The chromatic number of $\mathbb{R}^n$ with multiple forbidden distances

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Abstract

Let $A \subset \mathbb{R}_{>0}$ be a finite set of distances, and let $G_A(\mathbb{R}^n)$ be the graph with vertex set $\mathbb{R}^n$ and edge set $\{(x, y) \in \mathbb{R}^n : \|x - y\|_2 \in A\}$, and let $\chi(\mathbb{R}^n, A) = \chi(G_A(\mathbb{R}^n))$. Erdős asked about the growth rate of the $m$-distance chromatic number

$$\overline{\chi}(\mathbb{R}^n; m) = \max_{|A|=m} \chi(\mathbb{R}^n, A).$$

We improve the best existing lower bound for $\overline{\chi}(\mathbb{R}^n; m)$, and show that

$$\overline{\chi}(\mathbb{R}^n; m) \geq \left(\Gamma_{\chi} \sqrt{m+1} + o(1)\right)^n,$$

where $\Gamma_{\chi} = 0.79983...$ is an explicit constant. Our full result is more general, and applies to cliques in this graph. Let $\chi_k(G)$ denote the minimum number of colors needed to color $G$ so that no color contains a $(k+1)$-clique, and let $\overline{\chi}_k(\mathbb{R}^n; m)$ denote the largest value this takes for any distance set of size $m$. Using the partition rank method, we show that

$$\overline{\chi}_k(\mathbb{R}^n; m) > \left(\Gamma_{\chi} \sqrt{\frac{m+1}{k}} + o(1)\right)^n.$$

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1 | INTRODUCTION

The chromatic number of Euclidean space, $\chi(\mathbb{R}^n)$, is the minimum number of colors required to color $\mathbb{R}^n$ so that no two points at distance 1 have the same color. When $n = 2$, determining $\chi(\mathbb{R}^2)$ is known as the Hadwiger–Nelson problem [12, 14, 15], and the best existing bounds are

$$5 \leq \chi(\mathbb{R}^2) \leq 7,$$

where the lower bound is a recent improvement due to De Grey [7]. For large $n$, Frankl and Wilson proved that $\chi(\mathbb{R}^n)$ grows exponentially [10], and the best bounds are

$$(1.239 \cdots + o(1))^n < \chi(\mathbb{R}^n) \leq (3 + o(1))^n,$$

due to Raigorodskii [26], and Larman and Rogers [19], respectively.

Erdős [9] proposed a variant where the graph is based on multiple distances instead of one. For a set of distances $A$, let $G_A(\mathbb{R}^n)$ denote the graph with vertex set $\mathbb{R}^n$ and edge set $\{(x, y) \in \mathbb{R}^n : \|x - y\|_2 \in A\}$, and let

$$\chi(\mathbb{R}^n, A) = \chi(G_A(\mathbb{R}^n)).$$

The $m$-distance chromatic number of $\mathbb{R}^n$ is

$$\overline{\chi}(\mathbb{R}^n; m) = \max_{A : |A| = m} \chi(\mathbb{R}^n, A).$$

Note that by rescaling, for any set $A$ with $|A| = 1$, we have that

$$\overline{\chi}(\mathbb{R}^n; 1) = \chi(\mathbb{R}^n; A) = \chi(\mathbb{R}^n).$$

Erdős noted that $\frac{1}{m} \overline{\chi}(\mathbb{R}^2; m) \to \infty$ as $m \to \infty$ [29], and by considering the $n \times n$ grid in the plane, which has $Cn^2/\sqrt{\log n}$ distinct distances, it follows that for some constant $C > 0$,

$$\overline{\chi}(\mathbb{R}^2; m) \geq Cm \sqrt{\log m}.$$ (1.1)

Erdős asked whether $\overline{\chi}(\mathbb{R}^2; m)$ grows polynomially, and more generally whether we can determine $\overline{\chi}(\mathbb{R}^n; m)$ for any $m$, and $\chi(\mathbb{R}^n, A)$ for a given $A$ [29, Open Problem 10 and 42].

In higher dimensions, the Larman–Rogers bound implies the upper bound

$$\overline{\chi}(\mathbb{R}^n; m) \leq (3 + o_n(1))^m n,$$

(see [18, p. 740]) and Raigorodskii [27] proved that there exists $c_1, c_2 > 0$ such that

$$\overline{\chi}(\mathbb{R}^n; m) > (c_1 m)^{c_2 n}.$$  

Berdnikov [2–4] improved on Raigorodskii’s lower bound, and showed that any $c_2 < \frac{1}{2}$ is admissible. In this paper, we prove the following result:
Theorem 1. We have that

\[
\chi(R^n; m) \geq \left( \Gamma_{\chi} \sqrt{m + 1} + o_n(1) \right)^n,
\]

where

\[
\Gamma_{\chi} = \sqrt{\frac{\pi}{2}} \max_{x > 0} \frac{1 - e^{-x}}{\sqrt{x}} = 0.7998308498\ldots.
\]

This theorem is proven by examining the specific set of distances, \( A_m = \{1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{m}\} \), and showing that

\[
\chi(R^n, A_m) \geq \left( \Gamma_{\chi} \sqrt{m + 1} + o(1) \right)^n.
\]

All of the best lower bounds for \( \chi(R^n; m) \), including Erdős’s bound (1.1), use this specific set of distances, or a large subset of it. In [18], Kupavskii showed that

\[
\chi(R^n, A_m) \leq \left( 2(\sqrt{m + 1}) + o_n(1) \right)^n,
\]

and so Theorem 1 implies that for the set of distances \( A_m \), we understand exactly how \( \chi(R^n, A_m) \) grows with \( m \).

Our full result is more general, and applies to \( m \)-distance \( k \)-cliques.

1.1 \( k \)-cliques

Let \( \chi_k(G) \) denote the minimum number of colors needed to color \( G \) so that no color contains a \((k + 1)\)-clique. For a set of distances \( A \), define

\[
\chi_k(R^n, A) = \chi_k(G_A(R^n)),
\]

where as before, \( G_A(R^n) \) is the graph with vertex set \( R^n \) where two vertices are connected if their distance is in \( A \). Define

\[
\overline{\chi}_k(R^n; m) = \max_{A: |A| = m} \chi_k(R^n, A).
\]

In 1987, Frankl and Rödl [11, Theorem 1.18] proved that \( \chi_k(R^n) = \overline{\chi}_k(R^n; 1) \) grows exponentially with \( n \) for any \( k \). For \( m = 1 \) and general \( k \), the best lower and upper bounds, due to Sagdeev [28] and Prosanov [24], respectively, are

\[
\left( 1 + \frac{1}{2^{2k+4}} + o(1) \right)^n < \overline{\chi}_k(R^n; 1) \leq \left( 1 + \sqrt[2k+1]{k} + o(1) \right)^n.
\]

For \( k = 2 \), the case of an equilateral triangle, better quantitative bounds are known [22] using the slice rank method developed in [6, 8, 30]. Berdnikov [4] gave lower bounds for \( \overline{\chi}_k \) for \( m \to \infty \),
and proved that for any \( c_2 < \frac{1}{2} \), there exists \( c_1 > 0 \), such that for sufficiently large \( m \), \( \chi_k(\mathbb{R}^n; m) > (c_1 m)^{2^n} \). Using the partition rank method, introduced in [21, 23], we provide a quantitative lower bound for the case of cliques with multiple distances that improves on the best existing bounds.

**Theorem 2.** For \( k \geq 1 \), we have that

\[
\chi_k(\mathbb{R}^n, A_m) \geq \left( \Gamma \sqrt{\frac{m+1}{k} + o(1)} \right)^n,
\]

where \( \Gamma = \sqrt{\frac{\pi}{2} \max_{x > 0} \frac{1-e^{-x}}{\sqrt{x}}} \).

For large \( m \), the dependence on \( k \) in this lower bound is substantially better than in (1.6). This is because (1.6) is based on Frankl and Rödl’s approach [11, Theorem 1.18], which inductively applies a result for the \( k = 1 \) case. The right-hand side of (1.7) is nontrivial only when \( m + 1 > \Gamma^{-2} \), however the method used yields a nontrivial result for \( m \geq k \) (as stated in Theorem 2 above). For \( k \) larger than \( m \), the best lower bound comes from the one distance case due to Sagdeev [28].

### 1.2 Outline

Sections 2, 3, and 4 are devoted to the proof of Theorem 2. In Section 2, we provide an introduction to the partition rank method of [21, 23], and define the Distinctness Indicator Function which will be needed to generalize to \( k \geq 2 \). In Section 3, we prove a lower bound for \( \chi(\mathbb{R}^n, A_m) \) in terms of a maximization involving a truncated theta function. Define

\[
\theta(t) = \sum_{l=1}^{\infty} t^{l(l+1)/2} = 1 + t + t^3 + t^6 + \cdots
\]

and define the truncation

\[
\theta(t; l) = 1 + t + t^3 + t^6 + t^{10} + \cdots + t^{l(l+1)/2}.
\]

Using the partition rank method, we prove the following:

**Theorem 3.** Let \( m \geq 1 \) be given. For any \( l > 1 \), and any \( k \geq 1 \), we have that

\[
\chi_k(\mathbb{R}^n, A_m) \geq \max_{0 < i < 1} \theta \left( \frac{k}{m+1}; l \right)^i + o(1) \right)^n.
\]

The right-hand side above is non-trivial for any \( k \leq m \). In Section 4, we analyze the growth rate of the right-hand side of (4), and prove the following theorem:
Theorem 4. For $0 < \gamma < 1$,
\[
\max_{l \geq 1} \max_{0 < t < 1} \frac{\theta(t^\gamma; l)}{1 + t + \ldots + t^{l-1}} \geq \Gamma_\chi \sqrt{\frac{1}{\gamma}},
\]
(1.10)
where \(\Gamma_\chi = \sqrt{\frac{\pi}{2}} \max_{0 < u < \infty} \frac{1-e^{-u}}{\sqrt{u}}\).

To prove Theorem 4, we first analyze how
\[
\max_{0 < t < 1} \frac{\theta(t^\gamma; l)}{1 + t + \ldots + t^{l-1}}
\]
depends on \(l\), and prove that this quantity is bounded from below by \(\max_{0 < t < 1} (1-t)\theta(t^\gamma)\). The Poisson summation formula yields a functional equation for \(\theta(t)\), and we use this functional equation to lower bound \(\max_{0 < t < 1} (1-t)\theta(t^\gamma)\) and prove Theorem 4. Theorem 2 follows directly from Theorem 3 and Theorem 4.

In Section 5, we mention a handful of related open problems and discuss their significance. In the Appendix, we use the methods of this paper to prove Theorem 6, stated in Section 5, which relates a minimization involving the theta function of an even integral lattice to the constant in the double cap conjecture.

1.3 | Exact maximization

In Section 4, we show that when \(k = 1\), the right-hand side of Equation (1.9) is maximized for some \(l \leq 2m + 1\) (note that for larger \(m\), the value of \(l\) that maximizes Equation 1.9 satisfies \(l < 2m\)). This disproves Conjecture 1 of Gorskaya, Miticheva, Protasov, and Raigorodskii [13]. The explicit nature of Theorem 4 has some advantages. When \(k = m = 1\), this quantity is maximized when \(l = 3\), and we retrieve Raigorodskii’s lower bound
\[
(\chi(\mathbb{R}^n))^\frac{1}{n} \geq \max_{0 < t < 1} \frac{1 + t + t^2}{1 + t + \sqrt{3} t^\frac{3}{2}} = 1.23956674 \ldots,
\]
expressed as a maximization of an explicit rational function. For \(k = 1\) and small values of \(m\), we improve upon the calculations of Gorskaya, Miticheva, Protasov, and Raigorodskii [13] for \(\chi(\mathbb{R}^m, m)\). Define
\[
\zeta_m = \limsup_{n \to \infty} \left( \frac{1}{\chi(\mathbb{R}^n; m)} \right)^\frac{1}{n}.
\]
For \(m = 2\) and \(m = 3\), for example, Theorem 3 yields the lower bounds

|               | Lower bound from Theorem 3 | Lower bound from [13] |
|---------------|-----------------------------|------------------------|
| \(m = 2\)     | \(\zeta_2 \geq 1.466299\)  | \(\zeta_2 \geq 1.465869\) |
| \(m = 3\)     | \(\zeta_3 \geq 1.667508\)  | \(\zeta_3 \geq 1.667462\) |
For the case of $k$-simplicies, define
\[
\zeta^k_m = \limsup_{n \to \infty} \left( \chi_k(\mathbb{R}^n; m) \right)^{\frac{1}{n}}.
\]

Theorem 3 yields the following table for small values of $k, m$:

| $k = 1$ | $k = 2$ | $k = 3$ | $k = 4$ |
|---------|---------|---------|---------|
| $m = 1$ | $\zeta_1^1 \geq 1.239566$ | | |
| $m = 2$ | $\zeta_2^1 \geq 1.466299$ | $\zeta_2^2 \geq 1.118433$ | |
| $m = 3$ | $\zeta_3^1 \geq 1.667508$ | $\zeta_3^3 \geq 1.239566$ | $\zeta_3^3 \geq 1.083024$ |
| $m = 4$ | $\zeta_4^1 \geq 1.848150$ | $\zeta_4^2 \geq 1.356230$ | $\zeta_4^3 \geq 1.158048$ | $\zeta_4^4 \geq 1.063933$ |
| $m = 5$ | $\zeta_5^1 \geq 2.013079$ | $\zeta_5^2 \geq 1.466299$ | $\zeta_5^3 \geq 1.239566$ | $\zeta_5^5 \geq 1.118433$ |

## 2 THE PARTITION RANK

The partition rank method is a multivariable version of the linear algebraic method, and allows us to prove our main result for $(k + 1)$-cliques. The partition rank was introduced by the author in [21, 23], and we refer the reader to those papers for a more comprehensive overview. To motivate the definition, we first revisit the definition of tensor rank. For finite sets $X_1, \ldots, X_n$, an order $n$ tensor refers to a function $F : X_1 \times \cdots \times X_n \to \mathbb{F}$. This can be thought of as an $|X_1| \times \cdots \times |X_n|$ grid of elements of $\mathbb{F}$, and when $n = 2$, we call this a matrix, and when $n = 1$, a vector. The function $h : X_1 \times \cdots \times X_n \to \mathbb{F}$ is called a rank 1 function if it takes the form

\[
h(x_1, \ldots, x_n) = f_1(x_1)f_2(x_2) \cdots f_n(x_n),
\]

and the tensor rank of $F : X_1 \times \cdots \times X_n \to \mathbb{F}$ is defined to be the minimal $r$ such that

\[
F = \sum_{i=1}^{r} g_i,
\]

where $g_i$ are rank 1 functions. Given variables $x_1, \ldots, x_k$, and a set $S \subset \{1, \ldots, k\}, S = \{s_1, \ldots, s_m\}$, let $\tilde{x}_S$ denote the $m$-tuple

\[
x_{s_1}, \ldots, x_{s_m},
\]

so that for a function $g$ of $m$ variables, we have

\[
g(\tilde{x}_S) = g(x_{s_1}, \ldots, x_{s_m}).
\]

For example, if $k = 5$, and $S = \{1, 2, 4\}$, then $g(\tilde{x}_S) = g(x_1, x_2, x_4)$, and $f(\tilde{x}_{\{1, \ldots, 5\}\setminus S}) = f(x_3, x_5)$. A partition of $\{1, 2, \ldots, k\}$ is a collection $P$ of non-empty pairwise disjoint subsets of $\{1, \ldots, k\}$ such that

\[
\bigcup_{A \in P} A = \{1, \ldots, k\}.
\]

We say that $P$ is the trivial partition if it consists only of a single set, $\{1, \ldots, k\}$.  


**Definition 1.** Let $X_1, \ldots, X_k$ be finite sets, and let

$$h : X_1 \times \cdots \times X_k \to \mathbb{F}.$$ 

We say that $h$ has **partition rank** 1 if there exists some nontrivial partition $P$ of the variables $\{1, \ldots, k\}$ such that

$$h(x_1, \ldots, x_k) = \prod_{A \in P} f_A(x_A)$$

for some functions $f_A$. We say that $h$ has **slice rank** 1, if in addition one of the sets $A \in P$ is a singleton.

The tensor $h : X_1 \times \cdots \times X_k \to \mathbb{F}$ will have partition rank 1 if and only if it can be written in the form

$$h(x_1, \ldots, x_k) = f(x_S) g(x_T)$$

for some $f$, $g,$ and some $S, T \neq \emptyset$ with $S \cup T = \{1, \ldots, k\}$. Additionally, $h$ will have slice rank 1 if it can be written in the above form with either $|S| = 1$ or $|T| = 1$. In other words, a function $h$ has partition rank 1 if the tensor can be written as a non-trivial outer product, and it has slice rank 1 if it can be written as the outer product between a vector and a $k - 1$-dimensional tensor.

**Definition 2.** Let $X_1, \ldots, X_k$ be finite sets. The **partition rank** of

$$F : X_1 \times \cdots \times X_k \to \mathbb{F}$$

is defined to be the minimal $r$ such that

$$F = \sum_{i=1}^{r} g_i,$$

where $g_i$ have partition rank 1. The **slice rank** of $F$ is the minimal $r$ such that

$$F = \sum_{i=1}^{r} g_i,$$

where $g_i$ have slice rank 1.

The partition rank is the minimal rank among all possible ranks obtained from partitioning the variables. The slice rank can be viewed as the rank which results from the partitions of $\{1, \ldots, k\}$ into a set of size 1 and a set of size $k - 1$, and so we have that

$$\text{partition-rank} \leq \text{slice-rank}.$$ 

For two variables, the slice rank, partition rank, and tensor rank are equivalent since there is only one non-trivial partition of a set of size 2. For three variables, the partition rank and the slice rank
are equivalent, and for 4 or more variables, all three ranks are different. A key property of the partition rank is the following lemma, given in [23, Lemma 11], which generalizes [30, Lemma 1].

**Lemma 1.** Let $X$ be a finite set, and let $X^k$ denote the $k$-fold Cartesian product of $X$ with itself. Suppose

$$F : X^k \rightarrow \mathbb{F}$$

is the diagonal identity tensor, that is,

$$F(x_1, \ldots, x_k) = \delta(x_1, \ldots, x_k) = \begin{cases} 1 & x_1 = \cdots = x_k \\ 0 & \text{otherwise} \end{cases}.$$ 

Then,

$$\text{partition-rank}(F) = |X|.$$ 

**Proof.** See [23, Lemma 11].

2.1 | The distinctness indicator

Let $X$ be a finite set, $\mathbb{F}$ a field, and let $X^k = X \times \cdots \times X$ denote the Cartesian product of $X$ with itself $k$ times. $S_k$ acts on $X^k$ by permutation, that is, for $\sigma \in S_k$, $(x_1, \ldots, x_k) \in X^k$, we have the group action $\sigma \cdot (x_1, \ldots, x_k) = (x_{\sigma(1)}, \ldots, x_{\sigma(k)})$. For every $\sigma \in S_k$, define

$$f_\sigma : X \times \cdots \times X \rightarrow \mathbb{F}$$

to be the function that is 1 if $(x_1, \ldots, x_k)$ is a fixed point of $\sigma$, and 0 otherwise. We will make use of the following lemma, proven in [23]:

**Lemma 2.** Let $\text{Cyc} \subset S_k$, be the $k$-cycles in $S_k$, and define

$$H_k(x_1, \ldots, x_k) = \sum_{\substack{\sigma \in S_k \\ \sigma \notin \text{Cyc}.}} \text{sgn}(\sigma)f_\sigma(x_1, \ldots, x_k).$$

Then,

$$H_k(x_1, \ldots, x_k) = \begin{cases} 1 & \text{if } x_1, \ldots, x_k \text{ are distinct}, \\ (-1)^{k-1}(k-1)! & \text{if } x_1 = \cdots = x_k, \\ 0 & \text{otherwise}. \end{cases}$$

**Proof.** See [23, Lemma 15].

This function can be used to zero-out those tuples of vectors with repetitions. Suppose that we are interested in the size of the largest set $A \subset X$ that does not contain $k$ distinct vectors satisfying
some condition $\mathcal{K}$. Then, if $F_k : X^k \to \mathbb{F}$ is some function satisfying

$$F_k(x_1, \ldots, x_k) = \begin{cases} c_1 & \text{if } x_1, \ldots, x_k \text{ satisfy } \mathcal{K} \\ c_2 & \text{if } x_1 = \cdots = x_k \\ 0 & \text{otherwise,} \end{cases}$$

where $c_2 \neq 0$, then

$$I_k(x_1, \ldots, x_k) := F_k(x_1, \ldots, x_k)H_k(x_1, \ldots, x_k)$$

when restricted to $A^k$ will be a diagonal tensor, and hence by Lemma 1,

$$|A| \leq \text{partition-rank}(I_k).$$

Let

$$\delta(x_1, \ldots, x_k) = \begin{cases} 1 & \text{if } x_1 = \cdots = x_k \\ 0 & \text{otherwise} \end{cases}$$

(2.1)

and for a partition $P$ of $\{1, \ldots, k\}$ define

$$\delta_P(x_1, \ldots, x_k) = \prod_{A \in P} \delta_A(\vec{x}_A).$$

For every $\sigma$, we have that $f_{\sigma} = \delta_P$ for some partition $P$, it follows that we can write

$$H_k(x_1, \ldots, x_k) = \sum_{P \in \mathcal{P}_k} c_P \prod_{A \in P} \delta_A(\vec{x}_A),$$

(2.2)

where $\mathcal{P}_k$ is the set of non-trivial partitions of $\{1, \ldots, k\}$ and $c_P$ are constants. Furthermore, since the definition of $H$ specifically excludes the cycles, it follows that only non-trivial partitions will appear with non-zero coefficients in the sum above. That is, every product of delta functions contains two or more terms in the product.

## 3 Lower Bounds for $\chi_k(\mathbb{R}^n; m)$

Let $S \subset \mathbb{R}^n$ be a finite set such that $\|x - y\|_2^2$ is even for any $x, y \in S$. Let $p$ be a prime satisfying

$$p > \max_{x,y \in S} \frac{\|x - y\|_2^2}{2}.$$

Consider the polynomials $F_r : S^r \to \mathbb{F}_p$ defined by

$$F_r(x_1, \ldots, x_r) = \prod_{i < j} \left(1 - \left(\frac{\|x_i - x_j\|_2^2}{2}\right)^{p-1}\right).$$
and note that
\[
\deg F_r = 2\binom{r}{2}(p - 1). \tag{3.1}
\]

Our results are most naturally stated when the domain of \( F_r \) is a cube-like subset of a lattice.

**Definition 3.** Let \( v_1, \ldots, v_n \) be a basis of \( \mathbb{R}^n \). We say that a set \( A \subset \mathbb{R}^n \) is an \( l^n \)-cube if for some constant \( k \in \mathbb{R}^n \),
\[
A = \{ k + c_0v_0 + \cdots + c_nv_n : c_i \in \{0, \ldots, l\} \}.
\]

To prove the main result, we need to only consider the specific \( l^n \)-cube \( \{0,1,\ldots,l\}^n \), however we state our results in this section with this generality to prove Theorem 6 from Section 5 in the Appendix. Note that if \( S \) is any subset of an \( l^n \)-cube, then
\[
\dim\{f : S \to F, \deg f \leq d\} \leq \# \left\{ v \in \{0, \ldots, l\}^n : \sum_{i=1}^{n} v_i \leq d \right\}. \tag{3.2}
\]

Let \( A_m = \{1, \sqrt{2}, \ldots, \sqrt{m}\} \). Then,
\[
F_{k+1}(x_1, \ldots, x_{k+1}) = \begin{cases} 1 & \|x_i - x_j\|_2 \in \sqrt{2pA_m} \cup \{0\} \forall i, j \\ 0 & \text{otherwise} \end{cases}.
\]

In other words, \( F_{k+1} \) is non-zero if and only if \( x_1, \ldots, x_{k+1} \) form a (possibly degenerate) \( k \)-simplex with distances in \( \sqrt{2pA_m} \). Note that \( p \) is a prime that depends on \( m \). The distinctness indicator function from the previous section will help construct an indicator function that captures when \( x_1, \ldots, x_{k+1} \) are a non-degenerate simplex. Define the function
\[
J_k = H_{k+1}F_{k+1}. \tag{3.3}
\]

Then, \( J_k \) will satisfy
\[
J_k(x_1, \ldots, x_k) = \begin{cases} 1 & \text{if } x_1, \ldots, x_{k+1} \text{ are distinct and form a } k \text{-simplex,} \\ (-1)^k k! & \text{if } x_1 = \cdots = x_{k+1}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}
\]

The prime \( p \) increases with the diameter of the set, so if \( n \) is large enough, we have \( p > k \) and hence \((-1)^k k! \neq 0 \pmod{p}\). In particular, if \( A \subset S \) does not contain a \( k \)-simplex, and \( n \) is large enough to insure that \( p > k \), we have that
\[
J_k|_{A^{k+1}} = (-1)^k k! \cdot \delta(x_1, \ldots, x_{k+1}).
\]

That is, \( J_k \) restricted to \( A^{k+1} \) is diagonal, and hence by Lemma 1, \( |A| \leq \text{partition-rank}(J_k) \).

\[
\chi_k(\mathbb{R}^n, A_m) \geq \frac{|S|}{\text{partition-rank}(J_k)}.
\]
Lemma 3. Let $S \subset \mathbb{R}^n$ be a subset of an $l^n$-cube such that $\|x - y\|_2^2 \in 2\mathbb{Z}$ for all $x, y \in S$, and let $p > \max_{x, y \in S} \frac{1}{2} \|x - y\|_2^2$. Then, the function $J_k : S^{k+1} \to \mathbb{F}_p$ defined by $J_k = F_{k+1}H_{k+1}$ satisfies

$$\text{partition-rank}(J_k) \leq 2^{k+1} \cdot \# \left\{ v \in \{0, ..., l\}^n : \sum_{i=1}^n v_i \leq k(p-1) \right\}.$$

Proof. Let $\mathcal{P}_{k+1}$ denote the set of non-trivial partitions of $\{1, \ldots, k+1\}$. By Lemma 2 and (2.2), $H_{k+1}$ can be written as

$$H_{k+1} = \sum_{P \in \mathcal{P}_{k+1}} c_P \prod_{A \in P} \delta(\bar{x}_A),$$

where $c_P$ is a constant depending on the partition $P$. In the construction of $H_{k+1}$, only non-trivial partitions were included, and so we can split the product $J_k = H_{k+1}R_{k+1}$ into a linear combination of terms of the form

$$R_{k+1}(x_1, \ldots, x_{k+1}) \prod_{A \in P} \delta(\bar{x}_A), \quad (3.5)$$

where the product of delta functions term always contains two or more delta functions. For such a term, the product of delta functions forces many variables among $x_1, \ldots, x_{k+1}$ to be equal. For a set $A$ and any element $a \in A$, and a polynomial function $Q$ of $|A|$ variables, the product $\delta(\bar{x}_A)Q(\bar{x}_A)$ will be exactly equal to $\delta(\bar{x}_A)Q(x_a, x_a, \ldots, x_a) = \delta(\bar{x}_A)\bar{Q}(x_a)$, where $\bar{Q}$ is a single-variable polynomial obtained from $Q$ by making all the variables equal. For our purposes, $a_i$ need only be some representative member of $A_i$, where the choice is made in a way that is consistent across different partitions. For simplicity, we choose $a_i = \min(A_i)$ for $i \leq r - 1$ and $a_r = k + 1$. Let $P = \{A_1, \ldots, A_r\}$ be a non-trivial partition of $\{1, \ldots, k + 1\}$. Let $a_i = \min(A_i)$ denote the minimal element of $A_i$ for each $i \leq r - 1$, and suppose without loss of generality $k + 1 \in A_r$, and let $a_r = k + 1$. Then,

$$F_{k+1}(x_1, \ldots, x_{k+1}) \prod_{A \in P} \delta(\bar{x}_A) = F_r(x_{a_1}, \ldots, x_{a_r}) \prod_{A \in P} \delta(\bar{x}_A).$$

We can expand $F_r$ in the above as a sum of monomials of the form

$$\left( \delta(\bar{x}_{A_1})x_{a_{1,1}}^{\epsilon_{1,1}} \ldots x_{a_{1,n}}^{\epsilon_{1,n}} \right) \ldots \left( \delta(\bar{x}_{A_r})x_{a_{r,1}}^{\epsilon_{r,1}} \ldots x_{a_{r,n}}^{\epsilon_{r,n}} \right).$$

Since $F_r$ has degree $2\binom{r}{2}(p-1)$, for each such term, we must have

$$\sum_{i=1}^r \sum_{j=1}^n \epsilon_{i,j} \leq \deg F_r \leq \binom{r}{2}(p-1),$$

and so, there exists $1 \leq i \leq r$ such that

$$\sum_{j=1}^n \epsilon_{i,j} \leq (r-1)(p-1).$$
Proceeding in this manner for every non-trivial partition of \(\{1, \ldots, k+1\}\), it follows that we can decompose

\[
J_k = \sum_{A \subset \{1, \ldots, k+1\}} \delta(\vec{x}_A) \sum_{f \text{ deg } f \leq k(p-1)} f(x_a)G_f(\vec{x}_{[1,\ldots,k+1]\setminus A})
\]

where for each set \(a = \min A\), and \(G_f\) is some function, possibly depending on \(f\). This implies that

\[
\text{partition-rank}(J_k) \leq (2^{k+1} - 2) \dim \{ f \mid f : S \to \mathbb{F}_p \text{ and } \deg f \leq k(p-1) \},
\]

and the result follows by (3.2) since

\[
\dim \{ f \mid f : S \to \mathbb{F}_p \text{ and } \deg f \leq k(p-1) \} \leq \# \left\{ v \in \{0, \ldots, l\}^n : \sum_{i=1}^{n} v_i \leq k(p-1) \right\}. \quad \Box
\]

Remark 1. In the statement of Lemma 3, the constant in front depending on \(k\) is crude. Proceeding more carefully, taking into account each subset \(A \subset \{1, \ldots, k+1\}\), in a similar manner to [23, Lemma 22, Proposition 23], the bound can be improved to

\[
\text{partition-rank}(J_k) \leq (1 + o_k(1))\# \left\{ v \in \{0, \ldots, l\}^n : \sum_{i=1}^{n} v_i \leq k(p-1) \right\},
\]

where \(o_k(1)\) tends to zero as \(n \to \infty\) with a constant depending on \(k\). However, for our purposes, Lemma 3 as stated is sufficient.

**Proposition 1.** Let \(k \geq 1\), and let \(m \geq 1\) be the number of colors. Let \(S\) be a subset of an \(l^n\)-cube where \(\|x - y\|_2^2 \in 2\mathbb{Z}\) for all \(x, y \in S\). Suppose that the maximum distance between elements of \(S\) is

\[
d_{\text{max}} = \max_{x, y \in S} \frac{1}{2} \|x - y\|_2^2.
\]

Then,

\[
\chi(\mathbb{R}^n, A_m) \geq |S|2^{k+1} \max_{0 < t < 1} \frac{t^k d_{\text{max}} + \varepsilon_0}{(1 + t + \ldots + t)^n},
\]

where \(\varepsilon_0 \leq C(d_{\text{max}})^{0.525}\) for a fixed constant \(C > 0\), where \(A_m = \{1, \sqrt{2}, \ldots, \sqrt{m}\}\).

**Proof.** Let \(p\) be the smallest prime satisfying \(p > \frac{1}{m+1} d_{\text{max}}\). We will bound the maximum size of a set in \(\mathbb{R}^n\) avoiding any distances in \(\{\sqrt{2p}, \sqrt{4p}, \ldots, \sqrt{2mp}\}\), which is a scaling of \(A_m\). We have that

\[
p < \frac{1}{m+1} d_{\text{max}} + \varepsilon_0,
\]
where $\epsilon_0 \leq C \left( \frac{d_{\max}}{m+1} \right)^{0.525}$ due to the best bounds on prime gaps [1]. Let $J_k$ be defined as in (3.3). Then if $A \subset S$ does not contain a $(k + 1)$-clique, we must have that $J_k$ restricted to $A$ is diagonal, that is,

$$J_k |_{A^{k+1}} = \delta(x_1, \ldots, x_{k+1}).$$

By Lemma 1, this implies that

$$|A| \leq \text{partition-rank}(J_k),$$

and by Lemma 3, we have

$$|A| \leq 2^{k+1} \cdot \# \left\{ v \in \{0, \ldots, l\}^n : \sum_{i=1}^{n} v_i \leq \frac{k}{m+1} d_{\max} + \epsilon_0 \right\}.$$

Consider the expansion

$$\frac{(1 + t + \cdots + t^l)^n}{t^C}.$$

The terms in this expansion are in one-to-one correspondence with the vectors $v \in \{0, \ldots, l\}^n$, and each vector $v$ with $\sum_{i=1}^{n} v_i \leq C$ will have a coefficient of $t$ with a negative power. It follows that for $0 < t < 1$, we must have

$$\# \left\{ v \in \{0, \ldots, l\}^n : \frac{1}{n} \sum_{i=1}^{n} v_i \leq \frac{k}{m+1} d_{\max} + \epsilon_0 \right\} \leq \frac{(1 + t + \cdots + t^l)^n}{t^{k \frac{d_{\max} + \epsilon_0}{m+1}}},$$

and hence

$$|A| \leq 2^{k+1} \min_{0 < t < 1} \frac{(1 + t + \cdots + t^l)^n}{t^{k \frac{d_{\max} + \epsilon_0}{m+1}}}.$$

Let $\mathcal{A}$ be a coloring of $S$ without $(k + 1)$-cliques. That is, let $\mathcal{A}$ be collection of sets such that $S \subset \bigcup_{A \in \mathcal{A}} A$, and such that each $A \in \mathcal{A}$ does not contain a $(k + 1)$-clique of elements with pairwise distances in $\sqrt{2} p A_m$. Then, due to the upper bound on the possible size of each $A$, we must have that

$$|\mathcal{A}| \geq \frac{|S|}{2} 2^{k+1} \max_{0 < t < 1} \frac{t^{k \frac{d_{\max} + \epsilon_0}{m+1}}}{(1 + t + \cdots + t^l)^n}. \quad \Box$$

**Lemma 4.** Let $0 < t < 1$, and suppose $c = (c_0, \ldots, c_l) \in \mathbb{R}_{\geq 0}^l$ is fixed. Let

$$\mathcal{A}_{n,l} = \{ a \in \mathbb{N}^l : a_0 + \cdots + a_l = n, a_i \geq 0 \}$$

and

$$\mathcal{A}_{n,l}(c) = \{ a \in \mathcal{A}_{n,l} : a_i \leq c_j \text{ if } c_i \geq c_j \}. $$
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Then,

$$\max_{a \in A_{n,l}(c)} \left( \begin{array}{c} n \\ a_0, \ldots, a_l \end{array} \right) t^{c_0 a_0 + \cdots + c_l a_l} \geq \left( \frac{t^{c_0} + \cdots + t^{c_l}}{n^l} \right)^n.$$ 

Proof. The multinomial theorem states that

$$\sum_{a \in A_{n,l}} \left( \begin{array}{c} n \\ a_0, \ldots, a_k \end{array} \right) t^{c_0 a_0 + \cdots + c_l a_l} = \left( t^{c_0} + \cdots + t^{c_l} \right)^n,$$

and so

$$\max_{a \in A_{n,l}} \left( \begin{array}{c} n \\ a_0, \ldots, a_l \end{array} \right) \geq \frac{1}{|A_{n,l}|} \left( t^{c_0} + \cdots + t^{c_l} \right)^n.$$ 

We trivially have the bound $|A_{n,l}| \leq n^l$, since $a_i$ is determined by $a_0, \ldots, a_{i-1}$, and since there are at most $n$ choices for each of $a_0, \ldots, a_{i-1}$. What remains to be shown is that the maximal $a$ lies in the restriction of $A_{n,l}$ to $A_{n,l}(c)$. Indeed, since $0 < t < 1$, we have

$$t^{ci} \leq t^{cj}$$

if $c_i \geq c_j$, and so, for the maximal tuple $a$, we must have $a_i \leq a_j$, since if $a_i > a_j$, transposing $a_i$ and $a_j$ would increase the quantity being maximized. \qed

Using Lemma 4 and Proposition 1, we are now ready to prove Theorem 3.

Proof of Theorem 3. Let $S_n^a \subset \{0, 1, \ldots, l\}^n$ be a set where each element contains $a_i$ copies of element $i$, so that $\sum_{i=0}^l a_i = n$. Then,

$$|S_n^a| = \left( \begin{array}{c} n \\ a_0, \ldots, a_l \end{array} \right),$$

and every element of $S_n^a$ will have norm of the same parity, and hence any two elements $x, y \in S_n^a$ will have even distance squared distance. Due to the convexity of $x^2$, the maximum squared distance between two elements in $S_n^a$ occurs when the $l$s match up with the $0$s, the remaining $0$s match up with the $(k - 1)$s, the remaining $(k - 1)$s match up with the $1$s, and so on. For ease of indexing, let $\{b_i\}_{i=0}^l$ be a permutation of $a_i$, where $b_0 = a_l, b_{l-1} = a_0, b_{l-2} = a_{l-1}, b_{l-3} = a_1, \ldots$. Then, it is possible to match coefficients in precisely the way described above if for each $j$,

$$\sum_{i > j} (-1)^{j+i+1} b_i \leq b_j.$$ \hfill (3.6)

Assume $b_i \leq b_j$ for $i > j$, which implies this condition. Let

$$d_{\max} = \max_{x, y \in S_n^a} \frac{1}{2} \|x - y\|^2_2.$$
Then we have that
\[ d_{\text{max}} = l^2 b_l + (l - 1)^2 (b_{l-1} - b_l) + (l - 2)^2 (b_{l-2} - b_{l-1} + b_l) + \ldots \]
\[ = \sum_{i=0}^{l} (l - i)^2 \sum_{j=l-i}^{l} (-1)^{j+l-i} b_j. \]

Note that the assumption in (3.6) is equivalent to assuming that the inner sum is always non-negative. Rearranging the order of summation, we have
\[ d_{\text{max}} = \sum_{j=0}^{l} b_j \sum_{i=0}^{j} (-1)^{l-j-i} \]
\[ = \sum_{j=0}^{l} b_j \sum_{i=0}^{j} i^2 (-1)^{j+i} \]
\[ = \sum_{j=0}^{l} b_j \binom{j + 1}{2}, \]
where the final equality is due to the identity
\[ \sum_{i=0}^{j} (j - i)^2 (-1)^i = \binom{j + 1}{2}. \]

Thus, for \( S_n^a \subset \{0, 1, \ldots, l\}^n \) satisfying \( b_i \leq b_j \) for \( i > j \), we have
\[ d_{\text{max}} = \max_{x, y \in S_n^a} \frac{1}{2} \|x - y\|_2^2 = \sum_{j=0}^{l} b_j \binom{j + 1}{2}, \]
and hence by Proposition 1,
\[ \chi_k(\mathbb{R}^n, A_m) \geq 2^{k+1} |S_n^a| \max_{0 \leq l < 1} \frac{\sum_{j=0}^{l} c_j a_j + \varepsilon_0}{(1 + t + \cdots + t^l)^n}, \]
where the constants \( c_j \) are the binomials \( \binom{j + 1}{2} \). Maximizing over all possible sets \( S_n^a \), Lemma 4 implies
\[ \max_{a \in A_n, l} |S_n^a| \frac{1}{t^m+1} \sum_{j=0}^{l} c_j a_j \geq \frac{1}{n^l} \left( \sum_{j=0}^{l} t^{k(j+1)/m+1} \right)^n, \]
and hence
\[ \chi_k(\mathbb{R}^n, A_m) \geq 2^{k+1} \left( \max_{0 \leq l < 1} \frac{\theta(t^{k/m+1}, l+1) \frac{c_0}{t \pi}}{1 + t + \cdots + t^l} \right)^n. \]
Since $\frac{c_0}{n} \ll n^{-0.475}$, $t^{\frac{c_0}{n}} = 1 + o(1)$, and $2^{\frac{k+1}{n}} = 1 + O\left(\frac{1}{n}\right)$, we have

$$\chi_k(\mathbb{R}^n, A_m) \geq \left(\max_{0 < t < 1} \frac{\theta \left(\frac{k}{m+1}, l+1\right)}{1 + t + \cdots + t^l} + o(1)\right)^n,$$

which proves the theorem.

\[ \square \]

4 \hspace{1cm} ASYMMETRIC

In this section, we prove Theorem 4. Let

$$\theta(t) = \sum_{n=1}^{\infty} t^{\frac{n(n-1)}{2}},$$

and

$$\theta(t; l) = 1 + t + t^3 + t^6 + t^{10} + \cdots + t^{l^2},$$

and define

$$F_\gamma(t, l) = \frac{\theta(t^\gamma; l)}{1 + t + \cdots + t^{l-1}}.$$  

We begin by proving $\max_{0 < t < 1} F_\gamma(t, l)$ is non-trivial for any $0 < \gamma < 1$.

**Proposition 2.** For $0 < \gamma < 1$ we have

$$\max_{0 < t < 1} F_\gamma(t, l) > 1 \quad (4.1)$$

and

$$\max_{0 < t < 1} (1 - t)\theta(t^\gamma) > 1. \quad (4.2)$$

**Proof.** Let $\eta = \frac{1}{\gamma}$, and define $G_\eta(t, l) = F_\gamma(t^\eta, l)$. Then,

$$\max_{0 < t < 1} G_\eta(t, l) = \max_{0 < t < 1} F_\gamma(t, l).$$

Observe that $G_\eta(0, l) = \frac{l}{l} = 1$, and since $\eta > 1$,

$$\frac{d}{dt} G_\eta(t, l) \bigg|_{t=0} = 1.$$
This implies that there exists $\varepsilon > 0$ such $G_\eta(\varepsilon, l) > G_\eta(0, l) = 1$, Equation (4.1) follows. To prove the second part, let

$$G_\eta(t) = (1 - t^\eta)\theta(t).$$

Then, $G_\eta(0) = 1$, and

$$\frac{d}{dt} G_\eta(t) = (1 - t^\eta) \left( \sum_{n=2}^{\infty} \binom{n}{2} t^{(n-1)/2} \right) - \eta t^{\eta-1} \theta(t).$$

Since $\eta > 1$, we have $\frac{d}{dt} G_\eta(0) = 1$, and Equation (4.2) follows. □

Next we try to understand for which $l$, $\max_{0 < t < 1} F_\gamma(t, l)$ is largest, and then lower bound this quantity by a minimization in terms of $\theta(t)$.

**Proposition 3.** For $0 < \gamma < 1$,

$$\max_{0 < t < 1} F_\gamma(t, l)$$

is maximized by choosing $l < \frac{2}{\gamma}$.

**Proof.** Suppose $l \geq \frac{2}{\gamma}$, and let $0 < t_0 < 1$ be chosen so that $F_\gamma(t_0, l)$ is maximized. The final coefficient of $\theta(t_0^\gamma; l)$ is $t_0^{(\gamma)}$, and since $\gamma \left( \frac{l}{2} \right) \geq l - 1$, it follows that $t_0^{(\gamma)} \leq t_0^{l-1}$. For $a, b, c, d > 0$, if $\frac{c}{d} < \frac{a}{b}$, then $\frac{a+c}{b+d} < \frac{a}{b}$, and so since $F_\gamma(t_0, l) > 1$, by setting $a = \theta(t_0^\gamma, l - 1)$, $b = 1 + \cdots + t_0^{l-2}$, $c = t_0^{(\gamma)}$, $d = t_0^{l-1}$, it follows that

$$F_\gamma(t_0, l - 1) > F_\gamma(t_0, l),$$

which proves the proposition. □

**Proposition 4.** For $0 < \gamma < 1$, and for $l \geq \frac{2}{\gamma}$,

$$\max_{0 < t < 1} \frac{\theta(t^\gamma; l)}{1 + t + \cdots + t^{l-1}} \geq \max_{0 < t < 1} (1 - t)\theta(t^\gamma).$$

**Proof.** Let $0 < t_1 < 1$ be such that $(1 - t_1)\theta(t_1^\gamma)$ is maximized. For any $j \geq l$, $\gamma \left( \frac{j+1}{2} \right) \geq j$, and hence

$$\sum_{j \geq l} t_1^{\gamma(j+1)/2} \leq \sum_{j \geq l} t_1^{j}.$$
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since the inequality holds term by term. Since

$$(1 - t_1)\theta(t'_1) = \frac{\theta(t_1; l) + \sum_{j \neq l} t_j^{(j+1)}}{1 + \cdots + t_{l-1}^{l-1} + \sum_{j \neq l} t_j^{l}}$$

the result again follows from the fact that $\max_{0 < t < 1} F_\gamma(t, l) \geq 1$ if $a, b, c, d > 0$ and $\frac{c}{d} < \frac{a}{b}$, then

$$\frac{a+c}{b+d} < \frac{a}{b}.$$  \hfill \Box

To complete the proof of Theorem 4, $0 < \gamma < 1$, we need a general lower bound for

$$\max_{0 < t < 1} (1 - t)\theta(t').$$

To do so, we first use the Poisson summation formula to establish a functional equation for $\theta(t)$.

**Proposition 5.** For $x > 0$, we have

$$\theta(e^{-\pi x}) = \frac{e^{\pi x}}{\sqrt{2x}} \vartheta_4\left(e^{-\frac{\pi}{x}}\right),$$

where

$$\vartheta_4(q) = \left(1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2}\right)$$

is a Jacobi Theta function.

**Proof.** We have

$$\theta(e^{-2\pi x}) = \frac{1}{2} e^{\pi x} \vartheta_2(e^{-\pi x}),$$

where $\vartheta_2$ is the Jacobi Theta function

$$\vartheta_2(e^{-\pi x}) = \sum_{n=-\infty}^{\infty} e^{-\pi x \left(n + \frac{1}{2}\right)^2}.$$ 

Let $f(t) = e^{-\pi x (t + \frac{1}{2})^2}$, and consider the Fourier transform

$$\mathcal{F}(f(t))(\omega) = \int_{\mathbb{R}} e^{-2\pi i \omega f(t)} dt.$$ 

Then,

$$\mathcal{F}(f(t)) = \frac{1}{\sqrt{x}} \exp \left(-\frac{\pi \omega^2}{x} + i\pi \omega\right).$$
and hence by the Poisson summation formula
\[ \vartheta_2(e^{-\pi x}) = \frac{1}{\sqrt{x}} \vartheta_4 \left( e^{-\frac{\pi}{x}} \right). \]

Using this functional equation, we lower bound the maximum.

**Theorem 5.** For $0 < \gamma < 1$ we have
\[
\max_{0 < t < 1} (1 - t) \vartheta(t^{\gamma}) > \Gamma_{\chi} \sqrt{\frac{1}{\gamma}},
\]
where
\[
\Gamma_{\chi} = \sqrt{\frac{\pi}{2}} \max_{0 < u < \infty} \frac{1 - e^{-u}}{\sqrt{u}} = 0.7998308498\ldots.
\]

**Proof.** Let $\eta = \frac{1}{\gamma}$. By replacing $t$ with $t^{\eta}$, we need to show that for $\eta > 1$,
\[
\max_{0 < t < 1} (1 - t) \vartheta(t^{\eta}) > \Gamma_{\chi} \sqrt{\eta}.
\]

For $\eta < 1.56$ we have $\Gamma_{\chi} \sqrt{\eta} < 1$, and so the result holds trivially by (4.2). Substituting $t = e^{-\pi x}$, by Proposition 5, we have
\[
\max_{0 < t < 1} (1 - t^{\eta}) \vartheta(t) = \max_{0 < x < \infty} (1 - e^{-\pi \gamma x}) \frac{\pi x}{\sqrt{2x}} \vartheta_4 \left( e^{-\frac{2\pi}{x}} \right).
\]

Letting $x = \frac{u}{\pi \eta}$, this becomes
\[
\sqrt{\frac{\pi \eta}{2}} \max_{0 < u < \infty} \frac{1 - e^{-u}}{\sqrt{u}} \frac{u}{\sqrt{u}} \vartheta_4 \left( e^{-\frac{2\pi^2 \eta}{u}} \right).
\]

Since $\vartheta_4(q)$ is an alternating series,
\[
\vartheta_4 \left( e^{-\frac{2\pi^2 \eta}{u}} \right) \geq 1 - 2e^{-\frac{2\pi^2 \eta}{u}}.
\]

For $0 \leq u < 25 \eta$, we have
\[
e^{\frac{u}{\sqrt{2 \eta}}} \left( 1 - 2e^{-\frac{2\pi^2 \eta}{u}} \right) \geq 1,
\]
and so
\[
\max_{0 < t < 1} (1 - t^{\eta}) \vartheta(t) \geq \sqrt{\frac{\pi \eta}{2}} \max_{0 < u < 25} \frac{1 - e^{-u}}{\sqrt{u}}.
\]
The function \( \frac{1 - e^{-u}}{\sqrt{u}} \) is maximized at \( u = 1.25643 \ldots \), which implies that
\[
\max_{0 < u < 25} \frac{1 - e^{-u}}{\sqrt{u}} = \max_{0 < u < \infty} \frac{1 - e^{-u}}{\sqrt{u}} = 0.638172686 \ldots,
\]
and the result follows. \( \square \)

Theorem 4 then follows from Theorem 5 and Proposition 4.

5 | OPEN PROBLEMS

In this section, we discuss some open problems related to the chromatic number of \( \mathbb{R}^n \) with multiple forbidden distances.

5.1 | Sphere packing

Kupavskii’s upper bound from [18] can be restated in terms of the lattice sphere packing constant. Let \( \Delta_n \) denote the density of the densest lattice packing of spheres in \( \mathbb{R}^n \). The best bounds are (see [17])
\[
2^{-n(1+o(1))} \leq \Delta_n \leq 2^{-0.5990n}.
\]

Kupavskii proved that
\[
\chi(\mathbb{R}^n, [1, l]) \leq \frac{(l + 1 + o(1))^n}{\Delta_n}.
\]

Theorem 3 along with Kupavskii’s bound for \( l = \sqrt{m} \) implies that for \( A_m = \{ 1, \sqrt{2}, \ldots, \sqrt{m} \} \),
\[
\Gamma \chi \sqrt{m} + 1 \leq \frac{(\sqrt{m} + 1 + o(1))^n}{\Delta_n}.
\]

For \( n \) sufficiently large, this implies the upper bound
\[
\Delta_n \leq \frac{1 + o(1)}{\Gamma \chi^n}.
\]

Sadly this is a trivial result since \( \Gamma \chi < 1 \), however a 26% improvement to the constant in Theorem 1 would yield a non-trivial result.

Problem 1. Can the bound in Theorem 1 be improved to
\[
\chi(\mathbb{R}^n, A_m) \geq (C \sqrt{m} + o(1))^n
\]
for some \( C > 1 \), giving a new proof of a non-trivial upper bound for sphere packing?
5.2 | Upper bounds

Kupavskii’s upper bound for $\overline{\chi}(\mathbb{R}^n; m)$ is far from the lower bound established in this paper. Can any improvement be made? In particular, can the dependence on $m$ be reduced from exponential to polynomial?

Problem 2. Does there exist $c_1, c_2 > 0$ such that

$$\overline{\chi}(\mathbb{R}^n; m) \leq (c_1 m)^{c_2 n}?$$

Kupavskii’s upper bound for $\chi(\mathbb{R}^n, A_m)$ (1.5) differs from the Larman–Rogers bound when $m = 1$. Indeed, when $m = 1$, (1.5) becomes

$$\chi(\mathbb{R}^n) \leq (4 + o(1))^n.$$

Problem 3. Can Kupavskii’s upper bound for $\chi(\mathbb{R}^n, A_m)$ be improved in such a way that $m = 1$ lines up with the Larman–Rogers bound of $3^n$?

5.3 | Low dimensions

Let $g(n)$ denote the minimal number of distinct distances between $n$ points in the plane. Then, $\overline{\chi}(\mathbb{R}^2; m) \geq g^{-1}(m)$. Can this simple lower bound be improved at all for large $m$?

Problem 4. Does there exist $C > 1$ such that

$$\overline{\chi}(\mathbb{R}^2; m) \geq C g^{-1}(m)?$$

This is significantly weaker than Erdős’ question (see [29, Problem 10 and 42]) whether $\overline{\chi}(\mathbb{R}^2; m)$ grows polynomially in $m$. I believe that the answer to problem 4 is “yes,” but it seems like it would require a different approach to this problem.

5.4 | Theta functions and the double cap conjecture

The double cap conjecture states that the set $A \subset S^{n-1}$ with the largest volume that does not contain two orthogonal vectors is the union of two spherical caps on either side of the sphere. Let $V_{n-1}$ denote the maximum of $\text{vol}(A)/\text{vol}(S^{n-1})$ for any set $A \subset S^{n-1}$ avoiding two orthogonal vectors, and define

$$\mu_S = \limsup_{n \to \infty} (V_{n-1})^{\frac{1}{n}}.$$

Then, we have

$$\frac{1}{\sqrt{2}} \leq \mu_S \leq \frac{\sqrt{3}}{2},$$

where the lower bound comes from the double cap, and the upper bound is due to Raigorodskii [25]. Let $\Lambda \subset \mathbb{R}^d$ be an even integral lattice, and define

$$\theta_{\Lambda}(q) = \sum_{\lambda \in \Lambda} q^{-\|\lambda\|^2_2}.$$ 

Define

$$\mu_{\Lambda} = \left( \max_{0 < t < 1} \theta_{\Lambda}(t)(1 - t)^d \right)^{-\frac{1}{d}}.$$ 

The methods of this paper yield the following result:

**Theorem 6.** For any even integral lattice $\Lambda$

$$\mu_S \leq \mu_{\Lambda}. \quad (5.1)$$

(See the Appendix for a proof.) Let $D_k$ denote the elements of $\mathbb{Z}^k$ at even distance from the origin, and let $\mu_{\mathbb{Z}} = \lim_{k \to \infty} \mu_{D_k}$. Then by [5, chapter 4]

$$\theta_{D_k}(t) = \frac{1}{2}(\theta_3(t)^n + \theta_4(t)^n),$$

where $\theta_3(t) = \sum_{n=-\infty}^{\infty} t^{n^2}$ and $\theta_4(t) = \sum_{n=-\infty}^{\infty} (-1)^n t^{n^2}$, and taking the limit as $n \to \infty$, we have

$$\mu_{\mathbb{Z}} = \min_{0 < t < 1} (\theta_3(t)(1 - t))^{-1} = 0.883337 \ldots.$$ 

Let $\Lambda_{24}$ denote the Leech Lattice, and $E_8$ the $E_8$ lattice. Using [5, chapter 4], we have the following calculations:

$$\mu_{E_8} = 0.88406 \ldots$$

and

$$\mu_{\Lambda_{24}} = 0.88407 \ldots.$$ 

These all result in upper bounds for $\mu_S$ that are all worse than Raigorodskii’s bound of $\sqrt{3}/2 = 0.866 \ldots$.

**Problem 5.** Does there exist a lattice $\Lambda \subset \mathbb{R}^d$ such that

$$\mu_{\Lambda} < \mu_{\mathbb{Z}}?$$

If yes, it maybe be possible to work with a subset of such a lattice to obtain a new upper bound for the double cap conjecture.
APPENDIX: LATTICES AND THE DOUBLE CAP CONJECTURE

In Section 5, we looked at the double cap conjecture, and stated Theorem 6 relating the problem to the theta function of a lattice. In this Appendix, we prove Theorem 6, which is a similar proof to that of Theorem 3. Let \( \Lambda \) be an even integral lattice, and let \( v_1, \ldots, v_d \) be a basis for \( \Lambda \). For \( l \in \mathbb{N} \), define

\[
\Lambda_l = \{ c_1 v_1 + \cdots + c_d v_d : c_i \in \mathbb{Z}, -l \leq c_i \leq l \}.
\]

Let \( \ell = (|\Lambda| - 1)/2 \), and index the elements of \( \Lambda_l \) from \(-\ell\) to \( \ell \), where elements indexed by \(-i, i\) are antipodal for every \( i \neq 0 \). Then,

\[
\theta_{\Lambda}(t) = \sum_{\lambda \in \Lambda} t^{-\|\lambda\|_2^2}
\]

and given a choice of basis, we can define the truncation

\[
\theta_{\Lambda}(t, l) = \sum_{i=-\ell}^{\ell} t^{-\|v_i\|_2^2}.
\]

**Proposition A1.** For any even integral lattice \( \Lambda \) and any \( l \geq 1 \), we have

\[
\mu_S \leq \min_{0 < t < 1} \frac{1 + t + \cdots + t^{2l+1}}{\theta_{\Lambda}(t, l)^{\frac{1}{d}}}
\]

for some choice of basis \( v_1, \ldots, v_d \).

**Proof.** Choose a basis for \( \Lambda \). Consider the \( n \)-fold product of \( \Lambda_l \) with itself, \( \Lambda_l^n = \Lambda_l \times \cdots \times \Lambda_l \subset \mathbb{R}^{nd} \).

By the assumption that the lattice is integral and even, \( \frac{\|x-y\|_2^2}{2} \in 2\mathbb{Z} \) for every \( x, y \in \Lambda_l^n \). Let \( n \) be sufficiently large, and let \( S \subset \Lambda_l^n \) be such that each element contains \( a_i \) copies of element \( i \) and \(-i\) of \( \Lambda_l \). Due to the convexity of \( x^2 \), the maximum squared distance between two elements in \( S_n^a \) occurs when antipodal points are paired. It follows that the maximum squared distance will be

\[
\max_{x, y \in S} \frac{\|x - y\|_2^2}{2} = \sum_{i=-\ell}^{\ell} a_i \|v_i\|_2^2,
\]

and this occurs exactly when \( x, y \) are antipodal. Let \( S \) be the set \( S \) where directly antipodal points are removed, so that \( |S| > |S|/2 \), and

\[
d_{\text{max}} = \max_{x, y \in S} \frac{\|x - y\|_2^2}{2} < \sum_{i=-\ell}^{\ell} a_i \|v_i\|_2^2.
\]

Let

\[
p = \frac{1}{2} \left( \sum_{i=-\ell}^{\ell} a_i \|v_i\|_2^2 \right)
\]

(A.1)
and suppose that \( p \) is prime. Then, the polynomial \( F: \tilde{S} \times \tilde{S} \to \mathbb{F}_p \) defined by

\[
F(x, y) = 1 - \left( \frac{\|x - y\|_2^2}{2} \right)^{p-1}
\]

will be 1 if \( x = y \) or if \( x, y \) are at a right angle. The proof of Proposition 1 implies that if \( A \subset S \) satisfies

\[
|A| \geq 4 \min_{0 < t < 1} \frac{(1 + t + \ldots + t^{2l+1})^{nd}}{t^{d_{\text{max}}}},
\]

then it contains two elements \( x \neq y \) with \( F(x, y) = 1 \), that is two elements at a right angle. If we could maximize over all possible sets \( \tilde{S} \) while ignoring the constraint that \( p \) must be prime, then Lemma 4 would imply that

\[
\max_{\tilde{S}} |\tilde{S}|^n \sum_{i=-\ell}^{\ell} a_i \|v_i\|_2^2 \geq \frac{1}{nd} \left( \sum_{i=-\ell}^{\ell} t \|v_i\|_2^2 \right)^n \geq \frac{1}{nd} \theta_\Lambda(t, l)^n,
\]

and hence that there exists \( \tilde{S} \) such that if \( A \subset \tilde{S} \) does not contain two elements at a right angle, then

\[
\frac{|A|}{|\tilde{S}|} \leq 4nd \min_{0 < t < 1} \frac{(1 + t + \ldots + t^{2l+1})^{nd}}{\theta_\Lambda(t, l)^n}.
\]

Using bounds for prime gaps, we can modify Lemma 4 to handle the constraint that \( p \) is prime, while only varying \( n \) a small amount. This is due to the fact that small perturbations of size \( \ll n^{1-\varepsilon} \) to the individual coefficients \( a_i \) will not affect the final asymptotic, but do allow us to modify the quantity

\[
\frac{1}{2} \cdot \left( n \sum_{i=-\ell}^{\ell} a_i \|v_i\|_2^2 \right)
\]

sufficiently to insure that it is prime. In particular, let \( i \) be such that \( \alpha = \|v_i\|_2^2/2 \) is minimal. If we increase \( n \) by 2 and increase the coefficients \( a_i, a_{-i} \), corresponding to \( v_i \) and \( -v_i \), each by 1, we will increase the value of \( p \) by exactly \( 2\alpha \). Heilbronn [16] showed that for any fixed modulus \( q \), there exists \( \delta > 0 \) such that

\[
\pi(x + x^{1-\delta}; q, a) - \pi(x; q, a) \sim \frac{1}{\phi(q)} \frac{x^{1-\delta}}{\log x},
\]

where

\[
\pi(x; q, a) = \sum_{\substack{p \leq x \\backslash \|p\equiv a(q)\}} 1
\]
is the prime counting function, and $\phi(q)$ is the Euler–Totient function (see also [20, pp. 141–142]). Applying this theorem with the modulus $2\alpha$, by modifying $a_i, a_{-i}$ in the way described, it follows that there exists $\delta > 0$ and some constant $C_1$ depending on $v_1, \ldots, v_d$ and on $l$, such that there exists $m$ such that

$$n \leq m \leq n + C_1 n^{1-\delta}$$

with $p = \frac{1}{2} \cdot (\sum_{i=-\ell}^{\ell} a_i \|v_i\|_2^2)$ prime. Consequently, there is a choice of $a_i$ such that for any subset of $A \subset \tilde{S}$ of density

$$\frac{|A|}{|\tilde{S}|} \leq \left( \min_{0 < t < 1} \left( \frac{(1 + t + \cdots + t^{2l+1})^d}{\vartheta_{\Lambda}(t, l)} + o(1) \right)^n \right),$$

and it follows that

$$\mu_S \leq \min_{0 < t < 1} \frac{1 + t + \cdots + t^{2l+1}}{\vartheta_{\Lambda}(t, l) \frac{1}{d}} .$$

For any $l$, since $1 + t + \cdots + t^{2l+1} \leq \sum_{i=0}^{\infty} t^i$ for any $0 < t < 1$, it follows that

$$\min_{0 < t < 1} \frac{1 + t + \cdots + t^{2l+1}}{\vartheta_{\Lambda}(t, l) \frac{1}{d}} \leq \min_{0 < t < 1} \left( \vartheta_{\Lambda}(t, l) \frac{1}{d} (1 - t) \right)^{-1} ,$$

and so

$$\mu_S \leq \min_{0 < t < 1} \left( \vartheta_{\Lambda}(t, l) \frac{1}{d} (1 - t) \right)^{-1}$$

for every $l$. Taking the limit as $l \to \infty$, we obtain Theorem 3.

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