A PROOF OF SMALE’S MEAN VALUE CONJECTURE

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Abstract. A proof of Smale’s mean value conjecture from 1981 is given.

Connected with his investigations on the complexity of determining polynomial roots by Newton’s method, Steve Smale [2] considered difference quotients

$$D(\zeta, z) = \frac{p(\zeta) - p(z)}{\zeta - z},$$

where \( p \) is a non-constant polynomial, \( p'(\zeta) = 0 \) and \( z \neq \zeta \) an arbitrary complex number. He asked for some universal (i.e., valid for all such polynomials and all \( z \neq \zeta \)) constant \( K \) such that \( |D(\zeta, z)| \leq K|p'(z)| \) for at least one derivative zero \( \zeta \).

He proved in [2], using results on univalent functions, that this is true for \( K = 4 \) and conjectured \( K = 1 \) to be best possible.

Obviously one may without loss of generality assume that \( z = 0 \) and \( p(0) = 0 \). Then the question is to estimate the number \( \min \left\{ \left| \frac{p(\zeta)}{\zeta p'(0)} \right| : p'(\zeta) = 0 \right\} \). Note that the conjecture trivially holds for polynomials of degree one.

The conjectured bound 1 can be sharpened a little bit if we consider only polynomials of a fixed degree. Here we will prove the following:

Let \( p \in \mathbb{C}[z] \) be a polynomial of degree \( n > 1 \) with \( p(0) = 0 \) and \( p'(0) \neq 0 \). Then

$$\min \left\{ \left| \frac{p(\zeta)}{\zeta p'(0)} \right| : p'(\zeta) = 0 \right\} \leq \frac{n - 1}{n}.$$

Equality only occurs for \( p(z) = a_1z + a_nz^n \) with arbitrary \( a_1, a_n \in \mathbb{C} \setminus \{0\} \).

Let \( n > 1 \) be fixed and define \( \mathcal{F}_n \) as the class of \( n \)th degree monic complex polynomials \( p \) with \( p(0) = 0 \), \( p'(0) \neq 0 \) and \( p(\zeta) \neq 0 \) for all derivative zeros \( \zeta \) of \( p \).

Obviously it suffices to consider polynomials \( p \in \mathcal{F}_n \) in order to give a proof of Smale’s conjecture. For such \( p \) we define

$$\rho(p, \zeta) := \left| \frac{p(\zeta)}{\zeta p'(0)} \right|$$

and the associated number as

$$\rho(p) := \min \left\{ \rho(p, \zeta) : p'(\zeta) = 0 \right\}.$$

The zero \( \zeta_0 \) of \( p' \) is essential if

$$\rho(p) = \left| \frac{p(\zeta_0)}{\zeta_0 p'(0)} \right|.$$

Note that a polynomial may have more than one essential derivative zero. We call \( p \in \mathcal{F}_n \) simple if \( p''(\zeta) \neq 0 \) for all essential derivative zeros \( \zeta \) of \( p \).
A polynomial \( p_0 \in \mathcal{F}_n \) is maximal if \( \rho(p) \leq \rho(p_0) \) for all \( p \in \mathcal{F}_n \). Below we will determine the maximal polynomials in \( \mathcal{F}_n \). In the following we will prove:

**Theorem 1.** For each \( p \in \mathcal{F}_n \) there exists some \( q \in \mathcal{F}_n \) which zeros \( w_2, \ldots, w_n \neq 0 \) have the same modulus and it holds \( \rho(q) \geq \rho(p) \).

In order to prove theorem 1 we may without loss of generality assume that \( |z_j| \leq 1 \) holds for the zeros \( z_2, \ldots, z_n \neq 0 \) of \( p \), and equality is taken for at least one of them. Otherwise we consider the polynomial \( s^n p(z/s) \) with \( s = \max\{|z_2|, \ldots, |z_n|\} \). The associated number of this polynomial is the same than those of \( p \). Moreover we may assume that \( |z_n| < 1 \).

### 1. The basic idea

If \( p \in \mathcal{F}_n \) is a polynomial with the zeros \( z_2, \ldots, z_n \) besides 0 and the derivative zero \( \zeta \) with \( p(\zeta) \neq 0 \), then

\[
\frac{p'(\zeta)}{p}(\zeta) = 0 = \frac{1}{\zeta} + \sum_{j=2}^{n} \frac{1}{\zeta - z_j}.
\]

As explained we may provide that \( |z_j| \leq 1 \) for \( j = 2, \ldots, n \) and \( |z_n| < 1 \). We let \( z_2, \ldots, z_{n-1} \) be fixed and vary \( z_n \), i.e., we consider the polynomials

\[
Q(z, u) = (z - u) z \prod_{j=2}^{n-1} (z - z_j) = (z - u) q(z).
\]

We assume for the moment that \( \zeta \) is a zero of \( p' \), but not a zero of \( p'' \). The implicit function theorem (cf. [1]) shows the existence of a holomorphic function \( \zeta(u) \) with \( \zeta(z_n) = \zeta \) and \( \frac{\partial Q}{\partial z}(\zeta(u), u) \equiv 0 \), defined in a neighborhood of \( z_n \). If we move \( u \) along a path \( \gamma \) in \( \mathbb{C} \) starting in \( \gamma(0) = z_n \) then we have an unrestricted analytic continuation of \( \zeta(\gamma(t)) \) if \( \frac{\partial^2 Q}{\partial z^2}(\zeta(\gamma(t)), \gamma(t)) \neq 0 \) for all \( t \). If the path would meet these exceptional points, we would have at least a continuation of \( \zeta(\gamma(t)) \) which is at least continuous in such points. Note that the values of \( \zeta(\gamma(t)) \) with respect to this continuation move on the Riemann surface \( R \), which is defined by the equation \( Q'(z, u) = 0 \) (derivative with respect to \( z \)). We will discuss this surface in section 2.

It comes out (\( Q' \) denotes the derivative of \( Q \) with respect to \( z \))

\[
\frac{Q(\zeta(u), u)}{\zeta(u) Q'(0, u)} = \left( \frac{\zeta(u)}{u} - 1 \right) \prod_{j=2}^{n-1} \left( \frac{\zeta(u)}{z_j} - 1 \right)
\]

\[
= \left( \frac{\zeta(u)}{u} - 1 \right) q(\zeta(u)) \prod_{j=2}^{n-1} z_j^{-1} =: g(u, \zeta(u)).
\]

Note that \( \ln \rho(Q(., u), \zeta(u)) = \Re \log g(u, \zeta(u)) \).
Let \( p \in \mathcal{F}_n \) and \( \zeta \) be a (not necessarily essential) derivative zero of \( p \). As above let \( 0, z_2, \ldots, z_n \in \mathbb{C}_n^{\ast} \) be the zeros of \( p \) and \( |z_n| < 1 \). If \( \gamma : [0, 1] \to \mathbb{C} \) is a path with \( \gamma(0) = z_n, \gamma(1) = u \) we see

\[
\frac{d}{dt} \ln \rho(Q(\cdot, \zeta(t)), \zeta(\gamma(t))) = \frac{d}{dt} \Re \log g(\gamma(t), \zeta(\gamma(t))) = \frac{d}{dt} \frac{g(\gamma(t), \zeta(\gamma(t)))}{g(\gamma(t), \zeta(\gamma(t)))}.
\]

Note that \( \zeta(\gamma(t)) \) depends on the path \( \gamma \). So we have

\[
\ln \rho(Q(\cdot, u), \zeta(u)) - \ln \rho(p, \zeta) = \int_0^1 \frac{d}{dt} \ln f(\gamma(t), \zeta(\gamma(t))) dt = \Re \int_0^1 \frac{d}{dt} \log g(\gamma(t), \zeta(\gamma(t))) dt.
\]

The integrand can be calculated as

\[
\frac{d}{dt} \frac{g(\gamma(t), \zeta(\gamma(t)))}{g(\gamma(t), \zeta(\gamma(t)))} = \gamma'(t) \left( \frac{\zeta'(\gamma(t))\gamma(t) - \zeta(\gamma(t))}{(\zeta(\gamma(t)) - \gamma(t))\gamma(t)} + \frac{\zeta'(\gamma(t))q'(\zeta(\gamma(t)))}{q(\zeta(\gamma(t)))} \right),
\]

and this leads to

\[
\ln \rho(Q(\cdot, u), \zeta(u)) - \ln \rho(p, \zeta) = \Re \int_\gamma \frac{\zeta'(v)}{(\zeta(v) - v)} + \zeta'(v) \left( \frac{1}{\zeta(v) - v} + \frac{q'}{q}(\zeta(v)) \right) dv.
\]

The right hand side can be written as

\[
\Re \int_\gamma \frac{-1}{\zeta(v) - v} \cdot \frac{\zeta(v)}{v} + \zeta'(v) \left( \frac{1}{\zeta(v) - v} + \frac{q'}{q}(\zeta(v)) \right) dv.
\]

From (1) we obtain

\[
0 = \frac{Q'(\zeta(v), v)}{Q(\zeta(v), v)} = \frac{1}{\zeta(v)} + \frac{1}{\zeta(v) - v} + \frac{q'}{q}(\zeta(v)).
\]

It comes out

\[
\ln \rho(Q(\cdot, u), \zeta(u)) = \ln \rho(p, \zeta) - \Re \int_\gamma \frac{1}{\zeta(v) - v} \cdot \frac{\zeta(v)}{v} + \frac{\zeta'(v)}{\zeta(v)} dv,
\]

and therefore

(4) \[
\rho(Q(\cdot, u)), \zeta(u)) = \rho(p, \zeta) \cdot \left| \exp \left( - \int_\gamma \frac{1}{\zeta(v) - v} \cdot \frac{\zeta(v)}{v} + \frac{\zeta'(v)}{\zeta(v)} dv \right) \right|
\]

\[
= \rho(p, \zeta) \cdot \left| \frac{\zeta}{\zeta(u)} \right| \cdot \left| \exp \left( - \int_\gamma \frac{1}{\zeta(v) - v} \cdot \frac{\zeta(v)}{v} dv \right) \right|.
\]

2. The Riemann surface \( R \)

The Riemann surface \( R \) of the derivative zeros of \( Q \) is given by the equation

(5) \[
Q'(w) = q(w) + (w - u)q'(w) = 0.
\]

This (actually compact) manifold \( R \) consists of the points \( w \) (which are the derivative zeros of \( Q(\cdot, u) \), and the equation gives local uniformizations of \( R \), if the derivative of \( u = \varphi(w) := w + \frac{q}{q'}(w) \) with respect to \( w \) does not vanish (note that these branch points are also described by \( \frac{\partial^2 Q}{\partial z^2}(w, u) = 0 \)). So the points \( w \) where \( 2q'(w)^2 = \)
$q(w)q''(w)$ are branch points of the surface. This branch points play in fact no special role on the Riemann surface, their appearance depend on the special local coordinates, which are given by the defining equation (example: the surface of the square root is defined by $w^2 = u$ with 0 as a branch point; if we add this point, it is conformally equivalent to the plane resp. $\overline{\mathbb{C}}$). They can actually added as "normal" points to the surface and have simply connected neighborhoods on which local coordinates can be found.

$R$, as a compact surface, may be regarded as a $(n-1)$-sheeted covering of $\overline{\mathbb{C}}$, and $\varphi$ gives a canonically projection $R \to \overline{\mathbb{C}}$.

We define

$$f(u, \zeta(u)) := \frac{\zeta}{\zeta(u)} \exp \left( -\int_{\gamma_u} \frac{1}{\zeta(v) - v} \cdot \frac{\zeta(v)}{v} \, dv \right),$$

where $\gamma_u : [0, 1] \to \mathbb{C}$ with $\gamma_u(0) = z_n, \gamma_u(1) = u$ and $\zeta(\gamma_u(0)) = \zeta_0$ (some fixed derivative zero of $p$), $\zeta(\gamma_u(1)) = \zeta(u)$. By (6) we have

$$\rho(Q(u), \zeta(u)) = \rho(p, \zeta_0) \cdot |f(u, \zeta(u))|.$$

$f$ is, up to isolated singularities, a holomorphic function on $R$, because it has this property in the local coordinate $u \in \mathbb{C}$ (the case $u = \infty$ we discuss separately). The holomorphy is not obviously clear in the following cases.

(i) $Q'(0, u_0) = \frac{\partial Q}{\partial z}(0, u_0) = 0$, or

(ii) $Q(w_1, u_1) = 0$ (this includes the case $u = \zeta(u)$), or

(iii) $2q'(w_2)^2 = q(w_2)q''(w_2)$ (branch points)

(in case of (i) or (ii) the polynomial $Q$ does not belong to the class $\mathcal{F}_n$). We discuss this three cases.

Case (i): The polynomial $p$ has only simple zeros. So $Q'(0, u_0) = 0$ is only possible if $u = 0$. A direct calculation gives that $\rho(Q(., 0)) = \rho(Q(., 0), 0) = \frac{1}{2}$. Thus $f$ is has a removable singularity in $u = 0$ if $\zeta(u) = 0$. If $\varphi(w_0) = 0$, but $w_0 \neq 0$ (and thus is not essential for $Q(., 0)$) then we see that $f$ has a pole in $w_0$, because $\rho(Q(., w), w) \to \infty$ if $w \to w_0$.

Case (ii): The assumption implies that $Q(., u_1)$ has a multiple zero in the point $w_1$. This is only possible if $u_1$ is one of the zeros $z_2, \ldots, z_{n-1}$ of $p$ ($u = 0$ has already been discussed) and $u_1 = w_1$. By the definition we see that $\rho(Q(., u_1), w_1) = 0$ and $\rho(Q(., u_1), w) > 0$ if $\varphi(w) = u_1$ and $w \neq w_1$. So these singularities of $f$ are removable. Moreover we have $\rho(Q(., u_1), w_1) = 0$ in this case.

Case (iii): If $2q'(w_2)^2 = q(w_2)q''(w_2)$, then $w_2 \notin \{0, z_2, \ldots, z_{n-1}\}$, because $q$ has only simple zeros in these points. $f$ shows that $f$ is bounded in a neighborhood of the branch point $w_2$ on $R$. Again we conclude that $f$ has a removable singularity in this case.

We summarize:

**Lemma 1.** The function $f$ as defined in (6) is meromorphic on the Riemann surface $R' := \{w \in R : \varphi(w) \in \mathbb{C}\}$. It has poles exactly in the points $w \in R'$ with $\varphi(w) = 0$ and $w \neq 0$. The zeros of $f$ are the points $w \in R'$ with $w = \varphi(w) \in \{z_2, \ldots, z_{n-1}\}$. 
We can give an alternative representation of \( f \). It holds \( \rho(Q(.,u), \zeta(u)) = \frac{Q(\zeta(u),u)}{\zeta(u)Q'(0,u)} \).

From (7) we obtain that \( f(u, \zeta(u)) \) equals \( \frac{\zeta u q'(0)}{\rho(u)} Q(\zeta(u),u) \), up to a possible factor of modulus one. For \( u = z_n \) we see that this factor is one. By (5) we receive the representation:

\[
(8) \quad f(u, \zeta(u)) = \frac{z_n \zeta_0 q'(\zeta_0)}{q(\zeta_0)^2} \cdot \frac{q(\zeta(u))^2}{u \zeta(u) q'(\zeta(u))}.
\]

Finally we investigate the structure of \( R \) close to \( u = \infty \). The point infinity is no branch point of \( R \), because the function \( 1/\varphi(1/w) \) has in \( w = 0 \) the expansion \( w(\frac{a_1}{n} + a_2 w + \ldots) \).

For \( u \in E \) all zeros of \( Q(.,u) \) are contained in \( E \). By the Gauß-Lucas theorem we know that the zeros of the derivative \( Q'(z,u) = \frac{\partial Q}{\partial z}(z,u) \) lie in the convex hull \( C \) of the zeros. They are inner points of \( C \) with the only exception of multiple zeros of \( Q \). None of these derivative zeros in our case is of bigger order than 1. So the same argument gives that the zeros of the second order derivative \( Q''(z,u) = \frac{\partial^2 Q}{\partial z^2}(z,u) \) are points the open unit disk \( E \). So the same is true for the branch points of \( R \). To be more precise, all branch points \( w \) of \( R \) fulfill \( |\varphi(w)| < 1 \).

The subset \( D_1 \) of \( R \) with \( \varphi(D_1) = \overline{E} \) therefore contains all branch points.

As a consequence, the complement \( R \setminus D_1 \) (including \( \infty \)) consists of \( n-1 \) simply connected domains \( G_1, \ldots, G_{n-1} \). Let \( \zeta(u) \) be the function which is defined on \( G_k \) with respect to a fixed start point \( \zeta_0 \) with \( \varphi(\zeta_0) = z_n \). Then the mappings \( \Phi_k := \varphi|G_k = \varphi|G_k : G_k \rightarrow \{ u \in \mathbb{C} : |u| > 1 \} \) are conformal.

The boundaries of the domains \( G_j \) are pairwise disjoint. Each \( \partial G_j \) is mapped homeomorphically by \( \varphi \) on the unit circle.

It holds \( P(z,u) := \frac{Q(z,u)}{u} = (\frac{u}{a_1} - 1)q(z) \). The derivative zeros of \( P \) with respect to \( z \) are the same as those of \( Q \). For \( u \rightarrow \infty \) the polynomials \( P(z,u) \) tend locally uniformly to \( q(z) \). So, in this case, \( \zeta(u) \) tends to \( \infty \) on one \( G_k \), let us say on \( G_1 \). For \( k = 2, \ldots, n-1 \) it follows that each \( \zeta_j(u) \in G_k \) tends to some derivative zero \( \xi_k \) of \( q' \) if \( u \rightarrow \infty \).

2.1. \( \zeta(u) \) on \( G_1 \). From (8) we see that \( \zeta(u) \) has a pole in \( \infty \in G_1 \). The equation

\[
1 - \frac{u}{\zeta(u)} = - \frac{q(\zeta(u))}{\zeta(u)q'(\zeta(u))}
\]

gives that \( \frac{u}{\zeta(u)} \rightarrow \frac{n+1}{n} \). It holds

\[
f(u, \zeta(u)) = \frac{z_n \zeta_0 q'(\zeta_0)}{q(\zeta_0)^2} \cdot \frac{q(\zeta(u))^2}{u \zeta(u) q'(\zeta(u))} = \frac{z_n \zeta_0 q'(\zeta_0)}{q(\zeta_0)^2} \cdot \frac{q(\zeta(u))^2}{\zeta(u) q'(\zeta(u))} \cdot \frac{q(\zeta(u))}{u \zeta(u)} \cdot \frac{\zeta(u) q'(\zeta(u))}{\zeta(u) q'(\zeta(u))}.
\]

All fractions stay to be finite (and non zero) for \( u \rightarrow \infty \), except of the last one, which has a pole of order \( n-2 \) in \( \infty \), and so \( f \) has.

2.2. \( \zeta(u) \) on \( G_k \) for \( k > 1 \). In this cases \( \zeta(u) \) tends to some derivative zero \( \xi_k \) of \( q' \). From

\[
0 = \zeta(u)q'(\zeta(u)) - uq'(\zeta(u)) + q(\zeta(u))
\]

we conclude that \( uq'(\zeta(u)) \rightarrow q(\xi_k) \) if \( u \rightarrow \infty \). Now we see from (8) that \( f \) is holomorphic in \( \infty_k \in G_k \), and \( c_k := f(\infty_k) = \frac{z_n \zeta_0 q'(\zeta_0)}{q(\zeta_0)^2} \cdot q(\xi_k) \). Thus \( f \) is holomorphic
on $G_k$. Moreover $f$ does not vanish in $G_k$, because the zeros of $q$ are all in $\overline{E}$. But on the boundary (as well as on the boundary of $G_1$) there will be some zero, which comes from the zero(s) of $p$ on the unit circle.

3. Blowing up and pulling back

Let $r > 0$ and $p_r(z) = r^n p(z/r)$. If we start the considerations of the preceding section with $p_r$ instead of $p$ we have to replace the zeros $z_2, \ldots, z_n$ of $p$ by $rz_2, \ldots, rz_n$ and the derivative zeros $\zeta(u)$ by $r\zeta(u)$ as well as $q(z)$ by $r^{n-1}q(z/r)$. The variation is then

$$Q_r(z, u) := r^n Q(z/r, u) = z(z - ur) \cdot r^{n-1} q(z/r) = z(z - ur)^{n-1} \prod_{j=2}^{n-1} (z - z_j r).$$

Note that the zeros of $Q_r(., u)$ are the points $ru, rz_2, \ldots, rz_n$, and it has the derivative zeros $r\zeta(u)$, where $\zeta(u)$ denotes those of $Q(., u)$.

As already mentioned we have $\rho(p_r) = \rho(p)$ for all $r > 0$. Let $u_0$ be some complex number of modulus $r$. If $r$ is large enough me may provide that

$$|f(ru_0(r), r\zeta(u_0(r)))| > |c_k|/2$$

if $r\zeta(u_0(r)) \in G_2, \ldots, G_{n-1}$. If $r\zeta(u_0(r)) \in G_1$ we may, because of the pole of $f$ in $\infty_1 \in G_1$, assume that $|f(ru_0(r), r\zeta(u_0(r)))| > 1$. Now [8] and (7) show

$$\rho(Q_r(., u), r\zeta(u)) = r \cdot f(ru, r\zeta(u)).$$

So $\rho(Q_r(., u_0(r)), r\zeta(u_0(r))) > \rho(p, \zeta_0)$ for all sufficiently large $r$ and all derivative zeros of this polynomial. If $\zeta_0$ has been taken above as an essential derivative zero of $p$ this says that $\rho(Q_r(., u_0(r)), \zeta(u_0(r))) > \rho(p)$ for all derivative zeros $r\zeta(u_0(r))$ of $Q_r(., u_0(r))$. This gives, together with the remark above, $\rho(Q_r(., u_0(r))) > \rho(p_r) = \rho(p)$.

The polynomial $Q_r(., u_0(r))$ has all its zeros in $|z| \leq r$ and one zero more on the boundary of this disk than $p$ have on the unit circle (namely $u_0(r)$, in which $z_n$ has been changed). By $p^*(z) := r^{-n}Q_r(zr, u_0(r))$ we pull all the zeros back into the closed unit disk and so we found some polynomial, which has one zero more on the unit circle as $p$ and which fulfills $\rho(p^*) > \rho(p)$.

We can repeat this argument until we obtain a polynomial vanishing only on the unit circle and which associated number is bigger than that of $p$. This finishes the proof of theorem [1].

4. Proof of Smale’s conjecture

It remains to compare $\rho(p)$ for polynomials $p(z) = z \prod_{j=2}^{n}(z - z_j)$ with $|z_2| = \ldots |z_n| = 1$. For such polynomials Smale’s conjecture has already been proved by Tischler [3]. He also determined the maximal polynomials for this subclass of $F_n$ as $p(z) = a_1 z + a_n z^n$ with $a_1, a_n \in \mathbb{C} \setminus \{0\}$, and we have the result that these are indeed the only maximal polynomials in $F_n$. 
References

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