LOGARITHMIC GOOD REDUCTION AND THE INDEX

by

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Abstract. — Let $K$ be the fraction field of a complete discrete valuation ring, with algebraically closed residue field of characteristic $p > 0$. This paper studies the index of a smooth, proper $K$-variety $X$ with logarithmic good reduction. We prove that it is prime to $p$ in 'most' cases, for example if the Euler number of $X$ does not vanish, but (perhaps surprisingly) not always. We also fully characterise curves of genus $1$ with logarithmic good reduction, thereby completing classical results of T. Saito and Stix valid for curves of genus at least $2$.

1. Introduction

1.1. Context. — Let $K$ be the fraction field of a complete discrete valuation ring, with algebraically closed residue field $k$ of characteristic $p$. Fix a separable closure $K^s$ of $K$, and denote by $K^t$ the maximal tamely ramified extension of $K$ contained in $K^s$. Let $X$ be a smooth, proper $K$-variety (all $K$-varieties are assumed to be geometrically integral over $K$).

The index $i(X)$ is the smallest positive degree of a zero-cycle on $X$, or equivalently, the greatest common divisor of the degrees of all finite field extensions $L/K$ for which $X(L) \neq \emptyset$. This is an important arithmetic invariant of $X$, which has been well studied; let us mention three interesting and recent contributions to the topic, in chronological order.

Gabber–Liu–Lorenzini show in [4] how, over arbitrary Henselian discretely valued fields, $i(X)$ can be computed from the special fibre of a proper, regular model $\mathcal{X}$ for $X$ over $\mathcal{O}_K$, the ring of integers of $K$. Their approach is based on intersection theory and moving lemmata.

Esnault–Levine–Wittenberg study the index in [3] using Euler characteristics of coherent sheaves on $X$. Among other results, they prove the statement that if $X$ is a $K$-variety which is rationally connected and if moreover $p = 0$ or $p > \dim X + 1$, then $i(X) = 1$.

In [8], Kesteloot–Nicaise introduce the specialisation index $i_{sp}(X)$, an invariant which refines the index. They explain how to compute it starting from a log regular $\mathcal{O}_K$-model $\mathcal{X}$. Their work suggests the following natural question (also mentioned in [18], Introduction).

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1.2. Question. — Assume that $p > 0$. It is well-known that every positive integer prime to $p$ appears as the index of some smooth, proper $K$-variety $X$ with logarithmic good reduction, i.e. with a proper, log smooth model $X$ over $\mathcal{O}_K$. Conversely, if $X$ is a smooth, proper $K$-variety with logarithmic good reduction, is $\iota(X)$ necessarily prime to $p$? This is equivalent to asking whether $X(K^t) \neq \emptyset$ if $X$ has logarithmic good reduction.

It may seem natural to expect an affirmative answer to this question, by analogy with Hensel’s lemma for smooth families in the classical sense. This note will show that this expectation is too naïve: the answer is affirmative in “most” cases, but not always.

We should mention that the existence of quasi-sections in logarithmic geometry has been studied in a more general setting by Nakayama in [13]; in this note, we sacrifice generality in order to obtain precise answers to the question in a concrete, geometric setting.

1.3. Results. — We obtain two main results: a positive statement which is rather general, and some specific examples showing that this positive statement is essentially optimal.

Our first main result (proven in §3) gives sufficient conditions for the existence of a $K^t$-point. Denote by $\chi(X)$ the $\ell$-adic Euler characteristic $\sum_{i \geq 0} (-1)^i \dim_{\mathbb{Q}} H^i(X \times_K K^t, \mathbb{Q}_\ell)$, where $\ell \neq p$ is prime; this is an integer which does not depend on the choice of $\ell$.

**Theorem 1.1.** — Let $X$ be a smooth, proper $K$-variety with logarithmic good reduction. Assume that $\chi(X) \neq 0$. Then $X(K^t) \neq \emptyset$, i.e. $\iota(X)$ is prime to $p$.

The topological assumption on the ($\ell$-adic) Euler number may seem a bit strange, but it turns out to be necessary since Theorem 1.1 breaks down when $\chi(X) = 0$, even in the case of curves. Indeed, our second main result (proven in §5) yields a full characterisation of curves of genus 1 with logarithmic good reduction, completing the results obtained previously by T. Saito and Stix for curves of genus at least 2 in [16, 17, 19].

To state our result, recall that the period of a curve $C$ of genus 1 over $K$ is defined as the order of the class of $C$ in $H^1(K, \text{Jac} C)$. Since we assume the residue field $k$ to be algebraically closed, the Brauer group of $K$ vanishes, and hence the period is actually equal to the index $\iota(C)$ by a result of Lichtenbaum [10, Theorem 1].

**Theorem 1.2.** — Let $C$ be a curve of genus 1 over $K$. Denote its period by $m$ and let $\mathcal{C}$ be its minimal proper regular model over $\mathcal{O}_K$. The curve $C$ has logarithmic good reduction if and only if both of the following conditions are satisfied:

(a) the Galois action on $H^1(C \times_K K^t, \mathbb{Q}_\ell)$ is tamely ramified;
(b) if $p \mid m$, then $\text{Jac} C$ has good reduction and $\mathcal{C}$ is cohomologically flat over $\mathcal{O}_K$.

Recall that the (by now classical) results of T. Saito and Stix say that a curve $C$ of genus at least 2 over $K$ has logarithmic good reduction if and only if the Galois action on $H^1(C \times_K K^t, \mathbb{Q}_\ell)$ is tamely ramified. The same result is known to hold for elliptic curves, but condition (b) in our theorem shows that the situation is more delicate for curves of genus 1 which do not have a rational point. The whole crux of Theorem 1.2 is of course that there really exist curves $C$ of genus 1 over $K$ satisfying both conditions (a) and (b) and for which $C(K^t) = \emptyset$ (or equivalently, $p \mid m$): this follows from [15, 9.4.1.(iii)]. For the construction of such examples, the interested reader can consult the work of Katsura–Ueno [7] and Harbourne–Lang [6] on so-called tame and wild fibres of elliptic surfaces in positive characteristic; the construction in [7, Example 4.9] is particularly elegant.
1.4. Structure of the paper. — In §2 we will give a simple geometric description, using logarithmic differential forms, for the log smooth locus of a log regular scheme over a discrete valuation ring. This description will subsequently be used in §3 to prove that the geometry of a log smooth degeneration without any tamely ramified quasi-sections must be strongly restricted. Based on these considerations, we then prove Theorem 1.1.

In §4 we define an invariant which, still in the spirit of the criterion from §2, allows to measure the “defect” of log smoothness in the absence of tamely ramified quasi-sections. This proves useful in §5 for the trickier cases of the study of curves of genus 1, that is, for the study of torsors under elliptic curves with good reduction: it allows us to separate those torsors with logarithmic good reduction from those without.

1.5. Notation and conventions. — For generalities on logarithmic geometry, we refer to the foundational papers by Kazuya Kato, or to [13, §2] for a short summary of the notions needed in this paper. We will only use Zariski log structures and fs log schemes.

Given the spectrum $S$ of a discrete valuation ring, with generic point $\eta$ and closed point $s$, and a flat $S$-scheme $\mathcal{X}$, we denote by $\mathcal{X}^\dagger$ the log scheme obtained by equipping $\mathcal{X}$ with the natural log structure induced by the special fibre $\mathcal{X}_s$ (this log scheme is not fs in general). If $U$ denotes the generic fibre of $\mathcal{X} \rightarrow S$, and if $j : U \rightarrow \mathcal{X}$ denotes the corresponding open immersion, then this log structure is given by the inclusion $\mathcal{M}_\mathcal{X} := \mathcal{O}_\mathcal{X} \cap j_* \mathcal{O}_U^\circ \hookrightarrow \mathcal{O}_\mathcal{X}$. A scheme $\mathcal{X}$ smooth and proper over $\eta$ is said to have logarithmic good reduction if there exists a proper model $\mathcal{X}$ for $\mathcal{X}$ over $S$ with the property that $\mathcal{X}^\dagger$ is log smooth over $S^\dagger$.

Given a monoid $P$, we denote by $P^{\text{gp}}$ its group envelope, by $P^\times$ the subgroup of invertible elements, and by $P^\sharp$ the associated sharp monoid $P/P^\times$.

2. The log smooth locus of a log regular model

Let $R$ be a discrete valuation ring, with fraction field $K$ and perfect residue field $k$ of characteristic $p \geq 0$. Let $\pi$ be a uniformiser of $R$. Let $S = \text{Spec} R$, with generic point $\eta$ and closed point $s$. Let $\mathcal{X}$ be a flat $S$-scheme of finite type, with smooth and geometrically integral generic fibre. Assume that $\mathcal{X}^\dagger$ is log regular. The goal of this section is to understand the locus of points on $\mathcal{X}_s$ where the induced morphism $\mathcal{X}^\dagger \rightarrow S^\dagger$ of log schemes is log smooth.

Denote the Kato fan associated to $\mathcal{X}^\dagger$ by $F(\mathcal{X}^\dagger)$. There is a continuous map of monoidal spaces $\Pi : \mathcal{X}^\dagger \rightarrow F(\mathcal{X}^\dagger)$, which determines a standard stratification of $\mathcal{X}$ into finitely many locally closed subsets. Given $p \in F(\mathcal{X}^\dagger)$, denote by $U_p = \Pi^{-1}(\{p\})$ the corresponding locally closed subset of $\mathcal{X}$, equipped with the reduced subscheme structure. Each locally closed stratum $U_p$ is smooth, and its Zariski closure $\overline{V}_p$ with the reduced subscheme structure is normal. The boundary $\partial V_p = V_p \setminus U_p$ is a Weil divisor on $V_p$, and we equip $V_p$ with the divisorial log structure induced by this boundary divisor; then it becomes again log regular.

We denote by $\mathcal{M}_\mathcal{X} \hookrightarrow \mathcal{O}_\mathcal{X}$ the log structure on $\mathcal{X}^\dagger$. Take $p \in F(\mathcal{X}^\dagger)$. If $p$ is the generic point of $F(\mathcal{X}^\dagger)$, we simply set $m_p^\sharp = 1$. Otherwise, we denote by $m_p^\sharp$ the largest positive integer such that the image of 1 under the natural map $N \rightarrow \mathcal{M}_\mathcal{X}^\dagger$ is divisible by $m_p^\sharp$.

**Definition 2.1.** — Define $F_p(\mathcal{X}^\dagger)$ (resp. $F_p(\mathcal{X}^\dagger)$) to be the subset of $F(\mathcal{X}^\dagger)$ consisting of those $p$ such that $m_p^\sharp$ is divisible (resp. not divisible) by $p$. Let

$$X_p = \Pi^{-1}(F_p(\mathcal{X}^\dagger)) \quad \text{and} \quad X_p' = \Pi^{-1}(F_p(\mathcal{X}^\dagger)).$$

We refer to $X_p$ as the $p$-locus of $\mathcal{X}$, and to $X_p'$ as the $p'$-locus of $\mathcal{X}$. 

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Note that by the definition of the integers \( n_p \), above, the \( p' \)-locus \( X_{p'} \) contains the generic fibre of \( X \to S \), and therefore its complement \( X_p \) is contained in the special fibre.

**Remark 2.2.** — With notation as above, assume that \( p \) and \( q \) are points of \( F(X^\dagger) \), not equal to the generic point, such that \( p \) specialises to \( q \). If \( q \in F_p(X^\dagger) \), then also \( p \in F_p(X^\dagger) \). This means precisely that the \( p' \)-locus \( X_{p'} \) cuts out a closed subset of the special fibre.

As we will see shortly, \( X^\dagger \to S^\dagger \) is log smooth at all points of \( X_{p'} \), but for points of \( X_p \), the situation is less clear; our goal is to pin down the subset of \( X_p \) where \( X^\dagger \to S^\dagger \) is log smooth using a criterion involving differential forms. We will need the following notion:

**Definition 2.3.** — With notation as above, let \( p \in F_p(X^\dagger) \) be an arbitrary point. We define the \( p' \)-boundary \( \partial_{p'} V_p \) of \( V_p \) as the Weil divisor \( V_p \cap X_{p'} \) on \( V_p \); this is a subset of \( \partial V_p \).

Since \( V_p \) is log regular, or equivalently since \( V_p \) is log smooth over \( \Spec k \) equipped with the trivial log structure, the sheaf \( \Omega_{V_p/\Spec k}^1(\log \partial V_p) \) of differential forms on \( V_p \) with at most logarithmic poles along the boundary \( \partial V_p \) is a vector bundle on \( V_p \).

We will also consider the subsheaf

\[
\Omega_{V_p/k}^1(\log \partial_P V_p) \to \Omega_{V_p/k}^1(\log \partial V_p)
\]

consisting of differentials with at most logarithmic poles along the \( p' \)-boundary \( \partial_{p'} V_p \subseteq \partial V_p \).

We will show that for each \( p \in F_p(X^\dagger) \), there exists a canonical logarithmic 1-form

\[
\omega_p \in H^0(V_p, \Omega_{V_p/k}^1(\log \partial_P V_p)),
\]

the zero locus of which cuts out precisely the locus consisting of those points on \( V_p \) where \( X^\dagger \to S^\dagger \) is not log smooth. This generalises (in several ways) an idea used in \([13] \S 5\).

**Proposition 2.4.** — Let \( X \) be as above, let \( x \in X_p \), and let \( p \in F(X^\dagger) \) such that \( x \in U_p \).

1. If \( x \) belongs to the \( p' \)-locus \( X_{p'} \), then \( X^\dagger \to S^\dagger \) is log smooth at \( x \).
2. Assume that \( x \) belongs to the \( p \)-locus \( X_p \). Then there exist functions \( u, f \in A \) on an affine neighbourhood \( W = \Spec A \) of \( x \) such that \( u \) is a unit at \( x \) and \( \pi = uf^p \). Denote by \( I \) the ideal of \( A \) which defines \( V_p \cap W \) and by \( \pi \) the image of \( u \) in \( A/I \).

The 1-form \( \log \pi \) extends to a section \( \omega_p \in H^0(V_p, \Omega_{V_p/k}^1(\log \partial_P V_p)) \). Moreover, the morphism \( X^\dagger \to S^\dagger \) is log smooth at \( x \) if and only if \( \omega_p \) does not vanish at \( x \).

**Proof.** — The map \( N \to R \) given by \( 1 \mapsto \pi \) yields a chart for the log structure on \( S^\dagger \). Consider a chart for \( X^\dagger \to S^\dagger \) around \( x \), consisting of an affine neighbourhood \( W = \Spec A \) of \( x \), an fs monoid \( P \) and homomorphisms \( \varphi : N \to P \) and \( \varphi : P \to A \) for which \( \varphi(v(1)) = \pi \).

The inverse image under \( \varphi \) of the prime ideal of \( A \) which defines \( x \) is a prime ideal \( q \) of \( P \).

The complement \( P \setminus q \) is a face, so \( P/(P \setminus q) \) is a sharp fs monoid; saying that \( p \in F_p(X^\dagger) \) is equivalent to saying that the image of 1 under \( N \to P \to P/(P \setminus q) \) is divisible by \( p \).

Therefore if \( p \in F_p(X^\dagger) \), the cokernel of \( \varphi^{\text{sp}} : N^{\text{sp}} \to P^{\text{sp}} \) does not have \( p \)-torsion, simply because the cokernel of the composition \( N^{\text{sp}} \to P^{\text{sp}}/(P \setminus q)^{\text{sp}} \) does not have any. Hence Kato’s criterion implies that \( X^\dagger \to S^\dagger \) is log smooth at \( x \). This settles (1).

Let us now prove (2). Since \( p \in F_p(X^\dagger) \), there exists an element \( c \in P \setminus q \) such that \( v(1) - c \) is divisible by \( p \) in \( P^{\text{sp}} \). Therefore \( v(1) - c = pa \) for some \( a \in P^{\text{sp}} \). Take \( u = \varphi(c) \) and \( f = \varphi(a) \). Then \( u \) is a unit in \( O_{X,x} \) and \( \pi = uf^p \). The ideal \( I \) defining \( V_p \) is \( (q) \), the ideal generated by \( \varphi(q) : q \in q \). We consider the divisorial log structure on \( V_p \) induced by its boundary \( \partial V_p \); locally on \( V_p \cap W \), the map \( p : P \setminus q \to A/(q) \) gives a chart.
We have a natural map

\[ A/(q) \otimes_{\mathbb{Z}} (P \setminus q)^{gp} \to \left( \Omega^1_{V_p/k}(\log \partial V_p) \right) (V_p \cap W) : \pi \otimes t \mapsto \pi \log \varphi(t) \]

and the logarithmic 1-form \( \log \varphi \) is the image of \( \mathbf{T} \otimes c \) under this map. Choosing any other element \( c' \in P \setminus q \) such that \( v(1) - c' \) is divisible by \( p \) would yield the same 1-form, since \( c - c' \) is divisible by \( p \) in \( (P \setminus p)^{gp} \) and hence, since \( \mathbf{T} = 0 \) in \( A/(q) \),

\[ \mathbf{T} \otimes (c - c') = 0 \quad \text{in} \quad A/(q) \otimes_{\mathbb{Z}} (P \setminus q)^{gp}. \]

Therefore our construction is unambiguous.

The 1-form which we just defined on \( W = \text{Spec} A \) extends to a section

\[ \omega_p \in H^0(V_p, \Omega^1_{V_p/k}(\log \partial V_p)). \]

Indeed, the same construction can be carried out around any other point on \( V_p \) and the 1-forms defined locally in this way then automatically glue to a global log 1-form: given any other affine neighbourhood \( W' = \text{Spec} A' \) and a factorisation \( \pi = u'(f')p \) on \( W' \), we see that \( u f^p = u'(f')p \) on \( W \cap W' \) and hence \( \log \varphi = \log \varphi' \) on \( (W \cap W') \cap V_p \), since \( p = 0 \) on \( V_p \).

Let us check that \( \omega_p \) cannot have poles away from the \( p' \)-boundary \( \partial_{p'} V_p \). Locally on the affine neighbourhood \( W \), an irreducible component \( D \) of \( \partial V_p \) is defined by the ideal \( (q') \) of \( A \), where \( q' \) is a prime ideal of \( P \) which strictly contains \( q \). If \( q' \in \mathcal{P}_p(\mathcal{X}^t) \), there exists an element \( c \) in the submonoid \( P \setminus q' \) of \( P \setminus q \) such that \( p \) divides \( v(1) - c \). The local description of \( \omega_p \) given above then shows that \( \omega_p \) is actually holomorphic along \( D \). It follows that the only (potential) poles of \( \omega_p \) lie along the \( p' \)-boundary \( \partial_{p'} V_p \).

It remains to prove the last assertion of (2). We keep the notation used above. A slight refinement of Kato’s criterion (see e.g. [5, Theorem 12.3.37, Corollary 12.3.42]) says that the morphism \( \mathcal{X}^t \to \mathcal{S}^t \) is log smooth at \( x \) if and only if one can choose, étale locally around \( x \), an affine neighbourhood \( W = \text{Spec} A \), a toric monoid \( P \) and morphisms \( v : \mathcal{N} \to P \) and \( \varphi : P \to \Lambda \) for which \( \varphi(v(1)) = \pi \), such that the induced map

\[ \Phi : W \to \text{Spec} \mathcal{R}[P]/(\pi - v(1)) \]

is étale and the cokernel of \( v^{sp} : \mathcal{N}^{sp} \to \mathcal{P}^{sp} \) does not have \( p \)-torsion. However, a chart satisfying all of these conditions, except for possibly the last one, can always be found as soon as \( \mathcal{X}^t \) is log regular. Let us therefore show that for such a chart, the non-vanishing of \( \omega_p \) at \( x \) is equivalent to the fact that \( \text{coker} v^{sp} \) does not have \( p \)-torsion.

We take \( q \) as before, and we again choose \( c \in P \setminus q \) such that \( p \) divides \( v(1) - c \). Since \( \Phi \) is étale by assumption, so is the base change

\[ V_p \cap W \to \text{Spec} k[P \setminus q]. \]

It follows that in this case, the logarithmic 1-forms on \( V_p \cap W \) are given by

\[ \left( \Omega^1_{V_p/k}(\log \partial V_p) \right) (V_p \cap W) \cong A/(q) \otimes_{\mathbb{Z}} (P \setminus q)^{gp} \]

and the restriction of \( \omega_p \) to \( V_p \cap W \) corresponds to \( \mathbf{T} \otimes c \). Now \( \omega_p \) vanishes at \( x \) if and only if \( \mathbf{T} \otimes c = 0 \) in \( \kappa(x) \otimes_{\mathbb{Z}} (P \setminus q)^{gp} \),

where \( \kappa(x) \) is the residue field of the local ring \( \mathcal{O}_{X,x} \). This happens if and only if \( c \) is divisible by \( p \) in \( (P \setminus q)^{gp} \), which is equivalent to \( v(1) \) being divisible by \( p \) in \( P^{sp} \), i.e. to \( \text{coker} v^{sp} \) having \( p \)-torsion. Hence the vanishing of the differential form \( \omega_p \) at \( x \) is equivalent to the non-existence of a suitable Kato chart at \( x \). This proves (2). \( \square \)
Remark 2.5. — Let $\mathcal{X}$ be as above, and assume that $\mathcal{X}_p = \mathcal{X}_s$. Let $D = \mathcal{X}_p/p$, which is a Cartier divisor on $\mathcal{X}$ under our assumptions. What the proof of Proposition 2.4 actually gives in this specific case is the existence of a global section of $\Omega^1_{D/k}$, which, for each $p \in F_p(\mathcal{X}^\dagger)$, reduces to $\omega_p$ under the natural restriction map $\Omega^1_{D/k} \to \Omega^1_{\mathcal{V}_p/k}$.

3. Existence of tame points

We keep the notation from the previous section. We will present sufficient geometric conditions for the existence of a $K^t$-point on smooth, proper $K$-varieties with logarithmic good reduction (in particular, we give a proof of Theorem 1.1). We need the following fact:

Lemma 3.1. — Let $X$ be a smooth, proper $K$-variety, and let $X$ be a proper model for $X$ over $S$ such that $X^\dagger$ is log regular. Then $X(K^t) \neq \emptyset$ if and only if $X_p \neq X_s$.

This result does not seem to be available in the literature in this precise form, but it immediately follows from [8, Corollary 2.7] or from the much more general [4, Theorem 8.2]. Indeed, desingularising a (potentially singular) log regular model $X$ using suitably chosen log blow-ups will yield a proper regular model with strict normal crossings.

The following result generalises [18, Proposition 5.1].

Proposition 3.2. — Let $X$ be a smooth, proper $K$-variety, and let $X$ be a proper model for $X$ over $S$ such that $X^\dagger$ is log smooth over $S^\dagger$. Then for every point $p \in F_p(\mathcal{X}^\dagger)$, the Euler number $\chi(U_p)$ vanishes. In particular, $\chi(X_p) = 0$.

Proof. — Since $V_p$ is proper, the Euler number $\chi(U_p)$ — by which we mean the $\ell$-adic Euler characteristic, for any prime $\ell \neq p$ — can be computed as the positive or negative degree of the top Chern class of the vector bundle $\Omega^1_{V_p/k}(\log \partial V_p)$ on $V_p$; the sign depends on the dimension. This statement is well-known for smooth varieties with a strict normal crossings boundary. The log regular case then follows by means of a desingularisation by log blow-ups. Indeed, such a desingularisation does not change the interior of the log scheme, whence the Euler number remains the same. On the other hand, since the desingularisation map is log étale, it induces an isomorphism at the level of sheaves of logarithmic differentials.

Since $p \in F_p(\mathcal{X}^\dagger)$, Proposition 2.3 (2) now implies that $\Omega^1_{V_p/k}(\log \partial V_p)$ admits a nowhere vanishing global section, and hence its top Chern class vanishes; this then yields the first statement. The last statement follows immediately from the first one since $X_p$ is the disjoint union of all locally closed strata $U_p$, where $p$ ranges over $F_p(\mathcal{X}^\dagger)$.

The following corollary illustrates the fact that log smooth degenerations without any tamely ramified quasi-sections must show geometric behaviour which is strongly restricted; for example, there cannot be any maximally degenerate points on the special fibre:

Corollary 3.3. — With the notation of Proposition 3.2, assume that $X(K^t) = \emptyset$ holds, and let $p \in F(\mathcal{X}^\dagger)$ be any point different from the generic point of $F(\mathcal{X}^\dagger)$.

The closed stratum $V_p$ cannot be zero-dimensional. If $V_p$ is one-dimensional, then it is a curve of genus 1. If $V_p$ is two-dimensional, then it is not of general type.

Proof. — If $X(K^t) = \emptyset$, then $X_p = \mathcal{X}_s$ by Lemma 3.1 (and vice versa). Therefore every point of $F(\mathcal{X}^\dagger)$ different from the generic point belongs to the $p$-locus of $\mathcal{X}$. 
If $\dim V_p = 0$, then $\chi(V_p) \neq 0$, which contradicts Proposition 3.2 hence such a point $p$ cannot exist. If $\dim V_p = 1$, then (since obviously $\partial_p V_p = \emptyset$) it follows from Proposition 3.2 that $V_p$ must be a curve of genus 1. Similarly, if $\dim V_p = 2$, then since again $\partial_p V_p = \emptyset$, Proposition 2.4 implies that $V_p$ carries a nowhere vanishing holomorphic 1-form. It then follows from [9, Theorem 5.1] that $V_p$ is not of general type.

We are now also able to prove Theorem 1.1:

**Proof of Theorem 1.1.** — The assumptions of the theorem and [12, Corollary 0.1.1] together imply that the wild inertia of $\text{Gal}(K_s/K)$ acts trivially on $H^*(X \times_K K^s, \mathbb{Q}_\ell)$. Hence

$$\chi(X) = \chi^{\text{tame}}(X) := \sum_{i \geq 0} (-1)^i \dim \mathbb{Q}_\ell H^i(X \times_K K^s, \mathbb{Q}_\ell).$$

The right hand side can be computed by means of Nakayama’s description of nearby cycles [12, Theorem 3.5]. Computations of this type have been carried out by Nicaise in [14] and also by the second named author in [18, §3], where the tame monodromy zeta function was calculated. Since $k$ is algebraically closed, the group $\text{Gal}(K^t/K)$ is procyclic, topologically generated by an element $\varphi$. The tame monodromy zeta function is given by

$$\zeta_X^{\text{tame}}(t) = \prod_{m \geq 0} \det (t \cdot \text{Id} - \varphi | H^m(X \times_K K^t, \mathbb{Q}_\ell))^{(-1)^{m+1}}$$

and does not depend on the choice of generator $\varphi$ (as will be clear from (1)). Given a point $p \in F(X^t)$ of codimension 1, denote by $m_p'$ the biggest prime-to-$p$ divisor of the multiplicity $m_p$ of the corresponding component of $X_s$. Then [18] Corollary 3.8 yields

$$\zeta_X^{\text{tame}}(t) = \prod_p \left(t^{m_p'} - 1\right)^{-\chi(U_p)},$$

where the product ranges over all points of $F(X^t)$ of codimension 1. Hence we obtain

$$\chi^{\text{tame}}(X) = \sum_p m_p' \chi(U_p),$$

simply by taking negative degrees on both sides of (1). As before, the equalities $X(K^t) = \emptyset$ and $X_s = X_s$ are equivalent; if these hold, then Proposition 3.2 says that the right hand side of the equality (2), and hence also $\chi(X)$, vanishes. This proves the theorem.

**4. A numerical criterion**

As in the previous paragraphs, we assume that $X$ is a smooth, proper $K$-variety and that $X$ is a proper, flat model for $X$ over $S$ such that $X^t$ is log regular. Let $m = \gcd_{p \in F(X^t)} (m_p^t)$. The special fibre $X_s$ considered as a Weil divisor on $X$ may be written as

$$X_s = \sum_{p \in F(X^t)} m_p^t V_p,$$

where $F(X^t)^{(1)}$ denotes the set of points of codimension 1 in $F(X^t)$.

Let $n_p = m_p^t/m$ for all $p \in F(X^t)$ and consider the Weil divisor

$$D = \sum_{p \in F(X^t)^{(1)}} n_p V_p.$$
Since $X^+$ is log regular, $D$ is actually Cartier. Indeed, consider an affine open subscheme $U = \text{Spec } A$ of $X$ and a chart for the log structure consisting of an fs monoid $P$ and maps $v : N \to P$ and $\varphi : P \to A$ such that $\varphi(v(1)) = \pi$. Let $\varphi^* : P^\sharp \to A^\sharp$ be the associated homomorphism between sharp monoids. Since the image of $v(1)$ in $P^\sharp$ is divisible by $m$, we see that $\varphi^*(v(1)) \in A^\sharp$ is the $m$-th power of an element of $A^\sharp$ which defines $D$ on $U$.

We denote by $E$ the reduction of $X_n$, and let $\mathcal{L} = \mathcal{O}_X(D)|_E$. As an element of $\text{Pic } E$, the line bundle $\mathcal{L}$ has finite order dividing $m$. Indeed, $\mathcal{L}^\otimes m$ is the trivial line bundle since $mD = X_n$ and $\mathcal{O}_X(X_n) \cong \mathcal{O}_X$. Let us denote the order of $\mathcal{L}$ by $\mu$.

**Lemma 4.1.** — The ratio $m/\mu$ is a power of $p$.

**Proof.** — Let

$$I = \mathcal{O}_X(-D), I_0 = \sqrt{I}, I_n = I^n \quad \text{and} \quad N_n = I_n/I_{n+1}$$

for each $n \in \mathbb{Z}_{>0}$. We denote by $Y_n$ the closed subscheme of $X$ defined by $I_n$, and by $\mu_n$ the order of $\mathcal{O}_X(D)|_{Y_n}$ in $\text{Pic } Y_n$. Then $Y_0 = E$, and $\mu_0 = \mu$. For each $n \geq 0$, the exact sequence of sheaves of abelian groups on $X$ given by

$$0 \longrightarrow 1 + N_n \overset{\alpha}{\longrightarrow} \mathcal{O}_{Y_{n+1}}^\times \overset{\beta}{\longrightarrow} \mathcal{O}_{Y_n}^\times \longrightarrow 0$$

yields the exact sequence of abelian groups

$$H^1(Y_{n+1}, 1 + N_n) \longrightarrow \text{Pic } Y_{n+1} \overset{\alpha}{\longrightarrow} \text{Pic } Y_n.$$

Since $1 + N_n$ is $p$-torsion, the abelian group $H^1(Y_{n+1}, 1 + N_n)$ is $p$-torsion as well. It follows that the kernel of $\alpha$ is $p$-torsion, which implies that $\mu_{n+1} \mid p \mu_n$. If $r \in \mathbb{Z}$ is sufficiently large, then $\mu_r = m$ by [15] Lemme 6.4.4. Since $\mu_0 = \mu$, the result follows.

**Proposition 4.2.** — With notation as above, the following conditions are equivalent:

1. there exists a point $x$ on $X_n$ such that $X^+ \to S^+$ is log smooth at $x$;
2. the equality $m = \mu$ holds.

We need a preliminary result. Take an affine covering $(X_i)_{i \in I}$ of $X$ with the property that $\mathcal{O}_X(D)|_{X_i} \cong \mathcal{O}_{X_i}$ for all $i \in I$. Let $t_i$ be a defining function of $D$ on $X_i$ and take $u_i \in \mathcal{O}_X^\times(X_i)$ such that $\pi = u_i t_i^m$. Let $E_i = E \cap X_i$ and $t_i = u_i |_{E_i}$.

Furthermore, given $i, j \in I$, let $X_{ij} = X_i \cap X_j$ and take $t_{ij} \in \mathcal{O}_X^\times(X_{ij})$ such that $t_{ij} t_i = t_j$ on $X_{ij}$. Then $(t_{ij})_{i, j \in I}$ is a transition function for $\mathcal{O}_X(D)$. Let $E_{ij} = E \cap X_{ij}$ and $t_{ij} = t_{ij}|_{E_{ij}}$. Since $\mathcal{L}^\otimes p$ is trivial, we may take $(v_i \in \mathcal{O}_E^\times(E_i))_{i \in I}$ such that $(t_{ij})^p v_j = v_i$ on $E_{ij}$.

**Lemma 4.3.** — There exists $a \in \mathcal{O}_E^\times(E) = k^\times$ such that $\overline{\pi} = av_i^{m/\mu}$ on $E_i$ for all $i \in I$. In particular, if $m \neq p$, then $\overline{\pi} = 0$ in $\Omega_{k(E)/k}^1(E_i)$ for all $i \in I$.

**Proof.** — Since $t_{ij}^m u_j = u_i |_{X_{ij}}$ and $(t_{ij})^m v_j = v_i |_{E_{ij}}$, the equality

$$v_i^{m/\mu} / \overline{\pi} = v_j^{m/\mu} / \overline{\pi}$$

holds on $E_{ij}$ for all pairs of indices $i, j \in I$. Thus, there exists $a \in \mathcal{O}_E^\times(E) = k^\times$ such that $\overline{\pi} = av_i^{m/\mu}$ on $E_i$ for all $i \in I$, which proves the first statement. The second statement follows immediately from the first one and Lemma 4.4.
Proof of Proposition 4.2 — If $p$ does not divide $m$, then both statements are clearly true, the first one by Proposition 2.4(1) and the second one by Lemma 4.1.

Assume that $p$ divides $m$ (that is, $X_p = X_s$) and that (1) holds. Then (2) must hold as well: if not, then Lemma 4.3 and Proposition 2.4(2) would contradict each other.

Conversely, assume that (2) holds. By Proposition 2.4(2), it suffices to show that $d | a_i$ for some $i$. Assume for a moment that $d | a_i$, then both statements are clearly true, and generalises to the more general case where $X$ is a strict normal crossings divisor on $X$, and generalises to the more general case where $X$ is log regular using a simple polyhedral argument.

Since $L^{(i)}$ holds on $E_{ij}$, since $E$ is reduced, this implies that $L^{(ij)} = v_j$ on $E_{ij}$, and hence that $L^{j/|P|}$ is trivial. This contradicts our assumption that $m = \mu$. □

5. Logarithmic good reduction of curves of genus 1

In this final section, we will prove Theorem 1.2. For simplicity, we will assume the residue field $k$ to be algebraically closed (as we did in the introduction). Recall that if $X$ is a curve of positive genus over $K$, then among all proper regular models for $X$ over $R$ with strict normal crossings special fibre, there is a minimal one, the minimal sncd model.

Let us first show that to verify whether a given curve (of positive genus) has logarithmic good reduction, it suffices to consider this particular model.

Lemma 5.1. — Let $X$ be a smooth, proper curve of positive genus over $K$. If $X$ has logarithmic good reduction and if $X$ is its minimal sncd model, then $X \rightarrow S$ is log smooth.

Proof. — Let $X_1$ be a proper model for $X$ over $R$ such that $X_1 \rightarrow S$ is log smooth. There exists a desingularisation by log blow-ups $X_2 \rightarrow X_1$ such that the underlying scheme $X_2$ is regular. Then $X_2 \rightarrow S$ is still log smooth since log blow-ups are log étale.

By minimality, we get a morphism $X_2 \rightarrow X$ to the minimal sncd model. This morphism successively contracts $(-1)$-curves which intersect at most two other components of the special fibre, each of them in one point. If such a $(-1)$-curve meets two other components, the corresponding contraction is a log blow-up and preserves log smoothness. If it meets only one component, the contraction map is no longer a log blow-up. However, the multiplicities of the $(-1)$-curve and of the component which it intersects must be prime to $p$: if not, the
original morphism would not be log smooth at the intersection point by Corollary \ref{cor:log-smooth}. Hence the contraction again preserves log smoothness and $X^\dagger \to S^\dagger$ is log smooth, as required.

Let us first deal with the simpler cases of Theorem \ref{thm:log-smooth}. We denote by $m$ the period of $C$ and by $J$ its Jacobian. Let $J$ (resp. $C$) be the minimal sncd model of $J$ (resp. $C$) over $R$ with special fibre $J_s$ (resp. $C_s$). The existence of a Gal$(K^s/K)$-equivariant isomorphism

$$H^1(C \times_K K^s, \mathbb{Q}_\ell) \cong H^1(J \times_K K^s, \mathbb{Q}_\ell)$$

shows that $H^1(J \times_K K^s, \mathbb{Q}_\ell)$ is also tamely ramified. Since $J$ is an elliptic curve, $J_s$ satisfies Saito’s criterion \cite[Theorem 3.11]{saito-1998}, i.e. each component with multiplicity divisible by $p$ is a copy of $\mathbb{P}^1_k$ intersecting exactly two other components, with multiplicities prime to $p$. Hence $C_s$ satisfies the same property: indeed, it is a consequence of \cite[Theorem 6.6]{saito-1998} that the weighted dual graph of $C_s$ is obtained by multiplying the corresponding graph for $J_s$ by the integer $m$, which is prime to $p$ (see Remark \ref{rem:weighted-dual} below). This yields the result.

**Remark 5.3.** — We should note that \cite[Theorem 6.6]{saito-1998} deals with the types associated to the minimal regular models instead of the minimal sncd models considered here; however, it is not hard to see that the conclusion of [loc. cit.] remains valid for the minimal sncd models and the weighted dual graphs of their special fibres.

Hence the only remaining cases of Theorem \ref{thm:log-smooth} are those where the period $m$ is divisible by $p$. The case where the Jacobian has bad reduction is not difficult either:

**Lemma 5.4.** — Let $C$ be a smooth, proper curve of genus 1 over $K$. If its Jacobian $J$ has bad reduction, then $C$ does not have logarithmic good reduction.

**Proof.** — Let $J$ (resp. $C$) be the minimal sncd model of $J$ (resp. of $C$). Theorem \ref{thm:log-smooth} implies that it suffices to check that the morphism $X^\dagger \to S^\dagger$ is not log smooth. This follows from the combination of \cite[Theorem 6.6]{saito-1998} (applied to the minimal sncd model, see Remark \ref{rem:weighted-dual}) and Lemma \ref{lem:log-smooth}. Indeed, the morphism is certainly not log smooth at any point where two components of $C_s$ intersect, since the Euler number of a point is non-zero.

We can now finish the proof of Theorem \ref{thm:log-smooth} as follows.

**End of Proof of Theorem \ref{thm:log-smooth}.** — The only case which remains is the one where the period $m$ of $C$ is divisible by $p$ and the Jacobian $J$ has good reduction. Again considering the minimal sncd models, \cite[Proposition 8.1]{saito-1998} says that the $S$-group scheme $J$ acts on $C$, inducing a transitive action of $J(k)$ on $C(k)$. It follows that on the special fibre, the morphism $C^\dagger \to S^\dagger$ is either everywhere log smooth, or nowhere at all.

The reduction $D$ of the special fibre $C_s$ is isomorphic to an elliptic curve. Denote by $\mu$ the order of $N_{D/C}$ in Pic $D$. Using the above observation and Proposition \ref{prop:log-smooth} we see that $C^\dagger \to S^\dagger$ is log smooth (resp. nowhere log smooth) on $C_s$ if $m = \mu$ (resp. if $m \neq \mu$). However $m = \mu$ is equivalent to $C$ being cohomologically flat over $R$ by \cite[Corollary 2.3.3]{saito-1998}.

Let us close with some simple remarks.
Remark 5.5. — The condition $m = \mu$ forces the curve $D$ in the above proof to be ordinary: $\mathcal{N}_{D/X}^{\otimes \mu/p}$ yields non-trivial $p$-torsion on $\text{Pic}^0 D$, which cannot exist if $D$ is supersingular.

Remark 5.6. — A logarithmic version of the Néron–Ogg–Shafarevich criterion for good reduction of abelian varieties has been proven in [1]. It would be very interesting to have a full characterisation of torsors under abelian varieties with logarithmic good reduction.

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