Abstract. We introduce the $p$-adic analogue of Arakelov intersection theory on arithmetic surfaces. The intersection pairing in an extension of the $p$-adic height pairing for divisors of degree 0 in the form described by Coleman and Gross. It also uses Coleman integration and is related to work of Colmez on $p$-adic Green functions. We introduce the $p$-adic version of a metrized line bundle and define the metric on the determinant of its cohomology in the style of Faltings. It is possible to prove in this theory analogues of the Adjunction formula and the Riemann-Roch formula.

1. Introduction

The purpose of this paper is to create a $p$-adic analogue to the part of Arakelov theory that deals with arithmetic surfaces [Ara74, Fal84].

In the classical case of Arakelov theory the usual intersection pairings above finite primes are supplemented by a pairing at infinity, involving analysis on the resulting Riemann surface. Likewise, in the $p$-adic theory the same pairings at primes not above $p$ are supplemented by a pairing at $p$ involving “Coleman analysis” [CdS88, Col98, Bes02a] on the completions above $p$. This “local” part of the theory pauses most of the difficulties while using it to produce global results is fairly straightforward given the classical case.

We would like to clarify that this work has nothing to do with the work of Bloch, Gillet and Soule on “non-archimedean Arakelov theory”. Much like with the term “$p$-adic integration”, which is confusingly used for a number of non-related constructions, there is ample room for confusion here. The construction of non-archimedean Arakelov theory give real valued results, whereas $p$-adic Arakelov theory gives $p$-adic results.

Our starting point was the theory of $p$-adic height pairings, especially in the form of Coleman and Gross in [CG89]. The local part at $p$ of that theory gives the intersection index $\langle D, E \rangle$ for two divisors $D$ and $E$, of degree 0, on a complete curve over a $p$-adic field. The goal of the local theory is to describe an extension of this pairing, by giving a Green function $G(P, Q)$ with certain desirable properties, in such a way that

$$\langle \sum n_i P_i, \sum m_j Q_j \rangle = \sum n_i m_j G(P_i, Q_j).$$

To isolate a canonical as possible Green function, one needs some extra conditions. The most natural comes by introducing a notion of metrized line bundles. These are line bundles together with a function which behaves like a log (up to scaling) on the fibers and which is a Coleman function of a certain type. Having
this notion one can impose the analogue of the condition, which is satisfied by the canonical Green function in the classical theory, that the residue map defines a metric on the canonical bundle, which is admissible with respect to the Green function (or rather, the volume form). This extra condition indeed isolates a canonical choice of a Green function, up to a constant \( (7.1) \). It is possible to work with this definition but this becomes very cumbersome. We have therefore chosen another route, which runs in parallel with the classical theory.

We utilize the definition of the \( p \)-adic \( \bar{\partial} \) operator from [Bes02a] to define the curvature of a metrized line bundle. Once this is done, the Green function on a curve \( X \) is derived from the metric on the line bundle \( \mathcal{O}(\Delta) \), where \( \Delta \) is the diagonal in \( X \times X \), having a prescribe curvature similar to the one encountered in the classical theory. Unlike the classical theory we still need to impose the residue condition described before to obtain a unique choice up to constant. The relation with the Coleman-Gross height pairing now requires a proof.

The advantage of this approach is that we have a better understanding of the Green function as a function of two variables. This allows us to define the analogue of the Faltings volume on the determinant of cohomology.

A \( p \)-adic Green function on curves was previously defined by Colmez in [Col98], using abelian varieties. We related his theory with ours in Appendix B.

We then turn to the global theory, proving analogues of the adjunction formula and the Riemann-Roch formula. We are unfortunately unable at the moment to produce a result corresponding to the Noether formula because of the problem with the normalization of the Green function.

Still missing are applications. Unlike real heights, it is not clear to us what \( p \)-adic heights are good for except exact formulas. It seems to us though that research in classical Arakelov theory is leaning more and more towards exact formulas involving heights. Such results could eventually find \( p \)-adic analogues.

This word begun when the author was at the university of Münster and some crucial progress was made while visiting IHES. We would like to thank both institutions. We would also like to thank Amnon Yekutieli.

2. Review of \( p \)-adic integration

The \( p \)-adic analysis required for defining the part of Arakelov intersection “at infinity”, i.e., at \( p \), is given by the theory of Coleman integration. We use here the tools developed in [Bes02a]. However, the work of Vologodski [Vol01] is extremely useful here because it applies to the case of bad reduction as well, and because it works with the Zariski topology instead of with the rigid topology, which is particularly convenient for working with Arakelov geometry (one drawback is that it works only over finite extensions of \( \mathbb{Q}_p \), but this is all we need). Our goal in this section is to recall the constructions of [Bes02a] while showing how they work in the context of Vologodski’s work. The proofs are mostly easy modifications of the ones in [Bes02a] so we do not repeat them here.

We let \( K \) be a finite extension of \( \mathbb{Q}_p \). We choose a branch of the logarithm. In [Vol01] a branch is not chosen and the integration takes place in a ring containing a formal variable for the log of \( p \). In our setup we will choose a branch anyhow and the specialization to this situation is clear.

Let \( X \) be a smooth, geometrically connected algebraic variety over \( K \). Let \( (F, \nabla) \) be a unipotent connection over \( X \), i.e., a coherent sheaf \( F \) with an integrable
connection $\nabla$ which is a successive extension of trivial connections. The main result of Vologodski is

**Theorem 2.1** ([Vol01] Theorem B). for any two points $x, y \in X(K)$ there exists a canonical parallel translation isomorphism $v_{x,y} = v_{x,y}^F : F_x \to F_y$ of the fibers of $F$ over $x$ and $y$. This translation satisfying the following properties:

1. The translation $v_{x,x}$ is the identity.
2. For any 3 points $x, y, z$ we have $v_{y,z} \circ v_{x,y} = v_{x,z}$.
3. The translation $v_{x,y}$ is locally analytic in $x$ and $y$.
4. For the trivial connection on $O_X$ the translation carries 1 to 1.
5. For any map $T : E \to F$ of unipotent connections let $T_x$ and $T_y$ be the restrictions of $T$ to the fibers at $x$ and $y$. Then we have $T_y \circ v_{x,y}^E = v_{x,y}^F \circ T_x$.
6. For any two unipotent connections $E$ and $F$ we have $v_{x,y}^E \circ F = v_{x,y}^E \otimes v_{x,y}^F$.
7. For any $K$-morphism $f : X' \to X$, and unipotent connection $F$ on $X$ and any two points $x, y \in X'(K)$ we have $v_{f(x),y}^F \circ (f_x)_* = (f_y)_* \circ v_{f(x),f(y)}^F$, where $(f_x)_* : F_x \to (f_*F)_x$ is the pullback map (similarly with $x$ replaced by $y$).
8. Let $\sigma : \text{Spec}(L) \to \text{Spec}(K)$ be a finite map, where $L$ is another field. Let $X_L := X \times_{\text{Spec}(K)} \text{Spec}(L)$ and let, for a unipotent connection $F$ on $X$, $F_L$ denote the extension of scalars. For $x \in X(K)$ let $\sigma(x) \in X_L(L)$ be the corresponding point and let $\sigma : F_x \to (F_L)_{\sigma(x)}$ be the obvious map. Then the parallel translation is compatible with $\sigma$ in the sense that $\sigma \circ v_{x,y}^F = v_{\sigma(x),\sigma(y)}^F \circ \sigma$.

Note that property 5 is not stated as such in loc. cit. but it follows from other properties. Also note that the locally analytic nature of $v_{x,y}$ means that if $s \in F_x$, then $v(y) = v_{x,y} s$ is a locally analytic section of $F$.

**Definition 2.2** (Compare [Bes02a, Definition 4.1]). A (Vologodski) abstract Coleman function on $X$ with values in a locally free sheaf $F$ is a fourtuple $(M, \nabla, s, y)$ consisting of a unipotent connection $(M, \nabla)$ on $X$ together with a homomorphism $s \in \text{Hom}(M, F)$ (a sheaf but not a connection homomorphism) and a compatible system $y = (y_x)$ of elements $y_x \in M_x$ for every $x \in X(L)$, for any finite extension $L$ of $K$. This system should satisfy the following two conditions

1. For any two points $x_1, x_2 \in X(L) = X_L(L)$, parallel translation on $X_L$ takes $y_{x_1}$ to $y_{x_2}$.
2. For any map of fields $\sigma : L_1 \to L_2$ fixing $K$ for any point $x \in X(L_1)$ we have $\sigma(y_x) = y_{\sigma(x)}$.

A morphism between two abstract Coleman functions with values in $F$, $(M_i, \nabla_i, s_i, y_i)$, $i = 1, 2$, is a map $f : (M_1, \nabla_1) \to (M_2, \nabla_2)$ pulling back $s_2$ to $s_1$ and sending $y_1$ to $y_2$. A Coleman function with values in $F$ is a connected component of the category of abstract Coleman functions. We denote the connected component of $(M, \nabla, s, y)$ by $[M, \nabla, s, y]$. The collection of Coleman functions on $X$ with values in $F$ is denoted by $\mathcal{O}_{\text{Col}}(X, F)$. In particular we have set $\mathcal{O}_{\text{Col}}(X) := \mathcal{O}_{\text{Col}}(X, \mathcal{O}_X)$ and $\Omega^i_{\text{Col}}(X) := \mathcal{O}_{\text{Col}}(X, \Omega^i_{X/K})$.

A Coleman function can be interpreted as a set theoretic section of $F$ over $X(K)$ as follows: if $f$ corresponds to $(M, \nabla, s, y)$ then $f(x) = s(y_x)$. From the definition of Coleman functions it is clear that this function depends only on the
Proposition 2.6. The association 

\[ f^* : \mathcal{O}_{\text{Col}}(Y, \mathcal{F}) \rightarrow \mathcal{O}_{\text{Col}}(X, f^* \mathcal{F}), \]

which gives as particular cases a ring homomorphism \( f^* : \mathcal{O}_{\text{Col}}(Y) \rightarrow \mathcal{O}_{\text{Col}}(X) \) and maps \( f^* : \Omega^i_{\text{Col}}(Y) \rightarrow \Omega^i_{\text{Col}}(X) \) compatible with differentials.

**Proposition 2.6.** The association \( U \mapsto \mathcal{O}_{\text{Col}}(U, \mathcal{F}|_U) \) is a Zariski sheaf on \( X \).

**Proof.** The same proof as in [Bes02a, Proposition 4.21] works. \( \square \)
In [Bes02a Section 6] we defined an operator on a subspace of Coleman functions which we termed the \( p \)-adic \( \partial \)-operator. Here we recall this theory in the context of Vologodski’s theory. Everything works essentially without any change. We define the subspace \( \mathcal{O}_{\text{Col},n}(X,F) \) of \( \mathcal{O}_{\text{Col}}(X,F) \) to consist of all Coleman functions that have a representative \([M, \nabla, s, y]\) where \((M, \nabla)\) is a successive extension of at most \( n + 1 \) trivial connections (where here trivial means a direct sum of any number of copies of \( \mathcal{O}_X \) with its trivial connection). As in loc. cit. this space is locally described by iterated integrals involving at most \( n \) iterated integrals. For an open \( U \subset X \) we define

\[
H^\oplus_U := H^1_{\text{dR}}(U/K) \otimes_K \mathcal{F}(U) .
\]

This is a presheaf on \( X \) and there are interesting obstructions for gluing (see [Bes02a Corollary 6.9]). The map \( \bar{\mathcal{M}} \) have a representative \( E \) of horizontal sections of \( \mathcal{F} \) such that \( E \) sits in a short exact sequence

\[
0 \to E_1 \to E \to E_2 \to 0 ,
\]

such that \( E_1 \) and \( E_2 \) are trivial. The projection of the \( y_2 \) give a compatible system of horizontal sections of \( E_2 \). Since \( E_2 \) is trivial this system comes from a global horizontal section \( y_2 \) of \( E_2 \). The connection \((E, \nabla)\) gives an extension class \([E] \in \text{Ext}^1_{\text{dR}}(E_2, E_1)\). The horizontal section \( y_2 \) is an element of \( \text{Hom}_{\text{dR}}(\mathcal{O}_X, E_2) \). We can pullback the extension \([E]\) via \( y_2 \) to obtain \([E] \circ y_2 \in \text{Ext}^1_{\text{dR}}(\mathcal{O}_X, E_1)\). The homomorphism \( s \) restricts to \( s_1 \in \text{Hom}(E_1, \mathcal{F}) \) since \( E_1 \) is trivial the natural map

\[
\text{Hom}_{\text{dR}}(E_1, \mathcal{O}_X) \otimes \text{Hom}(\mathcal{O}_X, \mathcal{F}) \to \text{Hom}(E_1, \mathcal{F})
\]

is an isomorphism. Thus we may view \( s_1 \) as an element of the left hand side. There is a product \( \text{Hom}_{\text{dR}}(E_1, \mathcal{O}_X) \otimes \text{Ext}^1_{\text{dR}}(\mathcal{O}_X, E_1) \to \text{Ext}^1_{\text{dR}}(\mathcal{O}_X, \mathcal{O}_X) \). Taking this product in the first coordinate of \( s_1 \) with \([E] \circ y_2 \) we obtain

\[
\bar{\partial}(E, \nabla, s, y) := -([E] \circ y) \circ s' \in \text{Ext}^1_{\text{dR}}(\mathcal{O}_X, \mathcal{O}_X) \otimes \text{Hom}(\mathcal{O}_X, \mathcal{F}) = H^1_{\text{dR}}(X/K) \otimes \mathcal{F}(X)
\]

(note that we changed the sign from loc. cit. and that Proposition 6.8 there is actually true with the new sign). As in loc. cit. Proposition 6.4 the \( \bar{\partial} \)-operator is independent of the choice of the short exact sequence in which \( E \) sits and on the choice of a representative abstract Coleman function. Also, as in loc. cit. Proposition 6.5 this operator can be described when \( X \) is affine as sending \((\int \omega) \times s\) to \([\omega] \cdot s\), where \([\omega] \) is the cohomology class of \( \omega \).

**Proposition 2.7.** There is a short exact sequence,

\[
0 \to \mathcal{F}(X) \to \mathcal{O}_{\text{Col},1}(X,F) \xrightarrow{\bar{\partial}} H^\oplus_F(X) ,
\]

which is exact on the right if \( X \) is affine.

**Proof.** The proof is the same as in [Bes02a Proposition 6.6]. \( \square \)

A final observation regarding \( \bar{\partial} \), which is obvious from the above description, is the following.

**Lemma 2.8.** If \( l : \mathcal{F} \to \mathcal{G} \) is an \( \mathcal{O}_X \)-linear map and \( F \in \mathcal{O}_{\text{Col},1}(X,F) \), then \( l \circ F \in \mathcal{O}_{\text{Col},1}(X,G) \) and \( \bar{\partial}(l \circ F) = (\text{Id} \otimes l)(\bar{\partial}F) \).
From now on we will denote $H^\otimes_{\Omega^1_{\mathcal{X}/K}}$ simply by $H^\otimes$, as this is the case that will be used almost exclusively in this work.

Suppose now that $X$ is a smooth variety over $\overline{\mathbb{Q}}_p$. Then, working with models over finite extensions of $\mathbb{Q}_p$ and using functoriality, it is clear that we can extend all of the above constructions to the case $K = \overline{\mathbb{Q}}_p$. We can easily recover the field of definition of a Coleman function as follows.

**Proposition 2.9.** Let $X/K$ be a smooth variety, $\mathcal{F}$ a locally free sheaf defined over $K$, and let $\overline{X}$ and $\overline{\mathcal{F}}$ be the extension of these objects to the algebraic closure of $K$. We assume that we have chosen a branch of the logarithm defined over $K$. Let $f \in \mathcal{O}_\text{Col}(\overline{X}, \overline{\mathcal{F}})$ be a Coleman function and let $\sigma$ be an automorphism of $K$ over $K$. Then the function $f^\sigma$ defined by $f^\sigma(x) = \sigma(f(\sigma^{-1}(x)))$ is also a Coleman function on $\overline{X}$. If for any such $\sigma$ we have $f^\sigma = f$, then in fact $f$ is an extension of scalars from a function in $\mathcal{O}_\text{Col}(X, \mathcal{F})$.

**Proof.** It is obvious that $f^\sigma$ is a Coleman function. The second statement is proved using minimal models for Coleman functions in the same way that the sheaf property is proved. \qed

If $\pi : X \to Y$ is a finite covering of varieties over $\overline{\mathbb{Q}}_p$ and $f$ is a Coleman function on $X$ we can construct the trace of $f$ along $\pi$ down to $Y$. Surprisingly perhaps, this may not always be a Coleman function on $Y$. We will now describe a very simple situation where one can show that the trace is indeed a Coleman function. The problem seems interesting and deserves further study, but here we limit ourselves to a situation that suffices for our uses in this paper.

**Lemma 2.10.** Suppose $\pi : X \to Y$ is a finite covering and $\eta$ is a locally analytic one-form on $Y$ such that

1. $\omega := \pi^*\eta$ belongs to $\Omega^1_{\text{Col},1}(X)$, and
2. There exists $\alpha \in H^\otimes(Y)$ such that $\pi^*\omega = \partial\eta$.

Then in fact $\eta \in \Omega^1_{\text{Col},1}(Y)$ and $\partial\eta = \alpha$.

**Proof.** The problem is (Zariski) local on $Y$, which we may therefore assume affine. By Proposition 2.7 we can find $\eta' \in \Omega^1_{\text{Col},1}(Y)$ such that $\partial\eta' = \alpha$. It follows that $\partial(\omega - \pi^*\eta') = 0$ and therefore, again by Proposition 2.7, $\omega - \pi^*\eta' = \pi^*(\eta - \eta') \in \Omega^1(X)$. It follows that

$$\eta - \eta' = \frac{1}{\deg \pi} \text{tr}_\pi\pi^*(\eta - \eta') \in \Omega^1(Y),$$

which implies the result. \qed

**Proposition 2.11.** Suppose we have a diagram of finite maps $X' \xrightarrow{\pi'} X \xrightarrow{\pi} Y$ where the composition $X' \xrightarrow{\pi''} Y$ is a Galois covering with Galois group $G$. Let $F \in \mathcal{O}_\text{Col}(X)$ with $\partial F \in \Omega^1_{\text{Col},1}(X)$ (Here we use $\partial$ instead of $d$ so that the composed operator is $\partial\partial$, which is similar to the complex notation). Suppose there exists $\alpha \in H^\otimes(Y)$ such that

$$(\pi'')^*\alpha = \sum_{\sigma \in G} \sigma^*((\pi')^*\partial\partial F).$$

Then $\text{tr}_\pi(F) \in \mathcal{O}_\text{Col}(Y)$ and $\partial\partial \text{tr}_\pi(F) = \alpha / \deg(\pi')$. 
Proof. We have $tr_x(F) = tr_x((\pi')^*F) / \deg(\pi')$, so it suffices to consider the case $\pi' = \text{Id}$, $\pi = \pi''$. Applying Lemma 2.10 to $\eta = dtr_x(f)$ gives the result. □

3. The double index

In this section we recall the theory of the double index from [Bes00b, Section 4]. We must do a few things anew for the algebraic theory used in this paper.

Let $K$ be a field of characteristic 0. We consider the field of Laurent series in the variable $z$ over $F$, $\mathcal{M} = K((z))$, the polynomial algebra over $\mathcal{M}$ in the formal variable $\log(z)$, $A_{log} := \mathcal{M}[\log(z)]$, and the module of differentials $\Omega_{log} := A_{log} \cdot dz$. There is a formal derivative $d : A_{log} \to \Omega_{log}$ such that $d \log(z) = dz/z$ and it is an easy exercise in integration by parts to see that every form in $\Omega_{log}$ has an integral in $A_{log}$ in a unique way up to a constant. We distinguish in $A_{log}$ the subspace $A_{log,1} = \mathcal{M} + K \cdot \log(z)$ consisting exactly of all functions whose differential is in $\mathcal{M} \cdot dz$. To $F \in A_{log,1}$ we can associated the residue of its differential $\text{Res} dF \in K$. If $F \in A_{log,1}$, then $F \in \mathcal{M}$ if and only if $\text{Res} dF = 0$.

Definition 3.1. The double index is the unique anti-symmetric bilinear form $\langle \cdot, \cdot \rangle : A_{log,1} \times A_{log,1} \to K$ with the property that $\langle F, G \rangle = \text{Res} FdG$ whenever the left hand side has a meaning.

The existence of the double index relies on a trivial linear algebra lemma (Lemma 4.4 of [Bes00b]). In loc. cit. it is computed in a rigid analytic context and denoted $\text{ind}(F, G)$. Assume from now on that $K$ is a complete subfield of $\mathbb{C}_p$ and that a branch of the log on $K$ has been chosen. The following lemma is the algebraic analogue of Lemma 4.6 in loc. cit.

Lemma 3.2. With a subscript $z$ or $w$ denoting the variable, let $\alpha : \mathcal{M}_z \to \mathcal{M}_w$ be given by $\alpha(z) = \sum_{k=0}^{\infty} a_k w^k$, with $a_n \neq 0$, and extend $\alpha$ in the obvious way to $A_{log}$ and $\Omega_{log}$. Then $\langle \alpha(F), \alpha(G) \rangle_w = n(\alpha(F), \alpha(G))_z$.

Suppose now that $K$ is either $\mathbb{C}_p$ or $\overline{\mathbb{Q}}_p$. For both of these fields one has a “Coleman integration theory” of holomorphic forms. This means that for any smooth connected variety $X/K$ there is an integration map, $\omega \mapsto \int \omega$, from $\Omega^1(X)^{\text{div} = 0}$ to locally analytic $K$-valued functions on $X$ modulo the constant functions, which is an inverse to the differential $d$ and which is functorial with respect to arbitrary morphisms in the sense that for such a morphism $f$ we have $\int (f^* \omega) = f^* \int \omega$. We further require that on $\mathbb{P}^1$ we have $\int d\log(z) = \log(z)$ (in fact, it is not hard to show that the theory determines uniquely a branch of the log for which this relation holds). Such a theory over $K = \overline{\mathbb{Q}}_p$ exists as part of the more general theory described in Section 2. For $\mathbb{C}_p$, or more generally for complete subfields of $\mathbb{C}_p$ it follows from [Col98, Théorème 0.1].

Let $X$ be a proper smooth curve over $K$ and suppose $F$ and $G$ belong to $\mathcal{O}_{\text{Col}}(\mathcal{U})$, with $U \subset X$ open, and such that $dF, dG \in \Omega^1(\mathcal{U})$. Then we can compute the double index at every point $x \in X$ in an obvious way, as the last lemma shows that the computation will be independent of the choice of a local variable. The index is non-zero only for $x \in X - U$. Since the sum of the residues of $dF$ and $dG$ on $x \in X$ is 0, the sum of all these double indices does not change if we change $F$ or $G$ by a constant and therefore we obtain a global pairing

\begin{equation}
\langle dF, dG \rangle_{\text{gl}} := \sum_{x \in X} \langle F, G \rangle_x.
\end{equation}
Lemma 3.3. If both $dF$ and $dG$ are forms of the second kind, then $\langle dF, dG \rangle_{\text{gl}} = [dF] \cup [dG]$, where $[dF]$ and $[dG]$ denote the cohomology classes in $H^1_{\text{dR}}(X)$ of the corresponding forms.

Proof. This follows from the usual formula for cup products on curves. \hfill $\square$

In [Bes00b, Proposition 4.10] we proved, under the assumption that $X$ had good reduction, an extension of this cohomological formula for the pairing, where the forms are arbitrary, and in fact are even allowed to be defined only on a certain rigid open subset of $X$. For algebraic differentials of the third kind a similar result was proved in the good reduction case by Coleman [Col89, Theorem 5.2] and in general by Colmez [Col98, Théorème II.4.2]. Here we will prove in Theorem 3.9 the analogue of our result for meromorphic differentials but without assuming good reduction, thus generalizing the result of Colmez.

Proposition 3.4. For any rational function $f \in K(X)$ and any meromorphic form $\omega$ we have $\langle \omega, \log(f) \rangle_{\text{gl}} = 0$.

We first need a few auxiliary results.

Lemma 3.5. On $\mathbb{P}^1$ we have for any $a \in K$ that $\langle \log(t), \log(t-a) \rangle_{\text{gl}} = 0$.

Proof. The non-trivial local indices are at $0$, $\infty$ and $a$. At infinity we have 
$$\langle \log(t), \log(t-a) \rangle_{\infty} = \langle \log(t), \log(t) \rangle_{\infty} + \langle \log(t), \log(1 - \frac{a}{t}) \rangle_{\infty} = \log(1 - \frac{a}{\infty}) = 0$$
while $\langle \log(t), \log(t-a) \rangle_0 = -\log(0-a)$ and $\langle \log(t), \log(t-a) \rangle_a = \log(a)$, so the result is clear. \hfill $\square$

Lemma 3.6. If $f : X \to Y$ is a finite map of curves, then $\langle f^*dF, f^*dG \rangle_{\text{gl}} = \deg(f)\langle dF, dG \rangle_{\text{gl}}$.

Proof. This follows immediately from the fact that if $f(x) = y$ with multiplicity $n$, then we have $\langle f^*F, f^*G \rangle_x = n \langle F, G \rangle_y$ by Lemma 3.2. \hfill $\square$

Lemma 3.7. If $f : X \to Y$ is a finite map of curves and $\omega \in \Omega^1(K(Y))$, $\eta \in \Omega^1(K(X))$, then $\langle f^*\omega, \eta \rangle_{\text{gl}} = \langle \omega, tr_f \eta \rangle_{\text{gl}}$.

Proof. We may assume that $f$ is a Galois covering with Galois group $G$. Then we have 
$$\langle f^*\omega, \eta \rangle_{\text{gl}} = \frac{1}{|G|} \sum_{\sigma \in G} \langle \sigma^* f^* \omega, \sigma^* \eta \rangle_{\text{gl}} = \frac{1}{|G|} \langle f^* \omega, \sum_{\sigma \in G} \sigma^* \eta \rangle_{\text{gl}}$$
$$= \frac{1}{|G|} \langle f^* \omega, f^* tr_f \eta \rangle_{\text{gl}} = \langle \omega, tr_f \eta \rangle_{\text{gl}}.$$ \hfill $\square$

Proof of Proposition 3.4. The last lemma implies that $\langle \omega, \log(f) \rangle_{\text{gl}} = \langle tr_f \omega, \log(t) \rangle_{\text{gl}}$ on $\mathbb{P}^1$. Now, $tr_f \omega$ is a sum of a form of the second kind on $\mathbb{P}^1$ and a linear combination of forms $\log(t - a_i)$. By Lemma 3.3 we may assume that $tr_f \omega$ is of the second kind, and must therefore equal $dg$ for some $g \in K(\mathbb{P}^1)$. But $\langle dg, \log(t) \rangle_{\text{gl}} = \sum_{a \in \mathbb{P}^1} \text{Res}_a(g \log(t)) = 0$. \hfill $\square$
Definition 3.8. We define a map $\Psi' : \Omega^1(K(X)) \to H^1_{dR}(X)$ by the condition that for any form of the second kind $\omega$ on $X$ we have

$$\Psi'(\eta) \cup [\omega] = \langle \eta, \omega \rangle_{gl}.$$  

It follows immediately from Lemma 3.3 and Proposition 3.4 that $\Psi'$ is well defined and vanishes on log-differentials (i.e., of the form $d\log f$). From Lemma 3.3 we have $\Psi'(\eta) = [\eta]$ if $\eta$ is of the second kind. The main result of this section is the following theorem.

Theorem 3.9. For any two meromorphic differentials $\omega, \eta \in \Omega^1(K(X))$ we have

$$\langle \eta, \omega \rangle_{gl} = \Psi'(\eta) \cup \Psi'(\omega).$$

To explain why this is a generalization of the results mentioned above, we state the relation between the map $\Psi'$ and the logarithm on the universal vectorial extension of the Jacobian of $X$.

Let $\mathcal{T}$ be the group of differentials of the third kind on $X$ and let $\mathcal{T}_l$ be the subspace of log differentials.

Recall from [CG89, Proposition 2.5] that there exists a commutative diagram with exact rows

$$
\begin{array}{cccccc}
0 & \longrightarrow & \Omega^1(X) & \longrightarrow & \mathcal{T}/\mathcal{T}_l & \longrightarrow & J & \longrightarrow & 0 \\
0 & \longrightarrow & \Omega^1(X) & \longrightarrow & H^1_{dR}(X) & \longrightarrow & H^1_{dR}(X)/\Omega^1(X) & \longrightarrow & 0 \\
\end{array}
$$

where $J$ is the Jacobian of $X$, the map $\mathcal{T} \to J$ sends a differential of the third kind to its residue divisor and $\log_J$ is the logarithm from $J$ to its Lie algebra, which is isomorphic to $H^1_{dR}(X)/\Omega^1(X)$. The group $\mathcal{T}/\mathcal{T}_l$ is the group of $K$ points of the universal vectorial extension $G_X$ of $J$ and $\Psi$ is simply the logarithm for this group. To be precise, we are working here with $K$ points, which is fine for $K = \mathbb{C}_p$. For $K = \mathbb{Q}_p$ we are taking the limit of the corresponding map over all finite extensions of $\mathbb{Q}_p$. Note that for a commutative algebraic group $H$ the logarithm is characterized as the unique locally analytic group homomorphism whose differential at the identity element is the identity.

Since the map $\Psi'$ vanishes on $\mathcal{T}_l$ it induces a map $\Psi' : \mathcal{T}/\mathcal{T}_l \to H^1_{dR}(X)$.

Proposition 3.10. We have $\Psi' = \Psi$ on $\mathcal{T}/\mathcal{T}_l$.

To prove both the last proposition and our theorem we need to recall the notion of differentiation of differential forms with respect to a vector field (see [War83, 2.24] or [Yek95, Proposition 4.6]).

Definition 3.11. Let $Y$ be a space ($Y$ could be a scheme or an analytic space over some field or a $C^\infty$ space) and let $\partial/\partial t$ be a vector field on $Y$. Then the operator of differentiation with respect to $\partial/\partial t$ is defined on $\Omega^i(Y)$ by

$$(3.2) \frac{\partial}{\partial t} \omega = d(\omega|_{\partial t}) + (-1)^i d\omega|_{\partial t}$$

where $|_{\partial t}$ is the contraction operator.

This operator commutes with the exterior differential $d$. 
Lemma 3.12. Let $\omega$ be a meromorphic form on $X$. Let $T$ be a smooth parameter variety and let $(\eta_t)_{t \in T}$ be a family of forms of the third kind on $X$ parametrized by $T$ (see Appendix A for this notion) and $\partial/\partial t$ a vector field on $T$. Then $\partial/\partial t \eta_t$ is a form of the second kind, $\langle \omega, \eta_t \rangle_{gl}$ is locally analytic in $t$ and

$$\frac{\partial}{\partial t} \langle \omega, \eta_t \rangle_{gl} = \left\langle \omega, \frac{\partial}{\partial t} \eta_t \right\rangle_{gl}. $$

Proof. The fact that the derivative is of the second kind is proved in the Appendix. We check the formula at some $t_0 \in T$. We can find discs $D_i \subset X$, $i = 1, \ldots, n$ and a ball $B$ around $t_0$ such that for each $t \in B$ all the singular points of any of the forms $\eta_t$ are contained in some $D_i$. By Proposition 3.4, we may change $\omega$ by a linear combination of log differentials, so we may assume $\omega$ has no residues inside the $D_i$’s. By further reducing $D_i$ we may assume that $\int \omega$ is a meromorphic function on each $D_i$. In such a situation, the argument of the proof of Proposition 5.5 in [Bes00b] implies that for each $t \in B$ we may replace $\langle \omega, \eta_t \rangle_{gl}$ with $\sum_i \langle \int \omega, \int \eta_t \rangle_{e_i}$, where $e_i$ is an annulus around $D_i$ and the local index around an annulus is the one defined in [Bes00b, Proposition 4.5]. In this case, each double index $\langle \int \omega, \int \eta_t \rangle_{e_i}$ equals $\text{Res}_{e_i} ((\int \omega) \eta_t)$. By writing a Laurent expansion it is clear that each of these expressions commute with differentiation. □

Proof of Theorem 3.9. Let $\eta'$ be a form of the second kind representing $\Psi'(\eta)$. Then, by the definition of $\Psi'$ we have

$$\Psi'(\eta) \cup \Psi'(\omega) = \langle \eta', \omega \rangle_{gl},$$

so we must prove that

$$\langle \eta - \eta', \omega \rangle_{gl} = 0$$

for any $\omega$. Again by the definition of $\Psi'$ this is true for any $\omega$ of the second kind. Since $\langle \eta - \eta', \omega \rangle_{gl}$ is linear in $\omega$ it suffices to prove that it vanishes on forms of the third kind, and we already know that it vanishes on log-differentials by Proposition 3.10. We thus get an additive map

$$\mathcal{T}/\mathcal{T}_I \to K, \quad \omega \mapsto \langle \eta - \eta', \omega \rangle_{gl},$$

and we must show that it is the 0 map. Since $\mathcal{T}/\mathcal{T}_I = G_X(K)$, where $G_X$ is the universal vectorial extension of $J$, it suffices to show that the derivative of this map at the identity is 0. But this is clear by Lemma 3.12 because the derivative of a family of forms of the third kind is a form of the second kind $\omega$, for which (3.3) holds. □

Proof of Proposition 3.11. Since $\Psi'$ is clearly additive, it suffices to show that it is analytic near the origin and that its differential at 0 is the identity map when $\text{Lie}(G_X)$ is identified with $H^1_{dR}(X)$. This is equivalent to showing that for any form $\omega$ of the second kind the map $G_X \to K$, given by

$$\eta \mapsto \Psi'(\eta) \cup [\omega] = \langle \eta, \omega \rangle_{gl},$$

is analytic near the origin and its differential at 0 is given by cupping with $[\omega]$. By Lemma A.4 we can find an algebraic family of forms $\eta_t$ for $t$ in a neighborhood of 0 in $G_X$ such that $\eta_t$ represents $t$. This already shows that (3.4) is analytic near 0. The formula for the differential at 0 follows easily from Lemma 3.12 and Corollary A.5. □
The following corollary is the only result needed in the rest of the text.

**Corollary 3.13.** Suppose \( \omega \) is a form of the second kind while \( \eta \in \mathcal{T} \). Then
\[
\langle \omega, \eta \rangle_{\mathcal{G}} = [\omega] \cup \Psi(\eta).
\]

The following result is needed in [Bes02b]. We give it here since it uses the techniques of this section.

**Proposition 3.14.** Let \( X \) be a curve over \( \mathbb{C}_p \) with good reduction. Then the Coleman and Colmez integrals of algebraic forms on \( X \) coincide.

**Proof.** For forms of the second kind this follows from [Col85] as it is shown there that the Coleman integral can be obtained by pullback from an abelian variety, which is how the Colmez integral is defined. By linearity of both integrals it remains to consider a form of the third kind \( \eta \). Let \( F_1 \) and \( F_2 \) be the Colmez (respectively Coleman) integral for \( \eta \). Since both \( F_1 \) and \( F_2 \) differentiate to \( \eta \), \( F_1 - F_2 \) is a locally constant function on \( X \). For a divisor \( D \) of degree 0 on \( X \) let \( \beta(D) \) be \( F_1 - F_2 \) evaluated at \( D \). The function \( \beta \) is locally constant on the points in \( D \). It is furthermore additive. If we show that it vanishes on principal divisors it will give a locally constant additive function on the Jacobian and will have to vanish. So suppose \( D = (f) \) for a rational function \( f \). Both Coleman and Colmez integrals of \( d\log(f) \) are \( \log(f) \). It follows from Proposition 3.4 that \( \sum_{x \in X} \langle \log(f), F_1 \rangle_x = 0 \) and the same follows for \( F_2 \) by [Bes00b, Corollary 4.11] combined with the analysis in the proof of Proposition 5.5 in [Bes00b] that shows we may replace local indices on annuli by local indices at points. But \( \sum_{x \in X} \langle \log(f), F_1 - F_2 \rangle_x = \beta(D) \) so the proof is complete. \( \square \)

4. **Metrized line bundles over the \( p \)-adics**

In this section it is most convenient to work over an algebraically closed field, so we will always work over \( \overline{\mathbb{Q}_p} \): All varieties will be smooth and connected over \( \overline{\mathbb{Q}_p} \), and differential forms and de Rham cohomology will also be taken over \( \overline{\mathbb{Q}_p} \). We also fix a branch of the logarithm \( \log: \overline{\mathbb{Q}_p} \to \overline{\mathbb{Q}_p} \).

In classical Arakelov theory one of the basic notions is that of a metrized line bundle: A line bundle endowed with a metric. Often one does not use the absolute value of the section but rather the log of this absolute value. Experience has shown that the replacement for the log of an absolute value in the complex case is simply the \( p \)-adic logarithm in the \( p \)-adic case. This suggests the following definition.

**Definition 4.1.** Let \( X \) be a smooth variety and let \( \mathcal{L} \) be a line bundle over \( X \). A log function on \( \mathcal{L} \) is a function \( \log_\mathcal{L} \in \mathcal{O}_{\text{Col}}(\mathcal{L}^\times) \), where \( \mathcal{L}^\times := \text{Tot}(\mathcal{L}) - \{0\}, \text{Tot}(\mathcal{L}) \) being the total space of \( \mathcal{L} \), such that the following conditions are satisfied:

1. For any \( x \in X \), any \( v \) in the fiber \( \mathcal{L}_x \) and \( \alpha \in \mathbb{Q}_p \) one has \( \log_\mathcal{L}(\alpha v) = \log(\alpha) + \log_\mathcal{L}(v) \).
2. \( d\log_\mathcal{L} \in \Omega^1_{\text{Col},1}(\mathcal{L}^\times) \)

A function satisfying only the first condition will be called a pseudo log function. A line bundle together with a log function will be called a metrized line bundle (the more natural name of a log line bundle is rejected because of possible confusion with the theory of log schemes). There is an obvious notion of isometry of metrized line bundles.
Clearly the first condition implies that if \( s \) is a section of \( \mathcal{L} \) and \( f \) is a rational function on \( X \), then \( \log_{\mathcal{L}}(fs) = \log(f) + \log_{\mathcal{L}}(s) \). Adding a constant to a log function one obtains a new log function. This operation will be called scaling.

**Remark 4.2.** The second condition on a log function seems a bit arbitrary. Clearly, some condition should exist to make the log function amenable to the tools of p-adic analysis. A weaker condition could be that for one, hence any, invertible section \( s \) of \( \mathcal{L} \) in a Zariski open \( U \) the function \( \log_{\mathcal{L}}(s) \) on \( U \) is a Coleman function. This is clearly implied by our stronger condition. One reason for this condition is that we can use the theory of the p-adic \( \partial \) operator to define a kind of curvature for our log function. Another reason is that the canonical Green function we will define can be characterized, as we will see in Proposition 4.4, by different means, and satisfies this property. As a consequence, log functions admissible with respect to this Green function, in a sense we will define, have this property and these are the only log functions we are interested in.

**Definition 4.3.** If \( \mathcal{L} \) and \( \mathcal{M} \) are metrized line bundles then \( \mathcal{L} \otimes \mathcal{M} \) has a canonical log function \( \log_{\mathcal{L}} \otimes \log_{\mathcal{M}} \) defined by

\[
(\log_{\mathcal{L}} \otimes \log_{\mathcal{M}})(s \otimes t) = \log_{\mathcal{L}}(s) + \log_{\mathcal{M}}(t).
\]

Exactly as in the complex case, it turns out one can usually associate to a metrized line bundle a kind of curvature. Let \( \mathcal{L} \) be a metrized line bundle on \( X \). Since \( \text{dlog}_{\mathcal{L}} \in \Omega^1_{\text{Col},1}(\mathcal{L}^\times) \) by assumption we can take its \( \partial \) operator. We use the notation \( \partial \partial \log_{\mathcal{L}} \in H^\bullet(\mathcal{L}^\times) \) for the result instead of using \( \partial \partial \) to make the notation similar to the complex one.

**Proposition 4.4.** Suppose \( X \) is proper and let \( \pi : \mathcal{L}^\times \to X \) be the projection.

1. Suppose that \( \text{ch}_1(\mathcal{L}) \in \text{Im}(\cup : H^\bullet(X) \to H^\bullet_{dR}(X)) \). Then there exists a unique \( \text{Curve}(\mathcal{L}) \in H^\bullet(X) \) such that \( \pi^* \text{Curve}(\mathcal{L}) = \partial \partial \log_{\mathcal{L}} \). The element \( \text{Curve}(\mathcal{L}) \) is called the curvature form of the metrized line bundle \( \mathcal{L} \) and it satisfies the relation \( \partial \text{Curve}(\mathcal{L}) = \text{ch}_1(\mathcal{L}) \).

2. Conversely, suppose that \( \alpha \in H^\bullet(X) \) satisfies \( \cup \alpha = \text{ch}_1(\mathcal{L}) \). Then there exist a log function \( \log_{\mathcal{L}} \) on \( \mathcal{L} \) such that \( \text{Curve}(\mathcal{L}) = \alpha \).

**Proof.** Consider first the case of the trivial line bundle \( \mathcal{L} \) over \( X \), where we have \( \mathcal{L}^\times = X \times (\mathbb{A}^1 - \{0\}) \). The log function is then given by \( \log_{\mathcal{L}}(x,t) = \log_{\mathcal{L}}(1)(x) + \log(t) \). Since \( \partial \partial (\log(t)) = 0 \) it follows that in this case \( \partial \partial (\log_{\mathcal{L}}) = \pi^*(\partial \partial \log_{\mathcal{L}}(1)) \).

It is also clear that \( \partial \partial \log_{\mathcal{L}}(1) \) is characterized by this equation. Suppose now that \( \mathcal{L} \) is arbitrary and that \( \mathcal{U} = \{U_i\} \) is a covering of \( X \) over which \( \mathcal{L} \) is trivialized, so that we have nonvanishing sections \( s_i \) over \( U_i \). We obtain a 0-Cech cocycle \( i \mapsto \alpha_i := \partial \partial (\log_{\mathcal{L}}(s_i)) \), since \( \log_{\mathcal{L}}(s_i) - \log_{\mathcal{L}}(s_j) = \log(s_i/s_j) \) on \( U_i \cap U_j \) and its \( \partial \partial \) is 0. If this cocycle comes from \( \alpha \in H^\bullet(X) \) then \( \pi^* \alpha \) equals \( \partial \partial \log_{\mathcal{L}} \) on \( \pi^{-1}(U_i) \) for each \( i \) and thus on \( \mathcal{L}^\times \), proving the existence of the curvature. To show that we can indeed find such an \( \alpha \) we now use the criterion of [Bes00a, Corollary 6.9]. According to it, \( \alpha \) exists if the image of the cocycle \( i \mapsto \alpha_i \) under the map \( \Psi \) of loc. cit. Definition 6.7, is in the image of \( \cup \). It thus suffices to prove that this image is \( \text{ch}_1(\mathcal{L}) \). The map \( \Psi \) is computed as follows: For each \( i \) we need to find \( \omega_i \in \Omega^1_{\text{Col},1}(U_i) \) such that \( \partial \omega_i = \alpha_i \) and then consider the cocycle \( ij \mapsto \omega_i - \omega_j \). But in our case we can clearly take \( \omega_i = \text{dlog}_{\mathcal{L}}(s_i) \) and \( \omega_i - \omega_j = \text{dlog}(s_i/s_j) \). This last cocycle is well known to represent \( \text{ch}_1(\mathcal{L}) \) so the first assertion is proved, except
for uniqueness. This follows because clearly the map $H^\oplus(X) \to H^\oplus(U)$, when $U$ is open in $X$, is injective.

Suppose now that we are given $\Omega = \sum \theta^i \otimes \omega^i \in H^\oplus(X)$ such that $\cup \Omega = ch_1(\mathcal{L})$. Notice that $d\omega_i = 0$ since the $\omega_j \in \Omega^1(X)$ and $X$ is proper. We want to construct a log function on $\mathcal{L}$ whose curvature is $\Omega$. The first step is to write explicit cocycles representing the $\theta^i$ and then $\cup \Omega$. We do this using the affine covering $\mathcal{U} = \{U_j\}$.

With respect to this covering we can write

$$\theta^i = ((\eta^i_j \in \Omega^1(U_j)), (f^i_{jk} \in \mathcal{O}(U_{jk}))), \eta^i_j - \eta^i_k = df^i_{jk} \text{ on } U_{jk},$$

where $U_{jk} = U_j \cap U_k$. We make the comparison $\cup \Omega = ch_1(\mathcal{L})$ in $H^2(X, F^1\Omega^\bullet)$ instead of in the full $H^2_{dR}(X/K)$. Since $X$ is proper this forms a subspace and both sides of the equation belong to this subspace. Using again Čech cocycles we can write

$$H^2(X, F^1\Omega^\bullet) = \left\{((\chi_j \in \Omega^2(U_j), \zeta_{jk} \in \Omega^1(U_{jk}), \chi_j - \chi_k = d\zeta_{jk}, \zeta_{jk} - \zeta_{jl} + \zeta_{kl} = 0), \alpha_j \in \Omega^1(U_j)) \right\}.$$

With this representation we have

$$\cup \Omega = (\sum_i \eta^i_j \wedge \omega^i, \sum_i f^i_{jk} \omega^i)$$

and

$$ch_1(\mathcal{L}) = (0, d\log g_{jk}),$$

where $g_{jk} = s_j/s_k$. Therefore, the condition $\cup \Omega = ch_1(\mathcal{L})$ spells out as

$$\sum_i \eta^i_j \wedge \omega^i = d\alpha_j$$

$$\sum_i f^i_{jk} \omega^i = d\log g_{jk} + \alpha_j - \alpha_k,$$

for some $\alpha_j \in \Omega^1(U_j)$.

To define the log function it suffices to define $\log_{\mathcal{L}}(s_j)$. By assumption,

$$\bar{\partial}\partial \log_{\mathcal{L}}(s_j) = \Omega_{U_j} = \sum_i [\eta^i_j] \otimes \omega^i.$$

This implies that for some choices of $\gamma_j \in \Omega^1(U_j)$ and Coleman integrals $H^i_j = \int \eta^i_j$ we should have

$$d\log_{\mathcal{L}} s_j = \sum_i H^i_j \omega^i + \gamma_j.$$

We further need to have $\log_{\mathcal{L}}(s_j) - \log_{\mathcal{L}}(s_k) = \log(g_{jk})$ and differentiating this we get

$$\sum_i (H^i_j - H^i_k) \omega^i + \gamma_j - \gamma_k = d\log g_{jk}$$

(recall that $d\omega_i = 0$). Since $H^1(U, \mathbb{Q}_p) = 0$ we can arrange the constants of integration in such a way that $H^1_j - H^1_k = f^i_{jk}$. We then get

$$\sum_i (f^i_{jk} \omega^i + \gamma_j - \gamma_k) = d\log g_{jk}.$$
and this can be arranged by taking $\gamma_j = -\alpha_j$ in view of (4.1). So set

$$\delta_j = \sum_i H^j_i \omega^i - \alpha_j$$

with the choice of $H^j_i$ discussed before. We notice that these $\delta_j$ are closed forms. Indeed, since $d\omega^i = 0$ we have

$$d\delta_j = \sum_i \eta^j_i \wedge \omega^i - d\alpha_j = 0$$

by (4.1). Define now a Coleman form on $L \times \pi^{-1} U_j$ as follows: Choose an isomorphism $\pi^{-1} U_j \cong A^1 \times U_j$ in such a way that $s_j$ corresponds to the section 1 and define the form there by $\pi^* \delta_j + d\log(t)$. It now follows that these forms are closed and that they glue to give a closed Coleman form $\delta \in \Omega^{1}_{\text{Col}, L}$ which we can then integrate to obtain our required $\text{log}_L$. □

The behavior of log functions with respect to pullbacks is given by the following obvious result.

**Proposition 4.5.** Let $L$ be a line bundle on $Y$ with a log function $\text{log}_L$ and let $f : X \to Y$ be a morphism. Let $L' = f^* L$ and consider the map $\tilde{f} : L'^{\times} \to L^\times$. Then $\tilde{f}^* \text{log}_L$ is a log function on $L'$ whose curvature is $f^* \text{Curve}(\text{log}_L)$.

We want to consider the behavior of log functions with respect to norms. Suppose $\pi : X \to Y$ is a finite covering, $L$ is a line bundle on $X$ with a log function $\text{log}_L$. the norm of $L$ to $Y$, $\text{Norm}_\pi L$, acquires a natural pseudo log function $\text{Norm}_\pi \text{log}_L$ as follows: Its fiber over $y \in Y$ is $\bigotimes_{z \in \pi^{-1}(y)} L^\otimes_{\pi(x)}$, with $n_{\pi,x}$ the multiplicity of $\pi$ at $x$. Since each of the fibers $L_x$ has a log function the tensor product has one as well.

**Lemma 4.6.** If $L$ has a pseudo log function $\text{log}_L$ and the induced pseudo log function on $L^\otimes n$ is a log function, then so is $\text{log}_L$.

**Proof.** This is clear once taking a Čech covering. If $s$ is an invertible section of $L$ on $U$, then $s^\otimes n$ is an invertible section for $L^\otimes n$, and if $\text{log}_L \otimes n(s^\otimes n) = n \text{log}_L(s)$ is a Coleman function of the right type, then so is $\text{log}_L(s)$. □

**Proposition 4.7.** Suppose we have a diagram of finite maps $X' \xrightarrow{\pi'} X \xrightarrow{\pi} Y$ where the composition $X' \xrightarrow{\pi''} Y$ is a Galois covering with Galois group $G$. Let $L$ be a line bundle on $X$ with log function $\text{log}_L$. Suppose there exists $\alpha \in H^{1}(\pi'' Y)$ such that

$$(\pi'')^* \alpha = \sum_{\sigma \in G} \sigma^* ((\pi')^* \text{Curve}(\text{log}_L))$$

Then $\text{Norm}_\pi(\text{log}_L)$ is a log function on $\text{Norm}_\pi L$ and its curvature is $\alpha / \text{deg}(\pi')$.

**Proof.** We have a natural isomorphism

$$\text{Norm}_\pi \otimes (\pi^* L) \cong (\text{Norm}_\pi L) \otimes \text{deg}(\pi')$$

which is compatible with pseudo-log functions. By the previous lemma the proposition reduces to the case where $\pi' = \text{Id}$ and $\pi = \pi''$ is Galois. Let $L' = \text{Norm}_\pi L$ and let

$$L'' := \pi^* \text{Norm}_\pi L \cong \bigotimes_{\sigma \in G} \pi^* \sigma^* L.$$
The last isomorphism is in fact an isometry. The log function on $\mathcal{L}''$ has curvature
\[ \text{Curve}(\log_{\mathcal{L}''}) = \sum_{\sigma \in G} \sigma^* \text{Curve}(\log_{\mathcal{L}}) = \pi^* \alpha. \]

Let $\pi_Y : \mathcal{L}' \to Y$, $\pi_X : \mathcal{L}'' \to X$, be the projections. The map $\tilde{\pi} : \mathcal{L}'' \to \mathcal{L}'$ is finite. We have $\tilde{\pi}^* d \log_{\mathcal{L}''} = d \log_{\mathcal{L}'}$ and
\[ \tilde{\partial} d \log_{\mathcal{L}''} = \pi_X^* \text{Curve}(\log_{\mathcal{L}''}) = \pi_X^* \pi_Y^* \alpha = \tilde{\pi}^* \pi_Y^* \alpha. \]

It follows from Lemma 2.10 that $d \log_{\mathcal{L}'} \in \Omega_{\text{col},1}(\mathcal{L}')$ and $\tilde{\partial} d \log_{\mathcal{L}'} = \pi_Y^* \alpha$ so the curvature of $\log_{\mathcal{L}'}$ is $\alpha$.

5. The almost canonical Green function

We now construct the almost canonical Green function on a complete non-singular curve $X$ of positive genus $g$ over $\mathbb{Q}_p$. Taking the hint from classical Arakelov theory we define instead an almost canonical log function on $\mathcal{O}(\Delta)$ on $X \times X$. Here almost canonical means canonical up to scaling. The Green function is then simply $\log_{\mathcal{O}(\Delta)}(1)$ and it is therefore defined as a function of two variables up to a constant.

We fix splitting $H^1_{\text{dR}}(X) = W \oplus \Omega^1(X)$. This type of splitting occurs in the theory of $p$-adic height pairings.

Let $p : \mathcal{O}(\Delta)^\times \to X \times X$ and $\pi_1, \pi_2 : X \times X \to X$ be the obvious projections.

**Definition 5.1.** We define elements $\mu \in H^\otimes(X)$ and $\Phi \in H^\otimes(X \times X)$ as follows: Fix a basis $\{\omega_1, \ldots, \omega_g\}$ of $\Omega^1(X)$. Let $\{\bar{\omega}_1, \ldots, \bar{\omega}_g\} \subset W$ be a dual basis with respect to the cup product (i.e., $\text{tr}(\bar{\omega}_i \cup \omega_j) = \delta_{ij}$). Then we set
\[
\mu = \frac{1}{g} \sum_{i=1}^g \bar{\omega}_i \otimes \omega_i \in H^\otimes(X),
\]
\[
\Phi = \pi_1^* \mu + \pi_2^* \mu - \sum_{i=1}^g (\pi_1^* \bar{\omega}_i \otimes \pi_2^* \omega_i + \pi_2^* \bar{\omega}_i \otimes \pi_1^* \omega_i) \in H^\otimes(X \times X).
\]

**Lemma 5.2.** We have $\cup \Phi = \pi_1(\mathcal{O}(\Delta)) = \mathcal{G}(\Delta)$.

**Proof.** We have
\[
\cup \Phi = \pi_1^* (\cup \mu) + \pi_2^* (\cup \mu) - \sum_{i=1}^g (\pi_1^* \bar{\omega}_i \cup \pi_2^* \omega_i + \pi_2^* \bar{\omega}_i \cup \pi_1^* \omega_i).
\]

To prove that this is the cohomology class of the diagonal it suffices to show that $\text{tr}(\cup \Phi \cup \phi) = \text{tr} \Delta^* \phi$ for any $\phi \in H^2_{\text{dR}}(X \times X)$. By Künneth we can write such $\phi$ as a combination of forms of the following three forms: $\pi_1^*(\cup \mu)$, $\pi_1^* \bar{\omega}_i \cup \pi_2^* \omega_j$ or $\pi_2^* \bar{\omega}_i \cup \pi_1^* \omega_i$. Checking the formula in each of these three cases is straightforward.  

From the above lemma and Proposition 4.1 we obtain the existence of a log function on $\mathcal{O}(\Delta)$ with curvature $\Phi$. In the classical case the analogous relation already suffices to characterize the metric up to a constant. In our case however, this is not sufficient yet, since we can always modify our log function by $p^* f \phi$ for $\phi \in \Omega^1(X \times X)$. Again by Künneth such a form $\phi$ can be written as $\sum_{i=1}^2 \pi_i^* \phi_i$ with $\phi_i \in \Omega^1(X)$. 

For any choice of a log function \( \log_{O(\Delta)} \) with curvature \( \Phi \) we can define a corresponding Green function \( G \) by \( G = \log_{O(\Delta)}(1) \), where 1 is the canonical section of \( O(\Delta) \).

**Definition 5.3.** For any divisor \( D = \sum n_i P_i \) on \( X \) we define the Green function for \( D \) as \( G_D = \sum n_i G(P_i, \bullet) \). We define the canonical log function on \( O(D) \) by the condition \( \log_{O(D)}(1) = G_D \).

We can alternatively express this log function as follows: Suppose again \( D = \sum n_j P_j \). For \( P \in X \) let \( i_P : X \to X \times X \) be the map \( i_P(x) = (P, x) \). Since \( O(P) = i_P^*(O(\Delta)) \) we have

\[
O(D) = \otimes (i_{P_j}^* O(\Delta))^\otimes n_j
\]

and in this way \( O(D) \) inherits a log function from the log function \( \log_{O(\Delta)} \).

Notice that the Green function for a divisor of degree 0 is determined by what was already done without fixing \( \log_{O(\Delta)} \) any further. In fact, for principal divisors it is what one can expect.

**Proposition 5.4.** We have \( G_{(f)} = \log(f) + \text{Const} \).

**Proof.** This is equivalent to the following statement: The function \( f \) determines an isomorphism \( O((f)) \cong O_X \) and this isomorphism is an isometry up to a constant.

To prove this, consider the map \( f \times \text{Id} : X \times X \to \mathbb{P}^1 \times X \). Choose any \( x_0 \in X \) and consider the line bundle \( L = O(\Delta) \otimes \pi_1^*(O(x_0))^{-1} \) with its induced log function.

We have

\[
\text{Curve}(L) = \text{Curve}(O(\Delta)) - \pi_1^* \mu = \pi_2^* \mu - \sum_{i=1}^g (\pi_1^* \omega_i + \pi_2^* \omega_i + \pi_3^* \omega_i) .
\]

We want to show that this curvature has a trace via \( f \times \text{Id} \) and compute this trace.

We have a diagram \( X' \xrightarrow{\pi} X \to \mathbb{P}^1 \) where the composed map \( X' \to \mathbb{P}^1 \) is Galois, say with Galois group \( G \). This is then also true with respect to the base change \( X' \times X \to \mathbb{P}^1 \times X \). Let \( p_i, i = 1, 2 \) be the projections from \( X' \times X \) to its factors.

We now have

\[
\sum_{\sigma \in G} (\sigma \times \text{Id})^* (\pi \times \text{Id})^* \text{Curve}(L)
\]

\[
= |G| p_2^* \mu - \sum_{i=1}^g \left( \sum_{\sigma \in G} p_1^* \sigma^* \pi^* \omega_i + p_2^* \omega_i + \sum_{\sigma \in G} p_1^* \sigma \omega_i + p_2^* \sigma^* \pi^* \omega_i \right) = |G| p_2^* \mu .
\]

The two sums in the brackets are 0 because \( \sum \sigma^* \pi^* \eta \), with \( \eta = \omega_i \) or \( \bar{\omega}_i \), is a pullback from \( \mathbb{P}^1 \) and \( \Omega^1(\mathbb{P}^1) = H^1_{\text{dR}}(\mathbb{P}^1) = 0 \). Proposition 4.7 now implies that the induced quasi log function on \( L' := \text{Norm}_{f \times \text{Id}} L \) is a log function and its curvature is a multiple of \( \pi_2^* \mu \), where \( \pi_2 : \mathbb{P}^1 \times X \to X \) is the projection on the second factor. Consider now \( L'' = L' \otimes (i_0^* L')^{-1} \), where \( i_0 : \mathbb{P}^1 \times X \to \mathbb{P}^1 \times X \) is given by \( i_0(a, x) := (0, x) \). We give \( L'' \) the induced log function. The curvature of this log function is 0, which determines it up to the integral of an element of \( \Omega^1(\mathbb{P}^1 \times X) \leftrightarrow \Omega^1(X) \). Now, the restriction of \( L'' \) to 0 \times X is canonically trivial with trivial log function. By what was said before the conditions of trivial curvature and triviality on the restriction to 0 \times X determines the log function uniquely. On the other hand, the restriction of \( L'' \) to \( \infty \times X \) is up to scaling \( O((f)) \) with its log function. There is an isomorphism \( L'' \cong O_{\mathbb{P}^1 \times X} \). Indeed, suppose \( f(x_0) = a_0 \). The
We define the homomorphism \( \text{Corollary 5.5.} \) An isomorphism \( f \) is trivial after scaling. Thus, after scaling we obtained the same log function as \( L \). Oture is still 0. Since \( F(\infty, x)^{-1} \) is a constant multiple of \( f(x) \), so up to scaling \( f \) indeed induces an isometry.

**Corollary 5.5.** An isomorphism \( \mathcal{O}(D) \rightarrow \mathcal{O}(D') \) is an isometry up to constant.

**Definition 5.6.** We define the homomorphism \( \tau_{\log} : \overline{\mathbb{Q}}_p(X)^* \rightarrow \overline{\mathbb{Q}}_p \) by \( \tau_{\log}(f) = \log(f) - G(f) \).

The \( \tau_{\log} \) character is the \( p \)-adic analogue of the integral of the norm of a section. It will thus be used to associate the “infinite” fibers to the Arakelov divisor of a section of a line bundle.

**Definition 5.7.** Let \( \mathcal{L} \) be a line bundle on \( X \). The log function \( \log_{\mathcal{L}} \) is called admissible (with respect to the Green function \( G \)), if for each section \( s \) of \( \mathcal{L} \) the function \( \log_{\mathcal{L}}(s) - G_{\text{div}(s)} \) is constant. This constant will be denoted \( \tau_{\log_{\mathcal{L}}}(s) \).

**Lemma 5.8.** The condition for being admissible need only be checked on a single section. The function \( \tau_{\log_{\mathcal{L}}} \) satisfies the relation \( \tau_{\log_{\mathcal{L}}}(fs) = \tau_{\log}(f) + \tau_{\log_{\mathcal{L}}}(s) \) when \( f \in \overline{\mathbb{Q}}_p(X)^* \).

**Proof.** Immediate from Proposition 5.3.

**Lemma 5.9.** Any isomorphism between admissible metrized line bundles is an isometry up to scaling.

**Proof.** If \( T : \mathcal{L}_1 \rightarrow \mathcal{M} \) is an isomorphism let \( s \in \mathcal{L}(C) \). Since \( \tau_{\log_{\mathcal{L}}}(s) \) and \( \tau_{\log_{\mathcal{M}}}(T(s)) \) have the same divisor admissibility implies that \( \log_{\mathcal{L}}(s) - \log_{\mathcal{M}}(T(s)) \) is a constant. Since any other section is obtained from \( s \) via multiplication by a rational function it is easy to see that this constant is independent of \( s \).

For any divisor \( D \) the log function we defined on the line bundle \( \mathcal{O}(D) \) is clearly admissible. Note also that the log function \( \log_{\mathcal{L}} \otimes \log_{\mathcal{M}} \) on \( \mathcal{L} \otimes \mathcal{M} \) is admissible if both \( \log_{\mathcal{L}} \) and \( \log_{\mathcal{M}} \) are.

As a first step towards removing the degrees of freedom in the definition of \( \log_{\mathcal{O}(\Delta)} \) we can assume that it is symmetric. Indeed, since there is a canonical isomorphism between \( \mathcal{O}(\Delta) \) and \( \sigma^*\mathcal{O}(\Delta) \), where \( \sigma(x, y) = (y, x) \) we can consider, for any log function \( \log_{\mathcal{O}(\Delta)} \) as above the function \( (\log_{\mathcal{O}(\Delta)} + \sigma^*\log_{\mathcal{O}(\Delta)})/2 \). This still has the same curvature form since \( \Phi \) is symmetric. A symmetric log function is determined up to a function of the form \( \sum_{i=1}^2 p^i \pi_i^* \int_\phi \phi \in \Omega^1(X) \).

It is well known that \( \Delta^*\mathcal{O}(-\Delta) \) is canonically isomorphic to \( \omega_X \) - the canonical bundle on \( X \). Thus, given a choice of a log function on \( \mathcal{O}(\Delta) \) the line bundle \( \omega_X \) inherits a canonical log function. It therefore make sense to impose the condition that this log function is admissible with respect the Green function the same log function defined. This allows us to determine a canonical log function on \( \mathcal{O}(\Delta) \), hence a canonical Green function, up to a constant, as we will see in the next theorem.
Theorem 5.10. There exist a unique up to constant symmetric log function on $O(\Delta)$ with curvature $\Phi$ and such that the pulled back log function on $\omega_X$ is admissible with respect to the induced Green function.

Proof. We interpret the admissibility condition as follows: for $P \in X$ let $i_P : X \to X \times X$ be the map $i_P(x) = (P, x)$. We know that $\omega_X \cong O(D)$ for some divisor $D = \sum n_j P_j$. Since $O(P) = i^*_P O(\Delta)$ we can write the isomorphism as

$$ (\Delta^* O(\Delta))^{-1} \cong \otimes (i^*_P O(\Delta))^{\otimes n_j}. $$

For each choice of a log function both sides inherit a log function and we need to find one for which the isomorphism is an isometry up to a constant. First we notice that both sides have the same curvature. This is because $\Delta^* \Phi = (2 - 2g) \mu$ while $i^*_P \Phi = \mu$ and $\text{deg}(D) = 2g - 2$. (this is analogous to the classical theory). Thus, the differential of the difference of the log functions equal $\pi^* \omega$ for some $\omega \in \Omega^1(X)$. On the other hand, if we modify the log function by $\sum_{i=1}^d p^* \pi^* \phi$ the differential of the log function on the left hand side of (5.1) changes by $-2\phi$ while on the right hand side it changes by $\text{deg} D \cdot \phi = (2g - 2)\phi$. Since $g > 0$ by assumption we can solve this equation uniquely to obtain $d \log O(\Delta)$ uniquely, hence $\log O(\Delta)$ uniquely up to a constant. □

Corollary 5.11. We have $\bar{\partial} \partial G = \Phi|_{X \times X - \Delta}$.

In the course of proving Theorem 5.10 we saw the following.

Corollary 5.12. The curvature of an admissible log function on a line bundle $L$ is $\text{deg}(L) \cdot \mu$.

6. The Faltings volume on the determinant of cohomology

In classical Arakelov theory one defines a volume on the determinant of cohomology of a line bundle (or, more generally, of a vector bundle). This data then enters into the Riemann-Roch theorem. In [Fal84] Faltings constructs the volume for line bundles in an axiomatic way. It is also possible to obtain a volume using analysis (analytic torsion). Here we follow the approach of Faltings. It would be very interesting if one could also find a definition using an analogue of analytic torsion but we have no idea how to do this.

Let $X$ be a complete non-singular curve over $\mathbb{Q}_p$. We fix a Green function $G$ on $X$ out of the almost canonical class. Let $L$ be a line bundle on $X$. Recall that the determinant of cohomology of $L$ is given by

$$ \lambda(L) = \det(H^0(X, L)) \otimes \det(H^1(X, L))^{-1}, $$

where $\det$ is the top exterior power.

Proposition 6.1. There exist a correspondence

$$(L, \log_L) \mapsto \log \lambda(L) \log^{(\log_L)} \lambda(L)$$

from line bundles with an admissible log function with respect to $G$ to “metrized lines”, such that the following properties are satisfied:

1. an isometry $L \to L'$ induces an isometry $\lambda(L) \to \lambda(L')$,
2. The behavior with respect to scaling is such that

$$ \log^{(\log_L)} + \alpha = \log^{(\log_L)} + \chi(L) \cdot \alpha, $$

where $\chi(L)$ is the Euler characteristic of $L$. 
(3) The canonical isomorphism
\[ \lambda(\mathcal{O}(D)) \cong \lambda(\mathcal{O}(D - P)) \otimes \mathcal{O}(D)[P], \]
where \( \mathcal{O}(D)[P] \) is the fiber of \( \mathcal{O}(D) \) at \( P \), is an isomorphism.

Furthermore, these properties determine \( \log_{\lambda(\mathcal{L})} \) up to common scaling for all \( \mathcal{L} \) together.

**Proof.** The proof proceeds in a similar manner to the corresponding proof in [Fal84]. The uniqueness is clear since we can pass from one metrized line bundle to any other by either adding or deleting points or scaling, so fixing \( \log_{\lambda(\mathcal{L})} \) for one metrized line bundle \( \mathcal{L} \) determines it on all of them. Following Faltings again, fixing a divisor \( D \) of a log function on \( \mathcal{O} \) of a log function on \( \mathcal{O} \), we get a line bundle \( \mathcal{N} \) on \( X^r \) whose fiber at \( (P_1, \ldots, P_r) \) is \( \lambda(\mathcal{O}(E - \sum P_i)) \). The line bundle \( \mathcal{N} \) carries a pseudo-log function determined by our conditions. As shown by Faltings, \( \mathcal{N} \) is the pullback from \( \text{Pic}_{g-1}(X) \), the jacobian variety of line bundles of degree \( g - 1 \) on \( X \), of \( \mathcal{O}(-\Theta) \) under the map \( \varphi \) sending \((P_1, \ldots, P_r) \) to \( E - \sum P_i \), where \( \Theta \) is the theta divisor of line bundles with a global section. It suffices to show that the quasi-log function on \( \mathcal{N} \) is the pullback of a log function on \( \mathcal{O}(-\Theta) \).

The forms and cohomology classes \( \omega_i \) and \( \check{\omega}_i \) are pullbacks of classes, which we denote by the same notation, on \( \text{Pic}_{g-1}(X) \). It is known that
\[ ch_1(\mathcal{O}(-\Theta)) = - \sum_{i=1}^{g} \check{\omega}_i \cup \omega_i. \]

It follows from Proposition 4.4 that on \( \mathcal{O}(-\Theta) \) there exist a log function whose curvature form is
\[ \text{Curve}(\mathcal{O}(-\Theta)) = - \sum_{i=1}^{g} \check{\omega}_i \otimes \omega_i. \]

Let \( p_k : X^r \to X, 1 \leq k \leq r \) be the projection on the \( k \)th factor and let \( p_{km} \) be the projection on the \( k \) and \( m \) factors. Then the curvature on the pullback log function on \( \mathcal{N} \) is
\[ - \sum_{i=1}^{g} \left( \sum_{k=1}^{r} p_k^*(\check{\omega}_i) \right) \otimes \left( \sum_{k=1}^{r} p_k^*(\omega_i) \right). \]

We now show that the pseudo-log function on \( \mathcal{N} \) is indeed a log function and compute its curvature. Let \( \mathcal{N}_m, 0 \leq m \leq r \), be the line bundle whose fiber at \((P_1, \ldots, P_r)\) is \( \lambda(\mathcal{O}(E - \sum_{i=1}^{m} P_i)) \). Then \( \mathcal{N}_0 \) is the constant line bundle \( \lambda(\mathcal{O}(E)) \) and condition 3 implies an isometry \( \mathcal{N}_m = \mathcal{N}_{m-1} \otimes \mathcal{L}_m^{-1} \), where \( \mathcal{L}_m \) is the line bundle whose fiber at \((P_1, \ldots, P_r)\) is the fiber at \( P_m \) of \( \mathcal{O}(E - \sum_{i=1}^{m-1} P_i) \). We have
\[ \mathcal{L}_m = p_m^* \mathcal{O}(E) \otimes \bigotimes_{k=1}^{m-1} p_{km}^* \mathcal{O}(\Delta)^{-1}. \]

Thus we obtain an isometry
\[ \mathcal{N} \cong \lambda(\mathcal{O}(E)) \otimes \bigotimes_{m=1}^{r} p_m^* \mathcal{O}(E)^{-1} \otimes \bigotimes_{k<m} p_{km}^* \mathcal{O}(\Delta). \]
In particular, the pseudo-log function on $\mathcal{N}$ is indeed a log function and we may also compute its curvature, using Corollary 5.12 for the curvature of $\mathcal{O}(E)$, to be (here we follow [Lan88, p. 146])

$$\text{Curve}(\mathcal{N}) = -(r + g - 1) \sum_{m=1}^{r} p_{m}^{*} \mu + \sum_{k<m} p_{km}^{*} \Phi$$

$$= -(r + g - 1) \sum_{m=1}^{r} p_{m}^{*} \mu$$

$$+ \sum_{k<m} \left( p_{k}^{*} \mu + p_{m}^{*} \mu - \sum_{i=1}^{g} (p_{k}^{*} \bar{\omega}_{i} \otimes p_{m}^{*} \omega_{i} + p_{m}^{*} \bar{\omega}_{i} \otimes p_{k}^{*} \omega_{i}) \right).$$

The term $(r - 1) \sum_{m=1}^{r} p_{m}^{*} \mu$ cancels, leaving us with

$$= - \sum_{m=1}^{r} g \sum_{i=1}^{g} p_{m}^{*} (\bar{\omega}_{i} \otimes \omega_{i}) - \sum_{k<m} \sum_{i=1}^{g} (p_{k}^{*} \bar{\omega}_{i} \otimes p_{m}^{*} \omega_{i} + p_{m}^{*} \bar{\omega}_{i} \otimes p_{k}^{*} \omega_{i})$$

$$= - g \sum_{i=1}^{r} \left( \sum_{k=1}^{r} p_{k}^{*} (\bar{\omega}_{i}) \right) \otimes \left( \sum_{k=1}^{r} p_{k}^{*} (\omega_{i}) \right).$$

Thus, $\log_{\mathcal{N}}$ and $\varphi^{*} \log_{\mathcal{O}(-\Theta)}$ have the same curvature hence they differ by the integral of a holomorphic form $\omega$. However, both log functions are invariant with respect to the action of the symmetric group on $X$. It follows that $\omega = \sum_{i} p_{i}^{*} \omega'$ where $\omega' \in \Omega^{1}(X)$. But then $\omega$ can be pulled back from $\text{Pic}_{g-1}$ and therefore $\log_{\mathcal{N}}$ can be pulled back as well, which proved the result. \hfill \Box

7. RELATIONS WITH THE THEORY OF COLEMAN AND GROSS

In [CG89] Coleman and Gross define a $p$-adic height pairing on curves with good reduction above $p$ as a sum of local terms. In [Bes02b] we prove that this local height pairing is the same as the one defined by Nekovář in [Nek93]. The height pairing is defined for divisors of degree 0 and one expects that it coincides with the restriction to these divisors of the Arakelov intersection pairing. Once we define the Arakelov intersection we will prove that this is indeed the case. At the moment we can only prove that the local terms above $p$ agree.

We first recall the local theory in [CG89]. We reformulate slightly since Coleman and Gross work over $\mathbb{C}_{p}$, but we can easily work over $\mathbb{Q}_{p}$ instead. Also, their definition is only for curves with good reduction but once one has integration theory in the bad reduction case as well the extension is done verbatim. Let $X$ be again a complete non singular curve over $\mathbb{Q}_{p}$. Recall from Section 3 the space $\mathcal{T}$ of forms of the third kind on $X$, the subspace $\mathcal{T}_{l}$ of dlog forms and the map $\Psi : \mathcal{T}/\mathcal{T}_{l} \to H^{1}_{\text{dr}}(X)$.

The theory of Coleman and Gross depends, as does our theory, on the choice of a subspace $W \in H^{1}_{\text{dr}}(X)$ complementary to $\Omega^{1}(X)$ which is isotropic with respect to the cup product. In [CG89] this is not needed but is imposed if one wants to make the height pairing symmetric.

**Definition 7.1.** For any divisor $D$ of degree 0 on $X$ we let $\omega_{D} \in \mathcal{T}$ be the unique form satisfying $\text{Res}(\omega_{D}) = D$ and $\Psi(\omega_{D}) \in W.$
We will now determine the cohomology class of the form $\frac{\partial}{\partial P}$ Let $D$ be a divisor of degree $0$ on $X$ with disjoint supports. Then their pairing is given by $\langle D, E \rangle := \int_{E} \omega_D$. The integral in the definition is the Coleman integral of $\omega_D$ evaluated in the standard way on $E$.

It is now clear that the equality of the local height pairing and the Arakelov pairing at primes above $p$ follows from the following result.

**Theorem 7.3.** Let the space $\mathcal{W}$ be chosen. Then for any divisor $D$ of degree $0$ on $X$ we have $dG_D = \omega_D$.

**Proof.** We have $\tilde{\partial}(G_D)|_{X-D} = 0$ by Corollary 5.12 so that $dG_D$ is holomorphic outside $D$. It follows from the definition of $G_D$ that it has logarithmic singularities and its residue divisor is exactly $D$. Let $\omega_D := dG_D - \omega_D$. Then $\omega_D \in \Omega^1(X)$ for each $D$. Further, by Proposition 5.3 we have $\omega_D = 0$ for a principal divisor $D$, so the map $D \mapsto \omega_D$ factors through $\mathcal{W}$, where $\mathcal{W}$ is the Jacobian of $X$, and it is clearly additive. To prove that $\omega_D = 0$ it suffices to show that for any $w \in \mathcal{W}$ we have $w \cup \omega_D = 0$. The projection $\Psi$ is the identity on $\Omega^1(X)$ and by construction it maps $\omega_D$ to $w$. Since $\mathcal{W}$ is isotropic we have $w \cup \omega_D = w \cup \Psi(dG_D)$, and by Corollary 5.13 this equals $\langle w, dG_D \rangle_g$. The map $D \mapsto \langle w, dG_D \rangle_g$ is an additive map on $\mathcal{W}$, so it suffices to prove that it is locally analytic and its derivative is $0$. For any $w \in \mathcal{W}$ we consider the map $X^r \to J_0$, $(P_1, \ldots, P_r) \mapsto \sum P_i - rP_0$, for some $P_0$, which is surjective for sufficiently large $r$. It will suffice to show that the map $(P_1, \ldots, P_r) \mapsto \langle w, dG_{\sum P_i - rP_0} \rangle_g$ has zero derivative with respect to every $P_i$, and for this it suffices to check the derivative of $P \mapsto \langle w, dG_{P - P_0} \rangle_g$. Let $\partial/\partial P$ be a vector field on $X$. By Lemma 5.14 we have

$$\frac{\partial}{\partial P} \langle w, dG_{P - P_0} \rangle_g = \langle w, \frac{\partial}{\partial P} dG_{P - P_0} \rangle_g = \langle w, \frac{\partial}{\partial P} dG_P \rangle_g.$$ 

We will now determine the cohomology class of the form $(\partial/\partial P) dG_P$. The situation can be described as follows. We have on $X \times X - \Delta$ the Coleman form $dG_P(Q)$. We differentiate with respect to the vector field $\partial/\partial P$ on the first variable $P$ and then restrict to the fiber at $P$. Since $dG$ is closed we have by 3.2 $(\partial/\partial P) dG_P = d(\partial (dG_P))|_{\partial/\partial P}$. Since the retraction operator $|_{\partial/\partial P}$ is $\mathcal{O}_X$-linear we have by Lemma 2.12 that $\partial (dG_P) = (\partial dG)|_{\partial/\partial P}$, where the retraction on $H^1_{\text{dR}}(X \times X - \Delta) \otimes \Omega^1(X \times X - \Delta)$ operates on the second factor. By Corollary 5.14 and Definition 5.1 we have

$$\partial (dG_P) = \Phi|_{\partial/\partial P} = \frac{1}{g} \sum_{i=1}^{g} \pi_i \tilde{\omega}_i \otimes \pi_i^*(\omega_i|_{\partial/\partial P}) - \sum_{i=1}^{g} \pi_i^* \tilde{\omega}_i \otimes \pi_i^*(\omega_i|_{\partial/\partial P}),$$

which, when restricted to the fiber above $P$ yields $\sum \tilde{\omega}_i \otimes \alpha_i$, with $\alpha_i$ the constant $-(\omega_i|_{\partial/\partial P})(P)$. If we now represent the $\tilde{\omega}_i$ by forms of the second kind and restrict further to $U \subset X$ where all these forms are holomorphic, then it follows from Proposition 2.7 and the description of the $\partial$ operator on affine varieties that

$$(dG_P)|_{\partial/\partial P} = \sum \alpha_i \int \tilde{\omega}_i + f$$

where $f \in \mathcal{O}(U)$. Therefore,

$$d((dG_P)|_{\partial/\partial P})|_{\partial/\partial P} = \sum \alpha_i \tilde{\omega}_i + df,$$
whose cohomology class belongs to $W$ by the definition of the \( \bar{\omega} \). It follows that 
\((\partial/\partial P) dG_P\) is a form of the second kind representing a class in $W$ and the isotropy of $W$ completes the proof. \(\square\)

The following proposition demonstrates that the Green function is forced on us if we assume compatibility with the Coleman-Gross pairing, symmetry and a natural residue condition, analogous to the one we have in the classical theory. As mentioned in the introduction, this was our original approach. At the same time we provide a formula for the Green function using only Coleman integration in one variable.

**Proposition 7.4.** The canonical Green function $G$ is the unique function up to constant satisfying the following properties.

1. $G$ is symmetric
2. The induced height pairing \( (\cdot, \cdot) \) is the Coleman-Gross height pairing.
3. The following residue condition is satisfied: For any point $P$ the canonical map $\omega_X \otimes \mathcal{O}(P) \to (\mathcal{Q}_P)_P$, where $(\mathcal{Q}_P)_P$ is the skyscraper sheaf at $P$ with fiber $\mathcal{Q}_P$, given by $\omega \otimes f \to \text{Res}_P(f \omega)$, is an isometry.

**Proof.** That $G$ satisfies the 3 conditions follows from Theorem 7.3 and Theorem 5.10. For uniqueness we will in fact prove the following explicit formula for the Green function in terms of the Coleman-Gross height pairing: Choose $a$ and $b$ two points in $X$. Then we have

\[
G(P, Q) = \frac{1}{2g} \left( \int_{2gP - \text{div} \omega_2 - Q - b} \omega_{Q - b} + \int_{2gP - \text{div} \omega_1 - P - a} \omega_{P - a} \right)
\]

where $\omega_1$ (respectively $\omega_2$) is any form with log singularities at $P$ and $a$ (respectively $Q$ and $b$) and such that the log of its residues at these two points is the same.

Let $P$ be a point of $X$ and let $f$ be a local parameter at $P$. Let $\omega$ be a form with a simple pole at $P$. We can write $\omega = (1/f) f \omega$ and by condition (3) we should have

\[
\log(\text{Res}_P(\omega)) = (\log_{\mathcal{O}(P)}(1/f) + \log_{\omega_X}(f \omega))_P = (G_P + \log(f) - \log(f) + \log_{\omega_X}(\omega))_P = \lim_{z \to P} G_P(z) + \log_{\omega_X}(\omega)(z).
\]

Since the log function on $\omega$ is admissible we have

\[
\log(\text{Res}_P(\omega)) = \lim_{z \to P} G_P(z) + \log_{\omega_X}(\omega) = G_{\text{div}(\omega) + P}(P) + \log_{\omega_X}(\omega).
\]

Consider now any two points $P$ and $Q$ on $X$ and a differential $\omega$ with simple poles at both $P$ and $Q$ such that the logs of the residues of $\omega$ at $P$ and $Q$ are equal. We find the equations

\[
G_{\text{div}(\omega) + P + Q}(P) - G_{Q}(P) + \log_{\omega}(\omega) = G_{\text{div}(\omega) + P + Q}(Q) - G_{P}(Q) + \log_{\omega}(\omega)
\]

and from the symmetry of $G$ we get by subtracting

\[
G_{\text{div}(\omega) + P + Q}(P - Q) = 0,
\]

from which we get, again by symmetry

\[
G_{P - Q}(\text{div}(\omega) + P + Q) = 0.
\]
Note that the divisor \( \text{div}(\omega) + P + Q \) has degree \( 2g \). It follows that for any point \( x \) in \( X \) different from \( P \) and \( Q \) we have

\[
2gG_{P-Q}(x) = G_{P-Q}(2gx) = G_{P-Q}(2gx - (\text{div}(\omega) + P + Q))
\]

\[= \int_{2gx-(\text{div}(\omega)+P+Q)} \omega_{P-Q}.
\]

This gives us formula \((7.1)\) as follows: Write the bilinear height pairing as \( \langle \bullet, \bullet \rangle \).

We choose a constant for our Green function by insisting that \( G(a, b) = 0 \). Then

\[
2gG(P, Q) = 2g\langle P, Q \rangle = 2g(\langle P, Q-b \rangle + \langle b, P-a \rangle + \langle b, a \rangle)
\]

\[= \int_{2gP-\text{div} \omega_2-Q-b} \omega_{Q-b} + \int_{2gb-\text{div} \omega_1-P-a} \omega_{P-a} + 0.
\]

\(\square\)

8. Local theory over non algebraically closed fields

So far we found it more convenient to develop the local theory over \( \mathbb{Q}_p \). However, for Arakelov theory we will need to work with finite extensions of \( \mathbb{Q}_p \). In this section we collect all the necessary results needed for doing this.

Suppose now that \( K \) is a finite extension of \( \mathbb{Q}_p \) and that \( X/K \) is a smooth complete curve of genus \( \geq 1 \). We assume that the space \( W \) is also defined over \( K \).

Finally, we choose a branch of the logarithm defined over \( K \).

**Proposition 8.1.** There exist a canonical Green function \( G \) for \( X \) defined over \( K \).

It is defined up to a constant in \( K \).

**Proof.** The form \( dG \) is uniquely determined by the conditions spelled out in Theorem \(\text{(5.10)}\). If \( \sigma \in \text{Gal}(K/K) \), then all of these conditions are invariant under \( \sigma \), so \( \sigma(dG) = dG \). By Proposition \(\text{(2.9)}\) the form \( dG \) is in fact defined over \( K \), hence it has an integral defined over \( K \) and defined up to a constant in \( K \). \(\square\)

From now on we will assume that a Green function defined over \( K \) has been fixed.

**Corollary 8.2.** If \( L \) is a line bundle on \( X \), then there exists an admissible log function on \( L \) defined over \( K \). If \( D \) is a divisor on \( X \), then the canonical log function on \( \mathcal{O}(D) \) of Definition \(\text{(5.3)}\) is defined over \( K \).

Using our Green function we can define the local intersection pairing.

**Definition 8.3.** Let \( D \) and \( E \) be divisors on \( X \) with disjoint supports and let \( \overline{D} = \sum n_iP_i \) and \( \overline{E} = \sum m_jQ_j \) be their extensions to \( \overline{X} \). The **local intersection pairing** \( \langle D, E \rangle \in K \) is defined by

\[
\langle D, E \rangle = \sum n_im_jG(P_i, Q_j).
\]

Hidden in this definition is the fact that the pairing indeed takes values in \( K \), which follows trivially from the properties of \( G \).

Until the end of this section, we develop the relation between log functions and determinants. The following definition of log functions is a mere specialization of previous definitions for the case of dimension 0.
Definition 8.4. Let $K'$ be a finite extension of $K$. Let $V$ be a $K'$-line, i.e., a one-dimensional $K'$ vector space. A log function on $V$ gives, for any embedding $\tau : K' \hookrightarrow K$ fixing $K$ a log function $\log_\tau$ on $V_\tau := V \otimes_{K'} K$. Such a log function is said to be defined over $K$ if for any automorphism $\sigma \in \text{Gal}(K/K)$ we have $\sigma(\log_\tau(x)) = \log_\sigma(\sigma(x))$ for $x \in V_\tau$, where $\sigma : V_\tau \to V_{\sigma \tau}$ is the evident map.

Note that a log function on $V$ defined over $K'$ is simply a log function $\log_V : V \to K'$. Note also that if $\mathcal{L}$ is a metrized line bundle over a $K$-variety $X$ and $\log_{\mathcal{L}}$ is a log function on $\mathcal{L}$. Then the fiber of $\mathcal{L}$ over any closed points acquires in a natural way a log function over $K$.

We will now consider a $K'$-line $V$ as above. Since $V$ is a finite dimensional vector space over $K$, its determinant $\det V = \det_K V$ is defined. We want to obtain log functions on the determinants in certain situations.

Definition 8.5. Suppose that $U$ and $V$ are both $K'$-lines with log functions defined over $K'$. Then, the $K$-line $(U : V) := \det(U) \otimes \det(V)^{-1}$ has a log function defined over $K$ in the following way: Let $\alpha : U \to V$ be an isomorphism such that $\log_U \circ \alpha = \log_U + c$, where $c \in K'$. The isomorphism $\alpha$ induces by composition a canonical isomorphism $\beta : (U : V) \to K$ and we define $\log_{(U : V)} = \log \circ \beta - \text{tr}_{K'/K} c$.

The log function we defined is independent of the choices made. Indeed if $\alpha$ is changed to $c'\alpha$, then $c$ is changed to $c + \log(c')$ while $\beta$ is changed to $(\text{Norm}_{K'/K} c') \beta$ so $\log_{(U : V)}$ is unchanged. Another way of describing this log function is to say that if $\alpha$ is an isometry then $\beta$ is also an isometry while if we scale the log function on $U$ by $c$ then we scale the log function on $(U : V)$ by $\text{tr}_{K'/K} c$.

Now we consider extension of scalars.

Definition 8.6. Suppose $V$ is a $K'$-metrized line and $K''$ is a finite extension of $K'$. We define the log function on the $K''$-line $W := V \otimes_{K'} K''$ by $\log(v \otimes \alpha) = \log(v) + \log(\alpha)$.

Determinants behave in a well known way under extensions. Suppose $[K'' : K'] = n$. We have a canonical isomorphism

\begin{equation}
\det_K(W) \otimes \det_K(K'')^{-1} \cong (\det_K(V) \otimes \det_K(K'))^{\otimes n}
\end{equation}

defined as follows: Choose a $K'$-isomorphism $K'^{\otimes n} \cong K''$. This induces an isomorphism $V^{\otimes n} \cong V \otimes_{K'} K''$ and as consequence isomorphisms

\begin{align*}
(\det_K K')^{\otimes n} &\cong \det_K K'' \\
(\det_K V)^{\otimes n} &\cong \det_K (V \otimes_{K'} K''),
\end{align*}

from which (8.1) follows. It is easily verified that this isomorphisms is independent of the choices. The behavior with respect to log functions is also easily checked.

Proposition 8.7. In the situation described above the canonical isomorphism (8.1) is an isometry.

Proof. This is clear from the description above if we choose an isometry $K' \cong V$. Observing how the log functions change with respect to scaling finishes the proof.

For log functions defined over $K$ we can define the log function on the determinant, and not only on the quotient of two determinants.
Lemma 8.8. Suppose $K'$ is a finite extension of $K$, $V$ a $K'$-line equipped with a log function over $K$. Then the $K$-line $\det_K V$ has a unique log function satisfying the following property: The isomorphism

$$V \otimes_K \tilde{K} \cong \sum_{\sigma : K' \to \tilde{K}} V \otimes_{K', \sigma} \tilde{K}, \quad x \otimes \alpha \mapsto \sum x \otimes \alpha,$$

where the sum is over all embeddings $\sigma : K' \to \tilde{K}$ fixing $K$, induces an isomorphism

$$(\det_K V) \otimes_K \tilde{K} \cong \det_{\tilde{K}} V \otimes_K \tilde{K} \cong \otimes_{\sigma : K' \to \tilde{K}} V \otimes_{K', \sigma} \tilde{K}$$

and this isomorphism becomes an isometry with respect to the log functions on both sides.

Proof. Uniqueness is clear. To prove existence, let $x$ be a basis of $V$ over $K'$ and let $\{\beta_i\}$, $i = 1, \ldots, n$ be a basis of $K'$ over $K$. Then $\{\beta_i x\}$ is a basis of $V$ over $K$. Number the embeddings $\sigma_j$, $j = 1, \ldots, n$. We have

$$\beta_i x \mapsto \beta_i x \otimes 1 \mapsto \oplus_j \beta_i x \otimes 1 = \oplus_j \sigma_j(\beta_i) \cdot (x \otimes 1)_j$$

where the subscript $j$ is given to distinguish the different components. The basis $\wedge_i (\beta_i x)$ of $\det_K V$ is mapped to $\det(\sigma_j(\beta_i) \cdot \wedge_j (x \otimes 1))$ and this forces us to define

$$\log(\wedge_i (\beta_i x)) = \log(\det(\sigma_j(\beta_i))) + \sum_j \log_j ((x \otimes 1)_j),$$

where $\log_j$ is the log function on $V \otimes_{K', \sigma_j} \tilde{K}$. Since the log function is defined over $K$ we have $\log_j ((x \otimes 1)_j) = \sigma_j(\log(x))$ so we obtain

$$\log(\wedge_i (\beta_i x)) = \log(\det(\sigma_j(\beta_i))) + \text{tr}_{K/K} \log(x).$$

The proof will be complete if we show that the $\log(\det(\sigma_j(\beta_i))) \in K$. This is clear since applying an automorphism of $\tilde{K}$ over $K$ multiplies $\det(\sigma_j(\beta_i))$ by $\pm 1$ so the log is unchanged.

The log function just defined is easily seen to be compatible with the one defined in Definition 8.5 as follows:

Proposition 8.9. If $U$ and $V$ are two $K'$-lines with a log function defined over $K$. Let $(U : V)$ be the metrized $K$-line of Definition 8.3. Then the isomorphism $(U : V) = \det(U) \otimes \det(V)^{-1}$ is an isometry, with $\det(U)$ and $\det(V)$ having the log functions defined in Lemma 8.8.

Proposition 8.10. Let $K'$ be a finite extension of $K$. Let $\log$ be the log function on $K'$ obtained from the one on $K$. Let $V = K'$ with this log function (defined over $K$). The trace form induces an isomorphism $\det_K V \otimes \det_K V \to K$ and this is an isometry.

Proof. Let $\{\beta_i\}$ be a basis of $K'$ over $K$ and let $\beta_i^t$ be a dual basis with respect to the trace form. The log function we defined sends $\wedge_i \beta_i$ to $\log(\det(\sigma_j(\beta_i)))$, where $\sigma_j$ are the embeddings of $K'$ in $\tilde{K}$, and similarly with $\beta_i^t$ replacing $\beta_i$. The duality with respect to the trace form implies $(\sigma_j(\beta_i)) \cdot (\sigma_j(\beta_i^t))^t = I$, hence $\log(\wedge_i \beta_i) + \log(\wedge_i \beta_i^t) = 0$, which is what we want.
9. THE INTERSECTION PAIRING

We now combine the $p$-adic analysis of the previous chapters to obtain a $p$-adic Arakelov intersection pairing. For motivation to the setup introduced here the reader is encouraged to look at [CG89].

The general setup is as follows: $F$ is a number field and $p$ is a prime. We choose a “global log” - a continuous idele class character

$$\ell : \mathbb{A}_F^\times / F^\times \to \mathbb{Q}_p.$$ 

One deduces from $\ell$ the following data:

- For any place $v \nmid p$ we have $\ell_v(\mathcal{O}_{F_v}^\times) = 0$ for continuity reasons, which implies that $\ell_v$ is completely determined by the number $\ell_v(\pi_v)$, where $\pi_v$ is any uniformizer in $F_v$.
- For any place $v | p$ one can decompose

$$\mathcal{O}_{F_v}^\times \xrightarrow{\ell_v} \mathbb{Q}_p \xrightarrow{\log_v} F_v \xrightarrow{t_v} \mathbb{Q}_p$$

where $t_v$ is a $\mathbb{Q}_p$-linear map. We assume that $\ell_v$ is ramified in the sense that it does not vanish on $\mathcal{O}_{F_v}^\times$. It is then possible to extend $\log_v$ to $F_v^\times$ in such a way that the diagram above remains commutative when $\mathcal{O}_{F_v}^\times$ is replaced by $F_v^\times$, i.e., we have the decomposition

$$\ell_v = t_v \circ \log_v.$$

Note that for $v \nmid p$ and for any $\mathcal{O}_{F_v}$-ideal $I$ in $F_v$ we can define unambiguously

$$\ell_v(I) := \ell_v(\tau), \quad \tau \text{ is a generator of } I.$$

Let $X$ be an arithmetic surface over $\mathcal{O}_F$ (i.e., a proper regular curve over $\mathcal{O}_F$). Let $X_v := X \otimes_{\mathcal{O}_F} F_v$. We make the following additional choices for each $v | p$

- A space $W_v$ in $H_{dR}^1(X_v/F_v)$ complementary to $F^1$ as in Section 5.
- A choice of a Green function $G_v$, defined over $F_v$, out of the almost canonical one (choice of a constant).

**Definition 9.1.** A \textit{$p$-adic Arakelov divisor} (Arakelov divisor for short) on $X$ is a formal combination

$$D = D_{\text{fin}} + D_{\infty}, \quad \text{where } D_{\infty} = \sum_{v | p} \lambda_v X_v,$$

where $\lambda_v \in F_v$. Here, $X_v$ should be treated as a formal symbol. The group of all Arakelov divisors on $X$ is denoted $\text{Div}_{\mathcal{A}}(X)$.

**Definition 9.2.** Let $D$ and $E$ be two Arakelov divisors and suppose that the intersections of $D_{\text{fin}}$ and $E_{\text{fin}}$ with the generic fiber have disjoint supports. The \textit{Arakelov intersection pairing} of $D$ and $E$ is defined as

$$D \cdot E = \sum_v [D, E]_v$$

where the local intersection multiplicities $[D, E]_v \in \mathbb{Q}_p$ are defined by the following rules (extended by symmetry):
(1) If $v \nmid p$, then
\[ [D, E]_v = \ell_v(\langle D_{\text{fin}}, E_{\text{fin}} \rangle)_v \]
where $\langle D_{\text{fin}}, E_{\text{fin}} \rangle_v$ is the usual intersection multiplicity at $v$ of the finite parts of $D$ and $E$.

(2) If $v\mid p$, then we have
\[ [D, E]_v = t_v(\langle D, E \rangle_v) \]
where the intersection multiplicities $\langle D, E \rangle_v \in F_v$ are given by the following rules:

(a) if $w \neq v$, then $\langle D, \lambda_w X_w \rangle_v = 0$.
(b) $\langle \lambda_1 X_v, \lambda_2 X_v \rangle_v = 0$.
(c) if $D$ is a finite divisor, then $\langle D, \lambda X_v \rangle_v = \lambda \deg D_F$, where $D_F$ is the generic part of $D$.
(d) Suppose $D$ and $E$ are finite and let $D_v$ and $E_v$ be their images in $X_v$.
Then we have
\[ \langle D, E \rangle_v = \langle D_v, E_v \rangle , \]
where this last pairing is the one of Definition $8.3$ taken with respect to the Green function at $v$.

**Definition 9.3.** An Arakelov line bundle on $X$ is a line bundle $\mathcal{L}$ on $X$ together with a choice of an admissible metric on $\mathcal{L}_v$ for every $v \mid p$. The trivial Arakelov line bundle $\mathcal{O}_X$ is the line bundle $\mathcal{O}_X$ together with the canonical metric.

There is an obvious notions of isomorphisms of Arakelov line bundles and of the tensor product of them.

**Definition 9.4.** Let $\mathcal{L}$ be an Arakelov line bundle on $X$ and let $s$ be a rational section of $\mathcal{L}$. The Arakelov divisor $(s)$ of $s$ is defined as $(s) = (s)_{\text{fin}} + (s)_{\infty}$ where $(s)_{\text{fin}}$ is the usual divisor of $s$ and
\[ (s)_{\infty} = \sum_{v\mid p} \log_\mathcal{L}(s_v)X_v . \]
In particular, considering the case $\mathcal{L} = \mathcal{O}_X$ we obtain the Arakelov divisor of a rational function.

Clearly we have
\[ (s \otimes t) = (s) + (t) \]
for sections of two line bundles. In particular we have $(fg) = (f) + (g)$ for any two functions and $(fs) = (f) + (s)$ where $f$ is a rational function and $s$ a section of a line bundle.

**Definition 9.5.** The group of principal Arakelov divisors is the group
\[ \text{Prin}_{\text{Ar}}(X) := \{ (f) \mid f \in F(X)^\times \} . \]

**Lemma 9.6.** We Let $D$ be a finite divisor and suppose $v \nmid p$. Then $[D, (f)]_v = \ell_v(f(D))$.

**Proof.** Well known, see for example in [CG89] Proposition 1.2 and its proof. □

**Proposition 9.7.** If $f$ is a rational function and $D$ and Arakelov divisor, then
\[ D \cdot (f) = 0 \]
Proof. The only interesting case is when $D$ is finite, where we have
\[
D \cdot (f) = \sum_{v \mid p} |D, (f)|_v + \sum_{v \nmid p} (|D, (f)\text{fin}|_v + [D, t_{\log}(f_v)X_v]|_v)
\]
\[
= \sum_{v \mid p} f_v(f(D)) + \sum_{v \mid p} t_v(G(f_v)(D) + t_{\log}(f_v) \deg D_F)
\]
\[
= \sum_{v \mid p} f_v(f(D)) + \sum_{v \mid p} t_v(\log_v(f(D))) \quad \text{by Definition 5.6}
\]
\[
= \sum_{v \mid p} f_v(f(D)) + \sum_{v \mid p} t_v(f(D)) \quad \text{by (9.1)}
\]
\[
= 0
\]
since $\ell$ is an idele class character. \hfill \Box

**Definition 9.8.** The *Arakelov Chow group* is the quotient group
\[
\text{CH}_{\text{Ar}}(\mathcal{X}) := \text{Div}_{\text{Ar}}(\mathcal{X}) / \text{Prin}_{\text{Ar}}(\mathcal{X}).
\]

The following result is now standard

**Proposition 9.9.** There is a unique bilinear Arakelov intersection pairing on $\text{CH}_{\text{Ar}}(\mathcal{X})$ specializing to the previously defined intersection pairing for two divisors with disjoint supports on the generic fiber.

**Definition 9.10.** The *Arakelov Picard group* of $\mathcal{X}$ is the group $\text{Pic}_{\text{Ar}}(\mathcal{X})$ of isomorphism classes of line bundles on $\mathcal{X}$ with admissible metrics at primes above $p$.

**Definition 9.11.** Given an Arakelov divisor $D = D_{\text{fin}} + \sum \lambda_v X_v$, we define the metrized line bundle $\mathcal{O}(D)$ on $\mathcal{X}$ as follows: As a line bundle it is simply $\mathcal{O}(D_{\text{fin}})$ and if $v \mid p$, then the log function on $\mathcal{O}(D_{\text{fin}})_v = \mathcal{O}((D_{\text{fin}})_v)$ is the canonical one (Definition 5.3) scaled by $\lambda_v$.

The line bundle $\mathcal{O}(D)$ is admissible and it is clear that any admissible metrized line bundle is isomorphic to $\mathcal{O}(D)$ for some Arakelov divisor $D$. The following result is clear.

**Proposition 9.12.** There is an isomorphism $\text{Pic}_{\text{Ar}}(\mathcal{X}) \cong \text{CH}_{\text{Ar}}(\mathcal{X})$ given by the two inverse maps
\[
\mathcal{L} \mapsto c(\mathcal{L}), \quad D \mapsto \mathcal{O}(D).
\]
where
\[
(9.2) \quad c(\mathcal{L}) := (s), \ s \text{ a rational section of } \mathcal{L}.
\]

**Definition 9.13.** Let $\mathcal{N}$ be a metrized line bundle over $\mathcal{O}_F$, i.e., a locally free $\mathcal{O}_F$-module of rank 1 together with a choice, for each $v \mid p$, of a log function $\log_v$ on $\mathcal{N}_v := \mathcal{N} \otimes_{\mathcal{O}_F} F_v$. We define the *degree* of $\mathcal{N}$ as follows: Fix an isomorphism $\theta : F \stackrel{\sim}{\rightarrow} \mathcal{N} \otimes F$, which induces local isomorphisms $\theta_v : F_v \stackrel{\sim}{\rightarrow} \mathcal{N}_v$ for each $v$. Then we define
\[
\deg \mathcal{N} = \sum_{v \mid p} t_v(\log_v(\theta_v(1))) - \sum_{v \mid p} \ell_v(\theta_v^{-1}(\mathcal{N}_v)).
\]
It is very easy to see that this definition is independent of the choice of the isomorphism $\theta$.

We next generalize the notion of degree to line bundles over finite $\mathcal{O}_F$-schemes. We use here the theory of the determinant line bundle [KM76]. Suppose that $A$ is a finite integral $\mathcal{O}_F$-algebra and that $\mathcal{N}$ is a line bundle on $\text{Spec}(A)$. Let $L$ be the fraction field of $A$. Let $w$ be a place of $L$ above the place $v$ of $F$, lying above $p$. As before, the choice of log $v$ extends uniquely to a branch log $w$ on $L_w$.

**Definition 9.14.** We say that $\mathcal{N}$ is metrized if for any such $w$ we are given a log function on $\mathcal{N} \otimes_A L_w$. We say it is metrized over $F$ if for each such $w$ lying over $v$ this log function is defined over $F_v$.

**Definition 9.15.** Let $\mathcal{N}$ be a metrized line bundle on $A$. Then, the degree of $\mathcal{N}$, $\deg(\mathcal{N})$, is defined as the degree of the line bundle $\det_{\mathcal{O}_F} \mathcal{N} \otimes (\det_{\mathcal{O}_F} A)^{-1}$, where the log function on

$$((\det_{\mathcal{O}_F} \mathcal{N} \otimes (\det_{\mathcal{O}_F} A)^{-1}) \otimes_{\mathcal{O}_F} F_v = \bigotimes_{w|v} \det(\mathcal{N} \otimes_A L_w) \otimes (\det L_w)^{-1}$$

is the tensor product of the log functions of Definition 9.14.

**Proposition 9.16.** We have $\deg(\mathcal{N}_1 \otimes \mathcal{N}_2) = \deg(\mathcal{N}_1) + \deg(\mathcal{N}_2)$.

**Proof.** This is clear for line bundles on $\mathcal{O}_F$. For more general line bundles one can argue as follows: We can find sections $A \to \mathcal{N}_1$ and $A \to \mathcal{N}_2$ such that the supports of the cohomology of the resulting complexes are disjoint (choose the first section arbitrarily and choose the second to avoid the support of the first). It follows that the tensor product of the two complexes over $A$ is exact. Taking determinants we find $\det \mathcal{N}_1 \otimes \det \mathcal{N}_2 \cong \det(\mathcal{N}_1 \otimes \mathcal{N}_2) \otimes \det A$

or

$$\det \mathcal{N}_1 \otimes \det(A)^{-1} \otimes \det \mathcal{N}_2 \otimes \det(A)^{-1} \cong \det(\mathcal{N}_1 \otimes \mathcal{N}_2) \otimes \det(A)^{-1}.$$ 

It is easy to see that this isomorphism is an isometry, giving the result. \hfill \Box

**Proposition 9.17.** Suppose $f : \text{Spec}(B) \to \text{Spec}(A)$ is a surjective morphism of finite $\mathcal{O}_F$-schemes of degree $m$ and $\mathcal{N}$ is a metrized line bundle on $\text{Spec}(A)$. Then $\deg(f^* \mathcal{N}) = m \deg(\mathcal{N})$.

**Proof.** We need to compute $\det(\mathcal{N} \otimes_A B) \otimes \det(B)^{-1} = \det(B \otimes (A \to \mathcal{N}))^{-1}$ for every section $A \to \mathcal{N}$. Take an injection of $A$ modules $A^m \to B$ whose cokernel is supported on a finite number of points. By choosing the section $A \to \mathcal{N}$ appropriately, as we did in the proof of Proposition 9.16, we can replace $B$ by $A^m$ in the last equality to get $\det(\mathcal{N} \otimes_A B) \otimes \det(B)^{-1} = \det(A^m \otimes (A \to \mathcal{N}))^{-1} = (\det(\mathcal{N}) \otimes \det(A)^{-1})^m$.

It follows immediately from Proposition 9.17 that this is an isometry and the result follows. \hfill \Box

**Proposition 9.18.** Let $L/F$ be a finite extension of fields and let $\mathcal{O}_L$ be the ring of integers in $L$. Define an idele class character on $L$ by $\ell_L := \ell \circ N_{L/K}$. Let $\mathcal{N}$ be a line bundle on $\text{Spec}(\mathcal{O}_L)$. Then $\deg(\mathcal{N})$ is the same as the degree of $\mathcal{N}$ as an $\mathcal{O}_L$ bundle, computed with respect to $\ell_L$. 

**Definition 9.21.** Let \( \mathcal{M} \) be a line bundle over \( \text{Spec}(A) \) which is metrized over \( F \). Then the Euler characteristic of \( \mathcal{M} \) is defined to be
\[
\chi(\mathcal{M}) = \deg(\text{det}_{\mathcal{O}_F} \mathcal{M}),
\]
where \( \text{det}_{\mathcal{O}_F} \mathcal{N} \) is metrized according to the log functions obtained from Lemma 8.8.

The following two results are immediate consequences of Propositions 8.9 and 8.10 respectively.

**Proposition 9.20.** For \( \mathcal{N} \) as above we have \( \deg(\mathcal{N}) = \chi(\mathcal{N}) - \chi(A) \).

**Definition 9.21.** Let \( A \) and \( L \) be as above. The dualizing module of \( A \) over \( \mathcal{O}_F \) is given by
\[
W_{A/\mathcal{O}_F} := \{ b \in L : \text{tr}_{L/F}(bA) \subset \mathcal{O}_F \},
\]
metrized by the log functions induced from the inclusion into \( L \).

**Proposition 9.22.** We have \( \chi(A) = -\chi(W_{A/\mathcal{O}_F}) = -\frac{1}{2} \deg(W_{A/\mathcal{O}_F}) \).

**Proposition 9.23.** Let \( D \) be an Arakelov divisor on \( X \) and \( E = \text{Spec}(A) \subset X \) a horizontal divisor, with \( A \) finite over \( \mathcal{O}_F \). Then \( D \cdot E = \deg(\mathcal{O}(D)|_{E}) \).

**Proof.** This is easily checked for an infinite fiber, so by linearity we may assume that \( D \) is an irreducible subscheme of codimension 1, and by a moving lemma on the generic fiber that the intersection of \( D \) and \( E \) with the generic fiber have disjoint supports. It follows that \( \mathcal{L} := \mathcal{O}(D) \) has a global section \( \mathcal{O}_X \xrightarrow{s} \mathcal{L} \), and this diagram serves as a locally free resolution of \( \mathcal{O}_D \). It also follows that \( D \) and \( E \) have proper intersection. Let \( i : E \to X \) be the embedding and \( f : X \to \text{Spec}(\mathcal{O}_F) \) the structure map. We must compute the degree of the \( \mathcal{O}_F \)-line bundle
\[
\mathcal{M} := \text{det}(\mathcal{L}|_{E}) \otimes (\text{det} \mathcal{O}_E)^{-1} = \text{det}((\mathcal{O}_X \xrightarrow{s} \mathcal{L})|_{E}) .
\]
Here we implicitly must push down from \( E \) to \( \text{Spec}(\mathcal{O}_F) \) along the map which we can write as \( f \circ i \). The bundle \( \mathcal{M} \) has a canonical section \( s^\prime : \mathcal{O}_F \to \mathcal{M} \) induced by
the restriction to $E$ of the commutative diagram

$$
\begin{array}{ccc}
\mathcal{O}_X & \xrightarrow{s} & \mathcal{L} \\
\text{Id} & & \text{Id} \\
\downarrow & & \downarrow \\
\mathcal{O}_X & \xrightarrow{s} & \mathcal{O}_X
\end{array}
$$

and the obvious triviality of the determinant of the bottom row. Now we see that we can compute $\mathcal{M}$ as follows:

$$
\mathcal{M} = \text{det} \mathbb{R} f_* \mathbb{R} i_* (\mathcal{O}_X \to \mathcal{L})|_E
$$

$$
= \text{det} \mathbb{R} f_* ((\mathbb{R} i_* \mathcal{O}_E) \otimes (\mathcal{O}_X \to \mathcal{L}))
$$

by the projection formula

$$
= \text{det} \mathbb{R} f_* (\mathcal{F}^* \otimes (\mathcal{O}_X \to \mathcal{L}))
$$

where $\mathcal{F}^*$ is a locally free resolution of $\mathcal{O}_E$. Since the cohomology of $\mathcal{F}^* \otimes (\mathcal{O}_X \to \mathcal{L})$ is an $\mathcal{O}_X$-module supported exactly on the closed points of intersection between $D$ and $E$, and since for such modules the map $f_* \to \mathbb{R} f_*$ is a quasi-isomorphism, it follows that

$$
\mathcal{M} \cong \bigotimes_{x \in D \cap E} (\otimes_i \text{det}(f_* \text{Tor}^{O_X i}_i (\mathcal{O}_D, \mathcal{O}_E))(-1)^i).
$$

Replacing $\mathcal{O}_X \to \mathcal{L}$ by $\mathcal{O}_X \to \mathcal{O}_X$ it is clear that $s'$ is the alternating product of the maps induced by $0 \to \text{Tor}^{O_X i}_i (\mathcal{O}_D, \mathcal{O}_E)$. For any place $v$ the determinant of the map $0 \to \mathcal{O}_{F_v}/\pi_v^{k_v} \mathcal{O}_{F_v}$ tensored with $F_v$ is such that the inverse image of $\text{det}(\mathcal{O}_{F_v}/\pi_v^{k_v} \mathcal{O}_{F_v})$ is $\pi_v^{k_v} \mathcal{O}_{F_v}$ [KM78 Theorem 3 (vi)]. It follows that the isomorphism $\theta_v^{-1}(\mathcal{M}_v) = \pi_v^{k_v} \mathcal{O}_{F_v}, \quad k = \sum_{x \in D \cap E} \sum_i (-1)^i \text{length}(\text{Tor}^{O_X i}_i (\mathcal{O}_D, \mathcal{O}_E)) = \langle D, E \rangle_v$.

Now we turn to the infinite contributions. Suppose $v \mid p$ and write $E_v = \sum Q_j$.

Then

$$
\mathcal{M}_v = \text{det}((\mathcal{O}_{X_v} \xrightarrow{s} \mathcal{L}_v)|_{E_v}) = \otimes_j \text{det}((\mathcal{O}_{X_v} \xrightarrow{s} \mathcal{L}_v)|_{Q_j}).
$$

The isomorphism we have chosen with $\mathcal{F}_v$ is the tensor product of the isomorphism of the $j$-th term with $\mathcal{F}_v$, which is exactly the isomorphism which was used in Definition 5.3. By this definition it is easy to see that

$$
\log(\theta_v(1)) = \sum_j \text{tr}_{F_v(Q_j)/F_v} \log_{\mathcal{L}_v}(s(1))(Q_j) = \sum_j \text{tr}_{F_v(Q_j)/F_v} G_{D_v}(Q_j) = \langle D, E \rangle_v.
$$

This completes the proof.

\[\square\]

10. The adjunction formula and the Riemann-Roch theorem

In this section we would like to show how some of the main theorems of classical Arakelov theory have precise analogues in $p$-adic Arakelov theory. In fact, after the work of the previous sections, the proofs do not differ much from the proofs in the classical case. We have chosen to follow the treatment of Lang [Lan88].

We begin with the adjunction formula. Let $E \subset X$ be a horizontal curve, with $E = \text{Spec}(A)$ and $A$ finite over $\mathcal{O}_F$. Let $\omega_{E/\mathcal{O}_F}$ be the relative dualizing module. It is known that

$$
\omega_{E/\mathcal{O}_F} = (\omega_{X/\mathcal{O}_F} \otimes \mathcal{O}(E))|_E.
$$
Let \( F(E) \) be the function field of \( E \). The residue map gives an injection \( \text{Res} : \Gamma(E, \omega_{E/O_F}) \hookrightarrow F(E) \).

**Theorem 10.1** ([Lan88, Theorem 4.1, p. 94]). The image of \( \text{Res} \) is the dualizing module \( W_{A/O_F} \) of Definition 9.21.

Note that this dualizing module is taken in this definition without its metric. This metric figures in the next definition.

**Definition 10.2.** The \((p\text{-adic}) \) discriminant of \( E \) is \( d(E) = \deg(W_{A/O_F}) \).

This definition does not take into account the embedding of \( E \) in \( X \). Let now \( v \) be a prime above \( p \) and let \( \overline{F_v} \) be an algebraic closure of \( F_v \). By definition we have a Green function \( G_v \) on \( \mathcal{X} \otimes \overline{F_v} \). Over \( \overline{F_v} \) the divisor \( E \) splits as a sum of distinct points \( E \otimes \overline{F_v} = \sum_{j=1}^{e} P_j \).

**Definition 10.3.** The discriminant above \( v \) of \( E \) in \( \mathcal{X} \) is \( d_v(E, \mathcal{X}) = \sum_{i \neq j} G_v(P_i, P_j) \).

The infinite discriminant is defined as \( d_{\infty}(E, \mathcal{X}) = \sum_{v \mid p} d_v(E, \mathcal{X}) \).

The adjunction formula is now the following statement.

**Theorem 10.4.** Let \( E \) be a horizontal divisor on \( X \). Then

\[ \omega_{X/O_F} \cdot E + E \cdot E = \deg(W_{A/O_F}) \cdot (\omega_{E/O_F}). \]

**Proof.** From (10.1) and the fact that both \( \omega_{X/O_F} \) and \( \mathcal{O}(E) \) have natural metrics, we obtain a metric on \( \omega_{E/O_F} \). With respect to this metric it follows from Proposition 9.23 that

\[ \omega_{X/O_F} \cdot E + E \cdot E = \deg(\omega_{E/O_F}) \cdot (\omega_{X/O_F}). \]

As we defined them, \( \omega_{E/O_F} \) and \( W_{A/O_F} \) are the same module but with different metrics. The difference in their degree is thus the sum of the differences between their log functions. Consider a place \( v \mid p \) of \( F \) and a point \( P_i \) in \( E \otimes \overline{F_v} \) as before. The log function on the fiber at \( P_i \) of \( W_{A/O_F} \) is such that the residue map to \( \overline{F_v} \) is an isometry. On the other hand, the fiber at the same point of \( \omega_{X/O_F} \) is viewed as the fiber of \( \omega_{X/O_F} \otimes \mathcal{O}(\sum P_j) \). The log function on the fiber of \( \omega_{X/O_F} \otimes \mathcal{O}(P_i) \) is such that the residue is an isometry and the points \( P_j \) for \( j \neq i \) contribute an added term of \( G_v(P_i, P_j) \). The result is now clear. \( \square \)

Suppose now that \( \mathcal{X} \) is an arithmetic surface over \( O_F \) and that \( \mathcal{L} \) is a metrized line bundle over \( \mathcal{X} \). Let

\[ \mathcal{M} = \lambda(\mathcal{L}) := \det H^0(\mathcal{X}, \mathcal{L}) \otimes (\det H^1(\mathcal{X}, \mathcal{L}))^{-1}. \]

Then, for any place \( v \mid p \) of \( F \) we have \( \mathcal{M}_v = \lambda(\mathcal{L}_v) \) and by Proposition 6.1 was done before it acquires a log function. Thus, \( \mathcal{M} \) is a metrized line bundle on \( O_F \) and we can define

\[ \chi(\mathcal{L}) = \deg(\lambda(\mathcal{L})). \]

The following lemma is the \( p \)-adic analogue of a well known result in classical Arakelov theory, and is an immediate consequence of the multiplicativity of the
determinant and the behavior of the Faltings volume with respect to adding points given in part of Proposition 6.1.

**Lemma 10.5.** If $D$ is an Arakelov divisor and $E$ is a horizontal divisor, then

$$\chi(O(D + E)) = \chi(O(D)) + \chi(O(D + E)|_E) - \frac{1}{2}d_{\infty}(E, X).$$

**Theorem 10.6.** We have the following Riemann-Roch formula:

$$\chi(L) - \chi(O_X) = \frac{1}{2}L \cdot (L - \omega_X).$$

**Proof.** We follow the proof given by Lang. One can assume that $L$ is of the form $O(D)$ for some Arakelov divisor $D$. Then one checks that the validity of the theorem is unchanged if one adds or subtracts from $D$ a divisor. The two cases of a fiber at infinity and of a vertical divisor are essentially the same as in the classical case so we leave them for the reader. The case of adding a horizontal divisor is treated exactly as in Lang. We reproduce the proof to see that we have all the ingredients (with slightly different notation). We have

$$\chi(O(D)|_E) = \deg(O(D)|_E) + \chi(O_E) \quad \text{by Proposition 9.20}$$

$$= D \cdot E + \chi(O_E) \quad \text{by Proposition 9.22}$$

$$= D \cdot E - \frac{1}{2}d(E) \quad \text{by Proposition 9.22 and Definition 10.2}$$

$$= D \cdot E - \frac{1}{2}(E \cdot E + \omega_X/O_F \cdot E - d_{\infty}(E, X))$$

by the adjunction formula (Theorem 10.4). If we replace $O(D)$ by $O(D + E)$ the left hand side of the Riemann-Roch formula changes by

$$\chi(O(D + E)) - \chi(O(D)) = \chi(O(D + E)|_E) - \frac{1}{2}d_{\infty}(E, X) \quad \text{by Lemma 10.5}$$

$$= (D + E) \cdot E - \frac{1}{2}(E \cdot E + \omega_X/O_F \cdot E - d_{\infty}(E, X)) - \frac{1}{2}d_{\infty}(E, X)$$

$$= D \cdot E + \frac{1}{2}E \cdot (E - \omega_X/O_F),$$

which is exactly the amount by which the right hand side changes. \qed

**Remark 10.7.** It is evident from the proof of the Riemann-Roch theorem that it is too simple to depend on a particular normalization of the Green function $G$ or of the log function on the determinant of cohomology. This latter independence is clear. Here we would like to check the independence of the Green function directly, since this requires keeping careful track of all normalizations. Suppose then that we have two Green functions $G_1$ and $G_2 = G_1 + 1$ at the place $v$ and the Green functions at the other places are the same. It suffices to consider this case since all contributions will be linear in the constant $G_2 - G_1$. We check how the two sides of Theorem 10.6 change when we make this change, beginning with the right hand side. With respect to these two functions we have the following quantities that change: The intersection pairing $\langle \cdot, \cdot \rangle$, the Arakelov Chern class $c_i$ and the canonical class $\omega_i$, for $i = 1, 2$. Let $d = \deg(D)$ be the degree of $L$ on the generic fiber.

The relation between the intersection products is that

$$\langle D, E \rangle_2 = \langle D, E \rangle_1 + \deg(D_F) \cdot \deg(E_F).$$
Since $G_2(D, \bullet) = G_1(D, \bullet) + \deg D_F$ it follows from Definition 5.7 that the relation between the $\iota_{log}$ characters at the place $v$ is $\iota_{log,L,2} = \iota_{log,L,1} - \deg(L)$ and therefore 
\[ c_2(L) = c_1(L) - dX_v. \]

Finally, Proposition 7.4 implies that \( \log \omega_2 = \log \omega_2 - 1 \) from which it follows that 
\[ c_2(\omega_2) = c_1(\omega_1) - (2g - 2 + 1)X_v. \]

Thus, (twice) the right hand side of the Riemann-Roch formula changes as follows:
\[
\langle c_2(L), c_2(L) - c_2(\omega_2) \rangle_2 = \langle c_1(L) - dX_v, c_1(L) - c_1(\omega_1) - (d - 2g + 1)X_v \rangle_1 + d(d - 2g + 2) \\
= \langle c_1(L), c_1(L) - c_1(\omega_1) \rangle_1 - d(d - 2g + 2) - d(d - 2g + 1) + d(d - 2g + 2) \\
= d(c_1(L), c_1(L) - c_1(\omega_1))_1 - d(d - 2g + 1).
\]

Now we turn to the left hand side. We have
\[
\chi(L) - \chi(O) = (\chi(L) - \chi(O(D))) + (\chi(O(D)) - \chi(O))
\]
where $D$ is the finite part of the Chern class of $L$. The first summand reflects the different metric between $O(D)$ and $L$. The change of $G$ adds $d$ to $\log O(D)$, which, in view of (2) of Proposition 6.1, subtracts $(d + 1 - g)$ from the first summand. On the other hand, it follows from (3) of Proposition 6.1 that the second summand gets $1 + 2 + \cdots + d = d(d + 1)/2$ added. So overall the left hand side is reduced by
\[
(d + 1 - g)d - \frac{d(d + 1)}{2} = d \left( \frac{d(d + 1)}{2} - g \right).
\]

which is exactly what gets subtracted from the right hand side.

**Appendix A. The Universal Vectorial Extension of a Jacobian**

In this appendix we review the theory of the universal vectorial extension of the Jacobian of a curve and prove several algebraic results that will be required in the main text. Probably, everything is well known but we do not know of a reference. The general theory of vectorial extensions of abelian varieties is to be found in [Mum74]. It is utilized for Jacobians by Coleman in [Col90, Col91], but our treatment is independent of his.

Let $C$ be a curve over a base scheme $S$. Let $K(C)^\times$ be the sheaf
\[
\bigoplus_{\text{cod} x = 0} i_x K(x)^\times
\]
where $i_x$ is the embedding of $x$ in $C$ and $k(x)^\times$ is the multiplicative group of the residue field of $x$. We let $\mathcal{C}_{C/S}$ be the complex of sheaves
\[
\mathcal{O}_C^\times \xrightarrow{d} \Omega_{C/S} \oplus K(C)^\times, \quad d(f) = (d \log(f), f).
\]

**Definition A.1.** The space of differentials of the third kind on $C$ relative to $S$ is the group $H^1(C, \mathcal{C}_{C/S})$. 
To see why this definition captures differentials of the third kind we compute this cohomology with the help of a Zariski covering \( \{ U_i \} \) of \( C \). A one cocycle is given by
\[
(g_i \in K(C)^\times(U_i), \omega_i \in \Omega_{C/S}(U_i), f_{ij} \in O(U_{ij}))
\]
such that
\[
\omega_i - \omega_j = d\log(f_{ij}) \quad \text{and} \quad f_{ij} = \frac{g_i}{g_j}
\]
Given such a cocycle, we recover a form of the third kind by taking \( \omega_i - d\log(g_i) \) on \( U_i \) and noticing that the conditions guarantee that these glue together. The resulting form has by definition logarithmic singularities. More conceptually, when \( C \) and \( S \) are spectra of fields we have an isomorphism \( C_{C/S} \to \Omega^1_{C/S}[1] \) given by \( (\omega, g) \mapsto \omega - d\log(g) \). Thus we obtain from a form of the third kind a differential at the generic point by first restricting and then applying this isomorphism on cohomology.

In [MM74, I.3.1.7] the multiplicative de Rham complex of \( C/S \) is defined to be the complex \( \Omega^\times_{C/S} = (O_C^{\times} \xrightarrow{d\log} \Omega^1_{C/S}) \) (in loc. cit. it is extended further to the right, which we do not have to do). There is an obvious short exact sequence
\[
0 \to K(C)^\times[1] \to C_{C/S} \to \Omega^\times_{C/S} \to 0 .
\]
Taking cohomology we obtain the short exact sequence
\[
K(C)^\times \to H^1(C, C_{C/S}) \to H^1(\Omega^\times_{C/S}) \to 0 .
\]
It is known that when sheafifying the right term of the above sequence one obtains a functor represented by the universal vectorial extension \( G_X \) of the Jacobian \( J \) of \( C \). The map on the left sends a rational function \( f \) to the form of the third kind \(-d\log(f)\).

Recall from Section 3 the definition of differentiation of differential forms with respect to a vector field. We are going to refine this to a differentiation from a family of forms of the third kind to a family of forms of the second kind, a notion defined as follows.

**Definition A.2.** A family of forms of the second kind on \( C/S \) is an element of \( H^1(C, B_{C/S}) \), where \( B_{C/S} \) is the complex
\[
O_C \xrightarrow{d} \Omega^1_{C/S} \oplus K(C) , \quad d(f) = (df, f) .
\]
We have an obvious short exact sequence
\[
0 \to K(C)[1] \to B_{C/S} \to \Omega^\bullet_{C/S} \to 0 .
\]
Taking cohomology we obtain
\[
K(C) \to H^1(C, B_{C/S}) \to H^1(C, \Omega^\bullet_{C/S}) \to 0 .
\]
When \( S = \text{Spec}(K) \) this map describes the representation of the first de Rham cohomology of \( C \) as the quotient of the space of forms of the second kind by the differentials of rational functions.

**Definition A.3.** Let \( \partial/\partial t \) be a vector field on \( C \). We define a map
\[
\frac{\partial}{\partial t} : C_{C/S} \to B_{C/S} ,
\]
given in degree 0 by
\[ f \mapsto \frac{\partial f}{\partial t} \]
and in degree 1 by
\[ (\omega, f) \mapsto \left( \frac{\partial}{\partial t} \omega, \frac{\partial f}{\partial t} \right) . \]

It is easy to check that this is indeed a map of complexes. On \( H^1 \) it gives a map, which we continue to call \( \partial/\partial t \), from forms of the third kind to forms of the second kind. It is further easy to check that viewing both forms of the third and second kind as differential forms on the generic point, this map is just differentiation of forms with respect to the restriction of \( \partial/\partial t \) to this point.

**Proposition A.4.** Consider the family \( C[\epsilon]/S[\epsilon] \) where \( \epsilon^2 = 0 \). Then we have the following commutative diagram

\[
\begin{array}{c}
\text{Ker} \left( H^1(C, \Omega^\bullet_{C[S]/S}) \rightarrow H^1(C, \Omega^\bullet_{C/S}) \right) \\
\downarrow \\
\text{Ker} \left( H^1(C, \Omega^\bullet_{C[\epsilon]/S[\epsilon]}) \rightarrow H^1(C, \Omega^\bullet_{C/S}) \right) \\
\end{array}
\]

In this diagram the top horizontal map is differentiation composed with restriction to \( C \) and the vertical map on the right sends a differential of the third kind to its de Rham cohomology class.

**Proof.** Suppose we have a form of the third kind on \( C[\epsilon]/S[\epsilon] \) whose restriction to \( C/S \) is 0. The element \( \epsilon \) provides a canonical vector field on \( C[\epsilon] \) and we would like to compute the derivative of this form of the third kind with respect to \( \epsilon \). First we notice that there is a commutative diagram with split short exact sequence rows of sheaves on \( C \) (compare the proof of Proposition I.4.1.4 in [MM74])

\[
\begin{array}{c}
0 \rightarrow \Omega^\bullet_{C/S} \rightarrow \mathfrak{C}_{C[S]/S[\epsilon]} \rightarrow \mathfrak{C}_{C/S} \rightarrow 0 \\
0 \rightarrow \Omega^\bullet_{C/S} \rightarrow \Omega^\times_{C[\epsilon]/S[\epsilon]} \rightarrow \Omega^\times_{C/S} \rightarrow 0
\end{array}
\]

In this diagram, the vertical maps are the ones defined before. The top left horizontal map in degree 0 sends \( f \) to \( 1 + \epsilon f \) and in degree 1 sends \( \omega \) to \( \epsilon \omega \). It is now easy to see that the composed map

\[ \Omega^\bullet_{C/S} \rightarrow \mathfrak{C}_{C[S]/S[\epsilon]} \xrightarrow{\partial/\partial \epsilon} \mathfrak{B}_{C[S]/S[\epsilon]} \rightarrow \mathfrak{B}_{C/S} \rightarrow \Omega^\bullet_{C/S} \]

is the identity map. Indeed, in degree 0 it first sends \( f \) to \( 1 + \epsilon f \). Then log differentiating with respect to \( \epsilon \) sends this to \( f/(1 + \epsilon f) \) and this is then sent back to \( f \) by the map that kills \( \epsilon \) and this is sent to \( f \) again. In degree 1 a form \( \omega \) is sent to \( (\epsilon \omega, 0) \), differentiation with respect to \( \epsilon \) sends this to \( (\omega, 0) \), then to \( (\omega, 0) \) again and finally to \( \omega \). By taking cohomology we obtain the result. \( \square \)

**Corollary A.5.** Let \( C/K \) be a complete curve, \( T/K \) a variety, \( 0 \in T(K) \) a fixed point. Let \( G_C \) be the universal vectorial extension of \( J(C) \). Let \( \eta_t \in T \) be a family
of forms of the third kind on $C \times T/T$ and let $\rho : T \to G_C$ be the induced map. Suppose $\eta_0 = 0$, which implies $\rho(0) = 0$. Let $\partial / \partial t$ be a vector field on $T$. Then, the form of the second kind on $C$, $(\partial \eta / \partial t)|_{t=0}$, represents the de Rham cohomology class

$$(d\rho)(\partial / \partial t)|_{t=0} \in \text{Lie}(G_C) \cong H^1_{\text{dR}}(C/K).$$

**Proof.** This follows from the previous proposition by restricting to the infinitesimal neighborhood of 0 in $T$ and interpreting the result. \qed

Finally, to use the previous corollary, we want to show that we can at least locally lift elements of the universal vectorial extension to forms of the third kind.

**Lemma A.6.** Consider $C$, $T$ and 0 as in the corollary. Let $\rho : T \to G_C$ be a map. Then there exist a neighborhood $U$ of 0 in $T$ and a family of forms of the third kind $(\eta_t)_{t \in U}$ inducing $\rho_U$.

**Proof.** By [MM74] the map $\rho$ is locally induced by a line bundle $L$ on $T \times C$ together with a relative connection $\nabla$ on it. To obtain a family of forms of the third kind one takes a section $s$ of $L$ and computes the form of the third kind $\nabla(s) / s$. We may choose $U$ and the section in such a way that $s$ is invertible on $U$, hence the family of forms is defined on $U$. \qed

**Appendix B. Relations with the theory of Colmez**

In [Col98] Colmez developed a theory of $p$-adic integration using Abelian varieties. In this theory there is also a notion of Green functions. The purpose of this section is to compare this notion of Green functions with the one we have been developing here. Note that Colmez is working over $\mathbb{C}_p$ while we are working over $\overline{\mathbb{Q}}_p$.

Let $A$ be an abelian variety over $\overline{\mathbb{Q}}_p$. For each non-negative integer $n$ Colmez defines a correspondence $\Delta^{[n]} : A^{n+1} \to A$ (a kind of difference operator) as follows: Define, for $I \subset \{1, \ldots, n\}$, $m_I : A^{n+1} \to A$ by

$$m_I(x, h_1, \ldots, h_n) = x + \sum_{i \in I} h_i$$

Then,

$$\Delta^{[n]} := \sum_{I \subset \{1, \ldots, n\}} (-1)^{n-|I|} m_I^*.$$

The following easy lemma, taken from [Col98], gives an alternative recursive description of $\Delta^{[n]}$.

**Lemma B.1.** Let

$$\pi_n, m_n : A^{n+1} \to A^n,$$

$$\pi_n(x, h_1, \ldots, h_n) = (x, h_1, \ldots, h_{n-1}),$$

$$m_n(x, h_1, \ldots, h_n) = (x + h_n, h_1, \ldots, h_{n-1})$$

Then, $\Delta^{[n]}$ is given recursively by the formulas

$$\Delta^{[0]} = \text{Id}, \quad \Delta^{[n]} = (\pi_n^* - m_n^*) \circ \Delta^{[n-1]}.$$
It is immediate to see that for any $i \in \{1, \ldots, n\}$ restriction to $h_i = 0$ composed with $\Delta^{[n]}$ equals 0.

Now let $L$ be a line bundle on $A$. The Theorem of the cube implies the existence of an isomorphism

$$\Delta^{[3]} L \cong \mathcal{O}_{A^4}. \tag{B.1}$$

We normalize this isomorphism by requiring that it restricts to the identity isomorphism on each $\{h_i = 0\}$.

**Proposition B.2.** For any log function $\log L$ on $L$ the isomorphism $\Delta^{[3]} L \cong \mathcal{O}_{A^4}$ is an isometry of metrized line bundles.

**Proof.** Since $\Delta^{[n]}$ is a difference operator it follows easily that $\Delta^{[n]}$ kills $H^1_{\text{dR}}(A)$ for $n \geq 2$ and $\Delta^{[n]}$ kills $H^\otimes(A)$ for $n \geq 3$. In particular, the curvature of $\Delta^{[3]} L$ is 0. It follows that the differentials of the log functions on the two sides of (B.1) can differ by at most a holomorphic differential on $A^4$. This has the form $\sum_{i=0}^3 p_i^* \omega_i$ with $\omega_i \in \Omega^1(A)$ and $p_i : A^4 \to A$ the projection on the $i$th coordinate. But the restriction of this form to $\{h_i = 0\}$ is 0, showing that $\omega_i = 0$. Thus, (B.1) is an isometry up to scaling and again restricting to $\{h_i = 0\}$ shows that it is in fact an isometry. □

We now compare this result with Proposition I.2.8 of [Col98]. Let $s$ be a section of the line bundle $L$ and let $D$ be the divisor of $s$. For any log function $\log L$ on $L$ let $G_D = \log L(s)$. The Theorem of the cube implies that $\Delta^{[3]} D$ is a principal divisor and Colmez chooses a rational function $f_D^{(4)}$ normalized in such a way that its restriction to $\{h_i = 0\}$ is 1. This is clearly just the image of $\Delta^{[3]} s$ under the canonical choice of (B.1). It is now immediate that $\Delta^{[3]} G_D = \log f_D^{(4)}$, which is the defining property of the Green function of the divisor $D$ in Colmez’s definition. By the properties of Coleman integration it is easily seen that $G_D$ is locally analytic outside $D$ and has logarithmic singularities along $D$. It is therefore the Green function of Colmez. The kernel of the cup product map $\cup : H^\otimes(A) \to H^2_{\text{dR}}(A)$ is exactly $\text{Symm}^2 \Omega^1(A)$. Thus, different choices for $\log L$, and consequently for $G_D$, differ by the constant of integration, by the integral of a holomorphic form on $A$, and by integrals corresponding to elements of $\text{Symm}^2 \Omega^1(A)$, i.e., integrals of the form

$$\int (\omega \int \omega) = \frac{1}{2} \left( \int \omega \cdot \left( \int \omega \right) \right).$$

In other words, $G_D$ is unique up to a polynomial of degree 2 in the integrals of holomorphic forms on $A$, which are the logarithms of $A$ in Colmez’s terminology, and this is exactly the indeterminacy in Colmez’s Green functions. To sum up, we have proved

**Proposition B.3.** Let $L$ be a line bundle on $A$, $s$ a section of $L$ and $D$ the divisor of $s$. The collection of Green functions for $D$ defined by Colmez is the same as the collection of functions $\log L(s)$ for all possible log functions $\log L$ on $L$. In particular, the Green functions of Colmez are Coleman functions.

**References**

[Ara74] S. Ju. Arakelov. An intersection theory for divisors on an arithmetic surface. Izv. Akad. Nauk SSSR Ser. Mat., 38:1179–1192, 1974.
[Bes00a] A. Besser. Syntomic regulators and $p$-adic integration I: rigid syntomic regulators. *Israel Journal of Math.*, 120:291–334, 2000.

[Bes00b] A. Besser. Syntomic regulators and $p$-adic integration II: $K_2$ of curves. *Israel Journal of Math.*, 120:335–360, 2000.

[Bes02a] A. Besser. Coleman integration using the Tannakian formalism. *Math. Ann.*, 322(1):19–48, 2002.

[Bes02b] A. Besser. The $p$-adic height pairings of Coleman-Gross and of Nekovář. Preprint available as [math.NT/0209006](http://arxiv.org/abs/math.NT/0209006), 2002.

[CdS88] R. Coleman and E. de Shalit. $p$-adic regulators on curves and special values of $p$-adic $L$-functions. *Invent. Math.*, 93(2):239–266, 1988.

[CG89] R. Coleman and B. Gross. $p$-adic heights on curves. In *Algebraic number theory*, pages 73–81. Academic Press, Boston, MA, 1989.

[Col85] R. Coleman. Torsion points on curves and $p$-adic abelian integrals. *Annals of Math.*, 121:111–168, 1985.

[Col89] R. Coleman. Reciprocity laws on curves. *Compositio Math.*, 72(2):205–235, 1989.

[Col90] R. Coleman. Vectorial extensions of Jacobians. *Ann. Inst. Fourier (Grenoble)*, 40(4):769–783 (1991), 1990.

[Col91] R. Coleman. The universal vectorial bi-extension and $p$-adic heights. *Invent. Math.*, 103(3):631–650, 1991.

[Col98] P. Colmez. Intégration sur les variétés $p$-adiques. *Astérisque*, (248):viii+155, 1998.

[Fal84] G. Faltings. Calculus on arithmetic surfaces. *Ann. of Math. (2)*, 119(2):387–424, 1984.

[KM76] F. Knudsen and D. Mumford. The projectivity of the moduli space of stable curves. I. Preliminaries on “det” and “Div”. *Math. Scand.*, 39(1):19–55, 1976.

[Lan88] S. Lang. *Introduction to Arakelov theory*. Springer-Verlag, New York, 1988.

[MM74] B. Mazur and W. Messing. *Universal extensions and one dimensional crystalline cohomology*. Springer-Verlag, Berlin, 1974. Lecture Notes in Mathematics, Vol. 370.

[Nek93] J. Nekovář. On $p$-adic height pairings. In *Séminaire de Théorie des Nombres, Paris, 1990–91*, pages 127–202. Birkhäuser Boston, Boston, MA, 1993.

[Vol01] V. Vologodsky. Hodge structure on the fundamental group and its application to $p$-adic integration. Preprint available as [math.AG/0108109](http://arxiv.org/abs/math.AG/0108109), 2001.

[War83] F. W. Warner. *Foundations of differentiable manifolds and Lie groups*. Springer-Verlag, New York, 1983. Corrected reprint of the 1971 edition.

[Yek95] A. Yekutieli. Traces and differential operators over Beilinson completion algebras. *Compositio Math.*, 99(1):59–97, 1995.