RIGHT COIDEAL SUBALGEBRAS IN \( U_q^+ (so_{2n+1}) \)

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Abstract. We give a complete classification of right coideal subalgebras that contain all group-like elements for the quantum group \( U_q^+ (so_{2n+1}) \), provided that \( q \) is not a root of 1. If \( q \) has a finite multiplicative order \( t > 4 \), this classification remains valid for homogeneous right coideal subalgebras of the small Lusztig quantum group \( u_q^+ (so_{2n+1}) \). As a consequence, we determine that the total number of right coideal subalgebras that contain the coradical equals \( (2n)! \), the order of the Weyl group defined by the root system of type \( B_n \).

1. Introduction

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In the present paper, we continue the classification of right coideal subalgebras in quantized enveloping algebras started in [13]. We offer a complete classification of right coideal subalgebras that contain all group-like elements for the multiparameter version of the quantum group \( U_q^+ (so_{2n+1}) \), provided that the main parameter \( q \) is not a root of 1. If \( q \) has a finite multiplicative order \( t > 4 \), this classification remains valid for homogeneous right coideal subalgebras of the multiparameter version of the small Lusztig quantum group \( u_q^+ (so_{2n+1}) \). The main result of the paper establishes a bijection between all sequences \((\theta_1, \theta_2, \ldots, \theta_n)\) such that \( 0 \leq \theta_k \leq 2n - 2k + 1, 1 \leq k \leq n \) and the set of all (homogeneous if \( q^t = 1, t > 4 \)) right coideal subalgebras of \( U_q^+ (so_{2n+1}) \), \( q^t \neq 1 \) (respectively of \( u_q^+ (so_{2n+1}) \)) that contain the coradical. (Recall that in a pointed Hopf algebra, the group-like elements span the coradical.) In particular there are \((2n)!\) different right coideal subalgebras that contain the coradical. Interestingly this number coincides with the order of the Weyl group for the root system of type \( B_n \). In [13] we proved that the number of different right coideal subalgebras that contain the coradical in \( U_q^+ (sl_{n+1}) \) equals \((n + 1)!\), the order of the Weyl group for the root system of type \( A_n \). Recently B. Pogorelsky [16] proved that the quantum Borel algebra \( U_q^+ (g) \) for the simple Lie algebra of type \( G_2 \) has 12 different right coideal subalgebras over the coradical. This number also coincides with the order of the Weyl group of type \( G_2 \). Although there is no theoretical explanation why the Weyl group appears in these results, we state the following general hypothesis.

Conjecture. Let \( g \) be a simple Lie algebra defined by a finite root system \( R \). The number of different right coideal subalgebras that contain the coradical in a quantum Borel algebra \( U_q^+ (g) \) equals the order of the Weyl group defined by the root system \( R \), provided that \( q \) is not a root of 1.

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In Section 2, following [13], we introduce main concepts and formulate some known results that are useful for classification. In the third section we prove auxiliary relations in a multiparameter version of $U_q^+(\mathfrak{so}_{2n+1})$. In the fourth section we note that the Weyl basis of integer numbers from the interval $[1, n]$.

where

$$
\tau_n = 1 \text{ with only one exception being } \tau_n = q, \text{ where } g_{k_i} \text{ are suitable group-like elements. Interestingly, this coproduct formula differs from that in } U_q^+(\mathfrak{sl}_{2n+1}) \text{ by just one term (see formula (3.3) in [11]).}
$$

By means of the shuffle representation, in Theorem 4.3 we prove the explicit formula for the coproduct of these PBW-generators, which is the key result for the following further considerations:

$$
\Delta(u[k, m]) = u[k, m] \otimes 1 + g_{km} \otimes u[k, m] + \sum_{i=k}^{m-1} \tau_i(1 - q^{-2})g_{ki}u[i + 1, m] \otimes u[k, i],
$$

where

$$
\Delta(u[k, m]) = u[k, m] \otimes 1 + g_{km} \otimes u[k, m] + \sum_{i=k}^{m-1} \tau_i(1 - q^{-2})g_{ki}u[i + 1, m] \otimes u[k, i],
$$

where $\tau_i = 1$ with only one exception being $\tau_n = q$, while $g_{k_i}$ are suitable group-like elements. Interestingly, this coproduct formula differs from that in $U_q^+(\mathfrak{sl}_{2n+1})$ by just one term (see formula (3.3) in [11]).

In Section 5 we show that each homogeneous right coideal subalgebra in $U_q^+(\mathfrak{so}_{2n+1})$ or in $U_q^+(\mathfrak{so}_{2n+1})$ has PBW-generators of a special form, $\Phi^S(k, m)$, where $S$ is a set of integer numbers from the interval $[1, 2n]$. The polynomial $\Phi^S(k, m)$ is defined by induction on the number $r$ of elements in the set $S \cap [k, m - 1] = \{s_1, s_2, \ldots, s_r\}$, $k \leq s_1 < s_2 < \ldots < s_r < m$ as follows:

$$
\Phi^S(k, m) = u[k, m] - (1 - q^{-2}) \sum_{i=1}^{r} \alpha_{km}^i \Phi^S(1 + s_i, m)u[k, s_i],
$$

where $\alpha_{km}^i$ are scalars, $\alpha_{km}^i = \tau_i p(u(1 + s, m), u(k, s))^{-1}$. This implies that the set of all (homogeneous) right coideal subalgebras that contain the coradical is finite (Corollary [5.7].

In Sections 6 and 7 we singled out special sets $S$, called $(k, m)$-regular sets. In Proposition [4, 10] we establish a kind of duality for elements $\Phi^S(k, m)$ with regular $S$, which provides a powerful tool for investigation of PBW-generators for the right coideal subalgebras.

In Section 8 we define a root sequence $r(U) = (\theta_1, \theta_2, \ldots, \theta_n)$ in the following way. The number $\theta_i$ is the maximal $m$ such that for some $S$ the value of $\Phi^S(i, m)$ belongs to $U$, while the degree $x_i + x_{i+1} + \ldots + x_m$ of $\Phi^S(i, m)$ is not a sum of other nonzero degrees of elements from $U$. In Theorem [8.2] we show that the root sequence uniquely defines the right coideal subalgebra $U$ that contains the coradical.

In Section 9 we consider some important examples. Among them is the right coideal subalgebra generated by $\Phi^S(k, m)$ with regular $S$. We also analyze in detail the simplest (but not trivial, [2]) case $n = 2$.

In Section 10 we associate a right coideal subalgebra $U_\theta$ to each sequence of integer numbers $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$, $0 \leq \theta_i \leq 2n - 2i + 1$, so that $r(U_\theta) = \theta$. First, by downward induction on $k$ we define sets

$$
R_k \subseteq [k, 2n - k], \quad T_k \subseteq [k, 2n - k + 1], \quad 1 \leq k \leq 2n
$$
as follows. For $k > n$ we put $R_k = T_k = \emptyset$. Suppose that $R_i, T_i, k < i \leq 2n$ are already defined. Denote by $P$ the following binary predicate on the set of all
ordered pairs $i \leq j$:
\[ P(i, j) = j \in T_i \cup 2n - i + 1 \in T_{2n-i+1}. \]
If $\theta_k = 0$, then we set $R_k = T_k = \emptyset$. If $\theta_k \neq 0$, then by definition $R_k$ contains $\tilde{\theta}_k = k + \theta_k - 1$ and all $m$ satisfying the following three properties
\begin{itemize}
  \item[a)] $k \leq m < \tilde{\theta}_k$;
  \item[b)] $\neg P(m + 1, \tilde{\theta}_k)$;
  \item[c)] $\forall r(k \leq r < m) \ P(r + 1, m) \iff P(r + 1, \tilde{\theta}_k)$.
\end{itemize}
Further, we define an auxiliary set
\[ T_k^r = R_k \cup \bigcup_{s \in R_k} \{ a \mid s < a \leq 2n - k, \ P(s + 1, a) \}, \]
and put
\[ T_k = \begin{cases} T_k^r, & \text{if } (2n - R_k) \cap T_k^r = \emptyset; \\
T_k^r \cup \{ 2n - k + 1 \}, & \text{otherwise}. \end{cases} \]
Next, the subalgebra $U_\theta$ by definition is generated over $k[G]$ by values in $U_q(\mathfrak{so}_{2n+1})$ or in $u_q(\mathfrak{so}_{2n+1})$ of the polynomials $\Phi^T_k(k, m)$, $1 \leq k \leq m$ with $m \in R_k$.

Theorem 8.2 and Theorem 10.3 show that all right coideal subalgebras over the coradical have the form $U_\theta$.

In Section 11 we restate the main result in a slightly more general form. We consider homogeneous right coideal subalgebras $U$ such that the intersection $\Omega = U \cap G$ with the group $G$ of all group-like elements is a subgroup. In this case $U = k[\Omega]U_\theta$, where $U_\theta$ is a subalgebra generated by $g^{-1}a$ when $a = \Phi^T_k(k, m)$ runs through the above described generators of $U_\theta$.

The present paper extends [13] by similar methods and in a parallel way. However technically it is much more complicated. The proof of the explicit formula for comultiplication (Theorem 10.3) essentially depends on the shuffle representation given in Proposition 4.2 while the same formula for case $A_n$ is proved by a simple induction [11]. The elements $\Phi^S(k, m)$ that naturally appear as PBW-generators for right coideal subalgebras do not satisfy all necessary properties for further development. Therefore in Section 7 we have to introduce and investigate the elements $\Phi^S(k, m)$ with so called $(k, m)$-regular sets $S$. In Proposition 7.10 we establish a powerful duality for such elements. Interestingly, as a consequence of the classification, we prove that every right coideal subalgebra over the coradical is generated as an algebra by elements $\Phi^S(k, m)$ with $(k, m)$-regular sets $S$ (Corollary 10.4). The construction of $U_\theta$ is more complicated and it has an important new element, a binary predicate defined on the ordered pairs of indices. In [13] we relatively easy find a differential subspace generated by $\Psi^S(k, m)$ by virtue of the fact that this element is linear in each variable that it depends on. However the elements $\Phi^S(k, m)$ that appeared in the present work are not linear in some variables. In this connection we fail to find their partial derivatives in an appropriate form. Instead, in Theorem 9.8 using the root technique developed in Section 8, we find algebra generators of the right coideal subalgebra generated by $\Phi^S(k, m)$ with $(k, m)$-regular set $S$.

2. Preliminaries

PBW-generators. Let $A$ be an algebra over a field $k$ and $B$ its subalgebra with a fixed basis $\{ g_j \mid j \in J \}$. A linearly ordered subset $V \subseteq A$ is said to be a set of
PBW-generators of $A$ over $B$ if there exists a function $h : V \to \mathbb{Z}^+ \cup \infty$, called the **height function**, such that the set of all products
\[(2.1)\]
\[g_1 v_1^{n_1} v_2^{n_2} \cdots v_k^{n_k},\]
where $j \in J$, $v_1 < v_2 < \ldots < v_k \in V$, $n_i < h(v_i), 1 \leq i \leq k$ is a basis of $A$. The value $h(v)$ is referred to as the **height** of $v$ in $V$.

**Skew brackets.** Recall that a Hopf algebra $H$ is referred to as a **character Hopf algebra** if the group $G$ of all group like elements is commutative and $H$ is generated over $k[G]$ by skew primitive semi-invariants $a_i$, $i \in I$:
\[(2.2)\]
\[\Delta(a_i) = a_i \otimes 1 + g_i \otimes a_i, \quad g_i^{-1}a_i g = \chi^i(g)a_i, \quad g, g_i \in G,\]
where $\chi^i, i \in I$ are characters of the group $G$. By means of the Dedekind Lemma it is easy to see that every character Hopf algebra is graded by the monoid $G^*$ of characters generated by $\chi^i$:
\[(2.3)\]
\[H = \bigoplus_{\chi \in G^*} \otimes H^\chi, \quad H^\chi = \{a \in H \mid g^{-1}ag = \chi(g)a, \quad g \in G\}.\]

Let us associate a “quantum” variable $x_i$ to $a_i$. For each word $u$ in $X = \{x_i \mid i \in I\}$ we denote by $g_u$ or $\text{gr}(u)$ an element of $G$ that appears from $u$ by replacing each $x_i$ with $g_i$. In the same way we denote by $\chi^u$ a character that appears from $u$ by replacing each $x_i$ with $\chi^i$. We define a bilinear skew commutator on homogeneous linear combinations of words in $a_i$ or in $x_i, i \in I$ by the formula
\[(2.4)\]
\[[u, v] = uv - \chi^u(g_v)vu,\]
where sometimes for short we use the notation $\chi^u(g_v) = p_{uv} = p(u, v)$. Of course $p(u, v)$ is a bimultiplicative map:
\[(2.5)\]
\[p(u, vt) = p(u, v)p(u, t), \quad p(ut, v) = p(u, v)p(t, v), \quad \text{sqot}\]
The brackets satisfy the following Jacobi identity:
\[(2.6)\]
\[[[u, v], w] = [u, [v, w]] + p_{uv}^{-1}[[u, w], v] + (p_{vw} - p_{uv}^{-1})[u, w] \cdot v.\]
or, equivalently, in the other less symmetric form
\[(2.7)\]
\[[[u, v], w] = [u, [v, w]] + p_{uv}[u, w] \cdot v - p_{uv}v \cdot [u, w].\]

Jacobi identity (2.6) implies the following conditional identity
\[(2.8)\]
\[[u, v], w] = [u, [v, w]], \quad \text{provided that } [u, w] = 0.\]
By the evident induction on length this conditional identity admits the following generalization, see [13, Lemma 2.2].

**Lemma 2.1.** If $y_1, y_2, \ldots, y_m$ are homogeneous linear combinations of words such that $[y_i, y_j] = 0$, $1 \leq i < j - 1 < m$, then the bracketed polynomial $[y_1 y_2 \ldots y_m]$ is independent of the precise alignment of brackets:
\[(2.9)\]
\[[y_1 y_2 \ldots y_m] = [[y_1 y_2 \ldots y_s], [y_{s+1} y_{s+2} \ldots y_m]], \quad 1 \leq s < m.\]

The brackets are related by the product by the following ad-identities
\[(2.10)\]
\[[u \cdot v, w] = p_{uv}[u, w] \cdot v + u \cdot [v, w],\]
\[(2.11)\]
\[[u, v \cdot w] = [u, v] \cdot w + p_{uw}v \cdot [u, w].\]
In particular, if \([u, w] = 0\), we have

\[(2.12)\quad [u \cdot v, w] = u \cdot [v, w].\]

The antisymmetry identity transforms into the following two equalities

\[(2.13)\quad [u, v] = -p_{uv}[v, u] + (1 - p_{uv}p_{vu})u \cdot v\]

\[(2.14)\quad [u, v] = -p_{vu}^{-1}[v, u] + (p_{vu}^{-1} - p_{uv})v \cdot u.\]

In particular, if \(p_{uv}p_{vu} = 1\), the “color” antisymmetry, \([u, v] = -p_{uv}[v, u]\), holds.

The group \(G\) acts on the free algebra \(k\langle X \rangle\) by \(g^{-1}ug = \chi^u(g)u\), where \(u\) is an arbitrary monomial in \(X\). The skew group algebra \(G\langle X \rangle\) has the natural Hopf algebra structure

\[
\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i, \quad i \in I, \quad \Delta(g) = g \otimes g, \quad g \in G.
\]

We fix a Hopf algebra homomorphism

\[(2.15)\quad \xi : G\langle X \rangle \to H, \quad \xi(x_i) = a_i, \quad \xi(g) = g, \quad i \in I, \quad g \in G.
\]

**PBW-basis of a character Hopf algebra.** A *constitution* of a word \(u\) in \(G \cup X\) is a family of non-negative integers \(\{m_x, x \in X\}\) such that \(u\) has \(m_x\) occurrences of \(x\).

Certainly almost all \(m_x\) in the constitution are zero. We fix an arbitrary complete order, <, on the set \(X\). Normally if \(X = \{x_1, \ldots, x_n\}\), we set \(x_1 > x_2 > \ldots > x_n\).

Let \(\Gamma^+\) be the free additive (commutative) monoid generated by \(X\). The monoid \(\Gamma^+\) is a completely ordered additive monoid with respect to the following order:

\[(2.16)\quad m_1x_{i_1} + m_2x_{i_2} + \ldots + m_kx_{i_k} > m'_1x_{i_1} + m'_2x_{i_2} + \ldots + m'_kx_{i_k}
\]

if the first from the left nonzero number in \((m_1 - m'_1, m_2 - m'_2, \ldots, m_k - m'_k)\) is positive, where \(x_{i_1} > x_{i_2} > \ldots > x_{i_k}\) in \(X\). We associate a formal degree \(D(u) = \sum_{x \in X} m_xx \in \Gamma^+\) to a word \(u\) in \(G \cup X\), where \(\{m_x | x \in X\}\) is the constitution of \(u\). Respectively, if \(f = \sum \alpha_iu_i \in G\langle X \rangle\), \(0 \neq \alpha_i \in k\), then

\[(2.17)\quad D(f) = \max_i\{D(u_i)\}.
\]

On the set of all words in \(X\) we fix the lexicographical order with the priority from the left to the right, where a proper beginning of a word is considered to be greater than the word itself.

A non-empty word \(u\) is called a *standard word* (or Lyndon word, or Lyndon-Shirshov word) if \(vw > wv\) for each decomposition \(u = vw\) with non-empty \(v, w\). A *nonassociative* word is a word where brackets \([,]\) somehow arranged to show how multiplication applies. If \([u]\) denotes a nonassociative word, then by \(u\) we denote an associative word obtained from \([u]\) by removing the brackets. The set of *standard nonassociative* words is the biggest set \(SL\) that contains all variables \(x_i\) and satisfies the following properties.

1. If \([u] = [[v][w]] \in SL\), then \([v], [w] \in SL\), and \(v > w\) are standard.
2. If \([u] = [[v_1][v_2]][w] \in SL\), then \(v_2 \leq w\).

Every standard word has only one alignment of brackets such that the appeared nonassociative word is standard (Shirshov theorem [18]). In order to find this alignment one may use the following inductive procedure:

**Algorithm.** The factors \(v, w\) of the nonassociative decomposition \([u] = [[v][w]]\) are the standard words such that \(u = vw\) and \(v\) has the minimal length ([19], see also [14]).
Definition 2.2. A super-letter is a polynomial that equals a nonassociative standard word where the brackets mean (2.4). A super-word is a word in super-letters.

By Shirshov’s theorem every standard word \( u \) defines only one super-letter, in what follows we shall denote it by \([u]\). The order on the super-letters is defined in the natural way: \([u] > [v] \iff u > v\).

In what follows we fix a notation \( H \) for a character Hopf algebra homogeneous in each \( a_i \), see (2.2) and (2.15).

Definition 2.3. A super-letter \([u]\) is called hard in \( H \) provided that its value in \( H \) is not a linear combination of values of super-words of the same degree (2.17) in smaller than \([u]\) super-letters.

Definition 2.4. We say that a height of a hard super-letter \([u]\) in \( H \) equals \( h = h([u]) \) if \( h \) is the smallest number such that: first, \( p_{uu} \) is a primitive \( t \)-th root of \( 1 \) and either \( h = t \) or \( h = tl_r \), where \( l = \text{char}(k) \); and then the value of \([u]^h\) in \( H \) is a linear combination of super-words of the same degree (2.17) in less than \([u]\) super-letters. If there exists no such number, then the height equals infinity.

Theorem 2.5. ([7, Theorem 2]). The values of all hard super-letters in \( H \) with the above defined height function form a set of PBW-generators for \( H \) over \( k[G] \).

PBW-basis of a homogeneous right coideal subalgebra. The set \( T \) of PBW-generators for a homogeneous right coideal subalgebra \( U \), \( k[G] \subseteq U \subseteq H \), can be obtained from the PBW-basis given in Theorem 2.5 in the following way; see [12, Theorem 1.1].

Suppose that for a hard super-letter \([u]\) there exists a homogeneous element \( c \in U \) with the leading term \([u]^s\) in the PBW-decomposition given in Theorem 2.5:

\[
c = [u]^s + \sum_i \alpha_i W_i \in U,
\]

where \( W_i \) are the basis super-words starting with less than \([u]\) super-letters. We fix one of the elements with the minimal \( s \), and denote it by \( c_u \). Thus, for every hard super-letter \([u]\) in \( H \) we have at most one element \( c_u \). We define the height function by means of the following lemma.

Lemma 2.6. ([12, Lemma 4.3]). In the representation (2.18) of the chosen element \( c_u \) either \( s = 1 \), or \( p(u, u) \) is a primitive \( t \)-th root of \( 1 \) and \( s = t \) or (in the case of positive characteristic) \( s = t(\text{char} k)^r \).

If the height of \([u]\) in \( H \) is infinite, then the height of \( c_u \) in \( U \) is defined to be infinite as well. If the height of \([u]\) in \( H \) equals \( t \), then, due to the above lemma, \( s = 1 \) (in the PBW-decomposition (2.18) the exponent \( s \) must be less than the height of \([u]\)). In this case the height of \( c_u \) in \( U \) is supposed to be \( t \) as well. If the characteristic \( l \) is positive, and the height of \([u]\) in \( H \) equals \( tl_r \), then we define the height of \( c_u \) in \( U \) to be equal to \( tl_r / s \).

Proposition 2.7. ([12, Proposition 4.4]). The set of all chosen \( c_u \) with the above defined height function forms a set of PBW-generators for \( U \) over \( k[G] \).

We are reminded that the PBW-basis is not uniquely defined in the above process. Nevertheless the set of leading terms of the PBW-generators indeed is uniquely defined.
Definition 2.8. The degree $sD(c_u) \in \Gamma^+$ of a PBW-generator $c_u$ is said to be an U-root. An U-root $\gamma \in \Gamma^+$ is called a simple U-root if it is not a sum of two or more other U-roots.

The set of U-roots, and the set of simple U-roots are invariants for any right coideal subalgebra $U$.

Shuffle representation. If the kernel of $\xi$ defined in (2.14) is contained in the ideal $G(X)^{(2)}$ generated by $x_ix_j$, $i, j \in I$, then there exists a Hopf algebra projection $\pi : H \to k[G]$, $a_i \to 0$, $g_i \to g_i$. Hence by the Radford theorem [17] we have a decomposition in a biproduct, $H = A \# k[G]$, where $A$ is a subalgebra generated by $a_i$, $i \in I$, see [1] §1.5, §1.7.

Definition 2.9. In what follows we denote by $A$ the biggest Hopf ideal in $G(X)^{(2)}$, where as above $G(X)^{(2)}$ is the ideal of $G(X)$ generated by $x_ix_j$, $i, j \in I$. The ideal $A$ is homogeneous in each $x_i \in X$, see [1] Lemma 2.2.

If $\text{Ker} \xi = A$ or, equivalently, if $A$ is a quantum symmetric algebra (a Nichols algebra [1] §1.3, Section 2), then $A$ has a shuffle representation as follows.

The algebra $A$ has a structure of a braided Hopf algebra, [20], with a braiding $\tau(u \otimes v) = p(v, u)^{-1}v \otimes u$. The braided coproduct $\Delta^b$ on $A$ is connected with the coproduct on $H$ in the following way

$$\Delta^b(u) = \sum_{(u)} u^{(1)} \text{gr}(u^{(2)})^{-1} \otimes u^{(2)}.$$  

(2.19)  

The tensor space $T(V)$, $V = \sum x_i k$ also has a structure of a braided Hopf algebra. This is the quantum shuffle algebra $Sh_\tau(V)$ with the coproduct

$$\Delta^b(u) = \sum_{i=0}^m (z_1 \ldots z_i) \otimes (z_{i+1} \ldots z_m),$$  

(2.20)  

where $z_i \in X$, and $u = (z_1 z_2 \ldots z_{m-1} z_m)$ is the tensor $z_1 \otimes z_2 \otimes \ldots \otimes z_{m-1} \otimes z_m$ considered as an element of $Sh_\tau(V)$. The shuffle product satisfies

$$\sum_{u = u} (w)(x_i) = \sum_{u = v} p(x_i, v)^{-1}(ux_i v), \quad (x_i)(w) = \sum_{u = v} p(u, x_i)^{-1}(ux_i v).$$  

(2.21)  

The map $a_i \to (x_i)$ defines an embedding of the braided Hopf algebra $A$ into the braided Hopf algebra $Sh_\tau(V)$. This embedding is extremely useful for calculation of the coproduct due to formulae (2.19), (2.20).

Differential calculus. The free algebra $k\langle X \rangle$ has a coordinate differential calculus

$$\partial_j(x_i) = \delta_i^j, \quad \partial_i(uv) = \partial_i(u) \cdot v + \chi^u(g_i) u \cdot \partial_i(v).$$  

(2.22)  

The partial derivatives connect the calculus with the coproduct on $k\langle X \rangle$ via

$$\Delta(u) \equiv u \otimes 1 + \sum_i g_i \partial_i(u) \otimes x_i \quad (\text{mod } G(X) \otimes k\langle X \rangle^{(2)}),$$  

(2.23)  

where $k\langle X \rangle^{(2)}$ is the ideal generated by $x_ix_j$, $1 \leq i, j \leq n$.

Lemma 2.10. Let $u \in k\langle X \rangle$ be an element homogeneous in each $x_i$. If $p_{uu}$ is a $t$-th primitive root of 1, then

$$\partial_i(u^t) = p(u, x_i)^{t-1} [u, [u, \ldots [u, \partial_i(u)] \ldots]].$$  

(2.24)
Proof. First of all we note that the sequence \( p_{uu}, p_{uu}^2, \ldots, p_{uu}^{t-1} \) contains all \( t \)-th roots of 1 except the 1 itself. All members in this sequence are different. Hence we may write a polynomial equality

\[
(1 - x^t) = (1 - x) \prod_{s=1}^{t-1} (1 - p_{uu}^s x).
\]

Let us calculate the right hand side of (2.24). Denote by \( L_u, R_u \) the operators of left and right multiplication by \( u \) respectively. The right hand side of (2.24) has the following operator representation

\[
p(u, x_i)^{t-1} \left( \prod_{s=1}^{t-1} (L_u - Q p_{uu}^{s-1} R_u) \right),
\]

where \( Q = p(u, \partial_i(u)) = p_{uu} p(u, x_i)^{-1} \). Consider a polynomial

\[
f(\lambda) = \prod_{s=1}^{t-1} (1 - Q p_{uu}^{s-1} \lambda) \equiv \sum_{k=0}^{t-1} \alpha_k \lambda^k.
\]

Since the operators \( R_u \) and \( L_u \) commute, we may develop the multiplication in the operator product considering \( R_u \) and \( L_u \) as formal commutative variables:

\[
\prod_{s=1}^{t-1} (L_u - Q p_{uu}^{s-1} R_u) = L_u^{t-1} f\left( \frac{R_u}{L_u} \right) = \sum_{k=0}^{t-1} \alpha_k L_u^{t-1-k} R_u^k.
\]

Thus the right hand side of (2.24) equals

\[
p(u, x_i)^{t-1} \sum_{k=0}^{t-1} \alpha_k u^{t-1-k} \partial_i(u) u^k.
\]

Further, since \( Q = p_{uu} p(u, x_i)^{-1} \), the polynomial \( f \) has a representation

\[
f(\lambda) = \prod_{s=1}^{t-1} (1 - p_{uu}^s \xi),
\]

where \( \xi = \lambda p(u, x_i)^{-1} \). Taking into account (2.25), we get

\[
f(\lambda) = \frac{1 - \xi^t}{1 - \xi} = \frac{1 - \lambda^t p(u, x_i)^{-t}}{1 - \lambda p(u, x_i)^{-1}}
\]

\[
= 1 + \lambda p(u, x_i)^{-1} + \lambda^2 p(u, x_i)^{-2} + \cdots + \lambda^{t-1} p(u, x_i)^{1-t};
\]

that is, \( \alpha_k = p(u, x_i)^{-k} \), while the right hand side of (2.24) takes the form

\[
(2.26) \quad \sum_{k=0}^{t-1} p(u, x_i)^{t-1-k} u^{t-1-k} \partial_i(u) u^k.
\]

At the same time Leibniz formula (2.22) shows that \( \partial_i(u^t) \) also equals (2.26). \( \square \)

**MS-criterion.** The quantum symmetric algebra has a lot of nice characterizations. One of them says that this is the *optimal algebra* for the calculus defined by (2.22). In other words the above defined algebra \( \mathcal{A} \) is a quantum symmetric algebra (or, equivalently, Ker \( \xi = \mathcal{A} \)) if and only if all constants in \( \mathcal{A} \) are scalars.

For braidings of Cartan type this characterization was proved by A. Milinski and H.-J. Schneider in [15], and then generalized for arbitrary (even not necessarily invertible) braidings by the author in [10, Theorem 4.11]. Moreover, if \( X \) is finite,
then $\Lambda$ (as well as any differential ideal in $k(X)$) is generated as a left ideal by constants from $k(X)^{(2)}$, see [10] Corollary 7.8. Thus, we may formulate the following criterion that is useful for checking relations.

**Lemma 2.11.** (Milinski—Schneider criterion). Suppose that $\text{Ker}\xi = \Lambda$. If a polynomial $f \in k(X)$ is a constant in $\Lambda$ (that is, $\partial_i(f) \in \Lambda$, $i \in I$), then there exists $\alpha \in k$ such that $f - \alpha = 0$ in $A$.

Of course one can easily prove this criterion by means of (2.19), (2.20) and (2.23) using the above shuffle representation since (2.20) implies that all constants in the shuffle coalgebra are scalars.

**Quantum Borel algebra.** Let $C = ||a_{ij}||$ be a symmetrizable by $D = \text{diag}(d_1, \ldots, d_n)$ generalized Cartan matrix, $d_ia_{ij} = d_j a_{ji}$. Denote by $\mathfrak{g}$ a Kac-Moody algebra defined by $C$, see [5]. Suppose that parameters $p_{ij}$ are related by

$$p_{ii} = q^{d_i}, \quad p_{ij}p_{ji} = q^{d_{aij}}, \quad 1 \leq i, j \leq n.$$  

Denote by $g_j$ a linear transformation $g_j : x_i \mapsto p_{ij}x_i$ of the linear space spanned by a set of variables $X = \{x_1, x_2, \ldots, x_n\}$. By $\chi_i^1$ we denote a character $\chi^1 : g_j \mapsto p_{ij}$ of the group $G$ generated by $g_i$, $1 \leq i \leq n$. We consider each $x_i$ as a “quantum variable” with parameters $g_i$, $\chi_i^1$. As above we denote by $G(X)$ the skew group algebra with commutation rules $x_ig_j = p_{ij}g_jx_i$, $1 \leq i, j \leq n$. This algebra has a structure of a character Hopf algebra

$$\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i, \quad \Delta(g_i) = g_i \otimes g_i.$$  

In this case the multiparameter quantization $U_q^+(\mathfrak{g})$ of the Borel subalgebra $\mathfrak{g}^+$ is a homomorphic image of $G(X)$ defined by Serre relations with the skew brackets in place of the Lie operation:

$$[[\ldots[x_{ij}, x_{j}], x_j], \ldots, x_j] = 0, \quad 1 \leq i \neq j \leq n.$$  

By [6] Theorem 6.1] the left hand sides of these relations are skew-primitive elements in $G(X)$. Therefore the ideal generated by these elements is a Hopf ideal, while $U^+_q(\mathfrak{g})$ has a natural structure of a character Hopf algebra.

**Lemma 2.12.** ([13] Corollary 3.2]). If $q$ is not a root of 1 and $C$ is of finite type, then every subalgebra $U$ of $U_q^+(\mathfrak{g})$ containing $G$ is homogeneous with respect to each of the variables $x_i$.

**Definition 2.13.** If the multiplicative order $t$ of $q$ is finite, then we define $u_q^+(\mathfrak{g})$ as $G(X)/\Lambda$, where $\Lambda$ is the biggest Hopf ideal in $G(X)^{(2)}$, see Definition 2.11.

Since a skew-primitive element generate a Hopf ideal, $\Lambda$ contains all skew-primitive elements of $G(X)^{(2)}$. Hence relations (2.20) are still valid in $u_q^+(\mathfrak{g})$.

3. **Relations in quantum Borel algebra $U_q^+(g_{\alpha_{2n+1}})$.**

In what follows we fix a parameter $q$ such that $q^4 \neq 1$, $q^3 \neq 1$. If $C$ is a Cartan matrix of type $B_n$, relations (2.27) take up the form

$$p_{nn} = q, \quad p_{ii} = q^2, \quad p_{i+1}p_{i+1} = q^{-2}, \quad 1 \leq i < n;$$

$$p_{ij}p_{ji} = 1, \quad j > i + 1.$$
Proof. The element \( u \) are the same for \( U \) if \( \text{Lemma 3.4.} \)
The defining relations that are independent of \( p \)
Here we slightly modify Serre relations \( (2.29) \) so that the left hand side of each
relation is a super-letter. It is possible to do due to the following general relation
in \( k \langle X \rangle \), see \[9, Corollary 4.10]\):
\[
(3.5) \quad \prod_{i<n} [x_i, x_{i+1} + x_{i-1}] = \alpha \prod_{i<n} [x_{i+1}, x_{i-1}], \quad 0 \neq \alpha \in k,
\]
provided that \( p_{ij} = p_{ji}^{-1} \).

**Definition 3.1.** The elements \( u, v \) are said to be *separated* if there exists an index
\( j, 1 \leq j \leq n \), such that either \( u \in k(x_i | i < j) \), \( v \in k(x_i | i > j) \) or vise versa:
\( u \in k(x_i | i > j) \), \( v \in k(x_i | i < j) \).

**Lemma 3.2.** In the algebra \( U^+_q(\mathfrak{so}_{2n+1}) \) every two separated homogeneous in each
\( x_i \in X \) elements \( u, v \) (skew)commute, \( [u, v] = [v, u] = 0 \).

**Proof.** Relations \[3.2\] and conditional antisymmetry \[2.13\] show that \( [x_i, x_j] = [x_j, x_i] = 0 \), provided that \( |i - j| > 1 \). Now relations \[2.10\], \[2.11\] allow one to perform an evident induction. \( \square \)

Certainly the subalgebra of \( U^+_q(\mathfrak{so}_{2n+1}) \) generated over \( k[g_1, \ldots, g_{n-1}] \) by \( x_i \),
\( 1 \leq i < n \) is the Hopf algebra \( U^+_q(\mathfrak{sl}_n) \) dened by the Cartan matrix of type \( A_{n-1} \).
Let us replace just one parameter \( p_{nn} \leftarrow q^2 \). Then the quantum Borel algebra
\( U^+_q(\mathfrak{sl}_n) \) is a homomorphic image of \( G'(X) \) subject to relations
\[
(3.6) \quad [x_i, x_{i+1}], x_{i+1} = [x_i, x_{i+1}], x_{i+1} = [x_i, x_j] = 0, \quad j > i + 1.
\]
Here \( G' \) is the group generated by transformations \( g_1, \ldots, g_{n-1}, g'_n \), where \( g'_n(x_i) = g_n(x_i) \) with only one exception being \( g'_n(x_n) = q^2 x_n \).

**Lemma 3.3.** A linear in \( x_n \) relation \( f = 0, f \in k(X) \) is valid in \( U^+_q(\mathfrak{so}_{2n+1}) \) if
and only if it is valid in the above algebra \( U^+_q(\mathfrak{sl}_n) \).

**Proof.** The element \( f \) as an element of free algebra belongs to the ideal generated by
the defining relations that are independent of \( x_n \) or linear in \( x_n \). All that relations are the same for \( U^+_q(\mathfrak{so}_{2n+1}) \) and for \( U^+_q(\mathfrak{sl}_n) \).

**Lemma 3.4.** If \( u \) is a standard word, then either \( u = x_k x_{k+1} \ldots x_m, k \leq m \leq n \), or
\( [u] = 0 \) in \( U^+_q(\mathfrak{sl}_n) \). Here \( [u] \) is a nonassociative word with the standard alignment
of brackets, see Algorithm on page \[7\]

**Proof.** See the third statement of \[9, Theorem A_n\]. \( \square \)

As a corollary of the above two lemmas we can prove some relations in \( U^+_q(\mathfrak{so}_{2n+1}) \):
\[
(3.7) \quad [x_{k+1} x_k x_{k-1}], x_k] = 0, \quad [x_{k-1} x_k x_{k+1}], x_k] = 0, \quad k < n.
\]
Indeed, $x_{k-1} x_k x_{k+1} x_k$ is a standard word and the standard alignment of brackets is precisely $[[x_{k-1}, [x_k, x_{k+1}]], x_k]$. Hence (2.8) with Lemma 3.3 and Lemma 3.4 imply the latter relation.

The former relation reduces to the latter one by means of the replacement $x_i \leftarrow x_{n-i+1}$, $1 \leq i \leq n$, $k \leftarrow n-k+1$. It remains to note that the defining relations (3.9) are invariant under this replacement (see (3.5)), and again use Lemma 3.3 and Lemma 3.4.

**Definition 3.5.** In what follows we denote by $x_i$, $n < i \leq 2n$ the generator $x_{2n-i+1}$. Respectively, $u(k, m)$, $1 \leq k \leq m \leq 2n$ is the word $x_k x_{k+1} \cdots x_{m-1} x_m$, while $u(m, k)$ is the word $x_m x_{m-1} \cdots x_{k+1} x_k$. If $1 \leq i \leq 2n$, then by $\psi(i)$ we denote the number $2n - i + 1$, so that $x_i = x_{\psi(i)}$. We shall frequently use the following properties of $\psi$: if $i < j$, then $\psi(i) > \psi(j)$; $\psi(\psi(i)) = i$; $\psi(i + 1) = \psi(i) - 1$.

**Definition 3.6.** If $k \leq i < m \leq 2n$, then we denote
\begin{align*}
\sigma_k^m & \overset{df}{=} p(u(k, m), u(k, m)), \\
\mu_k^{m,i} & \overset{df}{=} p(u(k, i), u(i + 1, m)) \cdot p(u(i + 1, m), u(k, i)).
\end{align*}

Of course, one can find $\mu$’s and $\sigma$’s by means of (3.1), (3.2). It turns out that these coefficients depend only on $q$. More precisely,
\begin{equation}
\sigma_k^m = \begin{cases} q, & \text{if } m = n, \text{ or } k = n + 1; \\
q^4, & \text{if } m = \psi(k); \\
q^2, & \text{otherwise.}
\end{cases}
\end{equation}

Indeed, the bimultiplicativity of $p(-, -)$ implies that $\sigma_k^m = \prod_{k \leq s, t \leq m} p_{st}$ is the product of all coefficients of the $(m-k+1) \times (m-k+1)$-matrix $|p_{st}|$. By (3.1) all coefficients on the main diagonal equal $q^2$ with only two possible exceptions being $p_{nn} = q$, $p_{n+1,n+1} = q$. In particular, if $m < n$ or $k > n + 1$, then for non diagonal coefficients we have $p_{st} = 1$ unless $|s - t| = 1$, while $p_{s+1,s+1} = q^{-2}$. Hence $\sigma_k^m = q^{2(m-k+1)} \cdot q^{-2(k-m)} = q^2$. If $m = n$ or $k = n + 1$, then by the same reason we have $\sigma_k^m = q^{2(m-k)+1} \cdot q^{-2(k-m)} = q$. In the remaining case, $k \leq n < m$, we split the matrix in four submatrices as follows
\begin{equation}
\sigma_k^m = \sigma_k^s \cdot \sigma_s^{m-1} \cdot \prod_{k \leq s \leq n, n+1 \leq t \leq m} p_{st} \cdot \prod_{n+1 \leq s \leq m, k \leq t \leq n} p_{st}.
\end{equation}

According to Definition 3.5 we have $p_{st} = p_{\psi(s) t} = p_{s \psi(t)} = p_{\psi(s) \psi(t)}$. Therefore the third and fourth factors in (3.11) respectively equal
\begin{equation*}
\prod_{k \leq s \leq n, \psi(m) \leq t \leq n} p_{st}; \quad \prod_{\psi(m) \leq s \leq n, k \leq t \leq n} p_{st}.
\end{equation*}

In particular, if $\psi(m) = k$, then all four factors in (3.11) coincide with $\sigma_k^s = q$, hence $\sigma_k^m = q^4$. If $\psi(m) \neq k$, say $\psi(m) > k$, then we split the rectangular $A = [k, n] \times [\psi(m), n]$ in a union of the square $B = [\psi(m), n] \times [\psi(m), n]$ and the rectangular $C = [k, \psi(m) - 1] \times [\psi(m), n]$. Similarly the rectangular $A^* = [\psi(m), n] \times [k, n]$ is a union of the same square and the rectangular $C^* = [\psi(m), n] \times [k, \psi(m) - 1]$. Right coideal subalgebras 11
Certainly, if \((s, t) \in C\), then \(t - s > 1\) unless \(t = \psi(m) - 1\), \(s = \psi(m)\). Hence relations (3.2) imply
\[
\prod_{(s, t) \in C} p_{st}p_{ts} = p_{\psi(m) - 1, \psi(m)} p_{\psi(m)} = q^{-2}.
\]
At the same time \(\prod_{(s, t) \in B} p_{st} = \sigma_{\psi(m)}^n = q\). Finally, (3.11) takes the form
\[
\sigma_{\psi(m)}^m = q \cdot q \cdot \left( \prod_{(s, t) \in B} p_{st} \right)^2 \cdot \prod_{(s, t) \in C} p_{st}p_{ts} = q^2,
\]
which proves (3.10).
To find \(\mu\)'s we consider decomposition (3.11) with \(n - i\). Since \(p(-, -)\) is a bimultiplicative map, the product of the last two factors is precisely \(\mu_{m,i}^m\). In particular we have
\[
\mu_{m,i}^m = \sigma_k^m (\sigma_{k+1}^m)^{-1}.
\]
This formula with (3.10) allows one easily to find the \(\mu\)'s. More precisely, if \(m < \psi(k)\), then
\[
\mu_{m,i}^m = \begin{cases} 
q^{-4}, & \text{if } m > n, \ i = \psi(m) - 1; \\
q^{-2}, & \text{if } i = n; \\
q^{-4}, & \text{otherwise}.
\end{cases}
\]
If \(m = \psi(k)\); that is \(x_m = x_k\), then
\[
\mu_{m,i}^m = \begin{cases} 
q^4, & \text{if } i = n; \\
1, & \text{otherwise}.
\end{cases}
\]
If \(m > \psi(k)\), then the \(\mu\)'s satisfy \(\mu_{m,i}^m = \mu_{\psi(k), \psi(i)}^{-1}\), hence one may use (3.13):
\[
\mu_{m,i}^m = \begin{cases} 
q^{-4}, & \text{if } k \leq n, \ i = \psi(k); \\
q^{-2}, & \text{otherwise}.
\end{cases}
\]
We define the bracketing of \(u(k, m), k \leq m\) as follows.
\[
u[k, m] = \begin{cases} 
[[\ldots [x_k, x_{k+1}, \ldots, x_{m-1}, x_m] \ldots]], & \text{if } m < \psi(k); \\
[x_k, [x_{k+1}, \ldots, [x_{m-1}, x_m], \ldots]], & \text{if } m > \psi(k); \\
[\beta[u[n+1, m], u[k, n]], u[k, n]], & \text{if } m = \psi(k),
\end{cases}
\]

where \(\beta = -p(u(n+1, m), u(k, n))^{-1}\) normalizes the coefficient at \(u(k, m)\). Conditional identity (3.14) shows that the value of \(u[k, m]\) in \(U_q^+(\mathfrak{so}_{2n+1})\) is independent of the precise alignment of brackets provided that \(m \leq n\) or \(k > n\).

In what follows we denote by \(\sim\) the projective equality: \(a \sim b\) if and only if \(a = \alpha b\), where \(0 \neq \alpha \in k\).

**Lemma 3.7.** If \(t \notin \{k - 1, k\}\), \(t < n\), then \([u[k, n], x_t] = [x_t, u[k, n]] = 0\).

**Proof.** If \(t \leq k - 2\), then the equality follows from the second group of defining relations (3.3). Let \(k < t < n\). By (2.8) we may write
\[
[u[k, n], x_t] = [[u[k, t - 2], u[t - 1, n]], x_t] = [u[k, t - 2], [u[t - 1, n], x_t]].
\]
By Lemma 3.3 the element \([u[t - 1, n], x_t]\) equals zero in \(U_q^+(\mathfrak{so}_{2n+1})\) since the word \(u(t - 1, n)x_t\) is standard and the standard bracketing is precisely \([u[t - 1, n], x_t]\).
This element is linear in \(x_n\). Hence \([u[k, n], x_t] = 0\) in \(U_q^+(\mathfrak{so}_{2n+1})\) as well due to
Lemma 3.8. Since \( p(u(k, n), x_t)p(x_t, u(k, n)) = pt_{t+1}pt_{t-1} \cdot pt_{t+1}pt_{t-1} = 1 \),
the antisymmetry identity (2.13) applies.

**Lemma 3.8.** If \( t \notin \{ \psi(m) - 1, \psi(m) \} \), \( t < n < m \), then
\[
[x_t, u[n + 1, m]] = [u[n + 1, m], x_t] = 0.
\]

**Proof.** If \( t \leq \psi(m) - 2 \), then the required relation follows from the second group of relations (3.3). Let \( \psi(m) < t < n \). By Lemma 2.1 the value of \( u[n + 1, m] \) in \( U_q^+(so_{2n+1}) \) is independent of the alignment of brackets. In particular \( u[n + 1, m] = [w, [x_1+x_2, x_t-1]], v \), where \( w = u[n + 1, \psi(t) - 2], v = u[\psi(t) + 2, m] \). Since \( pt_{t+1}pt_{t-1} \cdot pt_{t+1}pt_{t-1} = 1 \), the antisymmetry identity (2.13) and the first of (3.7) imply \([x_t, [x_1+x_2, x_t-1]] \sim [x_1+x_2, x_t] = 0 \). It remains to note that \([x_t, w] = [w, x_t] = 0, [x_t, v] = [v, x_t] = 0 \) according to the second group of defining relations (3.8).

**Lemma 3.9.** If \( k \leq n < m \leq \psi(k) \), then the value in \( U_q^+(so_{2n+1}) \) of the bracketed word \([y_k x_{n+1} x_n + 2 \cdots x_m] \), where \( y_k = u[k, n] \), is independent of the precise alignment of brackets.

**Proof.** In order to apply (2.9) it suffices to check \([u[k, n], x_t] = 0, n + 1 < t \leq m \). Since the application of \( \psi \) change the order, we have \( k < \psi(m) \leq \psi(t) < n \). Hence taking into account \( x_t = x_{\psi(t)} \), one may use Lemma 3.8.

**Lemma 3.10.** If \( k < n < m \leq \psi(k) \), then the value in \( U_q^+(so_{2n+1}) \) of the bracketed word \([x_k x_{k+1} \cdots x_n y_m] \), where \( y_m = u[n + 1, m] \), is independent of the precise alignment of brackets.

**Proof.** To apply (2.9) we need \([x_t, u[n + 1, m]] = 0, k + 1 < t \leq n \). To get these equalities one may use Lemma 3.8.

**Lemma 3.11.** If \( m \neq \psi(k) \), \( k \leq i < n < m \), then
\[
[u[k, i], u[n + 1, m]] = [u[n + 1, m], u[k, i]] = 0
\]
unless \( i = \psi(m) - 1 \).

**Proof.** Denote \( u = u[k, i], w = u[n + 1, m] \). Relations (3.1), (3.2) imply \( p_{uw}p_{wu} = 1 \).

Hence by (2.13) we have \([u, w] = -p_{uw} [w, v] \).

If \( \psi(m) < k \), then by Lemma 3.8 we have \([x_t, u[n + 1, m]] = 0, k \leq t \leq i \). Hence \([u[k, i], u[n + 1, m]] = 0 \).

Suppose that \( \psi(m) > k \). If \( i < \psi(m) - 1 \), then due to the second group of defining relations (3.8) we have \([x_t, u[n + 1, m]] = 0, k \leq t \leq i \). Hence \([u[k, i], u[n + 1, m]] = 0 \).

Let \( \psi(m) \leq i < n \). If we denote \( u_1 = u[k, \psi(m) - 2], u_2 = u[\psi(m) - 1, i] \), then certainly \( u = [u_1, u_2] \) unless \( k = \psi(m) - 1, u = u_2 \). Since \([u_1, w] = 0 \), conditional Jacobi identity (2.8) implies that in both cases we need just to check \([u_2, w] = 0 \).

Let us put \( u_3 = [x_{\psi(m) - 1}, x_{\psi(m)}], u_4 = u[\psi(m) + 1, i] \). Then \( u_2 = [u_3, u_4] \) unless \( i = \psi(m), u_2 = u_3 \). By Lemma 3.8 we have \([x_t, u[n + 1, m]] = 0 \) for all \( t, \psi(m) < t < n \). Hence \([u_3, w] = 0 \). Now Jacobi identity (2.8) with \( u \leftrightarrow u_3, u \leftrightarrow u_4 \) shows that it suffices to prove the equality \([u_3, w] = 0 \).

Let us put \( u_1 = u[m + 1, m - 2], u_2 = [x_{m-1}, x_m] \). Then \( w = [u_1, u_2] \) unless \( m - 2 = n, w = w_2 \). (Recall that we are considering the case \( \psi(m) \leq i < n \), in particular \( \psi(m) \leq n - 1 \), and hence \( m \geq \psi(n - 1) = n + 2 \). We have \([u_3, w_1] = 0 \).
Therefore Jacobi identity \ref{2.6} with \(u \leftrightarrow u_3, v \leftrightarrow u_1, w \leftrightarrow w_2\) shows that it is sufficient to get the equality \([u_3, u_2] = 0\); that is, \([x_{t-1}, x_i], [x_{t+1}, x_i] = 0\) with \(t = \psi(m) < n\). Since \([x_{i-1}, x_i], x_i] = 0\) is one of the defining relations, the conditional identity \ref{2.8} implies \([x_{t-1}, x_i], [x_{t+1}, x_i] = [x_{t-1}x_{t+1}, x_i]\). It remains to apply the second of \ref{3.7}.

\begin{lemma}
If \(m \neq \psi(k), k \leq n < i < m\), then
\[\gamma[k, n], u[i + 1, m] = [u[i + 1, m], u[k, n]] = 0\]
unless \(i = \psi(k)\).
\end{lemma}

\begin{proof}
The proof is quite similar to the proof of the above lemma. It is based on Lemma 3.7 and the second of \ref{3.7} in the same way as the proof of the above lemma is based on Lemma 3.8 and the first of \ref{3.7}.
\end{proof}

\begin{corollary}
If \(m \neq \psi(k), k \leq n < m\), then in \(U_q^+(\mathfrak{so}_{2n+1})\) we have
\[\gamma[k, m] = [u[k, n], u[n + 1, m]] = \beta[u[n + 1, m], u[k, n]],\]
where \(\beta = -p(u(n + 1, m), u(k, n))^{-1}\).
\end{corollary}

\begin{proof}
Let us denote \(u = u[k, n], v = u[n + 1, m]\). Equalities \ref{3.13}, \ref{3.15} with \(i = n\) show that \(p_{uv}p_{vu} = \mu_{k,n}^2 = 1\) provided that \(m \neq \psi(k)\). Hence \([n, v] = uv - p_{uv}vu = -p_{uv}(v, u)\). This proves the second equality. To prove the first one we apply Lemma 3.9 if \(m < \psi(k)\), and Lemma 3.10 otherwise.
\end{proof}

\begin{proposition}
If \(m \neq \psi(k),\) then in \(U_q^+(\mathfrak{so}_{2n+1})\) for each \(i, k \leq i < m\) we have
\[\gamma[k, i], u[i + 1, m] = u[k, m]\]
with only two possible exceptions being \(i = \psi(m) - 1,\) and \(i = \psi(k)\).
\end{proposition}

\begin{proof}
If \(m \leq n\) or \(k \geq n + 1\), then the statement follows from \ref{2.9}. Thus we may suppose that \(m > n\).

If \(i = n\), Corollary 3.13 implies the required formula.

If \(i > n\), then Corollary 3.13 yields \(u[k, i] = [u[k, n], u[n + 1, i]]\), while by Lemma 3.12 we have \([u[k, n], u[i + 1, m]] = 0\). Hence \ref{2.8} implies
\[\gamma[k, n], u[n + 1, i], u[i + 1, m]] = [u[k, n], u[n + 1, i], u[i + 1, m]].\]

Now \ref{2.9} shows that \([u[n + 1, i], u[i + 1, m]] = [u[n, m]],\) and again Corollary 3.13 implies the required formula.

If \(i < n\), then Corollary 3.13 yields \(u[i + 1, m] = [u[i + 1, n], u[n + 1, m]]\), while by Lemma 3.11 we have \([u[k, i], u[n + 1, m]] = 0\). Hence \ref{2.8} implies
\[\gamma[k, i], u[i + 1, n], u[n + 1, m]] = [u[k, i], u[i + 1, n], u[n + 1, m]].\]

Now \ref{2.9} shows that \([u[k, i], u[i + 1, n]] = u[k, n],\) and again Corollary 3.13 implies the required formula.
\end{proof}

\begin{proposition}
If \(m \neq \psi(k), k \leq i < j < m, m \neq \psi(i) - 1, j \neq \psi(k),\) then \([u[k, i], u[j + 1, m]] = 0\). If additionally \(i \neq \psi(j) - 1,\) then \([u[j + 1, m], u[k, i]] = 0\).
\end{proposition}

\begin{proof}
If \(m \leq n\) or \(k > n\), then \(u[k, i]\) and \(u[j + 1, m]\) are separated by \(x_j\), hence the statement follows from Lemma 5.2.

If \(k \leq n < i\), then by Corollary 3.13 we have \(u[k, i] = [a, b]\) with \(a = u[k, n], b = u[n + 1, i]\). The second group of relations \ref{5.3} implies \([b, u[j + 1, m]] = 0\), while Lemma 3.12 implies \([a, u[j + 1, m]] = 0\). Hence by \ref{2.6} we get the required relation.
If $j < n \leq m$, then again by Corollary 3.13 we have $u[j + 1, m] = [a, b]$ with $a = u[j + 1, n]$, $b = u[n + 1, m]$. The second group of relations (3.3) implies $[u[k, i], a] = 0$, while Lemma 3.11 implies $[u[k, i], b] = 0$. Hence by (2.10) we get the required relation.

Assume $i \leq n \leq j$. If $i > \psi(j) - 1$, then, taking into account Lemma 3.3, one may apply Lemma 3.12 with $n \leftarrow i \leftarrow j$. Similarly, if $i < \psi(j) - 1$, one may apply Lemma 3.11 with $n \leftarrow \psi(j) - 1$. Let $i = \psi(j) - 1$. We may apply already considered case “$i > \psi(j) - 1$” to the sequence $k \leq i < j' < m$ with $j' = j + 1$ unless $j' = m$, or $j' = \psi(k)$. Thus $[u[k, i], u[j + 2, m]] = 0$, provided that $j + 1 \neq m$, $j + 1 \neq \psi(k)$. Lemma 2.11 implies

$$u[k, i], x_i] = [u[k, i - 2], [x_{i-1}, x_i, x_i]] = 0,$$

for inequality $i < j - 1$ and equality $i = \psi(j) - 1$ imply $i < n$. Now if $j + 1 \neq m$, $j + 1 \neq \psi(k)$, then using Lemma 2.11, we have

$$[u[k, i], u[j + 1, m]] = ([u[k, i], x_i, u[j + 2, m]]) = [u[k, i], u[j + 2, m]] = 0,$$

for $x_{j+1} = x_i$. The exceptional equality $j + 1 = \psi(k)$ implies $k = \psi(j) - 1 = i$. In this case, taking into account Lemma 2.1, we have

$$[x_i, u[j + 1, m]] = [[x_i, [x_i, x_i-1]], u[j + 3, m]] = 0.$$

The exceptional equality $j + 1 = m$ implies $u[1 + j, m] = x_m = x_i$, for $\psi(j + 1) = i$. Hence relation (3.18) applies. The equality $[u[k, i], u[j + 1, m]] = 0$ is proved.

Assume $i \neq \psi(j) - 1$. Definition (3.9) shows that

$$p(u[k, i], u[j + 1, m]) \cdot p(u[j + 1, m], u(k, i)) = \mu_k^{m,i} (\mu_k^{j,i})^{-1}.$$

Using (3.13) and (3.15) we shall prove that $\mu_k^{m,i} = \mu_k^{j,i}$. If $i = n$, then $\mu_k^{m,i} = \mu_k^{j,i} = 1$. Let $i \neq n$. If $m < \psi(k)$, then $\mu_k^{m,i} = q^{-2}$, for $i = \psi(m) - 1$ is equivalent to $m = \psi(i) - 1$. Similarly $\mu_k^{j,i} = q^{-2}$, for $j \neq \psi(i) - 1$, and $j \leq m < \psi(k)$.

If $m > \psi(k)$, and $i \neq \psi(k)$, then by (3.15) we have $\mu_k^{m,i} = q^{-2}$, while $\mu_k^{j,i} = q^{-2}$ in both cases: if $j < \psi(k)$ by (3.13), and if $j > \psi(k)$ by (3.15). Finally, if $i = \psi(k)$, then $j > i = \psi(k)$, hence (3.15) implies $\mu_k^{m,i} = \mu_k^{j,i} = q^{-4}$.

In order to get $[u[j + 1, m], u[k, i]] = 0$ it remains to apply (2.13).

4. PBW-generators of the quantum Borel algebra

**Proposition 4.1.** If $q^3 \neq 1$, $q^4 \neq 1$, then Values of the elements $u[k, m], k \leq m < \psi(k)$ form a set of PBW-generators for the algebra $U_q^+(\mathfrak{so}_{2n+1})$ over $k[G]$. All heights are infinite.

**Proof.** By [9] Theorem B, p. 211] the set of PBW-generators (the values of hard super-letters, see Theorem 2.9) consists of $[u_{km}], k \leq m \leq n$, and $[w_{ks}], 1 \leq k < s \leq n$, where $[u_{km}], [w_{ks}]$ are precisely the words $u(k, m), u(k, \psi(s))$ with the standard alignment of brackets (see Algorithm p. 5). By conditional identity (2.9) we have $[u_{km}] = u[k, m]$ in $U_q^+(\mathfrak{so}_{2n+1})$. According to [9] Lemma 7.8] the brackets in $[w_{ks}]$ are set by the following recurrence formulae:

$$\begin{align*}
[w_{ks}] &= [x_k [w_{k+1}s]], \\
[w_{k+1}] &= [w_{k+2}x_{k+1}],
\end{align*}$$

(4.1)

where by definition $w_{k+1} = u(k, n)$. We shall check the equality $[w_{ks}] = u(k, \psi(s))$ in $U_q^+(\mathfrak{so}_{2n+1})$.

If $k = n - 1, s = n$, then $w_{ks} = [x_{n-1}, x_n, x_n] = u[n - 1, n + 2]$. 

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If $k < s - 1$, then by (2.8) we have

$$\{x_k, [u[k + 1], u[n + 1, \psi(s)]\} = [u[k, n], u[n + 1, \psi(s)]]$$

for $[x_k, x_i] = 0$, $n + 1 \leq t \leq \psi(s)$. Thus, evident induction applies due to (3.17).

If $q$ is not a root of 1, then the fourth statement of Theorem 3.3, p. 211 shows that each skew-primitive element in $U_q^+(\mathfrak{so}_{2n+1})$ is proportional to either $x_i$, $1 \leq i \leq n$, or $1 - g, g \in G$. In particular $\xi(G(X)^{(2)})$ has no nonzero skew-primitive elements. At the same time due to the Heyneman-Radford theorem [4, 5 Corollary 5.3] every bi-ideal of a character Hopf algebra has a nonzero skew-primitive element. Therefore $\ker \xi = \Lambda$, while the subalgebra $A$ generated by values of $x_i, 1 \leq i \leq n$ in $U_q^+(\mathfrak{so}_{2n+1})$ has the shuffle representation given in Section 2.

If the multiplicative order of $q$ is finite, then by definition of $H = U_q^+(\mathfrak{so}_{2n+1})$ we have $\ker \xi = \Lambda$. Hence the subalgebra $A$ generated by values of $x_i, 1 \leq i \leq n$ in $U_q^+(\mathfrak{so}_{2n+1})$ has the shuffle representation also.

Recall that by $(u(m, k))$ we denote the tensor $x_m \otimes x_{m-1} \otimes \cdots \otimes x_k$ considered as an element of $Sh_+(V)$.

**Proposition 4.2.** Let $k \leq m \leq 2n$. In the shuffle representation we have

$$u[k, m] = \alpha_k^m \cdot (u(m, k)), \quad \alpha_k^m \overset{df}{=} \varepsilon_k^m (q^2 - 1)^{-m-k} \cdot \prod_{k \leq i < j \leq m} p_{ij},$$

where

$$\varepsilon_k^m = \begin{cases} 1, & \text{if } m \leq n, \text{or } k > n; \\ q^{-1}, & \text{if } k \leq n < m, \text{ or } k = \psi(k); \\ q^{-3}, & \text{if } m = \psi(k). \end{cases}$$

**Proof.** We use induction on $m - k$. If $m = k$, the equality reduces to $x_k = (x_k)$.

a. Consider firstly the case $m < \psi(k)$. By the inductive supposition we have $u[k, m - 1] = \alpha_k^{m-1} \cdot (w)$, $w = u(m - 1, k)$. Using (2.21) we may write

$$u[k, m] = \alpha_k^{m-1} \{(w)(x_m) - p(w, x_m) \cdot (x_m)(w)\}$$

where

$$\varepsilon_k^m \overset{df}{=} \alpha_k^{m-1} \sum_{uv = w} \{p(x_m, v)^{-1} - p(w, x_m)p(u, x_m)^{-1}\}(ux_mv).$$

Since $w = uv$, we have $p(w, x_m)p(u, x_m)^{-1} = p(v, x_m)$.

If $m \leq n$, then relations (3.2) imply $p(v, x_m)p(x_m, v) = 1$ with only one exception being $v = w$. Hence sum (4.3) has just one term. The coefficient at $(x_m, w) = (u(m, k))$ equals

$$\alpha_k^{m-1} p(w, x_m)(p(w, x_m)^{-1}p(x_m, w)^{-1} - 1) = \alpha_k^{m-1} p(w, x_m)(q^2 - 1),$$

which is required.

If $m = n + 1$, then still $p(v, x_m)p(x_m, v) = 1$ with two exceptions being $v = w$, and $v = u(n - 1, k)$. In both cases $(ux_mv)$ equals $(u(m, k))$. Hence the coefficient at $(u(m, k))$ in sum (4.3) equals

$$p(x_n, u(k, n - 1))^{-1} - p(u(k, n - 1), x_n) + p(x_n, u(k, n))^{-1} - p(u(k, n), x_n)$$

$$= p(w, x_{n+1})\{p_{n-1}^{-1}p_{n-1}^{-1}p_{nn}^{-1}p_{nn}^{-1} - p_{n}^{-1} + p_{n-1}^{-1}p_{nn}^{-1}p_{nn}^{-1}p_{n-1} - 1\}.$$
Due to (3.1), (3.2) we get \( \alpha^m = \alpha^{m-1}_k p(w, x_{n+1})(q^2 - 1)q^{-1} \), which is required.

Suppose that \( m > n + 1 \). In this case by definition \( x_m = x_t \), where \( t = \psi(m) < \psi(n + 1) = n \). Let \( v = u(s, k) \). If \( s < t - 1 \), then \( v \) depends only on \( x_i, i < t - 1 \), and relations (3.1), (3.2) imply \( p(v, x_m)p(x_m, v) = 1 \). If \( s > t, s \neq m - 1 \), then \( p(v, x_m)p(x_m, v) = p(t-k)p_{k+1}p_{k+1} + p_{k+1}p_{k+1}p_{k+1} = 1 \). Hence in (4.4) remains three terms with \( s = t - 1, s = t \), and \( s = m - 1 \). If \( v = u(t - 1, k) \) or \( v = u(t, k) \), then \( ux_m \) equals \( u(k, t)x_k^2u(t + 1, m - 1) \), while the coefficient at this tensor in sum (4.4) is

\[
p(x_t, u(k, t - 1))^{-1} - p(u(k, t - 1), x_t) + p(x_t, u(k, t))^{-1} - p(u(k, t), x_t)
\]

Thus in (4.4) remains just one term with \( v = u(m - 1, k) \). It has the required coefficient:

\[
\alpha^m_k = \alpha^{m-1}_k p(w, x_m)^{-1} - p(w, x_m) = \alpha^{m-1}_k p(w, x_m)(q^2 - 1).
\]

b). In perfect analogy we consider the case \( m > \psi(k) \). By the inductive supposition we have \( u[k + 1, m] = \alpha^m_{k+1} \cdot (w), w = u(m, k + 1) \). Using (2.21) we may write

\[
u[k, m] = \alpha^m_{k+1} \{ (x_k)(w) - p(x_k, w) \cdot (w)(x_k) \}
\]

(4.5)

Thus in (4.4) remains just one term with \( v = u(m - 1, k) \), and the coefficient equals

\[
\alpha^m_{k+1} p(x_k, w)(p(w, x_k)^{-1} p(x_k, w)^{-1} - 1) = \alpha^m_{k+1} p(x_k, w)(q^2 - 1),
\]

which is required.

If \( k = n \), then \( p(u, x_k)p(x_k, u) = 1 \) with two exceptions being \( u = w, u = u(m, n + 2) \). In both cases \((ux_kv)\) equals \((u(m, k))\), while the coefficient takes up the form

\[
p(w, x_n)^{-1} - p(x_n, w) + p(u(m, n + 2), x_n)^{-1} - p(x_n, u(m, n + 2))
\]

\[
= p(x_n, w) \{ p_{n-1}^{-1} p_{n-1}^{-1} p_{n-1}^{-2} - 1 + p_{n-1}^{-1} p_{n-1}^{-1} p_{n-1}^{-1} - p_{n-1}^{-1} \}.
\]

Due to relations (3.1), (3.2) we get \( \alpha^m = \alpha^{m+1}_n p(x_n, w)(q^2 - 1)q^{-1} \), which is required.

Suppose that \( k < n \). In this case \( x_k = x_t \) with \( m > t \); \( \psi(k) > \psi(n) = n + 1 \). Let \( u = u(m, s) \). If \( s > t \), then \( u \) depends only on \( x_i, i < k - 1 \), and relations (3.1), (3.2) imply \( p(x_k, u)p(u, x_k) = 1 \). If \( s < t - 1, s \neq k + 1 \), then \( p(x_k, w)p(u, x_k) = p_{k-1}^{-1} p_{k-1}^{-1} p_{k-1}^{-1} p_{k-1}^{-1} = 1 \). Hence in (4.4) remains three terms with \( s = t, s = t + 1 \), and \( s = k + 1 \). If \( u = u(m, t) \) or \( u = u(m, t + 1) \), then \( ux_kv = u(m, t + 1)x_k^2u(t - 1, k) \), while the coefficient at the corresponding tensor is

\[
p(u(m, t + 1), x_k)^{-1} - p(x_k, u(m, t + 1)) + p(u(m, t), x_k)^{-1} - p(x_k, u(m, t))
\]

\[
= p(x_k, u(m, t + 1)) \{ p_{k-1}^{-1} p_{k-1}^{-1} - 1 + p_{k-1}^{-1} p_{k-1}^{-1} p_{k-1}^{-1} - p_{k-1}^{-1} \} = 0.
\]

Thus in (4.4) remains just one term, and

\[
\alpha^m = \alpha^m_{k+1} p(w, x_k)^{-1} p(x_k, w) = \alpha^m_{k+1} p(x_k, w)(q^2 - 1).
\]

c). Let us consider the remaining case \( m = \psi(k) \). In this case \( x_m = x_k \). If \( k = n, m = n + 1 \), then \( u[n, n + 1] = -p_{n-1}^{-1} x_n = (1 - q^{-1})x_n^2 \), while in the
shuffle representation we have \((x_n)(x_n) = (1 + q^{-1})(x_n x_n)\). Hence \(u[n, n + 1] = (1 - q^{-2})(x_n x_{n+1})\), which is required: \((1 - q^{-2}) = q^{-3}. (q^{2} - 1) \cdot p_{nn}\).

If \(k \leq n\), we put \(u = u[n + 1, m]\), \(v = x_k\), \(w = u[k + 1, n]\). By definition (3.16) we have \(u[k, m] = \beta [u, v, w]\), where \(\beta = -p(u(n + 1, m), u(k, n))^{-1}\); that is, \(\beta = -p_{uvw}^{-1}\). Since \(u[n + 1, m] = [u[n + 1, m - 2], [x_{k+1}, x_k]]\), conditional identity (2.8) implies \([u, v] = [u[n + 1, m - 2], [[x_{k+1}, x_k], x_k]] = 0\). Thus \([u, v], w = 0\), and formula (2.7) yields

\[
\beta^{-1}u[k, m] = p_{uvx_k} \cdot [u, w] - p_{wuv}[u, w] \cdot x_k.
\]

Formula (3.16) implies \(\beta_1[u, w] = u[k + 1, m]\) with \(\beta_1 = -p_{uvw}^{-1}\). Hence considered above case b) allows us to find the shuffle representation \([u, w] = \alpha \cdot (z)\) with \(z = u(m, k + 1)\), and \(\alpha = -p_{uw} \alpha^m_{k+1}\). By (2.21) the shuffle representation of the right hand side of (4.6) is

\[
\alpha \sum_{s = u(m, k + 1)} (p_{uv} p(s, x_k) - p_{vw} p(x_k, y)^{-1}) \cdot (sx_ky).
\]

We have \(\beta_\alpha = -\beta_{p_{uw} \alpha^m_{k+1}} = p_{uv}^{-1} \alpha^m_{k+1}\), and

\[
p_{uv} p_{vw} = p_{k+1} p_{kk} p_{kk+1} = q^2
\]

since \(k < n\). Therefore we get

\[
u[k, m] = \alpha^m_{k+1} \sum_{s = u(m, k + 1)} (p(s, x_k)^{-1} - q^{-2} p(x_k, s)) \cdot (sx_ky).
\]

If \(s \not\in \{\emptyset, y, x_m, z = u(m, k + 1)\}\), then \(p(s, x_k) p(x_k, s) = p_{k+1} p_{kk} p_{kk+1} p_{kk} = q^2\), that is in (4.7) remains just three terms. If \(s = \emptyset\) or \(s = x_m\), then \((sx_ky) = (x_1 z)\) since \(x_m = x_k\). Hence the coefficient at \((x_1 z)\) in (4.7) equals \(1 - q^{-2} + p_{kk} - q^{-2} p_{kk} = 0\). Thus in (4.7) remains just one term with the coefficient

\[
\alpha^m_{k+1} (p(z, x_k)^{-1} - q^{-2} p(x_k, z)) = \alpha^m_{k+1} p(x_k, z) q^{-2}(q^2 - 1) = \alpha^m_k
\]

since \(p(z, x_k) \cdot p(x_k, z) = p_{kk} p_{kk+1} p_{ kk+1} \cdot p_{kk} p_{kk+1} + p_{kk+1} = 1\).

**Theorem 4.3.** In \(U_q^+(sl_2n+1)\) the coproduct on the elements \(u[k, m]\), \(k \leq m \leq 2n\) has the following explicit form

\[
\Delta(u[k, m]) = u[k, m] \otimes 1 + \sum_{i=k}^{m-1} \tau_i (1 - q^{-2}) g_k g_{k+1} \cdots g_i \cdot u[i + 1, m] \otimes u[k, i],
\]

where \(\tau_i = 1\) with only one exception being \(\tau_n = q\).

**Proof.** Formulae (1.2), (2.20), and (2.19) show that the coproduct has form (4.8), where \(\tau_i(1 - q^{-2}) = \alpha^m_k \alpha^m_{k+1}\). Therefore definition of \(\mu^m_k\) given in (3.9) and definition of \(\alpha^m_k\) given in (1.2) imply \(\tau_i(1 - q^{-2}) = \varepsilon_k^m (\varepsilon_k^m \varepsilon_{k+1}^m)^{-1}(q^2 - 1) \mu^m_k\), that is, \(\tau_i = \varepsilon_k^m (\varepsilon_k^m \varepsilon_{k+1}^m)^{-1} q^2 \mu^m_k\). By (3.12)
we have $\mu_k^{m,i} = \sigma_k^m(\sigma_k^i \sigma_{k+1}^m)^{-1}$. Using (3.3) and (4.3) we see that

\begin{equation}
\varepsilon_k^m \sigma_k^m = \begin{cases} q^2, & \text{if } m < n \text{ or } k > n + 1; \\
q, & \text{otherwise}.
\end{cases}
\end{equation}

Now it is easy to check that the $\tau$’s have the following elegant form

\begin{equation}
\tau_i = \varepsilon_k^m \sigma_k^m (\varepsilon_k^i \sigma_k^1)^{-1}(\varepsilon_{k+1}^m \sigma_{k+1}^1)^{-1} q^2 = \begin{cases} q, & \text{if } i = n; \\
1, & \text{otherwise}.
\end{cases}
\end{equation}

Interestingly, the coproduct formula differs from that in $U_q^+(\mathfrak{sl}_{2n+1})$ by just one term, see formula (3.3) in [11].

Now we are going to find PBW-generators for $u_q^+(\mathfrak{so}_{2n+1})$. To do this we need more relations in $U_q^+(\mathfrak{so}_{2n+1})$.

**Lemma 4.4.** If $k \leq m < \psi(k)$, then in the algebra $U_q^+(\mathfrak{so}_{2n+1})$ we have

\begin{equation}
[u[k, m], [u[k, m], u[k+1, m]] = 0.
\end{equation}

**Proof.** Suppose, first, that $m < \psi(k) - 1$. In this case both words $u(k, m)$ and $u(k + 1, m)$ are standard. The standard alignment of brackets for these words is defined by (4.1). However in Proposition 4.4 we have seen that $[u(k, m)] = u[k, m]$, and hence also $[u[k+1, m]] = u[k+1, m]$ in the algebra $U_q^+(\mathfrak{so}_{2n+1})$.

The word $w = u(k, m)u(k, m)u(k+1, m)$ is standard. Algorithm given on p. 5 shows that the standard alignment of brackets is precisely

\[[u[k, m], [u[k, m], u[k+1, m]]].\]

Hence the value of the super-word $[w]$ in $U_q^+(\mathfrak{so}_{2n+1})$ equals the left hand side of (4.11).

By Proposition 4.4 all hard super-letters in $U_q^+(\mathfrak{so}_{2n+1})$ are $[u(k, m)]$, $k \leq m < \psi(k)$. Hence $[w]$ is not hard. The multiple use of Definition 2.3 shows that the value of $[w]$ is a linear combination of the values of super-words in smaller than $[w]$ hard super-letters. Since $U_q^+(\mathfrak{so}_{2n+1})$ is homogeneous, each of the super-words in that decomposition has two hard super-letters smaller than $[w]$ and of degree 1 in $x_k$ (if a hard super-letter $[u(r, s)]$ is of degree 2 in $x_k$, then $r < k$ and $u(r, s) > w$).

At the same time all such hard super-letters are $[u(k, m + 1)]$, $[u(k, m + 2)]$, $[u(k, 2n - k)]$. Each of them has degree 2 in $x_{m+1}$ if $m \geq n$, and at least 1 if $m < n$.

Hence the super-word has degree in $x_{m+1}$ at least 4 if $m \geq n$, and at least 1 if $m < n$. However $w$ is of degree 3 in $x_{m+1}$ if $m \geq n$, and is independent of $x_{m+1}$ if $m < n$. Therefore the decomposition is empty, and $[w] = 0$.

Let, then, $m = \psi(k) - 1$. In this case $u(k+1, m)$ is not standard and we may not apply the above arguments. Nevertheless we shall prove in a similar way that $[u[k, 2n-k], x_t] = 0$, $k < t \leq n$. This would imply both $[u[k, 2n-k], u[k+1, 2n-k]] = 0$ and (4.11).

If $k + 1 < t < n$, then Lemma 3.7 and Lemma 3.8 imply

\[ [u[k, n], x_t] = [u[n+1, 2n-k], x_t] = 0. \]

Due to Corollary 3.13 we have $[u[k, 2n-k], x_t] = 0$.

If $t = k + 1$, we consider the word $v = u(k, 2n-k)x_{k+1}$. This is a standard word, and the standard alignment of brackets is $[v] = [u(k, 2n-k)] x_{k+1}$. Therefore the value of the super-letter $[v]$ equals $[u[k, 2n-k], x_{k+1}]$. At the same time $[v]$ does
not belong to the set of PBW-generators; that is, this is not hard. The multiple use of Definition 2.3 shows that the value of $[v]$ is a linear combination of the values of super-words in smaller than $[v]$ hard super-letters. Each of the super-words in that decomposition has a hard super-letter smaller than $[v]$ and of degree 1 in $x_k$. However there are no such super-letters. Thus the decomposition is empty and $[v] = 0$.

Let $t = n$. If $k = n - 1$, then $[u[k, 2n - k], x_n] = [[[x_{n-1}, x_n], x_n], x_n] = 0$ due to (3.3). If $k = n - 2$, we consider the word $u = u(k, 2n - k)x_n = x_{n-2}x_{n-1}x_nx_{n-1}x_n$. This is a standard word, while the super-letter $[u]$ is not hard. Again, there do not exists a hard super-letter smaller than $[u]$ and of degree 1 in $x_{n-2}$. Hence $[u] = 0$ in $U_q^+(\mathfrak{so}_{2n+1})$. The standard alignment of brackets is $[[x_{n-2}x_{n-1}x_n], x_{n-1}x_n]$. Hence we get $[[x_{n-2}, [x_{n-1}, x_n]], [x_{n-1}, x_n]] = 0.$

At the same time $[x_{n-2}, x_n] = 0$ and $[[[x_{n-1}, x_n], x_n], x_n] = 0$ imply $[[x_{n-2}, [x_{n-1}, x_n]], x_n] = 0$.

Conditional identity (2.8) yields $[[x_{n-2}, [x_{n-1}, x_n]], [x_{n-1}, x_n]] - [[x_{n-2}, [x_{n-1}, x_n]], x_n], [x_{n-1}, x_n]] = [[[x_{n-2}, [x_{n-1}, x_n]], x_n], x_n] = 0,$

which is required, for $[u[n-2, n+2], x_n] = [[[x_{n-2}, [x_{n-1}, x_n]], x_n], x_n], [x_{n-1}, x_n]] - [[[x_{n-2}, [x_{n-1}, x_n]], x_n], x_n]] = 0.$

Finally, suppose that $k < n - 2$. Denote $u_1 = u[k, n - 3], v_1 = u[n + 3, 2n - k]$, $w_1 = u[n - 2, n + 2]$. We have already proved that $[w_1, x_n] = 0$. The second group of relations (3.3) implies $[u_1, x_n] = 0, [v_1, x_n] = 0$. At the same time, due to Proposition 3.12 we have $u[k, 2n - k] = [u[k, n + 2], v_1]$ and $u[k, n + 2] = [u_1, w_1]$; that is $u[k, 2n - k] = [u_1, w_1], v_1]$. This certainly implies the required relation $[u_1, x_n] = 0$. $\square$

Proposition 4.5. If the multiplicative order $q$ of $q$ is finite, $t > 4$, then the values of $u[k,m]$, $k \leq m < \psi(k)$ form a set of PBW-generators for $U_q^+ (\mathfrak{so}_{2n+1})$ over $k[G]$. The height $h$ of $u[k,m]$ equals $t$ if $m = n$ or $t$ is odd. If $m \neq n$ and $t$ is even, then $h = t/2$. In all cases $u[k,m]^h = 0$ in $U_q^+ (\mathfrak{so}_{2n+1}).$

Proof. First we note that Definition 2.3 implies that a non-hard super-letter in $U_q^+ (\mathfrak{so}_{2n+1})$ is still non-hard in $U_q^+ (\mathfrak{so}_{2n+1})$. Hence all hard super-letters in $U_q^+ (\mathfrak{so}_{2n+1})$ are in the list $u[k, m], k \leq m < \psi(k)$. If, next, $u[k, m]$ is not hard in $U_q^+ (\mathfrak{so}_{2n+1})$, then by the multiple use of Definition 2.3 the value of $u[k, m]$ is a linear combination of super-words in hard super-letters smaller than $u[k, m]$. Since $U_q^+ (\mathfrak{so}_{2n+1})$ is homogeneous, each of the super-words in that decomposition has a hard super-letter smaller than $u[k, m]$ and of degree 1 in $x_k$. At the same time all such hard super-letters are in the list $u[k(m + 1)], [u[k, m + 2]], \ldots, [u(k, 2n - k)]$. Each of them has degree 2 in $x_{m+1}$ if $m \geq n$, and at least 1 if $m < n$. Hence the super-word has degree at least 2 if $m \geq n$, and at least 1 if $m < n$. However $u[k, m]$ is of degree 1 in $x_{m+1}$ if $m \geq n$, and is independent of $x_{m+1}$ if $m < n$. Therefore the decomposition is empty, and $u[k, m] = 0$. This contradicts Proposition 4.2 for $(u(m, k)) \neq 0$ in the shuffle algebra.

Denote for short $u = u[k,m]$. Equation (3.10) implies $p_{uu} = q$ if $m = n$ and $p_{uu} = q^2$ otherwise (recall that now $m < \psi(k)$). By Definition 2.4 the minimal possible value for the height is precisely the $h$ given in the proposition. It remains to
show that \( u^h = 0 \) in \( U^+_q(\mathfrak{so}_{2n+1}) \). By Lemma \( \ref{2.11} \) it suffices to prove that \( \partial_i(u^h) = 0 \), \( 1 \leq i \leq n \). Lemma \( \ref{2.10} \) yields

\[
\partial_i(u^h) = p(u, x_i)^{h-1}\left[ u, [u, \ldots [u, \partial_i(u)] \ldots] \right].
\]

Coproduct formula \( \ref{4.8} \) with \( \ref{2.23} \) implies

\[
\partial_i(u) = \begin{cases} 
(1 - q^{-2})\tau_k u[k + 1, m], & \text{if } i = k < m; \\
0, & \text{if } i \neq k; \\
1, & \text{if } i = k = m.
\end{cases}
\]

At the same time Lemma \( \ref{4.3} \) provides the relation \([u, [u, u[k + 1, m]]] = 0 \) in \( U^+_q(\mathfrak{so}_{2n+1}) \), and hence in \( U^+_q(\mathfrak{so}_{2n+1}) \) as well. Since always \( h > 2 \), we get the required equalities \( \partial_i(u^h) = 0 \), \( 1 \leq i \leq n \).

\[ \square \]

**Remark.** To prove \( \ref{4.8} \) we have used the shuffle representation. Therefore if \( q \) has a finite multiplicative order, then \( \ref{4.8} \) is proved only for \( u^+_{q}(\mathfrak{so}_{2n+1}) \). However we have seen that the kernel of the natural homomorphism \( U^+_q(\mathfrak{so}_{2n+1}) \to U^+_q(\mathfrak{so}_{2n+1}) \) is generated by the elements \( u[k, m]^h, k \leq m < \psi(k) \). Degree of \( u[k, m]^h \) in a given \( x_i \) is either zero or greater than 2. At the same time all tensors in \( \ref{4.8} \) have degree less than or equal to 2 in each variable. Therefore \( \ref{4.8} \), and hence \( \ref{4.12} \), are valid in \( U^+_q(\mathfrak{so}_{2n+1}) \) provided that \( q \) has a finite multiplicative order \( t > 4 \) as well.

5. **PBW-generators for right coideal subalgebras**

In what follows we denote by \( A_{k+1}, \ k < n \) a subalgebra of \( U^+_q(\mathfrak{so}_{2n+1}) \) or \( U^+_q(\mathfrak{so}_{2n+1}) \) generated by \( x_i, \ k < i \leq n \), respectively \( A \) is a subalgebra generated by all \( x_i, \ 1 \leq i \leq n \). Of course \( k[\{g_{k+1}, \ldots g_n \}]A_{k+1} \) may be identified with \( U^+_q(\mathfrak{so}_{2(n-k)+1}) \) or \( U^+_q(\mathfrak{so}_{2(n-k)+1}) \).

Suppose that a homogeneous element \( f \in k[X] \) is linear in the maximal letter \( x_k, \ 1 \leq k \leq n \) that it depends on: \( \deg_k(f) = 1, \deg_i(f) = 0, \ i < k \). Then in the decomposition of \( a = \xi(f) \) in the PBW-basis defined in Proposition \( \ref{4.1} \) or Proposition \( \ref{4.2} \) each summand has just one PBW-generator that depends on \( x_k \) since \( U^+_q(\mathfrak{so}_{2n+1}) \) and \( U^+_q(\mathfrak{so}_{2n+1}) \) are homogeneous in each \( x_i \). Moreover this PBW-generator, considered as a super-letter, starts by \( x_k \), hence it is the biggest super-letter of the summand. In particular this super-letter is located at the end of the basis super-word; that is, the PBW-decomposition takes up the form

\[
a = \sum_{i=k}^{2n-k} F_i u[k, i], \quad F_i \in A_{k+1}.
\]

**Definition 5.1.** The set \( \text{Sp}(a) \) of all \( i \) such tat in \( \ref{5.1} \) we have \( F_i \neq 0 \) is called the spectrum of \( a \).

Let \( S \) be a set of integer numbers from the interval \([1, 2n] \). We define a polynomial \( \Phi^S(k, m), \ 1 \leq k \leq m \leq 2n \) by induction on the number \( r \) of elements in the set \( S \cap [k, m - 1] = \{s_1, s_2, \ldots, s_r \}, \ k \leq s_1 < s_2 < \ldots < s_r < m \) as follows:

\[
\Phi^S(k, m) = u[k, m] - (1 - q^{-2}) \sum_{i=1}^{r} \alpha_{s_i km}^s \Phi^S(1 + s_i, m)u[k, s_i],
\]

where \( \alpha_{s_i km}^s = \tau_{sp}(u(1 + s, m), u(k, s))^{-1} \), while the \( \tau \)'s was defined in \( \ref{4.10} \).
We display the element $\Phi^S(k, m)$ schematically as a sequence of black and white points labeled by the numbers $k - 1, k, k + 1, \ldots m - 1, m$, where the first point is always white, and the last one is always black, while an intermediate point labeled by $i$ is black if and only if $i \in S$:

\[
\begin{array}{cccccccc}
k-1 & k & k+1 & k+2 & \ldots & m-2 & m-1 & m \\
\end{array}
\]

(5.3)

Sometimes, if $k \leq n < m$, it is more convenient to display the element $\Phi^S(k, m)$ in two lines putting the points labeled by indices $i, \psi(i)$ that define the same variable $x_i = x_{\psi(i)}$ in one column:

\[
\begin{array}{cccccccc}
& \cdots & \psi(i) & \cdots & n+1 \\
k-1 & \cdots & \psi(m) & \cdots & i & \cdots & n \\
\end{array}
\]

(5.4)

Below, to illustrate the notion of a regular set, we shall need a shifted representation that appears from (5.3) by shifting the upper line to the left by one step and putting the colored point labeled by $n_i$, if any, to the vacant position (so that this point appears twice in the shifted scheme):

\[
\begin{array}{cccccccc}
k-1 & \cdots & m & \cdots & \psi(m)-1 & \cdots & n+1 & \cdots & n \\
\end{array}
\]

(5.5)

If $k \leq m < \psi(k)$, then definition (5.3) shows that the spectrum of $\Phi^S(k, m)$ is contained in $S \cup \{m\}$, while its leading term is $u[k, m]$. However if $m \geq \psi(k)$, then Eq. (5.2) do not provide sufficient information even for the immediate conclusion that $\Phi^S(k, m) \neq 0$. In particular some of the factors $\Phi^S(1 + s_i, m)$ in (5.2) may be zero even if $k \leq m < \psi(k)$. Hence a priori the spectrum of $\Phi^S(k, m)$, $k \leq m < \psi(k)$ may be a proper subset of $S \cup \{m\}$.

By $\pi_{kl}$,

\[
\begin{array}{cccccccc}
1 \leq k \leq l < \psi(k) \quad \text{we denote a natural projection of } U^+_q(\mathfrak{so}_{2n+1}) \quad \text{or} \quad u^+_q(\mathfrak{so}_{2n+1}) \quad \text{onto } ku[k, l] \quad \text{with respect to the PBW-basis defined in Proposition 4.1 or 4.3 respectively.}
\end{array}
\]

Lemma 5.2. If $a \in A_{k+1}$, then $\pi_{kl}(au[k, i]) = 0$, $k \leq i < \psi(k)$ unless $a \in k$, $i = l$.

Proof. The PBW-decomposition $\tilde{a}$ of $a$ in basis defined in Proposition 4.1 or 4.3 involves only PBW-generators that belong to $A_{k+1}$. All of them are less than $u[k, i]$. Hence the PBW-decomposition of $au[k, i]$ is $\tilde{a}u[k, i]$. We have $\pi_{kl}(\tilde{a}u[k, i]) \neq 0$ only if $\tilde{a} \in k$, $i = l$.

Lemma 5.3. If $a \in A_{k+1}$, $k \leq l < \psi(k)$, then

\[
\Delta(au[k, i]) \cdot (id \otimes \pi_{kl}) = \begin{cases}
0, & \text{if } i < l; \\
ag_{kl} \otimes u[k, l], & \text{if } i = l; \\
(1 - q^{-2})ag_{kl}u[l + 1, i] \otimes u[k, l], & \text{if } i > l,
\end{cases}
\]

(5.6)

where by definition $g_{kl} = g(u[k, l]) = g_kg_{k+1}\cdots g_l$.

Proof. By means of (4.8) we have

\[
\Delta(au[k, i]) = \sum_{(\alpha), j} a^{(1)}(\alpha)g_{kj}u[j + 1, i] \otimes a^{(2)}u[k, j]
\]

for suitable $a_{\alpha} \in k$. By the above lemma we get $\pi_{kl}(a^{(2)}u[k, j]) = 0$ unless $a^{(2)} \in k$, $i = l$. It remains to apply explicit formula (4.8).
Lemma 5.4. If $k \leq l < m < \psi(k)$, then
\[
\Delta(\Phi^S(k, m)) \cdot (\id \otimes \pi_{kl}) = \begin{cases} 0, & \text{if } l \in S; \\ \tau_l (1-q^{-2}) g_{kl} \Phi^S(1+l, m) \otimes u[k, l], & \text{if } l \notin S. \end{cases}
\]

Proof. Let us apply $\Delta(\id \otimes \pi_{kl})$ to (5.2). Since $a_{kl} \overset{df}{=} \Phi^S(1+s_i, m) \in A_{k+1}$, we may use Lemma 5.3. We have $a_{kl} g_{kl} = \chi^{\alpha_i}(g_{kl}) a_i$, $\chi^{\alpha_i}(g_{kl}) = p(u(1+s_i, m), u(k,l))$. Thus, if $s_i > l$, then $\alpha^S_{km} \chi^{\alpha_i}(g_{kl}) = \alpha^S_{k+1+l, m}$, while if $s_i = l$, then $\alpha^S_{km} \chi^{\alpha_i}(g_{kl}) = \tau_l$. Now (5.6) implies the required relation. \qed

Lemma 5.5. Let $k \leq l < m < \psi(k)$, and $a \in A_{k+1}$ is a nonzero homogeneous element with $D(a) = D(u(1+l, m))$. Denote by $\nu_a$ any homogeneous projection $\nu_a : U^+_q(\mathfrak{s}\mathfrak{o}_{2n+1}) \to ak$. If $D(b) = D(u(1+i, m))$, then we have
\[
\Delta(bu[k, i]) \cdot (\id \otimes \nu_a) = \begin{cases} 0, & \text{if } l < i < m; \\ g_uu[k, l] \otimes a, & \text{if } i = l, b = a; \\ g_u b'u[k, i] \otimes a, & \text{if } i < l. \end{cases}
\]

Proof. All right hand side components of the tensors in (4.8) depend on $x_k$ with the only exception for the first summand. Since $\nu_a$ kills all elements with positive degree in $x_k$, we have
\[
\Delta(bu[k, i]) \cdot (\id \otimes \nu_a) = \sum_{(b)} b^{(1)} u[k, i] \otimes \nu_a(b^{(2)}).
\]
If $l < i < m$, then $D(b^{(2)}) \leq D(b) < D(a)$, hence $\nu_a(b^{(2)}) = 0$.
If $b = a$, $i = s$, then $D(b^{(2)}) = D(a)$ only if $b^{(1)} = g_u$, $b^{(2)} = a$.
If $i < l$, then (5.7) provides the third option given in the lemma. \qed

Proposition 5.6. If a right coideal subalgebra $U \supseteq k[G]$ of $U^+_q(\mathfrak{s}\mathfrak{o}_{2n+1})$ or $u^+_q(\mathfrak{s}\mathfrak{o}_{2n+1})$ contains a homogeneous element $c \in A$ with the leading term $u[k, m], k \leq m < \psi(k)$, then $\Phi^S(k, m) \in U$ for a suitable subset $S$ of the spectrum of $c$.

Proof. Every summand of the decomposition of $c$ in the PBW-basis defined in Proposition 4.1 or 4.5 has just one PBW-generator that depends on $x_k$, for $U^+_q(\mathfrak{s}\mathfrak{o}_{2n+1})$ and $u^+_q(\mathfrak{s}\mathfrak{o}_{2n+1})$ are homogeneous in each $x_i$. Moreover this PBW-generator, considered as a super-letter, starts by $x_k$, and hence it is the biggest super-letter of the summand. In particular that super-letter is located at the end of the basis super-word; that is, the PBW-decomposition takes up the form
\[
c = u[k, m] + \sum_{i=k}^{m-1} F_i u[k, i], \quad F_i \in A_{k+1}, \quad k \leq i < m.
\]
By definition $i$ belongs to the spectrum, ${\text{Sp}}(a)$, of $a$ if and only if $F_i \neq 0$. We may rewrite this representation in the following way:
\[
\Phi^{S_t}(k, m) + \sum_{i \in {\text{Sp}}(a), \ i < t} F_i u[k, i] \in U,
\]
where $t = m$, and by definition $S_m = \emptyset$. We shall prove that relation (5.9) with a given $t$, $k < t \leq m$, $S_t \subseteq {\text{Sp}}(a)$, and $t \leq \inf S_t$ implies a relation of the same type with $t \leftarrow l$, $S_t = S_l \cup \{l\}$, where $l$, as above, is the maximal $i$ in (5.9) such that
If the main parameter $F_i \neq 0$. Since certainly $l < t$, by downward induction this will imply (5.9) with $t = k$, $S = S_k \subseteq \Sp(a)$:

\[(5.10) \quad \Phi^S(k, m) \in U.\]

Let us apply $\Delta \cdot (\id \otimes \pi_{kl})$ to (5.9), where $\pi_{kl}$ is the projection onto $ku[k, l]$, and $l$ is the maximal $i$ in (5.9) with $F_i \neq 0$. By Lemma 5.3 we have $\Delta(F_i u[k, i]) \cdot (\id \otimes \pi_{kl}) = 0$, if $i < l$, while $\Delta(F_i u[k, l]) \cdot (\id \otimes \pi_{kl}) = F_i g_{kl} \otimes [k, l]$. Lemma (5.3) implies $\Delta(\Phi^S(k, m)) \cdot (\id \otimes \pi_{kl}) = \tau_l(1 - q^{-2})g_{kl}\Phi^S(1 + l, m) \otimes u[k, l]$. Since $U$ is a right coideal subalgebra that contains all group-like elements, we get

\[(5.11) \quad F_l + \chi^{F_i}(g_{kl})^{-1} \tau_l(1 - q^{-2}) \Phi^S(1 + l, m) = v \in U.\]

Further, consider any homogeneous projection $\nu_a$ with $a = F_l$. Let us apply $\Delta \cdot (\id \otimes \nu_a)$ to (5.9). Since $l < \inf S_k$, Lemma 5.3 and definition (5.2) imply $\Delta(\Phi^S(k, m)) \cdot (\id \otimes \nu_a) = 0$. Lemma 5.5 shows also that $\Delta(F_i u[k, i]) \cdot (\id \otimes \nu_a) = g_a u[k, l] \otimes a$, while $\Delta(F_i u[k, i]) \cdot (\id \otimes \nu_a) = g_a A'_i u[k, i] \otimes a, i < l$. Hence we arrive to the relation

\[(5.12) \quad u[k, l] + \sum_{i \in \Sp(a), i < l} F_i' u[k, i] = w \in U.\]

Relations (5.11), (5.12) imply

\[F_l u[k, l] = vw - \sum_{i \in \Sp(a), i < l} v F'_i u[k, i] - \chi^{F_i}(g_{kl})^{-1} \tau_l(1 - q^{-2}) \Phi^S(1 + l, m) \cdot u[k, l].\]

This allows one to replace $F_l u[k, l]$ in (5.9). Since according to definition (5.2) we have $\Phi^S(k, m) - \chi^{F_i}(g_{kl})^{-1} \tau_l(1 - q^{-2}) \Phi^S(1 + l, m) \cdot u[k, l] = \Phi^{S_i \cup \{l\}}(k, m)$, we get the required relation

\[\Phi^S(k, m) + \sum_{i \in \Sp(a), i < l} (F_i - v F'_i) u[k, i] \in U.\]

\[\Box\]

**Corollary 5.7.** If the main parameter $q$ is not a root of 1, then every right coideal subalgebra of $U_q^+(\mathfrak{so}_{2n+1})$ that contains the coradical has a set of PBW-generators of the form $\Phi^S(k, m)$. In particular there exists just a finite number of the right coideal subalgebras in $U_q^+(\mathfrak{so}_{2n+1})$ that contain the coradical. If $q$ has a finite multiplicative order $t > 4$, then this is the case for the homogeneous right coideal subalgebras in $u_q^+(\mathfrak{so}_{2n+1})$.

**Proof.** If $U$ is a right coideal subalgebra of $U_q^+(\mathfrak{so}_{2n+1})$ that contains $k[G]$, then by Lemma 2.12 it is homogeneous in each $x_i$. By Proposition 4.1 and Proposition 2.7 it has PBW-generators of the form (2.18):

\[(5.13) \quad c_u = u^s + \sum_i \alpha_i W_i \in U, \quad u = u[k, m], k \leq m \leq \psi(k).\]

By (5.10) we have $p_{uu} = \sigma^m_k = q^2$ if $m \neq n$, and $p_{uu} = q$ otherwise. Thus, if $q$ is not a root of 1, Lemma 2.6 shows that in (5.13) the exponent $s$ equals 1, while all heights of the $c_u$’s in $U$ are infinite.

If $q$ has a finite multiplicative order $t > 4$, then $u[k, m]^h = 0$ in $u_q^+(\mathfrak{so}_{2n+1})$, where $h$ is the multiplicative order of $p_{uu}$, see Proposition 4.5. By Lemma 2.6 in (5.13) we have $s \in \{1, h, h^2\}$. Since $u[k, m]^h = u[k, m]^{h'} = 0$, the exponent $s$ in (5.13) equals 1, while the height of $c_u$ in $U$ equals $h$. 

```
Since $U$ is homogeneous with respect to each $x_i \in X$, in both cases the PBW-generators of $U$ have the following form

\begin{equation}
(5.14) \quad c_u = u[k, m] + \sum \alpha_i W_i, \quad k \leq m \leq \psi(k),
\end{equation}

where $W_i$ are the basis super-words starting with less than $u[k, m]$ super-letters, $D(W_i) = D(u[k, m]) = x_k + x_{k+1} + \ldots + x_m$. By Proposition 5.4 we have $\Phi^S(k, m) \in U$. The leading term of $\Phi^S(k, m)$ equals $u[k, m]$, see definition (5.2). Hence we may replace $c_u$ with $\Phi^S(k, m)$ in the set of PBW-generators. The number of possible elements $\Phi^S(k, m)$ is finite. Hence the total number of possible sets of PBW-generators of the form $\Phi^S(k, m)$ is finite as well. 

\[ \square \]

6. Elements $\Phi^{[k,m-1]}(k, m)$. 

In this section we are going to prove the following relation in $U_q^+(\mathfrak{so}_{2n+1})$:

\begin{equation}
(6.1) \quad \Phi^{[k,m-1]}(k, m) = (-1)^{m-k} \left( \prod_{m \geq \psi(j) \geq k} p_{ij}^{-1} \right) \cdot u[\psi(m), \psi(k)],
\end{equation}

where as above $\psi(i) = 2n - i + 1$. The main idea of the proof is to use the Milinski-Schneider criterion (Lemma 2.11). To do this we need to find the partial derivatives of the both sides. In what follows by $\partial_i, 1 \leq i \leq 2n$ we denote the partial derivation with respect to $x_i$, see (2.22). In particular $\partial_i = \partial_{\psi(i)}$. Coproduct formula (4.8) with (2.23) implies

\begin{equation}
(6.2) \quad \partial_i(u[k, m]) = \begin{cases} 
(1 - q^{-2})\tau_k u[k + 1, m], & \text{if } x_i = x_k, \; k < m; \\
0, & \text{if } x_i \neq x_k; \\
1, & \text{if } x_i = x_k, \; k = m. 
\end{cases}
\end{equation}

This allows us easily to find the derivatives of the right hand side. By induction on $m - k$ we shall prove a similar formula

\begin{equation}
(6.3) \quad \partial_i(\Phi^{[k,m-1]}(k, m)) = \begin{cases} 
\beta_k^m \Phi^{[k,m-2]}(k, m - 1), & \text{if } x_i = x_m, \; k < m; \\
0, & \text{if } x_i \neq x_m; \\
1, & \text{if } x_i = x_m, \; k = m,
\end{cases}
\end{equation}

where $\beta_k^m = -(-1)^{m-2} \gamma_{km}^{m-1} = -(1 - q^{-2})\tau_{m-1} p(x_m, u(k, m - 1))^{-1}$. To simplify the notation we remark that $\Phi^{[k,m-1]}(k, m) = \Phi^S(k, m)$ for arbitrary $S$ that contains the interval $[k, m - 1]$. In particular in the above formula $\Phi^{[k,m-2]}(k, m - 1) = \Phi^{[k,m-1]}(k, m - 1)$.

If $x_i \neq x_m, x_i \neq x_k$, then (6.2) and the inductive supposition applied to definition (5.2) imply $\partial_i(\Phi^{[k,m-1]}(k, m)) = 0$.

If $x_i = x_k \neq x_m$, then $\partial_i = \partial_k$. Taking into account definition (5.2) we have

\begin{equation}
(6.4) \quad \partial_k(\Phi^{[k,m-1]}(k, m)) = \partial_k(u[k, m]) - (1 - q^{-2}) \sum_{i=k}^{m-1} \alpha_{km}^i \Phi^{[k,m-1]}(1 + i, m) u[k, i]),
\end{equation}

where $\alpha_{km}^i = \tau_i p(u(1 + i, m), u(k, i))^{-1}$, while the $\tau$’s have been defined in (4.10). By means of the inductive supposition, skew differential Leibniz formula (2.22), and (6.2) we may continue

\begin{equation}
(6.5) \quad = (1 - q^{-2})\tau_k \left( u[k + 1, m] - \tau_k^{-1} \alpha_{km}^k p(u(1 + k, m), x_k) \Phi^{[k,m-1]}(1 + k, m) \right),
\end{equation}

and

\begin{equation}
(6.6) \quad = (1 - q^{-2})\tau_k \left( u[k + 1, m] - \tau_k^{-1} \alpha_{km}^k p(u(1 + k, m), x_k) \Phi^{[k,m-1]}(1 + k, m) \right).
\end{equation}
\[-(1 - q^{-2}) \sum_{i=k+1}^{m-1} \alpha_{km}^i p(u(1 + i, m), x_k) \Phi^{[k,m-1]}(1 + i, m) u[k + 1, i].\]

Since obviously \(\alpha_{km}^i p(u(1 + k, m), x_k) = \tau_k, \) \(\alpha_{km}^i p(u(1 + i, m), x_k) = \alpha_{k+1,m}^i,\) definition (5.2) shows that the above expression is zero.

If \(x_i = x_m \neq x_k,\) then \(\partial_i = \partial_m.\) Again by definition (5.2), inductive supposition, skew differential Leibniz formula (2.22), and (6.2) we have

\[\partial_m(\Phi^{[k,m-1]}(k, m)) = -(1 - q^{-2}) \sum_{i=k}^{m-2} \alpha_{km}^i \beta_{1+i}^m \Phi^{[k,m-2]}(1 + i, m - 1) u[k, i].\]

(6.5)

By definition \(-(1 - q^{-2})\alpha_{km}^{m-1} = \beta_{km}.\) At the same time

\[\alpha_{km}^i \beta_{1+i}^m = \tau_p u(1 + i, m), u(k,i))^{-1} \cdot \{-1 - q^{-2}) \tau_{m-1} p(x_m, u(1 + i, m - 1))^{-1}\}
\[= -(1 - q^{-2}) \tau_{m-1} p(x_m, u(k, m - 1))^{-1} \cdot \tau_i p(u(1 + i, m - 1), u(k,i))^{-1} = \beta_{km}^m \cdot \alpha_{km}^{m-1}.\]

Thus, according to (5.2) the right hand side of (6.5) equals \(\beta_{km}^m \Phi^{[k,m-2]}(k, m - 1),\) which is required.

Finally, if \(x_i = x_m = x_k, \) \(k \neq m,\) that is \(m = \psi(k),\) then due to skew differential Leibniz formula (2.22) the derivative \(\partial_i(\Phi^{[k,m-1]}(k, m))\) equals the sum of the expression (6.4) with the right hand side of (6.5). It remains to note that (6.4) is still zero, while the right hand side of (6.5) still equals \(\beta_{km}^m \Phi^{[k,m-2]}(k, m - 1),\) Eq. (6.3) is completely proved.

Now we are ready to prove (6.1) by induction on \(m - k.\) If \(m = k\) both sides equal \(x_k.\) If \(k < m,\) then the derivatives \(\partial_i\) of both sides are zero with the only exception being \(x_i = x_m = x_{\psi(m)}).\) Due to (6.2) the derivative \(\partial_m\) applied to the right hand side of (6.1) equals

\[(-1)^{m-k} \left( \prod_{m > i > j \geq k} p_{ij}^{-1} \right) \left(1 - q^{-2}) \tau_{\psi(m)} \cdot u[\psi(m) + 1, \psi(k)].\]

Since \(\psi(m) = n\) if and only if \(m - 1 = n,\) formula (4.10) yields \(\tau_{\psi(m)} = \tau_{m-1}.\) At the same time (6.3) and the inductive supposition imply

\[\partial_m(\Phi^{[k,m-1]}(k, m)) = \beta_{km}^m (-1)^{m-1-k} \left( \prod_{m > i > j \geq k} p_{ij}^{-1} \right) u[\psi(m) + 1, \psi(k)] \cdot \]

(6.7)

By definition we have

\[\beta_{km}^m = -(1 - q^{-2}) \tau_{m-1} p(x_m, u(k, m - 1))^{-1} = -\beta_{m-1}^m \prod_{m > j \geq k} p_{mj}^{-1} \cdot\]

Thus (6.6) coincides with (6.7), and due to MS-criterion (6.1) is proved.

Remark. To prove (6.1) we have used the MS-criterion. Therefore if \(q\) has a finite multiplicative order \(t,\) relation (6.1) is proved only for \(u_{(s\lo 2n+1)}^t.\) However we have seen in Proposition (4.5) that if \(t > 4,\) then the kernel of the natural homomorphism \(U_q(\mathfrak{s\lo 2n+1}) \to U_q^+(\mathfrak{s\lo 2n+1})\) is generated by the elements \(u[k, m], h \geq 3.\) At the
same time all polynomials in \((7.1)\) have degree less than or equal to 2 in each variable. Therefore \((6.1)\) is valid in \(U_q^{+}(so_{2n+1})\) provided that \(t > 4\).

7. \((k, m)\)-regular sets

**Definition 7.1.** Let \(1 \leq k \leq n < m \leq 2n\). A set \(S\) is said to be white \((k, m)\)-regular if for every \(i\), \(k - 1 \leq i < m\), such that \(k \leq \psi(i) \leq m + 1\) either \(i\) or \(\psi(i) - 1\) does not belong to \(S \cup \{k - 1, m\}\).

A set \(S\) is said to be black \((k, m)\)-regular if for every \(i\), \(k \leq i < m\), such that \(k \leq \psi(i) \leq m + 1\) either \(i\) or \(\psi(i) - 1\) belongs to \(S \setminus \{k - 1, m\}\).

If \(m \leq n\), or \(k > n\) (or, equivalently, if \(u[k, m]\) is of degree \(\leq 1\) in \(x_n\)), then by definition each set \(S\) is both white and black \((k, m)\)-regular.

A set \(S\) is said to be \((k, m)\)-regular if it is either black or white \((k, m)\)-regular.

If \(k \leq n < m\) and \(S\) is white \((k, m)\)-regular, then \(n \notin S\), for \(\psi(n) - 1 = n\). If additionally \(m < \psi(k)\), then taking \(i = \psi(m) - 1\) we get \(\psi(i) - 1 = m\), hence the definition implies \(\psi(m) - 1 \notin S\). We see that if \(m < \psi(k)\), \(k \leq n < m\), then \(S\) is white \((k, m)\)-regular if and only if the shifted scheme of \(\Phi^k(k, m)\) given in \((5.5)\) has no black columns:

\[
\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\bullet & \cdots & \bullet & \cdots & \bullet & \cdots & \bullet \\
\circ & \cdots & \circ & \cdots & \circ & \cdots & \circ \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

In the same way, if \(m > \psi(k)\), then for \(i = \psi(k)\) we get \(\psi(i) - 1 = k - 1\), hence \(\psi(k) \notin S\). That is, if \(m > \psi(k)\), \(k \leq n < m\), then \(S\) is white \((k, m)\)-regular if and only if the shifted scheme \((5.5)\) has no black columns and the first from the left complete column is a white one.

\[
\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\bullet & \cdots & \bullet & \cdots & \bullet & \cdots & \bullet \\
\circ & \cdots & \circ & \cdots & \circ & \cdots & \circ \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

Similarly, if \(k \leq n < m\) and \(S\) is black \((k, m)\)-regular, then \(n \in S\). If additionally \(m < \psi(k)\), then taking \(i = \psi(m) - 1\) we get \(\psi(i) - 1 = m\), hence \(\psi(m) - 1 \in S\). We see that if \(m < \psi(k)\), \(k \leq n < m\), then \(S\) is black \((k, m)\)-regular if and only if the shifted scheme \((5.5)\) has no white columns and the first from the left complete column is a black one.

\[
\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\bullet & \cdots & \bullet & \cdots & \bullet & \cdots & \bullet \\
\circ & \cdots & \circ & \cdots & \circ & \cdots & \circ \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

If \(m > \psi(k)\), then for \(i = \psi(k)\) we get \(\psi(i) - 1 = k - 1\), hence \(\psi(k) \in S\). That is, if \(m > \psi(k)\), \(k \leq n < m\), then \(S\) is black \((k, m)\)-regular if and only if the shifted scheme \((5.5)\) has no white columns:

\[
\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\bullet & \cdots & \bullet & \cdots & \bullet & \cdots & \bullet \\
\circ & \cdots & \circ & \cdots & \circ & \cdots & \circ \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

At the same time we should stress that if \(m = \psi(k)\), then no one set is \((k, m)\)-regular. Indeed, for \(i = k - 1\) we have \(\psi(i) - 1 = m\). Hence both of the elements \(i, \psi(i) - 1\) belong to \(S \cup \{k - 1, m\}\), and therefore \(S\) is not white \((k, \psi(k))\)-regular.

If we take \(i = m\), then \(\psi(i) - 1 = k - 1\), and no one of the elements \(i, \psi(i) - 1\) belongs to \(S \setminus \{k - 1, m\}\). Thus \(S\) is neither black \((k, \psi(k))\)-regular.
Lemma 7.2. If $S$ is white $(k,m)$-regular, then values in $U_q^+(s_{2n+1})$ of the bracketed words $[u_r u_{r-1} \ldots u_1 u_0]$ and $[u_0 u_1 \ldots u_{r-1} u_r]$ are independent of the precise alignment of brackets.

Proof. Let $0 \leq i < j-1, j \leq r$. Assume $k \leq n < m$. The points $s_i$, and $\psi(1+s_i)$ form a column on the shifted scheme (7.1) or (7.2), for $s_i + \psi(1+s_i) = 2n$. Hence $\psi(1+s_i) = \psi(s_i) - 1$ is not a black point. In particular $s_{j+1} \neq \psi(1+s_i)$, $s_j \neq \psi(1+s_i)$. Similarly the points $s_{i+1}$, and $\psi(s_{i+1}) - 1$ form a column on the shifted scheme, and hence $s_{j+1} \neq \psi(s_{i+1}) - 1, s_j \neq \psi(s_{i+1}) - 1$.

We have $1 + s_i < s_{i+1} < s_j < s_{j+1}, s_{j+1} \neq \psi(1+s_i), s_{j+1} \neq \psi(s_{i+1}) - 1, s_j \neq \psi(1+s_i)$, and $s_j \neq \psi(s_{i+1}) - 1$. Therefore Proposition 3.14 with $k \leftarrow 1 + s_i$, $i \leftarrow s_{i+1}, j \leftarrow s_j$, $m \leftarrow s_{j+1}$ implies $[u_i, u_j] = [u_j, u_i] = 0$. If $m \leq n$ or $k > n$, then $u_i$ and $u_j$ are separated, hence still $[u_i, u_j] = [u_j, u_i] = 0$ due to Lemma 3.2. It remains to apply Lemma 7.1. □

Lemma 7.3. If $S$ is white $(k,m)$-regular, then $[u_0 u_1 \ldots u_{r-1} u_r] = u[k,m]$.

Proof. We use induction on $r$. If $r = 0$, the equality is clear. In the general case the inductive supposition yields $[u_0 u_1 \ldots u_{r-1}] = u[k,s_r]$, for $S$ is white $(k,s_r)$-regular. By Proposition 7.14 we have $[u[k,s_r], u_r] = u[k,m]$, unless $s_r = \psi(m) - 1$ or $s_r = \psi(k)$. However the white $(k,m)$-regularity implies that $\psi(m) - 1, \psi(k)$ are not black points. We are done. □

Lemma 7.4. If $S$ is white $(k,m)$-regular, then in the above notations we have

\begin{equation}
\Phi^S(k,m) = (-1)^r \prod_{r \geq i > j \geq 0} p(u_i, u_j)^{-1} [u_r u_{r-1} \ldots u_1 u_0].
\end{equation}

Proof. To prove the equality it suffices to check the recurrence relations (5.2) for the right hand side. We shall use induction on $r$. If $r = 0$, there is nothing to prove. By Lemma 7.3 we have $u[k,m] = [u_0 u_1 \ldots u_{r-1} u_r]$. The inductive supposition for the white $(k,m)$-regular set $S \setminus \{s_1\}$ takes up the form

\begin{equation}
(-1)^{r-1} \prod_{r \geq i > j \geq 0} p(u_i, u_j)^{-1} [u_r u_{r-1} \ldots u_2 u_0 u_1] = [u_0 u_1 u_2 \ldots u_r]
\end{equation}

\begin{equation}
-(1 - q^{-2}) \sum_{l=2}^{r} \alpha_{k,m}^{s_1} (-1)^{r-l} \prod_{r \geq i > j \geq l} p(u_i, u_j)^{-1} [u_r u_{r-1} \ldots u_l u_0 u_1 u_2 \ldots u_{l+1}].
\end{equation}

By definition $p(u_0, u_1) p(u_1, u_0) = \mu^2_{k,s_1}$, see Definition 3.6, while by (3.13), (3.15) we have $\mu^2_{k,s_1} = q^{-2}$, for the regularity condition implies $s_1 \neq n, s_1 \neq \psi(s_2) - 1, s_1 \neq \psi(k)$. Hence by (2.13) we may write

\[ p(u_0, u_1) [u_0 u_1] = -[u_1 u_0] + (1 - q^{-2}) u_1 \cdot u_0. \]

This implies

\[ p(u_1, u_0) [u_0 u_1 u_2 [u_0 u_1]] = -[u_r u_{r-1} \ldots u_2 u_1 u_0] + (1 - q^{-2}) ([u_r u_{r-1} \ldots u_2], u_1 \cdot u_0]. \]

Since $[u_i, u_0] = 0, i \geq 2$, ad-identity (2.11) yields

\[ [u_r u_{r-1} \ldots u_0] = [u_r u_{r-1} \ldots u_2 u_1] \cdot u_0. \]
Thus the left hand side of (7.6) reduces to
\[-1 \prod_{r \geq i, j \geq 0} p(u_i, u_j)^{-1} \cdot [u_r u_{r-1} \ldots u_2 u_1 u_0] + \mathfrak{A},\]
where
\[\mathfrak{A} = (1 - q^{-2}) (-1)^{r-1} \prod_{r \geq i > 0} p(u_i, u_0)^{-1} \prod_{r \geq i, j \geq 1} p(u_i, u_j)^{-1} \cdot [u_r u_{r-1} \ldots u_2 u_1] \cdot u_0.\]

At the same time \(\mathfrak{A}\) up to a sign coincides with the missing summand of the right hand side of (7.6) corresponding to \(l = 1\), for
\[\alpha_{s, k, m} = \tau_{s, p} u_r u_{r-1} \ldots u_1 u_0)^{-1} = \prod_{r \geq i > 0} p(u_i, u_0)^{-1}.\]

\[\square\]

**Corollary 7.5.** If \(S\) is white \((k, m)\)-regular, \(s \in S \cup \{n\}, k \leq s < m\), then
\[\Phi_S^S(k, m) = -p_{ab}^{-1} \Phi_S(1 + s, m), \Phi_S^S(k, s),\]
where \(a = u(1+s, m), b = u(k, s).\)

**Proof.** Let \(s = s_t, 1 \leq t \leq r\). Since by Lemma 7.2 the value of the bracketed word \([u_r u_{r-1} \ldots u_0]\) is independent of the precise alignment of brackets, we have \([u_r u_{r-1} \ldots u_0] = [u_r u_{r-1} \ldots u_t], [u_t \ldots u_0]\). It remains to apply Lemma 7.4.

Let \(k \leq s = n < m\). Since in a white regular set, \(n\) is always white, we can find \(j\) such that \(s_j < n < s_{j+1}\). Denote \(u_j' = u[1 + s_j, n], u''_j = u[n + 1, s_{j+1}]\). The points \(s_j\) and \(\psi(1 + s_j)\) form a column on shifted scheme (7.1) or (7.2). Hence \(\psi(1 + s_j)\) is a white point. In particular \(s_{j+1} \neq \psi(1 + s_j)\). Thus, by Corollary 3.13 with \(k \leftarrow 1 + s_j, m \leftarrow s_{j+1}\), we have \(u_j = [u_j', u_j''] = -p(u_j', u_j'')^{-1} [u''_j, u''_j].\)

Note that the value of the bracketed word
\[[u_r u_{r-1} \ldots u_{j+1} u''_j u_j' u_j u_{j-1} \ldots u_1 u_0]\]
is independent of the precise alignment of brackets. Indeed, Lemma 3.12 with \(k \leftarrow 1 + s_i, i \leftarrow s_i, m \leftarrow s_{i+1}\) says \([u_i, u_j'] = 0, \ i > j\), unless \(s_{i+1} = \psi(1 + s_j)\) or \(s_i = \psi(1 + s_j)\). However the points \(s_j\) and \(\psi(1 + s_j)\) form a column on the shifted scheme (7.1) or (7.2). Hence \(\psi(1 + s_j)\) is not a black point. In particular \(s_{i+1} \neq \psi(1 + s_j)\), and \(s_i \neq \psi(1 + s_j)\).

At the same time if \(i < j - 1\), then \(u_j'\) and \(u_i\), are separated by \(u_{j-1}\) (Definition 3.1), hence Lemma 3.1 implies \([u'_j, u_i] = 0\).

In perfect analogy we get \([u''_j, u_i] = 0, \ i < j,\) and \([u_i, u''_j] = 0, \ i > j + 1\). Thus Lemma 2.1 implies that (7.7) is independent of the precise alignment of brackets. In particular
\[[u_r u_{r-1} \ldots u_{j+1} u''_j u_j' u_j u_{j-1} \ldots u_1 u_0] = [u_r u_{r-1} \ldots u_{j+1} u''_j, [u_j' u_{j-1} \ldots u_1 u_0]].\]
It remains to apply Lemma 7.4.\[\square\]

**Lemma 7.6.** If \(k \leq t < m, t \notin S,\) then
\[\Phi_{S(t)}^S(k, m) - \Phi^S(k, m) = (q^{-2} - 1) p_{ab}^{-1} \tau_t \Phi^S(1 + t, m) \Phi^S(k, t),\]
where \(a = u(1+t, m), b = u(k, t).\)
Proof. We shall use induction on \( m - k \). If \( m = k \), there is nothing to prove. By definition \((5.2)\) we have
\[
\Phi^{S_{\psi}(t)}(k, m) - \Phi^S(k, m) = -(1 - q^{-2})\left\{ \tau_t p^{-1}_{ab} \Phi^S(1 + t, m) u[k, t] + \sum_{s_i < t} \tau_s p^{-1}_{ui,v_i}(\Phi^{S_{\psi}(t)}(1 + s_i, m) - \Phi^S(1 + s_i, m)) u[k, s_i] \right\},
\]
where \( u_i = u(1 + s_i, m), v_i = u(k, s_i) \). By means of the inductive supposition we may continue
\[
= (q^{-2} - 1) p^{-1}_{ab} \tau_t \Phi^S(1 + t, m) \cdot \left\{ u[k, t] - (1 - q^{-2}) \sum_{s_i < t} \tau_s p^{-1}_{ui,v_i} p^{-1}_{ab} \Phi^S(1 + s_i, t) u[k, s_i] \right\},
\]
where \( b_i = u(1 + s_i, t) \). It remains to note that
\[
p^{-1}_{ui,v_i} p^{-1}_{ab} = p(u(1 + s_i, t), u(k, s_i))^{-1},
\]
and use definition \((5.2)\).

\[\square\]

Corollary 7.7. If \( S \cup \{ t \} \) is white \((k, m)\)-regular, \( t \notin S \), \( k \leq t < m \), then
\[
(7.9) \quad \Phi^S(k, m) \sim \left[ \Phi^{S_{\psi}(t)}(k, t), \Phi^S(1 + t, m) \right].
\]

Proof. Denote \( A = \Phi^S(k, t), B = \Phi^S(1 + t, m) \). By Corollary 7.5 we have \( \Phi^{S_{\psi}(t)}(k, m) = -p^{-1}_{ab}[B, A] \). At the same time \( t \neq n \) (for \( S \cup \{ t \} \) is white \((k, m)\)-regular), and hence by Lemma 7.6 we get \( \Phi^{S_{\psi}(t)}(k, m) - \Phi^S(k, m) = (q^{-2} - 1)p^{-1}_{ab} BA \). These two equalities imply
\[
\Phi^S(k, m) = -p^{-1}_{ab}[B, A] - (q^{-2} - 1)p^{-1}_{ab} BA \tag{7.10}
\]

By definition \((3.6)\) we know that \( p_{ABpBA} = \mu^{m,t}_k \). In which case schemes \((7.1), (7.2)\), related to the white regular set \( S \cup \{ t \} \) show that \( t \neq \psi(m) - 1, t \neq n, t \neq \psi(k), m \neq \psi(k) \), for \( t, m \) are black points. Hence formulae \((3.13), (3.15)\) imply \( \mu^{m,t}_k = q^{-2} \). Thus, we get \( p_{ABpBA} = q^{-2} \); that is, \( q^{-2} p^{-1}_{BA} = p_{AB} \). Now \((7.10)\) reduces to \((7.9)\).

\[\square\]

Lemma 7.8. A set \( S \) is white \((k, m)\)-regular if and only if \( \psi(S) - 1 \) is black regular with respect to \((\psi(m), \psi(k)) \). Here by \( \psi(S) - 1 \) we denote \( \{\psi(s) - 1 | s \in S\} \), while the complement is related to the interval \([\psi(m), \psi(k) - 1] \).

Proof. Let us replace the parameter \( i \) with \( j = \psi(i) - 1 \) in the definition of regularity. Since \( \psi \) change the order, we have \( k - 1 \leq i < m \) is equivalent to \( \psi(k) + 1 \geq \psi(i) \geq \psi(m) \), that is \( \psi(k) \geq j \geq \psi(m) \). Similarly the condition \( k \leq \psi(i) \leq m + 1 \) is equivalent to \( \psi(k) \geq i \geq \psi(m) - 1 \). Since \( \psi(j) = i + 1 \), this is \( \psi(k) + 1 \geq \psi(j) \geq \psi(m) \).

The condition \( i \notin S \cup \{k - 1, m\} \) is equivalent to \( j \notin (\psi(S) - 1) \cup \{\psi(m) - 1, \psi(k)\} \), which, in turn, is equivalent to \( j \in (\psi(S) - 1) \\setminus \{\psi(m) - 1, \psi(k)\} \). In the same way \( \psi(i) - 1 \notin S \cup \{k - 1, m\} \) is equivalent to \( \psi(j) - 1 \in (\psi(S) - 1) \\setminus \{\psi(m) - 1, \psi(k)\} \).

\[\square\]

Lemma 7.9. A set \( S \) is black \((k, m)\)-regular if and only if \( \psi(S) - 1 \) is white \((\psi(m), \psi(k))\)-regular.
Proof. This follows from the above lemma under the substitutions \( k \leftarrow \psi(m),\ m \leftarrow \psi(k),\ S \leftarrow \psi(S) - 1 \).

Alternatively one may easily to check Lemma 7.8 and Lemma 7.9 by means of the scheme interpretation (6.1) (10). Indeed, the shifted representation for \( \Phi^T(\psi(m), \psi(k)) \), \( T = \psi(S) - 1 \) appears from one for \( \Phi^S(k, m) \) by changing the color of all points and switching the rows.

**Proposition 7.10.** If \( S \) is black \((k, m)\)-regular, then

\[
\Phi^S(k, m) = (-1)^{m-k} q^{-2r} \left( \prod_{m \geq i > j \geq k} p_{ij}^{-1} \right) \Phi^T(\psi(m), \psi(k)),
\]

where \( T = \psi(S) - 1 \) is a white \((\psi(m), \psi(k))\)-regular set with \( r \) elements, and as above by \( \psi(S) - 1 \) we denote \( \{ \psi(s) - 1 \mid s \in S \} \), while the complement is related to the interval \([\psi(m), \psi(k) - 1]\).

**Proof.** We shall use double induction on \( r \) and on \( m-k \). If \( m = k \), then the equality reduces to \( x_k = x_{\psi(k)} \). If for given \( k, m \) we have \( r = 0 \), then \( S \) contains the interval \([k, m - 1]\) and the equality reduces to (6.1).

Suppose that \( r > 0 \). We fix an element \( t \in T \). By the inductive supposition on \( r \) we get

\[
(7.11) \quad \Phi^{S\cup\{\psi(t) - 1\}}(k, m) = (-1)^{m-k} q^{-2(r-1)} \left( \prod_{m \geq i > j \geq k} p_{ij}^{-1} \right) \Phi^{T\setminus\{t\}}(\psi(m), \psi(k)).
\]

We have \( t \notin \psi(S) - 1 \), and hence \( \psi(t) - 1 \notin S \). In particular \( \psi(t) - 1 \neq n \), and \( \tau_{\psi(t)-1} = 1 \), see (14.10). Thus, relation (AN) with \( t \leftarrow \psi(t) - 1 \) implies

\[
(7.12) \quad \Phi^S(k, m) = \Phi^{S\cup\{\psi(t) - 1\}}(k, m) + (1 - q^{-2}) p_{ab}^{-1} a \cdot b,
\]

where \( a = \Phi^S(\psi(t), m) \), \( b = \Phi^S(k, \psi(t) - 1) \). Inductive supposition on \( m-k \) yields

\[
(7.13) \quad a = (-1)^{m-\psi(t)q^{-2r_1}} \left( \prod_{m \geq i > j \geq \psi(t)} p_{ij}^{-1} \right) \Phi^T(\psi(m), t),
\]

\[
(7.14) \quad b = (-1)^{\psi(t) - 1 - k} q^{-2r_2} \left( \prod_{\psi(t) > i > j \geq k} p_{ij}^{-1} \right) \Phi^T(1 + t, \psi(k)),
\]

where \( r_1 \) is the number of elements in \( T \cap [\psi(m), t - 1] \), and \( r_2 \) is the number of elements in \( T \cap [1 + t, \psi(k) - 1] \). Obviously \( r_1 + r_2 = r - 1 \). Therefore

\[
(7.13) \quad p_{ab}^{-1} ab = (-1)^{m-k-1} q^{-2(r-1)} \left( \prod_{m \geq i > j \geq k} p_{ij}^{-1} \right) cd,
\]

where \( c = \Phi^T(\psi(m), t) \), \( d = \Phi^T(1 + t, \psi(k)) \). Now (7.12) and (7.11) imply

\[
(7.14) \quad \Phi^S(k, m) = (-1)^{m-k} q^{-2(r-1)} \left( \prod_{m \geq i > j \geq k} p_{ij}^{-1} \right) \{ \Phi^{T\setminus\{t\}}(\psi(m), \psi(k)) - (1 - q^{-2}) cd \}.
\]
We have $t \neq n$, for $T$ is white regular. Hence relation (7.8) with $S \leftarrow T \setminus \{t\}$, $t \leftarrow t$, $k \leftarrow \psi(m)$, $m \leftarrow \psi(k)$ implies
\[
\Phi^T_s(t)(\psi(m), \psi(k)) = \Phi^T_s(\psi(m), \psi(k)) + (1 - q^{-2})p^{-1}_{dc}dc,
\]
and the expression in braces of (7.14) reduces to
\[
(7.15) \quad \Phi^T_s(\psi(m), \psi(k)) + (1 - q^{-2})p^{-1}_{dc}[d, c].
\]
At the same time Corollary 7.5 with $S \leftarrow T$, $s \leftarrow t$, $k \leftarrow \psi(m)$, $m \leftarrow \psi(k)$ shows that $p^{-1}_{dc}[d, c]$ is black (1 + $q^{-2}$)$\Phi^T(\psi(m), \psi(k))$. This allows us to continue (7.15):
\[
= \Phi^T(\psi(m), \psi(k)) - (1 - q^{-2})\Phi^T(\psi(m), \psi(k)) = q^{-2}\Phi^T(\psi(m), \psi(k)).
\]
In order to get the required relation it remains to replace the expression in braces of (7.14) with $q^{-2}\Phi^T(\psi(m), \psi(k))$.

**Corollary 7.11.** If $S$ is $(k, m)$-regular, then $\Phi^S(\psi(m), \psi(k))$ for a suitable $(\psi(m), \psi(k))$-regular set $T$.

**Proof.** If $S$ is black $(k, m)$-regular, we apply Proposition 7.10. If $S$ is white $(k, m)$-regular, we still may apply Proposition 7.10 with $S \leftarrow T$, $T \leftarrow S$ due to Lemma 7.9.

**Corollary 7.12.** Let $S$ be $(k, m)$-regular. If $m > \psi(k)$, then the leading term of $\Phi^S(\psi(m), \psi(k))$ is proportional to $u[\psi(m), \psi(k)]$. In particular always $\Phi^S(\psi(m), \psi(k)) \neq 0$.

**Proof.** If $m < \psi(k)$, then definition (5.2) shows that the leading term of $\Phi^S(\psi(m), \psi(k))$ in the PBW-decomposition is $u[k, m]$, hence $\Phi^S(\psi(m), \psi(k)) \neq 0$.

If $m > \psi(k)$, then Proposition 7.10 (with $T \leftarrow S$, $S \leftarrow T$ provided that $S$ is white regular) shows that $\Phi^S(\psi(m), \psi(k))$ is proportional to $\Phi^T(\psi(m), \psi(k)) \neq 0$, for $\psi(k) < \psi(\psi(m)) = m$.

**Corollary 7.13.** If $S$ is black $(k, m)$-regular, $t \notin S \setminus \{n\}$, $k \leq t < m$, then
\[
\Phi^S(\psi(m), \psi(k)) \sim [\Phi^S(\psi(t), \psi(k)), \Phi^T(\psi(m), \psi(t) - 1)].
\]

**Proof.** If $t \notin S \setminus \{n\}$, then $\psi(t) - 1 \in T \cup \{n\}$, where $T = \overline{\psi(S)} - 1$. By Proposition 7.10 we have $\Phi^S(\psi(m), \psi(k))$. Corollary 7.5 yields
\[
\Phi^T(\psi(m), \psi(k)) \sim [\Phi^T(\psi(t), \psi(k)), \Phi^T(\psi(m), \psi(t) - 1)].
\]
Since $t$ is a white point or $t = n$, the set $S$ is black $(k, t)$-regular and black $(1 + t, m)$-regular, see the shifted schemes (7.3), (7.4). Hence Proposition 7.10 implies $\Phi^S(\psi(t), \psi(k))$ for $\Phi^S(1 + t, m) \sim \Phi^T(\psi(m), \psi(t) - 1)$. We are done.

**Corollary 7.14.** If $S \setminus \{s\}$ is black $(k, m)$-regular, $s \in S$, $k \leq s < m$, then
\[
(7.16) \quad \Phi^S(\psi(m), \psi(k)) \sim [\Phi^S(\psi(1 + s, m), \Phi^S(\psi(s), \psi(s))].
\]

**Proof.** Follows from Lemma 7.7 and Proposition 7.10 in the similar way.
8. Root sequence

Our next goal is to show that the total number of right coideal subalgebras containing $k[G]$ is less than or equal to $(2n)! = 2^n \cdot n!$.

In what follows for short we shall denote by $[k : m]$, $k \leq m \leq 2n$ the element $x_k + x_{k+1} + \ldots + x_m$ considered as an element of the group $\Gamma^+$. Of course $[k : m] = [\psi(m) : \psi(k)]$. If $k \leq m < \psi(k)$, then $[k : m]$ is an $U_q^+(\mathfrak{so}_{2n+1})$-root since $u[k, m]$ is a PBW-generator for $U_q^+(\mathfrak{so}_{2n+1})$. The simple $U_q^+(\mathfrak{so}_{2n+1})$-roots are precisely the generators $x_k = [k : k]$, $1 \leq k \leq n$. To put it another way, the $U_q^+(\mathfrak{so}_{2n+1})$-roots form the positive part $R^+$ of the classical root system of type $B_n$, provided that we formally replace symbols $x_i$ with $\alpha_i$ (the Weyl basis for $R$, see [3, Chapter IV, §6, Theorem 7]).

We fix a notation $U$ for a (homogeneous if $q^l = 1$, $l > 4$) right coideal subalgebra of $U_q^+(\mathfrak{so}_{2n+1})$, $q^t \neq 1$ (respectively of $U_q^+(\mathfrak{so}_{2n+1})$) that contains $G$. The $U$-roots form a subset $D(U)$ of $R^+$. In this section we will see, in particular, that $D(U)$ uniquely defines $U$.

**Definition 8.1.** Let $\gamma_k$ be a simple $U$-root of the form $[k : m]$, $k \leq m < \psi(k)$ with the maximal $m$. Denote by $\theta_k$ the number $m - k + 1$, the length of $\gamma_k$. If there are no simple $U$-roots of the form $[k : m]$, $k \leq m < \psi(k)$ we put $\theta_k = 0$. The sequence $r(U) = (\theta_1, \theta_2, \ldots, \theta_n)$ satisfies $0 \leq \theta_k \leq 2n - 2k + 1$ and it is uniquely defined by $U$. We shall call $r(U)$ a root sequence of $U$, or just an $r$-sequence of $U$. By $\theta_k$ we denote $k + \theta_k - 1$, the maximal value of $m$ for the simple $U$-roots of the form $[k : m]$ with fixed $k$.

**Theorem 8.2.** For each sequence $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$, such that $0 \leq \theta_k \leq 2n - 2k + 1$, $1 \leq k \leq n$ there exists at most one (homogeneous if $q^l = 1$, $l > 4$) right coideal subalgebra $U \supseteq G$ of $U_q^+(\mathfrak{sl}_{n+1})$, $q^t \neq 1$ (respectively of $U_q^+(\mathfrak{sl}_{n+1})$) with $r(U) = \theta$.

The proof will result from the following lemmas.

**Lemma 8.3.** If $[k : m]$ is a simple $U$-root, then there exists only one element $a \in U$ of the form $a = \Phi^S(k, m)$.

**Proof.** Suppose that $a = \Phi^S(k, m)$ and $b = \Phi^{S'}(k, m)$ are two different elements from $U$. Then $a - b$ is not a PBW-generator for $U$ since its leading term with respect to the PBW-decomposition given in Proposition 7.1 is not equal to $u[k, m]$. Hence nonzero homogeneous element $a - b$ is a polynomial in PBW-generators of $U$. Thus $[k : m]$, being the degree of $a - b$, is a sum of $U$-roots, a contradiction. □

**Lemma 8.4.** Let $\Phi^S(k, m) \in U$, $k \leq m < \psi(k)$. Suppose that $\Phi^{S'}(k, m) \notin U$ for all subsets $S' \subset S$. If $j \notin S$, $k \leq j < m$, then $\Phi^S(1 + j, m) \in U$. If $j \in S$, $k \leq j < m$, then $\Phi^{S'}(k, j) \in U$ with $S'' \subseteq S \cap [k, j]$, in particular $[k : j]$ is an $U$-root.

**Proof.** If in (5.2) we have $\Phi^S(1 + s_i, m) = 0$, then the spectrum $\text{Sp}(a)$ of $a = \Phi^S(k, m)$ is a proper subset of $S \cup \{m\}$. By Proposition 5.4 there exists a subset $S' \subseteq \text{Sp}(a) \subset S$ such that $\Phi^{S'}(k, m) \in U$. This contradiction implies that $\Phi^S(1 + j, m) \neq 0$ for all $j \in S \cap [k, m - 1]$.

If $j \notin S$, then Lemma 5.4 implies $\Phi^S(1 + j, m) \in U$.

If $j \in S$, then we apply $\Delta(\text{id} \otimes \nu_a)$ with $a = \Phi^S(1 + j, m) \neq 0$ as defined in Lemma 5.5 to both sides of (5.2). Lemma 5.5 shows that the value of $\Delta(\Phi^S(1 + i, m) u[k, i]) = (\text{id} \otimes \nu_a)$ has the following three options. If $j < i < m$, this is zero. If $i = j$, this is
$g_{a}u[k,j] \otimes a$. If $i < r$, this is $g_{a}b_{i}u[k,i], b_{i}' \in A_{k+1}$. Since $\Delta(u[k,m]) \cdot (\id \otimes a) = 0$ due to (1.8), we get the following relation

$$b = u[k,j] + \sum_{i < j, i \in S} b_{i}u[k,i] \in U, \quad b_{i}' \in A_{k+1}.$$

By definition this means that $[k : j]$ is an $U$-root, while Proposition 3.6 implies $\Phi S''(k,j) \in U$ with $S'' \subseteq \text{Sp}(b) \subseteq S \cap [k,j]$. \hfill \Box

**Lemma 8.5.** If $[k : m]$ is a simple $U$-root, $k \leq m < \psi(k)$, then the minimal $S$ such that $\Phi S(k,m) \in U$ equals $\{ j \mid k \leq j < m, [k : j]$ is an $U$-root $\}$, and this is a $(k,m)$-regular set, see Definition 7.1.

**Proof.** Suppose that $S$ is not $(k,m)$-regular. Then we have $k \leq n < m$.

If $n$ is a white point, $n \notin S$, then by Lemma 8.4 we have $\Phi S''(1 + n, m) \in U$. Hence $[n + 1 : m] = [\psi(m) : n]$ is an $U$-root due to Corollary 7.12. Since $S$ is not white $(k,m)$-regular, on the shifted scheme (7.2) we can find a black column, say, $n + i \in S \cup \{m\}$, $n - i \notin S$.

By Lemma 8.4 applied to $\Phi S(n+1,m)$ we have $[n + 1 : n + i]$ is an $U$-root, while the same lemma applied to $\Phi S''(k,m)$ shows that $[k : n - i]$ is also an $U$-root. Now we have

$$[k : m] = [k : n] + [n + 1 : m] = [k : n - i] + [n + 1 : n + i] + [n + 1 : m]$$

is a sum of $U$-roots, a contradiction.

If $n$ is a black point, $n \in S$, then by Lemma 8.4 we have $\Phi S''(k,n) \in U$, and $[k : n]$ is an $U$-root. Since $S$ is not black $(k,m)$-regular, we can find $i, 1 \leq i \leq m - n$ such that $n + i \notin S \setminus \{m\}, n - i \notin S$, see (7.3). We have $n - i \notin S''$, for $S'' \subseteq S$. Hence Lemma 8.4 applied to $\Phi S''(k,n)$ implies that $[1 + n - i : n] = [n + 1 : n + i]$ is an $U$-root. The same lemma applied to $\Phi S''(k,m)$ shows that $\Phi S''(1 + n + i, m) \in U$.

Hence, due to Corollary 7.12, the element $[1 + n + i : m] = [\psi(m) : n + i]$ is an $U$-root as well. We get a similar contradiction:

$$[k : m] = [k : n] + [n + 1 : m] = [k : n] + [n + 1 : n + i] + [1 + n + i : m].$$

Due to Lemma 8.4 it remains to show that if $[k : j]$ is an $U$-root, then $j \in S$. Suppose in contrary that $j \notin S$. Then Lemma 8.4 implies $a = \Phi S(1 + j, m) \in U$.

If $S$ is $(1 + j,m)$-regular, or $1 + j < \psi(m)$, then $a \neq 0$ and $[1 + j : m]$ is an $U$-root, see Corollary 7.12. This is a contradiction, for $[k : m] = [k : j] + [1 + j : m]$.

Suppose, finally, that $S$ is not $(1 + j,m)$-regular and $1 + j \geq \psi(m)$. Since $S$ indeed is $(k,m)$-regular, this is possible only in two cases: $j = \psi(m) - 1$, or $n \notin S$, $\psi(j) - 1 \in S$, see the shifted scheme representations (7.2), (7.3).

In the former case by Lemma 8.4 either $\Phi S(1 + n, m) \in U$ (if $n \notin S$), or $\Phi S''(k,m) \in U$ and $\Phi S''(1 + j, m) \in U$, for $j \notin S'' \subseteq S$ (if $n \in S$). Therefore $[n + 1 : m] = [\psi(m), n] = [j + 1 : n]$ is an $U$-root due to Corollary 7.12. We get a contradiction $[k : m] = [k : \psi(m) - 1] + [\psi(m), n] + [n + 1 : m]$.

In the latter case similarly $\Phi S''(1 + n, m) \in U$ and $\Phi S''(1 + j, n) \in U$. Hence Corollary 7.12 implies that $[n + 1 : m], [1 + j : n]$ are $U$-roots. Again contradiction: $[k : m] = [k : j] + [1 + j, n] + [n + 1 : m]$. \hfill \Box

**Lemma 8.6.** If $[k : m] = \sum_{i=1}^{r+1}[l_{i} : m_{i}],$ $k \leq m \leq 2n,$ $l_{i} \leq m_{i} < \psi(l_{i})$, then it is possible to replace some of the pairs $(l_{i}, m_{i})$ with $(\psi(m_{i}), \psi(l_{i}))$ so that the given decomposition takes up the form

$$[k : m] = [1 + k_{0} : k_{1}] + [1 + k_{1} : k_{2}] + \ldots + [1 + k_{r} : m].$$
Lemma 8.7. If \( m \leq k \leq m + 1 : m \) is the maximal letter among \( \{x_{ij} | k \leq j \leq m \} \). Hence there exists at least one \( i \) such that, respectively, \( l_i = k \) or \( l_i = \psi(m) \). In the former case we may put \( k_1 = m_i \) and apply the inductive supposition to \([m_i + 1 : m]\). In the latter case we put \( k_1 = \psi(m_i) - 1 \). Then \([k_r + 1 : m] = [\psi(m_i) : \psi(l_i)]\). One may apply the inductive supposition to \([k : k_r]\). □

Lemma 8.7. If \([k : m] \neq \psi(k)\) is a sum of \( U\)-roots, then \([k : m]\) itself is an \( U\)-root.

Proof. Without loss of generality we may suppose that \( m \leq \psi(k)\), for \([k : m] = [\psi(m) : \psi(k)]\). By Lemma 8.6 we have a decomposition (8.1), where \([1 + k_i : k_{i+1}]\), \( 0 \leq i < r \) are \( U\)-roots. Increasing, if necessary, the number \( r \) we may suppose that all roots \([1 + k_i : k_{i+1}]\), \( 0 \leq i < r \) are simple.

If \( k_{i+1} < \psi(1 + k_i)\), then by Proposition 5.6 we find a set \( S_i \subseteq [1 + k_i, k_{i+1} - 1] \) such that \( \Phi^{S_i}(1 + k_i, k_{i+1}) \in U \). Moreover by Lemma 8.5 the set \( S_i \) may be taken to be \((1 + k_i, k_{i+1})\)-regular.

If \( k_{i+1} > \psi(1 + k_i)\), then of course \( \psi(1 + k_i) < \psi(k_{i+1})\), and again by Proposition 5.6 and Lemma 8.5 we find a \((\psi(k_{i+1}), \psi(1 + k_i))\)-regular set \( T_i \subseteq [\psi(k_{i+1}), \psi(1 + k_i) - 1] \) such that \( \Phi^{T_i}(\psi(k_{i+1}), \psi(1 + k_i)) \in U \). By Corollary 7.11 with \( S \leftarrow T_i \) we have \( \Phi^{T_i}(\psi(k_{i+1}), \psi(1 + k_i)) \sim \Phi^{S_i}(1 + k_i, k_{i+1})\), where \( S_i \) is \((1 + k_i, k_{i+1})\)-regular. Thus in all cases

(8.2) \[ f_i \overset{df}{=} \Phi^{S_i}(1 + k_i, k_{i+1}) \in U, \quad S_i \subseteq [1 + k_i, k_{i+1} - 1] \]

with regular \( S_i \) (we stress that this is a restriction on \( S_i \) only if \( 1 + k_i \leq n < k_{i+1} \)).

By Definition 2.8 we have to construct an element \( c \in U \) with the leading super-word \( u[k, m] \). Firstly we shall prove that for \( r = 1 \) the element \( c = [f_0, f_1] \) is such an element even if \([1 + k_i : k_{i+1}]\) are not necessarily simple roots, but \( S_i, i = 0, 1 \) are still regular sets.

There is the following natural diminishing process for the decomposition of a linear combination of super-words in the PBW-basis given in Theorem 2.3 and Propositions 4.1 4.5. Let \( W \) be a super-word. First, due to [7, Lemma 7] we decompose the super-word \( W \) in a linear combination of smaller monotonous super-words. Then, we replace each non-hard super-letter with the decomposition of its value that exists by Definition 2.3, and again decompose the appeared super-words in linear combinations of smaller monotonous super-words, and so on, until we get a linear combination of monotonous super-words in hard super-letters. If these super-words are not restricted, we may apply Definition 2.4 and repeat the process until we get only monotonous restricted words in hard super-letters.

This process shows that if a super-word \( W \) starts with a super-letter smaller than \( u[k, m] \), then all super-words in the PBW-decomposition of \( W \) do as well. Using this remark we shall prove the following auxiliary statement.

If \( k \leq i < j < m < \psi(k), m \neq \psi(i) - 1 \), then all super-words in the PBW-decomposition of \([u[k, i], \Phi^S(1 + j, m)]\) start with super-letters smaller than \([u[k, m]]\).

Indeed, by definition (5.2) we have

\[ \Phi^S(1 + j, m) = u[1 + j, m] + \sum_{m > s \geq 1 + j} \gamma_s \Phi^S(1 + s, m) \cdot u[1 + j, s], \quad \gamma_s \in k. \]
We use induction on $m - j$. By Proposition 3.15 we have $[u[k, l], u[1 + j, m]] = 0$, for inequalities $\psi(k) > m > j$ imply $j \neq \psi(k)$. Denote $u = u[k, l]$, $v = \Phi^S(1 + s, m)$, $w = u[1 + j, s]$. Relation (2.11) reads $[u, v, w] = [u, v] \cdot w + puvv [u, w]$. By the inductive supposition all super-words in the PBW-decomposition of $[u, v]$ start with smaller than $u[k, m]$ super-letters. Hence so do ones for $[u, v] \cdot w$. The element $v$ depends only on $x_i$, $i > k$. Therefore so do all super-letters in the PBW-decomposition of $v$, while the starting super-letters in the PBW-decomposition of $v \cdot [u, w]$ are still less than $u[k, m]$. Thus, all super-words in the PBW-decomposition of $[u[k, l], \Phi^S(1 + j, m)]$ start with super-letters smaller than $u[k, m]$. This proves the auxiliary statement.

Now we have

$$[f_1, f_2] = \left[\Phi^S_0(k, k_1) + \Phi^S_1(1 + k_1, m)\right]$$

$$= \left[u[k, k_1] + \sum_{k_1 > s \geq k} \gamma_s \Phi^S_0(1 + s, k_1) \cdot u[k, s], u[1 + k_1, m] + \sum_{m > l \geq 1 + k_1} \beta_l \Phi^S_1(1 + l, m) \cdot u[1 + j, l]\right]$$

$$= \left[u[k, m] + \sum_{m > l \geq 1 + k_1} \beta_l [u[k, k_1], \Phi^S_1(1 + l, m) \cdot u[1 + k_1, l]], \gamma_s [\Phi^S_0(1 + s, k_1) \cdot u[k, s], f_2]\right]$$

We see that each element in the latter sum has a non trivial left factor that depends only on $x_i$, $i > k$, this is either $\Phi^S_0(1 + s, k_1)$ or $f_2$. Hence all super-words in the PBW-decomposition of that elements start with super-letters smaller than $u[k, m]$. To check the former sum denote $u = u[k, k_1]$, $v = \Phi^S_1(1 + l, m)$, $w = u[1 + k_1, l]$. By (2.11) the general element in the sum is proportional to $[u, v, w] = [u, v] \cdot w + puvv [u, w]$. By the above auxiliary statement with $i \leftarrow k_1$, $j \leftarrow l$ all super-words in the PBW-decomposition of $[u, v]$ start with super-letters smaller than $u[k, m]$. Hence so do the ones for $[u, v] \cdot w$. The element $v$ depends only on $x_i$, $i > k$. Therefore the starting super-letters in the PBW-decomposition of $v \cdot [u, w]$ are less than $u[k, m]$ as well. Thus the leading term of $[f_0, f_1]$ indeed is $u[k, m]$. This completes the case $r = 1$.

Consider the general case. Denote by $t$ the index such that $1 + k_t \leq n \leq k_{t+1}$, if any. Recall that $S_t$ is either white or black $(1 + k_t, k_{t+1})$-regular, while each of $S_i$, $i \neq t$ is both white and black $(1 + k_i, k_{i+1})$-regular, for its degree in $x_n$ is less than or equal to 1. We shall consider four options for the regular set $S_t$ given in (7.1)-(7.4) separately.

1. $k_{t+1} < \psi(1 + k_t)$, and $S_t$ is white regular. Let $S = \cup_{i=0}^{t} S_i \cup \{k_i | 0 < i < t\}$. The set $S$ is white $(k, k_{t+1})$-regular since all complete columns on the shifted scheme (7.1) for $\Phi^S(k, k_{t+1})$ coincide with ones for $\Phi^{S_t}(k_t, k_{t+1})$. By Lemma 7.4 we have

$$\Phi^S(k, k_{t+1}) \sim \left[f_1 f_{t-1} \cdots f_2 f_1 f_0\right]$$

with an arbitrary alignment of brackets on the right hand side. In the same way consider the set $S' = \cup_{i=t+1}^{r} S_i \cup \{k_i | t + 1 < i < r\}$. This set is white $(1 + k_{t+1}, m)$-regular, for the shifted scheme (7.1) for $\Phi^{S'}(1 + k_{t+1}, m)$ has no complete columns at all. Lemma 7.4 yields

$$\Phi^{S'}(1 + k_{t+1}, m) \sim \left[f_r f_{r-1} \cdots f_{t+2} f_{t+1}\right].$$
Now we may apply the considered above case \( r = 1 \) with \( S_0 \leftarrow S, S_1 \leftarrow S' \), \( t_1 \leftarrow k_{t+1} \) Thus the leading super-word of the element
\[
(8.3) \quad c = \left[ [f_t f_{t-1} \ldots f_2 f_1 f_0], [f_r f_{r-1} \ldots f_{t+2} f_{t+1}] \right]
\]
equals \( u[k, m] \) and obviously \( c \in U \), for \( f_i \in U \), \( 0 \leq i \leq r \).

2. \( k_{t+1} > \psi(1 + k_t) \), and \( S_t \) is white regular. In perfect analogy we consider the sets \( S = \bigcup_{i=0}^{t-1} S_i \cup \{k_i \mid 0 < i < t - 1\} \), and \( S' = \bigcup_{i=t}^{r} S_i \cup \{k_i \mid t < i < r\} \). By the case \( r = 1 \) under the substitutions \( S_0 \leftarrow S, S_1 \leftarrow S', t_1 \leftarrow k_t \), we see that the required element is
\[
(8.4) \quad c = \left[[f_{t-1} f_{t-2} \ldots f_1 f_0], [f_r f_{r-1} \ldots f_{t+1} f_t] \right].
\]

3. \( k_{t+1} < \psi(1 + k_t) \), and \( S_t \) is black regular. Let \( S = \bigcup_{i=0}^{t-1} S_i \). The set \( S \) is black \((k, k_{t+1})\)-regular since all complete columns on the shifted scheme \( (7.3) \) for \( \Phi^S(k, k_{t+1}) \) coincide with ones for \( \Phi^{S_t}(k_t, k_{t+1}) \). No one of the points \( k_1, k_2, \ldots, r_r \) belongs to the set \( S \), see \( (5.2) \). Therefore by multiple use of Corollary \( (7.13) \) we have
\[
\Phi^{S'}(1 + k_{t+1}, m) \sim [f_{t+1} f_{t+2} \ldots f_r f_r].
\]
with an arbitrary alignment of brackets on the right hand side. In the same way consider the set \( S' = \bigcup_{i=t}^{r} S_i \). This set is black \((1 + k_{t+1}, m)\)-regular, for the shifted scheme \( (7.3) \) for \( \Phi^{S'}(1 + k_{t+1}, m) \) has no complete columns at all. The multiple use of Corollary \( (7.13) \) yields
\[
\Phi^{S'}(1 + k_{t+1}, m) \sim [f_{t+1} f_{t+2} \ldots f_r f_r].
\]

Now we may find \( c \) using the case \( r = 1 \) with \( S_0 \leftarrow S, S_1 \leftarrow S', t_1 \leftarrow k_{t+1} \) :
\[
(8.5) \quad c = \left[[f_0 f_1 \ldots f_{t-1} f_t], [f_{t+1} f_{t+2} \ldots f_r f_r] \right].
\]

4. \( k_{t+1} > \psi(1 + k_t) \), and \( S_t \) is black regular. In perfect analogy we consider the sets \( S = \bigcup_{i=0}^{t-1} S_i \), and \( S' = \bigcup_{i=t+1}^{r} S_i \). By the case \( r = 1 \) under the substitutions \( S_0 \leftarrow S, S_1 \leftarrow S', t_1 \leftarrow k_t \), we see that the required element is
\[
(8.6) \quad c = \left[[f_0 f_1 \ldots f_{t-2} f_{t-1}], [f_t f_{t+1} \ldots f_r f_r] \right].
\]
The proof is complete. \( \Box \)

**Lemma 8.8.** If \([k : m], k \leq m < \psi(k)\) is a simple \( U \)-root, \( k \leq j < m \), then \([k : j]\) is an \( U \)-root if and only if \([1 + j : m]\) is not a sum of \( U \)-roots.

**Proof.** If \([k, j]\) is an \( U \)-root, then \([1 + j : m]\) is not a sum of \( U \)-roots, for \([k : m] = [k : j] + [1 + j : m]\) is a simple \( U \)-root.

We note, first, that the converse statement is valid if the minimal \( S \), such that \( \Phi^S(k, m) \in U \), is \((1 + j, m)\)-regular. Indeed, in this case \( \Phi^S(1 + j, m) \neq 0 \) due to Corollary \( (7.12) \). By Lemma \( 8.3 \) the element \([k : j]\) is an \( U \)-root if and only if \( j \in S \). If \( j \notin S \), then by Lemma \( 8.4 \) we have \( a = \Phi^S(1 + j, m) \in U \). Hence the nonzero homogeneous element \( a \) is a polynomial in PBW-generators of \( U \). Thus \([1 + j : m]\), being degree of \( a \), is a sum of \( U \)-roots (by Lemma \( 5.7 \) this is even an \( U \)-root, for the regularity hypothesis implies \( \psi(1 + j) \neq m \)).

Suppose, next, that \( S \) is not \((1 + j, m)\)-regular, and \( j \notin S \). In this case \( 1 + j \leq n < m \). Moreover \( m \geq \psi(1 + j) \) since otherwise all complete columns in the shifted scheme \( (7.1), \( 7.4) \) of \( \Phi^S(1 + j, m) \) coincide with that of \( \Phi^S(k, m) \). Obviously in general only the first from the left complete column for \( \Phi^S(1 + j, m) \) may be different from a complete column for \( \Phi^S(k, m) \). Hence we have just the following three options:
1) \( \psi(1 + j) = m \); 2) \( \psi(1 + j) \in S \), while \( n \notin S \); 3) \( \psi(1 + j) \notin S \), while \( n \in S \).
1) In the shifted scheme of $\Phi^S(k, m)$, the point $j = \psi(m) - 1$ has the same color as $n$ does, see (7.1), (7.3); that is, $n$ is a white point. At the same time, since $S$ is always $(n + 1, m)$-regular, we already know that the point $n$ is white if and only if $[n + 1 : m]$ is an $U$-root. Thus $[n + 1 : m]$ is an $U$-root, while 
$[1 + j : m] = [1 + j : n] + [n + 1 : m] = 2[n + 1 : m]$ is a sum of two $U$-roots.

2) In the second case certainly $S$ is $(n + 1, m)$-regular. Hence $n \notin S$ implies that $[n + 1 : m]$ is an $U$-root. By Lemma 8.3 we have $\Phi^{S'}(k, \psi(1 + j)) \in U$ with $S' \subseteq S$, for $\psi(1 + j) \in S$. In particular, still $n \notin S$. Hence again the same lemma implies $a = \Phi^S(n + 1, \psi(1 + j)) \in U$. By Corollary 7.12 the leading super-word of $a$ equals $u[1 + j, n]$; that is, $[1 + j : n]$ is an $U$-root. Now $[1 + j : n] = [1 + j : n] + [n + 1 : m]$ is a sum of two roots, which is required.

3) By Lemma 8.4 we have $\Phi^{S''}(k, n) \in U$ with $S'' \subseteq S$, for $n \in S$. In particular, still $j \notin S''$. Hence the same lemma implies that $[1 + j : n]$ is an $U$-root. Since $\psi(1 + j) \notin S$, and obviously $S$ is $(\psi(1 + j), m)$-regular, we already know that $[1 + \psi(1 + j) : m] = [\psi(j) : m]$ is an $U$-root. Now we have $[1 + j : m] = [1 + j : n] + [n + 1 : \psi(1 + j)] + [\psi(j) : m]$ is a sum of $U$-roots, for $[n + 1 : \psi(1 + j)] = [1 + j : n]$. □

Lemma 8.9. A (homogeneous) right coideal subalgebra $U$ that contains $k[G]$ is uniquely defined by the set of all its simple roots.

Proof. Since obviously two subalgebras with the same PBW-basis coincide, it suffices to find a PBW-basis of $U$ that depends only on a set of simple $U$-roots. We note firstly that the set of all $U$-roots is uniquely defined by the set of simple $U$-roots. Indeed, if $[k : m]$ is an $U$-root, then it is a sum of simple $U$-roots. By Lemma 8.6 there exists a sequence $k = 0 \leq k_1 < \ldots < k_r < m = k_{r+1}$ such that $[1 + k_i : k_{i+1}]$, $0 \leq i \leq r$ are simple $U$-roots. Conversely, if there exists a sequence $k = 0 < k_1 < \ldots < k_r = m + 1$ such that $[1 + k_i : k_{i+1}]$, $0 \leq i < r$ are simple $U$-roots, then by Lemma 8.7 the element $[k : m]$ is an $U$-root. Of course the decomposition of $[k : m]$ in a sum of simple $U$-roots is not unique in general, however for the construction of the PBW-basis we may fix that decomposition for each non-simple $U$-root from the very beginning.

Now if $[k : m]$ is a simple $U$-root, Lemmas 8.3 and 8.5 show that the element $\Phi^S(k, m) \in U$ is uniquely defined by the set of simple $U$-roots. We include this element in the PBW-basis of $U$. If $[k : m]$ is a non-simple $U$-root with a fixed decomposition in a sum of simple $U$-roots, then we include in the PBW-basis the element $c$ defined in one of the formulae 8.3, 8.6 depending up the type of the decomposition.

□

Lemma 8.10. If for (homogeneous) right coideal subalgebras $U, U'$ containing $k[G]$ we have $r(U) = r(U')$, then $U = U'$.

Proof. By Lemma 8.9 it suffices to show that the $r$-sequence uniquely defines the set of all simple roots. We use the downward induction on $k$, the onset of a simple $U$-root. If $k = n$, then the only possible $\gamma = [n : n]$ is a simple $U$-root if and only if $\theta_n = 1$. Let $k < n$. By definition there do not exist simple $U$-roots of the form $[k : m]$, $m > \theta_k$, while $[k : \theta_k]$ is a simple $U$-root. If $m < \theta_k$, then by Lemma 8.8 the element $[k : m]$ is an $U$-root if and only if $[m + 1 : \theta_k]$ is not a sum of $U$-roots starting with a number greater than $k$. By inductive supposition the $r$-sequence defines all roots starting with a number greater than $k$. Hence by Lemma 8.8 the $r$-sequence defines the set of all $U$-roots of the form $[k : m]$, $m < \theta_k$ as well. Thus the $r$-sequence defines the set if all $U$-roots and the set of simple $U$-roots. □
9. Examples

In this section we find the simple roots for some fundamental examples of right coideal subalgebras. We keep all notations of the above section.

Example 9.1. Let \( U(k, m) \) be a right coideal subalgebra generated over \( k[G] \) by a single element \( u[k, m], k \leq m \leq \psi(k) \). By \[1.3\], the right coideal generated by \( u[k, m] \) is spanned by the elements \( g_\mu u[i + 1, m] \). Hence \( U(k, m) \) as an algebra is generated over \( k[G] \) by the elements \( u[i, m], k \leq i \leq m \). Respectively, the additive monoid of degrees of homogeneous elements from \( U(k, m) \) is generated by \([i : m], k \leq i \leq m \). In this monoid the indecomposable elements (by definition they are simple \( U(k, m) \)-roots) are precisely \([i : m], k \leq i \leq m, i \neq \psi(m) \). The length of \([i : m] \) equals \( m - i + 1 \). However, if \( i > \psi(m) \), then the maximal letter among \( x_j \), \( i \leq j \leq m \) is \( x_{\psi(m)} \), for \([i : m] = [\psi(m) : \psi(i)] \) with \( \psi(m) \leq \psi(i) < \psi(\psi(m)) \). Hence the maximal length of a simple root starting with \( \psi(m) \) equals \( m - (\psi(m) + 1) + 1 = 2(m - n) - 1 \), while there are no simple roots of the form \([k' : m'], k' \leq m' < \psi(k') \) with \( k' > \psi(m) \). Thus due to Definition \[5.1\] we have

\[
\theta_i = \begin{cases} 
  m - i + 1, & \text{if } k \leq i < \psi(m); \\
  2(m - n) - 1, & \text{if } k \leq i = \psi(m) \leq n; \\
  0, & \text{otherwise}.
\end{cases}
\]

The set \( \{u[i, m] | k \leq i \leq m, i \neq \psi(m)\} \) is a set of PBW-generators for \( U(k, m) \) over \( k[G] \).

Example 9.2. Let us analyze in details the simplest (but not trivial, \[2\]) case \( n = 2 \). Consider the following six elements \( w_1 = u[1, 3] = [x_1, x_2, x_2], w_2 = u[2, 4] = [x_2, x_2, x_1], w_3 = u[1, 2] = [x_1, x_2], w_4 = u[3, 4] = [x_2, x_1], w_5 = x_1, w_6 = x_2 \). Denote by \( U_j, 1 \leq j \leq 6 \) a right coideal subalgebra generated by \( w_j \) and \( k[G] \).

By means of \[9.1\] we have \( r(U_1) = (3, 1) \). Indeed, in this case \( k = 1, m = 3, \psi(m) = 2 \); hence \( \theta_1 = m - 1 + 1 = 3 \) according to the first option of \[9.1\], while \( \theta_2 = 2(m - n) - 1 = 1 \) due to the second option of \[9.1\].

In the same way \( r(U_2) = (3, 0) \), for in this case \( k = 2, m = 4, \psi(m) = 1 \); hence \( \theta_1 = 2(m - 2) - 1 = 3 \) according to the second option, while \( \theta_2 = 0 \) due to the third one.

In perfect analogy we have \( r(U_3) = (2, 1), r(U_4) = (2, 0), r(U_5) = (1, 0), r(U_6) = (0, 1) \). We see that all these six right coideal subalgebras are different. There are two more (improper) right coideal subalgebras \( U_7 = U_q^+(\mathfrak{so}_5), U_8 = k[G] \) with the \( r \)-sequences \((1, 1)\) and \((0, 0)\) respectively. Thus we have found all \((2n)! = 8\) possible right coideal subalgebras in \( U_q^+(\mathfrak{so}_5) \) containing \( G \). They form the following lattice:
Our next goal is to generalize formula (9.1) to an arbitrary right coideal subalgebra $U^S(k, m)$ generated over $k[G]$ (as a right coideal subalgebra) by a single element $\Phi^S(k, m)$ with a $(k, m)$-regular set $S$.

**Proposition 9.3.** If $S$ is $(k, m)$-regular, then the coproduct of $\Phi^S(k, m)$ has a decomposition

\[
\Delta(\Phi^S(k, m)) = \sum a^{(1)} \otimes a^{(2)},
\]

where degrees of the left components of tensors belong to the additive monoid $\Sigma$ generated by all $[1 + t : s]$ with $t$ being a white point ($t = k - 1$, or $t \notin S, k \leq t < m$), and $s$ being a black point ($s \in S \cap [k, m - 1]$, or $s = m$).

**Proof.** Let $S$ be white $(k, m)$-regular. Lemma 7.4 shows that $\Phi^S(k, m)$ is a linear combination of products (in different orders) of the elements $u_i = u[1 + s_i, s_i+1]$, $0 \leq i \leq r$. Hence by (4.3) the coproduct is a linear combination of products of the tensors

\[
u_i \otimes 1, \quad f_i \otimes u_i, \quad h_i u[1 + t_i, s_i+1] \otimes u[1 + s_i, t_i],
\]

where $s_i < t_i < s_i+1$, $f_i = \text{gr}(u_i)$, $h_i = \text{gr}(u[1 + s_i, t_i])$. Degrees of the left components of these tensors, except $u_i \otimes 1$, $i > 0$, belong to $\Sigma$. We stress that in each product there is exactly one tensor of (9.3) related to a given $i$.

Denote by $\Sigma'$ the additive monoid generated by all $[1 + t : s]$, where $t \notin S$, $k \leq t < m$, while $s$ is a black point. By induction on the number $r$ of elements in $S \cap [k, m - 1]$ we shall prove that there exists a decomposition (9.2) such that for each $i$ either $D(a^{(1)}) \in \Sigma'$ or $D(a^{(1)}) = [k : s] + \alpha$, where $s$ is a black point, and $\alpha \in \Sigma'$.

If $r = 0$, then $\Phi^S(k, m) = u[k, m]$, and the statement follows from (4.8).

If $r > 0$, then Corollary 7.5 implies that $\Phi^S(k, m) \sim [\Phi^S(1 + s_1, m), u[k, s_1]]$. By the inductive supposition we have $\Delta(\Phi^S(1 + s_1, m)) = \sum b^{(1)} \otimes b^{(2)}$, where either $D(b^{(1)}) = \alpha \in \Sigma'_1$, or $D(b^{(1)}) = [1 + s_1 : s] + \alpha, \alpha \in \Sigma'_1$ with $s$ being a black point on the scheme of $\Phi^S(1 + s_1, m)$, see (6.3). Here $\Sigma'_1$ is the $\Sigma'$ related to $\Phi^S(1 + s_1, m)$: the additive monoid generated by all $[1 + t : s]$, where $t \notin S$, $s_1 < t < m$, and $s$ is a black point. Certainly $\Sigma'_1 \subseteq \Sigma'$, for on the scheme of $\Phi^S(1 + s_1, m)$ there is just one point, $s_1$, that has color other than it has on the scheme of $\Phi^S(k, m)$.

By (4.8) the coproduct of $u_0 = u[k, s_1]$ is a linear combination of the tensors (6.3) with $i = 0$. Degree of the left components of the tensors of

\[
\left[ b^{(1)} \otimes b^{(2)}, h_0 u[1 + t_0, s_1] \otimes u[k, t_0] \right]
\]

equals either $[1 + t_0 : s_1] + \alpha$ or $[1 + t_0 : s_1] + [1 + s_1 : s] + \alpha = [1 + t_0 : s] + \alpha$. In both cases it belongs to $\Sigma'$ since $t_0$ is a white point in both schemes, and $t_0 \neq k - 1$.

In the same way, degree of the left components of the tensors of $[b^{(1)} \otimes b^{(2)}, u_0 \otimes 1]$ equals either $[k : s_1] + \alpha$ or $[k : s_1] + [1 + s_1 : s] + \alpha = [k : s] + \alpha$. In both cases the degree has a required form.

It remains to consider the skew commutator

\[
\left[ b^{(1)} \otimes b^{(2)}, f_0 \otimes u_0 \right] = b^{(1)} f_0 \otimes b^{(2)} u_0 - p(b^{(1)} b^{(2)}, u_0) f_0 b^{(1)} \otimes u_0 b^{(2)}.
\]

Degree of the left components of these tensors equals $D(b^{(1)})$. We shall prove that one of the following three options is valid: $[b^{(1)} \otimes b^{(2)}, f_0 \otimes u_0] = 0$, or $D(b^{(1)}) \in \Sigma'$, or $D(b^{(1)}) = [k : s] + \alpha, \alpha \in \Sigma'$ with black $s$. 

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The comments given around (9.3) show that there exists a sequence of elements 
\((t_i \mid 0 \leq i \leq r)\), such that \(s_i \leq t_i \leq s_{i+1}\), and

\[
(9.4) \quad D(b^{(1)}) = \sum_{i=1}^{r} [1 + t_i : s_{i+1}] \quad \text{and} \quad D(b^{(2)}) = \sum_{i=1}^{r} [1 + s_i : t_i],
\]

where formally \([1 + s_i : s_i] = [1 + s_{i+1} : s_{i+1}] = 0\). We consider separately the following two cases.

**Case 1.** \(t_1 > s_1\). Due to the first of (9.4), degree of \(b^{(1)}\) in \(x_{1+s_1}\) is less than or equal to 1. At the same time the equality \(D(b^{(1)}) = [1 + s_1 : s] + \alpha\) shows that this degree equals 1, and \(x_{1+s_1}\)-th component of \(\alpha\) is zero. Hence there exists \(i \geq 2\) such that \(t_i < \psi(1 + s_1) \leq s_{i+1}\). However, \(\psi(1 + s_1) = \psi(s_1) - 1\) is a white point, for \(S\) is white \((k,m)\)-regular. In particular \(\psi(1 + s_1) \neq s_{i+1}\); that is, \(\psi(1 + s_1) < s_{i+1}\). Now the nonempty interval \([1 + \psi(1 + s_1) : s_{i+1}] = [\psi(s_1) : s_{i+1}]\) must be covered by \(\alpha \in \Sigma'_1\). This is possible only if \(\alpha\) has a summand \(\alpha_1 = [\psi(s_1) : s_j], j \geq i + 1\), for degree of \(\Phi^S(1 + s_1, m)\) in each of \(x_1, \psi(s_1) \leq l \leq m\) equals 1, while the \(x_{\psi(s_1)−1}\)-th component of \(\alpha\) is zero (recall that \(x_{\psi(s_1)−1} = x_{1+s_1}\)). Thus, we have \(\alpha = \alpha_1 \in \Sigma'_1\).

If \(\psi(s_j) > k\), or, equivalently, \(s_j < \psi(k)\), then \(\psi(s_j) = 1\) is a white point, for \(\psi(1 + s_1) < s_{i+1} \leq s_j\) implies \(s_1 > \psi(s_j) = 1\). We have

\[
\alpha_1 [1 + s_1 : s] = [\psi(s_1) : s_j] + [1 + s_1 : s] = [\psi(s_j) : s] \in \Sigma'_1.
\]

Hence \(D(b^{(1)}) = (\alpha_1 + [1 + s_1 : s]) + (\alpha - \alpha_1) \in \Sigma'_1\), which is required.

If \(\psi(s_j) < k\), or, equivalently, \(s_j > \psi(k)\), then \(\psi(k)\) is a white point, see (7.2). Hence \([\psi(s_j) : k - 1] = [1 + \psi(k) : s_j] \in \Sigma'_1\), while

\[
\alpha_1 [1 + s_1 : s] = [\psi(s_j) : k - 1] + [k : s] = [\psi(s_j) : s] \in [k : s] + \Sigma'_1,
\]

and \(D(b^{(1)}) = (\alpha_1 + [1 + s_1 : s]) + (\alpha - \alpha_1) \in [k : s] + \Sigma'_1\).

Of course \(s_j \neq \psi(k)\) since \(S\) is white \((k,m)\)-regular, see (7.2).

**Case 2.** \(t_1 = s_1\). Assume, first, that the sequence \((t_i \mid 1 < i \leq r)\) does not contain the point \(\psi(s_1) - 1 = \psi(1 + s_1)\). We have seen, see comments around (9.3), that \(b^{(2)}\) is a product of the elements \(u[1 + s_1, t_i], i > 0\) in some order. For \(i = 1\) the tensor \(u_0 \otimes 1\) does participate in the construction of \(b^{(1)} \otimes b^{(2)}\) (recall that now \(t_1 = s_1\)).

By Proposition 3.4 with \(i \leftarrow s_1, j \leftarrow s_i m \leftarrow t_i\) we have \([u[1 + s_1, t_i], u_0] = 0\), \(i > 1\), for now \(t_i \neq \psi(s_1) - 1\) and \(s_i \neq \psi(k)\), see (7.2). Hence ad-identity (2.10) implies \([b^{(2)}, u_0] = 0\); that is, \(b^{(2)}u_0 = p(b^{(2)}, u_0)u_0b^{(2)}\). Since \(f_0 = gr(u_0)\), we have

\[
(b^{(1)} \otimes b^{(2)})(f_0 \otimes u_0) = b^{(1)}f_0 \otimes b^{(2)}u_0 = p(b^{(1)}b^{(2)}, u_0)(f_0 \otimes u_0)(b^{(1)} \otimes b^{(2)}).
\]

This equality in more compact form is \([b^{(1)} \otimes b^{(2)}, f_0 \otimes u_0] = 0\), which is one of the required options.

Suppose, next, that \(\psi(s_1) - 1 = t_i\) for a suitable \(i, 1 < i \leq r\). Due to the first of (9.4) degree of \(b^{(1)}\) in \(x_{1+s_1}\) equals 1, while the equality \(D(b^{(1)}) = [1 + s_1 : s] + \alpha\) implies that \(x_{1+s_1}\)-th component of \(\alpha\) is zero. At the same time \(t_i \neq s_{i+1}\), for \(t_i\) and \(s_1\) are located in the same column of the shifted scheme (7.1), (7.2). Hence, again due to the first of (9.4), the nonempty interval \([1 + t_i : s_{i+1}] = [\psi(s_1) : s_{i+1}]\) must be covered by \(\alpha \in \Sigma'_1\). This is possible only if \(\alpha\) has a summand \(\alpha_1 = [\psi(s_1) : s_j], j \geq i + 1\), for degree of \(\Phi^S(1 + s_1, m)\) in each of \(x_t, \psi(s_1) \leq l \leq m\) equals 1, while the \(x_{\psi(s_1)−1}\)-th component of \(\alpha\) is zero (recall that \(x_{\psi(s_1)−1} = x_{1+s_1}\)). Thus, we have \(\alpha - \alpha_1 \in \Sigma'_1\).
If \( \psi(s_j) > k \), or, equivalently, \( s_j < \psi(k) \), then \( \psi(s_j) - 1 \) is a white point, for \( \psi(1 + s_1) < s_{j+1} \leq s_j \) implies \( s_1 > \psi(s_j) - 1 \). We have
\[
\alpha_1 + [1 + s_1 : s] = [\psi(s_1) : s_j] + [1 + s_1 : s] = [\psi(s_j) : s] \in \Sigma'.
\]
Hence \( D(b^{(1)}) = (\alpha_1 + [1 + s_1 : s]) + (\alpha - \alpha_1) \in \Sigma' \), which is required.

If \( \psi(s_j) < k \), or, equivalently, \( s_j > \psi(k) \), then \( \psi(k) \) is a white point, see (7.2).

Hence \( [\psi(s_j) : k - 1] = [1 + \psi(k) : s_j] \in \Sigma' \), while
\[
\alpha_1 + [1 + s_1 : s] = [\psi(s_j) : k - 1] + [k : s] = [\psi(s_j) : s] \in [k : s] + \Sigma',
\]
and \( D(b^{(1)}) = (\alpha_1 + [1 + s_1 : s]) + (\alpha - \alpha_1) \in [k : s] + \Sigma' \). Of course \( s_j \neq \psi(k) \) since \( S \) is white \((k, m)\)-regular, see (7.2). This completes the proof for a white regular set \( S \).

If \( S \) is black \((k, m)\)-regular, then by Proposition 7.10 we have \( \Phi^S(k, m) \sim \Phi^T(\psi(m), \psi(k)) \), where \( T = \psi(S) - 1 \) is a white \((\psi(m), \psi(k))\)-regular set. If \( t, s \) are, respectively, white and black points for \( \Phi^S(k, m) \), then so do \( \psi(s) - 1, \psi(t) - 1 \) with respect to \( \Phi^T(\psi(m), \psi(k)) \). We have
\[
[1 + t : s] = [\psi(s) : \psi(1 + t)] = [1 + (\psi(s) - 1) : \psi(t) - 1].
\]
Hence \( \Phi^S(k, m) \) and \( \Phi^T(\psi(m), \psi(k)) \) define the same additive monoid \( \Sigma \). It remains to apply already proved statement to \( \Phi^T(\psi(m), \psi(k)) \).

\[\square\]

Corollary 9.4. If \( S \) is \((k, m)\)-regular, then all \( \mathbf{U}^S(k, m) \)-roots belong to the monoid \( \Sigma \) defined in the above proposition.

Proof. We are reminded that coassociativity of the coproduct implies that the left components of the tensor (9.2) span a right coideal. Hence \( \mathbf{U}^S(k, m) \) as an algebra is generated by the \( a^{(1)} \)'s and \( k[G] \). Hence the degrees of all homogeneous elements from \( \mathbf{U}^S(k, m) \) belong to \( \Sigma \). In particular all \( \mathbf{U}^S(k, m) \)-roots, being the degrees of PBW-generators, belong to \( \Sigma \) as well.

\[\square\]

Lemma 9.5. Let \( S \) be a white \((k, m)\)-regular set. An element \( [1 + t : s] \), \( t < s \) with white \( t \) and black \( s \) is indecomposable in \( \Sigma \) if and only if one of the following conditions is fulfilled:

a) The point \( \psi(1 + t) \) is not black (it is white or does not appear on the scheme at all).

b) On the shifted scheme all columns between \( t \) and \( s \) are white-black or black-white ones (in particular all of them are complete and \( n \notin [t, s] \)).

Proof. If no one of the conditions is fulfilled, then \( \psi(1 + t) \) is a black point and there exists \( j, t \leq j \leq s \), such that both \( j \) and \( \psi(1 + j) \) are white points on the scheme (white regular scheme has no black-black columns). Certainly \( j \neq t, j \neq s \).

We have
\[
[1 + t : j] = [\psi(j) : \psi(1 + t)] = [1 + \psi(1 + j) : \psi(1 + t)] \in \Sigma.
\]
Thus \( [1 + t : s] = [1 + t : j] + [1 + j : s] \) is a non trivial decomposition in \( \Sigma \).

Conversely. Assume that \( [1 + t : s] \) is decomposable in \( \Sigma \):
\[
[1 + t : s] = \sum_{i=1}^{r} [1 + l_i : s_i].
\]

Without loss of generality we may suppose that \( s_i \leq \psi(1 + l_i) \) due to \( [1 + l_i : s_i] = [\psi(s_i) : \psi(1 + l_i)] \). Moreover, if \( s_i = \psi(1 + l_i) \), then \( [1 + l_i : n] = [1 + n : s_i] \in \Sigma \), for
n is a white point (S is white regular). This allows one to replace \([1 + l_i : s_i]\) with \(2[1 + n : s_i]\) in (9.5). Thus we may suppose that \(s_i \leq \psi(1 + l_i)\) for all \(i\) in (9.5).

By Lemma 8.3 we find a sequence \(t = t_0 < t_1 < \ldots < t_r < s = t_{r+1}\) such that for each \(i\) either \(t_i\) is white and \(t_{i+1}\) is black points, or \(\psi(1 + t_{i+1})\) is white and \(\psi(1 + t_i)\) is black ones. In the former case to the index \(i\) we associate the sign “+,” while in the latter case to \(i\) we associate the sign “−.” It is clear that in the sequence of indices \(0, 1, 2, \ldots, r\) no one pair of neighbors has the same sign associated.

If, now, \(\psi(1 + t)\) is not a black point, then “+” is associated to the index 0. Hence “−” is associated to the index 1. In particular \(\psi(1 + t_1)\) is a black point. However \(t_1\) is also a black point. This is impossible, for \(S\) is white regular.

Assume that on the shifted scheme all columns between \(t\) and \(s\) are white-black or black-white ones. If \(t_1\) is a white point, then both of \(t_0 = t, t_1\) are white, while both of \(\psi(1 + t_1), \psi(1 + t_0)\) are black points; that is, no sign may be associated to

the index 0. Hence \(t_1\) is a black point, while \(\psi(1 + t_1)\) must be a white one. In this case “−” may not be associated to the index 1. Thus “+” is associated to 1. But then \(t_1\) is a white point, a contradiction. \(\square\)

**Lemma 9.6.** Let \(S\) be a black \((k, m)\)-regular set. An element \([1 + t : s]\), \(t < s\) with white \(t\) and black \(s\) is indecomposable in \(\Sigma\) if and only if one of the following conditions is fulfilled:

a) The point \(\psi(1 + s)\) is not white (it is black or does not appear on the scheme at all).

b) On the shifted scheme all columns between \(t\) and \(s\) are white-black or black-white ones (in particular all of them are complete and \(n \notin [t, s]\)).

**Proof.** The proof follows from the above lemma by means of Lemma 7.9 and Proposition 7.10. \(\square\)

**Lemma 9.7.** Let \(S\) be a \((k, m)\)-regular set. An element \(\alpha = [a : b]\) is a simple \(U^S(k, m)\)-root if and only if \(\alpha \in \Sigma\) and it is indecomposable in \(\Sigma\) (in particular \(\alpha = [1 + t : s]\), \(t < s\) with white \(t\) and black \(s\) determined in Lemmas 9.5 9.6).

**Proof.** Without loss of generality we may suppose that \(k \leq m < \psi(k)\) due to Proposition 7.10. We have already mentioned that all \(U^S(k, m)\)-roots belong to \(\Sigma\) (see Corollary 9.4).

Certainly \([k : m]\) is an \(U^S(k, m)\)-root, for \(\Phi^S(k, m) \in U^S(k, m)\). Since \(\psi(k - 1) - 1 = \psi(k) > m\), the point \(\psi(k - 1) - 1\) does not appear on the scheme of \(\Phi^S(k, m)\). If \(S\) is black \((k, m)\)-regular, then \(\psi(m - 1)\) is a black point, see (7.3). Hence Lemmas 9.6 and 9.7 show that in both cases \([k : m]\) is indecomposable in \(\Sigma\). Thus \([k : m]\) is a simple \(U^S(k, m)\)-root.

If \(s\) is a black point, then \([1 + s : m] \notin \Sigma\) (otherwise \([k : m]\) would be decomposable in \(\Sigma\)). In particular \([1 + s : m]\) is not a sum of \(U^S(k, m)\)-roots. By Lemma 8.8 the element \([k : s]\) is an \(U^S(k, m)\)-root (in particular Lemma 8.8 implies that \(S\) equals the minimal set \(S'\) such that \(\Phi^{S'}(k, m) \in U^S(k, m)\)). If additionally \([k : s]\) is indecomposable in \(\Sigma\), then it is a simple \(U^S(k, m)\)-root.

If \(t, s\) are, respectively, white and black points, \(k \leq t < s\), then by Lemma 8.4 we have \(\Phi^S(k, s) \in U^S(k, m)\) for a suitable (minimal) set \(S' \subseteq S\). Since \(t\) is still a white point for \(\Phi^{S'}(k, s)\), the same lemma applied to \(\Phi^{S''}(k, s)\) implies \(\Phi^{S''}(1 + t, s) \in U^S(k, m)\).
Proposition 9.9. Let \( \alpha \) be indecomposable in \( \Sigma \). Since by definition \( \Sigma \) is an additive monoid generated by elements of the form \( [1 + t : s] \) with white \( t \) and black \( s \), all indecomposable elements have that form: \( \alpha = [1 + t : s] \). If, first, \( [1 + t : s] \) satisfies property b) of Lemma 9.5 or Lemma 9.6 then \( n \not\in [t, s] \). Hence \( S'' \) (as well as any other set) is white and black \((1 + t, s)\)-regular. By Corollary 7.12 we have \( \Phi^{S''}(1 + t, s) \neq 0 \), hence \( [1 + t : s] \) is a \( U_S^S(k, m) \)-root. This is simple, for it is indecomposable in \( \Sigma \).

If, next, \( [1 + t : s] \) satisfies property a) of Lemma 9.5 or Lemma 9.6 then \( k : s \) also satisfies this property; that is, \( k : s \) is indecomposable in \( \Sigma \). In particular \( k : t \not\in \Sigma \), and hence \( k : t \) is not an \( U_S^S(k, m) \)-root. By Lemma 8.8 applied to the simple \( U_S^S(k, m) \)-root \( k : s \) we see that \( [1 + j : s] \) is a sum of \( U_S^S(k, m) \)-roots. Since \( [1 + j : s] \) is indecomposable in \( \Sigma \) and all roots belong to \( \Sigma \), the sum has just one summand; that is, \( [1 + j : s] \) is a simple \( U_S^S(k, m) \)-root.

Conversely, if \( \alpha \) is a simple \( U_S^S(k, m) \)-root, then by Corollary 9.4 we have \( \alpha \in \Sigma \). In particular \( \alpha \) is a sum of indecomposable in \( \Sigma \) elements. However we have already proved that each indecomposable in \( \Sigma \) element is an \( U_S^S(k, m) \)-root. Thus the sum has just one summand; that is, \( \alpha \) is indecomposable in \( \Sigma \).

\[ \square \]

Theorem 9.8. Let \( S \) be a white (black) \((k, m)\)-regular set. The right coideal subalgebra \( U_S^S(k, m) \) coincides with the subalgebra \( A \) generated over \( k[G] \) by all elements \( \Phi^S(1 + t, s) \) where \( t < s \) are, respectively, white and black points that satisfy one of the conditions of Lemma 9.5 (Lemma 9.6).

Proof. Of course we should show that \( \Phi^S(1 + t, s) \in U_S^S(k, m) \). To do this suppose firstly that \( s < \psi(1 + t) \). Denote by \( S' \) a minimal set such that \( \Phi^S(1 + t, s) \in U_S^S(k, m) \), see Lemmas 8.3, 9.7.

If \( a \in S \cap [1 + t, s - 1] \), then by definition \([1 + t : a] \in \Sigma \). Hence \([1 + t : a] \) is a sum of \( U_S^S(k, m) \)-roots. Lemma 8.7 applied to \([1 + t : s] \) shows that \([1 + t : a] \) itself is an \( U_S^S(k, m) \)-root (note that \( a \neq \psi(1 + t) \) since \( a < s < \psi(1 + t) \)). Thus Lemma 8.3 applied to \([1 + t : s] \) shows that \( a \in S' \); that is, \( S \cap [1 + t, s - 1] \subseteq S' \).

If \( b \in S' \), then by Lemma 8.3 applied to \([1 + t : s] \) the element \([1 + t : b] \) is an \( U_S^S(k, m) \)-root. In particular \([1 + t : b] \in \Sigma \). If \( b \notin S \), then by definition \([1 + b : s] \in \Sigma \), and we get a contradiction \([1 + t : s] = [1 + t : b] + [1 + b : s] \). Thus \( b \in S \); that is, \( S' = S \cap [1 + t, s - 1] \), and \( \Phi^S(1 + t, s) = \Phi^S(1 + t, s) \in U_S^S(k, m) \).

If \( s > \psi(1 + t) \), then by Proposition 7.10 we have \( \Phi^S(1 + t, s) \sim \Phi^T(\psi(s), \psi(1 + t)) \). Since certainly \( \psi(1 + t) < \psi(\psi(s)) \), we may apply already considered case: \( \Phi^T(\psi(s), \psi(1 + t)) \in U_T(\psi(m), \psi(k)) = U_S^S(k, m) \).

If \([a : b] \) is a non-simple \( U_S^S(k, m) \)-root, then it has a decomposition in sum of simple roots of the form \([1 + t : s] \). The element \( c \) defined in each of formuleas 8.3, 8.6, belongs to the subalgebra \( A \) generated by all \( \Phi^S(1 + t, s) \). Hence \( U_S^S(k, m) \) has PBW-generators from \( A \); that is, \( U_S^S(k, m) = A \).

The proved theorem allows one easily to find the root sequence for \( U_S^S(k, m) \) with regular \( S \). Due to Corollary 7.11 it suffices consider the case \( k \leq m < \psi(k) \) only.

Proposition 9.9. Let \( S \) be a white \((k, m)\)-regular set, \( k \leq m < \psi(k) \). The root sequence \((\theta_i, 1 \leq i \leq n) \) for \( U_S^S(k, m) \) has the following form in terms of the shifted
scheme of $\Phi^S(k, m)$:

$\theta_i = \begin{cases} 
0, & \text{if } i - 1 \text{ is not white;} \\
\psi(i) - a_i, & \text{if } i - 1 \text{ is white, } \psi(i) \text{ is black;} \\
b_i - i + 1, & \text{if } i - 1 \text{ is white, } \psi(i) \text{ is not black,}
\end{cases}

where $a_i$ is the minimal number such that $(a_i, \psi(a_i) - 1)$ is a white-white column, while $b_i$, $i \leq b_i < \psi(i)$, is the maximal black point, if any, otherwise $b_i = i - 1$ (hence $\theta_i = b_i - i + 1 = 0$).

Proof. An element $\alpha = [1 + t : s]$ given in Lemma 9.7 defines a simple $U^S(k, m)$-root starting with $i$ if either $i = 1 + t$ & $s < \psi(1 + t)$ or $s = \psi(i)$ & $s > \psi(1 + t)$.

If $i - 1$ is not a white point, then, of course, $i \neq 1 + t$, hence $s = \psi(i)$. The column $(s, i - 1) = (\psi(i), i - 1)$ is not a black-black one, for $S$ is white-regular. Therefore it is incomplete; that is, $t = i - 1$ does not appear in the scheme, a contradiction. Thus there are no simple $U^S(k, m)$-roots starting with $i$ at all, and $\theta_i = 0$.

Assume $i - 1$ is white, while $\psi(i)$ is black. In this case $[1 + n : \psi(i)]$ satisfies condition a) of Lemma 9.5. Hence $[i : n] = [\psi(n) : \psi(i)] = [1 + n : \psi(i)]$ is a simple $U^S(k, m)$-root starting with $i$. In particular $\theta_i > n - i$.

If $i = 1 + t$, $s < \psi(1 + t)$, then $[1 + t : s]$ does not satisfy condition a) of Lemma 9.5 for $\psi(1 + t) = \psi(i)$ is black. If it satisfies condition b), then the length of $[1 + t : s]$ is less than $n - i$.

If $s = \psi(i)$, $s > \psi(1 + t)$, then $[1 + t : s]$ satisfies condition a) of Lemma 9.5 if and only if $(t, \psi(t + 1))$ is a white-white column. In this case the length equals $s - (1 + t) + 1 = \psi(i) - t$. It has the maximal value if $t$ is minimal: $t = a_i$.

Assume $i - 1$ is white, while $\psi(i)$ is not black. In this case $s \neq \psi(i)$. Hence $i = 1 + t$, and $s$ is a black point such that $s < \psi(1 + t) = \psi(i)$. The length of $[1 + t : s]$ equals $s - t = s - i + 1$. It takes the maximal value if $s$ is the maximal black point such that $i \leq s < \psi(i)$; that is $s = b_i$. If all points in the interval $[i, \psi(i) - 1]$ are white, then there are no simple $U^S(k, m)$-roots starting with $i$ at all. Hence still $\theta_i = b_i - i + 1 = 0$. \hfill $\Box$

Proposition 9.10. Let $S$ be a black $(k, m)$-regular set, $k \leq m < \psi(k)$. The root sequence $(\theta_i, 1 \leq i \leq n)$ for $U^S(k, m)$ has the following form in terms of the shifted scheme of $\Phi^S(k, m)$:

$\theta_i = \begin{cases} 
0, & \text{if } i - 1 \text{ is not white, } \psi(i) \text{ is not black;} \\
\psi(i) - d_i, & \text{if } i - 1 \text{ is not white, } \psi(i) \text{ is black;} \\
\psi(i) - c_i, & \text{if } i - 1 \text{ is white,}
\end{cases}

where $c_i$ is the minimal number such that $(c_i, \psi(c_i) - 1)$ is a black-black column, while $d_i$, $i \leq d_i < \psi(i)$, is the minimal white point, if any, otherwise $d_i = \psi(i)$ (hence $\theta_i = \psi(i) - d_i = 0$).

Proof. The proof follows from Lemma 9.6 in a quite similar way like the proof of the above proposition follows from Lemma 9.5. \hfill $\Box$

Example 9.11. Consider a right coideal subalgebra $U(w)$ generated over $k[G]$ by the element $w = [(x_3, [x_3, x_2 x_1]), x_2]$ with $n = 3$ (recall that the value of $[x_3, x_2 x_1]$ in $U_q^+(so_7)$ is independent of the precise alignment of brackets, see 2.8). By definition 8.10 we have $[x_3, [x_3, x_2, x_1]] = u[3, 6]$, while Lemma 7.10 implies $[u[3, 6], x_2] \sim \Phi^{(2)}(2, 6)$. Here $\{2\}$ is a white $(2, 6)$-regular set, however $6 > \psi(2) = 5$. By Proposition 7.10 we have $\Phi^{(2)}(2, 6) \sim \Phi^{(1, 2, 3)}(1, 5)$. Since $5 > \psi(1) = 6$ and $\{1, 2, 3\}$ is a
black \((1,5)\)-regular set, to find the root sequence for \(U(w) = U^{1,2,3}(1,5)\), we may apply Proposition \(9.10\). The shifted scheme

\[
\begin{array}{ccc}
5 & 4 & 3 \\
0 & 1 & 2 & 3
\end{array}
\]

shows that \(c_1 = 1\), \(c_2 = 3\), \(c_3 = 3\), while \(d_1 = 4\), \(d_2 = 4\), \(d_3 = \psi(3) = 4\). If \(i = 1\), then \(i-1 = 0\) is a white point, and by the third option of \((9.7)\) we have \(\theta_1 = \psi(1) - c_1 = 5\). If \(i = 2\), then \(i-1 = 1\) and \(\psi(i) = 5\) are black points. Hence the second option of \((9.7)\) applies: \(\theta_2 = \psi(2) - d_2 = 5 - 4 = 1\). If \(i = 3\), then \(i-1 = 2\) is a black point, while \(\psi(i) = 4\) is a white point; that is, according to the first option of \((9.7)\) we have \(\theta_3 = 0\). Thus \(\theta(U(w)) = (5,1,0)\).

10. CONSTRUCTION

Our next goal is a construction of a right coideal subalgebra with a given root sequence

\[
(\theta) = (\theta_1, \theta_2, \ldots, \theta_n), \text{ such that } 0 \leq \theta_k \leq 2n - 2k + 1, \ 1 \leq k \leq n.
\]

We shall need the following auxiliary objects.

**Definition 10.1.** By downward induction on \(k\) we define sets \(R_k \subseteq [k, \psi(k) - 1]\), \(T_k \subseteq [k, \psi(k)]\), \(1 \leq k \leq 2n\) associated to a given sequence \((\theta)\) as follows. For \(k > n\) we put \(R_k = T_k = \emptyset\).

Suppose that \(R_i, T_i, k < i \leq 2n\) are already defined. We denote by \(P\) the following binary predicate on the set of all ordered pairs \(i \leq j\):

\[
P(i, j) \iff j \in T_i \lor \psi(i) \in T_{\psi(j)}.
\]

Of course it is defined only on pairs \((i, j)\) such that \(k < i \leq j < \psi(k)\) yet. We note that \(P(i, j) = P(\psi(j), \psi(i))\). Also it is useful to note that for given \(i, j\) one of the conditions \(j \in T_i\) or \(\psi(i) \in T_{\psi(j)}\) is false due to \(T_s \subseteq [s, \psi(s)]\), all \(s\), and \(T_s = \emptyset\) for \(s > n\) with the only exception being \(j = \psi(i)\) when these conditions coincide. In particular, we see that if \(j < \psi(i)\), then \(P(i, j)\) is equivalent to \(j \in T_i\).

If \(\theta_k = 0\), then we set \(R_k = T_k = \emptyset\). If \(\theta_k \neq 0\), then by definition \(R_k\) contains \(\hat{\theta}_k = k + \theta_k - 1\) and all \(m\) satisfying the following three properties

\[
\begin{align*}
& a) \ k \leq m < \hat{\theta}_k; \\
& b) \ \neg P(m + 1, \hat{\theta}_k); \\
& c) \ \forall r(k \leq r < m) \ P(r + 1, m) \iff P(r + 1, \hat{\theta}_k).
\end{align*}
\]

Further, we define an auxiliary set

\[
(10.4) \quad T'_k = R_k \cup \bigcup_{s \in R_k} \{ a \mid s < a < \psi(k), \ P(s + 1, a) \},
\]

and finally,

\[
(10.5) \quad T_k = \begin{cases} T'_k, & \text{if } \psi(R_k + 1) \cap T'_k = \emptyset; \\ T'_k \cup \{ \psi(k) \}, & \text{otherwise.} \end{cases}
\]

For example, the first step of the construction is as follows. If \(\theta_n = 0\), certainly we have \(R_n = T_n = \emptyset\). Since by definition \(\theta_n \leq 2n - 2n + 1 = 1\), there exists just one additional option \(\theta_n = 1\). In this case \(\hat{\theta}_n = n\), and \(R_n = \{ n \}\), while \(T'_n = R_n\). We have \(\psi(R_n + 1) \cap T'_n = \{ n \} \neq \emptyset\). Hence \(T_n = \{ n, \psi(n) \} = \{ n, n + 1 \}\).
Example 10.2. Assume $n = 3$, $\theta = (5, 1, 0)$. Since $\theta_3 = 0$, by definition $R_k = T_k = \emptyset$, $k \geq 3$.

Let $k = 2$. We have $\theta_2 = 1 \neq 0$, hence $\tilde{\theta}_2 = 2 + \theta_2 - 1 = 2 \in R_2$. Certainly there are no points $m$ that satisfy $k = 2 \leq m < \tilde{\theta}_2 = 2$; that is, $R_2 = \{2\}$. Eq. (10.4) yields

$$T'_2 = \{2\} \cup \bigcup_{s \in \{2\}} \{a \mid 2 < s < \psi(2) = 5, \ P(3,a) = \{2\} \}.  \tag{10.4}$$

We have $\psi(R_2 + 1) \cap T'_2 = \{4\} \cap \{2\} = \emptyset$, hence $T_2 = \{2\}$.

To find $R_1$ it is convenient to fix already known values of the predicate $P$ in tabulated form.

| $i \setminus j$ | 2 | 3 | 4 | 5 |
|----------------|---|---|---|---|
| 2              | $T$ | $F$ | $F$ | $F$ |
| 3              | $F$ | $F$ | $F$ |   |
| 4              |     | $F$ | $F$ |   |
| 5              |     |     |     | $T$ |

**Values of $P$**

We have $\theta_1 \neq 0$; that is, $\tilde{\theta}_1 = 1 + 5 - 1 = 5 \in R_1$.

There exist four points $m$ that satisfy $k = 1 \leq m < \tilde{\theta}_1 = 5$; they are 1, 2, 3, and 4. The point $m = 4$ does not satisfy condition b), for $P(5,5)$ is true. Hence $4 \notin R_1$. The points $m = 1, 2, 3$ satisfy condition b) since in the last column of the tableaux there is just one value “$T$”; this corresponds to $m + 1 = 5$.

Let us check condition c) for $m = 1$. The interval $1 = k \leq r < m = 1$ is empty. Therefore the equivalence c) is true (elements from the empty set satisfy all conditions, even $r \neq r$). Thus $1 \in R_1$.

Equivalence c) in terms of the tableaux of the values of $P$ means that the column corresponding to $j = m$ equals a subcolumn corresponding to $j = \tilde{\theta}_1 = 5$. This is indeed the case for $m = 3$, but not for $m = 2$. Thus $R_1 = \{1, 3, 5\}$.

To find $T'_1$ we have to check just two remaining points: $a = 2, 4$. From the tableaux we see that $P(x, 4)$ is always false, hence $4 \notin T'_1$. At the same time $P(s + 1, 2)$ is true for $s = 1 \in R_1$. Hence $2 \in T'_1$.

Finally, $\psi(R_1 + 1) \cap T'_1 = \{5, 3, 1\} \cap \{1, 2, 3, 5\} \neq \emptyset$, hence $T_1 = \{1, 2, 3, 5, 6\}$.

Thus, for $\theta = (5, 1, 0)$ we have $R_3 = T_3 = \emptyset$, $R_2 = T_2 = \{2\}$, $R_1 = \{1, 3, 5\}$, and $T_1 = \{1, 2, 3, 5, 6\}$.

Theorem 10.3. For each sequence $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$, such that

$$0 \leq \theta_k \leq 2n - 2k + 1, \quad 1 \leq k \leq n$$

there exists a homogeneous right coideal subalgebra $U \supseteq k[G]$ with $r(U) = \theta$. In what follows we shall denote it by $U_\theta$.

Proof. Denote by $U$ a subalgebra generated over $k[G]$ by values in $U_q(so_{2n+1})$ or in $u_q(so_{2n+1})$ of the following elements

$$\Phi^S(k, m), \quad 1 \leq k \leq m \in R_k, \quad S = T_k. \tag{10.6}$$

(For example, if $\theta = (5, 1, 0)$, then the generators are $x_1, x_2, [x_3x_2x_1], [[x_3[x_3x_2x_1]], x_2]$). We shall prove that $U$ is a right coideal subalgebra with $r(U) = \theta$. To attain these ends we need to check some properties of $R_k, T_k$, and $P$.

Claim 1. $P(k, m)$ is true if and only if there exists a sequence

$$k - 1 = k_0 < k_1 < \ldots < k_r < m = k_{r+1}, \tag{10.7}$$
such that for each \( i, \) \( 0 \leq i \leq r, \) either \( k_{i+1} \in R_{1+k_i} \) or \( \psi(1 + k_i) \in R_{\psi(k_i+1)}. \)

We shall use induction on \( m - k. \) If \( m = k, \) then the condition \( k \in T_k \) is equivalent to \( k \in T_k' \) for \( k \neq \psi(k). \) Eq. (10.7) implies, in turn, that \( k \in T_k' \) is equivalent to \( k \in R_k. \) Thus \( P(k, k) \) is equivalent to \( k \in R_k \lor \psi(k) \in R_{\psi(k)}; \) that is, we have sequence (10.7) with \( r = 0. \)

Assume, first, \( m \in T_k. \) If \( m \in R_k, \) we put \( k_1 = m + 1, \) \( r = 1. \)

If \( m \notin R_k, \) \( m \neq \psi(k), \) then by definition \( m \in T_k' \); that is, by (10.4) there exists \( s \in R_k, \) \( s < m \) such that \( \psi (s+1) \in R_{\psi(s+1)} \). Indeed, one may extend from the right sequence (10.7) corresponding to the pair \( (P_k, \psi(k)) \) with \( k \neq \psi(k) \). Conversely, suppose that we have sequence (10.7) with \( k_0 = s. \) One may extend this sequence from the left by \( k - 1 < k < s, \) for \( s \in R_k. \)

If \( m = \psi(k), \) then by definition \( \psi(s_1 + 1) \in T_k' \) for a suitable \( s_1 \in R_k. \) Of course, we have that \( P(k, \psi(s_1 + 1)) \) is true. Hence considered above case with \( m = \psi(s_1 + 1) \) yields sequence (10.7) with \( k_{r+1} = \psi(s_1 + 1). \) We may extend this sequence from the right by \( \psi(s_1 + 1) < \psi(k) = m \) since \( s_1 = \psi(1 + \psi(s_1 + 1)) \in R_{\psi(\psi(k))} = R_k. \)

Assume, next, \( \psi(k) \in T_{\psi(m)}. \) Since \( \psi(k) - \psi(m) = m - k, \) we may apply considered above case with \( k \leftarrow \psi(m), m \leftarrow \psi(k) \). Hence there exists sequence (10.7) with \( k_0 = \psi(m) - 1, k_{r+1} = \psi(k). \) Let us denote \( k'_i = \psi(k_i) - 1, 0 \leq i \leq r + 1. \) We have

\[
(10.8) \quad k - 1 = k'_{r+1} < k'_r < \ldots < k'_1 < k'_0 = m.
\]

In this case \( k'_i \in R_{1+k'_i+1} \) is equivalent to \( \psi(1 + k_i) \in R_{\psi(k_i+1)} \), while \( \psi(1 + k'_i+1) \in R_{\psi(k'_i)} \) is equivalent to \( k_{i+1} \in R_{1+k_i}. \)

Conversely, suppose that we have sequence (10.7). Without loss of generality we may suppose that \( m \leq \psi(k), \) otherwise we turn to (10.8). The inductive supposition implies that \( P(1 + k_1, m) \) is true. Moreover \( k_1 \in R_k. \) Indeed, otherwise we have \( \psi(k) \in R_{\psi(k_1)} \subseteq [\psi(k_1), k_1 - 1]. \) In particular \( \psi(k) < k_1, \) and hence \( k > \psi(k_1). \) However \( k_1 \leq m \leq \psi(k) \) implies \( \psi(k_1) \geq k. \) Now, if \( m \neq \psi(k), \) then definition (10.4) with \( s = k_1, a = m \) implies \( m \in T_k'. \)

Let \( m = \psi(k). \) In this case considering above sequence (10.7) we have \( k'_i \in R_k. \) By definition \( k'_i = \psi(k_i) - 1. \) Hence \( k_i \in R(k_i). \) At the same time definition (10.4) shows that \( k_r \in T_k' \) since the inductive supposition implies that \( P(k + 1, k_r) \) is true provided that \( r > 1, \) while if \( r = 1, \) then \( k_r = k_1 \in R_k. \) Thus definition (10.5) implies \( m = \psi(k) \in T_k. \)

Claim 2. If \( P(k, s) \) and \( P(s + 1, m), \) then \( P(k, m). \)

Indeed, one may extend from the right sequence (10.7) corresponding to the pair \( (k, s) \) by the sequence corresponding to \( (s + 1, m). \)

Claim 3. If \( P(k, m), \) then for each \( s, k \leq s < m \) either \( P(k, s) \) or \( P(s + 1, m). \)

We use induction on \( m - k. \) Without loss of generality we may suppose that \( m \leq \psi(k), \) for \( P(k, m) \) is equivalent to \( P(\psi(m), \psi(k)). \) By Claim 1 there exists sequence (10.7) with \( k_0 = k - 1, k_{r+1} = m. \) The same claim implies \( P(1 + k_1, m) \) provided that \( r \geq 1. \)

Since \( k \leq s < m, \) there exists \( i, 1 \leq i \leq r, \) such that \( k_i < s \leq k_{i+1}. \) If \( i \geq 1, \) then the inductive supposition applied to \( (1 + k_1, m) \) implies that either \( P(1 + k_1, s) \) or \( P(s + 1, m). \) In the latter case we have got the required condition. If \( P(1 + k_1, s) \) is true, then Claim 2 implies \( P(k, s) \), for \( P(k, k_1) \) is true according to Claim 1.
Thus, it remains to check the case $i = 0$; that is, $k \leq s \leq k_1$. In this case $k_1 \in R_k$.
Indeed, otherwise we have $\psi(k) \in R_{\psi(k_1)} \subseteq [\psi(k_1), k_1 - 1]$. In particular $\psi(k) < k_1$, and hence $k > \psi(k_1)$. However $k_1 \leq m \leq \psi(k)$ implies $\psi(k_1) \geq k$.

Claim 2 with $s \leftarrow 1 + k_1$, $k \leftarrow s + 1$ says that conditions $P(s + 1, k_1)$ and $P(1 + k_1, m)$ imply $P(s + 1, m)$ holds. Hence it is sufficient to show that either $P(k, s)$ or $P(s + 1, k_1)$ is true. If $s = k_1$, then of course $s = k_1 \in R_k$ yields $P(k, s)$. This allows us to replace $m$ with $k_1$ and suppose further that $m \in R_k$, $i = 0$. In this case condition (10.3) with $r \leftarrow s$ is “$P(s + 1, m) \iff P(s + 1, \tilde{\theta}_k)$. Therefore we have to consider only one case $m = \tilde{\theta}_k$.

Let us suppose that for some $s$, $k \leq s < \tilde{\theta}_k$ we have $\neg P(k, s)$ and $\neg P(s + 1, \tilde{\theta}_k)$.
By induction on $s$, in addition to the induction on $m - k$, we shall show that these conditions are inconsistent (more precisely they imply $s \in R_k$, which contradicts to $\neg P(k, s)$; see definition (10.4)).

Definition (10.3) with $m = k$ shows that $k \in R_k$ if and only if $\neg P(k + 1, \tilde{\theta}_k)$.
Since in our case $\neg P(s + 1, \tilde{\theta}_k)$, we have $s \in R_k$, provided that $s = k$.

Let $s > k$. Conditions (10.3) and (10.3) with $m \leftarrow s$ are valid. Suppose that (10.3) fails. In this case we may find a number $t$, $k \leq t < s$, such that $\neg(P(t + 1, s) \iff P(t + 1, \tilde{\theta}_k))$.
If $P(t + 1, s)$ but $\neg P(t + 1, \tilde{\theta}_k)$, then by the inductive supposition (induction on $s$) either $P(k, t)$ or $P(t + 1, \tilde{\theta}_k)$; that is, $P(k, t)$ is true. Claim 2 implies $P(k, s)$, a contradiction.

If $P(t + 1, \tilde{\theta}_k)$, but $\neg P(t + 1, \tilde{\theta}_k)$, then the inductive supposition of the induction on $m - k$ with $k \leftarrow t + 1, m \leftarrow \tilde{\theta}_k$ shows that either $P(t + 1, s)$ or $P(s + 1, \tilde{\theta}_k)$; that is, $P(s + 1, \tilde{\theta}_k)$, again a contradiction.

Thus $s$ satisfies all conditions (10.3), hence $s \in R_k$.

Claim 4. If $k \leq m < \tilde{\theta}_k$, then $m \in T_k$ if and only if $\neg P(m + 1, \tilde{\theta}_k)$.
First of all recall that condition $m \in T_k$ is equivalent to $P(k, m)$, for by definition $\tilde{\theta}_k < \psi(k)$.

According to Claim 3 one of the conditions $P(k, m)$ or $P(m + 1, \tilde{\theta}_k)$ always holds. If both of them are valid, then due to Claim 1 we find sequence (10.7) with $k_0 = k - 1, k_{r + 1} = m$, such that $k_{i + 1} \in R_{\psi(k_{i + 1})}, \psi(1 + k_i) \in R_{\psi(k_{i + 1})}, 0 \leq i \leq r$. Due to (10.3) we have $m \notin R_k$, and of course $\psi(k) \notin R_{\psi(m)}$, for $m \leq \tilde{\theta}_k < \psi(k)$. Hence $r > 1$.

Again by the first claim we get $P(1 + k_1, m)$. Since $k_1 \leq m < \psi(k)$, we have $\psi(k) \notin R_{\psi(k_1)}$. Hence $k_1 \in R_k$. Therefore $k_1$ satisfies condition (10.3), which is $P(1 + k_1, \tilde{\theta}_k)$. However Claim 2 shows that the conditions $P(1 + k_1, m)$ and $P(m + 1, \tilde{\theta}_k)$ imply $P(1 + k_1, \tilde{\theta}_k)$; a contradiction, that proves the claim.

Claim 5. The set $T_k$ is $(k, m)$-regular for all $m \in R_k$.
We may suppose that $k \leq n < m$ since otherwise we have nothing to prove. Assume, first, that $n$ is a white point; that is $n \notin T_k$, while scheme (11.1) has a black column, say $n - i \in T_k, n + i \in T_k, i > 0$. Condition $n + i \in T_k$ implies $P(k, n + i)$. Hence by Claim 3 with $m \leftarrow n + i, s \leftarrow n$ we have $P(k, n) \lor P(n + 1, n + i)$. However $n \notin T_k$ implies $\neg P(k, n)$, for $\psi(k) \notin T_{\psi(n)} = T_{n + 1} = \emptyset$. Hence $P(n + 1, n + i)$ is true. We have $P(n + 1, n + i) = P(\psi(n + i), \psi(n + 1)) = P(n - i + 1, n)$ is also true. Since $n - i \in T_k$ implies $P(k, n - i)$, Claim 2 with $s \leftarrow n - i, m \leftarrow n$ shows that $P(k, n)$ is true. This is a contradiction, for $n \notin T_k$ implies $\neg P(k, n)$.
Let, then, \( n \) be a black point; that is, \( n \in T_k \), while scheme (7.3) have a white column, say \( n-i \notin T_k \), \( n+i \notin T_k \), \( i > 0 \). Condition \( n-i \notin T_k \) implies \( \neg P(k, n-i) \), for \( T_{\psi(n-i)} = T_{n+i+1} = 0 \). By Claim 3 with \( m \leftarrow n \), \( s \leftarrow n-i \) we have \( P(n-i+1, n) \), for \( n \in T_k \) implies \( P(k, n) \). Hence \( P(n-i+1, n) = P(\psi(n), \psi(n-i+1)) = P(n+1, n+i) \) is true as well. At the same time Claim 4 with \( m \leftarrow n+i \) implies \( P(n+i+1, \tilde{\theta}_k) \), while Claim 2 with \( k \leftarrow n+1 \), \( s \leftarrow n+i \) implies \( P(n+1, \tilde{\theta}_k) \). Again Claim 4 with \( m \leftarrow n \) shows that \( n \notin T_k \), a contradiction.

It remains to show, next, that if \( n \in T_k \), then the first from the left complete column of (7.3) is a black one; that is \( \psi(m)-1 \notin T_k \). Let to the contrary \( \psi(m)-1 \notin T_k \). Then we have \( \neg P(k, \psi(m)-1) \), for \( T_{\psi(\psi(m)-1)} = T_{m+1} = 0 \). Claim 3 with \( s \leftarrow \psi(m)-1 \), \( m \leftarrow n \) implies \( P(\psi(m), n) \), while Claim 4 with \( m \leftarrow n \) implies \( \neg P(n+1, \tilde{\theta}_k) \). We see that the point \( r = n < m \) does not satisfy condition (10.2), for \( P(n+1, m) = P(\psi(n), m) = P(\psi(m), n) \) is true, while \( P(n+1, \tilde{\theta}_k) \) is false. Thus \( m \notin R_k \), a contradiction.

Claim 6. Let \( \tilde{U} \) be a subalgebra generated by all right coideals \( U_{T_k}(k, m) \), \( m \in R_k \). If \( 1 \leq a \leq b \leq 2n \), \( b \neq \psi(a) \), then \( P(a, b) \) is true if and only if \( [a : b] \) is an \( \tilde{U} \)-root.

In particular the set of all \( \tilde{U} \)-roots is \( \{ [k : m] \mid m \in T_k^r \} \).

Certainly \( \tilde{U} \) is a right coideal subalgebra that contains \( k[G] \). By Theorem 9.8 it is generated over \( k[G] \) by elements \( \Phi^{T_t}(1 + t, s) \), where \( t < s \) are, respectively, white and black points for \( \Phi^{T_t}(k, m) \); that is, \( t = k-1 \) or \( t \notin T_k \), and \( s = m \) or \( s \in T_k \).

In particular \( P(k, s) \) is true, while \( P(k, t) \) is false \( (\psi(k) \notin [t, \psi(t)] \supseteq T_{\psi(t)}) \), for \( k \leq t < s < \psi(k) \). Hence, by Claim 3 with \( s \leftarrow t \) we have \( P(1 + t, s) \).

If \( \gamma = [a : b] \), \( a \leq b < \psi(a) \) is an \( \tilde{U} \)-root, then, by definition, in \( \tilde{U} \) there exists a homogeneous element \( c_\gamma \in \tilde{U} \) of the form (5.4) of degree \( \gamma \). Since \( \tilde{U} \) is generated by \( \Phi^{T_t}(1 + t, s) \), the degree \( \gamma \) is a sum of degrees \( [1 + t : s] \) of the generators. In particular \( \gamma = \sum_i [a_i : b_i] \), where \( P(a_i, b_i) \) are true, and \( b_i \neq \psi(a_i) \). By Lemma 8.9 we may modify the decomposition of \( \gamma \) so that

\[
\gamma = [k_0 - 1 : k_1] + [1 + k_1 : k_2] + \ldots + [1 + k_r : k_{r+1}],
\]

where \( a - 1 = k_0 < k_1 < \ldots < k_r < b = k_{r+1} \), and for each \( i \), \( 0 \leq i \leq r \), still \( P(1 + k_i, k_{i+1}) \) is true. Now Claim 2 implies \( P(a, b) \). Hence \( b \in T_{\psi(a)} \) for \( a \leq b < \psi(a) \).

Conversely, if \( m \in T_k^r \), then by Claim 1 we have a sequence \( k - 1 = k_0 < k_1 < \ldots < k_r < b = k_{r+1} \), and for each \( i \), \( 0 \leq i \leq r \), still \( P(1 + k_i, k_{i+1}) \) is true. Now Claim 2 implies \( P(a, b) \) for \( a \leq b < \psi(a) \).

Claim 7. The set of all simple \( \tilde{U} \)-roots is \( \{ [k : m] \mid m \in R_k \} \). In particular \( r(\tilde{U}) = \emptyset \).

If \( \gamma = [k : m] \), \( k \leq m < \psi(k) \) is a simple \( \tilde{U} \)-root, then due to the above claim \( P(k, m) \) is true. Hence, according to Claim 1, we may find sequence (10.4). In this case \( \gamma = [k : k_1] + [1 + k_1 : k_2] + \ldots + [1 + k_r : m] \) is a sum of \( \tilde{U} \)-roots, for \( P(1 + k_i, k_{i+1}) \) is true by definition (10.2). Since \( \gamma \) is simple this is possible only if \( r = 0 \). Thus, \( m = k_1 \in R_k \), for \( \psi(k) \notin [m, \psi(m)] \supseteq R_{\psi(m)} \).

Conversely, let \( m \in R_k \). Then by definition (10.5) we have \( m \in T_k \). Claim 6 implies that \( [k : m] \) is an \( \tilde{U} \)-root. If it is not simple, then it is a sum of two or more
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Claim 8. \( \tilde{U} \) as an algebra is generated by \( k[G] \) and \( \Phi^T_k(k, m), m \in R_k; \) that is, \( \tilde{U} = U. \)

It suffices to note that \( U \) contains a set of PBW-generators for \( \tilde{U} \) over \( k[G]. \) If \( [k : m] \) is an \( \tilde{U} \)-root, then it is a sum of simple \( \tilde{U} \)-roots, \( [k : m] = \sum [k_i : m_i], m_i \in R_{k_i}. \) Elements \( f_i = \Phi^T_{k_i}(k_i, m_i) \) by definition belong to \( U. \) The PBW-generator corresponding to the root \( [k : m] \) can be taken to be a polynomials in \( f_i \) determined in one of the formulae \( \text{(8.3) - (8.6)} \) depending up of the type of the decomposition of \( [k : m] \) in a sum of simple roots.

Theorem 10.3 is completely proved. □

Corollary 10.4. Every (homogeneous if \( q^i = 1, t > 4 \) ) right coideal subalgebra \( U \) of \( U_q^+(\mathfrak{so}_{2n+1}), q^i \neq 1 \) (respectively of \( u_q^+(\mathfrak{so}_{2n+1}) \)) that contains \( G \) is generated as an algebra by \( G \) and a set of elements \( \Phi^S(k, m) \) with \( (k, m) \)-regular sets \( S. \)

Proof. Theorems 8.2 and 10.3 imply that \( U \) has the form \( U_\theta, \) where \( \theta \) is the root sequence. At the same time definition 10.6 shows that \( U_\theta \) as an algebra is generated by \( G \) and elements \( \Phi^T_k(k, m), m \in R_k. \) It remains to apply Claim 5. □

11. Right coideal subalgebras that do not contain the coradical

In this small section we restate the main result in a slightly more general form. More precisely we consider right coideal subalgebras in \( U_q^+(\mathfrak{so}_{2n+1}), \) (respectively in \( u_q^+(\mathfrak{so}_{2n+1}) \)) whose intersection with the coradical is a subgroup. We are reminded that for every submonoid \( \Omega \subseteq G \) the set of all linear combinations \( k[\Omega] \) is a right coideal subalgebra. Conversely if \( U_0 \subseteq k[G] \) is a right coideal subalgebra, then \( U_0 = k[\Omega] \) for \( \Omega = U_0 \cap G \) since \( a = \sum_i \alpha_i g_i \in U_0 \) implies \( \Delta(a) = \sum_j \alpha_j g_j \in U_0 \otimes k[G]; \) that is, \( \alpha_j g_j \in U_0. \)

Definition 11.1. For a sequence \( \theta = (\theta_1, \theta_2, \ldots, \theta_n) \), such that \( 0 \leq \theta_k \leq 2n-2k+1, \) \( 1 \leq k \leq n \) we denote by \( U_\theta \) a subalgebra with 1 generated by \( g_{km}^{-1}\Phi^S(k, m), \) where \( g_{km} = g(u(k, m)) \), and \( m \in R_k, S = T_k; \) see Theorem 10.3.

Lemma 11.2. The subalgebra \( U_\theta \) is a homogeneous right coideal, and \( U_\theta \cap G = \{1\}. \)

Proof. The subalgebra \( U_\theta \) is homogeneous since it is generated by homogeneous elements. Its zero homogeneous component equals \( k \) since among the generators just one, the unity, has zero degree.

Denote by \( B_\theta \) a \( k \)-subalgebra generated by \( \Phi^S(k, m) \) with \( m \in R_k, S = T_k. \) The algebra \( U_\theta \) is spanned by all elements of the form \( g_{a}^{-1}a, a \in B_\theta. \) Since \( U_\theta \) is a right coideal, for any homogeneous \( a \in B_\theta \) we have \( \Delta(a) = \sum g(a^{(2)})a^{(1)} \otimes a^{(2)} \) where \( a^{(1)} \in B_\theta, g_a = g(a^{(1)})g(a^{(2)}). \) Therefore \( \Delta(g_{a}^{-1}a) = \sum g(a^{(2)})^{-1}a^{(1)} \otimes g_a^{-1}a^{(2)} \) with \( g(a^{(1)})^{-1}a^{(1)} \in U_\theta. \) □

Lemma 11.3. If \( \Omega \) is a submonoid of \( G, \) then \( k[\Omega]U_\theta \) is a homogeneous right coideal subalgebra, and \( k[\Omega]U_\theta \cap G = \Omega. \) Moreover \( k[\Omega]U_\theta = k[\Omega']U_{\theta'} \) if and only if \( \Omega = \Omega' \) and \( \theta = \theta'. \)
Theorem 11.4. If $U_\theta$ is a homogeneous right coideal subalgebra of $U_q^+(\mathfrak{su}_{2n+1})$ such that $\Omega \subseteq U \cap G$ is a group, then $U = k[\Omega]U_\theta$ for a suitable $\theta$.

Proof. Let $u = \sum h_{ai} \in U$ be a homogeneous element of degree $\gamma \in \Gamma^+$ with different $h_i \in G$ and $a_i \in A$, where by $A$ we denote the $k$-subalgebra generated by $x_i$, $1 \leq i \leq n$. Denote by $\pi_\gamma$ the natural projection on the homogeneous component of degree $\gamma$. Respectively $\pi_\gamma$, $g \in G$ is a projection on the subspace $k\gamma$. We have $\Delta(u) = \mathcal{A}(\pi_\gamma \otimes \pi_{h_i}) = h_i a_i \otimes \pi_{h_i}$. Thus $h_{ai} \in U$.

By Theorem [10,3] and Theorem [8,2] we have $k[G]U = U_\theta$ for a suitable $\theta$. If $u = ha \in U$, $h \in G$, $a \in A$, then $\Delta(u) = \mathcal{A}(\pi_{hg} \otimes \pi_{ha}) = hg \otimes ha$. Therefore $hg \in U \cap G = \Omega$; that is, $u = \omega g_{a}^{-1}a \in \Omega$. Since $\Omega$ is a subgroup we get $g^{-1}a \in U$. It remains to note that all elements $g_{a}^{-1}a$, such that $ha \in U$ span the algebra $U_\theta$.

If $U \cap G$ is not a group, then $U$ may have a more complicated structure, see [13, Example 6.4].

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