REMARK ON GALOIS COHOMOLOGY

IGOR V. NIKOLAEV

Abstract. We recast the Galois cohomology of the variety $V$ over a number field $k$ in terms of the K-theory of a $C^*$-algebra $\mathcal{A}_V$ connected to $V$. It is proved that $V$ is isomorphic to $V'$ over $k$ (algebraic closure of $k$, resp.) if and only if $\mathcal{A}_V$ is isomorphic (Morita equivalent, resp.) to $\mathcal{A}_{V'}$. In particular, the Morita equivalent $C^*$-algebras $\mathcal{A}_V$ parametrize twists of the variety $V$. The case of rational elliptic curves is considered in detail.

1. Introduction

Let $V$ be a complex projective variety given by a homogeneous coordinate ring $\mathcal{A}$. If $k \subset \mathbb{C}$ is a subfield of complex numbers and $V(k)$ is the set of $k$-points of $V$, then the isomorphisms of $V(k)$ over $\mathbb{C}$ cannot be restricted to the field $k$ in general. When such a restriction fails, the variety $V(k)$ is called a twist of $V$. Equivalently, the varieties $V(k)$ and $V'(k)$ are not isomorphic over $k$, yet they are isomorphic over $\mathbb{C}$. The twists of $V(k)$ are classified in terms of the Galois cohomology [Serre 1997] [4, p. 123].

Recall that the Serre $C^*$-algebra $\mathcal{A}_V$ is defined as the norm closure of a self-adjoint representation of the twisted homogeneous coordinate ring of $V(k)$ by the bounded linear operators on a Hilbert space $\mathcal{H}$, see [Stafford & van den Bergh 2001] [6] and [3, Section 5.3.1]. The $C^*$-algebra $A$ is said to be Morita equivalent to $A'$, if $A \otimes \mathcal{K} \cong A' \otimes \mathcal{K}$, where $\mathcal{K}$ is the $C^*$-algebra of all compact operators on $\mathcal{H}$ and $\cong$ is an isomorphism of the $C^*$-algebras [Blackadar 1986] [1, Section 13.7.1]. (Notice that if $A \cong A'$ are isomorphic $C^*$-algebras, then $A$ is Morita equivalent to $A'$.) The correspondence $V(k) \mapsto \mathcal{A}_V$ is a functor, which maps $\mathbb{C}$-isomorphic varieties to the Morita equivalent Serre $C^*$-algebras [3, Theorem 5.3.3].

The aim of our note is a classification of the twists of $V(k)$ in terms of the Serre $C^*$-algebra $\mathcal{A}_V$. Namely, we prove that if the variety $V(k)$ is $k$-isomorphic to a variety $V'(k)$, then the Serre $C^*$-algebra $\mathcal{A}_V$ is isomorphic to $\mathcal{A}_{V'}$, while if $V(k)$ is $\mathbb{C}$-isomorphic to $V'(k)$, then the Serre $C^*$-algebra $\mathcal{A}_V$ is Morita equivalent to $\mathcal{A}_{V'}$, see corollary 1.2. To formalize our results, we need the following definitions.

Let $A$ be a unital $C^*$-algebra and denote by $V(A)$ the union of projections in all the $n \times n$ matrix $C^*$-algebra with entries in $A$ [Blackadar 1986] [1, Section 5]. Recall that projections $p, q \in V(A)$ are called equivalent, if there exists a partial isometry $u$ such that $p = uu^*$ and $q = uu^*$. The corresponding equivalence class is denoted by $[p]$. The equivalence classes of the orthogonal projections can be made to a semigroup with addition defined by the formula $[p] + [q] = [p+q]$. The Grothendieck completion of the semigroup to an abelian group, $K_0(A)$, is called the

2010 Mathematics Subject Classification. Primary 11G35, 14A22; Secondary 46L85.

Key words and phrases. twist, Serre $C^*$-algebras.
$K_0$-group of the algebra $A$. The functor $A \rightarrow K_0(A)$ maps the category of unital $C^*$-algebras into the category of abelian groups, so that projections in the algebra $A$ correspond to a positive cone $K_0^+(A) \subset K_0(A)$ and the unit element $1 \in A$ corresponds to an order unit $u \in K_0^+(A)$. An isomorphism class of the ordered abelian group $(K_0(A), K_0^+(A))$ is known as a dimension group. The dimension group $(K_0(A), K_0^+(A), u)$ with a fixed order unit $u$ is called a scaled dimension group [Blackadar 1986] [1, Section 6].

An AF-algebra $\mathcal{B}$ (Approximately Finite $C^*$-algebra) is the norm closure of an ascending sequence of the finite-dimensional $C^*$-algebras $M_n(C)$, where $M_n(C)$ is the $C^*$-algebra of the $n \times n$ matrices with entries in $C$ [Blackadar 1986] [1, Section 7.1]. The scaled dimension group $(K_0(\mathcal{B}), K_0^+(\mathcal{B}), u)$ is an isomorphism invariant of the algebra $\mathcal{B}$. In contrast, the dimension group $(K_0(\mathcal{B}), K_0^+(\mathcal{B}))$ is an invariant of the Morita equivalence of the AF-algebra $\mathcal{B}$ [Blackadar 1986] [1, Section 7.3]. The Serre $C^*$-algebra $\mathcal{A}_V$ is not an AF-algebra, but there exists a dense embedding $\mathcal{A}_V \hookrightarrow \mathcal{B}$, where $\mathcal{B}$ is an AF-algebra such that $(K_0(\mathcal{A}_V), K_0^+(\mathcal{A}_V)) \cong (K_0(\mathcal{B}), K_0^+(\mathcal{B}))$ [2, Lemma 3.1].

By $H^1(Gal(\overline{k}|k), Aut_C^{ab}(V))$ we understand the first Galois cohomology group [Serre 1997] [4, p. 123] of the extension $k \subset C$, where $Aut_C^{ab}(V)$ is the maximal abelian subgroup of the group of $C$-automorphisms of the variety $V(k)$. The $H^1(Gal(\overline{k}|k), Aut_C^{ab}(V))$ is a dimension group of stationary type, see lemma 3.1.

Our main results can be formulated as follows.

**Theorem 1.1.** $H^1(Gal(\overline{k}|k), Aut_C^{ab}(V)) \cong (K_0(\mathcal{A}_V), K_0^+(\mathcal{A}_V))$, where $\cong$ is an order-isomorphism of the dimension groups.

**Corollary 1.2.** Let $\mathcal{A}_V = F(V(k))$ and $\mathcal{A}_V' = F(V'(k))$, where $V(k)$ and $V'(k)$ are complex projective varieties over the field $k \subset C$. Then:

(i) $V(k)$ and $V'(k)$ are isomorphic over $k$ if and only if the Serre $C^*$-algebras $\mathcal{A}_V \cong \mathcal{A}_V'$ are isomorphic;

(ii) $V(k)$ and $V'(k)$ are isomorphic over $C$ if and only if the Serre $C^*$-algebras $\mathcal{A}_V$ and $\mathcal{A}_V'$ are Morita equivalent.

The article is organized as follows. In Section 2 we briefly review the Serre $C^*$-algebras and the Galois cohomology. Theorem 1.1 and corollary 1.2 are proved in Section 3. An illustration of corollary 1.2 can be found in Section 4.

### 2. Preliminaries

In this section we briefly review the Galois cohomology and the Serre $C^*$-algebras. We refer the reader to [Serre 1997] [4, Chapter I, §5] and [3, Section 5.3.1] for a detailed account.

#### 2.1. Serre $C^*$-algebras

Let $V$ be an $n$-dimensional complex projective variety endowed with an automorphism $\sigma : V \rightarrow V$ and denote by $B(V, L, \sigma)$ its twisted homogeneous coordinate ring [Stafford & van den Bergh 2001] [6]. Let $R$ be a commutative graded ring, such that $V = \text{Proj}(R)$. Denote by $R[t, t^{-1}; \sigma]$ the ring of skew Laurent polynomials defined by the commutation relation $b^\sigma t = tb$ for all $b \in R$, where $b^\sigma$ is the image of $b$ under automorphism $\sigma$. It is known, that $R[t, t^{-1}; \sigma] \cong B(V, L, \sigma)$. 
Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. For a ring of skew Laurent polynomials $R[t, t^{-1}; \sigma]$, consider a homomorphism:

$$\rho : R[t, t^{-1}; \sigma] \rightarrow \mathcal{B}(\mathcal{H}). \quad (2.1)$$

Recall that $\mathcal{B}(\mathcal{H})$ is endowed with a $*$-involution; the involution comes from the scalar product on the Hilbert space $\mathcal{H}$. We shall call representation $(2.1)$ $*$-coherent, if (i) $\rho(t)$ and $\rho(t^{-1})$ are unitary operators, such that $\rho^*(b) = \rho(t^{-1})$ and (ii) for all $b \in R$ it holds $\rho^*(b) = \rho^*(b^\sigma)$, where $\sigma$ is an automorphism of $\rho(R)$ induced by $\sigma$. Whenever $B = R[t, t^{-1}; \sigma]$ admits a $*$-coherent representation, $\rho(B)$ is a $*$-algebra. The norm closure of $\rho(B)$ is a $C^*$-algebra denoted by $\mathcal{A}_V$. We refer to the $\mathcal{A}_V$ as the Serre $C^*$-algebra of variety $V$.

### 2.2. Galois cohomology

Let $G$ be a group. The set $A$ is called a $G$-set, if $G$ acts on $A$ on the left continuously. If $A$ is a group and $G$ acts on $A$ by group morphisms, then $A$ is called a $G$-group. In particular, if $A$ is abelian, one gets a $G$-module.

If $A$ is a $G$-group, then a 1-cocycle of $G$ in $A$ is a map $s \mapsto a_s$ of $G$ to $A$ which is continuous and such that $a_{st} = a_s a_t$ for all $s, t \in G$. The set of all 1-cocycles is denoted by $Z^1(G, A)$. Two cocycles $a$ and $a'$ are said to be cohomologous, if there exists $b \in A$ such that $a' = b^{-1} a b$. The quotient of $Z^1(G, A)$ by this equivalence relation is called the first cohomology set and is denoted by $H^1(G, A)$. The class of the unit cocycle is a distinguished element $1$ in the $H^1(G, A)$. Notice that in general there is no composition law on the set $H^1(G, A)$. If $A$ is an abelian group, the set $H^1(G, A)$ is a cohomology group.

If $G$ is a profinite group, then

$$H^1(G, A) = \varprojlim H^1(G/U, A^U), \quad (2.2)$$

where $U$ runs through the set of open normal subgroups of $G$ and $A^U$ is a subset of $A$ fixed under action of $U$. The maps $H^1(G/U, A^U) \rightarrow H^1(G, A)$ are injective.

Let $k$ be a number field and $K$ the algebraic closure of $k$. Denote by $Gal(\bar{k}/k)$ the profinite Galois group of $k$. Let $V(k)$ be a projective variety over $k$ and $Aut V(k)$ the group of the $k$-automorphisms of $V(k)$.

**Lemma 2.1.** [Serre 1997] [4, p. 124] There exits a bijective correspondence between the twists of $V(k)$ and the set $H^1(Gal(\bar{k}/k), Aut V(k))$.

### 3. Proofs

#### 3.1. Proof of theorem 1.1

We shall split the proof in a series of lemmas.

**Lemma 3.1.** The $H^1(Gal(\bar{k}/k), Aut_{C^\text{ab}}(V))$ is a stationary dimension group.

**Proof.** It is known that the Galois group $Gal(\bar{k}/k)$ of the field extension $k \subset C$ is a profinite group. We denote by $U_j$ an infinite ascending sequence of the open normal subgroups of $Gal(\bar{k}/k)$. In other words,

$$Gal(\bar{k}/k) = \varprojlim U_j. \quad (3.1)$$

The corresponding Galois cohomology $(2.2)$ can be written in the form

$$H^1(Gal(\bar{k}/k), Aut_{C^\text{ab}}(V)) = \varprojlim H^1 \left( Gal(\bar{k}/k)/U_j, (Aut_{C^\text{ab}}(V))^{U_j} \right). \quad (3.2)$$
Let us show that the dimension group $H^1(Gal(\bar{k}|k), Aut_{C}^{ab}(V))$ is a stationary dimension group, i.e. $n_j$ and some positive group homomorphisms $\varphi_j$, where $Z^n$ is given the usual ordering

\[(Z^n)^+ = \{(x_1, x_2, \ldots, x_n) \in Z^n : x_j \geq 0\} \tag{3.5}\]

The dimension group is called stationary, if $n_j = n = Const$ and $\varphi_j = \varphi = Const$.

We shall define a group homomorphism $\varphi_j : H^1(Gal(\bar{k}|k)/U_j, (Aut_{C}^{ab}(V))^{U_j}) \to H^1(Gal(\bar{k}|k)/U_{j+1}, (Aut_{C}^{ab}(V))^{U_{j+1}})$ from a commutative diagram in Figure 1, where the injective homomorphisms $\alpha_j$ are defined by the formula (3.3). Note that $H^1(Gal(\bar{k}|k)/U_j, (Aut_{C}^{ab}(V))^{U_j}) \cong Z^{n_j}$ for an integer $n_j \geq 0$; the order on $Z^{n_j}$ is defined by (3.5). Moreover, it is easy to see that $\varphi_j$ preserves the the order, i.e. $\varphi_j$ is a positive homomorphism. Comparing formulas (3.2) and (3.4), we conclude that the cohomology group $H^1(Gal(\bar{k}|k), Aut_{C}^{ab}(V))$ is a dimension group.

Let us show that the dimension group $H^1(Gal(\bar{k}|k), Aut_{C}^{ab}(V))$ is a stationary dimension group, i.e. $n_j = n = Const$ and $\varphi_j = \varphi = Const$. Indeed, notice that the shift $j \mapsto j + 1$ in the RHS of formula (3.1) corresponds to an automorphism of the group $Gal(\bar{k}|k)$. (Namely, the inductive limits $\varinjlim U_j$ and $\varinjlim U_{j+1}$ generate isomorphic profinite groups $Gal(\bar{k}|k)$.) Such an automorphism gives rise to the shift automorphism of the Galois cohomology (3.2) and the corresponding dimension group (3.4). But the dimension group admits the shift automorphism if and only if it is a stationary dimension group; the latter fact follows from [Blackadar 1986] [1, Theorem 7.3.2] since the automorphisms of stationary AF-algebras are generated by the shift automorphism of the corresponding dimension group.

**Lemma 3.3.** $H^1(Gal(\bar{k}|k), Aut_{C}^{ab}(V)) \cong (K_0(\mathcal{A} V), K_0^+(\mathcal{A} V))$. 

---

**Figure 1.** The group homomorphism $\varphi_j$. 

\[
H^1 \left( Gal(\bar{k}|k)/U_j, (Aut_{C}^{ab}(V))^{U_j} \right) \xrightarrow{\varphi_j} H^1 \left( Gal(\bar{k}|k)/U_{j+1}, (Aut_{C}^{ab}(V))^{U_{j+1}} \right)
\]

\[
\alpha_j \xrightarrow{\alpha_{j+1}} H^1(Gal(\bar{k}|k), Aut_{C}^{ab}(V))
\]
Proof. Recall that there exists an embedding $\mathcal{A}_V \hookrightarrow \mathcal{B}$, where $\mathcal{B}$ is an AF-algebra such that $(K_0(\mathcal{A}_V), K_0^+(\mathcal{A}_V)) \cong (K_0(\mathcal{B}), K_0^+(\mathcal{B}))$ [2, Lemma 3.1]. Moreover, the $(K_0(\mathcal{B}), K_0^+(\mathcal{B}))$ is a stationary dimension group, ibid.

On the other hand, it is known that the Galois cohomology $H^1(\text{Gal}(\bar{k}|k), \text{Aut}_{\mathcal{C}}^{ab}(V))$ is a functor from the category of projective varieties $V(k)$ to a category of abelian groups [Serre 1997] [4]. We shall denote by $\mathcal{B}'$ an AF-algebra such that

$$(K_0(\mathcal{B}'), K_0^+(\mathcal{B}')) \cong H^1(\text{Gal}(\bar{k}|k), \text{Aut}_{\mathcal{C}}^{ab}(V)).$$

Since the Galois cohomology is the functor, we conclude that the AF-algebra $\mathcal{B}'$ contains a densely embedded coordinate ring of the variety $V(k)$. But any such ring must be Morita equivalent to the Serre $C^*$-algebra $\mathcal{A}_V$. In other words,

$$(K_0(\mathcal{B}'), K_0^+(\mathcal{B}')) \cong (K_0(\mathcal{A}_V), K_0^+(\mathcal{A}_V)).$$

The conclusion of lemma 3.3 follows from the formulas (3.6) and (3.7). □

Theorem 1.1 follows from lemma 3.3.

Remark 3.4. It is well known, that the $H^1(\text{Gal}(\bar{k}|k), \text{Aut}_{\mathcal{C}}^{ab}(V))$ is a torsion group, i.e. each element of the group has finite order, see e.g. [Serre 1997] [4, Chapter I, §2.2, Corollary 3]. On the other hand, the dimension group is unperforated, so that it is torsion free [Blackadar 1986] [1, Section 7.4]. However, this observation does not contradict lemma 3.3. To recover the structure of torsion group from the torsion free dimension group $(K_0(\mathcal{A}_V), K_0^+(\mathcal{A}_V))$, recall that the order unit $u \in K_0^+(\mathcal{A}_V)$ defines the scale $\Sigma(\mathcal{A}_V) = \{0 < x < u \mid x \in K_0^+(\mathcal{A}_V)\}$, i.e. a generating, hereditary and directed subset of the $K_0^+(\mathcal{A}_V)$, see [Blackadar 1986] [1, Section 6.1]. It is easy to see, that the scale $\Sigma(\mathcal{A}_V)$ is a torsion group. Indeed, we let $x \in \Sigma(\mathcal{A}_V)$ to be of order $k$, if $k$ is the least integer, such that $u < x^{k+1}$.

3.2. Proof of corollary 1.2.

Lemma 3.5. There exits a bijective correspondence between the elements of the abelian group $H^1(\text{Gal}(\bar{k}|k), \text{Aut}_{\mathcal{C}}^{ab} V(k))$ and a subset of the set of twists of the variety $V(k)$.

Proof. A restriction of the coefficient group $\text{Aut}_{\mathcal{C}} V(k)$ of the Galois cohomology $H^1(\text{Gal}(\bar{k}|k), \text{Aut}_{\mathcal{C}} V(k))$ to its unique maximal abelian subgroup $\text{Aut}_{\mathcal{C}}^{ab} V(k))$ defines an inclusion of the sets

$$H^1(\text{Gal}(\bar{k}|k), \text{Aut}_{\mathcal{C}}^{ab} V(k)) \subseteq H^1(\text{Gal}(\bar{k}|k), \text{Aut}_{\mathcal{C}} V(k)).$$

In view of the Serre’s Lemma 2.1, one gets from the inclusion (3.8) a bijection between the elements of the abelian group $H^1(\text{Gal}(\bar{k}|k), \text{Aut}_{\mathcal{C}}^{ab} V(k))$ and a subset of the set of twists of the variety $V(k)$. Lemma 3.5 is proved. □

Corollary 3.6. There exits a bijective correspondence between the elements of the dimension group $(K_0(\mathcal{A}_V), K_0^+(\mathcal{A}_V))$ and a subset of the set of twists of the variety $V(k)$.

Proof. This corollary is an implication of theorem 1.1 and lemma 3.5. □
Lemma 3.7. There exists a bijective correspondence between the set of all scaled dimension groups \((K_0(\mathcal{A}_V), K_0^+(\mathcal{A}_V), u)\) and a subset of the set of twists of the variety \(V(k)\).

Proof. It is known, that each \(u \in K_0^+(\mathcal{A}_V)\) can be taken for an order-unit of the scaled dimension group \((K_0(\mathcal{A}_V), K_0^+(\mathcal{A}_V), u)\) [Blackadar 1986] [1, Section 6.2]. Moreover, the obtained scaled dimension groups \((K_0(\mathcal{A}_V), K_0^+(\mathcal{A}_V), u)\) are distinct for different elements \(u \in K_0^+(\mathcal{A}_V)\) and any such group can be obtained in this way [Blackadar 1986] [1, Section 6.2]. Thus lemma 3.7 follows from the corollary 3.6. \(\square\)

Lemma 3.8. There exists a bijective correspondence between the set of all Morita equivalent but pairwise non-isomorphic Serre C\(^\ast\)-algebras \(\mathcal{A}_V\) and a subset of the set of twists of the variety \(V(k)\).

Proof. Recall that there exists an embedding \(A_V \hookrightarrow B\), where \(B\) is an AF-algebra, such that
\[
(K_0(\mathcal{A}_V), K_0^+(\mathcal{A}_V)) \cong (K_0(\mathcal{B}), K_0^+(\mathcal{B})).
\]
It is known that the dimension group \((K_0(\mathcal{B}), K_0^+(\mathcal{B}))\) is an invariant of the Morita equivalence of the AF-algebra \(\mathcal{B}\), while the scaled dimension group \((K_0(\mathcal{B}), K_0^+(\mathcal{B}), u)\) is an invariant of the isomorphism of \(\mathcal{B}\), see e.g. [Blackadar 1986] [1, Theorem 7.3.2]. In view of (3.9), the same is true of the Serre C\(^\ast\)-algebra \(\mathcal{A}_V\). Lemma 3.8 follows from the corollary 3.6 and lemma 3.7. \(\square\)

Corollary 1.2 follows from lemma 3.8 and the definition of a twist.

4. Rational elliptic curves

To illustrate corollary 1.2, we shall consider the case \(V(k) \cong \mathcal{E}(k)\), where \(\mathcal{E}(k)\) is a rational elliptic curve. We briefly review the related definition and facts.

4.1. Elliptic curves. By an elliptic curve we understand the subset of the complex projective plane of the form
\[
\mathcal{E}(k) = \{(x, y, z) \in \mathbb{C}P^2 \mid y^2 z = x^3 + Axz^2 + Bz^3\},
\]
where \(A, B \in k\) are some constants. Recall that the number \(j(\mathcal{E}) = 1728(4A^3)/(4A^3 + 27B^2)\) is an invariant of the \(\mathbb{C}\)-isomorphisms of the elliptic curve \(\mathcal{E}(k)\). The twists \(\mathcal{E}_t(k)\) of \(\mathcal{E}(k)\) are given by the equations
\[
\begin{align*}
  y^2 z &= x^3 + t^2 Axz^2 + t^3 Bz^3, \quad \text{if } j(\mathcal{E}) \neq 0, 1728 \\
  y^2 z &= x^3 + tAxz^2, \quad \text{if } j(\mathcal{E}) = 1728 \\
  y^2 z &= x^3 + tBz^3, \quad \text{if } j(\mathcal{E}) = 0,
\end{align*}
\]
where \(t \in k\), see e.g. [Silverman 1985] [5, Proposition 5.4]. It is easy to verify, that \(j(\mathcal{E}_t(k)) = j(\mathcal{E}(k))\).
4.2. **Noncommutative tori.** A $C^*$-algebra $\mathcal{A}_\theta$ on two generators $u$ and $v$ satisfying the relation $vu = e^{2\pi i \theta} uv$ for a constant $\theta \in \mathbb{R}$ is called the noncommutative torus. The well known Rieffel’s Theorem [3, Theorem 1.1.2] says that the algebra $\mathcal{A}_\theta$ is Morita equivalent to the algebra $\mathcal{A}_{\theta'}$, if and only if,

$$\theta' = \frac{a\theta + b}{c\theta + d} \quad \text{for a matrix } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

In contrast, the algebra $\mathcal{A}_\theta$ is isomorphic to the algebra $\mathcal{A}_{\theta'}$ if and only if $2(\theta - \theta') \in \mathbb{Z}$. Notice that in terms of the continued fraction

$$\theta = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}} := [a_0, a_1, a_2, \ldots]$$

attached to the parameter $\theta$, it means that the $\mathcal{A}_{\theta'}$ is Morita equivalent to the $\mathcal{A}_\theta$, if and only if, the continued fraction of $\theta'$ coincides with such of $\theta$ everywhere but a finite number of terms. In other words, an infinite tail of the corresponding continued fractions must be the same. Clearly, the $\mathcal{A}_{\theta'}$ is isomorphic to the $\mathcal{A}_\theta$, if and only if, the continued fraction of $\theta'$ coincides with such of $\theta$.

**Remark 4.1.** An infinite tail of continued fraction (4.4) is an invariant of the Morita equivalence of the algebra $\mathcal{A}_\theta$. Such a tail is an analog of the $j$-invariant of an elliptic curve.

4.3. **Twists of $E(k)$**. It is known that the Serre $C^*$-algebra of an elliptic curve $E(k)$ is isomorphic to the $\mathcal{A}_\theta$ [3, Theorem 1.3.1]. Moreover, if $k$ is a number field, then $\theta$ is a real quadratic number, i.e. the irrational root of a quadratic polynomial with integer coefficients, see [3, Theorem 1.4.1]. For such a number, the continued fraction (4.4) must be eventually periodic, i.e.

$$\theta = [a_0, \ldots, a_N; \overline{b_1, \ldots, b_n}],$$

where $(b_1, \ldots, b_n)$ is the minimal period of the fraction.

**Corollary 4.2.** The period $(b_1, \ldots, b_n)$ of the continued fraction (4.5) is an invariant of the twists (4.2), while $(a_0, \ldots, a_N)$ in (4.5) depend on the twist parameter $t \in k$ in (4.2).

**Proof.** Up to a cyclic permutation, the period $(b_1, \ldots, b_n)$ is the Morita invariant of the algebra $\mathcal{A}_\theta$, see remark 4.1. The corollary 4.2 follows from the corollary 1.2. □

**References**

1. B. Blackadar, *K-Theory for Operator Algebras*, MSRI Publications 5, Springer, 1986.
2. I. V. Nikolaev, *Langlands reciprocity for $C^*$-algebras*, Operator Theory, Functional Analysis and Applications 282 (2021), 515-528.
3. I. V. Nikolaev, *Noncommutative Geometry*, Second Edition, De Gruyter Studies in Math. 66, Berlin, 2022.
4. J.-P. Serre, *Galois Cohomology*: translated from the French by Patrick Ion, Springer, 1997.
5. J. H. Silverman, *The Arithmetic of Elliptic Curves*, GTM 106, Springer, 1985.
6. J. T. Stafford and M. van den Bergh, *Noncommutative curves and noncommutative surfaces*, Bull. Amer. Math. Soc. 38 (2001), 171-216.

1 Department of Mathematics and Computer Science, St. John’s University, 8000 Utopia Parkway, New York, NY 11439, United States.

Email address: igor.v.nikolaev@gmail.com