BAYESIAN ESTIMATION OF A BIVARIATE COPULA
USING THE JEFFREYS PRIOR

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Abstract: A bivariate distribution with continuous margins can be uniquely decomposed via a copula and its marginal distributions. We consider the problem of estimating the copula function and adopt a Bayesian approach. On the space of copula functions, we construct a finite dimensional approximation subspace which is parametrized by a doubly stochastic matrix. A major problem here is the selection of a prior distribution on the space of doubly stochastic matrices also known as the Birkhoff polytope. The main contributions of this paper are the derivation of a simple formula for the Jeffreys prior and showing that it is proper. It is known in the literature that for a complex problem like the one treated here, the above results are difficult to obtain. The Bayes estimator resulting from the Jeffreys prior is then evaluated numerically via Markov chain Monte Carlo methodology. A rather extensive simulation experiment is carried out. In many cases, the results favour the Bayes estimator over frequentist estimators such as the standard kernel estimator and Deheuvels’ estimator in terms of mean integrated squared error.

Key words and phrases: Birkhoff polytope, copula, doubly stochastic matrices, Jeffreys prior, Markov chain Monte Carlo, Metropolis-within-Gibbs sampling, non-parametric, objective Bayes.

1. Introduction

Copulas have received considerable attention over the last years because of their increasing use in multiple fields such as environmental studies, genetics, data networks and simulation. They are also currently one of the hot topics in quantitative finance and insurance, see for instance Genest et al. (2009). Since it is precisely the copula that holds the dependence structure among various random quantities, estimating a copula is part of many techniques employed in these fields. For instance, in Risk Measurement, the Value-at-Risk (VaR) is computed by simulating asset (log)returns from an estimated copula. Further financial examples in which copulas are estimated are provided in the books
written by Cherubini et al. (2004), and Trivedi and Zimmer (2005). In this paper, we provide new generic methodology for estimating copulas, and that, in a Bayesian framework.

Let us first recall that a bivariate copula $C$ is a distribution function on $S = [0,1] \times [0,1]$ with uniform margins. In particular, copulas are Lipschitz continuous and form an equicontinuous family. They are bounded by the so-called Fréchet-Hoeffding copulas, that is

$$\max(0, u + v - 1) \leq C(u,v) \leq \min(u,v), \quad \text{for all } (u,v) \in S.$$ 

Sklar’s Theorem states that a bivariate distribution $F$ is completely characterized by its marginal distributions $F_X, F_Y$ and its copula $C$. More precisely, we have the representation

$$F(x,y) = C(F_X(x), F_Y(y)), \quad \text{for all } (x,y) \in \mathbb{R}^2,$$  \hfill (1.1) 

where $C$ is well defined on $\text{Ran}(F_X) \times \text{Ran}(F_Y)$, see Nelsen (1999). In particular, the copula is unique if $F_X$ and $F_Y$ are continuous, and in this case, we have the following expression for the copula

$$C(u,v) = F(F_X^{-1}(u), F_Y^{-1}(v)), \quad \text{for all } (u,v) \in S.$$ \hfill (1.2) 

Let $\{(x_i, y_i), i = 1, \ldots, n\}$ be a sample, where every $(x_i, y_i)$ is a realization of the random couple $(X_i, Y_i), i = 1, \ldots, n$, with joint cumulative distribution function $F$, and continuous marginal distributions $F_X$ and $F_Y$. We consider the problem of estimating the copula $C$ by a copula $\hat{C}$, where $\hat{C}$ depends on the sample. In this problem, the individual marginal distributions are treated as nuisance parameters. The literature presents three generic approaches for estimating $C$, namely the fully parametric, the semiparametric and the nonparametric approaches. Below, we briefly describe each approach and emphasize on two nonparametric estimators, since we will subsequently compare our estimator with these.

**The fully parametric approach.** When parametric models for both the marginal distributions $F_X$ and $F_Y$ and for the copula function $C$ are specified, the likelihood of the sample $\{(x_i, y_i): i = 1, \ldots, n\}$ is computed via equation (1.1). In
principle, estimates can be jointly obtained for the marginal distribution parameters and for the copula. However, when joint estimation is computationally difficult, Joe (1997) proposes a two-step method in which the marginal distributions are estimated in a first stage and then plugged-in thereafter as the true margins, enabling the estimation of the copula function in a second step. This approach is called *Inference for Margins (IFM)*. The asymptotic efficiency of IFM is discussed in Joe (2005) by considering maximum likelihood estimates at both stages of the procedure. In a Bayesian setup, Silva and Lopes (2008) argue that under a deviance-based model selection criteria, the joint estimation of the marginal parameters and of the copula parameters is better than the two-step procedure.

*The semiparametric approach.* Here a parametric model is assumed for the copula function $C$ while the margins are kept unspecified. In this setup, Genest et al. (1995) have proposed to use $n/(n+1)$ times the empirical distributions as the estimates $\hat{F}_X$ and $\hat{F}_Y$ and a pseudo-likelihood estimator for $C$. The authors show that the resulting estimator is consistent and asymptotically normal. In Kim et al. (2007), comparisons are made between the fully parametric approach and the semiparametric approach proposed by Genest et al. (1995). More recently, in a Bayesian setup, Hoff (2008) proposes a general estimation procedure, via a likelihood based on ranks, that does not depend on any parameters describing the marginal distributions. The latter methodology can accommodate both continuous and discrete data.

*The nonparametric approach.* This approach exploits equation (1.2). Here we describe Deheuvels’ estimator and the kernel estimator. Let $\hat{F}$ be the empirical cumulative distribution function and let $\hat{F}_X^{-1}$ and $\hat{F}_Y^{-1}$ be the generalized inverses. We say that $\hat{C}$ satisfies the Deheuvels constraint provided that for all $i,j = 1,\ldots,n$,

$$
\hat{C}(i/n, j/n) = \hat{F}\left(\hat{F}_X^{-1}(i/n), \hat{F}_Y^{-1}(j/n)\right),
$$

$$
= (1/n) \sum_{k=1}^{n} 1(\text{rank}(x_k) \leq i, \text{rank}(y_k) \leq j). \text{ (Deheuvels’ constraint)}
$$
In Deheuvels (1979), the asymptotic behaviour of the class of copulas \( \hat{C} \) satisfying the Deheuvels constraint is described. Note that \( \hat{C} \) is sometimes called empirical copula in the literature, see Nelsen (1999) for instance. We propose an estimator that satisfies Deheuvels’ constraint in Lemma 3 which we call Deheuvels’ estimator henceforth. One nice property of this estimator is its invariance under strictly increasing transformations of the margins. In other words, if \( f \) and \( g \) are two strictly increasing functions, then Deheuvels’ estimator based on the original sample and the one based on the sample \( \{(f(x_i), g(y_i)) : i = 1, \ldots, n\} \) are identical. This is a desirable property for a copula estimator since it is inherent to copulas themselves.

In general, if \( \hat{F} \) is a smooth kernel estimator of \( F \) (\( \hat{F}_X \) and \( \hat{F}_Y \) are continuous say), then

\[
\hat{C}(u, v) = \hat{F} \left( \hat{F}_X^{-1}(u), \hat{F}_Y^{-1}(v) \right),
\]

for all \((u, v) \in S\), (kernel estimator)
is called a kernel estimator for \( C \). Asymptotic properties of Gaussian kernel estimators are discussed in Fermanian and Scaillet (2003), and the reader is referred to Charpentier et al. (2006) for a recent review.

Although both of the nonparametric estimators discussed above have good asymptotic properties, it is not the case for finite samples in general. In fact, these estimators give poor results for many types of dependency structures which is illustrated in Section 5. This could be a considerable inconvenience for practitioners working with small samples.

Our aim is to develop a Bayesian alternative for the estimation of \( C \) which circumvents this problem. Following Genest et al. (1995), the marginal distributions are kept unspecified when these are unknown, and we use \( n/(n+1) \) times the empirical distributions as their estimates. In view of this, our methodology can be called empirical Bayes. When the marginal distributions are known, they are transformed into uniform distributions, and in this case our procedure is purely Bayesian. In both cases, our estimator has the property of being invariant under monotone transformations of the margins, just like Deheuvels’ estimator.

Essentially, our model is obtained as follows. First, in Section 2 we construct an approximation subspace \( \mathcal{A} \subset \mathcal{C} \), where \( \mathcal{C} \) is the space of all copulas. This is achieved by considering a norm \( \| \cdot \| \) and setting a precision \( \epsilon > 0 \) so that for every
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For every $m > 1$, we construct a finite dimensional approximation subspace $\mathcal{A}_m \subset \mathcal{C}$. The construction of $\mathcal{A}_m$ uses a basis which forms a partition of unity. The elements of $\mathcal{A}_m$ are parametrized by a doubly stochastic matrix $P$. The choice of the basis is fixed while $P$ varies. The representation is given in expres-
sion (2.3). Furthermore, we give upper bounds on
\[ \sup_{C \in \mathcal{C}} \inf_{A \in \mathcal{A}_m} \| A - C \|_{\infty}. \]

A partition of unity is a set of nonnegative functions \( \varphi = \{ \varphi_i \}_{i=1}^m \), defined over the unit interval \([0, 1]\), such that \( m \varphi_i \) is a density for all \( i = 1, \ldots, m \), and
\[ \sum_{i=1}^m \varphi_i(u) = 1, \text{ for all } u \in [0,1]. \]

Particular examples are given by indicator functions
\[ \varphi_1 = 1_{[0,1/m]}, \varphi_i = 1_{((i-1)/m,i/m]}, \quad i = 2, \ldots, m, \tag{2.1} \]
and Bernstein polynomials
\[ \varphi_i = B_{i-1}^{m-1}, \quad i = 1, \ldots, m, \tag{2.2} \]
where
\[ B_i^m(u) = \binom{m}{i} u^i (1 - u)^{m-i}, \text{ for all } u \in [0,1]. \]

See Li et al. (1998) for more examples of partitions of unity. In the following, let \( \Phi = (\Phi_1, \ldots, \Phi_m)' \), where \( \Phi_i(u) = \int_0^u \varphi_i(t) \, dt \), for all \( u \in [0,1], \ i = 1, \ldots, m \) and let
\[ A_P(u,v) = m \Phi(u)' P \Phi(v), \quad \text{for all } (u,v) \in S, \tag{2.3} \]
where \( P \) is an \( m \times m \) doubly stochastic matrix. The following Lemma is straightforward to prove.

**Lemma 1.** For every doubly stochastic matrix \( P \), \( A_P \) is an absolutely continuous copula.

In view of the above result, we define the approximation space as
\[ \mathcal{A}_m = \{ A_P : P \text{ is a doubly stochastic matrix} \}. \]

The approximation order of \( \mathcal{A}_m \) is now discussed, it depends on the choice of the basis \( \Phi \). Let \( \mathcal{G}_m = \{(i/m,j/m) : i, j = 1, \ldots, m\} \), be a uniformly spaced grid on the unit square \( S \). For a given copula \( C \), let \( R_C = (C(i/m,j/m))_{i,j=1}^m \) be the
restriction of \( C \) on \( \mathcal{G}_m \). Let

\[
D = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
-1 & 1 & 0 & 0 & \cdots & 0 \\
0 & -1 & 1 & 0 & \cdots & 0 \\
0 & 0 & -1 & 1 & \cdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & 0 \\
0 & 0 & 0 & 0 & -1 & 1
\end{pmatrix},
\]

then \( P_C = mDR_CD' \) is a doubly stochastic matrix. Upper bounds for \( \| A_{PC} - C \|_\infty \) are given in the following Lemma.

**Lemma 2.** Let \( C \) be a copula and let \( A = A_{PC} \in \mathcal{A}_m \).

(a) For a model using indicator functions basis \([2.1]\), we have \( R_A = R_C \) and \( \| A - C \|_\infty \leq 2/m \).

(b) For a model using the Bernstein basis \([2.2]\), we have \( \| A - C \|_\infty \leq 1/\sqrt{m} \).

**Proof.** (a) A direct evaluation shows that \( R_A = R_C \). From the Lipschitz condition, if two copulas \( C_1 \) and \( C_2 \) satisfy the constraint \( R_{C_1} = R_{C_2} \), then \( \| C_1 - C_2 \|_\infty \leq 2/m \).

(b) First, it is well known that \( m\Phi' D = (B^m_1, \ldots, B^m_m) \). For any \((u, v) \in S\) consider \( X \) and \( Y \) two independent random variables, \( X \) has a binomial\((m, u)\) distribution and \( Y \) has a binomial\((m, v)\) distribution. Let \( \theta = (u, v) \). We have

\[
A(\theta) = E_\theta[C(X/m, Y/m)].
\]

Therefore,

\[
\sup_{\theta \in S} |A(\theta) - C(\theta)| = \sup_{\theta \in S} |E_\theta[C(X/m, Y/m) - C(u, v)]|,
\]

\[
\leq \sup_{\theta \in S} E_\theta[|C(X/m, Y/m) - C(u, v)|],
\]

\[
\leq \sup_{\theta \in S} E_\theta[|X/m - u| + |Y/m - v|],
\]

\[
= (2/m) \sup_{u \in (0,1)} E_u[|X - mu|].
\]
In Lemma 5 of the Appendix, we give the exact value of \( \sup_{u \in (0,1)} E_u[|X - mu|] \). However, a simple expression for an upper bound is given by H"older's inequality

\[
\sup_{u \in (0,1)} 2E_u[|X - mu|] / m \leq \sup_{u \in (0,1)} 2\sqrt{\text{Var}_u[X]} / m,
\]

\[
= 1 / \sqrt{m}.
\]

Bernstein copulas have appeared in the past literature and their properties have been extensively studied in [Sancetta and Satchell (2004) and Sancetta and Satchell (2001)]. However, in view of Lemma 2 and of the simplicity of indicator functions, we subsequently use the indicator functions basis given in (2.1) for \( \Phi \) in our model. Notice that in this situation, if \( \{(U_i, V_i)\}_{i=1}^n \) denotes a random sample of size \( n \), then

\[
N_{ij} = \sum_{k=1}^n \varphi_i(U_k)\varphi_j(V_k) \quad i, j = 1, \ldots, m,
\]

is a sufficient statistic with a multinomial\((n, m^{-1}P)\) distribution. So there is a connection between our problem and the problem of estimating the probabilities in a multinomial setup when the probabilities live in a constrained parameter space.

The following Lemma is used to define what we call Deheuvels’ estimator.

**Lemma 3.** Let \( \{(x_i, y_i) : i = 1, \ldots, n\} \) be a sample, and let \( R = (r_{ij}) \) be the \( n \times n \) matrix given by

\[
r_{ij} = (1/n) \sum_{k=1}^n 1(\text{rank}(x_k) \leq i, \text{rank}(y_k) \leq j), \quad \text{for } i, j = 1, \ldots, n.
\]

If we use the indicator basis (2.1) with \( m = n \) for \( \Phi \), then the copula

\[
\hat{C}_{DEH} = n^2 \Phi' DRD' \Phi, \quad \text{(Deheuvels’ estimator)}
\]

satisfies Deheuvels’ constraint.

### 3. The prior distribution

The choice of a prior concentrated on the approximation space is delicate. The prior distribution is specified on \( \mathcal{B} \), the set of doubly stochastic matrices
of order $m$, $m > 1$. Here, we adopt an objective point of view and derive the Jeffreys prior. We also discuss two representations of doubly stochastic matrices that can be useful for the specification of other prior distributions on $\mathcal{B}$.

The set $\mathcal{B}$ is a convex polytope of dimension $(m - 1)^2$. It is known in the literature as the Birkhoff polytope and has been the object of much research in the past years. For instance, computing the exact value of its volume is an outstanding problem in mathematics, it is known only for $m \leq 10$, see [Beck and Pixton (2003)].

The Fisher information matrix is obtained as follows. For $m > 1$, let $P \in \mathcal{B}$, and let $W = (1/m)P \in \mathcal{W}$. The copula (2.3) is a mixture of $m^2$ bivariate distribution functions

$$
A_P(u, v) = m\Phi(u)'P\Phi(v) = \Psi(u)'W\Psi(v) = \sum_{i=1}^{m} \sum_{j=1}^{m} w_{ij} \Psi_i(u)\Psi_j(v),
$$

where $W = (1/m)P \in \mathcal{W}$, and $\Psi_i(u) = \int_{0}^{u} \psi_i(t)dt$, for all $u \in [0, 1]$, with $\psi_i(\cdot) = m\varphi_i(\cdot), i = 1, \ldots, m$. The density $a_P$ of $A_P$ is thus

$$
a_P(u, v) = \sum_{i=1}^{m} \sum_{j=1}^{m} w_{ij} \psi_i(u)\psi_j(v),
$$

$$
= 1 + \sum_{i=1}^{m} \sum_{j=1}^{m} (w_{ij} - 1/m^2)\psi_i(u)\psi_j(v),
$$

$$
= 1 + \sum_{i=1}^{m-1} \sum_{j=1}^{m-1} (w_{ij} - 1/m^2)(\psi_i(u) - \psi_m(u))(\psi_j(v) - \psi_m(v)).
$$

The last equality expresses the fact that there are $(m - 1)^2$ free parameters in the model. By considering the indicator functions basis (2.1), for all $i_1, j_1, i_2, j_2 =$
\[ E \left[ -\frac{\partial^2 \log a_p(u, v)}{\partial w_{i1j1} \partial w_{i2j2}} \right] = \int_0^1 \int_0^1 \frac{(\psi_{i1}(u)\psi_{i2}(u) + \psi_{m1}(u)\psi_{m2}(v) + \psi_{m1}(v)\psi_{m2}(v))}{a_p(u, v)} du dv, \]

\begin{align*}
1/w_{i1j1} + 1/w_{i1m} + 1/w_{m1j1} + 1/w_{mm}, & \quad \text{if } i_1 = i_2, j_1 = j_2, \\
1/w_{i1m} + 1/w_{mm}, & \quad \text{if } i_1 = i_2, j_1 \neq j_2, \\
1/w_{m1j1} + 1/w_{mm}, & \quad \text{if } i_1 \neq i_2, j_1 = j_2, \\
1/w_{mm}, & \quad \text{if } i_1 \neq i_2, j_1 \neq j_2.
\end{align*}

Although the information matrix is of order \((m - 1)^2 \times (m - 1)^2\), the following result shows how to reduce the computation of its determinant to that of a matrix of order \((m - 1) \times (m - 1)\). The important reduction provided by (3.1) is greatly appreciated when running an MCMC algorithm which computes the determinant at every iteration. Most importantly, this expression enables us to derive the main result of this paper, that is Theorem 1. The proofs of these two results are quite technical, so we have put them in the Appendix.

**Lemma 4.** The Fisher information for \( W = (w_{ij})_{i,j=1,...,m} \in \mathcal{W} \) is given by
\[
I(W) = \frac{\det((1/m)I - mV'V)}{m^m \det D_0 \det D_1}, \tag{3.1}
\]
where
\[
V = (w_{ij})_{i=1,...,m; j=1,...,m-1},
\]
\[
D_0 = \text{diag}(w_{11}, ..., w_{1(m-1)}, ..., w_{(m-1)(m-1)}),
\]
and
\[
D_1 = \text{diag}(w_{mm}, w_{1m}, ..., w_{(m-1)m}, w_{m1}, ..., w_{m(m-1)}).
\]

**Theorem 1.** The Jeffreys prior \( \pi \propto I^{1/2} \) is proper.

Now, in order to specify different priors we can consider the two following representations.

The Hilbert space representation. Let \( \mathcal{B}_0 = \{ P - (1/m)11' : P \in \mathcal{B} \} \) and \( \mathcal{V} = \text{Span}(\mathcal{B}_0) \). Consider the inner product \( \langle V_1, V_2 \rangle = \text{tr}(V_1V_2') \) on \( \mathcal{V} \). Thus, \( \mathcal{V} \) is an \((m - 1)^2\) dimensional Hilbert space and an orthonormal basis is given by \( \{v_iv_j'\}_{i,j=1,...,m-1} \), with
\[
v_i = \frac{1}{\sqrt{i(i+1)}}(1,...,1,-i,0,0,...), \quad i = 1,...,m-1.
\]
Now, for all $P \in \mathcal{B}$, there exists a unique $(m - 1) \times (m - 1)$ matrix $\alpha$ such that
\[ P = m^{-1}11' + G\alpha G', \quad (3.2) \]
where $G$ is the $m \times (m - 1)$ matrix given by $G = (v_1, v_2, \ldots, v_{m-1})$. In this representation $\alpha = G'PG$. Therefore, if we let $\mathcal{B}' = G'\mathcal{B}G$, then we have a bijection between $\mathcal{B}$ and $\mathcal{B}'$. The set $\mathcal{B}'$ is a bounded convex subset of $\mathbb{R}^{(m-1)^2}$ with positive Lebesgue measure. From this, priors on $\mathcal{B}$ can be induced by priors on $\mathcal{B}'$, and later on, we shall refer to the uniform prior on the polytope $\mathcal{B}$ as the uniform distribution on $\mathcal{B}'$. The above representation is also particularly useful to construct a Gibbs sampler for distributions on the polytope.

The Birkhoff-von Neumann representation. Another decomposition is obtained by making use of the Birkhoff-von Neumann Theorem. Doubly stochastic matrices can be decomposed via convex combinations of permutation matrices. In fact, $\mathcal{B}$ is the convex hull of the permutation matrices and these are precisely the extreme points (or vertices) of $\mathcal{B}$. Furthermore, every $m \times m$ doubly stochastic matrix $P$ is a convex combination of at most $k = (m - 1)^2 + 1$ permutation matrices, see Mirsky (1963). In other words, if $\{\sigma_i\}_{i=1}^m$ is the set of permutation matrices and if $P \in \mathcal{B}$, then there exists $1 \leq i_1 < \cdots < i_k \leq m$ such that $P = \sum_{j=1}^k \lambda_j \sigma_{i_j}$, for some weight vector $(\lambda_{i_1}, \ldots, \lambda_{i_k})$ lying in the $(k - 1)$-simplex $\Lambda_k = \{(\lambda_1, \ldots, \lambda_k) : 0 \leq \lambda_j, \text{ for all } j \text{ and } \sum_{j=1}^k \lambda_j = 1\}$. A prior distribution over the polytope can be selected using a discrete distribution over the set $\{1 \leq i_1 < \cdots < i_k \leq m\}$ and a continuous distribution over the simplex $\Lambda_k$, such as a Dirichlet distribution. See also Melilli and Petris (1995) for work in this direction.

4. The MCMC algorithm

Let $\{(x_i, y_i), i = 1, \ldots, n\}$ be a sample, where each $(x_i, y_i)$ is a realization of the random couple $(X_i, Y_i)$, $i = 1, \ldots, n$, with dependence structure given by a copula $C$, and with continuous marginal distributions $F_X$ and $F_Y$. If the marginal distributions are known, then the transformed observations $\tilde{x}_i = F_X(x_i)$ and $\tilde{y}_i = F_Y(y_i), i = 1, \ldots, n$, are both samples from a uniform distribution on $(0, 1)$. If the marginal distributions are unknown, then we follow Genest et al. (1995) and consider the pseudo-observations $\tilde{x}_i = (n/(n + 1))\hat{F}_X(x_i)$ and $\tilde{y}_i =$
\[ \frac{n}{(n+1)} \hat{F}_Y(y_i), i = 1, \ldots, n, \] where \( \hat{F}_X \) and \( \hat{F}_Y \) are the empirical distributions.

The algorithm below describes the transition kernel for the Markov chain used to numerically evaluate the Bayesian estimator \( \hat{C} \) associated to the Jeffreys prior \( \pi \). The type of algorithm is called Metropolis-within-Gibbs, see for instance [Gamerman and Lopes (2006)]. An individual estimate is approximated by the sampling mean of the chain.

Let \( T \geq 1 \) be the length of the chain, and at each iteration \( t, 1 \leq t \leq T \), let \( P_t \) be the current doubly stochastic matrix. From representation (3.2) in the previous section,

\[ P_t - \left( \frac{1}{m} \right) 11' = \sum_{k=1}^{m-1} \sum_{l=1}^{m-1} \alpha_{kl} v_k v_l'. \]

Repeat for \( i, j = 1, \ldots, m - 1 \):

1. Select direction \( v_i v_j' \) and compute the interval \( \Gamma_{ij} \subset \mathbb{R} \) as follows:

1.1 For every \( p, q = 1, \ldots, m \), find the largest interval \( \Gamma_{ij}^{(p,q)} \) such that

\[ \gamma_{ij}^{(p)} v_i^{(p)} v_j^{(q)} \geq -1/m - \sum_{k=1}^{m-1} \sum_{l=1}^{m-1} \alpha_{kl} v_k^{(p)} v_l^{(q)}, \text{ for all } \gamma_{ij} \in \Gamma_{ij}^{(p,q)}. \]

1.2 Take \( \Gamma_{ij} = \bigcap_{p,q} \Gamma_{ij}^{(p,q)}. \)

2. Draw \( \gamma_{ij} \) from the uniform distribution on \( \Gamma_{ij} \), and set \( \beta_{ij} = \alpha_{ij} + \gamma_{ij} \) and \( \beta_{kl} = \alpha_{kl}, \) for every \( k \neq i, l \neq j \). The proposed doubly stochastic matrix is given by

\[ P_{t}^{\text{prop}} = \left( \frac{1}{m} \right) 11' + \sum_{k=1}^{m-1} \sum_{l=1}^{m-1} \beta_{kl} v_k v_l'. \]

3. Accept \( P_{t+1} = P_{t}^{\text{prop}} \) with probability

\[ \alpha(P_t, P_{t}^{\text{prop}}) = \min \left\{ 1, \frac{\pi(P_{t}^{\text{prop}}) L(P_{t}^{\text{prop}} | \tilde{x}, \tilde{y})}{\pi(P_t) L(P_t | \tilde{x}, \tilde{y})} \right\}, \] (4.1)

where \( L(\cdot | \tilde{x}, \tilde{y}) \) is the likelihood derived from expression (2.3).

Note that the above algorithm could also be used with any prior specified via the Hilbert space representation described in the previous section, including the uniform prior on the polytope \( \mathcal{B} \). In particular, it could be adapted to
draw random doubly stochastic matrices according to such priors by replacing the acceptance probability (4.1) with
\[ \alpha(P_t, P_t^{\text{prop}}) = \min \left\{ 1, \frac{\pi(P_t^{\text{prop}})}{\pi(P_t)} \right\}. \]
In order to further describe the Jeffreys prior, we use the algorithm to approximate the probability of the largest ball contained in \( B \). This ball has radius \( 1/(m - 1) \), where \( m > 1 \) is the size of the doubly stochastic matrix. Although this probability can be obtained exactly for the uniform distribution, we nevertheless approximate it using our algorithm, meanwhile providing some validation of the MCMC algorithm. Figure 4.1 shows the results we get for \( m = 4 \).

Notice that this probability is much smaller for the Jeffreys prior, because it distributes more mass towards the extremities of the polytope than the uniform prior does. This may also be observed by plotting the density estimates of the radius of the doubly stochastic matrix, that is the \( L_2 \)-distance of the doubly stochastic matrix from the centre of the polytope \( B \). These are shown in Figure 4.2.

4. Simulation experiments

The goal of the experiment is to study the performance of our estimator on artificial data sets generated from various known bivariate distributions. We provide evidence that the Bayesian estimator gives good results in general, or in other words, that the Jeffreys prior is a reasonable choice.

For every data set, the copula function is estimated. Three different dependence structures are considered, the first one is the Gaussian copula
\[ C_\rho(u, v) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v)), \quad |\rho| \leq 1, \]
where \( \Phi_\rho \) is the standard bivariate Gaussian cdf with correlation coefficient \( \rho \) and \( \Phi \) is the univariate standard normal distribution. See Figure 4.3(a). The second dependence structure that we consider is obtained by the following: let \((U, V)\) be a random vector with uniform margins with joint distribution \( C_\rho \), let \( W \) be an independent uniformly distributed random variable and consider the random vector \((U_c, V_c) = (U, V)\mathbf{1}(W \leq 1/2) + (U, 1 - V)\mathbf{1}(W > 1/2)\). The distribution of \((U_c, V_c)\) is given by
\[ C_{\rho,c}(u, v) = 1/2(C_\rho(u, v) - C_\rho(u, 1 - v) + u), \quad \text{for all} \ (u, v) \in S. \]
Figure 4.1: Convergence of 1000 parallel MCMC runs for the probability of the largest ball contained in the polytope $\mathcal{B}$ with $m = 4$. Shaded region represents the range of the entire set of approximations at each iteration. Above is the convergence for the probability in the case of the uniform distribution. The flat line, in this case, corresponds to the true probability $p \approx 0.0027$. Below is the same for the Jeffreys prior.
Figure 4.2: Plots of samples and density estimates of the radius ($L_2$-distance of the doubly stochastic matrix from the center of the polytope $\mathcal{B}$), on the interval $[0, q_{95}]$, where $q_{95}$ is the 95$^{th}$ quantile of its distribution. Above are results when sampling from the uniform prior and below from the Jeffreys prior. Here $m = 4$. 
Here, the index \( c \) is to highlight the “cross like” dependence structure, see Figure 4.3(b). A “diamond like” dependence structure is also considered, this is obtained by the transformation \((U_d, V_d) = (U_c + 1/2 \mod 1, V_c)\) which is distributed according to the copula

\[
C_{\rho,d}(u,v) = \begin{cases} 
C_{\rho,c}(u + 1/2, v) - C_{\rho,c}(1/2, v), & \text{if } u \leq 1/2, \\
C_{\rho,c}(u - 1/2, v) + v - C_{\rho,c}(1/2, v), & \text{if } u > 1/2.
\end{cases}
\]

See Figure 4.3(c) for an illustration of its density.

![Gaussian, Cross, Diamond](image)

Figure 4.3: Gaussian copula with \( \rho = 0.5 \), and corresponding cross and diamond dependence structures.

An extensive simulation experiment is carried out in two parts. In the first part, we consider the case of known marginal distributions and use bivariate data sampled from the copula models above. In the second part of the experiment, we
simulate an unknown margins situation. The data sets are generated from the same copulas, but a Student $t$ with 7 degrees of freedom and a chi-square with 4 degrees of freedom are considered as the first and second margins respectively.

In the experiment, 1000 samples of both sizes $n = 30$ and $n = 100$ are generated from each model. The Bayesian estimators from the Jeffreys prior and the uniform prior over the polytope are evaluated. For the uniform distribution, we take a uniform distribution on $\mathcal{B}'$, a subset of $\mathbb{R}^{(m-1)^2}$, and we use the bijection $B = m^{-1}11' + G\mathcal{B}'G'$ given in expression (3.2). The maximum likelihood estimator for $P$ in copula (2.3) is evaluated numerically. In every case, the order of the doubly stochastic matrix in our model is given by $m = 6$. We compare the performance of our estimators with Deheuvels’ estimator given in Theorem 3 and the Gaussian kernel estimator. The results we obtain in the first and second part of the simulation are given in Figures 4.4 and 4.5 respectively. These Figures report the values of the mean integrated squared errors

$$ \text{MISE}(\hat{C}) = E \left[ \int_0^1 \int_0^1 (\hat{C}(u,v) - C(u,v))^2 du dv \right], $$

for the five estimators as a function of the parameter $\rho$, for $0 \leq \rho \leq 1$.

As the results indicate, the Bayesian approach generally outperforms Deheuvels’ estimator and the kernel estimator. Unfortunately, this is not the case when the true model is the Gaussian copula $C_\rho$, for large values of $\rho$. As $\rho$ increases to 1, the true copula approaches the Fréchet-Hoeffding upper bound, also called the comonotone copula, corresponding to (almost sure) perfect positive linear dependence. Notice that for the Bayes estimators, the MLE and Deheuvels’ estimator, the invariance property mentioned in the introduction is reflected in the results since for each model, their MISE curves in both the known and the unknown margins cases are very similar. This is less the case for the kernel estimator. Notice also the resemblance in shape of the MISE for Deheuvels’ estimator and the kernel estimator in the unknown margins cases. Finally the performance of the MLE is worth mentioning, since in many cases it has the smallest MISE, especially for large values of $\rho$. This is explained because the MLE will go on the boundary of the parameter space easily, while the Bayes estimator will always stay away from the boundary with the type of priors that we have selected. However, if such an extreme case is to happen in a real life
Figure 4.4: Plots of MISE against $\rho$ in the known margins case. J: Bayes estimator using the Jeffreys prior, U: Bayes estimator using the uniform prior, MLE: maximum likelihood from our model, D: Deheuvels’ estimator, K: kernel estimator. Here $m = 6$. 
Figure 4.5: Plots of MISE against $\rho$ in the unknown margins case. J: Bayes estimator using the Jeffreys prior, U: Bayes estimator using the uniform prior, MLE: maximum likelihood from our model, D: Deheuvels’ estimator, K: kernel estimator. Here $m = 6$. 
problem, it is probable that the practitioner has some insight on the phenomenon beforehand, and may choose to work with a more appropriate (subjective) prior.

5. Discussion

Two points need to be further discussed. First, our methodology is purely Bayesian only when the marginal distributions are known. When these are unknown, our methodology is empirical Bayes. In fact, in this case we propose a two-step procedure by first estimating the margins via the empirical marginal distributions and then plugging them in as the true distributions thereafter. We have chosen to do this because it is common practice to do so, see [Genest et al. (1995)], because it is simple to implement, and because our estimator is consequently invariant under increasing transformations of the margins. One way to propose a purely Bayesian estimator by using our model for the copula, is to use finite mixtures for the margins. This way, if the densities used in the latter mixtures have disjoint supports, then the Jeffreys prior for the mixing weights has a simple form and is proper, see [Bernardo and Girón (1988)]. Now by selecting independent Jeffreys priors for the margins and for the copula, the resulting prior is proper as well.

Finally, our models given by the approximation spaces $\mathcal{A}_m$, $m > 1$ are called sieves by some authors, see [Grenander (1981)]. Using these sieves we can construct a nonparametric model for the copula which can, in some sense, respect the infinite-dimensional nature of the copula functions. In fact, if we take $\mathcal{A} = \bigcup_{m > 1} \mathcal{A}_m$, then $\mathcal{A}$ is dense in the space of copulas. Our Bayesian methodology can be easily adapted here. This can be achieved by selecting an infinite support prior for the model index $m$, and by using our methodology inside each model. The Bayesian estimator becomes an infinite mixture of the estimators proposed in this paper (one for each model $m$), where the mixing weights are given by the posterior probabilities of the models.

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Appendix

Lemma 5. Consider $X$, a binomial$(n, p)$ random variable. We have

$$
\sup_{0 \leq p \leq 1} E[|X - np|] = \begin{cases} 
1/B(1/2, (n+1)/2), & \text{if } n \text{ is odd}, \\
(1 - (n+1)^{-2})^{n/2} \left(1 + (n+1)^{-2}\right) 1/B(1/2, n/2), & \text{if } n \text{ is even}.
\end{cases}
$$

Proof. Let $g_n(p) = E[|X - np|]$. We have

$$
g_n(p) = \frac{2n!}{([np])!(n-1-[np])!} p^{[np]+1} (1-p)^{n-[np]} \text{ for all } n \geq 1, p \in [0, 1],
$$

where $[x] = \sup\{n: n \leq x, n \text{ is an integer}\}$ for all $x$. Therefore,

$$
\sup_{0 \leq p \leq 1} g_n(p) = \max_{0 \leq k \leq n-1} \sup_{\{p: [np]=k\}} g_n(p),
$$

$$
= \max_{0 \leq k \leq n-1} g_n\left(\frac{k+1}{n+1}\right).
$$

First of all, $\sup_{0 \leq p \leq 1} g_1(p) = g_1(1/2) = 1/2 = 1/B(1/2, 1)$. Assume that $n > 1$.

Let $h_n(k) = g_n\left(\frac{k+2}{n+1}\right) / g_n\left(\frac{k+1}{n+1}\right)$, for $k = 0, \ldots, n-2$. We have

$$
h_n(k) = \frac{(1 + (k+1)^{-1})^{k+2}}{(1 + (n-k-1)^{-1})^{n-k}} \text{ and } h_n(k) = \frac{1}{h_n(n-2-k)} \text{ for } k = 0, \ldots, n-2.
$$

However,

$$
\frac{d}{dt} \log \left(1 + \frac{1}{t}\right)^{t+1} = \log \left(1 + \frac{1}{t}\right) - \frac{1}{t} < 0 \text{ for all } t > 1.
$$

This implies that $h_n$ decreases on $\{0, \ldots, n-2\}$. Therefore,

$$
g_n\left(\frac{1}{n+1}\right) < \cdots < g_n\left(\frac{n+1/2}{n+1}\right) > \cdots > g_n\left(\frac{n}{n+1}\right) \text{ if } n \text{ is odd}
$$

$$
g_n\left(\frac{1}{n+1}\right) < \cdots < g_n\left(\frac{n/2}{n+1}\right) = g_n\left(\frac{n/2+1}{n+1}\right) > \cdots > g_n\left(\frac{n}{n+1}\right) \text{ if } n \text{ is even}.
$$

The final expression is obtained using the following identity:

$$
n! = 2^n \Gamma\left(\frac{n}{2} + 1\right) \Gamma\left(\frac{n+1}{2}\right) / \Gamma\left(\frac{1}{2}\right) \text{ for all } n \geq 0.
$$

\qed
Proof of Lemma 4. Here we show how to compute $I(W) = \det \left( E \left[ -\frac{\partial^2 \log a_p(u,v)}{\partial W^2} \right] \right)$ efficiently. First, notice that $A = \left( E \left[ -\frac{\partial^2 \log a_p(u,v)}{\partial W^2} \right] \right)$ can be written as $A = D_0^{-1} + CD_1^{-1}C'$, where

$$D_0 = \text{diag}(w_{11}, \ldots, w_{1(m-1)}, \ldots, w_{(m-1)(m-1)}),$$
$$D_1 = \text{diag}(w_{mm}, w_{1m}, \ldots, w_{(m-1)m}, w_{m1}, \ldots, w_{m(m-1)}),$$

and

$$C_{(m-1)^2 \times (2m-1)} = \begin{pmatrix}
1_{m-1} & 1_{m-1} & 01_{m-1} & \cdots & I_{m-1} \\
1_{m-1} & 01_{m-1} & 1_{m-1} & \cdots & I_{m-1} \\
1_{m-1} & 01_{m-1} & 01_{m-1} & \cdots & I_{m-1} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
1_{m-1} & 01_{m-1} & 01_{m-1} & \cdots & I_{m-1}
\end{pmatrix}.$$ 

Thus, $\det A = \det(D_1 + C'D_0C) / (\det D_0 \det D_1)$. If we let $B = (w_{ij})_{i,j=1,\ldots,m-1}$, then since $\sum_{i=1}^m w_{ij} = 1/m$, for all $j = 1, \ldots, m$, and $\sum_{j=1}^m w_{ij} = 1/m$ for all $i = 1, \ldots, m$,

$$D_1 + C'D_0C = \begin{pmatrix}
1 - 2(1/m - w_{mm}) & 1'B' & 1'B \\
B1 & (1/m)I & B \\
B'1 & B' & (1/m)I
\end{pmatrix}.$$ 

By elementary row and column operations, we get

$$\det(D_1 + C'D_0C) = \det \left( \begin{pmatrix}
1 & (1/m)1' & (1/m)1' \\
(1/m)1 & (1/m)I & B \\
(1/m)1 & B' & (1/m)I
\end{pmatrix} \right),$$

so that

$$\det(D_1 + C'D_0C) = \det \left( \begin{pmatrix}
(1/m)I & B \\
B' & (1/m)I
\end{pmatrix} - (1/m^2)1_{2(m-1)}1'_{2(m-1)} \right),$$

$$= \det \left( (1/m)(I - (1/m)11') \quad B - (1/m^2)11' \\
B' - (1/m^2)11' \quad (1/m)(I - (1/m)11') \right),$$

$$= \det \left( (1/m)(I - (1/m)11') \right) \det \left( [(1/m)(I - (1/m)11')]^{-1}B - (1/m^2)11' \right).$$
Finally, \( \det \left( \frac{1}{m} (I - \frac{1}{m} 11') \right) = \left( \frac{1}{m} \right)^m \) and \( \left[ \frac{1}{m} (I - \frac{1}{m} 11') \right]^{-1} = m (I + 11') \), thus
\[
\det(D + C'D_0C) = \left( \frac{1}{m} \right)^m \det((1/m) (I - mV'V)),
\]
where \( V = (w_{ij})_{i=1,...,m;j=1,...,m-1} \).

**Proof of Theorem 1**. We prove that the Jeffreys prior is proper. Consider the following partition of \( V \),

\[
V = (V_1 V_2 \cdots V_{m-1}),
\]
where each \( V_j \) is a vector, \( j = 1, \ldots, m - 1 \). The matrix \( (1/m)I - mV'V \) is symmetric nonnegative semi-definite, so that by Hadamard’s inequality, we have
\[
det((1/m)I - mV'V) \leq \left( \frac{1}{m} \right)^{m-1} \prod_{1 \leq i < k \leq m} w_{ij} w_{kj}.
\]

Let \( A = \{ \alpha_{ij} \in \{1/2, 1\} : i, j = 1, \ldots, m, \alpha_{+j} = m/2 + 1, j = 1, \ldots, m - 1 \text{ and } \alpha_{+m} = m/2 \} \). For any \( W \in \mathcal{W} \), we have
\[
\sqrt{I(W)} = \sqrt{\det((1/m)I - mV'V) / \prod_{i,j=1}^{m} w_{ij}},
\]
\[
\leq \sqrt{2^{m-1}/m} \left\{ \prod_{1 \leq i < k \leq m} w_{ij} w_{kj} \right\}^{1/2} / \prod_{i,j=1}^{m} w_{ij},
\]
\[
\leq \sqrt{2^{m-1}/m} \sum_{1 \leq i < k \leq m} \prod_{j=1}^{m} w_{ij} w_{kj} \left\{ \prod_{i,j=1}^{m} w_{ij} \right\}^{1/2},
\]
\[
= \sqrt{2^{m-1}/m} \sum_{\alpha \in A} \prod_{i,j=1}^{m} w_{ij}^{\alpha_{ij}-1}.
\]

We need to show that the integral of \( \prod_{i,j=1}^{m} w_{ij}^{\alpha_{ij}-1} \) is finite for all \( \alpha \in A \). The integration is made with respect to \( w_{ij}, i \lor j < m \), the free variables. For any
permutation matrices $P_1$ and $P_2$, the transformation $W \mapsto P_1 WP_2$ is a one to one transformation from $\mathcal{W}$ onto $\mathcal{W}$, and the Jacobian, in absolute value, is equal to one. Therefore, it is sufficient to verify that the integral of $\prod_{i,j=1}^m w_{ij}^{\alpha_{ij}-1}$ is finite for all $\alpha \in \mathcal{A}_0$, where $\mathcal{A}_0 = \{\alpha \in \mathcal{A} : \alpha_{m-1m} = \alpha_{mm} = 1\}$. The idea is to decompose the multiple integral into $m - 2$ iterated integrals over the sections given by

$$\mathcal{W}_k = \{w_{ij} \geq 0: i \land j = k, i \lor j \leq m, w_{k+} = w_{+k} = 1/m\}, \quad k = 1, \ldots, m - 2,$$

and

$$\mathcal{W}_{m-1} = \{w_{ij} \geq 0: i,j = m - 1, m, w_{m-1+} = w_{+m-1} = w_{m+} = w_{+m} = 1/m\}.$$ 

Here, the set $\mathcal{W}_1$ is fixed, the sets $\mathcal{W}_k$ are parameterized by $\{w_{ij} \geq 0: i \land j < k, i \lor j = k\}$, $k = 2, \ldots, m - 2$, and $\mathcal{W}_{m-1}$ is parameterized by $\{w_{ij} \geq 0: i \land j < m - 1, i \lor j = m - 1, m\}$. By Fubini’s Theorem, for any nonnegative function $f$, we can write

$$\int_{\mathcal{W}} f(W) \prod_{i,j=1}^{m-1} dw_{ij} = \int_{\mathcal{W}_1} \left\{ \cdots \int_{\mathcal{W}_{m-1}} \{f(W)dw_{m-1m-1}\} \cdots \right\} \prod_{i,j=1, i \lor j < m} dw_{ij}.$$ 

The next step consists in finding finite functions $c_k$, $k = 1, \ldots, m - 1$, on $\mathcal{A}_0$, such that

$$\int_{\mathcal{W}_k} \prod_{i,j=k, i \land j \leq m} w_{ij}^{\alpha_{ij}-1} \prod_{i,j=k, i \lor j < m} dw_{ij} \leq c_k(\alpha),$$

for all $\alpha \in \mathcal{A}_0$, uniformly on $\{w_{ij} \geq 0: i \land j < k, i \lor j = k\}$, for $k = 1, \ldots, m - 2$, and uniformly on $\{w_{ij} \geq 0: i \land j < m - 1, i \lor j = m - 1, m\}$, for $k = m - 1$. This will give us that

$$\int_{\mathcal{W}} \prod_{i,j \leq m} w_{ij}^{\alpha_{ij}-1} \prod_{i,j < m} dw_{ij} \leq \prod_{k=1}^{m-1} c_k(\alpha),$$

for all $\alpha \in \mathcal{A}_0$.

Let $a = 0 \lor \{\sum_{\ell<m-1} (w_{\ell m} - w_{m-1\ell})\} \lor \{\sum_{\ell<m-1} (w_{m \ell} - w_{\ell m-1})\}$ and $b = 1/m - (\sum_{\ell<m-1} w_{\ell m-1})$. If $a > b$, the set $\mathcal{W}_{m-1}$ is empty. Suppose that $\mathcal{W}_{m-1}$ is not empty and $\alpha \in \mathcal{A}_0$. Let $b_0 = 1/m - \sum_{\ell<m-1} w_{\ell m-1}$. 


We have
\[
\int_{\mathcal{W}_{m-1}} \prod_{i,j=m-1,m} w_{ij}^{\alpha_{ij}-1} dw_{m-1,m-1} = \int_a^b u^{\alpha_{m-1,m-1}-1}(b_0 - u)^{\alpha_{m,m-1}-1} du,
\]
\[
\leq \int_0^{b_0} u^{\alpha_{m-1,m-1}-1}(b_0 - u)^{\alpha_{m,m-1}-1} du,
\]
\[
= b_0^{\alpha_{m-1,m-1}+\alpha_{m,m-1}-1} B(\alpha_{m-1,m-1}, \alpha_{m,m-1}),
\]
\[
\leq B(\alpha_{m-1,m-1}, \alpha_{m,m-1}),
\]
\[
= c_{m-1}(\alpha).
\]

For \(k = 1, \ldots, m-2\) and \(\alpha \in \mathcal{A}_0\) we can take
\[
c_k(\alpha) = \left( B \left( \alpha_{kk}, \sum_{i=k+1}^m \alpha_{ik} \right) + B \left( \alpha_{kk}, \sum_{j=k+1}^m \alpha_{kj} \right) \right) \frac{\prod_{i>k} \Gamma(\alpha_{ik}) \prod_{j>k} \Gamma(\alpha_{kj})}{\Gamma(\sum_{i>k} \alpha_{ik}) \Gamma(\sum_{j>k} \alpha_{kj})}.
\]

The justification is given by Lemma 6, the following Lemma.

**Lemma 6.** If \(0 < a, b \leq 1, m \geq 3, \alpha > 0,\)
\[
\beta_j > 0, j = 1, \ldots, m-1, \text{ with } \beta = \sum_{j=1}^{m-1} \beta_j \geq 1,
\]
\[
\gamma_i > 0, i = 1, \ldots, m-1, \text{ with } \gamma = \sum_{i=1}^{m-1} \gamma_i \geq 1,
\]
and
\[
C = \left\{ w_{ij} \geq 0 : i \wedge j = 1, i \vee j \leq m, \sum_{j=1}^m w_{1j} = a, \sum_{i=1}^m w_{i1} = b \right\},
\]
then
\[
\int_C w_{11}^{\alpha_{11}-1} \prod_{j=2}^m w_{1j}^{\beta_{1j-1}-1} \prod_{i=2}^m w_{ij}^{\gamma_{i-1}-1} dw_{11} \prod_{j=2}^{m-1} dw_{1j} \prod_{i=2}^{m-1} dw_{i1} \leq \frac{(B(\alpha, \beta) + B(\alpha, \gamma)) \prod_{j=1}^{m-1} \Gamma(\beta_j) \prod_{i=1}^{m-1} \Gamma(\gamma_i)}{\Gamma(\beta) \Gamma(\gamma)}.
\]

**Proof.** Let
\[
K(a, b, \alpha, \beta, \gamma) = \int_0^{a \wedge b} w^{\alpha-1}(a - w)^{\beta-1}(b - w)^{\gamma-1} dw.
\]
If \( a < b \), then
\[
K(a, b, \alpha, \beta, \gamma) = \int_0^a w^{\alpha-1} (a - w)^{\beta-1} (b - w)^{\gamma-1} \, dw,
\]
\[
\leq b^{\gamma-1} \int_0^a w^{\alpha-1} (a - w)^{\beta-1} \, dw,
\]
\[
= a^{\alpha+\beta-1} b^{\gamma-1} B(\alpha, \beta) \leq B(\alpha, \beta).
\]
In the same way, if \( b < a \), then
\[
K(a, b, \alpha, \beta, \gamma) \leq B(\alpha, \gamma),
\]
so that
\[
K(a, b, \alpha, \beta, \gamma) \leq B(\alpha, \beta) + B(\alpha, \gamma).
\] (1)

Now, let \( W_{11} \) be a random variable on \((0, a \wedge b)\) with density
\[
\frac{1}{K(a, b, \alpha, \beta, \gamma)} w_{11}^{\alpha-1} (a - w_{11})^{\beta-1} (b - w_{11})^{\gamma-1},
\]
let \( (U_{12}, \ldots, U_{1m}) \) be a random vector distributed according to a Dirichlet(\( \beta_1, \ldots, \beta_{m-1} \)),
let \( (U_{21}, \ldots, U_{m1}) \) be distributed according to a Dirichlet(\( \gamma_1, \ldots, \gamma_{m-1} \)), and further assume independence between \( W_{11}, (U_{12}, \ldots, U_{1m}), \) and \( (U_{21}, \ldots, U_{m1}) \). Let
\[
W_{1j} = (a - W_{11}) U_{1j}, \quad j = 2, \ldots, m,
\]
and \( W_{i1} = (b - W_{11}) U_{i1}, \quad i = 2, \ldots, m. \)
From this construction, given \( W_{11} = w_{11} \), we have that \( (W_{12}, \ldots, W_{1m}) \) and
\( (W_{21}, \ldots, W_{m1}) \) are conditionally independent with conditional densities given respectively by
\[
\frac{1}{(a - w_{11})^{\beta-1} \prod_{j=1}^{m-1} \Gamma(\beta_j)} w_{12}^{\beta_1-1} \cdots w_{1m}^{\beta_{m-1}-1},
\]
with \( w_{1j} \geq 0, j = 2, \ldots, m, \sum_{2 \leq j \leq m} w_{1j} = a - w_{11} \) and
\[
\frac{1}{(b - w_{11})^{\gamma-1} \prod_{i=1}^{m-1} \Gamma(\gamma_i)} w_{21}^{\gamma_1-1} \cdots w_{m1}^{\gamma_{m-1}-1},
\]
with \( w_{i1} \geq 0, i = 2, \ldots, m, \sum_{2 \leq i \leq m} w_{i1} = b - w_{11} \). This construction together with inequality (1) implies the result, namely
\[
\int_C w_{11}^{\alpha-1} \prod_{j=2}^m w_{1j}^{\beta_j-1} \prod_{i=2}^m w_{i1}^{\gamma_i-1} \, dw_{11} \prod_{j=2}^m dw_{1j} \prod_{i=2}^m dw_{i1} \leq

(B(\alpha, \beta) + B(\alpha, \gamma)) \frac{\prod_{j=1}^{m-1} \Gamma(\beta_j) \prod_{i=1}^{m-1} \Gamma(\gamma_i)}{\Gamma(\beta) \Gamma(\gamma)}.
\]
\( \square \)
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