Origin of Antifields in the Batalin-Vilkovisky Lagrangian Formalism

J. Alfarose and P. H. Damgaard
CERN – Geneva

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Abstract

The antifields of the Batalin-Vilkovisky Lagrangian quantization are standard antighosts of certain collective fields. These collective fields ensure that Schwinger-Dyson equations are satisfied as a consequence of the gauge symmetry algebra. The associated antibracket and its canonical structure appear naturally if one integrates out the corresponding ghost fields. An analogous Master Equation for the action involving these ghosts follows from the requirement that the path integral gives rise to the correct Schwinger-Dyson equations.

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*Permanent address: Fac. de Fisica, Universidad Catolica de Chile, Casilla 306, Santiago 22, Chile.
1 Introduction

When it comes to the quantization of gauge theories in a Lagrangian formalism, the framework of Batalin-Vilkovisky \[1\] appears to be superior to all other available schemes. Not only is this Batalin-Vilkovisky formalism fairly straightforward to implement (and, e.g., string field theory can hardly be done without it \[3\]; see also \[4\]), it also gives, for free, a new canonical structure contained in what is known as the “antibracket”. With the help of this canonical formalism, the definition of a gauge-fixed quantum action can be formulated by means of one equation (and certain subsidiary conditions): the Master Equation \[1\]. In this formalism, it is thus as if the very definition of a full quantum field theory can carried through at an algebraic level. We believe that this is part of its great appeal.

Let us briefly recapitulate the basic ingredients\[1\]. Start with a set of fields \( \phi^A(x) \) of given Grassmann parity (statistics) \( \epsilon(\phi^A) = \epsilon_A \), and then introduce for each field a corresponding antifield \( \phi^*_A \) of opposite Grassmann parity \( \epsilon(\phi^*_A) = \epsilon_A + 1 \). The fields and antifields are taken to be canonically conjugate,

\[
(\phi^A, \phi^*_B) = \delta^A_B, \quad (\phi^*, \phi^*_B) = (\phi^*_A, \phi^*_B) = 0,
\]

within a certain graded bracket structure \( (\cdot, \cdot) \), the antibracket:

\[
(F, G) = \frac{\delta^r F \delta^l G}{\delta \phi^A \delta \phi^*_A} - \frac{\delta^r F \delta^l G}{\delta \phi^*_A \delta \phi^* A}.
\]

The subscripts \( l \) and \( r \) denote left and right differentiation, respectively. The summation over indices \( A \) includes an integration over continuous variables such as space-time \( x \), when required.

This antibracket is statistics-changing in the sense that

\[
\epsilon[(F, G)] = \epsilon(F) + \epsilon(G) + 1,
\]

and satisfies the following exchange relation:

\[
(F, G) = -(-1)^{(\epsilon(F)+1)(\epsilon(G)+1)}(G, F).
\]

Furthermore, one may verify that the antibracket acts as a derivation of the kind

\[
(F, GH) = (F, G)H + (-1)^{\epsilon(G)(\epsilon(F)+1)}G(F, H)
\]

\[
(FG, H) = F(G, H) + (-1)^{\epsilon(G)(\epsilon(H)+1)}(F, H)G,
\]

and satisfies a Jacobi identity of the form

\[
(-1)^{(\epsilon(F)+1)(\epsilon(H)+1)}(F, (G, H)) + \text{cyclic perm.} = 0.
\]

Some simple consequences of these relations are that \( (F, F) = 0 \) for any Grassmann odd \( F \), and \( (F, (F, F)) = ((F, F), F) = 0 \) for any \( F \).

\[1\] For some excellent recent reviews, see ref. \[2\].
The antifields $\phi^*_A$ are also given definite ghost numbers $gh(\phi^*_A)$, related to those of the fields $\phi^A$:

$$gh(\phi^*_A) = -gh(\phi^A) - 1.$$  (7)

The absolute value of the ghost number can be fixed by requiring that the action carries ghost number zero.

The Batalin-Vilkovisky quantization prescription can now be formulated as follows. First solve the equation

$$\frac{1}{2}(W,W) = i\hbar\Delta W$$  (8)

where

$$\Delta = (-1)^{a+1} \frac{\delta^r}{\delta \phi^A} \frac{\delta^r}{\delta \phi^*_A}.$$  (9)

This $W$ will be the “quantum action”, presumed expandable in powers of $\hbar$:

$$W = S + \sum_{n=0}^{\infty} \hbar^n M_n,$$  (10)

and a boundary condition is that $S$ in eq. (10) should coincide with the classical action when all antifields are removed, i.e., after setting $\phi^*_A = 0$. One can solve for the additional $M_n$-terms through a recursive procedure, order-by-order in an $\hbar$-expansion. To lowest order in $\hbar$ this is the Master Equation:

$$(S,S) = 0.$$  (11)

Similarly, one can view eq. (8) as the full “quantum Master Equation”.

A correct path integral prescription for the quantization of the classical theory $S[\phi^A, \phi^*_A = 0]$ is that one should find an appropriate “gauge fermion” $\Psi$ (the precise properties of which need not concern us yet) such that the partition function is given by

$$Z = \int [d\phi^A][d\phi^*_A] \delta(\phi^*_A - \frac{\delta^r \Psi}{\delta \phi^A}) \exp \left[\frac{i}{\hbar}W\right].$$  (12)

This prescription guarantees gauge-independence of the $S$-matrix of the theory. The “extended action” $S \equiv S_{ext}[\phi^A, \phi^*_A]$ is a solution of the Master Equation (11), and can be given by an expansion in powers of antifields $[4]$. After the elimination of the antifields by the $\delta$-function constraint in eq. (12), one can verify that the action is invariant under the usual BRST symmetry, which we will here denote by $\delta$.

While this formalism thus in a beautiful way encompasses the usual Lagrangian BRST quantization (as formulated in, e.g., ref. [5]), it in a still mysterious way also seems to supersede it. Is there any rationale behind this?

But there are other, simpler, questions as well:

- Why do we suddenly have to effectively double the number of fields involved (by introducing the antifields), and then immediately remove these fields again through the introduction of a gauge fermion $\Psi$, and the $\delta$-function prescription of eq.(12)?
• Is such a $\delta$-function prescription for removing the antifields the most general?

• In this formalism, the full quantum action will in general contain all powers of $\hbar$. On the surface, this seems different from more conventional BRST quantization methods. What is the origin of these quantum corrections?

• Denoting the usual BRST transformation by $\delta$, the Batalin-Vilkovisky formalism induces a cohomology based not on $\delta$ but on the “quantum deformation” acting on both fields and antifields, which is defined by the nilpotent $\sigma = \delta - i\hbar \Delta$. Such a split into classical and quantum mechanical parts of symmetries is unnatural; from where does it arise?

• Is it possible to give a step-by-step Lagrangian derivation of the quantization principle?

We shall try to answer these questions one by one. In the process of doing so, we will uncover and derive in a simple manner the Batalin-Vilkovisky formalism starting from two basic ingredients: A) Standard BRST Lagrangian quantization, and B) The requirement that the most general Schwinger-Dyson equations of the full quantum theory follow from a symmetry principle. We shall throughout, unless otherwise stated, assume that when ultraviolet regularization is required, a suitable regulator which preserves the relevant BRST symmetry exists.

Schwinger-Dyson equations will thus play a crucial rôle in this analysis. The idea is to enlarge the usual BRST symmetry in precisely such a way that both the usual gauge-symmetry Ward Identities and the most general Schwinger-Dyson equations both follow from the same BRST Ward Identities. The way to do this is known; it is a special case of collective field transformations that can be used to gauge arbitrary symmetries. In fact, almost all the needed ingredients can be found in ref. [8], but we have made an effort towards writing this paper in a self-contained manner. No familiarity with the Batalin-Vilkovisky formalism beyond what has been sketched above is required.

We start in section 2 with the simplest case possible, that of a quantum field theory without any internal gauge degrees of freedom. This will nevertheless allow us to derive most of the algebraic structure needed subsequently: the antibracket, the gauge-fixing principle, the origin of a “quantum” BRST algebra, and the Master Equation itself. One of the conclusions we shall draw is that the required quantum corrections to the BRST generator can be seen as an artifact of having integrated out only one ghost field instead of the usual procedure of either integrating out both ghosts and antighosts simultaneously, or leaving them both. In section 3 we turn to a more non-trivial example: A gauge theory with Yang-Mills symmetry. We show here how the same principle of requiring Schwinger-Dyson equations to follow from the BRST algebra leads to the additional ingredients of the Batalin-Vilkovisky formalism: The extended action, the general transformation laws for the antifields, and the gauge fixing as a consequence of a canonical transformation

\footnote{For a geometrical interpretation of some of these issues, see ref. [6].}
(within the antibracket) of both fields and antifields. Similarly, the so-called non-minimal action can be derived in a completely analogous manner. This section 3 is highly technical, and some readers may wish to jump immediately to section 4 where the scheme is explained in a more condensed manner, and where also various generalizations are discussed. In particular, cases that can not be straightforwardly derived by means of collective fields (as, e.g., theories with open gauge algebras), require special care. Demanding that Schwinger-Dyson equations follow as BRST Ward Identities for such theories will in general correspond to actions and functional measures that are not separately invariant under the BRST transformations. As a consequence, the analogous Master Equation will contain a quantum correction, – the origin of the quantum Master Equation of Batalin and Vilkovisky. In this precise sense, the quantum Master Equation is a necessary and sufficient requirement for a consistent quantization of the theory, even before any gauge fixing. Section 5 contains our conclusions. We have listed our conventions and some useful formulae in an appendix.

2 No gauge Symmetries

Consider a quantum field theory based on an action $S[\phi^A]$ without any internal gauge symmetries. Such a quantum theory can be described by a path integral\(^3\)

$$Z = \int [d\phi^A] \exp \left[ \frac{i}{\hbar} S[\phi^A] \right],$$

and the associated generating functional. There is no need to introduce sources for the present discussion.

Equivalently, such a quantum field theory is believed to be entirely described by the solution of the corresponding Schwinger-Dyson equations, once appropriate boundary conditions have been imposed. Schwinger-Dyson equations are therefore quantum mechanically exact statements about the theory. At the path integral level they follow from invariances of the measure. Let us for simplicity consider the case of a flat measure which is invariant under arbitrary local shifts, $\phi^A(x) \rightarrow \phi^A(x) + \varepsilon^A(x)$. We can gauge this symmetry by means of collective fields $\varphi^A(x)$: Suppose we transform the original field as

$$\phi^A(x) \rightarrow \phi^A(x) - \varphi^A(x),$$

then the transformed action $S[\phi^A - \varphi^A]$ is trivially invariant under the local gauge symmetry

$$\delta \phi^A(x) = \Theta(x), \quad \delta \varphi^A(x) = \Theta(x),$$

and the measure for $\phi^A$ in eq. (13) is also invariant. We next integrate over the collective field in the transformed path integral, using the same flat measure. The integration is of course very formal since it will include the whole volume of the gauge group.\(^4\) To cure

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\(^3\)Since one of the curious properties of the Batalin-Vilkovisky formalism is the combination of classical and quantum parts in the BRST cohomology, it is useful to keep track of factors of $\hbar$.

\(^4\)This situation is no different from usual path integral manipulations of gauge theories.
this problem, we gauge-fix in the standard BRST Lagrangian manner \[3\]. That is, we add to the transformed Lagrangian a BRST-exact term in such a way that the local gauge symmetry is broken. In this case an obvious BRST multiplet consists of a ghost-antighost pair \(c^A(x), \phi^*_A(x)\), and a Nakanishi-Lautrup field \(B_A(x)\):

\[
\begin{align*}
\delta \phi^A(x) &= c^A(x) \\
\delta \phi^*_A(x) &= c^A(x) \\
\delta c^A(x) &= 0 \\
\delta \phi^*_A(x) &= B_A(x) \\
\delta B_A(x) &= 0.
\end{align*}
\]

(16)

No assumptions will be made as to whether \(\phi^A\) are of odd or even Grassmann parity. We assign the usual ghost numbers to the new fields,

\[
gh(c^A) = 1, \quad gh(\phi^*_A) = -1, \quad gh(B_A) = 0,
\]

(17)

and the operation \(\delta\) is statistics-changing. The rules for operating with \(\delta\) are given in the Appendix.

Let us choose to gauge-fix the transformed action by adding to the Lagrangian a term of the form

\[
-\delta \left[ \phi^*_A(x) \phi^A(x) \right] = (-1)^{\ell(A)+1} B_A(x) \phi^A(x) - \phi^*_A(x)c^A(x).
\]

(18)

The partition function is now again well-defined:

\[
Z = \int [d\phi][d\varphi][d\phi^*][dc][dB] \exp \left[ \frac{i}{\hbar} \left( S[\phi - \varphi] - \int dx \left\{ (-1)^{\ell(A)} B_A(x) \phi^A(x) + \phi^*_A(x)c^A(x) \right\} \right) \right].
\]

(19)

Since the collective field has just been gauge fixed to zero, it may appear useful to integrate both it and the field \(B_A(x)\) out. We are then left with

\[
Z = \int [d\phi^A][d\phi^*_A][dc^A] \exp \left[ \frac{i}{\hbar} S_{\text{ext}} \right]
\]

\[
S_{\text{ext}} = S[\phi^A] - \int dx \phi^*_A(x)c^A(x),
\]

(20)

which obviously coincides with the original expression (13) apart from the trivially decoupled ghosts. But the remnant BRST symmetry is still non-trivial: We find it in the usual way by substituting for \(B_A(x)\) its equation of motion. This gives

\[
\begin{align*}
\delta \phi^A(x) &= c^A(x) \\
\delta c^A(x) &= 0 \\
\delta \phi^*_A(x) &= -\frac{\delta S}{\delta \phi^A(x)}.
\end{align*}
\]

(21)

The functional measure is also invariant under this symmetry, according to our assumption about the measure for \(\phi^A\), and assuming a flat measure for \(\phi^*_A\) as well. The Ward Identities following from this symmetry are the sought-for Schwinger-Dyson equations. Consider, for example, the identity \(0 = \langle \delta \{ \phi^*_A(x) F[\phi^A] \} \rangle\), where we have chosen \(F\)...
to depend only on $\phi^A$ just to ensure that the whole object carries overall ghost number zero (more general $F$’s can of course also be considered). After integrating over both ghosts $c^A$ and antighosts $\phi^*_A$, this Ward Identity can be written

$$\langle \frac{\delta F}{\delta \phi^A(x)} + \left( i \frac{\hbar}{\delta} \right) \frac{\delta S}{\delta \phi^A(x)} F[\phi^A] \rangle = 0 ,$$ (22)

that is, precisely the most general Schwinger-Dyson equations for this theory. The symmetry (21) can be viewed as the BRST Schwinger-Dyson algebra.

Consider now the equation that expresses BRST invariance of the extended action $S_{\text{ext}}$:

$$0 = \delta S_{\text{ext}} = \int dx \frac{\delta^r S_{\text{ext}}}{\delta \phi^A(x)} c^A(x) - \int dx \frac{\delta^r S_{\text{ext}}}{\delta \phi^*_A(x)} \frac{\delta S}{\delta \phi^A(x)}$$

$$= \int dx \frac{\delta^r S_{\text{ext}}}{\delta \phi^A(x)} c^A(x) - \int dx \frac{\delta^r S_{\text{ext}}}{\delta \phi^*_A(x)} \frac{\delta S}{\delta \phi^A(x)} .$$ (23)

In the last line we have used the fact that $S$ differs from $S_{\text{ext}}$ by a term independent of $\phi^A$. Using the notation of the antibracket (2), this is seen to correspond to a Master Equation of the form

$$\frac{1}{2}(S_{\text{ext}}, S_{\text{ext}}) = - \int dx \frac{\delta^r S_{\text{ext}}}{\delta \phi^A(x)} c^A(x) .$$ (24)

The ghosts $c^A$ play the rôle of spectator fields in the antibracket. But their appearance on the r.h.s. of the Master Equation ensures that the solution $S_{\text{ext}}$ will contain these fields.

If one prefers to view the Master Equation (24) as more fundamental, one can take this equation as the starting point. To find a perhaps more general solution, let us assume that it can be written as an expansion in ghosts and antighosts. Since $S_{\text{ext}}$ should have ghost number zero, we try a general expansion of the form

$$S_{\text{ext}}[\phi^A, \phi^*_A, c^A] = S[\phi^A] + \sum_{n=1}^\infty a_n \phi^*_A \cdots \phi^*_A c^A_1 \cdots c^A_n$$ (25)

with unknown coefficients $a_n$. We have imposed the boundary condition that $S_{\text{ext}}[\phi^A, 0, 0] = S[\phi^A]$. The most general expansion in ghosts and antighosts involves $\phi^A$-dependent coefficients $a_n$, but then the relevant Schwinger-Dyson BRST algebra is not guaranteed to leave the functional measure invariant. The Master Equation (24) will then have to be changed. There is no need to enter into a discussion of these complications at this stage (they will resurface when we discuss the general procedure in section 4). We shall therefore for the moment content ourselves with the expansion (25). It appears in any case at this stage superficially required to have $\phi^A$-independent coefficients $a_n$ if we insist on obtaining Schwinger-Dyson equations for the original theory described by $S[\phi^A]$, since otherwise the manipulation in eq. (23) would not be valid. Plugging (25) into the Master Equation (24), we can now compare order by order in the number of ghost and antighost fields. This immediately leads to $a_1 = 1$, which in turn implies that all higher coefficients vanish, i.e.,

$$a_1 = 1 , \quad a_n = 0 \quad \text{for all } n > 1 .$$ (26)
In this case of no internal gauge symmetries, the action (20) is therefore the unique solution to the Master Equation (24) with the boundary condition \( S_{\text{ext}}[\phi^A, \phi^*_A = 0, c^A = 0] = S[\phi^A] \) and with the new ghosts and antighosts decoupled from the classical action.

The extended action of Batalin and Vilkovisky does however not coincide with \( S_{\text{ext}} \) as defined above. For one thing, the action (20) contains the new ghost fields \( c^A \) that are not present in the Batalin-Vilkovisky formalism. But suppose we integrate only over these ghosts \( c^A(x) \), without integrating over the corresponding antighosts \( \phi^*_A(x) \). Then the partition function reads

\[
Z = \int [d\phi^A][d\phi^*_A] \delta (\phi^*_A) \exp \left[ \frac{i}{\hbar} S[\phi^A] \right].
\] (27)

What has happened to the BRST algebra? As far as symmetry transformations are concerned, the usual rule for replacing non-propagating fields which are being integrated out, is to use the corresponding equations of motion. This recipe is in general only correct for bosonic Gaussian integrals. For the present case of a ghost field \( c^A \) appearing linearly in the action before being integrated out, we can derive the correct substitution rule as follows. First, we should really phrase the question in a more precise manner. What we need to know is how to replace \( c \) inside the path integral, i.e., inside Green functions. (In the action it is of course not present, having been integrated out). This will automatically give us the correct transformation rules for those fields that are not integrated out. Consider the identity

\[
\int [dc] F(c^B(y)) \exp \left[ -\frac{i}{\hbar} \int dx \phi^*_A(x) c^A(x) \right] = F \left( i \hbar \frac{\delta \Lambda}{\delta \phi^*_B(y)} \right) \exp \left[ -\frac{i}{\hbar} \int dx \phi^*_A(x) c^A(x) \right].
\] (28)

Eq. (28) teaches us that it is not enough to replace \( c \) by its equation of motion \( (c(x) = 0) \); a “quantum correction” in the form of the operator \( \hbar \delta / \delta \phi^* \) must be added as well. The appearance of this operator is the final step towards unravelling the canonical structure in the formalism of Batalin and Vilkovisky. It also shows that even in this trivial case we have to include “quantum corrections” to BRST symmetries if we insist on integrating out only one ghost field, while keeping its antighost.

It is important that the operator \( \hbar \delta / \delta \phi^* \) in eq. (28) always acts on the integral (really a \( \delta \)-function) to its right.

We now make this replacement, having always in mind that it is only meaningful inside the path integral. For the BRST transformation itself we get, upon one partial integration

\[
\delta \phi^A(x) = i \hbar (-1)^{c^A} \frac{\delta}{\delta \phi^A(x)}
\]

\[
\delta \phi^*_A(x) = - \frac{\delta S}{\delta \phi^A(x)}.
\] (29)

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5The reader may well ask why we were allowed to use a similar simple rule in the previous example, eq. (19), where we integrated out a field \( (B_A) \) that only appeared linearly in the action. The reason is that we simultaneously integrated out the collective field \( \varphi^A \). Had we chosen to keep \( \varphi^A \), the substitution rule would indeed be more complicated, and in fact entirely analogous to the recipe we shall provide above.
We know from our derivation that this transformation leaves at least the combination of measure and action invariant. As a check, if we consider the same Ward Identity as above, based on 0 = ⟨δ{φ∗_A(x)F[φ^A]}⟩, we recover the Schwinger-Dyson equation (22).

The original $S[φ^A]$ can in this case be identified with the extended action of Batalin and Vilkovisky, and the antighost $φ^*_A$ is the antifield corresponding to $φ^A$. Because there are no internal gauge symmetries, the extended action turns out to be independent of the antifields. Although our $S_{\text{ext}}$ of eq. (20) cannot be identified with the extended action of Batalin-Vilkovisky, the Master Equation derived in eq. (23) is of course very similar to their corresponding Master Equation. Writing eq.(23) in terms of the antibracket is, as follows from the derivation, a little forced. It is done in order to facilitate the comparison.

Finally, it only remains to be seen what has happened to this bracket structure after having integrated out the ghost. It is convenient to keep the same notation as before, so that in this case the “extended” action is identically equal to the original action: $S_{\text{ext}} = S[φ^A]$. Let us then again consider the variation of an arbitrary functional $G$, this time only a function of $φ^A$ and $φ^*_A$. Inside the path integral (and only there!) we can represent the variation of $G$ as:

$$δG[φ^A, φ^*_A] = \int dx \frac{δ^r G}{δφ_A(x)} \left[ \frac{δ^l S_{\text{ext}}}{δφ^*_A(x)} + (ih)(-1)^{ε_A} \frac{δ^r}{δφ^*_A(x)} \right] - \int dx \frac{δ^r G}{δφ^*_A(x)} \frac{δ^l S_{\text{ext}}}{δφ_A(x)}, \quad (30)$$

where the derivative operator no longer acts on the $δ$-function of $φ^*_A$. We have kept the term proportional to $δ^l S_{\text{ext}}/δφ^*_A$, even though $S_{\text{ext}}$ in this simple case is independent of $φ^*_A$ (and that term therefore vanishes). It comes from the partial integration with respect to the operator $δ^l/δφ^*_A$, and is there in general when $S_{\text{ext}}$ depends on $φ^*_A$ (see section 4.2).

This equation precisely describes the “quantum deformation” of the classical BRST charge, as it occurs in the Batalin-Vilkovisky framework:

$$δG = (G, S_{\text{ext}}) - i\hbar ΔG . \quad (31)$$

The BRST operator in this form is often denoted by $σ$.

We have here introduced

$$Δ \equiv (-1)^{ε_A+1} \frac{δ^r δ^r}{δφ^*_A(x)δφ^A(x)} \quad (32)$$

which is identical to the operator (9) of Batalin and Vilkovisky. Again, this term arises as a consequence of the partial integration which allows us to expose the operator $δ/δφ^*_A(x)$ that otherwise (in a correct, but inconvenient representation) acts only on the $δ$-functional $δ(φ^*_A)$. In other words, the $δ$-function constraint on $φ^*_A(x)$ is now considered as part of the functional measure for $φ^*_A$.

Precisely which properties of the partially-integrated extended action are then responsible for the canonical structure behind the Batalin-Vilkovisky formalism? As we have seen, the crucial ingredients come from integrating out the Nakanishi-Lautrup fields $B_A$ and the ghosts $ε_A$. Integrating out $B_A$ changes the BRST variation of the antighosts $φ^*_A$ into $-δ^l S_{\text{ext}}/δφ^A$. This is an inevitable consequence of introducing collective fields as shifts of the original fields (and hence enforcing Schwinger-Dyson equations), and then
gauge-fixing them to zero. But this only provides half of the canonical structure, making, loosely speaking, \( \phi^A \) canonically conjugate to \( \phi^*_A \), but not vice versa. The rest is provided by integrating over the ghosts \( c^A \); at the linear level it changes the BRST variation of the fields \( \phi^A \) themselves into \( \delta S_{ext}/\delta \phi^*_A \). This in turn is again an inevitable consequence of having introduced the collective fields as shifts, and then having gauge-fixed them to zero.\(^6\)

The latter operation is, however, complicated by the fact that the fields \( \phi^*_A \) which are fixed in the process of integrating out \( c^A \) are chosen to remain in the path integral. This makes it impossible to discard the “quantum correction” to \( \delta S_{ext}/\delta \phi^*_A \). So, in fact, if one insists on keeping these antighosts \( \phi^*_A \), now seen as canonically conjugate partners of \( \phi^A \), the simple canonical structure is in this sense never truly realized.\(^7\)

We have seen these features only in what is the trivial case of no internal gauge symmetries. But as we shall show in the following two sections, they hold in greater generality.

### 3 Gauge Theories: Yang-Mills

Before giving a more general presentation of the equivalence between the Batalin-Vilkovisky scheme and the collective field approach, let us first describe another simple example: Yang-Mills theory. It will turn out to contain most of the still missing ingredients.

We start with the pure Yang-Mills action \( S[A_\mu] \), and define a covariant derivative with respect to \( A_\mu \) as

\[
D^{(A)}_\mu \equiv \partial_\mu - [A_\mu,].
\]  

(33)

Next, we introduce the collective field \( a_\mu \) which will enforce Schwinger-Dyson equations as a result of the BRST algebra:

\[
A_\mu(x) \rightarrow A_\mu(x) - a_\mu(x).
\]  

(34)

In comparison with the previous example, the only new aspect here is that the transformed action \( S[A_\mu - a_\mu] \) actually has two independent gauge symmetries. Because of the redundancy introduced by the collective field, we can write the two symmetries in different ways. To make contact with the Batalin-Vilkovisky formalism, we will choose a very particular version, the one corresponding to

\[
\delta A_\mu(x) = \Theta_\mu(x),
\]

\[
\delta a_\mu(x) = \Theta_\mu(x) - D^{(A-a)}_\mu \varepsilon(x),
\]  

(35)

which also shows the need for being careful in defining what we mean by a covariant derivative. (The general principle is that we choose the original gauge symmetry of the

\(^6\)Gauge-fixing the collective fields to zero implies linear couplings to the auxiliary fields and ghosts, respectively. This is one of the central properties of the extended action that leads to the canonical structure, and to the fact that the extended action \( S_{ext} \) itself is the (classical) BRST generator. However, this does not exclude the possibility that different gauge fixings of the shift symmetries could produce a more general formalism.

\(^7\)It will not help to integrate further, and remove the antifields as well. Then the canonical structure is lost yet again, and the “Master Equation” is then simply the condition that the gauge-fixed action be invariant under the BRST symmetry of the internal gauge invariances.
original field to be carried entirely by the collective field; the transformation of the original
gauge field is then always just a shift.) Although $\Theta(x)$ includes arbitrary deformations,
it only leaves the transformed field invariant, while of course the action is also invariant
under Yang-Mills gauge transformations of this transformed field itself. Hence the need
for including two independent gauge transformations.

We now gauge-fix these two gauge symmetries, one at a time, in the standard BRST
manner. As a start, we introduce a suitable multiplet of ghosts and auxiliary fields. We
need one Lorentz vector ghost $\psi_\mu(x)$ for the shift symmetry of $A_\mu$, and one Yang-Mills
ghost $c(x)$. These are of course Grassmann odd, and both carry the same ghost number

$$gh(\psi_\mu) = gh(c) = 1.$$  \hfill (36)

Next, we gauge-fix the shift symmetry of $A_\mu$ by removing the collective field $a_\mu$. This
leads us to introduce a corresponding antighost $A^*_\mu(x)$, Grassmann odd, and an auxiliary
field $b_\mu(x)$, Grassmann even. They have the usual ghost number assignments,

$$gh(A^*_\mu) = -1, \quad gh(b_\mu) = 0,$$  \hfill (37)

and we now have the nilpotent BRST algebra

$$ \begin{align*}
\delta A_\mu(x) &= \psi_\mu(x) \\
\delta a_\mu(x) &= \psi_\mu(x) - D^{(A-a)}_\mu c(x) \\
\delta c(x) &= -\frac{1}{2}[c(x), c(x)] \\
\delta \psi_\mu(x) &= 0 \\
\delta A^*_\mu(x) &= b_\mu(x) \\
\delta b_\mu(x) &= 0. \hfill (38)
\end{align*}$$

Fixing $a_\mu(x)$ to zero is achieved by adding a term

$$-\delta[A^*_\mu(x) a^\mu(x)] = -b_\mu(x) a^\mu(x) - A^*_\mu(x) \{\psi^\mu - D^{(A-a)}_\mu c(x)\}$$  \hfill (39)

to the Lagrangian. We shall follow the usual rule of only starting to integrate over pairs of ghost-antighosts in the partition function. With this rule we shall still keep $c(x)$
unintegrated (since we have not yet introduced its corresponding antighost), but we can
now integrate over both $\psi_\mu(x)$ and $A^*_\mu(x)$. This leads to the following extended, but not
yet fully gauge-fixed, action $S_{ext}$:

$$ \begin{align*}
Z &= \int [dA_\mu][da_\mu][d\psi_\mu][dA^*_\mu][db_\mu] \exp \left[ \frac{i}{\hbar} S_{ext} \right] \\
S_{ext} &= S[A_\mu - a_\mu] - \int dx \{b_\mu(x) a_\mu(x) + A^*_\mu(x) [\psi^\mu(x) - D^{(A-a)}_\mu c(x)]\} \hfill (40)
\end{align*}$$

This extended action is invariant under the BRST transformation (38). The full inte-
gration measure is also invariant. Of course, the expression above is still formal, since
we have not yet gauge fixed ordinary Yang-Mills invariance. (This is obvious from our
construction, but can also be checked explicitly. The easiest way is to first integrate out
\(a_\mu\) and \(b_\mu\); integration over \(A^*_\mu\) then finally leaves a trivial \(\psi_\mu\)-integral. What is left over is nothing but the starting point, the Yang-Mills action \(S[A_\mu]\) integrated over the original measure.

Furthermore, we are eventually going to integrate over the ghost \(c\), which already now appears in the extended action. If we insist that the Schwinger-Dyson equations involving this field, i.e., equations of the form

\[
0 = \int [dc] \frac{\delta^l}{\delta c(x)} \left[ F e^\frac{1}{\hbar} [\text{Action}] \right]
\]

are to be satisfied automatically (for reasonable choices of functionals \(F\)) by means of the full unbroken BRST algebra, we must introduce yet one more collective field. This new collective field, call it \(\tilde{c}(x)\), is Grassmann odd, and has \(gh(\tilde{c}) = 1\). Now shift the Yang-Mills ghost:

\[
c(x) \to c(x) - \tilde{c}(x).
\]

To fix the associated fermionic gauge symmetry, we introduce a new BRST multiplet of a ghost-antighost pair and an auxiliary field. We follow the same rule as before and let the transformation of the new collective field \(\tilde{c}\) carry the (BRST) transformation of the original ghost \(c\), viz.,

\[
\begin{align*}
\delta c(x) &= C(x) \\
\delta \tilde{c}(x) &= C(x) + \frac{1}{2}[c(x) - \tilde{c}(x), c(x) - \tilde{c}(x)] \\
\delta C(x) &= 0 \\
\delta c^*(x) &= B(x) \\
\delta B(x) &= 0.
\end{align*}
\]

It follows from (43) that ghost number assignments should be as follows:

\[
gh(C) = 2, \quad gh(c^*) = -2, \quad gh(B) = -1,
\]

and \(B\) is a fermionic Nakanishi-Lautrup field. Fields of the kind \(C\) and \(c^*\) are sometimes known as ghosts for ghosts. However, in contrast to conventional examples involving such ghosts, these are not a priori required in this context. They enter only when we by hand gauge the ghost-shift symmetry.

Next, gauge-fix \(\tilde{c}(x)\) to zero by adding a term

\[
- \delta[c^*(x)\tilde{c}(x)] = B(x)\tilde{c}(x) - c^*(x)\{C(x) + \frac{1}{2}[c(x) - \tilde{c}(x), c(x) - \tilde{c}(x)]\}
\]

to the Lagrangian. This gives us the fully extended action,

\[
S_{ext} = S[A_\mu - a_\mu] - \int dx \{b_\mu(x)a_\mu(x) + A^*_\mu(x)[\psi^\mu(x) - D^\mu_{(A-a)}(c(x) - \tilde{c}(x))] \\
- B(x)\tilde{c}(x) + c^*(x)(C(x) + \frac{1}{2}[c(x) - \tilde{c}(x), c(x) - \tilde{c}(x)])\},
\]

13
with the partition function so far being integrated over all fields appearing above, except for \( c \) (whose antighost \( \bar{c} \) still has to be introduced when we gauge-fix the original Yang-Mills symmetry).

The extended action and the functional measure is formally invariant under the following set of transformations:

\[
\begin{align*}
\delta A_\mu(x) &= \psi_\mu(x), & \delta \psi_\mu(x) &= 0, \\
\delta a_\mu(x) &= \psi_\mu(x) - D_\mu(a)[c(x) - \bar{c}(x)], & \delta c(x) &= C(x), \\
\delta A_\mu^*(x) &= b_\mu(x), & \delta b_\mu(x) &= 0, \\
\delta \bar{c}(x), &= C(x) + \frac{1}{2}[c(x) - \bar{c}(x), c(x) - \bar{c}(x)], & \delta \bar{C}(x) &= 0, \\
\delta c^*(x) &= B(x), & \delta B(x) &= 0. 
\end{align*}
\]

As the notation indicates, the fields \( A_\mu^*(x) \) and \( c^*(x) \) can be identified with the Batalin-Vilkovisky antifields of \( A_\mu(x) \) and \( c(x) \), respectively. These antifields are the usual antighosts of the collective fields enforcing Schwinger-Dyson equations through shift symmetries.

Note that the general rule of assigning ghost number and Grassmann parity to the antifields,

\[
gh(\phi_\Lambda^*) = -(gh(\phi^A) + 1), \quad \epsilon(\phi_\Lambda^*) = \epsilon(\phi^A) + 1, \tag{47}
\]

arises in a completely straightforward manner. Here, it is a simple consequence of the fact that the BRST operator raises ghost number by one unit (and changes statistics), supplemented with the usual rule that antighosts have opposite ghost number of the ghosts.

The extended action (46) above has more fields than the extended action of Batalin and Vilkovisky, and the transformation laws of what we identify as antifields do not match those of ref. [1]. In the form we have presented it, the BRST symmetry is nilpotent also off-shell. If we integrate over the auxiliary fields \( b_\mu \) and \( B \) (and subsequently over \( a_\mu \) and \( \bar{c} \)) the BRST symmetry becomes nilpotent only on-shell. The extended action then takes

---

It may be necessary to comment on the invariance of the measure. Since a path integral measure needs regularization, a statement about invariance or non-invariance under the BRST symmetry really requires a more careful analysis. But there is a difference between a formally invariant measure and a formally non-invariant one: A formally invariant measure remains invariant within a regularization scheme that does not break the symmetry, while very specific regularization schemes are required to enforce invariance of a formally non-invariant measure. To give a more precise meaning to the difference between formally invariant or non-invariant measures one can consider the case of a finite number of degrees of freedom, where no regularization is needed. Dimensional regularization is believed to leave, in cases where it is meaningful, all measures invariant under local transformations since such transformations normally lead to factors of \( \delta(0) \). At the level of Feynman diagrams, such factors are equated with zero in dimensional regularization. Of course, if for one reason or another one insists on using a regularization scheme that explicitly breaks the above BRST symmetry, then the measure – so defined – may be non-invariant even in the limit of removing the regulator. One should then try to add appropriate local counterterms to the action, counterterms whose BRST variations should precisely compensate for the non-invariance of the measure. This situation is no different from what happens in standard gauge-fixed Yang-Mills theory if one insists on regulators that break the usual Yang-Mills BRST symmetry. For a recent discussion of this, of true anomalies, and one-loop counterterms within the Batalin-Vilkovisky framework see ref. [9].
the following form:

\[ S_{\text{ext}} = S[A_\mu] - \int dx \{ A^*_\mu(x) [\psi^\mu(x) - D^\mu c(x)] + c^*(x) [C(x) + \frac{1}{2} [c(x), c(x)]] \}. \quad (48) \]

This differs from the extended action of Batalin and Vilkovisky by the terms involving the ghost fields \( \psi_\mu(x) \) and \( c^*(x) \). Comparing with the case of no gauge symmetries, this is exactly what we should expect. These ghost fields \( \psi_\mu \) and \( c^* \) ensure the correct Schwinger-Dyson equations for \( A_\mu \) and \( c \), respectively.

To find the corresponding BRST symmetry we use – as justified earlier – the equations of motion for the auxiliary fields \( b_\mu \) and \( B \), and use the \( \delta \)-function constraints on \( a_\mu \) and \( \bar{c} \). This gives

\[
\begin{align*}
\delta A_\mu(x) &= \psi_\mu(x) \\
\delta \psi_\mu(x) &= 0 \\
\delta c(x) &= C(x) \\
\delta C(x) &= 0 \\
\delta A^*_\mu(x) &= -\frac{\delta S_{\text{ext}}}{\delta A_\mu(x)} \\
\delta c^*(x) &= -\frac{\delta S_{\text{ext}}}{\delta c(x)}
\end{align*}
\]  

(49)

BRST invariance of the extended action (48) immediately implies that it satisfies a Master Equation we can write as

\[
\int dx \frac{\delta^r S_{\text{ext}}}{\delta A^*_\mu(x)} \frac{\delta^l S_{\text{ext}}}{\delta A_\mu(x)} + \int dx \frac{\delta^r S_{\text{ext}}}{\delta c^*(x)} \frac{\delta^l S_{\text{ext}}}{\delta c(x)} = \int dx \frac{\delta^r S_{\text{ext}}}{\delta A_\mu(x)} \psi_\mu(x) + \int dx \frac{\delta^r S_{\text{ext}}}{\delta c(x)} C(x). \quad (50)
\]

Note that this Master Equation precisely is of the form

\[
\frac{1}{2}(S_{\text{ext}}, S_{\text{ext}}) = -\int dx \frac{\delta S_{\text{ext}}}{\delta A^*_\mu(x)} c^A(x), \quad (51)
\]

with two ghost fields \( c^A \) that are just \( \psi_\mu \) and \( C(x) \).

To finally make contact with the Batalin-Vilkovisky formalism, let us integrate out the ghost \( \psi_\mu(x) \). As in section 2, we shall use an identity of the form

\[
\int [d\psi_\mu] F[\psi^\mu(y)] \exp \left[ -\frac{i}{\hbar} \int dx A^*_\mu(x) \{ \psi^\mu(x) - D^\mu c(x) \} \right] = F \left[ D^\mu c(y) + (i\hbar) \frac{\delta^l}{\delta A^*_\mu(y)} \right] \int [d\psi_\mu] \exp \left[ -\frac{i}{\hbar} \int dx A^*_\mu(x) \{ \psi^\mu(x) - D^\mu c(x) \} \right] \\
= \exp \left[ \frac{i}{\hbar} \int dx A^*_\mu(x) D^\mu c(x) \right] F \left[ (i\hbar) \frac{\delta^l}{\delta A^*_\mu(y)} \right] \delta(A^*_\mu). \quad (52)
\]

This shows that we should replace \( \psi_\mu(x) \) by its equation of motion, plus the shown quantum correction of \( \mathcal{O}(\bar{\epsilon}) \) which then acts on the rest of the integral, or equivalently, by

\footnote{We are dropping the subscript on the covariant derivative since no confusion can arise at this point.}
just the derivative operator (which then acts solely on the functional $\delta$-function). To get a useful representation of $\psi^\mu(y)$ we integrate the latter version of the identity by parts, thus letting the derivative operator act on everything except $\delta(A_\mu^\ast)$. This automatically brings down the equations of motion for $\psi^\mu$. (An equivalent series of steps can of course be made using the former version of the identity, and the result is the same).

Having in this manner integrated out $\psi^\mu$ and $C$, the partition function reads

$$Z = \int [dA_\mu][dA_\mu^\ast][dc^\ast]\delta(A_\mu^\ast)\delta(c^\ast) \exp \left[ \frac{i}{\hbar} S_{ext} \right]$$

$$S_{ext} = S[A_\mu] + \int dx\{A_\mu^\ast(x)D^\mu c(x) - \frac{1}{2}c^\ast[c(x),c(x)]\},$$

and the classical BRST symmetry follows as discussed above by substituting only the equations of motion for $\psi_\mu$ and $C$:

$$\delta A_\mu(x) = \frac{\delta S_{ext}}{\delta A_\mu^\ast(x)} = D_\mu c(x)$$

$$\delta c(x) = \frac{\delta S_{ext}}{\delta c^\ast(x)} - \frac{1}{2}[c(x),c(x)]$$

$$\delta A_\mu^\ast(x) = -\frac{\delta S_{ext}}{\delta A_\mu(x)}$$

$$\delta c^\ast(x) = -\frac{\delta S_{ext}}{\delta c(x)}. \quad (54)$$

This is the usual extended action of Batalin and Vilkovisky and the corresponding classical BRST symmetry. Of course, in the partition function the integrals over $A_\mu^\ast$ and $c^\ast$ are trivial. This is as it should be, because by integrating out these antighosts we should finally recover the starting measure and the still not gauge-fixed Yang-Mills action. What we have provided here is thus only a very precise functional derivation of the extended action. It shows that we can understand the extended action in the usual path integral framework, and that the integration measures for $A_\mu^\ast$ and $c^\ast$ (with the accompanying $\delta$-function constraints) are provided automatically.

Precisely because of this prescribed path integral measure for what are really the antifields $A_\mu^\ast$ and $c^\ast$, it may appear as if this extended action does not have much independent significance: All additional terms in the action actually vanish because of the $\delta$-function constraints. Related to this is the fact that we could have derived an identity equivalent to eq. (52) by first shifting

$$\psi^\mu(x) \rightarrow \psi^\mu(x) + D_\mu c(x) \quad (55)$$

(and the right hand side of eq. (52) would in that case not contain the $D_\mu c(x)$ part in the exponential). Similarly, $C$ could be shifted as

$$C(x) \rightarrow C(x) - \frac{1}{2}[c(x),c(x)], \quad (56)$$

and the extended action would then contain neither $A_\mu^\ast$ nor $c^\ast$. At the same time the BRST transformations (49) would be changed, giving again the equations of motion for the shift
ghosts $\psi^\mu$ and $C$ (which just corresponds to the usual Yang-Mills BRST symmetry). This ambiguity in the derivation of the extended action is directly related to the fact that the split of symmetries as in eq. (35) is arbitrary. The split given in eq. (35) is what leads to the extended action of Batalin and Vilkovisky; the prescription is to choose the collective fields to carry the original gauge (or BRST) symmetry in their transformations. There is a one-to-one correspondence between this ambiguity and the freedom to choose boundary conditions when solving the Batalin-Vilkovisky Master Equation.

There should be no special significance attached to the way we shuffle the internal symmetry transformations between the original fields and the collective fields. Since there is a direct correspondence between this and the choice of boundary conditions in the Batalin-Vilkovisky formalism, it must mean that these boundary conditions can be changed. This is indeed the case (as will be shown in the next section), but the cost one pays is that gauge fixing of the internal symmetries may then no longer correspond to canonical transformations within the antibracket (see below).

Another very important remark is the following. Although the classical BRST symmetry (54) is a symmetry of the action, it is simply not a correct symmetry of the theory because the functional measure is not invariant due to the presence of $\delta$-function constraints on the antifields. So this classical BRST symmetry is of very limited use. In particular, if we derive standard BRST Ward Identities based on eq. (54) we find that they are correct only up to $\mathcal{O}(\hbar)$. In terms of Schwinger-Dyson equations, these Ward identities correspond only to the trivial classical part (i.e. terms involving the equations of motion). Since it is not useful to do classical field theory this way, the split into a classical and a quantum part of the BRST symmetry is rather unfortunate. The full BRST symmetry of the extended action (53) follows from our derivation above:

\[
\begin{align*}
\delta A^\mu(x) &= D^\mu c(x) + i\hbar \frac{\delta^r}{\delta A^*_\mu(x)} \\
\delta c(x) &= -\frac{1}{2}[c(x), c(x)] - i\hbar \frac{\delta^r}{\delta c^*(x)} \\
\delta A^*_\mu(x) &= -\frac{\delta S_{ext}}{\delta A^*_\mu(x)} \\
\delta c^*(x) &= -\frac{\delta S_{ext}}{\delta c(x)}.
\end{align*}
\] (57)

Since this set of transformations involves operators, an interpretation is required. Namely, the replacement of ghost fields by equations of motion plus a quantum correction is only valid inside the path integral. It is indeed a symmetry of the combination of action and measure. As a check, the Ward Identities derived from this symmetry are the correct Schwinger-Dyson equations.

Let us stress once again that the quantum BRST symmetry (57) from the present point of view is an awkward but of course bona fide representation of the combined Yang-Mills and Schwinger-Dyson BRST algebra. It is obtained only on the insistence of integrating out ghost fields while keeping their corresponding antighosts. The original BRST symmetry (49) automatically includes classical and quantum effects simultaneously, as is usual for internal symmetries.
When we finally (after Yang-Mills gauge fixing, which we shall turn to shortly) also integrate over $c^*$ and $A^*_\mu$, the full left-over BRST transformations equal the usual Yang-Mills BRST transformations.

Obviously, the extended action is not yet very useful from the point of view of ordinary BRST gauge fixing. To unravel some of the mechanisms behind the Batalin-Vilkovisky scheme, it is nevertheless advantageous to keep the – at this point somewhat superfluous – antifields. In fact, it is even more useful to return to the formulation in eq. (48), where there is yet no split into a classical and a quantum part of the symmetry. Let us therefore take (48) as the starting action, and now just gauge-fix in a standard manner the Yang-Mills symmetry. Choosing, e.g., a covariant gauge, we therefore finally extend the BRST multiplet to include a Yang-Mills antighost $\bar{c}$ and a Nakanishi-Lautrup scalar $b$. For there to be no doubt, let us also note that these fields have

$$gh(\bar{c}) = -1, \quad gh(b) = 0.$$  (58)

The BRST transformations are the usual $\delta \bar{c}(x) = b(x)$ and $\delta b(x) = 0$. Gauge fixing to a covariant $\alpha$-gauge can be achieved by adding a term

$$\delta[\bar{c}(x)\{\partial_\mu A^\mu(x) - \frac{1}{2\alpha}b(x)\}] = b(x)\partial_\mu A^\mu(x) + \bar{c}(x)\partial_\mu \psi^\mu(x) - \frac{1}{2\alpha}b(x)^2$$  (59)

to the Lagrangian. The corresponding completely gauge-fixed extended action then reads

$$S_{ext} = S[A_\mu] - \int dx\{A^*_\mu(x)[\psi^\mu(x) - D^\mu c(x)] + c^*(x)(C(x) + \frac{1}{2}[c(x), c(x)])$$
$$ - b(x)\partial_\mu A^\mu(x) - \bar{c}(x)\partial_\mu \psi^\mu(x) + \frac{1}{2\alpha}b(x)^2\}. \quad (60)$$

Now integrate out $\psi^\mu$ and $C$. The result is indeed a partition function of the form suggested by the formalism of Batalin and Vilkovisky:

$$Z = \int[dA_\mu][dA^*_\mu][dc][dc^*][db]\delta(A^*_\mu + \partial_\mu \bar{c})\delta(c^*)e^{\frac{1}{\hbar}S_{ext}}$$

$$S_{ext} = S[A_\mu] + \int dx\{A^*_\mu(x)D^\mu c(x) - \frac{1}{2}c^*(x)[c(x), c(x)]$$
$$ + b(x)\partial_\mu A^\mu(x) - \frac{1}{2\alpha}b(x)^2\}. \quad (61)$$

Note that by adding the Yang-Mills gauge-fixing terms, the $\delta$-function constraint on the antifield $A^*_\mu$ has been shifted. Thus when doing the $A^*_\mu$-integral, we are in effect substituting not $A^*_\mu(x) = 0$ but

$$A^*_\mu(x) = -\partial_\mu \bar{c}(x) = \frac{\delta^r \Psi}{\delta A^\mu(x)}; \quad (62)$$

where $\Psi$ is defined as the term whose BRST variation is added to the action, i.e., in this particular case,

$$\Psi = \int dx\{\bar{c}(x)(\partial^\mu A_\mu(x) - \frac{1}{2\alpha}b(x))\}. \quad (63)$$
Upon doing the $A_\mu^*$ and $c^*$ integrals, one recovers the standard covariantly gauge-fixed Yang-Mills theory

$$S = S[A_\mu] + \int dx \{ \bar{c}(x) \partial^\mu D_\mu c(x) + b(x) \partial^\mu A_\mu(x) - \frac{1}{2\alpha} b(x)^2 \}.$$  \hfill (64)

The identification (62) strongly suggests that one can see gauge fixing as a particular canonical transformation involving new fields (the antighosts $\bar{c}$). But there are terms in eq. (64) (those involving $b(x)$) which do not immediately follow from this perspective. In the Batalin-Vilkovisky framework, this is resolved by noting that one can always add terms of new fields and antifields with trivial antibrackets. In this version of the gauge-fixing procedure, one returns to the (“minimally”) extended action of eq. (61) and extends it in a “non-minimal” way. In the Yang-Mills case, this includes an additional term in the action of the form

$$S_{nm} = \int dx \bar{c}^*(x) b(x) ,$$ \hfill (65)

with $\bar{c}^*$ and $b$ having the same ghost number and Grassmann parity:

$$gh(\bar{c}^*) = gh(b) = 0 ; \quad \epsilon(\bar{c}^*) = \epsilon(b) = 0.$$ \hfill (66)

As the notation indicates, these new fields $\bar{c}^*$ and $b$ are indeed just the antifield of $\bar{c}$, and the usual Nakanishi-Lautrup field, respectively.

Gauge fixing to the same gauge as above can then be achieved by the same gauge fermion (63) which now affects both $A_\mu^*$ and $\bar{c}^*$ within the antibracket. It can thus be seen as the canonical transformation that shifts $A_\mu^*$ and $\bar{c}^*$ from zero to

$$A_\mu^*(x) = \frac{\delta^r \Psi}{\delta A_\mu(x)} , \quad \bar{c}^*(x) = \frac{\delta^r \Psi}{\delta \bar{c}(x)}.$$ \hfill (67)

Since $\Psi$ does not depend on the antifields, this canonical transformation leaves all fields $A_\mu, c$ and $\bar{c}$ unchanged.

Can we understand the non-minimally extended action from our point of view too? Consider the stage at which we introduce the antighost $\bar{c}$. This field does not yet appear in the action, but we can of course still introduce a corresponding collective “shift” field $\bar{c}'$ for $\bar{c}$ as well. The corresponding BRST multiplet consists of a new “shift-antighost” $\lambda(x)$, an “anti-antighost” $\bar{c}^*(x)$, and the associated auxiliary field $B'(x)$:

$$\begin{align*}
\delta \bar{c}(x) &= \lambda(x) \\
\delta \bar{c}'(x) &= \lambda(x) - b(x) \\
\delta \lambda(x) &= 0 \\
\delta b(x) &= 0 \\
\delta \bar{c}^*(x) &= B'(x) \\
\delta B'(x) &= 0 .
\end{align*}$$ \hfill (68)

The assignments will then have to be exactly as in eq. (66), supplemented with $gh(B') = \epsilon(B') = 1$. We are again dealing with two symmetries, because the shifted field $\bar{c}(x) - \bar{c}'(x)$
itself can still be shifted by the usual Nakanishi-Lautrup field. Let us now gauge-fix this huge symmetry. We do it in the most simple manner by adding a term
\[- \delta [\bar{c}'(x)c'(x)] = B'(x)c'(x) - \bar{c}'(x)(\lambda(x) - b(x))\] (69)
to the Lagrangian. The integrals over \(B'\) and \(\bar{c}'\) are of course trivial and we are left with the non-minimally extended action for this theory plus, as expected, the corresponding term with the new ghost \(\lambda\). The final gauge-fixing of the Yang-Mills symmetry will now consist in adding, instead of eq. (59),
\[\delta [\bar{c}(x)\{\partial_\mu A_\mu(x) - \frac{1}{2\alpha} b(x)\}] = \lambda(x)\partial_\mu A_\mu(x) + \bar{c}(x)\partial_\mu \psi^\mu(x) - \frac{1}{2\alpha} \lambda(x)b(x) .\] (70)

Before Yang-Mills gauge fixing, the integral over \(\lambda(x)\) just gave a factor of \(\delta(\bar{c}^*)\). After adding the gauge-fixing term, this is changed:
\[\delta(\bar{c}^*) \to \delta(\bar{c}^*(x) - \partial_\mu A_\mu(x) + \frac{1}{2\alpha} b(x)) .\] (71)

Substituting this back into extended action, we recover the result (64). Note that this indeed can be viewed as a canonical transformation within the antibracket. All the correct \(\delta\)-function constraints are provided by the collective fields and their ghosts. Since the functional \(\Psi\) has been chosen to depend only on the fundamental fields, and not on the antifields, the fields \(A_\mu, c, c'\) and \(\bar{c}\) are all left untouched by this canonical transformation. Extending the action from the minimal to the non-minimal case is equivalent to demanding that also Schwinger-Dyson equations for \(\bar{c}(x)\) follow as Ward identities of the BRST symmetry. Since the antighost \(\bar{c}\) remains in the path integral after gauge fixing, it would indeed be very unnatural not to demand that correct Schwinger-Dyson equations for this field follow as well. As shown, this requirement automatically leads to the non-minimally extended action.

As for the functional measure, we have stressed earlier that we always assume the existence of a suitable regulator that preserves the pertinent BRST symmetry. We can make this statement a little more explicit by detailing the required symmetries of the measure in this Yang-Mills case. Before integrating out any fields, the measures for \(A_\mu, c, c'\) and \(\bar{c}\) should all be invariant under local shifts. For \(A_\mu\) this corresponds to the usual euclidean measure (and it is very difficult to imagine this shift symmetry being broken by any reasonable regulator), while for \(c, c'\) and \(\bar{c}\) this is consistent with the usual rules of Berezin integration. The measures for the three collective fields should in addition be invariant under what corresponds to usual Yang-Mills BRST transformations, a property that indeed holds formally. Finally, the measures for all antifields are only required to be invariant under local shifts. After having integrated out the auxiliary fields \(B_A\), invariance of these measures of the antifields is now non-trivial but one can check explicitly that it is formally satisfied. This should indeed be the case, because it is straightforward to check that the action remains invariant. Since at least the combination of measure and action must remain invariant after integrating out some of the fields, invariance of the measure is in this case formally guaranteed.
Let us finally point out that once the antighost $\bar{c}$ is being treated on equal footing with $A_\mu$ and $c$, a Master Equation of the form

$$\frac{1}{2}(S_{\text{ext}}, S_{\text{ext}}) = -\int dx \frac{\delta r S_{\text{ext}}}{\delta \phi^A(x)} c^A(x)$$

now holds with $\phi^A$ denoting all the fields that finally remain in the path integral: $A_\mu, c$ and $\bar{c}$. Similarly, the BRST algebra becomes, upon integrating out the collective fields $\phi^A$ and the auxiliary fields $B_A$, of the very simple form (21) we encountered already in the case of no internal gauge symmetries.

4 Generalizations

The two previous sections were useful for illuminating in familiar settings the way the Batalin-Vilkovisky formalism arises from more standard quantization principles. The main new requirement is that the internal symmetry algebra should contain what we call the Schwinger-Dyson BRST symmetry. At a formal level, this guarantees that the full quantum theory is entirely determined by the classical action and the complete BRST symmetry.

At this stage it may be useful to step back and extract the basic ingredients of the analysis in a more condensed and general manner. As should be clear already from the Yang-Mills case, just the insistence on incorporating the Schwinger-Dyson BRST symmetry is not in itself sufficient to guarantee that the canonical formalism of Batalin and Vilkovisky follows. The part of possible representations of gauge theories that fall into this canonical framework is small, and we have to tune carefully the prescription in order to regain the Batalin-Vilkovisky formalism upon integrating out certain ghost fields. One possible advantage of this fact is that, although we can reproduce known results based on the antifield-antibracket formalism, we know now that there is ample scope for generalizations.

Having included all space-time integrations in a detailed manner in the two previous sections, we shall here for convenience of notation drop these integrations, and, as in section 1, consider the summation convention of repeated indices to include space-time summations as well. Consider first a theory which is invariant under an irreducible set of gauge transformations that form a closed algebra:

$$\delta \phi^A = R^A_\alpha \epsilon^\alpha .$$

Both theories without gauge symmetries and, e.g., Yang-Mills theory fall into this category. The general procedure is now the following. Introduce collective fields such that Schwinger-Dyson equations for all fundamental fields (i.e., all the fields that will eventually appear in the gauge-fixed action) are satisfied as Ward Identities of a BRST symmetry. To simplify the notation, let us group all fundamental fields into the same $\phi^A$. In, e.g. the Yang-Mills case this field $\phi^A$ will then contain both $A_\mu, c$ and $\bar{c}$. When the functional measures are flat, enforcing Schwinger-Dyson equations can be done through shifts:

$$\phi^A \rightarrow \phi^A - \varphi^A$$
The action is now invariant under arbitrary deformations of the fundamental fields, all those fields that remain after the gauge fixing. The next step is to form the nilpotent BRST algebra that incorporates these shifts. To get to the antibracket formalism, one should let the fundamental fields transform just as those shifts, i.e., \( \delta\phi^A = c^A \). (The ghosts \( c^A \) are now shift ghosts; they should not be confused with, e.g., the usual Yang-Mills ghosts.) Furthermore, one should include the usual gauge (and BRST) symmetries in the variations of the collective fields. As we shall see shortly, these requirements are equivalent to certain boundary conditions in Batalin-Vilkovisky formalism. To complete the BRST multiplet one finally introduces the corresponding antighosts, here denoted by \( \phi_A^* \). The full nilpotent BRST transformations are then

\[
\begin{align*}
\delta\phi^A &= c^A \\
\delta\varphi^A &= c^A - R^A[\phi - \varphi] \\
\delta c^A &= 0 \\
\delta b^A &= 0 \\
\delta\phi_A^* &= B_A \\
\delta B_A &= 0,
\end{align*}
\] (75)

where \( R^A \) is the BRST generalization of the gauge generator of eq. (73) to the full set of both usual fields, usual ghosts and usual antighosts. It fulfills \( \delta R^A = 0 \).

The gauge-fixing procedure can now be done in one step. One fixes all collective fields to zero, and introduces a “gauge fermion” \( \Psi \) to fix the underlying gauge symmetry (if there is any). This is done by adding a term

\[-\delta[\phi_A^*\varphi^A - \Psi[\phi]] = (-1)^{A+1} B_A\varphi^A - \phi_A^*(c^A - R^A[\phi - \varphi]) + \frac{\delta^r \Psi}{\delta\phi^A} c^A\] (76)

to the action, which then reads

\[S_{g.f.} = S[\phi - \varphi] + (-1)^{A+1} B_A\varphi^A - \phi_A^*(c^A - R^A[\phi - \varphi]) + \frac{\delta^r \Psi}{\delta\phi^A} c^A.\] (77)

The final gauge-fixed partition function is thus of the simple form

\[Z = \int [d\phi][d\varphi][d\phi^*][dB] \exp \left[ \frac{i}{\hbar} S_{g.f.} \right],\] (78)

and both the gauge-fixed action and the functional measure are formally invariant under the nilpotent BRST symmetry above. The nilpotency holds also off-shell. The functional measures for \( \phi^A \) and \( \phi_A^* \) are formally invariant since they are presumed flat. The measure for \( \varphi^A \) should in addition be invariant under the transformations \( R^A \).

Integrating out \( B_A, \varphi^A \) and \( c^A \) leads to

\[
\begin{align*}
Z &= \int [d\phi][d\phi^*] \delta \left( \phi_A^* - \frac{\delta^r \Psi}{\delta \varphi^A} \right) \exp \left[ \frac{i}{\hbar} S_{g.f.} \right] \\
S_{g.f.} &= S[\phi] + \phi_A^* R^A.
\end{align*}
\] (79)
This coincides with the Batalin-Vilkovisky gauge-fixed action for this case. We see that the final replacement of the “antifields” $\phi^*_A$ with $\delta \Psi / \delta \phi^A$ is a direct consequence of having integrated out the shift-ghosts $c^A$.

To see how the antibracket formalism emerges in this more general setting, split up the gauge-fixing procedure in two steps. First gauge-fix the collective fields to zero, without gauge-fixing the underlying gauge symmetry. This is achieved by adding only

$$ - \delta [\phi^*_A \varphi^A] = (-1)^{c^A+1} B_A \varphi^A - \phi^*_A (c^A - R^A[\phi - \varphi]) $$

(80)
to the action, which then reads

$$ S_{\text{ext}} = S[\phi - \varphi] + (-1)^{c^A+1} B_A \varphi^A - \phi^*_A (c^A - R^A[\phi - \varphi]). $$

(81)

Now consider integrating out the auxiliary fields $B_A$ and the collective fields $\varphi^A$ first. The extended action then becomes

$$ S_{\text{ext}} = S[\phi] - \phi^*_A (c^A - R^A[\phi]) . $$

(82)

At the same time, this changes the BRST algebra into

$$ \delta \phi^A = c^A $$

$$ \delta c^A = 0 $$

$$ \delta \phi^*_A = - \frac{\delta S_{\text{ext}}}{\delta \phi^A} . $$

(83)

Demanding that the extended action $S_{\text{ext}}$ is invariant under this BRST symmetry is then equivalent to

$$ 0 = \delta S_{\text{ext}} = \frac{\delta^r S_{\text{ext}}}{\delta \phi^A} c^A - \frac{\delta^r S_{\text{ext}}}{\delta \phi^*_A} \delta \phi^A , $$

or, in other words, precisely a Master Equation of the general form

$$ \frac{1}{2} (S_{\text{ext}}, S_{\text{ext}}) = - \frac{\delta^r S_{\text{ext}}}{\delta \phi^A} c^A . $$

(84)

(85)

For irreducible closed gauge algebras this is the end of the story. The Master Equation (85) contains no terms of order $\hbar$ or higher. It is a classical equation that can be solved algebraically, without resort to $\hbar$-expansions. For theories with ultraviolet divergences this holds as long as the chosen regularization scheme respects the BRST symmetry, but the existence of such a regularization procedure has been the working assumption throughout. We refer again to the footnote in section 3 concerning this issue.

If one finally gauge-fixes the underlying gauge symmetry, one adds a term

$$ \delta \Psi[\phi] = \frac{\delta^r \Psi}{\delta \phi^A} c^A $$

(86)
to the action. The gauge-fixed partition function is then again of the form (79).

It is important that the “gauge fermion” $\Psi$ is a function of the fundamental fields only. This is because the BRST Schwinger-Dyson symmetry (83) is not nilpotent in general. A
term of the form $\delta \Psi$ is therefore not automatically BRST invariant. It will of course have vanishing expectation value, but the exponentiated form $\exp[\delta \Psi]$ will not in general, and its presence will affect Green functions. However, the operator $\delta$ of eq. (83) is nilpotent when acting on the fields only: $\delta^2 \phi^A = 0$. We are therefore permitted to add a (then BRST-exact) term of the form $\delta \Psi$, with $\Psi$ being a function of just the fields $\phi^A$. Adding this term will for reasonable choices of $\Psi$ not affect BRST-invariant expectation values. This means also that the appropriate Schwinger-Dyson equations are preserved in this procedure.

Because of the symmetry properties of the antibracket with respect to bosonic objects such as the action $S_{\text{ext}}$, we happened to extract the antibracket in the Master Equation (85) even though we in effect have only half of the canonical structure upon integrating out the collective fields $\phi^A$ and the auxiliary fields $B_A$. Integrating out also the shift ghosts $c^A$ while keeping the antighosts $\phi^*_A$ yields the missing ingredients of the full canonical structure behind the antibracket. The quantum corrections to the BRST symmetry which we discussed in detail in the two previous sections can, however, not be ignored. They distort the canonical structure by the quantum operator

$$i\hbar \Delta = (-1)^{c_A+1} (i\hbar) \frac{\delta r \delta r}{\delta \phi^*_A \delta \phi^A} .$$

(87)

We shall discuss this in some more detail in section 4.2.

The addition of the term from the “gauge fermion” $\Psi$ to the action can, as we have seen above, be viewed as a canonical transformation from $\phi^*_A = 0$ to $\phi^*_A = \delta^r \Psi / \delta \phi^A$. It is not an arbitrary canonical transformation however, because $\Psi$ must in general be a function of the fields $\phi^A$ only. The reason for this restriction is very clear in the present formulation, because the BRST operator is only nilpotent on the $\phi^*_A$-independent subspace.

In the case of an irreducible closed gauge algebra, the boundary conditions one imposes on the Master Equation have direct counterparts in the collective field formalism. Recall that we always have the freedom to shuffle internal gauge symmetries between the fundamental fields and the collective fields. Choosing to let the transformations of the fundamental fields be only arbitrary shifts (thereby lumping all internal transformations into the transformations of the collective fields) is equivalent to specifying the boundary conditions of Batalin and Vilkovisky.

The freedom in specifying the boundary conditions can be made quite manifest by means of another choice of transformations for the fundamental fields and the collective fields. Suppose that instead of eq. (75) we choose to let the collective fields transform only as shifts. The BRST symmetry is then

$$\begin{align*}
\delta \phi^A &= c^A + \mathcal{R}^A[\phi] \\
\delta \phi^*_A &= c^A \\
\delta c^A &= 0 \\
\delta b^A &= 0 \\
\delta \phi^*_A &= B_A \\
\delta B_A &= 0 ,
\end{align*}$$

(88)
instead of (75). Gauge fixing the collective fields to zero is now achieved by adding
\[ -\delta[\phi^*_A c^A - \Psi[\phi]] = (-1)^{q_A+1} B_A \phi^A - \phi^*_A c^A + \frac{\delta^r \Psi}{\delta \phi^A} (c^A + R^A[\phi]) \]  
(89)
to the action, which then reads
\[ S_{g.f.} = S[\phi - \varphi] + (-1)^{q_A+1} B_A \phi^A - \phi^*_A c^A + \frac{\delta^r \Psi}{\delta \phi^A} (c^A + R^A[\phi]) . \]  
(90)

The final gauge-fixed partition function is thus again of a simple form
\[ Z = \int [d\phi][d\varphi][d\phi^*][dB] \delta \left( \phi^* - \frac{\delta^r \Psi}{\delta \phi^A} \right) \exp \left[ \frac{i}{\hbar} S_{g.f.} \right] . \]  
(91)

Comparing with eq. (79), the only difference is that after integrating over the ghost fields \( c^A \), the antighosts \( \phi^*_A \) no longer appear in the action. Instead, there is now an extra term multiplying \( \delta \Psi/\delta \phi^a \), and the final answer is of course the same.

Note that the prescription is still to gauge fix by replacing \( \phi^*_A = 0 \) by \( \phi^*_A = \frac{\delta^r \Psi}{\delta \phi^A} \), but that now this replacement is entirely trivial since the extended action is \( \phi^*_A \)-independent. Instead, the action contains the additional term
\[ \delta \Psi[\phi] = \frac{\delta^r \Psi}{\delta \phi^A} R^A , \]  
(92)
which is just the usual gauge fixing term one adds to the classical action in standard Lagrangian BRST quantization. This choice of the extended action corresponds to the boundary condition for the Master Equation that gives us just the classical action as the solution after having integrated out the shift ghosts \( c^A \). Of course, this is also a valid boundary condition, but the disadvantage is that one is back in the standard Lagrangian BRST formulation and one must then perform further steps in order to gauge fix the action (by adding an appropriate term of the form \( \delta \Psi[\phi] \)). This is, however, all automatically achieved in one step by the collective field technique.

At the level of the Master Equation, the choice (88) corresponds to
\[ \frac{1}{2} (S_{ext},S_{ext}) = -\frac{\delta^r S_{ext}}{\delta \phi^A} (c^A + R^A) . \]  
(93)

Since all steps are otherwise identical, the solution to eq. (93) must be a solution to eq. (85) as well. This shows that whenever we have a solution \( S_{ext}[c^A,\ldots] \), also \( S_{ext}[c^A + R^A,\ldots] \) is a valid solution. It is trivial to see that in fact an arbitrary coefficient \( \alpha \) can be introduced as well:
\[ S_{ext}[c^A,\ldots] \rightarrow S_{ext}[c^A + \alpha R^A,\ldots] . \]  
(94)

This redundancy is nothing but the expression that the solution for the extended action is also automatically invariant under the internal BRST symmetry:
\[ \frac{\delta S_{ext}}{\delta \phi^A} R^A = 0 . \]  
(95)
The functional measures should be specified as well. It has been our working assumption that all measures of the fundamental fields are flat, i.e. invariant under arbitrary local shifts. If one is forced to quantize theories with more complicated measures, the symmetries of those measures determine the transformations from which Schwinger-Dyson equations follow. The corresponding Master Equation will then also look different, but the principle of imposing the Schwinger-Dyson equations at the level of Ward Identities can still be enforced.

The measure of the fundamental fields is therefore by construction invariant under the Schwinger-Dyson BRST transformation, and in the particular case of a flat measure they are invariant under the shifts (83) used in deriving the Master Equation (85). Invariances of the measure for the antifields $\phi^*_A$ under their transformation (83) is required as well. This is because in order for the Ward Identities to be satisfied, at least the combination of measure and action must be invariant. Since the action $S_{ext}$ of (82) itself is invariant, this is required for the measure too. If one chooses a different measure, the correct Schwinger-Dyson equations will not be recovered and the quantization scheme is therefore inconsistent. In the most common case of Grassmann-valued antifields $\phi^*_A$, the usual rules of Berezin integration correspond to functional measures that are invariant under local shifts of the $\phi^*_A$ fields. It turns out that such measures will also be invariant under the transformation (83) in the case of (irreducible) closed gauge algebras. The relevant Jacobian is

$$J = 1 - \frac{\delta^r}{\delta \phi^*_A} \left( \frac{\delta S_{ext}}{\delta \phi^A(x)^{\mu}} \right),$$

which actually equals unity due to the trace properties of these gauge algebras. (The parameter $\mu$ is an anticommuting $\epsilon$-number needed to define true variations from BRST transformations; see the appendix for the conventions). This is indeed only a consistency check, because it follows from our derivation in terms of collective fields that the measure must be invariant under this transformation. So invariance is a priori guaranteed. We shall discuss how to go beyond this case below.

After having determined the solution $S_{ext}$ of eq. (85) which satisfies the proper boundary conditions, both the measure and the action is therefore in this case by construction invariant under the BRST symmetry (83). Since it can be viewed as the result of having integrated out certain auxiliary fields, it is no surprise that it is not nilpotent off-shell in general. But the crucial property is that it is off-shell nilpotent when acting on functions of the fields $\phi^A$ only.

### 4.1 Quantum Master Equations

We have seen from the collective field method that for closed irreducible gauge algebras we get an extended action that can be split into a part independent of the new ghosts $c^A$, and a simple quadratic term of the form $\phi^*_A c^A$. Let us, for reasons that will become evident shortly, denote the part which is independent of $c^A$ by $S^{(BV)}$, i.e.:

$$S_{ext}[\phi, \phi^*, c] = S^{(BV)}[\phi, \phi^*] - \phi^*_A c^A.$$  

(97)
This action is invariant under the transformations
\[
\delta \phi^A &= c^A \\
\delta c^A &= 0 \\
\delta \phi_A^* &= -\frac{\delta^* S_{ext}}{\delta \phi^A}.
\tag{98}
\]

Moreover, the functional measure is also formally guaranteed to be invariant in this case. It follows that in this case the Ward Identities of the kind \(0 = \langle \delta[\phi^*_A F[\phi]] \rangle\) are the most general Schwinger-Dyson equations for the quantum theory defined by the classical action \(S[\phi]\).

But demanding that both the action \(S_{ext}\) and the functional measure be invariant under the BRST Schwinger-Dyson symmetry above is not the most general condition. To derive the correct Ward Identities we only need that just the combination of action and measure is invariant. In this subsection we want to discuss the more general case in which the set of transformations (98) still generate a symmetry of the combination of measure and action, but not of each individually. If we insist on a solution of the form (97), then the other property that is required, \(\langle \epsilon^A \phi_B^* \rangle = -i\hbar \delta_A^B\), follows automatically.

It thus remains to be found under what conditions the combination of the action and the measure remain invariant under the BRST Schwinger-Dyson symmetry. With an action \(S_{ext}\) of the form (97), we get
\[
\delta S_{ext} = \frac{\delta^r S^{(BV)}}{\delta \phi^A} c^A + \frac{\delta^r S^{(BV)}}{\delta \phi_A^*} \left( -\frac{\delta^* S_{ext}}{\delta \phi^A} - \frac{\delta^r (\phi_A^* c^A)}{\delta \phi_B^*} \left( -\frac{\delta^* S_{ext}}{\delta \phi_B^*} \right) \right)
= \frac{\delta^r S^{(BV)}}{\delta \phi^A} c^A + \frac{\delta^r S^{(BV)}}{\delta \phi_A^*} \left( -\frac{\delta^* S^{(BV)}}{\delta \phi^A} \right)
= -\frac{\delta^r S^{(BV)}}{\delta \phi_A^*} \frac{\delta^* S^{(BV)}}{\delta \phi^A} = \frac{1}{2} \left( S^{(BV)}, S^{(BV)} \right).
\tag{99}
\]

We will still assume that we are integrating over a flat euclidean measure for the fundamental field \(\phi^A\). This measure is formally invariant under the transformation (98). However, for a corresponding flat euclidean measure for \(\phi_A^*\), the Jacobian of the transformation (98) will in general be different from unity. As we already discussed above, the Jacobian equals
\[
J = 1 - \frac{\delta^r}{\delta \phi_A^*} \left( \frac{\delta^* S_{ext}}{\delta \phi^A(x) \mu} \right).
\tag{100}
\]

Thus to demand that the combination of measure and action remains invariant, we must in general require that
\[
\frac{1}{2} (S_{ext}, S_{ext}) = -\frac{\delta^r S_{ext}}{\delta \phi_A^*} c^A + i\hbar \Delta S_{ext},
\tag{101}
\]
which, assuming the form (97) – since we know that this is sufficient to guarantee the correct Schwinger-Dyson equations – reduces to the quantum Master Equation of Batalin and Vilkovisky:
\[
\frac{1}{2} (S^{(BV)}, S^{(BV)}) = i\hbar \Delta S^{(BV)}.
\tag{102}
\]
Let us emphasize that this equation follows even before possible gauge fixings. It is required in order that the general Schwinger-Dyson equations for the fundamental fields are satisfied, and is not postulated on only the requirement that the final functional integral be independent of the gauge-fixing function. However, gauge independence of the functional integral upon the addition of a term of the form \( \delta \Psi[\phi] \) now follows straightforwardly, since for a functional \( \Psi \) that depends only on the fields \( \phi \), we have \( \delta^2 \Psi[\phi] = 0 \).

It may then be worthwhile to leave the actual derivation of the quantum Master Equation

\[
\frac{1}{2}(S_{\text{ext}}, S_{\text{ext}}) = -\frac{\delta^r S_{\text{ext}}}{\delta \phi^A} c^A + i\hbar \Delta S_{\text{ext}},
\]  

(103)

behind. Instead one can, in the spirit of Batalin and Vilkovisky [1], take it as the starting point of a more algebraic approach to quantization. At least two questions should, however, first be answered. Which are the boundary conditions one should impose on this equation, and is eq. (97) the most general acceptable solution? Why does this Master Equation guarantee the correct quantization prescription?

Consider the last question first. We have seen that the Master Equation (103) follows from requiring a BRST symmetry of the form (98). If the solution to the Master Equation contains precisely one term of the form \( \phi^*_A c^A \) (as in eq. (97), this BRST symmetry guarantees that the Schwinger-Dyson equations for all the fundamental fields are satisfied at the formal level. (It can only be at the formal level, because without complete gauge fixing, the path integral formalism is still not totally well-defined.) Having the full set of Schwinger-Dyson equations satisfied in a well-defined manner can be viewed as the only independent means of defining what we mean by a correctly quantized theory.

To ensure that the Schwinger-Dyson equations are satisfied not only at the formal level, one must therefore specify additional boundary conditions on the solution for \( S_{\text{ext}} \). These boundary conditions must be imposed in such a way that the path integral becomes well-defined, without changing gauge-invariant Green functions. Once the path integral is well-defined, the BRST algebra ensures that all Schwinger-Dyson equations are satisfied, and the full theory is then by definition correctly quantized.

When the solution of the equation (103) is chosen to be of the form (97), correct boundary conditions can be copied directly from the formalism of Batalin and Vilkovisky. This follows from the equivalence between eqs. (101) and (102) in that case. The whole problem has then in effect reduced to the original Batalin-Vilkovisky formulation.

Imposing the condition that \( S_{\text{ext}} \) contains only the term \( \phi^*_A c^A \) and no higher orders in \( c^A \) is perhaps not the only possibility. One way of stating it is that the measures for \( \phi^*_A \) and \( c^A \) are not fixed by any overall principle. A general requirement is that at least

\[
\langle c^A \phi^*_B \rangle = -i\hbar \delta^A_B .
\]  

(104)

This is needed to ensure that the Schwinger-Dyson equations of the fundamental fields are satisfied. However, the Schwinger-Dyson equations for the original classical fields will in general be those associated with an action \( S_{\text{ext}} \), and not those of the classical action \( S \). An additional criterion is then required to select a solution which yields the correct Schwinger-Dyson equations for the classical fields. This makes the choice of correct
boundary conditions far more complicated, and we have not investigated this question in detail. By choosing a solution containing only the term $\phi_A^* c^A$ in the action, one is guaranteed that $\phi_A^*$ is set to zero before gauge fixing. Then all additional terms in $S_{ext}$ are in fact effectively zero, and the correct Schwinger-Dyson equations for the classical fields arise automatically as a consequence of the first Batalin-Vilkovisky boundary condition. This shows the extent to which the $\delta$-function “gauge” for $\phi_A^*$ is the most general valid condition. (Adding a term of the form $\delta \Psi$ to the action still yields correct Schwinger-Dyson equations for gauge invariant objects, since such a term does not alter gauge invariant Green functions.) In cases where the derivation can be based straightforwardly on the collective field technique one can certainly entertain the idea of more complicated gauge choices for the collective fields – gauge choices which could give rise to different valid prescriptions for the $\phi_A^*$-integrations.

4.2 Quantum BRST

In section 2 we noted that the usual BRST Schwinger-Dyson symmetry acquires a “quantum correction” if one insists on using the formalism where the new ghost fields $c^A$ have been integrated out of the path integral. As we saw already in the case of no gauge symmetries, this deforms the BRST operator:

$$\delta \rightarrow \sigma = \delta - i \hbar \Delta.$$  \hspace{1cm} (105)

The notation is not entirely precise, because the operator $\delta$ on the right hand side of this equation of course equals the operator $\delta$ on the left hand side only modulo those changes incurred by integrating out the ghosts $c^A$. But we keep it like this in order not to clog up the paper with yet more notation. After having integrated out the ghosts $c^A$, the BRST operator $\delta$ will become identical to the variation within the antibracket.

Since this quantum deformation involves the same operator $\hbar \Delta$ that in certain specific cases may modify the classical Master Equation, one might be led to believe that these two issues are related, i.e., that the “quantum BRST” operator should only be applied when there are, (or as a consequence of having) quantum corrections in the full gauge-fixed action. This is actually not the case, and we therefore find it useful to return briefly to the meaning of the quantum BRST operator, here denoted by $\sigma$.

Let us again choose the simplest solution to the Master Equation of the form (97). We emphasize that it is immaterial whether this extended action $S_{ext}$ satisfies the classical or quantum Master Equations. Since we are interested in seeing the effect of integrating out the ghosts $c^A$, consider, as in section 2, the expectation value of the BRST variation of an arbitrary functional $G = G[\phi^A, \phi_A^*]$:  

$$\langle \delta G[\phi, \phi^*] \rangle = Z^{-1} \int [d\phi][d\phi^*][dc] \delta G[\phi, \phi^*] \exp \left[ \frac{i}{\hbar} \left( S^{(BV)} - \phi_A^* c^A \right) \right]$$

$$= Z^{-1} \int [d\phi][d\phi^*][dc] \left\{ \frac{\delta^r G}{\delta \phi^A} c^A + \frac{\delta^r G}{\delta \phi_A^*} \left( -\frac{\delta^l S_{ext}}{\delta \phi^A} \right) \right\} \exp \left[ \frac{i}{\hbar} \left( S^{(BV)} - \phi_A^* c^A \right) \right]$$

$$= Z^{-1} \int [d\phi][d\phi^*] \left\{ \frac{\delta^r G}{\delta \phi^A} (i\hbar) \frac{\delta^l}{\delta \phi_A^*} \delta(\phi^*) - \frac{\delta^r G}{\delta \phi^A} \delta(\phi^*) \right\} \exp \left[ \frac{i}{\hbar} S^{(BV)} \right]$$

29
\[ \begin{align*}
&= Z^{-1} \int [d\phi][d\phi^*] \delta(\phi^*) \left\{ \frac{\delta^* G \delta S^{(BV)}}{\delta \phi^* A} + (i\hbar)(-1)^{\epsilon_A} \frac{\delta^* \delta G}{\delta \phi^*_A \delta \phi^A} - \frac{\delta^* G \delta S^{(BV)}}{\delta \phi^*_A \delta \phi^A} \right\} \\
&\quad \times \exp \left[ \frac{i}{\hbar} S^{(BV)} \right] \\
&= \langle (G, S^{(BV)}) - i\hbar \Delta G \rangle .
\end{align*} \tag{106} \]

The derivation given here corresponds to the path integral before gauge fixing, but it goes through in entirely the same manner in the gauge-fixed case. (The only difference is that the relevant \( \delta \)-function reads \( \delta(\phi^* - \delta \phi \Psi^/\delta \phi) \) instead of \( \delta(\phi^*); \) this does not affect the manipulations above).

The emergence of the “quantum correction” in the BRST operator is thus completely independent of the particular solution \( S^{(BV)}[\phi, \phi^*] \); it must always be included when one uses the formalism in which the ghosts \( c^A \) have been integrated out. The quantum BRST operator \( \sigma \) is unusual, because it appears only after functional manipulations inside the path integral.

Since by construction the partition function is invariant under \( \delta \) (when keeping the ghosts \( c^A \)) and \( \sigma \) (after having integrated out these ghosts), it follows that all expectation values involving these operators vanish:

\[ \langle \delta G[\phi, \phi^*] \rangle = 0 \] (107)

when keeping \( c^A \), and

\[ \langle \sigma G[\phi, \phi^*] \rangle = 0 \] (108)

when the \( c^A \) have been integrated out.

This of course holds for the action as well:

\[ \langle \delta S_{ext} \rangle = 0 ; \quad \langle \sigma S^{(BV)} \rangle = 0 \] . (109)

The first of these equations is trivially satisfied when \( S_{ext} \) satisfies the classical Master Equation, because then the variation \( \delta S_{ext} \) itself vanishes. This equation is then only non-trivially satisfied when \( \Delta S_{ext} \neq 0 \).

Since the two operations \( \delta \) and \( \sigma \) are equivalent in the precise sense given above, the same considerations should apply to the second equation. Indeed it does: When \( S^{(BV)} \) satisfies the classical Master Equation, \( \sigma S^{(BV)} = 0 \) at the operator level, while that equation is satisfied only in terms of expectation values when \( \Delta S^{(BV)} \neq 0 \).

Note that when \( \Delta S_{ext} \neq 0 \) (or \( \Delta S^{(BV)} \neq 0 \)), the quantum action is neither invariant under \( \delta \) nor \( \sigma \). The action precisely has to remain non-invariant in order to cancel the non-trivial contribution from the measure in that case. This is the origin of the factor 1/2 difference between the quantum Master Equation

\[ \frac{1}{2} (S^{(BV)}, S^{(BV)}) - i\hbar \Delta S^{(BV)} = 0 \] (110)

and the operator \( \sigma \) (when acting on \( S^{(BV)} \)):

\[ \langle (S^{(BV)}, S^{(BV)}) - i\hbar \Delta S^{(BV)} \rangle = 0 \] . (111)
The combination of these two equations yields the new identities

\[ \langle \Delta S^{(BV)} \rangle = 0, \quad \langle (S^{(BV)}, S^{(BV)}) \rangle = 0 \] (112)

which can also be verified directly using the path integral.

The operator \( \delta \) defines a BRST cohomology only on the subspace of fields \( \phi^A \); it is only nilpotent on that subspace. The operator \( \sigma \) is nilpotent in general: \( \sigma^2 = 0 \) (a consequence of having performed partial integrations in deriving it). However, the two operators share the same physical content.

4.3 Open Gauge Algebras

Having the quantum Master Equation available, it is of interest to lift the restriction to closed irreducible gauge algebras. Recall that in that case the functional measure is formally BRST invariant, and the classical Master Equation suffices. Directly related to this is the fact that this case can be dealt with straightforwardly through the use of certain collective fields. We believe that the case of reducible gauge symmetries can be treated in a rather similar fashion, but something really new enters when one considers theories with open gauge algebras. Here the quantum Master Equation seems to be the preferable starting point.

An open gauge algebra is characterized by closing only on-shell. Off-shell a new term, proportional to equations of motion appear. For this reason, the collective field formalism has to be modified in order to be valid at the quantum level. This means that it is not straightforward to introduce collective fields so as to enforce well-defined Schwinger-Dyson equations in that case.

Let us thus first follow the reasoning above, and view the Master Equation (103) as the key to the solution of this problem, independently of its derivation by means of collective fields. We know that this point of view is acceptable, because it automatically guarantees correct Schwinger-Dyson equations.

Some notation should be introduced. The gauge generators defined by the symmetry

\[ \delta \phi^A = R^A_e \varepsilon^a \] (113)

i.e. through

\[ \frac{\delta S}{\delta \phi^A} R^A_e = 0 \] (114)

for the classical fields \( \phi^A \), form (see, e.g. ref. [10]) an algebra

\[ \frac{\delta R^A_e}{\delta \phi^B} R^B_\beta - \frac{\delta R^A_\beta}{\delta \phi^B} R^B_e = -R^A_{\gamma j} f_{\alpha \beta} - \frac{\delta S}{\delta \phi^B} E^{AB}_{\alpha \beta} . \] (115)

We have for convenience restricted ourselves to the case of bosonic fields \( \phi^A \). The coefficients \( E^{AB}_{\alpha \beta} \) can be considered new additional generators of the algebra. For closed gauge algebras all \( E^{AB}_{\alpha \beta} \) vanish.
The solution to the Master Equation (103) can, if we again restrict ourselves to solutions of the form (97), be immediately read off from the solution to the quantum Master Equation of Batalin and Vilkovisky [1,10]. The result for the extended action is

$$S_{ext} = S[\phi] - \phi^*_A (c^A - R_A) + \frac{1}{4} \phi^*_A \phi^*_B F^{AB} \alpha^\alpha c^\beta,$$

where $c^\alpha$ is the usual ghost associated with the symmetry (113). To this must be added the quantum corrections. Open gauge algebras of higher degree can be treated similarly [1].

By adding an appropriate gauge-fixing function $\Psi$, and integrating over the ghost-antighost pairs $\phi^*_A, c^A$, we of course recover the known result. It is again important that $\Psi$ is a function of just the fields $\phi^A$, since only on that subspace is the BRST Schwinger-Dyson operator $\delta$ nilpotent. This is also the condition which ensures that in the process of gauge-fixing the antifields are changed from zero to $\delta r / \delta \phi^A$.

Although one can demand correct Schwinger-Dyson equations by construction in this manner, it is interesting that the standard collective field technique breaks down in this case. If we trace it back, we note that this is because demanding BRST invariance of the classical action by itself is too restrictive, and in fact not required. We only need to insist on correct Schwinger-Dyson equations for the full gauge-fixed theory. For open gauge algebras the transformations of the fields and their collective field partners should therefore not necessarily be required to leave the classical action invariant. Since at the level before integrating out the collective fields $\varphi^A$ and their Nakanishi-Lautrup fields $B_A$ one should have a full nilpotent BRST algebra, one can in fact derive the required transformation laws. This is highly analogous to the original general solution of De Wit and van Holten [11]. We believe that the correct procedure for introducing collective fields in this case could go along such lines, but we have not investigated this question in detail. It is also possible that it may require the introduction of additional fields, perhaps resulting from the gauging of “trivial” gauge symmetries related to the open algebra structure coefficients, a point of view that has been advocated by Hull [12]. In any case, we find it quite likely that a consistent interpretation in terms of suitable collective fields exists in this more general case as well.

Finally a few words about the quantum corrections to the gauge-fixed action. These may seem unusual from the point of view of more conventional BRST gauge fixing, where it ordinarily suffices to add to the classical action a term of the form $\delta \Psi$, with $\delta$ being the variation with respect to a nilpotent BRST operator. This, however, assumes that the functional measure remains invariant under the transformation. If it does not, then the BRST procedure of, e.g., ref. [11] must be supplemented. One can do this by trying to find a term whose BRST variation precisely equals the contribution from the measure. If this is not possible (because adding the extra term may modify the Jacobian), one must resort to a higher-order expansion in $\hbar$, at each order trying to correct for the change in the measure. This is the ordinary BRST quantization analogue of solving the general quantum Master Equation in the Batalin-Vilkovisky framework.
5 Conclusion

We have shown that a new set of ghost fields, $c^A$ in the notation of this paper, naturally belong to the Batalin-Vilkovisky Lagrangian quantization scheme. What in the language of Batalin and Vilkovisky are known as “antifields”, are the usual BRST partners (antighosts) of these new ghost fields.

The ghosts $c^A$ appear when one insists that Schwinger-Dyson equations should be satisfied at the level of BRST Ward Identities. After integrating out these ghosts $c^A$ one recovers the full scheme of Batalin-Vilkovisky: The antibracket, the quantum Master Equation, the quantum BRST generator etc. It is truly remarkable that almost the whole formalism could be developed \([1]\) without this additional background, and without a corresponding derivation from more known quantization principles. The only other clue to this formalism that we know of is the Zinn-Justin equation for Yang-Mills theory \([13]\).

Interestingly, the fundamental BRST symmetry for any quantum field theory of flat measures for the fields is not the usual BRST symmetry associated with (possible) internal gauge invariances, but rather

\[
\begin{align*}
\delta \phi^A &= c^A \\
\delta c^A &= 0 \\
\delta \phi^*_A &= -\frac{\delta^i S_{\text{ext}}}{\delta \phi^A}.
\end{align*}
\]

(117)

This BRST symmetry, which we have called the BRST Schwinger-Dyson symmetry contains no explicit reference to the internal gauge (or internal BRST) invariances among the fields $\phi^A$. Instead, all knowledge of the gauge transformations has been transferred into certain conditions the extended action $S_{\text{ext}}$ must fulfill. This corresponds to choosing appropriate boundary conditions for the solution to the Master Equation.

Only when we integrate out the ghost-antighost pair $c^A, \phi^*_A$ do we recover the usual internal BRST transformation for the fields themselves. In the simplest cases, these internal BRST symmetries are nothing but the equations of motion for the new ghost fields. Intuitively we can understand this late appearance of the standard internal BRST algebra from the fact that Schwinger-Dyson equations represent much more general statements about the quantum theory, above such details as the particular symmetries that leave the theory invariant.

A new Master Equation follows if one keeps the new ghosts. In those cases where we can derive the quantization scheme straightforwardly from collective fields, it is of the simple form

\[
\frac{1}{2}(S_{\text{ext}}, S_{\text{ext}}) = -\frac{\delta^i S_{\text{ext}}}{\delta \phi^A} c^A
\]

(118)

with no quantum corrections. These cases include all theories without internal gauge symmetries, and theories invariant under transformations satisfying a closed irreducible algebra. The functional measures are in those cases formally invariant.

When one cannot immediately derive the correct prescription through the use of collective fields, one can instead rely on the more general principle of satisfying Schwinger-Dyson equations in the quantum theory. The only input is then the Schwinger-Dyson
BRST symmetry above, and one must now demand that at least the combination of measure and action remains invariant under this symmetry (since this suffices to derive the Schwinger-Dyson equations as Ward Identities of this symmetry). For flat $A^A$-measures, the relevant Jacobian differs from unity if and only if $\Delta S_{\text{ext}} \neq 0$. In that case the correct quantum action $S_{\text{ext}}$ will not be invariant under the BRST symmetry (117), but will instead transform in precisely such a manner as to cancel the Jacobian from the measure. The Master Equation (118) must then be replaced by the quantum Master Equation

$$\frac{1}{2}(S_{\text{ext}}, S_{\text{ext}}) = -\frac{\delta^r S_{\text{ext}}}{\delta \phi^A} c^A + \hbar \Delta S_{\text{ext}}. \quad (119)$$

This equation follows directly from demanding that the Schwinger-Dyson equations are Ward Identities of the symmetry (117). The quantum Master Equation may also be required in situations where consistent regulators that preserve the relevant BRST symmetry cannot be found. This may be the case for anomalous theories $[9]$, or in larger generality. An example of how this may lead to “anomalous gauge fixing” has been given in ref. $[14]$ in more conventional BRST language, but it presumably has a corresponding interpretation in terms of the Batalin-Vilkovisky quantum Master Equation.

The “quantum deformation” of the BRST generator from $\delta$ to $\sigma = \delta - i\hbar \Delta$ is unrelated to the need in certain cases for having a quantum correction to the Master Equation, as in eq. (119) above. The quantum Master Equation is only required in those cases where the functional measure does not respect the BRST Schwinger-Dyson algebra. In contrast, the BRST symmetry operator is always $\sigma$ rather than $\delta$ if one insists on the formulation in which the ghosts $c^A$ have been integrated out, but the antighosts $\phi^*_A$ are kept. If one keeps the new ghosts $c^A$, there is no quantum deformation, and the usual BRST operator $\delta$ generates a genuine symmetry of the path integral. Stated differently, the “quantum corrections” to the BRST operator are already automatically included in the formulation in which the ghosts $c^A$ are kept. In particular, the Ward Identities obtained using this operator are the correct Schwinger-Dyson equations.

The simple interpretation of the antifields as antighosts introduced through the gauge fixing of a certain Schwinger-Dyson gauge symmetry is lost if one relies solely on the Master Equation to solve the problem. However, we find it quite likely that some modification of the usual collective field technique can be used in general to describe the $\phi^*$-fields as the antighosts required to fix the relevant gauge symmetry. To get to that stage will perhaps require the introduction of more field variables in cases of, e.g., open gauge algebras.

Our discussion has at no point touched the issue of field theoretic unitarity. It has been shown that when a certain locality condition $[15]$ is imposed on the solution to the Master Equation, unitarity can be guaranteed $[16]$.

From the present point of view, the antifields $\phi^*_A$ are not artificial devices with which to set up the correct quantization procedure. They are genuine fields of the path integral, on equal footing with $c^A$ and all the usual fields $\phi^A$. For the most straightforward gauge fixings we have considered here, these ghost-antighost pairs $c^A, \phi^*_A$ do not propagate (and for this reason are often easy to integrate out of the path integral). Depending on the circumstances, this can either be seen as corresponding to very well-behaved functional integrals, or as having to deal with rather singular propagators of the $\delta$-function kind.
Regulators can be introduced which preserve the Schwinger-Dyson equations, but which give rise to propagating fields $c^A$ and $\phi^*_A$. A simple one-dimensional example which readily generalizes to higher dimensions has been given in ref. [17]. As expected, the corresponding BRST Schwinger-Dyson algebra is, however, modified (“regularized”). The change in the BRST transformations can be straightforwardly derived by the collective field technique.

Inclusion of the new ghosts $c^A$ may be aesthetically appealing, but will they be helpful in developing the correct quantization prescription in cases that are not yet fully understood? String theory clearly comes to mind here. What is the rôle of these new ghosts in string field theory? Do they simplify the formalism? Recently Verlinde [4] has shown how to derive an equation very reminiscent of the Batalin-Vilkovisky quantum Master Equation for low-dimensional string theory. The derivation was noted to be almost identical to the derivation of Virasoro and $W$-constraints on the string partition functions. This is presumably no coincidence, since in both cases they express Schwinger-Dyson equations. Is it advantageous to include the analogues of the ghosts $c^A$ in this formulation?

The shift symmetries introduced through collective fields to ensure Schwinger-Dyson equations at the level of the BRST algebra are only very special cases of more general field-enlarging transformations that can be performed within the Feynman path integral. Since the particular choice of variables should have no influence on physical quantities, i.e. in this context $S$-matrix elements, one should be able to formulate the quantization prescription in a more coordinate-independent manner. This is the content of the field redefinition theorem. Enforcing Schwinger-Dyson equations in different field variables can be accomplished by suitable field transformations, for which the simple shift (and subsequent gauging to zero of the collective fields) that we have considered in this paper corresponds to the identity transformation. Switching between one set of fields to another may, independently of whether one uses a Lagrangian or Hamiltonian formulation, involve the addition of terms of up to order $\bar{\hbar}^2$ to the naively transformed action, but also these additional terms have a corresponding interpretation in the BRST Schwinger-Dyson algebra [17].

When the functional measures for the fundamental fields are non-trivial, the Master Equations will differ in form. The advantage of the present derivation is that we now know how to write down the corresponding Master Equations for theories of arbitrary path integral measures.

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Appendix

In this appendix we give some additional conventions, and list some useful identities. Repeatedly in this formalism one needs the notion of left and right derivatives. For
bosons only, this notion is not useful. But as soon as the Grassmann parity of the fields is kept arbitrary, it pays to use both left and right derivatives in appropriate places, in order to simplify expressions. The Leibniz rules for derivations of the left and right kind read

$$\frac{\delta_l(F \cdot G)}{\delta A} = \frac{\delta_l F}{\delta A} G + (-1)^{\epsilon_F \cdot \epsilon_A} F \frac{\delta_l G}{\delta A}$$  \hspace{1cm} (120)$$

and

$$\frac{\delta_r(F \cdot G)}{\delta A} = F \frac{\delta_r G}{\delta A} + (-1)^{\epsilon_G \cdot \epsilon_A} \frac{\delta_r F}{\delta A} G ,$$  \hspace{1cm} (121)$$

where $A$ denotes a field (or antifield) of arbitrary Grassmann parity $\epsilon_A$. Similarly, $\epsilon_F$ and $\epsilon_G$ are the Grassmann parities of the functionals $F$ and $G$.

Actual variations, let us denote them by $\bar{\delta}$ in contrast to the BRST transformations $\delta$ of the paper, are defined as follows:

$$F[A + \bar{\delta} A] - F[A] \equiv \bar{\delta} F \equiv \frac{\delta A}{\delta A} \equiv \frac{\delta_r F}{\delta A} \delta A .$$  \hspace{1cm} (122)$$

The commutation rule of two arbitrary fields is

$$A \cdot B = (-1)^{\epsilon_A \epsilon_B} B \cdot A ,$$  \hspace{1cm} (123)$$

and for actual variations one has the simple rule that

$$\bar{\delta}(F \cdot G) = (\bar{\delta} F) G + F(\bar{\delta} G)$$  \hspace{1cm} (124)$$

independent of the Grassmann parities $\epsilon_F$ and $\epsilon_G$. The rules (122) and (123) in conjunction lead to the useful identity

$$\frac{\delta_l F}{\delta A} = (-1)^{\epsilon_A(\epsilon_F + 1)} \frac{\delta_r F}{\delta A} .$$  \hspace{1cm} (125)$$

The BRST variations we have worked with in this paper correspond to right derivation rules. This is of course not imposed upon us, but it is convenient if we wish to compare our expressions with those of Batalin and Vilkovisky. It follows from requiring the actual variations to be related to the BRST transformations by multiplication of an anticommuting parameter $\mu$ from the right. This then provides us with very helpful operational rules for the BRST transformations $\delta$. In particular,

$$\bar{\delta} F \equiv (\delta F)\mu = \frac{\delta_r F}{\delta A} \delta A .$$  \hspace{1cm} (126)$$

Now, since

$$\delta F \equiv \frac{\delta_r \bar{\delta} F}{\delta \mu} ,$$  \hspace{1cm} (127)$$

it follows that

$$\delta F = \frac{\delta_r F}{\delta A} \delta A .$$  \hspace{1cm} (128)$$

From this it also follows directly that the BRST transformations act as right derivations:

$$\delta(F \cdot G) = F(\delta G) + (-1)^{\epsilon_G(\delta F) G} .$$  \hspace{1cm} (129)$$

These are the basic rules that are needed for the manipulations in the main text.
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