On the $ER(2)$-cohomology of some odd-dimesional projective spaces

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Abstract

Kitchloo and Wilson have used the homotopy fixed points spectrum $ER(2)$ of the classical complex-oriented Johnson-Wilson spectrum $E(2)$ to deduce certain non-immersion results for real projective spaces. $ER(n)$ is a $2^{n+2}(2^n-1)$-periodic spectrum. The key result to use is the existence of a stable cofibration $\Sigma^{\lambda(n)}ER(n) \to ER(n) \to E(n)$ connecting the real Johnson-Wilson spectrum with the classical one. The value of $\lambda(n)$ is $2^{2n+1} - 2^{n+2} + 1$. We extend Kitchloo-Wilson’s results on non-immersions of real projective spaces by computing the second real Johnson-Wilson cohomology $ER(2)$ of the odd-dimensional real projective spaces $RP^{16K+9}$. This enables us to solve certain non-immersion problems of projective spaces using obstructions in $ER(2)$-cohomology.

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1 Introduction

The spectrum $MU$ of complex cobordism comes naturally equipped with an action of $\mathbb{Z}/2$ by complex conjugation. Hu and Kriz in [4] have used this action to construct genuine $\mathbb{Z}/2$ equivariant spectra $ER(n)$ from the complex-oriented spectra $E(n)$. Kitchloo and Wilson in [7] have used the homotopy fixed point spectrum of this to solve certain non-immersion problems of real projective spaces. The homotopy fixed point spectrum $ER(n)$ is $2^{n+2}(2^n-1)$-periodic compared to the $2(2^n-1)$-periodic $E(n)$. The spectrum $ER(1)$ is $KO(2)$ and $E(1)$ is $KU(2)$.

Kitchloo and Wilson have demonstrated the existence of a stable cofibration connecting $E(n)$ and $ER(n)$,

\[ \Sigma^{\lambda(n)}ER(n) \xrightarrow{x} ER(n) \xrightarrow{} E(n) \]  

where $\lambda(n) = 2^{2n+1} - 2^{n+2} + 1$. This leads to a Bockstein spectral sequence for $x$-torsion. It is known that $x^{2^{n+1} - 1} = 0$ so there can be only $2^{n+1} - 1$ differentials. For the case of our interest $n = 2$ there are only 7 differentials.

From [5] we know that if there is an immersion of $RP^b$ to $\mathbb{R}^c$ then there is an axial map

\[ RP^b \times RP^{2^k-c-2} \to RP^{2^k-b-2}. \]  

For $b = 2n$ and $c = 2k$ Don Davis shows in [2] that there is no such map when $n = m + \alpha(m) - 1$ and $k = 2m - \alpha(m)$, where $\alpha(m)$ is the number of ones in the binary expression of $m$ by finding an obstruction to James’s map (2) in $E(2)$-cohomology. Kitchloo and Wilson get new non-immersion results by computing obstructions in $ER(2)$-cohomology. In this paper we extend Kitchloo-Wilson’s results by computing the $ER(2)$-cohomology of the odd projective space $RP^{16K+9}$. This will give us newer non-immersion results. The main results are the following.

Theorem 1.1. A 2-adic basis of $ER(2)^{8K+9}(RP^{16K+9,*})$ is given by the elements

\[ \alpha^k w^j, \ (k \geq 0, 1 \leq j \leq 8K + 4) \]

\[ v_2^j \alpha^k w^j, \ (k \geq 1, 1 \leq j \leq 8K + 4) \]
\[ \nu^j \eta^j, \quad 4 \leq j \leq 8K + 4 \]
\[ x\alpha^k \iota_{16K+9}, \quad x\nu^4 \alpha^k \iota_{16K+9}, \quad (k \geq 0) \]

**Theorem 1.2.** Let \( \alpha(m) \) be the number of ones in the binary expansion of \( m \). If \( (m, \alpha(m)) \equiv (6, 2) \) or \((1, 0) \) mod 8, \( RP^{2(m + \alpha(m) - 1)} \) does not immerse in \( \mathbb{R}^{2(2m - \alpha(m)) + 1} \).

This shall give us new non-immersions that are often new and different from those of [7] and [8]. Using Davis’s table [1] the first new result is \( RP^{213} \) does not immerse in \( RP^{214} \).

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### 2 The Bockstein spectral sequence

The results obtained in this section can be found in [7]. We reproduce it here for the convenience of the reader.

We have the stable cofibration

\[ \Sigma^{\lambda(n)} ER(n) \xrightarrow{x} ER(n) \xrightarrow{\partial} E(n) \]

where \( x \in ER(n)^{-\lambda(n)} \) and \( \lambda(n) = 2^{2n+1} - 2^{n+2} + 1 \). The fibration gives us a long exact sequence

\[ \begin{array}{ccc}
ER(n)^*(X) & \xrightarrow{x} & ER(n)^*(X) \\
\downarrow\partial & & \downarrow\rho \\
E(n)^*(X) & & \\
\end{array} \]  

where \( x \) lowers the degree by \( \lambda(n) \) and \( \partial \) raises the degree by \( \lambda(n) + 1 \). This leads to the Bockstein spectral sequence, which will completely determine \( M = ER(n)^*(X)/(x) \) as a subring of \( E(n)^*(X) \).

We know that \( x^{2n+1} - 1 = 0 \) so there can be only \( 2^{n+1} - 1 \) differentials.

By filtering \( M \),

\[ 0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_{2^{n+1} - 1} = M \]

by submodules

\[ M_r = \text{Ker} \left[ x^r : \frac{ER(n)^*(X)}{x} \to \frac{x^r ER(n)^*(X)}{x^{r+1}} \right] \]

so that \( M_r/M_{r-1} \) gives the \( x^r \)-torsion elements of \( ER(n)^*(X) \) that are non-zero in \( M \).

We collect the basic facts about the spectral sequence in the following theorem. \( E(n) \) is a complex oriented spectrum with a complex conjugation action. Denote this action by \( c \).
Theorem 2.1. [4, Theorem 4.2] In the Bockstein spectral sequence for ER(n)* (X)

1. The exact couple (3) gives rise to a spectral sequence, \( E^r \), of ER(n)*-modules, starting with

\[ E^1 \cong E(n)^*(X). \]

2. \( E^{2n+1} = 0 \)

3. \( \text{Im} \ d^r \cong M_r/M_{r-1} \).

4. The degree of \( d^r \) is \( r\lambda(n) + 1 \).

5. \( d^r(ab) = d^r(a)b + c(a)d^r(b) \)

6. \( d^1(z) = v_n^{-(2^n-1)}(1-c)(z) \) where \( c(v_1) = -v_1 \).

7. If \( c(z) = z \) in \( E^1 \), then \( d^1(z) = 0 \). If \( c(z) = z \) in \( E^r \) then \( d^r(z^2) = 0 \).

8. The following are all vector spaces over \( \mathbb{Z}/2 \):

\[ M_j/M_i, \ (j \geq i > 0) \text{ and } E^r, \ (r \geq 2). \]

Note that the image of \( ER(n)^*(X) \to E(n)^*(X) \) consists of targets of the differentials and therefore always have the differentials trivial on them. Also anything in the image is trivial under the action of \( c \).

Since \( ER(n)^*(-) \) is \( 2n+2(2^n-1) \)-periodic we will consider it as graded over \( \mathbb{Z}/(2^{n+2}(2^n-1)) \).

We have to do the same then for \( E(n)^*(-) \). We can do this by setting the unit \( v_n^{2n+1} = 1 \) in the homotopy of \( E(n) \). (This does not lose any information since we can always recover the original by inserting powers of \( v_n^{2n+1} \) to make the degrees match.)

\[ 2.1 \text{ The spectral sequence for } ER(2)^* \]

For the Bockstein spectral sequence of Theorem 2.1 to be useful, we need to know the ring \( ER(n)^* \). From now on we concentrate on the case \( n = 2 \). This spectral sequence begins with \( E^1 = ER(2)^* \) which is just a free \( \mathbb{Z}_2[v_1] \)-module on a basis given by \( v_i^j \) for \( 0 \leq i < 8 \). We are grading mod 48. Since all elements of \( E^1 \) have even degree and degree deg \( d^r = 17r+1 \), \( d^r = 0 \) for \( r \) even. As \( E^8 = 0 \), we only have \( d^1, d^3, d^5, d^7 \) to consider.

We have the differential \( d^1 \) acting as follows:

\[ d^1(v_2^{2s+1}) = v_2^{-3}(1-c)v_2^{2s+1} = v_2^{-3}2v_2^{2s+1} = 2v_2^{2s-2}. \]

Similarly \( d^1(v_2^{2s}) = 0 \).

However multiplication by \( v_1 \) doesn’t behave well with respect to this differential. The problem is that \( v_1 \in ER(2)^* \) does not lift to \( ER(2)^* \). We will need a substitute for \( v_1 \). We shall use the element \( \alpha = v_1^{ER(2)} \in ER(2)^{(2)} \) ([4, page 13]). The image of \( \alpha \) is \( v_1v_2^3 \in E(2)^{-32} \). Because \( v_2 \) is a unit this a good substitute for ordinary \( v_1 \). Furthermore this is invariant under \( c \) because it is in the image of the map from \( ER(2)^* \) and is a permanent cycle. Or we could just see this by observing that there are an even number of \( v \)’s. We rewrite the homotopy of \( E(2) \) as \( \mathbb{Z}_2[\alpha, v_2^{\pm 1}] \) but again set \( v_2^8 = 1 \).

Now back to the computation of \( d^1 \) on \( v_2^{2s+1} \) where the \( E_1 \) term is a free \( \mathbb{Z}_2[\alpha]\)-module on generators \( v_2^i, \ 0 \leq i \leq 8 \).

\[ d^1(\alpha^kv_2^{2s+1}) = d^1(\alpha^kv_2^{2s+1}) + c(\alpha^k)d^1(v_2^{2s+1}) = 0 + \alpha^k2v_2^{2s-2}. \]

But this really follows from the fact that we have a spectral sequence of \( ER(2)^* \)-modules. Thus the \( d^1 \)-cycles form a free \( \mathbb{Z}_2[\alpha] \)-module generated by \( \{1, v_2^{2}, v_2^{4} \} \) and the \( d^1 \)-boundaries form the free
submodule with basis \( \{ \alpha_0, \alpha_1, \alpha_2, \alpha_3 \} \), where \( \alpha_i = 2v_2^i \). In particular, \( \alpha_0 = 2 \). Thus \( E^2 = E^3 \) is the free \( \mathbb{F}_2[\alpha] \)-module with the basis (the images of) \( \{ 1, v_2^2, v_2^4, v_2^6 \} \).

By \( \mathbb{Z}_2 \), \( d^2(v_2^2) = \alpha v_2^4 \). Since \( d^2 \) is a derivation, \( d^2(v_2^6) = \alpha, \) and the only elements of \( E^4 \) are 1 and \( v_2^4 \). Since \( \deg d^2 = 38, d^0 = 0 \). We must have \( d^2(v_2^2) = 1 \) to make \( E^5 = 0 \).

We can read off \( M \) as a \( \mathbb{Z}_2[\alpha] \)-module of \( E^1 \). \( M_1 = \text{Im} d^1 \) is the free module on basis \( \{ \alpha, \alpha_1, \alpha_2, \alpha_3 \} \). \( M_3 \) is generated as a module by adding the elements \( \alpha \) and \( w = \alpha v_2^4 \), which make \( M_3/M_1 \) the free \( \mathbb{F}_2[\alpha] \)-module with basis \( \{ \alpha, w \} \). Finally, the only new element of \( M_7 \) is 1. The rest of the module structure is given by

\[
2\alpha = \alpha \alpha_0, 2w = \alpha \alpha_2, 2.1 = \alpha, 1 = \alpha
\]  

Further, \( M \) is a subring of \( E^1 \), generated by \( \alpha_1, \alpha_2, \alpha_3, \alpha \) and \( w \). The products not already given are

\[
\alpha_s \alpha_t = 2 \alpha_{s+t} \quad \text{(taking } s + t \text{ mod } 4)  
\]

\[
w \alpha_s = \alpha \alpha_{s+2} \quad \text{(taking } s + 2 \text{ mod } 4)  
\]

\[
w^2 = \alpha^2
\]

To obtain \( E^2(R)^+ \), we must unfilter \( M = E^2(R)^+/(x) \) and add the generator \( x \). We lift each generator \( \alpha_s \) and \( w \) to \( E^2(R)^+ \), keeping the same names, with the relations

\[
\alpha_s x = 0, \alpha x^3 = 0, wx^3 = 0, x^7 = 0.
\]

By sparseness, each \( \alpha_s \) and \( w \) lifts uniquely to \( E^2(R)^+ \); further, the module actions (4) and multiplications (5), (6), (7) also lift uniquely and hold in \( E^2(R)^+ \), not merely \( \text{mod}(x) \).

**Proposition 2.1.** \( \mathbb{Z}_2 \). \( E^2(R)^+ \) is graded over \( \mathbb{Z}/48 \). It is generated as a ring by elements, \( x, w, \alpha, \alpha_1, \alpha_2, \alpha_3 \) of degrees \(-17, -8, -32, -12, -24 \) and \(-36 \) respectively, with relations and products as listed above.

### 2.2 The Bockstein Spectral Sequence for \( E^2(R)^+(RP^\infty) \)

As always, we have the split short exact sequence

\[
0 \to E^2(R)^+(X, \ast) \to E^2(R)^+(X) \to E^2(R)^+(\ast) \to 0
\]

for any space \( X \) with basepoint \( \ast \). Now that we know \( E^2(R)^+(\ast) = E^2(R)^+ \), we will concentrate on \( E^2(R)^+(X, \ast) \). Nevertheless, we need to use the action of \( E^2(R)^+ \) on \( E^2(R)^+(X) \) and hence \( E^2(R)^+(X, \ast) \); furthermore, this action extends to actions of the Bockstein spectral sequences.

The \( E^2(R)^+ \)-cohomology of \( RP^\infty \) can be computed from the Gysin sequence

\[
\ldots E^2(R)^{(CP^\infty)} \longrightarrow E^1(R)^{(CP^\infty)} \longrightarrow E^2(R)^{(RP^\infty)} \longrightarrow E^2(R)^{(CP^\infty)} \ldots
\]

where \( E^2(R)^{(CP^\infty)} \simeq E^2(R)^{([x_2]), x_2 \in E^2(R)^{(CP^\infty)} \text{. From above } E^2(R)^{(RP^\infty)} \simeq E^2(R)^{([x_2])/([2](x_2))} \text{. Recall that } E^2(R)^{(RP^\infty)} = E^2(R)^{([u]/([2](u)) \text{ where we have } u \in E^2(R)^{([1]-(2)(RP^\infty)) \text{. We will replace the } x_2 \text{ by the image of } u \in E^2(R)^{([16](RP^\infty)) \text{ which we also call } u \in E^2(R)^{([16](RP^\infty)) \text{, which is really } v_2^3x_2 \text{. Likewise we replace the usual } v_1 \in E^2(R)^{-2} \text{ with } v_2^3v_1 = \alpha \in E^2(R)^{-32} \text{ which comes from } \alpha \in E^2(R)^{-32} \text{. The element } w \in E^2(R)^{-8} \text{ maps to } \alpha v_2^3 = v_2v_1 \in E^2(R)^{-8} \text{. These changes are necessary because } x_2 \text{ and } v_1 \text{ are not in the image of } E^2(R)-\text{cohomology.}}

We will describe our groups in terms of a \( 2\)-adic basis in the sense of \( \mathbb{Z}_2 \), i.e., a set of elements such that any element in our group can be written as a unique sum of these elements with coefficients 0 or 1 (where the sum is allowed to be a formal power series in \( u \)). In the ring \( E^2(R)^{(RP^\infty)} \) we have \( 2u = \alpha u^2 + \ldots \), therefore the \( 2\)-adic basis is given by \( v_2^i \alpha^k w^j, (0 \leq i < 8, 0 \leq k, i \leq j) \).
The original relation \([2](u) = 2u + x v_1 u^2 + x v_2 u^4\) for \(ER(2)\) converts to the relation

\[
[2](u) = 2u + x u^2 + f u^4
\]

since \(v_1\) is replaced by \(v_1(v_2^3)^{-1} = v_1 v_2^5 = \alpha\) and \(v_2\) is replaced by \(v_2(v_2^3)^{-3} = v_2^{-8} = 1\). Because \(2x = 0\), \(x\) times the relation \((9)\) gives us \(0 = x(\alpha u^2 + f u^4)\). Therefore from the point of view of \(x^1\)-torsion \(\alpha u^2\) can be replaced with \(u^4 + \ldots\). Similarly, if we multiply by \(x^3\) and use the relation \(x^3\alpha = 0\) we end up with \(x^3 u = 0\).

Since \(u \in E(2)^\ast(RP^\infty)\) is in the image from \(ER(2)^\ast(RP^\infty)\) our differentials commute with multiplication by \(u\) and also commute with multiplication by \(\alpha\). The \(d^3\) differential creates a relation coming from our relation \(0 = 2u + x(\alpha u^2 + f u^4)\) when \(2u\) is set to zero. So in \(E^2\), we have \(\alpha u^2 \equiv u^4 + \ldots\). The Bockstein spectral sequence goes like this:

**Theorem 2.2.** [5 Theorem 8.1]

\[E^1 = E(2)^\ast(RP^\infty, \ast)\] is represented by

\[v_2^a \alpha^k u^j \quad (0 \leq i \leq 7, \quad 0 \leq k, \quad 1 \leq j).\]

\[d^1(v_2^{2s-5} \alpha^k w^j) = 2v_2^{s} \alpha^k w^j = v_2^{s} \alpha^{k+1} w^{j+1} + \ldots\]

\[E^2 = E^3\] is given by:

\[v_2^{s} \alpha^k u, \quad v_2^{s} w^j \quad (2 \leq j, \quad 0 \leq s \leq 4, \quad 0 \leq k)\]

\[d^2(v_2^{4s-2} \alpha^k u) = v_2^{4s} \alpha^{k+1} u, \quad d^2(v_2^{4s-2} w^j) = v_2^{4s} \alpha w^j = v_2^{4s} w^{j+2} + \ldots\]

\[E^4 = E^5 = E^6 = E^7\] is given by:

\[v_2^{4} u^{(1-3)} \quad u^{(1-3)}\]

\[d^6(v_2^{4} u^{(1-3)}) = u^{(1-3)}\].

The \(x^4\)-torsion generators are given by:

\[\alpha_i \alpha^k u^j \quad (0 \leq i, \quad 0 \leq k, \quad 1 \leq j)\]

where \(\alpha_0 = 2\).

The \(x^3\)-torsion generators are given by:

\[\alpha^{k+1} u, \quad \alpha^{k} w^j \quad (k \geq 0), \quad u^j \quad (j \geq 4), \quad w^j \quad (j \geq 2).\]

The only \(x^7\)-torsion generators are

\[u^{(1-3)}\].

In degrees that are multiples of 8 (denoted \(8\ast\)), the description of \(ER(2)^\ast(RP^\infty, \ast)\) simplifies enormously. As \(x\) is the only generator whose degree is not a multiple of 4, and \(x^4\) kills everything except powers of \(u\), multiplication by powers of \(x\) produces no new elements in degree \(8\ast\).

**Corollary 2.1.** [5 Theorem 2.1] The homomorphism

\[ER(2)^{8\ast}(RP^\infty, \ast) \rightarrow E(2)^{8\ast}(RP^\infty, \ast)\]

is injective, and is almost surjective – the only elements not hit are \(v_2^{j} u^j\) for \(1 \leq j \leq 3\).
We shall compute the Bockstein spectral sequence associated to $ER(2)$ for the odd-dimensional projective space $RP^{16K+9}$. This gives us nonimmersion results which are often new and differ from those of [S]. We will have to introduce some new elements for the computations in the next section. The Atiyah-Hirzebruch spectral sequence for $ER(2)^*(RP^2)$ gives elements $x_1$ and $x_2$ in filtration degrees 1 and 2. As an $ER(2)^*$ module $ER(2)^*(RP^2)$ is generated by elements we will call $z_{-16}$ represented by $x_1$ and $z_2$ represented by $x_2$. The element $z_{-16} = u \in ER(2)^*(RP^2)$.

For the cofibration $S^1 \to RP^2 \to S^2$, the long exact sequence

$$
\begin{array}{ccc}
ER(2)^*(S^1) & \xrightarrow{i^*} & ER(2)^*(RP^2) \\
\downarrow{\partial} & & \downarrow{\rho^*} \\
ER(2)^*(S^2, \ast)
\end{array}
$$

is given by $\partial(i_1) = 2i_2, \rho^*(i_2) = z_2$, and $i^*(u) = x_1$.

We know that $\Sigma^{2n-2}ER(2)^*(RP^2, \ast) \simeq ER(2)^*(RP^{2n}/RP^{2n-2}, \ast)$. We have elements $z_{2n-18}, z_{2n} \in ER(2)^*(RP^{2n}/RP^{2n-2}, \ast)$. These elements map to $v_2^m u^n$ and $v_2^n u^n$ respectively in $E(2)^*(RP^{2n}/RP^{2n-2}, \ast)$ by [7] section 10].

3 \quad $ER(2)^*(RP^{16K+9}, \ast)$

In [7] and [8] Kitchloo and Wilson considered all even dimensional real projective spaces and the odd dimensions $16K + 1$. Our object is to extend the results to more odd-dimensional cases.

**Proposition 3.1.** The element $u^{8K+5} \in ER(2)^*(RP^{16K+10})$ maps to a non-zero element of $ER(2)^*(RP^{16K+9})$.

Note that this cannot happen for a complex oriented cohomology theory.

**Proof.** We use the exact sequence

$$
ER(2)^*(S^{16K+10}, \ast) \xrightarrow{q^*} ER(2)^*(RP^{16K+10}) \to ER(2)^*(RP^{16K+9})
$$

We only have to show that $u^{8K+5}$ is not in the image of $q^*$. Now $ER(2)^*(S^{16K+10}, \ast)$ is the free $ER(2)^*$-module generated by the element $i_{16K+10}$, and $q^*$ is known.

The structure of $ER(2)^*(RP^{16K+10})$ is given by [7] Theorem 13.2 and [7] Theorem 13.3. We give a complete description of $M$, where

$$
ER(2)^*(RP^{16K+10}, \ast)/x ER(2)^*(RP^{16K+10}, \ast) \simeq M \subset E(2)^*(RP^{16K+10}, \ast)
$$

as a submodule of $E(2)^*(RP^{16K+10}) = Z[2][\alpha, v_2^{+1}; u]/(u^{8K+6}, [2](u))$. We describe $M$ by specifying a 2-adic basis. As $d^r = 0$ for $r$ even, $M$ is filtered by $0 = M_0 \subset M_1 = M_2 \subset M_3 = M_4 \subset M_5 = M_6 \subset M_7 = M$, where $M_r/M_{r-1} \simeq \text{Im}d^r$. Elements of $M_r$ not in $M_{r-1}$ lift to $x^r$-torsion elements of $ER(2)^*(RP^{16K+10}, \ast)$.

As both $\alpha$ and $u$ come from $ER(2)$-cohomology, we may describe $M$ from section 2 as a filtered $\mathbb{Z}_2[\alpha, u]$-module. We write $z_i$ for various elements of $ER(2)^*(RP^{16K+10}, \ast)$ and $\tilde{z}_i$ for its image in $M$, where $t$ denotes the degree.

$\alpha^k u^j (ua_x) \subset M_1$ for $k \geq 0, 0 \leq j \leq 8K + 4, 0 \leq s \leq 3$, where $ua_x = 2v_2^{s}u = d^j (v_2^{2s+3} u)$. Note that $\alpha_0 = 2$.

$a^k (ua) \in M_2$ for $k \geq 0$, where $ua = d^3 (v_2^{2s} u)$.

$a^k (uw) \in M_3$ for $k \geq 0$, where $uw = d^3 (v_2^{3} u) = w v_2^{+3}$.

$\alpha^k u^j \beta_0 \subset M_4$ for $k \geq 0, 0 \leq j \leq 8K + 1$, where $\beta_0 = d^3 (v_2^{j} u^2) = \alpha u^2 \equiv u^4 + \ldots \mod 2 = \alpha_0$.

$\alpha^k u^j \beta_1 \subset M_5$ for $k \geq 0, 0 \leq j \leq 8K + 1$ where $\beta_1 = d^3 (v_2^{j} u^2) = w u^2$.

$\alpha^k \gamma_0 \subset M_6$ for $k \geq 0$, where $\gamma_0 = v_2 \alpha u^{8K+5} = d^3 (v_2^{j} u^{8K+5})$.

$\alpha^k \gamma_1 \subset M_7$ for $k \geq 0$, where $\gamma_1 = v_2^{j} u^{8K+5} = d^3 (v_2^{j} u^{8K+5})$. 


The odd part is a free $d$-module. Note that $\alpha z_{16K+10} = \gamma_0$ and $wz_{16K+10} = \gamma_1$.

Since $E$ and $\alpha$, $\alpha z_{16K-14}$ is $d$-cohomology of $RP^{16K+10}$.

Now in $ER(2)$-cohomology, $q^*i_{16K+10} = v_2u^{8K+5}$, from [7] (13.1). Since the degrees of $\alpha_2, \alpha, w$ and $x$ are $-12, 16, 40$ and $-17$ respectively, mod 48, the only elements in $ER(2)^*i_{16K+10}$ of degree $-16(8K+5)$ have the form $\alpha^kwx^2i_{16K+10}$. Then we have $q^*(\alpha^kwx^2i_{16K+10}) = v_2\alpha^kwx^2u^{8K+5}$, which lies in the ideal $(x)$ and is not the same as $u^{8K+5}$.

### 3.1 The Bockstein spectral sequence for $ER(2)^*(RP^{16K+9})$

We compute this cohomology by sandwiching it between the $ER(2)$-cohomology of $RP^{16K+10}$ and $RP^{16K+8}$, which we know, in the commutative diagram of exact sequences

$$ER(2)^*(S^{16K+10,*}) \to ER(2)^*(S^{16K+10,*})$$

$$\downarrow \rho'$$

$$ER(2)^*(RP^{16K+10,*}) \to ER(2)^*(RP^{16K+10,*}) \to ER(2)^*(RP^{16K+8,*}) \downarrow \rho'$$

$$\downarrow \rho'$$

$$ER(2)^*(RP^{16K+9,*}) \to ER(2)^*(RP^{16K+9,*}) \to ER(2)^*(RP^{16K+8,*}) \to ER(2)^*(RP^{16K+9,*})$$

$$\downarrow 2$$

$$ER(2)^*(S^{16K+9,*})$$

The $E_1$-term for the Bockstein spectral sequence is just $E(2)^*(RP^{16K+9,*})$, which decomposes [2] as

$$E(2)^*(RP^{16K+8,*}) \oplus E(2)^*(S^{16K+9,*})$$

via the maps

$$RP^{16K+8} \to RP^{16K+9} \to S^{16K+9}$$

(In general, $E(2)^*(RP^{2n}) = E(2)^*[u]/(u^{n+1},[2](u))$, as $E(2)$ is complex oriented and $E(2)^*$ has no 2-torsion.) The even part of the $E_1$-term has the 2-adic basis

$$v_2^i\alpha^ku^j \quad (0 \leq i \leq 7, \quad 0 \leq k, \quad 0 \leq j \leq 8K+4)$$

The odd part is a free $Z_{(2)}$-module with basis

$$v_2^i\alpha^k i_{16K+9} \quad (0 \leq i \leq 7, \quad 0 \leq q, \quad 0 \leq k)$$

Since $d^1$ is of even degree we can just read off our $d^1$ from [7] Theorem 13.2 for $RP^{16K+8}$ and section 5 of [7] for the $S^{16K+9}$ part.

$$d^1(v_2^{2a-5}\alpha^ku^j) = v_2^{2a}\alpha^ku^j = v_2^{2a+1}u^j$$

$$d^1(v_2^{2a+1}\alpha^k i_{16K+9}) = v_2^{2a+2}\alpha^k i_{16K+9}$$
Thus $E^2$ is given by
\[ v_2^{2k} \alpha^k u \quad (k \geq 0), \quad v_2^{2j} u^j \quad (1 \leq j \leq 8K + 4), \quad v_2^{2s+1} \alpha^k u^{8K+4} \quad (0 \leq k), \]
\[ v_2^{2s} \alpha^s i_{16K+9} \]

$d^2$ has odd degree 35. Since we have both odd and even degree elements in the $E^2$-term, $d^2$ might very well be non-trivial. If it is, then by naturality, it must have its source in the $R\bar{P}^{16K+8}$ part and target in the $S^{16K+9}$ part. Also, the source cannot be anything from the $E^2$-term for the BSS for $R\bar{P}^\infty$. For we know that $d^2$ is trivial there. Therefore the only possible sources are $v_2^{2s+1} \alpha^k u^{8K+4}$ with possible targets $v_2^{2s} \alpha^s i_{16K+9}$.

Now, since $v_2^2$ is a unit, if there is a $d^2$, it must be non-zero on $v_2^{-1} u^{8K+4}$ which has degree $+6 - 16(8K + 4)$ which is $-10 + 16K$ mod 48. The degree of the target must be this plus 35. The possible targets have degrees $-12s - 32k + 16K + 9$. The only solutions are $\alpha i_{16K+9}, \alpha^4 i_{16K+9}$ etc.

If $d^2$ is non-zero our guess would be $d^2(v_2^{-1} u^{8K+4}) = \alpha i_{16K+9}$. Thus we need to show that $x^2 \alpha i_{16K+9} = 0$ in order for our guess to be correct.

The left column of (11) shows that the $ER(2)^*\text{-module} ER(2)^*(RP^{16K+10}/R\bar{P}^{16K+8}, *)$ is generated by two elements $z_{16K+10} = \rho^*i_{16K+10}$ and $z_{16K-8}$, where $i^*z_{16K-8} = x_{16K+9}$.

We want to show that $x^2\alpha i_{16K+9} = 0$ in $ER(2)^*(RP^{16K+9}, *)$. This is the image of the same named element in $ER(2)^*(S^{20K+9}, *)$ which lifts to $x\alpha z_{20K-8} \in ER(2)^*(RP^{16K+10}, *)$. This maps to $x\alpha v_2^2 u^{8K+5}$ in $ER(2)^*(RP^{16K+10}, *)$ from [13,1]. Since $2x = 0$, we can use the relation $[2](u) = 2u + F u^2 + F u^3 = 0$ on $\alpha v_2^2$, and we get
\[ x\alpha v_2^2 u^{8K+5} = x\alpha v_2^2 u^{8K+7} + \ldots \]

The least power of $u$ which is zero in $ER(2)^*(RP^{16K+10})$ is $8K + 8$. We have noted that $u^{8K+7} = x^3 z_{16K+4}$, so that we have $x^5 v_2^2 z_{16K+4}$, which is non-zero and does not lift to $S^{16K+10}$. It follows that $x^2 \alpha i_{16K+9} \neq 0$.

However, if we multiply the whole calculation by $\alpha^3$, we get $\alpha^3 x^5 v_2^2 z_{16K+4} = 0$, as $x^3 \alpha = 0$. So $x^2 \alpha^4 i_{16K+9} = 0$, and we conclude that
\[ d^2(v_2^{2s+1} \alpha^k u^{8K+4}) = v_2^{2s+2} \alpha^k u^{8K+4} i_{16K+9} \]

Then, $E^3$ is given by
\[ v_2^{2s} \alpha^k u \quad (0 \leq k), \quad v_2^{2s} u^j \quad (1 \leq j \leq 8K + 4), \quad v_2^{2s} \alpha^{(0-3)} i_{16K+9}. \]

$d^3$ is even degree so the even and odd parts don’t mix under the differential. On both parts we already know the $d^3$ differential:
\[ d^3(v_2^{(6,2)} \alpha^k u) = v_2^{(0,4)} \alpha^{k+1} u \]
\[ d^3(v_2^{(6,2)} u^j) = v_2^{(0,4)} \alpha u^j = v_2^{(0,4)} u^{j+2} \quad (1 \leq j \leq 8K + 2) \]
\[ d^3(v_2^{4s-2} \alpha^{(0-2)} i_{16K+9}) = v_2^{4s} \alpha^{(1-3)} i_{16K+9} \]
\[ d^3(v_2^{4s} \alpha^{(0-3)} i_{16K+9}) = 0 \]

Thus $E^4$ is given by
\[ v_2^{(0,4)} u^{(1-3)}, \quad v_2^{(6,2)} u^{8K+3,8K+4}, \quad v_2^{4s} i_{16K+9}. \]

$d^4$ has degree 21, which is odd. So it must go from the $R\bar{P}^{16K+8}$ part to the $S^{16K+9}$ part. $d^4$ must be zero on anything in the image from $R\bar{P}^\infty$. So our non-zero differentials have possible sources $v_2^{(6,2)} u^{8K+3,8K+4}$, and possible targets $v_2^{2s} i_{16K+9}$ and $v_2^{(6,2)} \alpha^3 i_{16K+9}$. Let’s compute the degrees (mod 48).
\[ v_2^6 u^{8K+3} : -36 - 16(8K + 3) \equiv -36 + 16K \rightarrow -15 + 16K \]
\[ v_2^6 u^{8K+3} : -12 - 16(8K + 3) \equiv -12 + 16K \rightarrow 9 + 16K \]
\[ v_2^6 u^{8K+4} : -36 - 16(8K + 4) \equiv -4 + 16K \rightarrow 17 + 16K \]
\[ v_2^2 u^{8K+4} : -12 - 16(8K + 4) \equiv 20 + 16K \rightarrow 41 + 16K \]

Comparing with degrees of the possible targets we see that the only possible differentials are:
\[ d^4(v_2^{[6,2]} u^{8K+3}) = v_2^{(0,4)} i_{16K+9} \]

This must be true if \( i_{16K+9} \) is \( x^4 \)-torsion. We invoke the commutative diagram (11) used before.

Consider \( x^3 z_{16K-8} \) in \( ER(2)^*(RP^{16K+8}) \). The following diagram shows its images in the lower left hand square of the commutative diagram.

\[
\begin{array}{ccc}
 x^3 z_{16K-8} & \rightarrow & x^3 v_2^4 u^{8K+5} \\
 \downarrow & & \downarrow \\
 x^4 i_{16K+9} & \rightarrow & x^4 i_{16K+9}
\end{array}
\]

The element in the upper right-hand corner is zero since \( x^3 u^4 = 0 \). This shows \( i_{16K+9} \) is indeed \( x^4 \)-torsion. We obtain our \( E^5 \)-term
\[
v_2^{(0,4)} u^{(1-3)}, \quad v_2^{(6,2)} u^{8K+4}.
\]

\( d^6 \) has even degree. For dimensional reasons the differentials must be zero in the odd part, and \[7, \text{page } 23\] we get that
\[ d^6(u^{6}) = 0. \]
This shows \( i_{16K+9} \) is indeed
\[ d^6 = 0. \]

Since \( d^7 \) has even degree, the differential does not mix odd and even degrees. First of all in the even part we have
\[ d^7(v_2^4 u^{(1-3)}) = u^{(1-3)} \]
Also by mapping to \( RP^{16K+8} \) \[7, \text{page } 23\] we get that
\[ d^7(v_2^6 u^{8K+4}) = v_2^2 u^{8K+4} \]

We collect our results in the following theorem.

**Theorem 3.1.** The Bockstein spectral sequence for \( ER^*(RP^{16K+9}, \ast) \) is as follows:

\[ E^1 \]
\[ v_2^i \alpha^k u^j \quad (0 \leq i \leq 7, \quad 0 \leq k, \quad 0 \leq j \leq 8K + 4) \]
\[ v_2^i \alpha^k i_{16K+9} \quad (0 \leq i \leq 7, \quad 0 \leq k) \]
\[ d^1(v_2^{2s-5} \alpha^k u^j) = 2v_2^2 \alpha^k u^j = v_2^{2s} \alpha^{k+1} u^j + \ldots \quad (j \leq 8K + 3) \]
\[ d^1(v_2^{2s+1} \alpha^k i_{16K+9}) = 2v_2^{2s-2} \alpha^k i_{16K+9} \]

where \( v_2^2 \alpha^k i_{16K+9} \) generates a free \( \mathbb{Z}_2 \)-module in \( E^1 \).

\[ E^2 \]
\[ v_2^2 \alpha^k u \quad (k \geq 0), \quad v_2^2 u^j \quad (1 \leq j \leq 8K + 4), \quad v_2^{2s+1} \alpha^k u^{8K+4} \quad (0 \leq k), \]
\[ v_2^2 \alpha^k i_{16K+9} \]
\[ d^2(v_2^{2s+1} \alpha^k u^{8K+4}) = v_2^{2s+2} \alpha^{k+1} i_{16K+9} \]
\[ E^3 \]
\[ v_2^{2s} \alpha^k u \quad (0 \leq k), \quad v_2^{2s} u^j \quad (1 \leq j \leq 8K + 4), \quad v_2^{2s} \alpha^{0-3} i_{16K+9} \]
\[ d^3(v_2^{(6,2)} \alpha^k u) = v_2^{(0,4)} \alpha^{k+1} u \]
\[ d^3(v_2^{(6,2)} u^j) = v_2^{(0,4)} \alpha^j = v_2^{(0,4)} u^{j+2} \quad (2 \leq j \leq 8K + 2) \]
\[ d^3(v_2^{(8,2)} \alpha^{0-2} i_{16K+9}) = v_2^{4s} \alpha^{(1-3)} i_{16K+9} \]
\[ d^3(v_2^{4s} \alpha^{0-3} i_{16K+9}) = 0 \]

\[ E^4 \]
\[ v_2^{(0,4)} u^{(1-3)}, \quad v_2^{(6,2)} u^{(8K+3,8K+4)}, \quad v_2^{4s} i_{16K+9} \]
\[ d^4(v_2^{(6,2)} u^{8K+3}) = v_2^{(0,4)} i_{16K+9} \]

\[ E^5 = E^6 = E^7 \]
\[ v_2^{(0,4)} u^{(1-3)}, \quad v_2^{(6,2)} u^{8K+4}, \quad v_2^{(6,2)} \alpha^{3} i_{16K+9} \]
\[ d^7(v_2^{4s} u^{(1-3)}) = u^{(1-3)}, \quad d^7(v_2^{6s} u^{8K+4}) = v_2^{2s} u^{8K+4} \]

Next we identify all the elements in degree 8*.

**Theorem 3.2.** A 2-adic basis of \( ER(2)^{8s}(RP^{16K+9}, \ast) \) is given by the elements

- \( \alpha^k u^j \), \( k \geq 0, 1 \leq j \leq 8K + 4 \)
- \( v_2^{k}\alpha^k u^j \), \( k \geq 1, 1 \leq j \leq 8K + 4 \)
- \( v_2^{2s} u^j \), \( 4 \leq j \leq 8K + 4 \)
- \( x\alpha^k i_{16K+9} \), \( x\alpha^{k}i_{16K+9} \), \( k \geq 0 \)

**Proof.** The first classes of elements represent the images of differentials in the spectral sequence that do not involve \( i_{16K+9} \). As in [8], multiplication by powers of \( x \) leads to no new elements in degree 8*. Those images involving \( i_{16K+9} \) provide \( x^2, x^3 \), or \( x^4 \)-torsion, which may be multiplied by \( x \). \( \square \)

**Corollary 3.1.** There is an algebraic map

\[
ER(2)^{8s}(RP^{16K+9}) \rightarrow E(2)^{8s}(RP^{16K+10})
\]

which only misses the elements \( v_2^{2s} u^{(1-3)} \).

## 4 Non-Immersions

If \( RP^b \) immerses in \( \mathbb{R}^c \), James showed [3] that there is an axial map

\[ m : RP^b \times RP^{2^k-c-2} \rightarrow RP^{2^k-b-2} \]

for large \( L \) (meaning a map that is non-trivial on both axes). Specifically, to show that \( RP^{2n} \) does not immerse in \( \mathbb{R}^{2k+1} \) we need to prove there is no axial map

\[ m : RP^{2n} \times RP^{2^k-2n-3} \rightarrow RP^{2^k-2n-2} \tag{12} \]

Our strategy is to consider the class \( u \in ER(2)^{+(RP^{2^k-2n-2})} \), which satisfies \( u^{2k-1-n} = 0 \) when \( n \equiv 0 \) or \( 7 \) mod 8 [7]. Theorem 1.6]. We shall see that \( m^* u \) is known, in principle. If we can show that \( (m^* u)^{2k-1-n} \neq 0 \), we have a contradiction.
Davis [2] used this approach, by using the complex-oriented cohomology theory $E(2)$ to deduce that $RP^{2n}$ does not immerse in $\mathbb{R}^{2k}$ by showing there is no axial map

$$m : RP^{2n} \times RP^{2k-2} \to RP^{2k-2n-2}$$

when $n = m + \alpha(m) - 1$ and $k = 2m - \alpha(m)$ for some $m$, where $\alpha(m)$ denotes the number of 1’s in the binary expansion of $m$. We wish to improve this result to show that for certain $n$ and $k$, (12) does not exist.

There is an axial map $m : RP^\infty \times RP^\infty \to RP^\infty$, which is the restriction of the map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ induced by the tensor product of the canonical Real line bundles. Therefore $m^*u = u_1 + u_2 + u_2$, where $u_1, u_2$ and $u$ are the Chern classes for the three copies of $RP^\infty$.

If $m : RP^b \times RP^c \to RP^d$ is an axial map, the diagram

$$\begin{array}{ccc}
RP^b \times RP^c & \xrightarrow{m} & RP^d \\
\downarrow & & \downarrow \\
RP^\infty \times RP^\infty & \xrightarrow{m} & RP^\infty
\end{array}$$

commutes, as all axial maps $RP^b \times RP^c \to RP^\infty$ are homotopic. It follows that the same formula $m^*u = u_1 + u_2$ holds for this $m$. As the formal group law $F$ is a formal power series in $u_1$ and $u_2$ over $\mathbb{Z}[2][\alpha]$ and deg $u = -16$ and deg $\alpha = 16$, we are interested only in degrees that are multiples of 16. This simplifies our work, as $ER(2)^{16*}(RP^\infty) \to E(2)^{16*}(RP^\infty)$ is an isomorphism by [2].

We assume $k = 2 \mod 8$, so that $2^L - 2k - 2 = 16K + 10$ and we can use Theorem 3.2. Consider the diagram

$$\begin{array}{ccc}
ER(2)^{16*}(RP^\infty \times RP^\infty) & \to & E(2)^{16*}(RP^\infty \times RP^\infty) \\
\downarrow & & \downarrow \\
ER(2)^{16*}(RP^{2n} \times RP^{16K+10}) & \to & E(2)^{16*}(RP^{2n} \times RP^{16K+10}) \\
\downarrow & & \downarrow \\
ER(2)^{16*}(RP^{2n} \times RP^{16K+9}) & \to & E(2)^{16*}(RP^{2n} \times RP^{16K+9})
\end{array}$$

From Don Davis’ work, the image of $(u_1 + u_2)^{2L-1-n} \in ER(2)^{16*}(RP^\infty \times RP^\infty)$ in $E(2)^{16*}(RP^{2n} \times RP^{16K+10})$ is non-zero. We need to show that the image in $ER(2)^{16*}(RP^{2n} \times RP^{16K+9})$ is non-zero, (Note that we cannot use $E(2)$-cohomology for this purpose, as it is complex-oriented, which implies that $u^{8K+5} \in E(2)^{16*}(RP^{2n} \times RP^{16K+10})$ maps to zero in $E(2)^{16*}(RP^{16K+9})$).

The two end terms in $(u_1 + u_2)^{2L-1-n}$ are $u_1^{2L-1-n}$ and $u_2^{2L-1-n}$, which are plainly zero; all the other terms have the form $\lambda \alpha^k u_1^i u_2^j$, where $\lambda \in \mathbb{Z}[2]$, $k \geq 0$, $i \geq 1$, $j \geq 1$, and $i + j \geq 2L-1 - n$. Following [7], we may use the formulae

$$2u_1 = -\alpha u_1^2 + \ldots, \quad \alpha u_1 u_2 = \alpha u_1^2 u_2 + \ldots$$

and induction to reduce $(u_1 + u_2)^{2L-1-n}$ to a sum of distinct terms of the forms $\alpha^k u_1^i u_2^j$, with no numerical coefficient. (The first formula comes from $[2](u_1) = 0$; the second from $u_1[u_2](u_2) - u_2(u_1) = 0$.) Further, again by [7], $u_1^{n+1} = 0$ since $n \equiv 0$ or 7 mod 8. Then $\alpha^k u_1 u_2 = 0$, since we still have $i + 1 \geq 2L-1 - n$. We do not know (or need to know) exactly which terms are present; all we have to do is show that the monomials $u_1^i u_2^j$ (for $1 \leq i \leq n$ and $1 \leq j \leq 8K+5$) remain linearly independent in $ER(2)^{16*}(RP^{2n} \times RP^{16K+9})$, which we defer to the next section.

Meanwhile, let us review the various numerical conditions. We need $n = m + \alpha(m) - 1$, $k = 2m - \alpha(m), k \equiv 2 \mod 8$ and $n \equiv 0$ or 7 mod 8. So $2m - \alpha(m) \equiv 2$ and $m + \alpha(m) \equiv 0$ or 1. Solving these, we get $(m, \alpha(m)) = (6, 2)$ or $(1, 0)$.
Theorem 4.1. If \((m, \alpha(m)) \equiv (6, 2)\) or \((1, 0)\) mod 8, then \(RP^{2(m+\alpha(m)-1)}\) does not immerse in \(\mathbb{R}^{2(2m-\alpha(m)+1)}\).

4.1 Products with an odd space

As we discussed towards the end of the previous section, in order to complete the proof of Theorem 4.1 we need to show that the monomials \(u_1^i u_2^j\) (for \(1 \leq i \leq n\) and \(1 \leq j \leq 8K + 5\)) remain linearly independent in \(ER(2)^*(RP^{2n} \times RP^{16K+9})\). The argument presented in this section is similar to [3] section 11.

We shall look into the Bockstein spectral sequence for

\[
ER(2)^*(RP^{2n} \wedge RP^{16K+9}, *)
\]

where \(2n < 16K + 9\).

The \(E^1\)-term is the usual

\[
E(2)^*(RP^{2n} \wedge RP^{16K+9}, *) \approx E(2)^*(RP^{2n}, *) \otimes E(2)^*(RP^{16K+9}, *) \oplus \Sigma^{-16(8K+4)-1} E(2)^*(RP^{2n}, *)
\]

(from [3]) Also, we know that

\[
E(2)^*(RP^{16K+9}, *) \cong E(2)^*(RP^{16K+8}, *) \oplus E(2)^*(S^{16K+9}, *).
\]

Since \(E(2)^*(S^{16K+9}, *)\) is free it does not affect the Tor term, only the tensor product. So our \(E^1\)-term is:

\[
E(2)^*(RP^{2n}, *) \otimes E(2)^*(RP^{16K+8}, *) \oplus E(2)^*(RP^{2n}, *) \otimes E(2)^*(S^{16K+9}, *)
\]

\[\oplus \Sigma^{16K-17} E(2)^*(RP^{2n}, *)\]

A 2-adic basis for this is given by

\[
v_2^a \alpha^k u_1^i u_2^j (0 \leq k, \ 0 < i \leq n, \ s < 8)
\]

\[
v_2^a u_1^i u_2^j (0 < i \leq n, \ 1 < j \leq 8K + 4, \ s < 8)
\]

by the same reduction as before, and

\[
v_2^a \alpha^k i_{16K+9} (0 \leq k, \ 0 \leq i \leq n, \ s < 8)
\]

\[
v_2^a \alpha^k i_{16K+33} (0 \leq k, \ 0 \leq i \leq n, \ s < 8).
\]

We know that \(x_{16K+9}\) represents \(v_2^a u_2^6K+4\). So \(v_2^a x_{16K+9}\) represents \(u_2^6 K+5\). There is no differential on \(v_2^a i_{16K+9}\). Also there is no differential on \(v_2^a i_{16K+9}\). All we have to do is show that \(u_2^a i_{16K+9}\) is not in the image of \(d^1\). Since \(d^1\) has even degree we only have to worry about the odd degree elements since \(u_2^a i_{16K+9}\) is odd degree.

\(d^1\) has degree 18 so if it is to hit \(u_2^a i_{16K+9}\) it must start at some \(\alpha^k u_1^i z_{16K-33}\) because they are the only elements in the correct degree modulo 16. Then we would have \(d^1\) non-trivial on \(z_{16K-33}\).

In the Bockstein spectral sequence for \(ER(2)^*(RP^{16M+16} \wedge RP^{16K+10}, *)\), \(8M + 8 < 8K + 5\), we have from [7] Theorem 19.2, \(d^1(z_{16K-1}) = 0\). From [3] Theorem 1.2 we have that \(z_{16K-1}\) maps to \(u_1 z_{16K-33}\) in the spectral sequence for \(ER(2)^*(RP^{16M+16} \wedge RP^{16K+10}, *)\). Since this passes through the spectral sequence for \(ER(2)^*(RP^{16M+16} \wedge RP^{16K+9}, *)\), \(z_{16K-1}\) maps to \(u_1 z_{16K-33}\) here as well. So \(d^1(u_1 z_{16K-33}) = u_1 d^1(z_{16K-33}) = 0\).

\[
ER(2)^*(RP^{16K+16} \wedge RP^{16K+10}) \quad \xrightarrow{d^1} \quad ER(2)^*(RP^{16K+16} \wedge RP^{16K+8})
\]

\[
ER(2)^*(RP^{16M+16} \wedge RP^{16K+9})
\]

All elements killed by multiplication by \(u_1\) go to zero under the map to \(RP^{16K} \wedge RP^{16K+9}\), so our \(d^1(z_{16K-33})\) is zero.
Theorem 4.2. When $n \leq 8M < 8M + 8 < 8K + 5$, in

$$ER(2)^{16*}(RP^{2n} \wedge RP^{16K+9})$$

the element $u_1^n u_2^{8K+5}$ is non-zero.

References

[1] D.Davis, *Table of immersions and embeddings of real projective spaces*, http://www.math.lehigh.edu/~dmd1/imms.html

[2] D.Davis, *A strong immersion theorem for real projective spaces*, Ann. of Math., 120:510-520, 1984.

[3] Jesus González, W.S.Wilson, *The BP-theory of two-fold products of projective spaces.*, Homotopy, Homotopy Appl., 10(3), 2008, 181-192.

[4] P.Hu, I.Kriz, *Real-oriented Homotopy Theory and an analogue of the Adams-Novikov Spectral Sequence*, Topology, 40(2):317-399, 2001.

[5] I.M.James, *On the immersion problem of real projective spaces*, Bull. Amer. Math., Soc. 69:231-238, 1963.

[6] N.Kitchloo, W.S.Wilson, *On fibrations related to real spectra*. Proceedings of the Nishida fest (Kinosaki 2003), volume 10, Geometry and Topology Monographs, pages 237-244, 2007.

[7] N.Kitchloo, W.S.Wilson, *The second real Johnson-Wilson theory and non-immersions of RP^n*, Homology, Homotopy Appl., 10(3), 2008, 223-268.

[8] N.Kitchloo, W.S.Wilson *The second real Johnson-Wilson theory and non-immersions of RP^n*, Part 2, Homology, Homotopy Appl., 10(3), 2008, 269-290.

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