The K-Z Equation and the Quantum-Group Difference Equation in Quantum Self-dual Yang-Mills Theory

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Abstract

From the time-independent current $\tilde{J}(\tilde{y}, \tilde{k})$ in the quantum self-dual Yang-Mills (SDYM) theory, we construct new group-valued quantum fields $\tilde{U}(\tilde{y}, \tilde{k})$ and $\tilde{U}^{-1}(\tilde{y}, \tilde{k})$ which satisfy a set of exchange algebras such that fields of $\tilde{J}(\tilde{y}, \tilde{k}) \sim \tilde{U}(\tilde{y}, \tilde{k}) \partial \tilde{y} \tilde{U}^{-1}(\tilde{y}, \tilde{k})$ satisfy the original time-independent current algebras. For the correlation functions of the products of the $\tilde{U}(\tilde{y}, \tilde{k})$ and $\tilde{U}^{-1}(\tilde{y}, \tilde{k})$ fields defined in the invariant state constructed through the current $\tilde{J}(\tilde{y}, \tilde{k})$ we can derive the Knizhnik-Zamolodchikov (K-Z) equations with an additional spatial dependence on $\tilde{k}$. From the $\tilde{U}(\tilde{y}, \tilde{k})$ and $\tilde{U}^{-1}(\tilde{y}, \tilde{k})$ fields we construct the quantum-group generators — local, global, and semi-local — and their algebraic relations. For the correlation functions of the products of the $\tilde{U}$ and $\tilde{U}^{-1}$ fields defined in the invariant state constructed through the semi-local quantum-group generators we obtain the quantum-group difference equations. We give the explicit solution to the two point function.
One of the pressing problems in particle physics is to formulate four-dimensional (4-D) quantum field theory nonperturbatively and to find nonperturbative quantum-field-theoretical solutions to the Yang-Mills equations for strong interactions, as well as to quantum gravity. The path we have taken in this pursuit of nonperturbative results has been through the integrable-system method. The main important framework the integrable-system provides is the possibility of formulating the field theory in terms of group-valued local fields. This is a nontrivial starting point for 4-D gauge theory with nonvanishing curvatures. In this formulation the self-dual Yang-Mill (SDYM) theory is the simplest, yet important prototype to work out.[1]

In Ref. [2] we succeeded in formulating the quantum SDYM field theory in terms of the group-valued local quantum field \( \tilde{J} \), [3]. We obtained the interaction Hamiltonian of the \( \tilde{J} \) fields, derived the exchange algebras that the \( \tilde{J} \) fields satisfy, showed that the \( \tilde{J} \) fields are bimodule quantum fields and the \( R \) matrix of the exchange algebras satisfies the Yang-Baxter relations so that the products of the \( \tilde{J} \) fields satisfy associativity, and we developed normal-ordering procedure for the products of fields. From the \( \tilde{J} \) fields, we constructed local currents and their algebras. We found that the integrated currents in a particular spatial dimension and their current algebras are actually time-independent, i.e. the currents commute with the interaction Hamiltonian. This is a new feature in this 4-D quantum field theory. In Ref. [4], we constructed a time-independent local monodromy matrix \( \tilde{T} \circ \tilde{T} \) from this time-independent currents and derived its local exchange algebra \( R^T \circ \tilde{T} \circ \tilde{T} = \tilde{T} \circ \tilde{T} \circ \tilde{T} \) which contains two infinitesimal forms, the time-independent infinite local charge algebras and the infinite local Yangian algebras. In this paper we develop further the implications of these time-independent currents.

From the time-independent currents \( \tilde{J}(\bar{y}, \bar{k}) \), we construct new group-valued quantum fields \( \tilde{U}(\bar{y}, \bar{k}) \) and \( \tilde{U}^{-1}(\bar{y}, \bar{k}) \) that satisfy a set of exchange algebras such that fields of \( \tilde{J}(\bar{y}, \bar{k}) \sim \tilde{U}(\bar{y}, \bar{k}) \frac{\partial \bar{y}}{\partial \bar{y}} \tilde{U}^{-1}(\bar{y}, \bar{k}) \) satisfy the original time-independent current algebras. (Throughout the paper we use letters with superscript “ - ” to indicate quantum operator fields.) For the correlation functions of the products of the \( \tilde{U}(\bar{y}, \bar{k}) \) and \( \tilde{U}^{-1}(\bar{y}, \bar{k}) \) fields defined in the invariant state constructed through the current \( \tilde{J}(\bar{y}, \bar{k}) \) we can derive the Knizhnik-Zamolodchikov (K-Z) equations[5] with an additional spatial dependence on \( \bar{k} \). We can obtain the n-point correlation functions of the \( \tilde{U}(\bar{y}, \bar{k}) \) and \( \tilde{U}^{-1}(\bar{y}, \bar{k}) \) fields; they are expressible in terms of the correlation functions of the quantum WZNW theory in
two-dimensions (2-D) with coefficients being unknown functions of \( \bar{k} \), one of the additional spatial coordinates in four dimensions.

From the \( \tilde{U}(\bar{y}, \bar{k}) \) and \( \tilde{U}^{-1}(\bar{y}, \bar{k}) \) fields we can also construct the quantum-group generators — local, global, and semi-local — and their algebraic relations. For the correlation functions of the products of the \( \tilde{U} \) and \( \tilde{U}^{-1} \) fields defined in the invariant state constructed through the semi-local quantum-group generators, using the method given by Frenkel and Reshetikhin\[6\] we obtain the quantum-group difference equations. We give the explicit solution to the two point function.

With these results, we have exposed many important quantum integrability part of this 4-D interactive theory. As expected, the 4-D interactive theory is not, and should not be, as fully integrable as integrable systems in 2-D. However it is to important to find out the quantum-field-theoretical integrability properties of the theories which have many classical integrability properties. This experience in formulating the quantum SDYM from integralable-system point of view has now prepared us to investigate fuller 4-D field theories.

**The Quantum SDYM System: Hamiltonian, Exchange Algebras, Critical Exponents, Normal-Ordering, and Current Algebras**

First we briefly review the quantum SDYM theory formulated in our previous paper\[2\]. It is characterized by a quantum field Hamiltonian

\[
\bar{H}_{\text{int}} = -\alpha \int d\bar{y} \; a^2 \Sigma_k \Sigma_{\bar{k}} \left\{ \text{Tr} \left\{ (\partial_k \tilde{J} - \partial_{\bar{k}} \tilde{J}^{-1}) \right\} \right.
\]

\[+ \int_0^1 d\rho \text{Tr} \left\{ (\partial_\rho \tilde{J} - (\partial_k \tilde{J}^{-1}) (\partial_{\bar{k}} \tilde{J}^{-1}) (\partial_k \tilde{J}^{-1}) (\partial_{\bar{k}} \tilde{J}^{-1}) \right\} \}, \tag{1}
\]

where, in the case of \( \text{sl}(2) \), \( \tilde{J} = \tilde{J}(y, \bar{y}, k, \bar{k}) \) is a \( 2 \times 2 \) matrix with non-commuting operator-valued entries depending on the 4-d coordinates \( y, \bar{y}, k, \bar{k} \); and \( y \) is the time. (Here we present the theory with \( z \) and \( \bar{z} \) coordinates discretized: \( z = ka, \bar{z} = \bar{k}a, \) and \( a \equiv l/N \) is the lattice size.) The field \( \tilde{J} = \tilde{J}(\rho; y, \bar{y}, k, \bar{k}) \) depends on a parameter \( \rho \) and \( \tilde{J}(\rho = 1) = \tilde{J} \).

The quantum \( \tilde{J} \) fields satisfy the following exchange algebras:

\[
\tilde{J}_I(y, \bar{y}_1, k_1, \bar{k}_1) \tilde{J}_I(y, \bar{y}_2, k_2, \bar{k}_2) = 1 \delta_{I, II} \tilde{J}_I(y, \bar{y}_2, k_2, \bar{k}_2) \tilde{J}_I(y, \bar{y}_1, k_1, \bar{k}_1) R_{I, II}(q, \bar{y}_1 - \bar{y}_2) , \tag{2}
\]

where

\[
R_{I, II}(q, \bar{y}_1 - \bar{y}_2) = \mathcal{P}_{I, II} \left\{ [q]^{\Delta_1} \varepsilon(q_1 - q_2) p_{j_{12}=1}^{q} - [q]^{\Delta_0} \varepsilon(q_1 - q_2) p_{j_{12}=0}^{q} \right\} , \tag{3}
\]
\[ \varepsilon(\bar{y}_1 - \bar{y}_2) = -\frac{\ln(\bar{y}_1 - \bar{y}_2 + i\varepsilon) - \ln(\bar{y}_2 - \bar{y}_1 + i\varepsilon)}{\pi i}, \]  
(4)

and

\[ \varepsilon(\bar{y}_1 - \bar{y}_2) = \pm 1, \quad \text{for} \quad \bar{y}_1 \geq \bar{y}_2; \]

\[ \varepsilon(\bar{y}_1 - \bar{y}_2) = 0, \quad \text{for} \quad \bar{y}_1 = \bar{y}_2; \]

\[ q \equiv e^{-[\varepsilon/(4\alpha^2)]\delta_{k_1,k_2}\delta_{k_1,k_2}}, \]

(7)

where \( \alpha \) is the coefficient in front of the SDYM interaction Hamiltonian, Eq. (1); \( \Delta_1 = 2\frac{1}{2}(\frac{1}{2} + 1) - 1(1 + 1) = -1/2 \) and \( \Delta_0 = 2\frac{1}{2}(\frac{1}{2} + 1) - 0(0 + 1) = 3/2 \) are differences of conformal dimensions of two spin \( \frac{1}{2} \) fields minus that of a spin \( j_{12} = 1 \) and \( 0 \) fields, respectively; and the \( \mathcal{P}_{j_{12}} \)'s are the \( q \)-ed projection matrices projecting the two spin \( 1/2 \) states into \( j_{12} = 0 \) or \( 1 \), satisfying \( \mathcal{P}_{j_{12}} \mathcal{P}_{j_{12}}^q = \mathcal{P}_{j_{12}} \delta_{j_{12}j_{12}'} \). In the more explicit expressions

\[ \mathcal{P}_{j_{12}=1}^q = \text{diag}\{1, d \begin{pmatrix} q & 1 \\ 1 & q^{-1} \end{pmatrix}, 1\}, \]

(8)

where \( d \equiv 1/(q + q^{-1}) \). The \( q \)-ed singlet projection matrix is related to the triplet one by \( \mathcal{P}_{j_{12}=0}^q = 1 - \mathcal{P}_{j_{12}=1}^q \). The matrix \( P_{I,II} \) interchanges matrix in space \( I \) to \( II \) and vice versa, e.g., \( P_{I,II} \tilde{J}_I(y, \bar{y}_1, k_1, \bar{k}_1) \tilde{J}_{II}(y, \bar{y}_2, k_2, \bar{k}_2) = \tilde{J}_{II}(y, \bar{y}_1, k_1, \bar{k}_1) \tilde{J}_{I}(y, \bar{y}_2, k_2, \bar{k}_2) P_{I,II} \), and its explicit representation is

\[ P_{I,II} = \frac{1}{2} + \frac{1}{2} \sum_{a=1}^{\delta} \sigma_I^a \sigma_{II}^a = \mathcal{P}_{j_{12}=1} - \mathcal{P}_{j_{12}=0}; \]

where the \( \mathcal{P}_{j_{12}} \)'s are the un-\( q \)-ed ordinary projection matrices, i.e., Eq. (8) with \( q = 1 \). The subscripts \( I \) and \( II \) denote the tensor spaces that the operator matrices or \( c \)-number matrices operate on. This tensor notation saves us the trouble of writing out the indices of the matrix elements; in terms of the matrix elements, Eq. (2) reads

\[ \tilde{J}_{m_1,\alpha_1}(y, \bar{y}_1, k_1, \bar{k}_1) \tilde{J}_{m_2,\alpha_2}(y, \bar{y}_2, k_2, \bar{k}_2) = \delta_{m_1,l_1} \delta_{m_2,l_2} \delta_{\beta_2,\alpha_2} \tilde{J}_{l_2,\beta_2}(y, \bar{y}_2, k_2, \bar{k}_2) \tilde{J}_{l_1,\beta_1}(y, \bar{y}_1, k_1, \bar{k}_1) R_{\beta_1,\beta_2;\alpha_1,\alpha_2}(q, \bar{y}_1 - \bar{y}_2). \]

(9)

Using another fact \( 1_{I,II} = \mathcal{P}_{j_{12}=1} + \mathcal{P}_{j_{12}=0} \), we can easily prove that at \( \bar{y}_1 = \bar{y}_2 \), the exchange algebra Eq. (2) gives

\[ \mathcal{P}_{j_{12}} \tilde{J}_{I}(y, \bar{y}_1, k_1, \bar{k}_1) \tilde{J}_{II}(y, \bar{y}_1, k_1, \bar{k}_1) = \tilde{J}_{I}(y, \bar{y}_1, k_1, \bar{k}_1) \tilde{J}_{II}(y, \bar{y}_1, k_1, \bar{k}_1) \mathcal{P}_{j_{12}}^q, \]

(10)
where \( j_{12} = 0, 1 \). Eq. (10) implies \( \mathcal{P}_{j_{12}} \tilde{J}_I(y, \bar{y}_1, k_1, \bar{k}_1) \tilde{J}_H(y, \bar{y}_1, k_1, \bar{k}_1) \mathcal{P}^g_{j_{12}} = 0 \), for \( j_{12} \neq j'_{12} \).

This and the later development of the quantum group generators rely crucially on our interpretation of the \( R \) matrix at the coincidence point, Eq. (3). We denote \( R_{I,H}(q, \bar{y}_1 - \bar{y}_2) = R_{I,H}(q, +) \), for \( \bar{y}_1 - \bar{y}_2 > 0 \) and \( R_{I,H}(q, \bar{y}_1 - \bar{y}_2) = R_{I,H}(q, -) \), for \( \bar{y}_1 - \bar{y}_2 < 0 \). Note that \([R_{I,H}(q, +)]^{-1} = R_{I,I}(q, -); [R_{I,H}(q, -)]^{-1} = R_{II,I}(q, +)\).

The expression for \( \epsilon(\bar{y}_1 - \bar{y}_2) \), Eq. (4), indicates that the product \( \tilde{J}_I(\bar{y}_1) \tilde{J}_H(\bar{y}_2) \) has singularity at \( \bar{y}_1 - \bar{y}_2 = 0 \) with the definite critical exponents given by

\[
\mathcal{P}_{j_{12}} \tilde{J}_I(y, \bar{y}_1, k_1, \bar{k}_1) \tilde{J}_H(y, \bar{y}_2, k_1, \bar{k}_1) \mathcal{P}^g_{j_{12}} = (\bar{y}_1 - \bar{y}_2) \Delta_{j_{12}}(\ln q)/\pi i \{}: \mathcal{P}_{j_{12}} \tilde{J}_I(y, \bar{y}_1, k_1, \bar{k}_1) \tilde{J}_H(y, \bar{y}_2, k_1, \bar{k}_1) \mathcal{P}^g_{j_{12}} :\} .
\] (11)

(We call the power of the singularity \( \Delta_{j_{12}}(\ln q)/\pi i \) the critical exponent of the product of the two \( \tilde{J} \) fields.) This also defines the normal-order products to be those in the curly brackets; their Taylor expansions give the operator-product expansions.

We then defined the \( \tilde{J}^{-1} \) field by the following fixed-y-time equation

\[
\tilde{J}(y, \bar{y}, k, \bar{k}) \tilde{J}^{-1}(y, \bar{y}, k, \bar{k}) = 1 = \tilde{J}^{-1}(y, \bar{y}, k, \bar{k}) \tilde{J}(y, \bar{y}, k, \bar{k}) .
\] (12)

From Eqs. (12) and (2), we can easily show that the \( \tilde{J}^{-1} \) field satisfies the following fixed-y-time exchange algebras

\[
\tilde{J}^{-1}_I(y, \bar{y}_1, k_1, \bar{k}_1) \tilde{J}_H(y, \bar{y}_2, k_2, \bar{k}_2) = \tilde{J}_H(y, \bar{y}_2, k_2, \bar{k}_2) R_{I,H}^{-1}(q, \bar{y}_1 - \bar{y}_2) \tilde{J}^{-1}_I(y, \bar{y}_1, k_1, \bar{k}_1) ,
\] (13)

and

\[
\tilde{J}^{-1}_I(y, \bar{y}_1, k_1, \bar{k}_1) \tilde{J}_H^{-1}(y, \bar{y}_2, k_2, \bar{k}_2) = R_{I,I}^{-1}(q, \bar{y}_1 - \bar{y}_2) \tilde{J}^{-1}_I(y, \bar{y}_2, k_2, \bar{k}_2) \tilde{J}^{-1}_I(y, \bar{y}_1, k_1, \bar{k}_1) .
\] (14)

The construction of this \( \tilde{J}^{-1} \) field is crucial for us to develop the full content of the theory in terms of the group-valued fields.

From \( \tilde{H}_{int} \) and the exchange algebra Eq. (2), we derived the equation of motion

\[
\partial_y (\tilde{J} \partial_y \tilde{J}^{-1}) = \frac{\hbar}{i} [\tilde{H}, \tilde{J} \partial_y \tilde{J}^{-1}] = a^{-2} \partial_k (\tilde{J} \partial_k \tilde{J}^{-1}) .
\] (15)

From fields \( \tilde{J} \) and \( \tilde{J}^{-1} \), we constructed the \( \widehat{sl}(2) \) current

\[
\tilde{j}(y, \bar{y}, k, \bar{k}) \equiv 2a \tilde{J} \partial_y \tilde{J}^{-1}(y, \bar{y}, k, \bar{k}) .
\] (16)
We then showed that the following equations can be easily derived from the exchange algebras Eqs. (2), (13), and (14),

\[
\left[ \tilde{j}_I(y, \tilde{y}_1, k_1, \tilde{k}_1), \tilde{j}_H(y, \tilde{y}_2, k_2, \tilde{k}_2) \right] = -i\hbar M_{I,II} [\tilde{j}_H(y, \tilde{y}_2, k_2, \tilde{k}_2)] \delta(\bar{y}_1 - \bar{y}_2) \delta_{k_1 k_2}/a^2 \\
- i\hbar 2 \alpha M_{I,II} \delta'(\bar{y}_1 - \bar{y}_2) \delta_{k_1 k_2}/a^2 ,
\]

(17)

where \( M_{I,II} \equiv P_{I,II} - \frac{i}{2} = \frac{1}{2} \sum_{a=1}^3 \sigma^a \sigma^a \), \( \delta \) is the current algebras of the currents \( \tilde{j} \). Taking trace of Eq. (17) onto \( \lambda^a_I H^b_I \) we can easily obtain current algebras in terms of the Lie-components of the current \( [\tilde{y}^a(y_1), \tilde{y}^b(y_2)] = i\hbar \varepsilon^{abc} \tilde{J}^c(y_1) \delta(\bar{y}_1 - \bar{y}_2) \delta_{k_1 k_2}/a^2 - i\hbar^2 2 \alpha M_{I,II} \delta'(\bar{y}_1 - \bar{y}_2) \delta_{k_1 k_2}/a^2 \), where \( K \equiv i\pi[a^2ln(q)_{k_1=k_2, k_2=k_2}]^{-1} = -4\pi\alpha/\hbar \). Eq. (18) indicates that the left side of \( \tilde{J} \) forms the fundamental representation of the current \( \tilde{j} \); Eq. (14) indicates that the right side of \( \tilde{J}^{-1} \) forms the fundamental representation of the current \( \tilde{j} \). However, all these relations are fixed time relations because \( \tilde{j} \) varies with time \( y \), i.e., \( [\tilde{j}, \tilde{H}] \neq 0 \).

Notice that we can easily obtain the continuum form of the current algebra by taking \( a \to 0 \) and \( \delta_{k_1 k_2}/a \to \delta(\bar{z}_1 - \bar{z}_2) \) and \( \delta_{k_1 k_2}/a \to \delta(\bar{z}_1 - \bar{z}_2) \).

As pointed out in Ref. [2] the \( k \)-summed current

\[
\tilde{J}(\bar{y}, k) \equiv a \Sigma_k \tilde{j}(\bar{y}, k, \bar{k}) ,
\]

(20)
is constant in time \( y \). It can be shown by directly calculating \([\tilde{J}, \tilde{H}_{int}] = 0 \) or by summing the equation of motion in \( k \), Eq. (13). (Since \( \tilde{J} \) is time-independent we do not put \( y \) dependence in \( \tilde{J}(\bar{y}, k) \).) Summing in \( k \), notice that \( \Sigma_k \delta_{k_1 k_2} = 1 \), from Eqs. (17) to (19) we obtain the corresponding algebras,

\[
[\tilde{J}_I(\bar{y}_1, \tilde{k}_1), \tilde{J}_H(\bar{y}_2, \tilde{k}_2)] = -i\hbar M_{I,II} [\tilde{J}_H(\bar{y}_2, \tilde{k}_2)] \delta(\bar{y}_1 - \bar{y}_2) \delta_{\tilde{k}_1 \tilde{k}_2}/a - i\hbar 2 \alpha M_{I,II} \delta'(\bar{y}_1 - \bar{y}_2) \delta_{\tilde{k}_1 \tilde{k}_2}/a ,
\]

(21)

\[
[\tilde{J}_I(\bar{y}_1, \tilde{k}_1), \tilde{J}_H(\bar{y}_2, \tilde{k}_2, \tilde{k}_2)] = -i\hbar M_{I,II} \tilde{J}_H(\bar{y}_2, \tilde{k}_2, \tilde{k}_2) \delta(\bar{y}_1 - \bar{y}_2) \delta_{\tilde{k}_1 \tilde{k}_2}/a ,
\]

(22)

\[
[\tilde{J}_I(\bar{y}_1, \tilde{k}_1), \tilde{J}_H^{-1}(\bar{y}_2, \tilde{k}_2, \tilde{k}_2)] = i\hbar \tilde{J}_H^{-1}(\bar{y}_2, \tilde{k}_2, \tilde{k}_2) M_{I,II} \delta(\bar{y}_1 - \bar{y}_2) \delta_{\tilde{k}_1 \tilde{k}_2}/a ,
\]

(23)

Taking trace of Eq. (21) onto \( \lambda^a_I H^b_I \) we can easily obtain the current algebras in terms of the Lie-components of the current \( [\tilde{J}^a(y_1), \tilde{J}^b(y_2)] = i\hbar \varepsilon^{abc} \tilde{J}^c(y_1) \delta(\bar{y}_1 - \bar{y}_2) \delta_{\tilde{k}_1 \tilde{k}_2}/a -
\]
The commutator equations Eqs. (17) to (19) can be written out in the commonly used algebra with \( \hat{\pi}_I \) satisfying the following exchange algebras

\[
\hat{\pi}_I (\bar{y}_1, \bar{k}_1) \hat{\pi}_I (\bar{y}_2, \bar{k}_2) = \hat{\pi}_I (\bar{y}_2, \bar{k}_2) \hat{\pi}_I (\bar{y}_1, \bar{k}_1) R_I (q', \bar{y}_1 - \bar{y}_2) \ , \quad (25)
\]

and the \( \hat{\pi}_I \) and \( \hat{\pi}_I^{-1} \) fields satisfy the following exchange algebras

\[
\hat{\pi}_I (\bar{y}_1, \bar{k}_1) \hat{\pi}_I (\bar{y}_2, \bar{k}_2) = \hat{\pi}_I (\bar{y}_2, \bar{k}_2) \hat{\pi}_I (\bar{y}_1, \bar{k}_1) R_I (q', \bar{y}_1 - \bar{y}_2) \ , \quad (26)
\]

where

\[
q' \equiv e^{-i\pi/(4a_I a_l)} \delta_{k_1 k_2} = (q^{1/N})_{k_1 = k_2} \quad (28)
\]

and \( K' = i\pi [a \ln(q')]_{k_1 = k_2}^{-1} \). These exchange algebras guarantee the current algebras of \( \hat{\pi}_I \), Eq. (21), and also gives the following algebra

\[
[\hat{\pi}_I (\bar{y}_1, \bar{k}_1), \hat{\pi}_I (\bar{y}_2, \bar{k}_2)] = -i\hbar M_{I, II} \delta (\bar{y}_1 - \bar{y}_2) \delta_{k_1 k_2}/a \ , \quad (29)
\]

and the algebra with \( \hat{\pi}_I^{-1} (\bar{y}, \bar{k}) \). From

\[
2\pi i \frac{\delta (\bar{y}_1 - \bar{y}_2)}{\bar{y}_1 - \bar{y}_2 - i\varepsilon} - \frac{1}{\bar{y}_1 - \bar{y}_2 + i\varepsilon} \ , \quad (31)
\]

the commutator equations Eqs. (17) to (19) can be written out in the commonly used operator-product-expansion forms (which we leave as exercises for the reader).

Let us make the decomposition

\[
\hat{\pi}_I (\bar{y}_1, \bar{k}_1) = \hat{\pi}^+ (\bar{y}_1, \bar{k}_1) - \hat{\pi}^- (\bar{y}_1, \bar{k}_1) \ , \quad (32)
\]

with \( \hat{\pi}^\pm \) satisfying the following algebras

\[
[\hat{\pi}^+_I (\bar{y}_1, \bar{k}_1), \hat{\pi}^+_I (\bar{y}_2, \bar{k}_2)] = -i\hbar \frac{1}{2\pi} \frac{1}{\bar{y}_1 - \bar{y}_2 + i\varepsilon} \left[ M_{I, II} \left( \hat{\pi}^+_I (\bar{y}_1, \bar{k}_1) + \hat{\pi}^+_I (\bar{y}_2, \bar{k}_2) \right) \right] \delta_{k_1 k_2}/a \ , \quad (33)
\]
\[
[\tilde{\mathcal{J}}_I^+(\bar{y}_1, \bar{k}_1), \tilde{\mathcal{J}}_I^-(\bar{y}_2, \bar{k}_2)] = \frac{-\hbar}{2\pi} \frac{1}{y_1 - y_2 - i\varepsilon} [M_{I,I}, (\tilde{\mathcal{J}}_I^+(y_1, k_1) + \tilde{\mathcal{J}}_I^-(y_2, k_2))] \delta_{k_1 k_2}/a \\
- \frac{\hbar}{2\pi} \frac{1}{(y_1 - y_2 - i\varepsilon)^2} 2\alpha I_{I,I} \delta_{k_1 k_2}/a , \tag{34}
\]

\[
[\tilde{\mathcal{J}}_I^+(\bar{y}_1, \bar{k}_1), \tilde{U}_{II}(\bar{y}_2, \bar{k}_2)] = \frac{-\hbar}{2\pi} \frac{1}{y_1 - y_2 + i\varepsilon} M_{I,I} \tilde{U}_{II}(y_2, k_2) \delta_{k_1 k_2}/a , \tag{35}
\]

so that Eqs. (21), (29), and (30) are guaranteed.

Because of the singularities in the products of fields, we must prescribe the normal-ordering procedure and make consistency checks. The goal is to obtain the following relation:

\[
c \partial_y \tilde{U} = : \tilde{\mathcal{J}} \tilde{U} :, \tag{36}
\]

where : \( \tilde{\mathcal{J}} \tilde{U} := -\tilde{\mathcal{J}}^+ \tilde{U} + (\tilde{U}^{T\mu}(\tilde{\mathcal{J}}^-)^{T\mu})^{T\mu} \) and \( c \) is a constant to be determined by the normal-ordering procedure and consistency. Following the procedure used in Ref. [6], we define the vacuum state \(|0\rangle\) to be

\[
\tilde{\mathcal{J}}^- (\bar{y}, \bar{k}) |0\rangle = 0 , \quad \text{and} \quad \langle 0 | \tilde{\mathcal{J}}^+ (\bar{y}, \bar{k}) = 0 . \tag{37}
\]

To determine the coefficient \( c \), we take the commutator of \( \tilde{\mathcal{J}}^+ \) and each side of Eq. (36), namely \( \partial_y \tilde{U} \) and : \( \tilde{\mathcal{J}} \tilde{U} : \). Here we consider the case \( \bar{y}_1 \neq \bar{y}_2 \) and we suppress \( \varepsilon \).

\[
\left[ \tilde{\mathcal{J}}_I^+(\bar{y}_1, \bar{k}_1), \partial_{y_2} \tilde{U}_{II}(\bar{y}_2, \bar{k}_2) \right] = \frac{-\hbar}{2\pi} \frac{1}{y_1 - y_2} M_{I,I} \tilde{U}_{II}(y_2, k_2) \delta_{k_1 k_2}/a , \tag{38}
\]

\[
\left[ \tilde{\mathcal{J}}_I^+(\bar{y}_1, \bar{k}_1), - : \tilde{\mathcal{J}}_{II}(\bar{y}_2, \bar{k}_2) \tilde{U}_{II}(\bar{y}_2, \bar{k}_2) : \right] = \left[ \tilde{\mathcal{J}}_I^+(\bar{y}_1, \bar{k}_1), - \tilde{\mathcal{J}}_{II}^+(\bar{y}_2, \bar{k}_2) \tilde{U}_{II}(\bar{y}_2, \bar{k}_2) + \left( \tilde{U}_{II}(\bar{y}_2, \bar{k}_2) \right)^{T\mu} \left( \tilde{\mathcal{J}}_{II}^+(\bar{y}_2, \bar{k}_2) \right)^{T\mu} \right] \tag{39}
\]

\[
= \frac{-\hbar}{2\pi} \frac{2\alpha I}{(y_1 - y_2)^2} M_{I,I} \tilde{U}_{II}(y_2, k_2) \delta_{k_1 k_2}/a + \left( \frac{\hbar}{2\pi} \right)^2 2 \frac{2}{(y_1 - y_2)^2} M_{I,I} \tilde{U}_{II}(y_2, k_2) \delta_{k_1 k_2}/a^2 \tag{40}
\]

The consistency of (39) with Eqs. (38) and (30) requires \( c = (2\alpha I - \frac{\hbar^2}{2\pi a}) \) and we obtain

\[
(2\alpha I - \frac{\hbar^2}{2\pi a}) \partial_y \tilde{U}_I(\bar{y}, \bar{k}) = -\tilde{\mathcal{J}}_I^+(\bar{y}, \bar{k}) \tilde{U}_I(\bar{y}, \bar{k}) + \left( \tilde{U}_I(\bar{y}, \bar{k}) \right)^T \left( \tilde{\mathcal{J}}_I^+(\bar{y}, \bar{k}) \right)^T . \tag{41}
\]
Then using Eqs. (14), (15) and (17) and following the procedure given in Ref. [5], we can straightforwardly derive the K-Z equation for the $\tilde{U}$ fields,

$$
\left(-\frac{1}{2}(a K' + 2) \partial_{y_j} + \frac{1}{2} \sum_{k \neq j} \frac{M_{j,k}}{y_j - y_k} \delta_{k_j, k_k} \right) \langle 0 | \tilde{U}_I(y_1, \tilde{k}_1) \cdots \tilde{U}_N(y_n, \tilde{k}_n) | 0 \rangle = 0 ,
$$

or

$$
\left(-\frac{1}{2}(a K' + 2) \partial_{y_j} - \sum_{k \neq j} \sum_{a} \lambda_j^a \lambda_k^a \frac{\delta_{k_j, k_k}}{y_j - y_k} \right) \langle 0 | \tilde{U}_I(y_1, \tilde{k}_1) \cdots \tilde{U}_N(y_n, \tilde{k}_n) | 0 \rangle = 0 .
$$

Notice that taking away the $\tilde{k}$ dependence, we recover the K-Z equation of the quantum WZNW theory.

The solutions of this SDYM K-Z equation are expressible in terms of those of the WZNW K-Z equation, $\langle 0 | g_I(y_1) \cdots g_N(y_n) | 0 \rangle$, multiplied by unknown functions in $\tilde{k}$.

For example

$$
\langle 0 | \tilde{U}_I(y_1, \tilde{k}_1) \tilde{U}_I(y_2, \tilde{k}_2) | 0 \rangle = C_2(\tilde{k}_1) \langle 0 | \tilde{g}_I(y_1) \tilde{g}_I(y_2) | 0 \rangle_{(aK'+2)} \delta_{k_1, k_2}
$$

$$
\langle 0 | \tilde{U}_I(y_1, \tilde{k}_1) \tilde{U}_II(y_2, \tilde{k}_2) \tilde{U}_III(y_3, \tilde{k}_3) | 0 \rangle = C_3(\tilde{k}_1) \langle 0 | \tilde{g}_I(y_1) \tilde{g}_II(y_2) \tilde{g}_III(y_3) | 0 \rangle_{(aK'+2)} \delta_{k_1, k_2} \delta_{k_2, k_3}
$$

$$
+ \sum_{j \neq k \neq l}^{to3} C_{3j}(\tilde{k}_k, \tilde{k}_j) \langle 0 | \tilde{g}_I(y_1) \tilde{g}_II(y_2) \tilde{g}_III(y_3) | 0 \rangle_{(aK'+2)} \delta_{k_k, k_j} \delta_{k_j, k_l} (1 - \delta_{k_k, k_l})
$$

$$
+ C_{34}(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3) \langle 0 | \tilde{g}_I(y_1) \tilde{g}_II(y_2) \tilde{g}_III(y_3) \tilde{g}_IV(y_4) | 0 \rangle_{(aK'+2)} \delta_{k_1, k_2} \delta_{k_2, k_3} \delta_{k_3, k_4}
$$

$$
+ \sum_{j \neq k \neq l}^{to4} C_{4j}(\tilde{k}_k, \tilde{k}_j) \langle 0 | \tilde{g}_I(y_1) \tilde{g}_II(y_2) \tilde{g}_III(y_3) \tilde{g}_IV(y_4) | 0 \rangle_{(aK'+2)} \delta_{k_1, k_2} \delta_{k_2, k_3} \delta_{k_3, k_4}
$$

$$
\times (1 - \delta_{k_1, k_j}) (1 - \delta_{k_2, k_k}) (1 - \delta_{k_3, k_l})
$$

$$
+ \sum_{j \neq k \neq l}^{to4} C_{41,jkml}(\tilde{k}_j, \tilde{k}_l) \langle 0 | \tilde{g}_I(y_1) \tilde{g}_II(y_2) \tilde{g}_III(y_3) \tilde{g}_IV(y_4) | 0 \rangle_{(aK'+2)} \delta_{k_1, k_2} \delta_{k_2, k_3} \delta_{k_3, k_4}
$$

$$
\times (1 - \delta_{k_1, k_j}) \delta_{k_2, k_k} \delta_{k_3, k_l}
$$

$$
+ \sum_{j \neq k \neq l}^{to4} C_{41,jkml}(\tilde{k}_j, \tilde{k}_l) \langle 0 | \tilde{g}_I(y_1) \tilde{g}_II(y_2) \tilde{g}_III(y_3) \tilde{g}_IV(y_4) | 0 \rangle_{(aK'+2)} \delta_{k_1, k_2} \delta_{k_2, k_3} \delta_{k_3, k_4}
$$

$$
\times \delta_{k_1, k_k} \delta_{k_2, k_m} (1 - \delta_{k_1, k_j})
$$

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\[ + \sum_{j \neq k \neq l \neq m}^1 \text{to} \ 4 \ C_{42,jklm}(\bar{k}_j, \bar{k}_k, \bar{k}_l, \bar{k}_m)\langle 0 \mid \tilde{g}_J(\bar{y}_j) \mid 0 \rangle_{(aK'+2)} \langle 0 \mid \tilde{g}_K(\bar{y}_k) \mid 0 \rangle_{(aK'+2)} \times \langle 0 \mid \tilde{g}_L(\bar{y}_l) \tilde{g}_M(\bar{y}_m) \mid 0 \rangle_{(aK'+2)} \ (1 - \delta_{\bar{k}_j \bar{k}_k} \delta_{\bar{k}_l \bar{k}_m} (1 - \delta_{\bar{k}_k \bar{k}_l}) (1 - \delta_{\bar{k}_j \bar{k}_k}) \right) \\
+ C_{43}(\bar{k}_1, \bar{k}_2, \bar{k}_3, \bar{k}_4)\langle 0 \mid \tilde{g}_I(\bar{y}_1) \mid 0 \rangle_{(aK'+2)} \langle 0 \mid \tilde{g}_I'(\bar{y}_2) \mid 0 \rangle_{(aK'+2)} \times \langle 0 \mid \tilde{g}_I'(\bar{y}_3) \mid 0 \rangle_{(aK'+2)} \langle 0 \mid \tilde{g}_I'(\bar{y}_4) \mid 0 \rangle_{(aK'+2)} \right) \\
\times (1 - \delta_{\bar{k}_1 \bar{k}_2}) (1 - \delta_{\bar{k}_1 \bar{k}_3}) (1 - \delta_{\bar{k}_1 \bar{k}_4}) (1 - \delta_{\bar{k}_2 \bar{k}_3}) (1 - \delta_{\bar{k}_2 \bar{k}_4}) (1 - \delta_{\bar{k}_3 \bar{k}_4}) \right) . \tag{46} \]

Taking away the \( \bar{k} \)-dependence (and all \( \delta_{\bar{k}_i \bar{k}_j} = 1 \)), the correlation functions become precisely those of quantum WZNW theory.

These solutions indicate that we can obtain much information for this 4-D quantum field theory; however, we cannot obtain as much information as for the 2-D integrable system. The quantum SDYM theory has provided us a valuable example of how we can extract some exact and nonperturbative information from the theory.
Quantum-Group Current \( \tilde{J}^q(\bar{y}, \bar{k}) \) and \( \bar{y} \)-Global Quantum-Group Generators \( \tilde{G}(\bar{k}) \)

Similar to the construction of the current \( \tilde{J} \), Eq. (24), it is natural to construct the other current,

\[
\tilde{J}^q(\bar{y}, \bar{k}) \equiv 2\alpha_l \tilde{U}^{-1}(\bar{y}, \bar{k}) \partial_{\bar{y}} \tilde{U}(\bar{y}, \bar{k}),
\]

which we shall call the quantum-group current since it has the quantum-group index on both sides.

We can work out the algebraic relations among its matrix elements and with the fields \( \tilde{U} \) and \( \tilde{U}^{-1} \), like Eqs. (21) to (23) for the \( \tilde{J} \). They all have nice quantum-group interpretations. However, we find that \( \tilde{J}^q \) is not as useful a quantity as the current \( \tilde{J} \) in that it can not be used to develop its vacuum states and the corresponding differential equations as the current \( \tilde{J} \) was used to develop the K-Z equations. On the other hand we find that the following group-valued quantities, \( \tilde{G}(\bar{k}) \) and \( \tilde{G}^\Delta(\bar{y}, \bar{k}) \), are the appropriate quantum-group generators for further development of the theory.

The \( \bar{y} \)-global quantum-group generator, denoted by \( \tilde{G}(\bar{k}) \), is derived from the quantum-group current \( \tilde{J}^q \) of Eq. (47) by a path-ordered integration,

\[
\tilde{G}(\bar{k}) = \tilde{P} \exp \left( \int_{-\infty}^{\infty} d\bar{y} \; \tilde{U}^{-1}(\bar{y}, \bar{k}) \partial_{\bar{y}} \tilde{U}(\bar{y}, \bar{k}) \right) = \tilde{U}^{-1}(\bar{y} = -\infty, \bar{k}) \; \tilde{U}(\bar{y} = +\infty, \bar{k}) .
\]

Then, from the exchange algebras of the fields \( \tilde{U} \) and \( \tilde{U}^{-1} \), Eqs. (25) to (27), we can derive the algebraic relations among the matrix elements of \( \tilde{G}(\bar{k}) \) and with the fields \( \tilde{U} \) and \( \tilde{U}^{-1} \),

\[
\{ R_{II,I}(q', +) \; \tilde{G}_I(\bar{k}_1) \; R_{I,II}(q', +) \} \; \tilde{G}_H(\bar{k}_2)
= \tilde{G}_H(\bar{k}_2) \; \{ R_{II,I}(q', +) \; \tilde{G}_I(\bar{k}_1) \; R_{I,II}(q', +) \} ,
\]

\[
\tilde{G}_I(\bar{k}_1) \; U_H(\bar{y}_2, \bar{k}_2) = U_H(\bar{y}_2, \bar{k}_2) \; \{ R_{II,I}(q', +) \; \tilde{G}_I(\bar{k}_1) \; R_{I,II}(q', +) \} ,
\]

\[
U_H^{-1}(\bar{y}_2, \bar{k}_2) \tilde{G}_I(\bar{k}_1) = \{ R_{II,I}(q', +) \; \tilde{G}_I(\bar{k}_1) \; R_{I,II}(q', +) \} \; \tilde{U}_H^{-1}(\bar{y}_2, \bar{k}_2) .
\]

where \( R_{I,II}(q', +) \) is the \( R \)-matrix with \( \epsilon(\bar{y}_1 - \bar{y}_2) = +1 \). In these equations, we use the curly bracket to bracket elements together to make the content of the equation clearer. These three equations are the algebraic relations parallel to those of Eqs. (21), (29), and (30). Associativity of all these fields are true because the \( R \) matrix satisfies the Yang-Baxter relations (which can be shown after some quite involved algebras.)
The basic elements of the quantum-group generators \( \{ \tilde{c}_i(\tilde{k}); i = 3 \text{ and } \pm \} \) are related to the components of the matrix \( \tilde{G}(\tilde{k}) \) by

\[
\tilde{G}(\tilde{k}) \equiv \left( \begin{array}{cc} 1 & 0 \\ (1 - q^2) \tilde{c}_+(\tilde{k}) & 1 \\ 0 & q^{-\tilde{e}_3}(\tilde{k}) \end{array} \right) \left( \begin{array}{cc} q^{-\tilde{e}_3}(\tilde{k}) & 0 \\ 0 & q^\tilde{e}_3(\tilde{k}) \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & q^{-1} - q \\ 1 \end{array} \right),
\]

where the \( \tilde{c}_\pm(\tilde{k}) \) and \( q^{-\tilde{e}_3}(\tilde{k}) \) satisfy local quantum-groups algebras, which generalize those given in Ref. [5].

The \( \tilde{y} \)-Semi-local Quantum-Group Generator \( \tilde{G}^\Delta(\tilde{y}, \tilde{k}) \)

Changing the integration range in Eq. (48) to a semi-local region we obtain the \( \tilde{y} \)-Semi-local Quantum-Group Generator \( \tilde{G}^\Delta(\tilde{y}, \tilde{k}) \) quantum group generator:

\[
\tilde{G}^\Delta(\tilde{y}, \tilde{k}) \equiv \tilde{P} \exp \left( \int_{\tilde{y} - \Delta}^{\tilde{y} + \Delta} dy' \tilde{U}^{-1}(\tilde{y}', \tilde{k}) \partial_{y'} \tilde{U}(\tilde{y}', \tilde{k}) \right) = \tilde{U}^{-1}(\tilde{y} - \Delta, \tilde{k}) \tilde{U}(\tilde{y} + \Delta, \tilde{k}) .
\]

We can easily show that \( \tilde{G}^\Delta \) satisfies the following algebras,

\[
\{ R_{I,II}^1(q', \tilde{y}_1 - \tilde{y}_2) \tilde{G}^\Delta_I(\tilde{y}_1, \tilde{k}_1) R_{I,II}(q', \tilde{y}_1 - \tilde{y}_2 + 2\Delta) \} \tilde{G}^\Delta_H(\tilde{y}_2, \tilde{k}_2)
\]

\[
= \tilde{G}^\Delta_H(\tilde{y}_2, \tilde{k}_2) \{ R_{I,II}^1(q', \tilde{y}_1 - \tilde{y}_2 - 2\Delta) \tilde{G}^\Delta_I(\tilde{y}_1, \tilde{k}_1) R_{I,II}(q', \tilde{y}_1 - \tilde{y}_2) \} ,
\]

\[
\tilde{G}^\Delta_I(\tilde{y}_1, \tilde{k}_1) \tilde{U}^+_H(\tilde{y}_2, \tilde{k}_2)
\]

\[
= \tilde{U}^+_H(\tilde{y}_2, \tilde{k}_2) \{ R_{I,II}^1(q', \tilde{y}_1 - \tilde{y}_2 - \Delta) \tilde{G}^\Delta_I(\tilde{y}_1, \tilde{k}_1) R_{I,II}(q', \tilde{y}_1 - \tilde{y}_2 + \Delta) \} \tilde{U}^-_H^{-1}(\tilde{y}_2, \tilde{k}_2) .
\]

We next split the semi-local generator into the annihilation and creation parts following a procedure similar to that used in Ref. [4],

\[
\tilde{G}^\Delta_I(\tilde{y}, \tilde{k}) \equiv [\tilde{G}^\pm_I(\tilde{y}, \tilde{k})]^{-1} \tilde{G}^\mp_I(\tilde{y}, \tilde{k}) ,
\]

and \( \tilde{G}^\Delta\pm(\tilde{y}, \tilde{k}) \) satisfies the following exchange algebras

\[
R_{I,II}(q', \tilde{y}_1 - \tilde{y}_2 - \Delta) \tilde{G}^\pm_I(\tilde{y}_1, \tilde{k}_1) \tilde{G}^\mp_H(\tilde{y}_2, \tilde{k}_2)
\]

\[
= \tilde{G}^\pm_H(\tilde{y}_2, \tilde{k}_2) \tilde{G}^\pm_I(\tilde{y}_1, \tilde{k}_1) R_{I,II}(q', \tilde{y}_1 - \tilde{y}_2 - \Delta) ,
\]

\[
R_{I,II}(q', \tilde{y}_1 - \tilde{y}_2) \tilde{G}^\pm_I(\tilde{y}_1, \tilde{k}_1) \tilde{G}^\mp_H(\tilde{y}_2, \tilde{k}_2)
\]

\[
= \tilde{G}^\mp_H(\tilde{y}_2, \tilde{k}_2) \tilde{G}^\pm_I(\tilde{y}_1, \tilde{k}_1) R_{I,II}(q', \tilde{y}_1 - \tilde{y}_2 - 2\Delta) .
\]
\[
\tilde{U}_I(\tilde{y}_1, \tilde{k}_1) \tilde{G}_H^{\Delta \pm}(\tilde{y}_2, \tilde{k}_2) = \tilde{G}_H^{\Delta \pm}(\tilde{y}_2, \tilde{k}_2) \tilde{U}_I(\tilde{y}_1, \tilde{k}_1) R_{I,I}(q', \tilde{y}_1 - \tilde{y}_2 \pm \Delta)
\]

such that Eqs. (54) to (56) are true.

Notice that

\[
\left[ \sum_n \tilde{J}(\tilde{y} + n\Delta, \tilde{k}), \tilde{G}_H^{\Delta}(\tilde{y}, \tilde{k}) \right] = 0
\]

which manifests what we call the \(k\)-local \(\text{sl}(n) \otimes U^{1/\Delta}q[\text{sl}(n)]\) symmetry of the theory. For \(\Delta \to \infty\), Eq. (51) becomes \([\tilde{J}(\tilde{y}, \tilde{k}), \tilde{G}(\tilde{k})] = 0\), manifesting the \(\tilde{k}\)-local \(\text{sl}(n) \otimes Uq[\text{sl}(n)]\) symmetry of the theory. For \(\Delta \to 0\), Eq. (61) becomes \([\tilde{Q}(\tilde{k}), \tilde{J}^q(\tilde{y}, \tilde{k})] = 0\), where \(\tilde{Q}(\tilde{k}) \equiv \int_{-\infty}^{\infty} \tilde{J}(\tilde{y}, \tilde{k})d\tilde{y} = \lim_{\Delta \to 0} \sum_{n=-\infty}^{\infty}[\Delta \tilde{J}(\tilde{y} + n\Delta, \tilde{k})]\) and \(\tilde{J}^q(\tilde{y}, \tilde{k})\) is from the coefficient of the \(\Delta\)-term in the expansion of Eq. (53), manifesting the \(\tilde{k}\)-local \(\text{sl}(n) \otimes U_q^\infty[\text{sl}(n)]\) symmetry of the theory.

**Quantum-Group Difference Equation of the Correlation Functions Defined in the \(| 0_q \rangle\)-Vacuum**

Using Eq. (57), we rewrite Eq. (53) as

\[
\tilde{U}(\tilde{y} + \Delta, \tilde{k}) = \tilde{U}(\tilde{y} - \Delta, \tilde{k}) \tilde{G}_H^{\Delta}(\tilde{y}, \tilde{k}) = \tilde{U}(\tilde{y} - \Delta, \tilde{k})[\tilde{G}_H^{\Delta-}(\tilde{y}, \tilde{k})]^{-1} \tilde{G}_H^{\Delta-}(\tilde{y}, \tilde{k})
\]

Now we want to move \((\tilde{G}_H^{\Delta+}(\tilde{y}, \tilde{k}))^{-1}\) to the left of \(\tilde{U}(\tilde{y} - \Delta, \tilde{k})\), since we shall consider the vacuum expectation values of the \(\tilde{U}\) fields by the vacuum \(| 0_q \rangle\) defined by

\[
\tilde{G}_H^{\Delta-}(\tilde{y}, \tilde{k}) | 0_q \rangle = | 0_q \rangle, \quad \text{and} \quad \langle 0_q | \tilde{G}_H^{\Delta+}(\tilde{y}, \tilde{k}) = \langle 0_q | .
\]

To achieve that feat first we use Eq. (50) for \(\tilde{y}_1 = \tilde{y} - \Delta\) and \(\tilde{y}_2 = \tilde{y}\) and interchange its r.h.s. and l.h.s.,

\[
\tilde{G}_H^{\Delta+}(\tilde{y}, \tilde{k}) \tilde{U}_I(\tilde{y} - \Delta, \tilde{k}) R_{I,I}(0) = \tilde{U}_I(\tilde{y} - \Delta, \tilde{k}) \tilde{G}_H^{\Delta+}(\tilde{y}, \tilde{k}).
\]

Multiplying it by \((\tilde{G}_H^{\Delta+}(\tilde{y}, \tilde{k}))^{-1}\) from both sides, we obtain

\[
\tilde{U}_I(\tilde{y} - \Delta, \tilde{k}) R_{I,I}(0) (\tilde{G}_H^{\Delta+}(\tilde{y}, \tilde{k}))^{-1} = (\tilde{G}_H^{\Delta+}(\tilde{y}, \tilde{k}))^{-1} \tilde{U}_I(\tilde{y} - \Delta, \tilde{k}).
\]

Taking transpose in tensor \(I\) space, we obtain

\[
R_{I,I,I}(0) \tilde{U}^{T_I}(\tilde{y} - \Delta, \tilde{k}) (\tilde{G}_H^{\Delta+}(\tilde{y}, \tilde{k}))^{-1} = (\tilde{G}_H^{\Delta+}(\tilde{y}, \tilde{k}))^{-1} \tilde{U}^{T_I}(\tilde{y} - \Delta, \tilde{k}).
\]

Multiplying it by \((R_{I,I,I}(0))^{-1}\) from left, it becomes

\[
\tilde{U}^{T_I}(\tilde{y} - \Delta, \tilde{k}) (\tilde{G}_H^{\Delta+}(\tilde{y}, \tilde{k}))^{-1} = (R_{I,I,I}(0))^{-1} (\tilde{G}_H^{\Delta+}(\tilde{y}, \tilde{k}))^{-1} \tilde{U}^{T_I}(\tilde{y} - \Delta, \tilde{k}).
\]

Taking transpose in tensor \(I\) space, we obtain
\[ \tilde{U}_I(\bar{y} - \Delta, \bar{k}) \left( \tilde{G}_H^{\Delta+}(\bar{y}, \bar{k}) \right)^{-1} = \left( \left( R_{I,II}^{T_I}(0) \right)^{-1} \left( \tilde{G}_H^{\Delta+}(\bar{y}, \bar{k}) \right)^{-1} \tilde{U}_I^{T_I}(\bar{y} - \Delta, \bar{k}) \right)^{T_I}. \]

Taking trace in tensor II space, it becomes

\[ \tilde{U}_I(\bar{y} - \Delta, \bar{k}) \left( \tilde{G}_I^{\Delta+}(\bar{y}, \bar{k}) \right)^{-1} = Tr_{II} \left[ \left( \left( R_{I,II}^{T_I}(0) \right)^{-1} \left( \tilde{G}_H^{\Delta+}(\bar{y}, \bar{k}) \right)^{-1} \tilde{U}_I^{T_I}(\bar{y} - \Delta, \bar{k}) \right)^{T_I} P_{I,II} \right]. \]

Using the fact : \((X^T)^T = X\), we obtain

\[ = Tr_{II} \left[ \left( \left( \tilde{G}_H^{\Delta+}(\bar{y}, \bar{k}) \right)^{-1} \left( R_{I,II}^{T_I}(0) \right)^{-1} \tilde{U}_I^{T_I}(\bar{y} - \Delta, \bar{k}) \right)^{T_I} P_{I,II} \right]. \]

Using \(P_{I,II}^{T_I} = P_{I,II}\), it becomes

\[ = Tr_{II} \left[ \left( P_{I,II} \left( \tilde{G}_H^{\Delta+}(\bar{y}, \bar{k}) \right)^{-1} \left( R_{I,II}^{T_I}(0) \right)^{-1} \tilde{U}_I^{T_I}(\bar{y} - \Delta, \bar{k}) \right)^{T_I} P_{I,II} \right]. \]

Moving \(P_{I,II}\) inside, we obtain

\[ = \left( \left( \tilde{G}_I^{\Delta+}(\bar{y}, \bar{k}) \right)^{-1} \left( Tr_{II} P_{I,II} \left( \left( R_{I,II}^{T_I}(0) \right)^{-1} \tilde{U}_I^{T_I}(\bar{y} - \Delta, \bar{k}) \right)^{T_I} \right) \right)^{T_I}. \] (64)

we finally reach

\[ \tilde{U}(\bar{y} + \Delta, \bar{k}) = \left( \left( \tilde{G}_I^{\Delta+}(\bar{y}, \bar{k}) \right)^{-1} \right)^T \Upsilon \left( \tilde{U}(\bar{y} - \Delta, \bar{k}) \right)^T \tilde{G}_I^{\Delta-}(\bar{y}, \bar{k}) , \] (65)

where the superscript \(T\) means matrix transpose ( but the order of the operator stay the same) ; \(\Upsilon \equiv \frac{q^{+}q^{-1}}{q^{+}+q^{-}} \times \text{diag}(q^+, q^{-1})\), which results from

\[ \Upsilon_I = (Tr)_{II} \left( P_{I,II} \left( \left( R_{I,II}(q^+, 0) \right)^{-1} \right)^{T_I} \right) , \] (66)

where the superscripts \(T_I\) and \(T_{II}\) indicate the transpose of matrices in the tensor spaces \(I\) and \(II\) respectively.

Using Eqs. (19) and (50), we obtain the difference equation for correlation function

\[ \langle 0_q | \tilde{U}_I(\bar{y}_1, \bar{k}_1) \cdots \tilde{U}_L(\bar{y}_L + 2\Delta, \bar{k}_L) \cdots \tilde{U}_N(\bar{y}_n, \bar{k}_n) | 0_q \rangle = \langle 0_q | \tilde{U}_I(\bar{y}_1, \bar{k}_1) \cdots \tilde{U}_L(\bar{y}_L, \bar{k}_L) \cdots \tilde{U}_N(\bar{y}_n, \bar{k}_n) | 0_q \rangle \]

\[ \times R_{L,L-1}(q^+, \bar{y}_L - \bar{y}_{L-1}) \cdots R_{L,1}(q^+, \bar{y}_1 - \bar{y}_1) \Upsilon_L \bar{R}_{L,N}(q^+, \bar{y}_n - \bar{y}_n + 2\Delta) \]

\[ \cdots \bar{R}_{L,L+1}(q^+, \bar{y}_n - \bar{y}_{n+1} + 2\Delta) . \] (67)

For the special case of “\(2\Delta\)” being at \(\bar{y}_n\), Eq. (57) simplifies to the following cyclic relation

\[ \langle 0_q | \tilde{U}_I(\bar{y}_1, \bar{k}_1) \cdots \tilde{U}_N(\bar{y}_n + 2\Delta, \bar{k}_n) | 0_q \rangle = \langle 0_q | \tilde{U}_N(\bar{y}_n, \bar{k}_n) \tilde{U}_I(\bar{y}_1, \bar{k}_1) \cdots \tilde{U}_{N-1}(\bar{y}_{n-1}, \bar{k}_{n-1}) | 0_q \rangle \Upsilon_N . \] (68)
For the two point function, Eq. (69) becomes

\[
\langle 0_q | \tilde{U}_I(\bar{y}_1, \tilde{k}_1) \tilde{U}_H(\bar{y}_2 + 2\Delta, \tilde{k}_2) | 0_q \rangle = \langle 0_q | \tilde{U}_I(\bar{y}_1, \tilde{k}_1) \tilde{U}_H(\bar{y}_2, \tilde{k}_2) | 0_q \rangle R_{II, I}(q', \bar{y}_2 - \bar{y}_1) \Upsilon_H . \tag{69}
\]

Multiplying Eq. (69) from the right by \(P'_{j_{i2}=0}\) and using the fact \(\langle 0_q | \tilde{U}_I \tilde{U}_H | 0_q \rangle P'_{j_{i2}=0} = \langle 0_q | \tilde{U}_I \tilde{U}_H | 0_q \rangle\), which can be shown using the definition of \(| 0_q \rangle \) given by Eq. (63), whereby Eq. (69) becomes

\[
\langle 0_q | \tilde{U}_I(\bar{y}_1, \tilde{k}_1) \tilde{U}_H(\bar{y}_2 + 2\Delta, \tilde{k}_2) | 0_q \rangle = \langle 0_q | \tilde{U}_I(\bar{y}_1, \tilde{k}_1) \tilde{U}_H(\bar{y}_2, \tilde{k}_2) | 0_q \rangle q'^{-\Delta_0^c(\bar{y}_1 - \bar{y}_2)} \frac{q' + q'^{-1}}{q'^2 + q'^{-2}} , \tag{70}
\]

where the last factor on the right is from \(P'_{j_{i2}=0} R_{II, I}(\bar{y}_1 - \bar{y}_2) \Upsilon_H P'_{j_{i2}=0} = P'_{j_{i2}=0} q'^{-\Delta_0^c(\bar{y}_1 - \bar{y}_2)} (b/a)\) with \(b/a \equiv (q' + q'^{-1})/(q'^2 + q'^{-2}) = (2|q'|^2/[4]' q\) and the fact that \(P'_{j_{i2}=0}\) multiplying the vacuum expectation value becomes unit.

Its solution can be easily found and written in the following form:

\[
\langle 0_q | \tilde{U}_I(\bar{y}_1, \tilde{k}_1) \tilde{U}_H(\bar{y}_2, \tilde{k}_2) | 0_q \rangle = \delta_{\tilde{k}_1, \tilde{k}_2} A_0(\tilde{k}_1) \text{Exp} \left\{ -\left(\frac{\bar{y}_1 - \bar{y}_2}{2\Delta}\right) \ln \left(\frac{q' + q'^{-1}}{q'^2 + q'^{-2}}\right) + \left(\frac{\bar{y}_1 - \bar{y}_2}{2\Delta}\right) \right\} + 2 \sum_{n=1}^{\infty} \theta \left( -\left(\frac{\bar{y}_1 - \bar{y}_2}{2\Delta}\right) - n \right) \ln(q'^{\Delta_0}) + (1 - \delta_{\tilde{k}_1, \tilde{k}_2}) A_1(\tilde{k}_1, \tilde{k}_2), \tag{71}
\]

where \(A_0\) and \(A_1\) are arbitrary functions; \(\theta(x) = 0, \frac{1}{2} \) for \(x < 0, x = 0, x > 0\), respectively. This expression for the solution is continuous in the \(\bar{y}_1 - \bar{y}_2 > 0\) region. For expressing the solution in a function that is continuous in the \(\bar{y}_1 - \bar{y}_2 < 0\) region, we replace \(\sum_{n=1}^{\infty} \rightarrow \sum_{n=0}^{\infty}\) in the square bracket of the above equations.

Taking away the \(\tilde{k}\)-dependence (and all \(\delta_{\tilde{k}_i, \tilde{k}_j} = 1\)) , the correlation functions become precisely those of quantum WZNW theory \[8\].

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