A Lower Bound on the Crossing Number of Uniform Hypergraphs

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Abstract

In this paper, we consider the embedding of a complete \( d \)-uniform geometric hypergraph with \( n \) vertices in general position in \( \mathbb{R}^d \), where each hyperedge is represented as a \((d - 1)\)-simplex, and a pair of hyperedges is defined to cross if they are vertex-disjoint and contains a common point in the relative interior of the simplices corresponding to them. As a corollary of the Van Kampen-Flores Theorem, it can be seen that such a hypergraph contains \( \Omega\left(\frac{2^d \log d}{\sqrt{d}}\right) \binom{n}{2d} \) crossing pairs of hyperedges. Using Gale Transform and Ham Sandwich Theorem, we improve this lower bound to \( \Omega\left(\frac{2^d \log d}{\sqrt{d}}\right) \binom{n}{2d} \).

Keywords: Geometric Hypergraph; Crossing Simplices; Ham Sandwich theorem; Gale Transform

1 Introduction

Hypergraphs are a natural generalization of graphs. A hypergraph is a pair \((V, E)\) where \( V \) is a finite set and \( E \subseteq 2^V \) is a collection of subsets of \( V \)[3]. The elements of \( E \) are called hyperedges. Given \( n \) points in general position in \( \mathbb{R}^d \), a geometric \((i + 1)\)-uniform hypergraph is defined as a collection of \( i \)-dimensional simplices as hyperedges, induced by some \((i + 1)\)-tuples from the point set [3]. In this paper, we consider \( i = d - 1 \). A complete geometric \( d \)-uniform hypergraph on \( n \) vertices is represented as \( K_n^d \) in this paper. The case \( d = 2, i = 1 \) has been studied in detail in the literature. The crossing number \( Cr_d(H) \) of a geometric hypergraph \( H \) embedded in \( \mathbb{R}^d \), for some \( d \geq 2 \), as the minimum number of pairwise crossing edges in any of its straight-line drawings in \( \mathbb{R}^2 \), such that no three of its vertices lie on the same straight line.

We define the crossing number \( Cr_d(H) \) of a geometric hypergraph \( H \) embedded in \( \mathbb{R}^d \), for some \( d \geq 2 \), as the minimum possible number of pairwise crossings of its hyperedges. A pair of hyperedges overlap if they have a common point in the relative interior of the simplices corresponding to them. It can be easily seen that a pair of 2-simplices in \( \mathbb{R}^3 \) can overlap in two different ways. The first way, as shown in Figure [1] is called a crossing and the second way is called an intersection. Similarly in \( \mathbb{R}^d \), there are various ways in which a pair of hyperedges can overlap. In order to define the crossing number \( Cr_d(H) \) of a geometric hypergraph \( H \) embedded in \( \mathbb{R}^d \), we only count the crossing pair of hyperedges, i.e., a pair of overlapping hyperedges that have no vertices in common.

As defined earlier, \( Cr_2(K_n^2) \) denotes the number of crossing pair of edges in a straight-line drawing of \( K_n^2 \). The best known lower bound on this number is \( Cr_2(K_n^2) > (0.375 + \epsilon)\binom{n}{2} \), where \( \epsilon \approx 10^{-5} \)[2]. It is quite easy to show that the minimum number of pairwise crossing 2-simplices in a complete geometric 3-uniform hypergraph \( K_n^3 \) embedded in \( \mathbb{R}^3 \) is \( \binom{n}{6} \). This follows from the fact that any set of 6 vertices in a general position in \( \mathbb{R}^3 \) contains a pair of crossing hyperedges. (See the Geometric Van Kampen-Flores Theorem below.) For a general dimension \( d \geq 3 \), let us denote by \( c_d \) the minimum

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number of crossing pair of $((d - 1))$-simplices spanned by a set of $2d$ vertices placed in general position in $\mathbb{R}^d$. This implies $Cr_d(K_n^d) \geq c_d \binom{n}{2d}$.

In order to obtain a lower bound on $c_d$, we first use the geometric version of Van Kampen-Flores Theorem \[1, 6\].

Theorem 1. (Geometric Van Kampen-Flores Theorem) For any $k \geq 1$, any set of $2k + 3$ points in $\mathbb{R}^{2k}$ contains two disjoint subsets $A$ and $B$ such that the convex hulls of $A$ and $B$ have a point common in their relative interior, and $|A| = |B| = k + 1$.

If $d$ is even, i.e., $d = 2k$ for some $k$, this Theorem shows the existence of a crossing pair of $\frac{d}{2}$-simplices spanned by any set of $d + 3$ vertices selected out of $2d$ vertices placed in general position in $\mathbb{R}^d$. This crossing pair can be extended to crossing pairs of $(d - 1)$-simplices in $\binom{\frac{d+3}{2}}{2} = \Theta(\frac{2^d}{\sqrt{d}})$ ways. If $d$ is odd, i.e., $d = 2k' - 1$ for some $k'$, we map all these $d + 3$ vertices in $\mathbb{R}^d$ to $d + 3$ vertices in $\mathbb{R}^{d+1}$ by adding a 0 as the last coordinate of all these vertices. We also add one dummy vertex whose first $d$ coordinates are 0 each, and whose last coordinate is non-zero. By the Geometric Van Kampen-Flores Theorem, this set of $2k' + 3$ vertices in $\mathbb{R}^{2k'}$ contains a crossing pair of $\binom{\frac{d+4}{2}}{2}$-simplices. Note that neither of these simplices contains the dummy vertex, as it is the only vertex in the $(d + 1)$-st dimension and therefore can’t be involved in a crossing. Therefore, this crossing pair can be extended to crossing pairs of $(d - 1)$-simplices in $\binom{\frac{d+3}{2}}{2} = \Theta(\frac{2^d}{\sqrt{d}})$ ways.

1.1 Our Contribution

In Section 3, we first show that there are at least 4 crossing pairs of 3-simplices in a given set of 8 points in general position in $\mathbb{R}^4$. This implies that $Cr_4(K_n^4) \geq 4 \binom{n}{4}$. Thereafter, we use a similar idea to prove that $c_4 = \Omega(\frac{2^{4 \log d}}{\sqrt{d}})$. This implies that $Cr_4(K_n^d) = \Omega(\frac{2^{d \log d}}{\sqrt{d}}) \binom{n}{2d}$.

As far as we know, this is the first non-trivial lower bound obtained on $Cr_d(K_n^d)$. It is an exciting open problem to find out whether this lower bound is tight.
2 Techniques used

2.1 Gale Transformation

For a positive integer $m$, consider a set $A$ of $m + d + 1$ points $x_1, x_2, \ldots, x_{m + d + 1}$ in general position in $\mathbb{R}^d$. It is easy to see that the matrix

\[
M(x_1, x_2, \ldots, x_{m + d + 1}) = \begin{pmatrix}
    x_1^1 & x_1^2 & \cdots & x_1^{m + d + 1} \\
    x_2^1 & x_2^2 & \cdots & x_2^{m + d + 1} \\
    \vdots & \vdots & \ddots & \vdots \\
    x_{m + d + 1}^1 & x_{m + d + 1}^2 & \cdots & x_{m + d + 1}^{m + d + 1} \\
    1 & 1 & \cdots & 1
\end{pmatrix}
\]

has $m$ null vectors. Let these null vectors be $(a_1^1, a_1^2, \ldots, a_1^{m + d + 1}), \ldots, (a_m^1, a_m^2, \ldots, a_m^{m + d + 1})$. Consider the $m$-dimensional vectors $(a_1^1, a_1^2, \ldots, a_1^m), (a_2^1, a_2^2, \ldots, a_2^m), \ldots, (a_{m + d + 1}^1, a_{m + d + 1}^2, \ldots, a_{m + d + 1}^m)$. These $m + d + 1$ points (representing $m + d + 1$ vectors) in $\mathbb{R}^m$ are the Gale transform of $x_1, x_2, \ldots, x_{m + d + 1}$ in $\mathbb{R}^d$. We denote the Gale transform of $A$ as $D(A)$. We define a proper linear separation of the point set $D(A)$ to be a partition of $D(A)$ into two subsets $D_1(A)$ and $D_2(A)$ of size $\lceil \frac{m + d + 1}{2} \rceil$ and $\lfloor \frac{m + d + 1}{2} \rfloor$ respectively, by a hyperplane passing through origin. Then, we have the following:

**Lemma 1.** [4] There is a bijection between the crossing pairs of $\lfloor \frac{m + d + 1}{2} \rfloor$ and $\lceil \frac{m + d + 1}{2} \rceil$-simplices in $A$ and the proper linear separations in $D(A)$.

**Lemma 2.** [4] The $m + d + 1$ points in $A$ are in a general position in $\mathbb{R}^d$ if and only if every set of $m$ vectors, corresponding to a set of $m$ points from $D(A)$, spans $\mathbb{R}^m$.

2.2 Ham Sandwich Theorem

**Lemma 3.** [3] Let $C_1, C_2, \ldots, C_d \in \mathbb{R}^d$ be finite point sets. Then, there exists a hyperplane $h$ that simultaneously bisects $C_1, C_2, \ldots, C_d$, i.e., each of open half-spaces defined by $h$ has at most $\lfloor \frac{1}{2} |C_i| \rfloor$ points of $C_i$.

3 A Lower Bound on $Cr_4(K_8^d)$ and $Cr_d(K_{2d}^d)$

**Lemma 4.** There are at least 4 crossing pairs of 3-simplices spanned by a set of 8 points in general position in $\mathbb{R}^4$.

**Proof:** Given a set $A$ of 8 points in general position in $\mathbb{R}^4$, we consider the set $D(A)$ (the Gale transform) in $\mathbb{R}^3$ and obtain a lower bound on the number of proper linear separations of $D(A)$. By Lemma 1 this lower bound implies a lower bound on the number of crossing pairs of 3-simplices spanned by $A$. The Lemma 2 implies that the vectors in $D(A)$ are in a general position, i.e., any set of 3 vectors spans $\mathbb{R}^3$. Therefore, any hyperplane $h$ that passes through the origin would have at most two points from $D(A)$ on it. Any such hyperplane can be rotated so that we can make either of the two points go above or below $h$, while keeping the origin on it and maintaining the partitioning of the remaining points with respect to $h$.

To apply the Ham Sandwich theorem in $\mathbb{R}^3$, we assign the origin to $C_3$ and create 2 disjoint point sets $C_1$ and $C_2$ from the 8 points in $D(A)$. We use the colors $c_1, c_2$ and $c_3$ to identify the sets $C_1, C_2$ and $C_3$ respectively. We use $c_3$ to color the origin. We proceed in the following manner with the colorings of the points in $D(A)$ with $c_1$ and $c_2$, assuming that $D(A) = \{ p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8 \}$.

- We color all the points with $c_1$, and don’t color any of the points with $c_2$. It leads to a proper linear separation of $D(A)$ (through applying the Ham Sandwich theorem and rotating the partitioning hyperplane if required), which we assume to be $\{ \{ p_1, p_2, p_3, p_4 \}, \{ p_5, p_6, p_7, p_8 \} \}$, without any loss of generality.
Lemma 5. A similar argument can be used to prove the following result.

We color the points in first group of points \( \{p_1, p_2, p_3, p_4\} \) with \( c_1 \) and the second group of points \( \{p_5, p_6, p_7, p_8\} \) with \( c_2 \). It gives a new proper linear separation. Without any loss of generality, we can assume it to be \( \{p_1, p_2, p_5, p_6\}, \{p_3, p_4, p_7, p_8\} \). Note that the pairs \( \{p_1, p_2\}, \{p_3, p_4\}, \{p_5, p_6\} \) and \( \{p_7, p_8\} \) have stayed together in both these separations.

We color the points \( p_1 \) and \( p_2 \) with \( c_1 \) and the rest of the points with \( c_2 \). In the resulting proper linear separation, the points \( p_1 \) and \( p_2 \) get separated and hence this proper linear separation is a new one. Here, we have two cases: (i) all of the remaining pairs, i.e., \( \{p_3, p_4\}, \{p_5, p_6\} \) and \( \{p_7, p_8\} \), get separated, (ii) one of these remaining pairs gets separated and the rest two pairs are still together.

In the (ii)-nd case, we just color the two points in one of the unseparated pairs with \( c_1 \) and rest all with \( c_2 \). We get a new proper linear separation.

In the (i)-st case, we can assume without a loss of generality that the proper linear separation is: \( \{p_1, p_3, p_5, p_7\}, \{p_2, p_4, p_6, p_8\} \). Consider the set of points \( \{p_1, p_2, p_3, p_5\} \). In every proper linear separation obtained till now, three out of these four points have always been in the same partition. We color these points with \( c_1 \) and the rest with \( c_2 \). This gives a new proper linear separation. □

A similar argument can be used to prove the following result.

**Lemma 5.** There are \( \Omega\left(\frac{2^d \log d}{\sqrt{d}}\right) \) crossing pairs of \( (d - 1) \)-simplices spanned by a set of \( 2d \) points in general position in \( \mathbb{R}^d \).

**Proof:** Consider any set \( A' \) of \( d + 4 \) points from the given set of \( 2d \) points in general position in \( \mathbb{R}^d \). The Gale transform \( D(A') \) of these \( d + 4 \) points in \( \mathbb{R}^d \) is a set of \( d + 4 \) vectors in \( \mathbb{R}^3 \), such that any set of 3 of these vectors spans \( \mathbb{R}^3 \). Therefore, any hyperplane \( h \) that passes through the origin would have at most two points from \( D(A') \) on it. Any such hyperplane can be rotated so that we can make either of the two points go above or below \( h \), while keeping the origin on it and maintaining the partitioning of the remaining points with respect to \( h \).

We proceed as in Lemma 4 i.e., we assign the origin to \( C_3 \) and create 2 disjoint point sets \( C_1 \) and \( C_2 \) from the \( d + 4 \) points in \( D(A') \). Through applying the Ham Sandwich theorem and rotating the partitioning hyperplane if required, we get \( \Theta(\log d) \) proper linear separations of \( D(A') \). Each of these proper linear separations corresponds to a \( \lfloor \frac{d+1}{2} \rfloor \)-simplex crossing a \( \lfloor \frac{d+2}{2} \rfloor \)-simplex spanned by \( A \). Each of these crossing pairs can be extended to a crossing pair of \( (d - 1) \)-simplices in \( \binom{d-4}{d-1} = \Theta(\frac{2^d}{\sqrt{d}}) \) ways. □

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