Equivariant Diffusions on Principal Bundles

K. David Elworthy, Yves Le Jan and Xue-Mei Li

Let \( \pi : P \rightarrow M \) be a smooth principal bundle with structure group \( G \). This means that there is a \( C^\infty \) right multiplication \( P \times G \rightarrow P, \ u \mapsto u \cdot g \) say, of the Lie group \( G \) such that \( \pi \) identifies the space of orbits of \( G \) with the manifold \( M \) and \( \pi \) is locally trivial in the sense that each point of \( M \) has an open neighbourhood \( U \) with a diffeomorphism

\[
\tau_U : \pi^{-1}(U) \rightarrow U \times G
\]

over \( U \), which is equivariant with respect to the right action of \( G \), i.e. if \( \tau_u (b) = (\pi(b), k) \) then \( \tau_u (b \cdot g) = (\pi(b), kg) \). Assume for simplicity that \( M \) is compact. Set \( n = dim M \). The fibres, \( \pi^{-1}(x), x \in M \) are diffeomorphic to \( G \) and their tangent spaces \( VT_u P = ker T_u \pi \), \( u \in P \), are the ‘vertical’ tangent spaces to \( P \). A connection on \( P \), (or on \( \pi \)) assigns a complementary ‘horizontal’ subspace \( HTP \) of \( VT_u P \) in \( T_u P \) for each \( u \), giving a smooth horizontal subbundle \( HTP \) of the tangent bundle \( TP \) to \( P \). Given such a connection it is a classical result that for any \( C^1 \) curve: \( \sigma : [0, T] \rightarrow M \) and \( u_0 \in \pi^{-1}(\sigma(0)) \) there is a unique horizontal \( \tilde{\sigma} : [0, T] \rightarrow P \) which is a lift of \( \sigma \), i.e. \( \pi(\tilde{\sigma}(t)) = \sigma(t) \) and has \( \tilde{\sigma}(0) = u_0 \).

In his startling ICM article [8] Itô showed how this construction could be extended to give horizontal lifts of the sample paths of diffusion processes. In fact he was particularly concerned with the case when \( M \) is given a Riemannian metric \( \langle \cdot, \cdot \rangle_x \), \( x \in M \), the diffusion is Brownian motion on \( M \), and \( P \) is the orthonormal frame bundle \( \pi : OM \rightarrow M \). Recall that each \( u \in OM \) with \( u \in \pi^{-1}(x) \) can be considered as an isometry \( u : \mathbb{R}^n \rightarrow T_x M, \langle \cdot, \cdot \rangle_x \) and a

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horizontal lift $\tilde{\sigma}$ determines parallel translation of tangent vectors along $\sigma$

$$\parallel_t \equiv \parallel_{(\sigma)_t} : T_{\sigma(\cdot)} M \to T_{\sigma(t)} M$$

$$v \mapsto \tilde{\sigma}(t)(\tilde{\sigma}(0))^{-1} v.$$

The resulting parallel translation along Brownian paths extends also to parallel translation of forms and elements of $\wedge^p TM$. This enabled Itô to use his construction to obtain a semi-group acting on differential forms

$$P_t \phi = \mathbb{E}((\parallel_t^{-1})_*(\phi)) = \mathbb{E}\phi(\parallel_t -).$$

As he pointed out this is not the semi-group generated by the Hodge-Kodaira Laplacian, $\Delta$. To obtain that generated by the Hodge-Kodaira Laplacian, $\Delta$, some modification had to be made since the latter contains zero order terms, the so called Weitzenbock curvature terms. The resulting probabilistic expression for the heat semi-groups on forms has played a major role in subsequent development.

In [5] we go in the opposite direction starting with a diffusion with smooth generator $B$ on $P$, which is $G$-invariant and so projects to a diffusion generator $A$ on $M$. We assume the symbol $\sigma_A$ has constant rank so determining a subbundle $E$ of $TM$, (so $E = TM$ if $A$ is elliptic). We show that this set-up induces a ‘semi-connection’ on $P$ over $E$ (a connection if $E = TM$) with respect to which $B$ can be decomposed into a horizontal component $A^H$ and a vertical part $B^V$. Moreover any vertical diffusion operator such as $B^V$ induces only zero order operators on sections of associated vector bundles.

There are two particularly interesting examples. The first when $\pi : GLM \to M$ is the full linear frame bundle and we are given a stochastic flow $\{\eta_t : 0 \leq t < \infty\}$ on $M$, generator $A$, inducing the diffusion $\{u_t : 0 \leq t < \infty\}$ on $GLM$ by

$$u_t = T\eta_t(u_0).$$

Here we can determine the connection on $GLM$ in terms of the LeJan-Watanabe connection of the flow [12], [1], as defined in [6], [7], in particular giving conditions when it is a Levi-Civita connection. The zero order operators arising from the vertical components can be identified with generalized Weitzenbock curvature terms.

The second example slightly extends the above framework by letting $\pi : P \to M$ be the evaluation map on the diffeomorphism group $\text{Diff}M$ of $M$ given by $\pi(h) := h(x_0)$ for a fixed point $x_0$ in $M$. The group $G$ corresponds to the group of diffeomorphisms fixing $x_0$. Again we take a flow $\{\xi_t(x) : x \in M, t \geq 0\}$ on $M$, but now the process on $\text{Diff}M$ is just the right invariant process determined by $\{\xi_t : 0 \leq t < \infty\}$. In this case the horizontal lift to the diffeomorphism group of the diffusion $\{\xi_t(x_0) : 0 \leq t < \infty\}$ on $M$ is...
obtained by 'removal of redundant noise', c.f. [7] while the vertical process is a flow of diffeomorphisms preserving $x_0$, driven by the redundant noise.

Here we report briefly on some of the main results to appear in [5] and give details of a more probabilistic version Theorem 2.5 below: a skew product decomposition which, although it has a statement not explicitly mentioning connections, relates to Itô's pioneering work on the existence of horizontal lifts. The derivative flow example and a simplified version of the stochastic flow example are described in §3.

The decomposition and lifting apply in much more generality than with the full structure of a principal bundle, for example to certain skew products and invariant processes on foliated manifolds. This will be reported on later. Earlier work on such decompositions includes [4] [13].

§1. Construction

A. If $\mathcal{A}$ is a second order differential operator on a manifold $X$, denote by $\sigma^\mathcal{A} : T^*X \to TX$ its symbol determined by

$$df \left( \sigma^\mathcal{A}(dg) \right) = \frac{1}{2} \mathcal{A}(fg) - \frac{1}{2} \mathcal{A}(f)g - \frac{1}{2} f \mathcal{A}(g),$$

for $C^2$ functions $f, g$. The operator is said to be semi-elliptic if $df \left( \sigma^\mathcal{A}(df) \right) \geq 0$ for each $f \in C^2(X)$, and elliptic if the inequality holds strictly. Ellipticity is equivalent to $\sigma^\mathcal{A}$ being onto. It is called a diffusion operator if it is semi-elliptic and annihilates constants, and is smooth if it sends smooth functions to smooth functions.

Consider a smooth map $p : N \to M$ between smooth manifolds $M$ and $N$. By a lift of a diffusion operator $\mathcal{A}$ on $M$ over $p$ we mean a diffusion operator $\mathcal{B}$ on $N$ such that

$$(1) \quad \mathcal{B}(f \circ p) = (\mathcal{A}f) \circ p$$

for all $C^2$ functions $f$ on $M$. Suppose $\mathcal{A}$ is a smooth diffusion operator on $M$ and $\mathcal{B}$ is a lift of $\mathcal{A}$.

**Lemma 1.1.** Let $\sigma^\mathcal{B}$ and $\sigma^\mathcal{A}$ be respectively the symbols for $\mathcal{B}$ and $\mathcal{A}$. The following diagram is commutative for all $u \in p^{-1}(x)$, $x \in M$:

$$
\begin{array}{ccc}
T^*_u N & \xrightarrow{\sigma^\mathcal{B}_u} & T_u N \\
(Tp)^* & \downarrow & Tp \\
T^*_x M & \xrightarrow{\sigma^\mathcal{A}_x} & T_x M.
\end{array}
$$
B. Semi-connections on principal bundles. Let $M$ be a smooth finite dimensional manifold and $P(M, G)$ a principal fibre bundle over $M$ with structure group $G$ a Lie group. Denote by $\pi: P \to M$ the projection and $R_a$ the right translation by $a$.

**Definition 1.2.** Let $E$ be a sub-bundle of $TM$ and $\pi: P \to M$ a principal $G$-bundle. An $E$ semi-connection on $\pi: P \to M$ is a smooth sub-bundle $H^E TP$ of $TP$ such that

(i) $T_u \pi$ maps the fibres $H^E T_u P$ bijectively onto $E_{\pi(u)}$ for all $u \in P$.

(ii) $H^E TP$ is $G$-invariant.

Notes.

(1) Such a semi-connection determines and is determined by, a smooth horizontal lift:

$$h_u : E_{\pi(u)} \to T_u P,$$

such that

(i) $T_u \pi \circ h_u(v) = v$, for all $v \in E_x \subset T_x M$;

(ii) $h_{u \cdot a} = T_u R_a \circ h_u$.

The horizontal subspace $H^E T_u P$ at $u$ is then the image at $u$ of $h_u$, and the composition $h_u \circ T_u P$ is a projection onto $H^E T_u P$.

(2) Let $F = P \times V / \sim$ be an associated vector bundle to $P$ with fibre $V$. An element of $F$ is an equivalence class $[(u, e)]$ such that $(ug, g^{-1}e) \sim (u, e)$. Set $\bar{u}(e) = [(u, e)]$. An $E$ semi-connection on $P$ gives a covariant derivative on $F$. Let $Z$ be a section of $F$ and $w \in E_x \subset T_x M$, the covariant derivative $\nabla_w Z \in F_x$ is defined, as usual for connections, by

$$\nabla_w Z = u(d\bar{Z}(h_u(w))) \in \pi^{-1}(x) = F_x.$$

Here $\bar{Z} : P \to V$ is $\bar{Z}(u) = \bar{u}^{-1}Z(\pi(u))$ considering $\bar{u}$ as an isomorphism $\bar{u} : V \to F_{\pi(u)}$. This agrees with the 'semi-connections on $E$' defined in Elworthy-LeJan-Li [7] when $P$ is taken to be the linear frame bundle of $TM$ and $F = TM$. As described there, any semi-connection can be completed to a genuine connection, but not canonically.

Consider on $P$ a diffusion generator $B$, which is equivariant, i.e.

$$Bf \circ R_a = B(f \circ R_a), \quad \forall f, g \in C^2(P, R), \quad a \in G.$$ 

The operator $B$ induces an operator $A$ on the base manifold $M$ by setting

(2) $$A_f(x) = B(f \circ \pi)(u) \in \pi^{-1}(x), f \in C^2(M),$$

which is well defined since

$$B(f \circ \pi)(u \cdot a) = B((f \circ \pi))(u).$$
Let $E_x := \text{Image}(\sigma^A_x) \subset T_x M$, the image of $\sigma^A_x$. Assume the dimension of $E_x = p$, independent of $x$. Set $E = \cup_x E_x$. Then $\pi : E \to M$ is a sub-bundle of $TM$.

**Theorem 1.3.** Assume $\sigma^A$ has constant rank. Then $\sigma^B$ gives rise to a semi-connection on the principal bundle $P$ whose horizontal map is given by

$$h_u(v) = \sigma^B((T_u \pi)^* \alpha)$$

where $\alpha \in T^*_\pi(u) M$ satisfies $\sigma^A_x(\alpha) = v$.

**Proof.** To prove $h_u$ is well defined we only need to show $\psi(\sigma^B(T_u \pi^* (\alpha))) = 0$ for every 1-form $\psi$ on $P$ and for every $\alpha$ in $\ker \sigma^A_x$. Now $\sigma^A_x \alpha = 0$ implies by Lemma 1.1 that

$$0 = \alpha \sigma^A(\alpha) = (T \pi)^* (\alpha) \sigma^B((T \pi)^* (\alpha)).$$

Thus $T \pi^* (\alpha) \sigma^B(T \pi^* (\alpha)) = 0$. On the other hand we may consider $\sigma^B$ as a bilinear form on $T^* P$ and then for all $\beta \in T^*_u P$,

$$\sigma^B(\beta + t(T \pi)^* (\alpha), \beta + t(T \pi)^* (\alpha))$$

$$= \sigma^B(\beta, \beta) + 2t \sigma^B(\beta, (T \pi)^* (\alpha)) + t^2 \sigma^B((T \pi)^* \alpha, (T \pi)^* \alpha)$$

$$= \sigma^B(\beta, \beta) + 2t \sigma^B(\beta, (T \pi)^* (\alpha)).$$

Suppose $\sigma^B(\beta, (T \pi)^* (\alpha)) \neq 0$. We can then choose $t$ such that

$$\sigma^B(\beta + t(T \pi)^* (\alpha), \beta + t(T \pi)^* (\alpha)) < 0,$$

which contradicts the semi-ellipticity of $\mathcal{B}$.

We must verify (i) $T_u \pi \circ h_u(v) = v$, $v \in E_x \subset T_x M$ and (ii) $h_{u \cdot a} = T_u R_a \circ h_u$. The first is immediate by Lemma 1.1 and for the second use the fact that $T \pi \circ T R_a = T \pi$ for all $a \in G$ and the equivariance of $\sigma^B$.

§2. Horizontal lifts of diffusion operators and decompositions of equivariant operators

A. Denote by $C^\infty \Omega^p$ the space of smooth differential $p$-forms on a manifold $M$. To each diffusion operator $\mathcal{A}$ we shall associate a unique operator $\delta^A$. The horizontal lift of $\mathcal{A}$ can be defined to be the unique operator such that the associated operator $\tilde{\delta}$ vanishes on vertical 1-forms and such that $\tilde{\delta}$ and $\delta^A$ are intertwined by the lift map $\pi^*$ acting on 1-forms.

**Proposition 2.1.** For each smooth diffusion operator $\mathcal{A}$ there is a unique smooth differential operator $\delta^A : C^\infty(\Omega^1) \to C^\infty \Omega^0$ such that

$$\delta^A(f \phi) = df \sigma^A(\phi)_x + f \cdot \delta^A(\phi)$$
(2) \( \delta^A (df) = A(f) \).

For example if \( A \) has Hörmander representation

\[
A = \frac{1}{2} \sum_{j=1}^{m} \mathcal{L}_{X_i} \mathcal{L}_{X_j} + \mathcal{L}_{A}
\]

for some \( C^1 \) vector fields \( X^i, A \) then

\[
\delta^A = \frac{1}{2} \sum_{j=1}^{m} \mathcal{L}_{X_j} \iota_{X_j} + \iota_{A}
\]

where \( \iota_A \) denotes the interior product of the vector field \( A \) acting on differential forms.

**Definition 2.2.** Let \( S \) be a \( C^\infty \) sub-bundle of \( TN \) for some smooth manifold \( N \). A diffusion operator \( \mathcal{B} \) on \( N \) is said to be along \( S \) if \( \delta^B \phi = 0 \) for all 1-forms \( \phi \) which vanish on \( S \); it is said to be strongly cohesive if \( \sigma^B \) has constant rank and \( \mathcal{B} \) is along the image of \( \sigma^B \).

To be along \( S \) implies that any Hörmander form representation of \( \mathcal{B} \) uses only vector fields which are sections of \( S \).

**Definition 2.3.** When a diffusion operator \( \mathcal{B} \) on \( P \) is along the vertical foliation \( VTP \) of the \( \pi : P \rightarrow M \) we say \( \mathcal{B} \) is vertical, and when the bundle has a semi-connection and \( \mathcal{B} \) is along the horizontal distribution we say \( \mathcal{B} \) is horizontal.

If \( \pi : P \rightarrow M \) has an \( \mathcal{E} \) semi-connection and \( \mathcal{A} \) is a smooth diffusion operator along \( \mathcal{E} \) it is easy to see that \( \mathcal{A} \) has a unique horizontal lift \( \mathcal{A}^H \), i.e. a smooth diffusion operator \( \mathcal{A}^H \) on \( P \) which is horizontal and is a lift of \( \mathcal{A} \) in the sense of (1). By uniqueness it is equivariant.

**B.** The action of \( G \) on \( P \) induces a homomorphism of the Lie algebra \( \mathfrak{g} \) of \( G \) with the algebra of right invariant vector fields on \( P \): if \( \alpha \in \mathfrak{g} \),

\[
A^\alpha(u) = \left. \frac{d}{dt} \right|_{t=0} u \exp(t\alpha),
\]

and \( A^\alpha \) is called the fundamental vector field corresponding to \( \alpha \). Take a basis \( A_1, \ldots, A_k \) of \( \mathfrak{g} \) and denote the corresponding fundamental vector fields by \( \{A_i\} \).

We can now give one of the main results from [5]:

\[
\delta^A (df) = A(f).
\]
Theorem 2.4. Let $\mathcal{B}$ be an equivariant operator on $P$ with $\mathcal{A}$ the induced operator on the base manifold. Assume $\mathcal{A}$ is strongly cohesive. Then there is a unique semi-connection on $P$ over $E$ for which $\mathcal{B}$ has a decomposition

$$\mathcal{B} = \mathcal{A}^H + \mathcal{B}^V,$$

where $\mathcal{A}^H$ is horizontal and $\mathcal{B}^V$ is vertical. Furthermore $\mathcal{B}^V$ has the expression $\sum \alpha^i_j \mathcal{L}_{A^i_j} \mathcal{L}_{A^j_i} + \sum \beta^k \mathcal{L}_{A^k}$, where $\alpha^i_j$ and $\beta^k$ are smooth functions on $P$, given by $\alpha^k = \tilde{\omega}^k (\sigma^B (\tilde{\omega}^k))$, and $\beta^k = \delta^B (\tilde{\omega}^k)$ for $\tilde{\omega}$ any connection 1-form on $P$ which vanishes on the horizontal subspaces of this semi-connection.

We shall only prove the first part of Theorem 2.4 here. The semi-connection is the one given by Theorem 1.3, and we define $\mathcal{A}^H$ to be the horizontal lift of $\mathcal{A}$. The proof that $\mathcal{B}^V := \mathcal{B} - \mathcal{A}^H$ is vertical is simplified by using the fact that a diffusion operator $\mathcal{D}$ on $P$ is vertical if and only if for all $C^2$ functions $f_1$ on $P$ and $f_2$ on $M$

$$\mathcal{D}(f_1(f_2 \circ \pi)) = (f_2 \circ \pi)\mathcal{D}(f_1).$$

Set $\tilde{f}_2 = f_2 \circ \pi$. Note

$$(\mathcal{B} - \mathcal{A}^H)(f_1 \tilde{f}_2) = \tilde{f}_2(\mathcal{B} - \mathcal{A}^H)f_1 + f_1(\mathcal{B} - \mathcal{A}^H)\tilde{f}_2 + 2(df_1)\sigma^{\mathcal{B} - \mathcal{A}^H}(df_2).$$

Therefore to show $(\mathcal{B} - \mathcal{A}^H)$ is vertical we only need to prove

$$f_1(\mathcal{B} - \mathcal{A}^H)\tilde{f}_2 + 2(df_1)\sigma^{\mathcal{B} - \mathcal{A}^H}(df_2) = 0.$$

Recall Lemma 1.1 and use the natural extension of $\sigma^A$ to $\sigma^A : E^* \to E$ and the fact that by (3) $h \circ \sigma^A_x = \sigma^B(T_u \pi)^x$ to see

$$\sigma^{\mathcal{A}^H}(df_2) = (h \circ \sigma^A h^*) (df_2 \circ T \pi) = h \circ \sigma^A df_2 = \sigma^B(df_2 \circ T \pi) = \sigma^B(df_2),$$

and so $\sigma^{(\mathcal{B} - \mathcal{A}^H)}(df_2) = 0$. Also by equation (1)

$$(B - A^H)\tilde{f}_2 = Af_2 - A^H \tilde{f}_2 = 0.$$

This shows that $\mathcal{B} - \mathcal{A}^H$ is vertical.

Define $\alpha : P \to g \otimes g$ and $\beta : P \to g$ by

$$\alpha(u) = \sum \alpha^i_j(u)A_i \otimes A_j$$

$$\beta(u) = \sum \beta^k(u)A_k.$$
It is easy to see that $B^V$ depends only on $\alpha$, $\beta$ and the expression is independent of the choice of basis of $g$. From the invariance of $B$ we obtain

\[
\alpha(ug) = (ad(g) \otimes ad(g)) \alpha(u), \\
\beta(ug) = ad(g)\beta(u)
\]

for all $u \in P$ and $g \in G$.

C. Theorem 2.4 has a more directly probabilistic version. For this let $\pi : P \to M$ be as before and for $0 \leq l < r < \infty$ let $C(l, r; P)$ be the space of continuous paths $y : [l, r] \to P$ with its usual Borel $\sigma$-algebra. For such write $l_y = l$ and $r_y = r$. Let $C(*, *; P)$ be the union of such spaces. It has the standard additive structure under concatenation: if $y$ and $y'$ are two paths with $r_y = l_{y'}$ and $y(r_y) = y'(l_{y'})$ let $y + y'$ be the corresponding element in $C(l_y, r_{y'}; P)$. The basic $\sigma$-algebra of $C(*, *, P)$ is defined to be the pull back by $\pi$ of the usual Borel $\sigma$-algebra on $C(*, *, M)$.

Consider the laws $\{P_{l,r}^a : 0 \leq l < r, a \in P\}$ of the process running from $a$ between times $l$ and $r$, associated to a smooth diffusion operator $B$ on $P$. Assume for simplicity that the diffusion has no explosion. Thus $\{P_{l,r}^a, a \in P\}$ is a kernel from $P$ to $C(l, r; P)$. The right action $R_g$ by $g$ in $G$ extends to give a right action, also written $R_g$, of $G$ on $C(*, *, P)$. Equivariance of $B$ is equivalent to

\[
\pi_\star (P_{l,r}^a) = (R_g)_\star P_{l,r}^a
\]

for all $0 \leq l \leq r$ and $a \in P$. If so $\pi_\star (P_{l,r}^a)$ depends only on $\pi(a)$, $l$, $r$ and gives the law of the induced diffusion $A$ on $M$. We say that such a diffusion $B$ is basic if for all $a \in P$ and $0 \leq l < r < \infty$ the basic $\sigma$-algebra on $C(l, r; P)$ contains all Borel sets up to $P_{l,r}^a$ negligible sets, i.e. for all $a \in P$ and Borel subsets $B$ of $C(l, r; P)$ there exists a Borel subset $A$ of $C(l, r, M)$ s.t. $P_a(\pi^{-1}(A) \Delta B) = 0$.

For paths in $G$ it is more convenient to consider the space $C_{id}(l, r; G)$ of continuous $\sigma : [l, r] \to G$ with $\sigma(l) = id$ for `id' the identity element. The corresponding space $C_{id}(*, *, G)$ has a multiplication

\[
C_{id}(s, t; G) \times C_{id}(t, u; G) \longrightarrow C_{id}(s, u; G)
\]

\[(g, g') \mapsto g \times g'
\]

where $(g \times g')(r) = g(r)$ for $r \in [s, t]$ and $(g \times g')(r) = g(t)g'(r)$ for $r \in [t, u]$.

Given probability measures $Q$, $Q'$ on $C_{id}(s, t; G)$ and $C_{id}(t, u; G)$ respectively this determines a convolution $Q \ast Q'$ of $Q$ with $Q'$ which is a probability measure on $C_{id}(s, u; G)$.
Theorem 2.5. Given the laws \( \{p^l_r : a \in P, 0 \leq l < r < \infty\} \) of an equivariant diffusion \( B \) as above with \( A \) strongly cohesive there exist probability kernels \( \{p^{l,r}_a : a \in P\} \) from \( P \) to \( C(l, r; P) \), \( 0 \leq l < r < \infty \) and \( q^{l,r}_y \), defined \( p^{l,r}_a \) a.s. from \( C(l, r, P) \) to \( C(l, r; G) \) such that

(i) \( \{p^{l,r}_a : a \in P\} \) is equivariant, basic and determining a strongly cohesive generator.

(ii) \( y \mapsto q^{l,r}_y \) satisfies

\[
q^{l,y+r'}_y = q^{l,y}_y \circ q^{l',r'}_y
\]

for \( p^{l,y}_y \otimes p^{l',r'}_y \) almost all \( y, y' \) with \( r_y = l_{y'} \).

(iii) For \( U \) a Borel subset of \( C(l, r, P) \),

\[
p^{l,r}_a(U) = \int \int \chi_U(y \cdot g.) q^{l,r}_y(dy) p^{H,l,r}_a(dy).
\]

The kernels \( p^{H,l,r}_a \) are uniquely determined as are the \( \{q^{l,r}_y : y \in \mathbb{R}\} \), \( p^{H,l,r}_a \) a.s. in \( y \) for all \( a \) in \( P \). Furthermore \( q^{l,r}_y \) depends on \( y \) only through its projection \( \pi(y) \) and its initial point \( y_l \).

Proof. Fix \( a \) in \( P \) and let \( \{b_t : l \leq r \leq t\} \) be a process with law \( p^{l,r}_a \). By Theorem 2.4 we can assume that \( b \) is given by an s.d.e. of the form

\[
db_t = \tilde{X}(b_t) \circ dB_t + \tilde{X}^0(b_t)dt + A(b_t) \circ d\beta_t + V(b_t)dt
\]

where \( \tilde{X} : P \times \mathbb{R}^p \to TP \) is the horizontal lift of some \( X : M \times \mathbb{R}^p \to E \), \( \tilde{X}^0 \) is the horizontal lift of a vector field \( X^0 \) on \( M \), while \( A : P \times \mathbb{R}^1 \to TP \) and the vector field \( V \) are vertical and determine \( B^V \). Here \( B \) and \( \beta \) are independent Brownian motions on \( \mathbb{R}^p \) and \( \mathbb{R}^q \) respectively, some \( q \), and we are using the semi-connection on \( P \) induced by \( B \) as in Theorem 1.3.

Let \( \tilde{x}_t : l \leq t \leq r \) satisfy

\[
d\tilde{x}_t = \tilde{X}(\tilde{x}_t) \circ dB_t + \tilde{X}^0(\tilde{x}_t)dt
\]

so \( \tilde{x}_t \) is the horizontal lift of \( \{\pi(b_t) : l \leq t \leq r\} \). Then there is a unique continuous process \( \{g_t : l \leq t \leq r\} \) in \( G \) with \( g_t = id \) such that

\[
\tilde{x}_t g_t = b_t.
\]

We have to analyse \( \{g_t : l \leq t \leq r\} \). Using local trivialisations of \( \pi : P \to M \) we see it is a semi-martingale. As in [9], Proposition 3.1 on page 69,

\[
db_t = TR_{g_t}(\circ d\tilde{x}_t) + A g_t^{-1} \circ d g_t(b_t)
\]
giving \[ \tilde{\omega}(\circ db_t) = \tilde{\omega} \left( A_t^{-1} \circ dg_t(b_t) \right) = g_t^{-1} \circ dg_t \]
for any smooth connection form \( \tilde{\omega} : P \to g \) on \( P \) which vanishes on \( H^E TP \). Thus
\[ dg_t = TLg_t \tilde{\omega} \left( A_t d\tilde{g}_t \right) \circ d\beta_t + V(\tilde{x}_t g_t) dt \]
\[ g_t = id, \quad l \leq t \leq r. \]
(7)

For \( y \in C(l, r : P) \) let \( \{ g^y_t : l \leq t \leq r \} \) be the solution of
\[ dg^y_t = TLg^y_t \tilde{\omega} \left( A_t g^y_t \right) \circ d\beta_t + V(\tilde{x}_t g^y_t) dt \]
\[ g^y_t = id \]
(8)

(where the Stratonovich equation is interpreted with \( 'dy_t d\beta_t = 0' \)). Since \( \beta \) and \( B \) and hence \( \tilde{x} \) are independent we see \( g = g^\tilde{x} \) almost surely. For a discussion of some technicalities concerning skew products, see [16].

For \( y \in C(\ast, \ast : P) \) let \( \{ h(y)_t : l_y \leq t \leq r_y \} \) be the horizontal lift of \( \pi(y) \), starting at \( y_t \). This exists for almost all \( y \) as can be seen either by the extension of Itô’s result to general principal bundles, e.g. using (6), or by the existence of measurable sections using the fact that \( \mathcal{A}^H \) is basic. Define \( \mathbb{P}^\mathcal{H,l} \) to be the law of \( \tilde{x} \) above and \( Q^\mathcal{H,l} \) to be that of \( g^h(y) \). Clearly conditions (i) is satisfied.

To check (ii) take \( y \) and \( y' \) with \( r_y = l_{y'} \). Then
\[ h(y + y') = h(y) + h(y') \left( g_{r_y}^{h(y)} \right)^{-1}, \]
writing \( y = h(y)g^{h(y)} \) and \( y' = h(y')g^{h(y')} \). For \( r_y \leq t \leq r_{y'} \) this shows
\[ (y + y')_t = h(y')_t \left( g_{r_y}^{h(y)} \right)^{-1} g_t^{h(y + y')} \]
But \( (y + y')_t = y'_t = h(y')_t g_t^{h(y')} \) and so we have \( g^{h(y + y')} = g_{r_y}^{h(y)} g_t^{h(y')} \) for \( t \geq r_y \), giving \( g^{h(y + y')} = g^{h(y)} \times g^{h(y')} \) almost surely. This proves (ii).

For uniqueness suppose we have another set of probability measures denoted \( \mathbb{Q}^\mathcal{H,l} \) and \( \tilde{\mathbb{P}}^\mathcal{H,l} \) which satisfy (i), (ii), (iii). Since \( \{ \tilde{\mathbb{P}}^\mathcal{H,l} \}_a \) is equivariant and induces \( \mathcal{A} \) on \( M \) we can apply the preceding argument to it in place of \( \{ \mathbb{P}^\mathcal{H,l} \}_a \). However since it is basic the term involving \( \beta \) in the stochastic differential equation (6) must vanish. Since it is also strongly cohesive the vertical part \( V \) must vanish also and we have \( \tilde{\mathbb{P}}^\mathcal{H,l} = \mathbb{P}^\mathcal{H,l} \). On the other hand in the decomposition \( b_t = \tilde{x}_t g^\tilde{x}_t \) the law of \( g^\tilde{x} \) is determined by those of \( b \) and \( \tilde{x} \). but \( \mathbb{Q}^\mathcal{H,l} \) is the conditional law of \( g^\tilde{x} \) given \( \tilde{x} = y \) and so is uniquely determined as described. \( \blacksquare \)
In fact $Q^{t,r}_y$ is associated to the time dependent generator which at $g \in G$ and $t \in [l, r]$ is $\sum \alpha^{ij}(h(y)_tg)\mathcal{L}_{A_i}\mathcal{L}_{A_j} + \sum \beta^k(h(y)_tg)\mathcal{L}_{A_k}$ for $\alpha^{ij}$ and $\beta^k$ as defined in Theorem 2.4 while $\mathcal{P}^{H,l,r}$ is associated to $A^H$. 

§3. Stochastic flows and derivative flows

A. Derivative flows. Let $A$ on $M$ be given in Hörmander form

$$A = \frac{1}{2} \sum_{j=1}^{m} \mathcal{L}_{X^j} \mathcal{L}_{X^j} + \mathcal{L}_A$$

for some vector fields $X^1, \ldots, X^m, A$. As before let $E_x = \text{span}\{X^1(x), \ldots, X^m(x)\}$ and assume $\dim E_x$ is constant, $p$, say, giving a sub-bundle $E \subset TM$. The $X^1(x), \ldots, X^m(x)$ determine a vector bundle map of the trivial bundle $\mathbb{R}^m$

$$X : \mathbb{R}^m \rightarrow TM$$

with $\sigma^A = X(x)X(x)^*$. We can, and will, consider $X$ as a map $X : \mathbb{R}^m \rightarrow E$.

As such it determines (a) a Riemannian metric $\langle \cdot, \cdot \rangle_x : x \in M$ on $E$ (the same as that determined by $\sigma^A$) and (b) a metric connection $\nabla$ on $E$ uniquely defined by the requirement that for each $x \in M$,

$$\nabla_v X(e) = 0$$

for all $v \in T_x M$ whenever $e$ is orthogonal to the kernel of $T_x M$. Then for any differentiable section $U$ of $E$,

$$\nabla_v U = Y(x)d(Y(U(\cdot)))(v), \quad v \in T_x M,$$

where $Y$ is the $\mathbb{R}^m$ valued 1-form on $M$ given by

$$\langle Y_x(v), e \rangle_{\mathbb{R}^m} = \langle X(x)(e), v \rangle_x, \quad e \in \mathbb{R}^m, v \in E_x, x \in M$$

e.g. [7] where it is referred to as the LeJan-Watanabe connection in this context. By a theorem of Narasimhan and Ramanan [14] all metric connections on $E$ arise this way, see [15], [7].

For $\{B_t : 0 \leq t < \infty\}$ a Brownian motion on $\mathbb{R}^m$, the stochastic differential equation

$$dx_t = X(x_t) \circ dB_t + A(x_t)dt$$

determines a Markov process with differential generator $A$. Over each solution $\{x_t : 0 \leq t < \rho\}$, where $\rho$ is the explosion time, there is a ‘derivative’ process $\{v_t : 0 \leq t < \rho\}$ in $TM$ which we can write as $\{T\xi_t(v_0) : 0 \leq t < \rho\}$
with $T \xi_t : T_{x_0}M \to T_{x_t}M$ linear. This would be the derivative of the flow $\{ \xi_t : 0 \leq t < \rho \}$ of the stochastic differential equation when the stochastic differential equation is strongly complete. In general it is given by a stochastic differential equation on the tangent bundle $TM$, or equivalently by a covariant equation along $\{ x_t : 0 \leq t < \rho \}$:

$$Dv_t = \nabla X(v_t) \circ dB_t + \nabla A(v_t)dt$$

with respect to any torsion free connection. Take $P$ to be the linear frame bundle $GL(M)$ of $M$, treating $u \in GL(M)$ as an isomorphism $u : \mathbb{R}^n \to T_{\pi(u)}M$. For $u_0 \in GLM$ we obtain a process $\{ u_t : 0 \leq t < \rho \}$ on $GLM$ by

$$u_t = T \xi_t \circ u_0.$$ 

Let $B$ be its differential generator. Clearly it is equivariant and a lift of $A$.

A proof of the following in the context of stochastic flows, is given later. For $w \in E_x$, set

$$(11) Z^w(y) = X(y)Y(x)(w).$$

**Theorem 3.1.** The semi-connection $\nabla$ induced by $B$ is the adjoint connection of the LeJan-Watanabe connection $\tilde{\nabla}$ determined by $X$, as defined by (9), [7]. Consequently $\nabla_w V = L_{Z^w}V$ for any vector field $V$ and $w \in E$ also $\nabla_{V(x)}Z^w$ vanishes if $w \in E_x$.

In the case of the derivative flow the $\alpha, \beta$ of Theorem 2.4 have an explicit expression: for $u \in GLM$,

$$(12) \begin{cases} \alpha(u) = \frac{1}{2} \sum \left( u^{-1}(-)\tilde{\nabla} u(-)X^p \right) \otimes \left( u^{-1}(-)\tilde{\nabla} u(-)X^p \right) \\ \beta(u) = -\frac{1}{2} \sum u^{-1}\tilde{\nabla} \tilde{\nabla} u(-)X^p - \frac{1}{2} u^{-1}Ric^# u(-). \end{cases}$$

Here $\tilde{\nabla}$ is the curvature tensor for $\tilde{\nabla}$ and $\tilde{\nabla}^# : TM \to E$ the Ricci curvature defined by $\tilde{\nabla}^#(v) = \sum_{j=1}^p \tilde{\nabla}(v, e^j)e^j, v \in T_xM$.

Equivariant operators on $GLM$ determine operators on associated bundles, such as $\wedge^qTM$. If the original operator was vertical this turns out to be a zero order operator (as is shown in [5] for general principal bundles) and in the case of $\wedge^qTM$ these operators are the generalized Weitzenbock curvature operators described in [7]. In particular for differential 1-forms the operator is $\phi \mapsto \phi(Ric^# -)$. To see this, as an illustrative example, given a 1-form $\phi$
Iterating we have
\[ L_{\bar{A}_j}(\bar{\phi})(u) = \frac{d}{dt} \bar{\phi}(u \cdot e^{\bar{A}_j t})|_{t=0} = \frac{d}{dt} \phi_{\pi u}(u \cdot e^{\bar{A}_j t})|_{t=0} = \phi_{\pi u}(uA_j-) = \bar{\phi}(u)(A_j-) \]

as required, by using the map \( gl(n) \otimes gl(n) \to gl(n), S \otimes T \mapsto S \circ T \), and equation (12).

**B. Stochastic flows.** In fact Theorem 3.1 can be understood in the more general context of stochastic flows as diffusions on the diffeomorphism groups. For this assume that \( M \) is compact and for \( r \in \{1, 2, \ldots \} \) and \( s > r + \dim(M)/2 \) let \( D^s = D^s M \) be the \( C^\infty \) manifold of diffeomorphisms of \( M \) of Sobolev class \( H^s \), (for example see Ebin-Marsden [2] or Elworthy [3].) Alternatively we could take the space \( D^\infty \) of \( C^\infty \) diffeomorphisms with differentiable structure as in [11]. Fix a base point \( x_0 \) in \( M \) and let \( \pi : D^s \to M \) be evaluation at \( x_0 \). This makes \( D^s \) into a principal bundle over \( M \) with group the manifold \( V_{s_0} \) of \( H^s \)-diffeomorphisms \( \theta \) with \( \theta(x_0) = x_0 \), acting on the right by composition (although the action of \( D^{s+r} \) is only \( C^r \), for \( r = 0, 1, 2, \ldots \)).

Let \( \{ \xi^s_t : 0 \leq s \leq t < \infty \} \) be the flow of (10) starting at time \( s \). Write \( \xi_t \) for \( \xi^0_t \). The more general case allowing for infinite dimensional noise is given in [5]. We define probability measures \( \{ \mathbb{P}^{s,t}_\theta : \theta \in D^s \} \) on \( C([s,t]; M) \) be letting \( \mathbb{P}^{s,t}_\theta \) be the law of \( \{ \xi^s_t : 0 \leq s \leq t \} \) (These correspond to the diffusion process on \( D^s \) associated to the right-invariant stochastic differential equation on \( D^s \) satisfied by \( \xi^s_t : 0 \leq t < \infty \) as in [3].) These are equivariant and project by \( \pi \) to the laws given by the stochastic differential equation on \( M \). Assuming that these give a strongly cohesive diffusion on \( M \) we are essentially in the situation of Theorem 2.5.

Let \( K(x) : \mathbb{R}^m \to \mathbb{R}^m \) be the orthogonal projection onto the kernel of \( X(x) \), each \( x \in M \). set \( K^\perp(x) = id - K(x) \). Consider the \( D^\infty \)-valued process \( \{ \theta_t : 0 \leq t < \infty \} \) given by (or as the flow of)

\[ d\theta_t(x) = X(\theta_t(x))K^\perp(\theta_t(x_0)) \circ dB_t + X(\theta_t(x))Y(\theta_t(x_0))A(\theta_t(x_0)) \]
for given $\theta_0$ in $D^\infty$ and, define a $D^\infty_{x_0}$-valued process $\{g_t : 0 \leq t < \infty\}$ by

\begin{align}
dg_t &= T_{\theta_t}^{-1} \{X(\theta_t g_t) - \partial\partial_t g_t\} dB_t \\
 &\quad + A(\theta_t g_t) dt - X(\theta_t g_t - X(\theta_t x_0)) A(\theta_t x_0) dt
\end{align}

$g_0 = \text{id.}$

Set $x^\theta_t = \xi_t(\theta_0(x_0))$. Note that $\pi(\theta_t) = \theta_t(x_0) = x^\theta_t$ since

$X(\theta_t(x_0)) K_{\theta_t} = X(\theta_t(x_0))$ and

$X(\theta_t(x_0)) Y(\theta_t(x_0)) A(\theta_t(x_0)) = A(\theta_t(x_0))$.

Thus $\{\theta_t : 0 \leq t < \infty\}$ is a lift of $\{x^\theta_t, 0 \leq t < \infty\}$. It can be considered to be driven by the ‘relevant noise’, (from the point of view of $\xi(\theta_0(x_0))$, i.e. by the Brownian motion $\tilde{B}$, given by

$$
\tilde{B}_t = \int_0^t \langle(x^\theta_s)^{-1}K_{\theta_s} (x^\theta_s) dB_s
$$

where $\{\langle(x^\theta_s), 0 \leq s < \infty\}$ is parallel translation along $x^\theta_t$ with respect to the connection on the trivial bundle $M \times \mathbb{R}^m \rightarrow M$ determined by $K$ and $K_{\theta_t}$, so that

$\langle(x^\theta_s) : \mathbb{R}^m \rightarrow \mathbb{R}^m$

is orthogonal and maps the kernel of $X(\theta(x_0))$ onto the kernel of $X(x^\theta_s)$ for $0 \leq s < \infty$, see [7](chapter 3).

Correspondingly there is the ‘redundant noise’, the Brownian motion $\{\beta_t : 0 \leq t < \infty\}$ given by

$$
\beta_t = \int_0^t \langle(x^\theta_s)^{-1}K_{\theta_s} (x^\theta_s) dB_s.
$$

Then, as shown in [7](chapter 3),

(i) $\tilde{B}$. has the same filtration as $\{x^\theta_s : 0 \leq s < \infty\}$

(ii) $\beta$ and $\tilde{B}$ are independent

(iii) $dB_t = \langle_t dB_t + \langle_t d\tilde{B}_t$.

We wish to see how $g$ is driven by $\beta$. For this observe

$$
\int_0^t K(x^\theta_s) dB_s = \int_0^t K(x^\theta_s) dB_s + \int_0^t \Lambda(x^\theta_s) ds
$$

for $\Lambda : M \rightarrow \mathbb{R}$ given by the Stratonovich correction term. By (iii)

$$
\int_0^t K(x^\theta_s) dB_s = \int_0^t \langle_s dB_s = \int_0^t \langle_s d\beta_s
$$
since $\tilde{\mathcal{J}}$. is independent of $\beta$ by (i) and (ii). Thus equation (14) for $g$. can be written as

$$
\begin{align*}
 dg_t &= T\theta_t^{-1} \left\{ X(\theta_t g_t -) \tilde{\mathcal{J}}(\theta_t(x_0)) \circ d\beta_t + X(\theta_t g_t -) \Lambda(\theta_t(x_0))dt \\
&\quad + A(\theta_t g_t -)dt - X(\theta_t g_t -)Y(\theta_t x_0)A(\theta_t x_0)dt \right\}
\end{align*}
$$

and if we define

$$
\begin{align*}
 dg_t^\theta &= T\theta_t^{-1} \left\{ X(y_t g_t -) \tilde{\mathcal{J}}(y_t(x_0)) \circ d\beta_t + X(y_t g_t -) \Lambda(y_t(x_0))dt \\
&\quad + A(y_t g_t -)dt - X(y_t g_t -)Y(y_t x_0)A(y_t x_0)dt \right\}
\end{align*}
$$

for any continuous $y : [0, \infty) \to D^\infty$, we see, by the independence of $\beta$ and $\theta$ that $g. = g.^\theta$.

By Itô's formula on $D^s$, for $x \in M$,

$$
d(\theta_t g_t(x_0)) = (\circ d\theta_t)(g_t(x)) + T\theta_t(\circ dg_t^\theta(x)).
$$

Now

$$
T\theta_t(\circ dg_t^\theta(x)) = \left\{ X(\theta_t g_t(x)) K(\theta_t x_0) \circ dB_t \\
+ A(\theta_t g_t(x))dt - X(\theta_t g_t(x))Y(\theta_t x_0)A(\theta_t x_0)dt \right\}
$$

and so by (13) we see that $\theta_t g_t = \xi_t \circ \theta_0$, a.s.

Taking $\theta_0 = id$ we have

**Proposition 3.2.** The flow $\xi$. has the decomposition

$$
\xi_t = \theta_t g_t^\theta, \quad 0 \leq t < \infty
$$

for $\theta$ and $g.^\theta \equiv g. \text{ given by (13) and (14) above. For almost all } \sigma : [0, \infty) \to M \text{ with } \sigma(0) = x_0 \text{ and bounded measurable } F : C(0, \infty; D^\infty) \to \mathbb{R}

$$
\mathbb{E}\{ F(\xi.) | \xi.(x_0) = \sigma \} = \mathbb{E}\{ F(\tilde{\sigma} g.^\tilde{\sigma}) \}
$$

where $\tilde{\sigma} : [0, \infty) \to D^\infty$ is the horizontal lift of $\sigma$ with $\tilde{\sigma}(0) = id$.

To define the 'horizontal lift' above we can use the fact, from (i) above, that $\theta$. has the same filtration as $\xi.(x_0)$ and so furnishes a lifting map.

In terms of the semi-connection induced on $\pi : D^s \to M$ over $E$, from above, by uniqueness or directly, we see the horizontal lift

$$
\begin{align*}
 h_\theta : E_{\theta(x_0)} &\longrightarrow T_0 D^s \\
h_\theta(v) : M &\longrightarrow TM
\end{align*}
$$
is given by $h_{\theta}(v) = X(\theta(x))Y(\theta(x_0))v$ and the horizontal lift $\tilde{\sigma}$ from $\tilde{\sigma}_0$ of a $C^1$ curve $\sigma$ on $M$ with $\tilde{\sigma}_0(x_0) = \sigma_0$ and $\tilde{\sigma}(t) \in E_{\sigma(t)}$, all $t$, is given by

$$\frac{d}{dt} \tilde{\sigma}_t = X(\tilde{\sigma}_t-\sigma_t)Y(\sigma_t)\tilde{\sigma}_t$$

for $\tilde{\sigma}_0 = id$. The lift is the solution flow of the differential equation

$$\dot{y}_t = Z^{\tilde{\sigma}}(y_t)$$

on $M$.

For each frame $u : \mathbb{R}^n \to T_{x_0}M$ there is a homomorphism of principal bundles

$$(15) \quad \begin{array}{ccl} D^s & \to & GL(M) \\ \theta & \mapsto & T_{x_0}\theta \circ u. \end{array}$$

This sends $\{\xi_t : t \geq 0\}$ to the derivative process $T_x\xi_t \circ u$. (If the latter satisfies the strongly cohesive condition we could apply our analysis to this submersion $D^s \to GLM$ and get another decomposition of $\xi$.)

Results in Kobayashi-Nomizu [9] (Proposition 6.1 on page 79) apply to the homomorphism $D^s \to GL(M)$ of (15). This gives a relationship between the curvature and holonomy groups of the semi-connection $\hat{\nabla}$ on $GLM$ determined by the derivative flow and those of the connection induced by the diffusion on $D^s \to M$. It also shows that the horizontal lift $\{\tilde{x}_t : t \geq 0\}$ through $u$ of $\{x_t : t \geq 0\}$ to $GL(M)$ is just $T_{x_0}\theta_t \circ u$ for $\{\theta_t : t \geq 0\}$ the flow given by (13) with $\theta_0 = id$, i.e. the solution flow of the stochastic differential equation

$$dy_t = Z^{\circ dx_t}(y_t).$$

From this and Lemma 1.3.4 of [7] we see that $\hat{\nabla}$ is the adjoint of the LeJan-Watanabe connection determined by the flow, so proving Theorem 3.1 above. However the present construction applies with $GLM$ replaced by any natural bundle over $M$ (e.g. jet bundles, see Kolar-Michor-Slovak [10]), to give semi-connections on these bundles.

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K. D. Elworthy

*Mathematics Institute, Warwick University, Coventry CV4 7AL, UK*

Y. Le Jan

*Département de Mathématique, Université Paris Sud, 91405 Orsay, France*

Xue-Mei Li

*Department of Computing and Mathematics, The Nottingham Trent University, Nottingham NG7 1AS, UK.* e-mail address: xuemei.li@ntu.ac.uk