A class of reductions of the two-component KP hierarchy and the Hirota-Ohta equation

L.V. Bogdanov $^1$ and Lingling Xue$^2$

$^1$Landau Institute for Theoretical Physics RAS, 142432 Chernogolovka, Russia
$^2$Department of Mathematics, Ningbo University, Ningbo 315211, P.R. China

Abstract

We introduce a class of reductions of the two-component KP hierarchy, which includes the Hirota-Ohta equation hierarchy. The description of the reduced hierarchies is based on the Hirota bilinear identity and an extra bilinear relation characterising the reduction. We derive the reduction conditions in terms of the Lax operator and higher linear operators of the hierarchy, as well as in terms of the basic two-component KP system of equations.

1 Introduction

In this work we introduce a class of reductions of the two-component KP hierarchy, which includes the Hirota-Ohta equation hierarchy [1], [2] as the zero order reduction. In the scalar case such a class was introduced in [3], where it was demonstrated that the lowest order reductions engender the CKP and BKP hierarchies. In the two-component case similar approach, but for another type of symmetry, was developed in [4]. Our starting point is the $\partial$-dressing scheme, for which the definition of the class of reduction is rather transparent [5], and which can be used to construct a big stock of explicit solutions. However, the algebraic description of the reduced hierarchies is based on the Hirota bilinear identity and an extra bilinear relation characterising the reduction, and it doesn’t necessarily require the dressing scheme. We derive the reduction conditions in terms of the Lax operator and higher linear operators of the hierarchy. The basic system of the two-component KP hierarchy with additional symmetry constraint for the dynamics defining a special set of times is a closed system of equations with three independent variables $x$, $y$, $t$ for six scalar functions. Each reduction of the class gives a set of three differential relations containing derivatives with respect to $x$, $y$ for these functions, and a pair of different reductions produces a closed system of (1+1)-dimensional equations.
2 Nonlocal $\bar{\partial}$ problem and Hirota bilinear identity

First we recall a general setting to consider multicomponent KP hierarchy in the framework of the $\bar{\partial}$-dressing method. We start from a pair of adjoint canonically normalised matrix $\bar{\partial}$-problems

$$\frac{\partial}{\partial \lambda} \chi(\lambda, t) = \int_C d\nu \wedge d \bar{\nu} \chi(\nu) g(\nu, t) R(\nu, \lambda) g^{-1}(\lambda, t),$$

$$\frac{\partial}{\partial \lambda} \tilde{\chi}(\lambda, t) = -\int_C d\nu \wedge d \bar{\nu} g(\lambda, t) R(\lambda, \nu) g^{-1}(\nu; t) \tilde{\chi}(\nu, t).$$

(1)

We choose the following parametrization of the multicomponent loop group $\Gamma^+ N$ defining the dynamics of multicomponent KP hierarchy:

$$g(\lambda, t) = \exp \left( \sum_{\alpha=1}^N \sum_{n=1}^\infty P_{\alpha} \lambda^n t_n^{(\alpha)} \right),$$

(2)

where the projection matrices $P_{\alpha}$ form a basis of the commutative subalgebra of diagonal matrices,

$$(P_{\alpha})_{\beta \gamma} = \delta_{\alpha \beta} \delta_{\beta \gamma} \quad (\alpha, \beta, \gamma = 1, ..., N).$$

So we have $N$ infinite series of dynamical variables $t_{(\alpha) n}$.

The kernel $R(\lambda, \nu)$ is supposed to be equal to zero in some neighbourhood of infinity for both variables $\lambda, \mu$, for simplicity we suggest that the support of the kernel belongs to the product of unit disks. Then the functions $\chi(\lambda, t), \tilde{\chi}(\lambda, t)$ are analytic outside the unit disc, at infinity $\chi(\lambda, t) = I + \sum_{n=1}^\infty \chi_n(t) \lambda^{-n}, \tilde{\chi}(\lambda, t) = I + \sum_{n=1}^\infty \tilde{\chi}_n(t) \lambda^{-n}$ The problems (1) imply Hirota bilinear identity on the unit circle $S$

$$\oint S \chi(\nu; t) g(\nu, t) g^{-1}(\nu, t') \tilde{\chi}(\nu; t') d\nu = 0$$

(3)

In a more familiar form, for the Baker-Akhieser functions $\psi(\lambda; g) = \chi(\lambda) g(\lambda), \tilde{\psi}(\lambda; g) = g^{-1}(\lambda) \tilde{\chi}(\lambda)$, we have

$$\oint S \psi(\nu; t) \tilde{\psi}(\nu; t') d\nu = 0.$$  

(4)

We will also use the Cauchy-Baker-Akhieser function (kernel), defined by nonlocal $\bar{\partial}$-problems (1) with pole normalisation $(\lambda - \mu)^{-1}$ [6],

$$\frac{\partial}{\partial \lambda} \chi(\lambda, \mu; t) = 2\pi i (\lambda - \mu) + \int_C d\nu \wedge d \bar{\nu} \chi(\nu, \mu; t) g(\nu; t) R(\nu, \lambda) g^{-1}(\lambda; t),$$

$$\frac{\partial}{\partial \lambda} \tilde{\chi}(\lambda, \mu; t) = 2\pi i (\lambda - \mu) - \int_C d\nu \wedge d \bar{\nu} g(\lambda; t) R(\lambda, \nu) g^{-1}(\nu; t) \tilde{\chi}(\nu, \mu; t).$$

(5)
After simple calculations we obtain
\[ \oint \chi(\nu, \lambda; t)g(\nu; t)g^{-1}(\nu; t')\tilde{\chi}(\nu, \mu; t')d\nu = 0. \] (6)

It follows from (6) taken for \( t = t' \) that outside the unit disk with respect to both variables the function \( \chi(\lambda, \mu) \) is equal to \( -\tilde{\chi}(\mu, \lambda) \), so in fact this identity should be written for one function,
\[ \oint \chi(\nu, \lambda; t)g(\nu; t)g^{-1}(\nu; t')\chi(\mu, \nu; t')d\nu = 0. \] (7)

By similar calculations it is possible to prove that if both problems (5) are solvable, \( \chi(\lambda, \mu) = -\tilde{\chi}(\lambda, \mu) \) for all \( \lambda, \mu \) where they are commonly defined.

Taking \( \lambda \to \infty, \mu \to \infty \), we reproduce identity (3) for \( \chi(\lambda; t) = \chi(\lambda, \infty; t), \tilde{\chi}(\lambda; t) = -\chi(\infty, \lambda; t) \).

In terms of the Cauchy-Baker-Akhieser (CBA) function \( \Psi(\lambda, \mu; t) = g^{-1}(\mu, t)\chi(\lambda, \mu; t)g(\lambda, t) \)
the Hirota bilinear identity reads
\[ \oint \Psi(\nu, \lambda; t)\Psi(\mu, \nu; t')d\nu = 0. \] (8)

3 A class of reductions of the two-component KP hierarchy

For the two-component KP hierarchy
\[ g(\lambda, t) = \exp \left( \sum_{n=1}^{\infty} (P_1 \lambda^n t^{(1)}_n + P_2 \lambda^n t^{(2)}_n) \right), \] (9)

We consider a class of reductions
\[ R^T(-\lambda, -\mu)A\lambda^k = A\lambda^k JR(\mu, \lambda)J^{-1}, \] (10)

where \( J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), the matrix \( A \) is equal to \( I \) or \( \sigma_3 \), so it commutes or anticommutes with \( J \). This class of reduction requires an involution
\[ g(-\lambda, t) = Jg(\lambda, t)^{-1}J^{-1}, \] (11)

so the reduction condition is compatible with the dynamics only if \( t^{(1)}_{2n+1} = t^{(2)}_{2n+1} \) (for odd times) and \( t^{(1)}_{2n} = -t^{(2)}_{2n} \) (for even times). We introduce a new set of times \( t_n \), for even order \( t_{2k} = t^{(1)}_{2k} = -t^{(2)}_{2k} \) and for odd order \( t_{2k-1} = t^{(1)}_{2k-1} = t^{(2)}_{2k-1} \). The factor defining the dynamics of the kernel looks like
\[ g(\lambda, t) = \exp \left( \sum_{n=1}^{\infty} (I\lambda^{2n-1}t_{2n-1} + \sigma_3\lambda^{2n}t_{2n}) \right). \] (12)
For the first three times we will use the notations $x = t_1$, $y = t_2$, $t = t_3$.

In terms of the Baker-Akhiezer function the reduction (10) is characterised by an extra bilinear relation

$$\oint \psi(\nu; t) J A \lambda^k \psi^T(-\nu; t') d\nu = 0,$$

(13)

for the CBA function we have

$$\oint \Psi(\nu, \lambda; t) J A \lambda^k \Psi^T(-\nu, -\mu; t') d\nu = 0.$$

(14)

Reduction condition corresponding to the Hirota-Ohta hierarchy is given by (10) with $A = I$, $n = 0$,

$$R^T(-\lambda, -\mu) = JR(\mu, \lambda)J^{-1},$$

for the Baker-Akhiezer functions we have a condition

$$\tilde{\psi}(\lambda; t) = -J \psi^T(-\lambda; t)J,$$

(15)

and identity (13) reads

$$\oint \psi(\nu; t) J \psi^T(-\nu; t') d\nu = 0.$$

(16)

In terms of the the CBA function the reduction condition is

$$\psi^T(-\lambda, -\mu) = J \psi(\mu, \lambda)J^{-1}.$$

4 Two-component KP hierarchy

Having in mind the class of reductions (10), we will first derive linear problems and equations for the two-component KP hierarchy case with the involution (11), and then we will consider reduction conditions in terms of linear operators and equations.

Let us start with Hirota bilinear identity (6), (7) with the dependence on times defined by (12).

Linear operators

The action of operators $\partial_0 = \partial / \partial t_0$ on $\psi$, $\psi^* = \tilde{\psi}^T$ corresponds to the following operators (the Manakov operators) acting on $\chi$, $\chi^*$

$$D_{t_{2n-1}} \chi = \partial_{t_{2n-1}} \chi + \lambda^{2n-1} \chi, \quad D^*_{t_{2n-1}} \chi^* = \partial_{t_{2n-1}} \chi^* - \lambda^{2n-1} \chi^*;$$

$$D_{t_{2n}} \chi = \partial_{t_{2n}} \chi + \lambda^{2n} \chi \sigma_3, \quad D^*_{t_{2n}} \chi^* = \partial_{t_{2n}} \chi^* - \lambda^{2n} \chi^* \sigma_3;$$

4
Hirota bilinear identity implies that some differential operator \( \sum_{n,m} u_m^{(n)} \partial_x^n \partial_y^m \) acting on \( \psi \) gives zero iff for respective Manakov operator the result of action on \( \chi \) has zero projection to nonnegative powers of \( \lambda \),

\[
\left( \sum_{n,m} u_m^{(n)} D_n^m \chi \right)_+ = 0 \iff \sum_{n,m} u_m^{(n)} D_n^m \chi = 0 \iff \sum_{n,m} u_m^{(n)} \partial_x^n \partial_y^m \psi = 0
\]  

(17)

Using this observation, it is possible to construct linear operators of the hierarchy.

Let us start with the Lax operator. For the first three times \( x = t_1, y = t_2, t = t_3 \) the Manakov operators look like

\[
D_x \chi = \partial_x \chi + \lambda \chi, \quad D_y \chi = \partial_y \chi + \lambda \chi, \quad D_t \chi = \partial_t \chi + \lambda \chi,
\]

\[
D_x^* \chi^* = \partial_x \chi^* - \lambda \chi^*, \quad D_y^* \chi^* = \partial_y \chi^* + \lambda \chi^*, \quad D_t^* \chi^* = \partial_t \chi^* + \lambda \chi^*,
\]

Using (17), we obtain

\[
(D_y - \sigma_3 D_x^2) \chi = [\chi_1, \sigma_3] D_x + ([\chi_2, \sigma_3] - 2 \sigma_3 \chi_1) - [\chi_1, \sigma_3] \chi_1),
\]

\[
(D_y^* + \sigma_3 D_x^2) \chi^* = [\chi_1^*, \sigma_3] D_x^* - ([\chi_2^*, \sigma_3] + 2 \sigma_3 \chi_1^*) + [\chi_1^*, \sigma_3] \chi_1).
\]

Thus for the Baker-Akhiezer functions

\[
\partial_y \psi = \left( \sigma_3 \partial_x^2 + \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} \partial_x + U \right) \psi, \tag{18}
\]

\[
\partial_y \psi^* = \left( -\sigma_3 \partial_x^2 + \begin{pmatrix} 0 & f^* \\ g^* & 0 \end{pmatrix} \partial_x + U^* \right) \psi^*,
\]

where

\[
\begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} = [\chi_1, \sigma_3], \quad \begin{pmatrix} 0 & f^* \\ g^* & 0 \end{pmatrix} = [\chi_1^*, \sigma_3],
\]

\[
U = [\chi_2, \sigma_3] - 2 \sigma_3 \chi_1 - [\chi_1, \sigma_3] \chi_1,
\]

\[
U^* = [\chi_2^*, \sigma_3] + 2 \sigma_3 \chi_1^* + [\chi_1^*, \sigma_3] \chi_1.
\]

From Hirota identity taken for equal times we get

\[
\oint \chi (\nu; t) \chi^T (\nu; t) d\nu = 0,
\]

(19)

then \( \chi_1^* = -\chi_1^T \) and \( f = f^*, \ g = g^* \). Differentiating identity (3) with respect to \( x \) and taking it for equal times, we get

\[
\oint \chi^* (\nu; t) (\partial_x + \nu) \chi^T (\nu; t) d\nu = 0,
\]

(20)

that implies

\[
\chi_2^T (t) = -\chi_2^T - \chi_1^T + \chi_1^T \chi_1^T.
\]
Using this relation it is easy to demonstrate that L-operators (18) are (anti)adjoint (defining \((V \partial_x)^* = -\partial_x V^T\)),

\[
\partial_y \psi = B_2 \psi, \quad \partial_y \psi^* = -B_2^* \psi^*,
\]

(21)

where the operator \(B_2\) is defined by \((18)\).

Linear operators corresponding to the time \(t\) read

\[
\partial_t \psi = B_3 \psi, \quad \partial_t \psi^* = -B_3^* \psi^*,
\]

(22)

where

\[
B_3 = \partial_3 x + 3W \partial_x + W_1,
W = -\chi_1x, \quad W_1 = 3\chi_1x_1 - 3\chi_2x - 3\chi_1xx.
\]

Higher linear operators can be written as

\[
\partial_t \psi = B_n \psi, \quad \partial_t \psi^* = -B_n^* \psi^*,
\]

(23)

where

\[
B_{2m} = \sigma_3 \partial_x^{2m} + \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} \partial_x^{2m-1} + \sum_{k=0}^{2n-2-2m} U^{(2m)}(x_k) \partial_x^k,
\]

\[
B_{2m+1} = \partial_x^{2m+1} + (2m + 1)W \partial_x^{2m-1} + \sum_{k=0}^{2n-2-2m} W_k^{(2m+1)} \partial_x^k
\]

The coefficients of these operators can be expressed through the coefficients of expansion of the function \(\chi(\lambda; t)\). The Lax operator \((18)\) written in terms of this function gives the recursion formulae, expressing the coefficients of expansion \(\chi(t)\) through the coefficients of the Lax operator \(f, g\) and \(U\) (six scalar functions). Indeed,

\[
\chi_y - \sigma_3 \chi_{xx} = -\lambda^2[\chi, \sigma_3] + 2\lambda \sigma_3 \chi_x \chi + \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} (\chi_x + \lambda \chi) + U \chi,
\]

and in terms of coefficients of expansion we have

\[
\chi_ky - \sigma_3 \chi_{kxx} = -[\chi_{k+2}, \sigma_3] + 2\lambda \sigma_3 \chi_{kx} \chi + \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} (\chi_{kx} + \chi_{k+1}) + U \chi_k.
\]

To have a correct structure of recursion, it is necessary to split this equation into diagonal and antidiagonal part, then we obtain

\[
2\chi_{k+2}^a = -2\chi_{k+1+1}^a + \begin{pmatrix} 0 & -f \\ g & 0 \end{pmatrix} (\chi_{k+1}^d + \chi_{kxx}^a) - \sigma_3(U \chi_k)^a + \sigma_3 \chi_{kyy}^a - \chi_{kxx}^a,
\]

(24)

\[
2\chi_{k+1+1}^d = \begin{pmatrix} 0 & -f \\ g & 0 \end{pmatrix} (\chi_{k+1}^a + \chi_{kxx}^a) - \sigma_3(U \chi_k)^d + \sigma_3 \chi_{kyy}^d - \chi_{kxx}^d.
\]

(25)
Let us write down several terms of the recursion explicitly:
anti-diagonal part, $k = -1$,

$$2\chi^a_1 = \begin{pmatrix} 0 & -f \\ g & 0 \end{pmatrix},$$

diagonal part, $k = 0$,

$$2\chi^d_1 = -\frac{1}{2} fgI - \sigma_3 U^d,$$

anti-diagonal part, $k = 0$,

$$2\chi^a_2 = -2\chi^a_1 + \left(\begin{array}{c} 0 \\ -g \end{array}\right)\chi^d_1 - \sigma_3 U^a,$$

diagonal part, $k = 1$,

$$2\chi^d_2 = \left(\begin{array}{c} 0 \\ -g \end{array}\right)\left(\chi^a_2 + \chi^a_1\right) - \sigma_3 (U\chi^a_1) + \sigma_3 \chi^d_1y - \chi^d_{1xx}.$$

**Equations**

Compatibility condition for linear equations (21), (22) is given by the Zakharov-Shabat equation

$$\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial t} = [B_2, B_3],$$

which engenders a closed system of equations for matrix coefficients of the operators $B_2, B_3$:

$$U_t - W_{1y} - U_{xx} + \sigma_3 W_{1xx} + \left(\begin{array}{c} 0 \\ g \end{array}\right) W_{1x} + [U, W_1] - 3W U_x = 0,$$

$$\left(\begin{array}{c} 0 \\ g_1 \end{array}\right)W_{1x} + 3\sigma_3 W_{xx} - 3W \left(\begin{array}{c} 0 \\ f_x \end{array}\right) + 3\left(\begin{array}{c} 0 \\ g \end{array}\right) W_x - 3[W, U]$$

$$-[W_1, \left(\begin{array}{c} 0 \\ g \end{array}\right)] + 2\sigma_3 W_{1x} - 3W_y - 3U_{xx} = 0,$$

$$3U_{xx} + 3\left(\begin{array}{c} 0 \\ f_{xx} \end{array}\right) - 6\sigma_3 W_x + [W_1, \sigma_3] + 3[W, \left(\begin{array}{c} 0 \\ g \end{array}\right)] = 0,$$

$$\left(\begin{array}{c} 0 \\ g_x \end{array}\right) = [W, \sigma_3].$$

This system represents a two-component KP system for the times $x, y, t$. Having in mind the recursion relations (24), (25) and expressions for $W, W_1$ (22), we come to the conclusion that all the matrix functions in this system can be expressed through $f, g, U$, and the system should give a closed system of equations for six scalar functions.
5 Reductions

Hirota-Ohta equation hierarchy

The Hirota-Ohta equation hierarchy is a reduction of the two-component KP hierarchy defined by the condition \( \lambda \), which is equivalent to

\[
\lambda^* = \lambda^* \psi^* \psi = -J \psi^* \psi = J \psi,
\]

Then for linear operators \( B_n^* = JB_nJ \) (compare \( \psi \)), and the reduced operator \( B_2 \) \( (18) \) is of the form

\[
B_2 = \sigma_3 \partial_x^2 + 2 \left( \begin{array}{cc}
u & u \\ -\bar{v} & -u \end{array} \right) \psi,
\]

\( f = \chi_1 = uI \). Reduced system \( \psi \) reads

\[
U_t - W_{1y} - U_{xx} + \sigma_3 W_{1x} + [U, W_1] - 3uU_x = 0,
3\sigma_3 u_{xx} + 2\sigma_3 W_{1x} - 3I u_y - 3U_{xx} = 0,
3U_x - 6\sigma_3 u_x + [W_1, \sigma_3] = 0,
\]

where the second equation gives the expression for \( W_{1x} \) in terms of \( U \), the third equation is implied by the second. From the first equation we get an equation for one matrix \( U \) of the form \( U = u\sigma_3 + U^a \) (antidiagonal part), which, written in components, represents the Hirota-Ohta (coupled KP) system \( \psi \).

Other reductions

Let us consider another zero order reduction \( (10), (13) \), with \( A = \sigma_3, k = 0 \). Effectively that leads to the change of the matrix \( J \) to the matrix

\[
J' = \sigma_3 J = \left( \begin{array}{cc}
0 & 1 \\
1 & 0
\end{array} \right).
\]

Then

\[
\psi^* (\lambda; t) = J' \psi (\lambda^*; t) J',
\]

for linear operators \( B_n^* = -J'B_nJ' \). The reduced Lax operator \( B_2 \) \( (18) \) is of the form

\[
\partial_y \psi = \left( \sigma_3 \partial_x^2 + \left( \begin{array}{c}
0 \\
g
\end{array} \right) \left( \begin{array}{c}
f \\
g_x
\end{array} \right) \right) \psi + uI + \frac{1}{2} \left( \begin{array}{c}
g_x \\
g
\end{array} \right) \left( \begin{array}{c}
f_x \\
g_x
\end{array} \right) \psi.
\]

Thus the Lax operator depends on three functions \( f, g, u \) instead of six functions in the general case of the two-component KP hierarchy, and the matrix system \( \psi \) should give a closed system of equations for these three functions.
First order reductions

Let us consider the reduction (13) with \( A = I \), \( k = 1 \),
\[
\oint \psi(\nu; t) J \lambda \psi^T(-\nu; t') d\nu = 0.
\]

Taking this identity for equal times, we get
\[
\chi_2 J + J \chi_2^T - \chi_1 J \chi_1^T = 0
\]

Recalling the recursion relations (24), (25), we obtain three scalar differential relations for six functions \( f, g, U \). These relations represent a reduction for the system (26).

Higher reductions

Higher reductions may be considered in a similar way. In general, a reduction of arbitrary order (or adjoint reduction) represents a set of three scalar differential relations for six functions \( f, g, U \). A pair of reductions of different orders engenders a closed (1+1)-dimensional system for six functions, connected with some stationary reductions of the hierarchy.

Another way to characterise the reduction is the existence of intertwining differential operator \( A_k \) of the order \( k \), which defines a map from the wave functions of adjoint operators to the wave functions of basic linear operators. Similar idea was used in [4] to construct the differential reductions for the case of the two-dimensional Dirac operator. It is convenient to introduce a modified conjugation operation, for matrix differential operator \( B \) we define \( B^\dagger = JB^* J^{-1} \). This operation possesses standard properties \( (B^\dagger)^\dagger = B, (AB)^\dagger = B^\dagger A^\dagger \). We denote \( \psi^\dagger(\lambda; t) = J \psi^*(\lambda; t) J^{-1} \). Then the reduction is characterised by the existence of differential operator \( A_k \) of the order \( k \), such that for any wave function \( \phi \)
\[
(\partial_y + B_2^\dagger) \phi = 0 \Rightarrow (\partial_y - B_2) A_k \phi = 0.
\]

Algebraically, this condition is equivalent to the operator equation
\[
(\partial_y - B_2) A_k = A_k (\partial_y + B_2^\dagger), \quad (29)
\]
see [3] for the scalar case. Using this equation, it is possible to express the coefficients of operator \( A_k \) through the coefficients of the Lax operator and get a reduction condition in terms of the coefficients of the Lax operator (or the solution of the system (26)). This type of condition can be also used to define the reductions in the context of the Lax-Sato equations (the scalar case is considered in [3]).
6 Appendix. Reductions in terms of the Lax-Sato equations

Here we briefly describe the Lax-Sato picture of the two-component KP hierarchy with the times (12) [2] and of the class of reductions corresponding to the bilinear relation (13). In the scalar case reductions of this type are described in [3].

The Lax-Sato equations define the dynamics of pseudodifferential operators

\[ L = \partial + U_1\partial^{-1} + U_2\partial^{-2} + \ldots, \]
\[ M = \sigma_3 + V_1\partial^{-1} + V_2\partial^{-2} + \ldots, \]

where \( U_n, V_n \) are 2 \times 2 matrices, \( \partial = \partial_x \), \( \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), with the characteristic properties

\[ [L, M] = 0, \quad M^2 = 1 \]

For odd times:

\[ \frac{\partial L}{\partial t_{2n+1}} = [(L^{2n+1})_+, L], \quad \frac{\partial M}{\partial t_{2n+1}} = [(L^{2n+1})_+, M], \tag{30} \]

for even times:

\[ \frac{\partial L}{\partial t_{2n}} = [(L^{2n} M)_+, L], \quad \frac{\partial M}{\partial t_{2n}} = [(L^{2n} M)_+, M] \tag{31} \]

The Gelfand-Dickey reductions for this hierarchy are defined by the following conditions: for odd flows

\[ (L^{2n+1})_- = 0, \quad (L^{2n+1})_+ = D^{(2n+1)}, \]

where \( D^{(2n+1)} \) is a differential operator of the order \( 2n + 1 \) with matrix coefficients . For even flows:

\[ (L^{2n} M)_- = 0, \quad (L^{2n} M)_+ = D^{(2n)} \]

Introducing formal pseudodifferential dressing operator (connected to operator \( (1 + \hat{K})_+ \) used by Kakei)

\[ P = I + W_1\partial^{-1} + W_2\partial^{-2} + \ldots, \]

it is possible to express operators \( L \) and \( M \) as

\[ L = P\partial P^{-1}, \quad M = P\sigma_3 P^{-1} \]

The operators \( L \) and \( M \) defined this way evidently possess necessary characteristic properties. Dynamics of the dressing operator is defined by the Sato
\begin{align}
\frac{\partial P}{\partial t_{2n+1}} &= -(P\partial^{2n+1}P^{-1})_P, \\
\frac{\partial P}{\partial t_{2n}} &= -(P\partial^{2n}\sigma_3P^{-1})_P, \\
\end{align}

that implies (31), (32). To find a dressing operator starting from \(L\) (or \(M\)), one should solve a factorization problem \(LP = P\partial, MP = P\sigma_3\). Reduction to the Hirota-Ohta equation hierarchy is described by the conditions

\begin{align*}
L^* &= JLJ, \\
M^* &= JMJ, \\
P^* &= -JP^{-1}J.
\end{align*}

A class of reductions corresponding to bilinear relation (13) is defined by the conditions: for \(A = I\)

\[ (P\partial^nJP^*)_P = 0, \]

for \(A = \sigma_3\)

\[ (P\sigma_3J\partial^nP^*)_P = 0. \]

Introducing the differential operators \(A_k = P\partial^kP^\dagger\) (for \(A = I\)) or \(A_k = P\sigma_3\partial^kP^\dagger\) (for \(A = \sigma_3\)), where we use the notation \(P^\dagger = JP^*_J\), we obtain the relations

\begin{align*}
LA_k &= A_kL^\dagger, \\
MA_k &= A_kM^\dagger,
\end{align*}

and also relations of the form (29).

\section*{Acknowledgements}

The reported study was funded by RFBR and NSFC, project number 21-51-53017.

\section*{References}

[1] R. Hirota and Y. Ohta, Hierarchies of coupled soliton equations. I, J. Phys. Soc. Jpn. 60 (1991) 798-809.

[2] S. Kakei, Dressing method and the coupled KP hierarchy, Phys. Lett. A 264(6) 449–458 (2000)
[3] E.Date, M.Jimbo, M.Kashiwara and T.Miwa, KP Hierarchies of Orthogonal and Symplectic Type -Transformation Groups for Soliton Equations VI-, J. Phys. Soc. Jpn. 50, 3813-3818 (1981)

[4] L.V. Bogdanov and E.V. Ferapontov, Projective differential geometry of higher reductions of the two-dimensional Dirac equation, Journal of Geometry and Physics 52(3) (2004) 328–352

[5] V.E. Zakharov, S.V. Manakov, Reductions in systems integrable by the method of inverse scattering problem, Doklady Mathematics, 57(3), 471-474, (1998)

[6] L.V. Bogdanov, Analytic-Bilinear Approach to Integrable Hierarchies (Mathematics and its Applications vol 493), Dordrecht: Kluwer 1999