Cascades in helical turbulence

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The energy dissipation is from the energy spectrum $E(k)$ and using
\[ \langle \delta v^2 \rangle = -(4/5) \eta k^4 \]
states that the third order correlator associated with energy flux equals the mean energy dissipation. As noted recently in the case of helical flow a similar relation exists for the transfer of helicity leading to an other scaling relation for a third order correlator associated with the flux of helicity, $\langle \delta v^2 \rangle = (2/15) \eta k^2$, where $\eta$ is the mean dissipation of helicity. This relation is called the ‘two-fifteenth law’ due to the numerical prefactor. This establishes another non-trivial scaling relation for velocity differences in a turbulent helical flow.

The coexistence of cascades of energy and enstrophy is prohibited for high Reynolds number flow in 2D turbulence. The reason for this is that the enstrophy dominates at small scales such that the ratio of energy – to enstrophy dissipation vanishes for high Reynolds number flow. The Kolmogorov scale $k_Z^{-1}$ for enstrophy dissipation is determined from the energy spectrum $E(k) \sim k^{-3}$ and the kinematic viscosity $\nu$ by $\zeta = \nu \int k^2 dkk^2 E(k) \sim \nu k_Z^2 \Rightarrow k_Z \sim \nu^{-1/2}$. The energy dissipation is $\zeta = \nu \int k^2 dkk^2 E(k) \sim \nu \log k_Z \sim -(1/2) \nu \log \nu \rightarrow 0$ for $\nu \rightarrow 0$. Consequently energy is cascaded upscale in 2D turbulence.

The situation in 3D turbulence is different. Here coexisting cascades of energy and helicity are possible. However, the same type of dimensional argument as for the cascades of energy and enstrophy in 2D turbulence applies. The helicity density is $h = u_i \omega_i$, where $\omega_i = \epsilon_{ijk} \partial_j u_k$ is the vorticity. The mean dissipation of helicity is $D_H = \nu(\partial_j u_i \partial_j \omega_i)$. Disregarding signs this can spectrally be represented as

\[
D_H \sim \nu \int k_E dkk^2 E(k) \sim \nu k_E^{7/3} \sim \nu^{-3/4},
\]

where $k_E^{-1} = \eta$ is the Kolmogorov scale and we have used $E(k) \sim k^{-5/3}$ and $k_E \sim \nu^{-3/4}$. This means that for high Reynolds numbers flow the dissipation of helicity will grow as $R_e^{3/4}$. Since the mean dissipation of energy $\zeta$ and helicity $\eta$ are determined by the integral scale forcing the growth of helicity dissipation with Reynolds number is apparently in conflict with the assumption of a constant energy dissipation in the limit of vanishing viscosity. This is not a true problem because helicity is non-positive, and the viscous term in the equation for the helicity $\nu(u_i \partial_j \omega_i + \omega_i \partial_j u_i)$ can have either sign. So in the high Reynolds number limit there must either be a detailed balance between dissipation of positive and negative helicity or the energy cascade is blocked. In the rather artificial case of a shell model where only one sign of helicity is dissipated by hyper-viscosity, the energy cascade is indeed prevented all together similar to the case of forward energy cascade of energy in 2D turbulence.

In a helical flow $(-\eta \neq 0)$ the dissipation of helicity defines a scale different from the Kolmogorov scale $\eta$. This we will call the Kolmogorov scale $\xi$ for dissipation of helicity.

Following K41, the Kolmogorov scale $\eta$ for energy dissipation is obtained from $\zeta \sim \delta u^2 / \eta$ and using $\delta u^2 / \eta \approx \eta \sim (\nu^3 / \zeta)^{1/4}$, where $\delta u^2$ is a typical variation of the velocity over a scale $l$. The Kolmogorov scale $\xi$ for dissipation of helicity is defined as the scale where the helicity dissipation is of same order as the spectral helicity flux. With dimensional counting we have $\delta \sim \nu \delta u^2 / \xi^2$ and using $\delta u^2 \sim (\nu \xi)^{1/3}$ we obtain
\[ \xi \sim (\nu^3 \frac{\xi}{\lambda^3})^{1/7}. \]  

(2)

Now it is clear why (2) leads to a wrong conclusion for the mean dissipation of the helicity \( \overline{\epsilon} \). The integral will not be dominated by contributions from \( k_E \) but contributions from \( k_H = 1/\xi \),

\[ D_H = \overline{\epsilon} \sim \nu k_H^{7/3} \Rightarrow k_H \sim \nu^{-3/7}. \]  

(3)

The ratio of the two Kolmogorov scales is then \( (\eta/\xi) = (k_H/k_E) \sim \nu^{-3/7+3/4} = \nu^{9/28} \rightarrow 0 \) for \( \nu \rightarrow 0 \). Thus for high Reynolds number helical flow the small scales will always be non-helical and a pure helicity cascade is not possible.

On the other hand for scales \( l < \xi \) the ratio of dissipation of energy and helicity is proportional to \( l \), \( D_E/D_H \sim l \), which means that helicity dissipation dominates and the dissipation of positive and negative helicity must balance. The reason for the flow to be non-helical on small scales is different from the reason why the flow tends to be isotropic on small scales even though the integral scale is non-isotropic. The reason for the small scales to be isotropic is that the structure functions associated with the non-isotropic sectors scale with scaling exponents that are larger than those of the isotropic sector and thus becomes sub-leading for the flow at small scales independent of the dissipation [\( \overline{\epsilon} \)].

The physical picture for fully developed helical turbulence is then that \( \overline{\epsilon} \) and \( \overline{\delta} \) are solely determined by the forcing in the integral scale. There will then be an inertial range with coexisting cascades of energy and helicity with third order structure functions determined by the four-fifth – and the two-fifteenth laws. This is followed by an inertial range between \( \xi \) and \( \eta \) corresponding to non-helical turbulence, where the dissipation of positive and negative helicity vortices balance and the two-fifteenth law is not applicable.

In order to test these ideas in a model system we investigate the role of helicity and the structure of the helicity transfer in a shell model.

Shell models are toy-models of turbulence which by construction have second order inviscid invariants similar to energy and helicity in 3D turbulence. Shell models can be investigated numerically for high Reynolds numbers, in contrast to the Navier-Stokes equation, and high order statistics and anomalous scaling exponents are easily accessible. Shell models lack any spatial structures so we stress that only certain aspects of the turbulent cascades have meaningful analogies in the shell models. This should especially be kept in mind when studying helicity which is intimately linked to spatial structures, and the dissipation of helicity to reconnection of vortex tubes [\( \overline{\epsilon} \)]. So the following only concerns the spectral aspects of the helicity and energy cascades.

The most well studied shell model, the GOY model [\( \overline{\epsilon} \)], is defined from the governing equation,

\[ u_n = ik_n(u_{n+2}u_{n+1} - \frac{\epsilon}{\lambda} |u_{n+1}u_{n-1}| + \frac{\epsilon - 1}{\lambda^2} u_{n-1}u_{n-2})^* - \nu k_n^2 u_n + f_n \]  

(4)

with \( n = 1, \ldots, N \) where the \( u_n \)'s are the complex shell velocities. The wavenumbers are defined as \( k_n = \lambda^n \), where \( \lambda \) is the shell spacing. The second and third terms are dissipation and forcing. The model has two inviscid invariants, energy, \( E = \sum_n E_n = \sum_n |u_n|^2 \), and ‘helicity’, \( H = \sum_n H_n = \sum_n (\epsilon - 1)^{-n} |u_n|^2 \). The model has two free parameters, \( \lambda \) and \( \epsilon \). The ‘helicity’ only has the correct dimension of helicity if \( |\epsilon - 1|^{-n} = k_n \Rightarrow 1/(1 - \epsilon) = \lambda \). In this work we use the standard parameters \( (\epsilon, \lambda) = (1/2, 2) \) for the GOY model.

A natural way to define the structure functions of moment \( p \) is through the transfer rates of the inviscid invariants,

\[ S^E_p(k_n) = \langle (\Pi^E_n)^{p/3} \rangle k_n^{-p/3} \]  

(5)

\[ S^H_p(k_n) = \langle (\Pi^H_n)^{p/3} \rangle k_n^{-2p/3} \]  

(6)

The energy flux is defined in the usual way as \( \Pi^E_n = d/dt |\sum_{m=1}^n E_m| \) where \( d/dt |\sum m \) is the time rate of change due to the non-linear term in [\( \overline{\epsilon} \)]. The helicity flux \( \Pi^H_n \) is defined similarly. By a simple algebra we have the following expression for the fluxes,

\[ \langle \Pi^E_n \rangle = (1 - \epsilon) \Delta_n + \Delta_{n+1} = \overline{\epsilon} \]  

(7)

\[ \langle \Pi^H_n \rangle = (-1)^n k_n (\Delta_{n+1} - \Delta_n) = \overline{\delta} \]  

(8)

where \( \Delta_n = k_{n-1} Im \langle u_{n-1}u_{n+1} \rangle \), \( \overline{\epsilon} \) and \( \overline{\delta} \) are the mean dissipations of energy and helicity respectively. The first equalities hold without averaging as well. These equations are the shell model equivalents of the four-fifth – and the two-fifteenth law. Kadanoff et al. [\( \overline{\epsilon} \)] defined a third order structure function as,

\[ S^3_n = Im \langle u_{n-1}u_nu_{n+1} \rangle = \Delta_n/k_{n-1} \]  

(9)
to avoid the spurious (specific to the GOY model) period 3 oscillation. Using this we obtain from (6) and (8) a scaling relation for $S_n^3$,

$$S_n^3 = \frac{1}{(1 - \epsilon/2)k_n} (\overline{\delta} - (-1)^n \overline{\delta}/k_n). \quad (10)$$

The last term in the parenthesis is sub-leading with period two oscillations. When $\overline{\delta} = 0$ the sub-leading term disappears and the scaling from the four-fifth law is obtained, figure 1. The relation (8) is the scaling relation corresponding to the sub-leading term, which survives due to detailed cancellations between the two terms $\Delta_{n+1}$ and $\Delta_n$ of the leading term corresponding to (6). The case $\overline{\delta} = 0$ and $\overline{\delta} \neq 0$ would, aside from the period two oscillation, correspond to a helicity cascade with the scaling obtained from dimensional counting $u_n \sim k_n^{-2/3}$. However, this situation is, as we will show shortly, not realizable.

The mean dissipations $\overline{\epsilon}$ and $\overline{\delta}$ are from energy and helicity conservations identical to the mean energy and helicity inputs which from (6) are,

$$\overline{\epsilon} = \sum_n \langle f_n u^*_n \rangle + c.c. \quad (11)$$

and

$$\overline{\delta} = \sum_n (-1)^n k_n \langle f_n u^*_n \rangle + c.c., \quad (12)$$

so $\overline{\epsilon}$ and $\overline{\delta}$ are not independent. The forcing can be chosen in many ways. A natural choice is $f_n = f^0_n/u^*_n$, where $f^0_n$ is independent on the shell velocities. Then we have, $\overline{\epsilon} = \sum_{n<n_l} f^0_n$ and $\overline{\delta} = \sum_{n<n_l} (-1)^n k_n f^0_n$, $n_l$ indicates the end of the integral scale. By choosing the coefficients, stochastic or deterministic functions of time, this last sum can vanish identically, which is referred to as helicity free forcing. The simulations shown in figure 1 are performed with the forcing, $f^0_3 = 10^{-2} (1 + i)$ and $f^0_4 = -A f^0_3/\lambda$ with $A = 0$ and $A = 1$, corresponding to $(\overline{\epsilon}, \overline{\delta}) = (0.01, 0)$ and $(\overline{\epsilon}, \overline{\delta}) = (0.01, 0.08)$ respectively.

Helicity is not positive and is dissipated with opposite signs for odd and even shells. If we consider the third order structure function associated with the helicity transfer as defined by (6) we see (figure 2) period two oscillations of the leading term corresponding to (7). The case $\Delta_{n+1} = 0$ and $\Delta_n$ is integrable (solve for $y = u_n/u_{n+1}$), and has no finite time singularities. Using this forcing we performed a set of
simulations with constant energy input \( \bar{\varepsilon} = 0.01 \) and varying helicity input \( \bar{\delta} = (0.0001, 0.001, 0.005, 0.01, 0.08) \). In figure 4 the spectra of the absolute value of the helicity transfer normalized with \( \bar{\delta} \) are plotted against wave number normalized with \( K_H \). \( K_H \) is in each case calculated from (2), and a clear data collapse is seen.

In summary the simulations with the GOY shell model suggest a new Kolmogorov scale for helicity, always smaller than the Kolmogorov scale for energy. Thus there exist two inertial ranges in helical turbulence, a range smaller than \( K_H \) with coexisting cascades of energy and helicity where both the four-fifth - and the two-fifteenth law applies, and a range between \( K_H \) and \( K_E \) where the flow is non-helical and only the four-fifth law applies.

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FIGURE CAPTIONS

Fig. 1 The third order structure function $S_3$ as calculated from (9) in the cases $\bar{\delta} > 0$ (crosses) and $\bar{\delta} = 0$ (diamonds). In the case of helicity free forcing the modulus 2 oscillations disappears. In the two runs we have 25 shells, $\nu = 10^{-9}, f_n = 0.01(1 + i)(\delta_{n,2}/u_2^* - A\delta_{n,3}/2u_3^*)$ with $A = 0, 1$ respectively.

Fig. 2 The helicity flux $\langle \Pi_{Hn} \rangle$ in the case $\bar{\delta} > 0$. The same curve is multiplied by 1000 and over-plotted in order to see the inertial range. The period 2 oscillations in the helicity transfer comes from the helicity dissipation.

Fig. 3 The absolute values of the helicity flux $|\langle \Pi_{Hn} \rangle|$ (diamonds) show a crossover from the inertial range for helicity to the range where the helicity is dissipated. The line has a slope of $7/3$ indicating the helicity dissipation. The dashed lines indicate the helicity input $\bar{\delta}$. The crosses is the helicity flux in the case $\bar{\delta} = 0$ where there is no inertial range and $K_H$ coincides with the integral scale. The triangles are the energy flux $\langle \Pi_{En} \rangle$.

Fig. 4 Five simulations with constant viscosity $\nu = 10^{-9}$, constant energy input $\bar{e} = 0.01$ and varying helicity input $\bar{\delta} = (0.0001, 0.001, 0.005, 0.01, 0.08)$ are shown. The absolut values of the helicity flux $|\langle \Pi_{Hn} \rangle|$ divided by $\bar{\delta}$ is plotted against the wave number divided by $K_H = (\nu^2/2\bar{e}^3)_{-1/3}$, which is obtained from (8) neglecting O(1) constants. A clear data collapse is seen.
FIG. 4.