Coherence measures with respect to general quantum measurements

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Quantum coherence with respect to orthonormal bases has been studied extensively in the past few years. Recently, Bischof, et al. [Phys. Rev. Lett. 123, 110402 (2019)] generalized it to the case of general positive operator-valued measure (POVM) measurements. Such POVM-based coherence, including the block coherence as special cases, have significant operational interpretations in quantifying the advantage of quantum states in quantum information processing. In this work we first establish an alternative framework for quantifying the block coherence and provide several block coherence measures. We then present several coherence measures with respect to POVM measurements, and prove a conjecture on the l1-norm related POVM coherence measure.

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I. INTRODUCTION

Quantum coherence is a characteristic feature of quantum mechanics, with wide applications in superconductivity, quantum thermodynamics and biological processes. From a resource-theoretic perspective the quantification of quantum coherence has attracted much attention and various kinds of coherence measures have been proposed [1–15]. Let ρ be a density operator in d-dimensional complex Hilbert space H. Under a fixed orthonormal basis {(|i⟩)⟩}d−1 i=1 of H, the state ρ is called incoherent if ⟨i|ρ|j⟩ = 0 for any i ≠ j [1]. Otherwise ρ is called coherent. The coherence theory has achieved fruitful results in the past few years (for recent reviews see e.g. [16, 17]).

From another perspective, the orthonormal basis {(|i⟩)⟩}d−1 i=1 corresponds to a rank-1 projective measurement (von Neumann measurement) {⟨i| ⟨i|}d−1 i=1, and ⟨i|ρ|j⟩ = 0 is equivalent to |i⟩⟨i|ρ⟩⟨j| = 0. This observation leads one to view the coherence with respect to the orthonormal basis {(|i⟩)⟩}d−1 i=1 as the coherence with respect to the rank-1 projective measurement {⟨i| ⟨i|}d−1 i=1. Along this idea, the concept of coherence can be generalized to the cases of general measurements. Recently, Bischof, et al. [18] have generalized the concept of coherence to the case of general quantum measurements, i.e., positive operator-valued measures (POVMs). One motivation of this generalization is due to the fact that POVMs may be more advantageous compared to von Neumann measurement in many applications [19]. There are many important problems, such as the optimal way to distinguish a set of quantum states, involve POVM, rather than projective measurement. Moreover, the notion of coherence with respect to a general measurement can be embedded in a consistent resource theory, and such POVM-based coherence measures have interesting operational interpretations which quantify the advantage of quantum states in a quantum information protocol [20]. Refs. [18, 20] provided a way of generalizing coherence theory not in an orthonormal basis, but with a generic POVM. This effort has been started in [21, 22].

After establishing a framework for quantifying the POVM coherence [18, 20], Bischof, et al. developed [18, 20] a scheme by employing the Naimark extension to embed the POVM coherence into the block coherence proposed in [23] in a larger Hilbert space. The Naimark extension [24, 25] states that any POVM can be extended to a projective measurement in a larger Hilbert space. The block coherence was defined with respect to projective measurements, not necessarily rank-1. With this scheme, the relative entropy of POVM coherence Crel, the robustness POVM coherence Crot were proposed. Recently, the structures of different incoherent operations for POVM coherence were investigated [26]. For simplicity, we call the coherence theory with respect to fixed orthonormal bases the standard coherence theory. As the generalizations of the standard coherence, both the block coherence and the POVM coherence reduce to the standard coherence in the case of the von Neumann measurement.

In the present work, we establish an alternative framework for quantifying the block coherence and provide several block coherence measures. We then present several POVM coherence measures. Meanwhile, we also prove a conjecture raised recently.

II. ALTERNATIVE FRAMEWORK FOR QUANTIFYING BLOCK COHERENCE

A. Block incoherent states and block incoherent channels

The block coherence theory was introduced in [23]. We adopt the framework proposed in [20] for quantifying the
block coherence. Consider a quantum system $A$ associated with an $m$-dimensional complex Hilbert space $H$. One has partition $H = \bigoplus_{i=1}^{n} \pi_i$ into orthogonal subspaces $\pi_i$ of dimension $\dim \pi_i = m_i$, $\sum_{i=1}^{n} m_i = m$. Correspondingly, one gets a projective measurement $P = \{ P_i \}_{i=1}^{n}$, with each projector satisfying $P_i(H) = \pi_i$. A state $\rho$ on $H$ is called block incoherent (BI) with respect to $P$ if

$$P_i \rho P_j = 0, \quad \forall i \neq j,$$

or

$$\rho = \sum_{i=1}^{n} P_i \rho P_i.$$

We denote the set of all quantum states in $H$ by $S(H)$, and the set of all block incoherent quantum states by $\mathcal{I}_B(H)$. It is easy to check that

$$\mathcal{I}_B(H) = \{ \sum_{i=1}^{n} P_i \rho P_i | \rho \in S(H) \}. \quad (3)$$

A quantum channel is a completely positive and trace preserving (CPTP) linear map of quantum states [27]. A quantum channel $\phi$ is often expressed by the Kraus operators $\{ K_i \}$ satisfying $\sum_i K_i K_i^\dagger = \mathbb{1}$, where $\mathbb{1}$ is the identity operator on $H$. An expression of Kraus operators is called block incoherent if it admits an expression of Kraus operators $\{ K_i \}$ such that

$$P_i K_i \rho K_i^\dagger P_j = 0, \quad \forall l, \forall i \neq j \quad (4)$$

for any $\rho \in \mathcal{I}_B(H)$. Such an expression $\phi = \{ K_i \}$ is called a block incoherent decomposition of $\phi$. We denote the set of all quantum channels on $H$ by $\mathcal{C}(H)$, and the set of all block incoherent quantum channels by $\mathcal{C}_B(H)$.

The concept of block coherence can be extended to the multipartite systems via the tensor product of the Hilbert spaces of the subsystems, similar to the case of standard coherence theory [16]. For bipartite systems, let $A'$ be another quantum system associated with the $m'$-dimensional complex Hilbert space $H'$. Partitioning $H' = \bigoplus_{i=1}^{n'} \pi_i'$ into orthogonal subspaces $\pi_i'$ of dimension $\dim \pi_i' = m_i'$, one gets a projective measurement $P' = \{ P_i' \}_{i=1}^{n'}$ with each projector $P_i'$ satisfying $P_i'(H') = \pi_i'$. Correspondingly one has concepts such as $\mathcal{S}(H'), \mathcal{I}_B(H'), \mathcal{C}(H')$ and $\mathcal{C}_B(H')$. For the composite Hilbert space $H^{AA'} = H^{A} \otimes H^{A'}$, the set of all states $\rho^{AA'}$ on $H^{AA'}$ is called block incoherent if it admits an expression of Kraus operators $\{ K_i^{AA'} \}$ such that

$$(P_i \otimes P_j')(K_i^{AA'} \rho^{AA'} (K_i^{AA'})^\dagger (P_j \otimes P_j') = 0 \quad (6)$$

for all $l$ and $(i, i') \neq (j, j')$. We denote the set of all block incoherent channels on $\mathcal{C}(H^{AA'})$ by $\mathcal{C}_B(H^{AA'})$ and call such an expression $\phi^{AA'} = \{ K_i^{AA'} \}$ a block incoherent decomposition of $\phi^{AA'}$.

### B. An alternative framework for quantifying the block coherence

A framework for quantifying the block coherence has been established in [20]: any valid block coherence measure $C(\rho; P)$ with respect to the projective measurement $P$ should satisfy the conditions (B1-B4) below.

- **(B1) Faithfulness:** $C(\rho; P) \geq 0$ with equality if $\rho \in \mathcal{I}_B(H)$.
- **(B2) Monotonicity:** $C(\phi_{BI}(\rho); P) \leq C(\rho; P)$ for any $\phi_{BI} \in \mathcal{C}_B(H)$.
- **(B3) Strong monotonicity:** $\sum_i p_i C(\rho_i; P) \leq C(\rho; P)$ for any block incoherent decomposition $\phi_{BI} = \{ K_i \} \in \mathcal{C}_B(H)$ of $\phi_{BI}$, where $p_i = \text{tr}(K_i \rho K_i^\dagger)$, $p_i = \text{tr}(K_i^\dagger K_i)$.
- **(B4) Convexity:** $C(\sum_i p_i \rho_i; P) \leq \sum_i p_i C(\rho_i; P)$ for any states $\{ \rho_i \}$ and any probability distribution $\{ p_i \}$.

This framework coincides with the one in the standard coherence theory [1] if all $\{ P_i \}_{i=1}^{n}$ are rank-1. Note that (B3) and (B4) together imply (B2).

The framework of the standard coherence theory [1] had been modified by adding an additivity condition in [28]. For the block coherence theory, we add the following condition:

- **(B5) Block additivity:**

$$C(p_1 \rho_1 + p_2 \rho_2; P) = p_1 C(\rho_1; P) + p_2 C(\rho_2; P), \quad (7)$$

where $p_1 > 0$, $p_2 > 0$, $p_1 + p_2 = 1$, $\rho_1, \rho_2 \in S(H)$, and for any partition $P = \{ P_i \}$, $\{ P_k \}$, $\{ P_{k_1} \}$ and $\{ P_{k_2} \}$ such that $\{ k_1 \} \cup \{ k_2 \}$ and $\{ k_1 \} \cap \{ k_2 \} = \emptyset$ and $P_{k_1} P_{k_2} = 0$ for any $k_1$ and $k_2$.

With condition (B5), we have the following theorem, which establishes an alternative framework for quantifying the block coherence.

**Theorem 1.** The framework given by conditions (B1) to (B4) is equivalent to the one given by the conditions (B1), (B2) and (B5).

**[Proof]** We first prove that conditions (B1) to (B4) imply (B1), (B2) and (B5). Suppose that (B1) to (B4) are fulfilled. For the state $p_1 \rho_1 + p_2 \rho_2$ as given in (B5), we construct the BI channel $\phi_{BI} = \{ K_1, K_2 \}$ with $K_1 = \sum_{k_1} P_{k_1}$, $K_2 = \sum_{k_2} P_{k_2}$. We have $K_1(p_1 \rho_1 + p_2 \rho_2) K_1^\dagger = p_1 \rho_1$ and $K_2(p_1 \rho_1 + p_2 \rho_2) K_2^\dagger = p_2 \rho_2$. Then from (B3) we get

$$C(p_1 \rho_1 + p_2 \rho_2; P) \geq p_1 C(\rho_1; P) + p_2 C(\rho_2; P). \quad (8)$$
On the other hand, since \( p_1 p_1 \oplus p_2 p_2 = p_1 p_1 + p_2 p_2 \), from (B4) we get
\[
C(p_1 p_1 \oplus p_2 p_2; P) \leq p_1 C(p_1; P) + p_2 C(p_2; P).
\]
Combining (8) and (9) we get the condition (B5).

Next we prove that (B1), (B2) and (B5) imply (B1) to (B4). Suppose conditions (B1), (B2) and (B5) are satisfied. Let \( \{K_i\}_{i=1}^{n'} \in C_{BI}(H) \) be a BI decomposition associated to the system \( A \).

Let the state \( \rho_{AA'} = \rho \otimes |1\rangle\langle 1| \) undergo a BI channel such that
\[
\phi_{BI}^{AA'}(\rho_{AA'}) = \sum_{l} (K_l \otimes U_l) (\rho \otimes |1\rangle\langle 1|)(K_l^\dagger \otimes U_l^\dagger)
\]
\[
= \sum_{l} K_l \rho K_l^\dagger \otimes |l\rangle\langle l|,
\]
(10)
where
\[
U_l = \sum_{k=1}^{n'} |k + l - 1\rangle\langle k|
\]
are the unitary operators on \( A' \). From (B5), (10) gives rise to
\[
C(\sum_{l} K_l \rho K_l^\dagger \otimes |l\rangle\langle l|; P \otimes P') = \sum_{l} p_l C(\rho_l; P),
\]
(11)
where \( P \) and \( P' \) are rank-1 projective measurements, \( p_l = tr(K_l \rho K_l^\dagger) \), \( \rho_l = K_l \rho K_l^\dagger / p_l \), and we have used
\[
C(\rho \otimes |l\rangle\langle l|; P \otimes P') = C(\rho; P).
\]
(12)
According to (B2), (10) and (11) together imply (B3).

Now consider the following state
\[
\rho_{AA'} = \sum_{l=1}^{n'} p_l \rho_l \otimes |l\rangle\langle l|,
\]
(13)
with \( \{p_l\}_{l=1}^{n'} \) a probability distribution and \( \{\rho_l\}_{l=1}^{n'} \subset S(H) \), \( \{\langle l|\}_{l=1}^{n'} \) orthonormal basis of \( H' \). According to (B5), we have
\[
C(\sum_{l} p_l \rho_l \otimes |l\rangle\langle l|; P \otimes P') = \sum_{l} p_l C(\rho_l; P).
\]
(14)
Let \( \rho_{AA'} \) undergo a BI channel as
\[
\phi_{BI}^{AA'}(\rho_{AA'}) = \sum_{k=1}^{n'} (I^{AA'} \otimes |1\rangle\langle 1|) \rho_{AA'} (I^{AA'} \otimes |1\rangle\langle 1|)
\]
\[
= \sum_{j} p_j \rho_j \otimes |1\rangle\langle 1|.
\]
(15)
Similarly, (B2), (B5), (14) and (15) together imply (B4).

We have provided an alternative framework for block coherence by proving that the conditions (B1) to (B4) are equivalent to the conditions (B1), (B2) and (B5). The similar condition (B5) in the standard coherence has particular advantages in calculating coherence of block diagonal states [29]. The condition (B5) in the block coherence may also simplify the calculations of the block coherence for certain block diagonal states.

C. Several block coherence measures

Under the framework of block coherence above, we now provide several block coherence measures. Denote \( P = \{P_i\}_{i=1}^{n} \) a projective measurement on the Hilbert space \( H \). The following Propositions 1-5 provide block coherence measures, see the detailed proofs in Appendix.

Proposition 1. \( l_1 \) norm of coherence
\[
C_{l_1}(\rho, P) = \sum_{i \neq j} ||P_i \rho P_j||_{tr}
\]
is a block coherence measure, where \( ||M||_{tr} = tr \sqrt{M^\dagger M} \) denotes the trace norm of the matrix \( M \).

Proposition 2. For \( \alpha \in (0, 1) \cup (1, 2] \), coherence based on Tsallis relative entropy
\[
C_{T,\alpha}(\rho, P) = \frac{1}{\alpha - 1} \left( \sum_{i} tr[(P_i \rho^\alpha P_i)^{1/\alpha}] - 1 \right)
\]
is a block coherence measure.

In particular, we have

Corollary 1.
\[
\lim_{\alpha \to 1} C_{T,\alpha}(\rho, P) = (\ln 2)C_{rel}(\rho, P),
\]
where
\[
C_{rel}(\rho, P) = \text{tr}(\rho \log_2 \rho) - \sum_{i} \text{tr}[(P_i \rho P_i) \log_2 (P_i \rho P_i)],
\]
and \( \text{ln} \) is the natural logarithm.

Proposition 3. Modified trace norm of coherence
\[
C_{tr}(\rho, P) = \min_{\lambda > 0, \sigma \in I_B(H)} ||\rho - \lambda \sigma||_{tr}
\]
is a block coherence measure.

Proposition 4. Coherence weight
\[
C_{w}(\rho, P) = \min_{\sigma, \tau \in I_B(H), \sigma \in S(H)} \{ s \geq 0 \rho = (1 - s)\sigma + s \tau, \sigma \in I_B(H), \tau \in S(H) \}
\]
is a block coherence measure.

Proposition 5. For \( \alpha \in [\frac{1}{2}, 1) \), coherence based on sandwiched Rényi relative entropy
\[
C_{R,\alpha}(\rho, P) = 1 - \max_{\sigma \in I_B(H)} \left( \{ \text{tr}[(\rho^{\alpha/2} \sigma^{1/\alpha} P_i \rho^{1/2})^\alpha] \}^{1/\alpha} \right)
\]
is a block coherence measure.

When $P$ is a rank-1 projective measurement, $C_{R_1}(\rho, P)$ recovers the standard coherence measure $C_1(\rho)$ proposed in Ref. [1]. $C_{P_1,\alpha}(\rho, P)$ recovers the standard coherence measure proposed in Ref. [28], $C_{w}(\rho, P)$ recovers the standard coherence measure proposed in Ref. [31], $C_{R,\alpha}(\rho, P)$ recovers the standard coherence measure proposed in Ref. [14]. In particular, when $\alpha = \frac{1}{2}$,

\[
C_{R,\frac{1}{2}}(\rho, P) = 1 - \max_{\sigma \in I_{in}(H)} \text{tr}\left(\sqrt{\rho}\sigma\sqrt{\rho}\right)^2
\]

reverses the standard coherence measure proposed in Ref. [32] when $P$ is a rank-1 projective measurement.

III. COHERENCE MEASURES WITH RESPECT TO GENERAL QUANTUM MEASUREMENTS

We now study the coherence measures with respect to general quantum measurements [20]. A general measurement or a POVM on $d$-dimensional Hilbert space $H$ is given by a set of positive semidefinite operators $E = \{E_i\}_{i=1}^n$ with $\sum_{i=1}^n E_i = I_d$ the identity on $H$. Projective measurement and rank-1 projective measurement are the special cases of POVM. Suppose $E_i = A_i A_i^\dagger$ for any $i$. We also denote $E = \{A_i\}_{i=1}^n$ with $\sum_{i=1}^n A_i A_i^\dagger = I_d$. Note that $E_i = (U_i A_i)(U_i A_i)^\dagger$ for any unitary $\{U_i\}_{i=1}^n$.

A state $\rho$ is called an incoherent state with respect to $E$ if [18]

\[
E_i \rho E_j = 0, \quad \forall i \neq j.
\]

Note that this is equivalent to [18]

\[
A_i \rho A_j^\dagger = 0, \quad \forall i \neq j.
\]

The POVM incoherent channel is defined via the canonical Naimark extension [20]. For POVM $E = \{E_i = A_i A_i^\dagger\}_{i=1}^n$ on $d$-dimensional Hilbert space $H$, introduce an $n$-dimensional Hilbert space $H_R$ with $\{|i\}_{i=1}^n$ an orthonormal basis of $H_R$. A canonical Naimark extension $P = \{P_i\}_{i=1}^n$ of $E = \{E_i\}_{i=1}^n$ is described by a unitary matrix $V$ on $H_e = H \otimes H_R$ as [20]

\[
V = \sum_{i=1}^n A_i \otimes |i\rangle\langle j|,
\]

\[
\mathcal{T} = \{\mathcal{T}_i = I_d \otimes |i\rangle\langle i|\}_{i=1}^n,
\]

\[
P_i = V_i^\dagger \mathcal{T}_i V,
\]

with $\{A_{ij}\}_{i,j=1}^n$ satisfying

\[
\sum_{i=1}^n A_{ij} A_{ik} = \delta_{jk} I_d,
\]

\[
A_{11} = A_1.
\]

A channel $\phi \in C(H)$ is called a POVM incoherent (PI) channel if [20] $\phi$ allows a Kraus operator decomposition $\phi = \{K_i\}_i$ with $\sum_i K_i K_i^\dagger = I_d$ and there exists a BI channel $\phi' = \{K'_i\}_i \in C_{BI}(H_e)$ with respect to a canonical Naimark extension $P = \{P_i\}_i$ such that

\[
K_i \rho K_i^\dagger \otimes |1\rangle\langle 1| = K'_i (\rho \otimes |1\rangle\langle 1|) K'_i^\dagger, \quad \forall l,
\]

where $\{K'_i\}_i$ is a BI decomposition of $\phi'$. For such case we call $\{K_i\}_i$ a PI decomposition of $\phi$.

We denote the set of all PI states as $\mathcal{I}_P(H)$, and the set of all PI channels as $\mathcal{C}_P(H)$. Note that $\mathcal{I}_P(H)$ may be empty for some POVMs. Note also that such definition of PI operation does not depend on the choice of Naimark extension [20].

A coherence measure for states in Hilbert space $H$ with respect to a general quantum measurement $E = \{E_i\}_{i=1}^n$ should satisfy the following conditions (P1)-(P4) [20]:

(P1) Faithfulness: $C(\rho, E) \geq 0$, with equality if $\rho \in \mathcal{I}_P(H)$.

(P2) Monotonicity: $C(\phi P_1(\rho), E) \leq C(\phi, E), \quad \forall \phi P_1 \in C_{P_1}(H)$.

(P3) Strong monotonicity: $\sum_i p_i C(\rho_i, P) \leq C(\rho, P)$, where $\{K_i\}_i$ is a PI decomposition of a PI channel, $p_i = \text{tr}(K_i K_i^\dagger)$, $\rho_i = K_i \rho K_i^\dagger / p_i$.

(P4) Convexity: $C(\sum\rho_i p_i, E) \leq \sum p_i C(\rho, E)$, $\{\rho_i\}_i \subset \mathcal{S}(H)$, $\{p_i\}_i$ a probability distribution.

Note that the definitions of PI states and PI channels and the conditions (P1)-(P4) all include the projective measurements and the rank-1 projective measurements as special cases [20]. We emphasize that the framework of POVM coherence measure is about POVM $E = \{E_i\}_{i=1}^n$. Hence, any valid coherence measure in terms of $\{A_i\}_i$ should be invariant under the unitary transformation $\{A_i\}_i \rightarrow \{U_i A_i\}_i$, for any unitary $\{U_i\}_{i=1}^n$ [20].

An efficient scheme for constructing POVM coherence measures is as follows [18, 20]

\[
C(\rho, E) = C(\varepsilon(\rho), \mathcal{T}),
\]

where

\[
\varepsilon(\rho) = \sum_{ij=1}^n A_{ij} \rho A_{ij}^\dagger \otimes |i\rangle\langle j|,
\]

It can be checked that if $C(\rho_e, \mathcal{T})$ is a unitarily invariant block coherence measure satisfying conditions (B1) to (B4), then $C(\rho, E)$ defined above is a POVM coherence measure satisfying conditions (P1) to (P4) [20]. Here $\rho_e$ is any state on $H_e = H \otimes H_R$. The unitary invariance means that

\[
C(\rho_e, \mathcal{T}) = C(U \rho_e U^\dagger, U \mathcal{T} U^\dagger)
\]
for any unitary transformation $U$ on $H$. Employing this scheme and using Propositions 1 to 5, we obtain the following Theorem.

**Theorem 2.** Let $E = \{ E_i = A_i^\dagger A_i \}_{i=1}^n$ be a POVM on the Hilbert space $H$. The following quantities given in (1)-(5) are all POVM coherence measures with respect to $E$.

1. $l_1$ norm of coherence

$$C_{l_1}(\rho, E) = \sum_{i\neq j} ||A_i\rho A_j^\dagger||_{tr}. \tag{33}$$

2. For $\alpha \in (0, 1) \cup (1, 2]$, coherence based on Tsallis relative entropy

$$C_{T,\alpha}(\rho, E) = \frac{1}{\alpha - 1} \left\{ \sum_i \text{tr}[(A_i^\alpha \rho A_i^\dagger)]^\alpha - 1 \right\}, \tag{34}$$

and

$$\lim_{\alpha \to 1} C_{T,\alpha}(\rho, E) = (\ln 2)C_{rel}(\rho, E), \tag{35}$$

where

$$C_{rel}(\rho, E) = \text{tr}(\rho \log_2 \rho) - \sum_i \text{tr}[(A_i^\dagger \rho A_i) \log_2(A_i^\dagger \rho A_i^\dagger)]. \tag{36}$$

3. Modified trace norm of coherence

$$C_{tr}(\rho, E) = \min_{\lambda > 0, \sigma \in I(H_\rho)} ||\varepsilon(\rho) - \lambda \sigma||_{tr}. \tag{37}$$

4. Coherence weight

$$C_w(\rho, E) = \min_{\sigma \in I(H_\rho)} \{ s \geq 0 | \varepsilon(\rho) \geq (1-s)\sigma \}. \tag{38}$$

5. For $\alpha \in [\frac{1}{2}, 1)$, coherence based on sandwiched Rényi relative entropy

$$C_{R,\alpha}(\rho, E) = 1 - \max_{\sigma \in I(H_\rho)} \left\{ \text{tr}[(\varepsilon(\rho^{\frac{1}{\alpha}})\sigma^{\frac{-1}{\alpha}})]^\alpha \right\}^{-\frac{1}{\alpha}}. \tag{39}$$

**[Proof].** To prove the results of the Theorem 2, we need to use the results of the Propositions 1 to 5. Let $\{|i\rangle\}_{i=1}^n$ be an orthonormal basis for the Hilbert space $H_R$, and $U$ and $\varepsilon(\rho)$ be defined in Eqs. (27) and (31), respectively. Since $C(\rho, E)$ is a POVM coherence measure satisfying conditions (P1) to (P4) if $C_{\rho, \varepsilon}$ is a unitarily invariant block coherence measure satisfying conditions (B1) to (B4), we only need to prove the unitary invariance Eq. (32) and show that $C_{l_1}(\rho, E), C_{T,\alpha}(\rho, E), C_{tr}(\rho, E), C_w(\rho, E)$ and $C_{R,\alpha}(\rho, E)$ take the forms of Eqs. (33), (34), (37), (38) and (39) under Eq. (30), respectively.

1. We prove that $C_{l_1}(\rho, E)$ is unitarily invariant. For any unitary $U$ on $H$, we have

$$C_{l_1}(U\rho U^\dagger, U\mathcal{P}U^\dagger) = \sum_{i\neq j} ||U\mathcal{P}_iU^\dagger U\rho_i U^\dagger U\mathcal{P}_j U^\dagger||_{tr}$$

and

$$= \sum_{i\neq j} ||\mathcal{P}_i \rho_i \mathcal{P}_j||_{tr} = C_{l_1}(\rho, \mathcal{P})$$

where we have used the fact that the trace norm is unitarily invariant. It is easy to see that $C_{l_1}(\rho, E)$ have the form of Eq. (33).

2. It is easy to see that $C_{T,\alpha}(\rho, E)$ is unitarily invariant. Now we prove that $C_{T,\alpha}(\rho, E)$ has the form of Eq. (34) under Eq. (30).

For the unitary transformation $V$ defined in Eq. (26),

$$\varepsilon_V(\rho) = V(\rho \otimes |1\rangle\langle 1|)V^\dagger = \sum_{ij} A_i^\dagger \rho A_j^\dagger \otimes |i\rangle \langle j| = \varepsilon(\rho).$$

As a result,

$$\text{tr}[(\mathcal{P}_i \varepsilon_V(\rho) \mathcal{P}_i^\dagger)^{1/\alpha}]$$

$$= \text{tr}[(\mathcal{P}_i V(\rho^\alpha \otimes |1\rangle\langle 1|)V^\dagger \mathcal{P}_i)^{1/\alpha}]$$

$$= \text{tr}[(\mathcal{P}_i \sum_j A_j \rho^\alpha A_j^\dagger \otimes |j\rangle \langle k| \mathcal{P}_i)^{1/\alpha}]$$

$$= \text{tr}[(A_i^\dagger \rho A_i^\dagger \otimes |i\rangle \langle i|)^{1/\alpha}]$$

$$= \text{tr}[(A_i^\dagger \rho A_i^\dagger)^{1/\alpha}].$$

Hence, $C_{T,\alpha}(\rho, E)$ has the form of Eq. (34). Eq. (35) can be proved as Corollary 1.

3. It is easy to see that $C_{tr}(\rho, E)$ has the form of Eq. (37). Now we show that $C_{tr}(\rho, E)$ is unitarily invariant. Note that

$$C_{tr}(\rho, \mathcal{P}) = \min_{\lambda > 0, \sigma} ||\rho - \lambda \sum_{i=1}^n \mathcal{P}_i \sigma \mathcal{P}_i||_{tr},$$

where $\sigma$ is any density operator on $H$. For any unitary $U$ on $H$, we have

$$C_{tr}(U\rho U^\dagger, U\mathcal{P}U^\dagger) = \min_{\lambda > 0, \sigma} ||U\rho U^\dagger - \lambda \sum_{i=1}^n U\mathcal{P}_i U^\dagger \sigma U\mathcal{P}_i U^\dagger||_{tr},$$

$$= \min_{\lambda > 0, \sigma} ||\rho - \lambda \sum_{i=1}^n \mathcal{P}_i \sigma \mathcal{P}_i||_{tr},$$

$$= \min_{\lambda > 0, \sigma} ||\rho - \lambda \sum_{i=1}^n \mathcal{P}_i \sigma \mathcal{P}_i||_{tr},$$

where we have used the facts that trace norm is unitarily invariant and $\{\sigma : \sigma \in S(H)\} = \{U^\dagger \sigma U : \sigma \in S(H)\}$.

4. It is easy to see that $C_w(\rho, E)$ has the form of Eq. (38). Next we show that $C_w(\rho, \mathcal{P})$ is unitarily invariant. Note that

$$C_w(\rho, \mathcal{P}) = \min_{\sigma} \{ s \geq 0 | \rho \geq (1-s) \sum_{i=1}^n \mathcal{P}_i \sigma \mathcal{P}_i \},$$
where $\sigma$ is any density operator on $H$. For any unitary $U$ on $H$, we have

$$
C_w(U\rho U^\dagger, U\mathcal{P}U^\dagger) 
= \min_{\sigma} \{ s \geq 0 | \rho_z U^\dagger \geq (1 - s) \sum_{i=1}^n U\mathcal{P}_i U^\dagger \sigma U\mathcal{P}_i U^\dagger \}
$$

$$
= \min_{\sigma} \{ s \geq 0 | \rho_z \geq (1 - s) \sum_{i=1}^n \mathcal{P}_i \sigma \mathcal{P}_i \}
$$

$$
= \min_{\sigma} \{ s \geq 0 | \rho_z \geq (1 - s) \sum_{i=1}^n \mathcal{P}_i \sigma \mathcal{P}_i \}
$$

$$
= C_w(\rho_z, \mathcal{P}),
$$

which completes the proof.

(5). It is easy to see that $C_{R,\alpha}(\rho, E)$ has the form of Eq. (39). Similarly to the proof of (3), one can show that $C_w(\rho_z, \mathcal{P})$ is unitarily invariant.

We remark that the coherence measure $C_{l_1}(\rho, P)$ was proposed in [23]. In [20] the authors conjectured that $C_{l_1}(\rho, P)$ is a well defined POVM coherence measure satisfying the conditions (P1)-(P4). Combining with our result of proposition 1, we have strictly proved in Theorem 2 that $C_{l_1}(\rho, E)$ is indeed a well defined POVM coherence measure.

**IV. SUMMARY**

We have established an alternative framework for quantifying the coherence with respect to projective measurements, and provided several coherence measures with respect to projective measurements. We then obtained several coherence measures with respect to general POVM measurements, from which a conjecture has been verified concerning the coherence measure $C_{l_1}(\rho, E)$. The coherence with respect to POVM measurements has operational significance. Our results may highlight further investigations on the coherence of quantum states and the applications in quantum information processing.

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**Appendix**

A. Proof of Proposition 1

From the definition of BI state and the properties of trace norm, $C_{l_1}(\rho, P)$ satisfies the condition (B1). It satisfies the conditions (B4) and (B5) due to the properties of trace norm. Since (B3) and (B4) imply (B2), we only need to prove that $C_{l_1}(\rho, P)$ fulfills (B3).

For any BI channel $\phi$ with BI decomposition $\phi_{BI} = \{K_l\}_l$, $\sum_{l=1}^d K_l = I_d$, each $K_l$ has the form [20],

$$
K_l = \sum_{i=1}^n P_{f_l(i)} M_i P_i, 
$$

where $f_l(i)$ is a function on $\{i\}_{l=1}^n$, $M_i$ is a matrix on $H$. Denote $p_l = \text{tr}(K_l \rho K_l^\dagger)$, $\rho_l = K_l \rho K_l^\dagger / p_l$. We have

$$
\sum_l p_l C_{l_1}(\rho_l, P) = \sum_{l,i,j} ||P_l K_l \rho K_l^\dagger P_j||_{tr}
$$

$$
= \sum_{l,i,j} ||P_l K_l \rho P_i P_j K_l^\dagger P_j||_{tr}
$$

$$
\leq \sum_{l,i,j,i'\neq j'} ||P_l K_l \rho P_i P_j K_l^\dagger P_j||_{tr}
$$

$$
= \sum_{l,i,j,i'\neq j'} ||P_{f(i')} K_l \rho P_{f(i')} K_l^\dagger P_{f(i')}||_{tr}
$$

$$
= \sum_{l,i,j,i'\neq j'} ||P_{f(i')} K_l \sum_k s_{i'j'k} |\psi_{i'j'k}\rangle \langle \psi_{i'j'k}| P_{f(i')}||_{tr}
$$

$$
\leq \sum_{k,s_{i'j'k}} s_{i'j'k} ||P_{f(i')} K_l \sum_k |\psi_{i'j'k}\rangle \langle \psi_{i'j'k}| P_{f(i')}||_{tr}
$$

$$
= \sum_{k,s_{i'j'k}} s_{i'j'k} \sqrt{ \langle \psi_{i'j'k}| P_{f(i')} K_l \langle \psi_{i'j'k}| ||K_l^\dagger P_{f(i')} K_l||_{tr}}
$$

(A2)

(A3)

(A4)

(A5)
To go ahead, we need the lemmas below.

\begin{align}
\frac{1}{2} \sum_{k,i' \neq j'} s_{i'j'} \sqrt{\sum_l \langle \psi_{i'j'} | K_i^l P_{i'j'} K_l \psi_{i'j'} \rangle} \sqrt{\sum_l \langle \bar{\psi}_{i'j'} | K_i^l P_{i'j'} K_l \bar{\psi}_{i'j'} \rangle} \\
= \sum_{k,i' \neq j'} s_{i'j'} \sqrt{\sum_l K_i^l P_{i'j'} K_l | \psi_{i'j'} \rangle \langle \psi_{i'j'} |} \frac{1}{\sqrt{\sum_l | \langle \bar{\psi}_{i'j'} | I_m | \bar{\psi}_{i'j'} \rangle |}} \left( \sum_l K_i^l P_{i'j'} K_l | \bar{\psi}_{i'j'} \rangle \langle \bar{\psi}_{i'j'} | \right)
\end{align}

\begin{align}
\langle \psi_{i'j'} | I_m | \bar{\psi}_{i'j'} \rangle = \sum_l K_i^l P_{i'j'} K_l | \psi_{i'j'} \rangle \langle \psi_{i'j'} | \sqrt{\sum_l | \langle \bar{\psi}_{i'j'} | I_m | \bar{\psi}_{i'j'} \rangle |} \\
= \sum_{k,i' \neq j'} s_{i'j'} = \sum_{k,i' \neq j'} \|P_{i'j'} \rho P_{i'j'}\|_\infty = C_{1,k}(\rho, P).
\end{align}

Lemma 1. Hölder inequality.

Suppose \( \{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n \) are all positive real numbers, then

1) when \( \alpha \in (0, 1) \),
\[
\sum_{i=1}^n a_i b_i \leq \left( \sum_{i=1}^n a_i^\alpha \right)^{\frac{1}{\alpha}} \left( \sum_{i=1}^n b_i^{\frac{1}{\alpha}} \right)^{\frac{1}{\beta}}
\]
and the equality holds if and only if \( a_i^\alpha = b_i^{\frac{1}{\alpha}} \) for any \( i,j \)

2) when \( \alpha > 1 \),
\[
\sum_{i=1}^n a_i b_i \leq \left( \sum_{i=1}^n a_i^{\frac{1}{\alpha}} \right)^{\alpha} \left( \sum_{i=1}^n b_i^{\frac{1}{\beta}} \right)^{\beta - \alpha}
\]
and the equality holds if and only if \( a_i^\alpha = b_i^{\frac{1}{\alpha}} \) for any \( i,j \).

Lemma 2 (Ref. [34]). For \( r \times r \) positive semidefinite matrices \( M \) and \( N \), it holds that

\[
\sum_{j=1}^r \lambda_j^{1/2}(M) \lambda_j^{1/2}(N) \leq \text{tr}(MN) \leq \sum_{j=1}^r \lambda_j^{1/2}(M) \lambda_j^{1/2}(N),
\]

where \( \{\lambda_j^{1/2}(M)\}_j \) are the eigenvalues of \( M \) in decreasing order.

Now for \( \alpha \in (0, 1) \) and \( \sigma \in \mathcal{B}(H) \), we have

\[
\text{tr}(\rho^\alpha \sigma^{1-\alpha}) = \text{tr}[\rho^\alpha \sum_{i=1}^n (P_i \sigma P_i)^{1-\alpha}]
\]

\[
= \sum_{i=1}^n q_i^{1-\alpha} \text{tr}(\rho^\alpha \sigma_i^{1-\alpha}) \leq \left( \sum_{i=1}^n \text{tr}(\rho^\alpha \sigma_i^{1-\alpha}) \right)^{\frac{1}{\alpha}}
\]

where \( q_i = \text{tr}(P_i \sigma P_i) \), \( \sigma_i = P_i \sigma P_i / q_i \), the Hölder inequality has been used, and the equality holds if and only if there exists constant \( C \geq 0 \) such that \( q_i = C \left[ \text{tr}(\rho^\alpha \sigma_i^{1-\alpha}) \right]^{\frac{1}{\alpha}} \) for any \( i \). Furthermore,

\[
\text{tr}(\rho^\alpha \sigma_i^{1-\alpha})
\]
\[ D \leq \left\{ \sum_{j=1}^{m_i} \left[ \lambda_j^i (P_i \rho \sigma_i P_i^*) \right] \right\}^{1-\alpha} \]

\[ = \sum_{j=1}^{m_i} \left[ \lambda_j^i (P_i \rho \sigma_i P_i^*) \right] \left( \lambda_j^i (\sigma_i) \right)^{1-\alpha} \]

\[ \leq \left\{ \sum_{j=1}^{m_i} \left[ \lambda_j^i (P_i \rho \sigma_i P_i^*) \right] \right\}^{1-\alpha} \sum_{j=1}^{m_i} \left( \lambda_j^i (\sigma_i) \right)^{1-\alpha} \]

\[ = \left\{ \sum_{j=1}^{m_i} \left[ \lambda_j^i (P_i \rho \sigma_i P_i^*) \right] \right\}^{1-\alpha}, \quad \text{(A17)} \]

where the Lemma 1 and Lemma 2 have been used. It is easy to check that when

\[ \sigma = \frac{\sum_{i=1}^{n} (P_i \rho \sigma_i P_i^*)^{1-\alpha}}{\sum_{i=1}^{n} \text{tr}(P_i \rho \sigma_i P_i^*)^{1-\alpha}} \quad \text{(A18)} \]

Eq. (A11) achieves Eq. (A12). As a result we get Eq. (A12).

For \( \alpha > 1 \), we have

\[ \text{tr}(\rho^\alpha \sigma^{1-\alpha}) \]

\[ = \text{tr}(\rho^\alpha \sum_i (P_i \sigma P_i)^{1-\alpha}) \]

\[ = \sum_i q_i^{1-\alpha} \text{tr}(\rho^\alpha (\sigma_i)^{1-\alpha}) \]

\[ \geq \left\{ \sum_i \text{tr}(\rho^\alpha (\sigma_i)^{1-\alpha}) \right\}^{1-\alpha}, \quad \text{(A19)} \]

and the equality holds if and only if there exists a constant \( C \geq 0 \) such that \( q_i = C \text{tr}(\rho^\alpha (\sigma_i)^{1-\alpha})^{1-\alpha} \) for any \( i \).

Moreover,

\[ \text{tr}(\rho^\alpha (\sigma_i)^{1-\alpha}) \]

\[ = \text{tr}(\rho^\alpha P_i (\sigma_i)^{1-\alpha} P_i) \]

\[ \geq \sum_{j=1}^{m_i} \lambda_j^i (P_i \rho^\alpha P_i) (\sigma_i)^{1-\alpha} \]

\[ = \sum_{j=1}^{m_i} \lambda_j^i (P_i \rho^\alpha P_i) (\lambda_j^i (\sigma_i))^{1-\alpha} \]

\[ \geq \left\{ \sum_{j=1}^{m_i} \left[ \lambda_j^i (P_i \rho^\alpha P_i) \right] \right\}^{1-\alpha} \sum_{j=1}^{m_i} \left( \lambda_j^i (\sigma_i) \right)^{1-\alpha} \]

\[ = \left\{ \sum_{j=1}^{m_i} \left[ \lambda_j^i (P_i \rho^\alpha P_i) \right] \right\}^{1-\alpha}, \quad \text{(A20)} \]

In above derivation, we have used Lemma 1 and Lemma 2. Again, when \( \sigma \) takes the value in Eq. (A18), Eq. (A11) achieves Eq. (A12). As a result we get Eq. (A12).

From Eqs. (A9) and (A11) we see that \( D_{T,\alpha}(\rho) \geq 0 \) and \( D_{T,\alpha}(\rho) = 0 \) if and only if \( \rho \in \mathcal{I}_B(H) \). Then from Eq. (A12) we have

\[ \frac{\left\{ \sum_i \text{tr}(P_i \rho^\alpha P_i)^{1/\alpha} \right\}^{\alpha}}{\alpha - 1} \geq 0, \]

namely,

\[ \frac{\sum_i \text{tr}(P_i \rho^\alpha P_i)^{1/\alpha} \left( \text{tr}(P_i \rho^\alpha P_i)^{1/\alpha} \right)^{\alpha - 1}}{\alpha - 1} \geq 0, \]

with the equality holding if and only if \( \rho \in \mathcal{I}_B(H) \), which proves that \( C_{T,\alpha}(\rho, P) \) satisfies (B1).

For any \( \phi_{BI} \in \mathcal{C}_{BI}(H) \), from Eqs. (A10) and (A11) we have

\[ D_{T,\alpha}(\rho) = \min_{\sigma \in \mathcal{I}_{B}(H)} D_{T,\alpha}(\rho \| \sigma) = D_{T,\alpha}(\rho \| \sigma^*) \]

\[ \geq D_{T,\alpha}(\phi_{BI}(\rho \| \sigma^*)) \]

\[ \geq \min_{\sigma \in \mathcal{I}_{B}(H)} D_{T,\alpha}(\phi_{BI}(\rho \| \sigma)) = D_{T,\alpha}(\phi_{BI}(\rho)), \quad \text{(A21)} \]

where \( \sigma^* \in \mathcal{I}_B(H) \) such that \( \min_{\sigma \in \mathcal{I}_{B}(H)} D_{T,\alpha}(\rho \| \sigma) = D_{T,\alpha}(\rho \| \sigma^*) \).

From Eq. (A12), Eq. (A21) is equivalent to

\[ \frac{\sum_i \text{tr}(P_i \rho^\alpha P_i)^{1/\alpha} \left( \text{tr}(P_i \rho^\alpha P_i)^{1/\alpha} \right)^{\alpha - 1}}{\alpha - 1} \]

\[ \leq \frac{\sum_i \text{tr}(P_i \rho^\alpha P_i)^{1/\alpha} \left( \text{tr}(P_i \rho^\alpha P_i)^{1/\alpha} \right)^{\alpha - 1}}{\alpha - 1}, \]

which is further equivalent to

\[ \sum_i \text{tr}(P_i \rho^\alpha P_i)^{1/\alpha} \left( \text{tr}(P_i \rho^\alpha P_i)^{1/\alpha} \right)^{\alpha - 1} \]

\[ \leq \sum_i \text{tr}(P_i \rho^\alpha P_i)^{1/\alpha} \left( \text{tr}(P_i \rho^\alpha P_i)^{1/\alpha} \right)^{\alpha - 1}. \]

We then proved that \( C_{T,\alpha}(\rho, P) \) satisfies (B2).

Now we prove that \( C_{T,\alpha}(\rho, P) \) also satisfies (B5). Suppose \( \rho = \rho_1 \rho_2 \) as described in (B5). Then

\[ \sum_{i=1}^{n} \text{tr}(P_i \rho^\alpha P_i)^{1/\alpha} \]

\[ = \rho_1 \sum_{k_1} \text{tr}(P_{k_1} \rho_1^2 P_{k_1})^{1/\alpha} + \rho_2 \sum_{k_2} \text{tr}(P_{k_2} \rho_2^2 P_{k_2})^{1/\alpha} \]

\[ = \rho_1 \sum_{i=1}^{n} \text{tr}(P_i \rho_1^2 P_i)^{1/\alpha} + \rho_2 \sum_{i=1}^{n} \text{tr}(P_i \rho_2^2 P_i)^{1/\alpha}. \quad \text{(A22)} \]

Substituting (A22) into Eq. (17), we then proved that \( C_{T,\alpha}(\rho, P) \) satisfies (B5).

C. Proof of Corollary 1

Set \( \alpha = 1 + \varepsilon \). Consider the Taylor expansions around \( \varepsilon = 0 \),

\[ M^{1+\varepsilon} = M + \varepsilon M \ln M + o(\varepsilon^2), \]

\[ \ln(M + \varepsilon N) = \ln M + o(\varepsilon), \]

\[ \frac{1}{1 + \varepsilon} = 1 - \varepsilon + o(\varepsilon^2), \]
where $M$, $N$ are Hermitian matrices, $o(\varepsilon)$ denotes the infinitesimal term with the order $\varepsilon$ or higher around $\varepsilon = 0$. We have $P_1 \rho^a P_1 = P_1 (\rho + \varepsilon \ln \rho + o(\varepsilon^2)) P_1$. Therefore,

$$\text{tr}[(P_1 \rho^a P_1)^k] = \text{tr}[(P_1 \rho^a P_1)^{1-\varepsilon + o(\varepsilon^2)}] = \text{tr}(P_1 \rho^a P_1 - \varepsilon (P_1 \rho^a P_1) \ln (P_1 \rho^a P_1) + o(\varepsilon^2)) = \text{tr}(P_1 \rho P_1 + \varepsilon P_1 (\rho \ln \rho) P_1 - \varepsilon (P_1 \rho P_1) \ln (P_1 \rho P_1) + o(\varepsilon^2)).$$

Applying the L'Hospital's rule to Eq. (17), we have

$$\lim_{\alpha \to 1} C_{\alpha, \alpha}(\rho, P) = \lim_{\alpha \to 1} \frac{d}{d\alpha} \sum_i \text{tr}[(P_i \rho^a P_i)^{1/\alpha}] = \sum_i \text{tr}[P_i (\rho \ln \rho) P_i - (P_i \rho P_i) \ln (P_i \rho P_i)] = \text{tr}(\rho \ln \rho) - \sum_i \text{tr}[(P_i \rho P_i) \ln (P_i \rho P_i)] = (\ln 2) C_{\text{rel}}(\rho, P).$$

### D. Proof of Proposition 3

Obviously, the condition (B1) is satisfied. (B2) is also satisfied as a consequence of the fact that $|||M|||_{tr} \geq |||\phi(M)|||_{tr}$ for any CPTP map $\phi$ and any Hermitian matrix $M$ [35]. Concerning (B5), we consider $\rho = p_1 \rho_1 \oplus p_2 \rho_2$. Any $\sigma \in \mathcal{I}_{B}(H)$ can be written as

$$\sigma = q_1 \sigma_1 \oplus q_2 \sigma_2,$$

with $q_1 \geq 0$, $q_2 \geq 0$, $q_1 + q_2 = 1$, and $\sigma_1, \sigma_2 \in \mathcal{S}(H)$, $\sigma_1 \rho_1 = \sigma_2 \rho_2 = 0$ for any $k_1$ and $k_2$. It follows that

$$C(p_1 \rho_1 \oplus p_2 \rho_2, P) = \min_{\lambda > 0, q_1, \sigma_1, \sigma_2} |||p_1 \rho_1 \oplus p_2 \rho_2 - \lambda (q_1 \sigma_1 \oplus q_2 \sigma_2)|||_{tr}$$

$$= \min_{\lambda > 0, q_1, \sigma_1, \sigma_2} (p_1 |||p_1 - \frac{\lambda q_1}{p_1} \sigma_1|||_{tr} + p_2 |||p_2 - \frac{\lambda q_2}{p_2} \sigma_2|||_{tr})$$

$$= p_1 \min_{\lambda > 0, \sigma_1} |||p_1 - \frac{\lambda q_1}{p_1} \sigma_1|||_{tr}$$

$$+ p_2 \min_{\lambda > 0, \sigma_2} |||p_2 - \frac{\lambda q_2}{p_2} \sigma_2|||_{tr}$$

$$= p_1 C(p_1 \rho_1) + p_2 C(p_2 \rho_2),$$

where we have used the facts that $\sigma_1, \sigma_2 \in \mathcal{S}(H)$, $\{q_1, q_2\}$ is a probability distribution, $\lambda_1 = \frac{q_1}{p_1}$ and $\lambda_2 = \frac{q_2}{p_2}$. 

### E. Proof of Proposition 4

It can be proved that $C_w(\rho, P)$ fulfills the conditions (B1), (B3) and (B4) by using a similar way adopted in Ref. [31]. Here we equivalently prove that $C_w(\rho, P)$ fulfills (B1), (B2) and (B5). (B1) is evidently satisfied. To prove (B2), suppose $\{K_i\} \in \mathcal{C}_{B}(H)$ with $\{K_i\}$ a BI decomposition. Then there exists $\sigma \in \mathcal{I}_{B}(H)$ such that

$$\rho \geq \frac{1 - C_w(\rho, P)}{\sigma},$$

$$\sum_i K_i \rho K_i^\dagger \geq \frac{1 - C_w(\rho, P)}{\sum_i K_i \sigma K_i^\dagger}.$$

Since $\sum_i K_i \sigma K_i^\dagger \in \mathcal{I}_{B}(H)$, we obtain $C_w(\sum_i K_i \sigma K_i^\dagger, P) \leq C_w(\rho, P)$, which proves that (B2) is satisfied.

To prove (B5), let us consider again $\rho = p_1 \rho_1 \oplus p_2 \rho_2$ as described in (B5). Then there exists $\sigma \in \mathcal{I}_{B}(H)$ such that

$$\rho \geq \frac{1 - C_w(\rho, P)}{p_1} \sigma_1,$$

$$\rho \geq \frac{1 - C_w(\rho, P)}{p_2} \sigma_2,$$

$$C_w(\rho_1, P) \leq 1 - \frac{1 - C_w(\rho, P)}{p_1} q_1,$$

$$C_w(\rho_2, P) \leq 1 - \frac{1 - C_w(\rho, P)}{p_2} q_2,$$

$$p_1 C_w(\rho_1, P) + p_2 C_w(\rho_2, P) \leq C_w(\rho, P).$$

Conversely, there exist $\sigma_1', \sigma_2' \in \mathcal{I}_{B}(H)$ such that

$$\rho_1 \geq \frac{1 - C_w(\rho_1, P)}{p_1} \sigma_1',$$

$$\rho_2 \geq \frac{1 - C_w(\rho_2, P)}{p_2} \sigma_2'.$$

It follows that

$$p_1 \rho_1 \oplus p_2 \rho_2 \geq p_1 \frac{1 - C_w(\rho_1, P)}{p_1} \sigma_1' + p_2 \frac{1 - C_w(\rho_2, P)}{p_2} \sigma_2',$$

$$C_w(\rho, P) \leq p_1 C_w(\rho_1, P) + p_2 C_w(\rho_2, P).$$

Eqs. (A24) and (A25) imply (B5), which completes the proof.

### F. Proof of Proposition 5

This proof is a generalization of the proof for the Theorem 1 in Ref. [14]. For $\alpha \in \left[\frac{1}{2}, 1\right)$, $\sigma, \rho \in \mathcal{S}(H)$, the sandwiched Rényi relative entropy is defined as [36, 37],

$$F_{\alpha}(\sigma|\rho) = \frac{\ln \text{tr}[(\rho^{\frac{1-\alpha}{2\alpha}} \sigma \rho^{\frac{1-\alpha}{2\alpha}})^{\alpha}]}{\alpha - 1},$$
It is shown that [37, 38] for $\alpha \in [\frac{1}{2}, 1)$, $F_\alpha(\sigma||\rho) \geq 0$, where the equality holds if and only if $\sigma = \rho$. This is equivalent to that

$$\text{tr}[\rho^{\frac{1-\alpha}{2\alpha}} \sigma^{\frac{1-\alpha}{2\alpha}}] \leq 1,$$

and to that

$$\{\text{tr}[\rho^{\frac{1-\alpha}{2\alpha}} \sigma^{\frac{1-\alpha}{2\alpha}}]^{\frac{1}{1-\alpha}} \leq 1,$$

with the equality holding if and only if $\sigma = \rho$. This says that $C_{R,\alpha}(\rho, P)$ satisfies (B1).

For $\alpha \in [\frac{1}{2}, 1)$, it has been shown that [37, 39] for $\sigma, \rho \in S(H)$, and any CPTP map $\phi$,

$$F_\alpha(\phi(\sigma)||\phi(\rho)) \leq F_\alpha(\sigma||\rho).$$

This implies

$$\text{tr}[(\phi(\rho))^{\frac{1-\alpha}{2\alpha}} \phi(\sigma)(\phi(\rho))^{\frac{1-\alpha}{2\alpha}}] \geq \text{tr}[(\rho^{\frac{1-\alpha}{2\alpha}} \sigma^{\frac{1-\alpha}{2\alpha}})],$$

$$\{\text{tr}[(\phi(\rho))^{\frac{1-\alpha}{2\alpha}} \phi(\sigma)(\phi(\rho))^{\frac{1-\alpha}{2\alpha}}]^{\frac{1}{1-\alpha}} \geq \text{tr}[(\rho^{\frac{1-\alpha}{2\alpha}} \sigma^{\frac{1-\alpha}{2\alpha}})]^{\frac{1}{1-\alpha}}.$$

For any BI map $\phi_{BI}$, there exists $\sigma^* \in I_B(H)$ such that

$$\max_{\sigma \in I_B(H)} \{\text{tr}[(\rho^{\frac{1-\alpha}{2\alpha}} \sigma^{\frac{1-\alpha}{2\alpha}})] \}^{\frac{1}{1-\alpha}} = \{\text{tr}[(\rho^{\frac{1-\alpha}{2\alpha}} \sigma^{\frac{1-\alpha}{2\alpha}})] \}^{\frac{1}{1-\alpha}} \leq \{\text{tr}[(\phi_{BI}(\rho))^{\frac{1}{2\alpha}} \phi_{BI}(\sigma^*)(\phi_{BI}(\rho))^{\frac{1}{2\alpha}}] \}^{\frac{1}{1-\alpha}} \leq \max_{\sigma \in I_B(H)} \{\text{tr}[(\phi_{BI}(\rho))^{\frac{1}{2\alpha}} \phi_{BI}(\sigma)(\phi_{BI}(\rho))^{\frac{1}{2\alpha}}] \}^{\frac{1}{1-\alpha}}.$$

This proves that $C_{R,\alpha}(\rho, P)$ satisfies (B2).

Next we prove $C_{R,\alpha}(\rho, P)$ satisfies (B5). Consider $\rho = p_1 \rho_1 \oplus p_2 \rho_2$ as described in (B5). As any $\sigma \in I_B(H)$ can be written as Eq. (A23), it follows that

$$\max_{\sigma \in I_B(H)} \text{tr}[(\rho^{\frac{1-\alpha}{2\alpha}} \sigma^{\frac{1-\alpha}{2\alpha}})] = \max_{q_1, q_2} \{\max_{\sigma_1} \text{tr}[(\rho_1^{\frac{1-\alpha}{2\alpha}} \sigma_1^{\frac{1-\alpha}{2\alpha}})] \},$$

$$\{\max_{q_1, q_2} \text{tr}[(\rho_2^{\frac{1-\alpha}{2\alpha}} \sigma_2^{\frac{1-\alpha}{2\alpha}})] \},$$

where

$$t_1 = \max_{\alpha} \text{tr}[(\rho_1^{\frac{1-\alpha}{2\alpha}} \sigma_1^{\frac{1-\alpha}{2\alpha}})],$$

$$t_2 = \max_{\alpha} \text{tr}[(\rho_2^{\frac{1-\alpha}{2\alpha}} \sigma_2^{\frac{1-\alpha}{2\alpha}})],$$

and the Lemma 1 (note here $t_1 > 0$ and $t_2 > 0$) has been taken into account.

Consequently,

$$\max_{\sigma \in I_B(H)} \{\text{tr}[(\rho^{\frac{1-\alpha}{2\alpha}} \sigma^{\frac{1-\alpha}{2\alpha}})] \}^{\frac{1}{1-\alpha}} = \{\max_{\sigma \in I_B(H)} \text{tr}[(\rho^{\frac{1-\alpha}{2\alpha}} \sigma^{\frac{1-\alpha}{2\alpha}})] \}^{\frac{1}{1-\alpha}} = p_1 t_1 + p_2 t_2 \leq p_1 t_1^\frac{1}{1-\alpha} + p_2 t_2^\frac{1}{1-\alpha}.$$

This shows that $C_{R,\alpha}(\rho, P)$ satisfies (B5).

[1] T. Baumgratz, M. Cramer, and M. B. Plenio, Phys. Rev. Lett. 113, 140401 (2014).
[2] X. Yuan, H. Zhou, Z. Cao, and X. Ma, Phys. Rev. A 92, 022124 (2015).
[3] A. Winter and D. Yang, Phys. Rev. Lett. 116, 120404 (2016).
[4] C. Napoli, T. R. Bromley, M. Cianciaruso, M. Piani, N. Johnston, and G. Adesso, Phys. Rev. Lett. 116, 150502 (2016).
[5] U. Singh, L. Zhang, and A. K. Pati, Phys. Rev. A 93, 032125 (2016).
[6] L. Zhang, Journal of Physics A: Mathematical and Theoretical 50, 155303 (2017).
[7] B. Chen and S.-M. Fei, Quantum Inf. Process. 17, 107 (2018).
[8] K. Bu, U. Singh, S.-M. Fei, A. K. Pati, and J. Wu, Phys. Rev. Lett. 119, 150405 (2017).
[9] C.-s. Yu, Phys. Rev. A 95, 042337 (2017).
[10] Y. Sun, Y. Mao, and S. Luo, EPL (Europhysics Letters) 118, 60007 (2017).
[11] S. Luo and Y. Sun, Phys. Rev. A 96, 022130 (2017).
[12] S. Luo and Y. Sun, Phys. Rev. A 96, 022136 (2017).
[13] C. Xiong, A. Kumar, and J. Wu, Phys. Rev. A 98, 032324 (2018).
[14] J. Xu, Chinese Physics B 29, 10301 (2020).
[15] X.-N. Zhu, Z.-X. Jin, and S.-M. Fei, Quantum Inf. Process. 18, 179 (2019).
[16] A. Strzelczyk, G. Adesso, and M. B. Plenio, Rev. Mod. Phys. 89, 041003 (2017).
[17] M.-L. Hu, X. Hu, J. Wang, Y. Peng, Y.-R. Zhang, and H. Fan, Physics Reports 762-764, 1 (2018).
[18] F. Bischof, H. Kampermann, and D. Bruß, Phys. Rev. Lett. 123, 110402 (2019).
[19] M. Oszmaniec, L. Guerini, P. Wittek, and A. Acín, Phys. Rev. Lett. 119, 190501 (2017).
[20] F. Bischof, H. Kampermann, and D. Bruß, arXiv:1907.08574 (2019).
[21] N. Killoran, F. E. S. Steinhoff, and M. B. Plenio, Phys. Rev. Lett. 116, 080402 (2016).
[22] T. Theurer, N. Killoran, D. Egloff, and M. B. Plenio, Phys. Rev. Lett. 119, 230401 (2017).
[23] J. Åberg, arXiv:quant-ph/0612146 (2006).
[24] A. Peres, *Quantum Theory: Concepts and Methods*, Vol. 57 (Springer Science & Business Media, New York, 2006).
[25] T. Decker, D. Janzing, and M. Rötteler, Journal of Mathematical Physics 46, 012104 (2005).
[26] P. K. Dey, D. Chakraborty, P. Char, I. Chattopadhyay, and D. Sarkar, arXiv:1908.01882 (2019).
[27] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
[28] X.-D. Yu, D.-J. Zhang, G. F. Xu, and D. M. Tong, Phys. Rev. A 94, 060302 (2016).
[29] M.-J. Zhao, T. Ma, Z. Wang, S.-M. Fei, and R. Pereira, Quantum Inf. Process. 19, 104 (2020).
[30] H. Zhao and C.-s. Yu, Scientific Reports 8, 299 (2018).
[31] K. Bu, N. Anand, and U. Singh, Phys. Rev. A 97, 032342 (2018).
[32] A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, and G. Adesso, Phys. Rev. Lett. 115, 020403 (2015).
[33] A. E. Rastegin, Phys. Rev. A 93, 032136 (2016).
[34] A. W. Marshall, I. Olkin, and B. C. Arnold, *Inequalities: Theory of Majorization and Its Applications* (Springer, New York, 2011) p. 340.
[35] D. Pérez-García, M. M. Wolf, D. Petz, and M. B. Ruskai, Journal of Mathematical Physics 47, 083506 (2006).
[36] M. M. Wilde, A. Winter, and D. Yang, Communications in Mathematical Physics 331, 593 (2014).
[37] M. Müller-Lennert, F. Dupuis, O. Szehr, S. Fehr, and M. Tomamichel, Journal of Mathematical Physics 54, 122202 (2013).
[38] S. Beigi, Journal of Mathematical Physics 54, 122202 (2013).
[39] R. L. Frank and E. H. Lieb, Journal of Mathematical Physics 54, 122201 (2013).