I show that there is a unique and well behaved derivative expansion of an effective action at finite temperature. The result is true for all formalisms including the popular Closed Time Path and Imaginary Time methods.

The effective action has been a powerful weapon in the arsenal of quantum field theory for many years [1–4]. In most cases, effective actions are still too difficult to work with. One way of extracting physically useful results is to try a further approximation, the derivative expansion. In particular, for an effective action of a scalar field, the lowest order term of such an expansion is the effective potential or free energy, a foremost tool in the study of the different phases of a theory.

However it has been known for some time [5] that at any non-zero temperature the derivative expansion is not well behaved in that there is no one, unique answer [5–17]. This then destroys the link between the Free energy and the potential that controls the dynamical evolution of a scalar field such as one has many pictures of phase transitions.

Here I will describe how the existing calculations incorrectly use equilibrium Green functions to calculate such effective actions. When considering the behaviour of slowly varying fields in a thermal bath, i.e. the system described by a derivative expansion of an effective action, then one finds that one must use different types of Green function. This is then seen to give a unique derivative expansion of the effective action.

The example I will consider has two fields, \( \phi \) and \( \eta \), whose dynamics is described by the Lagrangian

\[
\mathcal{L}[\phi, \eta] = \frac{1}{2} \eta \Delta^{-1} \eta - \frac{1}{2} g \phi \eta^2 + \mathcal{L}[\phi]
\]

One obtains an effective action for the \( \phi \) field by integrating out the \( \eta \) field as follows

\[
Z = \int D\phi \, D\eta \exp\left(i \int d^4 x \mathcal{L}[\phi, \eta]\right)
\]

The validity of such further approximations, and indeed the usefulness of integrating out one field first rather than another, must always be justified by the nature of the physical problem. For instance if the \( \phi \) field has a mass much less than the mass of the \( \eta \) field then it makes sense to first integrate out the slow fluctuations of the \( \eta \) field. A standard example is the Euler-Heisenberg effective action for electromagnetic fields in a QED plasma where the \( \phi \) field would become the photon field, and the \( \eta \) field the electron field [3,13]. Given that the recipe I use to obtain an effective action for the simple model (1) is often appropriate in genuine physical problems, I will use it on (1) to illustrate generic problems and my solution to them.

From (3), the next step is to expand the \( \ln \) and I find

\[
S_{\text{eff}}[\phi] = \frac{i}{2} \text{Tr} \{ \ln [1 - g(\phi(x)) \Delta(x,y)] \}
\]

\[
S_{\text{eff}}[\phi] = S_{\text{eff}}^{(1)} + S_{\text{eff}}^{(2)} + \ldots
\]

\[
= -\frac{i g}{2} \text{Tr} \{ \phi(x) \Delta(x,y) \}
\]
\[-\frac{ig^2}{4} \text{Tr} \{ \phi(x) \Delta(x, y) \phi(y) \Delta(y, x) \} + \ldots \]  
\[= \Delta(x, x) \Delta(x, y) \Delta(x, y) \]  
\[\phi(x) \phi(y) \phi(x) \phi(y) \]  
\[\text{where the dashed lines represent the } \eta \text{ field propagator, } \Delta. \]  
I will study the second term in this series, \( S_{\text{eff}}^{(2)} \),  
\[S_{\text{eff}}^{(2)} = -\frac{ig^2}{4} \int d^4x \int d^4y \{ \phi(x) \Delta(x, y) \phi(y) \Delta(y, x) \} \]  
as this is the quadratic term in \( \phi \) and so contains corrections to the mass and kinetic terms of \( \phi \).

\( S_{\text{eff}}^{(2)} \) is still too complicated. As classical actions contain one space-time integral, I wish to keep one, say \( \int d^4x, \int d^4y \) of (9). We need therefore to do the second \( \int d^4y \) of (9). However, we see that we need to know \( \phi \) at both \( x \) (considered fixed) and \( y \) points and we are integrating over all \( y \) i.e. the effective action is non-local.

This is too much information to expect. We are more likely to know a lot about the field at one point in time and/or space. It is therefore convenient in many problems to express the field at any point in time or space, \( y \), in terms of the field and its derivatives at one given point \( x \). Thus we use a Taylor expansion to expand \( \phi(y) \) as an infinite series.

\[\phi(y) = \phi(x) + (y - x)^\mu \partial_\mu \phi(x) + \ldots \]  
\[= \exp \{ i(y - x)^\mu p_\mu \} \phi(x) \]  
where \( p_\mu = -i \partial_\mu \). Then \( S_{\text{eff}}^{(2)} \) of (9) is given in terms of an infinite number of \( x \) terms, \( \phi(x), \partial_\mu \phi(x) \) etc., the known \( \eta \) propagator \( \Delta(x - y) \), and a series of simple \( (y - x) \mu \) polynomials. This is the derivative expansion. Taken as a whole nothing is gained, non-locality is replaced by the need to know an infinite number of derivatives. Progress is made by truncating this derivative expansion so that we need specify only a finite number of local terms in a further approximation to our full effective action.

Under the above approximations the result is valid only in certain physical situations. The truncation of the \( \ln \) expansion in (7) means that we are restricted to weak coupling and/or weak fields. The truncation of derivative expansion means that we must consider fields varying slowly in time and space.

Putting this all together gives the expression

\[ S_{\text{eff}}^{(2)} = -\frac{g^2}{4} \int d^4x \phi(x) B(-i \partial_\mu) \phi(x) \]  
\[-iB(p) = \frac{(-ig)^2}{2} \int d^4k \frac{i \Delta(k) \Delta(k + p)}{(2\pi)^4} \]  
\[= \frac{1}{2} \int d^4x \phi(x) B(-i \partial_\mu) \phi(x) \]  
\[= \int d^4x [ B(p = 0) \phi(x)^2 \]  
\[+ \left( \frac{\partial B(p)}{\partial E} \right)_{p=0} \phi(x) \frac{\partial \phi(x)}{\partial \tau} \) \]  
\[+ \frac{1}{2} \left( \frac{\partial^2 B(p)}{\partial E^2} \right)_{p=0} \phi(x) \frac{\partial^2 \phi(x)}{\partial \tau^2} + \ldots \]  
\[ = \frac{g^2}{4} \int d^4x \int d^4y \{ \phi(x) \Delta(x, y) \phi(y) \Delta(y, x) \} \]  
\[\text{i.e. the four-momentum expansion of } B \text{ gives the derivative expansion of } S_{\text{eff}}^{(2)}. \]

The analysis above is standard at zero temperature [1–3]. Formally there seems to be no problem at finite temperature, one can imagine that the same Feynman diagrams are required but one just uses finite temperature rules. If one takes the retard bubble diagram, \( B = B_{\text{ret}} \), then all thermal field formalisms give

\[B_{\text{ret}}(p_\mu = (E, \vec{p})) = -\frac{g^2}{2} \int d^4k \frac{(2\pi)^3}{4\omega \Omega} \sum_{s_0, s_1 = \pm 1} \frac{s_0 s_1}{E + s_0 \omega + s_1 \Omega (e^{\beta(s_0 \omega + s_1 \Omega)} - 1)} \]  
\[= \frac{1}{E + s_0 \omega + s_1 \Omega (e^{\beta(s_0 \omega + s_1 \Omega)} - 1)(e^{\beta s_1 \Omega} - 1)} \]  
where
\[\omega = |\vec{k}^2 + m^2|^{1/2}, \quad \Omega = |(\vec{k} + \vec{p})^2 + m^2|^{1/2}. \]

However such equilibrium \( B \)'s do not have a unique momentum expansion at any non-zero temperature! This has been known for a long time, e.g. see Abrahams and Tsumeto [5]. I learned this from Fujimoto [7]. A careful calculation for the scalar retarded bubble gives [8,9,19]

\[\lim_{E, \vec{p} \to 0} B_{\text{ret}}(E, \vec{p}) = (T=0 \text{ part}) \]  
\[-\frac{g^2}{8\pi^2} \int_{m}^{\infty} d\omega N(\omega) \frac{k}{\omega^2 + v^2 \gamma^2 m^2} \]  
\[\simeq -\frac{g^2}{16\pi} \frac{1}{1 + \gamma m} \]  
where
\[N(E) = [\exp(\beta E) - 1]^{-1} \]
\[v = \lim_{E, \vec{p} \to 0} \frac{|\vec{p}|}{E} \]
\[\gamma = (1 - v^2)^{-1/2} \]

Only in the case \( v = \infty \) (\( E \to 0 \) before \( \vec{p} \to 0 \)) is the usual contribution to the effective potential or Free energy is recovered.

The solution comes when we realise that we may not use just the finite temperature Feynman rules in the zero temperature diagrams to obtain a finite temperature result in a general calculation including derivative
expansions of effective actions. Apart from anything else, a Feynman diagram does not represent a unique function at non-zero temperature there are many distinct types with significantly different values (retarded, time-ordered, thermal Wightman, \ldots) all of which all have their roles in different physical problems \cite{20}. One must always consider what the physical problem is at finite temperature and let that tell you which mathematical Green function can be used to extract the relevant physics.

In the case of derivative expansions of effective actions, everyone uses the equilibrium Green functions where all fields are assumed to be in equilibrium and periodic. Usually retarded Green functions are used \cite{11} though this is not always the case \cite{15,17}. The question is why are we using equilibrium Green functions? A truncated derivative expansion of a periodic field is not periodic. Such approximations to field $\phi(y)$ are not equilibrium fields.

$$\phi(\tau) = \phi(0) + \tau \phi'(0) + \ldots$$

periodic constant, linear in $\text{Im}(\tau)$ periodic not periodic

So you MUST NOT assume that external legs represent equilibrium fields!

The Green functions relevant to derivative expansions of effective actions are not generally equilibrium ones. Of course! We always wanted to look at how slowly varying fields (i.e. ones which are clearly not in equilibrium) behave in an equilibrium background - $\eta$ is in equilibrium, $\phi$ is not.

So let us repeat calculation but

- We will work with time not energy in the Feynman integrals. This means that we can do all path-ordered thermal field formalisms easily.

- The $\eta$ field is in equilibrium and I use the following form of the propagator \cite{18} valid for all path-ordered thermal field theory formalisms

$$\Delta(\tau, \tau'; \vec{k}) = \frac{-i}{2\pi} \int_{-\infty}^{\infty} dE \ e^{-iE(\tau-\tau')} \left[ \theta_C(\tau, \tau') + N(E) \right] \rho(E, \omega)$$

where $\theta_C$ is a contour theta function, $N(E)$ is the Bose-Einstein distribution \cite{20}, and $\rho$ is the spectral function. For the free scalar field $\eta$, $\rho$ is of the form $(\pi/\omega) [\delta(E - \omega) - \delta(E + \omega)]$ with $\omega$ given in \cite{17}.

- The $\phi$ field is not assumed to be periodic or in equilibrium so $\text{exp}\{i\beta E\} \neq 1$ etc. when $E$ is the energy or time derivative associated with a $\phi$ field. I only use \cite{11} when doing the $\int dy$.

This gives

$$S_{\text{eff}}^{(2)} = \frac{-ig^2}{4} \text{Tr} \{ \phi(x) \Delta(x, y) \phi(y) \Delta(y, x) \}$$

$$= \Delta(x, y) \int \Delta(x, y) \delta \phi(y) \phi(y)$$

$$= \frac{1}{2} \int_{\tau_1}^{\tau_2} d\tau \int d^3 \vec{p} \ (2\pi)^3 \phi(\tau, \vec{p}) B(E, p; \tau - \tau_1) \phi(\tau, -\vec{p})$$

$$B(E, \vec{p}; \tau - \tau_1) = \frac{g^2}{2} \int d^3 \vec{k} \sum_{s_0, s_1 = \pm 1} N(s_0 \omega) N(s_1 \Omega) (F + G)$$

$$\neq B_{\text{ret}}$$

where $E = -i \frac{\partial}{\partial \tau}$ and acts only on the right hand side $\phi$ field. The $\tau_1$ is the start of the path in complex time used to define path-ordered thermal field formalisms \cite{18}. It is the time at which the density matrix is given in closed-time path methods for instance. Since the derivative expansion requires an expansion of $B$ about $E, \vec{p} \rightarrow 0$ then we are interested in the $\Omega \rightarrow \omega$ limit. When $s_0 = -s_1$ in the sum in \cite{25} then $A \rightarrow 0$ and in this limit the denominators of $F$ and $G$ are zero. These $s_0 = -s_1, \omega - \Omega$ terms are produced by Landau damping processes.

This new result based on $B$ has several consequences:

1. $B$ gives unique derivative series. When we take $E, p \rightarrow 0$ then the Landau damping terms have $A \rightarrow 0$. However both $F$ and $G$ are well behaved in this limit because of the form of the numerators.

2. In contrast, the equilibrium $B_{\text{ret}}$ \cite{16} is equivalent to setting $\exp\{\beta E\} = 1$ in $B$ \cite{25}, so that in $B_{\text{ret}}$ $\exp\{A\} = \exp\{\pm \omega \pm \Omega\}$. Thus when we look at the Landau damping terms in $B_{\text{ret}}$ where $A \rightarrow 0$ we find that $G \rightarrow 0$, but $F \sim A^{-1}$. This divergence in the integrand (there is still an integration over loop momenta to be done) leads to the poor behaviour of the four-momentum expansion of $B_{\text{ret}}$.

3. The only point where the correct $B$ and old retarded equilibrium $B_{\text{ret}}$ agree is in the static limit

\cite{1}One must also manipulate the pure $\omega$ dependent half of the integral in $B$, switching variables to $\vec{k} + \vec{p}$ and integrating over this to get the form \cite{16}.
I have assumed that the classical $M$ when using its of equilibrium Green functions, yet there is no problem terms which cause problems in the zero momentum limit. It includes contributions from Landau Damping this is unique and independent of the order in which $B$. The same result emerges in both cases as it must. However some extensions of this work and Ray Rivers for continu-

We can now write out the derivative expansion. Limiting myself to first order in derivatives then I find that for any path-ordered thermal field formalism the effective quadratic part of the effective Lagrangian $\mathcal{L}$ is

$$\mathcal{L}_{\text{eff}}[\varphi] = (M^2 + \delta M^2)\varphi^2(\tau, \vec{x}) - i\beta \left( \frac{1}{2} \delta M^2 + \mu(\tau - \tau_i) \right) \varphi(\tau, \vec{x}) \frac{\partial}{\partial \tau} \varphi(\tau, \vec{x}) + \ldots$$

$$\delta M^2 = -g^2 \frac{N(\omega)}{8\pi^2} \int_0^{\infty} dk \frac{N^2(\omega)}{\omega^3} \left[ -2i\tau \omega e^{i\omega} + 1 \right]$$

$$\mu(\tau) = \frac{g^2}{32\pi^2} \int_0^{\infty} dk \frac{N^2(\omega)}{\omega^3} \left[ -2i\tau \omega e^{i\omega} + 1 \right]$$

I have assumed that the classical $\mathcal{L}[\varphi]$ of (1) gives an $M^2\varphi^2$ contribution to this order. The key point is that this is unique and independent of the order in which we take the zero energy, momentum/time, space derivative limit. It includes contributions from Landau Damping terms which cause problems in the zero momentum limits of equilibrium Green functions, yet there is no problem when using $B$. In fact equilibrium retarded Green functions such as $B_{\text{ret}}$ describe the response to a system to a delta function impulse, as this is what the usual linear response analysis tells us [18]. Thus such Green functions are relevant only for sudden changes. The derivative expansion of an effective action and $B$ are appropriate for slowly varying fields, a very different type of physical problem.

This analysis of the derivative expansion of an effective action for scalar fields has used a general path ordered approach to thermal field theory using the two scalar field model of (1). The same model has also been considered in detail for the pure imaginary-time formalism [21] and in the closed-time path method [22,23]. The same result emerges in both cases as it must. However the result is viewed very differently in the two distinct approaches. Other more sophisticated models also have been considered - scalars or an abelian gauge field coupled to fermions in a heat bath, models with chemical potentials, and in effective actions for weakly coupled superconductors [24] - with similar success.

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