THE HILBERT-SCHMIDT ANALYTICITY ASSOCIATED WITH INFINITE-DIMENSIONAL UNITARY GROUPS

OLEH LOPUSHANSKY

Abstract. The article is devoted to the problem of Hilbert-Schmidt type analytic extensions in Hardy spaces over the infinite-dimensional unitary matrix group endowed with an invariant probability measure. An orthogonal basis of Hilbert-Schmidt polynomials, reproducing kernels, integral formulas and boundary values are investigated.

1. Introduction

The problem of Hilbert-Schmidt type analytic extensions in Hardy spaces $H^2_\chi$ of complex functions over the infinite-dimensional unitary matrix group $U(\infty) = \bigcup \{U(m) : m \in \mathbb{N}\}$ is investigated. The group $U(\infty)$ irreducibility operates on a separable complex Hilbert space $\mathcal{E}$ and it endows with a projective limit $\chi = \lim_{m \to \infty} \chi_m$ of Haar measures $\chi_m$ on the $m$-dimensional unitary groups $U(m)$ so that $\chi$ is Radon probability. Furthermore, the measure $\chi$ is supported by an appropriate projective limit $\Omega = \lim_{m \to \infty} U(m)$ and it is invariant under a right action of $U^2(\infty) := U(\infty) \times U(\infty)$ on $U(\infty)$.

The measure $\chi$ on $\mathcal{E}$ was described by G. Olshanski and Y. Neretin. Whereas, the notion $\Omega$ relates to D. Pickrell’s space of virtual Grassmannian. Problems of the Hilbert-Schmidt infinite-dimensional analyticity were considered by T.A.W. Dwyer III in [3]. Hardy spaces of analytic complex functions were investigated in the works [2, 8, 12] and others. Spaces of integrable functions with respect to measures, invariant under infinite-dimensional groups, have been applied in the theory of determinantal point stochastic processes [1].

The paper presents following results. In Theorem 3.2, we describe an orthogonal basis in $H^2_\chi$ of power functions, indexed by means of Yang diagrams. Using this basis, in Theorem 4.2 the reproducing kernel $\pi_\chi$ is calculated. Further on, we define a dense embedding $\mathcal{I} : \Gamma \ni H^2_\chi$ of the symmetric Fock space $\Gamma$, defined via the Hilbert space $\mathcal{E}$, which equips $H^2_\chi$ by a Hilbert-Schmidt analytic infinite-dimensional complex structure. By means of $\mathcal{I}$, we establish in Theorem 6.3 an integral formula for analytic extensions of functions from $H^2_\chi$ on the open unit ball $\mathcal{B} \subset \mathbb{E}$. Weighted radial boundary values of these analytic extensions on $\mathcal{B}$ is described in Theorem 7.4.

2. Background on invariant measure

Let $U(m)$ ($m \in \mathbb{N}$) be the group of unitary ($m \times m$)-matrices with the unit $\mathbb{I}_m$. We endow the infinite-dimensional unitary groups $U(\infty)$ with the inductive topology under embeddings $U(m) \ni U(\infty)$ that to any $u_m \in U(m)$ assigns the matrix $\begin{bmatrix} u_m & 0 \\ 0 & 1 \end{bmatrix} \in U(\infty)$. Over $U(\infty)$ is defined the right action

$$u.g = w^{-1}uv, \quad u \in U(\infty), \quad g = (v, w) \in U^2(\infty)$$

(similarly, over $U(m)$ with $u \in U(m)$, $g = (v, w) \in U^2(m) := U(m) \times U(m)$). Following [9,10], every matrix $u_m \in U(m)$ with $m > 1$ we can write, as $u_m = \begin{bmatrix} z_{m-1} & a \\ b & t \end{bmatrix}$ so that $z_{m-1} \in U(m-1)$ and $t \in \mathbb{C}$.

It was proven that the Livšic-type mapping (which is not a group homomorphism)

$$\pi_{m-1}^m : u_m \mapsto u_{m-1} := \begin{cases} z_{m-1} - |a(1 + t)^{-1}b| \mathbb{I}_{m-1} : t \neq -1 \\ z_{m-1} \\ \end{cases}$$

from $U(m)$ onto $U(m-1)$ is Borel and surjective. Consider the projective limit $\Omega = \lim_{m \to \infty} U(m)$, taken with respect to $\pi_{m-1}^m$. The embedding $\rho : U(\infty) \ni \Omega$ to every $u_m \in U(m)$ assigns the stabilizing sequence.

1991 Mathematics Subject Classification. Primary 46T12; Secondary 46G20.

Key words and phrases. Measures on infinite-dimensional manifolds, orthogonal polynomials, Hardy spaces on infinite-dimensional domains, Hilbert-Schmidt analyticity.
\[ u = (u_i)_{i \in \mathbb{N}} \] (see [10] n.4) so that
\[
\rho: U(m) \ni u \mapsto u_m \in \mathcal{U}, \quad u = \begin{cases} 
\pi^m_{m-1}(u_m) & : \ i = m - 1, \\
u_m & : \ i = m, \\
u_m \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} & : \ i = m + 1.
\end{cases}
\]

Thus, the projections \( \pi_m: \mathcal{U} \ni u \mapsto u_m \in U(m) \) so that \( \pi_{m-1} = \pi^m_{m-1} \circ \pi_m \) are surjective. If \( U'(m) \subset U(m) \) is the subset of matrices which do not have \(-1\), as an eigenvalue, then \( U'(m) \) is open in \( U(m) \) and \( U(m) \setminus U'(m) \) is \( \chi_m \)-negligible. The restriction \( \pi^m_{m-1}: U'(m) \rightarrow U'(m-1) \) is continuous and surjective [10] Lem 3.11]. Let \( \mathcal{U}' = \lim U'(m) \) be taken with respect to these restrictions \( \pi^m_{m-1} |_{U'(m)} \). Using (2.1), the right action of \( U^2(\infty) \) over \( \mathcal{U} \) can be defined, as
\[
\pi_m(u,g) = w^{-1} \pi_m(u)v, \quad u \in \mathcal{U},
\]
where \( m \) is so large that \( g = (v, w) \in U^2(m) \) (see [10] Def 4.5]).

We endow every group \( U(m) \) by the probability Haar measure \( \chi_m \). As is known [9] Thm 1.6], the pullback of the measure \( \chi_{m-1} \) on \( U(m-1) \) under \( \pi^m_{m-1} \) coincides with the measure \( \chi_m \) on \( U(m) \), i.e.,
\[
\chi_{m-1} \circ \pi^m_{m-1} = \chi_m \quad \text{for all} \quad m \in \mathbb{N}.
\]

Following [10] Lem 4.8], [9] n.3.1], via of the Kolmogorov consistent theorem, we define on \( \mathcal{U} \) the probability measure \( \chi \), which is defined to be the projective limit under the mapping (2.2), i.e.,
\[
\chi = \lim \chi_m \quad \text{with} \quad \chi = \chi_m \circ \pi_m \quad \text{for all} \quad m \in \mathbb{N}.
\]

A complex function on \( \mathcal{U} \) we call cylindrical if it has the form
\[
f(u) = (f_m \circ \pi_m)(u), \quad u \in \mathcal{U}
\]
for a certain \( m \in \mathbb{N} \) and a complex function \( f_m \) on \( U(m) \) [10] Def 4.5]. By \( L^\infty_{\chi} \) we denote the closed complex linear hull of all cylindrical \( \chi \)-essentially bounded Borel functions (2.6) endowed with the norm \( \|f\|_{L^\infty_{\chi}} = \text{ess sup}_{u \in \mathcal{U}} |f(u)| \).

The measure (2.3) is probability and \( U^2(\infty) \)-invariant under the right actions (2.3) over \( \mathcal{U} \) [9] Prop 3.2]. Moreover, this measure is Radon so that
\[
\int_{\mathcal{U}} f(u,g) d\chi(u) = \int_{\mathcal{U}} f(u) d\chi(u), \quad g \in U^2(\infty), \quad f \in L^\infty_{\chi}
\]
and it satisfies the property: \( (\chi \circ \rho)(\mathcal{K}) = \chi_m(\mathcal{K}) \) for all compact set \( \mathcal{K} \) in \( U(\infty) \) such that \( \mathcal{K} \subset U(m) \) with an index \( m \in \mathbb{N} \) [7] Lem 1]. Using the invariant property (2.7) and the Fubini theorem (see [7] Lem 2), we obtain
\[
\int_{\mathcal{U}} f d\chi = \int_{\mathcal{U}} d\chi(u) \int_{U^2(m)} f(u,g) d(\chi_m \otimes \chi_m)(g),
\]

\[
\int_{\mathcal{U}} f d\chi = \frac{1}{2\pi} \int_{\mathcal{U}} d\chi(u) \int_0^{\pi} f(\exp(\imath \theta)u) d\theta.
\]

The closed complex linear hull of all \( \chi \)-square-integrable functions (2.6) endowed with the norm \( \|f\|_{L^2_{\chi}} = (\int_{\mathcal{U}} |f|^2 d\chi)^{1/2} \) is denoted by \( L^2_{\chi} \). It is clear that \( L^\infty_{\chi} \hookrightarrow L^2_{\chi} \) and
\[
\|f\|_{L^2_{\chi}} \leq \|f\|_{L^\infty_{\chi}}, \quad f \in L^\infty_{\chi}.
\]

3. Hard Spaces

Throughout \( \mathcal{E} \) is a separable complex Hilbert space with an orthonormal basis \{\( e_k: k \in \mathbb{N} \)\} and the scalar product \( \langle \cdot | \cdot \rangle \), and the norm \( \| \cdot \| = \langle \cdot | \cdot \rangle^{1/2} \). In what follows, \( \mathcal{B} = \{ x \in \mathcal{E}: \|x\| < 1 \} \) stands to the unit open ball. For any element \( x \in \mathcal{E} \) the following Fourier decomposition holds,
\[
x = \sum_{k \in \mathbb{N}} e_k x_k, \quad x_k = \langle x | e_k \rangle.
\]

Let \( \mathcal{E}^{\otimes n} \) stand to the complete \( n \)th tensor power of \( \mathcal{E} \) endowed with
\[
\langle x_1 \otimes \ldots \otimes x_n | \psi \rangle = \sum_{\tau} \langle x_1 | y_{1\tau} \rangle \ldots \langle x_n | y_{n\tau} \rangle, \quad \|\psi\| = \langle \psi | \psi \rangle^{1/2}
\]
for all $x_1 \otimes \ldots \otimes x_n$, $y_1 \otimes \ldots \otimes y_m \in E^\otimes n$ with $x_{ti}, y_{ti} \in E$ $(t = 1, \ldots, n)$ and for all finite sums $\psi = \sum y_{ti} \otimes \ldots \otimes y_{ti} \in E^\otimes n$. Put $E^\otimes 0 = \mathbb{C}$. An orthogonal basis in $E^\otimes n$ forms elements $e_{i_1} \otimes \ldots \otimes e_{i_n}$ with indices $(i) = (i_1, \ldots, i_n) \in \mathbb{N}^n$ where $i_k$ is repeated $\lambda_k$ times. Let

$$\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{N}^m, \quad m \leq n,$$

be a partition of $n \in \mathbb{N}$ such that $\lambda_k$ means multiplicity of $i_k$ in $(i_1, \ldots, i_n)$. All partitions $\lambda$ can be indexed by vectors $i = (i_1, \ldots, i_m)$ consisting of cosets $i_k = \{i_j \in (i_1, \ldots, i_n) : i_j = i_k\}$ with a representative $i_k$ of multiplicity $\lambda_k = \# i_k$. Thus, we can assume that $i = (i_1, \ldots, i_m) \in \mathbb{N}^m$, where

$$\mathbb{N}^m = \{(i_1, \ldots, i_m) \in \mathbb{N}^m : i_j \neq i_k, \forall j \neq k\}.$$

We identify partitions with Young diagrams, that is, $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{N}^m$ is a such that $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_m > 0$ and $|\lambda| = n$, where $|\lambda| := \lambda_1 + \ldots + \lambda_m$. By $\ell(\lambda) = m$ we denote the length of $\lambda$ defined as the number of rows in $\lambda$. Let $Y$ stand to all Young diagrams and $Y_n := \{\lambda \in Y : |\lambda| = n\}$. Assume that $Y$ includes the empty partition $\emptyset = (0, 0, \ldots)$. As $\sigma : \{1, \ldots, n\} \rightarrow \{\sigma(1), \ldots, \sigma(n)\}$ runs through all $n$-elements permutations, the symmetric complete $n$th tensor power $E^\otimes n$ is defined to be the codomain of the orthogonal projector

$$E^\otimes n \ni x_1 \otimes \ldots \otimes x_n \mapsto x_1 \otimes \ldots \otimes x_n := \frac{1}{n!} \sum_{\sigma} x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)} \in E^\otimes n.$$ 

Note that $x^\otimes n = x \otimes \ldots \otimes x = x \otimes \ldots \otimes x = x^\otimes n$. The symmetric Fock space is defined to be the orthogonal sum

$$\Gamma = \bigoplus_{n \in \mathbb{Z}_+} E^\otimes n, \quad \langle \psi | \phi \rangle = \sum_{n \in \mathbb{Z}_+} \langle \psi_n | \phi_n \rangle$$

for all elements $\psi = \bigoplus_n \psi_n, \phi = \bigoplus_n \phi_n \in \Gamma$ with $\psi_n, \phi_n \in E^\otimes n$.

As is well known, the following systems of symmetric tensors $e^{\otimes \lambda}_i := e^{\otimes \lambda_1}_i \otimes \ldots \otimes e^{\otimes \lambda_m}_i$, $(e^{\otimes \emptyset}_i = 1)$, indexed by diagrams $\lambda = (\lambda_1, \ldots, \lambda_m) \in Y$, $e^{\otimes \emptyset}_n = \bigcup_{\lambda \in Y_n} \{e^{\otimes \lambda}_i : i \in \mathbb{N}^m, \ m = \ell(\lambda)\}$, $e^{\otimes Y_n} := \bigcup_{n \in \mathbb{Z}_+} \{e^{\otimes \emptyset}_n : n \in \mathbb{Z}_+\}$, form orthogonal bases in $E^\otimes n$ and $\Gamma$, respectively, with the norms

$$\|e^{\otimes \lambda}_i\|^2 = \frac{\lambda!}{|\lambda|!}, \quad \lambda := \lambda_1! \cdot \ldots \cdot \lambda_m!.$$ 

Hence, any element $\psi \in \Gamma$ has the Fourier expansion

$$\psi = \sum_{(\lambda, i) \in Y \times \mathbb{N}^{(\ell(\lambda)} \hat{\psi}_{(\lambda, i)} e^{\otimes \lambda}_i, \quad \hat{\psi}_{(\lambda, i)} := \frac{|\lambda|!}{\lambda!} \langle \psi | e^{\otimes \lambda}_i \rangle.$$ 

For every $\psi \in E^\otimes n$, we can uniquely define, so-called, the Hilbert-Schmidt $n$-homogenous polynomial

$$\psi^*(x) := \langle x^\otimes n | \psi \rangle, \quad x \in E.$$ 

In fact, the polarization formula for symmetric tensor products (see [5, n.1.5])

$$z_1 \otimes \ldots \otimes z_n = \frac{1}{2^n n!} \sum_{1 \leq k \leq n} \sum_{\theta_k = \pm 1} \theta_1 \ldots \theta_n x^\otimes n, \quad x := \sum_{k=1}^{n} \theta_k z_k$$

$(z_1, \ldots, z_n \in E)$ implies that the $n$-homogenous polynomial $\langle x^\otimes n | \psi \rangle$ is uniquely defined via $\psi$, because the set $z_1 \otimes \ldots \otimes z_n$ is total in $E^\otimes n$.

Let $E_i$ be the $m$-dimensional subspace in $E$ spanned by $\{e_{i_1}, \ldots, e_{i_m}\}$ and $S_i := \{x \in E_i : \|x\| = 1\}$. The symbol $E_i^\otimes n$ (resp., $E_i^{\otimes 0}$) means the $n$th (resp., symmetric) tensor powers of $E_i$. Given an index $i \in \mathbb{N}^m$, we assign the appropriate $S_i$-valued mapping

$$\zeta_i : U \ni u \mapsto [\pi_m(u)] e_{i_1},$$

defined via the surjective projection $\pi_m : \U \ni u \mapsto \pi_m(u) \in U(m)$. Consider the corresponding system of cylindrical Borel functions

$$\varepsilon_{i_k}(u) := \langle \zeta_i(u) | e_{i_k} \rangle, \quad k = 1, \ldots, m,$$

where $\varepsilon_{i_k} := e_{i_k} \circ \zeta_i$. Using $\zeta_i$, we may define the $E_i^{\otimes 0}$-valued Borel mapping

$$\zeta_i^{\otimes 0} : U \ni u \mapsto \underbrace{\zeta_i(u) \otimes \ldots \otimes \zeta_i(u)}_n, \quad \zeta_i^{\otimes 0} = 1.$$
The following assertion is a direct consequence of [7, Lem 3].

**Lemma 3.1.** The equality $S_i = \{ \zeta_i(u) : u \in \mathcal{U} \}$ holds for all $i \in \mathbb{N}_r^m$. As a consequence, to every $\psi \in E^{\ominus_m}_i$ uniquely corresponds in $L^\infty_{\chi}$ the function

$$
\psi_\zeta(u) := \langle \zeta_i^{\ominus_m}(u) \mid \psi \rangle, \quad u \in \mathcal{U},
$$

having continuous restriction to $\mathcal{U}$. In particular, to every $\epsilon_\zeta^{\ominus_m} \in \mathfrak{c}^{\ominus_m}$ corresponds in $L^\infty_{\chi}$ the cylindrical function in the variable $u \in \mathcal{U}$,

$$
(3.5) \\
\epsilon_\zeta^{\lambda}(u) := \langle \zeta_i^{\ominus_m}(u) \mid \epsilon_\zeta^{\lambda} \rangle = \prod_{k=1}^{m} \langle \zeta_i(u) \mid \epsilon_{ik}^{\lambda} \rangle.
$$

**Lemma 3.1** straightway implies that the system of symmetric tensors $\epsilon_\zeta^{\lambda} \in \mathfrak{c}^{\ominus_m}$, indexed by diagrams $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathfrak{Y}$, uniquely defines the appropriate system

$$
\mathfrak{c}^{\lambda} := \bigcup_{\lambda \in \mathfrak{Y}} \{ \epsilon_\zeta^{\lambda} \} = \bigcup_{\lambda \in \mathfrak{Y}} \{ \epsilon_\zeta^{\lambda_1} \cdots \epsilon_\zeta^{\lambda_m} : \epsilon_\zeta^{\lambda} \in \mathfrak{c}^{\lambda} \}
$$

of $\chi$-essentially bounded cylindrical functions in the variable $u \in \mathcal{U}$, possessing continuous restrictions to $\mathcal{U}$. Evidently, $\epsilon_\zeta^{0} = 1$.

**Theorem 3.2.** For any $i \in \mathbb{N}_r^m$ and $\psi, \phi \in E^{\ominus_m}_i$, the following equality holds,

$$
(3.6) \\
\bigg( n + m - 1 igg) \int_{\mathcal{U}} \phi \bar{\psi} d\chi = \langle \psi \mid \phi \rangle.
$$

As a consequence, the system of cylindrical functions $\epsilon_\zeta^{\lambda} \in \mathfrak{c}^{\lambda}$, indexed via pairs $(\lambda, i) \in \mathfrak{Y} \times \mathbb{N}_r^m$ with $m = \ell(\lambda)$, is orthogonal in the space $L^2_{\chi}$ and

$$
(3.7) \\
\| \epsilon_\zeta^{\lambda} \|_{L^2_{\chi}} = \left( \frac{(m-1)!}{(m - 1 + |\lambda|)!} \right)^{1/2}.
$$

**Proof.** Briefly denote $\psi[v \circ \pi_m(u)] := \langle \{ [v \pi_m(u)] \epsilon_{1i} \}^{\ominus_m} \mid \psi \rangle$ with $\psi \in E^{\ominus_m}_i$ for all $v \in U(m)$ and $u \in \mathcal{U}$. Using (2.8), we have

$$
(3.8) \\
\int_{\mathcal{U}} \phi \bar{\psi} d\chi = \int_{U(m)} d\chi(u) \int_{U(m)} \phi[v \circ \pi_m(u)] \bar{\psi}[v \circ \pi_m(u)] d\chi_m(v)
$$

for all $\psi, \phi \in E^{\ominus_m}_i$, where the interior integral with the Haar measures $\chi_m$ are independent of $\pi_m(u) \in U(m)$. It is clear that

$$
\bigg| \int_{U(m)} \phi \bar{\psi} d\chi_m \bigg| \leq \sup_{v \in U(m)} \big| \psi[v \circ \pi_m(u)] \big| \leq \| \phi \| \| \psi \|
$$

for all $u \in \mathcal{U}$. Hence, the corresponding sesquilinear form in (3.8) is continuous on $E^{\ominus_m}_i$. Thus, there exists a linear bounded operator $A$ over $E^{\ominus_m}_i$ such that

$$
\langle A \psi \mid \phi \rangle = \int_{U(m)} \phi \bar{\psi} d\chi_m, \quad \psi, \phi \in E^{\ominus_m}_i.
$$

Show that $A$ commutes with all operators $\{ w^{\ominus_m} \in \mathcal{L}(E^{\ominus_m}_i) : w \in U(m) \}$ acting as $w^{\ominus_m}x^{\ominus_m} = (wx)^{\ominus_m}$, $(x \in E_i)$. Invariant properties (2.7) of $\chi_m$ under the right action (2.3) yields

$$
\langle (A \circ w^{\ominus_m}) \psi \mid \phi \rangle = \int_{U(m)} \langle (w^{\ominus_m})^{\ominus_m} \mid \phi \rangle \langle (w^{\ominus_m} \psi)^{\ominus_m} \mid (w^{\ominus_m} \phi) \rangle d\chi_m(v)
$$

$$
= \int_{U(m)} \langle (w^{-1}v^{\ominus_m})^{\ominus_m} \mid (w^{-1})^{\ominus_m} \phi \rangle \langle (w^{-1}v^{\ominus_m} \psi)^{\ominus_m} \mid (w^{-1}v^{\ominus_m} \phi) \rangle d\chi_m(v)
$$

$$
= \int_{U(m)} \langle (v^{\ominus_m})^{\ominus_m} \mid (w^{-1})^{\ominus_m} \phi \rangle \langle (v^{\ominus_m} \psi)^{\ominus_m} \mid (w^{-1}v^{\ominus_m} \phi) \rangle d\chi_m(v)
$$

$$
= \langle A \psi \mid (w^{-1})^{\ominus_m} \phi \rangle = \langle (w^{\ominus_m} \circ A) \psi \mid \phi \rangle,
$$

where $w^{-1}$ coincides with the hermitian adjoint matrix of $w$. Hence, the equality

$$
(3.9) \\
A \circ w^{\ominus_m} = w^{\ominus_m} \circ A, \quad w \in U(m)
$$
In particular, the subsystem of cylindrical functions $\varphi$ since $\lambda$ we obtain that the operator with all matrices of an irreducible representation is a constant multiple of the unit matrix. As a result, (3.13)

\[ H \] (3.12)

is irreducible. This means that there is no subspace in $E_n^\otimes n$ other than $\{0\}$ and the whole space which is invariant under the action of $[U(m)]^\otimes n$.

Let us suppose, on the contrary, that there is $\psi \in E_n^\otimes n$ such that the equality $\langle w, \psi \rangle = 0$ with $w \in U(m)$. Then the group $U(m)$ acts surjectively on $E_n$. Hence, by general homogeneity, we obtain $\langle x, \psi \rangle = 0$ for all elements $x \in E_n$. Applying the polarization formula (3.4), we get $\psi = 0$. Hence, (3.10) is irreducible.

Thus, we can apply to (3.10) the Schur Lemma [5, Thm 21.30]: a non-zero matrix which commutes with all matrices of an irreducible representation is a constant multiple of the unit matrix. As a result, we obtain that the operator $A$, satisfying (3.9), is proportional to the identity operator on $E_n^\otimes n$. Clearly, $A = \alpha_{(\lambda, t)} \cdot I_{E_n^\otimes n}$ with a constant $\alpha_{(\lambda, t)} > 0$. It follows

\[ \int_{U(m)} \phi \, \tilde{\psi}_t \, d\chi_m = \alpha_{(\lambda, t)} \langle \psi \mid \phi \rangle, \quad \phi, \psi \in E_n^\otimes n. \]

In particular, the subsystem of cylindrical functions $\varphi$ with a fixed $t \in N^m$ is orthogonal in $L_n^2$, since the corresponding system of tensors $\varphi^\otimes \lambda$ indexed by $\lambda \in \Psi_n$ with $\ell(\lambda) = m$ forms an orthogonal basis in $E_n^\otimes n$.

Remains to note that the set of all indices $i = (i_1, \ldots, i_m) \in N^m$ with arbitrary $m = \ell(\lambda)$ of the system $\varphi$ is directed regarding the set-theoretic embedding, i.e., for any $i, i'$ there exists $i''$ so that $i \cup i' \subset i''$. This fact and the above reasoning imply that the whole system $\varphi$ is also orthogonal in $L_n^2$.

Taking into account (3.2), we can choose $\phi_n = \psi_n = \varphi^\otimes \lambda \sqrt{n! / \lambda}$ in (3.11). As a result, we obtain

\[ \alpha_{(\lambda, t)} = \frac{n!}{\lambda!} \int_{U(m)} |\varphi^\otimes \lambda|^2 \, d\chi_m = \frac{n!}{\lambda!} \|\varphi^\otimes \lambda\|^2_{L_n^2}. \]

The well known formula [13, n.1.4.9] for the unitary $m$-dimensional group gives

\[ \int_{U(m)} |\varphi^\otimes \lambda|^2 \, d\chi_m = \frac{\lambda!(m - 1)!}{(n + m - 1)!}, \quad |\lambda| = n, \quad \ell(\lambda) = m. \]

Using the last two formulas, we have

\[ \alpha_{(\lambda, t)} = \frac{n!}{\lambda!} \int_{U(m)} |\varphi^\otimes \lambda|^2 \, d\chi_m = \frac{n!}{\lambda!} \frac{\lambda!(m - 1)!}{(n + m - 1)!} = \frac{n!(m - 1)!}{(n + m - 1)!}. \]

Combining (3.8) and (3.12), we get (3.10) and, as a consequence, (3.11).

**Definition 3.3.** By $H_n^2$ we denote the Hardy space over $U(\infty)$ defined as the $L_n^2$-closure of the complex linear span of the orthogonal system $\varphi$.

Let the space $H_n^2$ be the $L_n^2$-closure of the complex linear span of the orthogonal subsystem $\varphi_n := \{ \varphi^\otimes \lambda : (\lambda, t) \in \Psi_n \times N(\lambda) \}$ with a fixed $n \in \mathbb{Z}_+$.

**Corollary 3.4.** For any positive integers $n \neq k$ the orthogonality $H_n^2 \perp H_k^2 \subset L_n^2$ holds. As a consequence, the following orthogonal decomposition holds,

\[ H_n^2 = \mathbb{C} \oplus H_n^{\perp 1} \oplus H_n^{\perp 2} \oplus \ldots. \]

**Proof.** The orthogonal property $\varphi^\mu \perp \varphi^\lambda$ with $|\mu| \neq |\lambda|$ for any $\varphi \in N^\ell(\lambda)$ and $\lambda \in N^\ell(\mu)$ follows from (2.8), since

\[ \int \varphi^\mu \cdot \varphi^\lambda \, d\chi = \int \varphi^\mu \cdot \varphi^\lambda \left( e^{i|\mu|} u \cdot e^{i|\lambda|} \right) \, d\chi(u) \]

\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi^\mu \cdot \varphi^\lambda \, d\chi \int_{-\pi}^{\pi} \exp \left( i (|\mu| - |\lambda|) |\varphi| \right) \, d\varphi = 0 \]

for all $\lambda \in \Psi_n \times N(\lambda)$ and $\mu \in N^\ell(\mu)$ follows from (2.8). This yields $H_n^{\perp |\mu|} \perp H_n^{\perp |\lambda|}$ in the space $L_n^2$. 

\[ \square \]
4. Reproducing kernels

Let us construct the reproducing kernel of $H^2_\chi$. We refer to [14] regarding the reproducing kernels.

Lemma 4.1. The reproducing kernel of $H^2_{\chi}^n$, endowed with the $L^2_\chi$-norm, is

$$
\hat{h}_m(v, u) = \sum_{(m, n) \in \mathbb{N} \times \mathbb{N}^n} \left( \frac{n + m - 1}{n} \right) \langle \zeta_i(v) \mid \zeta_i(u) \rangle^n
= \sum_{(\lambda, \mu) \in \mathbb{Y}_n \times \mathbb{N}^n} \frac{\varepsilon^\lambda_i(v) \varepsilon^\mu_i(u)}{\| \varepsilon^\lambda_i \|^2_{L^2_\chi}}
$$

for all $v, u \in \mathcal{U}$ and $m = \ell(\lambda)$.

Proof. Note that $\hat{h}_0 \equiv 1$. Let us decompose any element $\zeta_i(u) \in \mathcal{S}_i$ in the Fourier sum $\zeta_i(u) = \sum_{k=1}^m \varepsilon_{i_k}(\zeta_i(u) \mid \varepsilon_{i_k})$. The tensor multinomial theorem yields the following Fourier decomposition in terms of the orthogonal basis in $\mathcal{F}_i^{\otimes n}$,

$$
\langle \zeta_i(u) \rangle^{\otimes n} = \left( \sum_{k=1}^m \varepsilon_{i_k}(\zeta_i(u) \mid \varepsilon_{i_k}) \right)^{\otimes n} = \sum_{\lambda \in \mathbb{Y}_n} \frac{n!}{\lambda!} \varepsilon_i^{\otimes \lambda} \varepsilon_i^\lambda(u).
$$

Using the formula (3.2), we obtain

$$
\langle \zeta_i(v) \mid \zeta_i(u) \rangle^n = \langle \langle \zeta_i(v) \rangle^{\otimes n} \mid \langle \zeta_i(u) \rangle^{\otimes n} \rangle
= \sum_{\lambda \in \mathbb{Y}_n} \left( \frac{n!}{\lambda!} \right)^2 \langle \varepsilon_i^{\otimes \lambda} \mid \varepsilon_i^\lambda \rangle \varepsilon_i^\lambda(v) \varepsilon_i^\lambda(u)
= \sum_{\lambda \in \mathbb{Y}_n} \frac{n!}{\lambda!} \varepsilon_i^\lambda(v) \varepsilon_i^\lambda(u).
$$

Multiplying the both sides by $\left( \frac{n + m - 1}{n} \right)$ and summing over $(m, n) \in \mathbb{N} \times \mathbb{N}^n$ with a fixed $n \in \mathbb{N}$, we get (4.1).

Via Theorem 3.1 the system $\varepsilon_i^{\otimes n}$ forms an orthogonal basis in $H^2_{\chi}$. Hence, applying (4.1), we have $\int_{\mathcal{U}} \hat{h}_m(v, u) \varepsilon_i^\lambda(u) \, d \chi(u) = \varepsilon_i^\lambda(v)$ for all $v \in \mathcal{U}$, i.e., the kernel (4.1) operates as the identity mapping in $H^2_{\chi}$.

Consequently, the equality

$$
\int_{\mathcal{U}} \hat{h}_m(v, u) \psi_i(v) \, d \chi(u) = \psi_i(v), \quad \psi_i \in H^2_{\chi}
$$

holds. Thus, the kernel (4.1) is reproducing in $H^2_{\chi}$.

Let us consider the following complex-valued kernel

$$
\hat{h}(v, u) = 1 + \sum_{(m, n) \in \mathbb{N} \times \mathbb{N}^n} \left[ 1 - \langle \zeta_i(v) \mid \zeta_i(u) \rangle \right]^{-m}, \quad v, u \in \mathcal{U}.
$$

Theorem 4.2. The kernel $\hat{h}$ is reproducing in the Hardy space $H^2_{\chi}$ and the following decomposition holds,

$$
\hat{h}(v, u) = \sum_{n \in \mathbb{Z}_+} \hat{h}_n(v, u), \quad u, v \in \mathcal{U}.
$$

Proof. As is well known (see e.g. [13] n.1.4.10), for any $m \in \mathbb{N}$

$$
\left[ 1 - \langle \zeta_i(v) \mid \zeta_i(u) \rangle \right]^{-m} = \sum_{n \in \mathbb{Z}_+} \left( \frac{n + m - 1}{n} \right) \langle \zeta_i(v) \mid \zeta_i(u) \rangle^n.
$$

Using this equality and summing (4.1) over $n \in \mathbb{Z}_+$, we obtain

$$
\hat{h}(v, u) = \sum_{n \in \mathbb{Z}_+} \sum_{(\lambda, \mu) \in \mathbb{Y}_n \times \mathbb{N}^n} \frac{\varepsilon^\lambda(v) \varepsilon^\mu(u)}{\| \varepsilon^\lambda \|^2_{L^2_\chi}} = \sum_{n \in \mathbb{Z}_+} \hat{h}_n(v, u),
$$

where $m := \ell(\lambda)$. Hence, (4.3) holds. Every element $f \in H^2_{\chi}$ decomposes to the $L^2_\chi$-convergent orthogonal series $f = \sum_{n} f_n$, where $f_n \in H^2_{\chi}$ is the orthogonal projection of $f$ in the decomposition (4.3).

Observing that the $L^2_\chi$-orthogonal property $\hat{h}_k(\cdot, u) \perp f_n(\cdot)$ with $n \neq k$ holds by virtue of Corollary 3.2, we obtain

$$
\int_{\mathcal{U}} \hat{h}(v, u) f(u) \, d \chi(u) = \sum_{n} \int_{\mathcal{U}} \hat{h}_n(v, u) f_n(u) \, d \chi(u) = \sum_{n} f_n(v) = f(v)
$$

for all $v \in \mathcal{U}$ and $f \in H^2_{\chi}$. Hence, the kernel (4.3) is reproducing in $H^2_{\chi}$. 

\[\square\]
5. The Hilbert-Schmidt analyticity

Recall (see e.g. [4]) that a function defined on an open domain in a normed space is Gâteaux analytic if its restrictions to all finite-dimensional affine subsets are analytic. If a Gâteaux analytic function is norm continuous then it calls analytic. Following [3], it may be said that a function is Hilbert-Schmidt analytic if its Taylor coefficients are Hilbert-Schmidt polynomials.

First note that the subset \( \{ x^{\otimes n} : x \in \mathbb{E} \} \) is total in \( \mathbb{E}^{\otimes n} \) by virtue of (3.4). This provides the total property of the subsets \( \{ (1 - x)^{-\otimes 1} : x \in \mathbb{B} \} \) in \( \Gamma \), where is denoted

\[
(1 - x)^{-\otimes 1} := \sum_{n \in \mathbb{Z}_+} x^{\otimes n}, \quad x^{\otimes 0} = 1.
\]

Elements \( \{ (1 - x)^{-\otimes 1} : x \in \mathbb{B} \} \) is called geometrical vectors in \( \Gamma \). It is clear that the \( \Gamma \)-valued function \( (1 - x)^{-\otimes 1} \) in the variable \( x \in \mathbb{B} \) is analytic, since

\[
\| (1 - x)^{-\otimes 1} \|^2 = \sum_{n \in \mathbb{Z}_+} \| x \|^2 = \frac{1}{1 - \| x \|^2} < \infty.
\]

Let us define the Hilbert space of analytic complex functions in the variable \( x \in \mathbb{B} \), associated with the Fock space \( \Gamma \), as follows

\[
H^2 = \{ \psi^*(x) = \langle (1 - x)^{-\otimes 1} \mid \psi \rangle : \psi \in \Gamma \}, \quad \| \psi^* \|_{H^2} := \| \psi \|
\]

for all \( x \in \mathbb{B} \). This description is correct, because the function \( \psi^* \) in the variable \( x \in \mathbb{B} \) is analytic in the Hilbert-Schmidt sense, as a composition of the analytic \( \Gamma \)-valued function \( (1 - x)^{-\otimes 1} \) in the variable \( x \in \mathbb{B} \) with the linear functional \( \langle \cdot \mid \psi \rangle \). Similarly, we define its closed subspace of \( n \)-homogenous Hilbert-Schmidt polynomials in the variable \( x \in \mathbb{E} \), as follows

\[
H^2_n = \{ \psi^*_n(x) = \langle x^{\otimes n} \mid \psi_n \rangle : \psi_n \in \mathbb{E}^{\otimes n} \}.
\]

Note that the spaces \( \Gamma \) and \( \mathbb{E}^{\otimes n} \) are conjugately isomorphic to \( H^2 \) and \( H^2_n \), respectively. Let us denote the appropriate anti-linear isometries, as

\[
A : \Gamma \ni \psi \rightarrow \psi^* \in H^2, \quad A : \mathbb{E}^{\otimes n} \ni \psi_n \rightarrow \psi^*_n \in H^2_n.
\]

Clearly, the following orthogonal decomposition holds,

\[
H^2 = \mathbb{C} \oplus H^2_1 \oplus H^2_2 \oplus \ldots .
\]

Using the polarization formula (3.4), to any indices \( \lambda \in \mathbb{Y}_n \) and \( i \in \mathbb{N}_{-}(\lambda) \), we can uniquely assign the Hilbert-Schmidt \( n \)-homogenous polynomial in \( H^2 \), defined via the Fourier coefficients \( x_k := e_k^*(x) = \langle x \mid e_k \rangle \) of an element \( x \in \mathbb{E} \), as

\[
x_k^\lambda := \langle x^{\otimes n} \mid e_i^\otimes \lambda \rangle, \quad \| x_k^\lambda \|_{H^2} := \| e_i^\otimes \lambda \|.
\]

Evidently, \( x_k^0 \equiv 1 \). The systems of Hilbert-Schmidt polynomials

\[
x^{\mathbb{Y}_n} := \bigcup_{\lambda \in \mathbb{Y}_n} \{ x_k^\lambda : i \in \mathbb{N}_{-}(\lambda) \}, \quad x^{\mathbb{Y}} := \bigcup \{ x^{\mathbb{Y}_n} : n \in \mathbb{Z}_+ \}
\]

form orthogonal bases in the spaces \( H^2_n \) and \( H^2 \), respectively.

Taking into account (3.2), the tensor multinomial theorem yields the following Fourier decompositions with respect to the basis \( e_i^\otimes \lambda^\otimes \mathbb{Y} \) in \( \Gamma \),

\[
(1 - x)^{-\otimes 1} = \sum_{(\lambda, i) \in \mathbb{Y} \times \mathbb{N}_{-}(\lambda)} x_k^\lambda \frac{e_i^\otimes \lambda}{\| e_i^\otimes \lambda \|^2}, \quad x \in \mathbb{B}.
\]

Hence, any function \( \psi^* \in H^2 \) has the Taylor expansion on \( \mathbb{B} \),

\[
\psi^*(x) = \langle (1 - x)^{-\otimes 1} \mid \psi \rangle = \sum_{(\lambda, i) \in \mathbb{Y} \times \mathbb{N}_{-}(\lambda)} \hat{\psi}_{(\lambda, i)} x_k^\lambda,
\]

where \( \hat{\psi}_{(\lambda, i)} \) is defined in (3.3) as the Fourier coefficient of \( \psi \in \Gamma \) with respect to the orthogonal basis \( e_i^\otimes \lambda^\otimes \mathbb{Y} \) in \( \Gamma \).
6. Integral Formulas

Consider the dense embedding $\mathcal{J}$ and its adjoint mapping $\mathcal{J}^*$, acted as

$$\mathcal{J}: \Gamma \ni f \mapsto H_2^\chi, \quad \mathcal{J}^*: H_2^\chi \to \Gamma,$$

respectively, that are uniquely defined by change of orthogonal bases

$$\mathcal{J}: \Gamma \ni e_i^{\odot \lambda} \mapsto e_i^{\lambda} \in H_2^\chi, \quad \lambda \in \mathcal{Y}_\lambda, \quad i \in \mathbb{N}_e(\lambda)$$

while keeping an appropriate orthogonal decomposition. Namely, we assign to every $\psi \in \Gamma$ the function $\mathcal{J}\psi \in H_2^\chi$ in such way that

$$\sum_{(\lambda, i) \in \mathcal{Y}_\lambda \times \mathbb{N}_e(\lambda)} \hat{\psi}(\lambda, i)e_i^{\odot \lambda} = \psi \mapsto \mathcal{J}\psi = \sum_{(\lambda, i) \in \mathcal{Y}_\lambda \times \mathbb{N}_e(\lambda)} \hat{\psi}(\lambda, i)e_i^{\lambda},$$

where in the left side is the Fourier decomposition of $\psi \in \Gamma$. Note that the equality $\langle \mathcal{J}e_i^{\odot \lambda} | f \rangle_{L_2^\chi} = \langle e_i^{\odot \lambda} | \mathcal{J}^* f \rangle$ with $f \in H_2^\chi$ implies that

$$\mathcal{J}^*: e_i^{\lambda} \mapsto e_i^{\odot \lambda} \|e_i^{\lambda}\|^2 \|e_i^{\odot \lambda}\|^{-2}.$$

**Lemma 6.1.** The operators $\mathcal{J}$ and $\mathcal{J}^*$ are contractive, i.e.,

$$\|\mathcal{J}\psi\|_{L_2^\chi} \leq \|\psi\| \quad (\psi \in \Gamma) \quad \text{and} \quad \|\mathcal{J}^* f\| \leq \|f\|_{L_2^\chi} \quad (f \in H_2^\chi).$$

**Proof.** The equalities (6.2) and (6.7) immediately yield

$$\|\mathcal{J}\psi\|_{L_2^\chi}^2 = \sum_{(\lambda, i) \in \mathcal{Y}_\lambda \times \mathbb{N}_e(\lambda)} |\hat{\psi}(\lambda, i)|^2 \|e_i^{\lambda}\|^2 \leq \sum_{(\lambda, i) \in \mathcal{Y}_\lambda \times \mathbb{N}_e(\lambda)} |\hat{\psi}(\lambda, i)|^2 \|e_i^{\odot \lambda}\|^2 = \|\psi\|^2$$

for all $(\lambda, i) \in \mathcal{Y}_\lambda \times \mathbb{N}_e(\lambda)$. By Theorem 3.2 the series (6.1) are orthogonal, so

$$\|\mathcal{J}\psi\|_{L_2^\chi}^2 = \sum_{(\lambda, i) \in \mathcal{Y}_\lambda \times \mathbb{N}_e(\lambda)} |\hat{\psi}(\lambda, i)|^2 \|e_i^{\lambda}\|^2 \leq \sum_{(\lambda, i) \in \mathcal{Y}_\lambda \times \mathbb{N}_e(\lambda)} |\hat{\psi}(\lambda, i)|^2 \|e_i^{\odot \lambda}\|^2 = \|\psi\|^2$$

in virtue of (6.3). For the adjoint operator $\mathcal{J}^*$ it is the same. \[\square\]

In what follows, we assign to any $x \in B$ the vector-valued functions

$$x_\mathcal{J}: \Omega \ni u \mapsto (\mathcal{J}x)(u), \quad \mathcal{J}(1 - x)^{-\odot 1} = (1 - x_\mathcal{J})^{-1}.$$

**Lemma 6.2.** The functions $x_\mathcal{J}$ and $(1 - x_\mathcal{J})^{-1}$ have values in $L_2^\chi$.

**Proof.** The first assertion is a consequence of (6.3). Applying $\mathcal{J}$ to the Fourier decompositions (3.1) and (5.4), we obtain

$$\mathcal{J}(1 - x)^{-\odot 1} = \sum_{(\lambda, i) \in \mathcal{Y}_\lambda \times \mathbb{N}_e(\lambda)} \|e_i^{\lambda}\|^2 = \sum_{n \in \mathbb{Z}_+} \left( \sum_{m \in \mathbb{N}} x_m e_m \right)^n \leq (1 - \|x\|^2)^{-1}.$$

Taking into account (6.2), we have

$$\|\mathcal{J}(1 - x)^{-\odot 1}\|_{L_2^\chi}^2 = \sum_{n \in \mathbb{Z}_+} \sum_{\lambda \in \mathcal{Y}_\lambda, i \in \mathbb{N}_e(\lambda)} \|x_i^{\lambda}\|^2 \|e_i^{\odot \lambda}\|^2 = \sum_{n \in \mathbb{Z}_+} \left( \sum_{m \in \mathbb{N}} |x_m|^2 \right)^n \leq (1 - \|x\|^2)^{-1}.$$

Hence, the function $(1 - x_\mathcal{J})^{-1}$ with $x \in B$ has values in $L_2^\chi$. \[\square\]

In particular, the following series with a fixed $n \in \mathbb{N}$,

$$x_\mathcal{J}^n = \left( \sum_{m \in \mathbb{N}} x_m e_m \right)^n = \sum_{(\lambda, i) \in \mathcal{Y}_\lambda \times \mathbb{N}_e(\lambda)} \|e_i^{\lambda}\|^2,$$

is absolutely convergent in $L_2^\chi$ for all $x \in E$.\[\square\]
**Theorem 6.3.** For any \( f = \sum_n f_n \in H^2_\chi \) with \( f_n \in H^{2,n}_\chi \) the complex function
\[
(\mathcal{J}^* f)^*(x) = \langle (1 - x)^{-\odot 1} | \mathcal{J}^* f \rangle
\]
so that \( (\mathcal{J}^* f_n)^*(x) = \langle x^{\odot n} | \mathcal{J}^* f_n \rangle \) belongs to the Hilbert space \( H^2 \) on \( \mathcal{B} \) and it has the integral representation
\[
(\mathcal{J}^* f)^*(x) = \int_{\mathcal{B}} f \frac{d\chi}{1 - x_r^3}
\]
with the Taylor coefficients at the origin,
\[
\frac{d^n}{dx^n} (\mathcal{J}^* f)^*(x) = \int_{\mathcal{B}} x^n f_n d\chi.
\]

*Proof.* Consider the Fourier decomposition with respect to the basis \( \varepsilon^\lambda \),
\[
f = \sum_{(\lambda,i) \in \mathcal{Y} \times N_i} \hat{f}(\lambda,i) \varepsilon^\lambda_i, \quad \hat{f}(\lambda,i) = \frac{1}{\|\varepsilon^\lambda_i\|^2} \int_{\mathcal{B}} f \varepsilon^\lambda_i d\chi,
\]
where \( \hat{f}(\lambda,i) \) are Fourier coefficients. Substituting coefficients to \( (\mathcal{J}^* f)^* \), as well as, using the orthogonality together with (5.4) and (6.4), we have
\[
(\mathcal{J}^* f)^*(x) = \sum_{(\lambda,i) \in \mathcal{Y} \times N_i} \hat{f}(\lambda,i)^* \varepsilon^\lambda_i x_i \|\varepsilon^\lambda_i\| L^2_\chi = \int_{\mathcal{B}} \sum_{(\lambda,i) \in \mathcal{Y} \times N_i} \frac{x_i^\lambda \varepsilon^\lambda_i}{\|\varepsilon^\lambda_i\|^2} \hat{f} d\chi = \int_{\mathcal{B}} \frac{\hat{f} d\chi}{1 - x_r^3}.
\]

Similarly, using (6.5), we obtain
\[
(\mathcal{J}^* f_n)^*(x) = \langle x^{\odot n} | \mathcal{J}^* f_n \rangle = \int_{\mathcal{B}} x^n f_n d\chi.
\]
Hence, (6.6) holds. Taking into account (6.6) and (6.8), we have
\[
(\mathcal{J}^* f)^*(x) = \int_{\mathcal{B}} f \frac{d\chi}{1 - x_r^3} = \sum_{n \in \mathbb{Z}_+} \alpha^n \int_{\mathcal{B}} x^n f_n d\chi, \quad |\alpha| \leq 1.
\]
The \( \Gamma \)-valued function \( x \mapsto (1 - x)^{-\odot 1} \) is analytic on \( \mathcal{B} \) and \( (\mathcal{J}^* f)^* \) is the composition of \( (1 - x)^{-\odot 1} \) with the linear functional \( \langle \cdot | \mathcal{J}^* f \rangle \) on \( \Gamma \). So, \( (\mathcal{J}^* f)^* \) is analytic on \( \mathcal{B} \). Differentiating \( (\mathcal{J}^* f)^* \) at \( x = 0 \) and using the \( n \)-homogeneity of derivatives, we obtain
\[
\frac{d^n}{d\alpha^n} \sum_{n \in \mathbb{Z}_+} \alpha^n \int_{\mathcal{B}} x^n f_n d\chi \bigg|_{\alpha = 0} = n! \int_{\mathcal{B}} x^n f_n d\chi.
\]
Hence, the functions (6.7) coincide with the Taylor coefficients of the analytic function \( (\mathcal{J}^* f)^* \) at the origin, that are uniquely defined on \( \mathcal{B} \).

\[\square\]

7. Weighted radial boundary values

Let us consider the functions \( f_r := \sum_n r^n f_n \in H^2_\chi \) for any \( f = \sum_n f_n \in H^2_\chi \) with \( f_n \in H^{2,n}_\chi \) and \( r \in [0,1] \). In accordance with Theorem 6.3, every function \( (\mathcal{J}^* f_r)^* \in H^2_\chi \) has the following analytic extension on \( \mathcal{B} \),
\[
(\mathcal{J}^* f_r)^*(x) = \int_{\mathcal{B}} f_r \frac{d\chi}{1 - x_r^3} = \int_{\mathcal{B}} f \frac{d\chi}{1 - r x_r^3} = (\mathcal{J}^* f)^*(rx), \quad x \in \mathcal{B}.
\]
Taking into account the anti-linear isometries (5.2), we obtain
\[
(\mathcal{J} \circ \mathcal{A}^{-1})(\mathcal{J}^* f_r)^* = (\mathcal{J} \circ \mathcal{J}^*) f_r \in H^2_\chi, \quad (\mathcal{J} \circ \mathcal{A}^{-1})(\mathcal{J}^* f_n)^* = (\mathcal{J} \circ \mathcal{J}^*) f_n \in H^{2,n}_\chi.
\]

**Theorem 7.1.** The \( (\mathcal{J} \circ \mathcal{A}^{-1}) \)-weighted radial boundary values of the analytic function \( (\mathcal{J}^* f_r)^* \in H^2_\chi \) coincide with \( f \in H^2_\chi \) in the following sense:
\[
\lim_{r \searrow 1} \| (\mathcal{J} \circ \mathcal{J}^*)(f_r - f) \|_{L^2_\chi} = 0.
\]

Moreover, the following equality holds,
\[
\| (\mathcal{J} \circ \mathcal{J}^*) f \|_{L^2_\chi} = \sup_{r \in (0,1)} \| (\mathcal{J} \circ \mathcal{J}^*) f_r \|_{L^2_\chi}.
\]
Proof. Lemma 6.1 implies that the operator $J \circ J^*$ is contractive. Moreover, $f_k \perp f_n$ as $n \neq k$ in $H^2_\chi$ by decomposition \(3.13\). It follows that

$$\| (J \circ J^*)(f_r - f) \|_{L^2_\chi} \leq \| f_r - f \|_{L^2_\chi} = \sum_{n \in \mathbb{Z}_+} (r^n - 1) f_n \|_{L^2_\chi} = \sum_{n \in \mathbb{Z}_+} (r^n - 1)^2 \| f_n \|_{L^2_\chi}^2$$

tends to zero, as soon as, $r \to 1$. This yields (7.1). On the other hand, the mapping $J \circ J^*$ keeps orthogonality in $H^2_\chi$. Hence, the equality

$$\| (J \circ J^*)f_r \|_{L^2_\chi}^2 = \sup_{r \in [0,1]} \sum_{n \in \mathbb{Z}_+} r^{2n} \| (J \circ J^*)f_n \|_{L^2_\chi}^2 = \| (J \circ J^*)f \|_{L^2_\chi}^2$$

yields the required formula (7.2).

References

[1] A. Borodin and G. Olshanski, Harmonic analysis on the infinite-dimensional unitary group and determinantal point processes, Ann. Math., 161 (2005), 1319–1422.
[2] B. Cole and T.W. Gamelin, Representing measures and Hardy spaces for the infinite polydisk algebra. Proc. London Math. Soc., 53 (1986), 112–142.
[3] T.A.W. Dwyer III, Partial differential equations in Ficscher-Fock spaces for the Hilbert-Schmidt holomorphy type. Bull. Amer. Math. Soc., 77(5) (1971), 725–739.
[4] T.W. Gamelin, Analytic functions on Banach spaces, in Complex Function Theory (Gauthier and Sabidussi ads.) Kluwer, 1994, 187–223.
[5] K. Floret, Natural norms on symmetric tensor products of normed spaces. Note di Matematica, 17, (1997), 153–188.
[6] E. Hewitt and K. A. Ross, Abstract Harmonic Analysis, Vol. 2. Springer, 1994.
[7] O. Lopushansky, Hardy type space associated with an infinite-dimensional unitary matrix group. Abstr. Appl. Anal., ID 810735 (2013), 7 pages.
[8] O. Lopushansky and A. Zagorodnyuk, Hardy type spaces associated with compact unitary groups. Nonlinear Anal.: Theory Meth. & Appl., 74(2), (2011), 556–572.
[9] Yu. A. Neretin, Hua type integrals over unitary groups and over projective limits of unitary groups. Duke Math. J., 114(2) (2002), 239–266.
[10] G. Olshanski, The problem of harmonic analysis on the infinite-dimensional unitary group. J. Funct. Anal., 205 (2003), 464–524.
[11] D. Pickrell, Measures on infinite-dimensional Grassmann manifolds. J. Funct. Anal., 70 (1987), 323–356.
[12] D. Pinaresco and I. Zaldueco, Integral representations of holomorphic functions on Banach spaces. J. Math. Anal. Appl., 308 (2005), 159–174.
[13] W. Rudin, Function theory in the unit ball of $\mathbb{C}^n$. Springer, 2008.
[14] S. Saitoh, Integral Transforms, Reproducing Kernels and Their Applications. Pitman Research Notes in Math., Ser. Vol. 369, Longman, 1997.

Faculty of Mathematics and Natural Sciences, Rzeszów University,
1 Pogonia str., 35-310 Rzeszów, Poland
E-mail address: ovlopusz@ur.edu.pl