Ricci flow on Orbifolds

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Abstract

In this paper, we study the behavior of Ricci flows on compact orbifolds with finite singularities. We show that Perelman’s pseudolocality theorem also holds on orbifold Ricci flow. Using this property, we obtain a weak compactness theorem of Ricci flows on orbifolds under some natural technical conditions. This generalizes the corresponding theorem on manifolds. As an application, we can use Kähler Ricci flow to find new Kähler Einstein metrics on some orbifold Fano surfaces. For example, if \( Y \) is a cubic surface with only one ordinary double point or \( Y \) is an orbifold Fano surface with degree 1 and every singularity on it is a rational double point of type \( A_k (1 \leq k \leq 6) \), then we can find a KE metric of \( Y \) by running Kähler Ricci flow.

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1 Introduction

An important object of Ricci flow is to find Einstein metrics on a given manifold. In the seminal paper [Ha82], Hamilton showed that starting from any metric with positive Ricci curvature on $M^3$, the normalized Ricci flow will always converge to an Einstein metric at last. In the set of Kähler manifolds, Kähler Ricci flow was developed as an important tool in search of KE (Kähler Einstein) metrics. In [Cao85], based on the fundamental estimate of Yau (Yau78), Cao showed the long time existence of Kähler Ricci flow and the convergence of Kähler Ricci flow when $c_1(M) \leq 0$. If $c_1(M) > 0$, $M$ is called Fano manifold. In this case, situations are much more delicate. $M$ may not have KE metric. So we cannot expect the convergence of the Kähler Ricci flow to a KE metric in general. If the existence of KE metric is assumed, Chen and Tian showed that Kähler Ricci flow converges exponentially fast toward the KE metric if the initial metric has positive bisectional curvature (cf. CT1, CT2). Using his famous $\mu$-functional, Perelman developed fundamental estimates along Kähler Ricci flow on Fano manifolds. He also claimed that the Kähler Ricci flow will always converge to the KE metric on any KE manifold. This result was generalized to manifolds with Kähler Ricci solitons by Tian and Zhu (TZ). If the existence of KE metric is not assumed, there are a lot of works toward the convergence of Kähler Ricci flow after G. Perelman’s fundamental estimates. For example, important progress can be found in (listed in alphabetical order) CLH, CST, CW1, CW2, Hei, PSS, PSSW1, PSSW2, Ru, RZZ, Se1, To, TZs and references therein.

Following Tian’s original idea of $\alpha_{\nu,k}$-invariant in Tian90 and Tian91, Chen and the author (c.f. CW3 and CW4) proved that the Kähler Ricci flow converges to a KE metric if one of the following conditions is satisfied.

A natural question is: can we generalize these results to Fano orbifolds and use Kähler Ricci flow to search the KE metrics on Fano orbifolds? In this paper, we answer this question affirmatively. We use Kähler Ricci flow as a tool to find new KE metrics on some orbifold Fano surfaces. However, before we can use the orbifold Kähler Ricci flow, we firstly need some general results of orbifold Ricci flows. So we generalize Perelman’s Ricci flow theory to orbifold case. The study of orbifold Ricci flow is pioneered by the work CWL, Wu, Lu1.

We have the following theorems.

**Theorem 1.** Suppose $Y$ is a Fano orbifold, $\{(Y, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow solution tamed by $\nu$. Then this flow converges to a KE metric if one of the following conditions is satisfied.
\[ \alpha_{\nu,1} > \frac{n}{\nu + 1} \cdot \]
\[ \alpha_{\nu,2} > \frac{n}{\nu + 1}, \quad \alpha_{\nu,1} > \frac{1}{2^{(n+1)\alpha_{\nu,2}}} \cdot \]

The tamedness condition originates from Tian’s work in [Tian90] (c.f. equation (0.3) of [Tian90]). Under Kähler Ricci flow, it’s first defined in [CW4]. A flow is called tamed by constant \(\nu\) if the function
\[
F(\nu, x, t) \triangleq \frac{1}{\nu} \log \sum_{\beta=0}^{N_\nu} |S_{\nu,\beta}^t|_{h^t}^2(x)
\]
is uniformly bounded on \(Y \times [0, \infty)\). Here \(\{S_{\nu,\beta}^t\}_{\beta=0}^{N_\nu}\) are orthonormal basis of \(H^0(K^\nu_Y)\), i.e.,
\[
\int_Y (S_{\nu,\alpha}^t, S_{\nu,\beta}^t)_{h^t} \omega^n_t = \delta_{\alpha,\beta}, \quad 0 \leq \alpha, \beta \leq N_\nu = \dim H^0(K^\nu_Y) - 1; \quad h = \det g_{ij}(t).
\]

Therefore, this theorem gives us a way to search KE metric by Kähler Ricci flow. \(\alpha_{\nu,k}\) are defined as (c.f. Definition 5.3 for more details)
\[
\alpha_{\nu,k} = \sup \{\alpha | \sup_{\varphi \in P_{\nu,k}} \int_Y e^{-2\alpha \varphi} \omega^n_0 < \infty\}
\]
where \(P_{\nu,k}\) is the collection of all functions of the form \(\frac{1}{2\nu} \log \left( \sum_{\beta=0}^{k-1} ||S_{\nu,\beta}||_{h^t}^2 \right)\) for some orthonormal basis \(\{\tilde{S}_{\nu,\beta}\}_{\beta=0}^{k-1}\) of a \(k\)-dimensional subspace of \(H^0(K^\nu_Y)\). Note that \(\alpha_{\nu,k}\) are algebraic invariants which can be calculated explicitly in many cases, the most important thing now is to show when the tamedness condition is satisfied.

**Theorem 2.** Suppose \(Y\) is a Fano surface orbifold, \(\{(Y, g(t)), 0 \leq t < \infty\}\) is a Kähler Ricci flow solution. Then there is a constant \(\nu\) such that this flow is tamed by \(\nu\).

According to these two theorems and the calculations done in [Kosta] and in [SY], we obtain the existence of Kähler Einstein metrics on some orbifold Fano surfaces.

**Corollary 1.** Suppose \(Y\) is a cubic surface with only one ordinary double point, or \(Y\) is a degree 1 del Pezzo surface having only Du Val singularities of type \(A_k\) for \(k \leq 6\). Starting from any metric \(\omega\) satisfying \([\omega] = 2\pi c_1(Y)\), the Kähler Ricci flow will converge to a KE metric on \(Y\). In particular, \(Y\) admits a KE metric.

Actually, both Theorem 1 and Theorem 2 have corresponding versions in [CW3] and [CW4]. Their proofs are also similar to the ones in [CW3] and [CW4].
Theorem 1 follows from the partial $C^0$-estimated given by the tamedness condition:

$$\left| \varphi(t) - \sup_M \varphi(t) - \frac{1}{\nu} \log \sum_{\beta=0}^{N_\nu} \left| \lambda_{\beta}(t) S_{\nu,\beta}^t \right|_{g_0}^2 \right| < C,$$  \hspace{1cm} (2)

where $\varphi(t)$ is the evolving Kähler potential, $0 < \lambda_0(t) \leq \lambda_1(t) \leq \cdots \leq \lambda_{N_\nu}(t) = 1$ are $N_\nu + 1$ positive functions of time $t$, $\{S_{\nu,\beta}^t\}_{\beta=0}^{N_\nu}$ is an orthonormal basis of $H^0(K_M^{-\nu})$ under the fixed metric $g_0$. Intuitively, inequality (2) means that we can control $\text{Osc}_M \varphi(t)$ by $\frac{1}{\nu} \log \sum_{\beta=0}^{N_\nu} \left| \lambda_{\beta}(t) S_{\nu,\beta}^t \right|_{g_0}^2$ which only blows up along intersections of pluri-anticanonical divisors. Therefore, the estimate of $\varphi(t)$ is more or less translated to the study of the property of pluri-anticanonical holomorphic sections, which are described by $\alpha_{\nu,k}$.

Theorem 2 can be looked as the combination of the following two lemmas.

**Lemma 1.** Suppose $Y$ is a Fano orbifold, $\{(Y^n, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow solution satisfying the following two conditions

- **No concentration:** There is a constant $K$ such that
  $$\text{Vol}_{g(t)}(B_{g(t)}(x, r)) \leq Kr^{2n}$$
  for every $(x, t) \in Y \times [0, \infty), r \in (0, K^{-1}]$.

- **Weak compactness:** For every sequence $t_i \to \infty$, by passing to subsequence, we have
  $$(Y, g(t_i)) \overset{C^\infty}{\to} (\hat{Y}, \hat{g}),$$
  where $(\hat{Y}, \hat{g})$ is a Q-Fano normal variety.

Then this flow is tamed by some big constant $\nu$.

Note that the Q-Fano normal variety is a normal variety with a very ample line bundle whose restriction on the smooth part is the plurianticanonical line bundle. The convergence $\overset{C^\infty}{\to}$ is the convergence in Cheeger-Gromov topology, i.e., it means that the following two properties are satisfied simultaneously:

- $d_{GH}(Y_i, \hat{Y}) \to 0$ where $d_{GH}$ is the Gromov-Hausdorff distance among metric spaces.
- For every smooth compact set $K \subset \hat{Y}$, there are diffeomorphisms $\varphi_i : K \to Y_i$ such that $\text{Im}(\varphi_i)$ is a smooth subset of $Y_i$ and $\varphi_i^*(g_i)$ converges to $\hat{g}$ smoothly on $K$.

**Lemma 2.** Suppose $Y$ is an orbifold Fano surface, $\{(Y, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow solution, then this flow satisfies the no concentration and weak compactness property mentioned in Lemma 1. Moreover, every limit space $(\hat{Y}, \hat{g})$ is a Kähler Ricci soliton.
The proof of Lemma 1 follows directly (c.f. Theorem 3.2 of [CW4]) if we have the continuity of plurianticanonical holomorphic sections — orthonormal bases of $H^0(K_{\hat{Y}}^{-\nu})$ (under metric $\hat{g}$) converge to an orthonormal basis of $H^0(K_Y^{-\nu})$ (under metric $g$) whenever $(Y, g_i)$ converge to $(\hat{Y}, \hat{g})$. Moreover, every orthonormal basis of $H^0(K_{\hat{Y}}^{-\nu})$ is a limit of orthonormal bases of $H^0(K_{\hat{Y}}^{-\nu})$ (under metric $\hat{g}$) whenever $(Y, g_i)$ converge to $(\hat{Y}, \hat{g})$. This fact is assured by Hörmand’s $L^2$-estimate of $\overline{\partial}$-operator and an a priori estimate of $|S|$ and $|\nabla S|$, where $S$ is a unit norm section of $H^0(K_Y^{-\nu})$ (c.f. Lemma 5.2 for the a priori bounds of sections, Theorem 3.1 of [CW4] for the continuity of holomorphic sections).

The proof of Lemma 2 is essentially based on Riemannian geometry. It is a corollary of the following Theorem 3. In fact, if we define $O(m, c, \sigma, \kappa, E)$ as the moduli space of compact orbifold Ricci flow solutions $\{(X_m, g(t)), -1 \leq t \leq 1\}$ whose normalization constant is bounded by $c$, scalar curvature bounded by $\sigma$, volume ratio bounded by $\kappa$ from below, energy bounded by $E$ (c.f. Definition 4.3), then this moduli space have no concentration and weak compactness properties.

Theorem 3. $O(m, c, \sigma, \kappa, E)$ satisfies the following two properties.

- No concentration. There is a constant $K$ such that
  \[
  \text{Vol}_{g(0)}(B_{g(0)}(x, r)) \leq Kr^m
  \]
  whenever $r \in (0, K^{-1})$, $x \in X$, $\{(X, g(t)), -1 \leq t \leq 1\} \in O(m, c, \sigma, \kappa, E)$.

- Weak compactness. If $\{(X_i, x_i, g_i(t)), -1 \leq t \leq 1\} \in O(m, c, \sigma, \kappa, E)$ for every $i$, by passing to subsequence if necessary, we have
  \[
  (X_i, x_i, g_i(0)) \overset{C^\infty}{\longrightarrow} (\hat{X}, \hat{x}, \hat{g})
  \]
  for some $C^0$-orbifold $\hat{X}$ in Cheeger-Gromov sense.

Actually, according to the fact that scalar curvature and $\int_Y |Rm|^2 \omega_2^2$ are uniformly bounded (c.f. Proposition 5.1) along Kähler Ricci flow on orbifold Fano surface, it is clear that $\{(Y, g(t + T)), -1 \leq t \leq 1\} \in O(4, 1, \sigma, \kappa, E)$ for every $T \geq 1$. Therefore Theorem 3 applies. In order to obtain Lemma 2 we need to show that the limit space $\hat{Y}$ is a Kähler Ricci soliton and every orbifold singularity is a $C^\infty$-orbifold point (c.f. Definition 4.1). The first property is a direct application of Perelman functional’s monotonicity (c.f. [Se1]), the second property follows from Uhlenbeck’s removing singularity method (c.f. [CS]).

Theorem 3 is a generalization of the corresponding weak compactness theorem in [CW3]. If we assume Perelman’s pseudolocality theorem (Theorem 10.3 of [Pe1]) holds in orbifold case, then its proof can be almost the same as the corresponding theorems in [CW3]. Therefore, an important technical difficulty of this paper is the following pseudolocality theorem.

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Theorem 4. There exists $\eta = \eta(m, \kappa) > 0$ with the following property.

Suppose $\{(X, g(t)), 0 \leq t \leq r_0^2\}$ is a compact orbifold Ricci flow solution. Assume that at $t = 0$ we have $|\hat{Rm}|(x) \leq r_0^{-2}$ in $B(x, r_0)$, and $\text{Vol} B(x, r_0) \geq \kappa r_0^m$. Then the estimate $|\hat{Rm}|_{g(t)}(y) \leq (\eta r_0)^{-2}$ holds whenever $0 \leq t \leq (\eta r_0)^2$, $d_{g(t)}(y, x) < \eta r_0$.

Note that $|\hat{Rm}|$ is defined as

\[ |\hat{Rm}|(x) = \begin{cases} |Rm|(x), & \text{if } x \text{ is a smooth point}. \\ \infty, & \text{if } x \text{ is a singularity}. \end{cases} \]

The proof of Theorem 4 is a combination of Perelman’s point selecting method and maximal principle. Note that the manifold version of Theorem 4 (Theorem 10.3 of [Pe1]) is claimed by Perelman without proof. This first written proof is given by Lu Peng in [Lu2] recently.

With Theorem 4 in hand, we can prove Theorem 3 as we did in [CW3]. However, we prefer to give a new proof. In [CW3], the proof of weak compactness theorem is complicated. A lot of efforts are paid to show the locally connectedness of the limit space. In other words, we need to show the limit space is an orbifold, not a manifold. We used bubble tree on space time to argue by contradiction. If we are able to construct bubble tree on a fixed time slice, then the argument will be much easier. In this paper, we achieve this by observing some stability of $\int |Rm|^2$ in unit geodesic balls.

In short, the new ingredients of this paper are listed as follows.

- We offer a method to find KE metrics on orbifold Fano surfaces.
- We give a simplified proof of weak compactness theorem, i.e., Theorem 3.
- We prove the pseudolocality theorem in orbifold Ricci flow.

It’s interesting to compare the two methods used in search of KE metrics: the continuity method and the flow method. Suppose $(M, g, J)$ is a Kähler manifold with positive first Chern class $c_1$, $\omega$ is the $(1, 1)$-form compatible to $g$ and $J$. The existence of KE metric under the complex structure $J$ is equivalent to the solvability of the equation

\[ \det(g_{ij} + \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}) = e^{-u - \varphi} \det(g_{ij}), \quad g_{ij} + \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} > 0, \]

where $u$ is a smooth function on $M$ satisfying

\[ u_{ij} = g_{ij} - R_{ij}, \quad \frac{1}{V} \int_M (e^{-u} - 1) \omega^n = 1. \]

In continuity method, we try to solve a family of equation $(0 \leq t \leq 1)$:

\[ \begin{cases} \det(g_{ij} + \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}) = e^{-u(t) - \varphi} \det(g_{ij}), \\ g_{ij} + \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} > 0. \end{cases} \]
In Kähler Ricci flow method, we try to show the convergence of the parabolic equation solution:

\[
\frac{\partial \varphi}{\partial t} = \log \frac{\det(g_{ij} + \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j})}{\det(g_{ij})} + \varphi + u.
\]

In both methods, the existence of KE metric is reduced to set up a uniform \(C^0\)-bound of the Kähler potential function \(\varphi\). If \(\alpha(M) > \frac{3}{n+1}\), then \(\varphi\) is uniformly bounded in either case (c.f. [Tian87], [Ru], [CW2]). If \(\alpha(M) \leq \frac{2}{n+1}\), we need more geometric estimates to show the uniform bound of \(\varphi\). Under continuity path, these geometric estimates are stated by Tian in [Tian90] and [Tian91] (c.f. inequality (0.3) of [Tian90] and inequality (5.2) of [Tian91]). In Kähler Ricci flow case, we used a similar statement and called it tamedness condition (c.f. equation (1)) for simplicity. If the continuity path or Kähler Ricci flow is tamed, then \(\varphi\) is uniformly bounded if \(\alpha_{\nu,k}(k = 1,2)\) is big enough.

However, if the complex structure is fixed, there are slight difference in obtaining the tamedness condition between these two methods. The tamedness condition of a Kähler Ricci flow maybe easier to verify under the help of Perelman’s functional. On a continuity path, the tamedness condition is conjectured to be true by Tian (c.f. Inequality (5.2.) of [Tian91]).

Let’s recall how to find the KE metric on Kähler surface \((M, J)\) whenever \(c_1^2(M) = 3\) and \((M, J)\) contains Eckhard point. It was first found by Tian in [Tian90] where he used continuity method twice. Note that on the differential manifold \(M \sim \mathbb{CP}^2 \# 6\mathbb{CP}^2\), all the complex structures such that \(c_1\) positive form a connected 4-dimensional algebraic variety \(\mathcal{J}\). Choose \(J_0 \in \mathcal{J}\) such that \(\alpha_G(M, J_0) > \frac{2}{3}\) for some compact group \(G \subset \text{Aut}(M, J_0)\) (e.g. Fermat surface). By continuity method, there is a KE metric \(g_0\) compatible with \(J_0\). Now connecting \(J_0\) and \(J\) by a family of complex structures \(J_t \in \mathcal{J}, 0 \leq t \leq 1\) such that \(J_1 = J\). Choose \(\tilde{g}_t\) be a continuous family of metrics compatible with \(J_t\). Let \(I\) be the collection of all \(t\) such that there exists a KE metric \(g_t\) compatible with \(J_t\). It’s easy to show that \(I\) is an open subset of \([0,1]\). In order to prove \(I = [0,1]\), one only need to show the closedness of \(I\). Let

\[(g_t)_{ij} = (\tilde{g}_t)_{ij} + \varphi_{ij}.
\]

Then it suffices to show a uniform bound of \(O_{SCM}\varphi(t)\) on \(I\). Since along this curve of complex structures, \(\alpha_{\nu,2}(M, J_t) > \frac{2}{3}, \alpha_{\nu,1}(M, J_t) \geq \frac{2}{3}\) for every \(t \in I \subset [0,1]\) (c.f. [SY], [ChS]), it suffices to show the tamedness condition (inequality (0.3) of [Tian90]) on the set \(I\). In fact, this tamedness condition is guaranteed by the weak compactness theorem of KE metrics on \(M\) (c.f. Proposition 4.2 of [Tian90]).

In Kähler Ricci flow method, we are unable to change complex structure. Inspired by the continuity method, we also reduce the boundedness of \(\varphi\) to the tamedness condition since \(\alpha_{\nu,2}(M, J) > \frac{2}{3}, \alpha_{\nu,1}(M, J) \geq \frac{2}{3}\). Now in order to show the tamedness condition, we need a weak compactness of time slices of a Kähler Ricci flow. This seems to be more difficult since each time slice is only a Kähler metric, not a KE metric, we therefore lose the regularity property of KE metrics. Luckily, under the help of Perelman’s estimates and pseudolocality theorem, we are able to show the weak compactness theorem (c.f. Theorem...
4.4 of [CW3]). Consequently the tamedness condition of the Kähler Ricci flow on \( M \) holds, so \( \varphi \) is uniformly bounded and this flow converges to a KE metric.

Once the weak compactness of time slices is proved, the disadvantage of Kähler Ricci flow becomes an advantage: we can prove the tamedness condition without changing complex structure. This is not easy to be proved under a continuity path when the complex structure is fixed. Suppose we have a differential manifold \( M \) whose complex structures with positive \( c_1 \) form a space \( \mathcal{J} \) satisfying

\[
\alpha_{\nu,1}(M, J) \leq \frac{n}{n+1}, \quad \forall J \in \mathcal{J}.
\]

Without using symmetry of the initial metric, we cannot apply continuity method directly to draw conclusion about the existence of KE metrics on \((M, J)\). However, Kähler Ricci flow can still possibly be applied. For example, if \((Y, J)\) is an Fano orbifold surface with degree 1 and with three rational double points of type \( A_5, A_2 \) and \( A_1 \). Then \( J \) is the unique complex structure on \( Y \) such that \( c_1(Y) > 0 \) (c.f. [Zhd], [YQ]). According to the calculations in [Kosta], we know \( \alpha_{\nu,1}(Y) = \frac{2}{3}, \alpha_{\nu,2} > \frac{2}{7} \). So we are unable to use continuity method directly to conclude the existence of KE metric on \((Y, J)\) because of the absence of tamedness condition. However, we do have this condition under Kähler Ricci flow by Theorem 2. Therefore, the Kähler Ricci flow on \((Y, J)\) must converge to a KE metric.

The organization of this paper is as follows. In section 2, we set up notations. In section 3, we go over Perelman’s theory on Ricci flow on orbifolds and prove the pseudolocality theorem (Theorem 4). In section 4, we give a simplified version of proof of weak compactness theorem (Theorem 3). In section 5, we give some improved estimates of pluricanonical line bundles and prove Theorem 1 and Theorem 2. At last, in section 6, we give some examples where our theorems can be applied. In particular, we show Corollary 1.

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2 Set up of Notations

**Definition 2.1.** A \( C^\infty(\mathcal{O}) \)-orbifold \((\hat{X}^m, \hat{g})\) is a topological space which is a smooth manifold with a smooth Riemannian metric away from finitely many singular points. At every singular point, \( \hat{X} \) is locally diffeomorphic to a cone over \( S^{m-1}/\Gamma \) for some finite subgroup \( \Gamma \subset SO(m) \). Furthermore, at such a singular point, the metric is locally the quotient of a smooth (continuous) \( \Gamma \)-invariant metric on \( B^m \) under the orbifold group \( \Gamma \).

A \( C^\infty(\mathcal{O}) \)-multifold \((\tilde{X}, \tilde{g})\) is a finite union of \( C^\infty(\mathcal{O}) \)-orbifolds after identifying finite points. In other words, \( \tilde{X} = \prod_{i=1}^N \hat{X}_i/\sim \) where every \( \hat{X}_i \) is an orbifold, the relation \( \sim \)
identifies finite points of $\prod_{i=1}^{N} \tilde{X}_i$.

For simplicity, we say a space is a Riemannian orbifold or orbifold (multifold) if it is a $C^\infty$-orbifold ($C^\infty$-multifold).

**Definition 2.2.** For a compact Riemannian orbifold $X^m$ without boundary, we define its isoperimetric constant as

$$I(X) \triangleq \inf_{\Omega} \frac{|\partial \Omega|}{\min\{ |\Omega|, |X \setminus \Omega| \}^{\frac{m-1}{m}}}$$

where $\Omega$ runs over all domains with rectifiable boundaries in $X$.

For a complete Riemannian orbifold $X^m$ with boundary, we define its isoperimetric constant as

$$I(X) \triangleq \inf_{\Omega} \frac{|\partial \Omega|}{|\Omega|^{\frac{m-1}{m}}}$$

where $\Omega$ runs over all domains with rectifiable boundaries in the interior of $X$.

**Definition 2.3.** A geodesic ball $B(p, \rho)$ is called $\kappa$-noncollapsed if $\frac{\text{Vol}(B(q, s))}{s^m} > \kappa$ whenever $B(q, s) \subset B(p, \rho)$.

A Riemannian orbifold $X^m$ is called $\kappa$-noncollapsed on scale $r$ if every geodesic ball $B(p, \rho) \subset X$ is $\kappa$-noncollapsed whenever $\rho \leq r$.

A Riemannian orbifold $X^m$ is called $\kappa$-noncollapsed if it is $\kappa$-noncollapsed on every scale $r \leq \text{diam}(X^m)$.

**Definition 2.4.** Suppose $(x,t)$ is a point in a Ricci flow solution. Then parabolic balls are defined as

$$P^+(x,t,r,\theta) = \{(y,s)|d_{g(t)}(y,x) \leq r, t \leq s \leq s + \theta\}.$$  
$$P^-(x,t,r,\theta) = \{(y,s)|d_{g(t)}(y,x) \leq r, t - \theta \leq s \leq s\}.$$  

Geometric parabolic balls are defined as

$$\tilde{P}^+(x,t,r,\theta) = \{(y,s)|d_{g(s)}(y,x) \leq r, t \leq s \leq s + \theta\}.$$  
$$\tilde{P}^-(x,t,r,\theta) = \{(y,s)|d_{g(s)}(y,x) \leq r, t - \theta \leq s \leq s\}.$$  

**Definition 2.5.** Suppose $x$ is a point in the Riemannian orbifold $X$. Then we define

$$|Rm| = \left\{ \begin{array}{ll} |\text{Rm}|(x), & \text{if } x \text{ is a smooth point}, \\ \infty, & \text{if } x \text{ is a singular point}. \end{array} \right.$$
3 Pseudolocality Theorem

3.1 Perelman’s Functional and Reduced Distance

Denote $\square = \frac{\partial}{\partial t} - \triangle$, $\square^* = -\frac{\partial}{\partial t} - \triangle + R$.

In our setting, every orbifold only has finite singularities. All the concepts in [Pe1] can be reestablished in our orbifold case. For example, we can define $W$-functional, reduced distance, reduced volume on orbifold Ricci flow.

**Definition 3.1.** Let $(X, g)$ be a Riemannian orbifold, $\tau > 0$ a constant, $f$ a smooth function on $X$. Define

$$W(g, \tau, f) = \int_X \{\tau(R + |\nabla f|^2) + f - n\}(4\pi \tau)^{-\frac{n}{2}} e^{-f} dv,$$

$$\mu(g, \tau) = \inf_{g \in (4\pi \tau)^{-\frac{n}{2}} e^{-f} dv = 1} W(g, \tau, f).$$

Since the Sobolev constant of $X$ exists, we know $\mu(g, \tau) > -\infty$ and it is achieved by some smooth function $f$.

Suppose $\{(X, g(t)), 0 \leq t \leq T\}$ is a Ricci flow solution on compact orbifold $X$, $u = (4\pi(T-t))^{-\frac{n}{2}} e^{-f}$ satisfies $\square^* u = 0$. Let $v = \{(T-t)(2\Delta f - |\nabla f|^2 + R) + f - m\}u$, then

$$\square^* v = -2(T-t)|R_{ij} + f_{ij} - \frac{1}{2(T-t)}g_{ij}|^2 u \leq 0.$$

This implies that

$$\frac{\partial}{\partial t} \int_X \{(T-t)(R + |\nabla f|^2) + f - m\}(4\pi \tau)^{-\frac{n}{2}} e^{-f} \mu(g, \tau) = \frac{\partial}{\partial t} \int_X v = \int_X \square^* v \leq 0.$$

It follows that $\mu(g(t), T-t)$ is nondecreasing along Ricci flow. From this monotonicity, we can obtain the no-local-collapsing theorem.

**Proposition 3.1.** Suppose $\{(X, g(t)), 0 \leq t < T_0\}$ is a Ricci flow solution on compact orbifold $X$, then there is a constant $\kappa$ such that the following property holds.

Under metric $g(t)$, if scalar curvature norm $|R| \leq r^{-2}$ in $B(x, r)$ for some $r < 1$, then $\text{Vol}(B(x, r)) \geq \kappa r^n$.

The proof of this proposition is the same as Theorem 4.1 in [Pe1] if $R$ is replaced by $|Rm|$. See [KL], [SeT] for the improvement to scalar curvature.

**Definition 3.2.** Fix a base point $p \in X$. Let $C(p, q, \bar{\tau})$ be the collection of all smooth curves $\{\gamma(\tau), 0 \leq \tau \leq \bar{\tau}\}$ satisfying $\gamma(0) = p, \gamma(\bar{\tau}) = q$. As in [Pe1], we define

$$\mathcal{L}(\gamma) = \int_0^{\bar{\tau}} \sqrt{\tau(R + |\gamma(\tau)|^2)} d\tau,$$

$$L(p, q, \bar{\tau}) = \inf_{\gamma \in C(p, q, \bar{\tau})} \mathcal{L}(\gamma),$$

$$l(p, q, \bar{\tau}) = \frac{L(p, q, \bar{\tau})}{2\sqrt{\bar{\tau}}}.$$
Like manifold case, $L(p, q, \bar{\tau})$ is achieved by some shortest $\mathcal{L}$-geodesic $\gamma$.

Under Ricci flow, since the evolution of distance is controlled by Ricci curvature. Definition 3.2 yields the following estimate (c.f. [Ye]).

**Proposition 3.2.** Suppose $|\text{Ric}| \leq Cg$ when $0 \leq \tau \leq \bar{\tau}$ for a nonnegative constant $C$. Then

$$e^{-2C\tau} \frac{d^2_{g(0)}(p, q)}{4\tau} - \frac{nC}{3} \leq l(p, q, \tau) \leq e^{2C\tau} \frac{d^2_{g(0)}(p, q)}{4\tau} + \frac{nC}{3} \tau.$$ 

Therefore, as $\tau \to 0$, $l(p, q, \tau)$ behaves like $\frac{d^2_{g(0)}(p, q)}{4\tau}$.

**Proposition 3.3.** Let $u(p, q, \tau)$ be the heat kernel of $\Box^*$ on $X \times [0, \bar{\tau}]$. As $q \to p$, $\tau \to 0$, we have

$$u(p, q, \tau) \sim (4\pi\tau)^{-\frac{n}{2}} e^{-\frac{d^2_{g(0)}(p, q)}{4\tau} + \log |\Gamma_p|}.$$ 

In the case the underlying space is a manifold, this approximation can be proved by constructing parametrix for the operator $\Box^*$ (c.f. [CLN] for detailed proof). This construction can be applied to orbifold case easily. See [DSGW] for the construction of parametrix of heat kernel on general orbifold under fixed metrics. This proposition is the combination of the corresponding theorems in [CLN] and [DSGW]. The proof method is the same, so we omit the proof for simplicity.

**Proposition 3.4.** $\Box^*\{(4\pi\tau)^{-\frac{n}{2}} e^{-l}\} \leq 0$.

**Proposition 3.5.** Suppose $h$ is the solution of $\Box h = 0$, then

$$\lim_{t \to 0} \int_X hv \leq -\log |\Gamma|h(p, 0).$$

**Proof.** Direct calculation shows that

$$\frac{\partial}{\partial t} \left( \int_X hv \right) = -\int_X h\Box^* v = 2\tau \int_X \left|R_{ij} + f_{,ij} - \frac{g_{ij}}{2\tau}\right|^2 uh \geq 0.$$ 

Therefore, $\lim_{t \to 0^+} \int_X hv$ exists if $\int_X hv$ is uniformly bounded as $t \to 0^-$. However, we can decompose $\int_X hv$ as
\[
\int_X hv = \int_X \left[ \tau (2\Delta f - |\nabla f|^2 + R) + f - n \right] uh \\
= (4\pi \tau)^{-\frac{n}{2}} \int_X \left[ \tau (2\Delta f - |\nabla f|^2 + R) + f - n \right] e^{-f} h \\
= (4\pi \tau)^{-\frac{n}{2}} \left\{ \int_X \left[ \tau (|\nabla f|^2 + R) + f - n \right] e^{-f} h - 2\tau \int_X h \Delta e^{-f} \right\} \\
= \int_X \left[ -2\tau \Delta h + (R\tau - n) h \right] u + \int_X \tau |\nabla f|^2 uh + \int_X f uh.
\]

Note that \( \int_X u \equiv 1 \). Term I is uniformly bounded. By the gradient estimate of heat equation, as in [Ni1], we have

\[
\tau |\nabla u|^2 \leq (2 + C_1 \tau) \{ \log \left( \frac{B u \tau}{n} \right) + C_2 \} 
\]

for some constants \( C_1, C_2 \). Together with \( \int_X u \equiv 1 \), this implies

\[
II = \int_X \tau |\nabla f|^2 uh \leq (2 + C_1 \tau) \{ \int_X (\log B + f + C_2 \tau) uh \} \leq C + 3 \int_X f uh,
\]

where \( C \) is a constant depending on \( X \) and \( h \). It follows that

\[
\int_X hv \leq C' + 4 \int_X f uh.
\]

In order to show \( \int_X hv \) have a uniform upper bound, it suffices to show that \( III = \int_X f uh \) is uniformly bounded from above.

Around \((p,0)\), the reduced distance \( l \) on \( X \) approximates \( \frac{d^2}{4\tau} \). (See [Pe1], [Ye] for more details.) As a consequence, we have

\[
\Box^* \left\{ (4\pi \tau)^{-\frac{n}{2}} e^{-l(y,\tau) + \log |\Gamma|} \right\} \leq 0, \quad \lim_{\tau \to 0} (4\pi \tau)^{-\frac{n}{2}} e^{-l(y,\tau) + \log |\Gamma|} = \delta_x(y).
\]

Then maximal principle implies that

\[
f(y,\tau) \leq l(y,\tau) - \log |\Gamma|.
\]

for every \( y \in X, 0 < \tau \leq 1 \).

Inequality (3) implies

\[
\limsup_{\tau \to 0} \int_X f uh \leq \limsup_{\tau \to 0} \int_X (l - \log |\Gamma|) uh \\
= -\log |\Gamma| h(p,0) + \limsup_{\tau \to 0} \int_X \frac{d^2}{4\tau} uh \\\n\leq \left( \frac{n}{2} - \log |\Gamma| \right) h(p,0).
\]
The last step holds since the expansion of $u$ around point $(p,0)$ tells us that
\[
\limsup_{\tau \to 0} \int_X \frac{d^2}{4 \tau} uh \leq h(p,0) \{ \int_{\mathbb{R}^n / \Gamma} \frac{|z|^2}{4} (4\pi)^{-\frac{n}{2}} e^{-\frac{|z|^2}{4\tau} + \log|\Gamma|} \} = \frac{n}{2} h(p,0).
\]

After the uniform upper bound of $\int_X vh$ is set up, by the monotonicity of $\int_X vh$, we see $\lim_{\tau \to 0} \int_X vh$ exists. Since $\frac{1}{\tau}$ is not integrable on $[0,1]$, for every $k$, there are small $\tau$'s such that
\[
\frac{\partial}{\partial t} \int_X vh \leq \frac{1}{k \tau}.
\]

So we can extract a sequence of $\tau_k \to 0$ such that
\[
\lim_{k \to \infty} 2 \tau_k^2 \int_X |R_{ij} + f_{,ij} - \frac{g_{ij}}{2\tau}|^2 uh = \lim_{k \to \infty} \tau_k \frac{\partial}{\partial t} \int_X vh \leq \lim_{k \to \infty} \frac{1}{k} = 0.
\]

H"older inequality and Cauchy-Schwartz inequality implies that
\[
\lim_{\tau_k \to 0} \tau_k \int_X (R + \Delta f - \frac{n}{2\tau_k}) uh \leq \lim_{\tau_k \to 0} \tau_k \int_X |R_{ij} + f_{,ij} - \frac{g_{ij}}{2\tau_k}|^2 uh
\]
\[
\leq \lim_{\tau_k \to 0} \{ \tau_k^2 \int_X |R_{ij} + f_{,ij} - \frac{g_{ij}}{2\tau_k}|^2 uh \}^{\frac{1}{2}} \cdot \{ \int_X uh \}^{\frac{1}{2}}
\]
\[
= 0.
\]

Therefore,
\[
\lim_{\tau \to 0} \int_X vh = \lim_{\tau_k \to 0} \int_X vh
\]
\[
= \lim_{\tau_k \to 0} \int_X [(R + \Delta f) - \frac{n}{2\tau_k}] uh - \lim_{\tau_k \to 0} \tau_k \int_X u \Delta h + \lim_{\tau_k \to 0} \int_X (f - \frac{n}{2}) uh
\]
\[
= \lim_{\tau_k \to 0} \int_X (f - \frac{n}{2}) uh
\]
\[
\leq - \log |\Gamma|h(p,0).
\]

Corollary 3.1. $v \leq 0$.

Theorem 3.1. Suppose $h$ is a nonnegative function, there is a large constant $K$ such that $\max\{\Box h, -\Delta h\} \leq K$ whenever $t \in [-K^{-1}, 0]$. Then
\[
\lim_{t \to 0} \int_X hv \leq - \log |\Gamma|h(p,0).
\]

Proof. The monotonicity of $\int_X v$ tells us that
\[
\int_X v \geq \int_X v|_{t=K^{-1}} \geq \mu(g(-K^{-1}), K^{-1}).
\]
whenever \( t \in [-K^{-1}, 0) \). The conditions \( \Box h \leq K \), \( v \leq 0 \) imply
\[
\frac{\partial}{\partial t} \left\{ \int_X hv \right\} = \int_X (v \Box h - h \Box^* v) \geq K \int_X v - \int_X h \Box^* v \geq C + 2\tau \int_X \left| R_{ij} + f_{,ij} - \frac{g_{ij}}{2\tau} \right|^2 uh
\]
where \( C = K\mu(g(-K^{-1}), K^{-1}), \tau = -t \). In other words,
\[
\frac{\partial}{\partial t} \{ C\tau + \int_X hv \} \geq 2\tau \int_X \left| R_{ij} + f_{,ij} - \frac{g_{ij}}{2\tau} \right|^2 uh \geq 0.
\]
By the same argument as in Proposition 3.5, \( C\tau + \int_X hv \) is uniformly bounded from above.
So the limit \( \lim_{\tau \to 0} \int_X hv = \lim_{\tau \to 0} C\tau + \int_X hv \) exists. There is a sequence \( \tau_k \to 0 \) such that
\[
2\tau_k^2 \int_X \left| R_{ij} + f_{,ij} - \frac{g_{ij}}{2\tau} \right|^2 \to 0.
\]
This yields that \( \lim_{\tau_k \to 0} \int_X (R + \Delta f - \frac{n}{2\tau_k})uh = 0 \). Note \( -\Delta h \leq K \), as in Proposition 3.5, we have
\[
\lim_{\tau_k \to 0} \int_X v h = \lim_{\tau_k \to 0} \left\{ \tau_k \int_X (R + \Delta f - \frac{n}{2\tau_k})uh - \tau_k \int_X u\Delta h + \int_X (f - \frac{n}{2})uh \right\}
\leq - \log |\Gamma|h(p, 0).
\]
\[\square\]

3.2 Proof of Pseudolocality Theorem

In this section, we fix \( \alpha = \frac{1}{10^{p_{,m}}} \).

Theorem 3.2 (Pseudolocality theorem). There exist \( \delta > 0, \epsilon > 0 \) with the following property. Suppose \( \{(X, g(t)), 0 \leq t \leq \epsilon^2 \} \) is an orbifold Ricci flow solution satisfying
- Isoperimetric constant close to Euclidean one: \( I(B(x, 1)) \geq (1 - \delta)I(\mathbb{R}^n) \)
- Scalar curvature bounded from below: \( R \geq -1 \) in \( B(x, 1) \).

under metric \( g(0) \). Then in the geometric parabolic ball \( \tilde{P}^+(x, 0, \epsilon, \epsilon^2) \), every point is smooth and \( |Rm| \leq \frac{\alpha}{t} + \epsilon^{-2} \).

Remark 3.1. The condition \( I(B(x, 1)) > (1 - \delta)I(\mathbb{R}^n) \) implies that there is no orbifold singularity in \( B(x, 1) \).

Proof. Define \( F(x, r) \triangleq \sup_{(y,t) \in \tilde{P}^+(x,0,r^2)} \{|\hat{R}m| - \frac{\alpha}{t} - r^{-2}\} \) where \( |\hat{R}m| \) is defined in Definition 2.5 Then the conclusion of the theorem is equivalent to \( F(x, \epsilon) \leq 0 \).

Suppose this theorem is wrong. For every \( (\delta, \eta) \in \mathbb{R}^+ \times \mathbb{R}^+ \), there are orbifold Ricci flow solutions violating the property. So we can take a sequence of positive numbers
$(\delta_i, \eta_i) \to (0, 0)$ and orbifold Ricci flow solutions $\{(X_i, x_i, g_i(t)), 0 \leq t \leq \eta_i^2\}$ satisfying the initial conditions but $F(x_i, \eta_i) > 0$.

Define $\epsilon_i$ to be the infimum of $r$ such that $F(x_i, r) > 0$. Since $x_i$ is a smooth point, we have $\eta_i > \epsilon_i > 0$. For every point $(z, t) \in \tilde{P}^+(x_i, 0, \epsilon_i, \epsilon_i^2)$, we have

$$|\tilde{Rm}|_{g_i(t)}(z) - \frac{\alpha}{t} - \epsilon_i^{-2} \leq |\tilde{Rm}|_{g_i(t_i)}(y_i) - \frac{\alpha}{t} - \epsilon_i^{-2} = 0$$

for some point $(y_i, t_i) \in \tilde{P}^+(x_i, 0, \epsilon_i, \epsilon_i^2)$.

Let $A_i = \alpha \epsilon_i^{-1} = \frac{1}{10 A \epsilon_i}$.

**Claim.** Every point in the geometric parabolic ball $\tilde{P}^+(x_i, 0, 4 A \epsilon_i, \epsilon_i^2)$ is a smooth point.

For convenience, we omit the subindex $i$. Suppose that $(p, s)$ is a singular point in $\tilde{P}^+(x, 0, \epsilon, \epsilon^2)$. Let

$$\eta(y, t) = \phi\left(\frac{d_{g(t)}(y, x) + 200 m \sqrt{t}}{10 A \epsilon}\right)$$

where $\phi$ is a cutoff function satisfying the following properties. It takes value one on $(-\infty, 1]$ and decreases to zero on $[1, 2]$. Moreover, $-\phi'' \leq 10 \phi, (\phi')^2 \leq 10 \phi$. Recall that

$$|\tilde{Rm}| \leq \frac{\alpha}{t} + \epsilon_i^{-2} \leq \frac{1 + \alpha}{t} < \frac{2}{t}$$

in the set $\tilde{P}^+(x, 0, \epsilon, \epsilon^2)$. In particular, every point in $B_{g(t)(x, \sqrt{t})}$ is smooth and satisfies $|Rm| < \frac{2}{t}$. This curvature estimate implies that (c.f. Lemma 8.3 (a) of [Pe1], it also holds in orbifold case.)

$$\Box d \geq -(m - 1)(\frac{2}{3} \cdot \frac{2}{t} \cdot \sqrt{\frac{t}{2}} + \sqrt{\frac{2}{t}}) > -4 m t^{-\frac{1}{2}},$$

where $d(\cdot) = d_{g(t)}(\cdot, x)$. Therefore, as calculated in [Pe1], we have

$$\Box \eta = \frac{1}{10 A \epsilon}(\Box d + 100 m t^{-\frac{1}{2}}) \phi' - \frac{1}{(10 A \epsilon)^2} \phi'' \leq \frac{10 \eta}{(10 A \epsilon)^2}.$$

Let $u$ be the fundamental solution of the backward heat equation $\Box^* u = 0$ and $u = \delta_p$ at point $(p, s)$. We can calculate

$$\frac{\partial}{\partial t} \int_X \eta u = \int_X (u \Box \eta - \eta \Box^* u) = \int_X u \Box \eta \leq \frac{1}{10 (A \epsilon)^2} \int_X u \eta \leq \frac{1}{10 (A \epsilon)^2} \int_X u = \frac{1}{10 (A \epsilon)^2}.$$

It follows that

$$\int_X \eta u \bigg|_{t=0} \geq \int_X \eta u \bigg|_{t=s} - \frac{s}{10 (A \epsilon)^2} \geq 1 - \frac{1}{10 A \epsilon^2}.$$
Similarly, if we let \( \bar{\eta}(y, t) = \phi(\frac{d_{g(t)}(y, x) + 200\epsilon\sqrt{t}}{5A}) \), we can obtain
\[
\int_{B(x, 10A\epsilon)} u \geq \int_X \bar{\eta}u \bigg|_{t=0} \geq 1 - \frac{10}{(5A)^2}.
\]
It forces that
\[
\int_{B(x, 20A\epsilon) \setminus B(x, 10A\epsilon)} \eta u \bigg|_{t=0} \leq 1 - (1 - \frac{10}{(5A)^2}) < A^{-2}.
\]
On the other hand, we have
\[
\frac{\partial}{\partial t} \int_X -\eta v = \int_X (-v\Box\eta + \eta\Box^* v)
\leq \int_X -v\Box\eta
\leq \frac{1}{10(4\epsilon)^2} \int_X -\eta v,
\]
where we used the fact \(-v \geq 0\) and \(\Box\eta \leq \frac{\eta}{10(4\epsilon)^2}\). This inequality together with Theorem 3.1 implies
\[
\int_X -\eta v \bigg|_{t=0} \geq e^{-\frac{\epsilon}{10(4\epsilon)^2}} \int_X -\eta v \bigg|_{t=s} \geq \log |\Gamma| \eta(x, s) e^{-\frac{\epsilon}{10(4\epsilon)^2}} > \frac{1}{2} \log |\Gamma| \geq \frac{1}{2} \log 2.
\]
Let \(\bar{u} = u\eta\) and \(\bar{f} = f - \log \eta\). At \(t = 0\), as in [Pe1], we can compute
\[
\frac{1}{2} \log 2 \leq -\int_X v\eta = \int_X \{(-2\Delta f + |\nabla f|^2 - R)s - f + m\} \eta u
= \int_X \{-s|\nabla \bar{f}|^2 - \bar{f} + m\} \bar{u} + \int_X \{s(\frac{|\nabla \eta|^2}{\eta} - R\eta) - \eta \log \eta\} u
\leq 10A^{-1} + 100\epsilon^2 + \int_X \{-s|\nabla \bar{f}|^2 - \bar{f} - m\} \bar{u}
\]
After rescaling \(s\) to be \(\frac{1}{2}\), we obtain
\[
\begin{align*}
\left\{\begin{array}{l}
\int_{B(x, 20A\epsilon)} \{\frac{1}{2}|\nabla \bar{f}|^2 + \bar{f} - m\} < -\frac{1}{2} \log 2.
1 - A^{-2} < \int_{B(x, 20A\epsilon)} \bar{u} \leq 1.
\end{array}\right.
\end{align*}
\]
This contradicts to the fact that \(B(x, 20A\epsilon) \subset B(x, \frac{1}{\sqrt{2\epsilon}})\) has almost Euclidean isoperimetric constant (c.f. Proposition 3.1 of [Ni2] for more details). So we finish the proof of the claim.

Now we can do as Perelman did in Claim 1 of the proof of Theorem 10.1 of [Pe1]. We can find a point \((\bar{x}, \bar{t})\) such that
\[
|Rm|_{g(\bar{t})}(z) \leq 4|Rm|_{g(\bar{t})}(\bar{x})
\]
whenever
\[(z,t) \in X_\alpha, \quad 0 < t \leq \bar{t}, \quad d_{g(t)}(z,x) \leq d_{g(\bar{t})}(\bar{x}, x) + A|\tilde{R}m|_{g(\bar{t})}(\bar{x})^{-\frac{1}{2}}\]
where \(X_\alpha\) is the set of pairs \((z,t)\) satisfying \(|\tilde{R}m|_{g(t)}(z) \geq \frac{\alpha}{2}\). Moreover, we also have \(d_{g(\bar{t})}(\bar{x}, x) < (2A + 1)\varepsilon\). Therefore, the geometric parabolic ball \(\tilde{P}^+(\bar{x}, 0, A\varepsilon, \bar{t})\) is strictly contained in the geometric parabolic ball \(\tilde{P}^+(x, 0, 4A\varepsilon, \varepsilon^2)\). Therefore, every point around \((\bar{x}, \bar{t})\) is smooth. We can replace \(|\tilde{R}m|\) by \(|Rm|\) and all the arguments of Perelman’s proof in [Pe1] apply directly. For simplicity, we only sketch the basic steps.

\[P^- (\bar{x}, \bar{t}, \frac{1}{10}AQ^{-\frac{1}{2}}, \frac{1}{4}aQ^{-1})\] is a parabolic ball satisfying \(|Rm| \leq 4Q = 4|Rm|_{g(\bar{t})}(\bar{x})\), every point in it is smooth. Then by blowup argument, we can show that there is a time \(\bar{t} \in [\bar{t} - \frac{1}{2}aQ^{-1}, \bar{t}]\), such that \(\int_{B_{g(\bar{t})}(\bar{x}, \sqrt{\bar{t} - \bar{t}})} v < -c_0\) for some positive constant \(c_0\), where \(v\) is the auxiliary function related to the fundamental solution \(u = (4\pi(\bar{t} - t))^{-\frac{1}{2}}e^{-f}\) of conjugate heat equation, starting from \(\delta\)-functions at \((\bar{x}, \bar{t})\). Under the help of cutoff functions, we can construct a function \(\tilde{f}\) satisfying
\[
\begin{align*}
&\int_{B(x, \frac{20A}{\sqrt{2}l})} \frac{1}{2} (|\nabla \tilde{f}|^2 + \tilde{f} - m) < -\frac{1}{2}c_0, \\
&1 - A^{-2} < \int_{B(x, \frac{20A}{\sqrt{2}l})} \tilde{u} \leq 1.
\end{align*}
\]
under the metric \(\frac{1}{2}g(0)\). Since \(B(x, \frac{20A}{\sqrt{2}l}) \subset B(x, \frac{1}{\sqrt{2}l})\) has almost Euclidean isoperimetric constant as \(\varepsilon \to 0, A \to \infty\), we know these inequalities cannot hold simultaneously! \(\Box\)

**Proposition 3.6.** Let \(\{(X, g(t)), 0 \leq t \leq 1\}\), \(x, \delta, \varepsilon\) be the same as in the previous theorem. If in addition, \(|Rm| < 1\) in the ball \(B(x, 1)\) at time \(t = 0\), then
\[|Rm|_{g(t)}(y) < (\alpha\varepsilon)^{-2}\]
whenever \(0 < t < (\alpha\varepsilon)^2\), \(\text{dist}_{g(t)}(y, x) < \alpha\).

**Proof.** Suppose not. There is a point \((y_0, t_0)\) satisfying
\[|Rm|_{g(t_0)}(y_0) \geq (\alpha\varepsilon)^{-2}, \quad 0 < t < (\alpha\varepsilon)^2, \quad d_{g(t_0)}(y_0, x) < \alpha.\]
Check if \(Q = |Rm|_{g(t_0)}(y_0)\) can control \(|Rm|\) of “previous and outside” points. In other words, check if the following property \(\bullet\) is satisfied.

\[\bullet:\quad |Rm|_{g(t)}(z) \leq 4Q, \quad \forall 0 \leq t \leq t_0, \quad d_{g(t)}(z, x) \leq d_{g(t_0)}(y_0, x) + Q^{-\frac{1}{2}}.\]
If not, there is a point \((z, s)\) such that
\[|Rm|_{g(s)}(z) > 4Q, \quad 0 < s \leq t_0, \quad d_{g(s)}(z, x) \leq d_{g(t_0)}(y_0, x) + Q^{-\frac{1}{2}}.\]
Then we denote \((z, s)\) as \((y_1, t_1)\) and check if the property \(\bullet\) is satisfied at this new base point. Now matter how many steps this process are performed, the base point \((y_k, t_k)\) satisfies
\[0 < t_k \leq t_0 < (\alpha\varepsilon)^2, \quad d_{g(t_k)}(y_k, x) < d_{g(t_0)}(y_0, x) + \sum_{l=0}^{k-1} 2^{-l}Q^{-\frac{1}{2}} < d_{g(t_0)}(y_0, x) + 2Q^{-\frac{1}{2}} < 3\alpha\varepsilon < \varepsilon.\]

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Namely, \((y_k, t_k)\) will never escape the compact set
\[
\Omega = \{(z, s) | 0 \leq s \leq (\alpha e)^2, \ d_{g(s)}(z, x) < 3\alpha\}
\]
which has bounded \(|Rm|\). During each step, \(|Rm|\) doubles at least. Therefore, this process must terminate in finite steps and property \(\clubsuit\) will finally hold. Without loss of generality, we can assume property \(\clubsuit\) holds already at the point \((y_0, t_0) \in \Omega\). Define
\[
P = \{(z, s) | 0 \leq s \leq t_0, \ d_{g(s)}(z, y_0) < Q^{-\frac{1}{2}}\}.
\]
Triangle inequality and property \(\clubsuit\) implies \(|Rm| \leq 4Q\) holds in \(P\). Let \(\hat{g}(t) = 4Qg(\frac{t}{4Q})\), we have
\[
P = \{(z, s) | 0 \leq s \leq 4Qt_0, \ d_{\hat{g}(s)}(z, y_0) < 2\}.
\]
From now on, we do all the calculation under the metric \(\hat{g}(t)\).

Define \(\eta(z, t) = 5\phi(d(z, y_0) + 100mt)\) where \(\phi\) is the same cutoff function as before. It equals 1 on \((-\infty, 0]\) and decreases to zero on \([1, 2]\). It satisfies \(-\phi'' \leq 10\phi\), \((\phi')^2 \leq 10\phi\). In \(P\), we calculate
\[
|\nabla \eta|^2 = 25(\phi')^2|\nabla d|^2 = 25(\phi')^2 \leq 250\phi = 50\eta,
\]
\[
\Box \eta = 5(\Box d + 100m)\phi' - 5\phi'' \leq -5\phi'' \leq 10\eta,
\]
\[
\Box \eta^{-4} = -4\eta^{-5}\Box \eta - 20\eta^{-6}|\nabla \eta|^2 \geq -40\eta^{-4} - 1000\eta^{-5} = (-40\eta^2 - 100\eta)(\eta^{-6}) \geq -6000(\eta^{-4})^\frac{3}{2}.
\]
On the other hand, in \(P\), we have
\[
\Box \{|Rm|^2(1 - \frac{t}{32\alpha})\} \leq 16|Rm|^3(1 - \frac{t}{32\alpha}) - \frac{1}{32\alpha}|Rm|^2 \\
\leq (16 - \frac{1+16t}{32\alpha})|Rm|^2 \\
\leq (16 - \frac{1+16t}{32\alpha})|Rm|^3 \\
\leq (16 - \frac{1+16t}{32\alpha})\{|Rm|^2(1 - \frac{t}{32\alpha})\}^\frac{3}{2} \\
\leq -6000\{|Rm|^2(1 - \frac{t}{32\alpha})\}^\frac{3}{2}.
\]
In these inequalities, we used the fact that \(|Rm| \leq 1\), \(16 - \frac{1+16t}{32\alpha} < -6000 < 0\) and \(1 - \frac{t}{32\alpha} > 0\) in \(P\). \(|Rm| \leq 1\) is guaranteed by the choice of \(P\). Recall \(\alpha = \frac{1}{10}\), so \(16 - \frac{1+16t}{32\alpha} < -6000\) is obvious. To prove \(1 - \frac{t}{32\alpha} > 0\), we note that \(Q = |Rm|_{g(t_0)}(y_0) < \frac{\alpha}{t_0} + \epsilon^2\), so we have
\[
Qt_0 < \alpha + t_0\epsilon^2 < \alpha(1 + \alpha), \quad 1 - \frac{t}{32\alpha} \geq 1 - \frac{4Qt_0}{32\alpha} > 1 - \frac{1+\alpha}{8} > 0.
\]
It follows that
\[
\Box \{|Rm|^2(1 - \frac{t}{32\alpha})\} < -6000\{|Rm|^2(1 - \frac{t}{32\alpha})\}^\frac{3}{2}
\]
in $P$. Therefore, $\eta^{-4}$ is a super solution of $\square F = -600F^{\frac{4}{3}}$, $|\text{Rm}|^2(1 - \frac{t}{32\alpha})$ is a sub solution of this equation. Moreover,

$$|\text{Rm}|^2(1 - \frac{t}{32\alpha}) < \frac{1}{4Q} \leq \frac{1}{4}(\alpha\epsilon)^2 < \frac{1}{625} < \eta^{-4}, \text{ whenever } t = 0.$$  

$$|\text{Rm}|^2(1 - \frac{t}{32\alpha}) < \infty = \eta^{-4}, \text{ whenever } d(z, y) = 2.$$  

Therefore, for every point in $P$, $|\text{Rm}|^2(1 - \frac{t}{32\alpha})$ is controlled by $\eta^{-4}$. In particular, under metric $\tilde{g}$, at point $(y_0, 4Qt_0)$, we have

$$|\text{Rm}|^2(y_0)(1 - \frac{4Qt_0}{32\alpha}) \leq \eta(y_0, 4Qt_0)^{-4} = \{5\phi(400mQt_0)\}^{-4} \leq 5^{-4}\{\phi(400m(1 + \alpha))\}^{-4} \leq 5^{-4}\{\phi(1)\}^{-4} = \frac{1}{625}$$

On the other hand, recall that $\alpha = \frac{1}{10^m}$, we have

$$|\text{Rm}|^2(y_0)(1 - \frac{4Qt_0}{32\alpha}) = \frac{1}{16}(1 - \frac{Qt_0}{8\alpha}) > \frac{1}{16}(1 - \frac{1 + \alpha}{8}) > \frac{1}{32}.$$  

It follows that $\frac{1}{32} < \frac{1}{625}$. Contradiction! 

As a corollary of this proposition, we can obtain the improved Pseudolocality theorem.

**Theorem 3.3 (Improved Pseudolocality Theorem).** There exists $\eta = \eta(m, \kappa) > 0$ with the following property.

Suppose $\{\{(X, g(t)), 0 \leq t \leq r_0^2\} \}$ is a compact orbifold Ricci flow solution. Assume that at $t = 0$ we have $|\text{Rm}|(x) \leq r_0^2$ in $B(x, r_0)$, and $\text{Vol} B(x, r_0) \geq \kappa r_0^m$. Then the estimate $|\text{Rm}|_{g(t)}(x) \leq (\eta r_0)^{-2}$ holds whenever $0 \leq t \leq (\eta r_0)^2$, $d_{g(t)}(x, x) < \eta r_0$.

**Remark 3.2.** Suppose $c_0 \geq -c$ is a constant, then the “normalized flow” $\frac{\partial g}{\partial t} = -\text{Ric} + c_0g$ is just a parabolic rescaling of the flow $\frac{\partial g}{\partial t} = -2\text{Ric}$. So Theorem 3.3 also hold for “normalized” Ricci flow solutions $\frac{\partial g}{\partial t} = -\text{Ric} + c_0g$. However, the constant $\eta$ will also depend on $c$ then.

## 4 Weak Compactness Theorem

In this section, $\kappa, E$ are fixed constants. $\kappa, \xi$ are small constants depending only on $\kappa$ and $m$ by Definition 4.1 and Definition 4.2.
4.1 Choice of Constants

**Proposition 4.1** (Bando, [Ban90]). There exists a constant $h_a = h_a(m, \kappa)$ such that the following property holds.

If $X$ is a $\kappa$-noncollapsed, Ricci-flat ALE orbifold, it has unique singularity and small energy, i.e., $\int_X |\nabla Rm|^2 d\mu < h_a$, then $X$ is a flat cone.

**Proposition 4.2.** Suppose $B(p, \rho)$ is a smooth, Ricci-flat, $\kappa$-noncollapsed geodesic ball and $\partial B(p, \rho) \neq \emptyset$. Then there is a small constant $h_b = h_b(m, \kappa) < \left(\frac{1}{2\kappa}ight)^{\frac{m}{2}}$ such that

$$\sup_{B(p, \frac{\rho}{2})} |\nabla^k Rm| \leq \frac{C_k}{\rho^{2+k}} \left\{ \int_{B(p, \rho)} |Rm|^\frac{m}{2} d\mu \right\}^{\frac{2}{m}}$$  \hspace{1cm} (5)

whenever $\int_{B(p, \rho)} |Rm|^\frac{m}{2} d\mu < h_b$. Here $C_k$ are constants depending only on the dimension $m$. In particular, $B(p, \rho)$ satisfies energy concentration property. In other words, if $|Rm|(p) \geq \frac{1}{2\rho^2}$, then we have

$$\int_{B(p, \frac{\rho}{2})} |Rm|^\frac{m}{2} d\mu > h_b.$$

**Definition 4.1.** Let $h \triangleq \min\{h_a, h_b\}$.

**Proposition 4.3.** There is a small constant $\xi_a(\kappa, m)$ such that the following properties hold. Suppose that $\{(X, g(t), 0 \leq t \leq 1)\}$ is a Ricci flow solution on a compact orbifold $X$ which is $\kappa$-noncollapsed. $\Omega \subset X$ and $|Rm|_{g(0)}(x) \leq \xi_a^{-\frac{3}{2}}$ for every point $x \in \Omega$. Then we have

$$|Rm|_{g(t)}(x) \leq \frac{1}{10000m^2} \xi_a^{-2}, \quad \forall \, x \in \Omega', \, t \in [0, 9\xi_a^2].$$

where $\Omega' = \{y \in \Omega | d_{g(0)}(y, \partial\Omega) > \xi_a^\frac{3}{2}\}$.

**Proof.** Suppose not. There are a sequence of $\{(X_i, g_i(t)), 0 \leq t \leq 1\}$, $x_i$, $\Omega'$, $\Omega_i$, and $\xi_i \to 0$ violating the statement.

Blowup them by scale $\xi_i^{-\frac{3}{2}}$, let $\tilde{g}_i(t) = g_i(\xi_i^{-\frac{3}{2}} t)$. We can choose a sequence of points $y_i \in \Omega_i'$, $t_i \in [0, 9\xi_i^2]$ satisfying

$$|Rm|_{\tilde{g}_i(t_i)}(y_i) \geq \frac{1}{10000m^2} \xi_i^{-\frac{1}{2}} \to \infty.$$  \hspace{1cm} (6)

Note that under metric $\tilde{g}_i(0)$, $|Rm| \leq 1$ in $B(y_i, 1)$, so inequality (6) contradicts the improved pseudolocality theorem! \hfill \Box

**Proposition 4.4.** Suppose $X$ is an orbifold which is $\kappa$-noncollapsed on scale 1, $|Rm| \leq 1$ in the smooth geodesic ball $B(x, 1) \subset X$. Then there is a small constant $\xi_b$ such that

$$\frac{\text{Vol}(B(y, r))}{r^m} > \frac{7}{8} \omega(m)$$

whenever $y \in B(x, \frac{1}{2})$ and $r < \xi_b^\frac{2}{3}$.  

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Definition 4.2. Define $\xi = \min\{\xi_a, \xi_b, (10000m^2E^2\pi)^{-1}\}$.

4.2 Refined Sequences

The main theorems of this section are almost the same as that of [CW3].

Definition 4.3. Define $O(m, c, \sigma, \kappa, E)$ as the moduli space of compact orbifold Ricci flow solutions $\{(X, g(t)), -1 \leq t \leq 1\}$ satisfying:

1. $\frac{\partial}{\partial t} g(t) = -\text{Ric}_g(t) + c_0 g(t)$ where $c_0$ is a constant satisfying $|c_0| \leq c$.
2. $|R|_{L^\infty(X \times [-1, 1])} \leq \sigma$.
3. $\frac{\text{Vol}_g(t)(B_{g(t)}(x, r))}{r^m} \geq \kappa$ for all $x \in X, t \in [-1, 1], r \in (0, 1]$.
4. $\{\#(\text{Sing}(X))\} \cdot h + \int_X |R_m|_{g(t)}d\mu_g(t) \leq E$ for all $t \in [-1, 1]$.

Clearly, in order this moduli space be really a generalization of the $\mathcal{M}(m, c, \sigma, \kappa, E)$ defined in [CW3], we need $m$ to be an even number.

We want to show the weak compactness and uniform isoperimetric constant bound of $O(m, c, \sigma, \kappa, E)$. As in [CW3], we use refined sequence as a tool to study $O(m, c, \sigma, \kappa, E)$. After we obtain the weak compactness theorem of refined sequence, the properties of $O(m, c, \sigma, \kappa, E)$ follows from routine blowup and bubble tree arguments. However, we’ll give a simpler proof of the weak compactness theorem of refined sequences.

As in [CW3], we define Refined sequence.

Definition 4.4. Let $\{(X_i^m, g_i(t)), -1 \leq t \leq 1\}$ be a sequence of Ricci flows on closed orbifolds $X_i^m$. It is called a refined sequence if the following properties are satisfied for every $i$.

1. $\frac{\partial}{\partial t} g_i = -\text{Ric}_{g_i} + c_0 g_i$ and $\lim_{i \to \infty} c_i = 0$.
2. Scalar curvature norm tends to zero: $\lim_{i \to \infty} \|R\|_{L^\infty(X_i \times [-1, 1])} = 0$.
3. For every radius $r$, there exists $N(r)$ such that $(X_i, g_i(t))$ is $\kappa$-noncollapsed on scale $r$ for every $t \in [-1, 1]$ whenever $i > N(r)$.
4. Energy uniformly bounded by $E$:

$$\{\#(\text{Sing}(X_i))\} \cdot h + \int_{X_i} |R_m|_{g_i(t)}d\mu_{g_i(t)} \leq E, \quad \forall \ t \in [-1, 1].$$

In order to show the weak compactness theorem for every refined sequence, we need two auxiliary concepts.
Definition 4.5. A refined sequence \( \{(X_i, g_i(t)), -1 \leq t \leq 1\} \) is called an E-refined sequence under constraint \( H \) if under metric \( g_i(t) \), we have
\[
\#\text{Sing}(B(x_0, Q^{-\frac{1}{2}})) \cdot h + \int_{B(x_0, Q^{-\frac{1}{2}})} |Rm| d\mu \geq h, \quad \forall t \in [t_0, t_0 + \xi^2 Q^{-1}],
\]
whenever \((x_0, t_0) \in X_i \times [-\frac{1}{2}, \frac{1}{2}]\) and \( Q = |Rm|_{g_i(t_0)}(x_0) \geq H \).

Definition 4.6. An E-refined sequence \( \{(X_i, g_i(t)), -1 \leq t \leq 1\} \) under constraint \( H \) is called an EV-refined sequence under constraint \((H, K)\) if under metric \( g_i(t) \), we have
\[
\frac{\text{Vol} B(x, r)}{r^m} \leq K
\]
for every \( i \) and \((x, t) \in X_i \times [-\frac{1}{4}, \frac{1}{4}]\), \( r \in (0, 1] \).

When meaning is clear, we omit the constraint when we mention E-refined and EV-refined sequences. Clearly, an E-refined sequence is a refined sequence whose center-part-solutions satisfy energy concentration property, an EV-refined sequence is an E-refined sequence whose center-part-solutions have bounded volume ratios. For convenience, we also call a pointed normalized Ricci flow sequence \( \{(X_i, x_i, g_i(t)), -1 \leq t \leq 1\} \) a \((E-, EV-)\)-refined sequence if \( \{(X_i, g_i(t)), -1 \leq t \leq 1\} \) is a \((E-, EV-)\)-refined sequence. Since volume ratio, energy are scaling invariants, blowing up a \((E-, EV-)\)-refined sequence at proper points generates a new \((E-, EV-)\)-refined sequence with smaller constraints.

Remark 4.1. The definition of refined sequence is the same as in [CW3]. However, the definition of \( E-, EV- \)-refined sequence here is a slight different. This is for the convenience of a simplified proof of the weak compactness theorem of refined sequences.

We first prove the weak compactness of EV-refined sequence.

Proposition 4.5 \((C^{1, \frac{1}{2}})-\text{Weak Compactness of EV-refined Sequence})\). Suppose that \( \{(X_i, x_i, g_i(t)), -1 \leq t \leq 1\} \) is an EV-refined sequence, we have
\[
(X_i, x_i, g_i(0)) \overset{C^{1, \frac{1}{2}}}{\longrightarrow} (X_\infty, x_\infty, g_\infty)
\]
where \( X_\infty \) is a Ricci flat ALE orbifold.

Proof. As volume ratio upper bound and energy concentration holds, it is not hard to see that
\[
(X_i, x_i, g_i(0)) \overset{C^{1, \frac{1}{2}}}{\longrightarrow} (X_\infty, x_\infty, g_\infty)
\]
for some limit metric space \( X_\infty \), which has finite singularity and it’s regular part is a \( C^{1, \frac{1}{2}} \) manifold. Moreover, by the improved pseudolocality theorem and the almost scalar flat property of the limit sequence, every smooth open set of \( X_\infty \) is isometric to an open set.
of a time slice of a scalar flat, hence Ricci flat Ricci flow solution. In other words, every open set of the smooth part of $X_\infty$ is Ricci flat. It’s not hard to see that

$$\sharp \text{Sing}(X_\infty) \cdot h + \int_{X_\infty} |Rm|^{\frac{m}{2}} d\mu \leq E.$$  

This energy bound forces that the tangent space of every singular point to be a flat cone, but maybe with more than one ends. Also, the tangent cone at infinity is a flat cone (c.f. [BKN], [An90], [Tian90]). In other words, $X_\infty$ is a Ricci flat, smooth, ALE multifold with finite energy. We need to show that this limit is an orbifold, i.e., for every $p \in X_\infty$, the tangent space of $p$ is a flat cone with a unique end. This can be done through the following two steps.

**Step 1.** Every singular point of $X_\infty$ cannot sit on a smooth component. In other words, suppose $p$ is a singular point of $X_\infty$, then there exists $\delta_0$ depending on $p$ such that every component of $\partial B(p, \delta)$ has nontrivial $\pi_1$ whenever $\delta < \delta_0$.

If this statement is wrong, we can choose $\delta_i \to 0$ such that $|\partial E_i| > \frac{7}{8} m \omega(m) \delta_i^{m-1}$ where $E_i$ is a component of $\partial B(p_i, \delta_i)$. By taking subsequence if necessary, we can choose $X_i \ni p_i \to p$ satisfying $|\partial E_i| > \frac{7}{8} m \omega(m) \delta_i^{m-1}$ where $E_i^i$ is some component of $\partial B(p_i, \delta_i)$. Moreover, we can let $p_i$ be the point with largest Riemannian curvature in $B(p_i, \rho)$ for some fixed small number $\rho$. Define

- $r_i \triangleq \sup \{ r | r < \delta_i, \text{ the largest component of } \partial B(p_i, r) \text{ has area ratio } \leq \frac{7}{8} m \omega(m) \}$
- $r_i' \triangleq \inf \{ r | \text{ the ball } B(p_i, r) \text{ has volume ratio } \leq \frac{3}{4} \omega(m) \}$.

We claim that $r_i' \leq C Q_i^{-\frac{1}{2}}$ where $Q_i = |Rm|(p_i)$ and $C$ is a uniform constant. Otherwise, by rescaling $Q_i$ to be 1 and fixing the central time slice to be time 0, we can take limit for a new EV-refined sequence

$$\{(X_i, p_i, \tilde{g}_i(t)), 0 \leq t \leq \xi^2 \} \overset{C^{1, \frac{1}{2}}}{\to} \{(\tilde{X}_\infty, p_\infty, \tilde{g}_\infty(t)), 0 \leq t \leq \xi^2 \},$$

where the limit is a stable (Ricci flat) Ricci flow solution on a complete manifold $\tilde{X}_\infty$. Moreover, the convergence is smooth when $t > 0$. Therefore, $(\tilde{X}_\infty, p_\infty, \tilde{g}_\infty(0))$ is isometric to $(\tilde{X}_\infty, p_\infty, \tilde{g}_\infty(\xi^2))$. This forces that $(\tilde{X}_\infty, p_\infty, \tilde{g}_\infty(\xi^2))$ satisfies

$$h \leq \int_{X_\infty} |Rm|^{\frac{m}{2}} d\mu \leq E, \quad \lim_{r \to \infty} \frac{\text{Vol}(B(p_\infty, r))}{r^m} \geq \frac{3}{4} \omega(m).$$

simultaneously. This is impossible! Therefore, $r_i' \leq C Q_i^{-\frac{1}{2}} \to 0$. This estimate of $r_i'$ implies that $r_i$ is well defined. Moreover, similar blowup argument shows that $\lim_{i \to \infty} r_i = \infty$.
Clearly, $r_i < \delta_i \to 0$. Rescale $r_i$ to be 1 to obtain a new EV-refined sequence
\[
\{(X_i^{(1)}, x_i^{(1)}, g_i^{(1)}(t)), -1 \leq t \leq 1\}
\]
where $x_i^{(1)} = p_i$. We have convergence
\[
(X_i^{(1)}, x_i^{(1)}, g_i^{(1)}(0)) \overset{C^{1,1}}{\to} (X_\infty^{(1)}, x_\infty^{(1)}, g_\infty^{(1)}).
\]
By our choice of $r_i$, for every $r > 1$, there is a component of $\partial B(x_\infty^{(1)}, r)$ whose area is at least $\frac{7}{8}m\omega(m)r^{m-1}$. Therefore, the ALE space $X_\infty^{(1)}$ has an end whose volume growth is greater than $\frac{7}{8}\omega(m)r^{m} > \frac{1}{2}\omega(m)r^{m}$. Detach $X_\infty$ as union of orbifolds. One of them must be ALE space whose volume growth at infinity is exactly $\omega(m)r^{m}$, the Ricci flatness forces that this ALE component is isometric to Euclidean space. Since one component of $\partial B(x_\infty^{(1)}, 1)$ has volume $\frac{7}{8}m\omega(m)$, $X_\infty^{(1)}$ itself cannot be Euclidean. Therefore, $X_\infty^{(1)}$ must contain a singular point which connects a Euclidean space. In other words, $X_\infty^{(1)}$ contains a singular point $q$ which sit in a smooth component.

If $X_\infty^{(1)}$ has more than one singularity, we can blowup at point $q$ as before and obtain a new bubble $X_\infty^{(2)}$. However, a fixed amount of energy (at least $h$) will be lost during this process. Therefore such process must stop in finite times. Without loss of generality, we can assume that $X_\infty^{(1)}$ has a unique singularity $q$. By the choice of $p_i$, $x_\infty^{(1)} = \lim_{i \to \infty} p_i$ must be singular if $X_\infty^{(1)}$ contains a singular point. It follows that $X_\infty^{(1)}$ has a unique singularity $x_\infty^{(1)}$. From the previous argument, we already know that $x_\infty^{(1)} = q$ connects a Euclidean space. Since $x_\infty^{(1)}$ is the unique singularity, every geodesic $\gamma$ connecting $x_\infty^{(1)}$ and some point $x$ in the Euclidean space must stay in that Euclidean space. Therefore, $\partial B(x_\infty^{(1)}, 1)$ has a component which is a standard sphere whose area is $m\omega(m) > \frac{7}{8}m\omega(m)$. So for large $i$, the largest component of $\partial B(p_i, r_i)$ has area strictly greater than $\frac{7}{8}m\omega(m)r_i^{m-1}$. This contradicts to the choice of $r_i$!

**Step 2.** Every singular point of $X_\infty$ has only one end. In other words, suppose $p$ is a singular point of $X_\infty$, then there exists $\delta_0$ depending on $p$ such that $\partial B(p, \delta)$ is connected whenever $\delta < \delta_0$.

Suppose not, there is a small $\delta$ such that $\partial B(p, \delta)$ is not connected. Choose $x, y$ in two different components of $\partial B(p, \delta)$. Let $\gamma$ be the shortest geodesic connecting $x$ and $y$. It must pass through $p$. Suppose $x_i, y_i, p_i \in X_i$, $\gamma_i \subset X_i$ satisfy
\[
x_i \to x, \quad y_i \to y, \quad p_i \to p, \quad \gamma_i \to \gamma
\]
where $\gamma_i$ is the shortest geodesic connecting $x_i, y_i$.

For every $z \in X_i$, we can define
\[
\mathcal{R}(z) \triangleq \sup\{r(|\text{Sing}(B(z, r))| \cdot h + \int_{B(z, r)} |Rm| d\mu \geq \frac{1}{2} h}\}
\]
under the metric $g_i(0)$. Clearly, $\mathcal{R}(z) = 0$ iff $z$ is singular. On $\gamma_i$, let $q_i$ be the point with the smallest $\mathcal{R}$ value and define $r_i = \mathcal{R}(q_i)$. Note that on orbifold $X_i$, every shortest geodesic
connecting two smooth points never pass through orbifold singularity. This implies that \( r_i = \| q_i \|_2 > 0 \). Clearly, \( r_i \rightarrow 0 \). Now, we rescale \( r_i \) to be 1 to obtain new EV-refined sequence \( \{(X_i, q_i, g_i^{(1)}(t)), -1 \leq t \leq 1\} \) and take limit

\[
(X_i, q_i, g_i^{(1)}(0)) \xrightarrow{C^{1,1}} (X_\infty, q_\infty, g_\infty^{(1)}).
\]

Clearly, \( X_\infty^{(1)} \) contains a straight line passing through \( q_\infty \) which we denote as \( \gamma_\infty \). After rescaling, every unit geodesic ball centered on a point of \( \gamma_i \) contains energy not more than \( \frac{1}{2} \). The energy concentration property forces that \( |Rm| \) is uniformly bounded around \( \gamma_i \). So \( \gamma_\infty \) is a straight line free of singular point. Detach \( X_\infty^{(1)} \) as union of orbifolds. Then \( \gamma_\infty \) must stay in one orbifold component. Therefore, there is an orbifold component containing a straight line. Then the splitting theorem for Ricci flat orbifolds applies and forces that component must be Euclidean space. Since \( X_\infty^{(1)} \) contains a Euclidean component. From Step 1, we know every singularity cannot stay on smooth component. Therefore, \( X_\infty^{(1)} \) itself must be the Euclidean space. So we actually have convergence

\[
\{(X_i, q_i, g_i^{(1)}(t)), 0 < t \leq \xi^2\} \xrightarrow{C^{\infty}} \{(X_\infty, q_\infty, g_\infty^{(1)}(t)), 0 < t \leq \xi^2\}.
\]

This forces that \( (X_\infty, q_\infty, g_\infty^{(1)}(t)) \) is Euclidean space for every \( t \in [0, \xi^2] \).

Now return to the choice of \( q_i \)

\[
\{\#Sing(B(q_i, r_i))\} \cdot \h + \int_{B(q_i, r_i)} |Rm| \frac{\omega}{m} d\mu = \frac{1}{2} \h.
\]

actually reads \( \int_{B(q_i, r_i)} |Rm| \frac{\omega}{m} d\mu = \frac{1}{2} \h \). There is a point \( q_i' \in B(q_i, r_i) \) satisfying

\[
Q_i' \triangleq |Rm|(q_i') > \left( \frac{\h}{2 \Vol(B(q_i, 2r_i))} \right)^{\frac{2}{m}} > \left( \frac{\h}{4 \omega(m)(2r_i)^m} \right)^{\frac{2}{m}} > \left( \frac{\h}{4^m \omega(m)} \right)^{\frac{2}{m}} r_i^{-2} \rightarrow \infty.
\]

On the other hand, the no-singularity-property of \( X_\infty^{(1)} \) implies \( Q_i' < C r_i^{-2} \) for some uniform constant \( C \). Therefore, we have

\[
\delta^2 \triangleq \left( \frac{\h}{4^m \omega(m)} \right)^{\frac{2}{m}} < Q_i'^2 \rightarrow C r_i^{-2} < \infty.
\]

In particular, \( Q_i' \rightarrow \infty \) and therefore the energy concentration property applies on \( q_i' \):

\[
\int_{B_{q_i'}(t) \cdot \| q_i' \|_2^{-\frac{1}{2}}} |Rm| \frac{\omega}{m} d\mu \geq \h, \quad \forall \ 0 \leq t \leq \xi^2(Q_i')^{-1}.
\]

Combining this with inequality (10) implies

\[
\int_{B_{q_i'}(t) \cdot \delta^{-1} r_i} |Rm| \frac{\omega}{m} d\mu \geq \h, \quad \forall \ 0 \leq t \leq \frac{\xi^2}{C r_i^2}.
\]
After rescaling, we have

$$\int_{B_{g_i(t_1)}(x, \delta^{-1})} |Rm|^{\frac{m}{r}} \, d\mu \geq h, \quad \ell = \frac{\xi^2}{C}. $$

The smooth convergence implies that the energy of \((X_\infty, g_\infty^{(1)}(\bar{t}))\) is not less than \(h\). This contradicts to the property that \((X_\infty, g_\infty^{(1)}(\bar{t}))\) is a Euclidean space!

Therefore, every singular point of \(X_\infty\) has a unique nontrivial end, i.e., it has tangent space \(\mathbb{R}^n/\Gamma\) for some nontrivial \(\Gamma\). So \(X_\infty\) is an orbifold.

**Proposition 4.6.** Every \(E\)-refined sequence is an \(EV\)-refined sequence.

The next thing we need to do is to improve the convergence topology from \(C^{1, \frac{1}{2}}\) to \(C^\infty\). In light of Shi’s estimate, the following backward pseudolocality property assures this improvement.

**Proposition 4.7.** Suppose \(\{(X_i, x_i, g_i(t)), -1 \leq t \leq 1\}\) is an \(E\)-refined sequence satisfying \(|Rm|_{g_i(0)}(x) \leq r^{-2}\) in \(B_{g_i(0)}(x, r)\) for some \(r \in (0, 1)\).

Then there is a uniform constant \(C\) depending on this sequence such that

$$|Rm|_{g_i(t)}(y) \leq C, \text{ whenever } (y, t) \in P^-(x_i, 0, \frac{1}{2} r, 9\xi^2 r^2).$$

**Proof.** Without loss of generality, we can assume \(r = 1\).

Suppose this statement is wrong, there are points \((y_i, t_i) \in P^-(x_i, 0, \frac{1}{2}, 9\xi^2)\) satisfying \(|Rm|_{g_i(t_i)}(y_i) \to \infty\). According to Proposition 4.5 and Proposition 4.6, we can take limit

$$(X_i, y_i, g_i(t_i)) \xrightarrow{C^{1, \frac{1}{2}}} (Y_\infty, y_\infty, h_\infty),$$

where \(Y_\infty\) is a Ricci flat ALE orbifold, \(y_\infty\) is a singular point.

**Claim.** \(B_{g_i(0)}(y_i, \frac{\xi^2}{2}) \subset B_{g_i(t_i)}(y_i, \lambda_m \xi^2)\) for large \(i\), where \(\lambda_m = 1 + \frac{1}{10000}\).

Actually, let \(\gamma\) be the shortest geodesic connecting \(y_i\) and \(p \in B_{g_i(0)}(y_i, \frac{\xi^2}{2})\) under metric \(g_i(0)\). By energy concentration property, under metric \(g_i(t_i)\), after deleting (at most) \(N = \left\lceil \frac{\xi^2}{4} \right\rceil\) geodesic balls of radius \(\xi^2\), the remainder set which we denote as \(\Omega_i\) has uniform Riemannian curvature bounded by \(\xi^{-\frac{3}{2}}\). Therefore, according to the choice of \(\xi\) (c.f. Proposition 4.3), we know \(|Rm|\) is uniformly bounded by \(\frac{1}{10000m^2\xi^{-2}}\) on \(\Omega_i' \times [t_i, 0]\) where

$$\Omega_i' = \{ x \in \Omega_i | d_{g_i(t_i)}(x, \partial\Omega_i) \geq \frac{\xi^2}{4} \}, \quad [t_i, 0] \subset [t_i, t_i + 9\xi^2].$$

As the change of length is controlled by integration of Ricci curvature over time, we have

$$\text{dist}_{g_i(t_i)}(p, y_i) \leq e^{\frac{1}{1000m}} \text{length}_{g_i(0)} \gamma + N \cdot 2\xi^2 \leq (e^{\frac{1}{1000m}} + N \cdot 2\xi^2) \xi^2 \leq \lambda_m \xi^2,$$
where the last step follows from the choice of $\xi$. The Claim is proved.

Since $y_i \in B_{g_i(0)}(x_i, \frac{1}{4})$, according to the choice of $\xi$, we have $\text{Vol}_{g_i(0)}(B_{g_i(0)}(y_i, \frac{1}{4})) > \frac{7}{8}\omega(m)\xi^2$. On the other hand, $C^1, \frac{1}{4}\text{-convergence}$ and volume comparison implies

$$\frac{\text{Vol}_{g_i(t_i)}(B_{g_i(t_i)}(y_i, \lambda_m\xi^2))}{(\lambda_m\xi^2)^m} \to \frac{\text{Vol}(B(y_\infty, \lambda_m\xi^2))}{(\lambda_m\xi^2)^m} \leq \lim_{r \to 0} \frac{\text{Vol}(B(y_\infty, r))}{r^m} = \frac{\omega(m)}{|\Gamma(y_\infty)|}.$$

As volume change is controlled by integration of scalar curvature which is tending to zero, we know

$$\lim_{i \to \infty} \frac{\text{Vol}_{g_i(0)}(B_{g_i(t_i)}(y_i, \lambda_m\xi^2))}{(\lambda_m\xi^2)^m} = \lim_{i \to \infty} \frac{\text{Vol}_{g_i(t_i)}(B_{g_i(t_i)}(y_i, \lambda_m\xi^2))}{(\lambda_m\xi^2)^m} \leq \frac{\omega(m)}{|\Gamma(p_\infty)|} \leq \frac{1}{4}\omega(m).$$

Therefore, for large $i$, we have

$$\frac{7}{8}\omega(m)\xi^2 < \text{Vol}_{g_i(0)}(B_{g_i(0)}(y_i, \frac{1}{4})) \leq \text{Vol}_{g_i(0)}(B_{g_i(t_i)}(y_i, \lambda_m\xi^2)) \leq \frac{3}{4}\omega(m)(\lambda_m)^m\xi^2.$$

It implies $e^{\frac{100}{m} - (1 + \frac{100}{m})^m} = \lambda_m^m > \frac{7}{8}$ which is impossible! This contradiction establish the proof of backward pseudolocality.

Using this backward pseudolocality theorem, we can improve $C^1, \frac{1}{4}\text{-convergence}$ to $C^\infty$-convergence.

**Proposition 4.8 (C^\infty-Weak Compactness of EV-refined Sequence).** Suppose that $\{(X_i, x_i, g_i(t)): -1 \leq t \leq 1\}$ is an EV-refined sequence, we have

$$(X_i, x_i, g_i(0)) \xrightarrow{C^\infty} (X_\infty, x_\infty, g_\infty)$$

where $X_\infty$ is a Ricci flat ALE orbifold.

**Proposition 4.9.** Every refined sequence is an E-refined sequence.

**Proof.** Suppose not. Then by delicate selecting of base points and blowup, we can find an E-refined sequence $\{(X_i, x_i, g_i(t)): -1 \leq t \leq 1\}$ under constraint 2 satisfying $|\text{Rm}|_{g_i(0)}(x_i) = 1$ and energy concentration fails at $(x_i, 0)$, i.e.,

$$\#(\text{Sing}(B_{g_i(t_i)}(x_i, 1))) + \int_{B_{g_i(t_i)}(x_i, 1)} |\text{Rm}|^{\frac{m}{2}} d\mu < h$$

for some $t_i \in [0, \xi^2]$. This means that, under metric $g_i(t_i)$, $B(x_i, 1)$ is free of singularity and $\int_{B(x_i, 1)} |\text{Rm}|^{\frac{m}{2}} d\mu < h$. The energy concentration property implies that $|\text{Rm}| \leq 4$ in $B(x_i, \frac{1}{4})$ under the metric $g_i(t_i)$. Therefore, by the backward pseudolocality, we have

$$|\text{Rm}|_{g_i(t)}(x) \leq 4C, \quad \forall (x, t) \in P^-(x_i, t_i, \frac{1}{4}, \frac{9}{4}\xi^2).$$
In particular, $|Rm|_{g_i(0)}(y) \leq 4C$ for every $y \in B_{g_i(t_i)}(x_i, \frac{1}{4})$. Based at $(x_i, t_i)$, we can take the smooth limit of $P^-(x_i, t_i, \frac{1}{8}, 2^2) \subset P^-(x_i, t_i, \frac{1}{4}, \frac{9}{4} \xi^2)$, which will be a Ricci flat Ricci flow solution. Therefore,

$$\lim_{i \to \infty} |Rm|_{g_i(t_i)}(x_i) = \lim_{i \to \infty} |Rm|_{g_i(0)}(x_i) = 1. \quad (11)$$

On the other hand, the sequence $\{(X_i, x_i, g_i(t)), -1 \leq t \leq 1\}$ is an EV-refined sequence by Proposition 4.6, the $C^\infty$-weak compactness theorem for EV-refined sequence (under constraint $(2, K_0)$) implies

$$(X_i, x_i, g_i(t)) \xrightarrow{C^\infty} (X_{\infty}, x_{\infty}, g_{\infty})$$

for some Ricci flat ALE orbifold $X_{\infty}$. Clearly, $B(x_{\infty}, 1)$ is free of singularity. So Moser iteration of $|Rm|$ implies that $|Rm|(x_{\infty}) < \frac{1}{2}$. It follows that $\lim_{i \to \infty} |Rm|_{g_i(t_i)}(x_i) \leq \frac{1}{2}$ which contradicts to equation (11)!

It follows directly the following theorem.

**Theorem 4.1 (Weak compactness of refined sequence).** Suppose $\{(X_i, x_i, g_i(t)), -1 \leq t \leq 1\}$ is a refined sequence. Then we have

$$(X_i, x_i, g_i(0)) \xrightarrow{C^\infty} (X_{\infty}, x_{\infty}, g_{\infty})$$

for some Ricci flat, ALE orbifold $X_{\infty}$.

### 4.3 Applications of Refined Sequences

After we obtain this smooth weak convergence, we can use refined sequence as a tool to study the moduli space $\mathcal{O}(m, c, \sigma, \kappa, E)$ which is defined at the beginning of this section. Using the same argument as in [CW3], we can obtain the following theorems.

**Theorem 4.2 (No Volume Concentration and Weak Compactness).** $\mathcal{O}(m, c, \sigma, \kappa, E)$ satisfies the following two properties.

- **No volume concentration.** There is a constant $K$ such that

  $$\text{Vol}_{g(0)}(B_g(x, r)) \leq Kr^m$$

  whenever $r \in (0, K^{-1}]$, $x \in X$, $\{(X, g(t)), -1 \leq t \leq 1\} \in \mathcal{O}(m, c, \sigma, \kappa, E)$.

- **Weak compactness.** If $\{(X_i, x_i, g_i(t)), -1 \leq t \leq 1\} \in \mathcal{O}(m, c, \sigma, \kappa, E)$ for every $i$, by passing to subsequence if necessary, we have

  $$(X_i, x_i, g_i(0)) \xrightarrow{C^\infty} (\hat{X}, \hat{x}, \hat{g})$$

  for some $C^0$-orbifold $\hat{X}$ in Cheeger-Gromov sense.
Theorem 4.3 (Isoperimetric Constants). There is a constant $\iota = \iota(m, c, \sigma, \kappa, E, D)$ such that the following property holds.

If $\{ (X, g(t)), -1 \leq t \leq 1 \} \in \mathcal{O}(m, c, \sigma, \kappa, E)$ and $\text{diam}_{g(0)}(X) < D$, then

$$I(X, g(0)) > \iota.$$ 

Theorem 4.5 of [CW3] can be improved as $\mathcal{O}_S(m, \sigma, \kappa, E, V)$—the moduli space of compact gradient shrinking Ricci soliton orbifolds—is compact.

5 Kähler Ricci Flow on Fano Orbifolds

5.1 Some Estimates

All the estimates developed under the Kähler Ricci flow on Fano manifolds hold for Fano orbifolds. We list the important ones and only give sketch of proofs if the statement is not obvious.

Proposition 5.1 (Perelman, c.f. [SeT]). Suppose $\{ (Y^n, g(t)), 0 \leq t < \infty \}$ is a Kähler Ricci flow solution on Fano orbifold $Y^n$. There are two positive constants $B, \kappa$ depending only on this flow such that the following two estimates hold.

1. Under metric $g(t)$, let $R$ be the scalar curvature, $-u$ be the normalized Ricci potential, i.e.,

$$Ric - \omega_{\varphi(t)} = -\sqrt{-1} \partial \bar{\partial} u, \quad \frac{1}{V} \int_Y e^{-u} \omega_{\varphi(t)}^n = 1.$$ 

Then we have

$$\|R\|_{C^0} + \text{diam } Y + \|u\|_{C^0} + \|\nabla u\|_{C^0} < B.$$ 

2. Under metric $g(t)$, $\frac{\text{Vol}(B(x, r))}{r^{2n}} > \kappa$ for every $r \in (0, 1)$, $(x, t) \in Y \times [0, \infty)$.

Proof. When scalar curvature norm $|R|$ is uniformly bounded, the second estimate becomes a direct corollary of the general noncollapsing theorem. So we only need to show the first estimate. The proof is almost the same as the manifold case.

First, note that Green’s function exists on every compact orbifold, and Perelman’s functional behaves the same as in manifold case. Same as in [SeT], we can apply Green’s function and Perelman’s functional to obtain a uniform lower bound of $u(t)$ where $-u(t)$ is the normalized Ricci potential.
Second, since \( u(t) \) is uniformly bounded from below, we can find a big constant \( B \) such that \( u + 2B > B \). Then maximal principle tells us that there is a constant \( C \) such that

\[
\frac{\Delta u}{u + 2B} < C, \quad \frac{\|
abla u\|^2}{u + 2B} < C.
\]

By the second inequality, we know \( u \) is Lipschitz. Therefore, \( u \) will be bounded whenever diameter is bounded.

Third, diameter is bounded. Suppose that diameter is unbounded, we can find a sequence of annulus \( A_i = B_{g(t_i)}(x_i, 2^{i+2}) \setminus B_{g(t_i)}(x_i, 2^{i-2}) \) such that the following properties hold.

- The closure \( \overline{A_i} \) contains no singular point.
- \( \text{Vol}_{g(t_i)}(A_i) \to 0. \)
- Under metric \( g(t_i) \), \( \frac{\text{Vol}(B(x_i, 2^{i+2}) \setminus B(x_i, 2^{i-2}))}{\text{Vol}(B(x_i, 2^{i+1}) \setminus B(x_i, 2^{i-1}))} < 2^{10n} \).

The reason we can do this is that \( Y \) contains only finite singularities. Then by taking a proper cutoff function whose support is in \( A_i \), we can deduce that Perelman’s functional \( \mu(g_0, \frac{1}{2}) \) must tend to \(-\infty\). Impossible! \( \square \)

The following estimates on orbifolds are exactly the same as the corresponding estimates on manifolds.

**Proposition 5.2** ([Zhu], [Ye]). \( \{(Y^n, g(t)), 0 \leq t < \infty\} \) is a Kähler Ricci flow on Fano orbifold \( Y^n \). Then there is a uniform Sobolev constant \( C_S \) along this flow. In other words, for every \( f \in C^\infty(Y) \), we have

\[
\left( \int_Y |f|^2 \omega^n \right)^{\frac{1}{2}} \leq C_S \left\{ \int_Y |\nabla f|^2 \omega^n + \frac{1}{V^n} \int_Y |f|^2 \omega^n \right\}.
\]

**Proposition 5.3** (c.f. [Fu2], [TZ]). \( \{(Y^n, g(t)), 0 \leq t < \infty\} \) is a Kähler Ricci flow on Fano orbifold \( Y^n \). Then there is a uniform weak Poincaré constant \( C_P \) along this flow. Namely, for every nonnegative function \( f \in C^\infty(Y) \), we have

\[
\frac{1}{V^n} \int_Y f^2 \omega^n \leq C_P \left( \frac{1}{V^n} \int_Y |\nabla f|^2 \omega^n + \frac{1}{V^n} \int_Y f^2 \omega^n \right)^{\frac{1}{2}}.
\]

**Proposition 5.4** (c.f. [PSS], [CW2]). By properly choosing initial condition, we have

\[
\|\phi\|_{C^0} + \|\nabla \phi\|_{C^0} < C
\]

for some constant \( C \) independent of time \( t \).
Proposition 5.5 ([CW2]). There is a constant $C$ such that
\[
\frac{1}{V} \int_Y (-\varphi) \omega^n \leq n \sup_Y \varphi - \sum_{i=0}^{n-1} \int_Y \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^{n-1-i} + C.
\] (12)

Proposition 5.6 ([Ru], c.f. [CW2]). \{($Y^n, g(t)$), $0 \leq t < \infty$\} is a Kähler Ricci flow on Fano orbifold $Y^n$. Then the following conditions are equivalent.

- $\varphi$ is uniformly bounded.
- $\sup_Y \varphi$ is uniformly bounded from above.
- $\inf_Y \varphi$ is uniformly bounded from below.
- $\int_Y \varphi \omega^n$ is uniformly bounded from above.
- $\int_Y (-\varphi) \omega^n$ is uniformly bounded from above.
- $I_{\omega} (\varphi)$ is uniformly bounded.
- $\text{Osc}_Y \varphi$ is uniformly bounded.

5.2 Tamed Condition by Two Functions: $F$ and $G$

This subsection is similar to the corresponding part in [CW3]. However, we compare different metrics on the line bundle to study the tamedness condition.

Along the Kähler Ricci Flow, we have
\[
\omega_{\varphi(t)} = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi(t), \quad \sqrt{-1} \partial \bar{\partial} \dot{\varphi}(t) = \omega_{\varphi(t)} - \text{Ric}_{\omega_{\varphi(t)}}
\]

For simplicity, we omit the subindex $t$. Let $h$ be the metric on $K^{-1}_Y$ induced directly by the metric on $Y$, i.e., $h = \det g_{ij}$. Let $l = e^{-\dot{\varphi}} h$. Clearly, we have
\[
-\sqrt{-1} \partial \bar{\partial} \log |S|^2_l + \sqrt{-1} \partial \bar{\partial} \log |S|^2_h = \sqrt{-1} \partial \bar{\partial} \dot{\varphi} = \omega_\varphi - \text{Ric}_{\omega_\varphi}
\]

It follows that $\sqrt{-1} \partial \bar{\partial} \log |S|^2_l = \omega_\varphi$.

Definition 5.1. Choose \{\(T^t_{\nu,\beta}\)\}_{\beta=0}^{N_\nu} as orthonormal basis of $H^0(K^{-\nu}_Y)$ under the metric $h^\nu$. Then
\[
F(\nu, x, t) = \frac{1}{\nu} \log \sum_{\beta=0}^{N_\nu} |T^t_{\nu,\beta}|^2_{h^\nu}(x),
\]
\[
G(\nu, x, t) = \sum_{\beta=0}^{N_\nu} |\nabla T^t_{\nu,\beta}|^2_{h^\nu}(x)
\]
are well defined functions on $Y \times [0, \infty)$.

We call the flow is tamed by $\nu$ if $F(\nu, \cdot, \cdot)$ is a bounded function on $Y \times [0, \infty)$.
Remark 5.1. If \( Y \) is an orbifold, \( K_Y^{-\nu} \) is a line bundle if and only if \( \nu \) is an integer multiple of the Gorenstein index of \( Y \). We call such \( \nu \) as appropriate. In this note, we always choose \( \nu \) as appropriate ones.

Clearly, \( G = \Delta e^{\nu F} - \nu R e^{\nu F} \). Fix \((x,t)\), by rotating basis, we can always find a section \( T \) such that
\[
\int_Y |T|^2_{h^\nu(t)} \omega^n = 1, \quad e^{\nu F(x,t)} = |T|^2_{h^\nu(t)}(x).
\]
There also exists a section \( T' \) such that
\[
\int_Y |T'|^2_{h^\nu(t)} \omega^n = 1, \quad G(x,t) = |\nabla T'|^2_{h^\nu(t)}(x).
\]

Definition 5.2. Choose \( \{ S^t_{\nu,\beta} \}_{\beta=0}^{N_\nu} \) as orthonormal basis of \( H^0(K_Y^{-\nu}) \) under the metric \( l^\nu \). Then
\[
\mathcal{F}(\nu,x,t) = \frac{1}{\nu} \log \sum_{\beta=0}^{N_\nu} |S^t_{\nu,\beta}|^2_{l^\nu}(x),
\]
\[
\mathcal{G}(\nu,x,t) = \sum_{\beta=0}^{N_\nu} |\nabla S^t_{\nu,\beta}|^2_{l^\nu}(x).
\]
are well defined functions on \( Y \times [0, \infty) \).

Similarly, \( G = \Delta e^{\nu F} - \nu R e^{\nu F} \). Fix \((x,t)\), by rotating basis, there are unit norm sections \( S \) and \( S' \) such that
\[
\int_Y |S|^2_{l^\nu(t)} \omega^n = 1, \quad e^{\nu F(x,t)} = |S|^2_{l^\nu(t)}(x);
\]
\[
\int_Y |S'|^2_{l^\nu(t)} \omega^n = 1, \quad G(x,t) = |\nabla S'|^2_{l^\nu(t)}(x).
\]

At point \((x,t)\), we have
\[
e^{\nu F} = |S|^2_{l^\nu(t)} = e^{-\bar{\nu} \phi} |S|^2_{h^\nu(t)} = e^{-\bar{\nu} \phi} \frac{|S|^2_{h^\nu(t)}(x)}{\int_Y |S|^2_{h^\nu(t)} \omega^n} \int_Y |S|^2_{h^\nu(t)} \omega^n \leq e^{\nu (F - \bar{\nu} \phi + |\phi|_{C^0})} \leq e^{2\nu B} e^{\nu F}.
\]

Similarly, we can do the other way and it follows that
\[
F - 2B \leq \mathcal{F} \leq F + 2B.
\]

Therefore, a flow is tamed by \( \nu \) if and only if \( \mathcal{F}(\nu,\cdot,\cdot) \) is uniformly bounded on \( Y \times [0, \infty) \). However, the calculation under the metric \( l^\nu \) is easier in many cases. Some estimates in [CW4] can be improved.

1The calculation under the metric \( l^\nu \) was first suggested to the author by Tian.
Lemma 5.1. There is a uniform constant $A = A(B, C_S, n)$ such that

$$|S|_\nu < A\nu \frac{n}{2},$$

$$|
abla S|_\nu < A\nu^\frac{n+1}{2},$$

whenever $S \in H^0(Y, K_Y^{-\nu})$ is a unit norm section (under the metric $l''$).

Proof. For simplicity, we omit subindex $l''$ in the proof. Note $\triangle_{\omega_\nu}|S|^2 = |\nabla S|^2 - n\nu|S|^2$, the proof of inequality (13) follows directly the proof of Lemma 3.1 in [CW4]. So we only prove inequality (14).

Direct calculation shows that

$$\triangle_{\omega_\nu}|S|^2 = |\nabla \nabla S|^2 - (n+2)\nu|\nabla S|^2 + n\nu^2|S|^2 + R_{ij}S_jS_j$$

$$= |\nabla \nabla S|^2 - [(n+2)\nu - 1]|\nabla S|^2 + n\nu^2|S|^2 - \phi_{ji}S_iS_j.$$ (15)

Note that $S_{ij} = -\nu S_{gij}$, integration under measure $\omega_\nu$ implies

$$\int_Y |\nabla \nabla S|^2 = -n\nu^2 + [(n+2)\nu - 1]\int_Y |\nabla S|^2 + \int_Y \phi_{ji}S_iS_j$$

$$= n\nu[(n+1)\nu - 1] - \int_Y \phi_iS_{ij}S_j + n\nu \int \phi_iS_iS.$$ In view of $|\phi| \leq \mathcal{B}$, Hölder inequality implies

$$\int_Y |\nabla \nabla S|^2 \leq \mathcal{B} \left( \int_Y |\nabla \nabla S|^2 \right)^\frac{n}{2} + n\nu \left( \int_Y |\nabla S|^2 \right)^\frac{n}{2} + n\nu[(n+1)\nu - 1]$$

$$= \sqrt{n\nu} \mathcal{B} \left( \int_Y |\nabla \nabla S|^2 \right)^\frac{n}{2} + n\nu[(n+1)\nu - 1]$$

$$\leq \frac{1}{2} \int_Y |\nabla \nabla S|^2 + \frac{1}{2} n\nu \mathcal{B}^2 + n\nu[(n+1)\nu - 1].$$

It follows that

$$\int_Y |\nabla \nabla S|^2 \leq C\nu^2,$$

for some constant $C = C(n, \mathcal{B})$. Combining with the fact $\int_Y |\nabla \nabla S|^2 = n\nu^2$, Sobolev inequality implies

$$(\int_Y |\nabla S|^{\frac{2n}{n-2}}) \frac{2}{n-2} \leq C\nu^2.$$ (16)

Fix $\beta > 1$, multiplying $-|\nabla S|^{2(\beta-1)}$ to both sides of equation (15), we have

$$\frac{4(\beta-1)}{\beta^2} \int_Y |\nabla |\nabla S|^{\beta}|^2$$

$$= -\int_Y (n\nu^2|S|^2 + |\nabla \nabla S|^2)|\nabla S|^{2(\beta-1)} + [(n+2)\nu - 1]\int_Y |\nabla S|^{2\beta}$$

$$+ \int_Y \phi_{ij}S_iS_j|\nabla S|^{2(\beta-1)}$$
Note that
\[
\int_Y \dot{\varphi}_{i\bar{j}} S_{i\bar{j}} |\nabla S|^{2(\beta-1)}
= -\int_Y \dot{\varphi}_{i\bar{j}} (S_{i\bar{j}} S_{j\bar{j}} + S_{j\bar{i}} S_{i\bar{j}}) |\nabla S|^{2(\beta-1)} - (\beta - 1) \int_Y \dot{\varphi}_{i\bar{j}} S_{i\bar{j}} (S_{k\bar{j}} S_{\bar{k}j} + S_{k\bar{i}} S_{\bar{j}i}) |\nabla S|^{2(\beta-2)}
\leq \nu [\beta - 1 + n] \int_Y \dot{\varphi}_{i\bar{j}} S_{i\bar{j}} |\nabla S|^{2(\beta-1)} + B\beta \int_Y |\nabla \nabla S| |\nabla S|^{2\beta-1}
\]

Hölder inequality and Schwartz inequality yield that
\[
B\nu [\beta - 1 + n] \left\{ \int_Y |S|^2 |\nabla S|^{2(\beta-1)} \right\}^{\frac{1}{2}} \left\{ \int_Y |\nabla S|^{2\beta} \right\}^{\frac{1}{2}}
+ B\beta \left\{ \int_Y |\nabla \nabla S|^2 |\nabla S|^{2(\beta-1)} \right\}^{\frac{1}{2}} \left\{ \int_Y |\nabla S|^{2\beta} \right\}^{\frac{1}{2}}
\leq n\nu^2 \left( \left\{ \int_Y |S|^2 |\nabla S|^{2(\beta-1)} \right\} + \left\{ \int_Y |\nabla \nabla S|^2 |\nabla S|^{2(\beta-1)} \right\} \right)
+ \frac{B^2(\beta - 1 + n)^2}{4n} + \frac{B^2\beta^2}{4n\nu^2} \int_Y |\nabla S|^{2\beta}.
\]

If \( \beta \geq \frac{1}{n-1} \), combining previous three inequalities implies
\[
\int_Y |\nabla |\nabla S|^\beta | \leq C\beta(\beta^2 + \nu) \int_Y |\nabla S|^{2\beta}.
\]

In light of Sobolev inequality, we have
\[
\left( \int_Y |\nabla S|^\beta \right)^{\frac{n-k}{n}} \leq C_S \left\{ \int_Y |\nabla S|^\beta \right\}^{\frac{n-k}{n}} \leq C\beta(\beta^2 + \nu) \int_Y |\nabla S|^{2\beta}.
\]

Let \( k_0 \) be the number such that \( \lambda^{2k_0} \geq \nu > \lambda^{2(k_0-1)} \) where \( \lambda = \frac{n}{n-1} \), we have
\[
\left( \int_Y |\nabla S|^\beta \right)^{\frac{n-k_0}{n}} \leq \begin{cases} C\beta^3 & \text{if } \beta > \lambda^{k_0}, \\ (C\nu)\beta & \text{if } \beta \leq \lambda^{k_0}. \end{cases}
\]

Iteration implies
\[
\begin{align*}
\| |\nabla S|^2 \|_{L^\infty} &\leq C\sum_{k=1}^{\infty} \lambda^{-k} \lambda^{\sum_{k=1}^{\infty} k\lambda^{-k}} \| |\nabla S|^2 \|_{L^{\lambda k_0}}, \\
\| |\nabla S|^2 \|_{L^{\lambda k_0}} &\leq (C\nu)\sum_{k=1}^{\infty} \lambda^{-k} \| |\nabla S|^2 \|_{L^{\lambda}}.
\end{align*}
\]

Since \( \sum_{k=1}^{k_0} \lambda^{-k} < \sum_{k=1}^{\infty} \lambda^{-k} = n - 1 \), combining these inequalities with inequality (10) gives us
\[
\| |\nabla S|^2 \|_{L^\infty} \leq Cl^{n+1}.
\]

This proves inequality (14). \( \square \)
Similarly, by sharpening the constants in Lemma 3.2 of [CW4], we obtain

**Lemma 5.2.** There is a uniform constant \( A = A(B, C_S, n) \) such that
\[
\begin{align*}
|S|_{h'} &< A\nu^{\frac{2}{3}}, \\
|\nabla S|_{h'} &< A\nu^{\frac{4}{3}},
\end{align*}
\]
whenever \( S \in H^0(Y, K_{Y}^{-\nu}) \) is a unit norm section (under the metric \( h' \)).

Lemma 5.1 and Lemma 5.2 clearly implies the following estimates.

**Corollary 5.1.** There is a uniform constant \( A = A(B, C_S, n) \) such that
\[
\begin{align*}
\max\{\mathcal{F}, F\} &\leq \frac{\log A + n\log\nu}{\nu}, \\
\max\{\mathcal{G}, G\} &\leq A\nu^{n+1}.
\end{align*}
\]

**Proposition 5.7.** Along the flow, \( F \) satisfies
\[
\begin{align*}
\frac{\partial}{\partial t}F &= -\dot{\phi} + \int_Y (\dot{\phi} - \Delta\phi)e^{\nu F}\omega^n_{\phi}^\nu, \\
\Delta F &= -n + \frac{1}{n}\int_Y (e^{-\nu F}G - \nu|\nabla F|^2) \geq -n, \\
\Box F &= -\dot{\phi} + \int_Y (\dot{\phi} - \Delta\phi)e^{\nu F}\omega^n_{\phi}^\nu + (\nu|\nabla F|^2 - \frac{1}{n}e^{-\nu F}G).
\end{align*}
\]

**Proof.** At \( t = t_0 \), suppose \( \{\beta\}_{\beta=0}^{N_0} \) are orthonormal holomorphic sections of \( H^0(Y, K_{Y}^{-1}) \) under the metric \( l'\nu(t_0) \). Assume \( \{a_{\alpha\beta}(t)S_{\beta}\}_{\alpha=0}^{N_0} \) are orthonormal holomorphic sections at time \( t \) under the metric \( l'\nu(t) \). By the uniformization condition, we have
\[
\begin{align*}
a_{\alpha\beta}(t_0) &= \delta_{\alpha\beta}, \\
\delta_{\alpha\gamma} &= a_{\alpha\beta}a_{\gamma\xi} \int_Y \langle S_{\beta}, S_{\xi}\rangle\omega^n_{\phi}^\nu, \\
0 &= a_{\alpha\beta}a_{\gamma\xi} \int_Y \langle S_{\beta}, S_{\xi}\rangle\omega^n_{\phi}^\nu + a_{\alpha\beta}\dot{a}_{\gamma\xi} \int_Y \langle S_{\beta}, S_{\xi}\rangle\omega^n_{\phi}^\nu \\
&\quad + a_{\alpha\beta}\dot{a}_{\gamma\xi} \int_Y (\nu\dot{\phi} + \Delta\phi)\langle S_{\beta}, S_{\xi}\rangle\omega^n_{\phi}^\nu.
\end{align*}
\]
In particular, at \( t = t_0 \), using sum convention we have
\[
\begin{align*}
0 &= \dot{a}_{\alpha\gamma} + \dot{a}_{\gamma\alpha} + \int_Y (\nu\dot{\phi} + \Delta\phi)\langle S_{\alpha}, S_{\gamma}\rangle\omega^n_{\phi}^\nu, \\
\frac{\partial}{\partial t}e^{\nu F}\bigg|_{t=t_0} &= \frac{\partial}{\partial t} \left( a_{\alpha\beta}\dot{a}_{\alpha\gamma} \langle S_{\beta}, S_{\gamma}\rangle \right)_{t=t_0} \\
&= \dot{a}_{\alpha\beta} \langle S_{\beta}, S_{\alpha}\rangle + \dot{a}_{\alpha\gamma} \langle S_{\alpha}, S_{\gamma}\rangle + (\nu\dot{\phi})\langle S_{\alpha}, S_{\alpha}\rangle.
\end{align*}
\]

Fix $x \in Y$, at $t = t_0$, there is a unit norm section $S$ such that $|S|^2_{\nu}(x) = e^{\nu F}$. Let $S_0 = S$, then we have

$$\left. \frac{\partial}{\partial t} e^{\nu F} \right|_{t=t_0} = e^{\nu F}(\dot{a}_{00} + \dot{a}_{00} - \nu \dot{\varphi}) = e^{\nu F}(\int_Y (\nu \dot{\varphi} - \triangle \varphi)e^{\nu F} \omega^n_\varphi - \nu \dot{\varphi})$$

On the other hand,

$$\Delta e^{\nu F} = \langle \nabla S_\alpha, \nabla S_\alpha \rangle - n \nu \langle S_\alpha, S_\alpha \rangle = \mathcal{G} - nve^{\nu F}.$$ 

It follows that

$$\left(\frac{\partial}{\partial t} - \triangle \right) e^{\nu F} = e^{\nu F}\left\{ \int_Y (\nu \dot{\varphi} - \triangle \varphi)e^{\nu F} \omega^n_\varphi + \nu(n - \varphi) \right\} - \mathcal{G}.$$ 

Similarly, we can have

$$\left. \frac{\partial}{\partial t} e^{\nu F} \right| = e^{\nu F}\left\{ \nu \Delta \varphi - (\nu + 1) \int_Y \frac{\partial}{\partial t} \Delta \varphi e^{\nu F} \omega^n_\varphi \right\}$$

$$= e^{\nu F}\left\{ \nu(n - R) - (\nu + 1) \int_Y (n - R)e^{\nu F} \omega^n_\varphi \right\},$$

$$\Delta e^{\nu F} = G - \nu \text{Re} e^{\nu F},$$

$$\Box e^{\nu F} = e^{\nu F}\left\{ n \nu - (\nu + 1) \int_Y (n - R)e^{\nu F} \omega^n_\varphi \right\} - \mathcal{G}.$$

From the evolution equation of $e^{\nu F}$ and $\nu F$, we can easily obtain the evolution equation of $F$ and $F$. \hfill $\square$

**Remark 5.2.** The advantage of $F$ appears when the evolution equation is calculated. Every term in $\frac{\partial F}{\partial t}$ is a geometric quantity. Suppose that $\int_0^\infty \int_Y (R - n) \omega^n_\varphi dt < \infty$ and $\int_0^\infty (R_{\text{max}}(t) - n) dt < \infty$, then $F$ must be bounded from below and the flow is tamed.

When we consider the convergence of metric space, the smooth convergence of $g_{ij}$ will automatically induce the smooth convergence of $h^\nu = \det(g_{ij})^\nu$. Therefore, we prefer to use $h^\nu$ as the more natural metric of $K_Y^{\nu}$ under the Kähler Ricci flow.

Since Hörmansdorfer’s estimate holds in the orbifold case. The bound in Lemma 4.2 implies the convergence of plurianticanonical sections when the underlying orbifolds converge.

**Proposition 5.8.** Suppose $Y$ is a Fano orbifold, $\{(Y, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow without volume concentration. Let $t_i$ be a sequence of time such that $(Y, g(t_i)) \overset{C^{\infty}}{\rightarrow} (\hat{Y}, \hat{g})$ for some Q-Fano normal variety $(\hat{Y}, \hat{g})$. Then for any fixed positive integer $\nu$ (appropriate for both $Y$ and $\hat{Y}$), the following properties hold.

1. If $S_i \in H^0(Y, K_Y^{\nu})$ and $\int_Y |S_i|^2_{h^{\nu}(t_i)} \omega^n_\varphi(t_i) = 1$, then by taking subsequence if necessary, we have $\hat{S} \in H^0(\hat{Y}, K_{\hat{Y}}^{\nu})$ such that

$$S_i \overset{C^{\infty}}{\rightarrow} \hat{S}, \quad \int_{\hat{Y}} |\hat{S}|^2_{h^{\nu}} \omega^n = 1.$$
2. If $\tilde{S} \in H^0(\tilde{Y}, K^{-\nu}_Y)$ and $\int_{\tilde{Y}} \left| \tilde{S} \right|^2_{h^\nu} = 1$, then there is a subsequence of sections $S_i \in H^0(Y_i, K^{-\nu}_{Y_i})$ satisfying
$$\int_{Y_i} \left| S_i \right|^2_{h^\nu(t_i)} = 1, \quad S_i \xrightarrow{C^\infty} \tilde{S}.$$  

Using this property, we can justify the tamedness condition by weak compactness exactly as Theorem 3.2 of [CW4].

**Theorem 5.1.** Suppose $Y$ is a Fano orbifold, $\{(Y, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow without volume concentration. Suppose this flow satisfies weak compactness, i.e., for every sequence $t_i \to \infty$, by passing to subsequence, we have
$$\left( Y, g(t_i) \right) \xrightarrow{C^\infty} (\tilde{Y}, \tilde{g}),$$
where $(\tilde{Y}, \tilde{g})$ is a $Q$-Fano normal variety.

Then this flow is tamed by a big constant $\nu$.

As mentioned in the introduction. Suppose $Y$ is an orbifold Fano surface, $\{(Y, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow solution. Then this flow has no volume concentration and satisfies weak compactness theorem. Under the help of Perelman’s functional, every weak limit $(\tilde{Y}, \tilde{g})$ must satisfy Kähler Ricci soliton equation on its smooth part. On the other hand, the soliton potential function has uniform $C^1$-norm bound since it is the smooth limit of $-\dot{\varphi}(t_i)$. Therefore Uhlenbeck’s removing singularity method applies and we obtain $(\tilde{Y}, \tilde{g})$ is a smooth orbifold which can be embedded into $\mathbb{C}P^N$ by line bundle $K_{\tilde{Y}}^{-\nu}$ for some big $\nu$ (c.f. [Bai]). Then the following Theorem from [51] directly.

**Theorem 5.2.** Suppose $Y$ is an orbifold Fano surface, $\{(Y, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow solution. Then there is a big constant $\nu$ such that this flow is tamed by $\nu$.

### 5.3 Properties of Tamed Flow

Follow [Tian91], we define

**Definition 5.3.** Let $\mathcal{P}_{G,\nu,k}(Y, \omega)$ be the collection of all $G$-invariant functions of form
$$\frac{1}{2\nu} \log \left( \sum_{\beta=0}^{k-1} \left\| \tilde{S}_{\nu,\beta} \right\|^2_{h^\nu} \right),$$
where $\tilde{S}_{\nu,\beta} \in H^0(K^{-\nu}_Y)$ satisfies
$$\int_Y \left( \tilde{S}_\alpha, \tilde{S}_\beta \right)_{h^\nu} = \delta_{\alpha\beta}, \quad 0 \leq \alpha, \beta \leq k - 1 \leq \dim(K^{-\nu}_Y) - 1; \quad h = \det g^\omega.

Define
$$\alpha_{G,\nu,k} \doteq \sup \left\{ \alpha \mid \sup_{\varphi \in \mathcal{P}_{G,\nu,k}} \int_Y e^{-2\alpha \varphi} \omega^n < \infty \right\}.$$  
If $G$ is trivial, we denote $\alpha_{\nu,k}$ as $\alpha_{G,\nu,k}$, denote $\mathcal{P}(\nu, k)$ as $\mathcal{P}(G, \nu, k).$

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The next definition follows [DK].

**Definition 5.4.** Let $Y$ be a complex orbifold and $f$ is a plurisubharmonic function and $f \in L^1(Y)$. For any compact set $K \subset Y$, define

$$
\alpha_K(f) = \sup\{c \geq 0 : e^{-2cf} \text{ is } L^1 \text{ on a neighborhood of } K\},
$$

This $\alpha_K(f)$ is called the complex singularity exponent of $f$ on $K$.

If $f \in \mathcal{P}(\nu, k)$ and $\alpha < \alpha_{\nu,k}$, we have $\int_Y e^{-2\alpha f} < \infty$ by definition. Since the set $\mathcal{P}(\nu, k)$ is compact in $L^1(Y)$-topology (actually in $C^\infty$ topology). By the semicontinuity property proved in [DK], we see there is a uniform constant $C_{\alpha,\nu,k}$ such that

$$
\int_Y e^{-2\alpha f} < C_{\alpha,\nu,k}, \quad \forall f \in \mathcal{P}(\nu, k).
$$

Suppose the flow is tamed by $\nu$. By rotating basis, we can choose $\{S^t_{\nu,\beta}\}_{\beta=0}^{N_{\nu}}$ and $\{\tilde{S}^t_{\nu,\beta}\}_{\beta=0}^{N_{\nu}}$ as orthonormal basis of $H^0(K^{-\nu}_Y)$ under the metric $h^\nu(t)$ and $h^\nu(0)$ respectively, and they satisfy

$$
S^t_{\nu,\beta} = a(t)\lambda(t)\tilde{S}^t_{\nu,\beta}, \quad 0 < \lambda_0(t) \leq \lambda_1(t) \leq \cdots \leq \lambda_{N_{\nu}}(t) = 1.
$$

As in [CW4], we have the partial $C^0$-estimate

$$
\left| \varphi - \sup_Y \varphi - \frac{1}{\nu} \log \sum_{\beta=0}^{N_{\nu}} |\lambda(t)\tilde{S}^t_{\nu,\beta}|^2_{h^\nu_0} \right| < C.
$$

This yields

$$
\int_Y e^{-\alpha(\varphi - \sup_Y \varphi)} \omega^n < C \int_Y \left( \sum_{\beta=0}^{N_{\nu}} |\lambda(t)\tilde{S}^t_{\nu,\beta}|^2_{h^\nu_0} \right)^{-\frac{\alpha}{\nu}} \omega^n
$$

$$
< C \int_Y \left( \sum_{\beta=N_{\nu}-k+1}^{N_{\nu}} |\lambda_\beta(t)\tilde{S}^t_{\nu,\beta}|^2_{h^\nu_0} \right)^{-\frac{\alpha}{\nu}} \omega^n
$$

$$
\leq C \lambda^{-\frac{2\alpha}{\nu}}_{N_{\nu}-k+1} \int_Y \left( \sum_{\beta=N_{\nu}-k+1}^{N_{\nu}} |\tilde{S}^t_{\nu,\beta}|^2_{h^\nu_0} \right)^{-\frac{\alpha}{\nu}} \omega^n
$$

$$
< C C_{\alpha,\nu,k} \lambda^{-\frac{2\alpha}{\nu}}_{N_{\nu}-k+1}.
$$

Plug in the equation $\dot{\varphi} = \log \frac{\omega^n}{\omega^\varphi} + \varphi + u_\omega$ and note that $\dot{\varphi}, u_\omega$ are bounded, we have

$$
\int_Y e^{(1-\alpha)\varphi + \alpha \sup Y \varphi} \omega^n < C'(\alpha, \nu, k) \lambda^{-\frac{2\alpha}{\nu}}_{N_{\nu}-k+1}
$$

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The convexity of exponential function implies
\[
(1 - \alpha) \frac{1}{V} \int_Y \varphi \omega^n + \alpha \sup_Y \varphi < C''(\alpha, \nu, k) - \frac{2\alpha}{\nu} \log \lambda_{\nu, k+1}.
\]
whenever \( \alpha < \alpha_{\nu, k} \). Using this estimate, we can obtain the following two convergence theorems as in [CW4].

**Theorem 5.3.** Suppose \( \{(Y^n, g(t)), 0 \leq t < \infty\} \) is a Kähler Ricci flow tamed by \( \nu \). If \( \alpha_{\nu, 1} > \frac{n}{n+1} \), then \( \varphi \) is uniformly bounded along this flow. In particular, this flow converges to a KE metric exponentially fast.

**Proof.** Choose \( \alpha \in \left( \frac{n}{n+1}, \alpha_{\nu, 1} \right) \). Put \( k = 1 \) into inequality (19), we have
\[
(1 - \alpha) \frac{1}{V} \int_Y \varphi \omega^n + \alpha \sup_Y \varphi < C(\alpha, \nu).
\]
Together with \( \frac{1}{V} \int_Y (\varphi \omega^n) \leq n \sup_Y \varphi + C \), it implies
\[
\{\alpha - n(1 - \alpha)\} \sup_Y \varphi < C.
\]
As \( \alpha > \alpha_{\nu, 1} > \frac{n}{n+1} \), we have \( \alpha - n(1 - \alpha) > 0 \), this yields that \( \sup_Y \varphi \) is uniformly bounded from above. Therefore, \( \varphi \) is uniformly bounded. \( \square \)

**Theorem 5.4.** Suppose \( \{(Y^n, g(t)), 0 \leq t < \infty\} \) is a Kähler Ricci flow tamed by \( \nu \). If \( \alpha_{\nu, 2} > \frac{n}{n+1} \) and \( \alpha_{\nu, 1} > \frac{1}{2 - (n+1)\alpha_{\nu, 2}} \), then \( \varphi \) is uniformly bounded along this flow. In particular, this flow converges to a KE metric exponentially fast.

**Proof.** We argue by contradiction. Suppose that \( \varphi \) is not uniformly bounded.

Then there must be a sequence of \( t_i \) such that \( \sup_Y \varphi(t_i) \to \infty \). We claim that
\[
\lambda_{\nu, 1}(t_i) \to 0.
\]
Otherwise, \( \log \lambda_{\nu, 1}(t_i) \) is uniformly bounded. Choose \( \alpha \in \left( \frac{n}{n+1}, \alpha_{\nu, 2} \right) \). Combining
\[
\frac{1}{V} \int_Y (\varphi \omega^n) \leq n \sup_Y \varphi + C
\]
and the inequality (19) in the case \( k = 1 \), the same argument as in the proof of Theorem 5.3 implies that \( \sup_Y \varphi(t_i) \) is uniformly bounded. This contradicts to our assumption!

Note that \( \mathbb{R} \)-coefficient Poincaré duality holds on orbifold, singularities on \( Y \) are isolated which can be included in small geodesic balls with few contribution to integration. Since \( \lambda_{\nu, 1}(t_i) \to 0 \), as in [Tian91], for every small \( \delta > 0 \), we have
\[
\frac{1}{V} \int_Y \sqrt{-1} \partial X_{t_i} \wedge \bar{\partial} X_{t_i} \wedge \omega^{n-1} \geq -\frac{(1 - \delta)}{\nu} \log \lambda_{\nu, 1}(t_i) - C
\]
for large $i$. Here $X_{t_{i}} = \frac{1}{\nu} \log \sum_{i=0}^{N_{i}} |\lambda_{\beta}(t_{i}) \tilde{z}_{t_{i}}^{i}|^{2}$. For notation simplicity, we omit the subindex $t_{i}$ from now on. It follows that

$$\sum_{j=0}^{n-1} \int_{Y} \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^{n-1-j} \sum_{j=0}^{n-1} \int_{Y} \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^{n-1} \geq \sum_{j=0}^{n-1} \int_{Y} \sqrt{-1} \partial X \wedge \bar{\partial} X \wedge \omega^{n-1} - C \geq -(1 - \delta) \cdot \left(\frac{n - 1}{\nu}\right) \cdot \log \lambda_{N_{i} - 1} - C.$$

Plug this into inequality (19) in the case $k = 1$, we arrive

$$(1 - \alpha) \frac{1}{V} \int_{Y} \phi \omega^{n}_{\nu} + \alpha \sup_{Y} \varphi < C(\alpha, \nu) + \frac{1}{1 - \delta} \cdot \frac{2\alpha}{(n - 1)} \sum_{i=0}^{n-1} \int_{Y} \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^{i} \wedge \omega^{n-1-i}.$$

Combining it with

$$\frac{1}{V} \int_{Y} (-\varphi) \omega^{n}_{\nu} \leq n \sup_{Y} \varphi - \sum_{i=0}^{n-1} \int_{Y} \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^{i} \wedge \omega^{n-1-i} + C$$

we have

$$(2A - 1 - \alpha) \frac{1}{V} \int_{Y} (-\varphi) \omega^{n}_{\nu} < \alpha(2A - 1) \sup_{Y} \varphi + C.$$
6 Some Applications and Examples

The following theorem is a direct corollary of Theorem 5.2, Theorem 5.3 and Theorem 5.4.

**Theorem 6.1.** Suppose that $Y$ is an orbifold Fano surface such that one of the following two conditions holds for every large integer $\nu$,

- $\alpha_{\nu,1} > \frac{2}{3}$.
- $\alpha_{\nu,2} > \frac{2}{3}$, $\alpha_{\nu,1} > \frac{1}{2 - \frac{1}{3\alpha_{\nu,2}}}$.

Then $Y$ admits a KE metric.

There are a lot of orbifold Fano surfaces where Theorem 6.1 can be applied. For simplicity, we only consider the good case: every singularity is a rational double point. This kind of orbifolds are called Gorenstein log del Pezzo surfaces.

Let’s first recall some definitions.

**Definition 6.1.** Suppose that $X$ is a normal variety and $D = \sum d_i D_i$ is a $\mathbb{Q}$-cartier divisor on $X$ such that $K_X + D$ is $\mathbb{Q}$-cartier and let $f: Y \rightarrow X$ be a birational morphism, where $Y$ is normal. We can write

$$K_Y \sim_{\mathbb{Q}} f^*(K_X + D) + \sum a(X, D, E)E.$$ 

The discrepancy of the log pair $(X, D)$ is the number

$$\text{discrep}(X, D) = \inf\{a(X, D, E) | E \text{ is exceptional divisor over } X\}.$$

The total discrepancy of the log pair $(X, D)$ is the number

$$\text{totaldiscrep}(X, D) = \inf\{a(X, D, E) | E \text{ is divisor over } X\}.$$

We say that the log pair $K_X + D$ is

- Kawamata log terminal (or log terminal) if and only if $\text{totaldiscrep}(X, D) > -1$.
- log canonical if $\text{discrep}(X, D) \geq -1$.

Assume now that $X$ is a variety with log terminal singularities, let $Z \subset X$ be a closed subvariety and let $D$ be an effective $\mathbb{Q}$-Cartier divisor on $X$. Then the number

$$\text{lct}_Z(X, D) = \sup\{\lambda \in \mathbb{Q} | \text{the log pair } (X, \lambda D) \text{ is log canonical along } Z\}.$$ 

Let $x$ be a point in $X$, $f$ be a local defining holomorphic function of divisor $D$ around $x$, then we have

$$\text{lct}_x(X, D) = \alpha_x(\log f)$$

where $\alpha_x(\log f)$ is the singularity exponent of plurisubharmonic function $\log f$ around point $x$. (c.f. definition 5.4).
Definition 6.2.

\[ \text{let}_\nu(X) = \inf\{\text{let}(X, \frac{1}{\nu} D) | D \text{ effective } \mathbb{Q} \text{-divisor on } X \text{ such that } D \in \lfloor -\nu K_X \rfloor \}. \]

The global log canonical threshold of \( X \) is the number

\[ \text{let}(X) = \inf\{\text{let}(X, D) | D \text{ effective divisor of } X \text{ such that } D \sim_{\mathbb{Q}} -K_X \}. \]

It’s not hard to see \( \text{let}_\nu(X) = \alpha_{\nu,1} \). According to the proof of Demailly (c.f. [ChS], [SYl]), we know

\[ \alpha(X) = \text{let}(X) = \lim_{\nu \to \infty} \text{let}_\nu(X) = \lim_{\nu \to \infty} \alpha_{\nu,1}. \]

Therefore, we have

\[ \infty = \alpha_{\nu,N_{\nu}+1} \geq \cdots \geq \alpha_{\nu,3} \geq \alpha_{\nu,1}(X) = \text{let}_\nu(X) \geq \text{let}(X) = \alpha(X). \]

The calculation of \( \alpha_{\nu,k} \) is itself a very interesting problem (c.f. [SYl], [ChS]). Here we will use some results calculated in [Kosta].

Lemma 6.1 ([Kosta]). Let \( Y \) be a Gorenstein log del Pezzo surface, every singularity of \( Y \) is of type \( A_k \). Suppose \( Y \) satisfies one of the following conditions.

- \( Y \) has only singularities of type \( A_1 \) or \( A_2 \) and \( K_2^Y = 1 \).
- \( Y \) has one singularity of type \( A_5 \) and \( K_2^Y = 1 \).
- \( Y \) has one singularity of type \( A_6 \) and \( K_2^Y = 1 \).

Then \( \alpha_{\nu,1} \geq \frac{2}{3} \) and \( \alpha_{\nu,2} > \frac{2}{3} \).

Proof. The proof argues case by case and the main ingredients are contained in [Kosta] already. For simplicity, we only give a sketch proof of the second case.

If \( f \in \mathcal{P}(\nu, 1) \) and \( \alpha_x(f) \leq \frac{2}{3} \), one can show that \( f = \frac{1}{\nu} \log |S|_{h_0^\nu}^2 \) for some \( S \in H^0(K_Y^{-\nu}) \). Moreover, \( x \) is the unique singularity of type \( A_5 \) and \( S = (S')^{-\nu} \) for some \( S' \in H^0(K_Y^{-1}) \). \( Z(S') \) is the unique divisor passing through \( x \) such that \( \text{let}_x(Y, Z(S')) = \frac{2}{3} \).

For every \( \varphi \in \mathcal{P}(\nu, 2) \), we have \( e^{2\nu\varphi} = e^{2\nu\varphi_1} + e^{2\nu\varphi_2} \) where

\[ \varphi_1 = \frac{1}{\nu} \log |S_1|_{h_0^\nu}^2, \quad \varphi_2 = \frac{1}{\nu} \log |S_2|_{h_0^\nu}^2, \quad \int_Y (S_1, S_2)_{h_0^\nu}^\nu \omega_0^n = 0. \]

Clearly, for every point \( y \in Y \), we have

\[ \alpha_y(\varphi) \geq \max\{\alpha_y(\varphi_1), \alpha_y(\varphi_2)\} > \frac{2}{3}. \]

Since \( \alpha_y(\varphi_1), \alpha_y(\varphi_2) \) can only achieve finite possible values, we have

\[ \inf_{y \in Y, \varphi \in \mathcal{P}(\nu, 2)} \alpha_y(\varphi) > \frac{2}{3}. \]

By the compactness of \( Y \) and the semicontinuity property proved in [DK], we have the inequality \( \alpha_{\nu,2} > \frac{2}{3} \). 

\[ \square \]
Therefore, Theorem 6.1 applies and we know KE metrics exist on such orbifolds \( Y \) in Lemma 6.1. Together with Theorem 1.6 of [Kosta] and Theorem 5.1 of [SY], we have proved the following theorem.

**Theorem 6.2.** Suppose \( Y \) is a cubic surface with only one ordinary double point, or \( Y \) is a degree 1 del Pezzo surface having only Du Val singularities of type \( A_k \) for \( k \leq 6 \). Starting from any metric \( \omega \) satisfying \([\omega] = 2\pi c_1(Y)\), the Kähler Ricci flow will converge to a KE metric on \( Y \). In particular, \( Y \) admits a KE metric.

**Remark 6.1.** If we consider \( \alpha_{G,\nu,k} \) instead of \( \alpha_{\nu,k} \) for some finite group \( G \subset \text{Aut}(Y) \), it’s still possible to study the existence of KE metrics on degree 1 Gorenstein log Del Pezzo surfaces with \( A_7 \) or \( A_8 \) singularities.

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