The noncommutative residue and canonical trace in the light of Stokes’ and continuity properties

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Abstract

We show that the noncommutative residue density, resp. the cut-off regularised integral are the only closed linear, resp. continuous closed linear forms on certain classes of symbols. This leads to alternative proofs of the uniqueness of the noncommutative residue, resp. the canonical trace as linear, resp. continuous linear forms on certain classes of classical pseudodifferential operators which vanish on brackets.

The uniqueness of the canonical trace actually holds on classes of classical pseudodifferential with vanishing residue density which include non integer order operators in all dimensions and odd-class (resp. even-class) operators in odd (resp. even) dimensions. The description of the canonical trace for non integer order operators as an integrated global density on the manifold is extended to odd-class (resp. even-class) operators in odd (resp. even) dimensions on the grounds of defect formulae for regularised traces of classical pseudodifferential operators.

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Introduction

The uniqueness of the noncommutative residue originally introduced by Adler and Manin in the one dimensional case was then extended to all dimensions by Wodzicki in [W1] (see also [W2] and [K] for a review) and proved independently by Guillemin [G2]. Since then other proofs, in particular a homological proof on symbols in [BG] and various extensions of this uniqueness result were derived, see [FCLS] for a generalisation to manifolds with boundary, see [S] for a generalisation to manifolds with conical singularities (both of which prove uniqueness up to smoothing operators), see [L] for an extension to log-polyhomogeneous operators as well as for an argument due to Wodzicki to get uniqueness on the whole algebra of classical operators, see [Po2] for an extension to Heisenberg manifolds.

In contrast to the familiar characterisation of the noncommutative residue as the unique trace on the algebra of all classical pseudodifferential operators, only recently was the focus drawn on a characterisation of the canonical trace as the unique linear extension of the ordinary trace to non integer order classical pseudodifferential operators which vanishes on non integer order brackets.

We revisit and slightly improve these results handling the noncommutative residue and the canonical trace on an equal footing via a characterisation of closed linear forms on certain classes of symbols.

\[1\] The authors of [MSS] actually extended the uniqueness to odd-class, resp. even-class operators in odd, resp. even dimensions.
A cornerstone in our approach is the requirement that a linear form satisfies Stokes’ property (or be closed in the language of noncommutative geometry) on a certain class of symbols i.e. that it vanishes on partial derivatives in that class. The vanishing on derivatives is a natural requirement in view of the fact that any distribution on $\mathbb{R}^n$ with vanishing derivatives is proportional to the ordinary integral on $\mathbb{R}^n$; it serves here to characterise its unique closed extension given by the cut-off regularised integral $\int_{\mathbb{R}^n}$ defined by Hadamard finite parts. This leads to a characterisation of the noncommutative residue on symbols (Theorem 1) on the one hand and the cut-off regularised integral on symbols (Theorem 2) on the other hand.

The link between the vanishing on brackets of a linear functional on classical operators and the vanishing on partial derivatives of a linear functional on symbols can best be seen from the simple formula

\[ \text{Theorem 2} \]

The vanishing on derivatives is a natural requirement in the context of the fact that any distribution on $\mathbb{R}$ vanishes on partial derivatives in that class. The vanishing on derivatives is a natural requirement in Kontsevich and Vishik’s [KV] work (see also [CM]) and coincides up to a smoothing symbol with a sum of derivatives of symbols whose order is 1+ the order of the original symbol. This is why we then consider classes of operators with vanishing residue density in order to carry out the linear extensions.

The vanishing of the residue density which therefore plays a crucial role for uniqueness issues, arises once more for existence issues. This indeed turns out to be an essential ingredient in section 2, where we show that the canonical trace is well defined as an integrated global density on certain classes of classical pseudodifferential operators, such as odd-class operators in odd dimensions and even-class operators in even dimensions. To do so, we approximate the operator under study along a holomorphic path of classical operators and use a defect formula for regularised traces derived in [PS].

This is carried out along of a line of thought underlying Guillemin’s [G2], Wodzicki’s [W2] and later Kontsevich and Vishik’s [KV] work (see also [CM]); a classical $\Psi DO$ $A$ is embedded in a holomorphic family $z \mapsto A(z)$ with $A(0) = A$, the canonical trace of which yields a meromorphic map $z \mapsto \text{TR}(A(z))$. These authors focus on the important case of $\zeta$-regularisation $A^\zeta(z) = A Q^{-\zeta}$ built from some admissible elliptic operator $Q$ with positive order. In particular, if $A$ has non integer order then $z \mapsto \text{TR}(A^\zeta(z))$ is meromorphic at $z = 0$, the canonical trace density of $A$ is globally defined and integrates over $M$ to the canonical trace $\text{TR}(A)$ of $A$ which coincides with $\lim_{z \to 0} \text{TR}(A^\zeta(z))$ independently of the choice of $Q$.

Similar continuity results hold for odd-class (resp. even-class) operators $A$ in odd (resp. even) dimensions; it was observed in [KV] (resp. [Gr]) that for an odd-class elliptic operator $Q$ with even positive order close enough to a positive self-adjoint operator, and $A$ and odd-(resp. even-) class operator in odd (resp. even) dimensions, the map $z \mapsto \text{TR}(A^\zeta(z))$ is holomorphic at $z = 0$ and $\text{Tr}_{(-1)}(A) := \lim_{z \to 0} \text{TR}(A^\zeta(z))$ is independent of the choice of $Q$.

As a straightforward application of defect formulae both on the symbol and the operator level derived in [PS], we extend these results to any holomorphic family $A(z)$ with non constant affine order such that $A = A(0)$ and $A'(0)$ lie in the odd- (resp. even-) class. We infer from there that in odd (resp. even) dimensions

1. the map $z \mapsto \text{TR}(A(z))$ is holomorphic at $z = 0$,
2. the canonical trace density is globally defined for any odd- (resp. even-) class operator $A$, and integrates over $M$ to the canonical trace $\text{TR}(A)$,
3. $\text{TR}(A) = \lim_{z \to 0} \text{TR}(A(z))$ independently of any appropriate (see above initial conditions) choice of the family the family $A(z)$.

This shows in particular that both Kontsevich and Vishik’s (resp. Grubb’s) extended trace $\text{Tr}_{(-1)}$ on odd- (resp. even-) class operators in odd (resp. even) dimensions and the symmetrised trace $\text{Tr}^{\text{sym}}$...
introduced by Braverman in [B] on odd-class operators in odd dimensions coincide with the canonical trace TR.

To sum up, the characterisation we provide of the noncommutative residue and of the canonical trace on the grounds of a characterisation of closed linear forms on certain classes of symbols sheds light on common mechanisms that underly their uniqueness. It brings out the importance of the closedness requirement on the underlying functionals on symbols, which was already implicit in the homological proofs of the uniqueness of the residue. In the case of the canonical trace it further puts forward the role of the vanishing of the residue on the symbol level and of the residue density on the operator level which also turns out to be an essential ingredient for existence issues.

The paper is organised as follows:

1. **Uniqueness: characterisation of closed linear forms on symbols**
   (a) Notations
   (b) Stokes’ property versus translation invariance
   (c) A characterisation of the noncommutative residue and its kernel
   (d) A characterisation of the cut-off regularised integral \( \int_{\mathbb{R}^n} \) in terms of Stokes’ property

2. **Existence: The canonical trace on odd- (resp. even-) class operators in odd (resp. even) dimensions**
   (a) Notations
   (b) Classical symbol valued forms on an open subset
   (c) The noncommutative residue on classical pseudodifferential operators
   (d) The canonical trace on non integer order operators
   (e) Holomorphic families of classical pseudodifferential operators
   (f) Continuity of the canonical trace on non integer order pseudodifferential operators
   (g) Odd-class (resp. even-class) operators embedded in holomorphic families
   (h) The canonical trace on odd- (resp. even-) class operators in odd (resp. even) dimensions

3. **Uniqueness: Characterisation of linear forms on operators that vanish on brackets**
   (a) Uniqueness of the noncommutative residue
   (b) Uniqueness of the canonical trace
1 Uniqueness: Characterisation of closed linear forms on symbols

We characterise the noncommutative residue and the cut-off regularised integral in terms of a closedness requirement on linear forms on classes of classical symbols with constant coefficients on $\mathbb{R}^n$.

1.1 Notations

We only give a few definitions and refer the reader to [Sh, T, Tr] for further details on classical pseudodifferential operators.

For any complex number $a$, let us denote by $S^a_{c.c}(\mathbb{R}^n)$ the set of smooth functions on $\mathbb{R}^n$ called symbols with constant coefficients, such that for any multiindex $\beta \in \mathbb{N}^n$ there is a constant $C(\beta)$ satisfying the following requirement:

$$|\partial^\beta \sigma(\xi)| \leq C(\beta)(1 + |\xi|)^{\text{Re}(a) - |\beta|}$$

where $\text{Re}(a)$ stands for the real part of $a$, $|\xi|$ for the euclidean norm of $\xi$. We single out the subset $CS^a_{c.c}(\mathbb{R}^n) \subset S^a_{c.c}(\mathbb{R}^n)$ of symbols $\sigma$, called classical symbols of order $a$ with constant coefficients, such that

$$\sigma(\xi) = \sum_{j=0}^{N-1} \chi(\xi) \sigma_{a-j}(\xi) + \sigma_N(\xi)$$

(1)

where $\sigma_N \in S^{a-N}_{c.c}(\mathbb{R}^n)$ and where $\chi$ is a smooth cut-off function which vanishes in a small ball of $\mathbb{R}^n$ centered at 0 and which is constant equal to 1 outside the unit ball. Here $\sigma_{a-j}$, $j \in \mathbb{N}_0$ are positively homogeneous of degree $a - j$.

The ordinary product of functions sends $CS^a_{c.c}(\mathbb{R}^n) \times CS^b_{c.c}(\mathbb{R}^n)$ to $CS^{a+b}_{c.c}(\mathbb{R}^n)$ provided $b - a \in \mathbb{Z}$; let

$$CS_{c.c}(\mathbb{R}^n) = \bigcup_{a \in \mathbb{C}} CS^a_{c.c}(\mathbb{R}^n)$$

denote the algebra generated by all classical symbols with constant coefficients on $\mathbb{R}^n$. Let

$$CS^{-\infty}_{c.c}(\mathbb{R}^n) = \bigcap_{a \in \mathbb{C}} CS^a_{c.c}(\mathbb{R}^n)$$

be the algebra of smoothing symbols. We write $\sigma \sim \sigma'$ for two symbols $\sigma, \sigma'$ which differ by a smoothing symbol.

We also denote by $CS^{<p}_{c.c}(\mathbb{R}^n) := \bigcup_{\text{Re}(a) < p} CS^a_{c.c}(\mathbb{R}^n)$, the set of classical symbols of order with real part $< p$ and by $CS^{\leq p}_{c.c}(\mathbb{R}^n) := \bigcup_{a \in \mathbb{C}} CS^a_{c.c}(\mathbb{R}^n)$ the set of non integer order symbols.

We equip the set $CS_{c.c}(\mathbb{R}^n)$ of classical symbols of order $a$ with a Fréchet structure with the help of the following semi-norms labelled by multiindices $\beta$ and integers $j \geq 0$, $N$ (see [H]):

$$\sup_{\xi \in \mathbb{R}^n} (1 + |\xi|)^{-\text{Re}(a) + |\beta|} \|\partial^\beta \sigma(\xi)\|;$$

$$\sup_{\xi \in \mathbb{R}^n} (1 + |\xi|)^{-\text{Re}(a) + N + |\beta|} \|\partial^\beta (\sum_{j=0}^{N-1} \chi(\xi) \sigma_{a-j})(\xi)\|;$$

$$\sup_{|\xi| = 1} \|\partial^\beta \sigma_{a-j}(\xi)\|.$$

$CS_{c.c}^{-\infty}(\mathbb{R}^n)$ is equipped with the natural induced topology so that a linear $\rho$ which extends to continuous linear maps $\rho_a$ on $CS^a_{c.c}(\mathbb{R}^n)$ for any $a \in \mathbb{Z} \cap (-\infty, -K]$ (with $K$ some arbitrary positive number) is continuous.

We borrow from [MMP] (see also [LI]) the notion of $\Psi DO$ -valued form.

Definition 1 Let $k$ be a non negative integer, $a$ a complex number. We let

$$\Omega^k CS^a_{c.c}(\mathbb{R}^n) = \{\alpha \in \Omega^k(\mathbb{R}^n), \sum_{I \subset \{1, \ldots, n\}, |I| = k} \alpha_I(\xi) d\xi_I \}$$

with $\alpha_I \in CS^{a-|I|}_{c.c}(\mathbb{R}^n)$
denote the set of order a classical symbol valued forms on $\mathbb{R}^n$ with constant coefficients. Let

$$\Omega^k CS_{c,c}(\mathbb{R}^n) = \{ \alpha \in \Omega^k(\mathbb{R}^n), \quad \alpha = \sum_{I \subset \{1, \ldots, n\}, |J|=k} \alpha_I(\xi) \, d\xi_I \quad \text{with} \quad \alpha_I \in CS_{c,c}(\mathbb{R}^n) \}$$

denote the set of classical symbol valued k-forms on $\mathbb{R}^n$ of all orders with constant coefficients.

The exterior product on forms induces a product $\Omega^k CS_{c,c}(\mathbb{R}^n) \times \Omega^l CS_{c,c}(\mathbb{R}^n) \to \Omega^{k+l} CS_{c,c}(\mathbb{R}^n)$; let

$$\Omega CS_{c,c}(\mathbb{R}^n) := \bigoplus_{k=0}^{\infty} \Omega^k CS_{c,c}(\mathbb{R}^n)$$

stand for the $\mathbb{N}_0$ graded algebra (also filtered by the symbol order) of classical symbol valued forms on $\mathbb{R}^n$ with constant coefficients.

We also consider the sets $\Omega^k CS_{c,c}^\mathbb{Z}(\mathbb{R}^n) := \bigcup_{a \in \mathbb{Z}} \Omega^k CS_{c,c}^a(\mathbb{R}^n)$ of integer order classical symbols valued k-forms and $\Omega^k CS_{c,c}^\mathbb{Z}(U) := \bigcup_{a \in \mathbb{Z}} \Omega^k CS_{c,c}^a(U)$ of non integer order classical symbol valued k-forms. Clearly, $\Omega CS_{c,c}^\mathbb{Z}(\mathbb{R}^n) := \bigoplus_{k=0}^{\infty} \Omega^k CS_{c,c}^\mathbb{Z}(\mathbb{R}^n)$ is a subalgebra of $\Omega CS_{c,c}(\mathbb{R}^n)$.

**Definition 2** Let $\mathcal{S} \subset CS_{c,c}(\mathbb{R}^n)$ be a set containing smoothing symbols. We call a linear form $^2\rho : \mathcal{S} \to \mathbb{C}$ singular if it vanishes on smoothing symbols, and regular otherwise.

A linear form $\rho : \mathcal{S} \to \mathbb{C}$ extends to a linear form $\tilde{\rho} : \Omega \mathcal{S} \to \mathbb{C}$ defined by

$$\tilde{\rho}(\alpha(\xi) \, d\xi_{i_1} \wedge \cdots \wedge d\xi_{i_k}) := \rho(\alpha) \, \delta_{k-n},$$

with $i_1 < \cdots < i_k$. Here we have set

$$\Omega^k \mathcal{S} := \{ \sum_{|I|=k} \alpha_I(\xi) \, d\xi_I, \quad \alpha_I \in \mathcal{S} \}.$$

Exterior differentiation on forms extends to symbol valued forms (see (5.14) in [LP]):

$$d : \Omega^k CS_{c,c}(\mathbb{R}^n) \to \Omega^{k+1} CS_{c,c}(\mathbb{R}^n)$$

$$\alpha(\xi) \, d\xi_{i_1} \wedge \cdots \wedge d\xi_{i_k} \mapsto \sum_{i=1}^{n} \partial_i \alpha(\xi) \, d\xi_{i_1} \wedge d\xi_{i_1} \wedge \cdots \wedge d\xi_{i_k}.$$

We call a symbol valued form $\alpha$ closed if $d\alpha = 0$ and exact if $\alpha = d\beta$ where $\beta$ is a symbol valued form; this gives rise to the following cohomology groups

$$H^k CS_{c,c}(\mathbb{R}^n) := \{ \alpha \in \Omega^k CS_{c,c}(\mathbb{R}^n), \quad d\alpha = 0 \} / \{ d\beta, \beta \in \Omega^{k-1} CS_{c,c}(\mathbb{R}^n) \}.$$

We call a symbol valued form $\alpha$ closed “up to a smoothing symbol” if $d\alpha \sim 0$ and exact “up to a smoothing symbol” if $\alpha \sim d\beta$ where $\beta$ is a symbol valued form. Since $\alpha \sim d\beta \Rightarrow d\alpha \sim 0$, this gives rise to the following cohomology groups

$$H^k_{\sim} CS_{c,c}(\mathbb{R}^n) := \{ \alpha \in \Omega^k CS_{c,c}(\mathbb{R}^n), \quad d\alpha \sim 0 \} / \{ \alpha \sim d\beta, \beta \in \Omega^{k-1} CS_{c,c}(\mathbb{R}^n) \}.$$

The next two paragraphs are dedicated to the description of the set of top degree forms which are exact “up to smoothing operators” (see Corollary [1]). The uniqueness of the residue as a closed singular linear form on the algebra of symbols then follows (see Theorem [1]).

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$^2$By linear we mean that $\rho(\alpha_1 \sigma_1 + \alpha_2 \sigma_2) = \alpha_1 \rho(\sigma_1) + \alpha_2 \rho(\sigma_2)$ whenever $\sigma_1, \sigma_2, \alpha_1, \alpha_2$ lie in $\mathcal{S}$. 

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1.2 Stokes’ property versus translation invariance

**Lemma 1** Let \( \rho : S \subset CS_{c,c}(\mathbb{R}^n) \to \mathbb{C} \) be a linear form. The following two conditions are equivalent:

\[
\exists j \in \{1, \cdots, n\} \quad \text{s.t.} \quad \sigma = \partial_j \tau \in S \quad \implies \quad \rho(\sigma) = 0 \quad \alpha = d\beta \in \Omega^n S \quad \implies \quad \tilde{\rho}(\alpha) = 0.
\]

**Proof:** We first show that the second condition follows from the first one. Since \( \tilde{\rho} \) vanishes on forms of degree \( < n \) we can assume that \( \alpha \) is a homogeneous form of degree \( n \) and show that the first condition implies that \( \tilde{\rho}(\alpha) = 0 \). Write \( \alpha = d(\sum |j| = n-1 \beta_j d\xi_{i_j} \wedge \cdots \wedge d\xi_{i_1}) = \sum_{i=1}^{n} \sum_{|j| = n-1} \partial_i \beta_j d\xi_i \wedge d\xi_{i_1} \wedge \cdots \wedge d\xi_{i_{n-1}} \) then \( \tilde{\rho}(\alpha) = \sum_{i=1}^{n} \sum_{|j| = n-1} \rho(\partial_i \beta_j) \) vanishes by the first condition.

Conversely, if \( \sigma = \partial_i \tau \) then

\[
\alpha = \sigma(\xi) d\xi_1 \wedge \cdots \wedge d\xi_n = \partial_i \tau(\xi) d\xi_1 \wedge \cdots \wedge d\xi_n = (-1)^{i-1} d\left( \tau(\xi) d\xi_1 \wedge \cdots \wedge d\xi_{i-1} \wedge \hat{d}\xi_i \wedge d\xi_{i+1} \wedge \cdots \wedge d\xi_n \right) = d\left( (-1)^{i-1} \tau(\xi) d\xi_1 \wedge \cdots \wedge d\xi_{i-1} \wedge \hat{d}\xi_i \wedge d\xi_{i+1} \wedge \cdots \wedge d\xi_n \right)
\]

is an exact form \( \alpha = d\beta_i \) where we have set \( \beta_i := (-1)^{i-1} \tau_i(\xi) d\xi_1 \wedge \cdots \wedge d\xi_{i-1} \wedge d\xi_{i+1} \wedge \cdots \wedge d\xi_n \). If the second condition is satisfied then \( \tilde{\rho} \circ d(\beta_i) = 0 \) from which the first condition \( \rho \circ \partial_i(\tau) = 0 \) follows. \( \square \)

Following the terminology used in noncommutative geometry, we set the following definitions.

**Definition 3** A linear form \( \tilde{\rho} : \Omega S \subset \Omega CS_{c,c}(\mathbb{R}^n) \to \mathbb{C} \) is closed when it satisfies the equivalent conditions of Lemma 1. We also say by extension that \( \rho \) is closed if \( \tilde{\rho} \) is or with the analogy with the ordinary integral in mind, that \( \rho \) satisfies Stokes’ property.

**Remark 1**

1. A closed linear form \( \tilde{\rho} : \Omega CS_{c,c}(\mathbb{R}^n) \to \mathbb{C} \) clearly induces a linear form \( H^\bullet CS_{c,c}(\mathbb{R}^n) \to \mathbb{C} \).

2. When \( \rho \) is singular, closedness of \( \tilde{\rho} \) is equivalent to the fact that

\[
\alpha \sim d\beta \implies \tilde{\rho}(\alpha) = 0.
\]

A closed singular linear form therefore induces a linear form \( H^\bullet CS_{c,c}(\mathbb{R}^n) \to \mathbb{C} \).

Closedness turns out to be equivalent to translation invariance for any linear map on classical symbols which fulfills Stokes’ property on symbols of negative enough order. This extends results of [MMP].

**Proposition 1** Let \( S \subset CS_{c,c}(\mathbb{R}^n) \) be a set stable under translations and derivatives such that there is some positive integer \( K \)

\[
CS_{c,c}^{< -K}(\mathbb{R}^n) \subset S.
\]

Let \( \rho : S \to \mathbb{C} \) be a linear map with the Stokes’ property on \( CS_{c,c}^{< -K}(\mathbb{R}^n) \) i.e.

\[
\exists j \in \{1, \cdots, n\} \quad \text{s.t.} \quad \sigma = \partial_j \tau \in S \cap CS_{c,c}^{< -K}(\mathbb{R}^n) \implies \rho(\sigma) = 0.
\]

Then for any \( \sigma \in S \) we have

\[
\rho(\partial_j \sigma) = 0 \quad \forall j \in \{1, \cdots, n\} \quad \text{(closedness condition)}
\]

\[
\iff \rho(t^*_\eta \sigma) = \rho(\sigma) \quad \forall \eta \in \mathbb{R}^n \quad \text{(translation invariance)}.
\]

**Proof:** The proof borrows ideas from [MMP].

Let \( \sigma \in CS_{c,c}(\mathbb{R}^n) \) and let us write a Taylor expansion of the map \( t^*_\eta \sigma \) in a neighborhood of \( \eta = 0 \). There is some \( \theta \in [0, 1] \) such that

\[
t^*_\eta \sigma = \sum_{|\alpha| = 0}^{N-1} \partial^\alpha \sigma \frac{\theta^\alpha}{\alpha!} + \sum_{|\alpha| = N} \partial^\alpha t^*_\eta(\sigma) \frac{\theta^\alpha}{\alpha!}.
\]

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Choosing $N$ large enough for $\partial^n t_{\theta n}(\sigma)$ to be of order $<-n$, it follows from the linearity of $\rho$ that

$$\rho(t^*_\eta \sigma) = \sum_{|\alpha|=0}^{N-1} \rho(\partial^\alpha \sigma) \frac{\eta^\alpha}{\alpha!} + \sum_{|\alpha|=N} \rho(\partial^n t_{\theta n}(\sigma)) \frac{\eta^\alpha}{\alpha!}$$

$$= \sum_{|\alpha|=0}^{N-1} \rho(\partial^\alpha \sigma) \frac{\eta^\alpha}{\alpha!} + \sum_{|\alpha|=N} \rho(\partial_j \partial^\beta t_{\theta n}(\sigma)) \frac{\eta^\alpha}{\alpha!}$$

$$= \sum_{|\alpha|=0}^{N-1} \rho(\partial^\alpha \sigma) \frac{\eta^\alpha}{\alpha!}$$

so that

$$\rho(t^*_\eta \sigma) - \rho(\sigma) = \sum_{|\alpha|=1}^{N-1} \rho(\partial^\alpha \sigma) \frac{\eta^\alpha}{\alpha!}$$

from which the result follows.

Here we set $\partial^n = \partial_j \circ \partial^\beta$ for some multiindex $\beta$ whenever $|\alpha| \neq 0$ and, choosing $N$ large enough so that the remainder term is of order $<-K$ we used the assumption that $\rho$ verifies Stokes’ property on $CS_{c,c}^{-K}(\mathbb{R}^n)$.

### 1.3 A characterisation of the noncommutative residue and its kernel

We show that the noncommutative residue is the unique singular linear form on classical symbols on $\mathbb{R}^n$ with constant coefficients which fulfills Stokes’ property. This is based on results of [FGLS] and [GL] (see also [L] for a generalisation to logarithmic powers) as well as results of [MMP].

We henceforth and throughout the paper assume that the dimension $n$ is larger or equal two.

**Definition 4** The noncommutative residue is a linear form on $CS_{c,c}(\mathbb{R}^n)$ defined by

$$\text{res}(\sigma) := \frac{1}{\sqrt{2\pi}} \int_{S^{n-1}} \sigma_{-n}(\xi) d\mu_S(\xi)$$

where

$$d\mu_S(\xi) := \sum_{j=1}^n (-1)^{j-1} \xi_j \eta_1 \wedge \cdots \wedge \eta_{j-1} \wedge d\xi_{j} \wedge \cdots \wedge d\xi_n$$

denotes the volume measure on $S^{n-1}$ induced by the canonical measure on $\mathbb{R}^n$.

The noncommutative residue fulfills Stokes’ property.

**Proposition 2** [MMP] (see also [LT]) The noncommutative residue vanishes on symbols which are partial derivatives in $CS_{c,c}(\mathbb{R}^n)$ up to some smoothing operator:

$$\sigma \sim \partial_i \tau \implies \text{res} \circ \sigma = 0 \quad \forall i = 1, \ldots, n, \quad \forall \tau \in CS_{c,c}(\mathbb{R}^n).$$

Equivalently, its extension $\widetilde{\text{res}}$ to classical symbol valued forms on $\mathbb{R}^n$ is closed.

**Proof**: Assume that $\sigma \sim \partial_i \tau$. Since res vanishes on smoothing symbols, we can assume that $\sigma = \partial_i \tau$ for some $\tau \in CS_{c,c}(\mathbb{R}^n)$ then $\sigma_{-n} = \partial_i \tau_{-n+1}$.

We have $d\mu_S(\xi) = \iota_X(\Omega)(\xi)$ where $\Omega(\xi) := \eta_1 \wedge \cdots \wedge d\xi_n$ is the volume form on $\mathbb{R}^n$ and $X := \sum_{i=1}^n \xi_i \partial_i$ is the Liouville field on $\mathbb{R}^n$. Since the map $\xi \mapsto \sigma_{-n+1}(\xi) \eta_1 \wedge \cdots \wedge \eta_{i-1} \wedge d\xi_{i} \wedge \cdots \wedge d\xi_n$ (where $d\xi_i$ means we have omitted the variable $\xi_i$) is invariant under $\xi \mapsto t\xi$ for any $t > 0$, we have $\mathcal{L}_X(\sigma_{-n+1}(\xi) \eta_1 \wedge$
\[ \cdots \wedge \hat{d}_i \wedge \cdots \wedge d_{n} = 0. \]

Using Cartan’s formula \( \mathcal{L}_X = d \circ \iota_X + \iota_X \circ d \) we write

\[
\text{res}(\sigma) = \int_{S^{n-1}} \sigma_{-n}(\xi) \iota_X(\Omega)(\xi) \\
= \int_{S^{n-1}} \iota_X(\partial_{\tau_{n+1}}(\xi) \Omega)(\xi) \\
= (-1)^{n-1} \int_{S^{n-1}} \iota_X \circ d(\tau_{n+1}(\xi)) d\xi_1 \wedge \cdots \wedge d\xi_n(\xi) \\
= (-1)^{n-1} \int_{S^{n-1}} d \circ \iota_X(\tau_{n+1}(\xi)) d\xi_1 \wedge \cdots \wedge d\xi_n(\xi) \\
= 0,
\]

where we have used the ordinary Stokes’ formula in the last equality. \( \square \)

The description of homogeneous symbols as sum of partial derivatives induces a similar description “up to smoothing symbols” for all classical symbols with vanishing residue. The following elementary result is very useful for that purpose.

**Lemma 2** (Euler’s theorem) For any homogeneous functions \( f \) of degree \( a \) on \( \mathbb{R}^n - \{0\} \)

\[
\sum_{i=1}^{n} \xi_i \partial_i f = a f.
\]

**Proof:**

\[
\sum_{i=1}^{n} \partial_i(f(\xi)) \xi_i = \frac{\partial}{\partial t}_{|t=1} f(t \xi) = \frac{\partial}{\partial t}_{|t=1} t^a f(\xi) = a f(\xi).
\]

\( \square \)

The following proposition collects results from [FGLS] (see Lemma 1.3).

**Proposition 3** Any symbol \( \sigma \in CS^{a}_{c,c}(\mathbb{R}^n) \) with vanishing residue

\[
\text{res}(\sigma) = \int_{S^{n-1}} \sigma_{-n}(\xi) \, d\xi = 0
\]

is up to some smoothing symbol, a finite sum of partial derivatives, i.e. there exist symbols \( \tau_i \in CS^{a}_{c,c}(\mathbb{R}^n) \), \( i = 1, \cdots, n \) such that

\[
\sigma \sim \sum_{i=1}^{n} \partial_i \tau_i.
\]  

In particular, given a linear form \( \rho : CS^{a}_{c,c}(\mathbb{R}^n) \to \mathbb{C} \),

\[
\text{rho} \text{ is singular and satisfies Stokes’ property} \implies \text{Ker(res)} \subseteq \text{Ker}(\rho).
\]  

**Proof:** Equation (4) clearly follows from equation (3) since \( \rho \) is assumed to vanish on smoothing symbols.

To prove (3) we write \( \sigma \sim \sum_{j=0}^{\infty} \chi \sigma_{a-j} \) with \( \sigma_{a-j} \in C^\infty(\mathbb{R}^n - \{0\}) \) homogeneous of degree \( a-j \).

- If \( a-j \neq -n \) it follows from Lemma 2 that the homogeneous function \( \tau_{i,a-j+1} = \frac{\xi_i \sigma_{a-j}}{a+n-j} \) is such that \( \sum_{i=1}^{n} \partial_i \tau_{i,a-j+1} = \sigma_{a-j} \) since \( \sum_{i=1}^{n} \partial_i(\sigma_{a-j}(\xi) \xi_i) = (a+n-j) \sigma_{a-j}(\xi) \).

- We now consider the case \( a-j = -n \). In polar coordinates \( (r, \omega) \in \mathbb{R}^{+} \times S^{n-1} \) the Laplacian reads \( \Delta = -\sum_{i=1}^{n} \partial_i^2 = -r^{1-n} \partial_r(r^{n-1} \partial_r) + r^{-2} \Delta_{S^{n-1}} \). Since \( \Delta(f(\omega)r^{2-n}) = r^{-n} \Delta_{S^{n-1}} \) we have \( \Delta(f(\omega)r^{2-n}) = \sigma_{-n}(r \omega) \Rightarrow \Delta_{S^{n-1}} f = (\sigma_{-n})_{S^{n-1}} \). Setting \( F(\omega) := f(\omega) r^{2-n} \) it follows that the equation \( \Delta F = \sigma_{-n} \) has a solution if and only if \( \sigma_{-n} \in \text{Ker}(\Delta_{S^{n-1}}) \) i.e. if \( \text{res}(\sigma) = 0 \). In that case, \( \sigma_{-n} = \sum_{i=1}^{n} \partial_i \tau_{i,-n+1} \) where we have set \( \tau_{i,-n+1} := \partial_i F \).
Let \( \tau_i \sim \sum_{j=1}^{\infty} \chi \tau_{i,a-j+1} \) then

\[
\sigma \sim \sum_{i=0}^{n} \sum_{j=0}^{\infty} \chi \partial_i \tau_{i,a-j+1} \sim \sum_{i=1}^{n} \partial_i \tau_i
\]

(5)

since \( \partial_i \chi \) has compact support so that the difference \( \sigma - \sum_{i=1}^{n} \partial_i \tau_i \) is smoothing. Since the \( \tau_i \) are by construction of order \( a+1 \), statement (3) of the proposition follows. \( \square \)

The following proposition gives a characterisation of the kernel of the noncommutative residue.

**Corollary 1** Top degree symbol valued forms which are exact up to smoothing symbols coincide with forms with vanishing (extended) residue:

\[
\text{Ker}\left( \text{res}_{\Omega^{n-1}CS_c,c}(\mathbb{R}^n) \right) = \{ \alpha \sim d\beta, \quad \beta \in \Omega^{n-1}CS_c,c(\mathbb{R}^n) \}
\]

(6)

which implies that

\[
H^n_{\infty}CS_c,c(\mathbb{R}^n) := \{ \alpha \in \Omega^nCS_c,c(\mathbb{R}^n), \quad d\alpha \sim 0 \} / \text{Ker}\left( \text{res}_{\Omega^nCS_c,c}(\mathbb{R}^n) \right).
\]

**Proof:** Equation (6) clearly follows from the first part of the assertion.

Let us turn to the first part of the assertion and prove (6). By Proposition 2 we know that exact forms lie in the kernel of the residue up to smoothing symbols.

To prove the other inclusion, let

\[
\alpha = \sum_{|J| \geq 0} \alpha_J(\xi) d\xi_{i_1} \wedge \cdots \wedge d\xi_{i_J}
\]

(we can choose \( i_1 < \cdots < i_J \) without loss of generality) has vanishing residue \( \tilde{\text{res}}(\alpha) = 0 \). Then either \( |J| < n \) or \( |J| = n \) in which case \( i_1 = 1, \cdots, i_n = n \) and \( \tilde{\text{res}}(\alpha) = \text{res}(\alpha_n) = 0 \). In that case, we can apply Proposition 3 to \( \sigma := \alpha_n \) and write:

\[
\sigma(\xi) d\xi_1 \wedge \cdots \wedge d\xi_n \sim \sum_{i=1}^{n} \partial_i \tau_i(\xi) d\xi_1 \wedge \cdots \wedge d\xi_{n-1} \wedge d\xi_i + d\xi_{i+1} \wedge \cdots \wedge d\xi_n
\]

\[
\sim d\left( \sum_{i=1}^{n} (-1)^{i-1} \tau_i(\xi) d\xi_1 \wedge \cdots \wedge d\xi_{i-1} \wedge d\hat{\xi}_i \wedge d\xi_{i+1} \wedge \cdots \wedge d\xi_n \right)
\]

which proves (6). \( \square \)

**Theorem 1** Any singular linear form \( \rho : CS_c,c(\mathbb{R}^n) \rightarrow \mathbb{C} \) with Stokes’ property is proportional to the residue, i.e. \( \rho = c \cdot \text{res} \) for some constant \( c \).

Equivalently, any closed singular linear form \( \tilde{\rho} : \Omega CS_c,c(\mathbb{R}^n) \rightarrow \mathbb{C} \) is proportional to the residue extended to forms, i.e. \( \tilde{\rho} = c \cdot \tilde{\text{res}} \) for some constant \( c \).

**Proof:** By Proposition 3 \( \rho \) satisfies Stokes’ property implies that \( \rho \) vanishes on \( \text{Ker}(\text{res}) \).

Since \( \sigma - \text{res}(\sigma) \frac{\sqrt{2\pi}}{\text{Vol}(S^{n-1})} |\xi|^{-n-1} \chi(\xi) \) vanishing residue\( \tilde{\text{res}} \) we infer that \( \rho(\sigma) = \text{res}(\sigma) \frac{\sqrt{2\pi}}{\text{Vol}(S^{n-1})} \rho(\xi) \rightarrow |\xi|^{-n-1} \chi(\xi) \) from which the statement of the theorem follows setting \( c := \frac{\sqrt{2\pi}}{\text{Vol}(S^{n-1})} \rho(\xi) \rightarrow |\xi|^{-n} \chi(\xi) \).

Since \( \rho \) vanishes on smoothing symbols by assumption, this constant is independent of the choice of \( \chi \). \( \square \)

\(^3\text{Here as before } \chi \text{ is a smooth cut-off function which vanishes in a neighborhood of 0 and is identically 1 outside the unit ball.}\)
1.4 A characterisation of the cut-off regularised integral $\int_{\mathbb{R}^n}$ in terms of Stokes’ property

Let us recall the construction of a useful extension of the ordinary integral given by the cut-off regularised integral.

For any $R > 0$, $B(0, R)$ denotes the ball of radius $R$ centered at 0 in $\mathbb{R}^n$. We recall that given a symbol $\sigma \in CS_{c.c.}(\mathbb{R}^n)$, the map $R \mapsto \int_{B(0,R)} \sigma(\xi) \, d\xi$ has an asymptotic expansion as $R \to \infty$ of the form (here we use the notations of (11):

$$\int_{B(0,R)} \sigma(\xi) \, d\xi \sim_{R \to \infty} \alpha_0(\sigma) + \sum_{j=0, a - j + n \neq 0}^{\infty} \sigma_{a-j} R^{a-j+n} + \text{res}(\sigma) \cdot \log R$$

where $\alpha_0(\sigma)$ is the constant term given by:

$$\int_{\mathbb{R}^n} \sigma(\xi) \, d\xi := \int_{\mathbb{R}^n} \sigma (N)(\xi) \, d\xi + \sum_{j=0}^{N-1} \chi(\xi) \sigma_{a-j}(\xi) \, d\xi$$

$$- \sum_{j=0, a - j + n \neq 0}^{N-1} \frac{1}{a - j + n} \int_{S^{n-1}} \sigma_{a-j}(\omega) \, d\mu_S(\omega).$$

This cut-off integral $\int_{\mathbb{R}^n}$ defines a linear form on $CS_{c.c.}(\mathbb{R}^n)$ which extends the ordinary integral in the following sense; if $\sigma$ has complex order with real part smaller than $-n$ then $\int_{B(0,R)} \sigma(\xi) \, d\xi$ converges as $R \to \infty$ and

$$\int_{\mathbb{R}^n} \sigma(\xi) \, d\xi = \int_{\mathbb{R}^n} \sigma(\xi) \, d\xi.$$

As it is the custom for the ordinary integral we use the same symbol $\int_{\mathbb{R}^n}$ for its extension to forms so that:

$$\int_{\mathbb{R}^n} \alpha(\xi) \, d\xi_1 \wedge \cdots \wedge d\xi_k := \left( \int_{\mathbb{R}^n} \alpha \right) \delta_{k-n},$$

where we have assumed that $i_1 < \cdots < i_k$

**Remark 2** Since the cut-off regularised integral $\int_{\mathbb{R}^n}$ coincides on symbols of order $< -n$ with the ordinary integral which vanishes on partial derivatives, $\rho := \int_{\mathbb{R}^n}$ fulfills the assumptions of Proposition 7 with $S = CS_{c.c.}(\mathbb{R}^n)$. Consequently, translation invariance of $\int_{\mathbb{R}^n}$ is equivalent to closedness:

$$\int_{\mathbb{R}^n} \partial_j \sigma = 0 \quad \forall j \in \{1, \cdots, n\} \quad \text{(closedness condition)}$$

$$\iff \int_{\mathbb{R}^n} i^*_\eta(\sigma) = \int_{\mathbb{R}^n} \sigma \quad \forall \eta \in \mathbb{R}^n \quad \text{(translation invariance)}.$$

We investigate its closedness: unfortunately, the cut-off regularised integral does not in general satisfy Stokes’ property.

**Proposition 4** [MMP] For any $\tau \in CS_{c.c.}(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \partial_i \tau(\xi) \, d\xi = (-1)^{i-1} \int_{|\xi|=1} \tau_{-n+1}(\xi) \, d\xi_1 \wedge \cdots \wedge d\xi_i \wedge \cdots \wedge d\xi_n.$$
\[ \begin{align*}
\int_{\mathbb{R}^n} \partial_i \tau(\xi) \, d\xi &= \lim_{R \to \infty} \int_{B(0,R)} \partial_i \tau(\xi) \, d\xi \\
&= \lim_{R \to \infty} \int_{B(0,1)} \partial_i \tau(R\xi) \, d\xi \\
&= \lim_{R \to \infty} \int_{B(0,1)} \partial_i (\tau(R\xi)) \, d\xi \\
&= (-1)^{i-1} \lim_{R \to \infty} \int_{B(0,1)} d \left( \tau(R\xi) \, d\xi_1 \wedge \cdots \wedge d\xi_i \wedge \cdots \wedge d\xi_n \right) \\
&= (-1)^{i-1} \lim_{R \to \infty} \int_{B(0,1)} |\xi|=1 \tau(\xi) \, d\xi_1 \wedge \cdots \wedge d\xi_i \wedge \cdots \wedge d\xi_n \\
&= (-1)^{i-1} \int_{|\xi|=1} \tau_{-1-n}(\xi) \, d\xi_1 \wedge \cdots \wedge d\xi_i \wedge \cdots \wedge d\xi_n
\end{align*} \]

in view of [3]. \( \square \)

However, the cut-off regularised integral does obey Stokes’ property on specific types of symbols.

**Corollary 2** We have

\[ \sigma(\xi) = \partial_i \tau(\xi) \Rightarrow \int_{\mathbb{R}^n} \sigma(\xi) \, d\xi = 0 \quad \forall i \in \{1, \cdots, n\} \]

in the following cases:

1. if \( \sigma \) has non integer order,
2. if \( \sigma \) has integer order \( a \) and \( \sigma_{a-j}(-\xi) = (-1)^{a-j} \sigma(\xi) \) \( \forall j \in \mathbb{N}_0 \) in odd dimension
3. if has integer order \( a \) and \( \sigma_{a-j}(-\xi) = (-1)^{a-j+1} \sigma(\xi) \) \( \forall j \in \mathbb{N}_0 \) in even dimension.

**Proof:** By Proposition [4]

\[ \int_{\mathbb{R}^n} \sigma(\xi) \, d\xi = (-1)^{i-1} \int_{|\xi|=1} \tau_{1-n}(\xi) \, d\xi_1 \wedge \cdots \wedge d\xi_i \wedge \cdots \wedge d\xi_n. \]

1. If \( \sigma \) has non integer order, then so has \( \tau \) which implies that \( \tau_{1-n} = 0 \) so that \( \int_{\mathbb{R}^n} \sigma(\xi) \, d\xi \) vanishes.
2. For any holomorphic family\(^4\) \( \sigma(z) \) in \( \mathcal{C}_\alpha(z) \) with non constant affine holomorphic order \( \alpha(z) \) and such that \( \sigma(0) = \sigma \) we have by \([FS]\) (see \([2]\) in Section 2)

\[ \lim_{z \to 0} \int_{\mathbb{R}^n} \sigma(z) = \int_{\mathbb{R}^n} \sigma - \frac{1}{\alpha'(0)} \int_{S^{n-1}} \left( \partial_\xi \sigma |_{z=0} \right)_{-n} \, d\mu_S. \]

We apply this to \( \sigma(z) = \partial_i (\tau(z)) \) with \( \tau(z)(x) = \chi(x) \tau(x) |\xi|^{-2} \) for some smooth cut-off function \( \chi \) which vanishes in a neighborhood of 0 and is identically one outside the open unit ball. By the first part of the corollary, since \( \sigma(z) \) has non integer order outside a discrete set of complex numbers (which correspond to the poles of \( \int_{\mathbb{R}^n} \sigma(z) \) we have \( \int_{\mathbb{R}^n} \sigma(z) = \int_{\mathbb{R}^n} \partial_i (\tau(z)) = 0 \) as a meromorphic map. On the other hand, since \( \left. \left( \partial_\xi \sigma |_{z=0} \right)_{-n} \right|_{S^n} = \left. \left( \tau \partial_\xi \log |\xi| \right) \right|_{S^n} = - \left. \left( \tau_{-n} \right) \right|_{S^n} \)

it follows that \( \int_{S^{n-1}} \left( \partial_\xi \sigma |_{z=0} \right)_{-n} \, d\mu_S = \int_{S^{n-1}} \tau_{-1-n} \, d\mu_S. \) But this last quantity vanishes whenever \( \tau_{-n-1} \) is an even function i.e whenever \( \sigma_{n-2} \) is an odd function. This holds in odd dimension if \( \sigma_{a-j}(-\xi) = (-1)^{a-j} \sigma(\xi) \) or in even dimension if \( \sigma_{a-j}(-\xi) = (-1)^{a-j+1} \sigma(\xi) \) so that in both of these cases \( \int_{\mathbb{R}^n} \sigma = 0. \)

\( \square \)

\(^4\)We refer the reader to Section 2 for the notion of holomorphic family of symbols.
Theorem 2 Let $\mathcal{S}$ be a subset of $CS_{c,c}(\mathbb{R}^n)$ such that

$$CS_{c,c}^{-\infty}(\mathbb{R}^n) \subset \mathcal{S} \subset \text{Ker}(\text{res}).$$

Then by Proposition 3

$$\sigma \in \mathcal{S} \cap CS_{c,c}^{a}(\mathbb{R}^n) \implies \exists \tau_i \in CS_{c,c}^{a+1}(\mathbb{R}^n) \text{ s.t. } \sigma \sim \sum_{i=1}^{n} \partial_i \tau_i$$

with $\sigma \sim \sum_{j=0}^{\infty} \chi \sigma_{a-j}$ and $\tau_i \sim \sum_{j=0}^{\infty} \chi \tau_{i,a-j+1}, i = 1, \cdots, n.$

If for any $\sigma \in \mathcal{S}$ the $\tau_i$ and $\chi \tau_{i,a+1-j}, j \in \mathbb{N}_0$ can be chosen in $\mathcal{S}$ then any linear form $\rho : \mathcal{S} \rightarrow \mathbb{C}$ which satisfies Stokes' property is entirely determined by its restriction to $CS_{c,c}^{\leq-K}(\mathbb{R}^n)$ for any positive integer $K \leq n$.

Equivalently, under the same conditions any closed linear form $\tilde{\rho} : \Omega \mathcal{S} \rightarrow \mathbb{C}$ is entirely determined by its restriction to $\Omega CS_{c,c}^{\leq-K}(\mathbb{R}^n)$ for any positive integer $K \leq n$.

In particular, if $\int_{\mathbb{R}^n}$ satisfies Stokes' property on $\mathcal{S}$ and $\rho$ is continuous\(^5\) on $\mathcal{S} \cap CS_{c,c}^{a}(\mathbb{R}^n)$ for any complex number $a$, then there is a constant $c$ such that

$$\rho = c \cdot \int_{\mathbb{R}^n}.$$  

Remark 3 In practice $\mathcal{S}$ can be described in terms of some condition on the homogeneous components of the symbol and therefore automatically satisfies the requirements of the theorem.

Proof: We write a symbol $\sigma \in CS_{c,c}^{a}(\mathbb{R}^n)$

$$\sigma = \sum_{j=0}^{N-1} \chi \sigma_{a-j} + \sigma_{(N)}$$

with $N$ any integer chosen large enough so that $\sigma_{(N)}$ has order $<-n$. Here $\chi$ is a smooth cut-off function which vanishes in a neighborhood of $0$ and is one outside the unit ball. As before, the $\sigma_{a-j}$ are positively homogeneous of degree $a-j$.

By linearity of $\rho$ we have:

$$\rho(\sigma) = \sum_{j=0}^{N-1} \rho(\chi \sigma_{a-j}) + \rho(\sigma_{(N)}). \quad (9)$$

Let now $\sigma \in \mathcal{S}$. Since by the assumption on $\mathcal{S}$ the symbol $\sigma$ has vanishing residue we can write as in the proof of Proposition 3 $\sigma_{a-j} = \partial_{i_j} \tau_{a+1-j}$ for some $i_j \in \{1, \cdots, n\}$ and some homogeneous symbol $\tau_{a+1-j}$. By the closedness condition $\rho((\partial_{i_j} \chi) \tau_{a+1-j}) = 0$ so that

$$\rho(\chi \sigma_{a-j}) = \rho(\chi \partial_{i_j} \tau_{a+1-j}) = -\rho((\partial_{i_j} \chi) \tau_{a+1-j}).$$

Summing over $j = 1, \cdots, N-1$ we get:

$$\rho(\sigma) = - \sum_{j=0}^{N-1} \rho((\partial_{i_j} \chi) \tau_{a+1-j}) + \rho(\sigma_{(N)}). \quad (10)$$

Another choice of primitive $\tilde{\tau}_{a+1-j} = \tau_{a+1-j} + c_j$ modifies this expression by $c_j \sum_{j=0}^{N-1} \rho((\partial_{i_j} \chi)$ which vanishes.

Since $N$ can be chosen arbitrarily large, formula (10) shows that $\rho$ is uniquely determined by its expression on symbols of arbitrarily negative order.

\(^5\)i.e. its restriction to symbols of constant order is continuous.
Thus $\rho$ is determined by its restriction to $\bigcap_{K \geq n} CS_{c.c.}^{-K}(\mathbb{R}^n) = CS_{c.c.}^{-\infty}(\mathbb{R}^n)$. This restriction is continuous as a result of the continuity of the restriction of $\rho$ to any $CS_{c.c.}^{a}(\mathbb{R}^n)$. Thus $\rho$ restricted to $CS_{c.c.}^{-\infty}(\mathbb{R}^n)$ can be seen as a tempered distribution with vanishing derivatives at all orders. Such a distribution is a priori of the form $f \mapsto \int_{\mathbb{R}^n} f(\xi) \phi(\xi) \, d\xi$ for some smooth function $\phi$; since all its derivatives vanish, $\phi$ is constant so that $\rho$ restricted to smoothing symbols is proportional to the ordinary integral $\int_{\mathbb{R}^n}$.

From the above discussion we conclude that two closed and continuous (on symbols of constant order) linear forms $\rho_1$ and $\rho_2$ on a set $\mathcal{S}$ which satisfy the assumptions of the theorem are proportional.

The cut-off regularised integral is continuous on symbols of constant order. Thus, if it has Stokes’ property on the set $\mathcal{S}$, we infer from the above uniqueness result that $\rho$ is proportional to $\int_{\mathbb{R}^n}$. \[\square\]

Here are some examples of subsets of $CS_{c.c}(\mathbb{R}^n)$ which fulfill the assumptions of Theorem 2 and on which the cut-off regularised integral $\int_{\mathbb{R}^n}$ satisfies Stokes’ property in view of Corollary 2.

**Example 1** The set $CS_{c.c.}^{\mathbb{Z}}(\mathbb{R}^n)$ of non integer order symbols.

**Example 2** In odd dimension $n$ the set

$$CS_{c.c.}^{\text{odd}}(\mathbb{R}^n) := \{\sigma \in CS_{c.c.}^{\mathbb{Z}}(\mathbb{R}^n), \quad \sigma_{a-j}(-\xi) = (-1)^{a-j}\sigma_{a-j}(\xi) \quad \forall \xi \in \mathbb{R}^n\}$$

of odd-class symbols.

**Example 3** In even dimension $n$ the set

$$CS_{c.c.}^{\text{even}}(\mathbb{R}^n) := \{\sigma \in CS_{c.c.}^{\mathbb{Z}}(\mathbb{R}^n), \quad \sigma_{a-j}(-\xi) = (-1)^{a-j+1}\sigma_{a-j}(\xi) \quad \forall \xi \in \mathbb{R}^n\}$$

of even-class symbols.

From these examples we get the following straightforward application of Theorem 2.

**Corollary 3** Any closed linear form on $CS_{c.c.}^{\mathbb{Z}}(\mathbb{R}^n)$, resp. $CS_{c.c.}^{\text{odd}}(\mathbb{R}^n)$ in odd dimensions, resp. $CS_{c.c.}^{\text{even}}(\mathbb{R}^n)$ in even dimensions is determined by its restriction to symbols of arbitrarily negative order. If it is continuous on symbols of constant order, it is proportional to the cut-off regularised integral $\int_{\mathbb{R}^n}$.

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*I thank E. Schrohe for drawing my attention to this point.*
2 Existence: The canonical trace on odd- (resp. even-) class operators in odd (resp. even) dimensions

We show that the canonical trace density $TR_x(A)\,dx$ defines a global density in odd (resp. even) dimensions for odd-(even-) class operators $A$ which integrates over the manifold to the (extended) canonical trace

$$TR(A) := \frac{1}{\sqrt{2\pi}} \int_M dx \, TR_x(A).$$

To do so, on the grounds of results of [PS], we carry out a continuous extension along holomorphic paths $z \mapsto A(z) \in C\ell(M, E)$ such that $A(0) \in C\ell^{odd}(M, E)$ and $A'(0) \in C\ell^{odd}(M, E)$ (resp. $A(0) \in C\ell^{even}(M, E)$ and $A'(0) \in C\ell^{even}(M, E)$), and show that the extension is independent of the holomorphic path, thereby extending results of [KV] and [Gr]. Along the way we define the noncommutative residue on the algebra of classical pseudodifferential operators as well as the canonical trace on non integer order classical pseudodifferential operators.

2.1 Notations

Let $U$ be a connected open subset of $\mathbb{R}^n$ where as before we assume that $n > 1$.

For any complex number $a$, let $S^a_{\text{cpt}}(U)$ denote the set of smooth functions on $U \times \mathbb{R}^n$ called symbols with compact support in $U$, such that for any multiindices $\beta, \gamma \in \mathbb{N}^n$, there is a constant $C(\beta, \gamma)$ satisfying the following requirement:

$$|\partial^\beta_\xi \partial^\gamma_\sigma(x, \xi)| \leq C(\beta, \gamma)(1 + |\xi|)^{\text{Re}(\sigma) - |\beta|}$$

where $\text{Re}(\sigma)$ stands for the real part of $\sigma$, $|\xi|$ for the euclidean norm of $\xi$. We single out the subset $CS^a_{\text{cpt}}(\mathbb{R}^n) \subset S^a_{\text{cpt}}(\mathbb{R}^n)$ of symbols $\sigma$, called classical symbols of order $a$ with compact support in $U$, such that

$$\sigma(x, \xi) = \sum_{j=0}^{N-1} \chi(\xi) \sigma_{a-j}(x, \xi) + \sigma_{(N)}(x, \xi) \quad (11)$$

where $\sigma_{(N)} \in S^{a-N}_{\text{cpt}}(U)$ and where $\chi$ is a smooth cut-off function which vanishes in a small ball of $\mathbb{R}^n$ centered at 0 and which is constant equal to 1 outside the unit ball. Here $\sigma_{a-j}(x, \cdot), j \in \mathbb{N}_0$ are positively homogeneous of degree $a - j$.

Let

$$CS^{-\infty}_{\text{cpt}}(U) = \bigcap_{a \in \mathbb{C}} CS^a_{\text{cpt}}(U)$$

be the set of smoothing symbols with compact support in $U$; we write $\sigma \sim \tau$ for two symbols that differ by a smoothing symbol.

We equip the set $CS^a_{\text{cpt}}(U)$ with a Fréchet structure with the help of the following semi-norms labelled by multiindices $\alpha, \beta$ and integers $j \geq 0$, $N$ (see [H]):

$$\sup_{x \in U, \xi \in \mathbb{R}^n} (1 + |\xi|)^{-\text{Re}(\alpha) + |\beta|} \| \partial^\alpha_\xi \partial^\beta_\sigma(x, \xi) \|;$$

$$\sup_{x \in U, \xi \in \mathbb{R}^n} (1 + |\xi|)^{-\text{Re}(\alpha) + N + |\beta|} \| \partial^\alpha_\xi \partial^\beta_\sigma \left( \sigma - \sum_{j=0}^{N-1} \chi(\xi) \sigma_{a-j} \right)(x, \xi) \|;$$

$$\sup_{x \in U, |\xi|=1} \| \partial^\alpha_\xi \partial^\beta_\sigma \sigma_{a-j}(x, \xi) \|. $$

The star product

$$\sigma \star \tau = \sum_{\alpha} \frac{(-i)^{|\alpha|}}{\alpha!} \partial^\alpha_\sigma \partial^\alpha_\tau \quad (12)$$

of symbols $\sigma \in CS^a_{\text{cpt}}(U)$ and $\tau \in CS^b_{\text{cpt}}(U)$ lies in $CS^{a+b}_{\text{cpt}}(U)$ provided $a - b \in \mathbb{Z}$.

Let

$$CS_{\text{cpt}}(U) = \bigcup_{a \in \mathbb{C}} CS^a_{\text{cpt}}(U)$$

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denote the algebra generated by all classical symbols with compact support in \( U \). We denote by 
\[ \text{CS}_{\text{cpt}}^{a}(U) := \bigcup_{\text{Re}(\alpha) < p} \text{CS}_{\text{cpt}}^{\alpha}(U), \]
the set of classical symbols of order with real part \( < p \) with compact support in \( U \), by 
\[ \text{CS}_{\text{cpt}}^{\infty}(U) := \bigcup_{\alpha \in \mathbb{Z}} \text{CS}_{\text{cpt}}^{\alpha}(U) \]
the algebra of integer order symbols, and by 
\[ \text{CS}_{\text{cpt}}^{\infty}(U) := \bigcup_{\alpha \in \mathbb{Z}} \text{CS}_{\text{cpt}}^{\alpha}(U) \]
the set of non integer order symbols with compact support in \( U \).

We shall also need to consider the set introduced in [KV]
\[ \text{CS}_{\text{cpt}}^{\text{o}}(U) := \{ \sigma \in \text{CS}_{\text{cpt}}^{\infty}(U), \sigma_{a-j}(-\xi) = (-1)^{a-j} \sigma_{a-j}(\xi) \quad \forall (x, \xi) \in T^{*}U \} \]
of odd-class (also called even-even in [Gr]) symbols and the set introduced by G. Grubb (under the name even-odd)
\[ \text{CS}_{\text{cpt}}^{\text{e}}(U) := \{ \sigma \in \text{CS}_{\text{cpt}}^{\infty}(U), \sigma_{a-j}(-\xi) = (-1)^{a-j+1} \sigma_{a-j}(\xi) \quad \forall (x, \xi) \in T^{*}U \} \]
of even-class symbols with compact support in \( U \).

Whereas \( \text{CS}_{\text{cpt}}^{\text{o}}(U) \) is stable under the symbol product [12], \( \text{CS}_{\text{cpt}}^{\text{e}}(U) \) is not since the product of two even symbols is odd. Similarly, one can check that the product of an odd and an even symbol is odd, two properties which conflict with the intuition suggested by the terminology even/odd suggested by [KV] (hence the alternative terminology used by Grubb).

The above definitions extend to non scalar symbols. Given a finite dimensional vector space \( V \) and any \( a \in \mathbb{C} \) we set
\[ \text{CS}_{\text{cpt}}^{a}(U, V) := \text{CS}_{\text{cpt}}^{a}(U) \otimes \text{End}(V) \]
Similarly, we define \( \text{CS}(U, V) \), \( \text{CS}_{\text{cpt}}^{\infty}(U, V) \), \( \text{CS}_{\text{cpt}}^{\infty}(U, V) \) and \( \text{CS}_{\text{cpt}}^{(a+b)}(U, V) \), \( \text{CS}_{\text{cpt}}^{a}(U, V) \) from \( \text{CS}_{\text{cpt}}^{a}(U) \), \( \text{CS}_{\text{cpt}}^{\infty}(U) \), \( \text{CS}_{\text{cpt}}^{\infty}(U) \) and \( \text{CS}_{\text{cpt}}^{(a+b)}(U, V) \), \( \text{CS}_{\text{cpt}}^{a}(U, V) \).

[Remark 4] Note that \( \sigma \in \text{CS}_{\text{cpt}}^{a}(U, V) \implies \text{tr}(\sigma) \in \text{CS}^{a}(U) \) where \( \text{tr} \) stands for the trace on matrices.

Similar properties hold for \( \text{CS}(U, V) \), \( \text{CS}_{\text{cpt}}^{\infty}(U, V) \) and \( \text{CS}_{\text{cpt}}^{(a+b)}(U, V) \), \( \text{CS}_{\text{cpt}}^{a}(U, V) \).

Let \( M \) be an \( n \)-dimensional closed connected Riemannian manifold (as before \( n > 1 \)). For \( a \in \mathbb{C} \), let \( C^{a}(M) \) denote the linear space of classical pseudodifferential operators of order \( a \), i.e. linear maps acting on smooth functions \( C^{\infty}(M) \), which using a partition of unity adapted to an atlas on \( M \) can be written as a finite sum of operators
\[ A = \text{Op}(\sigma(A)) + R \]
where \( R \) is a linear operator with smooth kernel and \( \sigma(A) \in \text{CS}_{\text{cpt}}^{a}(U) \) for some open subset \( U \subset \mathbb{R}^{n} \).

Here we have set
\[ \text{Op}(\sigma)(u) := \int_{\mathbb{R}^{n}} e^{i(x-y,\xi)} \sigma(x, \xi) u(y) \ dy \ d\xi \]
where \( (\cdot, \cdot) \) stands for the canonical scalar product in \( \mathbb{R}^{n} \).

The star product [12] on classical symbols with compact support induces the operator product on (properly supported) classical pseudodifferential operators since \( \text{Op}(\sigma \ast \tau) = \text{Op}(\sigma) \text{Op}(\tau) \). It follows that the product \( AB \) of two classical pseudodifferential operators \( A \in C^{a}(M) \), \( B \in C^{b}(M) \) lies in \( C^{a+b}(M) \) provided \( a - b \in \mathbb{Z} \). Let us denote by \( C^{a}(M) = \langle \bigcup_{a \in \mathbb{C}^{\text{odd}}} C^{a}(M) \rangle \) the algebra generated by all classical pseudodifferential operators acting on \( C^{\infty}(M) \).

Given a finite rank vector bundle \( E \) over \( M \) we set \( C^{a}(M, E) := C^{a}(M) \otimes \text{End}(E) \), \( C^{(a+b)}(M, E) := C^{a+b}(M) \otimes \text{End}(E) \).

[Remark 5] Note that if \( A \in C^{a}(M, E) \), in a local trivialisation \( E|_{U} \simeq U \times V \) over an open subset \( U \) of \( M \), the map \( (x, \xi) \mapsto \sigma(A)(x, \xi) \) lies in \( \text{CS}^{a}(U, V) \).

\( C^{a}(M, E) \) inherits a Fréchet structure via the Fréchet structure on classical symbols of order \( a \).

The algebras \( C^{(a+b)}(M, E), C^{\infty}(M, E), C^{\text{odd}}(M, E), C^{\text{even}}(M, E) \) are defined similarly using trivialisations of \( E \) from \( \text{CS}_{\text{cpt}}^{a}(U) \), \( \text{CS}_{\text{cpt}}^{\infty}(U) \), \( \text{CS}_{\text{cpt}}^{\text{odd}}(U) \) and \( \text{CS}_{\text{cpt}}^{\text{even}}(U) \).
2.2 Classical symbol valued forms on an open subset

The notations introduced in paragraph 1.1 for symbols on $\mathbb{R}^n$ with constant coefficients easily extend to symbols with support in an open subset of $U \subset \mathbb{R}^n$ with varying coefficients. Let $U$ be a connected open subset of $\mathbb{R}^n$ as before. We borrow from [MMP](see also [LP]) the following notations and some of the subsequent definitions.

**Definition 5** Let $k$ be a non negative integer, $\alpha$ a complex number. We let
\[ \Omega^k CS^a_{\text{cpt}}(U) = \{ \alpha \in \Omega^k(T^*U), \alpha = \sum_{I,J \subseteq \{1, \ldots, n\}, |I|+|J|=k} \alpha_{IJ}(x, \xi) \, d\xi_I \wedge dx_J \] 
with $\alpha_{IJ} \in CS^{a-|I|}(U)$
denote the set of order $k$ classical symbol valued forms on $U$ with compact support. Let
\[ \Omega^k CS_{\text{cpt}}(U) = \{ \alpha \in \Omega^k(T^*U), \alpha = \sum_{I,J \subseteq \{1, \ldots, n\}, |I|+|J|=k} \alpha_{IJ}(x, \xi) \, d\xi_I \wedge dx_J \] 
with $\alpha_{IJ} \in CS(U)$
denote the set of classical symbol valued $k$-forms on $U$ of all orders with compact support.

The exterior product on forms combined with the star product on symbols induces a product $\Omega^k CS^a_{\text{cpt}}(U) \times \Omega^k CS_{\text{cpt}}(U) \to \Omega^{k+1} CS^a_{\text{cpt}}(U)$; let
\[ \Omega CS_{\text{cpt}}(U) := \bigoplus_{k=0}^{\infty} \Omega^k CS_{\text{cpt}}(U) \]
stand for the $\mathbb{N}_0$ graded algebra (also filtered by the symbol order) of classical symbol valued forms on $U$ with compact support.

We shall also consider the sets $\Omega^k CS^a_{\text{cpt}}(U) := \bigcup_{a \in \mathbb{Z}} \Omega^k CS^a_{\text{cpt}}(U)$ of integer order classical symbols valued $k$-forms, $\Omega^k CS^a_{\text{cpt}}(U) := \bigcup_{a \in \mathbb{Z}} \Omega^k CS^a_{\text{cpt}}(U)$ of non integer order classical symbol valued $k$-forms, respectively $\Omega^k CS^a_{\text{odd}}(U)$, resp. $\Omega^k CS^a_{\text{even}}(U)$ of odd- resp. even-) classical symbol valued $k$-forms.

Exterior differentiation on forms extends to symbol valued forms (see (5.14) in [LP]):
\[ d : \Omega^k CS_{\text{cpt}}(U) \to \Omega^{k+1} CS_{\text{cpt}}(U) \]
\[ \alpha_{IJ}(x, \xi) \, d\xi_I \wedge dx_J \mapsto \sum_{i=1}^{2n} \partial_i \alpha_{IJ}(\xi) \, du_i \wedge d\xi_I \wedge dx_J, \]
where $u_i = \xi_i, \partial_i = \partial_{\xi_i}$ if $1 \leq i \leq n$ and $u_i = x_i, \partial_i = \partial_{x_i}$ if $n+1 \leq i \leq 2n$.

As before, we call a symbol valued form $\alpha$ closed if $d\alpha = 0$ and exact if $\alpha = d\beta$ where $\beta$ is a symbol valued form; this gives rise to the following cohomology groups
\[ H^k CS_{\text{cpt}}(U) := \{ \alpha \in \Omega^k CS_{\text{cpt}}(U), \, d\alpha = 0 \} / \{ d\beta, \, \beta \in \Omega^{k+1} CS_{\text{cpt}}(U) \}. \]

Let $D(U) \subset CS_{\text{cpt}}(U)$ be a set containing smoothing symbols. We call a linear form $\rho : D(U) \to \mathbb{C}$ singular if it vanishes on smoothing symbols, and regular otherwise.

A linear form $\rho : D(U) \to \mathbb{C}$ extends to a linear form $\hat{\rho} : \Omega D(U) \to \mathbb{C}$ defined by
\[ \hat{\rho}(\alpha_{IJ}(x, \xi) \, d\xi_i \wedge \ldots \wedge d\xi_{i|I|} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j|J|}) := \rho(\alpha_{IJ}) \delta_{|I|+|J|-2n}, \]
with $i_1 < \cdots < i_{|I|}, \, j_1 < \cdots < j_{|J|}$. Here we have set
\[ \Omega^k D(U) := \{ \sum_{|I|+|J|\leq k} \alpha_{IJ}(x, \xi) \, d\xi_I \wedge dx_J, \, \alpha_{IJ} \in D(U) \}. \]

This is a straightforward generalisation of Lemma[1]

---

1By linear we mean that $\rho(\alpha_1 \sigma_1 + \alpha_2 \sigma_2) = \alpha_1 \rho(\sigma_1) + \alpha_2 \rho(\sigma_2)$ whenever $\sigma_1, \sigma_2, \alpha_1 \sigma_1 + \alpha_2 \sigma_2$ lie in $D(U)$. 

[1]
Lemma 3 Let $\rho : \mathcal{D}(U) \subset CS_{cpt}(U) \rightarrow \mathbb{C}$ be a linear form. The following two conditions are equivalent:

\[ \exists i, j \in \{1, \ldots, n\} \quad \text{s.t.} \quad \sigma = \partial_\xi \tau \in \mathcal{D}(U) \quad \text{or} \quad \sigma = \partial_x \tau \in \mathcal{D}(U) \implies \rho(\sigma) = 0 \]

\[ \alpha = d\beta \in \Omega^p\mathcal{D}(U) \implies \tilde{\rho}(\alpha) = 0 \]

As before we call closed a linear form $\tilde{\rho}$ obeying the second condition and by extension $\rho$ is then also said to be closed. We also say that $\rho$ satisfies Stokes’ condition.

Remark 6 A closed linear form $\tilde{\rho}$ on $\Omega CS_{cpt}(U)$ induces a linear form $\tilde{\rho} : H^\bullet CS_{cpt}(U) \rightarrow \mathbb{C}$. 

Proposition 5 A linear form $\rho : \mathcal{D}(U) \subset CS_{cpt}(U) \rightarrow \mathbb{C}$ is closed whenever

\[ \rho(\{\sigma, \tau\}_\star) = 0 \quad \forall \sigma, \tau \in CS_{cpt}(U), \quad \text{s.t.} \quad \{\sigma, \tau\}_\star \in \mathcal{D}(U) \]

where we have set:

\[ \{\sigma, \tau\}_\star := \sum_{\alpha} \frac{(-i)^{[\alpha]}}{\alpha!} (\partial_\xi^\alpha \sigma \partial_x^\alpha \tau - \partial_x^\alpha \sigma \partial_\xi^\alpha \tau) . \]

Proof: If the linear form is closed, we can perform integration by parts and write:

\[ \rho(\{\sigma, \tau\}_\star) = \sum_{\alpha} \frac{(-i)^{[\alpha]}}{\alpha!} \rho(\partial_\xi^\alpha \sigma \partial_x^\alpha \tau - \partial_x^\alpha \sigma \partial_\xi^\alpha \tau) \]

\[ = \sum_{\alpha} \frac{(-i)^{[\alpha]}}{\alpha!} \rho(\partial_\xi^\alpha \sigma \partial_x^\alpha \tau - \partial_x^\alpha \sigma \partial_\xi^\alpha \tau) \]

\[ = 0. \]

Conversely, if the linear form vanishes on brackets $\{\cdot, \cdot\}_\star$ contained in $\mathcal{D}(U)$ then for any $\sigma \in CS_{cpt}(U)$ such that $\partial_\xi \sigma = i \{\sigma, \xi\}_\star \in \mathcal{D}(U)$ we have

\[ \rho(\partial_\xi \sigma) = i \rho(\{\xi, \sigma\}_\star) = 0 \]

and similarly for any $\sigma \in CS_{cpt}(U)$ such that $\partial_x \sigma = i \{x, \sigma\}_\star \in \mathcal{D}(U)$ we have

\[ \rho(\partial_x \sigma) = i \rho(\{x, \sigma\}_\star) = 0. \]

\[ \square \]

2.3 The noncommutative residue

Definition 6 The noncommutative residue of a symbol $\sigma \in CS_{cpt}(U)$ is defined by

\[ \text{res}(\sigma) := \frac{1}{(2\pi)^n} \int_U dx \int_{S^{n-1}} \sigma_{-n}(x, \xi) \mu S(\xi) = \frac{1}{\sqrt{2\pi}} \int_U \text{res}_x(\sigma) d x \]

where $\text{res}_x(\sigma) := \frac{1}{\sqrt{2\pi}} \int_{S^{n-1}} \sigma_{-n}(x, \xi) \mu S(\xi)$ is the residue density at point $x$ and where as before

\[ \mu S(\xi) := \sum_{j=1}^n (-1)^j \xi_j d\xi_1 \wedge \cdots \wedge d\xi_j \wedge \cdots \wedge d\xi_n \]

denotes the volume measure on $S^{n-1}$ induced by the canonical measure on $\mathbb{R}^n$.

Lemma 4 The noncommutative residue is a singular closed linear form on $CS_{cpt}(U)$ which restricts to a continuous map on each $CS_{cpt}^a(U), a \in C$. 

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Remark 7. It follows from the definition that the residue is continuous on each classical symbols with compact support to build a noncommutative residue on classical operators on a closed manifold $M$ introduced by Wodzicki [W1] (see also [G1]).

Definition 7. The noncommutative residue of $A \in \text{Cl}(M, E)$ is defined by

$$\text{res}(A) := \frac{1}{(2\pi)^n} \int_M dx \int_{S^*_xM} \text{tr}_x(\sigma(A))_{-n}(x, \xi) \mu_S(\xi) = \frac{1}{\sqrt{2\pi}} \int_M \text{res}_x(A) \, dx$$

where $\text{res}_x(A) := \frac{1}{\sqrt{2\pi}} \int_{S^*_xM} \text{tr}_x(\sigma(A))_{-n}(x, \xi) \mu_S(\xi)$ is the residue density at point $x$ and as before

$$\mu_S(\xi) := \sum_{j=1}^n (-1)^j \xi_j \wedge \cdots \wedge \partial^\xi j \wedge \cdots \wedge d\xi_n$$

denotes the volume measure on the cotangent sphere $S^*_xM$ induced by the canonical measure on the cotangent space $T^*_xM$ at point $x$. Here $\text{tr}_x$ stands for the fibrewise trace on the vector bundle $E|_x$.

Remark 7. It follows from the definition that the residue is continuous on each $\text{Cl}^a(M, E), a \in \mathbb{C}$.

We derive the cyclicity of the residue on operators from Stokes’ property of the residue on symbols.

Proposition 6. \[ \text{res}([A, B]) = 0 \quad \forall A, B \in \text{Cl}(M, E). \]

Proof: The product of two $\Psi$DOs $A, B$ in $\text{Cl}(M, E)$ reads

$$AB = \sum_{|\alpha| \leq N} \frac{(-i)^{|\alpha|}}{\alpha!} \text{Op}(\partial^\alpha x z \sigma(A) \partial^\alpha y \sigma(B)) + R_N(AB) \tag{13}$$

for any integer $N$ and with $R_N(AB)$ of order $\leq a + b - N$ where $a, b$ are the orders of $A, B$ respectively. Hence

$$[A, B] = \sum_{|\alpha| \leq N} \frac{(-i)^{|\alpha|}}{\alpha!} \text{Op}(\partial^\alpha x z \sigma(A) \partial^\alpha y \sigma(B) - \partial^\alpha y \sigma(B) \partial^\alpha x z \sigma(A)) + R_N([A, B])$$

with similar notations.

Applying the noncommutative residue on either side, choosing $N$ such that $a + b - N < -n$ we have

$$\text{res}([A, B]) = \sum_{|\alpha| \leq N} \frac{(-i)^{|\alpha|}}{\alpha!} \int_M dx \int_{\mathbb{R}^n} \text{tr}_x(\partial^\alpha x z \sigma(A) \partial^\alpha y \sigma(B) - \partial^\alpha y \sigma(B) \partial^\alpha x z \sigma(A)) \, d\xi + \text{res}(R_N([A, B]))$$

$$= \sum_{|\alpha| \leq N} \frac{(-i)^{|\alpha|}}{\alpha!} \int_M dx \int_{\mathbb{R}^n} \text{tr}_x(\partial^\alpha x z \sigma(A) \partial^\alpha y \sigma(B) - \partial^\alpha y \sigma(B) \partial^\alpha x z \sigma(A)) \, d\xi$$

$$= 0.$$  

In the last identity we used Stokes’ property for residue on symbols to implement repeated integration by parts combined with the fact that the residue vanishes on symbols of order $< -n$ and the cyclicity of the ordinary trace on matrices. \(\square\)

---

\(^8\)Note that this continuity holds only on symbols of constant order; it breaks down if one lets the order vary.
2.4 The canonical trace on non integer order operators

The cut-off regularised integral extends to $\text{CS}_{\text{cpt}}(U)$.

**Definition 8** For any $\sigma \in \text{CS}_{\text{cpt}}(U)$ the cut-off regularised integral of $\sigma$ is defined by

\[
\int_{T_{\text{cpt}}} \sigma := \int_{U} dx \int_{T_{\text{cpt}} U} d\xi \sigma(x, \xi).
\]

It extends to pseudodifferential symbol valued forms by

\[
\int_{T_{\text{cpt}}} \alpha_{IJ} d\xi_I \wedge dx_J := \left( \int_{T_{\text{cpt}}} \alpha_{IJ} \right) \delta_{|J| + |I| = 2n}
\]

where $d\xi_I := d\xi_{i_1} \wedge \cdots \wedge d\xi_{i_k}$ with $i_1 < \cdots < i_k$ and $dx_J := dx_{j_1} \wedge \cdots \wedge dx_{j_l}$ with $j_1 < \cdots < j_l$.

**Lemma 5** The cut-off regularised integral is a linear form on $\text{CS}_{\text{cpt}}(U)$ which restricts to a continuous linear form on each $\text{CS}_{\text{cpt}}^a(U)$ and satisfies Stokes’ property on non integer order symbols:

\[
\left( \exists j = 1, \cdots , n, \quad \sigma = \partial_{x_j} \tau \quad \text{or} \quad \sigma = \partial_{\xi_j} \tau \quad \text{with} \quad \sigma \in \text{CS}_{\text{cpt}}^{a\ell}(U) \right) \Rightarrow \int_{U \times \mathbb{R}^n} \sigma = 0.
\]

Equivalently, it extends to a linear form on $\Omega \text{CS}_{\text{cpt}}(U)$ which restricts to a continuous linear form on each $\Omega \text{CS}_{\text{cpt}}^a(U)$ and is closed on non integer order symbols valued forms:

\[
\left( \alpha = d\beta \in \Omega \text{CS}_{\text{cpt}}^{a\ell}(U) \right) \Rightarrow \int_{T_{\text{cpt}}} \alpha = 0.
\]

**Proof:** We prove the first statement. The continuity follows from the continuity of the cut-off regularised integral on $\text{CS}_{\text{cpt}}^a(\mathbb{R}^n)$ for any $a \in \mathbb{C}$. Similarly, Stokes’ property follows from Stokes’ property of the ordinary integral on $\text{CS}_{\text{cpt}}^{a\ell}(U)$ combined with the fact that the cut-off regularised integral $\int_{\mathbb{R}^n}$ vanishes on derivatives $\partial_{\xi_j}$ of non integer order symbols as a result of Proposition[1,2] ☐

Using a partition of unity, one can patch up the cut-off regularised integral of symbols with compact support to a canonical trace on non integer order classical pseudo-differential operators [KV].

**Definition 9** The canonical trace is defined on $\text{Cl}^{a\ell}(M, E)$ by

\[
\text{TR}(A) := \frac{1}{(2\pi)^n} \int_{M} dx \int_{T^*_M} \text{tr}_x (\sigma(A)(x, \xi)) \ d\xi = \frac{1}{\sqrt{2\pi}} \int_{M} \text{TR}_x(A) \ dx
\]

where $\text{TR}_x(A) := \frac{1}{(2\pi)^n} \int_{T^*_M} \text{tr}_x (\sigma(A)(x, \xi)) \ d\xi$ is the canonical trace density at point $x$.

The canonical trace is tracial on non integer order operators as a consequence of Stokes’ property for cut-off regularised integrals on non integer order symbols.

**Proposition 7** Let $A \in \text{Cl}(M)$, $B \in \text{Cl}(M, E)$ be two classical operators with non integer order such that their bracket $[A, B]$ also has non integer order. Then

\[
\text{TR}([A, B]) = 0.
\]

**Proof:** The product of $A$ and $B$ on $M$ reads

\[
AB = \sum_{|\alpha| \leq N} \frac{(-i)^{|\alpha|}}{\alpha!} \text{Op}(\partial_x^\alpha \sigma(A) \partial_x^\alpha \sigma(B)) + R_N(A B)
\]

for any integer $N$ and with $R_N(A B)$ of order $< a + b - N$ where $a, b$ are the orders of $A, B$ respectively. Hence

\[
[A, B] = \sum_{|\alpha| \leq N} \frac{(-i)^{|\alpha|}}{\alpha!} \text{Op} \left( \partial_x^\alpha \sigma(A) \partial_x^\alpha \sigma(B) - \partial_x^\alpha \sigma(B) \partial_x^\alpha \sigma(A) \right) + R_N([A, B])
\]
with similar notations.

When the bracket \([A, B]\) has non integer order, we can apply the canonical trace on either side and write:

\[
\text{TR} ([A, B]) = \sum_{|\alpha| \leq N} \frac{(-i)^{|\alpha|}}{\alpha!} \int_M dx \int_{\mathbb{R}^n} \text{tr}_x \left( \partial_{\xi}^\alpha \sigma(A) \partial_{\xi}^\alpha \sigma(B) - \partial_{\xi}^\alpha \sigma(B) \partial_{\xi}^\alpha \sigma(A) \right) d\xi + \text{tr} (R_N([A, B]))
\]

\[
= \sum_{|\alpha| \leq N} \frac{(-i)^{|\alpha|}}{\alpha!} \int_M dx \int_{\mathbb{R}^n} \text{tr}_x \left( \partial_{\xi}^\alpha \sigma(A) \partial_{\xi}^\alpha \sigma(B) - \partial_{\xi}^\alpha \sigma(A) \partial_{\xi}^\alpha \sigma(B) \right) d\xi + \text{tr} (R_N([A, B]))
\]

\[
= \text{tr} (R_N([A, B])).
\]

In the last identity, we used Stokes’ property for cut-off regularised integrals on non integer order symbols (see Lemma \(\diamond\)) to implement repeated integration by parts in order to show that the integral term on the r.h.s. vanishes using the fact that the ordinary trace on matrices is cyclic.

Thus we have

\[
\text{TR} ([A, B]) = \text{tr} (R_N([A, B]))
\]

with \(R_N([A, B])\) of order \(< a + b - N\).

Since \(N\) can be chosen arbitrarily large, we have \(\text{TR} ([A, B]) = \text{tr} (R_\infty([A, B]))\) for some smoothing operator \(R_\infty([A, B])\).

On the other hand, for any smoothing operators \(S, T\) the operators \([S, B]\) and \([A, T]\) are smoothing and a direct check using the kernel representation of these operators shows that \(\text{TR} ([S, B]) = \text{tr} ([S, B]) = 0\) and similarly, \(\text{TR} ([A, T]) = \text{tr} ([A, T]) = 0\). It follows that \(\text{TR} ([A + S, B + T]) = \text{TR} ([A, B])\) leading to

\[
\text{tr} (R_\infty([A + S, B + T])) = \text{tr} (R_\infty([A, B]))
\]

for any smoothing operators \(S, T\). But this means that the bilinear form \((A, B) \mapsto \text{tr} (R_\infty([A, B]))\) is purely symbolic, namely that it depends only on a finite number of homogeneous components of the symbols of \(A\) and \(B\), which by its very construction is clearly not the case unless it vanishes. This proves that \(\text{tr} (R_\infty([A, B])) = 0\) and hence that \(\text{TR} ([A, B]) = 0\). \(\square\)

2.5 Holomorphic families of classical pseudodifferential operators

The notion of holomorphic family of classical pseudodifferential operators first introduced by Guillemin in [GI] and extensively used by Kontsevich and Vishik in [KV] generalises the notion of complex power \(A^z\) of an elliptic operator developed by Seeley [Se], the derivatives of which lead to logarithms.

**Definition 10** Let \(\Omega\) be a domain of \(\mathbb{C}\) and \(U\) an open subset of \(\mathbb{R}^n\). A family \((\sigma(z))_{z \in \Omega} \subset \mathcal{CS}(U)\) is holomorphic when

(i) the order \(\alpha(z)\) of \(\sigma(z)\) is holomorphic on \(\Omega\).

(ii) For \((x, \xi) \in U \times \mathbb{R}^n\), the function \(z \mapsto \sigma(z)(x, \xi)\) is holomorphic on \(\Omega\) and \(\forall k \geq 0, \partial_{x_j} \sigma(z) \in S^{\alpha(z) + \varepsilon}(U)\) for all \(\varepsilon > 0\).

(iii) For any integer \(j \geq 0\), the (positively) homogeneous component \(\sigma_{\alpha(z) - j}(z)(x, \xi)\) of degree \(\alpha(z) - j\) of the symbol is holomorphic on \(\Omega\).

The derivative of a holomorphic family \(\sigma(z)\) of classical symbols yields a holomorphic family of symbols, the asymptotic expansions of which a priori involve a logarithmic term.

**Lemma 6** The derivative of a holomorphic family \(\sigma(z)\) of classical symbols of order \(\alpha(z)\) defines a holomorphic family of symbols \(\sigma'(z)\) of order \(\alpha(z)\) with asymptotic expansion:

\[
\sigma'(z)(x, \xi) \sim \sum_{j=0}^{\infty} \chi(\xi) \left( \log |\xi| \sigma'_{\alpha(z) - j,1}(z)(x, \xi) + \sigma'_{\alpha(z) - j,0}(z)(x, \xi) \right) \quad \forall (x, \xi) \in T^* U \quad (14)
\]

for some smooth cut-off function \(\chi\) around the origin which is identically equal to 1 outside the open unit ball and positively homogeneous symbols

\[
\sigma'_{\alpha(z) - j,0}(z)(x, \xi) = |\xi|^{\alpha(z) - j} \frac{\partial_z}{|\xi|} \left( \sigma_{\alpha(z) - j}(z)(x, \frac{\xi}{|\xi|}) \right), \quad \sigma'_{\alpha(z) - j,1}(z) = \alpha'(z) \sigma_{\alpha(z) - j}(z)(x, \xi) \quad (15)
\]

of degree \(\alpha(z) - j\).
Proof: We write
\[ \sigma(z)(x, \xi) \sim \sum_{j=0}^{\infty} \chi(\xi) \sigma_{\alpha(z)-j}(z)(x, \xi). \]

Using the positive homogeneity of the components \( \sigma_{\alpha(z)-j} \) we have:
\[
\partial_z \left( \sigma_{\alpha(z)-j}(z)(x, \xi) \right) \\
= \partial_z \left( |\xi|^{|\alpha(z)-j|} \sigma_{\alpha(z)-j}(z)(x, \frac{\xi}{|\xi|}) \right) \\
= \left( \alpha'(z) |\xi|^{|\alpha(z)-j|} \sigma_{\alpha(z)-j}(z)(x, \frac{\xi}{|\xi|}) \right) \log |\xi| + |\xi|^{|\alpha(z)-j|} \partial_z \left( \sigma_{\alpha(z)-j}(z)(x, \frac{\xi}{|\xi|}) \right)
\]
which shows that \( \partial_z \left( \sigma_{\alpha(z)-j}(z)(x, \xi) \right) \) has order \( \alpha(z) - j \). Thus
\[
\partial_z \left( \sigma_N(z)(x, \xi) \right) = \sigma'(z)(x, \xi) - \sum_{j<N} \chi(\xi) \partial_z \left( \sigma_{\alpha(z)-j}(z)(x, \xi) \right)
\]
lies in \( \mathcal{S}^{\alpha(z)-N+\varepsilon}(U) \) for any \( \varepsilon > 0 \) so that \( \sigma'(z) \) is a symbol of order \( \alpha(z) \) with asymptotic expansion:
\[
\sigma'(z)(x, \xi) \sim \sum_{j=0}^{\infty} \chi(\xi) \sigma'_{\alpha(z)-j}(z) \quad \forall (x, \xi) \in T^*U
\]
where
\[
\sigma'_{\alpha(z)-j}(z)(x, \xi) := \log |\xi| \sigma'_{\alpha(z)-j,1}(z)(x, \xi) + \sigma'_{\alpha(z)-j,0}(z)(x, \xi)
\]
for some positively homogeneous symbols
\[
\sigma'_{\alpha(z)-j,0}(z)(x, \xi) := |\xi|^{|\alpha(z)-j|} \partial_z \left( \sigma_{\alpha(z)-j}(z)(x, \frac{\xi}{|\xi|}) \right)
\]
and
\[
\sigma'_{\alpha(z)-j,1}(z)(x, \xi) := \alpha'(z) \sigma_{\alpha(z)-j}(z)(x, \xi)
\]
of degree \( \alpha(z) - j \).

On the other hand, differentiating the asymptotic expansion \( \sigma(z)(x, \xi) \sim \sum_{j=0}^{\infty} \chi(\xi) \sigma_{\alpha(z)-j}(z)(x, \xi) \) w.r. to \( z \) yields
\[
\sigma'(z)(x, \xi) \sim \sum_{j=0}^{\infty} \chi(\xi) \partial_z \left( \sigma_{\alpha(z)-j}(z)(x, \xi) \right).
\]

Hence,
\[
\partial_z \left( \sigma_{\alpha(z)-j}(z)(x, \xi) \right) = \sigma'_{\alpha(z)-j}(z)(x, \xi) = |\xi|^{|\alpha(z)-j|} \partial_z \left( \sigma_{\alpha(z)-j}(z)(x, \frac{\xi}{|\xi|}) \right) + \alpha'(z) \sigma_{\alpha(z)-j}(x, \xi) \log |\xi|
\]
as announced. \( \square \)

Correspondingly we recall the notion of holomorphic classical pseudodifferential operators.

**Definition** 11 A family \( (A(z))_{z \in \Omega} \in \mathcal{C}l(M, E) \) is holomorphic if in any local trivialisation we can write \( A(z) \) in the form \( A(z) = Op(\sigma(A(z))) + R(z) \), for some holomorphic family of symbols \( \sigma(A(z)) \) and some holomorphic family \( \{R(z)\}_{z \in \Omega} \) of smoothing operators i.e. given by a holomorphic family of smooth Schwartz kernels.

It follows from (14) and (15) that
\[
\partial_z \left( \sigma(A(z))_{\alpha(z)-j} \right)(x, \xi) = \sigma_{\alpha(z)-j}(A'(z))(x, \xi)
\]
\[
= \alpha'(z) \sigma_{\alpha(z)-j}(A(z))(x, \xi) \log |\xi| + |\xi|^{|\alpha(z)-j|} \partial_z \left( \sigma_{\alpha(z)-j}(A(z))(x, \frac{\xi}{|\xi|}) \right)(x, \xi).
\]
Example 4 Given an admissible operator \( A \in \mathcal{Cl}(M, E) \) with spectral cut \( \theta \) an operator \( A_\theta \) has no eigenvalue on the ray \( \{ re^{i\theta}, r \geq 0 \} \) in which case it is elliptic, and such that the spectrum of \( A \) does not meet the open ray \( \{ re^{i\theta}, r > 0 \} \). In that case, following Seeley [Sc], one can define the complex power \( A_\theta^z \) which yields a holomorphic family \( z \mapsto A_\theta^z \) in \( \mathcal{Cl}(M, E) \).

2.6 Continuity of the canonical trace on non integer order classical pseudodifferential operators

It follows from the very definition of the canonical trace that it is continuous w.r. to the Fréchet topology on \( \mathcal{Cl}(M, E) \) for every \( a \notin \mathbb{Z} \). In this paragraph we discuss its continuity on (holomorphic) families of varying order.

The following proposition collects results from [KV] and [PS].

Proposition 8 Let \( \sigma(z) \in CS(U) \) (resp. \( A(z) \in \mathcal{Cl}(M, E) \)) be a holomorphic family of order \( \alpha(z) \) such that \( \alpha'(0) \neq 0 \). The map

\[
z \mapsto \int_{T^* U} \sigma(z)(x, \xi) \, d\xi
\]

(resp. \( z \mapsto \text{tr}(A(z)) \)) is holomorphic on \( \alpha^{-1}(]-\infty, -n[) \) and extends to a meromorphic map \( z \mapsto \int_{T^* U} \text{tr}_x (\sigma(z)(x, \xi)) \, d\xi \) (resp. \( z \mapsto \text{TR}(A(z)) \)) on the complex plane with simple poles and

\[
\text{Res}_{z=0} \int_{T^* U} \sigma(z)(x, \xi) \, d\xi = -\frac{1}{\alpha'(0)} \text{res}_x (\sigma(0)(x, \xi)),
\]

( resp. \( \text{Res}_{z=0} \text{TR}(A(z)) = -\frac{1}{\alpha'(0)} \text{res}(A(0)). \) )

Furthermore, if \( \alpha(z) \) is affine in \( z \) [PS]

\[
\text{fp}_{z=0} \int_{T^* U} \sigma(z)(x, \xi) \, d\xi = \int_{T^* U} \text{tr}_x (\sigma(0)(x, \xi)) \, d\xi - \frac{1}{\alpha'(0)} \int_{S^* U} \sigma'(0)(x, \cdot) \, d\xi \quad \forall x \in U
\]

( resp. \( \text{fp}_{z=0} \text{TR}(A(z)) = \int_M dx \left( \text{TR}_x(A(0)) - \frac{1}{\alpha'(0)} \text{res}_x (A'(0)) \right) \) )

Corollary 4 The canonical trace on non integer order operators is continuous along holomorphic families with affine order. In other words, for any holomorphic family \( A(z) \in \mathcal{Cl}(M, E) \) with affine order \( \alpha(z) \) such that \( A(0) \in \mathcal{Cl}^{\mathbb{Z}}(M, E) \)

\[
\lim_{z \to 0} \text{TR}(A(z)) = \text{TR}(A(0)).
\]

Proof: We can assume that the order \( \alpha(z) \) of \( A(z) \) satisfies \( \alpha'(0) \neq 0 \) for otherwise the order is constant in which case we already know that the canonical trace is continuous at 0.

If \( \alpha'(0) \neq 0 \), the map \( z \mapsto \text{TR}(A(z)) \) is holomorphic at \( z = 0 \) since by equation \( \text{11} \)

\[
\text{Res}_{z=0} \text{TR}(A(z)) = -\frac{1}{\alpha'(0)} \text{res}(A(0)).
\]
which vanishes as a result of the non integrality of the order of $A(0)$. Since the derivative $A'(0)$ at $z = 0$ has same order $\alpha(0)$ as $A(0)$ which is non integer by assumption, $A'(0)$ also has vanishing residue density so that by equation (21) we have:

$$fp_{z=0}\text{TR}(A(z)) = \frac{1}{\sqrt{2\pi}} \int_M \text{TR}_x(A(0)) \, dx = \text{TR}(A(0)).$$

\[ \square \]

### 2.7 Odd- (resp. even-) class operators embedded in holomorphic families

For any integer $a$, the condition $\sigma_{a-j}(x, -\xi) = (-1)^{a-j} \sigma_{a-j}(x, \xi)$ $\forall j \in \mathbb{N} \cup \{0\}$ which characterises a classical symbol of order $a$ that lies in the odd-class, extends to log-polyhomogeneous symbols of logarithmic type $1$:

$$\sigma(x, \xi) \sim \sum_{j=0}^{a} \chi(\xi) \sigma_{a-j}(x, \xi); \quad \sigma_{a-j} = \sigma_{a-j}^0 + \sigma_{a-j}^1 \log |\xi|$$

with $\sigma_{a-j}^i(x, \cdot), i = 0, 1$ positively homogeneous of degree $a - j$. One requires that

$$\sigma_{a-j}(x, -\xi) = (-1)^{a-j} \sigma_{a-j}(x, \xi) \quad \forall j \in \mathbb{N} \cup \{0\} \quad \forall (x, \xi) \in T^*U$$

or equivalently that

$$\sigma_{a-j}^i(x, -\xi) = (-1)^{a-j} \sigma_{a-j}^i(x, \xi) \quad \forall j \in \mathbb{N} \cup \{0\} \quad \forall (x, \xi) \in T^*U$$

for both $i = 0$ and $i = 1$.

**Proposition 9** [B] Given an admissible operator $A \in Cl^{\text{odd}}(M, E)$ with positive order $a > 0$ and spectral cuts $\theta$ and $\theta - a\pi$, the symmetrised logarithm

$$A'(0) = \frac{\log A + \log \theta - a\pi}{2} A$$

where we have set $A_0(z) := \frac{A_0^+ + A_0^-}{2}$, lies in the odd-class.

**Remark 8** When the order $a$ of $A$ is even, then $A_0 = A_0 - a\pi$ so that $A_0(z) = A_0^+$ and $A_0'(0) = \log A$. This yields back the known fact [KV] that the logarithm of an odd-class admissible PDO with even order lies in the odd-class.

**Proof:** Recall that the homogeneous components of the symbol of $A_0^+$ are

$$\sigma_{az-j}(A_0^+)(x, \xi) = \frac{j}{2\pi} \int_{\Gamma_o} \lambda^0 \sigma_{a-j}(x, \xi, \lambda) \, d\lambda \quad (22)$$

with

$$q_{-a} = (\sigma(A) - \lambda)^{-1}$$

and

$$q_{-a-j} = -q_{-a} \left( \sum_{k+l+j \geq 0} \frac{1}{\alpha!} \sigma_{a-k}^l(D_x^a q_{a-k}) \right)$$

the positively homogeneous components of the resolvent $(A - \lambda I)^{-1}$. In other words, these components $q_{-a-j}$ are positively homogeneous in $(\xi, \lambda^2)$ i.e. for $t > 0$, for $(x, \xi) \in T^*M$,

$$q_k(x, t\xi, t^2\lambda) = t^k q_k(x, \xi, \lambda) \quad \forall t > 0. \quad (23)$$

If $A \in Cl^a(M, E)$ lies in the odd-class, this extends to any real number $t$ since we have [KV] par. 2

$$q_k(x, -\xi, (-1)^a \lambda) = (-1)^k q_k(x, \xi, \lambda). \quad (24)$$
Now, assume that \( \text{Re} \, z < 0 \). A Cauchy integral gives
\[
\sigma_{za-j}(A^z_\theta)(x, -\xi) = \frac{i}{2\pi} \int_{\Gamma_\theta} \lambda^z_\theta q_{-a-j}(x, -\xi, \lambda) \, d\lambda
\]
\[
= (-1)^{a+j} \frac{i}{2\pi} \int_{\Gamma_\theta} \lambda^z_\theta q_{-a-j}(x, \xi, (-1)^{a} \lambda) \, d\lambda
\]
where \( \Gamma_\theta \) is an appropriate contour around the spectrum of \( A \) along the ray \( L_\theta \).

By a change of variable, we obtain
\[
\sigma_{az-j}(A^z_\theta)(x, -\xi) = (-1)^{a+j} \frac{i}{2\pi} \int_{\Gamma_{a-\pi}} (e^{-i\pi \mu})^z_\theta \mu q_{-a-j}(x, \xi, \mu) \, d(e^{-i\pi \mu})
\]
\[
= (-1)^{a+j} e^{-i\pi \mu} \frac{i}{2\pi} \int_{\Gamma_{a-\pi}} q_{-a-j}(x, \xi, \mu) \, d\mu
\]
\[
= (-1)^{j} e^{i\pi \mu} \sigma_{az-j}(A^z_\theta)(x, \xi).
\]

Thus
\[
\sigma_{az-j}(A^z_\theta)(x, -\xi) = e^{i\pi(az-j)} \sigma_{az-j}(A^z_\theta)(x, \xi).
\] (25)

Since both the left and the right hand side of this equality are analytic in \( z \), we conclude that the equality holds for all \( z \in \mathbb{C} \). Similarly,
\[
\sigma_{az-j}(A^z_\theta)(x, -\xi) = e^{i\pi(az-j)} \sigma_{az-j}(A^z_\theta)(x, \xi).
\] (26)

Differentiating (25) w.r. to \( z \) on either side yields:
\[
\sigma_{az-j}(\partial_z A^z_\theta)(x, -\xi) = \partial_z (\sigma_{az-j}(A^z_\theta)(x, -\xi))
\]
\[
= \partial_z (e^{i\pi(az-j)} \sigma_{az-j}(A^z_\theta)(x, \xi))
\]
\[
= i\pi a e^{i\pi(az-j)} \sigma_{az-j}(A^z_\theta)(x, \xi) + e^{i\pi(az-j)} \sigma_{az-j}(\partial_z A^z_\theta)(x, \xi).
\] (28)

Similarly, differentiating (25) on either side yields:
\[
\sigma_{az-j}(\partial_z A^z_\theta)(x, -\xi) = \partial_z (\sigma_{az-j}(A^z_\theta)(x, -\xi))
\]
\[
= \partial_z (e^{i\pi(az-j)} \sigma_{az-j}(A^z_\theta)(x, \xi))
\]
\[
= -i\pi a e^{i\pi(az-j)} \sigma_{az-j}(A^z_\theta)(x, \xi) + e^{i\pi(az-j)} \sigma_{az-j}(\partial_z A^z_\theta)(x, \xi).
\] (30)

Combining equations (27) and (29) yields at \( z = 0 \):
\[
\sigma_{-j}(A'(0))(x, -\xi) = \frac{\sigma_{-j}\left(\left(\partial_z A^z_\theta\right)_{z=0}\right)(x, -\xi) + \sigma_{-j}\left(\left(\partial_z A^z_\theta\right)_{z=0}\right)(x, -\xi)}{2}
\]
\[
= \frac{i\pi a \delta_{j,0} + (-1)^j \sigma_{-j}\left(\left(\partial_z A^z_\theta\right)_{z=0}\right)(x, \xi) - i\pi a \delta_{j,0} + (-1)^j \sigma_{-j}\left(\left(\partial_z A^z_\theta\right)_{z=0}\right)(x, \xi)}{2}
\]
\[
= (-1)^j \sigma_{-j}(A'(0))(x, \xi)
\]
so that \( A'(0) \) lies in the odd-class.
Example 5. Take $M$ a Riemannian manifold and $A = \Delta_g$ the Laplace Beltrami operator with $\theta = \frac{\pi}{2}$. It has order 2 and lies in the odd-class since it is a differential operator. Then $A_{\theta}^g(0) = \log_{\frac{\pi}{2}} \Delta_g$ lies in the odd-class.

Example 6. Let $M$ be a spin manifold and $E = S \otimes W$ a twisted bundle with $S$ the spinor bundle and $W$ an exterior vector bundle over $M$. From a twisted connection $\nabla^E = \nabla^S \otimes 1 + 1 \otimes \nabla^W$ on $E$, with $\nabla^S$ the connection on $S$ induced by the Levi-Civita connection on $M$, $\nabla^W$ a connection on $W$, one can build the corresponding twisted Dirac operator $D = c \cdot \nabla^E$. Here $c$ stands for the Clifford multiplication on the Clifford bundle $E$. $D$ is an admissible operator of order 1 which lies in the odd-class since it is a differential operator. Moreover, since it is self-adjoint, its spectrum is real so that it has spectral cut $\theta := \frac{\pi}{2}$ and $\theta - \pi = -\frac{\pi}{2}$.

Take the order one differential operator $A = D$. It lies in the odd-class and so does its symmetrised logarithm $\frac{\log_{\frac{\pi}{2}} D + \log_{\frac{\pi}{2}} D}{2}$.

Corollary 5. Provided there is an admissible operator $Q \in C^q(M, E)$ with positive order $q$ and spectral cuts $\theta$ and $\theta - q\pi$, then any operator $A \in C^{q+1}(M, E)$ (resp. $A \in C^{q+1}(M, E)$) can be embedded in a holomorphic family

$$A_{\theta}^Q(z) := A \frac{Q_\theta^z + Q_\theta^{q\pi}}{2}$$

such that $\left(\partial_z A_{\theta}^Q(z)\right)_{|z=0}$ lies in the odd-class (resp. even-class).

Proof: This follows from Proposition 2 applied to $Q$ combined with the stability of the odd-class under products (resp. the fact that the product of an even and odd-class operator is even).

Let us focus on the odd-class case, since the proof in the even-class case goes in a similar manner.

By Proposition 2, $Q_\theta'(0)$ lies in the odd-class. Since

$$\left(\partial_z A_{\theta}^Q(z)\right)_{|z=0} = A Q_\theta'(0),$$

applying

$$\sigma(AB) = \sum_{\alpha} \frac{(-1)^{\alpha}}{\alpha!} \partial^\alpha_x \sigma(A) \partial^\alpha_x \sigma(B)$$

to $A$ and $B = Q_\theta'(0)$ yields that $\left(\partial_z A_{\theta}^Q(z)\right)_{|z=0}$ lies in the odd-class since $A \in C^{q+1}(M, E)$. \hfill \Box

2.8 The canonical trace on odd (resp. even)-class operators in odd (resp. even) dimensions

In the sequel, $M$ is an odd-(resp. even-) dimensional manifold. Let $\pi : E \rightarrow M$ be a vector bundle over $M$ such that there is an admissible operator $Q \in C^{q+1}(M, E)$ with positive order $q$ and spectral cuts $\theta$ and $\theta - q\pi$.

Remark 9. In view of the above examples, these assumptions are fulfilled in very natural geometric setups.

Theorem 3. The canonical trace $\text{TR}$ extends continuously to $C^{q+1}(M, E)$ in odd dimensions (resp. $C^{q+1}(M, E)$ in even dimensions) in the following manner.

Let $M$ be odd (resp. even) dimensional. For any holomorphic family $A(z) \in C^q(M, E)$ with non constant affine order such that both $A(0)$ and $A'(0)$ lie in $C^{q+1}(M, E)$ (resp. $C^{q+1}(M, E)$),

1. the map $z \mapsto \text{TR}(A(z))$ is holomorphic at $z = 0$,

2. $\int_M \left(\int_{T^*_x M} \text{tr}_x \sigma(A(0))(x, \xi) \, d\xi\right) \, dx$ defines a global density on $M$ so that

$$\text{TR}(A(0)) = \frac{1}{(2\pi)^n} \int_M \left(\int_{T^*_x M} \text{tr}_x \sigma(A(0))(x, \xi) \, d\xi\right) \, dx$$

is well-defined,
3. \(\lim_{z=0} \text{TR}(A(z)) = \text{TR}(A(0))\).

**Proof:** We carry out the proof in odd dimensions for odd-class operators. The proof goes similarly in the even dimensional case for even-class operators. Since the noncommutative residue vanishes on \(C\ell^{\text{odd}}(M,E)\) and \(A(0) \in C\ell^{\text{odd}}(M,E)\), we have \(\text{res}(A(0)) = 0\). It follows from (18) that the complex residue \(\text{res}(z=0)\) vanishes so that the map \(z \mapsto \text{TR}(A(z))\) is holomorphic at \(z = 0\). We now apply (20) to \(\sigma(z) := \sigma(A(z))\); since \(A'(0)\) lies in the odd-class, it has vanishing residue density \(\text{res}_{x}(A'(0))\). Consequently,

\[
\lim_{z=0} \int_{T_{x}^{*}M} \text{tr}_{x}(\sigma(A(z))(x,\xi)) \, d\xi = \int_{T_{x}^{*}M} \text{tr}_{x}(\sigma(A)(x,\xi)) \, d\xi = \sqrt{2\pi} \text{TR}_{x}(A(0)) \quad \forall x \in M.
\]

Since the l.h.s gives rise to a globally defined density \(\text{fp}_{z=0} \int_{T_{x}^{*}M} \sigma(A(z))(x,\xi) \, d\xi\) and the right hand side give rise to a globally defined density \(\text{TR}_{x}(A(0))\) so does the right hand side give rise to a globally defined density \(\text{TR}_{x}(A(0))\). Integrating over \(M\) yields the existence of \(\text{TR}(A(0))\) and:

\[
\lim_{z \to 0} \text{TR}(A(z)) = \frac{1}{\sqrt{2\pi}} \int_{M} \text{TR}_{x}(A(0)) \, dx = \text{TR}(A(0)).
\]

\(\square\)

**Corollary 6** Any operator \(A \in C\ell^{\text{odd}}(M,E)\) in odd dimensions (resp. \(A \in C\ell^{\text{even}}(M,E)\) in even dimensions) has well-defined canonical trace

\[
\text{TR}(A) = \int_{M} dx \left( \int_{T_{x}^{*}M} \text{tr}_{x}(\sigma_{A}(x,\xi)) \right) \, d\xi
\]

and \(\text{TR}(A) = \lim_{z \to 0} \text{TR}(A(z))\) for any holomorphic family \(A(z)\) with non constant affine order such that \(A(0) = A\) and \(A'(0)\) lie in \(C\ell^{\text{odd}}(M,E)\) (resp. \(C\ell^{\text{even}}(M,E)\)). In particular,

- Kontsevich and Vishik’s (resp. Grubb’s) extended canonical trace \([KV]\) (resp. \([Gr]\)) on odd-class (resp. even-class) operators in odd (resp. even) dimensions,

  \[
  A \mapsto \text{Tr}_{(-1)}(A) := \lim_{z \to 0} \text{TR}(A Q_{\frac{z}{2}})
  \]

  with \(Q \in C\ell^{\text{odd}}(M,E)\) an admissible operator of even positive order close enough to a positive self-adjoint operator,

- the symmetrised trace introduced by Braverman \([B]\) on odd-class operators in odd dimensions

  \[
  A \mapsto \text{Tr}_{\text{sym}}(A) := \lim_{z \to 0} \text{TR}(A Q(z))
  \]

  with \(Q \in C\ell^{\text{odd}}(M,E)\) an admissible operator of any positive order \(q\) and with spectral cuts \(\theta\) and \(\theta - q\pi\),

  coincide with the canonical trace.

We also recover as a side result the fact \([KV]\) that the extended canonical trace vanishes on brackets of odd-class operators.

**Corollary 7** In odd dimensions, for any operators \(A \in C\ell^{\text{odd}}(M,E)\), \(B \in C\ell^{\text{odd}}(M,E)\) we have

\[
\text{TR}([A,B]) = 0.
\]
**Proof:** With the notations of Proposition 8, the holomorphic family 
\[ C(z) = [A_Q^0(z), B_Q^0(z)] \]
has derivative \( C'(0) = [A_Q'\theta(0), B] + [A, B Q_0'(0)] \) at \( z = 0 \). It lies in the odd class as a result of the stability of the odd-class under products. The result then follows from applying Theorem 3 to the holomorphic family \( C(z) \). Since \( TR(C(z)) \) vanishes as a meromorphic map as a consequence of the vanishing of the canonical trace on non integer order brackets, taking finite parts as \( z \to 0 \) we find that

\[ 0 = TR([A, B]). \]
3 Uniqueness : Characterisation of linear forms that vanish on operator brackets

We prove that the noncommutative residue on the algebra of classical pseudodifferential operators, the canonical trace on the set of non integer order ones, or in odd (resp. even) dimensions on the classes of odd-(resp. even-) class operators, are the unique (possibly continuous) linear forms that vanish on brackets. The classes to which the canonical trace naturally extends have in common that their operators have symbols with vanishing residue density.

3.1 Uniqueness of the noncommutative residue

We use the notations of section 2; in particular $U$ is an open connected subset of $\mathbb{R}^n$.

Proposition 10 Any singular linear form $\rho : CS_{\text{cpt}}(U) \to \mathbb{C}$ which restricts to a continuous map on $CS_{\text{cpt}}^0(U)$ for any $a \in \mathbb{C}$ and which fulfills Stokes’ property is proportional to the noncommutative residue.

Equivalently, any closed singular linear form $\tilde{\rho} : \Omega CS_{\text{cpt}}(U) \to \mathbb{C}$ which restricts to a continuous linear form on $\Omega CS_{\text{cpt}}^0(U)$ for any $a \in \mathbb{C}$ is proportional to the noncommutative residue $\tilde{\rho}$ extended to forms.

Proof: By similar arguments as in the case of symbols with constant coefficients we can check that the two statements are equivalent. Let us prove the first one. For any fixed $f \in C_{\text{cpt}}^\infty(U)$ the map $\rho_f : \sigma \mapsto \rho(f \sigma)$ defines a singular linear form on $CS_{\text{c},c}(\mathbb{R}^n)$ which vanishes on derivatives in $\xi$ since we have $\rho(f \partial_\xi \sigma) = \rho(\partial_\xi (f \sigma)) = 0$. By Theorem 1 it follows that there is a constant $c(f)$ such that $\rho(f \sigma) = c(f) \text{res}(\sigma)$ for any $\sigma \in CS_{\text{c},c}(\mathbb{R}^n)$. Since $f \mapsto c(f)$ is continuous, $c : f \mapsto c(f)$ lies in $(C_{\text{c},c}(U))'$. A general symbol $\sigma \in CS_{\text{c},c}(U)$ can be approximated by linear combinations of tensor products $f \otimes \sigma$ with $f \in C_{\text{cpt}}^\infty(U), \sigma \in CS_{\text{c},c}(\mathbb{R}^n)$. It follows from the continuity of $\rho$ that there is a distribution $F \in (C_{\text{c},c}^\infty(U))$ such that $\rho(\sigma) = F(\text{res}(\sigma(x,\cdot)))$ for any $\sigma \in CS_{\text{c},c}(U)$. This distribution being continuous, it reads $F(f) = \int_U \psi(x) f(x) \, dx$ for some $\psi \in C^\infty(U)$ so that

$$\rho(\sigma) = \int_U \psi(x) \text{res}(\sigma(x,\cdot)) \, dx.$$  

But since $\rho$ is closed by assumption, for any $\sigma = f \otimes \tau$ with $\tau \in CS_{\text{c},c}(\mathbb{R}^n)$ and $f \in C^\infty(U)$ we have

$$0 = \rho(\partial_{x_i} (f \otimes \tau)) = \rho(\partial_{x_i} f \otimes \tau).$$

Choosing $\tau$ with non vanishing residue and integrating by parts implies that

$$\int_U \partial_{x_i} (\psi(x) f(x)) \, dx = 0 \quad \forall f \in C_{\text{cpt}}^\infty(U)$$

so that $\psi$ is a constant $c$ and $\rho(\sigma) = c \int_U \text{res}(\sigma(x,\cdot)) \, dx$ is proportional to the noncommutative residue. □

We now derive from the characterisation of the residue on symbols in terms of Stokes’ property, the uniqueness (up to a multiplicative constant) of the residue as a trace on $\mathcal{C}\ell(M)$ which restricts to continuous linear forms on each $\mathcal{C}\ell^n(M)$. It uses the following lemma.

Lemma 7 ([Po1] Lemma 3.20 and [Po2] Lemma 4.4.) Any smoothing operator $A \in \mathcal{C}\ell(M)$ can be written as a finite sum of brackets $\sigma = \sum_{i=1}^n [x_i, B_i]$ with $B_i \in \mathcal{C}\ell^{n+1}(M)$.

Proof: We briefly sketch the proof which we take from [Po1] and [Po2]. A smoothing operator $R$ has smooth kernel $k_R(x,y)$ so that $k_R(x,y) - k_R(x,x)$ is smooth and vanishes on the diagonal. It follows that there are smooth functions $k_1, \cdots, k_n$ such that $k_R(x,y) = k_R(x,x) + \sum_{j=1}^n (x_j - y_j) k_j(x,y)$. Let $Q$ be the operator defined by the kernel $k_Q(x,y) = k_R(x,x)$ and let $R_j, j = 1, \cdots, n$ be the smoothing operators defined by the kernels $k_j(x,y)$ then $R = Q + \sum_{j=1}^n [x_j, R_j]$. 

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Set $H_j(x, y) := y_j |y|^2 k_Q(x, x)$ and let $Q_j$ be the operator with kernel $(x, y) \mapsto H_j(x, x - y)$; by proposition 2.7 in [Po2], it is a classical pseudodifferential operator of order $-n + 1$. Since

$$\sum_{j=1}^{n} (x_j - y_j) H_j(x, x - y) = \sum_{j=1}^{n} \frac{(x_j - y_j)^2}{|x - y|^2} k_R(x, x) = k_Q(x, y),$$

it follows that $Q = \sum_{j=1}^{n} [x_j, Q_j]$.

Since $R_j$ are smoothing and $Q_j$ are of order $-n + 1$ the result of the lemma follows. □

**Theorem 4** Any linear form $L : C\ell(M) \to \mathbb{C}$ which restricts to continuous linear forms on $C\ell^{a}(M)$ for any $a \in \mathbb{C}$ and which vanishes on brackets

$$L([A, B]) = 0 \quad \forall A, B \in C\ell(M)$$

is proportional to the noncommutative residue.

**Proof:** By Lemma 4 such a linear form $L$ vanishes on smoothing operators.

Given a local chart $(U, \phi)$ on $M$, the map

$$\rho_\phi := L \circ \phi^* \circ \text{Op}$$

then defines a singular linear form on $CS_{\text{cpt}}(\phi(U))$.

For any $\sigma \in CS_{\text{cpt}}(\phi(U))$ and for any $x_j, j = 1, \cdots, n$ corresponding to the coordinates in the local chart $(U, \phi)$ we have\footnote{We borrow this observation from [MSS] who use it to prove the uniqueness of the extension of the ordinary trace on trace-class operators to non integer order operators.}

$$(\text{Op}(\partial_{\xi_j} \sigma) u) (x) = \int_{\mathbb{R}^n} e^{i(x, \xi)} \partial_{\xi_j} \sigma(x, \xi) \hat{u}(\xi) d\xi = -i \left( \text{ad}_x \circ \text{Op}(\sigma) u \right) (x) \quad \forall u \in C^\infty_{\text{cpt}}(U).$$

Furthermore,

$$\rho_\phi \circ \partial_{\xi_j} = L \circ \phi^* \circ \text{Op} \circ \partial_{\xi_j} = -i L \circ \phi^* \circ \text{ad}_{x_j} \circ \text{Op} = -i L \circ \text{ad}_{x_j} \circ \phi^* \circ \text{Op}.$$

Since $L$ vanishes on brackets, $\rho_\phi$ vanishes on derivatives $\partial_{\xi_j} \tau$. Similarly, for any $u \in C^\infty_{\text{cpt}}(U)$

$$(\text{Op}(\partial_{x_j} \sigma) u) (x) = \int_{\mathbb{R}^n} e^{i(x, \xi)} \partial_{x_j} \sigma(x, \xi) \hat{u}(\xi) d\xi$$

$$= \partial_{x_j} \int_{\mathbb{R}^n} e^{i(x, \xi)} \sigma(x, \xi) \hat{u}(\xi) d\xi - i \int_{\mathbb{R}^n} \xi_j e^{i(x, \xi)} \sigma(x, \xi) \hat{u}(\xi) d\xi$$

$$= \partial_{x_j} \int_{\mathbb{R}^n} e^{i(x, \xi)} \sigma(x, \xi) \hat{u}(\xi) d\xi - \int_{\mathbb{R}^n} e^{i(x, \xi)} \sigma(x, \xi) \partial_{x_j} \hat{u}(\xi) d\xi$$

$$= \left[ \partial_{x_j}, \text{Op}(\sigma) \right] u(x).$$

Furthermore,

$$\rho_\phi \circ \partial_{x_j} = L \circ \phi^* \circ \text{Op} \circ \partial_{x_j} = L \circ \phi^* \circ \text{ad}_{x_j} \circ \text{Op} = L \circ \text{ad}_{x_j} \circ \phi^* \circ \text{Op}.$$

Since $L$ vanishes on brackets it follows that $\rho_\phi$ vanishes on derivatives $\partial_{x_j} \tau$ and therefore satisfies Stokes’ property.

By Proposition 10, $\rho_\phi$ which is continuous on each $CS_{\text{cpt}}^a(\phi(U))$ as a result of the continuity of $L$ on
each $\text{Cl}^p(M)$, is therefore proportional to the noncommutative residue so that there is a constant $c_\rho$ such that

$$
\rho_\phi(\sigma) = L(\phi^* \text{Op}(\sigma)) = c_\rho \cdot \text{res}(\sigma) \quad \forall \sigma \in CS_{\text{cpt}}(\phi(U)).
$$

We can now use a partition of unity $(U_i, \chi_i)_{i \in I}$ subordinated to an atlas $(U_i, \phi_i)_{i \in I}$ on $M$ to write any operator $P \in \text{Cl}(M)$ as a finite sum of localised operators $P = \sum_{i \in I} P_i$ with $P_i := \chi_i \cdot P \chi_i$. We can further assume that $P_i = \phi_i^* \text{Op}(p_i)$ with $p_i \in CS_{\text{cpt}}(\phi(U_i))$. It follows from the first part of the proof that $L(P_i) = \rho_{\phi_i}(p_i) = c_{\phi_i} \cdot \text{res}(p_i)$ so that by linearity of $L$, we have $L(P) = \sum_{i \in I} L(P_i) = \sum_{i \in I} c_{\phi_i} \cdot \text{res}(p_i)$. But since the l.h.s is globally defined, the r.h.s is independent of the local chart; it follows that $L(P) = c \cdot \text{res}(P)$ for some constant $c \in \mathbb{C}$. \(\Box\)

### 3.2 Uniqueness of the canonical trace

**Proposition 11** Let $\mathcal{D}(U)$ be a subset of $CS_{\text{cpt}}(U)$ containing smoothing symbols which is stable under multiplication by smooth functions:

$$C_{\text{cpt}}^\infty(U) \cdot \mathcal{D}(U) \subset \mathcal{D}(U).$$

Let

$$S := \{ \sigma \in CS_{\text{c.c.}}(\mathbb{R}^n), \: f \cdot \sigma \in \mathcal{D}(U) \forall f \in C_{\text{cpt}}^\infty(U) \}.$$

We further assume that $C_{\text{cpt}}^\infty(U) \otimes (S \cap CS_{\text{c.c.}}^a(\mathbb{R}^n))$ is dense in $\mathcal{D}(U) \cap CS_{\text{cpt}}^a(U)$ for any $a \in \mathbb{C}$ and that it fulfills the requirements of Theorem 4.

Then, any continuous linear form $\rho : \mathcal{D}(U) \to \mathbb{C}$ which satisfies Stokes’ property is proportional to the cut-off regularised integral:

$$\exists c \in \mathbb{C}, \quad \rho(\sigma) = c \cdot \int_{T \cdot U} \sigma \quad \forall \sigma \in \mathcal{D}(U).$$

**Remark** Since $S$ fulfills the requirement of Theorem 2 we have $S \subset \text{Ker}(\text{res})$ and hence $\text{res}(f \cdot \sigma) = 0 \forall f \in C_{\text{cpt}}^\infty(U), \forall \sigma \in S$. By a density argument using the continuity of the residue on symbols of constant order, this implies that $\text{res}(f \cdot \sigma) = 0 \forall f \in C_{\text{cpt}}^\infty(U), \forall \sigma \in \mathcal{D}(U)$. The requirements of the proposition therefore imply that symbols in $\mathcal{D}(U)$ have vanishing residue density $\text{res}_x(\sigma(x, \cdot)) = 0 \forall \sigma \in \mathcal{D}(U) \forall x \in U$.

**Proof:** We closely follow the proof of Proposition 10.

For a fixed $f \in C_{\text{cpt}}^\infty(U)$ the map $\sigma \mapsto \rho(f \sigma)$ defines a continuous linear form on $S$ which vanishes on derivatives in $\xi$ since we have $\rho(f \partial_\xi \sigma) = \rho(\partial_\xi (f \sigma)) = 0$ for any smooth function $f \in C_{\text{cpt}}^\infty(U)$. By Theorem 2 it follows that there is a constant $c(f)$ such that $\rho(f \sigma) = c(f) \cdot \int_{\mathbb{R}^n} \sigma$ for any $\sigma \in S$ and for any $f \in C_{\text{cpt}}^\infty(U)$. Since $CS_{\text{c.p.}}^a(U) \otimes (S \cap CS_{\text{c.c.}}^a(\mathbb{R}^n))$ is dense in $\mathcal{D}(U) \cap CS_{\text{cpt}}^a(U)$ for any $a \in \mathbb{C}$ and since $\rho$ is continuous on $\mathcal{D}(U) \cap CS_{\text{cpt}}^a(U)$ it follows that

$$\rho(\sigma) = F \left( \int_{\mathbb{R}^n} \sigma (x, \cdot) \right)$$

for some continuous distribution $F : f \mapsto \int_U f(x) \phi(x) dx$ with $\phi \in C^\infty(U)$. From the closedness of $\rho$ we infer that $\rho(\partial_\xi f \sigma) = F (\partial_\xi f \int_{\mathbb{R}^n} \sigma) = 0$ for any $\sigma \in S$ and any $f \in C_{\text{cpt}}^\infty(U)$. Choosing $\sigma$ such that $\int_{\mathbb{R}^n} \sigma \neq 0$ implies that $F(\partial_\xi f) = 0$ and hence that $\phi$ is constant and

$$\rho(\sigma) = c \cdot \int_{U} \int_{\mathbb{R}^n} \sigma = c \cdot \int_{U \times \mathbb{R}^n} \sigma \quad \forall \sigma \in \mathcal{S}(U).$$

\(\Box\)

**Example 7** $\mathcal{D}(U) := CS_{\text{c.p.}}^{\text{ct}}(U)$ satisfies the assumptions of the proposition. Indeed in that case $S = CS_{\text{c.c.}}^a(\mathbb{R}^n)$ fulfills the requirements of Theorem 2 and $C_{\text{cpt}}^\infty(U) \otimes CS_{\text{c.c.}}^a(\mathbb{R}^n)$ is dense in $CS_{\text{cpt}}^a(U)$ for any non integer order $a$.

\(\text{10}\) i.e. its restriction to $\mathcal{D}(U) \cap CS_{\text{cpt}}^a(U)$ is continuous for any $a \in \mathbb{C}$. 30
Example 8 If the dimension $n$ is odd then

$$D(U) := CS_{cpt}^{odd}(U) = \{ \sigma \in CS_{cpt}(U), \quad \sigma_{a-j}(x,-\xi) = (-1)^{a-j} \sigma_{a-j}(x,\xi) \quad \forall x \in U \quad \text{with} \quad a = \text{ord} \sigma \}$$

fulfills the assumptions of Proposition 11 with $U$ since $a = \text{ord} \sigma$.

Proof: Let us first observe that given any local chart $(U, \phi)$ on $M$

$$S_\phi := \{ \sigma \in CS_{c.c.}(\mathbb{R}^n), \quad \phi^* \text{Op}(f \cdot \sigma) \in D(M) \quad \forall f \in C_{c.p.t}^\infty(\phi(U)) \}$$

fulfills the assumptions of Theorem 5.

Then any continuous linear form $L : D(M) \to \mathbb{C}$ which vanishes on brackets:

$$L ([A,B]) = 0 \quad \forall A,B \in C\ell(M) \quad \text{s.t.} \quad [A,B] \in D(M)$$

is proportional to the canonical trace:

$$\exists c \in \mathbb{C}, \quad L(A) = c \cdot \text{TR}(A) \quad \forall A \in D(M).$$

Remark 11 According to Remark 10, it follows from the assumption 2 that $D(M)$ is contained in

$$\text{Ker} (\text{res})_{loc} (M) := \{ A \in C\ell(M), \quad \text{s.t.} \quad f \cdot A \in \text{Ker} (\text{res}) \quad \forall f \in C^\infty(M) \}$$

which corresponds to the linear space of operators $A \in C\ell(M)$ with vanishing residue density i.e.:

$$\text{Ker} (\text{res})_{loc} (M) = \{ A \in C\ell(M), \quad \text{s.t.} \quad \sigma_A(x,\cdot) \in \text{Ker} (\text{res}_x) \quad \forall x \in M \}$$

since

$$\text{res} (f A) = 0 \quad \forall f \in C^\infty(M)$$

$$\Leftrightarrow \quad \int_M f(x) \text{res}_x (\sigma_A)(x,\cdot) \, dx = 0 \quad \forall f \in C^\infty(M)$$

$$\Leftrightarrow \quad \text{res}_x (\sigma_A)(x,\cdot) = 0 \quad \forall x \in M.$$

Proof: Since the proof closely follows that of Theorem 4 we do not repeat some of the steps common to the two proofs.

Let us first observe that given any local chart $(U, \phi)$ on $M$ the set

$$D(\phi(U)) := \{ \sigma \in CS_{cpt}(\phi(U)), \quad \phi^* \circ \text{Op}(\sigma) \in D(M) \}$$

fulfills the assumptions of Proposition 11 with $U$ replaced by $\phi(U)$ and with $\mathcal{S}$ replaced by $\mathcal{S}_\phi$ as in the statement of the theorem.

---

11 i.e. which restricts to a continuous map on $D(M) \cap C\ell^a(M)$ for any $a \in \mathbb{C}$.

12 By linear we mean here that $L(\alpha A + \beta B) = \alpha L(A) + \beta L(B)$ whenever $A, B, \alpha A + \beta B \in D(M)$.
From a linear form $L$ on $\mathcal{D}(M)$ which obeys the requirements of the theorem we can build the linear form

$$\rho_\phi := L \circ \phi^* \circ \text{Op}$$

on $\mathcal{D}(\phi(U))$ which obeys the requirements of Proposition $10$. Hence $\rho_\phi$ is proportional to the cut-off regularised integral so that there is a constant $c_\phi$ such that

$$\rho_\phi(\sigma) = L(\phi^* \text{Op}(\sigma)) = c_\phi \cdot \int_{T^* \phi(U)} \sigma \quad \forall \sigma \in \mathcal{D}(\phi(U)).$$

As before, using a partition of unity to write any operator $P \in \mathcal{C}_\ell(M)$ as a finite sum of localised operators $P = \sum_{i \in I} P_i$ with $P_i := \chi_i P \chi_i$, with $P_i = \phi_i^* \text{Op}(p_i)$ with $p_i \in C_\text{cpt}(\phi_i(U_i))$ we infer that $L(P_i) = \rho_{\phi_i}(p_i) = c_{\phi_i} \cdot \int_{T^* \phi_i(U_i)} p_i$ so that by linearity of $L$

$$L(P) = \sum_{i \in I} L(P_i) = \sum_{i \in I} c_{\phi_i} \cdot \int_{T^* \phi_i(U_i)} p_i.$$

But since the l.h.s is globally defined, the r.h.s is independent of the local chart; it follows that $L(P) = c \cdot \int_{T^* M} \sigma(P) = c \cdot \text{TR}(P)$ for some constant $c \in \mathbb{C}$. $\square$

Here are a few known examples of sets $\mathcal{D}(M)$ which obey assumptions 1 and 2 of the above theorem. In particular, they lie in $\text{Ker}(\text{res})_{\text{loc}}(M)$.

**Example 10** The set $\mathcal{C}_\ell^{\mathbb{Z}}(M)$ of non integer order classical pseudodifferential operators on $M$.

**Example 11** The set $\mathcal{C}_\ell^{\text{odd}}(M)$ of odd-class operators on odd dimensional manifolds $M$ introduced in [KV] (see also [Gr] where such operators are called even-even).

$$\mathcal{C}_\ell^{\text{odd}}(M) = \{ A \in \mathcal{C}_\ell^{\mathbb{Z}}(M), \quad \sigma(A) \sim \sum_{j=0}^{\infty} \chi \sigma_{a-j}(A), \quad \sigma_{a-j}(A)(x, -\xi) = (-1)^{a-j} \sigma_{a-j}(A)(x, \xi) \}$$

of odd-class operators on odd dimensional manifolds $M$ (see [Gr] where such operators are called odd-oDD).

**Example 12** The set $\mathcal{C}_\ell^{\text{even}}(M)$ of even-class operators on even dimensional manifolds $M$ (see [Gr] where such operators are called even-even).

$$\mathcal{C}_\ell^{\text{even}}(M) = \{ A \in \mathcal{C}_\ell^{\mathbb{Z}}(M), \quad \sigma(A) \sim \sum_{j=0}^{\infty} \chi \sigma_{a-j}(A), \quad \sigma_{a-j}(A)(x, -\xi) = (-1)^{a-j+1} \sigma_{a-j}(A)(x, \xi) \}$$

of even-class operators on even dimensional manifolds $M$.

Applying Theorem $11$ to $\mathcal{D}(M) = \mathcal{C}_\ell^{\mathbb{Z}}(M)$, resp. $\mathcal{D}(M) = \mathcal{C}_\ell^{\text{odd}}(M)$ in odd dimensions, resp. $\mathcal{D}(M) = \mathcal{C}_\ell^{\text{even}}(M)$ in even dimensions, leads to the following uniqueness result.

**Corollary 8** The canonical trace is (up to a multiplicative constant) the unique linear form on $\mathcal{C}_\ell^{\mathbb{Z}}(M)$, resp. $\mathcal{C}_\ell^{\text{odd}}(M)$ in odd dimensions, resp. $\mathcal{C}_\ell^{\text{even}}(M)$ in even dimensions which is continuous on operators of constant order and which vanishes on brackets that lie in $\mathcal{C}_\ell^{\mathbb{Z}}(M)$, resp. $\mathcal{C}_\ell^{\text{odd}}(M)$ in odd dimensions, resp. $\mathcal{C}_\ell^{\text{even}}(M)$.

**Remark 12** In the course of the proof we showed that the vanishing of $L$ on brackets $[P_a, -]$ implies Stokes’ property for $\rho_\phi$. Conversely, Stokes’ property for $\rho_\phi$ implies that $L(\phi^* \text{Op}(\sigma)) := \rho_\phi(\sigma)$ vanishes on brackets $[x_i, -]$ and $[\partial x_i, -]$ contained in $\mathcal{D}(M)$. But this implies that $L$ vanishes on brackets $[P_U, -] \in \mathcal{D}(M)$ where $P_U$ is the localisation of any classical pseudodifferential operator. Stokes’ property on symbols and the vanishing on brackets of operators are therefore equivalent.
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