Modified non-linear Schrödinger models, infinite tower of conserved charges and dark solitons

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Abstract. We show that the quasi-integrability concept holds for the modified defocusing NLS model with dark soliton solutions and it exhibits the new feature of an infinite sequence of alternating conserved and asymptotically conserved charges. For the special case of two dark soliton solutions, where the field components are eigenstates of a space-reflection symmetry, the first four and the sequence of even order charges are exactly conserved in the scattering process of the solitons. We perform extensive numerical simulations and consider the scattering of dark solitons for the cubic-quintic NLS model with potential \( V = \eta I^2 - \epsilon I^3 \) and the saturable type potential satisfying \( V'[I] = 2\eta I - \frac{\epsilon q}{I + I^q} \), \( q \in \mathbb{Z}_+ \), with a deformation parameter \( \epsilon \in \mathbb{R} \) and \( I = |\psi|^2 \). The saturable NLS supports elastic scattering of two soliton solutions for a wide range of values of \( \{\eta, \epsilon, q\} \). Our results may find potential applications in several areas of non-linear science, such as the Bose-Einstein condensation.

1. Introduction

Certain non-linear field theory models with relevant physical applications possess soliton-like solutions and it is difficult to know a priori if they are in fact true solitons [1]. The so-called quasi-integrability concept has recently been put forward in the context of certain deformations of the integrable sine-Gordon (SG), Bullough-Dodd (BD) and non-linear Schrödinger (NLS) models [2, 3, 4]. The idea is that many non-integrable theories possess solitary wave solutions that behave similar to solitons, i.e. the scattering of such solitons basically preserve their shapes and velocities.

In this context, our aim is to predict the results of solitary wave collisions and test the quasi-
integrability concept in deformed defocusing NLS models. The deformed focusing NLS with bright soliton solutions and the structures responsible for the phenomenon of quasi-integrability have been discussed in [3]. It has been shown that this model possesses an infinite number of asymptotically conserved charges. On the other hand, the deformed defocusing NLS with dark soliton solutions also presents the above remarkable properties, however in this case we can say even more [4]. There are some soliton like solutions which present a special space-reflection symmetry for any time, so this property implies the exact conservation of the sequence of the even order charges, which in the absence of this special symmetry would be conserved only asymptotically, as in the models studied in [2, 3]. However, it seems to be that such parity property is a sufficient but not a necessary condition in order to have the sequence of exactly conserved charges. In fact, it has been shown, by numerical simulations, that there are certain soliton like configurations without this symmetry which also exhibit such conserved charges [4]. So, these properties constitute the distinguishing new features associated to the deformed defocusing NLS with dark soliton solutions, as compared to the previous quasi-integrable models [2, 3].

In order to simulate the dark soliton collisions the so-called time-splitting cosine pseudo-spectral finite difference (TSCP) and the time-splitting finite difference with transformation (TSFD-T) methods [5] have been used. In fact, these methods allowed us to improve in several orders of magnitude the accuracy in the computation of the charges and anomalies presented in [3].

The paper is organized as follows. In the next section we discuss the defocusing NLS model, its 2-dark soliton solutions and the space-reflection parity symmetry. In section 3, we discuss the first conservation laws. In section 4 we construct analytically the quasi-conserved charges and anomalies based on analytical results. It is introduced an anomalous zero-curvature equation based on the $sl(2)$ Kac-Moody (KM) loop algebra. It is defined the first non-trivial quasi-conservation law. In section 5 we present the results of our numerical simulations and the computations of the anomaly $\beta^{(4)}$ for dark soliton scattering of the modified NLS model. We present the results of the deformations with potentials $V = \eta|\psi|^4 - \frac{\epsilon}{9}|\psi|^6$ and $V'[I] = 2\eta I - \epsilon \frac{I^3}{1+\gamma}$. In section 6 we present some conclusions.

2. The non-linear Schrödinger model (NLS)

The NLS model has many applications in non-linear physics: optics, condensed matter (e.g. Bose-Einstein condensation-BEC), plasma physics, hydrodynamics, 2D gravity, analogs of gravitation in BECs, etc. The NLS model is a well known integrable system defined by

$$i \frac{\partial}{\partial t} \psi(x,t) + \frac{\partial^2}{\partial x^2} \psi(x,t) - 2\eta |\psi(x,t)|^2 \psi(x,t) = 0, \quad \psi(x,t) \in C. \quad (2.1)$$

Let us summarize some properties of the eq. (2.1):
- For $\eta < 0$, one has the focusing NLS, which supports bright soliton solutions.
- For $\eta > 0$, it is the defocusing NLS, which possesses dark soliton solutions [6].
- The model (2.1) is an integrable system with reach mathematical and physical structures: N-soliton solutions, infinite number of conserved charges, solvable by the inverse scattering method (IST), Backlund transformations, (bi-tri-) Hamiltonian formulation, etc.
2.1. (Defocusing) Non-linear Schrödinger model and dark solitons

The 1-dark soliton of the defocusing NLS eq. (2.1) becomes [4, 7]

$$\psi(x, t) = \frac{|\psi_0|}{2} \exp \left[ i(kx + wt + x_0) \right] \left\{ 1 + y + (y - 1) \tanh \left[ \frac{\theta(x, t) + \bar{\theta}(x, t)}{2} \right] \right\} \quad (2.2)$$

This solution possesses three arbitrary real parameters, say $|\psi_0|, k$ and the phase associated to $y = e^{2i\delta}$. The intensity function $|\psi|$ moves at the velocity $v = -2p_1, (p \equiv p_R + ip_I)$, which is the velocity of the dark soliton. The dark soliton approaches constant amplitude $|\psi_0|$ as $|x| \to \pm \infty$. As $x$ varies from $-\infty$ to $+\infty$ the soliton acquires a $2\delta$ phase. We restrict $-\pi/2 < \delta < \pi/2$. Moreover, at the center of the soliton, one has that the intensity becomes $|\psi|_{\text{center}} = |\psi_0| \cos \delta$. This center intensity is lower than the asymptotic amplitude $|\psi_0|$ and this property characterizes a dark soliton. Notice that this center intensity is controlled by the parameter $\delta$; i.e. this parameter defines the “darkness” of the soliton.

2.2. Space-reflection parity transformation

Let us write the function $\psi$ as $\psi = \sqrt{R}e^{i\frac{\varphi}{2}}$ such that $\varphi/2 = (kx + wt + x_0) + \phi + \delta$. It is interesting to see the behaviour of the fields $R$ and $\phi$ for the two-soliton solutions possessing a space-reflection symmetric amplitude $R(-\bar{x}, \bar{t}) = R(\bar{x}, \bar{t})$ and phase $\phi(-\bar{x}, \bar{t}) = \phi(\bar{x}, \bar{t})$ for any given shifted space and time, $\bar{x} = x - x_p$, and $\bar{t} = t - t_p$. Consider the next 2-dark soliton solution [4, 7]

$$R(\bar{x}, \bar{t}) = \frac{1}{4}|\psi_0|^2 \left[ N(x_1, t_1)N(x_1, t_1) \right]^{*}; \quad x_1 = 2p_1R\bar{x}, \quad t_1 = 4p_1Rt_1$$

$$\phi(\bar{x}, \bar{t}) = \arctan \left[ \frac{\sin 2\delta_1 \sinh t_1}{e^{-\Delta/2} \cosh x_1 + \cosh t_1} \right], \quad (2.3)$$

$$x_0 = -\Delta - 2\Re(\theta_0^1) - 2\Re(\theta_0^2) \quad \text{and} \quad t_0 = \frac{\Re(\theta_0^2) - \Re(\theta_0^1)}{8p_1Rt_1} \quad (2.4)$$

In Fig. 1 we plot the functions $R$ and $\phi$ for three successive times. Note that they are symmetric under the reflection $\bar{x} \to -\bar{x}$ for any value of the variable $\bar{t} = t$. Therefore, it is clear that the above solution satisfies

$$P_x: \quad R(-\bar{x}, \bar{t}) \to R(\bar{x}, \bar{t}), \quad \phi(-\bar{x}, \bar{t}) \to \phi(\bar{x}, \bar{t}) \quad (2.5)$$

This type of solution will be associated to the even order ($n \geq 4$) exactly conserved charges as discussed below. In our numerical simulations we will verify the properties for this kind of solutions of deformed NLS models, as well as for configurations without this type of symmetry.
Figure 1. (color online) The amplitude $R = |\psi|^2$ (a) and phase $\phi$ (b) ($\psi = \sqrt{Re^{i\phi}}$) with space-reflection symmetries for 2-solitons sent at $v_1 = -v_2 = 2, v_s = 4, |\psi_0| = 2, \eta = 1$, for initial $t_i$ (green), collision $t_c$ (dashed) and final $t_f$ (red) times, respectively. At $t_c$ the solitons have merged and their phases cancel each other. After collision, $t_f$, each soliton again reveals its phase rise or fall.

3. Conservation laws

The infinite number of conserved charges of the integrable NLS model satisfy

$$\frac{dQ^{(n)}}{dt} = 0, \quad n = 0, 1, 2, 3, ... \quad (3.6)$$

The first conserved charges of the NLS model become

$$Q^{(0)} = -i \int dx \left[ \bar{\psi} \partial_x \psi - \psi \bar{\partial}_x \bar{\psi} \right], \quad (3.7)$$

$$Q^{(1)} = 2\eta \int_{-\infty}^{+\infty} dx |\psi|^2 \quad (3.8)$$

$$Q^{(2)} = 2\eta \int_{-\infty}^{+\infty} dx \left[ \bar{\psi} \partial_x \psi - \psi \bar{\partial}_x \bar{\psi} \right], \quad (3.9)$$

$$Q^{(3)} = \int_{-\infty}^{+\infty} dx \left[ |\partial_x \psi|^2 + \frac{1}{2} \eta |\psi|^4 \right]. \quad (3.10)$$

These charges correspond to the conservation of the physical quantities: $Q^{(0)} =$ phase; $Q^{(1)} =$ normalization; $Q^{(2)} =$momentum; $Q^{(3)} =$energy. In the case of dark solitons some charges need to be renormalized; e.g. $Q^{(1)} = 2 \int_{-\infty}^{+\infty} dx |\psi|^2 \left( 1 - \frac{|\psi_0|^2}{|\psi|^2} \right)$. Since NLS is an integrable system, it possesses infinite number of conserved charges $Q^{(n)}, n = 0, 1, 2, ...$ and N-dark soliton solutions.

4. Quasi-conserved charges and anomalies: analytical results

4.1. (Anomalous) zero-curvature equation in the $sl(2)$ Kac-Moody (KM) loop algebra

The analytical study of the properties of the deformed NLS model (2.1) can be done using the well known techniques from the integrable field theories. We follow the developments and notations
put forward in [2, 3] on quasi-integrability. Then, one considers an anomalous zero curvature representation of the deformed NLS model (2.1) with the deformed gauge connections (Lax pair) \( \{ A_x, A_t \} \) given for the NLS complex field \( \psi(\tilde{\psi}) \) and deformed potential \( V(|\psi|^2) \).

\[
\begin{align*}
A_x &= -i T_3^1 + \tilde{\gamma} \tilde{\psi} T_+^0 + \gamma \psi T_+^0; \\
A_t &= i T_3^2 + i \frac{\delta V}{\delta |\psi|^2} T_+^0 - (\tilde{\gamma} \tilde{\psi} T_+^1 + \gamma \psi T_+^1) - i(\gamma \partial_x \tilde{\psi} T_+^0 - \gamma \partial_x \psi T_+^0), \\
F_{xt} &= \partial_t A_x - \partial_x A_t + [A_x, A_t] \\
&= XT_3^0 + i \gamma \left[ -i \partial_t \tilde{\psi} + \partial_x^2 \tilde{\psi} - \tilde{\psi} \frac{\delta V}{\delta |\psi|^2} \right] T_+^0 - i \gamma \left[ i \partial_t \psi + \partial_x^2 \psi - \psi \frac{\delta V}{\delta |\psi|^2} \right] T_0^0 \\
X &= -i \partial_x \left[ \frac{\delta V}{\delta |\psi|^2} - 2 \eta |\psi|^2 \right], \quad \eta \equiv \tilde{\gamma} \gamma.
\end{align*}
\]

We have used the \( \hat{sl}(2) \) affine KM algebra generators: \( \{ T_\pm^\alpha, T_\mp^\beta \} \). The anomaly is defined by \( X \), and the non-vanishing curvature by \( F_{xt} \). Therefore, for a deformation of the NLS model defined by

\[
i \partial_t \psi + \partial_x^2 \psi - \psi \frac{\delta V}{\delta |\psi|^2} = 0, \quad (4.11)
\]

one has that \( F_{xt} \neq 0 \). For the usual NLS one has: \( V(|\psi|^2) = \gamma \gamma (|\psi|^2)^2 \), which implies the vanishing of the anomaly \( X \equiv 0 \). Therefore, one has the usual result \( F_{xt} \equiv 0 \), for NLS field configurations.

In order to study the quasi-conservation laws it has been applied the abelianization procedure, which amounts to gauge transform \( A_\mu \rightarrow a_\mu = g A_\mu g^{-1} + \partial_\mu g g^{-1} \) such that the next anomalous conservation laws arise [3]

\[
\frac{\partial}{\partial t} a_\alpha^{(3,-n)} - \frac{\partial}{\partial x} a_\alpha^{(3,-n)} = X a^{(3,-n)}; \quad n = 0, 1, 2, ...
\]

(4.12)

So, one has asymptotically conserved charges

\[
\frac{dQ^{(n)}}{dt} = \beta^{(n)}; \quad Q^{(n)} = -i \int_{-\infty}^{\infty} dx a_\alpha^{(3,-n)}; \quad \beta^{(n)} = -i \int_{-\infty}^{\infty} dx X a^{(3,-n)}.
\]

(4.13)

For the usual (focusing or defocusing) NLS one has \( \beta^{(n)} = 0 \Rightarrow Q^{(n)} = \text{constant}, \ n = 0, 1, 2, 3, 4, 5, ..... \)

4.2. First non-trivial quasi-conservation law

Except for relevant renormalization procedures, the form of the first three charges \( Q^{(i)}, (i = 0, 1, 2) \) of (3.7)-(3.9) remain the same for the deformed NLS model (2.1). The next order charge \( Q^{(3)} \) also requires a renormalization procedure. In fact, the renormalized Hamiltonian associated to the dark soliton itself for \( k = 0 \) becomes [6]

\[
H = \int_{-\infty}^{+\infty} dx \left\{ |\partial_x \psi|^2 + \int_{|\psi_0|^2}^{2} dI \left[ V'[I] - V'[|\psi_0|^2] \right] \right\}.
\]

(4.14)
So, the renormalized Hamiltonian of the usual NLS model \((V = \eta |\psi|^4\)) can be written as

\[
H_{NLS} = \int_{-\infty}^{+\infty} dx \left\{ |\partial_x \psi|^2 + \eta (|\psi|^2 - |\psi_0|^2)^2 \right\}.
\] (4.15)

The next fourth order charge \(Q^{(4)}\) presents the first non-trivial anomaly, i.e.

\[
\frac{dQ^{(4)}}{dt} = \beta^{(4)}.
\] (4.16)

The charge \(Q^{(4)}\) and anomaly \(\beta^{(4)}\), respectively, in the parametrization \(\psi = \sqrt{R}e^{i\varphi/2}\) are

\[
Q^{(4)} = \frac{\eta}{4} \int_{-\infty}^{+\infty} dx \left[ 12\eta R^2 \partial_x \varphi + 3\partial_x \varphi \left( \frac{\partial_x R^2}{R} \right)^2 + R\left( \frac{\partial_x \varphi}{x} \right)^3 - 4R^3 \right]
\] (4.17)

\[
\beta^{(4)} \equiv -\eta \int_{-\infty}^{+\infty} dx [V''(R) - 2\eta] \left\{ \frac{3}{4} \partial_x R^2 \left( \partial_x \varphi \right)^2 - \partial_x (\partial_x R)^2 + \frac{3}{2} \left( \frac{\partial_x R}{R} \right)^3 \right\}
\] (4.18)

\[
\int_{-\infty}^{+\infty} dx \gamma(x, t), \quad \gamma(x, t) \equiv \text{anomaly density}
\] (4.19)

\[
Q^{(4)}(t_0) - Q^{(4)}(0) = \int_{t_0}^{t_0} \beta^{(4)}(t) dt = \int_{0}^{t_0} dt \int_{-\infty}^{+\infty} dx \gamma(x, t).
\] (4.20)

Notice that for soliton solutions with the symmetry (2.5) the anomaly density of \(\gamma(x, t)\) becomes an odd function of \(x\). Therefore, the anomaly \(\beta^{(4)}\) would vanish for soliton configurations presenting this type of symmetry. This would imply the exact conservation of the charge \(Q^{(4)}\) for this type of soliton configurations.

5. Examples and numerical results

Next we will consider some modified NLS models (4.11) and numerically simulate the behaviour of the quasi-conservation law (4.16) under the scattering of two dark solitons.

5.1. Modified (defocusing \(\eta > 0\)) non-linear Schrödinger (MNLS)

Next we discuss some particular modifications (defocusing \(\eta > 0\)) of the NLS equations (MNLS) defined for the deformation parameter \(\epsilon\).

**Cubic-quintic potential NLS:** \(V(|\psi|^2) = \eta (|\psi|^2)^2 - \frac{\epsilon}{6} (|\psi|^2)^3\)

\[
i \frac{\partial}{\partial t} \psi(x, t) + \frac{\partial^2}{\partial x^2} \psi(x, t) - \left[ 2\eta |\psi(x, t)|^2 - \frac{\epsilon}{2} (|\psi(x, t)|^2)^2 \right] \psi(x, t) = 0, \quad \psi(x, t) \in C.
\] (5.21)

**Saturable-type potential:** \(V(|\psi|^2) = 2\eta |\psi|^2 - \frac{\epsilon(|\psi|^2)^q}{1 + (|\psi|^2)^q}, \quad q \in \mathbb{Z}\)

\[
i \frac{\partial}{\partial t} \psi(x, t) + \frac{\partial^2}{\partial x^2} \psi(x, t) - \left[ 2\eta |\psi(x, t)|^2 - \frac{\epsilon(|\psi(x, t)|^2)^q}{1 + (|\psi(x, t)|^2)^q} \right] \psi(x, t) = 0.
\] (5.22)

Notice that in the both cases one has that in the limit \(\epsilon = 0\) one recovers the usual NLS model.
The cubic-quintic modified NLS system (5.21) possesses the solitary waves
\[
\Phi_\pm(z) = \frac{\xi_1 + r\xi_2 \tanh^2[k^\pm(z - z_0)]}{1 + r \tanh^2[k^\pm(z - z_0)]}
\]
where
\[
r = \frac{|\psi_0|^2 - \xi_1}{\xi_2 - |\psi_0|^2}, \quad k^\pm = \sqrt{\frac{\epsilon}{6} \sqrt{(\pm)(\xi_2 - |\psi_0|^2)(|\psi_0|^2 - \xi_1)}},
\]
\[
\xi_1 = \frac{B - \sqrt{B^2 - 6\psi_0^2\epsilon}}{2\epsilon}, \quad \xi_2 = \frac{B + \sqrt{B^2 - 6\psi_0^2\epsilon}}{2\epsilon}, \quad B \equiv 6\eta - 2\epsilon|\psi_0|^2.
\]

The signs \(\pm\) in \(\Phi_\pm\) and \(\Theta_\pm\) correspond to the case \(\epsilon > 0\) and \(\epsilon < 0\), respectively. The parameter \(\xi_1\) is always positive and \(\xi_2\) becomes positive (negative) for \(\epsilon > 0\) (\(\epsilon < 0\)). For \(\epsilon > 0\) they satisfy \(\xi_1 < |\psi_0|^2 < \xi_2\), whereas for \(\epsilon < 0\) it holds \(\xi_2 < \xi_1 < |\psi_0|^2\). Notice that \(k^\pm\) characterizes the inverse soliton width and \(\sqrt{\xi_1}\) is the minimum intensity (dip) of the dark soliton. The maximum intensity of the dark soliton approaches \(\Phi_2(\pm\infty) = |\psi_0|\). Moreover, for \(v = 0\) the parameter \(\xi_1\) vanishes and the dark soliton becomes a black soliton.

For the second deformed model (5.22), with \(\epsilon \neq 0, q \in \mathbb{Z}_+\), we do not know any analytical expression for its solitary wave; however, one can generate numerically a solitary wave.

One can study the dark soliton collisions numerically for the both deformed NLS models and compare the outcomes with the properties for the integrable NLS model. Then, we can take two one dark solitons located some distance apart as the initial condition for our numerical simulations of soliton scattering.

In the Figs. 2-5 we present the numerical simulation of the collision of two dark solitons in the cubic-quintic NLS model (5.21). The two-soliton collision with equal and opposite velocities (equal amplitudes) and parameter \(\epsilon = -0.01\) is shown in Fig. 2. The anomaly density \(\gamma(x, t)\) in (4.19) has been plotted (Fig. 3) as function of the space variable \(x\) for three successive times. The anomaly (4.18) (Fig. 4) and the time integrated anomaly (4.20) as functions of time (Fig. 5) are plotted.

One notices the vanishing of the anomaly, and therefore the exact conservation of the charge \(Q(4)\), within numerical accuracy, for this type of soliton configuration. In addition, one has the qualitative realization of the symmetry (2.5) and the confirmation of our expectations for the vanishing of the anomaly, since the anomaly density \(\gamma(x, t)\) in (4.19) becomes an odd function for this type of soliton configuration (see Fig. 3).

In the Figs. 6-9 we present the numerical simulation of soliton collisions in the saturable-type potential NLS model (5.22). The two-soliton collision with equal and opposite velocities (equal amplitudes) and parameter \(\epsilon = 0.1\) is shown in the Fig. 6. The anomaly density \(\gamma(x, t)\) in (4.19) has been plotted as function of the space variable \(x\) for successive times (see Fig. 7). The two-soliton
Figure 2. The reflection of two dark solitons of the cubic-quintic NLS model is plotted for $\epsilon = -0.01$, $|\psi_0| = 6$, $\eta = 2.5$. The initial gray solitons ($t_i$ = green line) travel in opposite direction with velocity $v \approx 1.97\sqrt{2}$. They partially overlap ($t_c$ = blue line) in their closest approximation and then reflect to each other. The gray solitons after collision are plotted as a red line ($t_f$). Note that $v < v_s/2$ ($v_s \approx 13.89\sqrt{2}$).

Figure 3. The anomaly density $\gamma(x,t)$ plotted for three successive times ($t_i$, $t_c$ and $t_f$).

collision with equal and opposite velocities (equal amplitudes) and parameter $\epsilon = 0.1$. The relevant anomaly (4.18) and the integrated anomaly (4.20) are plotted in the Figs. 8 and 9, respectively.

Similarly, in this case the anomaly vanishes, and the charge $\hat{Q}^{(4)}$ is conserved, within numerical accuracy, for this type of soliton configuration. In addition, one has the qualitative realization of the symmetry (2.5) and the vanishing of the anomaly, as we have anticipated above based solely on symmetry arguments.

Some comments are in order here: First, the deformed models (5.21)-(5.22) have dark solitary waves and they present the properties of quasi-integrable models. Second, they present infinite tower of exactly conserved charges [4]: $Q^{(0)}, Q^{(1)}, Q^{(2)}, Q^{(3)}, \hat{Q}^{(4)}, \hat{Q}^{(6)}, \hat{Q}^{(8)}$, .... Third, they present infinite tower of asymptotically-conserved charges: $\hat{Q}^{(5)}, \hat{Q}^{(7)}, \hat{Q}^{(9)}$, .... such that

$$\frac{d\hat{Q}^{(n')}}{dt} = \beta^{(n')}, \int_{-\infty}^{+\infty} dx \beta^{(n')} = 0 \Rightarrow \hat{Q}^{(n')}(x = +\infty) = \hat{Q}^{(n')}(x = -\infty), \; n' = 5, 7, 9, ... \quad (5.28)$$

Fourth, one can argue that they exhibit solitary waves resembling to solitons, i.e. preserve their properties after collision.
6. Conclusion

The modified defocusing NLS models have been studied analytically and numerically. The charges of the MNLS models split into two subsets:

- A first subset of asymptotically conserved charges for 2-dark soliton solutions possessing space-time reflection parity symmetry [3].

- A second subset of infinite tower of exactly conserved charges for two-dark soliton solutions possessing space reflection parity symmetry [4].

We have computed numerically the first non-trivial anomaly $\beta^{(4)}$ of the $Q^{(4)}$ charge quasi-conservation law. We have verified that this anomaly vanishes, and consequently the exact conservation of the charge $Q^{(4)}$ holds for various 2-soliton configurations, within numerical accuracy. The only explanation we have found, so far, for the exact conservation of the even order charges ($n \geq 4$), is the space-reflection parity symmetry of 2-dark soliton solutions.

In [4] we have found that even for two-soliton solutions with different velocities (different amplitudes) which do not satisfy the space-reflection symmetry, the anomaly $\beta^{(4)}$ vanishes, within the numerical accuracy. Then, we may argue that the parity property (2.5) is not the cause of
Figure 6. Transmission of two dark solitons of the saturable NLS model with $q = 3$ and equal velocities are plotted for $\epsilon = 0.1$, $|\psi_0| = 6$, $\eta = 2.5$. The initial gray solitons ($t_i=$green line) travel in opposite direction with velocities $v_1 = -6.42\sqrt{2}$ (right soliton), $v_2 = 6.42\sqrt{2}$ (left soliton). They completely overlap ($t_c=$blue line) and then transmit to each other. The gray solitons after collision are plotted as a red line ($t_f$). Note that $|v_1| + v_2 > v_s$ ($v_s = 7.1\sqrt{2}$).

Figure 7. The anomaly density $\gamma(x,t)$ plotted for three successive times ($t_i$, $t_c$ and $t_f$).

the exact conservation of the charges, but according to our analytical calculations it is a sufficient condition for these phenomena to happen. Further research work is necessary to settle such questions which involve the non-linear dynamics of the scattering. So, the symmetries involved in the quasi-integrability phenomenon deserve further investigation and they may have relevant applications in many areas of non-linear sciences.
Figure 8. The anomaly $\beta^{(4)}(t)$.

Figure 9. The time integrated anomaly $\int dt' \beta^{(4)}(t')$.

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