KP Solitons and Mach Reflection in Shallow Water

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Abstract

This talk gives a survey of our recent studies on soliton solutions of the Kadomtsev-Petviashvili equation with an emphasis on the Mach reflection problem in shallow water.

1. Shallow water waves: Basic equations

We consider a surface wave on water which is assumed to be irrotational and incompressible (see for examples [24, 1]). We are mainly interested in a long wave phenomena, and assume zero surface tension, that is, we ignore the gravity-capillary waves. Let us denote the following scales:

\[
\begin{align*}
\lambda_0 & \sim \text{horizontal length scale = typical wave length} \\
h_0 & \sim \text{vertical length scale = asymptotic water depth} \\
a_0 & \sim \text{nonlinear scale = typical wave amplitude}
\end{align*}
\]

The non-dimensional variables \{x, y, z, t, \eta, \phi\} with the corresponding physical variables with the space variables \((\tilde{x}, \tilde{y}, \tilde{z})\), the amplitude of the wave \(\tilde{\eta}\), and the velocity potential \(\tilde{\phi}\) are defined as

\[
\begin{align*}
\tilde{x} &= \lambda_0 x, & \tilde{y} &= \lambda_0 y, & \tilde{z} &= h_0 z, & \tilde{t} &= \frac{\lambda_0}{c_0} t, \\
\tilde{\eta} &= a_0 \eta, & \tilde{\phi} &= a_0 h_0 \frac{\lambda_0}{c_0} \phi.
\end{align*}
\]

Then the shallow water equation in the non-dimensional form is given by

\[
\begin{align*}
\phi_{zz} + \beta \Delta \phi &= 0, & & \text{for } 0 < z < 1 + \alpha \eta, \\
\phi_z &= 0, & & \text{at } z = 0, \\
\phi_t + \frac{1}{2} \alpha |\nabla \phi|^2 + \frac{1}{2 \beta} \phi_z^2 + \eta &= 0, & & \text{at } z = 1 + \alpha \eta, \\
\eta_t + \alpha \nabla \phi \cdot \nabla \eta &= \frac{1}{\beta} \phi_z,
\end{align*}
\]

where \(\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\) and \(\Delta = \nabla^2\), the two-dimensional Laplace operator, for \((x, y) \in \mathbb{R}^2\).

The parameters \(\alpha\) and \(\beta\) are given by

\[
\alpha = \frac{a_0}{h_0} \quad \text{and} \quad \beta = \left(\frac{h_0}{\lambda_0}\right)^2.
\]

The weak nonlinearity implies \(\alpha \ll 1\), and the weak dispersion (or long wave assumption) implies \(\beta \ll 1\). With a small parameter \(\epsilon \ll 1\), we assume \(\alpha \sim \beta = O(\epsilon)\).

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First we note, from the first two equations, that \( \phi \) can be written formally in the form,

\[
\phi(x, y, z, t) = \cos \left( z \sqrt{\frac{\beta}{\Delta}} \right) \psi = \psi - \frac{\beta^2}{2} \Delta \psi + \frac{\beta^2 \gamma^2}{24} \Delta^2 \psi + \mathcal{O}(\epsilon^3),
\]

where \( \psi(x, y, t) = \phi(x, y, 0, t) \). Then the equations at the surface \( z = 1 + \alpha \eta \) give a Boussinesq-type system,

\[
\begin{align*}
\psi_t + \eta \cdot \frac{\alpha}{2} |\nabla \psi|^2 - \frac{\beta}{2} \Delta \psi_t & \quad + \frac{\alpha \beta}{2} \left( (\Delta \psi)^2 + \nabla \psi \cdot \nabla (\Delta \psi) - 2 \eta \Delta \psi_t \right) + \frac{\beta^2}{24} \Delta^2 \psi_t = \mathcal{O}(\epsilon^3). \quad (1.2) \\
\eta_t + \Delta \psi + \alpha \nabla \cdot (\eta \nabla \psi) - \frac{1}{6} \Delta^2 \psi & \quad - \frac{\alpha \beta}{2} \left( \nabla \eta \cdot \nabla (\Delta \psi) + \eta \Delta^2 \psi \right) + \frac{\beta^2}{120} \Delta^3 \psi = \mathcal{O}(\epsilon^3). \quad (1.3)
\end{align*}
\]

We now derive the KP equation with higher order corrections which will be a key equation for a physical application. The KP equation is obtained under the assumption with a weak dependence in the \( y \)-direction (quasi-two-dimensionality), and we introduce a small parameter \( \gamma \) so that the \( y \)-coordinate is scaled as

\[
\zeta := \sqrt{\gamma} y, \quad \text{with} \quad \gamma = \mathcal{O}(\epsilon).
\]

We also consider a far field with the scaled coordinates \( \xi = x - t \) and \( \tau = et \) (i.e. a unidirectional approximation). Then eliminating \( \eta \) in the equations (1.2) and (1.3), we obtain the KP equation for the function \( v := \psi_\xi(\xi, \zeta, \tau) \) with higher order corrections up to \( \mathcal{O}(\epsilon^2) \),

\[
2 \epsilon v + 3 \alpha v_v \xi + \frac{\beta}{3} v_{\xi\xi} + \gamma D^{-1} v_{\zeta\zeta} + \frac{19}{180} \beta^2 v_{\xi\xi\xi\xi} + \alpha \beta \left( \frac{15}{6} v v_{\xi\xi\xi} + \frac{53}{12} v_{\xi v_{\xi\xi}} \right) + \frac{\beta \gamma}{2} v_{\zeta\zeta\zeta\zeta} - \frac{\gamma^2}{4} D^{-3} v_{\zeta\zeta\zeta\zeta}
\]

\[
+ \alpha \gamma \left( \frac{5}{4} v D^{-1} v_{\zeta\zeta} + 2 v_v D^{-1} v_{v\xi} - \frac{3}{4} D^{-1} (v^2)_{\zeta\zeta} + \frac{1}{2} v_{vD^{-2}v_{\zeta\zeta}} \right) = \mathcal{O}(\epsilon^3). \quad (1.5)
\]

where \( D^{-1} := \partial_\xi^{-1} \) is a formal integral operator. The first line of this equation of the order \( \mathcal{O}(\epsilon) \) is the KP equation, and the terms in the second and third lines are the \( \epsilon^2 \)-order corrections to the KP equation. The wave amplitude \( \eta \) is then given by

\[
\eta = v + \frac{\alpha}{4} v^2 - \frac{\beta}{3} v_{\xi\xi} + \frac{\gamma}{2} D^{-2} v_{\zeta\zeta} + \mathcal{O}(\epsilon^2). \quad (1.6)
\]

**Remark 1.1** In terms of physical coordinates, the KP equation is given by

\[
\left( \tilde{\eta}_t + c_0 \tilde{\eta}_z + \frac{3c_0}{2h_0} \tilde{\eta}_{\tilde{z}} + \frac{c_0 h_0^2}{6} \tilde{\eta}_{\tilde{z}\tilde{z}} \right) + c_0 \frac{1}{2} \tilde{\eta}_{\tilde{y}\tilde{y}} = 0.
\]

As a particular solution, we have a solitary wave solution in the coordinate perpendicular to the wave crest, \( \tilde{\chi} = \tilde{x} \cos \Psi_0 + \tilde{y} \sin \Psi_0, \)

\[
\tilde{\eta} = a_0 \text{sech}^2 \sqrt{\frac{3a_0}{4h_0 \cos^2 \Psi_0}} \left[ \tilde{\chi} - c_0 \cos \Psi_0 \left( 1 + \frac{a_0}{2h_0} + \frac{1}{2} \tan^2 \Psi_0 \right) \tilde{t} - \chi_0 \right]. \quad (1.7)
\]
where \( a_0 > 0 \), \( \Psi_0 \) and \( \tilde{x}_0 \) are arbitrary constants. Recall that the KP equation is derived under the assumption of quasi-two dimensionality, that is, \( \gamma = \tan^2 \Psi = O(\epsilon) \), and the solution \((1.7)\) becomes unphysical for the case with a large angle. That is, the width of the solitary wave depends on the propagation direction, \( \Psi_0 \), which should be corrected when we apply the KP equation to a physical problem. In this talk, I will explain how this can be fixed using a normal form theory concerning the higher order corrections to the KP equation, and show that some of the exact solutions of the KP equation can be used to describe two-dimensional interaction phenomena in shallow water, called the Mach reflection.

2. Normal form for the KP equation with higher order corrections

In order to study the higher order corrections, we define a normal form for \((1.5)\). For this purpose, we first put \((1.5)\) into a canonical form of the KP equation with the change of variables,

\[
\xi = \sqrt{\beta} X, \quad \zeta = \sqrt{\beta} \gamma Y, \quad \tau = -\frac{3\epsilon\sqrt{\beta}}{2} T, \quad v = \frac{2}{3\alpha} u.
\]

Then \((1.5)\) becomes

\[
4u_T = 6uu_X + u_{XXX} + 3D^{-1}u_{YY} \\
+ \frac{19}{60}u_{xxxxx} + \frac{5}{3}uu_{XXX} + \frac{53}{6} u_Xu_{XX} + \frac{3}{2}u_{XY} - \frac{3}{4}D^{-3}u_{YY} \\
+ \frac{5}{2}uD^{-1}u_{YY} + 4u_Y D^{-1}u_Y - \frac{3}{4}D^{-1}(u^2)_{YY} + u_X D^{-2}u_{YY} + O(\epsilon^2).
\]

(2.1)

Note here we have the orders \( u \sim O(\epsilon) \), \( \partial_X \sim O(\epsilon^{\frac{1}{2}}) \), \( D^{-1} \sim O(\epsilon^{\frac{1}{2}}) \), and \( \partial_Y \sim O(\epsilon) \). Here the new variables \((X,Y,T)\) are related to the physical ones with

\[
\tilde{x} - c_0 \tilde{t} = h_0 X, \quad \tilde{y} = h_0 Y, \quad \tilde{t} = \frac{3h_0}{2c_0} T. \quad (2.2)
\]

The wave amplitude \( \eta \) in terms of \( u \) is given by

\[
\alpha \eta = \frac{2}{3} u + \frac{1}{9} u^2 - \frac{2}{9} u_X + \frac{1}{3} D^{-2} u_{YY} + O(\epsilon^3)
\]

Hereafter we use the lower case letters \((x,y,t)\) for \((X,Y,T)\), and the KP variables can be converted to the physical variables directly through the relations \((2.2)\).

2.1. The normal form for the KdV equation

Before discussing a normal form for \((1.5)\), we give a brief summary of the result in \([9, 10, 5]\) for the case of the KdV equation. Taking \( \partial_Y u = 0 \) in \((2.1)\), we have the KdV equation with higher order corrections,

\[
4u_t = 6uu_x + u_{xxx} + \left( \frac{19}{60}u_{xxxxx} + \frac{5}{3}uu_{xxx} + \frac{53}{6}u_xu_{xx} \right) + O(\epsilon^2).
\]

(2.3)

In \([9, 10, 5]\), we found that one can transform \((2.3)\) with a formal change of variable, a Lie exponential transformation \( e^{V \varphi} \) with the generating function \( \varphi \) (see \([5]\)),

\[
u = e^{V \varphi} \circ U = U + \left( U_{xx} + \frac{4}{3} u^2 + \frac{1}{2} U_x D^{-1} U \right) + O(\epsilon^3), \quad (2.4)
\]
into the equation which we referred to as the normal form of \((2.3)\),

\[ 4U_t = 6UU_x + U_{xxx} + \frac{19}{60} \left( U_{xxxx} + 10UU_{xxx} + 20U_xU_{xx} + 30U^2U_x \right) + \mathcal{O}(\epsilon^2). \]

The point here is that the higher order term of \(\mathcal{O}(\epsilon^2)\) in this equation is the 5th order symmetry of the KdV equation, and hence the normal form is integrable up to \(\mathcal{O}(\epsilon^2)\). That is, we have an integrability not only at the KdV of \(\mathcal{O}(\epsilon^5)\) but also at the next order correction of \(\mathcal{O}(\epsilon^7)\). This is true even for the general form of the higher order correction,

\[ \alpha_1 u_{xxxxx} + \alpha_2 uu_{xxx} + \alpha_3 u_u u_{xx} + \alpha_4 u^2 u_x. \]

with arbitrary coefficients \(\alpha_1, \ldots, \alpha_4\). This implies that any weakly nonlinear long-wave equation whose leading order is approximated by the KdV equation is asymptotically integrable up to the next order approximation \([9, 10, 5]\).

Note also that the normal form admits one-soliton solution in the form,

\[ U = A_0 \text{sech}^2 \sqrt{\frac{A_0}{2}(x + x_0(t))}, \]

where \(x_0(t)\) is determined by \(\frac{dx_0}{dt} := C_{hKdV} = \frac{1}{2} A_0 + \frac{19}{60} A_0^2 + \mathcal{O}(\epsilon^3)\). Then the solution of the higher order KdV equation \((2.3)\) is given by the transformation \((2.4)\), i.e.

\[ u = A_0 S^2 + \left( A_0^2 S^2 - \frac{2}{3} A_0^2 S^4 \right) + \mathcal{O}(\epsilon^3). \]

where \(S := \text{sech}\sqrt{\frac{A_0}{2}(x + x_0)}\). Notice that the amplitude \(\eta\) of \((1.6)\) for the KdV case is given by

\[ \alpha \eta = \frac{2}{3} A_0 S^2 + \frac{2}{9} A_0^2 S^2 + \frac{1}{3} A_0^2 S^4 + \mathcal{O}(\epsilon^3) \]  \((2.5)\)

2.2. Normal form of the KP equation

Now we define a normal form of \((2.1)\) by a Lie exponential transform,

\[ u = e^{V_x} U = U + \left( \beta_1 U_{xx} + \beta_2 U^2 + \beta_3 U_x D^{-1} U + \beta_4 D^{-2} U_{yy} \right) + \mathcal{O}(\epsilon^3), \]

where \(\beta_i\)'s are determined such a way that the transformed equation (normal form) has a “good” property. First, we require that the normal form is reduced to the KdV-normal form when we have \(\partial_y u = 0\). That is, from \((2.4)\), we have \(\beta_1 = 1, \beta_2 = \frac{4}{3}, \beta_3 = \frac{1}{2}, \) but \(\beta_4\) remains free. Then we require that the normal form admits a solitary wave in the form of one-soliton solution,

\[ U = A_0 \text{sech}^2 \sqrt{\frac{A_0}{2}(x + x_0(y,t))}. \]

This determines uniquely \(\beta_4 = \frac{1}{2}\), and the normal form of \((2.1)\) is given by

\[ 4U_t = 6UU_x + U_{xxx} + 3D^{-1}U_{yy} \]

\[ + \frac{19}{60} \left( U_{xxxx} + 10UU_{xxx} + 20U_xU_{xx} + 30U^2U_x \right) + \frac{3}{2} U_{xxyy} - \frac{3}{4} D^{-3} U_{yyyy} \]

\[ - UD^{-1} U_{yy} + 7U_y D^{-1} U_y + \frac{1}{4} D^{-1} (U^2)_{yy} + \frac{5}{2} U_{x} D^{-2} U_{yy} + \mathcal{O}(\epsilon^2). \]  \((2.6)\)
Then the phase $x_0(y, t)$ satisfies
\[
4 \frac{\partial x_0}{\partial t} = 2A_0 + 3 \left( \frac{\partial x_0}{\partial y} \right)^2 + \frac{19}{15} A_0^2 + 3 A_0 \left( \frac{\partial x_0}{\partial y} \right)^2 - \frac{3}{4} \left( \frac{\partial x_0}{\partial y} \right)^4 + \mathcal{O}(\epsilon^3).
\]
Setting the angle $\Psi_0$ of the soliton with $\frac{\partial x_0}{\partial y} = \tan \Psi_0$, we have the velocity $C_{hKP} = \frac{\partial x_0}{\partial t}$ of the KP soliton with the higher order corrections. The corresponding solitary wave solution $u$ is then given by
\[
u = A_0 S^2 + \left( A_0^2 S^2 + \frac{1}{2} A_0 \tan^2 \Psi_0 S^2 - \frac{2}{3} A_0^2 S^4 \right) + \mathcal{O}(\epsilon^3)
\]
and the wave amplitude $\eta$ of (1.6) is given by
\[
\alpha \eta = \frac{2}{3} A_0 S^2 + \frac{2}{3} \tan^2 \Psi_0 S^2 + \frac{2}{9} A_0^2 S^2 + \frac{1}{3} A_0^2 S^4 + \mathcal{O}(\epsilon^3)
\]
\[
= \frac{2}{3} [A_0] S^2 + \frac{2}{9} [A_0]^2 S^2 + \frac{1}{3} [A_0]^2 S^4 + \mathcal{O}(\epsilon^3),
\]
where $[A_0] := A_0(1 + \tan^2 \Psi_0) = A_0/\cos^2 \Psi_0$ (cf. (1.7)). Notice that the correction to quasi-two-dimensional approximation can be absorbed into the amplitude of the KdV equation (cf. (2.5)). This formula gives the relation between the observed amplitude $\eta$ from numerical simulation (or experiment) of shallow water wave system and the KP amplitude $A_0$.

3. The KP solitons
Here we give a brief summary of the soliton solutions of the KP equation [2, 11, 12],
\[
-4u_t + 6uu_x + u_{xxx} + D^{-1}u_{yy} = 0.
\]
(3.1)
We write the solution of the KP equation in the $\tau$-function form,
\[
u(x, y, t) = 2\partial_x^2 \ln \tau(x, y, t).
\]
(3.2)
where the $\tau$-function is assumed to be the Wronskian determinant with $N$ functions $f_i$’s (see for examples [22, 6]),
\[
\tau = \text{Wr}(f_1, f_2, \ldots, f_N).
\]
(3.3)
Here the functions $\{f_1, \ldots, f_N\}$ form a set of linearly independent solutions of the linear equations,
\[
\partial_y f_n = \partial_x^2 f_n, \quad \partial_t f_n = \partial_x^3 f_n.
\]
In particular, we consider a set of finite dimensional solutions for $\{f_1, f_2, \ldots, f_N\}$,
\[
f_n(x, y, t) = \sum_{m=1}^M a_{n,m} E_m(x, y, t) \quad \text{with} \quad E_m = \epsilon^{\kappa_m} := \exp(\kappa_m x + \kappa_m^2 y + \kappa_m^3 t).
\]
Thus this type of solution is characterized by the ordered parameters $\{\kappa_1 < \cdots < \kappa_M\}$ and the $N \times M$ matrix $A := (a_{n,m})$ of rank($A$) = $N$, that is, we have
\[
(f_1, f_2, \ldots, f_N) = (E_1, E_2, \ldots, E_M) A^T.
\]
(3.4)
Note that \( \{E_1, E_2, \ldots, E_M\} \) gives a basis of \( \mathbb{R}^M \) and \( \{f_1, f_2, \ldots, f_N\} \) spans an \( N \)-dimensional subspace of \( \mathbb{R}^M \). This means that the \( A \)-matrix can be identified as a point on the real Grassmann manifold \( \text{Gr}(N, M) \) (see [11, 2]). More precisely, let \( M_{N \times M}(\mathbb{R}) \) be the set of all \( N \times M \) matrices of rank \( N \). Then \( \text{Gr}(N, M) \) can be expressed as

\[
\text{Gr}(N, M) = \text{GL}_N(\mathbb{R}) \backslash M_{N \times M}(\mathbb{R}).
\]

where \( \text{GL}_N(\mathbb{R}) \) is the general linear group of rank \( N \). This expression means that other basis \( (g_1, \ldots, g_N) = (f_1, \ldots, f_N)H \) for any \( H \in \text{GL}_N(\mathbb{R}) \) spans the same subspace, that is, \( A \to H^T A \) (\( \text{GL}_N(\mathbb{R}) \) acting from the left). Notice here that the freedom in the \( A \)-matrix with \( \text{GL}_N(\mathbb{R}) \) can be fixed by expressing \( A \) in the reduced row echelon form (RREF).

Now using the Binet-Cauchy Lemma for the determinant, the \( \tau \)-function of (3.3) can be expressed in the form,

\[
\tau(x, y, t) = \sum_{J \in (\mathbb{Z}^N)} \Delta_J(A) E_J(x, y, t),
\]

where \( \Delta_J(A) \) is the \( N \times N \) minor of the \( A \)-matrix with \( N \) columns marked by \( J = \{j_1, \ldots, j_N\} \in (\mathbb{Z}^M) \), the set of \( N \)-subindices of the set \( [M] := \{1, \ldots, M\} \), and \( E_J(x, y, t) \) is given by

\[
E_J(x, y, t) = \text{Wr}(E_{j_1}, \ldots, E_{j_N}) = \prod_{l < m} (\kappa_{j_m} - \kappa_{j_l}) E_{j_1} \cdots E_{j_N}.
\]

We are also interested in non-singular solutions. Since the solution is given by \( u = 2\partial_x^2(\ln \tau) \), the non-singular solutions are obtained by imposing the non-negativity condition on the minors,

\[
\Delta_J(A) \geq 0, \quad \text{for all} \quad J \in (\mathbb{Z}^M) / (\mathbb{Z}^N).
\]

This condition is sufficient for the non-singularity of the solution for any initial data (see [15] for the necessary condition for the regularity). We call a matrix \( A \) having the condition (3.6) totally non-negative matrix, referred to as TNN matrix, and the set of those matrices forms totally non-negative Grassmannian, denoted by \( \text{Gr}^+(N, M) \) (see e.g. [21, 14]).

**Example 3.1** Let us express one line-soliton solution in our setting. Here we also introduce some notations to describe the soliton solutions. One soliton solution is obtained by the \( \tau \)-function with \( M = 2 \) and \( N = 1 \), i.e. \( \tau = f_1 = a_{11} E_1 + a_{12} E_2 \), with \( A = (a_{11}, a_{12}) \). Since the solution \( u \) is given by (3.2), one can assume \( a_{11} = 1 \) and denote \( a_{12} = a > 0 \). Then we have

\[
u = 2\partial_x^2 \ln \tau = \frac{1}{2} (\kappa_1 - \kappa_2)^2 \text{sech}^2 \left( \frac{1}{2} (\theta_1 - \theta_2 - \ln a) \right).
\]

Thus the solution is localized along the line \( \theta_1 - \theta_2 = \ln a \), hence we call it line-soliton solution. We emphasize here that the line-soliton appears at the boundary of two regions where either \( E_1 \) or \( E_2 \) is the dominant exponential term, and because of this we also call this soliton a \([1, 2]\)-soliton solution. We refer to each of these asymptotic line-solitons
as the \([i, j]\)-soliton. The \([i, j]\)-soliton solution with \(i < j\) has the same (local) structure as the one-soliton solution, and can be described as follows

\[
u = A_{[i,j]} \sech^2 \frac{1}{2} \left( K_{[i,j]} \cdot x - \Omega_{[i,j]} t + \Theta_{[i,j]}^0 \right)
\]

with some constant \(\Theta^0_{[i,j]}\). The amplitude \(A_{[i,j]}\), the wave-vector \(K_{[i,j]}\) and the frequency \(\Omega_{[i,j]}\) are then expressed by

\[
A_{[i,j]} = \frac{1}{2}(\kappa_j - \kappa_i)^2, \quad K_{[i,j]} = (\kappa_j - \kappa_i, \kappa_j^2 - \kappa_i^2), \quad \Omega_{[i,j]} = \kappa_j^3 - \kappa_i^3.
\]

Note that \(K_{[i,j]}\) and \(\Omega_{[i,j]}\) satisfy the soliton-dispersion relation, 

\[
4\Omega_{[i,j]} K_{x,[i,j]}^2 = (K_{x,[i,j]}^2)^4 + 3(K_{y,[i,j]}^2)^2.
\]

The direction of the wave-vector \(K_{[i,j]} = (K_{x,[i,j]}, K_{y,[i,j]})\) is measured in the counterclockwise rotation from the \(y\)-axis, and it is given by

\[
\frac{K_{y,[i,j]}}{K_{x,[i,j]}} = \tan \Psi_{[i,j]} = \kappa_i + \kappa_j,
\]

that is, \(\Psi_{[i,j]}\) gives the angle between the line \(K_{[i,j]} \cdot x = \text{const}\) and the \(y\)-axis.

Figure 3.1: One line-soliton solution of \([i, j]\)-type and the corresponding chord diagram. The upper oriented chord represents the part of \([i, j]\)-soliton for \(y \gg 0\) and the lower one for \(y \ll 0\).

4. Classification of soliton solutions

We now present the classification theorems of the soliton solutions generated by the \(\tau\)-function defined in (3.5). Here we consider the matrix \(A\) to be in RREF, and we also assume that \(A\) is irreducible as defined below:

**Definition 4.1** An \(N \times M\) matrix \(A\) is irreducible if each column of \(A\) contains at least one nonzero element, or each row contains at least one nonzero element other than the pivot once \(A\) is in RREF. So the irreducibility implies that we consider only derangements (i.e. no fixed points) of the permutation.

Our classification scheme of the soliton solutions is given by identifying the asymptotic line-solitons as \(y \to \pm \infty\). We denote a line-soliton solution by \((N_-, N_+)-\text{soliton}\) whose asymptotic form consists of \(N_-\) line-solitons as \(y \to -\infty\) and \(N_+\) line-solitons for
y \rightarrow \infty$ in the $xy$-plane. The next Proposition provides a general result characterizing the asymptotic line-solitons of the $(N_-, N_+)$-soliton solutions (the proof can be found in [2]):

**Proposition 4.1** Let $\{e_1, e_2, \ldots, e_N\}$ and $\{g_1, g_2, \ldots, g_{M-N}\}$ denote respectively, the pivot and non-pivot indices associated with an irreducible, $N \times M$, TNN A-matrix. Then the soliton solution obtained from the $\tau$-function in (3.5) with this A-matrix has the following structure:

(a) For $y \gg 0$, there are $N$ line-solitons of $[e_n, j_n]$-type for some $j_n$.

(b) For $y \ll 0$, there are $(M - N)$ line-solitons of $[i_m, g_m]$-type for some $i_m$.

An important consequence of Proposition 4.1 is that it defines the pairing map $\pi : [M] \rightarrow [M]$ with

$$\begin{align*}
\pi(e_n) &= j_n, \quad n = 1, 2, \ldots, N, \\
\pi(g_m) &= i_m, \quad m = 1, 2, \ldots, M - N.
\end{align*}$$

Recall that $\{e_n\}_{n=1}^N$ and $\{g_m\}_{m=1}^{M-N}$ are respectively, the pivot and non-pivot indices of the A-matrix and form a disjoint partition of $[M]$. Then the unique index pairings in Proposition 4.1 imply that the map $\pi$ is a permutation of $M$ indices, i.e. $\pi \in S_M$, the symmetric group of permutations. Furthermore, since $\pi(e_n) = j_n > e_n$, $n = 1, \ldots, N$ and $\pi(g_m) = i_m < g_m$, $m = 1, \ldots, M - N$, $\pi$ defined by (4.1) is a permutation with no fixed point, i.e. derangements. Yet another feature of $\pi$ is that it has exactly $N$ excedances defined as follows: an element $l \in [M]$ is an excedance of $\pi$ if $\pi(l) > l$. The excedance set of $\pi$ in (4.1) is the set of pivot indices $\{e_1, e_2, \ldots, e_N\}$. Then we have the following characterization for the line-soliton solution of the KP equation [2].

**Theorem 4.2** Let $A$ be an $N \times M$, TNN, irreducible matrix which corresponds to a point in the non-negative Grassmannian $Gr^+(N, M)$. Then the $\tau$-function (3.5) associated with this A-matrix generates an $(M - N, N)$-soliton solutions. The $M$ asymptotic line-solitons associated with each of these solutions can be identified via a pairing map $\pi$ defined by (4.1). The map $\pi \in S_M$ is a derangement of the index set $[M]$ with $N$ excedances given by the pivot indices $\{e_1, e_2, \ldots, e_N\}$ of the A-matrix in RREF.

Theorem 4.2 provides a unique parametrization of each TNN Grassmannian cell in terms of the derangement of $S_M$. One should note that Theorem 4.2 does not give us the indices $j_n$ and $i_m$ in the $[e_n, j_n]$ and $[i_m, g_m]$ line-solitons. The specific conditions that an index pair $[i, j]$ identifies an asymptotic line-soliton are obtained by identifying the dominant exponential in each domain in the $xy$-plane. To visualize those asymptotic solitons, we define a chord diagram:

**Definition 4.3** A chord diagram associated to a derangement $\pi \in S_M$ is defined as follows: Consider a line segment with $M$ marked points by the numbers $\{1, \ldots, M\}$ in the increasing order from the left.

(a) If $i < \pi(i)$ (expedience), then draw a chord joining $i$ and $\pi(i)$ on the upper part of the line.

(b) If $j > \pi(j)$ (deficiency), then draw a chord joining $j$ and $\pi(j)$ on the lower part of the line.
Then Proposition 4.1 and Theorem 4.2 imply that each chord diagram identifies the types of solitons appearing in the asymptotic regions \(|y| \gg 0\). It is also useful to consider those marked points as the values of \(k\)-parameters, so that for each chord joining \(i\) and \(j\), its length gives the amplitude and the sum of the joined points gives the inclination of the line-soliton of \([i,j]\)-type. Figure 4.1 illustrates the time evolution of an example of (3,3)-soliton solution. The chord diagram shows all asymptotic line-solitons for \(y \rightarrow \pm \infty\).

\[
A = \begin{pmatrix} 1 & 0 & -1 & -1 & 0 & 2 \\ 0 & 1 & 2 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}
\]

Figure 4.1: An example of (3,3)-soliton solution. The permutation of this solution is \(\pi = (451263)\). The \(k\)-parameters are chosen as \((\kappa_1, \kappa_2, \ldots, \kappa_6) = (-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2})\).

(See [14] for a classification of those patterns.)

5. Shallow water waves: The Mach reflection

In this last section, we discuss a real application of the exact soliton solutions of the KP equation described in the previous sections to the Mach reflection phenomena in shallow water. Let us first discuss some exact solutions relevant to this phenomena.

5.1. Exact soliton solutions from \(\text{Gr}^+(2,4)\)

Recall that the \(\tau\)-function for \(\text{Gr}^+(2,4)\) is given by

\[
\tau(x, y, t) = \sum_{1 \leq i < j \leq 4} \Delta_{i,j}(A) E_{i,j}(x, y, t)
\]

where \(\Delta_{i,j}(A) \geq 0\) is the Plücker coordinates, the \(2 \times 2\) minors consisting of \(i\)-th and \(j\)-th columns of the \(2 \times 4\) \(A\)-matrix, and \(E_{i,j} = \text{Wr}(e^{\theta_i}, e^{\theta_j}) = (\kappa_j - \kappa_i)e^{\theta_i + \theta_j}\) (note \(E_{i,j} > 0\) with the order \(\kappa_i < \kappa_j\)). Proposition 4.1 shows that \(\tau\)-function (5.1) generates a soliton solution which consists of at most two line-solitons for both \(|y| \gg 0\). We consider the following two types: one consists of two line-solitons of \([1,2]\) and \([3,4]\) for both \(|Y| \gg 0\), which is called O-type soliton (“O” stands for original, see [11]); the other one consists of \([1,3]\) and \([3,4]\) line-solitons for \(Y \gg 0\) and \([1,2]\) and \([2,4]\) line-solitons for \(Y \ll 0\). Theorem 4.2 indicates that this soliton can be referred as to \((3142)\)-type, because those line-solitons represent a permutation \(\pi = (1\ 2\ 3\ 4)\). Figure 5.1 illustrates the contour plots of O-type and (3142)-type solutions in the \(xy\)-plane, and the corresponding chord diagrams. The upper chords represent the asymptotic
solitons \([i, j]\) for \(y \gg 0\), and the lower chords for the asymptotic solitons \([i, j]\) for \(y \ll 0\). The \(A\)-matrices for those solutions are respectively given by

\[
A_{O} = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 0 & 1 & b \end{pmatrix}, \quad A_{(3142)} = \begin{pmatrix} 1 & a & 0 & -c \\ 0 & 0 & 1 & b \end{pmatrix},
\]

where \(a, b, c > 0\) are constants determining the locations of the solitons (see [2]). Notice that the \(\tau\)-function for the \((3142)\)-type contains five exponential terms, and the \(\tau\)-function for O-type with \(c = 0\) in (5.2) contains only four terms.

Let us fix the amplitudes and the angles of the solitons in the positive \(x\) regions for both O- and (3142)-types, so that those solutions are symmetric with respect to the \(x\)-axis (see Figure 5.1):

\[
A_0 \equiv \begin{cases} A_{[1,2]} = A_{[3,4]} & \text{(O type)} \\ A_{[1,3]} = A_{[2,4]} & \text{(3142 type)} \end{cases}
\]

\[
\Psi_0 \equiv \begin{cases} -\Psi_{[1,2]} = \Psi_{[3,4]} > 0 & \text{(O type)} \\ -\Psi_{[1,3]} = \Psi_{[2,4]} > 0 & \text{(3142 type)} \end{cases}
\]

Then we express the \(\kappa\)-parameters in terms of \(A_0\) and \(\tan \psi_0\) with \(\kappa_1 = -\kappa_4\) and \(\kappa_2 = -\kappa_3\) (due to the symmetry): In the case of O-type, we have

\[
\kappa_1 = -\frac{1}{2} \left( \tan \psi_0 + \sqrt{2A_0} \right), \quad \kappa_2 = -\frac{1}{2} \left( \tan \psi_0 - \sqrt{2A_0} \right).
\]

The ordering \(\kappa_2 < \kappa_3\) then implies \(\tan \psi_0 > \sqrt{2A_0}\). On the other hand, for the (3142)-type, we have

\[
\kappa_1 = -\frac{1}{2} \left( \tan \psi_0 + \sqrt{2A_0} \right), \quad \kappa_2 = \frac{1}{2} \left( \tan \psi_0 - \sqrt{2A_0} \right).
\]

The ordering \(\kappa_2 < \kappa_3\) implies \(\tan \psi_0 < \sqrt{2A_0}\). Thus, if all the solitons in the positive \(x\)-region have the same amplitude \(A_0\) for both O- and (3142)-types, then an O-type solution arises when \(\tan \Psi_0 > \sqrt{2A_0}\), and a (3142)-type when \(\tan \Psi_0 < \sqrt{2A_0}\). Then the limiting value at \(\kappa_2 = \kappa_3\) (= 0) defines the critical angle \(\Psi_c\),

\[
\tan \Psi_c := \sqrt{2A_0}.
\]
Note that at the critical angle, i.e. $\kappa_2 = \kappa_3$, the $\tau$-function has only three exponential terms, and this gives a “$Y$”-shape resonant solution as Miles noted [19].

One should note that for (3142)-type solution, the solitons in the negative $x$-region are smaller than those in the positive region, i.e. $A_{[3,4]} = A_{[1,2]} = \frac{1}{2}\tan^2 \Psi_0 < A_0$, and the angles of those in the negative $x$-regions do not depend on $\Psi_0$ and $\Psi_{[3,4]} = -\Psi_{[1,2]} = \Psi_c$. Two sets of three solitons $\{[1,3], [1,4], [3,4]\}$ and $\{[2,4], [1,4], [1,2]\}$ are both in the soliton resonant state. These properties of the (3142)-type solution are the same as those of Miles’s asymptotic solution for the Mach reflection in shallow water [19]. However one should note that the $[1,4]$-soliton corresponding to the Mach stem becomes a soliton solution only in an asymptotic sense. Then the exact solution of (3142)-type can provide an estimate of a propagation distance, at which the amplitude is sufficiently developed.

5.2. The Mach reflection in shallow water

In [19], J. Miles considered an oblique interaction of two line-solitons using O-type solutions. He observed that resonance occurs at the critical angle $\Psi_c$, and when the initial oblique angle $\Psi_0$ is smaller than $\Psi_c$, the O-type solution becomes singular. He also noticed a similarity between this resonant interaction and the Mach reflection found in shock wave interaction (see for example [3, 24]). This may be illustrated by the left figure of Figure 5.2 where an incidence wave shown by the vertical line is propagating to the right, and it hits a rigid wall with the angle $\Psi_0$. If the angle $\Psi_0$ is large, the reflected wave behind the incidence wave has the same angle $\Psi_0$, i.e. a regular reflection occurs. However, if the angle is small, then an intermediate wave called the Mach stem appears as illustrated in Figure 5.2. The Mach stem, the incident wave and the reflected wave interact resonantly, and those three waves form a resonant triplet. The right panel in Figure 5.2 illustrates the wave propagation which is equivalent to that in the left panel, if one ignores the effect of viscosity on the wall (i.e. no boundary layer). These reflection patterns are well explained by the exact solutions of O- and (3142)-types [16, 25]. In the next section, we give a brief history of studies on the Mach reflection.

Figure 5.2: The Mach reflection. The left figure illustrates an incidence wave propagating parallel to the wall with the mirror image. The right figure is an equivalent system to the left one. The resulting wave pattern shown here is a (3142)-soliton solution. The angle $\Phi$ becomes zero if the initial angle satisfies $\Psi_0 \geq \Psi_c$, i.e. no stem.
5.3. Previous numerical results

One of the most interesting things of the Mach reflection is that the KP theory predicts an extraordinary four-fold amplification of the stem wave at the critical angle \[\Psi_0 = \frac{\Psi_0}{\sqrt{2A_0}}\] \[k = \tan \Psi_0 / \sqrt{2A_0} \tag{5.7}\].

Then from the KP theory, the amplification factor \(\alpha_0 = u_{\text{max}} / A_0\) is given by \[\alpha_0 = \begin{cases} (1 + k)^2, & \text{for } k < 1, \\ 4 \left(1 + \sqrt{1 - k^2}\right), & \text{for } k > 1. \end{cases} \tag{5.8}\]

Several laboratory and numerical experiments tried to confirm the formula (5.8), in particular, the four-fold amplification at the critical value \(\kappa = 1\) (see for example \[\kappa = 1\] \[\kappa = 1\] \[\kappa = 1\], \[\kappa = 1\], \[\kappa = 1\]). In \[\kappa = 1\], Funakoshi made a numerical simulation of the Mach reflection problem using the system of equations equivalent to the Boussinesq-type system (1.2) and (1.3) up to the first order. He considered the initial wave with a small amplitude \(a_i = 0.05\), and concluded that his results agree very well with the resonantly interacting solitary wave solution predicted by Miles. However his results on the amplification parameter \(\alpha\) are slightly shifted to the lower values of the original Miles parameter \(k_M = \Psi_0 / \sqrt{3a_i}\).

Tanaka in \[\kappa = 1\] then re-examined Funakoshi’s results for higher amplitude incidence waves with \(a_i = 0.3\) using the high-order spectral method. He noted that the effect of large amplitude tends to prevent the Mach reflection to occur, and all the parameters such as the critical angle \(\Psi_0\) are shifted toward the values corresponding to the regular reflection (i.e. O-type). For example, he obtained the maximum amplification factor \(\alpha = 2.897\) at \(k_M = 0.695\). Because of the quasi-two dimensional approximation, i.e. \(\kappa = 1\), Miles in his paper \[\kappa = 1\] replaced \(\tan \Psi_0\) by \(\Psi_0\), and then in \[\kappa = 1\] and \[\kappa = 1\], the authors continued on to use this replacement. Then their computations with rather large values of \(\Psi_0\) gave significant shifts of the parameter \(\kappa\). In the next section, we re-evaluate their results using the new parameters obtained from the normal form.

5.4. Re-evaluation of the previous results using the corrected parameters obtained by the normal form

We recall that the KP equation is derived under the assumptions of quasi-two dimensionality, weak dispersion and weak nonlinearity. Then for a physical application of the KP theory, one needs to include higher order corrections to those assumptions using the normal form (2.6). We then convert the observed amplitude in the numerical simulations (or the experiments) to the corresponding KP amplitude via the formula (2.7).

Here we just summarize the results: Let \(a_i\) be the amplitude of the incident wave used in the numerical simulation (or the experiment), and \(\Psi_i = \Psi_0\) be the inclination of the incidence wave. Also let \(\alpha\) be the amplification factor obtained from the numerical simulation (or the experiment). Then the Miles parameter in terms of the observed amplitude is given by

\[k = \tan \Psi_0 / \sqrt{2A_0} = \frac{\sqrt{1 + \sqrt{1 + 5a_i \tan \Psi_0}}}{\sqrt{6a_i \cos \Psi_0}} \tag{5.9}\]
Figure 5.3: Numerical results of the amplification factor $\alpha$ versus the parameter $\kappa$. The circles show Funakoshi’s result [4], the squares show Tanaka’s result [23]. The black dots shows the experimental results by Yeh et al [16].

The corrected amplification factor (in the KP coordinate) is given by

$$\alpha_0 = \begin{cases} 
\frac{\alpha(1 + \sqrt{1 + 5\alpha_i})}{(1 + \sqrt{1 + 5\alpha_i}) \cos^2 \Psi_0} & \text{if } k < 1 \\
\alpha & \text{if } k > 1
\end{cases}$$

We then re-evaluate their results with the new formulae (5.9) and (5.10), and the results are shown in Figure 5.3. Since Funakoshi’s simulations are based on small amplitude incidence waves, his results agree quite well with the KP predictions. Tanaka’s results are also in good agreement with the KP theory except for the cases near the critical angle (i.e. $k = 1$), where the amplification parameter $\alpha_0$ gets close to 3. This region clearly violates the assumption of the weak nonlinearity. However, the original plots of Tanaka’s are significantly improved with those formulae (5.9) and (5.10). The black dots in Figure 5.3 indicate the results of recent laboratory experiments done by Yeh and his colleagues [25, 16]. The laboratory experiments are performed using 7.3 m long and 3.6 m wide wave tank with a water depth of 6.0 cm. In the talk, I will show the details of the experiments including some movies and pictures of several real shallow water solitons. I will also discuss a stability problem of solitons solutions, and present some numerical simulations which indicate a convergence of KP solutions to certain exact soliton solutions similar to the case of the KdV equation [13, 8] (see Figure 5.4).

Figure 5.4: Numerical simulation of with the V-shaped initial wave. The solution converges to (3142)-type exact solution. Notice a completion of a chord diagram.
References

[1] M. J. Ablowitz and H. Segur, *Solitons and the inverse scattering transform*, SIAM Studies in Applied Mathematics, (SIAM, Philadelphia/1981).

[2] S. Chakravarty and Y. Kodama, Soliton solutions of the KP equation and application to shallow water waves, *Stud. Appl. Math.*, **123** (2009) 83-151.

[3] R. Courant and K. O. Friedrichs, *Supersonic flow and shock waves*, (Intersciences Publ., New York, 1948).

[4] M. Funakoshi, Reflection of obliquely incident solitary waves, *J. Phys. Soc. Jpn.*, **49** (1980) 2371-2379.

[5] Y. Hiraoka and Y. Kodama, Normal form and solitons, *Lect. Notes Phys.*, **767** (2009) 175-214.

[6] R Hirota, *The Direct Method in Soliton Theory* (Cambridge University Press, 2004)

[7] B. B. Kadomtsev and V. I. Petviashvili, On the stability of solitary waves in weakly dispersive media, *Sov. Phys. - Dokl.* **15** (1970) 539-541.

[8] C.-Y. Kao, and Y. Kodama, Numerical study on the KP equation for non-periodic waves, *Math. Comp. Sim.* **82** (2012) 1185-1218.

[9] Y. Kodama, Normal forms for weakly dispersive wave equations, *Phys. Lett. A*, **112** (1985) 193-196.

[10] Y. Kodama, On solitary-wave interaction, *Phys. Lett. A*, **123** (1987) 276-282.

[11] Y. Kodama Young diagrams and N-soliton solutions of the KP equation, *J. Phys. A: Math. Gen.* **37** (2004) 11169-90.

[12] Y. Kodama, KP solitons in shallow water, *J. Phys. A: Math. Theor.* **43** (2010) 434004 (54pp).

[13] Y. Kodama, M. Oikawa and H. Tsuji, Soliton solutions of the KP equation with V-shape initial waves, *J. Phys. A: Math. Theor.*, **42** (2009) 312001 (9pp).

[14] Y. Kodama and L. Williams, KP solitons and total positivity for the Grassmannian, (arXiv:1106.0023).

[15] Y. Kodama and L. Williams, A Deodhar decomposition of the Grassmannian and the regularity of KP solitons, (arXiv:1204:6446).

[16] W. Li, H. Yeh and Y. Kodama, On the Mach reflection of a solitary wave: revisited, *J. Fluid Mech.*, **672** (2011) 326-357.

[17] W. K. Melville, On the Mach reflection of a solitary wave, *J. Fluid Mech.*, **98** (1980) 285-297.

[18] J. W. Miles, Obliquely interacting solitary waves, *J. Fluid Mech.*, **79** (1977) 157-169.

[19] J. W. Miles, Resonantly interacting solitary waves, *J. Fluid Mech.*, **79** (1977) 171-179.

[20] P. H. Perroud, The solitary wave reflection along a straight vertical wall at oblique incidence, Institute of Engineering Research, Wave Research Laboratory, Tech. Rep. 99/3, University of California, Berkeley (1957) 93pp.

[21] A. Postnikov, Totally positivity, Grassmannians and networks, (arXiv:CO/0609764).

[22] J. Satsuma, A Wronskian representation of N-soliton solutions of nonlinear evolution equations, *J. Phys. Soc. Japan*, **46** (1979) 356-360.

[23] M. Tanaka, Mach reflection of a large-amplitude solitary wave, *J. Phys. Mech.*, **248** (1993) 637-661.

[24] G. B. Whitham, *Linear and nonlinear waves*, A Wiley-interscience publication (John Wiley & Sons, New York, 1974).

[25] H. Yeh, W. Li and Y. Kodama, Mach reflection and KP solitons in shallow water, *Eur. Phys. J. : Special Topics*, **185** (2010) 97-111.