Supersymmetric Gauge Anomaly with General Homotopic Paths

S. James Gates, Jr.†
Department of Physics, University of Maryland
College Park, MD 20742-4111 USA

Marcus T. Grisaru,‡§ Marcia E. Knutt¶
Physics Department, McGill University
Montreal, QC Canada H3A 2T8

Silvia Penati∥
Dipartimento di Fisica dell’Università di Milano-Bicocca
and INFN, Sezione di Milano, piazza delle Scienze 3, I-20126 Milano, Italy

Hiroshi Suzuki∗∗
Department of Mathematical Sciences, Ibaraki University, Mito 310-8512, Japan

ABSTRACT

We use the method of Banerjee, Banerjee and Mitra and minimal homotopy paths to compute the consistent gauge anomaly for several superspace models of SSYM coupled to matter. We review the derivation of the anomaly for $N = 1$ in four dimensions and then discuss the anomaly for two-dimensional models with (2,0) supersymmetry.

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†gates@wam.umd.edu
‡grisaru@brandeis.edu
§On leave from Brandeis University
¶knutt@physics.mcgill.ca
∥Silvia.Penati@mi.infn.it
∗∗hsuzuki@mito.ipc.ibaraki.ac.jp
1 Introduction

Over the years, the construction of the superspace consistent (chiral) anomaly for four-dimensional supersymmetric Yang-Mills theories has attracted the attention of many authors [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. In particular McArthur and Osborn [12] have discussed the construction of the consistent anomaly by applying to supersymmetric theories the method used by Leutwyler [16] for non-supersymmetric theories. In this method one begins with the regularized gauge variation of the one-loop effective action \( \Gamma \) for chiral fields in a background gauge field, defined in terms of the determinant of the kinetic operator. Due to the regularization, \( (\delta \Gamma)_{\text{reg}} \neq \delta(\Gamma_{\text{reg}}) \) and consequently the (anomalous) gauge variation does not satisfy the Wess-Zumino consistency condition [17]; one must supplement it by the variation of a local expression in order to obtain the consistent anomaly. Equivalently, the covariant current must be supplemented by a local term, the so-called Bardeen-Zumino current [18]. The construction of the latter involves the choice of a homotopic path which connects the gauge field to zero. Depending on the choice of path one obtains different, but equally valid, forms of the consistent anomaly.

The superspace gauge field \( V \) enters in the action for matter superfields in the form \( e^V \), and various authors have used a homotopy of the form \( e^V \to e^{yV} \) with \( y \) varying between 0 and 1. While this choice has some advantages, it has one serious disadvantage: whereas the gauge variation \( \delta e^V = i(\bar{\Lambda}e^V - e^V\Lambda) \) is relatively simple, the variation of the homotopically extended exponential \( e^{yV} \) is extremely complicated. Therefore, in ref. [15], we proposed a different homotopy \( e^V \to 1 + y(e^V - 1) \) whose gauge variation is simple and leads to a relatively simple form of the consistent anomaly. We followed the McArthur-Osborn procedure, constructing the four-dimensional \( N = 1 \) consistent anomaly as the sum of the covariant anomaly, obtained from \( (\delta \Gamma)_{\text{reg}} \), augmented by consistency terms.

A few years ago, Banerjee, Banerjee and Mitra [19] proposed a simple field-theoretic prescription which is equivalent to the Leutwyler method but has the advantage of constructing in one step the consistent anomaly. In this prescription one starts with a regularized form of the effective action before computing its variation. Consequently the variation automatically satisfies the consistency condition. This procedure was applied to the four-dimensional \( N = 1 \) Yang-Mills theory by Ohshima, Okuyama, Suzuki and Yasuta [14] and was shown to be equivalent to the McArthur-Osborn result. It still requires a choice of homotopy, and these authors made the standard choice with \( e^{yV} \) so that the final form of the consistent anomaly is still complicated.

In this paper we apply the procedure of refs. [15, 14] to a number of superspace examples, using the "minimal" homotopy of [13]. To begin with, we reconsider the anomaly for four-dimensional \( N = 1 \) SSYM coupled to chiral superfields. We
also show how the consistent anomalies for different homotopic paths are related by gauge transformations of a local counterterm. We then turn to some examples in two-dimensional theories: $N = 2, (2, 2)$ with chiral superfields coupled to $(2, 2)$ SSYM, and $(2, 0)$ theories with chiral scalar or chiral spinor superfields. In the first of these examples we find, as expected, no anomaly while the anomalies in the latter two cases, when considered together, cancel as expected since the corresponding multiplets appear in the decomposition of a $(2, 2)$ chiral superfield into $(2, 0)$ superfields. We also show how these $(2, 0)$ anomalies, when considered in WZ gauge, reduce to the standard gauge anomaly for fermions in two dimensions. We have attempted to make the paper as pedagogical as possible, at the risk of including some well-known facts. Some of these appear in appendices.

2 Consistent Anomaly for N=1 SSYM in Four Dimensions

In this section we repeat, in order to make the paper self-contained, the calculation of the anomaly as described in ref. [14]. The only essential difference is the use of a different homotopy class. We consider the gauge anomaly resulting from the one-loop effective action $\Gamma[V]$ of a massless chiral superfield coupled to an external gauge superfield

$$S = \int d^8z \Phi e^V \Phi \quad ,$$

(2.1)

We define the effective action by generalizing the prescription of [14] as follows. We introduce the one-parameter family of gauge superfields which smoothly interpolates between the identity and $e^V$

$$g(y, V) \quad , \quad y \in [0, 1] \quad ,$$

(2.2)

where $g(y = 0, V) = 1$ and $g(y = 1, V) = e^V$. We also define the corresponding deformed superspace Yang-Mills gauge theory by the rule

$$e^V \rightarrow g(y, V) \quad ,$$

(2.3)

substituting all appearances of $e^V$ by $g$. As long as $g^\dagger = g$, this deformation preserves supersymmetry. The action (2.1) is thus replaced for general $y$ by

$$S \rightarrow S_y = \int d^8z \Phi g^\dagger \Phi g \quad , \quad e^{-\Gamma} \equiv \int e^{-S} \rightarrow e^{-\Gamma_y} \equiv \int [D\Phi]e^{-S_y} \quad ,$$

(2.4)

and in chiral representation

$$\nabla_\alpha \equiv g^{-1}D_\alpha g \quad , \quad \bar{\nabla}_{\dot{\alpha}} \equiv \bar{D}_{\dot{\alpha}} \quad , \quad i\nabla_\alpha \dot{\alpha} = \{\nabla_\alpha, \bar{\nabla}_{\dot{\alpha}}\} \quad ,$$

(2.5)
These deformed covariant derivatives preserve the constraints of the usual formulation of 4D, \( N = 1 \) gauge theories for all values of \( y \). We thus have a definition of 4D, \( N = 1 \) supersymmetric Yang-Mills theory that extends throughout the familiar “cone-construction” often used in discussion of homotopies. We also define the deformed chiral/antichiral d’Alembertians

\[
\Box_+ = \nabla^2 \nabla^2 + \nabla^2 \nabla^2 - \nabla^i \nabla^2 \nabla_{\dot{\alpha}} = \Box - i \nabla^\alpha \nabla_\alpha - \frac{i}{2} (\nabla^\alpha \nabla_\alpha) , \\
\Box_- = \nabla^2 \nabla^2 + \nabla^2 \nabla^2 - \nabla^\alpha \nabla^2 \nabla_\dot{\alpha} = \Box - i \nabla^\dot{\alpha} \nabla_{\dot{\alpha}} - \frac{i}{2} (\nabla^\dot{\alpha} \nabla_{\dot{\alpha}}) , \\
\mathcal{W}_\alpha = i D^2(g^{-1} D_\alpha g) , \quad \mathcal{W}_{\dot{\alpha}} = i g^{-1} D^2(\bar{g} D_{\dot{\alpha}} g)^{-1} g , \\
\Box = \frac{1}{2} \nabla^\alpha \nabla_{\alpha \dot{\alpha}}
\]

(2.6)

The effective action of the original theory \( (y = 1) \) is obtained as \( \Gamma = \Gamma_{y=0} + \int_0^1 dy \partial_y \Gamma_y \) with, from (2.4), \( \partial_y \Gamma_y = \langle \partial_y S_y \rangle \), namely

\[
\Gamma[V] = \int_0^1 dy \int d^8 z \partial_y g(z) \langle \frac{\delta S_y}{\delta g(z)_{ij}} \rangle ,
\]

(2.7)

where we have set \( \Gamma_{y=0} = 0 \). This is a formal expression and we have to specify a regularization for the gauge current \( \langle \delta S_y / \delta g \rangle \). We regularize it “gauge covariantly” by treating the homotopy \( g \) as we would treat the usual exponential of the gauge superfield \( e^V \),

\[
\langle \frac{\delta S_y}{\delta g(z)_{ij}} \rangle = \lim_{z' \to z} \langle \Phi(z) \bar{\Phi}(z') \rangle_{ji} \\
\equiv \lim_{z' \to z} \left( e^{\Box_+/M^2} 1/\Box_+ \nabla^2 \nabla g^{-1} \right)_{ji} (z - z') \\
= - \lim_{z' \to z} \left( \int_0^\infty dt e^{\Box_+/M^2} \nabla^2 \nabla g^{-1} \right)_{ji} (z - z') .
\]

(2.8)

As derived in Appendix B, the full propagator for the chiral superfield is

\[
\langle \Phi(z) \bar{\Phi}(z') \rangle_{ji} = \delta_{ij} \nabla^2 \frac{1}{\Box_+} \nabla^2 g^{-1} \delta^{(8)}(z - z') ,
\]

(2.9)

where now the derivative operators are deformed as in (2.5) and we have regularized the propagator by introducing the factor \( e^{\Box_+/M^2} \), \( M \) being an ultraviolet cutoff.

The equations (2.7) and (2.8) are our definition of the regularized one-loop effective action.\(^{††}\)

\(^{††}\)The gauge covariance of (2.5) under the formal gauge transformation \( \delta g = i(\hat{\Lambda} g - g \Lambda) \) considerably simplifies the calculation of the anomaly as emphasized in (14). Note that this is not the gauge transformation on \( g \) induced by \( \delta e^V \equiv i(\hat{\Lambda} e^V - e^V \Lambda) \).
Let us consider an arbitrary infinitesimal variation of the gauge superfield \( \delta V \). The associated variation of the effective action (2.7) is given by

\[
\delta \Gamma[V] = \int_0^1 dy \int d^8z \delta \partial_y g(z)_{ij} \left\langle \frac{\delta S_y}{\delta g(z)_{ij}} \right\rangle + \int_0^1 dy \int d^8z \int d^8z'' \partial_y g(z)_{ij} \delta g(z'')_{kl} \frac{\delta}{\delta g(z)_{ij}} \left\langle \frac{\delta S_y}{\delta g(z)_{kl}} \right\rangle
\]

(2.10)

In the first term the variation \( \delta \) commutes with the \( y \)-derivative, \( \delta \partial_y g = \partial_y \delta g \), because \( \delta \) is generated by \( \delta = \int d^8z \delta V(\delta/\delta V) \) which is independent of \( y \). Thus after integration by parts on \( y \), we have

\[
\delta \Gamma[V] = \int d^8z (\delta e^{V(z)})_{ij} \left\langle \frac{\delta S}{\delta e^{V(z)}} \right\rangle_{ij} + \int_0^1 dy \int d^8z \int d^8z'' \delta g(z'')_{kl} \partial_y g(z)_{ij} \times \left[ \frac{\delta}{\delta g(z'')_{kl}} \left\langle \frac{\delta S_y}{\delta g(z)_{ij}} \right\rangle - \frac{\delta}{\delta g(z)_{ij}} \left\langle \frac{\delta S_y}{\delta g(z'')_{kl}} \right\rangle \right]
\]

\[
\equiv L - \frac{1}{16\pi^2} \int_0^1 dy X(y) \quad .
\]

(2.11)

The first term \( L \) corresponds to the covariant gauge current and is independent of the homotopic path. The second term \( X \), which carries \( g \) dependence, corresponds to the Bardeen-Zumino terms. Note that eq. (2.11) is a variation of the effective action (2.7) and thus is automatically consistent.

### 2.1 The Covariant Anomaly

From the expression for the propagator, the covariant term can be written as

\[
L = \int d^8z \lim_{z' \to z} \delta e^{V(z)} e^{\Box_z + M^2} \frac{1}{\Box_+} \nabla^2 \nabla^2 e^{-V(z)} \delta^{(8)}(z - z')
\]

\[
= \int d^8z \lim_{z' \to z} e^{-V} \delta e^V e^{\Box_z + M^2} \frac{1}{\Box_-} \nabla^2 \nabla^2 \delta^{(8)}(z - z') \quad ,
\]

(2.12)

where we have brought the factor of \( e^{-V} \) around to the front since we are dealing essentially with a trace, both in group theory labels and superspace (throughout the paper we neglect the \( \text{Tr} \) symbol for the trace on the group indices reinserting it only in the final results). For a gauge transformation we have

\[
e^{-V} \delta e^V = -i(\Lambda - e^{-V} \bar{\Lambda} e^V) \equiv -i(\Lambda - \bar{\Lambda}) \quad .
\]

(2.13)
This equation allows the introduction of a holomorphic/anti-holomorphic separation of the variation operator [13]

\[ e^{-V} \delta e^V \equiv e^{-V} \delta_R^1 e^V + e^{-V} \delta_R e^V \rightarrow e^{-V} \delta_R e^V = -i\Lambda, \quad e^{-V} \delta_R^1 e^V = i\bar{\Lambda}. \tag{2.14} \]

We consider the part dependent on \( \Lambda \) – this is equivalent to the use of the \( \delta_R^1 \)-operator to replace the \( \delta \)-operator. The \( \bar{\Lambda} \) part can be obtained by hermitean conjugation.

From the superspace integral we pull out a factor of \( d^2 \bar{\theta} = \nabla^2 \). Noting that one must take the limit \( z' \rightarrow z \) first, and using the chirality property of the various quantities in the integrand, we observe that \( \nabla^2 \) can only act on the \( z' \) appearing in the \( \delta \)-function. Using \( \nabla^2\nabla^2 \delta^{(8)}(z-z') \) \( \nabla^2 = \nabla^2\nabla^2 \delta^{(8)}(z-z') = \Box_+ \nabla^2 \delta^{(8)}(z-z') \)

we obtain

\[ L = -i \int d^6 z \lim_{z' \rightarrow z} \Lambda e^{\Box_+/M^2} \nabla^2 \delta^{(8)}(z-z'). \tag{2.15} \]

We write \( \delta^{(8)}(z-z') = \frac{M^4}{(2\pi)^2} \int d^4 k e^{i M k (x-x')} \delta^{(4)}(\theta-\theta') \). The derivatives act on the \( x \)-variable so that the \( \exp(-i M k x') \) factor can be written in front. We then pull the \( \exp(i k x) \) factor through the derivatives picking up factors of \( M k \) along the way from the space-time derivatives \( \partial_a \) [20, 21, 22] and subsequently take the limit \( x' \rightarrow x \).

We obtain\(^\dagger\)

\[ L = \frac{-i}{8 \pi^2} \int d^4 x d^2 \theta \Lambda \lim_{\theta' \rightarrow \theta} \int d^4 k e^{-k^2 + i k^a \nabla_a/M + \Box/M^2 - i W^a \nabla_a/M^2 - i(\nabla^a W_a)/(2M^2)} \nabla^2 \delta^{(4)}(\theta-\theta') \]

\((k^2 = \frac{1}{2} k^a \bar{k}_{\alpha\dot{\alpha}})\). In the limit one obtains a zero result except from terms, in the expansion of the exponential, that can produce a factor of \( \nabla^2 \) which together with the \( \nabla^2 \) remove the \( \delta^{(4)}(\theta-\theta') \) factor, i.e. from the second order term \( 1/2!(W^a \nabla_a)^2 \). These terms also produce a factor of \( 1/M^4 \) which cancels the \( M^4 \) factor. One can now take the limit \( \theta' \rightarrow \theta \), remove the regulator, \( M \rightarrow \infty \), perform the \( k \)-integration of the remaining \( e^{-k^2} \) factor, and obtain the final form of the covariant anomaly

\[ L = -\frac{i}{8 \pi^2} \int d^6 z \operatorname{Tr} \left[ \Lambda W^a W_a \right] + \text{h.c.} \tag{2.17} \]

### 2.2 The Consistency Terms

We consider

\[ \int d^4 z'' \delta g(z'')_{kl} \frac{\delta}{\delta g(z'')}_{kl} \left\langle \frac{\delta S_y}{\delta g(z)} \rightangle. \]

\(^\dagger\)There is a slight subtlety here. Because the derivative \( \partial_a \) appears also in the covariant spinor derivative \( \nabla_a = \partial_a + \frac{1}{2} \bar{\theta}^\alpha \partial_a \theta_{\alpha} \) one also generates exponentials of \( -\frac{1}{2M} W^a \bar{\theta}^a k_{\alpha\dot{\alpha}} \). One can argue, or show explicitly, that these terms do not contribute [13].
\[
= -\delta \lim_{z' \to z} \left( \int_{1/M^2}^{\infty} dt e^{\partial^+_t \nabla^2 \nabla^2 g^{-1}} \right) \delta^{(8)}(z - z'), \tag{2.18}
\]

where the covariant derivatives correspond to the deformed theory. As follows from eq. (2.5), in chiral representation the dependence on \( y \) in (2.18) only comes from \( \Box_+ \) and \( \nabla^2 \). Using

\[
\delta g^{-1} = -g^{-1} \delta g g^{-1},
\]

\[
\delta \nabla_{\alpha} = \delta (g^{-1} D_{\alpha} g) = [\nabla_{\alpha}, g^{-1} \delta g],
\]

\[
\delta \nabla^2 g^{-1} = -g^{-1} \delta g \nabla^2 g^{-1},
\]

\[
\delta e^{\Box_+ t} = \int_0^t dse^{\Box_+ s} \delta \Box_+ e^{\Box_+(t-s)}, \tag{2.19}
\]

and the fact that when acting on a chiral quantity \( \Box_+ = \nabla^2 \nabla^2 \), we obtain in (2.11)

\[
\int_0^1 dy \int d^8 z \partial_y g(z) \delta \left[ -\int_{1/M^2}^{\infty} dt e^{\partial^+_t \nabla^2 \nabla^2 g^{-1}} \delta^{(8)}(z - z') \right]
= \int_0^1 dy \int d^8 z \partial_y g(z) \left[ \int_{1/M^2}^{\infty} dt e^{\partial^+_t \nabla^2 g^{-1}} \delta g \nabla^2 g^{-1}
- \int_{1/M^2}^{\infty} dt \int_0^t dse^{\Box_+ s} \nabla^2 [\nabla^2, g^{-1} \delta g] e^{\Box_+(t-s)} \nabla^2 g^{-1} \right] \delta^{(8)}(z - z'). \tag{2.20}
\]

In the last term, only the \(-g^{-1} \delta g \nabla^2 \) term from the commutator is relevant. The other order leads to a term which, after using the trace property of the whole expression as well as a change of variables \( s \to t - s \), cancels a corresponding term from the expression in (2.11) with \( z \) and \( z'' \) interchanged. Then one manipulates the last term as follows:

\[
\int_{1/M^2}^{\infty} dt \int_0^t dse^{\Box_+ s} \nabla^2 g^{-1} \delta g \nabla^2 e^{\Box_+(t-s)} \nabla^2 g^{-1}
= \int_{1/M^2}^{\infty} dt \int_0^t dse^{\Box_+ s} \nabla^2 g^{-1} \delta g \nabla^2 e^{\nabla^2(t-s)} \nabla^2 g^{-1}
= \int_{1/M^2}^{\infty} dt \int_0^t dse^{\Box_+ s} \nabla^2 g^{-1} \delta g \frac{\partial}{\partial t} e^{\Box_-(t-s)} \nabla^2 g^{-1} \tag{2.21}
\]

Now, one integrates by parts the \( \partial_t \) derivative. One obtains one term which cancels the first term in (2.20) and one is left with

\[
- \int_0^1 dy d^8 z \int_0^{1/M^2} ds \partial_y g(z) e^{\Box_+ s} \nabla^2 g^{-1} \delta g e^{\Box_-(1/M^2-s)} \nabla^2 g^{-1} \delta^{(8)} (z - z')
= - \int_0^1 dy d^8 z \int_0^{1/M^2} d\beta \frac{1}{M^2} g^{-1} \partial_y g e^{\beta \Box_+/M^2} \nabla^2 g^{-1} \delta g e^{(1-\beta) \Box_-/M^2} \nabla^2 \delta^{(8)} (z - z'). \tag{2.22}
\]
This expression, and the corresponding one with $z \leftrightarrow z''$, is manipulated in the same manner we treated the covariant anomaly. We give some details in Appendix D (see also \[14\]). One introduces a momentum basis for the $\delta^{(4)}(x - x')$ factor, producing an $M^2$ factor and factors of $Mk$ in the various exponentials. One investigates then what survives in the limit $M^2 \to \infty$ and $\theta' \to \theta$. This time factors from the expansion of the exponentials proportional to $W^\alpha \nabla_\alpha/M^2$, $\nabla_\alpha W_\alpha/M^2$, $\tilde{W}^\alpha \nabla_\alpha/M^2$ and $\nabla_\alpha \tilde{W}_\alpha/M^2$ give the relevant contributions, cancelling the overall $M^2$ factor. The final result takes the form

$$X(y) = - 16\pi^2 (\delta_1 \langle \delta_2 S_y \rangle - \delta_2 \langle \delta_1 S_y \rangle)\bigg|_{M \to \infty} = \frac{2i}{\partial_y} \int_0^1 dz \left( [\nabla^\alpha h_2, \nabla_\alpha] + \left\{ h_2, \tilde{\nabla} \tilde{W}^\alpha \right\} \right) = \frac{2i}{\partial_y} \int_0^1 dz \left( h_2 \left[ \tilde{\nabla} h_2, \nabla \right] - h_2 \left[ \nabla^\alpha h_1, \nabla_\alpha \right] \right), \quad (2.23)$$

where $\delta_1 g \equiv \delta g$ and $\delta_2 g \equiv \delta_y g$, $h_1 \equiv g^{-1} \delta g$ and $h_2 \equiv g^{-1} \partial_y g$. Moreover we have defined $\nabla_\alpha A \equiv \{ \nabla_\alpha, A \}$ for any scalar or spinor object $A$ in the adjoint representation of the gauge group. The covariant derivatives depend explicitly on $y$ according to their definition \[(2.5)\]. The result in eq. \[(2.23)\] thus reproduces the formulae \[(2.25)\] and \[(2.28)\] of \[12\] for general homotopic paths. Note that our definition of the effective action \[(2.7)\] and \[(2.8)\] is identical to the effective action eq. \[(2.31)\] of \[12\], because the variations of both formulas coincide, as eqs. \[(2.11)\] and \[(2.23)\] show.

The contribution $X$ of the Bardeen-Zumino current depends on the homotopic path $g$. An interesting property of $X$ is \[14\]

$$X(y)|_{\delta=\delta} = 16\pi^2 \partial_y \int d^8 z \tilde{\delta} g(z)_{ij} \left\langle \frac{\delta S_y}{\delta g(z)} \right\rangle \bigg|_{M \to \infty} = \frac{2i}{\partial_y} \int d^8 z \text{Tr} \Lambda W^\alpha W_\alpha + \text{h.c.}, \quad (2.24)$$

which follows from the formal gauge covariance of \[(2.8)\] under $\tilde{\delta}$ as defined in the footnote below eq. \[(2.9)\]. We note that the right hand side of this equation is (a $y$-derivative of) the covariant anomaly with the substitution \[(2.2)\].

Different choices of the homotopic path \[(2.2)\] lead to different forms of the consistent anomaly but two choices $g$ and $g'$ have to be related to each other by a gauge transformation of a local counterterm. In fact, it is easy to find the counterterm in the present prescription. We introduce the one-parameter deformation of the path

$$\gamma(y, u) = \gamma(y, u = 0) = g$$

such that $\gamma(y, u = 0) = g$ and $\gamma(y, u = 1) = g'$. We then apply the identity (we set $\delta_1 \gamma = \delta \gamma$, $\delta_2 \gamma = \partial_y \gamma$ and $\delta_3 \gamma = \partial_\alpha \gamma$)

$$\delta_3 \left( \delta_1 \langle \delta_2 S_y \rangle - \delta_2 \langle \delta_1 S_y \rangle \right) = \delta_1 \left( \delta_3 \langle \delta_2 S_y \rangle - \delta_2 \langle \delta_3 S_y \rangle \right) + \delta_2 \left( \delta_1 \langle \delta_3 S_y \rangle - \delta_3 \langle \delta_1 S_y \rangle \right), \quad (2.26)$$

$$\gamma(y, u) = \gamma(y, u = 0) = g$$

such that $\gamma(y, u = 0) = g$ and $\gamma(y, u = 1) = g'$. We then apply the identity (we set $\delta_1 \gamma = \delta \gamma$, $\delta_2 \gamma = \partial_y \gamma$ and $\delta_3 \gamma = \partial_\alpha \gamma$)

$$\delta_3 \left( \delta_1 \langle \delta_2 S_y \rangle - \delta_2 \langle \delta_1 S_y \rangle \right) = \delta_1 \left( \delta_3 \langle \delta_2 S_y \rangle - \delta_2 \langle \delta_3 S_y \rangle \right) + \delta_2 \left( \delta_1 \langle \delta_3 S_y \rangle - \delta_3 \langle \delta_1 S_y \rangle \right), \quad (2.26)$$

$$\gamma(y, u) = \gamma(y, u = 0) = g$$

such that $\gamma(y, u = 0) = g$ and $\gamma(y, u = 1) = g'$. We then apply the identity (we set $\delta_1 \gamma = \delta \gamma$, $\delta_2 \gamma = \partial_y \gamma$ and $\delta_3 \gamma = \partial_\alpha \gamma$)

$$\delta_3 \left( \delta_1 \langle \delta_2 S_y \rangle - \delta_2 \langle \delta_1 S_y \rangle \right) = \delta_1 \left( \delta_3 \langle \delta_2 S_y \rangle - \delta_2 \langle \delta_3 S_y \rangle \right) + \delta_2 \left( \delta_1 \langle \delta_3 S_y \rangle - \delta_3 \langle \delta_1 S_y \rangle \right), \quad (2.26)$$
to eq. (2.23). Integrating over $u$, we have

$$
\delta \Gamma[V]_{g'} - \delta \Gamma[V]_g = \delta \left( \frac{-1}{16\pi^2} \right) \int_0^1 du \int_0^1 dy \int d^8 z \left( \frac{1}{2} i \text{Tr} \left[ \bar{\gamma}^{-1} \partial_u \gamma \left[ D_\alpha (\gamma^{-1} \partial_y \gamma), \bar{\mathcal{W}}^\alpha \right] - \gamma^{-1} \partial_y \gamma \left[ D^\alpha (\gamma^{-1} \partial_u \gamma), \mathcal{W}_\alpha \right] \right) \right),

(2.27)
$$

where we have used the fact that the deformation (2.25) keeps the endpoints $\gamma(y = 0, u) = 1$ and $\gamma(y = 1, u) = e^V$ fixed. The right hand side is indeed the variation of a local term.

A possible choice of the path in (2.2) is [12, 13, 14]

$$
g \equiv e^V.

(2.28)
$$

The advantage of this choice is that the resulting anomaly, as given by the last line of (2.23) is automatically proportional to the anomaly coefficient $d_{abc} = \text{Tr} [T^a \{ T^b, T^c \}]$ where $T^a$ is the representation matrix of the Lie algebra, $g$ being an element of the Lie algebra (as are, then, $h_1$, $h_2$, $\mathcal{W}_\alpha$ and $\bar{\mathcal{W}}^\alpha$). The disadvantage of the choice (2.28) is that the resulting expression of the consistent anomaly involves $\delta g$ which cannot be expressed in terms of $e^V$ and geometric objects but ends up to be a quite complicated function of $V$.

In ref. [15], use of the 4D, $N = 1$ supersymmetric Yang-Mills gauge theory “minimal” homotopy operator

$$
g \equiv 1 + y ( e^V - 1 ) ,

(2.29)
$$

was advocated. This homotopy is minimal in the sense that it satisfies the requisite boundary conditions and it satisfies Newton’s second law as well, both with respect to the exponential of the gauge superfield and the homotopic coordinate $y$

$$
\frac{\partial^2 g}{\partial e^V \partial e^V} = 0 , \quad \frac{\partial^2 g}{\partial y^2} = 0 .

(2.30)
$$

Note that the first of these is valid at the endpoints for all relevant homotopies. The minimal homotopy extends its validity for all values of $y$. Since the gauge variation of the minimal homotopy is expressible in terms of $e^V$,

$$
\delta g = - iy ( e^V \Lambda - \bar{\Lambda} e^V ) ,

(2.31)
$$

this choice yields a simple expression (see (2.32)-(2.34) below) of the consistent gauge anomaly [15]. In the notation of ref. [15], with this minimal choice the 4D, $N = 1$ supersymmetric Yang-Mills Bardeen-Gross-Jackiw consistent anomaly can be expressed as the imaginary part of a superaction, $\mathcal{A}_{BGJ} = \text{Im} [\bar{\mathcal{A}}_{BGJ}]$ where

$$
\bar{\mathcal{A}}_{BGJ} = \frac{1}{4\pi^2} \int d^4 x d^2 \theta d^2 \bar{\theta} \, \mathcal{P}(\Lambda ; e^V) ,

(2.32)
$$

8
with \[ \{ \] 

\[
\mathcal{P}(\Lambda; e^V) = \text{Tr} \left[ \Lambda \left( \Gamma^\alpha W_\alpha - \int_0^1 dy y \left( [\mathcal{W}^\alpha, \pi_\alpha] e^V \mathcal{G} + \{ \tilde{\mathcal{W}}^\alpha, 1 - e^V \mathcal{G} \} \tilde{\pi}_\alpha \right) \right) \right],
\]

(2.33)
in terms of the quantities

\[
\pi_\alpha \equiv e^V \mathcal{G}^2 \Gamma_\alpha, \quad \tilde{\pi}_\alpha \equiv e^V \mathcal{G} \tilde{\Gamma}_\alpha \mathcal{G},
\]

\[
W_\alpha \equiv [D^2(\mathcal{G}D_\alpha \mathcal{G}^{-1})], \quad \tilde{\mathcal{W}}_{\dot{\alpha}} \equiv \mathcal{G} [D^2(\mathcal{G}^{-1} \tilde{D}_{\dot{\alpha}} \mathcal{G})] \mathcal{G}^{-1},
\]

(2.34)

\[
\mathcal{G} \equiv \left[ 1 + y(e^V - 1) \right]^{-1}, \quad W_\alpha = W_\alpha(y = 1).
\]

According to the result in eq. (2.27) any other form for the anomaly differs from this by terms that are the gauge variation of a local functional.

By reducing to components the expression (2.32) it is easy to prove that the bosonic component coincides with the well known result [18].

### 3 The (2,2) Model in Two Dimensions

This model is described by essentially the same action and superfields as in four dimensions. Therefore, the propagator is still given in eq. (2.9) and we use its regularization as in (2.8). The calculation of the anomaly follows exactly the same steps, but with one important difference: since we are working in two dimensions the introduction of the momentum basis

\[
\delta^{(2)}(x - x') = \frac{M^2}{(2\pi)^2} \int d^2k e^{iMk(x - x')},
\]

(3.35)

only produces the overall factor of \( M^2 \) instead of \( M^4 \) as in four dimensions. As a consequence the complete variation of the effective action vanishes in the limit \( M \to \infty \). As expected, there is no gauge anomaly for the (left-right symmetric) (2,2) theory in two dimensions, since the theory is not chiral.

### 4 BGJ Anomaly in the (2,0) SUSY Yang–Mills Theory

In a (2,0) supersymmetric theory, matter can be described by chiral scalar or spinor superfields (see Appendix B for details). We compute the gauge anomaly for the supersymmetric Yang–Mills theory due to the presence of chiral matter assigned
either to a scalar or to a spinor superfield. Since the sum of the actions for the two cases gives rise to the non–anomalous \((2,2)\) theory a consistency check of our calculations will be to show that the expressions for their gauge anomalies sum up to zero.

The \((2,0)\) SSYM theory is described by two quantities (see Appendix B), a prepotential \(V\) which appears in the “plus” covariant derivatives (in chiral representation) \(\nabla_+ = e^{-V} D_+ e^V\), \(\nabla_+ = D_+\), \(\nabla_\pm = -i\{\nabla_+, \nabla_+\}\), and an independent gauge connection \(\Gamma_\pm\) which appears in \(\nabla_\pm = \partial_\pm - i\Gamma_\pm\). As in the 4D case, we use a general homotopy defined by a function \(g(y, V)\), \(y \in [0,1]\) such that \(g(0, V) = 1\) and \(g(1, V) = e^V\). The deformed theory is obtained by substituting \(e^V\) with \(g\) everywhere. In particular, the deformed covariant derivatives are

\[
\nabla_+ = g^{-1} D_+ g \quad \text{,} \quad \nabla_+ = D_+ \quad \text{,} \quad i\nabla_\pm = \{\nabla_+, \nabla_+\} \quad \text{,} \quad \nabla_\pm = \partial_\pm - i\Gamma_\pm .
\]

However, the supersymmetry algebra does not constrain the homotopic extension of the vector connection \(\Gamma_\pm \rightarrow \Gamma_\pm\); any independent choice of \(\Gamma_\pm\) is compatible with the SUSY algebra.

In the formal evaluation of the anomaly given in this section we do not specify the particular extension of the “minus-minus” vector connection, thus obtaining a final answer written in terms of the homotopic object \(\Gamma_\pm\). We postpone to the next section the general discussion on the possible structure of \(\Gamma_\pm\).

### 4.1 The Chiral Spinor Anomaly

We begin by studying the gauge anomaly for the \((2,0)\) SSYM due to chiral matter described by a spinor superfield. From the minimally coupled action \((\text{B.21})\) we read its homotopic extension

\[
S = - \int d^4 z \, \bar{\chi} g \chi .
\]

The effective action for the original theory can be written as

\[
\Gamma = \int_0^1 dy \int d^4 z \partial_y g(z) \left\langle \frac{\delta S}{\delta g(z)} \right\rangle .
\]

Now we compute the gauge variation of \((\text{4.38})\)

\[
\delta \Gamma = \int_0^1 dy \int d^4 z \left[ \partial_y \delta g(z) \left\langle \frac{\delta S}{\delta g(z)} \right\rangle + \partial_y g(z) \delta \left\langle \frac{\delta S}{\delta g(z)} \right\rangle \right] ,
\]

where in the variation of the expectation value we have to vary both \(g\) and \(\Gamma_\pm\).
Integrating by parts on $y$ one gets the integrated term
\[
\int d^4 z \delta g(z) \left. \left( \frac{\delta S}{\delta g(z)} \right) \right|_{y=1}, \tag{4.40}
\]
which gives the covariant anomaly, and
\[
\int_0^1 dy \int d^4 z d^4 z'' \left[ \partial_y g(z) \delta g(z'') \left( \frac{\delta}{\delta g(z')} \left( \frac{\delta S}{\delta g(z)} \right) - \frac{\delta}{\delta g(z)} \left( \frac{\delta S}{\delta g(z'')} \right) \right) + [\partial_y g(z) \delta \tilde{\Gamma}_z(z'') - \delta g(z) \partial_y \tilde{\Gamma}_z(z'') \frac{\delta}{\delta \tilde{\Gamma}_z(z'')} \left( \frac{\delta S}{\delta g(z)} \right)] \right], \tag{4.41}
\]
which is the consistency piece.

The expectation value in the previous expressions must be thought as being suitably regularized. Proceeding as in the four dimensional case, we write
\[
\left\langle \frac{\delta S}{\delta g(z)} \right\rangle = - \left\langle \frac{\delta}{\delta g} \int d^4 z \bar{\chi} - g \chi \right\rangle = \lim_{z' \to z} \left\langle \chi_- (z) \bar{\chi} - (z') \right\rangle
\]
\[
= i \lim_{z' \to z} e^{-V/2} \int_{\Delta_z} \frac{1}{\Delta_+} \nabla_+ \bar{\nabla}_+ \nabla_+ g^{-1} \delta^{(4)} (z - z')
\]
\[
= -i \lim_{z' \to z} \int_1^{\infty} dt e^{-V/2} \int_{\Delta_z} \frac{1}{\Delta_+} \nabla_+ \bar{\nabla}_+ \nabla_+ g^{-1} \delta^{(4)} (z - z') . \tag{4.42}
\]
Here the extended expression for the spinor propagator (C.29) has been used.

4.1.1 The Covariant Anomaly

From eq. (1.40) and the expression in the second line of (1.42) the covariant anomaly is given by (we bring the $g^{-1}$ around because we have basically a trace)
\[
\delta \Gamma_{\text{cov}} = i \int d^4 z \lim_{z' \to z} (g^{-1} \delta g) \left. e^{-V/2} \int_{\Delta_z} \frac{1}{\Delta_+} \nabla_+ \bar{\nabla}_+ \nabla_+ g^{-1} \delta^{(4)} (z - z') \right|_{y=1} . \tag{4.43}
\]
Since $g^{-1} \delta g|_{y=1} = i(\tilde{\Lambda} - \Lambda)$ where $\tilde{\Lambda} \equiv e^{-V} \Lambda e^V$, we split the above expression into a sum of two pieces proportional to $\Lambda$ and $\tilde{\Lambda}$ respectively, which are the hermitian conjugate of one another. Therefore, we only concentrate on the $\Lambda$ piece.

In the $\Lambda$–term we pull out of the superspace integration measure a spinor derivative $\nabla_+$ which acts on both $z$ and $z'$ due to the limit $z' \to z$. When it acts on the $z$ variable, it is acting from the left on chiral objects and the result is zero. Instead, when it acts on the $z'$ variable, it is just acting on the $\delta$-function from the right, and
using the chain of identities \( \nabla_{z'} \delta^{(4)}(z - z') = \ldots \delta^{(4)}(z - z') \nabla_{z'} = - \ldots \nabla_z \delta^{(4)}(z - z') \). We obtain

\[
\delta \Gamma_{cov} = - \int d^2 x d\theta^+ \lim_{z' \to z} \Lambda e^{\Box_+/M^2} \frac{1}{\Box_+} \nabla_+ \nabla_\mp \nabla_+ \nabla_+ \delta^{(4)}(z - z') = -i \int d^2 x d\theta^+ \lim_{z' \to z} \Lambda e^{\Box_+/M^2} \nabla_+ \delta^{(4)}(z - z') , \tag{4.44}
\]

where we have legitimately cancelled the \( 1/\Box_+ \). In the limit we will get zero unless we can pull out of the exponential a factor of \( \nabla_+ \) to act on the \( \delta^{(2)}(\theta - \theta') \). To this end, we write again

\[
\delta^{(4)}(z - z') = \delta^{(2)}(x - x') \delta^{(2)}(\theta - \theta') = M^2 \int \frac{d^2 k}{(2\pi)^2} e^{i Mk(x-x')} \delta^{(2)}(\theta - \theta') , \tag{4.45}
\]

and pull the \( e^{i Mkx} \) factor through the derivatives before taking the limit \( x' \to x \). The result of this operation is that again the various derivatives are shifted by factors of \( Mk \), and using the explicit form of \( \Box_+ \) the exponential becomes

\[
\exp \left[ -k^2 + ik^a \nabla_a / M + \Box / M^2 - 1/2M^2(\nabla_+ W_+ - \nabla_+ W_-) - W_- \nabla_+ / M^2 \right] \tag{4.46}
\]

\( (k^2 = k_+ k_-) \). Expanding in powers of \( 1/M \) only the last term in the exponential will contribute in the limit \( \theta' \to \theta, M \to \infty \), when expanded to first order. The only thing that survives in the exponential is just the \( k^2 \) term so the \( k \) integral can be explicitly performed and gives a factor of \( \pi \). We obtain then, for the covariant anomaly,

\[
\delta \Gamma_{cov} = \frac{i}{4\pi} \int d^2 x d\theta^+ \text{Tr} (\Lambda W_-) + \text{h.c.} . \tag{4.47}
\]

This result can also be obtained by a very simple supergraph calculation of the relevant two-point function for the one-loop gauge field effective action due to a chiral spinor loop.

### 4.1.2 The Consistency Terms

We now focus on the consistency term (4.41) where we use the last line in eq. (4.42) for the expectation value \( \langle \delta S/\delta g \rangle \). In that expression the spinorial derivatives are homotopically extended according to eq. (4.36) and the vector connection \( \Gamma_\mp \) is independently extended to \( \tilde{\Gamma}_\mp \).

We will make use of the following identities

\[
\frac{\delta}{\delta g} \nabla_+ \tilde{\nabla}_\mp \nabla_+ g^{-1} = \frac{\delta}{\delta g} \nabla_+ \tilde{\nabla}_\mp g^{-1} D_+ = -\nabla_+ \tilde{\nabla}_\mp g^{-1} \frac{\delta g}{\delta g} g^{-1} D_+ = -\nabla_+ \tilde{\nabla}_\mp g^{-1} \frac{\delta g}{\delta g} \nabla_+ g^{-1} , \tag{4.48}
\]
so that

\[
\partial_y g \frac{\delta}{\delta g} \nabla_+ \nabla_+ g^{-1} = -\nabla_+ \nabla_+ (g^{-1} \partial_y g) \nabla_+ g^{-1} \\
\delta g \frac{\delta}{\delta g} \nabla_+ \nabla_+ g^{-1} = -\nabla_+ \nabla_+ (g^{-1} \delta g) \nabla_+ g^{-1}.
\] (4.49)

Similarly

\[
\partial_y \tilde{\Gamma}_\pm \frac{\delta}{\delta \Gamma_\pm} \nabla_+ \nabla_+ g^{-1} = -i \nabla_+ \partial_y \tilde{\Gamma}_\pm \nabla_+ g^{-1} \\
\delta \tilde{\Gamma}_\pm \frac{\delta}{\delta \Gamma_\pm} \nabla_+ \nabla_+ g^{-1} = -i \nabla_+ \delta \tilde{\Gamma}_\pm \nabla_+ g^{-1}.
\] (4.50)

Also we have

\[
\delta \int_{1/M^2}^\infty e^{\Box_+ t} = \int_{1/M^2}^\infty dt \int_0^t dse^{\Box_+ s} \delta \Box_+ e^{\Box_+ (t-s)}
\]

where \(\Box_+\) is given in (4.28) and \(\delta \Box_+\) acting on a chiral expression is

\[
\delta \Box_+ \rightarrow -i \delta \nabla_+ \nabla_+ = -\nabla_+ \delta \nabla_+ - i \nabla_+ \nabla_+ [\nabla_+, (g^{-1} \delta g)]
\] (4.52)

Using all the previous identities it is now easy to compute the consistent terms in (4.41). We begin by considering the first line in (4.41). Leaving aside an overall \(\int_0^\infty dy \int d^4 z \lim_{z' \to z} \delta^{(4)}(z - z')\) it can be written as

\[
-i (g^{-1} \delta g) \int_{1/M^2}^\infty e^{\Box_+ t} \nabla_+ \nabla_+ (g^{-1} \partial_y g) \nabla_+ + i (g^{-1} \partial_y g) \int_{1/M^2}^\infty e^{\Box_+ t} \nabla_+ \nabla_+ (g^{-1} \delta g) \nabla_+ \\
-i (g^{-1} \partial_y g) \int_{1/M^2}^\infty dt \int_0^t dse^{\Box_+ s} \nabla_+ \nabla_+ [\nabla_+, (g^{-1} \delta g)] e^{\Box_+ (t-s)} \nabla_+ \nabla_+ \nabla_+ (-i) \\
+i (g^{-1} \delta g) \int_{1/M^2}^\infty dt \int_0^t dse^{\Box_+ s} \nabla_+ \nabla_+ [\nabla_+, (g^{-1} \partial_y g)] e^{\Box_+ (t-s)} \nabla_+ \nabla_+ \nabla_+ (-i)
\] (4.53)

Now, it is easy to check that from the commutators such as \([\nabla_+, (g^{-1} \delta g)]\) only the second terms survives, since in the first terms one can use the cyclic property of the trace to show that the corresponding terms from the last two lines in (4.53) are identical and cancel. Therefore, we are left with

\[
-i (g^{-1} \delta g) \int_{1/M^2}^\infty e^{\Box_+ t} \nabla_+ \nabla_+ (g^{-1} \partial_y g) \nabla_+ \\
-i (g^{-1} \delta g) \int_{1/M^2}^\infty dt \int_0^t dse^{\Box_+ s} \nabla_+ \nabla_+ (g^{-1} \partial_y g) \nabla_+ e^{\Box_+ (t-s)} (-i) \nabla_+ \nabla_+ \nabla_+ - \delta \leftrightarrow \partial_y
\] (4.54)
The second term of this expression can be rewritten as

\[
-i(g^{-1} \delta g) \int_{1/M^2}^\infty dt \int_0^t ds e^{(1-s)\Box} \nabla_+ \nabla_0 (g^{-1} \partial_y g) \nabla_+ e^{(1-t)\Box} (-i) \nabla_+ \nabla_+ \nabla_+ \\
= -i(g^{-1} \delta g) \int_{1/M^2}^\infty dt \int_0^t ds e^{(1-s)\Box} \nabla_+ \nabla_0 (g^{-1} \partial_y g) \nabla_+ \frac{\partial}{\partial t} e^{-\Box} \nabla_+ \nabla_+ \\
= -i(g^{-1} \delta g) \int_{1/M^2}^\infty dt \int_0^t ds e^{(1-s)\Box} \nabla_+ \nabla_0 (g^{-1} \partial_y g) \frac{\partial}{\partial t} e^{(1-t)\Box} \nabla_+ ,
\]

(4.55)

where \( \Box \) has been defined in (C.30). Integrating the \( t \)-derivative by parts one gets the integrated term, and another term where the derivative acts on the upper limit of the \( s \)-integral. This second term cancels the first line of (4.54). Therefore we are left with

\[
i(g^{-1} \delta g) \int_0^{1/M^2} ds e^{(1-s)\Box} \nabla_+ \nabla_0 (g^{-1} \partial_y g) e^{(1-(1/M^2-s))} \nabla_+ \\
-i(g^{-1} \partial_y g) \int_0^{1/M^2} ds e^{(1-s)\Box} \nabla_+ \nabla_0 (g^{-1} \delta g) e^{(1-(1/M^2-s))} \nabla_+ .
\]

(4.56)

Again, besides an integration over \( y \), a functional trace is understood; equivalently there is a \( \delta^{(4)} (z-y) \) which is treated as in the case of the covariant anomaly. Introducing a momentum basis as before and sending \( M^2 \to \infty \) after performing the \( k \)-integration we finally obtain

\[
-\frac{i}{4\pi} \int_0^1 dy \int d^4z \text{Tr} \left[ (g^{-1} \delta g) \nabla_0 (g^{-1} \partial_y g) - (g^{-1} \partial_y g) \nabla_0 (g^{-1} \delta g) \right] .
\]

(4.57)

The second term can be integrated by parts into the first one.

This is the contribution to the consistent anomaly from the first line in eq. (4.53). Now we look at the second line. Substituting the regularized expression (4.42) for the expectation value we have

\[
-i(g^{-1} \partial_y g (z)) \frac{\delta}{\delta \Gamma_0 (z')} \int dt e^{(1-t)\Box} \nabla_0 \nabla_0 \nabla_+ - \delta \leftrightarrow \partial_y \\
= - (g^{-1} \partial_y g (z)) \int dt e^{(1-t)\Box} \nabla_0 \nabla_0 \\
+ i(g^{-1} \partial_y g (z)) \int dt \int ds e^{(1-s)\Box} \nabla_0 \delta \Gamma_0 \nabla_0 e^{(1-t)\Box} \nabla_0 \nabla_+ \\
- \delta \leftrightarrow \partial_y .
\]

(4.58)

Here we have used \( \delta \nabla_0 = -i \delta \Gamma_0 \). Now one plays standard games exactly as before. Without giving the details the final result from the second line in (4.53) is

\[
-\frac{1}{4\pi} \int_0^1 dy \int d^4z \text{Tr} \left[ (g^{-1} \partial_y g) \delta \Gamma_0 - (g^{-1} \delta g) \partial_y \Gamma_0 \right] .
\]

(4.59)
Adding the contributions from (4.47) and (4.57) the final expression for the (2,0) BGJ anomaly due to a chiral spinor superfield is

\[
\mathcal{A}_{\text{BGJ}} = i \frac{4\pi}{4\pi} \int d^2x d\theta^+ \text{Tr} (\Lambda W_-) - i \frac{4\pi}{4\pi} \int d^2x d\theta^+ \text{Tr} (\Lambda\overline{W}_-)
- \frac{i}{2\pi} \int_0^1 dy \int d^4z \text{Tr} \left[ (g^{-1}\partial g)\overline{\nabla}_z (g^{-1}\partial_y g) \right]
- \frac{1}{4\pi} \int_0^1 dy \int d^4z \text{Tr} \left[ (g^{-1}\partial_y g) \delta \overline{\Gamma}_z - (g^{-1}\partial g) \partial_y \overline{\Gamma}_z \right] ,
\]

where in (4.57) an integration by parts on the second term has been performed. Here we have defined \( \overline{W}_- \equiv -(W_-)^\dagger \).

### 4.2 The (2,0) Chiral Scalar Anomaly

The evaluation of the gauge anomaly due to matter in a chiral scalar superfield follows exactly the same procedure as in the spinor case. Therefore, in this section we only describe the main steps of the calculation without giving details.

The supersymmetric action for a chiral scalar coupled to a gauge field is given in (4.61). We homotopically extend \( e^V \rightarrow g \) as well as \( \Gamma_- \rightarrow \tilde{\Gamma}_- \) obtaining

\[
S = -i \int d^2x d^2\theta \overline{\Phi} g \tilde{\nabla}_z \Phi .
\]

We write the effective action as

\[
\Gamma = \int_0^1 dy \int d^4z \left[ \partial_y g(z) \left\langle \frac{\delta S}{\delta g(z)} \right\rangle + \partial_y \tilde{\Gamma}_z(z) \left\langle \frac{\delta S}{\delta \tilde{\Gamma}_z(z)} \right\rangle \right] .
\]

In contradistinction to the spinor case, now the action (4.61) depends explicitly on \( \tilde{\Gamma}_- \) so that computing the gauge variation one gets a more complicated expression

\[
\delta \Gamma = \int_0^1 dy \int d^4z \left[ \partial_y g(z) \left\langle \frac{\delta S}{\delta g(z)} \right\rangle + \partial_y \tilde{\Gamma}_z(z) \left\langle \frac{\delta S}{\delta \tilde{\Gamma}_z(z)} \right\rangle \right] + \partial_y \delta \tilde{\Gamma}_z(z) \left\langle \frac{\delta S}{\delta \tilde{\Gamma}_z(z)} \right\rangle .
\]

As before, we integrate by parts on \( y \) obtaining the integrated term

\[
\int d^4z \left[ \delta g(z) \left\langle \frac{\delta S}{\delta g(z)} \right\rangle + \delta \tilde{\Gamma}_z(z) \left\langle \frac{\delta S}{\delta \tilde{\Gamma}_z(z)} \right\rangle \right]_{y=1} ,
\]

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which represents the covariant anomaly, and

\[
\int dy \int d^4 z d^4 z' \left[ \partial_y g(z) \delta g(z') \left( \frac{\delta}{\delta g(z')} \left\langle \frac{\delta S}{\delta g(z)} \right\rangle - \frac{\delta}{\delta g(z)} \left\langle \frac{\delta S}{\delta g(z')} \right\rangle \right) \\
+ \partial_y g(z) \delta \tilde{\Gamma}_{\pm}(z') \left( \frac{\delta}{\delta \tilde{\Gamma}_{\pm}(z')} \left\langle \frac{\delta S}{\delta \tilde{\Gamma}_{\pm}(z)} \right\rangle - \frac{\delta}{\delta \tilde{\Gamma}_{\pm}(z)} \left\langle \frac{\delta S}{\delta \tilde{\Gamma}_{\pm}(z')} \right\rangle \right) \\
+ \partial_y \tilde{\Gamma}_{\pm}(z) \delta \tilde{\Gamma}_{\pm}(z') \left( \frac{\delta}{\delta \tilde{\Gamma}_{\pm}(z)} \left\langle \frac{\delta S}{\delta \tilde{\Gamma}_{\pm}(z')} \right\rangle - \frac{\delta}{\delta \tilde{\Gamma}_{\pm}(z')} \left\langle \frac{\delta S}{\delta \tilde{\Gamma}_{\pm}(z)} \right\rangle \right) \\
+ \partial_y \tilde{\Gamma}_{\pm}(z) \delta g(z') \left( \frac{\delta}{\delta g(z')} \left\langle \frac{\delta S}{\delta \tilde{\Gamma}_{\pm}(z)} \right\rangle - \frac{\delta}{\delta g(z)} \left\langle \frac{\delta S}{\delta \tilde{\Gamma}_{\pm}(z')} \right\rangle \right) \right]
\]

which is the consistency piece.

### 4.2.1 The Covariant Anomaly

We look first at the covariant term

\[
\delta \Gamma_{\text{cov}} = \int d^4 z \left( \delta g \left\langle \frac{\delta S}{\delta g} \right\rangle + \delta \tilde{\Gamma}_{\pm} \left\langle \frac{\delta S}{\delta \tilde{\Gamma}_{\pm}} \right\rangle \right) \bigg|_{y=1}
\]

\[
= - \lim_{z' \to z} \int d^4 z \left[ i \delta g \nabla_{\pm} \left\langle \Phi(z) \bar{\Phi}(z') \right\rangle + \delta \tilde{\Gamma}_{\pm} \left\langle \Phi(z) \bar{\Phi}(z') \right\rangle g(z') \right] \bigg|_{y=1}.
\] (4.66)

In this case the regularized expression for the scalar propagator (C.16) is given by

\[
\left\langle \Phi(z) \bar{\Phi}(z') \right\rangle = e^{\square_{\pm} / M^2} \nabla_{\pm} \frac{1}{\square_+} \nabla_+ g^{-1} \delta^{(4)}(z - z')
\]

\[
= - \int_{\frac{1}{M^2}}^{\infty} dt \nabla_+ e^{\square_{\pm} t} \nabla_+ g^{-1} \delta^{(4)}(z - z').
\] (4.67)

The expression (4.66) is evaluated at \(y = 1\) so the gauge variations are the usual ones, chosen to ensure the invariance of the classical action (see (B.22))

\[
g^{-1} \delta g \bigg|_{y=1} = i \tilde{\Lambda} - i \Lambda,
\]

\[
\delta \tilde{\Gamma}_{\pm} \bigg|_{y=1} = \delta \Gamma_{\pm} = \nabla_\pm \Lambda \quad ,
\] (4.68)

where again we have defined \(\tilde{\Lambda} \equiv e^{-V} \tilde{\Lambda} e^V\).

Therefore, with some obvious integration by parts, the expression for the covariant anomaly can be written as

\[
\delta \Gamma_{\text{cov}} = \int d^4 z \text{Tr} \left[ \left( -i g^{-1} \delta g \nabla_{\pm} e^{\square_{\pm} / M^2} \nabla_{\pm} \frac{1}{\square_+} \nabla_+ - \delta \Gamma_{\pm} e^{\square_{\pm} / M^2} \nabla_{\pm} \frac{1}{\square_+} \nabla_+ \right) \delta^{(4)}(z - z') \right]
\]

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From the explicit expression (C.13) for $\Box_+^2$ one realizes that in the limit $z' \to z$ the only nonvanishing contribution comes from the $W_+ \nabla_+$ term in the exponential and the final result for the covariant anomaly is

$$
\delta \Gamma_{\text{cov}} = - \frac{i}{4\pi} \int d^2 x d\theta^+ \text{Tr} (\Lambda W_-) + \text{h.c.} \quad .
$$

This expression only differs by an overall sign from the chiral anomaly (4.47) for the spinor case. Therefore, in the $(2,2)$ theory they cancel, consistent with the absence of anomalies for that theory.

### 4.2.2 The Consistency Terms

To evaluate the consistency terms we use the following identities for the homotopically extended derivative operators

$$
\delta g \nabla_+ = \left[ \nabla_+ , g^{-1} \delta g \right] ,
$$

$$
\delta g \nabla_+ g^{-1} = - g^{-1} \delta g \nabla_+ g^{-1} ,
$$

$$
\delta_g \Box_+ = - i \nabla_+ \left[ \nabla_+ , g^{-1} \delta g \right] \nabla_+ - i \left[ \nabla_+ , g^{-1} \delta g \right] \nabla_+ \nabla_+ ,
$$

$$
\delta \tilde{\Gamma}_\pm = - i \delta \tilde{\Gamma}_\pm \quad ,
$$

$$
\delta \tilde{\Gamma}_\pm \Box_+ = - \nabla_+ \nabla_+ \delta \tilde{\Gamma}_\pm - \nabla_+ \delta \tilde{\Gamma}_\pm \nabla_+ .
$$

Therefore, inserting the regularized expression (4.67) in (4.63) and evaluating the gauge variations we have (we omit an overall $\int_0^1 dy \int d^4 z \lim_{z' \to z}$ and a $\delta^4(z - z')$)

$$
\int d^4 z' \delta g(z') \frac{\delta}{\delta g(z')} \left\langle \frac{\delta S}{\delta g(z)} \right\rangle = - i \int_0^{\infty} dt \int_0^{t} ds \nabla_+ \nabla_+ e^{\Box_+ t} g^{-1} \delta g \nabla_+ g^{-1}
$$

$$
+ \int_0^{\infty} dt \int_0^{t} ds \nabla_+ \nabla_+ e^{\Box_+ s} \left[ \nabla_+ , g^{-1} \delta g \right] \nabla_+ \nabla_+ e^{\Box_+ (t-s)} \nabla_+ g^{-1}
$$

$$
\int d^4 z' \delta \tilde{\Gamma}_\pm^\ast (z') \frac{\delta}{\delta \tilde{\Gamma}_\pm (z')} \left\langle \frac{\delta S}{\delta g(z)} \right\rangle = \int dt \delta \tilde{\Gamma}_\pm e^{\Box_+ t} \nabla_+ g^{-1}
$$

$$
- i \int dt \int_0^{t} ds \nabla_+ \nabla_+ e^{\Box_+ s} \nabla_+ \delta \tilde{\Gamma}_\pm e^{\Box_+ (t-s)} \nabla_+ g^{-1}
$$

$$
\int d^4 z' \delta \tilde{\Gamma}_\pm^\ast (z') \frac{\delta}{\delta \tilde{\Gamma}_\pm (z')} \left\langle \frac{\delta S}{\delta \tilde{\Gamma}_\pm (z)} \right\rangle = - \int dt \int_0^{t} ds \nabla_+ e^{\Box_+ s} \nabla_+ \delta \tilde{\Gamma}_\pm e^{\Box_+ (t-s)} \nabla_+ \nabla_+ .
$$
\[
\int d^4 z'' \delta g(z'') \frac{\delta}{\delta g(z'')} \left\langle \frac{\delta S}{\delta \Gamma_\pm(z)} \right\rangle = \int dt \nabla_+ e^{\Box t} [\nabla_+ g^{-1} \delta g] \\
- i \int dt \int_0^t ds \nabla_+ e^{\Box s} [\nabla_+ g^{-1} \delta g] \nabla_+ e^{\Box (t-s)} \nabla_+ . \quad (4.72)
\]

Writing everything down one generates a large number of terms. Using the cyclicity of the trace and writing them somewhat symbolically, after many cancellations we have

\[
- g^{-1} \partial_y g \left[ i \nabla_\pm \nabla_\mp e^{\Box g^{-1} \delta g} + \nabla_+ \delta \Gamma_\mp \nabla_\pm e^{\Box} + i \nabla_+ g^{-1} \delta g \nabla_\pm e^{\Box} \\
+ \nabla_+ \nabla_\pm e^{\Box} g^{-1} \delta g \nabla_\pm \nabla_\pm e^{\Box} \nabla_+ - i \nabla_+ \nabla_\pm e^{\Box} \nabla_\pm \delta \Gamma_\mp \nabla_\pm e^{\Box} \\
+ \nabla_+ \nabla_\pm e^{\Box} g^{-1} \delta g \nabla_\pm \nabla_\pm e^{\Box} \nabla_+ \right] \\
- \partial_y \delta \Gamma_\mp \left[ + \nabla_\pm e^{\Box} g^{-1} \delta g \nabla_+ - i \nabla_\pm e^{\Box} g^{-1} \delta g \nabla_+ \nabla_\pm e^{\Box} \nabla_+ \right] . \quad (4.73)
\]

In this expression integration over \( s \) and \( t \) is understood, as above, and the exponentials are also essentially as above.

We now manipulate the double integral terms (those with two factors of \( e^{\Box} \) i.e. \( \ldots e^{\Box s} \ldots e^{\Box (t-s)} \ldots \)) in the usual fashion by writing

\[
\nabla_\pm \nabla_\mp e^{\Box (t-s)} \nabla_+ = i \frac{\partial}{\partial t} e^{\Box (t-s)} \nabla_+ , \quad (4.74)
\]

where \( \Box_- \) has been defined in \((C.17)\). Performing integration by parts on \( t \), we then obtain

\[
\int_0^{1/M^2} ds \left[ i (g^{-1} \partial_y g) \nabla_\pm e^{\Box s} (g^{-1} \delta g) e^{\Box (1/M^2-s)} \nabla_+ \\
+ i (g^{-1} \partial_y g) e^{\Box (1/M^2-s)} \nabla_+ (g^{-1} \delta g) \nabla_\pm e^{\Box s} \\
+ (g^{-1} \partial_y g) e^{\Box (1/M^2-s)} \nabla_+ \delta \Gamma_\mp \nabla_\pm e^{\Box s} \\
+ \partial_y \delta \Gamma_\mp \nabla_\pm e^{\Box s} (g^{-1} \delta g) e^{\Box (1/M^2-s)} \right] , \quad (4.75)
\]

leading to, in the usual manner,

\[
\delta \Gamma_{cons} = \frac{1}{4\pi} \int_0^1 dy \left[ 2i (g^{-1} \delta g) \nabla_\pm (g^{-1} \partial_y g) + (g^{-1} \partial_y g) \delta \Gamma_\mp - \partial_y \delta \Gamma_\mp (g^{-1} \delta g) \right] . \quad (4.76)
\]

Again, this expression differs from the spinor one \((4.57)\) and \((4.59)\) by an overall sign so that in the \((2,2)\) theory they exactly cancel. Summing the covariant and the consistent terms the gauge anomaly in the chiral scalar case is then given by minus the result in \((4.60)\).
5 Reduction to Components in WZ Gauge

In ref. [15] we showed that the supersymmetric anomaly (2.32) contains the correct bosonic component by reducing it to components in the Wess–Zumino gauge. In this section we show that the anomaly (4.60) reduces correctly to the corresponding component result.

In the bosonic case, the 2D Yang-Mills covariant derivative is defined in terms of a two–components vector \((A_{\pm}, A_{\mp})\) as
\[
\nabla_{\pm} = \partial_{\pm} - i A_{\pm}, \quad \nabla_{\mp} = \partial_{\mp} - i A_{\mp},
\]
\[
[\nabla_{\pm}, \nabla_{\mp}] = -i F_{\pm\mp} = -i (\partial_{\pm} A_{\mp} - \partial_{\mp} A_{\pm} - i [A_{\pm}, A_{\mp}]). \quad (5.77)
\]
We consider a theory of chiral fermions minimally coupled to a set of gauge fields
\[
S = i \int d^2 x \bar{\zeta} \nabla_{\mp} \zeta. \quad (5.78)
\]
This action is invariant under the gauge transformations
\[
\delta_G(\lambda) \zeta_- = i \lambda \zeta_- , \quad \delta_G(\lambda) \bar{\zeta}_+ = -i \bar{\zeta}_+ \lambda , \quad \delta_G(\lambda) A_{\pm} = \partial_{\pm} \lambda + i [\lambda, A_{\pm}] . \quad (5.79)
\]
Following the discussion in [15], we compute algebraically the 2D consistent anomaly by directly solving the Wess–Zumino consistency condition
\[
\delta_G(\lambda_1) A_{BGJ}(\lambda_2) - \delta_G(\lambda_2) A_{BGJ}(\lambda_1) = -i A_{BGJ}([\lambda_1, \lambda_2]) . \quad (5.80)
\]
To this end, we consider a set of basis monomials defined as
\[
M_0(\lambda) = \frac{1}{2} \epsilon^{ab} \text{Tr}\{\lambda F_{a\pm}\} , \quad M_2(\lambda) = \epsilon^{ab} \text{Tr}\{\lambda A_a A_b\} . \quad (5.81)
\]
In terms of light–cone coordinates they read
\[
M_0(\lambda) = \text{Tr}\{\lambda F_{a\pm}\} = \text{Tr}\{\lambda (\partial_{\pm} A_a - \partial_{a} A_{\pm} - i [A_a, A_{\pm}])\} ,
\]
\[
M_2(\lambda) = \text{Tr}\{\lambda (A_a A_b - A_b A_a)\} . \quad (5.82)
\]
We look for a solution of the WZ consistency condition as linear combination of the two previous monomials. Imposing the condition (5.80) on the general structure
\[
c_0 M_0(\lambda) + c_2 M_2(\lambda) , \quad (5.83)
\]
we find a solution for \(c_2 = ic_0\). Therefore, this leads to the identification
\[
A_{BGJ}(\lambda) \equiv C_0 \int d^2 x \left( M_0 + i M_2 \right)
\]
\[
= C_0 \int d^2 x \text{Tr}\{\lambda (\partial_{\pm} A_a - \partial_{a} A_{\pm})\} , \quad (5.84)
\]
with $C_0$ an overall normalization. The comparison with the perturbative calculation gives $C_0 = -(1/8\pi)$. In 2D, the consistent anomaly has the same form as the abelian anomaly.

The consistent anomaly is always defined up to the variation of a local functional. In the two dimensional case, an alternative expression for the anomaly is

$$A_{BGJ}(\lambda) = -2C_0 \int d^2 x \, \text{Tr}\{ \lambda \partial_+ A_\pm \} ,$$

which differs from (5.84) by the variation

$$\delta \int d^2 x \, \text{Tr}(A_\pm A_\mp) = -\int d^2 x \, \text{Tr}[ \lambda (\partial_\mp A_\pm + \partial_\pm A_\mp) ] .$$

We now consider the supersymmetric expression (4.60) for the (2,0) anomaly written as a sum of four contributions

$$A_{BGJ} = \left( \frac{1}{4\pi} \right) (A_1 + A_2 + A_3 + A_4) ,$$

where we have defined

$$A_1 = i \int d^2 x d\theta^+ \text{Tr}(\Lambda W_+) + \text{h.c.} ,$$

$$A_2 = -2i \int d^2 x d\theta \int_0^1 dy \, \text{Tr}\left[ (g^{-1} \delta g) \nabla_+(g^{-1} \partial_y g) \right] ,$$

$$A_3 = -\int d^2 x d\theta \int_0^1 dy \, \text{Tr}\left[ (g^{-1} \partial_y g) \delta \tilde{\Gamma}_\mp \right] ,$$

$$A_4 = \int d^2 x d\theta \int_0^1 dy \, \text{Tr}\left[ (g^{-1} \delta g) \partial_y \tilde{\Gamma}_\mp \right] ,$$

and we perform the reduction separately for the four terms.

The WZ gauge is defined by $V| = D_+ V| = 0$. In this gauge the bosonic components are

$$\Gamma_+ | = D_+ D_+ V| = A_+ ,$$

$$\Gamma_0 | = A_0 = 0 ,$$

$$[\nabla_+ , \nabla_0] | = \left( \nabla_+ W_+ - \nabla_0 W_- \right) | = -iF_+ .$$

with $F_{+\mp}$ as above. Moreover, using the conditions defining the WZ gauge, the chirality of $\Lambda$ and the identity $\Lambda| = \bar{\Lambda}| = \lambda$, one finds

$$g^{-1} \delta g| = iy \, g^{-1} \left[ \bar{\Lambda} e^V - e^V \Lambda \right] | = 0 ,$$

$$g^{-1} \partial_y g| = g^{-1} (e^V - 1) | = 0 ,$$

$$D_+ D_+ (g^{-1} \delta g) | = -y \left( \partial_+ \lambda - i[A_+, \lambda] \right) = -y \nabla_+ \lambda .$$

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In order to perform the reduction to components we need to choose a particular function \( \tilde{\Gamma} \). Some indications on the particular structure of this function can be obtained by defining the extended \((2,0)\) SYM theory as the reduction of an extended \((2,2)\) SYM theory. In fact, in the latter theory the supersymmetry algebra constrains the particular dependence on \( g \) of both vector connections in \( \nabla_\mp \) and \( \nabla_\pm \), once the homotopic path has been fixed in the definition of the extended spinorial derivatives (see eq. (4.36)). If we call \( \mathcal{H} \) the homotopy path of the \((2,2)\) theory \((\mathcal{H}|_{\theta^- = \theta^+ = 0} = g)\), in chiral representation the extended \((2,0)\) connection is obtained from the \((2,2)\) one, as an extension of (B.18)

\[
\tilde{\Gamma}_- = D_\pm (\mathcal{H}^{-1} D_\mp \mathcal{H})|_{\theta^- = \theta^+ = 0}.
\]  

(5.91)

Computing the gauge variation and the \( y \)-derivative in the \((2,2)\) theory and performing the complete reduction to the component theory, for the bosonic components we find

\[
\tilde{\Gamma}_- | = y A_- = y \Gamma_-, \\
\delta \tilde{\Gamma}_- | = y \partial_- \lambda - iy [A_-, \lambda] = y \nabla_- \lambda = y \delta \Gamma_- |,
\]

\[
\partial \tilde{\Gamma}_- | = A_+ = \Gamma_+ |.
\]  

(5.92)

The previous identities strongly suggest the following general structure for the homotopic extension of \( \Gamma_- \) directly in \((2,0)\) superspace

\[
\tilde{\Gamma}_- = f(y) \Gamma_- + \Delta \Gamma_- ,
\]  

(5.93)

where \( f(0) = 0 \) and \( f(1) = 1 \), consistently with the choice of the boundary conditions for \( g \), and \( \Delta \Gamma_- | = 0 \). We note that for \( f(y) = y \) we obtain the extension coming from the \((2,2)\) theory, whereas \( f(y) = 1 \) and \( \Delta \Gamma_- = 0 \) corresponds to no extension for \( \Gamma_- \).

We perform the reduction of (4.60) to the bosonic components by assuming for \( \tilde{\Gamma}_- \) the form (5.93). We first consider the covariant term \( \tilde{A}_1 \). Keeping only the bosonic components we have

\[
\tilde{A}_1 = i \int d^2 x \operatorname{Tr}(\Lambda D_+ W_-)| + \text{h.c.} = - \int d^2 x \operatorname{Tr}(\lambda F_{\mp \mp}) = \int d^2 x \operatorname{Tr} \{ \lambda (\partial_- A_\mp - \partial_\mp A_- + i [A_\pm, A_-]) \} ,
\]  

(5.94)

where we have used \( | = \bar{\Lambda} | = \lambda \) and eq. (5.77).

For the first consistent term \( \tilde{A}_2 \), using the identities in (5.90) we find

\[
\tilde{A}_2 = -2i \int d^2 x \int_0^1 dy y D_+ D_\mp \operatorname{Tr} \left[ g^{-1} \delta g \nabla_- (g^{-1} \partial_y g) \right] | \to 0 .
\]  

(5.95)
Therefore it never contributes to the bosonic anomaly, independent of the choice of the homotopic path.

Using eq. (5.92) the $A_3$ term gives (again neglecting fermionic terms)

$$A_3 = - \int d^2 x \int_0^1 dy D_+ D_+ \text{Tr} \left[ g^{-1} (e^V - 1) \delta \bar{\Gamma}_\pm \right]$$

$$\rightarrow - \int d^2 x \int_0^1 dy \text{Tr} \left[ (D_+ D_+ V) \delta \bar{\Gamma}_\pm \right]$$

$$= \int d^2 x \int_0^1 dy f(y) \text{Tr} \{ A_+ (\partial_\mp \lambda - i[A_\pm, \lambda]) \}$$

$$= \int_0^1 dy f(y) \int d^2 x \text{Tr} \left\{ \lambda \left( - \partial_\pm A_\pm - i[A_\pm, A_\pm] \right) \right\} . \quad (5.96)$$

Finally, using eqs. (5.90, 5.92) in the last consistent term $A_4$ we have

$$A_4 = \int d^2 x \int_0^1 dy D_+ D_+ \text{Tr} \left[ g^{-1} \delta g \partial_y \bar{\Gamma}_\pm \right]$$

$$\rightarrow \int d^2 x \int_0^1 dy \text{Tr} \left[ [D_+ D_+ (g^{-1} \delta g)] \partial_y \bar{\Gamma}_\pm \right]$$

$$= \int d^2 x \int_0^1 dy y f'(y) \text{Tr} \{ (-\partial_\pm \lambda + i[A_\pm, \lambda]) A_\pm \}$$

$$= \left( 1 - \int_0^1 dy f(y) \right) \int d^2 x \text{Tr} \left\{ \lambda \left( \partial_\pm A_\pm - i[A_\pm, A_\pm] \right) \right\} . \quad (5.97)$$

Summing the three nonvanishing contributions it is easy to see that the terms proportional to $[A_\pm, A_\pm]$ cancel independent of the choice of $f$ and we are left with

$$A_{BGJ}^{bos.}(\lambda) = \frac{1}{4\pi} \int d^2 x \text{Tr} \left\{ \lambda \left[ \partial_\pm A_\pm - \int_0^1 dy f(y) \left( \partial_\pm A_\pm + \partial_\mp A_\pm \right) \right] \right\} . \quad (5.98)$$

According to eq. (5.86) the second term is cohomologically trivial and can be neglected in the final expression for the anomaly which turns out to be in agreement with (5.85) (or equivalently (5.84)).

From the above result we can conclude that the choice of a particular function $f$ affects the final answer only by terms which are the variation of a local functional. Therefore, we are allowed to choose in (5.93) any regular function in $[0, 1]$ with correct boundary conditions. The most general homotopically extended $(2, 0)$ theory depends on the choice of two independent paths $g$ and $\bar{\Gamma}_\pm$. In particular, one could choose $f(y) = 1$ which corresponds to performing the entire derivation of the supersymmetric anomaly without homotopically extending $\Gamma_\pm$. This might be a priori expected in the case of a chiral spinor superfield since $\Gamma_\pm$ never couples to the spinor. Instead in the case of a chiral scalar it is not an obvious result.

Another choice that may ultimately prove of use in other contexts is $f(y) = y$ for which we find agreement between the perturbative result and the solution of
the 2D WZ consistency condition given in (5.84). This choice of bosonic homotopy for the minus-minus connection resembles the minimal supersymmetric homotopy (2.29) in that it satisfies equations analogous to (2.30) and is the one obtained by the truncation described in (5.92). For this choice, utilizing the minimal supersymmetric homotopy implies that we can re-write the last three terms in (5.87) and thus express the holomorphic consistent (2,0) Yang-Mills anomaly, as in ref. [15], in the form of the imaginary part of a superaction, $\mathcal{A}_{BGJ} = \Im[\tilde{A}_{BGJ}]$ where

$$\tilde{A}_{BGJ} = -\left(\frac{1}{2\pi}\right) \int d^2x d^2\theta \, \mathcal{P}_\Xi(\Lambda; e^V) ,$$

(5.99)

with the 2D, $N = (2,0)$ $\mathcal{P}$-superfunction given by

$$\mathcal{P}_\Xi(\Lambda; e^V) = \text{Tr}\left\{ \Lambda \left( \Gamma_\Xi - \int_0^1 dy y \left[ \Gamma_\Xi e^V G + [\Gamma_\Xi, G (e^V - 1)] \right] (1 - 2y e^V G) \right. \\
+ \left. i \left[ \partial_\Xi(G (e^V - 1)) \right] (1 - 2e^V G) \right\} .$$

(5.100)

where again $G = [1 + y(e^V - 1)]^{-1}$.

### 6 Conclusions and Summary

In this paper, we have presented examples of the construction of the superspace consistent anomaly using the method of Banerjee, Banerjee and Mitra [19]. We have shown that the 4D, $N = 1$ supersymmetric extension of the consistent anomaly has a preferred choice of homotopy which leads to vast simplification in the form of the BGJ non-Abelian 4D, $N = 1$ consistent anomaly action. We believe that this choice, which we call the minimal homotopy [15], yields substantial clarity to the issue of 4D, $N = 1$ supersymmetric anomalies over previous results in the literature [1]-[14]. We have also shown that any other choice of homotopy utilized to write the 4D, $N = 1$ supersymmetric BGJ action yields a non-minimal form of the anomaly that differs from the minimal one only by the gauge variation of a local counterterm. This provides a proof of a previous assertion [15] that homotopic non-minimality was responsible for the opacity of most discussions of this topic. Our discussion has also pointed out the differential equation (2.30) that singles out the minimal homotopy from all others. This provides a finer definition of the minimal homotopy.

Turning to the issue of the universality of the role of the minimal homotopy in supersymmetric gauge theories, we have explored this outside of 4D, $N = 1$ theories. For 2D, $N = (2,2)$ models, involving only chiral matter coupled to ordinary vector multiplets, all such anomalies are found to vanish as expected from the fact that these are vector-like theories.
For 2D, $N = (2,0)$ models, anomalies are expected and our study has determined their form. In this context, the minimal homotopy was also found to lead to a simple expression. We also found that the $(2,0)$ reduction of the homotopic extension of the 2D, $N = (2,2)$ (the analog of (2.5)) naturally leads to the use of the minimal homotopical extension for the connection in the non-supersymmetric sector of the theory.

We end this by again noting the fact that the existence of the minimal homotopy obeying (2.30), playing a special role in 4D, $N = 1$ theories, raises a question of whether the five-dimensional embedding space $(x, y)$ may be playing a more subtle role than may have been suspected.
A  Four Dimensional $N = 1$ Superspace Conventions

In four dimensions we use $Superspace$ notations and conventions supplemented by the conjugation rule $(D_a)^* = -\bar{D}_{\dot{a}}$, $(\bar{D}^\alpha)^* = \bar{D}^{\dot{\alpha}}$ and $(\partial_a)^* = \partial_{\dot{a}}$ ($a = \alpha \dot{\alpha}$).

In chiral representation, the superspace Yang–Mills covariant derivatives $\nabla_{\hat{A}} \equiv (\nabla_\alpha, \nabla_{\dot{\alpha}}, \nabla_a)$ with $\nabla_{\hat{A}} = D_{\hat{A}} - i\Gamma_{\hat{A}}$ are defined in terms of a real vector superfield $V$ as

$$\nabla_\alpha \equiv e^{-V}D_\alpha e^V, \quad \nabla_{\dot{\alpha}} \equiv \bar{D}_{\dot{\alpha}}, \quad \nabla_a \equiv -i\{\nabla_\alpha, \nabla_{\dot{\alpha}}\} \quad (A.1)$$

The corresponding spinorial field strengths are (the field strength $W_{\dot{\alpha}}$ below is the conjugate of $W_\alpha$; note that $\bar{\Gamma}^a - \Lambda$ should not be confused with $\Gamma^a$),

$$W_\alpha = \bar{D}^2\Gamma_\alpha = -i\bar{D}^2(e^{-V}D_\alpha e^V), \quad W_{\dot{\alpha}} = D^2\bar{\Gamma}_{\dot{\alpha}} = i\bar{D}^2(e^V\bar{D}_{\dot{\alpha}}e^{-V}) \quad (A.2)$$

The infinitesimal gauge transformations are generated by chiral and antichiral superfield parameters $\Lambda$, $\bar{\Lambda}$ ($\bar{D}_{\dot{a}}\Lambda = 0$, $D_a\bar{\Lambda} = 0$) given by

$$\delta e^V = i[\bar{\Lambda}e^V - e^V\Lambda], \quad \delta e^{-V} = i[\Lambda e^{-V} - e^{-V}\bar{\Lambda}],$$
$$\delta \Gamma_{\hat{A}} = D_{\hat{A}}\Lambda + i[\Lambda, \Gamma_{\hat{A}}], \quad \delta W_\alpha = i[\Lambda, W_\alpha] \quad (A.3)$$

The action describing scalar matter coupled to Yang–Mills is

$$S = \int d^8z \bar{\Phi}e^V\Phi \quad (A.4)$$

where $\bar{D}_{\dot{a}}\Phi = D_\alpha\Phi = 0$. It is invariant under gauge transformations (A.3) supplemented by

$$\delta \Phi = i\Lambda\Phi, \quad \delta \bar{\Phi} = -i\bar{\Phi}\Lambda \quad (A.5)$$

B  $N = 2$ Superspace and Superfields in Two Dimensions

B.1  $(2, 2)$ Superspace

The two–dimensional superspace description of a $(2, 2)$ theory is essentially the same as $N = 1$ in four dimensions with a suitable identification of the spinorial coordinates. It is described in terms of two light–cone coordinates $(x^+, x^-)$ and four spinor coordinates $(\theta^+, \theta^+, \theta^-, \theta^-)$ where plus and minus label the two chiral sectors.
An untwisted scalar $(2,2)$ supersymmetric theory is described by the superspace action

\[ S = \int d^2 x d^4 \theta \Phi \Phi , \]  

(B.1)

where $\Phi$ and $\Phi$ are chiral and antichiral fields respectively ($D_+ \Phi = D_- \Phi = D_\Phi = 0$) and

\[ d^4 \theta \equiv d\theta^+ d\bar{\theta} \bar{\theta} d\theta^- \equiv D_+ D_- D_\Phi . \]  

(B.2)

A twisted version of the theory is also consistent [24, 25] in terms of a twisted scalar superfield defined by $D_+ \Phi = D_- \Phi = 0$.

The minimal coupling to Yang–Mills fields is realized by introducing covariant derivatives. Also in this case there exist two alternative formulations [24, 25] in terms of Yang–Mills multiplets and twisted Yang–Mills multiplets, respectively. However, we consider only the version described by the constraints

\[
\begin{align*}
\{ \nabla_+ , \nabla_+ \} &= 0 , & \{ \nabla_+ , \nabla_- \} &= 0 , \\
\{ \nabla_+ , \nabla_- \} &= 0 , & \{ \nabla_+ , \nabla_- \} &= iW , \\
\{ \nabla_+ , \nabla_- \} &= 0 , & \{ \nabla_+ , \nabla_- \} &= -iW , \\
\{ \nabla_+ , \nabla_+ \} &= i\nabla_+ , & \{ \nabla_+ , \nabla_+ \} &= i\nabla_- , \\
[\nabla_+ , \nabla_+ ] &= 0 , & \{ \nabla_+ , \nabla_- \} &= 0 , \\
[\nabla_+ , \nabla_+ ] &= -\nabla_- W , & \{ \nabla_+ , \nabla_+ \} &= \nabla_+ W , \\
[\nabla_+ , \nabla_+ ] &= -iF .
\end{align*}
\]  

(B.3)

The superfields $W$ and $F$ satisfy the identities

\[
\begin{align*}
\nabla_- W &= \nabla_+ W = 0 , \\
F &= [\nabla_+ \nabla_- W - \nabla_+ \nabla_- W] .
\end{align*}
\]  

(B.4)

The solution to the constraints, as in 4D, can be expressed in terms of the hermitean prepotential $V$.

**B.2 (2,0) Superspace**

The chiral $(2,0)$ ($(0,2)$) theory can be obtained as a reduction of the $(2,2)$ theory by setting $\theta^- = \theta^\Phi = 0$ ($\theta^+ = \theta^{\Phi} = 0$).

From the action (B.1), by explicitly integrating on $(\theta^-, \theta^\Phi)$ one obtains

\[ S = \int d^2 x d\theta^+ d\theta^- [ -D_\Phi D_\Phi - i\Phi \partial_\Phi \Phi ] \bigg|_{\theta^- = \theta^\Phi = 0} , \]  

(B.5)
where the chirality constraints have been used. The two terms represent the actions for a chiral \((2,0)\) spinor superfield \(\chi_\cdot \equiv D_\cdot \Phi|_{\dot{\theta}^+ = \dot{\theta}^- = 0}\) and a chiral \((2,0)\) scalar \(\Phi \equiv \Phi|_{\theta^+ = \theta^- = 0}\), respectively.

In the \((2,0)\) sector we use the following conventions for functional derivatives. For the chiral scalar superfield we define

\[
\frac{\delta}{\delta \Phi(z')} \int d^2 x d\theta^+ f(\Phi(z)) = f'(\Phi(z')) \quad , \quad (B.6)
\]

\[
\frac{\delta}{\delta \bar{\Phi}(z')} \int d^2 x d\theta^+ f(\bar{\Phi}(z)) = f'(\bar{\Phi}(z')) \quad . \quad (B.7)
\]

It follows that the functional derivatives are

\[
\frac{\delta \Phi(z)}{\delta \Phi(z')} = D_+ \delta^{(4)}(z - z') \quad , \quad \frac{\delta \Phi(z)}{\delta \bar{\Phi}(z')} = -D_+ \delta^{(4)}(z - z') \quad , \quad (B.8)
\]

where the derivatives on the r.h.s. act on \(z\) and we have defined

\[
\delta^{(4)}(z - z') \equiv \delta^{(2)}(x - x') \delta^{(2)}(\theta - \theta') \equiv \delta^{(2)}(x - x') (\theta^+ - \theta^+')(\theta^+ - \theta^+) \quad . \quad (B.9)
\]

Using these conventions the scalar propagator is given by

\[
\langle \Phi(z)\bar{\Phi}(z') \rangle \equiv \int D\Phi D\bar{\Phi} e^{-S\Phi(z)\bar{\Phi}(z')} = -\frac{\delta}{\delta J(z)} \frac{\delta}{\delta \bar{J}(z')} \mathcal{W}[J, \bar{J}] \bigg|_{J = \bar{J} = 0} \quad , \quad (B.10)
\]

where

\[
\mathcal{W}[J, \bar{J}] = \int D\Phi D\bar{\Phi} \exp \{-S - \int d^2 x d\theta^+ J\Phi + \int d^2 x d\theta^+ \bar{\Phi}\bar{J} \} \quad . \quad (B.11)
\]

In the previous expression the sources \(J\) and \(\bar{J}\) are spinors and anticommute with the measure.

In the case of the chiral spinor superfield, using the same definitions \(B.7\) for the functional derivatives we obtain

\[
\frac{\delta \chi_\cdot(z)}{\delta \chi_\cdot(z')} = -D_+ \delta^{(4)}(z - z') \quad , \quad \frac{\delta \bar{\chi}_\cdot(z)}{\delta \bar{\chi}_\cdot(z')} = D_+ \delta^{(4)}(z - z') \quad , \quad (B.12)
\]

where the derivatives on the r.h.s. act on \(z\). The propagator is then defined as

\[
\langle \chi_\cdot(z)\bar{\chi}_\cdot(z') \rangle \equiv \int D\chi_\cdot D\bar{\chi}_\cdot e^{-S\chi_\cdot(z)\bar{\chi}_\cdot(z')} = -\frac{\delta}{\delta J(z)} \frac{\delta}{\delta \bar{J}(z')} \mathcal{W}[J, \bar{J}] \bigg|_{J = \bar{J} = 0} \quad , \quad (B.13)
\]

with

\[
\mathcal{W}[J, \bar{J}] = \int [D\chi_\cdot D\bar{\chi}_\cdot] \exp \{-S + \int d^2 x d\theta^+ J\chi_\cdot - \int d^2 x d\theta^+ \bar{J}\bar{\chi}_\cdot \} \quad . \quad (B.14)
\]
and scalar sources.

The (2, 0) Yang–Mills theory can also be obtained as a reduction of the (2, 2) theory, setting \( \theta^- = \theta^+ = 0 \). In this case the constraints are

\[
\begin{align*}
[\nabla_+, \nabla_+] &= [\nabla_+, \nabla_+] \quad , \\
[\nabla_+, \nabla_+] &= i \nabla_+ \quad , \\
[\nabla_+, \nabla_-] &= i W_- \quad , \\
[\nabla_-, \nabla_-] &= -i W_+ \quad , \\
[\nabla_+, \nabla_-] &= \nabla_+ W_- - \nabla_+ W_+ .
\end{align*}
\]

(B.15)

In chiral representation, they are solved by

\[
\begin{align*}
\nabla_+ &= e^{-V} D_+ e^V \quad , \\
\nabla_- &= D_+ \quad ,
\end{align*}
\]

(B.16)

where \( V \) is the (2, 0) vector multiplet, \( V \equiv V|_{\theta^- = \theta^+ = 0} \). From the previous relations it follows

\[
\Gamma_+ = i(e^{-V} D_+ e^V) \quad , \quad \Gamma_- = 0 \quad , \quad \Gamma_- = -i D_+ \Gamma_+ \quad ,
\]

(B.17)

The connection \( \Gamma_- \) is an independent superfield. However, if the theory is obtained by reduction of the (2, 2) theory one finds

\[
\Gamma_- = D_- (e^{-V} D_- e^V)|_{\theta^- = \theta^+ = 0} . \tag{B.18}
\]

Moreover, one also finds

\[
\begin{align*}
W_- &= D_+ \Gamma_- \quad , \\
W_- &= \partial_\bar{\theta} \Gamma_+ - D_+ \Gamma_- + i[\Gamma_+, \Gamma_-] = \partial_\bar{\theta} \Gamma_+ - \nabla_+ \Gamma_- .
\end{align*}
\]

(B.19)

These two superfields satisfy the chirality constraints \( \nabla_+ W_- = \nabla_+ W_- = 0 \).

As described at the beginning of this section, possible (2, 0) matter fields are chiral scalars \( \Phi \) and chiral spinors \( \chi_- \). The relevant actions for matter coupled to gauge fields are then

\[
\begin{align*}
S_\Phi &= -i \int d^2xd^2\bar{\theta} \Phi e^V \nabla_\bar{\theta} \Phi = -i \int d^2 xd^2\bar{\theta} \Phi e^V (\partial_\bar{\theta} - i \Gamma_-) \Phi \quad , \tag{B.20} \\
S_\chi &= -i \int d^2 xd^2\bar{\theta} \chi_- e^V \chi_- . \tag{B.21}
\end{align*}
\]

They are invariant under gauge transformations

\[
\begin{align*}
\Phi &\rightarrow e^{iA} \Phi \quad , \quad \bar{\Phi} \rightarrow \bar{\Phi} e^{-iA} \quad , \\
\chi_- &\rightarrow e^{iA} \chi_- \quad , \quad \bar{\chi}_- \rightarrow \bar{\chi}_- e^{-iA} \quad , \\
e^V &\rightarrow e^{iA} e^V e^{-iA} \quad , \\
\nabla_\bar{\theta} &\rightarrow e^{iA} \nabla_\bar{\theta} e^{-iA} . \tag{B.22}
\end{align*}
\]

The total action \( S_\Phi + S_\chi \) gives (2, 2) matter coupled to Yang–Mills fields.
C  Matter Propagators

In this appendix we derive the expressions for the exact propagators of the \( N = 1 \) scalar superfield in four dimensions coupled to gauge fields and for the \( (2,0) \) chiral scalar and spinor superfields coupled to 2D Yang–Mills fields. From the four dimensional propagator we immediately read also the result for a scalar multiplet coupled to Yang–Mills in the \( (2,2) \) theory.

C.1 N=1 Scalar Superfield

Here we review the calculation of the covariant propagator for a chiral scalar minimally coupled to Yang–Mills in four dimensions. One way to find the propagator is to determine the Schwinger-Dyson equation it satisfies [22]. Taking into account the explicit expression (A.4) for the scalar action, we write the functional integral

\[
\langle \overline{\Phi}(z) \rangle = \int D\Phi D\overline{\Phi} \exp\left[ -\int d^8w \overline{\Phi}(w)e^V \Phi(w) \right] \cdot \overline{\Phi}(z),
\]

(C.1)

and make a change of variable \( \overline{\Phi} \to \overline{\Phi} + \delta \overline{\Phi} \) under which the functional integral doesn’t change. One gets then

\[
0 = \int D\Phi D\overline{\Phi} \exp\left[ -\int d^8w \overline{\Phi}(w)e^V \Phi(w) \right] \cdot \left[ -\int d^8u \delta \overline{\Phi}(u)e^V \Phi(u)\overline{\Phi}(z) + \delta \overline{\Phi}(z) \right].
\]

(C.2)

Now taking the functional derivative \( \delta/\delta \overline{\Phi}(z') \) and reinterpreting the functional integral as giving an expectation value, we obtain

\[
\int d^8u D^2\delta^{(8)}(u-z') e^V \langle \Phi(u)\overline{\Phi}(z) \rangle - D^2\delta^{(8)}(z-z') = 0,
\]

(C.3)

or

\[
D^2e^V \langle \Phi(z')\overline{\Phi}(z) \rangle = D^2\delta^{(8)}(z'-z).
\]

(C.4)

Now, multiplying by \( e^{-V} \) on the left we have

\[
\nabla^2 \langle \Phi(z')\overline{\Phi}(z) \rangle = e^{-V} D^2\delta^{(8)}(z'-z) = \nabla^2 e^{-V} \delta^{(8)}(z'-z).
\]

(C.5)

We proceed then multiplying by \( \nabla^2 \) and extending \( \nabla^2\nabla^2 \) acting on the chiral superfield to the invertible operator

\[
i\Box + \nabla^2\nabla^2 + \nabla^2\nabla^2 - \nabla^a\nabla^2\nabla_{\dot{a}} = \Box - iW^a\nabla_{\dot{a}} - \frac{i}{2}(\nabla^a W_\alpha)\]

where

\[
\Box \equiv \frac{1}{2}\nabla^{\dot{a}\dot{a}}\nabla_{\dot{a}\dot{a}}.
\]

(C.6)

(C.7)
We obtain the scalar propagator (we interchange $z$ and $z'$)

$$
\langle \Phi(z)\Phi(z') \rangle = \nabla^2 \frac{1}{\Box_+} \nabla^2 e^{-V} \delta^{(8)}(z - z') ,
$$

(C.8)

where we have used the identity $\nabla^2 \Box_+ = \Box_+ \nabla^2$. The previous expression also gives the covariant scalar propagator for the $(2, 2)$ theory. In the main text we have also used $\Box_- = \Box - i\overline{W}^\alpha \nabla_\alpha - \frac{i}{2}(\nabla^3 \overline{W}_3)$.

### C.2 $(2, 0)$ Scalar Superfield

We compute the two–dimensional propagators by following a procedure analogous to the one used in the 4D case. From the explicit expression (B.20) for the scalar action, we write the functional integral

$$
\langle \overline{\Phi}(z) \rangle = \int D\Phi D\overline{\Phi} \exp\left[i \int d^4w \overline{\Phi}(w)e^{V} \nabla_w \Phi(w)\right] \cdot \overline{\Phi}(z) .
$$

(C.9)

Now making a change of variable $\overline{\Phi} \rightarrow \overline{\Phi} + \delta \overline{\Phi}$ under which the functional integral doesn’t change and taking the functional derivative $\delta/\delta \overline{\Phi}(z')$, we obtain

$$
i \int d^4u D_+ \delta^{(4)}(u - z')e^{V} \nabla_u \langle \Phi(u)\Phi(z) \rangle + D_+\delta^{(4)}(z - z') = 0 ,
$$

or

$$
i D_+ e^{V} \nabla_+ \langle \Phi(z')\Phi(z) \rangle = D_+\delta^{(4)}(z - z') = - D_+\delta^{(4)}(z' - z) .
$$

(C.11)

We then multiply by $\nabla_+ e^{-V}$ on the left to obtain

$$
i \nabla_+ \nabla_+ \nabla_+ \langle \Phi(z')\Phi(z) \rangle = - \nabla_+ e^{-V} D_+\delta^{(4)}(z' - z) = - \nabla_+ \nabla_+ e^{-V}\delta^{(4)}(z' - z) .
$$

(C.12)

The derivative operator on the l.h.s. can be suitably extended to an invertible operator

$$
i \Box_+ \equiv \nabla_+ \nabla_+ \nabla_+ + \nabla_+ \nabla_+ \nabla_+ = i\left[ \Box + \frac{1}{2}(\nabla_+ W_3 + \nabla_+ W_3) - W_3 \nabla_+ \right] ,
$$

(C.13)

where

$$
\Box \equiv \frac{1}{2}(\nabla_+ \nabla_+ + \nabla_+ \nabla_+) .
$$

(C.14)

Therefore, for the scalar propagator we obtain (we interchange $z$ and $z'$)

$$
\langle \Phi(z)\Phi(z') \rangle = \frac{1}{\Box_+} \nabla_+ \nabla_+ e^{-V}\delta^{(4)}(z - z') .
$$

(C.15)
Since the identity $\nabla \Box_+ = \Box_+ \nabla_+$ holds, in the previous equation one can interchange the two operators and write
\[
\langle \Phi(z)\bar{\Phi}(z') \rangle = \nabla_+ \frac{1}{\Box_+} \nabla_+ e^{-V(4)}(z - z') .
\] (C.16)

We also define
\[
i\Box_- \equiv \nabla_+ \nabla_+ + \nabla_+ \nabla_+ \nabla_+ = i\left[\Box_+ + \frac{1}{2}(\nabla_+ W_+ - \nabla_+ W_- + W_- \nabla_+)\right] .
\] (C.17)

This operator satisfies $\nabla_+ \Box_- = \Box_- \nabla_+$.

An alternative way to compute the propagator is to use the standard procedure by completing the square in the functional integral and perform the gaussian integration. Let us consider
\[
W = \int [D\Phi_c D\bar{\Phi}_c] \exp\left\{ i \int d^2x d^2\theta \Phi_c \nabla_+ \Phi_c - \int d^2x d^2\theta^+ J_c \Phi_c + \int d^2x d^2\theta^+ \bar{\Phi}_c \bar{J}_c \right\} ,
\] (C.18)

where $\Phi_c$ and $\bar{\Phi}_c$ are covariantly chiral and antichiral superfields respectively and $J_c$, $\bar{J}_c$ the corresponding sources (covariantly chiral and antichiral spinors). To compute the functional integral we write
\[
\int d^2x d^2\theta^+ J_c \Phi_c = \int d^2x d^2\theta^+ \frac{i}{\Box_+} \nabla_+ \nabla_+ \Phi_c ,
\]
\[
\int d^2x d^2\theta^+ \bar{\Phi}_c \bar{J}_c = \int d^2x d^2\theta^+ \bar{\Phi}_c \nabla_+ \nabla_+ \bar{J}_c ,
\] (C.19)

where we have used the operators (C.13,C.17). Then, we have
\[
W = \int [D\Phi_c D\bar{\Phi}_c] \exp\left\{ i \int d^2x d^2\theta \left[ (\Phi_c - J_c \frac{1}{\Box_+} \nabla_+) \nabla_+ (\Phi_c + \nabla_+ \frac{1}{\Box_-} \bar{J}_c) + J_c \frac{1}{\Box_+} \nabla_+ \nabla_+ \nabla_+ \frac{1}{\Box_-} \bar{J}_c \right] \right\}
\] (C.20)

If we now define
\[
\Phi'_c \equiv \Phi_c + \nabla_+ \frac{1}{\Box_-} \bar{J}_c , \quad \bar{\Phi}'_c \equiv \bar{\Phi}_c - J_c \frac{1}{\Box_+} \nabla_+ .
\] (C.21)

the chirality constraints are maintained, $\nabla_+ \Phi'_c = 0$ and $\bar{\Phi}'_c \nabla_+ = 0$, and we can perform the gaussian integral obtaining
\[
W \sim \exp\left\{ i \int d^2x d^2\theta J_c \frac{1}{\Box_+} \nabla_+ \nabla_+ \nabla_+ \frac{1}{\Box_-} \bar{J}_c \right\} ,
\] (C.22)

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We now compute the propagator using the definition (B.10)

\[
\langle \Phi_c(z)\bar{\Phi}_c(z') \rangle \equiv -\left. \frac{\delta}{\delta J_c(z)} \frac{\delta}{\delta \bar{J}_c(z')} W \right|_{J_c = \bar{J}_c = 0} = -i\nabla_+ \frac{1}{\Box_+} \nabla_+ \nabla_+ \frac{1}{\Box_-} \nabla_+ \delta^{(4)}(z - z') \quad (C.23)
\]

\[
\nabla_+ \frac{1}{\Box_+} \nabla_+ \nabla_+ \frac{1}{\Box_-} \nabla_+ \delta^{(4)}(z - z') = \nabla_+ \frac{1}{\Box_+} \nabla_+ \delta^{(4)}(z - z') .
\]

Going from the first to the second line we get a minus sign from interchanging \( J_c \) with \( \delta \delta \bar{J}_c \) (remember that \( J_c \) and \( \bar{J}_c \) are spinors) and a minus sign from the functional derivative as given in (B.12), resulting in no change in sign.

Now, remembering that in chiral representation \( \Phi_c = \Phi \) and \( \bar{\Phi}_c = \bar{\Phi} e^V \), from eq. (C.23) it is easy to infer the result (C.16).

### C.3 Spinor Propagator

We now consider a chiral spinor coupled to Yang–Mills fields, described by the action (B.21) and compute the propagator by following the same procedure of the scalar case. We start with

\[
\langle \bar{\chi}_+(z) \rangle = \int D\chi_- D\bar{\chi}_- \exp \left[ \int d^4 w \bar{\chi}_-(w) e^V \chi_-(w) \right] \cdot \bar{\chi}_+(z) \quad . \quad (C.24)
\]

and perform a shift \( \bar{\chi}_- \rightarrow \bar{\chi}_- + \delta \bar{\chi}_- \). We obtain

\[
0 = \int D\chi_- D\bar{\chi}_- \exp \left[ \int d^4 w \bar{\chi}_-(w) e^V \chi_-(w) \right] \times
\]

\[
\int d^4 u \delta \bar{\chi}_-(u) e^V \chi_-(u) \bar{\chi}_-(z) + \delta \bar{\chi}_-(z) \quad . \quad (C.25)
\]

Taking the functional derivative \( \delta/\delta \bar{\chi}_-(z') \) and following the same steps as before, we obtain

\[
D_+ e^V \langle \chi_-(z') \bar{\chi}_+(z) \rangle = -D_+ \delta^{(4)}(z' - z) . \quad (C.26)
\]

or (multiplying by \( e^{-V} \) and exchanging \( z \) and \( z' \))

\[
\nabla_+ \langle \chi_-(z) \bar{\chi}_+(z') \rangle = -\nabla_+ e^{-V} \delta^{(4)}(z - z') \quad . \quad (C.27)
\]

Then we apply \( \nabla_+ \nabla_- \). Defining the invertible operator (note that it is different from the corresponding one in (C.13))

\[
i\Box_+ \equiv \nabla_+ \nabla_+ \nabla_- + \nabla_- \nabla_+ \nabla_+ = i[\Box + \frac{1}{2}(\nabla_+ W_- - \nabla_- W_+)] , \quad (C.28)
\]
where $\Box$ is given in (C.14), we obtain
\[
\langle \chi_-(z) \chi_-(z') \rangle = \nabla_+ \frac{i}{\Box_+} \nabla_\pm e^{-V} \delta^{(4)}(z - z').
\] (C.29)

In the text we have also made use of the following operator
\[
i \Box_- \equiv \nabla_+ \nabla_\mp + \nabla_\mp \nabla_+ \nabla_\mp = i \left[ \Box - \frac{1}{2} (\nabla_+ W_\mp + \nabla_\mp W_-) + W_\mp \nabla_\mp \right].
\] (C.30)

## D Evaluation of the 4D consistent anomaly terms

We give here some details in the derivation of (2.23) from the last line in (2.22)
\[
- \int_0^1 dy \int d^8 z \int_0^1 d\beta \frac{1}{M^2} g^{-1} h_2 e^{\beta \Box_+ / M^2} \nabla^2 h_1 e^{(1-\beta) \Box_- / M^2} \nabla^2 \delta^{(8)}(z - z').
\] (D.1)

with $h_2 = g^{-1} \delta g$ and $h_2 = g^{-1} \partial_y g$. As described in the main text, we write
\[
\delta^{(8)}(z - z') = \frac{M^4}{(2\pi)^4} \int d^4 k e^{i M k(x - x')} \delta^{(4)}(\theta - \theta')
\] (D.2)

and we pull the exponential through the various derivatives, after which we take the limit $x' \to x$. We obtain (aside from some explicit $\theta$ terms that can be argued away [14]),
\[
- \frac{M^2}{(2\pi)^4} \int_0^1 dy \int d^8 z \int_0^1 d\beta \frac{1}{M^2} h_2 e^{-\beta [k^2 + \cdots + \nabla^\alpha \nabla_\alpha / M^2 + \frac{1}{2} (\nabla^\alpha \nabla_\alpha / M^2)]} 
\cdot \nabla^2 h_1 e^{-(1-\beta) [k^2 + \cdots + \nabla^\alpha \nabla_\alpha / M^2 + \frac{1}{2} (\nabla^\alpha \nabla_\alpha / M^2)] \nabla^2 \delta^{(4)}(\theta - \theta')
\] (D.3)
\[
\Rightarrow \frac{i}{(2\pi)^4} \int d^4 k e^{-k^2} \int_0^1 dy \int d^8 z \int_0^1 d\beta \left\{ \beta [h_2 \nabla^\alpha \nabla_\alpha + \frac{i}{2} (\nabla^\alpha \nabla_\alpha)] \nabla^2 h_1 \nabla^2 
+ (1 - \beta) h_2 \nabla^2 h_1 [\nabla^\alpha \nabla_\alpha + \frac{i}{2} (\nabla^\alpha \nabla_\alpha)] \nabla^2 \right\} \delta^{(4)}(\theta - \theta')
\]

where we have expanded the exponential, discarded a divergent term which is cancelled by the $z \leftrightarrow z''$ term in the original expression for the anomaly, and kept terms that can contribute in the limit $\theta \to \theta'$. The $k$ and $\beta$ integrations can now be performed.

After using two factors of $\nabla_\alpha$ and $\nabla_\alpha$ to remove the $\theta$'s, and subtracting the contribution with $z$ and $z''$ interchanged (which is equivalent to interchanging $h_1$ and $h_2$), we obtain
\[
\frac{i}{8\pi^2} \int_0^1 dy \int d^8 z \left[ h_2 \nabla^\alpha (\nabla_\alpha h_1) + \frac{1}{2} h_2 (\nabla^\alpha \nabla_\alpha) h_1 - h_2 \nabla^\alpha (h_1 \nabla_\alpha) + \frac{1}{2} h_2 h_1 (\nabla^\alpha \nabla_\alpha) 
- h_1 \nabla^\alpha (\nabla_\alpha h_2) + \frac{1}{2} h_1 (\nabla^\alpha \nabla_\alpha) h_2 + h_1 \nabla^\alpha (h_1 \nabla_\alpha) - \frac{1}{2} h_1 h_2 (\nabla^\alpha \nabla_\alpha) \right]
\] (D.4)
Using the cyclicity of the (group theory) trace, this expression can be simplified to the one given in (2.23).
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