On the well-posedness of the Cauchy problem for Fokker–Planck–Kolmogorov equations with potential terms on arbitrary domains

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We study the Cauchy problem for Fokker–Planck–Kolmogorov equations with unbounded and degenerate coefficients. Sufficient conditions for the existence and uniqueness of solutions are indicated.

Keywords Fokker–Planck–Kolmogorov equation; Cauchy problem; diffusion process

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1. Introduction

In this paper we study the Cauchy problem for the Fokker–Planck–Kolmogorov equation

\[ \partial_t \mu = \partial_x \partial_{x_j} \left( a^{ij} \mu \right) - \partial_{x_i} \left( b^i \mu \right) + c \mu, \quad \mu|_{t=0} = \nu. \tag{1.1} \]

Throughout the paper summation over all repeated indices is meant.

Equations of this type for transition probabilities of diffusion processes were first derived by Kolmogorov in his famous paper [14]. In the same paper, the question about the existence and uniqueness of probability solutions was posed (the case \( c = 0 \)). The classical works [1, 11, 21, 22, 23] deal with such equations with smooth coefficients, having at most linear growth at infinity.

Equations with integrable and Sobolev coefficients in the class of bounded Borel measures have been intensively studied in the past decade. For the variational approach to (1.1) in the case of unit diffusion matrix, a gradient drift and \( c = 0 \), see [13]. The existence and uniqueness of solutions given by flows of probability measures in the case where \( c = 0 \), the diffusion matrix is nondegenerate and Sobolev regular and the drift is integrable have been studied in [3, 4, 5, 8]. The papers [10, 17, 18] deal with equations with degenerate diffusion matrices. In particular, the solvability of the Cauchy problem for equations with a degenerate Sobolev regular \( A \) in the class of densities under certain growth restrictions on the lower order terms has been proved in [17]. Relations between the \( L^1 \)- and \( L^\infty \)-uniqueness of semigroups, Liouville-type theorems and the uniqueness of the \( L^1 \)-solution to the Cauchy problem for the Fokker–Planck–Kolmogorov equation have been studied in [10, 24].

We note that in the papers mentioned above only equations on all of \( \mathbb{R}^d \) have been considered. However, the problem of existence and uniqueness of solutions to the Cauchy problem for the Fokker–Planck–Kolmogorov equation with irregular coefficients on an arbitrary domain \( D \) is also of great interest. For example in [12], diffusion processes on an arbitrary domain \( D \subset \mathbb{R}^d \) were studied, in particular, the process on \( D = (-1, 1) \) with generator

\[ Lu(x) = 2^{-\alpha} |1 - |x||^{\alpha} u''(x) + \left( \frac{\tan(-\pi x/2) + \text{sgn} x}{\pi} \right) u'(x), \quad \alpha > 0. \]

The main difference between our results and the known ones is that we consider equations with potential terms on arbitrary domains with an arbitrary probability measure as the initial data, but even for \( c = 0 \) and \( D = \mathbb{R}^d \) our results are new. We extend the sufficient conditions for the existence of solutions obtained in [3] to the case of nondegenerate equations without restrictions on the smoothness of the diffusion matrix under the assumption that the drift and potential are locally bounded. Also, we impose no global restrictions on the coefficients to prove the existence. In the case \( c = 0 \), our method of constructing solutions differs from the ones used in the papers mentioned above, namely, first we construct a subprobability solution (this step is usually much easier) and then employ a Lyapunov function to ensure that the constructed solution is a probability solution.

Our uniqueness results are mostly extensions of the results in [8] to the case of equations with potential terms on arbitrary domains. However, an important difference is that we have managed to eliminate the assumption (which was crucial in the paper cited) that the Lyapunov functions involved are globally Lipschitzian.
To be more specific, we prove that under rather broad assumptions about coefficients \(a^{ij}, b^i,\) and \(c,\) the existence of a Lyapunov function \(V\) (i.e., \(V \in C^2(D)\) and \(V(x) \to +\infty\) as \(x \to \partial D\)) such that  
\[ a^{ij}(x,t)\partial_x x_i \partial_x x_j V(x) + b^i(x,t)\partial_x x_i V(x) + c(x,t)V(x) \leq K + KV(x), \quad K > 0 \]
ensures both the existence and uniqueness of a solution to the Cauchy problem (1.1) given by a flow of subprobability measures \(\mu_t\) such that the identity  
\[ \mu_t(D) = \nu(D) + \int_0^t \int_D c(x,s) \, d\mu_s \, ds \]
holds. If \(c = 0,\) then the measures \(\mu_t\) are probability measures. In particular, if \(D = \mathbb{R}^d\) and \(V(x) = |x|^2/2,\) then the existence and uniqueness is enough to have the inequality  
\[ \text{tr} A(x,t) + (b(x,t), x) + |x|^2 c(x,t)/2 \leq K + K|x|^2. \]

Thus, in the case of equations with unbounded coefficients we give an answer to the question about the existence and uniqueness of solutions to the Cauchy problem (1.1) posed by Kolmogorov in [14].

We now proceed to the definitions and exact statements.

Let \(T > 0\) and let \(D\) be an arbitrary open set in \(\mathbb{R}^d.\) We assume that along with the domain \(D\) an increasing sequence of bounded open sets \(D_k\) is given such that for every \(k\) the closure \(\overline{D_k}\) of \(D_k\) belongs to \(D_{k+1}\) and \(\bigcup_{k=1}^{\infty} D_k = D.\) For example, if \(D = \mathbb{R}^d,\) then for \(D_k\) the ball of radius \(k\) centered at the origin can be taken.

We shall say that a locally finite Borel measure \(\mu\) on the strip \(D \times (0,T)\) is given by a flow of Borel measures \((\mu_t)_{t \in (0,T)}\) if, for every Borel set \(B \subset D,\) the mapping \(t \mapsto \mu_t(B)\) is measurable and for every function \(u \in C_0^\infty(D \times (0,T))\) one has  
\[ \int_{D \times (0,T)} u(x,t) \, d\mu = \int_0^T \int_D u(x,t) \, d\mu_t \, dt. \]

Obviously, the last identity extends to all functions of the form \(fu,\) where \(u\) is as before and \(f\) is \(\mu\)-integrable on every compact set in \((0,T).\) For example, the transition probabilities \(\mu_t(B) = P(x_t \in B)\) of a stochastic process \(x_t\) in \(D\) define a measure \(\mu = \mu_t \, dt\) on \(D \times (0,T).\)

Set  
\[ L \varphi = a^{ij} \partial_{x_i} \partial_{x_j} \varphi + b^i \partial_{x_i} \varphi + c \varphi, \]
where \(a^{ij}, b^i, c\) are Borel functions on \(D \times [0,T]\) and \(A = (a^{ij})\) is a symmetric non-negative definite matrix (called the diffusion matrix), i.e., \(a^{ij} = a^{ji},\) \((A(x,t)y,y) \geq 0\) for all \((x,t) \in D \times [0,T]\) and all \(y \in \mathbb{R}^d.\) The mapping \(b\) is called the drift coefficient and \(c\) is called the potential.

We shall say that a measure \(\mu = (\mu_t)_{t \in (0,T)}\) satisfies the Cauchy problem (1.1) if \(a^{ij}, b^i\) and \(c\) belong to \(L^1(\overline{D_k} \times J, |\mu|)\) for each domain \(D_k\) and each interval \(J \subset (0,T)\) and for every function \(\varphi \in C_0^\infty(D)\) the following identity holds:  
\[ \int_D \varphi \, d\mu_t - \int_D \varphi \, d\nu = \lim_{\varepsilon \to 0^+} \int_\varepsilon^t \int_D L \varphi \, d\mu_s \, ds \]
for a.e. \(t \in (0,T).\) We note that in general the set of points \(t\) for which the identity (1.2) holds depends on \(\varphi.\) If the function \(t \mapsto \int_D \varphi \, d\mu_t\) is continuous on \((0,T),\) then identity (1.2) holds for all \(t \in [0,T).\) If one has the inclusion \(L \varphi \in L^1(D \times [0,T])\), then  
\[ \lim_{\varepsilon \to 0^+} \int_\varepsilon^t \int_D L \varphi \, d\mu_s \, ds = \int_0^t \int_D L \varphi \, d\mu_s \, ds. \]

We shall also use another definition of a solution, which is, however, equivalent to the previous one (see [8]). Namely, the measure \(\mu = (\mu_t)_{0 \leq t < T}\) satisfies \(\partial_t \mu = L^* \mu\) if  
\[ \int_0^T \int_D [\partial_t u + Lu] \, d\mu_t \, dt = 0 \quad \forall u \in C_0^\infty(D \times (0,T)). \]

The measure \(\mu = (\mu_t)_{0 \leq t < T}\) satisfies the initial condition \(\mu_t|_{t=0} = \nu\) if, for each function \(\varphi \in C_0^\infty(D),\) there exists a full Lebesgue measure set \(J_\varphi \subset (0,T)\) such that  
\[ \lim_{t \to 0^+, t \in J_\varphi} \int_D \varphi \, d\mu_t = \int_D \varphi \, d\nu. \]

As before, if the function \(t \mapsto \int_D \varphi \, d\mu_t\) is continuous on \((0,T),\) then \(J_\varphi = (0,T).\)

We shall always assume that \(c \leq 0.\) This assumption can obviously be replaced with \(c \leq c_0\) for some number \(c_0.\) Indeed, in order to remove \(c_0\) it suffices to consider \(e^{-c_0 t} \mu_t\) in place of \(\mu_t.\)
We study the existence and uniqueness in the class $M_\nu$ of measures $\mu$ given by flows of such nonnegative measures $(\mu_t)_{0 \leq t < T}$ that $\mu$ is a solution to the Cauchy problem \[1\] on $L^1(D \times (0, T), \mu)$ and for a.e. $t \in (0, T)$ the inequality
\[
\mu_t(D) \leq \nu(D) + \int_0^t \int_D c(x, s) \mu_s(dx) \, ds
\] holds.

The main goal of this paper is to obtain sufficient conditions, under which the set $M_\nu$ consists of exactly one element. Moreover, we are interested in conditions that admit unbounded and degenerate coefficients.

2. Existence results

In the present section we prove several existence theorems under different assumptions about the coefficients.

**Theorem 2.1.** Suppose that $c \leq 0$, for each $k \in \mathbb{N}$ the coefficients $a^{ij}$, $b^i$ and $c$ are bounded on $D_k \times [0, T]$ and there exist positive numbers $m_k$ and $M_k$ such that the inequality
\[
m_k |y|^2 \leq (A(x, t)y, y) \leq M_k |y|^2
\] holds for all $y \in \mathbb{R}^d$ and $(x, t) \in D_k \times [0, T]$. Then, for every probability measure $\nu$, the set $M_\nu$ is non-empty.

**Proof.** We divide the proof into several steps.

1. We set $a^{ij}(x, t) = 0$, $b^i(x, t) = 0$ and $c(x, t) = 0$ if $t \notin [0, T]$ or if $t \in [0, T]$, but $x \notin D$. Let $\omega$ be a homogenization kernel, i.e.,
\[
\omega \in C_0^\infty(\mathbb{R}^{d+1}), \quad \omega \geq 0, \quad \int_{\mathbb{R}^{d+1}} \omega(x, t) \, dx \, dt = 1.
\]

Set $\omega(x, t) = \varepsilon^{-d-1} \omega(x \varepsilon^{-1}, t \varepsilon^{-1})$. Let $I_n$ be the indicator of $D_n \times [0, T]$. Let
\[
a^{ij}_n = (a^{ij}I_n + \delta^{ij}(1 - I_n)) \ast \omega_{1/n}, \quad b^i_n = (b^i I_n) \ast \omega_{1/n}, \quad \text{and} \quad c_n = (c I_n) \ast \omega_{1/n}.
\]

It is quite obvious that for every fixed $n$ the functions $a^{ij}_n$, $b^i_n$, and $c_n$ are smooth and uniformly bounded together with all their derivatives. Moreover, $(A_n(x, t)y, y) \geq |y|^2 \min \{m_n, 1\}$ for all $x$ and $y$, here $m_n$ is a number from the assumptions of the theorem corresponding to the set $D_n \times [0, T]$. Finally, for every $k$ and $p \geq 1$ the sequences $a^{ij}_n$, $b^i_n$, $c_n$ converge in $L^p(D_k \times [0, T])$ to the functions $a^{ij}$, $b^i$ and $c$ respectively.

We extend the measure $\nu$ by zero outside $D$ to a measure on $\mathbb{R}^d$. Let $\eta_n \in C_0^\infty(D)$ be a sequence of non-negative functions such that $\eta_n \, dx$ are probability measures on $\mathbb{R}^d$ weakly convergent to $\nu$.

On $\mathbb{R}^d \times [0, T]$ we consider the Cauchy problem
\[
\partial_t u_n = \partial_x \partial_x a^{ij}_n u_n - \partial_x (b^i_n u_n) + c_n u_n, \quad u_n|_{t=0} = \eta_n.
\]

Let us rewrite it as
\[
\partial_t u_n = a^{ij}_n \partial_x a^{ij}_n u_n + (2 \partial_x a^{ij}_n - b^i_n) \partial_x u_n + (\partial_x a^{ij}_n - b^i_n + q_n) u_n, \quad u_n|_{t=0} = \eta_n.
\]

Here all the coefficients are smooth and bounded for each $n$. It is well-known (see [1], Chapter 1, §7, Theorem 12) that there exists a smooth bounded non-negative classical solution $u_n \in L^1(\mathbb{R}^d \times [0, T])$. Further, a classical solution $\{u_n\}$ is also a weak solution in sense of distributions, thus for every function $\psi \in C_0^\infty(\mathbb{R}^d)$ one has
\[
\int_{\mathbb{R}^d} \psi(x) u_n(x, t) \, dx = \int_{\mathbb{R}^d} \psi(x) \eta_n(x) \, dx + \int_0^t \int_{\mathbb{R}^d} L_n(x, s) \psi u_n(x, s) \, dx \, ds.
\]

Let $\zeta \in C_0^\infty(\mathbb{R}^d)$ be such that $\zeta(x) = 1$ if $|x| \leq 1$, $\zeta(x) = 0$ if $|x| > 2$, $|\zeta| \leq 1$ and it has two bounded derivatives. Substitute $\psi(x) = \zeta(x/N)$ in the equality \[2\] and let $N$ go to infinity. For every fixed $n$ the functions $a^{ij}_n$, $b^i_n$, $c_n$ are globally bounded, thus Lebesgue’s dominated theorem yields
\[
\int_{\mathbb{R}^d} u_n(x, t) \, dx = \int_{\mathbb{R}^d} \eta_n(x) \, dx + \int_0^t \int_{\mathbb{R}^d} c_n(x, s) u_n(x, s) \, dx \, ds.
\]

In particular, the measures $u_n(x, t) \, dx$ are subprobability measures for all $t \in [0, T]$. Taking into account that $\eta_n = 0$ outside $D$ and $c_n \leq 0$, we obtain
\[
\int_D u_n(x, t) \, dx \leq \int_D \eta_n(x) \, dx + \int_0^t \int_D c_n(x, s) u_n(x, s) \, dx \, ds.
\]
We observe that for fixed $k$ and sufficiently large $n$ the estimate $(A_n(x, t)y, y) \geq \|y\|^2 \min\{m_k+1, 1\}$ holds for all $(x, t) \in D_k \times [0, T]$ and $y \in \mathbb{R}^d$. Indeed,
\[
(a^ij_{I_n} + (1 - I_n)\delta^{ij}) * \omega_{1/n}(x, t) = (a^ij_{I_{k+1}} + (1 - I_{k+1})\delta^{ij}) * \omega_{1/n}(x, t)
\]
if $(x, t) \in D_k \times [0, T]$ and $n$ is large enough so that $\sup(y, \tau) \mapsto \omega_{1/n}(y - x, \tau - t) \subset D_{k+1} \times (-1, T + 1)$. Similarly, for fixed $n$ (large enough) and $k$ one has
\[
\|a_{n}^ij\|_{L^\infty(D_k \times [0, T])} \leq \|a\|_{L^\infty(D_{k+1} \times [0, T])} + 1, \quad \|b_{n}^k\|_{L^\infty(D_k \times [0, T])} \leq \|b\|_{L^\infty(D_{k+1} \times [0, T])},
\]
and
\[
\|c_{n}\|_{L^\infty(D_k \times [0, T])} \leq \|c\|_{L^\infty(D_{k+1} \times [0, T])}.
\]
Due to [3, Corollary 3.2], for every $k > 2$ one has
\[
\int_{D_k \times [Tk^{-1}, T(1-k^{-1})]} u_n^{(d+1)/d} \, dx \, dt \leq C_k,
\]
where $C_k$ depends only on $m_{k+1}$, $\|a\|_{L^\infty(D_{k+1} \times [0, T])}$, $\|b\|_{L^\infty(D_{k+1} \times [0, T])}$, $\|c\|_{L^\infty(D_{k+1} \times [0, T])}$ and does not depend on $n$.

Since the unit ball in $L^{(d+1)/d}$ is weakly compact, for every $k > 2$ one can extract from $\{u_n\}$ a subsequence weakly convergent in $L^{(d+1)/d}(D_k \times [Tk^{-1}, T(1-k^{-1})])$. Without loss of generality, using the diagonal procedure, we can assume that $\{u_n\}$ converges weakly to a non-negative function $u$ that belongs to $L^{(d+1)/d}(D_k \times [Tk^{-1}, T(1-k^{-1})])$ for every $k$.

3. As we have shown above, for every $k$ the coefficients $a_{n}^ij$, $b_{n}^k$ and $c_{n}$ are uniformly bounded (with respect to $n$) on $D_k \times [0, T]$. Let $\psi \in C_0^\infty(D)$. Then supp $\psi \subset D_k$ for some $k$ and there exists a number $C(\psi)$ (independent of $n$) such that
\[
\left| \int_D \psi(x) u_n(x, t) \, dx - \int_D \psi(x) u_n(x, s) \, dx \right| = \left| \int_s^t \int_D \{ L_n \psi(x, \tau) \} u_n(x, \tau) \, dx \, d\tau \right| \leq C(\psi) |t - s|
\]
for all $n$ and $s, t \in [0, T]$. Hence the functions
\[
f_n(t) := \int_D \psi(x) u_n(x, t) \, dx
\]
are Lipschitzian with constant $C(\psi)$ independent of $n$. Thus, this is a uniformly bounded and equicontinuous family of functions for every fixed $\psi$. Hence, by to Arzelà–Ascoli theorem there exists a subsequence, uniformly convergent on $[0, T]$. We observe that in the space $L^{(d+1)/d}([0, T])$ the same subsequence converges to
\[
f(t) := \int_D \psi(x) u(x, t) \, dx
\]
and, since the weak and uniform limits coincide a.e., every subsequence of $\{f_n\}$ (and thus the whole sequence as well) converges uniformly to the same Lipschitzian function $f$, which coincides with $f$ on a full Lebesgue measure set in $[0, T]$. Obviously this set depends on $\psi$. Denote it by $\tilde{T}(\psi)$.

4. Let us show that $u$ constructed at Step 2 is a solution to (1.1). Fix $\psi \in C_0^\infty(\mathbb{R}^d)$ with support in some $D_k$. We have $L \psi \in L^\infty(D_k \times [0, T])$ by the assumptions of the theorem. Moreover, there is a number $C_k$ such that $\sup_n \|L_n \psi\|_{L^\infty(D_k \times [0, T])} \leq C_k$, and $L_n \psi$ converges to $L \psi$ in $L^p(D_k \times [0, T])$ for every $p \geq 1$. Take $t$ in the full measure set $\tilde{T}(\psi)$; for every element of this set convergence
\[
\int_D \psi(x) u_n(x, t) \, dx \to \int_D \psi(x) u(x, t) \, dx
\]
takes place. Let $0 < s < t$. One has
\[
\left| \int_D \psi(x) u_n(x, t) \, dx - \int_D \psi(x) \eta_n(x) \, dx - \int_s^t \int_D L_n \psi u_n(x, \tau) \, dx \, d\tau \right| = \left| \int_D \psi(x) u_n(x, s) \, dx - \int_D \psi(x) \eta_n(x) \, dx \right| \leq C(\psi)s,
\]
where $C(\psi)$ is independent of $n$ and $s$. Hence,
\[
\left| \int_D \psi(x) u_n(x, t) \, dx - \int_D \psi(x) \eta_n(x) \, dx - \int_s^t \int_D L_n \psi(x, \tau) u_n(x, \tau) \, dx \, d\tau \right| \leq C(\psi)s. \quad (2.3)
\]
We observe that
\[
\lim_{n \to \infty} \int_s^t \int_D L_n \psi(x, \tau) u_n(x, \tau) \, dx \, d\tau = \int_s^t \int_D L \psi(x, \tau) u(x, \tau) \, dx \, d\tau.
\]
Indeed,
\[
\left| \int_s^t \int_D (L_n \psi(x, \tau)) u_n(x, \tau) \, dx \, d\tau - \int_s^t \int_D (L \psi(x, \tau)) u(x, \tau) \, dx \, d\tau \right| \\
\leq \| L_n \psi - L \psi \|_{L^{d+1}(D \times [s, t])} \| u_n \|_{L^{d+1/d}((D \times [s, t]))} \\
+ \left| \int_s^t \int_D (L \psi(x, \tau)) u_n(x, \tau) \, dx \, d\tau - \int_s^t \int_D (L \psi(x, \tau)) u(x, \tau) \, dx \, d\tau \right|,
\]
where the first summand in the right-hand side tends to zero due to convergence of $L_n \psi$ to $L \psi$ and the uniform norm boundedness of $\{u_n\}$, shown above. The second summand tends to zero by the weak convergence of $\{u_n\}$ and the boundedness of $L \psi$. Thus, letting $n \to \infty$ in (2.3), we obtain
\[
\left| \int_D \psi(x) u(x, t) \, dx - \int_D \psi(x) \, d\nu - \int_s^t \int_D L \psi u \, dx \, d\tau \right| \leq C(\psi) s.
\]
Letting $s$ to zero, we arrive at the equality
\[
\int_D \psi(x) u(x, t) \, dx = \int_D \psi(x) \, d\nu + \int_0^t \int_D L \psi u \, dx \, d\tau.
\]
Hence, the function $u$ is a non-negative solution to the Cauchy problem (1.1).

5. Let us show that the measure $u(x, t) \, dx \, dt$ is a solution in the class $\mathcal{M}_\nu$. We recall that $c_n \leq 0$ and $\eta_n \, dx$ are probability measures. Due to (2.2), for every function $\psi \in C_0^\infty(D)$, $0 \leq \psi \leq 1$ the following inequality holds:
\[
\int_D \psi(x) u_n(x, t) \, dx - \int_0^t \int_D \psi(x) c_n(x, s) u_n(x, s) \, dx \, ds \leq 1. \tag{2.4}
\]
Let $\psi_N \in C_0^\infty(D)$, $0 \leq \psi_N \leq 1$ and $\psi_N(x) = 1$ if $x \in D_N$. Let $t$ also belong to the full measure set $T = \cap_{N \in \mathbb{N}} T(\psi_N)$, i.e., for all $N \in \mathbb{N}$ the following convergence takes place:
\[
\int_D \psi_N(x) u_n(x, t) \, dx \to \int_D \psi_N(x) u(x, t) \, dx.
\]
Substituting such $\psi_N$ and $t$ into (2.4) and letting $n \to \infty$, we obtain
\[
\int_D \psi_N(x) u(x, t) \, dx - \int_0^t \int_D \psi_N(x) c(x, s) u(x, s) \, dx \, ds \leq 1.
\]
Finally, letting $N \to \infty$ and applying Fatou’s lemma, we arrive at the required inequality
\[
\int_D u(x, t) \, dx - \int_0^t \int_D c(x, s) u(x, s) \, dx \, ds \leq 1 = \nu(D).
\]
This completes the proof.

\textbf{Remark 2.2.} Due to \cite{6} Corollary 3.2], under the assumptions of Theorem 2.1 every solution $\mu$ in $\mathcal{M}_\nu$ is given by a density $\varrho \in L_{loc}^{(d+1)/d}(D \times (0, T))$ with respect to Lebesgue measure.

The assumptions of local boundedness can be weakened, but under the additional assumption that the elements of the matrix $A$ are Sobolev regular. The next theorem was proved in \cite{3} in the case $c = 0$ under the assumption of existence of a Lyapunov function. We give here a different and shorter proof and do not impose global restrictions on the coefficients.

\textbf{Theorem 2.3.} Let $p > d + 2$. Assume that for every $k$ we have $a^{ij}(\cdot, t) \in W_{loc}^{1,1}(D_k)$,
\[
\sup_{t \in (0, T)} \| a^{ij}(\cdot, t) \|_{W_{loc}^{1, p}(D_k)} < \infty
\]
and $(A(x, t) y, y) \geq m_k |y|^2$ for all $(x, t) \in D_k \times [0, T]$, $y \in \mathbb{R}^d$ and some $m_k > 0$. Assume also that $b \in L^p(D_k \times [0, T])$ and $c \in L^{p/2}(D_k \times [0, T])$ for each number $k$. Then for every probability measure $\nu$ the set $\mathcal{M}_\nu$ is not empty.

\textbf{Proof.} 1. Exactly as at Step 1 of the proof of the previous theorem, we construct sequences of smooth bounded functions $a^{ij}_n, b^i_n$ and $c_n$ such that for every $D_k$ one has
\[
\lim_{n \to \infty} \| a^{ij}_n - a^{ij} \|_{L_p(D_k \times [0, T])} = 0, \quad \lim_{n \to \infty} \| b^i_n - b^i \|_{L_{p/2}(D_k \times [0, T])} = 0, \quad \lim_{n \to \infty} \| c_n - c \|_{L_{p/2}(D_k \times [0, T])} = 0,
\]
in particular the norms $\| a^{ij}_n \|_{L_p(D_k \times [0, T])}, \| b^i_n \|_{L_{p/2}(D_k \times [0, T])}, \| c_n \|_{L_{p/2}(D_k \times [0, T])}$ are bounded uniformly in $n$. Moreover, $c_n \leq 0$ and $(A_n(x, t) y, y) \geq \min\{m_k, 1\}$ for all $n > k$. We extend the measure $\nu$ by zero.
outside $D$ to a measure on $\mathbb{R}^d$. Let $\eta_n \in C_0^\infty(D)$ be a sequence of non-negative functions such that $\eta_n \, dx$ are probability measures on $\mathbb{R}^d$ weakly convergent to $\nu$.

Let $\{u_n\}$ be a smooth bounded solution of the Cauchy problem

$$\partial_t u_n = \partial_x \partial_{x_j} (a_n^{ij} u_n) - \partial_x (b_n^i u_n) + c_n u_n, \quad u_n|_{t=0} = \eta_n,$$

and $u_n(x,t) \, dx$ are subprobability measures for every $t$ and

$$\int_{\mathbb{R}^d} u_n(x,t) \, dx = \int_{\mathbb{R}^d} \eta_n(x) \, dx + \int_0^t \int_{\mathbb{R}^d} c_n(x,t) u_n(x,t) \, dx \, dt.$$

2. We choose a convergent subsequence in $\{u_n\}$. Due to \cite{6 Corollary 3.9}, for every $k > 2$ the following estimate on the H"{o}lder norm holds:

$$\|u_n\|_{C^\alpha(D_k \times [Tk^{-1}, T(1-k^{-1})])} \leq C_k,$$

where $\alpha \in (0, 1)$ and $C_k$ are independent of $n$. The Arzel"{a}–Ascoli theorem, the diagonal method and a passage to a subsequence enable us to conclude that the sequence $\{u_n\}$ converges uniformly to some function $u$ on $D_k \times [Tk^{-1}, T(1-k^{-1})]$ for every $k$. It is obvious that $u$ is a non-negative continuous function. Let us show that $u$ satisfies (11).

Let $\psi \in C_0^\infty(D)$. Then $\text{supp} \psi \subset D_k$ for some $k$. The uniform convergence of $\{u_n\}$ immediately yields that

$$\lim_{n \to \infty} \int_D \psi(x) u_n(x,t) \, dx = \int_D \psi(x) u(x,t) \, dx$$

for all $t \in (0,T)$. Now let $0 < s < t < T$. Observe that

$$\left| \int_s^t \int_D L_n \psi u_n \, dx \, d\tau - \int_s^t \int_D L \psi u \, dx \, d\tau \right| \leq \|L_n \psi - L \psi\|_{L^1(D_k \times [s,t])} \|u_n\|_{L^\infty(D_k \times [s,t])} + \|L \psi\|_{L^1(D_k \times [s,t])} \|u - u_n\|_{L^\infty(D_k \times [s,t])},$$

where the first summand in right-hand side tends to zero by convergence of $a_n^{ij}$, $b_n^i$ and $c_n$ to $a^{ij}$, $b^i$ and $c$, respectively. The uniform convergence of $\{u_n\}$ to $u$ yields convergence of the second summand to zero. Hence,

$$\lim_{n \to \infty} \int_s^t \int_D L_n \psi u_n \, dx \, d\tau = \int_s^t \int_D L \psi u \, dx \, d\tau.$$

Thus, letting $n \to \infty$, we obtain

$$\int_D \psi(x) u(x,t) \, dx = \int_D \psi(x) u(x,s) \, dx + \int_s^t \int_D L \psi u \, dx \, d\tau$$

for all $s,t \in (0,T)$.

3. Let us justify the limit as $s \to 0$.

Let $0 < \tau < T$ and $y \in C_0^\infty(D)$. We extend the function $y$ by zero outside $D$. Let also $w_{n,\tau}$ be a solution of the adjoint problem

$$\partial_t w_{n,\tau} + a_n^{ij} \partial_{x_j} \partial_{x_i} w_{n,\tau} + b_n^i \partial_{x_i} w_{n,\tau} + c_n w_{n,\tau} = 0, \quad w_{n,\tau}|_{\tau = t} = y.$$

Let $\zeta \in C_0^\infty(\mathbb{R})$ be such that $\zeta(x) = 1$ if $|x| \leq 1$, $\zeta(x) = 0$ if $|x| > 2$, $|\zeta| \leq 1$ and let $\zeta$ have two bounded derivatives. Multiplying the adjoint equation by $\zeta_N u_n$ and integrating by parts and then letting $N$ to infinity, we obtain

$$\int_D y(x) u_n(x,\tau) \, dx = \int_D w_{n,\tau}(x,0) \eta_n(x) \, dx,$$

in the latter equality we have taken into account that $\text{supp} \eta_n \subset D$. By \cite{15 Part III, Theorem 10.1}, for every ball $U \subset \mathbb{R}^d$ the estimate

$$\|w_{n,\tau}(x,0) - y(x)\|_{L^\infty(U)} \leq C(U) \tau^\alpha$$

holds with $C$ and $\alpha$ independent of $a$. Thus,

$$\left| \int_D y(x) u_n(x,\tau) \, dx - \int_D y \, dv \right| \leq \int_D |w_{n,\tau}(x,0) - y(x)| \eta_n(x) \, dx + \left| \int_D y(x) \eta_n(x) \, dx - \int_D y \, dv \right| \leq C \tau^\alpha + \int_D y(x) \eta_n(x) \, dx - \int_D y \, dv \right| \leq C \tau^\alpha + \int_D y(x) \eta_n(x) \, dx - \int_D y \, dv \right|.$$
Letting $n \to \infty$, we obtain the estimate
\[
\left| \int_D y(x)u(x, \tau) \, dx - \int_D y \, dv \right| \leq C\tau^\alpha,
\]
which yields that
\[
\lim_{\tau \to 0} \int_D y(x)u(x, \tau) \, dx = \int_D y \, dv.
\]
Finally, the inequality
\[
\int_D u(x, t) \, dx - \int_0^t \int_D c(x, s)u(x, s) \, dx \, ds \leq 1 = \nu(D)
\]
can be justified exactly in the same way at Step 5 of the proof of Theorem 2.1. □

**Remark 2.4.** By [6] Corollary 3.9, under the assumptions of Theorem 2.3 every solution $\mu$ in $\mathcal{M}_\nu$ is given by a locally Hölder continuous density $\varphi$ with respect to Lebesgue measure.

Now we proceed to the case of a degenerate matrix $A$.

**Theorem 2.5.** Let the coefficients $a^{ij}$, $b^j$ and $c$ be continuous in $x$, measurable in $t$ and bounded on $D_k \times [0, T]$ for each $k$. Assume that the diffusion matrix $A$ is symmetric and $(A(x, t)y, y) \geq 0$ for all $(x, t) \in D \times [0, T]$ and $y \in \mathbb{R}^d$. Then, for every probability measure $\nu$, the set $\mathcal{M}_\nu$ is not empty.

**Proof.** We use the well-known method of "vanishing viscosity".

1. Introduce the operator $L_\varepsilon := \varepsilon \Delta + L$ for every $\varepsilon > 0$ and consider the Cauchy problem
\[
\partial_t \mu_t = L_\varepsilon^* \mu_t, \quad \mu|_{t=0} = \nu. \tag{2.5}
\]
Obviously, the hypotheses of Theorem 2.1 are fulfilled. Then, for every $n$, the problem (2.5) with $\varepsilon = 1/n$ has a solution $\mu^n$ given by a flow of subprobability measures $(\mu^n_t)_{t \in (0, T)}$ on $D$ for which the following inequality holds:
\[
\mu^n_t(D) \leq \nu(D) + \int_0^t \int_D c(x, s)\mu^n_s(dx) \, ds.
\]
2. Choosing a convergent subsequence of solutions.

There exists a subsequence of indices $n_i$ such that the measures $\mu^{n_i}$ converge weakly on each compact set $\overline{D_k}$ for each $t \in [0, T]$. To prove this, it suffices to apply Prokhorov’s theorem for any fixed compact set for a dense set in $S \in [0, T]$ and the diagonal method, and then, using again the diagonal method for compact sets $D_k$, we can extract a subsequence of measures that converges on all compact sets. Let us show that the constructed subsequence is a Cauchy sequence for every $t \in [0, T]$. Let $t \in [0, T]$, $s \in S$ and $\varphi \in C_0^{\infty}(\mathbb{R}^d)$. By the boundedness of the coefficients of $L$ on cylinders, one has
\[
\left| \int_D \varphi \, d\mu^{n_p}_t - \int_D \varphi \, d\mu^{n_k}_s \right| \leq \int_D \left| \varphi \, d\mu^{n_p}_t - \int_D \varphi \, d\mu^{n_p}_s \right| + \int_D \left| \varphi \, d\mu^{n_k}_s - \int_D \varphi \, d\mu^{n_k}_s \right| \leq 2C(\varphi) \cdot |t - s| + \int_D \left| \varphi \, d\mu^{n_p}_t - \int_D \varphi \, d\mu^{n_k}_s \right|.
\]
Given $\varepsilon > 0$, we can choose $s$ close enough to $t$ to make the first summand less than $\varepsilon/2$. Since the sequence $\mu^{n_i}$ converges, it is a Cauchy sequence, thus there exists a number $N$ such that for all $p, k > N$ the second summand is less than $\varepsilon/2$. Therefore, it is proved that for every function $\varphi \in C_0^{\infty}(\mathcal{D})$ the sequence of integrals $\int_D \varphi \, d\mu^{n_i}_t$ is a Cauchy sequence. Fix a number $k$. Since every function $f$ continuous on a compact set $\overline{D_k}$ can be uniformly approximated by functions in $C_0^{\infty}(\mathcal{D})$ on $\overline{D_k}$, the sequence of integrals $\int_{D_k} f \, d\mu^{n_i}_t$ is a Cauchy sequence. Hence for every $t$ the sequence $\mu^{n_i}_t$ converges weakly to some subprobability measure $\mu$ on each $D_k$. We observe that for every continuous function $f$ the mapping $t \mapsto \int_{D_k} f \, d\mu_t$ is Borel measurable on $[0, T]$ as a limit of measurable functions. Consider the class $\Phi$ of bounded Borel functions $\varphi$ on $D_k$, for which the mapping $t \mapsto \int_{D_k} \varphi \, d\mu_t$ is Borel measurable on $[0, T]$. The set $\Phi$ contains the algebra of continuous bounded functions on $D_k$ and is closed with respect to uniform and monotone limits. By the monotone class theorem (see [2] Theorem 2.2.12) the set $\Phi$ contains all bounded Borel functions on $D_k$. In particular, the mapping $t \mapsto \mu_t(B)$ is Borel measurable on $[0, T]$ for
each Borel set $B \subset D_k$. Since $k$ was arbitrary, this is true for every Borel subset of $D$. Let $\mu$ be the
measure given by a family $(\mu_t)_{t \in [0,T]}$. Obviously, $\mu^{n_k}$ converges weakly to $\mu$ on $D_k \times [0,T]$ for every $k$.

3. Passing to the subsequence constructed above, we conclude that the sequence $\mu^n = (\mu^n_t)_{t \in (0,t)}$ is such that $\mu^n$ converges weakly to $\mu = (\mu_t)_{t \in (0,T)}$ and $\mu^n_t$ converge weakly to $\mu_t$ for each $t$.

Let $\varphi \in C_0^\infty(D)$. For every $n$ one has

$$\int_D \varphi \, d\mu^n_t - \int_D \varphi \, d\mu_0 = \int_0^t \int_D L_{1/n} \varphi \, d\mu^n_s \, ds. \tag{2.6}$$

Since

$$\left| \int_0^t \int_D (L_{1/n} - L) \varphi \, d\mu^n_s \, ds \right| \leq \frac{T}{n} \sup_D |\Delta \varphi|,$$

by the weak convergence of $\mu^n$ to $\mu$ we conclude that

$$\lim_{n \to \infty} \int_0^t \int_D L_{1/n} \varphi \, d\mu^n_s \, ds = \int_0^t \int_D L \varphi \, d\mu_s \, ds,$$

and also for every $t$

$$\lim_{n \to \infty} \int_D \varphi \, d\mu^n_t = \int_D \varphi \, d\mu_t.$$

Hence one can let $n \to \infty$ in (2.6) and obtain

$$\int_D \varphi \, d\mu_t - \int_D \varphi \, d\mu_0 = \int_0^t \int_D L \varphi \, d\mu_s \, ds.$$

This means that $\mu$ is the required solution to the Cauchy problem (1.1). Finally, the inequality

$$\int_0^t \int_D u(x,t) \, dx - \int_0^t \int_D c(x,s)u(x,s) \, dx \, ds \leq 1 = \nu(D)$$

is justified as at Step 5 of the proof of Theorem 2.1. \qed

**Remark 2.6.** If we assume that the coefficients and the initial data are more regular, then one can construct a solution given by a density even in the case of a degenerate diffusion matrix. Such results were obtained for $D = \mathbb{R}^d$ in [9, 10, 17]. These results can be extended to the case of arbitrary domains $D$ as follows. Let $q \geq 1$. Suppose that for every $k$ the following assumptions are fulfilled: $b, c \in L^q(D_k \times [0,T])$, $a^{ij}(\cdot, t) \in W^{1,q}(D_k)$ and

$$\sup_{t \in [0,T]} \|a^{ij}\|_{W^{1,q}(D_k)} < \infty.$$

Let $p = (q - 1)/q$. Suppose that $(pc + (p - 1)\text{div} h)^+ \in L^1([0,T], L^\infty(D))$, where $h^i = \partial x_i a^{ij} - b^i$. Let also $\phi_0 \in L^p(D)$ and $\nu = \phi_0(x) \, dx$. Then, there exists a solution $\mu \in \mathcal{M}$ given by a density $\rho \in L^\infty([0,T], L^p(D))$.

The proof repeats practically verbatim the reasoning from [9] and [17]. We assume that the coefficients are locally smooth (for that if suffices to convolute the equation with a smooth kernel). Thus, the only problem is the unboundedness of the coefficients on $D$ and the degenerate diffusion matrix. Let $\psi_k \in C_0^\infty(D)$ and let $\psi_k(x) = 1$ if $x \in D_k$. Let $p > 1$. Consider the Cauchy problem $\partial_t u_k = L_k u_k$, $u_k|_{t=0} = u_0$, where

$$L_k = (\psi_k a^{ij} + k^{-1} \delta^{ij}) \partial x_i \partial x_j + \psi_k b^i \partial x_i + p^{-1}(p - 1)(2\partial x_i a^{ij} - b^i) \partial x_i \psi_k + a^{ij} \partial x_i \partial x_j \psi_k + \psi_k c.$$

Observe that there exists a solution $\{u_k\}$ of this Cauchy problem which is a smooth function and $u_k \in L^\infty([0,T], L^1(D))$. Then the inequality

$$\partial_t |u_k|^p \leq \partial x_i (\psi_k a^{ij} + k^{-1} \delta^{ij}) |u_k|^p - \partial x_i (\psi_k b|u_k|^p) + \psi_k (pc + (p - 1)\text{div} h)|u_k|^p$$

and the Gronwall inequality immediately yield that

$$\sup_{t \in [0,T]} \int_D |u_k(x,t)|^p \, dx \leq M$$

with a constant $M$ independent of $k$. Further in the standard way one can extract a subsequence from $u_k$ which converges to a solution $\psi$ of class $L^\infty([0,T], L^p(D))$.

To summarize this section, we find out when the identity

$$\mu_t(D) = \nu(D) + \int_0^t \int_{\mathbb{R}^d} c(x,s) \, d\mu_s \, ds \tag{2.7}$$

holds instead of an inequality (in particular, this means that if $c = 0$ and $\nu$ is a probability measure, then $\mu_t$ are also probability measures).
Theorem 2.7. Let $\mu = (\mu_t)_{0 < t < T} \in \mathcal{M}_\nu$ and $c \leq 0$. Suppose that there exists a function $V$ such that $V \in C^{2,1}(D \times (0, T)) \cap C(D \times [0, T])$, for every interval $[\alpha, \beta] \in (0, T)$

$$\lim_{k \to \infty} \inf_{D_{k \times [\alpha, \beta]}} V(x, t) = +\infty$$

and for some functions $K, H \in L^1((0, T))$ with $H \geq 0$, the following estimate holds:

$$\partial_t V(x, t) + LV(x, t) \leq K(t) + H(t)V(x, t).$$

Suppose also that $V(\cdot, 0) \in L^1(\nu)$. Then for a.e. $t \in (0, T)$

$$\mu_t(D) = \nu(D) + \int_0^t \int_D c(x, s) \, d\mu_t \, ds$$

and the estimate

$$\int_D V(x, t) \, d\mu_t \leq Q(t) + R(t) \int_D V(x, 0) \, d\nu$$

holds, where

$$R(t) = \exp \left( \int_0^t H(s) \, ds \right), \quad Q(t) = R(t) \int_0^t \frac{K(s)}{R(s)} \, ds.$$

Proof. Let $\zeta_N \in C^2([0, +\infty))$ be such that $0 \leq \zeta' \leq 1$, $\zeta'' \leq 0$, where $\zeta_N(s) = s$ if $s \leq N - 1$ and $\zeta(s) = N$ if $s > N + 1$. Let also $\eta \in C_0^\infty((0, T))$. For the function $u(x, t) = (\zeta_N(V(x, t)) - N)\eta(t)$ and the solution $\mu = (\mu_t)_{0 < t < T} \in \mathcal{M}_\nu$ we have

$$\int_0^T \int_D [\partial_t u + Lu] \, d\mu_t \, dt = 0,$$

which yields that

$$- \int_0^T \eta'(t) \int_D (\zeta_N(V(x, t)) - N) \, d\mu_t \, dt = \int_0^T \eta(t) \int_D L(\zeta_N(V(x, t)) - N) \, d\mu_t \, dt.$$

Since $\eta$ is arbitrary,

$$\frac{d}{dt} \int_D (\zeta_N(V(x, t)) - N) \, d\mu_t = \int_D L(\zeta_N(V(x, t)) - N) \, d\mu_t.$$

Hence there holds the identity

$$\int_D \zeta_N(V(x, t)) \, d\mu_t = \int_D \zeta_N(V(x, s)) \, d\mu_s +$$

$$\left( \mu_t(D) - \nu(D) - \int_0^t \int_D c(x, \tau) \, d\mu_{\tau} \, d\tau \right) N +$$

$$+ \int_0^t \int_D \left( \zeta''_N(V) \partial_t V + LV + \zeta''_N(V) |A\nabla V|^2 \right) \, d\mu_{\tau} \, d\tau +$$

$$+ \int_0^t \int_D c(\zeta_N(V) - \zeta'_N(V) V) \, d\mu_{\tau} \, d\tau.$$

Observing that $\zeta'_N(0) \leq \zeta_N(0)$, we arrive at

$$\int_D \zeta_N(V(x, t)) \, d\mu_t \leq \int_D \zeta_N(V(x, s)) \, d\mu_s +$$

$$\left( \mu_t(D) - \nu(D) - \int_0^t \int_D c(x, \tau) \, d\mu_{\tau} \, d\tau \right) N +$$

$$\int_0^t K(\tau) + H(\tau) \int_D \zeta_N(V(x, \tau)) \, d\mu_{\tau} \, d\tau,$$

Letting $s \to 0$, we obtain

$$\int_D \zeta_N(V(x, t)) \, d\mu_t \leq \int_D \zeta_N(V(x, 0)) \, d\nu +$$

$$\left( \mu_t(D) - \nu(D) - \int_0^t \int_D c(x, \tau) \, d\mu_{\tau} \, d\tau \right) N +$$

$$\int_0^t K(\tau) + H(\tau) \int_D \zeta_N(V(x, \tau)) \, d\mu_{\tau} \, d\tau. \quad (2.8)$$
Since \[ \mu_t(D) \leq \nu(D) + \int_0^t \int_D c(x, s) \, d\mu_s \, ds, \] the last inequality can be rewritten in the following way:
\[ \int_D \zeta_N(V(x, t)) \, d\mu_t \leq \int_D \zeta_N(V(x, 0)) \, d\nu + \int_0^t K(\tau) + H(\tau) \int_D \zeta_N(V(x, \tau)) \, d\mu_\tau \, d\tau. \]

Using Grownwall’s inequality, we have
\[ \int_D \zeta_N(V(x, t)) \, d\mu_t \leq Q(t) + R(t) \int_D \zeta_N(V(x, 0)) \, d\nu. \]

Letting \( N \to \infty \), we obtain the required estimate.

Moreover, if \( \mu_t(D) < \nu(D) + \int_0^t \int_D c(x, s) \, d\mu_s \, ds \), then, letting \( N \to \infty \) in (\ref{eq:inequality}), we obtain
\[ \int_D V(x, t) \, d\mu_t - \int_D V(x, 0) \, d\nu - \int_0^t K(\tau) + H(\tau) \int_D V(x, \tau) \, d\mu_\tau \, d\tau = -\infty, \]
which is impossible. Hence \( \mu_t(D) = \nu(D) + \int_0^t \int_D c(x, s) \, d\mu_s \, ds \). The theorem is proved. \( \square \)

Thus, in the case \( c = 0 \) to construct a probability solution it suffices to construct a subprobability solution (for that only a local regularity of the coefficients is needed). This solution is automatically a probability solution if there is a Lyapunov function.

Observe that the assumption \( V(\cdot, 0) \in L^1(\nu) \) is not restrictive: if there is some Lyapunov function, then there is a Lyapunov function integrable with respect to the initial condition. More precisely, the following generalization of a lemma from \cite{3} is true.

**Proposition 2.8.** Let \( \mu = (\mu_t)_{t \in [0, T]} \) be a solution to the Cauchy problem \( \partial_t \mu = \mu^* \mu \) and \( \mu|_{t=0} = \nu \), where \( \nu \) is a probability measure on \( D \). Suppose that there exists a non-negative function \( V \) such that \( V \in C^{2,1}([0, T]) \cap C([0, T]) \), for every interval \( [\alpha, \beta] \subseteq (0, T) \)
\[ \lim_{k \to \infty} \inf_{D_k \setminus D_{k-1} \times [\alpha, \beta]} V(x, t) = +\infty \]
and for some functions \( K, H \in L^1((0, T)) \) with \( H \geq 0 \), one has
\[ \partial_t V(x, t) + LV(x, t) \leq K(t) + H(t) V(x, t). \]

Then there exists a non-negative function \( W \in C^{2,1}([0, T]) \cap C([0, T]) \) such that for every interval \( [\alpha, \beta] \subseteq (0, T) \)
\[ \lim_{k \to \infty} \inf_{D_k \setminus D_{k-1} \times [\alpha, \beta]} W(x, t) = +\infty, \]
the following inequality holds: \( \partial_t W(x, t) + LW(x, t) \leq K(t) + H(t) W(x, t) \) holds and there is an inclusion \( W(x, 0) \in L^1(\nu) \).

**Proof.** Construct a non-negative function \( \theta \in L^1(\mathbb{R}) \) such that \( \theta(0) = 0 \), \( \lim_{r \to \infty} \theta(r) = +\infty \), \( 0 \leq \theta'(r) \leq 1 \), \( \theta''(r) \leq 0 \) and \( \theta(V(\cdot, 0)) \in L^1(\nu) \). For that it suffices to find a function \( \theta \) with the properties listed above and integrable with respect to the measure \( \sigma = \nu \circ W^{-1}(\cdot, 0) \). Find an increasing sequence of numbers \( z_k \) such that \( z_{k+1} - z_k \geq z_k - z_{k-1} \geq 1 \) and \( \sigma([z_k, \infty)) \leq 2^{-k} \). Let \( \theta_0 \) be a linear function on each interval \( [z_k, z_{k+1}] \) with \( \Theta_0(z_k) = k - 1 \). We obtain a \( \sigma \)-integrable increasing concave function \( \theta_0 \). However, it does not belong to the class \( L^1(\mathbb{R}) \). Take for \( \theta \) the function
\[ \theta(z) = \int_0^z g(s) \, ds, \quad g \in C^1(\mathbb{R}), \]
where \( g'(z) \leq 0 \) and \( g(z) = \theta_0'(z) \) if \( z \in (z_k, z_{k+1} - k^{-1}) \). Obviously, it is the required function. Further, taking into account that \( \theta \) is concave and that \( c \) is non-positive, we obtain
\[ \partial_t \theta(V) + L \theta(V) = \theta'(V) \theta(V) e^t + \theta''(V) (A \nabla V, \nabla V) + c \theta(V) \theta'(V) \leq K + H \theta(V). \]

The function \( W := \log(1 + \theta(V)) \) is the required function. \( \square \)

**Remark 2.9.** Modifying a bit the reasoning above, one can get sufficient conditions for \( \theta \) of another type. Let the function \( V \) be such that \( V \in C^2(D) \) and
\[ \lim_{k \to \infty} \inf_{D_k \setminus D_{k-1}} V(x) = +\infty. \]

Let \( \zeta \in C^\infty([0, +\infty)) \), \( \zeta(x) = 1 \) if \( x < 1 \) and \( \zeta(x) = 0 \) if \( x > 2 \).
Then the set $(H_1)$ holds for all $g \in VMO$. Suppose that Theorem 3.1. extends $g$, $y$ in some limiting procedure, we do not repeat this first step in details (which can be found in the listed papers).

Let the matrix $A(x, t) = (a^{ij}(x, t))_{1 \leq i,j \leq d}$ be symmetric and satisfy the following condition:

- (H1) for every $D_k \subset D$ there exist strictly positive numbers $m_k$ and $M_k$, such that the estimate

$$m_k|y|^2 \leq (A(x, t)y, y) \leq M_k|y|^2$$

holds for all $y \in \mathbb{R}^d$ and all $(x, t) \in D \times (0, T)$.

Let us recall the definition of the functional class VMO. Let $g$ be a bounded function on $\mathbb{R}^{d+1}$. Set

$$O(g, R) = \sup_{(x, t) \in \mathbb{R}^{d+1} \cap \bar{D}} \sup_{r \leq R} r^{-2} |U(x, r)|^{-2} \int_0^{t+r^2} \int_{y \in U(x, r)} |g(y, s) - g(z, s)| dy dz ds.$$ 

If $\lim_{R \to 0} O(g, R) = 0$, then the function $g$ is said to belong to the class $VMO_x(\mathbb{R}^{d+1})$.

If $g \in VMO_x(\mathbb{R}^{d+1})$, then one can always assume that $O(g, R) \leq w(R)$ for all $R > 0$, where $w$ is a continuous function on $[0, +\infty)$ and $w(0) = 0$.

Suppose that a function $g$ is defined on $D \times [0, T]$ and is bounded on $D_k \times [0, T]$ for every $k$. Let us extend $g$ by zero to all of $\mathbb{R}^{d+1}$. If for every function $\zeta \in C^0_c(D)$ the function $g\zeta$ belongs to the class $VMO_x(\mathbb{R}^{d+1})$, then we say that $g$ belongs to the class $VMO_{x,loc}(\mathbb{R}^d \times [0, T])$.

Set

$$L_0u = a^{ij}\partial_{x_i}\partial_{x_j}u + b^i\partial_{x_i}u.$$ 

**Theorem 3.1.** Suppose that $a^{ij} \in VMO_{x,loc}(D \times [0, T])$ and that the matrix $A = (a^{ij})$ satisfies condition (H1). Suppose that there is a function $V = V(x)$ such that $V \in C^2(D)$ and

$$\lim_{k \to \infty} \inf_{D_k \times [0, T]} V(x) = +\infty.$$ 

Then the set $M_\nu$ consisting of the measures $\mu \in M_\nu$ for which the functions $|L_0V|$, $|\sqrt{A}\nabla V|^2$ belong to $L^1(\mu, D_k \times [0, T])$ for every $k$ and

$$\lim_{N \to \infty} \int_0^T \int_{N \leq V \leq 2N} N^{-1}|L_0V| + N^{-2}|\sqrt{A}\nabla V|^2 \, d\mu = 0,$$

contains at most one element.
Proof. The proof is the same as the proof of Theorem 3.1 from [8], but there are some new aspects concerning the domain \(D\) and the term \(c\). More precisely, local part of the proof of Theorem 3.1 is saved but global part of the proof requires a new consideration.

We start with the case of the coefficients of class \(\bigcap_{k \in \mathbb{N}} C^\infty(D_k \times [0,T])\).

1. Let \(\varphi_N(x) = \eta(V(x)/N)\), where a nonnegative function \(\eta \in C^\infty([0, +\infty))\) is such that \(\eta(z) = 1\) for \(0 \leq z \leq 1\) and \(\eta(z) = 0\) for \(z > 2\). Moreover, \(0 \leq \eta \leq 1\) and there is a number \(K > 0\) such that the estimate \(|\eta'(z)|^2 \eta^{-1}(z) \leq K\) holds for all \(x \in \text{supp} \eta\). We observe that \(\varphi_N \in C^\infty_0(D)\), in particular, there exists a positive integer \(k_0 = k_0(N)\) such that \(\text{supp} \varphi_N \subset D_{k_0}\).

We redefine the functions \(a^{ij}, b^i, c\) outside \(D_{k_0} \times [0, T]\) so that the new functions are bounded together with all derivatives on \(\mathbb{R}^d \times [0, T]\).

2. Let \(\psi \in C^\infty_0(D), |\psi| \leq 1\) and let \(f\) be a solution of the adjoint Cauchy problem

\[
\partial_t f + L f = 0, \quad f|_{s=t} = \psi.
\]

Suppose that \(\sigma_1\) and \(\sigma_2\) are two solutions in the set \(M_\nu\). Set \(\mu = \sigma_1 - \sigma_2\). Multiplying the equation \(\partial_t \mu = L^* \mu\) by \(f \varphi_N\) and integrating, we arrive at the equality

\[
\int_D \varphi_N \, d\mu = \int_0^t \int_D 2(A \nabla \varphi_N, \nabla f) + f L_0\varphi_N \, d\mu_s \, ds.
\]

3. Set \(\sigma = (\sigma_1 + \sigma_2)/2\). Let us estimate

\[
\int_0^t \int_D \varphi_N |\nabla f|^2 \, d\sigma_s \, ds.
\]

Multiplying the equation \(\partial_t \sigma = L^* \sigma\) by \(f^2 \varphi_N\) and integrating, we arrive at the equalities

\[
\int_D \psi^2(x) \varphi_N(x) \, d\sigma + \int_D f^2(x, 0) \varphi_N(x) \, d\nu =
\]

\[
= 2 \int_0^t \int_D |\nabla f|^2 \varphi_N \, d\sigma_s \, ds + 2 \int_0^t \int_D \left( f(A \nabla f, \nabla \varphi_N) + f^2 L_0\varphi_N \right) \, d\sigma_s \, ds.
\]

Note that \(0 \leq \varphi_N(x) \leq 1\). By the maximum principle \(|f(x, s)| \leq \max_{x} |\psi(x)| \leq 1\). Using Cauchy’s inequality, we obtain the following estimate:

\[
\int_0^t \int_D \varphi_N |\nabla f|^2 \, d\sigma_s \, ds \leq 2 + \int_0^t \int_D |\nabla \varphi_N|^2 \varphi_N^{-1} + |L_0 \varphi_N| \, d\sigma_s \, ds.
\]

Thus,

\[
\int_D \psi \varphi_N \, d\mu \leq (1 + R_N)^{1/2} R_N^{1/2} + R_N,
\]

where

\[
R_N = \int_0^T \int_D |L_0 \varphi_N| + 2 \varphi_N^{-1} |\nabla \varphi_N| \, d\sigma_s \, ds.
\]

Finally, letting \(N \to \infty\), we arrive at the estimate

\[
\int_D \psi(x) \, d\mu_t \leq 0.
\]

Replacing \(\psi\) by \(-\psi\), we get the opposite inequality. Hence for a.e. \(t \in [0, T]\) we have

\[
\int_D \psi(x) \, d\mu_t = 0.
\]

Since \(\psi\) was an arbitrary function in \(C^\infty_0(\mathbb{R}^d)\) with the only restriction \(|\psi| \leq 1\), we conclude that \(\mu_t = 0\) and hence \(\sigma^1 = \sigma^2\).

In the general case (without assumptions about the smoothness of coefficients) one has to solve the Cauchy problems

\[
\partial_t f_n + L_n f_n = 0, \quad f_n|_{x=t} = \psi,
\]

in order to get an analogous estimate, where the coefficients of the operator \(L_n\) are smooth approximations of the coefficients of \(L\) on \(\text{supp} \varphi_N\) (more precisely see Theorem 3.1 [8]).
Example 3.2. The assumption of Theorem 3.1 is fulfilled for all measures \( \mu \in \mathcal{M}_\nu \) if
\[
(1 + V(x))^{-1}|L_0V(x, t)| + (1 + V(x))^{-2}|\sqrt{A}(x, t)\nabla V(x)|^2 \leq W(x, t),
\]
where \( W \in C^{1,1}(D \times (0, T)) \cap C(D \times [0, T]) \), \( W(\cdot, 0) \in L^1(\nu) \), for every interval \([\alpha, \beta]\) from \((0, T)\) one has
\[
\lim_{k \to \infty} \inf_{D_k \setminus D_{k-1} \times [\alpha, \beta]} W(x, t) = +\infty
\]
and for some functions \( K, H \in L^1((0, T)) \) with \( H \geq 0 \) the estimate
\[
\partial_t W(x, t) + LW(x, t) \leq K(t) + H(t)W(x, t)
\]
holds.

Indeed, by Theorem 2.7 we have \( W \in L^1(\mu, D \times [0, T]) \), hence
\[
\lim_{N \to \infty} \int_{0}^{T} \int_{N < V < 2N} W(x, t) d\mu = 0.
\]

Let us note that the assumptions of Theorem 3.1 admit practically any growth of the coefficients (the function \( W \) from the example above may grow arbitrarily fast), but we impose restrictions on \( |L_0V| \), i.e., we control the growth of \( L_0V \) from both sides. Moreover, we have proved the uniqueness only in the class \( \mathcal{M}_\nu \), but not for all measures from \( \mathcal{M}_\nu \). It is possible to eliminate these constraints in the case of more regular coefficients.

We shall assume now that along with (H1) we have

(H2) for every positive integer \( k \) there is a number \( \Lambda_k > 0 \) such that
\[
|a^{ij}(x, t) - a^{ij}(y, t)| \leq \Lambda_k|x - y|
\]
for all \( x, y \in D_k \) and \( t \in [0, T] \).

Let us recall some facts from [6] for completeness. The assumption (H1) ensures the existence of a density \( \varrho \) of the solution \( \mu \) with respect to Lebesgue measure. Moreover, if along with (H1) and (H2) we have \( b \in L^p_{\text{loc}}(D \times (0, T)) \) and \( c \in L^{p/2}_{\text{loc}}(D \times (0, T)) \) for some \( p > d + 2 \), then we can choose a version of \( \varrho \) continuous on \( D \times (0, T) \) such that for a.e. \( t \in (0, T) \) the function \( \varrho(\cdot, t) \) belongs to \( W^{1/p}(U) \) for every closed ball \( U \subset D \). Since for a.e. \( t \in (0, T) \) the measure \( \mu_t(dx) = \varrho(x, t) \mu(dx) \) is a subprobability measure on \( D \), Harnack’s inequality ensures that for every closed ball \( U \) from \( D \) and for every interval \( J \subset (0, T) \) there exists a number \( C > 0 \) such that \( \varrho(x, t) \geq C \) for all \( (x, t) \in U \times J \).

Below we deal with the continuous version of the density \( \varrho \).

We recall that, for every measure \( \mu \) given by a Sobolev density \( \varrho \) with respect to Lebesgue measure, its logarithmic gradient \( \beta_\mu \) with respect to the metric generated by the matrix \( A \) is defined by the following formula:
\[
\beta_\mu^i = \sum_{j=1}^{d}(\partial_{x_j} a^{ij} + a^{ij} \varrho^{-1} \partial_{x_j} \varrho).
\]

Further in this section we assume that the coefficients \( b \) and \( c \) are locally integrable with respect to Lebesgue measure on \( D \times (0, T) \) to power \( p \) and \( p/2 \), respectively, for some \( p > d + 2 \) and that the conditions (H1) and (H2) are fulfilled.

Moreover, we consider only the set \( \mathcal{M}_\nu \) of measures \( \mu \in \mathcal{M}_\nu \) satisfying the condition
\[
b \in L^2(\mu, D_k \times [0, T]) \quad \forall D_k.
\]

For example, the latter condition is fulfilled for the whole class \( \mathcal{M}_\nu \) in the case of a drift bounded on \( D_k \times [0, T] \) or in the case \( b \in L^s(D_k \times [0, T]) \) and \( \mu = \varrho dx dt \) with \( \varrho \in L^r(D_k \times [0, T]) \), where \( 2/s + 1/r = 1 \).

Suppose that there are two solutions to the Cauchy problem (1.1) in the class \( \mathcal{M}_\nu \) given by densities \( \sigma \) and \( \varrho \) with respect to Lebesgue measure. Then these densities are continuous on \( D \times (0, T) \). In addition, the functions \( \sigma \) and \( \varrho \) are strictly positive. Let \( v(x, t) = \sigma(x, t)/\varrho(x, t) \). The function \( v \) is continuous and positive on \( D \times (0, T) \).

Lemma 3.3. Suppose that for a.e. \( t \in (0, T) \) we have the estimates
\[
\int_D \varrho(x, t) dx = \nu(D) + \int_0^t \int_D c(x, s) \varrho(x, s) dx ds
\]
and
\[
\int_D \sigma(x, t) dx \leq \nu(D) + \int_0^t \int_D c(x, s) \sigma(x, s) dx ds.
\]
Suppose also that for every $\lambda > 0$ we have for a.e. $t \in (0, T)$

$$
\int_{\mathbb{R}^d} e^{\lambda(1-v(x,t))} \varrho(x,t) \, dx \leq 1. \tag{3.1}
$$

Then $v \equiv 1$, i.e., $\sigma = \varrho$.

**Proof.** Let $t$ be such that $\varrho(\cdot, t)$ and $\sigma(\cdot, t)$ satisfy the listed conditions and holds for every positive integer $\lambda$. We observe that the set of points $t$ for which this is not true is a set of zero Lebesgue measure. If there is a ball $U \subset D$ such that $v(x,t) \leq 1 - \delta$ for each $x \in U$ and some $\delta > 0$, then

$$
e^{\lambda \delta} \int_U \varrho \, dx \leq \int_U e^{\lambda(1-v(x,t))} \varrho(x,t) \, dx \leq 1.
$$

Letting $\lambda \to +\infty$, we obtain a contradiction. Hence, $v(x,t) \geq 1$ and $\sigma \geq \varrho$ for all $(x,t) \in D \times (0, T)$. Moreover,

$$
\int_D \varrho(x,t) \, dx \leq \int_D \sigma(x,t) \, dx, \quad \int_0^t \int_D |c(x,s)| \varrho(x,s) \, dx \, ds \leq \int_0^t \int_D |c(x,s)| \sigma(x,s) \, dx \, ds.
$$

We observe that

$$
\nu(D) = \int_D \varrho(x,t) \, dx + \int_0^t \int_D |c(x,s)| \varrho(x,s) \, dx \, ds \leq \int_D \sigma(x,t) \, dx + \int_0^t \int_D |c(x,s)| \sigma(x,s) \, dx \, ds = \nu(D).
$$

Hence we have

$$
\int_D \varrho(x,t) \, dx = \int_D \sigma(x,t) \, dx,
$$

which ensures $v \equiv 1$. The lemma is proved. \hfill \Box

The next lemma is crucial in our approach. In this lemma, it is possible to take $e^{\lambda(1-z)}$ and $e^{\lambda(1-z)} - e^{\lambda}$ for $f$.

**Lemma 3.4.** Let $\psi \in C_0^\infty(D)$, $\psi \geq 0$ and $0 < t < T$. Then the following estimate holds:

$$
\int_D f(v(x,t)) \varrho(x,t) \psi(x) \, dx \leq f(1) \int_D \psi(x) \, d\nu + \int_0^t \int_D \varrho f(v)L \psi \, dx \, ds. \tag{3.2}
$$

If, in addition, $(b - \beta_\mu) \varrho \in L^1(D_k \times (0, T))$ for every $k$, then

$$
\int_D f(v(x,t)) \varrho(x,t) \psi(x) \, dx \leq f(1) \int_D \psi(x) \, d\nu + \frac{1}{2} \int_0^t \int_D \varrho (A \nabla \psi, \nabla \psi) \psi^{-1} |f'(v)|^2 f''(v)^{-1} \, dx \, ds + \int_0^t \int_D f(v)(b - \beta_\mu, \nabla \psi) \varrho \, dx \, ds. \tag{3.3}
$$

**Proof.** The case $D = \mathbb{R}^d$ and $c = 0$ was considered in [19]. Since the estimate is local (only on supp $\psi$), there is no difference between arbitrary $D$ and $\mathbb{R}^d$. The addition of a new term $c$ does not give new difficulties and the reasonings are completely the same. \hfill \Box

**Theorem 3.5.** Suppose that (H1) and (H2) are fulfilled and

$$
b \in L^p_{loc}(D \times (0, T)), \quad c \in L^{p/2}_{loc}(D \times (0, T))
$$

for some $p > d + 2$. Assume also that there exists a function $V$ such that $V \in C^2(D)$ and

$$
\lim_{k \to \infty} \inf_{D_k} V(x) = +\infty.
$$

Suppose that at least one of the following conditions is fulfilled:

(i) for some measure $\mu \in \mathcal{M}_p$, one has

$$
\lim_{N \to \infty} \int_0^T \int_{N \leq v \leq 2N} N^{-1} |L_0 V| + N^{-2} |\sqrt{\lambda} \nabla V|^2 \, d\mu = 0,
$$

(ii) for some measure $\mu \in \mathcal{M}_p$, one has

$$
\lim_{N \to \infty} \int_0^T \int_{N \leq v \leq 2N} N^{-1} |(b - \beta_\mu) \nabla V| + N^{-2} |\sqrt{\lambda} \nabla V|^2 \, d\mu = 0,
$$



(iii) for some $K > 0$ and all $(x,t) \in D \times (0,T)$ the inequality
\[ LV(x,t) \leq K + KV(x) \]
holds. Then the class $M_\nu$ consists of at most one element.

Proof. Let us prove (i). Let a measure $\mu \in M_\nu$ satisfy the condition from (i) and have a density $\varrho$ with respect to Lebesgue measure. By Remark 2.3 the equality (2.7) holds for $\mu$. Suppose that there is yet another measure in $M_\nu$ given by a density $\sigma$. Set $v = \sigma/\varrho$. Let $\psi(x) = \zeta(V(x)/N)$, where $\zeta \in C^\infty_0(\mathbb{R}^d)$ is a nonnegative function such that $\zeta(x) = 1$ if $|x| \leq 1$ and $\zeta(x) = 0$ if $|x| > 2$, and there exists a number $M > 0$ such that for all $x \in \text{supp} \zeta$ one has
\[ |\zeta(x)| \leq M, \quad |\nabla \zeta(x)| \leq M, \quad |\nabla \zeta(x)|^2 \zeta^{-1}(x) \leq M. \]

Let $f(z) = e^{\lambda(1-z)}$. It is clear that $|f(z)| \leq e^\lambda$ if $z \geq 0$. Using (3.3) from Lemma 3.4 and the fact $c \leq 0$, we obtain
\[
\int_D e^{\lambda(1-v(x,t))} \varrho(x,t)\zeta(V(x)/N) \, dx \leq \int_D \zeta(V(x)/N) \, dv + e^\lambda M \int_0^t \int_{N \leq V \leq 2N} N^{-1}|L^0V| + N^{-2}|\sqrt{A}V|^2 \, dx.
\]

Letting $N \to +\infty$ and using Lemma 3.3 we obtain the required assertion.

The case (ii) can be treated similarly. Let us now prove (iii).

Suppose that there are two measures in $M_\nu$ given by densities $\sigma$ and $\varrho$ with respect to Lebesgue measure. According to Theorem 2.7, one has (2.7) for both measures. Set $v = \sigma/\varrho$. Let $\psi(x) = \zeta(N^{-1}V(x))$, where a nonnegative function $\zeta \in C^\infty_0(\mathbb{R})$ is such that $\zeta(0) = 1$, $\zeta(z) = 0$ if $|z| > 1$, $0 \leq \zeta \leq 1$ and, moreover, $\zeta''(z) \geq 0$ if $z > 0$.

Let $f(z) = e^{\lambda(1-z)} - e^\lambda$. Then $f(z) \leq 0$ and $|f(z)| \leq 2e^\lambda$ if $z \geq 0$. Observe that
\[ f(v) \zeta'LV \leq (K + KV)f(v)\zeta' \]
since $f(v)\zeta' \geq 0$. Using (3.2) from Lemma 3.4 we obtain
\[
\int_D (e^{\lambda(1-v(x,t))} - e^\lambda) \varrho(x,t)\zeta(N^{-1}V(x)) \, dx \leq (1 - e^\lambda) \int_D \zeta(N^{-1}V(x)) \, dv + 2e^\lambda MN^{-1} \int_0^t \int_{V < N} (K + KV) \varrho \, dx \, ds + \int_0^t \int_D \zeta(N^{-1}V(x))f(v(x,s))c(x,s)\varrho(x,s) \, dx \, ds.
\]

We observe that
\[
\lim_{N \to +\infty} N^{-1} \int_0^t \int_{V < N} (K + KV) \varrho \, dx \, ds = 0.
\]

Indeed, let $\gamma \in (0,1)$ and $N > \gamma^{-1}$, then
\[
N^{-1} \int_0^t \int_{V < N} (K + KV) \varrho \, dx \, ds \leq \gamma \int_0^t \int_{V < \gamma N} \varrho \, dx \, ds + K \int_0^t \int_{\gamma N < V < N} \varrho \, dx \, ds.
\]

Hence we have
\[
\lim_{N \to +\infty} N^{-1} \int_0^t \int_{V < N} (K + KV) \varrho \, dx \, ds \leq \gamma \int_0^t \int_D \varrho \, dx \, ds.
\]

Letting $\gamma \to 0$, we obtain the required observation. Thus, letting $N \to +\infty$, we obtain
\[
\int_D (e^{\lambda(1-v(x,t))} - e^\lambda) \varrho(x,t) \, dx \leq (1 - e^\lambda) \int_D dv + \int_0^t \int_D (e^{\lambda(1-v)} - e^\lambda)c(x,s)\varrho(x,s) \, dx \, ds.
\]

Since $c \leq 0$ and for a.e. $t \in (0,T)$ the identity
\[
\int_D \varrho(x,t) \, dx = \nu(D) + \int_0^t \int_D c(x,s)\varrho(x,s) \, dx \, ds,
\]
holds. Then for a.e. $t$ we have
\[
\int_{\mathbb{R}^d} e^{\lambda(1-v(x,t))} \varrho(x,t) \, dx \leq 1.
\]

Using Lemma 3.3 we complete the proof. □

A combination of Theorem 2.3 and Theorem 3.5 yields the following sufficient conditions for existence and uniqueness.
Theorem 3.6. Suppose that (H1) and (H2) hold and 
\[ c \in L^{p/2}_{loc}(D \times (0, T)), \quad b \in L^\infty(D_k \times [0, T]) \]
for some \( p > d + 2 \) and all \( k \). Assume that there exists a function \( V \) such that \( V \in C^2(D) \),
\[ \lim_{k \to \infty} \inf_{D_k \setminus D_{k-1}} V(x) = +\infty, \]
and for some number \( K > 0 \) and all \((x, t) \in D \times (0, T)\) one has inequality
\[ LV(x, t) \leq K + KV(x). \]
Then the class \( M_\nu \), where \( \nu \) is a probability measure on \( D \), consists of exactly one element \( \mu = (\mu_t)_{t \in (0, T)} \).
Moreover, for a.e. \( t \) the identity
\[ \mu_t(D) = \nu(D) + \int_0^t \int_D c(x,s) \, d\mu_s \, ds \]
holds. In particular, if \( c = 0 \), the measures \( \mu_t \) are probabilities for a.e. \( t \).

Remark 3.7. In Remark 2.6 we discussed a construction of a solution given by a density \( \rho \in L^\infty([0, T], L^p(D)) \) in the case of a degenerate diffusion matrix. Following [2] and [17], one can find sufficient conditions for the uniqueness of a solution. Suppose that, in addition to the conditions from Remark 2.6, there exists a function \( V \in C^2(D) \) such that
\[ \lim_{n \to \infty} \inf_{D_n \setminus D_{n-1}} V(x) = +\infty \]
and for some number \( K > 0 \) and all \((x, t) \in D \times (0, T)\) on has
\[ L_0 V(x, t) \geq -K - KV(x), \quad |V(x, t)| \leq KV(x), \]
where, as above, \( L_0 \psi = a^{ij} \partial_i \partial_j \psi + b_i \partial_i \psi \). Then a solution of class \( L^\infty([0, T], L^p(D)) \) is unique.

Indeed, one can show that, for any solution \( \rho \) to the Cauchy problem with zero initial condition and \( \psi \in C^\infty_0(D) \), the following identity holds:
\[ \int_D |\rho(x,t)|^p \psi(x) \, dx \leq \int_0^t \int_D \left( L_0 \psi + \psi (pc + (p-1) \text{div} h)^+ \right) |\rho|^p \, dx \, dt. \]
Let \( \psi_N(x) = \zeta(N^{-1}V(x)) \), where \( \zeta \in C^\infty_0(\mathbb{R}) \) is a nonnegative function such that \( \zeta(z) = 1 \) if \( |z| < 1 \) and \( \zeta(z) = 0 \) if \( |z| > 2, 0 \leq \zeta \leq 1 \), moreover, \( \zeta'(z) \leq 0 \). We observe that for some number \( C_1 > 0 \) and all \((x, t) \in D \times (0, T)\) we have
\[ L_0 \psi(x, t) = N^{-1} \zeta'(N^{-1}V(x)) L_0 V(x, t) + N^{-2} \zeta''(N^{-1}V(x)) \sqrt{A(x, t)} \nabla V(x)^2 \leq K_1. \]
Hence,
\[ \int_D |\rho(x,t)|^p \psi_N(x) \, dx \leq K_1 \int_0^t \int_{N^2 < \nabla x < 2N} |\rho|^p \, dx \, dt + \int_0^t \int_D \psi_N (pc + (p-1) \text{div} h)^+ |\rho|^p \, dx \, dt. \]
Letting \( N \to \infty \), we arrive at the equality
\[ \int_D |\rho(x,t)|^p \, dx \leq \int_0^t \int_D (pc + (p-1) \text{div} h)^+ |\rho|^p \, dx \, dt. \]
Grownwall’s inequality yields that
\[ \int_D |\rho(x,t)|^p \, dx = 0 \]
and \( \rho \equiv 0 \). This means exactly the uniqueness of a solution.

Remark 3.8. Let \( c = 0 \), let \( a^{ij}, b^i \in C(D) \), and let \( \det A \) be nonvanishing. Assume also that a function \( V \) is such that \( V \in C^2(D) \) and
\[ \lim_{n \to \infty} \inf_{D_n \setminus D_{n-1}} V(x) = +\infty. \]
Suppose that for some number \( K > 0 \) and some number \( n \) the estimate \( LV(x) \leq -KV \) holds for every \( x \in D \setminus D_n \). Then, for every probability measure \( \nu \), there exists a unique solution to the Cauchy problem \( \mu = (\mu_t)_{t \in (0, +\infty)} \) given by probability measures \( \mu_t \). Moreover, the solution is ergodic, i.e., the measures
\[ \sigma_t(dx) = t^{-1} \int_0^t \mu_s(dx) \, ds \]
converge weakly to a probability solution \( \mu \) of the stationary equation \( L^* \mu = 0 \) on \( D \) as \( t \to +\infty \).
The existence and uniqueness of a solution on \((0, +\infty)\) follow from the theorems above. The solution is a probability solution by the fact that \(c = 0\) and the existence of a Lyapunov function. The Grönwall’s inequality and a reasoning, similar to the proof of Theorem 2.7 yield that
\[
\int_D V(x) \, d\sigma_t = t^{-1} \int_0^t \int_D V(x) \, d\mu_s \, ds \leq K_1,
\]
where \(K_1\) does not depend on \(t\). Hence, the family \(\sigma_t\) is uniformly tight and each sequence \(\sigma_{t_n}\) contains a subsequence weakly convergent to some measure \(\sigma\). Obviously, \(\sigma\) satisfies the equation \(L^*\sigma = 0\). By the uniqueness of a probability solution \(\sigma\) the whole subsequence \(\sigma_{t_n}\) converges to it (the uniqueness for stationary equations follows from the existence of a Lyapunov function and in the case of an arbitrary domain \(D\) can be justified in the same way as in [7] for \(D = \mathbb{R}^d\)).

**Remark 3.9.** Suppose that all conditions sufficient for the existence and uniqueness in the class \(\mathcal{M}_\nu\) are fulfilled for \(\mu\). Let \(\mu_{s,t}\) denote the solution of the Cauchy problem with the initial condition \(\mu_{|t=s} = \delta_y\). We observe that for every Borel set \(B\) the mapping \((s, y) \mapsto \mu_{s,y}(B)\) is measurable as a limit of measurable mappings which correspond to the solutions \(\mu_{s,y}^n\) of the approximating problems with smooth coefficients (in the existence theorems the solution is constructed in exactly the same way; the fact that the constructed solution is unique ensures convergence of the whole sequence and not only of some subsequence). Moreover, \(\mu_{s,y}^n\) satisfy the Kolmogorov–Chapman equations (see, for example, [11]). This ensures that \(\mu_{s,y}\) also satisfies these equations.

To conclude, we shall consider some more examples. We begin with the example from [12], which has already been mentioned in the introduction. We point out once again that this example motivated our investigation.

**Example 3.10.** Let \(\nu\) be a probability measure on \(D = (-1, 1)\). Given \(\alpha > 0\), we consider the Cauchy problem
\[
\partial_t \mu_t = \frac{1}{2} \partial_{xx} \left( |1 - |x||^{2\alpha} \mu \right) - \partial_x \left( \left( \text{tg} \left(\frac{-\pi x}{2}\right) + \text{sgn} x \right) \mu \right), \quad \mu_0 = \nu.
\]

(3.4)

Note that the coefficients in the above equations are rather irregular. The drift coefficient is discontinuous at \(x = 0, x = 2k + 1\). Moreover, the diffusion coefficient does not satisfy the linear growth condition for \(\alpha > 1/2\) and is not Hölder continuous with exponent \(1/2\) if \(\alpha < 1/2\).

Let us show that this Cauchy problem has a unique solution on \((0, \infty)\) follow from the theorems above. The solution is ergodic in the above sense. Indeed, due to (3.5), there exist a positive constant \(K_2\) and a number \(k \in \mathbb{N}\) such that on \(D \setminus D_k\) one has \(LV \leq -K_2 \cdot V\). According to Remark 3.8 this inequality ensures the weak convergence of measures
\[
\sigma_t(dx) = \frac{1}{t} \int_0^t \mu_s(dx) \, ds
\]
to a probability measure \(\sigma\) on \(D\) which solves the stationary equation \(L^*\sigma = 0\), as \(t \to +\infty\).
Example 3.11. Let \( \nu \) be a probability measure on \( D = \mathbb{R}^d \). Consider the Cauchy problem
\[
\partial_t \mu_x = \partial_x \partial_x_j (a_{ij} \mu) - \partial_x_i (b_i \mu) + c \mu, \quad \mu|_{t=0} = \nu.
\]
Suppose that (H1) and (H2) are fulfilled and that
\[
c \in L^{p/2}_{\text{loc}}(\mathbb{R}^d \times (0, T)), \quad b \in L^\infty(B(0, k) \times [0, T])
\]
for some \( p > d+2 \) and all \( k \), where \( B(0, k) \) is the ball of radius \( k \) centered at the origin. Let \( V(x) = |x|^2/2 \). Then the condition \( LV \leq K + KV \) takes the form
\[
\text{tr} A(x, t) + (b(x, t), x) + |x|^2 c(x, t)/2 \leq K + K|x|^2
\]
for all \( (x, t) \in \mathbb{R}^d \times [0, T] \). If the latter inequality holds, then the set \( \mathcal{M}_p \) consists of exactly one element.

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