Permutation Symmetry $S_3$ and 
VEV Structure of Flavor-Triplet Higgs Scalars

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Abstract

A model with flavor-triplet Higgs scalars $\phi_i$ ($i = 1, 2, 3$) is investigated under a permutation symmetry $S_3$ and its symmetry breaking. A possible $S_3$ breaking form of the Higgs potential whose vacuum expectation values $v_i = \langle \phi_i \rangle$ satisfy a relation $v_1^2 + v_2^2 + v_3^2 = \frac{2}{3}(v_1 + v_2 + v_3)^2$ is investigated, because if we suppose a seesaw-like mass matrix model $M_e = mM^{-1}m$ with $m_{ij} \propto \delta_{ij}v_i$ and $M_{ij} \propto \delta_{ij}$, such a model can lead to the well-known charged lepton mass relation $m_e + m_\mu + m_\tau = \frac{2}{3}(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2$.

1 Introduction

Of the observed mass spectra of the fundamental particles, quarks and leptons, the charged lepton mass spectrum seems to give a promising clue to the unified understanding of quarks and leptons, because the observed charged lepton masses satisfy a very simple mass relation \[ m_e + m_\mu + m_\tau = \frac{2}{3}(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2, \] with remarkable precision. The mass formula (1.1) can give an excellent prediction of the tau lepton mass value \[ m_\tau = 1776.97 \text{ MeV}, \] from the observed electron and muon mass values $[4]$, $m_e = 0.51099892 \text{ MeV}$ and $m_\mu = 105.658369 \text{ MeV}$ (c.f. the observed value $[4]$ $m_\tau = 1776.99^{+0.20}_{-0.26} \text{ MeV}$). This excellent agreement seems to be beyond a matter of accidental coincidence, so that we should consider the origin of the mass formula (1.1) seriously. Several authors $[6, 7, 8]$ have challenged to give an explanation of the mass formula (1.1) from a geometrical point of view. However, up to the present, the theoretical basis of the mass formula (1.1) is still not clear. (For a review, for example, see Ref. $[8, 9]$.)

The charged lepton mass formula (1.1) has the following peculiar features:
(a) The mass formula is described in terms of the root squared masses $\sqrt{m_{ei}}$.
(b) The formula is well satisfied at a low energy scale rather than at a high energy scale.
(c) The mass formula is invariant under the exchanges $\sqrt{m_{ei}} \leftrightarrow \sqrt{m_{ej}}$.

The feature (a) suggests that the charged lepton mass spectrum is given by a bilinear form on the basis of some mass-generation mechanism. For example, in Refs. $[10, 11, 12, 13]$, the formula (1.1) has been discussed on the basis of a seesaw-like mechanism $[13]$: \[ M_e = mM_E^{-1}m^t, \]
where $M_E$ is a heavy charged lepton mass matrix $M_E \propto \text{diag}(1,1,1)$, and $m$ is given by $m \propto \text{diag}(v_1, v_2, v_3)$ ($v_i$ are vacuum expectation values (VEVs) of flavor-triplet scalars $\phi_i$). This idea that mass spectrum is due not to the structure of the Yukawa coupling constants, but to the VEV structure of Higgs scalars $\phi_i$ at a low energy scale is very attractive as an explanation of the feature (b). We have to seek for a model where the VEVs $v_i$ satisfy the following relation

$$v_1^2 + v_2^2 + v_3^2 = \frac{2}{3} (v_1 + v_2 + v_3)^2.$$  \hspace{1cm} (1.4)

The feature (c) suggests that the Higgs potential is invariant under a permutation symmetry $S_3$. Such attempts to understand the charged lepton mass spectrum from the VEV structure of flavor-triplet scalars are found in Refs.\[3, 11, 12\]. The basic idea is as follows: We consider the following $S_3$ invariant Higgs potential

$$V = \mu^2 \sum_i (\bar{\phi}_i \phi_i) + \frac{1}{2} \lambda_1 \left( \sum_i (\bar{\phi}_i \phi_i) \right)^2 + \lambda_2 (\bar{\phi}_\pi \phi_\pi + \bar{\phi}_\eta \phi_\eta),$$  \hspace{1cm} (1.5)

where $(\bar{\phi}_i \phi_i) = \phi_i^- \phi_i^+ + \phi_i^0 \phi_i^0$ ($i = 1, 2, 3$), and $(\phi_\pi, \phi_\eta)$ and $\phi_\sigma$ are a doublet and a singlet in the real basis of $S_3$, respectively:

$$\phi_\pi = \frac{1}{\sqrt{2}} (\phi_1 - \phi_2),$$

$$\phi_\eta = \frac{1}{\sqrt{6}} (\phi_1 + \phi_2 - 2\phi_3),$$

$$\phi_\sigma = \frac{1}{\sqrt{3}} (\phi_1 + \phi_2 + \phi_3).$$  \hspace{1cm} (1.6)

Although, in Refs.\[3, 11, 12\], we have regarded the fields $\phi_i$ as SU(2)$_L$ doublet scalars, we can also consider another model which gives the seesaw form (1.3). For example, by considering a model shown in Fig. 1, we can regard the fields $\phi_i$ as SU(2)$_L$ singlet scalars. Hereafter, the SU(2)$_L$ structure in the Higgs potential will be neglected. The fields $\phi_i$ denote both cases, a case of the SU(2)$_L$ doublets and a case of the SU(2)$_L$ singlets.

The conditions that the potential (1.5) takes the minimum lead to the relation for the VEVs $v_i \equiv \langle \phi_i^0 \rangle$

$$|v_\sigma|^2 = |v_\pi|^2 + |v_\eta|^2 = \frac{-\mu^2}{2\lambda_1 + \lambda_2}.$$  \hspace{1cm} (1.7)

Therefore, from the relation

$$\bar{\phi}_1 \phi_1 + \bar{\phi}_2 \phi_2 + \bar{\phi}_3 \phi_3 = \bar{\phi}_\pi \phi_\pi + \bar{\phi}_\eta \phi_\eta + \bar{\phi}_\sigma \phi_\sigma,$$  \hspace{1cm} (1.8)

we obtain

$$|v_1|^2 + |v_2|^2 + |v_3|^2 = |v_\pi|^2 + |v_\eta|^2 + |v_\sigma|^2 = 2 |v_\sigma|^2 = 2 \left( \frac{v_1 + v_2 + v_3}{\sqrt{3}} \right)^2.$$  \hspace{1cm} (1.9)
Thus, we obtain the relation (1.4).

However, note that (i) the Higgs potential (1.5) which is invariant under the permutation symmetry $S_3$ is not a general form of the $S_3$ invariant Higgs potential, and (ii) it cannot give a relation between $v_\pi = \langle \phi_\pi \rangle$ and $v_\eta = \langle \phi_\eta \rangle$, because the $S_3$ invariant Higgs potential (1.5) is also invariant under the permutation between $\phi_\pi$ and $\phi_\eta$ (we cannot determine the values $v_\pi$ and $v_\eta$ individually). We have to investigate a potential term which violates a $\phi_\pi \leftrightarrow \phi_\eta$ symmetry (also breaks the $S_3$ symmetry) but keeping the relation (1.4).

In order to obtain the charged lepton mass relation (1.1), we have to build a model with a seesaw-type mass matrix (1.3). A recent attempt to build such a model will be found in Ref.[14]. However, the purpose of the present paper is not to investigate such a seesaw mass matrix model. The purpose of the present paper is to discuss the Higgs potential form which gives the structure (1.4).

## 2 $S_3$ symmetric Higgs potential

In this section, we discuss a general form of the $S_3$ symmetric Higgs potential. However, for mass terms, we confine ourselves to the case $\mu^2 \sum \bar{\phi}_i \phi_i$ as given in Eq. (1.5). We consider an $S_3$-invariant general form only for the dimension-four terms. In general, we have two scalars $(\bar{\phi}_\sigma \phi_\sigma)$ and $(\bar{\phi}_\pi \phi_\pi + \bar{\phi}_\eta \phi_\eta)$ and one pseudo-scalar $(\bar{\phi}_\pi \phi_\eta - \bar{\phi}_\eta \phi_\pi)$ [15], so that we write the general form of the $S_3$-invariant Higgs potential as follows:

$$
V = \mu^2 \left( \bar{\phi}_1 \phi_1 + \bar{\phi}_2 \phi_2 + \bar{\phi}_3 \phi_3 \right) + \frac{1}{2} \lambda_\sigma (\bar{\phi}_\sigma \phi_\sigma)^2 + \frac{1}{2} \lambda_+ (\bar{\phi}_\pi \phi_\pi + \bar{\phi}_\eta \phi_\eta)^2 \\
+ \frac{1}{2} \lambda_- (\bar{\phi}_\pi \phi_\eta - \bar{\phi}_\eta \phi_\pi)^2 + \lambda_2 (\bar{\phi}_\sigma \phi_\sigma)(\bar{\phi}_\pi \phi_\pi + \bar{\phi}_\eta \phi_\eta) \\
+ \lambda_3 \left[ (\bar{\phi}_\pi \phi_\pi)(\bar{\phi}_\eta \phi_\eta) + (\bar{\phi}_\pi \phi_\eta)(\bar{\phi}_\pi \phi_\sigma) + (\bar{\phi}_\eta \phi_\pi)(\bar{\phi}_\pi \phi_\sigma) - (\bar{\phi}_\eta \phi_\eta)(\bar{\phi}_\eta \phi_\sigma) + h.c. \right].
$$

(2.1)

Here, for simplicity, we have denoted the case that $\phi_i$ are SU(2)$_L$ singlets. If the fields $\phi_i$ are SU(2)$_L$ doublets, the general form includes, for example, $(\bar{\phi}_\sigma \phi_\pi)(\bar{\phi}_\pi \phi_\sigma) + (\bar{\phi}_\sigma \phi_\eta)(\bar{\phi}_\eta \phi_\sigma)$ in addition to $(\bar{\phi}_\sigma \phi_\sigma)(\bar{\phi}_\pi \phi_\pi + \bar{\phi}_\eta \phi_\eta)$, and so on.

The Higgs potential (2.1) cannot, in general, lead to the relation (1.4). Especially, the $\lambda_3$-terms badly spoil the relation (1.4). Only when $\lambda_3 = 0$, the potential (2.1) leads to a simple relation

$$
\frac{v_\pi^2 + v_\eta^2}{v_\pi^2} = \frac{\lambda_2 - \lambda_\sigma}{\lambda_2 - \lambda_+} v_\sigma^2,
$$

(2.2)

so that we get the relation

$$
v_1^2 + v_2^2 + v_3^2 = \frac{1}{3} K (v_1 + v_2 + v_3)^2,
$$

(2.3)

where

$$
K = 1 + \frac{\lambda_2 - \lambda_\sigma}{\lambda_2 - \lambda_+}.
$$

(2.4)
Thus, the case with
\[ \lambda_3 = 0, \quad \lambda_\sigma = \lambda_+ \neq \lambda_2, \]
can give the relation (1.4).

The Higgs potential (2.1) with the conditions (2.5) is essentially identical with the Higgs potential (1.5). (Hereafter, we will use the expression (1.5) as the \( S_3 \)-invariant Higgs potential which can gives the relation (1.4).) The characteristic of the Higgs potentials (1.5) [and also (2.1) with the constraints (2.5)] which can give the relation (1.4) is that it is invariant under the replacement
\[ (\bar{\phi}_\sigma \phi_\sigma) \leftrightarrow (\bar{\phi}_\pi \phi_\pi + \bar{\phi}_\eta \phi_\eta). \]

As we stated in Sec.1, the potential (1.5) cannot give the difference between \( v_\pi \) and \( v_\eta \) and it generates a massless scalar, because the potential (1.5) is invariant under an SU(2)-flavor symmetry for the basis \( (\phi_\pi, \phi_\eta) \). We have to introduce a symmetry breaking of the SU(2)-flavor.

3 S\(_3\) symmetry breaking in the Higgs potential

In this section, we investigate an \( S_3 \) symmetry breaking term which does not spoil the relation (1.4).

By the way, when we define parameters \( z_i \) by \( v_i = z_i v \) with the normalization condition \( z_1^2 + z_2^2 + z_3^2 = 1 \), the parameters \( z_i \) which satisfy the relation
\[ z_1^2 + z_2^2 + z_3^2 = 1 = \frac{2}{3} (z_1 + z_2 + z_3)^2, \]
are explicitly expressed as follows [11]:
\[ z_1 = \frac{1 - \sqrt{1 - \varepsilon}}{\sqrt{6}}, \quad z_2 = \frac{2 + \sqrt{1 - \varepsilon} - \sqrt{3} \sqrt{1 + \varepsilon}}{2 \sqrt{6}}, \quad z_3 = \frac{2 + \sqrt{1 - \varepsilon} + \sqrt{3} \sqrt{1 + \varepsilon}}{2 \sqrt{6}}, \]
where the expression (3.2) has been so taken as to give \( m_e \to 0 \) in the limit of \( \varepsilon \to 0 \), and the value of \( \varepsilon \) is
\[ \varepsilon = 0.079072, \]
from the observed values of the charged lepton masses. Also, for the parameters \( z_a = v_a/v \) (\( a = \pi, \eta, \sigma \)), we obtain
\[ z_\pi = -\frac{1}{4} \left( \sqrt{3} \sqrt{1 - \varepsilon} - \sqrt{1 + \varepsilon} \right), \quad z_\eta = -\frac{1}{4} \left( \sqrt{3} \sqrt{1 + \varepsilon} + \sqrt{1 - \varepsilon} \right), \quad z_\sigma = \frac{1}{\sqrt{2}}. \]

We must to seek for the potential form which gives \( z_\pi \neq z_\eta \) keeping the relation (3.1).

As an example of such an \( S_3 \) symmetry breaking term, in Ref. [12], a term
\[ V_{SB} = \lambda_{SB} \left[ \xi_\pi (\bar{\phi}_\pi \phi_\pi) - \xi_\eta (\bar{\phi}_\eta \phi_\eta) \right]^2, \]
with \( \xi_\pi \neq \xi_\eta \) has been suggested. However, in this paper, we would like to consider a case that the symmetry is softly broken. Therefore, in this section, we consider a symmetry breaking in the mass term in the Higgs potential (1.5).
The Higgs potential is invariant under the SU(2)-flavor symmetry for the basis \((\phi_\pi, \phi_\eta)\), i.e. under the transformation

\[
\begin{pmatrix}
\phi'_\pi \\
\phi'_\eta
\end{pmatrix} =
\begin{pmatrix}
c & -s \\
s & c
\end{pmatrix}
\begin{pmatrix}
\phi_\pi \\
\phi_\eta
\end{pmatrix},
\]

(3.6)

where \(s = \sin \theta\) and \(c = \cos \theta\). Now, we want to fix the mixing (3.6) to a special basis. Therefore, we take a symmetry breaking form

\[
V_{SB} = \mu_{SB}^2 (\bar{\phi}'_\pi \phi'_\pi) = \mu_{SB}^2 (c \bar{\phi}_\pi - s \bar{\phi}_\eta)(c \phi_\pi - s \phi_\eta).
\]

(3.7)

This does not mean that the SU(2)-flavor invariant potential (1.5) is also given in terms of the new basis \((\phi'_\pi, \phi'_\eta, \phi_\sigma)\). We assume that the potential \(V\), Eq. (1.5), (hereafter, we call it \(V_0\)) is still given in terms of the basis \((\phi_\pi, \phi_\eta, \phi_\sigma)\), while, only for \(V_{SB}\), it is given by the form (3.7). Therefore, we regard the Higgs potential

\[
V = V_0 + V_{SB}
\]

(3.8)

as a function of the fields \((\phi_\pi, \phi_\eta, \phi_\sigma)\), and the mixing parameter \(\theta\) as a fundamental parameter in the present model.

Then, we obtain

\[
\begin{align*}
\left[ \mu^2 + \lambda_1 (|v_\pi|^2 + |v_\eta|^2 + |v_\sigma|^2) + \lambda_2 |v_\sigma|^2 \right] v_\pi + \mu_{SB}^2 c (c v_\pi - s v_\eta) &= 0, \\
\left[ \mu^2 + \lambda_1 (|v_\pi|^2 + |v_\eta|^2 + |v_\sigma|^2) + \lambda_2 |v_\sigma|^2 \right] v_\eta - \mu_{SB}^2 s (c v_\pi - s v_\eta) &= 0, \\
\left[ \mu^2 + \lambda_1 (|v_\pi|^2 + |v_\eta|^2 + |v_\sigma|^2) + \lambda_2 (|v_\pi|^2 + |v_\eta|^2) \right] v_\sigma &= 0,
\end{align*}
\]

(3.9-3.11)

from the conditions \(\partial V / \partial \bar{\phi}_\pi = 0\), \(\partial V / \partial \bar{\phi}_\eta = 0\), and \(\partial V / \partial \bar{\phi}_\sigma = 0\), respectively. Therefore, for \(\lambda_1 \neq 0\), \(\lambda_2 \neq 0\), and \(\mu_{SB}^2 \neq 0\), we obtain the relations

\[
|v_\pi|^2 + |v_\eta|^2 = |v_\sigma|^2,
\]

(3.12)

and

\[
\frac{s}{c} = \frac{v_\pi}{v_\eta}.
\]

(3.13)

The relation (3.12) leads to the relation (1.4). The relation (3.13) means

\[
\tan \theta = \frac{\sqrt{3}(z_2 - z_1)}{2z_3 - z_2 - z_1} = \frac{\sqrt{3}(\sqrt{m_\mu} - \sqrt{m_e})}{2\sqrt{m_\tau} - \sqrt{m_\mu} - \sqrt{m_e}},
\]

(3.14)

which gives a numerical result

\[
\theta = 12.7324^\circ, \quad \text{i.e.} \quad \sin \theta = 0.220398.
\]

(3.15)

Note that the mixing parameter value (3.15) is in an excellent agreement with the observed Cabibbo mixing angle, i.e. \(|V_{us}| = 0.2200 \pm 0.0026\) [4]. At present, this is an accidental coincidence, because we have not yet discussed a model of the quark mixing. (A formula for the
Cabibbo angle similar to Eq. (3.14) is found in Ref. [16]. However, this coincidence will become a hint for seeking for the quark mixing model. In the present stage, the value (3.15) of $\theta$ is only a phenomenological result from the observed charged lepton mass spectrum.

4 Concluding remarks

In conclusion, we have investigated a Higgs potential which gives the VEV structure (1.4). We have considered the following scenario for the Higgs potential:

(i) First, we consider SU(3)-flavor symmetric Higgs potential.
(ii) The SU(3) symmetric Higgs potential is broken by the $\lambda_2$-terms as shown in Eq. (1.5), but it is still invariant under the $S_3$ flavor symmetry.
(iii) The $S_3$ symmetric potential (1.5) is softly broken by the term (3.7) with a phenomenological mixing parameter $\theta$.

Then, we can obtain a realistic charged lepton mass spectrum for the parameter value (3.15). We consider that those symmetry breakings are explicitly broken at a high energy scale. In the present low energy phenomenology, we do not discuss the origin of those symmetry breakings.

The present model is a multi-Higgs model, so that the model basically induces the flavor-changing neutral currents (FCNC). However, the Higgs scalars $\phi_i$ in the present model do not couple to the quarks and leptons directly. Symbolically speaking, in the seesaw mass matrix model $M_f = m_L M_F^{-1} m_R$, the FCNC effects through the exchange of the Higgs scalars $\phi_{Li}$ are suppressed by the order of $(M_F^{-1} m_R)^2$. Therefore, we will be able to avoid the FCNC problem from the present seesaw model.

Finally, we would like to emphasize that the relation (1.4) can be obtained independently of the explicit values of the parameters $\lambda_1$, $\lambda_2$, and $\mu^2_{SB}$ in the Higgs potential. The relation (1.4) is determined only by the form (parameter-independent structure) of the Higgs potential. The explicit values of $z_i$ are dependent only on the value of $\varepsilon$ (so that on the value of the mixing parameter $\theta$). The magnitude of the parameter $\varepsilon$ of the $S_3$ violation should be understood from more fundamental theory in future. We believe that the charged lepton mass spectrum will be described only in terms of fundamental constants without adjustable parameters, while quark and neutrino mass matrices will be described in terms of such fundamental constants and some phenomenological parameters.

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References

[1] Y. Koide, Lett. Nuovo Cimento 34, 201 (1982).
[2] Y. Koide, Phys. Rev. D28, 252 (1983)
[3] Y. Koide, Mod. Phys. Lett. A5, 2319 (1990).
[4] S. Eidelman et al. (Particle Data Group), Phys. Lett. B592, 1 (2004).
[5] R. Foot, hep-ph/9402242

[6] S. Esposito and P. Santorelli, Mod. Phys. Lett. A10, 3077 (1995), hep-ph/9603369

[7] N. Li and B.-Q. Ma, Phys. Lett. B609, 309 (2005).

[8] A. Rivero and A. Gsponer, hep-ph/0505220 (2005).

[9] Y. Koide, hep-ph/0506247

[10] Y. Koide and H. Fusaoka, Z. Phys. C71, 459 (1996).

[11] Y. Koide and M. Tanimoto, Z. Phys. C72, 333 (1996).

[12] Y. Koide, Phys. Rev. D60, 077301 (1999).

[13] The seesaw mechanism for charged particles is known as the “universal seesaw mechanism”: Z. G. Berezhiani, Phys. Lett. 129B, 99 (1983); Phys. Lett. 150B, 177 (1985); D. Chang and R. N. Mohapatra, Phys. Rev. Lett. 58,1600 (1987); A. Davidson and K. C. Wali, Phys. Rev. Lett. 59, 393 (1987); S. Rajpoot, Mod. Phys. Lett. A2, 307 (1987); Phys. Lett. 191B, 122 (1987); Phys. Rev. D36, 1479 (1987); K. B. Babu and R. N. Mohapatra, Phys. Rev. Lett. 62, 1079 (1989); Phys. Rev. D41, 1286 (1990); S. Ranfone, Phys. Rev. D42, 3819 (1990); A. Davidson, S. Ranfone and K. C. Wali, Phys. Rev. D41, 208 (1990); I. Sogami and T. Shinohara, Prog. Theor. Phys. 66, 1031 (1991); Phys. Rev. D47, 2905 (1993); Z. G. Berezhiani and R. Rattazzi, Phys. Lett. B279, 124 (1992); P. Cho, Phys. Rev. D48, 5331 (1994); A. Davidson, L. Michel, M. L, Sage and K. C. Wali, Phys. Rev. D49, 1378 (1994); W. A. Ponce, A. Zepeda and R. G. Lozano, Phys. Rev. D49, 4954 (1994).

[14] Y. Koide, hep-ph/0508301

[15] For example, N. Haba and K. Yoshioka, hep-ph/0511108 (2005).

[16] Y. Koide, Phys. Rev. Lett. 47, 1241 (1981).
Fig. 1 Seesaw mass-generation of the charged leptons, where flavor-triplet scalars $\phi_i$ are singlets of SU(5).