ON THE EXISTENCE OF SOLUTIONS FOR FRENKEL-KONTOROVA MODELS ON QUASI-CRYSTALS

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ABSTRACT

We focus on the recent development of the investigation on the equilibria of the Frenkel-Kontorova models subjected to potentials generated by quasi-crystals. We present a specific one-dimensional model with an explicit equivariant potential driven by the Fibonacci quasi-crystal. For a given positive number \( \theta \), we show that there are multiple equilibria with rotation number \( \theta \), e.g., a minimal configuration and a non-minimal equilibrium configuration. Some numerical explorations of finding these equilibria are provided.

1 Introduction

We consider the Frenkel-Kontorova models with quasi-periodic potentials. One may refer to [1] for several physical interpretations of such models.

In order to give a unified picture of known results, let’s take one-dimensional quasi-crystals case for example (see [2] for more advanced studies for general quasi-crystals). These models describe the dislocations of particles deposited over a substratum given by the quasi-crystals (see [3]). That is, we model the position of the \( i \)-th particle by \( x_i \in \mathbb{R} \) and the total energy is formally composed of the summation over all \( i \in \mathbb{Z} \) of spring potential \( \frac{1}{2}(x_i - x_{i+1})^2 \) between the nearest neighbors \( i \)-th and the \((i + 1)\)-th particles, and the potential \( V(x_i) \) from interaction of the \( i \)-th particle with the quasi-periodic substratum:

\[
S\left( (x_i)_{i \in \mathbb{Z}} \right) := \sum_{i \in \mathbb{Z}} \left[ \frac{1}{2}(x_i - x_{i+1})^2 + V(x_i) \right].
\]

In this article, we concentrate on the case that the potential function \( V \) are the pattern equivariant function (see Definition 10 for detailed definition) and aim to discuss existed approaches to obtain the existence of equilibria. Related problems of finding equilibria in the cases of other types of quasi-periodic functions are discussed in [4, 5] and references therein.

The aperiodicity will bring us the main difficulties such as loss of compactness and less results are known. To the best of our knowledge, we now recall the following known results in the literature. In [6], under the assumptions that the potential function is large enough, the author uses the idea of the anti-integrable limits to obtain multiple equilibria with any prescribed rotation number or without any rotation numbers. This approach works also for the higher dimensional generalization of the quasi-periodic Frenkel-Kontorova model but require that the system is far away from integrable.

On the other hand, [7] considers the one-dimensional Fibonacci quasi-crystals without extra assumption on the system whether it is integrable or not, and uses mainly topological methods to establish the existence of the minimal configurations with given rotation numbers. Note that dimension one is rather essential here and minimal configuration, which is also a special equilibrium, has more properties than those of the equilibria. One may refer to [8] for other methods for searching for minimal configurations.

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We construct explicitly the potential \( V \) driven by the Fibonacci quasi-crystals and try to do more numerical simulations in the future work.

Our goal here is to provide several approaches to tackle variational problem for the Frenkel-Kontorova models on quasi-crystals. To be more precise, we aim to find the minimal configurations and non-minimal equilibrium configurations of the Frenkel-Kontorova model with quasi-periodic potentials.

In order to give a simple and self-contained proof and we fix at the beginning a positive number \( \theta = (3\tau + 1)/2 \) where \( \tau = (\sqrt{5} + 1)/2 = 1.618 \cdots \). We state the main results as follows:

**Theorem 1** For the Frenkel-Kontorova model with the potential \( V \) (which is defined in Section 2.3),

(i) there exists a minimal configuration with the rotation number \( \theta \),

(ii) there exists a non-minimal equilibrium configuration with the rotation number \( \theta \).

But so far, the higher dimension analogues of the above results are rarely known yet. We will take the specific example as a cornerstone of the theory of Frenkel-Kontorova models on quasi-crystals. We present several numerical simulations and hope to imply some intuitive guesses.

**Organization of the article**

In Section 2, we will introduce some necessary fundamentals about quasicrystals and the variational problem for the Frenkel-Kontorova models. In particular, we will give an example of the quasi-crystals, the Fibonacci chain, and an example of the Frenkel-Kontorova models with quasi-periodic potentials.

Section 3 are devoted to the proof of Theorem 1. Numerical simulations for equilibrium configurations are provided.

In fact, in Section 3.1, we will obtain the minimal configurations for Frenkel-Kontorova models on the one-dimensional Fibonacci quasi-crystals with slight modifications compared with those in [7], that is, item (i) of Theorem 1. The authors there use mainly topological methods to establish the existence of the minimal configurations with given rotation numbers.

We will apply the idea of anti-integrable limits to prove the existence of equilibrium configurations of type \( h \) in Section 3.2. Then, we will prove that these equilibrium configurations are non-minimal, which completes the proof of item (ii) of Theorem 1.

## 2 Preliminaries

We collect several standard notions of quasi-crystals in Section 2.1. In Section 2.2, we introduce a variational problem by which we give the notions of the solutions for the Frenkel-Kontorova models we are looking for. The standard Frenkel-Kontorova models motivated by dislocations in solids and its extensions are introduced in Section 2.2.

### 2.1 Quasicrystals

The \( d \)-dimensional Euclidean space is denoted by \( \mathbb{R}^d \), and is equipped with its usual metric topology and vector space structure. The origin \( 0 \in \mathbb{R}^d \) is the identity. The open ball of radius \( r \) around \( x \) is written as \( B_r(x) \). The closure of a set \( A \) is denoted by \( \overline{A} \). The cardinality of \( A \) is denoted by \( \text{Card}(A) \). If \( A \) is Lebesgue-measurable, the Lebesgue measure of \( A \) is denoted by \( \lambda(A) \). Countable subsets in \( \mathbb{R}^d \) are called point sets.

A point set \( \Lambda \subset \mathbb{R}^d \) is uniformly discrete if there is a real \( r > 0 \) such that \( (x + B_r(0)) \cap (y + B_r(0)) = \emptyset \) holds for all distinct \( x, y \in \Lambda \). Here, the plus is the Minkowski sum. \( \Lambda \) is relative dense if there is a real \( R > 0 \) such that \( \Lambda + B_R(0) = \mathbb{R}^d \).

**Definition 1 (Delone sets)** A point set \( \Lambda \subset \mathbb{R}^d \) is Delone if it is both uniformly discrete and relative dense.

A cluster of point set \( \Lambda \) is the intersection \( K \cap \Lambda \) for some compact set \( K \subset \mathbb{R}^d \). If \( K \) is convex, we also call a cluster \( K \cap \Lambda \) a patch. Two clusters \( P_1 \) and \( P_2 \) are said to be equivalent if there exists a vector \( v \in \mathbb{R}^d \) such that \( P_1 + v = P_2 \).

**Definition 2 (FLC)** A point set \( \Lambda \subset \mathbb{R}^d \) has finite local complexity (FLC) if for any \( M > 0 \), \( \Lambda \) possesses only finitely many equivalence classes of clusters with diameters smaller than \( M \).

**Definition 3 (repetitive)** A point set \( \Lambda \subset \mathbb{R}^d \) is repetitive if for any cluster \( P \subset \Lambda \), there is an \( R > 0 \) such that any ball with radius \( R \) contains a cluster equivalent to \( P \).
We will divide the construction of the Fibonacci quasi-crystal into the following three steps.

**Definition 4 (non-periodic)** A point set \( \Lambda \subset \mathbb{R}^d \) is non-periodic when there no non-zero \( t \in \mathbb{R}^d \) such that \( t + \Lambda = \Lambda \).

**Definition 5 (quasicrystal)** A point set \( \Lambda \subset \mathbb{R}^d \) is called a quasicrystal if it is FLC, repetitive and non-periodic.

Let \( K \subset \mathbb{R}^d \) be a compact set and fix the cluster \( P = \Lambda \cap K \) of a point set \( \Lambda \). The quotient
\[
\text{Card}\{ t \in B_r(\alpha) \mid (-t + \Lambda) \cap K = P \}
\]
is the number of clusters of type \( P \) per volume, with reference point in \( B_r(\alpha) \). If the quotient converges as \( r \to \infty \) with \( \alpha \in \mathbb{R}^d \) fixed, we call the limit the absolute frequency of the cluster \( P \) in \( \Lambda \) and denote it by \( \text{Freq}_\alpha(P) \). If the convergence is uniform in \( \alpha \), the limit is called the uniform absolute frequency of \( P \) and is denoted by \( \text{Freq}(P) \).

2.1.1 The Fibonacci chain as a specific one-dimensional quasi-crystal

We will divide the construction of the Fibonacci quasi-crystal into the following three steps.

**Step 1. Construct for a one-sided Fibonacci word.** Consider a two-letter alphabet \( \{a, b\} \) and the free group \( \langle a, b \rangle \) generated by letters \( a \) and \( b \). A substitution rule \( \rho \) on \( \{a, b\} \) is an endomorphism of \( \langle a, b \rangle \). Consider the substitution rule
\[
\rho : \begin{align*}
a &\mapsto ab \\
b &\mapsto a
\end{align*}
\]
and define a sequence \( \{u^{(i)}\}_{i \in \mathbb{Z}^+} \) of finite words by starting with the word \( u^{(1)} = a \) and iterating by \( u^{(i+1)} = \rho(u^{(i)}) \) for any \( i \geq 1 \):
\[
a \mapsto ab \mapsto aba \mapsto abaab \mapsto abaababaabaab \mapsto \cdots.
\]
By induction, we have
\[
u^{(i+2)} = u^{(i+1)} u^{(i)} \quad \text{for all } i \geq 1.
\]
Thus as \( i \to +\infty \), \( \{u^{(i)}\}_{i \in \mathbb{Z}^+} \) converges in the product topology of \( \{a, b\}^\mathbb{Z}^+ \) to an infinite word
\[
u := u_0 u_1 u_2 \cdots := abaababaabaab \cdots
\]
where \( u_i \in \{a, b\} \). The limit \( u \) is a fixed point of \( \rho \) and is called the one-sided Fibonacci word. Although \( u \) has some similar properties of quasicrystals, here we prefer the two-sided counterpart of \( u \), since it is better suited for our purposes.

**Remark** Notice that the length of \( u^{(i)} \) is the Fibonacci number \( f_i \) defined by \( f_1 = 1 \), \( f_2 = 2 \) and \( f_{i+2} = f_i + f_{i+1} \) for all \( i \geq 1 \). As a convention, let \( f_0 = 1 \), \( f_{-1} = 0 \) and \( \tau = (1 + \sqrt{5})/2 \). It is easy to show that
\[
f_i = \frac{\tau^{i+1} - (1 - \tau)^{i+1}}{\sqrt{5}}.
\]
Furthermore, the number of letter \( a \) in \( u^{(i)} \) is \( f_{i-1} \) and the number of \( b \) is \( f_{i-2} \).

**Step 2. Construct for a two-sided Fibonacci word.** Define a sequence \( \{w^{(i)}\}_{i \in \mathbb{N}} \) of finite words by \( w^{(i)} = u^{(i)} | u^{(i)} \): \[
a | a \mapsto ab | ab \mapsto aba | aba \mapsto abaab | abaabab \mapsto \cdots,
\]
where the vertical line marks the reference point. In fact, for all \( i \geq 3 \), suppressing the last two letters of \( u^{(i)} \) creates a palindrome \( \overline{u^{(i)}} \). Thus we have
\[
w^{(i)} = \begin{cases} 
\overline{a^{-1} b \overline{u^{(i)}} b a} u^{(i)} & \text{if } i \text{ is odd}, \\
\overline{a^{-1} b \overline{u^{(i)}} b a} u^{(i)} & \text{if } i \text{ is even}, 
\end{cases}
\]
for all \( i \geq 3 \), where \( \overline{u^{(i)}} \) denotes the reversal of \( u^{(i)} \). Therefore \( \{w^{(i)}\}_{i \in \mathbb{Z}^+} \) has two limit points in the product topology of \( \{a, b\}^\mathbb{Z} \) that are a 2-cycle under the substitution \( \rho \):
\[
\overline{a b} | u := \cdots abaababaababaababaababaababaababa \cdots
\]
\[
\overline{a b} | u = \cdots abaababaababaababaababaababaababa \cdots.
\]
We consider the space $\mathbb{R}^Z$ where $S$ (ii). Obviously, $S$ has non-zero period, then $u = x^N$ for some finite word $x$. Then the frequency of the letter $a$ in $u$ would be the quotient of the number of letter $a$ in $x$ divided by the length of $x$, which is rational. But on the other hand, the frequency of the letter $a$ is

$$\lim_{i \to \infty} \frac{\text{the number of letter } a \text{ in } u^{(i)}}{\text{the length of } u^{(i)}} = \lim_{i \to \infty} \frac{f_{i-1}}{f_{i}} = \frac{1}{\tau}$$

and is irrational, which is a contradiction. □

2.2 The variational framework for the Frenkel-Kontorova model

We consider the space $\mathbb{R}^Z$ of bi-infinite sequences of reals with the product topology. An element $x \in \mathbb{R}^Z$ is denoted by $(x_i)_{i \in \mathbb{Z}}$ and is sometimes called a configuration.

Given a function $H : \mathbb{R}^2 \to \mathbb{R}$ we extend $H$ to arbitrary finite segments $(x_j, \ldots, x_k)$, $j < k$, of configuration $x \in \mathbb{R}^Z$ by

$$H(x_j, \ldots, x_k) := \sum_{i=j}^{k-1} H(x_i, x_{i+1}).$$

We say that the segment $(x_j, \ldots, x_k)$ is minimal with respect to $H$ if

$$H(x_j, \ldots, x_k) \leq H(x^*_j, \ldots, x^*_k)$$

for all $(x^*_j, \ldots, x^*_k)$ with $x_j = x^*_j$ and $x_k = x^*_k$.

Definition 6 (minimal configuration) A configuration $x \in \mathbb{R}^Z$ is minimal if every finite segment of $x$ is minimal.

Definition 7 (stationary configuration) If $H$ is $C^2$, we say that $x \in \mathbb{R}^Z$ is a stationary configuration or an equilibrium if

$$\partial_2 H(x_{i-1}, x_i) + \partial_1 H(x_i, x_{i+1}) = 0 \text{ for all } i \in \mathbb{Z}.$$
We aim to build some special equivariant potential generated by the Fibonacci quasi-crystals. We will first introduce the following facts.

In order to investigate the properties the equilibria, we introduce the following two notions.

**Definition 8 (rotation number)** Let \( \rho \in \mathbb{R} \), a configuration \( x \in \mathbb{R}^\mathbb{Z} \) has a rotation number equal to \( \rho \) if the limit

\[
\lim_{i \to \pm\infty} \frac{x_i}{i} = \rho.
\]

**Definition 9 (type of configurations)** Let \( h : \mathbb{Z} \to \mathbb{R} \). A configuration \( x \in \mathbb{R}^\mathbb{Z} \) is of type \( h \) if

\[
\sup_{i \in \mathbb{Z}} |x_i - h(i)| < \infty.
\]

It is easy to see that the type of configurations is a more general notion than the rotation number. For instance, take \( h(i) = \rho i \), and then any configuration of type \( h \) has a rotation number \( \rho \).

### 2.3 The equivariant potential

We aim to build some special equivariant potential generated by the Fibonacci quasi-crystals. We will first introduce the following notion of “equivariant” in our setting.

**Definition 10** For any point set \( \Lambda \subset \mathbb{R}^d \), we say a continuous function \( f : \mathbb{R}^d \to \mathbb{R} \) is \( \Lambda \)-equivariant if there exists \( R > 0 \) such that if

\[
(\Lambda - x) \cap B_R(0) = (\Lambda - y) \cap B_R(0)
\]

then \( f(x) = f(y) \).

Given a real \( x \geq 0 \), we denote

\[
\alpha(x) = \sup\{y \in S \mid y \leq x\}, \quad \beta(x) = \inf\{y \in S \mid y > x\}.
\]

It is easy to see that both \( \alpha \) and \( \beta \) are right-continuous step functions. Thinking of \( u^{(i)} \) as a left-closed and right-open interval in \( \mathbb{R} \) for all \( i \in \mathbb{Z}^+ \), one can always find some \( i(x) \in \mathbb{Z}^+ \) such that \( x \in u^{(i(x))} \setminus u^{(i(x)-1)} \) with \( u^{(0)} = \emptyset \). Or in other words, due to (2), we have

\[
|u^{(i)}| = f_{i-1} + (f_i - f_{i-1}) = \tau^i \quad \text{for all} \quad i \in \mathbb{Z}^+,
\]

and so we get either

\[
0 \leq x < \tau \quad \text{or} \quad \tau^{n_1} \leq x < \tau^{n_1+1} \quad \text{for some} \quad n_1 \in \mathbb{Z}^+.
\]

Let us consider the later case, and we have \( 0 \leq x - \tau^{n_1} < \tau^{n_1-1} \). Therefore, considering \( x - \tau^{n_1} \) instead of \( x \), we get either

\[
0 \leq x - \tau^{n_1} < \tau \quad \text{or} \quad \tau^{n_2} \leq x - \tau^{n_1} < \tau^{n_2+1} \quad \text{for some} \quad n_2 \in \mathbb{Z}^+.
\]

Hence, we could repeat the above procedure finitely many times and obtain either (i) \( 0 \leq x < \tau \) or (ii) there exist infinitely many \( n_1, n_2, \ldots, n_r \in \mathbb{Z}^+ \) such that

\[
0 \leq x - \tau^{n_1} - \tau^{n_2} - \cdots - \tau^{n_r} < \tau.
\]

This means that \( x \) is covered either by the associated closed interval of \( u^{(1)} = a \) or \( u^{(n_1)} u^{(n_2)} \cdots u^{(n_r)} v \) where \( v \) is an unknown word. By [1, 4], it would be easy to see that \( v \) could be \( a, b, ba \) and \( bb \) but not an empty word just by the choice of \( n_r \) and its interval is right-open.

We could obtain the following facts.

**Lemma 2** If \( n_r > 1 \), then \( v = a \). If \( n_r = 1 \), then \( v = b \). Moreover, we have

\[
\beta(x) = \begin{cases} 
\alpha(x) + 1, & \text{if} \quad n_r = 1 \\
\alpha(x) + \tau, & \text{otherwise},
\end{cases}
\]

where \( \alpha(x) = \begin{cases} 0, & \text{if} \quad 0 \leq x < \tau \\
\tau^{n_1} + \tau^{n_1} + \cdots + \tau^{n_r}, & \text{if} \quad x \geq \tau.
\end{cases} \)
Proof: One could immediately have the representation of $\alpha(x)$. It then suffices to analyze the position of $x$ in the quasi-crystal into the following two cases:

Case 1. If $n_r = 1$ and so $n_r + 1 = 2$, we have either $x - \tau^{n_1} - \cdots - \tau^{n_{r-1}} - \tau^{n_r} = 0$, that is $x$ is just both the right end point of the associated interval of $a$ and the left end point of the associated interval of $b$ or $0 < x - \tau^{n_1} - \cdots - \tau^{n_{r-1}} - \tau^{n_r} < \tau$, that is $x$ is inside the associated interval of $b$. In both cases, we have $\beta(x) - \alpha(x) = 1$.

Case 2. If $n_r > 1$, we obtain $u^{(n_r+1)} = u^{(n_r)} u^{(n_r-1)}$ due to (1), that is, $x$ is covered by $u^{(n_1)} \cdots u^{(n_{r-1})} u^{(n_r)}$ but not $u^{(n_1)} \cdots u^{(n_{r-1})} u^{(n_r)}$. Therefore, $v$ must be a front part of $u^{(n_r-1)}$. Then the first letter of $v$ is $a$ and so $\beta(x) - \alpha(x) = \tau$.

$\blacksquare$

Notice that we have defined $\alpha(x)$ and $\beta(x)$ only for $x \geq 0$. Since $w = \overline{aba}|u$, we can extend $\alpha(x)$ and $\beta(x)$ on the entire real line $\mathbb{R}$ by

$$
\alpha(x) = \begin{cases} 
-\beta(-x), & \text{if } -\tau^2 \leq x < 0 \\
-\beta(-x - \tau^2) - \tau^2, & \text{if } x < -\tau^2
\end{cases}
$$

and

$$
\beta(x) = \begin{cases} 
-\alpha(-x), & \text{if } -\tau^2 \leq x < 0 \\
-\alpha(-x - \tau^2) - \tau^2, & \text{if } x < -\tau^2
\end{cases}
$$

Let $\zeta : \mathbb{R} \to \mathbb{R}$ be a function given by

$$
\zeta(x) = \begin{cases} 
\frac{64}{27} (3|x| - 1)^2 (96|x| - 11), & \text{if } x \in \left(-\frac{1}{3}, -\frac{1}{4}\right) \cup \left(\frac{1}{4}, \frac{1}{3}\right), \\
-64x^2 + \frac{160}{27}, & \text{if } x \in \left[-\frac{1}{4}, \frac{1}{4}\right], \\
0, & \text{otherwise.}
\end{cases}
$$
It is easy to check that $\zeta$ is $C^1$ everywhere and $C^2$ except $x = \pm 1/3$. The potential $V : \mathbb{R} \to \mathbb{R}$ of the interaction is defined by

$$V(x) = \begin{cases} 
\zeta(x - \alpha(x)), & \text{if } 2x \leq \alpha(x) + \beta(x), \\
\zeta(x - \beta(x)), & \text{if } 2x > \alpha(x) + \beta(x).
\end{cases}$$

Obviously, a sufficient condition of $V(x) = V(y)$ is $\alpha(x) - x = \alpha(y) - y$ and $\beta(x) - x = \beta(y) - y$.

**Lemma 3** $V$ is $S$-equivariant with range 1.

**Proof** Fix $x, y \in \mathbb{R}$. if

$$(S - x) \cap B_1(0) = (S - y) \cap B_1(0),$$

then

$$S \cap B_1(x) - x = S \cap B_1(y) - y.$$

If $\alpha(x) - x > -1$, then $\alpha(x) \in S \cap B_1(x)$. Then $\alpha(y) \in S \cap B_1(y)$ and $\alpha(x) - x = \alpha(y) - y$. 
If $\alpha(x) - x \leq -1$, then $\beta(x) - x > x - \alpha(x) \geq 1$. Since $\alpha(x), \beta(x) \in S$, we have $\beta(x) - \alpha(x) = \tau$. Similarly, we also have $\beta(y) - \alpha(y) = \tau$. Thus $\beta(x) - x = \alpha(x) - x + \tau \leq \tau - 1 < 1$. Then $\beta(x) - x = \beta(y) - y$. Hence $\alpha(x) - x = (\beta(x) - x) - (\beta(x) - \alpha(x)) = (\beta(y) - y) - (\beta(y) - \alpha(y)) = \alpha(y) - y$. Hence we always have $\alpha(x) - x = \alpha(y) - y$. Similarly, $\beta(x) - x = \beta(y) - y$. □

3 The explicit one-dimensional example and its multiple solutions

Our goal in this section is to present the Frenkel-Kontorova model with a special potential driven by the one-dimensional quasi-crystal and one can obtain minimal configurations, equilibrium configurations and multiple solutions with or without any rotation number.

3.1 Minimal configurations

In this section, we aim to find a minimal configuration with rotation number $(3\tau + 1)/2$. The construction is rather concrete although minimal configurations with given rotation number exist due to [7].

3.1.1 Local shapes of the Fibonacci chain

For any $x \in \mathbb{R}$, the translation $S - x$ of $S$ is still a quasicrystal. Let $S + \mathbb{R} = \{S - x \mid x \in \mathbb{R}\}$ be the collection of all translations of $S$.

Let $P$ be a given patch of quasicrystal $S$ and $U \subset \mathbb{R}$. The cylinder set $\Omega_{P,U}$ is the set of all quasicrystals in $S + \mathbb{R}$ that contain a copy of $P$ translated by an element of $U$. That is,

$$\Omega_{P,U} := \{T \in S + \mathbb{R} \mid P - u \text{ is a patch in } T \text{ for some } u \in U\}.$$  

In particular, we denote $\Omega_P = \Omega_{P,(0)}$. It is easy to check that $\Omega_{P,U} = \bigcup_{u \in U}(\Omega_P - u)$.

For any integer $l \geq 1$, let

$$c_l = S \cap [-\tau^{2l}, \tau^{2l}],$$  
$$\varepsilon_{l,1} = S \cap [-\tau^{2l}, \tau^{2l+2}],$$  
$$\varepsilon_{l,2} = S \cap [\tau^{2l-1}, 2\tau^{2l+2}] - \tau^{2l+1}.$$  

Sometimes we use the corresponding words to represent them, for example:

$$c_1 = ba|ab,$$  
$$\varepsilon_{1,1} = ba|abaab,$$  
$$\varepsilon_{1,2} = ba|ababaab;$$  

$$c_2 = ababa|abaab,$$  
$$\varepsilon_{2,1} = ababa|abaababaabaab,$$  
$$\varepsilon_{2,2} = ababa|abaababaababaabaab.$$  

Let $C_l := \Omega_{c_l}, \varepsilon_{l,1} := \Omega_{\varepsilon_{l,1}}$ and $\varepsilon_{l,2} := \Omega_{\varepsilon_{l,2}}$. Obviously, $\varepsilon_{l,1}$ and $\varepsilon_{l,2}$ are subsets of $C_l$.

Remark The definitions of $c_l, \varepsilon_{l,1}$ and $\varepsilon_{l,2}$ above is motivated as following. Let $c_l$ is represented by $w^{(2l)}$. Consider the function

$$L^l : C_l \to \mathbb{R}, T \mapsto \inf\{t > 0 \mid T - t \in C_l\}$$

which takes each $T$ to the least positive $L^l(T)$ such that $T - L^l(T) \in C_l$. The function $L^l$ measures the distance between every two adjacent patches translated from $c_l$. We will show that the distance can only take two values $\tau^{2l+1}$ and $2\tau^{2l+2}$, i.e. the range of $L^l$ is $\{\tau^{2l+1}, 2\tau^{2l+2}\}$, see Lemma 4. Then let $\varepsilon_{l,1}$ and $\varepsilon_{l,2}$ are the two patches whose head and tail are both $c_l$.

Lemma 4 For each $l \geq 1$,

(i) $C_l = \varepsilon_{l,1} \sqcup \varepsilon_{l,2}$,

(ii) $S + \mathbb{R} = \Omega_{\varepsilon_{l,1},[0,\tau^{2l+1}]} \sqcup \Omega_{\varepsilon_{l,2},[0,\tau^{2l+2}]}$.  

8
On the existence of solutions for Frenkel-Kontorova models on quasi-crystals

**Proof.** Firstly, we show that the range of $L^1$ is \{\(\tau^{2l+1}, \tau^{2l+2}\). Since \(w\) is a fixed point of \(\rho^2\), we have that \(c_{l+1} = \rho^2(c_l)\). Hence \(L^{l+1}(T) = \tau^2 L^l(T)\) for all \(T \in C_{l+1}\). We only need to show the claim when \(l = 1\). Notice that \(\varepsilon_{1,1} = baababab = c_1abab\), we have \(L^1(\varepsilon_{1,1}) = \{\tau^3\}\). Since the distance between the first \(b\) of \(c_1\) and the second is \(\tau^3, \tau^{-3}\) is minimal. It is easy to see that \(L^1(\varepsilon_{1,2}) = \{\tau^4\}\). Since \(bb\) cannot occur in \(w\), \(\tau^4\) is the second least in the range of \(L\). If there are more than one letters between two patches translated from \(c_1\), the patch \(P\) formed by these letters must be like \(ababa\cdots ab\) since \(bb\) and \(aa\) cannot appear in \(P\). If so, the patch \(c_1PC_1\) contains \(ababab\) which cannot appear in \(S\) since \(aaa\) cannot appear. Hence the range of \(L^1\) is \(\{\tau^3, \tau^4\}\). It is obvious that

\[
C_1 = (L^1)^{-1}(\tau^3) \sqcup (L^1)^{-1}(\tau^4) = \varepsilon_{1,1} \sqcup \varepsilon_{1,2}.
\]

Since \(\varepsilon_{1+1,i} = \rho^2(\varepsilon_{1,i}), i \in \{1, 2\}\), by induction, we get (i).

Consider all patches in \(S\) translated from \(c_1\). They decide a point set of \(\mathbb{R}\):

\[
S^l := \{x \in \mathbb{R} \mid P - x = c_l \text{ for some patches } P \in S\}.
\]

For any \(x \in \mathbb{R}\), let \(\alpha_l(x)\) be the largest number in \(S^l\) such that \(\alpha_l(x) \leq x\) and \(\beta_l(x)\) be the least number in \(S^l\) such that \(\beta_l(x) > x\). We know that \(\beta_l(x) - \alpha_l(x)\) is \(\tau^{2l+1}\) or \(\tau^{2l+2}\). If \(\beta_l(x) - \alpha_l(x) = \tau^{2l+1}\), then

\[
(S - \alpha_l(x)) \cap [-\tau^{2l}, \tau^{2l+2}] = \varepsilon_{l,1}.
\]

If \(\beta_l(x) - \alpha_l(x) = \tau^{2l+2}\), then

\[
(S - \alpha_l(x)) \cap [-\tau^{2l}, \tau^{2l+2} + \tau^{2l}] = \varepsilon_{l,2}.
\]

For any \(T = S - x \in S + \mathbb{R}\), \(T - (x - \alpha_l(x)) = S - \alpha_l(x)\) thus \(S + \mathbb{R} \subset \Omega_{\varepsilon_{l,1}, [0, \tau^{2l+1}]} \sqcup \Omega_{\varepsilon_{l,2}, [0, \tau^{2l+2}]}\). The rest of the proof is trivial. \(\square\)

### 3.1.2 The frequencies of the local shapes

If \(\beta_l(x) - \alpha_l(x) = \tau^{2l+1}\), then we denote the corresponding word of \(S \cap [\alpha_l(x), \beta_l(x)]\) by \(A_l\). If \(\beta_l(x) - \alpha_l(x) = \tau^{2l+2}\), then we denote the corresponding word of \(S \cap [\alpha_l(x), \beta_l(x)]\) by \(B_l\). For example,

\[
A_1 =aba, \quad B_1 = ababa, \quad A_2 = abaababa, \quad B_2 = abaababaababa.
\]

Notice that \(A_2 = A_1B_1\) and \(B_2 = A_1B_1B_1\). Since \(\varepsilon_{1+1,i} = \rho^2(\varepsilon_{1,i}), i \in \{1, 2\}\), we have \(A_{l+1} = \rho^2(A_l)\) and \(B_{l+1} = \rho^2(B_l)\). Hence we have \(A_{l+1} = A_1B_l\) and \(B_{l+1} = A_1B_lB_l\) for all \(l \geq 1\) by induction. Let

\[
M := \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \text{the number of } A_l \text{ in } A_{l+1} & \text{the number of } B_l \text{ in } A_{l+1} \\ \text{the number of } A_l \text{ in } B_{l+1} & \text{the number of } B_l \text{ in } B_{l+1} \end{pmatrix}
\]

Then

\[
M^n = \frac{1}{\sqrt{5}} \begin{pmatrix} \tau^{2n+1} + \tau^{2n-1} & -\tau^{-2n} + \tau^{2n} \\ -\tau^{-2n} + \tau^{2n} & \tau^{-2n-1} + \tau^{2n+1} \end{pmatrix} = \begin{pmatrix} \text{the number of } A_l \text{ in } A_{l+n} & \text{the number of } B_l \text{ in } A_{l+n} \\ \text{the number of } A_l \text{ in } B_{l+n} & \text{the number of } B_l \text{ in } B_{l+n} \end{pmatrix}
\]

Consider the two limits

\[
\lim_{n \to \infty} \frac{\text{the number of } A_l \text{ in } A_{l+n}}{\text{the range of } A_{l+n}} = \lim_{n \to \infty} \frac{\text{the number of } A_l \text{ in } B_{l+n}}{\text{the range of } B_{l+n}} = \frac{1}{\sqrt{5}\tau^{2l+2}}.
\]

Since the word of \(S \cap [-\tau^4, \tau^3]\) is \(B_1|A_1\). The absolute frequency of \(A_l\) at 0 is the limit \(1/(\sqrt{5}\tau^{2l+2})\) above. By the same way, the absolute frequency of \(B_l\) at 0 is \(1/(\sqrt{5}\tau^{2l+1})\).

### 3.1.3 The construction of a minimal configuration

For any integer \(l \geq 1\), let \(B_l\) be an oriented 1-dimensional branched manifold in \(\mathbb{R}^2\) which consists of two circles \(\gamma_{l,1}\) and \(\gamma_{l,2}\) tangent at a single point \(R_l\). The circumferences of \(\gamma_{l,1}\) and \(\gamma_{l,2}\) are \(\tau^{2l+1}\) and \(\tau^{2l+2}\). Given two points \(A\) and \(B\) on the same circle, the oriented length of the arc from \(A\) to \(B\) is denoted by \(d(A, B)\). Let \(m_{l,i}(x)\) denote the point on \(\gamma_{l,i}\) such that \(d(O, m_{l,i}(x)) = x\), \(i \in \{1, 2\}\).
For any \( l \geq 1 \), the map \( \kappa_l : B_{l+1} \to B_l \) is defined by

and the map \( \pi_l : \mathbb{R} \to B_l \) is defined by

\[
\pi_l(x) = \begin{cases} 
  m_{l,1}(x - \alpha_l(x)), & \text{if } \beta_l(x) - \alpha_l(x) = \tau^{2l+1} \\
  m_{l,2}(x - \alpha_l(x)), & \text{if } \beta_l(x) - \alpha_l(x) = \tau^{2l+2} 
\end{cases}
\]

Since \( A_{l+1} = A_l B_l \) and \( B_{l+1} = A_l B_l B_l \) for all \( l \geq 1 \), \( \kappa_l \circ \pi_{l+1} = \pi_l \). \( \pi_l \) is a covering map from \( \mathbb{R} \) to \( B_l \) and \( \pi_l(S^1) = \{ R_l \} \). For each point \( y \in B_l \), its inverse image \( (\pi_l)^{-1}(y) \) is a point set in \( \mathbb{R} \) and for any point \( x \) in \( (\pi_l)^{-1}(y) \), \( S \cap B_l(x) \) is the same patch up to translations. By lemma 3 the value of \( V \) is the same on \( (\pi_l)^{-1}(y) \). Thus we can define the potential on \( B_l \) by \( \tilde{V} := V \circ (\pi_l)^{-1} : B_l \to \mathbb{R} \). We can also define \( \tilde{H} : \gamma_{l,1} \times \gamma_{l,1} \cup \gamma_{l,2} \times \gamma_{l,2} \to \mathbb{R} \) which maps \((A, B)\) to \( \frac{1}{2}d(A, B)^2 \).

Now we construct the minimal configuration.

**Step 1:** Fix \( l = 1 \). For each \( i \in \{1, 2\} \), let \( b_{1,i} \) be a point of \( \gamma_{1,i} \backslash \{R_1\} \). Then we can calculate the potential energy of finite segment \((R_1, b_{1,i}, R_1)\):

\[
E(R_1, b_{1,i}, R_1) = \tilde{H}(R_1, b_{1,i}) + \tilde{V}(b_{1,i}) = d(R_1, b_{1,i})^2 - \tau^{2+i}d(R_1, b_{1,i}) + V(d(R_1, b_{1,i})) + \tau^{4+2i}/2 + V(0).
\]

Since \( V(\tau^{2+i}/2) = 0 \), it is easy to show that \( E(R_1, b_{1,i}, R_1) \) is minimal when \( d(R_1, b_{1,i}) = \tau^{2+i} \). Hence we let \( b_{1,i} = m_{1,i}(\tau^{2+i}) \) which is the antipodal point of \( R_1 \) on \( \gamma_{1,i} \). \((\pi_l)^{-1}(\{R, b_{1,1}, b_{1,2}\})\) is a discrete subset of \( \mathbb{R} \) that we can order as a bi-infinite increasing sequence \((\theta_{1,n})_{n\in\mathbb{Z}}\) with \( \theta_{1,0} = 0 \). Then \((\theta_{1,n})_{n\in\mathbb{Z}}\) is a configuration on \( \mathbb{R} \) whose potential energy is minimal on each segment \([\alpha_l(x), \beta_l(x)]\).

**Step 2:** For all \( l \geq 1 \), for each \( i \in \{1, 2\} \), let \( \{b_{l,i,j} \mid 1 \leq j \leq N_{l,i} - 1\} \) be \( N_{l,i} - 1 \) different points of \( \gamma_{l,i} \backslash \{R_l\} \) whose order indexed by \( j \) is the same as the orientation of \( \gamma_{l,i} \). \( N_{l,i} \) can be calculate by

\[
\begin{pmatrix} N_{l+1,1} \\ N_{l+1,2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} N_{l,1} \\ N_{l,2} \end{pmatrix}, N_{l,1} = N_{l,2} = 2.
\]

In fact, \((N_{l,1}, N_{l,2})^T = (2f_{2l-2}, 2f_{2l-1})^T\). Then we can calculate the energy potential of \((R_l, b_{l,i,1}, \ldots, b_{l,i,N_{l,i}-1}, R_l)\) and choose the \( N_{l,i} - 1 \) points in \( \gamma_{l,i} \) such that the energy is minimal. \((\pi_l)^{-1}(\{R, b_{l,i,j} \mid 1 \leq j \leq N_{l,i} - 1\})\) is a discrete subset of \( \mathbb{R} \) that we can order as a bi-infinite increasing sequence \((\theta_{l,n})_{n\in\mathbb{Z}}\) with \( \theta_{l,0} = 0 \). Then \((\theta_{l,n})_{n\in\mathbb{Z}}\) is a configuration on \( \mathbb{R} \) whose potential energy is minimal on each segment \([\alpha_l(x), \beta_l(x)]\).

**Step 3:** Now we get a sequence \(((\theta_{m,n})_{n\in\mathbb{Z}})_{m\in\mathbb{Z}}\) of configurations. In the following we will show that it is in a compact subset of \( \mathbb{R}^Z \) with the product topology and its accumulation point is a minimal configuration with rotation number \((3\tau + 1)/2\).

### 3.1.4 Combinatorics of minimal configurations

**Proposition 1** (see [10]) Let \((\theta_1, \ldots, \theta_n)\) be a minimal segment and let \( I \) be an interval in \([\theta_1, \theta_n]\), then there exists an integer \( n \in \mathbb{N} \) such that for any pair of disjoint intervals \( I_1 = I + u_1 \) and \( I_2 = I + u_2 \) in \([\theta_1, \theta_n]\) which satisfy that for each \( \theta \) in \( I \) and \( k = 1, 2 \):

\[
S \cap B_1(\theta) + u_k = S \cap B_1(\theta + u_k)
\]

each interval \( I_k \) contains either \( n, n + 1 \) or \( n + 2 \) atoms of the minimal segment.

**Corollary** For each \( l \geq 1, i = 1, 2 \), the difference between the numbers of atoms of a minimal configuration on any two sheets over \( \gamma_{l,i} \) is not larger than 2.
3.1.5 The proof of item (i) of Theorem \[8\]

**Lemma 5** For each \( l \geq 1 \), \((\theta_{l,n})_{n \in \mathbb{Z}}\) has rotation number \((3\tau + 1)/2\).

**Proof** Let \( n_{l,i} \) be the number of times \( \pi_l((\theta_{l,0}, \theta_{l,n})) \) covers completely the circle \( \gamma_{l,i} \). Then

\[
n_{l,1}\tau^{2l+1} + n_{l,2}\tau^{2l+2} \leq \theta_{l,n} - \theta_{l,0} \leq n_{l,1}\tau^{2l+1} + n_{l,2}\tau^{2l+2} + 2\tau^{2l+2}
\]

and

\[
n_{l,1}\bar{N}_{l,1} + n_{l,2}\bar{N}_{l,2} \leq n \leq n_{l,1}\bar{N}_{l,1} + n_{l,2}\bar{N}_{l,2} + 2N_{l,2}.
\]

Thus

\[
\frac{n_{l,1}\tau^{2l+1} + n_{l,2}\tau^{2l+2}}{n_{l,1}\bar{N}_{l,1} + n_{l,2}\bar{N}_{l,2}} \leq \frac{\theta_{l,n}}{n} \leq \frac{n_{l,1}\tau^{2l+1} + n_{l,2}\tau^{2l+2} + 2\tau^{2l+2}}{n_{l,1}\bar{N}_{l,1} + n_{l,2}\bar{N}_{l,2}}
\]

Then the rotation number \( \rho_l \) of \((\theta_{l,n})_{n \in \mathbb{Z}}\) is the limit (if exists):

\[
\lim_{n \to +\infty} \frac{n_{l,1}\tau^{2l+1} + n_{l,2}\tau^{2l+2}}{n_{l,1}\bar{N}_{l,1} + n_{l,2}\bar{N}_{l,2}}.
\]

When \( n \) goes to \( +\infty \) the quantity

\[
\frac{n_{l,i}}{n_{l,1}\bar{N}_{l,1} + n_{l,2}\bar{N}_{l,2}}
\]

goes to the absolute frequency of \( A_l \) if \( i = 1 \) or of \( B_l \) if \( i = 2 \). Hence

\[
\rho_l = \frac{1}{\text{Freq}_0(A_l)\bar{N}_{l,1} + \text{Freq}_0(B_l)\bar{N}_{l,2}} = \frac{3\tau + 1}{2}.
\]

**Lemma 6** There exists \( M > 0 \) such that for any \( l \geq 1 \), \( n \in \mathbb{Z} \), we have

\[
\theta_{l,n+1} - \theta_{l,n} \leq M.
\]

**Proof** Let us prove this lemma by contradiction. Let \( M(m) = 2|B_m| = 2\tau^{2m+3} \), where \( m \geq 2 \). Then there exists \( l(m) \) and \( n(m) \) such that

\[
\theta_{l(m),n(m)+1} - \theta_{l(m),n(m)} > M(m).
\]

Then there exists minimal segment

\[
(\theta_{l(m),n_1}, \ldots, \theta_{l(m),n(m)}, \theta_{l(m),n(m)+1}, \ldots, \theta_{l(m),n_2})
\]

such that

\[
\gamma_{m,i} \subset \pi_l([\theta_{l(m),n(m)}, \theta_{l(m),n(m)+1}]) \text{ for some } i.
\]

Since \( \kappa_{m-1}(\gamma_{m,1}) = B_{m-1} \), function \( \kappa_{m-1} \) on each side then we have

\[
B_{m-1} \subset \pi_{m-1}([\theta_{l(m),n(m)}, \theta_{l(m),n(m)+1}])
\]

By proposition \[1\] the number of atoms in any segment of \((\theta_{l,m},n)_{n \in \mathbb{Z}}\) is smaller than \( 3(n_{l(m),1} + n_{l(m),2} + 2) \). This is

\[
n_{l(m),1}\bar{N}_{l(m),1} + n_{l(m),2}\bar{N}_{l(m),2} \leq 3(n_{l(m),1} + n_{l(m),2} + 2) \text{ for all } m \geq 2.
\]

Figure 3: The red points are the graph of \((\theta_{5,n})_{n \in [-50,50]}\) and the green line is \( y = (3\tau + 1)/2x \).
This is a contradiction. □

Consider the product topology on $\mathbb{R}^\mathbb{Z}$. By lemma [6] for each $l \geq 1$, the distance of two connected points in $(\theta_{l,n})$ is bounded. All the configurations $((\theta_{m,n})_{n \in \mathbb{Z}})_{m \in \mathbb{N}}$ are contained in a compact subset of $\mathbb{R}^\mathbb{Z}$. Thus there exists an accumulation point $(\theta_{\infty,n})_{n \in \mathbb{Z}}$.

**Theorem 2** $(\theta_{\infty,n})_{n \in \mathbb{Z}}$ is a minimal configuration with rotation number $(3\tau + 1)/2$.

**Proof** Obviously $(\theta_{\infty,n})_{n \in \mathbb{Z}}$ is minimal. We just need to show its rotation number is $(3\tau + 1)/2$. Let $n_{\infty,l,i}$ be the number of times $\pi_l((\theta_{\infty,0}, \theta_{\infty,n}))$ covers completely the circle $\gamma_{l,i}$. Then

$$n_{\infty,l,1}\tau^{2l+1} + n_{\infty,l,2}\tau^{2l+2} \leq \theta_{\infty,n} - \theta_{\infty,0} \leq n_{\infty,l,1}\tau^{2l+1} + n_{\infty,l,2}\tau^{2l+2} + 2\tau^{2l+2}$$

and by proposition [1]

$$n_{\infty,l,1}(N_{l,1} - 2) + n_{\infty,l,2}(N_{l,2} - 2) \leq n \leq n_{\infty,l,1}(N_{l,1} + 2) + n_{\infty,l,2}(N_{l,2} + 2) + 2(N_{l,2} + 2).$$

Thus

$$\frac{n_{\infty,l,1}\tau^{2l+1} + n_{\infty,l,2}\tau^{2l+2}}{n_{\infty,l,1}(N_{l,1} + 2) + n_{\infty,l,2}(N_{l,2} + 2) + 2(N_{l,2} + 2)} \leq \frac{\theta_{\infty,n}}{n} \leq \frac{n_{\infty,l,1}\tau^{2l+1} + n_{\infty,l,2}\tau^{2l+2} + 2\tau^{2l+2}}{n_{\infty,l,1}(N_{l,1} + 2) + n_{\infty,l,2}(N_{l,2} - 2)}$$

When $n$ goes to $+\infty$ the quantity

$$\frac{n_{\infty,l,1}N_{l,1} + n_{\infty,l,2}N_{l,2}}{n_{\infty,l,1}(N_{l,1} + 2) + n_{\infty,l,2}(N_{l,2} + 2) + 2(N_{l,2} + 2)}$$

go to the absolute frequency of $A_i$ if $i = 1$ or of $B_l$ if $i = 2$. Then the rotation number $\rho_\infty$ of $(\theta_{\infty,n})_{n \in \mathbb{Z}}$ satisfies

$$\frac{1}{\text{Freq}_0(A_i)(N_{l,1} + 2) + \text{Freq}_0(B_i)(N_{l,2} + 2)} \leq \rho_\infty \leq \frac{1}{\text{Freq}_0(A_i)(N_{l,1} - 2) + \text{Freq}_0(B_i)(N_{l,2} - 2)}$$

for all $l \geq 1$. Let $l \to +\infty$,

$$\rho_\infty = \frac{3\tau + 1}{2}.$$
For all $u \in \Pi$, since
\[ |(\Delta u)_i| \leq |(\Delta u)_i - (\Delta g)_i| + |(\Delta g)_i| \leq 4 \sup_{i \in \mathbb{Z}} |u_i - g(i)| + 2\tau \leq \frac{64\tau}{31}, \]
we get $|\Phi(u)_i - g(i)| = \frac{1}{128} |(\Delta u)_i| \leq \frac{\tau}{62}$, which means that $\Phi$ is well-defined. $\Phi$ is a contraction, for
\[ |\Phi(u)_i - \Phi(u')_i| = \frac{1}{128} |(\Delta u)_i - (\Delta u')_i| \leq \frac{1}{62} \sup_{i \in \mathbb{Z}} |u_i - u'_i|. \]

By the fixed-point principle, $\Phi$ has a unique fixed point $u$ satisfying
\[ u_i = -\frac{1}{128} (\Delta u)_i + g(i) \]  \hspace{1cm} (6)
for all $i \in \mathbb{Z}$. Notice that $u_i$ is in the support of the quadratic part of $V$. Multiplying 128 of both sides of (3), one obtains (5), which says $u$ is an equilibrium configuration. Notice that $|u_i - g(i)| \leq \frac{\tau}{62}$ and $|g(i) - h(i)| < \frac{\tau}{2}$ for all $i \in \mathbb{Z}$, the rotation number of $u$ is $(3\tau + 1)/2$, the slope of $h$.

Next, let’s calculate the equilibrium configuration $u$ we constructed. Let $\alpha = 1/128$, then (4) is equivalent to
\[ \alpha u_{i-1} + (1 - 2\alpha) u_i + \alpha u_{i+1} = a_i \]
for all $i \in \mathbb{Z}$. Let $T$ be a tridiagonal operator defined by
\[ Te_i = \alpha e_{i-1} + (1 - 2\alpha) e_i + \alpha e_{i+1}, \quad i \in \mathbb{Z} \]
where $\{ e_i : i \in \mathbb{Z} \}$ is the orthonormal basis of the sequence space. Consider the truncation matrix
\[ T_n = \begin{pmatrix}
1 & -2\alpha & 0 & \cdots & 0 & 0 \\
\alpha & 1 - 2\alpha & \alpha & \cdots & 0 & 0 \\
0 & \alpha & 1 - 2\alpha & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 - 2\alpha & \alpha \\
0 & 0 & 0 & \cdots & \alpha & 1 - 2\alpha \\
\end{pmatrix}_{n \times n}, \]

By the Theorem 3.1. of [11], $T_n$ is invertible. By the Corollary 6.2 of [10], $\{ \| T_n^{-1} e_n \| \}$ and $\{ \| T_n^{-1} e_n \| \}$ are bounded. Then use the Theorem 3.1 of [12],
\[ u = \lim_{n \to \infty} T_n^{-1} e_n, \]
where $y_n = (g(-n), \ldots, g(0), \ldots, g(n))^T$ is the truncation of $(g(i))_{i \in \mathbb{Z}}$.

**Remark** In fact, the method of finding equilibrium configuration in this article can be applied for all $h$ satisfying $|(\Delta h)_i| < \infty$. For example, let
\[ h_1(i) = \begin{cases} 
  i^2, & \text{if } i \geq 0, \\
  -i^2, & \text{if } i < 0.
\end{cases} \]
In this case, $\lim_{n \to \pm \infty} h_1(i)/i = \pm \infty$. Hence the equilibrium configuration has no rotation number.

The following theorem is a special case of Theorem 1 in [6].

**Theorem 3** Let $h : \mathbb{Z} \to \mathbb{R}$ satisfies $|(\Delta h)_i| < \infty$ for all $i \in \mathbb{Z}$. Then there exists a $\lambda_0$ such that for any $\lambda > \lambda_0$, there exists an equilibrium configuration $u$ of type $h$ with respect to
\[ H_\lambda(\xi, \eta) = \frac{1}{2} (\xi - \eta)^2 + \lambda V(\xi). \]

Moreover, in our context, we could show in addition that there are also non-minimal equilibrium configurations:

**Theorem 4** Let $h : \mathbb{Z} \to \mathbb{R}$ satisfies $|(\Delta h)_i| < \infty$ for all $i \in \mathbb{Z}$. Then there exists two reals $\lambda_1 > \lambda_0$ such that for any $\lambda > \lambda_0$, there exists an equilibrium configuration $u$ of type $h$ with respect to
\[ H_\lambda(\xi, \eta) = \frac{1}{2} (\xi - \eta)^2 + \lambda V(\xi). \]
In particular, if $\lambda > \lambda_1$, the equilibrium configuration obtained above is non-minimal.
In fact, the idea we used in Section 3.2 to construct the equilibrium configuration is from the paper [6]. Hence, Theorem 4 implies the non-minimality of the configuration we constructed in Section 3.2.

To show Theorem 4, we just need a lemma:

**Lemma 7** For any \( \lambda > -4/\zeta''(0) = 1/32 \), if \( u \) is an equilibrium configuration with respect to \( H_\lambda \) and each component \( u_i \) lies in the quadratic part of \( V \), then \( u \) is non-minimal.

**Proof** Fix \( \lambda \) and \( u \) that meet the conditions. Since each component \( u_i \) lies in the quadratic part of \( V \), we can define the sequence \( g(i) \) as the closest turning point of \( V \) to \( u_i \). Notice that \( -\Delta u_i + \lambda V'(u_i) = 0 \) (7) and \( V'(g(i)) = 0 \) for all \( i \in \mathbb{Z} \), then there exists some index \( i_0 \) such that \( u_{i_0} \neq g(i_0) \), otherwise \( \Delta u_i = 0 \) and thus \( u \) is linear. If \( u \) is linear, since each \( u_i \) as a turning point of \( V \) lies in the Fibonacci chain \( S \), the frequency of the letter \( a \) in \( S \) is equal to the one in \( [u_0, u_1] \), which is rational. However, the frequency of the letter \( a \) in \( S \) is \( 1/\tau \), which was calculated in the proof of Lemma 1. Without loss of generality, suppose that \( i_0 = 0 \). In order to show that \( u \) is non-minimal, we show that \( (u_{-1}, u_0, u_1) \) is non-minimal. That is, we find a \( u'_0 := u'_0(\lambda, u) \) such that

\[
H_\lambda(u_{-1}, u'_0) + H_\lambda(u'_0, u_1) < H_\lambda(u_{-1}, u_0) + H_\lambda(u_0, u_1).
\]

(8)

Before we give the value of \( u'_0 \), there are some relations among \( u_0, g(0) \) and \( \bar{u} := (u_{-1} + u_1)/2 \). Since \( u_0 \) lies in the support of the quadratic part of \( V \), from Taylor’s formula and the condition of \( \lambda \), we have

\[
\lambda V'(u_0) = \lambda V''(g(0))(u_0 - g(0)) = \lambda \zeta''(0)(u_0 - g(0)) \begin{cases} < -4(u_0 - g(0)) & \text{if } u_0 > g(0); \\ > -4(u_0 - g(0)) & \text{if } u_0 < g(0). \end{cases}
\]

(9)

Then from (7) and (9), we know that

\[
\bar{u} = u_0 + \frac{\lambda V'(u_0)}{2} \begin{cases} < -u_0 + 2g(0) & \text{if } u_0 > g(0); \\ > -u_0 + 2g(0) & \text{if } u_0 < g(0). \end{cases}
\]

(10)
That is,\[ |u_0 - g(0)| < |\bar{u} - g(0)|. \tag{11} \]

And from (10) we can see that the sign of $u_0 - g(0)$ is different from the sign of $\bar{u} - g(0)$, thus we have
\[ |\bar{u} - g(0)| < |\bar{u} - u_0|. \tag{12} \]

Now, we discuss three different cases and give the value of $u_0'$ in each case.

**Case 1.** If $|\bar{u} - g(0)| \leq 1/2$, where $1/2$ is half of the minimal distance between two adjacent turning points of $V$. Let $u_0' = \bar{u}$. Consider the function $F(x) = \frac{1}{2}(u_0 - x)^2 + \frac{1}{2}(x - u_0)^2$. $F(x)$ is a quadratic function and takes minimum at $\bar{u}$. Hence,
\[ \frac{1}{2}(u_0 - \bar{u})^2 + \frac{1}{2}(\bar{u} - u_0)^2 \leq \frac{1}{2}(u_0 - u_1)^2 + \frac{1}{2}(u_0 - u_1)^2. \tag{13} \]

Since $|\bar{u} - g(0)| \leq 1/2$ and $V$ is increasing on the interval $[g(0) - 1/2, g(0)]$, decreasing on the interval $[g(0), g(0) + 1/2]$, and especially, strictly monotone on its quadratic parts, by (11), we have
\[ V(u_0') < V(u_0). \tag{14} \]

Multiply both sides of the inequality (14) by $\lambda$ and add the inequality (13), then we have (6).

**Case 2.** If $\bar{u} - g(0) > 1/2$. Let $u_0' = g(0) + 1/3$, where $1/3$ is the radius of each bump of $V$. Then
\[ V(u_0') = 0 \leq V(u_0). \tag{15} \]

By (12), we have
\[ |\bar{u} - u_0'| = |\bar{u} - g(0) - 1/3| < |\bar{u} - g(0)| < |\bar{u} - u_0|. \]

Then using the monotonicity of $F(x)$, we have
\[ \frac{1}{2}(u_0 - \bar{u})^2 + \frac{1}{2}(\bar{u} - u_0)^2 < \frac{1}{2}(u_0 - u_1)^2 + \frac{1}{2}(u_0 - u_1)^2. \tag{16} \]

Multiply both sides of the inequality (15) by $\lambda$ and add the inequality (16), then we have (8).

**Case 3.** If $\bar{u} - g(0) < -1/2$, let $u_0' = g(0) - 1/3$. The rest of the proof is the same as the one in Case 2.

**Proof of Theorem 4** For any $h$ with $|\langle \Delta h \rangle_i| < \infty$ for all $i \in \mathbb{Z}$, Theorem 3 ensures the existence of equilibrium configuration of type $h$ with respect to $H_\lambda$ for any $\lambda$ larger than some $\lambda_0$. Let $\lambda_1 = \max\{\lambda_0, -4/\zeta''(0)\}$. Then for any $\lambda > \lambda_1$, the equilibrium configuration $u$ whose existence is proved in Theorem 3 is non-minimal by Lemma 7 as long as we admit that each $u_i$ lies in the quadratic part of $V$.

To solve the problem of whether each $u_i$ can lie in the quadratic part of $V$, let us see the proof of Theorem 3 carefully. In our example of $V$, a sufficient condition is that the radius of the space that the contraction functions is smaller than 1/4. A possible construction of the contraction is
\[ \Pi_\lambda = \left\{ u : |u_i - g(i)| \leq \frac{2 \max\{|a|, |b|\} + \sup_{i \in \mathbb{Z}} |\langle \Delta h \rangle_i|}{-\lambda \zeta''(0) - 4} \text{ for all } i \in \mathbb{Z} \right\}, \]
\[ \Phi_\lambda : \Pi_\lambda \to \Pi_\lambda, \quad u \mapsto \left( \frac{1}{\lambda \zeta''(0)} \langle \Delta u \rangle_i + g(i) \right)_{i \in \mathbb{Z}}. \]

For all $u \in \Pi_\lambda$, since
\[
|\langle \Delta u \rangle_i| \leq |\langle \Delta u \rangle_i - (\Delta g)\rangle_i| + |(\Delta g)\rangle_i - (\Delta h)\rangle_i| + |(\Delta h)\rangle_i| \\
\leq 4 \sup_{i \in \mathbb{Z}} |u_i - g(i)| + 4 \sup_{i \in \mathbb{Z}} |g(i) - h(i)| + \sup_{i \in \mathbb{Z}} |(\Delta h)\rangle_i| \\
\leq 4 \cdot \frac{2 \max\{|a|, |b|\} + \sup_{i \in \mathbb{Z}} |\langle \Delta h \rangle_i|}{-\lambda \zeta''(0) - 4} + 4 \cdot \frac{\max\{|a|, |b|\}}{2} + \sup_{i \in \mathbb{Z}} |(\Delta h)\rangle_i| \\
= -\lambda \zeta''(0) \cdot \frac{2 \max\{|a|, |b|\} + \sup_{i \in \mathbb{Z}} |\langle \Delta h \rangle_i|}{-\lambda \zeta''(0) - 4},
\]
we get
\[ |\Phi_\lambda(u)_i - g(i)| = \left| \frac{1}{\lambda \zeta''(0)} \langle \Delta u \rangle_i \right| \leq \frac{2 \max\{|a|, |b|\} + \sup_{i \in \mathbb{Z}} |\langle \Delta h \rangle_i|}{-\lambda \zeta''(0) - 4}, \]
On the existence of solutions for Frenkel-Kontorova models on quasi-crystals

which means that $\Phi_\lambda$ is well-defined. $\Phi_\lambda$ is a contraction, for

$$|\Phi_\lambda(u)_i - \Phi_\lambda(u')_i| = \frac{1}{-\lambda\zeta''(0)}|\Delta u_i - (\Delta u')_i| \leq \frac{4}{-\lambda\zeta''(0)}\sup_{i\in\mathbb{Z}}|u_i - u'_i|.$$  

Let the radius of $\Pi_\lambda$ smaller than $1/4$, we get a restriction on $\lambda$:  

$$\lambda > 4 \cdot 2 \max\{|a|,|b|\} + \sup_{i\in\mathbb{Z}}|\Delta h_i| + 1 = \frac{2\tau + 1 + \sup_{i\in\mathbb{Z}}|\Delta h_i|}{32}.$$  

Hence let

$$\lambda_1 = \max \left\{ \lambda_2, \frac{2\tau + 1 + \sup_{i\in\mathbb{Z}}|\Delta h_i|}{32} \right\},$$

and then for any $\lambda > \lambda_1$, there exists a non-minimal equilibrium configuration of type $h$ with respect to $H_\lambda$. \hfill \Box

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