MEASURING SINGULARITY OF GENERALIZED MINIMIZERS FOR CONTROL-AFFINE PROBLEMS

MANUEL GUERRA\(^1\) AND ANDREY SARYCHEV\(^2\)

Abstract. An open question contributed by Yu. Orlov to a recently published volume "Unsolved Problems in Mathematical Systems and Control Theory", V.D. Blondel & A. Megretski (eds), Princeton Univ. Press, 2004, concerns regularization of optimal control-affine problems. These noncoercive problems in general admit 'cheap (generalized) controls' as minimizers; it has been questioned whether and under what conditions infima of the regularized problems converge to the infimum of the original problem. Starting with a study of this question we show by simple functional-theoretic reasoning that it admits, in general, positive answer. This answer does not depend on commutativity/noncommutativity of controlled vector fields. It depends instead on presence or absence of a Lavrentiev gap.

We set an alternative question of measuring "singularity" of minimizing sequences for control-affine optimal control problems by so-called degree of singularity. It is shown that, in the particular case of singular linear-quadratic problems, this degree is tightly related to the "order of singularity" of the problem. We formulate a similar question for nonlinear control-affine problem and establish partial results. Some conjectures and open questions are formulated.

Keywords: optimal control-affine problem, regularization, generalized control, singular linear-quadratic optimal control problem, order of singularity, Lavrentiev phenomenon

1. Introduction

The following open question, suggested by Yu. Orlov, appeared in a recently published volume by V. Blondel et. al. \[19\].

Consider an optimal control problem.

\[ J_T^0(u(\cdot)) = \int_0^T x(t)'Px(t)dt \to \min, \]

\[ \dot{x} = f(x) + G(x)u, \quad x(0) = x_0. \]

\( T \in [0, +\infty] \) is fixed, \( P \) denotes a symmetric definite positive matrix, \( f \) is a smooth vector field and \( G(x) = (g_1(x), g_2(x), ..., g_k(x)) \) is an array of smooth

---

\(^{1}\) The first author has been partially supported by Fundação para a Ciência e a Tecnologia (FCT), Portugal, co-financed by the European Community Fund FEDER/POCI via Research Center on Optimization and Control (CEOC) of the University of Aveiro, Portugal. The second author has been partially supported by MIUR, Italy via PRIN 2006019927.
vector fields. An endpoint condition
\[ x(T) = x_T \]
can be added when \( T < \infty \).

Consider a regularization of this problem, which amounts to minimization of the penalized functional
\[ J^T_\varepsilon(u(\cdot)) = \int_0^T x(t)'Px(t) + \varepsilon |u(t)|^2 dt \to \min, \]
calculated along the trajectories of (2). The question put by Yu. Orlov in [19] was whether and under what assumptions
\[ \lim_{\varepsilon \to 0^+} \min_u J^T_\varepsilon(u) = \inf_u J^T_0(u). \]
We show that the answer to this question is positive in almost all cases. Further, the result holds for every nonnegative penalization (not necessarily quadratic) that one may chose to regularize the functional (1).

We suggest an alternative question. In our opinion it is not the values of the infima which should be studied, but rather the asymptotics of the regularized functionals along minimizing sequences of \( J^T_0 \). Indeed, it is quite general phenomenon that, for generic data \( \lim_{m \to \infty} \|u^{(m)}\|_{L_2} = +\infty \) holds for any sequence \( \{u^{(m)} \in L_{\infty,loc}, m \in \mathbb{N}\} \) such that \( \lim_{m \to \infty} J^T_0(u^{(m)}) = \inf_u J^T_0(u) \).

We trust that the minimal rate of growth of a sequence \( \{\|u^{(m)}\|_{L_2}, m \in \mathbb{N}\} \) that can be achieved when \( J^T_0(u^{(m)}) \leq \inf_u J^T_0(u) + \frac{1}{m} \), is an important property characterizing the degree of singularity of problem (1)-(2) or (1)-(2)-(3).

For the particular case of singular linear-quadratic problems we are able to fully characterize all types of singularities that occur. For nonlinear control-affine case we provide partial answers.

This paper is organized as follows. In Section 2 we answer Yu. Orlov’s question in finite-horizon and infinite-horizon settings and introduce an extension of this question demonstrating its interrelation with Lavrentiev phenomenon in calculus of variations and optimal control. In Section 3 we introduce the notion of "degree of singularity" (Definition 3.1) and set the problem of 'measuring singularity' of generalized minimizers. In Section 4 we give a full characterization of possible values of the degree of singularity for singular linear-quadratic problem in finite-horizon and infinite-horizon settings (Theorems 4.1, 4.2, 4.3). In Section 5 we introduce the case of non-linear control-affine problems (1)-(2)-(3). The general driftless case is solved in Section 6 (Theorem 6.1). In Section 7 we provide an upper estimate of the degree of singularity for the case, where the cost is positive state-quadratic and controlled vector fields \( g_i \) commute: \( [g_i, g_j] = 0, \forall i, j \) (Theorem 7.1). In Section 8 we provide some evidence for existence of a better estimate for the commutative case and illustrate by example. The proofs of several results discussed in Sections 4-7 are quite technical and are collected in Section 9.
A brief exposition of part of these results has appeared in [10].

Partial results for generic control-affine problems (1)-(2)-(3) with non-commuting inputs will be the object of a separate publication.

We are grateful to an anonymous referee who brought to our attention some additional bibliographic references and whose stimulating questions and remarks allowed us to (hopefully) improve the presentation especially in what regards Section 2.

2. Convergence of regularized functionals and Lavrentiev phenomenon

In this Section we answer Yu. Orlov’s question in a slightly more general setting. Namely, we will consider in place of (4), the functional penalized by $\varepsilon \rho(t,u(t))$

$$J_T^\varepsilon (u(\cdot)) = \int_0^T x(t)'Px(t) + \varepsilon \rho(t,u(t)) dt,$$

where $\rho : [0,T] \times \mathbb{R}^k \mapsto [0, +\infty]$ is a nonnegative Borel function. We denote by $U_\rho$ the set of admissible controls for the problem (5)-(2) (5)-(2)-(3):

$$U_\rho = \left\{ u : [0,T] \mapsto \mathbb{R}^k | u \text{ is measurable, } \int_0^T \rho(t,u(t)) dt < +\infty \right\},$$

provided we set $J_T^\varepsilon(u) = +\infty$ for any $u \in U_\rho$ for which (2) does not admit solution in the interval $[0,T]$. It is clear that $U_\rho = L^\infty_p[0,T]$, whenever $\rho(t,u) = |u|^p$ in (6) with $p \in [1, +\infty[$, as it is often the case.

Basically, a positive answer to Yu. Orlov’s question is contained in the following result.

**Theorem 2.1.** Let $U_\rho$ be a class of admissible controls defined by (6), $J_0^T(u)$, $J_T^\varepsilon(u)$ be the original cost functional (1) and the regularized cost functional (5), respectively. Then

$$\lim_{\varepsilon \to 0^+} \inf_{u \in U_\rho} J_T^\varepsilon(u) = \inf_{u \in U_\rho} J_T^0(u). \quad \square$$

**Proof.** Fix $u^{(m)} \in U_\rho$, a minimizing sequence for $J_0^T$. Without loss of generality one may think that $J_T^\varepsilon(u^{(m)}) \leq \inf J_T^0 + 1/m$. Let $\int_0^T \rho(t,u(t)) dt = \nu_m$. Then,

$$\inf J_0^T \leq \inf J_T^\varepsilon \leq J_T^\varepsilon(u^{(m)}) = J_T^0(u^{(m)}) + \varepsilon \nu_m \leq \inf J_0^T + 1/m + \varepsilon \nu_m.$$

Taking $\varepsilon_m = \nu_m^{-1}/m$ we conclude that

$$\inf J_0^T \leq \inf J_{\varepsilon_m}^T \leq J_{\varepsilon_m}^T(u^{(m)}) \leq \inf J_0^T + 2/m$$

and hence (7) holds. \quad \square

**Theorem 2.1** has the following immediate Corollary:
Corollary 2.2. If \( u_\varepsilon(\cdot) \in U_\rho \) are minimizers of the regularized problems (5)-(2) or (5)-(2)-(3), then
\[
\lim_{\varepsilon \to 0^+} J_0^T(u_\varepsilon) = \inf_{u \in U_\rho} J_0^T(u). \quad \square
\]

Note that the main point of regularizing a singular problem (1)-(2) or (1)-(2)-(3) is to obtain a similar problem possessing a minimizer in a suitable space of regular controls. Existence results for optimal control problems with control-affine dynamics typically require superlinear growth of the integrand in the cost functional, as \(|u| \to \infty\) (see e.g. [4]), i.e. \(\rho(t,|u|)/|u| \to \infty\) as \(|u| \to \infty\), uniformly with respect to \(t \in [0,T]\). Therefore, a penalty of type \(\rho(t,u) = |u|^{1+\eta} \) (\(\eta > 0\), constant) typically guarantees existence of solution for the regularized problem, while a penalty of type \(\rho(t,u) = |u|\) may fail to do it. It is also natural to assume that all the classes \(U_\rho\) of controls (see (6)) are contained in \(L^k_1[0,T]\); otherwise it is hard to verify existence of trajectories of the control-affine system (2).

An interesting extension of the original question formulated in [19] would be to admit not only the possibility of regularizing the functional but also of 'regularizing' its domain of definition.

This would mean introducing two classes of controls \(U \supset U_\rho\), considering the functional (1) on \(U\) while considering the regularized functionals (5) in a smaller class of 'more regular' controls \(U_\rho\). Here \(U\) can be any suitable class of controls, not necessarily defined by an equality of type (6).

Our extended question would be whether
\[
\lim_{\varepsilon \to 0^+} \inf_{u \in U_\rho} J_\varepsilon^T(u) = \inf_{u \in U} J_0^T(u) ?
\]

This question turns out to be tightly related (and in fact equivalent) to another prominent issue of the calculus of variations and optimal control - the Lavrentiev phenomenon (see [4] for a brief account and historical remarks).

Recall that a functional \(J_0^T(u)\) defined on a class \(U \supset U_\rho\) exhibits Lavrentiev phenomenon or possesses \(\mathcal{U} - U_\rho\) Lavrentiev gap if
\[
\inf_{u \in U} J_0^T(u) < \inf_{u \in U_\rho} J_0^T(u) .
\]

The following elementary result shows that validity of (8) is equivalent to nonoccurence of Lavrentiev phenomenon for \(J_0^T\).

**Theorem 2.3.** Equality (8) holds if and only if \(J_0^T\) does not possess \(\mathcal{U} - U_\rho\) Lavrentiev gap. \(\square\)

**Proof.** Whenever we have equality in place of strict inequality in (9) there exists a minimizing sequence \(u^{(m)} \in U_\rho\) such that \(\lim_{m \to \infty} J_0^T(u^{(m)}) = \inf_{u \in U} J_0^T(u)\).

Now (8) is concluded in the same way as (7) has been concluded in the proof of the Theorem 2.1.

Note that by Theorem 2.1 \(\lim_{\varepsilon \to 0^+} \inf_{u \in U_\rho} J_\varepsilon^T(u) = \inf_{u \in U_\rho} J_0^T(u)\). By direct computation \(\lim_{\varepsilon \to 0^+} \inf_{u \in U_\rho} J_\varepsilon^T(u) \geq \inf_{u \in U} J_0^T(u)\). Whenever the last inequality is
strict we immediately conclude the presence of Lavrentiev gap \( \inf_{u \in \mathcal{U}_p} J^T_0(u) > \inf_{u \in \mathcal{U}} J^T_0(u) \). □

Thus we have completely reduced the validity of the equality [8] above to the nonoccurrence of \( \mathcal{U} - \mathcal{U}_p \) Lavrentiev gap for the optimal control problem (1)-(2) or (1)-(2)-(3).

Lavrentiev phenomenon has been mainly studied for the classical problem of Calculus of Variations, Some partial results regarding occurrence of this phenomenon for optimal control problems are known; see [5, 21] where there are examples of Lavrentiev phenomenon occurring for variational problems with higher-order derivatives; these problems can be interpreted as Lagrange problems with linear dynamics.

The Lavrentiev phenomenon is seen more as a rarity; the above cited results certainly involve more sophisticated cost functionals than the quadratic functional (1), though the dynamics involved are linear autonomous in contrast to (2).

In the case of finite horizon \( T < +\infty \), nonoccurrence of \( L^k_1[0,T] - L^k_\infty[0,T] \) Lavrentiev gap in (1)-(2) or (1)-(2)-(3) can be easily proved. To see this, consider a minimizing sequence \( \{u^{(m)} \in L^k_1[0,T], m \in \mathbb{N}\} \) of the functional \( J^T_0(u) \). Recall that the input/trajectory mapping \( u(\cdot) \mapsto x_u(\cdot) \) is continuous (on some \( L^k_1[0,T] \)-neighborhood of each \( u^{(m)} \)) with respect to \( L^k_1[0,T] \) metric of \( u \)'s and \( L^\infty_\infty[0,T] \)-metric of \( x_u \)'s. Then the map \( u \mapsto J^T_0(u) = \int_0^T x_u' P x_u dt \) is continuous. As we know the functions from \( L^k_1[0,T] \) are approximable in \( L_1 \)-metric by functions from \( L^\infty_\infty[0,T] \). Hence, taking proper approximations of the functions \( \{u^{(m)}\} \) we can construct for \( J^T_0(u) \) a minimizing sequence \( \{\tilde{u}^{(m)}\} \) of functions from \( L^k_\infty[0,T] \). Therefore

\[
\inf_{u \in L^k_\infty[0,T]} J^T_0(u) = \inf_{u \in L^k_1[0,T]} J^T_0(u),
\]

which implies equality in [8] according to Theorem 2.3. Thus we proved the following

**Theorem 2.4.** Consider the problem (1)-(2) (1)-(2)-(3) with finite horizon \( T < +\infty \). Equality [8] holds for any classes of controls \( L^k_1[0,T] \supset \mathcal{U} \supset \mathcal{U}_p \supset L^\infty_\infty[0,T] \). □

We are not aware of any results on occurrence/nonoccurrence of Lavrentiev phenomenon for infinite horizon. We provide below conditions which can be imposed on the control system (2) in order to guarantee the lack of Lavrentiev gap for the problem (1)-(2) with \( T = +\infty \) and validity of equality [8] for a pair \( L^k_{1,loc}, L^k_p[0, +\infty[ \).

**Definition 2.1.** The control affine system (2) is said to be locally stabilizable of order \( \alpha \) if there exists a Lipschitzian feedback \( \tilde{u}(x) \), and a constant \( C < +\infty \) such that \( \tilde{u}(0) = 0 \) and \( |x(t; x_0)| \leq C|x_0|(t + 1)^{-\alpha} \) holds for every \( x_0 \) in some neighborhood of the origin. Here \( x(t; x_0) \) is the trajectory of the ODE

\[
\dot{x} = f(x) + G(x)\tilde{u}(x), \quad x(0) = x_0.
\]
Theorem 2.5. Assume the horizon to be infinite: $T = +\infty$. If the system (2) is locally stabilizable of order $\alpha > \frac{1}{p}$, with $p \in [1, 2]$, then $J_0^\infty$ does not have $L_{1,\text{loc}} - (L_p \cap L_\infty)$ Lavrentiev gap, i.e.

$$\inf_{u \in L^k_0[0, +\infty] \cap L^k_\infty[0, +\infty]} J_0^\infty(u) = \inf_{u \in L^k_{1,\text{loc}}} J_0^\infty(u).$$

For $p \in [2, +\infty]$ the equality holds provided $\alpha > \frac{1}{2}$. \(\square\)

Proof. In the case $\inf_{u \in L^k_{1,\text{loc}}} J_0^\infty(u) = +\infty$, the theorem holds trivially. Hence we only need to consider the case when $\inf_{u \in L^k_{1,\text{loc}}} J_0^\infty(u) < +\infty$.

Fix $\varepsilon > 0$. For each $u \in L^k_{1,\text{loc}}$ let $x_u$ denote the corresponding trajectory of system (2). There exists $\tilde{u} \in L^k_{1,\text{loc}}$ such that $\int_0^\infty x_\tilde{u}(t)'P x_\tilde{u}(t)dt < \inf_{u \in L^k_{1,\text{loc}}} J_0^\infty + \varepsilon < +\infty$. Then $\lim_{t \to +\infty} \int_t^{+\infty} x_\tilde{u}(\tau)'P x_\tilde{u}(\tau)d\tau = 0$. Since $P$ is positive there must exist a sequence $\{t_j\} \to +\infty$ for which $x_\tilde{u}(t_j) \to 0$. Since $u \mapsto x_u(\cdot)$ is a continuous mapping from $L^k_1[0, t_j]$ into $L^k_\infty[0, t_j]$, it follows by density of $L^k_\infty[0, t_j]$ that there exist controls $u_j \in L^k_\infty[0, t_j]$ such that

$$\int_0^{t_j} x_{u_j}(t)'P x_{u_j}(t)dt \leq \int_0^{t_j} x_\tilde{u}(t)'P x_\tilde{u}(t)dt + \varepsilon \leq \inf_{u \in L^k_{1,\text{loc}}} J_0^\infty(u) + 2\varepsilon;$$

$$x_{u_j}(t_j) \to 0.$$

Concatenate the trajectory $x_{u_j}(\cdot)$ with the trajectory $y_j(t)$ starting at $x_{u_j}(t_j)$ and driven by the feedback control $\tilde{u}$ from Definition 2.1. For every sufficiently large $j \in \mathbb{N}$ there holds

$$|y_j(t)'P y_j(t)| \leq C_1|x_{u_j}(t_j)|^2(1 + t - t_j)^{-2\alpha}, \quad \forall t \geq t_j.$$

This implies

$$\int_{t_j}^{+\infty} y_j(t)'P y_j(t)dt \leq \frac{C_1}{2\alpha - 1} |x_{u_j}(t_j)|^2.$$

Since the feedback control $\tilde{u}(x)$ is Lipschitzian and $\tilde{u}(0) = 0$, then $|\tilde{u}(y_j(t))| \leq C_2|y_j(t)| \leq C_3|x_{u_j}(t_j)|(1 + t - t_j)^{-\alpha}$, and hence for some $M < +\infty$, $\int_{t_j}^{+\infty} |\tilde{u}(y_j(t))|^p dt < M$, holds for all sufficiently large $j \in \mathbb{N}$. This proves that the control,

$$\hat{u}_j(t) = \begin{cases} u_j(t) & \text{if } t \leq t_j; \\
\tilde{u}(y_j(t)) & \text{if } t > t_j,
\end{cases}$$

is of class $L^k_p[0, +\infty] \cap L^k_\infty[0, +\infty]$. Evaluating the functional $J_0$ along the corresponding concatenated trajectory $x_{\hat{u}_j}(\cdot)$ we conclude that

$$J_0(\hat{u}_j) \leq \inf_{u \in L^k_{1,\text{loc}}} J_0 + 2\varepsilon + \frac{C_1}{2\alpha - 1} |x_{u_j}(t_j)|^2.$$

Choosing \( j \) sufficiently large we thus construct a control \( u_\varepsilon(\cdot) \in L^k_p[0, +\infty] \) for which
\[
J_0(u_\varepsilon) \leq \inf_{u \in L^k_p[0, +\infty]} J_0 + 3\varepsilon.
\]
Taking \( \varepsilon \to 0 \) we arrive to a minimizing sequence of \( p \)-integrable controls. The proof is completed by application of Theorem 2.1. □

The following corollary follows immediately from Theorems 2.3 and 2.5:

**Corollary 2.6.** Assume the horizon to be infinite: \( T = +\infty \). If the system (2) is locally stabilizable of order \( \alpha > \frac{1}{p} \), with \( p \in [1, 2] \), then
\[
\lim_{\varepsilon \to 0^+} \inf_{u \in L^k_p[0, +\infty]} \left( J_0^\infty(u) + \varepsilon \|u\|_{L^k_p[0, +\infty]}^p \right) = \inf_{u \in L^k_p[0, +\infty]} J_0^\infty(u).
\]
For \( p \in [2, +\infty[ \) the equality holds provided \( \alpha > \frac{1}{2} \). □

Note that the convergence issue for regularized functionals is settled by elementary functional-theoretic arguments and the answer does not depend on commutativity assumptions for the controlled vector fields and other issues typically involved in the study of generalized controls. We would like to formulate now a different problem related to the system (2), which will be central point of our contribution.

3. Degree of singularity. Problem setting

In what follows we consider our optimal control problem with finite or infinite horizon.

Due to lack of coercivity, "classical" minimizers for (1)-(2)-(3) do not, in general, exist. It is known that for generic boundary conditions, minimizing sequences of classical controls usually converge to some 'generalized controls' which may contain impulses or more complex singularities. For such problems quasioptimal (\( \varepsilon \)-minimizing) controls \( u^\varepsilon \) are known to exhibit high-gain highly-oscillatory behavior. It is expected \( \lim_{\varepsilon \to 0^+} \|u^\varepsilon\|_{L^2} = +\infty \) to hold for any minimizing sequence \( \{u^\varepsilon\} \). Still the asymptotics of growth of \( \|u^\varepsilon\|_{L^2} \) varies from problem to problem and therefore this asymptotics can be used for measuring the degree of singularity of the problem. This is also a problem of practical importance, because suboptimal controls are harder to realize in practice when "good" approximations of \( \inf J_0^T \) require 'too high' gain and 'too fast' oscillation.

In order to address this question, we introduce the following measure of "singular behavior" of a problem (1)-(2)-(3).

**Definition 3.1.** In the finite horizon case the degree of singularity of the problem (1)-(2)-(3) is
\[
\sigma_T = \limsup_{\varepsilon \to 0^+} \frac{\inf \left\{ \ln \|u\|_{L^2} : J_0^T(u) \leq \inf J_0^T + \varepsilon, |x_u(T) - x_T| < \varepsilon \right\}}{\ln \frac{1}{\varepsilon}}.
\]
In the infinite horizon case the degree of singularity of the problem (1)-(2) is

\[ \sigma_\infty = \limsup_{\varepsilon \to 0^+} \frac{\inf \left\{ \ln \|u\|_{L_2} : J_0^\infty(u) \leq \inf J_0^\infty + \varepsilon \right\}}{\ln \frac{1}{\varepsilon}}. \]

Our main goal from now on will be computation of degree of singularity for various optimal control-affine problems. In the next Section we provide a complete analysis for singular linear quadratic problems.

4. SINGULAR LINEAR-QUADRATIC CASE

In this section we discuss the relationship between the degree of singularity \( \sigma_T \) and the structure of generalized minimizers in the singular linear-quadratic case. We believe this relation provides a compelling evidence for the usefulness of degree of singularity for measuring singular behavior of minimizing sequences.

In [9], a definition of order of singularity for LQ problem has been introduced and it was shown that singular linear-quadratic problems can be classified according to it. This order of singularity is an integer \( r \leq n \), \( n \) being the dimension of the state space. If \( \inf_u J_0^T(u) > -\infty \), then a problem of order \( r \) admits a generalized minimizer in the Sobolev space \( H_{-r} \).

We will show that the degree of singularity \( \sigma_T \) from Definition 3.1 is tightly related to the order of singularity of a problem. For (singular) LQ problems with state-quadratic integrand \( x'Px \) and \( P > 0 \) it is shown that \( \sigma_T = \frac{1}{2} \), while order of singularity equals 1. For an LQ problem, with more general functional (11), order of singularity \( r \) and generic boundary data, \( \sigma_T = r - \frac{1}{2} \). When nongeneric boundary conditions are imposed, one can show that \( \sigma_T \) admits values from a finite set. These values correspond to a stratification of the space of boundary data, which is related to results in [13], [14], [22] and [9].

The content of Subsections 4.1 to 4.4 below is essentially a brief sketch of the results contained in [9], which are essential for the computation of \( \sigma_T \). Subsection 4.5 contains an important technical result (Proposition 4.10) regarding approximation of distributions from Sobolev space \( H_{-r} \). This result is applied in Subsection 4.6 to computation of the values of degree of singularity \( \sigma_T \).

4.1. Assumptions. Along this Section the controlled dynamics (2) is linear time-invariant:

\[ \dot{x} = Ax + Bu, \quad x(0) = x_0. \]

The end-point condition is (3). The cost functional, we consider, will be more general than (1):

\[ J_0^T(u) = \int_0^T x'_u P x_u + 2u'Qx_u + u'R u \, d\tau, \]

(10)
where $R$ is a symmetric nonnegative matrix. If $R$ is positive definite, then there exists analytic minimizing control for this problem (at least for sufficiently small $T$) and hence the degree of singularity $\sigma_T = 0$. Therefore the case of interest is the singular one, where $R$ possesses a nontrivial kernel.

We assume the following to hold.

**Assumption 4.1.** Let the matrices $A, B, P, Q, R$ in (10), (11) be such that for $x_0 = x_T = 0$ and each $T \in [0, +\infty]$, there exists a subspace $S^+_T$, of finite codimension in $L_2^k[0,T]$, such that $J^T_0 > 0$ on $S^+_T \setminus \{0\}$ (the subspace $S^+_T$ and its codimension may depend on $T$).

Assumption 4.1 may look not very natural, but as we will now explain, it is closely related to finiteness of $\inf_u J^T_0$.

If there exist $T \in [0, +\infty]$ and an infinite-dimensional subspace, $S^- \subset L_2^k[0,T]$ such that $J^T_0(u) < 0$ in $S^- \setminus \{0\}$, then one can prove that $\inf_u J^T_0(u) = -\infty$ holds for every $T > 0$ and any boundary conditions. Thus finiteness of $\inf_u J^T_0$ requires that for each $T \in [0, +\infty]$ there exist some subspace of finite codimension in $L_2^k[0,T]$ where $J^T_0$ is non-negative. In this case, the only way in which Assumption 4.1 can fail is when the quadratic form $u \mapsto J^T_0(u)$ has infinite-dimensional kernel. In [9] it is shown how this kernel can be "factored out". Naturally, Assumption 4.1 will hold after such a factorization.

Resuming, we may think of Assumption 4.1 as of a version of the more intuitive

**Assumption 4.2.** $\inf_u J^T_0(u) > -\infty$ holds for each boundary data $(x_0, x_T)$ (for each initial data $x_0$, when $T = +\infty$).

Assumptions 4.1 and 4.2 are closely related but not equivalent. Using the first one is more convenient from the technical viewpoint. A complete study of problems (10)-(3)-(11) which satisfy $\inf_u J^T_0(u) > -\infty$ can be found in [9]. Therein it is shown how Assumption 4.1 can be checked using only linear algebra computations.

Note that, in the finite-horizon case neither $P$ nor the quadratic form $(x, u) \mapsto x'Px + 2u'Qx + u'Ru$ need to be nonnegative for $\inf_u J^T_0(u) > -\infty$ to hold. In the infinite-horizon case we will require this latter nonnegativity.

4.2. Desingularization of LQ problems. Provided Assumption 4.1 holds, a singular linear-quadratic problem (10)-(3)-(11) can be reduced to a regular problem, i.e. to an LQ problem with quadratic cost, which is strictly convex with respect to control. This is done by the following multistep procedure (for a detailed account of a more general procedure without Assumption 4.1 see [9]).

Let $\phi : L_{2,\text{loc}} \mapsto L_{2,\text{loc}}$ be the primitivization:

$$\phi u(t) = \int_0^t u(\tau) \, d\tau, \quad \forall u \in L_{2,\text{loc}}.$$
By choosing a suitable coordinate system in the space of control variables, we may assume without loss of generality that the nonnegative matrix $R$ is of the form $R = \left( \begin{array}{cc} R_{0,0} & 0 \\ 0 & 0 \end{array} \right)$, where $R_{0,0} \in \mathbb{R}^{k_0 \times k_0}$, $R_{0,0} > 0$. We consider the corresponding splitting of the vectors $u = (u_0, u_1) \in \mathbb{R}^k$, $u_0 \in \mathbb{R}^{k_0}$ and of the matrices

\[ B = (B_{0,0}, B_{0,1}) \in \mathbb{R}^{n \times k}, \ B_{0,0} \in \mathbb{R}^{n \times k_0}, \ Q = \left( \begin{array}{c} Q_{0,0} \\ Q_{0,1} \end{array} \right) \in \mathbb{R}^{k \times n}, \ Q_{0,0} \in \mathbb{R}^{k_0 \times n}. \]

Let us introduce the operator $\gamma : L_{2,loc}^k \mapsto L_{2,loc}^k$

\[ \gamma u = \left( u_0 + R_{0,0}^{-1} (Q_{0,0} B_{0,1} - B_{0,0}^0 Q_{0,1}') \phi u_1, \ \phi u_1 \right). \]

The following Proposition represents the trajectory $x_{x_0,\gamma u}$ and the value of the functional $J_0(u)$ via solution of an LQ problem, which is 'less singular'. Due to it the representation below is called desingularization procedure.

**Proposition 4.1** \((9)\). For every $x_0 \in \mathbb{R}^n$, $u \in L_2^k [0, T]$, there holds:

\[ x_{x_0,u} (t) = x_{x_0,\gamma u} (t) + B_{0,1} \phi u_1 (t), \]

\[ J_{x_0} (u) = \int_0^T x_{x_0,\gamma u}^T P x_{x_0,\gamma u} + 2 (\gamma u)' Q_1 x_{x_0,\gamma u} + (\gamma u)' R_1 \gamma u \, d\tau + \]

\[ + \int_0^T u_1' (Q_{0,1} B_{0,1} - B_{0,1}^0 Q_{0,1}') \phi u_1 \, d\tau + \]

\[ + 2 \phi u_1 (T)' Q_{0,1} x_{x_0,\gamma u} (T) + \phi u_1 (T)' Q_{0,1} B_{0,1} \phi u_1 (T), \]

where $x_{x_0,v}^1$ denotes the trajectory of the system

\[ \dot{x} = Ax + B_1 v, \quad x(0) = x_0, \]

\[ B_1 = (B_{0,0}, B_{1,1}); \]

\[ B_{1,1} = \left( A - B_{0,0} R_{0,0}^{-1} Q_{0,0} \right) B_{0,1} + B_{0,0} R_{0,0}^{-1} B_{0,1}^0 Q_{0,1}'; \]

\[ Q_1 = \left( \begin{array}{c} Q_{0,0} \\ Q_{1,1} \end{array} \right); \]

\[ Q_{1,1} = B_{0,1}' \left( P - Q_{0,0} R_{0,0}^{-1} Q_{0,0} \right) - Q_{0,1} \left( A - B_{0,0} R_{0,0}^{-1} Q_{0,0} \right); \]

\[ R_1 = \left( \begin{array}{cc} R_{0,0} & 0 \\ 0 & \tilde{R}_1 \end{array} \right); \quad \tilde{R}_1 = Q_{1,1} B_{0,1} - B_{1,1} Q_{0,1}'. \]

For Assumption \((4.4)\) to hold, there must be

\[ Q_{0,1} B_{0,1} - B_{0,0}' Q_{0,1} = 0 \]

and $R_1 \geq 0$. □
If Assumption 4.1 holds for the 5-ple \((A, B, P, Q, R)\), then it also will hold for the 5-ple \((A, B_1, P, Q_1, R_1)\), which corresponds to the desingularized problem. The quadratic form \((x, u) \mapsto x'Px + 2u'Qx + u'Ru\) is nonnegative if and only if the quadratic form \((x, u) \mapsto x'P_1x + 2u'Q_1x + u'R_1u\) is. If \(R_1\) has a nontrivial kernel, then we can repeat the procedure obtaining a sequence \((A, B_i, P, Q_i, R_i)\), \(i = 1, 2, \ldots\). The following Proposition states that this sequence must be finite.

**Proposition 4.2.** If Assumption 4.1 holds, then there exists an integer \(r \leq n\) such that \(R_r > 0\).

In [9] the integer \(r\) is called **order of singularity** of the problem (10)-(11).

Without loss of generality, we may assume that the coordinates of the space of control variables are such that the matrices \((B_i, Q_i, R_i)\), obtained at each desingularization step, have block structure

\[
R_i = \text{diag} (R_{0,0}, R_{1,1}, \ldots, R_{i,i}, 0),
\]
\[
B_i = (B_{0,0} B_{1,1} \cdots B_{i,i} B_{i+1} \cdots B_{i,r}),
\]
\[
Q_i' = (Q'_{0,0} Q'_{1,1} \cdots Q'_{i,i} Q'_{i+1} \cdots Q'_{i,r}).
\]

Let us introduce operator \(\gamma_r = (\gamma_{r,0}, \gamma_{r,1}, \ldots, \gamma_{r,r}) : L^k_{2,\text{loc}} \mapsto L^k_{2,\text{loc}}\) as follows

\[
\gamma_{r,i}u = \phi^i u_i + R_{i,i}^{-1} \sum_{i \leq j < l \leq r} \left( Q_{i,j} B_{j,l} - B'_{i,j} Q'_{j,l} \right) \phi^{j+1} u_l,
\]
\[
\gamma_{r,r}u = \phi^r u_r.
\]

This operator is used in the next Subsection to introduce a suitable topology in the space of generalized controls.

Applying Proposition 4.1 consequently \(r\) times, we arrive to the following corollary.

**Proposition 4.3.** The trajectory \(x_{x_0, u}\) can be represented as

\[
x_{x_0, u} = x_{x_0, \gamma_r u}^r + \sum_{0 \leq i < j \leq r} B_{i,j} \phi^{i+1} u_j,
\]

where \(x_{x_0, \gamma_r u}^r\) denotes the trajectory of the system

\[
\dot{x} = Ax + B_r \gamma_r u, \quad x(0) = x_0.
\]
If Assumption 4.1 holds and $T < +\infty$, then the functional $J_0^T$ can be represented as

$$J_0^T(u) = \int_0^T x_{\gamma u}' P x_{\gamma u} + 2\gamma u'Q_r x_{\gamma u} + \gamma u' R_r \gamma u \, dt + 2 \sum_{0 \leq i < j \leq r} \phi^{i+1} u_j(T)' Q_{i,j} x_{\gamma u}(T) + \left( \sum_{0 \leq i < j \leq r} \phi^{i+1} u_j(T)' Q_{i,j} \right) \left( \sum_{0 \leq i < j \leq r} B_{i,j} \phi^{i+1} u_j(T) \right),$$

where $R_r > 0$. □

It turns out that the nonintegral terms in the latter representation of $J_0^T$ depend only on $T$, $x_0$ and $x_T$, and hence

$$J_0^T(u) = \int_0^T x_{\gamma u}' P x_{\gamma u} + 2\gamma u'Q_r x_{\gamma u} + \gamma u' R_r \gamma u \, dt + C^T_r (x_0, x_T),$$

where $C^T_r (x_0, x_T)$ is quadratic with respect to $(x_0, x_T)$.

Remark 4.1. Recall that in the infinite-time horizon version of the problem we assume nonnegativeness of the quadratic form $(x, u) \mapsto x'Px + 2u'Qx + u'Ru$. In this case the nonintegral terms vanish and the functional takes form

$$J_0^\infty(u) = \int_0^\infty x_{\gamma u}' P x_{\gamma u} + 2\gamma u'Q_r x_{\gamma u} + \gamma u' R_r \gamma u \, dt.$$ 

4.3. Weak norms, generalized controls and trajectories. Consider the following norms in the space of inputs $L^2_k[0, T]$ and in the space of trajectories $L^2_n[0, T]$ (we embed absolutely continuous trajectories in $L^2_n$):

$$\|u\|_{\gamma, [0, T]} = \|\gamma u\|_{L^2_k[0, T]}, \quad u \in L^2_k[0, T];$$

$$\|u\|_{\tau, [0, T]} = \|\gamma u\|_{L^2_k[0, T]} + \sum_{1 \leq i \leq j \leq r} \|\phi^i u_j(T)\|, \quad u \in L^2_k[0, T];$$

$$\|x\|_{H^n_k, [0, T]} = \|\phi^n x\|_{L^2_n[0, T]}, \quad x \in L^2_n[0, T].$$

Fix $T \in ]0, +\infty[$ and let

- $U_{\gamma, [0, T]}$ be the topological completion of $L^2_k[0, T]$ with respect to $\|\cdot\|_{\gamma, [0, T]}$;
- $U_{\tau, [0, T]}$ be the topological completion of $L^2_k[0, T]$ with respect to $\|\cdot\|_{\tau, [0, T]}$;
- $H^n_{\gamma, [0, T]}$ be the topological completion of $L^2_n[0, T]$ with respect to $\|\cdot\|_{H^n_{\gamma, [0, T]}}$.

The following holds true (see [21])
Proposition 4.4. The input-to-trajectory map \( u \mapsto x_{x_0,u} \) is uniformly continuous with respect to the norm \( \| \cdot \|_{\gamma_r[0,T]} \) of inputs and the norm \( \| \cdot \|_{H^a_{-(r-1)}[0,T]} \) of trajectories. The functional \( J^T_r(u) \) is locally uniformly continuous in the norm \( \| \cdot \|_{\gamma_r[0,T]} \) of inputs. □

As a corollary we obtain.

Proposition 4.5. The input-to-trajectory map \( u \mapsto x_{x_0,u} \) admits a unique continuous extension with domain \( \mathcal{U}_{\gamma_r[0,T]} \) and range in \( H^a_{-(r-1)}[0,T] \), while the functional \( J^T_r(u) \) admits a unique continuous extension onto \( \mathcal{U}_{\gamma_r[0,T]} \). These extensions can be defined by equalities (14) and (13), respectively. □

We denote \( \mathcal{U}_{\gamma_r[0,+,\infty]} \) the topological completion of \( L^{k}_{2,loc} \) with respect to convergence in all the norms \( \| \cdot \|_{\gamma_r[0,T]} \), (with \( r \) fixed like in Proposition 4.4) and \( T \) ranging in \( [0, +\infty[ \). Similarly, \( H^a_{\gamma_r[0,+,\infty]} \) is the topological completion of \( L^{k}_{2,loc} \) with respect to convergence in all the norms \( \| \cdot \|_{H^a_{-r}([0,T]} \) \( (T < +\infty, r \) fixed).

Proposition 4.6. The map \( u \mapsto x_{x_0,u} \) admits a unique continuous extension onto \( \mathcal{U}_{\gamma_r[0,+,\infty]} \). If the quadratic form \( (x,u) \mapsto x'Px + 2u'Qx + u'R u \) is nonnegative, then the map \( u \mapsto J^+_{0,\infty}(u) \) admits a unique extension onto \( \mathcal{U}_{\gamma_r[0,+,\infty]} \). □

Note that any function \( v \in L^k_{2,loc} \) defines uniquely a generalized control \( u \in \mathcal{U}_{\gamma_r[0,T]} \) such that \( \gamma_r u = v \) and \( \phi_i u_j(T) = V_{i,j}, 1 \leq i \leq j \leq r \) (\( V_{i,j} \) fixed constant vectors of suitable dimensions); the metric in the space \( u \in \mathcal{U}_{\gamma_r[0,T]} \) is induced by \( L^2 \)-metric in the space of \( v = \gamma_r u \).

4.4. Desingularized LQ problems and generalized solutions. According to Proposition 4.3 the problem (10)-(3)-(11) can be transformed into the following regular LQ problem with the control \( v = \gamma_r u \):

\[
J^T_{red}(v) = \int_0^T x'Px + 2v'Qx + v'Ru v dt \to \min,
\]

\[
\dot{x} = Ax + Bu, \quad v \in L^k_{2}[0,T], \quad x(0) = x_0,
\]

\[
x(T) \in x_T + \text{span}\{B_{i,j}, 0 \leq i < j \leq r\},
\]

with the endpoint condition (19) being dropped in case \( T = +\infty \). Since \( R_r > 0 \), classical existence theory applies to (17)-(18)-(19).

Definition 4.1. A functional \( \theta \) defined in a normed space \( (X, \| \cdot \|) \) is said to be quadratically coercive if there are constants \( a \in \mathbb{R}, b > 0 \) such that

\[
\theta(\xi) \geq a + b\|\xi\|^2, \quad \forall \xi \in X. \quad □
\]

Due to the relationship \( v = \gamma_r u \), between the new and the original controls, the following results hold true for (10)-(3)-(11).

Proposition 4.7. Let Assumption 4.1 hold. Then:
i) for each sufficiently small $T > 0$, the functional $J^T_0(u)$ is quadratically coercive on \( \{ u \in \mathcal{U}_{\gamma_r}[0,T] : \phi^i u_j(T) = 0, \ 1 \leq i \leq j \leq r \} \);

ii) for any $T > 0$ the functional $J^T_0(u)$ is quadratically coercive on a subspace of finite codimension in $\mathcal{U}_{\gamma_r}[0,T]$;

iii) If $T = +\infty$, and the quadratic form \( u \mapsto x'Px + 2u'Qx + u'Ru \) is nonnegative, then the functional $J^\infty_0(u)$ is quadratically coercive on $\mathcal{U}_{\gamma_r}[0,+\infty]$. \( \Box \)

Using Proposition 4.7 classical existence theory and the relationship $v = \gamma_r u$ we obtain the following:

**Proposition 4.8.** Let Assumption 4.1 hold.

For the finite-horizon case: there exists $T_0 \in ]0, +\infty[$ such that

i) For each $T \in ]0, T_0[$, and every $x_T$ accessible from $x_0$, problem (10)-(3)-(11) admits a unique generalized solution, \((\hat{u}, \hat{x}) \in \mathcal{U}_{\gamma_r}[0,T] \times H^{n-(r-1)}_{(-1)}[0,T] \);

ii) For any $T > T_0$, and every $x_T$ accessible from $x_0$, \( \inf_u J^T_0 = -\infty \) holds;

For the infinite-horizon case:

iii) If the quadratic form \( (x, u) \mapsto x'Px + 2u'Qx + u'Ru \) is nonnegative and system (10) is feedback stabilizable, then problem (10)-(3)-(11) admits a unique generalized solution, \((\hat{u}, \hat{x}) \in \mathcal{U}_{\gamma_r}[0, +\infty[ \times H^{n-(r-1)}_{(-1)}[0, +\infty[. \) \( \Box \)

For the proofs of these results, see [9]. We will now briefly discuss the structure of generalized optimal solutions.

For regular linear-quadratic problems any optimal control $v^*(\cdot)$ satisfies the Pontryagin maximum principle and is analytic. From the desingularization procedure (Proposition 4.3) there follows that the corresponding optimal generalized control for the original singular LQ problem (10)-(3)-(11) satisfies the relationship $\gamma_r u^* = v^*$, with $\gamma_r$ defined by (13). Hence it is a sum of an analytic function and a distribution of order $\leq r$. This distribution is supported at the points $t = 0$ and $t = T$.

The corresponding generalized optimal trajectory is analytic on $]0, T[$ and may happen to be discontinuous at points $t = 0$ and $t = T$. The generalized trajectory ”jumps” at $t = 0$ from $x(0) = x_0$ to the point

\[
x(0^+) = x_0 + \sum_{0 \leq i < j \leq r} B_{i,j} \phi^{i+1} u_j(0^+).
\]

For $t \in ]0, T[$ it coincides with the analytic curve

\[
x(t) = x^r_{x_0, \gamma_r u^*}(t) + \sum_{0 \leq i < j \leq r} B_{i,j} \phi^{i+1} u_j(t),
\]
where $x_{x_0, \gamma_r u}^r$ is the trajectory of $(15)$ driven by the analytic control $\gamma_r u$. The generalized trajectory terminates with a "jump" from the point

$$x(T^-) = x_{\gamma_r u}^r(T) + \sum_{0 \leq i < j \leq r} B_{i,j} \phi^i u_j(T^-)$$

to the point

$$x(T) = x_{\gamma_r u}^r(T) + \sum_{0 \leq i < j \leq r} B_{i,j} \phi^i u_j(T) = x_T$$

(the vectors $\phi^i u(0^+)$ and $\phi^i u(T^-)$ are well defined for all $i \geq 0$).

Assumption 4.1 guarantees that for any jump $x(t^+_0) - x(t^-_0)$ belonging to

$$\text{span}\{B_{i,j}, 0 \leq i < j \leq r\}$$

there exists a unique distribution $\Delta \in \mathcal{U}_{\gamma_r [0, +\infty)}$ supported at $\{t_0\}$, such that

$$x(t^+_0) - x(t^-_0) = \sum_{0 \leq i < j \leq r} B_{i,j} \phi^i u_j \Delta_j.$$

The following result states an important property of the optimal synthesis for problem (10)-(11)-(3) (see [13], [14] and a result in [18, Ch. 6], which claims a minimizing control to be "generically" a sum of a continuous control with impulses of different orders located at the initial and the final point of the time interval).

**Proposition 4.9.** Let Assumption 4.1 hold and the infimum of the problem (10)-(3)-(11) be finite. Consider the subspace $B_r = \text{span}\{B_{i,j}, 0 \leq i < j \leq r\}$. For all

$$\tilde{x}_0 \in x_0 + B_r, \quad \tilde{x}_T \in x_T + B_r,$$

the problem with boundary conditions $x(0) = \tilde{x}_0$, $x(T) = \tilde{x}_T$ admits a generalized optimal solution (control and trajectory). For each boundary data from the sets $\tilde{x}_0$ there exists a generalized optimal solution coinciding in the interval $]0, T]$ with the analytic arc of the solution for the data $(x_0, x_T)$. □

Suppose Assumption 4.1 to hold. Let $m = \text{rank} (B, AB, ..., A^{n-1}B)$, and let $p = \text{dim} \text{span}\{B_{i,j}, 0 \leq i < j \leq r\}$. Fix $T \in ]0, +\infty]$ and let $X \subset \mathbb{R}^{2n}$ denote the set of pairs $(x_0, x_T)$ for which the problem $(10)-(11)-(3)$ possesses classical optimal solution, $(u, x_u) \in L^2_b[0, T] \times AC^n[0, T]$. Propositions 4.8 and 4.9 imply that, if $T > 0$ is sufficiently small, then $X$ is a linear subspace of dimension $n + m - 2p$. Further, existence and uniqueness of generalized optimal solutions implies that any pair $(x_0, x_T) \in \mathbb{R}^{2n}$, such that $x_T$ is reachable from $x_0$, admits a unique decomposition

$$x_0, x_T = (\tilde{x}_0, \tilde{x}_T) + \sum_{0 \leq i < j \leq r} \begin{pmatrix} B_{i,j} \\ 0 \end{pmatrix} \alpha_{i,j} + \sum_{0 \leq i < j \leq r} \begin{pmatrix} 0 \\ B_{i,j} \end{pmatrix} \beta_{i,j},$$
with \((\bar{x}_0, \bar{x}_T) \in X\), and \(\alpha_{i,j}, \beta_{i,j}\) being vectors of appropriate dimensions. The discontinuities of the generalized optimal trajectories are computed as
\[
x(0^+) - x_0 = \sum_{0 \leq i < j \leq r} B_{i,j} \alpha_{i,j},
\]
\[
x_T - x(T^-) = \sum_{0 \leq i < j \leq r} B_{i,j} \beta_{i,j}.
\]

### 4.5. Approximation of distributions
The following Proposition gives the asymptotics of approximations of some important distributions by square-integrable functions.

**Proposition 4.10.** Consider a \(m\)-th order distribution of the form
\[
v = \delta^{(m-1)} + \sum_{i=0}^{m-2} \alpha_i \delta^{(i)},
\]
where \(\alpha_i \in \mathbb{R}, i = 0, 1, ..., m - 2,\) and \(\delta^{(i)}\) denotes the \(i\)-th generalized derivative of Dirac’s ”delta function”.

For every integer \(p \geq m\) there holds
\[
\lim_{\eta \to 0^+} \inf \frac{\log \|u\|_{L^2[0,T]} + \|u-e\|_{H^{-p}[0,T]}}{\log \frac{1}{\eta}} = \frac{2m-1}{2(p-m)+1}. \quad \square
\]

The rather technical proof of Proposition 4.10 can be found in Subsection 9.1.

### 4.6. Degree of singularity of LQ problems
For the finite horizon case \((T < +\infty)\) the following result holds

**Theorem 4.1.** Consider the problem (10)- (3)-(11). Suppose Assumption 4.1 holds and let \((x_0, x_T), (\bar{x}_0, \bar{x}_T), \alpha_{i,j}, \beta_{i,j}\) be as in (21).

1. If \(\alpha_{i,j} = 0, \beta_{i,j} = 0, 0 \leq i < j \leq r\), and the optimal control is not identically zero, then \(\sigma_T = 0\); if the optimal control is zero then \(\sigma_T = -\infty\);
2. If either \(\alpha_{i,j} \neq 0\) or \(\beta_{i,j} \neq 0\) for some \((i, j)\), then
\[
\sigma_T = \max_{0 \leq i < j \leq r} \frac{i + 1/2}{2(j - i) - 1}. \quad \square
\]

Theorem 4.1 together with the decomposition (21) results in the following description of the geometry of singularity of LQ problems.

**Theorem 4.2.** The space \(\mathbb{R}^n \times \mathbb{R}^n\) of boundary data admits a stratification into linear subspaces. The directing linear subspaces of strata are spanned by the columns of the matrices
\[
\begin{pmatrix}
B_{i,j} & 0 \\
0 & B_{i,j}
\end{pmatrix}, \quad 0 \leq i < j \leq r,
\]
with $B_{i,j}$ defined in Subsection 4.2. Different strata correspond to different distributional components of optimal generalized controls. For generic boundary data (the largest stratum) optimal generalized controls contain distributional components of the form $\delta^{(r-1)}$; the degree of singularity equals $r - 1/2$. When passing to the strata of smaller dimensions the order of singularity $\sigma_T$ decreases, admitting values from the list

$$\left\{ \frac{i + 1/2}{2(j - i) - 1} : 0 \leq i < j \leq r \right\}.$$  □

The infinite horizon case ($T = +\infty$) is analogous to the finite horizon case; one just has to deal with stratification of the space of initial data.

Let $X_0 \subset \mathbb{R}^n$, denote the set of initial points $x_0$ for which problem (10)-(11) has a classical optimal solution, $(u, x_u) \in L^2_T(0, +\infty) \times AC^\infty(0, +\infty)$. If the quadratic form $(x, u) \mapsto x^T P x + 2u^T Q x + u^T R u$ is nonnegative and system (10) is feedback stabilizable, then Theorem 4.9 guarantees that $X_0$ is a linear subspace of dimension $n - p$. Further, existence and uniqueness of generalized optimal solutions implies that any initial point $x_0 \in \mathbb{R}^n$ admits a unique decomposition

$$x_0 = \tilde{x}_0 + \sum_{0 \leq i < j \leq r} B_{i,j} \alpha_{i,j},$$

with $\tilde{x}_0 \in X_0$, and $\alpha_{i,j}$ being vectors of appropriate dimensions. The discontinuities of the generalized optimal trajectories are

$$x(0^+) - x_0 = \sum_{0 \leq i < j \leq r} B_{i,j} \alpha_{i,j}.$$

Thus we have for the infinite horizon case the following analogous of Theorem 4.1:

**Theorem 4.3.** Consider the problem (10)-(11), with $T = +\infty$. Let Assumption 4.7 hold, the quadratic form $(x, u) \mapsto x^T P x + 2u^T Q x + u^T R u$ be nonnegative and system (10) be stabilizable by linear feedback. Then

1. If $\alpha_{i,j} = 0$, $0 \leq i < j \leq r$, and the optimal control is nonzero, then $\sigma_\infty = 0$; $\sigma_\infty = -\infty$, if the optimal control is zero;
2. If $\alpha_{i,j} \neq 0$ for some $(i, j)$, then

$$\sigma_\infty = \max_{0 \leq i < j \leq r} \frac{i + 1/2}{2(j - i) - 1},$$  □

**Remark 4.2.** Since the system (11) is linear, it follows that it is stabilizable if and only if it is stabilizable by a linear feedback. Therefore it is stabilizable if and only if it is stabilizable of order 1 in the sense of Definition 2.1.

**Proof of Theorem 4.1.** Notice that $\alpha_{i,j} = 0$, $\beta_{i,j} = 0$ for all $(i, j)$ such that $0 \leq i < j \leq r$ if and only if the optimal control is an analytic function in $[0, T]$. Thus, assertion (1) follows immediately.
Otherwise the optimal control is the sum of an analytic function with a distribution concentrated at points 0, T. We will first prove that $\sigma_T$ can not exceed the value (22).

For each $j \in \{1, \ldots, r\}$ let $p_j = \max \{i : \alpha_{i,j} \neq 0 \text{ or } \beta_{i,j} \neq 0\}$. Let $\hat{u} = (\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_r)$ denote the generalized optimal control. Since (21) holds, then by equality (14) $u_j$ contains a distributional component of order $p_j + 1$, supported at $\{0, T\}$. Proposition 4.10 guarantees the existence of $\{u_{r, \eta}, \eta > 0\}$ such that

$$\|\phi^r (u_{r, \eta} - \hat{u}_r)\|_{L_2[0, T]} = O (\eta), \quad \|u_{r, \eta}\|_{L_2[0, T]} = O (\eta^{-\frac{2(p_j+1)}{2(p_j-1)}}).$$

Suppose that for some $j \geq 1$ we can chose $\{(u_{j+1, \eta}, u_{j+2, \eta}, u_{r, \eta}), \eta > 0\}$ such that

$$\sum_{q=j+1}^{r} \|\gamma_{r,q} (u_{\eta} - \hat{u})\|_{L_2[0, T]} = O (\eta),$$

$$\sum_{q=j+1}^{r} \|u_{q, \eta}\|_{L_2[0, T]} = O (\eta^{-\max_{q>j} \frac{2p_j+1}{2(q-p_j)-1}}).$$

Since

$$\gamma_{r,j} \hat{u} = \phi^j \hat{u}_j + R_{j,j}^{-1} \sum_{j \leq i < l \leq r} (Q_{j,j} B_{i,l} - B_{j,j}' Q_{i,l}') \phi^{i+1} \hat{u}_l$$

are square-integrable and all distributional components of $\hat{u}$ are supported at $\{0, T\}$, then for some constants $V_i^0, V_i^T, i = 1, 2, \ldots, p_j$, and some square-integrable function, $w$ there holds

$$\hat{u}_j (t) + R_{j,j}^{-1} \sum_{j \leq i < l \leq r} (Q_{j,j} B_{i,l} - B_{j,j}' Q_{i,l}') \phi^{i+1} \hat{u}_l (t) = w (t) + \sum_{i=1}^{p_j} \left( V_i^0 \delta^{i-1} (t) + V_i^T \delta^{i-1} (t - T) \right).$$

Proposition 4.10 guarantees that one can chose square integrable functions $\Delta_{i, \eta}$ such that

$$\|\delta^{i-1} - \Delta_{i, \eta}\|_{H_{-j}[0, T]} = O (\eta),$$

$$\|\Delta_{i, \eta}\|_{L_2[0, T]} = O (\eta^{-\frac{2i+1}{2i+1}}), \quad i = 1, 2, \ldots, p_j.$$

Let

$$u_{j, \eta} = w + \sum_{i=1}^{p_j} \left( V_i^0 \Delta_{i, \eta} (t) + V_i^T \Delta_{i, \eta} (t - T) \right) -$$

$$- R_{j,j}^{-1} \sum_{j \leq i < l \leq r} (Q_{j,j} B_{i,l} - B_{j,j}' Q_{i,l}') \phi^{i+1} u_{l, \eta}.$$
This choice of \( u_{j, \eta} \) guarantees \( \| \gamma_{r,j} (u_{\eta} - \hat{u}) \|_{L_2[0,T]} = O(\eta) \) and
\[
\| v_j - \left( w + \sum_{i=1}^{p_j} (V^0_i \Delta i,\eta (t) + V^T_i \Delta i,\eta (t - T)) \right) \| = \nonumber
= O \left( \sum_{i=j+1}^{r} \| v_{i,\eta} \|_{H^{-1}} \right) = O \left( \eta^{-\frac{2p_j - 1}{2(j-i) - 1}} + \sum_{i=j+1}^{r} \| v_{i,\eta} \|_{H^{-1}} \right).
\]

This proves existence of a family of square-integrable controls, \( \{ u_{\eta}, \eta > 0 \} \) such that
\[
\| u_{\eta} - \hat{u} \|_{\gamma_r} = O(\eta), \quad \| u_{\eta} \|_{L_2[0,T]} = O \left( \eta^{-\max_1 \leq q \leq r} \frac{2p_q + 1}{2(q-p_q) - 1} \right)
\]
and therefore,
\[
\sigma_T \leq \max_{0 \leq i < j \leq r, (\alpha_{i,j} \neq 0 \text{ or } \beta_{i,j} \neq 0)} \frac{i + 1/2}{2(j-i) - 1}.
\]

In order to prove the inverse inequality, pick the greatest value \( m \) of those \( j \) for which \( \frac{2p_j - 1}{2(i-p_i) + 1} = \max_{1 \leq i \leq r} \frac{2p_i - 1}{2(i-p_i) + 1} \). Suppose there exists a family of controls
\[
\{ v_{\eta} = (v_{0,\eta}, v_{1,\eta}, \ldots, v_{r,\eta}), \ \eta > 0 \}
\]
such that
\[
\| \hat{u} - v_{\eta} \|_{\gamma_r} = O(\eta); \quad \lim_{\eta \to 0^+} \| v \|_{L_2} \eta^\frac{2pm - 1}{2(m-p_m) + 1} = 0.
\]

By Proposition 4.10 there exist constants \( C_1, C_2 \) such that
\[
C_1 \eta^{-\frac{2pm - 1}{2(m-p_m) + 1}} \leq \nonumber
\leq \| v_m + R_{m,1} m \sum_{m \leq i < j \leq r} (Q_{m,m} B_{i,j} - B'_{m,m} Q'_{i,j}) \phi^{-m+1} v_j \|_{L_2} \leq C_2 \| v \|_{L_2},
\]
which is a contradiction.

5. Singularity of nonlinear control-affine problems

While we have managed to provide exhaustive analysis of the degree of singularity for singular linear quadratic problems, similar questions for control-affine nonlinear problem \( (1)-(2)-(3) \) still need to be answered.

**Question.** Is the set of possible values of degree of singularity \( \sigma_T \) for control affine problems \( (1)-(2)-(3) \) finite? Is the value of \( \sigma_T \) semiinteger for generic boundary data? Does this set of numbers correspond to a (local) stratification of the state space? ☐

There is research activity related to this question.
It is important to correlate the degree of singularity $\sigma_T$ with the order of singularity introduced by Kelley, Kopp and Moyer in [15] (see also [16]) for singular extremals of optimal control problems.

Another recent study carried out in the context of sub-Riemannian geometry and motion planning, invokes concepts of entropy and complexity (see [12, 8]). The connection of these notions on one side with the degree of singularity on the other side is yet to be clarified.

In the following Sections we study the degree of singularity for generalized minimizers of control-affine problem (1)-(2)-(3) with positive state-quadratic cost.

Apparently commutativity/noncommutativity of controlled vector fields affects the value of order of singularity $\sigma_T$ a great deal. The connection between the commutativity/noncommutativity and generalized minimizers is an established fact [19, 20, 2, 3]. Yet it is not well understood, how the Lie structure is revealed in the properties of generalized minimizers. The commutativity assumption is not immediately apparent in the linear-quadratic case, but in fact, (12) is the commutativity condition for the class of singular problems discussed in the previous Section.

Here is the list of results on nonlinear control-affine problems, which appear in the following sections.

We start with control-linear (= driftless) case and prove (Section 6) that for generic boundary data degree of singularity equals $\frac{1}{2}$ independently of commutativity/noncommutativity of inputs.

In Section 7 we establish an upper estimate $\sigma_T \leq 3/2$ for the degree of singularity of control-affine problems (1)-(2)-(3) with commuting controlled vector fields and positive state-quadratic cost (Theorem 7.1). In Subsections 7.1-7.4 we provide a sketch of the proof of this result. The proof of the main result in Subsection 7.3 can be found in [11]. A full proof for the material in Subsection 7.4 can be found in Subsection 9.3 below.

In Section 8 we provide some evidence which allows us to conjecture that the degree of singularity in the commutative case should be $\leq 1$. An example is examined.

6. NON-COMMUTATIVE DRIFTLESS CASE: GENERAL RESULT

It turns out that the non-commutative driftless case

$$J_0^T(u(\cdot)) = \int_0^T x(t)'Px(t)dt \to \min, \quad P > 0,$$

$${\dot{x}} = \sum_{j=1}^r g^ju_j, \quad x(0) = x_0, \quad x(T) = x_T,$$

(23)

is rather simple. In this case for generic boundary data the value of the order of singularity equals $\sigma_T = 1/2$, i.e. it does not depend on the Lie structure of the system of vector fields $\{g^1, \ldots, g^r\}$. 
Theorem 6.1. Let \( A_{x_0} \) be the attainable set (in the driftless case the orbit) of the control system (23). Let \( x_T \in A_{x_0} \) and

\[
\alpha = \inf \{ x' P x \mid x \in A_{x_0} \}. 
\]

Then:

i) the infimum \( \inf J_0^T = \alpha T \);

ii) the degree of singularity \( \sigma_T \geq \frac{1}{2} \) unless \( x'_0 P x_0 = x'_T P x_T = \alpha \);

iii) if the infimum (24) is attained then the degree of singularity \( \sigma_T \leq \frac{1}{2} \).

□

A detailed proof can be found in Subsection 9.2. Here is a brief idea for the case when the system \( \{ g^1, \ldots, g^r \} \) has complete Lie rank. Then, roughly speaking, generalized optimal trajectory consists of three 'pieces': an initial 'jump', which brings it to the origin \( 0 \), a constant piece \( x(t) \equiv 0 \), \( t \in [0, T] \), and a final 'jump' to the end point \( x(T) = x_T \). Evidently \( \inf J_0^T = 0 \) and a simple homogeneity based argument shows that \( \sigma_T \leq \frac{1}{2} \).

To prove that in fact \( \sigma_T = \frac{1}{2} \) whenever \( x_0 \neq 0 \) or \( x_T \neq 0 \), assume that \( x_0 \neq 0 \) (the case \( x_T \neq 0 \) is analogous), fix \( \varepsilon > 0 \) and take a control \( u_\varepsilon \), such that

\[
J_0^T(u_\varepsilon) = \int_0^T x'_{u_\varepsilon}(t) P x_{u_\varepsilon}(t) dt < \inf J_0^T + \varepsilon = \varepsilon.
\]

Consider the set \( \{ x \in \mathbb{R}^n \mid x' P x \leq \frac{1}{2} x'_0 P x_0 \} \); let \( \rho \) be the distance from \( x_0 \) to this set. Since \( x' P x \) is positive definite, one concludes from the inequality (25), that there exists \( t_\varepsilon < \frac{2\rho}{x'_0 x_0} \) such that \( x_{u_\varepsilon}(t_\varepsilon)^* P x_{u_\varepsilon}(t_\varepsilon) < (1/2)x'_0 P x_0 \).

Then, by Cauchy-Schwarz inequality, the control needed to achieve \( x_{u_\varepsilon}(t_\varepsilon) \) from \( x_0 \) in time \( t_\varepsilon \), must satisfy the estimate \( \| u_\varepsilon \|_{L^2[0, T]}^2 \geq C \varepsilon^{-1} \), for some \( C > 0 \).

7. Degree of singularity for control-affine systems: commuting inputs

In this section we discuss the degree of singularity of optimal control problems of type (1)-(2)-(3) with \( T < +\infty \).

In what follows we denote by \( e^{tF} \) the flow of the smooth field \( F \); for each point \( x_0 \in \mathbb{R}^n \) the curve \( t \mapsto e^{tF} x_0 \) is the unique solution of the differential equation

\[
\dot{x} = F(x), \quad x(0) = x_0.
\]

For fixed \( t \) the map \( x \mapsto e^{tF} x \) is a local diffeomorphism in a neighborhood of any point \( x_0 \) such that \( e^{tF} x_0 \) exists.

For every (local) diffeomorphism \( P : \mathbb{R}^n \to \mathbb{R}^n \), and any vector field \( F \), we denote by \( AdPF \) the (local) field defined as

\[
AdPF(x) = (DP(x))^{-1} F(P(x)),
\]

where \( DP(x) \) denotes the Jacobian matrix of \( P \) evaluated at the point \( x \).
We keep the notation introduced in Section 4, according to which \( \phi \) denotes primitivization (i.e., \( \phi u(t) = \int_0^t u(\tau) \, d\tau, \forall u \in L_{1,\text{loc}} \)).

We introduce several assumptions.

**Assumption 7.1.** The fields \( f, g_i, i = 1, \ldots, k \) are complete. The controlled vector fields \( g_i \) span \( k \)-dimensional involutive distribution; for simplicity we assume that \([g_i, g_j] \equiv 0\) holds for all \( i, j \). \( \Box \)

The following three assumptions regard conditions on the growth of \( f, g^i \) and of their derivatives.

**Assumption 7.2.** For any compact set \( K \subset \mathbb{R}^n \)

\[
\lim_{|v| \to +\infty} |e^{Gv} x| = +\infty,
\]

uniformly with respect to \( x \in K \). \( \Box \)

**Assumption 7.3.** For any compact set \( K \subset \mathbb{R}^n \)

\[
\lim_{|v| \to +\infty} \frac{|\partial}{\partial x} \left( \left| (e^{Gv} x) \right| P (e^{Gv} x) \right) |e^{Gv} x|^2 = 0,
\]

uniformly with respect to \( x \in K \). \( \Box \)

**Assumption 7.4.** For each compact set \( K \subset \mathbb{R}^n \) there exists a function \( \gamma : [0, +\infty] \to \mathbb{R} \) bounded below, such that:

\[
\begin{align*}
\text{i)} & \quad \lim_{s \to +\infty} \frac{\gamma(s)}{s} = +\infty; \\
\text{ii)} & \quad \left| e^{Gv} x \right|^2 \geq \gamma \left( \left| (Ad (e^{Gv}) f) (x) \right| + \left| \frac{\partial}{\partial x} (Ad (e^{Gv}) f) (x) \right| \right), \\
& \quad \forall (x, v) \in K \times \mathbb{R}^k. \Box
\end{align*}
\]

As we will see below, Assumption **7.1** allows us to use a reduction procedure analogous to the procedure employed in Section 4 for the treatment of singular linear-quadratic problem, while Assumptions **7.2, 7.3** and **7.4** guarantee existence of minimizers in a suitable class of generalized controls.

The Assumptions **7.3** and **7.4** are somehow less explicitly formulated. In the two following remarks we formulate more particular growth conditions onto vector fields \( f \) and \( g^i \) and their flows which guarantee fulfillment of the two Assumptions.

**Remark 7.1.** Assume that the drift vector field satisfies the growth condition

\[
|f(y)| \leq \psi(|y|).
\]
For the flow $e^{Gv}$ generated by the controlled vector fields, we require

\[ |De^{Gv}x| = o\left(|e^{Gv}x|\right), \tag{26} \]

\[ \left|(De^{Gv}x)^{-1}\right| = o\left(\frac{|e^{Gv}x|^2}{\psi(|e^{Gv}x|)}\right), \tag{27} \]

\[ |D^2e^{Gv}x| = O\left(\frac{\psi(|e^{Gv}x|)}{|e^{Gv}x|^3}\right), \tag{28} \]

as $|v| \to \infty$, uniformly for $x$ belonging to any fixed compact $K$.

A typical choice of $\psi$ which guarantees completeness of $f$ would be

\[ \psi(|y|) = k(1 + |y|). \]

In this case (26) and (27) take the form

\[ |De^{Gv}x| + \left|(De^{Gv}x)^{-1}\right| = o\left(|e^{Gv}x|\right), \]

while (28) would mean that $|D^2e^{Gv}x| |e^{Gv}x|^2$ is uniformly (with respect to $x$ from a fixed compact $K$) upper bounded for all $v$. □

Another case we had in mind is described in the following

Remark 7.2. In the particular case of constant vector fields $g_i$, $i = 1, 2, ..., k$ the condition (ii) of the Assumption 7.4 reads

\[ |v|^2 \geq \gamma (|f(x + Gv)| + |Df(x + Gv)|), \quad \forall (x,v) \in K \times \mathbb{R}^k, \]

i.e. $|f|$ and $|Df|$ must exhibit subquadratic growth along the directions spanned by $g_i$, $i = 1, 2, ..., k$.

It is straightforward to check that Assumptions 7.2 and 7.3 hold if the fields $g_i$, $i = 1, 2, ..., k$ are linearly independent. □

Our main result in the commutative case is the following.

Theorem 7.1. If Assumptions 7.1, 7.2, 7.3 and 7.4 hold for problem (1)-(2)-(3), then

\[ \sigma_T \leq \frac{3}{2} \]

for $T < +\infty$ and generic boundary conditions. □

The rest of this Section contains sketch of the proof of Theorem 7.1. The feature which distinguishes the proof is an unbounded set of control parameters. In this context some components of the proof gain (in our opinion) an independent interest. Among those is Theorem 7.2 on existence and Lipschitzian regularity of relaxed minimizing trajectories in the case of unbounded controls. We discuss this question in details in [11]. Another important issue is Proposition 7.3 on approximation of relaxed trajectories by ordinary ones and on estimates of the variation of the approximants; this result is proved in Subsection 9.3.
7.1. Proof of Theorem [7.1]: desingularization. First we proceed with a "desingularizing transformation", which appeared in [1] under the name of 'reduction' and proved to be useful for analysis of control-affine systems with unconstrained controls.

Proposition 7.1. Under Assumption 7.1 the following holds:

\[ x_u(t) = e^{G\phi u(t)} y_{\phi u}(t), \quad \forall t \in [0, T], \ u \in L_\infty[0, T], \]

where \( x_u \) denotes the trajectory of the system

\[ \dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \]

\( y_v \) denotes the trajectory of the system

\[ \dot{y}(t) = \left( Ad\left(e^{Gv(t)}\right) f\right)(y(t)), \quad y(0) = x_0. \]

The substitution \( x = e^{G\phi u(t)} y \) (sometimes called in the literature Goh transform) leads us to the following 'desingularized' problem

\[ \dot{y}(t) = \left( Ad\left(e^{Gv(t)}\right) f\right)(y(t)), \]

\[ J_r(v) = \int_0^T \left(e^{Gv(t)} y(t)\right)' P \left(e^{Gv(t)} y(t)\right) dt \rightarrow \min, \]

with boundary conditions

\[ y(0) = x_0, \quad y(T) = e^{GV} x_T, \quad V \in \mathbb{R}^k. \]

Notice that in the case when the fields \( g_i, i = 1, 2, ..., k \) are constant we have \( e^{Gv} y = y + Gv, \left( Ad\left(e^{G}\right) f\right) (y) = f(y + Gv) \). Therefore, the reduced cost functional (30) takes the form

\[ J_r(v) = \int_0^T y(t)' P y(t) + 2v(t)' G' P y(t) + v(t)' GtPGv(t) dt. \]

Thus, in this case the reduction procedure achieves desingularization in the sense that the integrand in (32) exhibits quadratic growth with respect to the new control \( v \) (provided \( G \) is full rank). This does not hold in the general case with nonconstant \( G \), but it can be reasonably expected that it is a fairly generic outcome. However, one important feature of the nonlinear optimal control problem (29)-(30)-(31) is its lack of convexity with respect to \( v \). If we introduce the notation

\[ \left( \tilde{f}_0(y, v), \tilde{f}(y, v) \right) = \left( (e^{Gv} y)' P (e^{Gv} y), \left( Ad\left(e^{G}\right) f\right) (y) \right), \]

then, for generic \( y \in \mathbb{R}^n \) (fixed) the set

\[ \Gamma(y) = \left\{ \left( y_0, \tilde{f}(y, v) \right) : y_0 \geq \tilde{f}_0(y, v), \ v \in \mathbb{R}^m \right\} \subset \mathbb{R} \times \mathbb{R}^n \]

An anonymous referee brought to our attention a publication by Yu. Orlov [17] where such transform has been introduced (without Lie algebraic notation) with the scope of passing from control problems with measure-like controls to classical control problems. This construction is described by the same author in [18, Ch. 4].
is in general nonconvex, even in the case when \( G \) is constant. For example, for scalar \( v \) the set \( \left\{ \tilde{f}(y, v), v \in \mathbb{R} \right\} \) is a curve in \( \mathbb{R}^n \).

It follows that classical minimizers for the reduced problem (29)-(30)-(31) typically fail to exist. Instead, existence of relaxed minimizers can be established under the Assumptions 7.1, 7.2, 7.3 and 7.4.

7.2. Proof of Theorem 7.1: digression on relaxed controls. Recall that a relaxed control can be seen as a family \( t \mapsto \eta_t \) of probability measures in the space of \( v \in \mathbb{R}^k \), such that \( t \mapsto \eta_t \) is measurable in the weak sense with respect to \( t \in [0, T] \).

**Definition 7.1.** Consider a nonempty set \( A \subset \mathbb{R}^k \). We denote by \( \mathcal{M}_A \) the set of inner regular probability measures in \( \mathbb{R}^k \) with compact support contained in \( A \). We denote by \( \mathcal{M}_A[0, T] \) the set of functions \( t \mapsto \eta_t \), \( t \in [0, T] \), where:

i) \( \eta_t \in \mathcal{M}_A \), \( \forall t \in [0, T] \);

ii) For any continuous function, \( g : \mathbb{R}^{1+k} \rightarrow \mathbb{R} \), the function \( t \mapsto \int_{\mathbb{R}^k} g(t, u) \eta_t(du) \) is measurable. \( \square \)

Let \( \delta_v \) denote the Dirac measure in \( \mathbb{R}^k \) concentrated at the point \( v \). Obviously, the space of measurable essentially bounded functions can be embedded in the space \( \mathcal{M}_{\mathbb{R}^k}[0, T] \) through the map \( v(t) \mapsto \delta_{v(t)} \).

The dynamic equation (29) and cost functional (30) corresponding to a relaxed control become

\[
\dot{y}(t) = \int_{\mathbb{R}^k} \left( \text{Ad}(e^{Gv}) \cdot f \right)(y(t)) \eta_t(du),
\]

\[
J_r(\eta) = \int_0^T \int_{\mathbb{R}^k} (e^{Gv}y(t))' P(e^{Gv}y(t)) \eta_t(du) dt \rightarrow \min.
\]

It is clear that the equations above coincide with (29) and (30) in the case when \( \eta_t = \delta_{v(t)} \) holds at almost every \( t \in [0, T] \) for some function \( v \in L^k_{\infty}[0, T] \).

We use the short notation \( \langle \eta, f \rangle \) to indicate the averaging of \( f \) by the measure \( \eta \). I.e., for \( g : \mathbb{R}^{1+k} \rightarrow \mathbb{R}, \eta \in \mathcal{M}_A[0, T], \langle \eta_t, g(t, \cdot) \rangle \) denotes the function \( \langle \eta_t, g(t, \cdot) \rangle = \int_{\mathbb{R}^k} g(t, u) \eta_t(du), \quad t \in [0, T] \).

The following proposition (see [7]) relates relaxation with the convexification of the right-hand side.

**Proposition 7.2.** Consider a \( C_1 \)-map \( X : \mathbb{R}^k \rightarrow \mathbb{R}^n \), and a nonempty set \( A \subset \mathbb{R}^k \). Then,

\[
\{ (\eta, X), \ \eta \in \mathcal{M}_A \} = \text{conv} \{ X(v), \ v \in A \}. \quad \square
\]
Below we will consider a more restricted class of controls (called sometimes Gamkrelidze relaxed or chattering controls) of special form

\( \eta_t = \sum_{j=1}^{N} p_j(t) \delta_{\omega_j(t)} \), \( p_j(t) \geq 0 \), \( \sum_{j=1}^{N} p_j(t) \equiv 1 \).

It is clear that any measurable essentially bounded control can be identified with such a control with \( N = 1 \). The dynamic equation and functional corresponding to these controls become

\[ \dot{y}(t) = \sum_{j=1}^{N} p_j(t) \left( \text{Ad} \left( e^{G_{\omega_j}(t)} \right) f \right) (y(t)); \]

\[ J_r(\eta) = \int_{0}^{T} \sum_{j=1}^{N} p_j(t) \left( e^{G_{\omega_j}(t)} y(t) \right)' P \left( e^{G_{\omega_j}(t)} y(t) \right) dt \to \min. \]

Looking at Proposition 7.2 it is easy to understand why the class of controls (35) suffices: each vector of the convex hull in (34) can be represented as a finite convex combination of \( \left( \text{Ad} \left( e^{G_{\omega_j}} \right) f \right) (y) \); the same is valid for the functional. By virtue of Carathéodory theorem the combination need not have more than \( n + 2 \) summands.

7.3. Existence of relaxed minimizers and Lipschitzian regularity of optimal relaxed trajectories. The existence of minimizing relaxed controls in the case where the set of control parameters is bounded is a well known result closely related to classical A.F. Filippov’s existence theorem (see [7, Ch.8]). Our treatment involves controls without any a priori bound. Existence results for relaxed minimizers in this case, are referred to in [4, Ch. 11], but we are not aware of any previous results on Lipschitzian regularity of relaxed minimizing trajectories. In a separate paper [11] we present a technically involved proof of Lipschitzian regularity of relaxed minimizers. In our present setting the result in [11] takes the special form:

**Theorem 7.2.** Under Assumptions 7.1, 7.2, 7.3 and 7.4, the reduced problem (29)-(30)-(31) admits a minimizer, \( \eta_t = \sum_{j=1}^{n+2} p_j(t) \delta_{\omega_j(t)} \). Any such minimizer satisfies the Pontryagin maximum principle and, provided it does not correspond to a strictly abnormal extremal, there exists a constant \( M < +\infty \) such that

\[ \left| \left( \text{Ad} \left( e^{G_{\omega_j}(t)} \right) f \right) (y_\eta(t)) \right| \leq M, \]

\[ \left( e^{G_{\omega_j}(t)} y_\eta(t) \right)' P \left( e^{G_{\omega_j}(t)} y_\eta(t) \right) \leq M \]

hold for \( j = 1, 2, \ldots, n + 2 \) and almost every \( t \in [0, T] \).
Remark 7.3. For generic boundary conditions the relaxed optimal trajectory corresponds to a normal extremal. Therefore, it follows from inequality (38) that generic optimal relaxed trajectories are Lipschitz continuous. Inequality (39) together with Assumption 7.3 imply that the functions \( v_j, j = 1, 2, ..., n + 2 \) on the corresponding Gamkrelidze control are essentially bounded. □

7.4. Approximation of relaxed minimizers by absolutely-continuous controls. The rest of the proof of Theorem 7.1 consists of two approximation steps. In the first step we approximate the relaxed minimizer \( \eta \) of the reduced problem (29)-(30)-(31) by a piecewise constant control \( w_\varepsilon(\cdot) \) in such a way that the trajectory and the functional of (29), driven by \( w_\varepsilon(\cdot) \), are \( \varepsilon \)-close to the trajectory and the functional of (36). Such approximating controls can be chosen in such a way that the number of discontinuities is bounded by \( \frac{\text{Const.}}{\varepsilon} \).

The second step consists of approximating the piece-wise constant controls \( w_\varepsilon(\cdot) \) by absolutely continuous controls \( v_\varepsilon(\cdot) \), whose derivative \( u_\varepsilon(\cdot) = \dot{v}_\varepsilon(\cdot) \) becomes \( \varepsilon \)-minimizing control for the original problem (1)-(2)-(3). This second approximants can be obtained by altering the function \( w_\varepsilon \) at intervals of length \( \varepsilon^2 \) containing the points of discontinuity.

Using this two-step approximation we are able to prove the following proposition (see Subsection 9.3 below).

Proposition 7.3. Consider \( \eta = \sum_{j=1}^{n+2} p_j(t) \delta_{\varphi(t)} \), a Gamkrelidze minimizer for the reduced problem (22)-(30)-(31), satisfying (38), (39) for some \( M < +\infty \), let \( y_\eta \) denote the corresponding solution to the dynamical equation (30).

There exists a family of absolutely continuous piecewise linear controls \( v_\varepsilon \), \( \varepsilon > 0 \) such that

i) \( J_r(v_\varepsilon) = J_r(\eta) + O(\varepsilon) \) when \( \varepsilon \to 0^+ \);

ii) The trajectories of the reduced system (29) satisfy \( \|y_{v_\varepsilon} - y_\eta\|_{L_\infty[0,T]} = O(\varepsilon) \) when \( \varepsilon \to 0^+ \);

iii) The trajectories of the original system (2) corresponding to the controls \( u_\varepsilon = \frac{d}{dt}v_\varepsilon \) satisfy \( x_{u_\varepsilon}(T) = x_T + O(\varepsilon) \) when \( \varepsilon \to 0^+ \);

iv) \( \|\frac{d}{dt}v_\varepsilon\|_{L_2[0,T]}^2 = O\left(\frac{1}{\varepsilon^2}\right) \) when \( \varepsilon \to 0^+ \). □

The proof of Theorem 7.1 follows by noting that

\[
\sigma_T \leq \lim_{\varepsilon \to 0^+} \frac{\ln \left(\text{Const.}\|u_\varepsilon\|_{L_2[0,T]}\right)}{\ln \frac{1}{\varepsilon}},
\]

where \( u_\varepsilon = \frac{d}{dt}v_\varepsilon \) is the family of controls described in Proposition 7.3. This yields the estimate \( \sigma_T \leq \lim_{\varepsilon \to 0^+} \frac{\text{Const.} \ln \varepsilon^{-\frac{3}{2}}}{\ln \varepsilon} = \frac{3}{2} \).
8. Degree of singularity for input-commutative control-affine system: conjecture and example

In the previous Section we provide an upper bound for the degree of singularity by showing how to construct a minimizing sequence with asymptotics \( \sigma_T = \frac{3}{2} \). However we believe that this upper bound is not sharp and we provide the following conjecture for a sharp estimate:

**Conjecture 8.1.** Under the assumptions of Theorem [7.1]

\[
\sigma_T \leq 1. \quad \square
\]

Our conjecture relies on the proof of Proposition [7.3] which is key fragment of the proof of Theorem [7.1] We trust that our two-step approximation procedure can be improved: there exists a piecewise continuous control \( w_\varepsilon(\cdot) \) with \( \leq O(\varepsilon^{-1}) \) intervals of continuity, such that the end-point of the trajectory and the value of the functional of (30) driven by \( w_\varepsilon(\cdot) \) are \( \varepsilon^2 \)-close to the end-point of the trajectory and the value of the functional of (37).

If this holds true then, by modifying this approximant in intervals of length \( \varepsilon^3 \) instead of \( \varepsilon^2 \) we obtain a family of square-integrable controls \( u_\varepsilon = \frac{dw_\varepsilon}{dt} \) satisfying the estimate \( \| u_\varepsilon \|_{L_2} = O(\varepsilon^{-2}) \). Then, by virtue of majoration (40) we conclude that \( \sigma_T \leq 1 \).

Another possibility for sharpening the upper estimate of degree of singularity is related to the second approximation step described in the previous subsection. This step can be formalized as the following problem of best approximation.

**Problem 8.1.** Let \( B_M = \{ u : \| u \|_{L_2[0,T]} \leq M \} \) denote the ball of radius \( M \) in the space of square-integrable functions. Given a piecewise-continuous (or just essentially bounded) function \( \varphi : [0,T] \rightarrow \mathbb{R} \), find the asymptotics (the rate of decay) of the distance

\[
\rho_{L_p}(\varphi, B_M) = \inf \left\{ \| \varphi - \phi u \|_{L_p[0,T]} : u \in B_M \right\},
\]

as \( M \rightarrow +\infty \) (for fixed \( p \in \mathbb{N} \)). \( \square \)

The following example shows that, at least in some cases, the bound \( \sigma_T \leq 1 \) is tight.

**Example 8.1.** Consider optimal control-affine problem

\[
\dot{x} = f(x) + g^1(x)u_1, \quad x = (x_1, x_2, x_3),
\]

\[
f = x_1 \frac{\partial}{\partial x_2} + \gamma(x_1)(x_1^2 - 1) \frac{\partial}{\partial x_3}, \quad g_1 = \frac{\partial}{\partial x_1}
\]

\[
J_0^1 = \int_0^1 (x_1^2 + x_2^2 + x_3^2) dt \rightarrow \text{min},
\]

\[
x(0) = 0, \quad x(1) = 0,
\]

where \( \gamma(x) \) is a smooth function supported at \([-2, 2] \subset \mathbb{R}, 0 \leq \gamma(x) \leq 1 \) and \( \gamma(x) \equiv 1 \) on \([-3/2, 3/2] \). \( \square \)
In coordinates the dynamics of the problem is
\begin{equation}
\dot{x}_1 = u_1, \quad \dot{x}_2 = x_1, \quad \dot{x}_3 = \gamma(x_1)(x_1^2 - 1).
\end{equation}

To estimate the infimum of this problem note that
\begin{equation}
0 = \int_0^1 \gamma(x_1(t)) \left( x_1(t)^2 - 1 \right) dt \leq \frac{1}{2} \int_0^1 x_1(t)^2 dt - 1,
\end{equation}
and hence $J_0^1 \geq 1$. Now we construct a minimizing sequence of controls $u_N(\cdot)$, such that
\begin{equation}
J_1^0(\cdot) \to 1 \text{ as } N \to +\infty.
\end{equation}

First take the indicator function $p(t)$ of the interval $[0, 1]$ and construct a piecewise-constant function
\begin{equation}
q_N(t) = \sum_{j=0}^{2N-1} (-1)^j p(2Nt - j);
\end{equation}

$N$ being a large integer. Its intervals of constancy have lengths equal to $(2N)^{-1}$.

Then, alter the function $q_N$ on the subintervals $[0, N^{-3}]$, $[j(2N)^{-1} - N^{-3}, j(2N)^{-1} + N^{-3}]$, $j = 1, \ldots, 2N - 1$ and $[1 - N^{-3}, 1]$, transforming it into a piecewise-linear continuous function $q_N^c(t)$ with boundary values: $q_N^c(0) = q_N^c(T) = 0$.

Taking $x_1(t) = q_N^c(t)$ and substituting it into second and third equations of (41), we conclude that the corresponding solution satisfies the conditions $x_2(1) = 0$, $x_3(1) = O(N^{-2})$, as $N \to +\infty$. Besides
\begin{equation}
\int_0^1 (x_1^2 + x_2^2 + x_3^2) dt - 1 = O(N^{-2}), \quad \text{as } N \to +\infty.
\end{equation}
The $L_2$-norm of the corresponding control $\|u_1(t)\|_{L_2} = \|q_N^c(t)\|_{L_2}$ admits an estimate
\begin{equation}
\|u_1(t)\|_{L_2} \simeq ((N^3)^2N^{-3}N)^{1/2} = N^2
\end{equation}
as $N \to +\infty$. Therefore the order of singularity satisfies $\sigma_1 \leq 1$. \(\square\)

9. Proofs

9.1. Proof of Proposition 4.10. In order to prove Proposition 4.10 we will use the following variant of Tchebyshev inequality:

Lemma 9.1. Let $u \in L_2$ and $\lambda$ denote Lebesgue measure. Then
\begin{equation}
\|u\|_{L_2} < \eta \Rightarrow \forall \varepsilon > 0 : \lambda \{ x : |u(x)| \geq \varepsilon \} < \frac{\eta^2}{\varepsilon^2}.
\end{equation}

It suffices to prove Proposition 4.10 for distributions $v = \delta^{(m-1)}$. Notice that for any $p \geq m$:
\begin{equation}
\phi^p \delta^{(m-1)} (t) = \frac{t^{p-m}}{(p-m)!}, \quad \text{a.e. } t \in [0, T].
\end{equation}
Fix $\eta > 0$, and consider $u \in L_2[0, T]$ such that $\|u - v\|_{H^p[0, T]} < \eta$. It follows from (12) that $\lambda \left\{ t \in [0, T] : \left| \phi^p u(t) - \frac{\theta^p}{(p-m)!} \right| \geq \varepsilon \right\} < \frac{\eta^2}{\varepsilon^2}$. Then, for every $\theta > 0$ sufficiently small and provided $\eta > 0$ is small, there exist $\theta_{0,i} \in \mathbb{R}$ such that $\phi^p u(\theta_{0,i}) = \theta_{0,i}$, $i = 1, 2, \ldots, 2^{p-m}$ such that $\left| \phi^p u(\theta_{0,i}) - \frac{\theta^p}{(p-m)!} \right| < \varepsilon$.

By the mean-value theorem, there exist

$$\theta_{1,i} \in (2i - 1) \theta, 2i \theta - \frac{\eta^2}{\varepsilon^2}$$

such that

$$\left| \phi^{p-1} u(\theta_{1,i}) - \frac{\theta^{p-1}}{(p-m-1)!} \right| = \left| \phi^p u(\theta_{0,2i}) - \frac{\theta^p}{(p-m)!} \right| - \left| \phi^p u(\theta_{0,2i-1}) - \frac{\theta^p}{(p-m)!} \right|$$

$$= \frac{\left| \phi^p u(\theta_{0,2i}) - \theta^p + \frac{\theta^p}{(p-m)!} \right| - \left| \phi^p u(\theta_{0,2i-1}) - \theta^p + \frac{\theta^p}{(p-m)!} \right|}{\theta_{0,2i} - \theta_{0,2i-1}} \leq \frac{2\varepsilon}{\theta - \frac{\eta^2}{\varepsilon^2}}$$

Proceeding by induction we establish existence of $\theta_{p-m} \in [\theta, \theta_{p-m} + \frac{\eta^2}{\varepsilon^2}]$ such that

$$\left| \phi^m u(\theta_{p-m}) - 1 \right| < \frac{2^{p-m}\varepsilon}{(\theta - \frac{\eta^2}{\varepsilon^2})^{p-m}}$$

This implies

$$\phi^m u(\theta_{p-m}) > 1 - \frac{2^{p-m}\varepsilon}{(\theta - \frac{\eta^2}{\varepsilon^2})^{p-m}}$$

Once again, mean-value theorem guarantees the existence of $\theta_{p-m+1} \in [0, \theta_{p-m} + \frac{\eta^2}{\varepsilon^2}]$ such that

$$\phi^{m-1} u(\theta_{p-m+1}) = \frac{\phi^m u(\theta_{p-m})}{\theta_{p-m}} \geq \frac{1}{2^{p-m}\theta + \frac{\eta^2}{\varepsilon^2}} \left( 1 - \frac{2^{p-m}\varepsilon}{(\theta - \frac{\eta^2}{\varepsilon^2})^{p-m}} \right).$$

Repeating the same argument, one proves existence of $\theta_{p-1} \in [0, \theta_{p-m} + \frac{\eta^2}{\varepsilon^2}]$ such that

$$\phi u(\theta_{p-1}) \geq \frac{1}{(2^{p-m}\theta + \frac{\eta^2}{\varepsilon^2})^{m-1}} \left( 1 - \frac{2^{p-m}\varepsilon}{(\theta - \frac{\eta^2}{\varepsilon^2})^{p-m}} \right).$$
Applying Schwarz’s inequality, we conclude
\[
\frac{1}{(2p-m\theta + \frac{\eta^2}{\varepsilon^2})^{m-1}} \left(1 - \frac{2p-m\varepsilon}{\theta - \frac{\eta^2}{\varepsilon^2}}^{p-m}\right) \leq \sqrt{\theta - 2p} \int_0^{\theta/p-1} u(\tau)^2 \, d\tau < \sqrt{2p-m\theta + \frac{\eta^2}{\varepsilon^2}} \|u\|_{L^2[0,T]},
\]
and
\[
\|u\|_{L^2[0,T]} \geq \frac{1}{(2p-m+1)^{m-\frac{1}{2}}} \left( \frac{2p-m\varepsilon}{\theta - \frac{\eta^2}{\varepsilon^2}}^{p-m} \right). \tag{43}
\]
Now, for \( \varepsilon = \eta^{2/(p-m)+1} \), \( \theta = 2p-m+1 \eta^{2/(p-m)+1} \), inequality (43) reduces to
\[
\|u\|_{L^2[0,T]} \geq \frac{1}{(2p-m+2+1)^{m-\frac{1}{2}}} \eta^{2/(p-m)+1}.
\]
This proves existence of a constant \( C > 0 \) such that
\[
\inf \left\{ \log \|u\|_{L^2[0,T]} : \|u - v\|_{H^{-p}[0,T]} < \eta \right\} > \frac{2m-1}{2(p-m)+1} \log \frac{1}{\eta} + \log C,
\]
for all sufficiently small \( \eta > 0 \). Hence
\[
\lim_{\eta \to 0^+} \inf \left\{ \log \|u\|_{L^2[0,T]} : \|u - v\|_{H^{-p}[0,T]} < \eta \right\} \geq \frac{2m-1}{2(p-m)+1}.
\]
To prove the converse inequality, we consider piecewise polynomial functions \( \psi_{\eta} : [0,T] \mapsto \mathbb{R} \):
\[
\psi_{\eta}(t) = \begin{cases} 
\sum_{i=0}^{p-1} \frac{\alpha_i t^{p-i}}{(p-i)!} \eta^{m-i} & \text{if } t \in [0,\eta]; \\
\frac{t^{p-m}}{(p-m)!} & \text{if } t > \eta,
\end{cases}
\]
and make unique choice of constants \( \alpha_0, \alpha_1, \ldots, \alpha_{p-1} \in \mathbb{R} \) in such a way that \( \psi_{\eta} \) becomes \((p-1)\)-times differentiable with absolutely continuous \((p-1)^{th}\) derivative. One can check that \( \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{p-1}) \) is the unique solution of the linear system \( M \alpha = b \), where
\[
M = \begin{pmatrix} 
\frac{1}{p!} & \frac{1}{(p+1)!} & \frac{1}{(p+2)!} & \cdots & \frac{1}{(2p-1)!} \\
\frac{1}{(p-1)!} & \frac{1}{p!} & \frac{1}{(p+1)!} & \cdots & \frac{1}{(2p-2)!} \\
\frac{1}{(p-2)!} & \frac{1}{(p-1)!} & \frac{1}{p!} & \cdots & \frac{1}{(2p-3)!} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \frac{1}{2!} & \frac{1}{3!} & \cdots & \frac{1}{p!}
\end{pmatrix},
\]
\[ b' = \left( \frac{1}{(p - m)!} \frac{1}{(p - m - 1)!} \ldots 1 \ 0 \ldots 0 \right). \]

It follows that \( \alpha \) does not depend on \( \eta \).

Let \( u = \frac{d\phi}{dt} \). Then,

\[
\|u\|_{L^2[0,T]}^2 = \int_0^T \left( \sum_{i=0}^{p-1} \alpha_i \frac{t^i}{i!\eta^{m+i}} \right)^2 dt = \frac{C_1}{\eta^{2m-1}},
\]

\[
\|u - v\|_{H^{-p}[0,T]}^2 = \int_0^T \left( \sum_{i=0}^{p-1} \alpha_i \frac{t^{p+i}}{(p+i)!\eta^{m+i}} - \frac{t^{p-m}}{(p-m)!} \right)^2 dt = C_2\eta^{2(p-m)+1},
\]

where \( C_1, C_2 \) are positive constants. This shows that there exists a constant \( C \) such that

\[
\inf \left\{ \log \|u\|_{L^2[0,T]} : \|u-v\|_{H^{-p}[0,T]} < \eta \right\} \leq \frac{2m-1}{2(p-m)+1} \log \frac{1}{\eta} + C \log \frac{1}{\eta}.
\]

9.2. Degree of singularity for noncommutative driftless case (proof of Theorem 6.1). In the following, we consider the cost functional

\[ J^T(u) = \int_0^T x'Px \ dt, \]

to be minimized along the trajectories of the system

\[ \dot{x} = \sum_{i=1}^k g_i(x) u_i, \quad x(0) = x_0. \]

Let \( \mathcal{A}_{x_0} \) denote the set of points \( x \in \mathbb{R}^n \) which can be reached from \( x_0 \) through trajectories of (44).

Assertion i) of Theorem 6.1 is obvious. Hence we only need to prove assertions ii) and iii). We start with assertion iii).

Proposition 9.1. If \( x_T \in \mathcal{A}_{x_0} \) and the quadratic form \( x \mapsto x'Px \) admits a minimum in \( \mathcal{A}_{x_0} \), then \( \sigma_T \leq \frac{1}{2}. \)

Proof. Fix \( \dot{x} \in \mathcal{A}_{x_0} \), such that \( \dot{x}'P\dot{x} \leq x'Px \) holds for all \( x \in \mathcal{A}_{x_0} \), and consider the controls \( u_0, u_T \in L^k_2[0,1] \), such that:

1. The trajectory generated by \( u_0 \) starting at \( x(0) = x_0 \) satisfies \( x(1) = \dot{x} \);
2. The trajectory generated by \( u_T \) starting at \( x(0) = \dot{x} \) satisfies \( x(1) = x_T \).

Then, we consider the sequence of controls

\[ u_n(t) = nu_0(nt) \chi_{[0,\frac{1}{n}]}(t) + nu_T(1-n(T-t)) \chi_{[T-\frac{1}{n},T]}(t). \]

A simple computation shows that

\[ x_{nu_0(nt)}(t) = x_{u_0(nt)}(t) \]
Proof. The quadratic form

\[ J_n(t) = \int_0^t |u_0(\xi)|^2 d\xi + \int_{T-t}^T |u_T - n(1 - n(1 - T - t))|^2 dt \]

holds for all \( t \in \left[ T - \frac{1}{n}, T \right] \). Therefore, for all sufficiently large \( n \in \mathbb{N} \), \( u_n \) satisfies the boundary condition \( x_{u_n}(T) = x_T \). Now,

\[
\|u_n\|_{L^2(0,T)}^2 = n^2 \left( \int_0^1 |u_0(nt)|^2 dt + \int_{T-\frac{1}{n}}^T |u_T - n(1 - n(1 - T - t))|^2 dt \right) = n^2 \left( \int_0^1 |u_0(t)|^2 \frac{1}{n} dt + \int_0^1 |u_T(t)|^2 \frac{1}{n} dt \right) = n \left( \|u_0\|_{L^2(0,1)}^2 + \|u_T\|_{L^2(0,1)}^2 \right)
\]

Also,

\[
J_n^T(u_n) = \int_0^1 x_{u_0}(nt)P_{x_u_0}(nt) dt + \int_{T-\frac{1}{n}}^T x_{u_T}(1 - n(1 - T - t))P_{x_u_T}(1 - n(1 - T - t)) dt = 
\]

\[
\int_0^1 x_{u_0}(nt)P_{x_u_0}(nt) dt - \hat{x}'P\hat{x} dt + \hat{J}_0^T + 
\]

\[
+ \int_{T-\frac{1}{n}}^T x_{u_T}(1 - n(1 - T - t))P_{x_u_T}(1 - n(1 - T - t)) - \hat{x}'P\hat{x} dt = 
\]

\[
\hat{J}_0^T + \frac{1}{n} \int_0^1 x_{u_0}(t)P_{x_u_0}(t) - \hat{x}'P\hat{x} dt + \frac{1}{n} \int_0^1 x_{u_T}(t)P_{x_u_T}(t) - \hat{x}'P\hat{x} dt = 
\]

\[
= \hat{J}_0^T + \text{const.} \frac{1}{n},
\]

where \( \hat{J}_0^T = \inf_u J_n^T(u) \). This shows that,

\[
\inf_{\|u\|_{L^2(0,T)}} \left\{ \int_0^T |u(t)|^2 dt : J_n^T(u) \leq \hat{J}_0^T + \frac{1}{n} \right\} \leq \inf_{\|u\|_{L^2(0,T)}} \left\{ \int_0^T |u(t)|^2 dt : J_n^T(u) \leq \hat{J}_0^T + \frac{1}{n} \right\}
\]

\[
= \text{const.} \frac{1}{n}. \] By letting \( n \) go to \(+\infty\), we prove the result. \( \square \)

**Proposition 9.2.** Suppose that \( x_T \in A_{x_0} \) and there exists \( x \in A_{x_0} \) such that

\[
(45) \quad x'Px < \max \left\{ x_0'Px_0, x'_T P x_T \right\}.
\]

Then, \( \sigma_T \geq \frac{1}{2} \). \( \square \)

**Proof.** The quadratic form \( x \mapsto x'Px \) admits a minimum in the closure of \( A_{x_0} \). Let \( \hat{x} \in \overline{A}_{x_0} \) be such a minimizer. Assumption (45) is equivalent to state that \( \hat{x}'P\hat{x} < \max \left\{ x_0'Px_0, x'_T P x_T \right\} \). Without loss of generality, we suppose that \( \hat{x}'P\hat{x} < x_0'Px_0 \) holds. Then, there exist \( \delta > 0, \rho > 0 \) such that \( |x - x_0| \geq \rho \) holds whenever \( x'Px < \hat{x}'P\hat{x} + \delta \) holds. Consider some fixed \( \delta \) and \( \rho \) as above. For each \( \varepsilon \in ]0, \delta^2[ \), let \( u_\varepsilon \in L^2_T(0,T) \) denote a control satisfying

\[
(46) \quad J_n^T(u_\varepsilon) < \hat{J}_0^T + \varepsilon.
\]

This last condition implies

\[
\int_0^T (\hat{x}'_u P_{xu_\varepsilon} - \hat{x}'P\hat{x}) dt < \varepsilon.
\]
Since \( x'_{u_\epsilon}(t) \) \( P_{x_{u_\epsilon}}(t) \geq \hat{\alpha} \hat{P} \hat{x} \) holds for all \( t \), this implies
\[
\lambda \left\{ t \in [0,T] : x'_{u_\epsilon}(t) P_{x_{u_\epsilon}}(t) - \hat{\alpha} \hat{P} \hat{x} \geq \delta \right\} < \frac{\varepsilon}{\delta}.
\]
Here \( \lambda \) denotes Lebesgue measure in \( \mathbb{R} \). It follows that there exists \( t_\varepsilon \in [0, \frac{T}{\delta}] \) such that \( x'_{u_\epsilon}(t_\varepsilon) P_{x_{u_\epsilon}}(t_\varepsilon) - \hat{\alpha} \hat{P} \hat{x} < \delta \). This implies that \( |x_{u_\epsilon}(t_\varepsilon) - x_0| \geq \rho \).

Let
\[
\hat{t}_\varepsilon = \min \left\{ t \in [0,T] : |x_{u_\epsilon}(t) - x_0| \geq \rho \right\} ;
\]
\[
M = \max \left\{ \sum_{i=1}^{k} |g_i(x)| : |x - x_0| \leq \rho \right\}.
\]

It is clear that \( \hat{t}_\varepsilon < \frac{\varepsilon}{\delta} \) and \( M < +\infty \). Therefore, we have the estimates
\[
\rho = |x_{u_\epsilon}(\hat{t}_\varepsilon) - x_0| = \left| \int_{0}^{\hat{t}_\varepsilon} \sum_{i=1}^{k} g_i(x_{u_\epsilon}(t)) u_{i,\varepsilon}(t) \, dt \right| \leq \int_{0}^{\hat{t}_\varepsilon} \sum_{i=1}^{k} |g_i(x_{u_\epsilon}(t))| \times |u_{\varepsilon}(t)| \, dt \leq M \int_{0}^{\hat{t}_\varepsilon} |u_{\varepsilon}(t)| \, dt \leq M \sqrt{\hat{t}_\varepsilon} \|u_{\varepsilon}\|_{L^2[0,T]} \leq \frac{M \sqrt{\varepsilon}}{\delta} \|u_{\varepsilon}\|_{L^2[0,T]}.
\]

This shows that
\[
\|u_{\varepsilon}\|_{L^2[0,T]} \geq \frac{\rho \sqrt{\delta}}{M \sqrt{\varepsilon}} = \frac{\text{Const}}{\sqrt{\varepsilon}}.
\]

Since \( u_{\varepsilon} \) is an arbitrary control satisfying (46), it follows that
\[
\inf \left\{ \ln \|u\|_{L^2[0,T]} : J_T^f(u) < J_T^f + \varepsilon \right\} \geq \frac{1}{2} \ln \varepsilon + \frac{\text{Const}}{\ln \varepsilon},
\]
which proves the result. \( \square \)

9.3. **Proof of Proposition 7.3** We start with an auxiliary lemma, which establishes Lipschitz continuity of the input-to-trajectory map of system (29) with respect to so called relaxation metric in the space of time-variant vector fields.

**Definition 9.1.** Let \( \mathcal{O} \subset \mathbb{R}^r \) be a nonempty open set and let \( \mathcal{F} \) be a set of time-variant vector fields \( \mathcal{F} : [0,T] \times \mathcal{O} \rightarrow \mathbb{R}^n \).

\( \mathcal{F} \) is said to be locally uniformly Lipschitzian with respect to \( x \) if for every compact \( K \subset \mathcal{O} \) there exists a constant \( m < +\infty \) such that
\[
|F(t,x') - F(t,x)| \leq m |x' - x|,
\]
holds for every \( F \in \mathcal{F}, t \in [0,T], x, x' \in K \).

\( \square \)

\( ^2 \) see [7, Chapter 4] for a more general definition of uniformly Lipschitzian sets with \( m \) depending on \( t \).
Lemma 9.2. Consider a family of time-variant vector fields $\mathcal{F}$, locally uniformly Lipschitzian with respect to $x$. Fix $F_0 \in \mathcal{F}$ and suppose that the solution of the differential equation

$$\dot{x}(t) = F_0(t, x(t)), \quad x(0) = x_0$$

(denoted by $x_{F_0}$) is defined for $t \in [0, T]$. For every $F \in \mathcal{F}$ we define the 'deviation'

$$\Delta_{F_0,F} = \sup_{t \in [0,T]} \left| \int_0^t F(\tau, x_{F_0}(\tau))d\tau - \int_0^t F_0(\tau, x_{F_0}(\tau))d\tau \right|. \tag{47}$$

If $\Delta_{F_0,F}$ is sufficiently small then the solution of the differential equation

$$\dot{x}(t) = F(t, x(t)), \quad x(0) = x_0$$

is defined for $t \in [0, T]$ and satisfies

$$\|x_F - x_{F_0}\|_{L_\infty[0,T]} \leq e^{mT} \Delta_{F_0,F}, \tag{48}$$

where $m < +\infty$ is a constant independent of $F$. □

Remark 9.1. Note the collocation of the norm beyond the integral sign in (47). This characterizes the so called relaxation metrics in comparison with integral metrics. For example $\Delta_{0,F}$ in (47) can become small if $F$ does not depend on $x$ and is fast oscillating with respect to $t$ (e.g. $F(t, x) = \cos Nt$, with $N$ large). □

Proof. Like in the standard proof of continuous dependence of solutions on the right-hand side of ODE's, we can assume without loss of generality that all fields $F \in \mathcal{F}$ vanish outside some bounded open set containing the compact curve $x_{F_0}$. Therefore, we can assume that all fields are complete and we only need to prove that (48) holds.

Let $m < +\infty$ denote the Lipschitz constant of the family $\mathcal{F}$. A simple computation shows that

$$\|x_F - x_{F_0}\|_{L_\infty[0,T]} \leq \left| \int_0^t F(\tau, x_F(\tau)) - F_0(\tau, x_{F_0}(\tau)) \, d\tau \right| \leq$$

$$\leq \left| \int_0^t F(\tau, x_F(\tau)) - F(\tau, x_{F_0}(\tau)) \, d\tau \right| +$$

$$+ \left| \int_0^t F(\tau, x_{F_0}(\tau)) - F_0(\tau, x_{F_0}(\tau)) \, d\tau \right| \leq$$

$$\leq m \int_0^t |x_F(\tau) - x_{F_0}(\tau)| \, d\tau + \Delta_{F_0,F}.$$ 

Therefore (48) follows by application of Gronwall inequality. □

The following Lemma is a strengthened version of the well known Gamkrelidze Approximation Lemma [6, 7, Ch. 3]:
Lemma 9.3. Consider a controlled field $F(\cdot, \cdot) : \mathbb{R}^{n+k} \rightarrow \mathbb{R}^n$, continuously differentiable with respect to all variables. Fix a compact set $K \subset \mathbb{R}^k$ and a relaxed control supported in $K$, $\eta \in \mathcal{M}_K [0, T]$, such that $x_\eta$ (the trajectory of the system $\dot{x}(t) = (F(x(t), \cdot), \eta)$, $x(0) = x_0$) is defined for all $t \in [0, T]$. There exists a sequence of piecewise constant controls $\left\{v_N : [0, T] \rightarrow K\right\}_{N \in \mathbb{N}}$, such that:

i) each $v_N$ has at most $(n + 2)N$ points of discontinuity;

ii) $\|x_{v_N} - x_\eta\|_{L^\infty[0,T]} = O\left(\frac{1}{N}\right)$ as $N \rightarrow \infty$. □

Proof. Since $F(\cdot, \cdot) : \mathbb{R}^{n+k} \rightarrow \mathbb{R}^n$ is assumed to be continuously differentiable, it follows that the set

$$\left\{ \langle F(\cdot, \cdot), \nu_t \rangle : \nu \in \mathcal{M}_K [0, T] \right\}$$

is locally uniformly Lipschitzian. Therefore, due to Lemma 9.2, we only need to show that there exists a sequence of piecewise constant controls such that each $v_N$ has at most $(n + 2)N$ points of discontinuity and $\Delta_{(F,\eta),F(\cdot,v_N)} = O\left(\frac{1}{N}\right)$ as $N \rightarrow \infty$.

Fix $N > 1$, and let $t_i = \frac{iT}{N}$, $i = 0, 1, 2, ..., N$. Due to Proposition 7.2,

$$\frac{N}{T} \int_{t_{i-1}}^{t_i} \langle F(x_\eta(t_{i-1}), \cdot), \eta_t \rangle \ dt \in \text{conv} \left\{ F(x_\eta(t_{i-1}), v) : v \in K \right\}$$

holds and the Carathéodory theorem guarantees the existence of $v_{i,j}^N \in K$, $p_{i,j}^N \geq 0$ such that

$$\sum_{j=1}^{n+2} p_{i,j}^N = 1;$$

$$\frac{T}{N} \sum_{j=1}^{n+2} p_{i,j}^N F(x_\eta(t_{i-1}), v_{i,j}^N) = \int_{t_{i-1}}^{t_i} \langle F(x_\eta(t_{i-1}), \cdot), \eta_t \rangle \ dt.$$

We construct the piecewise continuous control

$$v_N(t) = \sum_{i=1}^{N} \sum_{j=1}^{n+2} v_{i,j}^N \chi_{[t_{i-1} + \sum_{s=1}^{i-1} p_{s,j}^N + \sum_{j=1}^{n+2} p_{s,j}^N, t_i]} (t),$$

where $\chi_{[a,b]} (t)$ denotes the characteristic function of the interval $[a, b]$. Now,

$$\Delta_{(F,\eta),F(\cdot,v_N)} = \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \left( F(x_\eta(t), v_N(t)) - \langle F(x_\eta(t), \cdot), \eta_t \rangle \right) \ dt =$$

$$= \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \left( F(x_\eta(t), v_N(t)) - F(x_\eta(t_{i-1}), v_N(t)) \right) \ dt +$$

$$+ \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \left( F(x_\eta(t_{i-1}), v_N(t)) - \langle F(x_\eta(t_{i-1}), \cdot), \eta_t \rangle \right) \ dt +$$

$$+ \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \left( \langle F(x_\eta(t_{i-1}), \cdot), \eta_t \rangle - \langle F(x_\eta(t), \cdot), \eta_t \rangle \right) \ dt.$$
Since all fields being considered form a locally uniformly Lipschitzian set and $x_\eta$ is Lipschitzian with respect to time, there exists a constant $L < +\infty$, independent of $N$, such that

$$\left| \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} (F(x_\eta(t), v_N(t)) - F(x_\eta(t_{i-1}), v_N(t))) \ dt \right| \leq$$

$$\leq \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} L |x_\eta(t) - x_\eta(t_{i-1})| \ dt \leq$$

$$\leq \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} L^2 (t - t_{i-1}) \ dt = \frac{L^2 T^2}{2N}$$

holds for every sufficiently large $N$. The same argument gives the similar inequality

$$\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \left( \langle F(x_\eta(t_{i-1}), \cdot), \eta \rangle - \langle F(x_\eta(t), \cdot), \eta \rangle \right) \ dt \leq \frac{L^2 T^2}{2N}.$$  

Finally,

$$\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} (F(x_\eta(t_{i-1}), v_N(t)) - \langle F(x_\eta(t), \cdot), \eta \rangle) \ dt =$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{n+2} \int_{t_{i-1}+\sum_{k=1}^{j-1} p_{i,j,k} T}^{t_{i-1}+\sum_{k=1}^{j} p_{i,j,k} T} F(x_\eta(t_{i-1}), v_{i,j}^N) \ dt -$$

$$- \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \langle F(x_\eta(t_{i-1}), \cdot), \eta \rangle \ dt =$$

$$= \sum_{i=1}^{N} \left( \frac{T}{N} \sum_{j=1}^{n+2} p_{i,j} F(x_\eta(t_{i-1}), v_{i,j}^N) - \int_{t_{i-1}}^{t_i} \langle F(x_\eta(t_{i-1}), \cdot), \eta \rangle \ dt \right) = 0,$$

which proves that $v_N(t)$ has the desired property. \qed

For the second approximation step we will use the following lemma concerning approximation of piece-wise continuous controls by absolutely continuous controls.

**Lemma 9.4.** Consider a controlled field $F(\cdot, \cdot) : \mathbb{R}^{n+k} \mapsto \mathbb{R}^n$, continuously differentiable with respect to all variables. Fix compact sets $K_1 \subset \mathbb{R}^n$, $K_2 \subset \mathbb{R}^k$.

For every piecewise constant control $v : \{0, T\} \mapsto K_2$ with $N$ points of discontinuity such that $x_v$ (the trajectory of the system $\dot{x} = F(x,v)$, $x(0) = x_0$) lies in $K_1$ and every sufficiently small $\varepsilon > 0$ there exists a continuous piecewise linear control $w_\varepsilon : \{0, T\} \mapsto \text{conv} (\{0\} \cup K_2)$ such that:

i) $\|x_{w_\varepsilon} - x_v\|_{L_\infty[0,T]} \leq CN \varepsilon$;

ii) $\|w_\varepsilon\| \leq \frac{CN \varepsilon}{\varepsilon}$.
The piecewise linear control \( C \) therefore holds. Fix a small \( \varepsilon > 0 \) as \( C < +\infty \) is a constant depending only on the sets \( K_1, K_2 \). □

**Proof.** Fix \( v \) satisfying the assumptions of the Lemma. Let \( 0 = t_0 < t_1 < t_2 < \ldots < t_N = T \) be the points of discontinuity of \( v \). \( v \) can be represented as

\[
v(t) = \sum_{i=1}^{N} v_i \chi_{[t_{i-1}, t_i]}(t).
\]

Fix a small \( \varepsilon > 0 \) and let

\[
i_0 = 0, \quad v_{i_0} = 0, \quad i_j = \begin{cases} \min\{i : t_i \geq t_{i-1} + \varepsilon\}, & \text{if } \{i : t_i \geq t_{i-1} + \varepsilon\} \neq \emptyset, \\ N, & \text{if } \{i : t_i \geq t_{i-1} + \varepsilon\} = \emptyset. \end{cases}
\]

The piecewise linear control

\[
w_{\varepsilon}(t) = \frac{1}{\varepsilon} \sum_{j=1}^{N} (v_{i_j} - v_{i_{j-1}}) t \chi_{[t_{i_{j-1}}, t_{i_j}]}(t) + v_{i_j} \chi_{[t_{i_j} + \varepsilon, t_{i_j}]}(t)
\]

takes values on \( \text{conv}\ (\{0\} \cup K_2) \) and differs from \( v \) only on the union of intervals \( \bigcup_{j=1}^{N} [t_{i_{j-1}}, t_{i_j}] \). Since \( K_2 \) is compact, there exists a constant \( C_1 < +\infty \) such that \( |v_{i_j} - v_{i_{j-1}}| < C_1 \) holds for \( j = 1, 2, \ldots, N \). Therefore,

\[
\|w_{\varepsilon}\|^2_{L_2[0,T]} = \sum_{j=1}^{N} \frac{|v_{i_j} - v_{i_{j-1}}|^2}{\varepsilon^2} \leq \frac{C_1 N}{\varepsilon}.
\]

The Lemma guarantees that the inequality

\[
\|x_{w_{\varepsilon}} - x_v\|_{L_\infty[0,T]} \leq e^{mT} \sup_{t\in[0,T]} \left| \int_{0}^{t} F(x_v, w_{\varepsilon}) - F(x_v, v) \, d\tau \right|
\]

holds provided the right-hand side is sufficiently small.

Since \( x_v \) lies in \( K_1 \), \( v \) lies in \( K_2 \) and \( w_{\varepsilon} \) lies in \( \text{conv}\ (\{0\} \cup K_2) \), there exists a constant \( C_2 < +\infty \) such that

\[
\sup_{t\in[0,T]} \left| \int_{0}^{t} F(x_v, w_{\varepsilon}) - F(x_v, v) \, d\tau \right| 
\]

\[
\leq \sum_{j=1}^{N} \int_{t_{i_{j-1}}}^{t_{i_j} + \varepsilon} |F(x_v, w_{\varepsilon}) - F(x_v, v) \, d\tau| \leq N C_2 \varepsilon.
\]

Therefore the Lemma holds for \( C \geq \max\{C_1, e^{mT} C_2\} \). □

To conclude the proof of Proposition [7,3] we consider the augmented state \( z(t) = (J^*_f, y(t)) \) with dynamics

\[
\dot{z}(t) = \sum_{i=1}^{n+2} p_i(t) F(z(t), v^i(t)) = \sum_{i=1}^{n+2} p_i(t) \left( \begin{array}{c} \tilde{f}_0(y(t), v^i(t)) \\ \tilde{f}(y(t), v^i(t)) \end{array} \right),
\]

Here \( x_{w_{\varepsilon}} \) denotes the trajectory of the system \( \dot{x} = F(x, v) \), \( x(0) = x_0 \) and \( C < +\infty \) is a constant depending only on the sets \( K_1, K_2 \). □

Here \( \tilde{f}_0 \) and \( \tilde{f} \) are the continuous extensions of \( f_0 \) and \( f \) defined by (49)
with \( \tilde{f}_0, \tilde{f} \) defined by (33).

Under the assumptions of Proposition 7.3, there exists a compact set \( K \subset \mathbb{R}^k \) such that
\[
\eta = \sum_{i=1}^{\tilde{n}+2} p_i \delta_{v_i} \in \mathcal{M}_K[0, T].
\]

Let \( N_\varepsilon = O(\frac{1}{\varepsilon}) \) when \( \varepsilon \to 0^+ \). Due to Lemma 9.3 there exist piecewise constant controls \( \{w_\varepsilon : [0, T] \mapsto K\}_{\varepsilon > 0} \) such that \( v_\varepsilon \) has \( O(\frac{1}{\varepsilon}) \) points of discontinuity and the corresponding trajectories of (49) satisfy
\[
\|z_{w_\varepsilon} - z_\eta\|_{L_\infty[0, T]} = O(\varepsilon), \quad \text{when } \varepsilon \to 0^+.
\]

Due to Lemma 9.4 there exist continuous piecewise linear controls \( \{v_\varepsilon : [0, T] \mapsto \text{conv}(\{0\} \cup K)\}_{\varepsilon > 0} \) such that
\[
\|z_{v_\varepsilon} - z_{w_\varepsilon}\|_{L_\infty[0, T]} = O(\varepsilon), \quad \|\dot{v}_\varepsilon\|_{L_2[0, T]} = O(\varepsilon^{-3}),
\]
when \( \varepsilon \to 0^+ \). It follows that
\[
\|z_{v_\varepsilon} - z_\eta\|_{L_\infty[0, T]} \leq \|z_{v_\varepsilon} - z_{w_\varepsilon}\|_{L_\infty[0, T]} + \|z_{w_\varepsilon} - z_\eta\|_{L_\infty[0, T]} = O(\varepsilon).
\]
This shows that conditions i), ii) and iv) of Proposition 7.3 can be satisfied.

Due to Lemma 9.3, \( y_{v_\varepsilon}(T) \) lies \( \varepsilon \)-close to \( y_\eta(T) \), which lies in the integral manifold of \( G \) that contains \( x_T \). Therefore, \( v_\varepsilon \) can be modified (without changing the magnitude of the estimates above) in such a way that
\[
\|x_T - e^{Gv_\varepsilon(T)}y_{v_\varepsilon}(T)\| = O(\varepsilon), \quad \text{when } \varepsilon \to 0^+.
\]
This concludes the proof.

References

[1] A.A. Agrachev & A.V. Sarychev, On reduction of smooth control systems. Math. USSR Sbornik, 58(1987). 15–30.
[2] A. Bressan, On Differential Systems with Impulsive Controls. Rend. Semin. Univ. Padova, 78(1987). 227–235.
[3] A. Bressan & F. Rampazzo, Impulsive Control Systems without Commutativity Assumptions. J. Optimization Theory and Applications, 81(1994). 435–457.
[4] L. Cesari, Optimization. Theory and applications. problems with ordinary differential equations. Springer-Verlag, New York–Heidelberg–Berlin (1983).
[5] C.-W. Cheng & V.Mizel, On the Lavrentiev phenomenon for optimal control problems with second-order dynamics. SIAM J. Control and Optimization, 34(1996). 2172–2179.
[6] R.V. Gamkrelidze, On some extremal problems in the theory of differential equations with applications to the theory of optimal control. J. Soc. Ind. Appl. Math., Ser. A: Control, 3(1965). 106–128.
[7] R. V. Gamkrelidze, Principles of Optimal Control Theory. Plenum Press (1978).
[8] J.P. Gauthier & V.M. Zakalyukin, On the Motion Planning Problem, Complexity, Entropy and Nonholonomic Interpolation. J. Dynamical Control Systems, 12(2006). 371–404.
[9] M. Guerra, Highly Singular L-Q Problems: Solutions in Distribution Spaces. J. Dynamical Control Systems, 6(2000). 265–309.
[10] M. Guerra & A. Sarychev, Approximation of Generalized Minimizers and Regularization of Optimal Control Problems. In: *Lagrangian and Hamiltonian Methods For Nonlinear Control 2006* (F. Bullo & K. Fujimoto, Eds.). Proceedings from the 3rd IFAC Workshop, Nagoya, Japan, July 2006. Springer-Verlag, Lecture Notes in Control and Information Sciences, **366**(2007). 269–279.

[11] M. Guerra & A. Sarychev, Existence and Lipschitzian regularity for relaxed minimizers. In: *Mathematical Control Theory and Finance* (A. Sarychev et al., Eds.). Springer-Verlag (2008). 231–250.

[12] F. Jean, Entropy and Complexity of a Path in SR Geometry, Entropy and complexity of a path in sub-Riemannian geometry. *ESAIM Control Optim. Calc. Var.*. **9**(2003). 485–508.

[13] V. Jurdjevic, Geometric Control Theory. *Cambridge University Press* (1997).

[14] V. Jurdjevic & I. A. K. Kupka, Linear systems with singular quadratic cost. *Preprint* (1992).

[15] H.J. Kelley, R.E. Kopp & H.G. Moyer, Singular extremals. In: *Topics on Optimization* (G. Leitmann, Ed.). *Academic Press, New York* (1967). 63–101.

[16] A.J. Krener, The high order maximum principle and its application to singular extremals. *SIAM J. Control Optimization*, **15**(1977). 256–293.

[17] Yu. Orlov, Vibrocorrect differential equations with measures, *Matematicheskie zametki*, **38**(1985). 110–119.

[18] Yu.V. Orlov, Theory of Optimal Systems with Generalized Controls (in Russian). *Nauka, Moscow* (1988).

[19] Yu. Orlov, Does cheap control solve a singular nonlinear quadratic problem?. In: *Unsolved Problems in Mathematical Systems and Control Theory* (Vincent D. Blondel & Alexandre Megretski, Eds.). *Princeton University Press* (2004). 111–113.

[20] A.V. Sarychev, Nonlinear Systems with Impulsive and Generalized Function Controls. In: *Proceedings of a IIASA Workshop held in Sopron, Hungary, June 1989* (C. I. Byrnes & A. Kurzhansky, Eds.). *Birkhäuser, Boston* (1991). 244–257.

[21] A.V. Sarychev, First and Second-Order Integral Functionals of the Calculus of Variations Which Exhibit the Lavrentiev Phenomenon. *J. Dynamical and Control Systems*, **3**(1997).565-588.

[22] J.C. Willems, A. Kitapçı & L.M. Silverman, Singular optimal control: a geometric approach. *SIAM J. Control Optimization*, **24**(1986). 2. 323–337.

---

1Instituto Superior de Economia e Gestão, Technical University of Lisbon, R. do Quelhas 6, 1200-781 Lisboa, Portugal, mguerra@iseg.utl.pt, 2Dipartimento di Matematica per le Decisioni, Università di Firenze, via C.Lombroso 6/17, 50134 - Firenze (FI), Italy, asarychev@unifi.it