Approximate Verification of Partially Observable Discrete Time Stochastic Hybrid Systems

Kendra Lesser, Student Member, IEEE, Meeko Oishi, Member, IEEE

Abstract

Assuring safety in discrete time stochastic hybrid systems is particularly difficult when only partial observations are available. We first review a formulation of the probabilistic viability (i.e. safety) problem under noisy hybrid observations as a dynamic program. Two methods for approximately solving the dynamic program are presented. The first method approximates the hybrid system as an equivalent finite state Markov decision process, so that the information state is a probability mass function. The second approach approximates an indicator function over the safe region using radial basis functions, to represent the information state as a Gaussian mixture. In both cases, we discretize the hybrid observation process and generate a sampled set of information states, then use point-based value iteration to under-approximate the viability probability. We obtain error bounds and convergence results in both cases, assuming additive Gaussian noise in the continuous state dynamics and observations. We compare the performance of the finite state and Gaussian mixture approaches on a simple numerical example.

I. INTRODUCTION

Stochastic hybrid systems provide a modeling framework well-suited for a wide range of applications. They allow for versatile dynamics that incorporate codependent discrete and continuous states, often exhibited in systems that may switch between different modes of operation, and account for probabilistic uncertainty in those dynamics. Such a flexible framework is particularly important in the context of safety verification, where the assessment of a system’s ability to meet rigorous safety requirements must be as accurate as possible. Indeed, viability, and similarly reachability, analysis (determining whether a system’s state stays within a given safe region and/or reaches a target set within some finite time horizon) for hybrid systems has been studied extensively [1], [2], [3], [4], [5].

Noisy or incomplete measurements of the state must be taken into account for safety verification. While there has been some work on deterministic hybrid systems with incomplete information [6] or uncertain hybrid systems with
the assumption of a worst-case disturbance [7], viability analysis of a partially observable stochastic hybrid system has been approached only recently [8], [9], and only theoretically; there are currently no computational results for viability or reachability properties of partially observable stochastic hybrid systems.

Computational results for analysis of perfectly observable stochastic hybrid systems are also limited. The viability problem for discrete time stochastic hybrid systems (DTSHS) is a multiplicative cost stochastic optimal control problem [4], which can equivalently be formulated as a Markov decision process (MDP). Unfortunately, solutions via dynamic programming [10] require evaluation of a value function over all possible states, which is infinite when those states are continuous. Discretization procedures can be employed to impose a finite number of states, as in [11] and [12], which present rigorous uniform and adaptive gridding methods for verification of DTSHS. Other approximate solution strategies include approximate dynamic programming, where the value function of the dynamic program is approximated by a set of basis functions, as in [13]. Even so, current applications are limited to those with only a few discrete and continuous states.

The viability problem for a partially observable DTSHS (PODTSHS) can similarly be formulated as a partially observable MDP (POMDP). However, POMDPs are plagued by dimensionality on an even greater scale than MDPs. The common approach to solving POMDPs is to replace the growing history of observations and actions by a sufficient statistic, often called the belief state, which, for a POMDP with an additive cost function, is the distribution of the current state conditioned on all past observations and actions [10]. This belief state is treated as the perfectly observed true state, and MDP solution methods can then be applied. However, with a continuous state space, the belief state is a continuous function defined over an infinite domain, and it is impossible to enumerate over all such functions. Therefore the study of efficient, approximate solutions to POMDPs is essential.

Although finding the solution to a general POMDP is hard [14], many algorithms for approximating solutions to finite state POMDPs have been developed. These mainly rely on point-based value iteration (PBVI) schemes that only consider a subset of the belief space to update the value function used in the dynamic program (for a survey of PBVI algorithms, see [15]). Because the value function is piecewise-linear and convex [16] (and so equivalently represented by a finite set of so-called $\alpha$-vectors), sampling from the belief state provides a systematic way of storing a finite subset of those vectors. Such methods must be tailored to continuous state POMDPs because of the dimensionality of the belief state.

Often the continuous state can be discretized and approximately solved as a finite state POMDP using PBVI methods. Depending on the dimensionality and behavior of the system, it may be more informative or computationally more efficient to preserve the continuity of the state space. Many existing methods for continuous state POMDPs assume the belief state is Gaussian (e.g. [17], [18]), and represent the belief state in a parameterized form which is then discretized and solved as a discrete state MDP. When the belief state cannot adequately be represented using a single Gaussian, a Gaussian mixture may be used instead. An equivalent point-based algorithm for continuous-state POMDPs using Gaussian mixtures is presented in [19], and demonstrated on a stochastic hybrid system with hidden modes in [20].

The viability problem for a PODTSHS further complicates the already difficult problem of solving continuous state POMDPs. As shown in both [8] and [9], the belief state of the PODTSHS is not the conditional distribution
of the current state of the system, but must also include the distribution of a binary variable that indicates whether the state of the system has remained within a safe region up to the previous time step. This, coupled with the stochastic hybrid system dynamics, makes accurately representing the belief state as a single Gaussian impossible.

Therefore, as the first investigation into approximate probabilistic safety verification for PODTSHS, we formulate the viability problem for a PODTSHS as a partially observable Markov decision process, and investigate representations of the belief state in either vector or Gaussian mixture form through finite- and continuous-state approximations to the PODTSHS. These representations allow us to exploit point-based methods developed for POMDPs, by sampling from the belief space and approximating the value function with a finite set of observations.

In this paper we make several contributions to the solution of safety verification problems for PODTSHS. First, we validate the use of POMDP solution techniques for viability analysis of a PODTSHS, by demonstrating that the value function is convex and admits an $\alpha$-function representation similar to the piecewise-linear $\alpha$-vector representation of a finite state POMDP. Second, we present a finite state approximation to the DTSHS (extended from [11]) that allows the belief state to take vector form under certain conditions, and show convergence for the approximation. Third, we preserve the continuity in the hybrid state space through a Gaussian mixture representation for the belief state, and approximate the indicator function that represents the safe region using Gaussian radial basis functions. In this case, we provide an error bound as a function of the $L^1$ error of the indicator function approximation. We outline a solution method that converges to the true solution from below, using either the finite or continuity-preserving belief state. Finally we demonstrate both approaches on a simple temperature regulation problem.

The rest of the paper is organized as follows. Section II defines the viability problem for a PODTSHS, and describes the related problem of optimal control of POMDPs. Section III justifies the use of POMDP solution techniques, and gives the finite and Gaussian mixture approximations to the viability problem for a PODTSHS as well as error bounds. Section IV describes the use of point-based approximation techniques, through sampling of belief states and discretization of the observations. Section V provides preliminary numerical results, and Section VI provides concluding remarks and directions for future work.

II. BACKGROUND

A. PODTSHS Model

A hybrid system is characterized by a set of both discrete and continuous states with interacting dynamics: the discrete state may affect the evolution of the continuous dynamics, and the continuous dynamics may affect changes in the discrete state. In the case of a DTSHS, both the discrete and continuous dynamics may be characterized by stochastic kernels, the product of which determines the stochastic transition kernel governing the combined discrete/continuous state of the system. For a PODTSHS, it is assumed that only an observation process is available to the controller, of the form $y_n = (y^x_n, y^q_n)$, where $y^x_n$ is associated with $x_n$, and $y^q_n$ with $q_n$. The observations are stochastic, and governed by independent stochastic kernels that are combined to produce a probability measure on the full observation process.
Definition 1. (Partially Observable Discrete Time Stochastic Hybrid System \( \mathcal{H} \)). A PODTSHS is a tuple \( \mathcal{H} = (\mathcal{X}, \mathcal{Q}, \mathcal{Y}^x, \mathcal{Y}^q, \mathcal{U}, T_x, T_q, T_{y^x}, T_{y^q}, \rho) \) where

1) \( \mathcal{X} \subseteq \mathbb{R}^n \) is a set of continuous states
2) \( \mathcal{Q} = \{q_1, q_2, \ldots, q_{N_q}\} \) is a finite set of discrete states with cardinality \( N_q \), with \( \mathcal{S} = \mathcal{X} \times \mathcal{Q} \) the hybrid state space
3) \( \mathcal{Y}^x \subseteq \mathbb{R}^n \) is a set of continuous observations
4) \( \mathcal{Y}^q \subseteq \mathcal{Q} \) is a set of discrete observations, with \( \mathcal{Y} = \mathcal{Y}^x \times \mathcal{Y}^q \) the hybrid observation space
5) \( \mathcal{U} \) is a finite, bounded set of possible control inputs, that affects discrete and continuous state transitions
6) \( T_x : \mathcal{B}(\mathbb{R}^n) \times \mathcal{Q} \times \mathcal{S} \times \mathcal{U} \to [0, 1] \) is a Borel-measurable stochastic kernel that assigns a probability measure to \( x_{n+1} \) given \( s_n = (x_n, q_n), u_n, q_{n+1} \) for all \( n \): \( T_x(dx_{n+1} \in B|q_{n+1}, s_n, u_n) \) where \( B \in \mathcal{B}(\mathbb{R}^n) \), the Borel \( \sigma \)-algebra on \( \mathbb{R}^n \)
7) \( T_q : \mathcal{Q} \times \mathcal{Q} \times \mathcal{U} \to [0, 1] \) is a discrete transition kernel that assigns a probability distribution to \( q_{n+1} \) given \( q_n, u_n \) for all \( n \): \( T_q(q_{n+1}|q_n, u_n) \)
8) \( T_{y^x} : \mathcal{B}(\mathbb{R}^n) \times \mathcal{X} \to [0, 1] \) is a continuous Borel-measurable observation function that assigns a probability distribution to observation \( y^x_n \) given state \( x_n \) for all \( n \): \( T_{y^x}(dy^x_n \in B|x_n) \)
9) \( T_{y^q} : \mathcal{Y}^q \times \mathcal{Q} \times \mathcal{U} \to [0, 1] \) is a discrete observation function that assigns a probability distribution to \( y^q_n \) given \( q_n \) and \( u_n \) for all \( n \): \( T_{y^q}(y^q_n|q_n, u_n) \)
10) \( \rho : \mathcal{B}(\mathbb{R}^n) \times \mathcal{Q} \to [0, 1] \) is an initial Borel-measurable density lying in the space of all probability measures on \( \mathcal{S} \), such that \( \rho(dx_0 \in B, q) \)

Kernels \( T_x \) and \( T_q \) can be combined for ease of notation to produce one hybrid state transition kernel

\[
\tau_s(ds_{n+1}|s_n, u_n) = T_x(dx_{n+1}|q_{n+1}, x_n, q_n, u_n)T_q(q_{n+1}|q_n, u_n)
\] (1)

We assume the continuous state \( x \) obeys the affine difference equation

\[
x_{n+1} = Ax_n + g(q_n, u_n, q_{n+1}) + v_n.
\] (2)

Matrix \( A \) is assumed invertible, and \( v_n \) are independent and identically distributed Gaussian random variables for all \( n \), \( v_n \sim \mathcal{N}(0, \mathcal{V}) \). Therefore kernel \( T_x \) admits a Gaussian density, with \( T_x(dx' \in B|q', x, q, u) = \int_B \phi(dx'; Ax + g(q, u, q'), \mathcal{V}) \). The function \( \phi \) is used to represent a Gaussian probability density function (pdf); \( \phi(x; \mu, \Sigma) \) is equal to a Gaussian pdf with mean \( \mu \) and covariance \( \Sigma \) evaluated at \( x \).

The observation functions can be combined as well, to produce one observation function

\[
\tau_y(dy_n|s_n, u_n) = T_{y^x}(dy^x_n|x_n)T_{y^q}(y^q_n|q_n, u_n).
\] (3)

We assume that the continuous observation \( y^x_n \) is equal to the state \( x_n \), disrupted by additive Gaussian noise \( w_n \sim \mathcal{N}(0, \mathcal{W}) \).

\[
y^x_n = x_n + w_n
\] (4)

The observation function \( T_{y^x} \) therefore also has a Gaussian density, with \( T_{y^x}(dy \in B|x) = \int_B \phi(dy;x, \mathcal{W}) \). The discrete observation space is defined for simplicity as \( \mathcal{Y}^q = \mathcal{Q} \).
Finally, the initial density $\rho$ is assumed Gaussian in $x$: $\rho(x, q) = Q_0(q)\phi(x; \mu_0, \Sigma_0)$ such that $\sum_{q \in Q} \int_X \rho(x, q) \, dx = 1$.

Because $T_{y^r}, T_x$, and $\rho$ are Gaussian, and $\mathcal{U}$ is finite and bounded, the following Lipschitz properties hold.

\[
\begin{align*}
\|T_x(x'|q', s, u) - T_x(x'|q, s, u)\| &\leq h_x^{(1)} \|x' - \bar{x}\| \\
\|T_x(x'|q', x, q, u) - T_x(x'|q, x, q, u)\| &\leq h_x^{(2)} \|x - \bar{x}\| \\
\|T_{y^r}(y|x) - T_{y^r}(\bar{y}|x)\| &\leq h_y^{(1)} \|y - \bar{y}\| \\
\|T_{y^r}(y|x) - T_{y^r}(\bar{y}|\bar{x})\| &\leq h_y^{(2)} \|x - \bar{x}\|
\end{align*}
\]

(5)

We define the maximum values of the densities associated with $T_x$ and $T_{y^r}$ as $\phi_v^*$ and $\phi_w^*$, respectively, with $\phi_v^* = (2\pi)^{-\frac{n}{2}}|\mathcal{V}|^{-\frac{1}{2}}$ and $\phi_w^* = (2\pi)^{-\frac{n}{2}}|\mathcal{W}|^{-\frac{1}{2}}$.

Definition $\Pi$ produces a probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P}^\pi)$ with $\Omega$ the state space $\mathcal{S}^N \times \mathcal{Y}^N$, $\mathcal{B}(\Omega)$ the Borel $\sigma$-algebra on $\Omega$, and $\mathbb{P}^\pi$ a probability measure based on $\rho$, $\tau_s$, $\tau_y$, and a control policy $\pi$ that selects control inputs $\mathcal{U}$ at each time step. The filtrations $\mathcal{G}_n$ and $\mathcal{Y}_n$ are generated by the sequences $\{s_0, \ldots, s_n, y_1, \ldots, y_{n-1}\}$ and $\{y_1, \ldots, y_n\}$, respectively. The available information at time $n$ is written as $i_n = (u_0, \ldots, u_{n-1}, y_1, \ldots, y_n) \in \mathcal{I}_n = \mathcal{U}^n \times \mathcal{Y}^n$ ($\mathcal{U}^n = \mathcal{U} \times \ldots \times \mathcal{U}$ the $n$-times Cartesian product of $\mathcal{U}$, and similarly for $\mathcal{Y}^n$). The control policy is $\pi = (\mu_0, \ldots, \mu_{N-1})$ with $\mu_n : \mathcal{I}_n \rightarrow \mathcal{U}$.

B. Viability Problem

Next, we present a cost function to analyze viability properties of the PODTSHS, i.e. the ability of the state to remain within some safe or desired region of the state space. We want to find both a control policy that maximizes the probability of the state remaining within that desired set, as well as an estimate of that probability. As in [4], this problem can be formulated using stochastic optimal control notation. For a compact Borel set $K \subseteq \mathcal{X} \times \mathcal{Q}$, terminal time $N$, and predefined policy $\pi$, define the cost function as

\[
p_{\text{safe}}(\pi, \rho; K) = \mathbb{P}_\pi^\pi[s_n \in K \forall n = 0, \ldots, N|\rho].
\]

(6)

Since for a random variable $X$ and event $A$, $\mathbb{P}[X \in A] = \mathbb{E}[\mathbf{1}_A(X)]$, with $\mathbb{E}$ denoting expected value and the indicator function $\mathbf{1}_A(X) = 1$ if $X \in A$ and $\mathbf{1}_A(X) = 0$ otherwise, (6) is rewritten as in [4]:

\[
p_{\text{safe}}(\pi, \rho; K) = \mathbb{E}^\pi \left[ \prod_{n=0}^{N} \mathbf{1}_K(s_n) \right].
\]

(7)

The expected value is taken with respect to the measure $\mathbb{P}_\pi^\pi$, hence the notation $\mathbb{E}^\pi$. We want to maximize $p_{\text{safe}}(\pi, \rho; K)$ with respect to the control policy $\pi$. The set $\Pi$ of admissible policies will be restricted to non-randomized policies, i.e. $\mu_n(i_n)$ maps to a single control $u$ with probability 1. The viability probability and optimal policy $\pi^*$ are given by

\[
p_{\text{safe}}(\rho; K) = \sup_{\pi \in \Pi} p_{\text{safe}}(\pi, \rho; K),
\]

(8)

\[
\pi^* = \arg \sup_{\pi \in \Pi} \left\{ p_{\text{safe}}(\pi, \rho; K) \right\}.
\]

(9)

We can now formally define the problem we wish to solve.
Problem 1. Consider a PODTSHS $\mathcal{H}$ (defined in Definition 1). Given a safe set $K$ and time horizon $N$ we would like to

1) compute the maximal probability \[8\] of remaining within $K$ for $N$ time steps, and

2) compute the optimal policy $\pi^*$ given by \[9\].

If the maximal probability and optimal policy cannot be computed exactly (which is quite likely \[14\]), an approximation that produces a suboptimal policy and lower bound on the maximal viability probability is desired.

C. Computing Optimal Control Policies for POMDPs

We provide an overview of POMDPs and efficient approximation techniques for optimal control, as required for our solution to Problem 1. POMDPs provide a framework for analyzing a controlled discrete time system, in which the controller is designed to optimize a known objective. The state evolves stochastically and is Markovian (e.g., the state at the next time step depends only on the current state and action). Further, the controller cannot directly observe the state of the system, and only has access to an observation process. We consider a POMDP with discrete states, actions, and observations, and an additive cost function. The theory and solution techniques for this type of POMDP provide the foundation for our extension to a PODTSHS and the solution of Problem 1.

Given a function $R$ that assigns a reward to states and control inputs at each time $n$, the goal is to maximize the expected sum of rewards over a (possibly infinite) time horizon $N$ by optimally choosing a sequence of control actions $\pi = \{u_1, u_2, \ldots\}$.

$$\max_{\pi} \mathbb{E} \left[ \sum_{n=0}^{N} R(s_n, u_n) \right]$$ (10)

The optimal control at time $n$ is based on a belief state that summarizes all available information up to time $n$, as opposed to a recorded history of all past actions and observations. The belief state is a sufficient statistic for the set of all observations and actions $\{u_1, \ldots, u_{n-1}, y_1, \ldots, y_n\}$ because it condenses all information necessary for making optimal decisions \[10\]. For additive reward function $R$, the belief state is a density function that describes the probability of being in state $s$ given all past observations and actions, $b(s_n) = P[s_n | u_1, \ldots, u_{n-1}, y_1, \ldots, y_n]$. By treating the belief state as the true state of the system, \[10\] can be equivalently solved using a perfect state information Markov decision process. An optimal policy $\pi^*$ for the POMDP is then defined in terms of the belief state, and maps beliefs to actions: $\pi^* : B \to \mathcal{U}$.

The optimal policy for a particular belief state can be found using a dynamic program. At each time step, the dynamic program maximizes a value function that describes the cumulative reward from time $n$ to the final time $N$, defined recursively as

$$V^*_n(b) = \max_{u \in \mathcal{U}} \left\{ \sum_s R(s, u) b(s) \sum_y V^*_{n+1} (M_{y,u} b) \mathbb{P}(y|u, b) \right\},$$ (11)

with the transition operator $M_{y,u} b$ that provides the next belief state $b_{n+1}$ given the current observation, action, and belief state

$$(M_{y,u} b) (s') = \frac{T_y(y|s', b, u) \sum_{s \in \mathcal{S}} T_s(s'|s, u) b(s)}{\mathbb{P}(y|b, u)},$$ (12)
and likelihood of the observation given by
\[ P(y|b, u) = \sum_{s \in S} b(s) \sum_{s' \in S} T_s(s'|s, u)T_y(y|s', u), \]  
with \( T_s \) and \( T_y \) the stochastic transition functions corresponding to finite state space \( S \) and observation space \( \mathcal{Y} \), respectively. Sondik [16] first showed that for a finite horizon \( N < \infty \), the value function at each time \( n \) is piecewise-linear and convex in the belief state, and thus can be expressed as
\[ V_n^*(b) = \max_{\alpha_n^i \in \Gamma_n} \sum_s \alpha_n^i(s)b(s). \]  
The functions \( \alpha_n^i \in \Gamma_n \), or “\( \alpha \)-vectors”, represent a policy tree that starts from a specific action \( u \) and state \( s \), and specifies optimal actions conditioned on observations for time steps \( n + 1 \) to \( N \). The \( \alpha \)-vectors thus characterize the current value of being in state \( s \) and taking action \( u \), plus the expected sum of future rewards assuming all subsequent actions are chosen optimally. Because each \( \alpha \)-vector is associated with a specific action, by picking the \( \alpha \)-vector that maximizes \( \sum_s \alpha_n^i(s)b(s) \), we also define the optimal policy for belief \( b \) at time \( n \).

Calculating the value function and optimal policy requires knowledge of the complete sets of \( \alpha \)-vectors, \( \Gamma_n \), for all \( n \). The \( \alpha \)-vectors at time \( n \) are computed recursively from the \( \alpha \)-vectors calculated at time \( n + 1 \). For each action, we observe one of \( |\mathcal{Y}| \) observations (where \(| \cdot |\) indicates the cardinality of the set), and for each observation there is a subsequent \( \alpha \)-vector defined at time \( n + 1 \), resulting in \( |\mathcal{Y}| \Gamma_{n+1} |\mathcal{Y}| \) \( \alpha \)-vectors at time \( n \). Hence using the \( \alpha \)-vector representation to optimize a POMDP is often infeasible, because the number of \( \alpha \)-vectors grows exponentially.

An approximate solution can be obtained though point-based value iteration (PBVI), in which a lower bound of the value function is computed using a finite subset \( B \subseteq \mathcal{B} \). The general idea is to generate a collection of points \( b \in B \), and for each of these points, estimate the value function via a “backup” operation. PBVI approaches are distinguished by how \( B \) is selected [21], [15].

We summarize the most common method of under-estimating the value function, assuming \( B \) has already been selected. One \( \alpha \)-vector is generated for each belief point \( b^i \in B \), \( B = (b^0, b^1, \ldots, b^m) \), so that \( \tilde{\Gamma}_n = (\alpha_n^0, \alpha_n^1, \ldots, \alpha_n^m) \) for all \( n \). We assume that an \( \alpha \)-vector \( \alpha_n^i \) corresponding to \( b^i \) will apply to all belief points in a region around \( b^i \) (i.e. for any \( b \) in a neighborhood of \( b^i \) the same action will likely be optimal). Hence the value at some \( b \) not necessarily in \( B \) can be approximated by
\[ V_n^*(b) \approx \max_{\alpha_n^i \in \tilde{\Gamma}_n} \sum_s \alpha_n^i(s)b(s) \]  
as in [14] but with a restricted set \( \tilde{\Gamma}_n \subseteq \Gamma_n \). The set \( \tilde{\Gamma}_n \) is generated recursively from \( \tilde{\Gamma}_{n+1} \), but without enumeration over all possible combinations of observations and subsequent \( \alpha \)-vectors in \( \tilde{\Gamma}_{n+1} \), by using the following backup operation for each \( b \in B \).

\[ \text{backup}(b) = \operatorname{arg} \max_{\alpha_n^i \in \tilde{\Gamma}_n} \sum_{s \in S} \alpha_n^i(s)b(s) \]  

The overall PBVI algorithm then consists of selecting a set of belief points \( B \), and repeatedly applying [15] to each element of \( B \). In the case of a finite horizon of length \( N \), the backup operator will be applied to \( B \) \( N \) times, and for an infinite horizon, the backup operator will be applied until some tolerance level is reached (for example, where \( \|V_{n+1}(b) - V_n(b)\| < \epsilon \).
are discrete, the number of $\alpha$-functions is finite, and the value function is piecewise-linear and convex under the $\alpha$-function representation. Further, summations over $S$ are replaced by integrals, hence (15) is written as

$$\text{backup}(b) = \arg \max_{\alpha_i \in \Gamma_n} (\alpha_i^T, b).$$

We maintain this notation in our derivations, where in the case of a hybrid state space with continuous state $x$ and discrete state $q$, $\langle f, g \rangle = \sum_q \int f(x, q) g(x, q) \, dx$ for well-defined functions $f$ and $g$.

III. REFORMULATION USING A POMDP

We exploit PBVI to solve Problem 1 by transforming it into an optimal control problem for a POMDP. Hence we first show the viability problem for $\mathcal{H}$ can be reduced to a dynamic program, despite a non-standard belief state. We then show that the $\alpha$-functions and belief states can be approximately represented in closed form (as either vectors or Gaussian mixtures) and that finite collections of each may be generated and used to approximate (8), similar to a point-based POMDP solver.

We present two approximations of Problem 1 for the PODTSHS $\mathcal{H}$: The first discretizes $S$ to produce a finite state POMDP, and the second preserves continuity in $S$ by using a Gaussian mixture approach, thus characterizing the PODTSHS by a collection of weights, means, and covariances.

A. Validity of POMDP Formulation

Since the viability problem is naturally expressed with a multiplicative cost function [4], and the belief state $b$ for an additive cost POMDP is therefore not applicable, we derived a sufficient statistic $\eta = (\eta_0, \ldots, \eta_N)$ for Problem 1 in [9]. This sufficient statistic produces a modified conditional distribution of the current state that includes the probability that all past states are in the safe set $K$.

$$\eta_n(\rho, i_n) = \mathbb{E}^\pi \left[ 1_q(\eta_n) 1_x(x_n) \prod_{i=0}^{n-1} 1_K(s_i) \bigg| \rho, i_n \right]$$

(16)

We define the information state as the function $\sigma_n(x_n, q_n) \in \Sigma \subseteq L^1$ (where $L^1$ is the space of integrable functions) such that $\eta_n(\rho, i_n) = \sigma_n$, which is distinct from the belief state (e.g. the conditional distribution of the current state). We showed that $\sigma_n$ updates recursively with a bounded linear operator $\Phi$ (for proof see [9])

$$\sigma_0 = \rho$$

$$\sigma_n = \Phi_{y_n, u_{n-1}} \sigma_{n-1}$$

(17)

where $\Phi_{y, u}\sigma$ is given by

$$\Theta_{y, u}(s') = \frac{1}{\mathbb{P}(y|\sigma, u)} \tau_\sigma(y|s', u) \int_K \tau_\sigma(s'|s, u) \sigma(s) \, ds.$$  

(18)

In comparing (12) to (17), and (13) to (18), the latter integrates over the compact hybrid set $K$, as opposed to a summation over finite set $S$.

We define a dynamic programming recursion over $\sigma$ as

$$V^*_N(\sigma) = \langle \sigma, 1_K \rangle$$

$$V^*_n(\sigma) = \max_{u \in \mathcal{U}} \mathbb{E}^\pi \left[ V^*_{n+1}(\Phi_{y, u}\sigma) \right]$$

(19)
with solution $V_0^* (\rho) = p_{safe}^N (\rho, K)$. The optimal policy is $\pi^* = (\mu^*_0, \ldots, \mu^*_{N-1})$, with

$$\mu^*_n (\sigma_n) = \arg \max_{u \in U} V_n^* (\sigma_n)$$  \hspace{1cm} (20)

for all $n \in [0, N]$.

**Lemma 1.** For any $n$, the value function (19) can be written as

$$V_n^* (\sigma) = \sup_{\sigma \in \Gamma_n} \left[ \alpha_n^i, \sigma \right].$$

Lemma 1 is shown by induction. For brevity, we only give an outline of the proof.

**Sketch of Proof:** From (19), we see that $\Gamma_N = 1_K (s)$. From the recursion in (19), we have

$$V_n^* (\sigma) = \max_{u \in U} \left[ \sup_{\sigma \in \Gamma_{n+1}} \left[ \int Y \left[ \alpha_{n+1}^i (s') \tau_y (y | s', u) \int K (ds | s, u) \sigma (ds) \right] \right] \right].$$  \hspace{1cm} (21)

For each individual $y$, $u$, and $\sigma$, we seek the $\alpha_{n+1}^i \in \Gamma_{n+1}$ that maximizes the expression within brackets in (22). Hence we express $V_n^*$ in terms of an intermediate function $\alpha_{y,u,\sigma}$.

$$V_n^* (\sigma) = \max_{u \in U} \left[ \int Y \left[ \alpha_{y,u,\sigma} (s) \right] dy \right],$$  \hspace{1cm} (23)

$$\alpha_{y,u,\sigma} (s) = \int S \alpha_{n+1}^* (y) \tau_y (y | s', u) \int K (ds | s, u) 1_K (s)$$  \hspace{1cm} (24)

with $*(y)$ denoting the index $i$ of the $\alpha$-function in $\Gamma_{n+1}$ that maximizes the bracketed expression in (22) for observation $y$. The set of $\alpha$-functions at time $n$ is

$$\Gamma_n = \bigcup_{\sigma \in \Sigma} \left\{ \int Y \alpha_{y,u,\sigma} dy \right\}$$  \hspace{1cm} (25)

with $u^*$ the control inputs chosen according to (23).

**Lemma 2.** The value function (19) is convex in $\sigma$ for all $n = 0, \ldots, N$, $\sigma_1, \sigma_2 \in L^1$ and $0 \leq \lambda \leq 1$:

$$V_n^* (\lambda \sigma_1 + (1-\lambda) \sigma_2) \leq \lambda V_n^* (\sigma_1) + (1-\lambda) V_n^* (\sigma_2).$$

Lemma 2 is shown by construction using the representation of the value function given in Lemma 1. Since Lemmas 1 and 2 show that the value function (19) is convex and admits an $\alpha$-function representation, $\mathcal{H}$ is amenable to POMDP solution techniques. Note, however, that Lemma 1 is not useful for solving Problem 1 directly, since $\Gamma_n$ is not finite and the $\alpha$-functions and information states have no common structure.

**B. Finite State Approximation**

We first consider a finite state POMDP (11), whose solution converges to the true viability probability (8) and optimal policy (9). The state space $S$ is discretized to obtain a vector representation of $\alpha$ and $\sigma$. The observation
space is unchanged (i.e., hybrid), because the set of observations only affects the finiteness of sets $\Gamma_n$ and $\Sigma_n$. We defer discussion of producing finite collections of $\Gamma_n$ and $\Sigma_n$ to section $\text{[IV]}$.

Given compact safe set $K \in B(S)$, let $K = \bigcup_{q \in Q} K_q \times \{q\}$. Denote $\lambda = \max_{q \in Q} L(K_q)$, the finite Lebesgue measure of $K_q \subset \mathbb{R}^n$. Each $K_q$ is partitioned into a finite number of subsets, so that $K_q = \bigcup_{i=1}^{m_q} K_{i,q}$, with $K_{i,q}$ pairwise disjoint (i.e., $K_{i,q} \cap K_{j,q} = \emptyset$ for all $i \neq j$), $K_i,q \in B(\mathbb{R}^n)$. Finally, let $\delta_{i,q}$ be the diameter of partition $K_{i,q}$ so that $\delta_{i,q} = \sup \{ \| x - \overline{x} \| : x, \overline{x} \in K_{i,q} \}$, with $\delta = \max_{i,q} \delta_{i,q}$.

The partition of $K$ is denoted by $G^* = \bigcup_{i=1,\ldots,m_q} G_{i,q}$, with $G_{i,q} = K_{i,q} \times \{q\}$. Each element $G_{i,q}$ has a representative point $(x^i,q)$ and the set $K_{i,q} = \{(x^i,q) : i = 1, \ldots, m_q, q \in Q\}$ is the discrete representation of $K$. We do not consider here how the points $(x^i,q)$ are selected, but an example is provided in Section $\text{[V]}$. The function $\xi : K \rightarrow K_{\delta}$ maps a state $s \in G_{i,q}$ to its representative point $(x^i,q)$ and the function $\Xi : K_{\delta} \rightarrow K$ is the set-valued map from discrete point $(x^i,q)$ to its corresponding set $G_{i,q}$. The discrete state space is defined as $Z_{\delta} = K_{\delta} \cup \{\psi_s\}$, with $\psi_s$ a single variable that represents all states $s \in S \setminus K$.

Definition 2. (POMDP approximation to PODTSHS, $\hat{\mathcal{H}}$. The POMDP approximation is a tuple $\hat{\mathcal{H}} = (Z_{\delta}, \mathcal{Y}, \mathcal{U}, \tau_{\delta}, \tau_y, \rho_{\delta})$ where

1. $Z_{\delta}$ is a finite set of discrete states
2. $\mathcal{Y}$ is as defined in Definition 1
3. $\mathcal{U}$ is as defined in Definition 1
4. $\tau_{\delta} : Z_{\delta} \times \mathcal{U} \times Z_{\delta} \rightarrow [0,1]$ is a discrete state transition function that assigns probabilities to elements of $Z_{\delta}$
5. $\tau_y$ is as defined in Definition 1
6. $\rho_{\delta} : Z_{\delta} \rightarrow [0,1]$ is a function that assigns probabilities to elements of $Z_{\delta}$ at time zero

We define the transition function as

$$
\tau_{\delta}^{\delta}(z'|z,u) = \begin{cases} 
\tau_s(\Xi(z'))|z,u), & \text{if } z' \in K_{\delta} \text{ and } z \in K_{\delta} \\
1 - \sum_{z \in K_{\delta}} \tau_s(\Xi(z)|z,u), & \text{if } z' = \psi_s \text{ and } z \in K_{\delta} \\
1, & \text{if } z' = \psi_s \text{ and } z = \psi_s \\
0, & \text{if } z' \in K_{\delta} \text{ and } z = \psi_s
\end{cases}
$$

with $\sum_{z \in Z_{\delta}} \tau_{\delta}^{\delta}(z'|z,u) = 1$, and the initial distribution $\rho_{\delta}$ on $Z_{\delta}$ as

$$
\rho_{\delta}(z) = \begin{cases} 
\rho(\Xi(z)), & \text{if } z \in K_{\delta} \\
1 - \sum_{z \in K_{\delta}} \rho(\Xi(z)) & \text{if } z = \psi_s
\end{cases}
$$

Recall from $\text{[I]}$ that $\tau_s(\Xi(z')|z,u) = T_s(K_{i,q}|q',z,u)T_q(q'|q,u)$, and $T_s$ evaluated over Borel set $K_{i,q'}$ is a Gaussian density integrated over set $K_{i,q'}$. The discrete probability space is $(\Omega_{\delta}, \sigma(\Omega_{\delta}), \mathbb{P}_{\delta}^{\pi_{\delta}})$ with $\Omega_{\delta} = \mathbb{Z}^{N+1}_\delta \times \mathcal{Y}^N$, $\sigma(\Omega_{\delta})$ the $\sigma$-algebra on $\Omega_{\delta}$, and $\mathbb{P}_{\delta}^{\pi_{\delta}}$ the probability measure uniquely defined by $\rho_{\delta}$, $\tau_y$, $\tau_{\delta}$, and a control policy $\pi_{\delta} = (\mu_{0\delta}^\delta, \ldots, \mu_{N-1\delta}^\delta)$, $\mu_{\delta}^\delta : \Sigma_{\delta} \rightarrow \mathcal{U}$, with $\Sigma_{\delta}$ the set of all information states $\sigma_{\delta}$.
We further define the operator \( \Phi_{y,u}^{\delta,\sigma} \) and the intermediate vector \( \alpha_{y,u,\sigma}^{\delta} \) as
\[
\Phi_{y,u}^{\delta,\sigma}(z') = \frac{1}{\P(y|\sigma,u)} \tau_y(y|z',u) \sum_{z \in K_{\delta}} \tau_{\sigma}^{\delta}(z'|z,u) \sigma_{\delta}(z) \tag{28}
\]
\[
\alpha_{y,u,\sigma}^{\delta}(z) = \sum_{z' \in K_{\delta}} \alpha_{n+1,\delta}^{\sigma}(z') \tau_{\sigma}(y|z',u) \tau_{\sigma}^{\delta}(z'|z,u) 1_{K_{\delta}}(z) \tag{29}
\]
for \( y \in \mathcal{Y}, u \in \mathcal{U}, z', z \in Z_{\delta} \). The viability problem for \( \hat{\mathcal{H}} \) is
\[
\sup_{\pi_{\delta} \in \Pi_{\delta}} p_{\text{safe}}^N(\pi_{\delta}, \rho_{\delta}; K_{\delta}) = \sup_{\sigma_{\delta} \in \Pi_{\delta}} [z_{n} \in K_{\delta}, \forall n \in [0,N]] \tag{30}
\]
To solve (30), we formulate the information state \( \sigma_{\delta} \) and the value function \( V_{n,\delta}^* : \sigma_{\delta} \rightarrow [0,1] \) for \( n = 0, \ldots, N \).

The discrete information state represents a probability mass function over \( Z_{\delta} \), and can be expressed as an integral over an equivalent density (just as \( \tau_{\sigma}^{\delta}(z'|z,u) = \tau_{\sigma}(z|z,u) \))
\[
\sigma_{n,\delta}(z) = \begin{cases} 
\int_{\Xi(z)} \hat{\sigma}_{n}(ds), & \text{if } z \in K_{\delta} \\
\int_{S\setminus K} \hat{\sigma}_{n}(ds), & \text{if } z = \psi_{s}
\end{cases} \tag{31}
\]
with \( \hat{\sigma}_{n}(s) \) given by
\[
\hat{\sigma}_{n}(s') = \begin{cases} 
\rho(s'), & \text{if } n = 0 \\
\left( \Phi_{y,u} \hat{\sigma}_{n-1} \right)(s') = \frac{1}{\P(y|\sigma_{n-1},u)} \tau_{y}(y|\xi(s'),u) \int_{K_{\delta}} \tau_{\sigma}(s'|\xi(s),u) \hat{\sigma}_{n-1}(ds), & \text{if } n > 0
\end{cases} \tag{32}
\]
This can be verified by substituting the expression for \( \tau_{\sigma}^{\delta} \) in terms of \( \tau_{\sigma} \) into (28) and using an induction argument.

The value function is
\[
V_{n,\delta}^*(\sigma_{\delta}) = \sum_{z \in K_{\delta}} \sigma_{\delta}(z) \tag{33}
\]
\[
V_{n+1,\delta}^*(\Phi_{y,u}^{\delta,\sigma_\delta}) = \max_{u \in \mathcal{U}} \int_{\mathcal{Y}} V_{n+1,\delta}^*(\Phi_{y,u}^{\delta,\sigma_\delta}) \P(dy|\sigma_{\delta},u) \tag{31}
\]
The maximum probability of remaining within \( K_{\delta} \) over \( N \) time steps (30) is
\[
p_{\text{safe}}^N(\rho_{\delta}; K_{\delta}) = V_{0,\delta}^*(\rho_{\delta}). \tag{34}
\]

We now show that the viability probability for the finite state approximation \( \hat{\mathcal{H}} \) converges to the true solution as grid size parameter \( \delta^x \) tends to zero. To do so, we first describe the error between the continuous information state \( \sigma \) and the vector approximation \( \sigma_{\delta} \).

1) Information State Approximation Error: We first characterize the relationship between the densities \( \sigma_{\delta} \) and \( \sigma \) in the following theorem.

**Theorem 1.** The density \( \sigma \) defined in (32) satisfies
\[
|\sigma_{n}(s) - \hat{\sigma}_{n}(s)| \leq \eta_{n}^x \delta^x
\]
for all \( s \in \mathcal{S}, \sigma_{n} \in \Sigma, \) and \( \eta_{n}^x \) given by
\[
\eta_{n}^x = \sum_{i=1}^{n} c_{1,i} \left( \prod_{j=i+1}^{n} c_{2,j} \right),
\]
with \( c_{1,i} = \min\left\{ \frac{1}{\P(y|\sigma_{n-1},u)}, \frac{1}{\P(y|\sigma_{n-1},u)} \right\} [\phi_{y}^{x}(2) + \phi_{u}^{x}(2)], \)
\( c_{2,j} = \min\left\{ \frac{1}{\P(y|\sigma_{n},u)}, \frac{1}{\P(y|\sigma_{n},u)} \right\} \phi_{u}^{x} N_{\lambda}. \)
Proof: By induction. At time $n = 0$, $\sigma_0(s) = \rho(s) = \hat{\sigma}_0(s)$ and the inequality is trivially satisfied. For all $i = 0, \ldots, n$, assume $|\sigma_i(s) - \hat{\sigma}_i(s)| \leq \eta_i^\alpha \delta^r$. At time $i = n + 1$, for any $y \in \mathcal{Y}$ and any $u \in \mathcal{U}$,

$$
|\sigma_{n+1}(s') - \hat{\sigma}_{n+1}(s')| \leq \min \left\{ \frac{1}{P(y|\sigma_n, u)}, \frac{1}{P(y|\hat{\sigma}_n, u)} \right\} \left[ \tau_y(y|s', u) \int_K \tau_s(s'|s, u)\sigma_n(ds) \right.
$$

\[ - \tau_y(y|\xi(s'), u) \int_K \tau_s(s'|\xi(s), u)\hat{\sigma}_n(ds) \right].

(35)

We add and subtract $\tau_y(y|\xi(s'), u) \int_K \tau_s(s'|s, u)\sigma_n(ds)$ and $\tau_y(y|\xi(s'), u) \int_K \tau_s(s'|\xi(s), u)\sigma_n(ds)$ and apply the triangle inequality.

$$
|\sigma_{n+1}(s') - \hat{\sigma}_{n+1}(s')| \leq \min \left\{ \frac{1}{P(y|\sigma_n, u)}, \frac{1}{P(y|\hat{\sigma}_n, u)} \right\} \left[ \tau_y(y|s', u) - \tau_y(y|\xi(s'), u) \right]
$$

\[ \int_K \tau_s(s'|s, u)\sigma_n(ds) \]

\[ + \tau_y(y|\xi(s'), u) \int_K \tau_s(s'|\xi(s), u)\sigma_n(ds) \]

\[ + \tau_y(y|\xi(s'), u) \int_K \tau_s(s'|\xi(s), u)\sigma_n(ds) \]

\[ \leq \min \left\{ \frac{1}{P(y|\sigma_n, u)}, \frac{1}{P(y|\hat{\sigma}_n, u)} \right\} \sup_{s' \in K} \left[ T_y(y|q', u) |T_y| y^\alpha| x' - T_y| y^\alpha| \xi(x')| \right] \phi_v^\ast

\[ + \phi_w^\ast |\sigma_n - \hat{\sigma}_n| \infty \int_K \tau_s(s'|\xi(s), u) ds \]

\[ \leq \min \left\{ \frac{1}{P(y|\sigma_n, u)}, \frac{1}{P(y|\hat{\sigma}_n, u)} \right\} + \left[ h_y^{(2)} \|x' - \xi(x')\| \phi_v^\ast + \phi_w^\ast h_x^{(2)} \|x' - \xi(x')\| \right]

\[ + \phi_w^\ast |\sigma_n - \hat{\sigma}_n| \infty \sum_{q \in Q} T_q(q'| q, u) \int_{K_q} T_x(x'|\xi(s), u) dx \]

(36)

Since $T_x$ is bounded by $\phi_v^\ast$, and the Lebesgue measure of $K_q$ is at most $\lambda$, we obtain

$$
|\sigma_{n+1}(s') - \hat{\sigma}_{n+1}(s')| \leq \min \left\{ \frac{1}{P(y|\sigma_n, u)}, \frac{1}{P(y|\hat{\sigma}_n, u)} \right\} \left[ h_y^{(2)} \phi_v^\ast \delta^r + \phi_w h_x^{(2)} \delta^r + \phi_w^\ast N_q \lambda |\sigma_n - \hat{\sigma}_n| \infty \right]

\[ \leq \min \left\{ \frac{1}{P(y|\sigma_n, u)}, \frac{1}{P(y|\hat{\sigma}_n, u)} \right\} \left[ h_y^{(2)} \phi_v^\ast \delta^r + \phi_w h_x^{(2)} \delta^r + \phi_w^\ast N_q \lambda \left( \eta_n^\alpha \delta^r \right) \right].

Combining terms gives the desired result.

\[ \Box \]

2) Convergence of (34) to (8): The value function (19) requires integrating over spaces $\mathcal{Y}$ and $\mathcal{S}$ of unbounded size. To prove convergence of the value function $V_{n,\delta}^\ast$ to $V_n^\ast$, we must show that these integrals are bounded.

Consider the following two lemmas regarding integration of $T_y^\alpha$ and $T_x$ over unbounded sets $\mathcal{Y}_x$ and $\mathcal{X}$, respectively.

Lemma 3. For any $x, \overline{x} \in K_{i,q}$, for all $i = 1, \ldots, m_q$, $q \in \mathcal{Q}$, the following holds:

$$
\int_{\mathcal{Y}} |T_y^\alpha(y^\alpha| x) - T_y^\alpha(y^\alpha| \overline{x})| dy^\alpha \leq \left[ \beta_{1,i}^y h_y^{(2)} + \beta_{2}^y \right] \delta_l^{x,q}
$$

with $\beta_{1,i}^y = \int_{\mathcal{Y}} \|y^\alpha - x\|_h^2 \leq w^\ast \|y^\alpha - x\|_{\mathcal{Y}, \mathcal{X} \in K_{i,q}}$, $1 dy^\alpha$ and $\beta_{2}^y = \phi \overline{x} \sqrt{T_m^2 \over \sigma_l \overline{x}}$. In other words, $\beta_{1,i}^y$ is the Lebesgue measure of region $K_{i,q}$ scaled by $\sqrt{w^\ast}$ in all directions, with $w^\ast$ the largest singular value of $\mathcal{W}$ (so if $K_{i,q}$ is a hyperrectangle, each side will increase by a factor of $2\sqrt{w^\ast}$).
Proof: We exploit properties of the derivative of a Gaussian distribution, which bounds the Lipschitz constants for \( T_{y^*} \) from above. The constant \( h_y^{(2)} \) is the maximum value of the derivative of \( \phi(y^*; x, \mathcal{W}) \) with respect to \( x \):

\[
\| \frac{\partial \phi}{\partial x} \|_2 \leq \frac{1}{(2\pi)^{\frac{3}{2}} |\mathcal{W}|^\frac{1}{2} w_{\sigma}^*} \| x - y^* \|_2 e^{-\frac{|x-y^*|^2}{2w_{\sigma}}}. \tag{37}
\]

Since \( \| \mathcal{W} \|_2 = |\rho(\mathcal{W})| = w_{\sigma}^* \) (with \( \rho(\mathcal{W}) \) the largest eigenvalue of \( \mathcal{W} \)), the maximum of (37) occurs at \( \| x - y^* \| = \sqrt{w_{\sigma}} \).

Although \( \| \frac{\partial \phi}{\partial x} \| \leq h_y^{(2)} \), we create a tighter bound for the case in which \( \| x - y^* \| \) is greater than \( \sqrt{w_{\sigma}} \) (for \( y^* \in \mathcal{Y} \) such that there exists \( x \in K_{i,q} \) for which \( \| x - y^* \|_2 = \sqrt{w_{\sigma}} \), the upper bound \( h_y^{(2)} \) is attained) using the following function.

\[
\eta_y(y^*) = \max_{x \in K_{i,q}} \left\{ \frac{1}{(2\pi)^{\frac{3}{2}} |\mathcal{W}|^\frac{1}{2} w_{\sigma}^*} \| x - y^* \|_2 e^{-\frac{|x-y^*|^2}{2w_{\sigma}}} \right\}. \tag{38}
\]

Then,

\[
\int_{\mathcal{Y}^*} |T_{y^*}(y^*|x) - T_{y^*}(y^*|x)| dy^* \leq \int_{\mathcal{Y}^*} \eta_y(y^*) \| x - \pi \| dy^* \tag{39}
\]

\[
\leq \delta_{\pi,q}^{\pi} \int_{\{y^*:\|x-y^*\|_2 \leq w_{\sigma}^*, y^* \in \mathcal{Y}^*, x \in K_{i,q}\}} \eta_y(y^*) dy^* + \delta_{\pi,q}^{\pi} \int_{\{y^*:\|x-y^*\|_2 > w_{\sigma}^*, y^* \in \mathcal{Y}^*, x \in K_{i,q}\}} \eta_y(y^*) dy^*
\]

\[
= \delta_{\pi,q}^{\pi} \delta_{\pi,q}^{\pi} + \delta_{\pi,q}^{\pi} \int_{\{y^*:\|x-y^*\|_2 > w_{\sigma}^*, y^* \in \mathcal{Y}^*, x \in K_{i,q}\}} \eta_y(y^*) dy^* \tag{40}
\]

We use the change of variable \( v = \| x^* - y^* \|_2 \), with \( x^* = \arg \min_{x \in K_{i,q}} \| x - y^* \| \), to rewrite the second term of (40), and apply an identity for integrals of polynomials.

\[
\int_{\{y^*:\|x-y^*\|_2 > w_{\sigma}^*, y^* \in \mathcal{Y}^*, x \in K_{i,q}\}} \eta_y(y^*) dy^* = \frac{1}{(2\pi)^{\frac{3}{2}} |\mathcal{W}|^\frac{1}{2} w_{\sigma}^*} \int_{|v|^2 < \frac{2w_{\sigma}^*}{e}} \frac{1}{2} v^2 e^{-\frac{v^2}{2w_{\sigma}^*}} dv
\]

\[
\leq \frac{1}{(2\pi)^{\frac{3}{2}} |\mathcal{W}|^\frac{1}{2} w_{\sigma}^*} \int_{0}^{\frac{2w_{\sigma}^*}{e}} \frac{1}{2} v^2 e^{-\frac{v^2}{2w_{\sigma}^*}} dv
\]

\[
\leq \frac{1}{(2\pi)^{\frac{3}{2}} |\mathcal{W}|^\frac{1}{2} w_{\sigma}^*} \frac{w_{\sigma}^*}{2} \sqrt{2w_{\sigma}^* \pi} \tag{41}
\]

Inserting (41) into (40) proves the lemma.

A similar result holds for the integral of \( T_x \) over \( \mathcal{X} \).

Lemma 4. For any \( x, \pi \in K_{i,q} \) for all \( i = 1, \ldots, m_q, q \in \mathcal{Q} \), the following holds:

\[
\int_{\mathcal{X}} |T_x(x'|q', x, q, u) - T_x(x'|q', \pi, q, u)| dx \leq \left[ \beta_{1,i} h_x^{(2)} + \beta_{2,i} \right] \delta_{\pi,q}
\]

with \( \beta_{1,i} = \int_{\{x':|x' - Ax - g(q,u,q')|_2 \leq v^*_{i,q}, x' \in \mathcal{X}, x \in K_{i,q}\}} 1 dx \) and \( \beta_{2,i} = \frac{a^*_q}{2} \sqrt{2v_{i,q} \pi} \), with \( a^*_{q} \) the largest singular value of \( A \).

The proof follows that of Lemma 3 with mean and covariance appropriate to \( T_x \).

In order to show convergence of (34) to (8), we need some additional definitions. First, similarly to \( \hat{\sigma} \), we define piecewise constant function \( \hat{\alpha} \) as \( \hat{\alpha}(s) = \alpha_{n,\hat{\sigma}}(\xi(s)) \), so that

\[
\hat{\alpha}(s) = \int_s^\infty \int_{\mathcal{Y}} \alpha_n^{(y)}(s') \tau_y(dy|\xi(s'), u) \tau_x(ds'|\xi(s), u) 1_{K_{\hat{\sigma}}}(\xi(s)). \tag{42}
\]
We also define $\hat{\alpha}_n(s)$ in the same manner as $\hat{\alpha}_n(s)$, except that it is directly related to $\alpha_n(s)$, i.e., uses the same optimal control input $u$, and the same combination of $\alpha_{n+1}$-functions (determined by $*(y)$). In other words, $\alpha_n(s)$ is identical to $\alpha_n(s)$ in terms of the optimal policy tree from time $n$ to $N$, but the values are calculated using $\tau_y(y|\xi(s'), u)$ and $\tau_s(s'|\xi(s), u)$.

$$\hat{\alpha}_n^i(s) = \int_S \int_Y \hat{\alpha}_{n+1}^i(s') \tau_Y(dy|s', u^i) \tau_s(ds'|\xi(s), u^i) 1_{K_\beta}(\xi(s)),$$

for a specific $\alpha$-function $i$ associated with $\alpha_n^i$. The superscript $i$ for $u^i$ and $i(y)$ indicates that the same choice of $u$ and combination of $\alpha_{n+1}^i(s)$ are used for both $\alpha_n^i(s)$ and $\hat{\alpha}_n^i(s)$. A bound on the difference between $\alpha_n^i(s)$ and $\hat{\alpha}_n^i(s)$ is given in the following lemma.

**Lemma 5.** For any $n \in [0, N]$, and any function $\alpha_n^i(s) \in \Gamma_n$, the associated function $\hat{\alpha}_n^i(s)$ defined in (43) satisfies

$$|\alpha_n^i(s) - \hat{\alpha}_n^i(s)| \leq (N - n) N_q \left[ \beta_1^y h_y^{(2)} + \beta_1^z h_z^{(2)} + \beta_2^y + \beta_2^z \right] \delta^x$$

for all $s \in S$. The constants $\beta_1$ and $\beta_2$ are equal to $\max_{i=1, \ldots, m_q, q \in Q} \beta_1^y$ and $\max_{i=1, \ldots, m_q, q \in Q} \beta_2^y$ from Lemmas 3 and 4 respectively.

**Proof:** By induction. At time $N$,

$$|\alpha_N^i(s) - \hat{\alpha}_N^i(s)| = \int_S (1_{K_1} - 1_{K_\beta}(\xi(s))) ds = 0$$

since for any $s \in K$, by definition $\xi(s) \in K_\beta$. Assume for all $j = N - 1, \ldots, n + 1$, $|\alpha_j^i(s) - \hat{\alpha}_j^i(s)| \leq (N - n) N_q \left[ \beta_1^y h_y^{(2)} + \beta_1^z h_z^{(2)} + \beta_2^y + \beta_2^z \right] \delta^x$. For $j = n$,

$$|\alpha_n^i(s) - \hat{\alpha}_n^i(s)| = \int_S \int_Y \alpha_{n+1}^i(s') \tau_Y(dy|s', u^i) \tau_s(ds'|s, u^i) 1_{K_1}(s)$$

$$- \int_S \int_Y \hat{\alpha}_{n+1}^i(s') \tau_Y(dy|s', u^i) \tau_s(ds'|s, u^i) 1_{K_\beta}(\xi(s))$$

$$\leq \int_S \int_Y \left| \alpha_{n+1}^i(s') - \hat{\alpha}_{n+1}^i(s') \right| \tau_Y(dy|s', u^i) \tau_s(ds'|s, u^i) 1_{K_1}(s)$$

$$+ \int_S \int_Y \hat{\alpha}_{n+1}^i(s') \left| \tau_Y(dy|s', u^i) - \tau_Y(dy|\xi(s'), u^i) \right| \tau_s(ds'|s, u^i) 1_{K_\beta}(\xi(s))$$

$$+ \int_S \int_Y \hat{\alpha}_{n+1}^i(s') \tau_Y(dy|\xi(s'), u^i) \left| \tau_s(ds'|s, u^i) - \tau_s(ds'|\xi(s), u^i) \right| 1_{K_\beta}(\xi(s))$$

$$\leq \left| \alpha_{n+1}^i(s') - \hat{\alpha}_{n+1}^i(s') \right| + N_q \left[ \beta_1^y h_y^{(2)} + \beta_2^y \right] \delta^x + N_q \left[ \beta_1^z h_z^{(2)} + \beta_2^z \right] \delta^x$$

The second term of (46) simplifies according to Lemma 3 and noting that $\alpha(s)$ represents a probability that is bounded above by one. The third term simplifies according to Lemma 4. The term $1_{K_1}(s)$ does not affect the bound, and only indicates that both $\alpha_n(s)$ and $\hat{\alpha}_n(s)$ are equal to zero for $s \notin K$. Applying the induction hypothesis to (47) gives the desired result.

We now can show convergence of the approximate viability probability over the discretized state space to the true viability probability.
Theorem 2. For any time \( n \in [0, N] \), and any \( \sigma \in \Sigma, \sigma_\delta \in \Sigma_\delta \), the error between the value function \( V \) and the value function \( \hat{V} \) based on the finite state approximation is bounded above by

\[
|V_n^*(\sigma_n) - V_n^*(\sigma_{n,\delta})| \leq \eta_n^\delta \delta^x
\]

with \( \eta_n^\delta = N_q \lambda \eta_n^\sigma + (N - n) N_q (\beta_1^y h_y^{(2)} + \beta_1^x h_x^{(2)} + \beta_2^y + \beta_2^x) \).

Specifically, the viability probability for PODTSHS \( \mathcal{H} \) over time horizon \( N \) satisfies

\[
|p_{\text{safe}}(\rho; K) - p_{\text{safe}}^N(\rho; K_\delta)| \leq \left[ N_q N (\beta_1^y h_y^{(2)} + \beta_1^x h_x^{(2)} + \beta_2^y + \beta_2^x) \right] \delta^x.
\]

Proof: By construction. At any time \( n \in [0, N] \), given \( \sigma_n \in \Sigma \) and \( \sigma_{n,\delta} \in \Sigma_\delta \), we can rewrite the value function evaluated at \( \alpha \) in terms of \( \alpha \)-functions.

\[
|V_n^*(\sigma_n) - V_n^*(\sigma_{n,\delta})| = \sup_{\alpha_n^i \in \Gamma_n} \langle \alpha_n^i, \sigma_n \rangle - \sup_{\alpha_{n,\delta}^i \in \Gamma_{n,\delta}} \langle \alpha_{n,\delta}^i, \sigma_{n,\delta} \rangle
\]

(48)

\[
= |\langle \alpha_n^k, \sigma_n \rangle - \langle \alpha_{n,\delta}^k, \sigma_{n,\delta} \rangle| \leq |\langle \alpha_n^k, \sigma_n \rangle - \langle \hat{\alpha}_n^k, \sigma_n \rangle| + |\langle \hat{\alpha}_n^k, \sigma_n \rangle - \langle \hat{\alpha}_{n,\delta}^k, \sigma_{n,\delta} \rangle| + \int \int |\alpha_n^k(s) - \hat{\alpha}_n^k(s)| ds + \int |\sigma_n(ds) - \hat{\sigma}_n(ds)|
\]

(53)

Applying Lemma 5 to the first term of (53), and noting that the integral in the second term is in fact taken over \( K \) rather than \( S \) since \( \hat{\alpha}_n^k(s) \) is zero for all \( s \notin K \), we obtain

\[
|V_n^*(\sigma_n) - V_n^*(\sigma_{n,\delta})| \leq (N - n) N_q \left[ \beta_1^y h_y^{(2)} + \beta_1^x h_x^{(2)} + \beta_2^y + \beta_2^x \right] \delta^x + N_q \lambda \eta_n^\sigma
\]

(54)

which completes the proof.

Theorem 2 shows that the finite state approximation \( \hat{\mathcal{H}} \) provides a means to approximately compute \( \hat{V} \) through the viability probability for \( \hat{\mathcal{H}} \). As \( \delta^x \to 0 \), the finite state viability probability \( \hat{V} \) converges to the true value \( V \), and the policy \( \pi^*_\delta \) converges to \( \pi^* \).

C. Gaussian Mixture Approximation

We now consider a different approximation by representing the information state \( \sigma \) and \( \alpha \)-functions from Lemma 1 as Gaussian mixtures. That is, the information states and \( \alpha \)-functions are each characterized by a finite set of weights, means, and covariances, dependent on the discrete mode \( q \).

Difficulty arises from the incorporation of the indicator function \( 1_K \) in (24) and (18). Integration over the compact set \( K \) rather than all of \( S \) violates the preservation of the Gaussian form of \( \sigma \) under operator \( \Phi_{y,u} \), and similarly
for the $\alpha$-functions. To preserve the Gaussian mixture structure, we therefore propose a radial basis function (RBF) approximation $[22]$ to the indicator function, using Gaussians as the basis function. For each $K_q$, we set

$$
1_{K_q}(x) \approx \sum_{i=1}^{I_q} w_i(q) \phi(x; \mu_i(q), \Sigma_i(q))
$$

(55)

using the most general form of the RBF with covariance $\Sigma_i$ rather than $\epsilon I$. For simplicity we will denote $\phi(x; \mu_i(q), \Sigma_i(q))$ by $\phi_i(x)$. This approximation is valid since the RBFs are dense in $L^p [22]$, i.e. given any function $f$ in $L_p$, a weighted combination of RBFs can approximate $f$ to arbitrary accuracy given enough components, and $1_K$ is in $L^1$.

However, the discontinuity in $1_{K_q}$ produces the Gibbs phenomenon at the boundary of $K_q$ in the RBF approximation. Although these oscillations will always exist for a finite number of components, they could possibly be mitigated $[23]$. The oscillations can be constrained to a smaller region of $K$ (shorter wavelength) with the addition of more components, indicating that the $L^p$ error can be reduced but the pointwise error may not. Because we are interested only in integrating over $K$, this works to our advantage.

We define a new operator $\Phi^g$ and a new $\alpha$-function $\alpha^g_{y,u,\sigma}$ by inserting the RBF approximation (55) into (18) and (24), respectively.

$$
\left( \Phi^g_{y,u,\sigma} \right) (s') = \frac{1}{P(y|\sigma_g,u)} \tau_y(y|s',u) \sum_{q \in Q} \sum_{i=1}^{I_q} w_i(q) \phi_i(x) \tau_x(s'|s,u) \sigma_g(s) dx
$$

(56)

$$
\alpha^g_{y,u,\sigma}(s) = \int_S \alpha^{y(g)}_{n+1,g}(s') \tau_y(y|s',u) \tau_x(ds'|s,u) \sum_{i=1}^{I_q} w_i(q) \phi_i(x)
$$

(57)

We presume continuous observations, as in Section [III-B] (the inclusion of a finite number of observations will be addressed in Section [IV]). We provide two lemmas stating that the operator $\Phi^g_{y,u}$ (56) and the $\alpha$-function update (57) preserve the Gaussian mixture representation of $\sigma_{n,g}$ and $\alpha_{n,g}$ for all $n$.

**Lemma 6.** The operator $\Phi^g_{y,u}$ (56) is closed under Gaussian mixtures, i.e. for $\sigma_g$ a Gaussian mixture with $L$ components, $\Phi^g_{y,u,\sigma_g}$ is also a Gaussian mixture with $N_q I_q L$ components for any $u \in U$, $y \in Y$.

**Lemma 7.** The expression (57) is closed under Gaussian mixtures, i.e. if $\alpha^{y(g)}_{n+1,g}$ is a Gaussian mixture with $M$ components, $\alpha^g_{y,u,\sigma}$ is also a Gaussian mixture with $N_q I_q M$ components, for any $u \in U$, $y \in Y$, $\sigma \in \Sigma$.

The proofs of Lemmas 6 and 7 are straightforward and can be shown through extensive manipulation of products of Gaussian pdfs, and so are omitted for brevity. Lemma 6 implies that we can approximate $\sigma$ through a Gaussian mixture and use the equivalent update operator $\Phi^g_{y,u}$, hence the Gaussian mixture approximation of $\sigma$ is

$$
\sigma_{0,g}(x,q) = \sigma_0(x,q) = Q_0(q) \phi(x; \mu_0, \Sigma_0)
$$

$$
\sigma_{n,g}(x,q) = \sum_{l=1}^{L} w_{l,n}^g(q) \phi(x; \mu_{l,n}^g(q), \Sigma_{l,n}^g(q))
$$

(58)
Similarly, the Gaussian mixture approximation of any $\alpha$-function is written:

$$\alpha_{N,g}(x,q) = \sum_{i=1}^{I_q} w_i(q) \phi_i(x)$$

$$\alpha_{n,g}(x,q) = \sum_{m=1}^{M} w_{m,n}^\alpha \phi(x; \mu_{m,n}^\alpha(q), \Sigma_{m,n}^\alpha(q))$$

(59)

The weights, means, and covariances are defined recursively. Their exact representations are lengthy, but again easy to derive through manipulations of Gaussians, and so are omitted.

Note that although the Gaussian mixture representation of $\alpha_{g,y,u,\sigma}$ has a finite number of components given that the representation of $\alpha_{n+1,g}$ is finite, the actual $\alpha$-function, $\alpha_{n,g}$, is expressed as the integral of $\alpha_{g,y,u,\sigma}$ over $\mathcal{Y}$. Therefore, without the assumption that $\mathcal{Y}$ is finite, $\alpha_{n,g}$ must have an infinite number of components (by breaking the integral over $\mathcal{Y}$ into a summation over regions of size $\Delta \subset \mathcal{Y}$ and taking the limit as $\Delta \to 0$). We take some liberty in overlooking this discrepancy, because it does not affect the proofs in this section. We impose a finite set $\mathcal{Y}$ in Section IV, which makes the Gaussian mixture representation of the $\alpha$-functions indeed valid, and discuss additional error implications.

The viability problem for the Gaussian mixture approximation is defined as

$$\sup_{\pi_g \in \Pi_g} p_{\text{safe}}^N(\pi_g, \rho_g; K_g) = \sup_{\pi_g \in \Pi_g} \mathbb{P}^\pi_g \left[ s_n \in K_g, \forall n \in [0, N] \right]$$

$$= V_{\rho_g}^*$$

(60)

with $K_g$ an approximation of $K$ according to (55). The value function $V_{n,g}^*(\sigma_{n,g})$ is described through the recursion

$$V_{N,g}^*(\sigma_g) = \sum_{\mathcal{Q}} \int_X \sum_{i=1}^{I_q} w_i(q) \phi_i(x) \sigma_g(x,q) \, dx$$

$$V_{n,g}^*(\sigma_g) = \max_{u \in U} \int_{\mathcal{Y}} V_{n+1,g}^*(\Phi_{g,y,u}^\sigma) \mathbb{P}(dy|\sigma_g, u)$$

(61)

Since $T_x$, $T_{y^*}y$, and $\rho$ are Gaussian, the Gaussian mixture representation of $\alpha$ and $\sigma$ are exact, aside from the approximation of $1_K$ using RBFs. To quantify the error incurred from calculating $V_{n,g}^*$ as opposed to $V_{\rho}^*$ (from integration of (55) over $\mathcal{S}$ rather than over $K$), we define the error

$$\epsilon_t = \left\| 1_K - \sum_{i=1}^{I_q} w_i(q) \phi_i(x) \right\|_{L^1(\mathcal{S})}$$

(62)

We additionally constrain the RBF approximation (55). The weights $w_i(q)$ must satisfy the following three conditions.

$$\int_X \sum_{i=1}^{I_q} w_i(q) \phi_i(x) \, dx \leq \int_{K_g} 1 \, dx \forall q \in \mathcal{Q}$$

$$\int_{\mathcal{S}} \sum_{i=1}^{I_q} w_i(q') \phi_i(x') \tau_{s'}(s'|s,u) \, ds' \leq 1 \forall s \in \mathcal{S}, s' \in \mathcal{S}, u \in \mathcal{U}$$

$$\int_{\mathcal{S}} \sum_{i=1}^{I_q} w_i(q) \phi_i(x) \tau_{s}(s'|s,u) \, ds \leq 1 \forall s \in \mathcal{S}, s' \in \mathcal{S}, u \in \mathcal{U}$$

(63)
The first condition assures that \( \int_{\mathcal{S}} \sum_{i=1}^{t_q} w_i(q) \phi_i(x) \, ds \leq N_q \lambda. \) The second and third conditions assure that no probability exceeds one. All conditions are easily satisfied by calculating the optimal weights and reducing them slightly if necessary.

1) Information State Approximation Error: The error between \( \sigma \) and \( \sigma_g \) is stated in terms of the \( L^1 \) norm on \( \mathcal{S} \), although a nearly identical result is available for the pointwise error.

**Theorem 3.** The Gaussian mixture approximation \( \sigma_{n,g} \) of \( \sigma \) satisfies

\[
\| \sigma_n - \sigma_{n,g} \|_1 \leq \gamma_n^\sigma \epsilon_I
\]

for any \( n \in [0, N] \), \( y \in \mathcal{Y} \), and \( u \in \mathcal{U} \), with \( \gamma_n^\sigma = \max_{l \in 1, \ldots, L \in Q} (2 \pi)^{-\frac{d}{2}} |\Sigma_l^\sigma(q)|^{-\frac{1}{2}} \).

**Proof:** By induction. At time zero, \( \sigma_{0,g}(s) = \sigma_0(s) \), so that \( \| \sigma_0 - \sigma_{0,g} \|_1 = 0 \). Assume that \( \| \sigma_i - \sigma_{i,g} \|_1 \leq \gamma_i^\sigma \epsilon_I \) for all \( i = 1, \ldots, n \). Then at time \( n+1 \) we have, for some \( y \in \mathcal{Y} \) and \( u \in \mathcal{U} \),

\[
\| \sigma_{n+1} - \sigma_{n+1,g} \|_1 \leq \int_{\mathcal{S}} \tau_y(y|s', u) \int_{\mathcal{U}} \left| 1_K(s)| \sigma_n(ds) - \sum_{i=1}^{t_q} w_i(q) \phi_i(x) \sigma_{n,g}(ds) \right| \tau_x(ds'|s, u)
\]

\[
\leq \phi^*_w \left[ \int_{\mathcal{S}} \left| 1_K(s) \sigma_n(ds) - 1_K(s) \sigma_{n,g}(ds) \right| \right]
\]

\[
+ \int_{\mathcal{S}} \left| 1_K(s) \sigma_{n,g}(ds) - \sum_{i=1}^{t_q} w_i(q) \phi_i(x) \sigma_{n,g}(ds) \right| \right]
\]

\[
\leq \phi^*_w \left[ \| \sigma_n - \sigma_{n,g} \|_1 + \sum_{i=1}^{t_q} w_i(q) \phi_i(x) \right] \| \sigma_{n,g}(ds) \right]
\]

(64)

The first term on line (64) follows because the integral over \( K \) is less than the integral over all of \( \mathcal{S} \), since \( K \) is a compact subset of \( \mathcal{S} \). The induction hypothesis completes the proof.

2) Convergence of \( \alpha \) to \( \alpha^- \): As for the proof of Theorem 2, we define the function \( \tilde{\alpha}^{i,y}_{n,g}(s) \) which utilizes the same policy tree as \( \alpha_n(s) \) for a specific \( \alpha^i_n(s) \in \Gamma_n \).

\[
\tilde{\alpha}^{i,y}_{n,g}(s) = \int_{\mathcal{S}} \int_{\mathcal{Y}} \tilde{\alpha}^{i,y}_{n+1,g} \tau_y(y|s', u^i) \tau_x(ds'|s, u^i) \sum_{i=1}^{t_q} w_i(q) \phi_i^*(x)
\]

(65)

with \( u^i \) the optimal control input associated with \( \alpha^i_n(s) \) and \( i(y) \) indicating that \( \tilde{\alpha}^{i(y)}_{n+1,g}(s) \) is chosen according to the indices selected by \( * \) for \( \alpha_n(s) \). The following lemma describes the relationship between \( \alpha_n(s) \) and \( \tilde{\alpha}_n(s) \).

**Lemma 8.** For any \( n \in [0, N] \), and any \( \alpha^i_n(s) \in \Gamma_n \), the associated function \( \tilde{\alpha}^{i,y}_{n,g}(s) \) defined in (65) satisfies

\[
\| \alpha_n(s) - \tilde{\alpha}^{i,y}_{n,g} \|_{L^1(\mathcal{S})} \leq \left( \sum_{k=n}^{N} (\lambda \phi_k^*)^{N-k} \right) \epsilon_I
\]

**Proof:** By induction. At time \( N \),

\[
\| \alpha_N(s) - \tilde{\alpha}_{N,g} \|_{L^1(\mathcal{S})} = \int_{\mathcal{S}} \left| 1_K(s) - \sum_{i=1}^{t_q} w_i(q) \phi_i^*(x) \right| ds
\]

(66)

\[
= \epsilon_I
\]

(67)
and the result is satisfied. Assume for \( j = N - 1, \ldots, n + 1 \) that \( \| \alpha_j^*(s) - \tilde{\alpha}_{j,g}^*(s) \|_1 \leq \left( \sum_{k=j}^{N} (\lambda \phi^*_x)^{N-k} \right) \epsilon_I \) for any \( \alpha_j^* \in \Gamma_j \). Then for \( j = n \),

\[
\| \alpha_n^*(s) - \tilde{\alpha}_{n,g}^*(s) \|_1 = \int_S \left( \int_S \int_S \alpha_{n+1}^*(s') \tau_y(dy|s', u^i) \tau_s(ds'|s, u^i) \right) \left( \sum_{i=1}^{l_y} w_i(q) \phi_i^l(x) \right) ds
\]

\[
\leq \int_S \left( \int_S \int_S \alpha_{n+1}^*(s') - \tilde{\alpha}_{n+1,g}^*(s') \right) \tau_y(dy|s', u^i) \tau_s(ds'|s, u^i) \left( \sum_{i=1}^{l_y} w_i(q) \phi_i^l(x) \right) ds \leq \lambda \phi^*_x \left( \sum_{k=n+1}^{N} (\lambda \phi^*_x)^{N-k} \right) \epsilon_I + \epsilon_I
\]

Theorem 4. For any time \( n \in [0, N] \), and any \( \sigma \in \Sigma, \sigma_g \in \Sigma_g \), the error between the value function \( V_n^*(\sigma, K) \) given \( \sigma \) and the value function \( \tilde{V}_n^*(\sigma_n, g) \) given \( \sigma_n \) using the Gaussian mixture approximation is bounded above by

\[
|V_n^*(\sigma) - \tilde{V}_n^*(\sigma_n, g)| \leq \gamma_n^\alpha \epsilon_I
\]

with \( \gamma_n^\alpha = \left( \sum_{k=n}^{N} (\lambda \phi^*_x)^{N-k} \right) \phi^*_x \gamma_n^\sigma + \gamma_n^\alpha \).

Specifically, the viability probability for PODTSHS \( H \) over time horizon \( N \) satisfies

\[
|P_{\text{safe}}^N(\rho; K) - P_{\text{safe}}^N(\rho_g; K_g)| \leq \left[ \sum_{k=0}^{N} (\lambda \phi^*_x)^{N-k} \phi^*_{\sigma,0} \right] \epsilon_I.
\]

Proof: By construction. For any time \( n \in [0, N] \), given \( \sigma_n \in \Sigma \) and \( \sigma_n, g \in \Sigma_g \), we can rewrite the value function evaluated at \( \sigma \) in terms of \( \alpha \)-functions.

\[
|V_n^*(\sigma_n) - \tilde{V}_n^*(\sigma_n, g)| = \sup_{\alpha_n^k \in \Gamma_n} (\alpha_n^k, \sigma_n) - \sup_{\alpha_{n,g}^l \in \Gamma_{n,g}} (\alpha_{n,g}^l, \sigma_n, g)
\]

(73)
As in the proof of Theorem 2 assume without loss of generality that \( \langle \alpha^k_n, \sigma_n \rangle \geq \langle \alpha^l_{n,g}, \sigma_{n,g} \rangle \).

\[
|V^*_n(\sigma_n) - V^*_{n,g}(\sigma_{n,g})| = |\langle \alpha^k_n, \sigma_n \rangle - \langle \alpha^l_{n,g}, \sigma_{n,g} \rangle| \\
\leq |\langle \alpha^k_n, \sigma_n \rangle - \langle \tilde{\alpha}^k_n, \sigma_n \rangle| + |\langle \tilde{\alpha}^k_n, \sigma_n \rangle - \langle \alpha^l_{n,g}, \sigma_{n,g} \rangle| \\
\leq \int_S |\alpha^k_n(s) - \tilde{\alpha}^k_{n,g}(s)| \sigma_n(ds) + \int_S |\tilde{\alpha}^k_n(s) - \alpha^l_{n,g}(s)| \sigma_{n,g}(ds) | \\
\leq \left( \sum_{k=n}^N (\lambda \phi(v)^{N-k}) \right) \phi_{\sigma,n} \epsilon_I + \gamma_n \epsilon_I
\] (79)

Theorem 4 shows that the convergence of the Gaussian mixture approximation of both \( \sigma \) and the value function depends on the \( L^1 \) error between the indicator function over \( K \) and the RBF approximation (55), rather than the pointwise error. Although the pointwise error may not converge to zero for a finite number of components in the RBF, the integral of the error can be small, as we will show in Section V.

IV. APPROXIMATE NUMERICAL SOLUTION WITH LOWER BOUND

A numerical solution of Problem 1 via either a discrete or Gaussian mixture approximation additionally requires sets \( \tilde{\Gamma}_n \) and \( \Sigma \) to be finite, whereas we have sets of infinite size because of the uncountable nature of \( \mathcal{Y} \). However, a lower bound on the viability probabilities (34) and (60) can still be obtained, by characterizing the error that results from using \( \tilde{\Gamma}_n \subset \Gamma_n \) and \( \tilde{\Sigma} \subset \Sigma \), finite collections of \( \alpha \)-functions and information states, respectively.

We again exploit point-based approximation methods described in Section II-C. We examine the generation of subsets of the information states and \( \alpha \)-functions, and prove that each guarantees a lower bound to the viability probability of whichever approximation of Section III we choose. In contrast to most point-based solvers, we do not assume a finite set of observations, and hence discretize the observations merely for the computation of the \( \alpha \)-functions. Combining belief space sampling with discretized observations assures a lower bound to the viability probability.

A. Sampling from the information space

We characterize the error from using a sampled subset of \( \Sigma \) for performing backup operations (as in (15)). Presume that a finite number of information states has been generated according to one of the many methods available (15). We generate a finite set \( \tilde{\Gamma}_n \) of \( \alpha \)-functions, one for each \( \sigma \in \tilde{\Sigma} \). The convexity of the value functions guarantees that the subset \( \tilde{\Gamma}_n \) provides a lower bound on \( V^*_n \). Further, we can show that the error between the approximate value function \( \tilde{V}^*_n \) and the true value function \( V^*_n \) depends on how densely we sample \( \Sigma \).

The value function \( \tilde{V}^*_n \) is formally defined as \( \tilde{V}^*_n(\sigma) = \sup_{\tilde{\alpha}_n \in \tilde{\Gamma}_n} \langle \tilde{\alpha}_n, \sigma \rangle \) with

\[
\tilde{\alpha}_n(s) = \int_{\mathcal{Y}} \int_{\mathcal{S}} \tilde{\alpha}^*_n(s') \tau_y(dy|s', u) \tau_s(ds'|s, u) \mathbf{1}_K(s) \\
\tilde{\alpha}^*_n(y) = \arg \left\{ \sup_{\tilde{\alpha}_{n+1} \in \tilde{\Gamma}_{n+1}} \int_{\mathcal{S}} \tilde{\alpha}_{n+1}(s') \tau_y(y|s', u) \tau_s(ds'|s, u) \sigma(ds) \right\}
\] (80)
so that $\hat{V}_n^*$ is characterized by the finite set $\hat{\Gamma}_n$ at each time step. We also define an intermediate value function $\tilde{V}_n^* = \sup_{\hat{\alpha}_n \in \hat{\Gamma}_n} \langle \hat{\alpha}_n, \sigma \rangle$ that generates $\hat{\Gamma}_n$, recursively from $\Gamma_{n+1}$, i.e. that introduces one point-based backup from the full set $\Gamma_{n+1}$. Then $\hat{\alpha}_n$ is written as a function of $\alpha_{n+1}^*$ rather than $\alpha_n^*$, with

$$\alpha_{n+1}^*(s') = \arg \left\{ \sup_{\alpha_{n+1} \in \Gamma_{n+1}} \int_{\mathcal{S}} \alpha_{n+1}(s') \tau_y(y|s', u) \int_{K} \tau_s(ds'|s, u)\sigma(ds) \right\}. \quad (81)$$

Finally, let $\delta^\sigma$ be the maximum $L^1$ distance between points in $\tilde{\Sigma}$ and points in $\Sigma$.

$$\delta^\sigma = \sup \inf_{\tilde{\sigma} \in \tilde{\Sigma}, \sigma \in \Sigma} \| \tilde{\sigma} - \sigma \|_1 \quad (82)$$

In the following, we do not distinguish between the vector and Gaussian mixture representations of $\sigma$ and $\alpha$, because the results apply to both cases.

**Lemma 9.** For any $n \in [0, N]$ and $\sigma \in \Sigma$, the error introduced in one iteration of point-based value iteration is at most $\delta^\sigma$.

$$|\tilde{V}_n^*(\sigma) - V_n^*(\sigma)| \leq \delta^\sigma$$

We now use Lemma 9 to derive a bound between the true value function and the point-based approximation at any time $n$.

**Theorem 5.** For a set of information states $\Sigma$, sampled set $\hat{\Sigma}$, and any time $n \in [0, N]$ and any $\sigma \in \Sigma$, the error from using point-based value iteration versus full value iteration is bounded above by

$$|\hat{V}_n^*(\sigma) - V_n^*(\sigma)| \leq (N - n)\delta^\sigma.$$ 

Thus the error between the point-based approximation and the actual value function is directly proportional to how densely $\hat{\Sigma}$ is sampled, and converges to zero as $\tilde{\Sigma}$ approaches $\Sigma$. The proofs of Lemma 9 and Theorem 5 are a straightforward extension of those appearing in [21], and are omitted for brevity.

### B. Calculating the $\alpha$-functions

Over the uncountably infinite space $\mathcal{Y}$, we cannot calculate $\alpha_{y,u,\sigma}$ for all $y \in \mathcal{Y}$, despite a finite set of $u$ and $\sigma$. We therefore compute a subset of $\alpha_{y,u,\sigma}$ for the finite set $y^i$, to approximate $\alpha_n$ as $\alpha_n(s) \approx \sum_{y^i} \alpha_{y^i,u,\sigma}(s)$. We discretize $\mathcal{Y}$ in a similar fashion to the discretization of $\mathcal{S}$ in Section III-B.

However, since $\mathcal{Y}^x$ is not compact, we consider an expanded set $\overline{\mathcal{K}} = \bigcup_{y^i \in Q} \overline{K}_{y^i} \supset K$ defined so that the probability of observing a value $y$ for $s \in K$ that is outside of $\overline{\mathcal{K}}$ is approximately zero, i.e. $\tau_y(\mathcal{Y}\setminus \overline{\mathcal{K}}|s \in K, u) < \epsilon$, $\epsilon \ll 1$. For example, $\overline{K}_{y^i} = \{x + \hat{w} : x \in K_{y^i}, ||\hat{w}|| \leq 3w^*\}$ with $w^*$ the largest diagonal entry of $W$ results in the probability of observing any $y$ outside of $\overline{\mathcal{K}}$ as less than 0.003, which can be essentially dismissed with minimal impact on resulting calculations. The sets $\overline{K}_{y^i}$ are divided into disjoint subsets $\overline{K}_{i,y^i} \cup_{i=1}^{l_y} \overline{K}_{i,y^i} = \overline{K}_{y^i}$. We also define $\overline{\psi}_{y} = \overline{\mathcal{K}} \setminus \overline{\mathcal{K}}$, such that $\bigcup_{i=1}^{l_y} \overline{K}_{i,y^i} \times \{\overline{\psi}_y\} = \mathbb{R}^n$.

The partition of $\overline{\mathcal{K}}$ is denoted $\overline{Q}^\mathcal{Y} = \bigcup_{y^i \in Q} \overline{Q}_{i,y^i}^\mathcal{Y}$, with $\overline{Q}_{i,y^i}^\mathcal{Y} = \{ \overline{K}_{i,y^i} \times y^i : i = 1, \ldots, l_y, y^i \in Q \}$, with maximum diameter $\delta^\mathcal{Y} = \max_{i,y^i} \delta_{i,y^i}^\mathcal{Y}$. Each partition $\overline{Q}_{i,y^i}$ has a representative element $(y^{i-1,y^i}, y^i)$ and a set $\overline{y}_d = \{(y^{i-1,y^i}, y^i) : i = 1, \ldots, l_y, y^i \in Q \}$. The function
$\theta : \mathcal{Y} \to \mathcal{Y}_\delta$ maps observation $y \in \mathcal{Y}$ to its representative value $(y^{x_i,y^q}, y^q)$; the set-valued function $\Theta : \mathcal{Y}_\delta \to \overline{\mathcal{S}}$ maps the point $(y^{x_i,y^q}, y^q)$ to the corresponding set $G_{x_i,y^q}$.

The finite observation space is $W_\delta = \mathcal{Y}_\delta \cup \{\psi_y\}$. For the finite state approximation, the transition function $\tau^\delta_y : W_\delta \times K_\delta \times \mathcal{U} \to [0,1]$ is defined as

$$
\tau^\delta_y(w|z,u) = \begin{cases} 
\tau_y(\Theta(w)|z,u), & \text{if } w \in \mathcal{Y}_\delta \\
1 - \sum_{\pi \in \mathcal{Y}_\delta} \tau_y(\Theta(\pi)|z,u), & \text{if } w = \psi_y 
\end{cases}.
$$

(83)

For the Gaussian mixture approximation, we define the transition function $\tau^\delta_y$ in the same fashion as (83), but with

$$
\tau_y(\Theta(w)|z,u) \approx T_{y^q}(y^q|q,u) \sum_{j=1}^{M_y} c_j \phi_j(y^q_{x_i,y^q}; x, W)
$$

(84)

so that the $\alpha$-functions will also be Gaussian mixtures at each time step. Note that $w = (y^{x_i,y^q}, y^q), y^q_{x_i,y^q}$ is a set of mesh points inside $G_{x_i,y^q}$ associated with $w$, and $c_j$ is a weight proportional to the mesh spacing (determined, e.g., by the trapezoidal rule for numerical integration).

1) Discretized observations for finite state approximation: We use $\hat{\Gamma}_{n,\delta}$ and $\hat{V}^*_n,\delta$ to denote the approximation using a finite subset of $\Gamma_{n,\delta}$, with the important distinction that the subset is now generated by a finite collection of observations (as opposed to $\hat{\Sigma}$).

The value function is then

$$
\hat{V}^*_{n,\delta} = \sup_{\hat{\alpha}_{n,\delta} \in \hat{\Gamma}_{n,\delta}} \sum_{z \in K_\delta} \hat{\alpha}_{n,\delta}(z)\sigma_\delta(z),
$$

with

$$
\hat{\alpha}_{n,\delta}(z) = \sum_{w \in W_\delta} \sum_{z' \in K_\delta} \hat{\alpha}_{n+1,\delta}^{*(w)}(z')\tau^\delta_y(w|z',u)\tau^\delta_s(z'|z,u)
$$

$$
\hat{\alpha}_{n+1,\delta}^{*(w)}(z') = \arg \left\{ \sup_{\hat{\alpha}_{n+1,\delta} \in \hat{\Gamma}_{n+1,\delta}} \sum_{z \in K_\delta} \hat{\alpha}_{n+1,\delta}(z')\tau^\delta_y(w|z',u)\tau^\delta_s(z'|z,u)\sigma_\delta(z) \right\}.
$$

(85)

Similarly to (81), $\hat{V}^*_{n,\delta}(\sigma_\delta) = \sup_{\hat{\alpha} \in \hat{\Gamma}_{n,\delta}} \sum_z \hat{\alpha}_{n,\delta}(z)\sigma_\delta(z)$ is the intermediate value function, with $\hat{\alpha}_{n,\delta}$ calculated using $\hat{\alpha}_{n+1,\delta}^{*(w)} \in \Gamma_{n+1,\delta}$ (as opposed to $\hat{\Sigma}_{n+1,\delta}$).

We can bound the error introduced in one iteration of approximating the $\alpha$-vectors through discretized observations.

**Lemma 10.** For any time $n \in [0,N]$ and $\sigma_\delta \in \Sigma_\delta$, the error between $V^*_{n,\delta}(\sigma_\delta)$ and $\hat{V}^*_{n,\delta}(\sigma_\delta)$ satisfies

$$
V^*_{n,\delta}(\sigma_\delta) - \hat{V}^*_{n,\delta}(\sigma_\delta) \leq \frac{N_y \lambda \lambda^1}{N} + \frac{\epsilon}{N}
$$

given that the discretized observations $w$ are chosen so that

$$
\sum_{z' \in K_\delta} \alpha_{n+1,\delta}^{*(w)}(z')\tau^\delta_y(w|z',u) > \sum_{z' \in K_\delta} \alpha_{n+1,\delta}^{*(w)}(z')\tau^\delta_y(w|z',u)|\Theta(w)|,
$$

and with $\lambda$ the largest Lebesgue measure of sets $\overline{K}_y$.
for any \( n, \delta \) for any time \( \alpha \). Given discretized observation process \( W_{\delta} \), the set \( \hat{V} \) of \( \alpha \)-functions calculated using the discretized observations.

**Lemma 10** leads to the following theorem regarding the error between \( V_{n,\delta}^* (\sigma_\delta) \) (based on continuous observations) and \( \hat{V}_{n,\delta} (\sigma_\delta) \) (based on discretized observations). We again use the notation \( \hat{V} \) to indicate that \( \hat{V} \) is represented by the set \( \hat{H} \) of \( \alpha \)-functions calculated using the discretized observations.

**Theorem 6.** Given discretized observation process \( W_{\delta} \) with transition function (83), for any time \( n \in [0, N] \), the error between \( V_{n,\delta}^* (\sigma_\delta) \) calculated according to \( \mathcal{Y} \) and \( \hat{V}_{n,\delta} (\sigma_\delta) \) calculated according to \( W_{\delta} \) satisfies

\[
V_{n,\delta}^* (\sigma_\delta) - \hat{V}_{n,\delta} (\sigma_\delta) \leq (N - n)Nq\bar{\lambda}h_y^{(1)} \delta^y + \frac{(N - n)\epsilon}{N}
\]

for any \( \sigma_\delta \in \Sigma_\delta \), with \( \bar{\lambda} \) the largest Lebesgue measure of sets \( \bar{K} y^\gamma \).

Specifically, the viability probability for \( \hat{H} \) satisfies

\[
p^N_{\text{safe}} (\rho_\delta; K_\delta) - \hat{V}_{n,\delta}^* (\rho_\delta) \leq N Nq\bar{\lambda}h_y^{(1)} \delta^y + \epsilon.
\]
Proof: By induction. At time $N$, $V_{n,\delta}^{*}(\sigma_{\delta}) = \hat{V}_{n,\delta}^{*}(\sigma_{\delta})$ since $\Gamma_{N} = \hat{\Gamma}_{N} = 1_{K_{g}}$. Assume for all $i = n + 1, \ldots, N - 1$ that $V_{n,\delta}^{*}(\sigma_{\delta}) - \hat{V}_{n,\delta}^{*}(\sigma_{\delta}) \leq (N - i)\lambda_{q}\lambda_{y}^{(1)}\delta^{y} + \frac{(N - i)\varepsilon}{N}$. Then, at time $n$,

$$V_{n,\delta}^{*}(\sigma_{\delta}) - \hat{V}_{n,\delta}^{*}(\sigma_{\delta}) = \langle \alpha_{n,\delta}^{*}, \sigma_{\delta} \rangle - \langle \hat{\alpha}_{n,\delta}^{*}, \sigma_{\delta} \rangle$$

$$= \langle \alpha_{n,\delta}^{*}, \sigma_{\delta} \rangle - \langle \hat{\alpha}_{n,\delta}^{*}, \sigma_{\delta} \rangle + \langle \hat{\alpha}_{n,\delta}^{*}, \sigma_{\delta} \rangle - \langle \hat{\alpha}_{n,\delta}^{*}, \sigma_{\delta} \rangle$$

$$\leq N_{q}\lambda_{y}^{(1)}\delta^{y} + \int_{\Gamma_{n+1,\delta}} \sup_{y} \langle \alpha_{n+1,\delta}, \Phi_{y}^{\delta}, \sigma_{\delta} \rangle \mathbb{P}(dy|\sigma_{\delta}, \hat{u}^{*})$$

$$- \int_{\Gamma_{n+1,\delta}} \sup_{y} \langle \hat{\alpha}_{n+1,\delta}, \Phi_{y}^{\delta}, \sigma_{\delta} \rangle \mathbb{P}(dy|\sigma_{\delta}, \hat{u}^{*}) + \frac{\varepsilon}{N}$$

$$\leq N_{q}\lambda_{y}^{(1)}\delta^{y} + N_{q}\lambda_{y}^{(1)}\delta^{y} + \int_{\Gamma_{n+1,\delta}} \sup_{y} \langle \hat{\alpha}_{n+1,\delta}, \Phi_{y}^{\delta}, \sigma_{\delta} \rangle \mathbb{P}(dy|\sigma_{\delta}, \hat{u}^{*}) + \frac{\varepsilon}{N}$$

$$\leq N_{q}\lambda_{y}^{(1)}\delta^{y} + \frac{\varepsilon}{N} + (N - n - 1)\frac{\varepsilon}{N}$$

Combining terms completes the proof.

2) Discretized observations for Gaussian mixture approximation: The results of discretizing the observations for the Gaussian mixture abstraction are nearly identical to those for the finite state abstraction. The main difference arises in approximating the integral $\tau_{y}(\Theta(w)|s', u)$ with a Gaussian sum: To ensure the approximate value function provides a lower bound to $V_{n,\sigma}^{*}$, we must under-approximate the integral $\tau_{y}(\Theta(w))$ for each $w$. We again define $\hat{V}_{n,\sigma}^{*}$ similarly to $\hat{V}_{n,\delta}^{*}$, with

$$\hat{\alpha}_{n,\sigma}(s) = \sum_{w \in W_{s}} \int_{S} \hat{\alpha}_{n+1,\sigma}(s')\tau_{y}(w|s', u)\tau_{s}(ds|s, u) \sum_{i=1}^{I_{q}} w_{i}(q'_{i})\phi_{i}(x')$$

and $\hat{V}_{n,\sigma}^{*}$ is the intermediate value function that finds the optimal $\hat{\alpha}_{n+1,\sigma}(s') \in \Gamma_{n+1,\sigma}$, rather than in $\Gamma_{n+1,\sigma}$. We can bound the error between $V_{n,\sigma}^{*}$ and $\hat{V}_{n,\sigma}^{*}$, and between $V_{n,\sigma}^{*}$ and $\hat{V}_{n,\sigma}^{*}$, equivalently to Lemma 10 and Theorem 6, respectively.

Lemma 11. For any time $n \in [0, N]$ and $\sigma_{g} \in \Sigma_{g}$, the error between $V_{n,\sigma}^{*}(\sigma_{g})$ and $\hat{V}_{n,\sigma}^{*}(\sigma_{g})$ satisfies

$$V_{n,\sigma}^{*}(\sigma_{g}) - \hat{V}_{n,\sigma}^{*}(\sigma_{g}) \leq N_{q}\lambda_{y}^{(1)}\delta^{y} + \frac{\varepsilon}{N}$$

given that the observations $w$ are chosen so that

$$\int_{S} \alpha_{n+1,\sigma}(s')\tau_{y}(w|s', u)\tau_{s}(ds|s, u) \sum_{i=1}^{I_{q}} w_{i}(q'_{i})\phi_{i}(x') > \int_{S} \alpha_{n+1,\sigma}(s')\tau_{y}(w|s', u)\tau_{s}(ds|s, u) \sum_{i=1}^{I_{q}} w_{i}(q'_{i})\phi_{i}(x')$$

and with $\lambda$ the largest Lebesgue measure of sets $K_{y}$.

Theorem 7. Given discretized observation process $W_{\delta}$ with transition function [84], for any time $n \in [0, N]$, the error between $V_{n,\sigma}^{*}(\sigma_{g})$ calculated according to $\mathcal{Y}$ and $\hat{V}_{n,\sigma}^{*}(\sigma_{g})$ calculated according to $W_{\delta}$ satisfies

$$V_{n,\sigma}^{*}(\sigma_{g}) - \hat{V}_{n,\sigma}^{*}(\sigma_{g}) \leq (N - n)\lambda_{y}^{(1)}\delta^{y} + \frac{(N - n)\varepsilon}{N}$$

for any $\sigma_{g} \in \Sigma_{g}$, with $\lambda$ the largest Lebesgue measure of sets $K_{y}$. 
Specifically, the viability probability for the Gaussian mixture approximation satisfies

\[ p^N_{\text{safe}}(\rho_g; K_\delta) - \bar{V}^*_{0,q}(\rho_g) \leq NN_q\bar{h}^{(1)}_g \bar{d}^g + \epsilon. \]

The proofs of Lemma 10 and Theorem 6 follow directly from the proofs of Lemma 11 and Theorem 7.

To summarize, given either the finite state or Gaussian mixture approximation, we can subsequently 1) sample \( y \) from \( \mathcal{Y} \) and \( u \) from \( \mathcal{U} \) to generate a subset \( \tilde{\Sigma}_\delta \) or \( \tilde{\Sigma}_g \), and 2) discretize \( \mathcal{Y} \) and use the set \( W_{\delta,y} \) to calculate \( \bar{\alpha}_{\delta, w, u, \sigma, \delta} \) or \( \bar{\alpha}_{\delta, w, u, \sigma, g} \), which are then used to generate \( \bar{\alpha}_{\delta, n, \sigma, \delta} \in \tilde{\Gamma}_{n, \delta} \) and \( \bar{\alpha}_{\delta, n, g} \in \tilde{\Gamma}_{n, g} \). Using sets \( \tilde{\Sigma}_n, \delta \) and \( \tilde{\Gamma}_{n, \delta} \) in place of \( \Sigma_n, \delta \) and \( \Gamma_{n, \delta} \) provides a lower bound to the viability probability \( p^N_{\text{safe}}(\rho_\delta; K_\delta) \) that converges to \( p^N_{\text{safe}}(\rho_\delta; K_\delta) \) as \( \delta^\sigma \) and \( \delta^y \) approach zero (and similarly for \( \tilde{\Sigma}_n, g \) and \( \tilde{\Gamma}_{n, g} \)).

V. Numerical Example

The temperature regulation problem is a benchmark example for hybrid systems, and a stochastic version with perfect state information is presented in [4]. We consider the case of one heater, which can either be turned on to heat one room, or turned off. The temperature of the room at time \( n \) is given by the continuous variable \( x_n \), and the discrete state \( q_n = 1 \) indicates the heater is on at time \( n \), and \( q_n = 0 \) denotes the heater is off. The stochastic difference equation governing the temperature is given by

\[ x_{n+1} = (1 - b)x_n + c q_{n+1} + b x_a + v_n \]

with constants \( b = 0.0167 \), \( c = 0.8 \), and \( x_a = 6 \), and \( v_n \) i.i.d. Gaussian random variables with mean zero and variance \( v^2 \). The control input is given by \( u_n \in \mathcal{U} \) with \( \mathcal{U} = \{0,1\} \), but the chosen control is not always implemented with probability 1. Instead, \( q_n \) is updated probabilistically, dependent on \( u_{n-1} \) and \( q_{n-1} \), with transition function \( T_q(q_{n+1}|q_n, u_n) \). So while function \( \mu_n(\sigma_n) \) deterministically returns a single control input, control input \( u_n = \mu_n(\sigma_n) \) may not always be implemented.

To model this as a partially observable problem, assume the actual temperature is unknown, and only a noisy measurement is available to the controller. The controller does, however, know whether the heater is on or off at time \( n \) (i.e. \( q_n \) is perfectly observed). The observation \( y_n = y^x_n \) is given by \( y^x_n = x_n + w_n \), with \( w_n \) i.i.d. Gaussian random variables with mean zero and variance \( w^2 \).

It is desirable to keep the temperature of the room between 17.5 and 22 degrees Celsius at all times, hence the safe region \( K = [17.5, 22] \) does not depend on the discrete state \( q_n \) (so \( 1_K(s) = 1_K(x) \)). We consider the viability probability of remaining within \( K \) for \( N = 5 \) time steps given initial temperature distribution \( \rho \) normally distributed with varying mean \( \mu_0 \in K \) and variance \( \Sigma_0 = 1 \). The initial mode is given as \( q_0 = 0 \). The finite state and Gaussian mixture approximations are used in a PBVI algorithm in the style of Perseus [19].

We consider a uniform grid \( \delta^x_{i,q} = \delta^x \) constant for all \( i, q \) over the region \( K \subset \mathbb{R} \) for the finite state approximation, with representative points at the end-point of each grid cell. For example, setting \( \delta^x = 0.1 \) gives \( x^{1,q} = 17.5, x^{2,q} = 17.6, \ldots \) for \( q = 0 \) and \( q = 1 \), and a total of \( m_q = 45 \) cells \( K_{i,q} \). The function \( \xi(x,q) \) maps \( q \) to itself, and maps \( x \) to the nearest \( x^{i,q} \) in absolute value.

The Gaussian mixture approximation utilizes an RBF approximation of the indicator function calculated using MATLAB’s \texttt{gmdistribution} function. We used a reduction process to limit the number of components of each \( \alpha \) and
for the Gaussian mixture approximation. Similar Gaussians are combined into a single component based on the \(L^2\) distance between functions [24]. Each mixture was limited to 30 components to reduce overall computation time without overly sacrificing accuracy. This number can easily be changed, however, depending on the importance of speed versus accuracy.

Both approximations employ a sampled set \(\tilde{\Sigma}\) and a finite set of observations to calculate the \(\alpha\)-functions for the PBVI algorithm. To generate the set \(\tilde{\Sigma}\), we initialized a set of 40 states \(\sigma_0\) normally distributed with variance \(\Sigma_0\) and mean \(\mu_0\) randomly chosen uniformly on \(K\). Each \(\sigma_0\) was updated according to \(\Phi_{g,v,u}\) with \(u\) chosen randomly and \(y\) sampled from the corresponding \(\sigma_0\) (i.e. \(y \sim \mathbb{P}(y|\sigma_0, u)\)). This process was repeated \(N\) times, so that for each time step we had a set of 40 sampled \(\sigma\)s. The finite set of observations were produced by a uniform grid over \(K = [16, 24]\), again using end-points as the representative observations.

To compare performance of the finite state and Gaussian mixture approximations, we present computation times and viability probability estimates for each, with varying \(\delta^x, \delta^y,\) and number of components in the indicator approximation. Viability probabilities for varying initial distributions \(\rho\) are presented in Figs. 1a and 1b for the finite state approximation and Gaussian mixture approximation, respectively. The optimal policy at time zero is shown for varying \(\rho\) in Figs. 2a and 2b for the finite and Gaussian approximations, respectively. Computation times for the finite state approximation are given in Table I and for the Gaussian mixture approximation in Table II.

![Fig. 1](image1.png)

(a) Comparison of viability probabilities over varying initial distribution \(\rho = \phi(x; \mu_0, 1)\) and \(q_0 = 0\), using the finite state approximation (a) and Gaussian mixture approximation (b). In both (a) and (b) \(\delta^y = 0.5\). Fig. (a) compares probabilities for \(\delta^x = 0.1\) (black dashed line) and \(\delta^x = 0.01\) (red solid line). Fig. (b) compares probabilities for \(I_q = 10\) (black dashed line) and \(I_q = 30\) (red solid line). The refinement of \(\delta^x\) and increase in \(I_q\) has a small impact on the viability probabilities. The finite state approximation estimates higher probabilities for \(\mu_0\) in the interior of \(K\) than the Gaussian mixture approximation.

We also show sample RBF approximations to the indicator function \(1_K\) in Fig. 3 with varying numbers of components \(I_q\). The \(L^1\) error between the RBF approximation and \(1_K\) for varying \(I_q\) is shown in Fig. 4. As the number of components increases, the approximation becomes more accurate, although as seen in Fig. 3 oscillations remain at the endpoints of \(K\). The increasing accuracy is most apparent in Fig. 4 and demonstrates the convergence towards zero of the \(L^1\) error with increasing \(I_q\).
Fig. 2: Comparison of optimal control inputs as a function of $\rho = \phi(x; \mu_0, 1)$ with $q_0 = 0$, using the finite state approximation (a) and Gaussian mixture approximation (b). In both (a) and (b), $\delta^y = 0.5$. Fig. (a) compares control inputs for $\delta^x = 0.1$ (black dashed line) and $\delta^x = 0.01$ (red solid line). Fig. (b) compares control inputs for $I_q = 10$ (black dashed line) and $I_q = 30$ (red solid line), which in this case are the same. All approaches produce a threshold policy that turns the heater off for $\mu_0 > 18.7$, except the finite approximation with $\delta^x = 0.1$, which turns the heater off for $\mu_0 > 18.8$.

We show viability probabilities for $\delta^y = 0.5$. Decreasing $\delta^y$ causes a slight increase in the viability probabilities, as expected, but there is not a significant improvement in the probability estimates, although as seen in Tables I and II the increase in computation time is significant. This is likely problem-specific, and the value of $\delta^y$ may have a greater impact for some applications.

The viability probability estimates for the finite state approximation are in general greater than for the Gaussian mixture approximation. The mixture reduction method employed, as well as the indicator function approximation, make the Gaussian method seemingly less accurate than the finite state approximation. However, over a finer mesh $\delta^x$, the finite state method results in greater computation time. Although the coarse grid produces similar results to the fine grid ($\delta^x = 0.1$ versus $\delta^x = 0.01$), in higher dimensional problems the number of grid cells becomes prohibitive even when $\delta^x$ is large, and the Gaussian mixture approximation may be more computationally tractable. All scenarios produce a nearly identical optimal threshold policy based on the initial mean $\mu_0$, indicating that an optimal policy may be computed fairly quickly using any of the above methods.

| $\delta^x$ | $\delta^y$ | Comp. time (s) |
|------------|------------|----------------|
| 0.1        | 1.0        | 50.5           |
| 0.01       | 1.0        | 8961.1         |
| 0.01       | 0.5        | 205.1          |
| 0.01       | 0.1        | 1599.8         |
| 0.01       | 0.1        | 15343.7        |
| 0.01       | 0.01       | 108591.3       |

TABLE I: Computation times using PBVI with finite state approximation, for varying continuous state spacing $\delta^x$ and discretized observation spacing $\delta^y$.

Interestingly, increasing the number of components in the RBF approximation to the indicator function only slightly improves the viability estimates of the Gaussian mixture approximation, although the $L^1$ error from
\[
\begin{array}{cccc}
I_q = 10 & I_q = 30 \\
\hline
\delta y = 1.0 & \delta y = 0.5 & \delta y = 1.0 & \delta y = 0.5 \\
\hline
\text{Comp. time (s)} & 365.5 & 1625.9 & 1865.0 & 5586.1 \\
\end{array}
\]

TABLE II: Computation times using PBVI with Gaussian mixture approximation, for varying number of components \(I_q\) for RBF approximation to \(1_K\) and discretized observation spacing \(\delta y\).

Increasing the number of components to 30 drops significantly. This may be caused by the mixture reduction technique, leading to a loss in the added benefit of an increased number of components when that number is again reduced. However, although the \(L^1\) error with \(I_q = 10\) is large, we obtain viability estimates that are quite similar to the finite state approximation. This requires further investigation, but may help in decreasing computation time without losing significant accuracy by choosing \(I_q\) to be small.

VI. Conclusion

We have presented the first numerical results for verification of a partially observable DTSHS, via two approximations that enable the use of a well-known POMDP optimization technique. The first approximation discretizes the state space over a compact set \(K\) and enables a vector representation of the information states and \(\alpha\)-functions. The second approximates the indicator function over compact set \(K\) using a finite set of Gaussian radial basis functions and enables a Gaussian mixture representation of the information states and \(\alpha\)-functions. We can apply point-based value iteration to either approximation, and guarantee a lower bound to the viability probability, which is proven to converge to the true viability probability of the original PODTSHS. A simple numerical example shows that both methods provide similar viability estimates. The finite state approximation is faster when a coarse discretization is used, but quickly becomes slower than the Gaussian mixture approximation with a finer discretization. Therefore, although the Gaussian mixture produces lower viability estimates, it may be better suited to higher dimensional problems.

![Comparison between \(1_K(x)\) (in black, dashed line) to RBF approximation (red, solid line) for (a) \(I_q = 10\) components, (b) \(I_q = 30\) components, (c) \(I_q = 100\) components, and (d) \(I_q = 400\) components. As the number of components increases, the approximation improves, although oscillations at the endpoints remain.](image-url)
Although we present a linear system with additive Gaussian noise, both approximations may be extended to non-Gaussian systems. Convergence results for the finite state approximation apply to arbitrary transition kernels $T_x$ and $T_y$. The Gaussian mixture approximation further requires approximating $T_x$ and $T_y$ with Gaussian mixtures, and introduces additional error. We also focus on the viability problem, although the computational techniques presented will apply to other verification properties such as reachability, reach-avoid objectives, etc. by modifying the information state and $\alpha$-functions slightly. We are currently working to formally extend these results to other verification objectives and more complex applications. Because both methods are relatively slow, we plan to continue to refine them to decrease computation time, which is possible through the use of more sophisticated existing point-based solvers. We are also exploring more efficient computation by exploiting problem structure, through the use of adaptive gridding schemes and other representations of $\alpha$ and $\sigma$ beyond vectors and Gaussian mixtures.

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