Inf-convolution and optimal risk sharing with arbitrary sets of risk measures

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Abstract

The inf-convolution of risk measures is directly related to risk sharing and general equilibrium, and it has attracted considerable attention in mathematical finance and insurance problems. However, the theory is restricted to finite (or at most countable in rare cases) sets of risk measures. In this study, we extend the inf-convolution of risk measures in its convex-combination form to an arbitrary (not necessarily finite or even countable) set of alternatives. The intuitive principle of this approach is to regard a probability measure as a generalization of convex weights in the finite case. Subsequently, we extensively generalize known properties and results to this framework. Specifically, we investigate the preservation of properties, dual representations, optimal allocations, and self-convolution.

Keywords: Risk measures, Inf-convolution, Risk sharing, Representations, Optimal allocations.

1 Introduction

The theory of risk measures has attracted considerable attention in mathematical finance and insurance since the seminal paper by [Artzner et al. 1999]. The books by Pflug and Römisch (2007), Delbaen (2012), Rüschendorf (2013), and Föllmer and Schied (2016) are comprehensive expositions of this subject. In these studies, a key topic is the inf-convolution of risk measures, which is directly related to risk sharing and general equilibrium. These problems may be connected with regulatory capital reduction, risk transfer in insurance–reinsurance contracts, and several other applications in classic studies such as (Borch, 1962), (Arrow, 1963), (Gerber, 1978), and (Buhlmann, 1982), as well as more recent research as in (Landsberger and Meilijson, 1994), (Dana and Meilijson, 2003), and (Heath and Ku, 2004).

Formally, the inf-convolution of risk measures is defined as

\[ \square_{i=1}^{n} \rho^i(X) = \inf \left\{ \sum_{i=1}^{n} \rho^i(X^i) : \sum_{i=1}^{n} X^i = X \right\}, \]

\[ \sum_{i=1}^{n} \rho^i(X^i) \]

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where $X$ and $X^i$, $i = 1, \cdots, n$, belong to some linear space of random variables over a probability space, and $\rho^i$, $i = 1, \cdots, n$, are risk measures, which are functional on this linear space. By induction, it is evident that $\boxn\rho^i(X) = \rho^1 \boxi \rho^n(X)$. By using a slightly modified version, convex combinations, which represent weighting schemes, may be considered as follows:

$\Lambda = \{\lambda_1, \cdots, \lambda_n\} \in [0, 1]^n$, $\sum_{i=1}^n \lambda_i = 1$; this modified version is defined as

$$\rho^\Lambda_{\text{conv}}(X) = \inf \left\{ \sum_{i=1}^n \lambda_i \rho^i(X^i) : \sum_{i=1}^n \lambda_i X^i = X \right\}.$$ 

Letting $\hat{\rho}^i = \lambda_i \rho^i$, $i = 1, \cdots, n$ immediately implies that $\rho^\Lambda_{\text{conv}}(X)$ has the same properties as the standard $\boxn\rho^i$.

Convex risk measures, as initially proposed by Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2002), have recently attracted considerable attention in the context of inf-convolutions, as in several other areas of risk management. This subject is explored in (Barrieu and El Karoui, 2005), (Burgert and Rüschendorf, 2006), (Burgert and Rüschendorf, 2008), (Jouini et al., 2008), (Filipović and Svindland, 2008), (Ludkovski and Rüschendorf, 2008), (Ludkovski and Young, 2009), (Acciaio and Svindland, 2009), (Acciaio, 2009), (Tsakalakos, 2009), (Dana and Le Van, 2010), (Delbaen, 2012), and (Kazi-Tani, 2017). These studies present a detailed investigation of the properties of inf-convolution as a risk measure per se, as well as optimality conditions for the resulting allocations.

Beyond the usual approach of convex risk measures, some studies have been concerned with inf-convolution in relation to specific properties, as in (Acciaio, 2007), (Grechuk et al., 2009), (Grechuk and Zabarankin, 2012), (Carlier et al., 2012), (Mastrogiacomo and Rosazza Gianina, 2015), and (Li et al., 2020), particular risk measures, as the recent quantile risk sharing in (Embrechts et al., 2018a, 2018b), (Embrechts et al., 2018a), (Weber, 2018), (Wang and Ziegel, 2018), and (Li et al., 2019), or even specific topics, as in (Wang, 2016) and (Liebrich and Svindland, 2019). However, these studies, with or without convexity, are restricted to finite (or at most countable in rare cases) sets of risk measures.

In this study, we extend the convex-combination-based inf-convolution of risk measures to an arbitrary (not necessarily finite or countable) set of alternatives. Specifically, we consider a collection of risk measures $\rho_\mathcal{I} = \{\rho^i, i \in \mathcal{I}\}$, where $\mathcal{I}$ is a nonempty set. Then, by considering a measure (probability) $\mu$ over the power set of $\mathcal{I}$, we obtain the generalized version of the convex inf-convolution as follows:

$$\rho^\mu_{\text{conv}}(X) = \inf \left\{ \int_\mathcal{I} \rho^i(X^i) d\mu : \int_\mathcal{I} X^i d\mu = X \right\}.$$ 

The intuitive principle of this approach is to regard a probability $\mu$ as a generalization of convex weights in the finite case.

Subsequently, we extensively generalize known properties and results to this framework. More specifically, we investigate the preservation of properties of $\rho_\mathcal{I}$, dual representations, optimal allocations, and self-convolution. Technical difficulties arise from the use of Lebesgue integration and distinguish the results in this study from those corresponding to finite $\mathcal{I}$. Of course, owing to the extent of the related literature, we do not intend to be exhaustive. To
the best of our knowledge, there is no study in this direction. The study by Right (2019) also considers an arbitrary set of risk measures and investigates the properties of combinations of the form $\rho = f(\rho_I)$, where $f$ is a combination function over a linear space generated by the outcomes of $\rho_I(X) = \{\rho_i(X), i \in I\}$. Clearly, the inf-convolution is not suitable for such a framework, as $X$ is fixed.

The remainder of this paper is organized as follows. In Section 2 we present preliminaries regarding notation, and briefly provide background material on the theory of risk measures. In Section 3 we present the proposed approach and results regarding the preservation of financial continuity properties of the set of risk measures. In Section 4, we prove results regarding notation, and briefly provide background material on the theory of risk measures. In Section 5, we explore optimal allocations by considering general results regarding the existence, comonotonic improvement, and law invariance of solutions, as well as the comonotonicity and flatness of distributions. In Section 6 we explore the special topic of self-convolution and its relation to regulatory arbitrage.

2 Preliminaries

We consider an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$. All equalities and inequalities are in the $\mathbb{P}$-a.s. sense. Let $L^0 = L^0(\Omega, \mathcal{F}, \mathbb{P})$ and $L^\infty = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ be the spaces of (equivalence classes under $\mathbb{P}$-a.s. equality) finite and essentially bounded random variables, respectively. We define $1_A$ as the indicator function for an event $A \in \mathcal{F}$. We identify constant random variables with real numbers. A pair $X, Y \in L^0$ is called comonotone if $(X(w) - X(w'))(Y(w) - Y(w')) \geq 0$, $\forall w, w' \in \Omega$. We denote by $X_n \to X$ convergence in the $L^\infty$ essential supremum norm $\|\cdot\|_\infty$, whereas $\lim_{n \to \infty} X_n = X$ indicates $\mathbb{P}$-a.s. convergence. The notation $X \succeq Y$ indicates second-order stochastic dominance, that is, $E[f(X)] \leq E[f(Y)]$ for any non-decreasing convex function $f : \mathbb{R} \to \mathbb{R}$. In particular, $E[X|\mathcal{F}] \succeq X$ for any $\sigma$-algebra $\mathcal{F}' \subseteq \mathcal{F}$.

Let $\mathcal{P}$ be the set of all probability measures on $(\Omega, \mathcal{F})$. We denote by $E_{\mathbb{Q}}[X] = \int_{\Omega} X d\mathbb{Q}$, $F_{X,\mathbb{Q}}(x) = \mathbb{Q}(X \leq x)$, and $F_{X,\mathbb{Q}}^{-1}(\alpha) = \inf \{x : F_{X,\mathbb{Q}}(x) \geq \alpha\}$, the expected value, the (increasing and right-continuous) probability function, and its quantile for $X$ with respect to $\mathbb{Q} \in \mathcal{P}$. We write $X \overset{\mathbb{Q}}{\succeq} Y$ when $F_{X,\mathbb{Q}} = F_{Y,\mathbb{Q}}$. We drop subscripts indicating probability measures when $\mathbb{Q} = \mathbb{P}$. Furthermore, let $\mathcal{Q} \subset \mathcal{P}$ be the set of probability measures that are absolutely continuous with respect to $\mathbb{P}$, with Radon–Nikodym derivative $d\mathbb{Q}/d\mathbb{P}$. We denote the topological dual $(L^\infty)^*$ of $L^\infty$ by $ba$, which is defined as the space of finitely additive signed measures (with finite total variation norm $\|\cdot\|_{TV}$) that are absolutely continuous with respect to $\mathbb{P}$; moreover, we let $ba_{1,+} = \{m \in ba : m \geq 0, m(\Omega) = 1\}$.

**Definition 2.1.** A functional $\rho : L^\infty \to \mathbb{R}$ is called a risk measure. It may have the following properties:

(i) **Monotonicity:** If $X \leq Y$, then $\rho(X) \geq \rho(Y)$, $\forall X, Y \in L^\infty$.

(ii) **Translation invariance:** $\rho(X + C) = \rho(X) - C$, $\forall X, Y \in L^\infty$, $\forall C \in \mathbb{R}$.

(iii) **Convexity:** $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y)$, $\forall X, Y \in L^\infty$, $\forall \lambda \in [0, 1]$. 

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(iv) Positive homogeneity: $\rho(\lambda X) = \lambda \rho(X)$, $\forall X, Y \in L^\infty$, $\forall \lambda \geq 0$.

(v) Law invariance: If $F_X = F_Y$, then $\rho(X) = \rho(Y)$, $\forall X, Y \in L^\infty$.

(vi) Comonotonic additivity: $\rho(X + Y) = \rho(X) + \rho(Y)$, $\forall X, Y \in L^\infty$ with $X, Y$ comonotone.

(vii) Loadedness: $\rho(X) \geq -E[X]$, $\forall X \in L^\infty$.

(viii) Limitedness: $\rho(X) \leq -\text{ess inf } X$, $\forall X \in L^\infty$.

A risk measure $\rho$ is called monetary if it satisfies (i) and (ii), convex if it is monetary and satisfies (iii), coherent if it is convex and satisfies (iv), law invariant if it satisfies (v), comonotone if it satisfies (vi), loaded if it satisfies (vii), and limited if it satisfies (viii). Unless otherwise stated, we assume that risk measures are normalized in the sense that $\rho(0) = 0$. The acceptance set of $\rho$ is defined as $A_\rho = \{X \in L^\infty : \rho(X) \leq 0\}$.

In addition to the usual norm-based continuity notions, $\mathbb{P}-a.s.$ pointwise continuity notions are relevant in the context of risk measures.

**Definition 2.2.** A risk measure $\rho : L^\infty \to \mathbb{R}$ is called

(i) Fatou continuous: If $\lim_{n \to \infty} X_n = X$ implies that $\rho(X) \leq \liminf_{n \to \infty} \rho(X_n)$, $\forall \{X_n\}_{n=1}^\infty$, $X \in L^\infty$.

(ii) Continuous from above: If $\lim_{n \to \infty} X_n = X$, with $\{X_n\}$ being non-increasing, implies that $\rho(X) = \lim_{n \to \infty} \rho(X_n)$, $\forall \{X_n\}_{n=1}^\infty$, $X \in L^\infty$.

(iii) Continuous from below: If $\lim_{n \to \infty} X_n = X$, with $\{X_n\}$ being non-decreasing, implies that $\rho(X) = \lim_{n \to \infty} \rho(X_n)$, $\forall \{X_n\}_{n=1}^\infty$, $X \in L^\infty$.

(iv) Lebesgue continuous: If $\lim_{n \to \infty} X_n = X$ implies that $\rho(X) = \lim_{n \to \infty} \rho(X_n)$, $\forall \{X_n\}_{n=1}^\infty$, $X \in L^\infty$.

For more details regarding these properties, we refer to the classic books mentioned in the introduction. We also have the following dual representations.

**Theorem 2.3** (Theorem 2.3 in [Delbaen, 2002], Theorem 4.33 in [Föllmer and Schied, 2016]). Let $\rho : L^\infty \to \mathbb{R}$ be a risk measure. Then,

(i) $\rho$ is a Fatou-continuous convex risk measure if and only if it can be represented as

$$
\rho(X) = \sup_{Q \in \mathcal{Q}} \left\{ E_Q[-X] - \alpha_{\rho}^{\text{min}}(Q) \right\}, \forall X \in L^\infty,
$$

where $\alpha_{\rho}^{\text{min}} : \mathcal{Q} \to \mathbb{R}_+ \cup \{\infty\}$, defined as $\alpha_{\rho}^{\text{min}}(Q) = \sup_{X \in A_\rho} E_Q[-X]$, is a lower semi-continuous (in the total-variation norm) convex function that is called penalty term.

(ii) $\rho$ is a Fatou-continuous coherent risk measure if and only if it can be represented as

$$
\rho(X) = \sup_{Q \in \mathcal{Q}_\rho} E_Q[-X], \forall X \in L^\infty,
$$

where $\mathcal{Q}_\rho \subseteq \mathcal{Q}$ is a nonempty, closed, and convex set that is called the dual set of $\rho$. 


Remark 2.4. Without the assumption of Fatou continuity, the representations in the previous theorem should be considered over \( ba_{1,+} \) instead of \( Q \), but with the supremum being attained. Moreover, for convex risk measures, we can define certain subgradients using Legendre–Fenchel duality (i.e., convex conjugates), as follows:

\[
\partial \rho(X) = \{ m \in ba_{1,+} : \rho(Y) - \rho(X) \geq E_m[-(Y - X)] \forall Y \in L^\infty \} = \{ m \in ba_{1,+} : E_m[-X] - \alpha^\text{min}_\rho(m) \geq \rho(X) \},
\]

\[
\partial \alpha^\text{min}_\rho(m) = \{ X \in L^\infty : \alpha^\text{min}_\rho(n) - \alpha^\text{min}_\rho(m) \geq E_{(n-m)}[-X] \forall n \in ba \} = \{ X \in L^\infty : E_m[-X] - \rho(X) \geq \alpha^\text{min}_\rho(m) \},
\]

where, by abuse of notation, we define \( E_m[-X] = \int_{\Omega} -X dm \) as the bilinear-form integral of \( X \in L^\infty \) with respect to \( m \in ba_{1,+} \). The negative sign in the expectation above is used to maintain the (anti) monotonicity pattern of risk measures. We note that these subgradient sets could be empty if we consider only \( Q \) instead of \( ba_{1,+} \). Moreover, by Theorem 2.3 we could replace the last inequalities by equalities. Further, it is immediate that \( X \in \partial \alpha^\text{min}_\rho(m) \) if and only if \( m \in \partial \rho(X) \).

Example 2.5. Examples of risk measures:

(i) Expected loss (EL): This is a Fatou-continuous, law-invariant, comonotone, coherent risk measure defined as \( EL(X) = -E[X] = -\int_0^1 F_X^{-1}(s)ds \). We have that \( A_{EL} = \{ X \in L^\infty : E[X] \geq 0 \} \) and \( Q_{EL} = \{ P \} \).

(ii) Value at risk (VaR): This is a Fatou-continuous, law-invariant, comonotone, monetary risk measure defined as \( VaR^\alpha(X) = -F_X^{-1}(\alpha) \), \( \alpha \in [0,1] \). We have the acceptance set \( A_{VaR^\alpha} = \{ X \in L^\infty : P(X < 0) \leq \alpha \} \).

(iii) Expected shortfall (ES): This is a Fatou-continuous, law-invariant, comonotone, coherent risk measure defined as \( ES^\alpha(X) = \frac{1}{\alpha} \int_0^\alpha VaR^\alpha(X)ds \), \( \alpha \in (0,1] \) and \( ES^0(X) = VaR^0(X) = -\text{ess inf} X \). We have \( A_{ES^\alpha} = \{ X \in L^\infty : \int_0^\alpha VaR^\alpha(X)ds \leq 0 \} \) and \( Q_{ES^\alpha} = \{ Q \in Q : \frac{dQ}{dx} \leq \frac{1}{\alpha} \} \).

(iv) Entropic risk measure (Ent): This is a Fatou-continuous, law-invariant, convex risk measure defined as \( Ent^\gamma(X) = \frac{1}{\gamma} \log \left( E \left[ e^{-\gamma X} \right] \right) \), \( \gamma \geq 0 \). Its acceptance set is \( A_{Ent^\gamma} = \{ X \in L^\infty : E[e^{-\gamma X}] \leq 1 \} \), and the penalty term is \( \alpha^\text{min}_{Ent^\gamma}(Q) = \frac{1}{\gamma} E \left[ \frac{dQ}{dx} \log \left( \frac{dQ}{dx} \right) \right] \).

(v) Maximum loss (ML): This is a Fatou-continuous, law-invariant, coherent risk measure defined as \( ML(X) = -\text{ess inf} X = F_X^{-1}(0) \). We have \( A_{ML} = \{ X \in L^\infty : X \geq 0 \} \) and \( Q_{ML} = Q \).

If law invariance is satisfied, as is the case in most practical applications, interesting features are present.

Theorem 2.6 (Theorem 2.1 in (Jouini et al., 2006) and Proposition 1.1 in (Svindland, 2010)).

Let \( \rho : L^\infty \rightarrow \mathbb{R} \) be a law-invariant, convex risk measure. Then, \( \rho \) is Fatou continuous.
Theorem 2.7 (Theorem 4.3 in Bäuerle and Müller, 2006, Corollary 4.65 in Föllmer and Schied, 2016). Let \( \rho : L^\infty \to \mathbb{R} \) be a law-invariant, convex risk measure. Then, \( X \succeq Y \) implies that \( \rho(X) \leq \rho(Y) \).

Theorem 2.8 (Theorems 4 and 7 in Kusuoka, 2001, Theorem 4.1 in Acerbi, 2002, Theorem 7 in Fritelli and Rosazza Gianin, 2005). Let \( \rho : L^\infty \to \mathbb{R} \) be a risk measure. Then

(i) \( \rho \) is a law-invariant, convex risk measure if and only if it can be represented as

\[
\rho(X) = \sup_{m \in M} \left\{ \int_{(0,1]} ES^\alpha(X)dm - \beta^\text{min}_\rho(m) \right\}, \forall X \in L^\infty,
\]

where \( M \) is the set of probability measures on \((0,1]\), and \( \beta^\text{min}_\rho : M \to \mathbb{R}_+ \cup \{\infty\} \) is defined as \( \beta^\text{min}_\rho(m) = \sup_{X \in A_\rho} \int_{(0,1]} ES^\alpha(X)dm \).

(ii) \( \rho \) is a law-invariant, coherent risk measure if and only if it can be represented as

\[
\rho(X) = \sup_{m \in M_\rho} \int_{(0,1]} ES^\alpha(X)dm, \forall X \in L^\infty,
\]

where \( M_\rho = \left\{ m \in M : \int_{(a,1]} \frac{1}{v}dm = F^{-1}_\rho(1-u), Q \in Q_\rho \right\} \).

(iii) \( \rho \) is a law-invariant, comonotone, coherent risk measure if and only if it can be represented as

\[
\rho(X) = \int_{(0,1]} ES^\alpha(X)dm
\]

\[
= \int_0^1 VaR^\alpha(X)\phi(\alpha)d\alpha
\]

\[
= \int_{-\infty}^0 (g(P(-X \geq x) - 1)dx + \int_0^\infty g(P(-X \geq x))dx, \forall X \in L^\infty,
\]

where \( m \in M_\rho, \phi : [0,1] \to \mathbb{R}_+ \) is non-increasing and right-continuous, with \( \phi(1) = 0 \) and \( \int_0^1 \phi(u)du = 1 \), and \( g : [0,1] \to [0,1] \) is non-decreasing and concave, with \( g(0) = 0 \) and \( g(1) = 1 \). We have that \( \int_{[u,1]} \frac{1}{v}dm = \phi(u) = g'_+(u) \forall u \in [0,1] \).

Remark 2.9. (i) Functionals with representation as in (iii) of the last theorem are called spectral or distortion risk measures. This concept is related to capacity set functions and Choquet integrals. In this case, the dual set can be understood as \( Q_\rho = \{ Q \in Q : Q(A) \leq g(P(A)), \forall A \in F \} \), which is the core of \( g \). If \( \phi \) is not non-increasing (and thus \( g \) is not concave), then the risk measure is not convex and cannot be represented as combinations of ES.

(ii) Without law invariance, we can (see, for instance, Theorem 4.94 and Corollary 4.95 in
related to (Ω is a nonempty set. We consider the measurable space (ω, F, µ) in this manner to avoid measurability issues. We could assume that the maps ω → X, ∀ X ∈ L∞, where X: F → [0, 1] is a normalized (c(∅) = 0 and c(Ω) = 1), monotone (if A ⊆ B then c(A) ≤ c(B)), submodular (c(A ∪ B) + c(A ∩ B) ≤ c(A) + c(B)) set function that is called capacity and is defined as c(A) = ρ(−1A) ∀ A ∈ F, and ba1,+ = {m ∈ ba1,+: m(A) ≤ c(A) ∀ A ∈ F}.

3 Proposed approach

Let ρI = {ρi: L∞ → R, i ∈ I} be some (a priori specified) collection of risk measures, where I is a nonempty set. We consider the measurable space (I, G), where G = 2I is the power set of I. In this space, we understand equalities, inequalities, and limits in the pointwise sense. We select G in this manner to avoid measurability issues. We could assume that the maps i → Xi(ω) are measurable for any ω ∈ Ω and every family {Xi ∈ L∞, i ∈ I}; this is necessary in order that our main functional be well defined. However, this would imply that any function I → R is measurable. Thus, the power set may be selected.

We define V as the set of probability measures in (I, G). Unless otherwise stated, we consider the fixed probability space (I, G, µ). To avoid confusion concerning measure-theoretic concepts related to (Ω, F) and (I, G), when a statement pertains to the latter, we make explicit the dependence on µ. In this study, all statements are understood to be true pointwise in I; however, in several cases, it would be sufficient if they were true in the µ – a.s. sense.

We use the notations {Xi ∈ L∞, i ∈ I} = {Xi, i ∈ I} = {X i | i ∈ I} = {Xi} for families indexed over I; these families should be understood as generalizations of n-tuples. For any X ∈ L∞, we define its allocations as

$$\mathbb{A}(X) = \left\{ \{X^i\} | i \in I: \int_I X^i(\omega)d\mu = X(\omega) \ P - a.s., \int_I E_m[X^i]d\mu = E_m[X] \forall m \in ba_{1,+} \right\}.$$  

Evidently, ω → ∫IXi(ω)dμ defines a random variable in L∞ for any {Xi}i∈I ∈ A(X), X ∈ L∞. We note that the identity ∫IXi dμ = X should then be understood in the P – a.s. sense. For technical reasons, we require that integrals over (Ω, F) and (I, G) should be interchanged. For finite I, this is always the case. We have that A(X) ≠ ∅ for any X because we can select X = X, ∀ i ∈ I. We also note that {Xi}i∈I ∈ A(X + Y) is equivalent to {Xi − Y}i∈I ∈ A(X) for any X, Y ∈ L∞. Furthermore, if {Xi}i∈I ∈ A(X) and {Y i}i∈I ∈ A(Y), then {aXi + bY i} ∈ A(aX + bY) for any a, b ∈ R and X, Y ∈ L∞. We now define the core functional in our study.

**Definition 3.1.** Let ρI = {ρi: L∞ → R, i ∈ I} be a collection of risk measures and µ ∈ V. The µ-weighted inf-convolution risk measure is a functional $\rho_{\text{conv}}^\mu: L^\infty \rightarrow R \cup \{-\infty, \infty\}$ defined
Proposition 3.3. We have that

\[ \rho^\mu_{\text{conv}}(X) = \inf \left\{ \int_I \rho^i(X) d\mu: \{X^i\}_{i \in I} \in \mathcal{A}(X) \right\}. \tag{3.1} \]

Remark 3.2. (i) We defined risk measures as functionals that only assume finite values. By abuse of notation, we will also consider \( \rho^\mu_{\text{conv}} \) to be a risk measure, and we will provide conditions whereby it is finite. For example, if \( \rho_I \) consists of risk measures pointwise bounded above, as is the case for monetary risk measures, then \( \rho^\mu_{\text{conv}} < \infty \). This is true because for any \( X \in L^\infty \), we have that \( \rho^\mu_{\text{conv}}(X) \leq \int_X \rho^i(X) d\mu < \infty \). Another case is when \( \rho^\mu_{\text{conv}} \) is convex and \( \rho^\mu_{\text{conv}} < \infty \). Then, it is finite if and only if \( \rho^\mu_{\text{conv}}(0) > -\infty \), which is a well-known fact from convex analysis. Moreover, we note that normalization is not directly inherited from \( \rho_I \); indeed, \( \rho^\mu_{\text{conv}}(0) \leq 0 \).

(ii) We could have considered using two distinct probability measures \( \mu, \nu \in \mathcal{V} \) in our formulation, one for the integral and the other for the allocations. However, this would introduce unnecessary complexity into the framework, without a clear gain. The same can be said regarding non-additive measures in the sense of Dennerberg [1994] instead of \( \sigma \)-additive probabilities in \( \mathcal{V} \).

The following proposition provides simple but interesting and useful properties of \( \rho^\mu_{\text{conv}} \).

**Proposition 3.3.** We have that

(i) \( \rho^\mu_{\text{conv}}(X) = \inf \left\{ \int_I \rho^i(X - X^i) d\mu: \{X^i\}_{i \in I} \in \mathcal{A}(0) \right\}, \forall X \in L^\infty. \)

(ii) If \( \rho_I \) consists of risk measures satisfying the positive homogeneity condition, then \( \rho^\mu_{\text{conv}}(X) \leq \rho^\mu(X) \mu \text{ a.s.}, \forall X \in L^\infty \). In particular, \( \rho^\mu_{\text{conv}} < \infty \).

(iii) If \( \rho_I \) consists of uniformly bounded risk measures, then the map \( \mathcal{V} \to \mathbb{R} \cup \{-\infty, \infty\} \) defined as \( \mu \to \rho^\mu_{\text{conv}}(X) \) is continuous with respect to the total-variation norm for any \( X \in L^\infty \).

**Proof.** (i) We note that \( \int_I X^i d\mu = X \) if and only if \( \int_I (X - X^i) d\mu = \int_I (X^i - X) d\mu = 0 \). Thus, by letting \( Y^i = X - X^i \), \( \forall i \in I \), we have that

\[ \rho^\mu_{\text{conv}}(X) = \inf \left\{ \int_I \rho^i(Y) d\mu: \{Y^i\}_{i \in I} \in \mathcal{A}(0) \right\}. \]

(ii) We assume, toward a contradiction, that there is \( X \in L^\infty \) such that \( \mu(A_X) > 0 \), where \( A_X = \{i: \rho^\mu_{\text{conv}}(X) > \rho^i(X)\} \). Let \( \{Y^i\}_{i \in I} \) be such that \( Y^i = (\mu(A_X))^{-1} X \) for \( i \in A_X \), and \( Y^i = 0 \) otherwise. Then, \( \{Y^i\}_{i \in I} \in \mathcal{A}(X) \). Let \( M_X = \inf\{k \in \mathbb{R}: \mu(\rho^i(X) > k|A_X|) = 0\} \), which is similar to the essential supremum of \( i \to \rho^i(X) \) restricted to \( A_X \). Thus, by positive homogeneity and the definition of \( \rho^\mu_{\text{conv}} \) and \( A_X \), we have for \( j \in A_X \) that

\[ \rho^j(X) < \rho^\mu_{\text{conv}}(X) \leq \int_I \rho^i(Y) d\mu = \frac{1}{\mu(A_X)} \int_{A_X} \rho^i(X) d\mu \leq M_X \mu \text{ a.s.} \]

As this is valid for any \( j \in A_X \), we conclude that \( M_X \leq (\mu(A_X))^{-1} \int_{A_X} \rho^i(X) d\mu \leq M_X \), which implies \( \mu(M_X = \rho^j(X)|A_X|) = 1 \). Then, for \( j \in A_X \), we obtain that \( \rho^j(X) < \int_{A_X} \rho^i(Y) d\mu = \rho^j(X) \), which is a contradiction. Hence, \( \rho^\mu_{\text{conv}}(X) \leq \rho^j(X) \mu \text{ a.s.}, \forall X \in L^\infty \). In this case, for any \( X \in L^\infty \), we have that \( \rho^\mu_{\text{conv}}(X) \leq \text{ess inf}_\mu \rho^i(X) < \infty. \)
(iii) Let \( X \in L^\infty \), \( \{\mu_n\} \subset \mathcal{V} \) such that \( \mu_n \to \mu \in \mathcal{V} \) with respect to the total variation norm. Let \( M \) be the uniform bound of \( \rho_I \), that is, \( |\rho^I(X)| \leq M < \infty, \forall X \in L^\infty, \forall i \in I \). Let \( \mathbb{H}^\nu(X) = \{X^i\}_{i \in I} : \int_I X^i(\omega)d\nu = X(\omega) \mathbb{P} - a.s., \int_I E_m[X^i]d\mu = E_m[X] \forall m \in ba_+ \} \) for \( \nu \in \mathcal{V} \), and let \( \|\cdot\|_{TV} \) denote the total variation norm in \( \mathcal{V} \). Then, we obtain the following:

\[
\lim_{n \to \infty} |\rho^\mu_{conv}(X) - \rho^\mu_{conv}(X)| \\
= \lim_{n \to \infty} \inf \left\{ \int_I \rho^i(X^i)d\mu : \{X^i\}_{i \in I} \in \mathbb{H}^\mu(X) \right\} - \inf \left\{ \int_I \rho^i(X^i)d\mu_n : \{X^i\}_{i \in I} \in \mathbb{H}^{\mu_n}(X) \right\} \\
\leq \lim_{n \to \infty} \sup \left\{ \left| \int_I \rho^i(X^i)d\mu - \int_I \rho^i(X^i)d\mu_n \right| : \{X^i\}_{i \in I} \right\} \\
\leq \lim_{n \to \infty} \sup \{ M\|\mu - \mu_n\|_{TV} : \{X^i\}_{i \in I} \} \\
= \lim_{n \to \infty} M\|\mu - \mu_n\|_{TV} = 0.
\]

Hence, the map \( \mu \to \rho^\mu_{conv}(X) \) is continuous with respect to the total variation norm for any \( X \in L^\infty \).

We now present a result regarding the preservation by \( \rho^\mu_{conv} \) of financial properties of \( \rho_I \).

**Proposition 3.4.** If \( \rho_I \) consists of risk measures with the monotonicity, translation invariance, convexity, positive homogeneity, loadedness, or limitedness property, then \( \rho^\mu_{conv} \) has this property as well.

**Proof.** (i) Monotonicity: Let \( X \geq Y \). Then, there is \( Z \geq 0 \) such that \( X = Y + Z \). In this case, we have

\[
\rho^\mu_{conv}(X) = \inf \left\{ \int_I \rho^i(X^i)d\mu : \{X^i\}_{i \in I} \in \mathbb{A}(Y + Z) \right\} \\
\leq \inf \left\{ \int_I \rho^i(Y^i + Z^i)d\mu : \{Y^i\}_{i \in I} \in \mathbb{A}(Y), \{Z^i\}_{i \in I} \in \mathbb{A}(Z), Z^i \geq 0 \forall i \in I \right\} \\
\leq \inf \left\{ \int_I \rho^i(Y^i)d\mu : \{Y^i\}_{i \in I} \in \mathbb{A}(Y) \right\} = \rho^\mu_{conv}(Y).
\]

(ii) Translation invariance: For any \( C \in \mathbb{R} \), we have that

\[
\rho^\mu_{conv}(X + C) = \inf \left\{ \int_I \rho^i(X^i)d\mu : \{X^i - C\}_{i \in I} \in \mathbb{A}(X) \right\} \\
= \inf \left\{ \int_I \rho^i(Y^i + C)d\mu : \{Y^i\}_{i \in I} \in \mathbb{A}(X) \right\} = \rho^\mu_{conv}(X) - C.
\]
(iii) Convexity: For any $\lambda \in [0, 1]$, we have that
\[
\begin{align*}
\lambda \rho_{\text{conv}}^{\mu}(X) + (1 - \lambda) \rho_{\text{conv}}^{\mu}(Y) &= \inf_{\{X^i\}_{i \in \mathcal{I}}, \{Y^i\}_{i \in \mathcal{I}} \in \mathcal{A}(X)} \int_{\mathcal{I}} [\lambda \rho^i(X^i) + (1 - \lambda) \rho^i(Y^i)]d\mu \\
&\geq \inf_{\{X^i\}_{i \in \mathcal{I}}, \{Y^i\}_{i \in \mathcal{I}} \in \mathcal{A}(X)} \int_{\mathcal{I}} [\rho^i(\lambda X^i) + (1 - \lambda) \rho^i(Y^i)]d\mu \\
&\geq \inf \left\{ \int_{\mathcal{I}} \rho^i(Z^i)d\mu : \{Z^i\}_{i \in \mathcal{I}} \in \mathcal{A}(\lambda X + (1 - \lambda)Y) \right\} \\
&= \rho_{\text{conv}}^{\mu}(\lambda X + (1 - \lambda)Y).
\end{align*}
\]

(iv) Positive homogeneity: For any $\lambda \geq 0$, we have that
\[
\lambda \rho_{\text{conv}}^{\mu}(X) = \inf \left\{ \int_{\mathcal{I}} \rho^i(\lambda Y^i)d\mu : \{Y^i\}_{i \in \mathcal{I}} \in \mathcal{A}(X) \right\} = \rho_{\text{conv}}^{\mu}(\lambda X).
\]

(v) Loadedness: We fix $X \in L^\infty$ and note that for any $\{X^i\}_{i \in \mathcal{I}} \in \mathcal{A}(X)$, we have that
\[
\int_{\mathcal{I}} \rho^i(X^i)d\mu \geq \int_{\mathcal{I}} E[-X^i]d\mu = E[-X].
\]

By taking the infimum over $\mathcal{A}(X)$, we obtain that $\rho_{\text{conv}}^{\mu}(X) \geq -E[X]$.

(vi) Limitedness: We fix $X \in L^\infty$. By the monotonicity of the integral, we have that
\[
\rho_{\text{conv}}^{\mu}(X) \leq \int_{\mathcal{I}} \rho^i(X)d\mu \leq -\text{ess inf } X.
\]

Remark 3.5. (i) Concerning the preservation of subadditivity, that is, $\rho(X + Y) \leq \rho(X) + \rho(Y)$, the result follows by an argument analogous to that for convexity, but with $X + Y$ instead of $\lambda X + (1 - \lambda)Y$. We note that in this case, we have normalization because $\rho_{\text{conv}}^{\mu}(0) \leq 0$, whereas $\rho_{\text{conv}}^{\mu}(X) \leq \rho_{\text{conv}}^{\mu}(X) + \rho_{\text{conv}}^{\mu}(0)$, which implies $\rho_{\text{conv}}^{\mu}(0) \geq 0$. If the risk measures of $\rho_{\mathcal{I}}$ are loaded, we also have normalization because $0 = \rho(0) \geq \rho_{\text{conv}}^{\mu}(0) \geq E[-0] = 0$. Of course, in the case of positive homogeneity, we also obtain normalization. Furthermore, regarding limitedness, we have that $\rho_{\text{conv}}^{\mu}(X) \leq \|X\|_\infty < \infty$ for any $X \in L^\infty$.

(ii) Liu et al. [2020] showed that, in the usual case of finite $\mathcal{I}$, law invariance is preserved under other continuity conditions that primarily rely on the existence of countable iid uniform random variables, which is always the case for atomless spaces. However, we are unable to extend their result, as in our framework, arbitrarily many such random variables would be required. Nonetheless, if convexity/coherence and Fatou continuity are assumed for $\rho_{\mathcal{I}}$ and $\rho_{\text{conv}}^{\mu}$, respectively, then law invariance is preserved, as shown in Theorem 4.9. This is based on dual representations for $\rho_{\text{conv}}^{\mu}$.

(iii) Regarding the preservation of comonotonic additivity, let $X, Y \in L^\infty$ be a comonotone
pair. Then, $\lambda X,(1 - \lambda)Y$ is also comonotone for any $\lambda \in [0,1]$. We note that for any risk measure $\rho$, comonotonic additivity implies $\rho(kX) = k\rho(X)$ for any rational number $k \geq 0$. Then, by arguing as in the proof of (iii) and (iv) of the last proposition with $\lambda = \frac{1}{2}$, we have that

$$
\rho^\mu_{conv}(X + Y) = 2\rho^\mu_{conv} \left( \frac{X}{2} + \frac{Y}{2} \right) \leq 2 \left( \frac{1}{2} \rho^\mu_{conv}(X) + \frac{1}{2} \rho^\mu_{conv}(Y) \right) = \rho^\mu_{conv}(X) + \rho^\mu_{conv}(Y).
$$

Thus, we obtain subadditivity for comonotone pairs. If, additionally, convexity (and hence coherence) and Fatou continuity for $\rho_I$ and $\rho^\mu_{conv}$, respectively, are assumed, comonotonic additivity is preserved, as shown in Theorem 4.9 and Corollary 4.11.

In the following, we focus on the preservation by $\rho^\mu_{conv}$ of continuity properties of $\rho_I$.

**Proposition 3.6.** We have that

(i) If $\rho_I$ consist of Lipschitz-continuous risk measures, then $\rho^\mu_{conv}$ is also Lipschitz continuous.

(ii) If $\rho_I$ consists of uniformly bounded, monotonic, and continuous-from-below risk measures, then $\rho^\mu_{conv}$ is continuous from below (and monotone).

**Proof.** (i) For each $i \in \mathcal{I}$, we have that $|\rho^i(X) - \rho^i(Y)| \leq C^i \|X - Y\|_\infty$, $C^i \in \mathbb{R}^*_+$. Let $K = \int_{\mathcal{I}} C^i d\mu > 0$. Thus,

$$
|\rho^\mu_{conv}(X) - \rho^\mu_{conv}(Y)|
= \left| \inf \left\{ \int_{\mathcal{I}} \rho^i(X - X^i) d\mu : \{X^i\}_{i \in \mathcal{I}} \in \mathcal{H}(0) \right\} - \inf \left\{ \int_{\mathcal{I}} \rho^i(Y - X^i) d\mu : \{X^i\}_{i \in \mathcal{I}} \in \mathcal{H}(0) \right\} \right|
\leq \sup \left\{ \left| \int_{\mathcal{I}} [\rho^i(X - X^i) - \rho^i(Y - X^i)] d\mu : \{X^i\}_{i \in \mathcal{I}} \in \mathcal{H}(0) \right| \right\} \leq K \|X - Y\|_\infty.
$$

(ii) Let $\{X_n\}_{n=1}^\infty \subset L^\infty$ be non-decreasing such that $\lim_{n \to \infty} X_n = X \in L^\infty$. By the monotonicity of $\rho_I$ and by the dominated convergence theorem, which is valid by the uniform boundedness of $\rho_I$, we have that

$$
\lim_{n \to \infty} \rho^\mu_{conv}(X_n) = \inf_n \left\{ \inf_{\{X^i\} \in \mathcal{H}(0)} \int_{\mathcal{I}} \rho^i(X_n - X^i) d\mu \right\}
\leq \inf_{\{X^i\} \in \mathcal{H}(0)} \int_{\mathcal{I}} \rho^i(X_n - X^i) d\mu
= \int_{\mathcal{I}} \left[ \inf_n \rho^i(X_n - X^i) \right] d\mu
= \rho^\mu_{conv}(X).
$$

**Remark 3.7.** (i) In our framework, uniform continuity is not always preserved because by arguing as in (i) above, we would have $|\rho^\mu_{conv}(X) - \rho^\mu_{conv}(Y)| \leq \int_{\mathcal{I}} \epsilon^i d\mu$. However, the right-hand term can be strictly greater than the arbitrary $\epsilon > 0$ in the definition of uniform continuity. Nonetheless, Lemma 1 in (Liu et al., 2020) shows that uniform continuity is preserved when $\mathcal{I}$ is finite, whereas Example 6 in that paper provides a case where ordinary continuity is not preserved.
(ii) It is important to note that Fatou continuity is not preserved even when \( I \) is finite, as 
\[
\lim_{n \to \infty} X_n = X \text{ does not imply the existence of } \{X^i_n\}_{i \in I} \in \mathcal{A}(X_n) \forall \ n \in \mathbb{N} \text{ and } \{X^i\}_{i \in I} \in \mathcal{A}(X) \text{ such that } \lim_{n \to \infty} X^i_n = X^i, \forall i \in I.
\] Accordingly, one should be careful when dual representations are considered.

(iii) The uniform boundedness of the risk measures of \( \rho_I \) as an assumption for the preservation of continuity from below can be restrictive. It would suffice to have \( \rho^i(X) \leq f(i) \forall X \in L^\infty \mu - \text{a.s.} \) for some integrable \( f : I \to \mathbb{R} \) and use the monotone convergence theorem, which also is restrictive. If the risk measures of \( \rho_I \) are convex, we will show in Corollary 4.4 that the preservation of continuity from below, which implies Fatou continuity for convex risk measures, is achieved without the assumption of uniform boundedness.

Robustness is a key concept in the presence of model uncertainty. It implies small variation in the output functional when there is misspecification. For risk measures, see, for instance, (Cont et al., 2010), (Kratschmer et al., 2014), and (Kiesel et al., 2016). We now present a formal definition.

**Definition 3.8.** Let \( d \) be a pseudo-metric on \( L^\infty \). Then, a risk measure \( \rho : L^\infty \to \mathbb{R} \) is called \( d \)-robust if it is continuous with respect to \( d \).

**Remark 3.9.** Convergence in distribution in the set of bounded random variables (i.e., convergence with respect to the Levy metric) is pivotal in the presence of uncertainty regarding distributions, as in the model risk framework. It is well known (see, for instance, the papers mentioned at the beginning of this subsection) that a non-convex risk measure is upper semi-continuous with respect to the Levy metric. By Proposition 4.4 if each member of \( \rho_I \) is a convex risk measure, then so is \( \rho^\mu_{\text{conv}} \). Thus, robustness with respect to this metric is ruled out.

In light of Proposition 3.6 we have that the continuity of the risk measures in \( \rho_I \) with respect to \( d \) is not generally preserved by \( \rho^\mu_{\text{conv}} \). Consequently, the same is true for \( d \)-robustness. Nonetheless, under stronger assumptions, we have the following corollary regarding the preservation of robustness.

**Corollary 3.10.** If \( \rho_I \) consists of risk measures that are Lipschitz continuous with respect to \( d \), then \( \rho^\mu_{\text{conv}} \) is \( d \)-robust.

**Proof.** The proof is analogous to that of (i) in Proposition 3.6.

## 4 Dual representations

To obtain dual representations for \( \rho^\mu_{\text{conv}} \), the following auxiliary result is required; it ensures that supremum and integral can be interchanged.

**Lemma 4.1.** [Lemma 4.3 in (Righi, 2019a)] Let \( (I, \mathcal{G}, \mu) \) be a probability space, and \( h^i : \mathcal{Y} \to \mathbb{R}, i \in I, \) be a collection of \( (\mu - \text{a.s.}) \) bounded functionals on a nonempty space \( \mathcal{Y} \) such that \( i \to h^i(y_i) \) is \( \mathcal{G} \)-measurable for any \( \{y_i \in \mathcal{Y}, i \in I\} \). Then,
\[
\int_I \sup_{y \in \mathcal{Y}_i} h^i(y) d\mu = \sup_{\{y_i \in \mathcal{Y}_i, i \in I\}} \int_I h^i(y_i) d\mu,
\]
where for any \( i \in \mathcal{I} \): \( \mathcal{X}_i \subseteq \mathcal{Y}, \mathcal{X}_i \neq \emptyset \), and \( \sup_{y \in \mathcal{X}_i} h^i(y) \) is \( \mathcal{G} \)-measurable.

We now present the main results regarding the representation of \( \rho_{conv}^\mu \) for convex cases.

**Theorem 4.2.** Let \( \rho_\mathcal{I}^\mu \) be a collection of convex risk measures. We have that

(i) The minimal penalty term of \( \rho_{conv}^\mu \) is bounded:

\[
\alpha_{\rho_{conv}^\mu}^\min (m) \leq \int_\mathcal{I} \alpha_{\rho^i}^\min (m) d\mu, \forall m \in ba_{1,+}. \tag{4.1}
\]

Furthermore, \( m \rightarrow \int_\mathcal{I} \alpha_{\rho^i}^\min (m) d\mu \) is non-negative, convex, and lower semi-continuous in the total variation norm.

(ii) If (4.1) becomes an equality, then

\[
\mathcal{A}_{\rho_{conv}^\mu} \supseteq cl(A_\mu), \mathcal{A}_\mu = \{ X \in L^\infty : \exists \{ X_i \}_{i \in \mathcal{I}] \in \mathcal{A}(X) s.t. X_i \in \mathcal{A}_{\rho^i} \mu - a.s. \}. \tag{4.2}
\]

Moreover, \( \mathcal{A}_\mu \) is not dense in \( L^\infty \) if and only if \( \mathcal{A}_\mu \neq L^\infty \).

**Proof.**

(i) By Proposition 3.4, we have that \( \rho_{conv}^\mu \) is a finite, convex risk measure. Moreover, if \( \int_\mathcal{I} \alpha_{\rho^i}^\min (m) d\mu = \infty \), then the claim follows immediately; otherwise, we have the following:

\[
\int_\mathcal{I} \alpha_{\rho^i}^\min (m) d\mu = \int_\mathcal{I} \left( \sup_{X \in L^\infty} \{ E_m [-X] - \rho^i(X) \} \right) d\mu
\]

\[
= \sup_{X \in L^\infty} \left\{ \int_\mathcal{I} (E_m[-X] - \rho^i(X)) d\mu \right\}
\]

\[
\geq \sup_{X \in L^\infty} \sup_{X \in \mathcal{A}(X)} \left\{ \int_\mathcal{I} (E_m[-X] - \rho^i(X)) d\mu \right\}
\]

\[
= \sup_{X \in L^\infty} \left\{ E_m[-X] - \inf \left\{ \int_\mathcal{I} \rho^i(X) d\mu : \{ X_i \}_{i \in \mathcal{I}] \in \mathcal{A}(X) \} \right\} \right\}
\]

\[
= \sup_{X \in L^\infty} \{ E_m[-X] - \rho_{\rho_{conv}^\mu}^\mu \} = \alpha_{\rho_{conv}^\mu}^\min (m).
\]

We have used Lemma 4.1 for the interchange of supremum and integral, which is valid because \( X \rightarrow E_m[-X] - \rho^i(X) \) is bounded by \( \alpha_{\rho^i}^\min (m) < \infty \) \( \mu - a.s. \). We have also used that \( \int_\mathcal{I} E_m [X] d\mu = E_m [X] \) for any \( m \in ba_{1,+} \) and \( \{ X_i \}_{i \in \mathcal{I}] \in \mathcal{A}(X) \).

Regarding the properties of \( m \rightarrow \int_\mathcal{I} \alpha_{\rho^i}^\min (m) d\mu \), non-negativity is straightforward, whereas convexity follows from the monotonicity of the integral and the convexity of each \( \alpha_{\rho^i}^\min \) because for any \( \lambda \in [0,1] \) and \( m_1, m_2 \in ba_{1,+} \), we have that

\[
\int_\mathcal{I} \alpha_{\rho^i}^\min (\lambda m_1 + (1-\lambda) m_2) d\mu \leq \int_\mathcal{I} \left[ \lambda \alpha_{\rho^i}^\min (m_1) + (1-\lambda) \alpha_{\rho^i}^\min (m_2) \right] d\mu
\]

\[
= \lambda \int_\mathcal{I} \alpha_{\rho^i}^\min (m_1) d\mu + (1-\lambda) \int_\mathcal{I} \alpha_{\rho^i}^\min (m_2) d\mu.
\]

Furthermore, by Fatou’s lemma (which can be used because each \( \alpha_{\rho^i}^\min \) is bounded from below by 0) and by the lower semi-continuity of each \( \alpha_{\rho^i}^\min \) with respect to the total
Remark 4.3 (i) The inequality \( \alpha_{\rho_{\text{conv}}}^\text{min}(m) \leq \int_I \alpha_{\rho}^\text{min}(m) d\mu \) is not in general an equality, which is the case for finite \( I \), owing to complications arising from Lebesgue integration. More specifically, we may have \( \{X^i\}_{i \in I} \notin \cup_{X \in L^\infty} \mathcal{A}(X) \), whereas for finite \( I \), this is not true because \( L^\infty \) is a vector space.

(ii) Under the assumption of Fatou continuity for both the risk measures in \( \rho^\mu_\mathcal{I} \) and \( \rho^\mu_{\text{conv}} \), the claims in Theorem 4.2 should be adapted by replacing the finitely additive measures \( m \in ba_{+,1} \) by probabilities \( Q \in \mathcal{Q} \). Moreover, weak* topological concepts should replace the corresponding strong (norm) topological concepts.

(iii) The weighted risk measure is a functional \( \rho^\mu : L^\infty \to \mathbb{R} \) defined as \( \rho^\mu(X) = \int_I \rho^\mu(X) d\mu \). Theorem 4.6 in (Righi, 2019b) states that, assuming Fatou continuity, \( \rho^\mu \) can be represented using a convex (not necessarily minimal) penalty defined as

\[
\alpha_{\rho^\mu}(Q) = \inf \left\{ \int_I \alpha_{\rho_i}^\text{min}(Q^i) d\mu : \int_I Q^i d\mu = Q, Q^i \in \mathcal{Q} \forall i \in I \right\}.
\]

By the duality of convex conjugates, \( \alpha_{\rho^\mu} = \alpha_{\rho^\mu}^\text{min} \) if and only if \( \alpha_{\rho^\mu} \) is lower semi-continuous. In contrast to (i) above, \( \int_I Q^i d\mu \) defines (see Lemma 4.5 in (Righi, 2019b)) a probability for any \( Q^i \in \mathcal{P} \). Nonetheless, this represents a connection between weighted and inf-convolution functions for arbitrary \( I \), as in the traditional finite case.
The following corollary provides interesting properties regarding the normalization, finiteness, preservation, and dominance of \( \rho_{\text{conv}}^\mu \).

**Corollary 4.4.** Let \( \rho_I \) be a collection of convex risk measures. Then

(i) If \( \left\{ m \in \text{ba}_{1,+}: \int_I \alpha_{\rho}^{\min}(m) d\mu < \infty \right\} \neq \emptyset \), then \( \rho_{\text{conv}}^\mu \) is finite. Assuming that equality holds in (4.1), if \( \rho_{\text{conv}}^\mu \) is finite, then \( \left\{ m \in \text{ba}_{1,+}: \alpha_{\rho}^{\min}(m) d\mu < \infty \right\} \neq \emptyset \). Moreover, if \( \rho_{\text{conv}}^\mu \) is finite, then \( \mathcal{A}_\mu \) is not dense in \( L^\infty \).

(ii) If \( \left\{ m \in \text{ba}_{1,+}: \alpha_{\rho}^{\min}(m) = 0 \mu - \text{a.s.} \right\} \neq \emptyset \), then \( \rho_{\text{conv}}^\mu \) is normalized. The converse is true if equality holds in (4.1).

(iii) If \( \rho: L^\infty \to \mathbb{R} \) is a convex risk measure with \( \rho(X) \leq \rho^I(X) \mu - \text{a.s.}, \forall X \in L^\infty \), then \( \rho(X) \leq \rho_{\text{conv}}^\mu(X), \forall X \in L^\infty \). Moreover, assuming coherence of \( \rho_I \), (4.1) becomes an equality.

(iv) If \( \rho: L^\infty \to \mathbb{R} \) is a convex risk measure with \( \rho^I(X) = \rho(X), \mu - \text{a.s.}, \forall X \in L^\infty \), then \( \rho_{\text{conv}}^\mu(X) = \rho(X), \forall X \in L^\infty \).

(v) Assuming that equality holds in (4.1), if \( \rho_I \) consists of risk measures that are continuous from below, then \( \rho_{\text{conv}}^\mu \) is Lebesgue continuous.

**Proof.**

(i) By Proposition 3.4, we have \( \rho_{\text{conv}}^\mu < \infty \). If \( \left\{ m \in \text{ba}_{1,+}: \int_I \alpha_{\rho}^{\min}(m) d\mu < \infty \right\} \neq \emptyset \), then there exists \( m \in \text{ba}_{1,+} \) such that

\[-\infty < E_m[-X] - \int_I \alpha_{\rho}^{\min}(m) d\mu \leq E_m[-X] - \alpha_{\rho_{\text{conv}}^\mu}^{\min}(m) \leq \rho_{\text{conv}}^\mu(X).

Moreover, if equality holds in (4.1), let \( \left\{ m \in \text{ba}_{1,+}: \alpha_{\rho}^{\min}(m) < \infty \mu - \text{a.s.} \right\} = \emptyset \). Thus, \( \alpha_{\rho_{\text{conv}}^\mu}^{\min}(m) = \int_I \alpha_{\rho}^{\min}(m) = \infty, \forall m \in \text{ba}_{1,+} \). Hence, \( \rho_{\text{conv}}^\mu(X) = -\infty, \forall X \in L^\infty \). Furthermore, if \( \mathcal{A}_\mu \) is dense in \( L^\infty \), then by (ii) in Theorem 4.2, we have \( \mathcal{A}_\mu = L^\infty \). Thus, for any \( X \in L^\infty \), we have \( \rho_{\text{conv}}^\mu(X) = \inf\left\{ m \in \mathbb{R}, X + m \in L^\infty \right\} = -\infty, \forall X \in L^\infty \).

(ii) Let \( \left\{ m \in \text{ba}_{1,+}: \alpha_{\rho}^{\min}(m) = 0 \mu - \text{a.s.} \right\} \neq \emptyset \) and \( m' \) in this set. Then, \( 0 \leq \alpha_{\rho_{\text{conv}}^\mu}^{\min}(m') \leq \int_I \alpha_{\rho}^{\min}(m') d\mu = 0 \). Hence,

\[ \rho_{\text{conv}}^\mu(0) = -\inf_{m \in \text{ba}_{1,+}} \alpha_{\rho_{\text{conv}}^\mu}^{\min}(m) = 0. \]

For the converse relation, let \( \left\{ m \in \text{ba}_{1,+}: \alpha_{\rho}^{\min}(m) = 0 \mu - \text{a.s.} \right\} = \emptyset \). Thus, for any \( m \in \text{ba}_{1,+} \), we have that \( \mu \left( \left\{ i: \alpha_{\rho}^{\min}(m) > 0 \right\} \right) > 0 \). Then,

\[ \rho_{\text{conv}}^\mu(0) = -\min_{m \in \text{ba}_{1,+}} \alpha_{\rho_{\text{conv}}^\mu}^{\min}(m) < 0, \]

which is a contradiction.

(iii) By Theorem 2.3, we have that \( \alpha_{\rho}^{\min}(m) \geq \alpha_{\rho}^{\min}(m) \mu - \text{a.s.} \) for any \( m \in \text{ba}_{1,+} \). Thus, by
Remark 4.5. Thus, assuming coherence, we recover the corresponding result for finite $I$ regarding the penalty term $\alpha_{\rho\Conv}^{\min}$. Moreover, as in this case, $\rho(X) \leq \rho_{\Conv}^{\mu}(X) \leq \rho^i(X) \mu - a.s., \forall X \in L^\infty$ for any convex risk measure $\rho$, we can understand $\rho_{\Conv}^{\mu}$ as the “lower-convexification” of the non-convex risk measure $\essinf_{\mu} \rho^i$ in the sense that the former is the largest convex risk measure that is dominated by the latter.

We now present the main results regarding the representation of $\rho_{\Conv}^{\mu}$ for coherent cases. Henceforth in this section, we focus on the Fatou continuous case, as it is the standard for such representations.

**Theorem 4.6.** Let $\rho_I$ be a collection of Fatou-continuous, coherent risk measures, and let $\rho_{\Conv}^{\mu}$ be Fatou continuous. Then,

(i) $\rho_{\Conv}^{\mu}$ is finite, and its dual set is defined as

$$Q_{\rho_{\Conv}^{\mu}} = \{ Q \in Q : Q \in Q_{\rho^i} \mu - a.s. \} .$$  

In particular, $\{ Q \in Q : \alpha_{\rho^i}^{\min}(Q) = 0 \mu - a.s. \} \neq \emptyset$.

(ii) The acceptance set of $\rho_{\Conv}^{\mu}$ is defined as

$$A_{\rho_{\Conv}^{\mu}} = \text{clconv} (A) = \{ X \in L^\infty : \mu \{ X \in A \} > 0 \} ,$$  

where ...
where \( \text{clconv} \) denotes the closed convex hull, and \( cl^* \) is the closure with respect to the weak* topology.

**Proof.**  (i) By Proposition 3.4, we have that \( \rho^\mu_{\text{conv}} \) is a finite, Fatou-continuous, coherent risk measure. Moreover, by Corollary 4.4 equality holds in (i). By Theorems 2.3 and 4.2 we obtain that

\[
Q_{\rho^\mu_{\text{conv}}} = \left\{ Q \in \mathcal{Q} : \int_{\mathcal{I}} \alpha^\rho_{\mu, \min}(Q) d\mu = 0 \right\} \\
= \left\{ Q \in \mathcal{Q} : \alpha^\rho_{\mu, \min}(Q) = 0 \text{ a.s.} \right\} \\
= \left\{ Q \in \mathcal{Q} : Q \in \rho^\mu_{\text{conv}} \text{ a.s.} \right\}.
\]

The convexity and closedness of \( Q_{\rho^\mu_{\text{conv}}} \) follow from the convexity and lower semicontinuity of \( \alpha^\rho_{\mu, \min} \). Furthermore, if \( \left\{ Q \in \mathcal{Q} : \alpha^\rho_{\mu, \min}(Q) = 0 \text{ a.s.} \right\} = \emptyset \), then by Corollary 4.4 \( \rho^\mu_{\text{conv}}(0) < 0 \), which contradicts coherence.

(ii) We recall that, by Theorem 2.3 for any Fatou-continuous, coherent risk measure \( \rho : L^\infty \to \mathbb{R} \), we have \( X \in \mathcal{A}_\rho \) if and only if \( E_Q[-X] \leq 0 \forall Q \in \mathcal{Q}_\rho \). Thus,

\[
Q_{\rho^\mu_{\text{conv}}} = \left\{ Q \in \mathcal{Q} : \mu (i : E_Q[-X] \leq 0, \forall X \in \mathcal{A}_\rho_i) = 1 \right\} \\
= \left\{ Q \in \mathcal{Q} : E_Q[-X] \leq 0, \forall X \in L^\infty \text{ s.t. } \mu (i : X \in \mathcal{A}_\rho_i) > 0 \right\} \\
= \left\{ Q \in \mathcal{Q} : E_Q[-X] \leq 0, \forall X \in \mathcal{A}_\rho \right\} \\
= \left\{ Q \in \mathcal{Q} : E_Q[-X] \leq 0, \forall X \in \text{clconv}(\mathcal{A}_\rho) \right\} = Q_{\rho_{\text{clconv}}(\mathcal{A}_\rho)}.
\]

The fact that considering the closed convex hull does not affect the fourth equality above is because the map \( X \to E_Q[X] \) is linear and continuous for any \( Q \in \mathcal{Q} \). Thus, \( \rho^\mu_{\text{conv}} = \rho_{\text{clconv}}(\mathcal{A}_\rho) \). Hence, \( A^\mu_{\text{conv}} = A_{\text{clconv}}(\mathcal{A}_\rho) = \text{clconv}(\mathcal{A}_\rho) \). We note that \( \mathcal{A}_\rho \) is nonempty, monotone (in the sense that \( X \in \mathcal{A}_\rho \) and \( Y \geq X \) implies \( Y \in \mathcal{A}_\rho \)), and a cone, as this is true for any \( \mathcal{A}_\rho \). Moreover, it is evident that \( \mathcal{A}_\rho \subseteq \mathcal{A}_\mu \) for normalized risk measures in \( \rho_\mathcal{I} \). Thus, by Theorem 4.2 we have that \( A^\mu_{\text{conv}} = \text{clconv}(\mathcal{A}_\rho) \subseteq cl^*(\mathcal{A}_\rho) \subseteq A^\mu_{\text{conv}} \).

\( \square \)

**Remark 4.7.** Assuming that \( \rho_\mathcal{I} \) consists of coherent risk measures and that \( \mathcal{I} \) is finite, we recover the well-known fact \( Q_{\rho^\mu_{\text{conv}}} = \bigcap_{i \in \mathcal{I}} Q_{\rho_i} \). Moreover, in light of Corollary 4.3 we have, under the hypotheses of Theorem 4.3, that \( \rho^\mu_{\text{conv}} \) is finite and normalized if and only if the condition \( \left\{ Q \in \mathcal{Q} : \alpha^\rho_{\mu, \min}(Q) = 0 \text{ a.s.} \right\} \neq \emptyset \) is satisfied. Moreover, both assertions are equivalent to \( \mathcal{A}_\mu \) not being weak*-dense in \( L^\infty \). The intuition for \( \mathcal{A}_\rho \) is that some position is acceptable if it is acceptable for any relevant (in the \( \mu \) sense) members of \( \rho_\mathcal{I} \). This is not particularly restrictive and hence suitable for inf-convolution.

We also have the following corollary regarding the dual set of coherent \( \rho^\mu_{\text{conv}} \).

**Corollary 4.8.** Let \( \rho_\mathcal{I} \) be a collection of Fatou-continuous, coherent risk measures, and let \( \rho^\mu_{\text{conv}} \) be Fatou continuous. Then, \( \bigcap_{i \in \mathcal{I}} Q_{\rho_i} \subseteq Q_{\rho^\mu_{\text{conv}}} \) for any \( A \in \mathcal{G} \) with \( \mu(A) = 1 \).
Proof. Let \( Q \in \bigcap_{i \in I} Q_{\rho_i} \) for some \( A \in \mathcal{G} \) with \( \mu(A) = 1 \). If the intersection is empty, then the result is obvious. For any \( \{X^i\}_{i \in I} \in \mathcal{A}(X) \), we have that \( E_Q[-X^i] \leq \rho^i(X^i), \forall i \in A \). In this case, we obtain \( E_Q[-X] = \int_I E_Q[-X^i]d\mu \leq \int_I \rho^i(X^i)d\mu \). By taking the infimum over \( \mathcal{A}(X) \) on the right side, we obtain \( E_Q[-X] \leq \rho^\mu_{\text{conv}}(X) \). As this is valid for any \( X \in L^\infty \), we have \( \alpha_{\rho^\mu_{\text{conv}}}^\min(Q) = 0 \). Hence, \( Q \in Q_{\rho^\mu_{\text{conv}}} \).

We now focus on dual representations under the assumption of law invariance and comonotonic additivity.

**Theorem 4.9.** Let \( \rho_I \) be a collection of convex, law-invariant risk measures, and let \( \rho^\mu_{\text{conv}} \) be Fatou continuous. Then,

(i) \( \rho^\mu_{\text{conv}} \) is finite and normalized.

(ii) If equality holds in (4.1), \( \rho^\mu_{\text{conv}} \) is law invariant with penalty term

\[
\beta^\min_{\rho^\mu_{\text{conv}}}(m) = \int_I \beta^\min_{\rho_i}(m)d\mu, \forall m \in \mathcal{M}.
\]

(iii) If, in addition, \( \rho_I \) consists of comonotone risk measures, then \( \rho^\mu_{\text{conv}} \) is comonotone, and its distortion function is

\[
g = \text{ess inf}_\mu g^i,
\]

where \( g^i \) is the distortion of \( \rho^i \) for each \( i \in I \).

**Proof.** (i) By Proposition 3.21 we have that \( \rho^\mu_{\text{conv}} \) is a Fatou-continuous, convex risk measure with \( \rho^\mu_{\text{conv}} < \infty \). Regarding finiteness and normalization, we note that \( \rho(X) \geq -E[X], \forall X \in L^\infty \) for normalized, convex, law-invariant risk measures by second-order stochastic dominance. Thus,

\[
\alpha^\min_{\rho^i}(\mathcal{P}) = \sup_{X \in L^\infty} \{ E[-X] - \rho(X) \} \leq \sup_{X \in L^\infty} \{ \rho(X) - \rho(X) \} = 0, \forall i \in I.
\]

By the non-negativity of the penalty terms, we have \( \alpha^\min_{\rho^i}(\mathcal{P}) = 0, \forall i \in I \). Hence, by Theorem 4.2 and Corollary 4.4, we conclude that \( \rho^\mu_{\text{conv}} \) is normalized and finite.

(ii) By Theorem 4.59 in (Föllmer and Schied, 2016), we have that the penalty functions of Fatou-continuous, convex risk measures are law invariant in the sense that \( \frac{dQ_1}{d\mathcal{P}} \sim \frac{dQ_2}{d\mathcal{P}} \) implies that \( \alpha^\min_{\rho^i}(Q_1) = \alpha^\min_{\rho^i}(Q_2), \forall Q_1, Q_2 \in \mathcal{P} \) if and only if \( \rho \) is law invariant. By Theorem 4.2 for any \( Q_1, Q_2 \in \mathcal{P} \) with \( \frac{dQ_1}{d\mathcal{P}} \sim \frac{dQ_2}{d\mathcal{P}} \), we have that

\[
\alpha^\min_{\rho^\mu_{\text{conv}}}(Q_1) = \int_I \alpha^\min_{\rho_i}(Q_1)d\mu = \int_I \alpha^\min_{\rho_i}(Q_2)d\mu = \alpha^\min_{\rho^\mu_{\text{conv}}}(Q_2).
\]

Thus, \( \rho^\mu_{\text{conv}} \) is law invariant. Moreover, the penalty term can be obtained by an argument similar to that in (i) of Theorem 4.2 by considering the representation in (2.3) and noticing
that, by Theorems 2.3 and 2.8 we have that for any \( m \in \mathcal{M} \), there is \( Q' \in Q \) such that

\[
\int_{(0,1]} ES^\alpha(X) dm = \sup \left\{ E_Q[-X]: \frac{dQ}{dP} \sim \frac{dQ'}{dP}, \int_{(u,1]} \frac{1}{v} dm = F_{d\rho^{-1}}^{-1}(1-u) \right\}
\]

\[
= E_{Q'}[-X], \quad \forall \ X \in L^\infty.
\]

(iii) By Theorems 2.8, 4.2, and 4.6, as well as Remark 2.9, we have, recalling that \( \rho_{\text{conv}}^\mu \) is finite and Fatou continuous, that

\[
Q_{\rho_{\text{conv}}^\mu} = \{ Q \in Q: Q(A) \leq g_i(P(A)) \mu - \text{a.s.} \forall A \in \mathcal{F} \}
\]

\[
= \{ Q \in Q: Q(A) \leq \text{ess inf}_\mu g_i(P(A)) \forall A \in \mathcal{F} \}.
\]

By the properties of the essential infimum and \( \{g_i\}_{i \in \mathcal{I}} \), we obtain that \( g: [0,1] \rightarrow [0,1] \) is non-decreasing and concave, and it satisfies \( g(0) = 0 \) and \( g(1) = 1 \). Thus, \( \rho_{\text{conv}}^\mu \) can be represented as a Choquet integral using (2.7), which implies that it is comonotone.

\[
\square\]

**Remark 4.10.** By Proposition 3.6 we can drop the Fatou-continuity assumption for \( \rho_{\text{conv}}^\mu \) if uniform boundedness and continuity from below are assumed for the risk measures in \( \rho_{\mathcal{I}} \). By Corollary 4.3 the same is true if equality holds in (4.1) without uniform boundedness. Moreover, by an argument similar to that in (ii) in the last theorem, we have that for any \( m \in \mathcal{M} \), there is \( Q' \in Q \) with

\[
\beta_{\rho_{\text{conv}}^\mu}(m) = \sup \left\{ \alpha_{\rho_{\text{conv}}^\mu}(Q): \frac{dQ}{d\mu} \sim \frac{dQ'}{d\mu}, \int_{(u,1]} \frac{1}{v} dm = F_{d\rho^{-1}}^{-1}(1-u) \right\} = \alpha_{\rho_{\text{conv}}^\mu}(Q').
\]

As a direct consequence of (iii) in the last theorem, if \( \rho^i = ES^{\alpha^i}, \alpha^i \in [0,1] \forall i \in \mathcal{I} \), with \( \alpha = \text{ess sup}_\mu \alpha^i \), then \( \rho_{\text{conv}}^\mu(X) = ES^{\alpha^i}(X), \forall X \in L^\infty \).

Regarding comonotone additivity, (ii) in Theorem 4.9 remains true if we drop the law invariance of \( \rho_{\mathcal{I}} \), as shown in the following corollary.

**Corollary 4.11.** Let \( \rho_{\mathcal{I}} = \{ \rho^i: L^\infty \rightarrow \mathbb{R}, i \in \mathcal{I} \} \) be a collection of convex, comonotone risk measures, and let \( \rho_{\text{conv}}^\mu: L^\infty \rightarrow \mathbb{R} \cup \{-\infty, \infty\} \) be defined as in (3.1). Then, \( \rho_{\text{conv}}^\mu \) is finite, normalized, and comonotone, and its capacity function is

\[
c(A) = \text{ess inf}_\mu c^i(A), \quad \forall A \in \mathcal{F},
\]

(4.7)

where \( c^i \) is the capacity of \( \rho^i \) for each \( i \in \mathcal{I} \).

**Proof.** We note that, by Proposition 3.4 we have \( \rho_{\text{conv}}^\mu < \infty \) and \( \rho_{\text{conv}}^\mu(0) = 0 > -\infty \). Let the set function \( c: \mathcal{F} \rightarrow [0,1] \) be defined as \( c(A) = \text{ess inf}_\mu c^i(A) \), where \( c^i \) is the capacity related to \( \rho^i \) for each \( i \in \mathcal{I} \). Thus, by an argument similar to that in Theorem 4.9 if we consider \( m_{\alpha,1,+}^i = \{ m \in ba_{1,+}: m(A) \leq c(A) \forall A \in \mathcal{F} \} \), then the reasoning in Remark 2.9 implies that the claim is true.

\[
\square\]
5 Optimal allocations

An interesting feature of traditional finite inf-convolution is capital allocation. Highly relevant concepts are Pareto optimality and risk sharing, which are defined as follows.

**Definition 5.1.** We call \( \{X^i\}_{i \in I} \in \mathcal{A}(X) \)

(i) Optimal for \( X \in L^\infty \) if \( \int_I \rho^i(X^i)d\mu = \rho^\text{conv}_i(X) \).

(ii) Pareto optimal for \( X \in L^\infty \) if for any \( \{Y^i\}_{i \in I} \in \mathcal{A}(X) \) such that \( \rho^i(Y^i) \leq \rho^i(X^i) \mu - \text{a.s.} \), we have \( \rho^i(Y^i) = \rho^i(X^i) \mu - \text{a.s.} \).

(iii) Optimal risk sharing for \( X \in L^\infty \) if it is Pareto optimal and \( \rho^i(X^i) \leq \rho^i(Y^i) \mu - \text{a.s.} \) for any \( \{Y^i\}_{i \in I} \in \mathcal{A}(X) \).

**Remark 5.2.** (i) If \( \rho^\text{conv}_i \) is normalized, \( \{X^i = 0\}_{i \in I} \) is optimal for 0. This implies the condition that if \( \{X^i\}_{i \in I} \in \mathcal{A}(0) \) and \( \rho^i(X^i) \leq 0 \mu - \text{a.s.} \), then \( \rho^i(X^i) = 0 \mu - \text{a.s.} \), and therefore \( \{X^i = 0\}_{i \in I} \) is also Pareto optimal for 0. This can be understood as a non-arbitrage condition.

(ii) In the paradigm of rational expectations, any allocation that is not an optimal risk sharing is rejected because the splitting procedure strictly increases risk. We note that any optimal allocation must be Pareto optimal, and that a risk sharing rule is also a Pareto-optimal allocation.

If \( \rho_I \) consists of monetary risk measures and \( I \) is finite, Theorem 3.1 in [Jouini et al., 2008] shows that optimal and Pareto-optimal allocations coincide. In the following proposition, we extend this result to the context of arbitrary \( I \).

**Proposition 5.3.** Let \( \rho_I \) be a collection of monetary risk measures. We have that

(i) \( \{X^i\}_{i \in I} \in \mathcal{A}(X) \) is optimal for \( X \in L^\infty \) if and only if it is Pareto optimal for \( X \in L^\infty \).

(ii) If \( \{X^i\}_{i \in I} \in \mathcal{A}(X) \) is Pareto optimal for \( X \in L^\infty \) and \( \pi^i = \rho^i(X) - \rho^i(Y^i) \), \( \mu - \text{a.s.} \), for some \( \{Y^i\}_{i \in I} \in \mathcal{A}(X) \), then \( \{X^i + C^i\}_{i \in I} \) such that \( \int_I C^i d\mu = 0 \) is optimal risk sharing for \( X \in L^\infty \) if and only if \( \pi^i \leq C^i \mu - \text{a.s.} \).

**Proof.** (i) The “only if” part is straightforward, as in Remark 5.2. For the “if” part, let \( \{X^i\}_{i \in I} \in \mathcal{A}(X) \) be not optimal for \( X \in L^\infty \). Then, there is \( \{Y^i\}_{i \in I} \in \mathcal{A}(X) \) such that \( \int_I \rho^i(Y^i)d\mu < \int_I \rho^i(X^i)d\mu \). Let \( k^i = \rho^i(X^i) - \rho^i(Y^i) \mu - \text{a.s.} \) and \( k = \int_I k^i d\mu > 0 \). Moreover, \( \{Y^i - k^i + k\}_{i \in I} \in \mathcal{A}(X) \) and

\[
\int_I \rho^i(Y^i - k^i + k)d\mu < \int_I \rho^i(Y^i - k^i + k)d\mu = \int_I \rho^i(X^i)d\mu.
\]

Hence, \( \{X^i\}_{i \in I} \) is not Pareto optimal for \( X \).

(ii) We claim that if \( \{X^i\}_{i \in I} \) is optimal for \( X \in L^\infty \), then so is \( \{X^i + C^i\}_{i \in I} \), where \( C^i \in \mathbb{R} \mu - \text{a.s.} \), and \( \int_I C^i d\mu = 0 \). To prove this, we note that

\[
\int_I \rho^i(X^i + C^i)d\mu = \int_I \rho^i(X^i)d\mu - \int_I C^i d\mu = \rho^\text{conv}_i(X).
\]
By the Pareto optimality of \( \{X^i\}_{i \in I} \), we have that
\[
\int_I \pi^i d\mu = \rho^\mu_\text{conv}(X) - \int_I \rho^i(Y^i) d\mu \leq 0.
\]

Thus, \( \{X^i + C^i\}_{i \in I} \) is also Pareto optimal. Hence, it is an optimal risk sharing for \( X \in L^\infty \) if and only if \( \rho^i(X^i - C^i) \leq \rho^i(Y^i) \mu - \text{a.s.} \), which, by translation invariance, is equivalent to \( \pi^i \leq C^i \mu - \text{a.s.} \).

\( \square \)

**Remark 5.4.** The determination of an optimal risk sharing reduces to the characterization of a Pareto-optimal \( \{X^i + C^i\}_{i \in I} \) such that \( \int_I C^i d\mu = 0 \). This is always the case for \( \rho_I \) consisting of law-invariant, convex risk measures, as will be proved in Theorem 5.10. Thus, in light of the last proposition, under these circumstances, we can treat the three definitions for optimality as equivalent.

We now determine a necessary and sufficient condition for optimality in the case of convex risk measures.

**Theorem 5.5.** Let \( \rho_I \) be a family of convex risk measures. \( \{X^i\}_{i \in I} \in \mathcal{A}(X) \) is optimal for \( X \in L^\infty \) if \( \{m \in ba_{1,+} : m \in \partial \rho^i(X^i) \mu - \text{a.s.} \} \neq \emptyset \). The converse is true if equality holds in (4.1).

**Proof.** We claim that \( \{m \in ba_{1,+} : m \in \partial \rho^i(X^i) \mu - \text{a.s.} \} \subseteq \partial \rho^\mu_\text{conv}(X) \) for any \( \{X^i\}_{i \in I} \in \mathcal{A}(X) \). To verify this, let \( m' \in \{m \in ba_{1,+} : m \in \partial \rho^i(X^i) \mu - \text{a.s.} \} \). Then,
\[
E_{m'}[-X] - \alpha^\text{min}_{\rho^\mu_\text{conv}}(m') \geq \int_I \left( E_{m'}[-X^i] - \alpha^\text{min}_{\rho^i}(m') \right) d\mu \geq \int_I \rho^i(Y^i) d\mu \geq \rho^\mu_\text{conv}(X),
\]
which implies \( m' \in \partial \rho^\mu_\text{conv}(X) \). We assume \( \{m \in ba_{1,+} : m \in \partial \rho^i(X^i) \mu - \text{a.s.} \} \neq \emptyset \). Then, let \( m' \in \{m \in ba_{1,+} : m \in \partial \rho^i(X^i) \mu - \text{a.s.} \} \subseteq \partial \rho^\mu_\text{conv}(X) \). We have that
\[
\int_I \rho^i(Y^i) d\mu = \int_I \left( E_{m'}[-X^i] - \alpha^\text{min}_{\rho^i}(m') \right) d\mu \leq E_{m'}[-X] - \alpha^\text{min}_{\rho^\mu_\text{conv}}(m') = \rho^\mu_\text{conv}(X).
\]
Hence, \( \{X^i\}_{i \in I} \) is optimal for \( X \in L^\infty \).

Regarding the converse, for any \( m' \in \partial \rho^\mu_\text{conv}(X) \), we obtain by an argument similar to that in Theorem 4.12 the following:
\[
\rho^\mu_\text{conv}(X) = E_{m'}[-X] - \int_I \alpha^\text{min}_{\rho^i}(m') d\mu
\]
\[
= E_{m'}[-X] + \inf_{Y \in L^\infty} \inf_{Y^i \in \mathcal{A}(Y)} \left( E_{m'}[(X - Y)] + \rho^i(X - Y) d\mu \right)
\]
\[
\leq \inf_{Y \in L^\infty} \inf_{Y^i \in \mathcal{A}(Y)} \int_I \left( E_{m'}[Y^i] + \rho^i(X - Y^i) \right) d\mu
\]
\[
\leq \inf_{Y^i \in \mathcal{A}(0)} \int_I \rho^i(X - Y^i) d\mu = \rho^\mu_\text{conv}(X).
\]
Thus, \( \rho^\mu_\text{conv}(X) = \int_I \rho^i(Y^i) d\mu \) if and only if \( \exists m' \in ba_{1,+} \) such that \( \rho^i(Y^i) = E_{m'}[-X] + \alpha^\text{min}_{\rho^i}(m') \mu - \text{a.s.} \). Hence, \( m' \in \{m \in ba_{1,+} : m \in \partial \rho^i(Y^i) \mu - \text{a.s.} \} \).
Remark 5.6. If $\mathcal{I}$ is finite, we can replace the optimality condition by $\bigcap_{i \in \mathcal{I}} \partial \rho^i(X^i) \neq \emptyset$. For arbitrary $\mathcal{I}$, if equality holds in (11), $\{m \in ba_{1,+} : m \in \partial \rho^i(X^i) \mu - a.s.\} = \partial \rho^i_{\text{conv}}(X)$ if and only if $\{X^i\}_{i \in \mathcal{I}} \in \mathcal{A}(X)$ is optimal for $X \in L^\infty$. By the definition of Legendre–Fenchel convex-conjugate duality, this condition is equivalent to the existence of $m \in ba_{1,+}$ such that $X^i \in \partial \rho^i_{\text{min}}(m) \mu - a.s.$

Under the assumption of law invariance, it is well known that, for finite $\mathcal{I}$, the minimization problem has a solution under co-monotonic allocations (see, for instance, Theorem 3.2 in (Jouini et al., 2008), Proposition 5 in (Dana and Meilijson, 2003), or Theorem 10.46 in (Ruschendorf, 2013)). For the extension to general $\mathcal{I}$, we should extend some definitions and results regarding comonotonicity. We note that if $\mathcal{I}$ is finite, these are equivalent to their traditional counterparts.

Definition 5.7. $\{X^i\}_{i \in \mathcal{I}}$ is called $\mathcal{I}$-comonotone if every pair $(X^i, X^j)$ is comonotone $\mu \times \mu - a.s.$, that is, $\mu \times \mu(\{(i, j) \in \mathcal{I} \times \mathcal{I} : (X^i, X^j) \text{ is comonotone}\}) = 1$.

Lemma 5.8. $\{X^i\}_{i \in \mathcal{I}}$ is $\mathcal{I}$-comonotone if and only if there exists a class of functions $\{h^i : \mathbb{R} \to \mathbb{R}, i \in \mathcal{I}\}$ that are (in the $\mu - a.s.$ sense) Lipschitz continuous and non-decreasing, and they satisfy $X^i = h^i \left( \int_{\mathcal{I}} X^i d\mu \right)$ and $\int_{\mathcal{I}} h^i(x) d\mu = x, \forall x \in \mathbb{R}$. In particular, if $\{X^i\}_{i \in \mathcal{I}} \in \mathcal{A}(X)$ is $\mathcal{I}$-comonotone, then $F_{X^{-1}}(\alpha) = \int_{\mathcal{I}} F_{X^i}^{-1}(\alpha d\mu(\alpha), \forall \alpha \in [0, 1]$.

Proof. The “if” part is trivial. For the “only if” part, let $\{X^i\}_{i \in \mathcal{I}}$ be $\mathcal{I}$-comonotone, $X = \int_{\mathcal{I}} X^i d\mu$, and $X(\Omega) = \{x \in \mathbb{R} \cup \{-\infty, \infty\} : \exists \omega \in \Omega \text{ s.t. } X(\omega) = x\}$. Then, for any fixed $\omega \in \Omega$, there is a family $\{x^i = X^i(\omega) \in \mathbb{R} : i \in \mathcal{I}\}$ such that $X(\omega) = x = \int_{\mathcal{I}} x^i d\mu$. Moreover, we define $\tilde{h}^i(x) = x^i \mu - a.s.$ in $X(\Omega) \setminus \{-\infty, \infty\}$. We claim that this decomposition is unique. Indeed, if there are $\omega, \omega' \in \Omega$ such that $\int_{\mathcal{I}} X^i(\omega) d\mu = x = \int_{\mathcal{I}} X^i(\omega') d\mu$, we then obtain $\int_{\mathcal{I}} (X^i(\omega) - X^i(\omega')) d\mu = 0$. Assuming $\mathcal{I}$-comonotonicity, we have that $X^i(\omega) = X^i(\omega') \mu - a.s.$ Consequently, the map $x \to \int_{\mathcal{I}} \tilde{h}^i(x) d\mu = \text{Id}(x)$ is well defined. Regarding the non-decreasing behavior of $\tilde{h}^i$, let $x, y \in X(\Omega)$ with $x \leq y$. Then, there are $\omega, \omega'$ such that $\int_{\mathcal{I}} X^i(\omega) d\mu = x \leq y = \int_{\mathcal{I}} X^i(\omega') d\mu$, which implies $\int_{\mathcal{I}} (X^i(\omega) - X^i(\omega')) d\mu \leq 0$. Comonotonicity implies that this relation is equivalent to $h^i(x^i) = X^i(\omega) \leq X^i(\omega') = h^i(y^i) \mu - a.s.$ Concerning Lipschitz continuity, as for any $\delta > 0$, the pair $(X^i + \delta, X^i)$ is comonotone, we obtain $0 \leq h^i(x + \delta) - h^i(x) \leq \delta$, $\mu - a.s.$ It remains to extend $\{h^i\}$ from $X(\Omega) \setminus \{-\infty, \infty\}$ to $\mathbb{R}$. We first extend it to $\partial l(X(\Omega) \setminus \{-\infty, \infty\})$. If $x \in bd(X(\Omega) \setminus \{-\infty, \infty\})$ is only a one-sided boundary point, then the continuous extension poses no problem, as non-decreasing functions are involved. If $x$ can be approximated from both sides, then Lipschitz continuity implies that the left- and right-sided continuous extensions coincide. The extension to $\mathbb{R}$ is performed linearly in each connected component of $\mathbb{R} \setminus cl(X(\Omega) \setminus \{-\infty, \infty\})$ so that the condition $\int_{\mathcal{I}} h^i(x) d\mu = x$ is satisfied. Then, the main claim is proved. Moreover, let $\{X^i\}_{i \in \mathcal{I}} \in \mathcal{A}(X)$ be $\mathcal{I}$-comonotone. Then, $x \to \int_{\mathcal{I}} \tilde{h}^i(x) d\mu$ is also Lipschitz continuous and non-decreasing. We recall that $F_{X^{-1}}(\alpha) = g(F_{X}^{-1}(\alpha))$ for any non-decreasing function $g : \mathbb{R} \to \mathbb{R}$. Then, for any $\alpha \in [0, 1]$, we obtain

$$F_X^{-1}(\alpha) = F_Z^{-1}(\alpha \int_{\mathcal{I}} h^i(F^{-1}_X(\alpha)) d\mu = \int_{\mathcal{I}} F^{-1}_X(\alpha) d\mu.$$
We now prove the following comonotonic-improvement theorem for arbitrary $\mathcal{I}$.

**Theorem 5.9.** Let $X \in L^\infty$. Then, for any $\{X^i\}_{i \in \mathcal{I}} \in \mathcal{A}(X)$, there is an $\mathcal{I}$-comonotone \{Y^i\}_{i \in \mathcal{I}} \in \mathcal{A}(X)$ such that $Y^i \succeq X^i \mu - \text{a.s.}$

**Proof.** Let $\mathcal{F}_n$ be the $\sigma$-algebra generated by $\{\omega \colon k2^{-n} \leq X(\omega) \leq k2^n\} \subset \Omega$ for $k > 0$, $X_n = E[X|\mathcal{F}_n]$, and $X^i_n = E[X^i|\mathcal{F}_n] \forall i \in \mathcal{I}$. Then, $\lim_{n \to \infty} X_n = X$, $\lim_{n \to \infty} X^i_n = X^i$ for any $i \in \mathcal{I}$, and $X^i_n \succeq X^i$ for each $n$ each $i$. By the arguments in Proposition 1 in (Landsberger and Meilijson, 1994) or Proposition 10.45 in (Rüschendorf, 2013), we can conclude that every allocation of $X$ taking a countable number of values is dominated by a comonotone allocation. Thus, by Lemma 5.8 for any $n \in \mathbb{N}$, there are Lipschitz-continuous, non-decreasing functions $\{h^i_n \colon \mathbb{R} \to \mathbb{R}, i \in \mathcal{I}\}$ with $\int_\mathcal{I} h^i_n d\mu = Id$ such that $Y^i_n = h^i_n(X^i_n) \succeq X^i_n \mu - \text{a.s.}$ We note that these functions constitute a bounded, closed, equicontinuous family. Then, as Ascoli’s theorem, there is a subsequence of $\{h^i_n\}$ that converges uniformly on $[\text{ess inf } X, \text{ess sup } X]$ to the Lipschitz-continuous and non-decreasing $h^i$ in the $\mu - \text{a.s.}$ sense. Thus, $\int_\mathcal{I} h^i d\mu = Id$ on $[\text{ess inf } X, \text{ess sup } X]$. We then have $Y^i = h^i(X) \succeq X^i \mu - \text{a.s.}$ by considering uniform limits. Finally, by Lemma 5.8 we obtain that $\{Y^i\}_{i \in \mathcal{I}}$ is $\mathcal{I}$-comonotone. It remains to show that $\{Y^i\}_{i \in \mathcal{I}}$ belongs to $\mathcal{A}(X)$. To this end, we note that as the functions are Lipschitz continuous and non-decreasing, $(i, \omega) \to h^i(X(\omega)) \leq \|X\|_\infty < \infty$. Thus, $\int_{\mathcal{I} \times \Omega} |h^i(X(\omega))| d(\mu \times m) \leq \|X\|_\infty < \infty$, $\forall m \in ba_{1,+}$. Finally, by Theorem 3.1 in (Appling, 1974) (which is an analogue of Fubini’s theorem for finitely additive measures), we have $\int_\mathcal{I} E_m[Y^i] d\mu = E_m \left[\int_\mathcal{I} h^i(X) d\mu\right] = E_m[X]$. 

We are now in a position to extend the existence of optimal allocations to our framework of law-invariant, convex risk measures.

**Theorem 5.10.** Let $\rho_\mathcal{I}$ be a collection of law-invariant, convex risk measures. Then,

(i) For any $X \in L^\infty$, there is an $\mathcal{I}$-comonotone optimal allocation.

(ii) In addition, if $\rho_\mathcal{I}$ consists of risk measures that are strictly monotone with respect to $\succeq$, then every optimal allocation for any $X \in L^\infty$ is $\mathcal{I}$-comonotone.

**Proof.** (i) By Theorem 5.9, we can restrict the minimization problem to $\mathcal{I}$-comonotonic allocations, as, by Theorem 2.7, law-invariant risk measures preserve second-order stochastic dominance. Let $\{Y^i_n = h^i_n(X) \in L^\infty, i \in \mathcal{I}\}_n$ be a sequence of optimal allocations for $X$, where $h^i_n \colon [\text{ess inf } X, \text{ess sup } X] \to \mathbb{R}$ are non-decreasing, bounded, and Lipschitz-continuous functions. By an argument similar to that in Theorem 5.9, we have that $h^i$ is the uniform limit (after passing to a subsequence if necessary) of $\{h^i_n\}$. Thus, $Y^i_n = h^i_n(X) \to h^i(X) = Y^i$. By continuity in the essential supremum norm, we have that $\lim_{n \to \infty} |\rho_\mathcal{I}(Y^i_n) - \rho_\mathcal{I}(Y^i)| = 0$. As $h^i$ is the uniform limit of $\{h^i_n\}$, we have that $\rho_\mathcal{I}^{\text{conv}}(X) = \int_\mathcal{I} \rho_\mathcal{I}(Y^i_n) d\mu \to \int_\mathcal{I} \rho_\mathcal{I}(Y^i) d\mu$.

Hence, $\{Y^i\}_{i \in \mathcal{I}}$ is the desired optimal allocation.
(ii) We recall that strict monotonicity implies that if \( X \succeq Y \) and \( X \not\succeq Y \), then \( \rho^i(X) < \rho^i(Y) \) \( \mu \)-a.s. for any \( X, Y \in L^\infty \). Let \( \{X^i\}_{i \in \mathcal{I}} \) be an optimal allocation for \( X \in L^\infty \). Then, by Theorem 5.9 there is an \( \mathcal{I} \)-comonotone allocation \( \{Y^i\}_{i \in \mathcal{I}} \in \mathcal{A}(X) \) such that

\[
\rho_{\text{conv}}^i(X) = \int_{\mathcal{I}} \rho^i(X^i) d\mu \geq \int_{\mathcal{I}} \rho^i(Y^i) d\mu.
\]

Thus, \( \{Y^i\}_{i \in \mathcal{I}} \) is also optimal. If \( X^i = Y^i \) \( \mu \)-a.s., then we have the claim. If \( \mu(X^i \neq Y^i) > 0 \), then we have \( \mu \left( \rho^i \left( \frac{Y^i + X^i}{2} \right) < \rho^i(X^i) \right) > 0 \), contradicting the optimality of \( \{X^i\}_{i \in \mathcal{I}} \). Hence, every optimal allocation for \( X \) is \( \mathcal{I} \)-comonotone.

\[\square\]

Remark 5.11. The examples in (Jouini et al., 2008) and (Delbaen, 2006) show that law invariance is essential to ensure the existence of an \( \mathcal{I} \)-comonotone solution as above. However, the uniqueness of this optimal allocation is not ensured. In the special case where \( \rho_{\mathcal{I}} \) consists of strictly convex functionals, we have uniqueness up to scaling. To verify this, we assume, toward a contradiction, that both \( \{X^i\}_{i \in \mathcal{I}} \) and \( \{Y^i\}_{i \in \mathcal{I}} \) are optimal allocations for \( X \in L^\infty \) such that \( \mu((X^i - Y^i) \not\in \mathbb{R}) > 0 \). We note that for any \( \lambda \in [0, 1] \), the family \( \{Z^i = \lambda X^i + (1 - \lambda Y^i)\}_{i \in \mathcal{I}} \) is in \( \mathcal{A}(X) \). However, we would have

\[
\int_{\mathcal{I}} \rho^i(Z^i) d\mu < \lambda \int_{\mathcal{I}} \rho^i(X^i) d\mu + (1 - \lambda) \int_{\mathcal{I}} \rho^i(Y^i) d\mu = \rho_{\text{conv}}^i(X),
\]

which contradicts the optimality of both \( \{X^i\}_{i \in \mathcal{I}} \) and \( \{Y^i\}_{i \in \mathcal{I}} \) for \( X \).

We have the following corollary regarding subdifferential and optimality conditions.

Corollary 5.12. Let \( \rho_{\mathcal{I}} \) be a collection of law-invariant, convex risk measures. Then, for any \( m \in ba_{1,+} \), we have

\[
\partial \alpha^m_{\rho_{\text{conv}}^i}(m) \supseteq \left\{ X \in L^\infty : \exists \{X^i\}_{i \in \mathcal{I}} \in \mathcal{A}(X) \text{ s.t. } X^i \in \partial \alpha^m_{\rho^i}(m) \mu \text{-a.s.} \right\}.
\]

The converse inclusion is true if equality holds in (4.1).

Proof. We fix \( m \in ba_{1,+} \). If \( X \in \{X \in L^\infty : \exists \{X^i\}_{i \in \mathcal{I}} \in \mathcal{A}(X) \text{ s.t. } X^i \in \partial \alpha^m_{\rho^i}(m) \mu \text{-a.s.} \} \), let \( \{X^i\}_{i \in \mathcal{I}} \in \mathcal{A}(X) \) such that \( X^i \in \partial \alpha^m_{\rho^i}(m) \mu \text{-a.s.} \). Then,

\[
E_m[-X] - \rho_{\text{conv}}^i(X) \geq \int_{\mathcal{I}} \left( E_m[-X^i] - \rho^i(X^i) \right) d\mu = \int_{\mathcal{I}} \alpha^m_{\rho^i}(m) d\mu \geq \alpha^m_{\rho_{\text{conv}}^i}(m).
\]

Thus, \( X \in \partial \alpha^m_{\rho_{\text{conv}}^i}(m) \). For the converse relation, if \( \partial \alpha^m_{\rho_{\text{conv}}^i}(m) = \emptyset \), then the claim is immediately obtained. Let then \( X \in \partial \alpha^m_{\rho_{\text{conv}}^i}(m) \). By (ii) of Theorem 5.10 there is an optimal allocation \( \{X^i\}_{i \in \mathcal{I}} \) for \( X \). By Theorem 5.9 and Remark 5.6 we have that \( X^i \in \partial \alpha^m_{\rho^i}(m) \mu \text{-a.s.} \).

Then, \( X \in \{X \in L^\infty : \exists \{X^i\}_{i \in \mathcal{I}} \in \mathcal{A}(X) \text{ s.t. } X^i \in \partial \alpha^m_{\rho^i}(m) \mu \text{-a.s.} \} \). \[\square\]

If \( \rho_{\mathcal{I}} \) consists of comonotone, law-invariant, convex risk measures, then we can prove an additional result regarding the connection between optimal allocations and the notion of flatness for quantile functions. To this end, the following definitions and lemma are required.
Definition 5.13. Let $g^1, g^2$ be two distortions with $g^1 \leq g^2$. A quantile function $F_X^{-1}, X \in L^\infty$, is called flat on $\{x \in [0,1] : g^1(x) < g^2(x)\}$ if $dF_X^{-1} = 0$ almost everywhere on $\{g^1 < g^2\}$ and $(F_X^{-1}(0^+) - F_X^{-1}(0))(g^2(0^+) - g^1(0^+)) = 0$.

Lemma 5.14 (Lemmas 4.1 and 4.2 in [Jouini et al., 2008]). Let $\rho: L^\infty \to \mathbb{R}$ be a law-invariant, comonotone, convex risk measure with distortion $g$; moreover, let $m \in ba_{1,+}$ have a Lebesgue decomposition $m = Z_m \mathbb{P} + m^s$ into a regular part with density $Z_m$ and a singular part $m^s$. Then,

(i) $g_m: [0,1] \to \mathbb{R}$ defined as $g_m(0) = 0$ and $g_m(t) = \|m^s\|_{TV} + \int_0^t F_{Z_m}^{-1}(1-s)ds, 0 < t \leq 1$, is a concave distortion.

(ii) For any $m \in \partial \rho(X)$, we have that $X$ and $-Z_m$ are comonotone. Moreover, the measure $m'$ such that $Z_{m'} = E[Z_m | X]$ belongs to $\partial \rho(X)$.

(iii) $\partial \rho(X) = \{m \in ba_{1,+} : g_m \leq g, F_X^{-1} \text{ is flat on } \{g_m < g\}\}$.

Theorem 5.15. Let $\rho^\mu I$ consist of law-invariant, comonotone, convex risk measures with distortions $\{g^i\}_{i \in I}$; moreover, let $\rho_{\text{conv}}^\mu$ be Fatou continuous, and $\{X^i\}_{i \in I} \in \mathcal{h}(X)$ be $I$-comonotone. If $F_{X^i}^{-1}$ is flat on $\{\text{ess inf}_\mu g^i < g^i\} \cap \{dF_{X^i}^{-1} > 0\} \mu$-a.s., then $\{X^i\}_{i \in I}$ is an optimal allocation for $X \in L^\infty$. The converse is true if $\int_I \|X^i\|_\infty d\mu < \infty$.

Proof. By Theorem 4.9 let $g = \text{ess inf}_\mu g^i$ be the distortion of $\rho_{\text{conv}}^\mu$. Let $U$ be a $[0,1]$-uniform random variable (the existence of which is ensured because the space is atomless) such that $X = F_X^{-1}(U)$. We define $m \in ba_{1,+}$ by $m = g(0^+)\delta_0(U) + g'(U)|_{[0,1]}(U)$, where $\delta_0$ is the Dirac measure at 0. It is easily verified using (i) in Lemma 5.14 that $g_m = g$. Moreover, let $\{X^i\}_{i \in I}$ be an $I$-comonotone optimal allocation for $X$ (the existence of which is ensured by Theorem 5.10). Thus, in the $\mu$-a.s. sense, $g_m \leq g^i, -Z_m$ is comonotone with $X^i$, and, by hypothesis, $F_{X^i}^{-1}$ is flat on $\{g_m < g^i\} \cap \{dF_{X^i}^{-1} > 0\}$. By Lemma 5.8 we have $\{dF_{X^i}^{-1} = 0\} = \{\alpha \in [0,1] : dF_{X^i}^{-1}(\alpha) = 0 \mu$-a.s.\}. Thus, $\{g_m < g^i\} \cap \{dF_{X^i}^{-1} = 0\} = \emptyset \mu$-a.s. By (iii) of Lemma 5.14 we have that $m \in \partial \rho(X^i) \mu$-a.s. Then, by Theorem 5.12 we obtain that $\{X^i\}_{i \in I}$ is an optimal allocation.

For the converse, by Theorem 5.5 and Remark 5.6 we have that $X^i \in \partial \rho_{\text{conv}}^\mu(m') \mu$-a.s. for some $m' \in ba_{1,+}$. By Corollary 5.12 we have that $X \in \partial \rho_{\text{conv}}^\mu(m')$, and by convex-conjugate duality, $m' \in \partial \rho_{\text{conv}}^\mu(X)$. Thus, (ii) in Lemma 5.14 we obtain that $m \in ba_{1,+}$ such that $Z_m = E[Z_m | X]$ belongs to $\partial \rho_{\text{conv}}^\mu(X)$ and $\partial \rho_{\text{conv}}^\mu(X) = \{m \in ba_{1,+} : m \in \partial \rho(X^i) \mu$-a.s.\}. By Theorem 2.8 and (iii) of Lemma 5.14 we have that $\rho(X^i) = \int_0^1 \text{VaR}_\alpha(X^i) g_m(\alpha) d\mu \mu$-a.s. As $\{X^i\}_{i \in I}$ is an optimal allocation, Theorem 4.9 and Lemma 5.8 imply that

$$\int_0^1 \text{VaR}_\alpha(X^i) g_m(\alpha) d\alpha = \rho_{\text{conv}}^\mu(X)$$

$$= \int_I \int_0^1 \text{VaR}_\alpha(X^i) g_m(\alpha) d\alpha d\mu$$

$$= \int_I \int_0^1 \text{VaR}_\alpha(X^i) d\mu g_m(\alpha) d\alpha$$

$$= \int_0^1 \text{VaR}_\alpha(X) g_m(\alpha) d\alpha.$$
We can apply the Fubini–Tonelli theorem because

\[
\int_{\mathcal{I} \times [0,1]} \text{VaR}^\alpha(X^i)g_m'(\alpha)(d\mu \times d\lambda) \leq \int_{\mathcal{I}} \|X^i\|_\infty d\mu < \infty,
\]

where \( \lambda \) is the Lebesgue measure. Then, we have \( \int_0^1 (g_m(\alpha) - g(\alpha)) dF_X^{-1}(t) = 0 \). As \( m \in \partial \rho_{\text{conv}}(X) \), (iii) in Lemma 5.14 implies that \( g_m \leq g \), and therefore \( g_m = g \) in \( \{dF_X^{-1} > 0\} \). Hence, \( F_X^{-1} \) is flat on \( \{g < g^i\} \cap \{dF_X^{-1} > 0\} \) \( \mu \)-a.s. \( \square \)

**Remark 5.16.** We also have in this context that for any optimal allocation \( \{X^i\}_{i \in \mathcal{I}} \), \( F_X^{-1} \) is flat on \( \{g_i \neq g_j\} \cap \{dF_X^{-1} = 0\} \) for any \( j \neq i \) in the \( \mu \)-a.s. sense. To see this, let \( t \in \{g_j < g_i\} \cap \{dF_X^{-1} = 0\} \). We note that, by Lemma 5.8, \( \{dF_X^{-1} = 0\} = \{\alpha \in [0,1] : dF_X^{-1}(\alpha) = 0 \mu \text{-a.s.}\} \). Then, by Lemma 5.14 \( g_m(t) < g_i(t) \), and thus \( dF_X^{-1} \) is flat at \( t \). By comonotonicity and Lemma 5.8 the same is true for \( dF_X^{-1} \). By repeating the argument for \( t \in \{g_j > g_i\} \cap \{dF_X^{-1} = 0\} \), we prove the claim.

A relevant concept in the present context is the dilated risk measure, which is stable under inf-convolution and has a dilatation property with respect to the size of a position. In this particular situation, we can provide explicit solutions for optimal allocations even without law invariance. We now define this concept and extend some interesting related results to our framework.

**Definition 5.17.** Let \( \rho : L^\infty \to \mathbb{R} \) be a risk measure, and \( \gamma > 0 \) be a real parameter. Then, the dilated risk measure with respect to \( \rho \) and \( \gamma \) is a functional \( \rho_\gamma : L^\infty \to \mathbb{R} \) defined as

\[
\rho_\gamma(X) = \gamma \rho \left( \frac{1}{\gamma} X \right). \quad (5.1)
\]

**Remark 5.18.** A typical example of dilated measure is the entropic \( \text{Ent}_\gamma \) with \( \text{Ent}_1 \) as basis. It is evident that, for convex risk measures, \( \alpha_{\rho_\gamma}^{\text{min}} = \gamma \alpha_{\rho}^{\text{min}} \). Moreover, a convex risk measure is coherent if and only if \( \rho = \rho_\gamma \) pointwise for any \( \gamma > 0 \). Moreover, under normalization, \( \lim_{\gamma \to \infty} \rho_\gamma \) defines the smallest coherent risk measure that dominates \( \rho \).

**Proposition 5.19.** We have that

(i) \( (\rho_{\text{conv}}^\mu)_\gamma = \inf \{ \int_{\mathcal{I}} (\rho^i)'(X^i) d\mu : \{X^i\}_{i \in \mathcal{I}} \in \mathcal{A}(X) \} \) for any \( \gamma > 0 \).

(ii) Let \( \{\gamma > 0\}_{i \in \mathcal{I}} \) with \( \gamma = \int_{\mathcal{I}} \gamma^i d\mu > 0 \). If \( \rho^i = \rho_\gamma \) \( \mu \)-a.s., then \( \left\{ \frac{1}{\gamma} X \right\}_{i \in \mathcal{I}} \) is optimal for \( X \in L^\infty \). Moreover, if \( \rho \) is a convex risk measure, then \( \rho_{\text{conv}}^\mu \geq \rho_\gamma \).

**Proof.** (i) For any \( \gamma > 0 \) and \( X \in L^\infty \), we have that

\[
\inf \left\{ \int_{\mathcal{I}} (\rho^i)'(X^i) d\mu : \{X^i\}_{i \in \mathcal{I}} \in \mathcal{A}(X) \right\} = \gamma \inf \left\{ \int_{\mathcal{I}} \rho^i \left( \frac{1}{\gamma} X^i \right) d\mu : \{X^i\}_{i \in \mathcal{I}} \in \mathcal{A}(X) \right\} = \gamma \inf \left\{ \int_{\mathcal{I}} \rho^i (Y^i) d\mu : \{Y^i\}_{i \in \mathcal{I}} \in \mathcal{A} \left( \frac{1}{\gamma} X \right) \right\} = \gamma \rho_{\text{conv}}^\mu \left( \frac{1}{\gamma} X \right) = (\rho_{\text{conv}}^\mu)_\gamma(X).
\]
(ii) We obtain that

\[
\int I \rho^i \left( \frac{\gamma^i}{\gamma} X \right) d\mu = \int I \gamma^i \rho \left( \frac{1}{\gamma^i} \frac{\gamma^i}{\gamma} X \right) d\mu = \gamma \rho \left( \frac{1}{\gamma} X \right) = \rho_\gamma(X).
\]

Moreover, under the additional conditions, we obtain

\[
\alpha_{\rho_{\text{conv}}}^{\min}(m) \leq \int I \gamma^i \alpha_{\rho}^{\min}(m) d\mu = \gamma \alpha_{\rho}^{\min}(m) = \alpha_{\rho_\gamma}^{\min}(m), \quad \forall m \in \text{ba}.
\]

Thus, \( \rho_{\text{conv}} \geq \rho_\gamma \).

**Remark 5.20.** We note that for any \( X \in L^\infty \), \( \{\gamma^i \gamma^{-1} X\}_{i \in I} \) is \( I \)-comonotone, which is in consonance with Theorem 5.10 when the risk measures in \( \rho_I \) are law invariant. Furthermore, we have that if \( \rho^i = \text{Ent}_{\gamma^i}, \gamma^i > 0, \mu \text{-a.s.} \), then \( \rho_{\text{conv}}^{\rho}(X) \geq \text{Ent}_{\gamma}(X), \forall X \in L^\infty \). If equality holds in (4.1), we obtain an equation for the last claim.

### 6 Self-convolution and regulatory arbitrage

In this section, we consider the special case \( \rho^i = \rho, \forall i \in I \). In this situation, we have that \( \rho_{\text{conv}}^{\rho} \) is a self-convolution. This concept is highly important in the context of regulatory arbitrage (as in [Wang, 2016]), where the goal is to reduce the regulatory capital of a position by splitting it. The difference between \( \rho(X) \) and \( \rho_{\text{conv}}^{\rho}(X) \) is then obtained by a simple rearrangement (sharing) of risk. We now adjust this concept to our framework.

**Definition 6.1.** The regulatory arbitrage of a risk measure \( \rho \) is a functional \( \tau_\rho: L^\infty \to \mathbb{R}_+ \cup \{\infty\} \) defined as

\[
\tau_\rho(X) = \rho(X) - \rho_{\text{conv}}^{\rho}(X), \quad \forall X \in L^\infty.
\]

Moreover, \( \rho \) is called

(i) free of regulatory arbitrage if \( \tau_\rho(X) = 0, \forall X \in L^\infty \);

(ii) of finite regulatory arbitrage if \( \tau_\rho(X) < \infty, \forall X \in L^\infty \);

(iii) of partially infinite regulatory arbitrage if \( \tau_\rho(X) = \infty \) for some \( X \in L^\infty \);

(iv) of infinite regulatory arbitrage if \( \tau_\rho(X) = \infty, \forall X \in L^\infty \).

**Remark 6.2.** As \( \infty > \rho \geq \rho_{\text{conv}}^{\rho} \), we have that \( \tau_\rho \) is well defined. Our approach is different from that in [Wang, 2016] because we consider not only an arbitrary set \( I \) but also “convex” inf-convolutions instead of sums. Moreover, the approach by Wang involves a countably infinite self-convolution that considers varying \( n \in \mathbb{N} \):

\[
R(X) = \inf \left\{ \sum_{i=1}^{n} \rho(X^i), n \in \mathbb{N}, X^i \in L^\infty, i = 1, \ldots, n, \sum_{i=1}^{n} X^i = X \right\}
\]

\[
= \lim_{n \to \infty} \inf \left\{ \sum_{i=1}^{n} \rho(X^i), X^i \in L^\infty, i = 1, \ldots, n, \sum_{i=1}^{n} X^i = X \right\}.
\]
This distinction leads to differences. For instance, in that approach, \( \text{Ent}_\gamma \) is of limited regulatory arbitrage, whereas in ours (see Proposition 6.6 below), we have \( \tau_{\text{Ent}_\gamma} = 0 \), that is, \( \text{Ent}_\gamma \) is free of regulatory arbitrage. Moreover, it is evident that \( \tau_{EL} = 0 \) pointwise because \( \int_I EL(X^i) d\mu = EL(X), \ \forall \{X^i\}_{i \in I} \in \mathcal{A}(X) \).

It was proved in (Wang, 2016) that \( \text{VaR}^\alpha \) is of infinite regulatory arbitrage. The following proposition adapts this to our framework.

**Proposition 6.3.** Let \( \alpha \in (0, 1] \). If there is a partition of \( I \) consisting of \( k + 1 \) non-null sets such that \( \frac{1}{k} < \alpha \), then \( \text{VaR}^\alpha \) is of infinite regulatory arbitrage.

**Proof.** Let \( \{A_j, j = 1, \cdots, k+1\} \) be some partition of \( I \) with cardinality \( k + 1 \) such that \( \frac{1}{k} < \alpha \). Then \( k > 1 \). Moreover, let \( \{B_j, j = 1, \cdots, k\} \) be a partition of \( \Omega \) such that \( \mathbb{P}(B_j) = \frac{1}{k} \) for any \( j = 1, \cdots, k \). We note that as \( (\Omega, \mathcal{F}, \mathbb{P}) \) is atom-less, such a partition always exists. For fixed \( X \in L^\infty \) and some arbitrary real number \( m > 0 \), let \( \{X^i\}_{i \in I} \) be defined as

\[
X^i(\omega) = \begin{cases} 
\frac{m(1 - k1_{B_j}(\omega))}{(k-1)\mu(A_j)}, & \text{for } i \in A_j, j = 1, \cdots, k, \\
\frac{X}{\mu(A_{k+1})}, & \text{for } i \in A_{k+1},
\end{cases}
\]

for any \( \omega \in \Omega \). Thus, \( \{X^i\}_{i \in I} \in \mathcal{A}(X) \) because for any \( \omega \in \Omega \), the following is true:

\[
\int_I X^i(\omega) d\mu = \sum_{j=1}^{k} \int_{A_j} \left[ \frac{m(1 - k1_{B_j}(\omega))}{(k-1)\mu(A_j)} \right] d\mu + \int_{A_{k+1}} \left[ \frac{X(\omega)}{\mu(A_{k+1})} \right] d\mu
\]

\[
= \frac{m}{k-1} \sum_{j=1}^{k} \frac{\mu(A_j)(1 - k1_{B_j}(\omega))}{\mu(A_j)} + \frac{\mu(A_{k+1})X(\omega)}{\mu(A_{k+1})}
\]

\[
= \frac{m}{k-1}(k-k) + X(\omega) = X(\omega).
\]

Similarly, we obtain that \( \int_I E_m[X^i] d\mu = E_m[X] \ \forall \ m \in ba_{1,+} \). Furthermore, we note that for \( i \in A_j, j = 1, \cdots, k \), we have that

\[
\mathbb{P}(X^i < 0) = \mathbb{P}\left(1_{B_j} > \frac{1}{k}\right) = \mathbb{P}(1_{B_j} = 1) = \mathbb{P}(B_j) = \frac{1}{k} < \alpha.
\]

Thus, \( \text{VaR}^\alpha(X^i) < 0 \). In fact, for \( i \in A_j, j = 1, \cdots, k \), we have that \( \text{VaR}^\alpha(-1_{B_j}) = 0 \) and thus

\[
\text{VaR}^\alpha(X^i) = \frac{m}{(k-1)\mu(A_j)}(k\text{VaR}^\alpha(-1_{B_j}) - 1) = -\frac{m}{(k-1)\mu(A_j)} < 0.
\]

As

\[
\int_I \text{VaR}^\alpha(X^i) d\mu = \sum_{j=1}^{k} \int_{A_j} \text{VaR}^\alpha(X^i) d\mu + \int_{A_{k+1}} \text{VaR}^\alpha(X^i) d\mu = \text{VaR}^\alpha(X) - \frac{mk}{k-1},
\]

\( \text{VaR}^\alpha(X) < \infty \), and \( m > 0 \) is arbitrary, we obtain that

\[
\rho_{\text{Conv}}(X) \leq \text{VaR}^\alpha(X) - \lim_{m \to \infty} \frac{mk}{k-1} = -\infty.
\]
Hence, we conclude that $\tau_\rho(X) = \infty$ for any $X \in L^\infty$, which implies that $\text{VaR}^\alpha$ is of infinite regulatory arbitrage. \hfill \Box

Remark 6.4. (i) The idea is that if any position $X$ could be split into $k + 1$ random variables with $\frac{1}{k} < \alpha$, then we would obtain an arbitrarily smaller weighted $\text{VaR}^\alpha$. In the framework in (Wang, 2016), it is always possible to obtain a countable division of any position owing to the nature of the functional $R$ in Remark 6.2.

(ii) We note that smaller values of $\alpha$ require a richer structure on $\mathcal{I}$ to allow the regulatory arbitrage strategy. This is in fact desired, as such values represent riskier scenarios. For example, for $\alpha = 0.01$, which is the demanded level for regulatory capital in Basel accords, a cardinality of at least $k + 1 = 102$ would be necessary for the required partition.

(iii) This result can be extended to the more general framework of $\rho_\mathcal{I} = \{\text{VaR}^\alpha_i, i \in \mathcal{I}\}$ when $\alpha^* = \text{ess inf}_\mu \alpha^i > 0$. As $\text{VaR}^\alpha_i \leq \text{VaR}^\alpha^* \mu - a.s.$, if there is a partition of $\mathcal{I}$ consisting of $k + 1$ non-null sets such that $\frac{1}{k} < \alpha^*$, then

$$\rho_\mathcal{I}^{\text{conv}}(X) = \inf_{\{X^i\} \in \mathcal{A}(X)} \left\{ \int_{\mathcal{I}} \text{VaR}^\alpha_i(X^i)d\mu \right\} \leq \inf_{\{X^i\} \in \mathcal{A}(X)} \left\{ \int_{\mathcal{I}} \text{VaR}^\alpha^*(X^i)d\mu \right\} = -\infty.$$ 

In fact, analogous reasoning is valid for any choice of risk measures $\rho_\mathcal{I}$ dominated by $\text{VaR}^\alpha^*$.

We now state more general results regarding $\tau_\rho$ in our framework. To this end, the following property of risk measures is required.

**Definition 6.5.** A risk measure $\rho: L^\infty \to \mathbb{R}$ is called $\mathcal{I}$-convex if

$$\int_{\mathcal{I}} \rho\left(\int_{\mathcal{I}} X^i d\mu\right) \leq \int_{\mathcal{I}} \rho\left(X^i\right) d\mu$$

for any $\{X^i\} \in \cup_{X \in L^\infty} \mathcal{A}(X)$.

**Theorem 6.6.** We have the following for a risk measure $\rho$:

(i) $\rho$ is $\mathcal{I}$-convex if and only if it is free of regulatory arbitrage.

(ii) If $\rho$ is a convex risk measure, then it is free of regulatory arbitrage. Moreover, if $(\mathcal{I}, \mathcal{G}, \mu)$ is atomless and $\rho$ is free of regulatory arbitrage, then it is convex.

(iii) If $\rho$ is subadditive, then it is at most of finite regulatory arbitrage.

(iv) Let $\rho_1, \rho_2: L^\infty \to \mathbb{R}$ be risk measures such that $\rho_1 \leq \rho_2$. If $\rho_1$ is of finite regulatory arbitrage, then $\rho_2$ is not of infinite arbitrage. Moreover, if $\rho_2$ is of infinite (or partially infinite) regulatory arbitrage, then so is $\rho_1$.

(v) If $\rho$ satisfies the positive homogeneity condition, then $\tau_\rho(0) > 0$ if and only if $\tau_\rho(0) = \infty$.

(vi) If $\rho$ is loaded, then it is not of infinite regulatory arbitrage. If, in addition, it has the limitedness property, then it is of finite regulatory arbitrage.

**Proof.** We note that as we consider finite risk measures, it holds that $\tau_\rho(X) = \infty$ if and only if $\rho_\mathcal{I}^{\text{conv}}(X) = -\infty$. Then,
We begin with the claim that if $\rho$ is $\mathcal{I}$-convex, and let $X \in L^\infty$. Then, $\rho(X) \leq \int_{\mathcal{I}} \rho(X^i) d\mu$ for any $\{X^i\}_{i \in \mathcal{I}} \in A(X)$. By taking the infimum over $A(X)$, we obtain $\rho^\mu_{\text{conv}}(X) \leq \rho(X) \leq \rho^\mu_{\text{conv}}(X)$. For the converse, we obtain $\rho(X) = \rho^\mu_{\text{conv}}(X) \leq \int_{\mathcal{I}} \rho(X^i) d\mu$ for any $\{X^i\}_{i \in \mathcal{I}} \in A(X)$, which is $\mathcal{I}$-convex.

By Corollary 4.4, we have that $\rho^\mu_{\text{conv}}(X) = \rho(X)$, $\forall X \in L^\infty$. As a direct consequence, we obtain that $\rho$ is free of regulatory arbitrage. Moreover, we assume that $(\mathcal{I}, \mathcal{G}, \mu)$ has no atoms and $\rho$ is free of regulatory arbitrage. By (i), $\rho$ is $\mathcal{I}$-convex. Let now $\lambda \in [0, 1]$, and $\mathcal{I}_1, \mathcal{I}_2$ be a partition of $\mathcal{I}$ such that $\mu(\mathcal{I}_1) = \lambda$ and $\mu(\mathcal{I}_2) = 1 - \lambda$. Such a partition always exists when $(\mathcal{I}, \mathcal{G}, \mu)$ is atom-less. Then, for some $X \in L^\infty$ such that $X = \lambda X^1 + (1 - \lambda) X^2$, let $X^i = X^1$, $\forall i \in \mathcal{I}_1$ and $X^i = X^2$, $\forall i \in \mathcal{I}_2$. Thus, $\rho(X) \leq \int_{\mathcal{I}} \rho(X^i) d\mu = \lambda \rho(X^1) + (1 - \lambda) \rho(X^2)$.

We begin with the claim that if $\rho$ is subadditive, then it is of partially infinite regulatory arbitrage if and only if it is of infinite regulatory arbitrage. By Proposition 3.4 and Remark 3.5, we have that $\rho^\mu_{\text{conv}}$ is also subadditive and normalized. We need only show that partially infinite regulatory arbitrage implies infinite regulatory arbitrage. Let $X \in L^\infty$ be such that $\tau_\rho(X) = \infty$. As $\rho$ is finite, it holds that $\rho^\mu_{\text{conv}}(X) = -\infty$. Let now $Y \in L^\infty$. We have that $\rho^\mu_{\text{conv}}(Y) \leq \rho^\mu_{\text{conv}}(X) + \rho^\mu_{\text{conv}}(Y - X) = -\infty$. Thus, $\rho$ is of infinite regulatory arbitrage. However, we have that $\tau_\rho(0) = \rho(0) - \rho^\mu_{\text{conv}}(0) = 0 < \infty$. Then, $\rho$ is not of infinite regulatory arbitrage and, by the previous claim, it is not of partially infinite regulatory arbitrage either. Thus, $\rho$ is at most of finite regulatory arbitrage.

It is evident that, in this case, we have, by abuse of notation, $(\rho_1)^\mu_{\text{conv}} \leq (\rho_2)^\mu_{\text{conv}}$. If $\rho_1$ is of finite regulatory arbitrage, then $-\infty < (\rho_1)^\mu_{\text{conv}}(X) \leq (\rho_2)^\mu_{\text{conv}}(X)$, $\forall X \in L^\infty$. Thus, $\rho_2$ is also of finite regulatory arbitrage. If now $\rho_2$ is of infinite regulatory arbitrage, then $(\rho_1)^\mu_{\text{conv}}(X) \leq (\rho_2)^\mu_{\text{conv}}(X) = -\infty$, $\forall X \in L^\infty$. Thus, $\rho_1$ is also of finite regulatory arbitrage. For partially infinite regulatory arbitrage, the reasoning is analogous.

We need only prove the “only if” part because the converse is automatically obtained. As $\rho(0) = 0$, $\tau_\rho(0) > 0$ implies $\rho^\mu_{\text{conv}}(0) < 0$. Then, there is $\{X^i\}_{i \in \mathcal{I}} \in A(0)$ such that $\rho^\mu_{\text{conv}}(0) \leq \int_{\mathcal{I}} \rho(X^i) d\mu < 0$. As $\{\lambda X^i\}_{i \in \mathcal{I}} \in A(0)$ $\forall \lambda \in \mathbb{R}_+$, by the positive homogeneity of $\rho$, we obtain that

$$
\rho^\mu_{\text{conv}}(0) \leq \lim_{\lambda \to \infty} \int_{\mathcal{I}} \rho(\lambda X^i) d\mu = \lim_{\lambda \to \infty} \lambda \int_{\mathcal{I}} \rho(X^i) d\mu = -\infty.
$$

Hence, $\tau_\rho(0) = \rho(0) - \rho^\mu_{\text{conv}}(0) = \infty$.

By Proposition 3.4, we have that $\tau_{\rho^\mu_{\text{conv}}}$ inherits loadedness and limitedness from $\rho$. The loadedness of $\rho$ implies normalization of $\rho^\mu_{\text{conv}}$, and therefore $\tau_\rho(0) = 0$. Thus, $\rho$ is not of infinite regulatory arbitrage. If, in addition, $\rho$ is limited, then for any $X \in L^\infty$, we have that $\tau_\rho(X) \leq E[X] - \text{ess inf } X < \infty$. Hence, we obtain finite regulatory arbitrage for $\rho$. 

\qed
Remark 6.7. (i) In the approach in [Wang, 2016], $\tau_{\rho}$ is always subadditive, whereas in our case, this is not ensured. This fact alters most results and arguments, as it is crucial in his study. Moreover, (iii), (iv), (v), and (vi) would remain true if we considered the general framework of arbitrary $\rho_I$ and made the adaptation $\tau_{\rho_I} = \rho^\mu - \rho^\mu_{\text{conv}}$.

(ii) A remarkable feature is that it is possible to identify $\tau_{\rho}$ as a deviation measure in the sense of Rockafellar et al. (2006), Rockafellar and Uryasev (2013), Righi and Ceretta (2016), Righi (2019a), and Righi et al. (2019). For instance, the bound for $\tau_{\rho}$ in the proof of (vi) is known as lower-range dominance for deviation measures.

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