POSITIVITY CRITERIA FOR LOG CANONICAL DIVISORS
AND HYPERBOLICITY

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Abstract. Let $X$ be a complex projective variety and $D$ a reduced divisor on $X$. Under a natural minimal condition on the singularities of the pair $(X, D)$, which includes the case of smooth $X$ with simple normal crossing $D$, we ask for geometric criteria guaranteeing various positivity conditions for the log-canonical divisor $K_X + D$. By adjunction and running the log minimal model program, natural to our setting, we obtain a geometric criterion for $K_X + D$ to be numerically effective as well as a geometric version of the cone theorem, generalizing to the context of log pairs these results of Mori. A criterion for $K_X + D$ to be pseudo-effective with mild hypothesis on $D$ follows. We also obtain, assuming the abundance conjecture and the existence of rational curves on Calabi-Yau manifolds, an optimal geometric sharpening of the Nakai-Moishezon criterion for the ampleness of a divisor of the form $K_X + D$, a criterion verified under a canonical hyperbolicity assumption on $(X, D)$. Without these conjectures, we verify this ampleness criterion with assumptions on the number of ample and non-ample components of $D$.

1. Introduction

Various notions of hyperbolicity have played important guiding roles in geometry and analysis throughout the centuries. In complex geometry, an intrinsic notion via the non-degeneracy of a holomorphically invariant pseudo-distance was formulated by S. Bloch. 2000 Mathematics Subject Classification. 32Q45, 14E30.

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Kobayashi in the late 1960’s from which arose one of his first conjectures, which asks that the canonical bundle of a compact complex manifold be ample provided that the manifold is hyperbolic, see the last section of [10]. This conjecture remains open above dimension one and, in the projective category, in dimension three and above. In the compact complex category, a well known criterion of Brody characterizes hyperbolicity by the absence of non-constant holomorphic images of \( \mathbb{C} \) (Brody hyperbolicity) and is the most natural notion in our context, see [14, Chap. 3]. Of pertinence here is the fundamental and intimately related fact in algebraic geometry that the canonical bundle of a complex projective manifold is numerically effective in the absence of rational curves. This fact is a direct corollary of the celebrated bend and break theorem of Mori in his solution of the Hartshorne conjecture, a theorem that was later refined to the utmost in his establishing the cone theorem. We are motivated by these same problems generalized to the setting of quasi projective varieties via a suitable category of singular projective pairs.

We now provide a quick complex hyperbolic geometric perspective to our problem, and to its formulation, via the conjecturally equivalent notions (again by S. Kobayashi, see [14, Chap. 4] and [11, Chap. 9]) of measure hyperbolicity and volume hyperbolicity with some modern ingredients thrown in. Without going into their definitions, the key point is that, for a complex space, it is measure hyperbolic if it is hyperbolic while it is projective and volume hyperbolic only if it is of general type (op. cit.). Thus, modulo the (known) facts about projective varieties of general type and their canonical models, the conjectured equivalence of these notions would give an affirmation of our problem above.

In the case of a pair \((X, D)\), where \(X\) is projective and \(D\) is a reduced divisor on \(X\), the natural notion of hyperbolicity would be that of hyperbolic embedding of \(X \setminus D\) in \(X\). When the pair \((X, D)\) consists of a smooth projective variety \(X\) and a simple normal crossing divisor \(D\) on \(X\), this notion is equivalent to the Brody hyperbolicity with respect to the irreducible decomposition of \(D\) in the sense given in \(\S 3\). Thus modulo the equivalence of measure hyperbolicity and being of log general type for \(X \setminus D\), and in a setting such that \(K_X + D\) can be used to define \(X \setminus D\) to be of log general type, hyperbolic embedding of \(X \setminus D\) in \(X\) should similarly imply the ampleness of \(K_X + D\).

We start with some preliminaries before stating our main results. We work over the field \(\mathbb{C}\) of complex numbers. An algebraic variety \(X\) is called Brody hyperbolic = BH (resp. Mori hyperbolic = MH) if the following hypothesis (BH) (resp. (MH)) is satisfied:

\(\text{(BH)}\) Every holomorphic map from the complex line \(\mathbb{C}\) to \(X\) is a constant map.

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1 For such a pair, that Brody hyperbolicity with respect to the irreducible decomposition of \(D\) implies hyperbolic embedding is a central observation of M. Green via Brody’s reparametrization lemma while the converse result is due to M. Zaidenberg, see [11] (3.6.13), (3.6.19)].
Every algebraic morphism from the complex line $\mathbb{C}$ to $X$ is a constant map. 

(Equivalently, no algebraic curve in $X$ has normalization equal to $\mathbb{P}^1$ or $\mathbb{C}$.)

Clearly, (BH) $\Rightarrow$ (MH).

Consider a pair $(X, D)$ of an (irreducible) algebraic variety $X$ and a (not necessarily irreducible or equi-dimensional) Zariski-closed subset $D$ of $X$. When $X$ is an irreducible curve and $D$ a finite subset of $X$, the pair $(X, D)$ is Brody hyperbolic (resp. MH) if $X \setminus D$ is Brody hyperbolic (resp. MH). Inductively, for an algebraic variety $X$ and a Zariski-closed subset $D$ of $X$, with $D = \bigcup_i D_i$ the irreducible decomposition, the pair $(X, D)$ is Brody hyperbolic (resp. MH) if $X \setminus D$ and the pairs $(D_k, D_k \cap (\bigcup_{i \neq k} D_i))$ for all $k$, are so. Note that the Zariski-closed subset $D_k \cap (\bigcup_{i \neq k} D_i)$ may not be equi-dimensional in general.

Here, it is crucial from the perspective of hyperbolic geometry that $D$ is a divisor and all the $D_k$'s are at least $\mathbb{Q}$-Cartier (so as to have equidimensionality above for example). But we will work with a bit less from the outset as demanded by our general setup.

The pair $(X, D)$ is called projective if $X$ is a projective variety and log smooth if further $X$ is smooth and $D$ is a reduced divisor with simple normal crossings.

From now on, we always assume that $D$ is a reduced Weil divisor for a pair $(X, D)$.

Recall that a divisor $F$ on a smooth projective variety $X$ is called numerically effective (nef), if $F.C \geq 0$ for all curves $C$ on $X$. We consider the following conjecture:

**Conjecture 1.1.** Let $(X, D)$ be a log smooth projective Mori hyperbolic (resp. Brody hyperbolic) pair. Then the log canonical divisor $K_X + D$ is nef (resp. ample).

Our main theorems below give an affirmative answer to Conjecture 1.1 but with further assumptions for the ampleness of $K_X + D$, such as the ampleness of at least $n-2$ irreducible Cartier components of $D$ for $n = \dim (X, D) := \dim X$. To precise them, we will need to refer to the following two standard conjectures on the structure of algebraic varieties.

**Conjecture 1.2. Abundance($l$):** Let $(X, D)$ be an $l$-dimensional log smooth projective pair whose log canonical divisor $K_X + D$ is nef. Then some positive multiple of $K_X + D$ is base point free, i.e. $K_X + D$ is semi-ample.

**Conjecture 1.3. CY($m$):** Let $X$ be an $m$-dimensional simply connected nonsingular projective variety with trivial canonical line bundle, i.e. a Calabi-Yau manifold. Then $X$ contains a rational curve.

We remark that Abundance($l$) is known to hold for $l \leq 3$ (even for dlt or lc pairs, see [2,1] and CY($m$) is known to hold for $m \leq 2$ (see [8], [17]).
Let $D$ be a reduced divisor on a projective variety $X$ and $\mathcal{D}$ the collection of terms in its irreducible decomposition $D = \sum_{i=1}^{s} D_i$. A stratum of $D$ is a set of the form $\bigcap_{i \in I} D_i$ ($= X$ when $I = \emptyset$) for some partition $\{1, \ldots, s\} = I \bigsqcup J$. We call a $\mathcal{D}$-rational curve a rational curve $\ell$ in a stratum $\bigcap_{i \in I} D_i$ of $D$ such that the normalization of $\ell \setminus (\bigcup_{j \in J} D_j)$ contains the complex line $\mathbb{C}$. We call a $\mathcal{D}$-algebraic 1-torus a rational curve $\ell$ in a stratum $\bigcap_{i \in I} D_i$ of $D$ such that $\ell \setminus (\bigcup_{j \in J} D_j)$ has the 1-dimensional algebraic torus $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ as its normalization. We call a closed subvariety $W \subseteq X$ a $\mathcal{D}$-compact variety if $W$ lies in some stratum $\bigcap_{i \in I} D_i$ of $D$ and $W \cap \bigcup_{j \in J} D_j = \emptyset$. A $\mathcal{D}$-compact variety $W \subseteq X$ is called a $\mathcal{D}$-compact rational curve, a $\mathcal{D}$-torus, a $\mathcal{D}$-$\mathbb{Q}$-torus, or a $\mathcal{D}$-$\mathbb{Q}$-CY variety if $W$ is a rational curve, an abelian variety, a $\mathbb{Q}$-torus, or a $\mathbb{Q}$-CY variety, respectively.

Here a projective variety is called a $\mathbb{Q}$-torus if it has an abelian variety as its finite étale (Galois) cover. A projective variety $X$ with only klt singularities (see [2.1] or [13]) is called a $\mathbb{Q}$-CY variety if some positive multiple of its canonical divisor $K_X$ is linearly equivalent to the trivial divisor. A simply connected $\mathbb{Q}$-CY surface with only Du Val singularities is called a normal $K3$ surface. A $\mathbb{Q}$-torus is an example of a smooth $\mathbb{Q}$-CY.

**Theorem 1.4.** Let $X$ be a smooth projective variety of dimension $n$ and $D$ a reduced divisor on $X$ with simple normal crossings. Then the closure $\overline{\mathrm{NE}}(X)$ of effective 1-cycles on $X$ is generated by the $(K_X + D)$-non negative part and at most a countable collection of extremal $\mathcal{D}$-rational curves $\{\ell_i\}_{i \in \mathbb{N}}$, $N \subseteq \mathbb{Z}$:

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X + D \geq 0} + \sum_{i \in \mathbb{N}} \mathbb{R}_{\geq 0}[\ell_i].$$

Further, $-\ell_i, (K_X + D)$ is in $\{1, 2, \ldots, 2n\}$, and is equal to 1 when $\ell_i$ is not a $\mathcal{D}$-compact rational curve. In particular, if the pair $(X, D)$ is Mori hyperbolic then $K_X + D$ is nef.

**Theorem 1.5.** Let $X$ be a smooth projective variety of dimension $n$ and $D$ a reduced divisor on $X$ with simple normal crossings. Fix an $r$ in $\{1, 2, \ldots, n\}$. Assume that Abundance($l$) holds for $l \leq r$ and that $D$ has at least $n - r + 1$ irreducible components amongst which at least $n - r$ are ample. If $K_X + D$ is not ample, then either it has non-positive degree on a $\mathcal{D}$-rational curve or on a $\mathcal{D}$-algebraic 1-torus, or some positive multiple of it is linearly equivalent to the trivial divisor on a smooth $\mathcal{D}$-$\mathbb{Q}$-CY variety $T$ of dimension $< r$. We can take $T$ to be a $\mathbb{Q}$-torus if further CY($m$) holds for $m < r$.

In particular, if Abundance($l$) and CY($m$) hold for $l \leq r$ and $m < r$ and the pair $(X, D)$ is Brody hyperbolic then $K_X + D$ is ample.

Since the two conjectures are known for $r = 3$ (see the remark above), an immediate corollary is the following generalization, to arbitrary dimensions for pairs, of results that were only known up to dimension two in the case $D = 0$ concerning hyperbolic varieties.
Theorem 1.6. Let \( X \) be a smooth projective variety of dimension \( n \) and \( D \) a reduced divisor on \( X \) with simple normal crossings such that \( D \) has at least \( n - 2 \) irreducible components amongst which at least \( n - 3 \) are ample. If \( K_X + D \) is not ample, then either it has non-positive degree on a \( D \)-rational curve or on a \( D \)-algebraic 1-torus, or some positive multiple of it is linearly equivalent to the trivial divisor on a \( D \)-torus \( T \) with \( \dim T \leq 2 \). In particular, if \( (X, D) \) is Brody hyperbolic, then \( K_X + D \) is ample.

We remark that Theorems 1.4, 1.6 and 1.5 are special cases of Theorems 3.1 and 3.2 (see also Theorem 3.3 Remark 5.3(1)) where we allow the pair \( (X, D) \) to be singular and \( D \) to be augmented with a fractional divisor \( \Gamma \) in a setting that is natural to our approach. Theorem 1.4 is also obtained in McQuillan-Pacienza [16] by a different method, but not our more general result in the singular case, Theorem 3.1. That article obtained the results in the setting of complete intersection “stacks” by a direct analysis of Mori’s bend and break procedure and includes for the most part our refined geometric version of the cone theorem for smooth pairs, Theorem 1.4 which we obtain however in our more general singular (dlt) setting, Theorem 3.1 and by a different and independent method.

By running the minimal model program, an easy consequence of this cone theorem is the following criterion for \( K_X + D \) to be pseudo-effective (the weakest form of positivity in birational geometry) under a mild condition on \( D \). It is a special case of Theorem 3.4.

Theorem 1.7. Let \( X \) be a smooth projective variety and \( D \) a nonzero simple normal crossing reduced divisor no component of which is uniruled. Then \( K_X + D \) is not pseudo-effective if and only if \( D \) has exactly one irreducible component and \( X \) is dominated by \( D \)-rational curves \( \ell \) with \( -\ell \cdot (K_X + D) = 1 \), \( \ell \cong \mathbb{P}^1 \) and \( \ell \setminus \ell \cap D \cong \mathbb{C} \) for a general \( \ell \).

In particular, \( K_X + D \) is pseudo-effective for a log smooth pair \( (X, D) \) such that \( D \) contains two or more non-uniruled components.

Key to our proof in the presence of Cartier boundary divisors is Kawamata’s result on the length of extremal rays, a fundamental result in the subject (see Lemma 4.3). Our proofs of Theorems 3.2 ~ 3.3 are inductive in nature and reduce the problem to questions on adjoint divisors in lower dimensions by adjunction. The log minimal model program (LMMP) is run formally without going into its technical details. Our inductive procedure is naturally adapted to answer some fundamental questions concerning adjoint divisors.

2Here by a singular pair, we will assume that it is a dlt pair (see §2.1 below). The assumption is natural (and in many respects the most general) as our proof is by induction on dimension from running the LMMP for singular pairs. It implies an explicit adjunction formula (Lemma 2.2) and that \( k \)-fold intersections of components of \( D \) are of pure codimension-\( k \) in \( X \), crucial in our inductive procedure.
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2. Preliminary results and the case of dimension two

We use the notation and terminology in [13], which is a basic reference for the log minimal model program (LMMP). We run the LMMP in our proofs, but detailed techniques or knowledge of LMMP are not required as precise references are always given.

2.1. Let $L$ be a $\mathbb{Q}$-Cartier divisor on a projective variety $X$, i.e. some integer multiple of $L$ is Cartier. $L$ is pseudo-effective if its class in the Néron-Severi space $\text{NS}_\mathbb{R}(X) := \text{NS}(X) \otimes \mathbb{R}$ is in the closure of the cone generated by the classes of effective divisors in $\text{NS}_\mathbb{R}(X)$. $L$ is numerically effective (or nef) if $\deg(L|_C) \geq 0$ for every curve $C$ on $X$. For two $\mathbb{Q}$-Cartier divisors $L_i$, we write $L_1 \sim_\mathbb{Q} L_2$ (linear equivalence) for some $m \geq 1$.

Let $\Gamma = \sum_{i=1}^s a_i \Gamma_i$ be a divisor on $X$ with $\Gamma_i$ distinct irreducible divisors and $a_i \in \mathbb{Q}$. Its integral part is given by $\lfloor \Gamma \rfloor := \sum_{i=1}^s \lfloor a_i \rfloor \Gamma_i$ where $\lfloor a_i \rfloor$ is the integral part of $a_i$.

Recall that a pair $(X, \Delta)$ of an effective Weil $\mathbb{Q}$-divisor $\Delta$ on a normal variety $X$ is called divisorial log terminal = dlt (resp. Kawamata log terminal = klt, or log canonical = lc) if for some log resolution (resp. for one (or equivalently for all) log resolution), the discrepancy of every exceptional divisor (resp. of every divisor) on the resolution satisfies $> -1$ (resp. $> -1$, or $\geq -1$). In all cases, $K_X + \Delta$ is $\mathbb{Q}$-Cartier. A dlt pair $(X, \Delta)$ is klt if and only if $\lfloor \Delta \rfloor = 0$. In case $\dim X = 2$, if $(X, \Delta)$ is dlt, then $X$ is $\mathbb{Q}$-factorial (so that all Weil divisors are $\mathbb{Q}$-Cartier). We refer to [13, Definition 2.34] and [3, §3] for details.

The following adjunction formula is key for our inductive process, see [3, Propositions 3.9.2 and 3.9.4] for a concise statement, and [12, Proposition 16.6, 16.7] for the proof. The pair in Lemma 2.2 being dlt implies by adjunction that all irreducible components $D_i$ ($1 \leq i \leq s$) of $D$ as well as those of their intersections are normal varieties, see [13, Corollary 5.52]. $D_1$ intersects $D - D_1$ transversally at a general point of their intersection (see [12, Proposition 16.6.1.1]). In particular, $(D - D_1)|_{D_1}$ is a well-defined Weil divisor.

Lemma 2.2. Let $X$ be a normal variety, $D$ a reduced Weil divisor and $\Gamma$ an effective Weil $\mathbb{Q}$-divisor such that the pair $(X, D + \Gamma)$ is dlt. Let $D_1 \subseteq D$ be an irreducible component. Then $D_1$ is a normal variety and there is an effective Weil $\mathbb{Q}$-divisor $\Delta$ such that

$$(K_X + D + \Gamma)|_{D_1} = K_{D_1} + (D - D_1)|_{D_1} + \Delta,$$
\((D - D_1)|_{D_1}\) is a reduced Weil divisor and the pair \((D_1, (D - D_1)|_{D_1} + \Delta)\) is dlt.

**Remark 2.3.** For a log smooth pair \((X, D)\), it is easily seen that if \((X, D)\) is Brody (resp. Mori) hyperbolic then so are the pairs \((D_1, (D - D_1)|_{D_1})\) and \((F, D|_F)\), where \(D_1\) is any irreducible component of \(D\) and \(F\) is a connected general fibre of a fibration \(X \to Y\) onto a normal variety \(Y\). By Lemma 2.2, the same claims hold if \((X, D)\) is dlt instead of log smooth (used in Lemma 4.1). We also note that if \((X, D)\) is dlt then so is \((F, D|_F)\).

Indeed, take a resolution \(\tau : X' \to X\) and the induced fibration \(X' \to Y\) with general fibre \(F'\). Then \(K_{F'} = K_{X'|F'}\) by adjunction. Since \(F\) is normal and chosen to meet transversally the singular locus \(\text{Sing} X\) of \(X\) at the general points of \(\text{Sing} X\) and since \(\tau_*K_{F'} = K_F\), we also have \(K_{X|F} = K_F\). We can now compare the discrepancies of \((X, D)\) and \((F, D|_F)\) to see that the latter is still dlt (e.g. by applying [13, Lemma 5.17(1)] repeatedly).

**Proposition 2.4.** Let \(X\) be a projective variety of dimension \(n \leq 2\), \(D\) a reduced divisor and \(\Gamma\) an effective Weil \(\mathbb{Q}\)-divisor such that the pair \((X, D + \Gamma)\) is dlt. Suppose that \(K_X + D + \Gamma\) is not ample and has strictly positive degree on every \(D\)-rational curve and \(D\)-algebraic 1-torus. Then either \((K_X + D + \Gamma)|_\ell \sim_{\mathbb{Q}} 0\) for a \(D\)-compact curve \(\ell\) which is smooth elliptic, or \(D = \Gamma = 0\) and \(X\) is either a simple abelian surface or a normal K3 surface with \(s \geq 1\) singularities (all Du Val) and infinitely many elliptic curves.

In particular, if the pair \((X, D)\) is Brody hyperbolic, then \(K_X + D + \Gamma\) is ample.

### 3. Main results for singular pairs, consequences and remarks

Theorems 3.1, 3.2 and 3.4 below include Theorems 1.4, 1.5 and 1.7 as special cases.

Let \(X\) be a normal projective variety and \(D\) a reduced Cartier divisor on \(X\). The pair \((X, D)\) is called BH, or MH with respect to a (Cartier) decomposition of \(D = \sum_{i=1}^s D_i\), if each \(D_i\) is a reduced (Cartier) divisor on \(X\), not necessarily irreducible, and both \(X \setminus D\) and \((\cap_{i \in I} D_i) \setminus (\cup_{j \in J} D_j)\) are respectively BH, or MH for every partition of \(\{1, \ldots, s\} = I \bigsqcup J\).

Note that, for a log smooth pair \((X, D)\), the respective definitions of hyperbolicity given in §1.1 are equivalent to the definitions of hyperbolicity above with respect to the irreducible decomposition of \(D\).

**Theorem 3.1.** Let \(X\) be a projective variety of dimension \(n\), \(D\) a reduced divisor that is the sum of a collection \(D = \{D_i\}_{i=1}^s\) of reduced Cartier divisors and \(\Gamma\) an effective Weil \(\mathbb{Q}\)-divisor on \(X\) such that the pair \((X, D + \Gamma)\) is divisorial log terminal. Then the closure
NE(X) of effective 1-cycles on X is generated by the \((K_X + D + \Gamma)\)-non negative part and at most a countable collection of extremal \(D\)-rational curves \(\{\ell_i\}_{i \in \mathbb{N}}\), \(N \subseteq \mathbb{Z}:\)

\[
\text{NE}(X) = \text{NE}(X)_{K_X + D + \Gamma} + \sum_{i \in \mathbb{N}} \mathbb{R}_{\geq 0}[\ell_i].
\]

Furthermore, \(-\ell_i.(K_X + D + \Gamma)\) is in \((0, 2n]\), and is even in \((0, 2)\) if \(\ell_i\) is not \(D\)-compact. In particular, if the pair \((X, D)\) is Mori hyperbolic with respect to the Cartier decomposition \(D = \sum_{i=1}^s D_i\) then \(K_X + D + \Gamma\) is nef.

One may take \(\Gamma = 0\) in Theorems 3.1–3.3. Such a \(\Gamma\) naturally appears in our inductive procedure as the ‘different’ in the adjunction formula Lemma 2.2.

We remark that Abundance(l) (for dlt pairs) and CY(m) always hold for \(l \leq 3\) and \(m \leq 2\) ([8], [17]) and that a normal K3 surface has infinitely many elliptic curves ([17]).

**Theorem 3.2.** Let \(X\) be a projective variety of dimension \(n\), \(D\) a reduced Weil divisor and \(\Gamma\) an effective Weil \(\mathbb{Q}\)-divisor on \(X\) such that the pair \((X, D + \Gamma)\) is divisorial log terminal (dlt). Assume one of the following three conditions.

1. \(n \leq 2\).
2. \(n = 3\), \(D\) is nonzero and Cartier.
3. \(n \geq 4\), the pair \((X, D + [\Gamma])\) is log smooth, there is an \(r \in \{1, 2, \ldots, n\}\) such that \(D\) has at least \(n - r + 1\) irreducible components amongst which at least \(n - r\) are ample and Abundance(l) (for dlt pairs) holds for \(l \leq r\).

Then \(K_X + D + \Gamma\) is ample, unless either \((K_X + D + \Gamma)\) has non-positive degree on a \(\mathcal{D}\)-rational curve or on a \(\mathcal{D}\)-algebraic 1-torus, or the restriction \((K_X + D + \Gamma)|_T \sim_{\mathbb{Q}} 0\) for a \(\mathcal{D}\)-\(\mathbb{Q}\)-CY variety \(T\). When \(T\) is singular, it is a normal K3 surface and \(n \in \{2, 3\}\). Otherwise, \(T\) can be taken to be an abelian surface or an elliptic curve in Cases (1), (2) and, if further CY(m) holds for \(m < r\), taken to be a \(\mathbb{Q}\)-torus with \(\dim T < r\) in Case (3).

In particular, if the pair \((X, D)\) is Brody hyperbolic then \(K_X + D + \Gamma\) is ample, provided that either \(n \leq 3\) or CY(m) holds for \(m < r\).

**Theorem 3.3.** Let \(X\) be a projective variety of dimension \(n\), \(D\) a reduced divisor that is the sum of a collection \(\mathcal{D} = \{D_i\}_{i=1}^s\) of reduced Cartier divisors for some \(s \geq 1\) and \(\Gamma\) an effective Weil \(\mathbb{Q}\)-divisor on \(X\) such that the pair \((X, D + \Gamma)\) is divisorial log terminal. Assume one of the following two conditions.

1. \(n = 4\), \(s \geq 2\) and \(D_1\) is irreducible and ample.
2. \(n \geq 4\), \(s \geq n - r + 1\) for some \(r \in \{1, 2, \ldots, n\}\), \(D_j\) is ample for all \(j \leq n - r + 1\) and Abundance(l) (for dlt pairs) holds for \(l \leq r\).

Then \(K_X + D + \Gamma\) is ample, unless either \((K_X + D + \Gamma)\) has non-positive degree on a \(\mathcal{D}\)-rational curve or on a \(\mathcal{D}\)-algebraic 1-torus, or the restriction \((K_X + D + \Gamma)|_T \sim_{\mathbb{Q}} 0\) for
some $\mathcal{D}$-$\mathbb{Q}$-CY variety $T$ ($\dim T < r$ in Case (2)). $T$ is smooth when the pair $(X, D + \lfloor \Gamma \rfloor)$ is log smooth and can be taken to be a $\mathbb{Q}$-torus when further $\text{CY}(m)$ holds for $m < r$.

**Theorem 3.4.** Let $X$ be a $\mathbb{Q}$-factorial projective variety, and $D$ a nonzero reduced Weil divisor with $D = \sum_{i=1}^{s} D_{i}$ the irreducible decomposition such that the pair $(X, D)$ is divisorial log terminal.

(1) Assume that no component of $D$ is uniruled and $K_{X} + D$ is not pseudo-effective. Then $D$ has exactly one irreducible component and $X$ is dominated by $\mathcal{D}$-rational curves $\ell$ with $-\ell.(K_{X} + D) = 1$, $\ell \cong \mathbb{P}^{1}$ and $\ell \setminus \ell \cap D \cong \mathbb{C}$ for a general $\ell$.

(2) Conversely, assume that $X$ has at worst canonical singularities and $X$ is dominated by $\mathcal{D}$-rational curves. Then $K_{X} + D$ is not pseudo-effective.

Of interest here is that the above geometric criterion for the pseudo-effectivity of $K_{X} + D$ in Theorem 3.4 is obtained naturally from Theorem 3.1 by running the log minimal model program for the pair $(X, D)$. In fact, by running the log minimal model program for a normal surface pair as in the proof of Proposition 2.4, we get the following slightly stronger consequence. It is noteworthy that Keel-M$^{\text{K}}$Kernan [9] has obtained the same conclusion in dimension two without the assumption of the absence of $\mathcal{D}$-rational curves in $D$. See §4.8 for the justification of these pseudo-effectivity criteria and for some generalizations.

**Proposition 3.5.** Let $X$ be a projective surface and $D$ a nonzero reduced Weil divisor with $D = \sum_{i=1}^{s} D_{i}$ the irreducible decomposition such that the pair $(X, D)$ is divisorial log terminal and there are no $\mathcal{D}$-rational curves in $D$. Assume that $K_{X} + D$ is not pseudo-effective. Then $D$ has exactly one irreducible component, and $X$ is dominated by $\mathcal{D}$-rational curves $\ell$ with $-\ell.(K_{X} + D) = 1$, $\ell \cong \mathbb{P}^{1}$ and $\ell \setminus \ell \cap D \cong \mathbb{C}$ for a general $\ell$.

**N.B.** In Theorem 3.3 (1), $D_{1}$ needs to be irreducible so that Theorem 3.2 (2) can be applied: if $D_{1} = D_{11} + D_{12}$ is reducible then the restriction $D_{2|D_{11}}$ might be zero. The ampleness assumption on some $D_{i}$’s in various theorems is due to the same reason.

4. **Proofs of Theorems**

In this section, we prove the results in §3.

Let $n \geq 1$. Let $(X, D + \Gamma)$ be a pair as in Theorem 3.1 (resp. 3.2 (3), or 3.3 (2)) but with $\dim X = n + 1$. Let $Y$ be an irreducible component of $D + \lfloor \Gamma \rfloor$. By Lemma 2.2

$$(K_{X} + D + \Gamma)|_{Y} = K_{Y} + (D + \lfloor \Gamma \rfloor - Y)|_{Y} + \Delta = K_{Y} + D_{Y} + \Gamma_{Y}$$

where $\Delta \geq 0$ and the pair $(Y, D_{Y} + \Gamma_{Y})$ is dlt. Here, if a Cartier decomposition $D = \sum_{i=1}^{s} D_{i}$ is involved, we define $D_{Y} := (\sum_{i \neq k} D_{i})|_{Y}$ and $\Gamma_{Y} := (D_{k} - Y + \lfloor \Gamma \rfloor)|_{Y} + \Delta$ when
with the above assumption and notation, the pair $(Y, D_Y + \Gamma_Y)$ satisfies the respective conditions in Theorems 3.1, 3.2, and 3.3 (2), and $\dim Y = n$.

The result below follows from Proposition 2.4 and Lemma 4.1.

Corollary 4.2. Let $(X, D)$ be a projective Brody hyperbolic pair with $\dim X \leq 3$ and $\Gamma \geq 0$ a Weil $\mathbb{Q}$-divisor such that the pair $(X, D + \Gamma)$ is dlt. Then the restriction $(K_X + D + \Gamma)|_G$ is an ample divisor on $G$ for every irreducible component $G$ of $D + [\Gamma]$.

Kawamata’s theorem, [6, Theorem 1], on the length of extremal rays is key to us.

Lemma 4.3. (see [6, Theorem 1]) Let $(X, \Delta)$ be a dlt pair and $g : X \to Y$ the contraction of a $(K_X + \Delta)$-negative extremal ray $R = \mathbb{R}_{>0}[\ell]$. Let $E$ be an irreducible component of the $g$-exceptional locus $\text{Exc}(g)$, i.e., the set of points at which $g$ is not locally an isomorphism. Let $d := \dim E - \dim g(E)$. Then we have:

(1) $E$ is covered by a family of (extremal) rational curves $\{\ell_t\}$ such that $g(\ell_t)$ is a point (and hence the class $[\ell_t] \in R$). In particular, $E$ is uniruled.

(2) Suppose further that $d = 1$. Then the $\ell_t$ in (1) can be chosen such that $-\ell_t.(K_X + \Delta) \leq 2d$.

Proof. If $(X, \Delta)$ is klt, then the lemma is just [6, Theorem 1]. When $(X, \Delta)$ is dlt, we have by [13, Proposition 2.43] that for any ample divisor $H$ on $X$, there is a constant $c > 0$ (depending on $H$) such that, for every $0 < \varepsilon << 1$, one can find a divisor $\Delta'$ on $X$ with $\Delta' \sim_{\mathbb{Q}} \Delta + \varepsilon cH$ and $(X, \Delta')$ klt. We choose $\varepsilon$ small enough such that $\ell$ is still $(K_X + \Delta')$-negative. Now [6] applies and $E$ is covered by $g$-contractible rational curves $\ell_t$ with $-\ell_t.(K_X + \Delta') \leq 2d$.

Suppose further that $d = 1$. Then these curves $\ell_t$ are irreducible components of the fibres of $g|_E : E \to g(E)$. Hence we may assume that this family $\{\ell_t\}$ is independent of the $\varepsilon$. Let $\varepsilon \to 0$. The lemma follows.

We remark that Lemma 4.3 holds without the assumption that $d = 1$. This can be extracted from [15, Theorem 10-3-1] after replacing definition of dlt there by ours, which is that of [13], and using [7, Theorem 1-2-5] to drop the $\mathbb{Q}$-factorial assumption there.

4.4. Proof of Theorem 3.1
Let \((X, D + \Gamma)\) be as in Theorem 3.1. The case \(n = \dim X = 1\) is clear. Thus we may assume that \(n \geq 2\). We proceed by induction on \(n\). By the LMMP for \((X, D + \Gamma)\) (see [4, Theorem 1.1]), the closed cone \(\overline{NE}(X)\) is generated by the closed cone \(\overline{NE}(X)_{K_X + D + \Gamma \geq 0}\) and countably many extremal rays \(R = \mathbb{R}_{\geq 0}[\ell]\) where \(\ell\) is a rational curve with \(-\ell.(K_X + D + \Gamma) \in (0, 2n]\). Let \(f : X \to X_1\) be the contraction of an extremal ray \(\mathbb{R}_{\geq 0}[\ell]\), and \(\text{Exc}(f) \subseteq X\) the \(f\)-exceptional locus. Let \(\lceil \Gamma \rceil\) be the integral part of \(\Gamma\).

If \(\ell \subseteq G\) for an irreducible component \(G\) of \(D + \lceil \Gamma \rceil\), then \(0 > \ell.(K_X + D + \Gamma) = \ell.(K_X + D + \Gamma)|_G\). By the induction on \(n\) and the adjunction in Lemma 4.1, the class \(\lceil \ell \rceil\) (on \(G\)) is parallel to \([\ell'] + [\ell'']\) where \([\ell']\) is a \((K_X + D + \Gamma)|_G\)-non negative effective class on \(G\) and \([\ell''] \subseteq G\) is a \(\mathcal{D}\)-rational curve with \(-\ell''.(K_X + D + \Gamma)|_G\) in \((0, 2(n - 1)]\) (and even in \((0, 2]\) when \(\ell''\) is not \(\mathcal{D}\)-compact). Since \(\mathbb{R}_{\geq 0}[\ell]\) is extremal, \([\ell]\) on \(X\) is parallel to (the image of \(X\) of) \([\ell'']\). We are done.

Therefore, we may add the extra assumption that \(\ell \not\subseteq (D + \lceil \Gamma \rceil)\) for any \([\ell] \in R\), and hence \(E_i \not\subseteq (D + \lceil \Gamma \rceil)\) for every irreducible component \(E_i\) of \(\text{Exc}(f)\). Since \(\ell \not\subseteq D\), \(0 > \ell.(K_X + D + \Gamma) \geq \ell.(K_X + \Gamma)\). Hence \(R\) is also a \((K_X + \Gamma)\)-negative extremal ray, and \((X, \Gamma)\) is dlt; see [13, Corollary 2.39] (to be used later).

Since \(D\) is Cartier, \(D_{|E_1}\) is an effective Cartier divisor on \(E_1\). If \(D = 0\) or if \(D \cap E_1 = \emptyset\), then the \(\ell_i\) in Lemma 4.3 is a \((K_X + D + \Gamma)\)-negative \(\mathcal{D}\)-compact rational curve. If \(f_{|E_1 \cap D}\) contracts a curve, also denoted by \(\ell\), to a point on \(X_1\), then \([\ell] \in R\) and \(\ell \subseteq D\), contradicting to our extra assumption. Thus, \(f_{|E_1 \cap D}\) is a finite morphism. Hence \(\dim f(E_1) \geq \dim f(E_1 \cap D) = \dim E_1 \cap D = \dim E_1 - 1\), so \(\dim f(E_1) = \dim E_1 - 1\).

By Lemma 4.3, \(E_1\) is covered by \(f\)-contractible (extremal) rational curves \(\ell\) such that \(-\ell.(K_X + \Gamma) \leq 2\). Now \(0 > \ell.(K_X + D + \Gamma) \geq -2 + \ell.D\). Hence, if \(\nu : \tilde{\ell} \to \ell\) is the normalization, then \(2 > \ell.D = \nu^*(D_{|\tilde{\ell}})\). So, \(D\) being Cartier and integral, the normalization of \(\ell \setminus D\) contains \(\mathbb{C}\). Thus \(\ell\) is a \(\mathcal{D}\)-rational curve. Further, \(0 < -\ell.(K_X + D + \Gamma) \leq 2 - \ell.D \leq 2\) (the latter being \(< 2\) when \(\ell\) is not \(\mathcal{D}\)-compact). This proves Theorem 3.1.

4.5. Proof of Theorems 3.2 and 3.3.

We proceed by induction on \(n = \dim X\). The case \(n = 1\) is clear. For the case of non-Cartier boundary, see Proposition 2.4. Thus we may assume that \(n \geq 2\) and \(D = \sum_i D_i\) is the decomposition into Cartier integral divisors (or you may assume \(n \geq 3\)). By the argument in [14,4], we may assume that \(K_X + D + \Gamma\) is nef in both Theorems 3.2 and 3.3.

**Lemma 4.6.** Let the assumptions be as in Theorems 3.2 and 3.3, assume the validity of these theorems in dimension \(n - 1\) and of Theorem 3.2 in dimension 3. Suppose that \(D_1\) is ample (true if \(n > r\) or in the case of Theorem 3.3(1)). Then \(K_X + D + \Gamma\) is ample.
Proof. Suppose on the contrary that \( K_X + D + \Gamma \) is not ample. Then, by Kleiman’s ampleness criterion, there is a nonzero class \([\ell]\) in the closure \( \overline{\text{NE}}(X) \) of effective 1-cycles on \( X \) (with \( \mathbb{R}\)-coefficients) such that \( \ell.(K_X + D + \Gamma) = 0 \). Since \( D_1 \subseteq D \) is ample, write \( D = D_\varepsilon + \Delta_\varepsilon \) with an ample \( \mathbb{Q}\)-Cartier divisor \( \Delta_\varepsilon = \varepsilon_1 D_1 + \varepsilon_2 D \) for some \( \varepsilon_i \in (0,1) \).

Now \((X, D_\varepsilon + \Gamma)\) is still dlt (see [13, Corollary 2.39]). Note that
\[
0 = \ell.(K_X + D + \Gamma) = \ell.(K_X + D_\varepsilon + \Gamma) + \ell.\Delta_\varepsilon > \ell.(K_X + D_\varepsilon + \Gamma).
\]

By the cone theorem [4, Theorem 1.1], \( \ell \) is parallel to \( \ell' + \ell'' \) for some class \([\ell']\) in \( \overline{\text{NE}}(X) \) and a \((K_X + D_\varepsilon + \Gamma)\)-negative extremal rational curve \( \ell'' \). Note that the nef divisor \( K_X + D + \Gamma \) is perpendicular to \( \ell \) and hence
\[
0 = \ell''.(K_X + D + \Gamma) = \ell'.(K_X + D + \Gamma).
\]

Let \( g : X \to X_2 \) be the contraction of the \((K_X + D_\varepsilon + \Gamma)\)-negative extremal ray \( \mathbb{R}_{>0}[\ell'] \). If \( \ell'' \) lies in an irreducible component \( G \) of \( D + [\Gamma] \), then \( 0 = \ell''.(K_X + D + \Gamma) = \ell.(K_X + D + \Gamma)|_G \), which contradicts the ampleness result in lower dimension by the inductive assumption as \( \dim G < \dim X \) (see Lemma 4.1 and note that, when \( n = 4 \), we reduce to Theorem 3.2(2) for all cases of Theorems 3.2 and 3.3). Thus we may assume that \( \ell'' \not\subseteq (D + [\Gamma]) \). So \( \ell'' \cap (D + [\Gamma]) \) is a non-empty finite set and no irreducible component \( E_1 \) of \( \text{Exc}(g) \) is contained in \( D + [\Gamma] \). Since we may assume to reach a contradiction that \( X \) has no \((K_X + D + \Gamma)\)-non-positive \( \mathcal{D}\)-rational curves, \( \ell'' \cap D \) is a non-empty finite set.

Thus \( 0 = \ell''.(K_X + D + \Gamma) > \ell''.(K_X + \Gamma) \). Hence \( \mathbb{R}_{>0}[\ell''] \) is also a \((K_X + \Gamma)\)-negative extremal ray. By the argument in §4.3 and noting that \( \ell'' \not\subseteq D \),
\[
\dim g(E_1) \geq \dim g(E_1 \cap D) = \dim E_1 \cap D = \dim E_1 - 1.
\]

By Lemma 4.3, \( E_1 \) is covered by rational curves \( \ell'' \) with \(-\ell''.(K_X + \Gamma) \leq 2 \). Moreover \( 0 = \ell''.(K_X + \Gamma) \geq -2 + \ell''.D \). Thus \( \ell''.D \leq 2 \), and \( \ell'' \) is a \((K_X + D + \Gamma)\)-non-positive \( \mathcal{D}\)-rational curve or \( \mathcal{D}\)-algebraic 1-torus as in §4.3. Contradiction.

By Lemma 4.6 and the assumption of Theorems 3.2 and 3.3, we may assume that \( n = r \), \( D \neq 0 \) and \( D \) is Cartier. This also includes the case of Theorem 3.2(2) where \( n = 3 \). (The case \( n = 2 \) with \( D = 0 \) is taken care of via Remark 5.3 by the same argument given below, or by referring to Proposition 2.4.) Since we have proved the nefness of \( K_X + D + \Gamma \), by the abundance assumption for dlt pairs of dimension \( n = r \), there is a fibration \( h : X \to Z \) with connected general fibre \( F \) such that
\[
K_X + D + \Gamma = h^*H \quad \text{and} \quad (K_X + D + \Gamma)|_F = h^*H|_F \sim_\mathbb{Q} 0
\]
for some ample \( \mathbb{Q}\)-divisor \( H \) on \( Z \). We divide into three cases: Case (I) \( \dim Z = 0 \), Case (II) \( 0 < \dim Z < \dim X \) and Case (III) \( \dim Z = \dim X \).
Case(I) dim Z = 0. Then \( K_X + D + \Gamma \sim Q 0 \). Thus for an irreducible component \( G \) of \( D \), we have \( (K_X + D + \Gamma)|_G \sim Q 0 \). There is such a \( G \) if \( D \) has at least \( n - r + 1 \geq 1 \) Cartier components as in Theorem 3.2 or 3.3.

Applying the adjunction as in Lemma 4.1 and replacing \( X \) by \( G \) (of dimension \( n - 1 = r - 1 \)) and further by an irreducible component of some \( D_1 \cap D_2 \cap \ldots \), we may assume that either \( \dim X \leq 1 \) which is clear; or \( D = 0 \), \( (X, \Gamma) \) is dlt and \( K_X + \Gamma \sim Q 0 \).

Consider the latter case. If \( \Gamma \neq 0 \), then \( X \) is uniruled and every rational curve \( C \) on \( X \) is a \( D \)-compact rational curve with \( (K_X + D + \Gamma)|_C \sim Q 0 \), which is the excluded case.

If \( \Gamma = 0 \), then \( K_X \sim Q 0 \) and hence \( X \) is \( D \)-compact variety which is \( Q \)-CY.

N.B. If the initial pair \((X, D + [\Gamma])\) is log smooth, then \( X \) (and subsequent namesakes of the pairs) are (log) smooth. We now have \( K_X \sim Q 0 \). By the Beauville-Bogomolov decomposition, either \( X \) is a \( Q \)-torus, or it has a Calabi-Yau manifold as its finite étale cover. Thus \( CY(m) \) (with \( m < r = n \)) implies, in the latter case, the existence of a \( D \)-compact rational curve \( C \) on the initial \( X \) with \( (K_X + D + \Gamma)|_C \sim Q 0 \).

Case(II) \( 0 < \dim Z < \dim X \). Then \( K_F + (D + \Gamma)|_F = (K_X + D + \Gamma)|_F = h^* H|_F \sim Q 0 \) as \( F \) is a fibre of \( h \). So Theorems 3.2 and 3.3 follow by the same arguments as for Case (I).

Case(III) \( \dim Z = \dim X \). Then \( K_X + D + \Gamma \) is nef and big. We may assume that \( h : X \rightarrow Z \) is birational but not isomorphic, \( \emptyset \neq \text{Exc}(h) \subset X \) the exceptional locus of \( h \), and \( \ell \subseteq \text{Exc}(h) \) a curve contracted by \( h \). Then \( (K_X + D + \Gamma)|_{\ell} = h^* H|_{\ell} \sim Q 0 \).

If \( \ell \subseteq G \) for an irreducible component \( G \) of \( D + [\Gamma] \), then \( (K_X + D + \Gamma)|_G \) is not ample and we use the abundance assumption on \( G \) and divide into Cases (I) - (III) again to conclude Theorems 3.2 and 3.3 by the induction on dimension.

Thus we may assume that \( \ell \not\subseteq (D + [\Gamma]) \). Hence \( \ell \cap (D + [\Gamma]) \) is a finite set and \( \ell . D \geq 0 \). So, no irreducible component of \( \text{Exc}(h) \) is contained in \( D + [\Gamma] \).

Consider first the case that \( \ell . (K_X + \Gamma) < 0 \) for some \( h \)-contractible curve \( \ell \). Then \( \ell \) is parallel to \( \ell' + \ell'' \) for some class \([\ell'] \in \overline{NE}(X) \) and a \((K_X + \Gamma)\)-negative extremal rational curve \( \ell'' \) by the cone theorem (see [4 Theorem 1.1]). Note that the nef divisor \( K_X + D + \Gamma \) is perpendicular to \( \ell \) and hence

\[ 0 = \ell'' . (K_X + D + \Gamma) = \ell' . (K_X + D + \Gamma). \]

Thus \( \ell'' \not\subseteq (D + [\Gamma]) \) by the same argument as above for \( \ell \) (for later use).

Let \( h_1 : X \rightarrow Z_1 \) be the contraction of the extremal ray \( \mathbb{R}_{>0}[\ell''] \). Since \( \ell''. h^* H = 0 \) with \( H \) ample, every irreducible component \( E_1 \) of the exceptional locus of \( h_1 \) is a subset of \( \text{Exc}(h) \) and hence is not contained in \( D + [\Gamma] \). As in the proofs of Lemma 4.6 and 4.14 and by Lemma 4.3 \( E_1 \) is covered by rational curves \( \ell'' \) with \( -\ell''. (K_X + \Gamma) \leq 2 \). Now \( 0 = \ell''. (K_X + D + \Gamma) \geq -2 + \ell''. D \). So \( \ell''. D \leq 2 \) and Theorems 3.2 and 3.3 hold as in 4.14.
Consider now the case that $\ell.(K_X + \Gamma) \geq 0$ for every curve $\ell$ on $X$ contracted by $h$. Then $0 = \ell.(K_X + D + \Gamma) \geq \ell.D \geq 0$. So $\ell.(K_X + \Gamma) = \ell.D = 0$ and $\ell \cap D = \emptyset$. Thus $\text{Exc}(h) \cap D = \emptyset$. By the construction of the birational morphism $h : X \to Z$, $K_X + D + \Gamma = h^*(K_Z + h_*(D + \Gamma))$. So when $h$ is a small contraction, a ‘good log resolution’ for the dlt pair $(X, D + \Gamma)$ as in $[13]$ Theorem 2.44(2) is also a ‘good log resolution’ for $(Z, h_*(D + \Gamma))$ and hence the latter is also dlt. Thus by $[5]$ Corollary 1.5], every fibre of $h : X \to Z$ is rationally chain connected. Hence the above $\ell$ can be chosen to be a rational curve away from $D$, and is a $D$-compact rational curve with $(K_X + D + \Gamma)|_\ell = h^*H_\ell \sim_Q 0$, which is the excluded case.

Therefore, we may assume that $h$ contracts an irreducible divisor $S \subset \text{Exc}(h)$. By Lemma $4.7$ below, a general fibre $F_S$ of $h_{|S} : S \to h(S)$ is uniruled. Since $S \cap D \subset \text{Exc}(h) \cap D = \emptyset$, every rational curve $(F \supseteq \ell) \subseteq F_S \subseteq S$ is a $D$-compact rational curve with $(K_X + D + \Gamma)|_\ell \sim_Q 0$, which is the excluded case. This proves Theorems $3.2$ and $3.3$ modulo Lemma $4.7$ below applied to $K_V + B = K_X + D + \Gamma$.

**Lemma 4.7.** Let $V$ be a normal projective variety and $B$ an effective Weil $\mathbb{Q}$-divisor with coefficients $\leq 1$ such that $K_V + B$ is $\mathbb{Q}$-Cartier. Let $\varphi : V \to W$ be a birational morphism and $S \subset V$ an irreducible divisor. Suppose that $\varphi$ contracts $S$, $S \not\subset [B]$ and $C.(K_V + B) \leq 0$ for a general curve $C$ on $S$ contracted by $\varphi$. Then a general fibre $F_S$ of $\varphi_{|S} : S \to \varphi(S)$ is a uniruled variety.

**Proof.** Let $\sigma_1 : V' \to V$ be a dlt blowup of the pair $(V, B)$ (see $[1]$ Theorem 10.4]) with $E_{\sigma_1}$ the reduced exceptional divisor and $B'$ the sum of $E_{\sigma_1}$ and the proper transform $\sigma' B$ of $B$. Then the pair $(V', B')$ is $\mathbb{Q}$-factorial and dlt and we have

$$K_{V'} + B' + E' = \sigma_1^*(K_V + B).$$

for a $\sigma_1$-exceptional effective divisor $E'$. Let $S' := \sigma_1^* S$ be the proper transform of $S$, which is now $\mathbb{Q}$-Cartier. Write $B' := aS' + \overline{B}'$ for some $a \in [0,1)$ such that $S'$ is not a component of $\overline{B}'$. Let $\sigma_2 : V'' \to V'$ be a dlt blowup of the pair $(V', \overline{B'} + S')$. Thus, if $E_{\sigma_2}$ is the reduced exceptional divisor and $\overline{B}''$ the sum of $E_{\sigma_2}$ and $\sigma_2^* \overline{B}'$, then there is a $\sigma_2$-exceptional divisor $E'' \geq 0$ such that the pair $(V'', \overline{B}'' + S'')$ is $\mathbb{Q}$-factorial dlt with $S'' := \sigma_2^* S'$ and such that

$$K_{V''} + \overline{B}'' + S'' + E'' = \sigma_2^*(K_{V'} + \overline{B}' + S').$$

Let $C' \subset S'$ and $C'' \subset S''$ be the strict transforms of the general curve $C \subset S$ (contracted by $\varphi$). By the adjunction formula given in Lemma $2.2$ we have a $\mathbb{Q}$-Cartier adjoint divisor $(K_{V''} + \overline{B}'' + S'')|_{S''} = K_{S''} + \Delta$ on the normal variety $S''$ for some $\Delta \geq 0$ and
that the pair \((S'', \Delta)\) is dlt. Thus
\[
C''.(K_{S''} + \Delta) \leq C''.(K_{V''} + \overline{B''} + S'' + E'')|_{S''} =
\]
\[
C''.\sigma'_2(K_{V''} + B' + S')|_{S''} = C'.(\sigma'_1(K_V + B) - E') + (1 - a)C'.S' \leq C.(K_V + B) + (1 - a)C'.S' \leq (1 - a)C'.S'.
\]

Since the composition \(V'' \to V \to W\) is birational and contracts \(S'\), by the well-known negativity lemma (see e.g. [1, Lemma 3.6.2]), one can choose general curves \(C'\) such that \(C'.S' < 0\) and that a general fibre \(F_{S''}\) of the fibration \(\varphi_{S''}\) on \(S'\) (resp. \(F_{S''}\) of \(\varphi_{S''}\) on \(S''\)) induced by \(\varphi_{S}\) is covered by such curves \(C'\) (resp. \(C''\)). Thus \(C''.(K_{S''} + \Delta) < 0\).

Since \((S'', \Delta)\) is dlt, for a general fibre \(F_{S''}\) of the fibration \(\varphi_{S''}\), we have \((K_{S''} + \Delta)|_{S''} = K_{F_{S''}} + \Theta\) with \((F_{S''}, \Theta)\) dlt. Now \(C''.(K_{F_{S''}} + \Theta) < 0\) with \(C'' \subseteq F_{S''}\) covering \(F_{S''}\). Thus \(F_{S''}\) (and hence \(F_S\)) are uniruled. Indeed, let \(\tau : \tilde{F} \to F_{S''}\) be a resolution and write
\[
K_{F} + \tau^*(\Theta) + E' = \tau^*(K_{F_{S''}} + \Theta) + E''
\]

with \(\tau\)-exceptional effective divisors \(E', E''\). Since \(C''.(K_{F_{S''}} + \Theta) < 0\) for curves \(C''\) covering \(F_{S''}\), \(K_{F_{S''}} + \Theta\) on \(F_{S''}\) and \(\tau^*(K_{F_{S''}} + \Theta)\) on \(\tilde{F}\) are not pseudo-effective (see [2, Theorem 0.2]). Thus \(K_{\tilde{F}}\) is not pseudo-effective, for otherwise the left hand side of the displayed equation above would be pseudo-effective, and pushing forward the right side would give the pseudo-effectivity of \(K_{F_{S''}} + \Theta\), a contradiction. Therefore, \(\tilde{F}\) (and hence its image \(F_S\) on \(S\)) are uniruled by the well-known uniruledness criterion of Miyaoka-Mori.

This completes the proof of Lemma 4.7 and also of Theorems 3.2 and 3.3. \(\square\)

4.8. Proof of Theorem 3.4 and Proposition 3.5

Theorem 3.4(2) is well known (see e.g. [9, Lemma 5.11]).

We now prove Theorem 3.4(1). Set \(n := \dim X\). Since \((X, D)\) is dlt, [13, Proposition 2.43] implies that \((X, D_t)\) is klt for some \(D_t \sim_\QQ D + \Gamma + tH\) with \(H\) an ample divisor and \(t \to 0^+\). Since \(K_X + D\) is not pseudo-effective, the \(H\)-directed LMMP in [1, Proof of Corollary 1.3.3] implies the existence of a composition
\[
X = X_0 \to X_1 \to \cdots \to X_m = W
\]
of divisorial contractions and flips of \((K_X + D(i))\)-negative extremal rays with \(D(i)\) the push-forward of \(D\) and the existence of a Fano contraction \(\gamma : W \to Y\) of a \((K_W + B)\)-negative extremal ray \(\RR_{>0}[\ell]\) with \(F\) a general fibre, \(B := D(m)\). Note that \((X_i, D(i))\) is still \(\QQ\)-factorial and dlt. Also \(-(K_W + B)\) is relatively ample over \(Y\) and \(-(K_W + B)|_F = -(K_F + B|_F)\) is ample by the definition of Fano contraction. The divisors contracted by the birational map \(f = f_{m-1} \circ \cdots \circ f_0 : X \dasharrow W\) being uniruled (see e.g. [5, Corollary 1.5]), no component of \(D\) is contracted by \(f\) by assumption. Let \(B = \sum B_i\) be the
irreducible decomposition. It follows that no \( B_i \) is uniruled and \( B \) has the same number of irreducible components as \( D \). By Lemma 2.2, each \( B_i \) is normal and for some \( \Delta_i \geq 0 \),

\[
(K_W + B)|_{B_i} = K_{B_i} + (B - B_i)|_{B_i} + \Delta_i.
\]

Suppose that \( \gamma|_{B_i} : B_1 \to \gamma(B_1) \) is not generically finite so that its general fibre can be taken as the extremal curve \( \ell \). Since \(- (K_W + B)\) is relative ample over \( Y \), we have

\[
0 > \ell.(K_W + B) = \ell.(K_W + B)|_{B_i} \geq \ell.(K_{B_i} + \Delta_i).
\]

So \( K_{B_i} + \Delta_i \) is not pseudo-effective. Hence \( B_1 \) is uniruled, contradicting the assumption. Thus, we may assume that \( \gamma|_{B_i} : B_i \to \gamma(B_i) \) is generically finite for every \( i \). In particular, \( n - 1 \geq \dim Y \geq \dim \gamma(B_i) = \dim B_i = n - 1 \). So \( \dim Y = n - 1 = \dim \gamma(B_i) \) and \( \gamma(B_i) = Y \). Also \( \dim F = 1 \) and hence \( F \cong \mathbb{P}^1 \). The ampleness of \(- (K_F + B|_F)\) implies that \( |B \cap F| = 1 \). So \( B = B_1 \), i.e., \( B \) (and therefore \( D \)) is irreducible.

Let \( \Sigma \subset X_m \) be the image of the union of the exceptional locus and the indeterminacy locus of all \( f_i \) (0 \( \leq \) \( i \) < \( m \)). Then \( \dim \Sigma \leq n - 2 \). So \( \gamma(\Sigma) \) is a proper subset of \( Y \). Thus \( U_m := X_m \setminus (D(m) \cup \Sigma) \) is covered by \( F \setminus F \cap D(m) \cong \mathbb{C} \) for a general fibre \( F \) of \( \gamma \).

Regarding \( U_m \) as an open subset of \( X \setminus D \), \( X \) is dominated by \( \mathcal{D} \)-rational curves (\( F \cong F_0 \subset X \) which are the strict transforms of general \( F \subset X_m \) with \( F \cap \Sigma = \emptyset \). Further, \( -F_0.(K_X + D) = -F.(K_{X_m} + D(m)) = 2 - 1 = 1 \). Theorem 3.4 is proved.

We now prove Proposition 3.5. It follows from the same argument as that given in the proof of Proposition 2.4 applied to the LMMP of extremal \((K_X + D)\)-negative extremal birational contractions: \( X = X_0 \to \cdots \to X_m \). Indeed, each pair \((X_i, D(i))\), with \( D(i) \) the push-forward of \( D \) on \( X_i \), still satisfies the condition of Proposition 3.5. Also each \( X_i \to X_{i+1} \) is the contraction of a rational curve \( \ell_i \) which is not a component of \( D(i) \) (due to the absence of \( \mathcal{D} \)-rational curves), \( \ell_i \) is homeomorphic to \( \mathbb{P}^1 \) and meets \( D(i) \) at at most two points (see the description before N.B. in the proof of Proposition 2.4). Thus \( D(m) \) has the same number of irreducible components as \( D \). We may assume that there is a fibration \( X_m \to Y \) with a general fibre \( F \) such that \(- (K_{X_m} + D(m))|_F \) is ample.

As in the proof of Theorem 3.4 if \( \dim Y = 1 \), \( D \) is irreducible and the inverse image \( F_0 \) (\( \cong F \cong \mathbb{P}^1 \)) on \( X \) of \( F \) is a \( \mathcal{D} \)-rational curve with \( F_0 \setminus D \cong \mathbb{C} \) and such \( F \)'s cover \( X \). If \( \dim Y = 0 \), take an irreducible component \( B_1 \) of \( D(m) \) and we have the ample divisor \(- (K_{X_m} + D(m))|_{B_1} \) on \( B_1 \). Its positive degree in the same calculation as in (*) of the proof of Proposition 2.4 shows that \( B_1 \) is a \( \mathcal{D}(m) \)-rational curve and that the proper transform of \( B_1 \) on \( X \) is a \( \mathcal{D} \)-rational curve in \( D \) by the contraction process of \( X \to X_m \). This contradicts the assumption and Proposition 3.5 is proved.
We remark that Theorem 3.4 (1) is generalizable to higher dimensions via our techniques (assuming there are no \(D\)-rational curves in \(D\)), a result obtained directly via bend and break for higher dimensional smooth pairs in [16]. Indeed, with a little more work, one can show that Theorem 3.4 (1) is true for a threefold pair, assuming only that there are no \(D\)-rational curves in the one dimensional stratum of \(D\) and that no component of \(D\) is dominated by \(D\)-rational curves. We leave this as an exercise for the reader.

5. Appendix: Proof of Proposition 2.4

In this appendix, we prove Proposition 2.4. Let \(n := \dim X\). When \(n = 1\), it is clear. Assume that \(n = 2\). Since \((X, D + \Gamma)\) is dlt, so is \((X, D)\); \(X\) is \(\mathbb{Q}\)-factorial and klt (see [13, Propositions 4.11 and 2.41, Corollary 2.39]).

Assume that \(K_X + D + \Gamma\) is not nef. By the cone theorem [13, Theorem 3.4.7], there is a \((K_X + D + \Gamma)\)-negative extremal rational curve \(\mathcal{L}\). Let \(\xi : X \to Z\) be the corresponding contraction with \(F\) a general fibre. Then \(\xi\) is either a Fano or a divisorial contraction.

Consider first the case that \(\xi\) is a Fano contraction. Then \(-(K_X + D + \Gamma)|_F\) is ample. Thus by the adjunction formula of Lemma 2.2 if \(\dim Z = 1\) (resp. \(\dim Z = 0\) and \(D \neq 0\)), we have anti-ample divisor \((K_X + D + \Gamma)|_F = K_F + (D + \Gamma)|_F\) (resp. of \((K_X + D + \Gamma)|_G = K_G + (D - G)|_G + \Delta\) with \(\Delta \geq 0\), \(G\) an irreducible component of \(D\)), so \(F\) (resp. \(G\)) is a \(D\)-rational curve having negative intersection with \(K_X + D + \Gamma\). When \(\dim Z = 0\) and \(D = 0\), the ampleness of \(-(K_X + D + \Gamma)\) implies that \(X\) is uniruled, and hence every rational curve \(C\) on \(X\) is a \(D\)-compact rational curve with \(C.(K_X + D + \Gamma) < 0\).

Consider the remaining case that \(\xi\) is a divisorial (i.e., birational) contraction. Since \((X, D + \Gamma)\) is dlt, so are \((Z, \xi_* (D + \Gamma))\) and \((Z, \xi_* D)\). We have the following two cases.

If the rational curve \(\ell\) is contained in a component of \(D\), written \(\ell \subseteq D\) by abuse of notation, then, by Lemma 2.2 it would be normal and (transversally) meet the other components of \(D\) at at most one point by the calculation:

\[
(*): 0 > \ell.(K_X + D + \Gamma) = \deg(K_X + D + \Gamma)|_\ell = \\
\deg(K_\ell + (D - \ell)|_\ell + \Delta) \geq \deg(K_\ell + (D - \ell)|_\ell) = -2 + \ell.(D - \ell).
\]

Hence \(\ell\) is a \(D\)-rational curve having negative intersection with \(K_X + D + \Gamma\). Indeed, recall that \(X\) is \(\mathbb{Q}\)-factorial, so the intersections here make sense. Also, every intersection point \(P\) of \(\ell\) with other components of \(D\) is in the smooth locus of \(X\) and \(D\) is of normal crossing at \(P\) by the classification of dlt singularities \((X, D)\) (see [13, Theorem 4.15 (1)])).

Consider now the case that \(\ell \not\subseteq D\). A simple check via the short but full classification of the dlt pair \((Z, \xi_* D)\) by Alexeev [12, §3] or equivalently [13, Theorems 4.7 and 4.15] shows that \(\ell\) is a rational curve homeomorphic to \(\mathbb{P}^1\) (whose inverse on the minimal resolution of \(X\) is a tree of smooth rational curves) which meets \(D\) at at most two points with the
case of two points occurring only when \( \xi(\ell) \) is a smooth point of \( Z \) at which \( \xi_*(D + \Gamma) \) is of normal crossing (see Case (1) of [13, Theorem 4.15]). Thus \( \ell \) is a \( D \)-rational curve or \( D \)-algebraic 1-torus having negative intersection with \( K_X + D + \Gamma \). We note that non-Cartier \( D \) can and does occur in the two point case, at least locally, and it would be interesting to see if we can rule out this case via global considerations. These also follow from the proof of Claim 5.1 below.

N.B. This is where the Cartier-ness of \( D \) would be required if we wanted to deduce the nefness of \( K_X + D + \Gamma \) just from the Mori hyperbolicity of \( (X, D) \) (see Remark 5.2).

Thus we may assume that \( K_X + D + \Gamma \) is nef. By the abundance theorem for dlt pairs and \( n \leq 3 \) (see [8]), there is a morphism \( \sigma : X \rightarrow Y \) with a connected general fibre \( F \) and an ample divisor \( H \) on \( Y \) such that \( K_X + D + \Gamma = \sigma^*H \). We have \( (K_X + D + \Gamma)|_F = \sigma^*H|_F \sim_{\mathbb{Q}} 0 \). We divide into: Case (I) \( \dim Y = 0 \), Case (II) \( \dim Y = 1 \) and Case (III) \( \dim Y = 2 \).

Case (I) \( \dim Y = 0 \), i.e., \( K_X + D + \Gamma \sim_{\mathbb{Q}} 0 \). If \( G \) is an irreducible component of \( D \), then \( (K_X + D + \Gamma)|_G \sim_{\mathbb{Q}} 0 \) and \( G \) is a \( D \)-rational curve, a \( D \)-algebraic 1-torus or a \( D \)-torus.

Thus we may assume that \( D = 0 \) and \( K_X + \Gamma \sim_{\mathbb{Q}} 0 \). If \( \Gamma \neq 0 \), then \( X \) is uniruled and every rational curve on \( X \) is a \( D \)-compact rational curve having zero intersection with \( K_X + D + \Gamma \). Suppose that \( \Gamma = 0 \) so that \( K_X \sim_{\mathbb{Q}} 0 \). Let \( X'' \rightarrow X \) be the global index-one cover (unramified in codimension-one) so that \( K_{X''} \sim 0 \) and \( X'' \) has at worst canonical singularities. The minimal resolution \( \hat{X} \) of \( X'' \) satisfies \( K_{\hat{X}} \sim 0 \). Thus Proposition 2.4 follows from the surface theory (see Remark 5.3 below). Indeed, a (smooth) K3 surface has at least one rational curve and infinitely many elliptic curves (see [17]).

Case (II) \( \dim Y = 1 \). Then \( (K_X + D + \Gamma)|_F \sim_{\mathbb{Q}} 0 \). Hence a general fibre \( F \) is either a \( D \)-rational curve, or a \( D \)-algebraic 1-torus, or a \( D \)-torus.

Case (III) \( \dim Y = 2 \), i.e., \( \sigma \) is birational. We may assume that \( \sigma \) is not isomorphic and hence contracts a curve \( \ell \). As \( K_X + D + \Gamma = \sigma^*(K_Y + \sigma_*(D + \Gamma)) \), the pair \( (Y, \sigma_*(D + \Gamma)) \) is also lc. Of course, \( (K_X + D + \Gamma)|_\ell \sim_{\mathbb{Q}} 0 \). The proposition in this case follows from Claim 5.1 below, without using powerful machineries.

Alternatively, as in §4.5 via LMMP, we can reduce to the case of Lemma 4.7 and conclude that \( \ell \) (= \( S \) there) is a \( D \)-compact rational curve.

Claim 5.1. At least one of the following cases holds.

(C1) \( \ell \) is a component of \( D \); \( \ell \) is either a (smooth) elliptic curve disjoint from \( D - \ell \), or a smooth rational curve and (transversally) meet the other components of \( D \) at at most two points.

In the rest, \( \ell \) is not a component of \( D \).
(C2) The geometric genus \( g(\ell) \leq 1 \) (with equality holding true only when \( \ell \) is a (smooth) elliptic curve and \( P = \sigma(\ell) \) is a non-klt point of \( Y \)). \( \ell \) is disjoint from \( D \).

(C3) \( \ell \) is a rational curve homeomorphic to \( \mathbb{P}^1 \) and meets \( D \) at at most two points.

(C4) There is a rational curve \( D_1 \) in \( D \) which is homeomorphic to \( \mathbb{P}^1 \) and meets the other components of \( D \) at at most one point (this case needs to be added only when \( P = \sigma(\ell) \) is a non-klt point of \( Y \)).

We prove Claim 5.1. When \( \ell \) is a component of \( D \), we have case (C1) by the calculation in (*) above, with ‘0 >’ replaced by ‘0 =’.

So assume that \( \ell \) is not a component of \( D \). Let \( \eta : X' \to X \) be the minimal resolution and \( \text{Exc}(\eta) \) the exceptional divisor. Let \( D' = \eta D, \Gamma' = \eta \Gamma \) and \( \ell' = \eta \ell \) be the proper transforms of \( D, \Gamma \) and \( \ell \), respectively. Write \( K_X + E_K = \eta^*K_X \), for some \( \eta \)-contractible divisor \( E_K \). Since \( \eta \) is a minimal resolution, \( E_K \geq 0 \). Write \( \eta^*D = D' + E_D \) and \( \eta^*\Gamma = \Gamma' + E_\Gamma \), with \( E_D \) and \( E_\Gamma \) effective and \( \eta \)-contractible. Then

\[
K_{X'} + D' + \Gamma' + E = \eta^*(K_X + D + \Gamma)
\]

where \( E = E_K + E_D + E_\Gamma \). In particular, \( \text{Supp}(D' + E) \supseteq \eta^{-1}(D) \).

Write \( \Gamma = a\ell + \Gamma_1 \) and hence \( \Gamma' = a\ell' + \Gamma'_1 \), where \( a \in [0, 1] \) and \( \Gamma_1 \) (resp. \( \Gamma'_1 \)) does not contain \( \ell \) (resp. \( \ell' \)). Since \( \ell' \) is contracted by \( \eta \circ \sigma \), \( (\ell')^2 < 0 \). Hence \( (a - 1)\ell'^2 \geq 0 \). Note that \( \ell \) is perpendicular to \( \sigma^*(K_Y + \sigma_*(D + \Gamma)) = K_X + D + \Gamma \). So

\[
0 = \ell.(K_X + D + \Gamma) = \ell'.\eta^*(K_X + D + \Gamma) = \ell'.(K_{X'} + D' + \Gamma' + E)
\]

\[
= \ell'.(K_{X'} + \ell' + D' + \Gamma'_1 + E) + (a - 1)\ell'^2
\]

\[
\geq 2p_a(\ell') - 2 + \ell'.(D' + \Gamma'_1 + E) \geq 2p_a(\ell') - 2.
\]

Thus \( g(\ell) \leq p_a(\ell') \leq 1 \). Further, if \( p_a(\ell') = 1 \), then the inequalities in the above display all become equalities. So \( a = 1 \), and \( 0 = \ell' \cap (D' + E) \supseteq \ell' \cap \eta^{-1}(D) \). This is case (C2).

We may now assume that \( p_a(\ell') = 0 \), i.e., \( \ell' \cong \mathbb{P}^1 \). Since \( P = \sigma(\ell) \) is a log canonical singularity of \( (Y, \sigma_*(D + \Gamma)) \) and hence the reduced divisor \( \sigma_*(D) \) contains at most two analytic branches at \( P \) by [13, Theorem 4.15], \( \ell \) meets \( D \) at at most two points. The exceptional divisor \( E_P \) of the minimal resolution of \( P \) is given in [13, Theorem 4.7, 1–5]. The rational curve \( \ell \) is homeomorphic to \( \mathbb{P}^1 \) and hence we are in case (C3) unless Case (1) in [13, Theorem 4.7] occurs, i.e., \( E_P \) is a rational curve with a simple node and the connected component \( \Sigma \) of \( D + \ell \) containing \( \ell \) is contracted to the point \( P \).

We consider the latter case. \( \Sigma \) is obtained from the nodal curve \( E_P \) by blowing up some points on it and then blowing down some curves in the inverse of \( E_P \). Hence \( \Sigma \) is the image of a divisor \( \Sigma'' \) on a blowup \( X'' \) of \( X \) such that \( \Sigma'' \) is of simple normal crossing and consists of smooth rational curves, \( \Sigma'' \) contains only one loop (which is simple), and \( X'' \to X \) is the contraction of some rational trees contained in \( \Sigma'' \), \( X \) being klt. Thus,
either $\ell$ has a node and case (C2) or C(4) occurs, or $\ell$ is homeomorphic to $\mathbb{P}^1$ and case (C3) occurs. This proves Claim 5.1 and also Proposition 2.4.

**Remark 5.2.** The same proof (see N.B.) shows that when $D$ is Cartier, if $K_X + D + \Gamma$ is not nef then it has strictly negative degree on a $D$-rational curve.

**Remark 5.3.** (1) By the surface theory, a $\mathbb{Q}$-CY surface has either a rational curve or smooth elliptic curves, or it is a simple abelian surface or a a normal $K3$ surface with $s \geq 1$ singularities (all Du Val) and infinitely many (singular) elliptic curves (see [17]).

(2) We define a variety $Y$ to be *algebraical Brody hyperbolic* (ABH) if no algebraic curve in $Y$ has an elliptic curve, $\mathbb{P}^1$, $C$ or $C^* = C \setminus \{0\}$ as normalization.

By the proof of Proposition 2.4, if we weaken BH to ABH in the assumption there, then either $K_X + D + \Gamma$ is ample, or $D = \Gamma = 0$ and $X$ is a klt surface with $K_X \sim_{\mathbb{Q}} 0$. In the latter case, $X$ is a simple abelian surface by (1) above since $X$ is ABH.

### 6. Concluding remarks

As remarked in the introduction, although hyperbolicity should imply the ampleness of the canonical divisor on general conjectural principles and therefore hyperbolic embeddedness that of the log canonical divisor, hyperbolic embeddedness is in general not the same as the usual notion of a Brody hyperbolic pair when singularities are present unless the boundary divisor can be decomposed into pieces that are Cartier up to some factor. This is the reason we restrict ourselves to the case of $\mathbb{Q}$-Cartier boundary divisors, automatically satisfied for the case of dlt surfaces but not in the more general lc case for surfaces. In fact, $\mathbb{Q}$-factoriality is natural to many considerations in the MMP and some people even assume it implicitly from the outset. Nevertheless, the ampleness of the canonical divisor is a much weaker condition than hyperbolicity and it is very possible that one can dispense with the $\mathbb{Q}$-Cartierness of the boundary in the definition of Brody hyperbolic pairs and still have ampleness of the canonical divisor even though it would be beyond “conjectural” limits of hyperbolic geometry. Also, the natural method of reducing the problem to lower dimensions is via adjunction and that is essentially the only process we have depended on and so it is likely that one can push the method further to general frameworks where adjunction can work. Therefore, although it is very likely from our consideration for the case of surfaces that the main conjectures for Mori/Brody hyperbolic pairs generalize to the case of $\mathbb{Q}$-factorial lc pairs, it is a very interesting question as to whether one can deal with the non-$\mathbb{Q}$-factorial case in general.
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