THE VOLUME FLUX GROUP AND NONPOSITIVE CURVATURE

by Pablo Suárez-Serrato

Abstract. We show that every closed nonpositively curved manifold with non-trivial volume flux group has zero minimal volume, and admits a finite covering with circle actions whose orbits are homologically essential. This proves a conjecture of Kedra–Kotschick–Morita for this class of manifolds.

Let $M$ be a closed smooth manifold and $\mu$ a volume form on $M$. Denote by $\text{Diff}^\mu$ the group of $\mu$–preserving diffeomorphisms of $M$, and by $\text{Diff}_0^\mu$ its identity component. The $\mu$–flux homomorphism $\text{Flux}_\mu$, from the universal covering $\tilde{\text{Diff}}_0^\mu$ to the $(n-1)$–cohomology group $H^{n-1}(M; \mathbb{R})$, is defined by the formula

$$\text{Flux}_\mu([\phi_t]) = \int_0^1 [i_{\dot{\phi}_t} \mu] dt.$$ 

It induces a homomorphism

$$\text{Flux}_\mu : \pi_1(\text{Diff}_0^\mu) \to H^{n-1}(M; \mathbb{R})$$

whose image is the volume flux group $\Gamma_\mu \subset H^{n-1}(M; \mathbb{R})$. The $\mu$–flux homomorphism descends to a homomorphism

$$\text{Flux}_\mu : \text{Diff}_0^\mu \to H^{n-1}(M; \mathbb{R})/\Gamma_\mu.$$ 

For a closed connected smooth Riemannian manifold $(M, g)$, let $\text{Vol}(M, g)$ denote the volume of $g$ and let $K_g$ be its sectional curvature. We define the minimal volume of $M$ following J Cheeger and M Gromov [2]:

$$\text{MinVol}(M) := \inf_g \{ \text{Vol}(M, g) : |K_g| \leq 1 \}$$
The minimal volume is a very sensitive invariant, it was first observed by L Bessières [1] that its value may depend on the differentiable structure of \((M, g)\). Indeed, D Kotschick [6] has shown that even the vanishing of \(\text{MinVol}(M)\) can detect changes in the smooth structure of \(M\).

The investigation of relationships between the volume flux group of \(M\) and various invariants which bound \(\text{MinVol}(M)\) from below was set in motion by J Kedra, D Kotschick and S Morita [5]. They put forward the idea that if a closed manifold has non-trivial volume flux group then its minimal volume should vanish. The aim of this note is to verify that statement for closed manifolds which carry a metric of nonpositive sectional curvature.

The volume flux group \(\Gamma_\mu\) is independent of the form \(\mu\)—see section 3 of [5]—so it can be considered as an invariant of the manifold \(M\) itself.

**Theorem 1.** Every closed nonpositively curved manifold with non-trivial volume flux group has zero minimal volume.

The technique we will use to show that the minimal volume vanishes is an \(\mathcal{F}\)-structure, which was introduced by Gromov as a generalisation of an \(S^1\)-action.

**Definition 2.** An \(\mathcal{F}\)-structure on a closed manifold \(M\) is given by the following conditions.

1. A finite open cover \(\{U_1, \ldots, U_N\}\)
2. \(\pi_i: \tilde{U}_i \to U_i\) a finite Galois covering with group of deck transformations \(\Gamma_i\), \(1 \leq i \leq N\)
3. A smooth torus action with finite kernel of the \(k_i\)-dimensional torus, \(\phi_i: T^{k_i} \to \text{Diff}(\tilde{U}_i), 1 \leq i \leq N\)
4. A homomorphism \(\Psi_i: \Gamma_i \to \text{Aut}(T^{k_i})\) such that \(\gamma(\phi_i(t)(x)) = \phi_i(\Psi_i(\gamma)(t))(\gamma x)\) for all \(\gamma \in \Gamma_i\), \(t \in T^{k_i}\) and \(x \in \tilde{U}_i\)
5. For any finite sub-collection \(\{U_{i_1}, \ldots, U_{i_l}\}\) such that \(U_{i_1} \cap \ldots \cap U_{i_l} \neq \emptyset\) the following compatibility condition holds: let \(\tilde{U}_{i_1 \ldots i_l}\) be the set of points \((x_{i_1}, \ldots, x_{i_l}) \in \tilde{U}_{i_1} \times \ldots \times \tilde{U}_{i_l}\) such that \(\pi_{i_1}(x_{i_1}) = \ldots = \pi_{i_l}(x_{i_l})\). The set \(\tilde{U}_{i_1 \ldots i_l}\) covers \(\pi_{i_j}^{-1}(U_{i_1 \ldots i_l}) \subset \tilde{U}_{i_j}\) for all \(1 \leq j \leq l\), then we require that \(\phi_{ij}\) leaves \(\pi_{i_j}^{-1}(U_{i_1 \ldots i_l})\) invariant and it lifts to an action on \(\tilde{U}_{i_1 \ldots i_l}\) such that all lifted actions commute.
Volume flux and nonpositive curvature

An $\mathcal{F}$–structure is said to be pure if all the orbits of all actions at a point, for every point have the same dimension. We will say an $\mathcal{F}$–structure is polarised if the smooth torus action $\phi_i$ above are fixed point free for every $U_i$. The existence of a polarised $\mathcal{F}$–structure on a manifold $M$ implies the minimal volume $\text{MinVol}(M)$ is zero by the main result of Cheeger and Gromov [2], the interested reader is invited to consult the illuminating examples found there as well.

The attentive reader will notice that the definition of an $\mathcal{F}$–structure above is different from the sophisticated one found in [2]. Despite this, it is sufficiently practical to be implemented and also satisfies the properties needed in the proof that a polarised $\mathcal{F}$–structure forces the minimal volume to vanish, which can be consulted in [2].

Proof of Theorem 1. Let $M$ be a compact nonpositively curved manifold whose volume flux group is not trivial. The fundamental group $\pi_1(M)$ has non-trivial centre $Z$, a proof can be found in [3, Theorem 15] and compared with work of A Fathi [4, Proposition 5.1] and the various references and attributions contained therein. In this case $M$ admits a finite covering space $M^*$ diffeomorphic to $T^k \times N$, where $T^k$ is a flat torus of dimension $k$ and $N$ is a compact nonpositively curved manifold as was shown by P Eberlein [3].

Even though it may seem that this already implies $\text{MinVol}(M) = 0$, for completeness will now show how to construct a pure polarised $\mathcal{F}$–structure on $M$; since the torus $T^k$ splits off $M^*$ smoothly and the action of $T^k$ on itself as a factor of $M^*$ is compatible with the covering transformation in the required sense. This will also provide an example of a detailed construction of a pure polarised $\mathcal{F}$–structure.

Represent $M$ as $H/\Gamma$, where $H$ is simply connected and $\Gamma$ is a properly discontinuous group of isometries of $H$ which acts freely. The space $H$ decomposes into $H_1 \times H_2$, where $H_1 = \mathbb{R}^k$ is a Euclidean space of dimension $k = \text{rank}(Z)$.

Notice that every element $\gamma$ of $\Gamma$ is of the form $\gamma = \gamma_1 \times \gamma_2 \in \text{Iso}(H_1) \times \text{Iso}(H_2)$, here $\text{Iso}(H_i)$ denotes the group of isometries of $H_i$ (see Lemma 1 in [3], and the subsequent discussion). Let $p_i : \Gamma \rightarrow \text{Iso}(H_i)$ denote the projection homomorphisms, then $\Gamma_1 = p_1(\Gamma)$ acts by translations on $H_1$ and $\Gamma_2 = p_2(\Gamma)$ is a discrete subgroup of $\text{Iso}(H_2)$. So $Z \subset \Gamma_1$ and $H_1/Z$ is a compact flat torus $T^k$. The projection $p : H_1 \rightarrow T^k$ allows us to define $\rho : \Gamma_2 \rightarrow T^k$ by setting $\rho(p_2 \gamma) = p(p_1 \gamma)$. The function $\rho$ is well defined since $\ker(p) = \ker(p_2) = Z$. The centre $Z$ can also be thought of as a set of vectors in $H_1$, as $Z \subset \Gamma_1$. In this way $Z$ acts on $H_1$ by translations.
Recall that $\Gamma_2$ has trivial centre and that $M$ is isometric to $(T^k \times H_2)/\Gamma_2$ [3], here $\Gamma_2$ acts on $T^k \times H_2$ by $\psi(s, h) = (\rho(\psi)s, \psi(h))$ with $(s, h)$ in $T^k \times H_2$ and $\psi$ in $\Gamma_2$. This can be read from the diagram

\[
\begin{array}{ccc}
H = H_1 \times H_2 & \overset{p \times Id}{\longrightarrow} & T^k \times H_2 \\
\downarrow & & \downarrow q \\
M = H/\Gamma & \overset{F}{\longrightarrow} & (T^k \times H_2)/\Gamma_2
\end{array}
\]

where $F$ is defined so that the diagram commutes.

In $\Gamma_2$ there exists a finite index subgroup $\Gamma_0$—denoted by $\Gamma_2^*$ in [3]—which makes the following diagram commute.

\[
\begin{array}{ccc}
H & \longrightarrow & T^k \times (H_2/\Gamma_0) := M^* \\
\downarrow & & \downarrow \\
(T^k \times H_2)/\Gamma_2 = M
\end{array}
\]

Define $q^*: T^k \times (H_2/\Gamma_0) = M^* \rightarrow M$ as in the previous diagram and denote by $\Gamma^*$ the group of deck transformations of $q^*$ seen as a covering map. Notice that $\Gamma^* \subset \Gamma_2$.

The function $\rho$ is defined on all of $\Gamma_2$, so we can also consider the restriction of $\rho$ to $\Gamma^*$.

We are now in a position to verify that this construction gives $M$ a pure polarised $\mathcal{F}$–structure, and hence $\text{MinVol}(M) = 0$ as claimed. Let us check that every condition which guarantees the existence of a polarised $\mathcal{F}$–structure is met.

1. Take $U = M$, as an open cover with a single set.
2. Define $\bar{U} := M^* = T^k \times N$, here $N = H_2/\Gamma_0$. The quotient map

\[ q^*: \bar{U} = T^k \times N \rightarrow (T^k \times H_2)/\Gamma_2 = M = U \]

is a finite Galois covering with deck transformation group $\Gamma^*$.
3. The $k$–torus $T^k$ acts smoothly and without fixed points on itself as a factor of $T^k \times (H_2/\Gamma_0)$. So we have the action

\[ \phi: T^k \rightarrow \text{Diff}(\bar{U}) = \text{Diff}(T^k \times N) \]

given by $\phi(t)(s, h) = (s + t, h)$.
4. We will use the function $\rho$ to define the automorphism $\Psi: \Gamma^* \rightarrow \text{Aut}(T^k)$, set $\Psi(\gamma) := \rho(\gamma)$. Let $t \in T^k$ and $x \in \bar{U} = T^k \times N \Rightarrow x = (s, h) \in T^k \times N$. Notice that for $s$ and $t$ in $T^k$ we have that $\rho(\gamma)(s + t) = (\rho(\gamma)s + \rho(\gamma)t)$ since $\rho(\gamma) \in T^k$. 

\[\boxed{4}\]
Volume flux and nonpositive curvature

We plug in this information to obtain the following equalities.

\[
\gamma(\phi(t)x) = \gamma(\phi(t)(s, h)) \\
= \gamma(s + t, h) \\
= (\rho(\gamma)(s + t), \gamma h) \\
= (\rho(\gamma)s + \rho(\gamma)t, \gamma h) = *
\]

\[
\phi(\Psi(\gamma)t)(\gamma x) = \phi(\rho(\gamma)t)(\gamma x) \\
= \phi(\rho(\gamma)t)(\rho(\gamma)s, \gamma h) \\
= (\rho(\gamma)s + \rho(\gamma)t, \gamma h) = *
\]

Therefore \(\gamma(\phi_i(t)(x)) = \phi_i(\Psi_i(\gamma)(t))(\gamma x)\) and the condition is satisfied.

(5) This condition does not need to be verified, because we only have one covering set.

Since the action of \(T^k\) on itself as a factor of \(T^k \times (H_2/\Gamma_0)\) is fixed point free, the above construction gives \(M\) a pure polarised \(\mathcal{F}\)–structure. □

We will state the contrapositive statement to the theorem because it strengthens Corollary 17 of [5]: Let \(M\) be a closed nonpositively curved manifold with positive minimal volume. Then the volume flux group \(\Gamma_\mu\) is trivial for every volume form \(\mu\) on \(M\).

Another noteworthy feature of the covering \(M^* \to M\) is that on \(M^* \cong T^k \times N\) elements of the volume flux group which come from rotations on the \(T^k\) factor are circle actions with homologically essential orbits. It is not yet clear if in the general case a non-trivial volume flux group implies existence of a circle action with homologically essential orbits at least in a multiple cover—compare with Remark 19 in [5]—but it is virtually so for every closed nonpositively curved manifold \(M\), since it is true for \(M^*\).

Acknowledgements: I wish to warmly thank Dieter Kotschick for a number of interesting conversations and for commenting on a previous version of this work.

References

[1] L. Bessières, *Un théorème de rigidité différentielle*, Comment. Math. Helv. 73 (1998), no. 3, 443–479.
[2] J. Cheeger and M. Gromov, *Collapsing Riemannian Manifolds while keeping their curvature bounded I*, Jour. Differential Geom. 23 (1986) 309–346.
[3] P. Eberlein, *A canonical form for compact nonpositively curved manifolds whose fundamental groups have nontrivial center*, Math. Ann. 260 (1982) no. 1, 23–29.

[4] A. Fathi, *Structure of the group of homeomorphisms preserving a good measure on a compact manifold*, Ann. Sci. École Norm. Sup. (4) 13 (1980), no. 1, 45–93.

[5] J. Kedra, D. Kotschick and S. Morita, *Crossed Flux Homomorphisms and Vanishing Theorems for Flux Groups*, GAFA 16 (2006) 1246–1273.

[6] D. Kotschick, *Entropies, Volumes and Einstein Metrics*, Preprint (2004) [arXiv:math.DG/0410215](http://arxiv.org/abs/math.DG/0410215).

Mathematisches Institut LMU, Theresienstrasse 39, München 80333 Germany.

CIMAT, CP: 36240, Guanajuato, Gto, México.

E-mail address: p.suarez-serrato@cantab.net