MASSES AND BOOST-INARIANT WAVE FUNCTIONS OF
HEAVY QUARKONIA FROM
THE LIGHT-FRONT HAMILTONIAN OF QCD

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A new scheme for calculating masses and boost-invariant wave functions of heavy quarkonia is developed in a light-front Hamiltonian formulation of QCD. Only the simplest approximate version with one flavor of quarks and an ansatz for the mass gap for gluons is discussed. The resulting spectra look reasonably good in view of the crude approximations made in the simplest version.

Keywords: bottomonium, charmonium, constituent, quark, gluon

1. Motivation for the LF Hamiltonian approach to QCD

The method for calculating masses and wave functions of heavy quarkonia that is reported here stems from the program of a weak-coupling expansion for Hamiltonians in light-front (LF) QCD. The LF form of dynamics was discovered by Dirac and continues to excite imagination of physicists. Many authors have rediscovered LF dynamics. A famous example concerns application to hard exclusive processes. Review articles provide other references concerning LF formalism. One reason of the great interest is that the field quantization on the front hyperplane leads to 7 kinematical generators of the Poincaré group, instead of only 6 kinematical generators in the standard form (3 momentum and 3 angular momentum operators). Another reason is that the vacuum problem in the LF formulation of quantum field theory appears intriguingly different from the standard version. The same two reasons propelled also the development of the renormalization group procedure for effective particles (RGPEP) that is the basis of the method discussed here. But there are two more reasons.

The first is that the LF Fock space of free bare particles can be intro-
duced before one constructs the concept of a quantum field operator and builds Hamiltonian interaction terms for the bare quanta using such field operators. This is useful when one attempts to mathematically define a theory of quarks and gluons that never appear as incoming or outgoing particles in scattering experiments but exist inside hadrons. Proceeding in this order, one can regulate the interaction terms in the LF Hamiltonian in a boost-invariant way. The regularization is accomplished using the relative transverse momenta and fractions of total "plus" momentum that the bare particles in interaction are carrying. The transverse and "plus" momenta are defined with respect to the direction of the front hyperplane, the latter conventionally defined by the condition \( x^+ = x^0 + x^3 = 0 \) in a frame of reference in which the front is moving along \( z \)-axis, extending in the transverse directions of coordinates \( x \). The transverse momenta of the particles are denoted by \( k^\perp \) and their "plus" momenta by \( k^+ = k^0 + k^3 \). The regulated interaction Hamiltonian for bare particles is invariant with respect to boosts along the \( z \)-axis and two additional boost-like transformations that can change transverse momenta to arbitrary values. It is also invariant with respect to two translations in the \( \perp \) directions, translation in the \( x^- \) direction, and rotations around the \( z \)-axis (typically directed along the beam, a dominant momentum transfer, or a suitable combination thereof depending on a scattering experiment, but in a complete theory the choice should not matter). Thus, the Hamiltonian has the same structure in a large class of frames of reference (7 dimensional). Consequently, one does not need to construct Hamiltonian counterterms that restore boost symmetry when one tries to quantitatively explain the mechanism by which masses, spins, and other quantum numbers of hadrons are formed. Most attractively, the basic Hamiltonian has the same structure in the rest frame of a hadron, where the constituent picture works, and in the infinite momentum frame, where the parton model works. The LF Hamiltonian approach raises hopes for conceptual and quantitative explanation of the constituent and parton models in a single and complete formulation of QCD.

The second reason is that one can take advantage of the concept of potentials acting at a distance between relativistic quarks and gluons, a feast not conceivable in the standard approach that is defined using objects distributed on a space-like hyperplane in space-time. Every interaction between two objects located at different points of such a hyperplane corresponds to a dynamical effect spreading faster than light and has to eventually cancel out in observables (this happens in perturbative QED but it is not clear how it may happen in non-perturbative QCD).
fore, it is common in the standard approach to consider only action and use local Lagrangian densities for field variables in a path-integral formula for transition amplitudes. Geometrical ideas such as strings and other nonlocal objects in multidimensional spaces are then used to regulate and explain the interaction terms in the Lagrangians. An additional argument for the Lagrangian approach is that it can incorporate variation of the metric in space-time and, hopefully, illuminate the problem of connection between particle dynamics and gravity\textsuperscript{13}. But if one leaves gravity aside as too weak to be of an immediate concern at the scale of hadronic binding mechanism, it is useful to observe that the LF Hamiltonian at $x^+ = 0$ can contain potential terms that act between particles separated by arbitrarily large distances and such interactions can obey the rule that dynamical effects do not spread faster than light. Namely, when the bare point-like particles have the same transverse positions, the four-dimensional space-time interval between them is zero no matter how large is their separation in the direction of $x^-$. In fact, the LF counterpart of the Coulomb interaction between two particles 1 and 2 on the LF is proportional to $|x_1^- - x_2^-|$ when $x_1^+ = x_2^+$, and otherwise vanishes. Precisely this type of interaction leads to a model of confinement in a 1+1 dimensional theory\textsuperscript{14}. It is clear that the LF Hamiltonians are very singular when transverse distances between charged point-like particles tend to zero, and the singular terms can involve entire functions of the $x^-$ distances between the particles.

Both reasons described above indicate that one needs a powerful ultraviolet renormalization technique for Hamiltonians in order to develop LF QCD (note that the Wilsonian concept of universality could help in identifying effective Hamiltonians irrespectively of many details in setting up the initial bare theory). A new technique has been invented\textsuperscript{15,16} and adopted in a general scheme of weak coupling expansion in LF QCD\textsuperscript{1}. More recently, the Hamiltonian approach has been redesigned in the form of RGPEP\textsuperscript{8}. The results reported below are obtained using RGPEP and an ansatz for a mass-gap for gluons.

2. Binding above threshold in heavy quarkonia

Since there is not enough room here to thoroughly explain the RGPEP in application to heavy quarkonia\textsuperscript{17} in comparison to other approaches, only the main steps are indicated. The central puzzle is how a systematic treatment of QCD can produce binding of quarks above threshold. QED describes quantum binding only below the mass threshold. So, how can binding above the threshold emerge in a relativistic quantum theory?
One begins from the Lagrangian for QCD with one flavor of quarks
\[ L = \bar{\psi} (i \not{D} - m) \psi - \frac{1}{4} F^{\mu \nu a} F_{\mu \nu}^a. \] (1)

A canonical LF procedure in gauge \( A^+ = 0 \) produces a Hamiltonian with many terms (constraint equations are solved explicitly)\(^5\)
\[ H_{\text{can}} = H_{\psi^2} + H_{A^2} + H_{\psi A^2} + H_{(\psi^2)^2} \]
\[ + \ H_{A^3} + H_{A^4} + H_{[\psi A^2]^2} + H_{[\partial AA]^2}(\psi^2). \] (2)

Each of these terms is an integral of the corresponding Hamiltonian density over the LF hyperplane with \( x^+ = 0 \).
\[ H_{\psi^2} = \frac{1}{2} \bar{\psi} \gamma^+ \frac{-\not{D} + m}{i \not{D}} \psi, \quad H_{A^2} = -\frac{1}{2} A^+ (\not{D} A^+) A^+, \]
\[ H_{\psi A^2} = g \bar{\psi} \not{A} \psi, \quad H_{(\psi^2)^2} = \frac{1}{2} g^2 \bar{\psi} \gamma^+ t^a \psi \frac{1}{(i \not{D})^2} \bar{\psi} \gamma^+ t^a \psi, \text{ etc.} \] (4)

The fields at \( x^+ = 0 \) are expanded into creation and annihilation operators for bare quarks and gluons, the measure is \( [k] = dk^+ d^2 k^\perp / (16 \pi^3 k^+) \):
\[ \psi = \sum_{\sigma c} \int [k] \left[ \chi_c u_{\kappa \sigma} b_{\kappa \sigma c} e^{-i k \cdot x} + \bar{\chi}_c v_{\kappa \sigma} d^\dagger_{\kappa \sigma c} e^{i k \cdot x} \right], \] (5)
\[ A^\mu = \sum_{\sigma c} \int [k] \left[ \epsilon^\mu_{\kappa \sigma} a_{\kappa \sigma c} e^{-i k \cdot x} + \bar{\epsilon}^\mu_{\kappa \sigma} a^\dagger_{\kappa \sigma c} e^{i k \cdot x} \right], \] (6)
\( c \) stands for color, \( \sigma \) for spin. The bilinear terms in \( H \) provide kinetic energies for the bare particles and the terms with more than two fields provide interactions that are regulated\(^1\) as indicated in Section 1. The regularization implies appearance of counterterms, \( H_{\text{CT}} \), that restore the dynamics that was cut off by the regularization. The full regulated Hamiltonian, \( H = [H_{\text{can}} + H_{\text{CT}}]_{\text{reg}} \) provides the initial condition for RGPEP (RGPEP is also used to determine \( H_{\text{CT}} \))\(^8\). The main step is to replace the canonical operators \( b, d, \) and \( a, \) or their hermitean conjugates in Eqs. (5) and (6), commonly denoted by \( q_{\text{can}} \), by unitarily equivalent operators that create or annihilate effective particles corresponding to the renormalization group parameter \( \lambda, q_{\lambda} = U_{\lambda} q_{\text{can}} U_{\lambda}^\dagger \), so that \( q_{\infty} = q_{\text{can}} \) and \( dH_{\lambda} / d\lambda = [T_{\lambda}, H_{\lambda}] \), where \( T_{\lambda} = dU_{\lambda} / d\lambda U_{\lambda}^\dagger \). Given the initial condition \( H_{\infty} = [H_{\text{can}} + H_{\text{CT}}]_{\text{reg}} \), one can systematically evaluate the Hamiltonian \( H_{\lambda} = H_{\infty} + \int_{\lambda}^\infty ds [T_s, H_s] \) in perturbation theory. \( H_{\lambda} \) is equal to \( H \) but it is expressed in terms of operators creating and annihilating effective particles of size \( 1/\lambda \) with respect to strong interactions. Since \( H_{\lambda} \) is expressed in terms of the creation and annihilation operators for effective quarks and gluons instead of the
bare canonical ones, it contains different interaction terms, including new
effective potentials. For $\lambda$ on the order of hadronic masses, the effective
particles are expected to correspond to the constituent quarks and gluons
that are used to describe hadrons in particle tables. The perturbative pro-
procedure for evaluating $H_\lambda$ is safe from genuine infrared singularities because
the RGPEP generator $T_\lambda$ is designed to exclude the possibility that energy
denominators in perturbation theory are significantly smaller than $\lambda$.

A quarkonium eigenvalue problem for the QCD Hamiltonian $H_\lambda$,
$(P^+H_\lambda - P^{\perp 2})|P\rangle = M^2|P\rangle$, is solved by first eliminating the eigenval-
ues $P^+$ and $P^{\perp}$ of three kinematical momentum operators $P_\lambda^+$ and $P^{\perp}_\lambda$ (these operators are also provided by RGPEP)
and obtaining an eigenvalue equation for the quarkonium mass $M$ (the center-of-mass motion is
eliminated from the eigenvalue problem exactly). Still, the eigenstate $|P\rangle$
is built from the virtual effective particles in the LF Fock space and ca rries
four-momentum $P$ with $P^- = (M^2 + P^{\perp 2})/P^+$. In terms of the effective
quark-antiquark, quark-antiquark-gluon, and other components:

$$|P\rangle = |Q_\lambda \bar{Q}_\lambda\rangle + |Q_\lambda \bar{Q}_\lambda g_\lambda\rangle + \ldots . \quad (7)$$

This expansion may converge, in distinction from the expansion of the same
state into canonical bare-particle sectors, because interactions in $H_\lambda$ are
limited to momentum transfers smaller than $\lambda$ by the form factors $f_\lambda$ that
appear in all interaction vertices in $H_\lambda$. The form factors are introduced
through the generator $T_\lambda$ of RGPEP. In the effective-particle basis, the
Hamiltonian $H_\lambda$ takes a matrix form

$$[H_\lambda] = \begin{bmatrix} T_3 + V_3 & Y \\ Y^\dagger & T_2 + V_2 \end{bmatrix} \rightarrow H_{2+3} = \begin{bmatrix} T_3 + \mu^2_{\text{ansatz}} & Y \\ Y^\dagger & T_2 + V_2 \end{bmatrix}, \quad (8)$$

in which dots denote couplings with sectors with more than 3 effective
constituents, $T$ refers to kinetic energy terms, $V$ to potentials, $Y$ to emission
of effective gluons by quarks, and 2 and 3 to the Fock components with 2
and 3 effective particles. The arrow indicates a truncation of the system to
sectors $|Q_\lambda \bar{Q}_\lambda\rangle$ and $|Q_\lambda \bar{Q}_\lambda g_\lambda\rangle$ only, which is done at the price of introducing
an ansatz for the gluon mass gap,

$$\mu^2_{\text{ansatz}} = \left(1 - \frac{\alpha^2}{\alpha_s^2}\right) \mu^2. \quad (9)$$

The ansatz is so designed that when the coupling constant $\alpha$ (this is the
effective coupling at some small scale $\lambda$) is extrapolated to a realistically
strong value $\alpha_s$, the ansatz will be removed and the true QCD interactions
can be recovered order-by-order in the weak coupling expansion in $\alpha$. The gap function $\mu^2$ is inserted in order to model the effect of all the non-abelian Coulomb potentials, $V_3$, that act in sector $|Q_\lambda \bar{Q}_\lambda g_\lambda\rangle$ and the interactions that produce couplings to additional sectors with more constituents (the dots). It is very unlikely that the first approximation in QCD should be $\mu^2 = 0$. But if $\mu^2 \neq 0$, the resulting dynamics in the $|Q_\lambda \bar{Q}_\lambda\rangle$ sector is described in the leading order in $\alpha$ by the eigenvalue equation $H_{QQ\lambda}|P\rangle = M^2|P\rangle$, where the effective quark-antiquark Hamiltonian has the form (qualitatively)

$$H_{QQ\lambda} = T_2 + V_2 + Y + \frac{1}{T_3 + \mu^2}Y.$$

(10)

The main point is that the gluon emission and absorption produces diverging (for small $q_z$) terms of the form $f_\lambda \frac{4m^2}{q_z^2} \frac{\mu^2}{T_3 + \mu^2} f_\lambda$, in which the momentum transfer $\vec{q}$ approaches zero. This happens also in the quark self-interaction terms. The net effect is positive, lifting the quark energy above threshold. In addition, the factor dependent on $\mu^2$ becomes 1 for small $\vec{q}$ irrespectively of the details of the ansatz for $\mu^2$. The final result is a harmonic oscillator potential that appears as a leading correction to the color Coulomb interaction at typical distances between the quarks (the Coulomb term appears with the Breit-Fermi spin factors). Technically, it is the harmonic oscillator term that leads to the binding above threshold, $M > 2m$, where $m$ is the mass ascribed to the quarks. Such effect is absent in positronium in QED because there are no Coulomb-like interactions between photons and electrons and no mass-gap for photons.

### 3. Masses and wave functions in the crudest approximation

The resulting eigenvalue equation for quarkonium wave function can be solved numerically and the mass spectrum depends on the choice of the coupling constant $\alpha$ and quark mass $m$ at some value of $\lambda$. The Breit-Fermi terms include three-dimensional $\delta$-functions that are smeared and made finite by the presence of the form factors $f_\lambda$. If one assumes $\alpha_{MZ} \sim 0.12$, the RGPEP evolution with one flavor of quarks in the same Hamiltonian scheme gives $\alpha \sim 0.326$ at $\lambda \sim 3.7$ GeV. Table 1 shows masses of $b\bar{b}$ quarkonia obtained for $\alpha = 0.326$ and $m = 4857$ MeV at $\lambda = 3699$ MeV, adjusted to fit masses of $\chi_1(1P)$ and $\chi_1(2P)$. The pattern of differences in the 4th column agrees with expectations in the new scheme. All details concerning this calculation and results for some other quarkonia can be found elsewhere. The oscillator frequency corresponding to Table 1 is $\omega = 182$ MeV. The bottom line is that the realistic value of $\alpha$ is near $1/3$ and
Table 1. Qualitative illustration of bottomonium masses.

| meson name | calculation (MeV) | experiment (MeV) | difference (MeV) |
|------------|------------------|-----------------|-----------------|
| Υ10865     | 10725            | 10865           | -140            |
| Υ10580     | 10464            | 10580           | -116            |
| Υ3S        | 10382            | 10269           | 7               |
| χ22P       | 10276            | 10256           | 0               |
| χ12P       | 10256            | 10232           | 27              |
| χ02P       | 10226            | 10007           | -111            |
| χ21P       | 9912             | 9912            | 0               |
| χ11P       | 9893             | 9893            | 0               |
| χ01P       | 9865             | 9859            | 5               |
| Υ2S        | 10012            | 10023           | -11             |
| ηc1S       | 9551             | 9460            | 91              |
| ηb1S       | 9510             | 9300            | 210             |

the new Hamiltonian approach to QCD can be further studied in a weak coupling expansion in the case of heavy quarkonia, including many effects in the complex relativistic color dynamics of virtual quarks and gluons.

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