ON GAUSSIAN KERNELS ON HILBERT SPACES AND KERNELS ON HYPERBOLIC SPACES

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ABSTRACT. This paper describes the concepts of Universal/Integrally Strictly Positive Definite/\(C_0\)-Universal for the Gaussian kernel on a Hilbert space. As a consequence we obtain a similar characterization for an important family of kernels studied and developed by Schoenberg and also on a family of spatial-time kernels popular on geostatistics, the Gneiting class, and its generalizations. Either by using similar techniques, or by a direct consequence of the Gaussian kernel on Hilbert spaces, we characterize the same concepts for a family of kernels defined on a Hyperbolic space.

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1. **Introduction**

The concept of a complex valued positive definite kernel has been permeating Mathematics since the beginning of the 20th century, especially after the seminal work [2], which laid down the connection between positive definite kernels and Reproducing Kernel Hilbert Spaces (RKHS). In applications (especially in Machine Learning), one of the main desirable properties on a RKHS is if it can approximate a target (but usually unknown) function. In this sense, the concepts of universality (ability to approximate continuous functions on compact sets) and $C_0$-universality (ability to approximate any $C_0$ function) are a basic requirement [8], [9].

Schoenberg in [25] proved a foundational result in metric geometry, by showing that a metric space $(X, D)$ can be isometrically embedded into some Hilbert space if and only if the kernel $e^{-tD^2(x,y)}$ is positive definite for every $t > 0$. For instance, spheres and hyperbolic spaces are not embedable, [17], [12], [14]. Later, this result was extended to a broader context, and it is usually presented as an equivalent definition for when a kernel $\gamma : X \times X \to \mathbb{C}$ is conditionally negative definite, by replacing $D^2(x,y)$ with $\gamma(x,y)$. One of the most important and widely used positive definite kernels is the Gaussian kernel $G_\sigma(x,y) = e^{-\sigma\|x-y\|^2}$ ($\sigma > 0$) defined on a Euclidean space $\mathbb{R}^m$, which is not only universal but in fact can approximate any differentiable function and its derivatives of any order on any compact set simultaneously [27].

The major aim of this article is to prove that the Strictly Positive Definite/Universal/Integrally Strictly Positive Definite/$C_0$-Universal are properties that occur not only on the Gaussian kernel but on a larger class among the Schoenberg kernels $e^{-\gamma(x,y)}$ ($\gamma$ is conditionally negative definite), being the characterization dependent on somewhat easily verifiable properties of the kernel $\gamma$. These results are presented on Section 4 and are achieved as a corollary of the results on Section 3 where we prove that the Gaussian kernel is Strictly Positive Definite/Universal/Integrally Strictly Positive Definite/$C_0$-Universal on any Hilbert space, by using several versions of the famous Stone-Weierstrass theorem, instead of the standard procedure by using the Fourier transform and its properties.

The Gaussian kernel also served as a building block to generate positive definite kernels on a product of spaces (also called spatio-temporal), being one of the most important examples (especially on geostatistics) the Gneiting class [18], initially proposed as a kernel on $\mathbb{R}^m \times \mathbb{R}^m$ and recently extended to $X \times \mathbb{R}^m$ [21]. Although having its popularity, none qualitative property of this family of kernels has been analyzed on the literature so far. On Section 5 we present a natural generalization of [21] and provide sufficient conditions for when this generalized family of kernels are Strictly Positive Definite/Universal/Integrally Strictly Positive Definite/$C_0$-Universal. The proofs are a consequence of the results on Section 3 together with analysis of when the Schur/Hadamard product of continuous positive definite kernels is Strictly Positive Definite/Universal/Integrally Strictly Positive Definite/$C_0$-Universal given that one of them satisfies this property, presented on Subsection 7.3.

We conclude the article on Section 6, where the focus is on kernels on Hyperbolic spaces and related types. A special family of positive definite kernels on hyperbolic spaces invariant by the hyperbolic distance, which shares some similarities with completely monotone functions,
are analyzed and the concepts of Strictly Positive Definite/Universal/Integrally Strictly Positive Definite/\( C_0 \)-Universal are fully characterized.

2. DEFINITIONS

A kernel \( K : X \times X \to \mathbb{C} \) is called **positive definite** if for every finite quantity of distinct points \( x_1, \ldots, x_n \in X \) and scalars \( c_1, \ldots, c_n \in \mathbb{C} \), we have that

\[
\int_X \int_X K(x, y) d\lambda(x) d\overline{\lambda}(y) = \sum_{\mu, \nu=1}^n c_\mu c_\nu K(x_\mu, x_\nu) \geq 0,
\]

where \( \lambda = \sum_{\mu=1}^n c_\mu \delta_{x_\mu} \). In addition, if the above double sum is zero only when all scalars \( c_\mu \) are zero, we say that the kernel is **strictly positive definite** (SPD). The set of measures on \( X \) used before are denoted by the symbol \( \mathcal{M}_0(X) \).

The reproducing kernel Hilbert space (RKHS) of a positive definite kernel \( K : X \times X \to \mathbb{C} \) is the Hilbert space \( \mathcal{H}_K \subset \mathcal{F}(X, \mathbb{C}) \), and it satisfies \( \langle F, K_y \rangle_{\mathcal{H}_K} = F(y) \), for every \( F : X \to \mathbb{C} \) that is an element of \( \mathcal{H}_K \) and \( [K_y] := K(x, y) \), \( \mathcal{M}_0(X) \).

Recall that for a locally compact space \( X \), the Banach space \( C_0(X) \) is defined as the set of continuous functions \( f : X \to \mathbb{C} \) such that for every \( \varepsilon > 0 \) there exists a compact set \( C_\varepsilon \) for which \( |f(x)| < \varepsilon \) for \( x \in X \setminus C_\varepsilon \), with norm given by \( \sup_{x \in X} |f(x)| \).

**Definition 2.1.** Let \( X \) be a Hausdorff space and \( K : X \times X \to \mathbb{C} \) be a positive definite kernel. We say that the kernel \( K \) is:

- **Universal**, if \( \mathcal{H}_K \subset C(X) \) and for every compact set \( C \subset X \), every continuous function \( g : C \to \mathbb{C} \) and every \( \varepsilon > 0 \) there exists \( f : X \to \mathbb{C} \in \mathcal{H}_K \) for which

\[
\sup_{x \in C} |f(x) - g(x)| < \varepsilon.
\]

In addition, when \( X \) is a locally compact space, we say that the kernel \( K \) is:

- **\( C_0 \)-universal**, if \( \mathcal{H}_K \subset C_0(X) \) and for every continuous function \( g \in C_0(X) \) and every \( \varepsilon > 0 \) there exists \( f : X \to \mathbb{C} \in \mathcal{H}_K \) for which

\[
\sup_{x \in X} |f(x) - g(x)| < \varepsilon.
\]

In other words, a kernel \( K : X \times X \to \mathbb{C} \) is universal if its RKHS are made of continuous functions that when restricted to any compact set \( C \subset X \) are dense on the Banach space \( C(C) \).

A kernel \( K : X \times X \to \mathbb{C} \) is \( C_0 \)-universal if its RKHS are made of \( C_0(X) \) functions that are dense on the Banach space \( C_0(X) \).

On the \( C_0 \) case we assume that \( X \) is locally compact in order to avoid pathological topologies. In \([6]\), it was presented the following criteria for \( \mathcal{H}_K \) to be a subset of \( C(X) \) and \( C_0(X) \):

**Proposition 2.2.** Let \( X \) be a Hausdorff space, \( K : X \times X \to \mathbb{C} \) be a positive definite kernel. Then:

(i) \( \mathcal{H}_K \subset C(X) \) if and only if the function \( x \in X \to K(x, x) \in \mathbb{C} \) is locally bounded and the function \( x \in X \to K(x, y) \in C(X) \), for every \( y \in X \).

(ii) \( \mathcal{H}_K \subset C_0(X) \) if and only if the function \( x \in X \to K(x, x) \in \mathbb{C} \) is bounded and the function \( x \in X \to K(x, y) \in C_0(X) \), for every \( y \in X \).
Although the definition for a positive definite kernel being universal ($C_0$-universal) is simple, it is important to have a condition for these properties when we do not have the description for the RKHS of a kernel. A direct consequence of [22], a kernel $K : X \times X \to \mathbb{C}$ for which $\mathcal{H}_K \subset C(X)$ is universal if and only if the only finite complex valued Radon measure of compact support $\lambda$ on $X$ such that

\begin{equation}
\int_X \int_X K(x,y) d\lambda(x) d\overline{\lambda}(y) = 0
\end{equation}

is the zero measure. We emphasize that the double integral in Equation 2.1 is always a nonnegative number because $K$ is positive definite and $\mathcal{H}_K \subset C(X)$. In order to simplify the notation, we denote by $\mathcal{M}_c(X)$ the set of finite complex valued Radon measures of compact support on a Hausdorff set $X$.

Similarly, by [28] a kernel $K : X \times X \to \mathbb{C}$ for which $\mathcal{H}_K \subset C_0(X)$ is $C_0$-universal if and only if the only finite complex valued Radon measure $\lambda$ on $X$ such that

\begin{equation}
\int_X \int_X K(x,y) d\lambda(x) d\overline{\lambda}(y) = 0
\end{equation}

is the zero measure. Again, we emphasize that the double integral in Equation 2.2 is always a nonnegative number because $K$ is positive definite and $\mathcal{H}_K \subset C_0(X)$. We denote by $\mathcal{M}(X)$ the set of finite complex valued Radon measures on a locally compact Hausdorff space $X$.

We recall that a finite Radon measure $\lambda$ on a Hausdorff space $X$ is a Borel measure for which its total variation $|\lambda|$ satisfy

(i) (Inner regular) $|\lambda|(E) = \sup\{|\lambda|(K), \ K \text{ is compact }, K \subset E\}$ for every Borel set $E$.

(ii) (Outer regular) $|\lambda|(E) = \inf\{|\lambda|(U), \ U \text{ is open }, E \subset U\}$ for every Borel set $E$.

where the outer regularity holds for every measurable set (instead of the usual definition on open sets) because the measure is finite. See section 7, especially Proposition 7.5 in [15] for more details.

Sometimes, the inclusion $\mathcal{H}_K \subset C_0(X)$ is difficult to verify, but the relation at Equation 2.2 is much simpler to analyze.

**Definition 2.3.** Let $X$ be a locally compact Hausdorff space, we say that a bounded positive definite kernel $K : X \times X \to \mathbb{C}$ for which $\mathcal{H}_K \subset C(X)$ is integrally strictly positive definite (ISPD) if the relation at Equation 2.2 is satisfied.

This definition is based on the one given in [29], and can be reinterpreted as $\mathcal{H}_K$ being dense on $L^1(|\lambda|, X)$ for every nonzero measure $\lambda \in \mathcal{M}(X)$. For some specific type of complex valued kernels, a good description of those who are ISPD were obtained in [7], [16], [29], especially the kernels on Euclidean spaces invariant by translations (more generally on a locally compact commutative group).

If the kernel $K$ is real valued, it is sufficient to test the double integrals for real valued measures in $\mathcal{M}(X)$. The concepts of SPD/Universality/$C_0$-Universality/ISPD also exists on the operator valued context [20], [5]. Since, we only use the matrix valued setting, we use the
simpler definition that a matrix valued kernel $K : X \times X \to M_\ell(C)$ is PD/SPD/Universal/C$_0$-Universal/ISPD if the scalar valued kernel $L : (X \times \{1, \ldots, \ell\}) \times (X \times \{1, \ldots, \ell\}) \to \mathbb{C}$ given by $L((x,i),(y,j)) = K_{i,j}(x,y)$ is PD/SPD/Universal/C$_0$-Universal/ISPD.

### 3. Gaussian kernel on Hilbert spaces

Throughout this Section $\mathcal{H}$ denotes a real Hilbert space.

**Theorem 3.1.** The Gaussian kernel $G_\sigma : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$, given by

$$G_\sigma(x,y) = e^{-\sigma\|x-y\|^2}$$

is SPD and universal for every $\sigma > 0$.

If $\mathcal{H}$ is infinite dimensional then it is not a locally compact space, so the concepts of C$_0$-universality and ISPD are not well defined for $G_\sigma$. However, we can analyse the kernel when restricted to a locally compact space. A key argument to achieve such characterization is a version of the Stone-Weierstrass Theorem for integrable functions proved in [13]. However, on [13] it is an hypothesis that the elements on the algebra of functions are Baire measurable, which is not clear to us if and how this hypothesis can be fulfilled. Being the main ingredient for the proof the inner regularity on all measurable sets, and every finite Radon measure satisfies this, we could still use the result on our setting. We prove this simple change of [13] at Section 7.6.

**Theorem 3.2.** Let $X \subset \mathcal{H}$ be locally compact. The Gaussian kernel

$$(x,y) \in X \times X \to G_\sigma(x,y) = e^{-\sigma\|x-y\|^2} \in \mathbb{R}$$

is ISPD.

Next we present a characterization for when the kernel $G_\sigma$ is C$_0(X)$-universal. The following structure result characterizes when the inclusion $\mathcal{H}_K \subset C_0(X)$ is satisfied.

**Lemma 3.3.** Let $X \subset \mathcal{H}$ be locally compact. The following conditions are equivalent

(i) There exists $z_0 \in X$ for which the function $G_{\sigma,z_0}(x) = e^{-\|x-z_0\|^2} \in C_0(X)$.

(ii) The function $G_{\sigma,z}(x) = e^{-\|x-z\|^2}$ is an element of $C_0(X)$ for every $z \in X$.

(iii) The inclusion $\mathcal{H}_{G_\sigma} \subset C_0(X)$ holds.

(iv) Every bounded and closed set on $X$ is a compact set on $X$.

Next theorem is a consequence of Theorem 3.2, however, we present a different proof for it, based on the C$_0$ version of the Stone-Weierstrass Theorem.

**Theorem 3.4.** Let $X \subset \mathcal{H}$ be locally compact. The Gaussian kernel

$$(x,y) \in X \times X \to G_\sigma(x,y) := e^{-\|x-y\|^2} \in \mathbb{R}$$

is $C_0(X)$-universal if and only if $\mathcal{H}_{G_\sigma} \subset C_0(X)$. 
The results in this section could be proved on a more general setting. By [25] a continuous function $g : [0, \infty) \to \mathbb{R}$ is such that the kernel

$$(x, y) \in \mathbb{R}^m \times \mathbb{R}^m \to g(\|x - y\|) \in \mathbb{R}$$

is positive definite for every $m \in \mathbb{N}$, if and only if $f(t) := g(\sqrt{t}) \in C^\infty((0, \infty))$ with $(-1)^n f^{(n)}(t) \geq 0$ for every $n \in \mathbb{N}$ (a function $f$ with these properties is called completely monotone), or equivalently that there exists a nonnegative measure $\lambda \in \mathcal{M}([0, \infty))$ for which

$$g(t) = \int_{[0, \infty)} e^{-rt^2} d\lambda(r).$$

Replacing the Gaussian kernel by a function of this type on Theorems 3.1, 3.2 and 3.4 is possible, whenever $g$ is not a constant function, or equivalently $\lambda((0, \infty)) > 0$. Lemma 3.3 is also possible whenever $g$ is not a constant function and relation (iv) is replaced by

(iv) Every bounded and closed set on $X$ is a compact set and $\lim_{t \to \infty} g(t) = 0$.

The argument that this generalization is indeed possible is a direct consequence of Theorem 3.7 in [20], and we do not present it.

4. Universality of Schoenberg-Gaussian Kernels

In [26] Schoenberg proved that a kernel $\gamma : X \times X \to \mathbb{R}$ is such that the kernel

$$(x, y) \in X \times X \to e^{-r\gamma(x, y)} \in \mathbb{R}$$

is positive definite for every $r > 0$ if and only if the kernel $\gamma$ is conditionally negative definite, that is, $\gamma$ is symmetric ($\gamma(x, y) = \gamma(y, x)$) and for every finite quantity of distinct points $x_1, \ldots, x_n$ and scalars $c_1, \ldots, c_n \in \mathbb{R}$, restricted to the hyperplane $\sum_{\mu=1}^n c_{\mu} x_{\mu} = 0$, it satisfies

$$\sum_{\mu, \nu=1}^n c_{\mu} c_{\nu} \gamma(x_{\mu}, x_{\nu}) \leq 0.$$

Since [2] it is known the strong connection between positive definite kernels and inner products on Hilbert spaces as well as conditionally negative definite kernels and norms on Hilbert spaces, since $\gamma : X \times X \to \mathbb{R}$ can be written as (Proposition 3.2 in [3])

$$\gamma(x, y) = \|h(x) - h(y)\|^2_{\mathcal{H}} + f(x) + f(y)$$

where $\mathcal{H}$ is a real Hilbert space and $h : X \to \mathcal{H}$, and $f : X \to \mathbb{R}$. This description allows us to understand the kernel $e^{-\gamma(x, y)}$ as a weighted version ($f$ may be nonzero) of a restriction of the Gaussian kernel defined on an (usually) infinite dimensional space.

An important relation to our purposes is if the function $h$ is injective (equivalently, if $2\gamma(x, y) > \gamma(x, x) + \gamma(y, y)$ for every $x, y \in X$). On this case there is a natural metric structure on $X$ provided by the norm on $\mathcal{H}$, being the distance

$$D_\gamma(x, y) := \sqrt{\gamma(x, y) - \frac{\gamma(x, x)}{2} - \frac{\gamma(y, y)}{2}}.$$
Naturally, a conditionally negative definite kernel with this property is called \textit{metrizable}. The set $X$ with the metric topology $D_γ$ is denoted as $X_γ$.

**Theorem 4.1.** Let $X$ be a Hausdorff space and $γ : X × X → \mathbb{R}$ be a continuous conditionally negative definite kernel. The kernel

$$(x, y) ∈ X × X → G_γ(x, y) := e^{-γ(x, y)} ∈ \mathbb{R}$$

is $\text{SPD}$ (universal) if and only if the kernel $γ$ is metrizable.

**Theorem 4.2.** Let $X$ be a locally compact Hausdorff space and $γ : X × X → \mathbb{R}$ be a continuous conditionally negative definite kernel that is metrizable and the function $x ∈ X → γ(x, x) ∈ \mathbb{R}$ is bounded. Suppose either:

i) The topologies of $X$ and $X_γ$ are equivalent

ii) $X_γ$ is a locally compact space and the function $x ∈ X_γ → γ(x, x)$ is continuous.

Then the kernel

$$(x, y) ∈ X_γ × X_γ → e^{-γ(x, y)} ∈ \mathbb{R}$$

is $\text{ISPD}$.

As an example of when the topologies of $X$ and $X_γ$ are equivalent, if $g : [0, ∞) → \mathbb{R}$ is a continuous function for which $\lim_{t → ∞} g(t) ∈ (0, ∞) \cup \{∞\}$, $g(0) = 0$, $g(t) ∈ (0, ∞)$ for $t ∈ (0, ∞)$ and such that the radial kernel $(x, y) ∈ \mathbb{R}^m × \mathbb{R}^m → g(∥x − y∥)$ is conditionally negative definite, then the metric generated from this kernel on $\mathbb{R}^m$ is equivalent to the Euclidean metric on $\mathbb{R}^m$.

On the second possibility of Theorem 4.2, we assumed the continuity of the function because there is no method to check the continuity of this function on the topology $X_γ$. We conclude this section by presenting a characterization for when the kernel $G_γ$ is $C_0(X)$-universal. The following structure result elucidates some aspects concerning the inclusion $\mathcal{H}_K ⊂ C_0(X)$.

**Lemma 4.3.** Let $X$ be a locally compact Hausdorff space and $γ : X × X → \mathbb{R}$ be a continuous negative definite metrizable kernel such that the function $x ∈ X → γ(x, x) ∈ \mathbb{R}$ is bounded.

(i) There exists $z_0 ∈ X$ for which the function $G_{γ,z_0}(x) = e^{-γ(z_0, x)}$ is an element of $C_0(X)$ if and only if $\mathcal{H}_{G_γ} ⊂ C_0(X)$.

(ii) If $X_γ$ is locally compact, then $\mathcal{H}_{G_γ} ⊂ C_0(X_γ)$ if and only if every bounded and closed subset in $X_γ$ is compact and the function $x ∈ X_γ → γ(x, x)$ is continuous.

**Theorem 4.4.** Let $X$ be a locally compact Hausdorff space and $γ : X × X → \mathbb{R}$ be a continuous conditionally negative definite kernel for which the function $x ∈ X → γ(x, x)$ is bounded. The kernel

$$(x, y) ∈ X × X → e^{-γ(x, y)} ∈ \mathbb{R}$$

is $C_0(X)$-universal if and only if $γ$ is metrizable and $\mathcal{H}_{G_γ} ⊂ C_0(X)$.

If $γ$ is metrizable and $X_γ$ is a locally compact space, then the kernel is $C_0(X_γ)$-universal if and only if $\mathcal{H}_{G_γ} ⊂ C_0(X_γ)$. 


5. Gneiting class and related kernels

Based on the results of the previous Sections we are able to prove qualitative properties of some important generalizations of the Gaussian (and related) kernel to a product of spaces. Although we could use Theorems 3.2 and 4.2 to obtain conditions for the ISPD case, we avoid such analysis in order to simplify the reading.

A popular example, especially on geostatistics, of such kernels is the Gneiting class \cite{Gneiting}, initially proposed as the family of positive definite kernels

\[
((u,x),(v,y)) \in (\mathbb{R}^m \times \mathbb{R}^m)^2 \to g(||u-v||^2)^{-m/2} \psi((\frac{||x-y||^2}{g(||u-v||^2)}) \in \mathbb{R}
\]

where \(g, \psi: [0, \infty) \to \mathbb{R}\) are continuous and nonconstant functions, \(g\) is a positive function, \(\psi\) is completely monotone and \(g\) is a Bernstein function. Several extensions and applications of this type of kernel have been proposed and proved \cite{Zweig}, \cite{Gneiting}. We focus on a generalization that encloses all of the above mentioned.

Let \(X\) be a Hausdorff space, \(\gamma: X \times X \to (0, \infty)\) be a continuous conditionally negative definite kernel and \(A: X \times X \to \mathbb{C}\) be a continuous kernel. Suppose that the kernel

\[
(u,v) \in X \times X \to C(u,v) := A(u,v)^{\gamma(u,v)^{m/2}} \in \mathbb{R}
\]

is positive definite. Under this hypothesis we define the kernel \(G_{A,\gamma}: (X \times \mathbb{R}^m) \times (X \times \mathbb{R}^m) \to \mathbb{C}\) as

\[
G_{A,\gamma}((u,x),(v,y)) := A(u,v)e^{-\|x-y\|^2/\gamma(u,v)}
\]

**Theorem 5.1.** The kernel \(G_{A,\gamma}\) is positive definite and continuous. If \(\gamma\) is a metrizable kernel, then \(G_{A,\gamma}\) is SPD (universal) if and only if \(A(u,u) > 0\) for every \(u \in X\).

When \(X\) is a finite set (on which the kernel \(G_{A,\gamma}\) can be understood as a matrix valued kernel on \(\mathbb{R}^m\)), it is possible to characterize when \(G_{A,\gamma}\) is SPD/universal even if \(\gamma\) is not metrizable. It is not clear if on the general setting of Theorem 5.1 the same approach is possible. As a consequence that the functions fulfilling Bochner’s Theorem are uniquely representable, the hypothesis that the kernel \(C\) is positive definite is in fact a necessary condition for \(G_{A,\gamma}\) be positive definite. Also, although we are not imposing that the kernel \(A\) is positive definite, it is positive definite because the kernel \(C\) is positive definite and \(\gamma^{-m/2}(u,v)\) as well by Lemma 7.9.

**Theorem 5.2.** The inclusion \(\mathcal{H}_{G_{A,\gamma}} \subset C_0(X \times \mathbb{R}^m)\) occurs if and only if \(\mathcal{H}_A \subset C_0(X)\).

If \(G_{\gamma}\) is ISPD for every \(t > 0\) and the kernel \(C\) is bounded, then the kernel \(G_{A,\gamma}\) is \(C_0(X \times \mathbb{R}^m)\)-universal if and only if \(\mathcal{H}_A \subset C_0(X)\) and \(A(u,u) > 0\) for every \(u \in X\).

When \(A(u,v) = \gamma(u,v)^{-m/2}\), more properties can be obtained. The kernel \(G_{\gamma^{-m/2},\gamma}\) is positive definite and is SPD (universal) if and only if \(\gamma\) is a metrizable kernel by Theorem 5.1 (the converse follows by a simple inspection of the interpolation matrix at the points \((u,0),(v,0)\)). Moreover, \(\mathcal{H}_{\gamma^{-m/2}} \subset C_0(X)\) if and only if \(\mathcal{H}_{G_{\gamma}} \subset C_0(X)\) for some \(t > 0\) (equivalently, for every \(t > 0\), so \(G_{\gamma^{-m/2},\gamma}\) is \(C_0(X \times \mathbb{R}^m)\)-universal if and only if \(\gamma\) is metrizable and \(\mathcal{H}_{G_{\gamma}} \subset C_0(X)\). In particular, if \(X = \mathbb{R}^m\) and \(\gamma(u,v) = g(||u-v||^2)\), where \(g: [0, \infty) \to (0, \infty)\) is a Bernstein
function, then the kernel \( G_{r^{-m/2}, y} \) is \( C_0 \)-universal if and only if the function \( g \) is unbounded. As a matter of fact, a kernel among the initially proposed Gneiting class in [13] is \( C_0 \)-universal if and only if the function \( g \) is unbounded and \( \lim_{t \to \infty} \gamma(t) = 0 \).

Another important set of kernels on geostatistics is the Matérn family

\[
(x, y) \in \mathbb{R}^m \times \mathbb{R}^m \to \mathcal{M}(\|x - y\|; \alpha, \nu) := \int_{(0, \infty)} e^{-\|x - y\|^2 t} \left( \frac{\alpha^2 t^{-\nu - 1}}{4} \Gamma(\nu) e^{-\alpha^2 / 4t} \right) dt
\]

which are positive definite for every \( m \in \mathbb{N} \), \( \mathcal{M}(0; \alpha, \nu) = 1 \) and also satisfies the equality \( \mathcal{M}(\|x\|; \alpha, \nu) = 2^{-1-\nu}(\|x\| \alpha)^\nu \mathcal{K}_\nu(\|x\| \alpha) / \Gamma(\nu) \), where \( \mathcal{K}_\nu \) denotes the modified Bessel function of the second kind of order \( \nu \) [11]. A matrix valued version of this family was proposed in [19] and later was generalized in [4] as the family of matrix valued kernels \( C_{i,j}^{\mathcal{M}, \psi} : (\mathbb{R}^m \times \mathbb{R}^m)^2 \to M_\ell(\mathbb{C}) \) given by

\[
C_{i,j}^{\mathcal{M}, \psi}((u, x), (v, y)) := c_{i,j} \frac{1}{\psi(\|u - v\|^2)} \mathcal{M}(\frac{\|x - y\|}{\psi(\|u - v\|^2)^{1/2}}; \alpha_i, j, \nu, j)
\]

where the function \( \psi : [0, \infty) \to \mathbb{R} \) is a positive Bernstein function, \( \alpha_i, j = ((\alpha_i^2 + \alpha_j^2)^{1/2}) \), \( \nu_i, j = \nu_i + \nu_j \), \( \alpha_i, \nu_i, \nu_j, j \in (0, \infty) \) and the matrix

\[
\begin{bmatrix}
\alpha_i^\nu & \alpha_j^\nu \\
\alpha_j^\nu & \alpha_i^\nu
\end{bmatrix}
\]

is positive semidefinite.

Similar to the definition of \( G_{A, \gamma} \), let \( X \) be a Hausdorff space, \( \gamma : X \times X \to (0, \infty) \) be a continuous conditionally negative definite kernel and \( A : X \times X \to M_\ell(\mathbb{C}) \) be a continuous matrix valued kernel. Suppose that the matrix valued kernel \( C : X \times X \to M_\ell(\mathbb{C}) \) defined as

\[
C_{i,j}(u, v) := A_{i,j}(u, v) \gamma^{m/2}(u, v) 2^{-\nu_i} \mathcal{K}_\nu(2\nu_i)^{1/2} 2^{-\nu_j} \mathcal{K}_\nu(2\nu_j)^{1/2} (\alpha_i + \alpha_j)^{\nu_i + \nu_j} / \Gamma(\nu_i + \nu_j)
\]

is positive definite. Under these hypothesis we define the kernel \( C^{A, \gamma} : (X \times \mathbb{R}^m)^2 \to M_\ell(\mathbb{C}) \), by

\[
C_{i,j}^{A, \gamma}((u, x), (v, y)) := A_{i,j}(u, v) \mathcal{M}(\frac{\|x - y\|}{\gamma(u, v)^{1/2}}; \alpha_i, j, \nu, j)
\]

where \( \alpha_i, j = ((\alpha_i^2 + \alpha_j^2)^{1/2}) \), \( \nu_i, j = \nu_i + \nu_j \), \( \alpha_i, \nu_i, \nu_j, j \in (0, \infty) \).

**Theorem 5.3.** The matrix valued kernel \( C^{A, \gamma} \) is positive definite and continuous.

*If \( \gamma \) is a metrizable kernel, then \( C^{A, \gamma} \) is SPD (universal) if and only if \( A_{i,i}(u, u) > 0 \) for every \( 1 \leq i \leq \ell \) and \( u \in X \) and \( \{ (i, j), (\alpha_i, \nu_i) = (\alpha_j, \nu_j) \} = \{ (i, i), 1 \leq i \leq \ell \}. \)

We point out that by the proof of Theorem 5.2 if \( G_{r, \gamma} \) is ISP\(D \) for every \( t > 0 \) and \( \inf_{u \in X} \gamma(u, u) > 0 \), then \( G_{r^{-m/2}, \gamma} \) is ISP\(D \). Also, if \( G_{r^{-m/2}, \gamma} \) is ISP\(D \) then it is a bounded kernel, and the boundedness is equivalent to \( \inf_{u \in X} \gamma(u, u) > 0 \).
Theorem 5.4. The inclusion \( \mathcal{H}_{\gamma} \subset C_0(X \times \mathbb{R}^m, \mathbb{C}^\ell) \) occurs if and only if \( \mathcal{H}_{\gamma} \subset C_0(X, \mathbb{C}^\ell) \). If \( G_{\gamma} \) is ISPD and the matrix valued kernel \( C^\gamma \) is \( C_0(X \times \mathbb{R}^m, \mathbb{C}^\ell) \)-universal if and only if \( \mathcal{H}_A \subset C_0(X, \mathbb{C}^\ell) \) and \( A_{i,u} > 0 \) for every \( u \in X \) and \( 1 \leq i \leq \ell \).

The definition of the matrix valued kernels \( C^\gamma \) is inspired on the Gneiting class, which turns out to be well defined only on Euclidean spaces of a bounded dimension. Being so, this definition does not take advantage that the Matern family is positive definite on all Euclidean spaces.

To surpass this problem, we define the matrix valued kernel \( \mathcal{M}_{A, \gamma} : (X \times \mathcal{H}) \times (X \times \mathcal{H}) \to M_\ell(\mathbb{C}) \) as

\[
[\mathcal{M}_{A, \gamma}((x,u),(y,v))]_{i,j} := A_{i,j}(u,v) \mathcal{M}(\|x-y\|; \gamma(u,v))^{1/2}, \quad v_i + v_j
\]

where \( \gamma : X \times X \to (0, \infty) \) is a continuous conditionally negative definite kernel, \( A : X \times X \to M_\ell(\mathbb{C}) \) is a continuous matrix valued kernel, \( v_i > 0 \) for every \( i \) and under the restriction that the matrix valued kernel \( C : X \times X \to M_\ell(\mathbb{C}) \) defined as

\[
C_{i,j}(u,v) := A_{i,j}(u,v) \frac{\gamma(u,v)^{v_i+v_j}}{\Gamma(v_i+v_j)}
\]

is positive definite.

Theorem 5.5. Let \( Z \subset \mathcal{H} \) be a locally compact space. The following properties holds.

(i) The matrix valued kernel \( \mathcal{M}_{A, \gamma} \) is positive definite and continuous.

(ii) If \( \mathcal{H}_{\gamma} \) is a metrizable kernel, then \( \mathcal{M}_{A, \gamma} \) is SPD (universal) if and only if the positive numbers \( v_i \) are distinct and \( A_{i,u} > 0 \) for every \( 1 \leq i \leq \ell \) and \( u \in X \).

(iii) The inclusion \( \mathcal{H}_{\gamma} \subset C_0(X \times \mathbb{R}^m, \mathbb{C}^\ell) \) occurs if and only if \( \mathcal{H}_A \subset C_0(X, \mathbb{C}^\ell) \).

(iv) If \( G_{\gamma} \) is ISPD for every \( i > 0 \) and the matrix valued kernel \( C \) is bounded, then the matrix valued kernel \( \mathcal{M}_{A, \gamma} \) is SPD.

6. Kernels and Hyperbolic Spaces

6.1. Isotropic kernels on real hyperbolic spaces. Let \( \mathbb{H}^m := \{(x,t) \in \mathbb{R}^m \times (0,\infty), \quad t^2 - \|x\|^2 = 1\} \) be the \( m \)-dimensional real Hyperbolic space and consider the bilinear form

\[
((x,t_x),(y,t_y)) \in \mathbb{H}^m \times \mathbb{H}^m \to [(x,t_x),(y,t_y)] := t_xt_y - \langle x,y \rangle \in [1, \infty),
\]

which satisfies the relation

\[
\cosh(d((x,t_x),(y,t_y))) = [(x,t_x),(y,t_y)].
\]

Where \( d \) is the geodesic distance in \( \mathbb{H}^m \). A kernel \( K : \mathbb{H}^m \times \mathbb{H}^m \to \mathbb{R} \) is called isotropic if its invariant by the group

\[
O(m,1) := \{ A \in M_{m+1}(\mathbb{R}), \quad [Ax,Ay] = [x,y] \text{ for every } x,y \in \mathbb{R}^{m+1}\},
\]
that is, $K(x,y) = K(Ax,Ay)$ for every $x,y \in \mathbb{H}^m$ and $A \in O(m,1)$. Similar to isotropic kernels on real spheres [10], if $K$ is an isotropic kernel on $\mathbb{H}^m$ there exists functions $f : [0,\infty) \to \mathbb{R}$, $g : [1,\infty) \to \mathbb{R}$ for which

$$K(x,y) = f(d(x,y)) = g([x,y]), \quad x,y \in \mathbb{H}^m.$$ 

At Section 8 of [12] (also page 174 of [3]), it is proved that if $g : [1,\infty) \to \mathbb{R}$ is a continuous function, the kernel

$$(x,t_x),(y,t_y)) \in \mathbb{H}^m \times \mathbb{H}^m \rightarrow g([(x,t_x),(y,t_y)]) \in \mathbb{R}$$

is positive definite for every $m \in \mathbb{N}$ if and only if the function $s \in [0,\infty) \rightarrow g(e^s) \in \mathbb{R}$ is completely monotone, or equivalently that there exists a nonnegative finite measure $\lambda \in \mathcal{M}([0,\infty))$ for which

$$g(s) = \int_{[0,\infty)} s^{-r}d\lambda(r), \quad s \in [1,\infty).$$

In terms of the function $f$, the expression is

$$f(t) = \int_{[0,\infty)} \operatorname{sech}(t)^rd\lambda(r).$$

In this subsection, we prove several qualitative properties for these kernels in a similar way as Section 3. In this sense, a real hyperbolic space is a set $\mathbb{H}$ for which there exists a Hilbert space $\mathcal{H}$ such that

$$\mathbb{H} = \{(x,t_x) \in \mathcal{H} \times (0,\infty), \quad t_x^2 - \|x\|^2 = 1\}.$$ 

The bilinear form $[\cdot,\cdot]$ is defined analogously, and $d(\cdot,\cdot) = \operatorname{arccosh}(\cdot,\cdot)$ defines a metric on $\mathbb{H}$.

The following Theorem is a version of Theorem 3.1 to the hyperbolic setting.

**Theorem 6.1.** The kernel

$$(z,w) \in \mathbb{H} \times \mathbb{H} \rightarrow H_r(z,w) := [z,w]^{-r} \in \mathbb{R}$$

is SPD (universal) for every $r > 0$.

A similar structure result characterizes when the inclusion $\mathcal{H}_H \subset C_0(X)$ is satisfied.

**Lemma 6.2.** Let $\mathbb{H}$ be a hyperbolic space, $X \subset \mathbb{H}$ be locally compact and $r > 0$. The following conditions are equivalent

(i) There exists $\xi_0 \in X$ for which the function $H_r,\xi_0(z) = [z,\xi]^{-r}$ is an element of $C_0(X)$.

(ii) The function $H_r,\xi(z) = [z,\xi]^{-r}$ is an element of $C_0(X)$ for every $\xi \in X$.

(iii) The inclusion $\mathcal{H}_H \subset C_0(X)$ holds.

(iv) Every bounded and closed set on $X$ is a compact set.

Similarly, a version of Theorem 3.2 and 3.3 to the hyperbolic setting also holds.

**Theorem 6.3.** Let $X \subset \mathbb{H}$ a locally compact subspace. The kernel

$$(z,w) \in X \times X \rightarrow H_r(z,w) := [z,w]^{-r} \in \mathbb{R}$$

is ISPD for every $r > 0$ and it is $C_0$-universal if and only if $\mathcal{H}_H \subset C_0(X)$. 

Similar to the comments made at the end of Section 3 if \( g : [1, \infty) \to \mathbb{R} \) is a continuous function for which the kernel
\[
(x, y) \in \mathbb{H}^m \times \mathbb{H}^m \to g([x, y]) \in \mathbb{R}
\]
is positive definite for every \( m \in \mathbb{N} \), replacing the kernel \([.,.]^{-r}\) on Theorems 6.1, 6.3 by the kernel \( g([.,.]) \) is possible, whenever \( g \) is not a constant function. Lemma 6.2 is also possible whenever \( g \) is not a constant function and relation (iv) is replaced by

\[(iv)' \quad \text{Every bounded and closed set on } X \text{ is a compact set and } \lim_{s \to \infty} g(s) = 0.\]
The argument that this generalization is possible is also a direct consequence of Theorem 3.7 in [20], and we do not present it.

6.2. Hyperbolic and log-conditional kernels. A kernel \( \beta : X \times X \to \mathbb{R} \) is called **hyperbolic** if there exists a Hyperbolic space \( \mathbb{H} \) and a function \( h : X \to \mathbb{H} \) for which \( \beta(x, y) = [h(x), h(y)] \). At [23] it is proved that a kernel \( \beta \) is hyperbolic if and only if \( \beta(x, x) = 1 \) for all \( x \in X \) and the kernel
\[
(x, y) \in X \times X \to \beta(x, z)\beta(y, z) - \beta(x, y) \in \mathbb{R}
\]
is positive definite for some \( z \in X \) (or equivalently, for every \( z \in X \)).

For example, if \( \gamma : X \times X \to \mathbb{R} \) is a conditionally negative definite kernel for which \( \gamma(x, x) = 0 \) for every \( x \in X \), then the kernel \( \beta(x, y) := 1 + \gamma(x, y) \) is hyperbolic, being a possible argument a verification that the kernel on Equation 6.4 is positive definite using the representation 4.3

Also, the kernel
\[
(x, y) \in \mathcal{H} \times \mathcal{H} \to \sqrt{1 + ||x||^2} \sqrt{1 + ||y||^2} - \langle x, y \rangle \in \mathbb{R}
\]
is hyperbolic on every Hilbert space \( \mathcal{H} \).

The relation between hyperbolic kernels and the functions \( s \in [1, \infty) \to s^{-r} \in \mathbb{R} \), \( r \in (0, \infty) \), is different from the relation between conditionally negative definite kernels and the functions \( s \in [0, \infty) \to e^{-sr} \in \mathbb{R} \), \( r \in (0, \infty) \).

If \( \gamma : X \times X \to \mathbb{R} \) is conditionally negative definite, by Schoenberg the kernel \( e^{-r\gamma(x,y)} \) is positive definite and the kernel \( r\gamma(x,y) \) is conditionally negative definite for every \( r > 0 \). However, if \( \beta : X \times X \to \mathbb{R} \) is a hyperbolic kernel, by Faraut and Harzallah the kernel \( \beta(x, y)^{-r} \) is positive definite for every \( r > 0 \), but the kernel \( \beta^r(x, y) \) is (with certainty) hyperbolic only for \( 1 \geq r > 0 \), [23]. What occurs is that by \( \beta \) being hyperbolic, \( \log(\beta(x, y)) \) is a conditionally negative definite kernel, and by Schoenberg this property is equivalent to the kernel \( e^{-r\log(\beta(x, y))} = \beta(x, y)^{-r} \) being positive definite for every \( r > 0 \). We say that a symmetric kernel \( L : X \times X \to [1, \infty) \) is **log-conditional** if the kernel \( \log L(x, y) \) is conditionally negative definite. Note that if \( L \) is log-conditional then so is \( L' \) for every \( r > 0 \).

Being so, a natural question is to analyse when the kernel \( L(x, y)^{-r} \) is SPD/ universal/ ISPD/ \( C_0 \)-universal, in a similar way as Section 4. However, since
\[
L(x, y)^{-r} = e^{-r\log L(x, y)}
\]
such characterizations are a consequence of the results proved on Section 4. This also includes the results from Subsection 6.1. For completion, we state these characterizations. Naturally, a log-conditional kernel $L$ is metrizable if $\log L$ is metrizable.

**Theorem 6.4.** Let $X$ be a Hausdorff space and $L : X \times X \to [1, \infty)$ be a continuous log-conditional kernel. Then the kernel

$$ (x, y) \in X \times X \to H_L(x, y) := L(x, y)^{-1} \in \mathbb{R} $$

is SPD (universal) if and only if the kernel $L$ is metrizable.

Similar to Section 4, the set $X$ with the metric induced by the conditionally negative definite kernel $\log L(x, y)$ is being denoted by $X_{\log L}$. If $L$ is a metrizable hyperbolic kernel, the hyperbolic metric $d_{\mathbb{H}}(x, y) := \text{arccosh} L(x, y)$ and the Hilbertian metric $d_{\mathbb{H}}(x, y) := \sqrt{\log L(x, y)}$ are equivalent because

$$ d_{\mathbb{H}} = \sqrt{\log \cosh d_{\mathbb{H}}}, \quad d_{\mathbb{H}} = \text{arccosh}(e^{(d_{\mathbb{H}})^2}) $$

and the functions $\sqrt{\log \cosh t}$, $\text{arccosh}(e^{t^2})$ are continuous on the the interval $[0, \infty)$.

**Theorem 6.5.** Let $X$ be a locally compact Hausdorff space and $L : X \times X \to [1, \infty)$ be a continuous log-conditional kernel that is metrizable and the function $x \in X \to L(x, x) \in \mathbb{R}$ is bounded. Suppose either:

i) The topologies of $X$ and $X_{\log L}$ are equivalent

ii) $X_{\log L}$ is a locally compact space and the function $x \in X_{\log L} \to \gamma(x, x)$ is continuous.

Then the kernel

$$ (x, y) \in X_{\log L} \times X_{\log L} \to H_L(x, y) := L(x, y)^{-1} \in \mathbb{R} $$

is ISPD.

**Theorem 6.6.** Let $X$ be a locally compact Hausdorff space and $L : X \times X \to [1, \infty)$ be a continuous log-conditional kernel. The kernel

$$ (x, y) \in X \times X \to H_L(x, y) := L(x, y)^{-1} \in \mathbb{R} $$

is $C_0(X)$-universal if and only if $L$ is metrizable and $\mathcal{H}_L \subset C_0(X)$.

Further, there exists $z_0 \in X$ for which the function $H_{L, z_0}(x) = L(x, z_0)^{-1}$ is an element of $C_0(X)$ if and only if $\mathcal{H}_L \subset C_0(X)$.
7. Proofs

7.1. Section 3

First, we state a few technical results that will be needed. If $K : X \times X \to \mathbb{C}$ is a positive definite kernel and $(\psi_i)_{i \in \mathcal{I}}$ is a complete orthonormal basis for $\mathcal{H}_k$, then it holds that

\begin{equation}
K(x, y) = \sum_{i \in \mathcal{I}} \psi_i(x) \overline{\psi_i(y)}, \quad x, y \in X.
\end{equation}

Lemma 7.1. Let $X$ be a compact Hausdorff space such that there exists a continuous conditionally negative definite metrizable kernel. Then $X$ is homeomorphic to a compact metric space and the RKHS of any continuous positive definite kernel on $X$ is a separable space.

Proof. Indeed, if $\gamma : X \times X \to \mathbb{R}$ is a continuous metrizable kernel, the metric

\[ D_\gamma(x, y) = \sqrt{\gamma(x, y) - \gamma(x, x)/2 - \gamma(y, y)/2} \]

is well defined. The inclusion $\iota : X \to X_\gamma$ is a continuous function because the kernel $D_\gamma$ is continuous. Conversely, since $X$ is compact and the topologies $X$ and $X_\gamma$ are Hausdorff, the inclusion must be a homeomorphism, and we can assume that $X$ is a compact metric space. In particular, $X$ is a separable space.

The conclusion that the RKHS of any continuous positive definite kernel on $X$ must be separable is a consequence that $X$ is a separable space as proved in page 130 in [30]. □

Lemma 7.2. Let $X$ be a locally compact Hausdorff space and $\lambda$ be a nonzero measure in $\mathcal{M}(X)$. Then there exists a sequence of nested compact sets $(\mathcal{C}_n)_{n \in \mathbb{N}}$ for which $\lambda(\mathcal{A}) = 0$ for every measurable set $\mathcal{A} \subset X - \bigcup_{n \in \mathbb{N}} \mathcal{C}_n$ and $\bigcup_{n \in \mathbb{N}} \mathcal{C}_n = \text{Supp}(\lambda)$.

In particular, if there exists a continuous conditionally negative definite metrizable kernel on $X$, then the set $\text{Supp}(\lambda)$ is separable (induced topology) and the RKHS of any continuous positive definite kernel on $X$ is separable when restricted to $\text{Supp}(\lambda)$.

Proof. On the first part we may assume that the measure $\lambda$ is nonnegative, because on the general case we can apply the result for the measures appearing on its Hahn decomposition.

Due to inner regularity, there exists a sequence of nested compact sets $\mathcal{D}_n$, for which $0 < \lambda(\mathcal{D}_n)$, $\lim_{n \to \infty} \lambda(\mathcal{D}_n) = \lambda(X)$. Define

\[ \mathcal{C}_n := \{ x \in \mathcal{D}_n, \text{ every open set that contains } x \text{ has positive measure} \} = \mathcal{D}_n \cap \text{Supp}(\lambda). \]

Then $\mathcal{C}_n$ is compact, $\lambda(\mathcal{C}_n) = \lambda(\mathcal{D}_n)$ and $\lambda(\bigcup_{n \in \mathbb{N}} \mathcal{C}_n) = \lambda(X)$. In particular, if $\mathcal{A} \subset X - \bigcup_{n \in \mathbb{N}} \mathcal{C}_n$ is a measurable set then $\lambda(\mathcal{A}) = 0$. If $x \in \text{Supp}(\lambda)$, then every open set that contains $x$ has positive measure, in particular it must intersect $\bigcup_{n \in \mathbb{N}} \mathcal{C}_n$, because otherwise it would have zero measure, and then $x \in \bigcup_{n \in \mathbb{N}} \mathcal{C}_n$. The fact that $\bigcup_{n \in \mathbb{N}} \mathcal{C}_n \subset \text{Supp}(\lambda)$ is a direct consequence of its definition.

If there exists a continuous conditionally negative definite metrizable kernel on $X$, by Lemma 7.1 each set $\mathcal{C}_n$ is separable (induced topology), but then the set $\text{Supp}(\lambda)$ is also separable. The conclusion that the RKHS of any continuous positive definite kernel on $X$ must be separable when restricted to $\text{Supp}(\lambda)$ is a consequence that $\text{Supp}(\lambda)$ is a separable space as proved in page 130 in [30]. □
Lemma 7.2 is the main reason why we do not need to impose that the Hilbert space $\mathcal{H}$ is separable.

The next two Lemmas are used to simplify some arguments throughout the paper. A proof for the first one can be found at [1] while the second is in [5].

**Lemma 7.3.** Let $X$ be a Hausdorff space, $K : X \times X \to \mathbb{R}$ be a continuous positive definite kernel and $f : X \to \mathbb{R}$ be a continuous function that is nowhere zero. The kernel

$$(x,y) \in X \times X \to K_f(x,y) := f(x)K(x,y)f(y) \in \mathbb{R}$$

is universal if and only if the kernel $K$ is universal. Further, if $X$ is a locally compact space and the functions $f$ and $1/f$ are bounded, then the kernel $K_f$ is ISPD if and only if the kernel $K$ is ISPD.

**Lemma 7.4.** Let $X$ and $\tilde{X}$ be Hausdorff spaces, $\tilde{K} : \tilde{X} \times \tilde{X} \to \mathbb{C}$ be an universal positive definite kernel and $h : X \to \tilde{X}$ be a continuous function. The positive definite kernel

$$(x,y) \in X \times X \to K(x,y) := \tilde{K}(h(x), h(y)) \in \mathbb{C}$$

is universal if and only if the function $h$ is injective. Similarly, if $X$ and $\tilde{X}$ are locally compact Hausdorff spaces and the kernel $\tilde{K}$ is ISPD, then the kernel $K$ is ISPD if and only if $h$ is an injective function.

The set $\mathbb{Z}_N^+$ stand for the space of functions $\mathbb{N} \to \mathbb{Z}_+$ and an element $\alpha$ on this space satisfies $|\alpha| = n$ if $\sum_{i \in \mathbb{N}} \alpha(i) = n$. If $x : \mathbb{N} \to \mathbb{R}$, then $x^{\alpha} := \prod_{n \in \mathbb{N}} x_i^{\alpha_i}$, where $x_i^0$ is always understood as been equal to 1.

**Proof of Theorem 3.1** Note that it is sufficient to prove the case $\sigma = 1/2$ by Lemma 7.4. Since

$$G_{1/2}(x,y) = e^{-(x,x)/2}e^{(x,y)}e^{-(y,y)/2},$$

Lemma 7.3 implies that the kernel $G_{1/2}$ is universal if and only if the kernel $e^{(x,y)}$ is universal. Let $\lambda \in \mathcal{M}_c(\mathcal{H})$, Lemma 7.1 implies that the RKHS of the dot kernel $\langle x,y \rangle$ is separable when restricted to the compact set $X := \text{Supp}(\lambda)$, that is, there exists a countable orthonormal set $(e_i)_{i \in \mathbb{N}}$, for which

$$\langle x,y \rangle = \sum_{i \in \mathbb{N}} \langle x,e_i \rangle \langle y,e_i \rangle := \sum_{i \in \mathbb{N}} x_i y_i, \quad x,y \in X.$$  

If $0 = \int_X \int_X e^{\langle x,y \rangle} d\lambda(x) d\lambda(y)$, then since the dot kernel is bounded on $X$, by the Lebesgue dominated convergence Theorem we obtain that

$$0 = \int_X \int_X e^{\langle x,y \rangle} d\lambda(x) d\lambda(y) = \sum_{n \in \mathbb{Z}_+} \frac{1}{n!} \int_X \int_X \langle x,y \rangle^n d\lambda(x) d\lambda(y).$$

Since

$$\langle x,y \rangle^n = \sum_{|\alpha| = n, \alpha \in \mathbb{Z}_N^+} \frac{n!}{\alpha!} x^{\alpha} y^{\alpha},$$
where $\alpha! = \prod_{i \in \mathbb{N}} \alpha_i!$, we have that
\[
0 = \int_X \int_X (\sum_{i \in \mathbb{N}} x_i y_i)^n d\lambda(x) d\lambda(y) = \sum_{|\alpha| = n, \alpha \in \mathbb{Z}_+^n} \frac{n!}{\alpha!} \int_X \int_X x^{\alpha} y^{\alpha} d\lambda(x) d\lambda(y)
= \sum_{|\alpha| = n, \alpha \in \mathbb{Z}_+^n} \frac{n!}{\alpha!} \left( \int_X x^{\alpha} d\lambda(x) \right)^2,
\]
then $\int_X x^{\alpha} d\lambda(x) = 0$ for every $\alpha \in \mathbb{Z}_+^n$. The algebra of continuous functions
\[
\mathcal{A} := \{ x \in X \to x^{\alpha} \in \mathbb{R}, \quad \alpha \in \mathbb{Z}_+^n \}
\]
separates points, because if $x, y$ are not separated by the algebra $\mathcal{A}$, then
\[
2 \langle x, y \rangle = 2 \sum_{i \in \mathbb{N}} x_i y_i = \sum_{i \in \mathbb{N}} x_i x_i + y_i y_i = \langle x, x \rangle + \langle y, y \rangle
\]
which can only occur if $x = y$. Since the constant function equal to 1 belongs to the algebra $\mathcal{A}$, the Stone-Weierstrass Theorem implies that $span \{ x \in X \to x^{\alpha} \in \mathbb{R} \}$ is dense on $C(X)$, and consequently $\lambda$ must be the zero measure, implying that the kernel $G_{1/2}$ is universal. \hfill \Box

**Proof of Theorem 3.2** The proof follows by similar arguments (also notation) as the one we used at the proof of Theorem 3.1 and several applications of the Lebesgue Dominated convergence Theorem. Again, it is sufficient to prove the case $\sigma = 1/2$ by Lemma 7.4. We focus on the converse relation, which is equivalent at the only measure $\lambda \in \mathcal{M}(X)$ (note that we are using the Borel sigma algebra $\mathcal{B}(X)$) such that
\[
(7.6) \quad 0 = \int_X \int_X e^{\langle x, y \rangle} e^{-\langle x, x \rangle/2} e^{-\langle y, y \rangle/2} d\lambda(x) d\lambda(y).
\]
is the zero measure. Lemma 7.2 implies that the RKHS of the dot kernel $\langle x, y \rangle$ is separable when restricted to the set $Z := Supp(\lambda)$, then there exists a countable orthonormal set $(e_i)_{i \in \mathbb{N}}$ in $\mathcal{H}$, for which
\[
\langle x, y \rangle = \sum_{i \in \mathbb{N}} \langle x, e_i \rangle \langle y, e_i \rangle := \sum_{i \in \mathbb{N}} x_i y_i, \quad x, y \in Z.
\]
Note that $|e^{\langle x, y \rangle} e^{-\langle x, x \rangle/2} e^{-\langle y, y \rangle/2}| \leq 1$ and
\[
\sum_{n=0}^m \frac{1}{n!} |\langle x, y \rangle^n e^{-\langle x, x \rangle/2} e^{-\langle y, y \rangle/2}| \leq e^{\langle x, x \rangle/2} e^{\langle y, y \rangle/2} \left( \sum_{n=0}^m \frac{1}{n!} \left( \frac{\langle x, x \rangle}{2} + \frac{\langle y, y \rangle}{2} \right)^n \right) \leq 1,
\]
so Equation 7.6 is equivalent at
\[
(7.7) \quad 0 = \int_Z \int_Z \langle x, y \rangle^n e^{-\langle x, x \rangle/2} e^{-\langle y, y \rangle/2} d\lambda(x) d\lambda(y), \quad n \in \mathbb{Z}_+.
\]
Since
\[ \langle x, y \rangle^n = \sum_{|\alpha| = n, \alpha \in \mathbb{Z}_+^n} \frac{n!}{\alpha!} x^\alpha y^\alpha, \quad x, y \in \mathbb{Z}, \]
Equation 7.7 is equivalent at
\[ (7.8) \quad 0 = \int_{\mathbb{Z}} x^\alpha e^{-\langle x, x \rangle / 2} d\lambda(x), \quad \alpha \in \mathbb{Z}_+^n, |\alpha| < \infty \]
because \( |\langle x, y \rangle^n e^{-\langle x, x \rangle / 2} e^{-\langle y, y \rangle / 2}| \leq n! \), also
\[ 2 \sum_{|\alpha| = n, \alpha \in \mathbb{Z}_+^n} \frac{n!}{\alpha!} |x^\alpha y^\alpha e^{-\langle x, x \rangle / 2} e^{-\langle y, y \rangle / 2}| \]
\[ \leq e^{-\langle y, y \rangle / 2} e^{-\langle x, x \rangle / 2} \sum_{|\alpha| = n, \alpha \in \mathbb{Z}_+^n} \frac{n!}{\alpha!} (x^{2\alpha} + y^{2\alpha}) \]
\[ \leq e^{-\langle y, y \rangle / 2} e^{-\langle x, x \rangle / 2} (\langle x, x \rangle^n + \langle y, y \rangle^n) \leq 2n!, \]
and consequently
\[ \int_{\mathbb{Z}} \int_{\mathbb{Z}} \langle x, y \rangle^n e^{-\langle x, x \rangle / 2} e^{-\langle y, y \rangle / 2} d\lambda(x) d\lambda(y) = \sum_{|\alpha| = n, \alpha \in \mathbb{Z}_+^n} \frac{n!}{\alpha!} \left( \int_{\mathbb{Z}} x^\alpha e^{-\langle x, x \rangle / 2} d\lambda(x) \right)^2. \]

○ (Affirmation 1) We claim that
\[ (7.9) \quad \int_{\mathbb{Z}} x^\alpha e^{-r\langle x, x \rangle / 2} d\lambda(x) = 0 \]
for every \( \alpha \in \mathbb{Z}_+^n \) with \( |\alpha| \leq \infty \) and \( r > 0 \). Indeed, Equation 7.8 implies that Equation 7.9 is valid for \( r = 1/2 \), and we use an induction type of argument to prove for the general case. Suppose that Equation 7.9 holds for a \( r' > 0 \), we claim that it also holds for every \( r \in (0, 2r') \). Indeed, for every \( \beta \in \mathbb{Z}_+^n \)
\[ \int_{\mathbb{Z}} x^{\alpha + \beta} e^{-r'\langle x, x \rangle} d\lambda(x) = 0. \]
By the Lebesgue dominated convergence Theorem
\[ 0 = \sum_{|\beta| = n, \beta \in \mathbb{Z}_+^n} \frac{n!}{\beta!} \int_{\mathbb{Z}} x^\alpha x^\beta e^{-r'\langle x, x \rangle} d\lambda(x) = \int_{\mathbb{Z}} x^\alpha \left( \sum_{|\beta| = n, \beta \in \mathbb{Z}_+^n} \frac{n!}{\beta!} x^\beta \right) e^{-r'\langle x, x \rangle} d\lambda(x) \]
\[ = \int_{\mathbb{Z}} x^\alpha \langle x, x \rangle^n e^{-r'\langle x, x \rangle} d\lambda(x). \]
In particular, by applying once again the Lebesgue dominated convergence Theorem, we obtain that for every $|s| < r'$

$$0 = \sum_{n \in \mathbb{Z}^+} \frac{s^n}{n!} \int_X \chi_n(x, x)^n \, d\lambda(x) = \int_X \chi (x) \left( \sum_{n \in \mathbb{Z}^+} \frac{s^n}{n!} \chi_n(x, x)^n \right) \, d\lambda(x)$$

$$= \int_X \chi \left( e^{(s - r')(x, x)} \right) \, d\lambda(x),$$

and so our claim is true on the interval $r \in (0, 2r')$ by choosing $s = r - r'$.

\(\bigcirc\) **Affirmation 2** We claim that

$$0 = \int_Z \prod_{\mu=1}^{m} \left( e^{\langle x, z_{\mu} \rangle} e^{-\langle x, x \rangle/2} \right) \, d\lambda(x),$$

for whichever $m \in \mathbb{N}$ and $z_1, \ldots, z_m \in Z$ (not necessarily distinct). Indeed, if $z = z_1 + \ldots + z_m$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\langle x, z \rangle^n}{2} \right) e^{-m(x, x)/2} \leq e^{\langle z, z \rangle/2} e^{-\langle m-1, x, x \rangle/2}$$

and

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\langle x, z \rangle^n}{2} \right) e^{-m(x, x)/2} \leq e^{\langle z, z \rangle/2} e^{-\langle m-1, x, x \rangle/2}.$$

Similarly,

$$\sum_{|\alpha|=n, \alpha \in \mathbb{Z}_+^n} \frac{n!}{\alpha!} \chi^{n} e^{-m(x, x)/2} \leq 2^n \langle x, x \rangle^n e^{-m(x, x)/2}$$

and

$$2 \sum_{|\alpha|=n, \alpha \in \mathbb{Z}_+^n} \frac{n!}{\alpha!} \chi^{n} e^{-m(x, x)/2} \leq \sum_{|\alpha|=n, \alpha \in \mathbb{Z}_+^n} \frac{n!}{\alpha!} \chi^{n} e^{-m(x, x)/2}.$$

The conclusion follows from Affirmation (1). Now, consider the algebra of functions generated by the set

$$\mathcal{A} := \text{span} \{ x \in Z \to e^{\langle x, z \rangle} e^{-\langle x, x \rangle/2} \in \mathbb{R}, \quad z \in Z \}.$$

Equation\[7,10\] implies that for every $h \in \mathcal{A}$ it holds that $\int_Z h(x) d\lambda(x) = 0$. Moreover:

(i) There exists $h \in \mathcal{A}$ for which $h(x) > 0$ for every $x \in Z$.

(ii) If $z_1, z_2 \in Z$, $B(z_1, R_1)$ and $B(z_2, R_2)$ are two disjoint open balls of $X$, there exists $h \in \mathcal{A}$ for which $h(x) > 0$ on $B(z_1, R_1) \cap Z$ and $h(x) < 0$ on $B(z_2, R_2) \cap Z$. 

For (i), take \( h \) as any of the functions \( x \rightarrow e^{(x,z)} e^{-(x,y)/2} \). As for (ii), define
\[
    h(x) = e^{-(z_1 z_1/2)} (e^{(x,z_1)} e^{-(x,y)/2} e^{R_1^2/2} - e^{-(z_2 z_2)/2} (e^{(x,z_2)} e^{-(x,y)/2} e^{R_2^2/2})
\]
\[
    = e^{-\|x-z_1\|^2/2 + R_1^2/2} - e^{-\|x-z_2\|^2/2 + R_2^2/2}.
\]

Theorem 7.10 implies that \( \mathcal{A} \) is dense on \( L^1(Z, |\lambda|) \). If \( P^+, N^- \) is a Hahn decomposition for the measure \( \lambda \), the continuous linear functional
\[
    h \in L^1(Z, |\lambda|) \rightarrow \int_Z h(x) |\chi_{P^+} (x) - \chi_{N^-} (x)| d|\lambda|(x) = \int_Z h(x) d\lambda(x) \in \mathbb{R}
\]
is zero on \( \mathcal{A} \), which can only happen if \( \chi_{P^+} = \chi_{N^-} \) on \( L^\infty(Z, |\lambda|) \), but then \( \lambda \) must be the zero measure, which concludes the proof. \( \square \)

**Proof of the Lemma 3.3.** Suppose that (i) holds, then for every \( \varepsilon > 0 \) there exists a compact set \( G_\varepsilon \) for which \( e^{-\|x-z_0\|^2} \leq \varepsilon \) for every \( x \in X \setminus G_M \). By the monotone properties of the function \( e^{-t} \), this relation is equivalent at for every \( M > 0 \) there exists a compact set \( G_M \) for which \( \|x-z_0\| \geq M \) for every \( x \in X \setminus G_M \). Relation (ii) follows by the inequality \( \|x-z\| \geq |||x-z_0\| - ||z-z_0||\). The converse is immediate Relations (ii) and (iii) are equivalent by Proposition 2.2.

If (iv) holds, then for every \( z \in X \) and \( \varepsilon > 0 \), the set \( \{ x \in X, e^{-\|x-z\|^2} \leq \varepsilon \} \) is bounded and closed on \( X \), so it must be compact by the hypnosis implying that the function \( G_{\sigma, \varepsilon} \subset C_0(X) \). The converse relation follows by the same argument as the first one presented, so we omit it. \( \square \)

**Proof of Theorem 3.4.** It is sufficient to prove the case \( \sigma = 1/2 \) by Lemma 7.4. If the kernel is \( C_0 \)-universal, by definition its necessary that \( \mathcal{H}_{1/2} \subset C_0(X) \). Conversely, if \( \mathcal{H}_{1/2} \subset C_0(X) \) we only need to to prove that the kernel is ISPD (on the sigma algebra \( \mathcal{B}(X) \) instead of \( \mathcal{B}(X^\gamma) \) as done in Theorem 3.2). We present a proof that does not involve Theorem 7.10 instead we use the \( C_0 \) version of the Stone-Weierstrass Theorem, which can be found at Section 4.7 at [15]. The arguments are the same as the one of Theorem 4.2 up to Equation 7.9 (we do not use Affirmation 2). The algebra of continuous functions on \( X \)
\[
    \mathcal{A} := \{ x^\alpha e^{r(x,y)}, r \in (0, \infty), \alpha \in \mathbb{Z}_+^N, |\alpha| < \infty \} \subset C_0(X).
\]
The function \( h(x) = e^{-(x,y)} \in \mathcal{A} \) is such that \( h(x) > 0 \) for every \( x \in X \), also, the algebra \( \mathcal{A} \) separates points because if \( x^\alpha e^{r(x,y)} = Y^\alpha e^{r(y,z)} \) for every \( \alpha \in \mathbb{Z}_+^N, |\alpha| < \infty \), then we must have that \( \langle x,y \rangle = \langle x,y \rangle = \langle y,y \rangle \), which can only occur if \( x = y \). As a direct consequence of the \( C_0 \) version of the Stone-Weierstrass Theorem we obtain that \( \lambda \) must be the zero measure, proving that the kernel \( G_{1/2} \) is \( C_0(X) \)-universal. \( \square \)

**Section 4.4**

**Proof of Theorem 4.7.** Since \( \gamma \) is a conditionally negative definite kernel, by Equation 4.3 there exists a Hilbert space \( \mathcal{H} \) and functions \( h : X \rightarrow \mathcal{H} \) and \( f : X \rightarrow \mathbb{R} \) for which
\[
    \gamma(x,y) = f(x) + \|h(x) - h(y)\|^2 + f(y)
\]
Note that that the functions $f, h$ are continuous because $f(x) = \gamma(x, x)/2$, and $\|h(x) - h(y)\| = \sqrt{\gamma(x, y) - f(x) - f(y)}$.

If the kernel $G_\gamma$ is universal, then the kernel $G_\gamma$ is SPD by definition.

If the kernel $G_\gamma$ is SPD the matrix
\[
\begin{bmatrix}
  e^{-\gamma(z)} & e^{-\gamma(z,w)} \\
  e^{-\gamma(w,z)} & e^{-\gamma(w,w)}
\end{bmatrix}
\]
is positive definite and its determinant is equal to $e^{-\gamma(z,w) - \gamma(w,w)} (1 - e^{\gamma(z,z) + \gamma(w,w) - 2\gamma(z,w)})$, which is nonzero for every $z \neq w$ if and only if $\gamma$ is metrizable.

It only remains to prove that if $\gamma$ is metrizable the kernel $G_\gamma$ is universal. Since
\[
G_\gamma(x, y) = e^{-f(x)} e^{-\|h(x) - h(y)\|^2} e^{-f(y)},
\]
Lemma 7.3 implies that the kernel $G_\gamma$ is universal if and only if the kernel $e^{-\|h(x) - h(y)\|^2}$ is universal. The metrizability is equivalent to the injectivity of the function $h$, being so, the kernel is universal by Proposition 7.4 and Theorem 3.1. \hfill \Box

**Proof of Theorem 4.2** By Equation 4.3, we have that
\[
\gamma(x, y) = f(x) + \|h(x) - h(y)\|^2 + f(y)
\]
and $D_\gamma(x, y) = \|h(x) - h(y)\|$. In both cases, the sets $X$ and $X_\gamma$ are locally compact, as well as $\{h(x), \quad x \in X\} \subset \mathcal{H}$, because this set is isometric with respect to $X_\gamma$. The functions $f, h$ are continuous in both cases. Since
\[
G_\gamma(x, y) = e^{-f(x)} e^{-\|h(x) - h(y)\|^2} e^{-f(y)},
\]
Lemma 7.3 implies that the kernel $G_\gamma$ is ISPD if and only if the kernel $e^{-\|h(x) - h(y)\|^2}$ is ISPD (because $e^{-f}$ and $e^{f}$ are bounded functions). The metrizability is equivalent to the injectivity of the function $h$, being so, the result is a consequence of Proposition 7.4 and Theorem 3.2. \hfill \Box

**Proof of the Lemma 4.3** Indeed, by the hypothesis, for every $\varepsilon > 0$ there exists a compact set $G_\varepsilon$ for which $e^{-\gamma(x, x_0)} \leq \varepsilon$ for every $x \in X \setminus G_\varepsilon$. By the monotone properties of the function $e^{-f}$ this relation is equivalent at for every $M > 0$ there exists a compact set $G_M$ for which $\gamma(x, z_0) \geq M$ for every $x \in X \setminus G_M$. Relation (i) follows by making use of the following inequality
\[
\gamma(x, z) \geq \left( \sqrt{\gamma(x, z_0) - \frac{\gamma(x, x_0)}{2} - \frac{\gamma(z_0, z_0)}{2} - D_\gamma(z_0, z)} \right)^2 + \frac{\gamma(x, x_0)}{2} + \frac{\gamma(z_0, z_0)}{2},
\]
which is a direct consequence of the triangle inequality of $X_\gamma$.

For the proof of (ii), the set $\{x \in X, \quad \gamma(x, z) \leq M\}$ is bounded and closed on $X_\gamma$ for every $M > 0$ and $z \in X$, because $x \in X \rightarrow \gamma(x, x)$ is continuous and this function is bounded. In particular this set is compact and this implies that $G_{\gamma, z} \subset C_0(X_\gamma)$.

Conversely, if $G_{\gamma, z} \subset C_0(X_\gamma)$, then the function $x \in X \rightarrow \gamma(x, z)$ is continuous and for every $M > 0$ there exists a compact set $G_M$ for which $\gamma(x, z) \geq M$ for every $x \in X \setminus G_M$. In particular, it holds that $\{x \in X, \quad \gamma(x, z) \leq M\} \subset G_M$, which implies that every bounded and closed subset
of the metric topology $X_γ$ is a compact set. The function $γ(x,x)$ is continuous because $γ(x,z) = D^2_γ(x,z) + γ(x,x)/2 + γ(z,z)$. □

**Proof of Theorem 4.4** If the kernel is $C_0(X)$-universal then it is SPD, which by Theorem 4.1 the kernel $γ$ must be metrizable. Also, $H_{Gγ} ⊂ C_0(X)$ by definition. Conversely, if $H_{Gγ} ⊂ C_0(X)$, $Gγ$ is $C_0$-universal if and only if it is ISPD. Since

$$Gγ(x,y) = e^{-f(x)} e^{-∥h(x)−h(y)∥^2} e^{-f(y)},$$

Lemma 7.3 implies that the kernel $Gγ$ is ISPD if and only if the kernel $e^{-∥h(x)−h(y)∥^2}$ is ISPD. Let $λ ∈ M(X)$ for which

$$\int_X \int_X e^{-∥h(x)−h(y)∥^2} dλ (x) dλ (y) = 0.$$ 

It is not possible to apply Theorem 4.4 in order to conclude the proof, because it is not clear if the set $\{h(x), \ x ∈ X\} ⊂ H$ is a locally compact space. But, by Lemma 7.2 there exists $\{φ_i\}_{i ∈ \mathbb{N}}$ an orthonormal set of the RKHS of the continuous dot kernel $(x,y) ∈ X × X → \langle h(x), h(y) \rangle$ for which

$$\langle h(x), h(y) \rangle = \sum_{i ∈ \mathbb{N}} φ_i (x) φ_i (y), \quad x, y ∈ Supp(λ).$$

With this representation, a change of notation in the proof of Theorem 3.4 (x_i to $φ_i (x)$) is sufficient to prove our claim and we omit it.

As for the second part of the theorem, it can be proved using similar arguments as the first part, with the addition of Theorem 4.2 (on this setting the set $\{h(x), \ x ∈ X\} ⊂ H$ is locally compact because is isometric with $X_γ$). □

**7.3. Products of positive definite kernels.** The Schur product Theorem asserts that if $p, q : X × X → \mathbb{C}$ are positive definite kernels, then their product kernel

$$(x,y) ∈ X × X → (p ∘ q)(x,y) := p(x,y)q(x,y)$$

is positive definite. This result is a direct consequence that the Hadamard Product of positive definite kernels is positive definite, where if $p : X × X → \mathbb{C}, q : Z × Z → \mathbb{C}$ are positive definite kernels, its Hadamard product is the kernel

$$((x,z), (y,w)) ∈ (X × Z) × (X × Z) → (p ⊗ q)((x,z), (y,w)) := p(x,y)q(z,w).$$

In this section we prove some results concerning the relation between the Schur/Hadamard product of kernels and the concepts of SPD/universality/ISPD. We also present a weighted version of these results.

We emphasize that in this section we implicitly assume that the domain of the kernels is a Hausdorff space on the SPD and universal settings, while it is a locally compact Hausdorff space on the ISPD setting. We also assume that an ISPD kernel is bounded.
Lemma 7.5. Let \( p, q : X \times X \to \mathbb{C} \) be continuous positive definite kernels. Suppose that the kernel \( p \) is SPD/universal/ISPD, then the kernel \( p \odot q \) is SPD/universal/ISPD if and only if the only measure \( \lambda \in \mathcal{M}_c(X) / \mathcal{M}_o(X) / \mathcal{M}(X) \) (respectively) for which
\[
\int_A \int_A q(x,y) d\lambda(x) \lambda(y) = 0,
\]
for every \( A \in \mathcal{B}(X) \) is the zero measure. In particular, if \( q(x,x) > 0 \) for every \( x \in X \), then the kernel \( p \odot q \) is SPD/universal/ISPD.

Proof. The proof for the three cases are identical, so we only focus on the ISPD case. Let \( \lambda \in \mathcal{M}(X) \) be such that
\[
\int_X \int_X p(x,y) q(x,y) d\lambda(x) \lambda(y) = 0.
\]
The continuous conditionally negative definite kernel
\[
(x,y) \in X \times X \to p(x,x) - p(x,y) - p(y,x) + p(y,y) \in \mathbb{R}
\]
is metrizable. Because of that, Lemma 7.2 implies that the kernel \( q \) can be written as \( q(x,y) = \sum_{k \in \mathbb{N}} q_k(x) \overline{q_k(y)} \) for \( x, y \in Z := \text{Supp}(\lambda) \), and then Equation 7.11 is equivalent to
\[
\int_Z \int_Z p(x,y) q_k(x) \overline{q_k(y)} d\lambda(x) \lambda(y) = 0, \quad k \in \mathbb{N}
\]
but the kernel \( p \) is ISPD, so the previous relation is equivalent to the measures \( q_k d\lambda \) (note that \( q_k d\lambda \) belongs to the same space of measures as \( \lambda \)) being zero for every \( k \in \mathbb{N} \). Using once again the series representation for \( q \), all the measures \( q_k d\lambda \) are zero if and only if
\[
\sum_{k \in \mathbb{N}} \left| \int_A q_k(x) d\lambda(x) \right|^2 = \int_A \int_A q(x,y) d\lambda(x) \lambda(y) = 0, \quad A \in \mathcal{B}(X),
\]
which proves our claim. Now, suppose in addition that \( q(x,x) > 0 \) for every \( x \in X \). In this case by the continuity of the function \( q \), for every \( z \in X \) there exists an open set \( U_z \) that contains \( z \) for which \( q(x,y) > 0 \) for every \( x, y \in U_z \). If \( X^{+, Re}, X^{-, Re}, X^{+, Im}, X^{-, Im} \) is a Hahn decomposition of the set \( X \) by the measure \( \lambda \), then \( U_z \cap X^{+, Re} \in \mathcal{B}(X) \) and
\[
\int_{U_z \cap X^{+, Re}} \int_{U_z \cap X^{+, Re}} q(x,y) d\lambda(x) \lambda(y) = 0.
\]
But the integrand is a positive function and the measure \( \lambda \) is nonnegative on \( X^{+, Re} \cap U_z \), which implies that this double integral is zero if and only if the measure \( \lambda \) is the zero measure on the set \( X^{+, Re} \cap U_z \). Suppose by an absurd that the measure \( \lambda^{+, Re} \) is nonzero and let \( \mathcal{C} \) be an arbitrary compact set on \( X \). Then there exists a finite set \( \{z_1, \ldots, z_n\} \subset X \) for which \( \mathcal{C} \subset \bigcup_{k=1}^n U_{z_k} \). Note that
\[
\lambda^{+, Re}(\mathcal{C}) = \lambda^{+, Re}(\mathcal{C} \cap X^{+, Re}) \leq \lambda^{+, Re}(\bigcup_{k=1}^n U_{z_k} \cap X^{+, Re}) \leq \sum_{k=1}^n \lambda^{+, Re}(U_{z_k} \cap X^{+, Re}) = 0,
\]
which is an absurd by the inner regularity of \( \lambda^{+, Re} \). \( \square \)
Now we focus on the Hadamard product. Before that we prove a measure theoretical Lemma that will simplify the arguments.

**Lemma 7.6.** Let \( X, Z \) be Hausdorff spaces and \( \lambda \in \mathcal{M}_b(X \times Z), \mathcal{M}_c(X \times Z), \mathcal{M}(X \times Z) \). If the function \( \phi : X \times Z \to \mathbb{C} \) is bounded and continuous then the function

\[
A \in \mathscr{B}(X) \to \lambda_\phi(A) := \int_{X \times Z} \chi_A(x) \phi(x,y) d\lambda(x,y) \in \mathbb{C}
\]

is a finite measure on \( \mathcal{M}_b(X), \mathcal{M}_c(X), \mathcal{M}(X) \).

**Proof.** We focus the arguments on the \( \mathcal{M}(X \times Z) \) case, being the others similar. The fact that \( \lambda_\phi \) is a measure is obtained by a direct application of the Lebesgue Dominated convergence Theorem. Note that \( \lambda_\phi \) is the linear combination of 16 measures (for instance, \( \chi_A(1_{\mathcal{M}}) d\lambda(1_{\mathcal{M}}) \)), so we can assume that \( \lambda \) is a nonnegative measure and \( \phi \) is a nonnegative function. In particular, if \( A \subset B \) then

\[
|\lambda_\phi(B) - \lambda_\phi(A)| = \int_{X \times Z} (\chi_B(x) - \chi_A(x)) \phi(x,y) d\lambda(x,y)
\leq \sup_{(x,y) \in X \times Z} |\phi(x,y)| (\lambda(B \times Z) - \lambda(A \times Z))
\]

and we prove the outer and inner regularity of \( \phi_\lambda \) by showing that the finite measure \( A \in \mathcal{B}(X) \to \lambda(A \times Z) \) is inner and outer regular.

\( \circ \) (Inner regular) Let \( \varepsilon > 0 \) and \( E \in \mathcal{B}(X) \). By the inner regularity of \( \lambda \) on the set \( E \times Z \) there exists a compact set \( C' \subset E \times Z \) for which \( \lambda(E \times Z) - \lambda(C') < \varepsilon \). The compact set

\[
C := \pi_1(C') \subset X \quad \text{(projection on the first variable)}
\]

is such that \( C' \subset C \times Z \subset E \times Z \), and then \( \lambda(E \times Z) - \lambda(C \times Z) < \varepsilon \).

\( \circ \) (Outer regular) Let \( \varepsilon > 0 \), \( E \in \mathcal{B}(X) \). By the inner regularity of \( \lambda \) there exists compact sets \( C_1 \) of \( X \) and \( C_2 \) of \( Z \) for which \( \lambda(X \times Z) - \lambda(C_1 \times C_2) < \varepsilon \). By the outer regularity of \( \lambda \) there exists an open set \( U \) that contains \( E \times C_2 \) and \( \lambda(U) - \lambda(E \times C_2) < \varepsilon \). For every \( x \in E \) there exists an open set \( U_x \) on \( X \) that contains \( x \) for which \( U_x \times C_2 \subset U \), because the set \( \{x\} \times C_2 \) is compact. Define the open set \( V = \bigcup_{x \in E} U_x \) of \( X \) and note that \( E \times C_2 \subset V \times C_2 \subset U \) and

\[
0 \leq \lambda(V \times Z) - \lambda(E \times Z) < 2\varepsilon + \lambda((V \times Z) \cap (C_1 \times C_2)) - \lambda((E \times Z) \cap (C_1 \times C_2))
\leq 2\varepsilon + \lambda((V \cap C_1) \times C_2) - \lambda((E \cap C_1) \times C_2)
\leq 4\varepsilon + \lambda(V \times C_2) - \lambda(E \times C_2) < 5\varepsilon.
\]

\( \square \)

On the next Lemma we use an equivalent condition for a positive definite kernel \( K : X \times X \to \mathbb{C} \) to be SPD/Universal/ISPD, which occurs if and only if the only measure \( \lambda \in \mathcal{M}_b(X)/\mathcal{M}_c(X)/\mathcal{M}(X) \) for which

\[
\int_X K(x,y) d\lambda(x) = 0, \quad y \in X
\]

is the zero measure \([22],[28]\).
Lemma 7.7. Let \( p : X \times X \to \mathbb{C} \) and \( q : Z \times Z \to \mathbb{C} \) be bounded positive definite continuous kernels. Suppose that the kernel \( p \) is SPD/universal/ISP D, then a measure \( \lambda \in \mathcal{M}(X \times Z)/\mathcal{M}(X) \) (respectively) satisfies

\[
\int_{X \times Z} \int_{X \times Z} p(x,y)q(u,v)d\lambda(x,u)d\overline{\lambda}(y,v) = 0
\]

if and only if

\[
\int_{Z} \int_{Z} q(u,v)d\lambda_A(u)d\overline{\lambda_A}(v) = 0,
\]

for every \( A \in \mathcal{B}(X) \) where \( \lambda_A \) is the measure in \( \mathcal{M}(Z)/\mathcal{M}(Z) \) (respectively) for which \( B \in \mathcal{B}(Z) \to \lambda_A(B) := \lambda(A \times B) \). In particular, the kernel \( p \otimes q \) is SPD/universal/ISP D if and only if the only the same occur with the kernels \( p, q \).

Proof. The proof for the three cases are identical, so we only focus on the ISPD case. Let \( \lambda \in \mathcal{M}(X \times Z) \) be such that

\[
\int_{X \times Z} \int_{X \times Z} p(x,y)q(u,v)d\lambda(x,u)d\overline{\lambda}(y,v) = 0
\]

for every \( (y,v) \in X \times Z \). Since \( p \) is an ISPD kernel and the measure

\[
A \in \mathcal{B}(X) \to \int_{X \times Z} \chi_A(x)q(u,v)d\lambda(x,u) \in \mathbb{C}
\]

is an element of \( \mathcal{M}(X) \) for every \( v \in Z \) by Lemma 7.6 then this measure is the zero measure for every \( v \in Z \). Note that the measure \( \lambda_A \) is an element of \( \mathcal{M}(Z) \) (this is obtained from Lemma 7.6 by reversing the roles of \( X \) and \( Z \) and taking \( \phi \) as the constant one function) and

\[
\int_{Z} q(u,v)d\lambda_A(u) = \int_{X \times Z} \chi_A(x)q(u,v)d\lambda(x,u) = 0, \quad v \in Z.
\]

In particular, if \( q \) is an ISPD kernel, the measure \( \lambda_A \) must be zero for every \( A \in \mathcal{B}(X) \), which implies that \( \lambda \) is the zero measure.

Conversely, if \( p \otimes q \) is an ISPD kernel \( p \) and \( q \) must also be integrally positive definite because

\[
\{ \lambda_1 \times \lambda_2, \quad \lambda_1 \in \mathcal{M}(X), \lambda_2 \in \mathcal{M}(Z) \} \subset \mathcal{M}(X \times Z)
\]

and Fubini-Tonelli Theorem. \( \square \)

Lastly, we prove a result that elucidates when a weighted sum of positive definite kernels is SPD/universal/ISP D.

Lemma 7.8. Let \( \Omega \) be a Hausdorff space, \( \eta \) be a nonnegative \( \sigma \)-finite Radon measure on it and a family of bounded positive definite kernels \( (p_w)_{w \in \Omega} \) on \( X \) such that \( p : \Omega \times (X \times X) \to \mathbb{C} \) is continuous. Suppose that the kernel

\[
(x,y) \in X \times X \to P(x,y) := \int_{\Omega} p_w(x,y)d\eta(w) \in \mathbb{C}
\]
is well defined and continuous. Then, the kernel $P$ is positive definite and a measure $\lambda \in \mathcal{M}_c(X)$ satisfy
\[
\int_X \int_X P(x,y) d\lambda(x) d\lambda(y) = 0
\]
if and only if
\[
\int_X \int_X p_w(x,y) d\lambda(x) d\lambda(y) = 0, \quad w \in Supp(\eta).
\]
If the kernel $P$ is bounded and the function $w \in \Omega \to \sup_{x \in X} p_w(x,x) \in \mathbb{R}$ is locally bounded, then the same relation occurs for $\lambda \in \mathcal{M}(X)$.

**Proof.** The fact that the kernel is positive definite is a consequence of the hypothesis that each kernel $p_w$ is positive definite. Now, let $\lambda \in \mathcal{M}_c(X)$, since the function $P$ is continuous it must be bounded on $Supp(\lambda)$. By Fubini-Tonelli we can change the order of integration
\[
\int_X \int_X P(x,y) d\lambda(x) d\lambda(y) = \int_\Omega \left[ \int_X \int_X p_w(x,y) d\lambda(x) d\lambda(y) \right] d\eta(w),
\]
because $2|p_w(x,y)| \leq p_w(x,x) + p_w(y,y)$ and $p \in L^1(\eta \times |\lambda| \times |\lambda|)$. The result we aim is a direct consequence that the function
\[
w \in \Omega \to P_\lambda(w) := \int_X \int_X p_w(x,y) d\lambda(x) d\lambda(y) \in \mathbb{R}
\]
is continuous and nonnegative. Indeed, it is nonnegative because the kernel $p_w$ is positive definite. For the continuity, since $\{w\} \times Supp(\lambda) \times Supp(\lambda)$ is compact set and $p$ is continuous, for every $\varepsilon > 0$ there exists an open neighborhood $U_w$ of $w$ for which $|p_w(x,y) - p_w(x,y)| < \varepsilon$ for all $x, y \in Supp(\lambda)$ and $w \in U_w$. So $|P_\lambda(w) - P_\lambda(w')| \leq \varepsilon(|\lambda|(X))^2$ which proves our claim. If $P$ is bounded, $X$ is locally compact and the function $w \in \Omega \to \sup_{x \in X} p_w(x,x) \in \mathbb{R}$ locally bounded, similar arguments can be used by replacing $Supp(\lambda)$ by a compact set $\mathcal{C}_\varepsilon$ for which $|\lambda|(X) - |\lambda|(\mathcal{C}_\varepsilon) < \varepsilon$. \hfill \Box

### 7.4. Section 5
Recall the formula
\[
e^{-\frac{||x||^2}{\sigma}} = \frac{1}{2m\pi^{m/2}} \sigma^{m/2} \int_{\mathbb{R}^m} e^{-ix\cdot\xi} e^{-\sigma||\xi||^2/4} d\xi, \quad x \in \mathbb{R}^m.
\]
Then
\[
G_{A,\gamma}((u,x),(v,y)) = \frac{1}{2m\pi^{m/2}} A(u,v) \gamma^{m/2} \int_{\mathbb{R}^m} e^{-i(x-y) \cdot \xi} e^{-\gamma(u,v)||\xi||^2/4} d\xi
\]
\[
= \frac{1}{2m\pi^{m/2}} C(u,v) \int_{\mathbb{R}^m} e^{-i(x-y) \cdot \xi} e^{-\gamma(u,v)||\xi||^2/4} d\xi
\]

**Proof of Theorem 5.1** The continuity of the kernel follows by its definition. It is positive definite because by the hypothesis, the kernel $C$ is positive definite and since the kernel $\gamma$ is conditionally negative definite, the kernel $e^{-i(x-y) \cdot \xi} e^{-\gamma(u,v)||\xi||^2/4}$ is positive definite for every $\xi \in \mathbb{R}$. If the kernel $G_{A,\gamma}$ is universal then it is SPD by definition. If the kernel $G_{A,\gamma}$ is SPD, then for
every \( u \in X \) we have that \( G_{\Lambda, \gamma}(u, 0), (u, 0) = A(u, u) > 0 \).

Now suppose that \( A(u, u) > 0 \) for every \( u \in X \). A measure \( \lambda \in \mathcal{M}(X \times \mathbb{R}^m) \) is such that

\[
\int_{X \times \mathbb{R}^m} \int_{X \times \mathbb{R}^m} G_{\Lambda, \gamma}((u, x), (v, y)) d\lambda(u, x) d\overline{\lambda}(v, y) = 0
\]

if and only if

\[
(7.12) \quad \int_{X \times \mathbb{R}^m} \int_{X \times \mathbb{R}^m} C(u, v) e^{-\|\xi\|^2/\gamma(u, v)/4} e^{-i(x-y)\xi} d\lambda(u, x) d\overline{\lambda}(v, y) = 0, \quad \xi \in \mathbb{R}^m.
\]

by Lemma 7.8. When \( \xi \neq 0 \), by the hypothesis on the kernel \( \gamma \), Theorem 4.1 and Lemma 7.5 implies that the kernel

\[
(u, v) \in X \times X \rightarrow C(u, v) e^{-\|\xi\|^2/\gamma(u, v)/4} \in \mathbb{C}
\]

is universal. By Lemma 7.7 we obtain that for every \( A \in \mathcal{B}(X) \) it holds that

\[
0 = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} e^{-i(x-y)\xi} d\lambda_A(x) d\overline{\lambda_A}(y) = |\lambda_A(\xi)|^2.
\]

for every \( \xi \in \mathbb{R}^m \setminus \{0\} \). Since the only finite measure on \( \mathcal{M}(\mathbb{R}^m) \) that satisfies this relation is the zero measure, we must have that \( \lambda(E \times B) = 0 \) for every \( B \in \mathcal{B}(X) \) and \( E \in \mathcal{B}(\mathbb{R}^m) \), which implies that \( \lambda \) is the zero measure and that the kernel \( G_{\Lambda, \gamma} \) is universal.

\[\square\]

**Proof of Theorem 5.2.** If \( \mathcal{H}_{G_{\Lambda, \gamma}} \subset C_0(X \times \mathbb{R}^m) \), then by Proposition 2.2 for every \( u \in X \) and \( x \in \mathbb{R}^m \), \( G_{\Lambda, \gamma}((u, x)(u, x)) = A(u, u) \) is a bounded function and \( (G_{\Lambda, \gamma}(u, x) \in C_0(X \times \mathbb{R}^m) \), but then \( A_u = (G_{\Lambda, \gamma}(u, x) \in C_0(X) \). By using Proposition 2.2 once again we have that \( \mathcal{H}_A \subset C_0(X) \).

Conversely, suppose that \( \mathcal{H}_A \subset C_0(X) \) and that \( A \) is bounded by 1, then for every \( v \in X \) and \( \varepsilon > 0 \) there exists a compact set \( \mathcal{C}_1, v \subset X \) for which \( |A(u, v)| < \varepsilon \) for \( u \in X \setminus \mathcal{C}_1, v \). Let \( M = \sup_{z \in \mathcal{C}_1, v \cup\{v\}} \gamma(z, z) \), then \( e^{-\|\xi\|^2/\gamma(u, v)} \leq e^{-\|\xi\|^2/M} \) for every \( u \in \mathcal{C}_1, v \) and \( \xi \in \mathbb{R}^m \), so if \( y \in \mathbb{R}^m \) and \( \mathcal{C}_2, y \) is a compact set for which \( e^{-\|x-y\|^2/M} < \varepsilon \) for every \( x \in \mathbb{R}^m \setminus \mathcal{C}_2, y \), the compact set \( \mathcal{C} := \mathcal{C}_1, v \times \mathcal{C}_2, y \) is such that \( |G_{\Lambda, \gamma}(u, x), (v, y))| < \varepsilon \) for every \( (u, x) \in X \times \mathbb{R}^m \setminus \mathcal{C} \), which concludes the argument.

Now, we focus on the second relation. If \( G_{\Lambda, \gamma} \) is \( C_0(X \times \mathbb{R}^m) \)-universal, then \( \mathcal{H}_{G_{\Lambda, \gamma}} \subset C_0(X \times \mathbb{R}^m) \) by definition and by the first part of the theorem we must have that \( \mathcal{H}_A \subset C_0(X) \). Also, \( A(u, u) > 0 \) for every \( u \in X \) by Theorem 5.1.

Conversely, note that we only need to prove that the kernel \( G_{\Lambda, \gamma} \) is ISPD. Since \( |G_{\Lambda, \gamma}| \leq \sup_{u \in X} A(u, u) < \infty \),

\[
G_{\Lambda, \gamma}((u, x), (v, y)) = \frac{1}{2m!\pi^{m/2}} \int_{\mathbb{R}^m} C(u, v) e^{-i(x-y)\xi} e^{-\gamma(u, v)}\|\xi\|^2/4 d\xi,
\]

and \( C \) is a bounded kernel, by Lemma 7.8 the kernel \( G_{\Lambda, \gamma} \) is ISPD if and only if the only measure \( \lambda \in \mathcal{M}(X \times \mathbb{R}^m) \) for which

\[
\int_{X \times \mathbb{R}^m} \int_{X \times \mathbb{R}^m} C(u, v) e^{-i(x-y)\xi} e^{-\gamma(u, v)}\|\xi\|^2/4 d\lambda(u, x) d\overline{\lambda}(v, y) = 0, \quad \xi \in \mathbb{R}^m
\]
is the zero measure. But, since $G_{tx}$ is ISPD for every $t > 0$, Lemma 7.7 together with the hypothesis that $C(u, u) > 0$ for every $u \in X$ implies that

$$0 = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} e^{-i(x-y) \cdot \xi} d\lambda_B(x) d\lambda_B(y) = |\widehat{\lambda_B}(\xi)|, \quad \xi \in \mathbb{R}^m \setminus \{0\}, \quad B \in \mathcal{B}(X).$$

Similar to the proof of Theorem 5.1, since the only measure on $\mathcal{M}(\mathbb{R}^m)$ that satisfies this relation is the zero measure, we must have that $\lambda(E \times B) = 0$ for every $B \in \mathcal{B}(X)$ and $E \in \mathcal{B}(\mathbb{R}^m)$, which implies that $\lambda$ is the zero measure and that the kernel $G_{A,Y}$ is ISPD.

Next Lemma is focused on the analysis on the kernel $\gamma(u, v)^{-1}$ on a broader context.

**Lemma 7.9.** Let $\gamma : X \times X \to (0, \infty)$ be a continuous conditionally negative definite kernel and $\nu_1, \ldots, \nu_\ell \in (0, \infty)$. Consider the matrix valued kernel

$$(u, v) \in X \times X \to \begin{bmatrix} \Gamma(v_i + v_j) \\ \gamma(u, v)^{v_i + v_j} \end{bmatrix}_{i,j=1}^\ell \in M_\ell(\mathbb{C})$$

(i) The matrix valued kernel is SPD if and only if

$$\{(i, j, u, v), \gamma(u, v) = \gamma(u, u) = \gamma(v, v) \text{ and } \nu_i = \nu_j\} = \{(i, i, u, u), 1 \leq i \leq \ell, u \in X\}.$$

(ii) If the kernel $\gamma$ is metrizable, then the matrix valued kernel is universal if and only if the $\nu_i$ are distinct.

(iii) If $G_{tx}$ is ISPD for every $t > 0$, then the matrix valued kernel is ISPD if and only if the $\nu_i$ are distinct and $\inf_{u \in X} \gamma(u, u) > 0$.

**Proof.** Indeed, if the sets are not equal and $(i, j, u, v)$ belongs to the left hand set but not to the right hand set, then the interpolation matrix of the kernel at the points $u, v$ ($\ell \times \ell$ matrix if $u = v$ and $2\ell \times 2\ell$ matrix if $u \neq v$) is not a positive definite matrix. Conversely, by the definition of the gamma function

$$\frac{\Gamma(v_i + v_j)}{\gamma(u, v)^{v_i + v_j}} = \int_{(0,\infty)} t^{v_i + v_j} e^{-\gamma(u, v)t} dt.$$ 

So, by Lemma 7.3 the kernel is SPD if and only if for any finite number of points $u_1, \ldots, u_m$ and scalars $c_{i,\mu} \in \mathbb{R}$ for which

$$\sum_{i,j=1}^\ell \sum_{\mu, \eta=1}^m c_{i,\mu} c_{j,\eta} t^{v_i + v_j} e^{-\gamma(u_\mu, u_\eta)t}, \quad t \in (0, \infty)$$

then all scalars $c_{i,\mu}$ are equal to zero. By Equation 4.3, we can write $\gamma(u_\mu, u_\eta) = f(u_\mu) + ||h(u_\mu) - h(u_\eta)||^2 + f(u_\eta)$. Consider the equivalence class $\mu \simeq \eta$ if $h(u_\mu) = h(u_\eta)$, which separates the set $\{1, \ldots, m\}$ on a finite number of disjoint sets $F_1, \ldots, F_m, m' \leq m$. Note that

$$0 = \sum_{i,j=1}^\ell \sum_{\mu, \eta=1}^m c_{i,\mu} c_{j,\eta} t^{v_i + v_j} e^{-\gamma(u_\mu, u_\eta)t} = \sum_{i,j=1}^\ell \sum_{\mu, \eta=1}^m c_{i,\mu} c_{j,\eta} t^{v_i + v_j} e^{-f(u_\mu)t} e^{-h(u_\mu) - h(u_\eta)} ||h(u_\mu) - h(u_\eta)||^2 t$$

$$= \sum_{a,b=1}^{m'} \sum_{i=1}^\ell \sum_{\mu \in F_a} c_{i,\mu} t^{v_i} e^{-f(u_\mu)t} \left( \sum_{j=1}^\ell \sum_{\eta \in F_b} c_{j,\eta} t^{v_i} e^{-f(u_\eta)t} \right) e^{-||h(x_a) - h(x_b)||^2 t}$$
where $x_1, \ldots, x_m$ are class representatives. By Theorem 5.1 we have that 
\[ \sum_{i=1}^{\ell} \sum_{\mu \in E_i} c_{i,\mu} t^{\nu} e^{-f(u_\mu)t} = 0 \] for every $t > 0$ and $1 \leq a \leq m$. Without loss of generalization suppose that $m' = m$ and note that the pairs $(v_\mu, f(u_\mu))$ are distinct by the hypothesis. If $Z := \{(v_\ell, f(u_\mu))\}$, $X_1 := \text{argmin}\{f(u_\mu), 1 \leq \mu \leq m\}$, $y_1 := \min\{f(\mu), 1 \leq \mu \leq m\}$ then the set of numbers $v_1 := \{(v, u), (v, u) \in (\{v\} \times X_1) \cap Z\}$ such that 
\[ 0 = \sum_{i=1}^{\ell} \sum_{\mu = 1}^{m} c_{i,\mu} t^{\nu} e^{-f(u_\mu)t} = \sum_{(v, u) \in v_1} c_{v,\mu} t^{\nu} + \sum_{i=1}^{\ell} \sum_{\mu \in X_1, \mu \notin X_1} c_{i,\mu} t^{\nu} e^{-f(u_\mu)+y_1}. \]

The first sum is a function that either is zero or diverges in module as $t \rightarrow \infty$, while the second sum is a function that goes to zero as $t$ goes to infinity, then we must have that each sum is the zero function on $(0, \infty)$. But, the function $\sum_{(v, u) \in v_1} c_{v,\mu} t^{\nu}$ being zero on $(0, \infty)$, either there are two equal exponents $\nu$, which does not occur by the hypothesis, or all coefficients are zero. By an induction argument all coefficients $c_{i,\mu}$ are zero and then the kernel is SPD.

As for relation (ii), since $e^{-\gamma(u,v)t}$ is an universal kernel for every $t > 0$, Lemma 7.8 and Lemma 7.7 implies that the matrix valued kernel is universal if and only if the only scalars $c_1, \ldots, c_\ell \in \mathbb{R}$ for which
\[ \sum_{i=1}^{\ell} c_{i} t^{\nu_i} = 0, \quad t \in (0, \infty) \]

are all equal to zero. The result is then a consequence that the set of functions $\{t^{\nu_1}, \ldots, t^{\nu_\ell}\}$ are linearly independent if and only if the exponents $\nu_1, \ldots, \nu_\ell$ are distinct.

Relation (iii) follows by similar arguments as relation (ii). The condition $\inf_{u \in X} \gamma(u,u) > 0$ is equivalent to the matrix valued kernel being bounded.

**Proof of Theorem 5.3** The continuity follows by the continuity of the functions involved. By the integral representation of $\mathcal{M}(\|x - y\|; \alpha, \nu)$ and the proof of Theorem 1 in [4] we have that
\[
C_{i,j}^\#: ((u,x), (v,y)) = A_{i,j}(u,v) \int_{(0, \infty)} e^{-\|x-y\|^2 t/\gamma(u,v)} \left( \frac{\alpha_{i,j}^2}{4} \right)^{v_i j} t^{\nu_i - 1 - \nu_j} e^{-\alpha_{i,j}^2/4t} dt = C_{i,j}(u,v) \frac{1}{\gamma^{m/2}(u,v)} \int_{(0, \infty)} e^{-\|x-y\|^2 t/\gamma(u,v)} m_{i,j}(t) dt,
\]

where $m_{i,j}(t) = m_{i}(m_{j}(t), m_{j}(t)) = \frac{\alpha_{i,j}^{2\nu_i}}{2^{m/2}\Gamma(2\nu_i)^{1/2}} t^{\nu_i - 1/2} e^{-\alpha_{i,j}^2/8t}$. The positivity of the kernel follows by this integral representation together with the hypothesis on the kernel $C$ and the fact that $G_{A,\gamma}$ is a positive definite kernel.

If the matrix valued kernel $C^{\#:\gamma}$ is SPD (universal) then $C_{i,j}^\#: ((0,u), (0,u)) = A_{i,j}(u,u) > 0$ for every $1 \leq i \leq \ell$ and $u \in X$. Also, if $i \neq j$ is such that $v_i = v_j$ and $\alpha_i = \alpha_j$ then the scalar valued kernels $C_{i,i}^\#: (0,0), C_{i,j}^\#: (0,u)$ are all equal, which does not occur if $C^{\#:\gamma}$ is a matrix valued SPD kernel.
In order to prove the converse we analyse the matrix valued kernel

\[(7.13) \quad ((u, x), (v, y)) \in (X \times \mathbb{R}^m)^2 \rightarrow \left[ \frac{1}{\gamma^{m/2}(u, v)} \int_{(0, \infty)} e^{-\|x-y\|^2/\gamma(u, v)} m_{i,j}(t) dt \right]_{i,j=1}^\ell \in M_\ell(\mathbb{C}). \]

Theorem 5.1 implies that the kernel \( \gamma(u, v)^{-m/2} e^{-\|x-y\|^2/\gamma(u, v)} \) defined on \( X \times \mathbb{R}^m \) is universal for every \( t > 0 \). Lemma 7.8 and Lemma 7.7 implies that the matrix valued kernel on Equation (7.13) is SPD (universal) if and only if the matrix (which is independent from \( u_0 \))

\[ \left[ \int_{(0, \infty)} m_{i,j}(t) dt \right]_{i,j=1}^\ell = \left[ \frac{A_{i,j}(u_0, u_0) \gamma^{m/2}(u_0, u_0)}{C_{i,j}(u_0, u_0)} \right]_{i,j=1}^\ell \]

\[= \left[ \frac{\alpha_i \gamma^v}{2^{-v} \Gamma(2v_j)1/2} \frac{\alpha_j r_v}{2^{-r} \Gamma(2r_j)1/2} \Gamma(v_i + v_j) \right]_{i,j=1}^\ell \]

is positive definite, which is characterized on Lemma 7.9.

Lemma 7.5 implies that the kernel \( C_{i,j} \gamma \) is SPD (universal) when the kernel on equation 7.13 is SPD(universal) and \( C_{i,j}(u, u) > 0 \) (or equivalently, \( A_{i,j}(u, u) > 0 \)) for every \( 1 \leq i \leq \ell \) and \( u \in X \).

**Proof of Theorem 5.4** The fact that if \( \mathcal{H}_{C, \gamma} \subset C_0(X \times \mathbb{R}^m, \mathbb{C}^\ell) \) then \( \mathcal{H}_A \subset C_0(X, \mathbb{C}^\ell) \) is similar to the one presented at Theorem 5.2. Conversely, if \( \mathcal{H}_A \subset C_0(X, \mathbb{C}^\ell) \) and the kernel \( A \) is bounded by 1, then for every \( v \in X \) and \( \varepsilon > 0 \) there exists a compact set \( \mathcal{C}_{v,e} \) for which \( |A_{i,j}(u, v)| < \varepsilon \) for every \( u \in X \setminus \mathcal{C}_{v,e} \) and \( 1 \leq i, j \leq \ell \). If \( M := \inf_{u \in \mathcal{C}_{v,e} \cup \{v\}} \gamma(u, u)^{1/2} \), then

\[ |\mathcal{M}(\|x-y\|/\gamma(u, v)^{1/2}; \alpha_{i,j, v_{i,j}})| \leq \mathcal{M}(\|x-y\|/M; \alpha_{i,j, v_{i,j}}), \quad u \in \mathcal{C}_{v,e}, 1 \leq i, j \leq \ell. \]

The \( \ell^2 \) functions on the right hand side of previous equation are in \( C_0(\mathbb{R}^m) \), so for every \( y \in \mathbb{R}^m \) there exists a compact set \( \mathcal{C}_{v,e} \) for which

\[ |\mathcal{M}(\|x-y\|/M; \alpha_{i,j, v_{i,j}})| < \varepsilon, \quad x \in \mathbb{R}^m \setminus \mathcal{C}_{v,e}, 1 \leq i, j \leq \ell, \]

and then

\[ |C_{i,j}((u, x), (v, y))| < \varepsilon, \quad (u, x) \in X \times \mathbb{R}^m \setminus \mathcal{C}_{v,e} \times \mathcal{C}_{v,e}, 1 \leq i, j \leq \ell. \]

Now, we focus on the second relation. If \( C_{A, \gamma} \) is \( C_0(X \times \mathbb{R}^m, \mathbb{C}^\ell) \)-universal, then \( \mathcal{H}_{C_{A, \gamma}} \subset C_0(X \times \mathbb{R}^m, \mathbb{C}^\ell) \) by definition and by the first part of the theorem we must have that \( \mathcal{H}_A \subset C_0(X, \mathbb{C}^\ell) \).

Also, \( A_{i,j}(u, u) > 0 \) for every \( u \in X \) and \( 1 \leq i \leq \ell \) by Theorem 5.1.

In order to prove the converse we analyse the matrix valued kernel

\[(7.14) \quad ((u, x), (y, v)) \in (X \times \mathbb{R}^m)^2 \rightarrow \left[ \frac{1}{\gamma^{m/2}(u, v)} \int_{(0, \infty)} e^{-\|x-y\|^2/\gamma(u, v)} m_{i,j}(t) dt \right]_{i,j=1}^\ell \in M_\ell(\mathbb{C}). \]

By the hypothesis and Lemma 7.4, the kernel \( \gamma(u, v)^{-m/2} e^{-\|x-y\|^2/\gamma(u, v)} \) defined on \( X \times \mathbb{R}^m \) is ISPD for every \( t > 0 \) (\( \|x-y\|^2 = \|\sqrt{t}x \| - (\sqrt{t}y) |^2 \)). Lemma 7.8 and Lemma 7.7 implies
that the matrix valued kernel on Equation \[7.14\] is ISPD if and only if the matrix (which is independent from \(u_0\))

\[
\left[ \int_{(0, \infty)} m_{i,j}(t) dt \right]_{i,j=1}^{\ell} = \left[ \frac{A_{i,j}(u_0, u_0) \gamma^{n/2}(u_0, u_0)}{C_{i,j}(u_0, u_0)} \right]_{i,j=1}^{\ell} = \left[ \frac{\alpha_i^{v_i} \alpha_j^{v_j} \Gamma(v_i + v_j)}{2^{-v_i} \Gamma(2v_i)^{1/2} 2^{-v_j} \Gamma(2v_j)^{1/2} (\alpha_i + \alpha_j)^{v_i + v_j}} \right]_{i,j=1}^{\ell}
\]

is positive definite, which is characterized on Lemma \[7.2\]. Lemma \[7.5\] implies that the kernel \(C_{i,j}^{M, \gamma}\) is ISPD when the kernel on Equation \[7.14\] is ISPD and \(A_{i,j}(u, u) > 0\) (or equivalently, \(A_{i,j}(u, u) > 0\)) for every \(1 \leq i \leq \ell\) and \(u \in X\).

**Proof of Theorem 5.5**

(i) The continuity follows by the continuity of the functions involved. By the integral representation of \(\mathcal{M}(\|x - y\|; \gamma, \nu)\), we have that

\[
[\mathcal{M}(\gamma, \nu)]_{i,j}(\gamma(x), (v, y)) = A_{i,j}(u, v) \int_{(0, \infty)} e^{-\|x - y\|^2 t} \left( \frac{\gamma(u, v)}{4} \right) \frac{t^{-1-v_i-v_j}}{\Gamma(v_i + v_j)} e^{-\gamma(u, v)/4t} dt = C_{i,j}(u, v) \int_{(0, \infty)} e^{-\|x - y\|^2 t} e^{-\gamma(u, v)/4t} t^{-1}(4t)^{-v_i}(4t)^{-v_j} dt,
\]

and the positivity of the kernel follows by this representation.

(ii) If \(A_{i,j}(u, u)\) is not a positive number for some \(u \in X\) and \(1 \leq i \leq \ell\) or the numbers \(v_1, \ldots, v_\ell\) are not distinct, it is immediate that the kernel is not SPD. Conversely, since \(C_{i,j}(u, u) > 0\) for every \(1 \leq i \leq \ell\) and \(u \in X\), by Lemma \[7.5\] in order to prove that the kernel is SPD/universal is sufficient to prove that the kernel defined by the integral on \((0, \infty)\) is SPD/universal.

The Gaussian kernel \(e^{-\|x - y\|^2 t}\) and the Schoenberg kernel \(e^{-\gamma(u, v)/t}\) are universal for every \(t \in (0, \infty)\), so by Lemma \[7.8\] and Lemma \[7.9\] the kernel defined by the integral on \((0, \infty)\) is SPD/universal if and only if the only scalars \(c_1, \ldots, c_\ell \in \mathbb{R}\) for which \(\sum_{i=1}^{\ell} c_i t^{-v_i} = 0\) for every \(t > 0\) are all equal to zero, which holds true because the numbers \(v_i\) are distinct.

(iii) If \(\mathcal{A} \subset C_0(X, \mathbb{C}^\ell)\) and the kernel \(A\) is bounded by 1, then for every \(v \in X\) and \(\varepsilon > 0\) there exists a compact set \(\mathcal{C}_{v, \varepsilon}\) for which \(|A_{i,j}(u, v)| < \varepsilon\) for every \(u \in X \setminus \mathcal{C}_{v, \varepsilon}\) and \(1 \leq i, j \leq \ell\). If \(M_1 := \inf_{u \in \mathcal{C}_{v, \varepsilon} \cup \{v\}} \gamma(u, u)\) and \(M_2 := \sup_{u \in \mathcal{C}_{v, \varepsilon} \cup \{v\}} \gamma(u, u)\), then

\[
[\mathcal{M}(\|x - y\|; \gamma, \nu)]_{i,j} \leq \left( \frac{M_2}{M_1} \right)^{v_i + v_j} \mathcal{M}(\|x - y\|; M_1, \alpha_i + \alpha_j),
\]

for every \(u \in \mathcal{C}_{v, \varepsilon}\), \(1 \leq i, j \leq \ell\). The \(\ell^2\) functions on the right hand side of previous equation are in \(C_0(\mathbb{R}^m)\), so for every \(y \in \mathbb{R}^m\) there exists a compact set \(\mathcal{C}_{y, \varepsilon}\) for which

\[
\left( \frac{M_2}{M_1} \right)^{v_i + v_j} \mathcal{M}(\|x - y\|; M_1, \alpha_i + \alpha_j) < \varepsilon, \quad x \in \mathcal{C}_{y, \varepsilon}, 1 \leq i, j \leq \ell,
\]
and then

$$||[\mathcal{M}_{A,Y}]_{i,j}((u,x),(v,y))|| < \varepsilon, \quad (u,x) \in X \times \mathbb{R}^m \setminus \mathcal{C}_{Y,F} \times \mathcal{C}_{Y,E}, 1 \leq i, j \leq \ell.$$ 

(iv) We focus on the proof of the converse relation. It is sufficient to prove that the kernel is ISPD by (iii). Since $C_{i,j}(u,u) > 0$ for every $1 \leq i, j \leq \ell$ and $u \in X$, by Lemma 7.5 in order to prove that the kernel is ISPD is sufficient to prove that the matrix valued kernel

$$\int_{(0,\infty)} e^{-\|x-y\|^2 t} e^{-\gamma(u,v)/4t} t^{-1}(4t)^{-\nu_y(4t)^{-\nu_y}} dt,$$

is ISPD. The Gaussian kernel $e^{-\|x-y\|^2 t}$ and the Schoenberg kernel $e^{-\gamma(u,v)/4t}$ are ISPD for every $t \in (0,\infty)$, so by Lemma 7.3 and Lemma 7.7 the kernel defined by the integral on $(0,\infty)$ is SPD/universal if and only if the only scalars $c_1, \ldots, c_n \in \mathbb{R}$ for which $\sum_{i=1}^{\ell} c_i t^{-\nu_y} = 0$ for every $t > 0$ are all equal to zero, which holds true because the numbers $\nu_y$ are distinct. \qed

7.5. Section 6. Even though the results in this section are a direct consequence of Section 4, as mentioned in Section 6.2, we present a direct proof for Theorem 6.3. We focus on the hyperbolic spaces $\mathbb{H}^m$ only to simplify the notation since several summations over multi-indexes are needed.

(Partial) Proof of Theorem 6.3 First, note that the kernel is bounded and

$$[(x,t_x),(y,t_y)]^{-r}_{\mathbb{H}^m} = (t_xt_y - (x,y))^{-r} = (\sqrt{1 + \|x\|^2} \sqrt{1 + \|y\|^2} - (x,y))^{-r} = (1 + \|x\|^2)^{-r/2} \left(1 - \frac{x}{\sqrt{1 + \|x\|^2}} \frac{y}{\sqrt{1 + \|y\|^2}}\right)^{-r} (1 + \|y\|^2)^{-r/2}.$$

Previous equation implies that $(x,t_x) \in \mathbb{H}^m \rightarrow [(x,t_x),(y,t_y)]^{-r}_{\mathbb{H}^m} \in \mathbb{R}$ belongs to $C_0(\mathbb{H}^m)$ for every $(y,t_y) \in \mathbb{H}^m$ because

$$[(x,t_x),(y,t_y)]^{-r}_{\mathbb{H}^m} \leq (1 + \|x\|^2)^{-r/2} \left(1 - \frac{||x||||y||}{\sqrt{1 + \|x\|^2} \sqrt{1 + \|y\|^2}}\right)^{-r} (1 + ||y||^2)^{-r/2},$$

Proposition 7.2 implies that $\mathcal{M}_{(-,\cdot)} \subset C_0(\mathbb{H}^m)$. By the homeomorphism $z = (x,t_x) \in \mathbb{H}^m \rightarrow x \in \mathbb{R}^m$, the kernel is $C_0$-universal if and only if the only measure $\lambda \in \mathcal{M}(\mathbb{R}^m)$ for which

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} (1 + \|x\|^2)^{-r/2} \left(1 - \frac{x}{\sqrt{1 + \|x\|^2}} \frac{y}{\sqrt{1 + \|y\|^2}}\right)^{-r} (1 + \|y\|^2)^{-r/2} d\lambda(x)d\lambda(y) = 0$$

is the zero measure. Since $|\langle x/\sqrt{1 + \|x\|^2}, y/\sqrt{1 + \|y\|^2} \rangle| < 1$ for every $x,y \in \mathbb{R}^m$, by the Taylor series of the hypergeometric functions $s \rightarrow (1 - s)^{-r}$, the following series is absolutely convergent for every $r \in \mathbb{R}$

$$\left(1 - \frac{x}{\sqrt{1 + \|x\|^2}} \frac{y}{\sqrt{1 + \|y\|^2}}\right)^{-r} = \sum_{k=0}^{\infty} \frac{(-r)^k}{k!} \left(\frac{x}{\sqrt{1 + \|x\|^2}} \frac{y}{\sqrt{1 + \|y\|^2}}\right)^k,$$
where \( \binom{a}{0} = 1, \binom{a}{1} = a \) and \( \binom{a}{k+1} = \frac{(a-k)(a)}{k} \) for every \( a \in \mathbb{R} \). So, if a finite measure \( \lambda \in \mathcal{M}(\mathbb{R}^m) \) is such that Equation 7.15 holds, then

\[
\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} (1 + \|x\|^2)^{-r/2} \left( \frac{x}{\sqrt{1 + \|x\|^2}}, \frac{y}{\sqrt{1 + \|y\|^2}} \right)^k (1 + \|y\|^2)^{-r/2} d\lambda(x) = 0
\]

for every \( k \in \mathbb{Z}_+ \), and consequently

\[
(7.16) \quad \int_{\mathbb{R}^m} x^\alpha (1 + \|x\|^2)^{-r/2 - |\alpha|/2} d\lambda(x) = 0
\]

for every \( \alpha \in \mathbb{Z}_+^m \). We claim that

\[
(7.17) \quad \int_{\mathbb{R}^m} x^\alpha (1 + \|x\|^2)^{-v - |\alpha|/2} d\lambda(x) = 0
\]

for every \( \alpha \in \mathbb{Z}_+^m \) and \( v > 0 \). To prove this relation we follow a similar path as the one we made at Equation 7.9 on the proof of Theorem 3.2. We already know that it holds for every \( \alpha \in \mathbb{Z}_+^m \) and \( v = r/2 \), our induction step is to prove that if it holds for every \( \alpha \in \mathbb{Z}_+^m \) and a \( u > 0 \), then it holds for every \( \alpha \in \mathbb{Z}_+^m \) and \( v \in (0, 2u) \). First, note that our induction hypothesis implies that

\[
\int_{\mathbb{R}^m} x^\alpha (1 + \|x\|^2)^{-u - |\alpha|/2} \left( \frac{\|x\|^2}{1 + \|x\|^2} \right)^k d\lambda(x) = 0
\]

for every \( \alpha \in \mathbb{Z}_+^m \) and \( k \in \mathbb{Z}_+ \). By the Taylor series expansion of the hypergeometric function \( s \to (1 - s)^{-u} \in \mathbb{R} \), the following series is absolutely convergent for every \( x \in \mathbb{R}^m \)

\[
(1 + \|x\|^2)^{u-v} = \left( 1 - \frac{\|x\|^2}{1 + \|x\|^2} \right)^{v-u} = \sum_{k=0}^{\infty} (-1)^k \binom{v-u}{k} \left( \frac{\|x\|^2}{1 + \|x\|^2} \right)^k,
\]

moreover the function \( x \in \mathbb{R}^m \to x^\alpha (1 + \|x\|^2)^{-v - |\alpha|/2} \in \mathbb{R} \) is bounded and also the function

\[
h(x) := \sum_{k=0}^{\infty} \left| x^\alpha (1 + \|x\|^2)^{-u - |\alpha|/2} (-1)^k \binom{v-u}{k} \left( \frac{\|x\|^2}{1 + \|x\|^2} \right)^k \right|.
\]

We separate the proof that the function \( h \) is bounded in two cases.

**Case 1:** When \( v < u \), the function \( h \) is bounded because \((-1)^k \binom{v-u}{k} \geq 0 \) for every \( k \in \mathbb{Z}_+ \), \( |x^\alpha (1 + \|x\|^2)^{-|\alpha|/2}| \leq 1 \) for every \( x \in \mathbb{R}^m \) and with these inequalities we obtain that \( h(x) \leq (1 + \|x\|^2)^{-u} (1 + \|x\|^2)^{v-u} = (1 + \|x\|^2)^{-v} \).

**Case 2:** When \( v > u \), let \( k_0 \in \mathbb{Z}_+ \) be such such that \( v - u - k_0 \geq 0 \) but \( v - u - k_0 - 1 < 0 \), then
\((-1)^k (1 + \|x\|^2)^{-u} h_k(x) \) \geq 0 for every \( k > k_0 \), so if \( h_k(x) := (1 + \|x\|^2)^{-u} (1 + \|x\|^2)^{-\alpha} \), then
\[
\begin{align*}
    h(x) &\leq \sum_{k=0}^{\infty} |h_k(x)| = \sum_{k=0}^{k_0} |h_k(x)| + (-1)^{k_0} \sum_{k=k_0+1}^{\infty} h_k(x) \\
    &= \sum_{k=0}^{k_0} |h_k(x)| + (-1)^{k_0} \left[ (1 + \|x\|^2)^{-u} (1 + \|x\|^2)^{-\alpha} - \sum_{k=0}^{k_0} h_k(x) \right] \\
    &\leq 2 \sum_{k=0}^{k_0} |h_k(x)| + (1 + \|x\|^2)^{-\alpha} ,
\end{align*}
\]
which proves that \( h \) is a bounded function because each \( h_k \) is a bounded function.
In particular, the Lebesgue dominated convergence Theorem implies that
\[
0 = \sum_{k=0}^{\infty} (-1)^k (1 + \|x\|^2)^{-u} \int_{\mathbb{R}^n} x^\alpha (1 + \|x\|^2)^{-u - |\alpha|/2} \left( \frac{\|x\|^2}{1 + \|x\|^2} \right)^k d\lambda(x) \\
= \int_{\mathbb{R}^n} x^\alpha (1 + \|x\|^2)^{-v - |\alpha|/2} d\lambda(x)
\]
which settles the proof of our claim. Now, consider the algebra of functions on \( C_0(\mathbb{R}^m) \)
\[
\mathcal{A} := \text{span}\{ x \in \mathbb{R}^m \rightarrow x^\alpha (1 + \|x\|^2)^{-v - \alpha/2} \in \mathbb{R}, \quad \alpha \in \mathbb{Z}^m_+, \quad \nu > 0 \}.
\]
The function \( h(x) = (1 + \|x\|^2)^{-1 - \alpha/2} \in \mathcal{A} \) is such that \( h(x) > 0 \) for every \( x \in \mathbb{R}^m \), also, the algebra \( \mathcal{A} \) separates points because if \( x^\alpha (1 + \|x\|^2)^{-1 - \alpha/2} = y^\alpha (1 + \|y\|^2)^{-1 - \alpha/2} \) for every \( \alpha \in \mathbb{R}^m \), then we must have that \( (x, t_x), (y, t_y) )_{\mathcal{F}_m} = 1 \), which can only occur if \( x = y \). By the Stone-Weierstrass Theorem the algebra of functions \( \mathcal{A} \) is dense on \( C_0(\mathbb{R}^m) \). Since our claim made at Equation (7.17) implies that for every \( h \in \mathcal{A} \), we have that \( \int_{\mathbb{R}^n} h(x) d\lambda(x) = 0 \), the measure \( \lambda \) must be the zero measure, which concludes the proof.

**Proof of Theorem 6.4** Since the kernel \( x, y \in \mathbb{H} \times \mathbb{H} \rightarrow [x, y] \in [1, \infty) \) is hyperbolic, \( \log[x, y] \) is a conditionally negative definite kernel. If \( 2 \log[x, y] = \log[x, x] + \log[y, y] \), then \( [x, y] = 1 \), which only occur when \( x = y \) because \( d_{\mathbb{H}}(x, y) = 0 \), implying that \( \log[x, y] \) is a metrizable kernel (note that the same property occurs on the kernel \( r \log[x, y] \), for \( r > 0 \)). Theorem 4.1 implies that the kernel
\[
(x, y) \in \mathbb{H} \times \mathbb{H} \rightarrow e^{-r \log[x, y]} = [x, y]^{-r} \in \mathbb{R}
\]
is universal (SPD) for every \( r > 0 \).

**Proof of Lemma 6.2** Suppose that \((i)\) holds, then for every \( \varepsilon > 0 \) there exists a compact set \( \mathcal{C}_\varepsilon \) for which \([z, \xi_0]^{-1} \leq \varepsilon \) for every \( x \in X \setminus \mathcal{C}_M \). By the monotone properties of the function \( \text{arccosh} \) this relation is equivalent at for every \( M > 0 \) there exists a compact set \( \mathcal{C}_M \) for which \( d_{\mathcal{F}}(z, \xi_0) \geq M \) for every \( z \in X \setminus \mathcal{C}_M \). Relation \((ii)\) follows by the inequality \( d_{\mathcal{F}}(z, \xi) \geq |d_{\mathcal{F}}(z, \xi_0) - d_{\mathcal{F}}(z, \xi_0)| \). The converse is immediate

Relations \((ii)\) and \((iii)\) are equivalent by Proposition 2.2.
If \((iv)\) holds, then for every \( \bar{\xi} \in X \) and \( M > 0 \), the set \( \{ x \in X, \quad d_{\mathbb{H}}(x, \bar{\xi}) \leq M \} \) is bounded and
closed on $X$, so it must be compact by the hypothesis implying that the function $H_{r_\xi} \in C_0(X)$. The converse relation follows by the same argument as the first one presented, so we omit it. \hfill $\square$

**Proof of Theorem 6.3.** Since the kernel $x, y \in X \times X \rightarrow [x, y] \in [1, \infty)$ is hyperbolic, $r \log[x, y]$ is a conditionally negative definite kernel that is metrizable for every $r > 0$. By the comments made at the Subsection 6.2, the Hilbertian metric on $X$ defined by the kernel $\log[\cdot, \cdot]$ is equivalent with the hyperbolic metric of $X$ inherited from $\mathbb{H}$, so $X_{r \log[\cdot, \cdot]}$ is a locally compact space and Theorem 4.2 implies that the kernel
\[(x, y) \in X_{r \log[\cdot, \cdot]} \times X_{r \log[\cdot, \cdot]} \rightarrow e^{-r \log[x, y]} = [x, y]^{-r} \in \mathbb{R}\]
is ISPD for every $r > 0$. The second claim on the Theorem is immediate. \hfill $\square$

Theorems 6.4, 6.5 and 6.6 are a direct consequence of the representation $L(x, y)^{-1} = e^{-\log L(x, y)}$ and Theorem 4.1, 4.2 and 4.4 respectively, so we omit the proof.

### 7.6. Dense algebras of bounded integrable functions on finite measures.

On this brief section we reprove the main result of [13], but under the assumption that the functions involved are Borel measurable instead of Baire measurable and the measure is Radon and finite instead of being $\sigma$-finite and Baire.

**Theorem 7.10.** Let $X$ be a locally compact Hausdorff space and $\lambda \in \mathcal{M}(X)$ be a nonzero nonnegative measure. Let $\mathcal{A}$ be an algebra of real valued functions in $L^1(\lambda)$ for which

(i) Every $h \in \mathcal{A}$ belongs to $L^\infty(X)$.

(ii) There exists $h \in \mathcal{A}$ for which $h(x) > 0$ almost everywhere on $\lambda$.

(iii) There exists a basis $(U_i)_{i \in \mathcal{I}}$ for the topology on $X$ such that if $U_i \cap U_j = \emptyset$ then for some $h_{i,j} \in \mathcal{A}$, $h_{i,j}(x) \geq 0$ for $x \in U_i$ and $h_{i,j}(x) \leq 0$ for $x \in U_j$.

Then the algebra $\mathcal{A}$ is dense on $L^1(\lambda)$

**Proof.** Relation (i) ensures that products of functions in $\mathcal{A}$ are elements of $L^1(\lambda)$ by the Holder’s inequality, which also implies that $\overline{\mathcal{A}}$ is an algebra on $L^1(\lambda)$

We show that $\mathcal{A}$ is dense in $L^1(\lambda)$ by showing that any continuous linear operator on $L^1(\lambda)$ that is zero on $\mathcal{A}$ is the zero operator. Indeed, let $I : L^1(\lambda) \rightarrow \mathbb{R}$ be a continuous operator that is zero on $\mathcal{A}$. Since $\lambda$ is finite there exists a function $\xi \in L^\infty(\lambda)$ for which

$I(\xi) = \int_X g(x) \xi(x) d\lambda(x) = \int_X g(x) \xi^+(x) d\lambda(x) - \int_X g(x) \xi^-(x) d\lambda(x)$. 

From this approach, we can assume that $\mathcal{A}$ is a closed vector space. By a similar argument as the one in Lemma 4.48 in [15] page 140, if $\phi, \psi \in \mathcal{A}$ then $\min(\psi, \phi), \max(\psi, \phi), \psi^+$ and $\psi^-$ belongs to $\mathcal{A}$.

We claim that the sets $X^+ := \{x \in X, \quad \xi(x) > 0\}$ and $X^- := \{x \in X, \quad \xi(x) < 0\}$ have $\lambda$ measure zero, which imply that $I$ is the zero functional. By the previous equality, it is sufficient to prove that $X^+$ has $\lambda$ measure zero.
By the inner regularity of \( \lambda \) on the sets \( X^+, X^- \), there exist two disjoint sequences of nested compact sets \( (\mathcal{C}_{+,n})_{n \in \mathbb{N}}, (\mathcal{C}_{-,n})_{n \in \mathbb{N}} \), for which
\[
\mathcal{C}_{+,n} \subset X^+, \lim_{n \to \infty} \lambda(\mathcal{C}_{+,n}) = \lambda(X^+), \quad \mathcal{C}_{-,n} \subset X^-, \lim_{n \to \infty} \lambda(\mathcal{C}_{-,n}) = \lambda(X^-).
\]
Since \( X \) is a Hausdorff space and the compact sets \( \mathcal{C}_{+,n} \) and \( \mathcal{C}_{-,n} \) are disjoint there exists disjoint open sets that separates them. Being the family of sets \( (U_i)_{i \in \mathcal{I}} \) from relation (iii) a basis for the topology on \( X \), for every \( n \in \mathbb{N} \) there exist finite sets \( F_{1,n}, F_{2,n} \subset \mathcal{I} \) such that
\[
\mathcal{C}_{+,n} \subset \bigcup_{i \in F_{1,n}} U_i, \quad \mathcal{C}_{-,n} \subset \bigcup_{j \in F_{2,n}} U_j
\]
and \( U_i \cap U_j = \emptyset \) if \( i \in F_{1,n} \) and \( j \in F_{2,n} \). The function \( h_n := \max_{i \in F_{1,n}} (\min_{j \in F_{2,n}} h_i, j(x)) \in \mathcal{A} \) and \( h_n(x) \leq 0 \) on \( \mathcal{C}_{-,n} \) and \( h_n(x) > 0 \) on \( \mathcal{C}_{+,n} \).

Since \( (\min(h_n, h))^+ \in \mathcal{A} \), we can suppose that \( 0 \leq h_n \leq h \), \( h(x) > 0 \) in \( \mathcal{C}_{+,n} \) and \( h(x) = 0 \) in \( \mathcal{C}_{-,n} \). The function \( k := \sup_{n \in \mathbb{N}} h_n \) is well defined and is an element of \( \mathcal{A} \), because the supremum over the set \( \{1, \ldots, m\} \) is an increasing sequence of functions (bounded by \( h \)) in \( \mathcal{A} \) and converges to \( k \) in \( L^1(\lambda) \) as \( m \) goes to infinity by the Lebesgue Dominated Convergence Theorem.

Note that \( k > 0 \) almost everywhere on \( X^+ \) and \( k = 0 \) almost everywhere on \( X^- \), however
\[
0 = I(k) = \int_X k(x) \xi^+(x) d\lambda(x) - \int_X k(x) \xi^-(x) d\lambda(x) = \int_X k(x) \xi^+(x) d\lambda(x),
\]
and \( k \xi^+ \geq 0 \), so \( k \xi^+ \) is the zero function on \( L^1(\lambda) \), which can only occur if \( \lambda(X^+) = 0 \). \( \square \)

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