UNEXPECTED HYPERSURFACES AND WHERE TO FIND THEM

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Abstract. This paper approaches the problem of studying unexpected hypersurfaces from the perspectives of algebra, geometry, representation theory and computation. We determine all \((n, d, m)\) for which there is an unexpected hypersurface of degree \(d\) in \(\mathbb{P}^n\) having a general point \(P\) of multiplicity \(m\) with respect to some finite set of points \(Z\). We also give new insight into where to look for the point sets \(Z\). The concept of an unexpected hypersurface (with a focus on plane curves) was introduced in a recent paper by Cook, Harbourne, Migliore and Nagel [CHMN], and those results allow one to recognize point sets \(Z \subset \mathbb{P}^2\) admitting unexpected curves once \(Z\) is in hand. But it is still an interesting question where to look for such \(Z\) in the first place. Recent work of Di Marca, Malara and Oneto [DMO] and of Bauer, Malara, Szemberg and Szpond [BMSS] give new results and examples in \(\mathbb{P}^2\) and \(\mathbb{P}^3\). In this paper we present two new methods for constructing unexpected hypersurfaces. One method applies in \(\mathbb{P}^n\) for \(n \geq 3\) and gives a broad range of examples, which we link to certain failures of the Weak Lefschetz Property. The other method uses root systems to construct new examples both in \(\mathbb{P}^2\) and \(\mathbb{P}^n\) for \(n \geq 3\). We also explain an observation of [BMSS], showing that the unexpected curves of [CHMN] are in some sense dual to their tangent cones at their singular point.

1. Introduction

Let \(K\) be an algebraically closed field of characteristic 0. Let \(R = K[\mathbb{P}^n] = K[x_0, \ldots, x_n]\) be the homogeneous coordinate ring of \(n\)-dimensional projective space. Consider distinct general points \(P_1, \ldots, P_r \in \mathbb{P}^n\) and positive integer multiplicities \(m_1, \ldots, m_r\). The fat point scheme \(X = m_1P_1 + \cdots + m_rP_r\) is the scheme defined by the homogeneous ideal \(I_X = \cap_i I_{P_i}^{m_i} \subset R\), where \(I_{P_i}\) is the ideal generated by all forms that vanish at \(P_i\). Given a homogeneous ideal \(I \subset R\), we denote by \([I]_d\) the \(K\)-vector space spanned by homogeneous forms in \(I\) of degree \(d\).

In [CHMN, Problem 1.4] the following problem was posed:

**Problem 1.1.** Characterize and then classify all quadruples \((n, Z, m, j)\) where \(Z = c_1Q_1 + \cdots + c_sQ_s\) for distinct points \(Q_i \in \mathbb{P}^n\), \(m = (m_1, \ldots, m_r)\) and \(X = m_1P_1 + \cdots + m_rP_r\) for general points \(P_i \in \mathbb{P}^n\), such that \(X\) fails to impose the expected number of conditions on \(V = [I_Z]_j\).

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In fact, it makes sense to pose the same problem for any subscheme $Z$ of $\mathbb{P}^n$, but at this early stage of study, the focus of most research up to now (as was the case in [CHMN]) has been on the case that $(n, Z, m, j) = (2, Z, (m_1), m_1 + 1)$, where $Z$ consists of a finite set of reduced points and $X = m_1 P$. Indeed, the case of greatest interest for us is still when $Z$ is a finite set of reduced points, but now in $\mathbb{P}^n$ more generally, and we obtain some of our results when $Z$ is a finite set of points by starting with a reduced variety $V$ of higher dimension and picking suitable points $Z$ on $V$ imposing independent conditions on hypersurfaces of degree $d$, for which the same unexpected hypersurface arises. Formalizing this idea, we say that a subvariety $Z \subset \mathbb{P}^n$ admits an unexpected hypersurface with respect to $X$ of degree $d$ if

$$\dim[I_Z \cap I_X]_d > \max \left\{ 0, \dim[I_Z]_d - \sum_{i=1}^r \binom{m_i - 1 + n}{n} \right\}.$$ 

That is, $Z$ admits an unexpected hypersurface with respect to $X$ of degree $d$ if the conditions imposed by $X$ on forms vanishing on $Z$ of degree $d$ are not independent. Our main focus will be when $X = mP$ is a single fat point with $P$ general, in which case we will also sometimes say that $Z$ admits an unexpected hypersurface with a general point $P$ of multiplicity $m$.

When $Z = \emptyset$, it is a long-standing open problem to characterize for which multiplicity vectors $m = (m_1, \ldots, m_r)$ and degrees $d$ there occur unexpected hypersurfaces. A conjectural characterization in the case of $n = 2$ is the content of the SHGH Conjecture [Se, Ha, G, Hi], for example. See also [LU] for a conjecture for $\mathbb{P}^3$.

As a means for approaching this problem, and motivated by an example in [DIV], the recent paper [CHMN] gave a careful analysis for the case where $Z$ is a reduced set of points in $\mathbb{P}^2$, $X$ is supported on a single point $P$, and the multiplicity of the unexpected curve at $P$ is one less than the degree of the unexpected curve. Extending results in [DIV] and [FV], [CHMN] studied unexpected curves in this context in $\mathbb{P}^2$ in connection to line arrangements and Lefschetz problems. Given an arrangement of lines in $\mathbb{P}^2$, the results of [CHMN] provide a means for determining whether the reduced scheme $Z$ of points dual to the lines admits an unexpected curve, but it is still very unclear which line arrangements to look at.

One of the best results in this regard is that of [DMO], completely characterizing the supersolvable line arrangements $\mathcal{A}$ such that the points $Z_\mathcal{A}$ dual to $\mathcal{A}$ admit an unexpected curve. Both supersolvable and non-supersolvable line arrangements were studied in [CHMN], and the latter can also give rise to unexpected curves, but it is not clear which ones do.

Moreover, apart from the examples of [BMSS], we are unaware of examples in $\mathbb{P}^n$ for $n > 2$ in the literature. It is a natural and interesting next step to work to understand the range of examples of unexpected hypersurfaces that can occur with an imposed singularity at a single general point in $\mathbb{P}^n$, both in dimension 2 and in higher dimensions, and to find structural connections between the geometry of a reduced finite set of points $Z$ in $\mathbb{P}^n$ for $n \geq 2$ and the existence of such an unexpected hypersurface. In §2 we show that a large class of examples is related to $Z$ lying on projective cones over codimension 2 subvarieties. It ends with an application to the question of when ideals generated by powers of linear forms fail the Weak Lefschetz Property. In §3 we show that root systems can give rise to unexpected hypersurfaces. There were already indications in [CHMN] (see also [I]) that hyperplane arrangements related to reflection groups sometimes give rise to unexpected hypersurfaces (see what was called the Fermat, Klein and Wiman arrangements in [CHMN]; these all come from complex reflection groups). We reinforce these indications here by finding additional examples coming from root systems of real reflection groups.
groups. These have the advantage of providing obvious candidates in higher dimension too. However, not all root systems seem to give rise to unexpected hypersurfaces; it would be an interesting project to understand what is special about those that do. In §4 we present initial results regarding a still mysterious duality between unexpected hypersurfaces having a single imposed general singular point $P$, and their tangent cone at $P$, first observed in [BMSS]. Finally, in §5 we present some open questions arising from our work.

Whereas [CHMN] studied unexpected plane curves of degree $m+1$ having a single imposed singular point of multiplicity $m$, an especially interesting aspect of the current paper is to relax the restriction that the multiplicity be one less than the degree. This greatly expands the universe of examples, as the following theorem shows; this is one of the main consequences of our work in this paper. (It is one of several interesting consequences of the careful analysis of cones over subvarieties of $\mathbb{P}^n$ of codimension two that we give in §2.)

**Theorem 1.2.** Denote by $d$ the degree of an unexpected hypersurface of some finite set of points $Z \subset \mathbb{P}^n$ and by $m$ its multiplicity at a general point $P$ in $\mathbb{P}^n$.

(i) If $n = 2$ then there exists some set $Z$ admitting such an unexpected curve if and only if $(d, m)$ satisfies $d > m > 2$.

(ii) If $n \geq 3$ then there exists a set $Z$ admitting such an unexpected hypersurface if and only if $(d, m)$ satisfies $d \geq m \geq 2$.

It is still very unclear what kinds of unexpected hypersurfaces can occur for each $d$ and $m$. One goal of this paper was to suggest new venues for where to find them. The title of this paper should not, however, be taken to mean that we have found all unexpected hypersurfaces (a title for a possible future paper could be “Unexpected hypersurfaces and where else to find them”). In addition, in contrast to what [CHMN] was able to do in $\mathbb{P}^2$, there are not yet good tools in higher dimension for rigorously verifying unexpectedness. In particular, we are able to give rigorous verifications of unexpectedness for the new examples coming from root systems only in some of the cases where we suspect that they occur.

**Notation.** For any subvariety (or subscheme) $V \subseteq \mathbb{P}^n$ we write $I_V \subseteq R$ for the saturated ideal of $V$ and $\mathcal{I}_V$ for the sheaf on $\mathbb{P}^n$ corresponding to $I_V$. For any integer function $h : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ the first difference $\Delta h$ is the backward difference $\Delta h(t) = h(t) - h(t-1)$, where we make the convention $h(-1) = 0$ (so $\Delta h(0) = h(0)$).

2. Cones

In this section we give a method for constructing examples of varieties $Z$ (not necessarily points) with unexpected hypersurfaces. Although by far the more interesting question is the problem of understanding the unexpected hypersurfaces arising from a finite set of points, one can also begin by asking whether a reduced, non-degenerate curve in $\mathbb{P}^3$ admits unexpected surfaces. We obtain the somewhat surprising fact that they always do! Using Bézout’s theorem we then translate this back to finite sets of points. We also extend this idea to $\mathbb{P}^n$. Finally, we find a connection to the well-studied question of when an ideal generated by powers of linear forms has the Weak Lefschetz Property, extending results of [DIV] who first noticed a connection between cones and WLP.

Our method involves cones. By a cone with vertex $P$ we mean a scheme $X$ such that for every point $Q$ in $X$ the line joining $P$ and $Q$ is in $X$. In particular, by Bézout, every hypersurface of degree $d$ with a point of multiplicity $d$ at a point $P$ is a cone with vertex $P$. 
It is not hard to show that a plane curve of degree $d$ in $\mathbb{P}^3$ does not admit an unexpected hypersurface with a point of multiplicity $d$ – instead, a point of multiplicity $d$ imposes the expected number of conditions on hypersurfaces of degree $d$ containing the plane curve (use the fact that a plane curve in $\mathbb{P}^3$ is a complete intersection, and the known Hilbert function for complete intersections). For non-degenerate curves the situation is very different, as we now show.

**Proposition 2.1.** Let $C$ be a reduced, equidimensional, non-degenerate curve of degree $d$ in $\mathbb{P}^3$ ($C$ may be reducible, disconnected, and/or singular but note that $d \geq 2$ since $C$ is non-degenerate, with $C$ being two skew lines if $d = 2$). Let $P \in \mathbb{P}^3$ be a general point. Then the cone $S_P = S_P(C)$ over $C$ with vertex $P$ is an unexpected surface of degree $d$ for $C$ with multiplicity $d$ at $P$. It is the unique unexpected surface of this degree and multiplicity.

**Proof.** We first check uniqueness. Let $F$ be a form defining a surface containing $C$, of degree $d$ with multiplicity $d$ at $P$. Let $\lambda$ be a line through $P$ and any point $Q$, of $C$. Then by Bézout, $F$ must vanish on all of $\lambda$. Thus the surface defined by $F$ is precisely $S_P$.

We now check unexpectedness. Let $D$ be a smooth plane curve of degree $d$.

**Claim 1:** The arithmetic genus, $g_C$, of $C$ is strictly less than that of $D$, which is $g_D = \binom{d-1}{2}$.

This argument is classical. Much of it is given in [Har] (when $C$ is irreducible) and in [Mi] Proposition 1.4.2, so no claim is made to originality; we include it here just for the reader’s convenience. Let $\Gamma$ be a general hyperplane section of $C$ by a hyperplane $H$ defined by a general linear form $L$. Let $I_{\Gamma|H}$ be the saturated ideal of $\Gamma$ in $H$. Let $\ell \gg 0$. Then

$$d\ell - g_C + 1 = h^0(O_C(\ell)) = h_{R/I_C}(\ell)$$

where $h_{R/I_C}(t)$ is the Hilbert function of $C$. On the other hand, for any integer $t$ we have the exact sequence

$$0 \to [I_C]_{t-1} \to [I_C]_t \to [I_{\Gamma|H}]_t \to H^1(I_C(t-1)) \to \ldots$$

(where $K$ is just the cokernel). Then after adding and subtracting some binomial coefficients and setting $R = R/L$, we obtain

$$\Delta h_{R/I}(t) = h_{R/I_{\Gamma|H}} + \dim [K]_t \geq h_{R/I_{\Gamma|H}}(t).$$

So since $\ell \gg 0$ we obtain

$$g_C = 1 + d\ell - h_{R/I_C}(\ell) = d\ell - \sum_{t=1}^{\ell} \Delta h_{R/I_C}(t) \leq d\ell - \sum_{t=1}^{\ell} h_{R/I_{\Gamma|H}}(t) = \sum_{t=1}^{\ell} \left[d - h_{R/I_{\Gamma|H}}(t)\right].$$
Now replace $C$ by $D$, and replace $\Gamma$ by the hyperplane section of $D$, which is a set of $d$ collinear points, say $A$. We have, similarly,

$$g_D = \sum_{t=1}^{t} \left[ d - h_{R/I_A}(t) \right] = \binom{d - 1}{2}.$$

It is clear that for any $t \geq 1$ we have

$$h_{R/I_A}(t) > h_{\bar{R}/I}(t)$$

so we obtain $g_C < g_D$. This completes the proof of Claim 1.

Now, by [GLP] Remark (1) (p. 497), we have $H^1(I_C(d)) = H^2(I_C(d)) = 0$, and we also have $H^1(I_D(d)) = H^2(I_D(d)) = 0$. Consider the exact sequence

$$0 \to [I_C]_d \to H^0(O_{\mathbb{P}^3}(d)) \to H^0(O_C(d)) \to H^1(I_C(d)) \to 0.$$

**Claim 2:** Let $D$ be a plane curve of degree $d$. Then $h^0(O_C(d)) > h^0(O_D(d))$.

Indeed,

$$d^2 - g_C + 1 = h^0(O_C(d)) - h^1(O_C(d)) = h^0(O_C(d)) - h^2(I_C(d)) = h^0(O_C(d))$$

and similarly for $D$, so thanks to Claim 1 we have

$$h^0(O_C(d)) > h^0(O_D(d))$$

and we have Claim 2.

Then

$$\dim[I_C]_d = \binom{d + 3}{3} - h^0(O_C(d)) \quad \text{(since } h^1(I_C(d)) = 0)$$

$$< \binom{d + 3}{3} - h^0(O_D(d))$$

thanks to Claim 2. To check that $S_P$ is an unexpected surface it is enough to note that

$$\dim[I_C]_d - \binom{d - 1 + 3}{3} < \binom{d + 3}{3} - h^0(O_D(d)) - \binom{d - 1 + 3}{3}$$

$$= \binom{d + 2}{2} - [d^2 - g_D + 1]$$

$$= 1$$

using the value of $g_D$ mentioned above.

**Remark 2.2.** The same argument that was used to show uniqueness in the last result also shows that if $C$ is a curve of degree $d$ in $\mathbb{P}^3$ then there does not exist a surface of degree $e \leq d - 1$ containing $C$ with a singularity of multiplicity $e$ at a general point, since Bézout would force $S_P$ to be a component of such a surface. This means that $\dim[I_C]_e \leq \binom{e - 1 + 3}{3}$ for all $1 \leq e \leq d$ (where the statement for degree $d$ is given in the proof).

In fact, the same argument works for a subvariety $V$ of $\mathbb{P}^n$ of codimension two and degree $d$, to show that $\dim[I_V]_e \leq \binom{e - 1 + n}{n}$ for $1 \leq e \leq d - 1$. This statement also extends to the case $e = d$, and the argument is contained in the proof of Proposition 2.4.
We will give several corollaries to Proposition 2.1. The first is to extend it to subvarieties of codimension two in $\mathbb{P}^n$.

**Lemma 2.3.** Let $V$ be a reduced, equidimensional, non-degenerate subvariety of $\mathbb{P}^n$ ($n \geq 4$) of codimension 2 (not necessarily irreducible). Let $H$ be a general hyperplane and let $W = V \cap H$. Then $W$ is non-degenerate in $H = \mathbb{P}^{n-1}$.

**Proof.** We have the exact sequence

$$0 \to [I_V]_0 \to [I_V]_1 \to [I_{W|H}]_1 \to H^1(I_V(0)) \to \ldots$$

We want to show that the third vector space in this exact sequence is zero. But $[I_V]_1 = 0$ since $V$ is non-degenerate. On the other hand, we claim that $H^1(I_V(0)) = 0$. This will complete the proof. But we also have the exact sequence

$$0 \to [I_V]_0 \to [R]_0 \to H^0(O_V(0)) \to H^1(I_V(0)) \to 0.$$

The first term is clearly zero. The second has dimension 1. The third has dimension 1 since $V$ is connected (being of codimension 2 and equidimensional) and reduced (by hypothesis). Thus the claim follows. □

**Proposition 2.4.** Let $V$ be a reduced, equidimensional, non-degenerate subvariety of $\mathbb{P}^n$ ($n \geq 3$) of codimension 2 and degree $d$ ($V$ may be reducible and/or singular but note that $d \geq 2$ since $V$ is non-degenerate, with $V$ being two codimension 2 linear spaces if $d = 2$). Let $P \in \mathbb{P}^n$ be a general point. Then the cone $S_P$ over $V$ with vertex $P$ is an unexpected hypersurface for $V$ of degree $d$ and multiplicity $d$ at $P$. It is the unique unexpected hypersurface of this degree and multiplicity.

**Proof.** The proof is by induction on $n$. The initial case is Proposition 2.1, so we can assume $n \geq 4$. Let $H$ be a general hyperplane through $P$ and let $W = V \cap H$. Since $P$ is general, we can assume that $H$ is general as well. By Lemma 2.3, $W$ is non-degenerate in $H = \mathbb{P}^{n-1}$, and it is also reduced and equidimensional. Let $T_P$ be the cone in $H$ over $W$ with vertex $P$. Thus by induction, $T_P$ is the unique hypersurface of degree $d$ containing $W$ with multiplicity $d$ at $P$, and it is unexpected.

Consider the exact sequence

$$0 \to [I_V]_{d-1} \to [I_V]_d \to [I_{W|H}]_d \to H^1(I_V(d-1)) \to \ldots$$

We have

$$\dim[I_V]_d = \dim[I_V]_{d-1} + \dim[I_{W|H}]_d - \dim[K]_d.$$

Since $\dim[I_Z]_d = \dim[I_V]_d$, we claim that

$$\dim[I_V]_d - \left( \frac{d-1+n}{n} \right) \leq 0.$$
so that $S_P$ is unexpected. We have
\[
\dim[I_V]_d - \binom{d-1+n}{n} = \dim[I_V]_{d-1} + \left( \dim[I_W|H]_d - \binom{d-1+(n-1)}{n-1} \right) \\
+ \binom{d-1+(n-1)}{n-1} - \dim[K]_d - \binom{d-1+n}{n}
\]
\[
\leq \dim[I_V]_{d-1} + \binom{d+n-2}{n-1} - \dim[K]_d - \binom{d+n-1}{n}
\]
\[
= \dim[I_V]_{d-1} - \binom{d+n-2}{n} - \dim[K]_d.
\]
But Remark 2.2 gives
\[
\dim[I_V]_{d-1} \leq \binom{d+n-2}{n},
\]
which completes the claim. Uniqueness follows in the same way as it did for Proposition 2.1.

The next few corollaries have analogs in higher projective space, but the statements are a bit cleaner for curves in $\mathbb{P}^3$.

**Corollary 2.5.** Let $C$ be a reduced, equidimensional, non-degenerate curve of degree $d$ in $\mathbb{P}^3$ (C may be reducible and/or singular). Let $P \in \mathbb{P}^3$ be a general point. Let $Z \subset C$ be any set of points on $C$ such that $[I_C]_d = [I_Z]_d$. Then the cone $S_P$ over $C$ with vertex $P$ is an unexpected surface of degree $d$ for $Z$ with multiplicity $d$ at $P$. It is the unique unexpected surface of this degree and multiplicity. In particular, we may choose $Z$ to impose independent conditions on forms of degree $d$.

**Proof.** It is immediate from the hypothesis that $[I_C]_d = [I_Z]_d$. \qed

**Corollary 2.6.** Let $C$ be a smooth, irreducible, non-degenerate curve of degree $d \geq 3$ in $\mathbb{P}^3$. Let $P \in \mathbb{P}^3$ be a general point. Let $Z \subset C$ be any set of at least $d^2 + 1$ points on $C$ (general or not). Then the cone $S_P$ over $C$ with vertex $P$ is an unexpected surface of degree $d$ for $Z$ with multiplicity $d$ at $P$. It is the unique unexpected surface of this degree and multiplicity.

**Example 2.7.** Let $C$ be a twisted cubic curve in $\mathbb{P}^3$. Then $d = 3$ and $g_C = 0$. Let $Z$ be a set of 10 points on $C$, so $[I_C]_3 = [I_Z]_3$ has dimension 10. In this case $\dim[I_Z]_3 - \binom{2+3}{3} = 10 - 10 = 0$ so we do not expect a hypersurface of degree 3 with multiplicity 3 at a general point containing the 10 points of $Z$. But in fact there is such an unexpected hypersurface, given by the cone over $C$ with vertex at a general point.

**Remark 2.8.** Corollary 2.5, Corollary 2.6, Corollary 2.14 and Corollary 2.20 all deal with the situation that we begin with a set of points lying on a variety $C$ of codimension two in $\mathbb{P}^n$, and have enough points so that $[I_C]_d = [I_Z]_d$. In fact this assumption can be relaxed, although the statement becomes a little bit less transparent so we retained this assumption. But notice that the fact that $C$ already admits an unexpected hypersurface of degree $d$ means that we only need a set of $\binom{d+n}{n} - \binom{d-1+n}{n}$ points on $C$ that impose independent conditions on forms of degree $d$, and this number can be much smaller than the number forced by the condition $[I_C]_d = [I_Z]_d$. 
For example, say \( C \) is a general smooth rational curve in \( \mathbb{P}^3 \) of degree 6. The Hilbert function of \( C \) is given by the sequence 1, 4, 10, 19, 25, 31, ... so the assumption that \([I_C]_6 = [I_Z]_6\) means we need \( Z \) to have at least 37 points of \( C \). Instead, suppose that \( Z \) is a sufficiently general set of \( \binom{6+3}{3} - \binom{5+3}{3} = 28 \) points on \( C \). Then the Hilbert function of \( Z \) is given by the sequence 1, 4, 10, 19, 25, 28, 28, ... and we still do not expect a hypersurface of degree 6 with a point of multiplicity 6 to contain \( Z \), but we know that the cone over \( C \) is such a hypersurface. Notice that in this case we not only have \([I_C]_6 \neq [I_Z]_6\) but even \([I_C]_5 \neq [I_Z]_5\).

**Corollary 2.9.** Let \( C \) be a non-degenerate union of \( d \) lines in \( \mathbb{P}^3 \). Let \( P \in \mathbb{P}^3 \) be a general point. Let \( Z \subset C \) be a set of \( d(d+1) \) points on \( C \) chosen by taking \( d+1 \) general points on each line. Then the cone \( S_P \) over \( C \) with vertex \( P \) is an unexpected surface of degree \( d \) for \( Z \) with multiplicity \( d \) at \( P \). It is the unique unexpected surface of this degree and multiplicity.

**Remark 2.10.** On the other hand, it is not the case that all sets of points in \( \mathbb{P}^3 \) (or any other projective space) admit an unexpected surface (resp. hypersurface) of some sort. Indeed, suppose \( Z \) is a general set of points in \( \mathbb{P}^3 \) and let us ask if there is any degree and multiplicity at a general point, in which \( Z \) admits an unexpected surface. By considering the conditions imposed first by the general multiple point and then by the general points \( Z \), we see that we must always get the expected number of conditions.

What is interesting is that in [CHMN] Corollary 6.8 it was shown that a set of points in linear general position in \( \mathbb{P}^2 \) does not admit an unexpected curve of degree \( d \) and multiplicity \( d-1 \) at a general point. Example 2.7 already shows that this does not extend to a set of points in linear general position in \( \mathbb{P}^3 \), if we weaken the condition on the multiplicity to allow multiplicity \( d \). We do not know if the precise result from [CHMN] continues to hold in higher dimensional projective spaces.

**Question 2.11.** Let \( Z \) be a non-degenerate set of points in linear general position in \( \mathbb{P}^n \), \( n \geq 3 \). Is it true that there does not exist an unexpected hypersurface of any degree \( d \) and multiplicity \( d-1 \) at a general point?

We next extend the cone construction in two different ways. First, we point out that Proposition 2.1 extends to surfaces in \( \mathbb{P}^3 \) of higher degree and higher multiplicity. At the end of this section we will apply this result to show the failure of the Weak Lefschetz Property for certain ideals of powers of linear forms in four variables.

**Corollary 2.12.** Let \( C \) be a reduced, equidimensional, non-degenerate curve of degree \( d \geq 2 \) in \( \mathbb{P}^3 \) (\( C \) may be reducible and/or singular). Let \( P \in \mathbb{P}^3 \) be a general point. Let \( k \geq d \) be a positive integer. Then \( C \) admits an unexpected surface of degree \( k \) with multiplicity \( k \) at \( P \).

**Proof.** Let \( Y = C \cup kP \). We want to show that

\[
\dim[I_C]_k - \binom{k+2}{3} < \dim[I_Y]_k.
\]

Modifying the calculation above, we know that

\[
\dim[I_C]_k = \binom{k+3}{3} - [dk - g_C + 1]
\]

So

\[
\dim[I_C]_k - \binom{k+2}{3} = \binom{k+2}{2} - dk + g_C - 1.
\]
On the other hand, we have a unique surface $S_P$ of degree $d$ with a singularity of multiplicity $d$ at the general point $P$, so by multiplying $S_P$ by an element of $[I_P^{k-d}]_{k-d}$ we always obtain a surface of degree $k$ with multiplicity $k$ at $P$. Thus

$$\dim[I_Y]_k \geq \binom{k - d + 2}{2}.$$ 

Thus combining, it is enough to show

$$\binom{k + 2}{2} - dk + g_C - 1 < \binom{k - d + 2}{2}.$$ 

A calculation shows that this is equivalent to

$$g_C < \frac{(d - 1)(d - 2)}{2},$$

which we showed in Claim 1 of Proposition 2.1. □

**Remark 2.13.** Although we do not state them explicitly, we get the analogous corollaries for “sufficiently many” points on $C$ that we got for Proposition 2.1, but now in higher degree. The key is to assume (directly or by a condition on the number of points) that $[I_C]_k = [I_Z]_k$.

We now give a different extension of the cone construction, allowing us to find unexpected hypersurfaces where the multiplicity is strictly less than the degree. It misses by 1 to be an answer to Question 2.11.

**Corollary 2.14.** Let $V$ be a reduced, equidimensional, non-degenerate subvariety of codimension two and degree $d$ in $\mathbb{P}^n$, $n \geq 3$. Let $S$ be a hypersurface of degree $e \geq 1$ not containing any irreducible component of $V$. Let $Y = V \cup S$. Let $Z \subset Y$ be a finite set of points such that $[I_Z]_{d+e} = [I_Y]_{d+e}$. Let $P$ be a general point in $\mathbb{P}^n$. Then $Z$ admits a unique unexpected hypersurface of degree $d + e$ with multiplicity $d$ at $P$. In particular, if $V$ is irreducible and $e \geq 2$ then we can take $Z$ to be points in linear general position which impose independent conditions on forms of degree $d + e$.

**Proof.** Let $F$ be the form defining $S$. Then

$$[I_Z]_{d+e} = [I_Y]_{d+e} = F \cdot [I_Y]_d$$

so

$$\dim[I_Z]_{d+e} - \binom{d + n - 1}{n} = \dim[I_Y]_d - \binom{d + n - 1}{n} < 1$$

as we saw in the proof of Proposition 2.4. Thus $S_P \cup S$ is an unexpected hypersurface of degree $d + e$ with a singular point of multiplicity $d$ at $P$. □

With the above results we can now give a complete answer to the following natural question. For which $d$ and $m$ does there exist a set of points $Z$ in $\mathbb{P}^2$ (resp. $\mathbb{P}^n$) such that $Z$ admits an unexpected curve (resp. hypersurface) of degree $d$ and multiplicity $m$ at a general point?

**Theorem 2.15.**

(i) There exists a finite set of points $Z \subset \mathbb{P}^2$ admitting an unexpected curve of degree $d$ and multiplicity $m$ at a general point if and only if $d > m > 2$.

(ii) For $n \geq 3$, there exists a finite set of points $Z \subset \mathbb{P}^n$ admitting an unexpected hypersurface of degree $d$ and multiplicity $m$ at a general point if and only if $d \geq m \geq 2$. 
Proof. For (i), we first note that there is an unexpected curve of degree $m+1$ and multiplicity $m$ at a general point for each $m \geq 3$. Indeed,

- $m \leq 2$. It was shown by Akesseh [A] that this occurs only in characteristic 2. (See also [FGST] for a different proof that shows that it does not occur in characteristic zero.)
- $m = 3$. This comes from the dual of the $B_3$ arrangement [DIV].
- $m = 4$. Consider the line arrangement defined by the linear factors of $xyz(x^3 - y^3)(x^3 - z^3)(y^3 - z^3)$.

This is a supersolvable arrangement, so Theorem 3.7 of [DMO] applies. In particular, the dual points give a reduced set of 12 points which admit an unexpected curve with $d = 5$ and $m = 4$.
- $m = 5$. This follows from [CHMN] Proposition 6.15, taking $k = 2$.
- $m \geq 6$. This comes from the points dual to the Fermat arrangement, by [CHMN] Proposition 6.15, taking $t \geq 5$.

It is clear that a set of points $Z$ in $\mathbb{P}^2$ cannot admit an unexpected curve whose degree and multiplicity at a general point $P$ are equal (unlike what we have seen in $\mathbb{P}^3$). Indeed, in this situation the line joining $P$ to any point of $Z$ must be a component of such an unexpected curve, so the unexpected curve is a cone with vertex $P$ over the points of $Z$. Let $d = \deg Z$ be the degree of this curve. Then $Z$ imposes independent conditions on curves of degree $d$, so $\dim [I_Z]_d = \binom{d+2}{2} - d$. Then

$$\dim [I_Z]_d - \left( \frac{(d - 1) + 2}{2} \right) = \left( \frac{d + 2}{2} \right) - d - \left( \frac{d + 1}{2} \right) = (d + 1) - d = 1$$

so in fact the cone is not unexpected. Thus $m < d$.

With this preparation, we use the same argument as was used to prove Corollary 2.14. Let $Z_0$ be a set of points admitting an unexpected curve of degree $m+1$ and multiplicity $m$ at a general point $P$ and let $A$ be a plane curve of degree $d - m - 1$ not passing through any point of $Z_0$. Since $P$ is general, it does not lie on $A$. Then choosing sufficiently many points on $A$ gives a unique unexpected curve of degree $d$ and multiplicity $m$ at $P$.

Part (ii) follows from Proposition 2.4 and Corollary 2.14.

We end this section with an application. There is an interesting interpretation of these results in terms of the Strong and Weak Lefschetz Properties. We first recall the definitions. For these definitions we maintain the assumptions on the polynomial ring $R$, but in fact all we need is that $K$ be an infinite field.

**Definition 2.16.** Let $R/I$ be an artinian $K$-algebra and let $L$ be a general linear form. Then $R/I$ satisfies the **Weak Lefschetz Property (WLP)** in degree $i$ if $\times L : [R/I]_i \rightarrow [R/I]_{i+1}$ has maximal rank, and we say that $R/I$ satisfies the **Strong Lefschetz Property (SLP)** in degree $i$ with range $k$ if $\times L^k : [R/I]_i \rightarrow [R/I]_{i+k}$ has maximal rank. We say that $R/I$ satisfies WLP (resp. SLP) if it does so for all $i$ (resp. for all $i$ and $k$).

Thus SLP failing in degree $i$ with range $k$ means that $\times L^k : [R/I]_i \rightarrow [R/I]_{i+k}$ does not have maximal rank, and WLP failing in degree $i$ is the same as SLP failing in degree $i$ with range 1. The result [DIV, Theorem 5.1] and the remarks that follow the theorem connect the failure of SLP in degree $i$ with range $k$ to the occurrence of a form on $\mathbb{P}^n$ of degree $d = i + k$ with a general point of multiplicity $i + 1$. In the specific case of $n = k = 2$, [CHMN, Theorem
Proposition 2.17. Let $L_1, \ldots, L_r$ be distinct linear forms on $\mathbb{P}^n$, and let $Z$ be the set of points in $\mathbb{P}^n$ dual to the hyperplanes defined by the $L_i$. Fix integers $d \geq m > 1$. Then the following are equivalent:

(a) $Z$ has an unexpected hypersurface of degree $d$ with a general point $P$ of multiplicity $m$;
(b) $R/(L_1^d, \ldots, L_r^d)$ fails SLP in degree $i = m - 1$ with range $k = d - m + 1$.

Proof. Let $L$ be a general linear form. Consider the exact sequence
\begin{equation}
\ldots \to [R/(L_1^d, \ldots, L_r^d)]_{m-1} \times L^{d-m+1} \to [R/(L_1^d, \ldots, L_r^d)]_d \to [R/(L_1^d, \ldots, L_r^d, L^{d-m+1})]_d \to 0.
\end{equation}
Let $P$ be the general point of $\mathbb{P}^n$ dual to $L$. By Macaulay duality [EI] and exactness,
\[
\dim[I_Z \cap I_P^m]_d = \dim[R/(L_1^d, \ldots, L_r^d, L^{d-m+1})]_d
\geq \dim[R/(L_1^d, \ldots, L_r^d)]_d - \dim[R/(L_1^d, \ldots, L_r^d)]_{m-1}
= \dim[I_Z]_d - \left(\frac{n + m - 1}{n}\right).
\]
Since (a) holds if and only if $\dim[I_Z \cap I_P^m]_d > \max(0, \dim[I_Z]_d - (\frac{n+m-1}{n}))$ if and only if $\dim[R/(L_1^d, \ldots, L_r^d, L^{d-m+1})]_d > \max(0, \dim[R/(L_1^d, \ldots, L_r^d)]_d - \dim[R/(L_1^d, \ldots, L_r^d)]_{m-1})$ if and only if (b) holds, the result follows. \qed

Note that a set of points in $\mathbb{P}^n$ dual to a set of general linear forms is, in particular, in linear general position. For a set of points in $\mathbb{P}^2$ in linear general position (i.e., no three on a line), [CHMN] shows that there is no unexpected curve of any degree, hence the corresponding ideals of powers of linear forms do not fail SLP in range 2. This is in contrast to the case for $n > 2$. Indeed, we now give a result showing failure of WLP for arbitrarily many linear forms in four variables whose dual points are in linear general position (but not general), followed in Corollary 2.20 by a similar but weaker result for forms in any number of variables $\geq 4$.

(For $\mathbb{P}^n$ with $n > 2$, a few papers have studied the question of WLP for ideals generated by powers of general linear forms, but most such results have focused on a small number of linear forms (e.g. [HSS], [MMN], [SS]).)

Corollary 2.18. Let $C$ be a reduced, irreducible, non-degenerate curve of degree $d \geq 3$ in $\mathbb{P}^3$. Let $k \geq d$ be a positive integer. Let $Z$ be any set of $m \geq dk + 1$ points of $C$. Let $L_1, \ldots, L_m$ be the linear forms dual to the points of $Z$. In particular, $L_1, \ldots, L_m$ can be chosen so that no four vanish on a point (i.e. the points of $Z$ are in linear general position). Then $R/(L_1^k, \ldots, L_m^k)$ fails the WLP in degree $k - 1$.

Proof. The statement about linear general position is immediate since $C$ is reduced, irreducible and non-degenerate. From Corollary 2.12 we know that $C$ has an unexpected surface of degree $k$ with a general point $P$ of multiplicity $k$, so we have $\dim[I_C \cap I_P^k]_k > \max(0, \dim[I_C]_k - (\frac{k+2}{3}))$. But by Bézout’s theorem we have $[I_Z]_k = [I_C]_k$ so we obtain $\dim[I_Z \cap I_P^k]_k > \max(0, \dim[I_Z]_k - (\frac{k+2}{3}))$, hence $Z$ has an unexpected surface of degree $k$.
with multiplicity $k$ at $P$, so $R/(L_1^k, \ldots, L_m^k)$ fails the WLP in degree $k - 1$ by Proposition 2.17.

\begin{prop}
Let $several$ degrees, but now it does $\not\equiv$ as claimed, but also in certain other degrees (depending on $k$). We have considered the case of 31 points on the twisted cubic as above, but allowed different $k$. When $k = 2$ the algebra has WLP. When $3 \leq k \leq 10$ it fails from degree $k - 1$ to degree $k$ as claimed, but also in certain other degrees (depending on $k$). For $k \geq 11$ it still fails in several degrees, but now it does not fail from degree $k - 1$ to $k$.

The following is a slightly weaker analog of Corollary 2.18 for $\mathbb{P}^n$.

\begin{corollary}
Let $n \geq 4$ and $k \geq 3$. Set
\[
f(n,k) = \begin{cases} 
\binom{n+k}{2} - \binom{n+k-2}{2} - \binom{n+\frac{k}{2}}{n} + \binom{n-2+\frac{k}{2}}{n} & \text{if } k \text{ is even} \\
\binom{n+k}{2} - \binom{n+k-2}{2} - 2\binom{n+\frac{k+1}{2}}{n} + 2\binom{n+\frac{k+1}{2}}{n-1} & \text{if } k \text{ is odd.}
\end{cases}
\]
Choose any integer $N \geq f(n,k)$. Then there exist linear forms $L_1, \ldots, L_N \in \mathbb{k}[x_0, \ldots, x_n] = R$ satisfying
\begin{itemize}
\item no $n+1$ of the linear forms have a common zero, and
\item $R/(L_1^k, \ldots, L_N^k)$ fails the WLP from degree $k - 1$ to degree $k$.
\end{itemize}
\end{corollary}

\begin{proof}
We recall that we can always find an irreducible, non-degenerate subvariety $V$ of codimension 2 and degree $k$ in $\mathbb{P}^n$. In fact, if $k$ is even we take $V$ as an intersection of a general form of degree $\frac{k}{2}$ and a general quadric. If $k$ is odd we take $V$ as a general arithmetically Cohen-Macaulay subscheme with a minimal free resolution of the form
\[
0 \to R(-\frac{k+2}{2})^2 \to R(-\frac{k+1}{2})^2 \oplus R(-2) \to R \to R/I_V \to 0
\]
(obtained, for example, by linking from a linear space of codimension 2). Using this sequence, or the Koszul resolution if $k$ is even, we obtain
\[
\dim[R/I_V]_k = f(n,k).
\]
Since $V$ is irreducible, it makes sense to speak of a general set of points on $V$. Let $Z$ be a general set of $N$ points on $V$. From the generality of $Z$ we have
\[
\dim[R/I_Z]_k = \min\{\dim[R/I_V]_k, |Z|\},
\]

hence $\dim[I_V]_k = \dim[I_Z]_k$. By Proposition 2.4 we then have that the cone over $V$ with vertex at a general point $P$ is an unexpected hypersurface of degree $k$ and multiplicity $k$ at $P$. Then the argument in the proof of Corollary 2.16 can be extended to our situation to show that the multiplication from degree $k - 1$ to degree $k$ by a general linear form has an unexpectedly large cokernel, i.e. maximal rank does not hold. Since $Z$ is general on $V$ and $V$ is irreducible and non-degenerate, $Z$ is a set of points in linear general position, and this implies the condition on the linear forms not to have a common zero.

The above results all focus on failure of the WLP. We can also give a result about failure of the SLP with ranges bigger than 1.
Corollary 2.21. Let $R = k[x_0, \ldots, x_n]$. Fix positive integers $d \geq m$. Then there exists an ideal $I = (L_1^d, \ldots, L_e^d)$ (for suitable $e$) for which

$$\times L^{d-m+1} : [R/I]_{m-1} \to [R/I]_d$$

fails to have maximal rank if and only if one of the following holds.

(i) We have $n = 2$ and $d > m > 2$.
(ii) We have $n \geq 3$ and $d \geq m \geq 2$.

In both cases this means that $R/I$ fails the SLP in degree $m - 1$ and range $d - m + 1$.

Proof. This follows immediately from Theorem 2.15 by applying Proposition 2.17. □

Remark 2.22. 1. Note that Corollary 2.21 (and similarly for the earlier results) is not necessarily about failure of surjectivity. The proof only shows that the cokernel of $\times L^{d-m+1}$ is bigger than expected, which means failure of surjectivity if $\dim[R/I]_{m-1} \geq \dim[R/I]_d$, and it means failure of injectivity if $\dim[R/I]_{m-1} \leq \dim[R/I]_d$.

2. As a result of Corollary 2.21 (i) we also recover the fact that such an algebra $R/I$ must have the WLP (this corresponds to the excluded case $d = m$). This is a special case of the main result of [SS].

3. Root system examples

The construction used in §2 to get an unexpected hypersurface of degree $d$ with a general point of multiplicity $m$ for a locus $Z$ in $\mathbb{P}^n$ for $n > 2$, is based on $[I_Z]_d$ having a positive dimensional base locus. There are, as we shall see, examples of finite point sets $Z$ with an unexpected hypersurface with $d = m$ and $n > 2$ such that the base locus of $[I_Z]_d$ is 0 dimensional. Thus our construction in §2 is not the end of the story, since unexpected hypersurfaces can arise in other ways. The question is where else can one look to find them?

In this section we find new habitats where unexpected hypersurfaces lurk, both for $n = 2$ and $n > 2$, at least some of which for $n > 2$ have the property that the base locus of $[I_Z]_d$ is 0-dimensional.

We first became aware of a set of points $Z$ admitting an unexpected curve from an example of [DIV]. The lines dual to $Z$ are shown in Figure 1. This example is interesting for a number of reasons. It is a simplicial real arrangement. (This means that the lines divide the real projective plane into triangles.) It is extremal. (If $t_k$ denotes the number of points where exactly $k$ lines meet, an inequality of Melchior [Me] for real arrangements of $d > 2$ lines with $t_d = 0$ states that $t_2 \geq 3 + \sum_{k>2}(k-3)t_k$. For $B_3$ we have $t_2 = 6$, $t_3 = 4$ and $t_4 = 3$, so equality holds.) It is free (meaning that if $F$ is the product of the linear forms defining the lines dual to the points of $Z$, then there are no second syzygies for the Jacobian ideal $J_F = (F_x, F_y, F_z)$; i.e., the syzygy bundle for $J_F$ is free.) Its unexpected curve has minimal degree in characteristic 0 (no unexpected curve in characteristic 0 has degree 3 or less [A, FGST]). It is a line arrangement coming from a root system. (The lines in $\mathbb{P}^2_\mathbb{R}$ correspond to the 2 dimensional vector subspaces in $\mathbb{R}^3$ orthogonal to the roots of the $B_3$ root system, under the bijective correspondence between lines in $\mathbb{P}^2_\mathbb{R}$ and planes through the origin in $\mathbb{R}^3$.) And it is a supersolvable arrangement.

A line arrangement in the projective plane is supersolvable if there is a so-called modular point (i.e., a point $P$ where two or more of the lines meet such that if $Q$ is any other point where two or more of the lines meet, then the line through $P$ and $Q$ is a line in the
Figure 1. The nine lines of the arrangement $B_3$ (the line $z = 0$ at infinity is not shown).

arrangement). Thus a supersolvable line arrangement includes the cone over its crossing points with vertex at any modular point. The multiplicity of a point with respect to a line arrangement is just the number of lines in the arrangement containing the point. When a line arrangement is supersolvable, every point of maximum multiplicity is modular (but not every modular point need have maximum multiplicity) [AT] (only the arXiv version includes the proof). For the line arrangement $B_3$, shown in Figure 1, the center point is modular and indeed has maximum multiplicity (no other crossing point has multiplicity more than 4). The result of [DMO] says the point scheme $Z_A$ dual to the lines of a supersolvable line arrangement $A$ has an unexpected curve of degree $m_A$ with respect to $X = (m_A - 1)P$ for a general point $P$ if and only if $2m_A < d_A$, where $d_A$ is the number of lines in the arrangement and $m_A$ is the maximum multiplicity of a point for $A$, and in this case the unexpected curve is unique. Since $m_{B_3} = 4$ and $d_{B_3} = 9$, it follows that $Z_{B_3}$ has a unique unexpected curve of degree 4.

The roots of $B_3$ can be defined as the integer vector solutions $(a, b, c) \in \mathbb{R}^3$ to $1 \leq a^2 + b^2 + c^2 \leq 2$. Geometrically, given a unit cube aligned with the coordinate axes of $\mathbb{R}^3$ and whose center is at the origin, these are the vectors pointing from the origin to the center of each face and to the midpoint of each edge. The roots in pairs correspond to points in the projectivization $\mathbb{P}_\mathbb{R}^2$ of $\mathbb{R}^3$. Thus the 18 roots give the 9 points of $Z_{B_3}$, and the lines of the line arrangement $B_3$ are just the projectivizations of the planes normal to the roots; these lines are the projective duals of the points of $Z_{B_3}$.

Given the interesting behavior of $B_3$, it is natural to look at other arrangements with similar properties. As noted above, [DMO] has done this for the case of supersolvable arrangements. Here we check what happens for arrangements coming from other root systems $A$ in $\mathbb{R}^n$, not only for $n = 3$ but also for $n > 3$. The set-up then is: a root system $A$ gives a finite set of vectors of $\mathbb{R}^n$ for some $n$. Each root gives a point in $\mathbb{P}^{n-1}$ and the set of these points for the given root system $A$ gives the point set we denote by $Z_A$. The codimension 1 linear subspaces normal to the roots define the hyperplanes of the arrangement corresponding to $A$ which we also refer to by $A$.

In principle, given a finite set of points $Z \subset \mathbb{P}^n$ and a general point $P = [a_0 : \cdots : a_n]$, to find an unexpected hypersurface for $Z + mP$ computationally one takes $P$ to be the generic point $P = [1 : \frac{a_1}{a_0} : \cdots : \frac{a_n}{a_0}]$ and works as usual in the homogeneous coordinate ring $S = \mathbb{K}(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0})[x_0, \ldots, x_n] = \mathbb{K}[\mathbb{P}^n_{\mathbb{K}(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0})}]$. (We will however abuse notation and typically use $[a_0 : \cdots : a_n]$ to denote $P$ and refer to it as a general point.) However, it can be convenient and more efficient to work in the bi-graded ring $R = \mathbb{K}[a_0, \ldots, a_n][x_0, \ldots, x_n]$. So we mention here a few clarifying but elementary remarks.
All of our rings of interest are contained in the field $\mathbb{K}(a_0, \ldots, a_n, x_0, \ldots, x_n)$, which is the field of fractions of a UFD. Thus given a form $F \in S$ of degree $d$ but which is not in $\mathbb{K}$, there is (up to scalars in $\mathbb{K}$) a unique factorization $F = BG/H$, where $B$ and $H$ are relatively prime forms in $\mathbb{K}[a_0, \ldots, a_n]$ and $G \in R$ is bi-homogeneous with no factors of bi-degree $(a, 0)$ with $a > 0$; its bi-degree is $(t, d)$, where $t = \deg(H) - \deg(B)$. We denote $G$ by $F^\circ$. Given any bi-homogeneous element $G \in R$ of bi-degree $(t, d)$, we denote $G/a_0^t$ by $G^\circ$. Note that $G^\circ \in S$ is homogeneous of degree $d$. It need not be true that $(G^\circ)^* = G$ (for example, $(a_1^t)^* = (a_1/a_0)^t = 1$ nor that $(F^\circ)^\circ = F$ (e.g., $(a_1 x_1/a_0)^* = x_1^* = x_1$), but we do have the following useful lemma.

**Lemma 3.1.** Let $S$ and $R$ be as above. Let $P$ be the point $[1: a_1/a_0 : \cdots : a_n/a_0] \in \mathbb{P}^n_{\mathbb{K}(a_1/a_0, \ldots, a_n/a_0)}$.

For any $m \geq 1$, let $I_{mP}$ denote the ideal $(I_P)^m$ in $S$, and let $J_{mP}$ denote the ideal $(J_P)^m$ in $R$, where $J_P = \{(a_i x_j - a_j x_i : i \neq j)\}$ (hence $(J_P)^m$ is generated by elements of bi-degree $(m, m)$). Now let $d > 0$ and $t \geq 0$, let $F \in S$ be a nonzero form of degree $d$, and let $G \in R$ be a nonzero, nonconstant bi-homogeneous form of bi-degree $(t, d)$.

(a) As ideals in $S$ we have $(F) = ((F^\circ)^\circ)$.

(b) If $F$ is irreducible, then so are $F^\circ$ and $(F^\circ)^\circ$.

(c) If $G$ is irreducible, then so are $G^\circ$ and $(G^\circ)^\circ$, and as ideals in $R$ we have $(G) = ((G^\circ)^\circ)$.

(d) If $F \in I_{mP}$, then $d \geq m$ and $F^\circ \in J_{mP}$ so we see that $F^\circ$ has bi-degree $(s, d)$ for some $s$ with $s \geq m$.

(e) If $G \in J_{mP}$, then $G^\circ \in I_{mP}$.

Proof. (a) Since $F = BF^\circ/H = B(F^\circ a_0^t/H)$, where $t = \deg(H) - \deg(B)$, we see that $F$ and $(F^\circ)^\circ$ differ by a unit factor in $S$.

(b) Note that $F$ and $(F^\circ)^\circ$ differ by a unit factor in $S$, so one is irreducible if and only if the other is. Also, $F^\circ$ by construction has no factors of bi-degree $(f, 0)$ with $f > 0$, so if $F^\circ$ fails to be irreducible, it factors as $F^\circ = AB$ where $A$ and $B$ have bi-degree $(a, d_A)$ and $(b, d_B)$ with $d_A, d_B > 0$ and $d_A + d_B = d$. But then $(F^\circ)^\circ = A^\circ B^\circ$ is a product of factors of positive degree and hence not irreducible.

(c) If $G^\circ$ is not irreducible, then $G^\circ = AB$ where both $A$ and $B$ have positive degree $d_A$ and $d_B$, so we have $G = \alpha A^\ast \beta B^\ast$ where $A^\ast$ and $B^\ast$ have bi-degrees $(a, d_A)$ and $(b, d_B)$, and $\alpha$ and $\beta$ have bi-degrees $(t_\alpha, 0)$ and $(t_\beta, 0)$. Since $R$ is a UFD, the denominators in the right hand side of the expression $G = \alpha A^\ast \beta B^\ast$ must cancel with factors of the numerators of the expression $\alpha A^\ast \beta B^\ast$ and this does not affect the values of $d_A$ or $d_B$, so we may assume that $\alpha, \beta, A^\ast, B^\ast$ all are in $R$ and $A^\ast$ and $B^\ast$ have bi-degrees $(a', d_A)$ and $(b', d_B)$, where the simplification might have changed $a$ and $b$ but will have left $d_a$ and $d_B$ unchanged. Since $d_A$ and $d_B$ are both positive, $G$ has nonunit factors so $G$ is not irreducible. Now note that irreducibility of $G^\circ$ implies that for $(G^\circ)^\circ$ by (b). Finally, by construction, if $G$ is irreducible with bi-degree $(t, d)$ where $d > 0$, then $G = c(G^\circ)^\circ$ for some nonzero $c \in \mathbb{K}$, hence $(G) = ((G^\circ)^\circ)$.

(d) Since $F \in I_{mP}$, by (a) we have $F^\circ/a_0^t = (F^\circ)^\circ \in I_{mP}$, where $(s, d)$ is the bi-degree of $F^\circ$, and since $d$ is the degree of $(F^\circ)^\circ$ and since $(F^\circ)^\circ \in I_{mP}$ we have $d \geq m$. Now let

$$D\left(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) = F^\circ\left(\frac{a_0}{a_0}, \frac{a_n}{a_0}, \frac{x_0}{x_0}, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) = \frac{F^\circ(a_0, \ldots, a_n, x_0, \ldots, x_n)}{a_0^s a_0^d x_0^e}.$$
so as a rational function $D$ has bi-degree $(0, 0)$, but as a polynomial in $\mathbb{K}(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0})[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}]$, $D$ has degree $\delta \leq d$. Moreover, since $x_0 \not\in I_P$, we have $D \in I_{mQ}$ where $Q$ is the point $Q = (\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0})$ in the affine open subset $\mathbb{K}(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0})^n$ away from $x_0 = 0$. Now translate $Q$ to the origin; i.e., consider

$$H(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) = D(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}, \frac{x_1}{x_0} + \frac{a_1}{a_0}, \ldots, \frac{x_n}{x_0} + \frac{a_n}{a_0}).$$

Note that

$$a_0^{s+d}_0 x_0^d H = a_0^{s+d} x_0^d F^*\left(\frac{a_0}{a_0}, \ldots, \frac{a_n}{a_0}, \frac{x_1}{x_0}, \frac{x_1}{x_0} + \frac{a_1}{a_0}, \ldots, \frac{x_n}{x_0} + \frac{a_n}{a_0}\right) = F^*(a_0, \ldots, a_n, a_0 x_0, a_0 x_1 + a_1 x_0, \ldots, a_0 x_n + a_n x_0) \in R$$

is a bi-homogeneous polynomial of bi-degree $(s, d, d)$.

Since $H$ has multiplicity $m$ at the origin (with respect to the variables $\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}$), $H$ is a sum of terms where each term consists of a monomial in $\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}$ of degree at most $m$ and at most $d$, times a polynomial in the variables $\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}$ of degree at most $s + d$. Thus each term is of the form

$$c(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0})\left(\frac{x_1}{x_0}\right)^{i_1} \cdots \left(\frac{x_n}{x_0}\right)^{i_n}$$

where $m \leq \sum_j i_j \leq d$ and $c$ has degree at most $s + d$. Thus multiplying by $a_0^{s+d} x_0^d$ clears the denominators. Translating back we recover $D$; i.e.,

$$D(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) = H(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}, \frac{x_1}{x_0} - \frac{a_1}{a_0}, \ldots, \frac{x_n}{x_0} - \frac{a_n}{a_0}),$$

but each term becomes

$$c(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0})\left(\frac{x_1}{x_0} - \frac{a_1}{a_0}\right)^{i_1} \cdots \left(\frac{x_n}{x_0} - \frac{a_n}{a_0}\right)^{i_n} = c(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0})\left(\frac{a_0 x_1 - a_1 x_0}{a_0 x_0}\right)^{i_1} \cdots \left(\frac{a_0 x_n - a_n x_0}{a_0 x_0}\right)^{i_n}$$

and multiplying by $a_0^{2d+s} x_0^d$ we obtain

$$C(a_0, \ldots, a_n)(a_0 x_1 - a_1 x_0)^{i_1} \cdots (a_0 x_n - a_n x_0)^{i_n}$$

where $C$ is homogeneous in the variables $a_i$ of degree $s + d$, and

$$(a_0 x_1 - a_1 x_0)^{i_1} \cdots (a_0 x_n - a_n x_0)^{i_n} \in J_{mP}.$$
Given a finite set of points $Z \subset \mathbb{P}^n$ we use the following script to check whether $Z$ has an unexpected hypersurface of degree $d$ vanishing on $mP$ for a general point $P = [a_0 : \cdots : a_n]$. Assuming that the computation of rank given by Macaulay2 is reliable, running the script on an example gives a rigorous proof of whether the example is unexpected or not.

The idea of the script is to construct the matrix $N$ expressing the conditions imposed on all forms $F$ of degree $d$ by the points of $Z$, together with the conditions imposed for $F$ to vanish to order $m$ at a point $P = [a_0 : \cdots : a_n]$ with indeterminate coordinates $a_i$ represented in the scripts with new variables. Thus if we enumerate the monomials of degree $d$ in $n + 1$ variables $x_i$ as $b_1, \ldots, b_{\binom{n+d}{n}}$, then $N$ will be a matrix with $|Z| + \binom{n+m-1}{n}$ rows and $\binom{n+d}{n}$ columns. A form $F = \sum_i c_i b_i$ vanishes on $Z$ and on $mP$ if and only if $Nc = 0$, where $c$ is the coefficient vector $c = (c_1, \ldots, c_{\binom{n+d}{n}})^T$. We can regard $N$ as consisting of two matrices, $Q_1$ and $Q_2$, where $Q_1$ comprises the top $|Z|$ rows of $N$ and gives the conditions imposed by the points of $Z$, and $Q_2$ comprises the bottom $\binom{n+m-1}{n}$ rows of $N$ giving the conditions imposed by the fat point $mP$. Thus the entries of $Q_1$ are scalars in the ground field, but the entries of $Q_2$ are the order $m - 1$ partials of the monomials $b_i$ evaluated at $P$. Then $Z$ has unexpected hypersurfaces of degree $d$ with a general point $P$ of multiplicity $m$ exactly when

$$\dim \ker N > \max(0, \left(\frac{n+d}{n}\right) - \text{rank}(Q_1) - \text{rank}(Q_2)),$$

and in this case the coefficient vectors $c$ of the unexpected hypersurfaces are precisely the nontrivial elements of $\ker N$.

The script which does this is given below. The section marked CODE BLOCK is where one puts in the list of the points of $Z$ (or where one puts in code needed to generate the list; see the examples). For now we exhibit code which works when the points of $Z$ are defined over the rationals. Subtleties arise when coordinates in an extension field are needed. We discuss that later.

Apart from output indicating current status of the computation, the script output indicates exactly when it finds an unexpected hypersurface. An output of the form $(n,d,m,edim,adim)$ means that there is a hypersurface for $Z$ in $\mathbb{P}^n$ of degree $d$ with a generic point of multiplicity $m$ (that is a point whose coordinates are variables); the vector space of unexpected forms has expected dimension $edim$ and actual dimension $adim$ (where $edim$ can be negative if the number of conditions imposed by $mp$ is greater than the dimension of $[I_Z^d]$). If for a given $n,d,m$ there is no output, then $Z$ has no unexpected hypersurface in $\mathbb{P}^n$ of degree $d$ with a general point $P$ of multiplicity $m$. We will refer to the script below as the universal script.

```plaintext
for n from 2 to 6 do { -- Loop over dim 2 to 6
R=QQ[x_0..x_n]; -- Define coordinate ring
S=frac(QQ[a_0..a_n]); -- Define ring for generic point
CODE BLOCK for Pts={...}; -- Insert the list of the points of Z here
print {"n","n","Pts","Pts"}; -- Print completion status indicator
for d from 2 to 6 do { -- Loop over degree
Md=flatten entries(basis(d,R)); -- Create deg d monomial basis
for m from 2 to d do { -- Loop over multiplicity m
print {"d","d","m","m"}; -- Print completion status indicator
Mm=flatten entries(basis(m-1,R)); -- Create deg m-1 monomial basis
A={}; -- A will contain the list of rows for transpose of matrix Q1
apply(Md,i->(N={};for s from 0 to #Pts-1 do N=N|{sub(i,matrix{Pts_s})};A=A|{N}));
```
D={}; -- D will contain the list of rows for transpose of matrix Q2
apply(Md,i->(N={};for s from 0 to #Mm-1 do N=N|{diff(Mm_s,i)};D=D|{N}));
Q1=transpose matrix A; -- Q1 is defined over R
Q2=transpose matrix D; -- Q2 is defined over R
M={};
for i from 0 to n do M=M|{a_i}; -- M is the coord vector for generic point
Q1S=sub(Q1,S); -- Q1 is now defined to be over S
Q2S=sub(Q2,matrix{M}); -- Swap x variables for a variables
N=Q1S||Q2S;
expdim=#Md - (rank Q1S) - (rank Q2S);
actdim=#Md-(rank N);
if actdim > expdim and actdim > 0 then
    print {"n","d","m","edim","adim"}}}}

One can recover the actual unexpected forms by putting in a line to print out the kernel of N. When the actual unexpected forms themselves are not needed, the script can be made more efficient. Note that the line Q2S=sub(Q2,matrix{M}) merely substitutes the variables \(a_i\) in for the variables \(x_i\) in the \((m-1)\) order partials. This doesn’t affect the rank. Also, the rank of the matrix over \(S\) after this substitution is the same as the rank of the matrix over \(R\) before the substitution. Thus if existence of and numerical data for unexpectedness is all that is needed, then the line S=frac(QQ[a_0..a_n]); can be deleted and the lines

M={};
for i from 0 to n do M=M|{a_i}; -- M is the coord vector for generic point
Q1S=sub(Q1,S); -- Q1 is now defined to be over S
Q2S=sub(Q2,matrix{M}); -- Swap x variables for a variables
N=Q1S||Q2S;
expdim=#Md - (rank Q1S) - (rank Q2S);
actdim=#Md-(rank N);
if actdim > expdim and actdim > 0 then
    print {"n","d","m","edim","adim"}}}}

can be changed to

N=Q1||Q2;
expdim=#Md - (rank Q1) - (rank Q2);
actdim=#Md-(rank N);
if actdim > expdim and actdim > 0 then
    print {"n","d","m","edim","adim"}}}}

It’s possible the script would run faster by evaluating the matrix \(Q_2\) of partials at a random point (with coordinates in the rationals or even in the integers) rather than at a generic point. To do so, replace the line

apply(Md,i->(N={};for s from 0 to #Mm-1 do N=N|{diff(Mm_s,i)};D=D|{N}));

with

G={};
v={};
F=random(1,R);
for i from 0 to n do (v=v|{diff(x_i,F)});
G=G|{v};
Table 1. Unexpected hypersurfaces arising from the root system $B_{n+1}$.

| $n$ | $d$ | $m$ | $edim$ | $adim$ |
|-----|-----|-----|--------|--------|
| 2   | 4   | 3   | 0      | 1      |
| 3   | 4   | 4   | $-1$   | 1      |
| 4   | 4   | 4   | 10     | 11     |
| 5   | 3   | 3   | $-1$   | 5      |
| 5   | 4   | 4   | 34     | 35     |
| 6   | 3   | 3   | 7      | 14     |
| 6   | 4   | 4   | 77     | 78     |

3.1. The root system $A_{n+1}$. The roots for $A_{n+1}$ are the $(n+1)(n+2)$ integer vectors in $\mathbb{R}^{n+2}$ having one entry of 1, one of $-1$ and the rest 0. We project these into $\mathbb{R}^{n+1}$ by dropping the last coordinate. Projectivizing then gives a set $Z \subset \mathbb{P}^n$ of $(\binom{n+1}{2})$ points. No unexpected hypersurfaces turned up for $2 \leq n \leq 6$, $2 \leq d \leq 6$, $2 \leq m \leq d$.

3.2. The root system $B_{n+1}$. The root system $B_{n+1} \subset \mathbb{R}^{n+1}$ consists of the $2(n+1)^2$ integer vectors $(a_1, \ldots, a_{n+1})$ such that $1 \leq a_1^2 + \cdots + a_{n+1}^2 \leq 2$. We thus have $|Z_{B_{n+1}}| = (n+1)^2$ for the corresponding set of points $Z_{B_{n+1}} \subset \mathbb{P}^n$. The CODE BLOCK here is:

```math
H = {}; W1 = subsets(n+1,1); apply(W1,s->(H=H|{x_(s_0)})); W2 = subsets(n+1,2); apply(W2,s->(H=H|{x_(s_0)+x_(s_1),x_(s_0)-x_(s_1)})); Pts = {}; for j from 0 to #H-1 do (v={};for i from 0 to n do (v=v|{diff(x_i,H_j)}); Pts=Pts|{v});
```

Checking $2 \leq n \leq 6$, $2 \leq d \leq 6$, $2 \leq m \leq d$ turned up seven cases of unexpected hypersurfaces, listed in Table 1. The case $(2, 4, 3, 0, 1)$ comes from the arrangement $B_3$ shown in Figure 1. Its unique unexpected curve was shown to be unexpected by other methods in [CHMN].

In the case of $B_4$ we get a previously unknown unique unexpected hypersurface. It has degree 4 with a general point $[a_0 : a_1 : a_2 : a_3]$ of multiplicity 4. Thus it is a cone at the point $[1 : 1 : 1 : 1]$. In this case $Z_{B_4}$ is the set of 16 points coming from the roots, but the vanishing locus of $[I_{Z_{B_4}}]_4$ is 0-dimensional, in contrast with the examples of §2. In fact, as the general point of multiplicity 4 moves around, the only points that lie on every one of the degree 4 unexpected surfaces are the 16 points of $Z_{B_4}$ together with the eight points $[1 : 1 : 1 : 1], [-1 : 1 : 1 : 1], [1 : -1 : 1 : 1], [1 : 1 : -1 : 1], [1 : 1 : 1 : -1], [-1 : -1 : 1 : 1], [-1 : 1 : -1 : 1], [1 : -1 : -1 : 1]$. (To compute this locus of 24 points, look at the ideal of
the coefficients of the monomials of the unexpected form $F(a, x)$ in $a = [a_0 : a_1 : a_2 : a_3]$; i.e., if $F(a, x)$ is the unexpected surface, write it as a polynomial in $a_i$ with coefficients which are polynomials in $x_j$. Take the ideal generated by these coefficient polynomials in $x_j$. They define the locus of points at which all of the unexpected surfaces vanish as the point $a$ moves around.

3.3. The root system $C_{n+1}$. Since $Z_{C_{n+1}} = Z_{B_{n+1}}$, this case is covered by $B_{n+1}$.

3.4. The root system $D_{n+1}$. The root system $D_{n+1}$ consists of the $2(n+1)n$ integer vectors $(a_1, \ldots, a_{n+1}) \in \mathbb{R}^{n+1}$ with $a_1^2 + \cdots + a_{n+1}^2 = 2$. The CODE BLOCK here is:

```plaintext
H={};
W2=subsets(n+1,2);
apply(W2,s->(H=H|{x_(s_0)+x_(s_1),x_(s_0)-x_(s_1)}));
Pts={};
for j from 0 to #H-1 do
  (v={}; for i from 0 to n do (v=v|{diff(x_i,H_j)});Pts=Pts|{v});
We checked $2 \leq n \leq 6$, $2 \leq d \leq 6$ and $2 \leq m \leq d$. The cases $(n,d,m,\text{edim, adim})$ that turned up were (3,3,3,−2,1), (3,4,4,3,4).

3.5. The root systems $E_{n+1}$, for $n = 5, 6, 7$. The root system $E_8$ is a set of 240 vectors consisting of all integer vectors $(a_1, \ldots, a_8) \in \mathbb{R}^8$ with $a_1^2 + \cdots + a_8^2 = 2$, (i.e., $D_8$) together with all vectors of the form $(1/2)(a_1, \ldots, a_8)$ where each entry $a_i$ is ±1 and $a_1 \cdots a_8 = 1$; thus $|Z_{E_8}| = 120$. The CODE BLOCK here is:

```plaintext
Pts={};
for j from 0 to #H-1 do
  (v={}; for i from 0 to n do (v=v|{diff(x_i,H_j)});Pts=Pts|{v});
```
(1, -1, -1, 1, 1, 1, -1, -1), (1, -1, -1, 1, 1, -1, 1, -1), (1, -1, -1, 1, -1, 1, 1, -1),
(1, -1, -1, 1, -1, 1, -1, 1), (1, -1, -1, -1, 1, 1, 1, -1), (1, -1, -1, 1, 1, -1, 1, -1),
(1, -1, -1, 1, 1, -1, -1, -1), (1, -1, -1, -1, 1, 1, 1, -1), (1, -1, -1, -1, 1, 1, -1, 1),
(1, -1, -1, -1, 1, 1, -1, -1), (1, -1, -1, -1, -1, 1, 1, 1), (1, -1, -1, -1, -1, 1, -1, 1),
(1, -1, -1, -1, -1, -1, 1, -1), (1, -1, -1, -1, -1, -1, -1, 1));

For $E_8$ we have $n = 7$. We checked $2 \leq d \leq 6$ and $2 \leq m \leq d$. The cases $(n, d, m, edim, adim)$ that arose were $(7, 4, 3, 174, 175), (7, 4, 4, 90, 99)$, and $(7, 5, 5, 342, 343)$.

The root system $E_7$ is the set of 126 elements of $E_8$ such that the first two coordinates are equal. Thus the CODE BLOCK for $E_7$ is obtained from that for $E_8$ by filtering out the cases where the first two coordinates are equal and then dropping the first coordinate to get a 7 element vector. We checked $2 \leq n \leq 6$, $2 \leq d \leq 6$ and $2 \leq m \leq d$. The only case $(n, d, m, edim, adim)$ that arose was $(6, 4, 4, 63, 64)$.

The root system $E_6$ is the set of 72 elements of $E_8$ such that the first three coordinates are equal. The CODE BLOCK in this case is obtained from that for $E_7$ by filtering out the cases where the first two coordinates are equal and then dropping the first coordinate to get a 6 element vector. We checked $2 \leq n \leq 6$, $2 \leq d \leq 6$ and $2 \leq m \leq d$ but did not find any unexpected hypersurfaces.

3.6. The root system $F_4$. The root system $F_4$ is the set of 48 vectors consisting of all vectors $(a_1, \ldots, a_4) \in \mathbb{R}^4$ such that each entry is $\pm 1$ and $1 \leq a_2^2 + \cdots + a_4^2 \leq 2$, or each entry is $\pm 1/2$. So here $n = 3$, $|Z_{F_4}| = 24$ and the CODE BLOCK is

$$
\text{Pts} = \{(1,1,0,0), (1,-1,0,0), (1,0,1,0), (1,0,-1,0), (1,0,0,1), (1,0,0,-1),
\{0,1,1,0\}, \{0,1,-1,0\}, \{0,1,0,1\}, \{0,1,0,-1\}, \{0,0,1,1\}, \{0,0,1,-1\},
\{1,0,0,0\}, \{0,1,0,0\}, \{0,0,1,0\}, \{0,0,0,1\}, \{1,1,1,1\}, \{1,1,-1,1\},
\{1,1,1,-1\}, \{1,1,-1,-1\}, \{1,-1,1,1\}, \{1,-1,-1,1\}, \{1,-1,1,-1\}, \{1,-1,-1,-1\}\};
$$

We checked $2 \leq n \leq 6$, $2 \leq d \leq 10$ and $2 \leq m \leq d$. The cases $(n, d, m, edim, adim)$ that arose were $(3, 4, 3, 2, 4), (3, 4, 4, -8, 1), (3, 5, 5, -3, 3), (3, 6, 6, 4, 7), (3, 7, 7, 12, 13)$.

3.7. The root system $H_n$, $n = 3, 4$. The root systems $H_3$ and $H_4$ are non-crystallographic. Figure 2 shows the 15 lines dual to the roots of the $H_3$ root system. We note that the $H_3$ line arrangement is simplicial but not supersolvable. Because the lines are not defined over $\mathbb{Q}$, for $H_3$ we must in the universal script replace

```plaintext
for n from 2 to 2 do {
R=QQ[x_0..x_n];
S=frac(QQ[a_0..a_n]);
by
for n from 2 to 2 do {
K=toField(QQ[t]/(t^2-5));
R=K[x_0..x_n];
S=frac(QQ[a_0..a_n]);
The CODE BLOCK is now
Pts={{0,0,1}, \{1,0,1\}, \{1,0,-1\}, \{0,1,1\},
\{0,1,-1\}, \{0,1,2 + t\}, \{0,1,-2 - t\}, \{1,0,2 + t\}, \{1,-1,0\}, \{1,1,0\},
\{t+3,-(2*t+4),3*t+7\}, \{2*t+4,-(t+3),-(3*t+7)\},
\{2*t+4,-(t+3),3*t+7\}, \{t+3,-(2*t+4),-(3*t+7)\}};
```
The unexpected curves that turned up for $n = 2$, $2 \leq d \leq 8$ and $2 \leq m \leq d$ were $(2, 6, 5, -2, 1), (2, 7, 6, 0, 2), (2, 8, 7, 2, 3)$.

The $H_4$ root system has 120 elements defined over $\mathbb{Q}[t]/(t^2 - t - 1)$ (see [BFGMV, St]). Thus $|Z_{H_4}| = 60$. For $H_4$ one must use

```plaintext
for n from 3 to 3 do {
K=toField(QQ[t]/(t^2-t-1));
R=K[x_0..x_n];
The CODE BLOCK is
Pts={{1,0,0,0}, {0,1,0,0}, {0,0,1,0}, {0,0,0,1}, {1,1,1,1}, {1,1,1,-1},
{1,1,-1,1}, {1,1,-1,-1}, {1,-1,1,1}, {1,-1,1,-1}, {1,-1,-1,1}, {1,-1,-1,-1},
{0,t,t^2,-1}, {0,t,t^2,-1}, {0,t,-t^2,1}, {0,t,-t^2,-1}, {0,t^2,1,t}, {0,t^2,1,-t},
{0,t^2,-1,t}, {0,t^2,-1,-t}, {1,1,t^2,0}, {1,1,t^2,-1}, {1,1,-t^2,0}, {1,1,-t^2,-1},
(3,6,3,14,15), (3,6,4,4,9), (3,6,5,−11,4), (3,6,6,−32,1).
4. BMSS Duality

A very interesting observation was made in [BMSS]. In the case of the unexpected quartic coming from the $B_3$ line arrangement, [BMSS] observed that $F^\ast(a,x)\in R$ has bi-degree $(3,4)$ and for each $x$ it defines three lines in the $a$ variables, and moreover that these three lines meet at the point $x$. In fact, given a form

$$F \in S = \mathbb{K}(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0})[x_0, \ldots, x_n]$$

defining an unexpected variety of degree $d$ with a general point $P$ of multiplicity $m$, one has by Lemma 3.1(d) that the bi-homogeneous form $F^\ast(a,x) \in R = \mathbb{K}[a_0, \ldots, a_n][x_0, \ldots, x_n]$ has bi-degree $(t, d)$ for some $t \geq m$. Thus one can regard $F^\ast(a,x)$ as defining a family of hypersurfaces in the $x$ variables, parameterized by $a$ (these are the unexpected hypersurfaces),

but one can also regard $F^*(a, x)$ as defining a more mysterious family of hypersurfaces in the $a$ variables, parameterized by $x$.

In the case of the $B_3$ unexpected quartic $F^*(a, x)$ with a general triple point, it follows from Lemma 3.1(d) that $F^*$ has multiplicity at least 3 in the $a$ variables at the point $a = x$ and hence has bi-degree $(s, 4)$ with $s \geq 3$. We see below why in fact $s = 3$; thus, given the point of multiplicity at least 3 at $a = x$, it must have multiplicity exactly 3, and for a general choice of $x$, $F^*(a, x)$ splits as a product of three forms linear in the $a$ variables, meeting at $a = x$.

In this section we will show that this phenomenon occurs in a range of cases, and that for these cases $F^*(x, a)$ defines the lines tangent to the branches of the curve $F^*(a, x)$ (see Figure 3). We will also study a similar duality for the hypersurfaces defined by our cone construction from §2.

Let $W \subset \mathbb{A}^n$ be a hypersurface in affine space defined by a reduced polynomial $F$. We first recall what the tangent cone is for $W$ at a point $P \in W$. Write $F$ as a sum $F = F_0 + F_1 + \cdots$ of polynomials $F_i \in I(P)^i$ where each $F_i$ is homogeneous of degree $i$ in coordinates centered at the point $P$. Let $j$ be the least index such that $F_j \neq 0$. Then $F_j = 0$ defines the tangent cone to $W$ at $P$. In characteristic 0, $F = F_0 + F_1 + \cdots$ is just a Taylor expansion of $F$ at $P$; the tangent cone is the term $F_j$ obtained by differentiating to order $j$ where $j$ is the multiplicity of $W$ at $P$.

One can also work projectively. Let $W \subset \mathbb{P}^n$ now be a hypersurface in projective space defined by a reduced polynomial $F$. For simplicity, assume the characteristic of $\mathbb{K}$ is 0. Given two polynomials $F$ and $G$, let $F \cdot G$ denote the action of differentiation, so

$$(2x_0^2x_1 + x_3) \cdot (x_0^3x_1^2x_3) = (2x_0^2x_1) \cdot (x_0^3x_1^2x_3) + x_3 \cdot (x_0^3x_1^2x_3) = 24x_0x_1x_3 + x_0^3x_1^2.$$

To compute the tangent cone of $W$ at a point $P$ of multiplicity $m$, let $\mu_j$ be an enumeration of the monomials of degree $m$. Let $c_j = \mu_j \cdot \mu_j$ be the factorial expression obtained by differentiating $\mu_j$ against itself (this is needed for the Taylor expansion). Then the tangent cone of $W$ at $P$ is defined by the degree $m$ form

$$H_P = \sum_j \frac{(\mu_j \cdot H)(P))\mu_j}{c_j}.$$

**Example 4.1.** As an example we consider the tangent cones for the cone construction of §2. Assume $Z$ is a finite set of points in $\mathbb{P}^n_{\mathbb{K}}$ and that the space $V \subset [S]_d$ of forms vanishing on the points has the property that there is (up to multiplication by a scalar) a unique form $F \in V$ with a point of multiplicity $d$ at a general point $P = [a_0 : \ldots : a_n] \in \mathbb{P}^n$. Applying an idea similar to that of Remark 2.8, remove points if necessary so that we are left with a subset $\{P_1, \ldots, P_r\}$ of $Z$ of $r = \binom{d + n}{n} - \binom{d - 1 + n}{n} - 1$ points. (Now $F$ is not unexpected.) Then $F^* \in \mathbb{K}[a_0, \ldots, a_n][x_0, \ldots, x_n]$ and by Lemma 3.1(d) $F^*$ is bi-homogeneous of bi-degree $(\delta, d)$ with $\delta \geq d$.

Let $M_j$ be an enumeration of the monomials in $T = \mathbb{K}[x_0, \ldots, x_n]$ of degree $d$. Let $m_i$ be an enumeration of the monomials in $T$ of degree $d - 1$. Let $t = \binom{n+1}{n}$ and $s = \binom{n+d-1}{n}$. Let $\Gamma$ be the matrix whose top $r$ rows are the values $M_j(P_i)$ of the monomials $M_j$ at the points $P_i$, and whose next (and bottom) $s$ rows are the values $(m_i \cdot M_j)(P_i)$, where as above the dot indicates the action of $T$ on partial differentiation. Note that the entries of $\Gamma$ are all in $\mathbb{K}[a_0, \ldots, a_n]$. Elements in the kernel of $\Gamma$ are coefficient vectors for forms vanishing at the points $P_i$ and having a point of multiplicity $d$ at $P = a$. The assumption that there
are \( r = t - s - 1 \) points \( P \) and that there is (up to multiplication by scalars) a unique form vanishing on the points with a point of multiplicity \( m \) at \( P \) means that \( \Gamma \) is a \((t-1) \times t\) matrix whose rank at a general point \( P \) is \( t-1 \). Since the entries of \( \Gamma \) are monomials of degree 0 or 1, we can divide the \( s \) rows having degree 1 monomials by \( a_0 \) and obtain a row equivalent matrix \( \Gamma' \) with entries in the field \( F = \mathbb{K}(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}) \). The kernel of \( \Gamma' \) has dimension 1, and for any nonzero vector \( v = (v_1, \ldots, v_t) \) in the kernel, we can take \( F = \sum v_i M_i \). Note that \((F^*)^o\) (and hence \( F^* \)) also are in the kernel, and so can be used in place of \( F \) and thus all have the same tangent cone for a general point \( P = a \). Each entry of \( v \) is in \( F \) so computing the tangent cone gives

\[
\sum_j \frac{((M_j \cdot F)(P)) M_j}{c_j} = \sum_{i,j} \frac{((M_j \cdot v_i M_i)(P)) M_j}{c_j} = \sum_i v_i \frac{((M_i \cdot M_j)(P)) M_i}{c_i} = \sum_i v_i M_i = F.
\]

I.e., \( F \) is its own tangent cone, which for a general point \( P = a \) is thus defined by \( F(a, x) = 0 \) (or equivalently \( F^*(a, x) = 0 \)).

We can also work over \( R \). Let \( \Gamma^* \) be the matrix obtained by appending \((M_1, \ldots, M_t)\) as a row at the bottom of \( \Gamma \), and let \( F^* \) be the matrix obtained by appending \((M_1, \ldots, M_t)\) as a row at the bottom of \( \Gamma \). Then \( G' = \det \Gamma^* \) is a multiple of \( F \) by a scalar in \( \mathbb{F} \) and we have \( a_0^* G' = G = \det \Gamma^* \in R \). Since \( G' = G/a_0^* \) is a scalar multiple of \( F \) by a scalar in \( \mathbb{F} \), it follows that \( G^* = F^* \). Moreover, we have \( G = C(a) G^* \) where \( C(a) \) is a polynomial describing for which points \( a \) the matrix \( \Gamma \) has less than full rank. (Suppose that we choose a point \( a = Q_1 \) such that there is a point \( x = Q_2 \) for which \( G^*(Q_1, Q_2) \neq 0 \). Then \( C(Q_1) = 0 \) if and only if \( \det \Gamma^* = G(Q_1, Q_2) = 0 \), which occurs if and only if the maximal minors of \( \Gamma \) all vanish; i.e., \( \Gamma \) has rank less than \( t-1 \).) As before, \( G \) is its own tangent cone, but it’s still not clear what \( F^* \) is or how \( F^*(a, x) \) is related to \( F^*(x, a) \).

Now assume that there is an irreducible variety \( W \) of degree \( d \) and codimension 2 such that for a general point \( P = a \) the locus \( F(a, x) = 0 \) is precisely the union of all lines through \( P \) and a point of \( W \), and that \( F(a, x) \) is irreducible (and hence so is \( F^*(a, x) \) by Lemma 3.1). Then we have \( F^*(a, x) = \pm F^*(x, a) \). Here’s why. For a general point \( P' = [a_0': \ldots: a_n'] \) of \( F(a, x) = 0 \) (and hence of \( F^*(a, x) = 0 \)), \( P' \) is on the cone through \( W \) having vertex \( P \), so \( P' \) is on the line through \( P \) and a point \( w \in W \). But then \( P \) is on the line through \( P' \) and \( w \), so \( P \) is on the cone \( F'(a', x) = 0 \) with vertex \( P' \) (hence on \( F^*(a', x) = 0 \)), so \( F^*(a', x) = 0 \). I.e., \( F^*(a', x) = 0 \) if and only if \( F^*(a, x) = 0 \). Thus the loci \( F^*(a, x) = 0 \) and \( F^*(x, a) = 0 \) intersect in a nonempty open subset of \( F(a, x) = 0 \). Since \( F^*(a, x) \) and \( F^*(x, a) \) are irreducible, we have \( F^*(a, x) = cF^*(x, a) \) for some scalar \( c \). But swapping variables again gives \( F^*(x, a) = cF^*(a, x) \) hence \( F^*(a, x) = c^2 F^*(a, x) \) so \( c = \pm 1 \). Thus in this case, without resorting to Lemma 3.1(d), we see \( F^*(a, x) \) has bi-degree \((d, d)\) and that the tangent cone to \( F^*(a, x) \) at a general point \( P = a \) is defined by \( F^*(x, a) \) (since here the tangent cone is defined by \( F^*(a, x) \) but \( F^*(a, x) \) and \( F^*(x, a) \) define the same locus).

**Question 4.2.** Is it always true for an unexpected hypersurface \( F(a, x) \) of degree \( d \) with a general point of multiplicity \( d \) that \( F^*(a, x) = \pm F^*(x, a) \)?

We now consider how \( F^*(a, x) \) and \( F^*(x, a) \) are related in the case of unexpected curves in the plane having degree \( m+1 \) and a general point \( P \) of multiplicity \( m \).

We begin with a lemma. Recall that \((x-x_1)(x-x_2)\cdots(x-x_n) = x^n + (-1)^{1} e_1 x^{n-1} + (-1)^2 e_2 x^{n-2} + \cdots + (-1)^{n} e_n x^0 \), where \( e_i = \sum_{1 \leq j_1 < j_2 < \cdots < j_i \leq n} x_{j_1} \cdots x_{j_i} \) is the so-called \( i \)th elementary symmetric polynomial. Let \( p_i = x_1^i + \cdots + x_n^i \) be the \( i \)th symmetric power sum.
Newton’s identities (also known as the Newton-Girard formulas) relate these as follows:

\[ e_1 = p_1, \]
\[ 2e_2 = e_1p_1 - p_2, \]
\[ 3e_3 = e_2p_1 - e_1p_2 + p_3, \]

etc., and in general

\[ ie_i = \sum_{j=1}^{i} (-1)^{j-1}e_{i-j}p_j. \]

Let \( f : \mathbb{K}^n \to \mathbb{K}^n \) be the map \( f(x_1, \ldots, x_n) = (e_1, \ldots, e_n) \) and let \( df = (\partial e_i/\partial x_j) \) be the matrix for the mapping on tangent spaces.

**Lemma 4.3.** Given the mapping \( f(x_1, \ldots, x_n) = (e_1, \ldots, e_n) \), then

\[ \det(df)(x_1, \ldots, x_n) = \pm \Pi_{i<j}(x_i - x_j) \]

so \( \det(df)(x_1, \ldots, x_n) \neq 0 \) if and only if \( x_i \neq x_j \) for all \( i \neq j \).

**Proof.** The proof is to show by applying row operations involving multiplying rows by \(-1\) and adding multiples of one row to another that one obtains the Vandermonde matrix.

Row \( n \) of the matrix \( df \) is the gradient of \( e_n \), namely \( \nabla(e_n) \). Using Newton’s identity we can rewrite this as \( \frac{1}{n} \nabla(\sum_{j=1}^{n} (-1)^{j-1} e_{n-j}p_j) = \frac{1}{n} \sum_{j=1}^{n} (-1)^{j-1} (\nabla(e_{n-j})p_j + e_{n-j}\nabla(p_j)) \).

Using row operations, we can (up to row equivalence) clear out the terms \( \nabla(e_{n-j})p_j \) since these are multiples of rows \( \nabla(e_i) \) higher up in the matrix. We can do the same now for row \( n - 1 \), and then row \( n - 2 \), etc.

Afterward, row 1 is unchanged (it is \( \nabla(e_1) = \nabla(p_1) = x_1 + \cdots + x_n \)), but row 2 is \( \frac{1}{2}(e_1\nabla(p_1) - \nabla(p_2)) \), so we can use row 1 to clear the term \( e_1\nabla(p_1) \) so that row 2 becomes \( -\nabla(p_2) \), at which point we can use rows 1 and 2 to clear row 3 so that row 3 becomes \( \nabla(p_3) \). Continuing in this way we eventually obtain a matrix row equivalent to \( df \) whose rows are \( \nabla(p_1) = (1, \ldots, 1), -\frac{1}{2}\nabla(p_2) = -(x_1, \ldots, x_n), \ldots, (-1)^{n-1}\frac{1}{n}\nabla(p_n) = (-1)^{n-1}(x_1^{n-1}, \ldots, x_n^{n-1}) \). Up to sign, this is the Vandermonde matrix, whose determinant is well known to be as claimed. \( \square \)

Let \( Z \subset \mathbb{P}^2 \) be a finite set of points admitting an unexpected curve \( C \) of degree \( m + 1 \) with a general point \( P = [a_0 : a_1 : a_2] \) of multiplicity \( m \). Let \( F \in S \) be the form defining \( C \) over the field \( \mathbb{K}(\frac{a_1}{a_0}, \frac{a_2}{a_0}) \). Examples suggest that \( F^* \) is bi-homogeneous of bi-degree \( (m, m+1) \), but our proof is restricted to the case that the line arrangement dual to \( Z \) is free. A forthcoming independent result of W. Trok [Tr] also obtains a similar result on bi-degree in the free case.

**Theorem 4.4.** Let \( Z \subset \mathbb{P}^2 \) be a finite set of points admitting an irreducible unexpected curve \( C = C_P \) of degree \( m + 1 \) with a general point \( P = [a_0 : a_1 : a_2] \) of multiplicity \( m \).

(a) The curve \( C_P \) is unique.

(b) Let \( F(a, x) \in S = \mathbb{K}(\frac{a_1}{a_0}, \frac{a_2}{a_0})[x_0, x_1, x_2] \) be the form defining \( C \) over the field \( \mathbb{K}(\frac{a_1}{a_0}, \frac{a_2}{a_0}) \). Assume that the lines dual to the points of \( Z \) comprise a free line arrangement. Then \( F^*(a, x) \) is bi-homogeneous of bi-degree \( (m, m+1) \). Furthermore, viewing \( F^*(a, x) \in \mathbb{K}[x_0, x_1, x_2][a_0, a_1, a_2] \), \( F^*(a, x) \) has multiplicity \( m \) in the \( a \) variables at the general point \( [x_0 : x_1 : x_2] \) (briefly we will say \( F^*(a, x) \) has a point of multiplicity \( m \) in the \( a \) variables at \( a = x \)).
(c) Assume that $F \in R = \mathbb{K}[a_0, a_1, a_2][x_0, x_1, x_2]$ is any bi-homogeneous form of bi-degree $(m, m + 1)$ such that $F(a, x)$ is reduced and irreducible for a general point $a = P$ and has multiplicity $m$ both in the $a$ variables at $a = x$ and in the $x$ variables at $x = a$. Then $F_P(a, x) = (-1)^m F(x, a)$ is the tangent cone at $x = P$ to the curve $F(P, x) = 0$ for $a = P$, where $F_P$ is defined in (4.1).

Proof. (a) Since $C$ is irreducible, then [CHMN] shows that $C$ is unique.

(b) By Lemma 3.1(d) we know that $F^*$ is bi-homogeneous of bi-degree $(m', m + 1)$ for some $m' \geq m$ and that $F^*$ has multiplicity at least $m$ in the $a$ variables at $a = x$. The conclusion follows from examining a parameterization given in [CHMN] to conclude that $m' \leq m$. Let $\Lambda$ be the product of the forms defining the lines dual to the points of $Z$. It is not hard to check that there are no unexpected curves for $Z$ with $|Z| < 3$, so after a change of coordinates if need be, we may assume $x_0 x_1$ divides $\Lambda$. Let $(s_0, s_1, s_2)$ be a syzygy of minimal degree (hence homogeneous of degree $m$; see [CHMN]) for the ideal $(\Lambda_{x_0}, \Lambda_{x_1}, \Lambda_{x_2})$, where $\Lambda_{x_i}$ denotes the partial with respect to $x_i$; thus

$$s_0 \Lambda_{x_0} + s_1 \Lambda_{x_1} + s_2 \Lambda_{x_2} = 0.$$

Define $\phi$ formally to be the vector whose components are given by the cross product

$$\phi = (\phi_0, \phi_1, \phi_2) = (s_0, s_1, s_2) \times (x_0, x_1, x_2).$$

Now let $\ell$ be the general line $a_0 x_0 + a_1 x_1 + a_2 x_2 = 0$. If $a_2 \neq 0$, we can parameterize $\ell$ by $(ta_2, -a_2 s, a_1 s - a_0 t)$, where $(t, s)$ are projective coordinates on $\mathbb{P}^1$, and by [CHMN] $\phi$ defines a birational map from $\ell$ to $C$.

Note that, since $x_0 x_1$ divides $\Lambda$, then $x_0$ divides $\Lambda_{x_0}$ and $\Lambda_{x_2}$ and $x_1$ divides $\Lambda_{x_0}$ and $\Lambda_{x_2}$, but $x_0$ does not divide $\Lambda_{x_0}$ and $x_1$ does not divide $\Lambda_{x_1}$. Setting $x_0 = 0$ in $s_0 \Lambda_{x_0} + s_1 \Lambda_{x_1} + s_2 \Lambda_{x_2} = 0$ gives

$$s_0 \Lambda_{x_0} = 0$$

so $x_0$ must divide $s_0$. Similarly, $x_1$ divides $s_1$.

Plugging $(x_0, x_1, x_2) = (ta_2, -a_2 s, a_1 s - a_0 t)$ into $\phi(x_0, x_1, x_2)$ we see from the definition of the cross product that $a_2$ divides $\phi_2(ta_2, -a_2 s, a_1 s - a_0 t)$, and since $x_i | s_i$ for $i = 0, 1$, we get that $x_i$ divides $\phi_i$ for $0, 1$ and thus after the substitution $(x_0, x_1, x_2) = (ta_2, -a_2 s, a_1 s - a_0 t)$ that $a_2$ divides $\phi_i$ for $i = 0, 1, 2$. Let $\psi = \phi_0 \phi_2$. Then $\psi$ is bi-homogeneous of degree $m - 1$ in the $a$ variables and degree $m$ in the variables $s$ and $t$.

We also get a parameterization of $C$ using the slope of lines through the point $P = a$. This induces an isomorphism from $E_P$ to $C$, where $E_P$ is the blow up of $P$. But $\psi = (\psi_0, \psi_1, \psi_2)$ induces a birational map $\ell$ to $C$. Composing gives an isomorphism $\ell$ to $E_P$, which is therefore linear. Thus after a change of coordinates fixing $P$ and $x_2 = 0$, we may assume that the parameterization of $C$ given by $\psi$ is the same as that given by slopes of lines through $P$. In particular, consider the line through the point $(a_1/a_2, a_1/a_2, 1)$ with slope $t/s$; in affine coordinates it is

$$\frac{x_1}{x_2} - \frac{a_1}{a_2} = t \left( \frac{x_0}{x_2} - \frac{a_0}{a_2} \right)$$

which we can rewrite as $(x_1 a_2 - a_1 x_2) s = t(x_0 a_2 - a_0 x_2)$. Setting $x_2 = 0$ gives the coordinates $a_2 t = x_1, a_2 s = x_0$ as the parameterization on $E_P$, and we can assume that the parameterization of $C$ given by mapping the point $(t, s)$ on $E_P$ to the point $(x_0 : x_1 : x_2)$ where the line $(x_1 a_2 - a_1 x_2) s = t(x_0 a_2 - a_0 x_2)$ meets $C$ (away from $P$) is the same point of $C$ as that given by $\psi(ta_2, -a_2 s, a_1 s - a_0 t)$. 


Thus the parameterization is such that
\[ [x_0 : x_1 : x_2] = [\psi_0(ta_2, -a_2s, a_1s - a_0t) : \psi_1(ta_2, -a_2s, a_1s - a_0t) : \psi_2(ta_2, -a_2s, a_1s - a_0t)], \]
and hence plugging \( s = x_0a_2 - a_0x_2 \) and \( t = x_1a_2 - a_1x_2 \) in to \((ta_2, -a_2s, a_1s - a_0t)\) gives \((a_2(x_1a_2 - a_1x_2), -a_2(x_0a_2 - a_0x_2), a_1x_0a_2 - a_0x_1a_2)\), and factoring out the \( a_2 \) we get \((x_1a_2 - a_1x_2, -(x_0a_2 - a_0x_2), a_1x_0 - a_0x_1)\), each component of which has bi-degree \((1, 1)\). So we now have parametric equations for \( C \),
\[
[x_0 : x_1 : x_2] = [\psi_0(v) : \psi_1(v) : \psi_2(v)],
\]
where \( v = (x_1a_2 - a_1x_2, -x_0a_2 + a_0x_2, a_1x_0 - a_0x_1) \), so each component of the right hand side has bi-degree \((m, m + 1)\). We get the following equations for \( C \) (bi-homogeneous of degree \((m, m + 2)\)):
\[
x_0\psi_1 - x_1\psi_0 = 0, \quad x_0\psi_2 - x_2\psi_0 = 0, \quad x_1\psi_2 - x_2\psi_1 = 0.
\]
The form \( F^*(a, x) \) defining \( C \) is a common divisor of these three equations. Thus \( F^* \) is bi-homogeneous of bi-degree \((m', m + 1)\) for some \( m' \leq m \). Thus \( F^* \) is bi-homogeneous of bi-degree \((m, m + 1)\), and since it has a point of multiplicity at least \( m \) at \( a = x \) but degree exactly \( m \), the multiplicity is also \( m \).

(c) Since \( F \) has degree \( m \) in the \( a \) variables, \( F \) defines a curve of degree \( m \) with a point of multiplicity \( m \) at \( a = x \), hence it defines a union of \( m \) lines through the point \( a = x \).

The question remains as to what these lines are. To answer this question, translate the point \( a = x \) to the point \([1 : 0 : 0]\). Doing this gives us
\[
G(a) = a_0^m F\left(1, \frac{a_1}{a_0} + \frac{x_1}{x_0}, \frac{a_2}{a_0} + \frac{x_2}{x_0}, \frac{1}{x_0}, \frac{x_2}{x_0} \right) \in \mathbb{K}\left[x_0, x_1, x_2 \right][a_0, a_1, a_2].
\]
In the \( a \) variables, this defines \( m \) lines meeting at \([1 : 0 : 0]\). Thus in the \( a \) variables \( G(a) \) is a homogeneous form of degree \( m \), but now the variable \( a_0 \), which we can regard as defining the line at infinity, does not appear. Note that neither \( a_1 \) nor \( a_2 \) is a factor of \( G(a) \), since if, say \( a_1 \) were a factor of \( G \), then for all choices of \( x, G = 0 \) includes the line from \([1 : 0 : 0]\) to \([0 : 0 : 1]\). Since the point \([0 : 0 : 1]\) is at infinity, it is fixed under the translation we employed, so \( F(a_0, a_1, a_2, x_0, x_1, x_2) = 0 \) includes the line from \([x_0 : x_1 : x_2]\) to \([0 : 0 : 1]\), hence \( x_0a_0 - x_0a_1 \) is a factor of \( F \), contradicting the assumption that \( C \) is irreducible.

Since \( a_0 \) does not appear in \( G(a) \) and \( a_1 \) and \( a_2 \) are not factors, we have \( G(a) = b_0a_0m + \cdots + b_mA_2^n \) with \( b_i \in \mathbb{K}\left[\frac{x_1}{x_0}, \frac{x_2}{x_0}\right] \) and \( b_0, b_m \neq 0 \). Setting \( a_2 = 1 \) in \( G \) gives \( g(a_1) = b_0a_0^n + \cdots + b_m \) and dividing by \( b_0 \) gives \( h(a_1) = g(a_1)/b_0 \in \mathbb{K}\left[\frac{x_1}{x_0}, \frac{x_2}{x_0}\right][a_1] \). Over an appropriate field extension of \( \mathbb{K}\left[\frac{x_1}{x_0}, \frac{x_2}{x_0}\right] \) this factors as \( h(a_1) = (a_1 - h_1) \cdots (a_1 - h_m) \). Note that the elements \( h_i \) are distinct. If not, then \( h \) and hence \( g \) has a multiple root, say \( h_i \), hence \( a_1 - h_i \) is a common factor of \( g \) and \( g' \) (or, taking more derivatives, of \( g^{(\mu - 1)} \)) and \( g^{(\mu)} \) if the root has multiplicity \( \mu > 1 \) and thus can be found using the Euclidean algorithm. Thus the linear factor \( a_1 - h_i \) is in \( \mathbb{K}\left[\frac{x_1}{x_0}, \frac{x_2}{x_0}\right][a_1] \) and so divides \( g \) over \( \mathbb{K}\left[\frac{x_1}{x_0}, \frac{x_2}{x_0}\right][a_1] \) and hence \( a_1 - h_i a_2 \) divides \( G(a) \) in \( \mathbb{K}\left[\frac{x_1}{x_0}, \frac{x_2}{x_0}\right][a_0, a_1, a_2] \) and thus \( F(a, x) \) as before has a corresponding factor linear in \( a \), contrary to assumption. Thus the roots \( h_i \) are distinct.

Since by Lemma 4.3 the mapping \( f \) expressing the coefficients of \( h \) as symmetric functions of the roots \( h_i \) has an invertible differential \( df \) (the roots being distinct), we can by the inverse function theorem [FG, p. 33] regard the roots \( h_i \) as homomorphic functions of the coefficients \( b_i/b_0 \) of \( h \). Since each \( b_i/b_0 \) is a rational (and so holomorphic) function of \( x_1/x_0 \) and \( x_2/x_0 \), we can regard the \( h_i \) as homomorphic functions of \( x_1/x_0 \) and \( x_2/x_0 \).

The roots \( h_i \) give the points at infinity where \( G(a, x) \) vanishes. Specifically these are \([0 : h_i : 1]\). These are unaffected by affine translations, so the points where \( F(a, x) \) vanishes
on $a_0 = 0$ are these same points $[0 : h_i : 1]$. Thus translating back we see $F(a, x)$ vanishes on the lines through $[x_0 : x_1 : x_2]$ and $[0 : h_i : 1]$, so the forms defining these lines divide $F(a, x)$. Specifically, we have $G(a) = b_0(a_1 - h_1 a_2) \cdots (a_1 - h_m a_2)$ so

$$b_0 \left( \frac{a_1}{a_0} - h_1 \frac{a_2}{a_0} \right) \cdots \left( \frac{a_1}{a_0} - h_m \frac{a_2}{a_0} \right) = G(a)/a_0^m = F \left( 1, \frac{a_1}{a_0} + \frac{x_1}{x_0}, \frac{a_2}{a_0} + \frac{x_2}{x_0}, 1, \frac{x_1}{x_0}, \frac{x_2}{x_0} \right)$$

translates back to

$$F(a, x) = x_0^{m+1} a_0^m b_0 \left( \frac{a_1}{a_0} - \frac{x_1}{x_0} - \left( \frac{a_2}{a_0} - \frac{x_2}{x_0} \right) h_1 \right) \cdots \left( \frac{a_1}{a_0} - \frac{x_1}{x_0} - \left( \frac{a_2}{a_0} - \frac{x_2}{x_0} \right) h_m \right) = x_0 b_0 \left( (a_1 x_0 - a_0 x_1) - (a_2 x_0 - a_0 x_2) h_1 \right) \cdots \left( (a_1 x_0 - a_0 x_1) - (a_2 x_0 - a_0 x_2) h_m \right) =$$

$$b_0 x_0 \begin{vmatrix} 0 & h_1(x) & 1 \\ x_0 & x_1 & x_2 \\ a_0 & a_1 & a_2 \end{vmatrix} \cdots \begin{vmatrix} 0 & h_m(x) & 1 \\ x_0 & x_1 & x_2 \\ a_0 & a_1 & a_2 \end{vmatrix}.$$

Consider a point $x = a$ where $b_0 x_0 \neq 0$. Then the branches of $F(a, x) = 0$ in a neighborhood of $x = a$ are defined by the vanishing of each factor $\lambda_i(x_0, x_1, x_2) = \begin{vmatrix} 0 & h_i(x) & 1 \\ x_0 & x_1 & x_2 \\ a_0 & a_1 & a_2 \end{vmatrix}$. The tangent line to the branch $\lambda_i(x) = 0$ at $x = a$ is defined by $((\nabla \lambda_i)|_{x=a}) \cdot (x_0, x_1, x_2)$. Since $\lambda_i = (a_1 - a_2 h_i) x_0 - a_0 x_1 + a_0 x_2 h_i$, the gradient $\nabla \lambda_i$ is

$$\begin{pmatrix} (a_1 - a_2 h_i), -a_0, a_0 h_i \end{pmatrix} + \begin{pmatrix} -a_2 h_0 x_0 + a_0 x_2 h_0, -a_2 h_1 x_0 + a_0 x_2 h_1, -a_2 h_2 x_0 + a_0 x_2 h_2 \end{pmatrix}.$$

The second term vanishes at $x = a$ so

$$(\nabla \lambda_i)|_{x=a} \cdot (x_0, x_1, x_2) = (a_1 - a_2 h_i(x), -a_0, a_0 h_i(x))|_{x=a} \cdot (x_0, x_1, x_2) = \begin{vmatrix} 0 & h_i(a) & 1 \\ x_0 & x_1 & x_2 \\ a_0 & a_1 & a_2 \end{vmatrix}.$$

So we see that the tangent cone to $F(a, x) = 0$ at a point $P = a$ such that $b_0 x_0 \neq 0$ is defined by the vanishing of

$$F_P(a, x) = b_0(a) a_0 \begin{vmatrix} 0 & h_1(a) & 1 \\ x_0 & x_1 & x_2 \\ a_0 & a_1 & a_2 \end{vmatrix} \cdots \begin{vmatrix} 0 & h_m(a) & 1 \\ x_0 & x_1 & x_2 \\ a_0 & a_1 & a_2 \end{vmatrix} =$$

$$(-1)^m b_0(a) a_0 \begin{vmatrix} 0 & h_1(a) & 1 \\ a_0 & a_1 & a_2 \\ x_0 & x_1 & x_2 \end{vmatrix} \cdots \begin{vmatrix} 0 & h_m(a) & 1 \\ a_0 & a_1 & a_2 \\ x_0 & x_1 & x_2 \end{vmatrix} = (-1)^m F(x, a).$$

We can now extend our result to unexpected curves that are unique but not necessarily irreducible.

**Corollary 4.5.** Let $Z \subset \mathbb{P}^2$ be a finite set of points admitting a unique unexpected curve $C$ of degree $m + 1$ with a general point $P = [a_0 : a_1 : a_2]$ of multiplicity $m$. Let $F(a, x) \in S$ be the form defining $C$ over the field $\mathbb{K}(\frac{a_0}{a_0}, \frac{a_2}{a_0})$ and assume $F^*$ is bi-homogeneous of bi-degree $(m, m + 1)$. Then $(F^*)_P(a, x) = (-1)^m F^*(x, a)$ defines the tangent cone to $C$ at $P$. 

\qed
Proof. It is shown in [CHMN] that $C = C' \cup \Lambda_1 \cup \cdots \cup \Lambda_r$ where $C'$ is unexpected for a subset $Z'$ of $Z$ and has degree $m+1-r$ with a general point $P$ of multiplicity $m-r$ where $r = |Z| - |Z'|$ and each $\Lambda_i$ is the line from $P$ to $p_i$ where $p_1, \ldots, p_r$ are the points of $Z$ not in $Z'$. Thus $F = GL_1 \cdots L_r$, where $G$ is the form defining $C'$ and $L_i$ is the bi-linear form defining $\Lambda_i$. So we have $F^* = G*L_1^* \cdots L_r^*$, but $L_i^* = L_i$ and $L_i(a, x) = -L_i(x, a)$, so $F^*(a, x) = (G*L_1 \cdots L_r)(a, x) = (-1)^m(G*L_1 \cdots L_r)(x, a)$. The tangent cone for $C$ is the tangent cone for $C'$ union with the lines $L_i$, so we also see that $(-1)^mG*(x, a)L_1^*(x, a) \cdots L_r^*(x, a) = F^*(x, a)$ defines the tangent cone for $C$. \hfill \Box

Example 4.6. The form $F(a, x)$ defining the unexpected quartic curve $C$ for the $B_3$ configuration is

$$F(a, x) = a_2^3x_0^3x_1 - a_2^3x_0x_1^3 - a_1^3x_0^3x_2 + (3a_0a_1^2 - 3a_0^2a_2)x_0^2x_1x_2 + (-3a_0^2a_1 + 3a_1a_2)x_0x_1^2x_2 + a_0^3x_1^3x_2 + (3a_0^2a_2 - 3a_1^2a_2)x_0x_1x_2^3 + a_1^3x_0x_2^3 - a_0^3x_1x_2^3 =$$

$$= (x_1^3x_2 - x_1x_2^3)a_0^3 - 3x_0x_1^2x_2a_0a_1 + 3x_0^2x_1^2x_2a_1^2 + (-x_0^3x_2 + x_0x_2^3)a_1^3 + 3x_0x_1^2x_2a_0^2a_2 - 3x_0x_1x_2^2a_1a_2 - 3x_0^2x_1x_2a_0^2 + 3x_0x_1^2x_2a_1^2 + (x_0^3x_1 - x_0x_1^3)a_2^3.$$

Substituting $a_0 = x_0 = 1$ and restricting to the line $L$ defined by $x_1 = a_1$ gives

$$F(1, a_1, a_2, 1, a_1, x_2) = a_1(a_1^2 - 1)(x_2 - a_2)^3,$$

while substituting $a_0 = x_0 = 1, a_1 = x_1$ gives

$$F(1, x_1, a_2, 1, x_1, x_2) = x_1(x_1^2 - 1)(x_2 - a_2)^3.$$

Thus we see that $F$ has a triple point both in terms of the $a$ variables and the $x$ variables. To see that the factors of $F(x, a)$ are just the lines tangent to the branches of $F(a, x)$, one can graph $F(a, x)$ and $F(x, a)$ on the same coordinate axes; see Figure 3.
5. Open Problems

In this short section we list some open problems stemming from this work.

1. Suppose $Z$ is a finite set of points which admits an unexpected curve of degree $d = m + 1$ having a general point $P$ of multiplicity $m$. If the arrangement of lines dual to $Z$ is not free, to what extent does BMSS duality still hold?

2. To what extent does BMSS duality hold in higher dimensions? For example, it holds for $B_4$ and $F_4$ with $d = m = 4$. In these cases the unexpected surfaces are defined by a form $F(a, x)$ of bi-degree $(4, 4)$ and the form for the tangent cone at $P$ is $F(x, a)$. It also holds for $D_4$ with $d = m = 3$. In this case the unexpected surfaces are defined by a form $F(a, x)$ of bi-degree $(3, 3)$ and the form for the tangent cone at $P$ is $-F(x, a)$.

3. Given an unexpected variety for a finite point set $Z$ having a general point $P$ of multiplicity $m$ and degree $d$, let $B_Z(P)$ be the base locus of $[I_Z + mP]_d$ and let $B_Z = \cap P B_Z(P)$, which we can refer to as the base locus associated to $Z$. What can be said about this associated base locus? For example, what is its dimension? If it is 0-dimensional, when is it strictly larger than $Z$?

4. What is special about the root systems having unexpected hypersurfaces? For example, why do the systems $A_{n+1}$ not seem to have any?

5. For the systems $B_{n+1}$, computational runs suggest there might be unexpected hypersurfaces with $d = m = 4$ for all $n \geq 3$ and for $d = m = 3$ for all $n \geq 5$. How can one prove this? And why only $3 \leq m \leq 4$?

6. Let $Z$ be a non-degenerate set of points in linear general position in $\mathbb{P}^n$, $n \geq 3$. Is it true that there does not exist an unexpected hypersurface of any degree $d$ and multiplicity $d - 1$ at a general point?

7. Is there a class of finite sets of points in $\mathbb{P}^n$ for $n \geq 3$ (or respectively a condition on $(d, m)$) for which the syzygy bundle plays a similar role, in the study of unexpected hypersurfaces, to that which it plays when $n = 2$ and $m = d - 1$, or is that purely a phenomenon for the plane?

8. Let $Z$ be a set of points in $\mathbb{P}^2$ admitting a (unique) unexpected curve of degree $m + 1$ with a general point of multiplicity $m$. Then $Z + mP$ does not impose independent conditions on forms of degree $m+1$, but for a suitable subset $Z'$ of $2m+2$ of the points of $Z$, $Z' + mP$ does impose independent conditions, and the curve is still unique (but it is no longer unexpected for $Z'$).

So suppose we consider more generally sets $Z$ of $2m + 2$ points such that there is a unique irreducible curve of degree $m + 1$ containing $Z$ and having a general point of multiplicity $m$ (but not necessarily unexpected). How does the bi-degree of $F^*$ depend on $Z$? Is there a connection between this bi-degree and the question of whether $Z$ extends to a set of points for which the curve is unexpected?

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