Dissipative self-interference and robustness of continuous error-correction to miscalibration

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(Dated: September 21, 2018)

We derive an effective equation of motion within the steady-state subspace of a large family of Markovian open systems (i.e., Lindbladians) due to perturbations of their Hamiltonians and system-bath couplings. Under mild and realistic conditions, competing dissipative processes destructively interfere without the need for fine-tuning and produce no dissipation within the steady-state subspace. In quantum error-correction, these effects imply that continuously error-correcting Lindbladians are robust to calibration errors, including miscalibrations consisting of operators undetectable by the code. A similar interference is present in more general systems if one implements a particular Hamiltonian drive, resulting in a coherent cancellation of dissipation.

Understanding how to reservoir-engineer [1] open quantum systems is important for the success of noisy intermediate-scale quantum (NISQ) [2] technologies. In this context, one often encounters the problem of experimentally controlling time-evolution within a particular subspace of states, e.g., in order to stabilize states [3–7] and phases of matter [8–12], generate gates using Zeno dynamics [13–16], or protect against unwanted errors [17–22]. Resolving this problem involves around variants of either perturbation theory or adiabatic elimination. In the case of interest here, one applies a perturbation $O$ to an unperturbed Lindbladian $\mathcal{L}$ [23–27] such that the resulting leading-order time-evolution within the steady-state subspace of $\mathcal{L}$ is governed by an effective Lindbladian $\mathcal{L}_{\text{eff}}$. In general, $\mathcal{L}_{\text{eff}}$ is difficult to put explicitly in Lindblad form since there is a complex interplay between dissipation and coherent evolution inherent in $\mathcal{L}$ and arising from $O$. Cases in which $\mathcal{L}_{\text{eff}}$ (to 1st [28–30] or 2nd [31–42] order in $O$) can be simplified are highly sought after since they yield physical intuition, are numerically tractable, and provide hydrogen-atom-like starting points for more complex scenarios. Due to the aforementioned complexity, such cases are scarce relative to the many combinations of steady-state structures [43], perturbation types [44, Sec. 6.1], and features of $\mathcal{L}$ [45, Sec. 2.1].

In this Letter, we derive an $\mathcal{L}_{\text{eff}}$ for arbitrary Hamiltonian and jump-operator perturbations to certain $\mathcal{L}$ admitting decoherence-free subspaces (DFS) [46–48], demonstrating surprising and (to an extent) generic interference effects. Being an extension of an effective operator formalism (EOF) [33] applicable to a variety of Rydberg [49–51], photonic [52, 53], and trapped-ion [22] platforms, our formalism and its predicted interference effects should be observable in and useful to many quantum technologies.

Minimal example.—To gain intuition into the interference effects, consider first a simple three-level system $|0\rangle, |1\rangle, |e\rangle$ [see Fig. 1(a)] where the excited level $|e\rangle$ resides at an energy $H = \delta |e\rangle\langle e|$ and decays into $|0\rangle$ under jump operator $F = \sqrt{\Gamma}|0\rangle\langle e|$ (with corresponding dissipator $D[F](\cdot) \equiv F(\cdot)F^\dagger - \frac{1}{2}[F^\dagger F, (\cdot)]$). The states $|0\rangle, |1\rangle$ form a DFS. Now, assume a small additional decay $|1\rangle \rightarrow |0\rangle$ arising from the same coupling to the bath as the $|e\rangle \rightarrow |0\rangle$ decay. Under such decay, $F \rightarrow F + f$ with perturbation $f = \sqrt{\gamma}|0\rangle\langle 1|$ and $\Gamma \gg \gamma$. Naturally, one would think that the leading-order $O(\gamma)$ dissipation due to $f$ will be $\mathcal{D}[f]$. However, our formalism identifies an additional $O(\gamma)$ effective process that interferes with this dissipation via the virtual transition $|1\rangle \rightarrow |e\rangle \rightarrow |0\rangle$. While neither the strong $(F)$ nor the weak $(f)$ dissipation alone cause the $|1\rangle \rightarrow |e\rangle$ part of that transition, perturbing $F'F \rightarrow (F + f)(F + f)$ in $\mathcal{D}[F]$ yields the term $F^\dagger f \propto |e\rangle\langle 1|$ which, when followed by $F$, produces that transition. That transition is also mediated by the inverse of the non-Hermitian “Kamiltonian” $K = (\delta - \frac{\gamma}{2} I)|e\rangle\langle e|$ governing evolution of $|e\rangle$.

Figure 1. (color online) (a) A three-level open system with unperturbed steady states $|0\rangle, |1\rangle$ (forming a DFS), Hamiltonian $H = \delta |e\rangle\langle e|$, and jump $F$ (black arrows). To leading order, a jump perturbation $f$ (red wavy arrow) induces two processes which destructively interfere with each other [see Eq. (1)]: one is simply $f$ itself while the other occurs via a virtual transition through $|e\rangle$ via $F^\dagger f$ (green dotted-dashed arrow). (b) Sketch of the block matrix formed by jumps $F^\ell$ satisfying the condition (4) necessary for a generalized interference effect. Each jump operator occupies its own block. The levels $\mathcal{F}$ represent the DFS while $\mathcal{D}$ are decaying via the unperturbed Lindbladian $\mathcal{L} = [H, F^\ell]$ (5). (c) Energy levels of a system satisfying the assumptions of the EJOF. The perturbations include Hamiltonian $(V; \text{blue dotted arrows})$ and jump perturbations $\{f^1, f^2\}$. 

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Leading-order dissipative evolution within the DFS is a superposition of both processes and is governed by $L_{\text{eff}}$ with effective jump operator

$$F_{\text{eff}} = f + \frac{i}{2} F K^{-1} F^\dagger f = \sqrt{\delta} - \frac{\delta}{2} |0\rangle \langle 1|.$$  \hspace{1cm} (1)

In the limit of large energy $\delta \gg \Gamma$, the virtual $|1\rangle \rightarrow |e\rangle \rightarrow |0\rangle$ transition is off-resonant, the second term in $F_{\text{eff}}$ goes to zero, and one reduces to the intuitive case ($F_{\text{eff}} = f$). However, when $\Gamma \gg \delta$, destructive interference between the two terms makes the effective dissipation disappear entirely ($F_{\text{eff}} = 0$). Although this cancellation can be understood nonperturbatively using dark-state physics [54], here we show that the perturbative interference holds much more generally than previously thought. Generalizing this three-level example, $F_{\text{eff}} = 0$ at zero energy for $f = \sqrt{\delta}|0\rangle \langle \psi|$ with any $|\psi\rangle$. Extending to four or more levels, we will see that $F_{\text{eff}} = 0$ for a much larger family of $(F, f)$.

**Generic cancellation.**—It is uncommon in Hamiltonian perturbation theory for a correction to be zero for any perturbation (unless a symmetry is present). In this example, we observe such a cancellation not due to symmetry, but to inherent destructive interference between generalizations of the two processes discussed above. Consider an $N = 2$-level system $\{0\}, \{1\}, \{e\}, \{|n\rangle\}$ with $\{0\}, \{1\}$ forming a DFS with corresponding projection $P = |0\rangle \langle 0| + |1\rangle \langle 1|$. To simplify notation, we partition operators $O$ into four corners [30]: $O_{\text{ff}} \equiv IP_{\text{ff}}O_{\text{ff}} \equiv IP_{\text{ff}}O_{\text{ff}} \equiv IP_{\text{ff}}O_{\text{ff}} = 0$ acting on the DFS, $O_{\text{ff}} \equiv IP_{\text{ff}}O_{\text{ff}}$ acting on the $N$ decaying states, the “lowering operator” $O_{\text{ff}} \equiv IP_{\text{ff}}O_{\text{ff}}$ mapping decaying states into the DFS, and the “raising operator” $O_{\text{ff}} \equiv IP_{\text{ff}}O_{\text{ff}}$ taking states out of the DFS. Assume no Hamiltonians ($H = 0$, for now) and an unperturbed jump $F = F_{\text{ff}}$, meaning that $F$ maps one directly into the DFS (red) from the decaying space (blue). This jump can have any combination of the $2N$ decay channels from the $N$ excited states into $\{0\}, \{1\}$, with the only restriction that it be surjective,

$$F (F^\dagger F)^{-1} F^\dagger = P_{\text{ff}}.$$ \hspace{1cm} (2)

Interestingly, randomly generated jumps do this: all a measure-zero set of $F = F_{\text{ff}}$ consisting of random entries [55] satisfy (2). For now, perturb $F$ with any small $f$ satisfying $f = f_{\text{ff}}$, i.e., any $f$ not mapping $P_{\text{ff}}$ into $P_{\text{ff}}$. Applying Eqs. (1,2) yields the effective jump

$$F_{\text{eff}} = f_{\text{ff}} - F (F^\dagger F)^{-1} F^\dagger f = f_{\text{ff}} - f_{\text{ff}} = 0$$ \hspace{1cm} (3)

to leading order in any jump perturbation $f_{\text{ff}}$. (We will later prove that $f_{\text{ff}}$ doesn’t participate at all.) Therefore, a random jump $F = F_{\text{ff}}$ perturbed by any small perturbation not mapping out of the DFS generically produces no leading-order dissipation within the DFS.

This cancellation can be extended to multiple unperturbed jumps $F^\ell$, granted that (2) holds for each $F^\ell$ and the additional “orthogonality” condition

$$F^\ell F^{\ell\dagger} = \delta_{\ell\ell'} F^\ell F^{\ell\dagger}$$ \hspace{1cm} (4)

is satisfied. This condition implies that a block matrix consisting of $\{F^\ell\}$ will look like Fig. 1(b). Conditions (2,4) imply that $K^{-1} = \sum_{\ell} (-\frac{1}{2} F^\ell F^{\ell\dagger})^{-1}$ and $F^\ell K^{-1} F^{\ell\dagger} \propto f_{\text{ff}}^\ell F^{\ell\dagger}$, yielding once again no dissipative evolution ($F_{\text{eff}} = 0$) for any $f_{\text{ff}}$. Having described the most interesting effect, we now state our general result—a formalism for tackling perturbations to a large class of Lindbladians.

**General result.**—Let the unperturbed Lindbladian $L$ consist of a Hamiltonian $H$ and jump operators $F^\ell$,

$$L (\cdot) = -i[H, \cdot] + \sum_{\ell} D(F^\ell) (\cdot).$$ \hspace{1cm} (5)

Consider coherent and dissipative perturbations, respectively,

$$H \rightarrow H + V \text{ and } F^\ell \rightarrow F^\ell + f^\ell.$$

Since $L$ governs the evolution of a system coupled to a bath [10, 56, 57], $V$ is a modification of the system Hamiltonian while $f^\ell$ modifies the system-bath coupling. If $L$ is a desired reservoir engineering operation, then $\{V, f^\ell\}$ can be thought of as uncontrollable coherent evolution and miscalibrations in the engineered dissipation, respectively. The resulting superoperator perturbation has terms both 1st and 2nd order in $\{V, f^\ell\}$, $O = O_1 + O_2$, and perturbation theory within the steady-state subspace yields the Lindbladian [58, Supplement]

$$L_{\text{eff}} = P_0 O_1 P_0 - P_0 O_1 L^{-1} O_1 P_0,$$

where $L^{-1}$ is the Drazin pseudoinverse [28, Eq. (D4)] and the asymptotic projection $P_0 = I - L L^{-1}$ (with $I$ identity) projects onto all steady states of $L$ [30, 45]. The above expression is not particularly illuminating as it is not in Lindblad form. However, since $L_{\text{eff}}$ is a Lindbladian, it must be expressible in terms of some effective Hamiltonian $H_{\text{eff}}$, jump operators $F_{\text{eff}}$, and/or a completely positive (CP) map $E_{\text{eff}}$ and its adjoint $E_{\text{eff}}^\dagger$ [24, Prop. 5], all depending on the unperturbed pieces $\{H, F^\ell\}$ and perturbations $\{V, f^\ell\}$. Generally, the expressions may not be simple and the dependence not explicit, but we are able to express $L$ in Lindblad form given the following assumptions. We assume $L$ admits a unique DFS $P_{\text{ff}}$ and that (A) the unperturbed Hamiltonian acts only on the decaying subspace ($H = H_{\text{ff}}$) and (B) unperturbed jump operators map decaying states directly into the DFS ($F^\ell = F^\ell_{\text{ff}}$). We assume these hold from now on, noting there are no restrictions on $(V, f^\ell)$; see Fig. 1(c) for an example. To simplify $L_{\text{eff}}$, we introduce Hamiltonians

$$K = H - \frac{i}{2} \sum_{\ell} F^{\ell\dagger} F^\ell$$ \hspace{1cm} (8a)

$$K_{\text{eff}} = V - \frac{i}{2} \sum_{\ell} (F^{\ell\dagger} f^\ell + f^\ell F^{\ell\dagger}).$$ \hspace{1cm} (8b)

As we have seen, $K = K_{\text{ff}}$ and its corresponding superoperator

$$\mathcal{K (\cdot)} \equiv -i \left[ K (\cdot) - (\cdot) K^\dagger \right]$$
govern evolution within \[ \mathcal{H} \] [45, Sec. 2.1.3]. As we will see shortly, pieces of the effective Hamiltonian \( K_{\text{eff}} = (K_{\text{eff}}) \) map one out of and into the DFS. We picked \( K_{\text{eff}} \) to depend only on \( \{V, f\} \) because \( \{V, f\} \) participate differently and \( \{V, f\} \) do not feature to this order. The resulting simplified \( L_{\text{eff}} \) is as follows [55].

**Proposition (EJOF).** Let \( \mathcal{L} \) be a Lindbladian with a unique DFS \( \mathcal{H} \) Hamiltonian \( \mathcal{H}_0 \), and jump operators \( \{f^\ell\} \) (5). Perturb \( \mathcal{L} \) with a Hamiltonian \( V \) and jump perturbations \( \{f^\ell\} \) (6).

The effective Lindbladian (7) within the DFS is

\[
L_{\text{eff}}(\cdot) = -[iH_{\text{eff}},\cdot] + \sum_\ell D[f^\ell_{\text{eff}}(\cdot)] + E_{\text{eff}}(\cdot) = \frac{1}{2}[E_{\text{eff}}^2(\cdot),(\cdot)],
\]

where the effective Hamiltonian, jumps, and CP map are

\[
H_{\text{eff}} = \frac{1}{2}(\mathcal{H} - K_{\text{eff}}^{-1}K_{\text{eff}}) + H.c.,
\]

\[
f^\ell_{\text{eff}} = f^\ell - F^\ell_{\text{eff}}K_{\text{eff}}^{-1}K_{\text{eff}}
\]

\[
f_{\text{eff}}(\cdot) = -\sum_{\ell,\ell'} F^\ell K^{-1}(f^\ell_{\text{eff}}(\cdot)f^\ell_{\text{eff}}^{\dagger}) F^{\ell'}.
\]

This effective jump-operator formalism (EJOF) reduces to the EOF [33] (see also [36, Lemma 3]) when \( f^\ell = 0 \) (and \( V = 0 \)). Therefore, the EOF, derived via adiabatic elimination, can alternatively be derived using time-independent perturbation theory [55].

The first term (10a) represents the resulting coherent evolution within the DFS. It consists of \( V_{\mathcal{H}} \) a 1st-order effect, and the effective Hamiltonian \( K_{\text{eff}}K_{\text{eff}}^{-1}K_{\text{eff}} + H.c. \) reminiscent of Hamiltonian perturbation theory. In the latter, \( K_{\text{eff}} \) maps states in the kernel \( \{\mathcal{H}\} \) of \( K \) into the range \( \{\mathcal{H}\} \) using both coherent \( \{V_{\mathcal{H}}\} \) and dissipative \( \{F^\ell_{\text{eff}}\} \) terms, returning via \( V_{\mathcal{H}} \) and \( F^\ell_{\text{eff}} \), respectively, with the “energy” denominator determined by \( K^{-1} \). Thus there are cross-terms consisting of leaving via dissipation and returning via a Hamiltonian and visa-versa.

Interestingly, the participating dissipative perturbation \( F_{\text{eff}} \) cannot map one out of the DFS, instead conspire with \( F^\ell \) to provide the dissipative analogue of \( V_{\mathcal{H}} \). A similar story occurs in the effective jump \( F^\ell_{\text{eff}} \) (10b) and is the key reason behind the highlighted cancellation. The first part of \( F^\ell_{\text{eff}} \) comes from the first piece \( \mathcal{D}[O_2] = \sum_\ell D[f^\ell_{\text{eff}}] \) in Eq. (7), which is itself a Lindbladian. However, the second piece \( \mathcal{D}[O_1]K^{-1}O_2 \), which surprisingly is not a Lindbladian, contributes the interference term \( F^\ell K^{-1}K_{\text{eff}} \). This term consists of leaving the DFS through \( K_{\text{eff}} \) and returning to the DFS via \( F^\ell \) while paying an “energy” penalty determined by the eigenvalues of \( K \). The third term (10c) \( [E^2_{\text{eff}}(\cdot) = \sum_\ell f^\ell_{\text{eff}} f^\ell_{\text{eff}}^{\dagger}] \) results from a nonzero \( f_{\text{eff}} \), mapping one out of the DFS and recovering via \( F^\ell \) with “energy” denominator determined by the superoperator \( K^{-1}(\cdot) \neq K^{-1}(\cdot)K^{-1} \).

This term has no analogue in Hamiltonian 2nd-order perturbation theory because it directly connects \( \mathcal{H} \) to \( \mathcal{H} \) via one instance of \( f_{\text{eff}} \). If \( K \) is diagonalizable, we can easily express \( K^{-1} \) using the eigendecomposition of \( K \). However, this formalism remains valid even for non-diagonalizable \( K \) [55].

**Coherent cancellation.**—In our previous examples, we assumed \( H = V = 0 \) since any initial coherent evolution spoils the interference effect. We now expand those examples to nonzero Hamiltonians \( H \neq 0 \neq V \), showing how to restore the interference spoiled by \( H \) with a judicious choice of \( V \). We maintain conditions (2.4) and let \( f^\ell = 0 \), so only \( \{H_{\text{eff}}, f^\ell_{\text{eff}}\} \) contribute to \( L_{\text{eff}} \). The presence of \( H \) in (8a) means that \( F^\ell K^{-1}F^{\ell'} \) is no longer the DFS identity and \( f^\ell_{\text{eff}} \neq 0 \). However, since the return to the DFS in \( f^\ell_{\text{eff}} \) occurs via dissipation only, \( V_{\mathcal{H}} \) does not contribute to \( E_{\text{eff}} \). Exploiting this effect, we pick

\[
V = \frac{i}{2} \sum_\ell (F^\ell f^\ell - F^{\ell'} f^{\ell'}) + \tilde{V}
\]

to cancel the \( F^\ell f^\ell \) term in \( K_{\text{eff}} \), leaving us with \( (K_{\text{eff}})_{\mathcal{H}} = \tilde{V}_{\mathcal{H}} \) that is dependent only on the coherent perturbation. Picking \( \tilde{V}_{\mathcal{H}} = K \sum_\ell (F^\ell f^\ell)^{-1}F^{\ell'} f^{\ell'} + H.c. \), the \( K \) out front cancels the \( K^{-1} \) in \( f^\ell_{\text{eff}} \) and removes \( f^\ell_{\text{eff}} \) via the same effect as that in Eq. (3). In other words, if \( H \neq 0 \), one can use a particular coherent perturbation to cancel leading-order effects due to unwanted jump perturbations \( f^\ell_{\text{eff}} \) [55].

**Universal dissipation.**—In a quick detour from canceling unwanted dissipation, let us instead use a customizable \( V \) to see what possible dissipation within \( \mathcal{H} \) we can generate (c.f. [35]). We assume to have full control over the perturbations, showing that restricting them to \( \{f_{\mathcal{H}}, f_{\mathcal{H}}\} \) allows universal dissipation within the DFS.

First, by letting \( \tilde{V} = 0 \) in Eq. (11), we cancel \( K_{\text{eff}} \)-dependent terms in both \( \{H_{\text{eff}}, f^\ell_{\text{eff}}\} \) (10a-b) and obtain \( L_{\text{eff}} = \{V_{\mathcal{H}}, f^\ell_{\text{eff}}\} \). Second, letting \( d \) be the dimension of the DFS, a general \( L_{\text{eff}} \) has \( d^2 - 1 \) jump operators \( \{f^\ell\} \). Therefore, if \( L \) has at least \( d^2 - 1 \) independent jump operators \( F^\ell \), \( L_{\text{eff}} \) generates any dissipation within \( \mathcal{H} \).

**Continuous error-correction.**—In conventional QEC, one starts out in a logical state located in the codespace \( \mathcal{H} \) and attempts to correct errors caused by an error channel \( \mathcal{E} \) by acting with a recovery channel \( \mathcal{R} \). Ideally, \( \mathcal{E} \) consists of correctable noise, so \( \mathcal{R} = \mathcal{I} \) [59, 60]; we focus on a similar case here. We consider a continuous QEC, where one has the ability to correct noise via an infinitesimal version of the recovery \( \mathcal{L} = \mathcal{R} - \mathcal{I} \) [61], whose jumps are the Kraus operators \( \{R^\ell\} \) of \( \mathcal{R} \) (and we additionally removed the DFS-identity Kraus operator \( R^0 \cong I_{\mathcal{H}} \)). Instead of perturbing \( \mathcal{L} \) with another Lindbladian representing external noise, we consider perturbations to the jump operators of \( \mathcal{L} \), which represent miscalibration of the recovery itself. Such noise is important since a recovery map is never perfect in real life. It consists of detectable errors \( f^\ell \) (which we assume are correctable by \( \mathcal{L} \)), undetectable errors \( f^\ell \) (which are not correctable since they act nontrivially within the codespace [60]), recovery errors \( f^\ell_{\text{eff}} \), and correctable errors \( f^\ell_{\text{eff}} \) in \( \mathcal{H} \). The EJOF shows that small imperfections of all types do not harm the quantum information.

Since \( \mathcal{L} = \mathcal{R} - \mathcal{I} \) comes from a recovery operation, each jump \( F^\ell \) is an isometry from a subspace of \( \mathcal{H} \) (corresponding
to a distinct error syndrome) into the codespace. Such $F^f$ automatically satisfy conditions (2,4) and, since $\mathcal{R}$ is a channel from $\mathcal{H}$ to $\mathcal{E}$, $\sum \mathcal{F}^f \mathcal{F}^f = 0$ [45, Sec. 2.1.4]. Let us further assume that miscalibrations mapping out of the codespace form a channel, $\mathcal{E}(\cdot) = \sum_i f^f_i (\cdot) f^f_i$, consisting of correctable noise, i.e., $\mathcal{R}\mathcal{E}(\rho) \propto \rho$ for all $\rho \in \mathcal{H}$. Application of the EJOE results in the following.

**Corollary.** Let $\mathcal{L} = \mathcal{R} - I$ with corresponding recovery channel $\mathcal{R}(\cdot) = \sum_i f^f_i (\cdot) f^f_i$ such that $\{f^f_0 = F^f_0\}$ satisfy conditions (2,4). Assume small miscalibrations $\{f^f\}$ in the recovery, $F^f \rightarrow F^f + f^f$, such that the pieces $\{f^f\}$ form a noise channel correctable by $\mathcal{R}$. To leading order, the miscalibrations $\{f^f\}$ do not induce errors within the codespace,

$$\mathcal{L}_{\text{eff}} = 0.$$  

To prove the above, we have to show that each line in Eq. (10) is zero. First, the simple structure of $\mathcal{L}$ lets us simplify all Hamiltonian inverses: $K = -\frac{1}{2} f^f_0$ and $\mathcal{R}[\rho] = -\mathcal{F}^f_0$ for any $\rho$. Plugging this into $H_{\text{eff}} = 0$. Similarly, and most surprisingly, the interference effect discussed above cancels the undetectable miscalibrations, yielding $f^f_{\text{eff}} = 0$ (10b). Lastly, simplifications to $K$ and the condition on $\{f^f_0\}$ yield the trivial CP map (10c), $\mathcal{E}_{\text{eff}}(\rho) = -\mathcal{R}K^{-1}\mathcal{E}(\rho) = \mathcal{R}\mathcal{E}(\rho) \propto \rho$ for all $\rho \in \mathcal{H}$.

The above robustness corollary shows that even undetectable miscalibrations $\{f^f\}$ in continuous error-recovery operations do not affect the codespace. Qualitatively, it is a statement that holds for any recovery $\mathcal{R}$ that maps into the codespace after one action and is applied rapidly (allowing us to consider $\mathcal{L} = \mathcal{R} - I$). For example, the statement holds for continuous recoveries for the three-qubit repetition [62] and binomial [63] codes. For the former, $I_{\mathcal{H}} = |000\rangle\langle 000| + |111\rangle\langle 111|$, and its jumps $F^f = I_{\mathcal{H}} F^f$ (for qubits $f \in \{1, 2, 3\}$ and $X, Y, Z$) the usual Pauli matrices) satisfy conditions (2,4). Terms $f^f \propto X^f$ are corrected by the continuous recovery while $f^f \propto Z^f$ are canceled out due to interference, despite being undetectable by the code. The terms $f^f \propto Y^f$ cannot be corrected since $X^f$ and $Y^f$ are not simultaneously correctable; picking a code correcting both solves this problem.

We cannot make the same statement about all recovery operations since the assumptions of the EJOE no longer hold. The assumption $F^f = F^f_0$ amounts to the jumps recovering all states (in their range) back into the codespace after one action. Another set of cases is where $F^f \neq F^f_0$ does not map all states immediately into the codespace, but instead keeps certain states uncorrected (i.e., in $\mathcal{H}$) after one action by, e.g., only correcting errors occurring in a localized region [45, Sec. 3.4]. Such systems include local recoveries for topological codes (45, Sec. 3.4) after one action by, e.g., only recovering all.

**Conclusion.**—We develop an effective jump-operator formalism to handle general perturbations to a particular class of Lindbladians relevant in quantum optics and error correction. We explicitly solve for the effective Lindbladian $\mathcal{L}_{\text{eff}}$ governing perturbation-induced evolution within the steady-state subspace of an unperturbed Lindbladian $\mathcal{L}$. Using this formalism, we uncover an interference effect that is a generalized version of the interference observed in dark-state physics. This interference occurs in generic Lindbladians of the type we study and can be applied to show that Lindbladian-based error-correction operations are robust to both detectable and undetectable calibration noise. While this interference is destroyed when the unperturbed system has a Hamiltonian piece, it can be reinstated with a certain Hamiltonian perturbation. This formalism also provides a simple way to realize universal Lindbladian simulation.

We thank Mikhail D. Lukin, Jacob P. Covey, Richard Kueng, John Preskill, Liang Jiang, Paola Cappellaro, and Mådălin Guță for illuminating discussions. We acknowledge financial support from the Walter Burke Institute for Theoretical Physics at Caltech (V.V.A.), the Korea Foundation for Advanced Studies (K.N.), and a Feodor-Lynen fellowship from the Alexander von Humboldt-Foundation (F.R.).
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[54] The jump $F + f$ annihilates the dark state $|\psi\rangle = \sqrt{\Gamma}|1\rangle - \sqrt{\gamma}|e\rangle$, which forms a DFS along with $|0\rangle$. For $\Gamma \gg \gamma$, $|\psi\rangle \propto |1\rangle$.

[55] See Supplemental Material for a proof of the EJOF and an ancillary Mathematica file for randomly generated examples.

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APPENDIX: PROOF OF THE EJOF

**Proposition.** Let \( \mathcal{L} \) be a Lindbladian with a unique DFS \( \mathbb{P} \). Hamiltonian \( H_{\mathbb{P}} \) and jump operators \( \{F^\ell, F^\ell\} \). Perturb \( \mathcal{L} \) with Hamiltonian \( V \) and jump perturbations \( \{f^\ell\} \). The effective Lindbladian \( (S5) \) within the DFS is

\[
\mathcal{L}_{\text{eff}} (\cdot) = -i[H_{\text{eff}} (\cdot)] + \sum_\ell D[F^\ell_{\text{eff}} (\cdot)] + E_{\text{eff}} (\cdot) - \frac{1}{2} \left\{ E^\ell_{\text{eff}} (I) , (\cdot) \right\}, \tag{S1}
\]

where the effective Hamiltonian, jumps, and CP map are

\[
H_{\text{eff}} = \frac{1}{2} \left( V_{\mathbb{P}} - K_{\text{eff}} K^{-1} K_{\text{eff}} \right) + H.c. \tag{S2a}
\]

\[
F^\ell_{\text{eff}} = f^\ell - F^\ell K^{-1} K_{\text{eff}} \tag{S2b}
\]

\[
E_{\text{eff}} (\cdot) = - \sum_\ell F^\ell K^{-1} \left( f^\ell (\cdot) F^\ell \right) F^{\ell - 1}. \tag{S2c}
\]

A similar proof of the EOF [33] using open-system perturbation theory was performed in Ref. [45], Sec. 4.3.5. The adjoint of a superoperator \( \mathcal{E} (\cdot) = \sum_\ell A_\ell (\cdot) B_\ell^\dagger \) is \( \mathcal{E}^\dagger (\cdot) \equiv \sum_\ell A_\ell^\dagger (\cdot) B_\ell \). The perturbation \( \mathcal{O} \) to \( \mathcal{L} \) consists of contributions from \( V \) and \( f^\ell \) [44, Sec. 6.1]. Let us conveniently split \( \mathcal{O} \) into various superoperators responsible for different processes. First, define the generalized commutator \( [A, B]^T \equiv AB - BA^T \) and Hamiltonians \( K = H - \frac{i}{2} \sum_\ell F^\ell F^\ell \) and \( K_{\text{eff}} \equiv V_{\mathbb{P}} - \frac{i}{2} \sum_\ell \left( f^\ell F + F^\ell f^\ell \right) \). Then, construct the superoperators

\[
\mathcal{V} (\cdot) = -i \left[ V_{\mathbb{P}} - \frac{i}{2} \sum_\ell (f^\ell F + F^\ell f^\ell) , (\cdot) \right]^* \tag{S3a}
\]

\[
\mathcal{K}_{\text{eff}} (\cdot) = -i [K_{\text{eff}}, (\cdot)]^* \tag{S3b}
\]

\[
\mathcal{F} (\cdot) = \sum_\ell \left( F^\ell (\cdot) f^\ell + f^\ell (\cdot) F^\ell \right). \tag{S3c}
\]

Split \( \mathcal{O} = O_1 + O_2 \) with \( O_1 \) containing one instance of either \( f^\ell \) or \( V \) in each term and \( O_2 \) containing two:

\[
O_1 = \mathcal{V} + \mathcal{K}_{\text{eff}} + \mathcal{F} \tag{S4a}
\]

\[
O_2 = \sum_\ell D[f^\ell]. \tag{S4b}
\]

Second-order perturbation theory within the DFS yields the effective Lindbladian [58, Supplement]

\[
\mathcal{L}_{\text{eff}} = \mathcal{P}_2 \mathcal{O} \mathcal{P}_2 - \mathcal{P}_2 O_1 K_{\text{eff}}^{-1} O_2 \mathcal{P}_2 \equiv \mathcal{T}_1 + \mathcal{T}_2. \tag{S5}
\]

We have simplified \( \mathcal{L}^{-1} \) to \( K^{-1} \) in the second term \( \mathcal{T}_2 \) due to the assumption that there is no additional dissipation within \( \mathbb{P} \). \( F^\ell_{\mathbb{P}} = 0 \) [45, Sec. 2.1.3]. As opposed to Hamiltonian perturbation theory, here the asymptotic projection \( \mathcal{P}_2 [30, 45] \) corresponds to a quantum channel arising from the infinite-time limit of evolution due to \( \mathcal{L} \). \( \mathcal{P}_2 = \lim_{t \to \infty} e^{t \mathcal{L}} \). This channel is trace-preserving, so it is not merely acting on the DFS since it has to map states initially in \( \mathbb{P} \) into the DFS. We use an analytical formula for it [30, Prop. 3], which for this particular DFS case is

\[
\mathcal{P}_2 (\cdot) = \mathcal{P}_{\mathbb{P}} (\cdot) - \mathcal{P}_{\mathbb{P}} \mathcal{L}_{\mathbb{P}}^{-1} (\cdot) = \mathcal{P}_{\mathbb{P}} (\cdot) - \sum_\ell F^\ell K^{-1} (\cdot) F^{\ell - 1}. \tag{S6}
\]

Above, the four-corners projection superoperators are \( \mathcal{P}_{\mathbb{P}} (\cdot) = I_{\mathbb{P}} (\cdot) I_{\mathbb{P}} \) and \( \mathcal{P}_{\mathbb{P}} (\cdot) = I_{\mathbb{P}} (\cdot) I_{\mathbb{P}} \), and \( \mathcal{A}_{\mathbb{P}} \equiv \mathcal{P}_{\mathbb{P}} \mathcal{A} \mathcal{P}_{\mathbb{P}} \) given any square combination \( \mathbb{P} \). Above, we have substituted \( \mathcal{L}^{-1} \) for \( K^{-1} \) and used \( \mathcal{P}_{\mathbb{P}} \mathcal{L}_{\mathbb{P}} (\cdot) = \sum_\ell F^\ell (\cdot) F^{\ell - 1} [45, \text{Eq. (2.8)}] \). We use this block notation to derive the EJOF, introducing the remaining four-corners projectors \( \mathcal{P}_{\mathbb{P}} (\cdot) = I_{\mathbb{P}} (\cdot) I_{\mathbb{P}} \) and \( \mathcal{P}_{\mathbb{P}} (\cdot) = I_{\mathbb{P}} (\cdot) I_{\mathbb{P}} \), noting that they are orthogonal and can add (e.g., \( \mathbb{P} \equiv \mathbb{P} + \mathbb{P} \)). Most importantly, note that

\[
\mathcal{P}_2 = \mathcal{P}_{\mathbb{P}} \mathcal{P}_{\mathbb{P}} = \mathcal{P}_{\mathbb{P}} \mathcal{P}_{\mathbb{P}} = \mathcal{P}_{\mathbb{P}} \mathcal{P}_{\mathbb{P}}, \tag{S7}
\]

so \( \mathcal{P}_2 \) maps all states into \( \mathbb{P} \) and destroys knowledge of all coherences \( \mathbb{P} \) between the DFS and the decaying states.
1. The term $T_1$

Inserting $1 = \mathcal{P}_\mathbb{B} + \mathcal{P}_\mathbb{B}$ and using Eq. (S7), we have

$$T_1 = (\mathcal{P}_\mathbb{B} \mathcal{P}_\mathbb{B}) \mathcal{O} (\mathcal{P}_\mathbb{B} \mathcal{P}_\mathbb{B}) = \mathcal{P}_\mathbb{B} \mathcal{P} \mathcal{P}_\mathbb{B} \mathcal{P}_\mathbb{B} = \mathcal{P}_\mathbb{B} \mathcal{P}_\mathbb{B} \mathcal{P} \mathcal{P}_\mathbb{B} \mathcal{P}_\mathbb{B} = \mathcal{O} \mathcal{P}_\mathbb{B} \mathcal{P} \mathcal{P}_\mathbb{B} \mathcal{P}_\mathbb{B} \mathcal{P}_\mathbb{B},$$

so we only need two superoperator elements, $\mathcal{O} \mathcal{P}_\mathbb{B} \mathcal{P} \mathcal{P}_\mathbb{B}$ and $\mathcal{P}_\mathbb{B} \mathcal{P} \mathcal{P}_\mathbb{B} \mathcal{P}_\mathbb{B}$, for this term. Note that we have applied $\mathcal{P}_\mathbb{B} \mathcal{P}_\mathbb{B} = \mathcal{P}_\mathbb{B}$ and replaced the rightmost $\mathcal{P}_\mathbb{B}$ with $\mathcal{P}_\mathbb{B}$ since the states we are perturbing are in $\mathbb{B}$. The former element is a projection of $\mathcal{O}$ onto the DFS while the latter is a leakage term into the decaying space. These elements are listed below for all of the terms $\mathcal{A} \in \{\mathcal{V}, \mathcal{K}_{\text{eff}}, \mathcal{F}, \mathcal{D}[f^\ell]\}$ of $\mathcal{O}$.

| $\mathcal{A}$ | $\mathcal{P}\mathcal{A}\mathcal{P}_\mathbb{B}$ | $\mathcal{P}_\mathbb{B}\mathcal{A}\mathcal{P}_\mathbb{B}$ | $\mathcal{P}_\mathbb{B}\mathcal{A}\mathcal{P}_\mathbb{B}$ |
|--------------|-----------------|-----------------|-----------------|
| $\mathcal{V}$ | $-i[\mathcal{V}_\mathbb{B}(\cdot)]$ | $0$ | $0$ |
| $\mathcal{K}_{\text{eff}}$ | $0$ | $0$ | $0$ |
| $\mathcal{F}$ | $0$ | $0$ | $0$ |
| $\mathcal{D}[f^\ell]$ | $\mathcal{D}[f^\ell](\cdot) - \frac{1}{2}[f^\ell, f^\ell](\cdot)$ | $f^\ell(\cdot) f^\ell$ | $f^\ell(\cdot) f^\ell$ |

Luckily, $\mathcal{P}_\mathbb{B}\mathcal{A}\mathcal{P}_\mathbb{B} = 0$ for $\mathcal{A} \in \{\mathcal{K}, \mathcal{V}\}$ due to the fact that their constituents act from one side at a time (the no-leak property; see [30, Sec. I.B]). Also, $\mathcal{P}_\mathbb{B} \mathcal{P}_\mathbb{B} = 0$ since its constituent $F^\ell = F^\ell$ cannot map one into $\mathbb{B}$ by construction. From the above table, we see that $\mathcal{V}$ contributes the first term in $H_{\text{eff}}$ (S2a) and $\mathcal{D}[f^\ell]$ contributes the dissipator $\mathcal{D}[f^\ell]$. We cannot yet combine all $f^\ell$ terms because we still need to act on $\mathcal{P}_\mathbb{B} \mathcal{D}[f^\ell] \mathcal{P}_\mathbb{B}(\cdot)$ with $\mathcal{P}_\mathbb{B}$ (S6):

$$\sum_\ell \mathcal{P}_\mathbb{B} \mathcal{D}[f^\ell] \mathcal{P}_\mathbb{B}(\cdot) = -\sum_\ell F^\ell K^{-1}_\text{eff} \left( f^\ell(\cdot) f^\ell(\cdot) \right) F^\ell \equiv \mathcal{E}_{\text{eff}}(\cdot).$$

This provides the first term for the $\mathcal{E}_{\text{eff}}$-dependent part of $L_{\text{eff}}$ (S1). To complete the derivation of $T_1$, we need to prove that $E_{\text{eff}}^\ell(I) = \sum_\ell f^\ell$. The anticommutator term should be $E_{\text{eff}}^\ell(\mathcal{K})$ since $\mathcal{E}_{\text{eff}}$ is a channel from $\mathbb{B}$ to itself, but padding with $\mathbb{I}_\mathbb{B}$ doesn’t make any difference and looks simpler. Note that $H = H_{\mathbb{B}}$ commutes with $\mathbb{I}_\mathbb{B}$ and so $\mathcal{K}^\ell(\mathbb{I}_\mathbb{B}) = -\sum_\ell F^\ell F^\ell$. Plugging this into $E_{\text{eff}}^\ell(I)$ cancels the $\mathcal{K}^{-1}$, yielding

$$E_{\text{eff}}^\ell(I) = \sum_\ell f^\ell \left( \mathcal{K}^{-1} \right)^{(\cdot)} \left( -\sum_\ell F^\ell F^\ell \right) f^\ell = \sum_\ell f^\ell \left( \mathcal{K}^{-1} \mathcal{K}^\ell \mathbb{I}_\mathbb{B} \right) f^\ell = \sum_\ell f^\ell f^\ell. $$

This provides the anticommutator term for the $\mathcal{E}_{\text{eff}}$-dependent part of $L_{\text{eff}}$ (S1). We are left with the $\mathcal{K}_{\text{eff}}$-dependent terms in $H_{\text{eff}}$ (S2a) and $F_{\text{eff}}$ (S2b), which come from $T_2$.

2. The term $T_2$

This term is more difficult since two actions of the perturbation are present. We likewise need to determine which superoperator elements are required for the calculation. Since $\mathcal{K}^{-1}$ does not act on $\mathbb{B}$ ($\mathcal{K}^{-1} = \mathcal{K}^{-1}$), the first part of $T_2$ is $\mathcal{K}^{-1} \mathcal{P}_\mathbb{B} = \mathcal{K}^{-1} \mathcal{P}_\mathbb{B} \mathcal{P}_\mathbb{B}$. However, we can see that $\mathcal{P}_\mathbb{B} \mathcal{O}_1 \mathcal{P}_\mathbb{B} = 0$ from the previous table, so only $\mathcal{K}^{-1}$ participates. Inserting this into $T_2$ and using Eq. (S7) yields

$$T_2 = -\mathcal{P}_\mathbb{B} \mathcal{O}_1 \mathcal{P}_\mathbb{B} + \mathcal{P}_\mathbb{B} \mathcal{P}_\mathbb{B} \mathcal{O}_1 \mathcal{P}_\mathbb{B} \mathcal{P}_\mathbb{B} \mathcal{K}^{-1} \mathcal{P}_\mathbb{B} \mathcal{O}_1 \mathcal{P}_\mathbb{B}. $$

Therefore, three elements are relevant; they are listed in the table below for all of the terms $\mathcal{A} \in \{\mathcal{V}, \mathcal{K}_{\text{eff}}, \mathcal{F}\}$ of $\mathcal{O}_1$:

| $\mathcal{A}$ | $\mathcal{P}_\mathbb{B}\mathcal{A}\mathcal{P}_\mathbb{B}$ | $\mathcal{P}_\mathbb{B}\mathcal{A}\mathcal{P}_\mathbb{B}$ | $\mathcal{P}_\mathbb{B}\mathcal{A}\mathcal{P}_\mathbb{B}$ |
|--------------|-----------------|-----------------|-----------------|
| $\mathcal{V}$ | $0$ | $0$ | $0$ |
| $\mathcal{K}_{\text{eff}}$ | $-i[(\mathcal{K}_{\text{eff}})\mathbb{B}(\cdot)]^*$ | $-i[(\mathcal{K}_{\text{eff}})\mathbb{B}(\cdot)]^*$ | $-i[(\mathcal{K}_{\text{eff}})\mathbb{B}(\cdot)]^*$ |
| $\mathcal{F}$ | $0$ | $0$ | $\sum_\ell (F^\ell(\cdot) f^\ell + f^\ell(\cdot) F^\ell)$ |

The first part $\mathcal{K}^{-1} \mathcal{P}_\mathbb{B} \mathcal{O}_1 \mathcal{P}_\mathbb{B} (S11)$ is shared by all terms, so we simplify it first by noting that the superoperator inverse $\mathcal{K}^{-1}$ can be written in terms of operator inverses due to the restriction $F^\ell = F^\ell$ [45, Eq. (2.8)],

$$K^{-1}(\cdot) = K^{-1}(\cdot) + K^{-1}(\cdot) = -i(\cdot) K^{-1} + iK^{-1}(\cdot) = i[K^{-1}(\cdot)]^*.$$
Plugging this and the first column of the above table into the first part of $T_2$ yields

$$K^{-1} (\mathcal{P}_\mathscr{a} \mathcal{Q}_\mathscr{b} \mathcal{P}_\mathscr{c}) (\cdot) = [K^{-1}, (K_{\text{eff}} \mathcal{P}_\mathscr{a}) (\cdot)]^* = K^{-1} K_{\text{eff}} (\cdot) + H.c., \quad (S13)$$

where we remember that the state $(\cdot) \in \mathfrak{H}$ and only $(K_{\text{eff}} \mathcal{P}_\mathscr{b})$ can map $(\cdot)$ into $\mathfrak{H}$ (so that $K^{-1}$ acts on the result). In the last equality, we let $(K_{\text{eff}} \mathcal{P}_\mathscr{b}) \to K_{\text{eff}}$ since adding $(K_{\text{eff}} \mathcal{P}_\mathscr{b})$ does not make any difference, i.e., $K^{-1}(K_{\text{eff}} \mathcal{P}_\mathscr{b}) = 0$. Now let us plug this simplified first part as well as all of the nonzero terms from the table into $T_2$:

$$T_2 = -\mathcal{P}_\mathscr{a} K_{\text{eff}} \mathcal{P}_\mathscr{b} + \mathcal{P}_\mathscr{a} \mathcal{F} \mathcal{P}_\mathscr{b} + \mathcal{P}_\mathscr{a} \mathcal{P}_\mathscr{b} K_{\text{eff}} \mathcal{P}_\mathscr{b} \left( K^{-1} K_{\text{eff}} (\cdot) + H.c. \right). \quad (S14)$$

We now determine the contribution coming from each of the three terms in the leftmost parentheses. Using the above table and substituting $(K_{\text{eff}} \mathcal{P}_\mathscr{b}) \to K_{\text{eff}}$ in the first line below, the first two terms are simple:

$$-\mathcal{P}_\mathscr{a} K_{\text{eff}} \mathcal{P}_\mathscr{b} \left( K^{-1} K_{\text{eff}} (\cdot) + H.c. \right) = i \left[ K_{\text{eff}} K^{-1} K_{\text{eff}} (\cdot) \right]^* \quad (S15a)$$

$$-\mathcal{P}_\mathscr{a} \mathcal{F} \mathcal{P}_\mathscr{b} \left( K^{-1} K_{\text{eff}} (\cdot) + H.c. \right) = - \sum_{\ell} (F^\ell K^{-1} K_{\text{eff}} (\cdot) f_{\ell}^\dagger + H.c.). \quad (S15b)$$

For the third term, note first this curious formula that we will use to eliminate the inverse coming from $\mathcal{P}_\mathscr{a} \mathcal{F} \mathcal{P}_\mathscr{b}$:

$$K^{-1} \left[i K^{-1} (\cdot) \right]^* = K^{-1} \mathcal{K} \left(K^{-1} (\cdot) K^{-1}\right) = K^{-1} (\cdot) K^{-1}\dagger \quad (S16)$$

for any operator $(\cdot) \in \mathfrak{H}$. Plugging in Eq. (S6) and applying the above formula yields

$$-\mathcal{P}_\mathscr{a} \mathcal{P}_\mathscr{b} K_{\text{eff}} \mathcal{P}_\mathscr{b} \left( K^{-1} K_{\text{eff}} (\cdot) + H.c. \right) = \sum_{\ell} F^\ell K^{-1} K_{\text{eff}} (\cdot) K_{\text{eff}}^\dagger K^{-1\dagger} F^\ell. \quad (S17)$$

3. Combining $T_1$ and $T_2$

Plugging the $\mathcal{E}_{\text{eff}}$-dependent terms (S9,S10), all $V_{\mathcal{P}_\mathscr{a}}$ and $f_{\ell}^\dagger$-dependent terms in the first table above, and Eqs. (S15,S17) yields the effective Lindbladian

$$\mathcal{L}_{\text{eff}} (\cdot) = -i[H_{\text{eff}} (\cdot), \cdot] + \mathcal{E}_{\text{eff}} (\cdot) + \left\{ \mathcal{E}_{\text{eff}} (f_{\ell}^\dagger) (\cdot), \cdot \right\} + i K_{\text{eff}} K^{-1} K_{\text{eff}} - H.c., (\cdot) \right\} \right]$$

$$+ \sum_{\ell} \mathcal{D}[f_{\ell}^\dagger (\cdot) + F^\ell K^{-1} K_{\text{eff}} (\cdot) K_{\text{eff}}^\dagger K^{-1\dagger} F^\ell - (F^\ell K^{-1} K_{\text{eff}} (\cdot) f_{\ell}^\dagger + H.c.)], \quad (S18)$$

where we have absorbed the Hermitian part of $K_{\text{eff}} K^{-1} K_{\text{eff}}$ into $H_{\text{eff}}$. Remarkably, the last term in the second line and the third line simplify to $\sum_{\ell} \mathcal{D}[F^\ell f_{\ell}^\dagger]$. Collecting all of the terms in the second line that act nontrivially from both sides into $F_{\text{eff}}$ makes this more clear, leaving only the anticommutator term $\sum_{\ell} F^\ell f_{\ell}^\dagger$ to be determined from the last term below:

$$\mathcal{L}_{\text{eff}} (\cdot) = -i[H_{\text{eff}} (\cdot), \cdot] + \mathcal{E}_{\text{eff}} (\cdot) - \left\{ \mathcal{E}_{\text{eff}} (f_{\ell}^\dagger) (\cdot), \cdot \right\} + \sum_{\ell} F^\ell f_{\ell}^\dagger \right[ F_{\text{eff}} - \frac{1}{2} \left\{ \sum_{\ell} f_{\ell}^\dagger f_{\ell} - i \left( K_{\text{eff}} K^{-1} K_{\text{eff}} - H.c., (\cdot) \right) \right\}. \quad (S19)$$

Let us now write $K_{\text{eff}} = V_{\mathcal{P}_\mathscr{a}} + \frac{i}{2} G_{\mathcal{P}_\mathscr{b}} + H.c.,$ where $G_{\mathcal{P}_\mathscr{b}} = \sum_{\ell} F^\ell f_{\ell}^\dagger$. We abbreviate $V_{\mathcal{P}_\mathscr{a}} \equiv (V_{\mathcal{P}_\mathscr{a}})^\dagger$ and similarly for $G$. Plugging this into $K_{\text{eff}}$ and simplifying yields

$$-i \left( K_{\text{eff}} K^{-1} K_{\text{eff}} - H.c. \right) = -i V_{\mathcal{P}_\mathscr{a}} \left( K^{-1} - K^{-1\dagger} \right) V_{\mathcal{P}_\mathscr{a}} + \frac{i}{2} \left( G_{\mathcal{P}_\mathscr{b}} \left( K^{-1} - K^{-1\dagger} \right) G_{\mathcal{P}_\mathscr{b}} \right)$$

$$- \frac{i}{2} G_{\mathcal{P}_\mathscr{b}} \left( K^{-1} + K^{-1\dagger} \right) V_{\mathcal{P}_\mathscr{a}} - \frac{i}{2} V_{\mathcal{P}_\mathscr{a}} \left( K^{-1} + K^{-1\dagger} \right) G_{\mathcal{P}_\mathscr{b}}. \quad (S20)$$

We obtain the same for $\sum_{\ell} F^\ell f_{\ell}^\dagger f_{\ell}^\dagger f_{\ell}^\dagger$ to finish the proof. For this, we have to use another curious identity that is proven using the definition of $K$,

$$\sum_{\ell} F^\ell f_{\ell}^\dagger f_{\ell}^\dagger f_{\ell}^\dagger = -i \left( K^{-1} - K^{-1\dagger} \right). \quad (S21)$$

Plugging this in, splitting $K_{\text{eff}}$ into $V_{\mathcal{P}_\mathscr{a}}$ and $G_{\mathcal{P}_\mathscr{b}}$, and simplifying yields

$$\sum_{\ell} F^\ell f_{\ell}^\dagger f_{\ell}^\dagger f_{\ell}^\dagger = -i \left( K_{\text{eff}} K^{-1} K_{\text{eff}} - H.c. \right). \quad (S22)$$