Small-scale kinematic dynamo and non-dynamo in inertial-range turbulence

Gregory L Eyink¹,³ and Antônio F Neto²,³

¹ Department of Applied Mathematics and Statistics, The Johns Hopkins University, USA
² Campus Alto Paraopeba, Universidade Federal de São João del-Rei, Brazil
E-mail: eyink@ams.jhu.edu and antfrannet@gmail.com

New Journal of Physics 12 (2010) 023021 (40pp)
Received 13 July 2009
Published 16 February 2010
Online at http://www.njp.org/
doi:10.1088/1367-2630/12/2/023021

Abstract. We investigate the Lagrangian mechanism of the kinematic ‘fluctuation’ magnetic dynamo in a turbulent plasma flow at small magnetic Prandtl numbers. The combined effect of turbulent advection and plasma resistivity is to carry infinitely many field lines to each space point, with the resultant magnetic field at that point given by the average over all the individual line vectors. As a consequence of the roughness of the advecting velocity, this remains true even in the limit of zero resistivity. We show that the presence of the dynamo effect requires sufficient angular correlation of the passive line vectors that arrive simultaneously at the same space point. We illustrate this in detail for the Kazantsev–Kraichnan model of the kinematic dynamo with a Gaussian advecting velocity that is spatially rough and white noise in time. In the regime where dynamo action fails, we also obtain the precise rate of decay of the magnetic energy. These exact results for the model are obtained by a generalization of the ‘slow-mode expansion’ of Bernard, Gawędzki and Kupiainen to non-Hermitian evolution. Much of our analysis applies also to magnetohydrodynamic turbulence.

³ Author to whom any correspondence should be addressed.
1. Introduction

The turbulent magnetic dynamo effect is of great importance in astrophysics and geophysics [1]. Many questions remain, however, about the basic mechanism of dynamo action, even for the kinematic stage when the seed magnetic field is weak and does not react back on the advecting velocity field. Stretching of field lines by a chaotic flow is, of course, the ultimate source of growth of magnetic field strength. Plasma resistivity $\eta$ in turn acts to damp the magnetic field. However, the dynamo cannot be understood as a simple competition between growth from stretching and dissipation from resistivity. For example, resistivity also plays a positive role in the dynamo effect through the reconnection of complex, small-scale field-line structures [2].

In addition, random advection may not lead to field growth in the limit of vanishing resistivity. Consider, for instance, the kinematic dynamo model of Kazantsev [3] and Kraichnan [4, 5] with a Gaussian random velocity that is delta correlated in time. In this model there is a dramatic dependence of the dynamo effect on the spatial rugosity of the velocity, as measured by the scaling exponent $0 < \xi < 2$ of the spatial two-point velocity correlation [3]. There exists a certain critical value $\xi_s$ such that for $\xi < \xi_s$, the kinematic dynamo effect exists only above a threshold value $Pr_c$ of the magnetic Prandtl number $Pr = \nu/\kappa$ [6, 7]. Here $\kappa = \eta c/4\pi$ is magnetic diffusivity, while $\nu$ is an effective viscosity associated with a ‘dissipation length’ $\ell_v$ of the velocity, above which scaling holds with exponent $\xi$ and below which the synthetic field becomes perfectly smooth. In this regime of extreme roughness of the advecting velocity field, there is no kinematic dynamo even as $\kappa, \nu \to 0$, if $Pr < Pr_c$.

On the contrary, in the Kazantsev–Kraichnan (KK) model for smoother velocity fields with $\xi > \xi_s$, there is a critical value $Re_{m,c}$ of the magnetic Reynolds number $Re_m = u_{rms} L / \kappa$, where
that the claims of Celani et al in time. Celani et al considered the covariance at time $t$ of the turbulent kinematic dynamo. In our opinion, important ideas have been contributed of the observed small-scale dynamo.

[12, 13]. Considerable debate still continues, however, about the precise nature and universality of the observed small-scale dynamo.

To resolve such subtle issues, a better physical understanding is required of the mechanism of the turbulent kinematic dynamo. In our opinion, important ideas have been contributed recently by Celani et al [14]. They pointed out that the existence of the dynamo effect in the KK model for space dimension $d = 3$ should be closely related to the angular correlation properties of material line vectors. They considered the covariance at time $t$ of two infinitesimal line vectors that are advected starting a distance $r$ apart at time 0. Celani et al argued that this correlation vanishes as $r$ decreases through the inertial scaling range or as $t \to \infty$, going to zero as a power $(r/t^{1/2})^{\xi}$. Here $\gamma = 2 - \xi$ and $\xi = \bar{\xi}(\xi)$ is the scaling exponent of a 'homogeneous zero mode' for the linear operator $\mathcal{M}_x^*\mathcal{N}_x^*$ that evolves the pair correlations of line elements forward in time. Celani et al [14] further claimed that the transition between dynamo regimes in the KK model for $d = 3$ corresponds exactly to the value $\xi_c = 1$, where $\bar{\xi}(\xi_c) = 0$ [14].

In this paper, we shall further investigate these questions. In the first place, we shall show that the claims of Celani et al [14] are not quite correct. It will be shown here that the specific correlation function proposed by those authors does not discriminate between dynamo and non-dynamo regimes. The scaling law that they proposed is valid, but holds over the entire range $0 < \xi < 2$ with a different zero mode and different scaling exponent than they had claimed. We shall show that quite a different correlation function of material line elements is necessary to serve as an ‘order parameter’ for the kinematic dynamo. The crucial difference is that the quantity introduced here measures the angular correlation of material line vectors that are advected to the same space point at time $t$. But still, why should there be any connection of roughness exponent $\xi$ with dynamo action? Individual field lines ought to be stretched and their field strengths increased for all values of $\xi$. We shall provide in this work a plausible physical explanation. Although individual lines may stretch due to chaotic advection, infinitely many magnetic field lines will arrive at each point of the fluid due to diffusion by resistivity and the final magnetic field will be the average value that results from reconnection and ‘gluing’ of field lines by resistivity. We shall show that too little angular correlation leads to large cancellations in this resistive averaging, with the net magnetic field suffering decay despite the growth of individual field lines.

We devote the remainder of our paper to a detailed study of the ‘failed dynamo regime’ in the KK model for $\xi < \xi_c$ and $Pr < Pr_c$. Part of our motivation is the speculation of [10, 11] that hydrodynamic turbulence at high magnetic Reynolds number but low $Pr$ resembles this parameter range of the KK model. A better understanding of this regime may be useful to
rule out its validity for hydrodynamic turbulence, based on astrophysical observations. We shall see, for example, that it implies a very rapid rate of decay of an initial seed magnetic field. Indeed, we show that in the KK model for rougher velocities, the decay of the magnetic field is not resistively limited, with dissipation rate non-vanishing even in the zero-resistivity limit $\kappa \to 0$, as long as $Pr < Pr_c$. There is a strong analogy with the anomalous decay of a turbulence-advected passive scalar, for which scalar dissipation is non-vanishing even in the limit of zero scalar diffusivity [15, 16]. The decay rate is instead determined by large-scale statistical conservation laws, associated with ‘slow modes’ of the scalar evolution operator. We show here that the decay of the magnetic field in the rough regime of the KK model is determined in the limit $\kappa \to 0$, $Pr < Pr_c$ by the ‘slow modes’ of the linear evolution operator $\mathcal{M}_\kappa^2$ for pairs of infinitesimal line elements. We shall establish these results by a formal extension of the slow-mode expansion of Bernard et al [17] to the case of non-Hermitian evolution operators, which is presented in the appendix. We shall furthermore determine all self-similar decay solutions of the magnetic field in the non-dynamo regime of the KK model, following [15] for the passive scalar. Unlike the scalar case, however, determining the decay law of the magnetic energy requires an additional step of matching these self-similar solutions to explicit resistive-range solutions. We shall use these results to discuss the physical mechanism of kinematic dynamo and, in particular, to relate our dynamo ‘order parameter’ to the process of ‘induction’ by a spatially uniform initial magnetic field. As we shall see, considerable insight can be obtained into the inner workings of the small-scale dynamo by considering also the situations where it fails.

2. The Kraichnan–Kazantsev dynamo and correlations of line elements

2.1. The kinematic dynamo

The evolution of the passive magnetic field $B(x, t)$ is governed by the induction equation

$$\partial_t B + (\mathbf{u} \cdot \nabla) B - (B \cdot \nabla) \mathbf{u} = \kappa \Delta B,$$

(1)

where $\mathbf{u} \equiv \mathbf{u}(x, t)$ is the advecting velocity field and $\kappa$ is the magnetic diffusivity. The magnetic field is taken to be solenoidal, assuming that there are no magnetic monopoles:

$$\nabla \cdot B = 0.$$  

(2)

Note that this condition is preserved by the evolution equation (1) if it is imposed at the initial time $t_0 = 0$. We have also assumed above that the advecting fluid is incompressible so that

$$\nabla \cdot \mathbf{u} = 0.$$  

(3)

For simplicity, we shall only consider this case hereafter.

For an incompressible fluid, one can represent the solution of the induction equation by a stochastic Lagrangian representation of the following form:

$$B(x, t) = \mathbb{E} \left[ B_0(\tilde{a}) \cdot \nabla \tilde{x}(\tilde{a}, t) \right]_{a = \tilde{a}(x, t)}.$$  

(4)

See [14, 18]. Here $\tilde{a}(x, t)$ are the ‘back-to-label maps’ for stochastic forward flows $\tilde{x}(a, t)$ solving the stochastic differential equation (SDE)

$$d\tilde{x}(a, t) = \mathbf{u}(\tilde{x}(a, t), t) dt + \sqrt{2\kappa} dW(t).$$  

(5)
We have assumed that \( u \) for a field \( B \) above equation for these are not quite 'material lines' in the traditional sense when of the magnetic correlation.

Here the prime symbol denotes a second Brownian motion with

\[
F_{i\ell}^j(a, a', 0|\mathbf{x}, \mathbf{x}', t) = \langle \tilde{F}_{i\ell}^j(a, 0|\mathbf{x}, t) \tilde{F}_{i\ell}^j(a', 0|\mathbf{x}', t) \rangle .
\]

For statistically homogeneous velocity and initial conditions, with \( C_{ij}(\mathbf{r}, t) \equiv \langle B_i(x, t)B_j(x', t) \rangle \) for \( \mathbf{r} = \mathbf{x} - \mathbf{x}' \), we obtain

\[
C_{ij}(\mathbf{r}, t) = \int d^d \rho \ C_{\ell\ell}(\rho, 0) \tilde{F}_{ij}^{\ell\ell}(\rho, 0|\mathbf{r}, t)
\]

with

\[
\tilde{F}_{i\ell}^{ij}(\rho, 0|\mathbf{r}, t) = \mathbb{E} \mathbb{E}' \left[ \left. \frac{\partial \tilde{x}^{ij}}{\partial a^k}(a + \rho, t) \frac{\partial \tilde{x}^{ij}}{\partial a^\ell}(a, t) \right|_{a = \tilde{a}(\mathbf{x}, t)} \delta^d(\tilde{a}(\mathbf{x} + \mathbf{r}, t) - \tilde{a}'(\mathbf{x}, t) - \rho) \right].
\]

Here the prime symbol denotes a second Brownian motion \( \mathbf{W}'(t) \) statistically independent of \( \mathbf{W}(t) \). Equation (9) was introduced by Celani et al [14] and heavily exploited in their analysis of the magnetic correlation.

Another closely related propagator was introduced by Celani et al [14] related to infinitesimal material line elements, which evolve according to the Lagrangian equation of motion:

\[
D_i \delta \ell = (\delta \ell \cdot \nabla) u.
\]

Note that the positions of line elements are assumed to move stochastically according to (5), so these are not quite ‘material lines’ in the traditional sense when \( \kappa > 0 \). The exact solution of the above equation for \( t > 0 \) is

\[
\delta \ell(t) = \delta \ell(0) \cdot \nabla \tilde{x}(a, t),
\]

with \( \tilde{x}(a, t) \) solving (5). Taking initial line elements \( \delta \ell_{ij}^k(0) = \delta_{ij}^k, \delta \ell_{ij}^\ell(0) = \delta_{ij}^\ell \) starting at positions \( a, a' \) displaced by \( \mathbf{r} = a' - a \), one may follow Celani et al [14] to define for statistically homogeneous turbulence

\[
F_{i\ell}^{ij}(\rho, t|\mathbf{r}, 0) = \langle \delta \ell_{ij}^k(t) \delta \ell_{ij}^\ell(t) \delta^d(\tilde{x}(\mathbf{t}) - \tilde{x}'(\mathbf{t}) - \rho) \rangle
\]

\[
= \mathbb{E} \mathbb{E}' \left[ \left. \frac{\partial \tilde{x}^{ij}}{\partial a^k}(a + \rho, t) \frac{\partial \tilde{x}^{ij}}{\partial a^\ell}(a, t) \right|_{a = \tilde{a}(\mathbf{x}, t)} \delta^d(\tilde{x}(a + \mathbf{r}, t) - \tilde{x}'(a, t) - \rho) \right].
\]
If we make the change of variables \( \mathbf{a} \rightarrow \mathbf{x} \) in the argument of the delta function of equation (10), then the Jacobian of this transformation of variables is 1 due to incompressibility. Therefore, one finds by comparison with (11) that

\[
\tilde{G}_{ij}^{kl}(\rho, 0|\mathbf{r}, t) = \mathbb{E} \mathbb{E}' \left[ \frac{\partial \tilde{x}^i}{\partial \tilde{a}^k}(\mathbf{a} + \rho, t) \frac{\partial \tilde{x}^j}{\partial \tilde{a}^l}(\mathbf{a}, t) \delta^d(\tilde{\mathbf{x}}(\mathbf{a} + \rho, t) - \tilde{\mathbf{x}}(\mathbf{a}, t) - \mathbf{r}) \right]
\]

\[
= F_{ij}^{kl}(\mathbf{r}, t|\rho, 0),
\]
equating the two propagators under interchange of arguments.

In our work below, an important role will also be played by the covariant vector given by the gradient \( \mathbf{G} = \nabla \theta \) of a passive scalar \( \theta \). The gradient satisfies the equation

\[
\partial_t \mathbf{G} + (\mathbf{u} \cdot \nabla) \mathbf{G} + (\nabla \mathbf{u}) \mathbf{G} = \kappa \Delta \mathbf{G},
\]

which is dual to equation (1) for the contravariant vector \( \mathbf{B} \) [19]. The above equation preserves the condition \( \mathbf{G} = \nabla \theta \), if this is imposed at time \( t_0 = 0 \). A stochastic Lagrangian representation also exists for the solution of this equation, which follows from \( \theta(\mathbf{x}, t) = \mathbb{E} \left[ \left[ \theta_0(\tilde{\mathbf{a}}(\mathbf{x}, t)) \right] \right] \) for the scalar field. Solved forward in time with \( \kappa > 0 \) this representation involves the matrix \( \tilde{\nabla} \tilde{\mathbf{a}}(\mathbf{x}, t) \).

However, taking \( \kappa \rightarrow -\kappa \) in (12) and solving backward from time \( t > 0 \) to time \( 0 \) yields the representation for the scalar \( \theta(\mathbf{a}, 0) = \mathbb{E}[\theta(\tilde{\mathbf{a}}(\mathbf{a}, t), t)] \) and for the \( i \) component of its gradient field:

\[
G_i(\mathbf{a}, 0) = \mathbb{E} \left[ \frac{\partial \tilde{x}^k}{\partial \tilde{a}^i}(\mathbf{a}, t) G_k(\tilde{\mathbf{x}}(\mathbf{a}, t), t) \right]
\]

\[
= \int d^d \mathbf{x} G_k(\mathbf{x}, t) \tilde{F}_{ik}^k(\mathbf{a}, 0|\mathbf{x}, t; \mathbf{u}).
\]

For statistically homogeneous velocity and initial conditions, we introduce the two-point correlation of the gradient field, \( G_{ij}(\rho, t) \equiv \left\langle G_i(\mathbf{a}, 0) G_j(\mathbf{a}', 0) \right\rangle \) with \( \rho = \mathbf{a} - \mathbf{a}' \). By the same arguments as previously

\[
G_{ij}(\rho, 0) = \int d^d \mathbf{r} G_{k\ell}(\mathbf{r}, t) \tilde{F}_{ij}^{k\ell}(\rho, 0|\mathbf{r}, t)
\]

\[
= \int d^d \mathbf{r} G_{k\ell}(\mathbf{r}, t) F_{ij}^{k\ell}(\mathbf{r}, t|\rho, 0),
\]

for positive times \( t > 0 \).

We shall generally avoid using the geometric language of differential forms and Lie derivatives in this paper, but a few brief remarks may be useful. For those unfamiliar with this formalism, a good introductory reference is [20]. The magnetic field \( \mathbf{B} \) discussed above is a 1-form, which is more properly represented by its Hodge dual \( \mathbf{B}^* \), a \((d - 1)\)-form. Equation (1) for \( \kappa = 0 \) is equivalent to \( \partial_t \mathbf{B}^* + \mathcal{L} \mathbf{B}^* = 0 \), where \( \mathcal{L} \) is the Lie derivative. The Lie derivative theorem thus implies that (1) for \( \kappa = 0 \) satisfies an analogue of the Alfvén theorem, with conserved flux of \( \mathbf{B} \) through \((d - 1)\)-dimensional material hypersurfaces. On the other hand, the field \( \mathbf{G} \) is a proper 1-form and equation (12) for \( \kappa = 0 \) is equivalent to \( \partial_t \mathbf{G} + \mathcal{L} \mathbf{G} = 0 \). The Lie derivative theorem thus implies that integrals of \( \mathbf{G} \) along material lines are conserved for \( \kappa = 0 \). Either of these equations could be regarded as a valid generalization of the kinematic dynamo problem to general space dimension \( d \) [21]. The non-gradient solutions \( \mathbf{G} \) of (12) are generalizations of the three-dimensional vector potential and the ‘magnetic flux’ is represented.

New Journal of Physics 12 (2010) 023021 (http://www.njp.org/)
by their line integrals around closed material loops. For \( d = 3 \), the non-gradient solutions of (12) are in one-to-one correspondence (up to gauge transformations) with the solenoidal solutions of (1), by the familiar relation \( \mathbf{B} = \nabla \times \mathbf{G} \).

We note finally that all of the results in this section hold for any random velocity field that is divergence-free and statistically homogeneous. Thus, they apply not only to the KK model discussed in the following sections but also to the kinematic dynamo problem for hydrodynamic turbulence governed by the incompressible Navier–Stokes equation.

2.2. White-noise velocity ensemble

In the KK model [3]–[5], the advecting velocity \( \mathbf{u}(\mathbf{x}, t) \) is taken to be a Gaussian random field with zero mean and second-order correlation delta function in time, given explicitly by

\[
\langle u^i(\mathbf{x}, t) u^j(\mathbf{x}, t') \rangle = [D_0 \delta^{ij} - S^{ij}(\mathbf{r})] \delta(t - t')
\]

with \( \mathbf{r} = \mathbf{x} - \mathbf{x} \). Under the incompressibility constraint \( \partial_i S^{ij}(\mathbf{r}) = 0 \) and supposing \( S^{ij}(\mathbf{r}) \) scales as \( \sim r^k \) for \( \ell_v \ll r \ll L \), one deduces for that range that

\[
S^{ij}(\mathbf{r}) = D_1 r^k \left[ (\xi + d - 1) \delta^{ij} - \xi \hat{r}^i \hat{r}^j \right],
\]

where \( \hat{r}^{i} = r^i/r \). Define ‘viscosity’ \( \nu = D_1 \ell_v^k \). Below we shall consider especially the limit \( \nu, \kappa \to 0 \) with \( \nu < \text{Pr}_{r, \kappa} \). This is the non-dynamo regime in the limit of infinite kinetic and magnetic Reynolds numbers. One of our main objectives is to understand better the geometric and statistical properties of this regime that lead to the failure of small-scale dynamo action.

In addition to the properties of statistical homogeneity, stationarity and incompressibility, the white-noise velocity ensemble is time-reflection symmetric. This implies

\[
\tilde{F}^{ij}_{k\ell}(\mathbf{r}, -t|\mathbf{\rho}, 0) = \tilde{F}^{ij}_{k\ell}(\mathbf{r}, t|\mathbf{\rho}, 0)
\]

and the similar property for \( F^{ij}_{k\ell}(\mathbf{\rho}, t|\mathbf{r}, 0) \). Combined with the other symmetries, this implies that

\[
F^{ij}_{k\ell}(\mathbf{\rho}, t|\mathbf{r}, 0) = \tilde{F}^{ij}_{k\ell}(\mathbf{r}, 0|\mathbf{\rho}, t) = \tilde{F}^{ij}_{k\ell}(\mathbf{r}, -t|\mathbf{\rho}, 0) = \tilde{F}^{ij}_{k\ell}(\mathbf{r}, t|\mathbf{\rho}, 0).
\]

The first line follows from incompressibility, the second line is due to time-translation invariance and the last equality follows from time-reflection symmetry. Thus, the two propagators are adjoints in the KK model.

Time-reflection symmetry has also an important implication for the evolution of the gradient field correlation. Note that time-translation invariance implies that equation (13) can be written for \( t > 0 \) as

\[
G_{ij}(\mathbf{\rho}, -t) = \int d^d r \ G_{k\ell}(\mathbf{r}, 0) F^{k\ell}_{ij}(\mathbf{r}, 0|\mathbf{\rho}, -t).
\]

Then time-reflection symmetry implies further that

\[
G_{ij}(\mathbf{\rho}, t) = \int d^d r \ G_{k\ell}(\mathbf{r}, 0) F^{k\ell}_{ij}(\mathbf{r}, 0|\mathbf{\rho}, t)
\]

for \( t > 0 \). Compare with equation (9) for the magnetic correlation. We see that the \( F \)-propagator in the KK model evolves forward in time the gradient correlation.
The most important property of the white-noise model is its Markovian character, which implies that time evolution of correlations is governed by second-order differential (diffusion) equations. For example, the \( n \)th order equal time correlation function \( C_{n,t^2-\cdots-t_n}^i \equiv \langle \prod_{a=1}^n B^{i_a}(X_a, t) \rangle \) satisfies an equation of the form \( \partial_t C_n = M_n C_n \). Expressions for the general \( n \)-body diffusion operators \( M_n \) can be found in [8], which can be obtained using the Itô formula as in [7] or, equivalently, by Gaussian integration by parts. In the limit \( n \to 0 \) all of these operators for general \( n \) become degenerate (singular) and homogeneous of degree \( -\gamma \) with \( \gamma = 2 - \xi \). Below we shall mainly consider \( n = 2 \) and thus write simply \( M \) for \( M_2 \). However, many of our considerations carry over to general \( n \), as will be noted explicitly below. Following the notations of [14], we write for \( n = 2 \):

\[
\partial_t C_{ij}^i (\mathbf{r}, t) = [M(\mathbf{r})]_{pq}^j C_{pq}^i (\mathbf{r}, t),
\]

with

\[
[M]^i_{pq} = \delta_i^q S^a\partial_a\partial_\beta - \delta_i^q S^a\partial_a\partial_\beta + \delta_i^q S^a\partial_a\partial_\beta + \delta_i^q S^a\partial_a\partial_\beta.
\]

The notation \( M(\mathbf{r}) \) indicates that \( \partial_i = \partial/\partial r^i \). Note that equation (17) has an invariant subspace satisfying \( \partial_i C_{ij}^i = \partial_j C_{ij}^i = 0 \).

It follows from (17) and (9) that the \( \bar{F} \)-propagator is the heat kernel of the adjoint operator

\[
[M^*]_{pq}^i = \delta_i^q S^a\partial_a S^\beta - \delta_i^q S^a\partial_a S^\beta + \delta_i^q S^a\partial_a S^\beta + \delta_i^q S^a\partial_a S^\beta,
\]

satisfying

\[
\partial_i \bar{F}_{ij}^{ij} (\mathbf{r}, 0|\mathbf{r}, t) = [M^* (\mathbf{r})]_{pq}^q \bar{F}_{ij}^{pq} (\mathbf{r}, 0|\mathbf{r}, t)
= [M(\mathbf{r})]_{pq}^j \bar{F}_{ij}^{pq} (\mathbf{r}, 0|\mathbf{r}, t).
\]

The propagator \( F \) is thus the heat kernel of \( M \):

\[
\partial_i F_{ij}^{kl} (\mathbf{r}, 0|\mathbf{r}, t) = [M(\mathbf{r})]_{pq}^j F_{ij}^{pq} (\mathbf{r}, 0|\mathbf{r}, t)
= [M^* (\mathbf{r})]_{ij}^{pq} F_{ij}^{pq} (\mathbf{r}, 0|\mathbf{r}, t).
\]

Because of the homogeneity of the operators \( M \) and \( M^* \) in the \( \nu, \kappa \to 0 \) limit, \( F \) satisfies the scaling relation

\[
F_{ij}^{kl} (\lambda \mathbf{r}, 0|\lambda \mathbf{r}, \lambda^\gamma t) = \lambda^{-d} F_{ij}^{kl} (\mathbf{r}, 0|\mathbf{r}, t),
\]

with an identical relation for the \( \bar{F} \)-propagator.

Finally, it follows from (16) that the gradient correlation satisfies

\[
\partial_i G_{ij} (\mathbf{r}, t) = [M^* (\mathbf{r})]_{pq}^q G_{pq} (\mathbf{r}, t).
\]

This equation has an invariant subspace of solutions of the form \( G_{ij} = -\partial_i \partial_j \Theta \) for a scalar correlation function \( \Theta (\mathbf{r}, t) \). Celani et al [14] have also introduced the quantity

\[
Q_{kl} (\mathbf{r}, t) = \int d^d \rho \ F_{kl}^{ij} (\mathbf{r}, t|\mathbf{r}, 0) \langle \delta \ell_k (t) \cdot \delta \ell'_l (t) \rangle,
\]

where on the right-hand side the line elements are initially unit vectors \( \delta \ell_k (0) = \hat{e}_k, \delta \ell'_l (0) = \hat{e}_l \) starting at positions displaced by \( \mathbf{r} \). This quantity measures the angular correlation of the
material line elements at times $t > 0$, as well as their growth in length. It follows from (21) that this quantity in the KK model satisfies

$$\partial_t Q_{\kappa \ell}(r, t) = \left[ M^*(r) \right]_{k\ell}^{pq} \hat{Q}_{pq}(r, t),$$

(25)

with the initial condition $Q_{\kappa \ell}(r, 0) = \delta_{k\ell}$, as already noted in [14]. This equation is identical to (23) for the gradient correlation and, furthermore, $Q_{\kappa \ell}(r, 0) = -\partial_t \partial_t \Theta(r, 0)$ with $\Theta(r, 0) = -(1/2)r^2$. Thus, $Q$ is of gradient type.

In this work, we restrict ourselves to conditions of statistical homogeneity, isotropy and parity invariance for all stochastic quantities. Thus, we can write the two-point correlation function of the magnetic field as

$$C_{ij} = C_L(r, t) \hat{r}^i \hat{r}^j + C_N(r, t)(\delta^{ij} - \hat{r}^i \hat{r}^j),$$

(26)

where $\hat{r}^i = r^i / r$. $C_L$ and $C_N$ are the longitudinal and transverse correlations, respectively. With the form of the velocity correlation in (15), the evolution equation (17) reduces to two coupled equations for $C_L$ and $C_N$. A lengthy but straightforward calculation gives

$$\partial_t C_L = D_1 r^\xi \left\{ (d - 1) \partial_r C_L + (d + 1) (d - \xi - 1) \frac{1}{r} \partial_r C_L + (d - 1) \left[ \xi^2 - \xi - 2(d - 1) \right] \frac{1}{r^2} C_L \right. \right.$$  
$$+ (d - 1) [(d + 1) \xi + 2(d - 1)] \frac{1}{r^2} C_N \left. \right\},$$

(27)

and

$$\partial_t C_N = D_1 r^\xi \left\{ (d - 1) \partial_r C_N + \left[ (d + 1) \xi + (d - 1)^2 \right] \frac{1}{r} \partial_r C_N + \left[ (d + 1) \xi^2 \right. \right.$$  
$$+ \left. (d^2 - 5) \xi - 2(d - 1) \right] \frac{1}{r^2} C_N + (\xi - 2) (\xi - 1) (d + \xi - 1) \frac{1}{r^2} C_L \right\},$$

(28)

respectively, when $v, \kappa \to 0$. For solenoidal solutions, such as for the magnetic field, it is easy to show that the two correlations are related by

$$C_N = C_L + \frac{1}{d - 1} r \partial_r C_L.$$ 

(29)

For example, see [19, 22]. The solutions satisfying this relation form an invariant subspace, with the evolution reducing to a single equation for $C_L$:

$$\partial_t C_L = D_1 r^\xi \left\{ (d - 1) \partial_r C_L + (2\xi + d^2 - 1) \frac{1}{r} \partial_r C_L + (d - 1) \xi(d + \xi) \frac{1}{r^2} C_L \right\}.$$ 

(30)

In the same manner, the general solution of (23) or (25) may be decomposed as

$$G_{ij} = G_L(r, t) \hat{r}_i \hat{r}_j + G_N(r, t)(\delta_{ij} - \hat{r}_i \hat{r}_j)$$ 

(31)

under assumptions of homogeneity, isotropy and reflection symmetry. Then $G_N$ and $G_L$ satisfy the following coupled equations for $v, \kappa \to 0$:

$$\partial_t G_L = (d - 1) D_1 r^\xi \left\{ \partial_r G_L + (3\xi + d - 1) \frac{1}{r} \partial_r G_L + (3\xi + 2d - 2) \left( \xi - 1 \right) \frac{1}{r^2} G_L \right. \right.$$  
$$+ \left. (d + \xi - 1) (\xi - 1) (\xi - 2) \frac{1}{r^2} G_N \right\}.$$ 

(32)
and
\[
\partial_t G_N = D_1 r^\xi \left\{ (d - 1) \partial_r G_N + [(d - 3) \xi + (d - 1)^2] \frac{1}{r} \partial_r G_N + [(d + 1)\xi + 2(d - 1)] \frac{1}{r^2} G_L + [(d - 1) \xi - 2] (\xi + d - 1) \frac{1}{r^2} G_N \right\},
\]
respectively. Gradient solutions satisfy the constraint
\[
G_L = G_N + r \partial_r G_N,
\]
where \( G_N = -\frac{1}{r} \partial_r \Theta \) in terms of the scalar correlation function \( \Theta \). In this invariant subspace of solutions, the dynamics reduces to a single equation for \( G_N \):
\[
\partial_t G_N = (d - 1) D_1 r^\xi \left[ \partial_r G_N + (2\xi + d + 1) \frac{1}{r} \partial_r G_N + (d + \xi) \frac{1}{r^2} G_N \right].
\]

2.3. Line-vector correlations

The kinematic dynamo effect is ultimately due to the stretching of magnetic field lines as they are passively advected by a chaotic velocity field. However, the properties of infinitesimal material line elements in the KK model are, at first sight, counterintuitive in this respect. In order to discuss stretching of individual lines, it must be understood that the velocity field is smoothed at very small scales \( \ell \ll \ell_\nu \). The inertial-range velocity structure function in (15) crosses over to a viscous-range form
\[
S^{ij}(r) = D_1 \ell_v^{-2} r^2 \left[ (d + 1) \delta^{ij} - 2 \hat{r}^i \hat{r}^j \right]
\]
for \( r \ll \ell_v \). The growth of line elements in such a smooth velocity field, white noise in time was derived by Kraichnan [23] to be exponential
\[
\langle \delta \ell^2(t) \rangle \approx e^{2\lambda t},
\]
with the Lyapunov exponent
\[
\lambda = \frac{1}{d + 2} \int_0^\infty dt \left( \frac{\partial u_i}{\partial x_j}(x, t) \frac{\partial u_i}{\partial x_j}(x, 0) \right)
= D_1 \ell_v^{d-2} d(d - 1).
\]
See also [24]–[26]. The ‘material’ line elements of relevance to the kinematic dynamo are subject to an additional Brownian motion proportional to \( \sqrt{2\kappa} \) in (5). However, this effect of molecular diffusivity \( \kappa \) corresponds just to changing the constant \( D_0 \) in the Kraichnan model velocity covariance (14) to \( D_0 + 2\kappa \). Since only velocity gradients enter (38), this result for the Lyapunov exponent still holds in the presence of molecular diffusivity.

It follows from (38) that line stretching is greater for smaller \( \xi \) and smaller \( \nu \). This may seem a bit perplexing, because the dynamo fails for \( \xi \) too small, in the range \( 0 < \xi \lesssim \xi^* \). In that regime, there is no dynamo action for \( \nu < Pr_c \kappa \) despite the fact that the stretching rate becomes larger as \( \nu \) decreases. The turbulent kinematic dynamo cannot be understood as a simple ‘competition’ between stretching and diffusion. What, then, can account for the presence of the dynamo in the range \( \xi < \xi < 2 \) of smoother velocities in the KK model and failure of the dynamo in the range \( 0 < \xi < \xi^* \) of very rough velocities? An intriguing suggestion has been made by Celani et al [14] that the existence of the dynamo effect can be characterized by the
angular correlations of material line elements. They proposed the function $Q$ defined in (24) as an ‘order parameter’ for the dynamo transition. As we shall demonstrate below, the principal conclusions of Celani et al [14] about $Q$ are erroneous and this quantity does not discriminate between dynamo and non-dynamo regimes in the KK model. However, our discussion will lead us to identify a different correlation property of infinitesimal material line vectors, which can indeed serve as an ‘order parameter’ for the dynamo.

The principal claims of [14] were as follows. First, in the non-dynamo regime for $0 < \xi < \xi_s$ with rough velocity, $Q$ exhibits a scaling law of correlations:

$$Q_{kl}(r, t) \sim (\text{const.}) \left( \frac{r}{(D_t t)^{1/\gamma}} \right)^{-\zeta} \bar{Z}_{kl}(\hat{r}),$$

(39)

for $r \ll \min\{(D_t t)^{1/\gamma}, L\}$. For finite $v, \kappa$, this relation holds in the inertial-convective range of scales $\max\{\ell_c, \ell_v\} \ll r \ll L$, with $\ell_c = (\kappa/D_t)^{1/\gamma}$ (assuming that $v < Pr_c\kappa$). This is an example of ‘zero-mode dominance’ [17]. Thus, the quantity $\bar{Z}_{kl}(r)$ is a homogeneous zero mode of the operator $M^*$, satisfying $[M^*]^{pq}_{kl} \bar{Z}_{pq} = 0$, with exponent $\zeta(\xi) > 0$ for $0 < \xi < \xi_s$. Intriguingly, it was found that $\zeta = \xi + 2$, where $\xi$ is the scaling exponent of the zero mode of $M$ which was shown in [21] to dominate in the magnetic correlation of the KK model for the same parameter range. It was furthermore claimed in [14] that $\zeta(\xi_s) = 0$. For $2 > \xi > \xi_s$, on the contrary, it was argued that $M^*$ develops point spectrum and that

$$Q_{kl}(r, t) \sim (\text{const.}) E_0^{\ell_l r} \bar{e}_{kl}(r),$$

(40)

where $E_0$ is the largest positive eigenvalue of $M^*$ and $\bar{e}_{kl}(r)$ is the corresponding eigenfunction. $E_0$ is numerically equal to the dynamo growth rate. Note in the limit $Pr \ll 1$ that $E_0 \propto 1/t_s = (D_t^2/\kappa)\gamma^{1/\gamma}$ [6], so that $\kappa$ must be kept nonzero (but with $Re_m > Re_{m,c}$). Thus, the ‘material line vectors’ in $Q$ are advected by velocity $u$ subject to Brownian noise proportional to $\sqrt{\kappa}$. The appropriate terms proportional to $\kappa$ must then be included in the diffusion operators $M$ and $M^*$ [14].

We shall show that the zero-mode dominance relation (39) does hold for $Q$, but with a different zero mode and different scaling exponent $\zeta$ than that claimed by [14]. Furthermore, the scaling relation (39) holds over the whole range $0 < \xi < 2$, assuming only that $\max\{\ell_c, \ell_v\} \ll r \ll L$, with an exponent $\zeta(\xi) = -\xi < 0$ that does not exhibit any qualitative change at the dynamo transition $\xi = \xi_s$. The exponential growth relation (40) does not hold for the quantity $Q$ anywhere over the range $0 < \xi < 2$, even if $Pr > Pr_c$ and $Re_m > Re_{m,c}$.

2.4. Zero-mode analysis

The basic tool of our investigation is a generalization of the slow-mode expansion of Bernard et al [17]. Those authors derived such an expansion for the propagator or heat kernel $P(t, r|t_0, r_0)$ that describes the evolution of a passive scalar in the Kraichnan white-noise velocity ensemble with covariance (14). However, the derivation of [17] was, in fact, axiomatic and applicable to the propagator for any non-positive, self-adjoint operator, with an absolutely continuous spectrum and homogeneous of degree $-\gamma$. This derivation showed that the slow-mode expansion follows from assumed meromorphic properties of the Mellin transform of the propagator and Green’s function of the operator. In the appendix, we generalize this axiomatic derivation to the case of the non-Hermitian operators $M$ and $M^*$, in the non-dynamo regime where both have absolutely continuous spectrum. We refer the reader to the appendix for details and here just state the essential results.
The operators $\mathcal{M}$ and $\mathcal{M}^*$ have two types of homogeneous zero modes: regular and singular. The regular zero modes are denoted $Z_{(a,p)}$ and $\tilde{Z}^{(a,p)}$ for $a = 1, 2, 3, \ldots$, respectively, with exponents $\zeta_a$ and $\bar{\zeta}_a$ whose real parts increase with $a$. These are ordinary functions that satisfy the conditions $\mathcal{M}Z_{(a,p)} = 0$ and $\mathcal{M}^*\tilde{Z}^{(a,p)} = 0$ globally. The singular zero modes $W_{(a)}$ and $\tilde{W}^{(a)}$ for $a = 1, 2, 3, \ldots$, instead have exponents $\omega_a$ and $\bar{\omega}_a$ whose real parts decrease with $a$ for $a = 1, 2, 3, \ldots$, respectively. These are distributions that satisfy the conditions $\mathcal{M}W_{(a)} = 0$ and $\mathcal{M}^*\tilde{W}^{(a)} = 0$ only up to contact terms. The scaling exponents of the two sets of zero modes are related by

$$\omega_a + \bar{\omega}_a = -d + \gamma, \quad \bar{\omega}_a + \bar{\zeta}_a = -d + \gamma.$$  \hfill (41)

Above each regular zero mode lies an ascending tower of slow modes $Z_{(a,p)}$ and $\tilde{Z}^{(a,p)}$ homogeneous of degree $\zeta_a$ and $\bar{\zeta}_a$ with scaling exponent of the smallest real part. Of course, the leading term may give a zero

$$W_{(a)} = Z_{(a,0)} = \bar{Z}^{(a,0)} = \bar{Z}^{(a,0)},$$

In terms of these quantities, there are short-distance expansions, for $\lambda \ll 1$, both for the $F$-propagator

$$F_{kk}^{ij}(\lambda r, t | \rho, 0) \sim \sum_{a,p \geq 0} \lambda^{-\zeta_a + \gamma p} Z_{(a,p)}(r) [\tilde{W}_{kk}^{(a,p)}(\rho, t)]^*.$$  \hfill (42)

and for the $\tilde{F}$-propagator

$$\tilde{F}_{kk}^{ij}(\lambda r, t | \rho, 0) \sim \sum_{a,p \geq 0} \lambda^{-\bar{\zeta}_a + \gamma p} \tilde{Z}_{(a,p)}(r) [W_{(a,p)}(\rho, t)]^*.$$  \hfill (43)

See the appendix for the details of derivation. Note that these asymptotic series are generally dominated by their leading terms for $a = 1$ and $p = 0$, corresponding to the regular zero mode with scaling exponent of the smallest real part. Of course, the leading term may give a zero contribution for various reasons and then subleading terms will dominate instead. In order to make use of this expansion, we must calculate explicitly the homogeneous zero modes of $\mathcal{M}$ and $\mathcal{M}^*$.

To find the isotropic and scale-invariant zero modes of $\mathcal{M}$, we substitute into (27) and (28) the forms

$$C_L = A_L r^\sigma, \quad C_N = A_N r^\sigma$$

giving the matrix equation

$$\begin{bmatrix} M_{LL} & M_{LN} \\ M_{NL} & M_{NN} \end{bmatrix} \begin{bmatrix} A_L \\ A_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

with

$$M_{LL} = (d - 1) \sigma (\sigma - 1) + (d - 1)(d - \xi - 1) \sigma + (d - 1)(\xi^2 - \xi - 2d + 2),$$

$$M_{LN} = (d - 1)(d + 1) \xi + 2(d - 1).$$

New Journal of Physics 12 (2010) 023021 (http://www.njp.org/)
\[ M_{NL} = (\xi - 2)(\xi - 1)(d + \xi - 1), \]
\[ M_{NN} = (d - 1)\sigma(\sigma - 1) + [(d + 1)\xi + (d - 1)^2]\sigma + (d + 1)\xi^2 + (d^2 - 5)\xi - 2d + 2. \]

Calculating the determinant
\[
\begin{vmatrix}
M_{LL} & M_{LN} \\
M_{NL} & M_{NN}
\end{vmatrix} = (d - 1)(\sigma - 2)(\sigma + d - 2)[(d - 1)\sigma^2 + (d^2 - d + 2\xi)\sigma + (d - 1)\xi(d + \xi)],
\]

one finds that the scaling exponents \( \sigma \) are
\[
\begin{align*}
\xi_1 &= \frac{-d}{2} - \frac{\xi}{d - 1} + \frac{d}{2} \left[ 1 - 4\xi \frac{(d - 2)(d + \xi - 1)}{d(d - 1)^2} \right]^{1/2} \\
\omega_2 &= \frac{-d}{2} - \frac{\xi}{d - 1} - \frac{d}{2} \left[ 1 - 4\xi \frac{(d - 2)(d + \xi - 1)}{d(d - 1)^2} \right]^{1/2},
\end{align*}
\]
\[ \xi_2 = 2, \quad \omega_1 = 2 - d. \]

Note that the set \( \xi_1, \omega_2 \) corresponds to the invariant subspace of solenoidal solutions, as may be verified by substituting the scaling ansatz for \( C_L \) into (30). The exponent \( \xi_1 \) coincides with that found by Vergassola [21] to dominate in the magnetic two-point correlation for a forced steady state at high magnetic Reynolds number and zero Prandtl number.

To find the isotropic and scale-invariant zero modes of \( M^* \), we likewise substitute into (32) and (33) the forms
\[
G_L = \tilde{A}_L r^{\tilde{\sigma}}, \quad G_N = \tilde{A}_N r^{\tilde{\sigma}}
\]
giving the matrix equation
\[
\begin{bmatrix}
\tilde{M}_{LL} & \tilde{M}_{LN} \\
\tilde{M}_{NL} & \tilde{M}_{NN}
\end{bmatrix}
\begin{bmatrix}
\tilde{A}_L \\
\tilde{A}_N
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
with
\[
\begin{align*}
\tilde{M}_{LL} &= (d - 1)\tilde{\sigma}(\tilde{\sigma} - 1) + (d - 1)(3\tilde{\xi} + d - 1)\tilde{\sigma} + (d - 1)(3\tilde{\xi} + 2d - 2)(\tilde{\xi} - 1), \\
\tilde{M}_{LN} &= (d - 1)(d + \tilde{\xi} - 1)(\tilde{\xi} - 1)(\tilde{\xi} - 2), \\
\tilde{M}_{NL} &= (d + 1)\tilde{\xi} + 2(d - 1), \\
\tilde{M}_{NN} &= (d - 1)\tilde{\sigma}(\tilde{\sigma} - 1) + ((d - 3)\tilde{\xi} + (d - 1)^2)\tilde{\sigma} + ((d - 1)\tilde{\xi} - 2)(d + \tilde{\xi} - 1).
\end{align*}
\]

Calculating the determinant
\[
\begin{vmatrix}
\tilde{M}_{LL} & \tilde{M}_{LN} \\
\tilde{M}_{NL} & \tilde{M}_{NN}
\end{vmatrix} = (d - 1)\tilde{\sigma}(\tilde{\sigma} + \tilde{\xi})(\tilde{\sigma} + \xi + d)[(d - 1)\tilde{\sigma}^2 + ((d - 4)(d - 1) + 2\xi(d - 2))\tilde{\sigma} + 2(d - 2)(d + \xi - 1)(\xi - 1)],
\]

*New Journal of Physics* 12 (2010) 023021 (http://www.njp.org/)
one finds that the scaling exponents $\tilde{\sigma}$ are

$$\tilde{\xi}_1 = -\xi, \quad \tilde{\omega}_2 = -(d + \xi),$$

$$\tilde{\xi}_2 = \frac{4 - d}{2} + \frac{2 - d}{d - 1} \xi + \frac{d}{2} \left[ 1 - 4\xi \frac{(d - 2)(d + \xi - 1)}{d(d - 1)^2} \right]^{1/2},$$

$$\tilde{\omega}_1 = \frac{4 - d}{2} + \frac{2 - d}{d - 1} \xi - \frac{d}{2} \left[ 1 - 4\xi \frac{(d - 2)(d + \xi - 1)}{d(d - 1)^2} \right]^{1/2}.$$
with \( \lambda = r/(D_1 t)^{1/\gamma} \) and \( \tilde{\rho} = \rho/(D_1 t)^{1/\gamma} \). Then using (43) for \( \lambda \ll 1 \) gives

\[
Q_{k\ell}(r, t) \sim \sum_{a, p \geq 0} \lambda^{x+yp} \tilde{Z}_{k\ell}^{(a, p)}(\tilde{r}) \left[ \int d^d \tilde{\rho} W_{(a, p)}(\tilde{\rho}, 1) \right].
\]

Note, however, that the space integral vanishes for all \( W_{(a, p)} \) in the solenoidal sector (see e.g. [6]). This follows, for any solenoidal correlation \( C \), from

\[
C^{ij}(r, t) = \delta_{k\ell} \partial_r A_{k\ell}(r, t) - \Delta A_{k\ell}(r, t),
\]

where \( A_{k\ell} \) is the correlation of the vector potential \( A \). Note that, in general dimension \( d \), \( B \) is a 1-form and \( A \) is a 2-form, related by the co-differential \( B = \delta A \). Since the solenoidal solutions \( W_{(a, p)} \) are associated in the expansion with the slow modes \( \tilde{Z}^{(a, p)} \) outside the gradient sector, all of these terms drop out in \( Q \). The result is the same as the slow-mode expansion carried out entirely in the gradient sector, with the leading term

\[
Q_{k\ell}(r, t) \sim C_2 \left( \frac{r}{(D_1 t)^{1/\gamma}} \right)^{1/2} \tilde{Z}_{k\ell}^{(1)}(\tilde{r}),
\]

for \( C_2 = \int d^d \tilde{\rho} W_{(2, 0)}(\tilde{\rho}, 1) \). This is the correct relation replacing relation (39) claimed in [14].

This same relation may be verified by appealing to the results of Eyink and Xin [15] on the self-similar decay of the passive scalar. Those authors found that there is a universal form of the self-similar decay solutions for the scalar correlation function at short distances:

\[
\Theta(r, t) \sim \frac{\partial^2 (t) - \chi(t)}{2\gamma d D_1} r^\gamma
\]

for \( r \ll (D_1 t)^{1/\gamma} \). See [15], equation (3.21). Here \( \chi(t) = -(1/2)(d/dt)\partial^2 (t) \) is the scalar dissipation rate. For general initial data with power-law decay of correlations in space, \( \Theta(r, 0) \sim r^{-\alpha} \), the decay rate is given by \( \partial^2 (t) \sim t^{-\alpha/\gamma} \) at long times [15]. Since \( Q_{k\ell}(r, 0) = \delta_{k\ell} \) corresponds to \( \Theta(r, 0) = -(1/2) r^2 \) with \( \alpha = 2 \), we recover from \( Q_{k\ell}(r, t) = -\delta_{k\ell} \partial_r \Theta(r, t) \) and equation (46) exactly the relation (45), with \( \tilde{Z}_{k\ell}^{(1)}(\tilde{r}) = \delta_{k\ell} - \xi k_\ell \tilde{r}_k \tilde{r}_\ell \). The latter result may be verified from equation (34) by substituting \( G_N = \tilde{A}_N r^{-\xi} \). This alternative derivation of (45) makes clear its validity over the whole range \( 0 < \xi < 2 \) and not just \( 0 < \xi < \xi_0 \). There can be no exponential growth relation for \( \Theta \), such as relation (40) proposed in [14]. It is true that the operator \( \mathcal{M}^* \) must have a positive eigenvalue whenever \( \mathcal{M} \) does so. However, the corresponding eigenfunctions must lie in the non-gradient sector. An exponential growth for \( Q_{k\ell}(r, t) = -\partial_r \partial_r \Theta(r, t) \) would require an exponential growth for the scalar correlation function \( \Theta(r, t) \), which does not occur.

---

4 Let us give this argument in more detail. The representation \( B = \delta A \) in dimension \( d \) means that \( B^i = \partial_j A^{ij} \), where \( A^{ij} = -A^{ji} \). The relation that replaces (44) in general dimension \( d \) is

\[
c^{ij}(r) = -\overline{\partial}_k \partial_r A^{k, ji}(r),
\]

where \( A^{k, ji} \) is the two-point correlation of the 2-form \( A \). The result that \( \int d^d r c^{ij}(r) = 0 \) follows if one assumes that the correlation function \( A^{k, ji}(r) \to 0 \) sufficiently rapidly as \( |r| \to \infty \). The result (44) for \( d = 3 \) is recovered from the relation \( A_{ij} = \epsilon^{ijk} A_k \) between the 2-form \( A^{ij} \) and the usual vector potential \( A_k \).
2.5. A dynamo order parameter

Based on the previous discussion, we will now propose an alternative definition of a line correlation that can serve as an ‘order parameter’ for the dynamo transition. Clearly, one should not integrate \( F_{ki}^ij(\rho, t|\mathbf{r}, 0) \) over \( \rho \), as this eliminates the solenoidal sector. We propose instead to set \( \rho = 0 \), defining

\[
R_{ki}(\mathbf{r}, t) = F_{ki}^ij(0, t|\mathbf{r}, 0) = \langle \delta \mathbf{\ell}_k(t) \cdot \delta \mathbf{\ell}'_i(t) \rangle_0, \tag{47}
\]

where, as before, the two-line elements are started with \( \delta \mathbf{\ell}_k(0) = \mathbf{\hat{e}}_k \), \( \delta \mathbf{\ell}'_i(0) = \mathbf{\hat{e}}_i \) and \( \mathbf{x}'(0) - \mathbf{x}(0) = \mathbf{r} \). In contrast to the quantity \( Q \) defined by \cite{14}, this quantity is directly related to the energy in the magnetic fluctuations, via the formula

\[
\langle B^2(t) \rangle = \int \! d^d \rho \; R_{ki}(\rho, t) e^{ik\ell}(\rho, 0). \tag{48}
\]

The above relation follows from (9) by setting \( r = 0 \) and summing over \( i = j \).

Because of the delta function in (47), the quantity \( R_{ki}(\mathbf{r}, t) \) measures the growth in magnitude and the angular correlation between those material line vectors that start at points displaced by \( \mathbf{r} \) and that arrive, stretched and rotated, at the same point at time \( t \). A quantity with even simpler geometric significance that might also serve as an ‘order parameter’ of the dynamo transition is

\[
R(t) = \frac{1}{d} \int \! d^d \mathbf{r} \; R_{kk}(\mathbf{r}, t) = \langle \delta \mathbf{\ell}(t) \cdot \delta \mathbf{\ell}'(t) \rangle_0. \tag{49}
\]

This is the covariance of two material line elements that started at any relative positions as identical unit vectors at time 0 and that ended at the same point at time \( t \). The notation \( \langle \cdot \rangle_0 \) denotes the conditional expectation over material lines that end at zero separation. Clearly \( R(0) = 1 \). As we shall discuss later on, \( R(t) \) has a direct interpretation in terms of the turbulent decay of an initially uniform magnetic field.

In this section, we shall focus instead on the more general function \( R_{ki}(\mathbf{r}, t) \). Just like the quantity \( Q \) defined in \cite{14}, \( R \) satisfies also the equation

\[
\partial_t R_{ki}(\mathbf{r}, t) = [M^*(\mathbf{r})]_{k\ell}^{pq} R_{pq}(\mathbf{r}, t). \tag{50}
\]

However, it has the initial value

\[
R_{ki}(\mathbf{r}, 0) = \delta_{k\ell} \delta^d(\mathbf{r}),
\]

which is non-gradient, unlike for \( Q \). Thus, \( R \) should experience exponential growth in the dynamo regime for \( 2 > \xi > \xi_s \),

\[
R_{ki}(\mathbf{r}, t) \sim C_0 e^{E_0 t} \tilde{\mathbf{\xi}}_{k\ell}(\mathbf{r}),
\]

proportional to the eigenfunction \( \tilde{\mathbf{\xi}}_{k\ell}(\mathbf{r}) \) of \( M^* \) with largest eigenvalue \( E_0 \). To demonstrate this, it is enough to show that the initial condition \( R_{ki}(\mathbf{r}, 0) \) gets a nonzero contribution from the eigenfunction \( \tilde{\mathbf{\xi}}_{k\ell}(\mathbf{r}) \). We may represent this initial state by an expansion

\[
R_{ki}(\mathbf{r}, 0) = \sum_{\alpha} C_{\alpha} \tilde{\mathbf{\xi}}_{k\ell}^\alpha(\mathbf{r}),
\]

where \( \tilde{\mathbf{\xi}}_{k\ell}^\alpha(\mathbf{r}) \) is the eigenfunction of \( M^* \) with eigenvalue \( E_\alpha \). Note that for the continuous spectrum, this is a generalized eigenfunction expansion where the sum over \( \alpha \) is a continuous
The coefficient $\mathcal{E}_a(0) = C_a = \langle \mathcal{E}_a, \mathcal{R}(0) \rangle = \int d^d r \mathcal{E}^{kk}_a(\mathbf{r}) \mathcal{R}_{kk}(\mathbf{r}, 0)$, where $\mathcal{E}_a^{kk}$ are the eigenfunctions of $\mathcal{M}$ with the same eigenvalue $E_a$. These form a bi-orthogonal set with the eigenfunctions $\tilde{\mathcal{E}}^{\nu}_a$ of $\mathcal{M}$. The coefficient $C_0$ corresponding to the eigenfunction $\tilde{\mathcal{E}}^{00}_{kk} = \mathcal{E}^{kk}_a(0)$ is nonzero because

$$C_0 = \int d^d r \mathcal{E}^{kk}_a(\mathbf{r}) \mathcal{R}_{kk}(\mathbf{r}, 0) = \mathcal{E}^{kk}_a(0),$$

where $\mathcal{E}^{kk}_a(0) \neq 0$ is (twice) the energy in the normalized dynamo state. Equation (50) follows.

Thus, unlike the quantity $Q$ proposed in [14], the line correlation $\mathcal{R}$ defined in (47) satisfies the exponential growth relation (50) in the dynamo regime. It would be of great interest to determine the spatial structure of the eigenfunction $\tilde{\mathcal{E}}_{kk}(\mathbf{r})$. Of course, this function must be of non-gradient type. It is known [6, 9] that the trace of the dual eigenfunction $\tilde{\mathcal{E}}(\mathbf{r}) = \mathcal{E}^{\nu\nu}(\mathbf{r})$ exhibits stretched-exponential decay of the form $\tilde{\mathcal{E}}(\mathbf{r}) \propto -\exp\left(-\beta(r/\ell_c)^{\nu/2}\right)$ for $r \gg \ell_c$.

A similar behavior for $\tilde{\mathcal{E}}(\mathbf{r}) = \tilde{\mathcal{E}}_{kk}(\mathbf{r})$ can be checked to be consistent with the dynamical equations, but a more careful investigation is required. This will be pursued elsewhere [27].

Because of the relation (48), one should expect quite a different time dependence for $\mathcal{R}_{kk}(\mathbf{r}, t)$ in the non-dynamo regime for $0 < \xi < \xi_c$. Formally, this dependence can be obtained from the slow-mode expansion of $\tilde{\mathcal{E}}^{ii}_{kk}(\lambda, \mathbf{r}, 1|0, 0)$ with $\lambda = r/(D_1 t)^{1/\nu}$. One obtains for $\lambda \ll 1$ that

$$\mathcal{R}_{kk}(\mathbf{r}, t) \sim \sum_{a, p} (D_1 t)^{-(d+\xi_a)/\nu} \bar{Z}^{i\nu}_{kk}(a, p) \left[W^{ii}_{(a, p)}(0, 1)\right]^\nu.$$

Thus, one expects $\mathcal{R}$ should exhibit a power-law decay in time, with the dominant terms given by the two isotropic zero modes

$$\mathcal{R}_{kk}(\mathbf{r}, t) \sim C_1(D_1 t)^{-(d+\xi_1)/\nu} \bar{Z}^{(1)}_{kk}(\mathbf{r}) + C_2(D_1 t)^{-(d+\xi_2)/\nu} \bar{Z}^{(2)}_{kk}(\mathbf{r}).$$

In all dimensions $d$ the exponent $\xi_2 > 0$ for $0 < \xi < 1$, becoming negative for $\xi > 1$. Thus, the first term with $\xi_1 = -\xi$ dominates for lower $\xi$ values. There is a critical dimension $d_c \approx 4.659$, given by the positive real root of the cubic polynomial $d^2 - 3d + 4d - 16$, above which it is instead true that $\xi_2 < \xi_1$ when $\xi > \xi_c$ with

$$\xi_c = \frac{\sqrt{(d^2 - 3d + 4)^2 + 8(d - 1)^2(d - 2) - (d^2 - 3d + 4)}}{2(d - 1)}.$$

Note that $\xi < \xi_c$, where the dynamo transition occurs. The latter value [7]

$$\xi_c = (d - 1) \left(\sqrt{\frac{d - 1}{2(d - 2)}} - \frac{1}{2}\right)$$

is the point at which $\tilde{\mathcal{E}}_2$ develops an imaginary part and the slow-mode expansion above breaks down.

---

We remark that related eigenfunction expansions hold for the heat kernels:

$$F_{ij}^{kk}(\mathbf{r}, 0|\rho, t) = F_{ij}^{kk}(\rho, 0|\mathbf{r}, t) = \sum_a \mathcal{E}^{a\nu}_{ij} E^{kk}_a(\mathbf{r}) \tilde{\mathcal{E}}^{\nu}_a(\rho).$$

New Journal of Physics 12 (2010) 023021 (http://www.njp.org/)
Unfortunately, the above argument is incorrect as it stands or, at least, incomplete. The slow-mode expansion of the propagators $F_{ij}^{k\ell}(\rho, t/\lambda, 0) = \bar{F}_{ij}^{k\ell}(\lambda r, t/\rho, 0)$ can only be applied in the scale-invariant inertial-convective range with all distances $\lambda r, \rho \gg \ell_x$. Thus, it is incorrect to apply this expansion to $R_{k\ell}(r, t)$ with separation $\rho = 0$. It turns out that the self-similar decay solutions $W_{ij}(\rho, t)$ that appear in the expansion (43) diverge to infinity in the limit $\rho \to 0$ through the convective range. Finite results are only obtained by a careful matching with resistive range solutions at the lengthscale $\rho \simeq \ell_x$. The consequence is that the coefficients $C_1, C_2$ in the decay law (51) have additional power-law dependence on both $\ell_x$ and $t$. In order to determine this behavior, we must undertake a careful study of magnetic energy decay.

3. Decay of the magnetic field

We now consider in detail the problem of the turbulent decay of the magnetic energy $\langle B^2(t) \rangle$ in the non-dynamo regime of the KK model, for $0 < \xi < \xi_s$ and $Pr < Pr_c$.

3.1. Discussion of the convective-range decay law

We begin by giving a simple, heuristic explanation of the decay law of the magnetic energy, for generic initial data of the magnetic field with rapid decay of correlations in space. The fundamental observation is that the zero modes $\bar{Z}^{(a)}(r)$, $a = 1, 2, 3, \ldots$, of the adjoint operator $\mathcal{M}^*$ give rise to statistical conservation laws in the evolution of the two-point correlation,

$$\bar{J}_a(t) \equiv \int d^d r \bar{Z}^{(a)}_{ij}(r)C^{ij}(r, t),$$

which satisfy

$$(d/dt)\bar{J}_a(t) = \int d^d r \bar{Z}^{(a)}_{ij}(r) \cdot [\mathcal{M}(r)]_{pq}^{ij}C^{pq}(r, t)$$

$${\} = \int d^d r [\mathcal{M}^*(r)]_{pq}^{ij} \bar{Z}^{(a)}_{ij}(r) \cdot C^{pq}(r, t) = 0.$$ 

Note, however, that only the non-gradient zero modes lead to nontrivial conservation laws because of the orthogonality of solenoidal and gradient correlations. The leading-order zero mode is thus $\bar{Z}^{(2)}$ found in the previous section, namely,

$$\bar{Z}^{(2)}_{ij}(r) = r^{\tilde{c}_2} \left[ \bar{A}^{(2)}_L \hat{r}_i \hat{r}_j + \bar{A}^{(2)}_N (\delta_{ij} - \hat{r}_i \hat{r}_j) \right],$$

with

$$\bar{A}^{(2)}_L = (\xi - 2)((d - 1)\tilde{c}_2 + (d - 3)(d + \xi - 1)],$$

$$\bar{A}^{(2)}_N = (d + 1)\xi + 2(d - 1).$$

This zero mode coincides with that found by Celani et al [14] for $d = 3$.

The corresponding conserved quantity $\bar{J}_2(t)$ plays the same role in the turbulent decay of the magnetic field as played by the ‘Corrsin invariant’ in the decay of the passive scalar [15, 16]. Assume, in fact, a self-similar decay law for the magnetic correlation

$$C^{ij}(r, t) = \bar{r}^2(t)\Gamma^{ij}(r/L(t)).$$
The length $L(t)$ is a large-distance correlation length or ‘integral length’ of the magnetic field. The quantity $\hat{h}(t)$ is a measure of the magnitude of the magnetic fluctuations at scale $L(t)$, which we term the magnetic amplitude. Just as for the scalar, the growth of the magnetic lengthscale $L(t)$ can be obtained dimensionally from

$$\frac{1}{L(t)} \frac{d}{dt} L(t) = D_1 L^{-\gamma}(t),$$

yielding

$$L(t) = [L^\gamma(0) + \gamma D_1 (t-t_0)]^{1/\gamma}.$$  \hfill (53)

To determine the decay rate requires a relation between $\hat{h}(t)$ and $L(t)$, which is provided by invariance of $\bar{J}_2$:

$$\bar{J}_2 = \hat{h}^2(t)L^{d+\xi_2}(t)C$$

with $C = \int d\rho \; \tilde{Z}_{ij}^{(2)}(\rho) \Gamma^{ij}(\rho)$. Thus, finally,

$$\hat{h}^2(t) \sim \bar{J}_2[L(t)]^{-(d+\xi_2)} \sim (t-t_0)^{-(d+\xi_2)/\gamma}$$

for $(t-t_0) \gg L^\gamma(0)/D_1$. The generic decay of the magnetic amplitude is predicted by this argument to be determined by the scaling exponent $\xi_2$, which decreases with increasing $\xi$ over the range $0 < \xi < \xi_*$. Thus, the decay rate is faster for rougher velocities and slower for smoother velocities. It is noteworthy that the decay law (55) is completely independent of the resistivity.

The above argument does not apply if $\bar{J}_2 = 0$. In that case, one can expect that invariants

$$\bar{J}_{a,p}(t) = \int d^d r \; \tilde{Z}_{ij}^{(a,p)}(r) \Gamma^{ij}(r, t),$$

from higher-order zero modes and slow modes of $\mathcal{M}^*$ (again in the non-gradient sector) will determine the decay rate. Note, for example, that $(d/dt)\bar{J}_{2,1}(t) = \bar{J}_2(t)$, so that $\bar{J}_{2,1}$ becomes invariant if $\bar{J}_2 = 0$. In [15], it was shown that there are two universality classes in the turbulent decay of the passive scalar for generic initial data with rapidly decaying correlations in space, depending upon whether the ‘Corrsin invariant’ $J_0$ from the constant zero mode is vanishing or non-vanishing. If $J_0 = 0$, then there exists a higher-order invariant $J_1 \neq 0$, associated with the first slow mode $r^\gamma$ in the tower above the constant zero mode, which determines the decay. Chaves et al [16] showed how this picture emerges from the slow-mode expansion of Bernard et al [17] and extends to the higher-order correlations of the scalar. In the following section, we shall present a similar treatment of the turbulent decay of the magnetic field, based on our generalized slow-mode expansion in the appendix.

There are essential differences, however, between the turbulent decay of a passive scalar and of a passive magnetic field. Whereas the scalar field has a finite limit as diffusivity $\kappa \to 0$, this is not true for the magnetic field, which even in the non-dynamo regime, tends to accumulate at the resistive scale [21]. As we shall see below, the scaling function $\Gamma^{ij}(\rho)$ grows with $\rho$ decreasing through the convective range. Thus, one cannot set $\rho = 0$ to interpret $\hat{h}^2(t)$ as the magnetic energy. A more correct interpretation of the magnetic amplitude is that $\hat{h}^2(t)/L^2(t) \simeq \langle |A(t)|^2 \rangle$, where $A$ is a vector potential (two-form) such that $B = \delta A$. The decay rate of the magnetic energy $\langle |B(t)|^2 \rangle$ cannot be obtained from purely ideal considerations, but requires an explicit matching of convective-range solutions with resistive-scale solutions. In the following two sections, we treat first the ideal, convective range problem with $\kappa \to 0$. 

*New Journal of Physics* **12** (2010) 023021 (http://www.njp.org/)
3.2. Self-similar decay for initial data with short-range correlations

Consider any initial two-point correlation function $C^{ij}(\mathbf{r}, 0)$ of the magnetic field, which decreases rapidly for large $r$. We shall demonstrate that the correlation $C^{ij}(\mathbf{r}, t)$ at much later times exhibits self-similar decay and determine the decay law. We use the propagator relation (9) and the symmetry properties of $\bar{F}$ to write

$$C^{ij}(\mathbf{r}, t) = \int d^d \rho \, C^{k\ell}(\rho, 0) \bar{F}^{ij}_{k\ell}(\rho, t|\mathbf{r}, 0)$$

$$= \lambda^d \int d^d \rho \, C^{k\ell}(\rho, 0) \bar{F}^{ij}_{k\ell}(\lambda \rho, 1|\mathbf{r}, 0), \quad (56)$$

with $\lambda = 1/(D_1 t)^{1/\gamma}$ and $\mathbf{r} = r/(D_1 t)^{1/\gamma}$. In the last line, we used the scaling property (22) for $\bar{F}$. Since $C^{k\ell}(\rho, 0)$ decays rapidly for $\rho \gg L(0)$, we may employ the slow-mode expansion (43) for $(D_1 t)^{1/\gamma} \gg L(0)$. Because $C^{k\ell}(\rho, 0)$ is solenoidal, only the non-gradient zero modes of $\mathcal{M}^s$ give a non-vanishing contribution.

The leading-order term, in general, is

$$C^{ij}(\mathbf{r}, t) \sim (D_1 t)^{-d(\bar{\zeta} + 2)/\gamma} \left( \int d^d \rho \, C^{k\ell}(\rho, 0) \bar{Z}^{(2)}_{k\ell}(\rho) \right) W^{ij}_{(2)} \left( \frac{\mathbf{r}}{(D_1 t)^{1/\gamma}}, 1 \right)$$

$$\sim \left( \int d^d \rho \, C^{k\ell}(\rho, 0) \bar{Z}^{(2)}_{k\ell}(\rho) \right) W^{ij}_{(2)}(\mathbf{r}, t). \quad (57)$$

In the last line, we have used the self-similarity property

$$W^{ij}_{(2)}(\lambda \mathbf{r}, \lambda^\gamma t) = \lambda^{-d(\bar{\zeta} + 2)} W^{ij}_{(2)}(\mathbf{r}, t).$$

We have also used the fact that $W^{ij}_{(2)}$ is a real-valued function. This will be demonstrated in the following section, where we shall derive the explicit functional form of all the self-similar decay solutions. We conclude that, as long as $J_2(0) \neq 0$, the generic magnetic correlation $C^{ij}(\mathbf{r}, t)$ with short-range initial data is proportional at long times to the self-similar decay solution $W^{ij}_{(2)}(\mathbf{r}, t)$.

It is important to demonstrate that the above scenario is statistically realizable [15]. We shall construct now a positive-definite covariance function for which $J_2(0) \neq 0$. This will also demonstrate the positive definiteness of the scaling solution $W^{ij}_{(2)}$, since the dynamics is realizability-preserving and the above argument shows that

$$\lim_{\lambda \to \infty} \lambda^{d(\bar{\zeta} + 2)} C(\lambda \mathbf{r}, \lambda^\gamma t) = J_2(0) \cdot W^{ij}_{(2)}(\mathbf{r}, t).$$

As a simple example we take, with $\mathcal{N} = (\sigma/\sqrt{2\pi})^d$,

$$C^{ij}(\rho, 0) = \mathcal{N} \int d^d k \, (k^2 \delta^{ij} - k^i k^j) \exp \left( -\frac{\sigma^2 k^2}{2} \right) e^{ik \cdot \rho}$$

$$= \left( -\Delta_\rho \delta^{ij} + \partial^{[i}_\rho \partial^{j]}_\rho \right) \exp \left( -\frac{\rho^2}{2\sigma^2} \right).$$

A bit of calculation shows for this example that

$$J_2(0) = (d - 1)(2\sigma^2)^{(d + \bar{\zeta} - 2)/2} S_{d-1} \Gamma \left( \frac{d + \bar{\zeta} - 2}{2} \right) \left[ \bar{A}^{(2)}_L - (\bar{\zeta} + 1) \bar{A}^{(2)}_N \right].$$
where \( S_{d-1} = 2\pi^{d/2} / \Gamma \left( \frac{d}{2} \right) \) is the hypersurface area of the unit sphere in \( d \)-dimensions and \( A_L^{(2)}, A_N^{(2)} \) are the coefficients given in the previous section. At generic values of \( d \) and \( \xi \), \( J_2(0) \neq 0 \). It is noteworthy that \( J_2(0) = 0 \) in the example above precisely at the point of degeneracy of zero modes, where \( \xi_1 = \tilde{\xi}_2 \). As discussed in section 2.3, this occurs for \( d > d_c = 4.659 \) at the single value \( \xi = \xi_c < \xi_\ast \). In fact, \( J_2(0) = 0 \) at this point for all initial data, because there is then a single zero mode of gradient-type satisfying \( A_L^{(2)} = (\tilde{\xi}_2 + 1) A_N^{(2)} \); see (34).

Of course, whenever \( J_2(0) = 0 \) then higher-order terms in the slow-mode expansion become dominant and a different self-similar solution \( W_{(a,p)}(r, t) \) becomes the long-time attractor. We shall defer to future work the study of this non-generic situation.

In the remainder of this section, we shall make some important comments about the generic case \( J_2(0) \neq 0 \). Our first observation is about the property of ‘quasi-equilibrium’. It was shown in [15, 16] that the short-distance scaling of the scalar structure function in the decay of the passive scalar is identical to the scaling of the scalar structure function in a forced steady state. This is the property of turbulence decay traditionally termed ‘quasi-equilibrium’. We show here a similar property for the turbulent decay of the magnetic field, using the slow-mode expansion, as in [16] for the scalar. We use the propagator \( F \), its scaling property (22), and the change of variables \( \tilde{\rho} = \lambda \rho \) with \( \lambda = 1/(D_1 t)^{1/\gamma} \) to write

\[
C^{ij}(r, t) = \int d^d \rho \, C^{kl}(\rho, 0) F_{k\ell}^{ij}(r, t|\rho, 0) = \int d^d \tilde{\rho} \, C^{kl}(\tilde{\rho}/\lambda, 0) F_{k\ell}^{ij}(\lambda r, 1|\tilde{\rho}, 0).
\]

We now employ the slow-mode expansion (42) of \( F \) for \( r \ll (D_1 t)^{1/\gamma} \) to conclude that

\[
C^{ij}(r, t) = \left( \frac{r}{(D_1 t)^{1/\gamma}} \right)^{\zeta_1} Z_{(i)}^{(j)}(\tilde{r}) \int d^d \tilde{\rho} \, C^{kl} \left( (D_1 t)^{1/\gamma} \tilde{\rho}, 0 \right) \tilde{W}_{k\ell}^{(1)\ast} \left( \tilde{\rho}, 1 \right).
\]

The scaling exponent \( \zeta_1 \) and zero mode \( Z_{(i)} \) are the same as found in [21] to determine the short-distance scaling of the magnetic correlation function in the forced steady state, which is just the ‘quasi-equilibrium’ property\(^6\). Since \( \zeta_1 < 0 \) for all \( 0 < \xi < \xi_\ast \), we see that \( C^{ij}(r, t) \) increases without bound as \( r \) decreases, in agreement with our earlier physical discussion. We shall confirm this result by an independent argument in the next section.

\(^6\) Note that (58) applies for small \( r \) at fixed times \( t \), whereas (57) applies at long times \( t \) for fixed \( r \). However, the two results agree in their common domain of validity for \( r, L(0) \ll (D_1 t)^{1/\gamma} \). This may be seen by applying (58) to \( W_{(2)} \) itself to obtain for \( r \ll (D_1 t)^{1/\gamma} \)

\[
W_{(2)}^{ij} \left( \frac{r}{(D_1 t)^{1/\gamma}}, 1 \right) \sim C \left( \frac{r}{(D_1 t)^{1/\gamma}} \right)^{\tilde{\zeta}_1} Z_{(i)}^{(j)}(\tilde{r}).
\]

This result is verified in section 3.3 with the explicit expression for \( W_{(2)} \). Substituting the above into (57) gives

\[
C^{ij}(r, t) \sim C(D_1 t)^{-(d + \xi_\ast \gamma)/\gamma} Z_{(i)}^{(j)}(r).
\]

The same result may be obtained by changing the integration variable in (58) back to \( \rho = \tilde{\rho}/(D_1 t)^{1/\gamma} \) and then employing the similar ‘quasi-equilibrium’ result

\[
\tilde{W}_{k\ell}^{(1)} \left( \frac{\rho}{(D_1 t)^{1/\gamma}}, 1 \right) \sim C \left( \frac{\rho}{(D_1 t)^{1/\gamma}} \right)^{\tilde{\zeta}_2} \tilde{Z}_{k\ell}^{(2)}(\tilde{\rho})
\]

substituted into (58).

New Journal of Physics 12 (2010) 023021 (http://www.njp.org/)
A second observation is that the above discussion—the demonstration of self-similar decay and quasi-equilibrium—carry over directly to the general \( n \)-point correlation function of the magnetic field. Note that

\[
C_n^{i_1i_2...i_n}(\mathbf{r}, t) = \left\{ B^{i_1}(\mathbf{x}_1, t)B^{i_2}(\mathbf{x}_2, t)\cdots B^{i_n}(\mathbf{x}_n, t) \right\} \\
= \int d^d\rho \ C_n^{j_1j_2...j_n}(\rho, 0)\tilde{F}_n^{i_1i_2...i_n}(\rho, 0|\mathbf{r}, t),
\]

where \( \mathbf{r} = (x_1, x_2, \ldots, x_n) \) and \( F_n \) is the \( n \)-body propagator. All the symmetries used in the previous argument hold for general \( n \), e.g. time reversal and \( F_n(\rho, t|\mathbf{r}, 0) = \tilde{F}_n(\mathbf{r}, 0|\rho, t) \). Note that due to space homogeneity only the separation of the variables matter (the absolute position of each particle is irrelevant) and we can work in the \((n-1)d\)-dimensional separation-of-variables sector. \( F_n \) then has the scaling property

\[
F_n(\lambda\rho, 0|\lambda\mathbf{r}, \lambda^\gamma t) = \lambda^{-d(n-1)}F_n(\rho, 0|\mathbf{r}, t).
\]

Finally, slow-mode expansions like (42) and (43) are valid for \( F_n \) and \( \tilde{F}_n \) for all integers \( n \). See [17] and the appendix for details. The whole analysis thus goes through as for \( n = 2 \) above and as in [16] for the scalar case.

### 3.3. General self-similar decay

To complement the previous discussion employing the slow-mode expansion, we shall here determine all possible self-similar decay solutions for the magnetic correlation function \( C \), following the analysis in [15] for the passive scalar. It is convenient to employ the longitudinal correlation \( C_L \) that satisfies equation (30). We introduce the self-similar ansatz

\[
C_L(r, t) = \tilde{h}^2(t)\Gamma\left( \frac{r}{L(t)} \right).
\]  

Substituting the ansatz (59) into equation (30) for \( C_L \), we arrive at, with \( \rho = r/L \),

\[
\frac{1}{D_1L^{-\gamma}(t)} \frac{2\dot{h}(t)}{\tilde{h}(t)}\Gamma(\rho) - \frac{1}{D_1L^{-\gamma}(t)} \frac{\dot{L}(t)}{L(t)}\rho\Gamma'(\rho) = (d-1)\rho^{\xi}\Gamma''(\rho) + (2\xi + d^2 - 1)\rho^{\xi-1}\Gamma'(\rho) + \xi(d-1)(d+\xi)\rho^{\xi-2}\Gamma(\rho).
\]

This implies that

\[
\frac{2\dot{h}(t)}{\tilde{h}(t)} = -\alpha D_1L^{-\gamma}(t),
\]

\[
\frac{\dot{L}(t)}{L(t)} = \beta D_1L^{-\gamma}(t)
\]

with constants \( \alpha \) and \( \beta \). We have freedom in choosing the value of \( \beta \) to fix the lengthscale; here we adopt \( \beta = 1 \). The equation for \( L(t) \) then becomes identical to (52) with solution (53). Combining (61) and (62) yields \( 2\dot{h}(t)/\tilde{h}(t) = -\alpha \dot{L}(t)/L(t) \) with solution

\[
\tilde{h}^2(t) = [L(t)]^{-\alpha}.
\]
Employing (61) and (62), equation (60) for the scaling function \( \Gamma \) becomes

\[
\rho \rho' \left[ \rho \Gamma'(\rho) + \alpha \Gamma(\rho) \right] = (d-1)\rho^2 \Gamma''(\rho) + (2\xi + d^2 - 1) \rho \Gamma'(\rho) + \xi(d-1)(d+\xi) \Gamma(\rho).
\]

(64)

Making the substitution \( x = -\rho' / \gamma(d-1) \) yields

\[
y^2 x^2 \Gamma_{xx} + \left[ \gamma \left( d + \gamma + \frac{2\xi}{d-1} \right) - y^2 x \right] x \Gamma_x + \left[ \xi(d+\xi) - \frac{\alpha \gamma}{d-1} \right] \Gamma = 0.
\]

An equation of this form can be solved by the Frobenius method (see e.g. [28], section 4.2).

According to the general theory, there are two independent solutions of the form \( \Gamma(x) = x^b \Phi(x) \), where \( b \) is a root of the indicial equation

\[
y^2 b(b-1) + \left( d + \gamma + \frac{2\xi}{d-1} \right) y b + \xi(d+\xi) = 0.
\]

If the two roots are distinct and do not differ by an integer, then the two functions \( \Phi \) are both analytic, given by convergent power series. Otherwise, only one solution must be analytic and the second may be an analytic function plus \( C \ln x \) times the first. In our case, it is easy to check that the roots of the indicial equation are just given by \( b = \zeta_1 / \gamma, \omega_2 / \gamma \) in terms of the scaling exponents of the zero modes of \( \mathcal{M} \). If we substitute \( \Gamma = x^{\zeta_1 / \gamma} \Phi \) into the equation for \( \Gamma \), we obtain the Kummer equation [29]

\[
x \Phi_{xx} + (c - x) \Phi_x - a \Phi = 0
\]

(65)

with

\[
a = \frac{\alpha + \zeta_1}{\gamma}, \quad c = \frac{1}{\gamma} \left( 2\zeta_1 + d + \gamma + \frac{2\xi}{d-1} \right).
\]

Both independent solutions can be obtained from this equation. The first is the Kummer function \( \Phi(a, c; x) \), an entire function given by the power series

\[
\Phi(a, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n x^n}{(c)_n n!}
\]

with \( (a)_n = a(a+1) \cdots (a+n-1) \). The other is the Kummer function of the second kind, \( \Psi(a, c; x) \), which is defined by a suitable linear combination of \( \Phi(a, c; x) \) and \( x^{1-c} \Phi(a-c+1, 2-c; x) \). See [29, section 6.5.6]. It is not hard to check that this second term corresponds to the root \( b = \omega_2 / \gamma \) of the indicial equation. However, we can argue as in [21] that matching the solutions in the convective range with those in the dissipation range permits only the regular zero mode as an admissible physical solution. Thus, we obtain \( \Gamma(x) = x^{\zeta_1 / \gamma} \Phi(a, c; x) \).

This result can be simplified somewhat by appealing to the relation

\[
\zeta_2 = \zeta_1 + \gamma + \frac{2\xi}{d-1},
\]

which follows by combining \( \zeta_1 + \omega_2 = -d - 2\xi / (d-1) \) and \( \omega_2 + \tilde{\zeta}_2 = -d + \gamma \). Note that the above relation generalizes the result \( \zeta_2 = \zeta_1 + 2 \) for \( d = 3 \) found in [14]. With this relation, we obtain \( c = (\zeta_1 + \tilde{\zeta}_2 + d) / \gamma \), so that

\[
\Gamma(\rho) = \rho^{\zeta_1} \Phi \left( \frac{\alpha + \zeta_1}{\gamma}, \frac{\zeta_1 + \tilde{\zeta}_2 + d}{\gamma}, -\frac{\rho^{\gamma}}{(d-1)\gamma} \right).
\]

(67)

All the self-similar solutions of equation (30) are given by the ansatz (59) with a scaling function of the form in (67) above and with \( L(t) \) and \( \bar{h}(t) \) given by equations (53) and (63),

\[http://www.njp.org/\]
respectively. Since $\Phi(0) = 1$ is finite, all of these self-similar solutions satisfy the condition of ‘quasi-equilibrium’, showing the same scaling $r^{c_1}$ for $r \ll L(t)$ as found in [21] for the forced steady state.

There are two distinct types of self-similar decay solutions corresponding to different choices of the free parameter $\alpha$. When $\alpha = \tilde{\alpha} + d + p \gamma$, for $p = 0$, $1$, $2$, . . . , then $a = c + p$ with $p = 0$, $1$, $2$, . . . . In this case

$$\Phi(c + p, c; -x) = \frac{p!}{(c)_p} L_p^{c-1}(x)e^{-x},$$

where $L_p^{c-1}(x)$ is the generalized Laguerre polynomial of degree $p$ (see [29, chapter 6]). This series of solutions has stretched-exponential decay in space. If, for example, we take $\alpha = \tilde{\alpha} + d$ corresponding to $p = 0$, then we obtain

$$\Gamma(\rho) = \rho^{c_1} \exp \left( -\frac{1}{d - 1} \frac{\rho^\gamma}{\gamma} \right).$$

The corresponding self-similar decay solution satisfies $C(\lambda r, \lambda^\gamma t) = \lambda^{-(d + \tilde{\alpha})} C(r, t)$. The $\alpha = \tilde{\alpha} + d$ solution thus coincides with the self-similar solution $W_{(2)}(r, t)$, which was shown in the previous section to describe the long-time decay of generic initial data with short-range correlations. More generally, the solutions with $\alpha = \tilde{\alpha} + d + \gamma p$ coincide with the self-similar solutions $W_{(2,p)}(r, t)$ for $p = 0$, $1$, $2$, . . . , which appear in the slow-mode expansion (43) of the adjoint propagator $F$.

For any other choice of $\alpha \neq \tilde{\alpha} + d + \gamma p$ with $p = 0$, $1$, $2$, . . . , one obtains instead a class of self-similar decay solutions with power-law decay of correlations at large distances. This follows from the asymptotic relation $\Phi(a, c; -x) \sim [\Gamma(c)/\Gamma(c - a)] x^{-a}$ for $\text{Re} x \to +\infty$, if $a \neq c + p$, $p = 0$, $1$, $2$, . . . [29, section 6.13.1]. Using the above relation together with (67), (59) and (63) gives for any self-similar solution with $\alpha \neq \tilde{\alpha} + d + \gamma p$, $p = 0$, $1$, $2$, . . . ,

$$C_L(r, t) \sim Ar^{-a}, \quad r \gg L(t),$$

where $A$ is a time-independent constant. This result is usually called the ‘permanence of the large-scale eddies’ in the turbulence literature. Note that for initial data with such power-law decay of correlations, the relation between $h(t)$ and $L(t)$ that determines the decay rate is obtained from this permanence, as $h^2(t) \simeq A[L(t)]^{-a}$, in agreement with (63). See [15] for more discussion.

### 3.4. Decay law of the magnetic energy

We are now ready to discuss the decay law for the magnetic energy:

$$E(t) = \frac{1}{2} \langle |B(t)|^2 \rangle = \text{Tr} \mathcal{C}(0, t).$$

Under the assumption of isotropic statistics made here, $E(t) = (d/2) C_L(0, t)$. Clearly, in order to evaluate this expression at $r = 0$, we must consider the matching of our convective range solution to the resistive scales. We may do this heuristically, as follows. We assume that, to leading order,

$$E(t) \simeq (d/2) C_L(\ell_x, t).$$
We then estimate the correlation function on the right by matching with the convective-range expression

\[ C_L(r, t) \simeq C_0 h^2(t) \left( \frac{r}{L(t)} \right)^{\zeta_1} \simeq C_0[L(t)]^{-(\alpha + \zeta_1)} r^\zeta_1 \]

for \( r \ll L(t) \) and some positive constant \( C_0 \). This yields

\[ E(t) \simeq C_1[L(t)]^{-(\alpha + \zeta_1)} \xi^\zeta_1 \]

for \( C_1 = (d/2)C_0 \). Although the energy magnitude increases as resistivity is lowered, the decay rate is independent of resistivity. Since \( \zeta_1 < 0 \), we see that the decay of magnetic energy \( E(t) \propto (t - t_0)^{-(\alpha + \zeta_1)/\gamma} \) is always slower than the decay of magnetic amplitude \( h^2(t) \propto (t - t_0)^{-\alpha/\gamma} \). For example, for the generic case with \( \alpha = \bar{\zeta} + d \), we obtain

\[ E(t) \propto (t - t_0)^{-c}, \]

with \( c = (\zeta_1 + \bar{\zeta} + d) / \gamma \).

The above argument is basically correct, but not fully rigorous. Since the time dependence of magnetic energy is crucial to the issue of whether dynamo action is present or not, we give a more systematic derivation by a standard method of matched asymptotic expansions (see, for instance, chapter V of [30]). The equation obeyed by \( C_L \) for \( \kappa > 0 \) is

\[ \partial_t C_L = [(d - 1) \rho^\epsilon \partial_r^2 C_L + (2 \xi + d^2 - 1) r^{\xi - 1} \partial_r C_L + \xi (d - 1) (d + \xi) r^{\xi - 2} C_L] + 2 \kappa \left[ \partial_r^2 C_L + (d + 1) \frac{1}{r} \partial_r C_L \right]. \]

(69)

See [7]. Substituting the self-similar ansatz (59), we obtain

\[ [\alpha \Gamma + \rho \Gamma_\rho] + [(d - 1) \rho^\epsilon \Gamma_\rho \rho + (2 \xi + d^2 - 1) \rho^{\xi - 1} \Gamma_\rho + \xi (d - 1) (d + \xi) \rho^{\xi - 2} \Gamma] + \epsilon^\xi \left[ \Gamma_\rho \rho + (d + 1) \frac{1}{\rho} \Gamma_\rho \right] = 0, \]

(70)

with \( \rho = r/L(t) \) and \( \epsilon \equiv \ell_\kappa / L(t) \). In the outer range where \( \rho = O(1) \), the dominant balance in equation (70) is between the first term from the time derivative, which acts like a forcing, and the second term from the turbulent advection. The third term may be neglected for small \( \epsilon \), yielding the leading-order equation for the outer solution. This is the same equation that was examined in the preceding section 3.3, where all self-similar solutions were found. Thus the outer solutions \( \Gamma_{out}(\rho) \) are given by formula (67) for any choice of \( \alpha \) and multiplied by an arbitrary constant \( C_{out} \). These solutions must now be matched to an appropriate inner solution in the resistive range.

We introduce the inner variable \( \sigma = r / \ell_\kappa \equiv \rho / \epsilon \) in equation (70) to obtain

\[ \epsilon^\gamma \left[ \alpha \Gamma + \sigma \Gamma_\sigma \right] + [(d - 1) \sigma^\epsilon \Gamma_{\sigma \sigma} + (2 \xi + d^2 - 1) \sigma^{\xi - 1} \Gamma_\sigma + \xi (d - 1) (d + \xi) \sigma^{\xi - 2} \Gamma] + \left[ \Gamma_{\sigma \sigma} + (d + 1) \frac{1}{\sigma} \Gamma_\sigma \right] = 0. \]

(71)

The dominant balance in (71) is between the second term from the turbulent advection and the third term from the molecular resistivity. To leading order, we can disregard the first term proportional to \( \epsilon^\gamma \) to obtain

\[ \sigma^2 \Gamma_{\sigma \sigma} + (d + 1) \sigma \Gamma_\sigma + \sigma^\xi \left[ (d - 1) \sigma^2 \Gamma_{\sigma \sigma} + (2 \xi + d^2 - 1) \sigma \Gamma_\sigma + \xi (d - 1) (d + \xi) \Gamma \right] = 0. \]

(72)
Making the change of variables \( y = -(d - 1)\sigma^\xi \) reduces this to a hypergeometric equation [29, chapter II]:

\[
y(1 - y)\Gamma_{yy} + [c_s - (a_s + b_s + 1)y] \Gamma_y - a_s b_s \Gamma = 0, \tag{73}
\]

where

\[
a_s + b_s = \frac{1}{\xi} \left( \frac{2\xi}{d - 1} + d \right), \quad a_s b_s = c_s = \frac{d + \xi}{\xi}. \tag{74}
\]

Up to an overall multiplicative constant, there is a unique solution \( F(a_s, b_s; c_s; y) \) of the above equation that is analytic in the region \( \arg(1 - y) < \pi \) of the complex \( y \)-plane. This hypergeometric function is given for \( |y| < 1 \) by the absolutely convergent power series,

\[
F(a_s, b_s; c_s; y) = \sum_{n=0}^{\infty} \frac{(a_s)_n (b_s)_n}{(c_s)_n} y^n \frac{n!}{n!}, \tag{75}
\]

if \( c_s \neq 0, -1, -2, \ldots \). The other independent solution, \( y^{1-c_1} F(a_s + 1 - c_s, b_s + 1 - c_s; 2 - c_s; y) \) [29, 2.9(17)], is singular at \( y = 0 \) and must be discarded. Because of the symmetry \( F(a_s, b_s; c_s; y) = F(b_s, a_s; c_s; y) \), we have freedom in choosing \( a_s \) and \( b_s \). Combining the equations in (74) yields a quadratic equation for \( a_s \)

\[
a_s^2 - \frac{1}{\xi} \left( \frac{2\xi}{d - 1} + d \right) a_s + \frac{d + \xi}{\xi} = 0
\]

and an identical equation for \( b_s \). It is easy to check that the roots are just \( -\zeta_1/\xi \) and \( -\omega_2/\xi \), where \( \zeta_1, \omega_2 \) are the scaling exponents found in section 2.3. We choose \( a_s = -\zeta_1/\xi \) and \( b_s = -\omega_2/\xi \). Thus, we obtain

\[
\Gamma_{\text{in}}(\sigma) = C_{\text{in}} F \left( \frac{-\zeta_1}{\xi}, \frac{-\omega_2}{\xi} ; \frac{-d + \xi}{\xi} ; -(d - 1)\sigma^\xi \right)
\]

for the inner solution, with an arbitrary constant \( C_{\text{in}} \). This solution gives the complete description in the resistive range, e.g. implying a magnetic energy spectrum \( \tilde{E}(k) \propto k^{-(1+\xi)} \) for \( \ell_k \ll 1 \).

To match this solution to the outer solution, we must find its asymptotic behavior for \( \sigma \gg 1 \). This is given by \( F(a_s, b_s; c_s; y) \sim \{\Gamma(c_s)\Gamma(b_s - a_s)/[\Gamma(b_s)\Gamma(c_s - a_s)]\}(-y)^{-a_s} \) as \( \text{Re} \ y \rightarrow -\infty \), for \( a_s < b_s, \ a_s \neq c_s + p \) with \( p = 0, 1, 2, \ldots \) (see [29, section 2.1.4(17)]) to be

\[
\Gamma_{\text{in}}(\sigma) \sim C_{\text{in}} \frac{\Gamma(c_s)\Gamma(b_s - a_s)}{\Gamma(b_s)\Gamma(c_s - a_s)} (d - 1)^{\xi_1/\xi} \cdot \sigma^{\xi_1}
\]

for \( \sigma \gg 1 \). This is the same power law as \( \Gamma_{\text{out}}(\rho) \sim C_{\text{out}} \rho^{\xi_1} \) for \( \rho \ll 1 \). Equating the inner and outer solutions \( \Gamma_{\text{in}}(\sigma) = \Gamma_{\text{out}}(\rho) \) in the overlap region \( \epsilon \ll \rho \ll 1 \) yields the relationship

\[
C_{\text{in}} = \frac{\Gamma(b_s)\Gamma(c_s - a_s)}{\Gamma(c_s)\Gamma(b_s - a_s)} (d - 1)^{\xi_1/\xi} \cdot \epsilon^{\xi_1} \cdot C_{\text{out}}.
\]

Note that the first factor is a numerical constant \( B(\xi) \) satisfying \( B(0) = d - 1 \) and \( B(\xi_1) = 0 \), and varying smoothly between those limits.

Finally, we obtain the magnetic energy from \( E(t) = (d/2)\tilde{h}^2(t)\Gamma_{\text{in}}(0) = (d/2)\tilde{h}^2(t)C_{\text{in}} \), which, with \( \epsilon = \ell_k/L(t) \) and \( \tilde{h}^2(t) = [L(t)]^{-\sigma} \), gives

\[
E(t) = C_1 [L(t)]^{-(\sigma + \xi_1)} \epsilon^{\xi_1} \ell_k^2
\]

for \( C_1 = (d/2)B(\xi) \cdot C_{\text{out}} \). This differs from the previous heuristic estimate only by a constant factor.
3.5. Dynamo order parameter and magnetic induction

The above arguments allow us to complete the analysis of the dynamo ‘order parameter’ \( R_{kl}(r, t) \) from section 2.5. By a similar matching argument as above, we can set

\[
\tilde{F}_{kk}^{ii}(r, t|0, 0) \simeq \bar{F}_{kk}^{ii}(r, t|\ell_k, 0),
\]

where on the right-hand side one employs the scale-invariant, convective-range solution. In this range, one can use scale homogeneity and the slow-mode expansion (43) to write

\[
\tilde{F}_{kk}^{ii}(r, t|\ell_k, 0) = \lambda^d \tilde{F}_{kk}^{ii}(\lambda r, 1|\lambda \ell_k, 0)
\]

\[
\sim \lambda^d \sum_{a, p} \lambda^{\zeta_a + \gamma(p)} \tilde{Z}_{kk}(a, p)(r) W_{(a, p)}(\lambda \ell_k, 1),
\]

with \( \lambda^{-1} \equiv (D_1 t)^{1/\gamma} \gg r \gg \ell_k \). Since \( W_{(a, p)}(\lambda \ell_k, 1) \simeq C_{(a, p)} (\lambda \ell_k)^{\zeta_1} \) it follows finally that

\[
R_{kl}(r, t) \equiv \tilde{F}_{kk}^{ii}(r, t|0, 0) \simeq \sum_{a, p} C_{(a, p)} \ell_k^{\zeta_1} (D_1 t)^{-(\zeta_a + \zeta_1 + d + \gamma p)/\gamma} \tilde{Z}_{kk}(a, p)(r)
\]

for \( r/(D_1 t)^{1/\gamma} \ll 1 \). We remark that the self-similar decay solutions \( W_{(a, p)} \) that appear in the slow-mode expansion are identified as those in the previous discussion with parameter \( \alpha = -\omega_0 + (p + 1)\gamma = \zeta_a + d + \gamma p \). It thus follows that \( W_{(a, p)}(0, t) \simeq (\text{const.})(D_1 t)^{-(\zeta_a + \zeta_1 + d + \gamma p)/\gamma} \ell_k^{\zeta_1} \).

If we retain only the two leading isotropic terms in the above expansion, we obtain a corrected version of the decay law (51) for the line correlations:

\[
R_{kl}(r, t) \sim C_1^1(D_1 t)^{-(d + \zeta_1 + \zeta_2)/\gamma} \ell_k^{\zeta_1} \tilde{Z}_{kk}^{(1)}(r) + C_2^2(D_1 t)^{-(d + \zeta_1)/\gamma} \ell_k^{\zeta_1} \tilde{Z}_{kk}^{(2)}(r).
\]

This result may be substituted into (48) to determine the magnetic energy \( \langle B^2(t) \rangle \) with initial correlations \( \bar{C}_{kl}(r, 0) \) that decay rapidly for \( r \gg L(0) \). Only the non-gradient zero mode \( \tilde{Z}_{kk}^{(2)} \) contributes for solenoidal correlations. In this way, we recover the generic decay law (68) for initial data with short-range correlations and \( J_2(0) \neq 0 \).

In section 2.5, we have introduced another possible ‘order parameter’ \( \mathcal{R}(t) \) obtained from \( R_{kl}(r, t) \) by summing over \( k = \ell \) and integrating \( r \) over all space. It is important to emphasize that the time dependence of \( \mathcal{R}(t) \) does not follow directly from that of \( R_{kl}(r, t) \). We cannot argue that \( \mathcal{R}(t) \) decays as a power in the non-dynamo regime, because the slow-mode expansion of \( R_{kl}(r, t) \) applies only for \( r \ll (D_1 t)^{1/\gamma} \), whereas the definition of \( \mathcal{R}(t) \) involves an integral over all \( r \). Also we cannot conclude that \( \mathcal{R}(t) \) grows exponentially in the dynamo regime, because this requires the condition \( \int d^d r \bar{E}_{kk}(r) \neq 0 \), which needs to be shown. However, we shall now show that our previous discussion of self-similar decay solutions determines the time dependence in the non-dynamo regime.

The ‘order parameter’ \( \mathcal{R}(t) \) that we defined in (49) can, in fact, be interpreted as the energy of a certain self-similar decay solution \( C_{(0)} \) corresponding to an initial condition, which is a random, statistically isotropic but spatially uniform magnetic field. Such a random magnetic field has a covariance of the form

\[
C_{ij}(r, 0) = A \delta^{ij}
\]

for a positive real number \( A \). A constant correlation such as the above would be invariant for an advected scalar, but it is not for a magnetic field. There is a well-known physical phenomenon of ‘shredding’ [1] or ‘induction’ [13] of a constant magnetic field due to the stretching term \( (\mathbf{B} \cdot \nabla) \mathbf{u} \) in the evolution equation. Thus, an initially constant magnetic field will develop a
The correlation at later times with the above initial condition is provided by (9), which yields
\[ C_{ij}^{(0)}(r, t) = A \int d^d \rho \ F_{kk}^{ij}(\rho, 0| r, t). \]
For the limit \( \kappa \to 0 \) in the KK model, the scaling relation (22) for \( \bar{F} \) then implies that
\[ C_{ij}^{(0)}(r, t) = C_{ij}^{(0)}(r, 0). \]
Thus, \( C_{ij} \) is a self-similar solution of \( \partial_t C_{ij} = \mathcal{M} C_{ij} \). It is clearly the self-similar solution with parameter \( \alpha = 0 \) in our general classification. On the other hand, if we take \( A = 1/d \), then also
\[ C_{ij}^{(0)}(r, t) = \frac{1}{d} \int d^d \rho \ F_{kk}^{ij}(r, t| \rho, 0) \]
\[ = \langle \delta \ell^i(t) \delta \ell^j(t) \rangle_r. \]
The latter expression denotes the correlation of line vectors, which started as the same random unit vector at time 0, at any pair of points, which ended up at time \( t \) at points displaced by \( r \). We should emphasize that this result is valid for any divergence-free advecting velocity field and thus applies as well to incompressible fluid turbulence. It immediately follows by summing over \( i = j \) and setting \( r = 0 \) that
\[ \mathcal{R}(t) = 2E_{(0)}(t), \]
where \( E_{(0)}(t) \) is the energy of the solution \( C_{(0)} \).

Our analysis in the previous section can be applied to describe the behavior of \( C_{(0)} \). The formula \( C_L(r, t) = \Gamma(r/L(t)) \) holds using the analytic expression (67) for \( \Gamma \) with \( \alpha = 0 \) and \( L(t) = (D_1 t)^{1/\gamma} \), valid for all \( r \gg \ell_k \). It is more interesting to consider various asymptotic behaviors. The ‘permanence of large eddies’ implies that
\[ C_L(r, t) \simeq A, \quad r \gg L(t). \]
In the convective range
\[ C_L(r, t) \simeq A \left( \frac{r}{L(t)} \right)^{\xi_1}, \quad \ell_k \ll r \ll L(t). \]
Finally, for \( r \to 0 \) and long times,
\[ E_{(0)}(t) \propto A \ell_k^{\xi_1} (D_1 t)^{|\xi_1|/\gamma}, \]
which grows with decreasing \( \kappa \) or increasing \( t \), but only as a modest power law. This result may be interpreted in terms of material-line correlations by setting \( A = 1/d \):
\[ \mathcal{R}(t) = \langle \delta \ell(t) \cdot \delta \ell'(t) \rangle_0 \propto \ell_k^{\xi_1} (D_1 t)^{|\xi_1|/\gamma}, \quad (77) \]
which implies that this quantity grows slowly with time.

It would be of great interest to determine the time dependence of \( \mathcal{R}(t) \) also in the dynamo regime. If the leading eigenfunction of \( \mathcal{M}^* \) satisfies \( \int d^d r \ \tilde{\mathcal{E}}_{kl}(r) = 0 \), then \( \mathcal{R}(t) \) need not grow exponentially. Note, for example, that the space integral of the dual eigenfunction \( \tilde{\mathcal{E}}_{ij}(r) \) does vanish, so the issue is not straightforward. In a following work [27] one of the present authors shows that, at least for space dimension \( d = 3 \), \( \int d^d r \ \tilde{\mathcal{E}}_{kl}(r) \neq 0 \), and thus \( \mathcal{R}(t) \simeq e^{E_{(0)}} \) in the small-Prandtl-number dynamo regime.
4. Final discussion

Our work leads to several important conclusions regarding the small-scale turbulent kinematic dynamo.

4.1. Breakdown of flux freezing and dynamo

In order to understand the turbulent dynamo process, a crucial fact is that magnetic field lines are not frozen into the plasma flow, even in the zero-resistance limit $\kappa \to 0$. Flux-freezing would imply that only a single field line is advected into each space point from the field configuration at an earlier time. In fact, infinitely many field lines are carried to each point by a combination of fluid advection and resistive diffusion [14, 18]. In the Kraichnan velocity ensemble, the probability for two line elements to arrive at the same point at time $t$ starting from points separated by $r$ at time 0 is $P(\mathbf{0}, t | \mathbf{r}, 0) \propto \exp(-r^2/\gamma D_I t)$ in the limit $\kappa \to 0$ and does not degenerate into a delta function $\delta^d(r)$ [17]. This is a manifestation of the phenomenon of ‘spontaneous stochasticity’ first pointed out by Bernard et al [17], which is due to the explosive separation of pairs of fluid particles undergoing turbulent Richardson diffusion. It was argued in [31] that this behavior as $\kappa \to 0$ holds, in general, for a turbulent plasma with a rough velocity field, so that Alfvén’s theorem on flux conservation remains as a stochastic law only.

The breakdown of flux-freezing in the case of rough velocity fields renders the turbulent kinematic dynamo an even more subtle problem than the laminar (or large Prandtl number) kinematic dynamo (for the latter, see e.g. [32]–[36]). For the very smooth velocities considered there ($\xi = 2$), Alfvén’s theorem holds in its usual form in the limit $\kappa \to 0$. However, for rougher velocities with rugosity exponent anywhere in the range $0 < \xi < 2$, an infinite number of field lines enters each point even in the zero-resistance limit. The resultant magnetic field is the resistive average over the field vectors of all of the individual lines. We have shown in this work that the presence of small-scale kinematic dynamo effect depends upon the existence of sufficient angular correlation between the individual field vectors. Thus, dynamo action occurs in the KK model for smoother velocities with $\xi_s < \xi < 2$ but not for rougher velocities with $0 < \xi < \xi_s$. This is true despite the fact that the stretching rate of individual field lines is much greater for $\xi$ smaller.

In section 2.3, we defined $R_{kk}(\mathbf{r}, t) = F_{kk}^{ij}(\mathbf{0}, t | \mathbf{r}, 0)$, which measures the correlation between line elements $\delta \ell_k(t)$ and $\delta \ell_i(t)$ at the same point at time $t$ which started out as unit vectors $\hat{e}_k$, $\hat{e}_i$ at distinct points separated by $r$ at time 0. We found that, in the non-dynamo regime of the KK model with $0 < \xi < \xi_s$, this quantity scales as (76)

$$R_{kk}(\mathbf{r}, t) \sim C \ell^d_k(D_I t)^{-d + \xi + \xi_0/\gamma} \tilde{Z}_{kk}(\mathbf{r}),$$

for $\ell_\nu$, $\ell_\kappa \ll r \ll (D_I t)^{1/\gamma}$. Here $\tilde{Z}$ is an appropriate zero mode of $M^*$ scaling as $\tilde{Z}_{kk}(r) \propto r^\zeta$, with $-d < \zeta < 0$. This correlation decays only as a power for $r$ increasing through the inertial-convective range, implying that line vectors initially separated by distances $\sim (D_I t)^{1/\gamma}$ contribute substantially to the final average. The correlation does not vanish as $\kappa \to 0$ but, in fact, increases as a moderate power of $\ell_\kappa$, demonstrating that infinitely many field lines continue to contribute in that limit. The result is, however, a correlation $R_{kk}(\mathbf{r}, t)$ slowly decaying in time. On the other hand, the lengths of the individual line elements $\langle \delta^2 \ell_k(t) \rangle$, $\langle \delta^2 \ell_i(t) \rangle$ grow...
exponentially in time as in (37) with rate \( \lambda \propto v/\ell_v^2 = 1/t_v \). The result is that
\[
\frac{R_{k\ell}(r, t)}{\sqrt{\langle \delta \ell_k^2(t) \rangle \langle \delta \ell_\ell^2(t) \rangle}} \to 0,
\]
(78)

exponentially rapidly either as \( t \to \infty \) or as \( \kappa \to 0 \) with \( v < Pr \kappa \). We conclude that the dynamo fails for a very rough velocity field because advected line vectors arrive at the same point with insufficient angular correlation. Although individual field lines are stretched to an incredible degree, resistive averaging of nearly uncorrelated lines leads to almost complete cancellation.

The situation is qualitatively different in the dynamo range with smoother velocities (\( \xi_s < \xi < 2 \)). In that case, we have from (50) that
\[
R_{k\ell}(r, t) \propto e^{E_0 t} \tilde{\xi}_{k\ell}(r),
\]
where \( E_0 \) is the dynamo growth rate. Since \( E_0 \propto 1/t_k \ll \lambda \propto 1/t_v \), for \( \lambda \) in (38), it is still true that the angular correlations (78) decay exponentially either as \( t \to \infty \) or as \( \kappa \to 0 \) with \( Pr \) small enough. However, the decay exponent is reduced by a finite amount. Enough correlations remain between line elements entering a point that the net magnetic field after resistive averaging can profit from stretching of individual lines and exponential growth of magnetic energy ensues.

4.2. Hydrodynamic and magnetohydrodynamic (MHD) turbulence

Much of the formalism of this paper carries over to the problem of kinematic dynamo for a weak seed magnetic field in hydrodynamic turbulence. The propagators \( \tilde{F}^{ij}_{k\ell}(\rho, t|0) = F^{ij}_{k\ell}(r, 0|\rho, t) \) give a fundamental description of the kinematic dynamo for any incompressible advecting flow. All of the results of section 2.1 apply, in general, in particular equations (9), (11) and (13), and also the relationship (48) in section 2.5 between magnetic energy and line-vector correlations. Any further simplifications from space homogeneity and time stationarity also apply where appropriate. On the other hand, some features of the KK model are quite special and do not apply more generally. The self-similarity property (22) of the propagators \( F \) and \( \tilde{F} \) does not carry over to hydrodynamic turbulence, because of small-scale intermittency of the advecting velocity field. Also, the statistics of forward and backward Lagrangian trajectories are not identical in hydrodynamic turbulence [37]. Thus, relations such as (16), which depend upon time-reversal symmetry, do not apply to the real turbulent dynamo. Lastly, the time evolution of the propagators \( F \) and \( \tilde{F} \) is, in general, non-Markovian and thus the simple diffusion equations such as (20) and (21) do not apply. One of the present authors (GE) is currently carrying out a numerical evaluation of these propagators for hydrodynamic turbulence, which will be reported elsewhere.

We expect that many of the ideas of this work will apply even to nonlinear MHD turbulence and the dynamo effect there. A stochastic form of flux-freezing and Alfvén’s theorem holds also for non-ideal (viscous and resistive) MHD [18]. We expect these conservation laws to remain stochastic in the limit \( \kappa \to 0, v \to 0 \) with \( Pr \) fixed [31]. However, there will be nontrivial differences from the kinematic problem studied here, due to backreaction of the magnetic field on the plasma flow via the Lorentz force. For example, in comparison with hydrodynamic turbulence, two-particle relative diffusion in MHD turbulence is observed to be suppressed in the direction transverse to the local magnetic field [38]. In principle, however, one can account for all such nonlinear effects by the presence of a second stochastic conservation law.
a modified Kelvin theorem [18, 39, 40]. We believe that ‘spontaneous stochasticity’ and the implied stochasticity of magnetic-line motion and flux conservation must play a central role in the understanding of MHD turbulence, dynamo and reconnection.

Acknowledgments

We thank W Hacker for a useful discussion of the matched asymptotics in section 3.4. The work of GE was partially supported by NSF grant AST-0428325 at the Johns Hopkins University. AFN was partially supported by CNPq-Brazil.

Appendix. Slow mode expansion for non-Hermitian evolution

Unlike for the passive scalar, the $n$-body evolution operators $\mathcal{M}_n$ for the passive magnetic field in the Kraichnan model are no longer even formally Hermitian, with $\mathcal{M}_n^* \neq \mathcal{M}_n$. Nevertheless, certain important properties of the scalar evolution operators remain true for $\mathcal{M}_n$ and $\mathcal{M}_n^*$: these are homogeneous of degree $-\gamma$, reality-preserving, and—in the non-dynamo regime—having an absolutely continuous spectrum on the negative real axis. As we shall show in the following, the above properties together with assumed analyticity conditions allow us to generalize the zero-mode and slow-mode expansions derived in [17] for the Hermitian case to pairs of non-Hermitian operators $\mathcal{M}$ and $\mathcal{M}^*$. Although our intended application is to the KK kinematic dynamo model, we shall carry out the derivation in an abstract, general setting. We shall employ the properties of $\mathcal{M}$ and $\mathcal{M}^*$ given above and also other properties that will be stated explicitly below. The entire argument is modeled very closely after that in [17], with just a few important differences that are stressed below.

A.1. Zero-mode expansion

We shall assume that the operators $\mathcal{M}$ and $\mathcal{M}^*$ act on $L^2(\mathbb{R}^d)$. The dimension $d$ need not be identified with the physical space dimension, as in the main text of the paper. (For example, if $d_S$ is the space dimension, then the $n$-body operators in the Kraichnan model act on $L^2(\mathbb{R}^d)$ with $d = nd_S$ or with $d = (n - 1)d_S$ in the translation-invariant sector.) Define Green’s functions for the operators by

$$
-M_x G(x, y) = -M_y^* G(x, y) = \delta^d(x - y),
-M_x^* \tilde{G}(x, y) = -M_y \tilde{G}(x, y) = \delta^d(x - y),
$$

(A.1)

where the subscript (x or y) indicates on which variable the operator acts. Note that these Green’s functions are both real-valued and, of course, $\tilde{G}(x, y) = G(y, x)$.

Our aim is to derive the following short-distance asymptotic expansion for $G$:

$$
G(x/L, y) \sim \sum_a L^{-\omega_a} f_a(x)[\tilde{g}_a(y)]^*, \quad L \gg 1,
$$

(A.2)

where * here denotes complex conjugation. The function $f_a$ is a regular zero mode of $\mathcal{M}$ with scaling dimension $\xi_a$, while $\tilde{g}_a$ is a singular zero mode of $\mathcal{M}^*$ with scaling dimension $\tilde{\omega}_a = -d + \gamma - \xi_a^*$. What dominates in the expansion (A.2) is the contributing zero mode whose scaling exponent $\xi_a$ has the smallest real part. Thus, we label the exponents according to the
magnitude of their real part, so that \( \text{Re} \, \xi_a > \text{Re} \, \xi_{a'} \) and \( \text{Re} \, \omega_a < \text{Re} \, \omega_{a'} \) for \( a > a' \). We derive also a similar expansion for the adjoint Green’s function

\[
\tilde{G}(x/L, y) \sim \sum_a L^{-\xi_a} \tilde{f}_a(x) [g_a(y)]^*, \quad L \gg 1, \tag{A.3}
\]

where now \( \tilde{f}_a \) is a regular zero mode of \( \mathcal{M}^* \) with scaling dimension \( \tilde{\xi}_a \), while \( g_a \) is a singular zero mode of \( \mathcal{M} \) with scaling dimension \( \omega_a = -d + \gamma - \xi_a^* \). We thus see that the homogeneous zero modes of the operators \( \mathcal{M} \) and \( \mathcal{M}^* \) come in pairs, \( (f_a, g_a) \) and \( (\tilde{f}_a, \tilde{g}_a) \), with related scaling exponents.

Following [17], we employ the Mellin transform, which is a unitary transformation between the spaces \( L^2(\mathbb{R}^d) \) and \( L^2(\text{Re} \, \sigma = -d/2) \otimes L^2(S^{d-1}) \) given by

\[
f(x) \mapsto \tilde{f}(\sigma, \hat{x}) = \int_{0}^{\infty} \lambda^{-\sigma-1} f(\lambda \hat{x}) d\lambda, \tag{A.4}
\]

with the inverse transform, for \( R = |x| \),

\[
f(x) = \frac{1}{2\pi i} \int_{\text{Re} \, \sigma = -d/2} R^\sigma \tilde{f}(\sigma, \hat{x}) d\sigma. \tag{A.5}
\]

The inner product of \( L^2(\text{Re} \, \sigma = -d/2) \otimes L^2(S^{d-1}) \) is

\[
(f, g) = \frac{1}{2\pi i} \int d\omega(\hat{x}) \int_{\text{Re} \, \sigma = -d/2} d\sigma [\tilde{f}(\sigma, \hat{x})]^* \tilde{g}(\sigma, \hat{x}). \tag{A.6}
\]

However, it is more convenient to write this as

\[
(f, g) = \frac{1}{2\pi i} \int d\omega(\hat{x}) \int_{\text{Re} \, \sigma = -d/2} d\sigma [\tilde{f}(\sigma^* - d, \hat{x})]^* \tilde{g}(\sigma, \hat{x}). \tag{A.7}
\]

Although \( [\tilde{f}(\sigma, \hat{x})]^* = [\tilde{f}(\sigma^* - d, \hat{x})]^* \) on the line \( \text{Re} \, \sigma = -d/2 \), the second expression is analytic in \( \sigma \) when \( \tilde{f}(\sigma, \hat{x}) \) is analytic. This form of the inner product allows one to shift integration contours in the complex \( \sigma \)-plane.

A key role in the analysis is played by the operator

\[
\mathcal{N} = R^{r/2} \mathcal{M} R^{-r/2}, \tag{A.8}
\]

which is homogeneous of degree zero. Since it thus commutes with the self-adjoint generator \( D = \frac{1}{i} (x \cdot \nabla_x + \frac{d}{2}) \) of dilatations, it is partially diagonalized under the Mellin transform:

\[
(Nf)^*(\sigma, \hat{x}) = \tilde{N}(\sigma) \tilde{f}(\sigma, \hat{x}),
\]

where \( \tilde{N}(\sigma) \) for each \( \sigma \) is an operator on \( L^2(S^{d-1}) \). Using \( \mathcal{M}^{-1} = R^{r/2} N^{-1} R^{-r/2} \), one straightforwardly derives the following fundamental identity for Green’s function \( G(x, y) = -\mathcal{M}^{-1}(x, y) \):

\[
G(x, y) = -\int_{\text{Re} \, \sigma = -(d/2) + i\gamma/2} \frac{d\sigma}{2\pi i} [R(x)]^\sigma \tilde{N}^{-1} \left( \sigma - \frac{\gamma}{2}, \hat{x}, \hat{y} \right) [R(y)]^{-d - \gamma - \sigma}. \tag{A.9}
\]

See [17]. We note that the shifts in \( \sigma \) arise because \( R^{r/2} \) acts as a translation by \( -\gamma/2 \) under the Mellin transform. The above identity is the key to deriving the zero-mode expansion for \( G \).

The main hypothesis is that the operator function \( \tilde{N}^{-1}(\sigma) \) is meromorphic in a wide vertical strip around the line \( \text{Re} \, \sigma = -d/2 \), whose only singularities are poles

\[
-\tilde{N}^{-1} \left( \sigma - \frac{\gamma}{2}, \hat{x}, \hat{y} \right) \simeq \frac{Z_0(\hat{x}, \hat{y})}{\sigma - \xi_a}
\]
at complex values $\zeta_a$, $a = 1, 2, \ldots$, in the strip. By moving the integration contour in (A.9) further and further to the right, one picks up successive pole contributions. This implies that Green’s function for large $L$ satisfies

$$G \left( \frac{x}{L}, y \right) = \sum_a L^{-\zeta_a} Z_a(x, y)$$

with the function

$$Z_a(x, y) \equiv [R(x)]^{\zeta_a} Z_a(\hat{x}, \hat{y}) [R(y)]^{-d+\gamma-\zeta_a},$$

which is homogeneous of degree $\zeta_a$ in $x$ and of degree $-d+\gamma-\zeta_a$ in $y$. From the definition of Green’s function, using $\mathcal{M}_x = L^{-\gamma} \mathcal{M}_{x'}$ with $x' = x/L$,

$$-L^{-\gamma} \delta^d \left( \frac{x}{L} - y \right) = \mathcal{M}_x G \left( \frac{x}{L}, y \right) = \sum_a L^{-\zeta_a} \mathcal{M}_x Z_a(x, y)$$

from which we obtain $\mathcal{M}_x Z_a(x, y) = 0$ for points off the diagonal. Likewise,

$$-\delta^d \left( \frac{x}{L} - y \right) = \mathcal{M}_y^* G \left( \frac{x}{L}, y \right) = \sum_a L^{-\zeta_a} \mathcal{M}_y^* Z_a(x, y)$$

from which we obtain $\mathcal{M}_y^* Z_a(x, y) = 0$ for points off the diagonal. We finally conclude that $Z_a(x, y)$ for fixed $y$ is a homogeneous zero mode of $\mathcal{M}_x$ of degree $\zeta_a$ and for fixed $x$ is a homogeneous zero mode of $\mathcal{M}_y^*$ of degree $-d+\gamma-\zeta_a$. If we assume that zero modes of a given scaling exponent are non-degenerate, as will generically be true, then we can write

$$Z_a(x, y) = f_a(x) \left[ \tilde{g}_a(y) \right]^*. $$

where $f_a$ is the unique scaling zero mode of $\mathcal{M}$ with exponent $\zeta_a$ and $\tilde{g}_a$ is the scaling zero mode of $\mathcal{M}_y^*$ with exponent $\tilde{\omega}_a = -d+\gamma-\zeta_a$. We have used here the fact that $\mathcal{M}_y^*$ is reality-preserving. This yields (A.2). The expansion (A.3) for $G$ is derived by an identical argument.

Although we shall not employ the corresponding large-distance expansion in this work, we make here a few remarks about it. Under the Mellin transform, the adjoint of $N^{-1}$ has the kernel

$$\tilde{N}^{-1} (\sigma; \hat{x}, \hat{y}) = \left[ \tilde{N}^{-1} (-\sigma^* - d; \hat{y}, \hat{x}) \right]^*. $$

This last relation reveals the important fact that if $\tilde{N}^{-1} (\sigma)$ has a pole at $\zeta_a$ then $\tilde{N}^{-1} (\sigma)$ has a pole at $\tilde{\omega}_a = -d+\gamma-\zeta_a^*$. Indeed, we have

$$\tilde{N}^{-1} \left( \sigma - \frac{\gamma}{2}; \hat{x}, \hat{y} \right) = \left[ \tilde{N}^{-1} \left( -d + \frac{\gamma}{2} - \sigma^*; \hat{y}, \hat{x} \right) \right]^*$$

$$= \left[ \tilde{N}^{-1} \left( -d + \gamma - \sigma^* - \frac{\gamma}{2}; \hat{y}, \hat{x} \right) \right]^*$$

$$\approx \left\{ \frac{-1}{\left( -d + \gamma - \sigma^* \right) - \zeta_a} f_a(\hat{y}) \left[ \tilde{g}_a(\hat{x}) \right]^* \right\}^*$$

$$\approx \frac{-1}{\tilde{\omega}_a - \sigma} \tilde{g}_a(\hat{x}) \left[ f_a(\hat{y}) \right]^*.$$
At this pole with \(\text{Re} \tilde{\omega}_a < -d/2\), the role of the regular and singular zero modes in the residue is reversed. By pushing the integration contour in (A.9) further and further to the left, one can thus derive a large-distance expansion for \(\bar{G}\). This can also be directly obtained from the short-distance expansion for \(G\) as follows:

\[
\bar{G}(Lx, y) = \bar{G}\left(Lx, L \frac{y}{L}\right)
\]

\[
= L^{\gamma-d} \bar{G}\left(x, \frac{y}{L}\right)
\]

\[
= L^{\gamma-d} \left[G\left(\frac{y}{L}, x\right)\right]^* + L^{\gamma-d} \sum_a L^{-\tilde{\omega}_a} \left[f_a(y)\right]^* \bar{g}_a(x)
\]

\[
= \sum_a L^{\tilde{\omega}_a} \bar{g}_a(x) \left[f_a(y)\right]^*
\]

for \(L \gg 1\). Of course, a similar expansion holds for \(G\).

### A.2. The slow-mode expansion: elementary arguments

Define the heat kernels

\[
P(x, t|x_0, t_0) = \langle x|e^{(t-t_0)M}|x_0\rangle,
\]

\[
P(x, t|x_0, t_0) = \langle x|e^{(t-t_0)M^*}|x_0\rangle,
\]

so that, obviously, \(\bar{P}(x, t|x_0, t_0) = P(x_0, t|x, t_0)\). We have the relations

\[
G(x, y) = \int_0^\infty dt \ P(x, t|y, 0)
\]

and

\[
\bar{G}(x, y) = \int_0^\infty dt \ \bar{P}(x, t|y, 0).
\]

Given the validity of the zero-mode expansions for \(G\) and \(\bar{G}\), one should expect that related expansions hold for \(P\) and \(\bar{P}\). We shall show that this is indeed true, with the asymptotic expansion analogous to (A.2) for \(L \gg 1\):

\[
P\left(\frac{x}{L}, t|x_0, 0\right) = \sum_{a,p \geq 0} L^{-\tilde{\omega}_a \gamma_p} f_{a,p}(x)[\bar{g}_{\gamma_p}(x_0, t)]^*.
\]

Here \(f_{a,p}\) are the tower of regular slow modes of \(M\), satisfying \(\lambda_0 f_{a,p} = f_{a,p+1} + f_{a,0} = f_a\). Also, \(\bar{g}_{\gamma_p}\) are solutions of \(\partial_t g_{\gamma_p}(x, t) = M^* \bar{g}_{\gamma_p}(x, t)\) with initial conditions \(\bar{g}_{\gamma_p}(x, 0) = \bar{g}_a(x)\) and \(\bar{g}_{\gamma_p+1}(x, 0) = -M^* \bar{g}_{\gamma_p}(x, 0)\). They satisfy the scaling relations

\[
\bar{g}_{\gamma_p}(\lambda x, \lambda^\gamma t) = \lambda^{\tilde{\omega}_a - \gamma_p} \bar{g}_{\gamma_p}(x, t)
\]

Note that the dominant contribution in (A.13) will generally come from the tower with minimum \(\text{Re}(\tilde{\omega}_a)\) and from the first (zero-mode) term \(p = 0\). There is an analogous expansion for \(\bar{P}\) with \(L \gg 1\):

\[
\bar{P}\left(\frac{x}{L}, t|x_0, 0\right) = \sum_{a,p \geq 0} L^{-\tilde{\omega}_a \gamma_p} \bar{f}_{a,p}(x)[\bar{g}_{\gamma_p}(x_0, t)]^*.
\]

with the roles of the operators \(M\) and \(M^*\) reversed.

New Journal of Physics 12 (2010) 023021 (http://www.njp.org/)
We shall derive the above expansions in this section and the next. Here we proceed by assuming that a general expansion exists for $L \gg 1$ of the form

$$P \left( \frac{X}{L}, t \big| x', 0 \right) \equiv \sum_{a} L^{-\rho_{a}} f_{a}(x) \left[ \tilde{g}_{a}(x', t) \right]^*. \quad (A.15)$$

We shall then identify the form this expansion must take. In the following section, we establish from a more fundamental point of view the existence of such an expansion.

First we substitute (A.15) into

$$\partial_{t} P(x, t|x', 0) = \mathcal{M}^{\alpha}_{x} P(x, t|x', 0)$$

obtaining

$$\sum_{a} L^{-\rho_{a}} f_{a}(x) \left[ \partial_{t} \tilde{g}_{a}(x', t) \right]^* = \sum_{a} L^{-\rho_{a}} f_{a}(x) \left[ \mathcal{M}^{\alpha}_{x} \tilde{g}_{a}(x', t) \right]^*$$

$$= \sum_{a} L^{-\rho_{a} + \gamma} \mathcal{M}_{x} f_{a}(x) \left[ \tilde{g}_{a}(x', t) \right]^*. \quad (A.16)$$

We see that whenever the asymptotic series contains a term proportional to $f_{a}(x)$ with scaling exponent $\rho_{a}$, it must also contain a term $\mathcal{M}_{x} f_{a}(x)$ with exponent $\rho_{a} - \gamma$ and then a term $\mathcal{M}_{x}^{2} f_{a}(x)$ with exponent $\rho_{a} - 2\gamma$ and so on. This cannot continue indefinitely, since, otherwise, there would be successively more and more divergent terms for $L \gg 1$. The only way that this sequence can terminate is if, eventually,

$$\mathcal{M}_{x}^{p+1} f_{a}(x) = 0$$

for some integer $p$. In that case, we see that $f_{a} = (-\mathcal{M}_{x})^{p} f_{a} \equiv f_{a, p}$ for some homogeneous zero mode $f_{a}$, and the expansion (A.15) contains the whole tower above that zero mode. All such towers associated with regular zero modes must appear because the condition (A.11) together with the zero-mode expansion for $G$ implies that

$$\sum_{a} L^{-\rho_{a}} f_{a}(x) \left[ \int_{0}^{\infty} dt \tilde{g}_{a}(x', t) \right]^* \equiv \sum_{a} L^{-\omega_{a}} f_{a}(x) \left[ \tilde{g}_{a}(x') \right]^*. \quad (A.17)$$

The expansion (A.15) thus must have precisely the form of equation (A.13) and we must only establish the properties of $\tilde{g}_{a} = \tilde{g}_{a, p}$. We note from (A.16) that

$$\partial_{t} \tilde{g}_{a, p} = \mathcal{M}_{x}^{p} \tilde{g}_{a, p}$$

$$= - \tilde{g}_{a, p+1}.$$

Also (A.17) implies that (away from the origin $x' = 0$)

$$\tilde{g}_{a, p-1}(x', 0) = - \int_{0}^{\infty} dt \partial_{t} \tilde{g}_{a, p-1}(x', t) = \int_{0}^{\infty} dt \tilde{g}_{a, p}(x', t) = 0$$

for $p = 1, 2, 3, \ldots$, whereas $\tilde{g}_{a, -1}(x', 0) = \tilde{g}_{a}(x')$, the singular zero mode of $\mathcal{M}_{x}$. Finally, the scaling properties of $\tilde{g}_{a, p}$ follow from the scaling property of $\tilde{g}_{a}$ and of the propagator, i.e. $\tilde{g}_{a}(\lambda x) = \lambda^{\omega_{a}} \tilde{g}_{a}(x)$ and $e^{\epsilon_{1} M_{x}^{*}}(\lambda x, \lambda y) = \lambda^{-d} e^{\epsilon_{1} M_{x}^{*}}(x, y)$, respectively. The expansion (A.14) for $\bar{P}$ is derived by an identical argument.
A.3. Slow-mode expansion: fundamental derivation

We shall now demonstrate the existence of the expansion (A.15) and verify by an independent argument its general properties discussed above. A key fact that we use is that the operators $\mathcal{M}$ and $\mathcal{M}^*$ both have spectrum absolutely continuous over the negative real axis. This assumption explicitly rules out the kinematic dynamo effect due to point spectrum on the positive real axis. Because of this assumed property, we may define

$$X = \log(-\mathcal{M}), \quad X^* = \log(-\mathcal{M}^*),$$

where the branch of the natural logarithm $\log(z)$ is defined with a cut along the negative real axis. Furthermore, because $\mathcal{M}$ and $\mathcal{M}^*$ are homogeneous of degree $-\gamma$, the operators $X$ and $X^*$ both satisfy the Heisenberg commutation relations

$$[D, X] = [D, X^*] = i\gamma I,$$

where $D$ is the self-adjoint generator of dilatations. We may decompose $X, X^*$ into Hermitian and skew-Hermitian parts, as

$$X = H + iK, \quad X^* = H - iK,$$

where $H, K$ are both Hermitian. In that case, we see that

$$[D, H] = i\gamma I, \quad [D, K] = 0.$$

We can now follow the arguments in [17] to infer that under the unitary Mellin transform

$$D \rightarrow \frac{1}{i} \left( \sigma + \frac{d}{2} \right),$$

$$H \rightarrow \hat{U}(\sigma)\gamma \partial_\sigma \hat{U}^{-1}(\sigma),$$

$$K \rightarrow \hat{K}_0(\sigma) = \hat{U}(\sigma)\hat{K}(\sigma)\hat{U}^{-1}(\sigma),$$

where $\text{Re}(\sigma) = -d/2$. As in [17], the operators $\hat{U}(\sigma)$ in equation (A.23) are unitary operators on $L^2(S^{d-1})$ and the result in equation (A.23) follows from the Stone–von Neumann theorem on uniqueness of representations of the Heisenberg algebra. The operators $\hat{K}_0(\sigma)$ in (A.24) are self-adjoint operators on $L^2(S^{d-1})$ and the result (A.24) is a consequence of the second half of (A.21)—commutativity of $D$ and $K$—so that $K$ leaves invariant the eigenspaces of $D$. It is convenient to introduce instead the self-adjoint operators $\tilde{K}(\sigma) = \hat{U}^{-1}(\sigma)\hat{K}_0(\sigma)\hat{U}(\sigma)$. Thus,

$$X \rightarrow \hat{U}(\sigma)[\gamma \partial_\sigma + i\hat{K}(\sigma)]\hat{U}^{-1}(\sigma),$$

$$X^* \rightarrow \hat{U}(\sigma)[\gamma \partial_\sigma - i\hat{K}(\sigma)]\hat{U}^{-1}(\sigma).$$

We now introduce the operators $\hat{L}(\sigma)$ on $L^2(S^{d-1})$ satisfying

$$\gamma \frac{d}{d\sigma} \hat{L}(\sigma) = -i\hat{K}(\sigma)\hat{L}(\sigma), \quad \hat{L}(0) = I,$$

$$\gamma \frac{d}{d\sigma} \hat{L}^{-1}(\sigma) = \hat{L}^{-1}(\sigma)i\hat{K}(\sigma), \quad \hat{L}^{-1}(0) = I.$$
The operators $\tilde{L}(\sigma)$ and $\tilde{L}^{-1}(\sigma)$ can be defined explicitly by ordered exponentials along the line $\sigma = -(d/2) + iv$:

$$\tilde{L}(\sigma) = \begin{cases} 
\text{Exp} \left[ \frac{1}{\gamma} \int_0^\nu d\nu' \tilde{K} \left( -\frac{d}{2} + iv' \right) \right], & \text{if } \nu \geq 0, \\
\bar{\text{Exp}} \left[ -\frac{1}{\gamma} \int_\nu^0 d\nu' \tilde{K} \left( -\frac{d}{2} + iv' \right) \right], & \text{if } \nu < 0,
\end{cases}$$

(A.29)

and

$$\tilde{L}^{-1}(\sigma) = \begin{cases} 
\bar{\text{Exp}} \left[ -\frac{1}{\gamma} \int_0^\nu d\nu' \tilde{K} \left( -\frac{d}{2} + iv' \right) \right], & \text{if } \nu \geq 0, \\
\text{Exp} \left[ \frac{1}{\gamma} \int_\nu^0 d\nu' \tilde{K} \left( -\frac{d}{2} + iv' \right) \right], & \text{if } \nu < 0.
\end{cases}$$

(A.30)

It follows that

$$\gamma \partial_\sigma + i\tilde{K}(\sigma) = \tilde{L}(\sigma)\gamma \partial_\sigma \tilde{L}^{-1}(\sigma),$$

(A.31)

$$\gamma \partial_\sigma - i\tilde{K}(\sigma) = \tilde{L}^{-1}(\sigma)\gamma \partial_\sigma \tilde{L}(\sigma).$$

(A.32)

Finally, combining (A.31), (A.32) with (A.25), (A.26), we obtain the mappings under the Mellin transform

$$X \longrightarrow \tilde{V}(\sigma)\gamma \partial_\sigma \tilde{V}^{-1}(\sigma),$$

(A.33)

$$X^* \longrightarrow \tilde{V}^{* -1}(\sigma)\gamma \partial_\sigma \tilde{V}^*(\sigma)$$

(A.34)

with

$$\tilde{V}(\sigma) = \tilde{U}(\sigma)\tilde{L}(\sigma), \quad \tilde{V}^*(\sigma) = \tilde{L}^*(\sigma)\tilde{U}^{-1}(\sigma).$$

(A.35)

This is the main result that we require.

The rest of the derivation of the slow-mode expansion follows the argument of [17], assuming that $\tilde{V}(\sigma)$ extends to a meromorphic operator-valued function of $\sigma$. We shall sketch here the main points. Note first that we can exponentiate the relations (A.33) and (A.34) to obtain

$$-\mathcal{M} = VR^{-\gamma}V^{-1}, \quad -\mathcal{M}^* = V^{* -1}R^{\gamma}V^*,$$

(A.36)

where we have defined the operators $V$ and $V^*$ by

$$(Vf)^-(\sigma, \hat{x}) \equiv \tilde{V}(\sigma) f(\sigma, \hat{x}), \quad (V^*f)^-(\sigma, \hat{x}) \equiv \tilde{V}^*(\sigma) f(\sigma, \hat{x}),$$

which are mutual adjoints. From the definition $\mathcal{N}^{-1} = R^{-\gamma/2}\mathcal{M}^{-1}R^{-\gamma/2}$ and (A.36), we see that

$$-\mathcal{N}^{-1} = (R^{-\gamma/2}VR^{\gamma/2})(R^{\gamma/2}VR^{-\gamma/2}),$$

which under the Mellin transform becomes

$$-\mathcal{N}^{-1} \left( \sigma - \frac{\gamma}{2} \right) = \tilde{V}(\sigma)\tilde{V}^{-1}(\sigma - \gamma).$$

(A.37)

Of course, we also have

$$-\mathcal{N}^{* -1} \left( \sigma - \frac{\gamma}{2} \right) = \tilde{V}^{* -1}(\sigma)\tilde{V}^*(\sigma - \gamma),$$

(A.38)

by an identical argument.

New Journal of Physics 12 (2010) 023021 (http://www.njp.org/)
One immediate consequence of (A.37) is that poles of $\tilde{N}^{-1}(\sigma - \gamma/2)$ can arise only from poles of $\tilde{V}(\sigma)$ or zeros of $\tilde{V}(\sigma - \gamma)$. Our main assumption will be that all of the poles of $\tilde{V}(\sigma)$ lie in the half-plane $\text{Re}\sigma > -d/2$ and all of its zeros lie in the half-plane $\text{Re}\sigma < -d/2$. Because of the adjoint relation

$$\tilde{V}^{* -1}(\sigma) = [\tilde{V}(-\sigma^* - d)]^{* -1},$$

we see that $\tilde{V}^{* -1}(\sigma)$ then enjoys the same property, with the poles of $\tilde{V}(\sigma)$ corresponding to zeros of $\tilde{V}^{* -1}(\sigma)$ and the zeros of $\tilde{V}(\sigma)$ corresponding to poles of $\tilde{V}^{* -1}(\sigma)$. The assumption on $\tilde{V}$ implies that all of the poles of $\tilde{N}^{-1}(\sigma - \gamma/2)$ for $\text{Re}\sigma > -d/2 + \gamma/2$ must arise from poles of $\tilde{V}(\sigma)$ with the form

$$\tilde{V}(\sigma) \approx \frac{1}{\sigma - \zeta_a} |f_a(\tilde{g}_a)| \tilde{V}(\sigma - \gamma),$$

in order to reproduce the known poles of $\tilde{N}^{-1}(\sigma - \gamma/2)$.

On the other hand, we can rewrite (A.37) as

$$\tilde{V}(\sigma + \gamma p) = -\tilde{N}^{-1}(\sigma + \gamma(p - \frac{1}{2})) \tilde{V}(\sigma + \gamma(p - 1)),$$

for $p = 1, 2, \ldots$. Let us assume that none of the poles of $\tilde{N}^{-1}(\sigma - \gamma/2)$ occur at points in the complex $\sigma$-plane with real parts differing by integer multiples of $\gamma$. This will hold generically. In that case, $\tilde{N}^{-1}(\sigma_a + \gamma(p - (1/2)))$ is a regular operator for all $p = 1, 2, \ldots$ and we may use the above relation to infer inductively a series of poles

$$\tilde{V}(\sigma) \approx \frac{1}{\sigma - \zeta_a - \gamma p} |f_{a,p}(\tilde{g}_a)| \tilde{V}(\sigma - \gamma),$$

for each $a = 1, 2, \ldots$ with

$$f_{a,p} = -\tilde{N}^{-1}(\sigma_a + \gamma(p - \frac{1}{2})) f_{a,p-1}$$

for $p = 1, 2, \ldots$ and $f_{a,0} = f_a$. It is not difficult to check that this coincides with the definition of $f_{a,p}$ given earlier.

Finally, we exponentiate one more time relations (A.36) to obtain

$$e^{tM} = V e^{-tR^{-1}V}^{-1}, \quad e^{tM^*} = V^{* -1} e^{-tR^{-1}V^*},$$

(A.41)

The first of these, under the Mellin transform, gives

$$(e^{tM}\varphi)(\hat{x}/L) = \frac{1}{\gamma} \int_{\text{Re}\sigma=-d/2} \frac{d\sigma}{2\pi i} L^{-\sigma} \int d\omega(\tilde{y}) \tilde{V}(\sigma; \hat{x}, \tilde{y}) \times \int_{\text{Re}\sigma'=-d/2-0} \frac{d\sigma'}{2\pi i} \prod_{s'\neq \sigma} \Gamma \left( \frac{\sigma - \sigma'}{\gamma} \right) (\tilde{V}^{-1}(\sigma') \tilde{\varphi})(\sigma', \tilde{y}).$$

Pushing the $\sigma$-integration contour further and further to the right gives the expansion for $L \gg 1$:

$$(e^{tM}\varphi)(\hat{x}/L) \approx \sum_{a,p} L^{-\sigma - \gamma p} f_{a,p}(x) (\tilde{g}_{a,p}(t), \varphi),$$

with a suitable definition of $\tilde{g}_{a,p}(t)$. See [17] for more details. The above is just an integrated form of the slow-mode expansion (A.13) for $P$. The slow-mode expansion (A.14) for $\tilde{P}$ follows.

New Journal of Physics 12 (2010) 023021 (http://www.njp.org/)
by an identical argument, in which the various terms arise from the poles of $\tilde{V}^{*-1}(\sigma)$ in the half-plane $\text{Re}\,\sigma > -d/2$.

There are similar large-distance expansions for $P$ and $\tilde{P}$, in which enter the ‘tunnels’ of singular slow modes. The terms in these expansions arise from the zeros of $\tilde{V}(\sigma')$ and $\tilde{V}^{*-1}(\sigma')$ in the half-plane $\text{Re}\,\sigma' < -d/2$ by moving $\sigma'$-integration contours to the left in a formula similar to the above. The reader may work out details. We shall just note here that the pole ($A.40$) of $\tilde{V}(\sigma)$ implies via the relation

$$\tilde{V}^{*}(\sigma) = \left[ V(-\sigma^* - d) \right]^*$$

the result

$$\tilde{V}^{*}(\sigma - \gamma) \approx \frac{1}{\omega_a - \sigma} \tilde{V}^{*}(\tilde{\omega}_a) | \tilde{g}_a |$$

(A.42)

and thus the zero of $\tilde{V}^{*-1}(\sigma)$ at $\sigma = \tilde{\omega}_a - \gamma$. This zero and the ‘tunnel’ of zeros beneath it give rise to the terms in the large-distance expansion of $\tilde{P}$.

In our discussion throughout the appendix, we have assumed that all the regular zero modes of $\mathcal{M}$ and $\mathcal{M}^*$ have scaling exponents $\sigma$ with $\text{Re}\,\sigma > -d/2 + \gamma/2$ and all the singular zero modes have scaling exponents with $\text{Re}\,\sigma < -d/2 - \gamma/2$. This is true in the KK model only for $d \geq 6$ and for $\xi$ not too large. For $d \leq 4$, the two primary ‘singular modes’ have exponents $\omega_1, \tilde{\omega}_1 \geq -d/2$ for all $\xi$ and, for sufficiently large $\xi$, these exponents even cross and become larger than $\zeta_1, \tilde{\zeta}_1$, respectively! For $d \geq 6$, it still happens that $\zeta_1 < -d/2$ and $\tilde{\omega}_1 > -d/2$ for sufficiently large $\xi$. These results are not consistent with the assumptions made in the derivation sketched above. Nevertheless, the zero- and slow-mode expansions seem to hold for all $d > 2$ and $0 < \xi < \xi^*$.

References

[1] Brandenburg A and Subramanian K 2005 Phys. Rep. 417 1–209
[2] Moffatt H K 1978 The oxymoronic role of molecular diffusivity in the dynamo process Woods Hole Oceanographic Institution Technical Report WHOI-78-67 (http://www.igf.fuw.edu.pl/KB/HKM/PDF/HKM_032.pdf)
[3] Kazantsev A P 1968 Sov. Phys.—JETP 26 1031–4
[4] Kraichnan R H and Nagarajan S 1967 Phys. Fluids 10 859–70
[5] Kraichnan R H 1968 Phys. Fluids 11 945–53
[6] Vincenzi D 2002 J. Stat. Phys. 106 1073–91
[7] Arponen H and Horvai P 2007 J. Stat. Phys. 129 205–39
[8] Rogachevski I and Kleerorin N 1997 Phys. Rev. E 56 417–26
[9] Boldyrev S and Cattaneo F 2004 Phys. Rev. Lett. 92 144501
[10] Schekochihin A A, Cowley S C, Maron J L and McWilliams J C 2004 Phys. Rev. Lett. 92 054502
[11] Schekochihin A A, Haugen N E L, Brandenburg A, Cowley S C, Maron J L and McWilliams J C 2005 Astrophys. J. 625 L115–8
[12] Iskakov A B, Schekochihin A A, Cowley S C, McWilliams J C and Proctor M R E 2007 Phys. Rev. Lett. 98 208501
[13] Schekochihin A A, Iskakov A B, Cowley S C, McWilliams J C, Proctor M R E and Yousef T A 2007 New J. Phys. 9 300
[14] Celani A, Mazzino A and Vincenzi D 2006 Proc. R. Soc. A 462 137–47
[15] Eyink G and Xin J 2000 J. Stat. Phys. 100 679–741
[16] Chaves M, Eyink G, Frisch U and Vergassola M 2001 Phys. Rev. Lett. 86 2305–8

New Journal of Physics 12 (2010) 023021 (http://www.njp.org/)
[17] Bernard D, Gawędzki K and Kupiainen A 1998 J. Stat. Phys. 90 519–69
[18] Eyink G L 2009 J. Math. Phys. 50 183102
[19] Saffman P 1963 J. Fluid Mech. 16 545–72
[20] Larsson J 2003 J. Plasma Phys. 69 211–52
[21] Vergassola M 1996 Phys. Rev. E 53 R3021–4
[22] Batchelor G K 1953 Homogeneous Turbulence (Cambridge: Cambridge University Press) p 14
[23] Kraichnan R H 1974 J. Fluid. Mech. 64 737–62
[24] Le Jan Y 1984 C. R. Acad. Sci., Paris I 299 947–9
[25] Le Jan Y 1985 Z. Warscheinlichkeitsntheorie verw. Gebiete 70 609–20
[26] Son D T 1999 Phys. Rev. E 59 R3811–4
[27] Eyink G 2010 Magnetic dynamo and magnetic induction in preparation
[28] Teschl G 2009 Ordinary Differential Equations and Dynamical Systems (Wien: Gerald Teschl) p 90
[29] Erdélyi A (ed) 1953 Higher Transcendental Functions vols I–III, Bateman Manuscript Project (Malabar, FL: Krieger) p 56
[30] Van Dyke M 1964 Perturbation Methods in Fluid Mechanics (New York: Academic) p 77
[31] Eyink G L 2007 Phys. Lett. A 368 486–90
[32] Kulsrud R M and Anderson S W 1992 Astrophys. J. 396 606–30
[33] Chertkov M, Falkovich G, Kolokolov I and Vergassola M 1999 Phys. Rev. Lett. 83 4065–8
[34] Schekochihin A A, Maron J L, Cowley S C and McWilliams J C 2002 Astrophys. J. 576 806–13
[35] Schekochihin A, Cowley S, Maron J and Malyskhin L 2002 Phys. Rev. E 65 016305
[36] Schekochihin A A, Boldyrev S A and Kulsrud R M 2002 Astrophys. J. 567 828–52
[37] Sawford B L, Yeung P K and Borgas M S 2005 Phys. Fluids 17 095109
[38] Homann H, Grauer R, Busse A and Müller W-C 2007 J. Plasma Phys. 73 122303
[39] Kuznetsov E A and Ruban V P 2000 Phys. Rev. E 61 831–41
[40] Bekenstein J D and Oron A 2000 Phys. Rev. E 62 5594–602