Report on A5. Computer Methods

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Abstract
Session A5 on numerical methods contained talks on colliding black holes, critical phenomena, investigation of singularities, and computer algebra.

1 Introduction

In this Chapter, I shall describe the current status of computer simulation and algebraic computing for general relativity and gravitation. Most of the effort, both as represented in this Workshop and world-wide, is focused on “numerical relativity,” the name usually given to large scale simulations of isolated gravitational systems. Elsewhere in these proceedings is the contribution by Lehner based on his plenary talk on the status of the colliding black hole (CBH) problem. This problem is extremely difficult but urgently needs to be solved. The urgency arises because one desired output of CBH simulations is a set of gravitational wave forms that can be used by the ground based gravitational wave detectors (LIGO, VIRGO, GEO600, and TAMA) now coming into operation. The difficulties arise on many fronts. These include (1) the computational spacetime grid is not the physical spacetime grid; (2) CBH spacetimes are singular within the horizons of the black holes. Note that simulations cannot be followed (much) past the time at which NaN (not a number) or INF (infinity) is recorded as a value anywhere in the computational grid; (3) Setting up initial data is highly nontrivial. In the process of solving the constraints, the connection between the actual initial system and the desired physical initial state is lost. For example, the “no incoming radiation” boundary condition is problematic; (4) Existing 3D CBH codes which have made considerable progress in dealing with (1) and (2) are plagued by instabilities which cause them to crash long before any physically relevant results are obtained. Note that a code that runs long enough can overcome the problem of radiation in the initial data by evolving until after it dissipates; (5) Identification of outgoing gravitational radiation is also nontrivial because radiation is a small effect thus requiring accurate simulations for a credible extraction and because many computational
grids must terminate before the radiation zone is reached. All of these issues are under intensive investigation and were discussed in this Workshop. Similar techniques have been developed to study the coalescence of neutron star binaries. Such simulations do not have to face the issues of singularity and horizon but do have the complication of matter.

The travails of the CBH simulations should not be taken to mean that all computer simulations in general relativity and gravitation have not yet been completely successful. In fact, rather the opposite is true. Most spectacular, of course, was the discovery by Choptuik of critical phenomena in the collapse of a spherically symmetric self-gravitating scalar field. Careful numerical techniques (including adaptive mesh refinement) allowed study of the critical system dividing scalar field configurations which eventually disperse from those which eventually form BHs. Scaling typical of that found in other phase transitions was observed. The critical solution itself (a zero mass naked singularity) displays periodic self-similarity — a phenomenon unknown outside gravitating systems. Choptuik’s original discovery sparked a large number of further and ongoing numerical and analytic studies of critical phenomena in gravitational collapse. Some of these were discussed in this Workshop.

Another area where there has been some success has been the numerical investigation of cosmological singularities. While an actual singularity cannot be handled on the computer, the regular behavior in the approach to it may be studied. If, asymptotically as the singularity is approached, some types of terms in Einstein’s equations dominate over others, it might be possible to characterize the approach to the singularity in a simple way. This was argued heuristically long ago by Belinskii, Lifshitz, and Khalatnikov (BKL) with their predictions of local Mixmaster dynamics (LMD) recently supported in the cosmological context with numerical experiment and, in some cases, mathematical proof. In fact, the synergy between the mathematical and numerical approaches in this area has been useful in understanding the relevant phenomenology.

Originally, this Workshop was designed to feature algebraic computing as well as computer simulation. However, the co-organizer, Jim Skea, was unable to attend. In the end, there was only a single talk on this subject where considerable progress was described toward the goal of automated classification of exact or perturbative solutions of Einstein’s equations.

In the rest of this Chapter, the highlights of the talks in each area will be given. For more details please see the indicated references.

2 Colliding Black Holes and Neutron Stars

Lousto presented an overview of the Lazarus Project to combine 3D simulations and perturbation approaches in an ambitious attempt to follow a BH collision and extract waveforms. This required the development of techniques to convert the coordinates suitable to the 3D simulation to those appropriate for the single black hole close-limit approximation. These simulations were checked by interchanging the use of 3D simulation and perturbation where both were valid.
In addition, the method was applied to a single Kerr BH where computed spurious radiation indicates the degree of error. This approach has made it possible to compute the first complete waveforms covering the post-orbital dynamics of a binary black hole system.

A number of talks were given on various techniques needed for standard (i.e., 3 + 1 formulation) 3D codes. It is currently believed that an important source of instability in 3D codes is constraint violating modes. Einstein’s equations may be written in an infinite number of ways by adding arbitrary multiples of the constraints to the dynamical equations of motion. The goal is to identify a formulation which will drive the system to the constraint hypersurface and thus eliminate these instabilities. One such approach was discussed by Shinkai.[2]

Pollney discussed work (with AEI collaborators) on BH excision and gauge techniques.[3] They describe recent progress in an implementation of excision for 3D black holes along with a set of gauge conditions which respond naturally to spacetime dynamics. Through the combined use of these techniques, they are able to produce accurate and long-lived evolutions of highly distorted, rotating black hole spacetimes. Accurate waveforms can be extracted.

Diener presented the results of the head-on CBH code test for 3D simulations. Koppitz reported on preliminary work in the construction of initial data sets. The objective is to consider thin sandwich, Kerr-Schild, and post-Newtonian approximation initial data using various implementations.

Bondarescu described how one might visualize BH horizons as they evolve in CBH simulations.[4] The 3D codes allow one to locate and track the evolution of apparent and event horizons in the coordinates of the simulation. One typically has little information about their real geometry. Previous studies in axisymmetric spacetimes have visualized horizons via an embedding in flat space. A new method which goes beyond axisymmetry was discussed. The method correctly reproduces known results and, for the first time, has allowed construction of embeddings of non-axisymmetric, distorted black holes.

Ashtekar described a method to extract physics from strong field regimes in numerical simulations using the isolated horizon (IH) framework.[5] If the IH (but not the spacetime) is axisymmetric, e.g., then, knowing just the intrinsic metric and one Newman-Penrose spin coefficient, the IH framework enables one to calculate the angular momentum and mass of the final black hole. These quantities refer only to the black hole in equilibrium and do not include the angular momentum and mass in matter fields or radiation outside the IH and thus could be found numerically.

Grandclément described a new approach to binary black hole evolution prior to the plunge phase.[6] One important feature of this phase of the evolution is the location of the innermost stable circular orbit (ISCO). This has been calculated both using post-Newtonian approximations and numerically from quasi-static equilibria with different results. If two orbiting bodies are far apart, the timescale associated with the gravitational radiation is much longer than the orbital period allowing one to assume that the two black holes are on exact circular orbits. Using multi-domain spectral methods, a sequence of two identical corotating black holes is computed. The ISCO appears as a turning point in the
total energy and angular momentum curves. Its position agrees well (for the first time) with the post-Newtonian values.

Several talks were given on characteristic codes. For a review of this approach, see [8]. Bishop described the initial value problem for a 3D characteristic code. The code is robust and stable and has been extended to include matter[9] although in a simple way that would not handle shocks. The code cannot evolve a whole spacetime, but requires data on an inner world tube. For this reason, it is important to identify initial data of the required type to represent a physical situation with strong gravitational fields.

D’Inverno and Pollney discussed lower dimensional code tests for a Cauchy characteristic matching (CCM) code. See (e.g.) [10]. The advantages of such a code are that it dispenses with an outer boundary condition and (since it uses a compactified coordinate) can yield global solutions in which gravitational waves can be identified unambiguously at future null infinity. Recently, a master vacuum axisymmetric CCM code has been completed. The main motivation for this work is to construct a three dimensional code possessing the characteristic, injection and extraction modules (now present in the 2D code) which can be attached to existing interior codes based on a finite grid.

A completely different approach involves a 3D code based on the conformal field equations developed by Friedrich. This work was presented by Hübner. He discussed the evolution of linear and (mildly) nonlinear gravitational waves. This approach aims at the numerical computation of the global structure of generic isolated systems in GR using well defined and unambiguous algorithms even out to and beyond future null infinity. The basic ideas and algorithms were developed by Hübner and Weaver in recent years and implemented by Hübner. Hübner[11] obtained the complete future of (the physical part of) the initial slice, illustrating a theorem by Friedrich, which states that for sufficiently weak initial data a regular point \( i^+ \) exists. However Hübner found that for higher amplitudes the gauge chosen in by Hübner results in code crashes which can be cured by some ad-hoc modification of the lapse. For still stronger (but still only mildly nonlinear) data, evolution up to \( i^+ \) would require the tuning of too many parameters to compensate for the lack of symmetry. Evolutions of axially symmetric initial data modeled in the spirit of “Brill waves”, and evolutions of Minkowski space in a static gauge were discussed. While the former evolution is numerically stable and quite robust, the latter exhibit instabilities which seem to be rooted in exponentially growing constraint violating modes inherent in the analytical formulation.

Siebel presented the results of a study using the characteristic approach to identify gravitational radiation. The implementation follows the characteristic initial value formulation of general relativity based on Bondi’s radiative metric. The study focuses on the evolution of neutron stars modeled by a \( N = 1 \) polytrope with fully relativistic hydrodynamics[12]. The code was used to study spherical relativistic stars with a scalar field to model the gravitational degrees of freedom. Depending on the specific neutron star model, the scalar field either induces oscillations of the star or makes it collapse to a black hole. The extracted frequencies of the oscillations agree very well with linear studies, ex-
cept, as expected, at the threshold of black hole formation. The code was also applied to evolutions of neutron stars in axisymmetry and to their interaction with gravitational waves.

Stark (with Lun) considered an alternative 3+1 scheme as applied to spherical perfect fluid collapse. The scheme utilizes the constraint equations contained in the Einstein field equations with those in the Bianchi identities to determine the 3-curvature tensor and the 3-covariant derivatives of the extrinsic curvature of the hypersurface. This yields a hyperbolic system of equations for the evolution of the gravitational potentials and hydrodynamical quantities. A numerical implementation of this system was used to model the Oppenheimer-Volkoff solution and the gravitational collapse of an initially homogeneous dust sphere to a BH.

Moreschi discussed modeling the collision of black holes with angular momentum with Robinson-Trautman (RT) spacetimes. These spacetimes contain purely outgoing gravitational radiation and decay to leave a Schwarzschild-like horizon. These RT spacetimes are natural candidates to study systems settling to a single black hole and may be used as background spacetimes for a perturbation treatment. Previous studies of this type give excellent agreement with numerical relativity results for the head-on collision, zero spin case. They are then extended to include angular momentum.

Reula presented results using the collapse of spherically symmetric self-gravitating scalar fields as a model problem to study numerical instabilities. There are instabilities present in this simple problem, and furthermore there are strong indications that, more generally, the instabilities are predominantly due to a longitudinal or Newtonian mode, namely the only one present in the spherically symmetric case. Free and constrained evolutions are compared. In a symmetric hyperbolic formulation is used, and certain freedom, still available in that setting, is used to suppress the main instability found there, to allow a stable propagation. However, their discretization scheme has first order numerical dissipation and the boundary-value problem for the system is not known to be well posed. Reula discussed an improved system that overcomes these problems, preserves the constraints, and reproduces several known results.

Gentle applied Regge calculus methods to the evolution of Brill waves. This is the first significant test of Regge calculus in a highly dynamic setting. An axisymmetric lattice is obtain by triangulating and collapsing a hypercubic grid which is aligned with a cylindrical polar coordinate system (see and references therein).

Neutron stars might act as a source of gravitational waves. Sperhake presented a new numerical approach to nonlinear oscillations in neutron stars. This approach rewrites the equations to cancel out the background terms in the numerical evolution of the perturbations. A comparison of the resulting perturbative code with a “standard” non-perturbative method was made. The perturbative scheme reproduces the expected harmonic oscillations with high accuracy, while the non-perturbative scheme produced spurious behavior. The new method is then used to accurately evolve initial data of arbitrary amplitude to investigate non-linear coupling of eigenmodes for a simplified neutron
star model.

Hawley (with Choptuik) explore the genericity of initial data for multi-scalar stars. They consider a class of general relativistic soliton-like solutions composed of multiple minimally coupled, massive, real scalar fields without self-interaction.

3 Numerical Investigation of Singularities

For reviews of critical phenomena see [18, 19]. Gundlach described what happens to the critical behavior of the gravitational collapse of a perfect fluid when there are departures from spherical symmetry. Does the type II critical behavior in the gravitational collapse of a $p = k \rho$ perfect fluid persist beyond the restriction to spherical symmetry? It was found that for $1/9 < k < 0.49$, all non-spherical perturbations decay. For $0.49 < k < 1$, there are one or several growing modes in the polar perturbations. For $0 < k < 1/9$, there is exactly one non-spherical growing mode. It is an $l = 1$ axial mode, linked to infinitesimal rotation. BH mass and BH angular momentum become a function of 2-parameter families of initial data, chosen so that one of the parameters is linked to rotation (and thus can be tuned to zero by axisymmetry). At the BH threshold they depend on the initial data through one universal function of one variable, a combination of these two parameters. The formula obtained is completely analogous to the dependence of the magnetization of a ferromagnet as a function of temperature and external magnetic field near its critical point.

Aichelburg presented results (including those of Lechner) on a new transition between discrete and continuous self-similarity in collapsing $SU(2)$ $\sigma$-models parametrized by a dimensionless coupling constant. [20] In the intermediate range, a competition between CSS and DSS solutions gives rise to new phenomena in the dynamical evolution of a critical search: the appearance of episodes of approximate CSS, where repeatedly the evolution approaches and departs from CSS before leading to black hole formation or dispersion. This picture is supported by the explicit numerical construction of the CSS and DSS solutions and its comparison with critical time evolutions.

Szpak studied critical behavior in a model nonlinear wave problem. [21] The motivation was to study the existence of an universal intermediate attractor in the dynamics at the threshold of forming singularity.

Hobill (with Webster) presented studies of trapped surface formation in Brill wave evolution to see if a naked singularity could form. The oblateness or prolateness of the system is determined by measuring the ratio of polar circumference to equatorial circumference for a surface with constant logarithmic radial coordinate. It is found that for initial data with positive amplitude the systems are prolate and for negative amplitude the systems are oblate. It is seen that, as the Brill wave amplitude is increased (thereby decreasing the radius of the outermost minimal surface), the location of the outermost trapped surface eventually coincides with the minimal surface thereby preventing the formation of a naked singularity. This behavior is in agreement with recent results of Garfinkle and Duncan. [22]
I presented a summary of our (with D. Garfinkle, J. Isenberg, V. Moncrief, M. Weaver) program of investigation of collapsing spatially inhomogeneous cosmologies. Significant recent progress has been made in rigorous studies of generic collapse in cases where one expects the approach to the singularity to be asymptotically velocity term dominated. However, one expects that an even larger class of spacetimes exhibits local Mixmaster dynamics (LMD) in the vicinity of the singularity. While numerical evidence for LMD in spatially inhomogeneous spacetimes with $T^2$ and $U(1)$ symmetries is compelling, rigorous results for spatially inhomogeneous cosmological spacetimes with LMD do not exist. New mathematical methods to handle LMD are needed.

4 Algebraic Computing: Invariant Classification in Maple

D’Inverno provided a summary of the program to classify solutions to Einstein’s equations automatically. Recent developments are in collaboration with Pollney, Skea, Araújo, de Albuquerque, and Roveda. A short review of the equivalence problem in general relativity was presented which included a description of the computer database of exact solutions whose central site is located at http://www.jim.dft.uerj.br. The database contains over 200 solutions of Einstein’s equations which have been classified with CLASSI, an extension of the general relativity system called SHEEP. In the next stage of this collaborative project attention has been focused on setting up a database which can easily be accessed and updated by the user community. The choice of platform is the general relativity package GRTensor which is an applications package of the computer algebra system Maple. The underlying algorithm is based on the Cartan-Karlhede invariant classification of geometries. In particular, new algorithms have been obtained for putting the Weyl spinor, Ricci spinor and general spinors into standard form and the derivative operators needed in the classification algorithm and the behavior of symmetric spinors under frame rotations have been investigated. More recent work was reported on a refinement of the JMS (Joly-MacCallum-Seixas) algorithm for completely determining the Segre type of a Ricci spinor. To date 30 of the 45 possible sub-cases have been completed and the algorithms have been implemented in GRTensor. In addition, the status of the work on obtaining the isometry group of a space-time (rather than simply its dimension as in the current database) was reported. It is currently possible to obtain the isometry group for spacetimes admitting a G2, G3 and some G4 symmetries. This may require the user to input limits on parameters and coordinates appearing in the line element, to overcome an ambiguity in determining the signature of a matrix occurring in the Bianchi classification. The goal is to complete the G4 case and extend the work to the G5, G6 and G7 cases. Work was also reported on the Maple package SPIDOR, which has been developed for obtaining spinor contractions. An example of its application is its ability to automatically generate the spinor curvature invariants built out.
of the Weyl spinor, its complex conjugate and the Ricci spinor up to the 20th order on a laptop. This package may be useful in relation to formalisms which have been developed to address particular issues in exact solutions.

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