Fokker-Planck equations for Marcus stochastic differential equations driven by Lévy processes

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Abstract

Marcus stochastic differential equations (SDEs) often are appropriate models for stochastic dynamical systems driven by non-Gaussian Lévy processes and have wide applications in engineering and physical sciences. The probability density of the solution to an SDE offers complete statistical information on the underlying stochastic process. Explicit formula for the Fokker-Planck equation, the governing equation for the probability density, is well-known when the SDE is driven by a Brownian motion. In this paper, we address the open question of finding the Fokker-Planck equations for Marcus SDEs in arbitrary dimensions driven by non-Gaussian Lévy processes. The equations are given in a simple form that facilitates theoretical analysis and numerical computation. Several examples are presented to illustrate how the theoretical results can be applied to obtain Fokker-Planck equations for Marcus SDEs driven by Lévy processes.

Key Words: Fokker-Planck equation, Marcus stochastic differential equations, Stochastic dynamical systems, non-Gaussian white noise, Lévy processes
1 Introduction

Stochastic differential equations (SDEs) have been widely used as mathematical models for stochastic dynamical systems [1, 2, 3]. Solutions of the SDEs, which are usually nowhere differentiable, are often interpreted in terms of some stochastic integral. The definition of the solutions may not be unique, because it depends on the specific stochastic integral involved.

Stochastic dynamical systems under excitation of Gaussian white noise are often modeled by SDEs driven by Brownian motions [1, 2], for which there are two popular definitions, namely, Itô SDEs and Stratonovich SDEs. While the former finds wide application in finance, econogy and biology, the latter turns out to be more appropriate models in engineering and physical sciences like mechanics and physics [2, 4].

Stochastic dynamical systems under excitation of non-Gaussian white noise are often modeled by SDEs driven by Lévy processes. The SDEs driven by Lévy processes can be defined in sense of Itô or Marcus. The Marcus SDEs [5, 6, 7, 8] can be regarded as the generalization of Stratonovich SDEs. In addition to the Stratonovich correction term, Marcus SDEs have an extra correction term due to the jumps. It is recently shown in [8] that the DiPaola-Falsone SDEs [9, 10], which are widely used in engineering and physics [11, 12], are actually equivalent to Marcus SDEs.

One of the main tasks in the research field of stochastic dynamical systems is to quantify how uncertainty propagates and evolves. Given the SDE model of a stochastic system, there are many methods available to achieve the goal. One of them is to study how the moments of the solution evolve in time [4]. Although the method of moments can provide some important information for the uncertainty, the statistical feature of the non-Gaussian distributions can not be fully captured by a finite number of moments in general. Another popular method is to obtain statistical information about the solution of the SDE path-wisely using Monte Carlo simulations [13]. However, the efficiency of this method is significantly limited by the accuracy and convergence rate of Monte Carlo simulation. To better quantify the uncertainty propagation and evolution, it is highly desirable to obtain the probability density of the solution, which contains the complete statistical information about the uncertainty.
Fokker-Planck equations, also known as Fokker-Planck-Kolmogorov equations or forward Kolmogorov equations, are deterministic equations describing how probability density functions evolve. For Itô or Stratonovich SDEs driven by Brownian motions, there are well established formulas to write down the corresponding Fokker-Planck equations [1, 2, 4]. For example, consider the following Itô SDE

\[
dX(t) = f(X(t))dt + \sigma(X(t))dB(t), \quad X(0) = x_0 \in \mathbb{R}^d, \tag{1}
\]

where \(X(t) = (X_1(t), X_2(t), \ldots, X_d(t))^T \in \mathbb{R}^d, f = (f_1, f_2, \ldots, f_d)^T : \mathbb{R}^d \to \mathbb{R}^d, \sigma = (\sigma_{ij})_{d \times n} : \mathbb{R}^d \to \mathbb{R}^{d \times n}\). \(B(t)\) is an \(n\)-dimensional Brownian motion, and \(f\) and \(g\) satisfy certain smoothness conditions. The probability density function \(p(x, t)\) for the solution \(X(t)\) in (1) can be expressed as [1]

\[
\frac{\partial p(x, t)}{\partial t} = -\sum_{i=1}^d \frac{\partial}{\partial x_i} [f_i(x)p(x, t)] + \sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial x_j} [D_{ij}(x)p(x, t)], \tag{2}
\]

where \(D_{ij}(x) = \sum_{k=1}^n \sigma_{ik}(x)\sigma_{kj}(x)\).

However, up to date, there is no formula available for the Fokker-Planck equations corresponding to SDEs driven by general non-Gaussian Lévy processes. The reason lies in that Fokker-Planck equations require explicit expressions for the adjoint of the infinitesimal generator associated with the solution of the SDEs. These explicit forms, however, are often unavailable for SDEs driven by non-Gaussian Lévy processes. For Itô SDEs driven by Lévy processes, although the general result is unknown, there are some special cases where the corresponding Fokker-Planck equations are available, e.g., Schertzer et al. [14] derived Fokker-Planck equations for Itô SDEs driven by \(\alpha\)-stable processes. There is little existing result for Fokker-Planck equations for Marcus SDEs. The only published result we can find so far is given by [15], where the authors have derived Fokker-Planck equations for one-dimensional Marcus SDEs driven by scalar Lévy processes under the very stringent condition that the noise coefficient is strictly nonzero. However, for Marcus SDEs, driven by multi-dimensional Lévy processes and with more general noise coefficients, a form of Fokker-Planck equations that is accessible for computation still remains an open problem.

Fokker-Planck equations have been a widely-used important tool to quantify the propagation of uncertainty in stochastic dynamical systems driven by Brow-
nian motions. In contrast, the unavailability of the Fokker-Planck equations for Marcus SDEs driven by Lévy processes poses as a significant obstacle on quantifying the uncertainty in stochastic dynamical systems under excitation of non-Gaussian Lévy noise. We note that since the Fokker-Planck equations are not available for Marcus SDEs with non-Gaussian Lévy noise, effort has been made to obtain approximate Fokker-Planck equations by stochastic averaging under the condition of small parameters, see among others.

The main objective of this paper is to derive the Fokker-Planck equations for multi-dimensional Marcus SDEs driven by Lévy processes.

Let \( L(t) \) denotes the \( \mathbb{R}^n \)-valued Lévy process characterized by the generating triplet \((b, A, \nu)\), where \( b \) is a vector in \( \mathbb{R}^n \), \( A = (A_{ij}) \) is a positive definite \( n \times n \) matrix, and \( \nu \) is a measure defined on \( \mathbb{R}^n \setminus \{0\} \) satisfying

\[
\int_{\mathbb{R}^n \setminus \{0\}} (|y|^2 \land 1) \nu(dy) < \infty. \tag{3}
\]

Here \( y = (y_1, y_2, \cdots, y_n) \) is a vector in \( \mathbb{R}^n \), \(|y| = \sqrt{y_1^2 + y_2^2 + \cdots + y_n^2} \) is the usual Euclidean norm of \( y \), and the operation ‘\( \land \)’ is defined as \( a \land b = \min\{a, b\} \).

By Lévy-Itô decomposition, the Lévy process \( L(t) \) can be expressed as

\[
L(t) = bt + B(t) + \int_{|y|<1} y \tilde{N}(t, dy) + \int_{|y|\geq1} y N(t, dy), \tag{4}
\]

where \( B(t) \) is the Brownian motion with the covariance matrix \( A \), and \( N(t, dy) \) is the Poisson random measure defined as

\[
N(t, Q)(\omega) = \#\{s | 0 \leq s \leq t; \Delta L(s)(\omega) \in Q\}, \tag{5}
\]

with \( \#\{\cdot\} \) representing the number of elements in the set ‘\( \cdot \)’. \( Q \) is a Borel set in \( \mathcal{B}(\mathbb{R} \setminus \{0\}) \), \( \Delta L(t) \) the jumps of \( L(t) \) at time \( t \) defined as \( \Delta L(t) = L(t) - L(t^-) \), and \( \tilde{N}(dt, dy) \) is the compensated Poisson measure defined as

\[
\tilde{N}(dt, dy) = N(dt, dy) - dt \nu(dy).
\]

Each component \( L_j(t) \) \((j = 1, 2, \cdots, n)\) of \( L(t) \) can be expressed as

\[
L_j(t) = b_j t + \sum_{k} \tau_{jk} B_k(t) + \int_{|y|<1} y_j \tilde{N}(t, dy) + \int_{|y|\geq1} y_j N(t, dy), \quad 1 \leq j \leq n, \tag{6}
\]

where \( b_j \) is the \( j \)-th component of vector \( b \), \( B_k(t) \) \((k = 1, 2, \cdots, n)\) are independent standard scalar Brownian motions and \( \tau = (\tau_{ij}) \) is a \( n \times n \) real matrix, which is related to the covariance matrix \( A \) by \( A = (A_{ij}) = \left( \sum_{k=1}^{n} \tau_{ik} \tau_{kj} \right) \).
Note that, throughout of this paper, if \( x \) represent an element in \( \mathbb{R}^n \), then \( x_j \) will be used to denote the \( j \)-th component of \( x \) without further claim.

Consider the SDE in sense of Marcus,

\[
dX(t) = f(X(t))dt + \sigma(X(t)) \circ dL(t), \quad X(0) = x_0 \in \mathbb{R}^d, \tag{7}
\]

where \( X(t) = (X_1(t), X_2(t), \cdots, X_d(t))^T \in \mathbb{R}^d, f = (f_1, f_2, \cdots, f_d)^T : \mathbb{R}^d \to \mathbb{R}^d, \sigma = (\sigma_{ij})_{d \times n} : \mathbb{R}^d \to \mathbb{R}^{d \times n}. \) The solution to the SDE (7) is interpreted as

\[
X(t) = X(0) + \int_0^t f(X(s))ds + \int_0^t \sigma(X(s)) \circ dL(s), \tag{8}
\]

where \( X(s-) \) is the left limit \( \lim_{u \to s, u < s} X(u) \), and \( \circ \) indicates Marcus integral \([5, 6, 7, 3]\) defined by

\[
\int_0^t \sigma(X(s-)) \circ dL(s) = \int_0^t \sigma(X(s-))dL(s) + \frac{1}{2} \int_0^t \tilde{\sigma}(X(s-))ds + \sum_{0 \leq s \leq t} [H(\Delta L(s), X(s-)) - X(s-) - \sigma(X(s-))\Delta L(s)]. \tag{9}
\]

Here \( \tilde{\sigma}(X(s-)) \) is a vector in \( \mathbb{R}^d \) with the \( i \)-th element as

\[
\tilde{\sigma}_i(X(s-)) = \sum_{m=1}^d \sum_{j=1}^n \sum_{l=1}^n \sigma_{ml}(X(s-)) \frac{\partial}{\partial x_m} \sigma_{ij}(X(s-)) A_{lj}, \quad i = 1, 2, \cdots, d. \tag{10}
\]

\( H : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^d \) is defined such that, for any \( u \in \mathbb{R}^d \) and \( v \in \mathbb{R}^n \), \( H(u, v) = \Phi(1) \) with \( \Phi : \mathbb{R} \to \mathbb{R}^d, r \to \Phi(r) \) being the solution to the initial value problem of the system of ordinary differential equations (ODEs)

\[
\frac{d\Phi(r)}{dr} = \sigma(\Phi(r))v, \quad \Phi(0) = u. \tag{11}
\]

It can be written in the form of components as

\[
\begin{align*}
\frac{d\Phi_i(r)}{dr} &= \sum_{j=1}^n \sigma_{ij}(\Phi_1(r), \Phi_2(r), \cdots, \Phi_d(r)) v_j, \\
\Phi(0) &= u_i, \quad i = 1, 2, \cdots, d. \tag{12}
\end{align*}
\]

The first term in the right hand side of (12) is the stochastic integral in sense of Itô. The second term is the correction term due to the continuous part of \( X(t) \), which is actually the correction term due to Stratonovich integral. The last term is the correction term due to the jumps of \( X(t) \). Note that Marcus integral reduces to Stratonovich integral when jumps are absent.
The goal of this paper is to derive the Fokker-Planck equations for the multidimensional Marcus SDEs \( (\mathbb{1}) \) driven by Lévy processes. Sections of this paper are organized as follows. We introduce the main results in section 2, give the proof of the main results in section 3, and present some examples in section 4 to illustrate how the main results is applied to obtain Fokker-Planck equations in some specific applications.

## 2 Main Result

Let \( p(x,t|X(0) = x_0) \) represent the probability density function for the solution \( X(t) \) of the SDE \( (\mathbb{1}) \), and for convenience, we drop the initial condition and simply denote it by \( p(x,t) \).

Throughout this work, we assume the following.

**Assumption (H1).** Probability density function \( p(x,t) \) for the solution \( X(t) \) defined in \( (\mathbb{1}) \) exists and is continuously differentiable with respect to \( t \) and twice continuously differentiable with respect to \( x \) for \( t \in \mathbb{R} \) and \( x \in \mathbb{R}^d \).

**Assumption (H2).** \( f : \mathbb{R}^d \to \mathbb{R}^d \) is continuously differentiable (i.e., \( f \in C^1(\mathbb{R}^d, \mathbb{R}^d) \)) and \( \sigma : \mathbb{R}^d \to \mathbb{R}^{d \times n} \) is Lipschitz and twice continuously differentiable (i.e., \( \sigma \in C^2(\mathbb{R}^d, \mathbb{R}^{d \times n}) \)).

The study of conditions for existence and regularity of the probability density for the solution of SDE \( (\mathbb{1}) \) is out of the scope of the current paper. We note that existence and regularity of probability densities for SDEs driven by Lévy processes are currently active research area. See [18, 19, 20] among others.

Our form of Fokker-Planck equation requires the mapping \( \tilde{H} \) defined below.

**Definition 1.** For \( u = (u_1, u_2, \cdots, u_d)^T \in \mathbb{R}^d \) and \( v = (v_1, v_2, \cdots, v_n)^T \in \mathbb{R}^n \), we introduce the mapping \( \tilde{H} \) that relies on \( \sigma = (\sigma_{ij}) \), the coefficient of the noise term in the SDE \( (\mathbb{1}) \) such that

\[
\tilde{H} : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^d, \quad (u, v) \mapsto \tilde{H}(u, v) = \Psi(1),
\]

where \( \Psi : \mathbb{R} \to \mathbb{R}^d, r \mapsto \Psi(r) \) is the solution of the ODEs

\[
\frac{d\Psi(r)}{dr} = -\sigma(\Psi(r))v, \quad \Psi(0) = u.
\]
\textbf{Theorem 1 (Main result).} Suppose the assumptions $H1$ and $H2$ hold, then the probability density function $p(x, t)$ for the solution $X(t)$ to the SDE \[(7)\] satisfies the following equation

\[
\frac{\partial p(x, t)}{\partial t} = -\sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left[ f_i(x) + \sum_{j=1}^{n} \sigma_{ij}(x)b_j + \frac{1}{2} \sum_{m=1}^{d} \sum_{j=1}^{n} \frac{\partial \sigma_{ij}(x)}{\partial x_m} \sigma_{ml}(x)A_{lj} \right] p(x, t) \\
+ \frac{1}{2} \sum_{i,m=1}^{d} \sum_{j,l=1}^{n} \frac{\partial^2}{\partial x_i \partial x_m} \left[ \sigma_{ij}(x)\sigma_{ml}(x)A_{lj}p(x, t) \right] \\
+ \int_{\mathbb{R}^n \setminus \{0\}} p(\tilde{H}(x, y), t) \left| \frac{\partial \tilde{H}(x, y)}{\partial x} \right| - p(x, t) + \sum_{i=1}^{d} \sum_{j=1}^{n} y_j I|y|<1(y) \frac{\partial}{\partial x_i} (\sigma_{ij}(x)p(x, t)) \right] \nu(dy),
\]

where $(b, A, \nu)$ is the generating triplet of the Lévy process $L(t)$, $\tilde{H}$ is defined in \[(13)\] and \[(14)\], and $\left| \frac{\partial \tilde{H}(x, y)}{\partial x} \right|$ is the Jacobian of $\tilde{H}(x, y)$ with respect to $x$, defined as

\[
\left| \frac{\partial \tilde{H}(x, y)}{\partial x} \right| = \begin{vmatrix} \frac{\partial \tilde{H}_1(x, y)}{\partial x_1} & \ldots & \frac{\partial \tilde{H}_1(x, y)}{\partial x_d} \\ \ldots & \ldots & \ldots \\ \frac{\partial \tilde{H}_d(x, y)}{\partial x_1} & \ldots & \frac{\partial \tilde{H}_d(x, y)}{\partial x_d} \end{vmatrix}. \tag{16}
\]

\section{Proof of the main result}

The following lemma reveals the relationship between the mapping $H$, as defined in \[(10)\] and \[(11)\], and $\bar{H}$, as defined in \[(13)\] and \[(14)\].

\textbf{Lemma 1.} If $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times n}$ is Lipschitz continuous and continuously differentiable (i.e., $\sigma \in C^1(\mathbb{R}^d, \mathbb{R}^{d \times n})$), then

(i) for all $v \in \mathbb{R}^n$, $H(\cdot, v) : \mathbb{R}^d \to \mathbb{R}^d$ is invertible with the inverse mapping as $\bar{H}(\cdot, v) : \mathbb{R}^d \to \mathbb{R}^d$, i.e., for all $u \in \mathbb{R}^d$ and $v \in \mathbb{R}^n$,

\[
\bar{H}(H(u, v), v) = H(\bar{H}(u, v), v) = u; \tag{17}
\]

(ii) both $H(\cdot, v)$ and $\bar{H}(\cdot, v)$ are continuously differentiable.

\textit{Proof of Lemma 1.} Note that $\sigma$ is Lipschitz continuous, then the solution of the ODE \[(11)\] or \[(14)\] exists globally and is unique.

Let $\xi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d, (r, u) \mapsto \xi(r, u) = \Phi(r)$ represent the flow of ODE \[(11)\] and $\xi_r : \mathbb{R}^d \to \mathbb{R}^d, u \mapsto \xi(r, u)$ be the time-$r$ mapping. The uniqueness of the
ODE solution implies that

$$\xi_0 = u, \quad \xi_{r_1 + r_2} = \xi_{r_1} \circ \xi_{r_2}, \quad \forall r_1, r_2 \in \mathbb{R}. \quad (18)$$

(18) indicates that time-u mapping $\xi_r$ is invertible and $\xi_{-r}^{-1} = \xi_{-r}$. Moreover, it is well known that the time-$r$ mapping associated with (11) or (14) is as smooth as $\sigma$, which follows from the dependence of the ODE solution on the initial value [21].

Since for all $v \in \mathbb{R}^n$, $H(\cdot, v)$ and $\tilde{H}(\cdot, v)$ are actually time-1 mapping associated with (11) and (14), respectively, and the flow for (14) is exactly the time reversal version of that for (11), we get the Lemma immediately. \[\square\]

The following Lemma can be found in many textbooks on theory of distributions. See [22] among others.

**Lemma 2.** Suppose $\gamma_1 \in C^0(\mathbb{R}^n)$ and $\gamma_2 \in C^0(\mathbb{R}^n)$, if $\forall \varphi \in C^\infty_0(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \varphi(x) \gamma_1(x) \, dx = \int_{\mathbb{R}^n} \varphi(x) \gamma_2(x) \, dx$, then $\forall x \in \mathbb{R}^n$, $\gamma_1(x) = \gamma_2(x)$.

**Proof of Theorem 1** It follows from (4) and (7) that, for $1 \leq i \leq d$,

$$dX_i(t) = f_i(X(t)) \, dt + \sum_{j=1}^{n} \sigma_{ij}(X(t-))b_j \, dt + \sum_{j,k=1}^{n} \sigma_{ij}(X(t-)) \tau_{jk} \, dB_k(t)$$

$$+ \frac{1}{2} \sum_{j,l=1}^{n} \sum_{m=1}^{d} \sigma_{ml}(X(t-))A_{ij} \frac{\partial}{\partial x_m} \sigma_{lj}(X(t-)) \, dt$$

$$+ \int_{|y|<1} [H_i(X(t-), y) - X_i(t-)] \tilde{N}(dt, dy)$$

$$+ \int_{|y|\geq1} [H_i(X(t-), y) - X_i(t-)] N(dt, dy)$$

$$+ \int_{|y|<1} [H_i(X(t-), y) - X_i(t-)] - \sum_{j=1}^{n} \sigma_{ij}(X(t-)) y_j \nu(dy) \, dt. \quad (19)$$

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By using (14) and the Itô formula, \( \forall \varphi \in C_0^\infty(\mathbb{R}^d) \), we can get

\[
\varphi(X(t + \Delta t)) - \varphi(X(t)) = \int_t^{t+\Delta t} \sum_{i=1}^{d} \left( \frac{\partial}{\partial x_i} \varphi(X(s)) \right) f_i(X(s)) \, ds + \int_t^{t+\Delta t} \sum_{i=1}^{n} \sum_{j=1}^{d} \left( \frac{\partial}{\partial x_i} \varphi(X(s)) \right) \sigma_{ij}(X(s))b_j \, ds \\
+ \int_t^{t+\Delta t} \sum_{i=1}^{d} \left( \frac{\partial}{\partial x_i} \varphi(X(s)) \right) \sum_{j,k=1}^{n} \sigma_{ij}(X(s))\tau_{jk} \, dB_k(s) \\
+ \frac{1}{2} \int_t^{t+\Delta t} \sum_{i,m=1}^{d} \sum_{j,l=1}^{n} \left( \frac{\partial}{\partial x_i} \varphi(X(s)) \right) \left( \frac{\partial}{\partial x_m} \sigma_{ij}(X(s)) \right) \sigma_{ml}(X(s))A_{jl} \, ds \\
+ \frac{1}{2} \int_t^{t+\Delta t} \sum_{i,m=1}^{d} \sum_{j,l=1}^{n} \left( \frac{\partial^2}{\partial x_i\partial x_m} \varphi(X(s)) \right) \sigma_{ij}(X(s))\sigma_{ml}(X(s))A_{jl} \, ds \\
+ \int_t^{t+\Delta t} \int_{|y| < 1} [\varphi(H(X(s), y)) - \varphi(X(s))] \tilde{N}(ds, dy) \\
+ \int_t^{t+\Delta t} \int_{|y| \geq 1} [\varphi(H(X(s), y)) - \varphi(X(s))] \tilde{N}(ds, dy) \\
+ \int_t^{t+\Delta t} \int_{|y| < 1} \left[ \varphi(H(X(s), y)) - \varphi(X(s)) - \sum_{i=1}^{d} \left( \frac{\partial}{\partial x_i} \varphi(X(s)) \right) \sigma_{ij}(X(s))y_j \right] \nu(dy)ds.
\]

(20)

Taking expectation at both sides of (20), we get

\[
\int_{\mathbb{R}^d} \varphi(x)p(x, t + \Delta t) \, dx - \int_{\mathbb{R}^d} \varphi(x)p(x, t) \, dx \\
= \int_t^{t+\Delta t} \int_{\mathbb{R}^d} \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \varphi(x) f_i(x)p(x, s) \, dx \, ds + \int_t^{t+\Delta t} \int_{\mathbb{R}^d} \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \varphi(x) \left( \sum_{j=1}^{n} \sigma_{ij}(x)b_j \right) p(x, s) \, dx \, ds \\
+ \frac{1}{2} \int_t^{t+\Delta t} \int_{\mathbb{R}^d} \sum_{i,m=1}^{d} \sum_{j,l=1}^{n} \frac{\partial^2}{\partial x_i \partial x_m} \varphi(x) \sigma_{ij}(x)\sigma_{ml}(x)A_{jl} p(x, s) \, dx \, ds \\
+ \frac{1}{2} \int_t^{t+\Delta t} \int_{\mathbb{R}^d} \sum_{i,m=1}^{d} \sum_{j,l=1}^{n} \frac{\partial^2}{\partial x_i \partial x_m} \varphi(x) \sigma_{ij}(x)\sigma_{ml}(x)A_{jl} p(x, s) \, dx \, ds \\
+ \int_t^{t+\Delta t} \int_{\mathbb{R}^d} \int_{|y| \geq 1} \varphi(H(x, y)) - \varphi(x) \nu(dy) \, dx \, ds \\
+ \int_t^{t+\Delta t} \int_{\mathbb{R}^d} \int_{|y| < 1} \left[ \varphi(H(x, y)) - \varphi(x) - \sum_{i=1}^{d} \left( \frac{\partial}{\partial x_i} \varphi(x) \right) \sigma_{ij}(x)y_j \right] \nu(dy) \, dx \, ds.
\]

(21)

To obtain (21), we have changed the orders of integrals justified by Fubini’s theorem, since \( \varphi \in C_0^\infty(\mathbb{R}) \). Moreover, we rely on the following facts for getting
\[ E \left\{ \int_t^{t+\Delta t} \sum_{i=1}^d \sum_{j,k=1}^n \frac{\partial}{\partial x_i} (\varphi(X(s-))) \sigma_{ij}(X(s-)) \tau_{jk} dB_k(s) \right\} = 0, \quad (22) \]

\[ E \left\{ \int_{|y|<1} [\varphi(H(X(s-), y)) - \varphi(X(s-))] \tilde{N}(ds, dy) \right\} = 0, \quad (23) \]

\[ E \left\{ \int_{|y| \geq 1} [\varphi(H(X(s-), y)) - \varphi(X(s-))] \tilde{N}(ds, dy) \right\} = 0, \quad (24) \]

and

\[ E \left\{ \int_t^{t+\Delta t} \int_{|y| \geq 1} [\varphi(H(X(s-), y)) - \varphi(X(s-))] N(ds, dy) \right\} = 0 + \int_{|y| \geq 1} [\varphi(H(x, y)) - \varphi(x) p(x, t)] \nu(dy) dx ds. \quad (25) \]

The first identity in (25) follows from (24).

Dividing both sides of (21) by \( \Delta t \), and taking the limit of \( \Delta t \to 0 \), we deduce that

\[ \int_{\mathbb{R}^d} \varphi(x) \frac{\partial p(x, t)}{\partial t} dx = \int_{\mathbb{R}^d} \sum_{i=1}^d \frac{\partial \varphi(x)}{\partial x_i} f_i(x)p(x, t) dx + \int_{\mathbb{R}^d} \sum_{i=1}^d \frac{\partial \varphi(x)}{\partial x_i} \left( \sum_{j=1}^n \sigma_{ij}(x) b_j \right) p(x, t) dx \]

\[ + \frac{1}{2} \int_{\mathbb{R}^d} \sum_{i,m=1}^d \frac{\partial \varphi(x)}{\partial x_i} \left( \sum_{j,l=1}^n \sigma_{ij}(x) \sigma_{ml}(x) A_{ij} \right) p(x, t) dx \]

\[ + \frac{1}{2} \int_{\mathbb{R}^d} \sum_{i,m=1}^d \frac{\partial^2 \varphi(x)}{\partial x_i \partial x_m} \left( \sum_{j,l=1}^n \sigma_{ij}(x) \sigma_{ml}(x) A_{jl} \right) p(x, t) dx \]

\[ + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} \left[ \varphi(H(x, y)) - \varphi(x) - \sum_{i=1}^d \frac{\partial \varphi(x)}{\partial x_i} \left( \sum_{j=1}^n \sigma_{ij}(x) y_j \right) I_{|y|<1}(y) \right] p(x, t) \nu(dy) dx. \quad (26) \]

Using integration by parts, we rewrite the first four terms at the right-hand side.
the variable $x$ and
\[ = \int z R \left\{ \partial \phi \right\} \cdots \]
\[ = \int z R \left\{ \partial \phi \right\} \cdots \]
\[ = \frac{1}{2} \int \phi(x) p(x, t) \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left( f_i(x) + \sum_{j=1}^{n} \sigma_{ij}(x) b_j + \frac{1}{2} \sum_{j,l=1}^{n} \frac{\partial \sigma_{ij}(x)}{\partial x_m} A_{ij} \right) dx, \] (27)

and
\[ = \frac{1}{2} \int \phi(x) \sum_{i=1}^{d} \sum_{j,l=1}^{n} \frac{\partial^2 \phi}{\partial x_i \partial x_m} (\sigma_{ij}(x) \sigma_{ml}(x) A_{ij}) p(x, t) dx \]
\[ = \frac{1}{2} \int \phi(x) \sum_{i=1}^{d} \sum_{j,l=1}^{n} \frac{\partial^2 \phi}{\partial x_i \partial x_m} (\sigma_{ij}(x) \sigma_{ml}(x) A_{ij} \nu(y)) \] dx. \quad (28)

The last integral in the RHS of (26) becomes
\[ = \int_{\mathbb{R}^d} \nu(dy) \int_{\mathbb{R}^d} \phi(H(x, y)) - \phi(x) - \sum_{i=1}^{d} \frac{\partial \phi}{\partial x_i} \left( \sum_{j=1}^{n} \sigma_{ij}(x) y_j \right) I_{|y| < 1} \] \(p(x, t)\) dx.
\[ = \int_{\mathbb{R}^d} \nu(dy) \int_{\mathbb{R}^d} \phi(H(x, y)) - \phi(x) - \sum_{i=1}^{d} \frac{\partial \phi}{\partial x_i} \left( \sum_{j=1}^{n} \sigma_{ij}(x) y_j \right) I_{|y| < 1} \] \(p(x, t)\) dx. \quad (29)

Let $z = H(x, y)$. By Lemma 1, we have $x = \tilde{H}(z, y)$. Making the changing
the variable $x = \tilde{H}(z, y)$, we have
\[ = \int_{\mathbb{R}^d} \phi(H(x, y)) p(x, t) dx = \int_{\mathbb{R}^d} \phi(z) p(\tilde{H}(z, y), t) \left| \frac{\partial \tilde{H}(z, y)}{\partial z} \right| \] dz, \quad (30)

where $\left| \frac{\partial \tilde{H}(z, y)}{\partial z} \right|$ is the Jacobian of $\tilde{H}(z, y)$ with respect to $z$. Also, we have
\[ = \int_{\mathbb{R}^d} \phi(z) \sum_{i=1}^{d} \frac{\partial \phi}{\partial x_i} \left( \sum_{j=1}^{n} \sigma_{ij}(x) y_j I_{|y| < 1} \right) \] \(p(x, t)\) dx
\[ = - \int_{\mathbb{R}^d} \phi(z) \sum_{i=1}^{d} \frac{\partial \phi}{\partial x_i} \left( \sum_{j=1}^{n} \sigma_{ij}(x) y_j I_{|y| < 1} \right) \] \(p(x, t)\) dx. \quad (31)
Substituting (30) and (31) into (29), we get
\[
\int_{\mathbb{R}^d} \varphi(x) \left[ \varphi(H(x,y)) - \varphi(x) - \sum_{i=1}^d \frac{\partial \varphi(x)}{\partial x_i} \left( \sum_{j=1}^n \sigma_{ij}(x)y_j \right) I_{|y|<1}(y) \right] p(x,t) \nu(dy)
\]
\[
= \int_{\mathbb{R}^d} \varphi(x) \left[ p(H(x,y),t) \frac{\partial H(x,y)}{\partial x} - p(x,t) + \sum_{i=1}^d \frac{\partial}{\partial x_i} \left( \sum_{j=1}^n \sigma_{ij}(x)y_j I_{|y|<1}(y)p(x,t) \right) \right] \nu(dy).
\]

Substituting (27), (28) and (32) into (26), we get
\[
\int_{\mathbb{R}^d} \varphi(x) \frac{\partial p(x,t)}{\partial t} \, dx
\]
\[
= - \int_{\mathbb{R}^d} \varphi(x) \sum_{i=1}^d \frac{\partial}{\partial x_i} \left[ f_i(x) + \sum_{j=1}^n \sigma_{ij}(x)b_j + \frac{1}{2} \sum_{m=1}^d \sum_{j=1}^n \frac{\partial \sigma_{ij}(x)}{\partial x_m} \sigma_{ml}(x)A_{ij} \right] p(x,t) \, dx
\]
\[
+ \frac{1}{2} \int_{\mathbb{R}^d} \varphi(x) \sum_{i,m=1}^d \frac{\partial^2}{\partial x_i \partial x_m} \left( \sum_{j=1}^n \sigma_{ij}(x)\sigma_{ml}(x)A_{ji} \right) p(x,t) \, dx
\]
\[
+ \int_{\mathbb{R}^d} \varphi(x) \, dx \int_{\mathbb{R}^n \setminus \{0\}} \left[ p(H(x,y),t) \left| \frac{\partial H(x,y)}{\partial x} \right| - p(x,t) + \sum_{i=1}^d \frac{\partial}{\partial x_i} \left( \sum_{j=1}^n \sigma_{ij}(x)y_j I_{|y|<1}(y)p(x,t) \right) \right] \nu(dy).
\]

The main result (15) follows from Lemma 2.

4 Examples

According to the main results presented in Theorem 1 for a given SDE (7) driven by the Lévy process $L(t)$ with the generating triplet $(b, A, \nu)$, we only need solving the ODE (14) for the mapping $\tilde{H}$ to obtain the corresponding Fokker-Planck equation.

The following lemma, which can be verified easily from the definition, will be used in the examples to obtain the generating triplet of an $\mathbb{R}^n$-valued Lévy process given each of its component being a scalar Lévy process with known triplet and being independent of each other.

Lemma 3. Let $\{L_i(t) \mid i = 1, 2, \ldots, n\}$ be $n$ independent scalar Lévy processes with $L_i(t)$ having the generating triplet $(b_i, A_i, \nu_i)$. Then the $\mathbb{R}^n$-valued process $L(t) = (L_1(t), L_2(t), \ldots, L_n(t))$ is a $\mathbb{R}^n$-valued Lévy process with the generating triplet $(b, A, \nu)$ and $b = (b_1, b_2, \ldots, b_n)$, $A = \text{diag}(A_1, A_2, \ldots, A_n)$,
and $\nu(dx_1, dx_2, \cdots, dx_n) = \sum_{i=1}^n \left( \nu_i(dx_i) \prod_{k=1, k \neq i}^n \delta_0(dx_k) \right)$, where $\delta_0(\cdot)$ is the Dirac measure on $\mathbb{R}$ centered at $0$.

Example 1. One-dimensional Marcus SDE driven by $\alpha$-stable process

Consider the following SDE,

$$dX(t) = f(X(t))dt + X(t) \diamond dL(t), \quad X(0) = x_0 \in \mathbb{R},$$

where $X(t) \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz and continuously differentiable, $L(t)$ is a symmetry $\alpha$-stable process with the generating triplet $(b, A, \nu)$ as $A = 0$ and $\nu(dy) = \frac{dy}{|y|^{1+\alpha}}$. Equation (34) can be written in the form of (7) with $d = 1$, $n = 1$, and $\sigma(X(t)) = X(t)$.

Solving (14), which becomes

$$d\Phi(r) dr = -\Phi(r)v, \quad \Phi(0) = u,$$

we get

$$\tilde{H}(u, v) = ue^{-v}.$$ (36)

By substituting $b$, $A$, $\nu$, and $\tilde{H}$ into (15), we get the following Fokker-Planck equation which governs the probability density function of $X(t)$ in SDE (34),

$$\frac{\partial p(x, t)}{\partial t} = -\frac{\partial}{\partial x} \left[ (f(x) + bx) p(x, t) \right] + \int_{|y| < 1} \frac{p(xe^{-y}, t)e^{-y} - p(x, t)}{|y|^{1+\alpha}} dy I_{|y| < 1}(y) dy.$$ (37)

Example 2. Nonlinear oscillator under excitation of combined Gaussian and Poisson white noise

Consider the second-order SDE

$$\ddot{Y}(t) + Y(t) = f(Y(t), \dot{Y}(t)) + \dot{B}(t) + \dot{Y}(t) \diamond \dot{C}(t),$$

where $Y(t) \in \mathbb{R}$, $f : \mathbb{R}^2 \to \mathbb{R}$, $B(t)$ is a scalar Brownian motion, and $C(t)$ is a compound Poisson process expressed as

$$C(t) = \sum_{i=1}^{N(t)} r_i.$$ (39)

Here $N(t)$ is a Poisson process with intensity parameter $\lambda$ and $\{r_i, i = 1, 2, \cdots, N(t)\}$ are i.i.d random variables with probability distribution function $\rho(dx)$. Note
that describes an stochastic oscillator. The first term in the left-hand side of (38) represents the acceleration, the second term represents elastic force, and the RHS represents all other forces. Let \( X_1(t) = Y(t), X_2(t) = \dot{Y}(t) \), then (38) can be written as

\[
\begin{cases}
    dX_1(t) = X_2(t)dt, \\
    dX_2(t) = -X_1(t)dt + dB(t) + X_2(t) \circ dC(t)
\end{cases}
\]

It can be expressed in the form of (7) with \( X(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} \), \( f(X(t)) = \begin{pmatrix} X_2(t) \\ -X_1(t) \end{pmatrix} \), \( \sigma(X(t)) = \begin{pmatrix} 0 & 0 \\ 1 & X_2(t) \end{pmatrix} \), and \( L(t) = \begin{pmatrix} B(t) \\ C(t) \end{pmatrix} \). It follows from Lemma 3 that \( L(t) \) is a two-dimensional Lévy process with the generating triplet \((b, A, \nu)\) as \( b = 0, A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \nu(dy_1, dy_2) = \lambda \delta_0(dy_1) \rho(dy_2) \). Now the ODE (14) becomes

\[
\begin{cases}
    \frac{d\Phi_1(r)}{dr} = 0, \\
    \frac{d\Phi_2(r)}{dr} = -\Phi_1(r) - v_2 \Phi_2(r)
\end{cases}
\]

with \( \Phi(0) = u \). It follows from (40) and (13) that

\[\tilde{H}((u_1, u_2)^T, (v_1, v_2)^T) = \begin{pmatrix} u_1 \\ u_1 K(-v_2) + e^{-v_2} u_2 \end{pmatrix},\]

where \( K(\cdot) \) is defined as

\[
K(x) = \begin{cases} 
    \frac{1 - e^x}{x}, & \text{for } x \neq 0 \\
    -1, & \text{for } x = 0.
\end{cases}
\]

Let \( p(x_1, x_2, t) \) be the joint probability density function of \((X_1(t), X_2(t))\). From Theorem 1, the governing equation for \( p \) is

\[
\begin{align*}
    \frac{\partial p}{\partial t}(x_1, x_2, t) &= -\frac{\partial}{\partial x_1}[x_2 p(x_1, x_2, t)] + \frac{\partial}{\partial x_2} \left[ \left( -x_1 + \frac{1}{2} x_2 \right) p(x_1, x_2, t) \right] \\
    &\quad + \frac{1}{2} \frac{\partial^2}{\partial x_2^2} [x_2^2 p(x_1, x_2, t)] \\
    &\quad + \lambda \int_{\mathbb{R} \setminus \{0\}} [p(x_1, x_1 K(-y_2) + e^{-y_2} x_2, t) e^{-y_2} - p(x_1, x_2, t)] \rho(dy_2).
\end{align*}
\]
Example 3. We consider the SDEs in $\mathbb{R}^2$.
\[
\begin{align*}
  dX_1(t) &= f_1(X_1(t), X_2(t))dt + X_2(t) \circ dL_1(t), \\
  dX_2(t) &= f_2(X_1(t), X_2(t))dt + X_1(t) \circ dL_2(t)
\end{align*}
\] (44)

with initial value $X_0 = x_0 \in \mathbb{R}^2$, where $L_1(t)$ and $L_2(t)$ are two independent $\alpha$-stable processes with generating triplets as $(1, 0, \nu_1)$ and $(1, 0, \nu_2)$, respectively. $\nu_1$ and $\nu_2$ are defined as $\nu_1(dy) = \frac{dy}{|y|^{1+\alpha_1}}$ and $\nu_2(dy) = \frac{dy}{|y|^{1+\alpha_2}}$.

Note that (44) can be expressed in the form of (7) with $\sigma(x) = \begin{pmatrix} x_2 & 0 \\ 0 & x_1 \end{pmatrix}$ and $L(t) = \begin{pmatrix} L_1(t) \\ L_2(t) \end{pmatrix}$. By Lemma 3, $L(t)$ is a two dimensional Lévy process with the generating triplet as $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\nu(dy_1, dy_2) = \delta_0(dy_1)\nu_2(dy_2) + \delta_0(dy_2)\nu_1(dy_1)$.

Solving ODE (14), which now becomes
\[
\begin{align*}
  \frac{d\Phi_1(r)}{dr} &= -v_1\Phi_2(r), \\
  \frac{d\Phi_2(r)}{dr} &= -v_2\Phi_1(r).
\end{align*}
\] (45)

with $(\Phi_1(0), \Phi_2(0)) = (u_1, u_2)$, we get the mapping $\tilde{H}$ as
\[
\tilde{H}((u_1, u_2)^T, (v_1, v_2)^T) = \begin{pmatrix} u_1\cos(v_1v_2) - v_1u_2\overline{\sin}(v_1v_2) \\ -v_2u_1\overline{\sin}(v_1v_2) + u_2\overline{\cos}(v_1v_2) \end{pmatrix}
\] (46)

where $\overline{\cos}(\cdot)$ and $\overline{\sin}(\cdot)$ are functions defined as
\[
\overline{\cos}(x) = \begin{cases} 
  \cosh(\sqrt{|x|}), & \text{for } x \geq 0, \\
  \cos(\sqrt{|x|}), & \text{for } x < 0
\end{cases}
\] (47)

and
\[
\overline{\sin}(x) = \begin{cases} 
  \sinh(\sqrt{|x|}), & \text{for } x > 0, \\
  1, & \text{for } x = 0, \\
  \sin(\sqrt{|x|}), & \text{for } x < 0
\end{cases}
\] (48)

By substituting $b$, $A$, $\nu$, and $\tilde{H}$ into (15), we get the following Fokker-Planck equation for the probability density function $p(x_1, x_2, t)$ of the solution...
(X_1(t), X_2(t))^T in SDE \text{ as}
\begin{align*}
\frac{\partial p(x_1, x_2, t)}{\partial t} &= -\frac{\partial}{\partial x_1} [(f_1(x_1, x_2) + x_2) p(x_1, x_2, t)] \\
&\quad - \frac{\partial}{\partial x_2} [(f_2(x_1, x_2) + x_1) p(x_1, x_2, t)] \\
&\quad + \int_{\mathbb{R}\setminus0} \left[ p(x_1, x_2 - y_2 x_1, t) - p(x_1, x_2, t) + \frac{\partial}{\partial x_2} (x_1 p(x_1, x_2, t) y_2 I|y_2|<1(y_2)) \right] \frac{dy_2}{|y_2|^{1+\alpha_2}} \\
&\quad + \int_{\mathbb{R}\setminus0} \left[ p(x_1 - y_1 x_2, x_2, t) - p(x_1, x_2, t) + \frac{\partial}{\partial x_1} (x_2 p(x_1, x_2, t) y_1 I|y_1|<1(y_1)) \right] \frac{dy_1}{|y_1|^{1+\alpha_1}}
\end{align*}
(49)

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