On quantum deformation of conformal symmetry: 
Gauge dependence via field redefinitions

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Abstract

The effective action in gauge theories is known to depend on a choice of gauge fixing conditions. This dependence is such that any change of gauge conditions is equivalent to a field redefinition in the effective action. In this sense, the quantum deformation of conformal symmetry in the $\mathcal{N} = 4$ super Yang-Mills theory, which was computed in 't Hooft gauge in hep-th/9808039 and hep-th/0203236, is gauge dependent. The deformation is an intrinsic property of the theory in that it cannot be eliminated by a local choice of gauge (although we sketch a field redefinition induced by a nonlocal gauge which, on the Coulomb branch of the theory, converts the one-loop quantum-corrected conformal transformations to the classical ones). We explicitly compute the deformed conformal symmetry in $R_\xi$ gauge. The conformal transformation law of the gauge field turns out to be $\xi$-independent. We construct the scalar field redefinition which relates the 't Hooft and $R_\xi$ gauge results. A unique feature of 't Hooft gauge is that it makes it possible to consistently truncate the one-loop conformal deformation to the terms of first order in derivatives of the fields such that the corresponding transformations form a field realization of the conformal algebra.
1 Introduction

In gauge theories, not all rigid symmetries of the classical action can be maintained manifestly throughout the quantization procedure, even in the absence of anomalies. As was demonstrated some years ago by van Holten [1] and also discussed in our recent paper [2], the problem of maintaining manifestly a rigid symmetry at the quantum level basically reduces to selecting covariant gauge fixing conditions. The latter cannot always be achieved, at least in the class of local gauge conditions. A prominent example is provided by conformal symmetry (or its supersymmetric extensions) in quantum Yang-Mills theories with identically vanishing beta-function such as the $\mathcal{N} = 4$ super Yang-Mills theory. It has been known since the early 1970’s (see, e.g., [3, 4, 5] for a more detailed discussion and references to the original publications) that, if the vacuum in such a theory is conformally invariant and the gauge field $A_m(x)$ transforms as a primary field with the canonical dimension $d_A = 1$, then the quantum theory is trivial since the gauge field two-point function is longitudinal, $\langle A_m(x_1) A_n(x_2) \rangle \propto \partial_m \partial_n \ln (x_1 - x_2)^2$. In other words, the gauge field has no physical transverse degrees of freedom, only purely gauge ones. This clearly demonstrates that no local conformally covariant gauge conditions exist. As was shown by Fradkin and Palchik [3], the generating functional in these theories is invariant under deformed special conformal transformations consisting of a combination of conformal transformations and compensating field-dependent gauge transformations; the conformal Ward identity associated with the deformed symmetry leads to a propagator with the correct transverse part.

The approach of [3] has recently been applied [9, 2] to evaluate leading quantum corrections to the deformed conformal transformation on the Coulomb branch of the $\mathcal{N} = 4$ super Yang-Mills theory. This has led to striking results which we summarize here. Classically, the action of the $\mathcal{N} = 4$ super Yang-Mills theory is invariant under linear conformal transformations which in the bosonic sector $\Phi^i = \{ A_m(x), Y_\mu(x) \}$, with $\mu = 1, \ldots, 6$, are:

$$
- \delta_c A_m = v A_m + \omega_m^n A_n + \sigma A_m , \quad - \delta_c Y_\mu = v Y_\mu + \sigma Y_\mu ,
$$

(1.1)

1In the seventies, several publications appeared (see [6] and references therein) where a nonlocal conformally covariant gauge condition was employed for computing some correlation functions in massless QED; in fact, manifest conformal covariance in this approach was achieved by accompanying any special conformal transformation by a gauge one, as in [3]. Another recipe for achieving a manifest conformal covariance in massless QED was [3] to use a version of Gupta-Bleuler quantization in conjunction with the higher derivative gauge condition $\Box \partial^m A_m = 0$ (introduced independently in [8]), which becomes conformally invariant when the Maxwell equation $\partial^m F_{mn} = 0$ is imposed.
where $v = v^m \partial_m$ is an arbitrary conformal Killing vector field,

$$\partial_m v_n + \partial_n v_m = 2 \eta_{mn} \sigma, \quad \sigma \equiv \frac{1}{4} \partial_m v^n, \quad \omega_{mn} \equiv \frac{1}{2} (\partial_m v_n - \partial_n v_m). \quad (1.2)$$

Quantum mechanically, the effective action is invariant under conformal transformations which in principle receive contributions at each loop order,

$$\Delta \Phi = \delta_c \Phi + \sum_{L=1}^{\infty} \hbar^L \delta (L) \Phi. \quad (1.3)$$

On the Coulomb branch of the $\mathcal{N} = 4$ super Yang-Mills theory, when the gauge group $SU(N + 1)$ is spontaneously broken to $SU(N) \times U(1)$, the one-loop deformation in the $U(1)$ sector reads (with $g$ the Yang-Mills coupling constant and $Y^2 = Y_\mu Y^\mu$)

$$\delta_{(L=1)} A_m = - \frac{Ng^2}{4\pi^2} (\partial^n \sigma) F_{mn} Y^2, \quad \delta_{(L=1)} Y_\mu = \frac{Ng^2}{4\pi^2} (\partial^n \sigma) \partial_n Y^\mu Y^2, \quad (1.4)$$

up to terms of second order in the derivative expansion. This deformation was computed in [2] in the framework of the background field approach and with the use of 't Hooft gauge. The scalar deformation, $\delta_{(L=1)} Y_\mu$, had previously been derived in [9]. Modulo a purely gauge contribution, the one-loop corrected transformations coincide with the rigid symmetry [10, 11] (in what follows, we set $\hbar = 1$ and introduce $R^4 = Ng^2/(2\pi^2)$)

$$\delta A_m = \delta_c A_m - \frac{R^4}{2Y^2} (\partial^n \sigma) F_{mn} + \partial_m \left( \frac{R^4}{2Y^2} (\partial^n \sigma) A_n \right), \quad (1.5)$$

$$\delta Y_\mu = \delta_c Y_\mu + \frac{R^4}{2Y^2} (\partial^n \sigma) \partial_n Y_\mu, \quad (1.6)$$

of a D3-brane embedded in $AdS_5 \times S^5$ with the action (we set $2\pi \alpha' = 1$ and ignore the Chern-Simons term, see, e.g. [12] for more detail):

$$S = - \frac{1}{g^2} \int d^4 x \left( \sqrt{-\det \left( \frac{Y^2}{R^2} \eta_{mn} + \frac{R^2}{Y^2} \partial_m Y_\mu \partial_n Y^\mu + F_{mn} \right)} - \frac{Y^4}{R^4} \right). \quad (1.7)$$

It was shown by Maldacena [10] that, assuming $SO(6)$ invariance along with supersymmetric non-renormalization theorems in the $\mathcal{N} = 4$ super Yang-Mills theory [13], the transformation law (1.4) uniquely fixes the scalar part of the D3-brane action (1.7). From the point of view of Yang-Mills theory, this low energy effective action results from summing up quantum corrections to all loop orders. Thus the one-loop deformation (1.6) of conformal symmetry allows us to get non-trivial multi-loop information about the effective action! This illustrates that the concept of deformed conformal symmetry is clearly
important and useful. On the other hand, one can ask the following natural question:

“Since the deformation (1.4) corresponds to a particular set of gauge conditions – ’t Hooft gauge - to what extent is it gauge independent?” In the present note, we address this question.

This paper is organized as follows. In section 2, we review some long established results (see, e.g., [14] and references therein) concerning the gauge dependence of the effective action in gauge theories. In particular, we provide a simple proof of the fact that any change of gauge fixing conditions is equivalent to a field redefinition in the effective action; a slightly different proof, based on the use of the BRST symmetry, has recently been given in [15] in the Matrix model context. This analysis is extended in section 3 to the background field quantization scheme, which is a convenient way to implement a manifestly gauge invariant definition of the effective action. In section 4, we specify the sufficient conditions for a rigid symmetry to become deformed at the quantum level. A general discussion of the quantum deformation of the conformal symmetry in \( \mathcal{N} = 4 \) SYM theory is provided. We also outline the construction of a nonlocal field redefinition, on the Coulomb branch of \( \mathcal{N} = 4 \) SYM theory, which converts the classical conformal transformation to the deformed one. The gauge dependence of the deformed conformal symmetry is analysed in section 5 by explicit calculations in \( R_\xi \) gauge, and we make some observations on the significance of ’t Hooft gauge. In section 6, we summarize our results.

2 Gauge dependence of the effective action

We will use DeWitt’s condensed notation [16, 17], which is by now standard in quantum field theory [18]; in particular, \( \Psi_{,i}[\Phi] \) denotes the variational derivative \( \delta \Psi[\Phi]/\delta \Phi^i \). For simplicity, we restrict attention to the case of bosonic gauge theories. Let \( S[\Phi] \) be the action of an irreducible gauge theory (following the terminology of [19]) describing the dynamics of bosonic fields \( \Phi^i \). The action is invariant, \( S[\Phi + \delta \Phi] = S[\Phi] \), under gauge transformations

\[
\delta \Phi^i = R^i_{\alpha}[\Phi] \, \delta \zeta^\alpha ,
\]

with \( R^i_{\alpha}[\Phi] \) the gauge generators and \( \delta \zeta^\alpha \) arbitrary local parameters of compact support. In what follows, the gauge algebra is assumed to be closed,

\[
R^i_{\alpha,j}[\Phi] \, R^j_{\beta}[\Phi] - R^i_{\beta,j}[\Phi] \, R^j_{\alpha}[\Phi] = R^i_{\gamma}[\Phi] \, f^\gamma_{\alpha\beta}[\Phi] ,
\]

together with the additional requirements on the gauge generators

\[
R^i_{\alpha,i}[\Phi] = 0 \ , \quad f^\beta_{\alpha\beta}[\Phi] = 0 ,
\]
which are naturally met in Yang-Mills theories.

Let $\Psi[\Phi]$ be a gauge invariant functional, $\Psi, i[\Phi] R^i_{\alpha}[\Phi] = 0$. Under the above assumptions, its chronological vacuum average $\langle \text{out} | T(\Psi[\Phi]) | \text{in} \rangle$ is known to have a functional integral representation of the form

$$
\langle \text{out} | T(\Psi[\Phi]) | \text{in} \rangle = N \int D\Phi \det(F[\Phi]) \Psi[\Phi] e^{i(S[\Phi] + S_{GF}[\chi[\Phi]])},
$$

(2.4)

where $\chi^\alpha[\Phi]$ are gauge conditions such that the Faddeev-Popov operator

$$
F^\alpha_{\beta}[\Phi] \equiv \chi^\alpha, i[\Phi] R^i_{\beta}[\Phi]
$$

(2.5)
is non-singular. The gauge fixing functional $S_{GF}[\chi]$ is chosen in such a way that the action $S[\Phi] + S_{GF}[\chi[\Phi]]$ is no longer gauge invariant. In perturbation theory, it is customary to choose $S_{GF}[\chi]$ to be of Gaussian form,

$$
S_{GF}[\chi] = \frac{1}{2} \chi^\alpha \eta_{\alpha\beta} \chi^\beta,
$$

(2.6)

with $\eta_{\alpha\beta}$ a constant non-singular symmetric matrix.

The chronological average $\langle \text{out} | T(\Psi[\Phi]) | \text{in} \rangle$ does not depend on the gauge conditions chosen,

$$
\langle \text{out} | T(\Psi[\Phi]) | \text{in} \rangle_{\chi + \delta \chi} = \langle \text{out} | T(\Psi[\Phi]) | \text{in} \rangle_{\chi},
$$

(2.6)

with $\delta \chi^\alpha[\Phi]$ a variation of the gauge conditions. An early proof of this fact [10, 20] (see also [2] for a recent review) is based on making the change of variables

$$
\Phi^i \rightarrow \Phi^i - R^i_{\alpha}[\Phi] \delta \zeta^\alpha[\Phi], \quad \delta \zeta^\alpha[\Phi] = (F^{-1}[\Phi])^\alpha_{\beta} \delta \chi^\beta[\Phi]
$$

(2.7)
in the functional integral

$$
\langle \text{out} | T(\Psi[\Phi]) | \text{in} \rangle_{\chi + \delta \chi} = N \int D\Phi \det(F'[\Phi] + \delta F'[\Phi]) \Psi[\Phi] e^{i(S'[\Phi] + S_{GF}[\chi'[\Phi] + \delta \chi'[\Phi]])},
$$

(2.8)

with $\delta F^\alpha_{\beta}[\Phi] = \delta \chi^\alpha, i[\Phi] R^i_{\beta}[\Phi]$, and then using eq. (2.3).

Let $W[J; \chi]$ be the generating functional of connected Green’s functions,

$$
e^{iW[J; \chi]} = N \int D\Phi \det(F[\Phi]) e^{i(S[\Phi] + S_{GF}[\chi[\Phi]] + J_\phi^i)},
$$

(2.9)

and $\Gamma[\phi; \chi]$ the effective action of the theory,

$$
\Gamma[\phi; \chi] = (W[J; \chi] - J_\phi \phi^i)|_{J=J[\phi;\chi]}, \quad \phi^i = \frac{\delta}{\delta J^i} W[J; \chi].
$$

(2.10)

Both $W[J; \chi]$ and $\Gamma[\phi; \chi]$ depend on the choice of gauge conditions. This dependence can readily be figured out by making the change of variables (2.7) in the functional integral representation for $W[J; \chi + \delta \chi]$. Then one gets

$$
W[J; \chi + \delta \chi] - W[J; \chi] = -J^i \langle R^i_{\alpha}[\Phi] (F^{-1}[\Phi])^\alpha_{\beta} \delta \chi^\beta[\Phi] \rangle,
$$

(2.11)
where the symbol \( \langle \ldots \rangle \) denotes the quantum average in the presence of the source,

\[
\langle A[\Phi] \rangle = e^{-iW[J,\chi]} \int D\Phi A[\Phi] \text{Det}(F[\Phi]) e^{i(S[\Phi] + SGF[\chi[\Phi]] + J, \Phi^i)} .
\] (2.12)

Since \( \delta W/\delta \lambda = \delta \Gamma/\delta \lambda \), where \( \lambda \) is any parameter in the theory, and since \( J_i = -\delta \Gamma/\delta \phi^i \), from eq. (2.11), we then derive the following final relation

\[
\Gamma[\phi; \chi + \delta \chi] = \Gamma[\phi + \delta \phi; \chi] , \quad \delta \phi^i[\phi; \chi] = \langle R^i_{\alpha}[\Phi] (F^{-1}[\Phi])^\alpha_\beta \delta \chi^\beta[\Phi] \rangle .
\] (2.13)

This relation shows that an infinitesimal change of gauge conditions, \( \chi[\Phi] \to \chi[\Phi] + \delta \chi[\Phi] \), is equivalent to a special nonlocal field redefinition, \( \phi^i \to \phi^i + \delta \phi^i[\phi; \chi] \), in the effective action. On the mass shell, \( \delta \Gamma/\delta \phi = 0 \), the effective action is gauge independent, and this is known to imply the gauge independence of the \( S \)-matrix (see, e.g. [14]).

### 3 Gauge dependence of the effective action in the background field approach

We now turn to discussing the issue of dependence of the effective action on gauge conditions in the framework of the background field formulation (see [16, 21] and references therein) which provides a manifestly gauge invariant definition of the effective action. For simplicity, our considerations will be restricted to Yang-Mills type theories in which the gauge generators are linear functionals of the fields,

\[
R^i_{\alpha,jk}[\Phi] = 0 .
\] (3.1)

In the background field approach, one splits the dynamical variables \( \Phi^i \) into the sum of **background** fields \( \phi^i \) and **quantum** fields \( \varphi^i \). The classical action \( S[\phi + \varphi] \) is then invariant under **background** gauge transformations

\[
\delta \phi^i = R^i_{\alpha}[\phi] \delta \zeta^\alpha , \quad \delta \varphi^i = R^i_{\alpha,j}[\phi^j + \delta \phi^j[\phi; \chi] \delta \zeta^\alpha ;
\] (3.2)

and **quantum** gauge transformations

\[
\delta \phi^i = 0 , \quad \delta \varphi^i = R^i_{\alpha}[\phi + \varphi] \delta \zeta^\alpha .
\] (3.3)

The background field quantization procedure consists of fixing the quantum gauge freedom, while keeping the background gauge invariance intact, by means of **background**
covariant gauge conditions $\chi^\alpha[\phi, \varphi]$. The effective action is given by the sum of all 1PI Feynman graphs which are vacuum with respect to the quantum fields. Defining the Faddeev-Popov operator

$$F^\alpha_\beta[\phi, \varphi] = \left(\frac{\delta}{\delta\varphi^i}\chi^\alpha[\varphi, \phi]\right) R^i_\beta[\phi + \varphi]$$  \hspace{1cm} (3.4)

and introducing a gauge fixing functional $S_{GF}[\chi]$, which is required to be invariant under the background gauge transformations, the generating functional of connected quantum Green’s functions, $W[J, \phi; \chi]$, is given by

$$e^{iW[J, \phi; \chi]} = N \int D\varphi \text{ Det}(F[\varphi, \phi]) e^{i(S[\phi + \varphi] + S_{GF}[\chi[\varphi, \phi]] + J \varphi^i)}.$$  \hspace{1cm} (3.5)

Its Legendre transform

$$\Gamma[\langle \varphi \rangle, \phi; \chi] = W[J, \phi; \chi] - J_i \langle \varphi^i \rangle, \hspace{1cm} \langle \varphi^i \rangle = \frac{\delta}{\delta J_i}W[J, \phi; \chi]$$  \hspace{1cm} (3.6)

is related to the effective action $\Gamma[\phi; \chi]$ as follows: $\Gamma[\phi; \chi] = \Gamma[\langle \varphi \rangle = 0, \phi; \chi]$. In other words, $\Gamma[\phi; \chi]$ coincides with $W[J, \phi; \chi]$ at its stationary point $J = J[\phi; \chi]$ such that $\delta W[J, \phi; \chi]/\delta J = 0$. By construction, $\Gamma[\phi; \chi]$ is invariant under the background gauge transformations.

To determine the dependence of $\Gamma[\phi; \chi]$ on $\chi$, one can start with the functional integral representation (3.5) for $W[J, \phi; \chi + \delta \chi]$, where $\delta \chi^\alpha[\varphi, \phi]$ is an infinitesimal change of the gauge conditions, and make in the integral the following replacement of variables:

$$\varphi^i \rightarrow \varphi^i - R^i_\alpha[\phi + \varphi] \left(F^{-1}[\varphi, \phi]\right)^\alpha_\beta \delta \chi^\beta[\varphi, \phi].$$  \hspace{1cm} (3.7)

This leads to

$$\Gamma[\phi; \chi + \delta \chi] - \Gamma[\phi; \chi] = \langle R^i_\alpha[\phi + \varphi] \left(F^{-1}[\varphi, \phi]\right)^\alpha_\beta \delta \chi^\beta[\varphi, \phi] \rangle \frac{\delta \Gamma[\langle \varphi \rangle, \phi; \chi]}{\delta \langle \varphi^i \rangle} \big|_{\langle \varphi \rangle = 0}. \hspace{1cm} (3.8)$$

Here the functional derivative $\delta \Gamma[\langle \varphi \rangle, \phi; \chi]/\delta \phi$ with the aid of the identity (see [2] and the last reference in [21]) for a derivation)

$$\delta \phi^i \frac{\delta \Gamma[\phi; \chi]}{\delta \phi^i} = \left\{ \delta \phi^i + \langle R^i_\alpha[\phi + \varphi] \left(F^{-1}[\varphi, \phi]\right)^\alpha_\beta \Delta \chi^\beta[\varphi, \phi] \rangle \right\} \frac{\delta \Gamma[\langle \varphi \rangle, \phi; \chi]}{\delta \langle \varphi^i \rangle} \big|_{\langle \varphi \rangle = 0}, \hspace{1cm} \Delta \chi^\alpha[\varphi, \phi] = \chi^\alpha[\phi - \delta \phi, \phi + \delta \phi] - \chi^\alpha[\varphi, \phi] \hspace{1cm} (3.9)$$

with $\delta \phi^i$ an arbitrary variation of the background fields. Eqs. (3.8) and (3.9) show that any change of the gauge conditions is equivalent to a nonlocal field redefinition in the effective action.
In this section, we briefly provide an overview of rigid anomaly-free symmetries of the effective action, see [2] for more details, and then give a general discussion of the quantum deformation of the conformal symmetry in $\mathcal{N} = 4$ SYM theory.

Let the classical action be invariant, $S[\Phi + \epsilon \Omega[\Phi]] = S[\Phi]$, with respect to a rigid transformation

$$\delta \Phi^i = \epsilon \Omega^i[\Phi], \quad \Omega^i[\Phi] = \Omega^i_j \Phi^j,$$

(4.1)

where $\Omega^i_j$ is a given field-independent operator and $\epsilon$ an arbitrary infinitesimal constant parameter. We will assume several additional properties of the structure of the gauge and global transformations:

$$\Omega^i_{,i}[\Phi] = 0,$$

(4.2)

$$R^i_{\alpha,j}[\Phi] \Omega^j[\Phi] - \Omega^i_{,j}[\Phi] R^j_{\alpha}[\Phi] = R^i_{\beta}[\Phi] f^\beta_{\alpha}[\Phi],$$

(4.3)

$$f^\alpha_{\alpha}[\Phi] = 0.$$

(4.4)

Eq. (4.2) ensures that the transformation $\Phi^i \rightarrow \Phi^i + \epsilon \Omega^i[\Phi]$ is unimodular. Eq. (4.3) implies that the commutator of a gauge transformation with a global symmetry transformation is a gauge transformation.

At the quantum level, one has to specify some set of gauge conditions, $\chi^\alpha[\Phi]$, and a gauge fixing functional, $S_{GF}[\chi]$. An additional assumption we make concerns the behaviour of the gauge conditions under the symmetry transformations. We assume

$$\delta \epsilon \chi^\alpha[\Phi] \equiv \epsilon \chi^\alpha_{,i}[\Phi] \Omega^i[\Phi] = \epsilon \left( \Lambda^\alpha_{\beta} \chi^\beta[\Phi] + \rho^\alpha[\Phi] \right),$$

(4.5)

with $\Lambda^\alpha_{\beta}$ a field independent operator. It will also be assumed that the homogeneous term on the right hand side leaves $S_{GF}[\chi]$ invariant, $S_{GF}[\chi^\alpha + \epsilon \Lambda^\alpha_{\beta} \chi^\beta] = S_{GF}[\chi^\alpha]$. Under all the above assumptions, the symmetry of the quantum theory can be shown [2] to be governed by the Ward identity

$$\Gamma_{,i}[\phi; \chi] \Omega^i[\phi] = \Gamma_{,i}[\phi; \chi] \langle R^i_{\alpha}[\Phi] (F^{-1}[\Phi])^\alpha_{\beta} \rho^\beta[\Phi] \rangle,$$

(4.6)

which is nothing but the condition of invariance under quantum mechanically corrected symmetry transformations.

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2It is sufficient for the purposes of the present paper to consider linear rigid classical symmetries only.
It is easy to generalize the Ward identity (4.4) to the background field formulation. Assuming that the background covariant gauge conditions $\chi^\alpha[\varphi, \phi]$ transform by the rule
\[
\delta \chi^\alpha[\varphi, \phi] = \chi^\alpha[\varphi + \epsilon \Omega[\varphi], \phi + \epsilon \Omega[\phi]] - \chi^\alpha[\varphi, \phi] = \epsilon \left( \Lambda^\alpha_\beta \chi^\beta[\varphi, \phi] + \rho^\alpha[\varphi, \phi] \right),
\]
and that the gauge fixing functional is invariant under (4.7) with $\rho^\alpha[\varphi, \phi] = 0$, one gets the following Ward identity
\[
\Omega^i[\varphi] \frac{\delta \Gamma[\varphi; \chi]}{\delta \phi^i} = \langle R^i_\alpha[\phi + \varphi] (F^{-1}[\varphi, \phi])^\alpha_\beta \rho^\beta[\varphi, \phi] \rangle \frac{\delta \Gamma[\langle \varphi \rangle, \phi; \chi]}{\delta \langle \phi^i \rangle} |_{\langle \varphi \rangle = 0},
\]
which has to be treated in conjunction with (3.9) to express the functional derivative $\delta \Gamma[\langle \varphi \rangle, \phi; \chi]/\delta \langle \varphi \rangle$ at $\langle \varphi \rangle = 0$ via $\delta \Gamma[\phi; \chi]/\delta \phi$.

Eq. (4.8) determines the true rigid symmetry of the effective action. In general, the correlation function in the right hand side of (4.8) is a nonlocal functional of the fields. If the effective action is computed in the framework of the derivative expansion, this correlation function can be represented as an infinite series of local terms with increasing number of derivatives. As a rule, this series cannot be truncated at a given order without spoiling the algebra of rigid symmetry transformations. As has been shown before, any change of gauge conditions is equivalent to a special nonlocal field redefinition in the effective action, the latter inducing a modification to the structure of symmetry transformations. Such nonlocal field redefinitions will always lead to a re-organization of the derivative expansion of the effective action. The freedom to choose gauge conditions can therefore be used to seek field redefinitions which are best adapted to the expression of symmetries in the context of the derivative expansion of the effective action. In the case of $\mathcal{N} = 4$ SYM, for example, use of ‘t Hooft gauge makes it possible to consistently truncate the one-loop deformation of conformal symmetry to the terms of first order in derivatives, as given in eq. (1.4), in that the corresponding transformations (1.5) and (1.6) form a field realization of the conformal algebra without the need to include the higher derivative terms in the modified quantum symmetry.

In principle, there is nothing wrong with the existence of a nonlocal gauge which, when implemented instead of ‘t Hooft gauge, would effectively convert the AdS conformal transformations (1.5) and (1.6) to the unmodified form (1.1); there is, however, one major problem with nonlocal gauges – it is not known how to consistently define quantum theory. We would like to sketch a field redefinition induced by such gauge conditions. Let $Y_\mu$ and $A_m$ be primary conformal scalar and vector fields of canonical mass dimension, $d_Y = d_A = 1$, such that $Y^2 = Y_\mu Y_\mu \neq 0$. Using their conformal transformation laws, given in
eq. (1.1), one readily derives the conformal variations of their descendants (including the field strength $F_{mn} = \partial_m A_n - \partial_n A_m$):

\begin{align}
- \delta (\partial_m Y_\mu) &= (v + 2\sigma) \partial_m Y_\mu + \omega_m \partial_n Y_\mu + (\partial_m \sigma) Y_\mu, \\
- \delta F_{mn} &= (v + 2\sigma) F_{mn} + \omega_m F_{pn} + \omega_n F_{mp}.
\end{align}

(4.9)

Now consider replacing the variables $Y_\mu$ and $A_m$ by new ones, $Y'_\mu$ and $A'_m$, which (i) have the same canonical dimension; (ii) are given by series in powers of derivatives of $Y_\mu$ and $A_m$; (iii) possess the $AdS_5 \times S^5$ transformations (1.5) and (1.6). To the leading order in derivatives of the fields, the new variables are\(^3\)

\begin{align}
Y_\mu &= Y_\mu - \frac{1}{4} R^4 \left\{ \frac{\partial^n Y_\mu}{Y^4} \partial_n Y^2 - Y_\mu \frac{\partial^n Y_\nu}{Y^4} \partial_n Y_\nu \right\} + O(\partial^3), \\
A_m &= A_m + \frac{1}{4} R^4 \left\{ \frac{F_{mn} \partial^n Y^2}{Y^4} - \partial_m \left( \frac{A_n \partial^n Y^2}{Y^4} \right) \right\} + O(\partial^3), \quad (4.10)
\end{align}

as can be checked with the use of (4.9). It is not difficult to convince oneself that such a field redefinition can be reconstructed order by order in the derivative expansion. Making the field redefinition (4.10) in the D3-brane action (1.7), one ends up with a higher derivative action which is invariant under the classical transformations (1.1).

5 Deformation of conformal symmetry in $R_\xi$ gauge

Here we illustrate the general analysis given in the preceding sections by explicit calculations of the quantum deformation of conformal symmetry in the so-called $R_\xi$ gauge. We are interested in the bosonic sector of the $\mathcal{N} = 4$ super Yang-Mills theory described by fields $\Phi^i = \{A_m(x), Y_\mu(x)\}$, where $m = 0, 1, 2, 3$ and $\mu = 1, \ldots, 6$. The classical action is

\begin{equation}
S[A, Y] = -\frac{1}{4g^2} \int d^4x \text{tr} \left( F_{mn}^2 + 2D^m Y_\mu D_m Y_\mu - [Y_\mu, Y_\nu] [Y_\mu, Y_\nu] \right), \quad (5.1)
\end{equation}

with $D_m = \partial_m + iA_m$, and is invariant under standard gauge transformations

\begin{align}
\delta A_m &= -D_m \tau = -\partial_m \tau - i [A_m, \tau], \\
\delta Y_\mu &= i [\tau, Y_\mu]. \quad (5.2)
\end{align}

The theory is quantized in the background field approach, i.e. by splitting the dynamical variables $\Phi^i$ into the sum of background fields $\phi^i = \{A_m(x), Y_\mu(x)\}$ and quantum fields

\(3\)There is also freedom to add terms containing factors of the free equations of motion, $\Box Y_\mu$ and $\partial^n F_{mn}$, which transform covariantly under the conformal group.
\( \varphi^i = \{a_m(x), y_\mu(x)\} \). In \( R_\xi \) gauge, the gauge conditions are

\[
\chi^{(\xi)} = \frac{1}{\sqrt{\xi}} D^m a_m + i \sqrt{\xi} [Y_\mu, y_\mu],
\]

(5.3)

where \( D_m \) are the background covariant derivatives, and \( \xi \) the gauge fixing parameter.

The gauge fixing functional, \( S_{GF} \), is the same as in \([2]\)

\[
S_{GF}[\chi^{(\xi)}] = -\frac{1}{2g^2} \int d^4x \, tr (\chi^{(\xi)})^2.
\]

(5.4)

The choice \( \xi = 1 \) corresponds to 't Hooft gauge implemented in our previous work \([2]\).

Under the combined conformal transformation (1.1) of the background and quantum fields, \( \chi^{(\xi)} \) changes as follows

\[
\delta_c \chi^{(\xi)} = -(v + 2\sigma) \chi^{(\xi)} + \frac{2}{\sqrt{\xi}} (\partial^m \sigma) a_m,
\]

(5.5)

and this transformation law is clearly of the form (4.7). The inhomogeneous term in (5.5) is the source of a quantum modification to the conformal Ward identity, which can be computed in \( R_\xi \) gauge by extension of the 't Hooft gauge calculation in \([2]\). Choosing a \( U(1) \) background which spontaneously breaks the gauge group \( SU(N+1) \) to \( SU(N) \times U(1) \), and retaining only terms of first order in derivatives, the one-loop modification to the conformal transformations is:

\[
\delta^{(\xi)} A_m = -\frac{Ng^2}{4\pi^2} \frac{(\partial^m \sigma) F_{mn}}{Y^2},
\]

(5.6)

\[
\delta^{(\xi)} Y_\mu = \frac{Ng^2}{4\pi^2} \frac{(\partial^m \sigma)(\partial_n Y_\mu)}{Y^2} \left[ \frac{\ln \xi}{(\xi - 1)^2} + \frac{\ln \xi}{(\xi - 1)} - \frac{1}{2(\xi - 1)} + \frac{1}{2} \right]
\]

\[
- \frac{Ng^2}{8\pi^2} \frac{(\partial^m \sigma)(\partial_n Y^2) Y_\mu}{Y^4} \left[ \frac{5}{2(\xi - 1)^2} + \frac{\ln \xi}{(\xi - 1)} - \frac{5}{2(\xi - 1)} + \frac{1}{4} \right].
\]

(5.7)

As can be seen, the gauge field transformation is the same as in 't Hooft gauge. In relation to the scalar transformation (5.7), the second square bracket in \( \delta^{(\xi)} Y_\mu \) vanishes in the limit \( \xi \to 1 \) ('t Hooft gauge), and the first square bracket gives 1, thus yielding the 't Hooft gauge result. The transformations (5.6) and (5.7) do not realize the conformal algebra for \( \xi \neq 1 \). This means that, when computing the quantum modification to the conformal transformations in \( R_\xi \) gauge, we have to take into account the terms of second and higher orders in derivatives of the fields.

Let us analyse a special case, \( \xi = 1 + \varepsilon \), with an infinitesimal parameter \( \varepsilon \). Then eq. (5.7) reduces to

\[
\delta^{(1+\varepsilon)} Y_\mu = \frac{Ng^2}{4\pi^2} \frac{(\partial^m \sigma)(\partial_n Y_\mu)}{Y^2} \left[ 1 - \frac{\varepsilon}{6} \right] - \frac{Ng^2}{8\pi^2} \frac{(\partial^m \sigma)(\partial_n Y^2) Y_\mu}{Y^4} \left[ \frac{\varepsilon}{3} \right].
\]

(5.8)
In accordance with the previous discussion, there should exist a field redefinition relating the fields $Y_\mu$ in $R_{1+\epsilon}$ gauge to those, $\tilde{Y}_\mu$, in 't Hooft gauge. It is

$$Y_\mu = \tilde{Y}_\mu + \epsilon \frac{Ng^2}{48\pi^2} \left\{ \frac{(\partial^n\tilde{Y}_\mu)(\partial^n\tilde{Y}_\nu)}{Y^4} + \frac{(\partial^n\tilde{Y}_\nu)(\partial^n\tilde{Y}_\mu)}{Y^4} \tilde{Y}_\mu \right\} + O(\partial^3). \quad (5.9)$$

6 Conclusion

In this paper, we have addressed several issues related to the gauge dependence of the quantum deformation of the conformal symmetry in $\mathcal{N} = 4$ SYM. In section 3, we have extended some well known results on the gauge dependence of the effective action to the case of background field quantization, and the result (3.8) demonstrates that a change of background covariant gauge conditions is equivalent to a nonlocal field redefinition. In section 4, we derived the quantum-corrected Ward identity (4.8) in the background field approach. The explicit one-loop calculations in $R_\xi$ gauge of the quantum modifications to conformal symmetry in $\mathcal{N} = 4$ SYM in section 5 highlight the very special nature of 't Hooft gauge for this theory. The modified conformal transformations (5.6) and (5.7) do not form a closed algebra when truncated at first derivative order in the derivative expansion, except in the case $\xi = 1$, namely 't Hooft gauge. Only in 't Hooft gauge, the truncated transformations coincide with the symmetry transformations (1.5) and (1.6) of a D3-brane embedded in $AdS_5 \times S^5$. Also striking is the fact that the quantum modification of the transformation of the gauge field is not modified by moving out of 't Hooft gauge to $R_\xi$ gauge. As was demonstrated in [2], 't Hooft gauge retains some memory of the origin of classical $\mathcal{N} = 4$ supersymmetric Yang-Mills theory as a dimensional reduction of ten-dimensional supersymmetric Yang-Mills theory. It would be of interest to determine if this is of any significance in relation to these observations.

The phenomenon of quantum deformation of rigid symmetries is quite general. Apart from conformal symmetry in $\mathcal{N} = 4$ super Yang-Mills theory, it is also worth mentioning here nice results on generalized conformal symmetry in D-brane matrix models [22, 13] and supersymmetry in Matrix theory [28].

We believe that the deformed conformal invariance in the $\mathcal{N} = 4$ super Yang-Mills theory should be crucial for a better understanding of numerous non-renormalization theorems which are predicted by the AdS/CFT conjecture and relate to the explicit structure of the low energy effective action in $\mathcal{N} = 4$ super Yang-Mills theory (see [24, 25, 26] for a more detailed discussion and additional references).
While this work was in the process of being written up, a paper appeared on the hep-th archive \[27\] where the techniques of nonlinear realizations were used to derive a field redefinition similar to the one introduced at the end of section 4.

**Acknowledgements.** Discussions with N. Dragon, O. Lechtenfeld, D. Lüst, B. Zupnik, and especially with J. Erdmenger and S. Theisen, are gratefully acknowledged. This work is partially supported by a University of Western Australia Small Grant and an ARC Discovery Grant. One of us (SMK) is thankful to N. Dragon and S. Theisen for kind hospitality extended to him at the Max Planck Institute for Gravitational Physics (Albert Einstein Institute), Golm and at the Institute for Theoretical Physics, Uni-Hannover. The work of SMK has been supported in part by the Alexander von Humboldt Foundation, the Max Planck Society and the German National Science Foundation.

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