We investigate the conditions under which an uncontrollable background processes may be harnessed by an agent to perform a task that would otherwise be impossible within their operational framework. This situation can be understood from the perspective of resource theory: rather than harnessing ‘useful’ quantum states to perform tasks, we propose a resource theory of quantum processes across multiple points in time. Uncontrollable background processes fulfil the role of resources, and a new set of objects called superprocesses, corresponding to operationally implementable control of the system undergoing the process, constitute the transformations between them. After formally introducing a framework for deriving resource theories of multi-time processes, we present a hierarchy of examples induced by restricting quantum or classical communication within the superprocess – corresponding to a client-server scenario. The resulting nine resource theories have different notions of quantum or classical memory as the determinant of their utility. Furthermore, one of these theories has a strict correspondence between non-useful processes and those that are Markovian and, therefore, could be said to be a true ‘quantum resource theory of non-Markovianity’.

I. INTRODUCTION

Before the invention of motors powered by hydrocarbon fuel, our ancestors were forced to rely on less tangible energy sources to power their voyages across the oceans. Sailing is a form of propulsion which works by actively harnessing the energy of an uncontrollable background process, namely the wind. Fast forward a few hundred years, and we are at the cusp of another technological revolution, which will be based on the logic of quantum mechanics. As quantum technology matures, understanding the scope of experimental control has become an area of significant focus. Most efforts to improve control over quantum systems have focussed on reducing the amount of influence which the environment can exert over the system; these efforts include error correction [1], decoupling [2], or simply engineering cleaner quantum systems. Our approach is entirely distinct from these, as we choose to make the most of the environment which will inevitably be present.

To see what ‘sailing’ through Hilbert space might look like, consider the following scenario. Several agents act in sequence on a quantum system with some goal in mind, by be it to extract work from the system, prepare it in a particular state, or to send messages to each other. Between actions, the system is subject to an uncontrollable noisy process – through interactions with its surroundings – which may be temporally correlated. Given that the agents may be limited in the actions they can perform, and the degree to which they can communicate to one another, how can we quantify their ability to achieve their goals given the background process?

An illustrative example of a (classical) process is given in Fig. 1. Here an agent wishes to turn on a light by converting a background process into useful work. The agent usually outsources this task to a contractor, who can use a wind turbine or a solar panel to generate electricity depending on the weather, stormy or sunny respectively. On the other hand, a wind turbine (solar panel) is useless on windless sunny (stormy) day. This example begs the question: under what conditions is it possible to extract useful work or information out of an uncontrollable background quantum process?

Quantum resource theories provide a framework in which the usefulness of objects for a particular task can be formally quantified. They are formulated in terms of a set of potential resources, and a set of allowed, or free, transformations between them. Usefulness is then determined by the set of other resources can be reached through these transformations. Quantum resource theories typically treat the set of quantum states (density operators) $\rho$ of a system as the resources, with completely positive trace preserving (CPTP) maps $\mathcal{E}$ as the transformations. Common examples of quantum resource theories, with restricted subsets of CPTP maps, are thermal operations [3–7], noisy operations [8, 9], and local or separable operations [10, 11].
By exploring various properties of quantum resources, such as asymptotic conversion [12] and rates of resource exchange [13], we can understand which properties of states are useful under a particular set of allowed operations. For example with local operations, entanglement is useful [14], whereas in thermodynamics, athermality is useful [3], and for stabiliser computation, magic states are useful [15].

Yet, these resource theories do not capture the usefulness of more general quantum processes, which take place over several points in time and which may involve temporal correlations mediated by an inaccessible environment i.e., non-Markovian memory. In fact, it appears that non-Markovianity is the norm rather than the exception; it must be accounted for in real quantum computers [16], in quantum metrology with noisy environments [17], and in many realistic quantum control tasks [18]. Furthermore, several researchers have suggested that non-Markovianity may be useful for certain tasks: it has been shown to improve the performance of quantum heat engines [19, 20], enhance quantum control [18, 21–23], reduce decoherence [24] and allow the perfect teleportation of mixed states [25]. However, there is no consensus on how to treat temporal correlations as a resource.

Here, we develop a framework for defining resource theories of multi-time quantum processes induced by limited experimental control, and use it to derive a family of theories in which non-Markovianity and related quantities become useful resources. While there have been numerous attempts to quantify the utility of non-Markovianity, giving rise to various resource theories [26, 27], our work is unique in that resource value is based on an operationally well defined framework for quantum processes, called the process tensor formalism [28–30], which accounts for all multi-time correlations. This framework is a useful subset of the more general frameworks of quantum networks [31] and higher order quantum maps [32].

We begin in the next section by introducing the process tensor, a description of non-Markovian processes in terms of higher order quantum maps, and then go on to use it in Sec. III to show how restrictions on experimental control can lead to meaningful resource theories, in which processes themselves play the role of resources. In Sec. IV, we consider the special case of restrictions on communication from past to future, demonstrating how different kinds of temporal correlations form a hierarchy of resources, before concluding in Sec. V. To start with, we will elucidate the general scenario we have in mind.

II. QUANTUM PROCESSES

Our description of quantum processes is an operational one: we explicitly account for what is within the control of some hypothetical agent, and what is not. There are two reasons why an agent may not have total control over a system $s$ of interest. Firstly, there may be degrees of freedom $e$ (the environment) with which the system interacts that are not directly accessible to the agent. Secondly, the agent may only have a limited capacity to influence the parts of the system that they can directly access, though they may be able to involve a separate ancillary system $a$ in their interactions with $s$.

We assume a setting where the agent can effectively act on $s$ and $a$ instantaneously at a series of discrete times, between which it interacts continuously with $e$. In this case, the actions of the agent, which may potentially include any physically allowed transformation (including doing nothing at all), can be represented by completely positive (CP) trace non-increasing maps $A$.

An ordered set of possibly restricted actions the agent is able to perform, conditional or otherwise

1 See Appendix D for a compact summary of the meanings of the various types of objects we use.
{A_{n-1}^{sa}, \ldots, A_0^{sa}} \) is called a control sequence; here the subscript denotes the ‘time-step’. However, the ancillary system \( a \) can act as a quantum memory in general, allowing the agent to effectively correlate their actions on \( s \) across time [28, 33]. To represent their multi-time action on the system, we can compose these actions on the ancillary space only, denoted by \( o_a \); this composition implies that there are no intermediate actions on the corresponding space. The result is a higher-order quantum map \( A_{n-1}^{sa} \), depicted in blue in Fig. 2, that encodes these correlations and acts on \( s \) alone, albeit at multiple times (here \( \rho_se \) is the initial state of the ancilla). When the actions can be applied unconditionally, this object satisfies a hierarchy of causality conditions and is referred to as a quantum comb [34]. Unlike these actions, interactions between \( s \) and \( e \) are outside of the agent’s influence, and need not be subject to the same limitations. Furthermore, the environment may become non-trivially correlated with \( s \) and \( e \) acting jointly on \( s \) and \( e \). The collection of these maps composed only over \( e \) (denoted by \( o_e \)) forms another quantum comb \( \mathbf{T}_{n:0} = \text{tr}_e \left\{ \mathcal{E}_{n-1}^{se} \circ \cdots \circ \mathcal{E}_0^{se} \right\} \), known as the process tensor. Given a potentially correlated initial state \( \rho_0^{se} \), the state after \( n \) actions can be written as

\[
\rho_n^{se} = \mathbf{T}_{n:0} [A_{n-1}^{sa}] = \text{tr}_a \left\{ \mathcal{E}_{n-1}^{se} \circ \cdots \circ \mathcal{E}_0^{se} \right\} \rho_0^{se}.
\]

The second equality indicates that the interleaved \( se \) and \( sa \) dynamics can be seen as the contraction of the two higher order quantum maps, as depicted in Fig. 2. The first is the control sequence \( A_{n-1}^{sa} \), and the second is the process tensor \( \mathbf{T}_{n:0} \) [30]. When the former can only be applied conditionally on some measurement outcome, the final state \( \rho_n^{se} \) will be subnormalised. The process tensor encodes all information about a quantum process which is not under the direct control of an agent, though a consistent set of maps \( \mathcal{E}_{j+1}^{se} \) and state \( \rho_n^{se} \), can be non-uniquely determined by the agent, in principle, through a generalised quantum process tomography [28]. While we consider a version here which maps control sequences to quantum states, our results apply equally well to other quantum combs, such as those with an additional quantum state as input, which can be seen as maps from control sequences to quantum channels.

It is usually convenient to represent the process tensor in the Choi form as a multipartite state instead of a multi-time-step evolution [28]. In this way, one can investigate its properties without being forced to specify an argument. The general form for a Choi state associated with a \( n \) step process tensor is

\[
\Upsilon_{n:0} = \text{tr}_e \left\{ \prod_{j=0}^{n-1} \left( \mathcal{E}_{j+1}^{se} \circ S^{a,0}_j \right) \bigotimes_{j=0}^{n-1} (\psi_{0,1}^{se}) \otimes \rho_0^{se} \right\},
\]

where \( S^{a,\beta} \) is a swap operation between subsystems \( a \) and \( \beta \), \( \rho_0^{se} \) is the system-environment initial state, \( \psi \) is a maximally entangled bipartite state, \( o \) and \( i \) index the two halves of the maximally entangled pair by whether they correspond to an output or input of \( \mathbf{T} \), and \( j \) indexes the step number. In our indexing, \( s = o_0 \), hence \( S^{(s,o_0)} = \mathbf{T} \).

The process tensor formalism is the quantum generalisation of classical stochastic processes [35]. As such it resolves several outstanding conundrums about quantum stochastic processes. For example, it provides an unambiguous necessary and sufficient condition for Markovianity of a quantum process [29, 30, 36]. The corresponding process tensor has the Choi state expressed in the form

\[
\Upsilon_{n:0}^{\text{Markov}} = \prod_{j=0}^{n-1} (A_{j+1,j}^{se}) \otimes \rho_0,
\]

where \( A_{j+1,j}^{se} \) is the Choi state of a map corresponding to the \( i+1 \)th leg of the process tensor [28]. More importantly, the process tensor enables the systematic exploration of the rich structure of quantum non-Markovian memory [37–48]. The formalism has also led to a pathway to generalise the theory of stochastic thermodynamics to quantum mechanics [49–53].

There are several other theories that share mathematical structure with the process tensor framework. Firstly, as noted above, the process tensor is a special case of the framework of quantum networks [31, 34, 54], which was originally derived as the most general representation of quantum circuit architectures. Beyond this there are causal automata/non-anticipatory channels [55, 56], which describe quantum channels with with memory; causal boxes [57] that enter into quantum networks with modular elements; operator tensors [58, 59] and superdensity matrices [60], employed to investigate quantum information in general relativistic space-time; and, finally, process matrices, used for quantum causal modelling [36, 61–63]; and the \( \epsilon \)-transducers used within the framework of computational mechanics [64, 65] to describe processes with active interventions. Quantum strategies [66–69] can also take on a similar operational structure to the process tensor when co-strategies are considered. Our results could be extended to the frameworks listed above, and any other framework for describing quantum processes as linear functionals.

The process tensor represents an uncontrollable background process, which like the weather in Fig. 1, can represent a resource. While the agent does not have any control over the process itself, she/he can choose how
to interact with it. In the next section, we will work within this structure to show that an agent’s repertoire of control operations can be used to derive a resource theory of multi-time processes.

III. MULTI-TIME PROCESSES AS RESOURCES

Before fully exploring transformations of process tensors, we will present another type transformation that is simpler but carries many of the important features we seek.

A. Preliminary Example: Supermaps and Resource Theories of Quantum Maps

While the majority of existing results on quantum resource theories pertain to states as resources, it is becoming increasingly apparent that dynamical objects such as channels and maps can be harnessed in much the same way to perform useful tasks. Applying resource theories to dynamical objects allows an experimenter to understand how changes to their control capabilities result in differing abilities to perform a particular desired task. Recently, a framework has been developed for resource theories of quantum maps (channels in particular) [70–72]. Prior to this, there were some specific results that look at channels as resources [73, 74], including a resource theory of memory [26]. Additionally, resource theories of entanglement in bipartite channels [75, 76], and asymmetric channel distinguishability [77] have been studied.

The central object in these theories are the so-called supermaps [54] \( S \), which enable transformations between resources (maps). Its action on a quantum map \( \mathcal{E}^s \), which in turn acts on state \( \rho^s \) to produce \( \sigma^s \) is

\[
\mathcal{E}^s[\rho^s] = \sigma^s, \quad S[\mathcal{E}^s] = \mathcal{E}^a, \quad \mathcal{E}^a[\rho^s] = \sigma^a. \tag{4}
\]

As depicted in Fig. 3, any deterministic supermap can be represented by

\[
S[\mathcal{E}^s][\rho^s] = \text{tr}_a \{ \mathcal{W}^a \circ \mathcal{E}^s \circ \mathcal{V}^a[\rho^s \otimes \rho^a] \}, \tag{5}
\]

where \( \mathcal{E}^s \) is a map on the main system \( s \) and is the argument of the supermap, while two supplementary maps \( \mathcal{V}^a \) and \( \mathcal{W}^a \) form a (non-unique) representation of the supermap itself, both acting on an additional ancillary subsystem \( a \). The first supplementary map \( \mathcal{V}^a \) acts on \( \rho^s \) prior to the application of \( \mathcal{E}^s \), and the second \( \mathcal{W}^a \) is applied to the output of \( \mathcal{E}^s \). These two maps can be thought of as pre- and post-manipulations in addition to the original map.

In this setting, free maps can then be defined as those reachable through allowed pre- and post-manipulations from any other map. In this way, the range of experimental control becomes a set of transformations on the object representing the dynamical process, in this case the map \( \mathcal{E}^s \). For example, when an agent can only implement trace preserving supermaps (superchannels)

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5 In this context, deterministic means that channels are mapped to channels, as opposed to arbitrary trace non-increasing CP maps; in other words, the supermaps can be realised unconditionally.
In general, \( A \) is undergoing. Instead can be harnessed. As mentioned in Sec. II, applying transformations that encode constraints on how they superprocess representing the interplay between process and control subnormalised. We can conclude that the diagram a measurement outcome, the final state \( \rho \) sequence cannot be implemented unconditionally [31].

\[
\text{superprocesses}
\]

Just as with the supermap, there is a dilated representation of any \( Z_{n,0} \) in terms of reversible maps \( \{W_j\} \) and \( \{V_j\} \) on an ancillary space, as proven in Theorem 24 of Chiribella et al. [78], under the assumption that there is a definite causal order between the control operations on the input and the output spaces. This allows us to rewrite the process tensor’s action on the control sequence as

\[
[T_{n,0}|Z_{n,0}|A'_{n,0}] = T_{n,0}[A'_{n,0}] = T_{n,0}[A_{n-1,0}],
\]

where \( A'_{n,0} = \text{tr}_w \{A_n^{a_1' \circ \cdots \circ a_0'} [\rho_{0}^{a_0}] \} \) is a control sequence on another system \( s' \) which, in analogy to \( A_{n-1,0} \), can be represented in terms of maps \( A_n^{a} \) on \( s \) and a further ancillary space \( \alpha' \). Expanding out the objects in Eq. (6), we arrive at

\[
[T_{n,0}|Z_{n,0}|A'_{n,0}] = \text{tr}_{|s \times a'} \{A_n^{a_1' \circ \cdots \circ a_0'} A_n^{e_0, a_0} \circ \cdots \circ M_{e_0, a_0}^{s_0} \circ A_0^{e_0, a_0} \circ W_{0}^{s_0} [ \rho_{0}^{s_0 a_0} ] \} = \rho_{n,0}^{a_1'},
\]

with

\[
M_{e_0, a_0}^{s_0} = \mathcal{W}_{s_0}^{e_0, a_0} \circ \mathcal{E}_{s_0}^{e_0, a_0} \circ \mathcal{V}_{s_0}^{e_0, a_0}.
\]

In general, \( A'_{n,0} \) can include a final measurement operation \( A_n^{a_1' \circ \cdots \circ a_0'} \) that occurs after the final se map \( \mathcal{E}_{n,n-1} \), to allow for the case where the original control sequence cannot be implemented unconditionally [31]. If the control sequence \( A'_{n,0} \) involves conditioning on a measurement outcome, the final state \( \rho_{n,0}^{a_1'} \) can be subnormalised. We can conclude that the diagram representing the interplay between process and control in Fig. 2 is equivalent to the one in Fig. 4. As such, the superprocess \( Z_{n,0} \) is itself a quantum comb with definite causal order alternating between its action on \( s \) and on \( s' \). We detail in Appendix A that the superprocess also has a convenient representation as a many-body quantum state through the Choi isomorphism (analogous to that for the process tensor).

We can construct a meaningful resource theory of multi-time quantum processes by placing restrictions on an agent’s control: The control sequences which the agent is capable of performing correspond to a set of superprocesses that relate them to a fiducial set of control sequences on \( s' \). These superprocesses can be regarded as free, and in turn through their dual action define the set of process tensors that can be obtained for free. Furthermore, when the process tensor is specified to not contain an initial state and only have one time-step, and the control operations are taken to be trivial, this expression reduces to the action of a supermap on a single quantum map. Resource theories of maps as in Sec. III A are a specific case of these more general theories.

An alternative picture of the superprocess is one where multiple agents with varying levels of control capabilities interact. The superprocess then represents the actions of an intermediary contracted by a client, who can only perform a limited set of operations on the \( s' \) system. This contractor interfaces directly with the process, which plays the role of a remote server, and returns the result to the client. From the client’s perspective, the joint system of contractor (superprocess) and server (underlying process) can be viewed as a new process (combining red and purple in Fig. 4): any control sequence the client could directly apply to the underlying process, they could equally apply to the transformed process. Equally, from the perspective of the underlying process, the combined control sequence and superprocess, can be viewed as a new control sequence (combining blue and purple in Fig. 4).

The latter picture could be straightforwardly gener-

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6 We will assume here that all Hilbert spaces are finite dimensional and avoid any of the potential complications with regards to dual spaces that a more general scenario would entail.
alised to scenarios where the system accessed by the client is very different from that which feeds directly into the server. It could also be extended to cases where the contractor’s actions are represented by superprocesses that take $n$ step processes to $m$ step ones. When $n < m$ the number of client actions can be increased by including more than one $V$-$W$ pair per step, and conversely when $n > m$ the main subsystem $s$ can be joined between $V$ at one step and $W$ at the next step.

C. Resource Theories of Multi-Time Processes

We now have all the necessary ingredients to formally define resource theories of multi-time processes:

**Definition 1** (Resource Theory of Multi-Time Processes). A resource theory of multi-time processes $\mathcal{R} = \{T, Z\}$ consists of two sets. The first set $T$ consists of uncontrollable background processes which an agent might be subject to, represented by process tensors $T_{n,0}$, while the second set $Z$ is that of superprocesses $Z_{n,0}$ which the agent is capable of implementing as transformations of background processes.

This induces a structure of convertibility of processes. In particular, it is natural to define a free process as one which can be obtained via an implementable superprocess acting on a background process which is operationally defined to be free. In a fully general category theoretic sense, free processes would behave like the identity element for the monoid operation that corresponds to resource composition, implying that it can be appended to any other resource without changing its value as a resource [79]. However, operationally it makes the most sense in a theory of multi-time processes to define free processes not as those which can always be appended for free, but those which can always be transformed to for free. We can shift to this notion of free-ness via maps which link every resource to a free resource. Many important quantum resource theories have these maps; for example, this is what thermalising a non-equilibrium state does. In our case, the agent applies a suitably noisy superprocess to the background process, such that the transformed background process loses its ability to carry information. We define the set of free processes to be those which can be obtained via a superprocess that maps every resource to a free one, or via the action of any free superprocess on the resultant output. Defined in such a way, the set of free processes will be a closed set, and satisfy the ‘golden rule of quantum resource theories’ presented in Ref. [14].

D. Monotones

We turn our attention to resolving one of the main questions underpinning this investigation: how does one quantify the utility of an uncontrollable background process? A class of monotones for multi-time process theories can be derived from the Choi representation of the process tensor.

**Theorem 1.** Given a resource theory of processes, any state distance $D(\cdot, \cdot)$ measure applied to the Choi states of process tensors satisfying contractivity under the action of superprocesses forms a monotone

$$M_T = \min_{T \in T_p} D(\Upsilon_T, \Upsilon_{T_p}),$$

where $\Upsilon_{T_p}$ are free process tensors in the Choi representation.

**Proof.** Since $D(\cdot, \cdot)$ is contractive under application of the superprocess:

$$D(\Upsilon_T, \Upsilon_{T_p}) \geq D(\Upsilon_T[Z], \Upsilon_{T_p[Z]}).$$
If we set $\Upsilon_T$ to be the closest free process to $\Upsilon_T$ this expression becomes
\[ D(\Upsilon_T, \Upsilon_{T_p}) \geq D(\Upsilon_{|T|Z}, \Upsilon_{|T_p|Z}). \] (11)
Since $Z$ is a free superprocess, then $\Upsilon_{|T_p|Z}$ must still be a free process; however, $\Upsilon_{|T_p|Z}$ need not be the closest free process to $\Upsilon_{|T|Z}$. Hence, there exists another free process $\Upsilon_{T_p}$ at least as close to $\Upsilon_{|T|Z}$
\[ D(\Upsilon_{|T|Z}, \Upsilon_{|T_p|Z}) \geq \min_{T_p} D(\Upsilon_{|T|Z}, \Upsilon_{T_p}), \] (12)
which gives us the final result:
\[ M_T(\Upsilon_T) \geq M_T(\Upsilon_{|T|Z}), \] (13)
for free superprocesses $Z$, proving that $M_T$ is a monotone.

The main difficulty associated with this family of monotones is specifying a $D(\cdot, \cdot)$ which is contractive under the action of superprocesses. Considering process tensors as states in the Choi representation, a superprocess is no more than a specific kind of CPTP map between states. As such, state distance measures which are contractive for all CPTP maps will automatically satisfy our needs. The trace distance and relative entropy distance on Choi states are two suitable choices in this regard [80]. A quantum strategy approach [77] has yielded a similar class of multi-time measures called generalised quantum strategy divergences. These coincides with our monotones when the optimisation is restricted to control sequences whose deterministic action has a Choi state proportional to the identity operator.

There are many more monotones which could be investigated for resource theories of quantum processes. For channel resource theories, a common monotone involves optimising some quantity over all possible input states [70, 71, 81]. The analogous family of monotones for process theories would correspond to an optimisation over control operations instead of initial states. We expect that numerous operationally important monotones would arise from different choices for the quantity which is optimised. A particularly promising avenue is to optimise the information retained about past states of the system. Such a monotone would correspond to how well an agent can preserve quantum information, given a background process with potentially non-Markovian noise.

A. Communication Maps

The communication restrictions on the superprocess can be set by replacing the identity channels between $\mathcal{V}$ and $\mathcal{W}$ in Fig. 4 by channels $\mathcal{C}$ and $\mathcal{K}$, as depicted in Fig. 5. The particular forms of these communication maps dictates what kind of information is able to propagate through the ancillary subsystems, i.e., systematically restrict or allow information transfer between different points in time. In this setting, the client solely acts on subsystem $s'$ with $\mathcal{A}_{s'}$. The communication maps $\mathcal{C}_{s'z'}$ and $\mathcal{K}_{sz}$ (as well as $\mathcal{W}_{zzs'}$ and $\mathcal{V}_{zs's'}$) make up the constrained superprocess. Hence, responsibility of communication and performing joint system-ancilla operations are delegated to the superprocess (contractor). Specifying this type of resource theory for quantum processes reduces to specifying the class of allowed $\mathcal{K}_{z}$ and $\mathcal{C}_{z+1}$.

We now use communication maps to enumerate the classes of allowed operations in our resource theories. There are three classes $\{\emptyset, \mathcal{B}, \mathcal{D}\}$ we will consider for each

IV. RESOURCE THEORIES OF PROCESSES WITH RESTRICTED COMMUNICATION

One of the most natural constraints on the structure of the superprocess is the connectivity of the ancillary subsystem, which may be restricted by constraining the information that flows between elementary maps $\mathcal{V}$ and $\mathcal{W}$ in Fig. 4. Restricted communication within the superprocess can be manifested in numerous kinds of operational scenarios. We make use of the client-contractor-server metaphor, first introduced in Sec. III B, to emphasise the important features of each scenario. Here, a client, interacting with a server, can only perform control operations on the system without access to a memory bearing subsystem, classical or quantum. However, the client can enlist a contractor to perform tasks requiring memory. As such, the contractor, who facilitates communication between the client and the server, is described by a superprocess.

In this setting, the contractor can only be useful to the client when their actions are more powerful than those of the client. Hence, while there is some freedom in choosing the resource theory of the client, their allowed operations must always be a subset of those permitted for the contractor. This can always be guaranteed if one chooses to take the actions of the client as local in time and uncorrelated, which is what we specify here. The contractor can then use their operations to interact with a server, corresponding to the process tensor. The set of free transformations on process tensors corresponds to the range of services that the contractor can provide the client. For instance, the contractor may charge a premium price for a service that involves quantum memory, and a lower price for only classical memory.
of $C_{\alpha+1:}\alpha$ and $K_{\alpha}$. The choices are: $\emptyset$ no communication, $B$ entanglement breaking communication (slightly more general than strictly classical communication [82]), and $D$ any quantum channel. These classes satisfy $\emptyset \subset B \subset D$. We will study the resource theory induced by each combination of $C_{\alpha+1:}\alpha$ and $K_{\alpha}$, resulting in an operational hierarchy of nine theories. For the sake of brevity, we will use $X^x$ acting on operators on the Hilbert space $x$ as a placeholder for either $C_{\alpha+1:}\alpha$ or $K_{\alpha}$ in the following cases.

($\emptyset$) To account for the absence of communication, $X^x$ consists of discarding the current state, and preparing an arbitrary fixed state. The form of $X^x$ is

$$X^x : \emptyset \rightarrow X^x[\rho^x] = \text{tr}\{\rho^x\} \tau^x,$$

where $\tau^x$ is some arbitrary state which the input is erased to. We refer to this as a fixed output channel.

($B$) Entanglement breaking communication corresponds operationally to a classical agent who may only interact with the quantum system by state measurement and preparation. An entanglement breaking channel can always be represented by a positive-operator valued measure (POVM) measurement with a subsequent (possibly correlated) state re-preparation [82]. Here, $X^x$ is an entanglement breaking channel on $x$ for $X^x : B \rightarrow X^x[\rho^x] = \sum_k \nu^x_k \text{tr}\{\Pi_k^x \rho^x\}$

($D$) In the fully quantum case, $X^x$ takes the form of any CPTP map acting on $x$:

$$X^x : D \rightarrow X^x[\rho^x] = \sum_i X_i^x \rho^x X_i^{x\dagger},$$

written in the Kraus form of a CPTP map on $x$ satisfying $\sum_i X_i^x X_i^{x\dagger} = 1$ with the $\{X_i^x\}$ otherwise general [83].

There may exist operational scenarios which not only allow for communication, but also pre-shared correlations. In these scenarios, in may be possible to carry types of information into the future which would be impossible with the specified type of communication alone. The most famous example of such a scenario is quantum teleportation: given pre-shared entanglement in a pair of qubits, the communication of two classical bits of information allows the state of one qubit to be teleported to the other [10]. If one allows for arbitrary amounts of classical communication (as in $B$) and supplementary entangled qubits, such an agent will be capable of any quantum communication, as in $D$. Consequently, this operational scenario should be classified as $D$. However, in the absence of any communication $\emptyset$, correlations cannot help the agent to perform new tasks.

In this subsection, we began by identifying where communication maps fit into the superprocess, and now we have enumerated three ways in which they might restrict the flow of information through time. Hence, the transformations in these resource theories can now be fully specified.
B. The Primitive Free Resource

To complete the specification of our resource theories, one more ingredient is required: free resources. We begin finding the set of free resources by specifying a single process which for operational reasons is defined a priori to have no value, as discussed in Sec. III C. We call this the primitive free process. Since the primitive free process is free, and free superprocesses are free, any process that can be reached by the application of free superprocesses to the primitive free process will also be free. As such, in a resource theory formed by communication restrictions \( (\mathcal{X}, \mathcal{Y}) \), described by superprocess \( Z_{n:0}^{(\mathcal{X}, \mathcal{Y})} \), the set of free processes \( T_{n:0}^{(\mathcal{X}, \mathcal{Y})} \) are defined as the action of the superprocess on the primitive resource:

\[
T_{n:0}^{(\mathcal{X}, \mathcal{Y})} := [T_{n:0}^{\text{prim}} Z_{n:0}^{(\mathcal{X}, \mathcal{Y})}].
\]

(17)

The complement of this set is the set of useful resources.

In all of the restricted communication resource theories, the natural choice for the primitive free process \( T_{n:0}^{\text{prim}} \) is an uncorrelated sequence of fixed output channels (plus an arbitrary initial state), whose Choi image of our previously defined primitive free resource is

\[
\|T_{n:0}^{(\mathcal{X}, \mathcal{Y})}\| A_{n-1:0} := [T_{n:0}^{\text{prim}} Z_{n:0}^{(\mathcal{X}, \mathcal{Y})}| A_{n-1:0}] = \text{tr}_{sz} \left\{ A_{n-1:0:0}^{n-1:0:0} \left( (\mathcal{M}^{szs}_{\alpha+1:0} \otimes \mathcal{C}^{szs}_{\alpha+1:0}) \circ \mathcal{V}_{\alpha}^{szs} \right) \circ \mathcal{W}_{0}^{szs} [\rho_{0}^{szs}] \right\},
\]

(19)

where now \( \mathcal{M}^{szs}_{\alpha+1:0} = \mathcal{W}_{\alpha}^{szs} \circ (\mathcal{C}_{\alpha+1:0}^{szs} \otimes \mathcal{C}_{\alpha+1:0}^{szs}) \circ \mathcal{V}_{\alpha}^{szs} \) (cf. Eqs. (7) and (8)). See Appendix B for more details on how \( \mathcal{C}_{\alpha+1:0}^{szs} \) determines the properties of \( \mathcal{M}^{szs}_{\alpha+1:0} \), and see Appendix C for an outline of how, along with \( \mathcal{K}_{sz}^{\alpha} \), this specifies the free processes of the theories.

1. \((\emptyset, \emptyset), (\emptyset, \mathcal{B}), \) and \((\emptyset, \mathcal{D})\)

In these theories, \( \mathcal{C}_{sz}^{\alpha} \) is a fixed output channel. For \( Z_{n:0}^{(\emptyset, \mathcal{X})} \) (top row of Fig. 6), independent of \( \mathcal{K}_{sz}^{\alpha} \), the right hand side of Eq. (19) becomes

\[
\text{tr}_{sz} \left\{ A_{n:0}^{n:0:0} \left( \nu_{\alpha}^{szs} \text{ tr}_{szs} \circ \mathcal{W}_{0}^{szs} [\rho_{0}^{szs}] \right) \right\}.
\]

(20)

To get here, from Eq. (19), we observe the fact that \( \mathcal{M}_{\alpha+1:0}^{szs} \) destroys any links in the system between the past and future via \( \mathcal{C}_{\alpha+1:0}^{szs} \) and \( \mathcal{C}_{\alpha+1:0}^{szs} \) are both.

state is denoted by

\[
\Upsilon_{n:0}^{\text{prim}} = \bigotimes_{j=0}^{n-1} (\tau_{j+1} \otimes I_{j}) \otimes \tau_{0},
\]

(18)

where \( \tau_{j} \) is an arbitrary state. Fixed output channels have zero information capacity, breaking any causal links in the environment between the past and the future. Thus, these processes can be considered to have no value or cost to agents.

C. Enumeration of Theories

We label a theory by the tuple \((\mathcal{X}, \mathcal{Y})\) where \( \mathcal{X}, \mathcal{Y} \in \{\emptyset, \mathcal{B}, \mathcal{D}\} \); \( \mathcal{X} \) denotes the type of communication in \( \mathcal{C} \), and \( \mathcal{Y} \) denotes that in \( \mathcal{K} \). The nine resultant theories are enumerated in Fig. 6.

In the next few subsections, we identify the full set of free resources in each theory. This is done by finding the image of our previously defined primitive free resource under all allowed superprocesses in that resource theory. As we have taken the background process to be the primitive free resource, the \( e \) subsystem is no longer relevant to our analysis, and will be omitted. As explained in Sec. IV A, the control operations \( A_{\alpha+1:0}^{szs} \) are taken to be local in time and uncorrelated, implying that they can be represented by fixed output channels. The general form for evolution under the primitive free resource after the action of a superprocess is

\[
\Upsilon_{n:0}^{(\emptyset, \mathcal{X})} = \bigotimes_{j=0}^{n-1} (\rho_{j+1} \otimes I_{j}) \otimes \rho_{0}, \quad \text{for } \mathcal{X} \in \{\emptyset, \mathcal{B}, \mathcal{D}\}
\]

(21)

where \( \rho_{j+1} \) is an arbitrary state equal to \( \text{tr}_{sz} \nu_{\alpha+1:0}^{szs} \).
While these three theories share a set of free resources, it is worth emphasising that they have different free superprocesses and therefore form distinct theories. In other words, different background processes will have different utility in each of them. When the background process is able to pass information in parallel to $\mathcal{C}_{\alpha+1:}\alpha$, they will react differently to information passed by the process in parallel to $K_{\alpha}$. For $(\emptyset,\emptyset)$, any communication is still useful, but for $(\emptyset,\mathcal{B})$ only quantum communication is valued. In the case of $(\emptyset,\mathcal{D})$, nothing further is useful.

Hence, all temporal correlations have resource value in these theories.

Now, $K_{\alpha}^{s\tau}$ is a fixed output channel, but $\mathcal{C}_{\alpha+1:}\alpha$ is entanglement breaking (Fig. 6 left column, middle row). The maps connecting adjacent $\mathcal{M}^{s\tau}_{\alpha+1:}\alpha$ in Eq. (19) become fixed output maps whose outputs are $(K_{\alpha}^{s\tau} \otimes A_{\alpha+1:}\alpha+1:}\alpha):\alpha \cdots \theta_{\alpha}^{s\tau} \otimes \eta_{\alpha}^{s\tau}$. While each step of the process is entanglement breaking, none of the steps depend on each other. This can be expressed in the Choi representation as a tensor product of independent entanglement breaking channels as in Eq. (3)

$$\Upsilon_{n:0}^{(\mathcal{B},\emptyset)} = \bigotimes_{j=0}^{n-1} \left( \Lambda_{j+1:}\alpha}^{s\tau} \right) \otimes \rho_0, \quad (22)$$

where $\Lambda_{j+1:}\alpha}^{s\tau}$ correspond to the Choi state of $\mathcal{M}^{s\tau}_{\alpha+1:}\alpha$. 
The free processes are Markovian but also entanglement breaking; fully quantum operations within the process are useful, as well as non-Markovianity.

3. \((\mathcal{D}, \emptyset)\)

Now, \(\mathcal{K}^{sz}\) is a fixed output channel, but \(\mathcal{C}^{sz'}\) is fully quantum (Fig. 6 left column, bottom row). The derivation of these free states is identical to the \((\mathcal{B}, \emptyset)\) case so we will not repeat it. However, since quantum communication is allowed in \(\mathcal{C}^{sz'}_{\alpha+1:0}\), the Choi state of free processes becomes

\[
\Upsilon_{n;0}^{(\mathcal{D}, \emptyset)} = \bigotimes_{j=0}^{n-1} (\Lambda_{j+1,j}^{\mathcal{D}}) \otimes \rho_0, \quad (23)
\]

where \(\Pi_{k_j}^{sz}\) is a POVM and \(\nu_{k_j}^{sz}\) is a re-preparation conditioned by that POVM, forming the entanglement breaking channel of \(\mathcal{K}^{sz}_{\alpha}\). \(\{k_j\}\) indexes the possible trajectories based on the set of outcomes of the POVM. If we define CP trace non-increasing maps \(\text{tr}_{sz} \{\Pi_{k_j}^{sz}, M_{\alpha+1,\alpha}^{sz}, \nu_{k_j}^{sz}\} = \mathcal{E}_{k_j+1, k_j}\), which must themselves be entanglement breaking, the expression becomes

\[
\sum_{\{k_j\}} tr_{sz'} \left\{ \bigotimes_{\alpha}^{n-1} \left( \mathcal{E}_{k_{\alpha+1}, k_{\alpha}}^{sz'} \circ \mathcal{A}_{\alpha}^{sz'} \circ \mathcal{W}_{0}^{sz'} \Gamma_{sz'} \right) \right\}, \quad (25)
\]

such that the process itself (excluding the \(\mathcal{A}_{\alpha}^{sz'}\)) can be expressed in the Choi representation as

\[
\Upsilon_{n;0}^{(\mathcal{B}, \emptyset)} = \sum_{\{k_j\}} p_{k_0} \bigotimes_{j=0}^{n-1} (\Lambda_{k_{j+1}, k_j}^{\mathcal{D}}) \otimes \rho_{k_0}, \quad (26)
\]

where \(\{k_j\}\) enumerates all possible trajectories to sum over, and \(p_{k_0}\) is the probability of measurement outcome \(k\) on the initial state. \(\Lambda_{k_{j+1}, k_j}^{\mathcal{D}}\) is the Choi form of an entanglement breaking CP trace non-increasing map \(\mathcal{E}_{k_{\alpha+1}, k_{\alpha}}^{sz'}\). The full expression for the free processes corresponds to a fully general entanglement breaking process – Markovian or otherwise. No quantum information can be retained within or between steps. Consequently, quantum entanglement and quantum memory are resources in these theories. The set of free processes in Eq. (26) contains all classical non-Markovian processes, see Ref. [84] for a detailed discussion.

4. \((\mathcal{B}, \emptyset)\)

In this case (Fig. 6 centre row, centre column), \(\mathcal{C}^{sz'}\) and \(\mathcal{K}^{sz}\) are entanglement breaking channels. Here, Eq. (19) resolves to a sum

\[
\sum_{\{k_j\}} tr_{sz} \left\{ \bigotimes_{\alpha}^{n-1} \left( \mathcal{E}_{k_{\alpha+1}, k_{\alpha}}^{sz'} \circ \mathcal{A}_{\alpha}^{sz} \circ \mathcal{W}_{0}^{sz} \Gamma_{sz} \right) \right\}, \quad (27)
\]

The difference to the previous cases is that now \(\Lambda_{k_{j+1}, k_j}^{\mathcal{D}}\) is the Choi state of a fully quantum CP trace non-increasing map. We define that a process has entanglement free memory (EFM) if its Choi state can be written as a convex combination of CP trace non-increasing maps as in Eq. (27).

With a process that has EFM, past dynamics can only influence future dynamics if that past is able to be communicated by an entanglement breaking map. Hence, such a process will appear to be Markovian under control operations that have differing action on entangled states but not on any separable states; they will seem non-Markovian under control operations that do have differing action on separable states. Another way to think of this is that free processes in this theory are non-Markovian with respect to classical information, but Markovian with respect to information that is uniquely quantum [85]. An important subset of these processes are those arising from interactions with
is an entanglement breaking channel, hence \( K \) also be any entanglement breaking channel. However, more generally the probability of realising a sequence of classical stochastic process. However, more generally the probability of realising a sequence of classical stochastic process. However, more generally the probability of realising a sequence of classical stochastic process. However, more generally the probability of realising a sequence of \( M \), the action of a segment of the transformed process tensor \( \Omega \) is a POVM and \( \nu_{\alpha+1,0} \) is a re-preparation conditioned by that POVM. Using this, we can express the action of a segment of the transformed process tensor as \( M_{\alpha+1,0}^{\rho_{xz}} = \sum_{k_{\alpha+1},m} p_{k_{\alpha+1},m} \rho_{m,k_{\alpha+1}} \). \( \mu_{xz}^{\rho_{xz},\alpha+1} = W_{\alpha+1}[\rho_{xz}^{\alpha+1}] \) is the output state of \( \mathcal{N}_{\alpha+1} \) and \( \text{tr}(\Omega_{\alpha+1,0}\rho_{xz}) = \text{tr}(\Omega_{\alpha+1,0}\rho_{xz}) \) is the probability of the \( k_{\alpha+1,0} \) outcome of the (modified) POVM corresponding to \( \Omega_{xz}^{\alpha+1} \). For \( \mathcal{Z}_{n,0} \), the right hand side of Eq. (19), in terms of the above objects, yields a sum of measurements and re-preparations

\[
\sum_{\{k_{\alpha+1},m\}} \text{tr}_{xz'} \left\{ \rho_{zx}^{\alpha+1} \right\} \prod_{\alpha=1}^{n-1} (p_{k_{\alpha+1,0},m_{\alpha-1}}) p_{k_{\alpha+1,0}} .
\]

where \( p_{k_{\alpha+1,0}} = \text{tr}_{xz'} \left\{ \Omega_{k_{\alpha+1,0}}^{\rho_{xz}} \rho_{xz}^{\alpha+1} \right\} \) represents the probability of an initial evolution on trajectory \( k \), while \( p_{k_{\alpha+1,0},m_{\alpha-1}} = \text{tr}_{xz'} \left\{ \Omega_{k_{\alpha+1,0}}^{\rho_{xz}} \rho_{xz}^{\alpha+1} \right\} \) represents the probability of evolution on trajectory \( k \) between steps \( \alpha - 1 \) and \( \alpha + 1 \). Consequently, the Choi state of a free process tensor in this theory has the form

\[
\Upsilon_{(\mathcal{B},\mathcal{D})} = \sum_{\{k_{\alpha+1},m\}} p_{k_{\alpha+1,0},m} \otimes \left( \Gamma_{j_{\alpha+1,j_{\alpha-1}}} \right) \otimes \Gamma_{k_{\alpha+1,0}} .
\]

Here, as in the \((\mathcal{D},\mathcal{B})\) theory, the \( \Gamma_{j_{\alpha+1,j_{\alpha-1}}} \) are Choi states of general CP trace non-increasing maps. However, here they have the same input and output spaces as the \( \mathcal{A}_{\alpha} \) control operations, meaning that they cannot be interpreted in terms of trajectories of interleaved \( \mathcal{E} \) and \( \mathcal{A} \) maps. Instead, each \( \Gamma_{j_{\alpha+1,j_{\alpha-1}}} \) implies a control operation dependent probability, and the full sequence determines a distribution over final states \( \rho_{k_{\alpha+1,0}} \). In this sense there is full quantum memory between adjacent legs of the free processes, but due to the entanglement breaking nature of \( C \), it cannot propagate more than one step. For this reason we call this kind of memory single-step quantum memory (SSQM).

In this case (Fig. 6 bottom row, right column), \( C_{xz} \) are quantum channels. Since there are no restrictions on \( C_{xz} \) and \( K_{xz} \), anything of the form of Eq. (19) can be achieved for free. This resource theory is trivial since every possible process in the theory is a free process; the Choi state \( \Upsilon_{(\mathcal{D},\mathcal{B})} \) can be that of any quantum process. Hence, nothing can be considered especially useful.

D. Summary of Results

As we are particularly interested in non-Markovianity, which is a property of memory, we define classical memory \( c \) and quantum memory \( q \) as

\[
(c, q) := \text{maximum number of steps retaining (classical,quantum) information}.
\]

The memory can take values \( \{0, 1, \infty\} \), which mean no information can be retained in time; information can be retained for at most one step; and information can be retained for the whole duration of the process. These quantities are formally related to the notion of Markov order, which has recently been generalised for quantum processes; there, the choice of instrument used for a measurement is taken into account in addition to the measurement outcomes [44, 45].

In all \( (\mathcal{D},\emptyset) \) theories, the free processes are only those which have no temporal correlations. However, once some temporal correlations are present, each of these theories differ, as any \( C \)-type communication can be better utilised if \( K \)-type communication is allowed. In the \( (\mathcal{B},\emptyset) \) and \( (\mathcal{D},\emptyset) \) theories, all of the free processes are Markovian, although in the former case only classical operations are allowed for free. On the other hand, \( (\mathcal{D},\emptyset) \) is a true resource theory of quantum non-Markovianity: there is a one-to-one correspondence between free-ness and Markovianity. The free processes in \( (\mathcal{B},\emptyset) \) correspond to agents who can carry out any multi-time entanglement breaking process for free, while every quantum process is free in \( (\mathcal{D},\emptyset) \). The free processes of \( (\mathcal{D},\emptyset) \) have entanglement free memory, and \( (\mathcal{D},\emptyset) \) has single step quantum memory. The length and quality of memory determines the type of process. In particular, classical non-Markovian processes are a strict subset of \( (\mathcal{B},\emptyset) \), which in turn is a subset of \( (\mathcal{D},\emptyset) \), \( (\mathcal{D},\emptyset) \), and \( (\mathcal{D},\mathcal{D}) \). The classes of free processes and the memory lengths are summarised in Table. II.

In our hierarchy, generally the Choi states of free processes satisfy some from of separability, while useful
ones have entanglement or other correlations, implying that these monotones will take typically the form of multipartite entanglement measures. In the simplest ($\emptyset$, $-\emptyset$) cases, the monotones are simply the $D(\cdot, \cdot)$ measure applied between the Choi state of interest and that of the primitive free resource, e.g. the relative entropy to the maximally mixed state. In the opposite extreme ($\mathcal{Z}$, $\emptyset$), the monotone is always zero since every process is free. Of particular importance, the monotone for ($\mathcal{Z}$, $\emptyset$) is a direct measure of the non-Markovianity of a process tensor as quantified in Ref. [29]. Furthermore, the monotone of ($\mathcal{Z}$, $\emptyset$) is a measure of a stricter notion of uniquely quantum non-Markovianity. The properties of utility for each resource theory are summarised in Table IV.

### Table I. Free Processes

| $c$ | $\emptyset$ | $\mathcal{Z}$ |
|-----|--------------|---------------|
| 0   | Fixed output, (0,0) | Fixed output, (0,0) |
| $\mathcal{R}$ Markovian EB, (1,0) | $\mathcal{E}$, (\infty, 0) | SSQM, (\infty, 1) |
| $\mathcal{D}$ Markovian, (1,1) | EFM, (\infty, 1) | CPTP, (\infty, \infty) |

Table II. The sets of free resources for the nine resource theories. Each row represents a different form of $c$, while each column represents a different form of $K$. The classical $c$ and quantum $q$ memory length of free processes is assigned as $(c, q)$ for the free processes of each case. A memory length of 0 implies that no information can be preserved temporally, while a memory length of 1 means that adjacent steps are able to depend on each other. A memory length of $\infty$ denotes the case where no restrictions exist on the memory. In the top row, the inability of the allowed superprocesses to carry information through time renders the processes constructed from them no more useful than the primitive free process itself. The left column contains theories where the free processes are all Markovian but with more control going from top to bottom. The diagonal contains the most general possible processes subject to each of the three types of constraints; either fixed output, entanglement breaking (EB), or completely positive and trace preserving (CPTP). ($\mathcal{Z}$, $\emptyset$) and ($\mathcal{R}$, $\mathcal{D}$) are the most atypical theories, as they have free processes which lie in a grey area between quantum and classical non-Markovianity. Their free processes have entanglement free memory (EFM), and single-step quantum memory (SSQM) respectively.

### E. Relation to Other Resource Theories

Our work here is based on a general framework for quantum processes which can be probed at multiple times. As such, we expect that many ‘resource theories of processes’ (see Ref. [14]), most of which take individual maps as resources, can be extended to encompass multi-step scenarios using our framework; this includes many important recent developments in resource theories of channels [70–72, 86]. Another completely distinct resource theoretic description of channels utilises discrimination tasks [87]. This approach elucidates the connection between channels being non-free and being useful for a particular task. This could be extended to the multi-time case with discrimination tasks for process tensors. Resource theoretic results on the entanglement of multipartite channels [75, 76, 88–90] have utilised the idea of quantum strategies [66–69], which allows for the possibility of dynamical resources being quantum combs. Discrimination of quantum strategies [77] has also been investigated, which involves transformations from strategies to strategies; this is in a similar spirit to our approach of transforming process tensors with superprocesses.

There has been a great deal of interest in the utility of non-Markovianity, giving rise to a plethora of different resource theories involving non-Markovianity, as well as other related properties. The resource theory described in Rosset et al. [26] captures similar behaviour to that of the ($\mathcal{Z}$, $\emptyset$) resource theories. There have also been resource theories of divisible operations [91, 92], which have provided numerous results which are related to the ($\mathcal{Z}$, $\emptyset$) theory in our hierarchy. This theory is formulated in terms of parameterised families of quantum maps (from density operators to density operators) rather than process tensors, and therefore cannot fully account for multi-time correlations. This inability to account for multi-time correlations is a common problem – many measures of non-Markovianity rely on the equivalence of Markovianity and CP divisibility, which have recently been shown to not always coincide [93]. A process matrix approach has been used to quantify the capability of processes to produce uniquely quantum effects, which has also led to a measure of non-Markovianity [94–96]. This method has also been used to study non-Markovian effects in real experimental data [96]. Finally, another class related theories non-Markovianity are explored in Refs. [27, 97]; here the Markov condition is derives from the conditional quantum mutual information between subsystems of multipartite states, as opposed to explicit temporal correlations. See Refs [44, 45] for relation between Markov chain states and quantum processes.

### V. CONCLUSION

At the heart of this investigation is the question: can an uncontrolled background process useful be useful to an agent? In order to answer this question, we have
Table III. Properties of Utility

| C | \emptyset | A | \emptyset |
|---|---|---|---|
| Correlations | Correlations | Correlations |
| Inter-E corr., intra-E enta. | Entanglement | Inter-E entanglement |
| Inter-E correlations | Inter-E entanglement | Nothing |

Table IV. The properties of utility which may be exhibited by resources in each of the nine resource theories. These are expressed as properties of the Choi state of a process tensor, and can be measured with monotones as in Sec. III D. In the top row, any correlations in the Choi state are useful, although differently useful in each theory. In every entry on the left column, correlations between individual partitions of the Choi state corresponding to legs of a process tensor are useful (memory between steps). In the top two entries of this column there are other properties which are also useful. However, in the bottom case only non-Markovianity is a resource, rendering it a true resource theory of quantum non-Markovianity. In the diagonal entries, any violations of the specified constraint is seen as a resource. In \((\emptyset, A)\) and \((\emptyset, \emptyset)\), only non-Markovianity that is explicitly quantum in nature can be useful.

presented a framework for resource theories of quantum processes. Here, the descriptor of the process, known as the process tensor, takes the role of resources, and a new construction – the superprocess – serves the purpose of transforming resources. Under this framework, a background process that can be simulated by agent actions on a pre-defined free process holds no value, while one that is not producible may have the potential to be used to perform tasks which were previously unavailable to the agent. We have used this framework to construct an operationally motivated hierarchy of nine theories corresponding to a realistic client-contractor-server scenario, and found the associated free processes and monotones. In many of these theories, notions of non-Markovianity are the main determinant of the value of a process. Furthermore, \((\emptyset, \emptyset)\) is a true theory of quantum non-Markovianity. The \((\emptyset, A)\) and \((A, \emptyset)\) free processes exhibited properties lying between classical and quantum non-Markovianity, implying that the processes of value exhibit a kind of non-Markovianity which is uniquely quantum.

While the background process is noisy, we assume that any map contained within the superprocess can be performed perfectly. We did limit control by employing our three classes of communication, but the effectiveness of the agent at implementing the allowed processes remains unlimited. Thus, a promising direction for future work is looking into theories which have more stringent restrictions. For example, \(V\) and \(W\) could be subject to their own channel resource theories. Furthermore, recent work on resource theories of measurements [98] could enable our theories which have the \(B\) class of communication to be further dissected.

There are numerous physical settings which might make use of a resource theory of multi-time processes. For example, the client-contractor-server scenario detailed in Sec. IV might have applications in cloud quantum computing, where quantum memory may be restricted on the server side. Superprocesses could also be used to model untrusted devices, allowing resource-theoretic investigations within the device independent paradigm, which has recently been extended to include multi-time causal processes [99]. More generally, super-processes provide a natural framework for investigating quantum control problems in a scenario where multiple interventions on a system are possible.

We hope that our formalism may enable known results to be applied in new contexts. One might seek construct a new multi-time process theory from state or channel theories which are already well understood. Furthermore, multi-time process resource theories are yet to be studied in the same kind of depth as state or channel theories. Properties such as robustness, resource distillation and dilution, and single shot vs asymptotic transformations are uncharted territory. We present an open invitation to sail the seas of Hilbert space using uncontrollable background processes.

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Appendix A: Superprocesses in the Choi Representation

The Choi state of an $n \rightarrow n$ superprocess is given by

$$\Psi_{n,0} = \text{tr}_{z} \left\{ \left( \bigotimes_{j=0}^{n-1} \left( S^{n,i_{j+1}} \circ W_{j+1}^{z \times s} \circ S^{n,i_{j}} \circ Y_{j}^{z \times s'} \right) \circ S^{n,i_{0}} \circ W_{0}^{z \times s'} \right) \right\} \left( \bigotimes_{j=0}^{n-1} \left( \psi_{i_{j+1},\sigma_{j+1}}^{z_{j+1}} \otimes \psi_{i_{j}^{z \times s'}}^{z_{j}} \otimes \psi_{s_{0}}^{z \times s} \otimes \rho_{0}^{z \times s} \right) \right). \tag{A1}$$

Our index system for Hilbert spaces (illustrated in Fig. 7) has three variables. The first is the time-step, denoted by the subscript. Additionally, the unprimed indices indicate that the Hilbert spaces are shared with the process tensor, while the primed indices correspond to Hilbert spaces shared with the control sequence. Finally, each Hilbert space is labelled by whether it corresponds to an input $i$ or output $o$ of the respective object that the superprocess connects to on that index.

Summing over like indices from our aforementioned expression for the Choi state of the process tensor (Eq. (2)) corresponds to the action of the superprocess on the process tensor to form a new effective background process. The action of an $n \rightarrow n$ superprocess on a length $n$ process tensor is:

$$\left[ T_{n,0} \right]_{n,0} = \text{tr}_{z,s} \left\{ \left( \mathbb{I}^{z \times s'} \otimes Y_{n,0}^{T} \right) \Psi_{n,0} \right\}, \tag{A2}$$

where the lack of time-step subscripts indicates that we are summing for all time-steps. $T$ indicates the partial
transpose of the Choi representation of the process tensor.

Appendix B: Intra-Step Behaviour

Here, we derive the how an individual map from a process tensor is affected by a superprocess in isolation. When reduced to act on only one map \( \mathcal{C}_{\alpha+1:0} \), a superprocess is just a supermap \( S_{\alpha+1:0} \). We take \( \mathcal{E}^{sc}_{\alpha+1:0} \) to be the primitive free resource, which is a fixed output map, and find its image under all allowed supermaps. The result hinges on the form of \( C_{\alpha+1:0} \), which is analysed for each of the three classes of communication. In the next Sec. C, we will outline how these results are applied to find the free processes in IV C.

1. \( \mathcal{C}_{\alpha+1:0} \) is a fixed output channel:

\[ S_{\alpha+1:0} \begin{bmatrix} \mathcal{E}^{sc}_{\alpha+1:0} & \rho_{szs}' \end{bmatrix} = \mathcal{W}_{\alpha+1} \circ (\mathcal{C}_{\alpha+1:0} \otimes \mathcal{E}^{sc}_{\alpha+1:0}) \circ \mathcal{V}_{szs} \begin{bmatrix} \rho_{szs}' \end{bmatrix} = \text{tr} \mathcal{W}_{\alpha+1} \begin{bmatrix} \sigma_{\alpha+1}^{szs} \end{bmatrix} \begin{bmatrix} \rho_{szs}' \end{bmatrix} = \delta_{szs} \]

Next, we turn our attention to the image of the primitive resource under any allowed supermaps is just other fixed output maps.

2. \( \mathcal{C}_{\alpha+1:0} \) is any quantum channel:

\[ S_{\alpha+1:0} \begin{bmatrix} \mathcal{E}^{sc}_{\alpha+1:0} & \rho_{szs}' \end{bmatrix} = \mathcal{W}_{\alpha+1} \circ (\mathcal{C}_{\alpha+1:0} \otimes \mathcal{E}^{sc}_{\alpha+1:0}) \circ \mathcal{V}_{szs} \begin{bmatrix} \rho_{szs}' \end{bmatrix} \begin{bmatrix} \rho_{sxz}' \end{bmatrix} = \delta_{szs} \]

Thus, the image of an arbitrarily useless bipartite quantum channel under all supermaps with quantum communication is all bipartite quantum channels. This procedure was to remove \( \mathcal{E}^{sc}_{\alpha+1:0} \) from mathematical consideration, and transplant it with \( \mathcal{C}_{\alpha+1:0} \) instead.

3. \( \mathcal{C}_{\alpha+1:0} \) is any entanglement breaking channel:

The case of entanglement breaking communication can be solved with a minor extension to what was done with quantum communication. Rather than letting \( \mathcal{C}_{\alpha+1:0} \) be any map, we write it as a POVM measurement and subsequent re-preparation on the ancilla subsystems.

\[ \mathcal{E}^{szs}_{\alpha+1:0} \begin{bmatrix} \rho_{szs}' \end{bmatrix} = \sum_k \mathcal{V}_{szs} \begin{bmatrix} \rho_{szs}' \end{bmatrix} \begin{bmatrix} \rho_{sxz}' \end{bmatrix} \begin{bmatrix} \rho_{szs}' \end{bmatrix} = \delta_{szs} \]

Appendix C: Inter-Step Behaviour

In the next section we took \( \mathcal{E} \) to be the fixed output map and observed how well \( \mathcal{C} \) was able to bypass the blockage in information flow. Now we turn our attention to the information flow between steps. Now, \( \mathcal{A} \) is taken to be the fixed output map, and \( \mathcal{K} \) is used to circumvent the information blockage.

This procedure is analogous to the intra-step case. Groupings of maps \( \mathcal{M} \) take the roles of \( \mathcal{V} \) and \( \mathcal{W} \). \( \mathcal{A} \) takes on the role of \( \mathcal{E} \), and \( \mathcal{K} \) takes the role of \( \mathcal{C} \). The rest of the mathematics is identical so we will not repeat it.

Appendix D: Notation summary

Shown in this section is a summary of notation for the most common objects featured in this work. The type of script used for each object holds a specific meaning:
Figure 7. Hilbert spaces of the superprocess. The top Hilbert spaces connect with the process tensor (unprimed), while the bottom ones connect to the control sequence (primed). The spaces are also labelled by whether they are incoming i or outgoing o from the perspective of the object connecting to the superprocess. There is also a subscript label for the time-step. s is the final output of the process.

Figure 8. A diagrammatic representation of our 'transplantation' procedure. The initial system state is passed through \( \mathcal{E}_{\alpha+1} \) instead of \( \mathcal{E}_{\alpha+1,0} \), and the ancilla simulates the role which was previously played by the environment. Hence, \( \mathcal{E}_{\alpha+1,0} \) is the sole determinant of what the resultant map can be.
| Object | Meaning |
|--------|---------|
| A      | Control sequence |
| A′     | Client/Fiducial control sequence |
| A''    | Client/Fiducial control operation |
| T      | Process tensor/background process |
| T'     | Transformed process |
| I      | Set of potential process tensors |
| E      | Individual step of the background process |
| Z      | Superprocess |
| Z*I    | Set of implementable superprocesses |
| V      | Pre-operation in dilated form of superprocess |
| W      | Post-operation in dilated form of superprocess |
| C      | Communication map in parallel to a process tensor step |
| K      | Communication map in parallel to client control operation |
| M      | Individual step of transformed background process |
| Z| A K | Left action, yielding effective background process |
| Z| A K | Right action, yielding full experimental control |
| Z| A K | Full dynamics, yielding output state |

Table V. Summary of notation for the most common objects featured in this work.