BOUNDARY DISTORTION ESTIMATES FOR HOLOMORPHIC MAPS

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Abstract. We establish some estimates of the angular derivatives from below for holomorphic self-maps of the unit disk \( \mathbb{D} \) at one and two fixed points of the unit circle provided there is no fixed point inside \( \mathbb{D} \). The results complement Cowen-Pommerenke and Anderson-Vasil’ev type estimates in the case of univalent functions. We use the method of extremal length and propose a new semigroup approach to deriving inequalities for holomorphic self-maps of the disk which are not necessarily univalent using known inequalities for univalent functions. This approach allowed us to receive a new Ossermans type estimate as well as inequalities for holomorphic self-maps which images do not separate the origin and the boundary.

1. Introduction

Let \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) be the unit disk, and let \( \text{Hol}(\mathbb{D}, \mathbb{D}) \) stand for the family of analytic self-maps of \( \mathbb{D} \). The family \( \text{Hol}(\mathbb{D}, \mathbb{D}) \) forms a semigroup with respect to the functional composition with the identity map as the unity. The study of fixed points of elements from \( \text{Hol}(\mathbb{D}, \mathbb{D}) \) always plays a prominent role in the theory of dynamical systems. We recall that a point \( \xi \in \hat{\mathbb{D}} \) from the closure \( \hat{\mathbb{D}} \) of \( \mathbb{D} \) is said to be a fixed point of an element \( \varphi \in \text{Hol}(\mathbb{D}, \mathbb{D}) \) if \( \lim_{r \to 1^-} \varphi(r\xi) = \xi \). In particular, if \( \xi \in \mathbb{T} := \partial \mathbb{D} \), then the above definition is equivalent to the assertion that the angular limit \( \angle \lim_{z \to \xi} \varphi(z) = \xi \) exists, i.e., \( \lim_{z \to \xi, z \in \Delta_{\xi}} \varphi(z) = \xi \) for any Stolz angle \( \Delta_{\xi} \) centered at \( \xi \), see, e.g., [15, Corollary 2.17, page 35]. Such points \( \xi \in \mathbb{T} \) are usually called boundary fixed points of \( \varphi \). Recall that the angular limit \( \varphi(\xi) \) exists for almost all \( \xi \in \mathbb{T} \), moreover, the exceptional set in \( \mathbb{T} \) is of capacity zero. The classification of the fixed points of \( \varphi \in \text{Hol}(\mathbb{D}, \mathbb{D}) \) can be performed regarding the value of the derivative \( |\varphi'(\xi)| \) in the case \( \xi \in \mathbb{D} \), or the value of the angular derivative

\[
\varphi'(\xi) := \angle \lim_{z \to \xi} \frac{\varphi(z) - \xi}{z - \xi},
\]

in the case \( \xi \in \mathbb{T} \), which is real and \( \varphi'(\xi) \in (0, +\infty) \cup \{\infty\} \) in this case, that follows from the Julia-Wolff lemma, see, e.g., [15, Proposition 4.13, page 82]. We

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recall that the angular derivative at a boundary point \( \xi \) exists if and only if the analytic function \( \varphi'(z) \) has the angular limit \( \lim_{z \to \xi} \varphi'(z) \), see, e. g., [15] Proposition 4.7, page 79. Whenever \( \varphi'(\xi) \neq \infty \), for a boundary fixed point \( \xi \), we say that \( \xi \) is a regular (boundary) fixed point. The regular fixed points can be attractive if \( \varphi'(\xi) \in (0, 1) \), neutral if \( \varphi'(\xi) = 1 \) or repulsive if \( \varphi'(\xi) \in (1, +\infty) \). Fixed points \( \xi \in \mathbb{D} \) with \( |\varphi'(\xi)| < 1 \) are also called attractive.

The existence of the angular derivative is a difficult problem in general, however the Julia-Wolff theory implies that in the case of \( \varphi \in \text{Hol}(\mathbb{D}, \mathbb{D}) \), the angular derivative \( \varphi'(\xi) \) exists (but perhaps infinite) at all points \( \xi \in \mathbb{T} \), where the angular limit \( \varphi(\xi) \) exists and \( |\varphi(\xi)| = 1 \). Furthermore, the mapping at the point \( \xi \) may be conformal (0 < \( |\varphi'(\xi)| < \infty \)) or twisting. The McMillan twist theorem, see [15] page 127, states that \( \varphi \) is conformal for almost all such points. A classical result by Denjoy and Wolff [8, 22], states that for a holomorphic self-map \( \varphi \) of the unit disk \( \mathbb{D} \) different from a (hyperbolic) rotation, there exists a unique fixed point \( \tau \in \mathbb{D} \) such that the sequence of the iterates \( (\varphi_n(z)) \), defined by \( \varphi_0(z) = z, \varphi_n(z) = \varphi(\varphi_{n-1}(z)), n = 1, 2, \ldots \), converges locally uniformly on \( \mathbb{D} \) to \( \tau \) as \( n \to \infty \). This point \( \tau \) is called the Denjoy-Wolff point of \( \varphi \) and it is the only fixed point of \( \varphi \) satisfying \( \varphi'(\tau) \in \mathbb{D} \). The point \( \tau \) is the only attractive fixed point of \( \varphi \) in the above multiplier sense.

The Julia-Carathéodory Theorem [4] and the Wolff Lemma [23] imply that if \( \varphi \) has no interior fixed point, then there exists a boundary fixed point \( \xi \) such that the angular derivative \( \varphi'(\xi) \) exists and \( \varphi'(\xi) \in (0, 1] \). The mapping \( \varphi \) is said to be of parabolic type if \( \varphi'(\xi) = 1 \), and of hyperbolic type if \( \varphi'(\xi) \in (0, 1) \). Otherwise, the mapping \( \varphi \) has an interior fixed point \( \tau \in \mathbb{D} \), and for each boundary fixed point \( \xi \in \mathbb{T} \), \( \varphi'(\xi) > 1 \). Quantification of the latter statement for the case \( \tau = 0 \) was first given by Unkelbash [20], and rediscovered by Osserman in [14] 60 years later. They proved that if \( \varphi \) has a regular boundary fixed point at 1, and \( \varphi(0) = 0 \), then

\[ \varphi'(1) \geq \frac{2}{1 + |\varphi'(0)|}. \]

In the study of the case of several fixed boundary points, a real breakthrough was made by Cowen and Pommerenke [7]. Summarising their results and adding recent progress by Elin, Shoikhet, Tarkhanov, and Bolotnikov [3, 9] we formulate the following theorem

**Theorem A.** Let \( \varphi \in \text{Hol}(\mathbb{D}, \mathbb{D}) \), and let \( \tau \) be the Denjoy-Wolff point of \( \varphi \) and \( \xi_1, \ldots, \xi_n \) be other possible distinct fixed points of \( \varphi \) in \( \mathbb{T} \). Then

- If \( \tau = 0 \), then

\[ \sum_{j=1}^{n} \frac{1}{\varphi'(\xi_j) - 1} \leq \text{Re} \frac{1 + \varphi'(0)}{1 - \varphi'(0)}; \]

- If \( \tau = 1 \) and \( \varphi'(1) \in (0, 1) \) (hyperbolic attractor), then

\[ \sum_{j=1}^{n} \frac{1}{\varphi'(\xi_j) - 1} \leq \frac{\varphi'(1)}{1 - \varphi'(1)}; \]
• If $\tau = 1$ (including $\varphi'(1) = 1$, parabolic attractor), then

$$
\sum_{j=1}^{n} \frac{|1 - \xi_j|^2}{\varphi''(\xi_j) - 1} \leq 2 \text{Re} \left( \frac{1}{\varphi(0)} - 1 \right);
$$

All estimates are sharp and the extremal functions satisfy some functional equations provided in [3, 7, 9].

In the case of univalent functions, the following theorem holds [7].

**Theorem B.** Let $\varphi \in \text{Hol}(D, D)$ be univalent with an attractive Denjoy-Wolff point $\tau \in \mathbb{T}$, and let $\xi_1, \ldots, \xi_n$ be $n$ different repulsive boundary fixed points of $\varphi$. Then,

$$
\sum_{k=1}^{n} \frac{1}{\log \varphi'(\xi_k)} \leq -\frac{1}{\log \varphi'(\tau)}.
$$

Moreover, this inequality is sharp.

A weighted version of this theorem was proved by Contreras, Díaz-Madrigal and Vasil’ev in [6].

For two boundary fixed points $\xi_1$ and $\xi_2$, without loss of generality, applying rotation, we assume that $\xi_1 = e^{-i\theta}$, $\xi_2 = e^{i\theta}$, and $\theta \in (0, \frac{\pi}{2}]$. The following result is a consequence of [7, Theorem 3.1].

**Theorem C.** Let $\varphi \in \text{Hol}(D, D)$ and let $\xi_1 = e^{-i\theta}$ and $\xi_2 = e^{i\theta}$ be fixed. Then,

$$
\varphi'(e^{i\theta})\varphi'(e^{-i\theta}) \geq \sup_{z \in \gamma} \left( 1 + \frac{4 \text{Im} \varphi(z)}{(1 - |\varphi(z)|^2)^2} \right),
$$

where $\gamma$ is the hyperbolic geodesic joining $\xi_1$ and $\xi_2$.

In this paper we are aimed at a sharp analogue of inequality (3) in Theorem A and (4) in Theorem C for univalent self-maps of the unit disk in the case of two fixed boundary points. The result is presented in Section 4. The method of the proof is based on the notion of digon and extremal partitions of a domain, which was already successfully applied for this type of problems with angular derivatives, see [1, 6, 16, 21].

In this paper we also present an approach which allows one to obtain estimates for functions from the general class $\text{Hol}(D, D)$ by means of estimates for univalent functions. The approach is based on the theory of semigroups of analytic functions.

In Section 3 we combine both methods presented in this paper. First we reprove a known sharp estimate

$$
\varphi'(1) \geq \text{Re} \frac{1 - \varphi(0)}{1 + \varphi(0)}
$$

for univalent functions with the fixed boundary point at 1 using moduli and extremal partitions. By doing this we demonstrate briefly the essence of the method we use to prove the result for function with two boundary fixed points. Then we use a semigroup approach to extend the inequality (5) to functions from $\text{Hol}(D, D)$.

In general, using this technique one can start with an estimate for univalent functions and arrive at different type of inequality for general analytic function. For
example, one can obtain an Osserman’s estimate for functions in $\text{Hol}(D, \mathbb{D})$ from Anderson-Vasil'ev inequality for univalent functions. This and other examples of transformation of inequalities is presented in Section 3.

2. Preliminaries

The definitions given here are specified for one digon in a domain in $\hat{\mathbb{C}}$. For general formulations in the case of admissible families of digons and extremal partitions of Riemann surfaces for weighted sums of the moduli, see [10, 12, 19, 21].

2.1. Reduced modulus of digon. Let $D$ be a hyperbolic simply connected domain in $\mathbb{C}$ with two finite fixed boundary points $a$, $b$ (maybe with the same support) on its piecewise smooth boundary. It is called a digon. Denote by $S(a, \varepsilon)$ a region that is the connected component of $D \cap \{|z - a| < \varepsilon\}$ with the point $a$ in its border. Denote by $D_{\varepsilon}$ the domain $D \setminus \{S(a, \varepsilon_1) \cup S(b, \varepsilon_2)\}$ for sufficiently small $\varepsilon_{1,2}$ such that there is a curve in $D_{\varepsilon}$ connecting the opposite sides on $S(a, \varepsilon_1)$ and $S(b, \varepsilon_2)$. Let $M(D_{\varepsilon})$ be the modulus of the family of paths in $D_{\varepsilon}$ that connect the boundary arcs of $S(a, \varepsilon_1)$ and $S(b, \varepsilon_2)$ when lie in the circumferences $|z - a| = \varepsilon_1$ and $|z - b| = \varepsilon_2$ (we choose a single arc in each circle so that both arcs can be connected in $D_{\varepsilon}$). If the limit

$$m(D, a, b) = \lim_{\varepsilon_{1,2} \to 0} \left( \frac{1}{M(D_{\varepsilon})} + \frac{1}{\varphi_a} \log \varepsilon_1 + \frac{1}{\varphi_b} \log \varepsilon_2 \right),$$

exists, where $\varphi_a = \sup \Delta_a$ and $\varphi_b = \sup \Delta_b$ are the inner angles, where $\Delta_a$ and $\Delta_b$ are the Stolz angles inscribed in $D$ at $a$ and $b$ respectively, then $m(D, a, b)$ is called the reduced modulus of the digon $D$. Various conditions guarantee the existence of this modulus, whereas even in the case of a piecewise analytic boundary there are examples [19] which show that this is not always the case. The existence of limit (6) is a local characteristic of the domain $D$ (see [19], Theorem 1.2). If the domain $D$ is conformal (see the definition in [15, page 80]) at the points $a$ and $b$, then ([19], Theorem 1.3) limit (6) exists. More generally, suppose that there is a conformal map $f(z)$ of the domain $S(a, \varepsilon_1) \subset D$ onto a circular sector, so that the angular limit $f(a)$ exists which is thought of as a vertex of this sector of angle $\varphi_a$. If the function $f$ has the finite non-zero angular derivative $f'(a)$ we say that the domain $D$ is also conformal at the point $a$ (compare [15, page 80]). If the digon $D$ is conformal at the points $a$ and $b$ then the limit (6) exists ([19], Theorem 1.3). It is noteworthy that Jenkins and Oikawa [11] in 1977 applied extremal length techniques to study the behaviour of a regular univalent map at a boundary point. Necessary and sufficient conditions were given for the existence of a finite non-zero angular derivative. Independently a similar result was obtained by Rodin and Warschawski [17].

The reduced modulus of a digon is not invariant under conformal mappings. The following result gives a change-of-variable formula, see, e.g., [21]. Let a digon $D$ with the vertices at $a$ and $b$ be so that limit (6) exists and the Stolz angles are $\varphi_a$ and $\varphi_b$. Suppose that there is a conformal map $f(z)$ of the digon $D$ (which is conformal at $a$ and $b$) onto a digon $D'$, so that there exist the angular limits $f(a)$ and $f(b)$ with
the inner angles \( \psi_a \) and \( \psi_b \) at the vertices \( f(a) \) and \( f(b) \) which we also understand as the supremum over all Stolz angles inscribed in \( D' \) with the vertices at \( f(a) \) or \( f(b) \), respectively. If the function \( f \) has the finite non-zero angular derivatives \( f'(a) \) and \( f'(b) \), then \( \varphi_a = \psi_a, \varphi_b = \psi_b \), and the reduced modulus \( m \) of \( D' \) exists and changes \( \{10, 12, 19, 21\} \) according to the rule

\[
m(f(D), f(a), f(b)) = m(D, a, b) + \frac{1}{\psi_a} \log |f'(a)| + \frac{1}{\psi_b} \log |f'(b)|.
\]

If we suppose, moreover, that \( f \) has the expansion

\[
f(z) = w_1 + (z - a)^{\psi_a/\varphi_a} (c_1 + c_2(z - a) + \ldots)
\]
in a neighborhood of the point \( a \), and the expansion

\[
f(z) = w_2 + (z - b)^{\psi_b/\varphi_b} (d_1 + d_2(z - b) + \ldots)
\]
in a neighborhood of the point \( b \), then the reduced modulus of \( D \) changes according to the rule

\[
m(f(D), f(a), f(b)) = m(D, a, b) + \frac{1}{\psi_a} \log |c_1| + \frac{1}{\psi_b} \log |d_1|.
\]

Obviously, one can extend this definition to the case of vertices with infinite support.

2.2. Extremal partition by digons. Let \( \Omega \) be a hyperbolic domain in \( \hat{\mathbb{C}} \) that has a finite number of hyperbolic and parabolic boundary components. We consider a family \( \mathcal{F} \) of digons \( D \) in \( \Omega \) with two fixed vertices \( a \) and \( b \) on \( \partial \Omega \), such that any arc connecting the vertices of \( D \in \mathcal{F} \) is not homotopic to a point of \( \Omega \). The boundary points are understood in the Carathéodory sense.

We require the digons from \( \mathcal{F} \) to be conformal at their vertices. A general theorem, see \[10, 12, 19, 21\], implies that any collection of admissible digons \( \mathcal{F} \) satisfies the inequality \( m(D, a, b) \geq m(D^*, a, b) \), with the equality sign only for \( D = D^* \). Here \( D^* \) is a strip domain in the trajectory structure of a unique quadratic differential \( Q(\zeta)d\zeta^2 \), and there is a conformal map \( g(\zeta), \zeta \in D^* \) that satisfies the differential equation

\[
\left( \frac{g'(\zeta)}{g(\zeta)} \right)^2 = 4\pi^2 Q(\zeta),
\]

and which maps \( D^* \) onto the strip \( \mathbb{C} \setminus [0, \infty) \). The critical trajectories of \( Q(\zeta)d\zeta^2 \) define in \( \Omega \) a strip domain \( D^* \) associated with \( \mathcal{F} \).

2.3. Semigroups of analytic functions. Let us recall that a \((one-parameter) semigroup of analytic functions\) is any continuous homomorphism \( \Phi : t \mapsto \Phi(t) = \varphi_t \) from the additive semigroup of non-negative real numbers into the composition semigroup of all analytic functions which map \( \mathbb{D} \) into \( \mathbb{D} \). That is, \( \Phi \) satisfies the following three conditions:

a) \( \varphi_0 \) is the identity in \( \mathbb{D} \),

b) \( \varphi_{t+s} = \varphi_t \circ \varphi_s \), for all \( t, s \geq 0 \),

c) \( \varphi_t(z) \) tends to \( z \) locally uniformly in \( \mathbb{D} \) as \( t \to 0 \).
It is well-known that the functions $\varphi_t$ are always univalent. If $a$ is the Denjoy-Wolff point of one of the functions $\varphi_{t_0}$, for some $t_0 > 0$, then $a$ is the Denjoy-Wolff point of all the functions of the semigroups, that is, all functions of a semigroup share the Denjoy-Wolff point. Moreover, if a point $\xi \in \partial \mathbb{D}$ is a boundary fixed point of $\varphi_{t_0}$ for some $t_0 > 0$, then it is a boundary fixed point of all $\varphi_t$.

Given a semigroup $(\varphi_t)_{t \geq 0}$, it is well-known (see [2, 18]) that there exists a unique analytic function $g: \mathbb{D} \to \mathbb{C}$ such that

$$\frac{d\varphi_t}{dt} = g(\varphi_t),$$

for all $z \in \mathbb{D}$ and $t \geq 0$, called the \textit{infinitesimal generator} of the semigroup $(\varphi_t)_{t \geq 0}$. The Berkson-Porta representation [2] assures that an analytic function $g: \mathbb{D} \to \mathbb{C}$ is the infinitesimal generator of a semigroup of analytic functions $(\varphi_t)_{t \geq 0}$ if and only if there exists a point $w \in \hat{\mathbb{D}}$ and an analytic function $p: \mathbb{D} \to \mathbb{C}$ with $\text{Re} p(z) > 0$ in $\mathbb{D}$, such that

$$g(z) = (w - z)(1 - \bar{w}z)p(z), \quad z \in \mathbb{D}.$$  

Such a representation is unique. If $(\varphi_t)$ is not the trivial group of the identity maps, then $w$ is either the Denjoy-Wolff point of the semigroup in the case where $(\varphi_t)_{t \geq 0}$ is not a group of hyperbolic rotations, or the unique interior fixed point, otherwise.

3. One fixed boundary point

3.1. Cowen-Pommerenke type inequality. In this section we will prove a known theorem by Cowen and Pommerenke showing, in particular, a method how one can obtain estimates for general functions from $\text{Hol}(\mathbb{D}, \mathbb{D})$ by means of estimates for univalent functions. At the same time we will show an application of the reduced moduli of digons and extremal partitions infinitesimal generators for semigroups.

\textbf{Theorem 1.} [7, Theorem 8.1] If $\varphi \in \text{Hol}(\mathbb{D}, \mathbb{D})$, and $\varphi(1) = 1$, then the sharp estimate

$$\varphi'(1) \geq \frac{1}{\text{Re} \frac{1 + \varphi(0)}{1 - \varphi(0)}}, \quad (10)$$

holds with the equality sign for the Möbius map

$$m(z) = \frac{1 + \bar{a}}{1 + a} \frac{z + a}{1 + z a}, \quad a \in \mathbb{D}, \quad a \frac{1 + \bar{a}}{1 + a} = \varphi(0).$$

\textbf{Corollary 1.} If $\varphi \in \text{Hol}(\mathbb{D}, \mathbb{D})$, and $\varphi(1) = 1$, then the sharp estimate

$$\varphi'(1) \geq \text{Re} \frac{1 - \varphi(0)}{1 + \varphi(0)}, \quad (11)$$

holds with the equality sign only for real values of $\varphi(0)$ and for the Möbius map

$$m(z) = \frac{z + \varphi(0)}{1 + z \varphi(0)}.$$
Proof. We start with the univalent case. Let us consider the family $F_0$ of digons $D_0$ in $\Omega = \mathbb{D} \setminus \{0\}$ with two vertices $1^+$ and $1^-$ over the same point $1$ and the equal angles $\pi/2$ at these vertices, such that any arc connecting $1^+$ and $1^-$ in $D_0$ starting at one of the vertices comes to the other in $\Omega$ making the round about the origin. Then

$$\min_{D_0 \in F} m(D_0, 1^+, 1^-) = m(D_0^*, 1^+, 1^-) = 0,$$

where $D_0^* = \mathbb{D} \setminus [0, 1)$, which is a strip domain in the trajectory structure of the quadratic differential

$$Q_0(z)dz^2 = \frac{1}{(z - 1)^2}dz^2, \quad z \in \Omega.$$

Let $\varphi \in \text{Hol}(\mathbb{D}, \mathbb{D})$ be an arbitrary univalent map, $\varphi(\xi) = \xi$, $\xi \in \partial \mathbb{D}$. Denote by $D_\xi^*$ the unit disk with a slit along the closed line segment connecting the origin and the point $\xi$. We regard $D_\xi^*$ as a digon with vertices at $\xi$. It can be obtained from the domain $D_0^*$ by rotation. The rotation transform does not change the reduced modulus of a digon. Therefore, the reduced modulus $m(D_\xi^*, \xi^+, \xi^-) = 0$.

We observe that $\sigma \circ \varphi(D_\xi^*)$ is an admissible domain in the problem of extremal partition (12), where

$$\sigma(z) = \frac{1 - \varphi(0)\xi}{\xi - \varphi(0)} \frac{z - \varphi(0)}{1 - \varphi(0)\xi}, \quad |\sigma'(\xi)| = \frac{1 - |\varphi(0)|^2}{|1 - \varphi(0)\xi|^2}.$$

This and relation (7) imply that

$$\frac{4}{\pi} \log |\varphi'(\xi)| + \frac{4}{\pi} \log \frac{1 - |\varphi(0)|^2}{|1 - \varphi(0)\xi|^2} \geq 0,$$

or

$$|\varphi'(\xi)| \geq \frac{|1 - \varphi(0)\xi|^2}{1 - |\varphi(0)|^2} \geq \text{Re} \frac{1 - \varphi(0)\xi}{1 + \varphi(0)\xi} = \frac{\xi - \varphi(0)}{\xi + \varphi(0)}.$$

When $\xi = 1$, the first inequality is equivalent to (10) and the last one is equivalent to (11). The uniqueness of the extremal function follows from the uniqueness of the extremal configuration.

**Remark 1.** We remark that

$$\text{Re} \frac{1 - \varphi(0)}{1 + \varphi(0)} \geq \frac{1 - |\varphi(0)|}{1 + |\varphi(0)|}.$$

Inequality (11) can be obtained with use of the theory of semigroups of holomorphic self-mappings of the unit disk.

Let $g$ be a generator of a one parameter semigroup $S = \{\varphi_t\}_{t \geq 0}$ having a boundary fixed point $\xi = \varphi_t(\xi) \in \partial \mathbb{D}$. If $g'(\xi)$ is finite, then it is a real number and $\varphi_t'(\xi) = e^{tg'(\xi)}$. Since $\varphi_t$ univalent for each $t \geq 0$, one can write by (13)

$$e^{tg'(\xi)} \geq \text{Re} \frac{\xi - \varphi_t(0)}{\xi + \varphi_t(0)}.$$
We calculate
\[ 1 - e^{tg'(\xi)} \leq \frac{1 - \text{Re} \frac{\xi - \varphi_t(0)}{\xi + \varphi_t(0)}}{t} = \frac{\text{Re} \frac{\xi + \varphi_t(0) - \xi - \varphi_t(0)}{\xi + \varphi_t(0)}}{t} = \frac{\text{Re} \frac{2\varphi_t(0)}{\xi + \varphi_t(0)}}{t} = 2\text{Re} \frac{\varphi_t(0)\overline{\xi}}{(1 + \varphi_t(0)\overline{\xi})t}. \]

Since \( \lim_{t \to 0^+} (\varphi_t(z) - z)/t = g(z) \) and \( \lim_{t \to 0^+} \varphi_t(z) = z, \ z \in \mathbb{D} \), we obtain
\[ \lim_{t \to 0^+} \frac{1 - e^{tg'(\xi)}}{t} \leq 2 \lim_{t \to 0^+} \frac{\text{Re} \varphi_t(0)\overline{\xi}}{t} = 2\text{Re} g(0)\overline{\xi} \]
or, finally,
\[ (14a) \quad -g'(\xi) \leq 2\text{Re} g(0)\overline{\xi}. \]

If \( \xi = 1 \), then \((14a)\) becomes
\[ (14b) \quad -g'(1) \leq 2\text{Re} g(0). \]

Let now \( \phi \) be any holomorphic function, \( \phi(\mathbb{D}) \subseteq \mathbb{D} \) with \( \phi(\xi) = \xi \in \partial \mathbb{D} \).

Consider the function \( g : \mathbb{D} \to \mathbb{C} \) defined by
\[ (15) \quad g(z) := (w - z)(1 - z\overline{w})\frac{\xi - \phi(z)}{\xi + \phi(z)}, \]
where \( w \in \partial \mathbb{D} \cup \mathbb{D} \) is chosen such that \( w \neq \xi \) and \( \phi \in \text{Hol} (\mathbb{D}, \mathbb{D}) \).

It follows by the Berkson-Porta formula (see, for example, [2, 18]) that \( g \) is a holomorphic generator and \( g(\xi) = 0 \). Therefore, one can use inequality \((14a)\).

We calculate again
\[ (16) \quad g'(\xi) = \lim_{z \to \xi} \frac{g(z)}{z - \xi} = \lim_{z \to \xi} (z - w)(1 - z\overline{w})\frac{\phi(z) - \xi}{z - \xi} \cdot \frac{1}{\xi + \phi(z)} = \frac{1}{2\xi} |1 - \overline{w}\xi|^2 \cdot \phi'(\xi). \]

In addition, \((15)\) implies that
\[ (17) \quad g(0) = w\frac{\xi - \phi(0)}{\xi + \phi(0)}. \]

We plug \((16)\) and \((17)\) into \((14a)\) and get
\[ (18) \quad -\frac{|1 - \overline{w}\xi|^2}{2} \phi'(\xi) \leq 2\text{Re} \left[ \xi w\frac{\xi - \phi(0)}{\xi + \phi(0)} \right]. \]

In particular, if \( \xi = 1 \), one can choose \( w = -1 \) to get the inequality
\[ \phi'(1) \geq \text{Re} \frac{1 - \phi(0)}{1 + \phi(0)} \]
which coincides with \((11)\). \(\square\)
3.2. **Osserman type inequality.** Combining our approach with the following result for univalent self-mappings of the disk established by Anderson and Vasil’ev [1], we derive some new estimate for holomorphic self-mappings which are not necessarily univalent that, in particular, improves Osserman’s result.

Let us define the Pick function

\[ p_\beta(z) = \frac{4\beta z}{(1 - z + \sqrt{(1 - z)^2 + 4\beta z})^2} = \beta z + \ldots, \]

that maps the unit disk \( \mathbb{D} \) onto \( \mathbb{D} \setminus (-1, -\beta/(1 + \sqrt{1 - \beta})^2] \). Set the Möbius transformation

\[ B_z(\zeta) = \frac{1 - \bar{z} \zeta - z}{1 - z 1 - \zeta \bar{z}}. \]

**Theorem 2.** [1] Let \( \varphi \in \text{Hol}(\mathbb{D}, \mathbb{D}) \) be a univalent function which is conformal at the boundary point \( \xi = 1 \), \( \varphi \chi_{z \to 1} = 1 \), and let \( \liminf_{z \to 1} \frac{1 - |\varphi(z)|}{1 - |z|} =: \alpha \) exists and is finite. Then for all \( z \in \mathbb{D} \),

\[ |\varphi'(z)| \geq \frac{1}{\alpha^2} \frac{(1 - |z|^2)^3}{|1 - |z||^4} \frac{|1 - \varphi(z)|^4}{(1 - |\varphi(z)|^2)^3}. \]

With a fixed \( z \in \mathbb{D} \) and \( \varphi(z) = w \), the equality sign is attained only for the function \( \varphi^* = B_w^{-1} \circ p_\beta \circ B_z \), where

\[ \beta = \frac{1}{\alpha^2} \frac{(1 - |z|^2)^2|1 - \varphi^*(z)|^4}{|1 - z|^4 (1 - |\varphi^*(z)|^2)^2}. \]

**Theorem 3.** Let \( \phi \in \text{Hol}(\mathbb{D}, \mathbb{D}) \) and suppose that \( \xi = 1 \) is its boundary regular fixed point. Then

\[ \phi'(1) \geq \frac{2}{\text{Re} \frac{1 - \phi(0)^2 + \phi'(0)}{(1 - \phi(0))^2}}. \]

**Proof.** Let \( S = \{ \varphi_t \}_{t \geq 0} \subset \text{Hol}(\mathbb{D}, \mathbb{D}) \) be a semigroup of conformal self-maps of \( \mathbb{D} \) generated by \( g \), and with a boundary regular fixed point \( \xi = 1 \). Since all functions \( \varphi_t \), \( t \geq 0 \), are univalent, they satisfy inequality (19). We rewrite it in the form

\[ \phi'(1) \geq \frac{1}{\text{Re} \frac{1 - \phi(0)^2 + \phi'(0)}{(1 - \phi(0))^2}} \frac{(1 - |z|^2)^3}{|1 - \varphi_t(z)|^4} \frac{|1 - \varphi_t(z)|^4}{|1 - z|^4 (1 - |\varphi_t(z)|^2)^3}. \]

Obviously, the map \( \varphi_t \big|_{t=0^+} = id \) gives the equality in (21). Then the same inequality holds for the \( t \)-derivatives of the both sides of (21) at the point \( 0^+ \). The derivative of the left-hand side at \( t = 0^+ \) becomes

\[ \frac{d}{dt} |\varphi_t'(1)|^2 \bigg|_{t=0^+} = \frac{d}{dt} e^{2g'(1)} \bigg|_{t=0^+} = 2g'(1). \]
In the right-hand side, the derivative is equal to the limit

\[
R = \lim_{t \to 0^+} \frac{1}{t} \frac{(1 - |z|^2)^3}{|\varphi'_t(z)|} |1 - \varphi_t(z)|^4 \frac{|1 - \varphi_t(z)||1 - z|^4 (1 - |\varphi_t(z)|^2)^3}{(1 - |\varphi_t(z)|^2)^3}
\]

\[
= \lim_{t \to 0^+} \frac{(1 - |z|^2)^3}{|\varphi'_t(z)||1 - z|^4 (1 - |\varphi_t(z)|^2)^3} t.
\]

Since for all \( z \in \mathbb{D} \), \( \varphi_t(z) - z \), and \( \frac{\varphi'_t(z) - 1}{t} = g'(z) \), the first term in the numerator is

\[
(1 - |z|^2)^6 |1 - \varphi_t(z)|^8 = (1 - |z|^2)^6 |(1 - z) + (z - \varphi_t(z))|^8
\]

\[
= (1 - |z|^2)^6 \left( |1 - z|^2 + 2\text{Re}(1 - \overline{z})(z - \varphi_t(z)) + o(t) \right)^4
\]

\[
= (1 - |z|^2)^6 \left( |1 - z|^4 + 4|1 - z|^4 \text{Re}(1 - \overline{z})(z - \varphi_t(z)) + o(t) \right)^2
\]

\[
= (1 - |z|^2)^6 \left( |1 - z|^8 + 8|1 - z|^8 \text{Re}(1 - \overline{z})(z - \varphi_t(z)) + o(t) \right)
\]

and the second term is

\[
|\varphi'_t(z)|^2 |1 - z|^8 (1 - |\varphi_t(z)|^2)^6
\]

\[
= |1 - z|^8 (1 + 2\text{Re}(\varphi'_t(z) - 1) + o(t)) (1 - |z|^2 - 2\text{Re}\overline{z}(\varphi_t(z) - z) + o(t))^6
\]

\[
= |1 - z|^8 (1 + 2\text{Re}(\varphi'_t(z) - 1)) \left( |1 - |z|^2|^6 - 12 (1 - |z|^2)^5 \text{Re}\overline{z}(\varphi_t(z) - z) \right) + o(t)
\]

\[
= |1 - z|^8 \left( |1 - |z|^2|^6 - 12 (1 - |z|^2)^5 \text{Re}\overline{z}(\varphi_t(z) - z) + 2 (1 - |z|^2)^6 \text{Re}(\varphi'_t(z) - 1) + o(t) \right)
\]

Hence,

\[
R = \lim_{t \to 0^+} \frac{8 (1 - |z|^2) \text{Re}(1 - \overline{z})(z - \varphi_t(z)) + 12\text{Re}\overline{z}(\varphi_t(z) - z) - 2 (1 - |z|^2) \text{Re}(\varphi'_t(z) - 1)}{2 (1 - |z|^2)t}
\]

\[
= -\frac{8 (1 - |z|^2) \text{Re}(1 - \overline{z})g(z) + 12\text{Re}\overline{z}g(z) - 2 (1 - |z|^2) \text{Re}g'(z)}{2 (1 - |z|^2)}.
\]

Consequently, for all \( z \in \mathbb{D} \),

\[
2g'(1) \geq -\frac{8 (1 - |z|^2) \text{Re}(1 - \overline{z})g(z) + 12\text{Re}\overline{z}g(z) - 2 (1 - |z|^2) \text{Re}g'(z)}{2 (1 - |z|^2)}.
\]

In particular, for \( z = 0 \),

\[
(23) \quad 2g'(1) \geq -\Re \left( g'(0) + 4g(0) \right).
\]
Suppose $\phi \in \text{Hol}(\mathbb{D}, \mathbb{D})$ with the regular fixed point $\xi = 1$. Then the function

$$g(z) = (1 - z)^{\frac{1 + \phi(z)}{1 - \phi(z)}}$$

is a generator of a semigroup with the boundary regular fixed point $\xi = 1$, and so it satisfies inequality (23). Simple calculations show that

$$g'(1) = -\frac{2}{\phi'(1)}, \quad g'(0) = \frac{2\phi'(0)}{(1 - \phi(0))^2} - 2\frac{1 + \phi(0)}{1 - \phi(0)}, \quad g(0) = \frac{1 + \phi(0)}{1 - \phi(0)}.$$

Substituting these expressions in (23), we have (20).

An analogous estimate for angular derivatives including the values of $\phi(0)$, $\phi'(0)$ and $\phi'(1)$, was established by Osserman in [14]. For the case of the boundary regular fixed point $\xi = 1$, we write his estimate in the form which is convenient to compare with our result:

$$(24) \quad \phi'(\xi) \geq \frac{2}{\frac{1 - |\phi(0)|^2}{(1 - |\phi(0)|)^2}} =: L.$$  

It is easy to see that our inequality improves this estimate. Moreover, Osserman’s estimate is the same for all boundary fixed points. Hence, if the Denjoy–Wolf point of $\phi$ is situated on the boundary of the unit disk, then $L \leq 1$ and, consequently, for all other (repelling) boundary fixed points, it does not give additional information.

Thus, in a simple example of the hyperbolic automorphism $\phi(z) = \frac{2z - 1}{2 - z}$ of $\mathbb{D}$ with fixed points at $\pm 1$, $\phi'(-1) = \frac{1}{3}$ and $\phi'(1) = 3$, Osserman’s estimate gives $\phi'(1) \geq \frac{1}{3}$ whereas (20) implies that $\phi'(1) \geq 3$.

Further, we present some estimates for holomorphic self-mappings of the unit disk whose images do not contain the origin, i.e., $0 \notin \phi(\mathbb{D})$. We also derive them from some inequalities for generators.

**Theorem 4.** Let $\phi \in \text{Hol}(\mathbb{D}, \mathbb{D})$. Suppose that $\phi(\mathbb{D})$ does not separate the origin and the boundary $\partial \mathbb{D}$ and $0 \notin \phi(\mathbb{D})$. If $\xi = 1$ is a boundary regular fixed point of $\phi$ then

$$(25) \quad \phi'(1) \geq -\frac{\ln|\phi(0)|}{2}.$$  

**Proof.** For a function $\phi$ which satisfies our assumptions, define the generator

$$(26) \quad g(z) = -z\frac{1 - \phi(z)^\frac{1}{n}}{1 + \phi(z)^\frac{1}{n}}, \quad z \in \mathbb{D},$$

where $w^{\frac{1}{n}}$ means the analytic branch of the root fixing 1. The generator $g$ has a regular null point at 1, and therefore, satisfies inequality (23). Substituting

$$g(0) = 0, \quad g'(0) = -\frac{1 - \phi(0)^\frac{1}{n}}{1 + \phi(0)^\frac{1}{n}}, \quad \text{and} \quad g'(1) = \frac{1}{n} \frac{\phi'(1)}{2},$$

we have $g'(1) = \frac{2}{\phi'(1)}$. Substituting these expressions in (23), we have (20). □
in (23), we have
\[ \phi'(1) \geq n \operatorname{Re} \frac{1 - \phi(0)^1}{1 + \phi(0)^1}, \]
and in the limit case, as \( n \to \infty \), we have (25). □

Notice, that if \( \phi(0) \in (0, 1) \), then one can show that inequality (25) improves the known estimate
\[ \phi'(1) \geq \frac{1}{1 + \phi(0)}, \]
which holds for holomorphic self-mappings of \( \mathbb{D} \) without an additional restriction on the image \( \phi(\mathbb{D}) \) (see [7]).

A similar method with generator (26) applied to the Harnack inequality for a holomorphic function \( p \) on \( \mathbb{D} \) with positive real part (see, for example, [18])
\[ \frac{1 - |z|}{1 + |z|} \operatorname{Re} p(0) \leq \operatorname{Re} p(z) \leq \frac{1 + |z|}{1 - |z|} \operatorname{Re} p(0), \]
gives again some estimate for holomorphic self-maps of the disk such that \( 0 \notin \phi(\mathbb{D}) \).

**Theorem 5.** Let \( \phi \in \operatorname{Hol}(\mathbb{D}, \mathbb{D}) \). Suppose that \( \phi(\mathbb{D}) \) does not separate the origin and the boundary \( \partial \mathbb{D} \) and \( 0 \notin \phi(\mathbb{D}) \). Then for all \( z \in \mathbb{D} \),
\[ |\phi(0)|^{\frac{1 + |z|}{1 - |z|}} \leq |\phi(z)| \leq |\phi(0)|^{\frac{1 - |z|}{1 + |z|}}. \]

**Proof.** Suppose that a function \( \phi \) satisfies our assumptions. Then for
\[ p(z) = \frac{1 - \phi(z)^1}{1 + \phi(z)^1}, \quad n \in \mathbb{N}, \quad z \in \mathbb{D}, \]
inequality (27) holds, and so
\[ \frac{1 - |z|}{1 + |z|} n \operatorname{Re} \frac{1 - \phi(0)^1}{1 + \phi(0)^1} \leq n \operatorname{Re} \frac{1 - \phi(z)^1}{1 + \phi(z)^1} \leq \frac{1 + |z|}{1 - |z|} n \operatorname{Re} \frac{1 - \phi(0)^1}{1 + \phi(0)^1}. \]
Since for all \( z \in \mathbb{D} \),
\[ \lim_{n \to \infty} n \frac{1 - \phi(z)^1}{1 + \phi(z)^1} = -\frac{\log \phi(z)}{2}, \]
we have
\[ -\frac{1 - |z|}{1 + |z|} \operatorname{Re} \log \phi(0) \leq -\operatorname{Re} \log \phi(z) \leq -\frac{1 + |z|}{1 - |z|} \operatorname{Re} \log \phi(0), \]
or which is the same,
\[ |\phi(0)|^{\frac{1 + |z|}{1 - |z|}} \leq |\phi(z)| \leq |\phi(0)|^{\frac{1 - |z|}{1 + |z|}}. \]
□
Observe, that the right-hand side of this inequality improves the known Lindelöf’s estimate [13] (which holds for all holomorphic self-mappings of $\mathbb{D}$ without an additional assumption on the image $\phi(D)$):

$$|\phi(z)| \leq \frac{|z| + |\phi(0)|}{1 + |z||\phi(0)|}.$$ 

Indeed, for each $a \in (0, 1)$, the real-valued functions

$$\varphi(x) = e^x \quad \text{and} \quad \psi(x) = \frac{(1 - x) + a(1 + x)}{(1 + x) + a(1 - x)}, \quad x \in (0, 1),$$

are strictly decreasing and coincide at the points $x = 0$ and $x = 1$, whereas $\varphi \left( \frac{1}{2} \right) < \psi \left( \frac{1}{2} \right)$. So, $\varphi(x) < \psi(x)$ for all $x \in (0, 1)$. Setting $x = \frac{1-|z|}{1+|z|}$ and $a = |\phi(0)|$, we have

$$|\phi(0)|^{1-|z|} \geq \frac{|z| + |\phi(0)|}{1 + |z||\phi(0)|}$$

for all $0 \neq z \in \mathbb{D}$.

4. Two Fixed Boundary Points

In this section we obtain estimates for angular derivatives of a univalent self-map of the unit disk with two fixed boundary points.

Let $\xi_1$ and $\xi_2$ be two different points of the unit circle $\mathbb{T}$. Let $\varphi: \mathbb{D} \to \mathbb{D}$ be univalent and $\varphi(\xi_j) = \xi_j, \ j = 1, 2$. We are interested in the lower estimate of the product $\varphi'(\xi_1)\varphi'(\xi_2)$ for functions $\varphi$ with a fixed value of $\varphi(0) \in \mathbb{D}$. Again without loss of generality, applying rotation, we assume that $\xi_1 = e^{-i\theta}$, $\xi_2 = e^{i\theta}$, and $\theta \in (0, \frac{\pi}{2})$.

In order to formulate the main result we need several preparatory definitions and notations. For every $a \in \mathbb{D}$ let us define a real-valued continuous function $\Phi(a)$ by

$$\Phi(a) = \begin{cases} \frac{1-|a|^2}{2(\cos \theta - Re a)} - \sqrt{1 + \left( \frac{1-|a|^2}{2(\cos \theta - Re a)} \right)^2} - \frac{1-|a|^2}{\cos \theta - Re a} \cos \theta, & \text{if } Re a < \cos \theta; \\ \cos \theta, & \text{if } Re a = \cos \theta; \\ \frac{1-|a|^2}{2(\cos \theta - Re a)} + \sqrt{1 + \left( \frac{1-|a|^2}{2(\cos \theta - Re a)} \right)^2} - \frac{1-|a|^2}{\cos \theta - Re a} \cos \theta, & \text{if } Re a > \cos \theta. \end{cases}$$

Geometrically, this function defines the intersection of the arc of a circle containing the points $a$ and $e^{\pm i\theta}$ with the interval $(-1, 1)$.

Denote by $\gamma_0$ the arc of the circle $\{(x, y): (x^2 + y^2) \cos \theta = x\}$ inside the unit disk $\mathbb{D}$, which contains the origin and the points $e^{\pm i\theta}$ in its closure. Observe that if $a \in \gamma_0$, then $\Phi(a) = 0$, and if $a$ is real, then $\Phi(a) = a$. Let us define three domains $U_1$, $U_2$, and $U_3$ of range of $\varphi(0)$ as in Fig.1. By $U_1$ we denote the segment between the unit circle to the right of the interval connecting the points $e^{\pm i\theta}$. By $U_2$ we denote the segment between the circle $\gamma_0$ to the left of the interval connecting the points $e^{\pm i\theta}$. Finally, by $U_3$ we denote the domain between the unit circle to the left from $\gamma_0$.

Now let us define the future extremal functions. Let $\zeta(z) = z + \frac{1}{z}$ denote the Joukowski map. For a fixed value of $x \in [0, 1)$ we define the Pick function $p^+_x(z)$ as
Figure 1. Domains $U_1$, $U_2$, and $U_3$ for $\varphi(0)$

A superposition of three functions $p_x^+(z) = \zeta^{-1} \circ u^+ \circ \zeta(z)$, where $u^+ (\zeta)$ is a Möbius map given by

$$u^+ (\zeta) = \frac{(1 + x^2)\zeta - 4x \cos \theta}{x\zeta + (1 - x)^2 - 2x \cos \theta}.$$ 

Observe that $p_x^+ (e^{\pm i\theta}) = e^{\pm i\theta}$ and the function $p_x^+ (z)$ maps the disk $\mathbb{D}$ onto the domain $\mathbb{D} \setminus (-1, r^+)$, where

$$r^+ = \frac{2\sqrt{2x(1 + x^2)\cos \theta} - x^2 - 2x \cos \theta - 1}{x^2 - 2x \cos \theta + 1}.$$ 

The angular derivative is

$$(p_x^+)'(e^{\pm i\theta}) = \frac{1 - 2x \cos \theta + x^2}{(1 - x)^2}.$$ 

Analogously, for a fixed value of $x \in (-1, 0]$ we define the Pick function $p_x^-(z)$ as a superposition of three functions $p_x^- (z) = \zeta^{-1} \circ u^- \circ \zeta(z)$, where $u^- (\zeta)$ is a Möbius map given by

$$u^- (\zeta) = \frac{-(1 + x^2)\zeta + 4x \cos \theta}{x\zeta - (1 - x)^2 - 2x \cos \theta}.$$ 

Observe that again $p_x^- (e^{\pm i\theta}) = e^{\pm i\theta}$ and the function $p_x^- (z)$ maps the disk $\mathbb{D}$ onto the domain $\mathbb{D} \setminus [r^- , 1)$, where

$$r^- = \frac{x^2 - 2x \cos \theta + 1 - 2\sqrt{2x(1 + x^2)\cos \theta}}{x^2 - 2x \cos \theta + 1}.$$ 

The angular derivative is

$$(p_x^-)'(e^{\pm i\theta}) = \frac{1 - 2x \cos \theta + x^2}{(1 + x)^2}.$$
We notice that if \( x = 0 \), then \( p_{\varphi}^+(z) = p_{\varphi}^-(z) = z \).

Finally, let us define the map \( m\mbox{ö}b(z,t) \), \( t \in (-\theta, \theta) \) given as a solution to the equation

\[
\frac{z - e^{-i\theta}}{1 - e^{-i\theta}} \frac{1 - e^{i\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{w - e^{-i\theta}}{w - e^{i\theta}} \frac{e^{it} - e^{i\theta}}{e^{it} - e^{-i\theta}},
\]

and a special value of \( t = t_0 = t_0(a) \) given a unique solution in the interval \( t \in (-\theta, \theta) \) to the equation

\[
\frac{a - e^{-i\theta}}{a - e^{i\theta}} \frac{1 - e^{i\theta}}{1 - e^{-i\theta}} = \frac{\Phi(a) - e^{-i\theta}}{\Phi(a) - e^{i\theta}} \frac{e^{it} - e^{i\theta}}{e^{it} - e^{-i\theta}}.
\]

By writing \( m\mbox{ö}b^{-1}(z,t) \) we mean the inverse with respect to \( z \).

Let us remind that if \( \varphi \in \text{Hol}(\mathbb{D}, \mathbb{D}) \) has no interior fixed points, then there exists a boundary Denjoy-Wolff point \( \xi_1 \) in which \( \varphi'(\xi_1) \in (0, 1] \). If there is another boundary fixed point \( \xi_2 \), then \( \varphi'(\xi_2) \geq 1 \). It is a simple consequence from Julia-Carathéodory-Wolff results that

\[
(29) \quad \varphi'(\xi_1)\varphi'(\xi_2) \geq 1,
\]

see also [7, Theorem 3.1]. The following theorem is a refinement of (29).

**Theorem 6.** If \( \varphi \in \text{Hol}(\mathbb{D}, \mathbb{D}) \) and univalent, \( \varphi(e^{\pm i\theta}) = e^{\pm i\theta} \), \( \theta \in (0, \frac{\pi}{2}) \) then the following sharp estimates hold:

(a) If \( \varphi(0) \in U_1 \cup U_2 \cup [e^{-i\theta}, e^{i\theta}] \) (lies to the right of \( \gamma_0 \)), then

\[
\sqrt{\varphi'(e^{i\theta})\varphi'(e^{-i\theta})} \geq \frac{1 - 2\Phi(\varphi(0)) \cos \theta + \Phi^2(\varphi(0))}{(1 - \Phi(\varphi(0)))^2};
\]

(b) If \( \varphi(0) \in U_3 \) (lies to the left of \( \gamma_0 \)), then

\[
\sqrt{\varphi'(e^{i\theta})\varphi'(e^{-i\theta})} \geq \frac{1 - 2\Phi(\varphi(0)) \cos \theta + \Phi^2(\varphi(0))}{(1 + \Phi(\varphi(0)))^2}.
\]

(c) If \( \varphi(0) \in \gamma_0 \), then \( \varphi'(e^{i\theta})\varphi'(e^{-i\theta}) \geq 1 \), or equivalently, the inequality (29) can not be refined.

In all three cases the estimates are sharp and the extremal value is given by a unique extremal function:

(a) \( m\mbox{ö}b^{-1}\left(p_{\varphi(\varphi(0))}^+(\varphi(0)), t_0(\varphi(0))\right) \);

(b) \( m\mbox{ö}b^{-1}\left(p_{\varphi(\varphi(0))}^-(\varphi(0)), t_0(\varphi(0))\right) \);

(c) \( m\mbox{ö}b^{-1}(\varphi(0)) \).

**Corollary 2.** If \( \varphi \in \text{Hol}(\mathbb{D}, \mathbb{D}) \) and univalent, \( \varphi(e^{\pm i\theta}) = e^{\pm i\theta} \), \( \theta \in (0, \frac{\pi}{2}) \) then the following sharp estimates hold:

- If \( \varphi(0) \in U_1 \), then

\[
\sqrt{\varphi'(e^{i\theta})\varphi'(e^{-i\theta})} \geq \frac{1 - 2\text{Re}\varphi(0) \cos \theta + (\text{Re}\varphi(0))^2}{(1 - \text{Re}\varphi(0))^2};
\]
• If \( \varphi(0) \in U_3 \), then
\[
\sqrt{\varphi'(e^{i\theta})\varphi'(e^{-i\theta})} \geq \frac{1 - 2\text{Re} \varphi(0) \cos \theta + (\text{Re} \varphi(0))^2}{(1 + \text{Re} \varphi(0))^2}.
\]

Observe that if \( \theta = \frac{\pi}{2} \), then the domain \( U_2 \) degenerates. The equality sign is attained for real values of \( \varphi(0) \) and for \( p_{\varphi(0)}^e(z) \) and \( p_{\varphi(0)}^f(z) \) respectively.

**Proof.** Let us assume first that \( \theta \in \left(0, \frac{\pi}{2}\right) \), and let \( \varphi \in \text{Hol}(\mathbb{D}, \mathbb{D}) \) fixes two points \( \xi_1 = e^{-i\theta} \) and \( \xi_2 = e^{i\theta} \), and let finally \( \varphi(0) \in U_1 \cup U_2 \cup [e^{-i\theta}, e^{i\theta}] \), see Fig 1.

Let us consider the family \( \mathcal{F}_{a_0} \) of digons \( D \) in \( \Omega = \mathbb{D} \setminus \{a_0\} \), \( 0 < a_0 < 1 \) with two vertices \( e^{\pm i\theta} \) and with the equal angles \( \pi \) at these vertices, such that any arc connecting \( e^{i\theta} \) and \( e^{-i\theta} \) in \( D \) is homotopic in \( \Omega \) to the arc \( \{e^{it}: t \in (\theta, \theta)\} \).

Then
\[
\min_{D \in \mathcal{F}_{a_0}} m(D, e^{i\theta}, e^{-i\theta}) = m(D_{a_0}^1, e^{i\theta}, e^{-i\theta}),
\]
where \( D_{a_0}^1 \) is a strip domain in the trajectory structure of the quadratic differential
\[
Q(w) dw^2 = \frac{A^2(w + 1)^2}{(w - e^{i\theta})^2(w - e^{-i\theta})^2 (w - a_0)(1 - a_0 w) dw^2}, \quad w \in \Omega,
\]
for some real value of \( A \).

Using elementary conformal maps and known reduced moduli of digons, see [21, page 33], we calculate the modulus of \( D_{a_0}^1 \) as
\[
m(D_{a_0}^1, e^{i\theta}, e^{-i\theta}) = \frac{2}{\pi} \log \frac{4(1 - \cos \theta) - 2a_0 \cos \theta + a_0^2}{(1 - a_0)^2}.
\]

The digon \( D_{a_0}^1 \) is mapped by the function \( \varphi \), satisfying the above properties, onto the digon \( \varphi(D_{a_0}^1) \) with the same vertices and angles at them, and \( \varphi(0) \in U_1 \). Unfortunately, it is not possible to apply symmetrization to this digon because the resulting object will not be a digon with the same vertices. Therefore, we apply another procedure.

Let us consider the following Möbius map \( w = m\tilde{b}(z, t) \)
\[
\frac{z - e^{-i\theta}}{z - e^{i\theta}} \frac{1 - e^{i\theta}}{1 - e^{-i\theta}} = \frac{w - e^{-i\theta}}{w - e^{i\theta}} \frac{e^{it} - e^{i\theta}}{e^{it} - e^{-i\theta}},
\]
which makes the correspondence \( 1 \rightarrow e^{it}, e^{\pm i\theta} \rightarrow e^{\pm i\theta} \). Fix a point \( a \in \mathbb{D} \) and consider the curve \( \gamma: (\theta, \theta) \rightarrow \mathbb{D} \) passing through \( a = \gamma(0) \) and defined by the equation
\[
\frac{a - e^{-i\theta}}{a - e^{i\theta}} \frac{1 - e^{i\theta}}{1 - e^{-i\theta}} = \frac{\gamma(t) - e^{-i\theta}}{\gamma(t) - e^{i\theta}} \frac{e^{it} - e^{i\theta}}{e^{it} - e^{-i\theta}}.
\]
We have \( \lim_{t \rightarrow \theta + 0} \gamma(t) = e^{-i\theta} \), and \( \lim_{t \rightarrow \theta - 0} \gamma(t) = e^{i\theta} \). Observe that \( \gamma \) is an arc of a circle centered on the point \( \left( \frac{1 - |a|^2}{2(\cos \theta - \text{Re} a)}, 0 \right) \).
and of radius
\[ \sqrt{1 + \left( \frac{1 - |a|^2}{2(\cos \theta - \operatorname{Re} a)} \right)^2 - \frac{1 - |a|^2}{\cos \theta - \operatorname{Re} a} \cdot \cos \theta}. \]

If \( \operatorname{Re} a = \cos \theta \), then the arc \( \gamma \) becomes the interval \( [e^{i\theta}, e^{-i\theta}] \). The arc \( \gamma \) intersects the real axis inside the unit disk at the point \( \gamma(t_0) = \Phi(a) \), and we denote \( t_0 = \gamma^{-1}(\Phi(a)) \).

We remark that if \( a \) is real, then \( \Phi(a) = a \). If \( a \in U_1 \cup U_2 \cup [e^{-i\theta}, e^{i\theta}] \), then \( \Phi(a) \in (0, 1) \). If \( a \in U_3 \), then \( \Phi(a) \in (-1, 0) \). If \( a \in \gamma_0 \), then \( \Phi(a) = 0 \).

The angular derivatives are
\[ w_z'(e^{-i\theta}, t_0) = \frac{\sin \frac{\theta-t_0}{2}}{\sin \frac{\theta+t_0}{2}}, \quad w_z'(e^{i\theta}, t_0) = \frac{\sin \frac{\theta+t_0}{2}}{\sin \frac{\theta-t_0}{2}}. \]

Let \( t_0 \) be defined as \( t_0 = \gamma^{-1}(\Phi(\varphi(0))) \). Observe that \( \Phi(\varphi(0)) \in (0, 1) \). The Möbius transformation \( m\circ b(z, t_0) \) maps the digon \( \varphi(D_0^1) \) onto the digon \( m\circ b(\varphi(D_0^1), t_0) \), which is admissible in the problem for the family of digons \( F^1_{\Phi(\varphi(0))} \). Due to admissibility we can write
\[ m(D_0^1, e^{i\theta}, e^{-i\theta}) + \frac{1}{\pi} \log \varphi'(e^{i\theta})\varphi'(e^{-i\theta}) + \frac{1}{\pi} \log w_z'(e^{i\theta}, t_0)w_z'(e^{-i\theta}, t_0) \geq m(D_0^1, e^{i\theta}, e^{-i\theta}). \]

Then we arrive at following inequality (a) of Theorem 1
\[ (30) \quad \sqrt{\varphi'(e^{i\theta})\varphi'(e^{-i\theta})} \geq \frac{1 - 2\Phi(\varphi(0)) \cos \theta + \Phi^2(\varphi(0))}{(1 - \Phi(\varphi(0))^2} \]

Now let \( \varphi(0) \in U_3 \), see Fig 1. Let us consider the family \( F_{\alpha_0}^2 \) of digons \( D \) in \( \Omega = \mathbb{D} \setminus \{a_0\} \), \( -1 < a_0 < 0 \) with two vertices \( e^{\pm i\theta} \) and with the equal angles \( \pi \) at these vertices, such that any arc connecting \( e^{i\theta} \) and \( e^{-i\theta} \) in \( D \) is homotopic in \( \Omega \) to the arc \( \{e^{it}: t \in (\theta, 2\pi - \theta)\} \).

Then
\[ \min_{D \in F_{\alpha_0}^2} m(D, e^{i\theta}, e^{-i\theta}) = m(D_{\alpha_0}^2, e^{i\theta}, e^{-i\theta}), \]
where \( D_{\alpha_0}^2 \) is a strip domain in the trajectory structure of the quadratic differential
\[ Q(w)dw^2 = \frac{A^2(w-1)^2}{(w-e^{i\theta})^2(w-e^{-i\theta})^2(w-a_0)(1-a_0w)}dw^2, \quad w \in \Omega, \]
for some real value of \( A \).

Using elementary conformal maps and known reduced moduli of digons, see [21, page 33], we calculate the modulus of \( D_{\alpha_0}^2 \) as
\[ m(D_{\alpha_0}^2, e^{i\theta}, e^{-i\theta}) = \frac{2}{\pi} \log \frac{4(1 - \cos \theta) - 2a_0 \cos \theta + a_0^2}{(1 + a_0)^2}. \]

Let \( t_0 \) be again defined as \( t_0 = \gamma^{-1}(\Phi(\varphi(0))) \). Observe that \( \Phi(\varphi(0)) \in (-1, 0) \). The Möbius transformation \( m\circ b(z, t_0) \) maps the digon \( \varphi(D_0^2) \) onto the digon \( m\circ b(\varphi(D_0^2), t_0) \), which is admissible in the problem for the family of digons \( F_{\Phi(\varphi(0))}^2 \). Due to admissibility we can write
\[ m(D_0^2, e^{i\theta}, e^{-i\theta}) + \frac{1}{\pi} \log \varphi'(e^{i\theta})\varphi'(e^{-i\theta}) + \frac{1}{\pi} \log w_z'(e^{i\theta}, t_0)w_z'(e^{-i\theta}, t_0) \geq m(D_0^2, e^{i\theta}, e^{-i\theta}). \]
Then we arrive at following inequality (b) of Theorem 1

\[\sqrt{\varphi'(e^{i\theta})\varphi'(e^{-i\theta})} \geq \frac{1 - 2\Phi(\varphi(0)) \cos \theta + \Phi^2(\varphi(0))}{(1 + \Phi(\varphi(0))^2}\]  

If \(\varphi(0) \in \gamma_0\), then \(\Phi(\varphi(0)) = 0\), and the sharp inequality

\[\varphi(e^{i\theta})\varphi(e^{-i\theta}) \geq 1\]

holds with the equality for the Möbius map \(m\overline{ob}(z, t_0)\), where \(t_0\) is defined by the formula

\[e^{it_0} = \frac{1 - \varphi(0)}{1 + (1 - 2 \cos \theta)\varphi(0)}\].

Observe that \(\varphi(0) \in \gamma_0\), hence the absolute value of the right-hand side of the above equation is one.

In order to prove the Corollary we consider the domains \(U_1, U_2,\) and \(U_3,\) as well as the arc \(\gamma_0,\) which are defined in Introduction. If \(\varphi(0) \in U_1,\) then \(\Phi(\varphi(0)) > \text{Re} \varphi(0) > \cos \theta\) and the inequality \[30\] may be strengthened as

\[\sqrt{\varphi'(e^{i\theta})\varphi'(e^{-i\theta})} \geq \frac{1 - 2\text{Re} \varphi(0) \cos \theta + (\text{Re} \varphi(0))^2}{(1 - \text{Re} \varphi(0))^2}\].

If \(\varphi(0) \in U_3,\) then \(\Phi(\varphi(0)) < \text{Re} \varphi(0) < 0\) and the inequality \[31\] may be strengthened as

\[\sqrt{\varphi'(e^{i\theta})\varphi'(e^{-i\theta})} \geq \frac{1 - 2\text{Re} \varphi(0) \cos \theta + (\text{Re} \varphi(0))^2}{(1 + \text{Re} \varphi(0))^2}\].

In both inequalities the equality sign is attained only if \(\varphi(0)\) is real.

In order to construct the extremal functions we observe that the function \(p^\pm_x\) satisfies the equation

\[\frac{(w \pm 1)^2}{(w - e^{i\theta})^2(w - e^{-i\theta})^2(w - x)(1 - wx)}dw^2 = \frac{(z \pm 1)^2}{(z - e^{i\theta})^2(z - e^{-i\theta})^2z}dz^2\]

in \(D_0^1\) or \(D_0^2\) respectively, i.e., map the extremal configuration for \(F^1_x\) onto the extremal configuration for \(F^2_x\). Further application of corresponding Möbius transforms for \(x = \Phi(\varphi(0))\) finishes the proof. \(\Box\)

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