The Fourier method for the linearized Davey-Stewartson I equation

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Abstract

The linearized Davey-Stewartson equation with varying coefficients is solved by Fourier method. The approach uses the inverse scattering transform for the Davey-Stewartson equation.

1 Introduction

The Davey-Stewartson equation (DS) is the well-known subject for investigations because of two causes. First, the equation models a nonlinear interaction between a long surface wave and a short surface wave [1]. Second, DS is integrable by an inverse scattering transform (IST) [2] and then one can investigate a solution structure in details.

In this work we consider a linear system of equations with varying coefficients, which come out when one linearizes the DS at a nonzero solution as a background. We develop the Fourier method for the above linear system. This method is based on IST results for DS [3, 4]. The basic functions, which are used for the Fourier expansion, are associated with a scattering problem for a Dirac system.

A realized approach one can see as generalizing on (2+1)-dimensions (two spatial variables and time) system of pioneer works of D.Kaup [5, 6], in which the Fourier method was formulated for linearized (1+1)-dimensional integrable equations. The obtained results make possible to study perturbations of the DS, in general putting out the equation from the class of integrable equations.

2 The Dirac equation and the basic functions

Here we consider a Goursat problem for the Dirac equation [3, 4]:

\[
\begin{pmatrix}
\frac{\partial}{\partial \xi} & 0 \\
0 & \frac{\partial}{\partial \eta}
\end{pmatrix}
\psi = -\frac{1}{2}
\begin{pmatrix}
0 & q_1 \\
q_2 & 0
\end{pmatrix}
\psi
\]  

(1)
Define a solutions of $(1)$ as follows $(2)$:

\[
\begin{align*}
\psi_{11}^+|_{\xi \to -\infty} &= \exp(ik\eta), & \psi_{12}^+|_{\xi \to -\infty} &= 0, \\
\psi_{21}^+|_{\eta \to -\infty} &= 0, & \psi_{22}^+|_{\eta \to -\infty} &= \exp(-ik\xi), \\
\psi_{11}^-|_{\xi \to -\infty} &= \exp(ik\eta), & \psi_{12}^-|_{\xi \to -\infty} &= 0, \\
\psi_{21}^-|_{\eta \to -\infty} &= 0, & \psi_{22}^-|_{\eta \to -\infty} &= \exp(-ik\xi).
\end{align*}
\]

Denote by $(\chi, \mu)_q$ a bilinear form:

\[
(\chi, \mu)_q = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\xi d\eta (\chi_{1\mu_1} q_2 + \chi_{2\mu_2} q_1),
\]

where $\chi_i$ and $\mu_i$ are elements of the columns $\chi$ and $\mu$.

Denote by $\phi^{(1)}$ and $\phi^{(2)}$ the solutions which are dual to $\psi_{11}$ (first column of the matrix $\psi^+$) and $\psi_{12}$ (second column of the matrix $\psi^-$) with respect to the bilinear form $(2)$.

It is known, that the solution of $(1)$ satisfies a nonlocal Riemann-Hilbert problem $(3)$. Formulate this problem for $\psi_{11}$ and $\psi_{12}$. Denote by $\psi^{(1)}$ a row $\{\psi_{11}, \psi_{12}\}$, then

\[\psi^{(1)} = E^{(1)}(ik\eta) + S[s]\psi^{(1)}.
\]

Here $E^{(1)}(z)$ is the first row of a matrix $E(z) = \text{diag}(\exp(z), \exp(-z))$, the operator $S[s]$ is defined by formula:

\[S[s]\psi^{(1)} = \left[ \exp(ik\eta) \left( \exp(-ik\eta) \int_{-\infty}^{\infty} ds_1(k,l) \psi_{12}(\xi, \eta, l) \right)^- \right. \quad \left. \exp(ik\xi) \left( \exp(-ik\xi) \int_{-\infty}^{\infty} ds_1(k,l) \psi_{11}(\xi, \eta, l) \right)^+ \right],
\]

where

\[
\left( f(k) \right)^{\pm} = \frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{dk'}{k' - (k \pm i0)}.
\]

Denote by $< \chi, \mu >$, a bilinear form:

\[< \chi, \mu > = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk dl (\chi_1(l)\mu_1(k) s_2(k,l) + \chi_2(l)\mu_2(k) s_1(k,l)),
\]

where $\chi_l$ is the element of the row $\chi$.

Let $\varphi^{(j)}$, $j = 1, 2$ be rows which are solution of the equations conjugated to the equation for the rows $\psi^{(j)} = \{\psi_{1j}, \psi_{2j}\}$ with respect to the bilinear form $(3)$.

**Theorem 1** Let $q_1$ and $q_2$ satisfy following conditions: $\partial^\alpha q_1, 2 \in L_1 \cap C$ at $|\alpha| \leq 3$. If $h_1(\xi, \eta)$ and $h_2(\xi, \eta)$ satisfy the conditions $\partial^\alpha h_1, 2 \in L_1 \cap C$ at $|\alpha| \leq 4$, then $h_1$ and $h_2$ may be represented in the form:

\[
\begin{align*}
h_1 &= \frac{1}{2} < \psi^{(1)}(\xi, \eta, l), \varphi^{(1)}(\xi, \eta, k) > _h, \\
h_2 &= \frac{1}{2} < \psi^{(2)}(\xi, \eta, l), \varphi^{(2)}(\xi, \eta, k) > _h,
\end{align*}
\]

(4)
where
\[
\hat{h}_1 = \frac{1}{4\pi} (\psi^+_1(\xi, \eta, k), \phi(1)(\xi, \eta, l))_n, \\
\hat{h}_2 = \frac{1}{4\pi} (\psi^-_1(\xi, \eta, k), \phi(2)(\xi, \eta, l))_n.
\]

Of course, if \(q_1 = q_2 = 0\), then the formulae (5) and (4) are ordinary Fourier transform with respect to two variables \(\xi, \eta \in \mathbb{R}\).

3 The Fourier method for the linearized Davey-Stewartson I equation.

We shall consider the Davey-Stewartson I equation:
\[
i \partial_t Q + (\partial^2_\xi + \partial^2_\eta)Q + (g_1 + g_2)Q = 0,
\]
\[
\partial_\xi g_1 = -\frac{\varepsilon}{2} \partial_\eta |Q|^2, \quad \partial_\eta g_2 = -\frac{\varepsilon}{2} \partial_\xi |Q|^2, \quad \varepsilon = \pm 1.
\]

Linearization of this equation on \(Q, g\) as a background gives:
\[
i \partial_t U + (\partial^2_\xi + \partial^2_\eta)U + (g_1 + g_2)U + (V_1 + V_2)Q = 0,
\]
\[
\partial_\xi V_1 = -\frac{\varepsilon}{2} \partial_\eta (Q\bar{U} + \bar{Q}U), \quad \partial_\eta V_2 = -\frac{\varepsilon}{2} \partial_\xi (Q\bar{U} + \bar{Q}U).
\]
The equation (6) is a compatible condition for (1) at \(q_1 = Q, q_2 = \varepsilon \bar{Q}\) and for a following system (7):
\[
\partial_t \psi = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\partial_\xi - \partial_\eta)^2 \psi + i \begin{pmatrix} 0 & q_1 \\ q_2 & 0 \end{pmatrix} (\partial_\xi - \partial_\eta) \psi + \begin{pmatrix} ig_1 \\ i\partial_\xi q_2 \\ -i\partial_\eta q_2 \end{pmatrix} \psi.
\]

Using the systems (1) and (8) one can prove the following statement:

**Theorem 2** Let \(U\) be the solution of (3) and \(Q\) be the solution of the Davey-Stewartson I equation with boundary conditions: \(g_1|_{\xi \to \infty} = 0\) and \(g_2|_{\eta \to \infty} = 0\), and \(U\) and \(Q\) satisfy the conditions of the theorem 1 for the functions \(h_1\) and \(q_1\) at \(\forall t \in [0, T]\) respectively, then:
\[
\partial_t \hat{U}_1 = i(k^2 + l^2) \hat{U}_1, \quad \partial_t \hat{U}_2 = -i(k^2 + l^2) \hat{U}_2,
\]
where \(\hat{U}_1 = \frac{1}{4\pi} (\psi^-_1(\xi, \eta, k))_U, \hat{U}_2 = \frac{1}{4\pi} (\psi^+_2(\xi, \eta, l))_U\).

**Inverse statement.** Let \(\hat{U}_1(t, k, l)\) and \(\hat{U}_2(t, k, l)\) be integrable with respect to \(k, l\) with factor \((1 + k^2)(1 + l^2)\) and satisfy equations (3), then the function
\[
U(\xi, \eta, t) = \frac{1}{\pi} < \psi_1(\xi, \eta, l), \varphi_1(\xi, \eta, k) >_{U(t)}
\]
satisfies linearized Davey-Stewartson I equation (3).

Solving of the system (3) is reduced to solving of the trivial equations (3) for the Fourier coefficients \(\hat{U}_{1,2}\). The theorems 1 and 2 give solving (3) by Fourier method.
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