu_i\}), \quad (1.8)$$

with

$$\mathcal{V}(\{\mu_i\}, \{\nu_i\}) = \sum_{0 \leq N_2 \leq N_3 \cdots \leq N_L \leq N} \epsilon_2^{N_2} \epsilon_3^{N_3} \cdots \epsilon_L^{N_L}, \quad \epsilon_i = \omega^{\mu_i - \nu_i}. \quad (1.9)$$

In section 2, we use the method in [8, chapter 11], to analyze the sum $\mathcal{V}(\{\mu_i\}, \{\nu_i\})$, followed by the analysis of $F(t, u)$ in sections 3, 4 and 5.

2. Reduction of the sum $\mathcal{V}(\{\mu_i\}, \{\nu_i\})$ by MacMahon method

We follow an idea of MacMahon described on page 555 in [8], to handle the inequality $0 \leq N_2 \leq N_3 \cdots \leq N_L$, by inserting new variables $\lambda_2, \lambda_3, \cdots, \lambda_{L-1}$ into (1.9), i.e.

$$\mathcal{V}(\{\mu_i\}, \{\nu_i\}) = \Omega \sum_{N_2=0}^{N} \sum_{N_3=0}^{N} \cdots \sum_{N_L=0}^{N} \epsilon_2^{N_2} \epsilon_3^{N_3} \cdots \epsilon_L^{N_L} \lambda_2^{N_2-N_3} \lambda_3^{N_3-N_4} \cdots \lambda_{L-1}^{N_{L-1}-N_L}. \quad (2.1)$$

By selecting only nonnegative powers of $\lambda_i$, which is denoted by the operator $\Omega_\ge$, sums with the inequality $0 \leq N_2 \leq N_3 \cdots \leq N_L$ can be replaced by independent sums over
Replacing with (2.8), we find
\[ \mathcal{V}(\{\mu_i\}, \{\nu_i\}) = \frac{1 - \epsilon_2/\lambda_2^{N+1}}{1 - \epsilon_2/\lambda_2} \cdots \frac{1 - \epsilon_j/(\lambda_j/\lambda)^{N+1}}{1 - \epsilon_j/(\lambda_j/\lambda)} \cdots \frac{1 - \epsilon_L/(\lambda_L^{N+1})}{1 - \epsilon_L/(\lambda_L)} \] (2.2)

It is easy to combine Lemma 11.2.3 and Proposition 11.3.1 in [8] in order to obtain
\[ \Omega \geq (1 - x\lambda)(1 - y_1/\lambda)(1 - y_2/\lambda) \cdots (1 - y_M/\lambda) \]
\[ = \frac{\lambda^{-\alpha}}{(1 - x)(1 - xy_1)(1 - xy_2) \cdots (1 - xy_M)}, \quad \alpha \geq 0. \] (2.3)

For the case with positive powers of \( \lambda \) in the numerator, the situation is rather different. We consider first
\[ \frac{\lambda^\alpha}{(1 - x\lambda)(1 - y/\lambda)^n} = \frac{\lambda^\alpha}{\sum_{m=0}^{\infty} (x\lambda)^m \sum_{\ell=0}^{\infty} \frac{(n)_{\ell}}{\ell!} y^\ell \lambda^{\alpha-\ell}} \]
\[ = \sum_{\ell=0}^{\alpha-1} \frac{(n)_{\ell}}{\ell!} y^\ell \sum_{m=0}^{\infty} x^m + \Omega \geq \sum_{m=0}^{\infty} (x\lambda)^m \sum_{\ell=\alpha}^{\infty} \frac{(n)_{\ell}}{\ell!} y^\ell \lambda^{\alpha-\ell}. \] (2.4)

Replacing \( m \to k = m - \ell + \alpha \) in the second term, this becomes
\[ \frac{\lambda^\alpha}{(1 - x\lambda)(1 - y/\lambda)^n} = \sum_{\ell=0}^{\alpha} \frac{(n)_{\ell}}{\ell!} y^\ell \frac{1}{1 - x} + \sum_{k=0}^{\infty} x^{k-\alpha} \sum_{\ell=\alpha}^{\infty} \frac{(n)_{\ell}}{\ell!} (xy)^\ell \] (2.5)
\[ = \frac{1}{1 - x} \left[ \frac{x^{-\alpha}}{(1 - xy)^n} + \sum_{\ell=0}^{\alpha-1} \frac{(n)_{\ell}}{\ell!} y^\ell (1 - x^{-\alpha+\ell}) \right]. \] (2.6)

Let the partial fraction decomposition be
\[ \frac{1}{R(t)} = \sum_{j=1}^{k} \sum_{i=1}^{n_j} \frac{a_{j,i}}{(1 - \bar{y}_j t)^i}, \quad \text{where} \quad R(t) = (1 - y_1 t)(1 - y_2 t) \cdots (1 - y_M t). \] (2.7)

Here, for given \( j \leq k \), \( 1/\bar{y}_j \) is one of the \( k \) distinct roots of \( R(t) \) with multiplicity \( n_j \) so that \( \sum_{j=1}^{k} n_j = M \). We may relate (2.7) to the symmetric functions \[11,12\] writing
\[ \frac{1}{R(t)} = \sum_{m=0}^{\infty} S_m t^m, \quad S_m = \sum_{1 \leq a_1 \leq a_2 \cdots \leq a_m \leq M} y_{a_1} y_{a_2} \cdots y_{a_m}. \] (2.8)

Expanding the right-hand side of the first equation in (2.7) as a series in \( t \), and comparing with (2.8), we find
\[ \frac{1}{R(t)} = \sum_{j=1}^{k} \sum_{i=1}^{n_j} \sum_{m=0}^{\infty} \frac{(i)_{m}}{m!} (\bar{y}_j t)^m, \quad \text{or} \quad S_m = \sum_{j=1}^{k} \sum_{i=1}^{n_j} a_{j,i} (i)_{m} / m! (\bar{y}_j)^m. \] (2.9)

We are now ready to prove the following proposition:
Proposition 1. For $\alpha > 0$, we find
\[
\Omega \geq \frac{\lambda^\alpha}{(1 - x \lambda)(1 - y_1 / \lambda)(1 - y_2 / \lambda) \cdots (1 - y_M / \lambda)} = \frac{\lambda^\alpha}{(1 - x \lambda)R(1/\lambda)}
\]
(2.10)
\[
x^{-\alpha} \geq \frac{1}{(1 - x)(1 - xy_1)(1 - xy_2) \cdots (1 - xy_M)} + \sum_{m=0}^{\alpha-1} S_m \frac{(1 - x^{m-\alpha})}{1 - x}
\]
(2.11)
\[
= \frac{1}{1 - x} \left[ \sum_{m=0}^{\alpha-1} S_m + \sum_{m=\alpha}^{\infty} S_m x^{m-\alpha} \right],
\]
(2.12)
where $S_m$ is the symmetric function defined in (2.8).

Proof. Substituting (2.7) into (2.10) and using (2.6), we find
\[
\Omega \geq \frac{\lambda^\alpha}{(1 - x \lambda)R(1/\lambda)} = \frac{1}{1 - x} \left[ \sum_{j=1}^{k} \sum_{i=1}^{n_j} a_{j,i} \left[ \frac{x^{-\alpha}}{(1 - xy_j)^{1-i}} \right] + \sum_{m=0}^{\alpha-1} \frac{(i)_m \bar{y}_m^m}{m!} (1 - x^{m-\alpha}) \right].
\]
(2.13)
From (2.7) and (2.9) we obtain (2.11); using the first equation in (2.8) we get (2.12). This completes the proof.

Now we define, for $2 \leq j \leq L - 1,$
\[
\mathcal{T}_j = \Omega \geq \left[ \frac{1 - (\epsilon_2 / \lambda_2)^N}{1 - \epsilon_2 / \lambda_2} \cdots \frac{1 - (\lambda_2 \epsilon_3 / \lambda_3)^{N^3}}{1 - \lambda_2 \epsilon_3 / \lambda_3} \cdots \frac{1 - (\lambda_j \epsilon_{j+1} / \lambda_{j+1})^{N^{j+1}}}{1 - \lambda_j \epsilon_{j+1} / \lambda_{j+1}} \right]
\]
(2.14)
with $\lambda_1 = \lambda_L = 1$. Then we use (2.3) and (2.12) to prove by induction the following:

Theorem 1. Let
\[
x_j = \epsilon_{j+1} / \lambda_{j+1}, \quad y_m^{(j-1)} = \prod_{i=0}^{m-1} \epsilon_{j-i}, \quad \text{for} \quad 1 \leq m \leq j - 1;
\]
\[
R_{j-1}(t) = \prod_{m=1}^{j-1} (1 - y_m^{(j-1)} t), \quad \frac{1}{R_{j-1}(t)} = \sum_{m=0}^{\infty} S_m^{(j-1)} t^m.
\]
(2.15)
Then
\[
\mathcal{T}_j = \frac{1}{(1 - x_j) R_{j-1}(x_j)} - \frac{x_j^{N+1}}{1 - x_j} \left[ \sum_{m=0}^{N} S_m^{(j-1)} + \sum_{m=N+1}^{\infty} S_m^{(j-1)} x_j^{m-N-1} \right].
\]
(2.16)

Proof. For $j = 2$, we can see from (2.14) that
\[
\mathcal{T}_2 = \Omega \geq \left[ \frac{1 - (y(1) / \lambda_2)^2}{1 - y(1) / \lambda_2} \right], \quad y^{(1)} = \epsilon_2, \quad x_2 = \frac{\epsilon_3}{\lambda_3}.
\]
(2.17)
It is straightforward to show by using (2.3) that
\[
\Omega \geq \left[ \frac{(y(1) / \lambda_2)^{N^2}}{1 - y(1) / \lambda_2} \right] = 0.
\]
(2.18)
Now we use (2.3) and (2.5) for \( n = 1 \) to find
\[
\mathcal{T}_2 = \frac{1}{1 - y(1)x_2} \left[ \sum_{j=0}^{N} (y(1))^m + \sum_{m=N+1}^{\infty} (y(1))^{m} x_2^{m-N-1} \right].
\] (2.19)

Since \( R_1(t) = 1 - ty(1), \) \( S_m(t) = (y(1))^m, \) we have proven (2.16) for \( j = 2. \) Next we assume (2.16) holds for \( j, \) and prove it for \( \mathcal{T}_{j+1}. \) Let
\[
y^{(j)}_1 = \epsilon_{j+1}, \quad y^{(j)}_{m+1} = \epsilon_{j+1} y^{(j-1)}_m = \prod_{i=0}^{m} \epsilon_{j-i+1}, \quad x_j = y^{(j)}_1/\lambda,
\] (2.20)
where \( \lambda = \lambda_{j+1}, \) so that
\[
(1 - x_j)R_{j-1}(x_j) = R_j(1/\lambda) = \prod_{m=1}^{j} (1 - y^{(j)}_m/\lambda).
\] (2.21)

From the definition in (2.14), we find
\[
\mathcal{T}_{j+1} = \Omega \left[ \mathcal{T}_j , \frac{1 - (\lambda x_{j+1})^{N+1}}{1 - \lambda x_{j+1}} \right] = \Omega \left[ \frac{1 - (\lambda x_{j+1})^{N+1}}{R_j(1/\lambda)(1 - \lambda x_{j+1})} \right] - \mathcal{Z}_{j+1},
\] (2.22)

where \( x_{j+1} = \epsilon_{j+2}/\lambda_{j+2} \) and
\[
\mathcal{Z}_{j+1} = \Omega \left( \frac{1 - (\lambda x_{j+1})^{N+1} (y^{(j)}_1/\lambda)^{N+1}}{(1 - y^{(j)}_1/\lambda)(1 - \lambda x_{j+1})} \left[ \sum_{m=0}^{N} S^{(j-1)}_m + \sum_{m=N+1}^{\infty} S^{(j-1)}_m (y^{(j)}_1/\lambda)^{m} \right] \right).
\] (2.23)

Since there are no positive powers of \( \lambda \) in the numerator, we can use (2.3) to show \( \mathcal{Z}_{j+1} = 0. \) Finally we use (2.3) and (2.12) to reduce (2.22) to
\[
\mathcal{T}_{j+1} = \frac{1}{R_j(x_{j+1})(1 - x_{j+1})} - \frac{x_{j+1}^{N+1}}{1 - x_{j+1}} \left[ \sum_{m=0}^{N} S^{(j)}_m + \sum_{m=N+1}^{\infty} S^{(j)}_m x_{j+1}^{m-N-1} \right].
\] (2.24)

This completes the proof. \( \square \)

Comparing (2.22) with (2.14), and using (2.15) so that \( x_{L-1} = \epsilon_L, \) we find from (2.16) that
\[
\mathcal{V}(\{\mu_i\}, \{\nu_i\}) = \mathcal{T}_{L-1} = \frac{1}{(1 - \epsilon_L)R_{L-2}(\epsilon_L)} - \frac{\epsilon_L^{N+1}}{1 - \epsilon_L} \left[ \sum_{m=0}^{N} S^{(L-2)}_m + \sum_{m=N+1}^{\infty} S^{(L-2)}_m \epsilon_L^{m-N-1} \right].
\] (2.25)

Using the equivalence of (2.12) and (2.11), and dropping the superscripts, we find the result
\[
\mathcal{V}(\{\mu_i\}, \{\nu_i\}) = \sum_{m=0}^{N} S_m \frac{\epsilon_m - \epsilon_L^{N+1}}{1 - \epsilon_L} = \sum_{\ell=0}^{N} \epsilon^\ell_L + \sum_{m=1}^{N} S_m \sum_{\ell=0}^{N-m} \epsilon_L^{m+\ell},
\] (2.26)
where $S_0 = 1$ and further $S_m$ defined in (2.8) are now given by

$$S_m = \sum_{1 \leq \ell \leq m} \sum_{\mu = 0}^{m-1} \sum_{\nu = 0}^{m-1} y_m \cdots y_{m-2} \cdots y_{m-1} \epsilon_{L-1-i}$$

with

$$y_m = \prod_{i=0}^{m-1} e^{L-1-i}, \quad \text{for} \quad 1 \leq m \leq L - 2.$$  (2.27)

Thus we have managed to relate the $L$-fold sum in (1.9) to the $m$-fold sums in (2.27) appearing in (2.26).

### 3. Reduction of $F(t, u)$

Define

$$R_{\ell, m} = \prod_{i=1}^{L} \sum_{\mu_i = 0}^{N-1} \sum_{\nu_i = 0}^{N-1} t^{\mu_i} u^{\nu_i} \epsilon^\ell_l S_m.$$  (3.1)

We find from (1.8) and (2.26) that

$$F(t, u) = \sum_{m=0}^{N} \sum_{\ell=m}^{N} R_{\ell, m} = \sum_{m=0}^{N} \sum_{\ell=0}^{m} R_{\ell, m}.$$  (3.2)

Since $S_0 = 1$, we substitute $\epsilon_i$ defined in (1.9) into (3.1) and carry out the summations to obtain

$$R_{\ell, 0} = \prod_{i=1}^{L} \sum_{\mu_i = 0}^{N-1} \sum_{\nu_i = 0}^{N-1} t^{\mu_i} u^{\nu_i} \epsilon^\ell_l S_0 = f_0^{L-1} f_{\ell},$$  (3.3)

where $f_\ell$ is defined in (1.6). Similarly, substituting (2.27) into (3.1), and using $\epsilon_i = \omega^{\mu_i - \nu_i}$, we find

$$R_{\ell, m} = \sum_{1 \leq a_1 \leq a_2 \cdots \leq a_m \leq L-2} \sum_{1=1}^{L} \sum_{\mu = 0}^{m-1} \sum_{\nu = 0}^{m-1} t^{\mu_i} u^{\nu_i} \epsilon^\ell_l y_{a_1} y_{a_2} \cdots y_{a_m}$$

$$= \sum_{1 \leq a_1 \leq a_2 \cdots \leq a_m \leq L-2} \sum_{1=1}^{L} \sum_{\mu = 0}^{m-1} \sum_{\nu = 0}^{m-1} (\omega^{\mu} t)^{\mu_i} \sum_{\nu_i = 0}^{m-1} (\omega^{-\mu} u)^{\nu_l} \sum_{\mu_l = 0}^{m-1} \sum_{\nu_l = 0}^{m-1} (\omega^{\mu_l} t)^{\mu_l} \sum_{\nu_l = 0}^{m-1}$$

$$\times \prod_{i=L-a_1}^{L-1-m} \sum_{i=0}^{m-1} (\omega^{i} t^{\mu_i} \sum_{i=0}^{m-1} (\omega^{-i} u^{\nu_i}) \sum_{i=0}^{m-1} \sum_{i=0}^{m-1} u^{\nu_i} \sum_{i=0}^{m-1} \sum_{i=0}^{m-1} u^{\nu_i}$$

$$= f_\ell \sum_{1 \leq a_1 \leq a_2 \cdots \leq a_m \leq L-2} \sum_{1=1}^{L} \sum_{\mu = 0}^{m-1} \sum_{\nu = 0}^{m-1} f_{a_1}^{a_1} f_{a_2}^{a_2} \cdots f_{a_{m-1}}^{a_{m-1}} f_{L-1-a_m}^{f_{L-1-a_m}}.$$  (3.4)

As the $a_i$ are now in the exponents, these functions are not the usual symmetric functions defined in (2.8).

### 3.1. Multiple sum $E_{m,k}(a, b)$

Define

$$E_{m,k}(a, b) = \sum_{a \leq a_1 \leq a_2 \cdots \leq a_k \leq b} \sum_{a_1}^{a_1} \sum_{a_2}^{a_2} \cdots \sum_{a_{m-1}}^{a_{m-1}} \left( \frac{f_{m-1}}{f_{m-2}} \right)^{a_1} \left( \frac{f_{m-2}}{f_{m-3}} \right)^{a_2} \cdots \left( \frac{f_{m-k}}{f_{m-k-1}} \right)^{a_k}.$$  (3.5)
Comparing with (3.4), we find

$$\mathcal{R}_{\ell,m} = f_0^L f_{t0} (f_t / f_0) \mathcal{E}_{m,m}(1, L - 2).$$

(3.6)

Letting \(a_i' = a_i - 1\) for \(i = 1, \cdots, k\) in (3.5), we can show

$$\mathcal{E}_{m,k}(a, b) = \sum_{n=0}^k \left( \frac{f_m}{f_{m-n}} \right)^a \mathcal{E}_{m-k,n}(a + 1, b).$$

(3.7)

Moreover, using the fact that the sum in (3.5) includes cases with \(a_1 = a_2 = \cdots = a_n = a\) and \(a + 1 \leq a_{n+1} \leq a_{n+2} \cdots \leq a_k \leq b\), for \(0 \leq n \leq k\), we find

$$\mathcal{E}_{m,k}(a, b) = \sum_{n=0}^k \left( \frac{f_m}{f_{m-n}} \right)^a \mathcal{E}_{m-k,n}(a + 1, b).$$

(3.8)

Next, using (3.7) and replacing \(n\) by \(k - n\) we rewrite this as

$$\mathcal{E}_{m,k}(a, b) = \left( \frac{f_m}{f_{m-k}} \right)^a \sum_{n=0}^k \mathcal{E}_{m-k+n,n}(a, b - 1).$$

(3.9)

From (3.6) and (3.9) we then find

$$\sum_{m=0}^\ell \mathcal{R}_{\ell,m} = f_0^L \sum_{m=0}^\ell \left( \frac{f_t}{f_0} \right)^a \mathcal{E}_{m,m}(1, L - 2) = f_0^L \mathcal{E}_{t0}(1, L - 1),$$

(3.10)

for \(0 \leq \ell \leq N\). Thus from (3.2) we have

$$\mathcal{F}(t, u) = \sum_{\ell=0}^N \sum_{m=0}^\ell \mathcal{R}_{\ell,m} = f_0^L \sum_{\ell=0}^N \mathcal{E}_{\ell,\ell}(1, L - 1) = f_0^L \mathcal{E}_{N,N}(1, L).$$

(3.11)

3.2. Multiple sum \(\mathcal{E}_{N,N}(1, L)\)

Noting from (1.6) that \(f_N = f_0\), we find from (3.5) that

$$\mathcal{E}_{N,N}(1, L) = \sum_{1 \leq a_1 \leq a_2 \cdots \leq a_N \leq L} f_{N-1}^{a_{N-1}} f_{N-2}^{a_{N-2}} \cdots f_0^{a_N + a_1},$$

(3.12)

where the exponents depend on differences of \(a_i\)'s, so that one of the summations can be carried out. The condition \(1 \leq a_1 \leq a_2 \cdots \leq a_N \leq L - 2\) may be broken up into the following cases:

- If \(a_1 = a_2 = \cdots = a_N \) for \(1 \leq a_1 \leq L\) in (3.12), the summand becomes 1 and the contribution of this case to \(\mathcal{E}_{N,N}(1, L)\) is \(C_1 = L\).
- When \(a_1 = \cdots = a_n < a_{n+1} = \cdots = a_N \) for \(n = 1, \cdots N - 1\), we find the contribution to the sum to be

$$C_2 = \sum_{n=1}^{N-1} \sum_{a_{n+1} = a_{n+1} = a_{n+1} + \cdots + a_n \leq}^L f_{N-n}^{a_{N-n}} f_0^{a_{N-n} + a_n}$$

$$= \sum_{n=1}^{N-1} \sum_{a_{n+1} = a_{n+1} = a_{n+1} + \cdots + a_n \leq}^L f_{N-n}^{a_{N-n}} f_0^{a_{N-n} + a_n}$$

$$= \sum_{n=1}^{N-1} \sum_{a_{n+1} = a_{n+1} = a_{n+1} + \cdots + a_n \leq}^L (L - \alpha) \sum_{\alpha=1}^{N-1} \left( \frac{f_{N-n}}{f_0} \right)^\alpha,$$

(3.13)
where \( \alpha = a_{n+1} - a_n \) and \( a = a_n \).

- When \( a_1 = \cdots = a_{n_1} < a_{n_1+1} = \cdots = a_{n_2} < \cdots = a_{n_k} < a_{n_k+1} = \cdots = a_N \) for \( 1 \leq n_1 < n_2 < \cdots < n_k \leq N - 1 \), it is straightforward to show that the contribution to the sum is

\[
C_k = \sum_{1 \leq n_1 < n_2 < \cdots < n_k \leq N-1} \sum_{a_1 = a_{n_1}}^{a_{n_1+1}} \sum_{a_2 = a_{n_2}}^{a_{n_2+1}} \cdots \sum_{a_k = a_{n_k}}^{a_{n_k+1}} f_{N-n_1}^{a_{n_2}-a_{n_1}} \times \cdots \times f_{N-n_k}^{a_{n_k}-a_{n_{k-1}}} f_0^{a_{n_1}-a_1}.
\]

Choosing new summation variables \( \alpha_i = a_{n_i+1} - a_{n_i} \) for \( 1 \leq i \leq k-1 \), \( \alpha_k = a_N - a_{n_1} \), and \( a = a_{n_1} \), we can carry out the sum over \( a \), \( 1 \leq a \leq L - \alpha_k \). Thus we find

\[
C_k = \sum_{1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq L} \sum_{i=1}^{N} \sum_{\alpha_1 < \alpha_2 < \cdots < \alpha_k \leq N-1} (L - \alpha_k) \mathcal{Y}_k(\{\alpha_i\}),
\]

where \( \{\alpha_i\} = (\alpha_1, \alpha_2, \cdots, \alpha_k) \) and

\[
\mathcal{Y}_k(\{\alpha_i\}) = \sum_{1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq N-1} \left[ \frac{f_{N-n_1}}{f_{N-n_2}} \right]^{\alpha_1} \left[ \frac{f_{N-n_2}}{f_{N-n_3}} \right]^{\alpha_2} \cdots \left[ \frac{f_{N-n_k}}{f_0} \right]^{\alpha_k}.
\]

Replacing \( N - n_i \) by \( n_{k-i+1} \) we may rewrite this as

\[
\mathcal{Y}_k = \mathcal{Y}_k(\{\alpha_i\}) = \sum_{1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq N-1} \left[ \frac{f_{n_{k-1}}}{f_{n_{k-2}}} \right]^{\alpha_1} \left[ \frac{f_{n_{k-2}}}{f_{n_{k-3}}} \right]^{\alpha_2} \cdots \left[ \frac{f_1}{f_0} \right]^{\alpha_k}.
\]

- Similarly, for \( a_1 \neq a_2 \neq \cdots \neq a_N \), we set \( \alpha_i = a_i - a_1 \) for \( 1 \leq i \leq N - 1 \), and carry out the summation over \( a_1 \) to find

\[
C_N = \sum_{1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{N-1} \leq L-2} (L - \alpha_{N-1}) \mathcal{Y}_{N-1},
\]

where

\[
\mathcal{Y}_{N-1} = \left[ \frac{f_{N-1}}{f_{N-2}} \right]^{\alpha_1} \left[ \frac{f_{N-2}}{f_{N-3}} \right]^{\alpha_2} \cdots \left[ \frac{f_1}{f_0} \right]^{\alpha_k}.
\]

This case is essentially the previous case with \( a_{n_i} \equiv a_i \).

Adding all the cases, we find from (3.11)

\[
\mathcal{F}(t, u) = f_0^L \mathcal{E}_{N,N}(1, L) = f_0^L \left[ L + \sum_{\alpha=1}^{L} (L - \alpha) \mathcal{Y}_1 + \sum_{1 \leq \alpha_1 < \alpha_2 \leq L} (L - \alpha_2) \mathcal{Y}_2 + \cdots + \sum_{1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{N-1} \leq L-2} (L - \alpha_{N-1}) \mathcal{Y}_{N-1} \right].
\]

4. Further simplification of \( \mathcal{F}(t, u) \) and \( \mathcal{G}(t, u) \)

Let

\[
\mathcal{J}_k = f_0^L \sum_{1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq L-1} (L - \alpha_k) \mathcal{Y}_k, \quad \mathcal{F}(t, u) = L f_0^L + \sum_{k=1}^{N-1} \mathcal{J}_k.
\]

where \( \mathcal{Y}_k \) is defined in (3.17). Using induction we now prove the following identity for partial \( \ell \)-fold sums appearing in (3.20) with (3.17) substituted:
Proposition 1. For $\ell \leq k$, we have

$$\mathcal{U}_\ell(a, b) \equiv \sum_{a_1 \leq a_2 \leq \ldots \leq a_\ell \leq b} \prod_{i=1}^{\ell} \left[ \frac{f_{n_k}}{f_{n_{k-1}}} \right]^{a_i} \left[ \frac{f_{n_{k-1}}}{f_{n_{k-2}}} \right]^{a_2} \ldots \left[ \frac{f_{n_{k-\ell}}}{f_{n_{k-\ell-1}}} \right]^{a_\ell}$$

$$= \sum_{i=0}^{\ell} \frac{(f_{n_{k-i}}/f_{n_{k-i-1}})^{b+1}(f_{n_k}/f_{n_{k-1}})^{a-1}}{\prod_{j=0,j \neq i}^{\ell}(f_{n_{k-i}}/f_{n_{k-j}} - 1)} \quad (4.2)$$

Proof. Consider $\ell = 1$. Then

$$\mathcal{U}_1(a, b) = \sum_{a_1 = a}^{b} \left[ \frac{f_{n_k}}{f_{n_{k-1}}} \right]^{a_1} = \frac{(f_{n_k}/f_{n_{k-1}})^{a}}{1 - (f_{n_k}/f_{n_{k-1}})} + \frac{(f_{n_k}/f_{n_{k-1}})^{b+1}}{(f_{n_{k-1}}/f_{n_k}) - 1}, \quad (4.3)$$

showing that (4.2) holds for $\ell = 1$.

Assuming (4.2) holds for $\ell$, we shall show it also holds for $\ell + 1$. Defining $g_i$ via the partial fraction decomposition

$$\frac{z^m}{\prod_{i=0}^{\ell}(z/f_{n_{k-i}} - 1)} = \sum_{i=0}^{\ell} g_i f_{n_{k-i}}^m, \quad g_i = \frac{1}{\prod_{j=0,j \neq i}^{\ell}(f_{n_{k-i}}/f_{n_{k-j}} - 1)}, \quad (4.4)$$

for $m \leq \ell$, we can rewrite (4.2), valid for $\ell$, as

$$\mathcal{U}_\ell(a, a_{\ell+1} - 1) = \sum_{i=0}^{\ell} g_i \left( \frac{f_{n_k}}{f_{n_{k-i}}} \right)^{a-1} \left( \frac{f_{n_{k-i}}}{f_{n_{k-\ell}}} \right)^{a_{\ell+1}}. \quad (4.5)$$

We then observe that $\mathcal{U}_\ell(a, a_{\ell+1} - 1) = 0$ for $a \leq a_{\ell+1} < a + \ell$, because $\sum_{i=0}^{\ell} g_i f_{n_{k-i}}^m = 0$ for $0 < m \leq \ell$. Therefore, we may extend the interval of summation over $a_{\ell+1}$ and find

$$\mathcal{U}_{\ell+1}(a, b) = \sum_{a_{\ell+1} = a}^{b} \mathcal{U}_\ell(a, a_{\ell+1} - 1) \left( \frac{f_{n_{k-\ell}}}{f_{n_{k-\ell-1}}} \right)^{a_{\ell+1}}$$

$$= \sum_{i=0}^{\ell} g_i \sum_{a_{\ell+1} = a}^{b} \left( \frac{f_{n_k}}{f_{n_{k-i}}} \right)^{a-1} \left( \frac{f_{n_{k-i}}}{f_{n_{k-\ell}}} \right)^{a_{\ell+1}} \left( \frac{f_{n_{k-\ell}}}{f_{n_{k-\ell-1}}} \right)^{a_{\ell+1}}. \quad (4.6)$$

Carrying out the summation over $a_{\ell+1}$, we find

$$\mathcal{U}_{\ell+1}(a, b) = \sum_{i=0}^{\ell} g_i \frac{(f_{n_{k-i}}/f_{n_{k-i-1}})^{a} - (f_{n_{k-i}}/f_{n_{k-\ell}})^{a+1}}{1 - f_{n_{k-i}}/f_{n_{k-\ell}}} \left( \frac{f_{n_k}}{f_{n_{k-i}}} \right)^{a-1}. \quad (4.7)$$

The $b$-independent part becomes the term $i = \ell + 1$ in (4.2) for $\mathcal{U}_{\ell+1}(a, b)$, as

$$\sum_{i=0}^{\ell} g_i \frac{1}{f_{n_{k-i}}/f_{n_{k-i-1}} - 1} = \frac{1}{\prod_{i=0}^{\ell}(f_{n_{k-i-1}}/f_{n_{k-i}} - 1)} \quad (4.8)$$

follows from (4.4). The terms $i = 0, \ldots, \ell$ are obtained rewriting

$$- \frac{g_i}{1 - f_{n_{k-i}}/f_{n_{k-\ell}}} = \frac{1}{\prod_{j=0,j \neq i}^{\ell+1}(f_{n_{k-i}}/f_{n_{k-j}} - 1)} \quad (4.9)$$

This shows the identity in (4.2) holds for $\ell + 1$, thus completing the proof. \(\square\)
From (4.1) and (3.17), we find

\[ J \]

Using (4.2) with \( \ell \leq a \), we may rewrite (4.12) as

\[ \sum_{a=1}^{L} (L - a) ax^a = x^L \sum_{a=1}^{L-1} a x^{-a} = \frac{x(x^L - 1)}{(x - 1)^2} - \frac{Lx}{x - 1}, \quad (4.11) \]

to carry out the sum over \( a_k \), we find

\[ J_k = f_0^L \sum_{1 \leq a_1 < a_2 < \ldots < a_k \leq L} \left[ \prod_{i=1}^{k} (f_{n_i}/f_0) \right] \frac{L f_0^L}{(f_{n_i}/f_0 - 1)} \]

\[ = \sum_{1 \leq a_1 < a_2 < \ldots < a_k \leq L} \left[ \prod_{i=1}^{k} \frac{f_0 (f_{n_i}/f_0 - 1)}{(f_{n_i}/f_0 - 1)(f_{n_i}/f_0 - 1)} \right] \left[ \frac{f_0^L - f_0^L}{(f_{n_i}/f_0 - 1)} - L f_0^L \right] \]

\[ = \sum_{\Omega_k} \left[ \prod_{r \in \Omega_k} \frac{f_r (f_{s}/f_0 - 1)}{(f_{s}/f_0 - 1)} \right] \left[ \frac{f_r^L - f_r^L}{(f_{s}/f_0 - 1)} - L f_r^L \right]. \quad (4.12) \]

where we let \( r = n_i \), so that \( 1 \leq i \leq k \) becomes \( r \in \Omega_k \), in which \( \Omega_k = \{n_1, n_2, \ldots, n_k\} \).

In the last step we multiplied the numerator and the denominator by the product over the complement of \( \Omega_k \), given by \( \Omega_k = \{1 \leq s \leq N - 1 | s \notin \Omega_k\} \). We now let

\[ A_r = (1 - t \omega^r)(1 - u \omega^{-r}), \quad f_r = (f_{n_i}/f_0) \]

\[ A_r \]

with \( f_r \) defined in (1.6). It is easy to evaluate

\[ \prod_{s=0, s \neq r}^{N-1} (A_s - A_r) = \prod_{s=0, s \neq r}^{N-1} (\omega^r - \omega^s) (t - \omega^{-r-s} u) = N(t^N - u^N)/(\omega^r t - \omega^{-r} u). \quad (4.14) \]

Consequently, if we let

\[ h(r) = \frac{\omega^r t - \omega^{-r} u}{t^N - u^N} \frac{f_r^L - f_r^L}{(f_{n_i}/f_0 - 1)} - L f_r^L \]

\[ h(r) \]

we may rewrite (4.12) as

\[ J_k = \sum_{\Omega_k} \left[ \prod_{r \in \Omega_k} A_r^{k-1} h(r) \right] \prod_{s \in \Omega_k} (A_s - A_r) = \sum_{\Omega_k} \left[ \prod_{r \in \Omega_k} A_r^{k-1} h(r) \right] \prod_{s \in \Omega_k} (A_s - A_r), \quad (4.16) \]
where we replaced the sum over \( r \in \Omega_k \) by the sum over the entire set \( \Omega_k \cup \bar{\Omega}_k \), for the product over \( s \) is identically zero if \( r \in \bar{\Omega}_k \); we also replaced the sum over all sets \( \Omega_k \) by the sum over all \( \bar{\Omega}_k \), as these are in one-to-one correspondence. Consider now the polynomial \([11, 12]\)

\[
\prod_{\ell=1}^{N-1} [z + (A_\ell - A_r)] = \sum_{\ell=0}^{N-2} z^{N-1-\ell} e_\ell, \quad e_\ell \equiv \sum_{1 \leq n_1 < n_2 < \cdots < n_\ell \leq N-1} \prod_{i=1}^\ell (A_{n_i} - A_r). \tag{4.17}
\]

In (4.16), the set \( \Omega_k \) has \( k \) elements; its complement \( \bar{\Omega}_k \) has \( N - 1 - k \) elements \( \bar{n}_i \) with \( 1 \leq \bar{n}_1 < \bar{n}_2 < \cdots < \bar{n}_{N-1-k} \). Therefore, the sum over \( \bar{\Omega}_k \) in (4.16) leads to \( e_{N-1-k} \), or more precisely

\[
J_k = \sum_{r=1}^{N-1} A_r^{-1} e_{N-1-k} h(r), \quad \text{and} \quad \sum_{k=1}^{N-1} J_k = \sum_{r=1}^{N-1} \sum_{k=1}^{N-1} A_r^{-1} e_{N-1-k} h(r). \tag{4.18}
\]

From (4.17) with \( z = A_r \), we find

\[
\sum_{k=1}^{N-1} A_r^{-1} e_{N-1-k} = A_r^{-1} \prod_{\ell=1}^{N-1} (A_r + A_\ell - A_r) = A_r^{-1} \prod_{\ell=1}^{N-1} A_\ell = f_0/A_r, \tag{4.19}
\]

where (4.13) and (1.6) have been used. Applying (1.7), (4.1), (4.15), (4.18), (4.19) and (1.6) in this order, we obtain

\[
G(t, u) = \frac{1}{N(t^N - u^N)} \sum_{r=1}^{N-1} (t \omega^r - u \omega^{-r}) \left[ \frac{f_r - f_0}{A_0 - A_r} - \frac{L f_0}{A_r} \right]. \tag{4.20}
\]

5. Final result for the sum \( G(t, u) \)

5.1. Useful identities

It is straightforward to show for \( k \geq 0 \) that

\[
\sum_{a=0}^{N-1} \frac{\omega^{-ak}}{1 - z\omega^a} = \sum_{n=0}^{\infty} z^n \sum_{a=0}^{N-1} \omega^{an-ak} = N \sum_{n=0}^{\infty} z^n \sum_{p=0}^{\infty} \delta_{n,k+pN} = \frac{N z^k}{1 - z^N}. \tag{5.1}
\]

Consequently, by substracting the 0th term, we find

\[
\sum_{a=1}^{N-1} \frac{\omega^{-ak}}{1 - z\omega^a} = \frac{N z^k}{1 - z^N} - \frac{1}{1 - z}. \tag{5.2}
\]

Taking the limit \( z \to 1 \), we get

\[
\sum_{a=1}^{N-1} \frac{\omega^{-ak}}{1 - \omega^a} = \frac{1}{2} (N - 1 - 2k). \tag{5.3}
\]

If \( -N \leq k < 0 \), we have to replace \( k \) by \( k + N \) in (5.2) and (5.3).
5.2. The sums $B_1$ and $B_2$

We can split the sum in (4.20), putting the contributions from the two terms within the square brackets into separate sums, writing

$$G(t, u) = (B_1 - B_2)/[N(t^N - u^N)].$$

Partial fraction decomposition yields

$$\frac{t \omega^r - u \omega^{-r}}{A_r} = \frac{t \omega^r - u \omega^{-r}}{(1 - t \omega^r)(1 - u \omega^{-r})} = \frac{1}{1 - t \omega^r} - \frac{1}{1 - u \omega^{-r}}. \tag{5.5}$$

Hence, we can use (5.2) with $k = 0$ to evaluate $B_2$ as

$$B_2 \equiv L f^L \sum_{r=1}^{N-1} \frac{t \omega^r - u \omega^{-r}}{A_r} = L f_0^L \left[ \frac{N(t^N - u^N)}{(1 - t^N)(1 - u^N)} - \frac{t - u}{(1 - t)(1 - u)} \right]. \tag{5.6}$$

We may also decompose

$$\frac{t \omega^r - u \omega^{-r}}{A_0 - A_r} = \frac{t}{t - u \omega^{-r}} + \frac{1}{\omega^r - 1}. \tag{5.7}$$

We note $f_0^L = Q(t \omega^r) Q(u \omega^{-r})$, see (1.4) and (1.6). Therefore, we may write

$$f_r^L - f_0^L = \sum_{Q=0}^{N-1} \sum_{\ell=0}^{m_Q} \sum_{P=0}^{m_P} \sum_{m=0}^{N-1} t^{\ell N + Q} u^{m N + P} \Lambda^Q_{\ell} \Lambda^P_m (\omega^{r(Q-P)} - 1), \tag{5.8}$$

where $m_Q \equiv \lfloor (N-1)L/N - Q/N \rfloor$ and a similar expression with $Q$ replaced by $P$, first introduced as (2.17) in [7]. Therefore, from (5.1) and (5.8), we find that

$$B_1 \equiv \sum_{r=1}^{N-1} (f_r^L - f_0^L) \left[ \frac{t \omega^r - u \omega^{-r}}{A_0 - A_r} \right]$$

$$= \sum_{Q=0}^{N-1} \sum_{\ell=0}^{m_Q} \sum_{P=0}^{m_P} \sum_{m=0}^{N-1} t^{\ell N + Q} u^{m N + P} \Lambda^Q_{\ell} \Lambda^P_m \sum_{r=1}^{N-1} (\omega^{r(Q-P)} - 1) \left[ \frac{t}{t - u \omega^{-r}} + \frac{1}{\omega^r - 1} \right]. \tag{5.9}$$

From (5.2), we can easily show

$$\sum_{r=1}^{N-1} (\omega^{r(Q-P)} - 1) \left[ \frac{t}{t - u \omega^{-r}} \right] = \begin{cases} N[(t/u)^{P-Q} - (t/u)^N], & P > Q, \\ (t/u)^N - 1, & P \leq Q. \end{cases} \tag{5.10}$$

Using (5.3), we find

$$\sum_{r=1}^{N-1} \frac{\omega^{r(Q-P)} - 1}{\omega^{r} - 1} = \begin{cases} P - Q, & P \geq Q, \\ P - Q + N, & P < Q. \end{cases} \tag{5.11}$$

The sums in (5.9), (5.10) and (5.11) are all identically zero for $P = Q$. Therefore, using (5.10) and (5.11), we can rewrite (5.9) as

$$B_1 = \sum_{\ell} \sum_{m} \left\{ \sum_{P>Q} t^{\ell N + Q} u^{m N + P} \Lambda^Q_{\ell} \Lambda^P_m \left[ N[(t/u)^{P-Q} - (t/u)^N]/(t/u)^N - 1 + P - Q \right] \\
+ \sum_{P<Q} t^{\ell N + Q} u^{m N + P} \Lambda^Q_{\ell} \Lambda^P_m \left[ N[(t/u)^{P-Q+N-1}]/(t/u)^N - 1 + P - Q \right] \right\}. \tag{5.12}$$
We may split \( \mathcal{B}_1 \) into two parts, one having the factor \( P - Q \)
\[
\mathcal{B}_1 = \beta_1 + \beta_2, \quad \beta_2 = \sum_{\ell} \sum_{m} \sum_{P > Q} \left[ \sum_{P > Q} + \sum_{P < Q} \right] \left[ t^{\ell N + Q} u^{m N + P} \Lambda^Q \Lambda^P_m (P - Q) \right]
= \sum_{\ell} \sum_{m} \sum_{P} \sum_{Q} \left[ t^{\ell N + Q} u^{m N + P} \Lambda^Q \Lambda^P_m (P - Q) \right], \tag{5.13}
\]
In the other part of (5.12), we interchange \( m \leftrightarrow \ell \) and \( P \leftrightarrow Q \) in the sums with \( P < Q \) to find
\[
\beta_1 = \frac{N}{t^N - u^N} \sum_{\ell} \sum_{m} \sum_{P > Q} \Lambda^Q \Lambda^P_m (t^{\ell N} u^{m N} - t^m N u^{\ell N}) (t^P u^{N+Q} + t^{N+Q} u^P). \tag{5.14}
\]
In (5.13), we let \( r = \ell N + Q \) and \( s = m N + P \) so that \( c_r = \Lambda^Q, c_s = \Lambda^P_m \) which are the coefficients of the polynomial in (1.4). Because \( P - Q = s - r + \ell N - mN \), we find
\[
\beta_2 = \sum_{r} \sum_{s} (s - r) c_r c_s t^r u^s + N \sum_{\ell} \sum_{m} \sum_{P} \sum_{Q} (\ell - m) \left[ t^{\ell N + Q} u^{m N + P} \Lambda^Q \Lambda^P_m \right]. \tag{5.15}
\]
The first term is denoted by \( \gamma_1 \) and second by \( \gamma_2 \), so that
\[
\beta_2 = \gamma_1 + \gamma_2, \quad \gamma_1 = \sum_{r} \sum_{s} (s - r) c_r c_s t^r u^s = u \mathcal{Q}(t) \mathcal{Q}'(u) - t \mathcal{Q}'(t) \mathcal{Q}(u). \tag{5.16}
\]
Using \( f_0^L = \mathcal{Q}(t) \mathcal{Q}(u) \), we may also write
\[
\gamma_1 = f_0^L \left[ u \frac{d \ln \mathcal{Q}(u)}{du} - t \frac{d \ln \mathcal{Q}(t)}{dt} \right] = L f_0^L \left[ \frac{N(\ell N - u^N)}{(1-t^N)(1-u^N)} - \frac{t-u}{(1-t)(1-u)} \right], \tag{5.17}
\]
which is identical to \( \mathcal{B}_2 \) in (5.6). We split the summation over \( P \) and \( Q \) of the second term \( \gamma_2 \) in (5.15) into \( P = Q, P > Q \) and \( P < Q \). For \( P < Q \), we interchange the variables \( P \leftrightarrow Q \) and \( m \leftrightarrow \ell \) to find
\[
\gamma_2 = \psi_1 + \psi_2, \quad \psi_1 = N \sum_{\ell} \sum_{m} \sum_{P=0}^{N-1} (\ell - m) t^{\ell N + P} u^{m N + P} \Lambda^P \Lambda^P_m, \tag{5.18}
\]
\[
\psi_2 = N \sum_{\ell} \sum_{m} \sum_{P > Q} (\ell - m) \Lambda^Q \Lambda^P_m (t^{\ell N + Q} u^{m N + P} - t^{m N + P} u^{\ell N + Q}). \tag{5.19}
\]
Since \( \gamma_1 = \mathcal{B}_2 \), we find
\[
\mathcal{B}_1 - \mathcal{B}_2 = \beta_1 + \gamma_1 + \gamma_2 - \mathcal{B}_2 = \beta_1 + \gamma_2 = \beta_1 + \psi_1 + \psi_2, \tag{5.20}
\]
so that (5.4) becomes
\[
\mathcal{G}(t,u) = (\beta_1 + \psi_1 + \psi_2)/[N(\ell N - u^N)]. \tag{5.21}
\]
5.3. The sum \( \psi_1 \)
We now split the summation over \( \ell \) and \( m \) in (5.18) into three parts: \( \ell > m \), \( \ell < m \) and \( \ell = m \). The summand is identically zero for \( \ell = m \) and combining the other two parts
by interchanging the variables $\ell \leftrightarrow m$ for the part with $\ell < m$, we obtain

$$
\psi_1 = N \sum_{P=0}^{N-1} \sum_{\ell=m}^{\ell+1} (\ell - m) \Lambda_{t \ell}^P \Lambda_m^P(tu)^P (t^{\ell N} u^{mN} - t^{mN} u^{\ell N})
$$

$$
= N(t^N - u^N) \sum_{P=0}^{N-1} \sum_{\ell=1}^{\ell-1} \sum_{m=0}^{\ell-1} \sum_{s=0}^{\ell-m-1} (\ell - m) \Lambda_{t \ell}^P \Lambda_m^P (tu)^{P+mN} t^{(\ell-m-1-s)N} u^{sN},
$$

(5.22)

where $m_P = \left\lfloor (N-1)L/N - P/N \right\rfloor$. We need to change the summation variables $m$ and $\ell$, so that we can read off the coefficients of $t^{L+P} u^{mN+P}$. To do so, we first interchange the order of summations to move the sum over $s$ to the left, followed by the change of summation variables to $\ell' = \ell - 1 - s$ and $m' = m + s$ and the move of the summation over $s$ back to the right. More explicitly,

$$
\psi'_1 \equiv \frac{\psi_1}{N(t^N - u^N)} = \sum_{P=0}^{N-1} \sum_{s=0}^{m_P-1} \sum_{\ell=s+1}^{m_P} \sum_{m=0}^{\ell-1-s} (\ell - m) \Lambda_{t \ell}^P \Lambda_m^P (t^{\ell N} + P u^{mN+P})
$$

(5.23)

$$
= \sum_{P=0}^{N-1} \sum_{s=0}^{m_P-1} \sum_{\ell'=s}^{m_P-1} \sum_{m'=s}^{m'} (\ell' - m' + 2s + 1) \Lambda_{t \ell'}^P \Lambda_{m'}^P (t^{\ell' N} + P u^{m'N+P})
$$

(5.24)

$$
= \sum_{P=0}^{N-1} \sum_{\ell=0}^{m_P-1} \sum_{m=0}^{m_P-1} t^{\ell N+P} u^{mN+P} \sum_{s=\max(0,m-\ell)}^{m} (\ell - m + 2s + 1) \Lambda_{t \ell+1}^P \Lambda_{m-s}^P,
$$

(5.25)

dropping the primes on $\ell$ and $m$ in the last step. In (5.24) we have $0 \leq s \leq m' \leq \ell' + s$, implying $\max(0, m - \ell) \leq s \leq m$ in (5.25).

It is easily seen that $\beta_1$ and $\psi_2$ vanish if $P = Q$. Therefore, from (5.21) we conclude that the coefficient of $t^{\ell N+P} u^{mN+P}$ in (5.25) gives

$$
G_{\ell N+P,mN+P} = \sum_{s=\max(0,m-\ell)}^{m} (\ell - m + 2s + 1) \Lambda_{t \ell+1}^P \Lambda_{m-s}^P,
$$

(5.26)

for all $0 \leq \ell, m \leq m_P - 1$. We note that (5.26) is identical to equation (37) in [9] for $m \leq \ell$. Making the substitution $s \rightarrow m - s$ we find the alternative expression

$$
G_{\ell N+P,mN+P} = \sum_{s=0}^{\min(m,\ell)} (\ell + m - 2s + 1) \Lambda_{t \ell+m+1-s}^P \Lambda_s^P,
$$

(5.27)

agreeing with (44) of [9] for $m \leq \ell$. Expression (5.27), and thus also (5.26), is symmetric in $\ell$ and $m$, as could be expected from the original definitions (1.1) and (1.2).

5.4. Evaluation of the sum $\psi_2 + \beta_1$

From (5.14) and (5.19) we find that the summands are identically zero for $m = \ell$. The summation over $m$ and $\ell$ can be split into the two contributions $\ell > m$ and $\ell < m$, resulting in

$$
\psi_2 + \beta_1 = \alpha_1 + \alpha_2,
$$

(5.28)
where

\[
\alpha_1 \equiv N \sum_{P > Q} \sum_{\ell > m} \Lambda_f^Q \Lambda_m^P \left[ (\ell - m)(t^{N+Q} u^{mN+P} - t^{mN+P} u^{\ell N+Q}) + (t^P u^{N+Q} - t^{N+Q} u^P) \sum_{s=0}^{\ell-m-1} t^{(\ell-s)N} u^{(m+s)N} \right], \tag{5.29}
\]

\[
\alpha_2 \equiv N \sum_{P > Q} \sum_{\ell < m} \Lambda_f^Q \Lambda_m^P \left[ (\ell - m)(t^{\ell N+Q} u^{mN+P} - t^{mN+P} u^{\ell N+Q}) - (t^P u^{N+Q} - t^{N+Q} u^P) \sum_{s=0}^{m-\ell-1} t^{(m-s)N} u^{(\ell+s)N} \right]. \tag{5.30}
\]

Using

\[
\sum_{s=0}^{\ell-m-1} 1 = \ell - m, \quad \sum_{s=0}^{\ell-m-1} t^{(\ell-s)N} u^{(m+s)N} = \sum_{s=0}^{\ell-m-1} t^{mN+P} u^{(\ell-s)N},
\]

we may rewrite (5.29) as

\[
\alpha_1 = N \sum_{P > Q} \sum_{\ell > m} \Lambda_f^Q \Lambda_m^P \sum_{s=0}^{\ell-m-1} (t^{\ell N+Q} u^{mN+P} - t^{mN+P} u^{\ell N+Q} + t^{(m+s)N+P} u^{(\ell-s)N+Q} - t^{(\ell-s)N+Q} u^{(m+s)N+P})
\]

\[
= N \sum_{P > Q} \sum_{\ell > m} \Lambda_f^Q \Lambda_m^P \sum_{s=0}^{\ell-m-1} (t^{\ell N+Q} u^{mN+P} + t^{mN+P} u^{(\ell-s)N+Q})(t^{sN} - u^{sN})
\]

\[
= N(t^N - u^N) \sum_{P > Q} \sum_{\ell > m} \Lambda_f^Q \Lambda_m^P \sum_{s=0}^{\ell-m-1} \left( t^{(\ell-s)N+Q} u^{mN+P} \sum_{r=0}^{s-1} t^{(s-1-r)N} u^{rN} + t^{mN+P} u^{(\ell-s)N+Q} \sum_{r=0}^{s-1} t^{rN} u^{(s-1-r)N} \right)
\]

\[
= N(t^N - u^N) \sum_{P > Q} \sum_{\ell > m} \Lambda_f^Q \Lambda_m^P \sum_{r=0}^{\ell-m-2} (\ell - m - 1 - r)
\]

\[
\times \left( t^{(\ell-1-r)N+Q} u^{(m+r)N+P} + t^{(m+r)N+P} u^{(\ell-1-r)N+Q} \right). \tag{5.32}
\]

Similarly we find

\[
\alpha_2 = N(t^N - u^N) \sum_{P > Q} \sum_{\ell < m} \Lambda_f^Q \Lambda_m^P \sum_{r=0}^{\ell-m-1} (m - \ell - r)
\]

\[
\times \left( t^{(\ell+r)N+Q} u^{(m-r-1)N+P} + t^{(m-r-1)N+P} u^{(\ell+r)N+Q} \right). \tag{5.33}
\]

In the same way as described in detail in subsection 5.3, we can obtain the coefficients of $t^{\ell N+Q} u^{mN+P}$, by first interchanging the order of the summations by moving the sum over $r$ to the left, then changing the summation variables, followed by interchanging the order of the summations in the reverse order. Letting $\alpha'_i \equiv \alpha_i / N(t^N - u^N)$, we find

\[
\alpha'_1 = \sum_{P > Q} \sum_{\ell=0}^{m_P - 1} \sum_{m=0}^{m_P - 1} \left( t^{\ell N+Q} u^{mN+P} + u^{\ell N+Q} t^{mN+P} \right)
\]
\[ \alpha_2' = \sum_{P>Q} \sum_{\ell=0}^{m_P-1} \sum_{m=0}^{m_P-1} (t^{\ell N + Q} u^{mN + P} + u^{\ell N + Q} t^{mN + P}) \]
\[ \times \sum_{r=\max(0,\ell-m)}^{m} (\ell - m + r)\Lambda^{Q}_{\ell+1+r} \Lambda^{P}_{m-r} \] \hspace{1cm} (5.34)

From (5.21) and (5.28), we may write
\[ G(t,u) = \psi_1' + \alpha_1' + \alpha_2'. \] \hspace{1cm} (5.36)

From (5.34) and (5.35) we find that the coefficient of \( t^{\ell N + Q} u^{mN + P} \) is the same as the one of \( u^{\ell N + Q} t^{mN + P} \), and for \( P > Q \) they are
\[ G_{\ell N + Q, mN + P} = G_{mN + P, \ell N + Q} = \]
\[ \sum_{r=\max(0,\ell-m)}^{m} (\ell - m + r)\Lambda^{Q}_{\ell+1+r} \Lambda^{P}_{m-r} + \sum_{r=\max(0,\ell-m)}^{\ell} (m + 1 - \ell + r)\Lambda^{Q}_{\ell-r} \Lambda^{P}_{m+r+1}. \] \hspace{1cm} (5.37)

Replacing \( r \to m - r \) in the first sum and \( r \to \ell - r \) in the second, we find
\[ G_{\ell N + Q, mN + P} = \sum_{r=0}^{\min(m,\ell)} [(\ell - r)\Lambda^{Q}_{\ell+m+1+r} \Lambda^{P}_{m+\ell+1-r} + (m + 1 - r)\Lambda^{Q}_{\ell+1+r} \Lambda^{P}_{m+\ell+1-r}], \] \hspace{1cm} (5.38)

which is identical to (3.7) in [1]. This complete the proof of our conjecture in [1].

6. Summary

Our previous work on the superintegrable chiral Potts model depended on two conjectures. In our previous paper [10], we made the conjecture that the Serre relation are satisfied and in [1] we conjectured a formula for the coefficients of the two variable generating function \( G(t,u) \). Both these conjectures were tested numerically. The validity of the Serre relations is to be expected on the basis of the work of Davies [13] and one should be able to construct a proof extending work by Nishino and Deguchi [14, 15, 16].

In this paper we have managed to prove the conjecture for \( G(t,u) \) in [1]. In section 2, the sum \( V(\{\mu_i\}, \{\nu_i\}) \) in (1.9) was analyzed using MacMahon’s method described in [8 chapter 11]. A theorem on symmetric functions [11, 12], namely (2.8) was used next to relate the \( L \)-fold sum in (1.9) to an \( m \)-fold sums in (2.27) by (2.26). In section 3 we related the sum \( F(t,u) \) in (1.8) by (3.2) to the sum \( R_{\ell,m} \) defined in (3.1), which was analyzed further in (3.4). In the remaining part of section 3, we got rid of the equality signs in the summation and were able to arrive at a simple formula for \( F(t,u) \) in (3.2) or (4.1). In section 4, we showed by induction the identity (4.2), allowing us to carry out the sums in (4.1). Again we used a well-known property of symmetric functions [11, 12] given in (4.17) to express \( G(t,u) \) in a much simpler form (4.20). Finally, in section 5, we analyzed the sum in (4.20), and proved the conjecture of [10].
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