Abstract

We study the entropy of entanglement of the ground state in a wide family of one-dimensional quantum spin chains whose interaction is of finite range and translation invariant. Such systems can be thought of as generalizations of the XY model. The chain is divided in two parts: one containing the first consecutive $L$ spins; the second the remaining ones. In this setting the entropy of entanglement is the von Neumann entropy of either part. At the core of our computation is the explicit evaluation of the leading order term as $L \to \infty$ of the determinant of a block-Toeplitz matrix with symbol

$$
\Phi(z) = \begin{pmatrix} i\lambda & g(z) \\ g^{-1}(z) & i\lambda \end{pmatrix},
$$

where $g(z)$ is the square root of a rational function and $g(1/z) = g^{-1}(z)$. The asymptotics of such determinant is computed in terms of multi-dimensional theta-functions associated to a hyperelliptic curve $\mathcal{L}$ of genus $g \geq 1$, which enter into the solution of a Riemann-Hilbert problem. Phase transitions for these systems are characterized by the branch points of $\mathcal{L}$ approaching the unit circle. In these circumstances the entropy diverges logarithmically. We also recover, as particular cases, the formulae for the entropy discovered by Jin and Korepin [14] for the XX model and Its, Jin and Korepin [12, 13] for the XY model.

Contents

1 Introduction 3
2 Statement of results 6
3 Quantum spin chains with anisotropic Hamiltonians 11
4 The von Neumann entropy and block-Toeplitz determinants 14

*A. Its was partially supported by the NSF grants DMS-0401009 and DMS-0701768. F. Mezzadri and M. Y. Mo acknowledge financial support by the EPSRC grant EP/D505534/1.
The Asymptotics of Block Toeplitz Determinants. Widom’s Theorem

The Wiener-Hopf factorization of Φ(z)

The asymptotics of $d \log D_L(\lambda)/d\lambda$ and $D_L(\lambda)$

The limiting entropy

Integrability at ±1. The final formula for the entropy

Critical behavior as roots of $g(z)$ approaches the unit circle

The contour $\Gamma(\epsilon)$ of the integral in equation (2.1). The bold lines ($-\infty, -1-\epsilon$) and $(1+\epsilon, \infty)$ are the cuts of the integrand $e(z, \lambda)$. The zeros of $D_L(\lambda)$ are located on the bold line $(-1,1)$.

The choice of cycles on the hyperelliptic curve $L$. The arrows denote the orientations of the cycles and branch cuts. Note that we have $\lambda_1 = z_1^{-1}$.

The location of one of the roots (2.4), say $\lambda_j$ determines the positions of other three: $\bar{\lambda}_j$, $1/\lambda_j$ and $1/\bar{\lambda}_j$. 

Appendix A. The density matrix of a subchain

Appendix B. The correlation matrix $C_M$

Appendix C. Thermodynamic limit of the correlation matrix $C_M$

Appendix D. The Riemann constant $K$

Appendix E. The cycle basis (10.18)

Appendix F. Solvability of the Wiener-Hopf factorization problem

List of Figures

1. The contour $\Gamma(\epsilon)$ of the integral in equation (2.1). The bold lines ($-\infty, -1-\epsilon$) and $(1+\epsilon, \infty)$ are the cuts of the integrand $e(z, \lambda)$. The zeros of $D_L(\lambda)$ are located on the bold line $(-1,1)$.

2. The choice of cycles on the hyperelliptic curve $L$. The arrows denote the orientations of the cycles and branch cuts. Note that we have $\lambda_1 = z_1^{-1}$.

3. The location of one of the roots (2.4), say $\lambda_j$ determines the positions of other three: $\bar{\lambda}_j$, $1/\lambda_j$ and $1/\bar{\lambda}_j$. 

References
The Jordan arc Σ connects all the branch points and extends to infinity on the left hand side of $\lambda_1$ and on the right hand side of $\lambda_{2n}$. All branch cuts belong to Σ and are denoted by $\Sigma_i$, while the intervals between the branch cuts are denoted by $\tilde{\Sigma}_i$.

The choice of cycles on the hyperelliptic curve $\mathcal{L}$. The arrows denote the orientations of the cycles and branch cuts.

As $\lambda_{2n} \to \lambda_{2n-1}$, integration around $\tilde{a}_n$ becomes a residue integral around $z = 1$.

Two pairs of roots, labelled according to the ordering (2.5), approaching the unit circle in the critical limit. We have $\lambda_{2(j+1)} \to \lambda_{2(2n-j)-1}$, $\lambda_{2n} \to \lambda_{2n+1}$ and $\lambda_{2j+1} \to \lambda_{2(2n-j)}$ respectively.

The choice of cycles on the hyperelliptic curve $\mathcal{L}$. The arrows denote the orientations of the cycles and branch cuts.

The curve going around $\lambda_2$.

The choice of cycles on the hyperelliptic curve $\mathcal{L}$. The arrows denote the orientations of the cycles and branch cuts.

1 Introduction

One dimensional quantum spin chains were introduced by Lieb et. al. [17] in 1961 as a model to study the magnetic properties of solids. Usually such systems depend on some parameter, e.g. the magnetic field. One of their most important features is that at zero temperature, when the system is in the ground state, as the number of spins tend to infinity they undergo a phase transition for a critical value of the parameter. As a consequence, the rate of the decay of correlation lengths changes suddenly from exponential to algebraic at the critical point. Furthermore, many examples of such chains are exactly solvable. Because of these reasons over the years the statistical mechanical properties of quantum spin chains have been investigated in great detail.

More recently, Osterloh et al. [20], and Osborne and Nielsen [22] realized that the existence of non-local physical correlations at a phase transition is a manifestation of the entanglement among the constituent parts of the chain. Entangled quantum states are characterized by non-local correlations that cannot be described by classical mechanics. Such correlations play an important role in the transmission of quantum information. It is therefore essential to be able to quantify entanglement. In its full generality this is still an open problem. However, when a physical system is in a pure state and is bipartite, i.e. is made of two separate parts, say A and B, a suitable measure of the entanglement shared between the two constituents is the von Neumann entropy of either part [2]. In this situation the Hilbert space of the whole system is $H_{AB} = H_A \otimes H_B$, where $H_A$ and $H_B$ are the Hilbert spaces associated to A and B respectively. Now, if $\rho_{AB}$ is the density matrix of the composite system, then the reduced density matrices of A and B are

$$\rho_A = \text{tr}_B \rho_{AB} \quad \text{and} \quad \rho_B = \text{tr}_A \rho_{AB},$$

(1.1)
where $\text{tr}_A$ and $\text{tr}_B$ are partial traces over the degrees of freedom $A$ and $B$ respectively. The entropy of the entanglement of formation is

$$S(\rho_A) = -\text{tr}_A \rho_A \log \rho_A = S(\rho_B) = -\text{tr}_B \rho_B \log \rho_B$$

(1.2)

In this paper we compute the entropy of entanglement of the ground state of a vast class of spin chains whose interaction among the constituent spins is non-local and translation invariant. These systems can be mapped into quadratic chains of fermionic operators by a suitable transformation and are generalizations of the XY model. We study the ground state of such systems, divide the chain in two halves and compute the von Neumann entropy in the thermodynamic limit of one of the two parts. If the ground state is not degenerate, then $\rho_{AB} = |\Psi_g\rangle \langle \Psi_g|$. At the core of our derivation of the entropy of entanglement is the computation of determinants of Toeplitz matrices for a wide class of $2 \times 2$ matrix symbols. The explicit expressions for such determinants were not available in the literature. The appearance of Toeplitz matrices and their invariants in the study of lattice models is a simple consequence of the translation invariance of the interaction among the spins. Thus, Toeplitz determinants appear in the computations of many other physical quantities like spin-spin correlations or the probability of the emptiness of formation, not only the entropy of entanglement. Therefore, our results have consequences that go beyond the application to the study of bipartite entanglement that we discuss.

Vidal et al. [23] were the first to investigate the entanglement of formation of the ground state of spin chains by dividing them in two parts. The models they considered were the XX, XY and XXZ model. They computed numerically the von Neumann entropy of one half of the chain and discovered that at a phase transition it grows logarithmically with its length $L$. Jin and Korepin [14] computed the von Neumann entropy of the ground state of the XX model using the Fisher-Hartwig formula for Toeplitz determinants. They showed that at the phase transition the entropy grows like $\frac{1}{3} \log L$, which is in agreement with the numerical observations of Vidal et al. For lattice systems that have a conformal field theory associated to it the logarithmic growth of the entropy was first discovered by Holzhey et al. [10] in 1994. This approach was later developed by Korepin [15], and by Calabrese and Cardy [4]. Its, Jin and Korepin [12, 13] determined the entropy for the XY model by computing an explicit formula for the asymptotics of the determinant of a block-Toeplitz matrix. They expressed the entropy of entanglement in terms of an integral of Jacobi theta functions.

Consider a $p \times p$ matrix-valued function on the unit circle $\Xi$:

$$\varphi(z) = \sum_{k=-\infty}^{\infty} \varphi_k z^k, \quad |z| = 1.$$ 

A block-Toeplitz matrix with symbol $\varphi$ is defined by

$$T_L[\varphi] = (\varphi_{j-k})_{0 \leq j,k \leq L-1}.$$
Furthermore, we shall denote its determinant by \( D_L = \det T_L[\varphi] \). The main ingredient of the computation of \( \text{Its}, \text{Jin and Korepin} \) was to use the Riemann-Hilbert approach to derive an asymptotic formula for the Fredholm determinant

\[
D_L(\lambda) = \det T_L[\varphi] = \det \left( I - K_L \right),
\]

where \( K_L \) is an appropriate integral operator on \( L^2(\Xi, \mathbb{C}^2) \). The symbol of the Toeplitz matrix \( T_L[\varphi] \) was

\[
\varphi(e^{i\theta}) = \begin{pmatrix}
  \frac{i\lambda}{\sqrt{|\alpha \cos \theta - 1 - i\gamma\alpha \sin \theta|}} & g(\theta) \\
  -g^{-1}(\theta) & \frac{i\lambda}{\sqrt{|\alpha \cos \theta - 1 - i\gamma\alpha \sin \theta|}}
\end{pmatrix},
\]

where

\[
g(\theta) = \frac{\alpha \cos \theta - 1 - i\gamma\alpha \sin \theta}{|\alpha \cos \theta - 1 - i\gamma\alpha \sin \theta|}.
\]

Keating and Mezzadri \[18, 19\] introduced families of spin chains that are characterized by the symmetries of the spin-spin interaction. The entropy of entanglement of the ground state of these systems, as well as other thermodynamical quantities like the spin-spin correlation function, can be determined by computing averages over the classical compact groups, which in turn means computing determinants of Toeplitz matrices or of sums of Hankel matrices. These models are solvable and can be mapped into a quadratic chain of Fermi operators via the Jordan-Wigner transformations. One of the main features of these families is that symmetries of the interaction can be put in one to one correspondence with the structure of the invariant measure of the group to be averaged over. If the Hamiltonian is translation invariant and the interaction is isotropic, then the relevant group over is \( \text{U}(N) \) equipped with Haar measure. In turn such averages are equivalent to Toeplitz determinants with a scalar symbol. These systems are generalizations of the XX model.

In this paper we consider spin chains whose interaction is translation invariant but the Hamiltonian is not isotropic. These are generalization of the XY model. The Fredholm determinant that we need to compute has the same structure as (1.3), but now the \( 2 \times 2 \) matrix symbol is

\[
\Phi(z) := \begin{pmatrix}
  \frac{i\lambda}{\sqrt{|p(z)/z^{2n}p(1/z)|}} & g(z) \\
  -g^{-1}(z) & \frac{i\lambda}{\sqrt{|p(z)/z^{2n}p(1/z)|}}
\end{pmatrix},
\]

where function \( g(z) \) is defined by

\[
g(z) := \sqrt{\frac{p(z)}{z^{2n}p(1/z)}}
\]

and \( p(z) \) is a polynomial of degree \( 2n \). We recover the XY model if we set

\[
p(z) = \frac{\alpha(1 - \gamma)}{2}z^2 - z + \frac{\alpha(1 + \gamma)}{2}.
\]

In the above equation \( \alpha = 2/h \), where \( h \) magnetic field, and \( \gamma \) measures the anisotropy of the Hamiltonian in the XY plane.
Figure 1: The contour $\Gamma(\epsilon)$ of the integral in equation (2.1). The bold lines $(-\infty, -1 - \epsilon)$ and $(1 + \epsilon, \infty)$ are the cuts of the integrand $e(1 + \epsilon, \lambda)$. The zeros of $D_L(\lambda)$ are located on the bold line $(-1, 1)$.

2 Statement of results

Following [14] and [13], we will identify the limiting von Neumann entropy for the systems that we study with the double limit

$$S(\rho_A) = \lim_{\epsilon \to 0^+} \left[ \lim_{L \to \infty} \frac{1}{4\pi i} \oint_{\Gamma(\epsilon)} e(1 + \epsilon, \lambda) \frac{d}{d\lambda} \log \left( D_L(\lambda)(\lambda^2 - 1)^{-L} \right) d\lambda \right]. \quad (2.1)$$

In the above formula $\Gamma(\epsilon)$ is the contour in figure 1. $D_L(\lambda)$ is the determinant of the block-Toeplitz matrix $T_L[\Phi]$ with symbol (1.5) and

$$e(x, \nu) := -\frac{x + \nu}{2} \log \left( \frac{x + \nu}{2} \right) - \frac{x - \nu}{2} \log \left( \frac{x - \nu}{2} \right). \quad (2.2)$$

The explicit Hamiltonians for the family of spin systems that we consider and their connection to formula (2.1) will be discussed in detail in sections 3 and 4.

One of the main objectives of this paper is to compute the double limit (2.1), which, as we shall see, can be expressed as an integral of multi-dimensional theta functions defined on Riemann surfaces. Thus, in order to state our main results, we need to introduce some definitions and notation.

Let us rewrite the function (1.6) as

$$g^2(z) = \prod_{j=1}^{2n} \frac{z - z_j}{1 - z_j z}.$$  

(2.3)
where the $z_j$'s are the $2n$ roots of the polynomial $p(z)$. This representation of $g(z)$ will be used throughout the paper. We fix the branch of $g(z)$ by requiring that $g(\infty) > 0$ on the real axis. The function $g(z)$ have jump discontinuities on the complex $z$-plane. In order to define its branch cuts we need to introduce an ordering of the roots $z_j$. Let

$$\{\lambda_1, \lambda_2, \ldots, \lambda_{4n}\} = \{z_1, \ldots, z_{2n}, z_{1}^{-1}, \ldots, z_{2n}^{-1}\}$$

(2.4)

where the above is merely an equality between sets, and we do not necessarily have, for example, $\lambda_i = z_i$. We order the $\lambda_i$'s such that

$$\text{Re}(\lambda_i) \leq \text{Re}(\lambda_j), \quad i < j$$

$$\text{Im}(\lambda_i) \leq \text{Im}(\lambda_j), \quad |\lambda_i|, |\lambda_j| < 1, \quad \text{Re}(\lambda_i) = \text{Re}(\lambda_j)$$

$$\text{Im}(\lambda_i) \leq \text{Im}(\lambda_j), \quad |\lambda_i|, |\lambda_j| > 1, \quad \text{Re}(\lambda_i) = \text{Re}(\lambda_j).$$

(2.5)

This ordering need not coincide with the ordering $z_j$'s. If necessary, we can always assume that one of the $z_j^{-1}$ has the smallest real part and set $\lambda_1 = z_j^{-1}$. This choice is equivalent to taking the transpose of $T_L[\Phi]$. The branch cuts for $g(z)$ are defined by the intervals $\Sigma_i$ joining $\lambda_{2i-1}$ and $\lambda_{2i}$:

$$\Sigma_i = [\lambda_{2i-1}, \lambda_{2i}], \quad i = 1, \ldots, 2n.$$ 

(2.6)

Therefore, $g(z)$ has the following jump discontinuities:

$$g_+(z) = -g_-(z), \quad z \in \Sigma_i,$$ 

(2.7)

where $g_{\pm}(z)$ are the boundary values of $g(z)$ on the left/right hand side of the branch cut.

Now, let $L$ be the hyperelliptic curve

$$L : w^2 = \prod_{i=1}^{4n}(z - \lambda_i).$$

(2.8)

The genus of $L$ is $g = 2n - 1$. We now choose a canonical basis for the cycles $\{a_i, b_i\}$ on $L$ as shown in figure 2 and define $d\omega_i$ to be 1-forms dual to this basis, i.e.

$$\int_{a_i} d\omega_j = \delta_{ij}, \quad \int_{b_i} d\omega_j = \Pi_{ij}.$$ 

(2.9)

Furthermore, let us define the $g \times g$ matrix $\Pi$ by setting $(\Pi)_{ij} = \Pi_{ij}$. The theta function $\theta : \mathbb{C}^g \rightarrow \mathbb{C}$ associated to $L$ is defined by

$$\theta(\tau) := \sum_{\vec{n} \in \mathbb{Z}^g} e^{i\pi \vec{n} \cdot \vec{\Pi} \cdot \vec{n} + 2i\pi \vec{s} \cdot \vec{n}}.$$ 

(2.10)

while the theta function with characteristics $\tau$ and $\delta$ is defined by

$$\theta \left[ \begin{array}{c} \tau \\ \delta \end{array} \right] (\vec{s}) := \exp \left( 2i\pi \left( \frac{\tau}{8} \cdot \Pi \cdot \tau + \frac{1}{2} \tau \cdot \vec{s} + \frac{1}{4} \vec{\tau} \cdot \vec{\delta} \right) \right) \theta \left( \frac{\vec{s} + \vec{\delta}}{2} + \Pi \frac{\vec{\tau}}{2} \right)$$

(2.11)

where $\tau$ and $\delta$ are $g$-dimensional complex vectors.

Our main results are summarised by the following two theorems.
Theorem 1. Let $H_\alpha$ be the Hamiltonian of the one-dimensional quantum spin system defined in equation (3.11). Let $A$ be the subsystem made of the first $L$ spins and $B$ the one formed by the remaining $M - L$. We also assume that the system is in a non-degenerate ground state $|\Psi_g\rangle$ and that the thermodynamic limit, i.e. $M \to \infty$, has been already taken. Then, the limiting (as $L \to \infty$) von Neumann entropy (2.1) is

$$S(\rho_A) = \frac{1}{2} \int_1^\infty \log \left( \frac{\theta(\beta(\lambda)\bar{e}' + \frac{\tau}{2}) \theta(\beta(\lambda)\bar{e}' - \frac{\tau}{2})}{\theta^2(\frac{\tau}{2})} \right) d\lambda,$$  

(2.12)

where $\bar{e}'$ is a $2n - 1$ vector whose last $n$ entries are 1 and the first $n - 1$ entries are 0.

The parameter $\tau$ in the argument of $\theta$ is introduced in section 6 and is defined in equation (6.11), while the expression of $\beta(\lambda)$ is

$$\beta(\lambda) := \frac{1}{2\pi i} \log \frac{\lambda + 1}{\lambda - 1}. \quad (2.13)$$

Theorem 1 generalizes the result by Its et al. [12, 13] for the XY model. In that case the genus of the of $\mathcal{L}$ is one, and the theta function in the integral reduces to the Jacobi theta function $\theta_3$. However, for the XY model the integral (2.12) can be expressed in term of the infinite series

$$S(\rho_A) = \sum_{m=-\infty}^{\infty} (1 + \mu_m) \log \frac{2}{1 + \mu_m} = 2 \sum_{m=0}^{\infty} e(1, \mu_m), \quad (2.14)$$

where the numbers $\mu_m$ are the solutions of the equation

$$\theta_3 \left( \beta(\lambda) + \frac{\sigma \tau}{2} \right) = 0 \quad (2.15)$$
and $\sigma$ is 0 or 1 depending on the strength of the magnetic field. The zeros of the one dimensional theta function are all known, so that the numbers $\mu_m$ can be described by the explicit formula

$$\mu_m = -i \tan \left( m + \frac{1 - \sigma}{2} \right) \pi \tau.$$  

Moreover, as it was shown by Peschel [21] (who also suggested an alternative heuristic derivation of equation (2.14) based on the work of Calabrese and Cardy [3]), the series (2.14) can be summed up to an elementary function of the complete elliptic integrals corresponding to the modular parameter $\tau$.

It is an open problem whether an analogous representation of the integral (2.12) exists for $g > 1$.

The next step consists of understanding what happens to formula (2.12) when we approach a phase transition. The hyperelliptic curve $L$, and hence all the parameters in the integral (2.12), are determined by the roots of the polynomial $p(z)$ which defines the symbol (1.5). In section 3 we discuss how the coefficients of $p(z)$ are related to the the Hamiltonians of the spin chains. In the case of the XY model $p(z)$ is given by equation (1.7); since the degree of $p(z)$ is two the roots $\lambda_j$ can be easily determined as a function of the parameters $\alpha$ and $\gamma$. It was shown by Calabrese and Cardy [4] that when $\alpha = 1$ — or the magnetic field $h = 2$ — the XY model undergoes a phase transition and the entropy diverges. Jin and Korepin [14] showed that when $\gamma$ approaches 0, i.e. the XY model approaches the XX model, and $\alpha \leq 1$, then the entanglement entropy diverges logarithmically. Its et. al. [12, 13] discovered that the divergence of the entropy for the XY and XX model corresponds to the roots (2.4) of (2.8) approaching the unit circle.

This phenomenon extends to the family of systems that we study. In other words, a phase transition manifests itself when pairs of roots of (2.8) approach the unit circle; one root in each pair is inside the unit circle, the other outside. As we shall see, in these circumstances the entropy of entanglement diverges logarithmically. From (2.4) we see that if $\lambda_j$ is a root of (2.8) so is $\lambda_j^{-1}$. Moreover, since (2.8) is a polynomial with real coefficients, if $\lambda_j$ is complex then $\overline{\lambda}_j$ and $\overline{\lambda}_j^{-1}$ will be roots of (2.8) too (see figure 3). Now, suppose that $\lambda_j$ approaches the unit circle and $|\lambda_j| < 1$, then $|\overline{\lambda}_j| > 1$ and $\overline{\lambda}_j^{-1}$ will also be approaching the unit circle with

$$\lambda_j - \overline{\lambda}_j^{-1} \to 0.$$  

At a phase transition the behavior of the entropy of entanglement is captured by

**Theorem 2.** Let the $m$ pairs of roots $\lambda_j, \overline{\lambda}_j^{-1}, j = 1, \ldots, m,$ approach together towards the unit circle such that the limiting values of $\lambda_j, \overline{\lambda}_j^{-1}$ are distinct from those of $\lambda_k, \overline{\lambda}_k^{-1}$ if $j \neq k$, then the entanglement entropy is asymptotic to

$$S(\rho_A) = -\frac{1}{6} \sum_{j=1}^{m} \log |\lambda_j - \overline{\lambda}_j^{-1}| + O(1), \quad \lambda_j \to \overline{\lambda}_j^{-1}, \quad j = 1, \ldots, m. \quad (2.16)$$
From the integral (2.1) it is evident that in order to prove theorems 1 and 2 we need an explicit asymptotic formula for the determinant $D_L(\lambda)$. Indeed, the following proposition gives us an asymptotic representation for the determinants of block-Toeplitz matrices whose symbols belong to the family defined in equations (1.5) and (1.6).

**Proposition 1.** Let $\Omega_\epsilon$ be the set

$$\Omega_\epsilon := \{ \lambda \in \mathbb{R} : |\lambda| \geq 1 + \epsilon \}. \quad (2.17)$$

Then the Toeplitz determinant $D_L(\lambda)$ admits the following asymptotic representation, which is uniform in $\lambda \in \Omega_\epsilon$:

$$D_L(\lambda) = (1 - \lambda^2)^L \frac{\theta \left( \beta(\lambda) \overline{\nu} + \frac{1}{\nu} \right) \theta \left( \beta(\lambda) \overline{\nu} - \frac{1}{\nu} \right)}{\theta^2 \left( \frac{\nu}{\overline{\nu}} \right)} \left( 1 + O \left( \rho^{-L} \right) \right), \quad L \to \infty, \quad (2.18)$$

Here $\rho$ is any real number satisfying the inequality

$$1 < \rho < \min\{|\lambda_j| : |\lambda_j| > 1\}.$$

**Remark 1.** The first factor in the right hand side of equation (2.18) corresponds to the “trivial” factor, $G[\Phi]$ of the general Widom’s formula (5.1), which we discuss in detail in section 5, while the ratio of the theta functions provides an explicit expression of the most interesting part of the formula — Widom’s pre-factor $E[\Phi] \equiv \det \left( T_\infty[\Phi] T_\infty[\Phi^{-1}] \right)$, which is given in formula (5.2).
Remark 2. The Asymptotic representation (2.18) is actually valid in a much wider domain of the complex plane \( \lambda \). Indeed, it is true everywhere away from the zeros of the right hand side, which, unfortunately, in the case of the genus \( g > 1 \) is very difficult to express in a simple closed form — one faces a very transcendental object, i.e. the theta-divisor. This constitutes an important difference between the general case and that one with \( g = 1 \) studied in \([12] \) and \([13] \), where the zeros of equation (2.15) can be easily evaluated.

3 Quantum spin chains with anisotropic Hamiltonians

The XY model is a spin-1/2 ferromagnetic chain with an exchange coupling \( \alpha \) in a constant transversal magnetic field \( h \). The Hamiltonian is \( H = hH_\alpha \) with \( H_\alpha \) given by

\[
H_\alpha = -\frac{\alpha}{2} \sum_{j=0}^{M-1} \left[ (1 + \gamma) \sigma_j^x \sigma_{j+1}^x + (1 - \gamma) \sigma_j^y \sigma_{j+1}^y \right] - \sum_{j=0}^{M-1} \sigma_j^z,
\]

where \( \{ \sigma^x, \sigma^y, \sigma^z \} \) are the Pauli matrices. The parameter \( \gamma \) lies in the interval \([0, 1]\) and measures the anisotropy of \( H_\alpha \). When \( \gamma = 0 \) \((3.1)\) becomes the Hamiltonian of the XX model. In the limit \( M \to \infty \) the XY model undergoes a phase transition at \( \alpha_c = 1 \).

It is well known that the Hamiltonian \((3.1)\) can be mapped into a quadratic form of Fermi operators and then diagonalized. To this purpose, we introduce the Jordan-Wigner transformations. Let us define

\[
m_{2l+1} = \left( \prod_{j=0}^{l-1} \sigma_j^z \right) \sigma_l^x \quad \text{and} \quad m_{2l} = \left( \prod_{j=0}^{l-1} \sigma_j^z \right) \sigma_l^y.
\]

The inverse relations are

\[
\sigma_l^x = i m_{2l+1}, \\
\sigma_l^x = \left( \prod_{j=0}^{l-1} i m_{2j} m_{2j+1} \right) m_{2l+1}, \\
\sigma_l^y = \left( \prod_{j=0}^{l-1} i m_{2j} m_{2j+1} \right) m_{2l}
\]

These operators obey the commutation relations \( \{m_j, m_k\} = 2\delta_{jk} \) but are not quite Fermi operator since they are Hermitian. Thus, we define

\[
b_l = (m_{2l+1} - im_{2l})/2 \quad \text{and} \quad b_l^\dagger = (m_{2l+1} + im_{2l})/2,
\]

which are proper Fermi operator as

\[
\{b_j, b_k\} = 0 \quad \text{and} \quad \{b_j, b_k^\dagger\} = \delta_{jk}.
\]
In terms of the operators $b_j$’s the Hamiltonian (3.1) becomes

$$H_{\alpha} = \frac{\alpha}{2} \sum_{j=0}^{M-1} \left[ b_j^\dagger b_{j+1} + b_j^\dagger b_j + \gamma \left( b_j^\dagger b_{j+1} - b_j b_{j+1} \right) \right] - 2 \sum_{j=0}^{M-1} b_j^\dagger b_j. \quad (3.4)$$

It turns out that the expectation values of the operators (3.2) with respect to the ground state $|\Psi_g\rangle$ are

$$\langle \Psi_g | m_k | \Psi_g \rangle = 0, \quad (3.5)$$
$$\langle \Psi_g | m_j m_k | \Psi_g \rangle = \delta_{jk} + i (C_M)_{jk}, \quad (3.6)$$

where the correlation matrix $C_M$ has the block structure

$$C_M = \begin{pmatrix}
C_{11} & C_{12} & \cdots & C_{1M} \\
C_{21} & C_{22} & \cdots & C_{2M} \\
\vdots & \vdots & \ddots & \vdots \\
C_{M1} & C_{M2} & \cdots & C_{MM}
\end{pmatrix} \quad (3.7)$$

with

$$C_{jk} = \begin{pmatrix} 0 & g_{j-k} \\
-g_{k-j} & 0 \end{pmatrix}.$$

For large $M$, the real numbers $g_l$ are the Fourier coefficients of

$$g(\theta) = \frac{\alpha \cos \theta - 1 - i \gamma \alpha \sin \theta}{|\alpha \cos \theta - 1 - i \gamma \alpha \sin \theta|}.$$

In other words, $C_M$ is a block-Toeplitz matrix with symbol

$$\varphi(\theta) = \begin{pmatrix} 0 & g(\theta) \\
-g^{-1}(\theta) & 0 \end{pmatrix}. \quad (3.8)$$

(We outline the derivations of formulae (3.5) and (3.6) for the family of systems (3.10) that we study in the appendices B and C.)

Equation (3.5) is a straightforward consequence of the invariance of $H_{\alpha}$ under the map $b_j \mapsto -b_j$; for the same reason the expectation value of the product of an odd number of $m_j$’s must be zero. Formula (3.6) was derived for the first time by Lieb et al. [17]. The expectation values of the product of an even number of the $m_j$’s can be computed using Wick’s theorem:

$$\langle \Psi_g | m_{j_1} m_{j_2} \cdots m_{j_{2n}} | \Psi_g \rangle = \sum_{\text{all pairings}} (-1)^p \prod_{\text{all pairs}} (\text{contraction of the pair}), \quad (3.9)$$

---

\footnote{This is strictly true only for open-end Hamiltonians. If we impose periodic boundary conditions, then the term $b_{M-1}^\dagger b_0$ in (3.4) should be replaced by $\prod_{j=0}^{M-1} \left( 2b_j^\dagger b_j - 1 \right) b_{M-1}^\dagger b_0$. However, because we are interested in the limit $M \to \infty$, the extra factor in front of $b_{M-1}^\dagger b_0$ can be neglected.}
where a contraction of a pair is defined by $\langle \Psi_g | m_j m_{j_n} | \Psi_g \rangle$ and $p$ is the signature of the permutation, for a given pairing, necessary to bring operators of the same pair next to one other from the original order. Many important physical quantities, including the von Neumann entropy and the spin-spin correlation functions, are expressed in terms of the expectation values (3.9).

In this paper we study generalizations of the Hamiltonian (3.4) that are quadratic in the Fermi operators and translation invariant. More explicitly, we consider the family of systems

$$H_\alpha = \alpha \left[ \sum_{j,k=0}^{M-1} b_{j}^\dagger A_{jk} b_{k} + \frac{\gamma}{2} \left( b_{j}^\dagger B_{jk} b_{k} - b_{j} B_{jk} b_{k} \right) \right] - 2 \sum_{j=0}^{M-1} b_{j}^\dagger b_{j}$$

with cyclic boundary conditions. In terms of Pauli operators this Hamiltonian becomes

$$H_\alpha = -\frac{\alpha}{2} \sum_{0 \leq j \leq k \leq M-1} \left[ (A_{jk} + \gamma B_{jk}) \sigma_j^x \sigma_k^x \left( \prod_{l=j+1}^{k-1} \sigma_l^z \right) + (A_{jk} - \gamma B_{jk}) \sigma_j^y \sigma_k^y \left( \prod_{l=j+1}^{k-1} \sigma_l^z \right) - \sum_{j=0}^{M-1} \sigma_j^z \right].$$

The translation invariance of the interaction implies that $A_{jk} = A_{j-k}$ and $B_{jk} = B_{j-k}$, and the cyclic boundary conditions force $A$ and $B$ to be circulant matrices. Furthermore, since $H_\alpha$ is a Hermitian operator, the matrices $A$ and $B$ must be symmetric and anti-symmetric respectively. Now, let us introduce two real functions,

$$a : \mathbb{Z}/M \mathbb{Z} \longrightarrow \mathbb{R} \quad \text{and} \quad b : \mathbb{Z}/M \mathbb{Z} \longrightarrow \mathbb{R},$$

such that

$$a(j-k) = \alpha A_{j-k} - 2 \delta_{j,k} \quad \text{and} \quad b(j-k) = \alpha B_{j-k}, \quad j, k \in \mathbb{Z}/M \mathbb{Z}. \quad (3.12)$$

Since $A$ is symmetric and $B$ anti-symmetric, we must have

$$a(-j) = a(j) \quad \text{and} \quad b(-j) = -b(j).$$

We shall consider systems with finite range interaction, which implies that there exists a fixed $n < M$ such that

$$a(j) = b(j) = 0 \quad \text{for} \quad j > n. \quad (3.13)$$

In the appendices B and C we derive the expectation values in the ground state of the Jordan-Wigner operators $m_j$'s. They have the same structure as the expectation values (3.5) and (3.6), but now in the limit as $M \to \infty$ the symbol (3.8) of the correlation matrix $C_M$ is replaced by

$$\Phi(z) = \begin{pmatrix} 0 & g(z) \\ -g^{-1}(z) & 0 \end{pmatrix}, \quad |z| = 1, \quad (3.14)$$
where
\[
q(z) = \sum_{j=-n}^{n} (a(j) - \gamma b(j)) z^j
\]  \hspace{1cm} (3.16)
\[
p(z) = z^n q(z).
\]  \hspace{1cm} (3.17)

4 The von Neumann entropy and block-Toeplitz determinants

We now concentrate our attention to study the entanglement of formation of the ground state \( |\Psi_g\rangle \) of the family of Hamiltonians (3.10). Since the ground state is not degenerate, the density matrix is simply the projection operator \( |\Psi_g\rangle \langle \Psi_g| \). We then divide the system into two subchains: the first one A containing \( L \) spins; the second one B, made of the remaining \( M - L \). We shall further assume that \( 1 \ll L \ll M \). This division creates a bipartite system. The Hilbert space of the whole system is the direct product \( \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \), where \( \mathcal{H}_A \) and \( \mathcal{H}_B \) are spanned by the vectors
\[
\prod_{j=0}^{L-1} (b_j^\dagger)^{r_j} |\Psi_{\text{vac}}\rangle \quad \text{and} \quad \prod_{j=L}^{M-L} (b_j^\dagger)^{r_j} |\Psi_{\text{vac}}\rangle, \quad r_j = 0, 1,
\]
respectively. The vector \( |\Psi_{\text{vac}}\rangle \) is the vacuum state, which is defined by
\[
b_j |\Psi_{\text{vac}}\rangle = 0, \quad j = 0, \ldots, M-1.
\]

Our goal is to determine the asymptotic behavior for large \( L \), with \( L = o(M) \), of the von Neumann entropy
\[
S(\rho_A) = -\text{tr}\rho_A \log \rho_A,
\]  \hspace{1cm} (4.1)
where \( \rho_A = \text{tr}_B \rho_{AB} \) and \( \rho_{AB} = |\Psi_g\rangle \langle \Psi_g| \).

It turns out that after computing the partial trace of \( \rho_{AB} \) over the degrees of freedom of \( B \), the reduced density matrix \( \rho_A \) can be expressed in terms of first \( L \) Fermi operators that generate a basis spanning \( \mathcal{H}_A \). As a consequence, only the submatrix \( C_L \) formed by the first \( 2L \) rows and columns of the correlation matrix (3.7) will be relevant in the computation of the entropy (4.1). Now, \( C_L \) is even dimensional and skew-symmetric. Furthermore, since
\[
g(e^{-i\theta}) = g(e^{i\theta})
\]
its Fourier coefficients are real, therefore there exists an orthogonal matrix \( V \) that block-diagonalizes \( C_L \):
\[
VC_L V^t = \bigoplus_{j=0}^{L-1} \nu_j \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]  \hspace{1cm} (4.2)
where the $\pm i\nu_j$s are imaginary numbers and are the eigenvalues of the block-Toeplitz matrix $C_L = T_L[\varphi]$, where $\varphi$ is the symbol (5.13).

Let us introduce the operators
\[ c_j = \frac{(d_{2j+1} - id_{2j})}{2}, \quad j = 0, \ldots, L - 1, \quad (4.3) \]
where
\[ d_j = \sum_{k=0}^{2L-1} V_{jk} m_k. \quad (4.4) \]

Since $V$ is orthogonal $\{d_j, d_k\} = 2\delta_{jk}$ and the $c_j$s are Fermi operators. Combining equations (4.2), (4.3) and (4.4), we obtain the expectation values
\[ \langle \Psi_g | c_j | \Psi_g \rangle = \langle \Psi_g | c_j c_k | \Psi_g \rangle = 0, \quad (4.5) \]
\[ \langle \Psi_g | c_j^\dagger c_k | \Psi_g \rangle = \delta_{jk} \frac{1 - \nu_j}{2}. \quad (4.6) \]
The reduced density matrix $\rho_A$ can be computed directly from these expectation values. We report this computation in appendix A. We have
\[ \rho_A = \prod_{j=0}^{L-1} \left( 1 - \frac{\nu_j}{2} c_j^\dagger c_j + \frac{1 + \nu_j}{2} c_j c_j^\dagger \right). \quad (4.7) \]

In other words, as equations (4.5) and (4.6) already suggest, these fermionic modes are in a product of uncorrelated states, therefore the density matrix is the direct product
\[ \rho_A = \bigotimes_{j=0}^{L-1} \rho_j \quad \text{with} \quad \rho_j = \frac{1 - \nu_j}{2} c_j^\dagger c_j + \frac{1 + \nu_j}{2} c_j c_j^\dagger. \quad (4.8) \]

Since $(1 + \nu_j)/2$ and $(1 - \nu_j)/2$ are eigenvalues of density matrices they must lie in the interval $(0, 1)$, therefore,
\[ -1 < \nu_j < 1, \quad j = 0, \ldots, L - 1. \]

At this point the entropy of the entanglement between the two subsystems can be easily derived from equation (4.1):
\[ S(\rho_A) = \sum_{j=0}^{L-1} e(1, \nu_j), \quad (4.9) \]
where $e(x, \nu)$ is defined in equation (2.2). Using the residue theorem, formula (4.9) can be rewritten as
\[ S(\rho_A) = \lim_{\epsilon \to 0^+} \frac{1}{4\pi i} \oint_{\Gamma(\epsilon)} \left( -1 \right)^L \sum_{j=0}^{L-1} \frac{2\lambda}{\lambda^2 - \nu_j^2} e(1 + \epsilon, \lambda) d\lambda \]
\[ = \lim_{\epsilon \to 0^+} \frac{1}{4\pi i} \oint_{\Gamma(\epsilon)} e(1 + \epsilon, \lambda) \frac{d \log D_L(\lambda)}{d\lambda} d\lambda \quad (4.10) \]
where $\Gamma(\epsilon)$ is the contour in figure [1] and

$$D_L(\lambda) = (-1)^L \prod_{j=0}^{L-1} (\lambda^2 - \nu_j^2)$$

is the determinant of the block-Toeplitz matrix $T_L[\Phi](\lambda)$ with symbol (1.5).

The integral (4.10) was introduced for the first time by Jin and Korepin [14] to compute the entropy of entanglement in the XX model. In this case $g^{-1}(\theta) = g(\theta)$ and $D_L(\lambda)$ becomes the determinant of a Toeplitz matrix with a scalar symbol. Keating and Mezzadri [18, 19] generalized it to lattice models where $D_L(\lambda)$ becomes an average over one of the classical compact groups. Its et al. [12, 13] computed the same integral for the XY model, for which $D_L(\lambda)$ is the determinant of a block-Toeplitz matrix with symbol (1.4).

Following the same approach of Its et al., in this paper we express $D_L(\lambda)$ as a Fredholm determinant of an integrable operator on $L^2(\Xi, \mathbb{C}^2)$ and solve the Riemann-Hilbert problem associated to it. This will give an explicit formula for $D_L(\lambda)$, which can then be used to compute the integral (4.10).

5 The Asymptotics of Block Toeplitz Determinants. Widom’s Theorem

A generalization of the strong Szegö’s theorem to determinants of block-Toeplitz matrices was first discovered by Widom [24, 25]. Consider a $p \times p$ matrix symbol $\varphi$ and assume that

$$||\varphi|| = \sum_{k=-\infty}^{\infty} ||\varphi_k|| + \left( \sum_{k=-\infty}^{\infty} |k| ||\varphi_k||^2 \right)^{1/2} < \infty.$$  

The norm that appear in the right-hand side of this equation is the Hilbert-Schmidt norm of the $p \times p$ matrices that occur. In addition, we shall require that

$$\det \varphi(z) \neq 0 \quad \text{and} \quad \Delta_{|z|=1} \arg \det \varphi(z) = 0.$$  

Widom showed that if one defines

$$G[\varphi] := \exp \left( \frac{1}{2\pi i} \int_{\Xi} \log \det \varphi(z) \frac{dz}{z} \right)$$

then

$$E[\varphi] := \lim_{L \to \infty} \frac{D_L[\varphi]}{G[\varphi]^{L+1}} = \det \left( T_\infty[\varphi] T_\infty[\varphi^{-1}] \right),$$  

where $T_\infty[\varphi]$ is a semi-infinite Toeplitz matrix acting on the Hilbert space of semi-infinite sequence of $p$-vectors:

$$l^2 = \left\{ \{v_k\}_{k=0}^{\infty} \mid v_k \in \mathbb{C}^p, \sum_{k=0}^{\infty} ||v_k||^2 < \infty \right\}.$$
Formulae (5.1) and (5.2) reduce to Szegő’s strong limit theorem when \( p = 1 \). Although this beautiful formula is very general, it is difficult to extract information from the right-hand side of equation (5.2) and determine formulae that can be used in the applications. The advantage of our approach is precisely to derive explicit formula for the leading order term of the asymptotics of block-Toeplitz determinants whose symbols \( \Phi(z) \) belong to the one-parameter family defined in (1.5).

A starting point of our analysis is the asymptotic representation of the logarithmic derivative (with respect to the parameter \( \lambda \)) of the determinant \( D_L(\lambda) = \det T_L[\Phi](\lambda) \) in terms of \( 2 \times 2 \) matrix-valued functions, denoted by \( U_\pm(z) \) and \( V_\pm(z) \), which solve the following Wiener-Hopf factorization problem:

\[
\Phi(z) = U_+(z)U_-(z) = V_-(z)V_+(z),
\]

(5.3)

\[
U_-(z) \quad \text{and} \quad V_-(z) \quad (U_+(z) \quad \text{and} \quad V_+(z)) \quad \text{are analytic outside (inside)}
\]

the unit circle \( \Xi \),

(5.4)

\[
U_-(\infty) = V_-(\infty) = I.
\]

(5.5)

Now, let us fix \( \epsilon > 0 \) and define the set

\[
\Omega_\epsilon := \{ \lambda \in \mathbb{R} : |\lambda| \geq 1 + \epsilon \}.
\]

(5.6)

In the next section we will show that for every \( \lambda \in \Omega_\epsilon \) the solution of the above Wiener-Hopf factorization problem exists, and the corresponding matrix functions, \( U_\pm(z) \) and \( V_\pm(z) \) satisfy the following uniform estimate:

\[
\frac{1}{\lambda} U_+(z), \quad \frac{1}{\lambda} V_+(z), \quad |U_-(z)|, \quad |V_-(z)| < C_\epsilon, \quad \forall z \in D_\pm, \quad \forall \lambda \in \Omega_\epsilon,
\]

(5.7)

where the notation \( D_+ (D_-) \) is used for the interior (exterior) of the unit circle \( \Xi \). Moreover, generalizing the approach of [12, 13] we will obtain the multidimensional theta function explicit formulae for the functions \( U_\pm(z) \) and \( V_\pm(z) \).

The asymptotic representation of the logarithmic derivative \( d \log D_L(\lambda)/d\lambda \) is given by the following theorem:

**Theorem 3.** Let \( \lambda \in \Omega_\epsilon \), and fix a positive number \( R > 0 \). Then, we have the following asymptotic representation for the logarithmic derivative of the determinant \( D_L(\lambda) = \det T_L[\Phi] \):

\[
\frac{d}{d\lambda} \log D_L(\lambda) = -\frac{2\lambda}{1 - \lambda^2} L + \frac{1}{2\pi} \int_{\Xi} \text{tr} \left[ \left( U'_+(z)U_+^{-1}(z) + V_+^{-1}(z)V'_+(z) \right) \Phi^{-1}(z) \right] dz + r_L(\lambda),
\]

(5.8)

where \( (\cdot) \) means the derivative with respect to \( z \), the error term \( r_L(\lambda) \) satisfies the estimate

\[
|r_L(\lambda)| \leq C \rho^{-L}, \quad \lambda \in \Omega_\epsilon \cap \{|\lambda| \leq R\}, \quad L \geq 1,
\]

(5.9)

and \( \rho \) is any real number such that \( 1 < \rho < \min\{|\lambda_j| : |\lambda_j| > 1\} \).
This theorem, without the error term estimate is a specification of one of the classical results of H. Widom \[24\] for the case of the matrix generators $\Phi(z)$ whose dependence on the extra parameter $\lambda$ is given by the equation

$$\Phi(z) \equiv \Phi(z; \lambda) = i\lambda I + \Phi(z; 0).$$

The estimate \[5,9\] of the error term as well as an alternative proof of the theorem itself in the case of curves of genus one is given in \[12\] and \[13\]. The method of \[12\] and \[13\] is based on the Riemann-Hilbert approach to the Toeplitz determinants \[5\] and on the theory of the integrable Fredholm operators \[11, 9\]; its extension to symbols \[1.5\], where the polynomial $p(z)$ entering in \[1.6\] is of arbitrary degree, is straightforward. Indeed, the following generalization of theorem 3 follows directly from the analytic considerations of \[13\].

**Theorem 4.** Suppose that the matrix generator $\Phi(z)$ is analytic in the annulus,

$$\mathcal{D}_\delta = \{1 - \delta < |z| < 1 + \delta\}.$$

Suppose also that $\Phi(z)$ depends analytically on an extra parameter $\mu$ and that it admits a Wiener-Hopf factorisation for all $\mu$ from a certain set $\mathcal{M}$. Finally, we shall assume that the matrix functions

$$\Phi(z), \Phi^{-1}(z), \frac{\partial \Phi(z)}{\partial \mu}, U_\pm(z), \text{ and } V_\pm(z)$$

are uniformly bounded for all $\mu \in \mathcal{M}$ and all $z$ from the respective domains, i.e. $\mathcal{D}_\delta$ in the case of $\Phi(z), \Phi^{-1}(z)$, and $\partial \Phi(z)/\partial \mu$, and $\mathcal{D}_\pm$ in the case of $U_\pm(z)$ and $V_\pm(z)$. Then, the logarithmic derivative of the determinant $D_L(\mu) = \det T_L[\Phi]$ has the following asymptotic representation:

$$\frac{d}{d\mu} \log D_L(\mu) = \frac{L}{2\pi i} \int_{\Xi} \text{tr} \left( \frac{\Phi^{-1}(z) \partial \Phi(z)}{\partial \mu} \right) \frac{dz}{z} + \frac{1}{2\pi i} \int_{\Xi} \text{tr} \left( (\Phi^{-1}(z))' \frac{\partial \Phi(z)}{\partial \mu} \right) \frac{dz}{z}$$

$$+ \frac{1}{2\pi i} \int_{\Xi} \text{tr} \left( U_+''(z)U_+^{-1}(z) \frac{\partial \Phi(z)}{\partial \mu} + V_+^{-1}(z) \Phi^{-1}(z) \frac{\partial \Phi(z)}{\partial \mu} + V_+''(z) \Phi^{-1}(z) \frac{\partial \Phi(z)}{\partial \mu} \right) \frac{dz}{z}$$

$$+ r_L(\mu),$$

where the error term $r_L(\mu)$ satisfies the uniform estimate

$$|r_L(\mu)| \leq C \rho^{-L}, \quad \mu \in \mathcal{M}, \quad L \geq 1,$$

and $\rho$ is any positive number such that $1 < \rho < 1 + \delta$.

This theorem, without the estimate of the error term and with much weaker assumptions on the generator $\Phi(z)$, is exactly the classical result of Widom from \[24\].
Remark 3. Denote
\[ u_\pm(z) = V_\pm^{-1}(z), \quad \text{and} \quad v_\pm(z) = U_\pm^{-1}(z), \]
so that
\[ \Phi^{-1}(z) = u_+(z)u_-(z) = v_-(z)v_+(z). \]
Then, equation (5.10) can be re-written in a more compact way:
\[
\frac{d}{d\mu} \log D_L(\mu) = \frac{L}{2\pi i} \int_{\Xi} \text{tr} \left( \Phi^{-1}(z) \frac{\partial \Phi(z)}{\partial \mu} \right) \frac{dz}{z} + i \frac{L}{2\pi} \int_{\Xi} \text{tr} \left( (u'_+(z)u_-(z) - v'_-(z)v_+(z)) \frac{\partial \Phi(z)}{\partial \mu} \right) dz + r_L(\mu).
\]
(5.12)

This form in which this result is formulated in [24].

Theorem 4 can be used to strengthen the statement of theorem 3 by removing the dependence of the constant \( C \) on \( R \) in the estimate (5.9). This leads to the following extension of theorem 3:

Theorem 5. Let \( \Omega_\epsilon \) be the set defined in (5.6) and let \( \Phi(z) \) be the symbol defined in (1.5). Then we have the following asymptotic representation of the logarithmic derivative of the determinant \( D_L(\lambda) = \det T_L[\Phi] \) for all \( \lambda \in \Omega_\epsilon \):
\[
\frac{d}{d\lambda} \log D_L(\lambda) = -\frac{2\lambda}{1 - \lambda^2} L + \frac{1}{2\pi} \int_{\Xi} \text{tr} \left[ (U'_+(z)U_+^{-1}(z) + V'_+(z)V_+^{-1}(z)) \Phi^{-1}(z) \right] dz + r_L(\lambda),
\]
(5.13)
where ('') means the derivative with respect to \( z \), the error term \( r_L(\lambda) \) satisfies the uniform estimate
\[
|r_L(\lambda)| \leq \frac{C}{\lambda^2 \rho^{-L}}, \quad \lambda \in \Omega_\epsilon, \quad L \geq 1
\]
(5.14)
and \( \rho \) is any real number such that \( 1 < \rho < \min\{|\lambda_j|: |\lambda_j| > 1\} \).

Proof. Let \( R > 1 + \epsilon \) and denote \( C_1 \) the constant \( C \) from estimate (5.9). Take now \( \lambda \in \Omega_\epsilon, \quad |\lambda| \geq R \) and set
\[
\mu = \frac{1}{\lambda} \in \mathcal{M} \equiv \left\{ \mu \in \mathbb{R} : |\mu| \leq \frac{1}{R} < \frac{1}{1 + \epsilon} \right\}.
\]
By trivial algebra, we arrive at
\[
\det D_L(\lambda) = (-\lambda^2)^L \det \tilde{D}_L(\mu),
\]
where \( \tilde{D}_L(\mu) \equiv \det T_L[\tilde{\Phi}] \) and
\[
\tilde{\Phi}(z) \equiv \frac{1}{i\lambda} \Phi(z) = I - i\mu \Phi(z; 0) \equiv \begin{pmatrix} 1 & -i\mu g(z) \\ i\mu g^{-1}(z) & 1 \end{pmatrix}.
\]
(5.15)
From this relation it also follows that

$$\frac{d}{d\lambda} \log \det D_L(\lambda) = \frac{2L}{\lambda} - \frac{1}{\lambda^2} \frac{d}{d\mu} \log \det \tilde{D}_L(\mu),$$

(5.16)

and hence the asymptotic analysis of the logarithmic derivative $d \log \det D_L(\lambda)/d\lambda$ for $|\lambda| \geq R$ is reduced to that one of the logarithmic derivative $d \log \det \tilde{D}_L(\mu)/d\mu$ for $\mu \in \mathcal{M} \equiv \{ \mu \in \mathbb{R} : |\mu| \leq \frac{1}{R} < \frac{1}{1+\varepsilon} \}.$

Firstly, we notice that for all $\mu \in \mathcal{M}$ and $z \in D_\delta$ the functions $\tilde{\Phi}(z)$, $\tilde{\Phi}^{-1}(z)$ and $\partial \tilde{\Phi}(z)/\partial \mu$ are uniformly bounded. Secondly, we have that

$$\tilde{\Phi}(z) = \frac{1}{i\lambda} \Phi(z) = \frac{1}{i\lambda} U_+(z)U_-(z) = \frac{1}{i\lambda} V_-(z)V_+(z),$$

and hence the matrix valued functions $\tilde{U}_+(z)$ and $\tilde{V}_+(z)$ defined by the relations

$$\tilde{U}_+(z) = \frac{1}{i\lambda} U_+(z), \quad \tilde{V}_+(z) = \frac{1}{i\lambda} V_+(z), \quad \tilde{U}_-(z) = U_-(z), \quad \tilde{V}_-(z) = V_-(z)$$

provide the Wiener-Hopf factorization of the generator $\tilde{\Phi}(z)$. Moreover, because of the estimates (5.7), the functions $\tilde{U}_\pm(z)$ and $\tilde{V}_\pm(z)$ are uniformly bounded for all $\mu \in \mathcal{M}$ and $z \in D_\pm$. Hence, all the conditions of theorem 4 are met, and we can claim the uniform asymptotic representation (5.10) of the logarithmic derivative of the determinant $\tilde{D}_L(\mu)$ with the symbols $\Phi$, $U$, and $V$ replaced by $\tilde{\Phi}$, $\tilde{U}$ and $\tilde{V}$ respectively. We shall also use the notation $\tilde{r}_L(\mu)$ and $C_2$ for the error term and constant $C$ from the corresponding estimate (5.11) respectively.

The specific form (5.15) of dependence of the generator $\tilde{\Phi}(z)$ on the parameter $\mu$ implies that

$$\tilde{\Phi}^{-1}(z) \frac{\partial \tilde{\Phi}(z)}{\partial \mu} = \frac{1}{1 - \mu^2} \begin{pmatrix} -\mu & -ig(z) \\ ig^{-1}(z) & -\mu \end{pmatrix},$$

(5.17)

and

$$(\tilde{\Phi}^{-1})'(z) \frac{\partial \tilde{\Phi}(z)}{\partial \mu} = \frac{1}{1 - \mu^2} \begin{pmatrix} -\mu g^{-1}(z)g'(z) & 0 \\ 0 & \mu g^{-1}(z)g'(z) \end{pmatrix}.$$ 

(5.18)

Hence

$$\text{tr} \left( \tilde{\Phi}^{-1}(z) \frac{\partial \tilde{\Phi}(z)}{\partial \mu} \right) = -\frac{2\mu}{1 - \mu^2} = \frac{2\lambda}{1 - \lambda^2}$$

and

$$\text{tr} \left( (\tilde{\Phi}^{-1})'(z) \frac{\partial \tilde{\Phi}(z)}{\partial \mu} \right) = 0$$

and equation (5.10) for the determinant $\tilde{D}_L(\mu)$ becomes

$$\frac{d}{d\mu} \log \tilde{D}_L(\mu) = \frac{2\lambda}{1 - \lambda^2} L.$$
+ \frac{1}{2\pi i} \int_{\Xi} \text{tr} \left( \tilde{U}'(z) \tilde{U}^{-1}(z) \frac{\partial \tilde{\Phi}(z)}{\partial \mu} \tilde{\Phi}^{-1}(z) + \tilde{V}'(z) \tilde{V}^{-1}(z) \frac{\partial \tilde{\Phi}(z)}{\partial \mu} \right) dz + \tilde{r}_L(\mu), \tag{5.19}

\text{with}
$$|\tilde{r}_L(\mu)| \leq C_2 \rho^{-L}, \quad \mu \in \mathcal{M}, \quad L \geq 1. \tag{5.20}$$

Observe now that equation (5.17) can be rewritten as
$$\tilde{\Phi}^{-1}(z) \frac{\partial \tilde{\Phi}(z)}{\partial \mu} = \frac{\partial \tilde{\Phi}(z)}{\partial \mu} \tilde{\Phi}^{-1}(z) = (\lambda I - i \lambda^2 \Phi^{-1}(z)).$$

This relation, together with the obvious fact that
$$\tilde{U}'(z) \tilde{U}^{-1}(z) = U'(z)U^{-1}(z) \quad \text{and} \quad \tilde{V}'(z) \tilde{V}^{-1}(z) = V^{-1}(z)V'(z),$$
allows to transform (5.19) into the asymptotic formula
$$\frac{d}{d\mu} \log \tilde{D}_L(\mu) = \frac{2\lambda}{1 - \lambda^2} L - \frac{\lambda^2}{2\pi} \int_{\Xi} \text{tr} \left[ (U'_+(z)U^{-1}_+(z) + V'_+(z)V^{-1}_+(z)) \Phi^{-1}(z) \right] dz + \tilde{r}_L(\mu). \tag{5.21}$$

The substitution of this relation into the right hand side of equation (5.16) yields the following asymptotic formula — which is complementary to the equation (5.8) —
$$\frac{d}{d\lambda} \log D_L(\lambda) = -\frac{2\lambda}{1 - \lambda^2} L + \frac{1}{2\pi} \int_{\Xi} \text{tr} \left[ (U'_+(z)U^{-1}_+(z) + V'_+(z)V^{-1}_+(z)) \Phi^{-1}(z) \right] dz + \tilde{r}_L(\lambda), \tag{5.22}$$
with the error term $r_L(\lambda)$ satisfying the estimate
$$|r_L(\lambda)| \leq \frac{C_2}{|\lambda|^2} \rho^{-L}, \quad \lambda \in \Omega_\epsilon \cap \{|\lambda| \geq R\}, \quad L \geq 1. \tag{5.23}$$

Choosing
$$C = \max \{C_1 R, C_2\},$$
we arrive at the statement of the theorem, but with a better estimate for the error term $r_L(\lambda)$ than that one in (5.14).

In order to improve the estimate (5.23), we notice that since $\tilde{\Phi}(z)$ becomes the identity matrix as $\mu \to 0$, the Wiener-Hopf factorization of $\tilde{\Phi}(z)$ exists for all $\mu$ from the small complex neighbourhood
$$\mathcal{M}_0 \equiv \{ \mu \in \mathbb{C} : |\mu| < \epsilon_0 \leq \frac{1}{R} \}.$$
of the point $\mu = 0$. In particular, this implies that the Wiener-Hopf factors, $\tilde{U}_\pm(z)$ and $\tilde{V}_\pm(z)$, admit an analytic continuation to the disc $\mathcal{M}_0$ and that the validity of the formulae (5.19) and (5.20) can be extended to the set $\mathcal{M}_0 \cup \mathcal{M}$.

Moreover, from equation (5.19) it follows that $\tilde{r}_L(\mu)$ is analytic in the disc $\mathcal{M}_0$ and that $\tilde{r}_L(0) = 0$. In order to see that the latter equality is true, one has to take into account that $\tilde{U}_\pm(z) = \tilde{V}_\pm(z) = I$ for all $z$ and $\mu = 0$ and the evenness of $\tilde{D}_L(\mu)$ as a function of $\mu$. Now, define

$$\hat{r}_L(\mu) = \frac{\tilde{r}_L(\mu)}{\mu}.$$  

The function $\hat{r}_L(\mu)$ is analytic in the disc $\mathcal{M}_0$ and satisfies the estimate (5.20) uniformly for $\mu \in C_\epsilon \equiv \{|\mu| = \epsilon_0\}$ and for any $0 < \epsilon' < \epsilon_0$. With the help of the Cauchy formula,

$$\hat{r}_L(\mu) = \frac{1}{2\pi i} \oint_{|\mu'| = \epsilon_0/2} \frac{\hat{r}_L(\mu')}{\mu' - \mu} \, d\mu',$$

we conclude that

$$|\hat{r}_L(\mu)| < C' \rho^L, \quad |\mu| \leq \epsilon_0/3, \quad L > 1$$

or

$$|\hat{r}_L(\mu)| < C' |\mu| \rho^L, \quad |\mu| \leq \epsilon_0/3, \quad L > 1.$$  

The last inequality combined with (5.20) allows to replace it by the estimate

$$|\hat{r}_L(\mu)| < C' |\mu| \rho^L, \quad \mu \in \mathcal{M}, \quad L > 1,$$

which, in turn, transforms estimate (5.23) into the estimate

$$|r_L(\lambda)| \leq \frac{C_2}{|\lambda|^2} \rho^{-L}, \quad \lambda \in \Omega_\epsilon \cap \{|\lambda| \geq R\}, \quad L \geq 1,$$

and hence yields the correction term as announced in (5.14). This completes the proof of the theorem. \hfill \square

### 6 The Wiener-Hopf factorization of $\Phi(z)$

In this section we will compute the Wiener-Hopf factorization of $\Phi(z)$. We will express the solution in terms of theta functions on a hyperelliptic curve $\mathcal{L}$.

From the equality

$$(1 - \lambda^2)\sigma_3 \Phi^{-1}(z) \sigma_3 = \Phi(z), \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
we can express $V$ in terms of $U$ as follows:
\[
V_-(z) = \sigma_3 U^{-1}_- \sigma_3 \\
V_+(z) = \sigma_3 U^+_\sigma_3(1 - \lambda^2), \quad \lambda \neq \pm 1. \tag{6.1}
\]
Therefore, we only need to compute $U(z)$. To do so, first note that $\Phi(z)$ can be diagonalized by the matrix
\[
Q(z) = \begin{pmatrix} g(z) & -g(z) \\
\bar{g}(z) & \bar{g}(z) \end{pmatrix}. \tag{6.2}
\]
Indeed, it is straightforward to see that
\[
\Phi(z) = Q(z) \Lambda Q^{-1}(z), \\
\Lambda = i \begin{pmatrix} \lambda + 1 & 0 \\
0 & \lambda - 1 \end{pmatrix}.
\]
The function $Q(z)$ has the following jump discontinuities on the $z$-plane:
\[
Q_+ = Q_-(z) \sigma_1, \quad z \in \Sigma_i, \\
\sigma_1 = \begin{pmatrix} 0 & 1 \\
1 & 0 \end{pmatrix},
\]
where the branch cuts $\Sigma_i$ are defined in (2.4), (2.5) and (2.6) and $Q_{\pm}(z)$ are the boundary values of $Q(z)$ to the left/right of $\Sigma_i$. It also has square-root singularities at each branch point with the following behavior:
\[
Q(z) = Q_{\pm i}(z) \begin{pmatrix} (z - z_i^{\pm 1})^{\frac{1}{2}} & 0 \\
0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\
1 & 1 \end{pmatrix}, \quad z \to z_i^{\pm},
\]
where $Q_{\pm i}(z)$ are functions that are holomorphic and invertible at $z_i^{\pm}$.

Let us define
\[
S(z) = U_-(z)Q(z)\Lambda^{-1}, \quad |z| \geq 1, \\
S(z) = U_+(z)^{-1}Q(z), \quad |z| \leq 1. \tag{6.3}
\]
By direct computation we see $S(z)$ is the unique solution of the following Riemann-Hilbert problem:
\[
S_+(z) = S_-(z) \sigma_1, \quad z \in \Sigma_i, \quad i = 1, \ldots, n \\
S_+(z) = S_-(z) \Lambda \sigma_1 \Lambda^{-1}, \quad z \in \Sigma_i, \quad i = n + 1, \ldots, 2n \\
\lim_{z \to \infty} S(z) = Q(\infty)\Lambda^{-1}, \tag{6.4}
\]
where, as before, $S_{\pm}(z)$ denotes the boundary values of $S(z)$ to the left/right of the branch cuts. The matrix function $S(z)$ is holomorphic and invertible everywhere, except on the
cuts $\Sigma_j$, where it has the jump discontinuities given in (6.4), and in proximity of the branch points, where it behaves like

$$S(z) = S_{\pm i}(z) \left( \begin{array}{cc} (z - z_i^\pm)^{\pm \frac{1}{2}} & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right), \quad z \to z_i^\pm, \quad |z| < 1,$$

(6.5)

$$S(z) = S_{\pm i}(z) \left( \begin{array}{cc} (z - z_i^\pm)^{\pm \frac{1}{2}} & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right)^{-1}, \quad z \to z_i^\pm, \quad |z| > 1.$$
has precisely \( g \) zeros located at the points \( d_i, \ i = 1, \ldots, g \). The vector \( K = (K_1, \ldots, K_g) \) is the Riemann constant

\[
K_j = \frac{2\pi i + \Pi_{jj}}{2} - \frac{1}{2\pi i} \sum_{l \neq j} \int_{\sigma_l} (d\omega_l(p)) \int_{z_1}^{p} d\omega_j.
\]

The hyperelliptic curve \( \mathcal{L} \) can be thought of as a branched cover of the Riemann sphere \( \mathbb{C} \cup \{\infty\} \). Indeed, a point \( p \in \mathcal{L} \) can be identified by two complex variables, \( p = (z, w) \), where \( w \) and \( z \) are related by equation (2.8). We shall denote by \( \mathbb{C}_1 \) the the Riemann sheet where \( g(\infty) > 0 \) on the real axis, and by \( \mathbb{C}_2 \) the other Riemann sheet in \( \mathcal{L} \). Thus, a function \( f \) on \( \mathcal{L} \) can be thought of as a function in two complex variables:

\[
f(p) = f(z, w).
\]

Consider the map

\[
T : \mathbb{C}/ \cup_{i=1}^{2n} \Sigma_i \to \mathcal{L}
\]

\[
T(z) = (z, w),
\]

where the branch of \( w \) is chosen such that \( (z, w) \) is on \( \mathbb{C}_1 \). A function \( f \) on \( \mathcal{L} \) then defines the function \( f \circ T \) on \( \mathbb{C}/ \cup_{i=1}^{2n} \Sigma_i \) by

\[
f \circ T(z) = f(z, w).
\]

For the sake of simplicity, and when there is no ambiguity, we shall write \( f(z) \) instead of \( f \circ T(z) \) and \( f(p) \) instead of \( f(z, w) \).

Abelian integrals on \( \mathcal{L} \) can be represented as integrals on the Riemann sheet with jump discontinuities. To do so, let us first define a Jordan arc \( \Sigma \) as in figure 4. Let \( f(z, w) \) be a function on \( \mathcal{L} \) and \( f(z) = f \circ T(z) \). Then an Abelian integral on \( \mathcal{L} \),

\[
I(p) = \int_{\lambda_1}^{p} f(p') dp',
\]

defines the following integral on \( \mathbb{C} \):

\[
I(z) = \int_{\lambda_1}^{z} f \circ T(z') dz',
\]

where the path of the integration does not intersect \( \Sigma/\{\lambda_1\} \). Such integral will in general have jump discontinuities along \( \Sigma \), and its value on the left hand side of \( \Sigma \) will be denoted by \( I(z)_+ \), while its value on the right hand side of \( \Sigma \) will be denoted by \( I(z)_- \).

Let \( \rho \) be the hyperelliptic involution that interchanges the two sheets of \( \mathcal{L} \), i.e.

\[
\rho(z, w) = (z, -w).
\]
The action of $\rho$ on $f(z)$ is given by
\[ \rho(f)(z) = f(z, -w) \]  
\[ \text{(6.8)} \]
i.e it is the function evaluated on $\mathbb{C}_2$. Similarly, the action of $\rho$ on an integral $I(z)$ is defined by
\[ \rho(I)(z) = \int_{\lambda_1}^{z} \rho(f)(z')dz' \]  
\[ \text{(6.9)} \]

Figure 4: The Jordan arc $\Sigma$ connects all the branch points and extends to infinity on the left hand side of $\lambda_1$ and on the right hand side of $\lambda_{4n}$. All branch cuts belong to $\Sigma$ and are denoted by $\Sigma_i$, while the intervals between the branch cuts are denoted by $\tilde{\Sigma}_i$.

From proposition \[ \text{2} \] we see that the composition of the Abel map $\omega$ with $\theta$ has the following jump discontinuities when considered as a function on $\mathbb{C}$:

**Lemma 2.** Let $z$ be a point on $\mathbb{C}$, and let $\Sigma$ be a Jordan arc joining all the branch cuts as in figure 4, then the quotient of theta functions has the following jump discontinuities on $\Sigma$:

\[ \frac{\theta(\omega(z) + A)}{\theta(\omega(z) + B)}_+ = \frac{\theta(\omega(z) + A)}{\theta(\omega(z) + B)}_- , \quad z \in \tilde{\Sigma}_j \]
\[ \frac{\theta(-\omega(z) + A)}{\theta(-\omega(z) + B)}_+ = e^{-2\pi i(A_{j-1} - B_{j-1})} \frac{\theta(-\omega(z) + A)}{\theta(-\omega(z) + B)}_- , \quad z \in \Sigma_j \]

where $A$ and $B$ are arbitrary $2n - 1$ vectors and $A_0 = B_0 = 0$.

**Proof.** The holomorphic differentials $d\omega_j$ are given by
\[ d\omega_i = \frac{P_i(z)}{w(z)}dz, \]
for some polynomial $P_i(z)$ of degree less than $2n - 1$ in $z$. This means that, under the action of $\rho$, $d\omega_i$ becomes $-d\omega_i$. In particular, we have
\[ \rho(\omega)(z) = -\omega(z) \]  
\[ \text{(6.10)} \]
where the action of \( \rho \) on \( \omega \) is given by \((6.8)\) and \((6.9)\).

We first consider the jumps across the gaps \( \bar{\Sigma}_j \). Take two distinct paths from \( \lambda_1 \) to a point \( z \in \bar{\Sigma}_j \). Assume also that both curves do not intersect \( \Sigma \) and that one extends to the left of \( \Sigma \), while the other to its right. The union of these paths lifts to a loop \( \bar{\gamma} \) on \( \mathcal{L} \). Moreover, \( \bar{\gamma} \) is a linear combinations of \( a \)-cycles, \( i.e. \)

\[
\bar{\gamma} = \sum_{i=1}^{g} N_i a_i,
\]

where the \( N_i \)'s are non-negative integers.

Therefore, we have

\[
\frac{\theta(\omega(z) + A)}{\theta(\omega(z) + B)}_+ \cdot \frac{\theta(\omega(z) + A)}{\theta(\omega(z) + B)}_+ = \exp\left(-2\pi i (A_{j-1} - B_{j-1})\right)
\]

\( k = \prod_{ki} \).

This proves the lemma.

We can now solve the Riemann-Hilbert problem \((6.1), (6.5)\). Let us define

\[
\tau := -2n \sum_{i=2}^{2n} \omega(z_i^{-1}) - K, \quad (6.11)
\]

\[
\Delta(z) := \int_{+\infty}^{z} d\Delta,
\]

where \( d\Delta \) is the normalized differential of third type with simple poles at \( \infty^\pm \) and residues \( \pm \frac{1}{2} \) respectively. In addition, we write

\[
\kappa := \left(\frac{1}{2\pi i} \int_{b_1} d\Delta, \ldots, \frac{1}{2\pi i} \int_{b_g} d\Delta\right).
\]
Proposition 3. Let $\infty^\pm$ be the points on $\mathcal{L}$ that projects to $\infty$ on $\mathbb{C}_1$. The unique solution of the Riemann-Hilbert problem (6.4), (6.5) is given by

$$S(z) = Q(\infty)\Lambda^{-1}(\infty)\Theta(z),$$

(6.12)

where entries of $\Theta(z)$ are given by

$$\Theta_{11}(z) = \sqrt{z - \lambda_1 e^{-\Delta(z)}} \frac{\theta (\omega(z) + \beta(z) \overline{e}^i - \kappa + \frac{\ell}{2})}{\theta (\omega(z) + \frac{\ell}{2})},$$

$$\Theta_{12}(z) = -\sqrt{z - \lambda_1 e^{-\Delta(z)}} \frac{\theta (\omega(z) - \beta(z) \overline{e}^i + \kappa - \frac{\ell}{2})}{\theta (\omega(z) - \frac{\ell}{2})},$$

$$\Theta_{21}(z) = -\sqrt{z - \lambda_1 e^{-\Delta(z)}} \frac{\theta (\omega(z) + \beta(z) \overline{e}^i + \kappa - \frac{\ell}{2})}{\theta (\omega(z) - \frac{\ell}{2})},$$

$$\Theta_{22}(z) = \sqrt{z - \lambda_1 e^{-\Delta(z)}} \frac{\theta (\omega(z) - \beta(z) \overline{e}^i - \kappa + \frac{\ell}{2})}{\theta (\omega(z) + \frac{\ell}{2})},$$

(6.13)

where and $\overline{e}$ is a $2n - 1$ dimensional vector whose last $n$ entries are 1 and the first $n - 1$ entries are 0. The branch cut of $\sqrt{z - \lambda_1}$ is defined to be $\Sigma/\Sigma_0$.

Proof. By using lemma 2, we see that $\Theta(z)$ has the following jump discontinuities

$$(\Theta_{11}(z))_+ = (\Theta_{12}(z))_-, \quad z \in \Sigma_i, \quad i = 1, \ldots, n$$

$$(\Theta_{12}(z))_+ = (\Theta_{11}(z))_-, \quad z \in \Sigma_i, \quad i = 1, \ldots, n$$

$$(\Theta_{21}(z))_+ = (\Theta_{22}(z))_-, \quad z \in \Sigma_i, \quad i = 1, \ldots, n$$

$$(\Theta_{22}(z))_+ = (\Theta_{21}(z))_-, \quad z \in \Sigma_i, \quad i = 1, \ldots, n$$

$$(\Theta_{11}(z))_+ = \frac{\lambda - 1}{\lambda + 1} (\Theta_{12}(z))_-, \quad z \in \Sigma_i, \quad i = n + 1, \ldots, 2n$$

$$(\Theta_{12}(z))_+ = \frac{\lambda + 1}{\lambda - 1} (\Theta_{11}(z))_-, \quad z \in \Sigma_i, \quad i = n + 1, \ldots, 2n$$

$$(\Theta_{21}(z))_+ = \frac{\lambda - 1}{\lambda + 1} (\Theta_{22}(z))_-, \quad z \in \Sigma_i, \quad i = n + 1, \ldots, 2n$$

$$(\Theta_{22}(z))_+ = \frac{\lambda + 1}{\lambda - 1} (\Theta_{21}(z))_-, \quad z \in \Sigma_i, \quad i = n + 1, \ldots, 2n$$

This means that $\Theta(z)$ has the same jump discontinuities as in (6.4). To see that $\Theta(z)$ has the singularity structure given by (6.5), note that the function

$$\tilde{U}_+ = Q(z)\Theta^{-1}(z), \quad |z| < 1$$

$$\tilde{U}_- = \Theta(z)\Lambda Q^{-1}(z), \quad |z| > 1$$

has no jump discontinuities across the branch cuts $\Sigma_j$. It can at only have singularities of order less than or equal to $\frac{1}{2}$ at the points $z_j^{\pm1}$. This means that, if it was singular
Entanglement entropy in quantum spin chains

at \( z_{j}^{\pm 1} \), then it would have jump discontinuities across \( \Sigma_{j} \) due to the branch point type singularities. Therefore it is holomorphic at the points \( z_{j}^{\pm 1} \). Hence, the function \( \Theta(z) \) must have the singularity structure of the form (6.5).

To show that \( S(z) \) has the correct asymptotic behavior at \( z = \infty \), we only need to prove that \( \Theta(z) \) is invertible at \( z = \infty \). The asymptotic behavior of \( \Theta(z) \) is given by

\[
\begin{align*}
\Theta_{11}(\infty) &= \theta \left( \omega(\infty) - \kappa + \beta(\lambda) \overline{\omega} + \frac{\tau}{2} \right) e^{-\Delta_{0}} \\
\Theta_{22}(\infty) &= \theta \left( \omega(\infty) - \kappa - \beta(\lambda) \overline{\omega} + \frac{\tau}{2} \right) e^{-\Delta_{0}} \\
\Theta_{12}(\infty) &= \Theta_{21}(\infty) = 0
\end{align*}
\]

where \( \Delta_{0} = \lim_{z \to \infty} \Delta(z) - \frac{1}{2} \log(z - \lambda_{1}) \).

We will now show that \( \omega(\infty) = \kappa \). Let \( \eta \) be a third type differential with simple poles at the points \( x_{i} \in \mathcal{L} \) and \( \tilde{\eta} \) be a holomorphic differential. Let \( \Pi^{i} \) and \( \tilde{\Pi}^{i} \) be their periods

\[
\begin{align*}
\int_{a_{i}} \eta &= \Pi^{i}, & \int_{b_{i}} \eta &= \Pi^{i+g} \\
\int_{a_{i}} \tilde{\eta} &= \tilde{\Pi}^{i}, & \int_{b_{i}} \tilde{\eta} &= \tilde{\Pi}^{i+g}
\end{align*}
\]

Now, by the Riemann bilinear relation [16] we have

\[
\sum_{i=1}^{g} \Pi^{i}\tilde{\Pi}^{i+g} - \Pi^{g+1}\tilde{\Pi}^{1} = 2\pi i \sum_{x_{i}} \text{Res}_{x_{i}}(\eta) \int_{p_{0}}^{x_{i}} \tilde{\eta},
\]

where \( p_{0} \) is an arbitrary point on \( \mathcal{L} \). By substituting \( \eta = d\Delta \) and \( \tilde{\eta} = d\omega_{j} \) for \( j = 1, \ldots, g \), we see that

\[
\kappa_{j} = \frac{1}{2} \left( \omega_{j}(\infty^{+}) - \omega_{j}(\infty^{-}) \right) = \omega_{j}(\infty),
\]

where the last equality follows from (6.10). Therefore, we obtain

\[
\begin{align*}
\Theta_{11}(\infty) &= \theta \left( \beta(\lambda) \overline{\omega} + \frac{\tau}{2} \right) e^{-\Delta_{0}} \\
\Theta_{22}(\infty) &= \theta \left( -\beta(\lambda) \overline{\omega} + \frac{\tau}{2} \right) e^{-\Delta_{0}} \\
\Theta_{12}(\infty) &= \Theta_{21}(\infty) = 0
\end{align*}
\]

Therefore \( \Theta(z) \) is invertible at \( \infty \) as long as

\[
\theta \left( \beta(\lambda) \overline{\omega} + \frac{\tau}{2} \right) \theta \left( -\beta(\lambda) \overline{\omega} + \frac{\tau}{2} \right) \neq 0.
\]

Thus, \( S(z) \) is the unique solution of the Riemann-Hilbert problem (6.14).
Remark 4. In appendix F, we will show that the Wiener-Hopf factorization is solvable for \( \beta(\lambda) \in i\mathbb{R} \), i.e. the Riemann-Hilbert problem (6.4) is solvable for these \( \beta(\lambda) \). This in turn implies that (6.15) is true for all \( \beta(\lambda) \in i\mathbb{R} \). Define (cf. (5.6))
\[
\Omega_\epsilon = \{ \lambda \in \mathbb{R} : |\lambda| \geq 1 + \epsilon \}.
\]
The function \( \lambda \rightarrow \beta(\lambda) \) maps \( \Omega_\epsilon \) onto the bounded subset \( \mathcal{N} \equiv \{ \alpha \in i\mathbb{R} : 0 < |\alpha| \leq \frac{1}{2\pi} \log(2\epsilon^{-1} + 1) \} \). By continuity, the inequality (6.15) is valid for all \( \alpha \) from the closure of \( \mathcal{N} \). This fact, together with the explicit formulae (6.12), (6.3) and (6.1) implies the uniform estimates which have been stated in (5.7) and used in the proof of theorem 5.

7 The asymptotics of \( d \log D_L(\lambda)/d\lambda \) and \( D_L(\lambda) \)

We are now ready to compute the derivative of the determinant \( D_L(\lambda) \). First we notice that in virtue of (6.1), equation (5.8) can be re-written as
\[
\frac{d}{d\lambda} \log D_L(\lambda) = -\frac{2\lambda}{1 - \lambda^2} L + \frac{1}{2\pi} \int_{|z|=1} \text{tr} \left[ U_+^\prime(z) U_+^{-1}(z) \left( \Phi^{-1}(z) - \sigma_3 \Phi^{-1}(z) \sigma_3 \right) \right] dz + r_L(\lambda). \tag{7.1}
\]
Define
\[
\Psi(z) := \Phi^{-1}(z) - \sigma_3 \Phi^{-1}(z) \sigma_3 = \frac{2}{1 - \lambda^2} \begin{pmatrix} 0 & -g(z) \\ g^{-1}(z) & 0 \end{pmatrix}.
\]
From equations (6.3) and (6.12) we have
\[
U_+^{-1}(z) = A \Theta(z) Q^{-1}(z), \quad U_+^\prime(z) = Q'(z) \Theta^{-1}(z) A^{-1} + Q(z) (\Theta^{-1})'(z) A^{-1},
\]
where we denote \( A = Q(\infty) \Lambda^{-1} \Theta^{-1}(\infty) \). Furthermore, from equation (6.2) we obtain
\[
Q^{-1}(z) = \frac{1}{2} \begin{pmatrix} g^{-1}(z) & -i \\ -g^{-1}(z) & -i \end{pmatrix}.
\]
Therefore, formula (7.1) transforms into the relation
\[
\frac{d}{d\lambda} \log D_L(\lambda) = -\frac{2\lambda}{1 - \lambda^2} L + \frac{i}{\pi(1 - \lambda^2)} \int_z \text{tr} \left[ \Theta^{-1} \frac{d}{dz} \Theta(z) \sigma_3 \right] dz + r_L(\lambda). \tag{7.2}
\]
We will now prove the following:
Entanglement entropy in quantum spin chains

Theorem 6. Let \( s(\lambda) \) be given by

\[
s(\lambda) = \frac{i}{\pi(1 - \lambda^2)} \int_{|z|=1} \alpha(z)dz,
\]

where the entries of the \( 2 \times 2 \) matrix \( \Theta(z) \) are given by \((6.13)\).

Then \( s(\lambda) \) can be written as

\[
s(\lambda) = -\frac{i}{\pi(1 - \lambda^2)} \frac{d}{d\lambda} \log \left( \theta \left( \beta(\lambda) \overline{\epsilon} + \frac{\tau}{2} \right) \theta \left( \beta(\lambda) \overline{\epsilon} - \frac{\tau}{2} \right) \right)
\]

Proof. To begin with, we would like to treat \( \alpha(z)dz \) as a 1-form on the hyperelliptic curve \( L \). We will show that it is, in fact, the holomorphic 1-form

\[
\alpha(z)dz = \sum_{i=1}^{2n-1} \partial_i \log \left( \theta \left( \beta(\lambda) \overline{\epsilon} + \frac{\tau}{2} \right) \theta \left( \beta(\lambda) \overline{\epsilon} - \frac{\tau}{2} \right) \right) d\omega_i
\]

where \( d\omega_i \) are the normalized holomorphic differentials on \( L \) and \( \partial_i \) is the partial derivative with respect to the \( i \)th argument.

Suppose this is true, then by deforming the contour of the integral \((7.3)\), we see that it can be written as

\[
s(\lambda) = -\frac{2n-1}{i} \int_{a_k} \alpha(z)dz
\]

To see that \( \alpha(z)dz \) is given by the corresponding 1-form, let us first compute \( \alpha(z)dz \). We have

\[
\alpha(z)dz = (\det \Theta(z))^{-1} \left( \Theta_{22}(z)\Theta'_{11}(z) - \Theta_{11}(z)\Theta'_{22}(z) \right.
\]

\[
-\Theta_{12}(z)\Theta'_{21}(z) + \Theta_{21}(z)\Theta'_{12}(z) \right) dz,
\]

where the prime denotes the derivative with respect to \( z \).
We can simplify equation (7.4) by observing that

\[
\begin{align*}
\Theta_{11}(z) &= h_1(z) \theta \left( \omega(z) + \beta(\lambda) e^r - \kappa + \frac{\tau}{2} \right) \\
\Theta_{22}(z) &= h_1(z) \theta \left( \omega(z) - \beta(\lambda) e^r - \kappa + \frac{\tau}{2} \right) \\
\Theta_{12}(z) &= h_2(z) \theta \left( \omega(z) - \beta(\lambda) e^r + \kappa - \frac{\tau}{2} \right) \\
\Theta_{21}(z) &= h_2(z) \theta \left( \omega(z) + \beta(\lambda) e^r + \kappa - \frac{\tau}{2} \right) \\
h_1(z) &= \sqrt{z - \lambda_1} \frac{e^{-\Delta(z)}}{\theta (\omega(z) + \frac{\tau}{2})} \\
h_2(z) &= -\sqrt{z - \lambda_1} \frac{e^{\Delta(z)}}{\theta (\omega(z) - \frac{\tau}{2})}.
\end{align*}
\]

Therefore, we have

\[
\begin{align*}
\Theta_{22}(z) \frac{\partial}{\partial z} (\Theta_{11}(z) - \Theta_{11}(z) \Theta_{22}(z)) &= (h_1(z))^2 (\theta_2 \theta_1' - \theta_1 \theta_2') \\
\Theta_{12}(z) \frac{\partial}{\partial z} (\Theta_{21}(z) - \Theta_{21}(z) \Theta_{12}(z)) &= (h_2(z))^2 (\theta_3 \theta_4' - \theta_4 \theta_3') ,
\end{align*}
\]

where the \( \theta_i \)'s are given by

\[
\begin{align*}
\theta_1 &= \theta \left( \omega(z) + \beta(\lambda) e^r - \kappa + \frac{\tau}{2} \right) \\
\theta_2 &= \theta \left( \omega(z) - \beta(\lambda) e^r - \kappa + \frac{\tau}{2} \right) \\
\theta_3 &= \theta \left( \omega(z) - \beta(\lambda) e^r + \kappa - \frac{\tau}{2} \right) \\
\theta_4 &= \theta \left( \omega(z) + \beta(\lambda) e^r + \kappa - \frac{\tau}{2} \right).
\end{align*}
\]

Now, the \( \theta_i' \)'s are just

\[
\theta_i' dz = \sum_{k=1}^{2n-1} \left( \partial_k \theta_i \right) d\omega_k.
\]

By substituting the right hand side of this equation into (7.4) we obtain

\[
\begin{align*}
\alpha(z) dz &= \det \Theta(z)^{-1} \sum_{k=1}^{2n-1} d\omega_k ( (h_1(z))^2 G^1_k(z) - (h_2(z))^2 G^2_k(z) ) \\
G^1_k(z) &= \theta_2 \partial_k \theta_1 - \theta_1 \partial_k \theta_2 \\
G^2_k(z) &= \theta_3 \partial_k \theta_4 - \theta_4 \partial_k \theta_3.
\end{align*}
\]

We would like to show that the expression

\[
\det \Theta(z)^{-1} ( (h_1(z))^2 G^1_k(z) - (h_2(z))^2 G^2_k(z) )
\]
is a constant. First note that, by considering the jump and singularity structure of $\det \Theta(z)$, we have

$$\det \Theta(z) = g(z) \det \Theta(\infty)g(\infty)^{-1},$$

where $g(z)$ is given by (2.3).

Since the $\Theta_{ij}(z)$’s have square root singularities at the $n$ points $z = z_j^{-1}$, the functions

$$(h_1(z))^2 G_k^1(z) - (h_2(z))^2 G_k^2(z)$$

can have at most simple poles at the points $(z_j)^{\pm 1}$, $j = 1, \ldots, 2n$. Near each of these points, they behave like

$$(h_1(z))^2 G_k^1(z) - (h_2(z))^2 G_k^2(z) = A_0^j + A_1^j(z - z_j)^{1/2} + O(z - z_j), \quad z \to z_j$$

$$(h_1(z))^2 G_k^1(z) - (h_2(z))^2 G_k^2(z) = B_0^j(z - z_j^{-1})^{-1} + B_1^j(z - z_j^{-1})^{-3/2} + O(1), \quad z \to z_j^{-1}$$

Since $\rho(\Delta)(z) = -\Delta(z)$, $\rho(\omega)(z) = -\omega(z)$ and $\rho(z - \lambda_1) = z - \lambda_1$, we have

$$\rho(h_1^2)(z) = h_2^2(z), \quad \rho(\theta_1)(z) = \theta_3(z), \quad \rho(\theta_2)(z) = \theta_4(z)$$

and

$$(h_1(z))^2 G_k^1(z) - (h_2(z))^2 G_k^2(z) = (h_1(z))^2 G_k^1(z) - \rho((h_1)^2 G_k^1)(z). \quad (7.5)$$

Since the action of $\rho$ on a Laurent series near a branch point $\lambda_j$ is given by

$$\rho \left( \sum_{k=-\infty}^{\infty} X_k(z - \lambda_j)^{\frac{k}{2}} \right) = \sum_{k=-\infty}^{\infty} X_k(-(z - \lambda_j))^{\frac{k}{2}},$$

by (7.5) we obtain $A_0^j = B_0^j = 0$ for all $j$. Hence, the function

$$\det \Theta(z)^{-1} \left( (h_1(z))^2 G_k^1(z) - (h_2(z))^2 G_k^2(z) \right)$$

does not have any pole on $\mathcal{L}$. To see that it does not have jumps too, let us consider

$$(h_1(z))^2 G_k^1(z) = (h_1(z))^2 (\theta_2 \partial_k \theta_1 - \theta_1 \partial_k \theta_2).$$

The periodicity of the term inside the brackets is given by proposition 2

$$\theta_1 \partial_k \theta_2(z + a_j) = \theta_1 \partial_k \theta_2$$
$$\theta_1 \partial_k \theta_2(z + b_j) = \theta_1 \partial_k \theta_2 e^{-2\pi i(2\omega_j(z) - 2\kappa_j + \tau_j + \Pi_j)}$$
$$\theta_2 \partial_k \theta_1(z + a_j) = \theta_2 \partial_k \theta_1$$
$$\theta_2 \partial_k \theta_1(z + b_j) = \theta_2 \partial_k \theta_1 e^{-2\pi i(2\omega_j(z) - 2\kappa_j + \tau_j + \Pi_j)}.$$
where \( \omega_j(z) = \int_{\lambda_j}^z d\omega_j \) is the \( j \)th component of the vector \( \omega(z) \). Hence the multiplicative factor picked up by \( G_k^1(z) \) after going around a \( b \)-cycle cancels exactly with the factor picked up by \( (h_1(z))^2 \). It follows that the function (7.6) does not have jumps on \( L \) too. Hence, they are holomorphic functions on \( L \) without any pole and must be constants. These constants can be computed by taking \( z = \infty \). In other words, they are given by (6.14). We therefore have

\[
\det \Theta(z)^{-1} \left( (h_1(z))^2 G_k^1(z) - (h_2(z))^2 G_k^2(z) \right) = \partial_k \log \left( \theta \left( \beta(\lambda) \overline{\epsilon} + \frac{\tau}{2} \right) \theta \left( \beta(\lambda) \overline{\epsilon} - \frac{\tau}{2} \right) \right)
\]

This proves the theorem. \( \Box \)

Theorem 6, in its turn, yields our main asymptotic result.

**Theorem 7.** Let \( \Omega_\epsilon \) be the domain of solvability (5.6). Then the logarithmic derivative of Toeplitz determinant \( D_L(\lambda) \) admits the following asymptotic representation, which is uniform in \( \lambda \in \Omega_\epsilon \).

\[
\frac{d}{d\lambda} \log D_L(\lambda) = -\frac{2\lambda}{1 - \lambda^2} L + \frac{d}{d\lambda} \log \left[ \theta \left( \beta(\lambda) \overline{\epsilon} + \frac{\tau}{2} \right) \theta \left( \beta(\lambda) \overline{\epsilon} - \frac{\tau}{2} \right) \right] + O \left( \frac{\rho - L}{\lambda^2} \right), \quad L \to \infty.
\]

(7.7)

Here \( \rho \) is any real number satisfying the inequality

\[
1 < \rho < \min\{|\lambda_j| : |\lambda_j| > 1\}.
\]

The uniformity of the estimate (7.7) with respect to \( \lambda \in \Omega_\epsilon \) allows its integration over \( \Omega_\epsilon \), which yields the equation

\[
\log \left( D_L(\lambda)(1 - \lambda^2)^{-L} \right) - \lim_{s \to \infty} \log \left( D_L(s)(1 - s^2)^{-L} \right) = \log \frac{\theta \left( \beta(\lambda) \overline{\epsilon} + \frac{\tau}{2} \right) \theta \left( \beta(\lambda) \overline{\epsilon} - \frac{\tau}{2} \right)}{\theta^2 \left( \frac{\tau}{2} \right)} + r(L),
\]

where \( r(L) = O \left( \rho - L \right) \) as \( L \to \infty \). Taking into account (4.11), the second term in the left hand side is zero. This proves Proposition 1.

8 The limiting entropy

Observe that equation (4.10) can also be rewritten as

\[
S_L(\rho_A) = \lim_{\epsilon \to 0^+} \frac{1}{4\pi i} \int_{\Gamma(\epsilon)} e(1 + \epsilon, \lambda) \frac{d}{d\lambda} \log \left( D_L(\lambda)(\lambda^2 - 1)^{-L} \right) d\lambda.
\]

(8.1)
The right hand side of this equation follows from
\[
\lim_{\epsilon \to 0^+} \oint_{\Gamma(\epsilon)} e(1 + \epsilon, \lambda) \frac{d}{d\lambda} \log(\lambda^2 - 1)^{-L} d\lambda = L \lim_{\epsilon \to 0^+} \oint_{\Gamma(\epsilon)} e(1 + \epsilon, \lambda) \frac{2\lambda}{1 - \lambda^2} d\lambda
\]
\[
= 2\pi i L \lim_{\epsilon \to 0^+} \left[ \text{res}_{\lambda=1} \left( e(1 + \epsilon, \lambda) \frac{2\lambda}{1 - \lambda^2} \right) + \text{res}_{\lambda=-1} \left( e(1 + \epsilon, \lambda) \frac{2\lambda}{1 - \lambda^2} \right) \right]
\]
\[
= 2\pi i L \lim_{\epsilon \to 0^+} \left( (2 + \epsilon) \log \frac{2 + \epsilon}{2} + \epsilon \log \frac{\epsilon}{2} \right) = 0.
\]
We identify the limiting entropy \( S(\rho_A) \) as the following double limit (cf. [13]),
\[
S(\rho_A) = \lim_{\epsilon \to 0^+} \left[ \lim_{L \to \infty} \frac{1}{4\pi i} \oint_{\Gamma(\epsilon)} e(1 + \epsilon, \lambda) \frac{d}{d\lambda} \log \left( D_L(\lambda)(\lambda^2 - 1)^{-L} \right) \right] d\lambda. 
\tag{8.2}
\]
We now want to apply theorem 7 and evaluate the large \( L \) limit in the right hand side of this equation. To this end we need first to replace the integration along the contour \( \Gamma(\epsilon) \) by the integration along a subset of the set \( \Omega \), where we can use the uniform asymptotic formula (7.7).

Let us define
\[
\delta(\lambda) := \frac{d}{d\lambda} \log \left( D_L(\lambda)(\lambda^2 - 1)^{-L} \right).
\]
The function \( \delta(\lambda) \) satisfies the following properties.

1. \( \delta(\lambda) \) is analytic outside of the interval \([-1, 1]\).
2. \( \delta(-\lambda) = -\delta(\lambda) \).
3. \( \delta(\lambda) = O(\lambda^{-3}), \quad \lambda \to \infty \).
4. \( \delta(\lambda) = O(\log |1 - \lambda^2|), \quad \lambda \to \pm 1 \).

Consider the identity
\[
\oint_{\Gamma(\epsilon)} e(1 + \epsilon, \lambda) \frac{d}{d\lambda} \log \left( D_L(\lambda)(\lambda^2 - 1)^{-L} \right) d\lambda \equiv \oint_{\Gamma(\epsilon)} e(1 + \epsilon, \lambda) \delta(\lambda) d\lambda.
\]
Property 1 allows us to replace the contour of integration \( \Gamma(\epsilon) \) by the large contour \( \Gamma' \) as depicted in figure 1, so that
\[
\oint_{\Gamma(\epsilon)} e(1 + \epsilon, \lambda) \delta(\lambda) d\lambda = \oint_{\Gamma'} e(1 + \epsilon, \lambda) \delta(\lambda) d\lambda.
\]
Simultaneously, property 3 allows to push \( R \to \infty \) in the right hand side of the last formula and hence re-write it as the relation,
\[
\oint_{\Gamma(\epsilon)} e(1 + \epsilon, \lambda) \delta(\lambda) d\lambda = \int_{-\infty}^{1-\epsilon} \delta(\lambda) \left[ -\frac{1 + \epsilon + \lambda}{2} \left( \log_+ \left( \frac{1 + \epsilon + \lambda}{2} \right) - \log_- \left( \frac{1 + \epsilon + \lambda}{2} \right) \right) \right] d\lambda
\]
$$\int_{1+\varepsilon}^{\infty} \delta(\lambda) \left[ -\frac{1+\varepsilon - \lambda}{2} \left( \log_+ \left( \frac{1+\varepsilon - \lambda}{2} \right) - \log_- \left( \frac{1+\varepsilon - \lambda}{2} \right) \right) \right] d\lambda. \quad (8.3)$$

Here \( \log_+ \left( \frac{1+\varepsilon + \lambda}{2} \right) \) and \( \log_- \left( \frac{1+\varepsilon + \lambda}{2} \right) \) denote, respectively, the upper and lower boundary values of the functions \( \log \left( \frac{1+\varepsilon + \lambda}{2} \right) \) on the real axis. We note that

$$\log_+ \left( \frac{1+\varepsilon + \lambda}{2} \right) - \log_- \left( \frac{1+\varepsilon + \lambda}{2} \right) = 2\pi i,$$

for all \( \lambda < -1 - \varepsilon \), and

$$\log_+ \left( \frac{1+\varepsilon - \lambda}{2} \right) - \log_- \left( \frac{1+\varepsilon - \lambda}{2} \right) = -2\pi i,$$

for all \( \lambda > 1 + \varepsilon \).

Therefore, equation (8.3) becomes

$$\oint_{\Gamma(\varepsilon)} e(1+\varepsilon, \lambda) \delta(\lambda) d\lambda = -\pi i \int_{-\infty}^{1-\varepsilon} (1+\varepsilon + \lambda) \delta(\lambda) d\lambda + \pi i \int_{1+\varepsilon}^{\infty} (1+\varepsilon - \lambda) \delta(\lambda) d\lambda = 2\pi i \int_{1+\varepsilon}^{\infty} (1+\varepsilon - \lambda) \delta(\lambda) d\lambda,$$  

(8.4)

where we have also taken into account the oddness of the function \( \delta(\lambda) \), i.e. property 2.

Recalling the definition of the function \( \delta(\lambda) \), we arrive at

$$\int_{\Gamma(\varepsilon)} e(1+\varepsilon, \lambda) \frac{d}{d\lambda} \log \left( D_L(\lambda)(\lambda^2 - 1)^{-L} \right) d\lambda = 2\pi i \int_{1+\varepsilon}^{\infty} (1+\varepsilon - \lambda) \frac{d}{d\lambda} \log \left( D_L(\lambda)(\lambda^2 - 1)^{-L} \right) d\lambda.$$  

(8.5)

The estimate (7.7) can be used in the right hand side of formula (8.5). This enables us to perform an explicit evaluation of the large \( L \) limit in (8.2) so that the formula for the entropy \( S(\rho_A) \) becomes

$$S(\rho_A) = \frac{1}{2} \lim_{\varepsilon \to 0^+} \left[ \int_{1+\varepsilon}^{\infty} (1+\varepsilon - \lambda) \frac{d}{d\lambda} \log \left( \theta \left( \beta(\lambda) \bar{\tau}^+ + \frac{\tau}{2} \right) \theta \left( \beta(\lambda) \bar{\tau}^- - \frac{\tau}{2} \right) \right) d\lambda \right]$$

$$= \frac{1}{2} \lim_{\varepsilon \to 0^+} \left[ \int_{1+\varepsilon}^{\infty} \log \frac{\theta \left( \beta(\lambda) \bar{\tau}^+ + \frac{\tau}{2} \right) \theta \left( \beta(\lambda) \bar{\tau}^- - \frac{\tau}{2} \right)}{\theta^2 \left( \frac{\tau}{2} \right)} d\lambda. \right] \quad (8.6)$$

To complete the evaluation of the entropy, we need to prove the existence of this limit.

### 9 Integrability at \( \pm 1 \). The final formula for the entropy

We will now proof the integrability of the function

$$\log \frac{\theta \left( \beta(\lambda) \bar{\tau}^+ + \frac{\tau}{2} \right) \theta \left( \beta(\lambda) \bar{\tau}^- - \frac{\tau}{2} \right)}{\theta^2 \left( \frac{\tau}{2} \right)}$$
at ±1.

First let us denote the real and imaginary parts of the period matrix \( \Pi \) by Re \( \Pi \) and Im \( \Pi \). Since the Im \( \Pi \) is non-singular, there exist a real vector \( \overrightarrow{v} \) such that

\[
\overrightarrow{e} = \text{Im} \overrightarrow{\Pi} \overrightarrow{v}.
\]

We now can write

\[
i \overrightarrow{e} = (\Pi - \text{Re} \Pi) \overrightarrow{v}.
\]

Let \( Q \) be a large real number, and let \( \overrightarrow{m} \) be an integer vector such that

\[
Q \overrightarrow{v} = \overrightarrow{m} + \overrightarrow{q},
\]

where the entries of \( \overrightarrow{q} \) are between 0 and 1.

In particular, we have

\[
\overrightarrow{m} = Q (\text{Im} \overrightarrow{\Pi})^{-1} \overrightarrow{e} - \overrightarrow{q}.
\]

Then, from the periodicity of the theta function (6.6), we see that

\[
\theta (iQ \overrightarrow{e} + \overrightarrow{c}_0) = \theta ((\overrightarrow{m} + \overrightarrow{q})^T (\Pi - \text{Re} \Pi) + \overrightarrow{c}_0)
\]

\[
= \exp \left( Q^2 \pi \left[ i \langle \overrightarrow{e}, (\text{Im} \overrightarrow{\Pi})^{-1} \text{Re} \Pi (\text{Im} \overrightarrow{\Pi})^{-1} \overrightarrow{e} \rangle 
\right.ight.
\]

\[
+ \langle \overrightarrow{e}, (\text{Im} \overrightarrow{\Pi})^{-1} \overrightarrow{e} \rangle 
\]

\[
- 2i\pi Q \left[ \langle \overrightarrow{e}, (\text{Im} \overrightarrow{\Pi})^{-1} \text{Re} \Pi \overrightarrow{q} \rangle + \langle \overrightarrow{e}, (\text{Im} \overrightarrow{\Pi})^{-1} \overrightarrow{c}_0 \rangle + 2i \langle \overrightarrow{e}, \overrightarrow{q} \rangle 
\]

\[
+ i\pi \left[ \langle \overrightarrow{q}, (\text{Im} \Pi) \overrightarrow{q} \rangle + 2 \langle \overrightarrow{q}, \overrightarrow{c}_0 \rangle 
\right)
\]

\[
\times \theta (-(\overrightarrow{m} + \overrightarrow{q})^T \text{Re} \Pi + \overrightarrow{c}_0 + \overrightarrow{q}^T \Pi)
\]

for some bounded constant \( \overrightarrow{c}_0 \).

Note that there exists an integer vector \( \overrightarrow{l} \) and real vector \( \overrightarrow{r} \) with entries between 0 and 1 such that

\[
(\overrightarrow{m} + \overrightarrow{q})^T \text{Re} \Pi = \overrightarrow{l} + \overrightarrow{r}.
\]

Therefore, we have

\[
\theta (-(\overrightarrow{m} + \overrightarrow{q})^T \text{Re} \Pi + \overrightarrow{c}_0 + \overrightarrow{q}^T \Pi) = \theta (-\overrightarrow{r} + \overrightarrow{c}_0 + \overrightarrow{q}^T \Pi).
\]

If log \( \theta (iQ \overrightarrow{e} + \overrightarrow{c}_0) \) is non-zero for all \( Q \), then from (9.2) we see that

\[
\log \theta (iQ \overrightarrow{e} + \overrightarrow{c}_0) = Q^2 \pi \left[ i \langle \overrightarrow{e}, (\text{Im} \overrightarrow{\Pi})^{-1} \text{Re} \Pi (\text{Im} \overrightarrow{\Pi})^{-1} \overrightarrow{e} \rangle 
\right.
\]

\[
+ \langle \overrightarrow{e}, (\text{Im} \overrightarrow{\Pi})^{-1} \overrightarrow{e} \rangle + 2i \frac{N(Q, c_0)}{Q^2} + O(Q^{-1}) \right], \quad Q \to \infty,
\]
where $N(Q, c_0)$ is an integer that depends on the branch of the logarithm. It may depend on $Q$ and $\tilde{c}_0$. This term arises because in the integral expression of the entropy,

$$\frac{1}{2} \int_{1+\epsilon}^{\infty} \log \frac{\theta (\beta(\lambda) \tilde{c} + \frac{z}{2}) \theta (\beta(\lambda) \tilde{c} - \frac{z}{2})}{\theta^2 \left( \frac{\tilde{c}}{2} \right)} d\lambda,$$

the branch of the logarithm must be chosen so that the integrand is continuous in $\lambda$. We shall determine the asymptotic behavior of $N(Q, c_0)$ as $Q \to \infty$.

Due to theorem 9, the inequality (6.15) is true when $\beta(\lambda) \in i\mathbb{R}$. Therefore, we can apply the above result to compute the asymptotic behavior of the integrand in (9.3):

$$\log \frac{\theta (\beta(\lambda) \tilde{c} + \frac{z}{2}) \theta (\beta(\lambda) \tilde{c} - \frac{z}{2})}{\theta^2 \left( \frac{\tilde{c}}{2} \right)} = -2\beta(\lambda)^2 \pi \left[ \langle \tilde{c}, (\text{Im } \Pi)^{-1} \tilde{c} \rangle + O(\beta(\lambda)^{-1}) \right].$$

(9.4)

Since $D_L(\lambda)$ in (4.11) is real and positive for $\lambda \in (1, \infty)$, and that $\log D_L(\lambda)(\lambda^2 - 1)^{-L}$ has to be zero at $\lambda = \infty$ (which is needed to deform the contour to obtain (8.6)), we see that $\log D_L(\lambda)$ has to be real for $\lambda \in (1, \infty)$. Therefore, the imaginary part of the leading order term in (9.4) must be zero. In particular, this means that

$$\langle \tilde{c}, (\text{Im } \Pi)^{-1} \text{Re } \Pi (\text{Im } \Pi)^{-1} \tilde{c} \rangle - 2 \frac{N(\beta(\lambda), \frac{z}{2}) + N(\beta(\lambda), -\frac{z}{2})}{\beta(\lambda)^2} = O(\beta(\lambda)^{-1}).$$

Thus, the asymptotic behavior of the integrand in (9.3) is

$$\log \frac{\theta (\beta(\lambda) \tilde{c} + \frac{z}{2}) \theta (\beta(\lambda) \tilde{c} - \frac{z}{2})}{\theta^2 \left( \frac{\tilde{c}}{2} \right)} = -2\pi \beta(\lambda)^2 \left( \langle \tilde{c}, (\text{Im } \Pi)^{-1} \tilde{c} \rangle + O(\beta(\lambda)^{-1}) \right), \quad \lambda \to 1^+.$$  

(9.5)

The left hand side of this equation is therefore integrable at $\lambda = 1^+$ and we can take the limit $\epsilon \to 0$ in (8.6) to obtain our final result for the entropy:

$$S(\rho_A) = \frac{1}{2} \int_{1}^{\infty} \log \frac{\theta (\beta(\lambda) \tilde{c} + \frac{z}{2}) \theta (\beta(\lambda) \tilde{c} - \frac{z}{2})}{\theta^2 \left( \frac{\tilde{c}}{2} \right)} d\lambda.$$ 

(9.6)

10 Critical behavior as roots of $g(z)$ approaches the unit circle

The purpose of this section is to prove theorem 2. We shall study the critical behavior of the entropy of entanglement as some pairs of the roots (2.4) approach the unit circle. As
we discussed in section 2, in each pair one root lies inside the unit circle, while the other outside. In this limit the entropy becomes singular. We shall study all the possible cases of such degeneracy, namely the following three:

1. the limit of two real roots approaching 1;
2. the limit of $2r$ pairs of complex roots approaching the unit circle;
3. the limit of $2r$ pairs of complex roots approaching the unit circle together with one pair of real roots approaching 1.

When pairs of roots in (2.4) approach the unit circle, the period matrix $\Pi$ in the definition of the theta function (2.10) becomes degenerate and some of its entries tend to zero. This will lead to a divergence in the sum (2.10) and hence a divergence in the entropy. It is very difficult to study such divergence directly from the sum (2.10). In order to compute such limits, we need to perform modular transformations to the theta functions. In particular, the following theorem from [7] will be used throughout the whole section.

**Theorem 8.** If the canonical bases of cycles $(\tilde{A} \; \tilde{B})$ and $(A \; B)$ are related by

$$
\begin{pmatrix}
\tilde{A} \\
\tilde{B}
\end{pmatrix} = Z
\begin{pmatrix}
A \\
B
\end{pmatrix} = \begin{pmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{pmatrix}
\begin{pmatrix}
A \\
B
\end{pmatrix},
$$

where the matrix $Z$ is symplectic i.e.

$$
Z^T \begin{pmatrix}
0 & -I_{2n-1} \\
I_{2n-1} & 0
\end{pmatrix} Z = \begin{pmatrix}
0 & -I_{2n-1} \\
I_{2n-1} & 0
\end{pmatrix},
$$

$$
Z^{-1} = \begin{pmatrix}
Z_{22}^T & -Z_{12}^T \\
-Z_{21}^T & Z_{11}^T
\end{pmatrix},
$$

then we have the following relations between the theta functions with different period matrices:

$$
\theta \left[ \frac{\varepsilon}{\varepsilon'} \right] (\xi, \Pi) = \varsigma \exp \left[ -\pi i \tilde{\xi}^T (-Z_{12}^T \tilde{\Pi} + Z_{22}^T)^{-1} Z_{12}^T \tilde{\xi} \right] \theta \left[ \frac{\tilde{\varepsilon}}{\tilde{\varepsilon}'} \right] (\tilde{\xi}, \tilde{\Pi}), \quad (10.1)
$$

where

$$
\tilde{\xi} = \left( (-Z_{12}^T \tilde{\Pi} + Z_{22}^T)^T \right) \xi \quad (10.2)
$$

and $\varsigma$ is a constant. The characteristics of the theta functions are related by

$$
\varepsilon = Z_{22}^T \tilde{\varepsilon} + Z_{12}^T \tilde{\varepsilon}' - \text{diag}(Z_{12}^T Z_{22})
$$

$$
\varepsilon' = Z_{21}^T \tilde{\varepsilon} + Z_{11}^T \tilde{\varepsilon}' - \text{diag}(Z_{11}^T Z_{21}),
$$

$$
\varsigma = Z_{11}^T \varsigma + Z_{12}^T \varsigma',
$$

$$
\varsigma' = Z_{22}^T \varsigma + Z_{21}^T \varsigma'.
$$
where \( \text{diag}(CD^T) \) is a column vector whose entries are the diagonal elements of \( CD^T \). The new period matrix is given by

\[
\tilde{\Pi} = (Z_{22}\Pi + Z_{21})(Z_{12}\Pi + Z_{11})^{-1} \tag{10.3}
\]

and the normalized one forms are related by

\[
d\tilde{\Omega} = \left((-Z_{12}^T\tilde{\Pi} + Z_{22}^T)^T\right) d\Omega \tag{10.4}
\]

\[
d\tilde{\Omega}^T = (d\bar{\omega}_1, \ldots, d\bar{\omega}_{2n-1})^T, \quad d\Omega^T = (d\omega_1, \ldots, d\omega_{2n-1})^T,
\]

which is the same transformation as in \((10.2)\).

Our aim is to find a good choice of basis \((\tilde{A} \quad \tilde{B})\) such that \(\theta(\tilde{\xi}, \tilde{\Pi})\) remains finite while some entries of \(\tilde{\Pi}\) tend to infinity as certain pairs of roots \(\lambda_j\) approach the unit circle. This would confine the divergence of the entropy within the exponential factor in \((10.1)\), which can be computed.

### 10.1 The limit of two real roots approaching 1

In this section the choice of the basis \((\tilde{A} \quad \tilde{B})\) described in theorem \(8\) is the one shown in figure 5. In the notation of theorem \(8\) the new basis \((\tilde{A} \quad \tilde{B})\) and the old one \((A \quad B)\) are related by

\[
\begin{pmatrix}
\tilde{A} \\
\tilde{B}
\end{pmatrix} = Z \begin{pmatrix} A \\ B \end{pmatrix}
\]

\[
Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} = \begin{pmatrix} 0 & -C_2 \\ C_1 & 0 \end{pmatrix}
\]

\[
\tilde{A}^T = (\tilde{a}_1, \ldots, \tilde{a}_{2n-1})^T, \quad \tilde{B}^T = (\tilde{b}_1, \ldots, \tilde{b}_{2n-1})^T
\]

\[
A^T = (a_1, \ldots, a_{2n-1})^T, \quad B^T = (b_1, \ldots, b_{2n-1})^T
\]

\[
(C_1)_{ij} = 1, \quad j \geq i, \quad (C_1)_{ij} = 0, \quad j < i
\]

\[
(C_2)_{ii} = 1, \quad (C_2)_{i,i-1} = -1, \quad (C_2)_{ij} = 0, \quad j \neq i, i-1
\]

\[
C_1 = (C_2^{-1})^T.
\]

Figure 5: The choice of cycles on the hyperelliptic curve \(\mathcal{L}\). The arrows denote the orientations of the cycles and branch cuts.
The relation between the two period matrices can be found using \( (10.3) \)

\[
\tilde{\Pi} = -C_1 \Pi^{-1} C_2^{-1}. \tag{10.6}
\]

To study the behavior of the entropy as the real roots \( \lambda_{2n} \to \lambda_{2n}^{-1} \), we need to know the behavior of the period matrix \( \tilde{\Pi} \) in this limit. Now, we have

\[
w_0 = \lim_{\lambda_{2n} \to \lambda_{2n}^{-1}} \left[ \frac{4n}{\prod_{i=1}^{4n} (z - \lambda_i)} = (z - 1) \prod_{i \neq 2n, 2n+1} (z - \lambda_i) \right]. \tag{10.7}
\]

Furthermore, as \( \lambda_{2n} \to \lambda_{2n}^{-1} \) the integration around \( \tilde{a}_n \) tends the residue at \( z = 1 \); the hyperelliptic curve \( \mathcal{L} \) becomes a singular hyperelliptic curve \( \mathcal{L}_0 \) of genus \( 2n - 2 \); the tilded basis of canonical cycles on this curve reduces to

\[
\tilde{A}_0^T = (\tilde{a}_1, \ldots, \tilde{a}_{n-1}, \tilde{a}_{n+1}, \ldots, \tilde{a}_{2n-1})^T, \\
\tilde{B}_0^T = (\tilde{b}_1, \ldots, \tilde{b}_{n-1}, \tilde{b}_{n+1}, \ldots, \tilde{b}_{2n-1})^T. \tag{10.8}
\]

The holomorphic 1-forms \( \tilde{\omega}_j \) tend to the following limit \( \tilde{\Pi} \):

\[
\tilde{\omega}_j^0 = \frac{\varphi_j(z)}{w_0} \frac{dz}{\tilde{a}_j},
\]

where \( \varphi_j(\lambda) \) are degree \( 2n - 2 \) polynomials determined by the normalization conditions

\[
\int_{\tilde{a}_j} \tilde{\omega}_k^0 = \delta_{kj}, \quad j \neq n \\
2\pi i \text{Res}_{z=1, w=w_0(1)} \tilde{\omega}_k^0 = \delta_{kn}.
\]

Therefore, the 1-forms \( \tilde{\omega}_k^0, \ k \neq n, \) become the holomorphic 1-forms that are dual to the basis \( \tilde{A}_0 \) on \( \mathcal{L}_0 \). Furthermore, \( \tilde{\omega}_n^0 \) becomes a normalized meromorphic 1-form with simple poles at the points above \( z = 1 \) on \( \mathcal{L}_0 \).

As in \( \Pi \), we see that the entries of the period matrix \( \tilde{\Pi} \) tend to the following limits:

\[
\lim_{\lambda_{2n} \to \lambda_{2n}^{-1}} \tilde{\Pi}_{jk} = \tilde{\Pi}_{jk}^0, \quad i, j \neq n, n \\
\tilde{\Pi}_{nn} = 2 \sum_{j=1}^{n} \int_{\lambda_{2j-1}}^{\lambda_{2j}} \tilde{\omega}_n = \frac{1}{\pi i} \log |\lambda_{2n}^{-1} - \lambda_{2n}| + O(1), \quad \lambda_{2n} \to \lambda_{2n}^{-1},
\]
where $\tilde{\Pi}^0_{ij}$ is finite for $i, j \neq n, n$.

Let us adopt the notation of theorem 8 and denote the argument of the theta function in the entropy (9.6) by $\xi$, that is

$$\xi = \beta(\lambda)^{-1} \pm \frac{\tau}{2}. \quad (10.9)$$

We will now compute the behavior of the argument $\tilde{\xi}$ in (10.1) with $\xi$ given by (10.9). We have

**Lemma 3.** Let $\xi$ be given by (10.9) and $\tilde{\xi}$ be

$$\tilde{\xi} = \left((-Z_{12}^T \tilde{\Pi} + Z_{22}^T)^T \right) \xi,$$

where $Z_{ij}$ are given by (10.5). Then as $\lambda_{2n} \to \lambda_{\ast}^{-1}$ we have

$$\tilde{\xi}_i = \beta(\lambda) \tilde{\Pi}_{in} \pm \eta_i, \quad i = 1, \ldots, 2n - 1, \quad (10.10)$$

where $\eta_i$ remains finite as $\lambda_{2n} \to \lambda_{\ast}^{-1}$.

**Proof.** To begin with, we will need to express $\tilde{\tau}$ in terms of the Abel map.

Recall that the term $\tilde{\tau}$ in (2.12) is given by

$$\tilde{\tau} = - \sum_{j=2}^{2n-1} \omega(z_j^{-1}) - K,$$

where $K$ is the Riemann constant. As in [6] (see also appendix D), the Riemann constant can be expressed as a sum of images of branch points under the Abel map. In particular, we have

$$K = - \sum_{j=2}^{2n} \omega(\lambda_{2j-1}).$$

Therefore we have

$$\tilde{\tau} = - \sum_{j=2}^{2n} \omega(z_j^{-1}) + \sum_{j=2}^{2n} \omega(\lambda_{2j-1})$$

Now by substituting (10.5) into (10.2) and make use of (10.4) and (10.6), we see that the argument $\tilde{\xi}$ in $\theta(\tilde{\xi}, \tilde{\Pi})$ can be expressed as follows

$$\tilde{\xi}_i = \beta(\lambda) \tilde{\Pi}_{in} \pm \left( \sum_{j=2}^{2n} \tilde{\omega}_i(z_j^{-1}) - \sum_{j=1}^{2n} \tilde{\omega}_i(\lambda_{2j-1}) \right), \quad i = 1, \ldots, 2n - 1, \quad (10.11)$$
where \( \tilde{\omega} \) is the Abel map with \( d\omega \) replaced by \( d\tilde{\omega} \) and \( \tilde{\omega}_i \) is the \( i \)th component of the map.

We would like to show that the term
\[
\sum_{j=2}^{2n} \tilde{\omega}_i (z_j^{-1}) - \sum_{j=1}^{2n} \tilde{\omega}_i (\lambda_{2j-1})
\]
in (10.11) remains finite as \( \lambda_{2n} \to \lambda_{2n}^{-1} \).

To see this, note that the set of points \( \{ z_j^{-1} \} \) must contain either one of the points \( \lambda_{2n} \) or \( \lambda_{2n}^{-1} \), but not both, while \( \{ \lambda_{2i-1} \} \) contains \( \lambda_{2i}^{-1} \) only. As \( \lambda_{2n} \to \lambda_{2n}^{-1} \), the terms \( \tilde{\omega}_n (\lambda_{2n}) \) and \( \tilde{\omega}_n (\lambda_{2n}^{-1}) \) in the sum in equation (10.11) will tend to \(-\infty\). However, since they appear in the sum with opposite signs, these contributions cancel and the quantity
\[
\sum_{j=2}^{2n} \tilde{\omega}_n (z_j^{-1}) - \sum_{j=1}^{2n} \tilde{\omega}_n (\lambda_{2j-1})
\]
remains finite as \( \lambda_{2n} \to \lambda_{2n}^{-1} \).

We can therefore write \( \tilde{\xi} \) as
\[
\tilde{\xi}_i = \beta(\lambda) \tilde{\Pi}_n \pm \eta_i, \quad i = 1, \ldots, 2n - 1
\]
where \( \eta_i \) remains finite as \( \lambda_{2n} \to \lambda_{2n}^{-1} \).

We are now ready to apply theorem 8 to compute the theta function as \( \lambda_{2n} \to \lambda_{2n}^{-1} \).

**Lemma 4.** In the limit \( \lambda_{2n} \to \lambda_{2n}^{-1} \) the theta function \( \theta(\xi, \Pi) \) behaves like
\[
\theta(\xi, \Pi) = \exp \left( \log |\lambda_{2n} - \lambda_{2n}^{-1}|/\beta^2(\lambda) + O(1) \right), \quad (10.12)
\]
where \( \xi \) is given by (10.9).

**Proof.** Firstly, let us use (10.1) and (10.6) to express \( \theta(\xi, \Pi) \) in terms of \( \theta(\tilde{\xi}, \tilde{\Pi}) \), we have
\[
\theta(\xi, \Pi) = \zeta \exp \left[ \pi i \xi^T \tilde{\Pi}^{-1} \tilde{\xi} \right] \theta \left( \tilde{\xi}, \tilde{\Pi} \right). \quad (10.13)
\]
Let us now use (10.10) to compute the asymptotic of the exponential term in (10.13). We obtain
\[
\xi^T \tilde{\Pi}^{-1} \tilde{\xi} = \sum_{i,j} \left( \tilde{\Pi}^{-1} \right)_{ij} \tilde{\xi}_i \tilde{\xi}_j.
\]
The behavior of the entries in \( \tilde{\Pi}^{-1} \) can be calculated by computing the determinant and the minors. We have
\[
\left( \tilde{\Pi}^{-1} \right)_{ij} = O(1), \quad \lambda_{2n} \to \lambda_{2n}^{-1}, \quad i, j \neq n
\]
\[
\left( \tilde{\Pi}^{-1} \right)_{nj} = O \left( \log^{-1} |\lambda_{2n} - \lambda_{2n}^{-1}| \right), \quad \lambda_{2n} \to \lambda_{2n}^{-1}, \quad j \neq n
\]
\[
\left( \tilde{\Pi}^{-1} \right)_{nn} = \pi i \log^{-1} |\lambda_{2n} - \lambda_{2n}^{-1}| + O \left( \log^{-2} |\lambda_{2n} - \lambda_{2n}^{-1}| \right), \quad \lambda_{2n} \to \lambda_{2n}^{-1}.
\]
Therefore, equation (10.14) becomes
\[
\pi i \sum_{i,j} \left( \tilde{\Pi}^{-1} \right)_{ij} \tilde{\xi}_i \tilde{\xi}_j = \log |\lambda_{2n} - \lambda_{2n}^{-1}| \beta^2(\lambda) + O(1), \quad \lambda_{2n} \to \lambda_{2n}^{-1}.
\] (10.14)

Next, we will use the definition (2.10) of the theta function to compute its limit as \( \lambda_{2n} \to \lambda_{2n}^{-1} \). We have,
\[
\theta(\tilde{\xi}, \tilde{\Pi}) = \sum_{m \in \mathbb{Z}} \exp \left[ \pi i \sum_{j \neq n} \tilde{\Pi}_{jn} m_j + 2\pi i \sum_{j \neq n} (\beta(\lambda) \tilde{\Pi}_{jn} \pm \eta_j) m_j + 2\pi i \tilde{\Pi}_{nn} (m_n^2 + 2\beta(\lambda)m_n) \pm 2\eta_n m_n \right].
\] (10.15)

Since
\[
\lim_{\lambda_{2n} \to \lambda_{2n}^{-1}} \text{Re}(2\pi i \tilde{\Pi}_{nn}) = -\infty
\]
and \( \beta(\lambda) \) is purely imaginary, we see that in the limit only the terms with \( m_n = 0 \) contribute to the sum. Therefore, equation (10.15) reduces to
\[
\lim_{\lambda_{2n} \to \lambda_{2n}^{-1}} \theta(\tilde{\xi}, \tilde{\Pi}) = \theta \left( \tilde{\xi}^0, \tilde{\Pi}^0 \right)
\] (10.16)
where the \( \hat{\xi}_n \) in the above equation means that the \( n^{th} \) entry of the vector is removed. The period matrix \( \tilde{\Pi}^0 \) is an \((2n - 2) \times (2n - 2)\) matrix obtained by removing the \( n^{th} \) row and \( n^{th} \) column of the period matrix \( \tilde{\Pi} \). Thus, the theta function \( \theta \left( \tilde{\xi}^0, \tilde{\Pi}^0 \right) \) remains finite as \( \lambda_{2n} \to \lambda_{2n}^{-1} \). This fact, together with (10.14), shows that \( \theta(\xi, \Pi) \) behaves like
\[
\theta(\xi, \Pi) = \varsigma \exp \left( \log |\lambda_{2n} - \lambda_{2n}^{-1}| \beta^2(\lambda) + O(1) \right) \theta \left( \tilde{\xi}^0, \tilde{\Pi}^0 \right), \quad \lambda_{2n} \to \lambda_{2n}^{-1}.
\]
Since \( \theta \left( \tilde{\xi}^0, \tilde{\Pi}^0 \right) \) and \( \varsigma \) remain finite as \( \lambda_{2n} \to \lambda_{2n}^{-1} \), the above equation becomes (10.12). This proves the lemma. \( \square \)

Finally, by substituting (10.12) into (9.6), we have
\[
S(\rho_A) = \frac{1}{2} \int_1^\infty \log \frac{\theta \left( \beta(\lambda) \bar{c} + \frac{z}{2} \right) \theta \left( \beta(\lambda) \bar{c} - \frac{z}{2} \right)}{\theta^2 \left( \frac{z}{2} \right)} d\lambda
\] \[
= \int_1^\infty (\log |\lambda_{2n} - \lambda_{2n}^{-1}| \beta^2(\lambda) + O(1)) d\lambda
\]
Since
\[
\int_1^\infty \beta^2(\lambda) d\lambda = -\frac{1}{6}
\]
\[
\lambda_{2(j+1)} \ldots \lambda_{2n} \quad \lambda_{2n+1} \ldots \lambda_{2(2n-j)-1}
\]
\[
\lambda_{2j+1} \ldots \lambda_{2n-1} \quad \lambda_{2n+2} \ldots \lambda_{2(2n-j)}
\]

Figure 7: Two pairs of roots, labelled according to the ordering (2.5), approaching the unit circle in the critical limit. We have \(\lambda_{2(j+1)} \rightarrow \lambda_{2(2n-j)-1}\), \(\lambda_{2n} \rightarrow \lambda_{2n+1}\) and \(\lambda_{2j+1} \rightarrow \lambda_{2(2n-j)}\) respectively.

we arrive at the following expression for the entropy of entanglement

\[
S(\rho_A) = -\frac{1}{6} \log |\lambda_{2n} - \lambda_{2n}^{-1}| + O(1), \quad \lambda_{2n} \rightarrow \lambda_{2n}^{-1}.
\]

10.2 The limit of complex roots approaching the unit circle

We will now study the case when \(2r\) pairs of complex roots approach each other towards the unit circle. Let \(\lambda_{2j+1}\) be a complex root with \(n - r \leq j \leq n - 1\). As we discussed in section 2, \(\lambda_{2j+1}, 1/\lambda_{2j+1}\) and \(1/\lambda_{2j+1}\) are roots too. The ordering (2.5) implies (see figure 7)

\[
\begin{align*}
\lambda_{2(j+1)} &= \lambda_{2n+1}^{-1} = \lambda_{2(2n-j)-1} = \lambda_{2(2n-j)}^{-1} \\
\lambda_{2(2n-j)} &= 1/\lambda_{2(j+1)} = \lambda_{2(2n-j)-1} = 1/\lambda_{2(j+1)}.
\end{align*}
\]

(10.17)

The critical limit occurs as \(\lambda_{2(j+1)} \rightarrow \lambda_{2(2n-j)-1}\). From the relations (10.17) this implies \(\lambda_{2j+1} \rightarrow \lambda_{2(2n-j)}\). Thus, in what follows we shall mainly discuss the limit \(\lambda_{2(j+1)} \rightarrow \lambda_{2(2n-j)-1}\).

10.2.1 Case 1: \(r < n\)

We now choose the tilded canonical basis of the cycles \((\tilde{A} \quad \tilde{B})\) as in figure 8. Namely, we have

\[
\begin{align*}
\tilde{a}_j &= a_j, \quad j < n - r, \quad j > n + r - 1 \\
\tilde{b}_j &= b_j, \quad j < n - r, \quad j > n + r - 1
\end{align*}
\]
\begin{align*}
\tilde{a}_{n-k} &= b_{n-k} - b_{n+k-1} + \sum_{j=n-k+1}^{n+k-2} a_j, \quad k = 1, \ldots, r \
\tilde{a}_{n+k} &= b_{n+k} - b_{n-k-1} + \sum_{j=n-k}^{n+k} a_j, \quad k = 0, \ldots, r - 1 \\
\tilde{b}_{n-k} &= b_{n-k} - \sum_{j=n-k}^{n+k-2} a_j - \sum_{j=n-r}^{n-k-1} (-1)^{n-k-j} (a_j - 2b_j), \quad k = 1, \ldots, r \\
\tilde{b}_{n+k} &= b_{n+k} + \sum_{j=n-r}^{n+k} (-1)^{n-k-j} (a_j - 2b_j), \quad k = 0, \ldots, r - 1.
\end{align*}

Figure 8: The choice of cycles on the hyperelliptic curve \( \mathcal{L} \). The arrows denote the orientations of the cycles and branch cuts.

We will show in appendix E that this is indeed a canonical basis of cycles. We can partition this basis as follows:

\[
(\tilde{A}) = \begin{pmatrix}
\tilde{a}^I \\
\tilde{a}^{II} \\
\tilde{a}^{III}
\end{pmatrix}
\]

\[
\tilde{a}^I_j = \tilde{a}_j, \quad 1 \leq k \leq n - r - 1 \\
\tilde{a}^{II}_j = \tilde{a}_{n-r+j-1}, \quad 1 \leq k \leq 2r \\
\tilde{a}^{III}_j = \tilde{a}_j, \quad n + r \leq k \leq 2n - 1.
\]

The relations among the \( b \)-cycles and the untilded basis are analogous.
If we write this relation in matrix form as in theorem 8, then the corresponding transformation matrix is given by

\[
\begin{pmatrix}
\tilde{A} \\
\tilde{B}
\end{pmatrix} = Z \begin{pmatrix}
A \\
B
\end{pmatrix} = \begin{pmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{pmatrix} \begin{pmatrix}
A \\
B
\end{pmatrix},
\]

where the blocks \( Z_{ij} \) can be written as

\[
Z_{ij} = \begin{pmatrix}
\delta_{ij} I_{n-r} & 0 & 0 \\
0 & C_{ij} & 0 \\
0 & 0 & \delta_{ij} I_{n-r}
\end{pmatrix},
\]

where \( I_{n-r} \) is the identity matrix of dimension \( n - r - 1 \) and the \( C_{ij} \)'s are the following \( 2r \times 2r \) matrices:

\[
(C_{11})_{kl} = \begin{cases}
1 & k + 1 \leq l \leq 2r - k, \\
0 & \text{otherwise},
\end{cases} & 1 \leq k \leq r
\]

\[
(C_{12})_{kl} = \delta_{kl} - \delta_{l,2r-k+1} & 1 \leq k, l \leq 2r
\]

\[
(C_{21})_{kl} = \begin{cases}
(-1)^{k-l+1} & 1 \leq l \leq k - 1; \\
-1 & k \leq l \leq 2r - k, \\
0 & 2r - k + 1 \leq l,
\end{cases} & 1 \leq k \leq r
\]

\[
(C_{22})_{kl} = \delta_{kl} + \begin{cases}
2(-1)^{k-l} & 1 \leq l \leq k - 1, \\
0 & \text{otherwise},
\end{cases} & 1 \leq k \leq r
\]

\[
(C_{21})_{kl} = \begin{cases}
(-1)^{l-k} & 1 \leq l \leq 2r - k, \\
0 & \text{otherwise},
\end{cases} & r + 1 \leq k \leq 2r
\]

These are matrices of the form

\[
C_{11} = \begin{pmatrix}
0 & 1 & 1 & \ldots & \ldots & 1 & 1 & 0 \\
0 & 0 & 1 & \ldots & \ldots & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 1 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
0 & \ldots & 0 & 1 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 1 & \ldots & \ldots & 1 & 1 & 1
\end{pmatrix}
\]

\[
C_{12} = I_{2r} - J_{2r}
\]

\[
J_{r} = \begin{pmatrix}
0 & \ldots & \ldots & 1 \\
0 & \ldots & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
1 & 0 & \ldots & 0
\end{pmatrix}
\]

(10.19)
As in section 10, some holomorphic 1-forms \( d\tilde{\omega}_j \) will become meromorphic as the roots approach the unit circle.

In this case, the holomorphic 1-form \( d\tilde{\omega}_j \), \( n-r \leq k \leq n+r-1 \) becomes a meromorphic 1-form with a simple pole at \( \lambda_2(j+1) \). All the other holomorphic 1-forms become normalized holomorphic 1-forms in the resulting surface.

In particular, we have the following:

**Lemma 5.** The entries of the period matrix \( \tilde{\Pi} \) behave like

\[
\lim_{\lambda_2(j+1) \to \lambda_2(2n-j)-1} \tilde{\Pi}_{ij} = \tilde{\Pi}_{ij}^0, \quad i \neq j
\]

\[
\lim_{\lambda_2(j+1) \to \lambda_2(2n-j)-1} \tilde{\Pi}_{jj} = \tilde{\Pi}_{jj}^0, \quad j > n + r - 1, \quad j < n - r
\]

\[
\lim_{\lambda_2(j+1) \to \lambda_2(2n-j)-1} \tilde{\Pi}_{jj} = \gamma_j + \tilde{\Pi}_{jj}^0, \quad n - r \leq j \leq n + r - 1
\]

where \( \tilde{\Pi}_{ij}^0 \) are finite.

Let us now consider the behavior of the terms \( \tilde{\xi} \) in (10.1):

**Lemma 6.** Let \( \xi \) be given by (10.9) and \( \tilde{\xi} \) be

\[
\tilde{\xi} = \left((\bar{Z}_{12}^T \tilde{\Pi} + Z_{22}^T)^T\right) \xi,
\]
where $Z_{ij}$ are given by (10.19). Then in the limit $\lambda_{2(j+1)} \rightarrow \lambda_{2(2n-j)-1}$ we have

$$
\tilde{\xi}_i = \eta_i^\pm, \quad i > n + r - 1, \quad i < n - r \\
\tilde{\xi}_i = \epsilon_i \beta(\lambda) \gamma_i + \eta_i^\pm, \quad n - r \leq i \leq n + r - 1,
$$

(10.21)

where $\eta_i^\pm$ remains finite as $\lambda_{2(j+1)} \rightarrow \lambda_{2(2n-j)-1}$.

Proof. Let

$$
Z_{12}^T \tilde{\Pi} - Z_{22}^T = \begin{pmatrix}
0 & 0 & 0 \\
0 & (I_{2r} - J_{2r}) D_r & 0 \\
0 & 0 & 0
\end{pmatrix} + W \\
D_r = \text{diag}(\gamma_{n-r}, \gamma_{n-r+1}, \ldots, \gamma_{n-r+1}, \gamma_{n-r})
$$

(10.22)

where $W$ is a matrix that remains finite as $\lambda_{2(j+1)} \rightarrow \lambda_{2(2n-j)-1}$. Then from (10.2) and (10.4), we see that $\tilde{\xi}$ is given by

$$
\tilde{\xi}_i = \beta(\lambda) \sum_{j=1}^{n-1} W_{n+j,i} + \frac{\tilde{\tau}_j}{2}, \quad i > n + r - 1, \quad i < n - r \\
\tilde{\xi}_i = \epsilon_i \beta(\lambda) \gamma_i + \beta(\lambda) \sum_{j=1}^{n-1} W_{n+j,i} + \frac{\tilde{\tau}_j}{2}, \quad n - r \leq i \leq n + r - 1,
$$

(10.23)

where

$$
\epsilon_i = 1, \quad i < n \\
\epsilon_i = -1, \quad i \geq n \\
\frac{\tilde{\tau}_j}{2} = \sum_{j=1}^{2n} \tilde{\omega}_i(z_j^{-1}) - \sum_{j=1}^{2n} \tilde{\omega}_i(\lambda_{2j-1}).
$$

Let $\lambda_{2(j+1)}$, $\lambda_{2(2n-j)-1}$ and $\lambda_{2j+1}$, $\lambda_{2(2n-j)}$, $n - r \leq j \leq n - 1$ be the pairs of points that approach each other. From their ordering we have $\lambda_{2(j+1)} = \lambda_{2(2n-j)-1}$ and $\lambda_{2j+1} = \lambda_{2(2n-j)-1}$.

For each fixed $j$, the point $\lambda_{2j+1}$ is a pole of $d\tilde{\omega}_{2n-j-1}$, while $\lambda_{2(2n-j)-1}$ is a pole of $d\tilde{\omega}_j$. Therefore, the Riemann constant behaves like

$$
\sum_{j=1}^{2n} \tilde{\omega}_i(\lambda_{2j-1}) = \frac{1}{2} \gamma_i + O(1), \quad \lambda_{2(j+1)} \rightarrow \lambda_{2(2n-j)-1} \quad n - r \leq i \leq n + r - 1.
$$

Moreover, among these 4 points there are exactly two points of the form $z_k^{-1}$ for some $k$. However, since $z_k$ are the roots of a polynomial with real coefficients, if $\lambda_j = z_k^{-1}$ for some $k$, then its complex conjugate $\bar{\lambda}_j$ is also of the form $z_k^{-1}$ for some $k'$. This means that either of the following is true:
1. Both $\lambda_{2(j+1)}$ and $\lambda_{2j+1}$ are of the form $z_k^{-1}$.

2. Both $\lambda_{2(2n-j)}$ and $\lambda_{2(2n-j)-1}$ are of the form $z_k^{-1}$.

Either way, we have

$$\sum_{j=1}^{2n} \tilde{\omega}_i(z_j^{-1}) = \frac{1}{2} \gamma_i + O(1), \quad n - r \leq i \leq n + r - 1.$$ 

Therefore, we can rewrite (10.23) as

$$\tilde{\xi}_i = \eta_i^\pm, \quad i > n + r - 1, \quad i < n - r$$

$$\tilde{\xi}_i = \epsilon_i \beta(\lambda) \gamma_i + \eta_i^\pm, \quad n - r \leq i \leq n + r - 1,$$

where $\eta_i^\pm$ remains finite as $\lambda_{2(j+1)} \to \lambda_{2(2n-j)-1}$. \hfill \Box

We now compute the behavior of the theta function $\theta(\xi, \Pi)$ in this limit.

**Lemma 7.** In the limit $\lambda_{2(j+1)} \to \lambda_{2(2n-j)-1}, \quad n - r \leq j \leq n + r - 1$, the theta function $\theta(\xi, \Pi)$ behaves like

$$\theta(\xi, \Pi) = \exp \left( 2\pi i \beta^2(\lambda) \sum_{j=n-r}^{n-1} \gamma_j + O(1) \right), \quad (10.24)$$

where $\xi$ is given by (10.9) and $\gamma_j$ by (10.20).

**Proof.** From (10.1) we see that

$$\theta(\xi, \Pi) = \varsigma \exp \left( \pi i \tilde{\xi}^T \left( Z_{12}^T \Pi - Z_{22}^T \right)^{-1} Z_{12}^T \tilde{\xi} \right) \theta \left[ \epsilon \epsilon' \right] (\tilde{\xi}, \tilde{\Pi}), \quad (10.25)$$

where the characteristics on the right hand side are obtained by solving the linear equations

$$\begin{align*}
\text{diag} \left( Z_{12}^T Z_{22} \right) &= Z_{22}^T \epsilon + Z_{12}^T \epsilon' \\
\text{diag} \left( Z_{11}^T Z_{21} \right) &= Z_{21}^T \epsilon + Z_{11}^T \epsilon'.
\end{align*}$$

The solution of this system is

$$\begin{align*}
\epsilon_j &= 0 \mod 2, \quad j = 1, \ldots, 2n - 1 \\
\epsilon'_j &= \begin{cases} 1 \mod 2, & n - r \leq j \leq n - 1; \\
0 \mod 2, & \text{otherwise}. \end{cases} \quad (10.26)
\end{align*}$$

Note that, from (2.11) and the periodicity properties of the theta function proposition 2, characteristics that differ by an even integer vector give the same theta function. That is

$$\theta \left[ \epsilon \epsilon' \right] (\xi, \Pi) = \theta \left[ \epsilon + \frac{2N}{\epsilon' + 2M} \right] (\xi, \Pi), \quad \overline{N}, \overline{M} \in \mathbb{Z}^{2n-1}$$
We will now compute the exponential term of (10.25). By performing rows and columns operations on $Z_{12}^T \tilde{\Pi} - Z_{22}^T$, we can transform its determinant into the form
\[
\det \left( Z_{12}^T \tilde{\Pi} - Z_{22}^T \right) = \det \left( \begin{pmatrix} 0_{n-r-1} & 0 & 0 \\ 0 & SD_r & 0 \\ 0 & 0 & 0_{n-r} \end{pmatrix} + W' \right) = 0, \\
S_{ij} = \begin{cases} 0, & 1 \leq i \leq r, \\ \delta_{ij}, & r + 1 \leq i \leq 2r, \end{cases}
\]
for some matrix $W'$ that remains finite as $\lambda_{2(j+1)} \rightarrow \lambda_{2(2n-j)-1}$.

This means that the leading order term of the determinant is of the order of $\prod_{k=n-r}^{n-1} \gamma_k$. That is
\[
\det \left( Z_{12}^T \tilde{\Pi} - Z_{22}^T \right) = D_r + O(\gamma_i^{r-1}), \quad \lambda_{2(j+1)} \rightarrow \lambda_{2(2n-j)-1},
\]
where the notation $O(\gamma_i^{r-1})$ means
\[
O(\gamma_i^{r-1}) = O\left( \prod_{i} \gamma_i^{\alpha_i} \right), \quad \sum_{i} \alpha_i \leq r - 1,
\]
Furthermore, $W'$ is the determinant of the $(2n - r - 1) \times (2n - r - 1)$ matrix formed by removing the $(n - r)^{th}$ up to the $(n - 1)^{th}$ rows and columns in $W'$.

Similarly, we see that the minors of $Z_{12}^T \tilde{\Pi} - Z_{22}^T$ cannot contain more than $r$ factors of $\gamma$. In particular, this means that the inverse matrix $\left( Z_{12}^T \tilde{\Pi} - Z_{22}^T \right)^{-1}$ is finite as $\lambda_{2(j+1)} \rightarrow \lambda_{2(2n-j)-1}$.

Therefore the inverse matrix $\left( Z_{12}^T \tilde{\Pi} - Z_{22}^T \right)^{-1}$ behaves like
\[
\left( Z_{12}^T \tilde{\Pi} - Z_{22}^T \right)^{-1} = X^0 + X^{-1} + O(\gamma_i^{-2}), \quad \lambda_{2(j+1)} \rightarrow \lambda_{2(2n-j)-1},
\]
where $X^{-1}$ is a term of order $-1$ in $\gamma_i$ and $X^0$ is a finite matrix.

From (10.28) and (10.22), we see that the leading order term of
\[
\left( Z_{12}^T \tilde{\Pi} - Z_{22}^T \right)^{-1} \left( Z_{12}^T \tilde{\Pi} - Z_{22}^T \right) = I_{2n-1}
\]
gives the following
\[
X^0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & (I_{2r} - J_{2r})D_r & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0,
\]
while the leading order term of
\[
\left( Z_{12}^T \tilde{\Pi} - Z_{22}^T \right) \left( Z_{12}^T \tilde{\Pi} - Z_{22}^T \right)^{-1} = I_{2n-1}
\]
gives
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & (I_{2r} - J_{2r}) D_r & 0 \\
0 & 0 & 0
\end{pmatrix} X^0 = 0.
\]
This implies that
\[
\begin{align*}
X_{i,j}^0 &= X_{i,2n-j-1}^0, \quad 1 \leq i \leq 2n-1, \quad n-r \leq j \leq n+r-1 \\
X_{i,j}^0 &= X_{2n-i-1,j}^0, \quad n-r \leq i \leq n+r-1, \quad 1 \leq j \leq 2n-1.
\end{align*}
\tag{10.30}
\]

The leading order term of the bilinear product in (10.25) then becomes
\[
\tilde{\xi}^T \left( Z_{12}^T \tilde{\Pi} - Z_{22}^T \right)^{-1} Z_{12}^T \tilde{\xi} = \beta^2(\lambda) \epsilon^T D_n X^{-1} \begin{pmatrix}
0 & 0 & 0 \\
0 & (I_{2r} - J_{2r}) & 0 \\
0 & 0 & 0
\end{pmatrix} D_n \epsilon \\
+ O(1), \quad \lambda_{2(j+1)} \rightarrow \lambda_{2(2n-j)-1}, \\
\epsilon_i &= 0, \quad i < n-r, \quad i > n+r-1, \\
\epsilon_i &= 1, \quad n-r \leq i < n, \\
\epsilon_i &= -1, \quad n \leq i < n+r-1 \\
D_n &= \begin{pmatrix}
0_{n-r-1} & 0 & 0 \\
0 & D_r & 0 \\
0 & 0 & 0_{n-r}
\end{pmatrix}.
\tag{10.31}
\]

Let us denote $\mathcal{P}$ by
\[
\mathcal{P} = X^{-1} \begin{pmatrix}
0 & 0 & 0 \\
0 & (I_{2r} - J_{2r}) & 0 \\
0 & 0 & 0
\end{pmatrix} D_n.
\]

Then constant term of (10.29) gives the following
\[
X^{-1} \begin{pmatrix}
0 & 0 & 0 \\
0 & (I_{2r} - J_{2r}) & 0 \\
0 & 0 & 0
\end{pmatrix} D_n + X^0 W = I_{2n-1}.
\]

By applying (10.30) to the above, we see that the entries of $\mathcal{P}$ are related by
\[
\mathcal{P}_{l,j} = \mathcal{P}_{2n-l-1,j} + \delta_{l,j} + \delta_{2n-l-1,j}, \quad n-r \leq l \leq n-1, \quad n-r \leq j \leq n+r-1.
\]
By substituting this back into (10.31), we see that the the exponential factor in (10.25) behaves like

$$
\tilde{\xi}^T \left( Z_{12}^T \tilde{\Pi} - Z_{22}^T \right)^{-1} Z_{12}^T \tilde{\xi} = 2\beta^2(\lambda) \sum_{j=n-r}^{n-1} \gamma_j + O(1).
$$

(10.32)

We will now show that the limit of the theta function with characteristics remains finite. By using the definition (2.10), we have

$$
\theta \left[ \epsilon \right] (\tilde{\xi}, \tilde{\Pi}) = \sum_{m_j \in \mathbb{Z}} \exp \left[ \pi i \sum_{j=n-r}^{n-1} \gamma_j \left( m_j + \frac{\epsilon_j}{2} \right) \left( 2\beta(\lambda) + m_j + \frac{\epsilon_j}{2} \right) \right] \times \left( -2\beta(\lambda) + m_{2n-j-1} + \frac{\epsilon_{2n-j-1}}{2} \right) + O(1), \quad \lambda_{2(j+1)} \to \lambda_{2(2n-j)-1}.
$$

As before, since $\beta(\lambda)$ is purely imaginary, only terms such that

$$
\left( m_j + \frac{\epsilon_j}{2} \right)^2 + \left( m_{2n-j-1} + \frac{\epsilon_{2n-j-1}}{2} \right)^2 = 0, \quad n-r \leq j \leq n-1,
$$

contribute. Recall that from (10.26) we have $\epsilon_j = \epsilon_{2n-j-1} = 0$, therefore

$$
m_j = m_{2n-j-1} = 0, \quad n-r \leq j \leq n-1.
$$

Thus, as before, the theta function with characteristics reduces to a $2n-2r-1$ dimensional theta function

$$
\lim_{\lambda_{2(j+1)} \to \lambda_{2(2n-j)-1}} \theta \left[ \epsilon \right] (\tilde{\xi}, \tilde{\Pi}) = \theta (\tilde{\xi}^0, \tilde{\Pi}^0), \quad (10.33)
$$

where the arguments on the right hand side are obtained from removing the $(n-r)^{th}$ up to the $(n+r-1)^{th}$ entries and that $\theta (\tilde{\xi}^0, \tilde{\Pi}^0)$ is finite in the limit.

By combining (10.32) and (10.33), we see that the theta function $\theta (\xi, \Pi)$ behaves like

$$
\theta (\xi, \Pi) = \zeta \exp \left( 2\pi i \beta^2(\lambda) \sum_{j=n-r}^{n-1} \gamma_j + O(1) \right) \theta (\tilde{\xi}^0, \tilde{\Pi}^0)
$$

This concludes the proof of the lemma.

Finally, from lemma 7 we see that the entropy (9.6) behaves like

$$
S(\rho_A) = -\frac{1}{3} \sum_{j=n-r}^{n-1} \log |\lambda_{2(j+1)} - \lambda_{2(2n-j)-1}| + O(1), \quad \lambda_{2(j+1)} \to \lambda_{2(2n-j)-1}.
$$
10.2.2 Case 2: \( r=n \)

We will now consider the case when \( r = n \). That is, all roots are complex and they all approach each other pairwise. The canonical basis will be chosen as in (10.18) but with \( r = n - 1 \), (not \( n \)) while the last elements in the basis are given by

\[
\tilde{a}_{2n-1} = b_{2n-1}, \quad \tilde{b}_{2n-1} = -a_{2n-1}.
\]

In other words, we have

\[
\tilde{a}_{n-k} = b_{n-k} - b_{n+k-1} + \sum_{j=n-k+1}^{n+k-2} a_j, \quad k = 1, \ldots, n-1
\]

\[
\tilde{a}_{n+k} = b_{n+k} - b_{n-k-1} + \sum_{j=n-k-1}^{n+k} a_j, \quad k = 0, \ldots, n-2
\]  

(10.34)

\[
\tilde{b}_{n-k} = b_{n-k} - \sum_{j=n-k}^{n+k-2} a_j - \sum_{j=1}^{n-k-1} (-1)^{n-k-j} (a_j - 2b_j), \quad k = 1, \ldots, n-1
\]

\[
\tilde{b}_{n+k} = b_{n+k} + \sum_{j=1}^{n-k-2} (-1)^{n-k-j} (a_j - 2b_j), \quad k = 0, \ldots, n-2
\]

\[
\tilde{a}_{2n-1} = b_{2n-1}, \quad \tilde{b}_{2n-1} = -a_{2n-1}.
\]  

(10.35)

As before, we can partition the basis as follows:

\[
(\tilde{A}) = \begin{pmatrix}
\tilde{a}^I \\
\tilde{a}^{II}
\end{pmatrix}
\]

\[
\tilde{a}^I_j = \tilde{a}_j, \quad 1 \leq k \leq n - r - 1
\]

\[
\tilde{a}^{II}_j = \tilde{a}_{2n-1}.
\]  

(10.36)

Furthermore, the \( b \)-cycles and the untilded basis are connected by analogous relations.

In the notation of theorem 9 we have

\[
\begin{pmatrix}
\tilde{A} \\ \tilde{B}
\end{pmatrix} = Z \begin{pmatrix}
A \\ B
\end{pmatrix} = \begin{pmatrix}
Z_{11} & Z_{12} \\ Z_{21} & Z_{22}
\end{pmatrix} \begin{pmatrix}
A \\ B
\end{pmatrix},
\]

where the transformation matrix \( Z \) can be written in block form according to the partition (10.36):  

\[
Z_{ij} = \begin{pmatrix}
C_{ij} & 0 \\ 0 & \mathcal{E}_{ij}
\end{pmatrix},
\]  

(10.37)

where \( C_{ij} \) are \( 2(n-2) \times 2(n-2) \) matrices defined as in (10.19), and \( \mathcal{E} \) is given by

\[
\mathcal{E}_{ij} = 0, \quad i = j
\]

\[
\mathcal{E}_{12} = 1, \quad \mathcal{E}_{21} = -1.
\]
By deformation of the contours, we see that the cycles $\tilde{a}_j$ become close loops around $\lambda_{2(j+1)}$ in the critical limit.

Let $\tilde{a}_0$ be the closed curve that becomes a loop around $\lambda_2$ as $\lambda_2 \to \lambda_{4n-1}$ (see figure 9). We have

$$\tilde{a}_0 = -b_{2n-1} + \sum_{j=1}^{2n-2} a_j$$

$$\tilde{a}_0 = -\tilde{a}_{2n-1} + \sum_{j=1}^{n-1} (-1)^{j+1} \tilde{a}_j + \sum_{j=n}^{2n-2} (-1)^j \tilde{a}_j$$

In particular, this means that in the limit, the 1-form $\tilde{\omega}_j$ will have a simple pole at $\lambda_{2(j+1)}$ with residue $\frac{1}{2\pi i}$ and a simple pole at $\lambda_2$ with residue $(-1)^{j+1} \frac{1}{2\pi i}$ for $1 \leq j \leq n-1$, $(-1)^j \frac{1}{2\pi i}$ for $n \leq j \leq 2n - 2$ and $-\frac{1}{2\pi i}$ for $j = 2n - 1$. Thus, we arrive at the following

**Lemma 8.** The entries of the period matrix behave like

$$\lim_{\lambda_{2(j+1)} \to \lambda_{2(2n-j)-1}} \tilde{\Pi}_{ij} = \tilde{\Pi}_{ij}^0, \quad i \neq j, \quad i, j \neq 2n - 1$$

$$\lim_{\lambda_{2(j+1)} \to \lambda_{2(2n-j)-1}} \tilde{\Pi}_{jj} = \gamma_j + \tilde{\Pi}_{jj}^0, \quad 1 \leq j \leq 2n - 2$$

$$\lim_{\lambda_{2(j+1)} \to \lambda_{2(2n-j)-1}} \tilde{\Pi}_{2n-1,2n-1} = 2\gamma_{2n-1} + \tilde{\Pi}_{2n-1,2n-1}^0$$

![Figure 9: The curve going around $\lambda_2$.](image-url)
\[
\lim_{\lambda_{2(j+1)} \to \lambda_{2(2n-j)-1}} \tilde{\Pi}_{j,2n-1} = (-1)^j \gamma_{2n-1} + \tilde{\Pi}_{j,2n-1}^0, \quad 1 \leq j \leq n-1
\]
\[
\lim_{\lambda_{2(j+1)} \to \lambda_{2(2n-j)-1}} \tilde{\Pi}_{j,2n-1} = (-1)^{j+1} \gamma_{2n-1} + \tilde{\Pi}_{j,2n-1}^0, \quad n \leq j \leq 2n-2
\]
\[
\tilde{\Pi}_{2n-1,j} = \tilde{\Pi}_{j,2n-1}
\]
\[
\gamma_j = \frac{1}{\pi i} \log |\lambda_{2(j+1)} - \lambda_{2(2n-j)-1}|
\]

where \(\tilde{\Pi}_{ij}\) are finite in the limit \(\lambda_{2(j+1)} \to \lambda_{2(2n-j)-1}\).

In this case, the argument \(\tilde{\xi}\) in (10.1) behaves as follows.

**Lemma 9.** Let \(\xi\) be given by (10.9) and \(\tilde{\xi}\) be

\[
\tilde{\xi} = \left(-Z_{12}^T \tilde{\Pi} + Z_{22}^T\right) \xi,
\]

where \(Z_{ij}\) are given by (10.37). Then in the limit \(\lambda_{2(j+1)} \to \lambda_{2(2n-j)-1}\) we have

\[
\tilde{\xi}_i = \sigma_i \beta(\lambda) \gamma_i + \eta_i^\pm, \quad 1 \leq i \leq 2n-1,
\]

\[
\sigma_i = \begin{cases} 
1 + (-1)^{i+1}, & 1 \leq i \leq n-1 \\
-1 + (-1)^{i+1}, & n \leq i \leq 2n-1
\end{cases}
\]

where \(\eta_i^\pm\) remains finite as \(\lambda_{2(j+1)} \to \lambda_{2(2n-j)-1}\).

**Proof.** In this case the matrix \(Z_{12}^T \tilde{\Pi} - Z_{22}^T\) takes the form

\[
Z_{12}^T \tilde{\Pi} - Z_{22}^T = \begin{pmatrix} (I_{2r} - J_{2r}) D_{n-1} & 0 \\ \overline{D}_{n-1} & 2 \gamma_{2n-1} \end{pmatrix} + W
\]

where \(W\) is a finite matrix as \(\lambda_{2(j+1)} \to \lambda_{2(2n-j)-1}\).

Therefore, \(\tilde{\xi}\) behaves like

\[
\tilde{\xi}_i = \sigma_i \beta(\lambda) \gamma_i + \beta(\lambda) \sum_{j=1}^{n-1} W_{n+j,i} \pm \frac{\tilde{\tau}_i}{2}, \quad 1 \leq i \leq 2n-1
\]

\[
\sigma_i = \begin{cases} 
1 + (-1)^{i+1}, & 1 \leq i \leq n-1 \\
-1 + (-1)^{i+1}, & n \leq i \leq 2n-1
\end{cases}
\]

\[
\frac{\tilde{\tau}_i}{2} = \sum_{j=1}^{2n} \tilde{\omega}_i (z_j^{-1}) - \sum_{j=1}^{2n} \tilde{\omega}_i (\lambda_{2j-1}).
\]

As in section 10.2.1, the leading order terms of \(\tilde{\xi}\) are zero. We can therefore rewrite \(\tilde{\xi}\) as

\[
\tilde{\xi}_i = \sigma_i \beta(\lambda) \gamma_i + \eta_i^\pm, \quad 1 \leq i \leq 2n-1,
\]

where \(\eta_i^\pm\) are finite in the limit.
The behavior of the theta function for this case is given by

**Lemma 10.** In the limit \( \lambda_{2(j+1)} \to \lambda_{2(2n-j)-1}, \) \( 1 \leq j \leq 2n-1, \) the theta function \( \theta(\xi, \Pi) \) behaves like

\[
\theta(\xi, \Pi) = \exp \left( 2\pi i \beta^2(\lambda) \sum_{j=1}^{n-1} \gamma_j + O(1) \right),
\]

where \( \xi \) is given by (10.9) and \( \gamma_j \) by lemma 8.

**Proof.** As in section 10.2.1, from (10.1) we have,

\[
\theta(\xi, \Pi) = \varsigma \exp \left( \pi i \tilde{\xi}^T \left( Z_{12}^T \tilde{\Pi} - Z_{22}^T \right)^{-1} Z_{12}^T \tilde{\xi} \right) \theta\left( \varepsilon', \tilde{\Pi} \right),
\]

where the characteristics on the right hand side are given by the same formula as before, with \( r \) replaced by \( n-1 \):

\[
\varepsilon_j = 0 \mod 2, \quad j = 1, \ldots, 2n-1 \\
\varepsilon'_j = \begin{cases} 1 \mod 2, & 1 \leq j \leq n-1; \\ 0 \mod 2, & \text{otherwise}. \end{cases}
\]

Since there is no non-zero matrix \( X_0 \) that is independent of \( \gamma_j \) such that the leading order term of

\[
\left( Z_{12}^T \tilde{\Pi} - Z_{22}^T \right) X_0
\]

is zero, we can write the inverse matrix \( \left( Z_{12}^T \tilde{\Pi} - Z_{22}^T \right)^{-1} \) as

\[
\left( Z_{12}^T \tilde{\Pi} - Z_{22}^T \right)^{-1} = X^{-1} + O(\gamma_i^{-2}), \quad \lambda_{2(j+1)} \to \lambda_{2(2n-j)-1}.
\]

where \( X^{-1} \) is a term that is of order \( -1 \) in the \( \gamma_j \).

Then, the leading order term of the bilinear product in (10.41) is

\[
\tilde{\xi}^T \left( Z_{12}^T \tilde{\Pi} - Z_{22}^T \right)^{-1} Z_{12} \tilde{\xi} = \beta^2(\lambda) \sigma^T D_n X^{-1} \begin{pmatrix} (J_{2n-2} - J_{2n-2}) \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} D_n \sigma
\]

\[
+ O(1), \quad \lambda_{2(j+1)} \to \lambda_{2(2n-j)-1},
\]

\[
\sigma_i = \begin{cases} 1 + (-1)^{i+1}, & 1 \leq i \leq n-1; \\ -(1 + (-1)^{i+1}), & n \leq i \leq 2n-1 \end{cases}
\]

\[
D_n = \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_2, \gamma_1, 2\gamma_{2n-1}).
\]

Let \( \tilde{\Pi}^1 \) be the leading order term of \( \tilde{\Pi} \):

\[
\tilde{\Pi}^1 = \begin{pmatrix} D_{n-1}^T & \overline{D}_{n-1}^T \\ D_{n-1} & 2\gamma_{2n-1} \end{pmatrix}.
\]
Equation (10.42) can now be rewritten as

\[ \tilde{\xi}^T \left( Z_{12}^T \tilde{\Pi} - Z_{22}^T \right)^{-1} Z_{12}^T \tilde{\xi} = \beta^2(\lambda) \epsilon^T \tilde{\Pi} X^{-1} \begin{pmatrix} (I_{2n-2} - J_{2n-2}) & 0 \\ 0 & 1 \end{pmatrix} \tilde{\Pi} \epsilon \]

\[ + O(1), \quad \lambda_{2(j+1)} \to \lambda_{2(2n-j)-1} \]

\[ \epsilon_i = 1, \quad 1 \leq i \leq n - 1 \]

\[ \epsilon_i = -1, \quad n \leq i \leq 2n - 1 \quad (10.43) \]

The constant term of

\[ (Z_{12}^T \tilde{\Pi} - Z_{22}^T)^{-1} (Z_{12}^T \tilde{\Pi} - Z_{22}^T) = I_{2n-1} \]

now gives

\[ X^{-1} \begin{pmatrix} I_{2n-2} - J_{2n-2} & 0 \\ 0 & 1 \end{pmatrix} \tilde{\Pi}^1 = I_{2n-1}. \]

By substituting this back into (10.43), we obtain

\[ \pi i \tilde{\xi}^T \left( Z_{12}^T \tilde{\Pi} - Z_{22}^T \right)^{-1} Z_{12}^T \tilde{\xi} = \sum_{j=1}^{2n-1} \log |\lambda_{2(j+1)} - \lambda_{2(2n-j)-1}| + O(1). \]

To complete the proof, note that in this case, the theta function in the right hand side of (10.41) becomes 1:

\[ \lim_{\lambda_{2(j+1)} \to \lambda_{2(2n-j)-1}} \theta \left[ \xi' \overline{\epsilon'} \right] (\tilde{\xi}, \tilde{\Pi}) = 1. \]

Therefore, we have

\[ \theta (\xi, \Pi) = \varsigma \exp \left( \pi i \sum_{j=1}^{2n-1} \gamma_j + O(1) \right), \quad \lambda_{2(j+1)} \to \lambda_{2(2n-j)-1}. \]

This completes the proof of the lemma. \( \square \)

Finally, by substituting (10.40) into (9.6), we find that the entropy behaves like

\[ S(\rho_A) = -\frac{1}{3} \sum_{j=1}^{2n-1} \log |\lambda_{2(j+1)} - \lambda_{2(2n-j)-1}| + O(1), \quad \lambda_{2(j+1)} \to \lambda_{2(2n-j)-1}. \]
10.3 Pairs of complex roots approaching the unit circle together with one pair of real roots approaching 1

The canonical basis used in this section is shown in figure 10:

\[
\tilde{a}_k = \begin{cases} 
-b_k + b_{k-1}, & k < n - r, \quad k > n + r \quad b_0 = 0 \\
\sum_{j=k}^{2n-1} a_j - \sum_{j=n-r}^{n+k-1} a_j, & k < n - r
\end{cases}
\]

\[
\tilde{b}_k = \sum_{j=k}^{n+k-2} a_j - \sum_{j=n-r}^{n+k} a_j, \quad k < n - r
\]

\[
\tilde{a}_{n-k} = b_{n-k} - b_{n+k-1} + \sum_{j=n-k+1}^{n+k-2} a_j, \quad k = 1, \ldots, r
\]

\[
\tilde{a}_{n+k} = b_{n+k} - b_{n-k-1} + \sum_{j=n-k}^{n+k-1} a_j, \quad k = 0, \ldots, r - 1
\]

\[
\tilde{b}_{n-k} = b_{n-k} + (-1)^{r-k} \sum_{j=n+r}^{2n-1} a_j - \sum_{j=n-k}^{n+k-2} a_j - \sum_{j=n-r}^{n-k-1} (-1)^{n-k-j} (a_j - 2b_j), \quad k = 1, \ldots, r
\]

\[
\tilde{b}_{n+k} = b_{n+k} + (-1)^{r-k} \sum_{j=n+r}^{2n-1} a_j + \sum_{j=n-r}^{n-k-2} (-1)^{n-k-j} (a_j - 2b_j), \quad k = 0, \ldots, r - 1
\]

\[
\tilde{a}_{n+r} = b_{n-r-1} - b_{n+r} + \sum_{j=0}^{r-1} (-1)^{r-j-1} (2b_{n+j} + a_{n+j} - 2b_{n-j-1} + a_{n-j-1})
\]

\[
\tilde{b}_k = \sum_{j=k}^{2n-1} a_j, \quad k \geq n + r.
\]

In the notation of theorem 8 the two bases are related by

\[
\begin{pmatrix} A \\ B \end{pmatrix} = Z \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}
\]

\[
Z_{11} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & C_{11} & 0 \\ 0 & T^{32} & 0 \end{pmatrix},
\]

\[
Z_{12} = \begin{pmatrix} -C_2^{m-r-1} & 0 & 0 \\ 0 & C_{12} & 0 \\ \gamma^{31} & \gamma^{32} & -C_2^{m-r-1} \end{pmatrix}
\]

\[
Z_{21} = \begin{pmatrix} C_1^{m-r-1} & 0 & \mathcal{U}_{13} \\ 0 & C_{21} & \mathcal{U}_{23} \\ 0 & 0 & C_1^{m-r-1} \end{pmatrix}
\]

\[
Z_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & C_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
where $C_{ij}$ are defined in (10.19) and $C_i^k$ are $k \times k$ matrix with entries defined as in (10.5). All the entries of the matrices $U^{13}$ are 1, while the entries of $V^{31}, V^{32}$ and $U^{23}$ are defined in

\[ T_{ij}^{32} = \delta_{i1}(-1)^{j+1}, \quad T_{i,2r-j+1}^{32} = T_{ij}^{32}, \quad 1 \leq j \leq r, \]
\[ V_{ij}^{31} = \delta_{i1}\delta_{j,n-r-1}, \]
\[ V_{ij}^{32} = 2(-1)^j \delta_{i1}, \]
\[ U_{ij}^{23} = (-1)^{j+1}. \]

Performing the same analysis as in section 10.2.1 we arrive at

**Lemma 11.** The entries of the period matrix $\tilde{\Pi}$ behave like

\[ \lim_{\lambda_{2(j+1)} \to \lambda_{2(2n-j)-1}} \tilde{\Pi}_{ij} = \tilde{\Pi}_{ij}^0, \quad i \neq j \]
\[ \lim_{\lambda_{2(j+1)} \to \lambda_{2(2n-j)-1}} \tilde{\Pi}_{ij} = \tilde{\Pi}_{ij}^0, \quad j > n + r, \quad j < n - r \]
\[ \lim_{\lambda_{2(j+1)} \to \lambda_{2(2n-j)-1}} \tilde{\Pi}_{jj} = \gamma_j + \tilde{\Pi}_{jj}^0, \quad n - r \leq j \leq n + r \]

\[ \gamma_j = \frac{1}{\pi i} \log |\lambda_{2(j+1)} \to \lambda_{2(2n-j)-1}|, \quad (10.45) \]

where $\tilde{\Pi}_{ij}^0$ are finite.

In this case, the argument $\tilde{\xi}$ is given by the following
Lemma 12. Let $\xi$ be given by (10.9) and $\tilde{\xi}$ be

$$\tilde{\xi} = \left( -Z_{12}^T \Pi + Z_{22}^T \right) \xi,$$

where $Z_{ij}$ are given by (10.44). Then in the limit $\lambda_{2(j+1)} \rightarrow \lambda_{2(2n-j)-1}$ we have

$$\tilde{\xi}_i = \eta_i^\pm, \quad i > n + r, \quad i < n - r$$
$$\tilde{\xi}_i = \epsilon_i \beta(\lambda) \gamma_i + \eta_i^\pm, \quad n - r \leq i \leq n + r - 1$$
$$\tilde{\xi}_{n+r} = \beta(\lambda) \gamma_{n+r} + \eta_{n+r}^\pm$$
$$\epsilon_i = 1, \quad i < n, \quad \epsilon_i = -1, \quad i > n - 1,$$

where $\eta_i^\pm$ remains finite as $\lambda_{2(j+1)} \rightarrow \lambda_{2(2n-j)-1}$, $n - r \leq j \leq n + r$.

The proof of this lemma follows from exactly the same type of argument as in section 10.2.1.

We will now compute the limit of the theta function.

Lemma 13. In the limit $\lambda_{2(j+1)} \rightarrow \lambda_{2(2n-j)-1}$, $n - r \leq j \leq n + r$, the theta function $\theta(\xi, \Pi)$ behaves like

$$\theta(\xi, \Pi) = \exp \left( 2\pi i \beta^2(\lambda) \sum_{j=1}^{n-1} \gamma_j + \beta^2(\lambda) \gamma_{n+r} + O(1) \right),$$

where $\xi$ is given by (10.9) and $\gamma_j$ by (10.45).

Proof. The characteristics in the theta function in (10.11) are once more given by (10.26). The matrix $Z_{12}^T \Pi - Z_{22}^T$ can now be written as

$$Z_{12}^T \Pi - Z_{22}^T = \begin{pmatrix}
0_{n-r-1} & 0 & 0 & 0 \\
0 & (I_{2r} - J_{2r})D_r & 0 & 0 \\
0 & 0 & \gamma_{n+r} & 0 \\
0 & 0 & 0 & 0_{n-r-1}
\end{pmatrix} + W$$

$$D_r = \text{diag}(\gamma_{n-r}, \gamma_{n-r+1}, \ldots, \gamma_{n-r+1}, \gamma_{n-r}),$$

where $W$ is finite in the limit and $0_{n-r-1}$ is the zero matrix of dimension $n - r - 1$.

As in section 10.2.1 by performing rows and columns operations on the matrix $Z_{12}^T \Pi - Z_{22}^T$, we see that the determinants has the following asymptotic behavior:

$$\det \left( Z_{12}^T \Pi - Z_{22}^T \right) = \gamma_{n+r} D_r + O(\gamma_i^r), \quad \lambda_{2(j+1)} \rightarrow \lambda_{2(2n-j)-1},$$

$$D_r = \mathcal{W} \prod_{k=n-r}^{n-1} \gamma_k,$$

where the notation $O(\gamma_i^r)$ was defined in equation (10.27) and $\mathcal{W}$ is some constant.
The inverse matrix \((Z_{12}^T\tilde{\Pi} - Z_{22}^T)^{-1}\) can now be written as in (10.28):

\[
(Z_{12}^T\tilde{\Pi} - Z_{22}^T)^{-1} = X^0 + X^{-1} + O(\gamma_i^{-2}), \quad \lambda_{2(j+1)} \rightarrow \lambda_{2(2n-j)-1},
\]

where the entries of the \(2r\) dimensional matrix \(X^0\) satisfy (10.30) with \((X^0)_{n+r,n+r} = 0\), and \(X^{-1}\) is a matrix of order \(-1\) in the \(\gamma_j\) with \((X^{-1})_{n+r,n+r} = \gamma_{n+r}^{-1}\).

Following exactly the same analysis in section 10.2, we see that the leading order term in the exponential factor in (10.25) is

\[
\tilde{\xi}^T (Z_{12}^T\tilde{\Pi} - Z_{22}^T)^{-1} Z_{12}^T\tilde{\xi} = \beta^2(\lambda) \left(2 \sum_{j=n-r}^{n-1} \gamma_j + \gamma_{n+r}\right) + O(1).
\]

We now look at the term \(\theta(\tilde{\xi}, \tilde{\Pi})\) in (10.1). As in section 10.2, we see that the theta function becomes \(2n - 2r - 2\) dimensional:

\[
\lim_{\lambda_{2(j+1)} \rightarrow \lambda_{2(2n-j)-1}} \theta \left(\xi, \Pi\right) = \theta(\tilde{\xi}^0, \tilde{\Pi}^0),
\]

where the arguments on the right hand side are obtained from removing the \((n - r)^{th}\) up to the \((n + r - 1)^{th}\) entries.

Therefore the theta function \(\theta(\xi, \Pi)\) behaves like

\[
\theta(\xi, \Pi) = \xi \exp \left(2\pi i / \beta^2(\lambda) \left(2 \sum_{j=n-r}^{n-1} \gamma_j + \gamma_{n+r}\right) + O(1)\right) \theta(\tilde{\xi}^0, \tilde{\Pi}^0).
\]

This completes the proof of the lemma.

By substituting (10.47) into (9.6), we see that the entropy is asymptotic to

\[
S(\rho_A) = -\frac{1}{3} \sum_{j=n-r}^{n-1} \log |\lambda_{2(j+1)} - \lambda_{2(2n-j)-1}| - \frac{1}{6} \log |\lambda_{2(n-r)} - \lambda_{2(n+r)+1}|
+ O(1), \quad \lambda_{2(j+1)} \rightarrow \lambda_{2(2n-j)-1}.
\]

This concludes the proof of theorem 2.

**Appendix A. The density matrix of a subchain**

Let \(\{|\psi_j\rangle\}\) be a basis of the Hilbert space \(\mathcal{H}\) of a system composed of two parts, A and B, so that \(\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B\). The density matrix of a statistical ensemble expressed in the basis \(\{|\psi_j\rangle\}\) is a positive Hermitian matrix given by

\[
\rho_{AB} = \sum_{jk} c_{jk} |\psi_j\rangle \langle \psi_k|,
\]
with the condition $\text{tr}_{AB}\rho_{AB} = 1$. Let us introduce the operators $S(j,k)$ and $\overline{S}(j,k)$ defined by the relations

$$S(j,k) = |\psi_j\rangle\langle\psi_k| \quad \text{and} \quad \overline{S}(j,k)S(k,l) = \delta_{jl} |\psi_j\rangle\langle\psi_j|.$$  

(In this formula repeated indices are not summed over.) Clearly, we have

$$c_{jk} = \text{tr}_{AB} \left[ \rho_{AB} \overline{S}(k,j) \right].$$

Let us now suppose that the Hamiltonian of our physical system is \(3.10\) and that the subsystem $P$ is composed of the first $L$ oscillators. Then a set of operators $S(j,k)$ for the subchain $P$ can be generated by products of the type $\prod_{j=1}^{L} G_j$, where $G_j$ can be any of the operators $\{c_j, c_j^\dagger, c_j c_j^\dagger, c_j^\dagger c_j\}$ and the $c_j$'s are Fermi operators that span $H_A$; it is straightforward to check that $\overline{S}(k,j) = \left(\prod_{j=1}^{L} G_j\right)^\dagger$. We then have

$$\rho_A = \sum_{\text{All the } S(l,k)} \text{tr}_P \left[ \rho_A \left( \prod_{j=1}^{L} G_j \right)^\dagger \prod_{j=1}^{L} G_j \right] = \sum_{\text{All the } S(l,k)} \text{tr}_B \left( \rho_{AB} \left( \prod_{j=1}^{L} G_j \right)^\dagger \prod_{j=1}^{L} G_j \right) = \sum_{\text{All the } S(l,k)} \text{tr}_{PQ} \left[ \rho_{AB} \left( \prod_{j=1}^{L} G_j \right)^\dagger \prod_{j=1}^{L} G_j \right].$$

Since $\rho_{AB} = |\Psi_g\rangle\langle\Psi_g|$, this expression simply reduces to

$$\rho_A = \sum_{\text{All the } S(l,k)} \langle \Psi_g | \left( \prod_{j=1}^{L} G_j \right)^\dagger |\Psi_g\rangle \prod_{j=1}^{L} G_j.$$  

The correlation functions in the above sum can be computed using Wick's theorem \(3.9\). Finally, if the correlations of the $c_j$'s are given by \(4.5\) and \(4.6\), we immediately obtain formula \(4.7\).

**Appendix B. The correlation matrix $C_M$**

The purpose of this appendix is to provide an explicit derivation of the expectation values

$$\langle \Psi_g | m_j m_k | \Psi_g \rangle$$  \hspace{1cm} (B.1)

when the dynamics is determined by the Hamiltonian \(3.10\).
First, we need to diagonalize $H_\alpha$, which is achieved by finding a linear transformation of the operators $b_j$ of the form

$$\eta_k = \sum_{j=0}^{M-1} \left( g_{kj} b_j + h_{kj} b_j^\dagger \right),$$

such that the Hamiltonian (3.10) becomes

$$H_\alpha = \sum_{k=0}^{M-1} |\Lambda_k| \eta_k^\dagger \eta_k + C,$$

where the coefficients $g_{kj}$ and $h_{kj}$ are real, the $\eta_k$s are Fermi operators and $C$ is a constant. The quadratic form (3.10) can be transformed into (B.3) by (B.2) if the system of equations

$$[\eta_k, H_\alpha] - |\Lambda_k| \eta_k = 0, \quad k = 0, \ldots, M - 1$$

has a solution. Substituting (3.10) and (B.2) into (B.4) we obtain the eigenvalue equations

$$|\Lambda_k| g_{kj} = \sum_{l=0}^{M-1} \left( g_{kl} \overline{A}_{lj} - h_{kl} \overline{B}_{lj} \right),$$

$$|\Lambda_k| h_{kj} = \sum_{l=0}^{M-1} \left( g_{kl} \overline{B}_{lj} - h_{kl} \overline{A}_{lj} \right),$$

where $\overline{A} = \alpha A - 2I$ and $\overline{B} = \alpha \gamma B$. These equations can be simplified by setting

$$\phi_{kj} = g_{kj} + h_{kj},$$

$$\psi_{kj} = g_{kj} - h_{kj},$$

in terms of which the equations (B.5) become

$$(\overline{A} + \overline{B})\phi_k = |\Lambda_k| \psi_k,$$

$$(\overline{A} - \overline{B})\psi_k = |\Lambda_k| \phi_k.$$  

Combining these two expressions, we obtain

$$(\overline{A} - \overline{B})(\overline{A} + \overline{B})\phi_k = |\Lambda_k|^2 \phi_k,$$

$$(\overline{A} + \overline{B})(\overline{A} - \overline{B})\psi_k = |\Lambda_k|^2 \psi_k.$$  

When $\Lambda_k \neq 0$, $\phi_k$ and $|\Lambda_k|$ can be determined by solving the eigenvalue equation (B.9), then $\psi_k$ can be computed using (B.7). Alternatively, one can solve equation (B.10) and then obtain $\phi_k$ from (B.8). When $\Lambda_k = 0$, $\phi_k$ and $\psi_k$ differ at most by a sign and can be deduced directly either from (B.7) and (B.8) or from (B.9) and (B.10).
Since $\overline{A}$ and $\overline{B}$ are real, the matrices $(\overline{A} - \overline{B})(\overline{A} + \overline{B})$ and $(\overline{A} + \overline{B})(\overline{A} - \overline{B})$ are symmetric and positive, which guarantees that all of their eigenvalues are positive. Furthermore, the $\phi_k$'s and $\psi_k$'s can be chosen to be real and orthonormal. As a consequence the coefficients $g_{kj}$ and $h_{kj}$ obey the constraints

$$\sum_{k=0}^{M-1} (g_{kj}g_{kl} + h_{kj}h_{kl}) = \delta_{jl}, \quad (B.11)$$

$$\sum_{k=0}^{M-1} (g_{kj}h_{kl} + h_{kj}g_{kl}) = 0, \quad (B.12)$$

which are necessary and sufficient conditions for the $\eta_k$'s to be Fermi operators.

The constant in equation (B.3) can be computed by taking the trace of $H_\alpha$ using the two expressions (3.10) and (B.3):

$$\text{tr}H_\alpha = 2^{M-1} \sum_{k=0}^{M-1} (\alpha A_{kk} - 2) = 2^{M-1} \sum_{k=0}^{M-1} |\Lambda_k| + 2^M C.$$ 

Therefore, we have

$$C = \frac{1}{2} \sum_{k=0}^{M-1} (\alpha A_{kk} - 2 - |\Lambda_k|).$$

We are now in a position to compute the contraction pair (B.1). Substituting (B.6) into (B.2) we have

$$\eta_k = \frac{1}{2} \sum_{j=0}^{M-1} (\phi_{kj}m_{2j+1} - i\psi_{kj}m_{2j}). \quad (B.13)$$

Since the $\phi_k$’s and $\psi_k$’s are two sets of real and orthogonal vectors, (B.13) can be inverted to give

$$m_{2j} = i \sum_{k=0}^{M-1} \psi_{kj} (\eta_k - \eta_k^\dagger) \quad (B.14)$$

$$m_{2j+1} = \sum_{k=0}^{M-1} \phi_{kj} (\eta_k + \eta_k^\dagger). \quad (B.15)$$

Since the vacuum state of the operators $\eta_k$ coincides with $|\Psi_g\rangle$, the expectation values (B.1) are easily computed from the expressions (B.14) and (B.15). We have

$$\langle \Psi_g | m_{2j} m_{2k} | \Psi_g \rangle = \sum_{l=0}^{M-1} \psi_{il} \psi_{lk} = \delta_{jk}, \quad (B.16)$$

$$\langle \Psi_g | m_{2j+1} m_{2k+1} | \Psi_g \rangle = \sum_{l=0}^{M-1} \phi_{il} \phi_{lk} = \delta_{jk}. \quad (B.17)$$
and
\[
\langle \Psi_g | m_{2j} m_{2k+1} | \Psi_g \rangle = \sum_{l=0}^{M-1} \psi_{lj} \phi_{lk}, \quad (B.18)
\]
\[
\langle \Psi_g | m_{2j+1} m_{2k} | \Psi_g \rangle = -i \sum_{l=0}^{M-1} \psi_{lk} \phi_{lj}. \quad (B.19)
\]

Finally, by introducing the real \( M \times M \) matrix
\[
(T_M)_{jk} = \sum_{l=0}^{M-1} \psi_{lj} \phi_{lk}, \quad j, k = 0, \ldots, M - 1 \quad (B.20)
\]
and combining the expressions \( B.16), \( B.17), \( B.18) \) and \( B.19) \) we obtain
\[
\langle \Psi_g | m_j m_k | \Psi_g \rangle = \delta_{jk} + i (C_M)_{jk}, \quad (B.21)
\]
where the matrix \( C_M \) has the block structure
\[
C_M = \begin{pmatrix}
C_{11} & C_{12} & \cdots & C_{1M} \\
C_{21} & C_{22} & \cdots & C_{2M} \\
\vdots & \vdots & \ddots & \vdots \\
C_{M1} & C_{M2} & \cdots & C_{MM}
\end{pmatrix} \quad (B.22)
\]
with
\[
C_{jk} = \begin{pmatrix}
0 & (T_M)_{jk} \\
-(T_M)_{kj} & 0
\end{pmatrix} \quad (B.23)
\]
We call \( C_M \) the correlation matrix. It is worth noting that because of the definition \( B.20) \), the matrix \( T_M \) contains all of the physical information relating to the ground state of \( H_\alpha \).

**Appendix C. Thermodynamic limit of the correlation matrix \( C_M \)**

In this appendix we prove the following

**Lemma 14.** Let \( H_\alpha \) be the Hamiltonian \( (3.10) \) and consider the correlation matrix \( B.22) \) associated to \( H_\alpha \). We have
\[
\lim_{M \to \infty} C_M = T_\infty[\Phi], \quad (C.1)
\]
where \( T_\infty[\Phi] \) is the semi-infinite block-Toeplitz matrix with symbol
\[
\Phi = \begin{pmatrix}
0 & g(e^{i\theta}) \\
g^{-1}(e^{i\theta}) & 0
\end{pmatrix},
\]
where the function \( g(z) \) is defined in \( 3.15) \).
Proof. From the definitions (3.15) and (3.16) we have that
\[ g(e^{-i\theta}) = g(e^{i\theta}) = g^{-1}(e^{i\theta}). \]

Thus, from equation (B.23) it suffices to show that
\[
\lim_{M \to \infty} (T_M)_{jk} = \frac{1}{2\pi} \int_0^{2\pi} g(e^{i\theta}) e^{-i(j-k)\theta} d\theta, \tag{C.2}
\]
where \( g(z) \) is defined in (3.15).

The first step consists in determining the vectors \( \phi_k \) and \( \psi_k \), and the numbers \( \Lambda_k \) via the eigenvalue equations (B.7), (B.8), (B.9), and (B.10). If we use the definitions (3.12), we can write
\[
(\overline{A} + \overline{B})_{jk} = a(j - k) + \gamma b(j - k) \quad \text{and} \quad (\overline{A} - \overline{B})_{jk} = a(j - k) - \gamma b(j - k).
\]

Two arbitrary circulant matrices commute and a common set of normalised eigenvectors is given by
\[
\psi_{kj} = \exp\left(\frac{2\pi i j k}{M}\right) \sqrt{\frac{1}{M}}, \quad j, k = 0, \ldots, M - 1, \tag{C.3}
\]
where the index \( j \) labels the component of the \( k \)-th eigenvector. As a consequence, the \( \psi_k \) are a set of common eigenvectors of both \( \overline{A} + \overline{B}(\overline{A} - \overline{B}) \) and \( \overline{A} - \overline{B} \). Now, combining equations (B.8) and (B.10) we can write
\[
\sum_{l=0}^{M-1} [a(j - l) - \gamma b(j - l)] \psi_{kl} = \Lambda_k \psi_{kj} = |\Lambda_k'| \phi_{kj}, \tag{C.4}
\]
with \( \phi_k = \psi_k \Lambda_k'/|\Lambda_k| \). Because both \( \phi_k \) and \( \psi_k \) are normalized, \( \Lambda_k'/|\Lambda_k| \) must be a complex number with modulo one and we can set \( \Lambda_k' = \Lambda_k \). The eigenvalues \( \Lambda_k \) can be computed by directly substituting the eigenvectors (C.3) into the left-hand side of (C.4) and using the parity properties of the functions \( a(j) \) and \( b(j) \). We obtain
\[
\Lambda_k = \begin{cases} 
\sum_{j=-(M-1)/2}^{(M-1)/2} (a(j) - \gamma b(j)) e^{ikj} & \text{if } M \text{ is odd} \\
\sum_{j=-M/2-1}^{M/2-1} (a(j) - \gamma b(j)) e^{ikj} + (-1)^l a(M/2) & \text{if } M \text{ is even},
\end{cases} \tag{C.5}
\]
where \( k \) does not denote an integer but the wave number
\[
k = \frac{2\pi l}{M}, \quad l = 0, \ldots, M - 1.
\]

We now define the matrix
\[
(T_M)_{jk} = \sum_{l=0}^{M-1} \overline{\psi}_{lj} \phi_{lk}. \tag{C.6}
\]
Note that for convenience we have used the complex eigenvectors \((C.3)\), while the matrix \((B.20)\) is defined in terms of the real eigenvectors of \((A-B)(A+B)\) and \((A+B)(A-B)\). However, these are related by the transformations
\[
\phi_k \mapsto U \phi_k \quad \text{and} \quad \psi_k \mapsto U \psi_k
\]
with the same unitary matrix \(U\). This mapping leaves the right-hand side of equation \((C.6)\) unchanged. Therefore, the two matrices \((B.20)\) and \((C.6)\) coincide.

The matrix \((C.6)\) now becomes
\[
(T_M)_{jl} = \frac{1}{2\pi} \sum_{k=0}^{2\pi(1-1/M)} \frac{\Lambda_k}{|\Lambda_k|} e^{-ik(j-l)} \Delta_k. \tag{C.7}
\]
For \(M\) large enough there exists an integer \(n < M\) such that
\[
a(j) = b(j) = 0 \quad \text{for} \quad j > n.
\]
Therefore,
\[
\lim_{M \to \infty} \Lambda_k(M) = q(e^{i\theta}) = \sum_{j=-n}^{n} (a(j) - \gamma b(j)) e^{ij\theta}.
\]
By taking the limit as \(M \to \infty\) of the left-hand side of equation \((C.7)\) we obtain \((C.2)\).

**Appendix D. The Riemann constant \(K\)**

In this appendix we will show that the Riemann constant \(K\) is given by
\[
K = -\sum_{j=2}^{2n} \omega(\lambda_{2j-1}).
\]
As in [6], let \(Q_1, \ldots, Q_{2n-1}\) be the zeros of the theta function \(\theta(\omega(z))\). Then the function
\[
\theta(\omega(z) - \sum_{j=1} \omega(Q_j) - K)
\]
has the same zeros as \(\theta(\omega(z))\). Therefore, the quotient of these two functions can be written as an Abelian integral of a holomorphic 1-form \(\nu\):
\[
\frac{\theta(\omega(z) - \sum_{j=1} \omega(Q_j) - K)}{\theta(\omega(z))} = \int_z^v \nu.
\]
Moreover, all the \(a\)-periods of \(\nu\) must vanish. Thus, the right hand side of the above equation is in fact a constant \(C\):
\[
\frac{\theta(\omega(z) - \sum_{j=1} \omega(Q_j) - K)}{\theta(\omega(z))} = C.
\]
Therefore, we have
\[
\sum_{j=1}^{\omega(Q_j)} = -K.
\]

We will now compute the values of \( \omega(\lambda_i) \) in the basis \( a_1, \ldots, a_{2n-1}, b_1, \ldots, b_{2n-1} \) and show that the \( 2n - 1 \) points \( \lambda_3, \ldots, \lambda_{4n-1} \) are the zeros of \( \theta(\omega(z)) \). We have
\[
\begin{align*}
\omega_j(\lambda_{2k+1}) &= \frac{1}{2} \Pi_{j,k}, \quad 0 < j < k \leq 2n - 1 \\
\omega_j(\lambda_{2k+1}) &= -\frac{1}{2} + \frac{1}{2} \Pi_{j,k}, \quad 0 < k \leq j \leq 2n - 1 \\
\omega_j(\lambda_k) &= \frac{1}{2} \Pi_{j,k-1}, \quad 0 < j < k \leq 2n \\
\omega_j(\lambda_k) &= -\frac{1}{2} + \frac{1}{2} \Pi_{j,k-1}, \quad 1 < k \leq j \leq 2n.
\end{align*}
\]

If we write \( \omega(\lambda_i) \) as
\[
\omega(\lambda_i) = \frac{1}{2} N_i + \frac{1}{2} PM_i,
\]
then, from the periodicity (6.6) of the theta function, we have
\[
\theta(\omega(\lambda_i)) = \exp(-\pi i \langle N_i, M_i \rangle) \theta(-\omega(\lambda_i)).
\]

Since \( \langle N_{2i+1}, M_{2i+1} \rangle \) are odd for \( 1 \leq i \leq 2n - 1 \), we see that \( \theta(\omega(\lambda_{2i+1})) = 0 \) and hence the \( g \) zeros of \( \theta(\omega(z)) \) are the points \( \lambda_3, \ldots, \lambda_{4n-1} \). Therefore, we have
\[
K = -\sum_{j=2}^{2n} \omega(\lambda_{2j-1}).
\]

**Appendix E. The cycle basis (10.18)**

In this appendix we will show that the basis defined in (10.18) are canonical. First note that, by direct computation, it is easy to check that the intersections between the \( a \)-cycles are zero
\[
\tilde{a}_{n-j-1} \cdot \tilde{a}_{n+l} = 0, \quad 0 \leq j, l \leq r - 1.
\]

We will now compute the other intersection numbers by induction.
First let us compute the intersection numbers between the tilded basis and the untilded basis. We have

\[ a_{n-k-1} \cdot \tilde{a}_{n-j-1} = \delta_{k,j} \]  (E.1)
\[ a_{n-k-1} \cdot \tilde{a}_{n+j} = -\delta_{k,j} \]  (E.2)
\[ a_{n+k} \cdot \tilde{a}_{n-j-1} = -\delta_{k,j} \]  (E.3)
\[ a_{n+k} \cdot \tilde{a}_{n+j} = \delta_{k,j} \]  (E.4)

\[ a_{n-k-1} \cdot \tilde{b}_{n-j-1} = \begin{cases} 1, & k = j; \\ 2(-1)^{k-j}, & j + 1 \leq k; \\ 0, & 0 \leq k \leq j - 1. \end{cases} \]  (E.5)
\[ a_{n-k-1} \cdot \tilde{b}_{n+j} = \begin{cases} 0, & 0 \leq k \leq j; \\ 2(-1)^{k-j}, & j + 1 \leq k. \end{cases} \]  (E.6)
\[ a_{n+k} \cdot \tilde{b}_{n-j-1} = 0 \]  (E.7)
\[ a_{n+k} \cdot \tilde{b}_{n+j} = \delta_{k,j} \]  (E.8)

\[ b_{n+k} \cdot \tilde{a}_{n-j-1} = \begin{cases} -1, & 0 \leq k \leq j - 1; \\ 0, & j \leq k. \end{cases} \]  (E.9)
\[ b_{n-k-1} \cdot \tilde{a}_{n-j-1} = \begin{cases} -1, & 0 \leq k \leq j - 1; \\ 0, & j \leq k. \end{cases} \]  (E.10)
\[ b_{n+k} \cdot \tilde{a}_{n+j} = \begin{cases} -1, & 0 \leq k \leq j; \\ 0, & j + 1 \leq k. \end{cases} \]  (E.11)
\[ b_{n-k-1} \cdot \tilde{a}_{n+j} = \begin{cases} -1, & 0 \leq k \leq j; \\ 0, & j + 1 \leq k. \end{cases} \]  (E.12)
\[ b_{n+k} \cdot \tilde{b}_{n-j-1} = \begin{cases} 1, & 0 \leq k \leq j - 1; \\ 0, & j \leq k. \end{cases} \]  (E.13)
\[ b_{n-k-1} \cdot \tilde{b}_{n-j-1} = \begin{cases} 1, & 0 \leq k \leq j; \\ (-1)^{k-j}, & j + 1 \leq k. \end{cases} \]  (E.14)
\[ b_{n-k-1} \cdot \tilde{b}_{n+j} = \begin{cases} 0, & 0 \leq k \leq j; \\ (-1)^{k-j}, & j + 1 \leq k. \end{cases} \]  (E.15)
\[ b_{n+k} \cdot \tilde{b}_{n+j} = 0 \]  (E.16)

where \( j, k \) range from 0 to \( r - 1 \).

Now, we have

\[ \tilde{b}_{n+r-1} = b_{n+r-1}. \]

Then from (E.9)-(E.16), we obtain the following intersection numbers:

\[ \tilde{b}_{n+r-1} \cdot \tilde{a}_j = \delta_{n+r-1,j}, \quad \tilde{b}_{n+r-1} \cdot \tilde{b}_j = 0. \]

Next, from (10.18) we have

\[ \tilde{b}_{n+k} + \tilde{b}_{n+k-1} = b_{n+k} + b_{n+k-1} + a_{n-k-1} - 2b_{n-k-1}. \quad k = 1, \ldots, r - 1 \]
From this relation and equation (E.1)-(E.16), we obtain
\[
\begin{align*}
(\tilde{b}_{n+k} + \tilde{b}_{n+k-1}) \cdot \tilde{a}_j &= -\delta_{j,n+k} - \delta_{j,n+k-1} \\
(\tilde{b}_{n+k} + \tilde{b}_{n+k-1}) \cdot \tilde{b}_j &= 0, \quad j = 1, \ldots, 2n - 1
\end{align*}
\]
Therefore, if we assume that \(\tilde{b}_{n+k}\) has the intersection numbers
\[
\begin{align*}
\tilde{b}_{n+k} \cdot \tilde{a}_j &= -\delta_{j,n+k} \\
\tilde{b}_{n+k} \cdot \tilde{b}_j &= 0, \quad j = 1, \ldots, 2n - 1,
\end{align*}
\]
then \(\tilde{b}_{n+k-1}\) will have the intersection numbers
\[
\begin{align*}
\tilde{b}_{n+k-1} \cdot \tilde{a}_j &= -\delta_{j,n+k-1} \\
\tilde{b}_{n+k-1} \cdot \tilde{b}_j &= 0, \quad j = 1, \ldots, 2n - 1, \quad 1 \leq k
\end{align*}
\]
Therefore, by induction we see that
\[
\begin{align*}
\tilde{b}_{n+k} \cdot \tilde{a}_j &= -\delta_{j,n+k} \\
\tilde{b}_{n+k} \cdot \tilde{b}_j &= 0, \quad j = 1, \ldots, 2n - 1, \quad k = 0, \ldots, r - 1. \quad (E.17)
\end{align*}
\]
We can now compute the intersection numbers of the \(\tilde{b}_{n-k-1}\). We have
\[
\tilde{b}_{n-k-1} - \tilde{b}_{n+k} = -\tilde{a}_n + a_{n+k}. \quad k = 0, \ldots, r - 1
\]
Therefore, by using (E.1)-(E.17) we obtain
\[
\begin{align*}
(\tilde{b}_{n+k} - \tilde{b}_{n-k-1}) \cdot \tilde{a}_j &= -\delta_{j,n+k} + \delta_{j,n-k-1} \\
(\tilde{b}_{n+k} - \tilde{b}_{n-k-1}) \cdot \tilde{b}_j &= 0, \quad j = 1, \ldots, 2n - 1
\end{align*}
\]
From (E.17), we see that the intersection numbers for the \(\tilde{b}_{n-k-1}\) are indeed given by
\[
\begin{align*}
\tilde{b}_{n-k-1} \cdot \tilde{a}_j &= -\delta_{j,n-k-1} \\
\tilde{b}_{n-k-1} \cdot \tilde{b}_j &= 0, \quad j = 1, \ldots, 2n - 1, \quad k = 0, \ldots, r - 1.
\end{align*}
\]

**Appendix F. Solvability of the Wiener-Hopf factorization problem**

We now show that the Wiener-Hopf factorization problem (5.3) is solvable when \(\beta(\lambda)\) is purely imaginary.

In other words, we have
Theorem 9. The following Riemann-Hilbert problem
\[
T_+(z) = \Phi(z)T_-(z), \quad |z| = 1
\]
\[
\Phi(z) = \begin{pmatrix}
  i\lambda & g(z) \\
  -g^{-1}(z) & i\lambda
\end{pmatrix}
\]
\hspace{1cm} (F.1)
where $T_+(z)$ is holomorphic for $|z| < 1$ and $T_-(z)$ is holomorphic for $|z| > 1$ with $T_-(\infty) = 1$ is solvable when $\beta(\lambda) \in i\mathbb{R}$.

Proof. We will use the vanishing lemma to prove this theorem. As in [8], we need to show that a certain singular integral operator is a bijection.

The solvability of the Riemann-Hilbert problem is related to the bijectivity of a singular integral operator. Let $C$ be the Cauchy operator
\[
C(f)(z) = \frac{1}{2\pi i} \int_{\Xi} \frac{f(s)}{s-z} \mathrm{d}s, \quad f \in L^2(\Xi)
\]
and let $C_+, C_-$ be its limit on the positive and negative side of the real axis
\[
C_+(f)(z) = \lim_{\epsilon \to 0} C(f)(z \pm i\epsilon), \quad z \in \Xi.
\]

Now, define the singular integral operator $C_\Phi$ as in [8],
\[
C_\Phi(f) = C_+ (f(I - \Phi^{-1}))
\]
\hspace{1cm} (F.2)
Suppose that $I - C_\Phi$ is invertible in $L^2(\Xi)$, and let $\mu = (I - C_\Phi)^{-1}C_+/I - \Phi^{-1})$: then the function
\[
\hat{T}(z) = I + C_+ ((I + \mu)(I - \Phi^{-1}))
\]
is a solution to the Riemann-Hilbert problem [F.1]. In fact, we have
\[
\hat{T}_+(z) = I + C_+(I - \Phi^{-1}) + C_\Phi \mu = I + \mu(z)
\]
\[
\hat{T}_-(z) = \hat{T}_+(z) - I - \mu(z) + \Phi^{-1}(z)(I + \mu(z)) = \Phi^{-1}(z)\hat{T}_+(z), \quad |z| = 1,
\]
where the second equation follows from the identity $C_+ - C_- = I$.

Therefore, in order to show that [F.1] is solvable when $\beta(\lambda) \in i\Xi$, we need to show that $I - C_\Phi$ is invertible in $L^2(\Xi)$.

Using standard analysis (see, e.g., [8]), we can show that the operator $C_\Phi$ is Fredholm and has index zero. Therefore we only need to show that its kernel is $\{0\}$.

Suppose that the kernel is non-trivial and let $(I - C_\Phi)\mu_0 = 0$. Then the function
\[
\hat{T}_0(z) = C [\mu_0(I - \Phi^{-1})]
\]
will solve the Riemann-Hilbert problem [F.1], but its asymptotic behavior will be
\[
\hat{T}_0(z) = O(z^{-1}), \quad z \to \infty.
\]
This means that the function \( R(z) = \hat{T}_0^\dagger(z^{-1})\hat{T}_0(z) \), where \( A^\dagger \) is the Hermitian conjugate of \( A \), is analytic outside the unit circle and behaves like \( O(z^{-2}) \) at infinity. Thus, by Cauchy’s theorem, we have

\[
\int_{\Xi} R_-(z)dz = 0.
\]

By making use of the jump conditions, we obtain

\[
\int_{\Xi} R_-(z)dz = \int_{\Xi} \left( \hat{T}_0^\dagger(z) \right)_+ \left( \hat{T}_0(z) \right)_- dz
= \int_{\Xi} \left( \hat{T}_0^\dagger(z) \right)_- \Phi^\dagger(z) \left( \hat{T}_0(z) \right)_- dz = 0 \quad (F.3)
\]

From (6.2), we see that the eigenvalues of \( \Phi(z) \) are \( i(\lambda + 1) \) and \( i(\lambda - 1) \). Therefore the matrix \( i\Phi(z) \) is Hermitian and is either positive definite or negative definite for \( \beta(\lambda) \in i\mathbb{R} \). This means that the boundary value of \( \left( \hat{T}_0(z) \right)_- \) on the unit circle is zero. In particular, it implies that \( \hat{T}_0(z) = 0 \) and hence the kernel of the singular integral operator \( I - C_\Phi \) is trivial. This concludes the proof of the theorem.

\[
\square
\]

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10 April 2008