UNIT-REGULARITY AND REPRESENTABILITY FOR SEMIARTINIAN ∗-REGULAR RINGS. ERRATUM

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ABSTRACT. We discuss whether a semiartinian ∗-regular ring $R$ is unit-regular; if, in addition, $R$ is subdirectly irreducible then it admits a representation within some inner product space.

0. ERRATUM

There is no proof of Thm. 7. since there is no proof of Fact 5.

1. INTRODUCTION

The motivating examples of ∗-regular rings, due to Murray and von Neumann, were the ∗-rings of unbounded operators affiliated with finite von Neumann algebra factors; to be subsumed, later, as ∗-rings of quotients of finite Rickart $C^*$-algebras. All the latter have been shown to be ∗-regular and unit-regular (Handelman [5]). Representations of these as ∗-rings of endomorphisms of suitable inner product spaces have been obtained first, in the von Neumann case, by Luca Giudici (cf. [6]), in general in joint work with Marina Semenova [9]. The existence of such representations implies direct finiteness [7]. In the present note we show that every semiartinian ∗-regular ring is unit-regular and a subdirect product of representables. This might be a contribution to the question, asked by Handelman (cf. [3, Problem 48]), whether all ∗-regular rings are unit-regular. We rely heavily on the result of Baccella and Spinosa [1] that a semiartinian regular ring is unit-regular provided that all its homomorphic images are directly finite. Also, we rely on the theory of representations of ∗-regular rings developed by Florence Micol [12] (cf. [9, 10]). Thanks are due to the referee for a timely, concise, and helpful report.

2. PRELIMINARIES: REGULAR AND ∗-REGULAR RINGS

We refer to Berberian [2] and Goodearl [3]. Unless stated otherwise, rings will be associative, with unit 1 as constant. A (von Neumann)
regular ring $R$ is such that for each $a \in R$ there is $x \in R$ such that $axa = a$; equivalently, every right (left) principal ideal is generated by an idempotent.

The socle $\text{Soc}(M)$ of a right $R$-module is the sum of all minimal submodules. For a ring $R$ define its Loewy series of right ideals $L_\alpha(R)$ by $L_0(R) = 0$. $L_{\alpha + 1} = \text{Soc}(R / L_\alpha(R))$, and $L_\alpha(R) = \bigcup_{\beta < \alpha} L_\beta(R)$ is $\alpha$ is a limit ordinal. $R$ has Loewy length $\alpha$ if $R = L_\alpha(R)$ with $\alpha$ minimal, provided that such exists. A ring $R$ with unit is (right) semiartinian if $R/M$ has nonzero socle for each right ideal of $R$; equivalently, $R$ has Loewy length $\alpha$ for some $\alpha$ - which must be of the form $\xi + 1$ since $R$ has unit 1. If $R$ is regular, then the $L_\alpha(R)$ are, moreover, ideals since left and right socle of a regular ring coincide \[1\].

A ring $R$ is directly finite if $xy = 1$ implies $yx = 1$ for all $x, y \in R$. A ring $R$ is unit-regular if for any $a \in R$ there is a unit $u$ of $R$ such that $aua = a$. Unit-regular rings are directly finite, in particular. The crucial fact to be used, here, is the following result of Baccella and Spinosa \[2\].

**Theorem 1.** A semiartinian regular ring is unit-regular provided all its homomorphic images are directly finite.

A $*$-ring is a ring $R$ endowed with an involution $r \mapsto r^*$. Such $R$ is $*$-regular if it is regular and $rr^* = 0$ only for $r = 0$. A projection is an idempotent $e$ such that $e = e^*$; we write $e \in P(I)$ if $e \in I$. A $*$-ring is $*$-regular if and only if for any $a \in R$ there is a projection $e$ with $aR = eR$; such $e$ is unique and obtained as $aa^+$ where $a^+$ is the pseudo-inverse of $a$. In particular, for $*$-regular $R$, each ideal $I$ is a $*$-ideal, that is, closed under the involution. Thus, $R/I$ is a $*$-ring with involution $a + I \mapsto a^* + I$ and a homomorphic image of the $*$-ring $R$. In particular, $R/I$ is regular; and $*$-regular since $aa^* + I$ is a projection generating $(a + I)(R/I)$.

If $R$ is a $*$-regular ring and $e \in P(R)$ then the corner $eRe$ is a $*$-regular ring with unit $e$, operations inherited from $R$, otherwise. For a $*$-regular ring, $P(R)$ is a modular lattice, with partial order given by $e \leq f \iff fe = e$, which is isomorphic to the lattice $L(R)$ of principal right ideals of $R$ via $e \mapsto eR$. In particular, $eRe$ is artinian if and only if $e$ is contained in the sum of finitely many minimal right ideals.

A $*$-ring is subdirectly irreducible if it has a unique minimal ideal, denoted by $M(R)$. Observe that $\text{Soc}(R) \neq 0$ implies $M(R) \subseteq \text{Soc}(R)$ since $\text{Soc}(R)$ is an ideal. For the following see Lemma 2 and Theorem 3 in \[8\].
Fact 2. If $R$ is a subdirectly irreducible $*$-regular ring then $eRe$ is simple for all $e \in P(M(R))$ and $R$ a homomorphic image of a $*$-regular sub-$*$-ring of some ultraproduct of the $eRe$, $e \in P(M(R))$.

3. Preliminaries: Representations

We refer to Gross [4] and Sections 1 of [9], 2–4 of [10]. By an inner product space $V_F$ we will mean a right vector space (also denoted by $V_F$) over a division $*$-ring $F$, endowed with a sesqui-linear form $\langle . | . \rangle$ which is anisotropic ($\langle v | v \rangle = 0$ only for $v = 0$) and orthosymmetric, that is, $\langle v | w \rangle = 0$ if and only if $\langle w | v \rangle = 0$. Let $\text{End}^*(V_F)$ denote the $*$-ring consisting of those endomorphisms $\varphi$ of the vector space $V_F$ which have an adjoint $\varphi^*$ w.r.t. $\langle . | . \rangle$.

A representation of a $*$-ring $R$ within $V_F$ is an embedding of $R$ into $\text{End}^*(V)$. $R$ is representable if such exists. The following is well known, cf. [11, Chapter IV.12]

Fact 3. Each simple artinian $*$-regular ring is representable.

The following two facts are consequences of Propositions 13 and 25 in [9] (cf. Micol [12, Corollary 3.9]) and, respectively, [7, Theorem 3.1] (cf. [8, Theorem 4]).

Fact 4. A $*$-regular ring is representable provided it is a homomorphic image of a $*$-regular sub-$*$-ring of an ultraproduct of representable $*$-regular rings.

Fact 5. Every representable $*$-regular ring is directly finite.

4. Main results

Theorem 6. If $R$ is a subdirectly irreducible $*$-regular ring such that $\text{Soc}(R) \neq 0$, then $\text{Soc}(R) = M(R)$, each $eRe$ with $e \in P(M(R))$ is artinian, and $R$ is representable.

Proof. Consider a minimal right ideal $aR$. As $R$ is subdirectly irreducible, $M(R)$ is contained in the ideal generated by $a$; that is, for any $0 \neq e \in P(M(R))$ one has $e = \sum_i r_i a s_i$ for suitable $r_i, s_i \in R$, $r_i a s_i \neq 0$. By minimality of $aR$, one has $a s_i R = a R$ and $r_i a s_i R = r_i a R$ is minimal, too. Indeed, $x \mapsto r_i x$ is an $R$-linear map of $a R$ onto $r_i a R \neq 0$. Thus, $e \in \sum_i r_i a R$ means that $e Re$ is artinian. By Facts 3 and 4, $R$ is representable.

It remains to show that $\text{Soc}(R) \subseteq M(R)$. Recall that the congruence lattice of $L(R)$ is isomorphic to the ideal lattice of $R$ ([13, Theorem 4.3] with an isomorphism $\theta \mapsto I$ such that $aR/0 \in \theta$ if and only if $a \in I$. In particular, since $R$ is subdirectly irreducible so is $L(R)$.
Choose \( e \in M(R) \) with \( eR \) minimal. Then for each minimal \( aR \) one has \( eR/0 \) in the lattice congruence \( \theta \) generated by \( aR/0 \). Since both quotients are prime, by modularity this means that they are projective to each other. Thus, \( aR/0 \) is in the lattice congruence generated by \( eR/0 \) whence \( a \) is in the ideal generated by \( e \), that is, in \( M(R) \).

**Theorem 7.** Every semiartinian \(*\)-regular ring \( R \) is unit-regular and a subdirect product of representable homomorphic images.

**Proof.** Consider an ideal \( I \) of \( R \). Then \( I = \bigcap_{x \in X} I_x \) with completely meet irreducible \( I_x \), that is, subdirectly irreducible \( R/I_x \). Since \( R \) is semiartinian one has \( Soc(R/I_x) \neq 0 \), whence \( R/I_x \) is representable by Theorem 6 and directly finite by Fact 5. Then \( R/I \) is directly finite, too, being a subdirect product of the \( R/I_x \). By Theorem 1 it follows that \( R \) is unit-regular.

5. Examples

It appears that semiartinian \(*\)-regular rings form a very special subclass of the class of unit-regular \(*\)-regular rings, even within the class of those which are subdirect products of representables. E.g. the \(*\)-ring of unbounded operators affiliated to the hyperfinite von Neumann algebra factor is representable, unit-regular, and \(*\)-regular with zero socle. On the other hand, due to the following, for every simple artinian \(*\)-regular ring \( R \) and any natural number \( n > 0 \) there is a semiartinian \(*\)-regular ring having ideal lattice an \( n \)-element chain and \( R \) as a homomorphic image.

**Proposition 8.** Every representable \(*\)-regular ring \( R \) embeds into some subdirectly irreducible representable \(*\)-regular ring \( \hat{R} \) such that \( R \cong \hat{R}/M(\hat{R}) \). In particular, \( \hat{R} \) is semiartinian if and only if so is \( R \).

The proof needs some preparation. Call a representation \( \iota : R \to \text{End}^*(V_F) \) large if for all \( a, b \in R \) with \( \text{im} \iota(b) \subseteq \text{im} \iota(a) \) and finite \( \dim(\text{im} \iota(a)/\text{im} \iota(b))_F \) one has \( \text{im} \iota(a) = \text{im} \iota(b) \).

**Lemma 9.** Any representable \(*\)-regular ring admits some large representation.

**Proof.** Inner product spaces can be considered as 2-sorted structures \( V_F \) with sorts \( V \) and \( F \). In particular, the class of inner product spaces is closed under formation of ultraproducts. Representations of \(*\)-rings \( R \) can be viewed as \( R\text{-}F\)-bimodules \( _RV_F \), that is as 3-sorted structures, with \( R \) acting faithfully on \( V \). It is easily verified that the class of representations of \(*\)-rings is closed under ultraproducts cf. [9, Proposition 13].
Now, given a representation $\eta$ of $R$ in $W_F$, form an ultrapower $\iota$, such that $\dim F_F'$ is infinite (recall that $F'$ is an ultrapower of $F$). Observe that $\End^*(V_{F'})$ is a sub-$*$-ring of $\End^*(V_F)$ and $\dim(U/W)_F$ is infinite for any subspaces $U \supseteq W$ of $V_{F'}$. Also, $S$ is an ultrapower of $R$ with canonical embedding $\varepsilon: R \to S$. Thus, $\varepsilon \circ \iota$ is a large representation of $R$ in $V_F$. □

Proof. of Proposition 8 In view of Lemma 9 we may assume a large representation $\iota$ of $R$ in $V_F$. Identifying $R$ via $\iota$ with its image, we have $R$ a $*$-regular sub-$*$-ring of $\End^*(V_F)$. Let $I$ denote the set of all $\varphi \in \End(V_F)$ such that $\dim(\text{im } \varphi)_F$ is finite. According to Micol [12, Proposition 3.12] (cf. Propositions 4.4 (i),(iii) and 4.5 in [10]) $R + I$ is a $*$-regular sub-$*$-ring of $\End^*(V_F)$, with unique minimal ideal $I$. By Theorem 6 one has $I = \text{Soc}(R + I)$. Moreover, $R \cap I = \{0\}$ since the representation $\iota$ of $R$ in $V_F$ is large. Hence, $R \cong (R + I)/I$. □

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