Positive solutions of viscoelastic problems

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Abstract

In 1, 2 or 3 dimensions a scalar wave excited by a non-negative source in a viscoelastic medium with a non-negative relaxation spectrum or a Newtonian response or both combined inherits the sign of the source. The key assumption is a constitutive relation which involves the sum of a Newtonian viscosity term and a memory term with a completely monotone relaxation kernel. In higher-dimensional spaces this result holds for sufficiently regular sources. Two positivity results for vector-valued wave fields including isotropic viscoelasticity are also obtained.

Notation.
\[ [a,b] := \{ x \in \mathbb{R} \mid a \leq x < b \} ; \]
\[ \mathbb{R}^+ = [0, \infty) ; \]
I: unit matrix;
\[ \langle k, x \rangle := \sum_{n=1}^{d} k_l x^l ; \]
\[ \theta(y) := \begin{cases} 1, & y > 0 \\ 0, & y < 0 \end{cases} ; \]
\[ \tilde{f}(p) := \int_0^\infty e^{-py} f(y) \, dy. \]
1 Introduction

Positivity of viscoelastic pulses was studied in a paper of Duff (1969). Duff assumed a special model with a rational complex modulus. Duff’s models are however loosely related to viscoelasticity and his assumptions are excessively restrictive.

In this paper a general scalar viscoelastic medium with the constitutive equation $\sigma = a \dot{e} + G(t) * \dot{e}$ with a completely monotone relaxation modulus $G$ and a non-negative Newtonian viscosity coefficient is studied. We show that a scalar viscoelastic wave field propagating in a $d$-dimensional medium and excited by a non-negative pulse is also non-negative provided $d \leq 3$. For higher dimensions and for non-zero initial data only wave fields excited by sufficiently regular sources are non-negative.

Positivity of viscoelastic signals can be considered as a test for the non-negative relaxation spectrum and for the presence of the Newtonian viscosity.

Positivity can be extended to matrix-valued fields, e.g. to Green’s functions of systems of PDEs. In Sec. 6 we consider a system of PDEs resembling the equations of motion of viscoelasticity with a CM relaxation kernel and prove that the Green’s function of this system of equations is positive-semidefinite. This result does not apply to general viscoelastic Green’s functions, which involve double gradients of positive semi-definite functions. In iso

2 Statement of the problem

In a hereditary or Newtonian linear viscoelastic medium a scalar field excited by positive source is non-negative. This applies to displacements in pure shear or to scalar displacement potentials. The key assumption about the material properties of the medium is a positive relaxation spectrum. The result holds for arbitrary spatial dimension.

We consider the problem:

$$
\rho D^2 u = a \nabla^2 D u + G(t) * \nabla^2 D u + s(t, x) \quad t \geq 0, \quad x \in \mathbb{R}^d
$$

(1)

with $s(t, x) = \theta(t) (c_1 + c_2 t) \delta(x)$ and the initial condition

$$
u(0, x) = u_0 \delta(x), \quad Du(0, x) = \dot{u}_0 \delta(x),$$

(2)

(Problem I) as well as $s(t, x) = c \delta(t) \delta(x)$ with a solution assumed to vanish for $t < 0$ (Problem II). It is assumed that $a \geq 0$ and $G$ is a completely monotone (CM) function.

The Laplace transform

$$
\tilde{u}(p, x) := \int_0^\infty e^{-pt} u(t, x) \, dt, \quad \text{Re} \, p > 0, \quad x \in \mathbb{R}^d
$$

(3)

satisfies the equation

$$
\rho p^2 \tilde{u}(p, x) = Q(p) \nabla^2 \tilde{u}(p, x) + g(p) \delta(x)
$$

(4)
where
\[ Q(p) := a p + p \tilde{G}(p) \]  \hspace{1cm} (5)

The function \( g \) is defined by the equation
\[ g_I(p) = \frac{1}{p} + p u_0 + \dot{u}_0 \]  \hspace{1cm} (6)
in Problem I and
\[ g_{II}(p) = 1 \]  \hspace{1cm} (7)
in Problem II.

## 3 Basic mathematical tools.

The classes of functions appropriate for viscoelastic responses are reviewed in detail in Seredynska & Hanyga (2009).

**Theorem 3.1** If the function \( \tilde{u}(\cdot, x) \) is completely monotone for every \( x \in \mathbb{R}^d \), then \( u(t, x) \geq 0 \) for every \( t \geq 0 \) and \( x \in \mathbb{R}^d \).

**Proof.** If \( \tilde{u}(\cdot, x) \) is completely monotone, then, in view of Bernstein’s theorem (Widder, 1946), for every \( x \in \mathbb{R}^d \) it is the Laplace transform of a positive Radon measure \( m_x \):
\[ \tilde{u}(\cdot, x) = \int_{[0, \infty]} e^{-p^*} m_x(ds) \]  \hspace{1cm} (8)

The Radon measure \( m_x \) is uniquely determined by \( \tilde{u}(\cdot, x) \), hence \( m_x(dt) = u(t, x)dt \) is a positive Radon measure. Hence, in view of continuity of \( u(\cdot, x) \), we have the inequality \( u(t, x) \geq 0 \) for all \( t \geq 0 \) and \( x \in \mathbb{R}^d \). \( \Box \)

The problem of proving that \( u(t, x) \) is non-negative is thus reduced to proving that \( \tilde{u}(\cdot, x) \) is completely monotone. The crucial step here is the realization that \( Q \) in (5) is a complete Bernstein function. We shall therefore recall some facts about Bernstein and complete Bernstein functions and their relations to completely monotone functions.

**Definition 3.2** A function \( f \) on \( \mathbb{R}_+ \) is said to be completely monotone (CM) if it is infinitely differentiable and satisfies the infinite set of inequalities:
\[ (-1)^n D^n f(y) \geq 0 \quad y > 0, \quad \text{for all non-negative integer } n \]

It follows from the definition and the Leibniz formula that the product of two CM functions is CM. A CM function can have a singularity at 0.

**Definition 3.3** A function \( f \) on \( \mathbb{R}_+ \) is said to be locally integrable completely monotone (LICM) if it is CM and integrable over the segment \([0, 1]\).
Definition 3.4 A function $f$ on $\mathbb{R}_+$ is said to be a Bernstein function (BF) if it is non-negative, differentiable and its derivative is a CM function.

Since a BF is non-negative and non-decreasing, it has a finite limit at 0. It can therefore be extended to a function on $\mathbb{R}_+$.

Every CM function $f$ is the Laplace transform of a positive Radon measure:

Theorem 3.5 (Bernstein’s theorem, Widder (1941))

$$f(t) = \int_{[0, \infty]} e^{-rt} \mu(dr)$$

(9)

It is easy to show that $f$ is a LICM if the Radon measure $\mu$ satisfies the inequality

$$\int_{[0, \infty]} \frac{\mu(dr)}{1 + r} < \infty$$

(10)

Theorem 3.6 (Jacob, 2001; Seredyńska & Hanyga, 2009) If $f, g$ are CM then the pointwise product $fg$ is CM.

Let $f$ be a Bernstein function. Since the derivative $Df$ of $f$ is LICM, Bernstein’s theorem can be applied. Upon integration the following integral representation of a general Bernstein function $f$ is obtained:

$$f(y) = a + by + \int_{[0, \infty]} \left[1 - e^{-r}y \right] \nu(dr)$$

(11)

where $a, b = Df(0) \geq 0$, and $\nu(dr) := \mu(dr)/r$ is a positive Radon measure on $\mathbb{R}_+$ satisfying the inequality

$$\int_{[0, \infty]} \frac{r \nu(dr)}{1 + r} < \infty$$

(12)

The constants $a, b$ and the Radon measure $\nu$ are uniquely determined by the function $f$.

Theorem 3.7 (Berg & Forst, 1973; Jacob, 2001) If $f$ is a CM function, $g$ is a BF and $g(y) > 0$ for $y > 0$ then the composition $f \circ g$ is a CM.

Corollary 3.8 (Berg & Forst, 1973; Jacob, 2001) If $g$ is a non-zero BF then $1/g$ is a CM function.

Note that the function $f(y) := \exp(-y)$ is CM but $1/f$ is not a BF.

Definition 3.9 A function $f$ is said to be a complete Bernstein function (CBF) if there is a Bernstein function $g$ such that $f(y) = y^2 \tilde{g}(y)$.
Theorem 3.10 (Jacob, 2001) A function $f$ is a CBF if and only if it satisfies the following two conditions:

1. $f$ admits an analytic continuation $f(z)$ to the upper complex half-plane; $f(z)$ is holomorphic and satisfies the inequality $\text{Im } f(z) \geq 0$ for $\text{Im } z > 0$;
2. $f(y) \geq 0$ for $y \in \mathbb{R}_+$.

The derivative $Dg$ of the Bernstein function $g$ is a LICM function $h$. Hence we have the following theorem:

Theorem 3.11 Every CBF $f$ can be expressed in the form

$$f(y) = y \tilde{h}(y) + ay$$

(13)

where $h$ is LICM and $a = g(0) \geq 0$. Conversely, for every LICM function $h$ and $a \geq 0$ the function $f$ given by (13) is a CBF.

Proof. For the first part, let $g$ be the BF in Definition 3.9 and let $h := Dg$. Since $\int_0^1 h(x) \, dx = g(1) - g(0) < \infty$, the function $h$ is LICM. For the second part, note that if $h$ is LICM, then $g(y) = a + \int_0^y h(s) \, ds$ is a BF and $f(y) = y^2 \tilde{g}(y)$.

Since the Laplace transform of a LICM function $h$ has the form

$$\tilde{h}(y) = \int_{[0, \infty]} \frac{\mu(dr)}{r + y}$$

(14)

where $\mu$ is the Radon measure associated with $h$, every CBF function $f$ has the following integral representation

$$f(y) = b + ay + y \int_{[0, \infty]} \frac{\mu(dr)}{r + y}$$

(15)

with arbitrary $a, b = \mu(\{0\}) \geq 0$ and an arbitrary positive Radon measure $\mu$ satisfying eq. (14). The constants $a, b$ and the Radon measure $\mu$ are uniquely determined by the function $f$.

Noting that $y/(y + r) = r [1/r - 1/(y + r)]$, we can also express the CBF $f$ in the following form

$$f(y) = b + ay + \int_0^\infty [1 - e^{-rz}] h(z) \, dz$$

where $h(z) := \int_{[0, \infty]} e^{-rz} m(dr) \geq 0$ and $m(dr) := r \mu(dr)$ satisfies the inequality

$$\int_{[0, \infty]} \frac{m(dr)}{r(r + 1)} < \infty$$
Let \( \nu(dz) := h(z) \, dz \). We have
\[
\int_{0,\infty} z \nu(dz) \frac{1}{1+z} = \int_0^\infty z h(z) \, dz = \int_{[0,\infty]} m(dr) \left[ \frac{1}{1} - e^r \Gamma(0, r) \right]
\]

Using the asymptotic properties of the incomplete Gamma function (Abramowitz & Stegun, 1970) it is possible to prove that the right-hand side is finite, hence the Radon measure \( \nu(dz) := h(z) \, dz \) satisfies inequality (12). We have thus proved an important theorem:

**Theorem 3.12** Every CBF is a BF.

However \( 1 - \exp(-y) \) is a BF but not a CBF.

The simplest example of a CBF is
\[
\varphi_a(y) := \frac{y}{y + a} \equiv y^2 \int_0^\infty e^{-sy} \left[ 1 - e^{-sa} \right] \, ds
\]

\( a \geq 0 \). It follows from eq. (15) that every CBF \( f \) which satisfies the conditions \( f(0) = 0 \) and \( \lim_{y \to \infty} f(y)/y = 0 \) is an integral superposition of the functions \( \varphi_a \). The CBF \( \varphi_a \) corresponds to a Debye element defined by the relaxation function \( G_a(t) = \exp(-at) \).

We shall need the following properties of CBFs:

**Theorem 3.13** (Jacob, 2001; Seredyńska & Hanyga, 2009)

1. \( f \) is a CBF if and only if \( y/f(y) \) is a CBF;
2. if \( f, g \) are CBFs, then \( f \circ g \) is a CBF.

The second statement follows easily from Theorem 3.10.

**Remark. 3.14** \( y^\alpha \) is a CBF if \( 0 < \alpha < 1 \), because
\[
y^\alpha - 1 = \frac{1}{\Gamma(1 - \alpha)} \int_0^\infty e^{-ys} s^{-\alpha} \, ds = \frac{1}{\Gamma(1 - \alpha) \Gamma(\alpha)} \int_0^\infty dz \int_0^\infty e^{-ys} e^{-zs} \, ds \, z^{\alpha-1} = \frac{\sin(\alpha \pi)}{\pi} \int_0^\infty \frac{z^{\alpha-1}}{y+z} \, dz
\]

and thus
\[
y^\alpha = \frac{y \sin(\alpha \pi)}{\pi} \int_0^\infty \frac{z^{\alpha-1}}{y+z} \, dz
\]

The sets of LICM functions and CBFs will be denoted by \( \mathcal{F} \) and \( \mathcal{C} \) respectively.
4 Positivity of one- and three-dimensional solutions.

Applying the results of the previous section, we get the following result:

**Theorem 4.1** If $a \geq 0$ and the relaxation modulus $G$ is CM then the function $Q$ defined by eq. (4) is a CBF.

The mapping $(a, G) \in \mathbb{R}_+ \times \mathfrak{F} \rightarrow Q \in \mathcal{C}$ defined by eq. (4) is bijective.

A one-dimensional solution of eq. (4) is given by

$$\tilde{u}_1(p, x) = U_1(p, |x|) := A(p) \exp(-B(p) |x|)$$

with $B(p) = \rho^{1/2} p/Q(p)^{1/2}$ and $A(p) = g(p)/[2B(p)]$ If $Q \in \mathcal{C}$, then $Q(y)^{1/2}$ is a composition of two CBFs, namely $y^{1/2}$ (Remark 3.14) and $Q$, hence it is a CBF by Theorem 3.13. The function $B(p)$ is a CBF by Theorem 3.13 and $1/B(p)$ is a CM function by Theorem 3.12 and Corollary 3.8.

The amplitude of the solution of Problem I is given by $A(p) = 1/[2 p B(p)] + \dot{u}_0/[2 B(p)]$. The first term is a CM function because it is the product of two CM functions. The second term is also CM, hence $A(p)$ is CM. The amplitude of the solution of Problem II $A(p) = 1/[2 B(p)]$ is also CM.

For every fixed $x$ the function $\exp(-B(p) |x|)$ is the composition of a CBF and the function $B$, which is a CBF and therefore a BF. By Theorem 3.7 the function $\exp(-B(\cdot) |x|)$ is CM. This proves that for $d = 1$ the solutions of Problem II and Problem I with $u_0 = 0$ are non-negative.

In a three-dimensional space the solution $\tilde{u}_3$ of (14) is given by the equation

$$\tilde{u}_3(p, x) = -\frac{1}{2 \pi r} \frac{\partial U_1(p, r)}{\partial r}$$

where $r = |x|$, so that

$$\tilde{u}_3(p, x) = \frac{1}{4 \pi r} A(p) B(p) \exp(-B(p) |x|)$$

But $A(p) B(p) = g(p)/2$. If $u_0 = 0$ then $g$ is CM. Hence $\tilde{u}_3(\cdot, x)$ is the product of two CM functions and thus CM.

5 Positivity of solutions in arbitrary dimension.

In an arbitrary dimension $d$

$$\tilde{u}_d(p, x) = \frac{g(p)}{(2\pi)^d Q(p)} \int e^{i k \cdot x} \frac{1}{\rho p^2/Q(p) + |k|^2} d d k$$

The above formula can be expressed in terms of MacDonald functions by using eq. (3) in Sec. 3.2.8 of Gel’fand & Shilov (1964):

$$\tilde{u}_d(p, x) = \frac{\rho^{d/4-1/2} g(p)^{d/2-1}}{(2\pi)^{d/2} Q(p)^{d/4+1/2}} r^{-(d/2-1)} K_{d/2-1} (B(p) r)$$
where $B(p)$ is defined in the preceding section.

The MacDonald function is given by the integral representation

$$K_{\mu}(z) = \int_0^\infty \exp(-z \cosh(s)) \cosh(\mu s) \, ds$$  \hspace{1cm} (19)

Since $\cosh(y)$ is a positive increasing function, it follows immediately that $K_{\mu}(z)$ is a CM function.

We shall need a stronger theorem on complete monotonicity of MacDonald functions.

**Theorem 5.1** [Miller & Samko, 2001].

The function $z^{1/2} K_{\mu}(z)$ is CM for $\mu \geq 1/2$. \hspace{1cm} \[\square\]

The proof of this theorem requires a lemma.

**Lemma 5.2** If $\alpha \geq 0$ then the function $(1 + 1/x)^\alpha$ is CM.

Proof. We begin with $0 \leq \alpha < 1$. Setting $t = 1/(xy)$ we have that

$$\frac{\alpha}{x^\alpha} \int_1^\infty \frac{dy}{y^{1+\alpha} (xy + 1)^{1-\alpha}} = \alpha \int_0^{1/x} \frac{t^{\alpha-1}}{(1/t + 1)^{1-\alpha}} \, dt = \alpha \int_1^{1+1/x} u^{\alpha-1} \, du = \left(1 + \frac{1}{x}\right)^\alpha$$

Since for each fixed value of $y > 0$ the function $(xy + 1)^{\alpha-1}$ is CM, the function $(1 + 1/x)^\alpha$ is also CM.

The function $1 + 1/x$ is CM, hence for every positive integer $n$ the function $(1 + 1/x)^n$ is CM. We can now decompose any positive non-integer $\alpha$ into the sum $\alpha = n + \beta$, where $n$ is a positive integer and $0 < \beta < 1$. Consequently

$$(1 + 1/x)^\alpha \equiv (1 + 1/x)^n (1 + 1/x)^\beta$$

is CM because it is a product of two CM functions. \hspace{1cm} \[\square\]

Proof of the theorem For $\mu > -1/2$ the MacDonald function has the following integral representation:

$$z^{1/2} K_{\mu}(z) = \sqrt{\frac{\pi}{2}} \frac{1}{\Gamma(1/2 - \mu)} e^{-z} \int_0^\infty e^{-s} s^{\mu-1/2} \left(1 + \frac{s}{2z}\right)^{\mu-1/2} \, ds, \quad z > 0$$  \hspace{1cm} (20)

[Gradshteyn & Ryzhik (1994), 8:432:8]. By Lemma 5.2 the integrand of the integral on the right-hand side is CM if $\mu \geq 1/2$. Hence the integral is the limit of sums of CM functions, therefore itself a CM function. Consequently, the function $z^{1/2} K_{\mu}(z)$ is the product of two CM functions, and thus it is CM

\[\text{\footnotesize 1} \text{The theorem is valid for } \mu \geq 0, \text{ see Miller & Samko (2001), but we do not need this fact.}\]
We now note that $\tilde{u}(p,x) = p^{(d-3)/2} g(p) F(p)$. We shall prove that $F(p)$ is the product of two CM functions of the argument $p$, viz. $Q(p)^{-(d+1)/4}$ and $L(z) := z^{1/2} K_{d/2-1}(z)$ with $z := B(p) r$, as well as a positive factor independent of $p$.

Lemma 5.3 If $Q$ is a CBF and $\alpha > 0$, then $Q(p)^{-\alpha}$ is CM.

Proof. Let $n$ be the integer part of $\alpha$, $\alpha = n + \beta$, $0 \geq \beta < 1$. $Q(p)^{-1}$ is CM (by Theorem 3.12 and Corollary 3.8) and therefore also $Q(p)^{-n}$ is CM. By Theorem 3.13 the function $Q(p)^{\beta}$ is a CBF, hence $1/Q(p)^{\beta}$ is CM. Consequently $Q(p)^{-\alpha}$ is CM. $\Box$

The lemma implies that the factor $Q(p)^{-(d+1)/4}$ is CM. Since the function $L$ is CM and we have already proved that $B(p)$ is BF, Theorem 3.7 implies that $L(B(p) r)$ is a CM function of $p$. For $d \leq 3$ the factor $p^{(d-3)/2}$ is also CM. Consequently, for $d \leq 3$ the solution $u(t,x)$ of Problem II is non-negative. The solution of the same problem with an arbitrary source of the form $s(t) \delta(x)$ and $s(t) \geq 0$ can be obtained by a convolution of two non-negative functions and therefore is also non-negative.

For $d \leq 5$ Problem I with $u_0 = \dot{u}_0 = 0$ has a non-negative solution if $c_1 > 0$.
For $d \leq 7$ Problem I has a non-negative solution if $c_1 = 0$ and $c_2 > 0$.

For $d > 3$ the fractional integral

$$I^\alpha u(t,x) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds, \quad t > 0$$

is non-negative provided $\alpha \geq (d-3)/2$ and $u_0 = 0$ or provided $\alpha \geq (d-1)/2$.

We summarize these results in a theorem.

Theorem 5.4 In a viscoelastic medium of dimension $d \leq 3$ with a constitutive relation

$$\sigma = a \dot{\epsilon} + G(t) \ast \epsilon, \quad a \geq 0; \quad G \in \mathbb{F}$$

Problem II as well as Problem I with the initial condition $u_0 = 0$ have non-negative solutions.

Under the same assumptions but for an arbitrary dimension $d > 3$ certain indefinite fractional time integrals of the solution are non-negative. For zero initial data Problem I has a non-negative solution if $d \leq 5$ and $c_1 > 0$, or if $d \leq 7$, $c_1 = 0$ and $c_2 > 0$.

6 Positivity properties of vector-valued fields.

It is interesting to examine the implications of CM relaxation kernels on positivity properties of vector fields. We shall prove that in a simple model complete monotonicity of a relaxation kernel implies that the Green’s function is positive semi-definite.
Unfortunately the tools developed in Sec. 3 fail for matrix-valued CM and Bernstein functions $Q(p)$ which do not commute with their derivatives. In particular, the product of two non-commuting matrix-valued functions need not be a CM function and the function $f \circ G$, where $f$ is CM and $G$ is a matrix-valued BF, need not be CM.

**Definition 6.1** A matrix-valued function $F : \mathbb{R}_+ \to \mathbb{R}^{n \times n}$ is said to be a CM function if it is infinitely differentiable and the matrices $(-1)^n D^n F(y)$ are positive semi-definite for all $y > 0$.

**Definition 6.2** A matrix-valued Radon measure $M$ is said to be positive if the matrix $\langle v, \int_{[0, \infty]} f(y) M(dy) v \rangle \geq 0$ for every vector $v \in \mathbb{R}^n$ and every non-negative function $f$ on $\mathbb{R}_+$ with compact support.

It is convenient to eliminate matrix-valued Radon measures by applying the following lemma (Hanyga & Seredynska, 2007):

**Lemma 6.3** Every matrix-valued Radon measure $M$ has the form $M(dx) = K(x) m(dx)$, where $m$ is a positive Radon measure, while $K$ is a matrix-valued function defined, bounded and positive semi-definite on $\mathbb{R}_+$ except on a subset $E$ such that $m(E) = 0$.

**Theorem 6.4** (Gripenberg et al., 1990) A matrix-valued function $F : \mathbb{R}_+ \to \mathbb{R}^{n \times n}$ is CM if and only if it is the Laplace transform of a positive matrix-valued Radon measure.

The following corollary will be applied to Green’s functions:

**Corollary 6.5** If $\tilde{R}(p) := \int_0^\infty e^{-pt} R(t) dt$ is a matrix-valued CM function then $R(t)$ is positive semi-definite for $t > 0$.

**Definition 6.6** A matrix-valued function $G : \mathbb{R}_+ \to \mathbb{R}^{n \times n}$ is said to be a Bernstein function (BF) if $G(y)$ is differentiable and positive semi-definite for all $y > 0$ and its derivative $DG$ is CM.

**Definition 6.7** A matrix-valued function $H : \mathbb{R}_+ \to \mathbb{R}^{n \times n}$ is said to be a complete Bernstein function (CBF) if $H(y) = y^2 G(y)$, where $G$ is an $n \times n$ matrix-valued BF.

The integral representation (15) of a CBF remains valid except that the Radon measure has to be replaced by a positive matrix-valued Radon measure $N(dr) = K(r) \nu(dr)$:

$$H(y) = B + y A + y \int_{[0, \infty[} \frac{K(r) \nu(dr)}{r+y}$$  \hspace{1cm} (21)
where the Radon measure $\nu$ satisfies the inequality
\[ \int_{[0, \infty]} \frac{\nu(dr)}{1 + r} < \infty \] (22)

the matrix-valued function $K(r)$ is positive semi-definite and bounded $\nu$-almost everywhere on $\mathbb{R}_+$ while $A, B$ are two positive semi-definite matrices. Every matrix-valued CBF $H$ can be expressed in the form
\[ H(y) = y \tilde{F}(y) + y A \] (23)

where $F$ is a matrix-valued LICM function.

We now consider the following problem
\[ \rho D^2 G = ADG + \nabla^2 DG + \delta(t) \delta(x) I, \quad t \geq 0, \quad x \in \mathbb{R}^d \] (24)

where $A$ is a positive semi-definite $n \times n$ matrix and $G$ is an $n \times n$ matrix-valued relaxation modulus.

If the relaxation modulus $G$ is a CM matrix-valued function then the function $Q(p) := p \tilde{G}(p)$ is a matrix-valued CBF. The function $Q$ is real and positive semi-definite, hence it is symmetric and has $n$ eigenvalues $q_k(p)$ and $n$ eigenvectors $e_k, k = 1, \ldots, n$. We shall now assume that the eigenvectors are constant:
\[ Q(p) = \sum_{k=1}^{n} q_k(p) e_k \otimes e_k \]

It is easy to see that the functions $q_k, k = 1, \ldots, n$, are CBFs.

The Laplace transform $\tilde{G}(p, x)$ of the Green function is given by the formula
\[ \tilde{G}(p, x) = \frac{1}{(2\pi)^d} \int e^{i(k,x)} \left[ p^2 I + |k|^2 Q(p) \right]^{-1} \, dk \equiv \sum_{k=1}^{n} \frac{1}{(2\pi)^d} \int e^{i(k,x)} \left[ p^2 + |k|^2 q_k(p) \right]^{-1} e_k \otimes e_k \equiv \sum_{k=1}^{n} g_k(p) e_k \otimes e_k \]

where
\[ g_k(p) := \frac{\rho^{d/4-1/2} p^{d/2-1}}{(2\pi)^{d/2} q_k(p)^{d/4+1/2}} r^{-(d/2-1)} K_{d/2-1}(B_k(p) r) \]

and $B_k(p) := \rho^{1/2} p / q_k(p)^{1/2}, k = 1, \ldots, n$. Assume for definiteness that $d \leq 3$.

The argument of Sec. [3] now leads to the conclusion that the functions $g_k, k = 1, \ldots, n$, are CM, hence the function $\tilde{G}(\cdot, x)$ is a matrix-valued CM function and therefore the Green function $G(t, x)$ is positive semi-definite for $t \geq 0, x \in \mathbb{R}^d$. In particular, we have the following theorem:
Theorem 6.8 Let $\rho \in \mathbb{R}_+, d \leq 3$, $s(t, x) = \delta(t) \delta(x) w$, where $w \in \mathbb{R}^n$.

If $G(s) = \sum_{k=1}^{n} G_k(s) e_k \otimes e_k$ and $A = \sum_{k=1}^{n} a_k e_k \otimes e_k$ with CM functions $G_k$ and real numbers $a_k \geq 0$, $k = 1, \ldots, n$, then the solution $u$ of the problem

$$\rho D^2 u = A D u + G * \nabla^2 D u + s(t, x), \quad t \geq 0, \quad x \in \mathbb{R}^d$$

satisfies the inequality

$$\langle u(t, x), w \rangle \geq 0, \quad t \geq 0, \quad x \in \mathbb{R}^d. \quad (25)$$

7 Positivity in isotropic viscoelasticity.

Consider now the Green’s function $G$ of a 3D isotropic viscoelastic medium. The function $G$ is the solution of the initial-value problem:

$$\rho D^2 G_{kr}(t, x) = G(t)_{klmn} * D G_{mr,nt} + \delta(t) \delta(x) \delta_{kr},$$

$$t > 0, \quad x \in \mathbb{R}^3, \quad k, r = 1, 2, 3 \quad (26)$$

with zero initial conditions, and

$$G_{klmn}(t) = \lambda(t) \delta_{kl} \delta_{mn} + \mu(t) \delta_{km} \delta_{ln} + \mu(t) \delta_{kn} \delta_{lm} \quad (27)$$

where the kernels $\lambda(t), \mu(t)$ are CM and $\rho \in \mathbb{R}_+$. The function $G$ with the components $G_{klmn}$ takes values in the linear space $\mathcal{S}$ of symmetric operators on the space $S$ of symmetric $3 \times 3$ matrices. It is easy to see that under our hypotheses this function is CM:

$$(-1)^n \langle e_1, G(t) e_2 \rangle \geq 0 \quad \text{for all } n = 0, 1, 2 \ldots$$

for every $e_1, e_2 \in S$, where $\langle v, w \rangle := v_{kl} w_{kl}$ is the inner product on $S$.

The Laplace transform $\tilde{G}$ of $G$ is given by the formula

$$G(p, x) = \frac{1}{p \rho} \left\{ \nabla \otimes \nabla \Delta^{-1} F_L(p, |x|) + [I - \nabla \otimes \nabla \Delta^{-1}] F_T(p, |x|) \right\}$$

where $\Delta := \nabla^2$,

$$F_L(p, r) := \frac{s_L(p)^2}{4\pi r} e^{-p^{1/2} s_L(p) r}$$

$$F_T(p, r) := \frac{s_T(p)^2}{4\pi r} e^{-p^{1/2} s_T(p) r}$$

and

$$s_L(p)^2 := \frac{\rho}{\lambda(p) + 2\mu(p)}$$

$$s_T(p)^2 := \frac{\rho}{\mu(p)}$$

(30)  (31)
Since \( q_L(p) = p/s_L(p) \) and \( q_T(p) = p/s_T(p) \) are CBFs, the functions \( p/q_L(p)^{1/2} = p^{1/2} s_L(p) \) and \( p/q_T(p)^{1/2} = p^{1/2} s_T(p) \) are BFs. Hence the exponentials in eqs (28–29) are CM functions of \( p \). Moreover the functions \( s_L(p)^2 \) and \( s_T(p)^2 \) are CM. It follows that the functions \( F_L(p, r) \) and \( F_T(p, r) \) are CM and therefore they are Laplace transforms of non-negative functions \( F_L(t, r) \) and \( F_T(t, r) \).

Their indefinite integrals \( f_L(t, r) := \int_0^t F_L(s, r) \, ds \) and \( f_T(t, r) := \int_0^t F_T(s, r) \, ds \) are also non-negative. The functions \( h_L(t, r) := \Delta^{-1} f_L(t, r) \, ds \), \( h_T(t, r) = \Delta^{-1} f_T(t, r) \) involve a convolution with a non-negative kernel and therefore are non-negative.

The Green’s function can be expressed in terms of these functions:

\[
G(t, x) = \frac{1}{\rho} \{ \nabla \cdot \nabla h_L(t, |x|) + [\Delta \mathbf{I} - \nabla \otimes \nabla] h_T(t, |x|) \} \tag{32}
\]

We shall use the notation \( v \geq 0 \) if \( v_k \geq 0 \) for \( k = 1, 2, 3 \).

**Theorem 7.1** Let \( u = \nabla \phi + \nabla \times \psi \) be the solution of the initial-value problem

\[
\rho D^2 u = G \ast \nabla^2 D u + s(t, x), \quad t \geq 0, \quad x \in \mathbb{R}^d \tag{33}
\]

with \( u(0, x) = 0 = Du(0, x) \) and \( s(t, x) = \nabla f(t, x) + \nabla \times g(t, x) \).

Then \( \nabla f(t, x) \geq 0 \) for all \( t \geq 0, \ x \in \mathbb{R}^3 \) implies that \( \nabla \phi(t, x) \geq 0 \) for all \( t \geq 0, \ x \in \mathbb{R}^3 \).

Similarly, \( \nabla \times g(t, x) \geq 0 \) for all \( t \geq 0, \ x \in \mathbb{R}^3 \) implies that \( \nabla \times \psi(t, x) \geq 0 \) for all \( t \geq 0, \ x \in \mathbb{R}^3 \).

Proof. Substitute \( u = \nabla \phi + \nabla \times \psi, \ s = \nabla f + \nabla \times g \) in the formula

\[
u(t, x) = \int_0^t \int \mathcal{G}(t - s, x - y) s(s, y) \, d_3 y \, d_3 x \, ds
\]

where \( \mathcal{G} \) is given by (32). Noting that \( \Delta^{-1} \) is a convolution operator commuting with \( \nabla \) and \( \nabla \Delta^{-1} \) \( \text{div} \) \( s = \nabla f \) we have

\[
\nabla \phi(t, x) = \frac{1}{\rho} \int f_L(t - s, |x - y|) (\nabla f)(s, y) \, d_3 y
\]

We now note that \( [\mathbf{I} - \Delta^{-1} \nabla \otimes \nabla] s = s - \nabla f = \nabla \times g \). Hence

\[
\nabla \times \psi(t, x) = \frac{1}{\rho} \int f_T(t - s, |x - y|) \nabla \times g(s, y) \, d_3 y
\]

The functions \( f_L \) and \( f_T \) are non-negative, hence the thesis follows.

\[ \square \]

### 8 Concluding remarks.

A non-negative source term excites a non-negative viscoelastic pulse. This result holds for scalar waves and for scalar potentials under the usual assumption that
the stress response is determined by a CM relaxation modulus $G$ or by a Newtonian term or both connected in parallel. The CM property of the relaxation modulus is a fairly general property of real viscoelastic media, equivalent to the assumption that the relaxation spectrum is non-negative. A generalization of positivity for vector-valued viscoelastic fields in viscoelastic media with the P class anisotropy (Hanyga, 2003) is sketched.

A particular example of a CBF is the rational function $F(p) = R(p)/S(p)$, where $R$ and $S$ are two polynomials with simple negative roots $\lambda_k$, $k = 1, \ldots, N$, $\mu_l$, $l = 1, \ldots, M$, $M = N$ or $N + 1$ satisfying the intertwining conditions:

$$0 \leq \lambda_1 < \mu_1 < \ldots < \mu_N < \lambda_{N+1}$$

(the last inequality is applicable only if $M = N + 1$) (Duff, 1969). A more general CBF is obtained by substituting in $F$ the CBF $p^\alpha$, with $0 < \alpha < 1$:

$$F_\alpha(p) = R(p^\alpha)/S(p^\alpha)$$

(Theorem 3.13). The choice of $Q = F_\alpha$ corresponds to a generalized Cole-Cole model of relaxation. For $N = M = 1$ the original Cole-Cole model (Cole & Cole, 1941; Bagley & Torvik, 1983) is recovered.

Anisotropic effects can be introduced by replacing the operator $\nabla^2$ by $g^{kl} \partial_k \partial_l$. If $h_{kl} g^{lm} = \delta_m^k$ then

$$\tilde{u}_d(p, x) = \sqrt{\det g} \frac{\rho^{d/4-1/2} g(p) \rho^{d/2-1} r^{-(d/2-1)} K_{d/2-1} \left(\rho^{1/2} pr/Q(p)^{1/2}\right)}{(2\pi)^d/2 Q(p)^{d/4+1/2}}$$

(34)

If $Q$ is a CBF then $u(t, x) \geq 0$. with $r := [h_{kl} x^k x^l]^{1/2}$, cf Gel’fand & Shilov (1964).

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15