Jacobians of chiral transformations and
two-dimensional bosonization

Jan B. Thomassen

Institut für Kernphysik, Technische Universität Wien
A–1040 Vienna, Austria

February 3, 1999

Abstract

We formulate a complete path integral bosonization procedure for any fermionic theory in two dimensions. The method works equally well for massive and massless fermions, and is a generalization of an approach suggested earlier by Andrianov. The classical action of the bosons in the bosonized theory is identified with $-i$ times the logarithm of the Jacobian of a local chiral transformation, with the boson fields as transformation parameters. Three examples, the Schwinger model, the massive Thirring model and massive non-Abelian bosonization, are worked out.

PACS numbers: 11.10.Kk; 11.15.Tk; 11.30.Rd

Keywords: Bosonization; Two-dimensional field theory; Chiral symmetry

1 Introduction

The derivation of the chiral Lagrangian of the strong interactions from QCD is still an open question. This Lagrangian describes the pseudoscalar octet – the $\pi$’s, $K$’s and the $\eta$ – in a way that is consistent with the scenario of spontaneously broken chiral symmetry with the pseudoscalars as the Goldstone bosons. Whatever the details of such a derivation may be, it is clear that one of its ingredients must be bosonization, simply because the $\pi$’s, $K$’s and $\eta$ are bosons.

There exists a physical and intuitively clear picture of how the chiral Lagrangian arises from QCD [2–4]. In ref. [2] Diakonov and Eides gave a formula for the classical action $W$ of a given configuration of pseudoscalar mesons $\pi^a(x)$ in the chiral limit. In our notation

$$e^{iW[\pi]} = \frac{\int DBDqD\bar{q} \exp i \left( S[B] + \int d^4x \bar{q} i D q \right)}{\int DBDqD\bar{q} \exp i \left( S[B] + \int d^4x \bar{q} e^{iH_{\gamma_5}} i D e^{iH_{\gamma_5}} q \right)},$$

where $B_\mu$ is the gluon field, $S$ is the gluon action, $q$ are the $u$, $d$, and $s$ quarks and $\Pi \equiv \pi^a \tau^a / f$, with $f = 93$ MeV the pseudoscalar decay constant. It is understood that the quarks both in

Supported by Fonds zur Förderung der wissenschaftlichen Forschung, P11387–PHY

E-mail: thomasse@kph.tuwien.ac.at

1 Compared to refs. [3, 4], the ratio in eq. (1) turns up side down – see below.
the numerator and in the denominator are regularized with a (Wess–Zumino consistent) vector current conserving scheme. This renders the Jacobian $J[\pi]$ of a local chiral rotation non-trivial, i.e. $W[\pi]$ is non-zero \[2\], and we can make the identification
\[
e^{iW[\pi]} = J[\pi]. \tag{2}
\]

The physical meaning of eq. (1) \[3\] is that the classical action $W$ of the pseudoscalars is the difference in “free energy” between chirally rotated QCD and QCD without this rotation. The pseudoscalars, being the parameters of the chiral rotations, are thus identified as Goldstone bosons of spontaneously broken chiral symmetry.

Eq. (1) has been taken as the basis for deriving a chiral Lagrangian from a cut-off version of QCD \[2–4\]. Furthermore, an interesting connection between the underlying idea of this formula and bosonization was pointed out \[3, 4\]. In particular, Andrianov noted that the Jacobian of a chiral rotation of non-Abelian fermions realizes bosonization in two dimensions (in the chiral limit) \[4\]. This motivates us to study this particular approach to two dimensional bosonization.

Of course, the literature on two dimensional bosonization is extensive, even if one restricts oneself to “path integral methods”. However, although these methods uses some of the same ingredients (Jacobians of chiral transformations, identification of the bosonic variables with parameters of chiral rotations, etc.) they are essentially different when massive fermions are considered. For example, some approaches uses path integrals in combination with expansions in Green’s functions. These can not be generalized to four dimensions since Green’s functions cannot in general be calculated explicitly in that case. This mass-term problem justifies a systematic treatment of Andrianov’s scheme.

In this paper, then, we formulate a complete path integral bosonization procedure for any fermionic theory in two dimensions, based on Jacobians of chiral rotations. It is a generalization of the bosonization procedure in the sense of Andrianov and can be used also for fermionic models with mass-terms. That is, fermions coupled to Abelian or non-Abelian $V$, $A$, $S$ and $P$ sources can be bosonized with this procedure. The bosonization is complete in the sense that a tensor is equivalent to a scalar in two dimensions.

The paper is organized as follows. In sec. 2 we formulate the bosonization procedure and give a general bosonization formula. We also discuss the sign of the action $W$. The ratio in eq. (1) is turned up side down compared to refs. \[3, 4\] in order to produce the right sign \[4, 4\], and we give a physical argument for why the ratio should be turned the way it does in eq. (1). We work out some examples in sec. 3 to demonstrate the power and correctness of the procedure. In sec. 4 we discuss some properties of the procedure and speculate on generalizations to four dimensions. General expressions in Minkowski space for the Jacobians of a chiral rotation are given in an appendix.

2 The bosonization procedure

Before we consider bosonization we give a physical interpretation of the sign of $W$ in eq. (1). As we have mentioned, the sign should be reversed \[4, 4\] compared to the expressions in refs. \[2, 3\].

A Grassmann integral over fermion fields describes a second quantized system of fermions with a filled Dirac sea\[2\]. The Dirac sea constitutes a perturbative vacuum for the fermions but is not necessarily the ground state. Thus, when the system is chirally rotated – which disturbs the Dirac sea – it is perfectly possible that the “free energy” is lowered, rather than increased. This is apparently what happens, at least for QCD and the two-dimensional models we shall

\[2\]See e.g. ref. \[7\] for a related discussion of the Dirac sea in the context of chiral anomalies.
Therefore, if we wish to describe the pseudoscalars with a positive “free energy” we must turn the ratio the way it does in eq. [3].

Let us now turn to two dimensional bosonization. The basic formula which we must consider is

\[ e^{iW[\theta]} = \frac{\int D\psi D\bar{\psi} \exp i \int d^2x \bar{\psi}[i\partial - \Gamma]\psi}{\int D\psi D\bar{\psi} \exp i \int d^2x \bar{\psi} e^{i\theta \gamma_5}[i\partial - \Gamma]e^{i\theta \gamma_5}\psi} \]  

where \( \psi \) is a Dirac fermion, \( \Gamma = V + A\gamma_5 + S + i\gamma_5P \) are external \( V, A, S \) and \( P \) sources and \( \theta \) is the pseudoscalar field – the parameter of chiral rotations. In the non-Abelian case \( \psi \) is in the fundamental representation of, say, \( SU(N) \) and all quantities are matrices, \( \theta \equiv \theta^a t^a \), etc.

The functional \( W[\theta] \) is, as we have noted, the classical action for the pseudoscalars. Thus, in order to get a quantum theory, we must path integrate over \( e^{iW[\theta]} \), leading to the partition function of the bosonized theory:

\[ Z[V,A,S,P] = \int D\theta e^{iW[\theta;V,A,S,P]} \]

where we have explicitly displayed the \( V, A, S \) and \( P \) dependence of \( W \). The integration measure should be properly normalized and have the appropriate invariance properties. For example, in the non-Abelian case we should use the Haar measure. The quantum theory of \( \theta \) is not automatically regularized. That part is left to the physicist who wishes to do calculations with the bosonic theory.

The bosonization procedure is then as follows. Calculate the Jacobian \( J[\theta] = e^{iW[\theta]} \) of the finite local chiral rotation with \( \theta(x) \) as the rotation angles. Then the partition function of the bosonized theory is found by path integrating \( J[\theta] \) over \( \theta \), using the appropriate measure for the integration.

It is important that we use a consistent regularization scheme that conserves vector currents in order to produce the correct action for the bosons. More precisely, we should use the most general regularization scheme with this requirement. We should not, for example, choose specific simplifying values for the free parameters of the scheme because we would then lose information. We will see examples of this in the next section.

For convenience and completeness we give the full Minkowski space expressions for the chiral Jacobians in an appendix.

### 3 Examples

We can now demonstrate that this bosonization procedure reproduces known results in the cases of the Schwinger model, the massive Thirring model and massive non-Abelian bosonization.

i) The Schwinger model

Due to the structural simplicity of this model, this example could be regarded as a warm-up. The partition function is

\[ Z = \int DAD\psi D\bar{\psi} \exp i \int d^2x \left( -\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}[i\partial - eA]\psi \right) \]

We choose Lorentz gauge for the photon field, \( \partial_\mu A^\mu = 0 \). Performing a chiral transformation of the fermion gives the Jacobian

\[ J[\theta] = \exp i \int d^2x \left( \frac{1}{2\pi} \partial_\mu \partial^\mu \theta - \frac{e}{\pi} A_\mu \epsilon^{\mu\nu} \partial_\nu \theta \right) \]
We bosonize the fermionic part of the partition function by replacing it by this Jacobian, leading to

\[
Z = \int DA \int D\theta \exp i \int d^2x \left( -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2\pi} \partial_\mu \theta \partial^\mu \theta - \frac{e}{\pi} A_\mu \epsilon^{\mu\nu} \partial_\nu \theta \right)
\]

\[
= \int DA \exp i \int d^2x \left( -\frac{1}{4} F_{\mu\nu}^2 + \frac{e^2}{2\pi} A^2 \right),
\]

which is appropriate for the description of a vector field with mass \( m = e/\sqrt{\pi} \).

\( \text{ii) The massive Thirring model [9–12]} \)

We will only consider the mass-term bosonization rules for simplicity and because it is the hardest part in the path integral formalism. Current bosonization rules can be obtained in the same way by coupling the Thirring fermion to external vector and axial vector sources.

The partition function with external scalar sources \( m(x) \) and \( m^\dagger(x) \) is

\[
Z[m, m^\dagger] = \int D\psi \overline{D}\psi \exp i \int d^2x \left( \overline{\psi} \left[ i\partial - m P_+ - m^\dagger P_- \right] \psi - \frac{1}{2} g j^2 \right)
\]

\[
= \int DB D\psi \overline{D}\psi \exp i \int d^2x \left( \overline{\psi} \left[ i\partial - B - m P_+ - m^\dagger P_- \right] \psi + \frac{1}{2} g B^2 \right)
\]

(8)

where \( P_\pm = \frac{1}{2}(1 \pm \gamma_5) \) and \( j_\mu = \overline{\psi} \gamma_\mu \psi \). The bosonization of the fermion is given by the chiral Jacobian:

\[
J[\theta] = \exp i \int d^2x \left( \frac{1}{2\pi} \partial_\mu \theta \partial^\mu \theta - \frac{1}{\pi} B_\mu \epsilon^{\mu\nu} \partial_\nu \theta \right.
\]

\[
+ \frac{1}{4\pi} \kappa_1 m (e^{2i\theta} - 1) + \frac{1}{4\pi} \kappa_1 m^\dagger (e^{-2i\theta} - 1)
\]

\[
+ \frac{1}{8\pi} m^2 (e^{4i\theta} - 1) + \frac{1}{8\pi} m^\dagger (e^{-4i\theta} - 1) \right).
\]

(9)

The coefficient \( \kappa_1 \) is an arbitrary mass which appears, for example, in the Pauli–Villars scheme [12, 13].

Bosonization is now completed by taking the path integral over \( \theta \):

\[
Z[m, m^\dagger] = \int DB D\theta \exp i \int d^2x \left( \frac{1}{2\pi} \partial_\mu \partial^\mu \theta - \frac{1}{\pi} B_\mu \epsilon^{\mu\nu} \partial_\nu \theta + \frac{1}{2g} B^2 \right.
\]

\[
+ \frac{1}{4\pi} \kappa_1 m (e^{2i\theta} - 1) + \frac{1}{4\pi} \kappa_1 m^\dagger (e^{-2i\theta} - 1)
\]

\[
+ \frac{1}{8\pi} m^2 (e^{4i\theta} - 1) + \frac{1}{8\pi} m^\dagger (e^{-4i\theta} - 1) \right)
\]

(10)

In the second expression we have rescaled \( \theta \),

\[
\varphi = \sqrt{\frac{1}{\pi} \left( 1 + \frac{g}{\pi} \right)} \theta,
\]

(11)

and introduced the parameter \( \beta \) by

\[
\frac{4\pi}{\beta^2} = 1 + \frac{g}{\pi}.
\]

(12)
We are allowed to add suitable polynomial counterterms in $m$ and $m^\dagger$ to the action. If we choose
\begin{equation}
\mathcal{L}_{ct} = \frac{1}{4\pi}\kappa_1(m + m^\dagger) + \frac{1}{8\pi}(m^2 + m^{\dagger 2})
\end{equation}
we can read off the bosonization rules:
\begin{align}
-\sigma_+ &= \frac{\kappa_1}{4\pi}e^{i\beta\varphi} + \frac{1}{4\pi}me^{2i\beta\varphi}, \\
-\sigma_- &= \frac{\kappa_1}{4\pi}e^{-i\beta\varphi} + \frac{1}{4\pi}m^\dagger e^{-2i\beta\varphi},
\end{align}
with $\sigma_\pm \equiv \bar{\psi}P_\pm \psi$. The presence of the $m$- and $m^\dagger$-dependent terms makes this a “field dependent” bosonization rule \[14\]. Modulo these terms – which generate contact terms in the Greens functions – eqs. \[14\] are the usual mass bosonization rules \[9, 11\].

We emphasize that we should not choose any special values for $\kappa_1$. In particular, we should not choose $\kappa_1 = 0$, since we would then lose the sine in the sine–Gordon equation of the bosonic theory. A similar arbitrary mass is also present in Coleman’s expressions \[1\].

iii) The massive non-Abelian case \[11, 15, 16\]

Here too we will only consider mass-term bosonization – bosonization of the currents, as we have already mentioned, is covered by the remark in ref. \[1\]. The partition function is
\begin{equation}
Z[m, m^\dagger] = \int D\psi D\bar{\psi} \exp i \int d^2x \left( \bar{\psi} [i\partial - mP_+ - m^\dagger P_-] \psi \right),
\end{equation}
where $\psi$ is now an $SU(N)$ multiplet and $m = m^a t^a, m^\dagger = m^{a\dagger} t^a$ are in the Lie algebra. From the chiral rotation we get the Jacobian
\begin{align}
J[U] &= \exp i \int d^2x \left( \frac{1}{8\pi} \text{tr} \partial_\mu U^\dagger \partial^\mu U + \frac{1}{12\pi} \int_0^1 dt \epsilon_{\mu\nu\tau} \text{tr} \left( \hat{U}^\dagger \partial^\mu \hat{U} \partial^\nu \hat{U} \partial^\tau \hat{U} \right) \right. \\
&\quad + \frac{1}{4\pi} \kappa_1 \text{tr} m(U - 1) + \frac{1}{4\pi} \kappa_1 \text{tr} m^\dagger (U^\dagger - 1) \\
&\quad + \frac{1}{8\pi} \text{tr} [mUmU - m^2] + \frac{1}{8\pi} \text{tr} [m^\dagger U^\dagger m^\dagger U - m^{\dagger 2}] \right)
\end{align}
where $U = e^{2i\theta}, \hat{U} = e^{2i\hat{\theta}},$ and our three-dimensional conventions are $x^2 \equiv t$, $g_{\mu\nu} = (+, -, -)$ and $\epsilon_{012} = 1$. Again, this leads to the required form for the bosonized theory \[16\] modulo contact terms.

4 Discussion and speculations

These examples demonstrates clearly that in two dimensions the procedure of calculating the Jacobian of a chiral rotation in the most general vector current conserving regularization scheme and then path integrating over it realizes bosonization. This bosonization procedure was suggested earlier by Andrianov \[1\] in the special case of massless non-Abelian fermions. In this paper we have displayed the full generality of the procedure.

Of course, only the information in the “uncharged sector” of the fermionic theory can be cast in a bosonized form, corresponding to Greens functions obtainable from the partition function by differentiation wrt. $V, A, S$ and $P$ sources. “Fermionic information” – that which is expressable in terms of Feynman diagrams with external fermion lines – can not be obtained from a bosonic theory.
Let us also point out that in a non-Abelian model we could also bosonize the singlet, or \( U(1) \), degree of freedom in addition to the \( SU(N) \) ones. The complete Jacobian is then just a product of the singlet and non-singlet Jacobians, and mixed Greens functions in the bosonized theory can be found accordingly.

One may ask why the bosonization procedure works since we are unable to actually derive the bosonic form of the partition function from its fermionic form. Under a chiral rotation the partition function transforms such that
\[
Z[\Gamma] = e^{iW[\theta;\Gamma]} Z[\theta;\Gamma]
\]
where \( Z[\theta;\Gamma] \) is the rotated partition function without the Jacobian. Since, due to bosonization,
\[
Z[\Gamma] = \int D\theta \ e^{iW[\theta;\Gamma]} = \int D\theta \frac{Z[\Gamma]}{Z[\theta;\Gamma]}
\]
we must have
\[
\int D\theta \frac{1}{Z[\theta;\Gamma]} = 1.
\]

It is necessary to prove this equation if we want to “prove” the bosonization procedure, but no such proof is known to us. The procedure does work, however, since it reproduces the bosonization rules that have been proved by other methods, for example by explicitly comparing the Greens functions in the fermionic and bosonic theories, respectively.

We can speculate that the chiral degrees of freedom somehow saturate the theory in two dimensions. This will also imply that the same procedure can only be approximately correct in four. If there are other symmetries of a fermionic theory in four dimensions that produces a Jacobian when the partition function is locally transformed, then the degrees of freedom associated with this transformation would also be important for bosonization. In fact, attempts at “bosonizing” QCD [2–4] according to eq. (1) using only chiral degrees of freedom gives a chiral Lagrangian with a quadratically divergent “kinetic term” for the pions. This violates the principle that physics should not depend on the regularization scheme. The inclusion of further degrees of freedom could be part of a cure for this problem. More generally, it appears to us not altogether improbable that if we could identify all such symmetries of a fermionic theory, we would be able to find a completely bosonized version of this theory.

**Acknowledgments** I would like to thank M. Faber, P.H. Damgaard and A.N. Ivanov for discussions and comments on the manuscript, and M.I. Eides, also for comments on the manuscript.

**Appendix**

In this appendix we give complete expressions for the Jacobians of chiral rotations in Minkowski space. The partition function is
\[
Z = \int D\psi D\bar{\psi} \exp i \int d^2x \bar{\psi} [i\partial - V - A\gamma_5 - S - i\gamma_5 P] \psi
\]
where all fields are either Abelian or non-Abelian. We are interested in the Jacobian of the change of variables
\[
\psi = e^{i\theta\gamma_5} \chi, \quad \bar{\psi} = \bar{\chi} e^{i\theta\gamma_5}.
\]
In the Abelian case, the Jacobian is

\[ J[\theta] = \exp i \int d^2 x \left( \frac{1}{2\pi} \partial_\mu \partial_\mu \theta - \frac{1}{\pi} V_\mu \epsilon^{\mu\nu} \partial_\nu \theta + \frac{1}{\pi} A_\mu \partial_\mu \theta ight. \\
+ \frac{1}{4\pi} \kappa_1 m (e^{2i\theta} - 1) + \frac{1}{4\pi} \kappa_1 m^\dagger (e^{-2i\theta} - 1) \\
+ \left. \frac{1}{8\pi} m^2 (e^{4i\theta} - 1) + \frac{1}{8\pi} m^\dagger (e^{-4i\theta} - 1) \right) \] (22)

where \( m \equiv S + iP \). In the non-Abelian case we get

\[ J[U] = \exp i \int d^2 x \left( \frac{1}{8\pi} \text{tr} \partial_\mu U^\dagger \partial_\mu U + \frac{i}{12\pi} \int_0^1 dt \epsilon_{\mu\nu\tau} \text{tr} \left( \bar{U}^\dagger \partial^\mu \bar{U} \partial^\nu \bar{U} \partial^\tau \bar{U} \right) \\
- \frac{1}{4\pi} \text{tr} \left[ R_\mu i \partial^\mu U U^\dagger - U^\dagger i \partial_\mu U L^\mu - \epsilon_{\mu\nu}(R^\mu i \partial^\nu U U^\dagger - U^\dagger i \partial^\nu L^\nu) \\
+ U^\dagger R_\mu U L^\mu - R_\mu L^\mu - \epsilon_{\mu\nu}(U^\dagger R^\mu U L^\nu - R^\mu L^\nu) \right] \] (23)

where we use standard LR-notation: \( R_\mu = V_\mu + A_\mu \) and \( L_\mu = V_\mu - A_\mu \). For the coefficient \( \kappa_1 \) and the other conventions, see the text.

References

[1] J. Gasser and H. Leutwyler, Ann. Phys. 158 (1984) 142; Nucl. Phys. B250 (1985) 465
[2] D.I. Diakonov and M.I. Eides, Pis'ma Zh. Eksp. Teor. Fiz. 38 (1983) 358 [JETP Lett. 38 (1983) 433]
[3] J. Balog, Phys. Lett. 149B (1984) 197
[4] A.A. Andrianov, Phys. Lett. 157B (1985) 425
[5] M.I. Eides, private communication
[6] A.A. Andrianov and Y. Novozhilov, Phys. Lett. 153B (1985) 422;
   A.A. Andrianov and Y.V. Novozhilov, Phys. Lett. B181 (1986) 129
[7] R. Jackiw, in Relativity, Groups and Topology II, Les Houches 1983, eds. B.S. DeWitt and R. Stora (North-Holland, Amsterdam, 1984)
[8] J. Schwinger, Phys. Rev. 128 (1962) 2425;
   R. Roskies and F.A. Schaposnik, Phys. Rev. D23 (1981) 558
[9] S. Coleman, Phys. Rev. D11 (1975) 2088
[10] A.V. Kulikov, Teor. Mat. Fiz. 54 (1983) 314 [Theor. Math. Phys. 54 (1983) 205];
    R.R. Gamboa Saraví, M.A. Muschietti, F.A. Schaposnik and J.E. Solomin, Ann. Phys. 157 (1984) 360;
    M.I. Eides, Phys. Lett. 153B (1985) 157;
    C.M. Naon, Phys. Rev. D31 (1985) 2035;
    L.C.L. Botelho, Phys. Rev. D33 (1986) 1195
[11] H. Dorn, Phys. Lett. 167B (1986) 86
[12] P.H. Damgaard, H.B. Nielsen and R. Sollacher, Nucl. Phys. B385 (1992) 227; Phys. Lett. B296 (1992) 132
[13] R.D. Ball, Phys. Rep. 182 (1989) 1
[14] P. Di Vecchia, B. Durhuus and J.L. Petersen, Phys. Lett. 144B (1984) 245
[15] A.M. Polyakov and P.B. Wiegmann, Phys. Lett. 131B (1983) 121
[16] E. Witten, Comm. Math. Phys. 92 (1984) 455;
     D. Gonzales and A.N. Redlich, Phys. Lett. 147B (1984) 150
[17] N.K. Pak and P. Rossi, Nucl. Phys. B250 (1985) 279