AN IDENTITY OF DISTRIBUTIVE LATTICES

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Abstract. In a finite distributive lattice $L$ we define two functions $s(\alpha) = \#\{\delta \in L | \delta \leq \alpha\}$ and $l(\alpha) = \#\{\delta \in L | \delta \geq \alpha\}$. In this present article we prove that the sum of these two functions over a finite distributive lattice are equal.

1. Introduction

In this present article we look at the functions $s(\alpha) = \#\{\delta \in L | \delta \leq \alpha\}$ which we will call “smaller” function and $l(\alpha) = \#\{\delta \in L | \delta \geq \alpha\}$ which we will call “larger” function, these functions are closely related to the rank $r(\alpha)$ and co-rank $cr(\alpha)$ functions of the lattice and clearly not same unless the lattice in question is a chain lattice. Even though it’s clear that $\sum_{\delta \in L} r(\delta) = \sum_{\delta \in L} cr(\delta)$ it is not trivial to demonstrate that “lower sum” $\sum_{\delta \in L} s(\delta)$ is equal to the “upper sum” $\sum_{\delta \in L} l(\delta)$. In this present article we provide an interesting proof of the fact. We give an example as below: Hasse diagram of a distributive lattice $L$ is given below, the nodes in the first graph denotes the numbers $s(\delta)$ and the numbers in the nodes of the second graph denotes $l(\delta)$. Even though the numbers are different in places the sum comes to 41 in both the cases.

Let us state below our main theorem that the lower sum and upper sum of a finite distributive lattice are equal.

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Theorem 1.1. For a finite distributive lattice \( L \) we have, \( \sum_{\delta \in L} s(\delta) = \sum_{\delta \in L} l(\delta) \)

We take a maximal join irreducible element \( \beta \in J \) and look at the pruned lattice, see [2,4] for a definition, \( L_\beta \). We then apply induction on the number of join irreducibles of the lattice \( L \). For a trivial lattice with \( \#J(L) = 1 \) the claim is obvious. Before the proof we need to recall few definitions and prove a few lemmas.

2. Definitions and Lemmas

Lemma 2.1. Let \( \theta > \delta \) be a cover in a distributive lattice \( L \), then there is a unique join irreducible \( \epsilon \) such that \( \theta = \delta \vee \epsilon \).

Proof. Using Birkhoff’s representation theorem [1] we write the elements \( \theta, \delta \in L \) as ideals in the poset of join irreducibles \( I_\theta, I_\delta \). Since \( \theta \) covers \( \delta \) we have \( I_\theta \setminus I_\delta = \{ \epsilon \} \) for a join irreducible element \( \epsilon \). It is clear that this epsilon serves the purpose. \( \square \)

Lemma 2.2. If \( \theta \vee \delta \geq \beta \) where \( \beta \) is a join irreducible then either \( \theta \) or \( \delta \) is larger than \( \beta \).

Proof. Since \( \theta \vee \beta \geq \delta \) we have \( I_\beta \subset I_\theta \cup I_\delta \). But we have \( \beta \in I_\beta \), \( \beta \) being a join irreducible. So we have \( \beta \in I_\theta \) or \( \beta \in I_\delta \). \( \square \)

For a lattice \( L \) and an elements \( \delta \in L \) let us define a sublattice \( \Gamma_\delta = \{ \beta \in L : \beta \sim \delta \} \). Let us also define a set of functions \( f, s, l \) on the lattice \( L \) taking values in \( \mathbb{N} \) as follows: \( s(\delta) = \# \{ \beta : \beta \leq \delta \} \) and \( l(\delta) = \# \{ \beta : \beta \geq \delta \} \) and \( f(\delta) = \# \Gamma_\delta = s(\delta) + l(\delta) - 1 \).

Let us recall from [2] the following theorem, where the number of non-comparable pairs of elements in a distributive lattice \( L \) was denoted by \( n(L) \).

Theorem 2.3. For a distributive lattice \( L \) we have
\[
n(L) = 1/2(\#L^2 - \sum_{\delta \in L} f(\delta))
\]

In light of the above theorem we would like to calculate the sum \( \sum_{\delta \in L} f(\delta) \). Using the functions \( s \) and \( l \) we can rewrite the sum as \( \sum_{\delta \in L} s(\delta) + \sum_{\delta \in L} l(\delta) - \#L \). So to understand the sum \( F = \sum_{\delta \in L} f(\delta) \) we have to understand the sums \( S = \sum_{\delta \in L} s(\delta) \) and \( L = \sum_{\delta \in L} l(\delta) \). The following theorem is one step towards understanding the two sums \( S \) and \( L \).

Let us start with a recollection of a concept called pruning, which will be essential for the induction step, from the paper [4]. Let \( \alpha \in L \) be a maximal join-irreducible element. Let us define a sublattice \( L_\alpha \) of \( L \).
which we called the pruned lattice of $\mathcal{L}$ with respect to the maximal
join irreducible $\alpha$.

**Definition 2.4.** The subset $\mathcal{L}_\alpha = \{\beta \in \mathcal{L} : \beta \not\leq \alpha\}$ is called the pruned
sublattice of the lattice $\mathcal{L}$ with respect to the maximal join irreducible
$\alpha$. So the subset $\mathcal{L}_\alpha$ consist of all the elements $\beta$ which are either
smaller than or non-comparable to $\alpha$.

Let us also recall the following definition of embedded sublattices from
[3].

**Definition 2.5.** A sublattice $D \subset \mathcal{L}$ is called an embedded sublattice
if for every $\theta, \delta \in D$ whenever there are elements $\alpha, \beta \in \mathcal{L}$ such that
$\alpha \vee \beta = \theta$ and $\alpha \wedge \beta = \delta$ then $\alpha, \beta \in D$.

In the lemma below we will establish that “pruned sublattice” is truly
a sublattice and is also embedded.

**Lemma 2.6.** The pruned lattice $\mathcal{L}_\alpha$ is a sublattice of
$\mathcal{L}$ that is embedded.

*Proof.* Pruned subset is a sublattice since if $x, y \in \mathcal{L}_\alpha$ then
$x \wedge y \not\leq \alpha$ and if $x \vee y \geq \alpha$ then by [2.2] we have either $x \geq \alpha$ or $y \geq \alpha$ which leads
to contradiction. For the embedded part if for $\theta, \delta$ we have $\theta \vee \delta$ and
$\theta \wedge \delta$ in $\mathcal{L}_\alpha$ then neither $x$ nor $y$ can be larger than $\alpha$ again by [2.2].

With the above notation in mind let us define a new set of functions,
let $\alpha$ be a maximal join irreducible and let $\mathcal{L}_\alpha$ be the corresponding
pruned sublattice. Then let us define

$$l_\alpha(\delta) = \# \{x \in \mathcal{L} \mid x \geq \delta\} \cap \mathcal{L}_\alpha,$$

$$s_\alpha(\delta) = \# \{x \in \mathcal{L} \mid x \leq \delta\} \cap \mathcal{L}_\alpha,$$

$$f_\alpha(\delta) = \# \Gamma_\delta \cap \mathcal{L}_\alpha.$$

**Lemma 2.7.** $l(\delta) = l_\alpha(\delta) + l(\alpha \cap \delta)$

*Proof.* From the definitions we have $l(\delta) - l_\alpha(\delta) = \# \{\beta : \beta \geq \alpha \text{ and } \beta \geq \delta\}$. But $\beta \geq \alpha$ and $\beta \geq \delta$ if and only if we have $\beta \geq \alpha \vee \delta$. So putting
all these together we get $l(\delta) - l_\alpha(\delta) + \# \{\beta : \beta \geq \alpha \vee \delta\}$. From
the definition of the function $l$ we write $\# \{\beta : \beta \geq \alpha \vee \delta\}$ as $l(\alpha \vee \delta)$ and
get the statement of the lemma. \[ \square \]

**Lemma 2.8.** $s(\delta) = s_\alpha(\delta) + \# \{\beta : \beta \geq \alpha \text{ and } \beta \leq \delta\}$. To simplify the
notations let us denote $\# \{\beta : \beta \geq \alpha \text{ and } \beta \leq \delta\}$ by $h_\alpha(\delta)$

*Proof.* Proof of this fact is straightforward from the definition. \[ \square \]

We want to show that the sum $I = \sum_{\delta \in \mathcal{L}} l(\delta) - s(\delta) = 0$. To show
that we want to use an induction argument on the size of the lattice
$\mathcal{L}$. Let $\alpha$ be a maximal join irreducible in the lattice $\mathcal{L}$ and $\mathcal{L}_\alpha$ be its
pruned lattice, let us denote the complement of the pruned lattice by
\[ \mathcal{X}_\alpha = \mathcal{L} \setminus \mathcal{L}_\alpha. \] Note that from the definition of the pruned lattice the structure of the set \( \mathcal{X}_\alpha \) can be written rather easily so we include that as a lemma.

**Lemma 2.9.** \( \mathcal{X}_\alpha = \{ \beta \in \mathcal{L} : \beta \geq \alpha \} \), is an embedded sub-lattice.

*Proof.* Clear from the definition 2.4. \( \Box \)

With the above definitions in mind we will split the sum

\[ I = \sum_{\delta \in \mathcal{L}} (l(\delta) - s(\delta)) \]

into two sums such as

\[ I = \sum_{\delta \in \mathcal{L}_\alpha} (l(\delta) - \sigma(\delta)) + \sum_{\delta \in \mathcal{X}_\alpha} (l(\delta) - s(\delta)). \]

For notational simplicity let us denote the first part of the sum namely \( \sum_{\delta \in \mathcal{L}_\alpha} (l(\delta) - \sigma(\delta)) \) as \( S_\alpha \) and the second part namely \( \sum_{\delta \in \mathcal{X}_\alpha} (l(\delta) - s(\delta)) \) as \( T_\alpha \). Thus we have \( I = S_\alpha + T_\alpha \), and observe that if we prove \( I = 0 \) then the main theorem 1.1 follows.

**Lemma 2.10.**

\[ S_\alpha = \sum_{\delta \in \mathcal{L}_\alpha} l(\alpha \lor \delta) \]

*Proof.*

\[ S_\alpha = \sum_{\delta \in \mathcal{L}_\alpha} (l(\delta) - s(\delta)) \]

\[ = \sum_{\delta \in \mathcal{L}_\alpha} l_\alpha(\delta) + l(\alpha \lor \delta) - s_\alpha(\delta) - h_\alpha(\delta) \]

(1)

But since \( \delta \in \mathcal{L}_\alpha \) we do not have any \( \beta \in \mathcal{L} \) which is larger than \( \alpha \) and smaller than \( \delta \), since that will imply \( \delta \geq \alpha \) contrary to our assumption. So the number \( h_\alpha(\delta) = \# \{ \beta : \beta \geq \alpha \text{ and } \beta \leq \delta \} \) is zero. Thus :

\[ S_\alpha = \sum_{\delta \in \mathcal{L}_\alpha} l_\alpha(\delta) + l(\alpha \lor \delta) - s_\alpha(\delta) \]

But by induction the sum \( \sum_{\delta \in \mathcal{L}_\alpha} (l_\alpha(\delta) - s_\alpha(\delta)) \) is zero. So we have \( S_\alpha = \sum_{\delta \in \mathcal{L}_\alpha} l(\alpha \lor \delta) \) as required. \( \Box \)

Let us now similarly investigate the number \( T_\alpha \).

**Lemma 2.11.** \( T_\alpha = -\sum_{\delta \in \mathcal{X}_\alpha} s_\alpha(\delta) \)

*Proof.*

\[ T_\alpha = \sum_{\delta \in \mathcal{X}_\alpha} (l(\delta) - s(\delta)) \]

\[ = \sum_{\delta \in \mathcal{X}_\alpha} (l_\alpha(\delta) + l(\alpha \land \delta) - s_\alpha(\delta) - h_\alpha(\delta)) \]

(2)
Since $\delta \geq \alpha$ the number $l_{\alpha}(\delta) = 0$ and $\alpha \land \delta = \delta$. So we rewrite the above equation to:

$$= \sum_{\delta \in X_{\alpha}} (l(d) - h_{\alpha}(\delta)) - \sum_{\delta \in X_{\alpha}} s_{\alpha}(\delta)$$

(3)

Looking at the sublattice $X_{\alpha}$ observe that the function $h_{\alpha}(\delta)$ is just the “smaller function” for the lattice $X_{\alpha}$. Hence using the induction hypothesis we get

$$\sum_{\delta \in X_{\alpha}} (l(d) - h_{\alpha}(\delta)) = 0$$

Thus:

$$T_{\alpha} = -\sum_{\delta \in X_{\alpha}} s_{\alpha}(\delta)$$

□

Let us summarise the above two lemmas in the following lemma:

**Lemma 2.12.**

$$I = S_{\alpha} + T_{\alpha} = \sum_{\delta \in L_{\alpha}} l(\alpha \lor \delta) - \sum_{\delta \in X_{\alpha}} s_{\alpha}(\delta)$$

**Definition 2.13.** For $\gamma \in X_{\alpha}$ let us denote $C_{\gamma} = \{x \in L_{\alpha} | x \lor \alpha = \gamma\}$

**Lemma 2.14.** For any $\gamma \in X_{\alpha}$, $C_{\gamma}$ is non-empty and is a sublattice of $L$.

**Proof.** Let $a$ be the maximal element in $L$ which is smaller than $\gamma$ but not larger than $\alpha$. If there exist no such element then every element $a$ smaller than $\gamma$ is also larger than $\alpha$, which will mean $\alpha$ is the smallest element in $L$. But that is not the case since $\alpha$ is a maximal join irreducible. Consider $a \lor \alpha$, since both $a$ and $\alpha$ are smaller than $\gamma$, $a \lor \alpha \leq \gamma$. If $\gamma = a \lor \alpha$ then we are done, and if $\gamma$ covers $a \lor \alpha$ then by 2.1 we have a join irreducible $\epsilon$ such that $\gamma = \epsilon \lor a \lor \alpha$ or $\gamma = (\epsilon \lor a) \lor \alpha$ which shows $\epsilon \lor a \in C_{\gamma}$. And if there is $\gamma_{1}$ such that $\gamma > \gamma_{1} > a \lor \alpha$, by induction on the rank of $\gamma$ we have an element $b$ such that $\gamma_{1} = b \lor \alpha$. But note that $\gamma > b$ and $b \not\leq \alpha$ and we have $\gamma > a \lor b$ and $a \lor b \not\leq \alpha$ which contradicts the maximality of $a$.

For the lattice part, if $x, y \in C_{\gamma}$ then $x \lor y \lor \alpha = x \lor \gamma$ but since $x \in C_{\gamma}$ we have $x \leq \gamma$ so $x \lor \gamma = \gamma$. Similarly for $(x \land y) \lor \alpha = (x \lor \alpha) \land (y \lor \alpha) = \gamma \land \gamma = \gamma$. □

**Definition 2.15.** Let $x_{\gamma}$ be the maximal element of $C_{\gamma}$.

**Lemma 2.16.** For each $\delta \in X_{\alpha}$ we have $s_{\alpha}(\delta) = s_{\alpha}(x_{\delta}) = s(x_{\delta})$
Proof. Since $\delta \in X_\alpha$ we have $\delta = x_\alpha \lor \alpha$ from the definition of $x_\delta$. And we know $s_\alpha(\delta) = \#\{\beta \in \mathcal{L}_\alpha : \beta \leq \delta\}$. But note by the following argument that the set $\{\beta \in \mathcal{L}_\alpha : \beta \leq \delta\}$ is equal to the set $\{\beta \in \mathcal{L}_\alpha : \beta \leq x_\delta\}$. Clearly if $\beta \leq x_\delta$ then $\beta \leq \delta = x_\delta \lor \alpha$ for the inclusion other way around see that if $y \in \mathcal{L}_\alpha$ and $y \leq \delta$ then if $y$ is non-comparable to $x_\delta$ then we have $x_\delta \leq y \lor x_\delta \leq \delta$ which contradicts the maximality of $x_\delta$ unless $y \lor x_\delta = \delta$. But $\delta \in X_\alpha$ which means $\delta \geq \alpha$ or $y \lor x_\delta \geq \alpha$ so by Lemma 2.2 we have either $y$ or $x_\delta$ larger than $\alpha$ which is a contradiction to the assumption that both $y$ and $x_\delta$ are in $\mathcal{L}_\alpha$. So it means $y$ is comparable to $x_\delta$ but it cannot be larger since it would violate the maximality of $x_\delta$ so we have $y \leq x_\delta$, or the two sets are equal.

Which means $s_\alpha(\delta) = \#\{\beta \in \mathcal{L}_\alpha : \beta \leq \delta\} = \#\{\beta \in \mathcal{L}_\alpha : \beta \leq x_\delta\} = s_\alpha(x_\delta)$. Now since $x_\delta$ is by definition in $\mathcal{L}_\alpha$ we have $s_\alpha(x_\delta) = s(x_\delta)$.

\[\sum_{\delta \in \mathcal{L}_\alpha} l(\alpha \lor \delta) = \sum_{\gamma \in X_\alpha} \#C_\gamma l(\gamma)\]

\[\text{Proof. If } \delta \in C_\gamma \text{ then by definition we have } \gamma = \alpha \lor \delta \text{ or } l(\alpha \lor \delta) = l(\gamma). \text{ So } \sum_{\delta \in \mathcal{L}_\alpha} l(\alpha \lor \delta) = \sum_{\gamma = \alpha \lor \delta \in X_\alpha} l(\gamma) \text{ we can do this change of variable since we have } \cup_{\gamma \in X_\alpha} C_\gamma = \mathcal{L}_\alpha. \text{ But } \sum_{\gamma = \alpha \lor \delta \in X_\alpha} l(\gamma) = \sum_{\gamma \in X_\alpha} \#C_\gamma l(\gamma)\]

Lemma 2.17.

\[\sum_{\gamma \in X_\alpha} \#C_\gamma l(\gamma) = \sum_{\gamma \in X_\alpha} s(x_\gamma)\]

\[\text{Proof. Let } A_\gamma \text{ be the set } \{\beta \in \mathcal{L} : \beta \leq x_\gamma\}. \text{ So } s(x_\gamma) = \#A_\gamma. \text{ On the set } A_\gamma \text{ let us give a relation } \sim \text{ as follows: } a, b \in A_\gamma \text{ we will call } a \sim b \text{ if and only if } a \lor \alpha = b \lor \alpha. \text{ Observe that this is an equivalence relation. Also observe that the equivalence classes are } C_\gamma = \{a \mid a \lor \alpha = \gamma_i\} \text{ where } \gamma_i \leq \gamma. \text{ Thus we have } s(x_\gamma) = \#A_\gamma = \sum_{\gamma_i \in X_\alpha, \gamma_i \leq \gamma} C_\gamma, \text{ if we sum the both sides up we get }\]

\[\sum_{\gamma \in X_\alpha} s(x_\gamma) = \sum_{\gamma \in X_\alpha} \sum_{\gamma_i \leq \gamma} \sum_{\gamma_i \in X_\alpha} C_\gamma, \text{ if we sum the both sides up we get }\]

\[= \sum_{\gamma \in X_\alpha} \#C_\gamma \sum_{y \geq \gamma} 1\]

Since each $C_\gamma$ is appearing $\sum_{y \geq \gamma} 1$ many times in the right hand sum. Now observe that $\sum_{y \geq \gamma} 1 = l(\gamma)$ so we can finally write:

\[\sum_{\gamma \in X_\alpha} s(x_\gamma) = \sum_{\gamma \in X_\alpha} \#C_\gamma l(\gamma)\]
Theorem 2.19. \( I = S_\alpha + T_\alpha = 0 \)

Proof.

\[
S_\alpha + T_\alpha = \sum_{\delta \in L_\alpha} l(\alpha \lor \delta) - \sum_{\alpha \in X_\alpha} s_\alpha(\delta) \\
= \sum_{\gamma \in X_\alpha} \# C_\gamma l(\gamma) - \sum_{\delta \in X_\alpha} s_\alpha(x_\delta)
\]

by 2.16 and 2.17

\[
= \sum_{\gamma \in X_\alpha} \# C_\gamma l(\gamma) - s(x_\gamma) = 0
\]

by 2.17 and after changing the running variable to a common variable.

And the last equality follows from 2.18 So we have proved the main theorem 1.1.

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