In this article, we address the problem of solving infinite-dimensional harmonic algebraic Lyapunov and Riccati equations up to an arbitrary small error. This question is of major practical importance for analysis and stabilization of periodic systems including tracking of periodic trajectories. We first give a closed form of a Floquet factorization in the general setting of $L^2$ matrix functions and study the spectral properties of infinite-dimensional harmonic matrices and their truncated version. This spectral study allows us to propose a generic and numerically efficient algorithm to solve infinite-dimensional harmonic algebraic Lyapunov equations up to an arbitrary small error.

We combine this algorithm with the Kleinman algorithm to solve infinite-dimensional harmonic Riccati equations and we apply the proposed results to the design of a harmonic $LQ$ control with periodic trajectory tracking.

I. INTRODUCTION

HARMONIC modeling and control is a topic of theoretical and practical interest for many application domains, such as energy management (including ac–dc and ac–ac power converters) or embedded systems to mention few. In a recent paper [6], a complete and rigorous mathematical framework for harmonic modeling and control has been proposed. Basically, the harmonic modeling of a periodic system leads to an equivalent time invariant model but of infinite dimension. The states of this model (also called phasors) are the coefficients obtained using a sliding Fourier decomposition. One of the main results of [6] establishes a strict equivalence between these two models and provides tools that allow us to reconstruct time trajectories from harmonic ones. In this framework, the analysis and harmonic control design are considerably simplified as any available method for time-invariant systems can be a priori applied.

Our objective is to provide tools for harmonic control to be used directly in the harmonic domain. The main difficulty is related to the infinite dimension nature of the obtained harmonic time invariant model. The associated infinite-dimensional harmonic state matrix is formed by the sum of a (block) Toeplitz matrix and a diagonal matrix. As a consequence, stability analysis or control design in this setting requires solving algebraic Lyapunov and Riccati equations of infinite dimension [6], [30]. This is an open and very challenging problem. In the time domain, there are methods dedicated to solving periodic Lyapunov and Riccati differential equations [5], [19], [25]. However, our framework is different and it is not easy to control harmonics in the time domain. The computational methods, we propose in this article can be associated efficiently with the harmonic control design methodology introduced in [6].

There is a huge literature concerning the study of infinite-dimensional Toeplitz matrices [1], [2], [11], [13], [14], [20], [22], [24] including contributions on solving quadratic matrix equations [3], [4]. However, these results cannot be used in our case as the matrix equations discussed in this article depend on harmonic state-space matrices. There are contributions that consider harmonic state-space matrices [7], [10], [15], [26], [27], [28] but most of the results in this literature are based on a Floquet factorization [12], [18], [23], [26], [27]. Floquet factorization is an existence result and, as such, it is not constructive [10], [12]. This is why Floquet factorization-based methods are mainly dedicated to analysis and it is very difficult to extend them to control design. As a consequence, solving infinite-dimensional harmonic algebraic Lyapunov and Riccati equations is a very challenging problem [28], [29].

The main objective of this article is to propose efficient algorithms with low computational burden and of interest for both analysis and control design. We first provide a simple and closed-form formula to determine a Floquet factorization in the general case of periodic and $L^2$ matrix functions. As the proposed Floquet factorization leads to a Jordan normal form representation of the harmonic state operator, it also solves the associated eigenvalue problem. This new result allows us to perform a detailed spectral analysis of the harmonic state-space matrix and its truncated version. In particular, it is shown that the harmonic state-space matrix is an unbounded operator on $l^2$ with a discrete spectrum and that a truncation of a Hurwitz harmonic matrix may never be Hurwitz, regardless of the truncation order.

To the best of authors’ knowledge, this is the first time that such a phenomenon is highlighted. It has a major impact in deriving tools for analysis and harmonic control design. As a consequence, if we consider a Hurwitz infinite-dimensional
harmonic matrix, there may be no positive definite solution to the associated truncated harmonic Lyapunov equation whatever the considered truncation order. To overcome this difficulty, we use Lyapunov symbolic equations associated with harmonic Lyapunov equations to provide an efficient algorithm that allows us to recover the solution of the infinite-dimensional harmonic Lyapunov equation up to an arbitrarily small error. We extend this result to solve infinite-dimensional harmonic Riccati equations and provide a Kleinman-like algorithm [16] in the harmonic framework. This generic result is made possible by the fact that we do not use a Floquet factorization to solve a harmonic Lyapunov equation at each step. To demonstrate that our results can be used for control design, we treat the problem of periodic trajectories tracking using a harmonic linear quadratic control as an illustrative example.

The rest of this article is organized as follows. We first give some mathematical preliminaries in the following section before stating in Section III, the problem we are interested in. In Section IV, we provide a complete and simple characterization of a Floquet factorization in the general case of $L^2$ matrix functions and analyze the spectral properties of the harmonic state-space operator and its truncated version. The main contribution of this article is detailed in Section V where an efficient algorithm to solve up to an arbitrarily small error infinite-dimensional harmonic Lyapunov equations is derived. This algorithm is extended to infinite-dimensional harmonic Riccati equations in Section VI. We illustrate the results of this article in Section VII where a design of a harmonic LQ control for a second-order linear time periodic (LTP) system is proposed. Finally, Section VIII concludes this article.

Notations: $\mathbb{Z}$ (resp. $\mathbb{Z}^+$ and $\mathbb{Z}^{++}$) denotes the set of integers (resp. positive and strictly positive integers). $\mathbb{J}$ is the complex unit. The transpose of a matrix $A$ is denoted $A^\top$ and $A^*$ denotes the complex conjugate transpose $A^* = A^\top$. The $n$-dimensional identity matrix is denoted $I_n$. The infinite identity matrix is denoted $I$. The $m \times n$ matrix of ones is denoted $1_{m,n}$. For $m \in \mathbb{Z}^{+} \cup \{\infty\}$, the flip matrix $J_m$ is the $(2m+1) \times (2m+1)$ matrix having 1 on the antidiagonal and zeros elsewhere. The product $\cdot$ refers to the Hadamard product (known also as element-by-element multiplication). $A \otimes B$ is the Kronecker product of two matrices $A$ and $B$. We denote by $\text{col}(X)$, the vectorization of a matrix $X$, formed by stacking the columns of $X$ into a single column vector. We use $\sigma^+$ to denote the largest singular value. $C^n$ denotes the space of absolutely continuous function, $L^p([a,b] \subset C^n)$ (resp. $\ell^p(C^n)$) denotes the Lebesgue spaces of $p$-integrable functions on $[a,b]$ with values in $C^n$ (resp. $p$-summable sequences of $C^n$) for $1 \leq p \leq \infty$. $L^2_{\text{loc}}$ is the set of locally $p$-integrable functions, i.e., on any compact set. The notation $f(t) = g(t)$ a.e. means almost everywhere in $t$ or for almost every $t$. To simplify the notations, $L^p([a,b])$ or $L^p$ will be often used instead of $L^p([a,b] \subset C^n)$. For example, $x \in L^2([a,b])$ means $x \in L^2([a,b] \subset C^n)$.

II. MATHEMATICAL PRELIMINARIES

We first start by recalling the definition of the sliding Fourier decomposition over a window of length $T$ and the so-called “coincidence condition” introduced in [6].

Definition 1: Let $x \in L^2_{\text{loc}}([0,\infty], \mathbb{C})$ be a complex valued function of time. Its sliding Fourier decomposition over a window of length $T$ is defined by the time-varying infinite sequence $X := F(x) \in C^{n}(\mathbb{R}, \ell^2(\mathbb{C}))$ whose components satisfy for $k \in \mathbb{Z}$

$$X_k(t) := \frac{1}{T} \int_{t-T}^{t} x(\tau)e^{-jk\omega}d\tau$$

with $\omega := \frac{2\pi}{T}$. If $x := (x_1, \ldots, x_n) \in L^2_{\text{loc}}([0,\infty], \mathbb{C}^n)$ is a complex valued vector function, then

$$X := F(x) = (F(x_1), \ldots, F(x_n))$$

The vector $X_k := (X_{1,k}, \ldots, X_{n,k})$ is called the $k$th phasor of $X$ where

$$X_{i,k}(t) := \frac{1}{T} \int_{t-T}^{t} x_\tau e^{-jk\omega}d\tau.$$

Definition 2: We say that $X$ belongs to $H$ if $X$ is an absolutely continuous function (i.e., $X \in C^{n}(\mathbb{R}, \ell^2(\mathbb{C}))$) and fulfills for any $k$ the following condition:

$$\dot{X}_k(t) = X_0(t)e^{-jk\omega} \quad \text{a.e.}$$

Similarly to the Riesz–Fisher theorem, which establishes a one-to-one correspondence between the spaces $L^2$ and $\ell^2$, the following “coincidence condition” establishes a one-to-one correspondence between the space $L^2_{\text{loc}}$ and the space $H$.

Theorem 1 (Coincidence Condition [6]): For a given $X \in L^2_{\text{loc}}([0,\infty], \mathbb{C}^n)$, there exists a representative $x \in L^2_{\text{loc}}([0,\infty], \mathbb{C}^n)$ of $X$, i.e., $X = F(x)$, if and only if $X$ belongs to $H$.

In the sequel, we provide some mathematical preliminaries related to block Toeplitz matrices and operator norms. These preliminaries are adaptations to our setting of some mathematical results borrowed from [2], [8], [11], [14], [17], [20], [22].

A. Finite and Infinite Toeplitz and Block Toeplitz Matrices

Consider a $T$-periodic $L^2([0,T], \mathbb{C})$ signal $a$, its associated Toeplitz matrix $T(a)$

$$T(a) := (t_{ij}), i, j \in \mathbb{Z}$$

such that $t_{ij} := a_{i-j}$ and its symbol (Laurent series) $a(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ where $a_k, k \in \mathbb{Z}$, are the phasors of $a(z)$. Define the semi-infinite Toeplitz matrix

$$T_{\infty}(a) := (t_{ij}), i, j \in \mathbb{Z}^+$$

such that $t_{ij} := a_{i-j}$ and let $a^+ := \sum_{k>0} a_k z^k$ and $a^- := \sum_{k<0} a_k z^{-k}$. We associate with $a^+$ and $a^-$ the following semi-infinite Hankel matrices:

$$H(a^+) := (h_{i,j}^+), i, j \in \mathbb{Z}^+, h_{i,j}^+ := a_{i+j-1}$$

$$H(a^-) := (h_{i,j}^-), i, j \in \mathbb{Z}^+, h_{i,j}^- := a_{i-j+1}.$$
An infinite-dimensional matrix as follows:

$$A := \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$$  

where the infinite matrices $A_{ij} := T(a_{ij})$, $i, j := 1, \ldots, n$, are the Toeplitz transformations of the entries $a_{ij}(t)$ of the matrix $A(t)$.

$$T(a_{ij}) := \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & a_{ij,0} & a_{ij,-1} & a_{ij,-2} \\ \cdot & a_{ij,1} & a_{ij,0} & a_{ij,-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdot & a_{ij,k} & \cdot & a_{ij,0} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

with $a_{ij,k} := \frac{1}{\pi} \int_{-T}^{T} a_{ij}(\tau)e^{-i\lambda k^2}d\lambda$.

In the sequel, to avoid confusions, for any $T$-periodic function $A \in L^2$, we denote by $\mathbf{A} := \mathcal{T}(A)$ its Fourier decomposition and by $\mathbf{A} := \mathcal{T}(A)$ its Toeplitz transformation. The $m$-truncation of the $n \times n$ block Toeplitz matrix $\mathbf{A}$ is defined by the $m$-truncation $T(a_{ij})_m$ of all its entries $(i, j)$. The symbol matrix $A(z)$ associated with a $n \times n$ block Toeplitz matrix is given by

$$A(z) := \begin{pmatrix} a_{11}(z) & a_{12}(z) & \cdots & a_{1n}(z) \\ a_{21}(z) & a_{22}(z) & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(z) & a_{n2}(z) & \cdots & a_{nn}(z) \end{pmatrix}$$

for $i, j := 1, \ldots, n$. In the same way, their subprincipal submatrices $H(A^{+})_{(p,q)}$, $H(A^{-})_{(p,q)}$ for $p, q > 0$ are obtained by considering the subprincipal submatrices of the entries $H(a_{ij})_{(p,q)}$ and $H(a_{ij})_{(p,q)}$ for $i, j := 1, \ldots, n$.

**Theorem 2**: Let $A(z)$, $B(z)$ be two symbol matrices and $C(z) := A(z)B(z)$. Then

$$\mathcal{T}_m(A)\mathcal{T}_m(B) = \mathcal{T}_m(C) - H(m,n)(A^+)H(m,n)(B^-)$$

and decomposing each term of the sum, that is

$$(\mathcal{T}_m(A)\mathcal{T}_m(B))_{ij} = \sum_{k=1}^{n} (\mathcal{T}_m(a_{ik})\mathcal{T}_m(b_{kj}))$$

where $a_{ik}(z) := a_{ik}(z)b_{kj}(z)$. The results follows for (4) and also for (3) using similar steps from the symbol formula $\mathcal{T}_m(a)\mathcal{T}_m(b) = \mathcal{T}_m(c) - H(a^+)H(b^-)$ (see [8, Prop. 1.3]).

An illustration of the abovementioned theorem is given in Fig. 2 for $n := 1$ with $a(z)$ and $b(z)$ Laurent polynomials of degree much less than $m$ so that $\mathcal{T}_m(a)$ and $\mathcal{T}_m(b)$ are banded. If $a(z) := \sum_{i=k}^{n} a_i z^i$ and $b(z) := \sum_{i=k}^{n} b_i z^i$ with $k$ much smaller than $m$, then the matrices $E^+ := H(m,n)(a^+)H(m,n)(b^-)$ and $E^- := J_m H(m,n)(a^-)H(m,n)(b^-)$ have disjoint supports located in the upper leftmost corner and in the lower rightmost corner, respectively. As a consequence, $\mathcal{T}_m(a)\mathcal{T}_m(b)$ can be represented as the sum of the Toeplitz matrix associated with $c(z)$ and two correcting terms $E^+$ and $E^-$. We end these preliminaries on block Toeplitz matrices by defining what we call left and right truncations and two results given without proofs as they follow from the block decompositions of Fig. 1.

**Definition 4**: The left $m$-truncation (resp. right $m$-truncation) of a $n \times n$ block Toeplitz matrix $A$ with blocks of infinite size...
is given by
\[
A_{m^+} := \begin{pmatrix}
A_{11_{m^+}} & A_{12_{m^+}} & \cdots & A_{1n_{m^+}} \\
A_{21_{m^+}} & A_{22_{m^+}} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
A_{n1_{m^+}} & \cdots & A_{nn_{m^+}} 
\end{pmatrix}.
\]

(resp. \(A_{m^-}\)) where \(A_{ij_{m^+}}, i, j := 1, \ldots, n\) are obtained by suppressing in the infinite matrices \(A_{ij}\) all the columns and lines having an index strictly smaller than \(-m\) (respectively, strictly greater than \(m\)). Finally, the \(m\)-truncation is obtained by applying successively a left and a right \(m\)-truncations.

**Proposition 1:** Let \(a(z)\) be a symbol and \(x := (x_k)_{k \in \mathbb{Z}}\) an infinite vector of complex numbers. Define the \(m\)-truncation of \(x\) by \(x|_{m} := (x_{-m}, \ldots, x_{m})\) and consider the semi-infinite vectors \(x_{1m} := (x_{-m+1}, x_{m+1}, \ldots)\) and \(x_{1m}^- := (\ldots, x_{-m-2}, x_{-m-1})\). Let \(x\) be the infinite vector given by \(\tilde{x} := (0, x_{0}, 0, \ldots)\). Then, the following relations hold true:
\[
T(a)\tilde{x} = \begin{pmatrix}
J_{\infty}H_{(m,\infty)}(a^-) \\
\mathcal{T}_m(a) \\
H_{(m,\infty)}(a^+)J_m
\end{pmatrix} x|_{m}
\]
\[
\mathcal{T}_m(a)x|_{m} = (T(a)x|_{m} - H_{(m,\infty)}(a^+)J_{\infty}x|_{m})
- J_mH_{(m,\infty)}(a^-)x|_{m}.
\]

The following proposition is a generalization of Proposition 1 to the case of \(n \times n\) block Toeplitz matrices.

**Proposition 2:** Let \(A(z)\) be an \(n \times n\) symbol matrix and \(x := (x_1, \ldots, x_n)\) a vector whose components \(x_i\) are infinite sequences \(x_i := (\ldots, x_{i-1}, x_0, x_i, 1, \ldots)\). Define \(x|_{m} := (x_1, \ldots, x_{m})\) the \(m\)-truncation of \(x\) where for \(i = 1, \ldots, n\), \(x|_{m-i} := (x_{i-n}, \ldots, x_{i-m})\). Define also the semi-infinite vectors \(x_{1m} := (x_{i+1}, x_{i+m+1}, \ldots)\) and \(x_{1m}^- := (\ldots, x_{i-2}, x_{i-1})\). Let \(\tilde{x} := (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)\) with \(\tilde{x}_i := (\ldots, 0, x_{i|0}, 0, \ldots)\) for any \(i = 1, \ldots, n\). Then, we have
\[
(\mathcal{T}(A)\tilde{x})_i = \sum_{j=1}^{n}(\mathcal{T}(a_{ij})\tilde{x})_j
\]
where \(\mathcal{T}(a_{ij})\tilde{x}_j\) is given by (5) and
\[
\mathcal{T}_m(A)x|_{m} = \sum_{j=1}^{n}(\mathcal{T}(a_{ij})x|_{m})
- H_{(m,\infty)}(a^+)J_{\infty}x|_{m} - J_mH_{(m,\infty)}(a^-)x|_{m}.
\]

**B. Operator Norms**

We provide here some results concerning operator norms to be used in the sequel. Recall that the norm of an operator \(M\) from \(\ell^p\) to \(\ell^q\) is given by
\[
\|M\|_{\ell^p\to\ell^q} := \sup_{\|X\|_{\ell^p} = 1} \|MX\|_{\ell^q}.
\]

This operator norm is submultiplicative, i.e., if \(M : \ell^p \to \ell^q\) and \(N : \ell^q \to \ell^r\) then \(\|NM\|_{\ell^p,\ell^r} \leq \|M\|_{\ell^p\to\ell^q}\|N\|_{\ell^q,\ell^r}\). If \(p = q\), we use the notation: \(\|M\|_{\ell^p} := \|M\|_{\ell^p,\ell^p}\).

**Definition 5:** Consider a vector \(x(t) \in L^2([0, T], \mathbb{C}^n)\) and define \(X := F(x)\) with its symbol \(X(z)\). The \(\ell^Q\)-norm of \(X(z)\) is given by
\[
\|X(z)\|_{\ell^Q} := \|X\|_{\ell^Q}
\]
where \(\|X\|_{\ell^Q} := (\sum_{k \in \mathbb{Z}} |X_k|^2)^{\frac{1}{2}}\).

**Theorem 3:** Let \(A \in L^2([0, T], \mathbb{C}^{n \times m})\). Then, \(A := T(A)\) is a bounded operator on \(\ell^2\) if and only if \(A \in L^\infty([0, T], \mathbb{C}^{n \times m})\). Moreover, we have the following:

1. the operator norm induced by the \(\ell^2\)-norm satisfies \(\|A(z)\|_{\ell^2} = \|A\|_{\ell^2} = \|A\|_{L^\infty}\); 2. the operator norm of the semi-infinite Toeplitz matrix satisfies: \(\|T(A)\|_{\ell^2} = \|A\|_{\ell^2}\); 3. the operator norm of the Hankel operators \(H(A^+), H(\mathcal{A}^+)\) satisfies: \(\|H(A^+)\|_{\ell^2} \leq \|A\|_{L^\infty}\) and \(\|H(\mathcal{A}^+)\|_{\ell^2} \leq \|A\|_{L^\infty}\); 4. the operator norm related to the left and right \(m\)-truncations satisfies: \(\|A_{m^+}\|_{\ell^2} = \|A_{m^-}\|_{\ell^2}\).

**Proof:** See [13, Part V pp. 562–574].

**III. Problem Statement**

To formulate the problem, we are interested in, we need to recall some key results from [6]. Under the “coincidence condition” of Theorem 1, it is established in [6] that any periodic system having solutions in Carathéodory sense can be transformed by a sliding Fourier decomposition into a time invariant system. For instance, consider \(T\)-periodic functions \(A(\cdot)\) and \(B(\cdot)\), respectively, of class \(L^2([0, T], \mathbb{R}^{n \times n})\) and let the linear time periodic system
\[
\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(0) = x_0.
\]
If \(x\) is a solution associated with the control \(u \in L^2_{\text{loc}}(\mathbb{R}, \mathbb{R}^{m \times n})\) of the linear time periodic system (7), then \(X := F(x)\) is a solution associated with \(U := F(u)\) of the linear time invariant system
\[
\dot{X}(t) = (A - \mathcal{N})X(t) + BU(t), \quad X(0) := F(x)(0)
\]
where \(A := T(A), B := T(B)\) and \(\mathcal{N} := I_{n \times n} \otimes \text{diag}(\mathbb{R}, k, k \in \mathbb{Z})\).
Reciprocally, if $X \in H$ is a solution of (8) with $U \in H$, then their representatives $(x, u)$ (i.e., $X = F(x)$ and $U = F(u)$) are solutions of (7). Moreover, for any $k \in \mathbb{Z}$, the phasors $X_k \in C^1(\mathbb{R}, \mathbb{C}^n)$ and $\tilde{X} \in C(\mathbb{R}, \ell^\infty(\mathbb{C}^n))$. As the solution $x$ is unique for the initial condition $x_0$, $X$ is also unique for the initial condition $X(0) := F(x(0))$. In addition, it is proved in [6] that one can reconstruct time trajectories from harmonic ones, that is

$$x(t) = F^{-1}(X)(t) := \sum_{k=\infty}^{t} X_k e^{j\omega_k t} + \frac{T}{2} \tilde{X}_0(t)$$

where $X_k = (X_{1,k}, \ldots, X_{n,k})$ for any $k \in \mathbb{Z}$.

In the same way, a strict equivalence between a periodic differential Lyapunov equation and its associated harmonic algebraic Lyapunov equation is also proved [6]. Namely, let $Q \in L^\infty([0, T])$ be a $T$-periodic symmetric and positive definite matrix function. $P$ is the unique $T$-periodic symmetric positive definite solution of the periodic differential Lyapunov equation

$$\dot{P}(t) + A^*(t)P(t) + P(t)A(t) + Q(t) = 0$$

if and only if $P := T(P)$ is the unique hermitian and positive definite solution of the harmonic algebraic Lyapunov equation

$$P(A - N) + (A - N)^*P + Q = 0$$

where $Q := T(Q)$ is hermitian positive definite and $A := T(A)$. Moreover, $P$ is a bounded operator on $\ell^2$ and $P$ is an absolutely continuous function.

These results are of great interest. Solving an algebraic Lyapunov equation rather than a periodic differential Lyapunov equation is worthwhile for analysis and control design provided coping with the infinite dimension nature of (11). The main difficulty is related to the diagonal matrix $N$ defined by (9), which is not a Toeplitz matrix nor a compact operator. Hence, the harmonic algebraic Lyapunov equation (11) cannot be expressed as a simple product of symbols as in the classical Toeplitz case [22]. In [28] and [29], the authors propose to use a Floquet factorization but the determination of this Floquet factorization is not so simple [15], [23], [31]. Furthermore, for control design purpose, it would not be adequate to proceed this way since the input matrix remains a full matrix with no particular and useful structure in the harmonic domain.

The main objective of this article is to show how the solution of the infinite-dimensional harmonic Lyapunov equation (11) can be obtained from a finite-dimensional problem up to an arbitrary error. As we will see, this is a practical result that avoids the computation of a Floquet factorization and reduces significantly the computation burden. We also extend our result to harmonic Riccati equations encountered in periodic optimal control. To this end, the characterization of the spectrum of the harmonic state operator $(A - N)$ is of major importance and plays a key role in the derivation of the main contributions of this article.

### IV. Spectral Properties of $(A - N)$

In this section, we provide a simple closed-form formula for a Floquet factorization, characterize the spectrum of the harmonic state operator $(A - N)$ and study the spectral properties of its truncated version. As noticed before, the harmonic state matrix $(A - N)$ is not Toeplitz because of $N$. This term has an important impact on the spectral properties of $(A - N)$. For instance, we know that the spectrum of a Toeplitz matrix $A$ is continuous [1], [20], [24] and bounded when $A(z)$ belongs to $\ell^2$. However, we will see in the sequel that the spectrum of $(A - N)$ is unbounded and discrete. We will also explain how this spectrum behaves when applying an $m$-truncation $(A_m - N_m)$.

#### A. Closed-Form Formula for a Floquet Factorization and Spectral Properties of $(A - N)$

Recall that the Floquet theorem [10], [28] states that for dynamical systems

$$\dot{x}(t) = A(t)x(t) \quad (12)$$

with $A(t)$ piecewise continuous and $T$-periodic, the state transition matrix $\Phi(t, 0)$ has a Floquet factorization $\Phi(t, 0) = W(t)e^{Qt}$, where $Q$ is a constant matrix and $W(t)$ is continuous in $t$, nonsingular and $T$-periodic in $t$. Moreover, the state transformation $z(t) := W(t)^{-1} x(t)$ leads to an LTI system:

$$\dot{z}(t) = Qz(t)$$

and the harmonic system associated with (12)

$$\dot{X} = (A - N)X$$

becomes

$$\dot{Z} = (Q - N)Z$$

with $Z := F(z)$ and $Q := T \otimes Q$. Unfortunately, this result is an existence result and, as such, it is not constructive. One may find algorithms to determine $W(t)$ and $Q$ as those proposed in [9] and [31]. Here, we show that a more simple characterization of a Floquet factorization can be obtained with $Q$ in a Jordan normal form and $W(t)$ easily determined as the solution of an initial value problem with explicit initial conditions. Moreover, our result is given with the assumption that the $T$-periodic matrix function $A$ belongs to $L^2$, which is more general than existing results.

When $A \in L^2([0, T], \mathbb{R}^{n \times n})$, the initial value problem defined by (12) and $x(0) := x_0$ admits a unique solution in the Carathéodory sense. We can define $n$ linearly independent fundamental solutions denoted $x^{(i)}(t)$ having $e_i$ as initial conditions. As a consequence, the Wronski matrix

$$\Phi(t, 0) = [x^{(1)}(t), \ldots, x^{(n)}(t)]$$

is the state transition matrix and for any time $t$, $x(t) = \Phi(t, 0)x_0$ is solution of the initial value problem. Moreover, $\Phi(t, 0)$ is nonsingular, absolutely continuous and, therefore, almost everywhere differentiable. This is important to characterize the eigenvalues and eigenvectors of the harmonic operator $(A - N)$ as shown in the following theorem for the case when $\Phi(T, 0)$ is nondefective.

**Theorem 4:** Assume that the $T$-periodic function $A(t)$ belongs to $L^2([0, T])$ and that $\Phi(T, 0)$ is nondefective. Let $\mu$ and $\phi$ be, respectively, an eigenvalue and an associated eigenvector of $\Phi(T, 0)$. Then, $\lambda$ and $V$ are an eigenvalue and an eigenvector of $(A - N)$

$$(A - N)V = \lambda V$$

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if and only if \( v := F^{-1}(V) \) is a T-periodic solution in the Carathéodory sense of the initial value problem

\[
\dot{v}(t) = (A(t) - \lambda \text{Id}_n)v(t) - v(0) := \phi \tag{14}
\]

where \( \lambda := \frac{1}{T} \log(\mu) \) (not necessarily its principal value).

**Proof:** Applying [6, Th. 4], it follows that a solution of (14) is a solution of

\[
\dot{V} = (A - \mathcal{N} - \lambda \mathcal{I})V \tag{15}
\]

where \( V := \mathcal{F}(v) \) and reciprocally [provided \( V \) is a trajectory of (15) that belongs to \( H \), see Definition 2]. If \( v \) is T-periodic then \( V = 0 \). Thus, \( \lambda \) and \( V \) are necessarily an eigenvalue and an eigenvector of \((A - \mathcal{N})\). Reciprocally, if \( \lambda \) and \( V \) are an eigenvalue and an eigenvector of \((A - \mathcal{N})\) this means that \( V = 0 \) in (15). As \( V \) is constant, it belongs trivially to \( H \). Hence, \( V \) admits an absolutely continuous and T-periodic representative \( v \) that satisfies (14) a.e. Now, consider an eigenvalue \( \mu \) and an associated eigenvector \( \phi \) of \( \Phi(T,0) \), then \( \Phi(T,0)\phi = \mu \phi \). Notice that \( \mu \) cannot be equal to zero since \( \Phi(T,0) \) is not singular. Define \( \lambda := \frac{1}{T} \log(\mu) \) (not necessarily as the principal value of \( \log(\mu) \)), then we have

\[
\phi = R(T,0)\phi \tag{16}
\]

with \( R(T,0) := e^{-\lambda T} \Phi(T,0) \). Moreover, as \( \Phi(T,0) \) is a.e. differentiable, we can write

\[
\dot{\Phi}(t,0) = A(t)\Phi(t,0) \text{ a.e.}
\]

Let \( R(t,0) := e^{-\lambda t} \Phi(t,0) \). We have

\[
\dot{R}(t,0) = -\lambda e^{-\lambda t}\Phi(t,0) + e^{-\lambda t}\dot{\Phi}(t,0) \tag{17}
\]

\[
= (A(t) - \lambda \text{Id}_n)R(t,0) \text{ a.e.} \tag{18}
\]

Hence, \( R(t,0) \) is the state transition matrix of the linear system (14). We conclude from (16) that the solution of the initial value problem (14) defined by such a \( \lambda \) and \( \phi \) is T-periodic.

To generalize this result to the case where \( \Phi(T,0) \) is defective, let us consider a Jordan normal form of the matrix \( \Phi(T,0) \) and assume that \((\mu_1, \ldots, \mu_n)\) and

\[
P_\Phi := [\phi_1, \ldots, \phi_n] \tag{19}
\]

are, respectively, the eigenvalues and the matrix formed by the generalized eigenvectors of \( \Phi(T,0) \). To ease the presentation, we assume without loss of generality that

\[
P_\Phi^{-1} \Phi(T,0)P_\Phi = \begin{pmatrix}
\mu & 1 & 0 & 0 \\
0 & \mu & \cdots & 0 \\
0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \mu
\end{pmatrix}
\]

with \( \phi_i \in \text{Ker}(\Phi(T,0) - \mu \text{Id})^2 \), for \( i = 1, \ldots, n \).

**Theorem 5:** Consider \( \lambda := \frac{1}{T} \log(\mu) \) (not necessarily the principal value) and let \( V_0 := 0 \) and \( v_0 := 0 \). For \( i = 1, \ldots, n \), \( V_i \) is a generalized eigenvector associated with \( \lambda_i \)

\[
(A - \mathcal{N})V_{i+1} = \lambda V_{i+1} + \frac{1}{T\mu}V_i \tag{20}
\]

if and only if \( v_i := F^{-1}(V_i) \) is a T-periodic solution in Carathéodory sense of the initial value problem

\[
\dot{v}_i(t) = (A(t) - \lambda \text{Id}_n)v_i(t) - \frac{1}{T\mu}v_{i-1}(t), \quad v_i(0) := \phi_i \tag{21}
\]

where \( \phi_i \) are provided by (19).

**Proof:** The strict equivalence between (20) and (21) is obtained following similar steps as in the proof of Theorem 4. For \( i := 1 \), as \( \phi_1 \) is an eigenvector of \( \Phi(T,0) \), the result is already proved (Theorem 4) and \( v_1(t) := R(t,0)v_1(0) \) with \( R(t,0) := e^{-\lambda t}\Phi(t,0) \) and \( \Phi \) given by (13). For \( i := 2 \), the solution

\[
v_2(t) = R(t,0)v_2(0) - \frac{t}{T\mu}v_{i-1}(t)
\]

is directly obtained from the formula:

\[
v_2(t) = R(t,0)v_2(0) - \frac{1}{T\mu} \int_0^t R(t,s)v_1(s)ds.
\]

As \( R(T,0) = \mu^{-1}\Phi(T,0) \), it follows that

\[
v_1(T) + \mu v_2(T) = \Phi(T,0)v_2(0)
\]

where \( v_1(T) = v_1(0) \). Thus, if \( v_2(0) := \phi_2 \) then it follows that \( v_2(T) = v_2(0) \), which proves that \( v_2(t) \) is T-periodic.

Now, assume that this property holds inductively until the index \( i - 1 \) and

\[
v_{i-1}(t) = R(t,0)v_{i-1}(0) - \frac{t}{T\mu}v_{i-2}(t)
\]

then, as

\[
v_i(t) = R(t,0)v_i(0) - \frac{1}{T\mu} \int_0^t R(t,s)v_{i-1}(s)ds
\]

it is straightforward to show that

\[
v_i(t) = R(t,0)v_i(0) - \frac{t}{T\mu}v_{i-1}(t).
\]

Thus, following the same reasoning as before, the conclusion on the periodicity of \( v_i \) follows by setting \( v_i(0) := \phi_i \).

We are now in position to give a closed-form formula for a Floquet Factorization.

**Theorem 6:** Assume that the T-periodic function \( A(t) \) belongs to \( L^2([0,T]) \) and \( \Phi(T,0) \) is given by (13). Consider for \( i := 1, \ldots, n \), the eigenvalues \( \mu_i \) and the generalized eigenvectors \( \phi_i \) of \( \Phi(T,0) \) and set \( \lambda_i \) to the principal value of \( \frac{1}{T} \log(\mu_i) \). Consider for each \( \lambda_i \), the solution \( v_i \) of the initial value problem with \( v_i(0) := \phi_i \), provided by Theorem 4 (or 5 if \( \Phi(T,0) \) is defective).

Then, a Floquet factorization is determined by \( W(t) := [v_1(t), \ldots, v_n(t)] \) and \( Q := \Lambda \) where \( \Lambda \) is a Jordan normal form given by, for \( i := 1, \ldots, n \), \( \Lambda(i,i) := \lambda_i \), \( \Lambda(i,i+1) := 0 \) or \( \frac{1}{T\mu} \) and zeros elsewhere. Moreover, the T-periodic and absolutely continuous matrices \( W \) and \( W^{-1} \) satisfy

\[
\dot{W}(t) = A(t)W(t) - W(t)\Lambda \text{ a.e.}
\]

\[
W^{-1}(t) = -W^{-1}(t)A(t) + \Lambda W^{-1}(t) \text{ a.e.}
\]

and the operator \( NW := T(W) \) is bounded on \( \ell^2 \), invertible and satisfies the eigenvalue problem

\[
W^{-1}(A - \mathcal{N})W = \Lambda \otimes \mathcal{I} - \mathcal{N}.
\]

In addition, taking \( z(t) := W^{-1}(t)x(t) \) transforms the LTP system \( \dot{x} = A(t)x \) into the LTI system

\[
\dot{z} = \Lambda z \text{ a.e.}
\]

**Proof:** For \( i := 1, \ldots, n \), consider for \( \lambda_i \) the principal value of \( \frac{1}{T} \log(\mu_i) \) only and let the T-periodic vectors \( v_i \) determined...
using Theorem 4 (or Theorem 5 if $\Phi(T, 0)$ is defective) with $V_i := F(v_i)$. Denote by
\[
A_k := \begin{pmatrix}
A_{11,k} & A_{12,k} & \cdots & A_{1n,k} \\
A_{21,k} & A_{22,k} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
A_{n1,k} & \cdots & \cdots & A_{nn,k}
\end{pmatrix}
\]
the $k$th phasors of $A$ and $V_i$, $k$th phasors of $V_i$. We have for any $k$
\[
\sum_{p \in \mathbb{Z}} A_{k-p}V_{i+1,p} - j\omega k V_{i+1,k} = \lambda_i V_{i+1,k} + s_i V_i, 
\]
with $s_i := \frac{1}{T\mu_i}$ or 0. Thus, an $m$-shift in the components of $V_i$, $V_{i+1}$ leads to
\[
\sum_{p \in \mathbb{Z}} A_{k-p}V_{i+1,p+m} - j\omega k V_{i+1,k+m} = (\lambda + j\omega m)V_{i+1,k+m} + s_i V_i, 
\]
which means that the $m$-shifted vector is also a generalized eigenvector associated with $\lambda + j\omega m$ (and not to $\lambda$). It follows that $T(v_i)$ is the set of all generalized eigenvectors associated with all values of $\frac{1}{T}\log(\mu_i)$ defined modulo $j\omega$. Now, let $W := [v_1, \ldots, v_n]$. Obviously, $W(t)$ satisfies (22). As $W(t)$ is a $T$-periodic and absolutely continuous matrix function, $W := T(W) = [T(v_1), \ldots, T(v_n)]$ is a constant and bounded operator on $\ell^2$ (see Theorem 3). Furthermore, using similar steps as in the proof of (see[6, Th. 5]), the $n \times n$ block Toeplitz matrix $W$ satisfies
\[
(A - N)W = W(\Lambda \otimes I - T).
\]
As $W$ solves the eigenvalue problem for all admissible eigenvalues and is invertible, the same holds true for $T(t)$. Since $W^{-1}(A - N) = (\Lambda \otimes I - T)W^{-1}$, using similar steps as in the proof of (see[6, Th. 5]), it is straightforward to establish that the absolutely continuous matrix function $W^{-1} := T^{-1}(W^{-1})$ satisfies (23).

Finally, let $x$ be a solution of $\dot{x} = A(t)x$ in Carathéodory sense and set $z(t) := W^{-1}(t)x(t)$. From (23), we have
\[
\dot{z}(t) = W^{-1}(t)z(t) + W^{-1}(t)\dot{x}(t) \text{ a.e.} \\
= \Lambda z(t) \text{ a.e.}
\]

**Corollary 1:** $(A - N)$ is nondefective if and only if $\Phi(T, 0)$ is nondefective  
*Proof:* As $z(t) = e^{AT}z_0$ and as $z(t) := W^{-1}(t)x(t) = W^{-1}(t)\Phi(t, 0)W(0)$, it follows that $e^{AT} = W^{-1}(t)\Phi(t, 0)W(0)$. For $t := T$ since $W$ is $T$-periodic, we have $W(0) = W(T)$ and $e^{AT} = W^{-1}(T)\Phi(T, 0)W(T)$.

(26)

Now if $(A - N)$ is nondefective, the eigenvalue problem corresponding to (24) is determined by a diagonal matrix $\Lambda$. Thus, $e^{AT}$ is diagonal and we conclude from (26) that $\Phi(T, 0)$ is nondefective. Reciprocally, if $\Phi(T, 0)$ is nondefective, Theorem 6 leads to (24) with $\Lambda$ diagonal.

The previous Theorem provides a simple characterization of a Floquet factorization, which is of interest for analysis purpose.

The fact that the input harmonic matrix remains a full matrix when applying a Floquet factorization makes this approach difficult to apply to design stabilizing state feedback control laws for example. Our choice is to push further the spectral analysis of the operator $(A - N)$ and analyze the impact of an $m$-truncation on the spectrum of $(A_m - N_m)$ in order to provide efficient algorithms that can also be used for harmonic control design. We start by the following corollary, which states that the spectrum of $(A - N)$ is unbounded and discrete.

**Corollary 2:** Assume that $(A - N)$ is nondefective. The spectrum of $(A - N)$ is given by the unbounded and discrete set
\[
\sigma(A - N) := \{\lambda_p + j\omega k : k \in \mathbb{Z}, \ p := 1, \ldots, n\}
\]
where $\lambda_p, p := 1, \ldots, n$ are not necessarily distinct eigenvalues.

*Proof:* Consider the unbounded diagonal operator $D := (\Lambda \otimes I - T)$. As the point spectrum $\sigma_p := \{\lambda_p + j\omega k : k \in \mathbb{Z}, \ p := 1, \ldots, n\}$ has no cluster points, it is a closed set and $\sigma_p \subset \sigma(D)$. Denote by $D_j$ the $j$th entry of the diagonal of $D$ and $e_j$ the $j$th vector of the basis. If $\zeta \notin \sigma_p$, then $D$ is defined by $Se_j := \frac{1}{\zeta - \lambda}e_j$ for any $j \in \mathbb{Z}$ is a bounded (diagonal) operator on $\ell^2$ whose inverse is $\zeta I - D$. Thus, $\zeta \notin \sigma(D)$ and $\sigma(D) = \sigma_p$.

The result of Corollary 2 holds also when $(A - N)$ is defective. This can be established by defining $J := (J \otimes I - N)$ with $J$ a Jordan normal form and showing that $\zeta I - J$ is invertible for any $\zeta \notin \sigma_p$. The invertibility of $\zeta I - J$ is proved recursively on the $n \times n$ blocks of $\zeta I - J$ noticing that each of these blocks is diagonal and using the matrix formula
\[
\begin{pmatrix}
A & B \\
0 & C
\end{pmatrix}^{-1} = \begin{pmatrix}
A^{-1} & -A^{-1}BC^{-1} \\
0 & C^{-1}
\end{pmatrix}.
\]

We discuss the properties of the inverse of $(A - N)$ in the following corollary.

**Corollary 3:** $A$ is invertible if and only if the operator $(A - N)$ is invertible. Moreover, $(A - N)^{-1}$ is bounded on $\ell^2$ and
\[
\sigma^+ := ||(A - N)^{-1}||_{\ell^2}
\]
where $\sigma^+ := \sup\{||(\lambda_p + j\omega k)^{-1}|| : k \in \mathbb{Z}, \ p = 1, \ldots, n\}$.

*Proof:* The result follows from the abovementioned theorem and noticing that the $\ell^2$ operator norm corresponds to the maximum singular value.

**Remark 1:** As shown in [6], if $x$ is the solution of $\dot{x} = A(t)x$ then $X := F(x) \in C^0(\mathbb{R}, \ell^p)$ and we have $(A - N)X(t) \in \ell^p$, for any $1 < p < +\infty$ and $(A - N)X(t) \in \ell^p$. Clearly, $(A - N)$ is not a bounded operator on $\ell^2$ while its inverse (if it exists) is bounded.

**B. Spectrum Analysis of $(A_m - N_m)$**

Here, we explain how the spectrum of $(A_m - N_m)$ is modified w.r.t. the spectrum of $(A - N)$ when performing an $m$-truncation on $(A - N)$. From now, we assume that the $T$-periodic matrix function $A(t)$ belongs to $L^\infty(\mathbb{R}, \mathbb{R}^{n \times n})$ or equivalently $A$ is a bounded operator on $\ell^2$. For simplicity reasons, we provide the results when the operator $(A - N)$ is nondefective but the results hold true in general.
Theorem 7: Assume that \( \Lambda(t) \in L^{\infty}([0, T]) \) and \( (A - N) \) is nondefective. Denote by \( \sigma := \{ \lambda_p + j\omega k : k \in \mathbb{Z}, p := 1, \ldots, n \} \), the spectrum of \( (A - N) \). Let \( (\Lambda_{m+} - N_{m+}) \) be a \( m \)-truncation of \( (A - N) \) according to Definition 4 and assume that it is nondefective, with an eigenvalues set denoted by \( \Lambda_{m+} \).

1) For \( \epsilon > 0 \), there exists an index \( j_0 \) such that for any eigenvalue \( \lambda \in \Lambda_{m+}^+(m) := \{ \lambda + j\omega k : k \in \mathbb{Z}, k \leq m + 1 - j_0, p := 1, \ldots, n \} \subset \sigma \)
\[
\| (A_{m+} - N_{m+} - \lambda I_{m+}) V_{m+} \|_2 < \epsilon \tag{27}
\]
where \( V_{m+} \) is the left \( m \)-truncation of the eigenvector associated with \( \lambda \).

2) The set \( \Lambda_{m+} \) can be approximated by the union of \( \Lambda_{m+}^+(m) \) and \( \Lambda_{m+}^-(m) \) that is
\[
\Lambda_{m+} \approx \Lambda_{m+}^+(m) \cup \Lambda_{m+}^-(m)
\]
where \( \Lambda_{m+}^+(m) \) is a finite subset of \( \Lambda_{m+} \). Moreover, any eigenvalue \( \lambda_{m+1} \), which belongs to the set \( \Lambda_{m+}^+(m + 1) \) is obtained by the relation: \( \lambda_{m+1} = \lambda_{m} + j\omega \) where \( \lambda_{m} \) belongs to \( \Lambda_{m+}^+(m) \).

Proof: It is sufficient to prove the theorem for \( n := 1 \). Indeed, as the difficulties are related to infinite Toeplitz matrices, if the result is established for \( n := 1 \), the same result holds for any finite \( n \) using ad hoc formulae of Proposition 2. Consider \( \mathcal{W} \) given by (24). When \( n := 1 \), the set of eigenvalues is given by \( \{ \lambda + j\omega k : k \in \mathbb{Z} \} \) for a given \( \lambda \) and the matrix \( \mathcal{W} \) reduces to \( \mathcal{W} := \mathcal{V} = T(v) \) with phasors denoted by \( \mathcal{V} \). Applying a left \( m \)-truncation and using Theorem 2, we have
\[
(A_{m+} - N_{m+}) V_{m+} = V_{m+} (\lambda I_{m+} - N_{m+}) = E^+ \tag{28}
\]
where \( E^+ = -\mathcal{H}(A^+) \mathcal{V} \). For \( j := 1, 2, \ldots \), the \( j \)-th column of \( E^+ \) is provided by \( E^+(j) := -\mathcal{H}(A^+) \mathcal{V}_{j} \) where \( \mathcal{V}_{j} := (\mathcal{V}_{j-1}, \mathcal{V}_{j-2}, \ldots, \ldots) \). Using Theorem 3, we have
\[
\| E^+(j) \|_2 \leq \| \mathcal{H}(A^+) \|_2 \| \mathcal{V}_{j} \|_2
\]
Therefore, for a given \( \epsilon > 0 \), there always exists an index \( j_0 \) such that \( \| E^+(j) \|_2 \leq \epsilon \) for \( j \geq j_0 \) since \( \| \mathcal{V}_{j} \|_2 \to 0 \) when \( j \to +\infty \), which establishes (27). The remaining \( j_0 \) eigenvalues form a finite subset \( \Lambda_{m+}^+(m) \) of \( \Lambda_{m+} \). Thus, if \( \lambda_2 \in \Lambda_{m+}^+(m) \) is an eigenvalue associated with its semi-infinite eigenvector \( V_{\lambda_2} \), it follows that
\[
(A_{m+} - N_{m+}) V_{\lambda_2} = \lambda_2 V_{\lambda_2}
\]
and it is straightforward to show that
\[
(A_{m+} - N_{m+}) V_{\lambda_2} = (\lambda_2 + j\omega) V_{\lambda_2}
\]
Therefore, any eigenvalue of the set \( \Lambda_{m+}^+(m + 1) \) is obtained from \( \lambda_2 \in \Lambda_{m+}^+(m) \) by adding \( j\omega \) and the associated semi-infinite eigenvector is obtained by shifting \( V_{\lambda_2} \).

Theorem 8: Assume that the matrix \( A(t) \in L^{\infty}([0, T]) \) and that \( (A - N) \) is nondefective with \( \sigma := \{ \lambda_p + j\omega k : k \in \mathbb{Z}, p := 1, \ldots, n \} \) its spectrum. Assume that the \( m \)-truncation \( (A_m - N_m) \) is nondefective with its eigenvalues set denoted by \( \Lambda_{m} \). For \( \epsilon > 0 \), there exists an \( m_0 \) such that for \( m \geq m_0 \),

1) there exists an index \( j_0 \) such that for any eigenvalue \( \lambda_1 \in \Lambda_1(m) \) defined by the subset \( \{ \lambda_p + j\omega k : k \leq j_0, p := 1, \ldots, n \} \) of \( \sigma \), the following relation is satisfied:
\[
\| (A_m - N_m - \lambda_1 I_m) V_j \|_2 \leq \epsilon \tag{28}
\]
where \( V_j \) is the \( m \)-truncation of the eigenvector associated with \( \lambda_1 \).

2) for any eigenvalue \( \lambda_2 \in \Lambda_2(m^+) \) or in \( \Lambda_2(m^-) := \Lambda_2(m^+) \) with \( \Lambda_2(m^+) \) defined in Theorem 7, the following relation is satisfied:
\[
\| (A_m - N_m - \lambda_2 I_m) V_j \|_2 \leq \epsilon
\]
where \( V_j \) is the \( m \)-truncation of the eigenvector associated with \( \lambda_2 \).

Then, the set \( \Lambda_m \) can be approximated by the union of the sets \( \Lambda_1(m), \Lambda_2(m) \) and \( \Lambda_2^+(m) \) that is
\[
\Lambda_m \approx \Lambda_1(m) \cup \Lambda_2(m) \cup \Lambda_2^+(m)
\]
Proof: As before, the proof is given for \( n := 1 \). The right \( m \)-truncation leads to a symmetric result of Theorem 7 for which \( \Lambda_2(m) = \Lambda_2^+(m) \). In case both right and left \( m \)-truncations are performed, applying Theorem 2 to (24) leads to
\[
(A_m - N_m) V_j - V_m (\lambda I_m - N_m) = -E_m
\]
where \( E_m := E^+ + E^m \) and
\[
E^+ := \mathcal{H}(m, m) (A^+) \mathcal{H}(m, m) \mathcal{V}^{-1}
\]
Notice that \( E_m(i, j) \) is simply obtained from \( E_m^+(i, j) \) by a central symmetry of index \( (m + 1, m + 1) \). As in the previous proof, for \( j := 1, 2, \ldots \), the \( \ell^2 \) norm of the \( j \)-th column of \( E^+ \) satisfies
\[
\| E^+(j) \|_2 \leq \| A(t) \|_2 \| \mathcal{V}_{j} \|_2
\]
and for a given \( \epsilon > 0 \), there exists an index \( j_0 \) such that \( m_0 \) is chosen sufficiently large such that for any \( j \geq j_0 \)
\[
\| E^+(j) \|_2 \leq \epsilon/2.
\]
By symmetry, the \( \ell^2 \)-norm of the columns of \( E_m^- \) is less than \( \epsilon/2 \) for \( j \leq 2(m + 1) - j_0 \). Thus, it follows that the \( \ell^2 \)-norm of the columns of \( E^- + E_m \) is less than \( \epsilon \) for \( j \) such that \( j_0 \leq j \leq 2(m + 1) - j_0 \). Consequently, (28) is satisfied.

Now it remains to show that the elements of \( \Lambda_2^+(m) \) and \( \Lambda_2(m) \) are eigenvalues of \( (A_m - N_m) \) up to an arbitrary small error. If \( \lambda_2 \in \Lambda_2^+(m) \) is an eigenvalue associated with an eigenvector \( V_2 \), it follows that
\[
(A_m - N_m) V_2 = \lambda_2 V_2.
\]
If a right \( m \)-truncation is applied on (29), then
\[
(A_m - N_m) V_m = \lambda_2 V_2 + E_m \tag{29}
\]
where \( E_m := -J_m \mathcal{H}(m, m) (A^+) \mathcal{V}^+ \) [see (6) in Proposition 1] and \( V_m^+ := (V_{m+1}, V_{m+2}, \ldots, \ldots) \). As before, we have \( \| E_m \|_2 \leq \| A(t) \|_2 \| \mathcal{V}_j \|_2 \| \mathcal{V}_{j} \|_2 \). Hence, there always exists an \( m_0 \) such that for \( m \geq m_0 \)
\[
\| E_m \|_2 \leq \epsilon
\]
that for any \( m \geq m_0 \), the matrix \( (A_m - N_m) \) is invertible. Moreover, \( \| (A_m - N_m)^{-1} \|_2 \) is uniformly bounded, i.e.,

\[
\sup_{m \geq m_0} \| (A_m - N_m)^{-1} \|_2 < +\infty.
\]

Proof: As \( \lim_{m \to +\infty} \min(\lambda^*_j(m)) = +\infty \), and as \( (A - N) \) is invertible, for sufficiently large \( m \), \( (A_m - N_m) \) is not singular and the eigenvalues of \( (A_m - N_m)^{-1} \) are uniformly bounded by \( \sup \{ |(\lambda_p - j\omega k)^{-1}| : k \in \mathbb{Z}, p := 1, \ldots, n \} \). Thus, \( \| (A_m - N_m)^{-1} \|_2 \) is uniformly bounded.

\[\text{C. Example}\]

Consider the following \( 2 \times 2 \) block Toeplitz matrix:

\[A := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\]

where the Toeplitz matrices \( A_{ij} \) are characterized by

\[a_{11} := (0.5, 0.6 - j, -1, 0.6 + j, 0.5)\]
\[a_{12} := (1.3 - 0.4j, -2.2 + 0.5j, -0.4, -2.2 - 0.5j, 1.3 + 0.4j)\]
\[a_{21} := (-0.3 - 0.6j, 0.4 + 0.7j, -0.1, 0.4 - 0.7j, -0.3 + 0.6j)\]
\[a_{22} := (-1.3 - 1.8j, 1.4 - 1.6j, -2, 1.4 + 1.6j, -1.3 + 1.8j)\]

with the underlined terms corresponding to the index 0. The eigenvalues of \( (A_m - N_m) \) are depicted in Fig. 3 for \( m := 20 \) (blue circles) and \( m := 40 \) (red stars). We clearly observe the sets \( \Lambda_1(m) \), \( \Lambda_2^1(m) \) and \( \Lambda_2^2(m) \). Notice that \( \Lambda_1(m) \) is defined by two eigenvalues (as expected) satisfying \( R(\lambda_1) \approx -0.3 \) and \( R(\lambda_2) \approx -2.7 \) as shown by the alignment of the eigenvalues along these vertical axes. Thus, \( (A - N) \) is Hurwitz while \( (A_m - N_m) \) is never Hurwitz for all \( m \) since \( \Lambda_2^1(m) \) and \( \Lambda_2^2(m) \) have eigenvalues with positive real parts. As mentioned, we see that the set \( \Lambda_2^2(40) \) is obtained from the set \( \Lambda_2^2(20) \) by a translation of \( j\omega 20 \) \( (\omega := 1) \).

This example illustrates the fact that having \( (A - N) \) Hurwitz, if we try to solve a Lyapunov equation with a truncated version \( (A_m - N_m) \), the solution would never be positive definite whatever \( m \). This motivates the following section devoted to solving harmonic Lyapunov equations.

\[\text{V. SOLVING HARMONIC LYAPUNOV EQUATION}\]

Taking benefit from the spectral analysis of the previous section, the objective here is to study how the infinite-dimensional harmonic Lyapunov equation (11) can be solved in a practice without invoking a Floquet factorization.

The following proposition introduces the symbol Lyapunov equation.

Proposition 4: Assume that \( A(t) \in L^\infty([0, T], \mathbb{R}^{n \times n}) \). The symbol harmonic Lyapunov equation is given by

\[A(z)P(z) + P(z)A(z) + (\mathbb{I}_{n,n} \otimes N(z)) \cdot P(z) + Q(z) = 0\]

(30)

where the symbol matrices \( A(z), Q(z), P(z) \) are given by (2) and where \( N(z) := \sum_{k=-\infty}^{+\infty} j\omega k z^k \).

Proof: Consider the harmonic Lyapunov equation (11). It is straightforward to show that the product \(-N^T P - P N = NP - PN\) is formed by \( n \times n \) blocks of Toeplitz matrices whose symbols for \( i, j := 1, \ldots, n \), are given by the Hadamard product \( N(z) \cdot P_{ij}(z) \) where \( P_{ij}(z) \) refers to the symbol associated the entry \((i,j)\) of the matrix \( P(t) \). Consequently, the symbol associated with \( NP - PN \) is

\[(\mathbb{I}_{n,n} \otimes N(z)) \cdot P(z)\]

where \( N(z) := \sum_{k=-\infty}^{+\infty} j\omega k z^k \). As it is assumed that \( A(t) \) is a real matrix, it implies in (1) that \( A_{ij} \) for \( i, j := 1, \ldots, n \), are Hermitian matrices, i.e., \( A^*_j = A_{ij} \), and thus, the symbols associated with \( A^* \) are given by the transpose of \( A(z) \). In view of this observation, replacing the Toeplitz matrix by its symbol ends the proof.

Looking at the previous symbol Lyapunov equation, we see that it is not possible to factorize \( P(z) \) to obtain a solution. In the following theorem, we show that if we try to solve a truncated version of (11), the resulting solution is not Toeplitz. As this Toeplitz property is required for the infinite-dimension case, an important practical consequence is the fact that the time counterpart \( P(t) \) does not exist and cannot be reconstructed using (10). For a better understanding, we show in the following theorem that the solution \( P_m \) obtained by solving the truncated harmonic Lyapunov equation differs from the solution of the infinite-dimensional harmonic Lyapunov equation by a correcting term \( \Delta P_m \).

Theorem 9: Consider finite-dimension Toeplitz matrices \( A_m := T_m(A) \) and \( Q_m := T_m(Q) \). The solution \( P_m \) of the Lyapunov equation

\[(A_m - N_m)^* P_m + P_m (A_m - N_m) + Q_m = 0\]

is given by \( P_m := P_m + \Delta P_m \) where \( P_m := T_m(P) \) with \( P(z) \) solution of (30) and \( \Delta P_m \) satisfies

\[(A_m - N_m)^* \Delta P_m + \Delta P_m = E^+ + E^- \]

with

\[E^+ := \mathcal{H}_{m,n}(A^*) \mathcal{H}_{n,m}(P^+) + \mathcal{H}_{m,n}(P^+) \mathcal{H}_{n,m}(A^*) \]
\[E^- := \mathcal{J}_{n,m}(\mathcal{H}_{m,n}(A^*) \mathcal{H}_{n,m}(P^+)) + \mathcal{H}_{m,n}(P^+) \mathcal{J}_{n,m}(A^*) \]

and \( 2n \geq \min d^2(\Lambda_1(z), \Lambda_2(z)) \).

Proof: The proof is obvious from Theorem 2 and noticing that \(-N^T P_m - P_m N_m \) does not give rise to a correction term since \( N \) is a diagonal matrix.

In practice, it is not clear how the Toeplitz part \( P_m \) of \( P_m \) can be extracted since the symbol \( P(z) \) is implicitly given by (30).
In fact, it can be shown that this linear problem is rank deficient and has infinitely many solutions. Thus, our aim is to prove that \( P \) can be determined up to an arbitrary small error. The necessity to determine \( P \) up to an arbitrary small error instead of \( P_m \) is crucial to prove stability of \((A - N)\). This is due to the fact that the matrix \((A_m - N_m)\) would never be Hurwitz for any \( m \) when \((A - N)\) is Hurwitz.

**Theorem 10:** Assume that \((A - N)\) is invertible. The phasor \( P := F(P) \) associated with the solution \( P := T(P) \) of the infinite-dimensional harmonic Lyapunov equation (11) is given by

\[
\text{col}(P) := - (\text{Id}_n \otimes (A - N)^* + \text{Id}_n \otimes A^*)^{-1} \text{col}(Q) \tag{32}
\]

where

\[
\text{Id}_n \otimes A := \begin{pmatrix}
\text{Id}_n \otimes A_{11} & \text{Id}_n \otimes A_{12} & \cdots & \text{Id}_n \otimes A_{1n} \\
\text{Id}_n \otimes A_{21} & \text{Id}_n \otimes A_{22} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
\text{Id}_n \otimes A_{n1} & \cdots & \cdots & \text{Id}_n \otimes A_{nn}
\end{pmatrix}
\tag{33}
\]

with \( \mathcal{N} \) given by (9) and where the matrix \( Q := F(Q) \).

**Proof:** Applying the well-known formula \((\text{Id}_n \otimes A + B^* \otimes \text{Id}_m)\text{col}(X) = \text{col}(C) \) associated with the Sylvester equation \( AX + XB = C \) to the case of the symbol Lyapunov equation (30), one gets

\[
(\text{Id}_n \otimes A(z)^{'} + A(z)^{'} \otimes \text{Id}_n)\text{col}(P(z)) + ((\text{Id}_n \otimes N(z)) \cdot \text{col}(P(z)) = -\text{col}(Q(z)).
\]

Notice that \( \text{col}((\text{Id}_n \otimes N(z)) \cdot \text{col}(P(z)) = \text{col}(\text{Id}_n \otimes N(z)) \cdot \text{col}(P(z)) = (\text{Id}_n \otimes N(z)) \cdot \text{col}(P(z)) \). Observe that the \( i \)th line, \( i = 1, \ldots, n^2 \), of this multinomial equation is given by

\[
\sum_{j=1}^{n^2} M_{ij}(z)P_j(z) + N(z) \cdot P_i(z) = -Q_i(z)
\]

where \( P_j(z), j = 1, \ldots, n^2 \) refers to the components of \( \text{col}(P(z)) \) and where the terms \( M_{ij}(z) := (\text{Id}_n \otimes A(z)^{'} + A(z)^{'} \otimes \text{Id}_n)_{ij} \) are determined from the expansions

\[
\text{Id}_n \otimes A(z)^{'} = \begin{pmatrix}
A(z) & 0 & 0 \\
0 & A(z) & 0 \\
0 & 0 & \ddots \\
0 & 0 & 0 & A(z)
\end{pmatrix}
\]

with \( A(z) \) provided by (2) and

\[
A(z)^{'} \otimes \text{Id}_n = \begin{pmatrix}
A_{11}(z)\text{Id}_n & A_{21}(z)\text{Id}_n & \cdots & A_{n1}(z)\text{Id}_n \\
A_{12}(z)\text{Id}_n & A_{22}(z)\text{Id}_n & \cdots & A_{n2}(z)\text{Id}_n \\
\vdots & \vdots & \ddots & \vdots \\
A_{1n}(z)\text{Id}_n & \cdots & \cdots & A_{nn}(z)\text{Id}_n
\end{pmatrix}
\]

Recall that \( \mathcal{N} := \text{Id}_n \otimes \text{diag}(j\omega k, k \in \mathbb{Z}) \). The symbol \((\text{Id}_n \otimes \mathcal{N})\) has coefficients corresponding to the matrix \( \text{Id}_n \otimes \mathcal{N} = -\text{Id}_n \otimes \mathcal{N}^* \). Replacing each symbol \( A_{ij}(z) \) in the abovementioned equation with their associated Toeplitz matrix leads to an equivalent equation involving the coefficients

\[
(\text{Id}_n \otimes (A - \mathcal{N}^*) + \text{Id}_n \otimes A^*)\text{col}(P) = -\text{col}(Q)
\]

where \( \text{Id}_n \otimes A \) is given by (33), \( P := F(P) \) and \( Q := F(Q) \). If \((A - \mathcal{N})\) is invertible, it follows necessarily that \((\text{Id}_n \otimes (A - \mathcal{N})^* + \text{Id}_n \otimes A^*)\) is also invertible, otherwise it contradicts the fact that the solution of the harmonic Lyapunov equation is uniquely defined.

We are now in position to state one of the main results of this article. To this end, define for any given \( m \) the \( m \)-truncated solution as

\[
\text{col}(\tilde{P}_m) := - (\text{Id}_n \otimes (A_m - N_m)^* + \text{Id}_n \otimes A_m^*)^{-1} \text{col}(Q_m)
\tag{34}
\]

with \( A_m := T_m(A), \) \( N_m := \text{Id}_n \otimes \text{diag}(j\omega k, |k| \leq m) \) and where \( \text{Id}_n \otimes A \) is defined by (33). The components of the \( m \)-truncated matrix \( Q_m \) are given by

\[
(Q_m)_{ij} := F_m(q_{ij}), \ i, j : = 1, \ldots, n
\]

with \( F_m(q_{ij}) \) the \( m \)-truncation of \( F(q_{ij}) \) obtained by suppressing all phasors of order \( |k| > m \).

**Theorem 11:** Assume that \( A(t) \in L^\infty([0\ T],\mathbb{R}^{n \times n}) \) and \((A - N)\) is invertible. For any given \( \epsilon > 0 \), there exists \( m_0 \) such that for any \( m \geq m_0 \)

\[
\|\text{col}(P - \tilde{P}_m)\|/\epsilon < \epsilon
\]

where \( P, \) given by (32), is the solution of the infinite-dimensional problem. Moreover

\[
\|P - \tilde{P}_m\|/\epsilon < \epsilon
\]

where \( \tilde{P}_m := T(\tilde{P}_m) \).

**Proof:** It is sufficient to prove the theorem for \( n = 1 \). In this case, \( A(z) = A_{11}(z) \) and \( A(z)^{'} = A(z) \). The symbol equation (30) reduces to: \( 2A(z)P(z) + N(z) \cdot P(z) + Q(z) = 0 \).

Now, observe that the \( k \)th coefficient of \( A(z)P(z) \) for \( k \in \mathbb{Z} \) is provided by

\[
(A(z)P(z))_k = \sum_{r \in \mathbb{Z}} A_{k-r}P_r
\]

while the \( k \)th coefficient of \( N(z) \cdot P(z) \) is given by

\[
(N(z) \cdot P(z))_k = j\omega k P_k.
\]

Thus, the symbol Lyapunov equation \( 2A(z)P(z) + N(z) \cdot P(z) + Q(z) = 0 \) can be rewritten equivalently by means to its coefficients as the infinite-dimensional linear system

\[
(2A - N)^* \textbf{P} = -\textbf{Q}
\tag{35}
\]

where \( \textbf{P} \) and \( \textbf{Q} \) are infinite vectors whose components are the coefficients of \( P(z) \) and \( Q(z) \) (or equivalently the phasors of their time counterpart \( P(t) \) and \( Q(t) \)). If an \( m \)-truncation on (35) is applied, we obtain

\[
(2A_m - N_m)^* \textbf{P}|_m = -\textbf{Q}|_m + E^* + E_m
\]

where the correcting term is given by \([\text{see (6) in Proposition 1}]\)

\[
E^*_m := -2H_{(m,\infty)}(A^*)\textbf{P}|_m
\]

\[
E_m := -2JH_{(m,\infty)}(A)\textbf{P}|_m
\]

with \( \textbf{P}|_m := (F_{m+1}, P_{m+2}, \ldots) \) and \( \textbf{P}|_m := (P_{-m-1}, P_{-m-2}, \ldots) \). As the operator \((A - \mathcal{N})\) is invertible, the matrix \( \Lambda \) in (25) is also invertible as well as \( 2A \). Consequently, the spectrum of \((2A - \mathcal{N})\) is given by \( \sigma(2A - \mathcal{N}) = \{2\lambda + j\omega k: k \in \mathbb{Z}\} \). Therefore, \((2A - \mathcal{N})\) is invertible as well as \((2A - \mathcal{N})^*\). Then, Corollary 4 implies that \((2A_m - N_m)^*\) is.
invertible for \( m \) sufficiently large. Now, define \( \tilde{P}_m \) the solution of the \( m \)-truncated problem by the following relation:

\[
\tilde{P}_m := -(2A_m - N_m)^{-1}Q_m.
\]

Therefore, we have

\[
\tilde{P}_m - P|_m = (2A_m - N_m)^{-1}(E_m^+ + E_m^-)
\]

with a \( \ell^2 \)-norm bounded by (see Theorem 3)

\[
\|\tilde{P}_m - P|_m\|_{\ell^2} \leq \|(2A_m - N_m)^{-1}\|_{\ell^2} \left(\|(E_m^+ + E_m^-)\|_{\ell^2}\right)
\]

As \( (2A_m - N_m)^{-1} \) is uniformly bounded (see Corollary 4), and as \( \|P\|_{\ell^2} = \|P\|_{\ell^2} \to 0 \) when \( m \to +\infty \), we conclude that for a given \( \epsilon > 0 \), there exists \( m_0 \) such that for any \( m \geq m_0 \),

\[
\|P - \tilde{P}_m\|_{\ell^2} < \epsilon/2 \quad \text{and} \quad \|P - P|_m\|_{\ell^2} < \epsilon/2
\]

as the phasors \( P_k \to 0 \) when \( k \to +\infty \). Finally, we obtain

\[
\|P - \tilde{P}_m\|_{\ell^2} < \epsilon
\]

assuming \( (\tilde{P}_m)_k := 0 \) for \( |k| > m \). To prove the last assertion, that is \( \|P - \tilde{P}_m\|_{\ell^2} < \epsilon \), notice that for \( n := 1 \), \( \text{col}(P - \tilde{P}_m) = P - \tilde{P}_m \). When \( n \geq 1 \), invoking similar steps as before yields \( \|\text{col}(P - \tilde{P}_m)\|_{\ell^2} < \epsilon \). The proof is completed invoking Proposition 3, for any \( n \).

**Remark 2:** Using the symbolic equation to derive the approximate solution leads us to a significant reduction of the computational burden since the linear problem defined by (34) is of dimension \( n(2m + 1) \) while the one defined by (31) is of dimension \( n^2(2m + 1)^2 \).

The following Corollary is interesting from a practical point of view in order to determine an accurate solution to the infinite harmonic Lyapunov equation from (34). Indeed, for a prescribed \( \epsilon > 0 \), it is sufficient to increase \( m \) in (34) until (36) is satisfied.

**Corollary 5:** For a given \( \epsilon > 0 \), there exists \( m_0 \) such that for any \( m \geq m_0 \), the symbol \( \tilde{P}_m(z) \) associated with \( P_m(z) \) satisfies

\[
|A(z)\tilde{P}_m(z) + \tilde{P}_m(z)A(z) + (1_n \otimes N(z)) \cdot \tilde{P}_m(z) + Q(z)||_{\ell^2} < \epsilon
\]

(36)

**Proof:** It is sufficient to prove the proof for \( n := 1 \). If we evaluate the symbolic equation with \( \tilde{P}_m(z) \), by construction of \( \tilde{P}_m(z) \), the result is given by

\[
A(z)\tilde{P}_m(z) + \tilde{P}_m(z)A(z) + (1_n \otimes N(z)) \cdot \tilde{P}_m(z) + Q(z)
\]

where

\[
E_m(z) := A(z)\tilde{P}_m(z) - (A(z)\tilde{P}_m(z))|_m - \tilde{P}_m(z)A(z) - (\tilde{P}_m(z)A(z))|_m
\]

is the product \((A(z)\tilde{P}_m(z))|_m\) the \( m \)-truncation of \((A(z)\tilde{P}_m(z))\). When \( n = 1 \), as \( A(z) = A' \) and as the coefficients of \( A(z) \) are complex scalar numbers, \( E(z) \) reduces to

\[
E_m(z) = 2(A(z)\tilde{P}_m(z) - (A(z)\tilde{P}_m(z))|_m).
\]

Thus, the nonzero coefficients of \( E_m(z) \) are of degree \( k \) for \( |k| > m \), and are given by the following equation (see (5), Prop. 1):

\[
E_m^+ := 2J_{\infty}\mathcal{H}_{(\infty,m)}(A^-)\tilde{P}_m
\]

\[
E_m^- := 2J_{\infty}\mathcal{H}_{(\infty,m)}(A^+)\tilde{P}_m
\]

Consider \( \tilde{m} := \frac{m}{2} \) (assuming \( m \) is an even number) and split \( J_{\infty}\mathcal{H}_{(\infty,m)}(A^-) \) as follows:

\[
J_{\infty}\mathcal{H}_{(\infty,m)}(A^-) = [M_1 \ M_2]
\]

where \( M_1 \) corresponds to the first \( \tilde{m} \) columns of \( J_{\infty}\mathcal{H}_{(\infty,m)}(A^-) \) and \( M_2 \) to its complement. Then, it follows

\[
\log_{10}|P_m(i, j) - P_m(i + 1, j + 1)|
\]

Fig. 4. \( \log_{10}|P_m(i, j) - P_m(i + 1, j + 1)| \). \( i, j := 1, \ldots, 2m \) of the Lyapunov solution \( P_m \) w.r.t. the truncation order \( m \). The axes are normalized.

Therefore, the \( \ell^2 \) norm satisfies

\[
\|M_2\|_{\ell^2} \leq \|\mathcal{H}(A_m^-)\|_{\ell^2}
\]

where \( A_m^- \) is the \( \tilde{m} \)-shifted symbol

\[
A_m^- := \sum_{k=1}^{\tilde{m}} A_{m-k} z^{-k}.
\]

Using Theorem 3, the following bounds can be established:

\[
\|M_1\|_{\ell^2} \leq 2 \|A\|_{L^\infty} \|\tilde{P}_1\|_{\ell^2}
\]

\[
\|M_2\|_{\ell^2} \leq 2 \|A\|_{L^\infty} \|\tilde{P}_2\|_{\ell^2}
\]

Since \( \|P_m - \tilde{P}_m\|_{\ell^2} \to 0 \) when \( m \to +\infty \) and since the phasors \( P_k \) of \( P \) vanish when \( k \to +\infty \), it follows that \( \|\tilde{P}_m\|_{\ell^2} = \left(\sum_{m=1}^{m/2} |\tilde{P}_m|^2\right)^{1/2} \to 0 \) when \( m \to +\infty \). On the other hand, we have \( \|\mathcal{H}(A_m^-)\|_{\ell^2} \to \tilde{m} \to +\infty \) since \( A_m^- \) \( \to 0 \). Therefore, for a given \( \epsilon > 0 \), there exists \( m_0 \) so that for \( m \geq m_0 \),

\[
\|E_m^-\|_{\ell^2} < \epsilon/2
\]

With similar steps, this is also the case for \( E_m^+ \) and the conclusion follows.

We illustrate the results of this section using a 1-dimensional example where the \( T \)-periodic state matrix is given by

\[
A(t) = -1 - \cos(t) + 2\sin(t) + \cos(2t) \quad (T = 2\pi)
\]

so that the associated symbol \( A(z) \) is given by

\[
A(z) = 2z^{-2} + (-1 + 2)z^{-1} + (-1 - 2)z + 2z^2
\]

and \( A_m \) is banded. Having fixed \( Q_m = \text{Id}_m \), if we attempt to solve the truncated harmonic Lyapunov equation (see Theorem 9), the obtained solution \( P_m \) is not Toeplitz. We show this in Fig. 4 where we evaluate
VI. SOLVING HARMONIC RICCATI EQUATIONS

Here, we combine the proposed algorithm for solving harmonic Lyapunov equations and the Kleinman algorithm [16] to solve harmonic Riccati equations. Recall that the Kleinman algorithm is a Newton-based algorithm that allows us to determine iteratively the unique positive definite solution of a standard algebraic Riccati equation.

Consider $T$-periodic symmetric positive definite matrix functions $R$ and $Q$ of $L^\infty$ class. Under the assumption that there exists $\eta > 0$ such that the set $\{ t : |\det(R(t))| < \eta \}$ is of zero measure, it is proved in [6] that $P$ is the unique $T$-periodic symmetric positive definite solution of the periodic Riccati differential equation

$$\dot{P}(t) + A'(t)P(t) + P(t)A(t) - P(t)BR^{-1}(t)B(t)'P(t) + Q(t) = 0$$

if and only if the matrix $P = T(P)$ is the unique hermitian and positive definite solution of the algebraic Riccati equation:

$$(A - N)'P + P(A - N) - PBR^{-1}B^*P + Q = 0 \quad (37)$$

where $Q = T(Q)$ is hermitian positive definite. Moreover, $P$ is a bounded operator on $\ell^2$.

Before generalizing the Kleinman algorithm to harmonic Riccati equations, we introduce the symbol Riccati equation and provide a link between a solution of a harmonic Riccati equation and a solution of the associated harmonic Lyapunov equation.

**Proposition 5:** $\mathcal{P}$ satisfies (37) if and only if $P(z)$ satisfies the symbol Riccati equation

$$A'(z)P(z) + P(z)A(z) + (\mathbb{1}_{n \times n} \otimes N(z)) \cdot P(z) - P(z)B(z)R^{-1}(z)B'(z)P(z) + Q(z) = 0 \quad (38)$$

where the operator $P(z)$ is bounded on $\ell^2$.

**Proof:** The result is obtained using similar steps to those in the proof of Proposition 4.

**Theorem 12:** Consider the harmonic Riccati equation (37) with $A(t) \in L^\infty([0, T], \mathbb{R}^{n \times n})$. Let $\mathcal{K} := -R^{-1}B^*P$, $\mathcal{Y} := K^*RK$, and $\mathcal{Y} := \mathcal{F}(\mathcal{Y})$ with $\mathcal{Y} := T^{-1}(\mathcal{Y})$. If $P := T(P)$ is solution of (37) then $\mathcal{P} := \mathcal{F}(P)$ satisfies

$$\text{col}(\mathcal{P}) = -M^{-1}\text{col}(Q + \mathcal{Y})$$

where $M := \mathbb{1}_{n \times n} \otimes (A - BK - N) + \mathbb{1}_{n \times n} \circ (A - BK)^*$ with $N$ defined by (9) and $Q := \mathcal{F}(Q)$.

**Proof:** If $P := T(P)$ is the solution of (37) then it is the unique solution of the Lyapunov equation

$$(A - BK - N)'P + P(A - BK - N) + Q + \mathcal{Y} = 0. \quad (39)$$

The result follows applying Theorem 10 to (39).

In the following theorem, we provide the algorithm to solve the infinite-dimensional harmonic Riccati equation (37) up to an arbitrary small error.

**Theorem 13:** Assume that $A(t) \in L^\infty([0, T], \mathbb{R}^{n \times n})$. For $k := 0, 1, 2, \ldots$ and for a sufficiently large $m_k := m(k)$, define $S_{m_k}(k)$ by

$$\text{col}(S_{m_k}(k)) := -M(k)^{-1}\text{col}(Q|_{m_k} + \mathcal{Y}(k)|_{m_k}) \quad (40)$$
the $m_k$-truncated unique solution of the algebraic Lyapunov equation
\[
(A(k) - N')^* S(k) + S(k)(A(k) - N') + Y(k) + Q = 0
\]
with $Y(k) := \mathcal{F}(Y(k)), Y(k) := T^{-1}(Y(k))$ and $Y(k)$ defined by
\[
Y(k) = K(k)^* RRK(k)
\]
and where $M(k) := I_d \otimes (A_m(k) - N_m(k))^* + I_d \circ A_m(k), A(k)$, its $m_k$-truncation $A_m(k)$ and $Y(k)$ are determined recursively by the symbols
\[
K(k)(z) := R^{-1}(z)B(z)^* S_{k-1}(z)
\]
\[
A_k(z) := A(z) - B(z)K_k(z)
\]
\[
Y_k(z) := K_k(z)^* R(z)K_k(z)
\]
in which $S_{k-1}(z)$ denotes the symbol associated with $S_{m_k-1}(k-1)$. Moreover, $K_0(z)$ is chosen such that the matrix $A(0) - N := A - BK_0 - N$ is Hurwitz.

Then, for $\epsilon > 0$ sufficiently small, if $m_k$ is chosen sufficiently large at each step, we have the following:

1. $\|S(k) - S_{m_k}(k)\|_{\ell^2} < \epsilon$ where $S(k)$ is the exact positive definite solution of (41) and $S_{m_k}(k) := T(F^{-1}(S_{m_k}(k)))$ with $S_{m_k}(k)$ given by (40);
2. $\mathcal{P} \leq S(k+1) \leq S(k) \leq \cdots \leq S(0)$ where $\mathcal{P}$ solves (37);
3. $S_{m_k}(k) > 0$ for any $k = 0, 1, \ldots$;
4. $\lim_{k \to +\infty} S_{m_k}(k) = \bar{S}$ with $\bar{m} = \lim_{k \to +\infty} m_k < +\infty$;
5. $\|S_{\infty} - \bar{S}\|_{\ell^2} \leq \epsilon$ where $S_{\infty} := \lim_{k \to +\infty} S(k)$ satisfies (37) with an error in $\ell^2$ norm given by
\[
\|A(0) - N')^* S_{\infty} + S_{\infty}(A(0) - N') - S_{\infty} BR^{-1}B^* S_{\infty} + \bar{Q}\|_{\ell^2} \leq \eta^2
\]
and $\eta := \|B(t)R^{-1}(t)B(t)^*\|_{L^\infty}$.

**Proof:** We use Theorem 11 in this proof as the related assumptions satisfied. For a given $\epsilon > 0$ and for $k := 0$, we have from (40), $S_{m_0}(0) := T(F^{-1}(S_{m_0}(0)))$, which differs from the exact solution $S(0)$ of (41) by
\[
\|S(0) - S_{m_0}(0)\|_{\ell^2} \leq \epsilon
\]
provided that $m_0$ is a sufficiently large number. Thus, $S(0)$ is positive definite, so is $S_{m_0}(0)$ provided that $\epsilon$ is small enough. Recall that if $A - BK - N$ is Hurwitz, the solution of (41) is provided by
\[
S_{k} := \int_0^{+\infty} e^{(A - BK - N)\tau}(Q + K'RK) e^{(A - BK - N)\tau} d\tau.
\]

Set $k := 1$ and consider $S_1$ the bounded operator on $\ell^2$ solution of (41) obtained with $K_1 := R^{-1}B^* S_{m_0}(0)$ and $A(1) := A - BK_1 - N$. Note that $S_1$ is well defined since $A - BK_1$ and $Y(1) + Q$ are bounded operators on $\ell^2$ (or equivalently $T^{-1}(A - BK_1)$ and $T^{-1}(Y(1) + Q)$ are $L^\infty$). Using similar steps as in the proof of [16], it can be established that
\[
S(0) - S(1) = \int_0^{+\infty} e^{A(1)\tau}(K(0) - K_1)^* R(K(0) - K_1) e^{A(1)\tau} d\tau \geq 0
\]
\[
S(1) - \mathcal{P} = \int_0^{+\infty} e^{A(1)\tau}(K(1) - K_1)^* R(K(1) - K) e^{A(1)\tau} d\tau \geq 0
\]
where $\mathcal{P}$ is the solution of the Riccati equation (37). Therefore, $0 < \mathcal{P} \leq S(1) \leq S(0)$ which proves that $A(1) := A - BK_1 - N$ is Hurwitz. Using Theorem 11, the approximated solution $\bar{S}_{m_1}(1) := T(F^{-1}(S_{m_1}(1)))$ where $S_{m_1}(1)$ is determined by (40), differs from $S(1)$ by $\|S(1) - \bar{S}_{m_1}(1)\|_{\ell^2} < \epsilon$ provided that $m_1$ is a sufficiently large number. Hence, $\bar{S}_{m_1}(1)$ is positive definite provided that $\epsilon$ is small enough.

Repeating, for $k := 2, 3, \ldots$ the abovementioned arguments, one gets:
1. $\mathcal{P} \leq S(k+1) \leq S(k) \leq \cdots \leq S(0)$;
2. $\|S(k) - \bar{S}_{m_k}(k)\|_{\ell^2} < \epsilon$;
3. for any $k, A(k) := A - BK_k - N$ is Hurwitz.

Recall that for any $k, S(k)$ are bounded operators on $\ell^2$. Using monotonic convergence of positive operators, it follows that $S_{\infty} := \lim_{k \to +\infty} S(k)$ exists with $S_{\infty}$ a bounded operator on $\ell^2$ satisfying
\[
(A - BK_{\infty} - N')^* S_{\infty} + S_{\infty}(A - BK_{\infty} - N') - S_{\infty} BR^{-1}B^* S_{\infty} + \bar{Q}\|_{\ell^2} \leq \eta^2
\]
and $\eta := \|B(t)R^{-1}(t)B(t)^*\|_{L^\infty}$. As by construction $\|K_{\infty} - \bar{K}_{\infty}\|_{\ell^2} \leq \|S_{\infty} - \bar{S}\|_{\ell^2} \leq \epsilon$ where $\epsilon := \|R^{-1}B'\|_{\ell^\infty}$ (see Theorem 3) and as $S_{\infty}$ and $K_{\infty}$ are bounded operators on $\ell^2$, we have necessarily that $S_{\infty}$ and $K_{\infty}$ are also bounded on $\ell^2$. It follows that $\bar{m}$ is a finite number. Indeed, as $S_{\infty}$ solves (43) and as the assumptions of Theorem 11 are satisfied, there exists a finite $m$ such that $\|S_{\infty} - \bar{S}\|_{\ell^2} \leq \epsilon$. Consequently, $\bar{m}$ is finite.

Now, taking the $\ell^2$-norm, we get
\[
\|A(0) - N')^* S_{\infty} + S_{\infty}(A(0) - N') - S_{\infty} BR^{-1}B^* S_{\infty} + \bar{Q}\|_{\ell^2} \leq \eta\|\bar{S}_{\infty} - S_{\infty}\|_{\ell^2}^2
\]
where $\eta$ is such that $\|BR^{-1}B'\|_{\ell^\infty} = \|B(t)R^{-1}(t)B(t)^*\|_{L^\infty} < \eta$ (see Theorem 3). We have by construction $\|\bar{S}_{\infty} - S_{\infty}\|_{\ell^2} \leq \epsilon$ and the conclusion follows.

This theorem shows that the algorithm returns a solution $\bar{S}_{\infty}$ that approximates in $\ell^2$-norm operator sense the solution of the algebraic harmonic Riccati equation (37) and this approximation is characterized by (42).

**Remark 3:** The choice of $m_k$ at each step $k$ must be sufficiently large to guarantee that $\|S(k) - S_{m_k}(k)\|_{\ell^2} < \epsilon$. This can be achieved by checking at each step a similar condition to the one provided in Corollary 5 using the symbol equation (38). Moreover, the algorithm requires an initial step where the initial gain must be chosen such that $A_0 - N := A - BK_0 - N$ is Hurwitz. This is not a major problem as one can use the pole placement algorithm proposed in [21] to design a stabilizing $K_0$.

**Remark 4:** Compared to [30] where an algorithm based on the iterative solution of the Lyapunov equation is proposed to
solve harmonic Riccati equations, the algorithm of Theorem 13 is more general as the matrices $A(t)$ and $B(t)$ belong to $L^\infty$. Moreover, it is assumed in [30] that $\mathcal{A} - \mathcal{N}$ is Hurwitz which is not the case here. Our algorithm applies to unstable harmonic matrices $\mathcal{A} - \mathcal{N}$ and it is also numerically more efficient with a significant reduction of the computational burden due to Theorem 11.

VII. HARMONIC LQ CONTROL DESIGN

Consider an LTP system defined by

$$
\dot{x} = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} x + \begin{pmatrix} b_{11}(t) \\ 0 \end{pmatrix} u
$$

(44)

where

$$
a_{11}(t) := 1 + \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin(\omega(2k+1)t)
$$

$$
a_{12}(t) := 2 + \frac{16}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos(\omega(2k+1)t)
$$

$$
a_{21}(t) := -1 + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sin \left( \omega kt + \frac{\pi}{4} \right)
$$

$$
a_{22}(t) := -2 \sin(2\omega t) - 2 \sin(6\omega t)
$$

$$
+ 2 \cos(6\omega t) + 2 \cos(10\omega t)
$$

$$
b_{11}(t) := 1 + 2 \cos(2\omega t) + 4 \sin(6\omega t) \text{ with } \omega := 2\pi.
$$

Observe that $a_{11}, a_{12},$ and $a_{21}$ are, respectively, square, triangular, and sawtooth signals and include an offset part. The associated Toeplitz matrix has an infinite number of phasors and is not banded. Moreover, this LTP system is unstable. The eigenvalues set is characterized by $\lambda = \{1 \pm j 1.64\}$. Let $Q := T(100I_{n})$ and $R := T(Id_{m})$. We want to solve the associated Harmonic Riccati Equation using an $m$-truncation. We perform the study with $m \in \{8, 16, 32, 64\}$.

Solving the $m$-truncated version of (37) gives a matrix $P_m$, which is not Toeplitz, as shown in Fig. 8. The associated gain $K_m$ is shown on Fig. 9. We observe that, for $m$ sufficiently large, the problem is mainly located at the upper left corner and lower right corner for both $P_m$ and $K_m$. As expected, increasing $m$ does not lead to a Toeplitz solution. Now, we apply our algorithm to obtain an approximation of the infinite-dimensional Toeplitz solution. The correcting term is shown on Figs. 10 and 11. When $m$ is chosen sufficiently large, we see that the correction terms for $P_m$ and $K_m$ are mainly located at upper left and lower right corners of the corresponding $n \times n$ and $m \times n$ blocks.

Looking at the phasors of the harmonic gain matrix $K_m := R^{-1}B^T S_m$ computed with a fixed $m_k := m$ (we do not adapt $m_k$ at each step $k$ as described in Theorem 13) and plotted in Fig. 12 w.r.t. $m$, we see that the obtained values converge and they vanish from $m \geq 32$. The significant values are obtained for small values of $m$. To show the effectiveness of the proposed approach, we consider an equilibrium of the harmonic system defined by

$$
0 = (\mathcal{A} - \mathcal{N}) X_{\text{ref}} + BU_{\text{ref}}
$$

(45)
and use (10) to reconstruct the associated $T$-periodic trajectory $x_{\text{ref}} = F^{-1}(X_{\text{ref}})$ and control $u_{\text{ref}} = F^{-1}(U_{\text{ref}})$. The control
\[
 u(t) := -K(t)(x(t) - x_{\text{ref}}(t)) + u_{\text{ref}}(t)
\]
where $K(t)$ is the $T$-periodic gain matrix given by $K(t) := \sum_{k=-m}^{m} K_k e^{j\omega_k t}$, stabilizes globally and asymptotically the unstable LTP system (44) on any $T$-periodic trajectory $x_{\text{ref}}(t) = F^{-1}(X_{\text{ref}})$ and $u_{\text{ref}}(t) = F^{-1}(U_{\text{ref}})$. To illustrate this, we plot on Fig. 13, the closed-loop response for three $T$-periodic reference trajectories. We start by $u_{\text{ref}}(t) = 0$, for $t < 4$, then $u_{\text{ref}} = 1 + \cos(2\pi t)$ for $4 \leq t < 8$ and for $t \geq 8$, we consider a desired steady state $X_d$ given by $F^{-1}(X_d)(t) := \left(\frac{1}{2}\cos(2\pi t), 0\right)$ and look for the nearest harmonic equilibrium, solution of the minimization problem $\min_{X_{\text{ref}}} ||X_d - X_{\text{ref}}||^2$, subject to (45). Clearly it can be observed that the provided state feedback allows us to track any $T$-periodic trajectory corresponding to any equilibrium of (45).

From a practical point of view, we see on this example that once a good approximation of the harmonic Riccati equation has been obtained for a sufficiently large $m$, a few number of coefficients $m_0 \leq m$ are needed to reconstruct the matrix gain $K(t) := \sum_{k=-m_0}^{m_0} K_k e^{j\omega_k t}$. This can be explained by the fact that the phasor gain modules vanish relatively quickly and can be approximated by $K_k \approx O\left(\frac{1}{k}\right)$ since $K(t) \in L^\infty$ (see Fig. 12).

**VIII. Conclusion**

In this article, a simple closed-form formula for a Floquet factorization in the general case of $L^2$ matrix functions as well as a detailed spectrum characterization of harmonic state-space operators and their $m$-truncations are provided. For any $m$, it has been proved that the spectrum of the truncated harmonic matrix may contain a part that does not converge to the spectrum of the original infinite-dimensional harmonic matrix. We built upon this analysis efficient solutions to solve infinite-dimensional harmonic Lyapunov and Riccati equations up to an arbitrarily small error. We recover the infinite-dimensional solution from a sequence of finite-dimensional problems and the computational burden is reduced from $n^2(2m+1)^2$ to $n(2m+1)$ where $n$ is the dimension of the LTP system and $m$ is the order of the truncation. Moreover, the design of a control law using the methodology presented in this article is not affected by truncation. The results are established invoking the symbolic equation. This methodology is illustrated on the design of a harmonic $LQ$ control with periodic trajectories tracking.

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