A WEIGHTED SOBOLEV REGULARITY THEORY OF THE PARABOLIC EQUATIONS WITH MEASURABLE COEFFICIENTS ON CONIC DOMAINS IN $\mathbb{R}^d$

KYEONG-HUN KIM, KIJUNG LEE, AND JINSOL SEO

ABSTRACT. We establish existence, uniqueness, and arbitrary order Sobolev regularity results for the second order parabolic equations with measurable coefficients defined on the conic domains $D$ of the type

$$D(M) := \left\{ x \in \mathbb{R}^d : \frac{x}{|x|} \in M \right\}, \quad M \subset S^{d-1}. \quad (0.1)$$

We obtain the regularity results by using a system of mixed weights consisting of appropriate powers of the distance to the vertex and of the distance to the boundary. We also provide the sharp ranges of admissible powers of the distance to the vertex and to the boundary.

1. Introduction

Various weighted Sobolev spaces have been used in the study of elliptic and parabolic equations, for instance, when the equations are defined on non-smooth domains (e.g. [1, 7, 13, 15, 18]), they are degenerate near the boundary of domains (e.g. [4, 16]), or they have rough external forces (e.g. [7, 11, 17]). In general, such irregular conditions combined with Dirichlet boundary condition cause the derivatives of solutions to blow up at the boundary and consequently one needs appropriate weights to understand the blow-up behaviors in view of regularity theory.

In this article we study the weighted Sobolev theory of the parabolic equation

$$u_t = \sum_{i,j=1}^d a^{ij}(t) u_{x^i x^j} + f, \quad t > 0 ; \quad u(0, \cdot) = 0 \quad (1.1)$$
given with zero boundary condition on the conic domain

$$D = D(M) := \left\{ x \in \mathbb{R}^d : \frac{x}{|x|} \in M \right\}.$$

Here, $M$ decides the shape of the domain and we assume that it is an open subset of $S^{d-1}$ with $C^2$ boundary; see Figure 1 in Section 2. The key point of this article is considering such domains which have smooth part and non-smooth part together. We also assume that the coefficients $a^{ij}(t)$ are merely measurable in $t$ and the external force $f$ can be very wild near the boundary of the domain.

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The second author was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) (No. NRF-2019R1F1A1068988).
Our interest on conic domains comes from the related theory of stochastic partial differential equations (SPDEs), especially stochastic parabolic equations. In this case the derivatives of the solutions are more sensitive near the boundary than the deterministic case, even near the smooth part of the boundary. We will give more explanation on this below.

To understand the behaviors of solutions near the boundary of conic domains, we use a new weight which actually is a unification of weights from the sources [11, 10, 17] and [13, 18, 19]. We obtain regularity results using a system of mixed weights consisting of appropriate powers of the distance to the vertex and of the distance to the boundary. Let

\[ \rho_o(x) := |x| \quad \text{and} \quad \rho(x) := d(x, \partial D) \]

denote the distance from \( x \in D \) to the vertex and to the boundary of \( D \), respectively.

We prove in this article that for any \( p \in (1, \infty) \) and \( n = 0, 1, 2, \ldots \), the estimate

\[
\int_0^T \int_D \left( |\rho^{-1} u|^p + |u_x|^p + |\rho u_{xx}|^p + \cdots + |\rho^{n+1} D^{n+2} u|^p \right) \rho_o^{-\theta} \rho^{\theta-d} \, dx \, dt \\
\leq N \int_0^T \int_D \left( |\rho f|^p + \cdots + |\rho^{n+1} D^n f|^p \right) \rho_o^{-\theta} \rho^{\theta-d} \, dx \, dt \tag{1.2}
\]

holds for the solution to equation (1.1) provided that

\[
d - 1 < \Theta < d - 1 + p, \quad p(1 - \lambda^+ \lambda^-) < \theta < p(d - 1 + \lambda^-). \tag{1.3}
\]

Here, \( \lambda^+ \) and \( \lambda^- \) are positive constants determined by \( \mathcal{M} \) and the operator \( \mathcal{L} = \sum_{ij} a^{ij}(t) D_{ij} \); see Definition 2.8 below and also see Proposition 2.11. As can be seen in estimate (1.2), our mixed weights help us measure the regularity of the solution near both the vertex and other boundary points. Note also that the second and higher derivatives of the solution satisfying (1.2) are allowed to blow up substantially fast near the boundary. Moreover, the external force \( f \) is allowed to be very wild near the boundary.

As we mentioned above, the main motivation of our interest in the weighted Sobolev spaces lies in the theory of SPDEs. It turns out (see [12]) that due to the incompatibility between random noises and Dirichlet boundary condition, the second and higher order derivatives of solutions to SPDEs blow up near the boundary; this behavior of the solutions occurs even on \( \mathbb{C}^\infty \) domains. Hence, we need an appropriate weight system to measure the derivatives near the boundary. In [6, 12] it is shown that if domains satisfy \( \mathbb{C}^1 \)-boundary condition, then the effects of such incompatibility can be described very accurately by a system of weights based solely on the distance to the boundary. As we may guess, with conic domains we need more subtle approach and it turns out that it is very appropriate for us to involve \( \rho \) and \( \rho_o \) in the manner presented in (1.2).

A preliminary and important step for the main result of [6, 12] on SPDEs on \( \mathbb{C}^1 \) domains was constructing the corresponding result on the deterministic equation, that is equation (1.1). This article is such work related to conic domains and the estimate (1.2) is the backbone of it. As a comparison, when the boundary is nice, the work was done in [7, 11] and the result is as follows: for the solution to equation (1.1) defined on a \( \mathbb{C}^1 \) domain \( \mathcal{O} \), it holds that

\[
\int_0^T \int_\mathcal{O} \left( |\rho^{-1} u|^p + |u_x|^p + |\rho u_{xx}|^p \right) \rho^{\theta-d} \, dx \, dt \leq N \int_0^T \int_\mathcal{O} |\rho f|^p \rho^{\theta-d} \, dx \, dt \tag{1.4}
\]
provided that
\[ p \in (1, \infty), \quad d - 1 < \Theta < d - 1 + p. \]
We also remark that if \( \partial \Omega \in C^{1,\delta}, \ \delta \in (0,1], \) then the second order derivative of solution can be estimated for wider range of \( \Theta: \) it is shown in [14] that
\[
\int_0^T \int_\Omega |\rho u_{xx}|^p \rho^{\Theta-d} \ dx \ dt \leq N \int_0^T \int_\Omega |\rho f|^p \rho^{\Theta-d} \ dx \ dt
\]
holds for \( d - 1 - \delta < \Theta < d - 1 + \theta. \) However, smooth domains do not yield wider range of \( \Theta \) for lower order derivatives of solution. That is, estimate (1.4) holds only for \( d - 1 < \Theta < d - 1 + p \) even on \( C^\infty \) domains (see [11]).

Now, what if domains do not have \( C^1 \) boundary? For instance, the boundary could have a vertex which makes the boundary of the domain non-smooth, i.e. a conic domain. Our interest on conic domains arises with a question which, in particular, asks if there can be an estimate on simple Lipschitz domains that makes estimates (1.4) a particular case for \( C^1 \) domains. It turns out that estimate (1.4) fails to hold in conic domains in general. Note that (1.2) with \( n = 0 \) and \( \Omega = \mathcal{D} \) may yield (1.4) if one can take \( \theta = \Theta. \) However, due to the ranges of \( \Theta \) and \( \theta \) in (1.3), taking \( \theta = \Theta \) is possible only if
\[ \Theta \in (d - 1, d - 1 + p) \cap (p(1 - \lambda_c^+), p(d - 1 + \lambda_c^-)) \]
Actually, an example in [5] shows that for any \( p > 4, \) there is a 2-dimensional conic domain of the type
\[
\mathcal{D} = \left\{ (r \cos \eta, r \sin \eta) \in \mathbb{R}^2 : r > 0, \eta \in \left( -\frac{\kappa}{2}, \frac{\kappa}{2} \right) \right\}
\]
with an appropriately chosen \( \kappa \in (0,2\pi) \) and a function \( u, \) a solution to the heat equation, such that estimate (1.4) fails even for \( \Theta = \bar{d}(= 2), \) as taking \( \theta = \Theta \) in (1.2) is not allowed for the constructed function \( u \) in the example. This example demonstrates that the presence of \( \rho_c \) in (1.2) is inevitable, and it also suggests that (1.2) and (1.4) are of different character.

In summary, the weight system based only on the distance to the boundary is insufficient to construct a regularity theory of SPDEs defined on general conic domains and one way or another we need a mixed weight like ours described above and we settle down with (1.2). In a subsequent article, based on the results presented in this article, we plan to construct the corresponding theory on SPDEs defined on the conic domains.

We also remark that if one formally replaces \( \rho_c \) with \( \rho \) in (1.2), then one sees
\[
\int_0^T \int_{\mathcal{D}} \left( |\rho^{-1}u|^p + |u_x|^p + |\rho_c u_{xx}|^p \right) \rho^{\Theta-d} \ dx \ dt \leq N \int_0^T \int_{\mathcal{D}} |\rho f|^p \rho^{\Theta-d} \ dx \ dt, \tag{1.6}
\]
which actually holds true (see [13, 18, 19]) for the same \( \theta \) satisfying (1.3). However, the weight system based only on the distance to the vertex provides poor regularity result near the boundary, and moreover it is not much useful in the study of SPDEs since higher derivatives of solutions to SPDEs can not be controlled without the help of weights related to the distance to the boundary. Hence, for this article and the subsequent one related to SPDEs, estimate (1.2) is essential.

Now, we shortly describe the main steps of the proof for estimate (1.2):
- We use a localization argument to control the higher order derivatives of solution in terms of lower order derivatives of solution and free terms. Consequently, the result of this step reduces the problem into obtaining appropriate estimate of the zero-th order derivative of solution. The idea of our localization argument is taken from [11] and modified in this article to handle Sobolev spaces with our mixed weights.
- We obtain the estimate of the zero-th order derivative of solution using the solution representation formula and a refined Green’s function estimate. We use direct but very delicate computations to derive the desired estimate. Such direct computation skill is borrowed from [13] and modified here to handle a mixed weight system.

Finally, we would like to add an important comment that the study on conic domains with \(d \geq 2\) is much involved than the case \(d = 2\). This article includes this task in Section 3.

This article is organized as follows. In Section 2 we introduce some properties of weighted Sobolev spaces and present our main result, Theorem 2.12. In Section 3 we estimate weighed \(L_p\) norm of the zero-th order derivative of the solution based on direct but highly nontrivial computations. The estimates of the derivatives of the solution are obtained in Section 4, and finally in Section 5 our main result is proved.

**Notation**

- We use := to denote a definition.

- For a measure space \((A, \mathcal{A}, \mu)\), a Banach space \(B\) and \(p \in [1, \infty)\), we write \(L^p_p(A, \mathcal{A}, \mu; B)\) for the collection of all \(B\)-valued \(\mathcal{A}\)-measurable functions \(f\) such that

  \[
  \|f\|^p_{L^p_p(A, \mathcal{A}, \mu; B)} := \int_A \|f\|^p_B \,d\mu < \infty.
  \]

  Here, \(\mathcal{A}\) is the completion of \(\mathcal{A}\) with respect to \(\mu\). The Borel \(\sigma\)-algebra on a topological space \(E\) is denoted by \(\mathcal{B}(E)\). We will drop \(A\) or \(\mu\) or even \(B\) in \(L^p_p(A, \mathcal{A}, \mu; B)\) when they are obvious from the context.

- \(\mathbb{R}^d\) stands for the \(d\)-dimensional Euclidean space of points \(x = (x^1, \ldots, x^d)\), \(B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\}\), \(\mathbb{R}^d_+ := \{x = (x^1, \ldots, x^d) : x^1 > 0\}\), and \(S^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}\).

- For \(\mathcal{O} \subset \mathbb{R}^d\), \(B_R^\mathcal{O}(x) := B_R(x) \cap \mathcal{O}\) and \(Q_R^\mathcal{O}(t, x) := (t - R^2, t) \times B_R^\mathcal{O}(x)\).

- \(\mathbb{N}\) denotes the natural number system, and \(\mathbb{Z}\) denotes the set of integers.

- For \(x, y \in \mathbb{R}^d\), \(x \cdot y := \sum_{i=1}^d x^i y^i\) denotes the standard inner product.

- For a domain \(\mathcal{O}\) in \(\mathbb{R}^d\), \(\partial \mathcal{O}\) denotes the boundary of \(\mathcal{O}\).

- For any multi-index \(\alpha = (\alpha_1, \ldots, \alpha_d)\), \(\alpha_i \in \{0\} \cup \mathbb{N}\),

  \[
  f_1 = \frac{\partial f}{\partial t}, \quad f_{x^i} = D_i f := \frac{\partial f}{\partial x^i}, \quad D^\alpha f(x) := D_1^{\alpha_1} \cdots D_1^{\alpha_d} f(x).
  \]

  We denote \(|\alpha| := \sum_{i=1}^d \alpha_i\). For the second order derivatives we denote \(D_j D_i f\) by \(D_{ij} f\). We often use the notation \(|g f_x|^p\) for \(|g|^p \sum_{i} |D_i f|^p\) and \(|g f_{xx}|^p\) for \(|g|^p \sum_{i,j} |D_{ij} f|^p\). We also use \(D^m f\) to denote arbitrary partial derivatives of order \(m\) with respect to the space variable.

- \(\Delta f := \sum_i D_i^2 f\), the Laplacian for \(f\).

- For \(n \in \{0\} \cup \mathbb{N}\), \(W^\alpha_p(\mathcal{O}) := \{ f : \sum_{|\alpha| \leq n} \int_\mathcal{O} |D^\alpha f|^p dx < \infty\}\), the Sobolev space.
• For a domain $\mathcal{O} \subseteq \mathbb{R}^d$, $C^\infty(\mathcal{O})$ is the space of infinitely differentiable functions with compact support in $\mathcal{O}$. $supp(f)$ denotes the support of the function $f$. Also, $C^\infty(\mathcal{O})$ denotes the the space of infinitely differentiable functions in $\mathcal{O}$.

• Throughout the article, the letter $N$ denotes a finite positive constant which may have different values along the argument while the dependence will be informed; $N = N(a,b,\cdots)$, meaning that $N$ depends only on the parameters inside the parentheses.

• $A \sim B$ means that there is a constant $N$ independent of $A$ and $B$ such that $A \leq N B$ and $B \leq N A$.

• $d(x,\mathcal{O})$ stands for the distance between a point $x$ and a set $\mathcal{O} \subset \mathbb{R}^d$.

• $a \lor b := \max\{a,b\}$ and $a \land b := \min\{a,b\}$.

• $1_U$ is the indicator function on $U$.

• We will use the following sets of functions (see [13]).

  - $\mathcal{V}(Q^O_R(t_0,x_0))$ : the set of functions $u$ defined at least on $Q^O_R(t_0,x_0)$ and satisfying
    \[
    \sup_{t \in (t_0-R^2,t_0]} \|u(t,\cdot)\|_{L^2(B^O_R(x_0))} + \|\nabla u\|_{L^2(Q^O_R(t_0,x_0))} < \infty.
    \]

  - $\mathcal{V}_{loc}(Q^O_R(t_0,x_0))$ : the set of functions $u$ defined at least on $Q^O_R(t_0,x_0)$ and satisfying
    \[
    u \in \mathcal{V}(Q^O_R(t_0,x_0)), \quad \forall r \in (0,R).
    \]

2. The Main Result on Conic Domains

Throughout this article we assume $d \geq 2$. Let $\mathcal{M}$ be a nonempty open set in $S^{d-1} := \{ x \in \mathbb{R}^d : |x| = 1 \}$ with $\overline{\mathcal{M}}^S \neq S^{d-1}$, where $\overline{\mathcal{M}}^S$ is the closure of $\mathcal{M}$ in $S^{d-1}$.

We define our conic domain in $\mathbb{R}^d$ by

\[
\mathcal{D} = \mathcal{D}(\mathcal{M}) := \left\{ x \in \mathbb{R}^d \setminus \{0\} \mid \frac{x}{|x|} \in \mathcal{M} \right\}.
\]

For example, when $d = 2$, for each fixed angle $\kappa \in (0, 2\pi)$ we can consider

\[
\mathcal{D} = \mathcal{D}(\kappa) := \left\{ (r \cos \eta, r \sin \eta) \in \mathbb{R}^2 \mid r \in (0, \infty), -\frac{\kappa}{2} < \eta < \frac{\kappa}{2} \right\}. \quad (2.1)
\]

In this article we study the regularity theory of the parabolic equation

\[
u_t = \sum_{i,j=1}^{d} a^{ij}(t)u_{x^i x^j} + f =: Lu + f, \quad t > 0, x \in \mathcal{D}; \ u(0,\cdot) = 0 \quad (2.2)
\]

under the Dirichlet boundary condition, where the diffusion coefficients $a_{ij}$ are real valued measurable functions of $t$, $a_{ij} = a_{ji}$, $i,j = 1,\ldots,d$, and satisfy the uniform parabolicity condition, i.e. there exist constants $\nu_1, \nu_2 > 0$ such that for any $t \in \mathbb{R}$ and $\xi = (\xi^1,\ldots,\xi^d) \in \mathbb{R}^d$,

\[
\nu_1 |\xi|^2 \leq \sum_{i,j} a_{ij}(t)\xi^i \xi^j \leq \nu_2 |\xi|^2. \quad (2.3)
\]

Now we specify our condition on $\mathcal{M}$. Since $\overline{\mathcal{M}}^S \neq S^{d-1}$, upon an appropriate rotation we may assume

\[
s_0 := (0,0,\cdots,0,-1) \notin \overline{\mathcal{M}}^S.
\]
Thus we can define the stereographic projection $\phi$ that maps the points of $S^{d-1} \setminus \{s_0\}$ onto the tangent plane at $-s_0$ which we identify with $\mathbb{R}^{d-1}$:

$$\phi : S^{d-1} \setminus \{s_0\} \to \mathbb{R}^{d-1}, \quad \phi(x', x^d) = \frac{2}{1 + x^d} x'$$

for $(x^1, \ldots, x^{d-1}, x^d) =: (x', x^d) \in S^{d-1} \setminus \{s_0\}$.

**Assumption 2.1.** The set $\mathcal{M}$ in $S^{d-1}$ is of class $C^2$, meaning that $\phi(\mathcal{M})$, the image of $\mathcal{M}$ under $\phi$, has $C^2$ boundary in $\mathbb{R}^{d-1}$.

To explain our main result in the frame of weighted Sobolev regularity, we introduce appropriate function spaces.

Recall

$$\rho_\circ(x) := |x|, \quad \rho(x) = \rho_D(x) := d(x, \partial D),$$

which denote the distances from a point $x \in D$ to the vertex 0 and to the boundary of $D$, respectively. For $p \in (1, \infty)$, $\theta \in \mathbb{R}$ and $\Theta \in \mathbb{R}$, define

$$L^p_{\rho_D, \rho}(D) := L^p(D, \rho_D^{\theta-\Theta} \rho^n \partial_D^{\Theta-d} dx; \mathbb{R}).$$

That is, $L^p_{\rho_D, \rho}(D)$ is the class of real-valued functions $f$ such that

$$\|f\|_{L^p_{\rho_D, \rho}(D)} := \left( \int_D |f|^p \rho_D^{\theta-\Theta} \rho^n \partial_D^{\Theta-d} dx \right)^{1/p} < \infty.$$

Note that $\rho_\circ^{\theta-\Theta} \rho^{\Theta-d} = \rho_\circ^{\Theta-d} \left( \frac{\rho}{\rho_\circ} \right)^{\Theta-d}$, which implies that our weight captures the dependence of functions on $\rho_\circ$ and the ratio $\frac{\rho}{\rho_\circ}$. With this building block, for $n \in \{0, 1, 2, \ldots\}$ we define the function spaces

$$K^n_{p, \theta, \Theta}(D) = \left\{ f : \|f\|_{K^n_{p, \theta, \Theta}(D)} := \sum_{|\alpha| \leq n} \|\rho_\circ^{\alpha} \partial_D^\alpha f\|_{L^p_{\rho_D, \rho}(D)} < \infty \right\}.$$

Note $K^n_{p, \theta, \Theta}(D) = L^p_{\rho_D, \rho}(D)$, and for any integer $n \in \{0, 1, 2, \ldots\}$

$$\|f\|_{K^n_{p, \theta+n, \Theta+n}(D)} = \sum_{|\alpha| \leq n} \|\rho_\circ^{\alpha+n} \partial_D^\alpha f\|_{L^p_{\rho_D, \rho}(D)}, \quad (2.4)$$
Below we list some basic properties of the spaces $K_{p,\theta,\Theta}^n(D)$. More properties are discussed in Section 4.

**Lemma 2.2.** Let $p \in (1, \infty)$, $\theta \in \mathbb{R}$, $\Theta \in \mathbb{R}$ and $n \in \{0, 1, 2, \ldots\}$.

(i) The space $K_{p,\theta,\Theta}^n(D)$ is a Banach space.

(ii) For any $\mu \in \mathbb{R}$

$$N^{-1}\|f\|_{K_{p,\theta,\Theta}^n(D)} \leq \|\rho_\mu^p f\|_{K_{p,\theta,\Theta}^{-\mu}D^n(D)} \leq N\|f\|_{K_{p,\theta,\Theta}^n(D)},$$

where $N = N(d, n, p, \mu)$.

(iii) If $n \geq 1$, then the differential operator $D_1 : K_{p,\theta,\Theta}^n(D) \to K_{p,\theta,\Theta}^{n-1}(D)$ is bounded for any $i = 1, \ldots, d$. Moreover, we have

$$\|D_1 f\|_{K_{p,\theta,\Theta}^{n-1}(D)} \leq \|f\|_{K_{p,\theta,\Theta}^n(D)},$$

for any multi-index $\alpha$ satisfying $|\alpha| \leq n$.

(iv) Let $R > 1$ and $f \in L_{1,loc}(D)$. If

$$\text{supp}(f) \subset V_R := \left\{x \in D : \frac{1}{R} < \rho(x) < R\right\},$$

then $f \in K_{p,\theta,\Theta}^n(D)$ if and only if $f \in W_p^n(D)$. Moreover

$$N^{-1}\|f\|_{K_{p,\theta,\Theta}^n(D)} \leq \|f\|_{W_p^n(D)} \leq N\|f\|_{K_{p,\theta,\Theta}^n(D)},$$

where $N = N(d, n, p, \Theta, R)$.

(v) $C_c^\infty(D)$ is dense in $K_{p,\theta,\Theta}^n(D)$.

**Proof.** The proof of (i) is straightforward and left to the reader. (ii) is due to the observation that

$$\sup_{x \in D} \left(\left|\rho_\mu^{|\alpha|}(x)(D_1 \rho_\mu^p(x))\right|\right) < \infty$$

holds for any $\mu \in \mathbb{R}$ and multi-index $\alpha$. (iii) follows the definition of the norm. (iv) is obvious since $\rho_\mu$ is bounded below and $\rho$ is bounded from above and below by positive constants on the support of $f$.

Let us prove (v). First note that by (ii), without loss of generality we may assume that $\theta = \Theta$. Let $f \in K_{p,\theta,\Theta}^n(D)$. Thus,

$$\|f\|_{K_{p,\theta,\Theta}^n(D)}^p = \sum_{|\alpha| \leq n} \int_D \left|\rho_\mu^{|\alpha|}D_1 \rho_\mu^p\right|^p \rho^{-\alpha}dx < \infty.

We choose a sequence of infinitely differentiable functions $\xi_m$ such that $\xi_m$ has support in $V_{2m}$, $0 \leq \xi_m \leq 1$, $\xi_m(x) \to 1$ as $m \to \infty$ for $x \in D$, and $|\beta|D^{\beta}\xi_m$ is uniformly bounded and goes to zero as $m \to \infty$ for any multi-index $\beta$ with $|\beta| \geq 1$. For instance, one can construct such functions as follows. Choose a nonnegative function $\xi \in C_c^\infty(\mathbb{R}^d)$ satisfying $\text{supp}(\xi) \subset B_1(0)$ and $\int_{\mathbb{R}^d} \xi(x)dx = 1$, and let

$$\xi^\epsilon := \frac{1}{\epsilon_d} \xi \left(\frac{\cdot}{\epsilon}\right)$$

for any $\epsilon > 0$. For $m \in \mathbb{N}$, let us define

$$\xi_m := 1_{D_1/m} * \xi^{\left(\frac{1}{m}\right)} - 1_{D_m} * \xi^{\left(\frac{1}{m}\right)},$$

where

$$D_\epsilon = \left\{x \in D : \rho(x) > \epsilon\right\}, \quad \epsilon > 0,$$
and \( \ast \) denotes the convolution of two functions involved. Then, as we intended, \( \xi_m \) satisfies 
\[ \text{supp}(\xi_m) \subset V_{2m} \]
and
\[ |D^\beta \xi_m(x)| \leq N m^{-|\beta|} (1_{V_{2m}} \setminus V_m)(x), \quad \forall x \in \mathbb{R}^d \]
for any multi-index \( \beta \) with \(|\beta| \geq 1\), where \( N \) is independent of \( x \) and \( m \).

Note that
\[ \lim_{m \to \infty} (1 - 1_{V_{2m}}) = \lim_{m \to \infty} (1_{V_{2m}} \setminus V_m) = 0 \]
pointwise. Hence, we get
\[ \lim_{m \to \infty} \| f - f \xi_m \|_{K_p^{n-1},0,\Theta}(D) = 0 \]
by Lebesgue’s dominating convergence theorem with a dominating function
\[ \sum_{|\alpha| \leq n} |p|^{|\alpha|} D^\alpha f \mid p^{\theta-d}. \]

Since \( \text{supp}(f \xi_m) \subset V_{2m} \), \( f \xi_m \) is in \( W_p^n(D) \). For each \( m \), by mollifying and cutting off, we choose \( g_{m,k} \in C_c^\infty(V_{4m}) \) such that \( g_{m,k} \to f \xi_m \) as \( k \to \infty \) in \( W_p^n(D) \) and hence in \( K_p^n,\Theta,\Theta(D) \) by (iv), meaning that we can choose \( g_m \in C_c^\infty(D) \) satisfying
\[ \| f \xi_m - g_m \|_{K_p^n,\Theta,\Theta(D)} \leq 2^{-m}. \]

Consequently, we get
\[ \limsup_{m \to \infty} \| f - g_m \|_{K_p^n,0,\Theta(D)} \leq \limsup_{m \to \infty} (\| f - f \xi_m \|_{K_p^n,\Theta,\Theta(D)} + 2^{-m}) = 0, \]
and (v) is proved.

Finally, we introduce our function space in which the solution \( u \) to equation (2.2) lies. For \( T \in (0, \infty) \), \( p \in (1, \infty) \), \( \theta \in \mathbb{R} \), \( \Theta \in \mathbb{R} \), and \( n \in \{0, 1, 2, \ldots\} \), we define the function spaces
\[ K_p^{n,0,\Theta,\Theta}(D, T) := \mathbb{L}_p ((0, T]; K_p^n,\Theta,\Theta(D)) \]
with \( \mathbb{L}_p(D, T) := \mathbb{K}_p^{0,\Theta,\Theta}(D, T) \).

**Remark 2.3.** By modifying the proof of Lemma 2.2(v), based on a mollification with respect to both time and space variables, one can prove that \( C_c^\infty ((0, T) \times D) \) is dense in \( \mathbb{K}_p^{n,0,\Theta,\Theta}(D, T) \).

Now we define our sense of solution together with the space for the source \( f \).

**Definition 2.4.** Let \( p \in (1, \infty) \), \( \theta \in \mathbb{R} \), \( \Theta \in \mathbb{R} \) and \( n \in \{0, 1, 2, \ldots\} \).

(i) We write \( u \in K_p^{n+2,\Theta,\Theta}(D, T) \) if \( u \in K_p^{n+2,0,\Theta,\Theta}(D, T) \) and there exists \( \tilde{f} \in \mathbb{K}_p^{n,\Theta,\Theta}(D, T) \) such that
\[ u_t = \tilde{f}, \quad t \in (0, T] ; \quad u(0, \cdot) = 0 \]
in the sense of distributions on \( D \), that is, for any \( \varphi \in C_c^\infty(D) \) the equality
\[ (u(t, \cdot), \varphi) = \int_0^t (\tilde{f}(s, \cdot), \varphi) ds \]
holds for all \( t \in (0, T] \). The norm in \( K_p^{n+2,\Theta,\Theta}(D, T) \) is defined by
\[ \| u \|_{K_p^{n+2,\Theta,\Theta}(D, T)} := \| u \|_{K_p^{n+2,0,\Theta,\Theta}(D, T)} + \| u_t \|_{K_p^{n,\Theta,\Theta}(D, T)}. \]
(ii) We say that $u$ is a solution to equation (2.2) in $K^{n+2}_{p,\theta+p,\Theta+p}(D,T)$ if the source $f$ is in $K^{n+2}_{p,\theta+p,\Theta+p}(D,T)$ and $u \in K^{n+2}_{p,\theta-p,\Theta-p}(D,T)$ satisfies

$$u_t = \mathcal{L}u + f, \quad t \in (0,T) \quad ; \quad u(0,\cdot) = 0$$

in the sense of distributions on $D$.

**Remark 2.5.** By Lemma 2.2 (iii), if $u \in K^{n+2}_{p,\theta-p,\Theta-p}(D,T)$, then $\mathcal{L}u \in K^{n+2}_{p,\theta+p,\Theta+p}(D,T)$. This supports Definition 2.4 (ii).

**Theorem 2.6.** For $p \in (1,\infty)$, $\theta \in \mathbb{R}$, $\Theta \in \mathbb{R}$, and $n \in \{0,1,2,\ldots\}$, $K^{n+2}_{p,\theta,\Theta}(D,T)$ is a Banach space.

**Proof.** The completeness of the space $K^{n+2}_{p,\theta,\Theta}(D,T)$ can be proved by repeating the argument in Remark 3.8 of [9], in which the completeness is proved for the special case of $D = \mathbb{R}^d_+$ and $\theta = \Theta$, meaning that only the distance to the boundary is involved in the weight, nevertheless the argument works for us. \qed

**Remark 2.7.** In Definition 2.4 the ranges of $\theta$ and $\Theta$ are still open. However, there is no guarantee yet that there is a solution in $K^{n+2}_{p,\theta-p,\Theta-p}(D,T)$ for arbitrary $p \in (1,\infty)$, $\theta \in \mathbb{R}$, $\Theta \in \mathbb{R}$, and $f \in K^{n+2}_{p,\theta+p,\Theta+p}(D,T)$. Particularly, since we are assuming zero Dirichlet boundary condition, the source function $f$ and $\theta, \Theta$ are needed to be appropriately chosen.

It turns out that the admissible range of $\theta$ for $f$ (and hence for $u$) is affected by the shape of the conic domain $D = D(M)$, the uniform parabolicity of the diffusion coefficients, the space dimension $d$, and the summability parameter $p$, while $\Theta$ depends only on $d$ and $p$.

To explain the admissible range of $\theta$ for equation (2.2) we need the following definition.

**Definition 2.8** (see Section 2 of [13]). One can refer to Section II for some of the notations below.

(i) By $\lambda_{-\mathcal{L}}$ we denote the supremum of all $\lambda \geq 0$ such that for some constant $K_0 = K_0(\mathcal{L},M,\lambda)$ it holds that

$$|u(t,x)| \leq K_0 \left(\frac{|x|}{R}\right)^\lambda \sup_{Q^D_R(t_0,0)} |u|, \quad \forall (t,x) \in Q^D_R(t_0,0) \quad (2.6)$$

for any $R > 0$, $t_0$ and $u$ belonging to $\mathcal{V}_{loc}(Q^D_R(t_0,0))$ and satisfying

$$u_t = \mathcal{L}u \text{ in } Q^D_R(t_0,0) \quad ; \quad u(t,x) = 0 \text{ for } x \in \partial D.$$

(ii) By $\lambda_{-\mathcal{L}}$ we denote supremum of all $\lambda \geq 0$ with above property for the operator

$$\mathcal{L} = \sum_{i,j} a_{ij}(-t)D_{ij}.$$

Although we consider one fixed operator $\mathcal{L}$ in this article, we want to pose the following definition as a preparation for our subsequent article on stochastic parabolic equations, for which, as we mentioned in the introduction, the result of this article will serve crucially. In the case of stochastic parabolic equations, the operator will be random and involve infinitely many operators. The definition is used to establish explicit dependency of constants appearing in our estimates.
Proposition 2.11. (i) Let $\mathcal{T}_{\nu_1,\nu_2}$ denote collection of all operators $\tilde{L} = \sum_{i,j=1}^d \tilde{a}_{ij}(t)D_{ij}$ such that $\tilde{A}(t) := (\tilde{a}_{ij}(t))_{d \times d}$ is measurable in $t$ and satisfies the uniform parabolicity condition (2.3).

(ii) By $\lambda_c(\nu_1,\nu_2)$ we denote the supremum of all $\lambda \geq 0$ such that for some constant $K_0 = K_0(\nu_1,\nu_2,\mathcal{M},\lambda)$ it holds that for any $\tilde{L} \in \mathcal{T}_{\nu_1,\nu_2}$, $R > 0$, $t_0$,

$$|u(t,x)| \leq K_0 \left( \frac{|x|}{R} \right)^\lambda \sup_{Q^R_R(t_0,0)} |u|, \quad \forall (t,x) \in Q^R_R(t_0,0), \quad (2.7)$$

provided that $u$ belongs to $\mathcal{V}_\text{loc}(Q^R_R(t_0,0))$ and satisfies

$$u_t = \tilde{L}u \quad \text{in } Q^R_R(t_0,0); \quad u(t,x) = 0 \quad \text{for } x \in \partial D.$$

Remark 2.10. (i) Note that the dependency of $K_0$ in Definition 2.9 is more explicit compared to that of Definition 2.8. By definitions, we have $\lambda^\pm_{c,\mathcal{L}} \geq \lambda_c(\nu_1,\nu_2)$ if $\mathcal{L} \in \mathcal{T}_{\nu_1,\nu_2}$.

(ii) The values of $\lambda^\pm_{c,\mathcal{L}}$ and $\lambda_c(\nu_1,\nu_2)$ do not change if one replaces $\frac{4}{\nu}$ in (2.6) and (2.7) by any number in $(1/2, 1)$ (see [13, Lemma 2.2]).

Both $\lambda^\pm_{c,\mathcal{L}}$ and $\lambda^\pm_{c,\mathcal{L}}$ definitely depend on $\mathcal{M}$ and $\mathcal{L}$. Below are some sharp estimates of $\lambda^+_{c,\mathcal{L}}$ and $\lambda^-_{c,\mathcal{L}}$. See [9, 13] for more informations.

Proposition 2.11. (i) If $\mathcal{L} = \Delta_x$, then

$$\lambda^\pm_{c,\mathcal{L}} = \frac{-d-2}{2} + \sqrt{\Lambda + \frac{(d-2)^2}{4}} > 0,$$

where $\Lambda$ is the first eigenvalue of Laplace-Beltrami operator with the Dirichlet condition on $\mathcal{M}$. In particular, if $d = 2$ and $\mathcal{D} = D^{(c)}$ (see (2.7)), then

$$\lambda^\pm_{c,\mathcal{L}} = \frac{\pi}{\kappa}.$$

(ii) Let $0 < \nu_1 < \nu_2$. Then for any $\tilde{L} \in \mathcal{T}_{\nu_1,\nu_2}$,

$$\lambda^\pm_{c,\mathcal{L}} \geq \lambda_c(\nu_1,\nu_2) \geq \left( \frac{-d-2}{2} + \sqrt{\frac{\nu_1}{\nu_2}} \sqrt{\Lambda + \frac{(d-2)^2}{4}} \right) \vee 0.$$

Proof. See [13, Theorem 2.4] (i) and [3, Theorem 3.2] for (ii). □

Here is the main result of this article. The proof is placed in Section 5.

Theorem 2.12. Let Assumption 2.7 and condition (2.3) hold, $p \in (1, \infty)$, and $n \in \{0, 1, 2, \ldots\}$. Also, assume that $\Theta \in \mathbb{R}$ and $\theta \in \mathbb{R}$ satisfy

$$d - 1 < \Theta < d - 1 + p \quad \text{and} \quad p(1 - \lambda^+_{c,\mathcal{L}}) < \theta < p(d - 1 + \lambda^-_{c,\mathcal{L}}).$$

Then for any $f \in \mathbb{R}^n_{p,\theta+p,\Theta+p}(D,T)$ there exists a unique solution $u$ in $\mathcal{K}^{n+2}_{p,\theta+p}(D,T)$ to equation (2.2). Moreover, the inequality

$$\|u\|_{\mathcal{K}^{n+2}_{p,\theta+p}(D,T)} \leq N \|f\|_{\mathbb{R}^n_{p,\theta+p,\Theta+p}(D,T)} \quad (2.8)$$

holds with a constant $N = N(\mathcal{M}, d, p, n, \theta, \Theta, \mathcal{L})$. Moreover, if

$$p(1 - \lambda_c(\nu_1,\nu_2)) < \theta < p(d - 1 + \lambda_c(\nu_1,\nu_2)), \quad (2.9)$$

then the constant $N$ in (2.8) depends only on $\mathcal{M}, d, p, n, \theta, \Theta, \nu_1$ and $\nu_2$. □
Remark 2.13. This is a good place to explain why the solution $u$ in Theorem 2.12 satisfied zero Dirichlet boundary condition. Under the assumption $d - 1 < \Theta < d - 1 + p$, [2, Theorem 2.8] implies that the trace operator is well defined for functions $u \in K^2_{p,\Theta-p}(D, T)$, and hence by Lemma 2.2 (v) we have $u|_{\partial D} = 0$.

Remark 2.14. Due to (2.4), estimate (2.8) certainly yields

$$
\sum_{|\alpha| \leq n + 2} \int_0^T \int_D |\rho|^{-1} D^{\alpha} u| p \rho^\Theta - d dx dt \\
\leq N \sum_{|\alpha| \leq n} \int_0^T \int_D |\rho|^{1+1} D^{\alpha} f| p \rho^\Theta - d dx dt.
$$

(2.10)

Remark 2.15. If $d = 2$, $D = D(\kappa)$ of (2.1), and $L = \Delta_x$, then the condition $p(1 - \lambda_{\kappa,L}^\kappa) < \Theta < p(d - 1 + \lambda_{\kappa,L}^\kappa)$ becomes

$$
p(1 - \frac{\pi}{\kappa}) < \Theta < p(1 + \frac{\pi}{\kappa}).
$$

(2.11)

If $\kappa = \pi$, then $D$ is a half space in $\mathbb{R}^2$. In this case the admissible range of $\Theta$ is $(0, 2p)$ which surely contains the range $(1, p + 1)$ of $\Theta$. Hence, we are safe to take $\Theta = \Theta$ in (2.10) and get

$$
\int_0^T \int_D (|\rho|^{-1} u| p + |u_x| p + |\rho u_{xx}|| p) \rho^\Theta - d dx dt \\
\leq N \int_0^T \int_D |\rho f| p \rho^\Theta - d dx dt,
$$

for any $\Theta \in (1, p + 1)$. This fits into the result of [11], and thus our result extends the main result in [11] up to the conic domains at least in two-dimensional space provided that

$$
\Theta \in \left(p(1 - \frac{\pi}{\kappa}), p(1 + \frac{\pi}{\kappa})\right).
$$

(2.12)

One can notice that, for any fixed $\Theta \in (1, 1 + p)$, (2.12) holds for all $p > 1$ if $\kappa \leq \pi$, and if $\kappa > \pi$ then (2.11) holds only for sufficiently small $p$. The bigger the angle $\kappa$ is, the less the summability of derivatives is.

Remark 2.16. Theorem 2.1 in [13] gives an $L_{p,q}$-estimate with the weight system involving only the distance to the vertex with the range of $\mu = \frac{\theta - d}{p} + 1$ given by

$$
2 - \frac{d}{p} - \lambda_{\kappa,L}^+ < \mu < d - \frac{d}{p} + \lambda_{\kappa,L}^-.
$$

(2.13)

(2.13) is the same as $p(1 - \lambda_{\kappa,L}^+ < \Theta < p(d - 1 + \lambda_{\kappa,L}^-)$ and the result with $p = q$ there fits into (2.10) with $\Theta = d$ and $n = 0$ since $\rho \leq \rho_o$.

3. Key Estimate

In this section we prove Lemma 3.1 below, which plays the key role when we prove our main result, Theorem 2.12 in Section 5.

Let $G(t, s, x, y)$ denote the Green’s function for the operator $\partial_t - L$ with the domain $D = D(M)$. By definition, $G$ is nonnegative and, for any fixed $s \in \mathbb{R}$ and $y \in D$, the function $v = G(\cdot, s, \cdot, y)$ satisfies

$$
v_t = Lv \text{ in } (s, \infty) \times D; \quad v = 0 \text{ on } (s, \infty) \times \partial D; \quad v(t, \cdot) = 0 \text{ for } t < s.
$$

Here is the main result of this section.
Lemma 3.1. Let \( p \in (1, \infty) \), and let \( \theta \in \mathbb{R} \) and \( \Theta \in \mathbb{R} \) satisfy
\[
P(1 - \frac{\lambda_c}{c} - \frac{1}{d}) < \theta < p(1 - \lambda_c - \frac{1}{d}) \quad \text{and} \quad d - 1 < \Theta < d - 1 + p.
\]
Then for any \( f \in L_{p,\theta+p,\theta+p}(\mathcal{D},T) \) and the function \( u \) defined by
\[
u(t,x) := \int_0^t \int_{\mathcal{D}} G(t,s,x,y) f(s,y) dy \, ds,
\]
\( u \) is in \( L_{p,\theta-p,\theta-p}(\mathcal{D},T) \), and the estimate
\[
\int_0^T \int_{\mathcal{D}} |\rho|^{p-1} u|^{p} \rho^{\theta-\Theta} \rho^{\theta-d} dx dt \leq N \int_0^T \int_{\mathcal{D}} |\rho f|^{p} \rho^{\theta-\Theta} \rho^{\theta-d} dx dt
\] (3.1)
holds with \( N = N(M,d,p,\theta,\Theta) \). Moreover, if
\[
P(1 - \frac{\lambda_c}{c} \nu_1, \nu_2) < \theta < p(1 - \lambda_c \nu_1, \nu_2),
\]
then the constant \( N \) in (3.1) depends only on \( M,d,p,\theta,\Theta,\nu_1 \) and \( \nu_2 \).

To prove Lemma 3.1, we need two quantitative lemmas below, Lemma 3.2 and Lemma 3.7.

Lemma 3.2. Let \( \alpha + \beta > 0 \), \( \beta > 0 \), and \( \gamma > 0 \). Then there exists a constant \( N(\alpha,\beta,\gamma) > 0 \) such that
\[
\int_0^\infty \frac{1}{(\sqrt{t} + a)^\alpha (\sqrt{t} + b)^\beta t^{1-\gamma}} dt \leq \frac{N}{a^{\alpha} b^{\beta}}
\] (3.2)
for any \( a \geq b > 0 \).

Proof. Multiplying both sides of (3.2) by \( a^{\alpha} b^{\beta} \), we see that is is enough to prove
\[
\int_0^\infty \left( \frac{a}{\sqrt{t} + a} \right)^\alpha \left( \frac{b}{\sqrt{t} + b} \right)^\beta \left( \frac{\sqrt{t}}{\sqrt{t} + b} \right)^\gamma \frac{dt}{t}
\]
is bounded by a constant \( N = N(\alpha,\beta,\gamma) \).

- **Case 1.** \( \alpha \geq 0 \).
  Since \( a > 0 \) and \( \alpha \geq 0 \), we have
  \[
  \left( \frac{a}{\sqrt{t} + a} \right)^\alpha \leq 1.
  \]
  Hence, we get
  \[
  \int_0^\infty \left( \frac{a}{\sqrt{t} + a} \right)^\alpha \left( \frac{b}{\sqrt{t} + b} \right)^\beta \left( \frac{\sqrt{t}}{\sqrt{t} + b} \right)^\gamma \frac{dt}{t} \leq \int_0^\infty \left( \frac{b}{\sqrt{t} + b} \right)^\beta \left( \frac{\sqrt{t} + b}{\sqrt{t} + b} \right)^\gamma \frac{dt}{t} = \int_0^\infty \frac{1}{(\sqrt{s} + 1)^{\beta+\gamma} s^{1-\frac{\gamma}{2}}} ds,
\]
where the last quantity follows the change of variable, \( s = t/b^2 \) and it is finite since \( 1 - \frac{\gamma}{2} < 1 \) and \( 1 + \frac{\gamma}{2} > 1 \).

- **Case 2.** \( \alpha < 0 \).
  Since \( \alpha < 0 \) and \( a \geq b > 0 \), we have
  \[
  \left( \frac{a}{\sqrt{t} + a} \right)^\alpha \leq \left( \frac{b}{\sqrt{t} + b} \right)^\alpha.
  \]
Hence,
\[
\int_0^\infty \left( \frac{a}{\sqrt{t} + a} \right)^\alpha \left( \frac{b}{\sqrt{t} + b} \right)^\beta \left( \frac{\sqrt{t}}{\sqrt{t} + b} \right)^\gamma \frac{1}{t} dt \\
\leq \int_0^\infty \left( \frac{b}{\sqrt{t} + b} \right)^{\alpha + \beta} \left( \frac{\sqrt{t}}{\sqrt{t} + b} \right)^\gamma \frac{1}{t} dt \\
\leq N(\alpha, \beta, \gamma)
\]

since \(\alpha + \beta > 0, \gamma > 0\) and hence we can use the argument of Case 1. \(\square\)

The following lemma is a particular result of Lemma 3.7 with \(D = \mathbb{R}^d\) and will be used in the proof of Lemma 3.7.

**Lemma 3.3.** Let \(\sigma > 0, \alpha + \gamma > -d, \gamma > -1\) and \(\beta, \omega \in \mathbb{R}\). Then there exists a constant \(N(d, \alpha, \beta, \gamma, \omega, \sigma) > 0\) such that
\[
\int_{\mathbb{R}^d} \frac{|y|^\alpha}{(|y| + 1)^\beta} \frac{|y'|^\gamma}{(|y'| + 1)^\omega} e^{-\sigma|x-y|^2} dy \leq N (|x| + 1)^{\alpha - \beta} (|x|^1 + 1)^{\gamma - \omega} \tag{3.3}
\]
for any \(x = (x^1, \ldots, x^d) \in \mathbb{R}^d\).

**Proof.** 1. Moving \((|x| + 1)^{\alpha - \beta} (|x|^1 + 1)^{\gamma - \omega}\) to the left hand side of (3.3), and then using the change of variables \(x - y \rightarrow y\), we note that it is enough to show that
\[
I(x) := \int_{\mathbb{R}^d} \frac{1}{(|x| + 1)^{\alpha - \beta} (|x|^1 + 1)^{\gamma - \omega}} \frac{|y|^\alpha}{(|y| + 1)^\beta} \frac{|y'|^\gamma}{(|y'| + 1)^\omega} e^{-\sigma|x-y|^2} dy
\]

is bounded by a constant \(N = N(d, \alpha, \beta, \gamma, \omega, \sigma)\), where
\[
f(x, y) = \left( \frac{|x - y| + 1}{|x| + 1} \right)^{\alpha - \beta} \left( \frac{|x^1 - y^1| + 1}{|x|^1 + 1} \right)^{\gamma - \omega} e^{-\sigma|y|^2}.
\]

By the observation
\[
\frac{|x - y| + 1}{|x| + 1} \leq |x| + |y| + 1 \leq |y| + 1, \quad \frac{|x| + 1}{|x - y| + 1} \leq |x| + |y| + 1 \leq |x - y| + 1 \leq |y| + 1,
\]
and the similar observation for \(x^1\) and \(y^1\), we get
\[
f(x, y) \leq (|y| + 1)^{|\alpha - \beta|} (|y^1| + 1)^{|\gamma - \omega|} e^{-\sigma|y|^2}
\]
\[
\leq N \left\{ (|y'| + 1)^{|\alpha - \beta|} (|y^1| + 1)^{|\gamma - \omega|} e^{-\sigma|y|^2} + (|y^1| + 1)^{|\alpha - \beta| + |\gamma - \omega|} e^{-\sigma|y^1|^2} \right\}
\]
\[
= N(d, \alpha, \beta) (\psi_1(y') \phi_1(y^1) + \psi_2(y') \phi_2(y^1)),
\]
where \(y = (y^1, y^2) \in \mathbb{R}^1 \times \mathbb{R}^{d-1}\) and
\[
\psi_1(y') = (|y'| + 1)^{|\alpha - \beta|} e^{-\sigma|y'|^2}, \quad \phi_1(y^1) = (|y^1| + 1)^{|\gamma - \omega|} e^{-\sigma|y^1|^2},
\]
\[
\psi_2(y') = e^{-\sigma|y'|^2} \quad \text{and} \quad \phi_2(y^1) = (|y^1| + 1)^{|\alpha - \beta| + |\gamma - \omega|} e^{-\sigma|y^1|^2}.
\]
Therefore, we have

\[ I(x) \leq N \sum_{i=1}^{2} \int_{\mathbb{R}^{d}} \left( \frac{|x - y|}{|x - y| + 1} \right)^{\alpha} \left( \frac{|x^{1} - y^{1}|}{|x^{1} - y^{1}| + 1} \right)^{\gamma} \psi_{i}(y')\phi_{i}(y) dy. \]

2. Noting that there exists a constant \( C = C(d, \alpha, \beta, \gamma, \omega, \sigma) > 0 \) such that

\[ \|\psi_{i}\|_{L_{1}(\mathbb{R}^{d-1})}, \max_{\mathbb{R}^{d-1}}|\psi_{i}|, \|\phi_{i}\|_{L_{1}(\mathbb{R})}, \max_{\mathbb{R}}|\phi_{i}| \leq C \quad \text{for } i = 1, 2, \]

we only need to show that there exists \( N(d, \alpha, \gamma, C) \) such that

\[ I'(x) := \int_{\mathbb{R}^{d}} \left( \frac{|x - y|}{|x - y| + 1} \right)^{\alpha} \left( \frac{|x^{1} - y^{1}|}{|x^{1} - y^{1}| + 1} \right)^{\gamma} \psi(y')\phi(y) dy \leq N \]

for all \( x \), provided that

\[ \|\psi\|_{L_{1}(\mathbb{R}^{d-1})}, \max_{\mathbb{R}^{d-1}}|\psi|, \|\phi\|_{L_{1}(\mathbb{R})}, \max_{\mathbb{R}}|\phi| \leq C \]

for some constant \( C > 0 \).

Case 1. \( \alpha > -d + 1 \).

Put

\[ I''(x, y) = \int_{\mathbb{R}^{d-1}} \left( \frac{|x - y|}{|x - y| + 1} \right)^{\alpha} \psi(y') dy'. \]

If \( \alpha \geq 0 \), we instantly get

\[ I''(x, y) \leq \int_{\mathbb{R}^{d-1}} \psi(y') dy' \leq C. \]

If \( -d + 1 < \alpha < 0 \), we also have

\[ \left( \frac{|x - y| + 1}{|x - y|} \right)^{-\alpha} \leq \left( 1 + \frac{1}{|x' - y'|} \right)^{-\alpha} \leq N(\alpha) (1 + |x' - y'|^{\alpha}) \]

for a constant \( N(\alpha) \). Hence, we get

\[ I''(x, y) \leq N \left( \int_{\mathbb{R}^{d-1}} \psi(y') dy' + \int_{\mathbb{R}^{d-1}} |x' - y'|^{\alpha} \psi(y') dy' \right) \]

\[ \leq N \left( 2 \|\psi\|_{L_{1}(\mathbb{R}^{d-1})} + \max_{\mathbb{R}^{d-1}}|\psi| \int_{|x' - y'| < 1} |x' - y'|^{\alpha} dy' \right) \]

\[ \leq N(d, \alpha, C), \]

and, for all \( \alpha > -d + 1 \), we have

\[ I'(x) \leq N(d, \alpha, C) \int_{\mathbb{R}} \left( \frac{|x^{1} - y^{1}|}{|x^{1} - y^{1}| + 1} \right)^{\gamma} \phi(y') dy'. \]

Then, keeping the condition \( \gamma > -1 \) in mind and using the similar argument above, we have

\[ I'(x) \leq N(d, \alpha, \gamma, C) \]

for all \( x \).

Case 2. \( \alpha \leq -d + 1 \).

Since \( \alpha \leq -d + 1 \) and \( \alpha + \gamma > -d \), we note

\[ \gamma + 1 > -\alpha - d + 1 \geq 0. \]
Choose any 
\[ \delta \in (-\alpha - d + 1, \gamma + 1) \subset (0, \infty). \]
Since \( \delta > 0 \), we have
\[ \left( \frac{|x^1 - y^1|}{|x^1 - y^1| + 1} \right)^\delta \leq \left( \frac{|x - y|}{|x - y| + 1} \right)^\delta. \]
Hence, we get
\[ I'(x) \leq \int_{\mathbb{R}^d} \left( \frac{|x - y|}{|x - y| + 1} \right)^{\alpha + \delta} \left( \frac{|x^1 - y^1|}{|x^1 - y^1| + 1} \right)^{\gamma - \delta} \psi(y') \phi(y) dy \leq N(d, \alpha, \gamma, C) \]
by \( \alpha + \delta > -d + 1, \gamma - \delta > -1, \) and the argument of Case 1.

In Lemma 3.3, the first coordinate \( x^1 \) plays the role of the distance between \( x \in \mathbb{R}^d \) and \( \partial \mathbb{R}^d_+ \). For our domain \( D = \mathcal{D}(\mathcal{M}) \) we need to generalize Lemma 3.3 with \( \rho(x) \), the distance between \( x \) and \( \partial D \). This will be done in Lemma 3.7. To prove Lemma 3.7, we use Lemma 3.3 and the following two auxiliary lemmas.

**Lemma 3.4.** Let \( \partial^S \mathcal{M} \) denote the boundary of \( \mathcal{M} \) in \( S^{d-1} \).

(i) For any \( x \in S^{d-1} \),
\[ d(x, \partial D) \leq d(x, \partial^S \mathcal{M}) \leq 2 \, d(x, \partial D). \]
(ii) Let \( 0 < \delta \leq 1 \) and \( x, y \in \mathbb{R}^d \setminus \{0\} \). If
\[ \frac{x \cdot y}{|x||y|} \leq (1 - \delta), \]
then \( \delta \left( |x|^2 + |y|^2 \right) \leq |x - y|^2. \)

**Proof.** The second claim \([\text{ii}]\) follows a direct calculation and we leave it to the reader.

Let us prove \([\text{i}]\). Take any \( x \in S^{d-1} \). The fact \( \partial^S \mathcal{M} \subset \partial D \) in \( \mathbb{R}^d \) instantly implies
\[ d(x, \partial D) \leq d(x, \partial^S \mathcal{M}). \]
For the other inequality, we consider two cases of \( d(x, \partial D) \) (\( \leq |x - 0| = 1 \)).

If \( d(x, \partial D) = 1 \), we have
\[ d(x, \partial^S \mathcal{M}) \leq 2 \leq 2 \, d(x, \partial D) \]
since \( x \in S^{d-1} \) and \( \mathcal{M} \subset S^{d-1} \).

If \( d(x, \partial D) < 1 \), then we note that there exists \( y \in \partial D \) satisfying \( |y| \neq 0 \) and \( |x - y| = d(x, \partial D) \). Take the unique \( \theta \in [0, \pi] \) satisfying \( x \cdot y = |x||y| \cos \theta = |y| \cos \theta \). Since \( ty \) is on \( \partial D \) for any \( t > 0 \), the function \( f(t) := |x - ty|^2 \), \( t > 0 \), has the minimum at \( t = 1 \). Using
\[ f(t) = 1 - 2t \, x \cdot y + t^2 |y|^2 = 1 - 2t |y| \cos \theta + t^2 |y|^2 = |y|^2 \left( t - \frac{\cos \theta}{|y|} \right)^2 + \sin^2 \theta, \]
we get
\[ |y| = \cos \theta, \quad \theta \in [0, \pi/2], \quad \text{and} \quad |x - y| = \sin \theta. \]
Hence, we have
\[ d(x, \partial^S \mathcal{M}) \leq \left| x - \frac{y}{|y|} \right| = \sqrt{2 - 2 \cos \theta} = 2 \sin \frac{\theta}{2} \leq 2 \sin \theta = 2|x - y| = 2 \, d(x, \partial D). \]
\[ \square \]
Recall that \( \overline{\mathcal{M}}^S \) is the closure of \( \mathcal{M} \) in \( S^{d-1} \), and \( s_0 = (0, \ldots, -1) \notin \overline{\mathcal{M}}^S \). Denote
\[
B^S_r(p) := B_r(p) \cap S^{d-1}, \quad p \in S^{d-1}.
\]

**Remark 3.5.** Denote \( R_0 := \frac{1}{2} d(s_0, \overline{\mathcal{M}}^S) \). Since \( \Omega := \phi(\mathcal{M}) \) is of class of \( C^2 \) in \( \mathbb{R}^{d-1} \), there exist constants \( r_0 \in (0, 1 \land R_0) \) and \( N_0 > 0 \) such that for any \( p \in \partial^S \mathcal{M} \) and \( V_p := \phi(B^S_{r_0}(p)) \subset \mathbb{R}^{d-1} \), there exists a \( C^2 \) bijective (flattening boundary) map \( \psi_p = (\psi^1_p, \ldots, \psi^{d-1}_p) \) from \( V_p \) onto a domain \( G_p := \psi_p(V_p) \subset \mathbb{R}^{d-1} \) satisfying the following:

(i) \( \psi_p(V_p \cap \Omega) = G_p \cap \mathbb{R}^{d-1}_+ \) and \( \psi_p(y_0) = 0 \), where \( y_0 = \phi(p) \) and \( \mathbb{R}^{d-1}_+ = \{ y = (y^1, \ldots, y^{d-1}) \in \mathbb{R}^{d-1} : y^1 > 0 \} \).

(ii) \( \psi_p(V_p \cap \partial \Omega) = G_p \cap \partial \mathbb{R}^{d-1}_+ \).

(iii) for any \( y \in V_p \),
\[
N_0^{-1} d(y, \partial \Omega) \leq |\psi^1_p(y)| \leq N_0 d(y, \partial \Omega).
\]

(iv) \( \|\psi_p\|_{C^2(V_p)} + \|\psi_p^{-1}\|_{C^2(G_p)} \leq N_0 \) and
\[
N_0^{-1} |y_1 - y_2| \leq |\psi_p(y_1) - \psi_p(y_2)| \leq N_0 |y_1 - y_2|, \quad \forall y_1, y_2 \in V_p.
\]

For the next lemmas, for open sets \( U \) of \( S^{d-1} \) and open sets \( V \) in \( \mathbb{R}^{d-1} \), we consider two types of domains \( \mathcal{D}(U) \) and \( \mathcal{D}(V) \) in \( \mathbb{R}^d \) defined by
\[
\mathcal{D}(U) := \left\{ x \in \mathbb{R}^d \setminus \{0\} : \frac{x}{|x|} \in U \right\},
\]
and
\[
\mathcal{D}(V) = \left\{ x = r(\xi^1, 1) = r(\xi^1, \cdots, \xi^{d-1}, 1) : r > 0, \xi^1 \in V \right\}.
\]
Obviously, \( x = (x', x^d) \in \mathcal{D}(V) \) if and only if \( x^d > 0 \) and \( \frac{x'}{x^d} \in V \).

Now we recall the stereographic projection \( \phi \) described in Section 2 and also take the constant \( r_0 \) from Remark 3.5. Then for any fixed \( p \in \partial^S \mathcal{M} \), let \( V_p := \phi(B^S_{r_0}(p)) \) and \( G_p := \psi_p(V_p) \) with the map \( \psi_p \) described in Remark 3.5. Then we can define the following two bijective maps:
\[
\Phi_p : \mathcal{D}(B^S_{r_0}(p)) \rightarrow \mathcal{D}(V_p)
\]
\[
(x) \rightarrow (|x| \phi\left(\frac{x}{|x|}\right), |x|),
\]
and
\[
\Psi_p : \mathcal{D}(V_p) \rightarrow \mathcal{D}(G_p)
\]
\[
(y', y^d) \rightarrow (y^d \psi_p\left(\frac{y'}{y^d}\right), y^d).
\]

Note
\[
\Psi_p \circ \Phi_p(x) = \left(|x| \psi_p \circ \phi\left(\frac{x}{|x|}\right), |x|\right).
\]

**Lemma 3.6.** There exists a constant \( N = N(\mathcal{M}, d) > 0 \) such that for any \( p \in \partial^S \mathcal{M} \) and the maps \( \Phi_p \) and \( \Psi_p \),

(i) \( N^{-1} |x - y| \leq |(\Psi_p \circ \Phi_p)(x) - (\Psi_p \circ \Phi_p)(y)| \leq N |x - y| \),

(ii) \( N^{-1} \leq |\det D(\Psi_p \circ \Phi_p)(x)| \leq N \),

(iii) \( N^{-1} d(x, \partial \Omega) \leq |(\Psi_p \circ \Phi_p)^1(x)| \leq N d(x, \partial D) \).
(iv) \( N^{-1} |x| \leq |(\Psi_p \circ \Phi_p)(x)| \leq N |x| \)
for all \( x, y \in D(B^S_{r_0}(p)) \), where \( D(\Psi_p \circ \Phi_p) \) is the Jacobian matrix function of \( \Psi_p \circ \Phi_p \).

Proof. We note that there exists a constant \( R_1 > 0 \), which depends only on the constants \( R_0 \) and \( N_0 \) in Remark 3.5 such that
\[
V_p, G_p \subset \{ x \in \mathbb{R}^{d-1} : |x| < R_1 \}
\]
for any \( p \in \partial^S \mathcal{M} \). For \( p \in \partial^S \mathcal{M} \), let us define \( U_p := B^S_{r_0}(p) \).

For any fixed \( p \in \partial^S \mathcal{M} \), by the definition of \( \Psi_p \) and Remark 3.5 for any \( \xi, \eta \in \mathcal{D}(V_p) \) we have
\[
|\Psi_p(\xi) - \Psi_p(\eta)| \leq |\xi - \eta| \left| \psi_p(\frac{\xi'}{\eta'}) \right| + |\xi - \eta| \left| \psi_p(\frac{\eta'}{\eta'}) \right| + |\xi - \eta|
\]
\[
\leq N_0 \left| \xi' - \eta' \right| + (R_1 + 1)|\xi - \eta|
\]
\[
\leq N_0 \left( |\xi' - \eta'| + \left| \frac{\eta'}{\eta'} \right| |\eta' - \xi'| \right) + (R_1 + 1)|\xi - \eta|
\]
\[
\leq N(N_0, R_1) |\xi - \eta|,
\]
where \( \xi = (\xi', \xi''), \eta = (\eta', \eta'') \), and \( N_0, R_1 \) together with \( R_0 \) are the constants from Remark 3.5. Adding the same calculation for \( \Psi_p^{-1} \), we find that there exists a constant \( N = N(\mathcal{M}) > 0 \) so that
\[
N^{-1} |\xi - \eta| \leq |\Psi_p(\xi) - \Psi_p(\eta)| \leq N |\xi - \eta|
\]
for any \( p \in \partial^S \mathcal{M} \) and \( \xi, \eta \in \mathcal{D}(V_p) \). On the other hand, by the definition of \( \Phi_p \), for fixed \( x, y \in \mathcal{D}(U_p) \) we get
\[
|\Phi_p(x) - \Phi_p(y)| \leq |x| \left| \phi\left( \frac{x}{|x|} \right) - \phi\left( \frac{y}{|y|} \right) \right| + |x| - |y| \left| \phi\left( \frac{y}{|y|} \right) \right| + |x| - |y|
\]
\[
\leq N(R_0) \left| x - \frac{x}{|y|} y \right| + (R_1 + 1)|x - y|
\]
\[
\leq N |x - y|,
\]
where \( N \) depends only on \( R_0 \) and \( R_1 \).

For the reverse inequality, we first note that \( \phi^{-1} \in C^\infty(\mathbb{R}^{d-1}) \). Hence, there exists a constant \( N = N(R_1, d) \) such that
\[
|\phi^{-1}(\xi') - \phi^{-1}(\eta')| \leq N |\xi' - \eta'|
\]
for all \( \xi', \eta' \in \mathbb{R}^{d-1} \) with \( |\xi'|, |\eta'| \leq R_1 \). By a similar calculation as above, we get
\[
|\Phi_p^{-1}(\xi) - \Phi_p^{-1}(\eta)| \leq N |\xi - \eta|
\]
for any \( \xi, \eta \in \mathcal{D}(V_p) \). This implies
\[
|x - y| \leq N|\Phi_p(x) - \Phi_p(y)|
\]
for any \( x, y \in \mathcal{D}(U_p) \).

Gathering all, we conclude that there exists a constant \( N = N(\mathcal{M}, d) > 0 \) such that
\[
N^{-1} |x - y| \leq |(\Psi_p \circ \Phi_p)(x) - (\Psi_p \circ \Phi_p)(y)| \leq N |x - y|
\]
for any \( p \in \partial^S \mathcal{M} \) and \( x, y \in \mathcal{D}(U_p) \).
By the result of [1], there exists a constant \( N \) such that
\[
\sup_{D(U_p)} |D(\Psi_p \circ \Phi_p)| + \sup_{D(U_p)} |D(\Phi_p^{-1} \circ \Psi_p^{-1})| \leq N.
\]
This gives
\[
N^{-1} \leq |\det D(\Psi_p \circ \Phi_p)(x)| \leq N, \quad \forall x \in D(B_{r_0}(p))
\]
for some constant \( N = N(M, d) > 0 \).

To prove (iii) and (iv), we first recall that
\[
(\Psi_p \circ \Phi_p)(x) = \left( |x| (\psi_p \circ \phi) \left( \frac{x}{|x|} \right), |x| \right).
\]

By Remark 3.3, we have
\[
(\Psi_p \circ \Phi_p)^1(x) = |x|(\psi_p^1 \circ \phi)(\hat{x}) \sim |x|d(\phi(\hat{x}), \partial \Omega)
\]
for \( x \in D(U_p) \), where \( \hat{x} := \frac{x}{|x|} \) and \( \Omega = \phi(M) \). Since, \( \phi(\partial^S M) = \partial \Omega \), the result of (i) and Lemma 3.4 give
\[
d(\phi(\hat{x}), \partial \Omega) \sim d(\hat{x}, \partial^S M) \sim d(\hat{x}, \partial D).
\]

Consequently, we get
\[
(\Psi_p \circ \Phi_p)^1(x) \sim |x|d(\phi(\hat{x}), \partial \Omega) \sim |x|d(\hat{x}, \partial D) = d(x, \partial D).
\]

All the comparabilities above depend only on \( M \) and \( d \).

This is due to
\[
|(\Psi_p \circ \Phi_p)(x)| \sim |x| \left( |(\psi_p \circ \phi)(\hat{x})| + 1 \right), \quad |(\psi_p \circ \phi)(\hat{x})| \leq C < \infty
\]
for \( x \in D(U_p) \), where the comparability relation and the constant \( C \) depend only on \( M, d \).

Lemma 3.7. Let \( \sigma > 0, \alpha + \gamma > -d, \gamma > -1 \) and \( \beta, \omega \in \mathbb{R} \). Then there exists a constant \( N(M, d, \alpha, \beta, \gamma, \sigma, \omega) > 0 \) such that
\[
\int_D \frac{|y|^{\alpha}}{(|y| + 1)^{\beta}} \frac{\rho(y)^\gamma}{(\rho(y) + 1)^\omega} e^{-\sigma|x-y|^2} dy \leq N \left( |x| + 1 \right)^{\alpha - \beta} \left( \rho(x) + 1 \right)^{\gamma - \omega}
\]
for any \( x \in D \).

Proof. 1. Take the constant \( r_0 \) in Remark 3.3 and let
\[
F := \left\{ p \in S^{d-1} : d(p, \partial^S M) \leq \frac{r_0}{4} \right\}.
\]
Then there exist a finite number of points \( p_1, \ldots, p_m \in \partial^S M \) such that
\[
F \subset \bigcup_{i=1}^{m} B_{r_0/2}(p_i).
\]

Denote
\[
D_0 = D(M \setminus F) = \left\{ z \in D : d\left( \frac{z}{|z|}, \partial^S M \right) > \frac{r_0}{4} \right\}
\]
and
\[
D_i := D\left( B_{r_0/2}(p_i) \right) = \left\{ z \in D : d\left( \frac{z}{|z|}, p_i \right) < \frac{r_0}{2} \right\}
\]
for $i = 1, \cdots, m$. It is obvious that

$$D = \bigcup_{i=0}^{m} D_i.$$  

Now, we fix $x \in D$ and consider two parts of $D$:

$$E_1(x) = \left\{ y \in D : \frac{x}{|x|} \cdot \frac{y}{|y|} > 1 - \delta \right\} \quad \text{and} \quad E_2(x) = \left\{ y \in D : \frac{x}{|x|} \cdot \frac{y}{|y|} \leq 1 - \delta \right\}$$

with $\delta = \frac{r_0^2}{128}$. Note that if $y \in E_1(x)$, then

$$\left| \frac{x}{|x|} - \frac{y}{|y|} \right| < \frac{r_0}{8}.$$  

Now, we consider

$$I(x) := \int_{D} \frac{|y|^\alpha}{(|y| + 1)^\beta} \rho(y)^\gamma \frac{e^{-\sigma|x-y|^2}}{(|y| + 1)^\omega} \, dy = I_1(x) + I_2(x)$$

where

$$I_i(x) = \int_{E_i(x)} \frac{|y|^\alpha}{(|y| + 1)^\beta} \rho(y)^\gamma \frac{e^{-\sigma|x-y|^2}}{(|y| + 1)^\omega} \, dy, \quad i = 1, 2.$$  

2. Estimation of $I_1(x)$.

- **Case 1.** $x \in D_0$.

For $y \in E_1(x)$,

$$d\left( \frac{y}{|y|}, \partial S M \right) \geq d\left( \frac{x}{|x|}, \partial S M \right) - \left| \frac{x}{|x|} - \frac{y}{|y|} \right| \geq \frac{r_0}{8}.$$  

Therefore, we get

$$d\left( \frac{x}{|x|}, \partial S M \right), d\left( \frac{y}{|y|}, \partial S M \right) \in \left[ \frac{r_0}{8}, 2 \right].$$

By the observation $\rho(x) = |x|/\rho(x)$ and Lemma 3.4, there exists a constant $N = N(M, d) > 0$ such that

$$N^{-1}|x| \leq \rho(x) \leq N|x| \quad \text{and} \quad N^{-1}|y| \leq \rho(y) \leq N|y|.$$  

By Lemma 3.3, we get

$$I_1(x) \leq N \int_{D_0} \frac{|y|^\alpha}{(|y| + 1)^\beta} e^{-\sigma|x-y|^2} \, dy \leq N \int_{\mathbb{R}^d} \frac{|y|^\alpha}{(|y| + 1)^\beta} e^{-\sigma|x-y|^2} \, dy \leq N (|x| + 1)^{\alpha - \beta + \gamma - \omega} \leq N (|x| + 1)^{\alpha - \beta} \rho(x) + 1)^{\gamma - \omega},$$

where $N$ depends only on $d$, $\alpha$, $\beta$, $\gamma$, $\omega$, $\sigma$ and $M$.

- **Case 2.** $x \in \bigcup_{i=1}^{m} D_i$.

Without loss of generality, we assume $x \in D_1$. Then, for $y \in E_1(x)$, we have

$$\left| \frac{x}{|x|} - p_1 \right| < \frac{r_0}{2} \quad \text{and hence} \quad \left| \frac{y}{|y|} - p_1 \right| < r_0.$$
Also, note that
\[ |x - y| \sim |(\Psi_{p_1} \circ \Phi_p)(x) - (\Psi_{p_1} \circ \Phi_p)(y)|, \]
\[ \rho(x) \sim (\Psi_{p_1} \circ \Phi_p)^{\dagger}(x), \quad \rho(y) \sim (\Psi_{p_1} \circ \Phi_p)^{\dagger}(y), \]
\[ |x| \sim |(\Psi_{p_1} \circ \Phi_p)(x)|, \quad |y| \sim |(\Psi_{p_1} \circ \Phi_p)(y)| \]
for \( y \in E_2(x) \), where all the comparability relations depend only on \( \mathcal{M}, d \). By the change of variables \( z = (\Psi_{p_1} \circ \Phi_p)(y) \), we have
\[
I_1(x) = \int_{E_1(x)} \frac{|y|^{\alpha}}{(|y| + 1)^{\beta} (\rho(y) + 1)^{\gamma}} \, e^{-\sigma|y|^2} \, dy \\
\leq \int_{D(\mathcal{B}_0^{\gamma}(p_1))} \frac{|y|^{\alpha}}{(|y| + 1)^{\beta} (\rho(y) + 1)^{\gamma}} \, e^{-\sigma|y|^2} \, dy \\
\leq N \int_{\mathbb{R}^d} \frac{|z|^{\alpha}}{(|z| + 1)^{\beta} (|z'| + 1)^{\gamma}} \, e^{-\sigma'|z'|^2} \, dz,
\]
where \( z^* := (\Psi_{p_1} \circ \Phi_p)(x) \) and \( N \) and \( \sigma' \) depend only on \( \mathcal{M}, d, \) and \( \sigma \). Lastly, since \( \alpha + \gamma > -d, \gamma > -1 \), Lemma 5.3 yields
\[
\int_{\mathbb{R}^d} \frac{|z|^{\alpha}}{(|z| + 1)^{\beta} (|z'| + 1)^{\gamma}} \, e^{-\sigma'|z'|^2} \, dz \leq N(|x| + 1)\alpha - \beta (x^*)^{\gamma - \omega} \\
\leq N(|x| + 1)\alpha - \beta (\rho(x) + 1)^{\gamma - \omega}
\]
with \( N = N(\mathcal{M}, d, \alpha, \beta, \gamma, \omega, \sigma) \). Hence, we get
\[
I_1(x) \leq N(|x| + 1)^{\alpha - \beta} (\rho(x) + 1)^{\gamma - \omega}.
\]

3. Estimation of \( I_2(x) \).
Since \( \frac{x}{|x|} \cdot \frac{y}{|y|} \leq 1 - \delta \) for \( y \in E_2(x) \), Lemma 5.4 gives
\[
e^{-\sigma|y|^2} \leq e^{-\sigma'|y|^2} \, e^{-\sigma'|y|^2}
\]
where \( \sigma' = \sigma \delta \). Therefore we have
\[
I_2(x) = \int_{E_2(x)} \frac{|y|^{\alpha}}{(|y| + 1)^{\beta} (\rho(y) + 1)^{\gamma}} \, e^{-\sigma|y|^2} \, dy \\
\leq e^{-\sigma'|y|^2} \int_{D(\mathcal{B}_0^{\gamma}(p_1))} \frac{|y|^{\alpha}}{(|y| + 1)^{\beta} (\rho(y) + 1)^{\gamma}} \, e^{-\sigma'|y|^2} \, dy \\
\leq e^{-\sigma'|y|^2} \sum_{i=0}^{m} \int_{D_i} \frac{|y|^{\alpha}}{(|y| + 1)^{\beta} (\rho(y) + 1)^{\gamma}} \, e^{-\sigma'|y|^2} \, dy.
\]
Following calculations used for \( I_1(x) \), we get
\[
I_2(x) \leq N(\alpha, \beta, \gamma, \omega, \sigma') e^{-\sigma'|x|^2}.
\]
Now, note that for any fixed \( \sigma_1 > 0, \sigma_2 \in \mathbb{R}, \)
\[
(a + 1)^{\sigma_2} e^{-\sigma_1 a^2}, \quad a > 0
\]
is bounded by a constant depending only on \( \sigma_1, \sigma_2 \), and also note
\[
1 \leq \rho(x) + 1 \leq |x| + 1.
\]
Putting
\[ \sigma_2 = \begin{cases} 
- (\alpha - \beta) & \text{if } \gamma - \omega > 0 \\
- (\alpha - \beta + \gamma - \omega) & \text{otherwise,}
\end{cases} \]
we conclude that there exists a constant \( N = N(\alpha, \beta, \gamma, \omega, \sigma') > 0 \) such that
\[ e^{-\sigma_2 |x|^2} \leq N (|x| + 1)^{\alpha - \beta} (\rho(x) + 1)^{\gamma - \omega}. \]
Hence, we obtain
\[ I_2(x) \leq N (|x| + 1)^{\alpha - \beta} (\rho(x) + 1)^{\gamma - \omega}, \]
where \( N \) depends only on \( \mathcal{M}, \alpha, \beta, \gamma, \omega, \) and \( \sigma \).

Next we introduce what we prepared in [3] for our main result of this article. It is a refined estimate of the Green’s function of the parabolic operator \( \partial_t - L \) with the domain \( D = D(M) \).

**Theorem 3.8.** Let \( \lambda^+ \in (0, \lambda_{c, \mathcal{L}}^+), \lambda^- \in (0, \lambda_{c, \mathcal{L}}^-) \), and denote \( K_0^+ := K_0(\mathcal{L}, \mathcal{M}, \lambda^+) \) and \( K_0^- := K_0(\mathcal{L}, \mathcal{M}, \lambda^-) \), where \( K_0 \) is from Definition 2.8. Then there exist positive constants \( N = N(\mathcal{M}, \nu_1, \nu_2, \lambda^\pm, K_0^\pm) \) and \( \sigma = \sigma(\nu_1, \nu_2) \) such that
\[ G(t, s, x, y) \leq N (t - s)^{d/2} \frac{R_{t-s,x}^{\lambda^+ - 1} R_{t-s,y}^{\lambda^- - 1} J_{t-s,x} J_{t-s,y} e^{-\sigma |x - y|^2}} \]
for any \( t > s, x, y \in D \). Moreover, if
\[ \lambda^+, \lambda^- \in (0, \lambda_c(\nu_1, \nu_2)), \]
then the constant \( N \) in (3.4) can depend only on \( \mathcal{M}, \nu_1, \nu_2, \) and \( \lambda^\pm \).

**Proof.** (3.4) holds due to [3] Theorem 2.6. If (3.5) holds, then by Definition 2.9 the constant \( K_0^\pm \) can be chosen such that it depends only on \( \mathcal{M}, \nu_1, \nu_2 \) and \( \lambda^\pm \). \( \square \)

We are ready to prove Lemma 3.1.

**Proof of Lemma 3.1**

We only prove the lemma for the case \( \lambda^+ \in (0, \lambda_{c, \mathcal{L}}^+) \) and \( \lambda^- \in (0, \lambda_{c, \mathcal{L}}^-) \). This is because the same proof works under condition (3.5) without any changes. The difference of the dependency of constant \( N \) in (3.4) is inherited from the constant \( N \) in (3.4).

1. Denote \( \mu := 1 + (\theta - d)/p \) and \( \alpha := 1 + (\Theta - d)/p \). By the range of \( \theta \) given in the statement, we can always find \( \lambda^+ < \lambda_c^+ \) and \( \lambda^- < \lambda_c^- \) satisfying
\[ 2 - \frac{d}{p} - \lambda^+ < \mu < d - \frac{d}{p} + \lambda^- . \]
Also, by the range of \( \Theta \) we have
\[ 1 - \frac{1}{p} < \alpha < 2 - \frac{1}{p} . \]
By the designed range of $\mu$ and $\alpha$, we can choose and fix the constants $\gamma_1$, $\gamma_2$, $\omega_1$, and $\omega_2$ satisfying
\[
-\frac{d-2}{p} < \gamma_1 < \lambda^+ - 2 + \mu + \frac{2}{p}, \quad 0 < \gamma_2 < \lambda^- + d - \frac{d}{p} - \mu,
\]
\[
\frac{1}{p} < \omega_1 < \alpha - 1 + \frac{2}{p}, \quad 0 < \omega_2 < 2 - \frac{1}{p} - \alpha.
\]
Since $\lambda^+ < \lambda^+_c$ and $\lambda^- < \lambda^-_c$, by Theorem 3.3 there exist constants $N, \sigma > 0$ such that
\[
G(t, s, x, y) \leq \frac{N}{(t-s)^{d/2}} e^{-\sigma \frac{|x-y|^2}{(t-s)^2}} J_{t-s,x} J_{t-s,y} R_{t-s,x}^{\lambda^+ - 1} R_{t-s,y}^{\lambda^- - 1}
\]
\[
= \frac{N}{(t-s)^{d/2}} e^{-\sigma \frac{|x-y|^2}{(t-s)^2}} R_{t-s,x}^{\gamma_1} \left( \frac{J_{t-s,x}}{R_{t-s,x}} \right)^{\omega_1} R_{t-s,y}^{\gamma_2} \left( \frac{J_{t-s,y}}{R_{t-s,y}} \right)^{\omega_2}
\]
\[
\times R_{t-s,x}^{\lambda^+ - \gamma_1} \left( \frac{J_{t-s,x}}{R_{t-s,x}} \right)^{1-\omega_1} R_{t-s,y}^{\lambda^- - \gamma_2} \left( \frac{J_{t-s,y}}{R_{t-s,y}} \right)^{1-\omega_2}
\]
for all $t > s$ and $x, y \in D$.

2. We set
\[
h(t, x) = |x|^\mu \left( \frac{\rho(x)}{|x|} \right)^{\alpha} f(t, x) \quad ; \quad h = \rho_0^{\mu-\alpha} \rho^\alpha f.
\]
Then, because of $\mu = 1 + (\theta - d)/p$ and $\alpha = 1 + (\Theta - d)/p$, the terms in estimate (3.1) turn into
\[
\int_0^T \int_D |\rho(x) f(t, x)|^p \rho_0^{\theta-d}(x) \left( \frac{\rho(x)}{\rho_0(x)} \right)^{\Theta-d} dx dt = \|h\|_{L_p([0,T] \times D)}^p,
\]
\[
\int_0^T \int_D |\rho^{-1}(x) u(t, x)|^p \rho_0^{\theta-d}(x) \left( \frac{\rho(x)}{\rho_0(x)} \right)^{\Theta-d} dx dt = \left\| \rho_0^{\mu-\alpha} \rho^{\alpha-2} u \right\|_{L_p([0,T] \times D)}.
\]
for the function
\[
u(t, x) := \int_0^t \int_D G(t, s, x, y) f(s, y) dy ds,
\]
and hence we need to show
\[
\left\| \rho_0^{\mu-\alpha} \rho^{\alpha-2} u \right\|_{L_p([0,T] \times D)} \leq N \|h\|_{L_p([0,T] \times D)}.
\]
We start with, using Hölder inequality,
\[
|u(t, x)| = \left| \int_0^t \int_D G(t, s, x, y) f(s, y) dy ds \right|
\]
\[
\leq \int_0^t \int_D G(t, s, x, y) |y|^{-\mu+\alpha} \rho(y)^{-\alpha} |h(s, y)| dy ds
\]
\[
\leq N \cdot I_1(t, x) \cdot I_2(t, x),
\]
where $q = p/(p-1)$,
\[
I_1^p(t, x) = \int_0^t \int_D \frac{1}{(t-s)^{d/2}} e^{-\sigma \frac{|x-y|^2}{(t-s)^2}} K_{1,1}(t-s, x) K_{1,2}(t-s, y) |h(s, y)|^p dy ds,
\]
and
\[
I_2^q(t, x) = \int_0^t \int_D \frac{1}{(t-s)^{d/2}} e^{-\sigma \frac{|x-y|^2}{(t-s)^2}} K_{2,1}(t-s, x) K_{2,2}(t-s, y) |y|^{-(\mu+\alpha)} \rho^{-\alpha}(y) dy ds
\]
Moreover, since \( \lambda \) obtain
This implies
Hence, we have
Lemma 3.7, we get
\[
K_{1,1}(t, x) = R_{t,x}^{\gamma_1} \left( \frac{J_{t,x}}{R_{t,x}} \right)^{\omega_1 p}, \quad K_{1,2}(t, y) = R_{t,y}^{\gamma_2} \left( \frac{J_{t,y}}{R_{t,y}} \right)^{\omega_2 p},
\]
K_{2,1}(t, x) = R_{t,x}^{(\lambda^+ - \gamma_1)q} \left( \frac{J_{t,x}}{R_{t,x}} \right)^{(1 - \omega_1)q}, \quad K_{2,2}(t, y) = R_{t,y}^{(\lambda^+ - \gamma_2)q} \left( \frac{J_{t,y}}{R_{t,y}} \right)^{(1 - \omega_2)q}.

3. In this step, we will show that there exists a constant \( N > 0 \) such that
\[
I_2(t, x) \leq N |x|^{-\mu + \alpha} \rho(x)^{-\alpha + \frac{2}{p}}.
\]
In particular, the right hand side is independent of \( t \).
Since \((\lambda^- - \mu - \gamma_2)q > -d \) and \((1 - \alpha - \omega_2)q > -1 \), by change of variables and Lemma 3.7 we get
\[
\frac{1}{(t-s)^{d/2}} \int_{D} e^{-\sigma \frac{|x-y|^2}{t-s}} K_{2,2}(t-s, y)|y|^{(-\mu + \alpha)q} |\rho(y)|^{\alpha q} dy
= (t-s)^{-\mu q/2} \int_{D} e^{-\sigma \frac{|x-y|^2}{t-s}} |y|^{(\lambda^- - \mu - \gamma_2 - 1 + \alpha + \omega_2)q} |\rho(y)|^{(1 - \alpha - \omega_2)q} dy
\leq N (|s| + \sqrt{t-s})^{(-\mu + \alpha)q} (\rho(x) + \sqrt{t-s})^{-\alpha q}.
\]
Hence, we have
\[
I_2^1(t, x) \leq N \int_{0}^{t} K_{2,1}(t-s, x) \left( |x| + \sqrt{t-s} \right)^{(-\mu + \alpha)q} (\rho(x) + \sqrt{t-s})^{-\alpha q} ds
\leq N \int_{0}^{\infty} \frac{|x|^{(\lambda^+ - 1 - \gamma_1 + \omega_1)q}}{(|x| + \sqrt{s})^{(\lambda^+ - 1 + \mu - \gamma_1 - \alpha + \omega_1)q}} \frac{\rho(x)^{(1 - \omega_1)q}}{(|x| + \sqrt{s})^{(\alpha + 1 - \omega_1)q}} ds.
\]
Moreover, since \((\lambda^+ + \mu - \gamma_1)q > 2 \) and \((\alpha + 1 - \omega_1)q > 2 \), by Lemma 3.2 we further obtain
\[
I_2^2(t, x) \leq N |x|^{(-\mu + \alpha)q} \rho(x)^{-\alpha q + 2}.
\]
This implies
\[
|u(t, x)| \leq N I_1(t, x) \cdot I_2(t, x) \leq N |x|^{-\mu + \alpha} \rho(x)^{-\alpha + \frac{2}{p}} I_3(t, x),
\]
and hence
\[
|x|^{\mu - \alpha} \rho(x)^{2 - 2}|u(t, x)| \leq N \rho(x)^{-\frac{2}{p}} I_1(t, x).
\]
4. Using this, we have
\[
\|\rho \overline{\mu} - \alpha \overline{\rho}^{\alpha - 2} u\|_{L^p(\Omega \times D)}^p \leq N \int_{0}^{T} \int_{D} |\rho(x)|^{-2} I_1^p(t, x) dx dt
= N \int_{0}^{T} \int_{D} I_3(s, y) \cdot |h(s, y)|^p dy ds,
\]
where
\[
I_3(s, y) = \int_{s}^{T} \int_{D} \frac{1}{(t-s)^{d/2}} e^{-\sigma \frac{|x-y|^2}{t-s}} K_{1,1}(t-s, x) K_{1,2}(t-s, y) \rho(x)^{-2} dx dt.
\]
Since $\gamma_1 p - 2 > -d$ and $\omega_1 p - 2 > -1$, by change of variables and Lemma [3.7]

$$I_3(s, y) = \int_0^T \frac{1}{(t-s)^{d/2}} K_{1,2}(t-s, y) \left( \int_{\mathcal{D}} e^{-\sigma \frac{|x|^2}{\Phi}} K_{1,1}(t-s, x) \rho(x)^{-2} \right) dt$$

$$\leq \int_0^\infty \frac{1}{t} K_{1,2}(t, y) \left( \int_{\mathcal{D}} \frac{|x|^{(\gamma_1-\omega_1)p}}{(1+|x|)^{(\gamma_1-\omega_1)p}} \frac{\rho(x)^{\omega_1 p-2}}{(\rho(x)+1)^{\omega_1 p}} e^{-\sigma \frac{|x|^2}{\Phi}} dx \right) dt$$

$$\leq N \int_0^\infty K_{1,2}(t, y) \left( \rho(y) + \sqrt{t} \right)^{-2} dt$$

$$= N \int_0^\infty \frac{|y|^{(\gamma_2-\omega_2)p}}{(1+|y|)^{\gamma_2-\omega_2})} \frac{\rho(y)^{\omega_2 p-2}}{(\rho(y)+\sqrt{t})^{\omega_2 p+2}} dt.$$

Lastly, owing to $\gamma_2 p > 0$ and $\omega_2 p > 0$, Lemma [3.2] gives

$$I_3(s, y) \leq N(M, d, p, \theta, \Theta, \mathcal{L}).$$

Hence, there exists a constant $N = N(M, d, p, \theta, \Theta, \mathcal{L}) > 0$ such that

$$\|\rho_0^{-\alpha} \rho^{-2 \alpha} u\|_{L^p_p([0,T] \times \mathcal{D})} \leq N \|h\|_{L^p_p([0,T] \times \mathcal{D})} = N \|\rho_0^{-\alpha} \rho^{-2 \alpha} f\|_{L^p_p([0,T] \times \mathcal{D})}$$

for any $f$ and the corresponding function $u$ in the form [3.4]. This inequality is [3.7]. The lemma is proved. \hfill \Box

4. ESTIMATE OF HIGH ORDER DERIVATIVES

In this section we estimate weighted $L^p$-norm of derivatives of solutions to equation (2.2). This result is presented in Theorem 4.6 and the proof is based on an alternative definition of $K_{p,\theta,\Theta}^n(\mathcal{D})$ introduced below.

We start with weighted Sobolev space $H_{p,\theta}^n(\mathcal{D})$ introduced in [7] [10] [11] [17]. For any $p \in (1, \infty)$ and $\Theta \in \mathbb{R}$, denote

$$L_{p,\theta,\Theta}(\mathcal{D}) := L_p(\mathcal{D}, \rho^{\Theta-d} dx; \mathbb{R}),$$

and by $H_{p,\theta}^n(\mathcal{D})$, $n = 0, 1, 2, \ldots$, we denote the space of all functions $f$ satisfying

$$\|f\|_{H_{p,\theta}^n(\mathcal{D})} := \sum_{|\alpha| \leq n} \|\rho^{\alpha} D^\alpha f\|_{L_p^p(\mathcal{D})} < \infty.$$

As described below, the space $H_{p,\theta}^n(\mathcal{D})$ enjoys another definition which suits our purpose well and also leads us to an alternative definition of $K_{p,\theta,\Theta}^n(\mathcal{D})$.

Let us fix an infinitely differentiable function $\psi$ (e.g. [13] Lemma 4.13) defined on $\mathcal{D}$ such that

$$N^{-1} \rho(x) \leq \psi(x) \leq N \rho(x), \quad \rho^m |D^m \psi_x| \leq N(m) < \infty, \quad m = 0, 1, 2, \ldots \quad (4.1)$$

For instance, by mollifying the indicator function of $\{x \in \mathcal{D} : e^{-1-k} < \rho(x) < e^{1-k}\} := G_k$ one can easily construct $\xi_k$ such that

$$\xi_k \in C_0^\infty(G_k), \quad |D^m \xi_k| \leq N(m)e^{mk}, \quad \sum_{k \in \mathbb{Z}} \xi_k(x) \sim 1,$$

and then one can take

$$\psi(x) := \sum_{k \in \mathbb{Z}} e^{-k} \xi_k(x).$$
We also fix a nonnegative function \( \zeta \in C_0^\infty(\mathbb{R}_+) \) satisfying
\[
\sum_{k=-\infty}^{\infty} \zeta(e^{k+t}) > c > 0, \quad \forall \, t \in \mathbb{R}.
\] (4.2)

Note that any non-negative function \( \zeta \in C_0^\infty(\mathbb{R}_+) \) satisfies \([4,2]\) if \( \zeta > 0 \) on \([e^{-1}, e]\).

Now, for \( k \in \mathbb{Z} \) and \( x \in D \) we define
\[
\zeta_k(x) = \zeta(e^k \psi(x)).
\]
Then \( \text{supp}(\zeta_k) \subset G'_k := \{ x \in D : e^{-k-k_0} < \rho(x) < e^{-k+k_0} \} \) with some integer \( k_0 > 0 \),
\[
\sum_{k=-\infty}^{\infty} \zeta_k(x) \geq \delta > 0,
\]
and
\[
\zeta_k \in C_0^\infty(G'_k), \quad |D^m \zeta_k(x)| \leq N(\zeta, m)e^{mk}.
\]

The following Lemma suggests us alternative definitions of \( H^n_p(\Theta, \overline{\Theta}) \) and \( K^n_p(\Theta, \overline{\Theta}) \).

From now on, if a function defined on \( D \) vanishes near the boundary of \( D \), then by a trivial extension we consider it as a function defined on \( \mathbb{R}^d \). Let \( H^n_p := W^n_p(\mathbb{R}^d) \), the usual Sobolev space on \( \mathbb{R}^d \) (see Introduction for notation).

**Lemma 4.1.** Let \( p \in (1, \infty), \, \Theta, \overline{\Theta} \in \mathbb{R}, \, \Theta \in \mathbb{R}, \) and \( n \in \{0, 1, 2, \cdots\} \).

(i) For any \( \eta \in C_c^\infty(\mathbb{R}_+) \) and \( f \in H^n_p(\overline{\Theta}), \)
\[
\sum_{k \in \mathbb{Z}} e^{k\Theta} ||\eta(e^{-k} \psi(e^{k} \cdot))f(e^{k} \cdot)||^p_{H^n_p} \leq N(\rho, \Theta, d, n, \eta)||f||^p_{H^n_p(\overline{\Theta})},
\]
(ii) The reverse inequality of (i) also holds if \( \eta \) satisfies \([11, Lemma 1.4]\).

(iii) \( f \in K^n_p(\Theta) \) if any only if \( |x|^\theta p f \in H^n_p(\overline{\Theta}) \) and
\[
||f||_{K^n_p(\Theta)} \sim |||x|^\theta f||_{H^n_p(\overline{\Theta})},
\]
where the equivalence relation depends only on \( \Theta, n, M \).

**Proof.** (i)-(ii). See \([17, Propositio 2.2]\) or \([11, Lemma 1.4]\). Below we give a short proof for reader’s convenience. If \( n = 0 \), then by the change of variables \( e^kx \to x, \)
\[
\sum_{k \in \mathbb{Z}} e^{k\Theta} ||\eta(e^{-k} \psi(e^{k} \cdot))f(e^{k} \cdot)||^p_{L_p} = \int_{\mathbb{R}^d} \left[ \sum_{k=-\infty}^{\infty} e^{k(\Theta-d)}||\eta(e^{-k} \psi(x))||^p \right] |f(x)|^p dx.
\]
Thus to prove (i), we only use the fact (see e.g. \([11, Remark 1.3]\)) that for any \( \eta \in C_c^\infty(\mathbb{R}_+), \)
\[
\sum_{k \in \mathbb{Z}} e^{k(\Theta-d)}||\eta(e^{-k} \psi(x))||^p \leq N\psi^{\Theta-d}(x),
\]
and the reverse inequality also holds if \( \eta \) satisfies \([4,2]\). The proof for \( n = 1, 2, \ldots \) is almost the same and mainly based on the change of variables \( e^kx \to x \). We leave the detail to the reader.

(iii) This follows from the fact \( H^n_p(\Theta) = K^n_p(\Theta) \) and Lemma 2.2 (ii). \( \square \)

Those alternative definitions will help us prove the main result of this section, **Theorem 4.6**. One issue is that in the proof we need the negative space \( K^{-1}_{1/p, \Theta, \overline{\Theta}}(\mathbb{R}^d) \) to be defined ahead. So, we extend the definition of the spaces for all \( n \in \mathbb{Z} \). For \( n \in \{-1, -2, \cdots\} \), let us define \( H^n_p \) as the dual space of \( H^{-n}_q \), where \( 1/p + 1/q = 1 \).
Definition 4.2. Let \( p \in (1, \infty), \theta \in \mathbb{R}, \Theta \in \mathbb{R}, \) and \( n \in \mathbb{Z}. \)

(i) We let \( H_{p,\psi}^n(D) \) denote the class of all distributions \( f \) on \( D \) such that
\[
\| f \|_{H_{p,\psi}^n(D)} := \sum_{k \in \mathbb{Z}} e^{k\theta} \| \zeta(e^{-k} \psi(e^k)) f(e^k) \|_{H_p^\infty} < \infty.
\]

(ii) We write \( f \in K_{p,\theta,\psi}^n(D) \) if and only if \( |x|^{(\theta-\Theta)/p} f \in H_{p,\psi}^n(D) \), and define
\[
\| f \|_{K_{p,\theta,\psi}^n(D)} := \| |(\theta-\Theta)/p(f(\cdot)) \|_{H_{p,\psi}^n(D)}
\]
with the newly defined \( H_{p,\psi}^n(D) \) in (i).

Then we have the following properties.

Lemma 4.3. Let \( p \in (1, \infty), \theta \in \mathbb{R}, \) and \( \Theta \in \mathbb{R}. \)

(i) The claims of Lemma 4.1(i)-(ii) hold for any \( n \in \mathbb{Z}. \)

(ii) For any \( \varepsilon > 0, \) and \( n_1, n_2, n_3 \in \mathbb{Z} \) with \( n_1 < n_2 < n_3, \)
\[
\| f \|_{H_{p,\psi}^{n_2}(D)} \leq \varepsilon \| f \|_{H_{p,\psi}^{n_3}(D)} + N(\varepsilon) \| f \|_{H_{p,\psi}^{n_1}(D)}.
\]

(iii) For any \( \mu \in \mathbb{R} \) and \( n \in \mathbb{Z}, \)
\[
\| \psi^\mu f \|_{H_{p,\psi}^n(D)} \sim \| f \|_{H_{p,\psi}^{n+\mu}(D)}, \quad \| \psi^\mu f \|_{K_{p,\theta,\psi}^n(D)} \sim \| f \|_{K_{p,\theta,\psi}^{n+\mu}(D)},
\]
where \( \psi \) is from (4.1).

(iv) Let \( n \in \mathbb{Z} \) and \( |a|^{(0)}_n := \sup_D \sum_{|\alpha| \leq |n|} \rho^{|\alpha|} |D^\alpha a| < \infty, \)
then
\[
\| af \|_{K_{p,\theta,\psi}^n(D)} \leq N(n, p, d) |a|^{(0)}_n \| f \|_{K_{p,\theta,\psi}^n(D)}.
\]

(v) For any \( n \in \mathbb{Z}, \)
\[
\| Df \|_{H_{p,\theta,\psi}^{n+1}(D)} \leq N(\| f \|_{H_{p,\theta,\psi}^{n+1}(D)}), \quad \| Dg \|_{K_{p,\theta,\psi}^{n+1}(D)} \leq N(\| g \|_{K_{p,\theta,\psi}^{n+1}(D)}),
\]
where \( N = N(d, p, n, \theta, \Theta, \mathcal{M}). \)

Proof. (i) See [17] Proposition 2.2. We also remark that (i) and (ii) are proved in [11] on \( \mathbb{R}_+^d = \{ x^1 > 0 \}. \) On \( \mathbb{R}_+^d \) we can take \( \psi(x) = \rho(x) = x^1, \) and therefore
\[
\zeta(e^{-k} \psi(e^k x)) = \zeta(x), \quad \eta(e^{-k} \psi(e^k x)) = \eta(x).
\]
For an alternative proof of (i)-(ii) on the conic domain \( D, \) it is enough to replace \( x^1 \) by our \( \psi \) and repeat the proof of [11] Lemma 1.4 word for word. All the arguments there go through due to (4.1).

(ii) Obviously we only need to prove the first assertion, and this assertion is an easy consequence of Definition 4.2 and the embedding inequality
\[
\| h \|_{H_{p,\psi}^{n_3}} \leq \varepsilon \| h \|_{H_{p,\psi}^{n_3} + N(\varepsilon, d, p)} \| h \|_{H_{p,\psi}^{n_1}}.
\]

(iii) Again we only prove the first relation. Also, since \( \mu, \Theta \in \mathbb{R} \) are arbitrary, it suffices to prove
\[
\| \psi^\mu f \|_{H_{p,\psi}^{n_1}(D)} \leq N(\| f \|_{H_{p,\psi}^{n_1+\mu}(D)}).
\]
By definition,
\[ \|\psi^\mu f\|^p_{H^p_{\theta,\Theta}(D)} = \sum_{k\in\mathbb{Z}} e^{k\Theta} \|\psi^\mu(e^k x) \zeta(e^{-k} \psi(e^k x)) f(e^k x)\|^p_{H^p_{\theta}} \]
\[ = \sum_{k\in\mathbb{Z}} e^{k(\Theta+\mu p)} \|e^{-k\mu} \psi^\mu(e^k x) \zeta(e^{-k} \psi(e^k x)) f(e^k x)\|^p_{H^p_{\theta}} \]
\[ = \sum_{k\in\mathbb{Z}} e^{k(\Theta+\mu p)} \|e^{-k\mu} \psi^\mu(e^k x) \eta(e^{-k} \psi(e^k x)) \zeta(e^{-k} \psi(e^k x)) f(e^k x)\|^p_{H^p_{\theta}}, \]
where \( \eta \in C_c^\infty(\mathbb{R}^+) \) such that \( \eta = 1 \) on the support of \( \zeta \). Note that \( \psi^\mu(e^k x) \sim e^{k\mu} \)
on the support of \( e^{-k} \psi(e^k x) \), and moreover using this and (4.1) one can check
\[ |e^{-k\mu} \psi^\mu(e^k x) \eta(e^{-k} \psi(e^k x))\|_{n} := \sum_{|\alpha| \leq |n|} \sup \|D^\alpha (e^{-k\mu} \psi^\mu(e^k x) \eta(e^{-k} \psi(e^k x)))\| \leq N(n) < \infty. \]

To prove (iii), we only need to use the classical result
\[ \|af\|_{H^p_{\theta}} \leq N(d, p, n)|a|_n \|f\|_{H^p_{\theta}}, \quad n \in \mathbb{Z}. \quad (4.4) \]
(iv) This can be proved as in the proof of (iii) using (4.4).
(v) By definition
\[ \|f_x\|^p_{H^p_{\theta,\Theta}(D)} = \sum_{k\in\mathbb{Z}} e^{k(\Theta+p)} \|\zeta(e^{-k} \psi(e^k x)) f_x(e^k x)\|^p_{H^p_{\theta}}, \]
Note
\[ \zeta(e^{-k} \psi(e^k x)) f_x(e^k x) = e^{-k}(\zeta(e^{-k} \psi(e^k x)) f(e^k x))_x - e^{-k} \zeta(e^{-k} \psi(e^k x)) f(e^k x). \]
Since \( D : H^{n+1}_p \to H^p_{\theta} \) is bounded and \( \| \cdot \|_{H^p_{\theta}} \leq \| \cdot \|_{H^{n+1}_p} \),
\[ \|f_x\|^p_{H^p_{\theta,\Theta}(D)} \leq N \sum_{k\in\mathbb{Z}} e^{k\Theta} \left( \|\zeta(e^{-k} \psi(e^k x)) f(e^k x)\|^p_{H^{n+1}_p} + \|\zeta(e^{-k} \psi(e^k x)) f(e^k x)\|^p_{H^{n+1}_p} \right). \]
This and the result of (i) prove the first assertion of (v). For the second assertion, we denote \( \xi := |x|^{(\theta-\Theta)/p} \) and observe
\[ \|g_x\|_{K^{n}_{p,\theta,\Theta+\mu p}(D)} := \|\xi g_x\|_{H^p_{\theta,\Theta}(D)} = \|\xi g_x\|_{H^{n+1}_p(D)} \leq N \|\xi g\|_{H^{n+1}_p(D)} + \|\xi g_x\|_{H^n_{\theta,\Theta}(D)} \]
\[ \leq N \|g\|_{K^{n+1}_p(D)} + N \|g\|_{K^{n}_{p,\theta}(D)} + N \|g\|_{\Theta} \xi \|g\|_{K^n_{p,\theta}(D)} \]
\[ \leq N \|g\|_{K^{n+1}_p(D)} + N \|g\|_{\Theta} \xi \|g\|_{K^n_{p,\theta}(D)}. \]
The last inequality above is due to (iii). Also, by (2.10)
\[ \|\psi^{-1} \xi x\|_{n} < \infty, \quad (4.5) \]
and therefore it is enough to apply the result of (iv). The lemma is proved. \[ \square \]

**Corollary 4.4.** Let \( p \in (1, \infty) \), \( \theta \in \mathbb{R} \), \( \Theta \in \mathbb{R} \), and \( n \in \mathbb{Z} \). Put \( \xi(x) = |x|^{(\theta-\Theta)/p} \). Then
\[ \|\xi^{-1} g_x\|_{K^{n}_{p,\theta,\Theta+\mu p}(D)} \leq N \|g\|_{K^{n}_{p,\theta,\Theta}(D)}, \]
and
\[ \|\xi^{-1} f_{xx}\|_{K^{n}_{p,\theta,\Theta+\mu p}(D)} \leq N \|f\|_{K^{n}_{p,\theta,\Theta+\mu p}(D)}, \]
where \( N = N(d, p, \theta, \Theta, n, M) \).
Proof. Due to the similarity, we only prove the first assertion. By Lemma 4.3 (iii),
\[ \|\xi^{-1}g\xi\|_{\mathcal{K}^n_{p,\theta+p,\Theta}(\mathcal{D})} \leq N\|\psi\xi^{-1}g\|_{\mathcal{K}^n_{p,\theta,\Theta}(\mathcal{D})}. \]
The assertion follows from (4.5) and Lemma 4.3 (iv). \(\square\)

The main result of this section, Theorem 4.6 below, is based on Definition 4.2 and the following result on \(\mathbb{R}^d\). For \(n \in \mathbb{N}\) we denote
\[ \mathbb{H}^n_p(T) := L_p((0,T);H^n_p), \quad \mathbb{L}_p(T) := \mathbb{H}^0_p(T). \]

Lemma 4.5. Let \(p \in (1,\infty)\) and \(n \in \{0,1,2,\ldots\}\). Also, let \(f \in \mathbb{H}^{n+1}_p(T)\) and \(u \in L_p(T)\) satisfy
\[ u_t = Lu + f, \quad t \in (0,T) \quad ; \quad u(0,\cdot) = 0 \]
in the sense of distributions on \(\mathbb{R}^d\). Then \(u \in \mathbb{H}^{n+1}_p(T)\) and
\[ \|u\|_{\mathbb{H}^{n+1}_p(T)} \leq N\|u\|_{L_p(T)} + N\|f\|_{\mathbb{H}^{n+1}_p(T)}, \]
where \(N = N(d,p,\nu_1,\nu_2)\) is independent of \(T\).

Proof. By e.g. [8, Theorem 1.1],
\[ \|u_{xx}\|_{\mathbb{H}^{n+1}_p(T)} \leq N(d,p,\nu_1,\nu_2)\|f\|_{\mathbb{H}^{n+1}_p(T)}. \]
This, (4.3), and the inequality
\[ \|u\|_{\mathbb{H}^{n+1}_p(T)} \leq N(\|u_{xx}\|_{\mathbb{H}^{n+1}_p(T)} + \|u\|_{\mathbb{H}^{n+1}_p(T)}) \]
yield the claim of the lemma. \(\square\)

The following is the main result of this section. One can refer to Section 2 for the definitions of the function spaces appearing in the statement. We remark that the theorem holds for any \(\theta,\Theta \in \mathbb{R}\).

Theorem 4.6. Let \(p \in (1,\infty), \theta,\Theta \in \mathbb{R}, \Theta \in \mathbb{R}, \text{ and } n \in \{0,1,2,\ldots\}\). Assume that \(f \in \mathcal{K}^n_{p,\theta+p,\Theta+p}(\mathcal{D},T)\) and \(u \in L_{p,\theta-p,\Theta-p}(\mathcal{D},T)\) satisfies
\[ u_t = Lu + f, \quad t \in (0,T) \quad ; \quad u(0,\cdot) = 0 \]
in the sense of distributions on \(\mathcal{D}\). Then \(u \in \mathcal{K}^{n+2}_{p,\theta-p,\Theta-p}(\mathcal{D},T)\), hence \(u \in \mathcal{K}^{n+2}_{p,\theta,\Theta}(\mathcal{D},T)\), and the estimate
\[ \|u\|_{\mathcal{K}^{n+2}_{p,\theta-p,\Theta-p}(\mathcal{D},T)} \leq N \left( \|u\|_{L_{p,\theta-p,\Theta-p}(\mathcal{D},T)} + \|f\|_{\mathcal{K}^n_{p,\theta+p,\Theta+p}(\mathcal{D},T)} \right) \tag{4.6} \]
holds, where \(N = N(M,p,n,\theta,\Theta,\nu_1,\nu_2)\) which is in particular independent of \(f, u, \text{ and } T\).

Proof. 1. Assume \(u \in \mathcal{K}^m_{p,\theta-p,\Theta-p}(\mathcal{D},T), m \in \{0,1,2,\ldots,n+1\}\). We prove that \(u \in \mathcal{K}^{m+1}_{p,\theta-p,\Theta-p}(\mathcal{D},T)\) and the estimate
\[ \|u\|_{\mathcal{K}^{m+1}_{p,\theta-p,\Theta-p}(\mathcal{D},T)} \leq N \left( \|u\|_{\mathcal{K}^m_{p,\theta-p,\Theta-p}(\mathcal{D},T)} + \|f\|_{\mathcal{K}^{m-1}_{p,\theta+p,\Theta+p}(\mathcal{D},T)} \right) \]
holds with \(N = N(M,p,n,\theta,\Theta,\nu_1,\nu_2)\).
Put $\xi(x) = |x|^{(\theta - \Theta)/p}$. Using Definition 4.2 and the change of variables $t \to e^{2kt}$, we have

$$
\|u\|_{L^p_{\mu, \xi, \sigma, \eta, \nu}(D, T)}^{p, m+1} = \sum_{k \in \mathbb{Z}} e^{k(\Theta - p)} \|\zeta(e^{-k}\psi(e^k \cdot))\xi(e^k \cdot) u(\cdot, e^k \cdot)\|_{L^p_{\mu, \xi, \sigma, \eta, \nu}(D, T)}^{p, m+1}
$$

and from (4.7) we get

$$
\sum_{k \in \mathbb{Z}} e^{k(\Theta - p + 2)} \|\zeta(e^{-k}\psi(e^k \cdot))\xi(e^k \cdot) u(\cdot, e^k \cdot)\|_{L^p_{\mu, \xi, \sigma, \eta, \nu}(D, T)}^{p, m+1}.
$$

(4.7)

Denote $v_k(t, x) := \zeta(e^{-k}\psi(e^k x))\xi(e^k x) u(2kt, e^k x)$, then $v_k$ satisfies

$$(v_k)_t = \mathcal{L}_k v_k + f_k, \quad t \leq e^{-2kT}; \quad v_k(0, \cdot) = 0$$

in the sense of distributions on $\mathbb{R}^d$, where

$$
\mathcal{L}_k := \sum_{i, j} a_{ij}^k(t) D_{ij}, \quad a_{ij}^k(t) := a^i_j(e^{2kt})
$$

and, with Einstein’s summation convention with respect to $i, j$,

$$
f_k(t, x) := e^{2k\zeta'(e^{-k}\psi(e^k x))\xi(e^k x)f(e^{2kt}, e^k x)} + e^{2k} a_{ij}^k(t) D_{ij} u(e^{2kt}, e^k x) \zeta''(e^{-k}\psi(e^k x)) D_i \psi(e^k x) D_j \xi(e^k x) + e^{2k} a_{ij}^k(t) u(e^{2kt}, e^k x) \zeta''(e^{-k}\psi(e^k x)) D_i \psi(e^k x) D_j \xi(e^k x)
$$

$$
+ e^{2k} a_{ij}^k(t) u(e^{2kt}, e^k x) \zeta''(e^{-k}\psi(e^k x)) D_i \psi(e^k x) D_j \xi(e^k x) + e^{2k} a_{ij}^k(t) u(e^{2kt}, e^k x) \zeta''(e^{-k}\psi(e^k x)) D_i \psi(e^k x) D_j \xi(e^k x)
$$

$$
= \sum_{l=0}^{6} f_k^l(t, x)
$$

$\zeta'$ and $\zeta''$ denote the first and second derivative of $\zeta$, respectively. We note that the operator $\mathcal{L}_k$ for any $k \in \mathbb{Z}$ satisfies the uniform parabolicity condition (2.3).

We can apply Lemma 4.3 and from (4.7) we get

$$
\|u\|_{L^p_{\mu, \xi, \sigma, \eta, \nu}(D, T)}^{p, m+1} \leq N \sum_{k \in \mathbb{Z}} e^{k(\Theta - p + 2)} \|v_k\|_{L^p_{\mu, \xi, \sigma, \eta, \nu}(e^{-2kT})}^{p, m+1}
$$

$$
+ N \sum_{l=0}^{6} \sum_{k \in \mathbb{Z}} e^{k(\Theta - p + 2)} \|f_k^l\|_{L^p_{\mu, \xi, \sigma, \eta, \nu}(e^{-2kT})}^{p, m+1}
$$

if $v_k \in L^p(e^{-2kT})$, $f_k^l \in \mathbb{H}^{m-1}(e^{-2kT})$, $k \in \mathbb{Z}$, $l = 0, 1, \ldots, 6$, are provided ahead.

Indeed, the change of variable $e^{2kt} \to t$ and Definition 4.2 yield

$$
\sum_{k \in \mathbb{Z}} e^{k(\Theta - p + 2)} \|v_k\|_{L^p_{\mu, \xi, \sigma, \eta, \nu}(e^{-2kT})}^{p, m+1}
$$

$$
= \sum_{k \in \mathbb{Z}} e^{k(\Theta - p)} \|\zeta(e^k \psi(e^k \cdot))\xi(e^k \cdot) u(\cdot, e^k \cdot)\|_{L^p_{\mu, \xi, \sigma, \eta, \nu}(T)}^{p, m+1} = \|u\|_{L^p_{\mu, \xi, \sigma, \eta, \nu}(D, T)}^{p, m+1},
$$

meaning especially $v_k \in L^p(e^{-2kT})$ for all $k$. 


Next we show $f'_k \in H_p^{m-1}(e^{-2kT})$ as follows. For $l = 0$, by Definition 4.2 and the change of variable $e^{2k t} \to t$, we have

$$\sum_{k \in \mathbb{Z}} e^{k(\theta-p+2)} \|f'_k\|_{H_p^{m-1}(e^{-2kT})}^p = \sum_{k \in \mathbb{Z}} e^{k(\theta+p)} \|\zeta(e^{-k}\psi(e^k))\xi(e^k)f(e^{2k}, e^k)\|_{H_p^{m-1}(T)}^p = \|f\|_{K_p^{m-1}(\mathbb{C}, T)}^p.$$ 

For $l = 1$, by Definition 4.2 and Lemma 4.3 (i), we get

$$\sum_{k \in \mathbb{Z}} e^{k(\theta-p+2)} \|f'_k\|_{H_p^{m-1}(e^{-2kT})}^p \leq N \sum_{k \in \mathbb{Z}, i,j} e^{k\theta} \|D_i u(-, e^k)\zeta(e^{-k}\psi(e^k))D_j \psi(e^k)\xi(e^k)\|_{H_p^{m-1}(T)}^p \leq N \|u_x\|_{K_p^{m-1}(\mathbb{C}, T)}^p \leq N \|u\|_{K_p^{m-1}(\mathbb{C}, T)}^p,$$

where the last two inequalities are due to (4.1), Lemma 4.3 (iv), and Lemma 4.3 (v). For $l = 2$, by definitions of norms, we have

$$\sum_{k \in \mathbb{Z}} e^{k(\theta-p+2)} \|f''_k\|_{H_p^{m-1}(e^{-2kT})}^p \leq N \sum_{k \in \mathbb{Z}, i,j} e^{k\theta} \|D_i u(-, e^k)\zeta(e^{-k}\psi(e^k))D_j \psi(e^k)\xi(e^k)\|_{H_p^{m-1}(T)}^p \leq N \|u_x\|_{K_p^{m-1}(\mathbb{C}, T)}^p \leq N \|u\|_{K_p^{m-1}(\mathbb{C}, T)}^p,$$

where the last inequality is due to Corollary 4.2. For other $l$ one can argue similarly.

We proved $f'_k \in H_p^{m-1}(e^{-2kT})$ for all $k$ and $l$. Moreover, we additionally proved

$$\sum_{l=0}^6 \sum_{k \in \mathbb{Z}} e^{k(\theta-p+2)} \|f'_k\|_{H_p^{m-1}(e^{-2kT})}^p \leq N \|u\|_{K_p^{m-1}(\mathbb{C}, T)}^p + N \|f\|_{K_p^{m-1}(\mathbb{C}, T)}^p,$$

Consequently, we have

$$\|u\|_{K_p^{m+1}(\mathbb{C}, T)}^p \leq N \|u\|_{K_p^{m-1}(\mathbb{C}, T)}^p + N \|f\|_{K_p^{m-1}(\mathbb{C}, T)}^p,$$

and this is exactly what we wanted to prove.

2. By applying the result of Step 1 for $m = 0, 1, \ldots, n + 1$ inductively and by the fact that $L_p(\mathbb{R}^d)$ is continuously embedded in $H_p^{-1}(\mathbb{R}^d)$, we conclude that $u$ belongs to $K_p^{m+2}(\mathbb{C}, T)$ and (4.6) holds. This ends the proof. \qed

5. The proof of Theorem 2.12

We start with a representation formula of the solution.

Lemma 5.1. Assume that $p \in (1, \infty)$, $p(1 - \lambda^{+}_{C, L}) < \theta < p(d - 1 + \lambda^{-}_{C, L})$, and $\Theta \in (d - 1, d - 1 + p)$. Let $f \in L_p, \Theta, \Theta + p(\mathbb{C}, T)$ and let $u \in K_p^{2, \Theta, \Theta}(\mathbb{C}, T)$ be a
solution to equation \(2.2\) with the source term \(f\). Then \(u = v\) in \(L_{p,q}^d(D,T)\), where \(v\) is the function defined by
\[
v(t,x) = \int_0^t \int_D G(t,s,x,y)f(s,y)dyds.
\]

**Proof.** Take a sequence of functions \(\xi_m\) from the proof of Lemma \(2.2\) and choose \(\eta \in C_0^\infty(\mathbb{R}^d)\) such that \(0 \leq \eta \leq 1\) and \(\eta(x) = 1\) for \(|x| \leq 1\). Denote
\[
\eta_m(x) := \xi_m(x)\eta(x/m).
\]
Then, by the choice of \(\xi_m\), one can easily check that \(\eta_m \in C_c^\infty(D)\), \(0 \leq \eta_m \leq 1\), \(\eta_m(x) \to 1\) as \(m \to \infty\) for \(x \in D\), and \(\rho_0^\beta D^\beta \eta_m\) is uniformly bounded and goes to zero as \(m \to \infty\) for any multi-index \(\beta\) with \(|\beta| \geq 1\).

Denote \(u_m = u\eta_m\), then it satisfies
\[
(u_m)_t = Lu_m + f_m + \eta_m, \quad t \in (0,T] ; \quad u_m(0,\cdot) = 0
\]
in the sense of distributions on \(D\), where
\[
f_m := f \eta_m + \alpha_{ij} \eta_m \frac{\partial}{\partial y} u_{ij} + \alpha_{ij} \eta_m \frac{\partial}{\partial x} u_{ij}
\]
with Einstein’s summation notation used on \(i,j\).

Note that, since \(\eta_m\) has compact support, we have
\[
\int_0^T \int_D (|\rho_0(u_m)_{xx}|^p + |(u_m)_x|^p + |\rho_0^{-1}u_m|^p + |\rho_0 f_m|^p) \rho_0^{d-1}dxdt < \infty.
\]
Thus, by Theorem 1.1 and Theorem 4.1 in \(13\), we have representation of \(u_m\), that is,
\[
u_m(t,x) = \int_0^t \int_D G(t,s,x,y)f_m(s,y)dyds,
\]
which gives
\[
\begin{align*}
u_m(t,x) - v(t,x) &= \int_0^t \int_D G(t,s,x,y) ((1 - \eta_m) f + \alpha_{ij} \eta_m \frac{\partial}{\partial y} u_{ij} + \alpha_{ij} \eta_m \frac{\partial}{\partial x} u_{ij})dyds.
\end{align*}
\]
Consequently, by Lemma \(3.1\) and Lemma \(13\) (iii), we get
\[
\begin{align*}
\|u_m - v\|_{L_{p,q}^d(D,T)} &\leq N\|((1 - \eta_m) f + \alpha_{ij} \eta_m \frac{\partial}{\partial y} u_{ij} + \alpha_{ij} \eta_m \frac{\partial}{\partial x} u_{ij})\|_{L_{p,q}^d(D,T)} \\
&\leq N\|((1 - \eta_m) f + u_{ij} \eta_m \frac{\partial}{\partial y} u_{ij} + u_{ij} \eta_m \frac{\partial}{\partial x} u_{ij})\|_{L_{p,q}^d(D,T)} + N\|u_{ij} \eta_m \frac{\partial}{\partial y} u_{ij} + u_{ij} \eta_m \frac{\partial}{\partial x} u_{ij}\|_{L_{p,q}^d(D,T)}.
\end{align*}
\]
Since \(\psi D\eta_m\) and \(\psi^2 D_{ij} \eta_m\) are uniformly bounded and go to zero as \(n \to \infty\), we conclude that the last three terms above go to zero, and therefore \(u_m \to v\) in \(L_{p,q}^d(D,T)\) as \(n \to \infty\). This and the fact \(u_m \to u\) in \(L_{p,q}^d(D,T)\) finish the proof of the lemma.

We are ready to prove our main result.
Proof of Theorem 2.12

Existence and the estimate. For the given \( f \in \mathbb{K}_{p,\theta+p,\Theta+\rho}(\mathcal{D}, T) \), choose functions \( f_m \in C_c^\infty((0, T) \times \mathcal{D}) \) such that \( f_m \to f \) as \( m \to \infty \) in \( \mathbb{K}_{p,\theta+p,\Theta+\rho}(\mathcal{D}, T) \). Then, by \cite[Theorem 4.1]{13}, the function

\[
u_m(t, x) := \int_0^t \int_{\mathcal{D}} G(t, x, s, y) f_m(s, y) dy ds
\]

is a solution to (2.2) with the source term \( f_m \).

Now, Lemma 3.1 comes in and, for any \( m, m' \in \mathbb{N} \), it says

\[
\|u_m\|_{L_{p,\theta-p,n-\rho}(\mathcal{D}, T)} \leq N\|f_m\|_{L_{p,\theta+p,\Theta+\rho}(\mathcal{D}, T)}
\]

and

\[
\|u_m - u_m'\|_{L_{p,\theta-p,n-\rho}(\mathcal{D}, T)} \leq N\|f_m - f_m'\|_{L_{p,\theta+p,\Theta+\rho}(\mathcal{D}, T)}.
\]

In particular, \( u_m \in L_{p,\theta-p,n-\rho}(\mathcal{D}, T) \). We emphasize that the dependency of \( N \) changes by the situations mentioned in Lemma 3.1. Then, thanks to Theorem 4.6 we have

\[
\|u_m\|_{\mathbb{K}_{p,\theta-p,n-\rho}(\mathcal{D}, T)} = \|u_m\|_{L_{p,\theta-p,n-\rho}(\mathcal{D}, T)} + \|(u_m)_t\|_{L_{p,\theta+p,\Theta+\rho}(\mathcal{D}, T)} \leq N\left(\|u_m\|_{L_{p,\theta-p,n-\rho}(\mathcal{D}, T)} + \|f_m\|_{L_{p,\theta+p,\Theta+\rho}(\mathcal{D}, T)}\right) \leq N\|f_m\|_{L_{p,\theta+p,\Theta+\rho}(\mathcal{D}, T)}.
\]

Similarly, we have

\[
\|u_m - u_m'\|_{\mathbb{K}_{p,\theta-p,n-\rho}(\mathcal{D}, T)} \leq N\|f_m - f_m'\|_{\mathbb{K}_{p,\theta+p,\Theta+\rho}(\mathcal{D}, T)}.
\]

It follows that \( u_m \) is a Cauchy sequence in \( \mathbb{K}_{p,\theta,\Theta}^{n+2}(\mathcal{D}, T) \) and there exists \( u \in \mathbb{K}_{p,\theta,\Theta}^{n+2}(\mathcal{D}, T) \) such that \( u_m \to u \) in \( \mathbb{K}_{p,\theta,\Theta}^{n+2}(\mathcal{D}, T) \). Moreover, since

\[
\mathcal{L} u_m \to \mathcal{L} u, \quad (u_m)_t \to (u)_t \quad \text{as} \quad m \to \infty \quad \text{in} \quad \mathbb{K}_{p,\theta+p,\Theta+\rho}(\mathcal{D}, T),
\]

we have \( u_t = \mathcal{L} u + f \) in \( \mathbb{K}_{p,\theta+p,\Theta+\rho}^{n+2}(\mathcal{D}, T) \). This handles the existence of a solution to equation (2.2) in \( \mathbb{K}_{p,\theta,\Theta}^{n+2}(\mathcal{D}, T) \). Moreover, estimate (5.2) follows from (5.2).

Uniqueness. Let \( w \in \mathbb{K}_{p,\theta,\Theta}^{n+2}(\mathcal{D}, T) \) be a solution to equation (2.2) with \( f \equiv 0 \). Then by Lemma 3.1 \( w \) coincides with \( v \) defined in (5.1), which is now identically zero. This handles the uniqueness. The theorem is proved. \( \square \)

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KYEONG-HUN KIM, DEPARTMENT OF MATHEMATICS, KOREA UNIVERSITY, ANAM-RO 145, SEONGBUK-GU, SEOUL, 02841, REPUBLIC OF KOREA
Email address: kyeonghun@korea.ac.kr

KIJUNG LEE, DEPARTMENT OF MATHEMATICS, AJOU UNIVERSITY, WORLDCUP-RO 206, YEONGTONG-GU, SUWON, 16499, REPUBLIC OF KOREA
Email address: kijung@ajou.ac.kr

JINSOL SEO, DEPARTMENT OF MATHEMATICS, KOREA UNIVERSITY, ANAM-RO 145, SEONGBUK-GU, SEOUL, 02841, REPUBLIC OF KOREA
Email address: seoj401@korea.ac.kr