CKS-SPACE IN TERMS OF GROWTH FUNCTIONS

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Abstract. A class of growth functions $u$ is introduced to construct Hida distributions and test functions. The Legendre transform $\ell_u$ of $u$ is used to define a sequence $\alpha(n) = (\ell_u(n)n!)^{-1}$, $n \geq 0$, of positive numbers. From this sequence we get a CKS-space. Under various conditions on $u$ we show that the associated sequence $\{\alpha(n)\}$ satisfies those conditions for carrying out the white noise distribution theory on the CKS-space. We show that $u$ and its dual Legendre transform $u^*$ are growth functions for test and generalized functions, respectively, in the characterization theorems.

1. CKS-space

Let $\{\alpha(n)\}_{n=0}^{\infty}$ be a sequence of positive numbers. The exponential generating functions $G_\alpha$ and $G_{1/\alpha}$ of the sequences $\{\alpha(n)\}$ and $\{1/\alpha(n)\}$ are defined by

$$G_\alpha(r) = \sum_{n=0}^{\infty} \frac{\alpha(n)}{n!} r^n, \quad G_{1/\alpha}(r) = \sum_{n=0}^{\infty} \frac{1}{n!\alpha(n)} r^n.$$  (1.1)

Later on we will impose various conditions from the following list:

(A1) $\alpha(0) = 1$ and $\inf_{n \geq 0} \alpha(n)n^\sigma > 0$ for some $\sigma \geq 1$.

(A2) $\lim_{n \to \infty} \left( \frac{\alpha(n)}{n!} \right)^{1/n} = 0$.

($\tilde{A}2$) $\lim_{n \to \infty} \left( \frac{1}{n!\alpha(n)} \right)^{1/n} = 0$.

(B1) $\limsup_{n \to \infty} \left( \frac{n!}{\alpha(n)} \right)^{1/n} \inf_{r > 0} \frac{G_\alpha(r)}{r^n} < \infty$.

($\tilde{B}1$) $\limsup_{n \to \infty} \left( n!\alpha(n) \right)^{1/n} \inf_{r > 0} \frac{G_{1/\alpha}(r)}{r^n} < \infty$.

(B2) The sequence $\gamma(n) = \frac{\alpha(n)}{n}$, $n \geq 0$, is log-concave, i.e., for all $n \geq 0$,

$$\gamma(n)\gamma(n+2) \leq \gamma(n+1)^2.$$

($\tilde{B}2$) The sequence $\left\{ \frac{1}{n!\alpha(n)} \right\}$ is log-concave.

(B3) The sequence $\{\alpha(n)\}$ is log-convex, i.e., for all $n \geq 0$,

$$\alpha(n)\alpha(n+2) \geq \alpha(n+1)^2.$$

(C1) There exists a constant $c_1$ such that for all $n \leq m$,

$$\alpha(n) \leq c_1^m \alpha(m).$$
(C2) There exists a constant $c_2$ such that for all $n$ and $m$,
\[ \alpha(n + m) \leq c_2^{n+m} \alpha(n)\alpha(m). \]

(C3) There exists a constant $c_3$ such that for all $n$ and $m$,
\[ \alpha(n)\alpha(m) \leq c_3^{n+m} \alpha(n + m). \]

Recently Cochran et al. [5] have used a sequence $\{\alpha(n)\}$ of positive numbers to define spaces of test and generalized functions on a white noise space. They assumed condition (A1) with $\sigma = 1$. But our (A1) is strong enough to imply that the space of test functions is contained in the $L^2$-space of the white noise measure.

In [5] conditions (A2) (B1) (B2) are considered. Condition (A2) is to assure that the function $G_\alpha$ in Equation (1.1) is an entire function. Condition (B1) is used for the characterization theorem of generalized functions in Theorems 5.1 and 6.1 [5]. Condition (B2) is shown to imply condition (B1) in Theorem 4.3 [5].

In the papers by Asai et al. [1] [2], conditions (˜A2) (˜B1) (˜B2) (B3) are considered. It can be easily checked that condition (A1) implies condition (˜A2). Condition (˜A2) is to assure that the function $G_{1/\alpha}$ in Equation (1.1) is an entire function. In [2] condition (B1) is used for the characterization theorem of test functions. Condition (B2) implies condition (B1), while obviously condition (B3) implies condition (B2).

In the paper by Kubo et al. [8], conditions (C1) (C2) (C3) are assumed in order to carry out the distribution theory for a CKS-space. As pointed out in [8], condition (C3) implies condition (C1).

An important example of $\{\alpha(n)\}$ is the sequence $\{b_k(n)\}$ of Bell’s numbers of order $k \geq 2$. The sequence $\{b_k(n)\}$ satisfies conditions (A1) (A2) (B1) (as shown in [5]), (B2) (B3) (as shown in [1]) (C1) (C2) (C3) (as shown in [8]). Therefore, Bell’s numbers satisfy all conditions in the above list.

The essential conditions for distribution theory on a CKS-space are (A1) (A2) (B2) (B3) (C1) (C2) (C3). All other conditions can be derived from these six conditions except for (B3). We have taken (B2) instead of (B3) for the following reason. The condition (B3) is rather strong and we do not know how to prove this condition for a growth function $u$ in Section 3. Fortunately, we do not need (B3) for white noise distribution theory.

Now, we briefly explain the CKS-space associated with a sequence $\{\alpha(n)\}$ of positive numbers. Let $E$ be a nuclear space with topology given by a sequence of inner product norms $\{| \cdot |_p\}_{p=0}^\infty$. Let $E_p$ be the completion of $E$ with respect to the norm $| \cdot |_p$. Assume the conditions:

(a) There exists a constant $0 < \rho < 1$ such that $| \cdot |_p \leq \rho | \cdot |_{p+1} \leq \rho^2 | \cdot |_{p+2} \leq \cdots$.

(b) For any $p \geq 0$, there exists some $q \geq p$ such that the inclusion mapping $i_{p,q} : E_q \to E_p$ is a Hilbert-Schmidt operator.

Let $E'$ be the dual space of $E$. Then we have a Gel’fand triple $E \subset E_0 \subset E'$. Let $\mu$ be the standard Gaussian measure on $E'$ and let $(L^2)$ denote the Hilbert space of complex-valued square integrable functions on $(E', \mu)$. Every $\varphi \in (L^2)$ can be uniquely represented by a sum of multiple Wiener integrals
\[ \varphi = \sum_{n=0}^\infty I_n(f_n), \quad f_n \in E_0^\otimes n. \]
Moreover, the \((L^2)\)-norm \(\|\varphi\|_0\) of \(\varphi\) is given as follows:

\[
\|\varphi\|_0^2 = \sum_{n=0}^{\infty} n! |f_n|^2.
\]

For a positive integer \(p\) such that \(\rho^{-2p} \geq \sigma\), or equivalently \(p \geq (-2 \log \rho)^{-1} \log \sigma\), define \(\|\cdot\|_{p,\alpha}\) by

\[
\|\varphi\|_{p,\alpha}^2 = \sum_{n=0}^{\infty} n! \alpha(n) |f_n|^2.
\]

We can use the above assumption (a) to show that

\[
\|\varphi\|_{p,\alpha}^2 \geq \left(\inf_{n \geq 0} \alpha(n) \sigma^n\right) \|\varphi\|_0^2.
\]

Thus if we define \([E_p]_\alpha = \{\varphi \in (L^2); \|\varphi\|_{p,\alpha} < \infty\}\), then condition (A1) implies that \([E_p]_\alpha \subset (L^2)\) for all \(p \geq (-2 \log \rho)^{-1} \log \sigma\). The space \([E]_\alpha\) of test functions on \(E^\prime\) is defined to be the projective limit of \([E_p]_\alpha; p \geq (-2 \log \rho)^{-1} \log \sigma\). Its dual space \([E]_\alpha^*\) is the space of generalized functions on \(E^\prime\). Then we get the following Gel'fand triple

\[
[E]_\alpha \subset (L^2) \subset [E]_\alpha^*.
\]

This Gel'fand triple was introduced by Cochran et al. in [3] and is often referred to as the CKS-space associated with the sequence \(\{\alpha(n)\}\). Note that \([E]_\alpha^* = \cup_p [E_p]_\alpha^*\) and the \([E_p]_\alpha^*\)-norm is given by

\[
\|\varphi\|_{p,1/\alpha}^2 = \sum_{n=0}^{\infty} \frac{n!}{\alpha(n)} |f_n|^2.
\]

For each \(\xi\) belonging to the complexification \(E_c\) of \(E\), define

\[
: e^{(\cdot,\xi)} : = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\xi^{\otimes n}).
\]

By Equations (1.2) and (1.3), we have

\[
\|: e^{(\cdot,\xi)} :\|_{p,\alpha}^2 = G_{1/\alpha}(|\xi|_p^2), \quad \|: e^{(\cdot,\xi)} :\|_{p,1/\alpha}^2 = G_{1/\alpha}(|\xi|^{-2}_p).
\]

The first equality shows that \(e^{(\cdot,\xi)} : \in [E]_\alpha\) for all \(\xi \in E_c\).

The \(S\)-transform of a generalized function \(\Phi\) in \([E]_\alpha^*\) is defined to be the function

\[
(S\Phi)(\xi) = \langle \Phi, e^{(\cdot,\xi)} : \rangle, \quad \xi \in E_c,
\]

where \(\langle \cdot, \cdot \rangle\) denotes the bilinear pairing of \([E]_\alpha^*\) and \([E]_\alpha\).

Under the condition (B1) it is proved in [3] that a complex-valued function \(F\) on \(E_c\) is the \(S\)-transform of a generalized function in \([E]_\alpha^*\) if and only if it satisfies an analyticity condition and the growth condition: There exist constants \(K, a, p \geq 0\) such that

\[
|F(\xi)| \leq KG_{\alpha}(|\xi|_p^2)^{1/2}, \quad \xi \in E_c.
\]

Here the inequality is motivated by the first equality in Equation (1.3).

On the other hand, under the condition (B1), it is shown in [3] that a complex-valued function \(F\) on \(E_c\) is the \(S\)-transform of a test function in \([E]_\alpha\) if and only
if it satisfies the analyticity condition and the growth condition: For any \( a, p \geq 0 \), there exists a constant \( K \geq 0 \) such that

\[
|F(\xi)| \leq KG^{1/\alpha}(a|\xi|_p^{-p})^{1/2}, \quad \xi \in \mathcal{E}_c.
\]  

(1.8)

This inequality is motivated by the second equality in Equation (1.5).

Observe that the functions \( G^\alpha \) and \( G^{1/\alpha} \) are defined by series as in Equation (1.1). In general it is impossible to find the sums of these series in closed forms. It is so even for the Kondratiev-Streit space \([6] [7] [9]\) when \( \alpha(n) = (n!)^\beta, \beta \neq 0 \). Thus it is desirable to find elementary functions which can be used as growth functions in Equations (1.7) and (1.8). This leads to the investigation by Asai et al. in the papers \([3] [4]\). We will explain the essential ideas in the rest of this paper.

2. Legendre and dual Legendre transforms

Let \( C_{+, \log} \) denote the set of all positive continuous functions \( u \) on \([0, \infty)\) satisfying the condition:

\[
\lim_{r \to \infty} \frac{\log u(r)}{\log r} = \infty.
\]

The Legendre transform \( \ell_u \) of \( u \in C_{+, \log} \) is defined to be the function

\[
\ell_u(t) = \inf_{r > 0} \frac{u(r)}{r^t}, \quad t \in [0, \infty).
\]

Below we state some properties of the Legendre transform. For the proofs, see the paper \([3]\).

**Theorem 2.1.** Let \( u \in C_{+, \log} \). Then

1. The function \( \ell_u \) is log-concave, i.e., for any \( r, s \geq 0 \) and \( 0 \leq \lambda \leq 1 \),

\[
\ell_u(\lambda r + (1 - \lambda)s) \geq \ell_u(r)^\lambda \ell_u(s)^{1-\lambda}.
\]

In particular, the sequence \( \{\ell_u(n)\}_{n=0}^\infty \) is log-concave.

2. For any nonnegative integers \( n \) and \( m \),

\[
\ell_u(0)\ell_u(n + m) \leq \ell_u(n)\ell_u(m).
\]

A positive continuous function \( u \) on \([0, \infty)\) is called \((\log, \exp)-\text{convex}\) if \( \log u(e^t) \) is convex on \( \mathbb{R} \).

**Theorem 2.2.** Let \( u \in C_{+, \log} \) be \((\log, \exp)-\text{convex}\). Then

1. \( \ell_u(t) \) is decreasing for large \( t \),
2. \( \lim_{t \to \infty} \ell_u(t)^{1/t} = 0 \),
3. \( u(r) = \sup_{t \geq 0} \ell_u(t)r^t \) for all \( r \geq 0 \).

Let \( k > 0 \). A positive continuous function \( u \) on \([0, \infty)\) is called \((\log, x^k)-\text{convex}\) if \( \log u(x^k) \) is convex on \([0, \infty)\).

**Theorem 2.3.** Let \( u \in C_{+, \log} \). We have the assertions:

1. \( u \) is \((\log, x^k)-\text{convex}\) if and only if \( \ell_u(t)^{kt} \) is log-convex.
2. If \( u \) is \((\log, x^k)-\text{convex}, \) then for any nonnegative integers \( n \) and \( m \),

\[
\ell_u(n)\ell_u(m) \leq \ell_u(0)2^{k(n+m)}\ell_u(n + m).
\]
Suppose \( u \in C_{+, \log} \) satisfies the condition \( \lim_{n \to \infty} \ell_u(n)^{1/n} = 0 \). We define its \( L \)-function by

\[
\mathcal{L}_u(r) = \sum_{n=0}^{\infty} \ell_u(n)r^n. \tag{2.1}
\]

Note that the function \( \mathcal{L}_u \) is an entire function. By Theorem 2.2(2), \( \mathcal{L}_u \) is defined for any (log, exp)-convex function \( u \in C_{+, \log} \).

**Theorem 2.4.**

1. Let \( u \in C_{+, \log} \) be (log, exp)-convex. Then its \( L \)-function \( \mathcal{L}_u \) is also (log, exp)-convex and for any \( a > 1 \),

\[
\mathcal{L}_u(r) \leq \frac{ea}{\log a} u(ar), \quad \forall r \geq 0.
\]

2. Let \( u \in C_{+, \log} \) be increasing and (log, \( x^k \))-convex. Then there exists a constant \( C \), independent of \( k \), such that

\[
u(r) \leq C \mathcal{L}_u(2^k r), \quad \forall r \geq 0.\]

Two positive functions \( u \) and \( v \) on \([0, \infty)\) are called equivalent if there exist positive constants \( c_1, c_2, a_1, a_2 \) such that

\[
c_1 u(a_1 r) \leq v(r) \leq c_2 u(a_2 r), \quad \forall r \geq 0.
\]

It can be easily checked that if \( u \in C_{+, \log} \) is increasing and (log, \( x^k \))-convex, then \( u \) is (log, exp)-convex. Thus by Theorem 2.4 the functions \( u \) and \( \mathcal{L}_u \) are equivalent for any increasing (log, \( x^k \))-convex function \( u \in C_{+, \log} \).

Next, let \( C_{+, 1/2} \) denote the set of all positive continuous functions \( u \) on \([0, \infty)\) satisfying the condition:

\[
\lim_{r \to \infty} \frac{\log u(r)}{\sqrt{r}} = \infty.
\]

The dual Legendre transform \( u^* \) of \( u \in C_{+, 1/2} \) is defined to be the function

\[
u^*(r) = \sup_{s \geq 0} e^{2\sqrt{rs}} u(s), \quad r \in [0, \infty).
\]

Note that \( C_{+, 1/2} \subset C_{+, \log} \). Below are some properties of the dual Legendre transform. For the proofs, see the paper [3].

**Theorem 2.5.** Let \( u \in C_{+, 1/2} \). Then \( u^* \) belongs to \( C_{+, 1/2} \) and is an increasing (log, \( x^2 \))-convex function on \([0, \infty)\).

**Theorem 2.6.** If \( u \in C_{+, 1/2} \) is (log, \( x^2 \))-convex, then the Legendre transform \( \ell_u^* \) of \( u^* \) is given by

\[
\ell_u^*(t) = \frac{e^{2t}}{\ell_u(t)^2 t^2}, \quad t \geq 0.
\]

Suppose \( u \in C_{+, 1/2} \) satisfies the condition \( \lim_{n \to \infty} (\ell_u(n)(n!)^2)^{-1/n} = 0 \). We define its \( L^\# \)-function by

\[
\mathcal{L}_u^\#(r) = \sum_{n=0}^{\infty} \frac{1}{\ell_u(n)(n!)^2} r^n. \tag{2.2}
\]
Note that the function $L_u^\#$ is an entire function. It can be checked that $L_u^\#$ is defined for any $(\log, x^2)$-convex function $u$ in $C_{+,1/2}$.

**Theorem 2.7.** Let $u \in C_{+,1/2}$ be $(\log, x^2)$-convex. Then the functions $u^*$, $L_u^*$, and $L_u^\#$ are equivalent.

3. CKS-space associated with a growth function

Let $u \in C_{+,\log}$ be a fixed function. Define a sequence of positive numbers by

$$
\alpha(n) = \frac{1}{\ell_u(n)n!}, \quad n \geq 0.
$$

(3.1)

Then from Equations (1.1) and (2.1) we see that

$$
G_{1/\alpha}(r) = \sum_{n=0}^{\infty} \frac{1}{n! \alpha(n)} r^n = \sum_{n=0}^{\infty} \ell_u(n) r^n = L_u(r).
$$

(3.2)

Moreover, by Equations (1.1) and (2.2), we have

$$
G_{\alpha}(r) = \sum_{n=0}^{\infty} \frac{\alpha(n)}{n!} r^n = \sum_{n=0}^{\infty} \frac{1}{\ell_u(n)(n!)^2} r^n = L_u^\#(r).
$$

(3.3)

Under various conditions on $u$, we will show that the corresponding sequence $\{\alpha(n)\}$ satisfies the six essential conditions $(A1)$ $(A2)$ $(B2)$ $(C2)$ $(C3)$ stated in Section 1. A weaker condition than $(B2)$, called near-$(B2)$, will be defined below.

**Lemma 3.1.** Let $u \in C_{+,\log}$. Suppose $\inf_{r>0} u(r) = 1$ and $u$ satisfies the condition

$$
\lim_{r \to \infty} \frac{\log u(r)}{r} < \infty.
$$

(3.4)

Then the sequence $\{\alpha(n)\}$ satisfies condition $(A1)$.

**Proof.** From the definition of the Legendre transform and the assumption, we get $\ell_u(0) = \inf_{r>0} u(r) = 1$. Hence $\alpha(0) = 1$. It is easy to check that Equation (3.4) holds if and only if there exist constants $a, c > 0$ such that $u(r) \leq ce^{ar}$. Therefore,

$$
\ell_u(n) = \inf_{r>0} \frac{u(r)}{r^n} \leq \inf_{r>0} \frac{ce^{ar}}{r^n} = ca^n \left( \frac{e}{n} \right)^n.
$$

This implies that $\alpha(n) = (\ell_u(n)n!)^{-1} \geq (nca^n)^{-1}(n/e)^n$. But from the Stirling formula, we have $n! \leq 2^{n/2}(n/e)^n$. Hence $\alpha(n) \geq (ce(a\sqrt{2})^n)^{-1}$. We can choose $\sigma = a\sqrt{2}$ to see that condition $(A1)$ is satisfied. $\square$

**Lemma 3.2.** Suppose $u \in C_{+,1/2}$ is $(\log, x^2)$-convex. Then the sequence $\{\alpha(n)\}$ satisfies condition $(A2)$.

**Proof.** By using the Stirling formula we get

$$
\frac{\alpha(n)}{n!} = \frac{1}{\ell_u(n)(n!)^2} \approx \frac{1}{2\pi n \ell_u(n)n^{2n}}, \quad \text{for large } n.
$$

(3.5)

By Theorem 2.7, the dual Legendre transform $u^*$ of $u$ belongs to $C_{+,1/2}$ (hence to $C_{+,\log}$) and is increasing and $(\log, x^2)$-convex (hence $(\log, \exp)$-convex.) Thus we can apply Theorem 2.2 to $u^*$ and by part (2) of that theorem,

$$
\lim_{n \to \infty} \ell_{u^*}(n)^{1/n} = 0.
$$

(3.6)
On the other hand, by Theorem 2.6
\[ \ell_u(n) = \frac{e^{2n}}{\ell_u(n)n^{2n}}. \]  
(3.7)

Therefore, from Equations (3.5) (3.6) (3.7), we easily see that \( (\alpha(n)/n!)^{1/n} \to 0 \) as \( n \to \infty \). Hence condition (A2) is satisfied.

Two sequences \( \{a(n)\} \) and \( \{b(n)\} \) are called equivalent if there exist positive constants \( K_1, K_2, c_1, c_2 \) such that for all \( n \)
\[ K_1c_1^n a(n) \leq b(n) \leq K_2c_2^n a(n). \]

A sequence is said to be nearly log-concave if it is equivalent to a log-concave sequence. With this concept we define a weaker condition than (B2).

- Near-(B2): The sequence \( \{\alpha(n)/n!\} \) is nearly log-concave.

Observe that near-(B2) condition can be stated in another way: The sequence \( \{\alpha(n)\} \) is equivalent to a sequence \( \{\delta(n)/n!\} \) which is log-concave.

In Section 1, we pointed out that condition (B2) implies condition (B1). It can be easily checked that condition near-(B2) also implies condition (B1).

Lemma 3.3. (1) Let \( u \in C_{+, \log} \) be \((\log, x^1)\)-convex. Then the sequence \( \{\alpha(n)\} \) satisfies condition (B2).

(2) Let \( u \in C_{+, \log} \) be \((\log, x^2)\)-convex. Then the sequence \( \{\alpha(n)\} \) satisfies condition near-(B2).

Proof. Let \( u \in C_{+, \log} \) be \((\log, x^1)\)-convex. By Theorem 2.3(1) the sequence \( \{\ell_u(n)n^n\} \) is log-convex and so
\[ \ell_u(n)n^n \ell_u(n+2)(n+2)^{n+2} \geq (\ell_u(n+1)(n+1)^{n+1})^2. \]

This inequality can be rewritten as
\[ \ell_u(n+1)^2 \leq \frac{n^n(n+2)^{n+2}}{(n+1)^{2(n+1)}} \ell_u(n)\ell_u(n+2). \]

But it can be easily verified that
\[ \frac{n^n(n+2)^{n+2}}{(n+1)^{2(n+1)}} \leq \left( \frac{n+2}{n+1} \right)^2. \]

Therefore,
\[ \ell_u(n+1)^2 \leq \left( \frac{n+2}{n+1} \right)^2 \ell_u(n)\ell_u(n+2). \]

Rewrite this inequality as
\[ \frac{1}{\ell_u(n)n!} \frac{1}{\ell_u(n+2)((n+2)!)^2} \leq \left( \frac{1}{\ell_u(n+1)((n+1)!)^2} \right)^2. \]

Note that \( \alpha(n)/n! = (\ell_u(n)n!)^{-1} \). Hence the last inequality shows that the sequence \( \{\alpha(n)/n!\} \) is log-concave, i.e., the sequence \( \{\alpha(n)\} \) satisfies condition (B2). This proves the first assertion of the lemma.

To prove the second assertion, let \( u \in C_{+, \log} \) be \((\log, x^2)\)-convex. By Theorem 2.3(1) the sequence \( \{\ell_u(n)n^{2n}\} \) is log-convex. But \( \alpha(n) = (\ell_u(n)n!)^{-1} \). Hence
the sequence \( \{ (\alpha(n)!)^{-1} n^{2n} \} \) is log-convex and so the sequence \( \{ \alpha(n)!/n^{2n} \} \) is log-concave.

On the other hand, note that by the Stirling formula the sequences \( \{ n! \} \) and \( \{ n^n \} \) are equivalent. This implies that the sequence \( \{ \alpha(n)/n! \} \) is equivalent to \( \{ \alpha(n)!/n^{2n} \} \), which has just been shown to be log-concave. Thus \( \{ \alpha(n)!/n! \} \) is nearly log-concave and so \( \{ \alpha(n)! \} \) satisfies condition near-(B2).

**Lemma 3.4.** Let \( u \in C_{+, \log} \). Then the sequence \( \{ \alpha(n)! \} \) satisfies condition (B2).

**Proof.** By Theorem 2.1(1) the sequence \( \{ \ell_u(n) \} \) is log-concave. Since \( \ell_u(n) = (n!\alpha(n))^{-1} \), the sequence \( \{ \alpha(n)!/n^2 \} \) is log-concave. Hence the sequence \( \{ \alpha(n)!/n! \} \) satisfies condition (B2).

**Lemma 3.5.** Let \( u \in C_{+, \log} \) be (log, \( x^2 \))-convex and \( \inf_{r>0} u(r) = 1 \). Then the sequence \( \{ \alpha(n)!/n! \} \) satisfies condition (C2).

**Proof.** From Theorem 2.3(2) with \( k = 2 \), we have

\[
\ell_u(n)\ell_u(m) \leq \ell_u(0)2^{m+n}\ell_u(n+m).
\]

But \( \ell_u(0) = \inf_{r>0} u(r) = 1 \) and \( \alpha(n) = (\ell_u(n)!)^{-1} \). Hence

\[
\frac{1}{\alpha(n)!n!} \leq 2^{n+m} \frac{1}{\alpha(n+m)!(n+m)!}.
\]

Note that \( n!m!/n+m! \leq 1 \). Hence \( \alpha(n+m) \leq 2^{n+m} \alpha(n)\alpha(m) \) and so the sequence \( \{ \alpha(n)!/n! \} \) satisfies condition (C2).

**Lemma 3.6.** Let \( u \in C_{+, \log} \) and suppose \( \inf_{r>0} u(r) = 1 \). Then the sequence \( \{ \alpha(n)!/n! \} \) satisfies condition (C3).

**Proof.** By Theorem 2.3(2) we have

\[
\ell_u(0)\ell_u(n+m) \leq \ell_u(n)\ell_u(m).
\]

Note that \( \ell_u(n) = (\alpha(n)!n!)^{-1} \) and \( \ell_u(0) = \inf_{r>0} u(r) = 1 \). Hence

\[
\alpha(n)m \leq \frac{(n+m)!}{n!m!} \alpha(n+m) \leq 2^{n+m} \alpha(n+m).
\]

This shows that the sequence \( \{ \alpha(n)!/n! \} \) satisfies condition (C3).

Now, we summarize the above lemmas to state a theorem which can be used for white noise distribution theory on a CKS-space arising from a growth function.

**Theorem 3.7.** Let \( u \in C_{+, 1/2} \) be a function satisfying the following conditions:

(U0) \( \inf_{r>0} u(r) = 1 \).
(U2) \( \lim_{r \to \infty} r^{-1} \log u(r) < \infty \).
(U3) \( u \) is (log, \( x^2 \))-convex.

Then the corresponding sequence \( \{ \alpha(n)!/n! \} \) satisfies conditions (A1), (A2), near-(B2), (B2), (C2), and (C3).

**Remark.** Define another condition on \( u \) by

(U1) \( u \) is increasing and \( u(0) = 1 \).
Obviously, condition (U1) implies condition (U0). We need this stronger condition in order to apply Theorem 2.4 (2) where \( u \) is assumed to be increasing.

Now, let \( u \in C_{+1/2} \) be a function satisfying conditions (U0) (U2) (U3). From this function \( u \) we define a sequence \( \{\alpha(n)\} \) of positive numbers by Equation (3.10). Then we use this sequence \( \{\alpha(n)\} \) to define a CKS-space as in Equation (3.9). The resulting Gel'fand triple is denoted by

\[
[E]_u \subset (L^2) \subset [E]^*_u. \tag{3.9}
\]

Let us consider the characterization of generalized functions in \( [E]^*_u \). Note that by Equation (3.3) the exponential generating function \( G_\alpha \) of \( \{\alpha(n)\} \) is given by \( G_\alpha = L^\#_u \). Hence the growth condition in Equation (1.7) can be stated as: There exist constants \( K, a, p \geq 0 \) such that

\[
|F(\xi)| \leq KL^\#_u(a|\xi|^2_p)^{1/2}, \quad \xi \in \mathcal{E}_c.
\]

Recall that by Theorem 2.3 the function \( L^\#_u \) is equivalent to the dual Legendre transform \( u^* \) of \( u \). Thus we can replace the growth function \( L^\#_u \) by \( u^* \) and we have the following theorem.

**Theorem 3.8.** Suppose \( u \in C_{+1/2} \) satisfies the conditions (U0) (U2) (U3). Then a complex-valued function \( F \) on \( \mathcal{E}_c \) is the \( S \)-transform of a generalized function in \( [E]^*_u \) if and only if it satisfies the conditions:

(a) For any \( \xi, \eta \in \mathcal{E}_c \), the function \( F(z\xi + \eta) \) is an entire function of \( z \in \mathbb{C} \).

(b) There exist constants \( K, a, p \geq 0 \), such that

\[
|F(\xi)| \leq Ku^*(a|\xi|^2_p)^{1/2}, \quad \xi \in \mathcal{E}_c. \tag{3.10}
\]

Next, we consider the characterization of test functions in \( [E]_u \). By Equation (3.4) we have \( G_{1/\alpha} = L^\#_u \). Hence the growth condition in Equation (1.8) can be stated as: For any \( a, p \geq 0 \), there exists a constant \( K \geq 0 \) such that

\[
|F(\xi)| \leq KL^\#_u(a|\xi|^2_p)^{1/2}, \quad \xi \in \mathcal{E}_c.
\]

Assume that \( u \) is increasing (so we need condition (U1)). Then we can apply Theorem 2.4 to see that \( L^\#_u \) is equivalent to the function \( u \). Thus we can replace the growth function \( L^\#_u \) by \( u \) and we have the next theorem.

**Theorem 3.9.** Suppose \( u \in C_{+1/2} \) satisfies the conditions (U0) (U2) (U3). Then a complex-valued function \( F \) on \( \mathcal{E}_c \) is the \( S \)-transform of a test function in \( [E]_u \) if and only if it satisfies the conditions:

(a) For any \( \xi, \eta \in \mathcal{E}_c \), the function \( F(z\xi + \eta) \) is an entire function of \( z \in \mathbb{C} \).

(b) For any constants \( a, p \geq 0 \), there exists a constant \( K \geq 0 \) such that

\[
|F(\xi)| \leq Ku(a|\xi|^2_p)^{1/2}, \quad \xi \in \mathcal{E}_c. \tag{3.11}
\]

At the end of this paper we give two examples to illustrate our method.

**Example 3.10.** Consider the Kondratiev-Streit space \( KS \) associated with the sequence \( \alpha(n) = (n!)^\beta, 0 \leq \beta < 1 \). If we apply the characterization theorems from \( KS \) and \( KS \), then we have to use the following growth functions:

\[
G_{\alpha}(r) = \sum_{n=0}^{\infty} \frac{1}{(n!)^1-\beta} r^n, \quad G_{1/\alpha}(r) = \sum_{n=0}^{\infty} \frac{1}{(n!)^{1+\beta}} r^n. \tag{3.12}
\]
But these growth functions are impractical since the series cannot be summed up in closed forms when $\beta \neq 0$. To overcome this difficulty, note that the sequence $\{n!\}$ is equivalent to the sequence $\{(n/e)^n\}$ by the Stirling formula. Hence the function $G_{1/\alpha}$ is equivalent to the following function

$$\theta(r) = \sum_{n=0}^{\infty} \left(\frac{e}{n}\right)^{(1+\beta)n} r^n.$$  

As shown in [3], $(e/n)^{(1+\beta)n} = \ell_u(n)$ for the function

$$u(r) = \exp\left[(1 + \beta)r^{1/(1+\beta)}\right].$$

Therefore $\theta = \mathcal{L}_u$ by Equation (2.1) and so $G_{1/\alpha}$ is equivalent to $\mathcal{L}_u$. Furthermore, by Theorem 2.4, the function $\mathcal{L}_u$ is equivalent to $u$. Hence the growth function $G_{1/\alpha}$ in Equation (3.12) can be replaced by the above function $u$.

We remark that the Kondratiev-Streit space turns out to be the same as the CKS-space arising from the function $u$. It is easy to check that the dual Legendre transform $u^*$ of $u$ is given by

$$u^*(r) = \exp\left((1 - \beta)r^{1/(1-\beta)}\right).$$

Hence the growth function $G_\alpha$ in Equation (3.12) can be replaced by $u^*$. Thus we have derived the growth functions $u$ and $u^*$ used by Kondratiev and Streit [6] [7].

**Example 3.11.** Let $u(r) = \exp_k(r) = \exp(\exp(\cdots(\exp(r))))$ be the $k$-th iteration of the exponential function. It is shown in [3] that the dual Legendre transform $u^*$ of $u$ is equivalent to the following function

$$v(r) = \exp\left[2\sqrt{r \log_{k-1} \sqrt{r}}\right].$$

where $\log_j$ is the function defined by

$$\log_1(r) = \log(\max\{r, e\}), \quad \log_j(r) = \log_1(\log_{j-1}(r)), \quad j \geq 2.$$  

The CKS-space arising from the function $v$ turns out to be the same as the one defined by the Bell numbers $\{b_k(n)\}$ of order $k$. The function $v$ serves as a growth function for the characterization theorem of test functions in this CKS-space. The function $v^*$ is equivalent to the function $u(r) = \exp_k(r)$. Hence $\exp_k$ can be used as a growth function for the characterization theorem of generalized functions in this CKS-space.

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