PRESSURE ROBUST MIXED METHODS FOR NEARLY INCOMPRESSIBLE ELASTICITY

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Abstract. Within the last years pressure robust methods for the discretization of incompressible fluids have been developed. These methods allow the use of standard finite elements for the solution of the problem while simultaneously removing a spurious pressure influence in the approximation error of the velocity of the fluid, or the displacement of an incompressible solid. To this end, reconstruction operators are utilized mapping discretely divergence free functions to divergence free functions. This work shows that the modifications proposed for Stokes equation by [12] also yield gradient robust methods for nearly incompressible elastic materials without the need to resort to discontinuous finite elements methods as proposed in [5].

1. Introduction

The Stokes equation for steady flow of an incompressible fluid is given as

\begin{equation}
-\nu \Delta u - \nabla p = f \quad \text{in } \Omega, \\
\nabla \cdot u = 0 \quad \text{in } \Omega, \\
u u = 0 \quad \text{on } \Omega,
\end{equation}

in a, polygonal, domain \(\Omega \subset \mathbb{R}^d; d = 2, 3\) for given data \(f \in L^2(\Omega)\) and \(\nu > 0\), where \(u\) denotes the fluid velocity and \(p\) denotes the pressure. Under the famous inf-sup condition for the finite element spaces \(V_h\) and \(Q_h\), the use of mixed finite elements allows to obtain discrete approximations \(u_h \in V_h\) and \(p_h \in Q_h\) satisfying an an error estimate of the form

\[ \|u - u_h\|_1 \leq c \frac{1}{\beta} \inf_{v_h \in V_h} \|u - v_h\|_1 + c \inf_{q_h \in Q_h} \|p - q_h\|_0, \]

see, e.g., [6]. Here \(\beta\) is the inf-sup constant associated to the choice of \(V_h\) and \(Q_h\), \(\| \cdot \|_1\) and \(\| \cdot \|_0\) denote the \(H^1\) and \(L^2\) norm on \(\Omega\), respectively. Further, here and throughout the paper \(c\) denotes a generic constant which is independent of all relevant quantities of the estimate but may take a different value at each appearance.

While the estimate yields asymptotically optimal orders without the need to utilize exactly divergence free finite element functions for the approximation of \(u_h\) the right hand side of the estimate hints towards an undesirable influence of the pressure on the approximation error of the velocity. In fact, it has been observed, e.g., in [12] that indeed complicated pressures can give rise to a large error in the velocity approximation, even in situations where the true velocity can be represented in the discrete space \(V_h\).
A potential remedy, allowing for arbitrary inf-sup stable element pairs while providing pressure independent velocity has been proposed by [12]. He proposed the use of reconstruction operators on the right hand side of the equation to map discretely divergence free functions to divergence free functions. This proposed method has been implemented to a range of problems and a variety of finite element pairs for the discretization of Stokes equation, such as non-conforming Crouzeix-Raviart element [13], Taylor-Hood and MINI elements with continuous pressure spaces [10], on rectangular elements [13], for embedded discontinuous Galerkin methods (EDG) [11]. For 3-d polyhedral domains with concave edges a pressure robust reconstruction is given in [1]. While the obtained convergence orders are optimal, the price to pay, for these methods is a loss of quasi optimality of the method due to Strang’s first lemma. Recently, [9] showed that a more involved construction of the reconstruction operator allows for a quasi-optimal discretization.

In this paper, we consider the extension of these results to nearly incompressible linear elasticity, e.g.,

\[-2\mu \nabla \cdot \varepsilon(u) - \lambda \nabla (\nabla \cdot u) = f \quad \text{in } \Omega,\]
\[u = 0 \quad \text{on } \partial \Omega,\]

where \(\varepsilon(u)\) denotes the symmetric gradient, and \(\mu, \lambda > 0\) are the Lamé parameters. To avoid the locking phenomenon, e.g., [4, Chapter VI.3], typically a mixed form

\[-2\mu \nabla \cdot \varepsilon(u) - \nabla p = f \quad \text{in } \Omega,\]
\[\nabla \cdot u - \frac{1}{\lambda} p = 0 \quad \text{in } \Omega,\]
\[u = 0 \quad \text{on } \partial \Omega,\]

is considered. Here the incompressible case, i.e., \(\lambda = \infty\), can easily be included by dropping the term \(-\frac{1}{\lambda} p\) in the second line. It is clear conceptually that the same difficulties as for the Stokes problem will occur in the incompressible limit. However, the treatment of the nearly incompressible case requires additional care. To this end, [5] defined a discretization to be “gradient robust”, if the influence of gradient forces \(f = \nabla \phi\) in the discrete solution vanishes sufficiently fast as \(\lambda \to \infty\). [5] showed that a standard mixed discretization of (2) is not gradient robust and provided a gradient robust hybrid discontinuous Galerkin (HDG) scheme. Within this article, we will show that mixed methods can be made gradient robust using the approach proposed by [12] for the mixed discretization of (2).

The rest of the paper is structured as follows. In Section 2, we introduce the notion of gradient robustness and discuss the discretization of (2). Next, in Section 3 we show that the proposed discretization is indeed gradient robust and provide error estimates. We conclude the paper with a series of examples highlighting the derived results in Section 4.

2. Gradient Robustness and Discretization

2.1. Gradient Robustness. We define the spaces \(V^0\) of divergence free function and its orthogonal complement \(V^\perp\) as

\[V^0 = \{ u \in H^1_0(\Omega; \mathbb{R}^d) : \nabla \cdot u = 0 \},\]
\[V^\perp = \{ u \in H^1_0(\Omega; \mathbb{R}^d) : a(u, v) = 0, \forall v \in V^0 \},\]

where for \(u, v \in V = H^1_0(\Omega; \mathbb{R}^d)\), we define the bilinear form (scalar product) \(a : V \times V \to \mathbb{R}\) by

\[a(u, v) = 2\mu(\varepsilon(u), \varepsilon(v)),\]
with the $L^2(\Omega)$-scalar product $(\cdot, \cdot)$. Now, any function $u \in V$ can be uniquely written as $u = u^0 + u^\perp \in V^0 \oplus V^\perp$.

Using Helmholtz decomposition, $f \in L^2(\Omega)$ can be uniquely decomposed as

$$f = \nabla \phi + w,$$

where $\nabla \phi \in H^1(\Omega)/\mathbb{R}$ is irrotational, $w$ is divergence free and both are orthogonal with respect to the $L^2(\Omega)$-scalar product, i.e.,

$$(w, \nabla \phi) = 0.$$

With these definitions, the decay of the influence of gradient forces, i.e., $w = 0$, onto the solutions $u$ of (2) can be quantified as the following result from [5, Theorem 1] shows:

**Lemma 1.** If $f \in H^{-1}(\Omega)$ is a gradient, i.e., $f = \nabla \phi$, for some $\phi \in L^2(\Omega)$. Then for the solution $u = u^0 + u^\perp$ of (2) it holds $u^0 = 0$ and

$$\|u\|_1 = \|u^\perp\|_1 \leq \frac{c}{\mu + \lambda} \|\phi\|_0.$$

In particular, $\|u\|_1 = O(\lambda^{-1})$ as $\lambda \to \infty$.

Since this bound need not hold for arbitrary discretizations, [5] introduced the following notion

**Definition 1.** A discretization of (2) is called gradient robust, if for any discretization parameter $h$ there is a constant $c_h$ such that the approximate solution $u_h$ satisfies

$$\|u_h\|_1 \leq \frac{c_h}{\lambda} \|\phi\|_0.$$

### 2.2. Abstract Discretization.

In order to discretize (2), we define a second bilinear form $b: Q \times V \to \mathbb{R}$, with $Q = L^2(\Omega)$, by

$$b(q, v) = (p, \nabla \cdot v).$$

Now we select subspaces $V_h \subset V$ and $Q_h \subset Q$ such that there is a positive constant $\beta$ satisfying the inf-sup condition

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{(q_h, \nabla \cdot v_h)}{\|q_h\|_0 \|v_h\|_1} \geq \beta.$$

Now, the standard, non gradient robust, weak formulation is given as follows: Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$a(u_h, v_h) + b(p_h, v_h) = (f, v_h) \quad \forall v_h \in V_h,$n

$$b(q_h, u_h) - \frac{1}{\lambda} (p_h, q_h) = 0 \quad \forall q_h \in Q_h.$$

Under the well known inf-sup condition (7) on $V_h$ and $Q_h$, the system (8) is uniquely solvable [6, Theorem 5.5.2]. Following [5, Proposition 5.5.3] the displacement error is thus bounded as follows:

$$\|u - u_h\|_1 \leq \frac{c}{\beta} \inf_{v_h \in V_h} \|u - v_h\|_1 + \frac{1}{\mu} (\frac{1}{\lambda} + 1) \inf_{q_h \in Q_h} \|p - q_h\|_0.$$

Following [12], we assume that there exists a reconstruction operator

$$\pi^{\text{div}}: V_h \to H^{1, \text{div}}(\Omega) = \{ v \in L^2(\Omega)^d : \nabla \cdot v \in L^2(\Omega) \},$$

to be specified later in Section 2.3 mapping discretely divergence free functions to divergence free functions. Then the modified problem is given as:

$$a(u_h, v_h) + b(p_h, v_h) = (f, \pi^{\text{div}} v_h) \quad \forall v_h \in V_h,$n

$$b(q_h, u_h) - \frac{1}{\lambda} (p_h, q_h) = 0 \quad \forall q_h \in Q_h.$$
Clearly, by construction, the modified problem (10) admits a solution under the same conditions as (8), since only the right hand side has been modified. In Theorem 4, we will see that the discretization (10) is gradient robust, under appropriate assumptions on $\pi^{\text{div}}$. Further, in Theorem 5, we show the gradient robust displacement error estimate

$$\|u - u_h\|_1 \leq c h^k \left( 1 + \sqrt{\frac{\mu}{\lambda}} \right) \|u\|_{k+1} + c \frac{h^k}{\mu \lambda} \|p\|_k,$$

where $\|\cdot\|_k$ denotes the norm on $H^k(\Omega)$ or $H^k(\Omega; \mathbb{R}^d)$; of course assuming sufficient regularity of $u$ and $p$ and approximation order of $V_h$ and $Q_h$.

### 2.3. Reconstruction Operator and Assumptions

The construction of the reconstruction operator $\pi^{\text{div}}$ proposed by [12] is based on the choice of a suitable subspace $M_h \subset H^{\text{div}}(\Omega)$ satisfying the commuting diagram in Figure 1 where $\pi^{L^2}$ denotes the $L^2$-projection onto $Q_h$. 

![Diagram](image)

**Figure 1.** Commutative diagram for the reconstruction operator $\pi^{\text{div}}$

The commuting diagram is equivalently expressed by the equation

$$b(q_h, \pi^{\text{div}} v_h) = b(q_h, v_h) \quad \forall v_h \in V_h, q_h \in Q_h,$$

holds. Moreover, defining

$$V^0_h = \{ v_h \in V_h : b(q_h, v_h) = 0 \forall q_h \in Q_h \},$$

$$H^{\text{div}, 0}_0(\Omega) = \{ v \in H^{\text{div}}(\Omega) : \nabla \cdot v = 0 \},$$

we require that the restriction of $\pi^{\text{div}}$ to discretely divergence free functions maps into divergence free functions, i.e.,

$$\pi^{\text{div}} : V^0_h \to H^{\text{div}, 0}_0(\Omega)$$

and further for any $v_h \in V_h$ it holds

$$\pi^{\text{div}} v_h \cdot n = 0 \quad \text{on} \ \partial \Omega.$$

Analogously to the continuous setting, we can define the orthogonal complement $V^\perp_h$ by

$$V^\perp_h = \{ u_h \in V_h : a(u_h, v_h) = 0, \forall v_h \in V^0_h \},$$

and the corresponding discrete decomposition $u_h = u^0_h + u^\perp_h \in V^0_h \oplus V^\perp_h$.

Before we continue, let us make some, generic assumptions on the considered spaces $V_h$ and $Q_h$ defined on a shape regular family $\mathcal{T}_h$ of decompositions of $\Omega$. 

Assumption 1. We assume, that for some \( k \geq 2 \) the finite element space \( V_h \) is equipped with an interpolation operator \( I_h : H^{k+1}(\Omega; \mathbb{R}^d) \rightarrow V_h \), satisfying
\[
\| I_h v - v \|_{k,T} \leq c h_T^{k+1} \| v \|_{k+1,T} \quad \forall v \in H^{k+1}(\Omega; \mathbb{R}^d), T \in \mathcal{T}_h
\]
where \( \| \cdot \|_{k,T} \) denotes the respective norm on the element \( T \), and \( h_T \) is the element diameter. For the space \( Q_h \), we assume that the \( L^2 \)-projection \( \pi^{L^2} = \pi_{k-1}^{L^2} : H^k(\Omega) \rightarrow Q_h \) satisfies
\[
\| I_h q - q \|_{0,T} \leq c h_T^{k} \| q \|_{k,T} \quad \forall q \in H^k(\Omega), T \in \mathcal{T}_h.
\]
Further, it is assumed that \( V_h \) and \( Q_h \) satisfy the inf-sup inequality (7). Finally, we assume that there exists a subspace \( \tilde{Q}_h \subset L^2(\Omega; \mathbb{R}^d) \) such that the respective \( L^2 \)-projection \( \pi^{L^2} = \pi_{k-2}^{L^2} \) satisfies
\[
\| I_h q - q \|_{0,T} \leq c h_T^{k-1} \| q \|_{k-1,T} \quad \forall q \in H^k(\Omega; \mathbb{R}^d), T \in \mathcal{T}_h.
\]
Further requirements on \( \tilde{Q}_h \) will be made in Assumption 3.

With these preparations, we can now state the additional assumptions on the recovery operator.

Assumption 2. We first assume, that the recovery operator satisfies the following orthogonality relation
\[
(v_h - \pi^{\text{div}} v_h, q) = 0 \quad \forall v_h \in V_h, q \in \tilde{Q}_h,
\]
where \( \tilde{Q}_h \subset L^2(\Omega; \mathbb{R}^d) \) is given in Assumption 3. Second, we assume the following local approximation property to hold
\[
\| \pi^{\text{div}} v_h - v_h \|_{0,T} \leq c h_T^m \| v_h \|_{m,T} \quad \forall v_h \in V_h, T \in \mathcal{T}_h, m = 0, 1.
\]

Before concluding the assumption, let us note that the assumptions can indeed be satisfied. To this end, we give an example which we will also use for the numerical results in Section 4.

Example 1. Let us assume that the domain can be decomposed into a family \( \mathcal{T}_h \) of shape regular rectangular \( (d = 2) \) or brick \( (d = 3) \) elements. For the space \( V_h = V_h^2 \), we consider, parametric, piecewise \( Q_h \) and globally continuous finite elements with \( k \geq 2 \). For the discretization of \( Q_h = Q_h^{k-1} \), we select the space of discontinuous piecewise \( P_{k-1} \) functions. Indeed these pairs satisfy the inf-sup condition [2, see, e.g., 3, Sec. 8.6.3 & 8.7.2] for \( k = 2 \), for arbitrary \( k \) [3, Sec. 3.2] or [15] for mapped pressure spaces. Moreover, [13, Sec. 4.2.1] showed that the choice \( M_h = BD M_k \) as space of Brezzi-Douglas-Marini elements yield the desired commuting diagram property [11] together with the canonical interpolation \( \pi^{\text{div}} \). Further, they showed [13, Lemma 2.1], that the restriction of \( \pi^{\text{div}} \) to discretely divergence free functions maps into divergence free functions, i.e.,
\[
\pi^{\text{div}} : \{ v_h \in V_h : b(q_h, v_h) \forall q_h \in Q_h \} \rightarrow \{ v \in H^{\text{div}}(\Omega) : \nabla \cdot v = 0 \}
\]
and further for any \( v_h \in V_h \) it holds
\[
\pi^{\text{div}} v_h \cdot n = 0 \quad \text{on } \partial \Omega.
\]

Remark 1. Indeed, [13] showed that [11] follow from a set of assumed orthogonality properties and surjectivity of divergence and normal traces from which suitable choices of \( M_h \) and constructions of \( \pi^{\text{div}} \) can be obtained.

3. Error Analysis

In this section, we proceed with error analysis of the modified weak form (10). We split the analysis in two parts for incompressible materials \( (\lambda = \infty) \) and nearly incompressible materials \( (\lambda \neq \infty) \).
3.1. Incompressible Materials. We proceed to the error analysis of incompressible materials, where \( \lambda = \infty \) and the term involving \( \frac{1}{\lambda} \) is dropped in (10). The analysis follows, at large, the arguments in [14] with some minor adjustments to the elasticity case.

**Theorem 2.** Let Assumptions 1 and 2 be satisfied and \( \lambda = \infty \). Then the solution \( (u, p) \in H^{k+1}(\Omega)^d \times H^k(\Omega) \) of the continuous problem (2) and the solution \( (u_h, p_h) \in V_h \times Q_h \) of (10) satisfy the error estimate

\[
|u - u_h|^2 \leq c \sum_{T \in T_h} h_T^2 |u|_{k+1,T}^2 \leq c h^2 \|u\|_{k+1},
\]

where \(| \cdot |\) denotes the \( H^1 \)-semi-norm.

Before proving the above theorem, we would like to prove an important lemma which is need to prove the theorem.

**Lemma 3.** Let Assumptions 1 and 2 be satisfied and \( \lambda = \infty \). Then for any functions \( u \in H^{k+1}(\Omega)^d \) and \( w_h \in V_h \) it is

\[
\left| (\nabla \cdot \epsilon(u), \pi^{\text{div}} w_h) + (\epsilon(u), \epsilon(w_h)) \right| \leq c \sum_{T \in T_h} h_T^2 |u|_{k+1,T} |w_h|_{1,T},
\]

where \(| \cdot |_{1,T}\) denotes the \( H^1 \)-semi-norm on \( T \).

**Proof.** We add and subtract \( (\nabla \cdot \epsilon(u), \pi^{\text{div}} w_h) \) on the left to obtain

\[
(\nabla \cdot \epsilon(u), \pi^{\text{div}} w_h) + (\epsilon(u), \epsilon(w_h)) = (\nabla \cdot \epsilon(u), \pi^{\text{div}} w_h - w_h) + (\epsilon(u), \epsilon(w_h)) + (\nabla \cdot \epsilon(u), w_h).
\]

Since \( \nabla \cdot \epsilon(u) \in L^2(\Omega; \mathbb{R}^d) \), we can apply the projection \( \pi^{\text{div}}_{k-2} \), from Assumption 1, to get \( \pi^{\text{div}}_{k-2} \nabla \cdot \epsilon(u) \in Q_h \). By the assumed orthogonality in (16), we have

\[
\left( \pi^{\text{div}}_{k-2} \nabla \cdot \epsilon(u), \pi^{\text{div}} w_h - w_h \right) = 0, \quad \forall w_h \in V_h.
\]

Using Assumption 1 and (17), we obtain, for the first summand on the right of (19),

\[
\left( \nabla \cdot \epsilon(u), \pi^{\text{div}} w_h - w_h \right) = \left( \nabla \cdot \epsilon(u) - \pi^{\text{div}}_{k-2} \nabla \cdot \epsilon(u), \pi^{\text{div}} w_h - w_h \right)
\leq \sum_{T \in T_h} \| \nabla \cdot \epsilon(u) - \pi^{\text{div}}_{k-2} \nabla \cdot \epsilon(u) \|_{0,T} \| \pi^{\text{div}} w_h - w_h \|_{0,T}
\leq \sum_{T \in T_h} c h_T^{k-1} |\nabla \cdot \epsilon(u)|_{k-1,T} |w_h|_{1,T}
\leq \sum_{T \in T_h} c h_T^k |u|_{k+1,T} |w_h|_{1,T}.
\]

For the last two summands of (19), we apply Gauss divergence theorem to get

\[
(\nabla \cdot \epsilon(u), w_h) + (\epsilon(u), \epsilon(w_h)) = \int_{\partial \Omega} \epsilon(u) \cdot n \, w_h \, ds = 0
\]

since \( w_h = 0 \) on \( \partial \Omega \). Combining (10) with the bounds (20) and (21) the assertion is shown. Q.E.D.

Now, we continue to prove Theorem 2.

**Proof.** (of Theorem 2) Let \( u_h \) be the solution of (10), with \( \lambda = \infty \), and let \( v_h \in V_h^0 \) be arbitrary. Defining \( w_h = u_h - v_h \in V_h^0 \) and applying the triangle inequality gives

\[
|u - u_h| = |u - w_h - v_h| \leq |u - v_h| + |w_h|.
\]
In view of the interpolation estimate in Assumption 1 we are left to estimate $|w_h|_1$. From Korn’s inequality, we have
\[ c|w_h|^2_1 = c\|w_h\|^2_1 \leq \|\varepsilon(w_h)\|^2_0. \]
From this, we conclude
\[ 2\mu c|w_h|^2_1 \leq a(w_h, w_h) \]
\[ = a(u_h - v_h, w_h) \]
\[ = a(u_h - v_h, w_h) \]
\[ = a(u_h - v_h + u - u, w_h) \]
\[ \leq |a(u - v_h, w_h)| + |a(u_h - u, w_h)|. \]
For the first summand on the right of (23) we use Cauchy-Schwartz inequality to get
\[ |a(u - v_h, w_h)| \leq 2\mu\|\varepsilon(u - v_h)\|_0\|\varepsilon(w_h)\|_0 \leq 2\mu|u - v_h|_1|w_h|_1. \]
Before we come to the bound of the second summand in (23), we make some preliminary calculations. Since $u_h$ is the solution of (10), choosing $v_h = w_h \in V_h^0$ gives
\[ a(u_h, w_h) = a(u_h, w_h) + b(p_h, w_h) = (f, \pi^\text{div} w_h). \]
Further, since $u$ is the solution to the equation (2) multiplication with $\pi^\text{div} w_h$ and integration yields
\[ -2\mu \int_\Omega \nabla \cdot \varepsilon(u) \pi^\text{div} w_h \, dx - \int_\Omega \nabla p \pi^\text{div} w_h \, dx = \int_\Omega f \pi^\text{div} w_h \, dx \]
by the compatibility of the reconstruction with the kernel of the divergence, i.e., (14), this gives
\[ -2\mu(\nabla \cdot \varepsilon(u), \pi^\text{div} w_h) = (f, \pi^\text{div} w_h). \]
Combining this with (25), we get
\[ a(u_h, w_h) = -2\mu(\nabla \cdot \varepsilon(u), \pi^\text{div} w_h). \]
Now, we can bound the second summand on the right of (23), using (26) we get
\[ |a(u_h - u, w_h)| = -2\mu(\nabla \cdot \varepsilon(u), \pi^\text{div} w_h) - 2\mu(\varepsilon(u), \varepsilon(w_h)) \]
\[ \leq 2\mu \left( |\nabla \cdot \varepsilon(u)|_{\pi^\text{div} w_h} + |\varepsilon(u)|_{\varepsilon(w_h)} \right). \]
By the previously shown lemma, i.e., (18), we can bound the right hand side to get
\[ |a(u_h - u, w_h)| \leq 2\mu c \sum_{T \in T_h} \left( h^k_T |u|_{k+1, T} |w_h|_1, T \right) \]
\[ \leq 2\mu c \left( \sum_{T \in T_h} h^k_T |u|_{k+1, T}^2 \right)^{1/2} |w_h|_1. \]
Now combining (23) with the two bounds (24) and (27), we get
\[ |w_h|_1 \leq |u - v_h|_1 + c \left( \sum_{T \in T_h} h^k_T |u|_{k+1, T}^2 \right)^{1/2}. \]
Substituting this in (22) yields
\[ |u - u_h|_1 \leq 2|u - v_h|_1 + c \left( \sum_{T \in T_h} h^k_T |u|_{k+1, T}^2 \right)^{1/2}. \]
To bound the best approximation error on $V_h^0$ in this inequality, we proceed using inf-sup condition as in [6], Chapter 2, (1.16) and the assumed interpolation estimate on $V_h$ in Assumption 4 to get the estimate

$$\inf_{v_h \in V_h} |u - v_h|_1 \leq c \inf_{v_h \in V_h} |u - v_h|_1 \leq c \left( \sum_{T \in \mathcal{T}_h} h_T^2|\nabla \pi h|_{L^2(T)}^2 \right)^{\frac{1}{2}}.$$  

Using this in (28) gives the desired estimate. □

3.2. Nearly Incompressible Materials. For the nearly incompressible case, i.e., $(\lambda \neq \infty)$, we start by assuming a gradient force $f = \nabla \phi$, for some $\phi \in L^2(\Omega)$. From Lemma 1 we have that the solution of (2) for such an $f$ is $u = 0$. The following result shows, that out mixed discretization (10) is gradient robust in the sense of Definition 1.

**Theorem 4.** Let Assumptions 1 and 2 be satisfied. If the right hand side $f \in H^{-1}(\Omega; \mathbb{R}^d)$ of equation (10) is a gradient field, i.e., $f = \nabla \phi$, for some $\phi \in L^2(\Omega)$, then the solution $(u_h, p_h) \in V_h \times Q_h$ of (10) with $\lambda \neq \infty$ satisfies

$$\|u_h\|_1 \leq \frac{1}{\lambda + \mu} \|\phi\|_0. \tag{29}$$

with a constant $c$ independent of $h$.

**Proof.** Consider $v_h = u_h$ in equation (10) with $f = \nabla \phi$. Then integration by parts for the right hand side, using the zero trace from (15), we get

$$a(u_h, u_h) + b(p_h, u_h) = -\langle \phi, \nabla \cdot \pi^\text{div} u_h \rangle. \tag{30}$$

Since $\nabla \cdot \pi^\text{div} u_h \in Q_h$ we can rewrite the right hand side as

$$\langle \phi, \nabla \cdot \pi^\text{div} u_h \rangle = (\pi^L \phi, \nabla \cdot \pi^\text{div} u_h). \tag{31}$$

Since $\pi^L \nabla \cdot u_h \in Q_h$, we can use it to test the second line in (10) giving

$$(\pi^L \nabla \cdot u_h, \pi^L \nabla \cdot u_h) = (\pi^L \nabla \cdot u_h, \nabla \cdot u_h)$$

$$= \frac{1}{\lambda} (p_h, \pi^L \nabla \cdot u_h)$$

$$= \frac{1}{\lambda} (p_h, \nabla \cdot u_h)$$

$$= \frac{1}{\lambda} b(p_h, u_h). \tag{32}$$

Substituting (31) and (32) in (30), we get

$$a(u_h, u_h) + \lambda (\pi^L \nabla \cdot u_h, \pi^L \nabla \cdot u_h) = - (\pi^L \phi, \nabla \cdot \pi^\text{div} u_h). \tag{33}$$

Now $\pi^L \phi \in Q_h$ and $u_h \in V_h$ hence, by (11), it holds

$$(\pi^L \phi, \nabla \cdot \pi^\text{div} u_h) = (\pi^L \phi, \nabla \cdot u_h).$$

Filling this into (33) gives

$$2\mu (\varepsilon(u_h), \varepsilon(u_h)) + \lambda \left( \pi^L \nabla \cdot u_h, \pi^L \nabla \cdot u_h \right) = - \left( \pi^L \phi, \pi^L \nabla \cdot u_h \right). \tag{34}$$

Using Cauchy-Schwartz inequality, we get

$$2\mu \|\varepsilon(u_h)\|_0^2 + \lambda \|\pi^L \nabla \cdot u_h\|_0^2 \leq \|\pi^L \phi\|_0 \|\pi^L \nabla \cdot u_h\|_0 \leq \|\phi\|_0 \|\pi^L \nabla \cdot u_h\|_0. \tag{35}$$

Now, to estimate the $H^1$-norm of $u_h$, we notice that by the choice of $f$ and (14), testing the first equation in (10) with a function $v_h \in V_h^0$ yields

$$a(u_h, v_h) = -b(p_h, v_h) = (\phi, \nabla \cdot \pi^\text{div} v_h) = 0$$

$$= -b(p_h, v_h) = (\phi, \nabla \cdot \pi^\text{div} v_h) = 0.$$
and thus $u_h \in V_h$. Hence by, e.g., [8] Lemma 3.58 it holds
\begin{equation}
||u_h||_1 \leq c_1 \frac{1}{\beta} ||\pi h^2 \nabla \cdot u_h||_0
\end{equation}
with the inf-sup constant $\beta$ from [7], since $\pi h^2 \nabla \cdot u_h \in Q_h$.

Using Korn’s inequality, (35), and (36), we get
\begin{equation}
(\mu + \lambda) ||u_h||_1^2 \leq c ||\epsilon(u_h)||^2_0 + \frac{\lambda c}{\beta} ||\pi h^2 \nabla \cdot u_h||_1^2
\end{equation}
\begin{equation}
\leq c ||\phi||_0 ||u_h||_1,
\end{equation}
and thus the assertion is shown.

**Theorem 5.** Let Assumptions [1] and [2] be satisfied. Then the solutions $(u, p) \in V \times Q$, of the problem (2) and $(u_h, p_h) \in V_h \times Q_h$ of (10) satisfy the error estimate
\begin{equation}
||u - u_h||_1 \leq c h^k \left(1 + \frac{1}{\lambda} \right) ||u||_{k+1} + c \frac{h^k}{\mu \lambda} ||p||_k,
\end{equation}
provided the regularity $(u, p) \in H^{k+1}(\Omega; \mathbb{R}^d) \times H^k(\Omega)$ is given.

**Proof.** As in the proof of Theorem 2 we could split the error
\begin{equation}
(u - u_h, p - p_h) = (u - v_h, p - q_h) + (v_h - u_h, q_h - p_h)
\end{equation}
with arbitrary $v_h \in V_h$ and $q_h \in Q_h$. However, as it will turn out to be useful, we will select $q_h = \pi h^2 p$ and $v_h$ as the elasticity projection of $u$, i.e., satisfying the following equation
\begin{equation}
(\epsilon(v_h), \epsilon(\varphi_h)) + b(\bar{p}_h, \varphi_h) = (\epsilon(u), \epsilon(\varphi_h)) \quad \forall \varphi_h \in V_h,
\end{equation}
b$(s_h, v_h) = b(s_h, u) \quad \forall s_h \in Q_h$.

Clearly, the solution to the continuous counterpart is $(v, \bar{p}) = (u, 0)$. Since the above equation is uniquely solvable, see, e.g., [3] Theorem 4.2.3], we have the orthogonality $b(\pi h^2 p - p_h, u - v_h) = 0$ and the approximation error satisfies, e.g., [3] Theorem 5.2.2].

\begin{equation}
||u - v_h||_1 + ||\bar{p} - \bar{p}_h|| \leq c \inf_{\varphi_h \in V_h} ||u - \varphi_h||_1 + c \inf_{s_h \in Q_h} ||0 - q_h||,
\end{equation}
which gives
\begin{equation}
||u - u_h||_1 \leq c \inf_{\varphi_h \in V_h} ||u - \varphi_h||_1
\end{equation}

Due to the interpolation estimates in Assumption [1] we are left with bounding $w_h = u_h - v_h \in V_h$ and $r_h = p_h - q_h \in Q_h$. We split $w_h = w_h^0 + w_h^\perp \in V_h^0 \oplus V_h^\perp$.

By definition of the bilinear forms $a$ and $b$, i.e., [3] and [6], and the first line in (10) and (2), the remainder $w_h$ and $r_h$ satisfy, for any discrete function $\varphi_h \in V_h$,
\begin{equation}
a(w_h, \varphi_h) + b(r_h, \varphi_h) = a(u_h - v_h, \varphi_h) + b(p_h - q_h, \varphi_h)
\end{equation}
\begin{equation}
= (f, \pi h^2 \nabla \varphi_h) + a(u - v_h, \varphi_h) + b(p - q_h, \varphi_h).
\end{equation}

Analogously, from the second line in (10) and (2), we get for arbitrary $s_h \in Q_h$
\begin{equation}
b(s_h, w_h) - \frac{1}{\lambda}(r_h, s_h) = b(s_h, u_h - v_h) - \frac{1}{\lambda}(p_h - q_h, s_h)
\end{equation}
\begin{equation}
= b(s_h, u_h) - \frac{1}{\lambda}(p_h, s_h) - (b(s_h, v_h) - \frac{1}{\lambda}(q_h, s_h))
\end{equation}
\begin{equation}
= b(s_h, u - v_h) - \frac{1}{\lambda}(p - q_h, s_h).
\end{equation}
Substituting this in (44), we get
\[ c\mu \|w_h\|^2 + \frac{1}{\lambda} \|r_h\|^2 \leq a(w_h, w_h) + \frac{1}{\lambda} (r_h, r_h) \]
\[ = a(w_h, w_h) + b(r_h, w_h) - b(r_h, w_h) + \frac{1}{\lambda} (r_h, r_h) \]
\[ = (f, \pi^{\text{div}} w_h - w_h) + a(u - v_h, w_h) \]
\[ + b(p - q_h, w_h) - b(r_h, u - v_h) - \frac{1}{\lambda} (p - q_h, r_h). \]

Using (18) and (2), we obtain a bound on \((f, \pi^{\text{div}} w_h - w_h)\) as follows
\[ (f, \pi^{\text{div}} w_h - w_h) = -2\mu (\nabla \cdot \varepsilon(u), \pi^{\text{div}} w_h - w_h) - (\nabla p, \pi^{\text{div}} w_h - w_h) \]
\[ = -2\mu (\nabla \cdot \varepsilon(u), \pi^{\text{div}} w_h) - 2\mu (\varepsilon(u), \varepsilon(w_h)) + b(p, \pi^{\text{div}} w_h - w_h) \]
\[ \leq c \sum_{T \in T_h} h_T^2 |u|_{k+1,T}^2 |w_h|_{1,T} + b(p, \pi^{\text{div}} w_h - w_h) \]
\[ \leq 2\mu c \left( \sum_{T \in T_h} h_T^2 |u|_{k+1,T}^2 \right)^{\frac{1}{2}} |w_h|_{1} + b(p, \pi^{\text{div}} w_h - w_h). \]

Substituting this in (44), we get
\[ c\mu \|w_h\|^2 + \frac{1}{\lambda} \|r_h\|^2 \leq 2\mu c \left( \sum_{T \in T_h} h_T^2 |u|_{k+1,T}^2 \right)^{\frac{1}{2}} |w_h|_{1} \]
\[ + \left( b(p, \pi^{\text{div}} w_h - w_h) + b(p - q_h, w_h) - b(r_h, u - v_h) \right) \]
\[ + \left( a(u - v_h, w_h) - \frac{1}{\lambda} (p - q_h, r_h) \right). \]

The last line can be estimated as
\[ a(u - v_h, w_h) - \frac{1}{\lambda} (p - q_h, r_h) \leq \frac{c\mu}{2} \|u - v_h\|^2 + \frac{c\mu}{2} \|w_h\|^2 + \frac{1}{2\lambda} \|p - q_h\|^2 + \frac{1}{2\lambda} \|r_h\|^2. \]

From (11), we have that \(b(q_h, \pi^{\text{div}} w_h - w_h) = 0\). Hence the second line in (45) becomes
\[ b(p, \pi^{\text{div}} w_h - w_h) + b(p - q_h, w_h) - b(r_h, u - v_h) \]
\[ = b(p - q_h, \pi^{\text{div}} w_h - w_h) + b(p - q_h, w_h) - b(r_h, u - v_h) \]
\[ = b(p - q_h, \pi^{\text{div}} w_h) - b(p - q_h, u - v_h) \]
\[ = b(\pi^{L^2} p - q_h, \pi^{\text{div}} w_h) - b(p - q_h, u - v_h) \]
\[ = b(\pi^{L^2} p - q_h, w_h) - b(p - q_h, u - v_h) \]
\[ = 0 \]

where we used the properties of the \(L^2\) projection \(\pi^{L^2}\), the commutative diagram (11) and \(\nabla \cdot A_h \subseteq Q_h\). Now, we utilize the choice \(q_h = \pi^{L^2} p\) to further simplify the representation of the second line in (45) to be
\[ b(p, \pi^{\text{div}} w_h - w_h) + b(p - q_h, w_h) - b(r_h, u - v_h) \]
\[ = b(\pi^{L^2} p - q_h, w_h) - b(p - q_h, u - v_h) \]
\[ = b(\pi^{L^2} p - p_h, u - v_h) \]
\[ = 0 \]
by our choice of \( v_h \). This provides the bound

\[
\frac{c\mu}{2} \| w_h \|^2 + \frac{1}{2\lambda} \| r_h \|^2 \leq 2\mu c \left( \sum_{T \in T_h} h_T^{2k} |u|_{k+1,T}^2 \right)^{\frac{1}{2}} \| w_h \|_1 
\]

(46)

\[
+ \frac{c\mu}{2} \| u - v_h \|^2 + \frac{1}{2\lambda} \| p - q_h \|^2.
\]

Of course (46) provides a bound on \( w_h \) but as it is suboptimal we continue by splitting \( w_h = w_h^0 + w_h^1 \).

We first bound \( \| w_h^0 \|_1 \). Consider \( c\mu \| w_h^0 \|_1 \) and using that \( a(w_h^0, w_h^0) = 0 \), we have, using (12), (42), and the choice of \( v_h \) as elasticity projection that

\[
c\mu \| w_h^0 \|^2 \leq a(w_h^0, w_h^0) = a(w_h, w_h^0) + b(r_h, w_h^0)
\]

\[
= (f, \pi \text{div} w_h^0 - w_h^0)
\]

\[
\leq (-2\nabla u \cdot \varepsilon(u) + \nabla p, \pi \text{div} w_h^0 - w_h^0)
\]

\[
\leq (-2\nabla u \cdot \varepsilon(u), \pi \text{div} w_h^0 - w_h^0) + (\nabla p, \pi \text{div} w_h^0 - w_h^0)
\]

\[
\leq (-2\nabla u \cdot \varepsilon(u), \pi \text{div} w_h^0 - \mu (\varepsilon(u), \varepsilon(w_h^0)))
\]

\[
\leq \mu (-2\nabla u \cdot \varepsilon(u), \pi \text{div} w_h^0) - \mu (\varepsilon(u), \varepsilon(w_h^0)).
\]

Thus, by Lemma [3] we conclude

\[
c\mu \| w_h^0 \|^2 \leq \mu c \left( \sum_{T \in T_h} h_T^{2k} |u|_{k+1,T} \| w_h^0 \|_{k+1,T} \right) \leq c\mu h^k \| u \|_{k+1} \| w_h^0 \|_1
\]

and hence

(47)

\[
\| w_h^0 \|_1 \leq c h^k \| u \|_{k+1}.
\]

For \( \| w_h^1 \|_1 \), we utilize \( w_h^1 \in V_h^1 \), i.e.,

\[
(\nabla \cdot w_h, q_h) = (\nabla \cdot w_h^1, q_h) \quad \forall q_h \in Q_h
\]

meaning

\[
\pi \text{L}^2 \nabla \cdot w_h = \pi \text{L}^2 \nabla \cdot w_h^1.
\]

Using [8] Lemma 3.58, we get

\[
\| w_h^1 \|_1 \leq \frac{c}{\beta} \| \pi \text{L}^2 \nabla \cdot w_h^1 \|_0
\]

\[
\leq \frac{c}{\beta} \| \pi \text{L}^2 \nabla \cdot u_h - \pi \text{L}^2 \nabla \cdot v_h \|_0
\]

\[
\leq \frac{c}{\beta} \| \frac{p_h}{\lambda} - \pi \text{L}^2 \nabla \cdot v_h \|_0
\]

from the definition of \( v_h \) as elasticity projection. Hence, noting that \( \nabla \cdot u = \frac{1}{\lambda} p \), we obtain

\[
\| w_h^1 \|_1 \leq \frac{c}{\beta \lambda} \| p_h - q_h \|_0 = \frac{c}{\beta \lambda} \| r_h \|_0.
\]

With this, we conclude from (46)

\[
\| r_h \|^2 \leq \lambda (\mu \| w_h \|^2 + \frac{1}{\lambda} \| r_h \|_0^2)
\]

\[
\leq c\mu \lambda h^{2k} \| u \|_{k+1}^2 + c \| p - q_h \|_0^2
\]

and thus

\[
\| w_h^k \|_1 \leq \frac{c}{\lambda} \| r_h \|_0
\]

(48)

\[
\leq c \sqrt{\frac{\mu}{\lambda}} h^k \| u \|_{k+1} + \frac{c}{\lambda} \| p - q_h \|_0.
\]
Now, we can bound $\|w_h\|_1$ using (47) and (48)

$$
\|w_h\|_1 \leq \|w_h^0\|_1 + \|w_h^\perp\|_1
\leq ch^k \|u\|_{k+1} + \frac{c}{\beta \lambda} r_h
\leq ch^k \|u\|_{k+1} + \frac{c}{\beta} \sqrt{\mu} h^k \|u\|_{k+1} + \frac{c}{\lambda} \|p - q_h\|_0
\leq c \left( 1 + \sqrt{\frac{\mu}{\lambda}} \right) h^k \|u\|_{k+1} + \frac{c}{\lambda} \|p - q_h\|_0.
$$

Finally, we arrive at the desired bound

$$
\|u - u_h\|_1 \leq \|u - v_h\|_1 + \|w_h\|_1
\leq c \left( 1 + \sqrt{\frac{\mu}{\lambda}} \right) h^k \|u\|_{k+1} + \frac{c}{\lambda} \|p\|_k
$$

by definition of $q_h$ and Assumption 1.

\[\square\]

4. Numerical Results

For our computation, we use DOpElab [7] based on the deal.II [2] finite element library. First, we present an example for incompressible materials.

**Example 2.** For the first numerical example, we consider a small variation of Example 5.1 in [12], where the displacement and pressure is given as

$$
u(x, y) = \begin{bmatrix} 200x^2(1-x)^2y(1-y)(1-2y) \\ -200y^2(10y)^2x(1-x)(1-2x) \end{bmatrix}
$$

$$
p(x, y) = -10 \left( x - \frac{1}{2} \right)^3 y^2 + (1-x)^3 \left( y - \frac{1}{2} \right)^3 + \frac{1}{8},
$$

for the incompressible linear elasticity equation

$$
-2\mu \nabla \cdot \varepsilon(u) + \nabla p = f,
$$

$$
\nabla \cdot u = 0
$$

with thus defined $f$. 

**Figure 2.** Comparing displacement error in $H^1$ norm vs. $\frac{1}{\mu}$ for Example 2 with and without gradient robust modification for $h = 2^{-3}$. 

\[\text{PDF figure} \]
Comparing (9) with Figure 2, we notice that the $H^1$-norm displacement error without interpolation grows linearly w.r.t $\frac{1}{\mu}$ as predicted due to the appearance of the pressure term $\frac{1}{\mu} \inf_{q_h \in Q_h} \| p - q_h \|_0$ in (9). For the gradient robust modification employing the interpolation onto the $BDM$ finite element space, the error is independent of $\mu$, highlighting the prediction of Theorem 2.

For future examples, we consider nearly incompressible materials given by equation (2).

**Example 3.** For the second numerical example, we set the right hand side $f = \nabla \phi; \phi = x^6 + y^6$ in equation (2), as 3 in [5; Example 2].

From Lemma 1, the solution for Example 3 is given as $u = 0$ and $p = x^6 + y^6$. From equation (29), we have the bound

$$\| u_h \|_1 \leq \frac{c}{\lambda + \mu} \| \phi \|_0$$

on the discrete function for a gradient robust discretization. For $\mu = 10^{-5}$, we have $\lambda + \mu \approx \lambda, \forall \lambda \geq 1$. Hence, we see a green line with positive slope in Figure 3a for the gradient robust method, while the non robust method shows an almost constant $\| u_h \|_1 \neq 0$. However, for $\lambda = 10^5$ we have $\frac{1}{\lambda + \mu} \approx c$(constant) $\forall 0 < \mu \leq 1$, which is seen in the flat green line in Figure 3b.

For non-gradient robust methods, we have

$$\| u_h \|_1 \leq \frac{c}{\mu} \left( \frac{1}{\lambda} + 1 \right) \| \phi \|_0$$

from equation (9). For $\mu = 10^{-5}$, the term $\left( \frac{1}{\lambda} + 1 \right) \to 1$ as $\lambda \to \infty$. The same is shown by the flat red line in Figure 3a. However, for $\lambda = 10^{-5}$, we have $\| u_h \|_1 \leq \frac{c}{\mu}$. Which is shown by the red line with negative slope in Figure 3b.

It should be noted in this example, that the line for the non-gradient robust $Q_2 \times DGP_1$ method coincides with the gradient robust modification. However, this effect is due to a too simple pressure. That indeed, the standard $Q_2 \times DGP_1$ method is not gradient robust is shown in the following example.
For the third numerical example, we consider the right hand side $f = \nabla \phi; \phi = -10(x - 0.5)^2 y^2 + (1 - x)^3 (y - 0.5)^3 - 1/8$ in Example 3.

Figure 3 shows our previous statement, that Example 3 had a pressure which is too simple to show the missing gradient robustness of the standard $Q_2 \times DGP_1$ discretization. Indeed, in this example, both $Q_2 \times Q_2$ and $Q_2 \times DGP_1$ discretization show the undesirable blowup for $\mu \to 0$ and the constant value as $\lambda \to \infty$, while the gradient robust modification shows the desired convergence.

Example 5. For the fourth example, we consider the nearly incompressible case ($\lambda \neq \infty$), i.e.,

$$-2\mu \nabla \cdot \varepsilon(u) + \nabla p = f,$$

$$\nabla \cdot u = \frac{1}{\lambda} p = 0$$

where we use the same $f$ as in Example 3.

In this example, for $\lambda = \infty$, the solution $u^\infty$ is known, i.e., it is given in (51). We denote the solution, for $\lambda \neq \infty$, as $u^\lambda$. We compute the error $\|u^\infty - u^\lambda\|$ in our numerical results, where $u^\lambda_h$ is the discrete approximated solution for a given value of $\lambda$. Since, Theorem 5 provides an estimate, for $\|u^\infty - u^\lambda\|$ only, we use the triangle inequality to get

$$\|u^\infty - u^\lambda\|_1 \leq \|u^\infty - u^\lambda\|_1 + \|u^\lambda - u^\lambda_h\|_1,$$

$$\leq \|u^\infty - u^\lambda\|_1 + c \left(1 + \sqrt{\frac{\mu}{\lambda}}\right) h^2 \|u\|_3 + \frac{ch^2}{\lambda} \|p\|_2. \quad (56)$$

Figure 5 follows the same pattern as Figure 4. However, there is a slight difference between Figures 5a and 4a which can be explained by (56). When $\lambda \to \infty$, we have

$$\|u^\infty - u^\lambda\|_1 + ch^2 \|u_h\|_3 \gg ch^2 \sqrt{\frac{\mu}{\lambda}} \|u\|_3 + \frac{ch^2}{\lambda} \|p\|_2.$$

So the estimate on $\|u^\infty - u^\lambda\|_1$ converges to $\|u^\infty - u^\lambda\|_1 \neq 0$ and thus saturates at a non-zero value contrary to the convergence in Figure 4a.
The temperature field is obtained as the solution to the stationary heat equation:

$$\Delta \theta = \frac{1}{\rho c_p} \nabla \cdot \mathbf{q}$$

where \(\rho\) is the density, \(c_p\) is the specific heat capacity, \(\mathbf{q}\) is the heat flux, and \(\Delta \theta\) is the temperature difference.

The boundary conditions are applied on both temperature and displacement. It is important to note that the thermal expansion coefficient \(\alpha\), which is given by

$$\alpha = \frac{1}{3} \frac{\lambda}{\mu}$$

where \(\lambda\) and \(\mu\) are the Lamé parameters, is used in the constitutive relation for the stress tensor in the thermo-elastic problem.

Example 6. Finally, we would like to compare our results with the thermo-elastic solids example given in [5, Section 6]. The gradient force \(\mathbf{f}\) is given by a temperature \(\theta\) as

$$\mathbf{f} = - (2\mu + 3\lambda) \alpha \nabla \theta.$$

The material used is a nearly incompressible hard rubber with Young’s Modulus \(E = 5 \times 10^7\,[\text{Pa}]\), Poisson ratio \(\nu = 0.4999\) and the thermal expansion coefficient \(\alpha = 8 \times 10^{-5}\,[1/\text{K}]\). Hence the Lamé parameters are \(\lambda = 8.332 \times 10^{10}\,[\text{Pa}]\) and \(\mu = 1.6667 \times 10^7\,[\text{Pa}]\). We take the domain \(\Omega = [0, L]^2\) with \(L = 0.1\,[\text{m}]\). The temperature field is obtained as the solution to the stationary heat equation:

$$-\nabla \cdot \gamma \nabla \theta = f,$$

where \(\gamma = 0.2\,[\text{W/(mK)}]\) is the thermal conductivity coefficient and \(f = 4 \times \exp(-40r^2)\,[\text{W/m}^3]\) is the heat source, with \(r^2 = (x-0.5L)^2 + (y-0.5L)^2\). Homogeneous Dirichlet boundary conditions are applied on both temperature and displacement. It is important to ensure that the solution satisfies these conditions.

From Figure 6a and 6b we can see, that \(\|u^\infty - u^\lambda\|_1\) converges to the constant \(\|u^\infty\|_1 + ch^2\|u^\lambda\|_1\) as \(\lambda \to \infty\) and \(\|u^\infty - u^\lambda\|_1 \to 0\) (since, \(\|u^\infty - u^\lambda\|_1 \to 0\)) as \(\lambda \to \infty\) and \(ch^2\|u^\lambda\|_1 \to 0\) as \(h \to 0\).

Figure 6. Comparing displacement error in \(H^1\) norm for Example 5 for \(\mu = 10^{-5}\)
to note that \( \theta \in H^1(\Omega) \) and thus \( f \in L^2(\Omega) \). For numerical computation, we additionally solve the temperature equation by a standard \( H^1 \)-conforming finite element discretization. Hence, the finite element spaces now consist of three components, the first two denote the displacement and pressure discretization as before. The third element, always \( Q_2 \), is used to solve the equation for the temperature \( \theta \).

In Figure 7, we can see that we achieve a well represented solution for the displacement with only 64 elements using a gradient robust method, and the magnitude is already captured with only 16 elements. In comparison, the non gradient robust methods require 256 and 1024 elements, respectively, to get a solution of similar shape and magnitude, see Figures 8 and 9.

\[\text{Figure 7. Displacement vector for different number of elements with } Q_2 \times DGP_1 \times Q_2 \text{ with BDM Interpolation} \]

\[\text{Figure 8. Displacement vector for different number of elements with } Q_2 \times Q_1 \times Q_2 \]
Figure 9. Displacement vector for different number of elements with $Q_2 \times DGP_1 \times Q_2$

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