Subtleties of the fermion measure in the presence of axion fields

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Abstract

It is known from path integral studies of the chiral anomaly that the fermion measure has to depend on gauge fields interacting with the fermion. It is pointed out here that in the presence of an axion field interacting with the fermion, it too is involved in the measure, with unexpected consequences for the utility of this field.
Introduction

While CP violation is observed in weak interactions, it is not known to occur in any other process. However, chiral mass terms for quarks led to apprehensions about the possibility of CP violation in strong interactions. It was because of these apprehensions that modifications of QCD were proposed to suppress imagined violations. The Peccei-Quinn approach [1] introduced an artificial chiral symmetry. While actual chiral symmetry is broken by the quark mass, a new chiral symmetry can be manufactured by coupling a new pseudoscalar field \( \varphi \) which absorbs the chiral transformation. The mass term is replaced by

\[
\bar{\psi} m e^{i\varphi} \gamma_5 e^{i\varphi} \gamma_5 \psi,
\]

(1)

which is invariant under the transformations

\[
\psi \rightarrow e^{i\alpha \gamma_5} \psi, \quad \bar{\psi} \rightarrow \bar{\psi} e^{i\alpha \gamma_5}, \quad \varphi \rightarrow \varphi - 2\alpha,
\]

(2)

which are respected by the kinetic terms of the action. Here \( \theta' \) is a phase in the mass term. There have been variations on this theme. The original interaction introduced by Peccei and Quinn was of the form

\[
\bar{\psi} \left[ e^{i\theta'} \Phi \frac{1 + \gamma_5}{2} + e^{-i\theta'} \Phi^{\dagger} \frac{1 - \gamma_5}{2} \right] \psi,
\]

(3)

where \( \Phi \) is a complex scalar field with a symmetry breaking potential. The chiral symmetry transformation is

\[
\Phi \rightarrow e^{-2i\alpha} \Phi, \quad \psi \rightarrow e^{i\alpha \gamma_5} \psi, \quad \bar{\psi} \rightarrow \bar{\psi} e^{i\alpha \gamma_5}.
\]

(4)

\( \Phi \) may be taken to be of the form \( p e^{i\varphi} \). The amplitude \( p \) of the scalar field acquires a vacuum expectation value because of symmetry breaking, which breaks the artificial chiral symmetry. It provides a massive boson. The phase \( \varphi \) is the zero mode of the potential and becomes the Goldstone boson. But this so-called axion [2], which acquires a mass because of the quark masses, has remained undetected [3]. How does this mechanism claim to remove P and T violation? Peccei and Quinn showed that \( \theta' + \varphi \) vanishes in the vacuum. Note that \( \theta' \) can be absorbed in \( \varphi \).

That analysis was mostly done before anomalies came to be properly appreciated. Now, unlike the vacuum angle \( \theta \), the quark mass parameter \( \theta' \) – which has to be clearly distinguished from it – simply leads to a redefinition of P, T. This occurs in classical field theory as shown earlier and explained below. That such symmetries are not anomalous in quantum field theory has been proved by Pauli-Villars regularization and also by constructing a \( \theta' \) dependent measure possessing the symmetries. Here, we discuss the possibility of a similar effect of \( \varphi \) on the measure. The question is whether \( \theta' \) or in the present instance \( \varphi \) can be removed from the action by a chiral transformation. Note that the chiral transformation has to be accompanied by a modification of the measure when the latter depends on \( \theta' \) or \( \varphi \).

Before discussing \( \varphi \), we shall review what is now known about \( \theta' \).
Parity of fermion fields in the context of $\theta'$, $\varphi$

It is advisable here to consider the parity operation of fermion fields. For fermions with real mass terms, one seeks invariance under

$$\psi(\vec{r}) \rightarrow P\psi(-\vec{r})$$

of the action

$$\int \bar{\psi}(i\gamma^\mu D_\mu - m)\psi.$$  

This implies that

$$P^\dagger P = 1, \quad -P^\dagger \gamma^0 \gamma^i P = \gamma^0 \gamma^i, \quad P^\dagger \gamma^0 P = \gamma^0$$  

whence

$$-P^\dagger \gamma^0 PP^\dagger \gamma^i P = \gamma^0 \gamma^i$$  

leading to

$$-P^\dagger \gamma^i P = \gamma^i.$$  

This is clearly satisfied by $P = \gamma^0$. As a consequence,

$$\bar{\psi}\psi \rightarrow \bar{\psi}(-\vec{r})\psi(-\vec{r})$$

$$\bar{\psi}\gamma_5\psi \rightarrow -\bar{\psi}(-\vec{r})\gamma_5\psi(-\vec{r})$$

which are thus seen to be scalar and pseudoscalar combinations, as is well known.

However, for a complex fermion mass term, the situation changes: One seeks invariance under

$$\psi(\vec{r}) \rightarrow P\psi(-\vec{r})$$

of the modified action

$$\int \bar{\psi}(i\gamma^\mu D_\mu - me^{i\theta'\gamma_5})\psi.$$  

This implies

$$P^\dagger P = 1, \quad -P^\dagger \gamma^0 \gamma^i P = \gamma^0 \gamma^i, \quad P^\dagger \gamma^0 e^{i\theta'\gamma_5} P = \gamma^0 e^{i\theta'\gamma_5}$$  

leading to

$$-P^\dagger \gamma^0 e^{i\theta'\gamma_5} PP^\dagger \gamma^i e^{i\theta'\gamma_5} P = \gamma^0 e^{i\theta'\gamma_5} \gamma^i e^{i\theta'\gamma_5}$$  

whence

$$-P^\dagger \gamma^i e^{i\theta'\gamma_5} P = \gamma^i e^{i\theta'\gamma_5}. $$
This is clearly satisfied by \( P = \gamma^0 e^{i\theta'} \gamma_5 = e^{-i\theta'} \gamma_5/2 \gamma^0 e^{i\theta'} \gamma_5/2 \). As a consequence,

\[
\bar{\psi} e^{i\theta'} \gamma_5 \psi \rightarrow \bar{\psi}(-\vec{r}) e^{i\theta'} \gamma_5 \psi(-\vec{r}) \\
\bar{\psi} \gamma_5 e^{i\theta'} \gamma_5 \psi \rightarrow \bar{\psi}(-\vec{r}) \gamma_5 e^{i\theta'} \gamma_5 \psi(-\vec{r})
\]  

(14)

(15)

showing how the scalar and pseudoscalar combinations are different now.

If an axion field is present, the situation is as follows: Now one seeks invariance under

\[
\psi(\vec{r}) \rightarrow P\psi(-\vec{r})
\]

(16)

of the action

\[
\int \bar{\psi}(i\gamma^\mu D_\mu - me^{i\varphi \gamma_5})\psi.
\]

(17)

This means that

\[
P^\dagger P = 1, \quad -P^\dagger \gamma^0 \gamma^i P = \gamma^0 \gamma^i, \quad P^\dagger \gamma^0 e^{i\varphi(\vec{r})} \gamma_5 P \rightarrow \gamma^0 e^{i\varphi(-\vec{r})} \gamma_5
\]

(18)

leading to

\[
- P^\dagger \gamma^0 e^{i\varphi(\vec{r})} \gamma_5 PP^\dagger \gamma^i e^{i\varphi(\vec{r})} \gamma_5 P \rightarrow \gamma^0 e^{i\varphi(-\vec{r})} \gamma_5 \gamma^i e^{i\varphi(-\vec{r})} \gamma_5
\]

(19)

whence

\[
- P^\dagger \gamma^0 e^{i\varphi(\vec{r})} \gamma_5 P \rightarrow \gamma^0 e^{i\varphi(-\vec{r})} \gamma_5
\]

(20)

This is now satisfied by \( P = \gamma^0 \) coupled with \( \varphi(\vec{r}) \rightarrow \varphi(-\vec{r}) \). This simply means that \( \varphi \) has to be a pseudoscalar field.

**Fermion measure involving \( \theta' \)**

Before investigating the effect of \( \varphi \) on the fermion measure, it is appropriate to review anomalies in the context of the phase \( \theta' \) in the mass.

Anomalies are often studied in euclidean spacetime. For a real fermion mass term, one has the gauge invariant fermion action

\[
S = \int \bar{\psi}(i\gamma^\mu D_\mu - m)\psi.
\]

(21)

With eigenfunctions \( \phi_n \) of \( \gamma^\mu D_\mu \) in euclidean spacetime, the fields are expanded [5]:

\[
\psi = \sum_n a_n \phi_n, \quad \bar{\psi} = \sum_n \bar{a}_n \phi_n^d
\]

(22)
and the fermion functional integral is defined as
\[ Z = \int DA \prod_n \int da_n \prod_n \int d\bar{a}_n e^{-S}. \] (23)

For a chiral transformation \( \psi \to e^{i\alpha \gamma^5} \psi, \quad \bar{\psi} \to \bar{\psi} e^{i\alpha \gamma^5} \), the coefficients transform as follows:
\[ a_n \to \sum_m \int \phi_n^\dagger e^{i\alpha \gamma^5} \phi_m a_m, \quad \bar{a}_n \to \sum_m \bar{a}_m \int \phi_m^\dagger e^{i\alpha \gamma^5} \phi_n, \] (24)
with a nontrivial Jacobian, calculated by regularizing the large (gauge invariant) eigenvalues of \( \gamma^\mu D_\mu \) as \( \exp[i \int \alpha g^2 F \bar{F} / 16\pi^2] \). This amounts to a breakdown of the chiral symmetry (even if \( m = 0 \)) by what is called the anomaly.

Now one has to recall why eigenfunctions of \( \gamma^\mu D_\mu \) are used here. It is to ensure the gauge invariance of the measure. Under gauge transformations,
\[ D \to UDU^{-1}, \quad \phi \to U\phi, \quad \psi \to U\psi \] (25)
so that \( \psi, \phi \) transform in the same way and \( a \) does not change under a gauge transformation. Similarly \( \bar{a} \) does not change because \( \bar{\psi}, \bar{\phi} \) transform in the same way. What about parity? Under parity for gauge fields,
\[ D(\vec{x}) \to \gamma^0 D(-\vec{x}) \gamma^0, \quad \phi(\vec{x}) \to \gamma^0 \phi(-\vec{x}), \quad \psi(\vec{x}) \to \gamma^0 \psi(-\vec{x}). \] (26)
Hence the measure is parity invariant for a real mass term.

For a complex mass term, the situation is complicated [6]: Under the parity operation for gauge fields,
\[ \phi_n(x_0, \vec{x}) \to \gamma^0 \phi_n(x_0, -\vec{x}), \quad \phi_n^\dagger(x_0, \vec{x}) \to \phi_n^\dagger(x_0, -\vec{x}) \gamma^0. \] (27)
so that
\[ e^{-i\theta' \gamma^5/2} \phi_n(x_0, \vec{x}) \to \gamma^0 e^{i\theta' \gamma^5/2} e^{-i\theta' \gamma^5/2} \phi_n(x_0, -\vec{x}), \] (28)
\[ \phi_n^\dagger(x_0, \vec{x}) e^{-i\theta' \gamma^5/2} \to \phi_n^\dagger(x_0, -\vec{x}) e^{-i\theta' \gamma^5/2} e^{i\theta' \gamma^5} \gamma^0, \] (29)
in which \( e^{-i\theta' \gamma^5/2} \phi_n \) is seen to have exactly the same parity transformation as the fermion. Hence one must now expand
\[ \psi = e^{-i\theta' \gamma^5/2} \sum_n a_n \phi_n, \quad \bar{\psi} = \sum_n \bar{a}_n \phi_n^\dagger e^{-i\theta' \gamma^5/2}, \] (30)
in order to obtain a measure that is not only gauge-invariant but also respects the rotated parity:
\[ Z = \int DA \prod_n \int da_n \prod_n \int d\bar{a}_n e^{-S}. \] (31)
One may check that under a chiral transformation,

\[ a_n \rightarrow \sum_m \int \phi_m^\dagger e^{ia\gamma^5} \phi_m a_m, \]  

(32)

\[ \bar{a}_n \rightarrow \sum_m \bar{a}_m \int \phi_m^\dagger e^{ia\gamma^5} \phi_n. \]  

(33)

This is the same transformation as before and yields the standard anomaly.

The consequence of having such a \( \theta' \) dependent measure is that the chiral transformation replacing \( e^{i\theta'\gamma^5/2} \psi \) by \( \psi \) removes \( \theta' \) from both the action and the measure. No gauge field term involving \( \theta' \) gets added to the action, i.e., the gluonic \( \theta \) term does not get altered in the process. This may also be understood as follows. It is true that the chiral transformation that removes \( \theta' \) from the action tends to change \( a, \bar{a} \) as in the derivation of the anomaly. But now the expressions of the fields in terms of \( a, \bar{a} \) depend on \( \theta' \), which has to be removed by the opposite transformation of the \( a, \bar{a} \). As both \( \theta' \) dependences are removed, the changes in \( a, \bar{a} \) cancel out, so that the measure does not change. This irrelevance of \( \theta' \) was proved first by using Pauli-Villars regularization [4].

**Fermion measure involving the axion field**

Finally we come to the main topic. What happens to the measure in the presence of an axion field? The field \( \varphi \) does not change the parity of \( \psi, \bar{\psi} \). As such, no change of the measure seems imperative. However, because of the oddness of the field under parity,

\[ e^{-i\varphi(x)\gamma^5/2} \phi_n(x) \rightarrow \gamma^0 e^{-i\varphi(-x)\gamma^5/2} \phi_n(-x), \]

\[ \phi_n^\dagger(x) e^{-i\varphi(x)\gamma^5/2} \rightarrow \phi_n^\dagger(-x) e^{-i\varphi(-x)\gamma^5/2} \gamma^0. \]  

(34)

Thus, \( e^{-i\varphi\gamma^5/2} \phi_n \) also has the required symmetry properties. In fact, any power of the exponential factor may be used in the expansion of \( \psi \). To decide what the appropriate factor is, note that the action has the combination \( e^{i(\theta'+\varphi)\gamma^5} \). This fixes the use of \( e^{-i(\theta'+\varphi)\gamma^5/2} \phi_n \) in expanding \( \psi \) when both the axion field and a phase are present.

\[ \psi = e^{-i(\theta'+\varphi)\gamma^5/2} \sum_n a_n \phi_n, \quad \bar{\psi} = \sum_n \bar{a}_n \phi_n^\dagger e^{-i(\theta'+\varphi)\gamma^5/2}, \]  

(35)

The consequence of this nontrivial measure involving the axion field is that this field cannot be transferred from the fermion sector of the action to the gauge field sector by a chiral transformation, as there has to be a corresponding change in the measure as well and this compensates the gauge field sector – exactly as in the case of \( \theta' \) reviewed above. The local chiral transformation replacing \( e^{i(\theta'+\varphi)\gamma^5/2} \psi \) by \( \psi \) essentially removes \( \theta' + \varphi \) from both the action
and the measure. There is a residue: the kinetic term of the fermion field is invariant only under global chiral transformations and not local ones, so that a derivative term results, $\bar{\psi} \gamma^\mu \gamma^5 \partial_\mu \phi \psi$. The upshot is that any $\theta$ term in the gauge field sector remains unaffected by chiral manipulations of $\phi$ just as it is unaltered by chiral manipulations of $\theta'$.

Conclusion

Mechanisms like the Peccei-Quinn scheme were invented to engineer the suppression of strong CP violation. Our detailed analysis of the symmetries of the functional measure shows the axion field to alter the fermion measure. It can then be essentially removed – apart from derivative terms – by chiral transformations. So it cannot modify the CP violation caused by a vacuum angle as expected earlier.

If the Peccei-Quinn mechanism fails, what can be done about the vacuum angle to avoid CP violation? The obvious solution is to set it equal to zero. This is not unnatural as shown in earlier work [4].

What remains then is a concern about the need for axions. Even if there is no rôle for axion fields in controlling CP violation, they may just happen to exist: this can presumably be determined by experiment and is beyond the reach of theory.

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