A Generalized Correlation Theorem for LTI Systems

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Abstract. In this paper we derive a generalized correlation theorem for linear time-invariant (LTI) system with wide-sense stationary (WSS) input. This theorem shows that if $z(n)$ and $x(n)$ are jointly WSS random signals, then $z(n)$ and $y(n)$, the output of $x(n)$ passing an LIT system $h(n)$, are also jointly WSS. In addition, the cross-correlation function (CCF) between $y(n)$ and $z(n)$ can be computed by the convolution of $h(n)$ and the CCF between $x(n)$ and $z(n)$. Many results of the correlations between the input and output of an LTI system are special cases of this theorem. Moreover, the correlations between the inputs and outputs of multiple LTI systems can be analysed by this theorem.

1. Introduction

Correlation functions and correlation operations are of great importance in numerous signal processing applications such as matched filtering, spectrum estimation, time-frequency analysis, and channel calibration [1-2]. The linear time-invariant (LTI) system with a wide-sense stationary (WSS) input is a basic model for communication and radar systems and plays an important role in receiver design [3]. The analysis and calculation of the correlations between different signals in LTI systems are the fundamental tasks for signal detection and estimation. In this paper we present a generalized result considering the cross-correlations of jointly WSS (JWSS) random signals passing through LTI systems. This new result greatly simplifies the analysis and calculation of the correlations between the input and output of LTI systems.

We will only consider the discrete-time systems and discrete-time WSS random signals. However, the results can be easily extended to their continuous-time counterparts.

2. Basic Concepts and Definitions

In this section we briefly review the basic concepts and definitions of the WSS random signals (processes). A discrete-time random signal $x(n)$ is called wide-sense stationary if (a) its mean $\mu_x = E[x(n)]$ is a constant independent of $n$ and (b) its auto-correlation function (ACF) depends only on the time difference, i.e.,

$$r_{xx}(n+m, n) = E[x(n+m)x^*(n)] = r_{xx}(m)$$

(1)
where the superscripts $(\cdot)^*$ denotes complex conjugate and the notation $E[\cdot]$ stands for mathematical expectation. Two random signals $x(n)$ and $y(n)$ are called JWSS if (a) each of them is WSS and (b) their cross-correlation function (CCF) depends only on the time difference, i.e.,

$$
    r_{xy}(n + m, n) = E[x(n + m)y^*(n)] = r_{xy}(m)
$$

(2)

The ACF and CCF have the following properties [4]

$$
    r_{xx}(m) = r_{xx}(-m), \quad r_{yy}(m) = r_{yy}(-m)
$$

(3)

The power spectral density (PSD, or auto-PSD) of a WSS signal $x(n)$ is defined as the discrete-time Fourier transform (DTFT) of its ACF:

$$
    P_{xx}(\omega) = \sum_{m=-\infty}^{\infty} r_{xx}(m) \exp(-j\omega m)
$$

(4)

where $j = \sqrt{-1}$ is the imaginary unit. The DTFT of a discrete-time signal $x(n)$ is denoted by $X(e^{j\omega})$ in the majority of the literature [5]. Here, for simplicity, we use $X(\omega)$ instead of $X(e^{j\omega})$ because we only consider discrete-time signals in this paper. The cross-power spectral density (cross-PSD) of two JWSS signals $x(n)$ and $y(n)$ is defined as the DTFT of their CCF:

$$
    P_{xy}(\omega) = \sum_{m=-\infty}^{\infty} r_{xy}(m) \exp(-j\omega m)
$$

(5)

In order to ensure the existence of DTFT, we make a mild assumption that $r_{xx}(m)$ and $r_{yy}(m)$ decay fast so as to be absolutely summable.

A discrete-time system is called bounded-input bounded-output (BIBO) stable if a bounded input produces a bounded output. An LTI system is BIBO stable if its impulse response $h(n)$ is absolutely summable [6], i.e.,

$$
    \sum_{n=-\infty}^{\infty} |h(n)| < \infty
$$

(6)

All LTI systems considered in this paper are assumed to be BIBO stable.

3. The Generalized Correlation Theorem

The main contribution of this paper described by theorem 2 is a generalization of the following correlation theorem.

**Theorem 1 (Correlation Theorem):** Suppose that a WSS signal $x(n)$ passes through a BIBO stable LTI system $h(n)$. Then the output

$$
    y(n) = h(n) \ast x(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)
$$

(7)

is also a WSS signal, where the operator $\ast$ represents the linear convolution. Moreover, $x(n)$ and $y(n)$ are JWSS.

**Proof:** This is a well-known result that can be found in the textbooks of statistical signal processing, such as [4].

**Theorem 2 (Generalized Correlation Theorem):** Suppose that $x(n)$ and $z(n)$ are two JWSS signals. If $y(n)$ is the output of a BIBO stable LTI system $h(n)$ with input $x(n)$, then $y(n)$ and $z(n)$ are JWSS and the CCF $r_{yz}(m)$ and $r_{zy}(m)$ are given by

$$
    r_{yz}(m) = h(m) \ast r_{xz}(m)
$$

(8)

$$
    r_{zy}(m) = h^*(-m) \ast r_{xz}(m)
$$

(9)

The results of this theorem are illustrated in figure 1.
Proof: Since $x(n)$ is a WSS random signal and $h(n)$ is an LTI system, we know from theorem 1 that $y(n)$ is also a WSS random signal. The CCF between $y(n)$ and $z(n)$ is given by

$$r_{yz}(n) = x(n) * h(n) * z(n)$$

where the first equality follows from the definition of CCF; the second equality from the definition of convolution; the third equality from the continuity of expectation (see appendices for more details); the fourth equality from the definition of CCF; and the fifth equality from the definition of convolution.

Therefore the CCF between $y(n)$ and $z(n)$ depends only on the time difference $m$, which proves that $y(n)$ and $z(n)$ are JWSS, and consequently (8) holds. It follows from (3) that

$$r_{yz}(m) = r_{yz}^*(-m) = h^*(-m) * r_{xz}^*(-m) = h^*(-m) * r_{xz}(m)$$

which completes the proof.

Corollary 1: Suppose that $x(n)$ and $z(n)$ are two JWSS signals. If $y(n)$ is the output of a BIBO stable system $h(n)$ with input $x(n)$, then the cross-PSDs $P_{yz}(\omega)$ and $P_{sy}(\omega)$ are given by

$$P_{yz}(\omega) = H(\omega) P_{xz}(\omega)$$
$$P_{sy}(\omega) = H^*(\omega) P_{xz}(\omega)$$

where

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) \exp(-j\omega n)$$

is the frequency response of $h(n)$.

Proof: Since $r_{yz}(m) = h(m) * r_{xz}(m)$, it follows from the convolution theorem of DTFT [6]:

$$x(m) * y(m) \xrightarrow{\text{DTFT}} X(\omega)Y(\omega)$$

that $P_{yz}(\omega) = H(\omega) P_{xz}(\omega)$. Since $r_{xz}(m) = r_{xz}^*(-m)$, it follows from the conjugate and time-reversed property of DTFT [6]:

$$x^*(-m) \xrightarrow{\text{DTFT}} X^*(\omega)$$

that $P_{xy}(\omega) = P_{xy}^*(\omega)$. By using the conjugate and time-reversed property again, we have
which completes the proof.

4. Applications of The Generalized Correlation Theorem

In this section we give some applications of theorem 2. First we study the ACF and CCF of an LTI system with WSS input shown in figure 2. Let \( z(n) = x(n) \), then we have

\[
\begin{align*}
    r_{yz}(m) &= h(m) * r_{xx}(m) \\
    r_{zy}(m) &= h^*(-m) * r_{xx}(m)
\end{align*}
\]

Let \( z(n) = y(n) \), then we have

\[
\begin{align*}
    r_{yy}(m) &= h(m) * r_{yy}(m) = h(m) * h^*(-m) + r_{xx}(m) = c_{bh}(m) * r_{xx}(m)
\end{align*}
\]

where

\[
c_{bh}(m) = h(m) * h^*(-m) = \sum_{k=-\infty}^{\infty} h(k) h^*(k - m)
\]

is the deterministic ACF of \( h(n) \). Note that \( h(n) \) is absolutely summable, it is square summable too. Thus \( c_{bh}(m) \) is finite following from the Schwarz inequality [6]. The correlation functions and power spectral densities of the signals shown in figure 2 are summarized in table 1, where the summation index is from \(-\infty\) to \(+\infty\).

![Figure 2](image)

**Figure 2.** One LTI system with WSS input.

| Table 1. Correlation functions and power spectral densities of the signals shown in figure 2. |
|------------------------------------------|
| Correlation functions | Power spectral densities |
|------------------------|-------------------------|
| \( r_{xx}(m) = E[x(n + m)x^*(n)] \) | \( P_{xx}(\omega) = \sum r_{xx}(m) \exp(-jm\omega) \) |
| \( r_{yx}(m) = h^*(-m) * r_{xx}(m) \) | \( P_{xy}(\omega) = H^*(\omega) P_{xx}(\omega) \) |
| \( r_{xy}(m) = h(m) * r_{xx}(m) \) | \( P_{yx}(\omega) = H(\omega) P_{xx}(\omega) \) |
| \( r_{yy}(m) = c_{bh}(m) * r_{xx}(m) \) | \( P_{yy}(\omega) = |H(\omega)|^2 P_{xx}(\omega) \) |

In the second example we consider the CCFs of two BIBO stable LTI systems with JWSS inputs \( x_1(n) \) and \( x_2(n) \), as shown in figure 3. This is a basic model for array signal processing [2]. Because \( x_1(n) \) and \( x_2(n) \) are JWSS and \( h_1(n) \) and \( h_2(n) \) are BIBO stable LTI systems, we know from theorem 2 that every pair of \( \{ x_1(n), x_2(n), y_1(n), y_2(n) \} \) are JWSS. For example, since \( x_1(n) \) and
Let $x(n) = x_2(n)$, $h(n) = h_2(n)$, $y(n) = y_2(n)$, and $z(n) = x_1(n)$, we have
\[
\begin{align*}
    r_{y_1x_1}(m) &= h_2(m) * r_{x_2x_1}(m) 
    \quad (22) \\
    r_{x_2y_2}(m) &= h_2^*(-m) * r_{x_2x_1}(m) 
    \quad (23)
\end{align*}
\]

Let $x(n) = x_1(n)$, $h(n) = h_1(n)$, $y(n) = y_1(n)$, and $z(n) = x_2(n)$, we have
\[
\begin{align*}
    r_{y_1x_2}(m) &= h_1(m) * r_{x_2x_1}(m) 
    \quad (24) \\
    r_{x_2y_2}(m) &= h_1^*(-m) * r_{x_2x_1}(m) 
    \quad (25)
\end{align*}
\]

Let $x(n) = x_2(n)$, $h(n) = h_2(n)$, $y(n) = y_2(n)$, and $z(n) = y_1(n)$, we have
\[
\begin{align*}
    r_{y_1y_2}(m) &= h_2(m) * r_{x_2x_1}(m) * h_2^*(-m) * r_{x_2x_1}(m) = c_{h_1h_2}(m) * r_{x_2x_1}(m) 
    \quad (26) \\
    r_{y_2y_2}(m) &= h_2^*(-m) * r_{x_2x_1}(m) * h_1^*(-m) * r_{x_2x_1}(m) = c_{h_1h_2}(m) * r_{x_2x_1}(m) 
    \quad (27)
\end{align*}
\]

where
\[
    c_{h_1h_2}(m) = h_p(m) * h_q^*(-m) = \sum_{k=-\infty}^{\infty} h_p(k)h_q^*(k-m) 
    \quad (28)
\]

are the deterministic CCFs between $h_p(n)$ and $h_q(n)$ for $p, q = 1, 2$. Because both of $h_1(n)$ and $h_1(n)$ are absolutely summable, $c_{h_1h_2}(m)$ is finite following from the Schwarz inequality. The correlation functions and power spectral densities of the signals shown in figure 3 are summarized in table 2, where the summation indices are from $-\infty$ to $+\infty$.

![Figure 3. Two LTI systems with JWSS inputs.](image)

**Table 2.** Correlation functions and power spectral densities of the signals shown in figure 3.

| Correlation functions | Power spectral densities |
|-----------------------|--------------------------|
| $r_{x_1x_1}(m) = E[x_1(n + m)x_1^*(n)]$ | $P_{x_1x_1}(\omega) = \sum_{m=-\infty}^{\infty} r_{x_1x_1}(m) \exp(-j\omega m)$ |
| $r_{x_2x_2}(m) = E[x_2(n + m)x_2^*(n)]$ | $P_{x_2x_2}(\omega) = \sum_{m=-\infty}^{\infty} r_{x_2x_2}(m) \exp(-j\omega m)$ |
| $r_{y_1y_2}(m) = h_2^*(-m) * r_{x_2x_1}(m)$ | $P_{y_1y_2}(\omega) = H_2^*(\omega)P_{x_2x_1}(\omega)$ |
| $r_{y_2y_2}(m) = h_1^*(-m) * r_{x_2x_1}(m)$ | $P_{y_2y_2}(\omega) = H_1^*(\omega)P_{x_2x_1}(\omega)$ |
| $r_{y_2x_2}(m) = h_2^*(m) * r_{x_2x_1}(m)$ | $P_{y_2x_2}(\omega) = H_2^*(\omega)P_{x_2x_1}(\omega)$ |
| $r_{y_2x_2}(m) = c_{h_1h_2}(m) * r_{x_2x_1}(m)$ | $P_{y_2x_2}(\omega) = H_2^*(\omega)H_1^*(\omega)P_{x_2x_1}(\omega)$ |
| $r_{y_2x_2}(m) = c_{h_1h_2}(m) * r_{x_2x_1}(m)$ | $P_{y_2x_2}(\omega) = H_2^*(\omega)H_1^*(\omega)P_{x_2x_1}(\omega)$ |
5. Conclusions
In this paper we proved a generalized correlation theorem for LTI system with WSS input, which simplifies the analysis and calculation of correlation functions between the inputs and outputs of LTI systems. Although only the discrete-time systems and discrete-time WSS random signals are discussed, the results can be easily extended to the continuous-time cases.

6. Appendices
There is a mistake in the majority of the signal processing literature that the reason why the expectation operator $\mathbb{E}$ and the infinite summation $\sum$ can be interchanged is that $\mathbb{E}$ is linear. It should be emphasized that the linearity does not ensure the interchangeability of $\mathbb{E}$ and $\sum$ [7]. It is the continuity of $\mathbb{E}$ that ensures the interchangeability of $\mathbb{E}$ and $\sum$. The proof of the third equality of (10) is given below. Let

$$y_k(n) = \sum_{k=-K}^{K} h(k)x(n-k)$$  \hspace{1cm} (29)

It follows from the absolute summability of $h(n)$ that $\lim_{K\to\infty} y_k(n) = y(n)$. Therefore,

$$\mathbb{E}[y(n+m)z^*(n)] = \mathbb{E}\left[\lim_{K\to\infty} y_k(n+m)z^*(n)\right]$$
$$= \lim_{K\to\infty} \mathbb{E}[y_k(n+m)z^*(n)]$$
$$= \lim_{K\to\infty} \mathbb{E}\left[\sum_{k=-K}^{K} h(k)x(n+m-k)z^*(n)\right]$$
$$= \lim_{K\to\infty} \sum_{k=-K}^{K} h(k)\mathbb{E}[x(n+m-k)z^*(n)]$$  \hspace{1cm} (30)

where the second equality follows from the continuity of $\mathbb{E}$ (in the mean square convergence [8]) and the fourth equality from the linearity of $\mathbb{E}$. Because $h(n)$ is absolutely summable, the summation in the fourth line of (30) is convergent. It is the third line of (10), which completes the proof.

Acknowledgments
This work was supported in part by the National Natural Science Foundation of China under Grant no. 61801368 and in part by the China Postdoctoral Science Foundation under Grant no. 2018M640990.

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