On the formulation of a Bäcklund Wahlquist-Estabrook transformation for a supersymmetric Korteweg-de Vries equation

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Abstract. We present a local Bäcklund Wahlquist-Estabrook (WE) transformation for a supersymmetric Korteweg-de Vries (KdV) equation. As in the scalar case, such type of transformation generates infinite hierarchies of solutions and also implicitly gives the associated (local) conserved quantities. A nice property is that every of such hierarchies admits a nonlinear superposition principle, starting for an initial solution, including as a particular case the multisolitonic solutions of the system. We discuss the symmetries of the system and we present in an explicit way its local conserved quantities with the help of the associated Gardner transformation.

1. Introduction

The Bäcklund transformation in the context of KdV equation was introduced by Wahlquist and Estabrook [1] soon after Hirota introduced his approach to generate multisolitonic solutions for the same equation [2]. The Bäcklund (WE) transformation for KdV equation consists of a system of partial differential equations for two auxiliary functions \( w \) and \( w' \). If \( w \) and \( w' \) solve the Bäcklund system then \( w \) and \( w' \) imply solutions of the proper KdV equation. In particular we can start with any solution \( w' \) of the Bäcklund system and solving for \( w \) we automatically get another solution of the KdV equation. Starting with the solution \( w' = 0 \), for example, the Bäcklund system allows to obtain in an iterative way multisolitonic solutions of the KdV equation. For this particular case the Bäcklund transformation generates and iterative procedure equivalent to the Hirota approach. The Bäcklund (WE) transformation may be thought as a nonlinear transformation from the space of solutions of the KdV equation into itself. Consequently it contains information on the symmetries of the KdV equation which are not a priori manifest. An important aspect is that one can obtain from the Bäcklund transformation information on the conserved quantities associated to these symmetries. In fact, we can obtain the sequence of infinite conserved local conserved quantities of the KdV equation from the Bäcklund transformation. The procedure to obtain them is to relate the difference \( w - w' \) to a solution of the Gardner equation \([3, 4, 5]\). It can be shown that \( w_z \) and \( w'_z \) are then solutions of the KdV equation obtained from the Gardner transformation. The existence of Bäcklund transformations is thus an important property of some integrable systems, which allows us to disentangle the hidden symmetries of them. The extension of Bäcklund transformations for
supersymmetric KdV equations [6, 7, 8, 9, 10, 11] is still an open problem in the subject. In [12] the Hirota approach has been extended for the supersymmetric KdV system analyzed by Mathieu in [6], arising from the KP hierarchy. So far an extension of it in the sense of Bäcklund (WE) has not been obtained.

In this work we introduce a Bäcklund transformation in the sense of (WE) for a supersymmetric KdV system. This system is invariant under Galileo transformations and hence it differs from the one in [6]. As we show in the next section the system reduces for a particular Grassmann algebra (with only one generator) to the integrable system considered in [13, 14, 15, 16, 17, 18]. The invariance under Galileo transformations is a fundamental property if we are seeking an application at low energy of the integrable system. The analysis of the supersymmetric model we consider in this work was not discussed in [6] because the equation for the even part of the superfield is exactly the KdV equation. However, its coupling to the odd part of the superfield is non-trivial and it has an interesting Poisson structure which was analyzed for a particular reduction in [18]. In addition, the supersymmetric model in [6] when expanded in terms of the basis of the Grassmann algebra yields exactly the KdV equation for the component of the superfield associated to the unitary element of the basis [19]. Consequently, in both supersymmetric systems, at some level, the KdV equation appears for some even component of the superfield.

Hence there are good reasons to analyze the supersymmetric model, invariant under Galileo transformations, we are going to consider in this work.

2. A supersymmetric Wahlquist-Estabrook (WE) transformation for the KdV equation

The supersymmetric extension of the Korteweg-de Vries equation we are going to consider is given by

$$\Phi_t + \Phi_{xxx} + (D\Phi)\Phi_x = 0. \quad (1)$$

In the context of equation (1) $\Phi$ is an odd superfield given by $\Phi(x, \theta) = \xi(x) + \theta u(x)$ and $D$ is the covariant derivative given by $D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x}$. This means that $\Phi$ can be viewed as a function of the real variables $x$ and $t$ (we do not write $t$ explicitly) with values on a given Grassmann algebra $\Lambda$ with one preferred generator $\theta$. It is a $N = 1$ supersymmetric extension for KdV equation. In that setting it is sufficient to take $u$ and $\xi$ to be even and odd respectively for every value of $x$.

Equation (1) is invariant under the supersymmetric transformation $\delta \Phi = \eta Q \Phi$ (where $\eta$ is an arbitrary odd parameter and $Q$ is given by $Q = \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial x}$) which in components reads as $\delta u(x) = \eta \xi'(x)$ and $\delta \xi(x) = \eta u(x)$.

We notice that (1) is equivalent to the equation given in [6] for the particular value of $a = 0$, the only one for which it is invariant under Galileo transformations and hence which may have a physical interpretation at low energies. The Galilean transformation for which (1) remains invariant is given by

$$x \rightarrow \tilde{x} = x + ct, \quad \tilde{\theta} = \theta, \quad (c \in \mathbb{R}),$$
$$t \rightarrow \tilde{t} = t,$$
$$\Phi \rightarrow \tilde{\Phi} = \Phi + c \theta.$$
Consequently, equation (1) remains invariant under the given Galileo transformation, in
distinction to the supersymmetric KdV equation discussed in [6] which does not have this
symmetry.

In this section we present a Bäcklund transformation for (1). We do so by introducing an
auxiliary supersymmetric equation in terms of a Casimir potential:

\[ \Omega_t + \Omega_{xxx} + \frac{1}{2} \Omega_x^2 = 0. \]  

In equation (2) \( \Omega \) is an even superfield potential for \( \Phi \) (or an even casimir potential for \( \Phi \)),
that is \( D \Omega = \Phi \) where \( \Omega \) is given by \( \Omega(x, \theta) = w(x) + \theta \zeta(x) \). We have that \( D \Omega = \Phi \) implies that
\( \zeta = \xi \) and \( w_x = u \). Equation (2) is equivalent to (1) after a redefinition of \( \Omega \).

Equation (1) in components reads as

\[ \begin{align*}
    u_t + u_{xxx} + uu_x &= 0 \\
    \xi_t + \xi_{xxx} + u\xi_x &= 0.
\end{align*} \]  

(3)

We also note that if we take the first derivative with respect to \( x \) in the second equation of (3)
and redefine \( \xi_x = \kappa \) we get the system

\[ \begin{align*}
    u_t + u_{xxx} + uu_x &= 0 \\
    \kappa_t + \kappa_{xxx} + (u\kappa)_x &= 0.
\end{align*} \]  

(4)

If we consider a Grassmann algebra with only one generator \( e \) we have

\[ \begin{align*}
    u &= u_1 \cdot 1 \\
    \xi &= \xi_1 e
\end{align*} \]  

(5)

for arbitrary real functions \( u_1, \xi_1 \) which we assume to be rapidly decreasing. In this case the
system (3) is simply given by

\[ \begin{align*}
    (u_1)_t + (u_1)_{xxx} + u_1(u_1)_x &= 0 \\
    (\xi_1)_t + (\xi_1)_{xxx} + u_1(\xi_1)_x &= 0
\end{align*} \]  

(6)

which is exactly the system in [17, 18] for \( \lambda = 0 \), see also [14, 15].

In what follows we propose, following Wahlquist and Estabrook, a method to generate
solutions for (2) and hence we can directly generate solutions for (1). To do this we propose a
(WE) transformation for (2), the system is given by

\[ \begin{align*}
    \Omega_x + \Omega'_x &= \eta - \frac{1}{12} (\Omega - \Omega')^2 \\
    \Omega_t + \Omega'_t &= \frac{1}{12} \left[(\Omega - \Omega')^2\right]_{xx} - \frac{1}{2} \Omega_x^2 - \frac{1}{2} (\Omega')^2.
\end{align*} \]  

(7)

To prove that the above system works like a (WE) transformation for (2) we first sum of the
second derivative of first equation of system (7) with the second equation of the same system,
obtaining directly

\[ Q(\Omega) + Q(\Omega') = 0, \]  

(8)

where

\[ Q(\Omega) = \Omega_t + \Omega_{xxx} + \frac{1}{2} \Omega_x^2. \]  

(9)

If we can prove also that \( Q(\Omega) - Q(\Omega') = 0 \) we automatically get that \( \Omega \) and \( \Omega' \) are solutions
for (2) and hence \( D \Omega = \Phi \) is solution for (1). To do this we differentiate the first equation of
(7) with respect to \( t \) and we then subtract to it the first derivative of the second equation of (7)
with respect to \( x \). We then obtain (using the compatibility condition \( \Omega_{xt} = \Omega_{tx} \), and the same for \( \Omega' \)) that
\[
-\frac{1}{6} (\Omega - \Omega')( \Omega - \Omega')_t = \frac{1}{6} \left[ (\Omega - \Omega')( \Omega - \Omega')_x \right]_{xx} - \Omega_x \Omega_{xx} - \Omega'_x \Omega'_{xx},
\]
which implies
\[
\frac{1}{6} \left[ (\Omega - \Omega')( \Omega - \Omega')_t + (\Omega - \Omega')( \Omega - \Omega')_{xxx} + (\Omega - \Omega')( \Omega - \Omega')_x \right] + \\
+ \frac{1}{3} (\Omega - \Omega')_x (\Omega - \Omega')_{xx} - \Omega_x \Omega_{xx} - \Omega'_x \Omega'_{xx} = 0.
\]

But we have that
\[
\frac{1}{2} (\Omega - \Omega')( \Omega - \Omega')_{xx} (\Omega - \Omega')_x - \Omega_x \Omega_{xx} - \Omega'_x \Omega'_{xx} = \\
= \frac{1}{2} \left[ \Omega_{xx} \Omega_x + \Omega'_{xx} \Omega'_x - \Omega_{xx} \Omega'_x - \Omega'_x \Omega'_{xx} \right] - \Omega_x \Omega_{xx} - \Omega'_x \Omega'_{xx} = \\
= -\frac{1}{2} \left( \Omega_{xx} + \Omega'_{xx} \right) (\Omega_x + \Omega'_x) = \\
= -\frac{1}{12} (\Omega - \Omega')( \Omega - \Omega')_x (\Omega_x + \Omega'_x) = \frac{1}{12} (\Omega - \Omega')(2 \Omega_x^2 - \Omega'_{xx}).
\]

We get then that
\[
\frac{1}{6} (\Omega - \Omega') \left[ Q(\Omega) - Q(\Omega') \right] = 0. \tag{10}
\]

If \( \Omega - \Omega' \) has in its expansion in terms of the Grassmann algebra generators a non zero coefficient relative to the identity generator, we get that \( Q(\Omega) = Q(\Omega') = 0 \) and from this and (8) we conclude that \( Q(\Omega) = Q(\Omega') = 0 \), or equivalently, \( \Omega \) and \( \Omega' \) satisfying (7) are solutions of (2).

We have thus proved that (7) is a (WE) transformation for equation (2).

3. Relating the Bäcklund transformation to the Gardner system
Let us analyze the two solutions \( \Omega \) and \( \Omega' \) of (7). The corresponding \( \Phi = D\Omega \) and \( \Phi' = D\Omega' \) are solutions of the supersymmetric KdV system (1). The integrability condition of (7) yields, assuming that the components of the identity on the Grassmann expansion is different from zero,
\[
Q(\Omega) - Q(\Omega') = 0.
\]

If we evaluate explicitly the left hand member of this equation and use the first equation in (7) we obtain
\[
\Omega - \Omega' = (\Omega - \Omega')_t + (\Omega - \Omega')_{xxx} + \frac{1}{2} \left[ \eta - \frac{1}{12} (\Omega - \Omega')^2 \right] (\Omega_x - \Omega'_x). \tag{11}
\]

We now define the superfield \( \chi \) as
\[
\chi = \frac{1}{2\epsilon} (\Omega - \Omega') + \frac{3}{\epsilon^2},
\]
where \( \epsilon \) is an even parameter related to \( \eta \) by \( \eta = \frac{3}{\epsilon^2} \).
The superfield \( \chi \) satisfies then
\[
\chi_t + \chi_{xxx} + \chi \chi_x - \frac{1}{6} \epsilon^2 \chi^2 \chi_x = 0. \tag{12}
\]
It is a parametric Gardner-type equation. Moreover, using the first of equations (7) we get
\[
\Omega_x + \Omega'_x = 2 \left( \chi - \frac{1}{6} \epsilon^2 \chi^2 \right). \tag{13}
\]
Combining (11) and (13) we obtain
\[
\Omega_x = \chi + \epsilon \chi_x - \frac{1}{6} \epsilon^2 \chi^2, \tag{14}
\]
\[
\Omega'_x = \chi - \epsilon \chi_x - \frac{1}{6} \epsilon^2 \chi^2. \tag{15}
\]
(14) and (15) define the Gardner transformation. Concluding, given \( \Omega \) and \( \Omega' \) a solution of the system (7), \( \chi \) is a solution of (12) and \( \Omega, \Omega' \) are related to \( \chi \) by the Gardner transformation. (14) and (15) are related by changing \( \epsilon \to -\epsilon \). The converse is also valid. Given a solution of the Gardner equation (12), the Gardner transformation (14) and (15) define \( \Omega \) and \( \Omega' \) which solve (7) and from them we obtain \( \Phi = D\Omega, \Phi' = D\Omega' \) both solutions of the supersymmetric KdV equation (1).

From (14) and (15) we can expand \( \chi \) as a formal series in \( \epsilon \). Since \( \int_{-\infty}^{+\infty} \chi dx \) is a conserved quantity all the coefficients of the expansion in \( \epsilon \) are conserved quantities. We then obtain an infinite sequence of conserved quantities. The first few of them are
\[
\begin{align*}
\int_{-\infty}^{+\infty} u dx, & \quad \int_{-\infty}^{+\infty} u \xi dx, \\
\int_{-\infty}^{+\infty} (u^2 \xi_x - 2u_x \xi_{xx}) dx, & \quad \int_{-\infty}^{+\infty} u^2 dx, \\
\int_{-\infty}^{+\infty} \left( \frac{1}{3} u^3 - (u_x)^2 \right) dx.
\end{align*}
\]

4. Solutions for the supersymmetric system
We finally obtain solutions for the system (7) starting with the trivial solution \( \Omega' = 0 \). In this case the first equation of (7) remains \( \Omega_x = \eta - \frac{1}{12} \Omega^2 \). This equation, written in components is
\[
\begin{align*}
w_x &= \eta_1 - \frac{1}{12} w^2 \\
\xi_x &= \eta_2 - \frac{1}{6} w \xi,
\end{align*}
\]
where \( \eta = \eta_1 + \theta \eta_2 \). For example, taking the Grassmann algebra with two generators \( e_1 \) and \( e_2 \) the previous system in components reads:
\[
\begin{align*}
w_{1x} &= \eta_1^1 - \frac{1}{12} w_1^2 \\
w_{12x} &= \eta_1^{12} - \frac{1}{6} w_1 w_{12} \\
\xi_1 &= \eta_2^1 - \frac{1}{6} w_1 \xi_1 \\
\xi_2 &= \eta_2^2 - \frac{1}{6} w_1 \xi_2,
\end{align*}
\]
where \( w = w_1 \cdot 1 + w_{12} e_1 e_2, \eta_1 = \eta_1^1 \cdot 1 + \eta_1^{12} e_1 e_2, \xi = \xi_1 e_1 + \xi_2 e_2 \) and \( \eta_2 = \eta_2^1 e_1 + \eta_2^2 e_2 \).

We obtain the solutions
\[
 w_1(\tau) = \sqrt{12 \eta_1^1} \left[ \frac{C_1 e^{\tau} - 1}{1 + C_1 e^{\tau}} \right] \tag{16}
\]
(where \( C_1 \) is an arbitrary real constant, \( r = \sqrt{\frac{1}{\eta_1^1}} \) and
\[
 w_{12}(\tau) = e^{\tau} \left( \frac{1}{C_1 e^{\tau} + 1} \right)^2 \left[ \eta_1^{12} \left( r C_1^2 e^{2\tau} + 2 C_1 \tau - r e^{-\tau} \right) + C_{12} \right] \tag{17}
\]
(where \( C_{12} \) is another arbitrary real constant), with similar expressions for \( \xi_1 \) and \( \xi_2 \).

To obtain the explicit dependance of \( t \) we use second equation of (7). After some calculations we have
\[
 \Omega_t = -\frac{1}{3} \eta \left( \eta - \frac{1}{12} \Omega^2 \right) \tag{18}
\]
which implies
\[
 \Omega_t + \frac{1}{3} \eta \Omega_x = 0 \tag{19}
\]
and thus
\[
 \Omega = \Omega \left( x - \frac{1}{3} \eta t \right). \tag{20}
\]

This means, with respect to (16) and (17), that \( \tau = x - \frac{1}{3} \eta t \).

We notice that the above solutions are wave solutions only in the case we have \( \eta_2 = 0 \).

We notice also that (16) implies the very well known solutions for KdV equation. That is, if \( C_1 \) is positive we have the one soliton solutions and if \( C_1 \) is negative we have (the first) singular solutions for KdV equation.

The above solutions can be iterated as in the scalar case using the Bäcklund Wahlquist-Estabrook transformation to generate infinite hierarchies of solutions.

5. Conclusions

We obtained a Bäcklund Wahlquist-Estabrook transformation for a supersymmetric KdV system. We found the explicit relation between the (WE) transformation and the associated Gardner map. We showed that the given system is Galilean invariant and in that sense it may have interesting physical approximation at low energies, in a similar way as the very well known KdV equation. We obtained solutions for the system with the help of the Bäcklund Wahlquist-Estabrook transformation and discussed its dependance in terms of the independent variables \( x \) and \( t \).

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