ON ESTIMATES FOR FULLY NONLINEAR ELLIPTIC EQUATIONS WITH NEUMANN BOUNDARY CONDITIONS ON RIEMANNIAN MANIFOLDS

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Abstract. We derive gradient and second order \textit{a priori} estimates for solutions of the Neumann problem for a general class of fully nonlinear elliptic equations on compact Riemannian manifolds with boundary. These estimates yield regularity and existence results.

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1. Introduction

Let $(\bar{M}^n, g)$ be a compact Riemannian manifold of dimension $n \geq 2$ with smooth boundary $\partial M$, and let $M$, $\nabla$ denote the interior of $\bar{M}$ and the Levi-Civita connection of $g$, respectively. For a function $u \in C^2(\bar{M})$ and a 2-tensor $A$ on $M$, let $\nabla^2 u$ denote the Hessian of $u$ and $\lambda[A] = (\lambda_1, \ldots, \lambda_n)$ the eigenvalues of $A$ with respect to the metric $g$. In this paper we are concerned with fully nonlinear elliptic equations of the form

\begin{equation}
(1.1) \quad f(\lambda[\nabla^2 u + \chi]) = \psi \quad \text{in } \bar{M}
\end{equation}

with Neumann boundary data

\begin{equation}
(1.2) \quad \nabla_\nu u = \varphi(x, u) \quad \text{on } \partial M,
\end{equation}

where $\nu$ denotes the interior unit normal to $\partial M$, and $f$ is a smooth symmetric function of $n$ variables defined in a symmetric open and convex cone $\Gamma$ in $\mathbb{R}^n$ containing $\Gamma_n$ with vertex at the origin, where

\begin{equation}
(1.3) \quad \Gamma_n \equiv \{\lambda \in \mathbb{R}^n : \text{all } \lambda_i > 0\}.
\end{equation}

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Following [2] we assume \( f \) to satisfy the fundamental structure conditions

\[
\frac{\partial f}{\partial \lambda_i} > 0 \text{ in } \Gamma, \quad 1 \leq i \leq n,
\]

(1.4)

\( f \) is a concave function in \( \Gamma \),

and

\[
\sup_{\partial \Gamma} f < \inf_M \psi
\]

(1.5)

where

\[
\sup_{\partial \Gamma} f = \sup_{\lambda_0 \in \partial \Gamma} \limsup_{\lambda \to \lambda_0} f(\lambda).
\]

A function \( u \in C^2(M) \) is called admissible if \( \lambda [\nabla^2 u + \chi] \in \Gamma \). Condition (1.4) ensures that (1.1) is elliptic for admissible solution \( u \in C^2(M) \), while (1.5) implies the function \( F \) defined by \( F(A) = f(\lambda[A]) \) is concave for \( A \in S^{n \times n} \) with \( \lambda[A] \in \Gamma \), where \( S^{n \times n} \) denotes the set of 2-tensors on \( M \). Condition (1.6) prevents equation (1.1) from being degenerate; see [2].

The most typical examples of form (1.1) are given by \( f = \sigma_k^{1/k} \) and \( f = (\sigma_k/\sigma_l)^{1/(k-l)} \), \( 1 \leq l < k \leq n \), defined on the cone

\[
\Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0 \text{ for } 1 \leq j \leq k \},
\]

(1.7)

where \( \sigma_k \) is the \( k \)-th elementary symmetric function

\[
\sigma_k(\lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}.
\]

(1.8)

There are other interesting functions satisfying (1.4) and (1.5) which naturally arise from important geometric problems.

The Neumann problem in uniformly convex domains in \( \mathbb{R}^n \) was treated by Lions-Trudinger-Urbas [17] for Monge-Ampère equation and recently by Ma-Qiu [18] for Hessian equations corresponding to \( f = \sigma_k^{1/k} \). Urbas [23, 24] studied the oblique boundary value problems for Hessian and curvature equations in two dimensions. Meanwhile, the Dirichlet problem has received much more attention and extensive study since work of Ivochkina [11] and Caffarelli-Nirenberg-Spruck [2]; see e.g. [4, 9, 10, 20, 28], etc.

Our primary goal in this paper is to establish \textit{a priori} estimates for admissible solutions of problem (1.1)-(1.2). For this we recall some notions and results from [9].
For $\sigma > \sup_{\partial \Gamma} f$, set $\Gamma^\sigma = \{ \lambda \in \Gamma : f(\lambda) > \sigma \}$. By conditions (1.4) and (1.5), the boundary of $\Gamma^\sigma$

$$\partial \Gamma^\sigma = \{ \lambda \in \Gamma : f(\lambda) = \sigma \},$$

which is a level hypersurface of $f$, is smooth and convex. Define for $\mu \in \Gamma$

$$S^\sigma_\mu = \{ \lambda \in \partial \Gamma^\sigma : \nu_\lambda \cdot (\mu - \nu) \leq 0 \}$$

and

$$C^+_\sigma = \{ \mu \in \Gamma : S^\sigma_\mu \text{ is compact} \}.$$

It was shown in [9] that $C^+_\sigma$ is open. We call $\partial C^+_\sigma$ the tangent cone at infinity of $\Gamma^\sigma$. The following assumption plays key roles in our results: there exist an admissible function $v \in C^2(\bar{M})$ satisfying

$$\lambda [\nabla^2 v + \chi] \in C^+_\psi.$$  

Throughout of this paper $\psi$ and $\varphi$ are assumed to be smooth functions, and let $u \in C^4(M) \cap C^3(\bar{M})$ be an admissible solution of the Neumann problem (1.1)-(1.2). Our first main result concerns the global second derivative estimates and may be stated as follows.

**Theorem 1.1.** Under conditions (1.4)-(1.6) and (1.9), there exists a constant $C$ depending $|u|_{C^1(M)}$ and other known data such that

$$(1.10) \quad \max_M |\nabla^2 u| \leq C \left( 1 + \max_{\partial M} |\nabla^2 u| \right).$$

If moreover,

$$\varphi = a(x)u + b(x),$$

where $a(x)$ and $b(x)$ are smooth functions, then

$$(1.11) \quad \max_M |\nabla^2 u| \leq C_1 \left( 1 + \max_M |\nabla u|^2 + \max_{\partial M} |\nabla^2 u| \right),$$

where $C_1$ is independent of $|\nabla u|_{C^0(M)}$.

Our next result concerns the gradient estimates.

**Theorem 1.2.** Assume in addition to (1.4)-(1.6) and (1.9) that

$$(1.12) \quad \sum f_i(\lambda)\lambda_i \geq -\omega_f(|\lambda^-|) \sum f_i \quad \text{in } \Gamma(\psi)$$

when $|\lambda|$ is sufficiently large, where $\lambda^- = (\lambda_1^-, \ldots, \lambda_n^-)$,

$$\Gamma(\psi) := \Gamma \cap \{ \inf_M \psi \leq f \leq \sup_M \psi \},$$
and \( \omega_f \geq 0 \) is a nondecreasing function satisfying the sublinear growth condition
\[
\lim_{t \to +\infty} \frac{\omega_f(t)}{t} = 0.
\]
Then
\[
\max_M |\nabla u| \leq C.
\]
for some constant \( C \) depending on \( |u|_{C^0(\bar{M})} \) and other known data.

Remark 1.3. It is not clear to the authors if (1.12) in fact always holds for some \( \omega_f \) satisfying (1.13); obviously it does for \( \omega_f(t) = t \).

Turning to the boundary estimates for second order derivatives, we need to strengthen our assumptions and impose restrictions to the underlying manifold and its boundary.

**Theorem 1.4.** Suppose that \((M^n, g)\) is locally conformally flat near boundary and \( \partial M \) is umbilic. Assume in addition to (1.4)-(1.6) and (1.9) that
\[
\sum f_i \lambda_i \geq 0 \quad \text{in } \Gamma(\psi)
\]
and that the function \( v \) satisfies
\[
\nabla \nu v \geq \sup_{a \leq t \leq b} \phi(x, t) + \epsilon_0 \quad \text{on } \partial M
\]
for some constant \( \epsilon_0 > 0 \) where \( a = \inf_M u, \ b = \sup_M u \). Suppose furthermore that there exists a function \( \underline{u} \in C^2(\bar{M}) \) satisfying
\[
\begin{cases}
\lambda [\nabla^2 \underline{u}] \in C^+_{\psi} \quad \text{in } M, \\
f(\lambda [\nabla^2 \underline{u}]) \geq \psi \quad \text{in } M, \\
u = 0 \quad \text{on } \partial M.
\end{cases}
\]
Then
\[
\max_{\partial M} |\nabla^2 u| \leq C
\]
for some constant \( C \) depending \( |u|_{C^1(\bar{M})} \) and other known data. If moreover,
\[
\phi = a(x)u + b(x)
\]
then
\[
\max_{\partial M} |\nabla^2 u| \leq C_1 \left( 1 + \max_M |\nabla u|^2 \right)
\]
where \( C_1 \) is independent of \( |\nabla u|_{C^0(\bar{M})} \).
More generally, we say \((\bar{M}^n, g)\) is \(\Gamma\)-admissible if there exists a function \(u \in C^2(\bar{M})\) with \(\lambda[\nabla^2 u] \in \Gamma\) in \(\bar{M}\) and \(u = 0\) on \(\partial M\). In particular, \(u\) is subharmonic, i.e. \(\Delta u \geq 0\) and therefore \(u \leq 0\) in \(\bar{M}\).

Remark 1.5. If the function \(\varphi\) in the Neumann condition (1.2) is independent of \(u\), in place of \(|u|_{C^0(\bar{M})}\) the constants \(C\) and \(C_1\) in the above estimates will only depend on
\[
\omega(u) := \sup_M u - \inf_M u.
\]

Remark 1.6. For \(f = \sigma_1^{1/k}, k \geq 2\), \(\partial \Gamma^\sigma\) is strictly convex and \(C^\sigma_\sigma = \Gamma_k\) for all \(\sigma > 0\). This follows from the property
\[
(1.20) \quad \lim_{R \to \infty} f(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n + R) = +\infty.
\]

Based on the estimates in Theorems 1.1, 1.2, 1.4 and the Evans-Krylov theorem due to Lieberman and Trudinger [16] we obtain the following existence result on the Neumann problem (1.1)-(1.2).

**Theorem 1.7.** Let \((M^n, g)\) be a smooth compact Riemannian manifold with smooth umbilic boundary \(\partial M\). Suppose that \(M\) is locally conformally flat near boundary and that for some constant \(\gamma_0 > 0\),
\[
(1.21) \quad \nabla_\nu u \geq \gamma_0.
\]
Assume that conditions (1.4)-(1.6), (1.9) and (1.15)-(1.17) hold with \(u\) furthermore satisfying
\[
(1.22) \quad f(\lambda[\nabla^2 u]) \geq \psi \quad \text{in} \quad \bar{M}
\]
and \(a, b\) in (1.16) given by
\[
(1.23) \quad a = \inf_M u + \frac{1}{\gamma_0} \inf_{x \in M} \varphi(x, 0), \quad b = \sup_M h + \frac{1}{\gamma_0} \sup_{x \in M} \varphi(x, 0),
\]
respectively, where \(h \in C^\infty(\bar{M})\) satisfies \(\Delta h + tr \chi = 0\) in \(\bar{M}\) and \(h = 0\) on \(\partial M\). Then the Neumann problem (1.1)-(1.2) admits a unique admissible solution \(u \in C^\infty(\bar{M})\).

Indeed, by (1.2) and (1.21) one immediately derives the lower and upper bounds by the maximum principle,
\[
(1.24) \quad \inf_M u + \frac{1}{\gamma_0} \inf_{x \in M} \varphi(x, 0) \leq u \leq \sup_M h + \frac{1}{\gamma_0} \sup_{x \in M} \varphi(x, 0).
\]
Consequently, one can apply the continuity method and the classical Schauder theory to prove the existence of an smooth admissible solution.
Equations of form (1.1), especially when $\chi$ and $\psi$ are allowed to depend on $u$ and $\nabla u$, naturally appear in interesting geometric problems such as Minkowski problem and the Christoffel-Minkowski problem. The Neumann problem for fully nonlinear equations on manifolds arises in the study of fully nonlinear versions of Yamabe problem on manifolds with boundary; we refer the reader to papers [26], [13] [12] [3] and references therein.

The rest of this paper is organized as follows. In Section 2 we shall prove gradient bounds for the admissible solutions. In Section 3 and Section 4 we shall derive the global and boundary estimates for second order derivative respectively.

2. Gradient estimates

In this section we derive the gradient estimates. In order to construct test functions, we shall first extend data on boundary to the whole manifold, which will also be used in later sections.

Let $d$ be the distance function to $\partial M$. Since $\partial M$ is smooth, $d$ is smooth in a neighborhood $M_{\delta_0} := \{ x \in M : d(x) < \delta_0 \}$ of $\partial M$ for $\delta_0 > 0$ sufficiently small. We modify $d$ outside $M_{\delta_0}/2$ into a smooth function such that

$$(\varphi_u + |\varphi_{uu}| + |\varphi_{uuu}|) d \leq \epsilon_0$$

for some sufficiently small constant $\epsilon \in (0, \delta_0/2]$ so in particular $2\varphi Ud \leq 1$; this will be used in the gradient estimates in this section. It is possible to do so by making $\delta_0$ smaller if necessary. So $\delta_0$ may depend on upper bounds of $\varphi_u + |\varphi_{uu}| + |\varphi_{uuu}|$.

Note that $\nu = \nabla d$ on $\partial \Omega \\nabla d$ denote the gradient of $d$. We may assume that $\nu$ has been extended to $\bar{M}$ by $\nu = \nabla d$. We shall also assume the function $\varphi$ has been smoothly extended to $\bar{M} \times \mathbb{R}$, and write $\tilde{\varphi}(x) = \varphi(x, u(x))$.

Let $\tilde{u} = u - \tilde{\varphi}d$, $w = 1 + |\nabla \tilde{u}|^2$ and $\phi = A - \tilde{u} - d$ where $A$ is a constant chosen sufficiently large so that $1 \leq \phi \leq 2A$ in $\bar{M}$. We assume the function $w\phi^{-1}$ attains a maximum at a point $x_0 \in \bar{M}$.

We first consider the case $x_0 \in \partial M$. Note that $\nabla_{\nu} \tilde{u} = 0$ on $\partial M$ by the boundary condition (1.2). We may choose orthonormal local frames $e_1, \ldots, e_n$ about $x_0$ so that $e_n = \nu$ and $\nabla_1 \tilde{u}(x_0) = |\nabla \tilde{u}(x_0)|$. Note that $d = 0$, $\nabla_1 d = 0$, $g(\nabla_n e_1, e_1) = 0$, and $\nabla_k \tilde{u} = 0$ for $k > 1$ at $x_0$. We have $\nabla_1 \tilde{u} = \nabla_1 u$,

$$\nabla_n (\nabla_1 \tilde{u}) = \nabla_{n1} \tilde{u} = \nabla_{1n} \tilde{u} = \nabla_1 (\nabla_n \tilde{u}) = -\tilde{\varphi} \nabla_1 d$$
and consequently,
\[ 0 \geq \phi \nabla w + w \nabla (\tilde{u} + d) \]
\[ = \phi \nabla \tilde{u} \nabla (\nabla \tilde{u}) + w \]
\[ = -\phi \tilde{\varphi} \nabla u \nabla d + w. \]

We derive a bound \( w(x_0) \leq C. \)

Suppose now that \( x_0 \in M. \) We choose orthonormal local frames about \( x_0 \) such that \( \nabla \epsilon = 0, \) and \( U_{ij} := \nabla u + \chi_{ij} \) is diagonal at \( x_0. \)

\[ \frac{\nabla_j w}{w} + \frac{\nabla_j \tilde{u} + \nabla_j d}{\phi} = 0 \]

and

\[ \phi F^{ii} \nabla_i w + w F^{ii}(\nabla_i \tilde{u} + \nabla_i d) \leq 0. \]

We calculate \( \nabla_j w = 2 \nabla_k \tilde{u} \nabla_{jk} \tilde{u}, \)
\[ \nabla_{ijk} \tilde{u} = (1 - \varphi_u d) \nabla_{ijk} u - d \nabla_{ijk} \varphi - \tilde{\varphi} \nabla_{ijk} d, \]
\[ = (1 - \varphi_u d) \nabla_{ijk} u - d \nabla_{ijk} \varphi - \tilde{\varphi} \nabla_{ijk} d, \]
\[ - (\nabla_{ij} d \varphi u - \tilde{\varphi} \nabla_{ij} d u) \]
\[ = \left[ 0 \right]_{W^1, \infty}, \]
\[ \nabla_{ijkl} \tilde{u} = (1 - \varphi_u d) \nabla_{ijkl} u - d \nabla_{ijkl} \varphi - \tilde{\varphi} \nabla_{ijkl} d, \]
\[ = (1 - \varphi_u d) \nabla_{ijkl} u - d \nabla_{ijkl} \varphi - \tilde{\varphi} \nabla_{ijkl} d, \]
\[ = (1 - \varphi_u d) \nabla_{ijkl} u - d \nabla_{ijkl} \varphi - \tilde{\varphi} \nabla_{ijkl} d, \]
\[ = \left[ 0 \right]_{W^1, \infty}, \]

Write equation \( (1.1) \) in the form
\[ F(U) := f(\lambda[U]) = \psi \]
for \( U = \nabla^2 u + \chi, \) and denote
\[ F^{ij} = \frac{\partial F}{\partial U_{ij}}(U), \]
\[ F^{ij,kl} = \frac{\partial^2 F}{\partial U_{ij} \partial U_{kl}}(U). \]
The matrix \( \{F^{ij}\} \) is positive definite by assumption (1.4) with eigenvalues \( f_1, \ldots, f_n \), and \( F \) is a concave function by (1.5); see [2]. Moreover, the following identities hold for \( \lambda[U] = (\lambda_1, \ldots, \lambda_n) \),

\[
F^{ij} U_{ij} = \sum f_i \lambda_i,
\]

and

\[
F^{ij} U_{ik} U_{kj} = \sum f_i \lambda_i^2.
\]

Differentiate equation (2.6), we obtain

\[
F^{ij} \nabla_k U_{ij} = \nabla_k \psi, \quad \text{for all } k.
\]

Note that \( \{F^{ij}\} \) is diagonal at \( x_0 \) since so is \( U_{ij} \). It follows that

\[
F^{ii} \nabla ii \tilde{u} \geq (1 - \varphi_u d) F^{ii} U_{ii} - C dw \sum F^{ii}
\]

and, by Schwarz inequality,

\[
F^{ii} \nabla ii w \geq 2(1 - \varphi_u d)^2 F^{ii} U_{ii}^2 - C |\varphi_{uuu}| dw^2 \sum F^{ii} - C dw F^{ii} |U_{ii}|
\]

\[
- C \varphi_u^2 d^2 w^2 \sum F^{ii} - C w^2 \sum F^{ii} - C \sqrt{w} F^{ii} |U_{ii}|
\]

\[
- C w \sum F^{ii} - 2(1 - \varphi_u d)^2 |\nabla \psi| \sqrt{w}.
\]

Assume \( |\nabla \tilde{u}| \geq |\nabla u|/4 \) and let \( I = \{i : n |\nabla_i \tilde{u}| \geq |\nabla \tilde{u}|\} \). We see that \( I \neq \emptyset \) and by (2.2) for \( i \in I \),

\[
(1 - \varphi_u d) U_{ii} \leq - \frac{w}{2\phi} + \frac{w}{|\nabla \tilde{u}|} + C \leq - \frac{w}{8A} + C \equiv -K.
\]

We shall assume \( w \) to be sufficiently large, and in particular \( K \geq \frac{w}{16A} \) (otherwise we are done) so that we may apply Theorem 2.17 (or Theorem 2.18 more directly) in [9] to \( \mu = \lambda(\nabla^2 v(x_0) + \chi(x_0)) \) and \( \lambda = \lambda(\nabla^2 u(x_0) + \chi(x_0)) \) to derive

\[
- \sum_{U_{ii} < 0} F^{ii} U_{ii} \geq F^{ii}(\nabla ii v - \nabla ii u) - F^{ii}(\nabla ii v + \chi ii) \geq \varepsilon - C \sum F^{ii}.
\]

Let \( J = \{i : U_{ii} \leq -K\} \) so \( I \subseteq J \). Clearly,

\[
F^{ii} U_{ii}^2 \geq K^2 \sum j F^{ii} \geq \frac{K^2}{n} \sum F^{ii}.
\]

Therefore,

\[
F^{ii} U_{ii}^2 \geq -K \sum j F^{ii} U_{ii} \geq - \frac{K}{n} \sum_{U_{ii} < 0} F^{ii} U_{ii} \geq \frac{\varepsilon K}{n} - CK \sum F^{ii}.
\]
Finally, note that \(1 - \varphi_u d \geq \frac{1}{2}\) and \(d\) small enough. By (2.11)-(2.14) and (2.3) we derive
\[
\left( c_0 K^2 - CK^2 + CK - C \right) \sum F^{ii} + c_0 \varepsilon K - CK^2 \leq w(\varphi_u d - 1) F^{ii} U^{ii}
\]
for some \(c_0 > 0\). This gives a bound \(K \leq C\), completing our proof of the gradient estimates, provided that \(f\) satisfies (1.12).

3. Global estimates for second order derivative

In this we shall derive the second order derivative estimate (1.10) in Theorem 1.1. Motivated by [17] and [18] we consider the following quantity. Let
\[
W(x, \xi) = \nabla_{\xi \xi} u + \chi_{\xi \xi} - 2 g(\xi, \nu) (\nabla_{\xi \xi} \tilde{\varphi} - \nabla_{\nu \xi} u + \chi_{\xi \nu})
\]
for \(x \in \bar{M}\) and \(\xi \in T_x M\), where \(\xi' = \xi - g(\xi, \nu) \nu\), \(\nu_{\xi'} = \nabla_{\xi'} \nu\) and \(\tilde{\varphi}(x) = \varphi(x, u)\); for convenience we shall write
\[
W'(x, \xi) = 2 g(\xi, \nu) (\nabla_{\xi} \tilde{\varphi} - \nabla_{\nu \xi} u).
\]
Let \(\eta = \eta(u, |\nabla u|)\) be a function to be determined. We consider
\[
W' = \max_{x \in M} \max_{\xi \in T_x M, |\xi| = 1} W e^\eta.
\]
Our goal is to derive a bound for \(W\) which we shall assume to be positive. Suppose that \(W\) is achieved at a point \(x_0 \in M\) for some unit vector \(\xi \in T_{x_0} M\). We shall consider separately two different cases: (a) \(x_0 \in M\) and (b) \(x_0 \in \partial M\). In this section we consider case (a) while case (b) will be treated in Section 4.
Assume now that \(x_0\) is an interior point. We choose smooth orthonormal local frames \(e_1, \ldots, e_n\) about \(x_0 \in M\) such that \(e_1 = \xi\), \(\nabla_i e_j = 0\) so \(\Gamma^k_{ij} = 0\) for all \(1 \leq i, j, k \leq n\), and \(U_{ij} = \nabla_{ij} u + \chi_{ij}\) is diagonal at \(x_0\). Denote \(W = W(x, e_1)\) and
\[
W' = W'(x, e_1) = 2 g(e_1, \nu) (\nabla_{e_1} \varphi + \varphi_u \nabla_{e_1} u - \nabla_{e_1} k d \nabla_k u),
\]
which are locally defined near \(x_0\). In what follows we modify the argument in [9] to derive a bound for \(W\) at \(x_0\); note that we can not use directly the result there though.
The function \(\log W + \eta\) attains its maximum at \(x_0\) and therefore for \(i = 1, \ldots, n\),
\[
\frac{\nabla_i W}{W} + \nabla_i \eta = 0,
\]
and
\[
\frac{\nabla_{ii} W}{W} - \left( \frac{\nabla_i W}{W} \right)^2 + \nabla_{ii} \eta \leq 0.
\]
Differentiating equation (2.6) twice we obtain at \(x_0\),

\[
F^{ii} \nabla_{11} U_{ii} + \sum F^{ij,kl} \nabla_1 U_{ij} \nabla_1 U_{kl} = \nabla_{11} \psi.
\] (3.5)

It follows that

\[
F^{ii} \nabla_{ii} U_{11} \geq - F^{ij,kl} \nabla_1 U_{ij} \nabla_1 U_{kl} + \nabla_{11} \psi - C(1 + |\nabla u| + W) \sum F^{ii}
\] (3.6)

and

\[
F^{ii} \nabla_{ii} W' \leq 2g(e_1, \nu) F^{ii} \nabla_{ii} (\varphi_u \nabla e_1^i u - \nabla e_1^i k d \nabla k u)
+ C(1 + |\nabla u|^2 + W) \sum F^{ii}
\leq C(1 + |\nabla u|^2 + W + Z) \sum F^{ii} + C,
\] (3.7)

where

\[
Z = |\varphi_{uuu}| |\nabla u|^3 + |\varphi_{uu}| |\nabla u| W.
\]

We now plug (3.6) and (3.7) in (3.4) to derive

\[
WF^{ii} \nabla_{ii} \eta \leq E + C_2 \sum F^{ii},
\] (3.8)

where \(C_2 = C(1 + |\nabla u|^2 + W + Z)\) and

\[
E = F^{ij,kl} \nabla_1 U_{ij} \nabla_1 U_{kl} + \frac{1}{W} \sum F^{ii} (\nabla_i W)^2.
\]

As in [9, 10] to estimate \(E\) we follow an idea of Urbas [25]. Let \(0 < s < 1\) to be chosen and

\[
J = \{ i : U_{ii} \leq -s U_{11} \},
K = \{ i : U_{ii} > -s U_{11}, s F^{ii} \geq F^{11} \},
L = \{ i : U_{ii} > -s U_{11}, s F^{ii} < F^{11} \}.
\]

It was shown by Andrews [1] and Gerhardt [6], also earlier by Caffarelli, Nirenberg and Spruck, that

\[
-F^{ij,kl} \nabla_1 U_{ij} \nabla_1 U_{kl} \geq \sum_{i \neq j} \frac{F^{ii} - F^{jj}}{U_{jj} - U_{ii}} (\nabla_1 U_{ij})^2.
\]
By (3.3) and Schwarz inequality we obtain

\[-F^{ij,kl} \nabla_i U_{ij} \nabla_k U_{kl} \geq 2 \sum_{i \geq 2} F^{ii} \frac{F^{11}}{U_{11}} (\nabla_i U_{i1})^2\]

\[\geq \frac{2(1-s)}{(1+s)U_{11}} \sum_{i \in K} F^{ii} (\nabla_i U_{i1})^2\]

\[\geq \frac{2(1-s)^2}{(1+s)U_{11}} \sum_{i \in K} F^{ii} ((\nabla_i U_{i1})^2 - C|\nabla u|^2/s)\]

\[\geq \frac{2(1-s)^3}{(1+s)U_{11}} \sum_{i \in K} F^{ii} (\nabla_i W)^2\]

\[-C \frac{1}{sU_{11}} \sum_{i \in K} F^{ii} (|\nabla u|^2 + (\nabla_i W')^2).\]

By straightforward calculations,

\[(3.10) \quad |\nabla W'|^2 \leq C(W^2 + |\nabla u|^2 + \varphi_{uu} |\nabla u|^4).\]

Next, we may assume \(|W'| \leq sU_{11}\) at \(x_0\) and fix \(s = 1/9\) so that

\[(3.11) \quad (1-s)U_{11} \leq W \leq (1+s)U_{11}\]

and

\[(3.12) \quad \frac{2(1-s)^3W}{(1+s)U_{11}} \geq \frac{2(1-s)^4}{(1+s)} \geq 1.\]

By (3.9)–(3.12) we obtain

\[(3.13) \quad E \leq \frac{1}{W} \sum_{i \in J \cup L} F^{ii} (\nabla_i W)^2 + C_3 \sum_{i \in K} F^{ii},\]

where

\[C_3 = CW + \frac{|\nabla u|^2 (1 + |\varphi_{uu}| |\nabla u|^2)}{W}.\]

Following [10] we take \(\phi(t) = -\log(1 - \gamma bt)\) and

\[\eta = \phi(|\nabla u|^2) + a(v - u),\]

where \(a > 0\) and \(\gamma \in (0, 1/2]\) are constant to be determined, \(b = 1/2b_1\) and

\[b_1 = \max_{M} (1 + |\nabla u|^2).\]

So

\[\frac{\gamma b}{2} \leq \phi'(|\nabla u|^2) = \sqrt{\phi''(|\nabla u|^2)} = \frac{\gamma b}{1 - b^2} \leq 2\gamma b.\]
We calculate
\begin{equation}
\nabla_i \eta_i = 2 \phi'(\nabla_k u \nabla_{ik} u + a \nabla_i (v - u)) = 2 \phi'(U_{ii} \nabla_i u) + a \nabla_i (v - u),
\end{equation}
\begin{equation}
\nabla_{ii} \eta = 2 \phi' (\nabla_{ik} u \nabla_{ik} u + \nabla_k u \nabla_{iik} u + 4 \phi'' (\nabla_k u \nabla_{ik} u)^2 + a \nabla_{ii} (v - u)).
\end{equation}

Therefore,
\begin{equation}
\sum_{i \in J \cup L} F_{ii} (\nabla_i \eta_i)^2 \leq C a^2 |\nabla u|^2 \sum_{i \in J \cup L} F_{ii} + C (\phi')^2 |\nabla u|^2 F_{ii} U_{ii}^2
\end{equation}
and
\begin{equation}
F_{ii} \nabla_{ii} \eta \geq \phi' F_{ii} U_{ii}^2 + 4 \phi'' F_{i} (\nabla_k u \nabla_{ik} u)^2 + a F_{ii} \nabla_{ii} (v - u)
- C \phi' |\nabla u|^2 \sum_{i} F_{ii} - C \phi' |\nabla u|.
\end{equation}

By (3.3), (3.8), (3.13), (3.16) and (3.17), we derive
\begin{equation}
\phi' F_{ii} U_{ii}^2 + a F_{ii} \nabla_{ii} (v - u) \leq C a^2 |\nabla u|^2 \sum_{i \in J \cup L} F_{ii} + C (\phi')^2 |\nabla u|^2 F_{ii} U_{ii}^2
+ C (1 + |\varphi_{uu}||\nabla u| + |\varphi_{uuu}||\nabla u|^3 W^{-1}) \sum_{i} F_{ii}
+ \frac{C |\nabla u|^2 (1 + |\varphi_{uu}||\nabla u|^2)}{W^2} \sum_{i} F_{ii} - \frac{\nabla_{11} \psi}{W} + C.
\end{equation}

Next, note that
\begin{equation}
F_{ii} U_{ii}^2 \geq F_{11} U_{11}^2 + \sum_{i \in J} F_{ii} U_{ii}^2 \geq F_{11} U_{11}^2 + s^2 U_{11}^2 \sum_{i \in J} F_{ii}.
\end{equation}

We now fix \( \gamma \leq 1/16C \) so that
\[ \phi' - C (\phi')^2 |\nabla u|^2 \geq \gamma b \frac{b}{2} - 4C \gamma^2 b = \frac{\gamma b (1 - 8C \gamma)}{2} \geq \frac{\gamma b}{4}. \]

By Theorem 2.17 in [9] we may fix \( a \geq A (1 + |\varphi_{uu}||\nabla u|) \) for \( A \) sufficiently large to obtain
\[ a W \leq C (|\varphi_{uuu}||\nabla u|^3 + |\nabla^2 \psi|) \]
or
\[ U_{11} (x_0) \leq C a (1 + |\nabla u|^2) \leq C (1 + |\nabla u|^2 + |\varphi_{uu}||\nabla u|^3). \]

Consequently,
\[ W \leq C (1 + |\nabla u|^2) + C (|\varphi_{uu} + |\varphi_{uuu}||\nabla u|^3). \]

where \( C \) is independent of \( |\nabla u|_{C^0(\bar{M})} \). The proof of Theorem 1.1 is complete.
4. Second derivative estimates on boundary

In this section we consider case (b) $x_0 \in \partial M$. First assume that $0 < |\xi'| < 1$ and let $\tau = |\xi'|^{-1}\xi'$. We have
\[
\nabla_{\xi\xi} u = \nabla_{\xi\xi'} u + 2g(\xi, \nu)(\nabla_{\xi'}(\nabla_{\nu} u) - \nabla_{\nu'} u) + (1 - |\xi'|^2) \nabla_{\nu\nu} u
\]
\[
= |\xi'|^2 \nabla_{\tau\tau} u + 2g(\xi, \nu)(\nabla_{\xi'}(\nabla_{\nu} u) - \nabla_{\nu'} u) + (1 - |\xi'|^2) \nabla_{\nu\nu} u.
\]

It follows that
\[
W(x_0, \xi) = |\xi'|^2 W(x_0, \tau) + (1 - |\xi'|^2) W(x_0, \nu)
\]
\[
\leq |\xi'|^2 W(x_0, \xi) + (1 - |\xi'|^2) W(x_0, \nu)
\]
which implies
\[
(4.1) \quad W(x_0, \xi) \leq W(x_0, \nu).
\]

Consequently, we only need to consider the following two cases: (i) $|\xi'| = 1$ and (ii) $|\xi'| = 0$.

Case (i) $|\xi'| = 1$. So $\xi$ is tangential to $\partial M$ at $x_0$. We choose smooth orthonormal local frames $e_1, \ldots, e_n$ around $x_0$ such that $e_n = \nu$ along $\partial M$ and $e_1 = \xi$ at $x_0$. Write $U_{11} = \nabla_{11} u + \chi_{11}$. At $x_0$ we have
\[
(4.2) \quad 0 \geq \nabla_n W + W \nabla_n \eta \geq \nabla_n U_{11} - \nabla_n W' + W \nabla_n \eta.
\]

Since $g(\xi, \nu) = 0$, we have $W' = 0$ and
\[
(4.3) \quad \nabla_n W' \geq -C(1 + |\nabla u|).
\]

By the boundary condition (1.2) we have for $k < n$,
\[
(4.4) \quad \nabla_{kn} u = \nabla_k(\nabla_n u) - \nabla_{k,e_n} u = \nabla_k \tilde{\varphi} + b_{kl} \nabla_l u,
\]
where $\{b_{kl}\}$ denotes the second fundamental form of $\partial M$, and
\[
\nabla_{11n} u = \nabla_{11}(\nabla_n u) - 2g(\nabla_{e_1} e_n, e_1) \nabla_{11} u - \nabla_{11 e_n} u
\]
\[
\geq \nabla_{11} \tilde{\varphi} + 2b_{11} \nabla_{11} u - C|\nabla u|
\]
\[
\geq (\varphi_u + 2b_{11}) \nabla_{11} u - C(1 + |\nabla u|^2).
\]

Next, note that $\nabla_{1k} u(x_0) = 0$ for all $k > 1$. Indeed, let
\[
e_k = e_1 \cos \theta + e_k \sin \theta.
\]
We have
\[
\nabla_{e_e e_\theta} u = \nabla_{11} u \cos^2 \theta + 2\nabla_{1k} u \cos \theta \sin \theta + \nabla_{kk} u \sin^2 \theta.
\]
It follows that
\[
\frac{\partial}{\partial \theta} \nabla_{\varepsilon \varepsilon \theta} u = (\nabla_{kk} u - \nabla_{11} u) \sin 2\theta + \nabla_{1k} u \cos 2\theta.
\]
and
\[
2\nabla_{1k} u = \frac{\partial}{\partial \theta} \bigg|_{\theta=0} \nabla_{\varepsilon \varepsilon \theta} u = 0.
\]
On the other hand, \( g(\nabla_n e_1, e_1) = 0 \) since \( |e_1| = 1 \). By (4.5) it follows that
\[
(4.6) \quad \nabla_n U_{11} = \nabla_n (\nabla_{11} u) + \nabla_n \chi_{11}
\]
\[
= \nabla_{111} u + \nabla_n \chi_{11}
\]
\[
\geq (\varphi_u + 2b_{11}) \nabla_{11} u - C(1 + |\nabla u|^2)
\]
\[
\geq (\varphi_u + 2b_{11}) W - C(1 + |\nabla u|^2).
\]

From Section 3
\[
\eta = \phi(|\nabla u|^2) + a(v - u),
\]
where \( a \) is a sufficiently large positive constant, \( v \) as in (1.9), and \( \phi \) is a nondecreasing function satisfying \( t\phi'(t) \leq 1 \). By (4.4),
\[
(4.7) \quad \nabla_n \eta = 2\phi' \nabla_k u \nabla_{nk} u + a \nabla_n (v - u)
\]
\[
\geq \epsilon_0 a + 2\phi' \varphi \nabla_{nn} u - C\phi'(1 + |\nabla u|^2).
\]

Finally, it follows from (4.2), (4.3), (4.6) and (4.7) that
\[
(4.8) \quad (\epsilon_0 a + 2\phi' \varphi \nabla_{nn} u + \varphi_u + 2b_{11} - C\phi'|\nabla u|^2) W \leq C(1 + |\nabla u|^2).
\]

By (4.10) below we derive
\[
(4.9) \quad W \leq C(1 + |\nabla u|)
\]
provided that \( a \) is sufficiently large, independent of \( |u|_{C^1(\bar{M})} \), so that
\[
\epsilon_0 a + 2\phi' \varphi \nabla_{nn} u + \varphi_u + 2b_{11} - C\phi'|\nabla u|^2 \geq 1.
\]

In order to derive a bound for \( W \) in both cases (i) and (ii), it is therefore enough to establish the double normal derivative estimate
\[
(4.10) \quad |\nabla_{\nu\nu} u| \leq C(1 + |\nabla u|) \quad \text{on } \partial M.
\]
The rest of this section is devoted to this estimate.
Consider an arbitrary point \(x_0 \in \partial M\). We choose smooth orthonormal local frames \(e_1, \ldots, e_n\) around \(x_0\) as before such that \(e_n = \nu\) along \(\partial M\). We shall use \(\rho(x)\) to denote the distance function from \(x\) to \(x_0\),

\[
\rho(x) \equiv \text{dist}_M(x, x_0),
\]

and let \(M_\delta = \{x \in M : \rho(x) < \delta\}\). Note that \(\nabla_{ij} \rho^2(x_0) = 2\delta_{ij}\). We may assume

\[
\{\delta_{ij}\} \leq \{\nabla_{ij} \rho^2\} \leq 3\{\delta_{ij}\}\]

in \(M_\delta\).

Differentiating equation (2.6) we obtain near \(x_0\),

\[
F_{ij} \nabla_{ij}(\nabla_k u) = F_{ij}(\nabla_{ijk} u + \Gamma^l_{jk} \nabla_l u + \Gamma^l_{ij} \nabla_l u)
\]

\[
= F_{ij} \nabla_{kij} u + F_{ij} R_{ijkl} \nabla_l u + 2F_{ij} \Gamma^l_{jk} \nabla_l u
\]

\[
= \nabla_k \psi + F_{ij} R_{ijkl} \nabla_l u + 2F_{ij} \Gamma^l_{jk} \nabla_l u,
\]

where \(R_{ijkl}\) and \(\Gamma^l_{jk}\) denote the Riemannian curvature tensor and Christoffel symbols, respectively.

We now make use of the assumptions that \(M\) is locally conformally flat near \(\partial M\) and \(\partial M\) is umbilic. It follows that when \(\delta\) is sufficiently small,

\[
\Gamma^l_{jn} = g(e_l, \nabla_j e_n) = -\kappa_d \delta_{jl} \quad \text{in } M_\delta,
\]

where \(\kappa_d(x)\) denotes the principal curvature of the level hypersurface of \(d\) passing through \(x \in M_\delta\). Therefore,

\[
F_{ij} \nabla_{ij}(\nabla_n u - \bar{\varphi}) = 2F_{ij} \Gamma^l_{jn} \nabla_l u - \varphi_u F_{ij} \nabla_{ij} u + Q
\]

\[
= -(2\kappa_d + \varphi_u) F_{ij} U_{ij} + Q,
\]

where

\[
|Q| \leq C(1 + |\nabla u|) \sum F^{ii} + C,
\]

and

\[
F_{ij} \nabla_{ij}(\nabla_n u - \bar{\varphi})^2 = 2(\nabla_n u - \bar{\varphi}) F_{ij} \nabla_{ij}(\nabla_n u - \bar{\varphi})
\]

\[
+ 2F_{ij} \nabla_i(\nabla_n u - \bar{\varphi}) \nabla_j(\nabla_n u - \bar{\varphi})
\]

\[
\geq F_{ij} U_{in} U_{nj} - C|F_{ij} U_{ij}| - C(1 + |\nabla u|) \sum F^{ij}.
\]

We now construct a barrier function as follows. Let

\[
H = A_1 u - A_2(d - Nd^2) - A_3 \rho^2,
\]
where $A_1$, $A_2$, $A_3$ and $N$ are positive constants to be chosen later; we shall assume $2N\delta \leq 1$ by fixing $\delta$ small after $N$ is determined. We first calculate

$$F^{ij} \nabla_{ij}(d - Nd^2) = (1 - 2Nd)F^{ij} \nabla_{ij}d - 2NF^{ij} \nabla_{ij}d \nabla_j d$$

\[(4.18)\]

$$\leq - 2NF^{ij} \nabla_{ij}d \nabla_j d + C \sum F^{ii}.$$  

At a fixed point $x \in \overline{M}_\delta$, let $\mu = \lambda[\nabla^2 u]$ and $\lambda = \lambda[U]$ denote the eigenvalues of $\nabla^2 u$ and $U$, respectively. By assumption (1.17) and Theorem 2.17 in [9], there are positive constants $R$ and $\epsilon$ depending such that

$$\sum f_i(\lambda)(\mu_i - \lambda_i) \geq \epsilon \left(1 + \sum f_i\right)$$

\[(4.19)\]

provided that $|\lambda| \geq R$.

We first assume $|\lambda| \leq R$. In this case, there are uniform bounds

$$0 < \alpha \leq f_i(\lambda) \leq \beta,$$

\[(4.20)\]

with $\alpha$, $\beta$ depending on $R$. Therefore, $F^{ij} \nabla_{ij}d \nabla_j d \geq \alpha$ since $|\nabla d| \equiv 1$. We now fix $N$ sufficiently large and then $\delta > 0$ such that $2N\delta \leq 1$ and, by (4.18),

$$F^{ij} \nabla_{ij}(d - Nd^2) \leq - \alpha N \sum F^{ii} \leq -\alpha^2 N.$$  

\[(4.21)\]

It is clear that $H(0) = 0$ and

$$H \leq -A_3\rho^2 \text{ in } M_\delta.$$  

Since $\nabla_n u - \tilde{\varphi} = 0$ on $\partial M$, we may fix $A_3$ and then $A_2$ large such that

$$|\nabla_n u - \tilde{\varphi}| \leq A_3\rho^2 \leq -H \text{ on } \partial M_\delta,$$

\[(4.22)\]

and

$$F^{ij} \nabla_{ij}(H \pm (\nabla_\nu u - \varphi)) \geq (A_2\alpha N - 3A_3 - C - CR) \sum F^{ii} - C \geq 0.$$  

\[(4.23)\]

Here we have used the fact that

$$F^{ij} \nabla_{ij}u \geq F(\nabla^2 u) - F(U) + \sum F^{ij} U_{ij} \geq 0$$

\[(4.24)\]

by the concavity of $f$ and assumptions (1.15) and (1.17).

Suppose now that $|\lambda| > R$. We note that by assumption (1.15) and the concavity of $f$,

$$0 \leq F^{ij} U_{ij} \leq F(U) - F(g) + \sum F^{ii} \leq \sum F^{ii} + C.$$  

By (4.19) we may finally fix $A_1$ such that

$$F^{ij} \nabla_{ij}(H \pm (\nabla_\nu u - \varphi)) \geq A_1 F^{ij} \nabla_{ij}u - C(A_2 + 3A_3 + 1) \sum F^{ii} - C \geq 0.$$  

\[(4.25)\]
Consequently, by the maximum principle we obtain
\[ H \pm (\nabla_\nu u - \varphi) \leq 0 \text{ in } M_\delta \]
and therefore,
\[ |\nabla_{nn} u| \leq \nabla_n H + |\nabla_n \varphi| \leq C(1 + |\nabla u|). \]
This completes the proof of Theorem 1.7.

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