A first-order optimization algorithm for statistical learning with hierarchical sparsity structure

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Abstract

In many statistical learning problems, it is desired that the optimal solution conforms to an a priori known sparsity structure e.g. for better interpretability. Inducing such structures by means of convex regularizers requires nonsmooth penalty functions that exploit group overlapping. Our study focuses on evaluating the proximal operator of the Latent Overlapping Group lasso developed by Jacob et al. [18]. We develop an Alternating Direction Method of Multiplier with a sharing scheme to solve large-scale instance of the underlying optimization problem efficiently. In the absence of strong convexity, linear convergence of the algorithm is established using the error bound theory. More specifically, the paper contributes to establishing primal and dual error bounds over an unbounded feasible set and when the nonsmooth component in the objective function does not have a polyhedral epigraph. Numerical simulation studies supporting the proposed algorithm and two learning applications are discussed.

Keywords— Proximal methods, error bound theory, Alternating Direction Method of Multipliers, hierarchical sparsity structure, latent overlapping group lasso.

1 Introduction

Convex sparsity-inducing regularization functions play an important role in different fields including machine learning, statistics, and signal processing [15]. Some well-known regularizers e.g. lasso [34] or group lasso [43] are commonly used in different learning frameworks to induce sparsity which allows simultaneous model fitting and feature selection. In contrast to lasso which assumes no a priori knowledge on sparsity pattern, group lasso assumes that variables belong to a priori known groups and the variables within a group tend to affect response similarly, i.e., all are simultaneously zero or none zero. This introduced the literature to more elaborate forms of zero/nonzero
patterns known as *structured sparsity* [2]. The focus of this paper is to address structured sparsities represented by a Directed Acyclic Graph (DAG) and the convex Latent Overlapping Group (LOG) lasso regularizer of [18] to induce such structure. To be more specific, we develop an optimization framework that allows incorporating this regularizer in large-scale learning problems for huge DAGs. In the remaining of this section, we discuss hierarchical structured sparsity following a DAG, convex regularizers to induce hierarchical sparsity structures, and the proximal mapping of the latent overlapping group lasso regularizer.

### 1.1 Hierarchical structured sparsity

Let $D = (S, E)$ be a DAG where $S = \{s_1, ..., s_N\}$ is the index set of nodes, and $E$ is the set of ordered pairs of node indices with an edge from the first to second element, e.g. $(s_i, s_j)$ is an edge from $s_i$ to $s_j$. Furthermore, let each node $i$ of the graph contains a set of $d_i$ variables where their indices are contained in $s_i$. We will refer to the variables in node $i$ by $\beta_{s_i}$.

The graph topology contains the sparsity structure between groups of variables in each node. For instance, consider the following two DAGs and assume the variables in a single node will be all simultaneously non-zero or all zero. The sparsity hierarchy introduced by the graph in Figure 1a is $(s_1 = 0) \Rightarrow (s_2 = 0)$ and $(s_2 \neq 0) \Rightarrow (s_1 \neq 0)$. Note that $s_j = 0$ means that all of the variables in $s_j$ are equal to zero; similarly, $(s_j \neq 0)$ means all of the variables in $s_j$ are nonzero. However, there are scenarios where a node has more than one ancestor, e.g. node $s_3$ in Figure 1b. Such scenarios can potentially be interpreted in two different ways. Under strong hierarchy assumption, all of the immediate ancestor nodes need to be nonzero for their descendent node to be nonzero, e.g., in Figure 1b $(s_3 \neq 0) \Rightarrow (s_1 \neq 0, s_2 \neq 0)$, and $(s_1 = 0 \text{ or } s_2 = 0) \Rightarrow (s_3 = 0)$. Under weak hierarchy assumption, for a descendent node to be nonzero it suffices that any of its immediate ancestors be nonzero, e.g., in Figure 1b $(s_3 \neq 0) \Rightarrow (s_1 \neq 0 \text{ or } s_2 \neq 0)$, and $(s_1 = 0, s_2 = 0) \Rightarrow (s_3 = 0)$ [10].

We are interested in statistical learning problems that require their solutions to follow given sparsity structures in form of DAGs. To be more specific, given a DAG $D$, we want to solve

$$
\min_{\beta} \{ \mathcal{L}(\beta) \mid \beta \in B, \ \text{supp}(\beta) \in D \} 
$$

(1)

where $\mathcal{L} : \mathbb{R}^d \to \mathbb{R}$ is a smooth, convex or nonconvex loss function with $d = \sum_{i=1}^N d_i$ being the problem dimension, $B \subseteq \mathbb{R}^d$ is a closed convex set, and with a slight abuse of notation, $\text{supp}(\beta) \in D$ denotes that the support of $\beta$ (index set of its nonzero elements) follows $D$ in the strong sense. One way to formulating $\text{supp}(\beta) \in D$ explicitly requires introducing binary variables. For instance, assuming one variable per node, formulating the hierarchy in Figure 1a shall be performed as

$$
z \epsilon \leq |\beta_1|, \ |\beta_2| \leq z \mu, \ z \in \{0, 1\},$$

Figure 1: Two Directed Acyclic Graphs (DAG)
where $\epsilon$ and $\mu$ are reasonably small and large numbers, respectively. Introduction of binary variables makes the optimization problem a Mixed Integer Program (MIP) [11] - see also [3, 4]. Finding the global optimal solution of large-scale MIPs for large DAGs is generally computationally challenging. We, however, would like to note some significant advances in MIP algorithms for statistical learning, specifically for feature selection) – see e.g. [8, 23, 1, 24, 7, 9]. Similar to using $\ell_1$ norm as a convex approximation to $\ell_0$ (pseudo) norm to induce sparsity, there are convex regularizers that promote hierarchical sparsity structures. Needless to mention, these approximation methods do not guarantee exact conformance of their solutions to given hierarchies, but, they allow solving high-dimensional problems.

### 1.2 Group Lasso with overlaps vs. Latent Overlapping Group lasso

There are mainly two convex regularizers to introduce hierarchical structured sparsity: 1. Group Lasso (GL) 2. Latent Overlapping Group Lasso (LOG) [42]. Given a set of groups of variables $G$, the GL regularizer is defined as

$$\Omega_{GL}(\beta) = \sum_{g \in G} w_g \|\beta_g\|$$  \hspace{1cm} (2)

where $w_g$ is a positive weight corresponding to group $g$ and $\beta_g \in \mathbb{R}^{|g|}$ is equal to $\beta$ for elements whose indices belongs to $g$ and zero for other elements, and the $\| \cdot \|$ is either an $\ell_2$ or $\ell_{\infty}$ norm. To induce hierarchical sparsity structure using the GL penalty, the groups should be defined in a descendants form, for instance, for the graph in Figure 1b the groups should be $G = \{s_3, s_4, \{s_1, s_3\}, \{s_2, s_3, s_4\}\}$ where $s_3 = \text{descendants}(D; s_3), \{s_1, s_3\} = \text{descendants}(D; s_1), s_4 = \text{descendants}(D; s_4), \text{and } \{s_2, s_3, s_4\} = \text{descendants}(D; s_2)$. The group lasso sets to zero a union of a subset of groups introduced in $G$. However, since there are overlaps between the groups defined in $G$, the support of the solution induced by GL is not necessarily a union of the groups. This is because of the fact that the complement of a union of a subset of groups is not necessarily a union of groups.

As an alternative to GL, Jacob et al. [18] introduced Latent Overlapping Group lasso (LOG) regularizer which is defined as

$$\Omega_{LOG}(\beta) = \inf_{\nu^{(g)}, \ g \in G} \left\{ \sum_{g \in G} w_g \|\nu^{(g)}\|_2 \text{ s.t. } \sum_{g \in G} \nu^{(g)} = \beta, \nu^{(g)}_{g^c} = 0 \right\}$$  \hspace{1cm} (3)

which sets to zero a subset of groups. Since $\beta$ is the sum of latent variables $\nu^{(g)} \in \mathbb{R}^d$, its support is the union of the groups of nonzero latent variables. Given a DAG $D$ with $N$ nodes, there exist $N$ groups in $G$ (i.e., $N = |G|$). To induce a hierarchical sparsity using the LOG penalty the group corresponding to each node contains the node indices of all its ancestors, i.e., $G = \text{ancestors}(D)$. For instance, the group set for the graph in Figure 1b is $G = \{s_1, s_2, \{s_1, s_2, s_3\}, \{s_2, s_4\}\}$ where $s_1 = \text{ancestors}(D; s_1), s_2 = \text{ancestors}(D; s_2), \{s_1, s_2, s_3\} = \text{ancestors}(D; s_3), \text{and } \{s_2, s_4\} = \text{ancestors}(D; s_4)$. Figure 2a shows a simple tree with three nodes, and the ancestor grouping scheme; Figure 2b shows the latent variables within the constraint in the LOG penalty. Recently, Yan et al. [42] performed a detailed comparison of GL vs. LOG regularizers. They showed that compared to LOG, GL sets to zero parameters which are deeper in the hierarchy. Hence, for DAGs with deep hierarchies it is very probable that GL sets to zero deeper variables which is undesirable. Furthermore, with LOG penalty, one has control over the solution support as it is a subset of columns of latent variables.
Figure 2: LOG penalty and the required groups to induce a tree structure

In the next section, we discuss solving statistical learning problems in the regularized form using the nonsmooth LOG penalty and propose solving them using proximal methods.

1.3 Proximal operator of the LOG penalty

Given a hierarchical sparsity structure represented by a graph $\mathcal{G}$, an approximate convex optimization problem to (1) is

$$\min_{\beta} \left\{ L(\beta) + \lambda \Omega_{\text{LOG}}(\beta) \quad \text{s.t.} \quad \beta \in \mathbb{R}^d \right\}$$

(4)

where $\mathcal{G}$ is a convex or nonconvex, differentiable loss function with Lipschitz continuous gradient, $\Omega_{\text{LOG}}$ is the LOG penalty introduced above with appropriately chosen groups, $\lambda > 0$ is a parameter that controls the tradeoff between the loss function and penalty, and $\mathcal{G} \subseteq \mathbb{R}^d$ is a closed convex set. Indeed, problem (4) is a convex nonsmooth program; hence, proximal methods are suitable to solve large instances of this problem [25, 5, 28].

Similar to gradient methods that require iterative evaluation of the gradient, proximal methods require iterative evaluation of the proximal operator [28]. Proximal operator of a function $\lambda \Omega(\beta)$ in general ($\lambda \Omega_{\text{LOG}}$ in our case) evaluated at $b \in \mathbb{R}^d$ is defined as

$$\text{prox}_{\lambda \Omega}(b) \triangleq \arg\min_{\beta \in \mathbb{R}^d} \left\{ \lambda \Omega(\beta) + \frac{1}{2} \|\beta - b\|^2_2 \right\}.$$  

(5)

Using the definition of $\Omega_{\text{LOG}}$, evaluating the proximal operator of the LOG penalty requires solving

$$\min_{\nu^{(g)} \in \mathbb{R}^d} \left\{ \lambda \sum_{g \in \mathcal{G}} w_g \|\nu^{(g)}\|_2 + \frac{1}{2} \sum_{g \in \mathcal{G}} \|\nu^{(g)} - b\|_2^2 \quad \text{s.t.} \quad \nu^{(g)} = 0, \; \forall g \in \mathcal{G} \right\},$$

(6)

over the latent variables $\nu^{(g)}$, $g \in \mathcal{G}$. A classic method to solve (6) is some implementation of the Block Coordinate Descent (BCD) algorithm [26, 39] – see Algorithm 1. Provided an algorithm to evaluate $\text{prox}_{\lambda \Omega_{\text{LOG}}}$ efficiently, one may use a proximal optimization method e.g. proximal gradient method [5, 25, 28] to solve (4). The main challenge is to evaluate the proximal operator of the LOG penalty (5) for large DAGs with large $|\mathcal{G}|$. The main drawbacks of the BCD algorithm to solve (5) are as follows. First, even though convergence of the BCD algorithm for nondifferentiable but separable functions has been established [35], to the best of our knowledge, the convergence rate of
Algorithm 1 Block Coordinate Descent (BCD) to solve $\text{prox}_{\Omega_{\text{LOG}}}(b)$

Require: $b$, $\lambda$, $w$, $G$

1: $\beta = 0$
2: $\nu^{(g)} = 0$, $\forall g \in G$
3: while stopping criterion not met do
4: for $g \in G$ do
5: $\beta \leftarrow \beta - \nu^{(g)}$
6: $\nu^{(g)} \leftarrow S_G(b_g - \beta_g, \lambda w_g)$
7: $\beta \leftarrow \beta + \nu^{(g)}$
8: end for
9: end while

Output: $\beta$

the algorithm for nonsmooth optimization is sublinear \[37, 30\]. Second, the BCD algorithm follows a Gauss-Siedel update rule and hence it cannot be parallelized. In this work, we introduce an efficient first-order method, based on Douglas-Rachford operator splitting, that can solve (5) over large graphs with fast, i.e., linear, rate of convergence.

1.4 Our contributions

A summary of our main contributions are as follows:

- We propose an ADMM algorithm with a sharing scheme to solve $\text{prox}_{\Omega_{\text{LOG}}}$ defined over large DAGs. The underlying DAG may have any general structure (e.g. not necessary to be a path graph) and the algorithm is guaranteed to converge to its optimal solution. Furthermore, the computationally itense subproblem of the algorithm (step 4 in Algorithm 3) can be run fully in parallel.

- We proved linear convergence of the algorithm given a sufficiently small stepsize in the absence of strong convexity. Establishing the linear convergence rate is based on the error bound theory (see e.g.\[36, 20, 44, 17\]).

  - The dual error bound is established in the presence of $\ell_2$-norm in the nonsmooth component of the objective function. A common key assumption in previous works requires the nonsmooth component to have a polyhedral epigraph. Our proof shows an approach to escape such assumption and enables further extension.

  - On the primal side, we rigorously prove the error bound for the augmented Lagrangian function. The main challenges are the presence of the dual variable $y$ and the splitting of $x$ into two blocks, i.e., $x^1$ and $x^2$, which requires a number of technical details, e.g., comparing the limiting behavior of $x^{1,k}$ and $x^{2,k}$. To the best of our knowledge, this is the first work showing error bound for augmented Lagrangian function with blockwise variables.

  - As another auxiliary lemma to show the linear rate of convergence, and using the proof in \[13\], we formally show the uniform boundedness (with respect to the stepsize) of both primal and dual updates generated by the algorithm.
The rest of the paper is organized as follows. In Section 2, we propose an ADMM algorithm with the sharing scheme to evaluate the proximal operator of the LOG penalty. Section 3 provides detailed convergence analysis of the proposed algorithm while some proofs are relegated to the Appendix. Section 4 contains two applications that use LOG penalty to induce sparsity structure related to topic modeling and breast cancer classification. Finally, Section 5 provides some concluding remarks.

1.5 Notation
Vectors are denoted by lowercase bold letter while matrices are denoted by uppercase letters. The identity matrix is denoted by \( \mathbf{I} \). Given a vector \( \mathbf{g} \subseteq \{1, \ldots, d\} \), \( \beta_{\mathbf{g}} \in \mathbb{R}^{|\mathbf{g}|} \) subsets \( \beta \) over the set \( \mathbf{g} \). We denote the \( j \)-th column of the matrix \( A \) by \( A_j \), similarly we denote \( A_i \). Furthermore, similar to the vector case, if \( \mathbf{g} \subseteq \{1, \ldots, n\} \), then \( A_{\mathbf{g}} \in \mathbb{R}^{n \times |\mathbf{g}|} \) subsets \( A \) over the columns indexed \( \mathbf{g} \). Inner product of two vectors is defined as \( \langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a}^\top \mathbf{b} \).

2 Evaluating the proximal operator of the LOG
Consider the optimization problem (6) to find the proximal operator of the LOG penalty. Given a general DAG \( \mathcal{G} \), each iteration of the BCD algorithm requires updating \( \lambda_g \), \( \forall g \in \mathcal{G} \). If \( |\mathcal{G}| \) is a large number, then per iteration complexity of the algorithm \( \mathcal{O}(\sum_{g \in \mathcal{G}} |g|) \) is costly. Furthermore, to the best of authors knowledge, convergence rates of the BCD algorithm for general nonsmooth optimization problems have not been well studied. However, convergence of the algorithm for problems where the nondifferentiable part is separable is established, cf. [35].

In this paper, we develop an Alternating Direction Method of Multiplier (ADMM) to solve the proximal operator of the LOG penalty. The algorithm is parallelizable which makes it suitable with the sharing scheme to evaluate the proximal operator of the LOG penalty. The algorithm is parallelizable which makes it suitable when the number of groups is very large. Define \( \mathbf{x} = [\lambda_g]_{g \in \mathcal{G}} \in \mathbb{R}^n \), \( n = \sum_{g \in \mathcal{G}} |g| \), be a long vector that contains the nonzero elements of \( \lambda_g \), \( g \in \mathcal{G} \), for some random order \( \mathcal{P} \) of the groups. Furthermore, let \( j(\cdot) : \mathcal{G} \to \{1, \ldots, n\} \) be the set map that associate a group \( g \in \mathcal{G} \) to its indices in vector \( \mathbf{x} \) given an order of the groups. For instance, for \( \mathcal{G} = \{\{1\}, \{1, 2\}, \{1, 3\}\} \) ordered as \( \mathcal{P} \) from left-to-right: 1. \( n = 5 \), 2. \( j_{\mathcal{P}}(\{1\}) = \{1\} \), \( j_{\mathcal{P}}(\{1, 2\}) = \{2, 3\} \), and \( j_{\mathcal{P}}(\{1, 3\}) = \{4, 5\} \). To simplify the notation, the group ordering \( \mathcal{P} \) is omitted. Finally, (6) can equivalently be written as

\[
\min_{\mathbf{x} \in \mathbb{R}^n} \lambda \sum_{g \in \mathcal{G}} \|W_g \mathbf{x}\|_2 + \frac{1}{2} \|M \mathbf{x} - \mathbf{b}\|_2^2, \tag{7}
\]

where \( W^g = w_g U^j(g) \), \( U^j(g) \in \mathbb{R}^{|g| \times n} \) such that \( [U^j(g)]_i g \in \mathcal{G} = \mathbf{I} \in \mathbb{R}^{n \times n} \), and \( M \in \mathbb{R}^d \times n \) sums elements of \( \mathbf{x} \) along each coordinate. Or, equivalently, (6) can be written as

\[
\min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x}) \triangleq \lambda \sum_{g \in \mathcal{G}} w_g \|x_{j(g)}\|_2 + \frac{1}{2} \|M \mathbf{x} - \mathbf{b}\|_2^2. \tag{8}
\]

Problem (8) is a convex (not strongly convex since \( M \) is not a full-column rank matrix), nonsmooth optimization program. This problem can also be solved using the proximal gradient method [44, 38].
We propose to solve this problem using the Alternating Direction Method of Multipliers (ADMM). First, splitting the problem into two blocks \( \Pi \), we have

\[
\min_{x^1, x^2 \in \mathbb{R}^n} \left\{ F(x^1, x^2) \triangleq \lambda \sum_{g \in \mathcal{G}} w_g \| x^1_{j(g)} \|_2 + \frac{\rho}{2} \| M x^2 - b \|_2^2, \quad \text{s.t.} \quad x^1 = x^2 \right\}. \tag{9}
\]

The augmented Lagrangian function for (9) is

\[
L_\rho(x^1, x^2, y) = \lambda \sum_{g \in \mathcal{G}} w_g \| x^1_{j(g)} \|_2 + \frac{\rho}{2} \| M x^2 - b \|_2^2 + \langle y, x^1 - x^2 \rangle + \frac{\rho}{2} \| x^1 - x^2 \|_2^2, \tag{10}
\]

where \( y \in \mathbb{R}^n \) is the Lagrange multiplier for the linear constraint \( x^1 = x^2 \), and \( \rho \geq 0 \) is a constant. Furthermore, the augmented dual function is given by

\[
g_\rho(y) = \min_{x^1, x^2 \in \mathbb{R}^n} \lambda \sum_{g \in \mathcal{G}} w_g \| x^1_{j(g)} \|_2 + \frac{\rho}{2} \| M x^2 - b \|_2^2 + \langle y, x^1 - x^2 \rangle + \frac{\rho}{2} \| x^1 - x^2 \|_2^2, \tag{11}
\]

which results into the the dual problem

\[
\max_{y \in \mathbb{R}^n} g_\rho(y). \tag{12}
\]

The ADMM iterates in the unscaled form (10) are

\[
x^{1,k+1}_{j(g)} \leftarrow \min_{x^1 \in \mathbb{R}^n} \lambda w_g \| x^1_{j(g)} \|_2 + \frac{\rho}{2} \| x^1_{j(g)} - x^{2,k}_{j(g)} \|_2^2, \quad \forall g \in \mathcal{G}, \tag{13}
\]

\[
x^{2,k+1} \leftarrow \min_{x^2 \in \mathbb{R}^n} \frac{1}{2} \| M x^2 - b \|_2^2 + \frac{\rho}{2} \| x^2 - x^{1,k+1} \|_2^2 - \frac{1}{\rho} \| y^{k+1} \|_2^2, \tag{14}
\]

\[
y^{k+1}_{j(g)} \leftarrow y^{k}_{j(g)} + \alpha(x^{1,k+1}_{j(g)} - x^{2,k+1}_{j(g)}), \quad \forall g \in \mathcal{G}, \tag{15}
\]

where \( \alpha \) is the dual stepsize. Algorithm 3 illustrates the resulting (unscaled) ADMM algorithm to evaluate the proximal map of the LOG penalty. Note that the subproblem (13) is parallelizable across groups, and the solution to each subproblem is available in the closed form. However, even though the update (14) has a closed-form solution, it involves inverting an \( n \times n \) matrix which generally requires \( O(n^3) \) operations, per iteration. Hence, for large DAGs where \( |\mathcal{G}| \) and hence \( n \) is large, the second update is very slow.

To deal with this issue, below we propose a sharing scheme that helps solving the second subproblem efficiently. First, we put the variables in the matrix form. Define \( X \in \mathbb{R}^{d \times |\mathcal{G}|} \) be a matrix that stacks \( x^{(g)} \), \( g \in \mathcal{G} \) where its columns are indexed by \( g \in \mathcal{G} \). Problem (6) can be written in the matrix form as

\[
\min_{X \in \mathbb{R}^{d \times |\mathcal{G}|}} \left\{ \lambda \sum_{g \in \mathcal{G}} w_g \| X_g \|_2 + \frac{1}{2} \| \sum_{g \in \mathcal{G}} X_g - b \|_2^2, \quad \text{s.t.} \quad (X_g)_g = 0 \forall g \in \mathcal{G} \right\}. \tag{16}
\]

Splitting the problem into two blocks, the problem is equivalent to

\[
\min_{X^1, X^2 \in \mathbb{R}^{d \times |\mathcal{G}|}} \left\{ \lambda \sum_{g \in \mathcal{G}} w_g \| X^1_g \|_2 + \frac{1}{2} \| \sum_{g \in \mathcal{G}} X^2_g - b \|_2^2, \quad \text{s.t.} \quad X^1 = X^2, \quad (X^1_g)_g = 0 \forall g \in \mathcal{G} \right\}. \tag{17}
\]
β

Furthermore, using (22) in (20), we get

in the matrix form - see [11] for details) to solve (17) are

The ADMM iterates in the scaled form (through defining \( U := (1/\rho)Y \) where \( Y \) is the dual variable in the matrix form - see [11] for details) to solve (17) are

\[
X^{1,k+1}_g \leftarrow \arg\min_{X^1_g \in \mathbb{R}^d} \left\{ \lambda w_g \|X^1_g\|_2 + \frac{\rho}{2} \|X^1_g - X^{2,k}_g + U^{k}_{g}\|_2^2 \right\} \quad \text{s.t.} \quad (X^1_g)_g = 0, \quad \forall g \in \mathcal{G},
\]

(18)

\[
X^{2,k+1}_g \leftarrow \arg\min_{X^2 \in \mathbb{R}^{d \times |\mathcal{G}|}} \frac{1}{2} \|\sum_{g \in \mathcal{G}} X^2_g - b\|_2^2 + \frac{\rho}{2} \sum_{g \in \mathcal{G}} \|X^2_g - X^{1,k+1}_g - U^{k}_g\|_2^2,
\]

(19)

\[
U^{k+1}_g \leftarrow U^k_g + (\alpha/\rho)(X^{1,k+1}_g - X^{2,k+1}_g), \quad \forall g \in \mathcal{G},
\]

(20)

Subproblem (18) is equivalent to solve

\[
\min_{X^{1,k+1}_g \in \mathbb{R}^d, \bar{x}^2 \in \mathbb{R}^d} \left\{ \frac{1}{2} \|g\bar{x}^2 - b\|_2^2 + \frac{\rho}{2} \sum_{g \in \mathcal{G}} \|X^{1,k+1}_g - U^{k}_g\|_2^2 \right\} \quad \text{s.t.} \quad \bar{x}^2 = (1/|\mathcal{G}|) \sum_{g \in \mathcal{G}} X^{2}_g.
\]

(21)

Minimizing over \( X^{2}_g \) with \( \bar{x}^2 \) fixed and using optimality conditions, we get

\[
X^{2}_g = \bar{x}^2 + X^{1,k+1}_g + U^{k}_g - (1/|\mathcal{G}|) \sum_{g \in \mathcal{G}} (X^{1,k+1}_g + U^{k}_g), \quad \forall g \in \mathcal{G}.
\]

(22)

Using (22) to solve (19) we get

\[
\bar{x}^2 = \frac{1}{|\mathcal{G}|} + \frac{\rho}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} (X^{1,k+1}_g + U^{k}_g).
\]

(23)

Furthermore, using (22) in (20), we get

\[
U^{k+1}_g = U^k_g + (\alpha/\rho)(\frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} (X^{1,k+1}_g + U^{k}_g) - \bar{x}^2), \quad \forall g \in \mathcal{G}.
\]

(24)

\[8\]
Algorithm 3 ADMM to solve $\text{prox}_{\lambda \Omega \text{LOG}}(b)$ in the scaled form with the sharing scheme

Require: $b, \lambda, \alpha, w, \forall g \in G$

1: $k = 0$, $U_g^0 = 0$, $X_g^{2,0} = 0 \forall g \in G$
2: while stopping criterion not met do
3: \hspace{1em} $k \leftarrow k + 1$
4: \hspace{2em} $X_g^{1,k+1} \leftarrow \text{prox}_{\lambda w_g \| \cdot \|_2}(X_g^{2,k} - U_g^k)$, $\forall g \in G$
5: \hspace{2em} $X_g^{1,k+1} \leftarrow 0$, $\forall g \in G$
6: \hspace{2em} $\bar{x}_g^{2,k+1} \leftarrow \frac{1}{|G| + \rho} (b + \frac{\rho}{|G|} \sum_{g \in G} (X_g^{1,k+1} + U_g^k))$
7: \hspace{2em} $U_g^{k+1} = U_g^k + (\alpha / \rho)(\frac{1}{|G|} \sum_{g \in G} (X_g^{1,k+1} + U_g^k) - \bar{x}_g^2)$, $\forall g \in G$.
8: end while
9: $\beta = \sum_{g \in G} X_g^{1,k+1}$
Output: $\beta$

3 Convergence analysis

Eckstein & Yao [13] show the iterates generated by two-block ADMM converges to some limiting points under certain conditions. Our setting follows their proposition. It is also straightforward to show such limiting points are optimal solutions. In this section, we establish linear convergence rate of the ADMM algorithm to solve the proximal operator of the LOG penalty [8] given a sufficiently small stepsize using the error bound theory.

3.1 Rate of convergence

Note that the objective function of (8) is not strongly convex since $M$ is not full column rank. To establish the linear convergence rate, we will use the error bound theory which is well-established for primal methods – see [36] and references therein. For a dual method, one needs to show that both primal and dual error bounds holds for the problem under investigation. As mentioned in the contributions, showing the dual error bound in the presence of $\ell_2$-norm in the objective function (which results in a second-order cone epigraph) is not trivial. Furthermore, in the absence of a bounded feasible region, boundedness of the iterates needs to be established. Finally, establishing primal error bound in the presence of the dual variable and under 2-block splitting is elaborate. All of these challenges are addressed in this section.

Let $X^* \subseteq \mathbb{R}^{2n}$ and $Y^* \subseteq \mathbb{R}^n$ denote the primal and dual optimal solution sets to (9) and (12), respectively. Let $X(y) \subseteq \mathbb{R}^{2n}$ denote the optimal solution set to the problem of minimizing the augmented Lagrangian function [11] given $y \in \mathbb{R}^n$. Note the augmented Lagrangian [10] is strongly convex in $x^1$ or $x^2$, but not jointly strongly convex. We use $x(y) = (x^1(y), x^2(y))^\top \in X(y)$ to represent a minimizer of [10] given $y$. Let $E \triangleq [I, -I] \in \mathbb{R}^{n \times 2n}$ and $M$ as in [8], and define

$$\ell(x^1, x^2) \triangleq \phi_b(Mx^2) + \psi_\rho(Ex)$$  \hspace{1em} (25)
where the functions \( \phi_b : \mathbb{R}^n \to \mathbb{R} \) and \( \psi : \mathbb{R}^n \to \mathbb{R} \) are defined as \( \phi_b(z) \triangleq \frac{1}{2} \|z - b\|_2^2 \) and \( \psi(z) \triangleq \frac{\rho}{2} \|z\|_2^2 \). For the simplicity of notation, the subscripts \( b \) and \( \rho \) are eliminated in the remaining of the manuscript. The following two properties are used in the subsequent analysis:

\[
\| \nabla x^2 \phi(Mx^2(y)) - \nabla x^2 \phi(Mx^2(\bar{y})) \|_2 \leq \| M^T M(x^2(y) - x^2(\bar{y})) \|_2 \leq L_\phi \| Mx^2(y) - Mx^2(\bar{y}) \|_2, 
\]

(26)

\[
\| \nabla x^2 \psi(Ex(y)) - \nabla x^2 \psi(Ex(\bar{y})) \|_2 \leq \rho \| E^T E(x(y) - x(\bar{y})) \|_2 \leq L_\psi \| Ex(y) - Ex(\bar{y}) \|_2, 
\]

(27)

where \( L_\phi = \| M \|_2 \) and \( L_\psi = \rho \sqrt{2} \).

Given \( y \in \mathbb{R}^n \), the optimization problem in (11) can equivalently be written as

\[
\min_{x^1 \in \mathbb{R}^n, x^2 \in \mathbb{R}^n, s \in \mathbb{R}|^G|} \lambda \sum_{g \in G} s_g + \frac{1}{2} \| Mx^2 - b \|^2_2 + \langle y, x^1 - x^2 \rangle + \frac{\rho}{2} \| x^1 - x^2 \|^2_2,
\]

(28)

s.t. \( w_g \| x^1_{j(g)} \|_2 \leq s_g, \quad \forall g \in G \).

The constraints in (25) are indeed second-order cones \( Q_g = \{ (x_g, s_g) \in \mathbb{R}|^G| \times \mathbb{R}_+ : \| x_g \|_2 \leq s_g \} \). The KKT system for the problem (28) is

\[
\begin{align*}
y_{j(g)} - w_g \mu_g + \rho(x^1_{j(g)} - x^2_{j(g)}) &= 0 \quad \forall g \in G, \\
M^T (Mx^2 - b) + \rho(x^2 - x^1) - y &= 0, \\
\lambda - \nu_g &= 0 \quad \forall g \in G, \\
w_g \| x^1_{j(g)} \|_2 \leq s_g \quad \forall g \in G, \\
\| \mu_g \|_2 \leq \nu_g \quad \forall g \in G, \\
\mu_g^* x^1_{j(g)} + \frac{s_g}{w_g} \nu_g &= 0 \quad \forall g \in G, 
\end{align*}
\]

(29)

where \( (\mu_g, \nu_g) \in \mathbb{R}|^G| \times \mathbb{R}_+ \) is the dual variable for the conic constraint. Hence, a pair \( ((x^1, x^2, s), (\mu, \nu)) \) is an optimal primal-dual pair if it satisfies (29). Using the KKT conditions, first we provide the dual error bound in Lemma 3.1.

**Lemma 3.1.** There exists \( \tau_d > 0 \) such that

\[
dist(y, Y^*) \leq \tau_d \| \nabla g_\rho(y) \|_2, 
\]

(30)

where \( g_\rho(y) \) is the augmented dual function defined in (11).

**Proof.** The framework of the proof was first proposed in [20] and also applied in [17] and require “locally upper Lipschitzian” property of polyhedral multifunction for the mapping induced by KKT conditions— see also [33, 40, 32, 22, 16]. However, due to presence of the conic constraints in (28), the resulting multifunction is not polyhedral anymore. Indeed, [10] showed that having a polyhedral graph is a necessary condition for the upper Lipschitzian property of the multifunction. The following proof uses problems specific structure to establish the dual error bound condition.

For any \( y \in \mathbb{R}^n \) and \( y^* \in Y^* \), consider the KKT conditions (29), we have

\[
\| y - y^* \|^2_2 = \| M^T M(x^2(y) - x^2(y^*)) + \rho(x^2(y) - x^1(y)) - \rho(x^2(y^*) - x^1(y^*)) \|^2_2 \\
= \| \nabla x^2 \phi(Mx^2(y)) - \nabla x^2 \phi(Mx^2(y^*)) + \nabla x^2 \psi(Ex(y)) - \nabla x^2 \psi(Ex(y^*)) \|^2_2 \\
\leq \| \nabla x^2 \phi(Mx^2(y)) - \nabla x^2 \phi(Mx^2(y^*)) \|^2_2 + \| \nabla x^2 \psi(Ex(y)) - \nabla x^2 \psi(Ex(y^*)) \|^2_2,
\]

\[
\leq \| \nabla x^2 \phi(Mx^2(y)) - \nabla x^2 \phi(Mx^2(y^*)) \|^2_2 + \| \nabla x^2 \psi(Ex(y)) - \nabla x^2 \psi(Ex(y^*)) \|^2_2,
\]
where the first equality follows from (29b), the second equality follows from the definition of $\phi(\cdot)$ and $\psi(\cdot)$ (defined below (25)), the third inequality follows from the triangle inequality. Hence, using (26) and (27), we have

$$
\|y - y^*\|_2^2 \leq \frac{L_G^2}{\rho} \|Mx^2(y) - Mx^2(y^*)\|_2^2 + L_\psi^2 \|Ex(y) - Ex(y^*)\|_2^2.
$$

(31)

Next, consider

$$
\|Mx^2(y) - Mx^2(y^*)\|_2^2 + \rho \|Ex(y) - Ex(y^*)\|_2^2
$$

$$
= \langle M^T Mx^2(y) - M^T Mx^2(y^*), x^2(y) - x^2(y^*) \rangle + \rho \langle E^T Ex(y) - M^T Mx(y^*), x(y) - x(y^*) \rangle
$$

$$
= \langle \nabla_x \phi(Mx^2(y)) - \nabla_x \phi(Mx^2(y^*)), x(y) - x(y^*) \rangle + \langle \nabla_x \psi(Ex(y)) - \nabla_x \psi(Ex(y^*)), x(y) - x(y^*) \rangle
$$

$$
= \langle \nabla_x \ell(x(y)) - \nabla_x \ell(x(y^*)), x(y) - x(y^*) \rangle
$$

$$
= \langle \nabla_x \ell(x(y)) - \nabla_x \ell(x(y^*)), x^1(y) - x^1(y^*) \rangle + \langle \nabla_x \ell(x(y)) - \nabla_x \ell(x(y^*)), x^2(y) - x^2(y^*) \rangle
$$

$$
= \sum_{g \in \mathcal{G}} \left\langle \nabla_{x_{j(g)}} \ell(x(y)) - \nabla_{x_{j(g)}} \ell(x(y^*)), x^1_{j(g)}(y) - x^1_{j(g)}(y^*) \right\rangle
$$

$$
+ \sum_{g \in \mathcal{G}} \left\langle \nabla_{x_{j(g)}} \ell(x(y)) - \nabla_{x_{j(g)}} \ell(x(y^*)), x^2_{j(g)}(y) - x^2_{j(g)}(y^*) \right\rangle
$$

$$
= \sum_{g \in \mathcal{G}} \left\langle w_g \mu_g(y) - y_{j(g)} - w_g \mu_g(y^*), x^1_{j(g)}(y) - x^1_{j(g)}(y^*) \right\rangle
$$

$$
+ \sum_{g \in \mathcal{G}} \left\langle y_{j(g)} - y^*_{j(g)}, x^2_{j(g)}(y) - x^2_{j(g)}(y^*) \right\rangle
$$

where the second and third equalities follow from the definitions of $\phi$, $\psi$, and $\ell$, in the forth and fifth equalities the gradient is expanded over each $x^1$ and $x^2$, and the last equality follows from (29a), (29b). Rearranging the terms in the last line above and using $x^1_{j(g)}(y^*) = x^2_{j(g)}(y^*) \forall g \in \mathcal{G}$, we get

$$
\|Mx^2(y) - Mx^2(y^*)\|_2^2 + \rho \|Ex(y) - Ex(y^*)\|_2^2
$$

$$
= \sum_{g \in \mathcal{G}} \left\langle w_g \mu_g(y) - w_g \mu_g(y^*), x^1_{j(g)}(y) - x^1_{j(g)}(y^*) \right\rangle
$$

$$
+ \sum_{g \in \mathcal{G}} \left\langle y_{j(g)} - y^*_{j(g)}, x^2_{j(g)}(y) - x^2_{j(g)}(y^*) \right\rangle,
$$

(32)

Since for all $g \in \mathcal{G}$, we have

$$
\left\langle w_g \mu_g(y) - w_g \mu_g(y^*), x^1_{j(g)}(y) - x^1_{j(g)}(y^*) \right\rangle
$$

$$
= w_g \mu_g(y)^T x^1_{j(g)}(y) - w_g \mu_g(y^*)^T x^1_{j(g)}(y) - w_g \mu_g(y)^T x^1_{j(g)}(y^*) + w_g \mu_g(y^*)^T x^1_{j(g)}(y^*)
$$

$$
= -w_g (s_g/w_g) \lambda - w_g \mu_g(y^*)^T x^1_{j(g)}(y) - w_g \mu_g(y)^T x^1_{j(g)}(y^*) - w_g (s_g/w_g) \lambda
$$

$$
\leq w_g \|\mu_g(y^*)\|_2 \|x^1_{j(g)}(y)\|_2 + w_g \|\mu_g(y)\|_2 \|x^1_{j(g)}(y^*)\|_2 - 2s_g \lambda
$$

$$
\leq 0,
$$

where the second equality follows from (29c) and (29d), the third inequality uses Cauchy-Schwarz.
inequality, and the last inequality follows from (29c), (29d), and (29e). Hence, we have
\[ \|Mx^2(y) - Mx^2(y^*)\|_2^2 + \rho\|Ex(y) - Ex(y^*)\|_2^2 \leq \sum_{g \in G} \left\langle y_{j(g)} - y_{j(g)}^*, x_{j(g)}^2(y) - x_{j(g)}^2(y^*) \right\rangle \]
\[ \leq \|y - y^*\| \|\nabla g_\rho(y)\|, \]  
where (33) uses nonpositivity of the first term in (32) (shown above), and (34) follows from Cauchy-Schwarz and the fact that \( \nabla g_\rho(y) = x^1(y) - x^2(y) \) — see e.g. [6] — Lemma 2.1. Finally, using (34) and (31), we get
\[ \|y - y^*\|_2^2 \leq \max\{L_{\rho}, L_{\rho}^2/\rho\}\left( \|Mx^2(y) - Mx^2(y^*)\|_2^2 + \rho\|Ex(y) - Ex(y^*)\|_2^2 \right) \]
\[ \leq \max\{L_{\rho}, L_{\rho}^2/\rho\}\|y - y^*\|_2 \|\nabla g_\rho(y)\|_2. \]  
Hence, we have
\[ \text{dist}(y, Y^*) \leq \|y - y^*\| \leq \max\{L_{\rho}, L_{\rho}^2/\rho\}\|\nabla g_\rho(y)\|_2. \]  
Hence, \( \tau_d \) in Lemma [3.1] is indeed function of \( \rho \) and \( M \). Next, in Lemma [3.2] below, we show the existence of a finite saddle point to the augmented Lagrangian function and that the sequence generated by the algorithm is uniformly bounded.

**Lemma 3.2.** Given the existence of a finite saddle point to the augmented Lagrangian function (10), for any \( \rho \) and \( \alpha \) such that \( 0 < \alpha < \rho \), the sequence \( \{x^{1,k}\}, \{x^{2,k}\} \) and \( \{y^k\} \) generated by the algorithm (13)-(15) is uniformly bounded.

**Proof.** Before proceeding with the proof of the boundedness of the iterates, we show the existence of a finite saddle point by the following argument. Consider
\[ \min_{x^1, x^2 \in \mathbb{R}^n} \left\{ \tilde{F}(x^1, x^2) = \lambda \sum_{g \in G} w_g \|x_{j(g)}^1\|_2 + \frac{1}{2}\|Mx^2 - b\|_2^2 + \frac{\rho}{2}\|x^1 - x^2\|_2^2, \text{ s.t. } x^1 = x^2 \right\}. \]  
The above problem is equivalent to (8) whose objective function is coercive and continuous. By Weierstrass’ Theorem (see e.g. [6]), we have finite optimal solution to (35) \( \{x^{1,*}, x^{2,*}\} \), i.e. \( \tilde{F}^* = \inf_{x^1, x^2 \in \mathbb{R}^n} \tilde{F}(x^1, x^2) \). Especially, \( x^{1,*} = x^{2,*} \). Consider \( L_\rho(x^1, x^2; y) = \tilde{F}(x^1, x^2) + \left\langle y, x^1 - x^2 \right\rangle \) and the corresponding dual function \( g_\rho(y) \). First, we have \( L_\rho(x^{1,*}, x^{2,*}; y) = \tilde{F}(x^{1,*}, x^{2,*}) \), for \( y \). By strong duality (see e.g. Prop. 5.2.1 in [6]), we know there is no duality gap. Furthermore, there exists at least one Lagrange multiplier \( y^* \), i.e. \( \tilde{F}^* = \inf_{x^1, x^2 \in \mathbb{R}^n} L_\rho(x^1, x^2; y^*) = g_\rho(y^*) \). We conclude \( \{x^{1,*}, x^{2,*}; y^*\} \) is a finite saddle point.

The idea for the proof of boundedness of the iterates is similar to Theorem 5.1 in [13]; however, we do not have the strong convexity assumption. Given a finite saddle point \( (x^{1,*}, x^{2,*}; y^*) \), define \( \hat{x}^{1,k} \triangleq x^{1,k} - x^{1,*}, \hat{x}^{2,k} \triangleq x^{2,k} - x^{2,*}, \hat{y}^k \triangleq y^k - y^* \) where \( x^{1,*} = x^{2,*} \). Establishing the boundedness of the sequence is equivalent to show that the sequence \( \{\|\hat{y}^k\|_2^2 + \alpha\rho\|\hat{x}^{2,k}\|_2^2 + \alpha(\rho - \alpha)\|\hat{x}^{1,k} - \hat{x}^{2,k}\|_2^2\} \) is non-increasing. From the convexity of the augmented Lagrangian function in \( x^1 \), we have
\[ \left\langle y^* + \rho(x^{1,*} - x^{2,*}), x^1 - x^{1,*} \right\rangle + \lambda \sum_{g \in G} w_g \|x_{j(g)}^1\|_2 - \lambda \sum_{g \in G} w_g \|x_{j(g)}^{1,*}\|_2 \geq 0, \forall x^1. \]  
\[ \text{(39)} \]
Furthermore, from the convexity of the augmented Lagrangian \(\mathcal{L}(x)\) in \(x^2\), we have
\[
\langle M^T (Mx^2 - b) - y^* + \rho (x^{2,*} - x^{1,*}), x^2 - x^{2,*} \rangle \geq 0, \forall x^2.
\] (40)

From the fact that \(L_\rho(x^1, x^2; y^*) \leq L_\rho(x^1, x^2; y^*), \forall x^1, x^2\), we have
\[
y^* = y^* + \alpha(x^{1,*} - x^{2,*}).
\] (41)

Similar to the arguments for \([39]-[41]\), from \([13]-[15]\), we have
\[
\langle \rho(x^{1,k+1} - x^{2,k}) + y^{k}_{j(g)}, x^{1}_{j(g)} - x^{1,k+1} \rangle + \lambda w_g \|x^{1}_{j(g)}\|_2^2 - \lambda w_g \|x^{1,k+1}\|_2 \geq 0, \forall g \in G, \forall x^{1}_{j(g)},
\] (42)
\[
\langle M^T (Mx^{2,k+1} - b) + \rho(x^{2,k+1} - x^{1,k+1} - y^k, x^2 - x^{2,k+1} \rangle \geq 0, \forall x^2,
\] (43)
\[
y^{k+1}_{j(g)} = y^k_{j(g)} + \alpha(x^{1,k+1} - x^{2,k+1}), \forall g \in G.
\] (44)

Since \(j(g) \cap j(\bar{g}) = \emptyset\) for all \(g, \bar{g} \in G\) such that \(g \neq \bar{g}\), from \([42]\) and \([44]\), we have:
\[
\langle \rho(x^{1,k+1} - x^{2,k}) + y^k, x^1 - x^{1,k+1} \rangle + \lambda \sum_{g \in G} w_g \|x^{1}_{j(g)}\|_2^2 - \lambda \sum_{g \in G} w_g \|x^{1,k+1}\|_2 \geq 0, \forall x^1,
\] (45)
\[
y^{k+1} = y^k + \alpha(x^{1,k+1} - x^{2,k+1}).
\] (46)

Setting \(x^1 = x^{1,k+1}\) in \([39]\), and \(x^1 = x^{1,*}\) in \([45]\) and adding them, we get
\[
\langle -\bar{y}^k + \rho(x^{2,k} - \bar{x}^{1,k+1}), \bar{x}^{1,k+1} \rangle \geq 0
\] (47)

Similarly, setting \(x^2 = x^{2,k+1}\) in \([40]\), and \(x^2 = x^{2,*}\) in \([43]\) and adding them, we get
\[
\langle M^T M\bar{x}^{2,k+1} - \bar{y}^k - \rho(\bar{x}^{1,k+1} - \bar{x}^{2,k+1}, -\bar{x}^{2,k+1}) \rangle \geq 0
\] (48)

Adding the left-hand-sides of \([47]\) to \([48]\) and rearranging the terms, we have,
\[
\langle -\bar{y}^k, x^{1,k+1} - x^{2,k+1} \rangle + \rho \langle x^{2,k} - \bar{x}^{1,k+1}, \bar{x}^{1,k+1} \rangle + \rho \langle \bar{x}^{1,k+1} - \bar{x}^{2,k+1}, \bar{x}^{2,k+1} \rangle - \|M\bar{x}^{2,k+1}\|_2^2
\]
\[
= \langle -\bar{y}^k, x^{1,k+1} - x^{2,k+1} \rangle + \rho \langle x^{2,k} - \bar{x}^{1,k+1}, x^{2,k+1} - \bar{x}^{2,k+1} \rangle - \|M\bar{x}^{2,k+1}\|_2^2
\]
\[
+ \rho \langle \bar{x}^{1,k+1} - \bar{x}^{2,k+1}, x^{2,k+1} \rangle - \|M\bar{x}^{2,k+1}\|_2^2
\]

Hence, we have
\[
\rho \langle \bar{x}^{2,k} - \bar{x}^{2,k+1}, \bar{x}^{1,k+1} \rangle - \|M\bar{x}^{2,k+1}\|_2^2 - \rho \|x^{2,k+1} - \bar{x}^{1,k+1}\|_2^2 \geq \langle \bar{x}^{1,k+1} - \bar{x}^{2,k+1}, \bar{y}^k \rangle;
\] (49)

From \([40]\), we have the following two inequalities:
\[
\bar{y}^{k+1} - \bar{y}^k = \alpha(x^{1,k+1} - x^{2,k+1})
\]
\[
\bar{y}^{k+1} + \bar{y}^k = \alpha(x^{1,k+1} - x^{2,k+1}) + 2\bar{y}^k
\]

Taking the inner product of the left terms together and the right terms together, we obtain
\[
\|\bar{y}^{k+1}\|^2_2 - \|\bar{y}^k\|^2_2 = \alpha^2\|x^{1,k+1} - x^{2,k+1}\| + 2\alpha \langle x^{1,k+1} - x^{2,k+1}, \bar{y}^k \rangle
\]
\[
\leq \alpha(\alpha - 2\rho)\|x^{1,k+1} - x^{2,k+1}\|_2^2 - 2\alpha\|M\bar{x}^{2,k+1}\|_2^2 + 2\alpha \rho \langle x^{2,k} - x^{2,k+1}, \bar{x}^{1,k+1} \rangle
\] (50)
where the inequality uses (49). Next, we will upper bound \( \langle \hat{x}^{2,k} - \hat{x}^{2,k+1}, \hat{x}^{1,k+1} \rangle \). Setting \( x^2 = \hat{x}^{2,k} \) in (43), we have
\[
\langle M^T (Mx^{2,k+1} - b) + \rho(x^{2,k+1} - x^{1,k+1}) - \hat{y}^k, x^{2,k} - x^{2,k+1} \rangle \geq 0.
\]
Setting \( k + 1 \) in (43) to \( k \), and \( x^2 = \hat{x}^{2,k+1} \), we have
\[
\langle M^T (Mx^{2,k} - b) + \rho(x^{2,k} - x^{1,k}) - \hat{y}^{k-1}, x^{2,k+1} - x^{2,k} \rangle \geq 0.
\]
Adding (52) and (53), we have
\[
\langle y^k - y^{k-1}, x^{2,k+1} - x^{2,k} \rangle - \rho(\|x^{2,k+1} - x^{2,k}\|^2_2 + \rho(\|x^{1,k+1} - x^{1,k}, x^{2,k+1} - x^{2,k}\)\)
\[
\geq \|M(x^{2,k+1} - x^{2,k})\|^2_2 \geq 0.
\]
From (46), we have \( y^k - y^{k-1} = \alpha(x^{1,k} - x^{2,k}) \). Using it in (54) and rearranging terms, we obtain
\[
\rho(\|x^{1,k+1} - x^{1,k}, x^{2,k+1} - x^{2,k}\)\)
\[
\geq \|x^{2,k+1} - x^{2,k}\|^2_2 - \alpha \langle x^{1,k} - x^{2,k}, x^{2,k+1} - x^{2,k}\)\).

Adding and subtracting \( x^{1,k} \) and \( x^{2,k} \) into each argument in (3.1) as needed, we have
\[
\rho(\|x^{1,k+1} - x^{1,k}, x^{2,k+1} - x^{2,k}\)\)
\[
\geq \|x^{2,k+1} - x^{2,k}\|^2_2 - \alpha \langle x^{1,k} - x^{2,k}, x^{2,k+1} - x^{2,k}\)\).

The term \( \langle \hat{x}^{2,k} - \hat{x}^{2,k+1}, \hat{x}^{1,k+1} \rangle \) can be transformed as following:
\[
\langle x^{2,k} - \hat{x}^{2,k+1}, \hat{x}^{1,k+1} \rangle
\]
\[
= \langle \hat{x}^{2,k} - \hat{x}^{2,k+1}, \hat{x}^{1,k+1} - \hat{x}^{1,k} + \hat{x}^{1,k} - \hat{x}^{2,k} + \hat{x}^{2,k} \rangle
\]
\[
= \langle \hat{x}^{2,k} - \hat{x}^{2,k+1}, \hat{x}^{1,k+1} - \hat{x}^{1,k} \rangle + \langle \hat{x}^{2,k} - \hat{x}^{2,k+1}, \hat{x}^{1,k} - \hat{x}^{2,k} \rangle + \langle \hat{x}^{2,k} - \hat{x}^{2,k+1}, \hat{x}^{2,k} \rangle
\]
\[
= \langle \hat{x}^{2,k} - \hat{x}^{2,k+1}, \hat{x}^{1,k+1} - \hat{x}^{1,k} \rangle + \langle \hat{x}^{2,k} - \hat{x}^{2,k+1}, \hat{x}^{1,k} - \hat{x}^{2,k} \rangle + \frac{1}{2}(\|\hat{x}^{2,k}\|^2_2 - \|\hat{x}^{2,k+1}\|^2_2 + \|\hat{x}^{2,k} - \hat{x}^{2,k+1}\|^2_2)
\]
\[
\leq \frac{1}{2}(\|\hat{x}^{2,k}\|^2_2 - \|\hat{x}^{2,k+1}\|^2_2 - \|\hat{x}^{2,k} - \hat{x}^{2,k+1}\|^2_2) + (1 - \frac{\alpha}{\rho}) \langle \hat{x}^{2,k} - \hat{x}^{2,k+1}, \hat{x}^{1,k} - \hat{x}^{2,k} \rangle
\]
\[
(56)
\]
where the last inequality follows from (55). Combining (50) and (56) and rearranging the terms, we obtain
\[
\|\hat{y}^{k+1}\|^2_2 + \alpha \rho(\|\hat{x}^{2,k+1}\|^2_2 + \|\hat{x}^{1,k+1} - \hat{x}^{2,k+1}\|^2_2 - (\|\hat{y}^k\|^2_2 + \alpha \rho(\|\hat{x}^{2,k}\|^2_2)
\]
\[
\leq -\alpha \rho(\|\hat{x}^{1,k+1} - \hat{x}^{2,k+1}\|^2_2 - 2\alpha \|M\hat{x}^{2,k+1}\|^2_2 - \alpha \rho(\|\hat{x}^{2,k} - \hat{x}^{2,k+1}\|^2_2 + 2\alpha(\rho - \alpha) \langle \hat{x}^{2,k} - \hat{x}^{2,k+1}, \hat{x}^{1,k} - \hat{x}^{2,k} \rangle
\]
\[
(57)
\]
By upper bounding the last term in (57) by the identity \( 2 \langle a, b \rangle \leq \|a\|^2_2 + \|b\|^2_2 \), we get
\[
\|\hat{y}^{k+1}\|^2_2 + \alpha \rho(\|\hat{x}^{2,k+1}\|^2_2 + \alpha(\rho - \alpha) \|\hat{x}^{1,k+1} - \hat{x}^{2,k+1}\|^2_2 - (\|\hat{y}^k\|^2_2 + \alpha \rho(\|\hat{x}^{2,k}\|^2_2)
\]
\[
\leq -\alpha \rho(\|\hat{x}^{1,k+1} - \hat{x}^{2,k+1}\|^2_2 - 2\alpha \|M\hat{x}^{2,k+1}\|^2_2 - \alpha \rho(\|\hat{x}^{2,k} - \hat{x}^{2,k+1}\|^2_2 \leq 0.
\]
\[
(58)
\]
We have shown that the sequence \( \{\|\hat{y}^k\|^2_2 + \alpha \rho(\|\hat{x}^{2,k}\|^2_2 + \alpha(\rho - \alpha) \|\hat{x}^{1,k} - \hat{x}^{2,k}\|^2_2\} \}_{k=1}^{\infty} \) is non-increasing. Once the initial point and saddle point are fixed, which are not related to \( \alpha \), then \( \{\|\hat{y}^k\|^2_2 + \alpha \rho(\|\hat{x}^{2,k}\|^2_2 + \alpha(\rho - \alpha) \|\hat{x}^{1,k} - \hat{x}^{2,k}\|^2_2\} \}_{k=1}^{\infty} \) is bounded by \( \|\hat{y}^0\|^2_2 + \rho^2(\|\hat{x}^{2,0}\|^2_2 + \frac{\alpha^2}{4} \|\hat{x}^{1,0} - \hat{x}^{2,0}\|^2_2) \). We concluded that the sequence \( \{x^{1,k}\}, \{x^{2,k}\} \) and \( \{y^k\} \) generated by (15) is uniformly bounded for any \( 0 < \alpha < \rho \). \qed
We also need to establish the primal error bound. Unlike the dual error bound, where the gradient of the augmented $g_p(y)$ nicely bounds the "error", i.e., the distance of a point to the optimal solution set, the primal function is not smooth and differentiable. Quantifying the error bound for nonsmooth functions is generally performed by the proximal gradient. For general surveys on error bounds, see [27] [45] and references therein.

**Definition 3.1.** Assume a convex function $f$ is decomposable as $f(x) = g(Ax) + h(x)$, where $g$ is a strongly convex and differentiable and $h$ is a convex (possibly nonsmooth) function, then we can define the proximal gradient of $f$ with respect to $h$ as

$$\nabla f(x) := x - \text{prox}_h(x - \nabla(f(x) - h(x))) = x - \text{prox}_h(x - A^T \nabla g(Ax))$$

If $h = 0$ then the proximal gradient $\nabla f(x)$ is equal to the gradient $\nabla f(x)$. In general, $\nabla f(x)$ can be used as the (extended) gradient for nonsmooth minimization $\min_{x \in \mathbb{R}^n} f(x)$. For instance, we have $\nabla f(x^*) = 0$ if and only if $x^*$ is a minimizer. For the Lagrangian function (10), the proximal gradient w.r.t. $x = (x^1,x^2)$ is defined as

$$\nabla_x L_\rho(x^1,x^2;y) := x - \text{prox}_\lambda \sum_{g \in \mathcal{G}} w_g \|x^1_{j(g)}\|_2 \left( x - \nabla_x \left( \frac{1}{2} \|Mx^2 - b\|_2^2 + \langle y, x^1 - x^2 \rangle + \frac{\rho}{2} \|x^1 - x^2\|_2^2 \right) \right),$$

which we split into $\nabla x^1 L_\rho$ and $\nabla x^2 L_\rho$ in the proof of Lemma 3.3.

**Lemma 3.3.** Assume that $(x^1,x^2;y)$ is in a compact set, then there exists $0 < \tau < +\infty$ and $\delta > 0$ such that

$$\text{dist}(x,X(y)) \leq \tau \|\nabla_x L_\rho(x^1,x^2;y)\|_2,$$

for all $(x^1,x^2;y)$ such that $\|\nabla_x L_\rho(x^1,x^2;y)\|_2 \leq \delta$. Furthermore, $\tau$ and $\delta$ are independent of $y$.

**Proof.** The proof extends the analysis of [36] and [44]. Since both works discuss primal methods, there are mainly two new ingredients in our proof: 1) dealing with the dual variable $y$, and 2) splitting $x$ into $x^1$ and $x^2$, where neither step is trivial. Given $y$, note that $X(y)$ can be written as $(X^1(y),X^2(y))$. For a fixed $y$, and for any sequence $\{(x^{1,k},x^{2,k};y) : x^{2,k} \notin X^2(y)\}_{k \geq 0}$, we define

$$r^{1,k} := \nabla x^1 L_\rho(x^{1,k},x^{2,k};y),$$
$$r^{2,k} := \nabla x^2 L_\rho(x^{1,k},x^{2,k};y) = M^T(Mx^{2,k} - b) - y + \rho(x^{2,k} - x^{1,k}),$$
$$\delta^k := \|x^{2,k} - x^{2,k}\|_2,$$
$$x^{1,k} := \text{argmin}_{x^1} \|x^{2,k} - x^1\|_2,$$
$$u^k := \frac{x^{2,k} - x^{2,k}}{\delta^k}.$$

Note that

$$\nabla x^1 L_\rho(x^1,x^2;y) = x^1 - \text{prox}_\lambda \sum_{g \in \mathcal{G}} w_g \|d_{j(g)}\|_2 \left( x^1 - y - \rho(x^1 - x^2) \right)$$

$$= x^1 - \text{argmin}_d \lambda \sum_{g \in \mathcal{G}} w_g \|d_{j(g)}\|_2 + \frac{1}{2} \|d - (x^1 - y + \rho(x^1 - x^2))\|_2^2$$

$$= \text{argmin}_d \lambda \sum_{g \in \mathcal{G}} w_g \|d_{j(g)} - x^1_{j(g)}\|_2 + \frac{1}{2} \|d - y - \rho(x^1 - x^2)\|_2^2,$$

15
where the second equality follows from the definition of the proximity operator and the third equality uses the transformation \( \mathbf{d} \triangleq \mathbf{x}^1 - \mathbf{d} \). Furthermore, for any group \( g \in \mathcal{G} \), we have

\[
\left( \nabla_{x^1} L_{\rho}(x^1, x^2; y) \right)_{j(g)} = \arg\min_{d_{j(g)}} \lambda w_g \| d_{j(g)} - x^1_{j(g)} \|^2 + \frac{1}{2} \| d_{j(g)} - y_{j(g)} - \rho(x^1_{j(g)} - x^2_{j(g)}) \|^2 \quad (68)
\]

\[
= \begin{cases} x^1_{j(g)}, & \text{if } \| x^1_{j(g)} - y_{j(g)} - \rho(x^1_{j(g)} - x^2_{j(g)}) \|_2 \leq \lambda w_g, \\ \gamma_g x^1_{j(g)} + (1 - \gamma_g)(y_{j(g)} + \rho(x^1_{j(g)} - x^2_{j(g)})), & \text{otherwise.} \end{cases}
\]

where \( \gamma_g = \lambda w_g / \| x^1_{j(g)} - y_{j(g)} \|_2 - \rho(x^1_{j(g)} - x^2_{j(g)}) \|_2 \). Note that the two cases from the soft-thresholding operator in (69) yield \( x^1_{j(g)} \) at the boundary \( \| x^1_{j(g)} - y_{j(g)} - \rho(x^1_{j(g)} - x^2_{j(g)}) \|_2 = \lambda w_g \), i.e., \( \nabla_{x^1} L_{\rho}(x^1, x^2; y)_{j(g)} \) is continuous in \((x^1, x^2, y)\).

To prove this lemma, we will first prove that it suffices to show that there exists \( 0 < \tau' < +\infty \) and \( \delta > 0 \) such that

\[
\text{dist}(x^2, X^2(y)) \leq \tau' \| \nabla_{x^2} L_{\rho}(x^1, x^2; y) \|_2, \quad (70)
\]

for all \((x^1, x^2, y)\) such that \( \| \nabla_{x^1} L_{\rho}(x^1, x^2; y) \|_2 \leq \delta \). Second, we will show (70).

Assume (70) holds. Given \((x^1, x^2, y)\), pick \((x^{1,*}, x^{2,*}) \in X(y)\), such that \( \text{dist}(x^{2,*}, x^{2,*}) = \text{dist}(x^2, X^2(y)) \), and \( x^{1,*} \) such that it satisfies the optimality condition (72). Recall that

\[
\nabla_{x^2} L_{\rho}(x^1, x^2; y) = M^T(Mx^2 - b) - y + \rho(x^{2,*} - x^{1,*}). \quad (71)
\]

Hence, from the optimality condition, we have

\[
\nabla_{x^2} L_{\rho}(x^{1,*}, x^{2,*}; y) = M^T(Mx^{2,*} - b) - y + \rho(x^{2,*} - x^{1,*}) = 0 \quad (72)
\]

Subtracting (72) from (71) and rearranging the terms, we obtain

\[
x^1 - x^{1,*} = \frac{1}{\rho}(M^T M + I)(x^2 - x^{2,*}) - \frac{1}{\rho} \nabla_{x^2} L_{\rho}(x^1, x^2; y). \quad (73)
\]

Thus,

\[
\text{dist}(x, X(y))^2 \leq \| x^1 - x^{1,*} \|^2 + \| x^2 - x^{2,*} \|^2 \quad (74)
\]

\[
\leq \left\| \left( \frac{1}{\rho} M^T M + I \right)(x^2 - x^{2,*}) \right\|^2 + \frac{1}{\rho} \| \nabla_{x^2} L_{\rho}(x^1, x^2; y) \|^2 + \| x^2 - x^{2,*} \|^2. \quad (75)
\]

Upper bounding \( \| x^2 - x^{2,*} \|^2 \) in (75) with (70), we have (59).

Next, we will show (70) by contradiction. Suppose (70) does not hold, then there exists a sequence \( \{(x^{1,k}, x^{2,k}; y) : x^{2,k} \notin X^2(y)\}_{k \geq 0} \) satisfying

\[
\| \nabla_{x^1} L_{\rho}(x^{1,k}, x^{2,k}, y) \|_2/\delta_k \to 0, \quad \text{and} \quad \| \nabla_{x^2} L_{\rho}(x^{1,k}, x^{2,k}, y) \|_2 \to 0. \quad (76)
\]

Note that

\[
\frac{\| r^{1,k} \| + \| r^{2,k} \| }{\sqrt{2}} \leq \| \nabla_{x^1} L_{\rho}(x^{1,k}, x^{2,k}; y) \| \leq \| r^{1,k} \| + \| r^{2,k} \|. \quad (77)
\]

where \( r^{1,k} \) and \( r^{2,k} \) are defined in (60) and (61), respectively. Hence, using the left inequality in (77), (76) implies

\[
\{ r^{1,k} \} \to 0, \quad \{ r^{2,k} \} \to 0, \quad \{ \frac{\| r^{1,k} \| + \| r^{2,k} \| }{\delta_k} \} \to 0. \quad (78)
\]
We will show that (78) does not hold. Since \((x^{1,k}, x^{2,k})\) is in a compact set, by passing to a subsequence if necessary, we can assume that \((x^{1,k}, x^{2,k}) \to (\bar{x}^1, \bar{x}^2)\). Since \(\{x^{1,k}\} \to 0\), and \(\{r^{2,k}\} \to 0\), then by the right inequality in (77), \(\nabla \rho(x^{1,k}, x^{2,k}; y) \to 0\). Furthermore, since \(\nabla \rho(x, x^2; y)\) is continuous, this implies \(\nabla \rho(\bar{x}^1, \bar{x}^2; y) = 0\). It further implies that \((\bar{x}^1, \bar{x}^2) \in X(y)\). Hence \(\delta^k \leq \|x^{2,k} - \bar{x}^2\| \to 0\), as \(k \to \infty\), so that \(\{x^{2,k}\} \to \bar{x}^2\). And based on (73), we have
\[
\{\bar{x}^{1,k}\} \to \bar{x}^1. \tag{79}
\]

Next, we claim there exists \(\kappa > 0\) such that,
\[
\|x^{2,k} - \bar{x}^{2,k}\| \leq \kappa \|M x^{2,k} - M \bar{x}^{2,k}\|, \quad \forall k \tag{80}
\]
Again, we argue by contraction. Suppose (80) does not hold, then by passing to a subsequence if necessary, we can assume
\[
\{\frac{\|M x^{2,k} - M \bar{x}^{2,k}\|}{\|x^{2,k} - \bar{x}^{2,k}\|}\} \to 0. \tag{81}
\]
This implies that \(\{M u^k\} \to 0\), where \(u^k\) is defined in (64). Note that \(\|u^k\| = 1\), we can assume \(u^k \to \bar{u} \neq 0\) (by further passing to a subsequence if necessary); hence, we have \(M \bar{u} = 0\) by continuity. Combining (71) and (78), we have
\[
M^T (M x^{2,k} - b) - y + \rho(x^{2,k} - x^{1,k}) = o(\delta_k). \tag{82}
\]
Furthermore,
\[
M^T (M \bar{x}^{2,k} - b) - y + \rho(\bar{x}^{2,k} - \bar{x}^{1,k}) = 0. \tag{83}
\]
Subtracting the above two equalities and using (81), we get
\[
x^{2,k} - \bar{x}^{2,k} = x^{1,k} - \bar{x}^{1,k} + o(\delta_k). \tag{84}
\]
Thus,
\[
\bar{u} = \lim_{k \to \infty} \frac{x^{2,k} - \bar{x}^{2,k}}{\delta_k} = \lim_{k \to \infty} \frac{x^{1,k} - \bar{x}^{1,k}}{\delta_k}. \tag{85}
\]
Since \(u^k \to \bar{u} \neq 0\), we have \(\langle u^k, \bar{u} \rangle > 0\) for \(k\) sufficiently large. Select \(k\) such that \(\langle u^k, \bar{u} \rangle > 0\) and let
\[
\bar{x}^{2,k} = \bar{x}^{2,k} + \epsilon \bar{u} \tag{86}
\]
for some \(\epsilon > 0\). We can show that for \(\epsilon > 0\) sufficiently small
\[
\bar{x}^{2,k} \in X^2(y), \tag{87}
\]
whose proof is relegated to Appendix A.1. Now, assume \(\bar{x}^{2,k} \in X^2(y^k)\) for \(\epsilon > 0\) sufficiently small. This leads to the following contradiction:
\[
\|x^{2,k} - \bar{x}^{2,k}\|_2 = \|x^{2,k} - \bar{x}^{2,k} - \epsilon \bar{u}\|_2 = \delta^k + \epsilon^2 - 2\epsilon (\langle u^k, \bar{u} \rangle < \delta^k \tag{88}
\]
for \(\epsilon\) sufficiently small, which contradicts the definition of \(x^{2,k}\) in (62). So (80) holds.
By (67), we have

\[ 0 \in \lambda \partial \sum_{g \in \mathcal{G}} w_g \| r_{j(g)}^{1,k} - x_{j(g)}^{1,k} \|_2 + \langle r^{1,k} - y - \rho (x^{1,k} - x^{2,k}), \rangle, \tag{86} \]

which is the optimal condition to

\[ r^{1,k} \in \text{argmin}_d \lambda \sum_{g \in \mathcal{G}} w_g \| d_{j(g)} - x_{j(g)}^{1,k} \|_2 + \langle r^{1,k} - y - \rho (x^{1,k} - x^{2,k}), d \rangle. \tag{87} \]

From (87), we have

\[
\begin{align*}
\lambda \sum_{g \in \mathcal{G}} w_g \| r_{j(g)}^{1,k} - x_{j(g)}^{1,k} \|_2 + \langle r^{1,k} - y - \rho (x^{1,k} - x^{2,k}), r^{1,k} \rangle \\
\leq \lambda \sum_{g \in \mathcal{G}} w_g \| x_{j(g)}^{1,k} \|_2 + \langle r^{1,k} - y - \rho (x^{1,k} - x^{2,k}), x^{1,k} - \bar{x}^{1,k} \rangle. \tag{88} \\
\end{align*}
\]

From \( \tilde{\nabla}_x L_{\rho}(\bar{x}^{1,k}, \bar{x}^{2,k}; y) = 0 \), we have

\[ 0 = \text{argmin}_d \lambda \sum_{g \in \mathcal{G}} w_g \| d_{j(g)} - x_{j(g)}^{1,k} \|_2 + \frac{1}{2} \| d - y - \rho (\bar{x}^{1} - \bar{x}^{2}) \|_2^2. \tag{89} \]

Similar to (86), we have

\[ 0 \in \lambda \partial \sum_{g \in \mathcal{G}} w_g \| \tilde{x}_{j(g)}^{1,k} \|_2 + (-y - \rho (\bar{x}^{1,k} - \bar{x}^{2,k})), \tag{90} \]

which is the optimal condition to

\[ 0 \in \text{argmin}_d \lambda \sum_{g \in \mathcal{G}} w_g \| d_{j(g)} - \tilde{x}_{j(g)}^{1,k} \|_2 + \langle -y - \rho (\bar{x}^{1,k} - \bar{x}^{2,k}), d \rangle. \tag{91} \]

From (91), we have

\[
\begin{align*}
\lambda \sum_{g \in \mathcal{G}} w_g \| \tilde{x}_{j(g)}^{1,k} \|_2 \\
\leq \lambda \sum_{g \in \mathcal{G}} w_g \| r_{j(g)}^{1,k} - x_{j(g)}^{1,k} \|_2 + \langle -y - \rho (\bar{x}^{1,k} - \bar{x}^{2,k}), \bar{x}^{1,k} + r^{1,k} - x^{1,k} \rangle. \tag{92} \\
\end{align*}
\]

Adding (88) and (92), and using (61), we obtain

\[
\begin{align*}
\langle r^{1,k} + r^{2,k}, r^{1,k} \rangle + \langle M^T M (x^{2,k} - \bar{x}^{2,k}), x^{1,k} - \bar{x}^{1,k} \rangle \\
\leq \langle r^{1,k} + r^{2,k}, x^{1,k} - \bar{x}^{1,k} \rangle + \langle M^T M (x^{2,k} - \bar{x}^{2,k}), r^{1,k} \rangle. \tag{93} \\
\end{align*}
\]

From (61), we have

\[ x^{1,k} - \bar{x}^{1,k} = \left( \frac{1}{\rho} M^T M + I \right) \left( x^{2,k} - \bar{x}^{2,k} \right) - \frac{1}{\rho} r^{2,k}. \tag{94} \]
Defining $A \triangleq \frac{1}{\rho} M^T M + I$ and using \cite{94} in \cite{93} and rearranging the terms, we obtain
\[
\langle r^{1,k} + r^{2,k}, r^{1,k} + \frac{1}{\rho} r^{2,k} \rangle + \langle M^T M(x^{2,k} - \bar{x}^{2,k}), A(x^{2,k} - \bar{x}^{2,k}) \rangle \\
\leq \langle r^{1,k} + r^{2,k}, A(x^{2,k} - \bar{x}^{2,k}) \rangle + \langle M^T M(x^{2,k} - \bar{x}^{2,k}), r^{1,k} + \frac{1}{\rho} r^{2,k} \rangle.
\] (95)

Let us consider term by term. We have
\[
\langle r^{1,k} + r^{2,k}, r^{1,k} + \frac{1}{\rho} r^{2,k} \rangle \geq \|r^{1,k}\|^2_2 + \frac{1}{\rho} \|r^{2,k}\|^2_2 - (\frac{1}{\rho} + 1) \|r^{1,k}\|_2 \|r^{2,k}\|_2.
\] (96)

Next, using \cite{80}, we have
\[
\langle M^T M(x^{2,k} - \bar{x}^{2,k}), A(x^{2,k} - \bar{x}^{2,k}) \rangle = \frac{1}{\rho} \|M^T M(x^{2,k} - \bar{x}^{2,k})\|^2_2 + \|M(x^{2,k} - \bar{x}^{2,k})\|^2_2 \\
\geq \kappa^2 \|x^{2,k} - \bar{x}^{2,k}\|^2_2.
\] (97)

Denote the largest eigenvalue of matrix $A$ by $L_1$, we have
\[
\langle r^{1,k} + r^{2,k}, A(x^{2,k} - \bar{x}^{2,k}) \rangle \leq L_1 \|r^{1,k} + r^{2,k}\|_2 \|x^{2,k} - \bar{x}^{2,k}\|_2.
\] (99)

Denote $L_2 \triangleq \max_{\|d\|=1} \|Md\|$, 
\[
\langle M^T M(x^{2,k} - \bar{x}^{2,k}), r^{1,k} + \frac{1}{\rho} r^{2,k} \rangle \leq L_2^2 \|r^{1,k} + \frac{1}{\rho} r^{2,k}\|_2 \|x^{2,k} - \bar{x}^{2,k}\|_2.
\] (100)

Combining the four inequalities above, we have
\[
\kappa^2 \|x^{2,k} - \bar{x}^{2,k}\|^2_2 + \|r^{1,k}\|^2_2 + \frac{1}{\rho} \|r^{2,k}\|^2_2 - (\frac{1}{\rho} + 1) \|r^{1,k}\|_2 \|r^{2,k}\|_2 \\
\leq (L_1 \|r^{1,k} + r^{2,k}\|_2 + L_2^2 \|r^{1,k} + \frac{1}{\rho} r^{2,k}\|_2) \|x^{2,k} - \bar{x}^{2,k}\|_2.
\] (101)

Denote $b \triangleq L_1 \|r^{1,k} + r^{2,k}\|_2 + L_2^2 \|r^{1,k} + \frac{1}{\rho} r^{2,k}\|_2$, $c \triangleq \|r^{1,k}\|^2_2 + \frac{1}{\rho} \|r^{2,k}\|^2_2 - (\frac{1}{\rho} + 1) \|r^{1,k}\|_2 \|r^{2,k}\|_2$. Using Vieta’s formula for second order polynomial, \cite{101} implies
\[
\|x^{2,k} - \bar{x}^{2,k}\|_2 \leq \frac{b + \sqrt{b^2 - 4kc^2}}{2k^2}.
\] (102)

Note that the right-hand-side of \cite{102} is $O(\|r^{1,k}\| + \|r^{2,k}\|)$, so \cite{102} contradicts \cite{78}, which says $\|r^{1,k}\| + \|r^{2,k}\| = o(\|x^{2,k} - \bar{x}^{2,k}\|)$.

So far, we have shown that for a fixed $y$, there exist exist $\tau$ and $\delta$ satisfying \cite{59} and the inequality below it, accordingly. From \cite{68} and \cite{61}, we know $\nabla_x L_\rho(x^1, x^2; y) = 0$, so $\text{dist}(x, \textbf{X}(y))$ is also continuous in $y$, which implies that we can define a continuous mapping from $y$ to $\tau$ and $\delta$. Note that from the above proof, we know that for any $y$, $\tau$ is finite, i.e., $\tau < \infty$, and $\delta > 0$. Hence, since $y$ is in a compact set, we can find $\bar{\tau} \triangleq \sup \{\tau\} < \infty$ and $\bar{\delta} \triangleq \inf \{\delta\} > 0$. This finishes the proof of the lemma. \hfill \Box
**Theorem 3.1.** The sequence generated by the ADMM algorithm to solve (9) converges linearly to an optimal primal-dual optimal solution provided that the stepsize $\alpha$ is sufficiently small.

**Proof.** With Lemmas 3.1 and 3.3, the proof follows from the proof of Theorem 3.1 in [17].

### 4 Numerical experiments

#### 4.1 Simulation studies

This section provides our numerical studies on the performance of the proposed algorithm to evaluate the proximal operator of the LOG penalty. We compare the convergence rate of the proposed ADMM algorithm with the sharing scheme, Algorithm 3, with few other algorithms including the Block Coordinate Descent (BCD) Algorithm 1, Random Block Coordinate Descent (RBCD) [31], Proximal Gradient Descent (PGM) which is the ISTA algorithm [5], and Accelerated PGM (PGM-ACC) which is the FISTA algorithm [5], to find the proximal mapping of the LOG penalty for different graphs on simulated data. Figure 3 shows the four DAGs considered in the study. Three graphs are trees with different structures with 101, 101, and 127 nodes and the last one is a random DAG with 8 nodes, previously considered in [42]. Each node represents a single parameter. For each simulation $b \in \mathbb{R}^n$ is sampled from $\mathcal{N}(0, I)$, $\lambda = 0.5$, and $w_g = |g|^{1/2}$. The step size of PGM and PGM-ACC is numerically tuned to be equal to 0.01. Figure 4 shows $\epsilon$-optimality vs. iteration $k$ for both algorithms averaged over 10 replications. All simulations in this section are run on a laptop with 2.6GHz CPU and 16 GB memory using only one thread.

As we can see, Algorithm 3 shows a linear convergence for all graphs which matches the theoretical upper bound discussed in Section 3.1. We want to reiterate that the objective function within the proximal evaluation of the LOG penalty, i.e. (8), is not strongly convex. Compared to the other algorithms, the proposed ADMM algorithm converges faster. Note that the discrepancy between the algorithms is more significant for larger DAGs with more parameters, Figure 4 (a)-(c). The performance of PGM and PGM-ACC might be improved by adding a line search step to each iteration which was not in the consideration of this paper.

Finally, we want to reiterate that the first update of the proposed ADMM algorithm either with the sharing scheme, Algorithm 3, or without, Algorithm 2, is parallelizable over all of the groups. Hence, in the presence of parallel computing resources, the computationally dominant step, the step (4) in Algorithm 3, can be parallelized over all groups $\mathcal{G}$. This allows fast evaluation of the proximal mapping of the LOG penalty for large graphs which is generally embedded within a master optimization algorithm – see e.g. Section 4.2.

#### 4.2 Application

The proposed algorithm allows efficient evaluation of the proximal operator of the LOG penalty. Evaluating the proximal operator is generally needed iteratively within a master optimization algorithm that tries to solve an underlying statistical learning problem. Note that the master problem might be a convex or nonconvex optimization problem. To demonstrate practicality of the proposed algorithm, in this section, we consider two statistical learning problems on topic modeling and classification. The topic modeling application is a dictionary learning problem for NeurIPS proceedings. The second application relates to a breast cancer classification problem using gene expression data.
Figure 3: Four different DAGs considered in the simulation study

Figure 4: Convergence rate of four DAGs
4.2.1 Topic modeling of NeurIPS proceedings

We are interested in solving the topic modeling problem represented as the dictionary learning problem (103) penalized with the LOG penalty. Introduction of the LOG penalty is to force the resulting topics to form a tree structure [19]. The underlying statistical learning problem can be written as

$$\min_{D \in D^+_1, A \in \mathbb{R}^{k \times n}} \sum_{j=1}^{n} \left[ \frac{1}{2} \|x^j - DA^j\|^2_2 + \lambda \Omega_{\text{LOG}}(A^j) \right]$$

where $X = [x^1, x^2, \cdots, x^n] \in \mathbb{R}^{m \times n}$ represents frequencies of $m$ words in $n$ articles and the $i$-th element of $x^j$ is the frequency of the $i$-th word in the $j$-th article. $D = [d^1, d^2, \cdots, d^k] \in D^+_1$ is the dictionary of $k$ topics to be learnt where $D^+_1 \triangleq \{ D \in \mathbb{R}^{m \times k} : \|d^j\|_1 \leq 1, j = 1, 2, \cdots, k \}$. Furthermore, $A \triangleq [\alpha^1, \alpha^2, \cdots, \alpha^n] \in \mathbb{R}^{k \times n}$ is the corresponding coefficients for each article such that $x^j \approx DA^j$.

Following the framework of [19], we solve (103) using an alternating minimization scheme, i.e., updating $D$ and $A$ one at a time while keeping the other one fixed. The $D$ update is done using a BCD algorithm, taking its columns as the blocks, using the algorithm of [21]. The $A$ update is solved by proximal methods. To be more specific, we apply PGM and PGM-ACC [5]. Within these two algorithms, to evaluate the proximal operator of the LOG penalty our proposed algorithm with the sharing scheme is implemented. These three nested algorithms are implemented for the NeurIPS proceedings from 1996 to 2015 [29]. The dataset contains $n = 1846$ articles with $m = 11463$ words that excludes stop words and words occurring less than 50 times. We set $k = 13$, $\lambda = 2^{-15}$, and followed the hierarchical structure proposed by [19] to induce a tree of topics - see Figure 6. The experiment is run on a cluster with 2.4GHz CPU and 128GB memory using 28 threads. Note that the $A$ update can be calculated in parallel for each $x^j$ and $\alpha^j$ over $n = 1846$ articles.

The real and CPU times for both implementations are recorded for analysis. The real time is the elapsed time that considers the parallel computation and CPU time reflects the total calculation effort of the 28 threads. Figure 5 shows the convergence behavior of the two proximal methods. As expected, the PGM-ACC method is faster than PGM in both real and CPU time. It is worth mentioning that CPU time is approximately 13 times of the real time, which theoretically should be 28 times. A possible explanation is the unbalances for $\alpha^j$ updates. Note that the waiting time for idle threads is close to zero in CPU time while that is not the case for the real time.

The resulting topics based on PGM and PGM-ACC algorithms are indeed the same. Figure 6 depicts the learnt hierarchal topics with the 7 most frequent words. The root is a general topic while the leafs are more specific and narrower topics.

4.2.2 Breast cancer classification

This section discusses fitting a logistic regression model penalized with the LOG penalty to classify breast cancer based on gene expression levels. It is known that genes functionalities are highly affected by their two-way interactions which might be a priori known based on a protein-protein network. Hence, to identify contributing genes for cancer metastasis, it is important to consider such structures.

We use the breast cancer dataset of [38] that consists of 8141 gene expression data for 78 metastatic and 217 non-metastatic patients. Following the experimental settings in [26], we build groups of genes based on the protein-protein network of [12]. Every two genes connected directly by
an edge in the network are assigned as a group. Although these groups do not represent hierarchical structure, LOG penalty is used to capture the relationship within groups.

The genes that are not contained in the network are eliminated and the 500 most correlated genes are selected. The learning problem involves minimizing the logistic loss function regularized with the LOG penalty where \( \lambda \) parameter is set equal to \( 10^{-3} \). The underlying learning problem is solved by PGM and PGM-ACC algorithms while the proximal operator of the LOG penalty is evaluated by our proposed ADMM algorithm. The experiment is run on a cluster with 2.4GHz CPU and 128GB memory using a single thread.

Validation of the classification performance with the LOG penalty for such a problem is performed e.g. in [26]; so, we only focus on the convergence behavior of the proposed algorithm. The left plot in Figure 7 shows the convergence for the two proximal methods.

We also examine the effect of the LOG penalty for gene selection. For visual convenience, we increase \( \lambda \) to from 0.001 to 0.05 to make the regression coefficients sparser and evaluate the relationships of selected and unselected genes. The right plot in Figure 7 is a subset of the network of 500 genes. Each node represents a gene and the edges are known a priori from the protein-protein network. Nonzero coefficients in the final model identify genes which are correlated with breast cancer metastasis. From this result, it is clear that connected genes are prone to be selected simultaneously which supports the rationality of the LOG penalty for this application.

5 Concluding remarks

The paper discusses an efficient algorithm to find the proximal mapping of the LOG penalty to induce hierarchical sparsity structure represented by a DAG. The sharing scheme for the underlying ADMM algorithm allows maximum parallelization over (potentially) many number of groups which allows solving large-scale instances of the underlying learning problems which could involve convex or nonconvex loss functions. On the theoretical side, the paper establishes linear rate of convergence in the absence of strong convexity. The rate analysis is performed through the error bound theory.
which is an elegant theory not well explored in the literature, specifically for dual methods. Hence, the manuscript touches a rather unexplored area of dual error bound theory and its potentials to establish faster convergence rates for first-order methods.

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Figure 7: Breast cancer classification: (Left) Convergence of PGM and PGM-ACC methods (Right) Part of the gene network: red nodes are nonzero genes while blue nodes are zero ones. Edge structures are known a priori from the protein-protein network - see [12]

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A Appendix

A.1 Proof of (84) in Lemma 3.3

The following proof is inspired by [36] and [44]. Denote \( t^k \triangleq y + \rho(x^{1,k} - x^{2,k}) \). Note that if we write (10) as a function of \( x^2 \) and \( z \triangleq x^1 - x^2 \), then the last term is strongly convex in \( z \). This implies that the value of \( \bar{t} \triangleq y + \rho(x^1 - x^2) \) is unique for \( \forall(x^1, x^2) \in X(y) \). Recall the definition in (63) and (83), we will show (84) is equivalent to

\[
0 \in \lambda \partial w_g \| (\bar{x}^{1,k} + \epsilon \bar{u})_{j(g)} \| + \bar{t}_{j(g)}, \quad \forall g.
\]

(104)

From the optimality condition of (10), we know \( \hat{t} \) and (83), we will show (84) is equivalent to

\[
\left\{ \begin{array}{l}
0 \in \lambda \partial \sum_{g \in G} w_g \| x^1_{j(g)} \|_2 + (y + \rho(x^1 - \bar{x}^2))_{j(g)}, \quad \forall g \\
0 = M^T(M \bar{x}^{2,k} - b) - (y + \rho(x^1 - \bar{x}^2)),
\end{array} \right.
\]

(105)

is satisfied for some \( x^1 \). From (83), we have

\[
M \bar{x}^2 = M \bar{x}^2
\]

since \( M \bar{u} = 0 \). So the second equality of (105) holds if and only if \( x^1 = \bar{x}^1 + \epsilon \bar{u} \). Since \( 0 = M^T(M \bar{x}^{2,k} - b) - (y + \rho(x^1 - \bar{x}^2)) \) holds by definitions (63) and (62), (105) is equivalent to

\[
0 \in \lambda \partial \sum_{g \in G} w_g \| \bar{x}^1_{j(g)} + \epsilon \bar{u} \|_2 + (y + \rho(x^1 - \bar{x}^2))_{j(g)}, \quad \forall g.
\]

(107)

Using \( \bar{t} \) to replace \((y + \rho(x^1 - \bar{x}^2))\), we have (104).

Based on (78) and (81), we have

\[
t^k - \bar{t} = M^T(M \bar{x}^{2,k} - \bar{x}^{2,k}) - r^{2,k} = o(\delta^k).
\]

(108)

By further passing to a subsequence if necessary, we can assume that, for each \( g \in G \), either

1. \( \| x^1_{j(g)} - t^k_{j(g)} \|_2 \leq \lambda w_g, \quad \forall k, \) or,
2. \( \| x^1_{j(g)} - t^k_{j(g)} \|_2 > \lambda w_g, \) and \( \bar{x}^{1,k}_{j(g)} \neq 0, \quad \forall k, \) or,
3. \( \| x^1_{j(g)} - t^k_{j(g)} \|_2 > \lambda w_g, \) and \( \bar{x}^{1,k}_{j(g)} = 0, \quad \forall k, \)

is true. We will show that in any of the three above cases, \( \bar{u}_{j(g)} \) is a certain multiple of \( \bar{t}_{j(g)} \) and then (104) is satisfied.

\[\text{In fact, from our discussion on the uniqueness of } y + \rho(x^1 - x^2) \text{ for } \forall(x^1, x^2) \in X(y), \text{ we can also conclude that } x^1 \text{ must be } \bar{x}^1 + \epsilon \bar{u}.\]
1. In this case, from (69), we know
\[ \bar{u}_{j(g)} = \lim_{k \to \infty} \frac{\bar{x}_{j(g)}^{1,k} - \bar{x}_{j(g)}^{1,k}}{\delta^k} = \lim_{k \to \infty} \frac{\bar{r}_{j(g)}^{1,k} - \bar{x}_{j(g)}^{1,k}}{\delta^k} = \lim_{k \to \infty} -\frac{\bar{x}_{j(g)}^{1,k}}{\delta^k} \] (109)

where the last equation comes from (78). Suppose that \( \bar{u}_{j(g)} \neq 0 \). (Otherwise, \( \bar{x}_{j(g)}^{2,k} = \bar{x}_{j(g)}^{2,k} \).) Then \( \bar{x}_{j(g)}^{1,k} \neq 0 \) for all \( k \) sufficiently large. From the optimality condition for (10), we have
\[ 0 = \lambda w_g \frac{\bar{x}_{j(g)}^{1,k}}{\| \bar{x}_{j(g)}^{1,k} \|_2} + \bar{t}_{j(g)}, \] (110)

for \( k \) sufficiently large. By continuity, we have \( \bar{u}_{j(g)} \) is a positive multiple of \( \bar{t}_{j(g)} \). Furthermore, \( \bar{x}_{j(g)}^{1,k} \) is a negative multiple of \( \bar{t}_{j(g)} \). Therefore, for \( \epsilon \) sufficiently small, (104) is satisfied.

2. In this case, since we assumed \( \bar{x}_{j(g)}^{1,k} \neq 0 \ \forall k \), (110) is always satisfied. It implies
\[ \bar{t}_{j(g)} = \lambda w_g \frac{\bar{x}_{j(g)}^{1,k} - \bar{x}_{j(g)}^{1,k}}{\| \bar{x}_{j(g)}^{1,k} - \bar{x}_{j(g)}^{1,k} \|_2}. \] (111)

From (69), we have
\[ \bar{r}_{j(g)}^{1,k} = \frac{\lambda w_g}{\| \bar{x}_{j(g)}^{1,k} - \bar{t}_{j(g)}^k \|_2} \bar{x}_{j(g)}^{1,k} + (\frac{\| \bar{x}_{j(g)}^{1,k} - \bar{t}_{j(g)}^k \|_2}{\| \bar{x}_{j(g)}^{1,k} - \bar{t}_{j(g)}^k \|_2} - \lambda w_g) \bar{t}_{j(g)}^k \]
\[ = \frac{\lambda w_g}{\| \bar{x}_{j(g)}^{1,k} - \bar{t}_{j(g)}^k \|_2} (\bar{x}_{j(g)}^{1,k} + \delta^k \bar{u}_{j(g)}^k + o(\delta^k)) + (\frac{\| \bar{x}_{j(g)}^{1,k} - \bar{t}_{j(g)}^k \|_2}{\| \bar{x}_{j(g)}^{1,k} - \bar{t}_{j(g)}^k \|_2} - \lambda w_g) (\bar{t}_{j(g)} + o(\delta^k)) \]
\[ = \frac{\lambda w_g \delta^k}{\| \bar{x}_{j(g)}^{1,k} - \bar{t}_{j(g)}^k \|_2} \bar{u}_{j(g)}^k + (\frac{\| \bar{t}_{j(g)}^k - \bar{x}_{j(g)}^{1,k} \|_2}{\| \bar{t}_{j(g)}^k - \bar{x}_{j(g)}^{1,k} \|_2} - \lambda w_g) (\bar{t}_{j(g)} + o(\delta^k)) \]
\[ = \frac{\lambda w_g \delta^k}{\| \bar{t}_{j(g)} - \bar{x}_{j(g)}^{1,k} \|_2} \bar{u}_{j(g)}^k + (\frac{\| \bar{t}_{j(g)}^k - \bar{x}_{j(g)}^{1,k} \|_2}{\| \bar{t}_{j(g)}^k - \bar{x}_{j(g)}^{1,k} \|_2} - \lambda w_g) (\bar{t}_{j(g)} + o(\delta^k)) \]
\[ + \frac{\lambda w_g}{\| \bar{t}_{j(g)} - \bar{x}_{j(g)}^{1,k} \|_2} (\bar{t}_{j(g)} - \bar{x}_{j(g)}^{1,k} - \delta^k \bar{u}_{j(g)}^k + o(\delta^k)) (\bar{t}_{j(g)} - \bar{x}_{j(g)}^{1,k} + o(\delta^k)) \]

where the second equality comes from (82) and (108). The forth equality follows from (111). Finally, we use (82) and (108) in the last equality. From the Taylor expansion of \( \| \cdot \|_2 \) and
given that \( \nabla_x \|x\|_2^{-1} = -x/\|x\|^2 \), we have
\[
\frac{1}{\|\hat{t}_{j(g)} - \hat{x}_{j(g)}^{1,k} - 0\|_2} = \frac{1}{\|\hat{t}_{j(g)} - \hat{x}_{j(g)}^{1,k}\|_2} - \frac{\langle \hat{t}_{j(g)} - \hat{x}_{j(g)}^{1,k}, -\delta k u_{j(g)} + o(\delta k) \rangle}{\|\hat{t}_{j(g)} - \hat{x}_{j(g)}^{1,k}\|^3}
\]
\[+ o(\| - \delta k u_{j(g)} + o(\delta k)\|_2)
\]
\[= \frac{1}{\|\hat{t}_{j(g)} - \hat{x}_{j(g)}^{1,k}\|_2} + \frac{\langle \hat{t}_{j(g)} - \hat{x}_{j(g)}^{1,k}, \delta k u_{j(g)} \rangle}{\|\hat{t}_{j(g)} - \hat{x}_{j(g)}^{1,k}\|^3} + o(\delta k).
\]

Using this back in the last equation for \( r_{j(g)}^{1,k} \) and rearranging the terms, we have
\[
r_{j(g)}^{1,k} = \frac{\lambda w_g \delta k}{\|\hat{t}_{j(g)} - \hat{x}_{j(g)}^{1,k}\|_2} - \frac{\lambda w_g \langle \hat{t}_{j(g)} - \hat{x}_{j(g)}^{1,k}, \delta k u_{j(g)} \rangle}{\|\hat{t}_{j(g)} - \hat{x}_{j(g)}^{1,k}\|_2}(\hat{t}_{j(g)} - \hat{x}_{j(g)}^{1,k}) + o(\delta k)
\]
\[= \frac{\lambda w_g \delta k}{\|\hat{t}_{j(g)} - \hat{x}_{j(g)}^{1,k}\|_2} - \frac{\langle \hat{t}_{j(g)} - \hat{x}_{j(g)}^{1,k}, \delta k u_{j(g)} \rangle}{\|\hat{t}_{j(g)} - \hat{x}_{j(g)}^{1,k}\|_2} \lambda w_g + o(\delta k),
\]
where the second equality uses (111). Multiplying both sides by \( \frac{\|\hat{t}_{j(g)} - \hat{x}_{j(g)}^{1,k}\|_2}{\lambda w_g \delta k} \) and using (78),(79) and \( \|\hat{t}_{j(g)}\|_2 = \lambda w_g \) (from (111)) yields in the limit
\[
0 = u_{j(g)} - \frac{\langle \hat{t}_{j(g)}, \hat{u}_{j(g)} \rangle}{\|\hat{t}_{j(g)}\|^2} \hat{t}_{j(g)}.
\]
(112)
Thus \( u_{j(g)} \) is a nonzero multiple of \( \hat{t}_{j(g)} \). In this case, since we assume \( \hat{x}_{j(g)}^{1,k} \neq 0 \), from (110), we know \( \hat{x}_{j(g)}^{1,k} \) is a negative multiple of \( \hat{t}_{j(g)} \). So (104) is satisfied for \( \epsilon \) sufficiently small.

3. In this case, we assume \( \hat{x}_{j(g)}^{1,k} = 0, \forall k \), from (79), we have \( \hat{x}_{j(g)}^{1,k} = 0 \). We also assume that \( \|x_{j(g)}^{1,k} - t_{j(g)}^{1,k}\|_2 > \lambda w_g \) for all \( k \), this implies \( \|\hat{t}_{j(g)}\|_2 \geq \lambda w_g \). From the optimality condition for (10) for \( x^1 \) we have
\[
0 = \hat{t}_{j(g)} + \lambda w_g \partial \|0\|_2,
\]
which implies \( \|\hat{t}_{j(g)}\|_2 \leq \lambda w_g \). Thus \( \|\hat{t}_{j(g)}\|_2 = \lambda w_g \). Then (69) implies
\[
r_{j(g)}^{1,k} = \left( \frac{\lambda w_g}{\|x_{j(g)}^{1,k} - t_{j(g)}^{1,k}\|_2} - \frac{\lambda w_g}{\|x_{j(g)}^{1,k} - t_{j(g)}^{1,k}\|_2} \right) t_{j(g)} ^k
\]
\[= \frac{\lambda w_g}{\|x_{j(g)}^{1,k} - t_{j(g)}^{1,k}\|_2} \hat{t}_{j(g)} + \frac{\lambda w_g \langle \hat{t}_{j(g)}, t_{j(g)}^{1,k} - x_{j(g)}^{1,k} \rangle}{\|t_{j(g)}\|^3} \hat{t}_{j(g)}
\]
\[+ o(\|t_{j(g)}^{1,k} - x_{j(g)}^{1,k} \|_2)
\]
\[= \frac{\lambda w_g}{\|x_{j(g)}^{1,k} - t_{j(g)}^{1,k}\|_2} \hat{t}_{j(g)} + \frac{\lambda w_g \langle \hat{t}_{j(g)}, x_{j(g)}^{1,k} \rangle}{\|t_{j(g)}\|^3} \hat{t}_{j(g)} + o(\delta k)
\]
where the second equality uses Taylor expansion similar to the case 2. The third equality follows from (108) and \( \{x^{1,k}_j(g)\} \to 0 \). Dividing both sides by \( \delta^k \) yield in the limit (112), where it uses
\[
\frac{x^{1,k}_j(g)}{\delta^k} = \{u^{k}_j(g) + o(\delta^k)\}/u_j(g).
\]
Since we assume \( \|x^{1,k}_j(g) - t^{k}_j(g)\| > \lambda w_g \) for all \( k \), we have the following equality from (113)
\[
0 = \lambda w_g \frac{r^{1,k}_j(g) - x^{1,k}_j(g)}{\|r^{1,k}_j(g) - x^{1,k}_j(g)\|_2} + r^{1,k}_j(g) - t^{k}_j(g).
\]
Suppose \( \bar{u}_j(g) \neq 0 \). Then \( u^{k}_j(g) = x^{1,k}_j(g) + o(\delta^k) / \delta^k \neq 0 \), for \( k \) sufficiently large. It implies that \( x^{1,k}_j(g) \neq 0 \), for \( k \) sufficiently large. Hence,
\[
\langle \bar{t}_j(g), \bar{u}_j(g) \rangle = \lim_{k \to +\infty} \langle t^{k}_j(g), u^{k}_j(g) \rangle
\]
\[
= \lim_{k \to +\infty} \left( r^{1,k}_j(g), \frac{x^{1,k}_j(g)}{\delta^k} \right) + \lambda w_g \frac{r^{1,k}_j(g) - x^{1,k}_j(g)}{\|r^{1,k}_j(g) - x^{1,k}_j(g)\|} \left( \frac{x^{1,k}_j(g)}{\delta^k} - \frac{x^{1,k}_j(g) - \bar{u}_j(g)}{\|x^{1,k}_j(g) - \bar{u}_j(g)\|} \right)
\]
\[
= \lim_{k \to +\infty} \frac{\lambda w_g}{\|r^{1,k}_j(g) - x^{1,k}_j(g)\|} \left( \frac{r^{1,k}_j(g)}{\delta^k}, \frac{x^{1,k}_j(g)}{\|x^{1,k}_j(g)\|} \right) - \|u^{k}_j(g)\|
\]
\[
= -\lambda w_g \|u^{k}_j(g)\|_2 < 0,
\]
where the second equality is based on (113) and \( \lim_{k \to +\infty} u^{k}_j(g) = \lim_{k \to +\infty} \frac{x^{1,k}_j(g)}{\delta^k} \), the third equality is based on \( \frac{r^{1,k}_j(g)}{\delta^k} \to 0 \) (by (78)), the forth equality is based on \( x^{1,k}_j(g) \neq 0 \) and \( \lim_{k \to +\infty} u^{k}_j(g) = \lim_{k \to +\infty} \frac{x^{1,k}_j(g)}{\delta^k} \), and, the fifth equality is based on \( r^{1,k}_j(g) \to 0 \) and \( \frac{r^{1,k}_j(g)}{\delta^k} \to 0 \) (by (78)). Finally, combined with (112), we obtain \( \bar{u}_j(g) \) is a negative multiplier of \( \bar{t}_j(g) \). Since \( x^{1,k}_j(g) = 0 \) in this case, (104) is satisfied.