Free Knots and Groups

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Virtual knot theory invented by Kauffman [10] is an important generalization of knot theory; some methods of classical knot theory can be generalized to virtual knot theory straightforwardly, and some other can not. [6]. On the other hand, there are lots of new constructions coming from virtual knot theory and similar theories, e.g., graph-links by Ilyutko and V.O.Manturov [9]. In the present paper we consider free knots, a thorough simplification of virtual knots, first introduced by Turaev [4] under the name of homotopy classes of Gauss words. Turaev conjectured the non-triviality of free knots, and the first examples of non-trivial free knots were constructed by the first named author, [5] and A.Gibson [3]. In [5], several important theorems about free knots were proved by using the notion of parity: a chord in the chord diagram is even if the number of chords it is linked with, is even; otherwise it is odd. Free knots are equivalence classes of Gauss diagrams (chord diagrams) by the relations corresponding to the three Reidemeister moves.

Non-triviality of free knots yields non-triviality of the underlying virtual knots and usually allows to improve many of virtual knot invariants (since the discovery of parity, several new applications appeared in [1, 2, 7].

Recently, a partial case of the invariant constructed in the present paper was proved to be a sliceness obstruction for free knots, see [8].

In the present paper we construct a simple and rather strong invariant of free knots, valued in a certain group (more precisely, there will be a group for every natural $m$, and the invariants for groups with greater $m$ naturally generalize invariants for smaller $m$). This invariant is constructed only out of the notion of parity, and in fact this invariant depends merely on the disposition of the chord ends rather than on the chord diagram itself.

Let $C$ be the segment of positive integers from 1 to a given integer $2n$. An unordered partition $a = \{(p_1, q_1), (p_2, q_2)\ldots (p_n, q_n)\}$ of $C$ (all numbers $p_i, q_j, i, j = 1, 2, \ldots, n$ are distinct) is called a chord diagram (with a base point). Each pair in $a$ is a chord, each element of pair is called an end of chord.

We say that two chords $(p_i, q_i), (p_j, q_j)$ are linked (resp., unlinked) depending on whether the following statement is true or false:

$$(p_i - p_j)(p_i - q_j)(q_i - p_j)(q_i - q_j) < 0.$$  

Here two chords $(p_1, p_2)$ and $(q_1, q_2)$ are linked whenever two half-circles connecting $(p_1, 0)$ to $(p_2, 0)$ and $(q_1, 0)$ to $(q_2, 0)$ in the upper half-plane have an intersection point. Note that the property of being linked does not change under the cyclic permutation of partitioned chord ends $1 \rightarrow 2 \rightarrow \cdots \rightarrow 2n \rightarrow 1$.

Fix a positive integer $m$. 

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Let \( g(p, b) \) be the number of all chords belonging to given set of chords \( b \) and linked with the chord \( p \). We denote by \( a_0 \) the following subset of the set \( a \):

\[
a_0 = \{ p \in a | g(p, a) \text{ is odd} \}
\]

We proceed by induction (for \( k \leq m - 1 \)) to define \( a_k \) as

\[
a_k = \{ p \in a \setminus \bigcup_{i<k} a_i | g(p, a \setminus \bigcup_{i<k} a_i) \text{ is odd} \}
\]

Here \( a \setminus \bigcup_{i<k} a_i \) denotes the complement to \( \bigcup_{i<k} a_i \) in \( a \).

Finally, \( a_m \) denotes the complement to \( \bigcup_{j=1}^{m-1} a_j \) in \( a \): these are all chords which survive after \( m \) consecutive operations of deleting odd chords.

We split each set \( a_k, k = 0, 1, \ldots, m - 1 \) into two disjoint subsets \( a_k = a'_k \cup a''_k \), by putting

\[
a'_k = \{ p \in a_k | g(p, a_k) \text{ is odd} \}
\]

. Having a chord diagram \( c \), we construct a word \( w(c) \) in the alphabet

\[
D = \{ a'_0, a''_0, a'_1, a''_1, \ldots, a''_{m-1}, a'_m \}
\]

as follows. The letter number \( k \) in the word \( w(c) \) to be constructed will be denoted by the same letter \( (a'_j, a''_j \) or \( a_j \) if \( j = n \)) as the subset of chords, the corresponding chord belongs to. We say that the end of a chord is marked with letter \( j \) from the alphabet \( D \).

Each word in \( D \) can be considered as an element of some group \( G \) generated by \( D \). Define the group \( G \) by generators from \( D \) and the following relations

\[
a'_0a'_0 = e, a''_0a''_0 = e, a'_1a'_1 = e, a''_1a''_1 = e, \ldots, a'_m a'_m = e, a''_m a''_m = e, a_m a_m = e,
\]

\[
a'_ia'_j = a'_j a'_i, i < j \quad a''_ia''_j = a''_j a''_i, i < j \quad a'_ia_m = a_m a'_i, i < m.
\]

Here \( e \) denotes the unity element in \( G \).

Here we abuse the notation by omitting the dependence of \( G \) on \( m \); here \( m \) is fixed once forever.

Now we are going to define free knots as equivalence classes of chord diagrams modulo “Reidemeister moves” on the set of all chord diagrams. The main statement of our work is

**Theorem 1.** If two chord diagrams \( c_1, c_2 \) are equivalent (represent the same long free knot) then we have

\[
w(c_1) = w(c_2)
\]

in \( G \).

A long free knot is an equivalence class of chord diagrams with a base point by Reidemeister moves. These Reidemeister moves correspond to usual Reidemeister moves applied to Gauss diagrams if we forget the information about arrows and signs corresponding to chords.

The first increasing (resp., decreasing) Reidemeister move is an addition (resp., removal) to given chord diagram \( c \) of a chord \( (p, q) \) such that \( |p - q| = 1 \). The second increasing (resp., decreasing) Reidemeister move is addition/removal a pair of “adjacent” chords. Two chords \( (p_i, q_i), (p_j, q_j) \) of
diagram $c$ are adjacent if $|\min(p_i, q_i) - \min(p_j, q_j)| = |\max(p_i, q_i) - \max(p_j, q_j)| = 1$ (in both cases, for the first and the second Reidemeister moves, after such an addition/removal, the remaining chords are renumbered accordingly).

We say that a triple of chords $(p_i, q_i), (p_j, q_j), (p_k, q_k)$ is completely adjoint if the six ends of these chords can be partitioned such a way that each pair contains two ends of different lower indices $i, j, k$ and the two numbers in the each pair differ by one, e.g., $|p_i - p_j| = |p_k - q_i| = |q_j - q_k| = 1$.

The third Reidemeister move is defined only for those diagrams, which contains a completely adjoint triple. It is easy to see that the six ends of a completely adjoint triple represent a set of integers looking like

$$ S = \{r, r+1, s, s+1, t, t+1\}, \tag{2} $$

and elements $r, r+1$ (and $s, s+1$ and $t, t+1$) belong to different chords. We define an involution $f : S \rightarrow S$ by setting

$$ r \leftrightarrow r + 1, s \leftrightarrow s + 1, t \leftrightarrow t + 1 \tag{3} $$

Now we define the transformation of a triple $T$ into triple $T'$ according the following rule: the set of chord ends of the triple $T$ coincides with that of triple $T'$, and a pair of integers $u, v \in S$ forms a chord in $T'$ whenever the pair $f(u), f(v)$ forms a chord in $T$. The third Reidemeister move transforms a given chord diagram including a completely adjoint triple $T$ into the diagram obtained by replacing the triple $T$ by $T'$ and leaving the remaining chords fixed.

So, we have completed the definition of a long free knot.

To prove Theorem 1, we check the invariance of $w(c)$ under Reidemeister’s moves.

For the first Reidemeister move $c_1 \rightarrow c_2$, $w(c_1)$ and $w(c_2)$ are obtained from each other by an addition/removal of a couple of consequitve identical letters (generators of the group), which yields the identity in the group.

In the case of the second Reidemeister move $c_1 \rightarrow c_2$, we add two pairs of identical letters, namely, follows that the words corresponding to diagrams $c_1, c_2$ look like

$$ w(c_1) = UVW; \quad w(c_2) = Uz_1z_2Vz_3z_4W, \tag{4} $$

where $U, V, W$ are some subwords form the word $w(c_1)$, and $z_1, z_2, z_3, z_4$ are letters from $D$ corresponding to the ends of $u, v$. We have $z_1 = z_2 = z_3 = z_4$, so $w(c_1) = w(c_2)$ in $G$.

For the third Reidemeister move, we get a word of the following type

$$ w(c_1) = U\alpha V/\beta W\gamma X, \tag{5} $$

constructed according to the above rules for the chord diagram $c_1$ and the word

$$ w(c_2) = U\delta V\epsilon W\zeta X, \tag{6} $$

constructed from the diagram $c_2$; the latter is obtained from $c_1$ by means of the third Reidemeister move.

Here each of $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ is a pair of generators of the group $G$ corresponding to the two adjacent chord ends in an adjoint triple.

Our goal is to show that in $G$ the following equalities hold: $\alpha = \delta, \beta = \epsilon, \gamma = \zeta$. 3
Every chord diagram $D$ containing a triple of completely adjoint chords has the following property: the number of odd chords in the triple is even, i.e., is equal to zero or two. This follows from the Pasch axiom of the Hilbert axiom system. (A chord $b \in c$ is even (odd) iff $g(b, c) \equiv 0 \mod 2$ (resp., $\equiv 1 \mod 2$)). An analogous property takes place for every of set from list $a_k, k = 0, 1, \cdots, m - 1$.

This means: if any of $a_k, k = 0, 1, \cdots, m - 1$ contains a completely adjoint triple, then the number of chords $b$ in the completely adjoint triple, satisfying the condition $g(b, a_k) \equiv 1 \mod 2$, is even.

Thus, we see that the triple of completely adjoint chords contains either two odd chords or zero odd chords.

Denote by $h(c)$ the chord diagram obtained from the diagram $c$ by deleting all odd chords in it. If $c_1$ has a completely adjoint triple without odd chords in it, then we pass to $h(c_1)$. The deletion of all odd chords leaves the property of a triple (of persistent chords) to be completely adjoint true. If $h(c_1)$ has a completely adjoint triple without odd chords in it then we pass to $h(h(c_1))$, and so on. As a result we get the following two options:

1. We obtain a diagram $c_1^*$ with exactly two odd chords in the triple.
2. We obtain a diagram $c_2^{**}$ without odd chords at all.

In the second case, all chords of the initial completely adjoint triple in $c_1$ have index $m$. So are the corresponding chords from $c_2$. So, each of the words $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ looks like $a_m \cdot a_m$, and the claim follows. So, the words $w(c_1)$ and $w(c_2)$ identically coincide.

It remains to consider the first case. In this case the sequence $c_1, h(c_1), h(h(c_1)), \ldots$ contains a chord diagram $c_1^*$ including a completely adjoint triple $T$ with exactly two odd chords. The six letters which mark the ends are coupled into elements $\alpha, \beta, \gamma \in G$. Without loss of generality, assume $\alpha$ is the the product of a pair of generators corresponding to two adjacent ends of odd chords. So, $\alpha$ is equal to one of the following: $a_k' \cdot a_k'$ or $a_k' \cdot a_k''$ or $a_k'' \cdot a_k''$ or $a_k'' \cdot a_k''$.

Notice that the indices of chords in $\delta$ will be the same as in $\alpha$. Now, it is easy to see that either the corresponding subword $\delta$ in $w(c_2)$ will be the same if $\alpha$ is equal to one of $a_k' \cdot a_k'$ or $a_k'' \cdot a_k''$. In case when $\alpha$ is a square of a generator $a_k'$ (resp., $a_k''$), the word $\delta$ is a square of the other generator $a_k''$ (resp., $a_k'$). So, $\alpha = \delta$ in $G$.

Now consider another segment of $w(c_1)$ (say, $\beta$) and the corresponding segment of $w(c_2)$ (in this case $\epsilon$).

In $\beta$ one letter is $a_k'$ or $a_k''$ and the other chord end with a higher index, then when passing from $\beta$ to $\epsilon$, the generators change their places and the generator with smaller index transforms $a_k' \leftrightarrow a_k''$, whence the generator with higher index remains the same.

So, the equality $\beta = \epsilon$ is one of the relations of $G$. Analogously, $\gamma = \zeta$.

The theorem is proved.

Chord diagrams considered above deal with so-called long free knots, i.e. free knots with a chosen initial point.

*Free knots* are equivalence classes of long free knots by the move which changes the initial point; this move acts by a cyclic permutation $1 \rightarrow 2 \rightarrow \cdots \rightarrow 2n \rightarrow 1 \rightarrow 2n$ on the set of partitioned points.

It is obvious that if $c_1$ is obtained from $c_2$ by such an operation then $w(c_1)$ and $w(c_2)$ are conjugate in $G$. This yields the following

**Corollary 1.** The conjugacy class $[w(\cdot)]$ in $G$ is the invariant of free knots.
For a given \( m \), the group \( G \) has a very simple Cayley graph. Namely, elements of \( G \) are in one-to-one correspondence with the set of points in Euclidean space \( \mathbb{R}^{m+1} \) such that their coordinates are integers and the last coordinate is 0 or 1.

Here, the origin of coordinates corresponds to the unit of the group \( G \). The right multiplication by element with low index \( k \) (i.e., \( a_k' \) or \( a_k'' \) or \( a_k \) if \( k = m \)) corresponds to one step shift along the coordinate number \( k \) defined as follows. Let \( x_1, x_2, ..., x_{m+1} \) be the coordinates of given point.

The multiplication by \( a_0' \) on the right (\( a_0'' \)) increases (decreases) the first coordinate \( x_1 \) if \( \sum_{s=1}^{m} x_s \) is even (odd), the multiplication by \( a_1' \) (\( a_1'' \)) increases (decreases) the second coordinate \( x_2 \) if \( \sum_{s=2}^{m} x_s \) is even (odd); the multiplication by \( a_k' \) (\( a_k'' \)) for \( k \leq m \) increases (decreases) the coordinate number \( x_{k+1} \) if \( \sum_{s=k}^{m} x_s \) is even (odd) and. Finally, \( a_m \) changes the coordinate \( x_{m+1} \) from zero to one and from one to zero.

In Fig. 1 we present a non-trivial free knot recognizable by the group with \( m = 1 \).

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