Research Article

A Classification of a Totally Umbilical Slant Submanifold of Cosymplectic Manifolds

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1. Introduction

The study of slant submanifolds in complex spaces was initiated by Chen as a natural generalization of both holomorphic and totally real submanifolds [1, 2]. Since then, many research papers have appeared concerning the existence of these submanifolds as well as on the geometry of the existent slant submanifolds in different known spaces (cf. [3, 4]). The slant submanifolds of an almost contact metric manifold were defined and studied by Lotta [4]. Later on, these submanifolds were studied by Cabrerozo et al. in the setting of Sasakian manifolds [3].

Recently, Şahin proved that a totally umbilical proper slant submanifold of a Kaehler manifold is totally geodesic [5]. Our aim in the present paper is to investigate slant submanifolds in contact manifolds. Thus, we study slant submanifolds of a cosymplectic manifold. We have shown that a totally umbilical slant submanifold \( M \) of a cosymplectic manifold \( M \) is either an anti-invariant submanifold or the dim \( M = 1 \) or the mean curvature vector \( H \in \Gamma(\mu) \), and then we have obtained an interesting result for a totally umbilical proper slant submanifold of a cosymplectic manifold.
2. Preliminaries

Let \( \bar{M} \) be a \((2n + 1)\)-dimensional manifold with \((1,1)\) tensor field \( \phi \) satisfying \([6]\):

\[
\phi^2 = -I + \eta \otimes \xi,
\]

where \( I \) is the identity transformation, \( \xi \) a vector field, and \( \eta \) a 1-form on \( \bar{M} \) satisfying \( \phi \xi = \eta \circ \phi = 0 \) and \( \eta(\xi) = 1 \). Then \( \bar{M} \) is said to have an almost contact structure. There always exists a Riemannian metric \( g \) on \( \bar{M} \) such that

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),
\]

for all vector fields \( X, Y \), on \( \bar{M} \). From (2.2), it is easy to observe that

\[
g(\phi X, Y) + g(X, \phi Y) = 0.
\]

The fundamental 2-form \( \Phi \) is defined as: \( \Phi(X, Y) = g(X, \phi Y) \). If \( [\phi, \phi] + d\eta \otimes \xi = 0 \), then the almost contact structure is said to be normal, where \( [\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] \). If \( \Phi = d\eta \), the almost contact structure is a contact structure. A normal almost contact structure such that \( \Phi \) is closed and \( d\eta = 0 \) is called cosymplectic structure. It is well known \([7]\) that the cosymplectic structure is characterized by

\[
\left( \bar{\nabla}_X \phi \right) Y = 0, \quad \left( \bar{\nabla}_X \eta \right) Y = 0,
\]

for all vector fields \( X, Y \), on \( \bar{M} \), where \( \bar{\nabla} \) is the Levi-Civita connection of \( g \). From the formula \( \bar{\nabla}_X \phi = 0 \), it follows that \( \bar{\nabla}_X \xi = 0 \).

Let \( M \) be submanifold of an almost contact metric manifold \( \bar{M} \) with induced metric \( g \) and let \( \nabla \) and \( \nabla^\perp \) be the induced connections on the tangent bundle \( TM \) and the normal bundle \( T^\perp M \) of \( M \), respectively. Denote by \( \mathcal{F}(M) \) the algebra of smooth functions on \( M \) and by \( \Gamma(TM) \) the \( \mathcal{F}(M) \)-module of smooth sections of a vector bundle \( TM \) over \( M \), then Gauss and Weingarten formulae are given by

\[
\bar{\nabla}_X Y = \nabla X Y + h(X, Y),
\]

\[
\bar{\nabla}_X N = -A_N X + \nabla^\perp_X N,
\]

for each \( X, Y \in \Gamma(TM) \) and \( N \in \Gamma(T^\perp M) \), where \( h \) and \( A_N \) are the second fundamental form and the shape operator (corresponding to the normal vector field \( N \)), respectively for the immersion of \( M \) into \( \bar{M} \). They are related as

\[
g(h(X, Y), N) = g(A_N X, Y),
\]

where \( g \) denotes the Riemannian metric on \( \bar{M} \) as well as the one induced on \( M \) \([8]\).
Abstract and Applied Analysis

For any $X \in \Gamma(TM)$, we write

$$\phi X = PX + FX,$$

(2.8)

where $PX$ is the tangential component and $FX$ is the normal component of $\phi X$. Similarly for any $N \in \Gamma(T^\perp M)$, we write

$$\phi N = tN + fN,$$

(2.9)

where $tN$ is the tangential component and $tN$ is the normal component of $\phi N$. If we denote the orthogonal complementary distribution of $F(TM)$ in $T^\perp M$ by $\mu$, then we have the direct sum

$$T^\perp M = F(TM) \oplus \mu.$$

(2.10)

We can see that $\mu$ is an invariant subbundle with respect to $\phi$. Furthermore, the covariant derivatives of the tensor fields $P$ and $F$ are defined as

$$\left(\nabla_X P\right)Y = \nabla_X PY - P\nabla_X Y,$$

$$\left(\nabla_X F\right)Y = \nabla^\perp_X FY - F\nabla_X Y,$$

(2.11)

for any $X, Y \in \Gamma(TM)$.

A submanifold $M$ is said to be invariant if $F$ is identically zero, that is, $\phi X \in \Gamma(TM)$ for any $X \in \Gamma(TM)$. On the other hand, $M$ is said to be anti-invariant if $P$ is identically zero, that is, $\phi X \in \Gamma(T^\perp M)$, for any $X \in \Gamma(TM)$.

A submanifold $M$ of an almost contact metric manifold $\overline{M}$ is called totally umbilical if

$$h(X, Y) = g(X, Y)H,$$

(2.12)

for any $X, Y \in \Gamma(TM)$. The mean curvature vector $H$ is given by

$$H = \sum_{i=1}^{m} h(e_i, e_i),$$

(2.13)

where $m$ is the dimension of $M$ and $\{e_1, e_2, \ldots, e_m\}$ is the local orthonormal frame on $M$. A submanifold $M$ is said to be totally geodesic if $h(X, Y) = 0$ for each $X, Y \in \Gamma(TM)$ and is minimal if $H = 0$ on $M$.

3. Slant Submanifolds

Throughout the section, we assume that $M$ is a slant submanifold of a cosymplectic manifold $\overline{M}$. We always consider such submanifold tangent to the structure vector field $\xi$. For each
nonzero vector $X$ tangent to $M$ at $x$, we denote by $0 \leq \theta(X) \leq \pi/2$, the angle between $\phi X$ and $T_x M$, known as the Wirtinger angle of $X$. If the Wirtinger angle $\theta(X)$ is constant, that is, independent of the choice of $x \in M$ and $X \in T_x M - \{\xi\}$, then $M$ is said to be a slant submanifold [4]. In this case the constant angle $\theta$ is called slant angle of the slant submanifold. Obviously if $\theta = 0$, $M$ is invariant and if $\theta = \pi/2$, $M$ is an anti-invariant submanifold. A slant submanifold is said to be proper slant if it is neither invariant nor anti-invariant submanifold.

If $M$ is a slant submanifold of an almost contact metric manifold, then the tangent bundle $TM$ is decomposed as

$$TM = \mathcal{D} \oplus \langle \xi \rangle,$$

(3.1)

where $\langle \xi \rangle$ denotes the distribution spanned by the structure vector field $\xi$ and $\mathcal{D}$ is the complementary distribution of $\langle \xi \rangle$ in $TM$, known as the slant distribution.

We recall the following result for a slant submanifold.

**Theorem 3.1** (see [3]). Let $M$ be a submanifold of an almost contact metric manifold $\overline{M}$, such that $\xi \in T_x M$. Then, $M$ is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that

$$P^2 = \lambda (-I + \eta \otimes \xi).$$

(3.2)

Furthermore, if $\theta$ is slant angle, then $\lambda = \cos^2 \theta$.

The following relations are straightforward consequence of (3.2):

$$g(PX, PY) = \cos^2 \theta [g(X, Y) - \eta(X) \eta(Y)],$$

(3.3)

$$g(FX, FY) = \sin^2 \theta [g(X, Y) - \eta(X) \eta(Y)],$$

(3.4)

for any $X, Y$ tangent to $M$.

Now, we prove the following.

**Theorem 3.2.** Let $M$ be a totally umbilical slant submanifold of a cosymplectic manifold $\overline{M}$. Then at least one of the following statements is true:

(i) $M$ is an anti-invariant submanifold;

(ii) $M$ is a 1-dimensional submanifold;

(iii) If $M$ is a proper slant submanifold, then $H \in \Gamma(\mu)$,

where $H$ is the mean curvature vector of the submanifold $M$.

**Proof.** Let $M$ be a totally umbilical slant submanifold of a cosymplectic manifold $\overline{M}$, then for any $X, Y \in \Gamma(TM)$, we have

$$h(PX, PY) = g(PX, PY) H.$$  

(3.5)

From (2.5) and (3.3), we deduce that

$$\nabla^\perp_{PX} PX - \nabla_{PX} PX = \cos^2 \theta \{g(X, X) - \eta(X) \eta(X)\} H.$$  

(3.6)
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Using (2.8) and the fact that \( M \) is cosymplectic we obtain that

\[
\phi \nabla_{PX} X - \nabla_{PX} FX - \nabla_{PX} PX = \cos^2 \theta \left\{ \|X\|^2 - \eta^2(X) \right\} H. \tag{3.7}
\]

Then from (2.5) and (2.6), we get

\[
\phi \nabla_{PX} X + \phi h(X, PX) + A_{FX} PX - \nabla_{PX}^\perp FX - \nabla_{PX} PX = \cos^2 \theta \left\{ \|X\|^2 - \eta^2(X) \right\} H. \tag{3.8}
\]

Thus by (2.8) and (2.12), we obtain

\[
P \nabla_{PX} X + F \nabla_{PX} X + g(PX, X)\phi H + A_{FX} PX - \nabla_{PX}^\perp FX - \nabla_{PX} PX = \cos^2 \theta \left\{ \|X\|^2 - \eta^2(X) \right\} H. \tag{3.9}
\]

Equating the normal components, we get

\[
F \nabla_{PX} X - \nabla_{PX}^\perp FX = \cos^2 \theta \left\{ \|X\|^2 - \eta^2(X) \right\} H. \tag{3.10}
\]

On the other hand, from (3.4), we have

\[
g(FX, FX) = \sin^2 \theta \left\{ g(X, X) - \eta(X)\eta(X) \right\}, \tag{3.11}
\]

for any \( X \in \Gamma(TM) \). Taking the covariant derivative of the above equation with respect to \( PX \), we obtain

\[
2g \left( \nabla_{PX} FX, FX \right) = 2\sin^2 \theta g \left( \nabla_{PX} X, X \right) - 2\sin^2 \theta \eta(X)g \left( \nabla_{PX} X, \xi \right) - 2\sin^2 \theta \eta(X)g \left( X, \nabla_{PX} \xi \right). \tag{3.12}
\]

Using the property of metric connection \( \nabla \), the last two terms of the right-hand side are cancelling each other, thus we have

\[
g \left( \nabla_{PX} FX, FX \right) = \sin^2 \theta g \left( \nabla_{PX} X, X \right). \tag{3.13}
\]

Then by (2.5) and (2.6), we derive

\[
g \left( \nabla_{PX}^\perp FX, FX \right) = \sin^2 \theta g(\nabla_{PX} X, X). \tag{3.14}
\]

Now, taking the inner product in (3.10) with \( FX \), for any \( X \in \Gamma(TM) \), then

\[
g(F \nabla_{PX} X, FX) - g \left( \nabla_{PX}^\perp FX, FX \right) = \cos^2 \theta \left\{ \|X\|^2 - \eta^2(X) \right\} g(H, FX). \tag{3.15}
\]
Then from (3.4) and (3.14), we obtain

\[-\sin^2 \theta \eta(X) \eta(\nabla_{PX} X) = \cos^2 \theta \left\{ \|X\|^2 - \eta^2(X) \right\} g(H, FX), \quad (3.16)\]

or

\[-\sin^2 \theta \eta(X) g(\nabla_{PX} X, \xi) = \cos^2 \theta \left\{ \|X\|^2 - \eta^2(X) \right\} g(H, FX). \quad (3.17)\]

Using (2.5), we derive

\[-\sin^2 \theta \eta(X) g\left( \nabla_{PX} X, \xi \right) = \cos^2 \theta \left\{ \|X\|^2 - \eta^2(X) \right\} g(H, FX). \quad (3.18)\]

Since \( \nabla \) is the metric connection, then the above equation can be written as

\[\sin^2 \theta \eta(X) g\left( X, \nabla_{PX} \xi \right) = \cos^2 \theta \left\{ \|X\|^2 - \eta^2(X) \right\} g(H, FX). \quad (3.19)\]

As \( \overline{M} \) is cosymplectic thus using the fact that \( \nabla_{PX} \xi = 0 \), the left hand side of the above equation vanishes identically, then

\[\cos^2 \theta \left\{ \|X\|^2 - \eta^2(X) \right\} g(H, FX) = 0. \quad (3.20)\]

Thus from (3.20), it follows that either \( \theta = \pi/2 \) or \( X = \xi \) or \( H \in \Gamma(\mu) \), where \( \mu \) is the invariant normal subbundle orthogonal to \( FTM \). This completes the proof. \( \Box \)

**Theorem 3.3.** Every totally umbilical proper slant submanifold \( M \) of a cosymplectic manifold \( \overline{M} \) is totally geodesic, provided \( \nabla_{\xi} H \in \Gamma(\mu) \), for any \( X \in TM \).

**Proof.** As \( \overline{M} \) is cosymplectic, then we have

\[\overline{\nabla}_{U} \phi V = \phi \overline{\nabla}_{U} V, \quad (3.21)\]

for any \( U, V \in \Gamma(T \overline{M}) \). Using this fact and formulae (2.5) and (2.8) we obtain that

\[\overline{\nabla}_{X} PY + \overline{\nabla}_{X} FY = P\nabla_{X} Y + F\nabla_{X} Y + \phi h(X, Y), \quad (3.22)\]

for any \( X, Y \in \Gamma(TM) \). Then from (2.5), (2.6) and (2.12), we get

\[\nabla_{X} PY + h(X, PY) - A_{Y} X + \nabla_{X} FY = P\nabla_{X} Y + F\nabla_{X} Y + g(X, Y)\phi H. \quad (3.23)\]
Taking the inner product in (3.23) with $\phi H$ and using the fact that $H \in \Gamma(\mu)$ (by Theorem 3.2), we obtain
\[ g\left(h(X, PY), \phi H\right) + g\left(\nabla_X^1 FY, \phi H\right) = g(X, Y)g(\phi H, \phi H). \tag{3.24} \]

Then from (2.2) and (2.12), we derive
\[ g(X, PY)g(H, \phi H) + g\left(\nabla_X^1 FY, \phi H\right) = g(X, Y)g(H, H). \tag{3.25} \]

That is,
\[ g\left(\nabla_X^1 FY, \phi H\right) = g(X, Y)\|H\|^2. \tag{3.26} \]

Now, we consider
\[ \nabla_X \phi H = \phi \nabla_X H, \tag{3.27} \]
for any $X \in \Gamma(TM)$. From (2.6), we obtain
\[ -A_{\phi H}X + \nabla_X^1 \phi H = \phi \left(-A_{H}X + \nabla_X^1 H\right). \tag{3.28} \]

Thus, on using (2.8), (2.9), we get
\[ -A_{\phi H}X + \nabla_X^1 \phi H = -PAH X - FAH X + t\nabla_X^1 H + f\nabla_X^1 H. \tag{3.29} \]

Taking the inner product with $FY$, for any $Y \in \Gamma(TM)$, then
\[ g\left(\nabla_X^1 \phi H, FY\right) = -g(FAH X, FY) + g\left(f\nabla_X^1 H, FY\right). \tag{3.30} \]

Since $f\nabla_X^1 H \in \Gamma(\mu)$, then by (3.4) the above equation takes the form
\[ g\left(\nabla_X^1 \phi H, FY\right) = -\sin^2 \theta \{ g(X, Y) - \eta(X) \eta(Y) \}. \tag{3.31} \]

Using (2.6), (2.7), and (2.12), we get
\[ g\left(\nabla_X H, FY\right) = -\sin^2 \theta \{ g(X, Y) - \eta(X) \eta(Y) \}\|H\|^2. \tag{3.32} \]

The above equation can be written as
\[ g\left(\nabla_X FY, \phi H\right) = \sin^2 \theta \{ g(X, Y) - \eta(X) \eta(Y) \}\|H\|^2. \tag{3.33} \]
Again using the fact that $H \in \Gamma(\mu)$, then by (2.6), we obtain

$$g\left(\nabla_XFY,\phi H\right) = \sin^2\theta \{g(X,Y) - \eta(X)\eta(Y)\}\|H\|^2.$$  \hspace{1cm} (3.34)

From (3.26) and (3.34), we derive

$$\left\{\cos^2\theta g(X,Y) + \sin^2\theta \eta(X)\eta(Y)\right\}\|H\|^2 = 0.$$ \hspace{1cm} (3.35)

Thus, (3.35) implies either $H = 0$ or $\theta = \tan^{-1}(\sqrt{-g(X,Y)/\eta(X)\eta(Y)})$, which is not possible, because the slant angle $\theta \in (0, \pi/2)$. Hence, $M$ is totally geodesic in $\bar{M}$. \hfill \Box

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