Asymptotic singular behaviour of inhomogeneous cosmologies in Einstein-Maxwell-dilaton-axion theory

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Abstract

We present the study of exact inhomogeneous cosmological solutions to a four-dimensional low energy limit of string theory containing non-minimal interacting electromagnetic, dilaton and axion fields. We analyze Einstein-Rosen solutions of Einstein-Maxwell-dilaton-axion equations and show, by explicitly taken the asymptotic limits, that they have asymptotically velocity-term dominated (AVTD) singularities.

1 Introduction

In the context of the so called string cosmologies there are several aspects of interest. One of them is the way they approach to the singularity as well as the asymptotic behaviour of spacetimes filled with coupled scalar and U(1) fields. Those matter fields arise naturally from low energy effective superstring theory [1] i.e. the Einstein-Maxwell-dilaton-axion (EMDA) system that contains the dilaton which is non-minimally coupled with the Maxwell and axion fields. The EMDA system has been discussed actively in the context of the black hole solutions and singularities, and possesses characteristic features different from the Einstein-Maxwell system [2].

In this paper we investigate the character of initial cosmological singularities in the presence of matter fields arising from the low energy superstring theory.
Belinskii, Khalatnikov and Lifshitz (BKL) [3] conjectured that the dynamics of nearby observers would decouple near the singularity for different spatial points. BKL also speculated that a generic singularity should exhibit such a local oscillatory behavior. There is a special case of the BKL conjecture called the asymptotically velocity-term-dominated (AVTD) singularity that is not described by an infinite sequence of Kasner epochs but by only one epoch. In this case no oscillatory approach to the singularity is observed. The characteristic feature of the AVTD singular behaviour is that all spatial derivative terms of the field equations are negligible sufficiently close to the singularity.

Another aspect of interest is to investigate the influence of nonminimal exponential coupling of the dilaton and axion to the $U(1)$-field on the character of the singularities. As it has been shown in [3] and [4] the minimally coupled scalar field can suppress the oscillatory behaviour; on the other hand, it has been shown numerically that magnetic Gowdy spacetimes are not AVTD but mixmaster [5]. It is therefore important to study the behavior of models containing both, scalar and electromagnetic fields, in the limit $t \to 0$.

Our aim is to verify BKL conjecture in non-vacuum and spatially inhomogeneous cases. To this end we shall consider Einstein-Rosen spacetimes.

The Einstein-Rosen models are important among other things because many of the non-perturbatively exact superstring backgrounds constructed from the gauged Wess-Zumino-Witten (WZW) models admit two Abelian isometries [6]. Hence, given our current knowledge of string cosmology and conformal field theory, the $G_2$ Einstein-Rosen cosmologies derived within the context of the low energy effective action represent a set of models that is closely related to many exact string solutions in four dimensions. Moreover, the Einstein-Rosen solutions include a number of spatially homogeneous Bianchi models as special cases and certain $G_2$ models may be viewed as inhomogeneous generalizations of those Bianchi cosmologies.

One subtle point in this treatment is that we are extrapolating asymptotic behavior of solutions that are valid in the weak coupling regime to regions near the big-bang singularity, for instance. In the strong coupling regime one expects that higher-order corrections to the perturbative theory, as well as non-perturbative string effects, should become increasingly important. Hence, in this regime the qualitative behavior of these solutions may deviate somewhat from that of solutions derived from the full M or string theory. However, there are reasons to believe that $G_2$ solutions should nevertheless provide a generic description of cosmological models in the vicinity of
a singularity. A major incentive for this comes from the long standing conjecture of Belinskii and Khalatnikov [3]. This states that on the approach to the cosmological singularity, the generalized Einstein-Rosen metrics may play the role of the leading-order approximation to the general solution of conventional Einstein gravity.

This paper is structured as follows: We present the EMDA field equations and solutions in Secs. 2 and 3, respectively. In Sec. 3 we identify the previous solution as an Einstein-Rosen spacetime and determine its kinematical parameters. We analyze the Raychaudhuri equation to verify the existence of singularity in Sec. 4. In Sec. 5 we perform the asymptotic analysis of the metric and the fields, and show that the approach to the singularity is of the form AVTD. In Sec. 6 we present the plane wave and cylindrically symmetric cases and show their approaching to $t = 0$. Finally, some conclusions are given in Sec. 7.

### 2 EMDA Einstein-Rosen solution

As we pointed out above, we shall consider solutions to the low energy effective theory for the heterotic string that arises as a dimensional reduction and truncations of the string theory in four dimensions under the following considerations: compactifications of six of the ten dimensions and omission of the arising massless fields in the obtained heterotic structure; only $U(1)$ charges are permitted. The dynamical equations of the resulting theory can be deduced from the action

$$S = \int d^4x \sqrt{-g} \{ R - 2(\partial \phi)^2 - \frac{e^\phi(\partial \kappa)^2}{2} - e^{-2\phi} F^2 - \kappa F_{\mu\nu} \tilde{F}^{\mu\nu} \},$$

(1)

where $R$ is the scalar Riemann curvature, $g_{\mu\nu}$ is the four-dimensional metric tensor, $F_{\mu\nu}$ is the electromagnetic antisymmetric tensor field, $\tilde{F}_{\mu\nu}$ its dual ($\tilde{F}_{\mu\nu} = -\frac{1}{2} \sqrt{-g} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$), $\phi$ is the dilaton scalar field and $\kappa$ is the axion field dual to the three index antisymmetric tensor field $H = -\frac{1}{2} e^{4\phi} \ast d\kappa$. The Einstein frame is related to the string frame by a conformal transformation: $G_{\mu\nu}(s) = e^{2\phi} g_{\mu\nu}(E)$, where $s$ and $E$ stand for string and Einstein, respectively.

Previous work on these lines includes solution generating techniques which permit to construct exact inhomogeneous solutions to the equations of low
energy string theory containing nonminimally coupled dilaton and electromagnetic fields (EMD) presented in [7]. Solutions to Einstein-dilaton-axion theory have been derived by applying the global symmetries of the string effective action to a generalized Einstein-Rosen metrics in [8]. An algorithm for generating families of inhomogeneous spacetimes with a massless scalar field was presented in [9].

In the Einstein-Maxwell-dilaton-axion (EMDA) theory, the dilaton $\phi$, axion $\kappa$, gravitational $g_{\mu\nu}$ and electromagnetic Maxwell $F_{\mu\nu}$ fields ought to fulfill the EMDA field equations derived from (1):

$$T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R,$$

$$T_{\mu\nu} = 2 \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} e^{4\phi} \partial_\mu \kappa \partial_\nu \kappa + 2 e^{-2\phi} [F_{\mu\sigma} F_{\nu}^{\sigma} - \frac{1}{4} g_{\mu\nu} F^2] - g_{\mu\nu} [\partial^\alpha \phi \partial_\alpha \phi + \frac{1}{4} e^{4\phi} \partial^\alpha \kappa \partial_\alpha \kappa],$$

$$0 = \nabla_\mu (e^{4\phi} \partial_\mu \kappa) - F_{\mu\nu} \tilde{F}^{\mu\nu},$$

$$J_\nu := \nabla_\mu (\kappa \tilde{F}^{\mu\nu} + e^{2\phi} F^{\mu\nu}).$$

In the next section we present EMDA solutions for metrics that possess two spacelike Killing vectors. The fundamental structural functions of these solutions are rational functions expressible as the ratio of polynomials of at most second degree in the coordinate variables. These solutions were obtained by straightforward integration of Eqs. (2)-(5) in [10].

### 2.1 Metric functions, fields and Weyl scalars

The metric, endowed with two Killing vectors: $\partial_x$ and $\partial_y$, is proposed in the form:

$$ds^2 = \frac{\Delta}{P} dp^2 - \frac{\Delta}{Q} dq^2 + \frac{P}{\Delta}(dx + Ndy)^2 + \frac{Q}{\Delta}(dx + Mdy)^2,$$

The solutions are given by
\[ P = \epsilon p^2 + 2np + \alpha, \]
\[ Q = \epsilon q^2 + 2mq - \alpha, \]
\[ M = \nu p^2 + 2bp, \]
\[ N = -\nu q^2 + 2\beta q, \]  \tag{7}

together with \( \epsilon = 1, 0, -1, \nu = 1, 0, -1, \Delta = M - N \). Being \( m, n, b, \beta, \alpha \) constants. There are distinct families of solutions depending on the value of the kinematical constant \( \epsilon = 1, 0, -1 \). The corresponding matter fields are given by

\[ e^{2\phi}(p, q) = \frac{\omega(p^2 + q^2)}{\Delta}, \]  \tag{8}
\[ \kappa(p, q) = \frac{2(bq + \beta p)}{\omega(p^2 + q^2)} + \kappa_0, \]  \tag{9}
\[ A_x(p, q) = -\frac{g_0 q - g_0 p}{\Delta}, \]  \tag{10}
\[ A_y(p, q) = \frac{\nu pq(q_0 p + g_0 q)}{\Delta}, \]  \tag{11}

where \( \kappa_0 \) and \( \omega \) are axion and dilaton parameters, respectively; while \( A_\mu \) is the electromagnetic potential and \( g_0 \) and \( q_0 \) are constants related to the electromagnetic field. To fulfill the EMDA equations, the constants \( m, n, b, \beta, \alpha, g_0, q_0, \kappa_0, \omega, \epsilon, \nu \) are restricted to satisfy the following algebraic conditions:

\[ \nu^2 q_0^2 = 2\omega \beta(\beta \epsilon + m\nu), \quad g_0 \beta - q_0 b = 0, \]  \tag{12}
\[ \nu^2 g_0^2 = 2\omega b(\beta \epsilon - n\nu), \quad n\beta + mb = 0, \]  \tag{13}

This constrictions arise in the process of solving the field equations with the metric functions proposed as polynomials. As a result, not all the constants can be considered as free parameters; from the eleven constants only five of them are free parameters.

The nonvanishing Weyl scalars are given by;

\[ \psi_1 = \psi_3 = -i\frac{\sqrt{PQ}}{2\Delta^3}(b^2 + \beta^2), \]  \tag{14}
\[
6\Delta^3\psi_2 = -\{(6\nu(n\nu - \epsilon b)[p(p^2 - 3q^2) + iq(q^2 - 3p^2)]
- 6\nu(m\nu + \epsilon\beta)[q(q^2 - 3p^2) - ip(p^2 - 3q^2)]
+ 2[2\epsilon(b^2 + \beta^2) + 3\nu(m\beta - nb)][(q^2 - p^2)
+ 12ipq[(be - n\nu)b + (\nu m + \epsilon\beta)\beta]
+ 4(nb - m\beta)(bp + \beta q) + 4\alpha(\beta^2 + b^2)}\right\},
\] (15)

Since the three Weyl scalars \(\psi_1, \psi_2, \psi_3\) are nonzero, the solution is of type G in the Petrov classification. From the above expressions it can be seen that it is the presence of the constants \(b\) and \(\beta\) (non-diagonal terms in (6)) that gives such structure, otherwise being a type D metric without axionic nor dilatonic fields; these fields become constant when \(b = 0 = \beta\). The invariants that can be constructed from the Weyl scalars are \(C^{(2)} = 6(\psi_2)^2 - 8\psi_1\psi_3\) and \(C^{(3)} = 12\psi_2\psi_1\psi_3 - 6(\psi_2)^3\). It can also be noted that such invariants will diverge provided the Weyl scalars do. This is the case if the function \(\Delta = M - N\) is zero, then essential singularities arise at those points.

3 Einstein-Rosen spacetimes

The line element (6) can be transformed into the generalized Einstein-Rosen form:

\[
ds^2 = e^f(\gamma_{ab}dx^adx^b),
\] (16)

where \(x^a = x, y\). All components are independent of the spatial coordinates \((x, y)\); \(f = f(z, t)\) determines the longitudinal part of the gravitational field. The metric of the surfaces of transitivity is \(\gamma_{ab}\) and the gradient \(K_{\mu} = \partial_{\mu}(det\gamma_{ab})^{1/2}\) determines the local behaviour of the model. If \(K_{\mu}\) is globally spacelike or null, the solutions represent cylindrical and plane gravitational waves, respectively. The cosmological models are characterized by a timelike \(K_{\mu}\) or when the sign of \(K_{\mu}K^{\mu}\) changes.

In terms of the metric functions \(\Delta, P\) and \(Q\) of (6) the Einstein-Rosen metric is

\[
ds^2 = \Delta(dz^2 - dt^2) + \frac{G}{\Delta}\{\chi(dx + Ndy)^2 + \chi^{-1}(dx + Mdy)^2\}
\] (17)

where \(\chi = \sqrt{P/Q}\) and \(G = \sqrt{PQ}\). The appropriate coordinate transformation \((p, q) \leftrightarrow (z, t)\) that leads from (6) to (17) depends on the values of
the constants $\epsilon, n, m, \alpha$. In Table 1 some cases are shown.

| $\epsilon$ | $p$ | $q$ | conditions |
|-----------|-----|-----|------------|
| 1         | $\sqrt{\alpha - n^2} \sinh z + n$ | $\sqrt{\alpha + m^2} \cosh t + m$ | $\alpha - n^2 > 0$, $\alpha + m^2 > 0$ |
| 1         | $e^z/2 - n$ | $e^t/2 - m$ | $\alpha = n^2$, $m^2 = -m^2 = 0$ |
| -1        | $\sqrt{\alpha + n^2} \sin z + n$ | $\sqrt{m^2 - \alpha} \sin t + m$ | $\alpha + n^2 > 0$, $m^2 - \alpha > 0$ |
| 1         | $e^z/2 - n$ | $\sqrt{\alpha + m^2} \cosh t - m$ | $\alpha = n^2$ |
| 1         | $\sqrt{n^2 + m^2} \cosh z + n$ | $e^t/2 - m$ | $\alpha = -m^2$ |
| 0         | $\frac{e^{z^2 - \alpha}}{2n}$ | $\frac{e^{t^2 + \alpha}}{2m}$ | $-1$ |

Table 1: Coordinate transformations $(p, q) \rightarrow (z, t)$ to pass from the line element (17) to the Einstein-Rosen form (17).

The calculus of the gradient of the transitivity surface shows that the three possibilities of interpreting the solution (17) are available as particular cases. We are interested in particular in the cosmological model obtained with the transformation $(p, q) \mapsto (z, t)$ given by

$$
p \mapsto \sqrt{\alpha - n^2} \sinh z - n, \quad q \mapsto \sqrt{\alpha + m^2} \cosh t - m.
$$

This transformation in the line element (17) gives

$$
\Delta ds^2 = \Delta^2 (dz^2 - dt^2) + (\alpha - n^2) \cosh^2 z(dx + Ndy)^2 + (\alpha + m^2) \sinh^2 t(dx + Mdy)^2,
$$

where $\Delta = M(z) - N(t)$ is a monotonically increasing function on $z$ and $t$, given by

$$
\Delta = \nu(\sqrt{\alpha - n^2} \sinh z - n)^2 + 2b(\sqrt{\alpha - n^2} \sinh z - n) + \nu(\sqrt{\alpha + m^2} \cosh t - m)^2 - 2\beta(\sqrt{\alpha + m^2} \cosh t - m).
$$

In our case $\Delta \neq 0$ over the whole spacetime and it implies that the invariants are regular as well. The fact that the invariants are regular and that the
spacetime has a nonvanishing acceleration, as it is shown in the next section, might be an indicative of absence of singularity. A nonvanishing acceleration may create a gradient of pressure which acts opposing gravitational attraction and this fact allows to avoid the focusing of the congruence in the Raychaudhuri equation. A singularity-free spacetime due to its acceleration was given in [11]. In the next subsection we characterize the cosmological model using kinematical parameters.

3.1 Kinematical parameters

The kinematical characteristics of the Einstein-Rosen spacetime (17) are determined with respect to the timelike vector $u^a = \frac{1}{\sqrt{\Delta}}\delta^a_t$, $u^au_a = -1$. The acceleration, expansion, deceleration and shear are given, respectively, by:

$$\dot{u}^a_z = \frac{M'}{2\Delta^2},$$

$$\Theta = \frac{\dot{Q}\Delta - \dot{N}Q}{2\Delta^{3/2}Q},$$

$$q = \frac{1}{(\dot{Q}\Delta - \dot{N}Q)^2}\{8\dot{N}Q^2 + 5\Delta^2\dot{Q}^2 - \Delta Q\dot{N}\dot{Q} + 6\Delta Q(Q\ddot{N} - \Delta\ddot{Q})\},$$

$$6\sigma^2 = \frac{\dot{Q}^2}{\Delta Q^2} + \frac{4\dot{Q}\dot{N}}{\Delta^2 Q} + \frac{\dot{N}^2(3P + 4Q)}{\Delta^3 Q},$$

(21)

being null the rotation or vorticity, $\omega_{ab} = 0$. For the cosmological model the kinematical parameters behave asymptotically as follows

$$\Theta(t \to 0) \to \infty, \quad \Theta(t \to \infty) \to 0.$$  (22)

$$\sigma(t \to 0) \to \infty, \quad \sigma(t \to \infty) \to 0.$$  (23)

$$q(t \to 0) \to \text{const}, \quad q(t \to \infty) \to \text{const}.$$  (24)

This behaviour is illustrated in Fig 1. The shear evolves towards isotropy for large times, $\sigma^2 \to 0$. The expansion, being infinite at the origin of time, it decreases as time passes. The deceleration is positive all the time and tends to be constant, this indicates that the spacetime expands in non accelerated way. Both, shear and expansion diverge at $t = 0$, indicating the presence of
Figure 1: Illustration of the behaviour of kinematical parameters for the cosmological solution. Deceleration parameter, $q$, shear, $\sigma$, and the expansion, $\Theta$, are shown. For this plot the constants have been fixed to $m = n = 2$, $b = -\beta = 1$, $\alpha = 5$ and $\nu = 1$.

a singularity; the analysis of the Raychaudhuri equation in the next section confirms that the spacetime has a spacelike singularity at $t = 0$.

4 Raychaudhuri equation

According to Raychaudhuri, the equation that governs the rate of change of expansion of timelike congruences, $\Theta$, is

$$\frac{d\Theta}{d\lambda} = -R_{ab}u^a u^b + 2\omega^2 - 2\sigma^2 - \frac{\Theta^2}{3} + \dot{u}_{,\alpha}^\alpha \quad (25)$$

where $u^\alpha$ is a tangent vector and $\lambda$ is the affine parameter of the congruence, $R_{ab}$ is the Ricci tensor, $\omega$ is the rotation or vorticity, $\sigma$ is the shear and $\dot{u}_{,\alpha}^\alpha$ is the acceleration of the congruence.

From Eq. (25) it can be seen that the expansion $\Theta$ of a timelike geodesic congruence with zero vorticity will monotonically decrease along a geodesic if, for any timelike vector $V^a$, $R_{ab}V^aV^b \geq 0$, i.e. if the strong energy condition (SEC) is hold. The term involving the Ricci tensor, $R_{ab}$, in Eq. (25) induces contraction of the geodesic lines, indicating that the focusing of neighbouring geodesics is unavoidable if SEC is fulfilled since the other terms on the right hand side of (25) are also negative, with exception of the acceleration;
however, as we shall see the positive term of the acceleration is defeated by
the term $R_{ab}u^au^b$. Substituting from (21) in Eq. (25) we obtain:

$$
R_{ab}u^au^b = \frac{1}{4Q^2\Delta}(\dot{Q}^2 - 2Q\ddot{Q}) - \frac{1}{2\Delta^3}(M'^2 + \frac{P}{Q}\ddot{N}^2)
+ \frac{1}{4\Delta^2}(M'\frac{P'}{P} - 3\dot{N}\frac{\dot{Q}}{Q} + 2M'' + 2\ddot{N}),
$$

(26)

Moreover, from the expression for $\dot{\Theta} = d\Theta/d\lambda$, Eq. (25), it can be seen
that in fact it diverges as $t$ approaches the origin provided the metric function
$Q$ be divergent,

$$
\dot{\Theta} = \frac{1}{2\Delta}\{\ddot{Q} - \frac{Q\dddot{Q}}{Q^2} + \frac{\dot{Q}\dot{N}}{2Q\Delta} - \frac{3\dot{N}^2}{2\Delta^2} - \frac{\dddot{N}}{\Delta}\}.
$$

(27)

In the cosmological case, $Q = (\alpha + m^2)\sinh^2 t$, therefore as $t \to 0,$
$\dot{\Theta} \approx -1/\sinh^2 t \to -\infty$, revealing the existence of a singularity as $t \to 0.$
This behaviour can be confirmed by writting the derivatives of the coordinates
with respect to an affine parameter and checking that their coefficients
diverge. By showing that the first derivatives are unbounded one concludes
that the corresponding geodesic curves $t(\tau), x(\tau), y(\tau), z(\tau)$, are incomplete
at $t = 0$ [12].

5 Asymptotic Behaviour

In the limit $t \to 0$ the fields become functions that depend on $z$ but otherwise
are constant,

$$
e^{2\phi}(t \to 0) \to \frac{\omega(A^2 + (\sqrt{\alpha + m^2} - m)^2)}{\Delta},
$$

$$
\kappa(t \to 0) \to \frac{2b(\sqrt{\alpha + m^2} - m) + 2\beta A}{\omega(A^2 + (\sqrt{\alpha + m^2} - m)^2)} + \kappa_0,
$$

$$
A_x(t \to 0) \to \frac{g_0A - g_0(\sqrt{\alpha + m^2} - m)}{\Delta},
$$

$$
A_y(t \to 0) \to \frac{\nu A(\sqrt{\alpha + m^2} - m)(g_0A + g_0(\sqrt{\alpha + m^2} - m))}{\Delta},
$$

(28)
In relation to the asymptotic behaviour as $t \to \infty$, both scalar fields, axion and dilaton tend to constant values as $t$ approaches infinity, therefore the fields decouple for large times,

$$
\begin{align*}
\phi(t \to \infty) & \to \frac{1}{2} \ln \frac{\omega}{\nu}, \\
\kappa(t \to \infty) & \to \kappa_0, \\
A_x(t \to \infty) & \to 0, \\
A_y(t \to \infty) & \to A_{g_0},
\end{align*}
$$

(29)

There exists a lower bound on the value of the dilaton field and this implies the existence of a lower, non-vanishing bound on the string coupling which, in the context of M theory, in turn implies the existence of a lower bound on the radius of the eleventh dimension [13]. Moreover, in the limits that the volume of the transverse space measured in the string frame becomes vanishingly small or arbitrarily large the axion field is constant, since $\Gamma = \det \Gamma_{ab} = e^{\phi} P Q$,

$$
\begin{align*}
\lim_{t \to 0} \Gamma & = (\alpha - n^2)(\alpha + m^2)(\cosh^2 z) t^2 \to 0, \quad \kappa \to \text{const}, \\
\lim_{t \to \infty} \Gamma & = \frac{\omega}{4\nu} (\alpha - n^2)(\alpha + m^2)(\cosh^2 z) e^{2t} \to \infty, \quad \kappa \to \kappa_0.
\end{align*}
$$

(30)

Thus the two-form potential effectively decouples from the field equations in these limits.

As was mentioned at the beginning, one of the most interesting aspects of analyzing cosmological models is to dilucidate how the spacetime behaves near the initial singularity and to determine if this is an oscillatory approach or a AVTD behaviour.

Narita, Torii and Maeda [14] studied the influence of the exponential dilaton-electromagnetic coupling on the character of initial singularities. They considered EMDA system with a $T^3$ Gowdy cosmology and using the Fushian algorithm they showed that those spacetimes have in general asymptotic velocity-term dominated singularities. Their results mean that the exponential coupling of the dilaton to the Maxwell field does not change the nature of the singularity. In what follows we arrive to the same conclusion for the Einstein-Rosen spaces by taking directly the limit $t \to 0$ in the metric (19).

Taking the limit $t \to 0$ with $z =$ constant, in the metric functions of (19) for the studied case we obtain the line element
\[ ds^2 = k(dz^2 - dt^2) + \frac{P(z)}{k}(dx + N dy)^2 + \frac{(\alpha + m^2)t^2}{k}(dx + M dy)^2, \quad (31) \]

where \( k = \text{const.} \) and the metric functions are (remind that \( z \) has been fixed to constant)

\[
\begin{align*}
    P(z) &= (\alpha - n^2) \cosh^2(z), \\
    M(z) &= \nu[\sqrt{\alpha - n^2}\sinh(z) - n]^2 + 2b[\sqrt{\alpha - n^2}\sinh(z) - n], \\
    N &= -\nu[\sqrt{\alpha + m^2} - m]^2 + 2b[\sqrt{\alpha + m^2} - m]. \quad (32)
\end{align*}
\]

The line element (31) corresponds to a Kasner metric with \( p_1 = p_3 = 0 \) and \( p_2 = 1 \). This establishes the AVTD behaviour of this model, extending the conclusion by Narita et al to Einstein-Rosen spaces.

### 6 Plane-symmetric Wave and Cylindrical spacetime

To establish a comparison of the behavior near the singularity, we take the limit as \( t \to 0 \) in the plane-symmetric wave spacetime and the cylindrical one associated to the metric (17). As quoted in Sec. 3 the local behavior of the Einstein-Rosen spacetime has to do with the character of the gradient of the transitivity surface. The spacetime (17) can be interpreted as a plane-symmetric wave space by using the coordinate transformation

\[ p \mapsto \exp z/2, \quad q \mapsto \exp t/2, \quad (33) \]

then the metric (17) takes the form

\[ ds^2 = \Delta(dz^2 - dt^2) + \frac{e^{2z}}{4\Delta}(dx + N dy)^2 + \frac{e^{2t}}{4\Delta}(dx + M dy)^2. \quad (34) \]

This metric behaves as a conformally plane spacetime when \( t \to 0 \):

\[ ds^2 = k(dz^2 - dt^2) + \frac{A^2}{4k}(dx + N_1 dy)^2 + \frac{1}{4k}(dx + M_1 dy)^2. \quad (35) \]

where \( A^2 = \exp 2z, \quad M_1 = \nu^2 \exp 2z/4 + \beta \exp z \) are constants for \( z = \text{const.} \) and \( N_1 = -\nu^2/4 + b \). The metric is completely regular at \( t \to 0 \); besides,
neither $\Theta$ nor $\sigma$ diverge at $t = 0$, in contrast with the cosmological case. Investigating the tendency of the change in expansion along a congruence, $\dot{\Theta}$, we found that it is finite for all the time range. However the possibility exists of a fine tuning of the axion and dilaton parameters, $b$ and $\beta$, that make $\Delta = 0$ and in that case the spacetime becomes singular.

The cylindrically symmetric case arises if instead we make the coordinate transformation,

$$ p \mapsto \sqrt{m^2 + n^2} \cosh z - n, \quad q \mapsto \exp t/2 - m. \quad (36) $$

In this case the norm of the gradient of the transitivity surface is always positive, $G_aG^a = e^{2t}(m^2 + n^2)/(4\Delta)$ and the spacetime can be interpreted as a cylindrical symmetric one; the form of the metric is:

$$ ds^2 = \Delta(dz^2 - dt^2) + \frac{(m^2 + n^2) \sinh^2 z}{\Delta} (dx + N dy)^2 + \frac{e^{2t}}{4\Delta} (dx + M dy)^2. \quad (37) $$

In this case the singularity at $z = 0$ is timelike.

7 Conclusions

We studied AVTD cosmological Einstein-Rosen spacetimes with EMDA fields. The matter fields show a tendency to decoupling for large times. Near the singularity the dynamics at different spatial points decouples and the metric has a spatially varying Kasner form. We conclude that the nonminimally coupling of the dilaton and axion to the Maxwell field does not change the nature of the singularity. In this sense we have extended to generalized Einstein-Rosen spaces the result, founded in Gowdy spaces by NTM, that exponential coupling of the scalar field does not necessarily lead to Mixmaster behavior.

Further analysis on solutions with plane-wave and cylindrical symmetry might be of interest as roughly is shown in Sec. 6. The asymptotics of the fields resemble the cosmological case, however, each model approaches $t = 0$ in a very different way.

References

[1] Shapere, A., Trivedi, S., Wilczek, F.: Dual Dilaton Dyons, Mod. Phys. Lett. A6 2677 (1991).
[2] Maeda, K., T. Torii, T., Narita, M.: Do naked singularities generically occur in generalized theories of gravity?, Phys. Rev. Lett. 81 5270 (1998).

[3] Belinskii, V., Khalatnikov, I.: Effect of scalar and vector fields on the nature of the cosmological singularity, Sov. Phys. JETP 36 591, (1973). Belinskii, V., Khalatnikov, I., Lifshitz, E.: A general solution of the Einstein equations with a time singularity, Adv. Phys.31 639 (1982).

[4] Berger, B.: Influence of scalar fields on the approach to a cosmological singularity, Phys. Rev. D 61 023508-1, (1999).

[5] Weaver, M., Isenberg, J., Berger, B. K.: Mixmaster behavior in inhomogeneous cosmological spacetimes, Phys. Rev. Lett. 80, 2984 (1998). Berger, B. K.: Hunting local mixmaster dynamics in spatially inhomogeneous cosmologies, Class. Quantum Grav. 21 581 (2004).

[6] Tseytlin, A. A.: Exact solutions of closed string theory, Class. Quantum Grav. 12 2365 (1995).

[7] Yazadjiev, S. S.: Exact inhomogeneous Einstein-Maxwell-Dilaton cosmologies, Phys. Rev. D 63 063510, (2001).

[8] Clancy, D., Feinstein, A., Lidsey, J. E., Tavakol, R.: Inhomogeneous Einstein-Rosen string cosmology, Phys. Rev D, 60, 043503, (1999).

[9] Lazkoz, R.: $G_1$ spacetimes with gravitational and scalar waves, Phys. Rev. D 60 104008 (1999).

[10] Breton, N.: Exact solutions in Einstein-Maxwell-Dilaton-Axion Theory, in Recent Developments in Gravitation, Proceedings of ERE99, Ed. by J. Ibáñez, Univ. País Vasco, España (2000), 179-184.

[11] Chinea, F. J., Fernández-Jambrina, L., Senovilla, J. M. M.: Singularity-free space-time, Phys. Rev. D 45, 481 (1992).

[12] Arnold, V. I.: Ordinary Differential Equations, (MIT Press, 1990).

[13] Witten, E.: String theory dynamics in various dimensions, Nucl. Phys. B443, 85 (1995).
[14] Narita, M., Torii, T., Maeda, K.: Asymptotic singular behaviour of Gowdy spacetimes in string theory, Class. Quantum Grav. 17 4597 (2000).