Galois hulls of cyclic serial codes over a finite chain ring

Sarra Talbi\(^a\), Aicha Batoul\(^a\), Alexandre Fotue Tabue\(^b\), Edgar Martínez-Moro\(^c\)

\(^a\)Faculty of Mathematics USTHB, University of Science and Technology of Algiers, Algeria
\(^b\)Department of Mathematics, HTTC Bertoua, The University of Ngaoundéré, Cameroon
\(^c\)Institute of Mathematics, University of Valladolid, Castilla, Spain

Abstract

In this paper we explore some properties of Galois hulls of cyclic serial codes over a chain ring and we devise an algorithm for computing all the possible parameters of the Euclidean hulls of that codes. We also establish the average \(p^r\)-dimension of the Euclidean hull, where \(F_{p^r}\) is the residue field of \(R\), and we provide some results of its relative growth.

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1. Introduction

The Euclidean hull of a linear code is defined to be the intersection of a code and its Euclidean dual. It was originally introduced by Assmus and Key\(^1\) to classify finite projective planes. Knowing the hull of a linear code is a key point to determine the complexity of some algorithms for investigating permutation of two linear codes and computing the automorphism group of the code, see [10, 15, 17]. In general, those algorithms have been proved to be very effective if the size of the Euclidean hull is small. In the case of codes over finite fields, Sendrier [16] established the number of linear codes of length \(n\) with a fix dimension Euclidean hull, also Skersys[19] discussed the average dimension of the Euclidean hull of cyclic codes. Later, Sangwisut et al. [18] determined the dimension of the Euclidean hull of cyclic and negacyclic codes of length \(n\) over a finite field. Furthermore, Jitman and Sangwisut [18] gave the average Euclidean hull dimension of negacyclic codes over a finite field. Recently, the concept of the Euclidean hulls has been generalized to cyclic codes of odd length over \(\mathbb{Z}_4\) by Jitman et al. [8] where the authors provided an algorithm to determine the type of the Euclidean hull of cyclic codes over \(\mathbb{Z}_4\).

An important class of linear codes over rings is the class of cyclic codes. They have been studied in a series of papers (see [4, 6, 7, 13, 14]). In particular, Dinho and Permouth [4] gave algebraic structure of simple root cyclic codes over finite chain rings \(R\). Martínez and Rúa [13] generalized these results to multivariable cyclic codes over \(R\). Free cyclic serial codes have been determined by using cyclotomic cosets and trace map over finite chain rings \(R\) by Fotue and Mouaha in [6]. It is clear that the Euclidean hull of cyclic codes is also cyclic, two special families of cyclic codes are of great interest, namely linear complementary dual codes, which are codes whose Euclidean hull is trivial (see for example [8]) and self-orthogonal codes, which are linear codes whose Euclidean hulls is the whole code (see for example [20]). These works motivate us to study the Galois hulls of cyclic codes over finite chain rings. In this paper, we focus on the study of the Galois hulls of cyclic codes of length \(n\) over a finite chain ring \(R\) of parameters \((p, r, a, e, r)\) such that \(n\) and \(p\) are coprime. This is the serial case stated in [13], i.e. the cyclic codes over \(R\) whose length \(n\) is coprime with \(p\) are serial modules over \(R\). We will generalize the techniques used in [9] (for \(\mathbb{Z}_4\)) to obtain the parameters and the average \(p^r\)-dimensions Euclidean hull of cyclic serial codes over finite chain rings.

The paper is organized as follows. In Section 2 some preliminary concepts and some basic results are recalled. In Section 3, we characterize Galois hulls of cyclic serial code over finite chain rings. Section 4 shows the parameters

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Email addresses: talbissarra@gmail.com (Sarra Talbi), a.batoul@hotmail.fr (Aicha Batoul), alexfotue@gmail.com (Alexandre Fotue Tabue), edgar.martinez@uva.es (Edgar Martínez-Moro)
and the \( p^r \)-dimensions of the Euclidean hull of cyclic serial codes. Finally, the average dimension of the Euclidean hull of cyclic serial codes is computed in Section 5.

2. Preliminaries

2.1. Chain rings

For an account on the results on finite rings in this section check [12]. Throughout this paper, \( p \) is a prime number, \( a, e, r \) are positive integers and \( \mathbb{Z}_{p^a} \) is the residue ring of integers modulo \( p^a \). \( R \) will denote a finite commutative chain ring of characteristic \( p^a \) and we will denote its maximal ideal by \( \mathcal{J}(R) \) and \( R^\times \) will denote its multiplicative group.

Note that since \( R \) is a chain ring it is a principal ideal ring, we will denote as \( \theta \in R \) a generator of \( \mathcal{J}(R) \) and the ideals of \( R \) form a chain under inclusion \( \{0\} = \mathcal{J}(R)^a \subseteq \mathcal{J}(R)^{a-1} \subseteq \cdots \subseteq \mathcal{J}(R) \subseteq R \) and \( \mathcal{J}(R) = \theta^b R \) for \( 0 \leq b < s \).

The ring epimorphism \( \pi : R \to R/\mathcal{J}(R) \cong \mathbb{F}_{p^r} \) naturally extends a ring epimorphism of polynomial rings from \( R[X] \) to \( \mathbb{F}_{p^r}[X] \) and on the other hand it naturally induces an \( R \)-module epimorphism from \( R^n \) to \( (\mathbb{F}_{p^r})^n \), we will abuse the notation and we will denote both mappings by \( \pi \).

A polynomial \( f \) is basic-irreducible over \( R \) if \( \pi(f) \) is irreducible over \( \mathbb{F}_{p^r} \). We will denote by denoted \( GR(p^a, r) \) the Galois ring of characteristic \( p^a \) and cardinality \( p^{ar} \), which is the quotient ring \( \mathbb{Z}_{p^a}[X]/\langle f \rangle \) where \( \langle f \rangle \in \mathbb{Z}_{p^a}[X] \) is the ideal generated by a monic basic-irreducible polynomial \( f \) of degree \( r \) over \( \mathbb{Z}_{p^a} \). The ring \( GR(p^a, r) \) is uniquely determined up to ring-isomorphism and for all positive integers \( r_1 \) and \( r_2 \), the ring \( GR(p^a, r_1) \) is a sub-ring of \( GR(p^a, r_2) \) if and only if \( r_1 \) divides \( r_2 \).

From the above setting for a given finite chain ring \( R \) there is a 5-tuple \((p, a, r, e, s)\) of positive integers, the so-called parameters of \( R \), such that \( R = GR(p^a, r)[\theta] \), and \( \langle \theta \rangle = \mathcal{J}(R) \), \( \theta^e \in p(\mathbb{Z}_{p^a}[\theta])^a \) and \( \theta^{e-1} \neq \theta^0 = 0 \). From now on, we will denote as \( S_d \) the subring of \( R \) such that \( S_d : = GR(p^a, d)[\theta] \) and \( d \) is a divisor of \( r \). The Teichmüller set of \( R \) will be denoted as \( \Gamma(R) \) and it is defined as \( \Gamma(R) = \{0\} \cup \{a \in R : a^{p^e-1} = 1\} \). It is the only cyclic subgroup of \( R^\times \) isomorphic to the multiplicative group of \( \mathbb{F}_{p^r} \). For each element \( a \) in \( R \), there is a unique \((a_0, a_1, \cdots, a_{s-1})\) in \( \Gamma(R)^s \) such that \( a = a_0 + a_1 \theta + \cdots + a_{s-1} \theta^{s-1} \).

Let \( R \) and \( S \) be two finite commutative chain rings such that \( S \subset R \) and \( 1_R = 1_S \), then we say that \( R \) is an extension of \( S \). Provided that \( J(R) \) and \( J(S) \) are their maximal ideals, we say that the extension is separable if \( J(S)R = J(R) \). The group \( G \) of all automorphism \( \gamma \) of \( R \) such that \( \gamma|_S \) is the identity is known as the Galois group of the extension. A separable extension is called Galois if \( \{r \in R : \gamma(r) = r, \forall \gamma \in G\} = S \). This condition is equivalent to the condition \( R \) is ring-isomorphic to \( S[X]/\langle f \rangle \), where \( f \) is a monic basic irreducible polynomial in \( S[X] \), see [21 Section 4][12 Theorem XIV.8].

We will denote by \( Aut_\mathbb{S}(R) \) the Galois group of the Galois extension \( S[R] \). Let \( d \) be positive divisor of \( r \), and let us consider \( S = \mathbb{Z}_{p^a}[\theta] \), \( R = \mathbb{G}R(p^a, r)[\theta] \), \( S_d = \mathbb{G}R(p^a, d)[\theta] \), and

\[
\mathbb{G}S(R) := \{ S_d : d \text{ is a divisor of } r \text{ and } \mathbb{Z}_{p^a}[\theta] \subseteq S_d \}.
\]

It is well known that \( Aut_\mathbb{S}(R) \) is a cyclic group generated by the Frobenius automorphism \( \sigma : R \to R \) given by

\[
\sigma\left( \sum_{t=0}^{s-1} a_t \theta^t \right) = \sum_{t=0}^{s-1} a_t \theta^t \theta^s, \text{ and therefore, the set } \mathbb{S}(Aut_\mathbb{S}(R)) \text{ of subgroups of } Aut_\mathbb{S}(R) \text{ is given by }
\]

\[
Sub(Aut_\mathbb{S}(R)) = \{ \langle \sigma^d \rangle : d \text{ is a divisor of } r \}.
\]

There is a Galois correspondence \((Stab, Fix)\) between \( \mathbb{G}S(R) \) and \( Sub(Aut_\mathbb{S}(R)) \) as follows \( Stab : \mathbb{G}S(R) \to Sub(Aut_\mathbb{S}(R)) \) and \( Fix : \mathbb{S}(Aut_\mathbb{S}(R)) \to \mathbb{G}S(R) \) where \( Stab(S_d) = \langle \sigma^d \rangle \) and \( Fix(\langle \sigma^d \rangle) = S_d \), where \( d \) is a divisor of \( r \) (see [6]). The diagram

![Diagram](image-url)
commutes, where $\overline{\sigma}$ denotes a generator $\text{Aut}_R(F_p(\overline{F}_q))$, $T_d = \sum_{i=0}^{\frac{d-1}{2}} \sigma^id$ and $\tau_d = \sum_{i=0}^{\frac{d-1}{2}} \pi^id$. For any $x = (x_1, \ldots, x_n) \in R^n$, and any matrix $G = (a_{ij})_{k \times n}$ over $R$, $\sigma$ will act on them component-wise as follows: $\sigma(x) = (\sigma(x_1), \ldots, \sigma(x_n))$.

2.2. Codes over chain rings

A linear code $C$ of length $n$ over the ring $R$ is defined to be a submodule of the $R$-module $R^n$. We will denote by $\{0\}$, the zero-submodule where $0 = (0, 0, \ldots, 0) \in R^n$. A linear code $C$ over $R$ is free if $C \cong R^k$ as $R$-modules for some positive integer $k$. The residue code of a linear code $C$ over $R$ is the linear code $\pi(C)$ over $F_q$ for all $x \in R^n$.

In [6], the authors introduced and studied the Galois closure of a linear code $C$ over $R$ of length $n$ as follows, $C \cup d(\ell) = \text{Ext}(\ell d(C))$, where $\text{Ext}(\ell d(C))$ is the linear code over $R$ of all $R$-combinations of codewords in the linear code $\ell d(C)$ over $S_d$. A linear code $C$ over $R$ is $(d^\ell)$-invariant, if $d^\ell(C) = C$, where $d$ is a divisor of $r$. Recall that for any linear code $C$ over $R$ of length $n$, its subring subcode is given by $\text{Res}_d(C) = C \cap (S_d)$. In [6] it is shown that $(d^\ell)$-invariant, if and only if, $\ell d(C) = \text{Res}_d(C)$ if and only if, $C = \text{Ext}(\text{Res}_d(C))$. For $\ell \in \{0, 1, \ldots, r-1\}$ we equip $R^n$ with the $\ell$-Galois inner-product defined as follows:

$$\langle u, v \rangle_\ell = \sum_{j=0}^{\ell-1} u_j \sigma^\ell (v_j), \quad \text{for all } u, v \in R^n.$$ 

When $\ell = 0$ it is just the usual Euclidean inner-product and if $r$ is even and $r = 2\ell$ it is the Hermitian inner-product. The $\ell$-Galois dual of a linear code $C$ over $R$ of length $n$, denoted $C^{\perp\ell}$, is defined to be the linear code

$$C^{\perp\ell} = \{u \in R^n : \langle u, c \rangle_\ell = 0, \text{ for all } c \in C\}.$$ 

If $C \subseteq C^{\perp\ell}$, then $C$ is said to be $\ell$-Galois self-orthogonal. Moreover, $C$ is to be $\ell$-Galois self-dual if $C = C^{\perp\ell}$. The two statements in Proposition 2 below follow immediately from the identity

$$\langle u, v \rangle_\ell = \langle u, \sigma^h(v) \rangle_{\ell-h} = \sigma^h (\langle \sigma^{\ell-h}(v), u \rangle_{\ell-h}),$$ 

for all $0 \leq h \leq \ell$, $u$ and $v$ in $R^n$ where the action is taken componentwise $\sigma^\ell(v) = (\sigma^\ell(v_0), \ldots, \sigma^\ell(v_{n-1}))$. The following proposition is a generalized Delsarte’s Theorem.

**Proposition 1.** ([6] Theorem 3.3) Let $C$ be a linear code over $R$ of length $n$. Then for any $\ell \in \{0, 1, \ldots, r-1\}$, $\ell d(C^{\perp\ell}) = (\text{Res}_d(C))^{\perp\ell}$.

The following proposition is a natural generalization to finite chain rings of [11] Proposition 2.2.

**Proposition 2.** Let $C$ be a linear code over $R$ of length $n$. Then

1. $(\sigma^h(C))^{\perp\ell} = \sigma^h(C^{\perp\ell})$, and $C^{\perp\ell} = \sigma^h(C^{\perp\ell-h})$, for any $0 \leq h \leq \ell$;
2. $(C^{\perp\ell})^{\perp h} = \sigma^{2r-\ell-h}(C)$, for all $0 \leq h, 0 \leq \ell, h \leq r-1$.

From Proposition 2 and [7] Theorem 3.1, we obtain the following result.

**Corollary 1.** Let $C$ and $C'$ be linear codes over $R$ of length $n$. Then

1. $(C + C')^{\perp\ell} = C^{\perp\ell} \cap C'^{\perp\ell}$;
2. $(C \cap C')^{\perp\ell} = C^{\perp\ell} \cap C'^{\perp\ell}$.

**Definition 1.** Let $C$ be a linear code over $R$. The $\ell$-Galois hull of $C$ will be denoted as $\mathcal{H}_\ell(C)$ and is the intersection of $C$ and its $\ell$-Galois dual

$$\mathcal{H}_\ell(C) = C \cap C^{\perp\ell}.$$ 

3
A linear code $C$ over $R$ is said to be $\ell$-Galois LCD if $H_\ell(C) = \{0\}$, and $C$ is said to be $\ell$-Galois self-orthogonal if $H_\ell(C) = C$. If we denote that for all $0 \leq \ell, h \leq r - 1$, we have $\sigma^h(H_\ell(C)) = H_\ell(\sigma^h(C))$, and $H_\ell(C) = H_{r-\ell}(C^{\perp_r})$. From generalized Delsarte’s Theorem in Proposition 1 it follows that $T_d(H_\ell(C)) = (\text{Res}_d(H_{r-\ell}(C)))^{\perp_r}$. Note that if $C$ is $(\sigma^t)$-invariant, then $H_\ell(C) = H_0(C)$.

From [14 Proposition 3.2 and Theorem 3.5], for any linear code $C$ over $R$ of length $n$, there is a unique $s$-tuple $(k_0, k_1, \cdots, k_{s-1})$ of positive integers, such that $C$ has a generator matrix in standard form

$$
\begin{pmatrix}
I_{k_0} & G_{0,1} & G_{0,2} & \cdots & G_{0,s-2} & G_{0,s-1} & G_{0,s} \\
0 & \theta G_{1,2} & \theta G_{1,3} & \cdots & \theta G_{1,s-2} & \theta G_{1,s-1} & \theta G_{1,s} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & \theta^{s-1} I_{k_{s-1}} & \theta^{s-1} G_{s-1,s}
\end{pmatrix},
$$

where $U$ is a permutation matrix and $0$ all zero matrices of suitable size. The elements in the $s$-tuple $(k_0, k_1, \cdots, k_{s-1})$ are called parameters of $C$ and the rank of $C$ is $k_0 + k_1 + \cdots + k_{s-1}$. From [14 Theorem 3.10], the parameters of $C^{\perp_r}$ are $(n - k, k_{s-1}, \cdots, k_2, k_1)$, where $k = \text{rank}_R(C)$. Note that $C$ is free if and only if $\text{rank}_R(C) = k_0$ and $k_1 = \cdots = k_{s-1} = 0$. The $q$-dimension of a linear code $C$ over $R$, denoted $\dim_q(C)$, is defined to be $\log_q(|C|)$.

Thus the $q$-dimension of a linear code $C$ over $R$ of parameters $(k_0, k_1, \cdots, k_{s-1})$ is $\sum_{t=0}^{s-1} (s-t)k_t$. Since $R$ is also a Frobenius ring, it follows that $\dim_q(C) + \dim_q(C^{\perp_r}) = sn$.

**Proposition 3.** Let $C$ and $C'$ be two linear codes over $R$ of the same length. Then

$$
\dim_q(C + C') = \dim_q(C) + \dim_q(C') - \dim_q(C \cap C').
$$

Moreover $\dim_q(H_\ell(C)) = \dim_q(H_{r-\ell}(C))$.

**Proof.** We have $C = \langle C \rangle \oplus (C \cap C')$ and $C + C' = \langle C \rangle \oplus C'$. It follows that $|C| = |\langle C \rangle| \times |C \cap C'|$ and $|C + C'| = |\langle C \rangle| \times |C'|$. Thus $|C + C'| = \frac{|C|}{|C \cap C'|} \times |C'|$. Therefore $\log_q(|C + C'|) = \log_q(|C|) - \log_q(|C \cap C'|) + \log_q(|C'|)$. From the definition of $q$-dimension of a linear code we have that $\dim_q(C + C') = \dim_q(C) + \dim_q(C') - \dim_q(C \cap C')$. Moreover,

$$
\dim_q(H_\ell(C)) = \dim_q((C + C^{\perp_r})^{\perp_r}), \text{ from Corollary } [1];
$$

$$
= sn - \dim_q(C + C^{\perp_r}), \text{ since } \dim_q(C + C^{\perp_r}) + \dim_q((C + C^{\perp_r})^{\perp_r}) = sn;
$$

$$
= sn - (\dim_q(C) + \dim_q(C^{\perp_r}) - \dim_q(H_{r-\ell}(C)));
$$

$$
= \dim_q(H_{r-\ell}(C)), \text{ since } \dim_q(C) + \dim_q(C^{\perp_r}) = sn.
$$

**Proposition 4.** Let $C$ be a free code over $R$ of length $n$ and $\ell$ be a positive integer. Then

1. $\dim_q(\sigma^t(C)) = s \times \text{rank}(\sigma^t(C)) = s \times \dim_q(\pi(\sigma^t(C)));
2. $\pi(C)^{\perp_r} = \pi(C^{\perp_r});
3. $\pi(H_\ell(C)) = H_\ell(\pi(C))$.

**Proof.** Since $C$ is free, a generator matrix for $\sigma^t(C)$ is $\begin{pmatrix} I_k & \sigma^t(A) \end{pmatrix}$, where $A$ is a $k \times (n-k)$-matrix over $R$ and $U$ is a permutation matrix. Thus $\begin{pmatrix} I_k & \pi(\sigma^t(A)) \end{pmatrix}$ is a generator matrix for $\pi(C)$. It follows that $|\sigma^t(C)| = q^{sk}$ and $\text{rank}(\sigma^t(C)) = \dim_q(\pi(\sigma^t(C))) = k$. This proves Item 1. Now to proves Item 2. The codes $\pi(C)^{\perp_r}$ and $\pi(C^{\perp_r})$ have the same parity matrix, which is $\begin{pmatrix} I_k & \pi(\sigma^t(A)) \end{pmatrix}$ and $\begin{pmatrix} I_k & \pi(\sigma^t(A)) \end{pmatrix}$. Hence $\pi(C)^{\perp_r} = \pi(C^{\perp_r})$. Item 3 is a consequence of the fact that the above diagram commutes, $\pi(H_\ell(C)) \subseteq H_\ell(\pi(C))$ and $\dim_q(\pi(H_\ell(C))) = \dim_q(\pi(H_\ell(\pi(C))))$. □
3. Galois hulls of cyclic serial codes

Let \( \mathbb{N} \) be the set of nonnegative integers and \( n \) be a positive integer such that \( \gcd(n, q) = 1 \). Set \([a; b] = \{a, a + 1, \ldots, b\} \) where \( (a, b) \in \mathbb{N}^2 \) such that \( a < b \). Let \( \Lambda, \beta \) be two subsets in \([0; n - 1]\), as usual, the opposite of \( \Lambda \) is \( -\Lambda = \{n - z : z \in \Lambda\} \) and its complementary is \( \overline{\Lambda} = \{z \in [0; n - 1] : z \notin \Lambda\} \). The set \( \Lambda \) is symmetric, if \( \Lambda = -\Lambda \) and the pair \( \{\Lambda, -\Lambda\} \) is asymmetric, if \( \Lambda = -\Lambda \). If \( u \in \mathbb{N} \setminus \{0\} \) then \( u\Lambda = \{i \in [0; n - 1] : (\exists z \in \Lambda)(uz \equiv i \pmod j)\} \).

A subset \( Z \) of \([0; n - 1]\) is a \( q \)-closed set modulo \( n \), if \( Z = qZ \). The smallest \( q \)-closed set modulo \( n \), contained a subset \( Z \) of \([0; n - 1]\) is \( \bigcup_{q \in \mathbb{N}} q^2 Z \) and we will denote it by \( \mathcal{C}_q(Z) \). The \( q \)-cyclotomic cosets modulo \( n \) are \( \mathcal{C}_q(z) \), where \( z \in [0; n - 1] \). We will take \( \mathcal{C}_q(\emptyset) = \emptyset \) by convention. It is clear that the \( q \)-cyclotomic cosets modulo \( n \) form a partition of \([0; n - 1]\), and any \( q \)-closure set modulo \( n \) is a union of \( q \)-cyclotomic cosets modulo \( n \). Denote \([0; n - 1]_q \), a subset of \([0; n - 1]\) such that \([0; n - 1] = \bigcup_{q \in [0,n-1]_q} \mathcal{C}_q(z) \). Let \( j \) be a divisor of \( n \) such that \( \gcd(j, q) = 1 \), we will use the following notation:

- \( \phi(\cdot) \) is the Euler totient function;
- \( \omega(n; q) \) the multiplicative order of \( q \) modulo \( j \);
- \( N_q = \{d \in \mathbb{N} \setminus \{0\} : (\exists i \in \mathbb{N} \setminus \{0\})(d \text{ divides } q^i + 1)\} \);
- \( \Lambda_j \) the set of symmetric \( q \)-cyclotomic cosets modulo \( n \) of size \( \omega j(q) \);
- \( \overline{\Lambda}_j \) the set of asymmetric pairs of \( q \)-cyclotomic cosets modulo \( n \) of size \( \omega d_j(q) \).

The following result is straightforward from Hensel’s Lemma \([12]\) the uniqueness of this basic-irreducible factorization.

**Lemma 1.** Let \( \delta \) be a generator of the cyclic multiplicative subgroup \( \Gamma(\mathcal{GR}(p^\alpha, m)) \setminus \{0\} \) of \( \mathcal{GR}(p^\alpha, m)^x \) where \( m = \omega d_n(q) \). The map

\[
\Omega: \{\mathcal{C}_q(Z) : Z \subseteq [0; n - 1]_q\} \to \{f \in \mathcal{GR}(p^\alpha, r)[X] : f \text{ is monic and } f|X^n - 1\}
\]

where \( \Omega(\emptyset) = 1 \), is bijective map. Moreover, for any \( z \in [0; n - 1] \) and for any sets \( \Lambda \) and \( B \) form by union of \( q \)-cyclotomic cosets modulo \( n \) we have

1. \( \Omega(\mathcal{C}_q(z)) \) is a monic basic-irreducible polynomial over \( \mathcal{GR}(p^\alpha, r) \) of degree \( |\mathcal{C}_q(z)| \);
2. \( \text{lcm}(\Omega(\Lambda), \Omega(B)) = \Omega(\Lambda \cup B) \) and \( \gcd(\Omega(\Lambda), \Omega(B)) = \Omega(\Lambda \cap B) \);
3. If \( \Lambda \cap B = \emptyset \), then \( \Omega(\Lambda \cup B) = \Omega(\Lambda) \Omega(B) \).

**Proposition 5.** \([18]\) Subsection 2.2) Let \( j \) be a divisor of \( n \) such that \( \gcd(j, q) = 1 \). Then

\[
\gamma(j; q) = |\Lambda_j| = \begin{cases} \frac{\omega(j)}{\omega d_j(q)}, & \text{if } j \in N_q; \\ 0, & \text{otherwise.} \end{cases}
\]

and \( \beta(j; q) = |\overline{\Lambda}_j| = \begin{cases} \frac{\omega(j)}{\omega d_j(q)}, & \text{if } j \notin N_q; \\ 0, & \text{otherwise.} \end{cases} \)

Moreover, \( \omega(n; q) = \sum_{i \in N_q} \gamma(i; q) + 2 \sum_{j \in N_q} \beta(j; q) \).

We will introduce the following notation

\[
E_n(q, s) = \mathcal{I}_n(q, s) \times (\mathcal{J}_n(q, s))^2.
\]
The elements in \( \mathcal{I}_n(q, s) \) are all the polynomials in \( \{ (u_{il}^{(a)})_{0 \leq a < s} \} \) where \( (u_{il}^{(a)})_{0 \leq a < s} \) are in \( \mathcal{E}_s \) and the indices \( i \) and \( j \) satisfy \( i | n, i \in \mathcal{N}_q \) and \( 1 \leq \gamma(i; q) \leq \beta(j) \), i.e.,

\[
(\{(u_{il}^{(a)})_{0 \leq a < s}\}^\circ) = \left( \{(u_{il}^{(a)})_{0 \leq a < s}\}_{1 \leq \gamma(i; q) \leq \beta(j)} \right)_{i | n, i \in \mathcal{N}_q} \in \mathcal{I}_n(q, s).
\]

Similarly, \( ((v_{ij}^{(a)})_{0 \leq a < s})^\circ = \left( \{(v_{ij}^{(a)})_{0 \leq a < s}\}_{1 \leq \gamma(i; q) \leq \beta(j)} \right)_{j | n, j \in \mathcal{N}_q} \in \mathcal{J}_n(q, s). \) Note that if \( s = 1 \), then \( \mathcal{E}_1 = \{0; 1\} \) and in this case, we write \( ((u_{il})^\circ) = ((u_{il}^{(a)})_{0 \leq a < 1}) \) and \( ((v_{ij})^\circ) = ((v_{ij}^{(a)})_{0 \leq a < 1}) \) for \( i, j \) positive integers such that \( i, j \in \mathcal{N}_q \).

In the sequel,

\[
\Lambda_i = \{G_{il} : 1 \leq l \leq \gamma(i; q)\} \quad\text{and}\quad \overline{\Lambda}_j = \{(F_{jh}, -F_{jh}) : 1 \leq h \leq \beta(j; q)\}.
\]

Of course, all the polynomials in \( \{\Omega(G_{il}) : 1 \leq l \leq \gamma(i; q)\} \) are basic-irreducible in \( R[X] \) of degree \( \omega_{\gamma(i; q)} \), and all the elements in \( \{\{\Omega(F_{jh}), \Omega(-F_{jh})\} : 1 \leq h \leq \beta(j; q)\} \) are of monic basic-irreducible reciprocal polynomials in \( R[X] \) of the same degree \( \omega_{\beta(j; q)} \). The basic-irreducible factorization of \( X^n - 1 \) in \( R[X] \) is given as

\[
X^n - 1 = \prod_{i | n, i \in \mathcal{N}_q} \left( \prod_{l = 1}^{\gamma(i; q)} \Omega(G_{il}) \right) \prod_{j | n, j \in \mathcal{N}_q} \left( \prod_{h = 1}^{\beta(j; q)} \Omega(F_{jh}) \cdot \Omega(-F_{jh}) \right). \tag{4}
\]

Thus, for any monic polynomial \( f \) dividing \( X^n - 1 \), there is uniquely \( ((u_{il}^\circ), (v_{ij}^\circ), (w_{jh}^\circ)) \) in \( \mathcal{E}_n(q, 1) \) such that

\[
f = \prod_{i | n, i \in \mathcal{N}_q} \left( \prod_{l = 1}^{\gamma(i; q)} \Omega(G_{il}) \right) \prod_{j | n, j \in \mathcal{N}_q} \left( \prod_{h = 1}^{\beta(j; q)} \Omega(F_{jh}) \cdot \Omega(-F_{jh}) \right), \tag{5}
\]

and inversely. Denote the right-hand of Eq. \( (5) \) by \( \partial(((u_{il})^\circ), (v_{ij})^\circ), (w_{jh})^\circ)) \). Note that \( \partial(((1)^\circ), ((1)^\circ), ((1)^\circ)) = X^n - 1, \partial(((0)^\circ), ((0)^\circ), ((0)^\circ)) = 1 \), for all \( f_1 = \partial(((u_{il})^\circ), (v_{ij})^\circ), (w_{jh})^\circ)) \) and \( f_2 = \partial(((u_{il})^\circ), (v_{ij}^\circ), (w_{jh}^\circ)) \), one notes that

\[
1 \in \mathbb{N} \quad\text{if}\quad f_1 f_2 = \partial(((\max(u_{il}, u_{il}^\circ))^{a}), ((\max(v_{ij}, v_{ij}^\circ))^{b}), ((\max(w_{jh}, w_{jh}^\circ))^{c}));
\]

\[
g \subseteq \mathbb{N} \quad\text{if}\quad f_1 f_2 = \partial(((\min(u_{il}, u_{il}^\circ))^{a}), ((\min(v_{ij}, v_{ij}^\circ))^{b}), ((\min(w_{jh}, w_{jh}^\circ))^{c})),
\]

and if all \( (u_{il} + u_{il}^\circ, v_{ij} + v_{ij}^\circ, w_{jh} + w_{jh}^\circ)s \) are in \( \{0; 1\}^3 \) then

\[
f_1 f_2 = \partial(((u_{il} + u_{il}^\circ), (v_{ij} + v_{ij}^\circ), (w_{jh} + w_{jh}^\circ)).
\]

A cyclic code \( C \) of length \( n \) over \( R \) is a linear code that is invariant under the transformation \( \tau((c_0, c_1, \cdots, c_{n-1})) = (c_{n-1}, c_0, \cdots, c_{n-2}) \). If we denote by \( \langle X^n - 1 \rangle \) the ideal of \( R[X] \) generated by \( X^n - 1 \), it is well-known that any cyclic code of length \( n \) over \( R \) can be represented as an ideal of the quotient ring \( R[X]/\langle X^n - 1 \rangle \) via the \( R \)-module isomorphism \( \Psi : R^n \rightarrow R[X]/\langle X^n - 1 \rangle \), where \( \Psi(c) = \Psi(c) + \langle X^n - 1 \rangle \) and

\[
\Psi : \quad R^n \quad\rightarrow\quad R[X]
\]

\[
\mathbf{u} = (u_0, u_1, \cdots, u_{n-1}) \quad\mapsto\quad \mathbf{u}(X) = u_0 + u_1 X + \cdots + u_{n-1} X^{n-1}
\]
which is an \( R \)-module homomorphism. We will slightly abuse notation, identifying vectors in \( R^n \) as polynomials in \( R[X] \) of degree less than \( n \), and vice versa when the context is clear. It is well-known that \( R[X]/(X^n - 1) \) is a principal ideal ring and \( C \) is a cyclic code of length \( n \) over \( R \) if and only if \( \Psi(C) \) is an ideal of \( R[X]/(X^n - 1) \), (see [4] and references therein).

Thus, the generator polynomial of a cyclic code \( C \) of \( R^n \), is the monic polynomial \( f \) in \( R[X] \) such that \( \Psi(C) = \langle f(x) \rangle \), where \( \langle f(x) \rangle \) is the ideal of \( R[X]/(X^n - 1) \) generated by \( f \). For a polynomial \( f \) of degree \( k \) its reciprocal polynomial \( X^k f(X^{-1}) \) will be denoted by \( f^* \) and if \( f \) is a factor of \( X^n - 1 \) we denote \( \bar{f} = \frac{X^n - 1}{f} \). A polynomial \( f \) is self-reciprocal if \( f = f^* \), otherwise \( f \) and \( f^* \) are called a reciprocal polynomial pair.

For any union \( A \) of q-cyclotomic cosets modulo \( n \), \( \Omega(A)^* = \Omega(-A) \) and \( \Omega(A) = \Omega(A) \). The \((s + 1)\)-tuple \((A_0, A_1, \ldots, A_s)\) is called to be an ordered \((q, s)\)-partition cyclotomic modulo \( n \), if \( A_0, A_1, \ldots, A_s \) are unions of \( q \)-cyclotomic cosets modulo \( n \) whose \( \{A_t : A_t \neq \emptyset, \quad 0 \leq t \leq s \} \) forms a partition of \([0; n - 1]\). Denote by \( \mathcal{R}_n(q, s) \) the set of ordered \((q, s)\)-partition cyclotomic modulo \( n \). Note that

\[
\mathcal{R}_n(q, s) = \left\{ \left( C_q^{-1}((\{0\})), C_q^{-1}((1)), \ldots, C_q^{-1}((s)) \right) : \lambda \in \{0; s\}^{[0; n-1]} \right\}.
\]

It follows that \( |\mathcal{R}_n(q, s)| = (s + 1)^{\omega(n, q)} \). Let \( \mathbf{A} = (A_0, A_1, \ldots, A_s) \) in \( \mathcal{R}_n(q, s) \). For a positive integer \( u \) we denote by \( u\mathbf{A} = (uA_0, uA_1, \ldots, uA_{s-1}) \). It is easy to see that \( p^\ell \mathbf{A} \in \mathcal{R}_n(q, s) \) for any \( 0 \leq \ell < r \). From [4] Theorems 3.4, 3.5 and 3.8, we have the following result.

**Lemma 2.** For any cyclic serial code \( C \) over \( R \) of length \( n \), there is a unique \((s + 1)\)-tuple \((A_0, A_1, \ldots, A_s)\) in \( \mathcal{R}_n(q, s) \) such that

\[
\Psi(C) = \bigoplus_{t=0}^{s-1} \theta^t \left\langle \Omega(A_{t+1}) \right\rangle = \left\{ \theta^t \prod_{a=1}^{s} \Omega(A_a) : 0 \leq t \leq s - 1 \right\}.
\]

Moreover, \( \Psi(C^\perp) = \bigoplus_{t=0}^{s-1} \theta^t \left\langle \Omega(-A_{t+1}) \right\rangle \).

Let \( \mathbf{A} \) be a union of \( q \)-cyclotomic cosets modulo \( n \). From now on, we will consider the code

\[
C(\mathbf{A}) = \left\{ \mathbf{c} \in R^{n} : \Omega(\mathbf{A}) \text{ divides } \Psi(\mathbf{c}) \right\},
\]

thus it is clear that \( \Psi(C(\mathbf{A})) = \langle \Omega(\mathbf{A}) \rangle(x) \).

**Remark 1.** Free cyclic serial codes over a finite chain ring have been studied in [6] using the cyclotomic cosets and the trace map. Note that \( C([0; n - 1]) = \{0\} \) and \( C(\emptyset) = R^n \). From Lemma 2 for any free cyclic serial code \( C \) of length \( n \) over \( R \) there exists a unique set \( \mathbf{A} \) which is a union of \( q \)-cyclotomic cosets modulo \( n \) such that \( C = C(\mathbf{A}) \).

Moreover, \( C(\mathbf{A})^\perp = C(-\mathbf{A}) \), the generator matrix of \( C(\mathbf{A}) \) is \( \Omega(\mathbf{A}) \), and \( x_{\forall \mathbf{a} \in R(C(\mathbf{A}))} = |\mathbf{A}| \).

**Proposition 6.** If \( \mathbf{A} \) and \( \mathbf{B} \) are unions of \( q \)-cyclotomic cosets modulo \( n \), then

1. \( \mathbf{A} \subseteq \mathbf{B} \) if and only \( C(\mathbf{A}) \subseteq C(\mathbf{B}) \);
2. \( C(\mathbf{A} \cap \mathbf{B}) = C(\mathbf{A}) \cap C(\mathbf{B}) \), and \( C(\mathbf{A} \cup \mathbf{B}) = C(\mathbf{A}) + C(\mathbf{B}) \);
3. \( \sigma^\ell(C(\mathbf{A})) = C(p^\ell \mathbf{A}) \) and \( C(\mathbf{A})^\perp = C(-p^\ell \mathbf{A}) \), for all \( 0 \leq \ell < r - 1 \).

**Proof.** Item (1) follows from the definition of \( C(\mathbf{A}) \) and \( C(B) \) and the fact that \( \mathbf{A} \subseteq \mathbf{B} \) if and only \( \Omega(\mathbf{B}) \) divides \( \Omega(\mathbf{A}) \). To prove (2), we note that since \( \mathbf{A} \subseteq \mathbf{B} \), we have \( C(\mathbf{A} \cap \mathbf{B}) \subseteq C(\mathbf{A}) \cap C(\mathbf{B}) \) and \( C(\mathbf{A} \cup \mathbf{B}) \subseteq C(\mathbf{A} \cup \mathbf{B}) \). Conversely, if \( \mathbf{c} \in C(\mathbf{A}) \cap C(\mathbf{B}) \) then \( \Omega(\mathbf{A}) \) and \( \Omega(\mathbf{B}) \) divide \( \Psi(\mathbf{c}) \). Thus \( 1_{cm}(\Omega(\mathbf{A}) \cap \Omega(\mathbf{B})) \) divides \( \Psi(\mathbf{c}) \). Now, \( 1_{cm}(\Omega(\mathbf{A}) \cap \Omega(\mathbf{B})) = \Omega(\mathbf{A} \cap \mathbf{B}) \), so we have \( C(\mathbf{A} \cap \mathbf{B}) \subseteq C(\mathbf{A} \cap \mathbf{B}) \). Since \( q \in \text{zd}(\Omega(\mathbf{A}) \cap \Omega(\mathbf{B})) = \Omega(\mathbf{A} \cap \mathbf{B}) = \Omega(\mathbf{A} \cup \mathbf{B}) \), hence \( C(\mathbf{A}) + C(\mathbf{B}) \supseteq C(\mathbf{A} \cup \mathbf{B}) \). To finish with the proof of the item (3), we have \( \sigma^\ell(C(\mathbf{A})) = \left\{ \mathbf{c} \in R^n : \sigma^\ell \left( \Omega(\mathbf{A}) \right) \text{ divides } \Psi(\mathbf{c}) \right\} \), thus \( \sigma^\ell(C(\mathbf{A})) = C(p^\ell \mathbf{A}) \), since \( \sigma^\ell(\Omega(\mathbf{X})) = \Omega(p^\ell \mathbf{X}) \). Finally, for any \( 0 \leq \ell < r - 1 \) we have

\[
C(\mathbf{A})^\perp = \left(\sigma^\ell(C(\mathbf{A}))^\perp\right)^\perp = \left(\sigma^\ell(C(p^\ell \mathbf{A}))^\perp\right)^\perp = \mathbf{C}(-p^\ell \mathbf{X}), \quad \text{from Remark 4}.
\]
Let $\mathbf{A} = (A_0, A_1, \ldots, A_s)$ and $\mathbf{B} = (B_0, B_1, \ldots, B_s)$ be elements in $\mathbb{R}_n(q,s)$. We will define the following set in $R^n$

$$C(\mathbf{A}) = \bigoplus_{i=0}^{s-1} \theta^i C(A_i).$$

Clearly, $C(\mathbf{A})$ is a cyclic serial code of length $n$ over $R$ and $\mathbf{A}$ is called the defining multiset of $C(\mathbf{A})$. The parameters of $C(\mathbf{A})$ are given by the entries in $([A_0], [A_1], \ldots, [A_s])$ and from Lemma 2 it follows that for any cyclic serial code $C$ over $R$ of length $n$, there is a unique defining multiset $\mathbf{A}$ in $\mathbb{R}_n(q,s)$ such that $C = C(\mathbf{A})$.

Let us denote by

$$\mathbf{A}^0 = (A_s, A_{s-1}, \ldots, A_0), \quad \mathbf{A} \cup \mathbf{B} = (E_0, E_1, \ldots, E_{s-1})$$

where $E_0 = A_0 \cup B_0$, and $E_t = (A_t \cup B_t) \setminus \bigcup_{i=0}^{t-1} (A_i \cup B_i)$ for all $0 < t \leq s$. It is easy to see that $\mathbf{A}^0$ and $\mathbf{A} \cup \mathbf{B}$ are in $\mathbb{R}_n(q,s)$. Moreover, $C^{s-t} = C(-p^t \mathbf{A}^0)$, and $\dim_q(C) = \sum_{i=0}^{s-1} (s-t) |A_t|$. Note that if $\mathbf{A} \cap \mathbf{B} = (\mathbf{A}^0 \cup \mathbf{B}^0)^0 = (E_0, E_1, \ldots, E_s)$, then $E_s = A_s \cup B_s$ and $E_{s-t} = (A_{s-t} \cup B_{s-t}) \setminus \bigcup_{i=0}^{s-1} (A_{s-i} \cup B_{s-i})$, for all $0 < t \leq s$.

**Proposition 7.** [Theorem 6] Let $\mathbf{A} = (A_0, A_1, \ldots, A_s)$ and $\mathbf{B} = (B_0, B_1, \ldots, B_s)$ be elements in $\mathbb{R}_n(q,s)$. Then $C(\mathbf{A}) + C(\mathbf{B}) = C(\mathbf{A} \cup \mathbf{B})$ and $C(\mathbf{A}) \cap C(\mathbf{B}) = C(\mathbf{A} \cap \mathbf{B})$.

**Corollary 2.** Let $\mathbf{A} = (A_0, A_1, \ldots, A_s)$ and $\mathbf{B} = (B_0, B_1, \ldots, B_s)$ be elements in $\mathbb{R}_n(q,s)$, and define $g_t = \prod_{a=t+1}^{a} \Omega(A_a)$ and $h_t = \prod_{a=t+1}^{a} \Omega(B_a)$, for all $0 \leq t < s$. Then

1. $\overline{\nu}(\mathbf{A}) = \langle \{ \theta^t g_t(x) : 0 \leq t < s \} \rangle$, and $\overline{\nu}(\mathbf{B}) = \langle \{ \theta^t h_t(x) : 0 \leq t < s \} \rangle$;
2. $\overline{\nu}(\mathbf{A} \cap \mathbf{B}) = \langle \{ \theta^t \circ \mu(g_t, h_t)(x) : 0 \leq t < s \} \rangle$.

**Proof.** We have $A \cap B = (E_0, E_1, \ldots, E_s)$, where $E_s = A_s \cup B_s$ and $E_{s-t} = (A_{s-t} \cup B_{s-t}) \setminus \bigcup_{i=0}^{s-1} (A_{s-i} \cup B_{s-i})$, for all $0 < t \leq s$. From Lemma 2 it follows that $\overline{\nu}(\mathbf{A}) = \langle \{ \theta^t g_t(x) : 0 \leq t < s \} \rangle$, and $\overline{\nu}(\mathbf{B}) = \langle \{ \theta^t h_t(x) : 0 \leq t < s \} \rangle$. Since $\overline{\nu}(\mathbf{A} \cap \mathbf{B}) = \overline{\nu}(\mathbf{A} \cap \mathbf{B})$, using again Lemma 2 and Proposition 7 it follows that

$$\overline{\nu}(\mathbf{A} \cap \mathbf{B}) = \langle \{ f_0(x), \theta f_1(x), \ldots, \theta^{s-1} f_{s-1}(x) \} \rangle,$$

where $f_t = \prod_{a=t+1}^{a} \Omega(E_a)$. Thus for all $0 \leq t < s$, $f_t = \Omega \left( \bigcup_{a=t+1}^{a} E_a \right)$ and $\bigcup_{a=t+1}^{a} E_a = \bigcup_{a=t+1}^{a} (A_{s-a+1} \cup B_{s-a+1})$. Then $f_t = \Omega \left( \bigcup_{a=t+1}^{a} (A_{s-a+1} \cup B_{s-a+1}) \right) = \lcm(g_t, h_t)$. \hfill $\Box$

**Theorem 1.** Let $\mathbf{A}$ be an $\mathbb{R}_n(q,s)$. Then

$$H_\ell(C(\mathbf{A})) = C(\mathbf{A} \cap -p^\ell \mathbf{A}^0).$$

**Proof.** Let $\mathbf{A}$ be an $\mathbb{R}_n(q,s)$ and $0 \leq \ell < r$. We have

$$H_\ell(C(\mathbf{A})) = C(\mathbf{A} \cap C(\mathbf{A})^{s-\ell}, \text{ from Definition 1});$$

$$= C(\mathbf{A}) \cap C(-p^\ell \mathbf{A}^0), \text{ since } C(\mathbf{A})^{s-\ell} = C(-p^\ell \mathbf{A}^0);$$

$$= C(\mathbf{A} \cap -p^\ell \mathbf{A}^0), \text{ from Proposition 7}.$$

$\Box$
**Example 3.1.** Let \( R = \mathbb{Z}_2 \cdot [\theta] \) with \( 1 \leq a \leq 2 \) be the finite chain ring of parameters \((2, a, 1, e, 2)\). Consider the 2-cyclotomic cosets modulo 7 given by \( \mathcal{C}_2(\{0\}) = \emptyset, \mathcal{C}_2(\{1\}) = \{1; 2; 4\}, \) and \( \mathcal{C}_2(\{3\}) = \{3; 5; 6\}. \) Note that \( \{\mathcal{C}_2(\{1\}), \mathcal{C}_2(\{3\})\} \) is an asymmetric set, and \( \mathcal{C}_2(\{0\}) \) is a symmetric set. Consider the cyclic serial code over \( R \) of length 7 with defining multisets \( \mathbf{A} = (\mathcal{C}_2(\{0\}), \mathcal{C}_2(\{3\}), \mathcal{C}_2(\{1\})) \).

Then \( -\mathbf{A}^\circ = (\mathcal{C}_2(\{3\}), \mathcal{C}_2(\{1\}), \mathcal{C}_2(\{0\})) \), and \( \mathcal{C}(\mathbf{A}) = \mathcal{C}(\mathcal{C}_2(\{0\}))) \oplus \theta \mathcal{C}(\mathcal{C}_2(\{3\})) \). Thus \( \mathcal{C}(\mathbf{A})^\perp = \mathcal{C}(-\mathbf{A}^\circ) = \mathcal{C}(\mathcal{C}_2(\{1\}) \oplus \theta \mathcal{C}(\mathcal{C}_2(\{3\})). \) Finally, \( \mathbf{A} \cap -\mathbf{A}^\circ = (F_0, F_1, F_2) \) where \( F_0 = \emptyset, F_1 = \mathcal{C}_2(\{3\}), \) and \( F_2 = \mathcal{C}_2(\{0; 1\}). \) Therefore \( \mathcal{H}_0(\mathcal{C}(\mathbf{A})) = \mathcal{C}(\mathbf{A} \cap -\mathbf{A}^\circ)) = \mathcal{C}(\emptyset, \mathcal{C}_2(\{3\}), \mathcal{C}_2(\{0; 1\})) = \theta \mathcal{C}(\mathcal{C}_2(\{0; 1\})). \)

### 3.1. Euclidean hulls

From now on, \( \ell = 0 \). The following result provides us a way of checking whether a given cyclic serial code \( D \) is the Euclidean hull of a cyclic code \( C \) or not. Of course, if \( \mathcal{H}_0(C) = D \), then the cyclic code \( D \) is serial if, and only if \( C \) the cyclic code is also a serial code. In the sequel, for each \( \mathbf{X} = (X_0, X_1, \ldots, X_s) \in \mathbb{R}_n(q, s), \) we will denote \( \Omega(X_a) = \partial \left( ((x_{a}^{(a)})^\circ), ((y_{j h}^{(a)})^\bullet), ((z_{j h}^{(a)})^\bullet) \right) \), for \( a \in \{0; 1; \ldots; s\}. \) Then \( \Omega(X_a) = \partial \left( ((x_{a}^{(a)})^\circ), ((y_{j h}^{(a)})^\bullet), ((z_{j h}^{(a)})^\bullet) \right), \) and for \( 0 \leq t \leq s \), \( \sum_{a=t+1}^{s} \Omega(X_a) = \partial \left( ((x_{a}^{(a)})^\circ), ((y_{j h}^{(a)})^\bullet), ((z_{j h}^{(a)})^\bullet) \right), \) where \( x_{a}^{[a]} = \sum_{a=t+1}^{s} x_{a}^{(a)}, y_{j h}^{[a]} = \sum_{a=t+1}^{s} y_{j h}^{(a)} \) and \( z_{j h}^{[a]} = \sum_{a=t+1}^{s} z_{j h}^{(a)} \). Note that, for all \( 0 \leq t < s \), we have \( x_{a}^{[a]} = x_{a}^{t+1} + x_{a}^{(t+1), t}, y_{j h}^{[a]} = y_{j h}^{t+1} + y_{j h}^{(t+1)} \), and \( z_{j h}^{[a]} = z_{j h}^{t+1} + z_{j h}^{(t+1)}. \) Since \( \partial(((1)^\circ), ((1)^\bullet), ((1)^\bullet)) = X^a = 1 = g_0 \partial(((x_{a}^{(a)})^\circ), ((y_{j h}^{(a)})^\bullet), ((z_{j h}^{(a)})^\bullet)), \) it follows that \( \sum_{a=0}^{s} x_{a}^{(a)} = \sum_{a=0}^{s} y_{j h}^{(a)} = \sum_{a=0}^{s} z_{j h}^{(a)} = 1. \) From Eqs. (5) and (6), there exists a unique

\[
(x^\circ, y^\bullet, z^\bullet) = \left( ((x_{a}^{(a)})_{0 \leq a < s}), ((y_{j h}^{(a)})_{0 \leq a < s}), ((z_{j h}^{(a)})_{0 \leq a < s}) \right)
\]

which is an element in \( \mathcal{E}_n(q, s) \) such that

\[
\overline{\mathbf{P}}(\mathcal{C}(\mathbf{X})) = \left\{ \theta^t \cdot \partial \left( ((x_{a}^{(a)})_{0 \leq a < s}), ((y_{j h}^{(a)})_{0 \leq a < s}), ((z_{j h}^{(a)})_{0 \leq a < s}) \right) : 0 \leq t \leq s - 1 \right\}.
\]

From Eqs. (4), (5), and (6), the following lemma follows.

**Lemma 3.** There is a bijection between the set \( \mathcal{C}(n; R) \) of cyclic serial codes of length \( n \) over \( R \) and the set \( \mathcal{E}_n(q, s). \)

When \( \ell = 0 \), and with the triple-sequences of a cyclic serial code, in comparing the two sides of Eq. (8) of Theorem 1 it obtains the following result.

**Corollary 3.** Let \((x^\circ, y^\bullet, z^\bullet) = \left( ((x_{a}^{(a)})_{0 \leq a < s}), ((y_{j h}^{(a)})_{0 \leq a < s}), ((z_{j h}^{(a)})_{0 \leq a < s}) \right) \) and

\[
(u, v, w) = \left( ((u_{a}^{(a)})_{0 \leq a < s}), ((v_{j h}^{(a)})_{0 \leq a < s}), ((w_{j h}^{(a)})_{0 \leq a < s}) \right)
\]

in \( \mathcal{E}_n(q, s) \) such that

\[
\overline{\mathbf{P}}(\mathcal{C}) = \left\{ \theta^t \cdot \partial \left( ((u_{a}^{(a)})_{0 \leq a < s}), ((v_{j h}^{(a)})_{0 \leq a < s}), ((w_{j h}^{(a)})_{0 \leq a < s}) \right) : 0 \leq t \leq s - 1 \right\},
\]

and

\[
\overline{\mathbf{P}}(\mathcal{D}) = \left\{ \theta^t \cdot \partial \left( ((u_{a}^{(a)})_{0 \leq a < s}), ((v_{j h}^{(a)})_{0 \leq a < s}), ((w_{j h}^{(a)})_{0 \leq a < s}) \right) : 0 \leq t \leq s - 1 \right\}.
\]

Then \( \mathcal{H}_0(C) = D \) if, and only if for all \( 0 \leq t \leq s - 1 \),

\[
\begin{align*}
\left\{ \begin{array}{l}
\left\{ u_{a}^{t+1} \right\} = \max \left\{ \sum_{a=t+1}^{s} u_{a}^{(a)}, \sum_{a=t+1}^{s} u_{a}^{(a-a)} \right\} ; \\
\left\{ v_{j h}^{t+1} \right\} = \max \left\{ \sum_{a=t+1}^{s} v_{j h}^{(a)}, \sum_{a=t+1}^{s} v_{j h}^{(a-a)} \right\} ; \\
\left\{ w_{j h}^{t+1} \right\} = \max \left\{ \sum_{a=t+1}^{s} w_{j h}^{(a)}, \sum_{a=t+1}^{s} w_{j h}^{(a-a)} \right\} .
\end{array} \right.
\end{align*}
\]
Moreover, for all $0 \leq t \leq s - 1$, if $2t \leq s - 1$, then $u_{tt}^{[n]} = \max \left\{ \sum_{a=t+1}^{s} x_{it}^{(a)} \mid x_{it}^{(a)} \in \{1; 0\}, \{0; 1\}, \{1; 1\} \right\}$, since $\sum_{a=0}^{s} x_{it}^{(a)} = 1$.

From Corollary 3.4, and Corollaries 3.5 and 3.6 in [9] can be naturally extended to finite chain rings of nilpotency index 2. The following remark gives this generalization.

**Remark 2.** Let $(x^{a}, y^{s}, z^{s}) = \left( (((x_{it}^{(a)}))_{0 \leq a < 2})^{s}, (((y_{jth}^{(a)}))_{0 \leq a < 2})^{s}, (((z_{jth}^{(a)}))_{0 \leq a < 2})^{s} \right)$ and

$$(u, v, w) = \left( (((u_{it}^{(a)}))_{0 \leq a < 2})^{s}, (((v_{jth}^{(a)}))_{0 \leq a < 2})^{s}, (((w_{jth}^{(a)}))_{0 \leq a < 2})^{s} \right)$$

in $\mathcal{E}_{n}(y, 2)$ such that

$$\mathcal{F}(C) = \left\{ \vartheta \left( ((x_{it}^{(a)}))^{s}, ((y_{jth}^{(a)}))^{s}, ((z_{jth}^{(a)}))^{s} \right), \theta \cdot \vartheta \left( ((x_{it}^{(a)}))^{s}, ((y_{jth}^{(a)}))^{s}, ((z_{jth}^{(a)}))^{s} \right) \right\},$$

and

$$\mathcal{F}(D) = \left\{ \vartheta \left( ((x_{it}^{(a)}))^{s}, ((y_{jth}^{(a)}))^{s}, ((z_{jth}^{(a)}))^{s} \right), \theta \cdot \vartheta \left( ((x_{it}^{(a)}))^{s}, ((y_{jth}^{(a)}))^{s}, ((z_{jth}^{(a)}))^{s} \right) \right\}.$$  

Then $\mathcal{H}_{0}(C) = D$ if, and only if $(x_{it}^{(1)}, x_{it}^{(2)}) \in \left\{ (0; 0), (0; 1), (1; 0), (1; 1), \right\}$, and

$$(y_{jth}^{(1)}, y_{jth}^{(2)}, z_{jth}^{(1)}, z_{jth}^{(2)}) \in \left\{ (0; 0; 0), (0; 1; 0), (1; 0; 0), (1; 1; 0), (0; 0; 1), (0; 1; 1), (1; 0; 1), (1; 1; 1) \right\}$$

for all $i, j, h$. Moreover,

1. $C$ is LCD if, and only if $(x_{it}^{(2)}) = x_{it}^{(1)} = y_{jth}^{(1)} = z_{jth}^{(1)}$ and $(x_{it}^{(2)}: y_{jth}^{(2)}: z_{jth}^{(2)}) \in \{ (0; 0), (0; 1), (1; 0), (1; 1) \}$, for all $i, j, h$.

2. $C$ is self-orthogonal if, and only if $(x_{it}^{(1)}) = (0; 0)$, and $(y_{jth}^{(1)}: z_{jth}^{(1)}) \in \{ (1; 0), (1; 1) \}$, for all $i, j, h$.

4. The $q$-dimensions of Euclidean hulls of cyclic serial codes

In this section, $C$ is a cyclic serial code of length $n$ over $R$ with triple-sequence

$$(x^{c}, y^{s}, z^{s}) = \left( (((x_{it}^{(a)}))_{0 \leq a < c})^{s}, (((y_{jth}^{(a)}))_{0 \leq a < c})^{s}, (((z_{jth}^{(a)}))_{0 \leq a < c})^{s} \right)$$

in $\mathcal{E}_{n}(g, s)$. Then

$$\mathcal{F}(C) = \left\{ \vartheta^{t} \cdot \vartheta \left( ((x_{it}^{(a)}))^{s}, ((y_{jth}^{(a)}))^{s}, ((z_{jth}^{(a)}))^{s} \right) : 0 \leq t \leq s - 1 \right\}.$$  

From Corollary 3.4

$$\mathcal{F}(\mathcal{H}_{0}(C)) = \left\{ \vartheta^{t} \cdot \vartheta \left( ((u_{it}^{(a)}))^{s}, ((v_{jth}^{(a)}))^{s}, ((w_{jth}^{(a)}))^{s} \right) : 0 \leq t \leq s - 1 \right\},$$
Case 1:

\[
\begin{cases}
    u^{[t]}_{il} = 1 - \min \left\{ \sum_{a=0}^{t} x^{(a)}_{il}, 1 - \sum_{a=0}^{s-t-1} x^{(a)}_{il} \right\}; \\
v^{[t]}_{jih} = 1 - \min \left\{ \sum_{a=0}^{t} y^{(a)}_{jih}, 1 - \sum_{a=0}^{s-t-1} z^{(a)}_{jih} \right\}; \\
w^{[t]}_{jih} = 1 - \min \left\{ \sum_{a=0}^{t} z^{(a)}_{jih}, 1 - \sum_{a=0}^{s-t-1} y^{(a)}_{jih} \right\};
\end{cases}
\]

for all \(0 \leq t \leq s - 1\). The following notations are important for the sequel of this paper. For all \(0 \leq t \leq s - 1\), \(1 \leq l \leq \gamma(i; q)\) and \(1 \leq h \leq \beta(j; q)\), denote by:

\[
\varepsilon^{(t)}_{jih} = v^{[t]}_{jih} + w^{[t]}_{jih}.
\]

Note that \(\varepsilon^{(t-1)}_{jih} = 2\). Let us consider now

\[
\Delta_{ul} = \sum_{t=0}^{s-1} (s-t)(u^{[t]}_{il} - u^{[t]}_{il}), \quad \text{and} \quad \Delta_{jh} = \sum_{t=0}^{s-1} (s-t)(\varepsilon^{(t-1)}_{jih} - \varepsilon^{(t)}_{jih}).
\]

Obviously, \(\Delta_{ul} = \sum_{t=1}^{s-1} \Delta^{(t)}_{ul}\), where \(\Delta^{(t)}_{ul} = \min \left\{ \sum_{a=0}^{t} x^{(a)}_{il}, 1 - \sum_{a=0}^{s-t-1} x^{(a)}_{il} \right\}\), and \(\Delta_{jh} = \sum_{t=0}^{s-1} \Delta^{(t)}_{jh}\), where

\[
\Delta^{(t)}_{jh} = \min \left\{ \sum_{a=0}^{t} y^{(a)}_{jih}, 1 - \sum_{a=0}^{s-t-1} z^{(a)}_{jih} \right\} + \min \left\{ \sum_{a=0}^{t} z^{(a)}_{jih}, 1 - \sum_{a=0}^{s-t-1} y^{(a)}_{jih} \right\}.
\]

Thus \(\Delta^{(t)}_{ul} = \sum_{t=1}^{s-1} \Delta^{(t)}_{ul}\), \(\varepsilon^{(t)}_{jih} = \sum_{h=1}^{\gamma(j; q)} \varepsilon^{(t)}_{jih}\), and \(\Delta^{(t)}_{jh} = \sum_{h=1}^{\beta(j; q)} \Delta^{(t)}_{jh}\). Note that for any integer \(t\), if \(t < 0\) then \(\varepsilon^{(t)}_{jih} = 2\beta(j; q)\).

Remark 3. Let \(0 \leq t \leq s - 1\).

1. \(\Delta^{(t)}_{ul} \in \{0, 1\}\) and \(\Delta^{(t)}_{jh} \in \{0, 1, 2\}\).

2. If \(0 < t < s\), then \(\Delta^{(t-1)}_{ul} \leq \Delta^{(t)}_{ul}\) and \(\Delta^{(t)}_{jh} \leq \Delta^{(t-1)}_{jh}\).

3. If \(2t < s\), then \(\Delta^{(t)}_{ul} = 0\) and \(\Delta^{(t)}_{jh} \leq 1\).

Lemma 4. Let \(j\) be a divisor of \(n\) such that \(j \notin N_q\). Then

\[
\begin{cases}
    0 \leq \varepsilon^{(t-1)}_{jih} - \varepsilon^{(t)}_{jih} \leq \beta(j; q) - (\varepsilon^{(t-2)}_{jih} - \varepsilon^{(t-1)}_{jih}), & \text{if } t < \left\lfloor \frac{s}{2} \right\rfloor; \\
    0 \leq \varepsilon^{(t)}_{jih} - \varepsilon^{(t-1)}_{jih} \leq 2 \left( \beta(j; q) - (\varepsilon^{(t-2)}_{jih} - \varepsilon^{(t-1)}_{jih}) \right), & \text{if } t \geq \left\lfloor \frac{s}{2} \right\rfloor.
\end{cases}
\]

Proof. Let \(0 \leq t \leq s - 1\) and \(\Delta^{(t)}_{jh} = \sum_{h=1}^{\beta(j; q)} \Delta^{(t)}_{jh}\). We have \(\varepsilon^{(t)}_{jih} = 2 - \Delta^{(t)}_{jh}\). From Remark 3 two cases are considered.

Let \(w^{(t-1)}_{j} = |\{h \in \mathbb{N} : 1 \leq h \leq \beta(j; q)\text{ and }\varepsilon^{(t-1)}_{jih} = 1\}|\).

Case 1: \(t < \left\lfloor \frac{s}{2} \right\rfloor\). We have \(\varepsilon^{(t)}_{jih} \in \{1, 2\}\), and \(\varepsilon^{(t-1)}_{jih} - \varepsilon^{(t)}_{jih} \in \{0, 1\}\), if \(\varepsilon^{(t-1)}_{jih} = 2\); \(\varepsilon^{(t-1)}_{jih} = 1\). Without loss of generality, we assume that \(\varepsilon^{(t-1)}_{jih} = 1\), for all \(h \in \{1, \ldots, w^{(t-1)}_{j}\}\). Thus

\[
\varepsilon^{(t-1)}_{jih} - \varepsilon^{(t)}_{jih} = \left( \sum_{h=1}^{\beta(j; q)} (\varepsilon^{(t-1)}_{jih} - \varepsilon^{(t)}_{jih}) \right) + \sum_{h=w^{(t-1)}_{j}+1}^{\beta(j; q)} (\varepsilon^{(t-1)}_{jih} - \varepsilon^{(t)}_{jih});
\]

\[
= 0 + \sum_{h=w^{(t-1)}_{j}+1}^{\beta(j; q)} (\varepsilon^{(t-1)}_{jih} - \varepsilon^{(t)}_{jih}), \quad \text{since } 0 \leq \varepsilon^{(t-1)}_{jih} - \varepsilon^{(t)}_{jih} \leq 1.
\]
Hence $0 \leq \varepsilon_j^{(t-1)} - \varepsilon_j^{(t)} \leq \beta(j; q) - \varpi_j^{(t-1)} \leq \beta(j; q) - (\varepsilon_j^{(t-2)} - \varepsilon_j^{(t-1)}).

**Case 2:** $t \geq \left[ \frac{q}{2} \right]$. We have
\[
\begin{align*}
\varepsilon_j^{(t-1)} - \varepsilon_j^{(t)} & \in \{0, 1, 2\}, & \text{if } \varepsilon_j^{(t-1)} \in \{1, 2\}; \\
\varepsilon_j^{(t-1)} - \varepsilon_j^{(t)} & = 0, & \text{if } \varepsilon_j^{(t-1)} = 0.
\end{align*}
\]
Without loss of generality, we assume that $\varepsilon_j^{(t-1)} = 0$, for all $h \in \{1, \ldots, \varpi_j^{(t-1)}\}$. Thus
\[
\varepsilon_j^{(t-1)} - \varepsilon_j^{(t)} = \left( \sum_{h=1}^{\varpi_j^{(t-1)}} (\varepsilon_j^{(t-1)} - \varepsilon_j^{(t)}) \right) + \left( \sum_{h=\varpi_j^{(t-1)}+1}^{\beta(j; q)} (\varepsilon_j^{(t-1)} - \varepsilon_j^{(t)}) \right);
\]

\[
= 0 + \left( \sum_{h=\varpi_j^{(t-1)}+1}^{\beta(j; q)} (\varepsilon_j^{(t-1)} - \varepsilon_j^{(t)}) \right), \text{ since } 0 \leq \varepsilon_j^{(t-1)} - \varepsilon_j^{(t)} \leq 2.
\]

Therefore $0 \leq \varepsilon_j^{(t-1)} - \varepsilon_j^{(t)} \leq 2(\beta(j; q) - \varpi_j^{(t-1)}) \leq 2 \left( \beta(j; q) - (\varepsilon_j^{(t-2)} - \varepsilon_j^{(t-1)}) \right)$.

\[\square\]

**Theorem 2.** The parameters of the Euclidean hull of a cyclic serial code over $R$ of length $n$ are given by $(k_0, k_1, \ldots, k_{s-1})$ where $2k_0 + k_1 + \cdots + k_{s-1} \leq n$,

\[
k_t = \sum_{i=1}^{\lfloor t/n \rfloor} \text{ord}_i(q) \cdot \mu_i^{(t)} + \sum_{j=1}^{\lfloor t/n \rfloor} \text{ord}_j(q) \cdot \nu_j^{(t)},
\]

with

\[
\begin{align*}
\left\{ \begin{array}{l}
\mu_i^{(t)} = 0, & \text{if } t < \left[ \frac{q}{2} \right]; \\
0 \leq \mu_i^{(t)} \leq \gamma(i; q), & \text{if } t \geq \left[ \frac{q}{2} \right],
\end{array} \right.
\]

and

\[
\left\{ \begin{array}{l}
\varepsilon_j^{(t)} = 0, & \text{if } n \in \mathcal{N}_q; \\
0 \leq \nu_j^{(t)} \leq \beta(j; q) - \nu_j^{(t-1)}, & \text{if } n \notin \mathcal{N}_q, \text{ and } t < \left[ \frac{q}{2} \right]; \\
0 \leq \nu_j^{(t)} \leq 2(\beta(j; q) - \nu_j^{(t-1)}), & \text{if } n \notin \mathcal{N}_q, \text{ and } t \geq \left[ \frac{q}{2} \right].
\end{array} \right.
\]

Moreover $\nu_j^{(t-1)} = 0$.

**Proof.** Let $(k_0, k_1, \ldots, k_{s-1})$ be the parameters of $\mathcal{H}_0(C)$. When $\mathcal{H}_0(C) = C$, we have $2k_0 + k_1 + \cdots + k_{s-1} \leq n$. Then for all $0 \leq t \leq s - 1$,

\[
k_t = \text{deg} \left( \partial \left( ((u_i^{[t-1]})^0), ((v_j^{[t-1]})^0), ((w_j^{[t]}))^0) \right) \right) - \text{deg} \left( \partial \left( ((u_i^{[t]})^0), ((v_j^{[t]})^0), ((w_j^{[t]})^0) \right) \right);
\]

\[
= \sum_{i=1}^{\lfloor t/n \rfloor} \text{ord}_i(q) \cdot \mu_i^{(t)} + \sum_{j=1}^{\lfloor t/n \rfloor} \text{ord}_j(q) \cdot (\varepsilon_j^{(t-1)} - \varepsilon_j^{(t)}), \quad \text{where } \mu_i^{(t)} = \sum_{l=1}^{\gamma(i; q)} (u_i^{[t-1]} - u_i^{[t]}).
\]

Since

\[
\left\{ \begin{array}{l}
\mu_i^{(t)} = 0, & \text{if } t < \left[ \frac{q}{2} \right]; \\
0 \leq \mu_i^{(t)} \leq \gamma(i; q), & \text{if } t \geq \left[ \frac{q}{2} \right],
\end{array} \right.
\]

it follows that

\[
\left\{ \begin{array}{l}
\mu_i^{(t)} = 0, & \text{if } 2t < s; \quad \text{On the other hand,}
\end{array} \right.
\]

one notes that if $n \in \mathcal{N}_q$, then all positive divisor of $n$ is in then $\mathcal{N}_q$. By Lemma 3, we obtain

\[
\left\{ \begin{array}{l}
\varepsilon_j^{(t)} = 0, & \text{if } n \in \mathcal{N}_q; \\
0 \leq \nu_j^{(t)} \leq \beta(j; q) - \nu_j^{(t-1)}, & \text{if } n \notin \mathcal{N}_q, \text{ and } t < \left[ \frac{q}{2} \right]; \\
0 \leq \nu_j^{(t)} \leq 2(\beta(j; q) - \nu_j^{(t-1)}), & \text{if } n \notin \mathcal{N}_q, \text{ and } t \geq \left[ \frac{q}{2} \right],
\end{array} \right.
\]

where $\nu_j^{(t)} = \varepsilon_j^{(t-1)} - \varepsilon_j^{(t)}$. Obviously $\nu_j^{(t-1)} = \varepsilon_j^{(t-2)} - \varepsilon_j^{(t-1)} = 0$.

\[\square\]

The previous discussion leads to the following algorithm.
Algorithm 1: Parameters of the Euclidean hull of a cyclic serial code of length $n$ over $R$.

Input: Length $n$, and a finite chain ring $R$ of parameters $(p, a, r, e, s)$ such that $\gcd(p, n) = 1$. 
Output: All possible $s$-tuples $(k_0, k_1, \ldots, k_{s-1})$.

1. if $n \in \mathcal{N}_q$ then
2. for $0 \leq t < s$ do
3. if $t < \left\lceil \frac{n}{2} \right\rceil$ then
4. $k_i = 0.$
5. else
6. For each $i \mid n$, compute $\text{ord}_i(q)$, and $\gamma(i; q)$,
7. therefore all the possible values of $k_t$, such that
8. 
9. $k_t = \sum_{i \mid n \in \mathcal{N}_q} \text{ord}_i(q) \cdot \mu_i^{(t)}$,
10. with $0 \leq \mu_i^{(t)} \leq \gamma(i; q)$.
11. return The possible parameters $(0, \ldots, 0, k_\left\lceil \frac{n}{2} \right\rceil, \ldots, k_{s-1})$ such that $k_\left\lceil \frac{n}{2} \right\rceil + \cdots + k_{s-1} \leq n$.
12. else
13. For each $i \mid n$, if $i \in \mathcal{N}_q$, then compute $\text{ord}_i(q)$, and $\gamma(i; q)$.
14. For each $j \mid n$, if $j \not\in \mathcal{N}_q$, then compute $\text{ord}_j(q)$, and $\beta(j; q)$.
15. if $0 \leq t < s$, do
16. compute $k_0 = \sum_{j \mid n \in \mathcal{N}_q} \text{ord}_j(q) \cdot \nu_j^{(0)}$, where $0 \leq \nu_j^{(0)} \leq \beta(j; q)$
17. else
18. while $0 < t < \left\lceil \frac{n}{2} \right\rceil$ do
19. For a fixed $\nu_j^{(t-1)}$ in $k_{t-1}$, compute $k_t = \sum_{j \mid n \in \mathcal{N}_q} \text{ord}_j(q) \cdot \nu_j^{(t)}$, where
20. $0 \leq \nu_j^{(t)} \leq \beta(j; q) - \nu_j^{(t-1)}$, if $2k_0 + k_1 + \cdots + k_t \leq n$ then
21. consider $k_t$, else
22. reject $k_t$
23. while $t \geq \left\lceil \frac{n}{2} \right\rceil$ do
24. For a fixed $\nu_j^{(t-1)}$ in $k_{t-1}$, compute $k_t = \sum_{i \mid n \in \mathcal{N}_q} \text{ord}_i(q) \cdot \mu_i^{(t)} + \sum_{j \mid n \in \mathcal{N}_q} \text{ord}_j(q) \cdot \nu_j^{(t)}$, where
25. $0 \leq \mu_i^{(t)} \leq \gamma(i; q)$ and $0 \leq \nu_j^{(t)} \leq 2 \cdot (\beta(j; q) - \nu_j^{(t-1)})$.
26. if $2k_0 + k_1 + \cdots + k_t \leq n$ then
27. consider $k_t$, else
28. reject $k_t$
29. return The possible parameters $(k_0, k_1, \ldots, k_{s-1})$. 

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Example 4.1. All possible parameters of Euclidean hulls of cyclic codes of length \( n = 11 \) over \( \mathbb{Z}_{27} \) are determined as follows.

1. The divisors of 11 are 1 and 11.
   a) We have 1 \( \in \mathcal{N}_3 \), so \( \text{ord}_1(3) = 1 \) and \( \gamma(1;3) = 1 \).
   b) We have 11 \( \notin \mathcal{N}_3 \), so \( \text{ord}_{11}(3) = 5 \) and \( \beta(11;3) = 1 \).

2. It follows that

\[
\begin{align*}
  k_0 &= 5\nu_{11}^{(0)}, \text{ where } 0 \leq \nu_{11}^{(0)} \leq 1 \\
  k_1 &= 5\nu_{11}^{(1)}, \text{ where } 0 \leq \nu_{11}^{(1)} \leq 1 - \nu_{11}^{(0)} \\
  k_2 &= \mu_{11}^{(2)} + 5\nu_{11}^{(2)} \text{ where } 0 \leq \mu_{11}^{(2)} \leq 1 \text{ and } 0 \leq \nu_{11}^{(2)} \leq 2(1 - \nu_{11}^{(1)}).
\end{align*}
\]

Hence, the all possible parameters \((k_0, k_1, k_2)\) of the Euclidean hulls of cyclic codes of length 7 over \( \mathbb{Z}_{8} \) are given in the following table

| \( k_0 \) | \( k_1 \) | \( k_2 \) |
|--------|--------|--------|
| 0      | 0      | 0, 1, 5, 6, 10, 11 |
| 5      | 0      | 1       |

Example 4.2. All the possible parameters \((k_0, k_1, k_2)\) of the Euclidean hull of a cyclic code of length 7 over \( \mathbb{Z}_{8} \) are determined as follows.

1. The divisors of 7 are 1 and 7.
   a) We have 1 \( \in \mathcal{N}_2 \), so \( \text{ord}_1(2) = 1 \) and \( \gamma(1;2) = 1 \).
   b) We have 7 \( \notin \mathcal{N}_2 \), so \( \text{ord}_7(2) = 3 \) and \( \beta(7;2) = 1 \).

2. It follows that

\[
\begin{align*}
  k_0 &= 3\nu_7^{(0)}, \text{ where } 0 \leq \nu_7^{(0)} \leq 1 \\
  k_1 &= 3\nu_7^{(1)}, \text{ where } 0 \leq \nu_7^{(1)} \leq 1 - \nu_7^{(0)} \\
  k_2 &= \mu_7^{(2)} + 3\nu_7^{(2)} \text{ where } 0 \leq \mu_7^{(2)} \leq 1 \text{ and } 0 \leq \nu_7^{(2)} \leq 2(1 - \nu_7^{(1)}).
\end{align*}
\]

Hence, the all possible parameters \((k_0, k_1, k_2)\) of the Euclidean hulls of cyclic codes of length 7 over \( \mathbb{Z}_{8} \) are given in the following table

| \( k_0 \) | \( k_1 \) | \( k_2 \) |
|--------|--------|--------|
| 0      | 0      | 0, 1, 3, 4, 6, 7 |
| 3      | 0      | 1       |

Example 4.3. The parameters of the Euclidean hulls of cyclic codes of length 21 over \( \mathbb{Z}_{8} \) are given by

1. The divisors of 21 are \( \{1, 3, 7, 21\} \).
   a) 1; 3 \( \in \mathcal{N}_2 \), we have \( \text{ord}_1(2) = 1, \text{ord}_3(2) = 2 \) and \( \gamma(1;2) = \gamma(3;2) = 1 \).
   b) 7; 21 \( \notin \mathcal{N}_2 \), we have \( \text{ord}_7(2) = 3, \text{ord}_{21}(2) = 6 \) and \( \beta(7;2) = \beta(21;2) = 1 \).

2. It follows that

\[
\begin{align*}
  k_0 &= 3\nu_7^{(0)} + 6\nu_3^{(0)}, \text{ with } 0 \leq \nu_j^{(0)} \leq 1, \text{ where } j \in \{7, 21\} \\
  k_1 &= 3\nu_7^{(1)} + 6\nu_3^{(1)}, \text{ with } 0 \leq \nu_j^{(1)} \leq 1 - \nu_j^{(0)}, \text{ where } j \in \{7, 21\} \\
  k_2 &= \mu_7^{(2)} + 2\mu_3^{(2)} + 3\nu_7^{(2)} + 6\nu_3^{(2)}, \text{ with } 0 \leq \mu_i^{(2)} \leq 1 \text{ and } 0 \leq \nu_j^{(2)} \leq 2(1 - \nu_j^{(1)}), \text{ where } i \in \{1, 3\}, \text{ and } j \in \{7, 21\}.
\end{align*}
\]

Hence, the all possible parameters \((k_0, k_1, k_2)\) of the Euclidean hulls of cyclic codes of length 21 over \( \mathbb{Z}_{8} \) are given in the following table

| \( k_0 \) | \( k_1 \) | \( k_2 \) |
|--------|--------|--------|
| 3      | 0      | 0, 1   |
Corollary 4. The set $\mathfrak{R}(n, s, q)$ of $q$-dimensions of the Euclidean hull of a cyclic serial code of length $n$ over $R$, is given by

$$\mathfrak{R}(n, s, q) = \left\{ \sum_{i \in N_q} \operatorname{ord}_i(q) \left( \sum_{t=1}^{s} \Delta_{it} \right) + \sum_{j \in N_q} \operatorname{ord}_j(q) \left( \sum_{h=1}^{\beta(j, q)} \triangle_{j h} \right) \mid 0 \leq \Delta_{it} \leq s - \left\lceil \frac{s}{2} \right\rceil \wedge 0 \leq \triangle_{j h} \leq s \right\}. $$

Proof. Let $C$ be a cyclic serial code of length $n$ over $R$ with triple-sequence

$$(x^0, y^0, z^0) = \left( \left( (x_{it}^{(a)})_{0 \leq a < s} \right)^\circ, \left( (y_{jh}^{(a)})_{0 \leq a < s} \right)^\circ, \left( (z_{jh}^{(a)})_{0 \leq a < s} \right)^\circ \right)$$

in $E_n(q, s)$. From Theorem 2 the parameters $(k_0, k_1, \ldots, k_{s-1})$ of $H_0(C)$ where for all $0 \leq t \leq s-1$,

$$k_t = \sum_{i \in N_q} \operatorname{ord}_i(q) \cdot \left( \sum_{j=1}^{s} \left( t_{it}^{[j]} - n_{it}^{[j]} \right) \right) + \sum_{j \in N_q} \operatorname{ord}_j(q) \cdot \left( \sum_{h=1}^{\beta(j, q)} (e_{j h}^{(t-1)} - e_{j h}^{(t)}) \right).$$

Thus the $q$-dimension of $H_0(C)$ is $\sum_{t=0}^{s-1} (s-t) k_t$. It follows that

$$\dim_q(C) = \sum_{i \in N_q} \operatorname{ord}_i(q) \cdot \left( \sum_{t=1}^{s} \Delta_{it} \right) + \sum_{j \in N_q} \operatorname{ord}_j(q) \cdot \left( \sum_{h=1}^{\beta(j, q)} \triangle_{j h} \right).$$

From Remark 3

$$\Delta_{it} = \sum_{t=0}^{s-1} \Delta_{it}^{(t)} = \sum_{t=\left\lceil \frac{s}{2} \right\rceil}^{s-1} \Delta_{it}^{(t)} \leq s - \left\lceil \frac{s}{2} \right\rceil,$$

and if $j \in N_q$ then $\triangle_{j h} = 0$. Otherwise,

$$\triangle_{j h} = \sum_{t=0}^{s-1} \triangle_{j h}^{(t)} = \sum_{t=0}^{s-1} \triangle_{j h}^{(t)} + \sum_{t=\left\lceil \frac{s}{2} \right\rceil}^{s-1} \triangle_{j h}^{(t)} \leq \max_{0 \leq k \leq s, k \neq \left\lceil \frac{s}{2} \right\rceil} \left\{ \left( \left\lceil \frac{s}{2} \right\rceil + b \right) + 2 \left( s - \left\lceil \frac{s}{2} \right\rceil - b \right) \right\} = s.$$
In this section, an explicit formula for $\mathbb{E}_R(n)$ and bounds are given in terms of $\mathbb{B}_{n,q}$ where

$$
\mathbb{B}_{n,q} = \deg \prod_{i \nmid n} \left( \prod_{l=1}^{\mathbb{g}(i,q)} \Omega(G_i) \right) = \sum \phi(i).
$$

Consider the maps

$$
\Delta: \mathcal{E}_s \to \mathbb{N}, \quad (x^{(0)}, \ldots, x^{(s-1)}) \mapsto \sum_{t=0}^{s-1} \min \left\{ \sum_{a=0}^{t} x^{(a)}; 1 - \sum_{a=0}^{s-t-1} x^{(a)} \right\},
$$

and $\Delta: \mathcal{E}_s \times \mathbb{N} \to \mathbb{N}$ defined as

$$
\Delta(y,z) = \sum_{t=0}^{s-1} \left( \min \left\{ \sum_{a=0}^{t} y^{(a)}; 1 - \sum_{a=0}^{s-t-1} z^{(a)} \right\} + \min \left\{ \sum_{a=0}^{t} z^{(a)}; 1 - \sum_{a=0}^{s-t-1} y^{(a)} \right\} \right),
$$

where $(y,z) = ((y^{(0)}, \ldots, y^{(s-1)}), (z^{(0)}, \ldots, z^{(s-1)}))$.

Let $\tau \in \mathbb{N}(n,s,q)$ an element in the set defined in Corollary 4. Then $\tau$ is the $q$-dimension of the Euclidean hull of a cyclic serial code of length $n$ over $R$. The following result gives the number of cyclic serial codes of length $n$ over $R$ whose Euclidean hulls have $q$-dimension $\tau$.

**Proposition 8.** Let $n$ be a positive integer such that $q|\mathbb{c}(n,p) = 1$ and $\tau \in \mathbb{N}(n,s,q)$ where $\mathbb{N}(n,s,q)$ is described in Corollary 4. The number $\varphi(n,\tau; R)$ of cyclic serial codes of length $n$ over $R$ whose Euclidean hulls have $q$-dimension $\tau$ is given by:

$$
\varphi(n,\tau; R) = \sum_{((\Delta_{il})^*),((\Delta_{jh})^*)}\prod_{i \nmid n} \sum_{l=1}^{\mathbb{g}(i,q)} \prod_{a \in \mathbb{N}_q} \psi_s(\Delta_{il}) \prod_{j \nmid n} \sum_{h=1}^{\mathbb{g}(j,q)} \rho_s(\Delta_{jh}),
$$

where

$$
\psi_s(\Delta_{il}) = |\{x \in \mathcal{E}_s : \Delta(x) = \Delta_{il}\}|, \quad \rho_s(\Delta_{jh}) = |\{(y,z) \in \mathcal{E}_s \times \mathcal{E}_s : \Delta(y,z) = \Delta_{jh}\}|,
$$

and

$$
\mathbb{T}(\tau) = \left\{ \left((\Delta_{il})^*,((\Delta_{jh})^*)\right) : \sum_{i \nmid n} \mathcal{O}(\mathcal{C}_i(q)) \left( \sum_{l=1}^{\mathbb{g}(i,q)} \Delta_{il} \right) + \sum_{j \nmid n} \mathcal{O}(\mathcal{C}_j(q)) \left( \sum_{h=1}^{\mathbb{g}(j,q)} \Delta_{jh} \right) = \tau \right\}.
$$

The above expression of $\mathbb{E}_R(n) = \sum_{\tau \in \mathbb{N}(n,s,q)} \mathbb{T}(\varphi(n,\tau; R))$, might lead to a tedious and lengthy computation. The remainder of the section will show an alternative simpler expression for the expected value.

**Lemma 5.** Consider the random variable $\Delta$ defined in (12) with uniform probability. The expected value $\mathbb{E}(\Delta)$ is given by:

$$
\mathbb{E}(\Delta) = \left[ \frac{s}{2} \right] \left( s - \left[ \frac{s}{2} \right] \right) \left[ \frac{s}{s+1} \right] = \left\{ \begin{array}{ll}
\frac{s^2}{2(s+1)}, & \text{if } s \text{ even;} \\
\frac{s}{s+1}, & \text{if } s \text{ odd.}
\end{array} \right.
$$

**Proof.** Let $t \in \{0; 1; \cdots; s-1\}$ and $x = (x^{(0)}, \ldots, x^{(s-1)}) \in \mathcal{E}_s$. Set

$$
\Delta^{(t)}(x) = \min \left\{ \sum_{a=0}^{t} x^{(a)}; 1 - \sum_{a=0}^{s-t-1} x^{(a)} \right\} \in \{0; 1\}.
$$
Then \( \Delta^{(t)} = 1 \) if and only if \( 2t \geq s \) and \( \sum_{a=t-s-t}^{t} x_{a}^{(a)} = 1 \). Thus for all \( \eta \in \mathbb{N} \), we have \( |\{ x \in \mathcal{E}_{s} : \Delta^{(t)} = \eta \}| = \begin{cases} 2t - s + 1, & \text{if } t \geq \left\lceil \frac{s}{2} \right\rceil \text{ and } \eta = 1; \\ 0, & \text{otherwise}. \end{cases} \)

Therefore,

\[
|\{ x \in \mathcal{E}_{s} : \Delta (x) = \eta \}| = \begin{cases} \sum_{t=\lceil \frac{s}{2} \rceil}^{s-1} (2t - s + 1), & \text{if } \eta = s - \left\lceil \frac{s}{2} \right\rceil; \\ 0, & \text{otherwise}. \end{cases}
\]

Since \( |\mathcal{E}_{s}| = s + 1 \) and \( P(\{ x \in \mathcal{E}_{s} : \Delta (x) = \eta \}) = \frac{|\{ \Delta (x) = \eta \}|}{|\mathcal{E}_{s}|} \), it follows that,

\[
E(\Delta) = \sum_{\eta \in \mathbb{N}} \eta P(\{ x \in \mathcal{E}_{s} : \Delta (x) = \eta \}) = \frac{\left\lceil \frac{s}{2} \right\rceil (s - \left\lceil \frac{s}{2} \right\rceil)}{s + 1}.
\]

\[\square\]

**Lemma 6.** Consider the random variable \( \Delta : \mathcal{E}_{s} \times \mathcal{E}_{s} \to \mathbb{N} \) defined in (13) with uniform distribution. The expected value \( E(\Delta) \) is given by

\[ E(\Delta) = \frac{s(2s + 1)}{3(s + 1)}. \]

**Proof.** From Corollary 4, for any \((y, z) \in \mathcal{E}_{s} \times \mathcal{E}_{s}, \) \( 0 \leq \Delta (y, z) \leq s \). Let \( \mathcal{E}_{s}(\eta) = \{ (y, z) \in \mathcal{E}_{s} \times \mathcal{E}_{s} : \Delta (y, z) = \eta \} \), for \( 0 \leq \eta \leq s \).

Now,

\[
|\mathcal{E}_{s}(\eta)| = \begin{cases} 2(\eta + 1), & \text{if } 0 \leq \eta \leq s - 1; \\ s + 1, & \text{if } \eta = s. \end{cases}
\]

Thus

\[
E(\Delta) = \frac{1}{(s + 1)^2} \sum_{\eta = 0}^{s} \eta |\mathcal{E}_{s}(\eta)| = \frac{1}{(s + 1)^2} \left( \sum_{\eta = 1}^{s-1} 2\eta(\eta + 1) + s(s + 1) \right) = \frac{s(2s^2 + 3s + 1)}{3(s + 1)^2}.
\]

\[\square\]

**Theorem 3.** The average \( q \)-dimension of the Euclidean hull of cyclic serial codes from \( C(n; R) \) is

\[
E_{R}(n) = \begin{cases} \frac{(2s+1)s}{6(s+1)} B_{n, q}, & \text{if } s \text{ even}; \\ \frac{(2s+1)s}{6(s+1)} n - \frac{(s+2)s}{12(s+1)} B_{n, q}, & \text{if } s \text{ odd.} \end{cases}
\]

where \( B_{n, q} = \sum_{i \mid n \atop i \in \mathbb{N}} \phi(i) \).
Proof. Let \( Y \) be the random variable that takes as value \( \dim_q(\mathcal{H}_0(C)) \) when we choose at random a cyclic serial code from \( \mathcal{C}(n; R) \) with uniform probability. Then \( \mathbb{E}(Y) = \mathbb{E}_R(n) \). By Lemma 3 there exists an one-to-one correspondence between \( \mathcal{C}(n; R) \) and \( \mathcal{E}_n(q, s) \). Therefore, choosing a cyclic serial code \( C \) from \( \mathcal{C}(n; R) \) their probabilities are identical. By Corollary 4 we obtain

\[
Y = \sum_{i \mid n, i \in \mathcal{N}_q} \sum_{j \mid n, j \notin \mathcal{N}_q} \left( \sum_{t=1}^{\gamma(i,q)} \Delta_{it} \right) + \sum_{j \mid n} \sum_{h=1}^{\beta(j,q)} \Delta_{jh}.
\]

For all \( i \) and \( j \) dividing \( n \) such that \( i \in \mathcal{N}_q \) and \( j \not\in \mathcal{N}_q \), from Lemmas 5 and 6 we note that \( \mathbb{E}(\Delta_{it}) = \mathbb{E}(\Delta) \) and \( \mathbb{E}(\Delta_{jh}) = \mathbb{E}() \). So, we get

\[
\mathbb{E}(Y) = \sum_{i \mid n, i \in \mathcal{N}_q} \sum_{j \mid n, j \notin \mathcal{N}_q} \left( \sum_{t=1}^{\gamma(i,q)} \mathbb{E}(\Delta) \right) + \sum_{j \mid n} \sum_{h=1}^{\beta(j,q)} \mathbb{E}(\Delta);
\]

\[
= \sum_{i \mid n, i \in \mathcal{N}_q} \sum_{j \mid n, j \notin \mathcal{N}_q} \phi(i) \mathbb{E}(\Delta_{it}) + \sum_{j \mid n} \mathbb{E}(\Delta_{jh});
\]

\[
= B_{n,q} \mathbb{E}(\Delta) + \frac{n - B_{n,q}}{2} \mathbb{E}();
\]

\[
= \frac{n}{2} \mathbb{E}() - B_{n,q} \cdot \left( \frac{1}{2} \mathbb{E}() - \mathbb{E}(\Delta) \right).
\]

From Lemmas 5 and 6 we have

\[
\mathbb{E}_R(n) = \begin{cases} 
\frac{(2s+1)s}{6(s+1)} n - \frac{(s+2)s}{12(s+1)} B_{n,q}, & \text{if } s \text{ even;} \\
\frac{(2s+1)s}{6(s+1)} n - \frac{s^2+2s+3}{12(s+1)} B_{n,q}, & \text{if } s \text{ odd.}
\end{cases}
\]

From [19], we have \( B_{n,q} = n \) if \( n \in \mathcal{N}_q \) and \( 1 \leq B_{n,q} \leq \frac{2n}{3} \) if \( n \not\in \mathcal{N}_q \). Thus

- If \( n \in \mathcal{N}_q \), then

\[
\mathbb{E}_R(n) = \begin{cases} 
\frac{s^2n}{4(s+1)}, & \text{if } s \text{ even;} \\
\frac{n(s-1)}{4}, & \text{if } s \text{ odd.}
\end{cases}
\]

- If \( n \not\in \mathcal{N}_q \), then

\[
\frac{(5s+1)n}{18(s+1)} \leq \mathbb{E}_R(n) \leq \frac{2n(2s+1)-s(s+2)n}{12(s+1)}, \quad \text{if } s \text{ even;}
\]

\[
\frac{(5s^2-3s-3)n}{18(s+1)} \leq \mathbb{E}_R(n) \leq \frac{2n(2s+1)-(s^2+2s+3)n}{12(s+1)}, \quad \text{if } s \text{ odd.}
\]

Remark 4. Note that \( \mathbb{E}_R(n) \) grows at the same rate as \( ns \) when \( s \) and \( n \) tend to infinity. Thus, the upper limit of the sequence \( \left( \frac{\mathbb{E}_R(n)}{ns} \right)_{(s, n) \in (\mathbb{N}\setminus\{0\})^2} \) is at most \( \frac{1}{4} \) and its lower limit is at least \( \frac{1}{18} \).

6. Conclusion

The Galois hulls of cyclic serial codes of length \( n \) over an arbitrary finite chain ring with parameters \( (p, r, a, e, s) \) have been investigated. Especially, the parameters and the average of the \( q \)-dimension of the Euclidean hull of cyclic codes are studied in terms of triple-sequences. The parameters and the average \( p' \)-dimensions of the Euclidean hulls of cyclic serial codes of arbitrary length have been determined as well. Asymptotically, it has been shown that the
average of $p^r$-dimension of the Euclidean hull of cyclic serial codes of length over $R$ grows the same rate as the length of the codes. An extension of this paper to the case of the hulls of cyclic or constacyclic codes over finite chain rings is an interesting research problem as well. It would be interesting to study the properties of Euclidean hulls of negacyclic serial codes.

References

[1] E.F. Assmus, J.D. Key, *Affine and projective planes*, Discrete Math. 83(2-3) (1990) 161-187.
[2] A. Batoul, K. Guenda, T.A. Gulliver, *On self-dual cyclic codes over finite chain rings*, Des. Codes Cryptogr. 70 (2014) 347-358.
[3] S. Bhowmick, A. Fotue-Tabue, E. Martínez-Moro, R. Bandi, S. Bagchi, *Do non-free LCD codes over finite commutative Frobenius rings exist*, Des. Codes Cryptogr. (2020).
[4] H. Q. Dinh, S. R. López-Permouth, *Cyclic and Negacyclic Codes Over Finite Chain Rings*, IEEE Trans. Inform. Theory, vol. 50, pp. 1728-1744, Nov. 2004.
[5] A. Fotue Tabue, E. Martínez-Moro, C. Mouaha, *Galois correspondence on linear codes over finite chain rings*, Discrete Mathematics (2019) 111653, https://doi.org/10.1016/j.disc.2019.111653
[6] A. Fotue-Tabue, C. Mouaha, *On the Lattice of Cyclic Linear Codes Over Finite Chain Rings*, Algebra and Discrete Mathematics 27 (2), 252-268 (2019)
[7] T. Honold, I. Landjev, *Linear Codes over Finite Chain Rings*, The electronic journal of combinatorics 7 (2000), #R11
[8] S. Jitman, E. Sangwisut, *The Average Hull Dimension of Negacyclic Codes over Finite Fields*, Math. Comput. Appl. 2018, 23, 41; doi:10.3390/mca2300041
[9] S. Jitman, E. Sangwisut, P. Udomkavanich, *Hulls of Cyclic Codes over $\mathbb{Z}_4$*, Discrete Mathematics (2019) 111621, https://doi.org/10.1016/j.disc.2019.111621.
[10] J. S. Leon, *Computing automorphism groups of error-correcting codes*, IEEE Trans. Inf. Theory 28(3) (1982) 496-511.
[11] H. Liu, X. Pan, *Galois hulls of linear codes over finite fields*, Des. Codes Cryptogr. (88) 241-255 (2020).
[12] B. R. McDonald *Finite Rings with Identity* Marcel Dekker Inc., New York (1974)
[13] E. Martínez-Moro, I. F. Rúa, *Multivariable Codes Over Finite Chain Rings: Serial Codes* SIAM J. Discrete Math., 20(4), 947-959.
[14] G. H. Norton, A. Salagean, *On the Structure of Linear and Cyclic Codes over a Finite Chain Ring*, AAECC 10, 489-506 (2000).
[15] E. Petrank, R.M. Roth, *Is code equivalence easy to decide?*, IEEE Trans. Inf. Theory 43(5) (1997) 1602-1604
[16] N. Sendrier, *On the dimension of the hull*, SIAM J. Appl. Math. 10 (1997) 282-293.
[17] N. Sendrier, *Finding the permutation between equivalent codes: the support splitting algorithm*, IEEE Trans. Inf. Theory 46(4) (2000) 1193-1203.
[18] E. Sangwisut, S. Jitman, S. Ling, P. Udomkavanich, *Hulls of cyclic and negacyclic codes over finite fields*. Finite Fields Appl. 2015, 33, 232-257.
[19] G. Skersys, *The average dimension of the hull of cyclic codes*, Discrete Appl. Math. 128 (1) (2003) 275-292.
[20] A.K. Singh, N. Kumar, K.P. Shum, *Cyclic self-orthogonal codes over finite chain rings*, Asian-Eur. J. Math. 11(2018) 1850078.
[21] E.A. Whelan *A note on finite local rings* Rocky Mountain J. Math., 22 (2) (1992), pp. 757-759.