Special geometry on the 101 dimensional moduli space of the quintic threefold.

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Abstract

A new method for explicit computation of the CY moduli space metric was proposed by the authors recently. The method makes use of the connection of the moduli space with a certain Frobenius algebra. Here we clarify this approach and demonstrate its efficiency by computing the Special geometry of the 101-dimensional moduli space of the quintic threefold around the orbifold point.

1 Introduction.

When compactifying the IIB superstring theory on a Calabi–Yau (CY) threefold $X$, one can write the low-energy effective theory in terms of the geometry of the CY moduli space [1]. More precisely, the effective Lagrangian of the vector multiplets in the superspace contains $h^{2,1}$ supermultiplets. Scalars from these multiplets take value in the target space $M$, which is a moduli space of complex structures on a CY manifold and is a special Kähler manifold itself [2, 3, 4]. Metric $G_{a\bar{b}}$ and Yukawa couplings $\kappa_{abc}$ on this space are given by the following formulae:

$$G_{a\bar{b}} = \partial_a \partial_{\bar{b}} K, \quad e^{-K} = -i \int_X \Omega \wedge \bar{\Omega},$$

$$\kappa_{abc} = \int_X \Omega \wedge \partial_a \partial_b \partial_c \Omega = \frac{\partial^3 F}{\partial z^a \partial z^b \partial z^c},$$

where

$$z^a = \int_{A_a} \Omega, \quad \frac{\partial F}{\partial z^a} = \int_{B^a} \Omega$$

are the period integrals of the holomorphic volume form $\Omega$ on $X$. Here $A_a$ and $B^a$ form the symplectic basis in $H_3(X, \mathbb{Z})$. We can rewrite the expression [5] for the Kähler potential using the periods as

$$e^{-K} = -i \Pi \Sigma \Pi^\dagger, \quad \Pi = (\partial F, \ z),$$

where matrix $(\Sigma)^{-1}$ is an intersection matrix of cycles $A_a$, $B^a$ equal to the symplectic unit. In practice, computation of periods in the symplectic basis is a very complicated problem and was done

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explicitly only in few examples. It is due to the fact, that it requires a case by case analysis and geometric description of the symplectic basis of cycles. Recently we proposed a method to easily compute the Kähler metric (and the symplectic basis) for a large class of CY manifolds which can be represented by specific hypersurfaces in weighted projective spaces. Our method does not require the knowledge of symplectic cycles, but instead use a structure of a Frobenius algebra associated with a CY of this class and its Hodge structure.

Namely, let a CY manifold \( X \) be given as a solution of an equation
\[
W(x, \phi) = W_0(x) + \sum_{k=1}^{k,1} \phi_k e_k(x) = 0
\]
in some weighted projective space, where \( W_0(x) \) is a quasihomogeneous function in \( \mathbb{C}^5 \) of weight \( d \) that defines an isolated singularity at \( x = 0 \), which is tightly related with the underlying \( N = 2 \) superconformal theory \( [13] \). The monomials \( e_k(x) \) also have weight \( d \) and correspond to deformations of the complex structure of \( X \).

Polynomial \( W_0(x) \) defines a Milnor ring \( R_0 \). Inside \( R_0 \) there exists a subring \( R_0^Q \) which is invariant w.r.t the action of the so-called quantum symmetry group \( Q \). This group acts on \( \mathbb{C}^5 \) diagonally, and preserves \( W(x, \phi) \). In the cases where all complex structure deformations can be represented by polynomial deformations, \( \dim R_0^Q = \dim H^3(X) \) and the ring itself has a Hodge structure \( R_0^Q = (R_0^Q)^0 \oplus (R_0^Q)^1 \oplus (R_0^Q)^2 \oplus (R_0^Q)^3 \) in correspondence with degrees of the elements 0, d, 2d, 3d. One can introduce an invariant pairing \( \eta \) on \( R_0^Q \). The pairing turns the ring to a Frobenius algebra \( [12] \) and plays an important role in the construction of our formula for \( e^{-K} \).

Also there exists a group of phase symmetries, which acts diagonally on \( \mathbb{C}^5 \) and preserves \( W_0(x) \). It acts naturally on the invariant ring \( R_0^Q \), and this action respects the Hodge decomposition of \( R_0^Q \). This allows to pick a basis \( e_\mu(x) \) in each of the Hodge decomposition components of \( R_0^Q \), which consists of eigenvectors of the phase symmetry group action.

Using the invariant ring \( R_0^Q \) and differentials \( D_\pm = \pm dW_0 \wedge \) we construct two groups of \( Q \)-invariant cohomology \( H^2_{D,\pm}(\mathbb{C}^5)_{\text{inv}} \). These groups inherit the Hodge structure from \( R_0^Q \). We can choose bases \( e_k(x) \) with \( d \) which also consist of eigenvectors of the phase symmetry group. As shown by Candelas \([11]\), elements of these cohomology groups are in correspondence with harmonic forms of \( H^3(X) \). This isomorphism allows to define a complex conjugation on \( R_0^Q \) on the invariant cohomology.

It turns out, that in the basis \( e_\mu(x) \) it reads
\[
\ast e_\mu(x) d^5 x = M^\mu_\nu e_\nu(x) d^5 x, \quad M^\mu_\nu = \delta_{\mu\nu} e_\mu e_\nu \sigma_\mu \sigma_\nu,
\]
where \( e_\mu(x) \) is the unique element of degree \( 3d \) in \( R_0^Q \), and \( \delta_{\mu\nu} e_\mu e_\nu \) is 1 if \( \mu = \nu \) and \( e_\mu \) and zero otherwise.

Having \( H^2_{D,\pm}(\mathbb{C}^5)_{\text{inv}} \) we define a relative invariant homology group \( H^2_{D,\pm,\text{inv}} := H_2(\mathbb{C}^5, W_0 = L, \text{ReL} \to \pm \infty)_{\text{inv}} \) inside a relative homology group \( H_2(\mathbb{C}^5, W_0 = L, \text{ReL} \to \pm \infty) \). For this purpose we use oscillatory integrals. Using the oscillatory integral pairing we define a cycle \( \Gamma^\pm_\mu \) in the basis of relative invariant homology to be dual to \( e_\mu(x) d^5 x \).

In last we define periods \( \sigma^\pm_\mu(\phi) \) to be oscillatory integrals over the basis of cycles \( \Gamma^\pm_\mu \). They are equal to periods of the holomorphic volume form \( \Omega \) on \( X \) in a special basis of cycles \( H_3(X, \mathbb{C}) \) with complex coefficients.

Due to the phase symmetry invariance, in the chosen basis of cycles \( \Gamma^\pm_\mu \) the formula for Kähler potential has the following diagonal form:
\[
e^{-K(\phi)} = \sum_{\mu} (-1)^{|\mu|} \sigma^\pm_\mu(\phi) A^\mu \sigma^\pm_\mu(\phi).
\]
On the other hand, as shown in \([8]\), matrix \( A = \text{diag}(A^\mu) \) is equal to the product of the matrix of the invariant pairing \( \eta \) in the Frobenius algebra \( R_0^Q \) and the real structure matrix \( M \) such that
\[
e^{-K(\phi)} = \sum_{\mu,\nu} \eta^\mu_\nu \sigma^\pm_\mu(\phi) M^\mu_\nu(\phi) \sigma^\pm_\nu(\phi).
\]
Matrix \( M \) can be represented as \( M = T^{-1} \bar{T} \), where \( T \) is a transition matrix from periods in some real basis of cycles \( Q^+\mu \) to periods \( \sigma^\pm_\mu(\phi) \). Actually the real structure matrix is nothing but matrix

\footnote{Actually, moduli space of a CY manifold is closely related with a Frobenius manifold \([12]\), and the Frobenius algebra, we use, is a tangent space to this manifold at one point.}
in our picture. 

denote $S(3)$ the knowledge of the geometry of the 1-dimensional moduli space of the quintic Kähler structures computed via the mirror symmetry in [5] it presumably gives the geometry of the full moduli space of Calabi-Yau quintic threefold.

In what follows we apply our method for the quintic threefold, however many things are true in the greater generality.

2 Hodge structure on the middle cohomology of the quintic

First of all we notice, that the formula (1) may be written in the arbitrary basis of cycles $q_\mu \in H_3(X, \mathbb{Z})$:

$$e^{-K(\phi)} = \omega_\mu(\phi) C^{\mu \nu} \overline{\omega_\nu(\phi)},$$

$$\omega_\mu(\phi) = \int_{Q_\mu} \Omega,$$

and $(C^{-1})_{\mu \nu} = q_\mu \cap q_\nu$.

Now let us specialize to the case where $X$ is a quintic threefold:

$$X = \{(x_1 : \cdots : x_5) \in \mathbb{P}^4 \mid W(x, \phi) = 0\},$$

where

$$W(x, \phi) = W_0(x) + \sum_{t=0}^{100} \phi_t e_t(x), \quad W_0(x) = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5$$

and $e_t(x)$ are the degree 5 monomials such that each variable has the power that is a non-negative integer less than four. Let us denote monomials $e_t(x) = x_1^{t_1} x_2^{t_2} x_3^{t_3} x_4^{t_4} x_5^{t_5}$ by its degree vector $t = (t_1, \cdots, t_5)$. Then there are precisely 101 of such monomials, which can be divided into 5 sets in respect to the permutation group $S_5$: $(1,1,1,1,1)$, $(2,1,1,1,0)$, $(2,2,1,0,0)$, $(3,1,1,0,0)$, $(3,2,0,0,0)$. In these groups there are correspondingly 1, 20, 30, 30, 20 different monomials. We denote $e_t(x) := e_{(1,1,1,1,1)}(x) = x_1 x_2 x_3 x_4 x_5$ to be the so-called fundamental monomial, which will be somewhat distinguished in our picture.

For this CY dim $H_3(X, \mathbb{Z}) = 204$ and period integrals have the form

$$\omega_\mu(x) = \int_{q_\mu} \frac{dx_1 \cdots dx_5}{\partial W(x, \phi)/\partial x_4} = \int_{Q_\mu} \frac{dx_1 \cdots dx_5}{W(x, \phi)},$$

where $q_\mu \in H_3(X, \mathbb{Z})$ and $Q_\mu \in H_5(\mathbb{C}^5 \setminus (W(x, \phi) = 0), \mathbb{Z})$ are the corresponding cycles. Cohomology groups of a Kähler manifold possess a Hodge structure $H^3(X) = H^{3,0}(X) \oplus H^{2,1}(X) \oplus H^{1,2}(X) \oplus H^{0,3}(X)$. Period integrals measure variation of the Hodge structure on $H^3(X)$ as the complex structure on $X$ varies with $\phi$. This Hodge structure variation is equivalent to the one on a certain ring which we will now describe.

3 Hodge structure on the invariant Milnor ring.

We can consider $W_0(x)$ as a singularity in $\mathbb{C}^5$. Then there is an associated Milnor (also Jacobi) ring

$$R_0 = \mathbb{C}[x_1, \cdots, x_5]/(\partial W).$$
We will identify its elements with unique smallest degree polynomial representatives. For the quintic threefold $X$ its Milnor ring $R_0$ is generated as a vector space by monomials where each variable has degree less than four, and $\dim R_0 = 1024$. Polynomial $W_0(x)$ is homogeneous and, in particular, $W_0(\alpha x_1, \ldots, \alpha x_5) = W_0(x_1, \ldots, x_5)$ for $\alpha^5 = 1$. This action preserves $W_0(x)$ and is trivial in the corresponding projective space and on $X$. Such a group with this action is called a quantum symmetry $Q$, in our case $Q \cong \mathbb{Z}_5$. $Q$ obviously acts on the Milnor ring $R_0$.

Now we define a subring $R_0^Q$ in the Milnor ring $R_0$,

$$R_0^Q := \{ e_\mu(x) \in R_0 \mid e_\mu(\alpha x) = e_\mu(x), \alpha^5 = 1 \},$$
to be a $Q$-invariant part of the Milnor ring.

It is multiplicatively generated by 101 fifth-degree monomials $e_i(x)$ from (4). More precisely, $R_0^Q$ consists of elements of degree 0, 5, 10 and 15, dimensions of the corresponding subspaces are 1, 101, 101 and 1. This degree filtration defines a Hodge structure on $R_0^Q$. Basically $R_0^Q$ is isomorphic to $H^3(X)$ and the isomorphism sends the degree filtration to the Hodge filtration on $H^3(X)$ [17].

Let us denote $\chi_j = g^{ik} x_{kj}$ as an extrinsic curvature tensor for the hypersurface $W(x, \phi) = 0$ in $\mathbb{P}^4$.

Then the isomorphism above can be written as a map from $R_0^Q$ to closed differential forms in $H^3(X)$:

$$1 \rightarrow \Omega_{ijk} \in H^3(X),$$

$$e_\mu(x) \rightarrow e_\mu(x(y)) \chi_j^1 \Omega_{ijk} \in H^{2,1}(X) \text{ if } |\mu| = 5,$$

$$e_\mu(x) \rightarrow e_\mu(x(y)) \chi_j^1 \chi_j^3 \Omega_{imk} \in H^{1,2}(X) \text{ if } |\mu| = 10,$$

$$e_\mu(x) = x_1^\mu x_2^\mu x_3^\mu x_4^\mu x_5^\mu \chi_j^1 \chi_j^5 \chi_j^5 \Omega_{imp} = \kappa \Omega \in H^{0,5}(X)$$

The details on this map can be found in [17, 19]. We also introduce the notation $e_\mu(x)$ for elements of the monomial basis of $R_0^Q$, where $\mu = (\mu_1, \ldots, \mu_5), \mu_1 \in \mathbb{Z}_5$, $e_\mu(x) = \prod x_1^{\mu_1}$ and $|\mu| = \sum \mu_1$ is the degree of $e_\mu(x)$. In particular, $\rho = (3, 3, 3, 3)$, that is $e_\rho(x)$ is a unique degree 15 element of $R_0^Q$.

There is a $\mathbb{Z}_5^5$ phase symmetry group acting diagonally on $\mathbb{C}_5$: $\alpha \cdot (x_1, \ldots, x_5) = (\alpha_1 x_1, \ldots, \alpha_5 x_5), \alpha_1^5 = 1$. This action preserves $W_0 = \sum x_5^2$. The mentioned above quantum symmetry $Q$ is a diagonal subgroup of the phase symmetries. Basis $\{ e_\mu(x) \}$ is an eigenbasis of the phase symmetry and each $e_\mu(x)$ has a unique weight. Note that phase symmetry preserves the Hodge decomposition.

One additional important fact is that on the invariant ring $R_0^Q$ there exists a natural invariant pairing turning it into a Frobenius algebra [12, 8]:

$$\eta_{\mu\nu} = \text{Res}_{x} e_\mu(x) e_\nu(x) \prod_i \delta_i W_0(x)$$

Up to an irrelevant constant for the monomial basis it is $\eta_{\mu\nu} = \delta_{\mu+i\nu,\rho}$. This pairing plays a crucial role in our construction.

Let us introduce a couple of differentials [20] on differential forms on $\mathbb{C}_5^5$: $D_\pm = d \pm dW_0(x)$. They define the cohomology groups $H^*_D(\mathbb{C}_5^5)$. The cohomologies are only nontrivial in the top dimension $H^3_D(\mathbb{C}_5^5) \cong R_0^Q$. The isomorphism $J$ has an explicit description

$$J(e_\mu(x)) = e_\mu(x) d^5 x, \ e_\mu(x) \in R_0.$$ We see, that $Q \cong \mathbb{Z}_5$ naturally acts on $H^3_D(\mathbb{C}_5^5)$ and $J$ sends the $Q$-invariant part $R_0^Q$ to $Q$-invariant subspace $H^3_{D, inv}(\mathbb{C}_5^5)$. Therefore, the latter space obtains the Hodge structure as well. Actually, this Hodge structure naturally corresponds to the Hodge structure on $H^3(X)$.

The complex conjugation acts on $H^3(X)$ so that $H^{\sigma}(X) = H^{\sigma}(X)$, in particular $H^{3,2}(X) = H^{1,2}(X)$. Through the isomorphism between $R_0^Q$ and $H^3(X)$ the complex conjugation acts also on the elements of the ring $R_0^Q$ as $e_\mu(x) = p_\mu e_{\mu-\rho}(x)$, where $p_\mu$ is a constant to be determined. In particular, differential form built from $e_\mu(x) + p_\mu e_{\mu-\rho}(x) \in H^3(X, \mathbb{R})$ is real and $p_\mu p_{\mu-\rho} = 1$.

4 Oscillatory representation and computation of $\sigma_\mu(\phi)$

Relative homology groups $H_3(\mathbb{C}_5^5, W_0 = L, \text{Re} L \rightarrow \pm \infty)$ have a natural pairing with $Q$-invariant cohomology groups $H^3_{D, inv}(\mathbb{C}_5^5)$:

$$\langle e_\mu(x) d^5 x, \Gamma^\pm \rangle = \int_{\Gamma^\pm} e_\mu(x) e^{\pm W_0(x)} d^5 x, \ H_3(\mathbb{C}_5^5, W_0 = L, \text{Re} L \rightarrow \pm \infty).$$
Using this we define two invariant homology groups $\mathcal{H}^\pm_{g, inv}$ as quotient of $H_2(C^5, W_0 = L, \text{Re}L \to \pm\infty)$ with respect to the subgroups orthogonal to $H_{D_2}^2(C^5)_{inv}$. Now we introduce bases $\Gamma^\pm_{\mu}$ in the homology groups $\mathcal{H}^\pm_{g, inv}$ using the duality with the bases in $H_{D_2}^2(C^5)_{inv}$:

$$\int_{\Gamma^\pm_{\mu}} e_\nu(x) e^{\pm W_0(x)} d^5 x = \delta_{\mu\nu}$$

and the corresponding periods

$$\sigma^\pm_{\alpha\mu}(\phi) := \int_{\Gamma^\pm_{\mu}} e_\alpha(x) e^{\pm W(x, \phi)} d^5 x,$$

$$\sigma^\pm_{\mu}(\phi) := \sigma^\pm_{\alpha\mu}(\phi)$$  \hspace{1cm} (6)

which are understood as series expansions in $\phi$ around zero.

Periods $\sigma^\pm_{\mu}(\phi)$ satisfy the same differential equation as periods $\omega_{\mu}(\phi)$ of the holomorphic volume form on $X$. Moreover, these sets of periods span same subspaces as functions of $\phi$. It follows, that we can define cycles $Q^\pm_{\mu} \in \mathcal{H}^\pm_{g, inv}$ such that

$$\int_{Q^\pm_{\mu}} e^{\pm W(x, \phi)} d^5 x = \int_{\gamma_{\mu}} \Omega = \int_{Q_{\mu}} \frac{d^5 x}{W(x, \phi)}$$  \hspace{1cm} (7)

and periods $\omega^\pm_{\alpha\mu}(\phi)$ are given by the integrals over these cycles analogous to (4).

With these notations the idea of computation of periods [21]

$$\sigma^\pm_{\mu}(\phi) = \int_{\Gamma^\pm_{\mu}} e^{\mp W(x, \phi)} d^5 x$$  \hspace{1cm} (8)

can be stated as follows.

To explicitly compute $\sigma^\pm_{\mu}(\phi)$, first we expand the exponent in the integral (4) in $\phi$ representing $W(x, \phi) = W_0(x) + \sum_\alpha \phi_\alpha e_\alpha(x)$

$$\sigma^\pm_{\mu}(\phi) = \sum_m \left( \prod_s \left( \frac{\pm \phi_s}{m_s!} \right) \right) \int_{\Gamma^\pm_{\mu}} \prod_s e_\alpha(x)^{m_s} e^{\mp W_0(x)} d^5 x.$$  \hspace{1cm} (9)

We note, that $\sigma^-_{\mu}(\phi) = (-1)^{|\mu|} \sigma^+_{\mu}(\phi)$, so we focus on $\sigma_{\mu}(\phi) := \sigma^+_{\mu}(\phi)$.

For each of the summands in (10) the form $\prod_s e_\alpha(x)^{m_s} d^3 x$ belongs to $H_{D_2}^2(C^5)_{inv}$, because it is $Q$–invariant. Therefore, we can expand it in the basis $e_\nu(x) d^3 x \in H_{D_2}^2(C^5)_{inv}$. Namely we always can find such a polynomial 4–form $U$, that

$$\prod_s e_\alpha(x)^{m_s} d^3 x = \sum_\nu C_\nu(m) e_\nu(x) d^3 x + D_\nu U.$$

Therefore for the integral in (10) we obtain

$$\int_{\Gamma^\pm_{\mu}} \prod_s e_\alpha(x)^{m_s} e^{\mp W_0(x)} d^5 x = C_\mu(m).$$

Writing (10) explicitly we have

$$\sigma_{\mu}(\phi) = \sum_m \left( \prod_s \frac{\phi_s^{m_s}}{m_s!} \right) \int_{\Gamma^\pm_{\mu}} \prod_s x_i^{m_s n_i} e^{-W_0(x)} d^5 x.$$  \hspace{1cm} (10)

Let $m_s n_i = 5n_i + \nu_i$, $\nu_i < 5$. Therefore we want to expand

$$\prod_i x_i^{5n_i + \nu_i} d^5 x = \sum_\nu c_\nu(x) e_\nu(x) d^5 x + D_\nu U.$$

Note that

$$D_+ \left( \frac{1}{5} x_1^{5n_i+k-4} f(x_2, \ldots, x_5) d x_2 \wedge \cdots \wedge d x_5 \right) =$$

$$= \left[ x_1^{5n_i+k} + \left( n + \frac{k-4}{5} \right) x_1^{5(n-1)+k} \right] f(x_2, \ldots, x_5) d^5 x$$  \hspace{1cm} (11)
Therefore in $D_+$ cohomology we have
\[ \prod_i x_i^{5n_i+\nu_i} d^5x = -\left( n_1 + \frac{\nu_1 - 4}{5} \right) x_1^{5(n_1-1)+\nu_1} \prod_{i=2}^5 x_i^{5n_i+\nu_i} d^5x, \quad \nu_i < 5. \] (12)

By induction we obtain
\[ \prod_i x_i^{5n_i+\nu_i} d^5x = (-1)^{\sum_i n_i} \prod_i \left( \frac{\mu_i + 1}{5} \right)^{n_i} \prod_i x_i^{\nu_i} d^5x, \quad \nu_i < 5. \] (13)

where $(\alpha)_{\alpha'} = \Gamma(a+n)/\Gamma(a)$.

Using (13) once again, we see that if any $\nu_i = 4$ then the differential form is trivial and the integral is zero. Hence, rhs of (13) is proportional to $e_{\nu}(x)$ and gives the desired expression. Plugging (4) into (3) and integrating over $\Gamma^+_\mu$ gives the answer
\[ \sigma_\mu(\phi) = \sigma^+_\mu(\phi) = \sum_{n_i \geq 0} \prod_i \left( \frac{\mu_i + 1}{5} \right)^{n_i} \prod_{m \in \Sigma_n} \frac{\phi^m_{\nu}}{m}! \],

where
\[ \Sigma_n = \{ m \mid \sum_s m_s s_i = 5n_i + \mu_i \} \]

Further we will also use the periods with slightly different normalization, which turn out to be convenient
\[ \hat{\sigma}_\mu(\phi) = \prod_i \Gamma \left( \frac{\mu_i + 1}{5} \right) \sigma_\mu(\phi) = \sum_{n_i \geq 0} \prod_i \Gamma \left( n_i + \frac{\mu_i + 1}{5} \right) \prod_{m \in \Sigma_n} \frac{\phi^m_{\nu}}{m}! \]. (14)

5 Computation of the Kähler potential

Pick any basis $Q^\pm_\mu$ of cycles with integer or real coefficients as in (4). Then for the Kähler potential we have the formula
\[ e^{-K} = \omega^\mu_\mu(\phi) C^{\nu\nu} \omega^-_\nu(\phi) \] (15)
in which the matrix $C^{\nu\nu}$ is related with the Frobenius pairing $\eta$ as
\[ \eta_{\alpha\beta} = \omega^\mu_{\alpha\mu}(0) C^{\nu\nu} \omega^-_\nu(0). \] (16)
The last expression is due to [22, 23]. Let also $T^\pm$ be a coordinate change matrix $Q^\pm_\mu = (T^\pm)_{\mu}^\nu Q^\pm_\nu$. Then $M = (T^-)^{-1} T^-$ is a real structure matrix, that is $M M = 1$ and by construction $M$ doesn’t depend on the choice of basis $Q^\pm_\mu$. $M$ is only defined by our choice of $\Gamma^+_\mu$.

In [24] we deduced from (3) and (4) the formula
\[ e^{-K(\phi)} = \sigma^+_\mu(\phi) \eta^{\lambda\mu} M_{\lambda\delta} \sigma^\delta_\nu(\phi) = \sigma_{\mu} A^{\nu\nu} \sigma_{\nu}, \] (17)

where $\eta^{\nu\nu} = \eta_{\nu\nu} = \delta_{\mu,\nu}$. In that papers our method to compute the real structure matrix $M$ used the knowledge of the periods in some basis $q_{\mu}$ computed using the residue formula and monodromy considerations. However, this method gives only 4 out of 204 linearly independent periods for the quintic threefold $X$.

Therefore we propose here a different method to find $M$.

Lemma 5.1. Inverse intersection matrix $A^{\nu\nu}$ in (3) is diagonal.

Proof. We may extend the action of the phase symmetry group to the action $\mathfrak{A}$ on the parameter space $\{ \phi_\nu \}$ such that $W = W_0 + \sum_\nu \phi_\nu e_\nu(x)$ is invariant under this new action. Each $e_\nu(x)$ has a unique weight under this group action.

Action $\mathfrak{A}$ can be compensated using the coordinate transformation and therefore is trivial on the moduli space of the quintic (implying that point $W = W_0$ is an orbifold point of the moduli space). In particular, $e^{-K} = \int_X \Omega \wedge \Omega$ is $\mathfrak{A}$ invariant. Consider
\[ e^{-K} = \sigma_{\mu} A^{\nu\nu} \sigma_{\nu}, \]
as a series in $\phi_\nu \wedge \Omega$. Each monomial has a certain weight under $\mathfrak{A}$. For the series to be invariant, each monomial must have weight 0. But weight of $\sigma_{\mu} \sigma_{\nu}$ equals to $\mu - \nu$ and due to non-degeneracy of weights of $\sigma_{\mu}$ only the ones with $\mu = \nu$ have weight zero. \qed
Thus, \( e^{-K} = \sum_{\rho} A^\mu |\sigma_\mu(\phi)|^2 \).

Moreover, the matrix \( A \) should be real and, because \( A = \eta \cdot M \), \( M \bar{M} = 1 \) and \( \eta_{\mu\nu} = \delta_{\mu+\nu,\rho} \), we have
\[
A^\mu A^{-\mu} = 1. \tag{18}
\]

**Monodromy considerations**

To fix the remaining 102 real numbers \( A^\mu \) we use monodromy invariance of \( e^{-K} \) around \( \phi_0 = \infty \). Fix some \( t = (t_1, t_2, t_3, t_4, t_5) \), \( |t| = 5 \) and let \( \phi_{v, t_1, 0} = 0 \), also consider only the first order in \( \phi_v \). Then the condition that period \( \sigma_\mu(\phi) \) contains only non-zero summands of the form \( \hat{\sigma}_0^0 \hat{\sigma}_v \) implies that \( \mu = t + \text{const} \cdot (1,1,1,1,1) \mod 5 \). For each \( t \) from the table below only such possibilities are \( \mu = t \) and \( \mu = \rho - t' = (3,3,3,3,3) - t' \), where \( t' \) denotes a vector obtained from \( t \) by permutation (written explicitly in the table below) of its coordinates.

Therefore, in this setting \( A^\mu \) becomes
\[
e^{-K} = \sum_{k=0}^{3} a_k |\hat{\sigma}_t(k,k,k,k,k)|^2 + a_t |\hat{\sigma}_t|^2 + a_{\rho-t'} |\hat{\sigma}_{\rho-t'}|^2 + O(\phi_t^5),
\]
where we used periods \( \hat{\sigma} \) from (4), \( a_t = A^t \prod_i \Gamma((t_1 + 1)/5)^2 \) and \( a_k, k = 0, 1, 2, 3 \) are already known \( \hat{\sigma} \). This expression should be monodromy invariant. We consider the effect of the transport of \( \phi_0 \) around \( \infty \). From the formula (4) we have
\[
F_1 = \hat{\sigma}_t(\phi_v, \phi_0) = g_t \phi_k F(a, b; a + b | (\phi_0/5)^3) + O(\phi_t^5),
\]
\[
F_2 = \hat{\sigma}_{\rho-t'}(\phi_v, \phi_0) = g_{\rho-t'} \phi_0^{-a-b} F(1 - a, 1 - b; 2 - a - b | (\phi_0/5)^3) + O(\phi_t^5),
\]
where \( g_t, g_{\rho-t'} \) are some constants. Explicitly for all different labels \( t \)

| \( t \)       | \( \rho - t' \)       | \( (a, b) \)       |
|------------|----------------------|-------------------|
| (2,1,1,1,0)| (3,2,2,2,1)          | (2,5,2/5)         |
| (2,2,1,0,0)| (3,3,2,1,1)          | (1,5,3/5)         |
| (3,1,1,0,0)| (0,3,3,2,2)          | (1,5,2/5)         |
| (3,2,0,0,0)| (1,0,3,3,3)          | (1,5,1/5)         |

and
\[
F(a, b; c | z) := \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} 2 F_1(a, b; c; z).
\]

When \( \phi_0 \) goes around infinity
\[
\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = B \cdot \begin{pmatrix} F_1 \\ F_2 \end{pmatrix},
\]
where (e.g. [24])
\[
B = \frac{1}{is(a + b)} \left( e^{|s(a-b)|} - e^{is(a+b)} \right) \left( 2e^{2is(a+b)}s(a)b - e^{|s(a-b)|} e^{2is(a+b)} - 2 \right).
\]

Here \( c(x) = \cos(\pi x), s(x) = \sin(\pi x) \). It is straightforward to show the following

**Proposition 1.**

\[
a_t |\hat{\sigma}_t|^2 + a_{\rho-t'} |\hat{\sigma}_{\rho-t'}|^2 = a_t \prod_i \Gamma\left( \frac{t_i + 1}{5} \right)^2 |\hat{\sigma}_t|^2 + a_{\rho-t'} \prod_i \Gamma\left( \frac{4 - t_i}{5} \right)^2 |\hat{\sigma}_{\rho-t'}|^2
\]
is \( B \)-invariant iff \( a_t = -a_{\rho-t'} \).

Due to symmetry we have \( a_{\rho-t'} = a_{\rho-t} \) in each case. From (3) it follows that the product of the coefficients at \( |\hat{\sigma}_t|^2 \) and \( |\hat{\sigma}_{\rho-t'}|^2 \) in the expression for \( e^{-K} \) should be 1:
\[
A^\mu \cdot A^t = a_{\rho-t'} \cdot a_t \prod_i \Gamma\left( \frac{t_i + 1}{5} \right)^2 \Gamma\left( \frac{4 - t_i}{5} \right)^2 = 1.
\]

Due to reflection formula \( a_t = \pm \prod_i \pi(t_i + 1)/5 \) up to a common factor of \( \pi \). The sign turns out to be minus for Kähler metric to be positive definite in the origin. Therefore
\[
A^\mu = (-1)^{\varphi(\mu)/5} \prod \Gamma\left( \frac{\mu_t + 1}{5} \right).
\]
Finally the Kähler potential becomes

\[ e^{-K(\phi)} = \sum_{\mu=0}^{203} (-1)^{\deg(\mu)/5} \prod \gamma \left( \frac{\mu_i + 1}{5} \right) |\sigma_\mu(\phi)|^2, \tag{19} \]

where \( \gamma(x) = \frac{\Gamma(\frac{x}{5})}{\Gamma(1 - \frac{x}{5})} \).

6 Real structure on the cycles \( \Gamma^\pm \)

Let cycles \( \gamma_\mu \in H_3(X) \) be the images of cycles \( \Gamma^\pm_\mu \) under the isomorphism \( H^+_5, \text{inv}^5 \cong H_3(X) \).

Complex conjugation sends \((2,1)\)-forms to \((1,2)\)-forms. Similarly it extends to a mapping on the dual homology cycles \( \gamma_\mu \). In the real basis of cycles a version of the formula (5) takes an especially simple form, because the real structure matrix \( M \) becomes an identity.

Lemma 6.1. Conjugation of homology classes has the following form: \( *\gamma_\mu = p_\mu \gamma_{\rho - \mu} \), where \( \rho = (3,3,3,3,3) \) is a unique maximal degree element in the Milnor ring.

Proof. We perform a proof for the cohomology classes represented by differential forms. For one-dimensional \( H^{1,0}(X) \) and \( H^{0,3}(X) \) it is obvious. Let

\[ \Omega_{2,1} := e_t(x) \chi_1^1 \Omega_{ijk} \in H^{2,1}(X). \]

Any element from \( H^{1,2}(X) \) is representable by a degree 10 polynomial \( P(x) \) as follows from (19) as

\[ \Omega_{2,1} = \Omega_{1,2} := P(x) \chi_1^1 \chi_\mu^m \Omega_{iok} \in H^{1,2}(X). \]

The group of phase symmetries modulo common factor acts by isomorphisms on \( X \). Therefore, it also acts on the differential forms. Lhs and rhs of the previous equation should have the same weight under this action, and weight of the lhs is equal \(-t\) modulo \((1,1,1,1,1)\). It follows that

\[ P(x) = p_t e_{p-1}(x) \]

with some constant \( p_t \).

Using this lemma and applying the complex conjugation of cycles to the formula (19) to obtain

\[ e^{-K} = \sum_\mu A^\mu |\sigma_\mu|^2 = \sum_\mu p_\mu^2 A^\mu |\sigma_{\rho - \mu}|^2, \]

it follows that \( A^\mu = \pm 1/p_\mu \). Now formula (19) implies

\[ p_\mu = \prod \gamma \left( \frac{1 - \mu_i}{5} \right). \tag{20} \]

7 Conclusions

The method for computing the Kähler potential on the CY moduli space from [9] modified in this paper does not require knowledge of periods in some real homology basis. Instead, we use some simple monodromy considerations to fix the real structure matrix. Another possible interesting method would be to determine this matrix by direct computation of coefficients (5) of the complex conjugation in the basis \( e_\mu(x) \). In this paper we use our modified method to compute Weil–Peterson metric on the whole 101-dimensional complex structure moduli space of the quintic threefold around the orbifold point [20]. Together with the computation of the moduli space geometry of the Kähler structures through the mirror map [20] it describes the Special geometry of all Ricci flat deformations of CY metric in the region.

Though we present our result for the quintic threefold, our method should be applicable to a bigger class of models, which are connected with Landau–Ginzburg description, in particular hypersurfaces in toric varieties. We plan to consider possible generalizations in the future publications.

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References

[1] P. Candelas, Gary T. Horowitz, Andrew Strominger, and Edward Witten. “Vacuum Configurations for Superstrings”. In: Nucl. Phys. B258 (1985), pp. 46–74. doi: 10.1016/0550-3213(85)90602-9

[2] Philip Candelas, Paul S. Green, and Tristan Hubsch. “Rolling Among Calabi-Yau Vacua”. In: Nucl. Phys. B330 (1990), p. 49. doi: 10.1016/0550-3213(90)90302-T

[3] Philip Candelas and Xenia de la Ossa. “Moduli Space of Calabi-Yau Manifolds”. In: Nucl. Phys. B355 (1991), pp. 455–481. doi: 10.1016/0550-3213(91)90122-E

[4] Andrew Strominger. “Special geometry”. In: Commun. Math. Phys. 133 (1990), pp. 163–180. doi: 10.1007/BF02096659

[5] Philip Candelas, Xenia C. De La Ossa, Paul S. Green, and Linda Parkes. “A Pair of Calabi-Yau manifolds as an exactly soluble superconformal theory”. In: Nucl. Phys. B359 (1991). [AMS/IP Stud. Adv. Math.9,31(1998)], pp. 21–74. doi: 10.1016/0550-3213(91)90292-6

[6] Albrecht Klemm and Stefan Theisen. “Recent efforts in the computation of string couplings”. In: Theor. Math. Phys. 95 (1993). [Teor. Mat. Fiz.95,293(1993)], pp. 583–594. doi: 10.1007/BF01017144 arXiv: hep-th/9210142 [hep-th]

[7] Philip Candelas, Anamaria Font, Sheldon H. Katz, and David R. Morrison. “Mirror symmetry for two parameter models. 2.” In: Nucl. Phys. B429 (1994), pp. 626–674. doi: 10.1016/0550-3213(94)90155-4 arXiv: hep-th/9403187 [hep-th]

[8] Philip Candelas, Xenia De La Ossa, Anamaria Font, Sheldon H. Katz, and David R. Morrison. “Mirror symmetry for two parameter models. 1.” In: Nucl. Phys. B416 (1994). [AMS/IP Stud. Adv. Math.1,483(1996)], pp. 481–538. doi: 10.1016/0550-3213(94)90322-0 arXiv: hep-th/9308083 [hep-th]

[9] Konstantin Aleshkin and Alexander Belavin. “A new approach for computing the geometry of the moduli spaces for a Calabi-Yau manifold”. In: (2017). arXiv: 1706.05342 [hep-th]

[10] Konstantin Aleshkin and Alexander Belavin. “Special geometry on the moduli space for the two-moduli non-Fermat Calabi-Yau”. In: (2017). arXiv: 1708.08362 [hep-th]

[11] Per Berglund and Tristan Hubsch. “A Generalized Construction of Calabi-Yau Models and Mirror Symmetry”. In: (2016). arXiv: 1611.10300 [hep-th]

[12] B. Dubrovin. “Integrable systems in topological field theory”. In: Nucl. Phys. B379 (1992), pp. 627–689. doi: 10.1016/0550-3213(92)90137-2

[13] Vladimir Arnold, Alexander Varchenko, and Sabir Gusein-Zade. Singularities of Differentiable Maps. Birkhauser, 1985.

[14] Wolfgang Lerche, Cumrun Vafa, and Nicholas P. Warner. “Chiral Rings in N=2 Superconformal Theories”. In: Nucl. Phys. B324 (1989), pp. 427–474. doi: 10.1016/0550-3213(89)90474-4

[15] Emil J. Martinec. “Algebraic Geometry and Effective Lagrangians”. In: Phys. Lett. B217 (1989), pp. 431–437. doi: 10.1016/0370-2693(89)90074-9

[16] Doron Gepner. “Exactly Solvable String Compactifications on Manifolds of SU(N) Holonomy”. In: Phys. Lett. B199 (1987), pp. 380–388. doi: 10.1016/0370-2693(87)90938-5

[17] P. Candelas. “Yukawa Couplings Between (2,1) Forms”. In: Nucl. Phys. B298 (1988), p. 458. doi: 10.1016/0550-3213(88)90351-3

[18] Per Berglund, Philip Candelas, Xenia De La Ossa, Anamaria Font, Tristan Hubsch, Dubravka Jancic, and Fernando Quevedo. “Periods for Calabi-Yau and Landau-Ginzburg vacua”. In: Nucl. Phys. B419 (1994), pp. 352–403. doi: 10.1016/0550-3213(94)90047-7 arXiv: hep-th/9308005 [hep-th]
[19] Philip Candelas and Sunny Kalara. “Yukawa Couplings for a Three Generation Superstring Compactification”. In: Nucl. Phys. B298 (1988), pp. 357–368. doi: 10.1016/0550-3213(88)90271-4

[20] Kyoji Saito. “The higher residue pairings $K_F^{(k)}$ for a family of hypersurface singular points”. In: Singularities, Proc. of symp. in pure math 40.2 (1983), pp. 441–463.

[21] Alexander Belavin and Vladimir Belavin. “Flat structures on the deformations of Gepner chiral rings”. In: JHEP 10 (2016), p. 128. doi: 10.1007/JHEP10(2016)128 arXiv: 1606.07376 [hep-th]

[22] Sergio Cecotti and Cumrun Vafa. “Topological antitopological fusion”. In: Nucl. Phys. B367 (1991), pp. 359–461. doi: 10.1016/0550-3213(91)90021-0

[23] Alessandro Chiodo, Hiroshi Iritani, and Yongbin Ruan. “Landau-Ginzburg/Calabi-Yau correspondence, global mirror symmetry and Orlov equivalence”. In: (2012). arXiv: 1201.0813 [math.AG]

[24] I. S. Gradshteyn and I. M. Ryzhik. Table of integrals, series, and products. Seventh. Translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger, With one CD-ROM (Windows, Macintosh and UNIX). Elsevier/Academic Press, Amsterdam, 2007, pp. xlviii+1171. ISBN: 978-0-12-373637-6; 0-12-373637-4.