Abstract The \( CP^2 \) model, with and without a generalized Hopf term, is studied using the collective coordinate approximation. In the spirit of this approximation, an ansatz is given which in previous numerical studies was seen to give a good parameterization of the numerical solution. The equations of motion for the collective coordinates are then solved analytically, for solitons close together and for solitons far apart. The solutions show how the generalized Hopf term changes the scattering angle which in its absence is 90°.
1 Introduction

σ-models in low dimensions have become an increasingly important area of research. In two Euclidean dimensions they appear to be the low dimensional analogues of four-dimensional Yang-Mills theories. They often arise as approximate models, in the context of both particle and solid state physics. They have recently been used in the construction of models of high $T_c$ superconductivity [1] and of the quantum Hall effect [2]. Moreover, they are simple examples of harmonic maps studied by differential geometers and, as such, are interesting in themselves. In addition, because of the nonlinearity of these models, their classical solutions represent structures which resemble solitons of (1+1) dimensional models and could become very useful in the description of many physical phenomena.

Recently, interesting scattering processes of these soliton-like objects were studied with the help of the collective coordinates approximation [3][4][5] and with the help of numerical simulations. The numerical simulations were performed for the $CP^1$ model [3][7], for its modification by the addition of potential-like and Skyrme-like terms [8][9], and most recently for the $CP^2$ model with and without a generalized Hopf term [10][11]. In Ref. 11, also the Ginibre et al. existence proof [12] and the Cauchy-Kowalewskyi theorem were used to study $CP^2$ soliton scattering. Although there is no mathematically rigorous justification for the collective coordinate approximation in the $CP^n$ model yet, Manton’s arguments, first put forward for the monopole motion [14], should also apply to the $CP^2$ soliton motion. In this paper we therefore use this approximation to study some aspects of the $CP^2$ soliton scattering. We again find the 90$^\circ$ scattering without, and a deviation from the 90$^\circ$ scattering, with the generalized Hopf term.

The paper is organized as follows. In section 2, we describe the $CP^2$ model without and with generalized Hopf term. We also formulate a Cauchy problem suitable for the description of solitons which merge at some stage of the scattering process. In section 3, the description of the scattering process in terms of collective coordinates is given. In section 4, we solve the equations for the collective coordinates for solitons close enough together, i.e. for short enough times before and after the collision. Solitons far apart are studied in section 5.
The model

The Lagrangian for the $CP^2$ model is,

$$L_0 = (D^{a}Z_{a})^*(D^{a}Z_{a}), \quad \mu = 0, 1, 2; \quad a = 1, 2, 3;$$

(2.1)

where $Z_{a}$ is a complex function of $(t, x^1, x^2)$ on $M_3$ (or a function of the variables $(t, z = \frac{1}{2}(x^1 + ix^2), z^*)$), and $D_{\mu}Z_{a} = \partial_{\mu}Z_{a} - Z_{b}^{*}(\partial_{\mu}Z^{b})Z_{a}$. Also, the $Z_{a}$ have to satisfy the condition $Z_{a}^{*}Z_{a} = 1$. $a, b, ..$ are raised and lowered with the metric $\eta = diag(+1, +1, +1)$. The space-time metric is $g = diag(+1, -1, -1)$. The equations of motion corresponding to (2.1) are,

$$D_{\mu}D^{\mu}Z_{a} + (D_{\mu}Z_{a})^*(D^{\mu}Z_{b})Z_{a} = 0,$$

(2.2)

together with $Z_{a}^{*}Z_{a} = 1$. In this paper, we will concentrate on solutions with winding number $k$, defined by

$$k = \frac{i}{2\pi} \int_{\mathbb{R}^2} \epsilon_{ijk}(D^{j}Z_{a})^*(D^{k}Z_{a}) \, d^{2}x,$$

(2.3)

equal to 2.

The initial data are calculated from the complex functions,

$$w_{1}(0, z, z^{*}) = \lambda z^{2}, \quad w_{2}(0, z, z^{*}) = \beta z,$$

$$\partial_{0}w_{1}(0, z, z^{*}) = \lambda v, \quad \partial_{0}w_{2}(0, z, z^{*}) = 0,$$

(2.4)

where $\lambda, \beta$ and $v$ are real positive constants, and where $w_{i}(t, z, z^{*})$ and $Z_{a}$ are related through

$$Z = \frac{1}{\sqrt{1 + |w_{1}|^2 + |w_{2}|^2}} \begin{pmatrix} 1 \\ w_{1} \\ w_{2} \end{pmatrix}.$$  

(2.5)

For these initial data the energy is finite and the winding number is 2. The initial data describe a head-on collision of solitons which 'coincide' at $t = 0$. Negative time is the time before, and positive time is the time after the collision. The condition, $|w_{1}|^2 + |w_{2}|^2 = 0$ at $z = 0$, which holds for the initial data, reflects the fact that the solitons merge at $t = 0$.

For $\partial_{0}w_{i}$ at $t = 0$, we have taken zero modes which give a head-on collision. This means that we don’t need any extra potential energy to go through the
‘ring’ which forms at $t = 0$. For small $v$, there is hardly any excess energy, and the slow motion approximation, or collective coordinate approximation, can be used. The idea of this approximation \cite{13} is to describe the solution at each fixed time in terms of a configuration with minimal potential energy. Then the action is minimized to obtain the collective coordinates, which parameterize these minimal-energy configurations, as functions of time.

For the $CP^2$ model the configurations with minimal potential energy, which for constant parameters are time independent solutions, are of the form (2.5) where $w_1$ and $w_2$ are rational functions of $z$ \cite{13}. These configurations can also be given in the form,

$$Z_a = \frac{f_a}{|f|} \quad \text{with} \quad |f| = \sqrt{f_a^* f_a},$$

(2.6)

where the $f_a$’s are polynomials in $z$. To relate the two descriptions use is made of the $U(1)$ symmetry of the theory. In terms of $f_a$, the equations of motion (2.2) read

$$\partial_\mu \partial^\mu f^a - \frac{f^a f^*_b \partial_\mu \partial^\mu f^b}{|f|^2} = \frac{2(\partial_\mu f^a) f^*_b \partial^\mu f^b}{|f|^2} - \frac{2 f^a f^*_b (\partial_\mu f^b) f^*_c \partial^\mu f^c}{|f|^4}. \quad (2.7)$$

The winding number can be expressed as

$$k = \frac{1}{2\pi} \int_0^{2\pi} \lim_{r \to \infty} \frac{f_a^* z \partial_z f_a}{|f|^2} d\theta,$$

(2.8)

where $z = re^{i\theta}$. This formula shows that for $k = 2$, we can restrict our attention to quadratic polynomials $f_a$.

In sections 4 and 5, we will also study the effect the generalized Hopf term,

$$\mathcal{L}_{Hopf} = \epsilon_{\mu\nu\rho}(D_\mu Z^a)^*(D_\nu Z_a)(Z^b \partial_\rho Z_b), \quad (2.9)$$

has on the scattering process. For the Lagrangian $\mathcal{L}_0 + \kappa \mathcal{L}_{Hopf}$, the equations of motion read,

$$(\delta_\rho^a - Z_a Z^{b*}) D_\mu D^\mu Z_b - 2\kappa \epsilon_{\mu\nu\rho} D_\rho Z_a((D_\mu Z^b)^* D_\nu Z_b) = 0,$$

(2.10)

where, as before, $Z_a^* Z^a = 1$. 

4
3 The collective coordinate ansatz

For the collective coordinate approximation, we take the static solutions, $Z_a(z, c_r)$, parameterized by parameters $c_r$, make the parameters time dependent and minimize within this ansatz. If we work with configurations of the form (2.6), this means that

$$f_a(t, z) = \sum_{l=0}^{n} c_a^n(t) z^n, \quad (3.1)$$

where $n = 2$, for a 2-soliton configuration.

Within the ansatz, the potential energy is a constant and the $CP^2$ kinetic energy is

$$T_0 = (D^0 Z^a)^*(D_0 Z_a) = \frac{1}{|f|^2}((\partial_0 f^*_a)(\partial^0 f^a) - \frac{1}{|f|^2} f^*_a(\partial_0 f^a) f^b(\partial^0 f^*_b)). \quad (3.2)$$

This yields,

$$\int_{\mathbb{R}^2} T_0 d^2 x = A_{ab}^{mn} c_a^m c_b^n = L \quad (3.3)$$

with,

$$A_{ab}^{mn} = \int_{\mathbb{R}^2} \frac{1}{|f|^2} (\delta_{ab} z^m z^n - \frac{1}{|f|^2} c_a^r c_b^s z^{n+r} z^{m+s}) d^2 x. \quad (3.4)$$

Before we minimize the action, $\int_{-\infty}^{\infty} L dt$, in the next two sections, we will simplify the ansatz (3.1). Our next arguments work for all initial data, including ours in (2.4), which satisfy the following conditions: $f^a = f^a(z), a = 1, 2, 3$, with $\partial_z f^a \neq 0$ for $a = 2, 3$; $\partial_0 f^k = 0, k = 1, 3$; $\partial_0 f^2 = \partial_0 f^2(z) \neq 0$. For $\kappa = 0$, we find $f^1 \partial_0^2 f^3 - f^3 \partial_0^2 f^1 = 0$ at $t = 0$. To derive this result, eq. (2.7), the conditions on the initial data and the formulas $-\partial_i \partial^i = \partial_z \partial_{z^*}$ and $-2(\partial_z f^a) \partial_i f^b = (\partial_z f^a) \partial_{z^*} f^b + (\partial_{z^*} f^a) \partial_z f^b$ for $a, b = 1, 2, 3$ are used. Now $f^1$ can be made real by a gauge transformation, and multiplying all $f^a$’s by $1/f^1$ we can achieve $f^1 = 1$ for all $t$. So at $t = 0$ we have $\partial_0^2 f^3 = 0$. We show next that $\partial_0^2 f^3 = 0$ for $t = 0$. For this we need again that the initial data depend on $z$ only.

To go further we need next that $\partial_0^2 f^2$ at $t = 0$ depends on $z$ only. This is not the case for the exact solution. In fact we obtain

$$\partial_0^2 f^2 = \frac{2}{|f|^2} f_2^*(\partial_0 f^2)^2 \quad (3.5)$$
at \( t = 0 \). However, within the confines of the collective coordinate approximation, we are neglecting contributions due to a \( z^* \) dependence of \( f_a \), and in that case we can actually show that all time derivatives of \( f^3 \) vanish at \( t = 0 \). The reasonable assumption of analyticity now leads to the time independence of \( f^3 \) for \( t > 0 \).

So far we have \( f^1 = 1 \), \( f^3 = \beta z \) and

\[
f^2 = \frac{a_1(t) z^2 + a_2(t) z + a_3(t)}{a_4(t) z + a_5(t)}
\]

(3.6)

If \( a_4 \) depends on time the kinetic energy, \( \int_{\mathbf{R}^2} T_0 d^2 x \), diverges. (The interpretation of this divergence is that a change in \( a_4 \) costs infinite energy.) So \( a_4 \) must be constant, and therefore zero because of the initial data. The kinetic energy also diverges for time dependent \( a_1 \). Hence we have \( a_1 = \lambda \) and \( f^2 = \lambda z^2 + \delta(t) z + \gamma(t) \). Next we want to show that \( \delta \) can be taken to be zero.

The Euler-Lagrange equations for \( L \) defined in eq. (3.3), the geodesic equations, are,

\[
A_{a2}^{mn} \dddot{c}_b + \frac{\partial A_{a2}^{mn}}{\partial c^n_c} \dot{c}_c^{\star} \dddot{c}_b + \frac{\partial A_{a2}^{mn}}{\partial c^{\star}_c} \dot{c}_b^{\star} \dddot{c}_b^{\star} - \frac{\partial A_{mn}^{r_\star}}{\partial c^n_m \dot{c}_c^{\star}} \dddot{c}_c^{\star} \dddot{c}_b = 0,
\]

(3.7)

with \( c^n_a(t) \) defined in (3.1). If \( c^n_2(t) = \gamma(t) \) and the other \( c^n_a \) are constant, then the equations reduce to,

\[
A_{a2}^{mn} \dddot{\gamma} + \frac{\partial A_{a2}^{mn}}{\partial \gamma} \dot{\gamma}^2 = 0.
\]

(3.8)

Here we have used that,

\[
\frac{\partial A_{a2}^{mn}}{\partial c^n_2} = \frac{\partial A_{22}^{mn}}{\partial c^n_m}.
\]

(3.9)

That (3.9) holds can be seen by taking the corresponding derivatives of the integrands which define the \( A \)'s in eq. (3.4).

If on the other hand we first set \( c^n_2 = \gamma(t) \) and the other \( c^n_a \) constant in eq. (3.3), and then minimize, we obtain,

\[
A_{22}^{mn} \dddot{\gamma} + \frac{\partial A_{22}^{mn}}{\partial \gamma} \dot{\gamma}^2 = 0.
\]

(3.10)
The equations (8) are now satisfied if,
\[
A_{a2}^{m0} \frac{\partial A_{22}^{00}}{\partial \gamma} = A_{22}^{00} \frac{\partial A_{a2}^{m0}}{\partial \gamma}. \tag{3.11}
\]
In our case, so far the ansatz has been reduced to \( f^1 = 1, f^2 = \lambda z^2 + \delta(t)z + \gamma(t), f^3 = \beta z \). For this ansatz, we have to solve the geodesic equations (3.7). A special solution is provided by \( \delta = 0 \), which satisfies the initial condition, and a function \( \gamma \) which satisfies (3.10). Now (3.11) holds trivially, since both the integrand of \( A_{a2}^{m0} \) and the integrand of \( \partial A_{22}^{00}/\partial \gamma \) have the symmetry \( I(\theta + \pi) = -I(\theta) \), as functions of the angle \( \theta \). Hence the corresponding integrals vanish. Therefore, finally our collective coordinate ansatz is
\[
f^1 = 1, \quad f^2 = \lambda z^2 + \gamma(t), \quad f^3 = \beta z. \tag{3.12}
\]

If the Hopf term is included, the Lagrange function, defined in (3.3), will acquire the extra term \( \kappa \int_{\mathbf{R}^2} \mathcal{L}_{\text{Hopf}} d^2 x \), where
\[
\mathcal{L}_{\text{Hopf}} = \epsilon^{\mu\nu\rho}(\partial_\mu f_a^*) (\partial_\nu f^a) Z_b^* \partial_\rho Z^b = \frac{\epsilon^{ij}}{2|f|^4} \left( (\partial_i f_a^*) (\partial_j f^a) (f_b^* \partial_0 f_b - f_b \partial_0 f_b^*) \right. \tag{3.13}
\]
\[
+ \left. ((\partial_0 f_a^*) (\partial_i f^a) - (\partial_i f_a^*)(\partial_0 f^a))(f_b^* \partial_j f^b - f_b \partial_j f_b^*) \right). \]

We will study the effect of the Hopf term for \( \kappa \) small enough so that the ansatz (3.12) still gives a reasonable approximation. If \( \kappa \) is not small, we do not expect a time independent \( f_3 \) to be a good approximation because
\[
\partial_0^2 f_3 = \frac{-i \kappa \beta v |\lambda|^2 |z|^2 (2 + |\beta|^2 |z|^2)}{(1 + |\beta|^2 |z|^2 + |\lambda|^2 |z|^4)^2} \tag{3.14}
\]
holds for our initial data at \( t = 0 \).

4 Approximate solution for solitons close together

The ansatz (3.12) leads to
\[
T_0 = \frac{1}{|f|^2} \left( 1 - \frac{|\lambda z^2 + \gamma|^2}{|f|^2} \right) \tilde{\gamma} \tilde{\gamma}^*. \tag{4.1}
\]
Note that $T_0$ is a rational function of $r$, where $z = re^{i\theta}$, and that the denominator is a product of quadratic functions in $r^2$. So the $r$ integration can be done, leading to functions of $\theta$ which have to be integrated numerically. We also obtain,

$$L_{\text{Hopf}} = \frac{i|\beta|^2}{4|f|^4}((\gamma^* - \lambda^* z^2) \dot{\gamma} - (\gamma - \lambda z^2) \dot{\gamma}^*) \quad (4.2)$$

Again this is a rational function of $r$ which can be integrated over $r$.

In this section, we concentrate on solitons close together, i.e., we choose $|\gamma|$ to be small. For small $|\gamma|$, we expand,

$$\frac{1}{|f|^2} = \frac{1}{1 + |\beta|^2 r^2 + |\lambda|^2 r^4} - \frac{\lambda \gamma^* r^2 + \lambda^* \gamma r^2}{(1 + |\beta|^2 r^2 + |\lambda|^2 r^4)^2} + O(|\gamma|^2). \quad (4.3)$$

Now we obtain,

$$\int_{\mathbb{R}^2} T_0 d^2 x = A \dot{\gamma} \dot{\gamma}^* + O(|\gamma|^3), \quad (4.4)$$

where

$$A = 2\pi \int_0^\infty dr \frac{r(1 + |\beta|^2 r^2)}{(1 + |\beta|^2 r^2 + |\lambda|^2 r^4)^2}. \quad (4.5)$$

The integral can be evaluated and given in terms of

$$g(\lambda, \beta) = \begin{cases} \frac{2}{\sqrt{|\beta|^4 - 4|\lambda|^4}} \log\left(\frac{|\beta|^2}{2|\lambda|}\right) + \frac{\sqrt{|\beta|^4 - 4|\lambda|^4}}{2|\lambda|} & \text{for } 2|\lambda| < |\beta|^2 \\ \frac{1}{|\lambda|} & \text{for } 2|\lambda| = |\beta|^2 \\ \frac{2}{\sqrt{4|\lambda|^2 - |\beta|^4}} \left(\frac{1}{2} - \arctan\left(\frac{|\beta|^2}{\sqrt{4|\lambda|^2 - |\beta|^4}}\right)\right) & \text{for } 2|\lambda| > |\beta|^2 \end{cases} \quad (4.6)$$

This yields,

$$A = \frac{\pi}{4|\lambda|^2 - |\beta|^4} \left(|\beta|^2 - (|\beta|^4 - 2|\lambda|^2)g(\lambda, \beta)\right) \quad (4.7)$$

for $|\beta|^2 \neq 2|\lambda|$. For $|\beta|^2 = 2|\lambda|$, the formula reduces to $A = 2\pi/(3|\lambda|)$.

We also have,

$$\int_{\mathbb{R}^2} \mathcal{L}_{\text{Hopf}} d^2 x = iB(\gamma^* \dot{\gamma} - \gamma \dot{\gamma}^*) + O(|\gamma|^3), \quad (4.8)$$

where

$$B = \frac{\pi |\beta|^2}{2} \int_0^\infty dr \left(\frac{r}{(1 + |\beta|^2 r^2 + |\lambda|^2 r^4)^2} + \frac{|\lambda|^2 r^5}{(1 + |\beta|^2 r^2 + |\lambda|^2 r^4)^3}\right). \quad (4.9)$$
Note that $\beta = 0$ ($CP^1$ embedding) implies $B = 0$. Again the integral can be evaluated, and we obtain

$$B = \frac{\pi |\beta|^2}{4(4|\lambda|^2 - |\beta|^2)^2} (|\beta|^2(|\beta|^4 - 7|\lambda|^2) + |\lambda|^2(10|\lambda|^2 - |\beta|^4)g(\lambda, \beta))$$  \hspace{1cm} (4.10)

for $|\beta|^2 \neq 2|\lambda|$. If $|\beta|^2 = 2|\lambda|$, the result reduces to $B = 11\pi/60$.

For small $|\gamma|$, we end up with the Lagrange function,

$$L = A\dot{\gamma}\dot{\gamma}^* + i\kappa B(\gamma^*\dot{\gamma} - \gamma\dot{\gamma}^*).$$  \hspace{1cm} (4.11)

Expressing $\gamma(t) = R(t)e^{i\sigma(t)}$ in terms of real functions, we obtain,

$$L = A(\ddot{R}^2 + R^2\dot{\sigma}^2) - 2\kappa BR^2\dot{\sigma},$$  \hspace{1cm} (4.12)

and the equations of motion,

$$A\ddot{R} - AR\dot{\sigma}^2 + 2\kappa BR\dot{\sigma} = 0,$$  \hspace{1cm} (4.13)

$$AR^2\ddot{\sigma} + 2AR\ddot{R} - 2\kappa BR\ddot{R} = 0.$$  \hspace{1cm} (4.14)

The second equation implies,

$$\dot{\sigma} = \frac{c_1 + \kappa BR^2}{AR^2}.$$  \hspace{1cm} (4.15)

If $c_1 \neq 0$, $\dot{\sigma} \to \infty$ as $R \to 0$. So we set $c_1 = 0$, and obtain $\dot{\sigma} = \kappa B/A$ and w.l.g. $\sigma = \kappa Bt/A$. Now the equation for $R$ reads,

$$A^2\ddot{R} + \kappa^2 B^2 R = 0,$$  \hspace{1cm} (4.16)

and, using $R(0) = 0$, we obtain,

$$R = c_2 \sin \frac{\kappa Bt}{A} = c_2 \frac{\kappa B}{A} t,$$  \hspace{1cm} (4.17)

for small $R$ (and $t$).

We now consider $|f|$. At $t = 0$, $|f|$ is radially symmetric and describes a 'crater-like' surface. When $t$ becomes nonzero, the surface 'buckles'. Consider, with $\lambda$ real for simplicity,

$$|f|^2 = 1 + |\beta|^2 r^2 + |\lambda|^2 r^4 + 2\lambda r^2 R(t) \cos(2\theta - \sigma(t)).$$  \hspace{1cm} (4.18)
The θ derivatives give us the ‘ridges’ and ‘valleys’, which are at \( \theta = n\pi/2 + \sigma/2 \). If for definiteness \( \lambda > 0 \) and \( c_2 > 0 \), the ‘valleys’ are at \( \frac{k_BT}{2A} \) and \( \pi + \frac{k_BT}{2A} \) for \( t < 0 \), and at \( \frac{3\pi}{2} + \frac{k_BT}{2A} \) and \( \frac{3\pi}{2} + \frac{k_BT}{2A} \) for \( t > 0 \) shortly before and shortly after the collision. For \( \kappa = 0 \), we again find 90° scattering, whereas for \( \kappa \neq 0 \), we find a deviation from this type of scattering.

5 Solitons at a distance

To study solitons which are not close together, \( T_0 \), from Eq.(4.1), and \( \mathcal{L}_{Hopf} \), from Eq.(4.2), must be integrated over \( \mathbb{R}^2 \). For \( T_0 \), the result of the \( r \) integration can be written as

\[
\int_{\mathbb{R}^2} T_0 d^2 x = \int_0^\infty d\theta \ a(\lambda, \beta, \gamma, \theta) \ \dot{\gamma} \dot{\gamma}^*.
\]

(5.1)

\( a(\lambda, \beta, \gamma, \theta) \) can be expressed in terms of

\[
b(\lambda, \beta, \gamma, \theta) = |\beta|^2 + \lambda \gamma^* e^{2i\theta} + \lambda^* \gamma e^{-2i\theta},
\]

(5.2)

\[
\Delta(\lambda, \beta, \gamma, \theta) = 4|\lambda|^2(1 + |\gamma|^2) - b^2,
\]

(5.3)

and

\[
h(\lambda, \beta, \gamma, \theta) = \begin{cases} 
\frac{1}{\sqrt{-\Delta}} \log \frac{b + \sqrt{-\Delta}}{b - \sqrt{-\Delta}} & \text{for } \Delta < 0 \\
\frac{2}{b} & \text{for } \Delta = 0 \\
\frac{2}{\sqrt{-\Delta}}(\pi - \arctan \frac{b}{\sqrt{-\Delta}}) & \text{for } \Delta > 0
\end{cases}
\]

(5.4)

as

\[
a(\lambda, \beta, \gamma, \theta) = \frac{|\beta|^2}{4|\lambda|^2(1 + |\gamma|^2)} + \frac{2|\lambda|^2 - |\beta|^2}{4|\lambda|^2 \Delta} \left[ 2|\lambda|^2 h(\lambda, \beta, \gamma, \theta) - \frac{b}{1 + |\gamma|^2} \right].
\]

(5.5)

The \( \theta \) integration and the integration of the Euler-Lagrange equations have to be done numerically.

For large \( |\gamma| \) though, we can obtain analytical results. We start by integrating the factor of \( \dot{\gamma} \dot{\gamma}^* \) on the right-hand side of (4.1) over regions close to \( \pm a \). Set \( \gamma = -\lambda a^2 \) and \( z = \pm a + b \) with \( 0 \leq |b| \leq |a|^{3/4} \). (Here and in our arguments below, there is a wide choice of limits of integration. Here for \( |b| \),
any power of $|a|$ between $\frac{1}{2}$ and 1 would do). Neglecting $b$ terms relative to $a$ terms, we get the integrals,

$$I_\pm = 4\pi \int_0^{[a]^{3/2}} \frac{1 + |\beta|^2|a|^2}{(1 + |\beta|^2|a|^2 + 4|\lambda|^2|a|^2|b|^2)^2} \, d|b|^2.$$  \hspace{1cm} (5.6)

Hence,

$$I_\pm = \frac{\pi}{|\lambda|^2|a|^2} - \frac{\pi(1 + |\beta|^2|a|^2)}{|\lambda|^2|a|^2(1 + |\beta|^2|a|^2 + 4|\lambda|^2|a|^{7/2})},$$

and for large $|a|$,

$$I = I_+ + I_- = \frac{2\pi}{|\lambda|^2|a|^2}.$$  \hspace{1cm} (5.7)

We now show that the rest of the integral is of lower order than (5.8) and can be neglected. This is an integral over $\mathbb{R}^2$ with two circles centered at $\pm a$ taken out. We split the integral into three integrals, $I_1$, $I_2$ and $I_3$. $I_1$ is the integral for $|z|$ between 0 and $|a|^{3/4}$. $I_2$ results from the integration for $|z|$ from $|a|^{3/4}$ to $|a|^2$, and $I_3$ is the integral over the rest of $\mathbb{R}^2$ outside the big circle of radius $|a|^2$. Starting with the integral $I_1$, we use the inequalities $|z \pm a| \geq |a|/2$, and take the lower bounds of these inequalities in the integrand. This yields,

$$I_1 \leq 4\pi \int_0^{[a]^{3/2}} \frac{1 + |\beta|^2|z|^2}{(1 + |\beta|^2|z|^2 + |\lambda|^2|a|^{4/16})^2} \, d|z|^2$$

$$\leq 4\pi \int_0^{[a]^{3/2}} \frac{1 + |\beta|^2|z|^2}{(1 + |\lambda|^2|a|^{4/16})^2} \, d|z|^2.$$  \hspace{1cm} (5.9)

So the bound on $I_1$ is of order $1/|a|^5$, and $I_1$ can be neglected.

The second integration area, which is an annulus with two small disks deleted, is split into two symmetric halves with a small disk deleted in each. In one half, $|z - a| \geq |a|^{3/4}$ and $|z + a| \geq |z|$; in the other, $|z + a| \geq |a|^{3/4}$ and $|z - a| \geq |z|$. By taking the lower bounds of these inequalities in the integrand, we get an upper bound for $I_2$. We now increase the value of this upper bound by including the two small disks. So we have

$$I_2 \leq 4\pi \int_{[a]^{3/2}}^{[a]^4} \frac{1 + |\beta|^2|z|^2}{(1 + |\beta|^2|z|^2 + |\lambda|^2|a|^{3/2}|z|^2)^2} \, d|z|^2 \frac{4\pi}{(|\beta|^2 + |\lambda|^2|a|^3)^2}$$

$$\times \left[ |\beta|^2 \log(1 + |\beta|^2|z|^2 + |\lambda|^2|a|^{3/2}|z|^2) - \frac{|\lambda|^2|a|^{3/2}}{1 + |\beta|^2|z|^2 + |\lambda|^2|a|^{3/2}|z|^2} \right]_{[a]^{3/2}}^{[a]^4}.$$  \hspace{1cm} (5.10)
For large $|a|$, this goes like $\frac{\log |a|}{|a|^4}$, and $I_2$ can be neglected. Finally, for the integral $I_3$, we use the inequalities $|z \pm a| \geq |z|/2$. This leads to the following upper bound,

$$I_3 \leq 4\pi \int_0^\infty \frac{1 + |\beta|^2 |z|^2}{(1 + |\beta|^2 |z|^2 + |\lambda|^2 |z|^4/16)^2} d|z|^2 \leq \frac{2^{10\pi} |\beta|^2}{|\lambda|^4 |a|^8}.$$  \quad (5.11)

To include the generalized Hopf term, we again perform the $r$ integration and obtain

$$\int_{\mathbb{R}^2} d^2 x \mathcal{L}_{Hopf} = \frac{1}{4} |\beta|^2 \int_0^{2\pi} d\theta \ c(\lambda, \beta, \gamma, \dot{\gamma}, \theta), \quad (5.12)$$

where

$$c(\lambda, \beta, \gamma, \dot{\gamma}, \theta) = \left[ \gamma^* \dot{\gamma} - \gamma \dot{\gamma}^* + \frac{b}{2|\lambda|^2} (\lambda^* \dot{\gamma} e^{-2i\theta} - \lambda \dot{\gamma}^* e^{2i\theta}) \right] \times \left[ \frac{2|\lambda|^2}{\Delta} h(\lambda, \beta, \gamma, \theta) - \frac{b}{\Delta (1 + |\gamma|^2)} \right]. \quad (5.13)$$

Again, the $\theta$ integration and the integration of the Euler-Lagrange equations have to be done numerically.

For large $|a|$, however, we obtain some analytical results. We must integrate $\mathcal{L}_{Hopf}$, given in (4.2), over $\mathbb{R}^2$. For large $|a|$, the leading term will again come from the integration over regions close to $\pm a$. As above we set $z = \pm a + b$ and integrate for $0 < |b|^2 < |a|^{3/2}$. Neglecting $b$ terms relative to $a$ terms, these integrals can be written as,

$$J_{\pm} = \int_{|a|^{3/2}}^{(|a|^{3/2})^2} \frac{4\pi i |\beta|^2 |\lambda|^2 |a|^2 (a^* \dot{a} - a \dot{a}^*)}{(1 + |\beta|^2 |a|^2 + 4|\lambda|^2 |a|^2 |b|^2)^2} d|b|^2. \quad (5.14)$$

To leading order in $|a|$, we obtain,

$$J = J_+ + J_- = \frac{2\pi i}{|a|^2} (a^* \dot{a} - a \dot{a}^*). \quad (5.15)$$

We now show that the integral over $\mathbb{R}^2$, with the two disks centered at $\pm a$ taken out, can be neglected compared to $J$. Again using the circles of radius $|z| = |a|^{3/4}$ and $|z| = |a|^2$, respectively, the integral is divided into three integrals, $J_1$, $J_2$ and $J_3$. With the help of the inequalities $|z \pm a| \geq |a|/2$ which we used above, we now get the bound,

$$|J_1| \leq \frac{2^{11\pi} |\beta|^2 |\dot{a}|}{|\lambda|^2 |a|^{7/2}}, \quad (5.16)$$
on the integral over the disk at the origin. $J_1$ is clearly of lower order than $J$. Integration over the annulus for $|a|^{3/2} < |z|^2 < |a|^4$, with the two disks deleted, leads to the bound,

$$|J_2| \leq \frac{4\pi|\beta|^2|\lambda|^2|a||\dot{a}|}{(|\beta|^2 + |\lambda|^2|a|^{3/2})^2} \times \left[ \log(1 + |\beta|^2|z|^2 + |\lambda|^2|a|^{3/2}|z|^2) + \frac{(1 - |a|^2|\beta|^2 - |\lambda|^2|a|^{7/2})}{1 + |\beta|^2|z|^2 + |\lambda|^2|a|^{3/2}|z|^2} \right]|a|^4.$$  
(5.17)

This bound is of order $|\dot{a}|/|a|^{3/2}$, and $J_2$ can be neglected. $J_3$ can be neglected as well, since

$$|J_3| \leq \frac{2^{10}\pi|\beta|^2|\dot{a}|}{|\lambda|^2|a|^7}. \quad (5.18)$$

If we use $|\dot{\gamma}|^2 = 4|\lambda|^2|a|^2|\dot{a}|^2$ and eqs. (5.8) and (5.15), we find the following Lagrange function for large $|a|$,

$$L = 8\pi\dot{a}\dot{a}^* + 2\pi\kappa\ln\frac{a}{a^*}.$$  
(5.19)

The Hopf term is the time derivative of $2\pi\kappa\ln\frac{a}{a^*}$ and does not contribute to the equations of motion. To leading order in the distance between the solitons, we have $a(t) = a_0 + vt$. This is the free motion seen in previous numerical studies.

6 Conclusions

In this paper, which a sequel to [11], we continued our investigation of the scattering of solitons in the $CP^2$ model in (2+1) dimensions. In [11] we showed that the addition of an extra term, the “generalised Hopf term” changes dynamics of the solitons; the 90° scattering seen in head-on collisions of $CP^2$ solitons in the absence of this term is replaced by a deflection, which for small values of the coefficient of this term is proportional to this coefficient. These expectations, based on the expansion of the field at the time of overlap of the two solitons, were shown to be born out by the
results of numerical simulations. The simulations have also shown that at larger distances the generalised Hopf term seems to play little role; all deflection of the solitons seems to be confined to the region when they are very close together.

In this paper we have looked at this problem from the point of view of collective coordinates. We have argued that in this problem we can approximate the field configuration by a simple polynomial expressions (with many terms vanishing) and then looked at the effects of the nonvanishing coefficient of the generalised Hopf term on the dynamics of these collective coordinates. We have shown that this approximation is self consistent and that it captures the main features of the data (i.e. the deflection from the $90^\circ$ scattering being proportional to the coefficient of the Hopf term). Surprisingly, the approximation also captures the effect of the Hopf term becoming irrelevant (for classical dynamics) when the solitons are well separated; in this case its contribution to the effective lagrangian becomes a total derivative. Moreover, the form of this total derivative is very similar to what is seen in the $CP^1$ model (though there this form is valid for all distances between solitons, including the cases when they overlap). Thus we see that well separated solitons in $CP^2$ model are not that different from solitons of the $CP^1$ model; only when they are close together the nature of target manifold begins to play an important role, and the generalised Hopf term brings out this difference.

In addition our work has given an extra support for the use of collective coordinates; they provide a good approximation to the full dynamics also when the generalised Hopf term is present.

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