Exceptional structure of the dilute A₃ model: E₈ and E₇ Rogers–Ramanujan identities

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Abstract

The dilute A₃ lattice model in regime 2 is in the universality class of the Ising model in a magnetic field. Here we establish directly the existence of an E₈ structure in the dilute A₃ model in this regime by expressing the 1-dimensional configuration sums in terms of fermionic sums which explicitly involve the E₈ root system. In the thermodynamic limit, these polynomial identities yield a proof of the E₈ Rogers–Ramanujan identity recently conjectured by Kedem et al. The polynomial identities also apply to regime 3, which is obtained by transforming the modular parameter by \( q \to 1/q \). In this case we find an A₁ × E₇ structure and prove a Rogers–Ramanujan identity of A₁ × E₇ type. Finally, in the critical \( q \to 1 \) limit, we give some intriguing expressions for the number of L-step paths on the A₃ Dynkin diagram with tadpoles in terms of the E₈ Cartan matrix. All our findings confirm the E₈ and E₇ structure of the dilute A₃ model found recently by means of the thermodynamic Bethe Ansatz.

1 Introduction

Recently, a Bethe Ansatz study [1] of the dilute A₃ lattice model [2, 3] has revealed a hidden E₈ structure. This establishes the expected relation between the dilute A₃ model and Zamolodchikov’s E₈ S-matrix of the critical Ising model in a field [4]. One of the drawbacks, however, of this Bethe Ansatz approach is that it relies heavily on the acceptance of a conjectured string structure of the Bethe Ansatz equations. In this letter we demonstrate the E₈ structure of the dilute A₃ model directly, without the use of a string hypothesis.

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In recent papers Melzer and Berkovich have shown that the 1-dimensional configuration sums of the ABF model admit a so-called fermionic representation in addition to the well-known bosonic forms of Andrews et al. Their motivation was in fact to prove Rogers–Ramanujan (RR) type identities for the \( \chi_{r,s}^{(h)} \) Virasoro characters associated with the minimal unitary models of central charge \( c = 1 - 6/h(h - 1) \) as conjectured by the Stony Brook group. In this letter we adopt a similar approach. Specifically, we rewrite the known bosonic expressions for the 1-dimensional configuration sums of the dilute A\(_3\) model in regime 2\(^+\) in terms of fermionic sums. These fermionic sums explicitly involve the E\(_8\) root system. In particular, in the thermodynamic limit, our “fermionic sum = bosonic sum” expressions yield precisely the E\(_8\) Rogers–Ramanujan identity for the \( \chi_{1,1}^{(4)} \) Virasoro character as given by Kedem et al.

Thermodynamic Bethe Ansatz computations were also carried out in regime 3\(^+\) of the dilute A\(_3\) model. In this case the model is known to decouple in the scaling limit into an Ising model and a \( \phi^{(5)}(2) \) perturbed minimal model. Accordingly, the Bethe Ansatz computations on the dilute A\(_3\) model in regime 3\(^+\) gives an A\(_1\) \( \times \) E\(_7\) structure leading to the correct central charge \( c = 1/2 + 7/10 = 6/5 \). These findings for regime 3\(^+\) are supported in this letter. By considering the thermodynamic limit of our “fermionic sum = bosonic sum” expressions, after carrying out the transformation \( q \rightarrow 1/q \), which maps regime 2\(^+\) onto regime 3\(^+\),[3] we find an A\(_1\) \( \times \) E\(_7\) Rogers–Ramanujan identity similar to the E\(_7\) identity for the \( \chi_{1,1}^{(5)} \) character conjectured by Kedem et al.

To conclude this letter, we point out some intriguing expressions for the number of walks on the adjacency graph of the dilute A\(_3\) model in terms of the E\(_8\) Cartan matrix.

### 2 Polynomial E\(_8\) Rogers–Ramanujan identity

Before we present the main results of this letter we need to introduce some notation.

We define the Gaussian multinomials or \( q \)-multinomials by

\[
\left[ \begin{array}{c} N \\ m_1, m_2, \ldots, m_n \end{array} \right]_q = \frac{(q)_N}{(q)_{m_1}(q)_{m_2} \cdots (q)_{m_n}(q)^{N-m_1-m_2-\cdots-m_n}},
\]

where \( (q)_m = \prod_{k=1}^{m} (1 - q^k) \) for \( m > 0 \) and \( (q)_0 = 1 \). Also, if \( I_{E_8} \) denotes the incidence matrix of E\(_8\) with the nodes labelled as in figure 1a, we define the following thermodynamic Bethe Ansatz (TBA) type systems:

\[
n + m = \frac{1}{2} (I_{E_8} m + (L - 1) e_1 + e_i) \quad i = 1, 2, \ldots, 8.
\]

Here \( n \) and \( m \) are 8-dimensional column vectors with integer entries \( n_i, m_i \), respectively, and \( e_i \) is a unit vector with components \( (e_i)_j = \delta_{i,j} \). The parameter \( L \) will be referred to as the system size. A pair of vectors \( n \) and \( m \) solving the \( i \)-th TBA type equation with system size \( L \) will be denoted by \( (n, m)_{L,i} \). Following ref. [1], we now define the fermionic functions \( F_i(L) \) by

\[
F_i(L) = \sum_{(n,m)_{L,i}} q^{n^T C_{E_8}^{-1} n} \prod_{j=1}^{8} \left[ \begin{array}{c} n_j + m_j \\ n_j \end{array} \right]_q,
\]

where \( C_{E_8} \) is the Cartan matrix of E\(_8\).
with $C_{E_8}$ the Cartan matrix of $E_8$ which is related to the incidence matrix by $(C_{E_8})_{i,j} = 2\delta_{i,j} - (I_{E_8})_{i,j}$.

Finally, we define the bosonic functions $B_{r,s}(L, a, b)$ by [3]

$$B_{r,s}(L, a, b) = \sum_{j,k=-\infty}^{\infty} \left\{ q^{12j^2+(4r+3s)j+k(k+8j+b-a)} \left[ \frac{L}{k, k + 8j + b - a} \right]_q - q^{12j^2+(4r+3s)j+rs+k(k+8j+b+a)} \left[ \frac{L}{k, k + 8j + b + a} \right]_q \right\}. $$

(2.4)

With these definitions our main assertion can be written as

$$F_1(L) = B_{1,1}(L, 1, 1).$$

(2.5)

Explicitly, this polynomial identity takes the form

$$\sum_{(n,m)_{L,1}} n^T C_{E_8}^{-1} n \prod_{i=1}^{8} \left[ \frac{n_i + m_i}{n_i} \right]_q$$

$$= \sum_{j,k=-\infty}^{\infty} \left\{ q^{12j^2+j+k(8j)} \left[ \frac{L}{k, k + 8j} \right]_q - q^{12j^2+7j+1+k(k+8j+2)} \left[ \frac{L}{k, k + 8j + 2} \right]_q \right\}. $$

(2.6)

which can be viewed as a finitization of the $E_8$ Rogers–Ramanujan identity of Kedem et al. [3]. Indeed, taking the limit $L \to \infty$, using the result [3]

$$\lim_{L \to \infty} \sum_{k=-\infty}^{\infty} q^{k(k+u)} \left[ \frac{L}{k, k + u} \right]_q = \frac{1}{(q)_{\infty}},$$

(2.7)

together with the simple formula $\lim_{N \to \infty} \left[ \frac{N}{m} \right]_q = 1/(q)_m$, gives

$$\sum_{n_1, \ldots, n_8=0}^{\infty} \frac{q^{n^T C_{E_8}^{-1} n}}{(q)_{n_1} \cdot \ldots \cdot (q)_{n_8}} = \frac{1}{(q)_{\infty}} \sum_{j=-\infty}^{\infty} \left\{ q^{12j^2+j} - q^{12j^2+7j+1} \right\}. $$

(2.8)

The RHS of this $E_8$ Rogers–Ramanujan identity is the usual (bosonic) Rocha-Caridi form for the $\chi^{(4)}_{1,1}$ Virasoro character [12]. The LHS is the fermionic counterpart conjectured by Kedem et al. [3].

Before we proceed to sketch a proof of identity (2.6), let us first explain how the above results relate to the $E_8$ structure of the dilute $A_3$ model. To this end we note that the bosonic side of equation (2.6) is precisely the expression for the 1-dimensional configuration sum $Y_L^{111}(q)$ of the dilute $A_3$ model in regime $2^+$ as computed in [3]. More generally, the configuration sums $Y_L^{a,b,c}(q)$ with $a, b, c \in \{1, 2, 3\}$ and $|b - c| \leq 1$ are defined via

$$Y_L^{\sigma_1 \sigma_{L+1} \sigma_{L+2}}(q) = \sum_{\sigma_2, \ldots, \sigma_L} q^{\sum_{j=1}^{L} jH(\sigma_j, \sigma_{j+1}, \sigma_{j+2})}. $$

(2.9)

The function $H$ herein follows directly from the Boltzmann weights of the dilute $A_3$ model by computing the ordered infinite field limit ($p \to 1$, $u/\epsilon$ fixed)

$$W \left( \begin{array}{cc} d & c \\ a & b \end{array} \right) = \frac{g_a g_c}{g_b g_d} e^{-2\pi u H(d, a, b)/\epsilon} \delta_{a,c} \quad \text{with} \quad g_a = e^{-2\lambda a^2/\epsilon}. $$

(2.10)
Here \( u \) is the spectral parameter, \( 3\lambda = 15\pi/16 \) is the crossing parameter and \( p = \exp(-\epsilon) \) is the nome of the elliptic function parametrization of the face weights \([3]\). A complete listing of the values of \( H(a, b, c) \) is given in (A.5) of ref. \([3]\). The occurrence of the particular configuration sum \( Y_{L}^{111}(q) \) in (2.6) can be understood from the fact that the configuration with all spins on the lattice taking the value 1 corresponds to the ground state of the model.

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Figure 1: (a) The Dynkin diagram of \( E_8 \). (b) The Dynkin diagram of \( E_7 \). (c) The incidence or adjacency graph of the dilute \( A_3 \) model.

### 2.1 Sketch of a proof of (2.7)

The proof of identity (2.6) is long and tedious so we will present it in full elsewhere. Here we only indicate the main ingredients of the proof and omit the detailed calculations.

Let us recall that the configuration sums \( Y_{L}^{a,b,c}(q) \) satisfy the recurrences \([3]\)

\[
Y_{L}^{a,b,c}(q) = q^L H(b^{-1},a,c) Y_{L-1}^{a,b+1,c}(q) + q^L H(b,a,c) Y_{L-1}^{a+1,b,c}(q) + q^L H(b+1,a,c) Y_{L-1}^{a,b,c}(q)
\]

subject to the initial condition

\[
Y_{1}^{a,b,c}(q) = q^H(a,b,c).
\]

Moreover, the recurrence relations together with the initial conditions uniquely determine the configuration sums \( Y_{L}^{a,b,c}(q) \). In view of (2.6) we will only consider the case \( a = 1 \). Apart from \( Y_{L}^{111}(q) \) we now also express the other \( Y_{L}^{1,b,c}(q) \) firstly in terms of fermionic sums and secondly in terms of bosonic sums. If we can then show that both the fermionic and bosonic expressions satisfy the recurrences (2.11) together with (2.12), the fermionic sums must equal the bosonic ones due to the uniqueness of solution to (2.11) and (2.12).

First, we list the bosonic expressions for \( Y_{L}^{1,b,c}(q) \):

\[
\begin{align*}
Y_{L}^{111}(q) & = B_{1,1}(L, 1, 1) \\
Y_{L}^{112}(q) & = q^L B_{1,1}(L, 1, 1) \\
Y_{L}^{121}(q) & = B_{3,1}(L, 1, 2) \\
Y_{L}^{122}(q) & = q^{L-1} B_{1,1}(L, 1, 2) + q^L (1 - q^L) B_{3,1}(L - 1, 1, 3) \\
Y_{L}^{123}(q) & = q^{2L-1} B_{1,1}(L, 1, 2) \\
Y_{L}^{132}(q) & = q^{L-1} B_{3,1}(L, 1, 3) \\
Y_{L}^{133}(q) & = q^{L-1} B_{3,1}(L, 1, 3) + q^{L-3} (1 - q^L) B_{1,1}(L - 1, 1, 2).
\end{align*}
\]
The proof that this solves (2.11) and (2.12) has been given in [3].

Second, we list fermionic expressions for $Y_L^{1bc}(q)$:

$$Y_L^{111}(q) = F_1(L)$$
$$Y_L^{112}(q) = q^L F_1(L)$$
$$Y_L^{121}(q) = \left(F_7(L) - (1 - q^L) F_1(L) - q^L (1 - q^L) F_1(L - 1) + q^{2L-1}(1 - q^{L-1}) F_1(L - 2)\right) / q^{L+1}$$
$$Y_L^{122}(q) = \left(F_7(L) - (1 - q^L) F_1(L) + q^{2L-1}(1 - q^{L-1}) F_1(L - 2)\right) / q$$
$$Y_L^{123}(q) = q^{2L-1} \left(F_7(L) + q^{L-1}(1 - q^{L-1}) F_1(L - 2)\right)$$
$$Y_L^{132}(q) = \left(F_7(L) - F_1(L) + q^L F_7(L - 1) + q^{L-1}(1 - q^{L-1}) F_7(L - 2) + q^{2L-2}(1 - q^{L-2}) F_1(L - 3) + q^{2L-4}(1 - q^{L-1})(1 - q^{L-3}) F_1(L - 4)\right) / q^{L+3}$$
$$Y_L^{133}(q) = q^{L-3} \left(F_7(L) - F_1(L) + F_7(L - 1) + q^{L-1}(1 - q^{L-1}) F_7(L - 2) + q^{L-2}(1 - q^{L-2}) F_1(L - 3) + q^{2L-4}(1 - q^{L-1})(1 - q^{L-3}) F_1(L - 4)\right).$$

These expressions are admittedly quite complicated and might very well be simplified. For example, we have chosen to express all configuration sums in terms of $F_1$ and $F_7$ only. Using easily verifiable recurrences of the type $F_1(L) - q^{2L-2} F_1(L - 2) = F_3(L - 1)$, $F_2(L) + q^{L+1}(1 - q^{L-2}) F_2(L - 2) = F_3(L - 1) + q^{L+1} F_1(L - 1)$, etc., one could conceivably find simpler forms for the above.

To prove the correctness of the fermionic solution to the recurrence relations we substitute (2.14) into (2.11). This gives seven identities which can be combined to yield the following two equations

$$F_1(L) - F_7(L - 1) - q^{L-1} F_1(L - 1) + q^{L-1}(1 - q^{L-1}) F_1(L - 2) - q^{2L-3}(1 - q^{L-2}) F_1(L - 3) = 0$$
$$F_7(L) - q^2 F_1(L - 1) - (1 + q^{L-1}) F_7(L - 1) + q^2(1 - q^{2L-4}) F_7(L - 2) + q^{L-1} F_1(L - 2) - q^L (1 - q^{L-2}) F_1(L - 3) + q^L(1 - q^{L-2})(1 - q^{L-3}) F_7(L - 3) = 0.$$

The actual proof of these final two equations will be omitted here, but we remark that they follow from elementary but tedious computations very similar to those carried out in [3]. The proof that (2.14) satisfies the initial conditions is a matter of straightforward case checking.

### 3 A₁ × E₇ Rogers–Ramanujan identity

In this section we consider the $L \to \infty$ limit of identity (2.6) after first replacing $q$ by $1/q$. The effect of this transformation on $q$ is to map from regime $2^+$ to regime $3^+$ of the dilute $A_3$ model. The critical behaviour of the model in this latter regime is described by [3, 11] a $c = 6/5$ conformal field theory given as a direct product of an Ising model ($c = 1/2$) and an $E_7$ theory with $c =$
2 rank $G/(g + 2) = (2)(7)/(18 + 2) = 7/10$. Hence it is to be expected that the above steps will result in an $A_1 \times E_7$ Rogers–Ramanujan identity.

To establish this we use two simple inversion formulas:

$$\left[ \frac{N}{m} \right]_{1/q} = q^{m(N-N)} \left[ \frac{N}{m} \right]_q \quad (3.1)$$

$$\left[ \frac{N}{m_1, m_2} \right]_{1/q} = q^{m_1^2+m_2^2-m_1-m_2} \left[ \frac{N}{m_1, m_2} \right]_q \quad (3.2)$$

Applying (3.2) to transform the bosonic RHS of (2.4) we obtain, after performing a shift on the summation variable $k$,

$$q^{(\mu-L)/2} \sum_{j,k=-\infty}^{\infty} q^{2k(\mu+\mu)} \left\{ q^{20j^2+j} \left[ \frac{(L-\mu)/2-4j-k, (L-\mu)/2+4j-k}{q} \right]_q 
-q^{20j^2+9j+1} \left[ \frac{(L-\mu)/2-4j-k-1, (L-\mu)/2+4j-k+1}{q} \right]_q \right\}. \quad (3.3)$$

Here the variable $\mu = 0, 1$ is given by the parity of $L$ via $(L - \mu)/2 \in \mathbb{Z}$. Multiplying by the factor $q^{L/2}$ and taking the thermodynamic limit using the result

$$\lim_{N \to \infty} \left[ \frac{2N}{N-a, N-b} \right]_q = \frac{1}{(q)_\infty(q)^{a+b}} \quad a + b \geq 0, \quad (3.4)$$

yields

$$q^{\mu/2} \sum_{k=0}^{\infty} \frac{q^{2k(\mu+\mu)}}{(q)_{2k+\mu}} \frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} \left\{ q^{20j^2+j} - q^{20j^2+9j+1} \right\} = q^{1/48+7/240} \chi_1(4) \chi_1(5). \quad (3.5)$$

This final expression indeed has the expected factorized form as alluded to before, with the second term being the $\chi_1(5)$ character corresponding to a $c = 7/10$ conformal field theory.

We now turn to the fermionic LHS of (2.4). After replacing $q$ with $1/q$, applying (3.1) and multiplying by the factor $q^{-L/2}$, the fermionic sum takes the form

$$\sum_n q^{(n-L e_1/2)^T C_{\psi}^{-1}(n-L e_1/2)} \prod_{i=1}^{8} \left[ \frac{n_i + m_i}{n_i} \right] \quad (3.6)$$

where the sum is an unrestricted sum over the components of $n$ and we regard the components $m_i$ as given in terms of $n_i$ by the TBA system (2.2). We split the sum into two parts with the restrictions

$$n_2 + n_4 + n_8 = \mu, 1 - \mu \quad (mod 2) \quad (3.7)$$

where $\mu = 0, 1$ gives the parity of $L$. After making the shifts $n_1 \to L/2 - n_1 - \ell$ and $n_1 \to L/2 - n_1 - 1/2 - \ell$, respectively, where

$$\ell = (3n_2 + 4n_3 + 5n_4 + 6n_5 + 4n_6 + 2n_7 + 3n_8)/2 \quad (3.8)$$
the fermionic sum can be written as
\[
\sum_{n_{2+n_{4}+n_{g}}=\mu \pmod{2}} q^{(2n_{1})^{2}/2} \pi \cdot E_{E_{7}}^{-1} \pi \left[ \frac{L/2 + 3(2n_{1})/2 - \ell}{2(2n_{1})} \right] \prod_{i=2}^{8} \left[ \frac{n_{i} + \overline{m}_{i}}{n_{i}} \right]_{q}
\]
\[
+ \sum_{n_{2+n_{4}+n_{g}}=1-\mu \pmod{2}} q^{(2n_{1}+1)^{2}/2} \pi \cdot E_{E_{7}}^{-1} \pi \left[ \frac{L/2 + 3(2n_{1} + 1)/2 - \ell}{2(2n_{1} + 1)} \right] \prod_{i=2}^{8} \left[ \frac{n_{i} + \overline{m}_{i}}{n_{i}} \right]_{q}.
\]  (3.9)

Here \( \overline{m}_{i}, \overline{m}_{s} \) satisfy the TBA system
\[
\pi + \overline{m} = \frac{1}{2} (\mathcal{I}_{E_{7}} - 1) e_{2} + e_{i} \quad i = 2, \ldots, 8,
\]  (3.10)

where \( \pi_{1} = 2n_{1}, 2n_{1} + 1 \) is even or odd, respectively. Combining the two sums into one sum and replacing \( \pi_{1} \) with \( n_{1} \) now gives
\[
\sum_{n_{1+n_{2}+n_{4}+n_{g}}=\mu \pmod{2}} q^{n_{1}^{2}/2} \pi \cdot E_{E_{7}}^{-1} \pi \left[ \frac{L/2 + 3n_{1}/2 - \ell}{2n_{1}} \right] \prod_{i=2}^{8} \left[ \frac{n_{i} + \overline{m}_{i}}{n_{i}} \right]_{q}.
\]  (3.11)

Taking the limit \( L \to \infty \) and equating this to the bosonic RHS gives the identity
\[
\sum_{n_{1+n_{2}+n_{4}+n_{g}}=\mu \pmod{2}} q^{n_{1}^{2}/2} \pi \cdot E_{E_{7}}^{-1} \pi \prod_{i=2}^{8} \left[ \frac{n_{i} + \overline{m}_{i}}{n_{i}} \right]_{q} = q^{1/20} \chi_{1+\mu,1}(q) \chi_{1,1}(q).
\]  (3.12)

This identity clearly has an \( A_{1} \times E_{7} \) structure and is very similar to the \( E_{7} \) identity of Kedem et al. [9]. Indeed, these results suggest that the \( c = 1/2 \) character should explicitly factor out of the LHS of this identity, but we have been unable to do this.

4 Some counting formulas

In this last section we list some fermionic expressions for the number of \( L \)-step paths on the dilute \( A_{3} \) adjacency graph \( \mathcal{G}_{dA_{3}} \), shown in figure 1c. These results come about by realizing that in the critical \( q \to 1 \) limit, the function \( B_{r,s}(L, a, b) \) counts the number of paths from \( a \) to \( b \) of length \( L \) on \( \mathcal{G}_{dA_{3}} \) [3]. In other words, in this limit, the function \( B_{r,s}(L, a, b) = (\mathcal{I}_{dA_{3}})^{L}_{a,b} \), with \( \mathcal{I}_{dA_{3}} \) the incidence matrix corresponding to \( \mathcal{G}_{dA_{3}} \). (We remind the reader that \( \lim_{q \to 1} \left[ m_{1,m_{2},\ldots,m_{n}} \right]_{q} = \left( m_{1,m_{2},\ldots,m_{n}} \right) \), the RHS being an ordinary multinomial.)

Setting \( q \to 1 \) in some of our “fermionic sum = bosonic sum” identities (not all of which are listed in this paper) we find
\[
F_{1}(L)|_{q=1} = (\mathcal{I}_{dA_{3}})^{L}_{1,1} \quad F_{2}(L)|_{q=1} = (\mathcal{I}_{dA_{3}})^{L}_{2,2} \quad F_{7}(L)|_{q=1} = (\mathcal{I}_{dA_{3}})^{L}_{1,2} \quad F_{8}(L)|_{q=1} = (\mathcal{I}_{dA_{3}})^{L+1}_{1,3}.
\]  (4.1)

As \( \mathcal{I}_{dA_{3}} \) has only four distinct entries these results are complete.
5 Summary and discussion

In this letter we have shown directly that the dilute A₃ model in regime 2⁺ exhibits a hidden E₈ structure. This was achieved by rewriting the known bosonic expressions for the 1-dimensional configuration sums in terms of fermionic sums involving the E₈ root system. Our results confirm recent work of ref. [1] where an E₈ structure was found using the Bethe Ansatz approach together with an appropriate string hypothesis. As a byproduct of our work, we prove E₈ [9] and A₁ × E₇ type Rogers-Ramanujan identities.

To conclude, we point out that a similar program can be carried out for the dilute A₄ and A₆ models [2]. In doing so we find that these models in regime 2 exhibit E₇ and E₆ structures, respectively. This again confirms the earlier findings of ref. [13] that these two models correspond to the exceptional S-matrices of Zamolodchikov and Fateev [14]. We hope to report these results together with the complete proof of identity (2.6) in a future publication.

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