Small Youden Rectangles and Their Connections to Other Row-Column Designs

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Abstract

In this paper we study Youden rectangles of small orders. We have enumerated all Youden rectangles for all small parameter values, excluding the almost square cases, in a large scale computer search.

For small parameter values where no Youden rectangles exist, we also enumerate rectangles where the number of symbols common to two columns is always one of two possible values. We refer to these objects as near Youden rectangles.

For all our designs we calculate the size of the autotopism group and investigate to which degree a certain transformation can yield other row-column designs, namely double arrays, triple arrays and sesqui arrays.

Finally we also investigate certain Latin rectangles with three possible pairwise intersection sizes for the columns and demonstrate that these can give rise to triple and sesqui arrays which cannot be obtained from Youden rectangles, using the transformation mentioned above.
1 Introduction

An \((n, k, \lambda)\) Youden rectangle (also commonly referred to as a Youden square) where \(n \geq k\) is a \(k \times n\) array on \(n\) symbols that satisfies the following two conditions:

1. There is no repeated symbol in any row or column, which we will call the \textit{Latin condition}.

2. The number of shared symbols between any two columns is always \(\lambda\), which we will call the \textit{design condition}.

As indicated by this choice of terminology, a Youden rectangle can be viewed as a special case of a \(k \times n\) Latin rectangle. We exclude the cases \(k = n\), \(k = n - 1\) and \(k = 1\), since for these parameter choices, all Latin rectangles trivially satisfy the second condition.

Clearly, each row will contain each symbol exactly once, and so the array will also be \textit{equireplicate}, that is, each symbol appears the same number of times, namely \(k\). As is well known, divisibility and double counting considerations easily give that in order for a Youden rectangle to exist, \(\lambda = \frac{k(k-1)}{n-1}\) must be an integer.

The reason for the use of the term ‘design’, is that when treating the columns of a Youden rectangle as sets of symbols, these sets by definition form the blocks of a \textit{symmetric balanced incomplete block design} (SBIBD). Conversely, it was proven by Hartley and Smith \cite{14} that the elements in the blocks of any SBIBD can be ordered to give a Youden rectangle. In fact, many different orderings are possible, so a single SBIBD will give rise to many different Youden rectangles. The early history of the study of Youden rectangles was chronicled by Preece \cite{11}.

Little has been done on complete enumeration of these objects, though in \cite{10} Youden rectangles with \(n \leq 7\) were classified by Preece, and in \cite{4} a full enumeration of mutually orthogonal (in the Latin rectangle sense) triples of Youden rectangles was completed for \(n \leq 7\). Note that orthogonal Youden rectangles are distinct from so-called multi-layered Youden rectangles \cite{12}. In the present paper, our main aim has been to perform a complete enumeration of Youden rectangles for as large parameters as possible.
The paper is structured as follows. In Section 2 we give some further basic notation and formal definitions. In Section 3 we state the questions guiding our investigation, and describe briefly the method and algorithms used together with some practical information regarding the computer calculations. In Section 4 we present the data our computer search resulted in, in particular the number of different Youden rectangles of some small orders. In Section 5 we analyze the constructed objects with regards to other types of row-column designs. Section 6 concludes.

2 Notation and Definitions

In the following, we use as symbol set \( \{0, 1, \ldots, n - 1\} \).

We call a Youden rectangle \emph{normalized} if it satisfies the following conditions:

(S1) (Ordering among columns) The first row is the identity permutation.

(S2) (Ordering among rows) The first column is \(0, 1, 2, \ldots, k - 1\).

Several different Youden rectangles can correspond to one normalized Youden rectangle, so normalization gives a notion of ‘equivalence’. We also employ the stronger equivalence notion of \emph{isotopism}, where two Youden rectangles \( Y_A \) and \( Y_B \) are said to be \emph{isotopic} if there exists a permutation \( \pi_s \) of the symbols, a permutation \( \pi_r \) of the rows and a permutation \( \pi_c \) of the columns such that when applying all three permutations to \( Y_A \), we get \( Y_B \). The equivalence concept isotopism is the most natural one when studying Youden rectangles. In particular, taking transposes does not map \( k \times n \) Youden rectangles to \( k \times n \) Youden rectangles.

Making this more formal, the following group of isotopisms acts on the set of \( k \times n \) Youden rectangles:

\[ G_{n,k} = S_k \times S_n \times S_n, \]

where \( S_k \) corresponds to a permutation of the rows, the first \( S_n \) corresponds to a permutation of the columns, and the last \( S_n \) corresponds to a permutation of the symbols. Two rectangles \( Y_A \) and \( Y_B \) of size \( k \times n \) are isotopic if there exists a \( g \in G_{n,k} \) such that \( g(Y_A) = Y_B \). The \emph{autotopism}
group of a Youden rectangle $Y$ is defined as $\text{Aut}(Y) := \{g \in G_{n,k} \mid g(Y) = Y\}$. When presenting examples, we use normalized representatives of autotopism classes.

3 Generating Data

In this section, we describe our computational work in general terms.

3.1 Guiding Questions

Our approach is complete enumeration by computer for as large parameter values as possible, and unless otherwise stated, we save all generated data. In particular, we do not only record the number of Youden rectangles found, but we save the objects themselves.

With some exceptions due to size restrictions, the data generated is available for download at [1]. Further details about the organization of the data are given there.

The following questions serve as guides for what data to generate.

(Q1) How many isotopism classes of $k \times n$ Youden rectangles are there?

(Q2) What is the order of the autotopism group of each $k \times n$ Youden rectangle?

(Q3) If some condition is relaxed, how many objects satisfying the relaxed conditions are there?

3.2 Implementation and Execution

We generated all rectangles by consecutively adding all possible columns, while observing that none of the conditions were violated. At suitable points, we reduced our list of partial objects by isotopism. Also, at selected stages, the list of partial objects was culled by running checks on whether they were at all extendible to a full Youden rectangle.
The algorithms used were implemented in C++ and run in a parallelized version on the Kebnekaise supercomputer at High Performance Computing Centre North (HPC2N).

The algorithm is divided into two parts. The first extends a given partial rectangle with \(k\) rows and \(t\) columns by one column, such that the new rectangle satisfies both the Latin condition and the design condition. More specifically, we first add a column with \(k\) different symbols. We then check that in the extended \(k \times (t + 1)\) rectangle, no symbols appear more than once in any row. We also check that the number of shared symbols between the added column and the \(t\) first columns is \(\lambda\). By checking all possible added columns, we find all extensions of the given \(k \times t\) rectangle.

The second part of the algorithm checks whether a received \(k \times (t+1)\) rectangle could be chosen as a normalized representative of an isotopism class.

When a full Youden rectangle has been received, we check the size of the autotopism group. The group of possible autotopism actions on a \(k \times n\) Youden rectangle is \(S_k \times S_n \times S_n\), so potentially, the number of actions we need to check is \(k! \cdot n! \cdot n!\).

Since we consider normalized rectangles this number can be reduced to \(n \cdot k! \cdot (n-k)!\), since once we have chosen the first column \((n\) options\) and row permutation \(\pi_r\) \((k!\) options\) we fix \(k\) symbols in the symbol permutation \(\pi_s\) \((so \ (n-k)!\) options remain\).

The running time grows quickly as the rectangle parameters grow. We completely enumerated Youden rectangles of sizes \(3 \times 7, 4 \times 7, 5 \times 11\) and \(4 \times 13\) in a few minutes on a standard desktop computer. On the other hand, computation on sizes \(6 \times 11\) and \(5 \times 21\) required high performance computers and significantly more time. Using a parallelized version of the algorithms, enumerating \(6 \times 11\) Youden rectangles took about 6000 core hours, which is a bit less than 1 year. The \(5 \times 21\) case required several hundred core years.

Our methods and code can be applied to larger parameter values as well, but here the number of Youden rectangles quickly becomes unmanageable.
4 Basic Results

We now turn to the results and analysis of our computational work.

4.1 The Number of Youden Rectangles

Our first result is an enumeration of Youden rectangles. In Tables 1 to 4 we present data on the number of non-isotopic Youden rectangles, sorted by the size of the autotopism groups.

It is relevant to compare these numbers with the number of Latin rectangles. One difficulty in doing so is finding comparable numbers, that is, where the same notion of equivalence has been applied. We have been unable to find a comprehensive source for the numbers of non-isotopic Latin rectangles. When no reduction at all is applied, there are 782,137,036,800 Latin rectangles of size $4 \times 7$, and only 512 Youden rectangles of the same size (note that this number is not given in any of the tables in the present paper). In [5], the numbers of reduced $n \times k$ Latin rectangles are given for $k \leq n$, $1 \leq n \leq 11$, that is, the number of Latin rectangles whose first row is the identity permutation. For comparison, there are 1,293,216 reduced Latin rectangles of size $4 \times 7$. This is not quite comparable to our notion of normalized Youden rectangles, since we additionally require the first column to be the $k$ first numbers in order.

For comparison, we enumerated all $4 \times 7$ non-isotopic Latin rectangles, and found 1,398 such rectangles, to be compared with 6 non-isotopic Youden rectangles of the same size. As we can see, the proportion of Latin rectangles that additionally satisfy the design condition is small.

We note again that the $3 \times 7$ and $4 \times 7$ Youden rectangles were completely classified by Preece [10], and that our enumerative results are in accordance with his classification.

From the tables, we see that clearly the most common autotopism group size is 1, but that there are also rare examples of rather symmetric Youden rectangles. One such example, a Youden rectangle of size $4 \times 13$, whose autotopism group size is 39 is presented in Figure 1. The autotopism group acts transi-
Table 1: The number of Youden rectangles with $n = 7$ sorted by autotopism group size.

| $(n, k, \lambda)$ | $(7,3,1)$ | $(7,4,2)$ |
|-------------------|-----------|-----------|
| #YR               | 1         | 6         |

| $|\text{Aut}|$ | 1 | 0 | 2 |
|----------------|---|---|---|
|                 | 3 | 0 | 3 |
|                 | 21| 1 | 1 |

Table 2: The number of Youden rectangles with $n = 11$ sorted by autotopism group size.

| $(n, k, \lambda)$ | $(11,5,2)$ | $(11,6,3)$ |
|-------------------|-----------|-----------|
| #YR               | 79 416    | 995 467 440 |

| $|\text{Aut}|$ | 1 | 77 694 | 995 421 832 |
|----------------|---|--------|-----------|
|                 | 2 | 1 423  | 40 831    |
|                 | 3 | 199    | 4 454     |
|                 | 4 | 45     | 124       |
|                 | 5 | 4      | 121       |
|                 | 6 | 38     | 62        |
|                 | 10| 3      | 3         |
|                 | 12| 7      | 10        |
|                 | 55| 1      | 1         |
|                 | 60| 2      | 2         |

Table 3: The number of Youden rectangles with $n = 13$ sorted by autotopism group size.

| $(n, k, \lambda)$ | $(13,4,1)$ |
|-------------------|-----------|
| #YR               | 20        |

| $|\text{Aut}|$ | 1 | 12 |
|----------------|---|----|
|                 | 3 | 7  |
|                 | 39| 1  |
Table 4: The number of Youden rectangles with $n = 21$ sorted by autotopism group size.

As is well known, taking a $(n, k, \lambda)$ difference set as first column and producing the remaining columns by developing this first column, that is, consecutively adding 1 to each entry, will produce a Youden rectangle. The autotopism group of the resulting Youden rectangle will then act transitively on the set of columns. We conclude that for $n = 7, 11, 13$, the very symmetric Youden rectangles we found, where the order of the autotopism group is divisible by the number of columns, correspond to those Youden rectangles generated from difference sets. The situation for $n = 21$ seems to be a bit more involved, and a complete analysis of the Youden rectangles with large autotopism groups is beyond the scope of this paper.

For larger parameters, that is, where there exist more than one corresponding SBIBD, it would also have been interesting to...
4.2 The Number of Near Youden Rectangles

For parameter sets where there do not exist Youden rectangles, i.e., where $\lambda$ as calculated by $\lambda = \frac{k(k-1)}{n-1}$ is not an integer, we have enumerated $k \times n$ rectangles satisfying the Latin condition, where the intersection sizes between symbol sets in columns are either $\lambda_1 = \lfloor \lambda \rfloor$ or $\lambda_2 = \lceil \lambda \rceil$, that is, the nearest smaller and larger integers. We will call such arrays near Youden rectangles (nYR). These arrays may be of some interest for experimental design purposes. An example of a $4 \times 6$ near Youden rectangle is given in Figure 2. Here, $\lambda_1 = 2$ and $\lambda_2 = 3$. For example, the first column intersects the second, third and fourth columns in 2 symbols, and the remaining columns in 3 symbols.

In Tables 5 to 9 we list complete data for the number of near Youden rectangles from $n = 5$ to $n = 9$ for sets of parameters where there are no Youden rectangles, sorted by the size of the autotopism groups. We have excluded the cases $k = 1$, $k = n - 1$ and $k = n$, since as observed above, for these cases all Latin rectangles are Youden rectangles as well. We also excluded the
| $(n, k, \lambda_1, \lambda_2)$ | $(5,2,0,1)$ | $(5,3,1,2)$ |
|-----------------------------|-------------|-------------|
| # nYR | 1 | 2 |
| $|\text{Aut}|$ | 2 | 0 | 1 |
| 10 | 1 | 1 |

Table 5: The number of near Youden rectangles with $n = 5$ sorted by autotopism group size.

| $(n, k, \lambda_1, \lambda_2)$ | $(6,2,0,1)$ | $(6,3,1,2)$ | $(6,4,2,3)$ |
|-----------------------------|-------------|-------------|-------------|
| #nYR | 2 | 2 | 34 |
| $|\text{Aut}|$ | 1 | 0 | 0 | 9 |
| 2 | 0 | 0 | 11 |
| 4 | 0 | 0 | 5 |
| 6 | 0 | 2 | 3 |
| 12 | 1 | 0 | 4 |
| 18 | 0 | 0 | 1 |
| 36 | 1 | 0 | 1 |

Table 6: The number of near Youden rectangles with $n = 6$ sorted by autotopism group size.

case $k = 7, n = 9$, for which the number of near Youden rectangles was deemed to be too large.

In Tables 10 to 13, we list data for the number of near Youden rectangles from $n = 10$ to $n = 13$ for as large $k$ as was feasible, with the same restrictions on parameter values as for $n = 5, \ldots, 9$.

We see that for fixed $n$ and growing $k$, at least for $n = 7$, $n = 11$ and $n = 13$, the number of near Youden rectangles grows faster than the number of Youden rectangles. The same holds for fixed $k$ and growing $n$. As with Youden rectangles, the majority of near Youden rectangles have trivial autotopism groups.
| $n$, $k$, $\lambda_1$, $\lambda_2$ | (7,2,0,1) | (7,5,3,4) |
|---|---|---|
| # nYR | 2 | 5 205 |
| | Aut | 1 | 0 | 4 889 |
| | | 2 | 0 | 307 |
| | | 4 | 0 | 8 |
| | | 14 | 1 | 1 |
| | | 24 | 1 | 0 |

Table 7: The number of near Youden rectangles with $n = 7$ sorted by autotopism group size.

| $n$, $k$, $\lambda_1$, $\lambda_2$ | (8,2,0,1) | (8,3,0,1) | (8,4,1,2) | (8,5,2,3) | (8,6,4,5) |
|---|---|---|---|---|---|
| # nYR | 3 | 4 | 285 | 6 688 | 21 956 009 |
| | Aut | 1 | 0 | 0 | 173 | 6 204 | 21 905 896 |
| | | 2 | 0 | 0 | 78 | 381 | 48 865 |
| | | 3 | 0 | 0 | 0 | 37 | 0 |
| | | 4 | 0 | 0 | 15 | 29 | 1 208 |
| | | 5 | 0 | 0 | 0 | 0 | 24 |
| | | 6 | 0 | 2 | 0 | 18 | 0 |
| | | 8 | 0 | 0 | 11 | 6 | 144 |
| | | 10 | 0 | 0 | 0 | 0 | 6 |
| | | 12 | 0 | 0 | 0 | 5 | 0 |
| | | 16 | 1 | 1 | 4 | 5 | 36 |
| | | 24 | 0 | 0 | 0 | 2 | 0 |
| | | 30 | 1 | 0 | 0 | 0 | 0 |
| | | 32 | 0 | 0 | 4 | 0 | 6 |
| | | 48 | 0 | 1 | 0 | 1 | 0 |
| | | 64 | 1 | 0 | 0 | 0 | 4 |

Table 8: The number of near Youden rectangles with $n = 8$ sorted by autotopism group size.
Table 9: The number of near Youden rectangles with \( n = 9 \) sorted by autotopism group size.

| \( (n, k, \lambda_1, \lambda_2) \) | \( (9,2,0,1) \) | \( (9,3,0,1) \) | \( (9,4,1,2) \) | \( (9,5,2,3) \) | \( (9,6,3,4) \) |
|-------------------------------|----------------|----------------|----------------|----------------|----------------|
| # nYR | 4 | 11 | 5 342 | 2 757 904 | 731 801 066 |
| Aut | 1 | 0 | 3 | 4 881 | 2 750 174 | 731 727 683 |
| | 2 | 0 | 1 | 355 | 7 148 | 69 733 |
| | 3 | 0 | 1 | 20 | 290 | 3 079 |
| | 4 | 0 | 0 | 54 | 177 | 312 |
| | 6 | 0 | 4 | 15 | 86 | 213 |
| | 8 | 0 | 0 | 3 | 7 | 0 |
| | 9 | 0 | 1 | 3 | 6 | 16 |
| | 12 | 0 | 0 | 8 | 6 | 18 |
| | 18 | 1 | 0 | 2 | 8 | 5 |
| | 36 | 1 | 0 | 0 | 1 | 4 |
| | 40 | 1 | 0 | 0 | 0 | 0 |
| | 54 | 0 | 1 | 0 | 0 | 1 |
| | 72 | 0 | 0 | 1 | 1 | 0 |
| | 108 | 0 | 0 | 0 | 0 | 2 |
| | 324 | 1 | 0 | 0 | 0 | 0 |

\( (n, k, \lambda_1, \lambda_2) \) represents the parameters of each near Youden rectangle configuration. The table lists the number of rectangles (nYR) for each configuration, sorted by the size of the autotopism group (Aut).
\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$(n, k, \lambda_1, \lambda_2)$ & (10,2,0,1) & (10,3,0,1) & (10,4,1,2) & (10,5,2,3) \\
\hline
\# nYR & 5 & 80 & 9 722 & 1 913 816 \\
\hline
1 & 0 & 48 & 9 288 & 1 907 844 \\
2 & 0 & 23 & 331 & 5 952 \\
3 & 0 & 4 & 72 & 0 \\
4 & 0 & 0 & 9 & 0 \\
5 & 0 & 0 & 2 & 4 \\
6 & 0 & 2 & 2 & 0 \\
10 & 0 & 3 & 9 & 16 \\
12 & 0 & 0 & 9 & 0 \\
20 & 1 & 0 & 0 & 0 \\
42 & 1 & 0 & 0 & 0 \\
48 & 1 & 0 & 0 & 0 \\
100 & 1 & 0 & 0 & 0 \\
144 & 1 & 0 & 0 & 0 \\
\hline
\end{tabular}
\caption{The number of near Youden rectangles with $n = 10$ sorted by autotopism group size.}
\end{table}

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|}
\hline
$(n, k, \lambda_1, \lambda_2)$ & (11,2,0,1) & (11,3,0,1) & (11,4,1,2) \\
\hline
\# nYR & 6 & 852 & 1 598 \\
\hline
1 & 0 & 759 & 1 597 \\
2 & 0 & 75 & 0 \\
3 & 0 & 12 & 0 \\
6 & 0 & 5 & 0 \\
11 & 0 & 1 & 1 \\
22 & 1 & 0 & 0 \\
48 & 1 & 0 & 0 \\
56 & 1 & 0 & 0 \\
60 & 1 & 0 & 0 \\
180 & 1 & 0 & 0 \\
192 & 1 & 0 & 0 \\
\hline
\end{tabular}
\caption{The number of near Youden rectangles with $n = 11$ sorted by autotopism group size.}
\end{table}
| $(n, k, \lambda_1, \lambda_2)$ | $(12,2,0,1)$ | $(12,3,0,1)$ | $(12,4,1,2)$ |
|-----------------------------|--------------|--------------|--------------|
| # nYR | 9 | 11 598 | 262 |
| | | | |
| $|\text{Aut}|$ | 1 | 0 | 11 174 | 182 |
| | 2 | 0 | 333 | 46 |
| | 3 | 0 | 35 | 16 |
| | 4 | 0 | 13 | 4 |
| | 6 | 0 | 27 | 10 |
| | 8 | 0 | 2 | 0 |
| | 12 | 0 | 5 | 4 |
| | 18 | 0 | 3 | 0 |
| | 24 | 1 | 4 | 0 |
| | 54 | 1 | 0 | 0 |
| | 64 | 1 | 0 | 0 |
| | 70 | 1 | 0 | 0 |
| | 72 | 0 | 2 | 0 |
| | 120 | 1 | 0 | 0 |
| | 144 | 1 | 0 | 0 |
| | 216 | 1 | 0 | 0 |
| | 768 | 1 | 0 | 0 |
| | 388 | 1 | 0 | 0 |

Table 12: The number of near Youden rectangles with $n = 12$ sorted by autotopism group size.
Table 13: The number of near Youden rectangles with $n = 13$ sorted by autotopism group size.

| $(n, k, \lambda_1, \lambda_2)$ | $(13,2,0,1)$ | $(13,3,0,1)$ |
|-------------------------------|--------------|--------------|
| # nYR | 10 | 169 262 |
| $|\text{Aut}|$ | 167 541 | 1 626 |
| 2 | 0 | 69 |
| 3 | 0 | 24 |
| 13 | 0 | 1 |
| 26 | 1 | 0 |
| 39 | 0 | 1 |
| 60 | 1 | 0 |
| 72 | 1 | 0 |
| 80 | 1 | 0 |
| 84 | 1 | 0 |
| 144 | 1 | 0 |
| 252 | 1 | 0 |
| 300 | 1 | 0 |
| 320 | 1 | 0 |
| 1296 | 1 | 0 |
5 Relations to Triple Arrays and Related Row-Column Designs

In this section, we present data and give some new theoretical results on the connection between Youden rectangles and double, triple and sesqui arrays.

5.1 Theoretical background

A \((v, e, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)\) triple array is an \(r \times c\) array on \(v\) symbols satisfying the following conditions:

(TA1) No symbol is repeated in any row or column.

(TA2) Each symbol occurs \(e\) times (the array is equireplicate).

(TA3) Any two distinct rows contain \(\lambda_{rr}\) common symbols.

(TA4) Any two distinct columns contain \(\lambda_{cc}\) common symbols.

(TA5) Any row and column contain \(\lambda_{rc}\) common symbols.

If we relax condition (TA5), the array is called a double array, and if condition (TA5) is expressly forbidden to hold, but all other conditions hold, we have a proper double array. If we relax condition (TA4), the array is called a sesqui array, and an array satisfying every condition except (TA4) we call a proper sesqui array. Triple arrays were introduced by Agrawal [2], though examples were known previously, and a good general introduction to triple and double arrays is given in [6]. Sesqui array were introduced in [3].

In discussing these designs we will find a new class of Latin rectangles useful.

Definition 5.1. A Latin rectangle with integer parameters \((n, k, \lambda)\), with \(\lambda = \frac{k(k-1)}{n-1}\) calculated from \(n\) and \(k\) as for a Youden rectangle, where the column intersections have sizes \(\lambda - 1, \lambda\) and \(\lambda + 1\) is called a 3-\(\lambda\) Latin rectangle.
Note that these objects are defined only for such \((n, k, \lambda)\) that allow Youden rectangles with these parameters, and that we require the intersection sizes to actually take on all these three values.

In [13] it was suggested that triple arrays could be constructed by taking an arbitrary Youden rectangle, removing one column and all symbols present in that column, and then exchanging the roles of columns and symbols, but in [15], the method was observed to be flawed. For ease of reference, we phrase this as follows.

Construction 5.2. For a given Youden rectangle \(Y\) and a column \(c\), let \(A\) be the array received from \(Y\) by first removing column \(c\) and all occurrences in \(Y\) of symbols present in \(c\), and then exchanging the roles of columns and symbols.

We say that a Youden rectangle \(Y\) is compatible with an array \(A\) if \(Y\) gives \(A\) via this construction for some suitable choice of column, and that \(Y\) yields \(A\).

Construction 5.2 was further investigated in [8], yielding among other the following results, reformulated to suit the terminology employed in the present paper:

Theorem 5.3 (Proposition 2 in [8]). Using Construction 5.2, any Youden rectangle always yields an array that satisfies conditions (TA1), (TA2) and (TA4), regardless of the choice of column.

In particular, when applied to a \((n, k, \lambda)\) Youden rectangle, Construction 5.2 yields an equireplicate \(r \times c = k \times (n-k)\) array on \(v = n-1\) symbols, with replication number \(e = k - \lambda\) and column intersection size \(\lambda_{cc} = \lambda\). We see then that Construction 5.2 may never (by definition of a proper sesqui array) yield a proper sesqui array, but it is possible that we would get the transpose of a proper \((n-k) \times k\) sesqui array.

Theorem 5.4 (Theorem 3 in [8]). Using Construction 5.2, any Youden rectangle with \(\lambda = 1\) always yields a double array for any choice of column, but never a triple array.

Theorem 5.5 (Theorem 7 in [8]). For any triple array \(T\) with \(v = r + c - 1\) and \(\lambda_{cc} = 2\), there exists a Youden rectangle (with \(k = r, n = v + 1, \lambda = 2\)) that yields \(T\) using Construction 5.2.
It was also conjectured in [8] that Theorem 5.5 would hold for triple arrays with $\lambda_{cc}$ larger than 2.

When applying Construction 5.2 to near Youden rectangles or 3-$\lambda$ Latin rectangles, removing a column together with all the symbols present in that column will leave a $k \times (n-1)$ equireplicate array with some empty cells. For a near Youden rectangle, the empty cells are distributed so that the number of empty cells in a column is either $\lambda_1$ or $\lambda_2$. For a 3-$\lambda$ Latin rectangle, the corresponding numbers of empty cells are $\lambda - 1$, $\lambda$ or $\lambda + 1$. If more than one value occurs for the number of empty cells in a column, the array will not be equireplicate after exchanging the roles of columns and symbols, since the number of appearances of a symbol in the resulting array will be the number of non-empty cells in the corresponding column.

However, it may be the case that the column removed has a single intersection size with all other columns, and in this case we will get an equireplicate array. It is therefore of interest to perform the same computations as for Youden rectangles. In the size range we have considered, we found no near Youden rectangles that gave double, triple or sesqui arrays, nor transposes of sesqui arrays.

However, the following theorem follows rather easily from results in [8].

**Theorem 5.6.** For any $(v, e, \lambda_{rr}, 1, \lambda_{rc} : r \times c)$ triple array $T$ with $v = r + c - 1$ (we will assume from now on that the relation $v = r + c - 1$ holds), there is a compatible $r \times (v - c)$ 3-$\lambda$ Latin rectangle $Y$ with column intersection sizes 0, 1 and 2.

The proof of this claim uses with the following result, where the RL-form $R$ of a triple array $T$ is the array that results from exchanging the roles of columns and symbols in the $T$.

**Theorem 5.7 (Corollary 1 in [8]).** In the RL-form $R$ of a triple array $T$ with $v = r + c - 1$, for any two columns $C_1$ and $C_2$, the sum of the number of common non-empty rows and the number of common symbols of $C_1$ and $C_2$ is constant, namely $e$, the replication number.
Proof of Theorem 5.6. Since the parameters of a $T$ are not all independent of each other (in particular, when $v = r + c - 1$, it holds that $\lambda_{cc} = r - c$, see [6]), we may also observe that when exchanging the roles of symbols and columns in a $T$, there will be $r - c = \lambda_{cc}$ empty cells in each column in $R$ (the number of rows in $T$ in which the corresponding symbol does not appear). Reasoning similarly, there will be $r - \lambda_{cc}$ empty cells in each row of $R$ (the number of columns in $T$ where the corresponding symbol does not appear).

For $\lambda_{cc} = 1$, Theorem 5.7 then implies that in $R$, each pair of columns shares 0 symbols (when their empty cells lie in the same row) or 1 symbol (when their empty cells lie in different rows).

With this information, given a triple array $T$, we can construct a Youden rectangle $Y$ compatible with $T$ by first taking the RL-form $R$ of $T$, and adding a new column $C_0$ with a set $S$ of $r$ new symbols, $s_1, s_2, \ldots, s_r$ in this order. To fill the empty cells in row $i$ in the RL-form, we then use the $r - 1$ symbols $S \setminus \{s_i\}$, in any order. This is the right number of symbols, since there are $r - 1$ empty cells in every row of $R$, and there will be no repeated symbol in any row or column.

The intersections between columns in $Y$ may now have three different sizes. As observed above, pairs of columns in $R$ shared either 0 symbols or 1 symbol, and after adding symbols to form $Y$, these numbers may have gone up by at most 1, since only one new symbol was added in each column.

An example of the construction in the above proof is given in Figure 3. Since Theorem 5.6 shows that the same transformation that we applied to Youden rectangles could yield interesting row-column designs when applied to a 3-$\lambda$ Latin rectangle, we have also included this in our computational studies.

5.2 Computational Results for Youden Rectangles

In this section, we report on the number of Youden rectangles which yield triple arrays, proper double arrays, or transposes of proper sesqui arrays, for all parameters $(n, k, \lambda)$ for which we have
(a) A $4 \times 9$ triple array $T$.

\[
\begin{array}{cccccccc}
0 & 2 & 1 & 4 & 5 & 6 & 8 & 7 & 10 \\
11 & 3 & 8 & 5 & 6 & 7 & 9 & 1 & 2 \\
5 & 7 & 4 & 9 & 3 & 11 & 0 & 10 & 8 \\
1 & 0 & 3 & 2 & 10 & 4 & 6 & 9 & 11 \\
\end{array}
\]

(b) The corresponding RL-form of $T$.

\[
\begin{array}{cccccccc}
0 & 2 & 1 & 3 & 4 & 5 & 7 & 6 & 8 \\
7 & 8 & 1 & 3 & 4 & 5 & 2 & 6 & 0 \\
6 & 4 & 2 & 0 & 1 & 8 & 3 & 7 & 5 \\
1 & 0 & 3 & 2 & 5 & 6 & 7 & 4 & 8 \\
\end{array}
\]

(c) A $3$-$\lambda$ Latin rectangle compatible with $T$.

\[
\begin{array}{cccccccccccc}
9 & 0 & 2 & 1 & 10 & 3 & 4 & 5 & 7 & 6 & 11 & 8 & 12 \\
10 & 9 & 7 & 8 & 1 & 11 & 3 & 4 & 5 & 2 & 6 & 12 & 0 \\
11 & 6 & 9 & 10 & 4 & 2 & 0 & 12 & 1 & 8 & 3 & 7 & 5 \\
12 & 1 & 0 & 3 & 2 & 5 & 9 & 6 & 10 & 11 & 7 & 4 & 8 \\
\end{array}
\]

Figure 3: Example of the construction in the proof of Theorem 5.6.

complete data, except for $(21, 5, 1)$ Youden rectangles, where the computing time required was too great. We note again that no near Youden rectangles yielded any double, triple or (transposes of) sesqui arrays.

We ran checks even for properties guaranteed by Theorems 5.3, 5.4 and 5.5. Computational results were compatible with those of these theorems, which can be taken as an independent indication of the correctness of the computations.

5.2.1 Triple Arrays

Among the possible parameters for Youden rectangles for which we have complete data, there are just two sets of parameters where there is a chance of producing triple arrays, namely $(11, 5, 2)$ and $(11, 6, 3)$. All Youden rectangles with $\lambda = 1$ are excluded by Theorem 5.4 and $(7, 4, 2)$ would give a $4 \times 3$ triple array, the
existence of which was excluded in [6].

In Table 14 for triple arrays and Table 16 for proper double arrays we give the following information:

1. The number of Youden rectangles that give a triple or double array via Construction 5.2 for at least one of its columns.

2. The total number of columns for which the construction yields a triple or double array (that is, Youden rectangles counted with ‘multiplicities’).

3. The number of non-isotopic triple or proper double arrays we observe appearing as a result of this operation.

\[
\begin{array}{|c|c|c|c|}
\hline
(n, k, \lambda) & \# \text{compatible YR} & \# \text{compatible columns} & \# \text{TA} \\
\hline
(11,5,2) & 52 & 52 & 7 \\
(11,6,3) & 826 & 826 & 7 \\
\hline
\end{array}
\]

Table 14: The number of Youden rectangles giving triple arrays.

The 5 × 6 triple arrays (and by taking transposes, also the 6 × 5 triple arrays) were completely classified into 7 isotopy classes in [9]. As predicted by Theorem 5.5 all 7 triple arrays appear in Table 14.

**Observation 5.8.** Each of the Youden rectangles which give a triple array does so using a unique column.

The 7 different triple arrays do not appear equally often. With classes numbered as in [9], the triple arrays appear with the frequencies given in Table 15. The sizes of the autotopism groups of the triple arrays (in the row labelled TA |Aut|) are taken from [9]. It seems that it is easier to produce those triple arrays that have smaller autotopism groups.

We investigated the autotopism group sizes of the Youden rectangles that produced triple arrays, but we observed no obvious patterns.
Table 15: The number of Youden rectangles giving each of the 7 classes of $5 \times 6$ triple arrays.

### 5.2.2 Proper Double Arrays

We also checked which Youden rectangles produced proper double arrays, and the results are given in Table 16. As predicted by Theorem 5.4, we see that all Youden rectangles with $\lambda = 1$ produced proper double arrays, for each column. For other values of $\lambda$, there is some indication that the proportion of compatible Youden rectangles decreases with growing $\lambda$, and that the most common case is that even in a compatible Youden rectangle, only one column is compatible.

| $(n, k, \lambda)$ | # compatible YR | # compatible columns | # DA |
|-------------------|-----------------|---------------------|------|
| (7,3,1)           | 1               | 7                   | 1    |
| (7,4,2)           | 6               | 18                  | 2    |
| (11,5,2)          | 44 012          | 64 949              | 17 642 |
| (11,6,3)          | 31 782 790     | 32 335 774          | 24 663 |
| (13,4,1)          | 20              | 260                 | 192  |

Table 16: The number of Youden rectangles giving proper double arrays.

We note also that for parameter pairs $(n, k, \lambda_1), (k, n - k, \lambda_2)$, the double arrays produced by the first have dimensions $k \times (n - k)$ and taking transposes yields a $(n - k) \times k$ double array, and vice versa. Despite this, we see different numbers of double arrays appearing through the construction both for the pair $(7, 3, 1), (7, 4, 2)$ and the pair $(11, 5, 2), (11, 6, 3)$. This would seem to indicate that there are double arrays that cannot be constructed using Construction 5.2.
We note that on the basis of these data, we can answer in the negative a question posed in [8], namely whether every Youden rectangle gives a double array using Construction 5.2 for some column. We phrase this as an observation.

**Observation 5.9.** There are Youden rectangles that cannot be used to produce double arrays by removing a column and all the symbols in that column, and then interchanging the roles of symbols and columns.

### 5.2.3 Transposes of Proper Sesqui Arrays

Using Construction 5.2, we checked for transposes of proper sesqui arrays, and the results are presented in Table 17.

| $(n, k, \lambda)$ | # compatible YR | # compatible columns | # SA$^T$ |
|-------------------|-----------------|----------------------|----------|
| (7,3,1)           | 0               | 0                    | 0        |
| (7,4,2)           | 1               | 3                    | 1        |
| (11,5,2)          | 0               | 0                    | 0        |
| (11,6,3)          | 8 234           | 8 234                | 34       |
| (13,4,1)          | 0               | 0                    | 0        |

Table 17: The number of Youden rectangles giving transposes of proper sesqui arrays.

We observe that transposes of sesqui arrays are relatively rare, and that the compatible $(11, 6, 3)$ Youden rectangles are only compatible for one single column each. The one compatible $(7, 4, 2)$ Youden rectangle given in Figure 4 together with the resulting transposed sesqui array.

### 5.2.4 Compatibility with Several Designs

In our data, we found some specimens of Youden rectangles exhibiting very good compatibility properties. To begin with, in Figure 4, we give a $(7, 4, 2)$ Youden rectangle which is compatible both with transposes of sesqui arrays, and with a proper double array.
(a) The Youden rectangle.

(b) The transposed sesqui array.

Figure 4: The unique $4 \times 7$ Youden rectangle compatible with the transpose of a sesqui array, with compatible columns marked by S, and a column compatible with a double array marked by D.

Figure 5: Example of a $5 \times 11$ Youden rectangle with maximum compatibility with respect to triple and proper double arrays. The column marked with T is compatible with a triple array, and the four columns marked with D are compatible with proper double arrays.

Further, some of the Youden rectangles that gave triple arrays of dimensions $5 \times 6$ and $6 \times 5$ also gave proper double arrays for some other columns. Examples with maximum number of columns compatible with double arrays are given in Figures 5 and 6.

Even for Youden rectangles with $\lambda \neq 1$, we found Youden rectangles that for each column are compatible with some proper double array.

In Figure 7, we give the unique $4 \times 7$ Youden rectangle where each column is compatible with a double array. For any column, the resulting double array is isotopic to the one given in Figure 7b. The Youden rectangle in Figure 7a has the largest autotopism.
Figure 6: Example of a $6 \times 11$ Youden rectangle with maximum compatibility with respect to triple and proper double arrays. The column marked with T is compatible with a triple array, and the column marked with D is compatible with a proper double array.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|----|
| 1 | 0 | 4 | 10| 5 | 7 | 8 | 2 | 3 | 6  | 9  |
| 2 | 3 | 6 | 9 | 7 | 8 | 0 | 10| 1 | 5  | 4  |
| 3 | 6 | 9 | 5 | 10| 0 | 4 | 1 | 2 | 8  | 7  |
| 4 | 7 | 3 | 6 | 8 | 9 | 2 | 5 | 10| 1  | 0  |
| 5 | 8 | 7 | 0 | 3 | 2 | 10| 6 | 9 | 4  | 1  |

| D | T |

Figure 6: Example of a $6 \times 11$ Youden rectangle with maximum compatibility with respect to triple and proper double arrays. The column marked with T is compatible with a triple array, and the column marked with D is compatible with a proper double array.

Figure 7: The unique $4 \times 7$ Youden rectangle where each column is compatible with a double array.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 2 | 4 | 5 | 3 | 6 | 0 |
| 2 | 4 | 3 | 6 | 5 | 0 | 1 |
| 3 | 5 | 6 | 1 | 0 | 2 | 4 |

| 0 | 1 | 3 |
|---|---|---|
| 1 | 2 | 5 |
| 2 | 4 | 0 |
| 3 | 5 | 4 |

(a) The Youden rectangle. (b) The double array.

Figure 7: The unique $4 \times 7$ Youden rectangle where each column is compatible with a double array.

group size, i.e., 21, and the autotopism group acts transitively on the columns. As observed above, this Youden rectangle can therefore be produced from a difference set. The double array has an autotopism group of size 3, which acts transitively on the columns.

For $n = 11$, the situation is a bit more complicated. In Figure 8, we give the two $5 \times 11$ examples we found, and in Figure 9, we give the unique $6 \times 11$ example.

The Youden rectangle in Figure 8a has an autotopism group of size 55, which acts transitively on the columns, and so comes from a difference set. All columns yield a double array isotopic to the one in Figure 10. The autotopism group size of this double array is 5, and it acts transitively on 5 of the columns, but keeps column 5 fixed.

The Youden rectangle in Figure 8b has an autotopism group
Figure 8: The only two $5 \times 11$ Youden rectangles where each column is compatible with a proper double array.

Figure 9: The unique $6 \times 11$ Youden rectangle where each column is compatible with a proper double array.

Figure 10: The double array produced from the $5 \times 11$ Youden rectangle with autotopism group size 55 given in Figure 8a.
of size 60, which acts transitively on two groups of columns, with 5 and 6 columns, respectively. All columns in the group with five columns yield the double array in Figure 11a and all columns in the group with six columns yield the double array in Figure 11b. The autotopism group size of these double arrays are 12 and 10, respectively, and the group action for the first one is transitive on the columns, while the autotopism group for the second one acts transitively on all columns except the second column, which is fixed.

Finally, the Youden rectangle in Figure 9 has an autotopism group of size 55, which acts transitively on the columns, and so comes from a difference set. All columns yield the same double array, given in Figure 12, which has an autotopism group size of 5, which acts transitively on the columns.

It is interesting to note that the Youden rectangles in Figures 7–9 that produce a single double array (up to isotopism) for all columns have autotopism groups that act transitively on the
| $(n,k)$ | $(7,3)$ | $(7,4)$ |
|--------|--------|--------|
| # 3-λ LR | 43 | 872 |
| $|\text{Aut}|$ | 1 | 18 | 756 |
| | 2 | 21 | 101 |
| | 3 | 1 | 10 |
| | 4 | 0 | 3 |
| | 5 | 2 | 1 |
| | 14 | 1 | 1 |

Table 18: The number of 3-λ Latin rectangles with $n=7$ sorted by autotopism group size.

columns. For an investigation of this topic, we refer the interested reader to [7].

5.3 Computational Results for 3-λ Latin Rectangles

As we noted earlier, 3-λ Latin rectangles both provide the missing source for the $\lambda = 1$ triple arrays and could potentially lead to additional row-column designs. In order to investigate this connection we have also generated all 3-λ Latin rectangles with $n=7$, but for larger $n$ we deemed full enumeration infeasible. The number of such rectangles is given in Table 18, sorted by the size of the autotopism groups.

In Table 19 we give the number of such rectangles that are compatible with some proper double array. The maximum number of columns which are compatible with a double array is 2. Among the resulting non-isotopic double arrays for $(7,3,1)$ and $(7,4,2)$, we see three different double arrays, when taking transposes into account. The rectangles in Figure 13 are examples where the two compatible columns yield non-isotopic arrays, as indicated by subscripts.

For 3-λ Latin rectangles we have also found two examples which are compatible with proper sesqui arrays, as indicated in Table 20. We also found transposes of proper sesqui arrays in
Table 19: The number of 3-$\lambda$ Latin rectangles giving proper double arrays.

| $(n, k, \lambda)$ | # compatible LR | # compatible columns | # DA |
|-------------------|-----------------|----------------------|------|
| (7,3,1)           | 6               | 8                    | 2    |
| (7,4,2)           | 97              | 104                  | 2    |

Figure 13: Two examples of 3-$\lambda$ Latin rectangles with two columns that are compatible with non-isotopic proper double arrays. Subscripted D indicate the resulting non-isotopic double arrays, taking transposes into account.

the case $4 \times 7$, as indicated in Table 21, and here the maximum number of columns was three. We include all the resulting sesqui arrays here (in normalized form) in Figures 14 and 15, since such arrays are scarce in the literature. We note that we only find two non-isotopic sesqui arrays $S_1$ and $S_2$, when taking transposes into account, and that $S_1$ in fact recurs from Figure 7b.

Table 20: The number of 3-$\lambda$ Latin rectangles giving proper sesqui arrays.

| $(n, k, \lambda)$ | # compatible LR | # compatible columns | # SA |
|-------------------|-----------------|----------------------|------|
| (7,3,1)           | 2               | 2                    | 2    |
| (7,4,2)           | 0               | 0                    | 0    |
Figure 14: The 3-\( \lambda \) Latin rectangles of size 3 \( \times \) 7 that give proper sesqui arrays, together with the corresponding sesqui arrays.

Figure 15: Example of a 3-\( \lambda \) Latin rectangle of size 4 \( \times \) 7 that gives transposes of proper sesqui arrays for three compatible columns, together with the corresponding non-isotopic transposed sesqui arrays \( S_1^T \) and \( S_2^T \).
\begin{tabular}{|c|c|c|c|}
\hline
$(n, k, \lambda)$ & # compatible LR & # compatible columns & # SAT $^T$ \\
\hline
(7,3,1) & 0 & 0 & 0 \\
(7,4,2) & 73 & 78 & 2 \\
\hline
\end{tabular}

Table 21: The number of 3-$\lambda$ Latin rectangles giving transposes of proper sesqui arrays.

6 Concluding remarks

With the computing time and storage available to us at present, we have exhausted the possibilities of complete enumeration of Youden rectangles. A further line of inquiry might be to enumerate some restricted class of Youden rectangles, satisfying some stronger conditions. Such conditions would have to go beyond the structure of the symbol intersections between columns, since by only employing the design condition, we can only distinguish between non-isotopic SBIBD:s.

In relation to triple, double and sesqui arrays, we would like to pose the following questions:

Question 6.1. For a given set of parameters, how many double arrays are there that cannot be constructed from any Youden rectangle by removing a column and all the symbols in that column, and then exchanging the roles of symbols and columns?

Question 6.2. For a given set of parameters, can every double, triple, and (transpose of) sesqui array be obtained from a Youden rectangle or a 3-$\lambda$ Latin rectangle by Construction 5.2?

Here one could of course extend the set of allowed intersection sizes in the Latin rectangle all the way up to $k$, so the focus is on whether a small span of intersection sizes suffices.

We hope to return to these questions in future work.

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