Research Article

Analysis of Reliable Solutions to the Boundary Value Problems by Using Shooting Method

Mohammad Asif Arefin®, Mahmuda Akhter Nishu®, Md Nayan Dhali®, and M. Hafiz Uddin®

Department of Mathematics, Jashore University of Science and Technology, Jashore 7408, Bangladesh

Correspondence should be addressed to Mohammad Asif Arefin; asif.math@just.edu.bd and M. Hafiz Uddin; mh.uddin@just.edu.bd

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This research aims to use the shooting method (SM) to find numerical solutions to the boundary value problems of ordinary differential equations (ODEs). Applied mathematics, theoretical physics, engineering, control, and optimization theory all have two-point boundary value problems. If the two-point boundary value problem cannot be solved analytically, numerical approaches must be used. The scenario in the two-point boundary value issue for a single second-order differential equation with prescribed initial and final values of the solution gives rise to shooting method. Firstly, the method is discussed, and some boundary value problems of ODEs are solved by using the proposed method. Obtained results are compared with the exact solution for the validation of the proposed method and represented both in graphical and tabular form. It has been found that the convergence rate of the shooting method to the exact solution is so high. As a finding of this research, it has been determined that the shooting method produces the best-fit numerical results of boundary value problems.

1. Introduction

Differential equations are well known to be the basis of most physical systems. Ordinary differential or partial differential equations are commonly used to modeled physical systems. In applied science and engineering, many linear problems, such as second-order ODEs with numerous types of boundary conditions, are solved analytically or numerically. Two-point BVPs arise in a wide range of situations where boundary layer theory in fluid dynamics, modeling chemical reactions, and heat power transfer are only a few examples. Boundary value problems have a wide range of applications in applied science and engineering, necessitating the development of faster and more reliable numerical methods. Shooting method have a significant advantage for solving nonlinear differential equations. The shooting techniques are quite broad and can be used to solve a wide range of differential equations. The equations do not have to be of certain types, such as even-order self-adjoint, in order for shooting methods to work. Many writers have tried a variety of methods to achieve higher accuracy more quickly. Application of the shooting method for the solution of second-order boundary value problems are elaborately discussed by Edun and Akinlabi [1]. The shooting technique and nonhomogeneous multi-point BVPs of second-order ODEs were reviewed by Kwong and Wong [2]. Wang et al. [3] looked into the shooting method’s application to second-order multipoint integral BVPs. The shooting procedure for the solution of two-point BVPs was discussed by Meade et al. [4]. Sharma et al. [5] used the Galerkin-Finite element approach to solve two-point BVPs numerically. Shooting method for a class of two-point singular nonlinear boundary value problems was discussed by Elgindi and Langer [6]. D. J. Jones [7] explored the shooting method for calculating eigenvalues of fourth-order two-point BVPs. In [8–10], authors discussed the shooting technique for linear and nonlinear BVPs. Hofstrand et al. [11] have discussed the bidirectional
shooting method for extreme nonlinear optics. The authors have discussed some numerical methods for solving the initial value problems in [12–14]. Ahmad and Charan [15] make an attempt to compare the finite difference approach to the shooting method. In [16], the authors described the sequential implementation of the linear shooting approach. Lijun Zhang et al. [17] also use the shooting method for solving their proposed governing equations. Al-Mdallal et al. [18] describe the collocation-shooting method for solving fractional boundary value problems. Sung N. [19] describes the nonlinear shooting method for two-point boundary value problems. The coupled nonlinear dimensionless ordinary differential equations have been solved numerically with the help of the shooting method [20]. It has been observed that many contemporary problems are effectively solved by this proposed method. As a consequence, Binfeng et al. [21] solve the hypersensitive optimal control problems by using a high-precision single shooting method. Ali Umit Keskin [22] discusses the shooting method for the solution of one-dimensional BVPs. S Abbasbandy and M. Hajiketabi [23] introduce a new Lie-group shooting method for solving nonlinear boundary value problems.

As a consequence, the foremost objective of this research is to solve two-point boundary value problems using a simple and proficient shooting method. In this article three BVPs of the ordinary differential equation are considered, find out the solution of these BVPs by the shooting method and finally compare them with the exact solution.

The remaining part of the paper will be structured as, in Section 2, we have described the proposed method. Numerical analysis and graphical representation of the proposed method are presented in Section 3. In Section 4, physical interpretation and result discussions are given. The conclusion is presented in the last section.

2. Description of the Method

The shooting approach reduces the boundary value issue to identifying the starting conditions that yield a root by viewing boundary conditions as a multivariate function of initial conditions at some point. The name of the shooting technique comes from a comparison to target shooting. We shoot the target and examine where it hits the target; based on the inaccuracies, we can modify our aim and shoot again in the hopes of hitting the target near to the mark. The shooting methods are designed to convert ODE boundary value issues into similar initial value problems, which may then be solved using appropriate methods. The shooting method is an iterative method that is well suited to solving any kind of BVPs of ODEs, regardless of the boundary conditions form. The shooting method is a computational method for solving two points boundary value problems of linear ODEs where the problem must be reduced as a system of an initial value problem [11]. We shoot trajectories in different directions before we find one, that has the desired boundary value.

There are three stages to the linear shooting method:

(1) The provided BVPs is split into two initial value problems (IVPs)
(2) Taylor’s series, Runge–Kutta method, or any other approach may be used to solve these two IVPs
(3) The required solution of the given BVP is a combination of these two solutions

Reduction to two IVPs [2]: Consider the function $f$ of the boundary value problem

$$y'' = f(x, y, y'), \quad \text{for } a \leq x \leq b,$$

with $y(a) = \alpha$, $y(b) = \beta$,

is continuous on the set, $D = \{(x, y, y') : \text{for } a \leq x \leq b, \text{with } -\infty \leq y \leq +\infty, -\infty \leq y' \leq +\infty \text{ and that partial derivatives } f_y \text{ and } f_y' \text{ are also continuous on } D \}$.

(1) $f_y(x, y, y') > 0$ for all $(x, y, y') \in D$, and
(2) A constant $M$ exists, with $|f_y(x, y, y')| \leq M$, for all $(x, y, y') \in D$

Then, the BVP has a unique solution. The differential equation $f''(x, y, y')$ is linear when the functions $p(x), q(x)$, and $r(x)$ exist with $f''(x, y, y') = p(x)y' + q(x)y + r(x)$ [24].

This type of problems occurs frequently, and in this stage, it can be simplified. Let linear BVP be

$$y'' = p(x)y' + q(x)y + r(x) \text{ for } a \leq x \leq b, \text{ with } y(a) = \alpha, y(b) = \beta,$$

satisfies.

(1) $p(x), q(x)$, and $r(x)$ are continuous on $[a, b]$
(2) $q(x) > 0$ is continuous on $[a, b]$

Then, the BVP has a unique solution.

To estimate the unique solution to this linear problem, we first investigate initial value problems.

$$y'' = p(x)y' + q(x)y + r(x),$$

for $a \leq x \leq b$, \hspace{1cm} (2)

with $y(a) = \alpha, y'(a) = 0$,

$$y'' = p(x)y' + q(x)y + r(x), \hspace{1cm} (3)$$

$$y'(a) = 1.$$

Two problems, have a solution, which is unique. Suppose $y_1(x)$ stands for the solution of (2). and suppose $y_2(x)$ stands for the solution of equation (3).

Consider that $y_2(b) \neq 0$

Define, $y(x) = y_1(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2(x)$. \hspace{1cm} (4)

Then, $y(x)$ will be the solution to the linear boundary value problem of equation (4).
First and foremost, keep in mind that, $y'(x) = y_1'(x) + \left(\frac{\beta - y_1(b)}{y_2(b)}\right)y_2'(x)$ and $y''(x) = y_1''(x) + \left(\frac{\beta - y_1(b)}{y_2(b)}\right)y_2''(x)$, putting $y_1''(x)$ and $y_2''(x)$ in this equation, which produces,

$$y'' = p(x)y_1' + q(x)y_1 + r(x) + \frac{\beta - y_1(b)}{y_2(b)}y_2.$$  

$$= p(x)y_1' + q(x)y_2,$$

$$= p(x)y_1' + q(x)y_2 + r(x).$$

Moreover, $y(a) = y_1(a) + \left(\frac{\beta - y_1(b)}{y_2(b)}\right)y_2(a) = \alpha + \left(\frac{\beta - y_1(b)}{y_2(b)}\right)y_2(b), 0 = \alpha$, and $y(b) = y_1(b) + \left(\frac{\beta - y_1(b)}{y_2(b)}\right)y_2(b), (a) = \beta$.

The shooting technique for linear equation depends on the replacement of the linear BVPs by the initial-value problems (2) and (3). There exist various methods for approximating the solutions of $y_1(x)$ and $y_2(x)$, equation(4) is used to estimate the solution to the BVPs once these approximations are ready to use.

3. Numerical Analysis and Graphical Representation of the Proposed Method

In this section, we have considered three boundary value problems of ODEs that we solved numerically using the shooting method and then compare the result with the exact solution. All the approximate results of different iterations, exact solutions, and error estimation of three problems are represented in tabular form (i.e., Table 1, 2, and 3). Approximate and exact solutions are also graphically described (i.e., Figure 1(a)–1(c), Figure 2(a)–2(c), and Figure 3(a)–3(c)) for better comprehension. Error analysis is also introduced graphically (i.e., Figure 1(d) and 1(e), Figure 2(d) and 2(e), and Figure 3(d) and 3(e)) were MATLAB and MS Excel are used. We use a bar diagram for the easy visualization of the error estimation. It has been found that after a certain number of iterations, we achieve the highest degree of accuracy, and then all subsequent iterations produce the same result. So, results of the first four iterations are represented here for better understanding. Programming software MATLAB is used for obtaining the results and graphs. Error analysis is also shown graphically using MATLAB and Microsoft Excel.
| Xi | Iteration 1 | Iteration 2 | Iteration 3 | Iteration 4 | Exact value | Error |
|---|---|---|---|---|---|---|
| 0.1 | 0.0998333 | 0.1186415 | 0.1186415 | 0.1186415 | 0.1186415 | 2.752E-08 |
| 0.2 | 0.198692 | 0.2360976 | 0.2360976 | 0.2360976 | 0.2360977 | 5.445E-08 |
| 0.3 | 0.29552 | 0.3511947 | 0.3511947 | 0.3511947 | 0.3511948 | 7.852E-08 |
| 0.4 | 0.389418 | 0.4627828 | 0.4627828 | 0.4627828 | 0.4627829 | 9.756E-08 |
| 0.5 | 0.4794252 | 0.5697469 | 0.5697469 | 0.5697469 | 0.569747 | 1.095E-07 |
| 0.6 | 0.564642 | 0.6710182 | 0.6710182 | 0.6710182 | 0.6710184 | 1.123E-07 |
| 0.7 | 0.6442172 | 0.765585 | 0.765585 | 0.765585 | 0.7655851 | 1.042E-07 |
| 0.8 | 0.7173556 | 0.8525024 | 0.8525024 | 0.8525024 | 0.8525025 | 8.365E-08 |
| 0.9 | 0.7833264 | 0.9309018 | 0.9309018 | 0.9309018 | 0.9309019 | 4.928E-08 |
| 1.0 | 0.8414705 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1E-16 |

**Figure 1: Continued.**

(a) Graphical Solution of Exact value

(b) Graphical Solution of Shooting Method

(c) Graphical Solution of Shooting Method with Exact solution

(d) Error Estimation
Figure 1: Graphical illustration of exact and shooting method with error estimations for example 1. (a) Diagrammatic depiction of exact value. (b) Diagrammatic depiction of shooting method. (c) Diagrammatic depiction of shooting method with exact solution. (d) Error estimation bar diagram using MS Excel. (e) Error estimation curve using MATLAB.

Figure 2: Continued.
Figure 2: Graphical illustration of exact and shooting method with error estimations for example 2. (a) Diagrammatic depiction of exact value. (b) Diagrammatic depiction of shooting method. (c) Diagrammatic depiction of shooting method with exact solution. (d) Error estimation bar diagram using MS Excel. (e) Error estimation curve using MATLAB.

Figure 3: Continued.
3.1 Numerical Example 1. Considering the boundary value problem \( \frac{d^2 y}{dx^2} = 2y + \cos(x), \quad 0 \leq x \leq \pi/2 \), and \( y(0) = -0.3, \quad y(\pi/2) = -0.1 \), where the exact solution is \( y(x) = -1/10 (\sin(x) + 3\cos(x)) \) [24].

Approximate solutions, exact solution, and errors are represented in Table 1 and approximate solution curve and exact solution curve are exhibited in Figures 1(a)–1(c). Error estimations are represented using MS Excel (Figure 1(d)) and using MATLAB (Figure 1(e)).

3.2 Numerical Example 2. Considering the boundary value problem \( d/dx(e^x y') + e^x y = x + (2 - x)e^x, \quad 0 \leq x \leq 1, \quad y(0) = y(1) = 0 \), use \( h = 0.1 \), where the exact solution of the problem is \( y(x) = (x - 1)(e^{-x} - 1) \) [24].

Approximate solutions, exact solution, and errors are represented in Table 2 and approximate solution curve and exact solution curve are exhibited in Figures 2(a)–2(c). Error estimations are represented using MS Excel (Figure 2(d)) and using MATLAB (Figure 2(e)).

3.3 Numerical Example 3. Considering the boundary value problem \( (d^2 y)/(dx^2) + y = 0, \quad y(0) = 0, \quad y(1) = 1 \), use \( h = 0.1 \), where exact solution of the problem is \( y(x) = \sin(x)/\sin(1) \) [24].

Approximate solutions, exact solution, and errors are represented in Table 3 and approximate solution curve and exact solution curve are exhibited in Figures 3(a)–3(c). Error estimations are represented using MS Excel (Figure 3(d)) and using MATLAB (Figure 3(e)).
4. Physical Interpretation and Result Discussion

Tables 1, 2, and 3 illustrate the approximate solution, exact solution, and error estimation for the shooting method of problem 1, problem 2, and problem 3, respectively. It has been found that after a certain number of iterations, we achieve the highest degree of accuracy, and then all subsequent iterations yield the same result. As a consequence, the results of the first four iterations are depicted here to aid comprehension. This result analysis reveals that the approach is in perfect agreement with the exact solution and the error term is almost negligible. For each problem, the approximate solution curve and exact solution curve are drawn separately. At the same time, both approximate and exact solution curves are drawn in the same frame to illustrate the accuracy level (i.e., Figures 1(c), 2(c), and 3(c)). From the error graphs (i.e., Figure 1(d) and 1(e), Figure 2(d) and 2(e), and Figure 3(d) and 3(e)) it is observed that the amount of error is so small, which is kind of negligible. The accuracy and efficiency level of the shooting approach for solving the BVPs of ODEs is extremely great, as shown by the tabulated and graphical results. It was also observed that, despite the fact that the process was the same, our findings were superior to those of Sung N. [19], Edun, and Akinlabi [1]. As a result, the shooting method may be effective in resolving critical boundary value issues involving ordinary differential equations. In a variety of boundary value problems and their contemporary problems, the shooting method plays a significant role in terms of accuracy while consuming little time.

5. Conclusion

In this article, we discuss the shooting method to solve ordinary differential equations of boundary value problems, in an efficient way. Three BVPs were solved using the shooting method, and the convergence rate of the shooting method was excellent as compared to the exact solution. The method’s competency can even be seen graphically. From the result tables it has been shown that, the error amount for the taken three problems are $2.09 \times 10^{-3}$, $6.31 \times 10^{-5}$, and $1 \times 10^{-17}$, respectively, which are almost negligible and best fitted for solving the BVPs. Since this error is so negligible, the results acquired using this method guarantee the maximum level of accuracy in any situation. The proposed approach also has a very short execution time. As this consequence, the shooting method is widely used in solving BVPs of ordinary differential equations because it is thought to be more robust, consistent, convergent, and reliable than all other methods. On elementary difficulties like the projectile problem, the shooting strategy can be highly effective. It has been implemented in numerous mathematical tools and may be easily expanded to provide a solution method for practically any boundary value problem based on its boundary conditions. We should anticipate that the shooting method can play a significant role in the field of engineering and applied sciences when critical boundary value problems of ODEs arise.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

Mohammad Asif Arefin was responsible for software, data curation, writing, investigation, conceptualization, and supervision. Mahmuda Akhter Nishu was responsible for software, data curation, writing, and formal analysis. Md. Nayan Dhali was responsible for validation and Investigation. M. Hafiz Uddin was responsible for writing-reviewing, editing, and validation.

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