Sharp lifespan estimates of blowup solutions to semilinear wave equations with time-dependent effective damping

Masahiro Ikeda*, Motohiro Sobajima† and Yuta Wakasugi‡

Abstract. In this paper we consider the initial value problem for the semilinear wave equation with an effective damping

\[
\begin{aligned}
\partial_t^2 u(x, t) - \Delta u(x, t) + b(t) \partial_t u(x, t) &= |u(x, t)|^{p-1} u(x, t), & (x, t) &\in \mathbb{R}^N \times (0, T), \\
u(x, 0) &= \varepsilon f(x), & x &\in \mathbb{R}^N, \\
\partial_t u(x, 0) &= \varepsilon g(x), & x &\in \mathbb{R}^N, \\
\end{aligned}
\]

(DW)

where \( N \in \mathbb{N}, 0 < b \leq C^1([0, \infty)), \) \( 1 < p \leq 1 + \frac{N}{2} \) and \( \varepsilon > 0 \) is a parameter describing the smallness of initial data. Here the coefficient \( b(t) \) of the damping term is assumed to be “effective”. The interest of this paper is to clarify the effective damping of the form \( b(t) = 1 + t^{\beta} \) \( (\beta \in (-1, 1)) \) and the threshold case \( b(t) = 1 + t \) in the sense of overdamping are discussed in Ikeda–Wakasugi [13] and Ikeda–Inui [9], respectively. In the present paper we discuss general damping terms with a certain assumption. The result of this paper is the sharp lifespan estimates of blowup solutions to (DW) including the typical case \( b(t) = 1 + t^{1/(1 + \log(1 + t))} \). The proof of upper bound of lifespan is a modification of the test function method given in [12] and the one of lower bound is based on the technique of scaling variables introduced in Gallay–Raugel [8] (for \( N = 1 \)) and Wakasugi [24] (for \( N \geq 2 \)).

Mathematics Subject Classification (2010): Primary: 35L71.
Key words and phrases: Wave equation with time-dependent damping, Small data blowup, Lifespan.

1 Introduction

In this paper we consider the blowup phenomena for the initial value problem of the semilinear wave equation with an effective damping of the form

\[
\begin{aligned}
\partial_t^2 u(x, t) - \Delta u(x, t) + b(t) \partial_t u(x, t) &= |u(x, t)|^{p-1} u(x, t), & (x, t) &\in \mathbb{R}^N \times (0, T), \\
u(x, 0) &= \varepsilon f(x), & x &\in \mathbb{R}^N, \\
\partial_t u(x, 0) &= \varepsilon g(x), & x &\in \mathbb{R}^N, \\
\end{aligned}
\]

(1.1)

where \( N \in \mathbb{N}, b \in C^1([0, \infty)), \) \( \varepsilon > 0 \) is a small parameter and \( f, g \) are given functions satisfying

\[(f, g) \in (H^1(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)) \times (L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)).\]

The term \( b(t) \partial_t u \) describes the damping effect which provides the reduction of the energy as a wave. Therefore the size of \( b(t) \) could affect to the profile of solution for sufficiently large \( t \). The interest of this paper is to clarify the effect of damping coefficient in terms of the behavior of the lifespan with respect to \( \varepsilon \).

* Department of Mathematics, Faculty of Science and Technology, Keio University, 3-14-1 Hiyoshi, Kohoku-ku, Yokohama, 223-8522, Japan/Center for Advanced Intelligence Project, RIKEN, Japan, E-mail: masahiro.ikeda@keio.jp/masahiro.ikeda@riken.jp
† Department of Mathematics, Faculty of Science and Technology, Tokyo University of Science, 2641 Yamazaki, Noda-shi, Chiba, 278-8510, Japan, E-mail: msoi@1984gmail.com
‡ Graduate School of Science and Engineering, Ehime University, 3, Bunkyo-cho, Matsuyama, Ehime, 790-8577, Japan, E-mail: wakasugi.yuta.v1@ehime-u.ac.jp.
The equation in (1.1) with \( b(t) \equiv 1 \) (without a nonlinear term) was introduced in Cattaneo [1] and Vernotte [22] to consider a model of heat conduction with finite propagation property. This equation is composed by “balance law” \( u_t = \text{div} q \) and “time-delayed Fourier law” \( \tau q_t + q = \nabla u \), where \( q \) is the heat flux and \( \tau \) is sufficiently small.

In the case \( b(t) \equiv 1 \), the equation (1.1) becomes the usual damped wave equation and therefore there are many previous works dealing with global existence and blowup of solutions to (1.1) with lifespan estimates (see e.g., Li–Zhou [17], Todorova–Yordanov [21], Nishihara [19], Ikeda–Wakasugi [13], Ikeda–Ogawa [10], Lai–Zhou [16]). As a summary, the Fujita exponent \( p = 1 + 2/N \) plays a role of critical exponent dividing the global existence and blowup of small solutions. The lifespan estimates are given as the following:

\[
\text{LifeSpan}(u) \sim \begin{cases} 
  C e^{-(p-1)(n+1)/2} & \text{if } 1 < p < 1 + \frac{2}{N}, \\
  \exp(C e^{-(p-1)}) & \text{if } p = 1 + \frac{2}{N}, \\
  \infty & \text{if } p > 1 + \frac{2}{N}
\end{cases}
\]

for sufficiently small \( \varepsilon > 0 \).

In the case \( b(t) = (1 + t)^{-\beta} \) with \( \beta \in (-1, 1) \), Lin–Nishihara–Zhai [18] found that the critical exponent in this case remains \( p = 1 + 2/N \). Later, the damping is generalized to the profile of \( b(t) \) as \( t \to \infty \) by D’Abbicco–Lucente [3] and D’Abbicco–Lucente–Reissig [4] and then the critical exponent remains \( p = 1 + 2/N \) again.

We have to mention that \( b(t) = (1 + t)^{-\beta} \) is so-called scale-invariant damping and in this case the effect of wave structure cannot be ignored in the sense of existence of global solutions. Actually, in Ikeda–Sobajima [11] a blowup result for \( 1 < p \leq p_S(N + \mu) \) is given for small damping case \( \mu \in (0, \frac{N}{N+2}) \), where \( p_S(n) \) is well-known Strauss exponent given by the positive root of the quadratic equation \( (n-1)p^2 - (n+1)p - 2 = 0 \). We also refer the reader to D’Abbicco [2] and D’Abbicco–Lucente–Reissig [5] for global existence results and determination of the critical exponent for the special case \( \mu = 2 \), respectively. In the scattering case \( b(t) = (1 + t)^{-\beta} \) with \( \beta > 1 \), Lai–Takamura [15] proved the blowup result for \( 1 < p < p_S(N) \), and therefore, in this case the damping term can be ignored.

On the other hand, if \( b(t) = (1 + t)^{-\beta} \) with \( \beta < -1 \), then the situation is completely different. In this case, according to the result by Ikeda–Wakasugi [14], the critical exponent disappears, that is, there exists small global solution of (1.1) for every \( p > 1 \). This phenomenon is so-called overdamping. This means that the case \( \beta = -1 \) can be regarded as the threshold for dividing effective and overdamping cases which is also discussed with Wirth [25, 26, 27, 28] for the linear equation.

Recently, Ikeda–Inui [9] gave the blowup result for the case \( b(t) = (1 + t)^{-\beta}, \beta \in [-1, 1] \) with the critical nonlinearity \( p = 1 + 2/N \) together with sharp lifespan estimates as follows

\[
\text{LifeSpan}(u) \sim \begin{cases} 
  \exp(C e^{-(p-1)}) & \text{if } b(t) = (1 + t)^{-\beta}, \beta \in (-1, 1), \ p = 1 + \frac{2}{N}, \\
  \exp \exp(C e^{-(p-1)}) & \text{if } b(t) = 1 + t, \ \beta = -1, \ p = 1 + \frac{2}{N}.
\end{cases}
\]

The first purpose of the present paper is to determine the critical exponent dividing the global existence and blowup of small solutions to (1.1) in more general damping coefficients including

\[ b(t) = 1 + t, \quad b(t) = (1 + t)(1 + \log(1 + t)). \]

The second is to give a sharp estimate for lifespan of blowup solutions to (1.1) in view of the small parameter \( \varepsilon > 0 \).

Our main result for the (implicit) upper bound of lifespan is as follows.
Theorem 1.1. Assume that

\[ b(t) > 0, \quad t \geq 0, \]
\[ b_0 := \limsup_{t \to \infty} \left( \frac{|b'(s)|}{b(s)} \right)^2 < 1 \]

and \((f, g) \in (H^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)) \times (H^1(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))\) with

\[ \int_{\mathbb{R}^N} f(x) \, dx + B_0 \int_{\mathbb{R}^N} g(x) \, dx > 0, \]

where

\[ B_0 := \int_0^\infty \exp \left( - \int_0^s \frac{b(\sigma)}{\sigma} \, d\sigma \right) \, ds < \infty \]

(which is valid under (1.2) and (1.3), see Lemma 2.1). If \(1 < p \leq 1 + \frac{2}{N}\), then there exists a positive constant \(C > 0\) such that the solution \(u_\varepsilon\) of (1.1) satisfies

\[ B(\text{LifeSpan}(u_\varepsilon)) \leq \begin{cases} Ce^{-\left(\frac{b_0}{b(\varepsilon)}\right)^{p^{-1}}} & \text{if } 1 < p < 1 + \frac{2}{N}, \\ \exp(Ce^{-p}) & \text{if } p = 1 + \frac{2}{N}, \end{cases} \]

where

\[ B(t) := \int_0^t \frac{1}{b(\sigma)} \, d\sigma. \]  

Additionally, if

\[ \frac{1}{b(t)} \notin L^1(0, \infty), \]

then small data blowup phenomena occurs.

The proof of Theorem 1.1 is done by using a test function method with a solution of the conjugate equation \(\partial_t^2 \Phi - \Delta \Phi - \partial_t(b(t)\Phi) = 0\) and rescaled cut-off functions. Also we use the idea for deriving upper bound of lifespan in [12].

Example 1. (1) The typical cases \(b(t) = (1 + t)^{-\beta}\) satisfy the condition (1.3) if \(\beta \in (-\infty, 1)\). In particular, we have \(b_0 = 0\) with \(\frac{|b'(s)|}{b(s)^p} = \beta(1 + t)^{-\beta-1}\). Also, \(b(t) = (1 + t)^{-\beta}\) satisfies the condition (1.5) if \(\beta \in [-1, \infty)\).

(2) In the scale invariant case \(b(t) = \mu(1 + t)^{-1} > 0\) with \(\mu > 1\), we see that \(b_0 = \mu^{-1} < 1\) and (1.3) holds.

It is worth noticing that the lifespan estimate in Theorem 1.1 is true even if we consider the following parabolic problem non-trivial initial data:

\[ \begin{aligned}
   b(t)\partial_t u(x, t) - \Delta u(x, t) = u(x, t)^p, \quad (x, t) \in \mathbb{R}^N \times (0, T), \\
   u(x, 0) = \varepsilon f(x) \geq 0, \quad x \in \mathbb{R}^N.
\end{aligned} \]  

This is clear if we consider the Fujita type equation with change of variables \(u(x, t) = v(x, B(t))\)

\[ \begin{aligned}
   \partial_s v(x, s) - \Delta v(x, s) = v(x, s)^p, \quad (x, s) \in \mathbb{R}^N \times (0, S), \\
   v(x, 0) = \varepsilon f(x) \geq 0, \quad x \in \mathbb{R}^N.
\end{aligned} \]  

3
To self-containedness, we would give a short proof of lower lifespan estimates in Appendix.

Next, we study the lower bound of lifespan of solutions to (1.1). In the following, we assume (1.2), (1.5), and the following stronger version of (1.3):

\[
\text{There exist } \gamma > 0 \text{ and } C > 0 \text{ such that } \frac{|b'(t)|}{b(t)^2} \leq C(t + 1)^{-\gamma} \text{ for } t > 0. \tag{1.8}
\]

We denote by \(H^{s,m}(\mathbb{R}^N)\) with \(s \in \mathbb{Z}_{\geq 0}\) and \(m \geq 0\) the weighted Sobolev space

\[
H^{s,m}(\mathbb{R}^N) = \left\{ f \in L^2(\mathbb{R}^N) ; \|f\|_{H^{s,m}} = \sum_{|\alpha| \leq s} \| (1 + |x|)^m \partial_x^\alpha f \|_{L^2} < \infty \right\}.
\]

We consider the initial data belonging to

\[
(f, g) \in H^{2,m}(\mathbb{R}^N) \times H^{1,m}(\mathbb{R}^N), \tag{1.9}
\]

where \(m\) satisfies

\[
m = 1 \ (N = 1), \quad m > \frac{N}{2} \ (N \geq 2). \tag{1.10}
\]

We remark that (1.9)–(1.10) imply \((f, g) \in (H^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)) \times (H^1(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)).\) For the nonlinearity, we assume

\[
1 < p < \infty \ (N = 1, 2), \quad 1 < p \leq \frac{N}{N-2} \ (N \geq 3). \tag{1.11}
\]

Under these assumptions, the existence of a unique local solution

\[
u_\varepsilon \in C([0, T); H^{2,m}(\mathbb{R}^N)) \cap C^1([0, T); H^{1,m}(\mathbb{R}^N))
\]

to (1.1) has already been proved by [24, Propositions 3.5, 3.6]. Thus, we define \(\text{LifeSpan}(u_\varepsilon)\) by the maximal existence time of the local solution. In this section, following the argument in [6, 24], we prove the sharp lower bound of the lifespan.

**Theorem 1.2.** Assume that (1.2), (1.5), (1.8), and (1.9)–(1.11) are satisfied. Then, there exist \(\varepsilon_0 > 0\) and \(C > 0\) such that for every \(\varepsilon \in (0, \varepsilon_0)\), one has

\[
B(\text{LifeSpan}(u_\varepsilon)) \geq \begin{cases} 
Ce^{-\left(\frac{1}{p-1} - \frac{N}{2}\right)^{-1}} & \text{if } 1 < p < 1 + \frac{2}{N}, \\
\exp\left(Ce^{-(p-1)}\right) & \text{if } p = 1 + \frac{2}{N}, \\
\infty & \text{if } p > 1 + \frac{2}{N}.
\end{cases} \tag{1.12}
\]

The proof is based on the method of scaling variables introduced by Gallay and Raugel [8]. In [24, 6], the global existence, asymptotic behavior, and the lower bound of the lifespan of solutions to the nonlinear problem (1.1) with \(b(t) = (1 + t)^\beta (-1 \leq \beta < 1)\) are studied, which is a typical example satisfying (1.2), (1.5) and (1.8).

**Remark 1.1.** From the proof of the supercritical case \(p > 1 + \frac{2}{N}\) with small additional argument, we can also have the asymptotic profile of the global solution \(u_\varepsilon\). More precisely, we can prove

\[
\|u_\varepsilon(\cdot, t) - \alpha^*(4\pi B(t))^{-\frac{N}{2}} e^{-\lambda t} \|_{L^2} = O(B(t)^{-\frac{N}{2}-1})
\]

with some \(\lambda > 0\) as \(t \to \infty\), where \(\alpha^* = \lim_{t \to \infty} \int_{\mathbb{R}^N} u_\varepsilon(x, t) \, dx\). For the detail, see [24, Section 3.9].
Here, we note that $b(t) = \mu(1 + t)^{-1}$ with $\mu > 0$ does not satisfy (1.8) (see Example 1 (2) above). Thus, this case is excluded for the lower bound of the lifespan.

To illustrate the result of the present paper, we give several lifespan estimates for the typical damping coefficients.

**Corollary 1.3.** Under the assumptions in Theorems 1.1 and 1.2, for respective cases, one has the following explicit bound of $\text{LifeSpan}(u_\epsilon)$:

(i) If $b(t) = (1 + t)^{\beta}$ with $\beta \in (-1, 1)$, then

\[
\text{LifeSpan}(u_\epsilon) \sim \begin{cases} 
C e^{-(1+\beta)^{-1} \left(\frac{1}{1+\beta} - \frac{\beta}{1}\right)^{-1}} & \text{if } 1 < p < 1 + \frac{2}{N}, \\
\text{exp}(C e^{-(p-1)^{-1}}) & \text{if } p = 1 + \frac{2}{N}.
\end{cases}
\]

(ii) If $b(t) = 1 + t$, then

\[
\text{LifeSpan}(u_\epsilon) \sim \begin{cases} 
\text{exp}(C e^{-(\frac{1}{p} - \frac{2}{N})^{-1}}) & \text{if } 1 < p < 1 + \frac{2}{N}, \\
\text{exp}(\text{exp}(C e^{-(p-1)^{-1}})) & \text{if } p = 1 + \frac{2}{N}.
\end{cases}
\]

(iii) If $b(t) = (1 + t)(1 + \log(1 + t))$, then

\[
\text{LifeSpan}(u_\epsilon) \sim \begin{cases} 
\text{exp}(\text{exp}(C e^{-(\frac{1}{p} - \frac{2}{N})^{-1}})) & \text{if } 1 < p < 1 + \frac{2}{N}, \\
\text{exp}(\text{exp}(C e^{-(p-1)^{-1}})) & \text{if } p = 1 + \frac{2}{N}.
\end{cases}
\]

(iv) If $b(t) = \prod_{k=1}^{n} \ell_k(t)$ ($n \geq 3$) with $\ell_1(t) = 1 + t$ and $\ell_{k+1}(t) = 1 + \log(\ell_k(t))$, then

\[
\text{LifeSpan}(u_\epsilon) \sim \begin{cases} 
\text{exp}^{[1]}(C e^{-(\frac{1}{p} - \frac{2}{N})^{-1}}) & \text{if } 1 < p < 1 + \frac{2}{N}, \\
\text{exp}^{[1]}(C e^{-(p-1)^{-1}}) & \text{if } p = 1 + \frac{2}{N},
\end{cases}
\]

where $\text{exp}^{[1]}(t) = \text{exp}(t)$ and $\text{exp}^{[k+1]}(t) = \text{exp}(\text{exp}^{[k]}(t))$.

(v) If $b(t) = \mu(1 + t)^{-1}$ with $\mu > 1$, then

\[
\text{LifeSpan}(u_\epsilon) \lesssim \begin{cases} 
C e^{-(1-1)^{-1}} & \text{if } 1 < p < 1 + \frac{2}{N}, \\
\text{exp}(C e^{-(p-1)^{-1}}) & \text{if } p = 1 + \frac{2}{N}.
\end{cases}
\]

**Remark 1.2.** In the case $b(t) = \mu(1 + t)^{-1}$ with $\mu > 1$, we do not obtain any lower bound for lifespan. The difficulty comes from the scale-invariant property of the damping term which breaks the advantage of the method of scaling variables in the proof of Theorem 1.2. Moreover, from another point of view, the upper lifespan estimate of solutions with scale-invariant damping has a wave-like profile as in [11] at least when $\mu \in (0, \frac{N^2 + N + 2}{N^2 + 2})$. If $\mu \geq \frac{N^2 + N + 2}{N^2 + 2}$, then we do not know whether the upper bound of lifespan in Corollary 1.3 (v) is sharp or not.

This paper is organized as follows: In Section 2, we collect important properties of the damping coefficient $b(t)$ and the profile of the solution to the linear conjugate equation of (1.1). Section 3 is devoted to prove upper bound for lifespan of solutions to (1.1) via a (time-rescaled) test function method with the solution of the linear conjugate equation. To close this paper, we give a proof of lower bound of lifespan of solutions to (1.1) via the method of scaling variables.
2 Preliminaries

Here we collect some basic properties of the damping coefficient $b(t)$ and the behavior of solution to

$$\partial_t^2 \Phi - \Delta \Phi - \partial_t(b(t)\Phi) = 0.$$ 

At the end, we introduce a family of cut-off functions with time-rescaling $s \sim \int_0^t b(\sigma)^{-1} d\sigma$.

2.1 Basic properties of the damping coefficient $b(t)$

First we prove some basic properties of $b(t)$, which we frequently use later.

**Lemma 2.1.** Assume that (1.2) and (1.3) are satisfied. Then one has

$$B_0 := \int_0^\infty \exp \left( - \int_0^s b(\sigma) d\sigma \right) ds < \infty,$$

$$\lim_{t \to \infty} \frac{1}{b(t)} \exp \left( - \int_0^t b(\sigma) d\sigma \right) = 0. \quad (2.2)$$

Assume further that (1.5) and (1.8) are satisfied. Then

$$\lim_{t \to \infty} \frac{1}{b(t)^2(B(t) + 1)} = 0. \quad (2.3)$$

**Proof.** By (1.3), there exist $t_0 > 0$ and $\delta > 0$ such that for every $t \geq t_0$,

$$\left| \frac{d}{dt} b(t)^{-1} \right| \leq 1 - \delta.$$

This yields that for every $t \geq t_1 = \max\{t_0, 2[\delta b(t_0)]^{-1}\}$,

$$b(t)^{-1} \leq b(t_0)^{-1} + (1 - \delta)(t - t_0) \leq (1 - \delta/2)t.$$

Therefore

$$\exp \left( - \int_0^t b(\sigma) d\sigma \right) \leq \exp \left( - \int_0^{\tau} b(\sigma) d\sigma \right) \times t_1^{(1-\delta/2)^{-1}} t^{-\gamma}.$$

This implies (2.1) and also (2.2).

On the other hand, assume (1.5) and (1.8). Taking $\tau > 1$ arbitrary, we see from (1.8) that

$$\bar{B}(t) = \frac{1}{b(t)^2(B(t) + 1)} = \frac{b(t)^{-2} + 2 \int_0^t b(\sigma)^{-1} \frac{d}{d\sigma} \left( b(\sigma)^{-1} \right) d\sigma}{B(t) + 1} \leq \frac{b(\tau)^{-2} + 2C(1 + \tau)^{-\gamma} B(t)}{B(t) + 1}.$$

Using (1.5), we deduce $\limsup_{t \to \infty} \bar{B}(t) \leq 2C(1 + \tau)^{-\gamma}$ and then (2.3) is shown. □
2.2 Construction of solutions to the conjugate equation

To find blowup phenomena, we will use the solution of the conjugate linear equation of (1.1)

\[ \partial_t^2 \Phi(x, t) - \Delta \Phi(x, t) - \partial_t (b(t) \Phi(x, t)) = 0. \]

In the current case, we can choose \( \Phi(x, t) = \Phi(t) \) (independent of \( x \)). The equation is reduced to

\[ \partial_t (\partial_t \Phi(t) - b(t) \Phi(t)) = 0. \]

The all solutions of the above equation are given by

\[ \Phi_{c_0, c_1}(t) = \exp \left( \int_0^t b(\sigma) \, d\sigma \right) \left[ c_0 + c_1 \int_0^t \exp \left( - \int_0^\sigma b(\tau) \, d\tau \right) \, d\sigma \right] \]

with \( c_0, c_1 \in \mathbb{R} \). Then we fix the parameters \( c_0, c_1 \in \mathbb{R} \) (in the former case) and collect the properties of \( \Phi \), which we use later.

**Lemma 2.2.** Assume that (1.2) and (1.3) are satisfied. Define for \( t \geq 0 \),

\[ \Phi(t) = \int_t^\infty \exp \left( - \int_0^\sigma b(\tau) \, d\tau \right) \, d\sigma. \]

Then \( \Phi \) is well-defined and satisfies the following properties:

(i) \( \Phi(0) = B_0 \), and \( \partial_t \Phi(t) - b(t)\Phi(t) = -1 \) for every \( t \geq 0 \).

(ii) There exist constants \( t_0 > 0 \), \( B_1 > 0 \) and \( B_2 > 0 \) such that for every \( t \geq t_0 \),

\[ \frac{B_1}{b(t)} \leq \Phi(t) \leq \frac{B_2}{b(t)} \]

(iii) For every \( t \geq t_0 \),

\[ |\partial_t \Phi(t)| \leq \frac{1 + b_0}{1 - b_0} \]

and in particular \( \partial_t \Phi \) is bounded in \([0, \infty)\).

**Proof.** First, in view of (2.1) in Lemma 2.1, we can choose

\[ c_0 = B_0 = \int_0^\infty \exp \left( - \int_0^\sigma b(\tau) \, d\tau \right) \, d\sigma > 0, \quad c_1 = -1 \]

and then \( \Phi_{c_0, c_1} \) is nothing but the function \( \Phi \) in this lemma. The assertion (i) is clear by the construction of \( \Phi_{c_0, c_1} \). For (ii), by integration by parts and (2.2) in Lemma 2.1 we have

\[
\Phi(t) = \int_t^\infty \frac{b(s)}{b(s)} \exp \left( - \int_t^\sigma b(\tau) \, d\tau \right) \, d\sigma = \left[ -\frac{1}{b(s)} \exp \left( - \int_t^\sigma b(\tau) \, d\tau \right) \right]_t^{\infty} - \int_t^\infty \frac{b'(s)}{b^2(s)} \exp \left( - \int_t^\sigma b(\tau) \, d\tau \right) \, d\sigma = \frac{1}{b(t)} - \int_t^\infty \frac{b'(s)}{b^2(s)} \exp \left( - \int_t^\sigma b(\tau) \, d\tau \right) \, d\sigma.
\]
This implies that
\[
\left| \Phi(t) - \frac{1}{b(t)} \right| \leq \int_t^\infty \frac{|b'(s)|}{b^2(s)} \exp \left( - \int_t^s b(\sigma) \, d\sigma \right) \, ds
\]
\[
\leq \sup_{s \geq t} \left( \frac{|b'(s)|}{b^2(s)} \right) \int_t^\infty \exp \left( - \int_t^s b(\sigma) \, d\sigma \right) \, ds
\]
\[
\leq \sup_{s \geq t} \left( \frac{|b'(s)|}{b^2(s)} \right) \Phi(t).
\]

Since (1.3) gives that there exists \( t_0 > 0 \) such that \( \sup_{s \geq b_0} \left( \frac{|b'(s)|}{b^2(s)} \right) \leq \frac{1+b_0}{2} < 1 \), we deduce
\[
\frac{2}{3 + b_0} \frac{1}{b(t)} \leq \Phi(t) \leq \frac{1}{1 - b_0} \frac{1}{b(t)}.
\]

Moreover, noting that
\[
|\partial_t \Phi(t)| = |b(t)\Phi(t) - 1| \leq \sup_{s \geq t} \left( \frac{|b'(s)|}{b^2(s)} \right) b(t)\Phi(t) \leq \frac{1 + b_0}{1 - b_0},
\]
we have (iii). \( \square \)

### 2.3 Choice of cut-off functions

The choice of the cut-off functions are based on that in Ikeda–Sobajima [12] with time rescaling.

Now we set two kinds of functions \( \eta \in C^2(0, \infty) \) and \( \eta^* \in L^\infty((0, \infty)) \) as follows:

\[
\eta(s) = \begin{cases} 
1 & \text{if } s \in [0, 1/2], \\
\text{is decreasing} & \text{if } s \in (1/2, 1), \\
0 & \text{if } s \notin [1, \infty),
\end{cases}
\]

\[
\eta^*(s) = \begin{cases} 
0 & \text{if } s \in [0, 1/2), \\
\eta(s) & \text{if } s \in [1/2, \infty).
\end{cases}
\]

**Definition 2.1.** For \( p > 1 \), we define for \( R > 0 \),

\[
\psi_R(x, t) = |\eta(s_R(x, t))|^{2p}, \quad (x, t) \in \mathbb{R}^N \times [0, \infty),
\]

\[
\psi_R'(x, t) = |\eta^*(s_R(x, t))|^{2p'}, \quad (x, t) \in \mathbb{R}^N \times [0, \infty)
\]

with

\[
s_R(x, t) = R^{-1} \left( 1 + |x|^2 + \int_0^t \Phi(\sigma) \, d\sigma \right).
\]

We also set

\[
P(R) = \left\{ (x, t) \in \mathbb{R}^N \times [0, \infty) \mid 1 + |x|^2 + \int_0^t \Phi(\sigma) \, d\sigma \leq R \right\}
\]

and \( t_R > 0 \) as

\[
1 + \int_0^{t_R} \Phi(\sigma) \, d\sigma = R.
\]

To deduce the upper bound for the solution to (1.1), we need the following lemma which is essentially given by [12]. We will only give a crucial idea of its proof.
Lemma 2.3. Let $\delta > 0$, $C_0 > 0$, $R_1 > 0$, $\theta \geq 0$ and $0 \leq w \in L^1_{\log}([0, T); L^1(\mathbb{R}^N))$. Assume that

$$\widehat{R}(T) := 1 + \int_0^T \Phi(\sigma) \, d\sigma > R_1$$

and for every $R \in [R_1, \widehat{R}(T))$, 

$$\delta + \int_{P(R)} w(x, t)\psi_R(x, t) \, dx \, dt \leq C_0 R^{-\theta} \left(\int_{P(R)} w(x, t)\psi_R^*(x, t) \, dx \, dt\right)^{\frac{1}{p}}. \quad (2.4)$$

Then $T$ has an (implicit) upper bound as follows:

$$\widehat{R}(T) \leq \begin{cases} R_1^{(p-1)\theta} + (\log 2)C_0^p \theta \delta^{-(p-1)} \frac{1}{1-\theta} & \text{if } \theta > 0, \\
\exp\left(\log R_1 + (\log 2)(p-1)^{-1}C_0 \delta^{-(p-1)}\right) & \text{if } \theta = 0. 
\end{cases}$$

Proof. Set

$$y(r) = \int_{P(r)} w(x, t)\psi_R^*(x, t) \, dx \, dt, \quad Y(R) = \int_0^R \frac{y(r)}{r} \, dr.$$ 

Then by (2.4), we can deduce

$$((\delta + (\log 2)Y(R))^p \leq C_0^p R^{1-\theta}Y'(R).$$

This gives the desired upper bound for $R$ and also for $1 + \int_0^T \Phi(\sigma) \, d\sigma$. \hfill $\square$

Lemma 2.4. Let $\psi_R$ and $\psi_R^*$ be as in Definition 2.1. Then $\psi_R$ and $\psi_R^*$ satisfy the following properties:

(i) If $(x, t) \in P(R/2)$, then $\psi_R(x, t) = 1$, and if $(x, t) \notin P(R)$, then $\psi_R(x, t) = 0$.

(ii) There exists a positive constant $C_1$ such that for every $(x, t) \in P(R)$,

$$|\partial_t \psi_R(x, t)| \leq C_1 R^{-1} \Phi(t)[\psi_R^*(x, t)]^{\frac{1}{p}}.$$

(iii) There exists a positive constant $C_2$ such that for every $(x, t) \in P(R)$,

$$|\Delta \psi_R(x, t)| \leq C_2 R^{-1} [\psi_R^*(x, t)]^{\frac{1}{p}}.$$

(iv) Further assume that (1.5). Then there exists a positive constant $C_3$ such that for every $(x, t) \in P(R)$,

$$|\partial_x^2 \psi_R(x, t)| \leq C_3 R^{-1} [\psi_R^*(x, t)]^{\frac{1}{p}}.$$

Proof. The assertion (i) is trivial by the definition. On the other hand, (ii) and (iii) follow from standard calculations:

$$|\partial_t \psi_R| = 2p'[\eta'(s_R)]^{2p'-1}[\eta'(s_R)]\partial_t s_R$$

$$\leq 2p'\|\eta'\|_{L^\infty}[\eta'(s_R)]^{2p'-1}\frac{\Phi(t)}{R}$$

$$\leq 2p'\|\eta'\|_{L^\infty}\Phi(t)[\psi_R^*]^{\frac{1}{p}}.$$
and
\[
|\Delta \psi_R| = 2p' \left| (2p' - 1)\eta'(s_R)^2 |\nabla s_R|^2 + \eta(s_R)\eta''(s_R)|\nabla s_R|^2 + \eta(s_R)\eta'(s_R)\Delta s_R \right| \psi_R^\ast \frac{1}{\psi_R}\nabla \psi_R
\leq 2p' \left( \frac{4(2p' - 1)\eta'(s_R)^2}{R^2} + 4\eta''|L^\infty| |x|^2 + \frac{2N\eta\eta''}{R} \right) |\psi_R^\ast| \frac{1}{\psi_R}
\]
with \(|x|^2 \leq R\) on supp \(\psi_R\). For (iv), we see that
\[
|\partial_t^2 \psi_R| = 2p' \left| (2p' - 1)(\eta'(s_R)\partial_t s_R)^2 + \eta(s_R)\eta''(s_R)(\partial_t s_R)^2 + \eta(s_R)\eta'(s_R)\partial_t^2 s_R \right| \psi_R^\ast \frac{1}{\psi_R}
\leq 2p' \left( (2p' - 1)\eta''|L^\infty| + \eta''|L^\infty| \right) \frac{\Phi(t)^2}{R^2} |\psi_R^\ast| \frac{1}{\psi_R} + 2p' \eta''|L^\infty| \frac{\Phi'(t)}{R} |\psi_R^\ast| \frac{1}{\psi_R}
\]
Here using \(1 + \int_0^t \Phi(\sigma) d\sigma \leq B_0\) and Lemma 2.2 (iii), we have
\[
\frac{\Phi(t)^2}{R} \leq \frac{B_0^2}{1 + \int_0^t \Phi(\sigma) d\sigma} \leq \frac{B_0^2 + 2||\Phi''||L^\infty| \int_0^t \Phi(\sigma) d\sigma}{1 + \int_0^t \Phi(\sigma) d\sigma} \leq \max\{B_0^2, 2||\Phi''||L^\infty\}.
\]
Hence we obtain (iii). \(\square\)

3 Upper bound of lifespan

In this section we prove Theorem 1.1.

Proof of Theorem 1.1. Let \(u\) be a solution to (1.1) in \([0, T]\) with \(T = \text{LifeSpan}(u)\). We assume \(T > t_{R_0}\) with large \(R_0\) determined later (otherwise the assertion is obvious). Multiplying the equation in (1.1) to \(\Phi(t)\psi_R(x, t)\) and using integration by parts, we have
\[
\int_{\mathbb{R}^N} |u|^p\Phi(t)\psi_R \, dx = \int_{\mathbb{R}^N} \left( \partial_t^2 u - \Delta u + b(t)\partial_t u \right) \Phi(t)\psi_R \, dx
\]
\[
= \frac{d}{dt} \left( \int_{\mathbb{R}^N} \partial_t u \Phi(t)\psi_R - u\partial_t \left( \Phi(t)\psi_R \right) + b(t)u \Phi(t)\psi_R \right) \, dx
\]
\[
+ \int_{\mathbb{R}^N} u \left( \partial_t^2 \left( \Phi(t)\psi_R \right) - \Delta \left( \Phi(t)\psi_R \right) - \partial_t \left( b(t)\Phi(t)\psi_R \right) \right) \, dx. \tag{3.1}
\]

It follows from Lemmas 2.2 and 2.4 that
\[
|\partial_t^2 \left( \Phi(t)\psi_R \right) - \Delta \left( \Phi(t)\psi_R \right) - \partial_t \left( b(t)\Phi(t)\psi_R \right)|
\leq 2|\Phi'(t)\partial_t \psi_R| + \Phi(t)|\partial_t^2 \psi_R| + \Phi(t)|\Delta \psi_R| + b(t)\Phi(t)|\partial_t \psi_R|
\leq 2C_1 \frac{\Phi(t)|\psi_R^\ast|^{1\frac{1}{p}} + C_3 \Phi(t)|\psi_R^\ast|^{1\frac{1}{p}} + C_2 \Phi(t)|\psi_R^\ast|^{1\frac{1}{p}} + B_2 C_1 \Phi(t)|\psi_R^\ast|^{1\frac{1}{p}}}{\frac{R}{R}}
\]
\[
\leq \frac{C_4}{\Phi(t)|\psi_R^\ast|^{1\frac{1}{p}}}
\]
with \(C_4 = 2C_1||\Phi''||L^\infty + C_3 + C_2 + B_2 C_1\). Therefore integrating (3.1) on \([0, t_R]\) and using the above estimate, we have
\[
j(R) e + \int_{\mathbb{R}^N} |u|^p \Phi(t)\psi_R \, dx \, dt \leq \frac{C_4}{\Phi(t)|\psi_R^\ast|^{1\frac{1}{p}}} \int_{\mathbb{R}^N} |u|^p \Phi(t)|\psi_R^\ast|^{1\frac{1}{p}} \, dx, \tag{3.2}
\]
where
\[ j(R) = \int_{\mathbb{R}^N} (f(x) + B_0 g(x)) \psi_R(x, 0) \, dx - B_0 \int_{\mathbb{R}^N} f(x) \partial_t \psi_R(x, 0) \, dx. \]

Noting that
\[ \|\partial_t \psi_R(\cdot, 0)\|_{L^\infty} = 2p' \sup_{x \in \mathbb{R}^N} \left( \left[ \eta^+ (s_R(x, 0))\right]^{2p' - 1} \eta'(s_R(x, 0)) \right)^{\Phi(0)/R} \leq 2p' \|\eta\|_{L^\infty} \Phi(0) \to 0 \]
as \( R \to \infty \), we see from the dominated convergence theorem that there exists \( R_0 > 0 \) such that for every \( R \geq R_0 \),
\[ j(R) \geq c_0 = \frac{1}{2} \int_{\mathbb{R}^N} (f(x) + B_0 g(x)) \, dx > 0. \]

Therefore by (3.2) with the Hölder inequality, we have
\[ c_0 \epsilon + \int_{P(R)} |u|^p \Phi(t) \psi_R \, dx \, dt \leq \frac{C_4}{R} \left( \int_{P(R)} \Phi(t) \, dx \, dt \right)^{\frac{p}{p' - 1}} \left( \int_{P(R)} |u|^p \Phi(t) \psi_R^p \, dx \, dt \right)^{\frac{p'}{p' - 1}}. \]

Since
\[ \int_{P(R)} \Phi(t) \, dx \, dt \leq \int_0^T \int_{B(0, \sqrt{R})} \Phi(t) \, dx \, dt = |S^{N - 1}| R^{1 + \frac{N}{2}}, \]

The last inequality gives
\[ c_0 \epsilon + \int_{P(R)} |u|^p \Phi(t) \psi_R \, dx \, dt \leq C_5 R^{-\left( \frac{1}{p' - 1} - \frac{N}{2} \right)} \left( \int_{P(R)} |u|^p \Phi(t) \psi_R^p \, dx \, dt \right)^{\frac{1}{p'}}. \]

Applying Lemma 2.3 with \( w(x, t) = |u(x, t)|^p \Phi(t) \), we deduce
\[ 1 + \int_0^T \Phi(t) \, d\sigma \leq \begin{cases} C e^{-\left( \frac{1}{p' - 1} - \frac{N}{2} \right)} \left( \frac{1}{p' - 1} \right) & \text{if } 1 < p < 1 + \frac{2}{N}, \\ \exp(C e^{(p - 1)}) & \text{if } p = 1 + \frac{2}{N}. \end{cases} \]

By Lemma 2.2 (ii), we obtain
\[ B(T) = \int_0^T \frac{1}{b(\sigma)} \, d\sigma \leq \begin{cases} C' e^{-\left( \frac{1}{p' - 1} - \frac{N}{2} \right)} \left( \frac{1}{p' - 1} \right) & \text{if } 1 < p < 1 + \frac{2}{N}, \\ \exp(C' e^{(p - 1)}) & \text{if } p = 1 + \frac{2}{N}. \end{cases} \]

The proof is complete. \( \square \)

### 4 Lower bound of lifespan

In this section, we discuss the lower bound of lifespan for (1.1). Since the following proof is the almost same as those of [24, 6], we give only the outline of the proof of Theorem 1.2.

In what follows, for simplicity, we denote by \( u \) the solution of (1.1) instead of \( u_\epsilon \). We first apply the changing variables
\[ y = (B(t) + 1)^{-1/2} x, \quad s = \log(B(t) + 1) \quad (4.1) \]
to the equation (1.1). Conversely, we also have \( t = t(s) = B^{-1}(e^s - 1) \). If we introduce a new unknown function \((v, w)\) by the relation

\[
  u(x, t) = (B(t) + 1)^{-N/2}v((B(t) + 1)^{-1/2}x, \log(B(t) + 1)),
  \]

\[
  u_t(x, t) = b(t)^{-1}(B(t) + 1)^{-N/2-1}w((B(t) + 1)^{-1/2}x, \log(B(t) + 1)),
\]

then we have the first order system

\[
\begin{aligned}
  &v_s - \frac{y}{2} \cdot \nabla v - \frac{N}{2} v = w, \\
  \frac{e^{-s}}{b(t(s))^2} \left( w_s - \frac{y}{2} \cdot \nabla w - \left( \frac{N}{2} + 1 \right) w \right) + w = \Delta v + \frac{b'(t(s))}{b(t(s))^2} w + e^{\frac{N(1+N/p)}{N}} v^p, \\
  v(y, 0) = v_0(y) = e f(y), \\
  w(y, 0) = w_0(y) = e g(y),
\end{aligned}
\]

\((y, s) \in \mathbb{R}^N \times (0, S)\),

\(y \in \mathbb{R}^N\). (4.2)

We first recall the local existence result.

**Proposition 4.1** (241). **Under the assumptions (1.2), (1.5), (2.3), and (1.8), there exists** \( S > 0 \) **depending only on the norm** \( \|v_0, w_0\|_{H^{2m} \times H^{1,m}} \) **such that the Cauchy problem (4.2) admits a unique strong solution** \((v, w)\) **satisfying**

\[
(v, w) \in C([0, S); H^{2m}(\mathbb{R}^N) \times H^{1,m}(\mathbb{R}^N)) \cap C^1([0, S); H^{1,m}(\mathbb{R}^N) \times H^{0,m}(\mathbb{R}^N)).
\]

**Moreover**, we have the almost global existence for small data, namely, for arbitrary fixed time \( S' > 0 \), by taking \( \epsilon \) sufficiently small, we can extend the solution to the interval \([0, S']\) with the estimate

\[
\|v(w)(S')\|_{H^{1,m} \times H^{0,m}} \leq C \|v_0, w_0\|_{H^{1,m} \times H^{0,m}}.
\]

**Finally**, we have the blow-up alternative, namely, if

\[
\text{LifeSpan}(v, w) = \sup \{ S \in (0, \infty); \text{there exists a unique strong solution} (v, w) \text{to (4.2) in} (0, S) \}
\]

**is finite**, then \( \lim_{s \to \text{LifeSpan}(v, w)} \|v, w(s)\|_{H^{1,m} \times H^{0,m}} = \infty \) **holds**.

For the proof of this proposition, we first prepare the local theory for the initial data in \( H^{1,m}(\mathbb{R}^N) \times H^{0,m}(\mathbb{R}^N) \) and then, we have the regularity of the solution (4.3) by the property of persistence regularity. For the detail, see [24, Proposition 3.6].

### 4.1 A priori estimate and the proof of Theorem 1.2

Our first goal is to obtain the following a priori estimate for the first order energy. For a constant \( s_0 \geq 0 \), we let for \( s \geq s_0 \),

\[
M(s) = \sup_{s_0 \leq \tau \leq s} \left( \|v(\tau)\|_{H^{1,m}}^2 + \frac{e^{-\sigma}}{b(\tau)^2} \|w(\tau)\|_{H^{0,m}}^2 \right).
\]

**Proposition 4.2.** **Under the assumptions** (1.2), (1.5), (2.3), (1.8), and (1.9)–(1.11) **there exist constants** \( s_0 \geq 0 \) **and** \( C > 0 \) **such that for any** \( s \geq s_0 \) **and for a solution** \((v, w)\) **to (4.2) on the interval** \([0, s]\), **we have the a priori estimate**

\[
M(s) \leq CM(s_0) + C \left\{ \begin{array}{ll}
  e^{N(1+2/N-p)/2}M(s)^p + e^{N(1+2/N-p)/2}M(s)^{p+1} & \text{if } 1 < p < 1 + \frac{2}{N}, \\
  M(s)^p + M(s)^{p+1} & \text{if } p = 1 + \frac{2}{N}, \\
  M(s)^p + M(s)^{p+1} & \text{if } p > 1 + \frac{2}{N}.
  \end{array} \right.
\]

(4.6)
We will give an outline of the proof of this proposition later. Here, we prove Theorem 1.2 from Proposition 4.2.

**Proof of Theorem 1.2.** Let \( s_0 \) be the constant given in Proposition 4.2. From Proposition 4.1, we have the unique local solution \((v, w)\) having \( s_0 < \text{LifeSpan}(v, w) \), provided that \( \varepsilon \) is sufficiently small. Moreover, by (4.4), we have the estimate

\[
M(s_0) \leq C \varepsilon^2 \| (v_0, w_0) \|^2_{H^{p+\alpha}}
\]

with some constant \( C > 0 \). Therefore, by (4.6), we obtain

\[
M(s) \leq C_0 \varepsilon^2 I_0 + C_1 \begin{cases}
\varepsilon^{N(1+2/(N-p))} M(s)^p + \varepsilon^{2N/(N-p)} M(s)^{p+1} & \text{if } 1 < p < 1 + \frac{2}{N}, \\
\varepsilon^{N(1+2/(N-p))} M(s)^p + M(s)^{p+1} & \text{if } p = 1 + \frac{2}{N}, \\
M(s)^p + M(s)^{p+1} & \text{if } p > 1 + \frac{2}{N},
\end{cases}
\]

(4.7)

where \( C_0, C_1 > 0 \) are some constants and \( I_0 = \| (v_0, w_0) \|^2_{H^{p+\alpha}} \).

We first consider the case \( 1 < p < 1 + 2/N \). Let \( S_1 = S_1(\varepsilon) \geq s_0 \) be the first time such that \( M(s) \) attains the value

\[
M(S_1) = 2C_0 \varepsilon^2 I_0. \quad (4.8)
\]

We note that the blow-up alternative in Proposition 4.1 ensures such a time \( S_1 \) exists if \( \text{LifeSpan}(v, w) < \infty \). We substitute \( s = S_1 \) into (4.7) to obtain

\[
C_0 \varepsilon^2 I_0 \leq 2C_1 \max \left\{ \varepsilon^{N(1+2/(N-p))S_1} M(S_1)^p, \varepsilon^{2N/(N-p)S_1} M(S_1)^{p+1} \right\}.
\]

From this and (4.8), we have

\[
\varepsilon^{-(\frac{1+2p}{N-p}-1)} \leq C \varepsilon^{S_1} \leq C (B(\text{LifeSpan}(u)) + 1),
\]

which implies (1.12) in the case \( 1 < p < 1 + 2/N \).

Next, we treat the case \( p = 1 + 2/N \). In this case, we take the time \( S_1 \) the same as (4.8) and use (4.7) to obtain

\[
C_0 \varepsilon^2 I_0 \leq 2C S_1 \max \left\{ (\varepsilon^2 I_0)^p, (\varepsilon^2 I_0)^{p+1} \right\} \leq CS_1 \varepsilon^{p+1},
\]

provided that \( \varepsilon \) is sufficiently small. Thus, we conclude

\[
\varepsilon^{-(p-1)} \leq CS_1 \leq C \log (B(\text{LifeSpan}(u)) + 1),
\]

which implies (1.12) in the case \( p = 1 + 2/N \).

Finally, we consider the case \( p > 1 + 2/N \). In this case, we have

\[
M(s) \leq C_0 \varepsilon^2 I_0 + C_1 \left( M(s)^p + M(s)^{p+1} \right).
\]

From this, we have the estimate

\[
M(s) \leq C \varepsilon^2
\]

for sufficiently small \( \varepsilon \). This and the blow-up alternative imply \( \text{LifeSpan}(u) = \infty \). \( \square \)
4.2 Outline of the proof of a priori estimate

We give an outline of the proof of Proposition 4.2. Let

\[ \alpha(s) = \int_{\mathbb{R}^N} v(y, s) \, dy \]

and

\[ \varphi_0(y) = (4\pi)^{-N/2} \exp\left( -\frac{|y|^2}{4} \right), \quad \psi_0(y) = \Delta_y \varphi_0(y). \]

We note that \( \alpha(s) \) makes sense, since \( v(s) \in H^{2,m}(\mathbb{R}^N) \subset L^1(\mathbb{R}^N) \) by (1.10). We decompose \((v, w)\) into

\[ v(y, s) = \alpha(s) \varphi_0(y) + f(y, s), \]
\[ w(y, s) = \frac{d\alpha}{ds}(s) \varphi_0(y) + \alpha(s) \psi_0(y) + g(y, s), \]

where \((f, g)\) are expected to be remainder terms. Noting that \( \Delta_y \varphi_0 = -\frac{y}{2} \cdot \nabla_y \varphi_0 - \frac{N}{2} \varphi_0 \), \( \int_{\mathbb{R}^N} \varphi_0(y) \, dy = 1 \), and the equation (4.2), we have

\[ \frac{d\alpha}{ds}(s) = \int_{\mathbb{R}^N} w(y, s) \, dy, \quad (4.9) \]
\[ \frac{e^{-s}}{b(t(s))^2} \frac{d^2 \alpha}{ds^2}(s) = \frac{e^{-s}}{b(t(s))^2} \frac{d\alpha}{ds}(s) - \frac{d\alpha}{ds}(s) + \frac{b'(t(s))}{b(t(s))^2} \frac{d\alpha}{ds}(s) + e^{\frac{s}{2}(1 + \frac{N}{2} - p)} \int_{\mathbb{R}^N} |v(y, s)|^p \, dy. \quad (4.10) \]

From this, we see that \((f, g)\) satisfies the system

\[ \begin{cases} f_s - \frac{y}{2} \cdot \nabla_y f - \frac{N}{2} f = g, \\ \frac{e^{-s}}{b(t(s))^2} \left( g_s - \frac{y}{2} \cdot \nabla_y g - \frac{N}{2} + 1 \right) g + g = \Delta_y f + \frac{b'(t(s))}{b(t(s))^2} g + h, \end{cases} \quad (4.11) \]

where

\[ h(y, s) = \frac{e^{-s}}{b(t(s))^2} \left( -2 \frac{d\alpha}{ds}(s) \psi_0(y) + \alpha(s) \left( \frac{y}{2} \cdot \nabla_y \psi_0(y) + \left( \frac{N}{2} + 1 \right) \psi_0(y) \right) \right) \]
\[ + \frac{b'(t(s))}{b(t(s))^2} \alpha(s) \psi_0(y) + e^{\frac{s}{2}(1 + \frac{N}{2} - p)} |f|^p - e^{\frac{s}{2}(1 + \frac{N}{2} - p)} \left( \int_{\mathbb{R}^N} |v|^p \, dy \right) \psi_0(y). \]

Moreover, by the definition of \((f, g)\) and the equation (4.11), we easily obtain

\[ \int_{\mathbb{R}^N} f(s, y) \, dy = \int_{\mathbb{R}^N} g(s, y) \, dy = \int_{\mathbb{R}^N} h(s, y) \, dy = 0. \quad (4.12) \]

In the following, based on the property (4.12), we derive energy estimates for \((f, g), \alpha, \) and \( \frac{d\alpha}{ds}. \)
4.3 Energy estimates for $N = 1$

We introduce
\[ F(y, s) = \int_{\infty}^{s} f(z, s) \, dz, \quad G(y, s) = \int_{\infty}^{s} g(z, s) \, dz, \quad H(y, s) = \int_{\infty}^{s} h(z, s) \, dz. \]

Here, we note that the property (4.12) implies
\[ \|F(s)\|_{L^2} \leq C\|y f(s)\|_{L^2}, \]
(see [24, Lemma 3.9]) and the same estimates hold for $G$ and $H$. Moreover, from the equation (4.11), we derive the following system for $F$ and $G$.
\[
\begin{aligned}
&F_y - \frac{y}{2} F_y = G, \\
&\frac{e^{-s}}{b(t(s))^2} (G_y - \frac{y}{2} G - G) + G = F_{yy} + \frac{b'(t(s))}{b(t(s))^2} G + H
\end{aligned}
\]  

We define
\[
E_0(s) = \int_{\mathbb{R}} \left[ \frac{1}{2} \left( F^2 + \frac{e^{-s}}{b(t(s))^2} G^2 \right) + \frac{1}{2} F^2 + \frac{e^{-s}}{b(t(s))^2} FG \right] dy,
\]
\[
E_1(s) = \int_{\mathbb{R}} \left[ \frac{1}{2} \left( F^2 + \frac{e^{-s}}{b(t(s))^2} G^2 \right) + f^2 + \frac{2 e^{-s}}{b(t(s))^2} f g \right] dy,
\]
\[
E_2(s) = \int_{\mathbb{R}} y^2 \left[ \frac{1}{2} \left( F^2 + \frac{e^{-s}}{b(t(s))^2} G^2 \right) + \frac{1}{2} F^2 + \frac{e^{-s}}{b(t(s))^2} f g \right] dy,
\]
\[
E_3(s) = \frac{1}{2} \frac{e^{-s}}{b(t(s))^2} \left( \frac{d\alpha}{ds}(s) \right)^2 + e^{-s/2} \alpha(s)^2,
\]
\[
E_4(s) = \frac{1}{2} \alpha(s)^2 + \frac{e^{-s}}{b(t(s))^2} \alpha(s) \frac{d\alpha}{ds}(s),
\]
and
\[
E_5(s) = \sum_{j=0}^{4} C_j E_j(s),
\]

where $C_j$ ($j = 0, \ldots, 4$) are constants such that $C_2 = C_3 = C_4 = 1$ and $1 \ll C_1 \ll C_0$. By a straightforward calculation, we can see that there exists sufficiently large $s_0 > 0$ such that $E_5(s)$ has the bound
\[
E_5(s) \sim \|f(s)\|^2_{H^{1,1}} + \frac{e^{-s}}{b(t(s))^2} \|g(s)\|^2_{L^{1,1}} + \alpha(s)^2 + \frac{e^{-s}}{b(t(s))^2} \left( \frac{d\alpha}{ds}(s) \right)^2
\]
for $s \geq s_0$. Furthermore, we have the following energy estimate.

**Lemma 4.3.** ([6, Lemma 4.4], [24, Lemmas 3.10–3.17]) There exists $s_0 > 0$ such that we have the energy identity
\[
\frac{d}{ds} E_5(s) + \frac{1}{2} \sum_{j=0}^{3} C_j E_j(s) + L_5(s) = R_5(s)
\]

15
for $s > 0$, where $L_5(s)$ has the lower bound

$$\|f(s)\|^2_{L^2} + \|g(s)\|^2_{L^p} + \left(\frac{d\alpha}{ds}(s)\right)^2 \leq C L_5(s)$$

for $s \geq s_0$, and $R_5(s)$ satisfies the estimate

$$|R_5(s)| \leq \frac{1}{2} L_5(s) + C e^{-\gamma s} E_5(s) + C e^{(3-p)s} E_5(s)^p + C e^{\frac{3-p}{1-p}} E_5(\alpha(s))^\alpha$$

for $s \geq s_0$, where $\gamma$ is given in (1.8).

Remark 4.1. (i) We can write down $L_5(s)$ and $R_5(s)$ explicitly (see [6, Lemma 4.4], [24, Lemmas 3.10–3.17]).

(ii) The term $e^{-\gamma s} E_5(s)$ comes from $b(t(s))^{-1} \frac{dt}{dt}(t(s)) \alpha(s) \frac{dt}{ds}(s)$, which appears in the remainder term $R_5(s)$, since we have $E_5(s) \geq \alpha(s)^2$ but we do not have $L_5(s) \geq \alpha(s)^2$.

The proof of the above lemma is the completely same as that of [24, Lemmas 3.10–3.17], which needs simple but tedious computations. Hence, we omit the detail.

4.4 Energy estimates for $N \geq 2$

When $N \geq 2$, we introduce

$$\hat{F}(\xi, s) = |\xi|^{-N/2-\delta} \hat{f}(\xi, s), \quad \hat{G}(\xi, s) = |\xi|^{-N/2-\delta} \hat{g}(\xi, s), \quad \hat{H}(\xi, s) = |\xi|^{-N/2-\delta} \hat{h}(\xi, s),$$

where $0 < \delta < 1$, and $\hat{f}(\xi, s)$ denotes the Fourier transform of $f(y, s)$ with respect to the space variable.

The following lemma is an improvement of [24, Lemma 3.11] by Fukuya [7].

Lemma 4.4. Let $m > N/2$ and $f(y) \in H^{0,m}(\mathbb{R}^N)$ be a function satisfying $\hat{f}(0) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} f(y)dy = 0$. Let $\hat{F}(\xi) = |\xi|^{-N/2-\delta} \hat{f}(\xi)$ with some $0 < \delta < 1$. Then, there exists a constant $C(N, m, \delta) > 0$ such that

$$\|F\|_{L^2} \leq C(N, m, \delta) \|f\|_{H^{0,m}} \quad (4.14)$$

holds.

Remark 4.2. In [24, Lemma 3.11], the $L^\infty$-$L^1$ estimate for the Fourier transform is used and then an extra restriction for $m$ is needed. In contrast, in [7] he gave a simple proof in which he used only the definition of Fourier transform and a basic inequality $|e^{-ix\xi} - 1| \leq |x|\xi$. These give the Hölder continuity of $\hat{f}$:

$$|\hat{f}(\xi) - \hat{f}(0)| = \left| \int_{\mathbb{R}^N} (e^{-ix\xi} - 1)f(x) \, dx \right| \leq 2^{1-\delta'}|\xi|^{\delta'} \int_{\mathbb{R}^N} |x|^{\delta'} |f(x)| \, dx, \quad (|\xi| < 1, 0 \leq \delta' \leq 1)$$

As a result, he could prove the assertion of Lemma 4.4 by assuming only the condition $m > N/2$ which is equivalent to the continuity of the inclusion $H^{0,m} \subset L^1$.

Also, by a direct calculation, one has

$$\|f\|_{L^2}^2 \leq \eta \|\nabla f\|_{L^2}^2 + C \|\nabla F\|_{L^2}^2$$

for any small $\eta > 0$, where the constant $C > 0$ depends on $\eta$ (see [24, (3.39)], [6, Lemma 4.6]).
In this case \( \hat{F} \) and \( \hat{G} \) satisfy the following system:

\[
\begin{aligned}
\left\{ \begin{array}{l}
\hat{F}_s + \frac{\xi}{2} \nabla_x \hat{F} + \frac{1}{2} \left( \frac{N}{2} + \delta \right) \hat{F} = \hat{G}, \\
e^{-s} \left( \hat{G}_s + \frac{\xi}{2} \nabla_x \hat{G} + \frac{1}{2} \left( \frac{N}{2} + \delta - 2 \right) \hat{G} \right) + \hat{G} = -|\xi|^2 \hat{F} + \frac{b'(t(s))}{b(t(s))} \hat{G} + \hat{h},
\end{array} \right. \\
(\xi, s) \in \mathbb{R}^N \times (0, S),
\end{aligned}
\]

We define the following energy

\[
E_0(s) = \text{Re} \int_{\mathbb{R}^N} \left( \frac{1}{2} |g|^2 |\hat{F}|^2 + \frac{e^{-s}}{b(t(s))} |\hat{G}|^2 \right) + \frac{1}{2} |\hat{F}|^2 + \frac{e^{-s}}{b(t(s))} |\hat{G}|^2 \, d\xi,
\]

\[
E_1(s) = \int_{\mathbb{R}^N} \left( \frac{1}{2} (\nabla_y f)^2 + \frac{e^{-s}}{b(t(s))} g^2 \right) + \left( \frac{N}{4} + 1 \right) \left( \frac{1}{2} s^2 + \frac{e^{-s}}{b(t(s))} f g \right) \, dy,
\]

\[
E_2(s) = \int_{\mathbb{R}^N} |\gamma|^{2m} \left( \frac{1}{2} (\nabla_y f)^2 + \frac{e^{-s}}{b(t(s))} g^2 \right) + \frac{1}{2} s^2 + \frac{e^{-s}}{b(t(s))} f g \, dy,
\]

\[
E_3(s) = \frac{1}{2} \frac{e^{-s}}{b(t(s))^2} \left( \frac{da}{ds}(s) \right)^2 + e^{-2s} \alpha(s)^2,
\]

\[
E_4(s) = \frac{1}{2} \alpha(s)^2 + \frac{e^{-s}}{b(t(s))^2} \alpha(s) \frac{da}{ds}(s),
\]

where \( \lambda \) is a parameter satisfying \( 0 < \lambda < \min \left\{ \frac{1}{2}, \frac{s}{2} - \frac{N}{4} \right\} \). Moreover, we define

\[
E_5(s) = \sum_{j=0}^{4} C_j E_j(s),
\]

with positive constants \( C_j \) (\( j = 0, \ldots , 4 \)) such that \( C_2 = C_3 = C_4 = 1 \) and \( 1 \ll C_1 \ll C_0 \). Then, as in the case of \( N = 1 \), there exists sufficiently large \( s_0 > 0 \) such that \( E_5(s) \) has the bound

\[
E_5(s) \sim \| f(s) \|_{H^{1, \infty}}^2 + \frac{e^{-s}}{b(t)} \| g(s) \|_{H^{1, \infty}}^2 + \alpha(s)^2 + \frac{e^{-s}}{b(t(s))^2} \left( \frac{da}{ds}(s) \right)^2
\]

for \( s \geq s_0 \). Moreover, in the same manner as the case \( N = 1 \), we have the following energy estimate.

**Lemma 4.5.** ([6, Lemma 4.7], [24, Lemmas 3.12–3.17]) There exist constants \( 0 < \lambda < \min \left\{ \frac{1}{2}, \frac{s}{2} - \frac{N}{4} \right\} \) and \( s_0 > 0 \) such that we have the energy identity

\[
\frac{d}{ds} E_5(s) + 2\lambda \sum_{j=0}^{3} C_j E_j(s) + L_5(s) = R_5(s),
\]

for \( s > 0 \), where \( L_5(s) \) has the lower bound

\[
\| f(s) \|_{H^{1, \infty}}^2 + \| g(s) \|_{H^{1, \infty}}^2 + \left( \frac{da}{ds}(s) \right)^2 \leq C L_5(s)
\]

for \( s \geq s_0 \), and \( R_5(s) \) satisfies the estimate

\[
| R_5(s) | \leq \frac{1}{2} L_5(s) + C e^{-\gamma s} E_5(s) + C e^{N(1+2(N-p))s} E_5(s)^p + C e^{\frac{s}{2} (1+2(N-p))} E_5(s)^{\frac{p}{2}}
\]

for \( s \geq s_0 \), where \( \gamma \) is given in (1.8).
4.5 Proof of Proposition 4.2

Let $\varepsilon_1 > 0$ be sufficiently small so that the local solution $(v, w)$ of (4.2) exists for $s > s_0$ (see Proposition 4.1). Therefore, we can apply Lemmas 4.3, 4.5. Putting

$$\Lambda(s) := \exp \left( -C \int_{s_0}^{s} e^{-\gamma \sigma} \, d\sigma \right),$$

which satisfies $\Lambda(s) \sim 1$, we have

$$\Lambda(s) E_5(s) \leq E_5(s_0) + C \int_{s_0}^{s} \left[ \Lambda(\sigma) e^{N(1+1/2-N)\sigma} E_5(\sigma)^p + \Lambda(\sigma) e^{\frac{2}{N}((1+1/2-N)\sigma + \frac{1}{\sigma})} \right] \, d\sigma.$$  

Consequently, if we define $M(s)$ by (4.5), then by calculating the integral, we easily obtain (4.6), which completes the proof.

Appendix: Lower bound of lifespan for the Fujita equation

We would like to give a short proof of lower bound of the lifespan of solutions to the standard semilinear heat equation of Fujita type:

$$\begin{cases}
\partial_t u(x, t) - \Delta u(x, t) = |u(x, t)|^p, & (x, t) \in \mathbb{R}^N \times (0, T), \\
u(x, 0) = f \varepsilon(x) & x \in \mathbb{R}^N.
\end{cases} \quad (4.15)$$

The following argument is based on Quittner and Souplet [20]. Put $0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and set

$$U(x, t) = h(t) \frac{1}{\varepsilon} e^{\Delta f \varepsilon}.$$  

Then we have

$$\partial_t - \Delta U(x, t) = -\frac{1}{p-1} h'(t) h(t) - \frac{p}{p-1} e^{\Delta f \varepsilon} h(t) (\partial_t - \Delta) e^{\Delta f \varepsilon} = -\frac{1}{p-1} h'(t) h(t) - \frac{p}{p-1} e^{\Delta f \varepsilon}.$$  

If $-\frac{1}{p-1} h'(t) h(t) - \frac{p}{p-1} e^{\Delta f \varepsilon} \geq |h(t)| - \frac{p}{p-1} e^{\Delta f \varepsilon} = |U(x, t)|^p$ and $h(0) \geq 1$, then $U(x, t)$ can be used as a super-solution of nonlinear problem (4.15). Solving $h'(t) = -(p - 1) |e^{\Delta f \varepsilon} |^p$ with $h(0) = 1$, we have

$$h(t) = 1 - (p - 1) \int_0^t |e^{\Delta f \varepsilon (s)} |^p \, ds, \quad t < t_e = \sup \{t > 0 \mid h(t) > 0\}.$$  

Then we have LifeSpan$(U) = t_e$. On the other hand, we have

$$\|e^{\Delta f \varepsilon (s)} \|_{L^p} \leq \min \left\{ \|f \|_{L^p}, C_N t^{-\frac{N}{2}} \|f \|_{L^1} \right\} = \varepsilon \min \left\{ \|f \|_{L^p}, C_N t^{-\frac{N}{2}} \|f \|_{L^1} \right\}.$$  

This implies that for $t > 1$,

$$\int_0^t \|e^{\Delta f \varepsilon (s)} \|^p \, ds = \int_0^1 \|e^{\Delta f \varepsilon (s)} \|^p \, ds + \int_1^t \|e^{\Delta f \varepsilon (s)} \|^p \, ds \leq \varepsilon \int_0^1 (\|f \|_{L^p}^p + C_N \|f \|_{L^p}^{p-1}) \, ds + \varepsilon \int_1^t \|f \|_{L^p}^p \, ds.$$  

Since by a continuity method, we can extend the solution $u$ until $t = \text{LifeSpan}(U)$ and then we have LifeSpan$(u) \geq \text{LifeSpan}(U)$. On the other hand, the above estimate yields

$$\text{LifeSpan}(U) \geq \begin{cases}
C \varepsilon^{-\frac{1}{p-1}} & \text{if } 1 < p < 1 + \frac{2}{N}, \\
\exp(C \varepsilon^{-1}) & \text{if } p = 1 + \frac{2}{N}.
\end{cases}$$
Acknowledgements

This work is partially supported by Grant-in-Aid for Young Scientists Research (B) No.16K17619, Grant-in-Aid for Young Scientists Research (B) No.15K17571, and Grant-in-Aid for Young Scientists Research (B) No.16K17625.

References

[1] C. Cattaneo, Sur une forme de l’´equation de la chaleur ´eliminant le paradoxe d’une propagation instan´an´ee, C. R. Acad. Sci. 247 (1958), 431–433.

[2] M. D’Abbicco, The threshold of effective damping for semilinear wave equations, Math. Methods Appl. Sci. 38 (2015), 1032–1045.

[3] M. D’Abbicco, S. Lucente, A modified test function method for damped wave equations, Adv. Nonlinear Stud. 13 (2013), 867–892.

[4] M. D’Abbicco, S. Lucente, M. Reissig, Semi-Linear wave equations with effective damping, Chin. Ann. Math., Ser. B 34 (2013), 345–380.

[5] M. D’Abbicco, S. Lucente, M. Reissig, A shift in the Strauss exponent for semilinear wave equations with a not effective damping, J. Differential Equations 259 (2015), 5040–5073.

[6] K. Fujiwara, M. Ikeda, Y. Wakisugi, Estimates of lifespan and blow-up rates for the wave equation with a time-dependent damping and a power-type nonlinearity, to appear in Funkcial. Ekvac. arXiv:1609.01035v1.

[7] K. Fukuya, Diffusion phenomena for semilinear wave equation with time-dependent damping, master thesis, 2018 (Japanese).

[8] Th. Gallay, G. Raugel, Scaling variables and asymptotic expansions in damped wave equations, J. Differential Equations 150 (1998), pp. 42–97.

[9] M. Ikeda, T. Inui, The sharp estimate of the lifespan for the semilinear wave equation with time-dependent damping, Differential Integral Equations 32 (2019), 1–36.

[10] M. Ikeda, T. Ogawa, Lifespan of solutions to the damped wave equation with a critical nonlinearity, J. Differential Equations 261 (2016), 1880–1903.

[11] M. Ikeda, M. Sobajima, Life-span of solutions to semilinear wave equation with time-dependent critical damping for specially localized initial data, Math. Ann. 372 (2018), 1017–1040.

[12] M. Ikeda, M. Sobajima, Sharp upper bound for lifespan of solutions to some critical semilinear parabolic, dispersive and hyperbolic equations via a test function method, Nonlinear Anal. 182 (2019), 57–74.

[13] M. Ikeda, Y. Wakisugi, A note on the lifespan of solutions to the semilinear damped wave equation, Proc. Amer. Math. Soc. 143 (2015), 163–171.

[14] M. Ikeda, Y. Wakisugi, Global well-posedness for the semilinear wave equation with time dependent damping in the overdamping case, to appear in Proc. Amer. Math. Soc., arXiv:1708.08044v1.
[15] N. A. Lai, H. Takamura, *Blow-up for semilinear damped wave equations with sub-Strauss exponent in the scattering case*, Nonlinear Anal. 168 (2018), 222–237.

[16] N. A. Lai, Y. Zhou, *The sharp lifespan estimate for semilinear damped wave equation with Fujita critical power in higher dimensions*, J. Math. Pures Appl. 123 (2019), 229–243.

[17] T.-T. Li, Y. Zhou, *Breakdown of solutions to □u + ut = |u|^{1+α}*, Discrete Contin. Dynam. Syst. 1 (1995), 503–520.

[18] J. Lin, K. Nishihara, J. Zhai, *Critical exponent for the semilinear wave equation with time-dependent damping*, Discrete Contin. Dyn. Syst. 32 (2012), 4307–4320.

[19] K. Nishihara, *Lp-Lq estimates for the 3-D damped wave equation and their application to the semilinear problem*, Seminar Notes of Math. Sci. 6, Ibaraki Univ., (2003), 69–83.

[20] P. Quittner, P. Souplet, *Superlinear parabolic problems. Blow-up, global existence and steady states*, Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser Verlag, Basel, 2007. xii+584 pp.

[21] G. Todorova, B. Yordanov, *Critical exponent for a nonlinear wave equation with damping*, J. Differential Equations 174 (2001), 464–489.

[22] P. Vernotte, *Les paradoxes de la théorie continue de l’équation de la chaleur*, Comptes Rendus 246 (1958), 3154–3155.

[23] Y. Wakasugi, *On the Diffusive Structure for the Damped Wave Equation with Variable Coefficients*, Doctoral thesis, Osaka University, 2014.

[24] Y. Wakasugi, *Scaling variables and asymptotic profiles for the semilinear damped wave equation with variable coefficients*, J. Math. Anal. Appl. 447 (2017), 452–487.

[25] J. Wirth, *Solution representations for a wave equation with weak dissipation*, Math. Meth. Appl. Sci. 27 (2004), 101–124.

[26] J. Wirth, *Wave equations with time-dependent dissipation I. Non-effective dissipation*, J. Differential Equations 222 (2006), 487–514.

[27] J. Wirth, *Wave equations with time-dependent dissipation II. Effective dissipation*, J. Differential Equations 232 (2007), 74–103.

[28] J. Wirth, *Scattering and modified scattering for abstract wave equations with time-dependent dissipation*, Adv. Differential Equations 12 (2007), 1115–1133.