We consider systems of particles hopping stochastically on $d$-dimensional lattices with space-dependent probabilities. We map the master equation onto an evolution equation in a Fock space where the dynamics are given by a quantum Hamiltonian (continuous time) or a transfer matrix resp. (discrete time). We show that under certain conditions the time-dependent two-point density correlation function in the $N$-particle steady state can be computed from the probability distribution of a single particle moving in the same environment. Focussing on exclusion models where each lattice site can be occupied by at most one particle we discuss as an example for such a stochastic process a generalized Heisenberg antiferromagnet where the strength of the spin-spin coupling is space-dependent. In discrete time one obtains for one-dimensional systems the diagonal-to-diagonal
transfer matrix of the two-dimensional six-vertex model with space-dependent vertex weights. For a random distribution of the vertex weights one obtains a version of the random barrier model describing diffusion of particles in disordered media. We derive exact expressions for the averaged two-point density correlation functions in the presence of weak, correlated disorder.

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I. INTRODUCTION

Stochastically hopping particles on a lattice represent a simple model for diffusion in various media. Of particular interest in recent years has been the influence of static disorder on the dynamics. The diffusion of a single particle on a lattice with random hopping rates is well understood \[1\] - \[7\]. Clearly, a system of interacting particles does not behave as a mere superposition of single particles. The effect of a hard core interaction on diffusion in ordered system has been intensively studied recently \[8\] - \[15\]. However the dynamics of exclusive particles in a random system poses a much more difficult problem. Some examples have been analyzed numerically \[16\] or analytically in a mean field approximation \[17\]. However, very few exact results are known.

The main aim of this paper is to derive properties of correlation functions for certain diffusive systems of exclusive particles (particles with a hard-core repulsion) in a disordered environment. We consider \(d\)-dimensional models where the diffusion of particles is determined by the presence of barriers of different strength. For these models the problem of calculating time-dependent (sometimes called “time-delayed”) correlation functions in the steady state of the many-particle system will be shown to be reducible to a one-particle problem.

We will use a description by means of stochastic occupation numbers on a lattice, the dynamics of which are given by a master equation in either discrete or continuous time. A very effective way of treating master equations is given by a Fock space method introduced by Doi \[18\], further extended and applied in \[14\] - \[26\]. In this formalism noninteracting particles are represented as bosons while the hard core interaction preventing the occupation of a lattice site by more than one particle gives rise to a fermionic character of the particles.

Besides its usefulness for solving various problems, the mapping of the classical system onto a Fock space opens deeper insight into the relations to certain quantum systems. So we find the diffusion of exclusive particles equivalent to a Heisenberg antiferromagnet. The correlations of occupation numbers correspond to spin-spin correlations in the antiferromagnet.
Such a mapping is particularly interesting for one-dimensional models where the Hamiltonian obtained for many equilibrium and non-equilibrium processes such as diffusion-limited chemical reactions or driven diffusive systems [23,24] turns out to be related to quantum Hamiltonians of integrable models. Thus part of the vast amount of knowledge that has accumulated for these models over the past decade and some of the methods applied for solving them such as the Bethe ansatz [10,23] are useful also for the classical systems from which the Hamiltonian was obtained and leads the way to new predictions.

One obtains quantum Hamiltonians for processes in continuous time, but one may also study stochastic processes in discrete time. In this case the mapping to a Fock space gives rise to a transfer matrix which again, for many interesting problems in one dimension, turns out to be related to the transfer matrix of integrable two-dimensional systems such as vertex models [8,11,12,27]. Here we will utilize this mapping for a study of correlated disorder in a one-dimensional many-particle random barrier model with exclusion. In this model the probability $p_x$ that a particle moves from a lattice site $x$ to its right neighbouring lattice site $x + 1$ is the same as for a jump from $x + 1$ to site $x$. However $p_x$ is space-dependent, i.e., different for each pair of sites $(x, x + 1)$. We define a parallel updating mechanism where these hopping rules are applied first to all pairs of sites $(2x - 1, 2x)$ and then in a next step to all shifted pairs $(2x, 2x + 1)$ as in Ref. [8]. The discrete time dynamics of this model are thus encoded in the diagonal-to-diagonal transfer matrix of a six vertex model with certain randomly chosen vertex weights.

Much is known about the (continuous-time) single-particle random barrier model with uncorrelated disorder, where all hopping rates $p_x$ are independently chosen random numbers with the same distribution $g(p)$. The averaged probability distribution of finding the particle has been studied in detail by many authors [1] - [6]. For distribution functions where the moments of the inverse hopping rates do not exist, a phase transition has been shown to take place [1,2,28]. Only a few results are known about systems with spatially correlated disorder [7,29].

We will study the behaviour of the averaged density two-point correlation function in
the steady state of the many particle system in an environment with correlated disorder. In Section 2 we show under which conditions this correlation function can be expressed in terms of the one-particle density distribution. This discussion is quite general and applies to ordered and disordered models in any dimension. In Section 3 we apply our result to systems where the corresponding quantum Hamiltonian turns out to become a generalized Heisenberg antiferromagnet with spatially dependent strength of the nearest-neighbour spin-spin coupling. We also briefly review some results on the one-dimensional case. In section 4 we present our results on the discrete-time formulation of the process, i.e., the 6-vertex model. We briefly indicate the existence of a new type of phase transition in the homogeneous model with equal and fixed hopping rates $p$ as they approach the value 1. Furthermore we derive exact expressions for the averaged correlation function for weak, correlated disorder and discuss the impact of the correlations on its behaviour as compared to the uncorrelated case. In Section 5 we summarize our results and in the Appendices we present some details of our calculations to Section 2 (in Appendix A) and we repeat and generalize the mapping of Ref. [8] of the diffusion problem to the 6-vertex model (in Appendix B).

II. THE TWO-POINT CORRELATION FUNCTION FOR A MANY-PARTICLE SYSTEM

In this section we will show how the two-point correlation function of a many particle system is related to the probability distribution of a single particle moving in the same environment.

The many particle system is assumed to be described by a set of occupation numbers on a d-dimensional lattice $\mathbf{n} = \{n_j\}$. Its dynamics are given by a master equation of the following form:

$$\partial_t F(\mathbf{n}, t) = H' F(\mathbf{n}, t)$$

(2.1)

where $H'$ is some linear operator acting on the $n_j$. We may also consider a discrete time
dynamics where the time derivative in Eq. (2.1) is replaced by a discrete difference. Instead
of \( H' \) a transfer matrix \( T' \) is then used (see Section 4).

According to Doi’s formalism the master equation is mapped onto an evolution equation
in a Fock space [18] - [21], [24]:

\[
\partial_t \langle F(t) \rangle = H \langle F(t) \rangle
\]

(2.2)
or a similar equation in case of a the discrete time. The solution of Eq. (2.2) can be written
as:

\[
\langle F(t) \rangle = U_t \langle F(0) \rangle \quad \text{with} \quad U_t = e^{Ht}.
\]

(2.3)
(For a discrete time dynamics \( U_t \) after \( t \) time steps is given by the \( t \)-th power
of the transfer matrix \( T \).)

A particular configuration \( n \) corresponds to a state \( \langle n \rangle = \prod_{j=1}^{L} (c_j^\dagger)^{n_j} \langle 0 \rangle \)
where \( \langle 0 \rangle \) is the vacuum containing no particles and \( L \) is the number of available lattice sites. A vector
in the Fock space is decomposed as \( \langle F(t) \rangle = \sum_{\langle n \rangle} F(n,t) \langle n \rangle \) and a scalar product is defined
by \( \langle n | m \rangle = \prod_{j=1}^{L} [\delta_{n_j,m_j} n_j!] \).

For noninteracting classical particles the creation operators \( c_j^\dagger \) acting on site \( i \) and their
adjoint operators \( c_j \) which annihilate particles obey bosonic commutation relations [18]. If
we consider a system of particles with exclusion, i.e., if at most one particle is allowed to be
on each lattice site, the creation and annihilation operators fulfill Pauli type commutation
rules: operators on the same lattice site have fermionic anticommutation relations whereas
operators for different lattice sites commute [21,24]. For simplicity we will refer to exclusive
particles as fermions.

Here we are interested in correlation functions. If the physical quantities \( A(n) \) and \( B(n) \)
are analytical functions of the occupation numbers we find [30]:

\[
\langle A(t)B(0) \rangle = \langle s | AU_t B | F(0) \rangle
\]

(2.4)
with \( \langle s | = \langle 0 | e^{\sum_j c_j^\dagger} \) and \( A, B \) are the corresponding functions of the particle
number operators \( n_j = c_j^\dagger c_j \). Because of the normalization condition
\[ \langle s | H = 0 \rangle \quad (2.5) \]

which has to be valid for any \( H \) describing a stochastic process we may insert \( U_{-t} \) between the first two terms on the r.h.s. of Eq. (2.4) and obtain an expression very similar to the one in ordinary quantum mechanics. Note that averages are linear in the probability distribution here, while they are bilinear in the state in quantum theory. For special problems, however, linearity and bilinearity may coincide (see (2.18)).

Of particular interest are two-point correlation functions of particle numbers in a steady state. In what follows we restrict ourselves to systems with particle conserving dynamics, such as diffusive systems. A two-point correlation function for an \( N \)-particle-system is given by

\[
G_N(x, y; t) = \langle n_x(t)n_y(0) \rangle_{st} \quad (2.6)
\]

where a subscript \( x \) denotes the label of the lattice site corresponding to a position \( x \) in space and \( n_x \) is the particle number in \( x \). Averaging is performed over the stationary \( N \)-particle state.

The connected two-point correlation function is defined by

\[
C_N(x, y; t) = \langle n_x(t)n_y(0) \rangle_{st} - \langle n_x \rangle_{st} \langle n_y \rangle_{st} \quad (2.7)
\]

where the arguments \( t \) and 0 are dropped in the second term, because \( \langle n_x(t) \rangle_{st} \) does not depend on time.

First we consider a system with one particle only. It can be described using a master equation (2.1) or much easier by means of the following probability distribution:

\[
P(x, t; y, 0) = P(x, t|y, 0)P(y, 0) \quad (2.8)
\]

which gives the probability of finding the particle in \( y \) at time 0 and in \( x \) at time \( t \). The first expression on the r.h.s. is the corresponding conditional probability. For the two-point correlation function defined above we obtain

6
\begin{align*}
G_1(x, y; t) &= \sum_{\{n_j\}, \{n'_j\}}^{(1)} n_x n'_y F(n, t; n', 0) \\
&= P(x, t|y, 0) P_{st}(y)
\end{align*}

(2.9)

where \( F(n, t; n', 0) \) is the combined probability of configuration \( n \) at time \( t \) and \( n' \) at time 0. (A superscript \((k)\) at the sum means that the sum runs over states with a total particle number \( k \) only). Assuming furthermore the steady state to be homogeneous we get:

\[
G_1(x, y; t) = \frac{1}{L} P(x, t|y, 0) .
\]

(2.10)

Our aim is to express \( C_N \) in terms of \( P(x, t|y, 0) \). First we will find relations between \( G_N \) and \( G_1 \), one valid in the fermionic case (with particle exclusion) and another one for bosons. We will start with the fermionic case. The calculations for bosons are analogous and are not shown here for sake of brevity.

**A. The fermionic case**

It is easy to check that in the fermionic case the operators

\[
S^+ = \sum_{j=1}^{L} c_j , \quad S^- = \sum_{j=1}^{L} c_j^\dagger , \quad S^z = \sum_{j=1}^{L} \left(-n_j + \frac{1}{2}\right)
\]

satisfy the commutation relations of \( SU(2) \).

In what follows we will study systems the dynamics of which are given by an \( SU(2) \)-symmetric Hamiltonian \( H \), i.e.,

\[
[H, S^\pm]_- = 0
\]

(2.12)

\[
[H, S^z]_- = 0 .
\]

(2.13)

(In the discrete time case \( H \) has to be replaced by the transfer matrix \( T \).)

As a consequence of this symmetry some interesting properties result:

(i) The number of particles in the system is a conserved quantity because of Eq. (2.13).

(ii) The Hamiltonian is hermitian.

Furthermore, because of the normalization condition (2.5) and Eq. (2.12) we find:
\[ H \langle N \rangle = 0 \quad \text{with} \quad \langle N \rangle = \frac{1}{N!} (S^-)^N \langle 0 \rangle \] 
(2.14)

and \[ \langle N | H = 0 \quad \text{with} \quad \langle N | = \frac{1}{N!} \langle 0 | (S^+)^N \].
(2.15)

Hence, the normalized N-particle state \((N < L)\)
\[ |N\rangle_{\text{norm}} = \left( \begin{array}{c} L \\ N \end{array} \right)^{-1} \langle N \rangle = \left( \begin{array}{c} L \\ N \end{array} \right)^{-1} \sum_{\{n_j\}}^{(N)} |n\rangle . \]
(2.16)
is a stationary solution of the problem assigning the same probability to any possible configuration of the \(n_j\) which has a total particle number \(N\). (The summation in (2.16) runs over states with \(n_j = 0, 1\) and \(\sum_{j=1}^{L} n_j = N\).) The averaged occupation numbers in state \(|N\rangle_{\text{norm}}\) are \(<n_x> = N/L\).

From Eq. (2.6) and (2.4) we find for the correlation function:
\[ G_{\text{ferm}}^N(x, y; t) = \langle s | n_x U_t n_y | N \rangle_{\text{norm}} \]
(2.17)
where the steady states (2.16) are used. Because of \(\langle s | = \sum_{N=0}^{L} \langle N |\), particle conservation and the orthogonality of the \(|N\rangle\) we get:
\[ \left( \begin{array}{c} L \\ N \end{array} \right) G_{\text{ferm}}^N(x, y; t) = \langle N | n_x U_t n_y | N \rangle \]
(2.18)
Note that expression (2.18) can be understood as an imaginary time scattering matrix of a quantum mechanical system since the Hamiltonian is hermitian.

Using the symmetry (2.12) the following recursion relation can be derived from Eq. (2.18) (see Appendix A, Eq. (A.8)):
\[ G_{\text{ferm}}^{N+1}(x, y; t) = \frac{L - N - 1}{L - N} G_{\text{ferm}}^N(x, y; t) + \frac{1}{L} \frac{N + 1}{L - N} . \]
(2.19)
From this an exact expression for \(G_N\) in terms of \(G_1\) can be derived, which reduces in the thermodynamic limit \(L, N \to \infty, \rho = N/L = \text{const.}\) to
\[ G_{\text{ferm}}^N(x, y; t) = N(1 - \rho) G_{\text{ferm}}^1(x, y; t) + \rho^2 \]
(2.20)
where \(\rho\) is the mean density of the N-particle system. Inserting Eq. (2.10) into Eq. (2.20) and using the definition (2.7) of the connected two-point correlation function we find:
\[
C_N^{\text{ferm}}(x, y; t) = \rho(1 - \rho)P(x, t|y, 0) .
\] (2.21)

Eq. (2.21) is symmetric with respect to \( \rho \leftrightarrow (1 - \rho) \). This reflects the particle-hole symmetry in a system where occupation numbers are restricted to 0 and 1 only. The amplitude of the correlation function has its maximum at \( \rho = 1/2 \).

Because the properties of a one-particle system are well known for many environments, Eq. (2.21) provides a useful tool to study many-particle systems in any dimension provided the \( SU(2) \)-symmetry holds. This will be demonstrated in the next section.

**B. The bosonic case**

Again we assume \( H \) to satisfy Eqs. (2.12) and (2.13) of the previous subsection. Note, however, that in the bosonic case the operators \( S^\pm, z \) (2.11) do not form a \( SU(2) \) algebra, instead one has a harmonic oscillator algebra with \( [S^+, S^-] = L \) where \( L \) is the number of sites in the system.

As in the fermionic case the total particle number is conserved, \( H = H^\dagger \) is hermitian and Eqs. (2.14) and (2.15) result from the symmetry (2.12). Only the normalization constants and the explicit form of the steady states are different:

\[
|N\rangle_{\text{norm}} = \frac{N!}{L^N} |N\rangle = \frac{N!}{L^N} \sum_{\{n_j\}}^{(N)} \left( \prod_{j=1}^{L} \frac{1}{n_j!} \right) |n\rangle .
\] (2.22)

These states correspond to homogeneous probability distributions as for fermions. If the system is in the state \( |N\rangle_{\text{norm}} \) any configuration with particle number \( N \) has a probability proportional to \( \prod_{j=1}^{L} \frac{1}{n_j!} \).

In a way analogous to that in section 2.1 we find a recursion relation

\[
G_{N+1}^{\text{boson}}(x, y; t) = \frac{N + 1}{N} G_{N}^{\text{boson}}(x, y; t) + \frac{N + 1}{L^2}
\] (2.23)

from which follows

\[
G_N^{\text{boson}}(x, y; t) = N G_1^{\text{boson}}(x, y; t) + \rho^2
\] (2.24)
in the thermodynamic limit. Instead of Eq. (2.21) we obtain for the connected two-point correlation function for bosons:

\[ C_N^{\text{boson}}(x, y, z) = \rho P(x, t \mid y, 0) \, . \]  

(2.25)

The probability on the r.h.s. is the same as in the fermionic case because there is no distinction between fermion and boson for a single particle system. The connected two-point correlation function differs from that in the fermionic case (2.21) in the factor \((1 - \rho)\) occurring in the latter one. This gives a quantitative description of the effect caused by the particle exclusion for the class of systems considered here. Of course, there is no particle-hole symmetry for the bosonic system.

III. MANY-PARTICLE DIFFUSION IN A RANDOM BARRIER ENVIRONMENT

Applying the results of the last section we will study many-particle diffusion in random environments which are constant in time. The one-dimensional system of interest is assumed to be described by a fermionic Hamiltonian (similar to [14,21]):

\[ H = \sum_j \{ p_j [c_{j+1}^\dagger c_j - c_{j+1} c_j^\dagger] + p_{j-1} [c_{j-1}^\dagger c_j - c_{j-1} c_j^\dagger] \} \]

(3.1)

\[ = \frac{1}{2} \sum_j p_j \sigma_j^+ \sigma_{j+1}^- - 1 \, \text{ with } \sigma_j = \{ c_j^\dagger + c_j, -i(c_j^\dagger - c_j), 1 - 2n_j \} \, . \]  

(3.2)

The components of \(\sigma_j^\dagger\) are the Pauli matrices. We see from Eq. (3.2) that \(H\) is the Hamiltonian of a generalized Heisenberg antiferromagnet with space-dependent spin-spin coupling. Hence the correlation functions calculated below have a physical meaning for this quantum system (see section 2.1).

In what follows we will concentrate on the classical system described by the evolution equation (2.2) with the Hamiltonian (3.1). The effect of particle exclusion is reflected in the terms \(c_{j+1} c_j^\dagger\) and in the Pauli commutation relations between the operators. The Hamiltonian of nonexclusive (bosonic) particles moving in the same environment is obtained from (3.1) by omitting the aforementioned factors.
The Hamiltonian (3.1) generalizes the problems investigated in [8,14] since it allows the hopping rates $p_j$ to be space dependent and stochastic. The general form (3.1) includes classical diffusion, diffusion in structured environments and in certain disordered media as e.g. in the random barrier medium. Other problems such as the random trap model are not described by (3.1) and cannot be treated in the simple way proposed here.

The Hamiltonian (3.1) is $SU(2)$ symmetric. (The corresponding bosonic Hamiltonian obeys the condition from section 2.2). Consequently, the relations (2.21) and (2.25) between the correlator in the $N$-particle sector and the probability distribution in the one-particle sector are valid in the thermodynamic limit. For a random environment (2.21) and (2.25) are valid for each realization of the environment and $\rho = N/L$ is the same for each realization. Hence, we can average the linear equations (2.21) and (2.25) over the environment (denoted by $X$). The result is:

$$C_{N}^{\text{ferm}}(x, y; t) = \rho(1 - \rho)P(x, t|y, 0)$$

$$C_{N}^{\text{boson}}(x, y; t) = \rho P(x, t|y, 0).$$

Eqs. (3.3) and (3.4) give the averaged two-point correlation functions for the many particle system provided $P(x, t|y, 0)$ is known. The latter quantity is the averaged solution of:

$$\partial_t P(x, t|y, 0) = p_x [P(x + 1, t|y, 0) - P(x, t|y, 0)] + p_{x-1} [P(x - 1, t|y, 0) - P(x, t|y, 0)].$$

Eq. (3.5) is obviously the master equation describing the motion of a single particle in the environment defined by Eq. (3.1). It can be formally derived as a correlation function (2.6) in the one particle sector (analogous to the procedure in section 4).

We briefly review some known results. The most simple case is that of classical diffusion, i.e., with deterministic $p_x = D \forall x$, for which $P(x, t|y, 0) = P(x, t|y, 0)$ is well-known from textbooks [31]. For $x, t \gg 1$ it is given by:

$$P(x, t|y, 0) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

(3.6)
and its Fourier-Laplace transform $\tilde{S}(k, \omega)$ is

$$\tilde{S}(k, \omega) = \frac{1}{\omega + Dk^2} \quad (3.7)$$

Note that $k^2$ in the denominator derives from the expansion of the Fourier transform $2(1 - \cos k)$ of the lattice Laplacian to lowest order in $k$. Keeping also higher terms leads to an effective frequency dependent diffusion constant $D(k) = (1 - k^2/12)D$ (see below). Eq. (3.6) combined with (3.3) and (3.4) gives the two-point correlation function for the many particle system in a deterministic, homogeneous environment of arbitrary dimension. The result for the fermionic case in one dimension is known from [8,23].

Next we consider a random barrier environment, where the hopping rates $p_x$ have the same probability distribution $g(p)$ for all sites $x$ and are not correlated. We denote the mean value of $p$ by $\overline{p}$ and its variance $(\overline{p} - p)^2$ by $\sigma^2$. (We shall neglect higher moments and powers of $\sigma^2$ in all calculations throughout this paper.) The mean value $\overline{p} = D$ is simply the diffusion constant of an ordered system with $g(p) = \delta(p - D)$.

For the single-particle problem in the case of weak, uncorrelated disorder, i.e., if all moments $\beta_m = \overline{p^{-m}}$ exist and the $p_x$ are independent random variables, the short wavelength - low frequency behaviour of the Fourier-Laplace transform $\tilde{S}(k, \omega)$ of $P(x, t|y, 0)$ is known [1] - [6]. One introduces a generalized diffusion constant $D(k, \omega)$ by writing

$$\tilde{S}(k, \omega) = \frac{1}{\omega + D(k, \omega)k^2} \quad (3.8)$$

and finds [3,4]

$$D(k, \omega) = D_0 + D_1 \sqrt{\omega} + D_2 \omega + \ldots$$

$$-k^2(E_0 + E_1 \sqrt{\omega} + \ldots) + \ldots \quad (3.9)$$

Here $D_j$ and $E_j$ are functions of the moments $\mu_l = \overline{(p^{l-1} - \overline{p}^{l-1})^l}$ of the inverse hopping rates. Expanding them around $D = \overline{p}$ and neglecting higher powers of $\sigma^2$ as well as higher moments $\sigma_l = \overline{(p - \overline{p})^l}$ one obtains
\[ D_0 = \overline{p^{-1}} \approx D - \sigma^2 D^{-1} \quad (3.10) \]
\[ D_1 = \frac{1}{2} D_0^{5/2} \mu_2 \approx \frac{1}{2} \sigma^2 D^{-3/2} . \quad (3.11) \]

Within the framework of our approximation \( D_2 \) is 0 and \( E_0 = D_0/12, E_1 = D_1/12 \) and we can write
\[ D(k, \omega) = D \left( 1 - \frac{k^2}{12} \right) \left( 1 - \frac{\sigma^2}{D^2} (1 - \frac{1}{2} \sqrt{D \omega}) \right) . \quad (3.12) \]

The prefactor \( 1 - k^2/12 \) has its origin as above (see Eq. (3.7)) in the Fourier transform of the lattice Laplacian and is present also in the ordered system. Up to order \( \sigma^2 \) the uncorrelated disorder affects only the time dependence of the correlation function. For \( x, t \to \infty \) and \( x^2/t \) fixed the averaged probability distribution \( \overline{P(x, t|y, 0)} \) has the same form as (3.6) with a diffusion constant \( D_0^{-1} = \overline{p^{-1}} \).

In the case of strong disorder (if some of the moments \( \beta_m \) do not exist) \( \overline{P(x, t|y, 0)} \) cannot be approximated by the classical diffusion distribution [1,2] and a phase transition occurs. This becomes apparent in the return probability \( \overline{P(x, t|x, 0)} \) [2,3] which is related via Eqs. (3.3) and (3.4) to the autocorrelation function of the many-particle system. Its large time behaviour is given by \( \overline{P(x, t|x, 0)} \propto t^{-\alpha} \) with \( \alpha = 1/2 \) for weak disorder and \( \alpha < 1/2 \) for strong disorder [1,2]. For more results on the random barrier model s. [1] - [4]. A similar approach can be used for studying many-particle diffusion in hierarchically structured media. Results for the single particle problem are given in [28,33,34].

Our discussion in Sec. 2 shows that all results derived for the probability distribution of the single particle process are also valid for the two-point correlation functions for many-particle systems given by (3.3) and (3.4). In the next section we are going to derive some new results on the effect of disorder correlations.

**IV. CORRELATION FUNCTIONS IN A RANDOM 6-VERTEX MODEL**

In this section we restrict ourselves to one space dimension only and consider discrete-time dynamics. It was shown by Kandel et al. that the diagonal-to-diagonal transfer matrix
of the 6-vertex model for a certain one parameter family of vertex weights describes diffusion of particles with exclusion in one dimension \[8\]. In this mapping the time evolution proceeds along one diagonal of a square lattice while the space extends along the diagonal perpendicular to the time-diagonal (see Fig.1). They consider the standard case where the vertex weights do not depend on the position of the vertex in the two-dimensional lattice. As in the preceding section we want to study space-dependent hopping rates. In the framework of the mapping to the vertex model on a square lattice this is achieved by introducing vertex weights which are constant on the time-diagonal of the lattice but vary along the space-diagonal. For the convenience of the reader we repeat this mapping in Appendix B and generalize it to arbitrary, space-dependent vertex weights.

The dynamics of the exclusion process which leads to the generalized transfer matrix of the 6-vertex model are defined as follows: We present the state of the system with \(L\) sites (\(L\) even) at time \(t\) by the quantity \(\mathbf{n}(t) = \{n_1(t), n_2(t), \ldots, n_L(t)\}\) where \(n_x(t)\) counts the number of particles on site \(x\) and can take the values 0 or 1. The time evolution consists of two steps. Suppose the system is in the state \(\mathbf{n}(t)\) with \(t\) an integer. In the first half-time step \(t \to t + 1/2\) we divide the chain of \(L\) sites into pairs of sites \((1, 2), (3, 4), \ldots, (L-1, L)\). If both sites in a pair \((2x-1, 2x)\) are occupied or empty then they remain so with probability 1. (We exclude the possibility of particle creation or annihilation.) If there is one particle and one hole then the particle hops to the unoccupied site in the pair with probability \(p_{2x}\) and remains where it was with probability \(1 - p_{2x}\). Note that the hopping probability in such a pair is the same for both directions, i.e., it does not depend on whether the particle is on site \(2x-1\) or on site \(2x\). These hopping rules are applied in parallel to all pairs in the chain. In the second half-time step \(t + 1/2 \to t + 1\) we shift the pairing of the chain by one lattice unit such that the pairs are now \((2, 3), \ldots, (L, 1)\) (we assume periodic boundary conditions). We apply the same rules as above, but the hopping probabilities in a pair \((2x, 2x+1)\) are now \(p_{2x}\). From these rules one can derive a master equation for the probability distribution \(F(n, t)\) \((2.1)\).

Instead of working with a master equation, we directly study the transfer matrix \(T\) which
encodes these hopping rules as discussed in section 2. We choose as a basis of the Fock space, the same basis as in section 3 where the presence of a particle corresponds to spin down and the absence of a particle corresponds to spin up. Then the transfer matrix is given by

\[ T = T^{\text{even}} T^{\text{odd}} = \prod_{j=1}^{L/2} T_{2j}(p_{2j}) \prod_{j=1}^{L/2} T_{2j-1}(p_{2j-1}) \]  

(4.1)

where

\[ T_j(p_j) = 1 - \frac{p_j}{2}(\sigma_j \sigma_{j+1}) \]  

(4.2)

with the Pauli matrices \( \sigma^x, \sigma^y, \sigma^z \). The time evolution operator \( T^t \) for \( t \) time steps is defined as the \( k \)th power of \( T \) if \( t = k \) is integer and as \( T^{\text{odd}} T^k \) if \( t = k + 1/2 \). We use periodic boundary condition and occasionally label the spatial indices from \(-L/2\) to \( L/2 - 1\). (Because of the periodic boundary condition they are defined mod \( L \).) The local transfer matrices \( T_j(p_j) \) act as unit operator on all sites except on the pair \((i, i+1)\). Each \( T_j \) commutes with the generators (2.11) of \( SU(2) \), so \( T \) is symmetric under the action of \( SU(2) \) and the time-dependent connected two-point correlation function in the steady state

\[ C_N(x, y; t) = \langle s | n_x T^t n_y | N \rangle - \rho^2 \]  

(4.3)

in the sector with \( N = \rho L \) particles is given by the correlation function \( G_1(x, y; t) \) in the same environment in the one particle sector as in (2.21). This is true for any choice of the \( p_j \) and we can restrict our discussion to the one particle sector. We shall omit the index 1 in the correlator and simply write \( G(x, y; t) \). In order to avoid boundary effects we shall furthermore work in the thermodynamic limit \( L \rightarrow \infty \). We denote the lattice constants in space and time direction by \( a_r \) and \( a_t \) respectively. On the square lattice one has \( a_r = a_t \), but we shall discuss later the case \( a_r \neq a_t \). The integer numbers \( r = R/a_r \) and \( t = \tau/(2a_t) \) measure the distance \( R \) in space and \( \tau \) in time direction resp. in units of the respective lattice constants. \( a_r \) and \( a_t \). Note that one full time step in the time evolution corresponds to \( \tau = 2a_t \).

We denote by \( | x \rangle \) the state with the particle being on site \( x \) and \( \langle x | \) is its transposed. The scalar product on this space is given by \( \langle x | y \rangle = \delta_{x,y} (\delta_{x,y} \) is the Kronecker symbol) and
the unit operator is $1 = \sum_x | x \rangle \langle x |$. The transfer matrix $A$ restricted to the one-particle sector reads

$$A = A^\text{even} A^\text{odd} = \sum_{j=1}^{L/2} A_{2j}(p_{2j}) \sum_{j=1}^{L/2} A_{2j-1}(p_{2j-1})$$

(4.4)

with local transfer matrices

$$A_j(p_j) = | j \rangle \langle j | + | j + 1 \rangle \langle j + 1 | - p_j(| j \rangle - | j + 1 \rangle)(| j \rangle - \langle j + 1 |) .$$

(4.5)

The steady state with eigenvalue 1 of $A$ is the vector $\langle a | = L^{-1/2} \sum_x | x \rangle$ and $| a \rangle$, which is $\langle s |$ restricted to the 1-particle sector, is its transpose. The particle number operator $n_x$ is simply given by $n_x = | x \rangle \langle x |$ and the correlation function (4.3) is the matrix element $\langle x | A^t | y \rangle = (A^t)_{x,y}$ of $A^t$. This power is defined in the same way as $T^t$:

$$A^t = \begin{cases} A^k & \text{if } t = k \\ A^\text{odd} A^k & \text{if } t = k + 1/2 \end{cases} .$$

(4.6)

With these definitions one can immediately derive a recursion relation for $G(x, y; t)$ w.r.t. $x$ and $t$. We have for integer values of $t = k$

$$G(x, y; t + 1/2) = \langle x | A^\text{odd} A^t | y \rangle .$$

(4.7)

Inserting Eq. (4.3) into this expression gives

$$G(x, y; t + 1/2) = \begin{cases} (1 - p_{x-1})G(x, y; t) + p_{x-1}G(x - 1, y; t) & x \text{ even} \\ (1 - p_x)G(x, y; t) + p_xG(x + 1, y; t) & x \text{ odd} . \end{cases}$$

(4.8)

In the same way one obtains

$$G(x, y; t + 1) = \begin{cases} (1 - p_x)G(x, y; t + 1/2) + p_xG(x + 1, y; t + 1/2) & x \text{ even} \\ (1 - p_{x-1})G(x, y; t + 1/2) + p_{x-1}G(x - 1, y; t + 1/2) & x \text{ odd} . \end{cases}$$

(4.9)

From these recursion relations which are the analogue of the recursion relation (3.5) together with the initial condition
\[ G(x, y; 0) = \delta_{x,y} \] (4.10)

one can compute the exact correlation function for arbitrary values of the local hopping probabilities \( p_x \).

The simplest non-trivial case is the homogeneous model \( p_x = \text{const.} = p \). In this case the system is invariant under translations by two lattice units and \( G(x, y; t) = G(x+2z, y+2z; t) \) depends only on whether \( y \) is even or odd and on the distance \( r = x - y \). For the special choice \( p = 1/2 \) the correlation function was computed (in a different way) by Kandel et al. [8] (see Appendix B).

We found that for arbitrary values of \( p \) the solution to the recursion relations (4.8)-(4.9) with initial condition (4.10) for \( x \) even and \( t \) integer is given by

\[
G(x, y; t) = p^{2t} \delta_{-r,2t} + \sum_{k=1}^{t-|r|/2} \binom{t-r/2}{k} \binom{t-1+r/2}{k-1} p^{2t-2k}(1-p)^{2k} \quad y \text{ even} \quad (4.11)
\]

\[
G(x, y; t) = \sum_{k=0}^{t-(r-1)/2} \binom{t-(r-1)/2}{k} \binom{t+(r-1)/2}{k} p^{2t-2k-1}(1-p)^{2k+1} \quad y \text{ odd} \quad (4.12)
\]

For \( x \) odd and \( t \) integer one finds

\[
G(x, y; t) = G(y, x; t) \quad y \text{ even} \quad (4.13)
\]

\[
G(x, y; t) = G(y + 1, x + 1; t) \quad y \text{ odd} \quad (4.14)
\]

The correlator for half-odd integer values of \( t \) is given by relations (4.8). For finite densities \( \rho = N/L \) one obtains the exact connected correlation function (in the thermodynamic limit) by multiplying the expressions (4.11) with \( \rho(1-\rho) \). Note that for \( p = 1/2 \) these expressions simplify considerably and we recover the result of Ref. [8] (see Eqs. (B.3) - (B.10) in Appendix B).

For large \( t \) and \( r \) (such that \( r^2/t \) remains finite) or, equivalently, for \( a_t = a_r^2 \to 0 \), one finds from (4.11)

\[
C_N(r, t) = \rho(1-\rho)(4\pi Dt)^{-1/2}e^{-r^2/4Dt} \quad (4.15)
\]

with the diffusion constant \( D = p/(1-p) \).
Replacing \( p \to pb \) and taking the limit \( b \to 0 \), one recovers the Heisenberg Hamiltonian (3.2) as \( H = \lim_{b \to 0} b^{-1}(T - 1) \). Similarly, taking the limit \( t \to \infty \) in (4.11) such that \( bt \) remains fixed leads to the correlation function (3.6) in the continuous time formulation with \( D = p \). On the other hand, for \( p = 1 \) the correlation function reduces to the \( \delta \)-function \( \delta_{r,2t} \). Thus in this case the correlation function is invariant under the scale transformations \( r \to \lambda r, t \to \lambda t \) corresponding to a dynamical exponent \( z = 1 \). The crossover as \( p \) approaches 1 (i.e. \( D \to \infty \)) will be discussed in a separate publication.

Now we study the average behaviour of the correlation function in a random environment. We assume all \( p_x \) to be distributed in the interval \( 0 \leq p_x \leq 1 \) with the same distribution function \( g(p_x) \) and introduce the quantity

\[
\Delta_x = p_x - \overline{p}
\]  

(4.16)

where \( \overline{p} \) denotes the mean value of the distribution \( g \). For the sake of technical simplicity we choose as mean value \( \overline{p} = 1/2 \) corresponding the diffusion constant \( D = 1 \). As long as \( \overline{p} \) is not close to 1, such a choice has no qualitative influence on the averaged correlation function. Furthermore we shall assume that the hopping probabilities of even and odd lattice sites are uncorrelated,

\[
\Delta_{2x-1}\Delta_{2y} = 0
\]  

(4.17)

while the correlations

\[
\overline{\Delta_{2x}\Delta_{2y}} = \overline{\Delta_{2x-1}\Delta_{2y-1}} = \sum_\nu h(2\nu)\delta_{r,2\nu}
\]  

(4.18)

depend only the absolute value of the distance \( r = |2y - 2x| \). The quantity \( h(0) = \sigma^2 \) appearing in the sum in the r.h.s. of Eq. (4.18) is the variance of \( \Delta_x \). Finally, we consider only distributions which are sharply centered around their mean value such that higher moments like \( \overline{\Delta_{2x}\Delta_{2y}\Delta_{2z}} \) etc. can be neglected in a perturbative expansion of the averaged correlation function in its moments.
With the definition (4.16) of the quantities $\Delta$, we write $A^{\text{odd}} = A_0^{\text{odd}} + \Delta^{\text{odd}}$ and analogously $A^{\text{even}} = A_0^{\text{even}} + \Delta^{\text{even}}$ where $A_0^{\text{odd(even)}}$ are the transfer matrices with all $p_x = 1/2$. The averaged correlation function is averaged matrix element

$$
\overline{G(x, y; t)} = (A^t)_{x,y} = \langle x | (A_0 + A_0^{\text{even} \Delta^{\text{odd}}} + \Delta^{\text{even}} A_0^{\text{odd}} + \Delta^{\text{even}} \Delta^{\text{odd}})^t | y \rangle .
$$

(4.19)

Neglecting all pieces with more than two Delta matrices in this expression and using $\Delta^{\text{even}} \Delta^{\text{odd}} = 0$ we obtain the lowest order correction to the correlation function in the presence of disorder

$$
\Gamma(x, y; t) = \overline{G(x, y; t)} - G^{(0)}(x, y; t)
$$

$$
= \sum_{l=0}^{n} \sum_{m=0}^{n-l} \langle x | A_0^{l} (A_0^{\text{even} \Delta^{\text{odd}}} A_0^{m} (A_0^{\text{even} \Delta^{\text{odd}}}))^{n-l-m} | y \rangle + \sum_{l=0}^{n} \sum_{m=0}^{n-l} \langle x | A_0^{l} (A_0^{\text{even} \Delta^{\text{odd}}} A_0^{m} (A_0^{\text{even} \Delta^{\text{odd}}}))^{n-l-m} | y \rangle .
$$

(4.20)

Here $G^{(0)}(x, y; t)$ denotes the correlator of the ordered system with $p = 1/2$ and $n = t - 2$. Some calculation shows that $\Gamma(x, y; t)$ can be expressed in terms of a sum of three-point correlation functions of the ordered system:

$$
\Gamma(x, y; t) = 16 \sum_{\nu} h(2\nu) \sum_{m=0}^{n} \left( (n - m) D_{\nu}(x, y; m) + \hat{D}_{\nu}(x, y; m) \right) D_{\nu}(0, 0; m + \frac{1}{2})
$$

(4.21)

with

$$
D_{\nu}(x, y; k) = \langle x | \left( A_0^k - A_0^{k+1} \right) | y + 2\nu \rangle
$$

(4.22)

$$
\hat{D}_{\nu}(x, y; k) = \langle x | \left( A_0^{k+1/2} - A_0^{k+1} \right) | y + 2\nu \rangle .
$$

(4.23)

Note that through the averaging, translational invariance is restored. Multiplying Eq. (4.21) by $\rho(1 - \rho)$ yields the lowest order correction to the averaged two-point correlation function.

It is easy to compute the autocorrelation function $\overline{G(0, 0; t)}$ if $h(2\nu) = \sigma^2 \delta_{\nu,0}$, i.e., in the absence of disorder correlations. From Eqs. (4.21), (4.22) and (4.23) one obtains

$$
\overline{G(0, 0; t)} = \left( 1 + 4\sigma^2 \frac{2t}{2t-1} \right) G^{(0)}(0, 0; t) .
$$

(4.24)
For large times this has the expected form (3.9), (3.10) for $D = 1$ and variance $4\sigma^2$: 
\[
G(0,0;t) \sim (4\pi D_0 t)^{-1/2} \text{ with } D_0 = 1 - 4\sigma^2.
\]

The behaviour of the correlation function for $r \neq 0$ and in the presence of disorder correlations becomes more transparent after a Fourier-Laplace transformation. The discrete Fourier-Laplace transform (see Appendix B) of $G(0)(x, y; t)$ is given by
\[
\tilde{S}(k, \omega) = \frac{1}{1 - e^{-\omega}\cos^2 k}
\]
while Fourier-Laplace transformation of $\Gamma(x, y; t)$ yields
\[
\tilde{\Sigma}(k, \omega) = \frac{4\sin^2 k(1 - e^{-\omega})}{1 - e^{-\omega}\cos^2 k} \left(\sigma^2(1 - \frac{1}{2}\sqrt{1 - e^{-\omega}}) - L(k, \omega)\right) \tilde{S}(k, \omega) .
\]

The function
\[
L(k, \omega) = \sqrt{1 - e^{-\omega}} \sum_{\nu=1}^{\infty} h(2\nu) \cos(2ik\nu) \left(\frac{e^{-\omega/2}}{1 + \sqrt{1 - e^{-\omega}}}\right)^{|2\nu|}
\]
on the r.h.s. of Eq. (4.26) is the contribution of the disorder correlation function $h(2\nu)$ to the generalized diffusion constant. Eqs. (4.21) and (4.26) are the main results of this section.

We would like to stress that up to this point all results are exact first order contributions, i.e., valid for arbitrary integer values of $r$ and $t \geq 0$ and arbitrary values of $k$ and $\omega \geq 0$.

Now we focus on the large distance behaviour and expand $\tilde{\Sigma}(k, \omega)$ up to order $\sqrt{\omega}$. For uncorrelated disorder, $L(k, \omega) = 0$, we recover the result (3.12) of Refs. [5,4,6] by expanding $\cos k$ and $\exp (-\omega)$ to first order in their respective arguments. In this case the generalized diffusion constant does not depend on the frequency. This changes for correlated disorder.

Assuming a decay of the form $h(2\nu) = \sigma^2 \exp(-|2\nu|/\xi)$ one obtains for the correction
\[
L(k, \omega) \approx \sigma^2 \sqrt{\omega} \frac{e^{-2(\sqrt{\omega} + \xi^{-1})} - \cos 2k}{\cosh(2\sqrt{\omega} + 2\xi^{-1}) - \cos 2k}.
\]

For $1 \ll \xi \ll \omega^{-1/2}$ this expression becomes
\[
L(k, \omega) \approx \sigma^2 \sqrt{\omega} \frac{\xi^{-1}}{\xi^{-2} + k^2} \to \sigma^2 \sqrt{\omega} \xi .
\]

This piece is independent of $k$ for low frequencies $k \propto \sqrt{\omega}$ in our expansion which considers only contributions of order $\sqrt{\omega}$.
\( \omega^{-1/2} \) plays the role of a crossover length scale where the correlation function changes its behaviour. For \( 1 \ll \omega^{-1/2} \ll \xi \) one obtains a \( k \)-dependent contribution in order \( \sqrt{\omega} \):

\[
L(k, \omega) \approx \sigma^2 \sqrt{\omega} \sqrt{\omega + k^2} = \sigma^2 \frac{\omega}{\omega + k^2} . \tag{4.30}
\]

For a decay of the disorder correlations of the form \( h(2\nu) \sim \sigma^2 |2\nu|^{-\alpha} \exp (-|2\nu|/\xi) \) the contribution of \( L(k, \omega) \) to \( D(k, \omega) \) is small for \( \alpha > 1 \), i.e., smaller than of order \( \sqrt{\omega} \). In this situation the contribution of the disorder correlations can be neglected and the system behaves as it was uncorrelated.

For \( 0 \leq \alpha < 1 \) and \( \xi \gtrsim \sqrt{\omega} \) the disorder contribution is larger than of order \( \sqrt{\omega} \) giving rise to a qualitative change in the frequency dependence of the diffusion constant and leading also to a \( k \)-dependence. As opposed to \( \alpha = 0 \) with infinite correlation length the contribution still vanishes as \( \omega, k^2 \to 0 \).

For \( h(2\nu) = \sigma^2 |2\nu|^{-1} \) (for \( \nu \geq 1 \)) one finds

\[
L(k, \omega) = \frac{1}{2} \sqrt{\omega} \ln \left( 1 + e^{-2(\omega+\xi^{-1})} - 2e^{-(\omega+\xi^{-1})} \cos 2k \right) \tag{4.31}
\]

from which the various limiting cases can be easily derived.

Finally we set \( R = ra \) and \( \tau = ta^2 \) and study the scaling limit \( a \to 0 \) keeping \( R \) and \( \tau \) fixed. \( \tilde{S}(k, \omega) \) becomes the well-known quantity

\[
\tilde{S}(k, \omega) = \frac{1}{\omega + k^2} \tag{4.32}
\]

while for the first order correction (4.26) we obtain

\[
\tilde{\Sigma}(k, \omega) = \frac{4k^2}{\omega + k^2} \left( \sigma^2 - \sqrt{\omega} \lim_{a \to 0} aL(ka, \omega a^2) \right) \tilde{S}(k, \omega) . \tag{4.33}
\]

From this expression we realize that the contribution of disorder correlations vanishes if their correlation length is finite. Furthermore, if they decay with a power law, \( h(2\nu) \sim |2\nu+1|^{-\alpha} \), then its contribution still vanishes for \( \alpha \geq 0 \). Only for an exponential decay \( h(2\nu) \sim |2\nu|^{-\alpha} \exp (-|2\nu|/\xi) \) with \( \alpha = 0 \), and an infinite correlation length \( \xi \gtrsim \omega^{-1/2} \), does one obtain a finite contribution (we study the limit \( \omega \to 0 \)). \( \alpha = 0 \) means that all fluctuating hopping
rates $p_{2x}$ would be equal to some quantity $p^{\text{even}}$ and all $p_{2x-1}$ would be equal to $p^{\text{odd}}$ and the averaged correlation function would be an average over semihomogeneous models with different hopping rates at even and odd time steps. On the other hand, $\alpha = \infty$ corresponds to completely uncorrelated choices of the hopping rates. However, periodic correlations of the type $\sigma^2 \cos 2\nu u$ or $\sigma^2 \sum_n \delta_{2\nu,n}$ or similar aperiodic types of behaviour also give a finite contribution to the averaged correlation function in the infinite time limit.

Our results show that with decreasing correlation $\tilde{\Sigma}(k, \omega)$ increases until one reaches $\alpha = 1$. For a stronger decay only the variance $\sigma^2$ of the distribution function is relevant.

V. CONCLUSION

We have studied systems of particles hopping stochastically on lattices of arbitrary dimension with space-dependent hopping probabilities. For exclusion models with SU(2)-invariant dynamics (see (2.11)-(2.13)) the time-dependent two-point correlation function of occupation numbers in the steady state is shown to be proportional to the probability distribution in space of a single particle moving in the same environment (see Eq. (2.21)). For bosonic systems we obtain an analogous result (2.25) if the time evolution operator commutes with the generators of the harmonic oscillator algebra. The factor of proportionality for fermions is different to the one for bosons. This reflects the effect of particle exclusion, i.e., of the hard core repulsion. Using known results for single particle diffusion in one-dimensional, disordered media, we obtain expressions for many-particle correlation functions for the same models.

The Fock space formalism we have applied reveals relations to quantum systems. In particular, a classical diffusive system of exclusive particles is shown to be equivalent to a generalized Heisenberg antiferromagnet.

Focussing on one-dimensional systems we study a version of the random barrier model with spatially correlated disorder. Its time evolution is given by the diagonal-to-diagonal transfer matrix of the 6-vertex model with a certain random choice of vertex weights. We
study the steady state and derive expressions (4.21) and (4.26) for the averaged time-dependent two-point density correlation function in the \( N \)-particle sector in the presence of weak disorder. These expression are exact in the lowest order of the expansion of the averaged function in the moments of distribution of the hopping rates. In the language of the two-dimensional vertex model these are arrow-arrow correlation functions in the plane.

We compare our results with known results for systems with uncorrelated disorder. If the correlation length \( \xi \) of the disorder correlations is finite or if the correlations \( h(r) \) have infinite range but decay faster than \( r^{-1} \), the contribution of the disorder correlations becomes negligibly small for large times and the system behaves like the system with uncorrelated disorder. For infinite-ranged disorder with a slower decay, \( h(r) \sim r^{-\alpha} \) where \( 0 \leq \alpha \leq 1 \), the correlation function changes its behaviour even after long times and the generalized diffusion constant \( D(k, \omega) \) becomes \( k \)-dependent even in the lowest order of the expansion. For uncorrelated disorder the diffusion constant depends only on \( \omega \) in this approximation.

Finally we studied the infinite time limit. For \( \alpha > 0 \) there is no contribution from the correlations, but for \( \alpha = 0 \), i.e., for a correlation that does not decay for \( r \rightarrow \infty \), we observe a qualitatively different behaviour.

An interesting open question is the applicability of the ideas of Section 2 to models with particles of different species.

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APPENDIX A: RECURSION RELATION FOR THE 2-POINT FUNCTION

Starting from Eq. (2.18) for the \( N \) particle correlation function and assuming the symmetry \( (2.12) \) we will prove the recursion relation \( (2.19) \).
For this we will use the following equations which can be proofed easily:

\[ S^{-} | N \rangle = (N + 1) | N + 1 \rangle \]  
(A.1)

\[ c_{j}^{\dagger} | N \rangle = n_{j} | N + 1 \rangle \]  
(A.2)

\[ c_{j} | N \rangle = (1 - n_{j}) | N - 1 \rangle \]  
(A.3)

\[ S^{+} | N \rangle = (L - N + 1) | N - 1 \rangle \]  
(A.4)

\[ [n_{j}, S^{-}] = c_{j}^{\dagger}. \]  
(A.5)

Writing down an expression similar to (2.18) for \( G^{\text{ferm}}_{N+1} \) and using Eq. (A.1) it results:

\[ (N + 1) \binom{L}{N+1} G^{\text{ferm}}_{N+1}(x, y; t) = \langle N + 1 | n_{x}U_{t}n_{y}S^{-} | N \rangle. \]  
(A.6)

By means of (i) and Eq. (A.3) we obtain:

\[ (N + 1) \binom{L}{N+1} G^{\text{ferm}}_{N+1}(x, y; t) = \langle N + 1 | S^{-}n_{x}U_{t}n_{y} | N \rangle \]
\[ + \langle N + 1 | c_{y}^{\dagger}U_{t}n_{y} | N \rangle \]
\[ + \langle N + 1 | n_{x}U_{t}c_{y}^{\dagger} | N \rangle. \]

Because of (A.2) and (A.3) and (A.4) this results in:

\[ N \binom{L}{N+1} G^{\text{ferm}}_{N+1}(x, y; t) = (L - N - 1) \binom{L}{N} G^{\text{ferm}}_{N}(x, y; t) + \langle N | U_{t}n_{y} | N \rangle. \]  
(A.7)

Because of the normalization condition (2.5) the second term on the r.h.s. of Eq. (A.7) is equal to \( N \binom{L}{N} \). Thus Eq. (A.7) results in the following recursion relation:

\[ G^{\text{ferm}}_{N+1}(x, y; t) = \frac{L - N - 1}{L - N} \frac{N + 1}{N} G_{N}(x, y; t) + \frac{1}{L} \frac{N + 1}{L - N}. \]  
(A.8)

**APPENDIX B: THE 6-VERTEX MODEL AS A DISORDERED DIFFUSIVE SYSTEM**

Here we repeat the mapping of Ref. [8] of a one-dimensional diffusion problem to a 6-vertex model and generalize it to a version of the random barrier model. Consider the 6-vertex model on a diagonal square lattice defined as follows: Place an up- or down-pointing
arrow on each link of the lattice and assign a non-zero Boltzmann weight to each of the vertices shown in figure 1. (All other configurations of arrows around an intersection of two lines, i.e., all other vertices, are forbidden.) The partition function is the sum of the products of Boltzmann weights of a lattice configuration taken over all allowed configurations. In the transfer matrix formalism up- and down-pointing arrows represent the state of the system at some given time \( t \) (Fig. 1). Each row of a diagonal square lattice is built by \( M \) of these vertices. Corresponding to the \( M \) vertices there are \( L = 2M \) sites in each row. The configuration of arrows in the next row above (represented by the upper arrows of the same vertices) then corresponds to the state of the system at an intermediate time \( t' = t + 1/2 \), and the configuration after a full time step \( t'' = t + 1 \) corresponds to the arrangement of arrows two rows above. Therefore each vertex represents a local transition from the state given by the lower two arrows of a vertex representing the configuration on sites \( j \) and \( j + 1 \) at time \( t \) to the state defined by the upper two arrows representing the configuration at sites \( j \) and \( j + 1 \) at time \( t + 1/2 \). The correspondence of the vertex language to the particle picture used in Section 4 can be understood by considering up-pointing arrows as particles occupying the respective sites of the chain while down-pointing arrows represent vacant sites, i.e., holes.

The diagonal-to-diagonal transfer matrix \( T \) acting on a chain of \( L \) sites (\( L \) even) of the six-vertex model with space dependent vertex weights as shown in figure 1 is then defined by

\[
T = \prod_{j=1}^{L/2} T_{2j} \cdot \prod_{j=1}^{L/2} T_{2j-1} = T^{\text{even}} T^{\text{odd}}. \tag{B.1}
\]

The matrices \( T_j \) act nontrivially on sites \( j \) and \( j + 1 \) in the chain, on all other sites they act as unit operator. All matrices \( T_j \) and \( T_{j'} \) with \( |j - j'| \neq 1 \) commute. For an explicit representation of the transfer matrix we choose a spin-1/2 tensor basis where the Pauli-matrix \( \sigma^z_j \) acting on site \( j \) of the chain is diagonal and spin down at site \( j \) represents a particle (up-pointing arrow) and spin up a hole (down-pointing arrow). In this basis \( n_j = \frac{1}{2}(1 - \sigma^z_j) \) is the projection operator on particles on site \( j \) and \( s^+_j = \frac{1}{2}(\sigma^x_j \pm i\sigma^y_j) \) (\( \sigma^{x,y,z} \) being the Pauli matrices) create \( (s^-_j) \) and annihilate \( (s^+_j) \) particles respectively. The matrices \( T_j \) in this basis
are defined by

\[
T_j = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 - p_j & p_j & 0 \\
0 & p_j & 1 - p_j & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}_{j,j+1}.
\]  

(B.2)

In the particle language the matrices \(T_j\) describe the local transition probabilities of particles moving from site \(j\) to site \(j + 1\) represented by the weights of the corresponding vertices. With these identifications the vertex model with a random choice of the numbers \(p_j\) in the interval \(0 \leq p_j \leq 1\) becomes a discrete-time version of the random barrier model.

The transfer matrix acts in parallel first on all odd-even pairs of sites \((2j - 1, 2j)\), then on all even-odd pairs. In a model with transfer matrix \(\tilde{T} = T^{\text{odd}}T^{\text{even}}\) one would start the time evolution at an intermediate half-odd integer time step and there will be no difference in the physical properties of these two systems. We assume periodic boundary conditions, i.e., we identify site \(L + 1\) with site 1.

The distinction between even and odd space coordinates becomes physically meaningful in the limiting case when all \(p_j\) are close to, or equal to one. Suppose \(p_j = 1\forall j\). Then particles which are on odd lattice sites at integer time steps move to right at a constant rate of two lattice units in space direction per full time step while particles move to left with same velocity which is the velocity of light in the system. Thus we have a system of non-interacting massless relativistic right- and left-movers. If the \(p_j\) are not equal but close to 1, one expects the particles to remain relativistic but massive. The homogeneous massive system is studied in detail in [32]. This feature of the vertex model makes it more interesting than the continuous-time formulation by the Hamiltonian (3.1) which allows only for nonrelativistic diffusion. In what follows we will use the term right(left)-movers for particles on odd (even) lattice sites also for the nonrelativistic case.

The time-dependent connected two-point correlation function for the homogeneous model with \(p = 1/2\) was computed by Kandel et al. [8] for full time steps in the continuum limit.
\[ L \to \infty. \] We quote their result and the corresponding expression for half-odd integer time intervals which are used in Section 4. One obtains with \( r = x - y \) and \( \rho = N/L \):

\( t \) integer:

\begin{align*}
\text{a) } x, y \text{ even} & \quad C_N(x, y; t) = \rho(1-\rho) \left( \frac{1}{2} \right)^{2t} \binom{2t-1}{t+r/2} \quad \text{(B.3)} \\
\text{b) } x \text{ odd, } y \text{ even} & \quad C_N(x, y; t) = \rho(1-\rho) \left( \frac{1}{2} \right)^{2t} \binom{2t-1}{t+(r-1)/2} \quad \text{(B.4)} \\
\text{c) } x \text{ even, } y \text{ odd} & \quad C_N(x, y; t) = \rho(1-\rho) \left( \frac{1}{2} \right)^{2t} \binom{2t-1}{t+(r-1)/2} \quad \text{(B.5)} \\
\text{d) } x, y \text{ odd} & \quad C_N(x, y; t) = \rho(1-\rho) \left( \frac{1}{2} \right)^{2t} \binom{2t-1}{t-r/2}. \quad \text{(B.6)}
\end{align*}

\( t \) half-odd integer:

\begin{align*}
\text{a) } x, y \text{ even} & \quad C_N(x, y; t) = \rho(1-\rho) \left( \frac{1}{2} \right)^{2t} \binom{2t-1}{t+(r-1)/2} \quad \text{(B.7)} \\
\text{b) } x \text{ odd, } y \text{ even} & \quad C_N(x, y; t) = \rho(1-\rho) \left( \frac{1}{2} \right)^{2t} \binom{2t-1}{t+r/2} \quad \text{(B.8)} \\
\text{c) } x \text{ even, } y \text{ odd} & \quad C_N(x, y; t) = \rho(1-\rho) \left( \frac{1}{2} \right)^{2t} \binom{2t-1}{t-r/2} \quad \text{(B.9)} \\
\text{d) } x, y \text{ odd} & \quad C_N(x, y; t) = \rho(1-\rho) \left( \frac{1}{2} \right)^{2t} \binom{2t-1}{t+(r-1)/2}. \quad \text{(B.10)}
\end{align*}

Because of the distinction of right- and left-movers and full and half-time steps we have to define carefully Fourier and Laplace transforms of space and time-dependent correlation functions. We define the dynamic structure function \( S(k, t) \) as the sum of Fourier transforms of the two-point correlation function between right-movers and left-movers:
\[ S(k, t) = \sum_x e^{ikx}(C(2x, 0; t) + C(2x + 1, 1; t)) \]  

(B.11)

From (B.3) one obtains for the correlation function \( C_N(x, y; t) \) of the ordered system with \( p = 1/2 \) at integer times

\[ S_0(k, t) = \rho(1 - \rho)(\cos k)^{2t} \]  

(B.12)

The discrete Laplace transform of a time-dependent quantity \( f(t) \) is defined as the sum

\[ \tilde{f}(\omega) = \sum_{t=0}^{\infty} e^{-\omega t} f(t) \]  

(B.13)

over full time steps only. For the dynamic structure function (B.12) we obtain

\[ \tilde{S}_0(k, \omega) = \frac{1}{1 - e^{-\omega} (\cos k)^2} \]  

(B.14)
[1] J.-P. Bouchard and A. Georges, Phys. Rep. 195, 127 (1990).

[2] J. W. Haus and K. W. Kehr, Phys. Rep. 68, 725 (1987).

[3] S. Alexander, J. Bernasconi, W. R. Schneider and R. Orbach, Rev. Mod. Phys. 53, 175 (1981).

[4] E. Hernandez-Garcia, M. A. Rodriguez, L. Pesquera and M. San Miguel, Phys. Rev. B 42, 10653 (1990).

[5] P.J.H. Denteneer and M.H. Ernst, Phys. Rev. B 29, 1755 (1984).

[6] R. Zwanzig, J. Stat. Phys. 28, 127 (1982).

[7] S. Havlin, M. Schwartz, R. Blumberg Selinger, A. Bunde and H. E. Stanley, Phys. Rev. A 40, 17 (1989).

[8] D. Kandel, E. Domany and B. Nienhuis, J. Phys. A, 23, L755 (1990).

[9] B. Derrida and M.R. Evans, J. Physique I 3, 311 (1993).

B. Derrida, M.R. Evans, V. Hakim and V. Pasquier, J. Phys. A: Math. Gen., in press (1993)

B. Derrida, M.R. Evans and D. Mukamel, Saclay preprint

[10] G. Schütz, J. Stat. Phys. 71, 485 (1993).

[11] G. Schütz, Phys. Rev. E 47, (in press) (1993).

[12] G. Schütz and E. Domany, J. Stat. Phys. 71, (in press) (1993).

[13] B. Derrida, E. Domany and D. Mukamel, J. Stat. Phys., 69, 667 (1992).

[14] H. Patzlaff, S. Sandow and S. Trimper, submitted to Physica A

[15] S. E. Esipov and T. J. Newman, J Stat. Phys. 70, 691 (1993).

[16] K. W. Kehr and O. Paetzold, Physica A 190, 1 (1992).
[17] S. Sandow and S. Trimper, submitted to J. Phys. A: Math. Gen.

[18] M. Doi, J. Phys. A: Math. Gen. 9, 1465 and 1479 (1976).

[19] P. Grassberger and M. Scheunert, Fortschr. Phys. 28, 547 (1980)

[20] L. Peliti, J. Physique 46, 1469 (1985).

[21] S. Sandow and S. Trimper, Europhys. Lett. 21(8), 799 (1993).

[22] S. Sandow and S. Trimper, J. Phys. A: Math. Gen. 26, (in press) (1993).

[23] L.-H. Gwa and H. Spohn, Phys. Rev. Lett., 68, 725 (1992).
L.-H. Gwa and H. Spohn, Phys. Rev. A, 46, 844 (1992).

[24] M. G. Rudavets, J. Phys. B: Condensed Matter 5, 1039 (1993).

[25] F. C. Alcaraz, M. Droz, M. Henkel and V. Rittenberg preprint 1992

[26] I. Jensen and R. Dickman, J. Stat. Phys. 71, 129 (1993).

[27] R.J. Baxter, *Exactly solved models in statistical mechanics*, New York, Academic Press, 1982.

[28] S. Teitel, D. Kutasov and E. Domany, Phys. Rev. B 28, 5711 (1983).

[29] A. Valle, M. A. Rodriguez and L. Pesquera, Phys. Rev. A 43, 948 (1991)

[30] Eq. (2.4) results from a discussion with H. Patzlaff and S. Trimper.

[31] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry*, Amsterdam, North-Holland Publishing Company (1981)
C. W. Gardiner, *Handbook of Stochastic Methods*, Berlin, Springer, (1983).

[32] U. Grimm and G. Schütz, in preparation.

[33] A. Maritan and A. Stella J. Phys. A: Math. Gen. 19, L269 (1986).

[34] D. S. Dean and K. M. Jansons J. Stat. Phys. 70, 1313 (1993).
[35] C. Destri and H.J. de Vega, Nucl. Phys. B, 290, 363 (1987).
FIG. 1. Allowed vertex configurations in the six-vertex model and their Boltzmann weights. Up-pointing arrows correspond to particles, down-pointing arrows represent vacant sites. In the dynamical interpretation of the model the Boltzmann weights give the transition probability of the state represented by the pair of arrows below the vertex to that above the vertex.