Non universality of fluctuations of outliers for Hermitian polynomials in a Wigner matrix and a spiked diagonal matrix

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Abstract

We study the fluctuations associated to the a.s. convergence of the outliers established in [13] of an Hermitian polynomial in a Wigner matrix and a spiked deterministic real diagonal matrix. Thus, we extend the non universality phenomenon established in [21] for additive deformations of Wigner matrices, to any Hermitian polynomial. The result is described using the operator-valued subordination functions of free probability theory.

Key words: Random matrices; Free probability; outliers; fluctuations; nonuniversality; Linearization; Operator-valued subordination.

1 Introduction

There is currently a quite precise knowledge of the asymptotic spectral properties (i.e. when the dimension of the matrix tends to infinity) of a number of “classical” random matrix models (Wigner matrices, Wishart matrices, invariant ensembles...). This understanding covers both the so-called global regime (asymptotic behavior of the spectral measure) and the local regime (asymptotic behavior of the extreme eigenvalues and eigenvectors, spacings...). We refer to the monographies [2, 5, 27, 29, 35, 38] for a thorough introduction to random matrix theory.

Practical problems (in the theory of statistical learning, signal detection etc.) naturally lead to wonder about the spectrum reaction of a given random matrix after a deterministic perturbation. For example, in the signal theory, the deterministic perturbation is seen as the signal, the perturbed matrix is perceived as a “noise”, and the question is to know whether the observation of the spectral properties of “signal plus noise” can give access to significant parameters on the signal. Theoretical results on these “deformed” random models may allow to establish statistical tests on these parameters. A typical illustration is the so-called BBP phenomenon (after Baik, Ben Arous, Péché [3]) which put forward outliers (eigenvalues that move away from the rest of the spectrum) and their Gaussian fluctuations for spiked covariance matrices.
Péché \cite{39} established Gaussian fluctuations for any outlier of a low rank additive deformation of a G.U.E. matrix. Fluctuations of outliers for additive finite rank deformations of non-Gaussian Wigner matrices have been studied in \cite{21, 22, 28, 40, 41}. It turns out that the limiting distribution depends on the localisation/delocalisation of the eigenvectors associated to the non-null eigenvalues of the perturbation. Note that in the G.U.E. case investigated by Péché \cite{39}, the eigenvectors of the perturbation are irrelevant for the fluctuations, due to the unitary invariance in Gaussian models. Let us illustrate this dependence on the eigenvectors of the perturbation in a very simple situation. Let

\[
\begin{pmatrix}
W_{ij}
\end{pmatrix}_{1 \leq i, j \leq N}
\]

be a $N \times N$ Hermitian Wigner matrix where \{\begin{align*}
W_{ii}, \sqrt{2R}W_{ij}, \sqrt{2I}W_{ij}
\end{align*}\}$_{1 \leq i < j}$ are independent identically distributed random variables with law $\mu$, $\mu$ is a symmetric distribution, with variance $\sigma^2$, and satisfies a Poincaré inequality (see the Appendix). Note that when $\mu$ is Gaussian, $W_N$ is a G.U.E matrix. Consider two finite rank perturbations of rank 1, with one non-null eigenvalue $\theta > \sigma$.

The first one $A_{N}^{(1)}$ is a matrix with all entries equal to $\theta/N$ (delocalized eigenvector associated to $\theta$). The second one $A_{N}^{(2)}$ is a diagonal matrix (localized eigenvector associated to $\theta$). The limiting spectral distribution of each matrix $M_{N}^{(i)} = W_N + A_{N}^{(i)}$ ($i = 1, 2$) is the semi-circular distribution

\[
d\mu_\sigma(t) = \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - t^2} 1_{[-2\sigma,2\sigma]}(t) dt.
\]

Nevertheless the largest eigenvalue $\lambda_1$ of each matrix $M_{N}^{(i)}$ ($i = 1, 2$) separates from the bulk and converges towards $\rho_\theta := \theta + \frac{\sigma^2}{\theta} (> 2\sigma)$. The fluctuations of $\lambda_1$ around $\rho_\theta$ are given as follows:

**Proposition 1.** 1. Delocalized case \cite{28}: The largest eigenvalue $\lambda_1(M_{N}^{(1)})$ have Gaussian fluctuations,

\[
\sqrt{N}(\lambda_1(M_{N}^{(1)}) - \rho_\theta) \overset{D}{\rightarrow} N(0, \sigma^2(1 - \sigma^2/\theta^2)).
\]

2. Localized case \cite{21}: The largest eigenvalue $\lambda_1(M_{N}^{(2)})$ fluctuates as

\[
\sqrt{N}(1 - \sigma^2/\theta^2)(\lambda_1(M_{N}^{(2)}) - \rho_\theta) \overset{D}{\rightarrow} \mu \ast N(0, \nu_\theta).
\]

where the variance $\nu_\theta$ of the Gaussian distribution depends on $\theta$ and the second and fourth moments of $\mu$.

Hence, for localized eigenvectors of the perturbation, the limiting distribution depends on the distribution of the entries of the Wigner matrix and thus, this uncovers a non universality phenomenon. This paper wants to extend such a non universality phenomenon for an additive deformation, to general polynomials in a Wigner matrix and a diagonal deterministic matrix. Free probability is a main tool to achieve this purpose.

Free probability theory was introduced by Voiculescu around 1983 motivated by the isomorphism problem of von Neumann algebras of free groups. He developed a noncommutative probability theory, on a noncommutative probability space, in which a new notion of freeness plays the role of independence in classical probability. Around 1991, Voiculescu \cite{44} threw a bridge connecting random matrix theory with free probability since he realized that the freeness property is also present for many classes of random matrices, in the asymptotic regime
when the size of the matrices tends to infinity. Since then, several papers aimed at developing the contribution of free probability theory to the analysis of the spectral properties of deformed ensembles and polynomials in random matrices. In particular, the main principle of subordination in free probability is emphasized as a main tool in the understanding of the localization of the outliers and the corresponding eigenvectors of many matricial models. It was the purpose of [20] to put forward an unified understanding based on subordination in free probability for studying the spectral properties of full rank deformations of classical Hermitian matrix models. This investigation relies notably on [23, 17, 18, 11, 19]. This universal understanding culminates in [13] dealing with non-commutative polynomials in random Hermitian matrices; this investigation is achieved by an even more general methodology based on a linearization procedure and operator-valued subordination properties.

The aim of this paper is to study the fluctuations associated to the a.s. convergence of the outliers described in [13] of an Hermitian polynomial in a Wigner matrix and a spiked deterministic Hermitian matrix (spiked means that the matrix has a fixed eigenvalue outside the support of its limiting spectral measure). Capitaine and Pêché [21] established Gaussian fluctuations for any outlier of a full rank additive deformation of a G.U.E. matrix using scalar-valued free probability theory. We generalize this result to any polynomial in a G.U.E. matrix and a deterministic Hermitian matrix which has a spike with multiplicity one, using operator-valued free probability theory. Moreover, considering any Hermitian polynomial in a non-Gaussian Wigner matrix and a deterministic real diagonal matrix which has a spike with multiplicity one, we establish that the limiting distribution of outliers is the classical convolution of a Gaussian distribution and the distribution of the entries of the Wigner matrix; thus, this extends the non universality phenomenon [2] established in [21] for additive deformations of Wigner matrices. The result is described in terms of operator-valued subordination functions related to a linearization of the noncommutative polynomial involved in the definition of our model. Therefore, we start by describing the necessary terminology and results concerning linearization procedure and free probability theory in Sections 2 and 3. In Section 4 we present our matrix model and main results (Theorem 3 and Corollary 1). Section 5 gathers several preliminary results that will be used in Section 6 to prove Theorem 3. An Appendix recalls some basic facts on Poincaré inequalities and concentration phenomenon that are used in some proofs.

2 A linearization trick

A powerful tool to deal with non commutative polynomials in random matrices or in operators is the so-called “linearization trick” that goes back to Haagerup and Thorbjørnsen [30] [31] in the context of operator algebras and random matrices (see [36]). We use the procedure introduced in [1, Proposition 3].

Given a polynomial \( P \in \mathbb{C}\langle X_1, \ldots, X_k \rangle \), we call linearization of \( P \) any \( L_P \in M_m(\mathbb{C}) \otimes \mathbb{C}\langle X_1, \ldots, X_k \rangle \) such that

\[
L_P := \begin{pmatrix} 0 & u \\ v & Q \end{pmatrix} \in M_m(\mathbb{C}) \otimes \mathbb{C}\langle X_1, \ldots, X_k \rangle
\]

where
1. \( m \in \mathbb{N} \),

2. \( Q \in M_{m-1}(\mathbb{C}) \otimes \mathbb{C}(X_1, \ldots, X_k) \) is invertible,

3. \( u \) is a row vector and \( v \) is a column vector, both of size \( m-1 \) with entries in \( \mathbb{C}(X_1, \ldots, X_k) \),

4. the polynomial entries in \( Q, u \) and \( v \) all have degree \( \leq 1 \),

5. 

\[
P = -uQ^{-1}v.
\]

It is shown in [1] that, given a polynomial \( P \in \mathbb{C}(X_1, \ldots, X_k) \), there exist \( m \in \mathbb{N} \) and a linearization \( L_P \in M_m(\mathbb{C}) \otimes \mathbb{C}(X_1, \ldots, X_k) \). The algebra of polynomials in non-commuting indeterminates \( X_1, \ldots, X_k \) becomes a *-algebra by anti-linear extension of \( (X_{i_1}X_{i_2} \cdots X_{i_l})^* = X_{i_l} \cdots X_{i_2}X_{i_1} \), \((i_1, \ldots, i_l) \in \{1, \ldots, k\}^l, l \in \mathbb{N} \setminus \{0\}\). It turns out that if \( P \) is self-adjoint, \( L_P \) can be chosen to be self-adjoint.

The well-known result about Schur complements (see [8], Chapter 10, Proposition 1) yields then the following invertibility equivalence.

**Lemma 1.** Let \( P = P^* \in \mathbb{C}(X_1, \ldots, X_k) \) and let \( L_P \in M_m(\mathbb{C}(X_1, \ldots, X_k)) \) be a linearization of \( P \) with the properties outlined above. Let \( e_1 \) be the \( m \times m \) matrix whose single nonnull entry equals one and occurs in the row 1 and column 1. Let \( y = (y_1, \ldots, y_k) \) be a \( k \)-tuple of self-adjoint operators in a unital \( C^* \)-algebra \( A \). Then, for any \( z \in \mathbb{C}, \ ze_{11} \otimes 1_A - L_P(y) \) is invertible if and only if \( z1_A - P(y) \) is invertible and we have

\[
(ze_{11} \otimes 1_A - L_P(y))^{-1} = \begin{pmatrix} (z1_A - P(y))^{-1} & * \\ * & * \end{pmatrix}.
\]

Beyond the equivalence described above, we will use the following bound.

**Lemma 2.** [13] Let \( z_0 \in \mathbb{C} \) be such that \( z_01_A - P(y) \) is invertible. There exist two polynomials \( T_1 \) and \( T_2 \) in \( k \) commutative indeterminates, with nonnegative coefficients, depending only on \( L_P \), such that

\[
\left\| (z_0e_{11} \otimes 1_A - L_P(y))^{-1} \right\| \\
\leq T_1(\|y_1\|, \ldots, \|y_k\|) \left\| (z_01_A - P(y))^{-1} \right\| + T_2(\|y_1\|, \ldots, \|y_k\|).
\]

Moreover, if the distance from \( z_0 \) to the spectrum of \( P(y) \) is at least \( \delta > 0 \), and for any \( i \in \{1, \ldots, k\}, \|y_i\| \leq C \), for some positive real numbers \( \delta \) and \( C \), then there exists a constant \( \varepsilon > 0 \), depending only on \( L_P, \delta, C \) such that the distance from 0 to the spectrum of \( (z_0e_{11} \otimes 1_A - L_P(y)) \) is at least \( \varepsilon \).

## 3 Free probability theory

### 3.1 Scalar-valued free probability theory

For the reader’s convenience, we recall the following basic definitions from free probability theory. For a thorough introduction to free probability theory, we refer to [18].
• A $C^*$-probability space, resp. a $W^*$-probability space, is a pair $(A, \phi)$ consisting of a unital $C^*$-algebra $A$, resp. of a unital von Neumann algebra, and a state $\phi$ on $A$ (i.e. a linear map $\phi : A \to \mathbb{C}$ such that $\phi(1_A) = 1$ and $\phi(aa^*) \geq 0$ for all $a \in A$), resp. a normal state. $\phi$ is a trace if it satisfies $\phi(ab) = \phi(ba)$ for every $(a, b) \in A^2$. A trace is said to be faithful if $\phi(aa^*) > 0$ whenever $a \neq 0$. An element of $A$ is called a noncommutative random variable.

• The noncommutative distribution of a family $a = (a_1, \ldots, a_k)$ of noncommutative random variables in a $C^*$-probability space $(A, \phi)$ is defined as the linear functional $\mu_a : P \mapsto \phi(P(a, a^*))$ defined on the set of polynomials in $2k$ noncommutative indeterminates, where $(a, a^*)$ denotes the $2k$-tuple $(a_1, \ldots, a_k, a_1^*, \ldots, a_k^*)$. For any self-adjoint element $a_1$ in $A$, there exists a probability measure $\nu_{a_1}$ on $\mathbb{R}$ such that, for every polynomial $P$, we have

$$\mu_{a_1}(P) = \int P(t) d\nu_{a_1}(t).$$

Then, we identify $\mu_{a_1}$ and $\nu_{a_1}$. If $\phi$ is faithful then the support of $\nu_{a_1}$ is the spectrum of $a_1$ and thus $|a_1| = \sup\{|z|, z \in \text{support}(\nu_{a_1})\}$.

• A family of elements $(a_i)_{i \in I}$ in a $C^*$-probability space $(A, \phi)$ is free if for all $k \in \mathbb{N}$ and all polynomials $p_1, \ldots, p_k$ in $2k$ noncommutative indeterminates, one has

$$\phi(p_1(a_{i_1}, a_{i_1}^*) \cdots p_k(a_{i_k}, a_{i_k}^*)) = 0$$

whenever $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{n-1} \neq i_k$ and $\phi(p_l(a_{i_l}, a_{i_l}^*)) = 0$ for $l = 1, \ldots, k$.

• A noncommutative random variable $x$ in a $C^*$-probability space $(A, \phi)$ is a standard semicircular variable if $x = x^*$ and for any $k \in \mathbb{N}$,

$$\phi(x^k) = \int t^k d\mu_{sc}(t)$$

where $d\mu_{sc}(t) = \frac{1}{\pi} \sqrt{4 - t^2}1_{[-2,2]}(t)dt$ is the semicircular standard distribution.

• Let $k$ be a nonnull integer number. Denote by $P$ the set of polynomials in $2k$ noncommutative indeterminates. A sequence of families of variables $(a_n)_{n \geq 1} = (a_1(n), \ldots, a_k(n))_{n \geq 1}$ in $C^*$-probability spaces $(A_n, \phi_n)$ converges, when $n$ goes to infinity, respectively in distribution if the map $P \in P \mapsto \phi_n(P(a_n, a_n^*))$ converges pointwise and strongly in distribution if moreover the map $P \in P \mapsto \|P(a_n, a_n^*)\|$ converges pointwise.

**Proposition 2.** [20 Proposition 2.1] Let $x_n = (x_1(n), \ldots, x_p(n))$ and $x = (x_1, \ldots, x_p)$ be $p$-tuples of variables in $C^*$-probability spaces, $(A_n, \phi_n)$ and $(A, \phi)$, with faithful states. Then, the following assertions are equivalent.

- $x_n$ converges strongly in distribution to $x$,
- for any self-adjoint variable $h_n = P(x_n)$, where $P$ is a fixed polynomial, $\mu_{h_n}$ converges in weak-* topology to $\mu_h$ where $h = P(x)$. Moreover, the support of $\mu_{h_n}$ converges in Hausdorff distance to the support of $\mu_h$, that is: for any $\epsilon > 0$, there exists $n_0$ such that for any $n \geq n_0$,

$$\text{supp}(\mu_{h_n}) \subset \text{supp}(\mu_h) + (-\epsilon, +\epsilon).$$

The symbol supp means the support of the measure.
Additive free convolution arises as a natural analogue of classical convolution in the context of free probability theory. For two Borel probability measures $\mu$ and $\nu$ on the real line, one defines the free additive convolution $\mu \⊞ \nu$ as the distribution of $a + b$, where $a$ and $b$ are free self-adjoint random variables with distributions $\mu$ and $\nu$, respectively. We refer to [15, 34, 43] for the definitions and main properties of free convolutions. Let us briefly recall the fundamental analytic subordination properties [16, 45, 47] of this convolution.

The analytic subordination phenomenon for free additive convolution was first noted by Voiculescu in [45] for free additive convolution of compactly supported probability measures. Biane [16] extended the result to free additive convolutions of arbitrary probability measures on $\mathbb{R}$. A new proof was given later, using a fixed point theorem for analytic self-maps of the upper half-plane [10]. Note that such a subordination property allows to give a new definition of free additive convolution [25]. Let us define the reciprocal Cauchy-Stieltjes transform $F_{\mu}(z) = 1/g_{\mu}(z)$, which is an analytic self-map of the upper half-plane, where $g_{\mu} : z \in \mathbb{C} \setminus \mathbb{R} \mapsto \int \frac{1}{z-t}d\mu(t)$. Given Borel probability measures $\mu$ and $\nu$ on $\mathbb{R}$, there exist a unique pair of analytic functions $\omega_1, \omega_2 : \mathbb{C}^+ \to \mathbb{C}^+$ such that

$$F_{\mu}(\omega_1(z)) = F_{\nu}(\omega_2(z)) = F_{\mu\nu}(z), \quad z \in \mathbb{C}^+. \quad (5)$$

Moreover $\lim_{y \to +\infty} \omega_j(iy)/iy = 1$, $j = 1, 2$ and

$$\omega_1(z) + \omega_2(z) - z = F_{\mu\nu}(z), \quad z \in \mathbb{C}^+.$$

In particular (see [10]), for any $z \in \mathbb{C}^+ \cup \mathbb{R}$ so that $\omega_1$ is analytic at $z$, $\omega_1(z)$ is the attracting fixed point of the self-map of $\mathbb{C}^+$ defined by

$$w \mapsto F_{\nu}(F_{\mu}(w) - w + z) - (F_{\mu}(w) - w).$$

A similar statement, with $\mu, \nu$ interchanged, holds for $\omega_2$.

In particular, according to (5), we have for any $z \in \mathbb{C}^+$,

$$g_{\mu\nu}(z) = g_{\mu}(\omega_1(z)) = g_{\nu}(\omega_2(z)). \quad (6)$$

### 3.2 Operator-valued free probability theory

There exists an extension, operator-valued free probability theory, which still shares the basic properties of free probability but is much more powerful because of its wider domain of applicability. The concept of freeness with amalgamation and some of the relevant analytic transforms were introduced by Voiculescu in [46].

**Definition 1.** Let $\mathcal{M}$ be an algebra and $\mathcal{B} \subset \mathcal{M}$ be a unital subalgebra. A linear map $E : \mathcal{M} \to \mathcal{B}$ is a conditional expectation if $E(ab) = b$ for all $b \in \mathcal{B}$ and $E(b_1ab_2) = b_1E(a)b_2$ for all $a \in \mathcal{M}$ and $b_1, b_2$ in $\mathcal{B}$. Then $(\mathcal{M}, E)$ is called a $\mathcal{B}$-valued probability space. If in addition $\mathcal{M}$ is a $C^*$-algebra, $\mathcal{B}$ is a $C^*$-subalgebra of $\mathcal{M}$ and $E$ is completely positive, then we have a $\mathcal{B}$-valued $C^*$-probability space.

**Example:** Let $(\mathcal{A}, \phi)$ be a non commutative probability space. Define

$$M_2(\mathcal{A}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathcal{A} \right\}, \quad E := \text{id}_2 \otimes \phi$$

that is
The operator upper half-plane of $A$ spectrum of $[47],[14]$ (see Theorem 5 p 259 [36]) Let there exist a unique pair of Fréchet (and thus also Gateaux) analytic maps $B$ on $H$ imaginary parts of $b$. For $C_C$ ($\ast$-probability space. Let

$$E \left[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] = \begin{pmatrix} \phi(a) & \phi(b) \\ \phi(c) & \phi(d) \end{pmatrix}.$$ 

$(M_2(A), E)$ is an $M_2(C)$-valued probability space. $(C \simeq C_1 A)$

As in scalar-valued free probability, one defines [46] freeness with amalgamation over $B$ via an algebraic relation similar to freeness, but involving $E$ and noncommutative polynomials with coefficients in $B$.

**Definition 2.** Let $(M, E : M \to B)$ be an operator-valued probability space. The $B$-valued distribution of $a \in M$ is given by all $B$-valued moments $E(ababa\cdots b_{n-1}a) \in B$, $n \in \mathbb{N}, b_0,\ldots, b_{n-1} \in B$.

Let $(A_i)_{i \in I}$ be a family of subalgebras with $B \subset A_i$ for all $i \in I$. The subalgebras $(A_i)_{i \in I}$ are free with respect to $E$ or free with amalgamation over $B$ if $E(a_1\cdots a_n) = 0$ whenever $a_j \in A_i$, $i_j \in I$, $E(a_j) = 0$, for all $j$ and $i_1 \neq i_2 \neq \cdots \neq i_n$.

Random variables in $M$ or subsets of $M$ are free with amalgamation over $B$ if the algebras generated by $B$ and the variables or the algebras generated by $B$ and the subsets, respectively, are so.

A centred $B$-valued semicircular random variable $s$ is uniquely determined by its variance $\eta : b \mapsto E(sbs)$; a characterization in terms of moments and cumulants via $\eta$ is provided by Speicher in [42].

The previous results of free subordination property in the scalar case are approached from an abstract coalgebra point of view by Voiculescu in [47] and this approach extends the results to the $B$-valued case. In [14], Belinschi, Mai and Speicher develop an analytic theory. In order to describe operator-valued subordination property, we need some notation. If $A$ is a unital $C^*$-algebra and $b \in A$, we denote by $\Re b = (b + b^*)/2$ and $\Im b = (b - b^*)/2i$ the real and imaginary parts of $b$, so $b = \Re b + i\Im b$. For a selfadjoint operator $b \in A$, we write $b \geq 0$ if the spectrum of $b$ is contained in $[0, +\infty)$ and $b > 0$ if the spectrum of $b$ is contained in $(0, +\infty)$. The operator upper half-plane of $A$ is the set $\mathbb{H}^+(A) = \{b \in A : \Im b > 0\}$.

**Theorem 1.** [47], [14] (see Theorem 5 p 259 [36]) Let $(M, E : M \to B)$ be an operator-valued $C^*$-probability space. Let $x_1, x_2 \in M$ be selfadjoint variables which are free with amalgamation over $B$.

There exist a unique pair of Fréchet (and thus also Gateaux) analytic maps $\omega_1, \omega_2 : \mathbb{H}^+(B) \to \mathbb{H}^+(B)$ such that, for all $b \in \mathbb{H}^+(B)$,

- $\Im \omega_j(b) \geq \Im b$, $j = 1, 2$;

- $E \left[ (b - (x_1 + x_2))^{-1} \right] = E \left[ (\omega_1(b) - x_1)^{-1} \right] = E \left[ (\omega_2(b) - x_2)^{-1} \right]$.

- $\left\{ E \left[ (\omega_1(b) - x_1)^{-1} \right] \right\}^{-1} + b = \left\{ E \left[ (\omega_2(b) - x_2)^{-1} \right] \right\}^{-1} + b$
  $\quad = \omega_1(b) + \omega_2(b)$.  

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Moreover, if \( b \in \mathbb{H}^+(\mathcal{B}) \), then \( \omega_1(b) \) is the unique fixed point of the map

\[
f_b : \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^+(\mathcal{B}), \quad f_b(w) = h_{x_2}(h_{x_1}(w) + b) + b
\]

where \( h_{x_i}(b) = E\left[(b - x_i)^{-1}\right]^{-1} - b \)

and \( \omega_1(b) = \lim_{k \to +\infty} f_b^k(w) \), for any \( w \in \mathbb{H}^+(\mathcal{B}) \).

The following result from [37] explains why the particular case \( \mathcal{B} = M_m(\mathbb{C}), \mathcal{M} = M_m(\mathcal{A}), E = \text{id}_m \otimes \phi \), where \((\mathcal{A}, \phi)\) is a non commutative probability space, is relevant in our work using linearizations of polynomials.

**Proposition 3.** Let \((\mathcal{A}, \phi)\) be a \( C^*\)-probability space, let \( m \) be a positive integer, and let \( x_1, \ldots, x_n \in \mathcal{A} \) be freely independent. Then the map \( \text{id}_m \otimes \phi : M_m(\mathcal{A}) \to M_m(\mathbb{C}) \) is a unit preserving conditional expectation, and \( \alpha_1 \otimes x_1, \ldots, \alpha_n \otimes x_n \) are free over \( M_m(\mathbb{C}) \) for any \( \alpha_i \in M_m(\mathbb{C}) \).

Now, if \( x \) is a standard scalar-valued semicircular centred non commutative random variable which is free from a selfadjoint variable \( a \) in some \( \mathcal{W}^*\)-probability space \((\mathcal{A}, \phi)\), then, for any Hermitian matrices \( \alpha, \beta \in M_m(\mathbb{C}) \), \( \alpha \otimes x \) is a \( M_m(\mathbb{C})\)-valued semicircular of variance \( \eta : b \mapsto \alpha \beta \alpha \) which is free over \( M_m(\mathbb{C}) \) from \( \beta \otimes a \) and the subordination function has a more explicit form (see [36] Chapter 9) and the end of the proof of Theorem 8.3 in [3]:

\[
\text{id}_m \otimes \phi \left[ (b \otimes 1_A - \alpha \otimes x - \beta \otimes a)^{-1} \right] = \text{id}_m \otimes \phi \left[ (\omega_m(b) \otimes 1_A - \beta \otimes a)^{-1} \right], \quad b \in \mathbb{H}^+(M_m(\mathbb{C})),
\]

where

\[
\omega_m(b) = b - \text{aid}_m \otimes \phi \left[ (b \otimes 1_A - \alpha \otimes x - \beta \otimes a)^{-1} \right] \alpha.
\]

Denote by \( \mathcal{N} \) the unital von Neumann algebra generated by \( M_m(\mathbb{C}) \) and \( \beta \otimes a \) and by \( E_{\mathcal{N}} \) the unique trace preserving conditional expectation of \( M_m(\mathcal{A}) \) onto \( \mathcal{N} \). Actually the following strengthened result [47] Theorem 3.8] holds:

\[
E_{\mathcal{N}} \left[ (b \otimes 1_A - \alpha \otimes x - \beta \otimes a)^{-1} \right] = (\omega_m(b) \otimes 1_A - \beta \otimes a)^{-1}.
\]

### 4 Assumptions and main results

**Assumptions on the Wigner matrix.**

We consider a \( N \times N \) Hermitian Wigner matrix \( W_N = (W_{ij})_{1 \leq i,j \leq N} \) such that the random variables \( \{W_{ii}, \sqrt{2}RW_{ij}, \sqrt{2}IW_{ij}\}_{1 \leq i < j} \) are independent identically distributed with law \( \mu \), \( \mu \) is a centered distribution, with variance 1, and satisfies a Poincaré inequality (see the Appendix). We set

\[
W_N = \begin{pmatrix} W_{11} & Y^* \\ Y & W_{N-1} \end{pmatrix},
\]

where \( Y^* = (W_{12}, \ldots, W_{1N}) \) and \( W_{N-1} \in M_{N-1}(\mathbb{C}) \).

**Assumptions on the deterministic matrix.**

We consider a deterministic real diagonal matrix \( A_N \):

\[
A_N = \text{diag}(\theta, A_{N-1})
\]
where \( \theta \in \mathbb{R} \) is independent of \( N \) and \( A_{N-1} \) is a \( (N-1) \times (N-1) \) deterministic diagonal matrix such that for any \( i = 1, \ldots, N-1 \), \((A_{N-1})_{ii} = d_i(N)\). We assume that \( A_{N-1} \in (M_{N-1}(\mathbb{C}), \frac{1}{N-1} \text{Tr}) \) converges strongly in distribution towards a non commutative random variable \( a \) in some \( \mathcal{W}^* \)-probability space \((\mathcal{A}, \phi)\), with \( \phi \) faithful (see Section 3.1 for the definition of strong convergence). Note that this implies that

\[
\sup_N \|A_{N-1}\| < +\infty,
\]

and, by Proposition 2 that, for all large \( N \), all the eigenvalues of \( A_{N-1} \) are in any small neighborhood of the spectrum of \( a \). We assume that \( \theta \) is such that \( \theta \notin \text{supp}(\mu_a) = \text{spect}(a) \).

Note that the previous assumptions yield that \( A_N \in (M_N(\mathbb{C}), \frac{1}{N} \text{Tr}) \) converges in distribution towards the non commutative random variable \( a \) and that, for \( N \) large enough, \( \theta \) is an eigenvalue of multiplicity 1 of \( A_N \).

**Matrix model.**

Fix a selfadjoint polynomial \( P \in \mathbb{C} < X_1, X_2 > \). The matrix model we are interested in is

\[
M_N = P \left( \frac{W_N}{\sqrt{N}} A_N \right).
\]

Denote by \( \lambda_i(M_N), i = 1, \ldots, N \), its eigenvalues and by

\[
\mu_{M_N} = \frac{1}{N} \sum_{i=1}^{N} \lambda_i(M_N)
\]

its empirical spectral measure. According to (2.10) in [12] and [2, Theorem 5.4.5], we have

\[
\lim_{N \to \infty} \mu_{M_N} = \mu_{P(x,a)}
\]

almost surely in the weak* topology, where \( x \) is a standard semicircular non commutative random variable in \((\mathcal{A}, \phi)\) (i.e \( d\mu_x = \frac{1}{2\pi} \sqrt{4-x^2} 1_{[-2,2]}(x) \)), \( a \) and \( x \) are freely independent, and \( \mu_{P(x,a)} \) denotes the distribution of \( P(x,a) \).

The set of outliers of \( M_N \) is calculated in [13] from the spike \( \theta \) of \( A_N \) using linearization and Voiculescu’s matrix subordination function [47] as follows. Choose a linearization \( L_P \) of \( P \) where \( L_P = \gamma \otimes 1 + \alpha \otimes X_1 + \beta \otimes X_2 \), \( \alpha, \beta, \gamma \) are selfadjoint matrices in \( M_m(\mathbb{C}) \), and let \( \omega_m \) be the subordination function associated to the semicircular operator-valued random variable \( \alpha \otimes x \) with respect to \( \beta \otimes a \), as defined by (5). According to Lemma 1 \( \omega_m \) extends as an analytic map \( z \mapsto \omega_m(ze^{11} - \gamma) \) to \( \mathbb{C} \setminus \text{supp}(\mu_{P(x,a)} \). For any \( \rho \notin \text{supp}(\mu_{P(x,a)}) \), define \( m(\rho) \) as the multiplicity of \( \rho \) as a zero of \( \det(\omega_m(\rho e^{11} - \gamma) - \theta \beta) \). [13] establishes the following.

**Theorem 2.** [13] There exists \( \delta_0 > 0 \) such that, for any \( 0 < \delta \leq \delta_0, \) a.s for large \( N \), there are exactly \( m(\rho) \) eigenvalues of \( P \left( \frac{W_N}{\sqrt{N}} A_N \right) \) in \( \rho - \delta; \rho + \delta \), counting multiplicity.

**Assumptions on \( \rho \).**

In this paper, we assume that there exists some real number

\[
\rho \notin \text{supp}(\mu_{P(x,a)}) = \text{spect}(P(x,a))
\]
such that $\rho$ is a zero with multiplicity one of
\begin{equation}
\det(\omega_m(\rho e_{11} - \gamma) - \theta \beta) = 0, \tag{10}
\end{equation}
that is such that
\begin{equation}
m(\rho) = 1. \tag{11}
\end{equation}

**Assumptions on $\varepsilon$.**
Throughout the paper $\varepsilon > 0$ is fixed such that
\begin{equation}
d(\rho, \text{spect}(P(x, a))) > \varepsilon \tag{12}
\end{equation}
and
\begin{equation}
\det(\omega_m(y e_{11} - \gamma) - \theta \beta) \neq 0, \text{ for any } y \in ]\rho - \varepsilon; \rho + \varepsilon[ \setminus \{\rho\}. \tag{13}
\end{equation}

**Main result.**
We first introduce events and objects needed to state our main result.

By strong asymptotic freeness of \[13\] and Proposition 2, almost surely for all large $N$, \(\text{spect} \left( P \left( \frac{W_N}{\sqrt{N}}, A_{N-1} \right) \right) \subset \{ y \in \mathbb{R}; d(y, \text{spect}(P(x, a))) \leq \varepsilon/2 \}.\) Thus,
\begin{equation}
\text{almost surely for all large } N, \ d \left( \rho, \text{spect} \left( P \left( \frac{W_{N-1}}{\sqrt{N}}, A_{N-1} \right) \right) \right) > \varepsilon/2. \tag{12}
\end{equation}

Define the event
\begin{equation}
\tilde{\Omega}_{N-1} = \left\{ d \left( \rho, \text{spect} \left( P \left( \frac{W_{N-1}}{\sqrt{N}}, A_{N-1} \right) \right) \right) > \varepsilon/2; \left\| \frac{W_{N-1}}{\sqrt{N}} \right\| \leq 3 \right\}. \tag{13}
\end{equation}

Note that according to Lemma 2 there exists $C_{\varepsilon} > 0$ such that
\begin{equation}
d \left( 0, \text{spect}(\rho e_{11} - \gamma) \otimes I_d - \alpha \otimes x - \beta \otimes a) \right) > C_{\varepsilon} \tag{14}
\end{equation}
and on $\tilde{\Omega}_{N-1}$
\begin{equation}
d \left( 0, \text{spect}(\rho e_{11} - \gamma) \otimes I_{N-1} - \alpha \otimes \frac{W_{N-1}}{\sqrt{N}} - \beta \otimes A_{N-1} \right) > C_{\varepsilon}. \tag{15}
\end{equation}

Let $\delta_0$ be as defined in Theorem 2. Set
\begin{equation}
\tau = \min(\delta_0, \varepsilon/4, C_{\varepsilon}/2). \tag{16}
\end{equation}

Define the event
\begin{equation}
\Omega_N = \tilde{\Omega}_{N-1} \cap \{ \text{card(spect}(M_N) \cap ]\rho - \tau; \rho + \tau[) = 1 \}. \tag{17}
\end{equation}

It readily follows from Theorem 2, \[12\] and Bai-Yin’s theorem (see \[7\] for the symmetric case and Theorem 5.1 in \[5\] for the Hermitian case) that
\begin{equation}
\lim_{N \to +\infty} I_{\Omega_N} = 1, \text{ a.s.} \tag{18}
\end{equation}
and then
\[ \mathbb{P}(\Omega_N) \to_{N \to +\infty} 1. \]

Now, define
\[ \lambda(N, \rho) = \begin{cases} \rho & \text{if } \text{spect}(M_N) \cap \rho - \tau; \rho + \tau = \emptyset \\ \max\{\text{spect}(M_N) \cap \rho - \tau; \rho + \tau\} & \text{else}. \end{cases} \tag{18} \]

On \( \Omega_N \), \( \lambda(N, \rho) \) is the unique eigenvalue of \( M_N \) which is located in \( ]\rho - \tau; \rho + \tau[ \). In this paper, we study the fluctuations of \( \lambda(N, \rho) \). Note that Theorem 2 readily implies that
\[ \lambda(N, \rho) \to_{N \to +\infty} \rho \text{ a.s..} \tag{19} \]

Let \( a_{N-1} \) be a selfadjoint noncommutative random variable in \( (A, \phi) \) whose distribution is \( \mu_{A_{N-1}} \) (meaning that \( \forall k \in \mathbb{N}, \frac{1}{N} \text{Tr}(A_{N-1}^{k}) = \phi((a_{N-1}^{k})) \) and which is free with the semicircular variable \( x \). According to [23], since \( A_{N-1} \) (and thus \( a_{N-1} \)) converges strongly to \( a \), we have, for all large \( N \),
\[ \text{spect}((\rho e_{11} - \gamma) \otimes 1_{A} - \alpha \otimes x - \beta \otimes a_{N-1}) \]
\[ \subset \text{spect}((\rho e_{11} - \gamma) \otimes 1_{A} - \alpha \otimes x - \beta \otimes a) + ] - C_{\epsilon}/4, C_{\epsilon}/4[, \tag{20} \]
and thus, using [14], for any \( z \in B(\rho, \tau) \),
\[ d(0, \text{spect}(ze_{11} - \gamma) \otimes 1_{A} - \alpha \otimes x - \beta \otimes a_{N-1})) > C_{\epsilon}/4. \tag{21} \]

Define for any \( \kappa \in \mathbb{H}_{+}(M_{m}(\mathbb{C})) \)
\[ \omega_{m}^{(N)}(\kappa) = \kappa - \alpha \text{id}_{m} \otimes \phi \left[(\kappa \otimes 1_{A} - \alpha \otimes x - \beta \otimes a_{N-1})^{-1}\right] \alpha. \tag{22} \]
\( \omega_{m}^{(N)} \) is the subordination function associated to the semicircular operator-valued random variable \( \alpha \otimes x \) with respect to \( \beta \otimes a_{N-1} \). According to Lemma [11], \( \omega_{m}^{(N)} \) extends as an analytic map \( z \mapsto \omega_{m}(ze_{11} - \gamma) \) to \( \mathbb{C} \setminus \text{supp}(\mu_{P(x,a_{N-1})}) \). \( \left(z \mapsto \det(\omega_{m}^{(N)}(ze_{11} - \gamma) - \theta \beta)\right)_{N \geq 1} \) is a bounded sequence in the set of analytic functions endowed with the uniform convergence on compact subsets of \( B(\rho, \tau) := \{z \in \mathbb{C}, |z - \rho| < \tau\} \); therefore, using moreover [15], by Hurwitz theorem, \( [10] \) yields that for any \( 0 < \tau' < \tau \), for all large \( N \), there exists one and only one \( \rho_{N} \) in \( B(\rho, \tau') \), such that
\[ \det(\omega_{m}^{(N)}(\rho_{N}e_{11} - \gamma) - \theta \beta) = 0, \tag{23} \]
and we have
\[ \rho_{N} \to_{N \to +\infty} \rho. \tag{24} \]
Moreover, necessarily \( \rho_{N} \) is real since [23] implies that \( \det(\omega_{m}^{(N)}(\rho_{N}e_{11} - \gamma) - \theta \beta) = 0 \).

Here is our main result.

**Theorem 3.** Define
\[ C_{m} = t \text{Com}(\omega_{m}(\rho e_{11} - \gamma) - \beta \theta), \tag{25} \]
\[ R_{\infty}(\rho e_{11} - \gamma) = (\rho e_{11} - \gamma) \otimes 1_{A} - \alpha \otimes x - \beta \otimes a)^{-1}, \tag{26} \]
\[ C_{\rho}^{(1)} = \text{Tr}_{m} \left( C_{m}[e_{11} + \alpha \text{id}_{m} \otimes \phi(R_{\infty}(\rho e_{11} - \gamma)(e_{11} \otimes 1_{A})R_{\infty}(\rho e_{11} - \gamma))\alpha] \right), \tag{27} \]
According to \cite[(4.6.6)]{9},
\[ \mu \]

Thus
\[ a \]

A random matrix
\[ W \]

where
\[ C \]

As an illustration, consider the random matrix
\[ X \]

Example
\[ (25) \]

and
\[ \otimes \]

The polynomial
\[ \lambda \]

of
\[ \chi \]

measure of
\[ W \]

converge uniformly to the compact support of
\[ \mu \]

and
\[ \lambda \]

a Gaussian distribution with mean 0 and variance
\[ v \].

Using the unitarily invariance of the distribution of a G.U.E. matrix, we can readily deduce the following result.

**Corollary 1.** Assume that
\[ W_N \]

is a G.U.E. matrix. Let
\[ A_N \]

be a deterministic Hermitian matrix such that its spectral measure
\[ \mu_{A_N} \]

weakly converges towards a compactly supported measure
\[ \theta \]

a spike of
\[ W \]

with multiplicity one whereas the other eigenvalues of
\[ A_N \]

converge uniformly to the compact support of
\[ \mu_a \]. Then, under the assumptions
\[ (10) \]

and
\[ (11) \]

\[ C^{(1)}_\rho \sqrt{N}(\lambda(N, \rho) - \rho_N) \]

converges in distribution to the classical convolution of the distribution of
\[ C^{(2)}_\rho \]

and
\[ W_{11} \]

and a Gaussian distribution with mean 0 and variance
\[ v_\rho \].

where
\[ \lambda(N, \rho), \rho_N, C^{(1)}_\rho, C^{(2)}_\rho, C_m, \text{ and } R_{\infty}(\rho e_{11} - \gamma) \]

are defined by
\[ (18), (23), (27), (28), (29) \]

and
\[ (30) \]

respectively.

**Example**

As an illustration, consider the random matrix
\[ M_N = A_N \frac{W_N}{\sqrt{N}} + \frac{W_N}{\sqrt{N}} A_N + \frac{W^2_N}{N}, \]

where
\[ W_N \]

is a Wigner matrix of size
\[ N \]

such that
\[ d\mu(x) = \frac{1}{2\sqrt{3}} \mathbb{1}_{[-\sqrt{3}, \sqrt{3}]}(x)dx \]

and
\[ A_N = \text{Diag}(\theta, 0, \ldots, 0), \quad \theta \in \mathbb{R} \setminus \{0\}. \]

According to
\[ (4.6.6) \]

\[ \mu \]

satisfies a Poincaré inequality. In this case, \[ A_N \]

has rank one, and thus
\[ a \]

is 0. It follows that the limit spectral measure
\[ \Pi \]

of
\[ M \]

is the same as the limit spectral measure of
\[ W^2_N/N \]. Thus, \[ \Pi \]

is the Marchenko-Pastur distribution with parameter 1:
\[ d\Pi(x) = \frac{\sqrt{(4-t)} t}{2\pi t} 1_{(0,4)}(t)dt. \]

The polynomial
\[ P \]

is
\[ P(X_1, X_2) = X_1 X_2 + X_2 X_1 + X_1^2, \quad a = 0 \]

and
\[ x \]

is the standard semicircular distribution. An economical linearization of
\[ P \]

is provided by
\[ L = \gamma \otimes 1 + \alpha \otimes X_1 + \beta \otimes X_2, \]

where
\[ \gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \alpha = \begin{bmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \]
Thus, here \( m = 3 \). Denote by
\[
G_{\Pi}(z) = \int_{0}^{1} \frac{1}{z-t} d\Pi(t) = \frac{z - \sqrt{z^2 - 4z}}{2z}, \quad z \in \mathbb{C} \setminus [0, 4],
\]
the Cauchy transform of the measure \( \Pi \). This function satisfies the quadratic equation
\[
zG_{\Pi}(z)^2 - zG_{\Pi}(z) + 1 = 0.
\]
Suppose now that \( t \notin [0, 4] \). Denoting by \( E = \text{id}_3 \otimes \phi : M_3(A) \to M_3(\mathbb{C}) \) the usual expectation, since \( a = 0 \), the function \( \omega_3 \) is computed as follows:
\[
\omega_3(t e_{11} - \gamma) = E((t e_{11} - \gamma - \alpha \otimes x)^{-1})^{-1}, \quad t \in \mathbb{R} \setminus [0, 4].
\]
The inverse of \( t e_{11} - \gamma - \alpha \otimes x \) is then calculated explicitly and application of the expected value to its entries yields
\[
\omega_3(t e_{11} - \gamma) = \begin{bmatrix}
\frac{1}{G_{\Pi}(t)} & 0 & 0 \\
0 & \frac{1}{2G_{\Pi}(t)} - 1 & \frac{1}{2G_{\Pi}(t)} + \frac{1}{2} \\
0 & \frac{1}{2G_{\Pi}(t)} - 1 & \frac{1}{2G_{\Pi}(t)} - \frac{1}{2}
\end{bmatrix}.
\]
The equation \( \det[3\theta - \omega(t e_{11} - \gamma)] = 0 \) is easily seen to reduce to
\[
\theta^2 G_{\Pi}(t)^2 - (1 - G_{\Pi}(t)) = 0.
\]
Thus, the matrix \( M_N \) exhibits one (negative) outlier when \( 0 < |\theta| \leq \sqrt{2} \)
\[
\rho_\theta^- = \frac{2\theta^4}{-(3\theta^2 + 1) - \sqrt{4\theta^2 + 1}(\theta^2 + 1)},
\]
and two outliers (one negative and one > 4) when \( |\theta| > \sqrt{2} \):
\[
\rho_\theta^\pm = \frac{2\theta^4}{-(3\theta^2 + 1) \pm \sqrt{4\theta^2 + 1}(\theta^2 + 1)};
\]
note that
\[
g_{\rho_\theta^\pm} = G_{\Pi}(\rho_\theta^\pm) = \frac{1}{2} + \frac{1}{2}(\theta^2 + 1) \pm \frac{\sqrt{4\theta^2 + 1}}{2\theta^2}.
\]
Let \( \rho \) be any of the two solutions \( \rho_\theta^+ \) and \( \rho_\theta^- \) and set
\[
g_{\rho} = G_{\Pi}(\rho).
\]
Note that since here \( a_{N-1} = a = 0 \), we have \( \rho_N = \rho \). After computations
\[
C_3 = \begin{pmatrix}
\frac{g_{\rho} - 1}{2} & \frac{g_{\rho} - 2}{2} & -g_{\rho} \\
\frac{g_{\rho} - 2}{2} & -\frac{1}{4} + \frac{1}{4} & -\frac{g_{\rho}}{2} + \frac{1}{2} \\
g_{\rho} & -\frac{g_{\rho}}{2} + \frac{1}{2} & -1
\end{pmatrix},
\]
\[
R_{\infty}(\rho e_{11} - \gamma) = \begin{pmatrix}
\frac{(\rho - x^2)^{-1}}{2} & \frac{x(\rho - x^2)^{-1}}{2} & x(\rho - x^2)^{-1} \\
\frac{x(\rho - x^2)^{-1}}{2} & \frac{x^2(\rho - x^2)^{-1}}{2} & 1 + \frac{1}{2}x^2(\rho - x^2)^{-1} \\
x(\rho - x^2)^{-1} & 1 + \frac{1}{2}x^2(\rho - x^2)^{-1} & x^2(\rho - x^2)^{-1}
\end{pmatrix},
\]
\[13\]
and then

\[ C_\rho^{(1)} = -\theta^2 g_\rho^2 \left( 1 + \int \frac{y}{(\rho - y)^2} d\Pi(y) \right) - \frac{1}{g_\rho^2} \int \frac{1}{(\rho - y)^2} d\Pi(y) < 0 \]

and

\[ v_\rho = -\frac{3}{5} \left( \theta^2 g_\rho + 2 \right)^2 + \theta^4 g_\rho^4 \int \frac{y^2}{(\rho - y)^2} d\mu_\Pi(y) + 2 \theta^2 g_\rho^2 \int \frac{1}{(\rho - y)^2} d\mu_\Pi(y) + 2 \theta^4 g_\rho^4 \int \frac{y}{(\rho - y)} d\mu_\Pi(y) \]

with

\[ \int \frac{y}{(\rho - y)^2} d\mu_\Pi(y) = -1 + \rho g_\rho, \]

\[ \int \frac{y}{(\rho - y)^2} d\mu_\Pi(y) = -g_\rho - \rho g'_\rho, \]

\[ \int \frac{y^2}{(\rho - y)^2} d\mu_\Pi(y) = 1 - 2\rho g_\rho - \rho^2 g'_\rho, \]

and \( g'_\rho = G'_\Pi(\rho) = \frac{g_\rho (1 - g_\rho)}{\rho (2g_\rho - 1)} \) (after differentiating the equation \( tG_\Pi(t)^2 - tG_\pi(t) + 1 = 0 \)). Thus,

\[ C_\rho^{(1)} = -\theta^4 g_\rho^4 + \frac{g'_\rho}{g_\rho^2} (g_\rho + 1), \quad C_\rho^{(2)} = -2\theta, \]

\[ v_\rho = -\frac{3}{5} \left( \theta^2 g_\rho + 2 \right)^2 + \frac{g'_\rho}{g_\rho^2} \left( 1 + \frac{7}{g_\rho} + \theta^2 \right) - 4\theta^2 g_\rho. \]

Now, set

\[ C = \left| \frac{C_\rho^{(2)}}{C_\rho^{(1)}} \right|, \quad \sigma^2 = \frac{v_\rho}{C_\rho^{(1)^2}}. \]

According to Theorem\[ 3 \] \( \sqrt{N} (\lambda(N, \rho) - \rho) \) converges in distribution to the probability measure with density function

\[ f(x) = \frac{1}{2\sqrt{6\pi C}\sigma} \int_{-\sqrt{3}C}^{\sqrt{3}C} \exp \left(-\frac{(x - t)^2}{2\sigma^2}\right) dt. \]
5 Preliminary results

5.1 Basic bounds and convergences

We start with straightforward bounds and convergences involving resolvents and that will be of basic use for the proof of Theorem 3. For any \( w \in M_m(\mathbb{C}) \) such that \( w \otimes I_{N-1} - \alpha \otimes \frac{W_{N-1}}{\sqrt{N}} - \beta \otimes A_{N-1} \), resp. \( w \otimes 1_A - \alpha \otimes x - \beta \otimes a \), is invertible, define

\[
R_{N-1}(w) = \left( w \otimes I_{N-1} - \alpha \otimes \frac{W_{N-1}}{\sqrt{N}} - \beta \otimes A_{N-1} \right)^{-1},
\]

resp.

\[
R_{\infty}(w) = (w \otimes 1_A - \alpha \otimes x - \beta \otimes a)^{-1}.
\]

Note that we have the following resolvent identities for any \( w_1 \) and \( w_2 \) in \( M_m(\mathbb{C}) \) such that the resolvents are defined:

\[
R_{N-1}(w_1) - R_{N-1}(w_2) = R_{N-1}(w_1) [(w_2 - w_1) \otimes I_{N-1}] R_{N-1}(w_2),
\]

\[
R_{\infty}(w_1) - R_{\infty}(w_2) = R_{\infty}(w_1) [(w_2 - w_1) \otimes 1_A] R_{\infty}(w_2).
\]

Lemma 3.  

- For any \( w \in M_m(\mathbb{C}) \) such that \( \Im w > 0 \), \( w \otimes I_{N-1} - \alpha \otimes \frac{W_{N-1}}{\sqrt{N}} - \beta \otimes A_{N-1} \), \( w \otimes 1_A - \alpha \otimes x - \beta \otimes a \) and \( w \otimes 1_A - \alpha \otimes x - \beta \otimes a_{N-1} \) are invertible and

\[
\|R_{N-1}(w)\| \leq \| (\Im w)^{-1} \|,
\]

\[
\|R_{\infty}(w)\| \leq \| (\Im w)^{-1} \|,
\]

\[
\left\| \left( w \otimes 1_A - \alpha \otimes x - \beta \otimes a_{N-1} \right)^{-1} \right\| \leq \| (\Im w)^{-1} \|.
\]

- Let \( \tilde{\Omega}_{N-1} \) be as defined by (13) and \( C_{\epsilon} \) be as in (14) and (15). Let \( z \) and \( z_0 \) be in \( \mathbb{C} \) such that, \( \Im z \Im z_0 \geq 0 \) and \( |z - \rho| + |z_0| < C_{\epsilon}/2 \).

Then, \((ze_{11} + z_0 I_{m} - \gamma) \otimes 1_A - \alpha \otimes x - \beta \otimes a_{m}\), \((ze_{11} + z_0 I_{m} - \gamma) \otimes 1_A - \alpha \otimes x - \beta \otimes a_{n-1}\) and \((ze_{11} + z_0 I_{m} - \gamma) \otimes I_{N-1} - \alpha \otimes \frac{W_{N-1}}{\sqrt{N}} - \beta \otimes A_{N-1}\) on \( \tilde{\Omega}_{N-1} \), are invertible and we have

\[
\left\| R_{N-1}(ze_{11} + z_0 I_{m} - \gamma) \right\|_{\tilde{\Omega}_{N-1}} \leq 2/C_{\epsilon},
\]

\[
\left\| R_{\infty}(ze_{11} + z_0 I_{m} - \gamma) \right\| \leq 2/C_{\epsilon},
\]

\[
\left\| \left( (ze_{11} + z_0 I_{m} - \gamma) \otimes 1_A - \alpha \otimes x - \beta \otimes a_{N-1} \right)^{-1} \right\| \leq 4/C_{\epsilon}.
\]

Moreover, for any \( t \) in the spectrum of \( a \), \( \omega_m(ze_{11} + z_0 I_{m} - \gamma) - t\beta \) is invertible and

\[
\left\| \left( \omega_m(ze_{11} + z_0 I_{m} - \gamma) - t\beta \right)^{-1} \right\| \leq 2/C_{\epsilon},
\]

and, for any \( t \) in the spectrum of \( A_{N-1} \), \( \omega_m^{(N)}(ze_{11} + z_0 I_{m} - \gamma) - t\beta \) is invertible and

\[
\left\| \left( \omega_m^{(N)}(ze_{11} + z_0 I_{m} - \gamma) - t\beta \right)^{-1} \right\| \leq 4/C_{\epsilon}.
\]
Proof. (33), (34) and (35) come from Lemma 3.1 (i) of [30]. (36), (37) and (38) easily follow from (11), (15), (20) and the following facts: if \( y \) is an invertible normal element in a \( C^* \)-algebra, then \( d(0, \text{spec}(y)) = 1/\|y^{-1}\| \) and for any other element \( y \), the distance between any element in the spectrum of \( y \) and the spectrum of \( y \) is smaller than \( \|y - \bar{y}\| \).

Now, one can easily deduce from (9) that, for almost surely,

\[
I_m \otimes 1_A = (\omega_m(ze_{11} + z_0 I_m - \gamma) \otimes 1_A - \beta \otimes a) E_N \left[ R_\infty (ze_{11} + z_0 I_m - \gamma) \right] = E_N \left[ R_\infty (ze_{11} + z_0 I_m - \gamma) \right] (\omega_m(ze_{11} + z_0 I_m - \gamma) \otimes 1_A - \beta \otimes a). \tag{41}
\]

Let \( t \) be in the spectrum of \( a \). Choose a character \( \chi \) of the commutative \( C^* \)-algebra \( \mathbb{C} < a > \) and denote by \( \chi_m : M_m(\mathbb{C}) < \beta \otimes a > \to M_m(\mathbb{C}) \) the algebra homomorphism obtained by applying \( \chi \) to each entry. Applying \( \chi_m \) to (41), we deduce that

\[
(\omega_m(ze_{11} + z_0 I_m - \gamma) - t\beta)^{-1} = \chi_m(E_N \left[ R_\infty (ze_{11} + z_0 I_m - \gamma) \right])
\]
so that (39) readily follows from (37). (40) can be prove similarly. \( \square \)

The following convergence results are quite straightforward consequences of asymptotic freeness of \( W_{N-1}/\sqrt{N} \) and \( A_{N-1} \).

\begin{lemma}
For any \( \Sigma, \Sigma_1, \Sigma_2 \) in \( M_m(\mathbb{C}) \) such that \( \exists C > 0, \|\Sigma\| \leq C, \|\Sigma_1\| \leq C, \|\Sigma_2\| \leq C \), almost surely,

1) \[
\mathbf{id}_m \otimes \text{tr}_{N-1}(R_{N-1}(\rho_N e_{11} - \gamma) (\Sigma \otimes I_{N-1}) R_{N-1}(\rho_N e_{11} - \gamma)) 1_{\Omega_{N-1}} \to_{N \to +\infty} \mathbf{id}_m \otimes \phi \left[ R_\infty(\rho e_{11} - \gamma) (\Sigma \otimes I_A) R_\infty(\rho e_{11} - \gamma) \right] \tag{42}
\]

2) \[
1_{\Omega_{N-1}} \times \text{tr}_{N-1} \left\{ \text{Tr}_m \otimes \mathbf{id}_{N-1} \left[ R_{N-1}(\rho_N e_{11} - \gamma) (\Sigma_1 \otimes I_{N-1}) \right] \times \text{Tr}_m \otimes \mathbf{id}_{N-1} \left[ R_{N-1}(\rho_N e_{11} - \gamma) (\Sigma_2 \otimes I_{N-1}) \right] \right\} \to_{N \to +\infty} \phi \left\{ \text{Tr}_m \otimes \mathbf{id}_{N-1} \left[ R_\infty(\rho e_{11} - \gamma) (\Sigma_1 \otimes I_A) \right] \times \text{Tr}_m \otimes \mathbf{id}_{N-1} \left[ R_\infty(\rho e_{11} - \gamma) (\Sigma_2 \otimes I_A) \right] \right\}, \tag{43}
\]

3) \[
\forall w \in M_m(\mathbb{C}), \exists w > 0, \omega_m^{(N)}(w) \to_{N \to +\infty} \omega_m(w). \tag{44}
\]
\[
\forall z \in \mathbb{C}, |z - \rho| < \tau, \omega_m^{(N)}(ze_{11} - \gamma) \to_{N \to +\infty} \omega_m(ze_{11} - \gamma). \tag{45}
\]
\[
\omega_m^{(N)}(\rho_N e_{11} - \gamma) \to_{N \to +\infty} \omega_m(\rho e_{11} - \gamma). \tag{46}
\]
\end{lemma}

Proof. We have for any self adjoint operators \( u \) and \( v \), for any \( w \in M_m(\mathbb{C}) \) such that \( \Im w > 0 \), for any non null integer \( p \),

\[
(w \otimes 1 - \alpha \otimes u - \beta \otimes v)^{-1} = \sum_{k=0}^{p-1} w^{-1} \otimes 1(\alpha w^{-1} \otimes u + \beta w^{-1} \otimes v)^k + (w \otimes 1 - \alpha \otimes u - \beta \otimes v)^{-1} (\alpha w^{-1} \otimes u + \beta w^{-1} \otimes v)^p. \tag{47}
\]
For any $K > 0$, define 
\[ \mathcal{O}_K = \{ w \in M_m(\mathbb{C}), \Im(w) > K I_m \}. \]
According to Lemma 3.1 (i) of [30], for any $w \in \mathcal{O}_K$, we have $\|w^{-1}\| \leq 1/K$. Let $0 < C < 1$. For any $\kappa > 0$, there exists $K = K(\kappa, \alpha, \beta) > 0$ such that if $w \in \mathcal{O}_K$, for any $u$ and $v$ such that $\|u\| \leq \kappa$ and $\|v\| \leq \kappa$ then 
\[ \|(\alpha w^{-1} \otimes u + \beta w^{-1} \otimes v)\| \leq C, \quad (48) \]
so that (using once more Lemma 3.1 (i) of [30])
\[ \sup_{w \in \mathcal{O}_K} \left\| (w \otimes 1 - \alpha \otimes u - \beta \otimes v)^{-1} (\alpha w^{-1} \otimes u + \beta w^{-1} \otimes v)^p \right\| \leq \frac{C^p}{K} \rightarrow_{p \rightarrow +\infty} 0. \]
Fix $K > 0$ such that (48) holds for $(u, v) = (x, a)$ and $(u, v) = \left( \frac{W_{N-1}}{\sqrt{N}}, A_{N-1} \right)$ on $\tilde{\Omega}_{N-1}$. Therefore, for any $\delta > 0$, we can find a polynomial $Q_w$ with coefficients in $M_m(\mathbb{C})$ depending on $w$ such that:
\[ \sup_{w \in \mathcal{O}_K} \left| R_{\infty}(w) - Q_w(x, a) \right| \leq \delta, \quad (49) \]
\[ \sup_{w \in \mathcal{O}_K} \left| R_{N-1}(w) - Q_w \left( \frac{W_{N-1}}{\sqrt{N}}, A_{N-1} \right) \right| \leq \delta. \quad (50) \]
Now, by the asymptotic freeness of $\frac{W_{N-1}}{\sqrt{N}}$ and $A_{N-1}$ (see [2] Theorem 5.4.5), we have that almost surely
\[ \text{id}_m \otimes \text{tr}_{N-1} \left\{ Q_w \left( \frac{W_{N-1}}{\sqrt{N}}, A_{N-1} \right) (\Sigma \otimes I_N) Q_w \left( \frac{W_{N-1}}{\sqrt{N}}, A_{N-1} \right) \right\} \rightarrow_{N \rightarrow \infty} \text{id}_m \otimes \phi \left\{ Q_w(x, a)(\Sigma \otimes 1_A) Q_w(x, a) \right\}. \quad (51) \]
Using (51), (49), (50), (33), (34), and (52) and
\[ \lim_{N \rightarrow +\infty} \mathbb{1}_{\tilde{\Omega}_{N-1}} = 1 \text{ a.s.}, \]
we can deduce that for $w \in \mathcal{O}_K$, $\text{id}_m \otimes \text{tr}_{N-1}(R_{N-1}(w)(\Sigma \otimes I_{N-1}) R_{N-1}(w))$ converges almost surely towards $\text{id}_m \otimes \phi(R_{\infty}(w)(\Sigma \otimes I_A) R_{\infty}(w))$, when $N$ goes to infinity. Let $\mathcal{O} = \{ w \in M_m(\mathbb{C}), \Im(w) > 0 \}$. The two functions
\[ \Phi_N(w) = \text{id}_m \otimes \text{tr}_{N-1} \left[ R_{N-1}(w) (\Sigma \otimes I_{N-1}) R_{N-1}(w) \right] \]
and
\[ \Phi_\infty(w) = \text{id}_m \otimes \phi \left[ R_{\infty}(w) (\Sigma \otimes I_A) R_{\infty}(w) \right] \]
are holomorphic on $\mathcal{O}$. Moreover,
\[ \|\Phi_N(w)\| \leq \|\Im(w)^{-1}\|^2 \|\Sigma\|. \]
It readily follows that $\Phi_N$ is a bounded sequence in the set of analytic functions on $\mathcal{O}$ endowed with the uniform convergence on compact subsets. Since moreover, almost surely, for any $t > K$, $t \in \mathbb{Q}$, $\Phi_N(itI_m)$ converges towards $\Phi(itI_m)$, we can apply Vitali’s theorem to conclude
that almost surely the convergence of \( \Phi_N \) towards \( \Phi_{\infty} \) holds on \( \mathcal{O} \). Of course, this convergence still holds on \(-\mathcal{O}\).

Let \( z \in \mathbb{C} \) be such that \( |z - \rho| \leq C_\epsilon/4 \) and set \( w = ze_{11} - \gamma \). Define \( \varepsilon_z = 1 \) if \( \Im z \geq 0 \) and \( \varepsilon_z = -1 \) else. For any \( q > 0 \), such that \( \frac{1}{|q|} \leq C_\epsilon/4 \), define \( w(q) = ze_{11} - \gamma + i\frac{q}{q} I_m \). Almost surely, \( \Phi_N(w(q)) \) converges towards \( \Phi_{\infty}(w(q)) \). Using the resolvent identities \((32)\) and \((31)\) on \( \Omega_{N-1} \), and the bounds \((36)\) and \((37)\), we easily deduce by letting \( q \) go to infinity that a.s.

\[
\text{id}_m \otimes \text{tr}_{N-1} [R_{N-1}(ze_{11} - \gamma) (\Sigma \otimes I_{N-1}) R_{N-1}(ze_{11} - \gamma)] I_{\Omega_{N-1}} \rightarrow_{N \to +\infty} \text{id}_m \otimes \phi [R_{\infty}(ze_{11} - \gamma) (\Sigma \otimes I_A) R_{\infty}(ze_{11} - \gamma)].
\]

Note that using \((24)\), \((52)\), the bound \((36)\), and the resolvent identity \((31)\) on \( \Omega_{N-1} \), \((53)\) readily follows from \((53)\) applied to \( z = \rho \).

\((33)\), \((44)\), \((45)\) and \((46)\) can be proved using similar ideas.

The proof of Theorem 3 that will be presented in Section 6 is based on the writing of the outlier in terms of a quadratic form involving the resolvent. Section 5.2 presents the central limit theorem for random quadratic forms involved in the proof whereas Section 5.3 gather results that will be used to prove that some terms are negligible.

### 5.2 Central limit theorem for random quadratic forms

**Proposition 4.** For any Hermitian \( m \times m \) matrix \( H \),

\[
\sqrt{N} \left( \frac{1}{N} \text{Tr}_m \left\{ H (\alpha \otimes Y^*) R_{N-1}(\rho Ne_{11} - \gamma) I_{\Omega_{N-1}} (\alpha \otimes Y) \right\} 
- \frac{1}{N} \text{Tr}_m \left\{ H \alpha \left[ \text{id}_m \otimes \text{tr}_N R_{N-1}(\rho Ne_{11} - \gamma) I_{\Omega_{N-1}} \right] \alpha \right\} \right)
\]

converges in distribution to a Gaussian variable with mean 0 and variance

\[
(\mathbb{E} (|y|^4) - 2) \int \left[ \text{Tr}_m \left( \alpha H \alpha (\omega_m(\rho e_{11} - \gamma) - t\beta)^{-1} \right) \right]^2 d\mu_\alpha(t)
+
\phi \left( \left[ \text{Tr}_m \otimes \text{id} \left\{ ((\rho e_{11} - \gamma) \otimes 1 \otimes x - \beta \otimes a)^{-1} (\alpha H \alpha) \otimes 1_A \right\} \right]^2 \right).
\]

**Proof.** We apply the following Proposition 3 to \( B = R_{N-1}(\rho Ne_{11} - \gamma) I_{\Omega_{N-1}} \) by using Proposition 7 below.

**Proposition 5.** Let \( m \) be a fixed integer number and \( \alpha \) be a Hermitian \( m \times m \) deterministic matrix that does not depend on \( N \). Let \( B \) be a random Hermitian \( mN \times mN \) matrix such that there exists \( C > 0 \) such that \( \|B\| \leq C \). Let us write \( B = \sum_{i,j=1}^{N} B_{ij} \otimes E_{ij} \) where \( B_{ij} \) are \( m \times m \) matrices and \( E_{ij} \) is the \( N \times N \) matrix whose single nonnull entry equals one and occurs in the row \( i \) and column \( j \). Assume that, for any \( p, q, p', q' \) in \( \{1, \ldots, m\}^4 \),

- \( \frac{1}{N} \sum_{i=1}^{N} (B_{ii})_{pq} (B_{ii})_{p'q'} \rightarrow_{N \to +\infty} \omega_{(p,q),(p',q')} \) in probability,
- \( \frac{1}{N} \sum_{i,j=1}^{N} (B_{ij})_{pq} (B_{ij})_{p'q'} \rightarrow_{N \to +\infty} \theta_{(p,q),(p',q')} \) in probability.
Let $X^T = (x_1, \ldots, x_N)$ be an independent vector of size $N$ which contains i.i.d. standardized entries with bounded fourth moment and such that $\mathbb{E}(x_i^2) = 0$. Let $H$ be a $m \times m$ deterministic Hermitian matrix that does not depend on $N$. Then, when $N$ goes to infinity,

$$
\frac{1}{\sqrt{N}} \text{Tr}_m \{ H [(\alpha \otimes X^*) B (\alpha \otimes X) - \text{aid}_m \otimes \text{Tr}_N(B)\alpha] \}
$$

converges in distribution to a Gaussian variable with mean 0 and variance

$$(\mathbb{E}(|x_1|^4) - 2) \sum_{p,q,p',q'=1}^m (\alpha H\alpha)_{qp} (\alpha H\alpha)_{q'p'} \omega(p,q),(p',q') + \sum_{p,q,p',q'=1}^m (\alpha H\alpha)_{qp} (\alpha H\alpha)_{q'p'} \theta(p,q),(p',q').$$

Proof. Note that

$$
\frac{1}{\sqrt{N}} \text{Tr}_m \{ H [(\alpha \otimes X^*) B (\alpha \otimes X) - \text{aid}_m \otimes \text{Tr}_N(B)\alpha] \} = \frac{1}{\sqrt{N}} \{ X^*BX - \text{Tr}B \}
$$

where $B = (B_{ij})_{1 \leq i,j \leq N}$ and $B_{ij} = \text{Tr}_m \alpha H\alpha B_{ij}$. The result follows from [6] or Theorem 5.2 in [21].

\[\Box\]

### Proposition 6

When it is defined, let us rewrite

$$
R_{N-1} = \sum_{i,j=1}^{N-1} (R_{N-1})_{ij} \otimes E_{ij},
$$

where $(R_{N-1})_{ij} \in M_n(\mathbb{C})$ and $E_{ij}$ is the $(N-1) \times (N-1)$ matrix whose single nonnull entry equals one and occurs in the row $i$ and column $j$. For any $w \in \mathbb{H}_n^+(\mathbb{C})$, we have that, almost surely,

$$
F_N(w) = \frac{1}{N-1} \sum_{i=1}^{N-1} [((R_{N-1}(w - \gamma))_{ii})_{pq}[(R_{N-1}(w - \gamma))_{ii}]_{p'q'} (54)
$$

$$
\rightarrow_{N \rightarrow +\infty} \int [\omega_m(\rho e_{11} - \gamma) - t\beta]^{-1]_{pq}[(\omega_m(\rho e_{11} - \gamma) - t\beta)^{-1}]_{p'q'} d\mu_a(t) \quad (55)
$$

Proof. First we are going to prove that almost surely,

$$
\frac{1}{N-1} \sum_{i=1}^{N-1} [((R_{N-1}(w - \gamma))_{ii})_{pq}[(R_{N-1}(w - \gamma))_{ii}]_{p'q'}
$$

$$
\quad - \frac{1}{N-1} \sum_{i=1}^{N-1} \mathbb{E}((R_{N-1}(w - \gamma))_{ii})_{pq} \mathbb{E}((R_{N-1}(w - \gamma))_{ii})_{p'q'} \rightarrow_{N \rightarrow +\infty} 0. \quad (56)
$$

Set $a_i = [(R_{N-1}(w - \gamma))_{ii}]_{pq}$ and $b_i = [(R_{N-1}(w - \gamma))_{ii}]_{p'q'}$. We have

$$
\frac{1}{N-1} \sum_{i=1}^{N-1} a_i b_i - \frac{1}{N-1} \sum_{i=1}^{N-1} \mathbb{E}(a_i) \mathbb{E}(b_i)
$$
\[
\begin{align*}
&= \frac{1}{N-1} \sum_{i=1}^{N-1} a_i b_i - \frac{1}{N-1} \sum_{i=1}^{N-1} \mathbb{E}(a_i b_i) \\
&\quad + \frac{1}{N-1} \sum_{i=1}^{N-1} \mathbb{E} \left\{ (a_i - \mathbb{E}(a_i))(b_i - \mathbb{E}(b_i)) \right\}.
\end{align*}
\]

Let us denote by \( M_{N-1}^{sa}(\mathbb{C}) \) the real vector space of \((N-1) \times (N-1)\) selfadjoint matrices with complex entries. Consider the linear isomorphism \( \Psi \) between \( M_{N-1}^{sa}(\mathbb{C}) \) and \( \mathbb{R}^{(N-1)^2} \) given by

\[
\Psi((a_{kl})_{1 \leq k,l \leq N-1}) = (a_{kk})_{1 \leq k \leq N-1}, (\sqrt{2} \Re a_{kl})_{1 \leq k \leq N-1, 1 \leq l \leq N-1}, (\sqrt{2} \Im a_{kl})_{1 \leq k \leq N-1, 1 \leq l \leq N-1}
\] (57)

for \((a_{kl})_{1 \leq k,l \leq N-1}\) in \( M_{N-1}^{sa}(\mathbb{C}) \). \( M_{N-1}(\mathbb{C})^{sa} \) is an Euclidean space with inner product given by \( \langle A, B \rangle = \text{Tr}_{N-1}(AB) \) and with norm

\[
\|A\|_e = (\text{Tr} A^2)^{1/2}.
\]

We shall identify \( M_{N-1}^{sa}(\mathbb{C}) \) with \( \mathbb{R}^{(N-1)^2} \) via the isomorphism \( \Psi \). Note that under this identification the norm \( \| \cdot \|_e \) on \( M_{N-1}^{sa}(\mathbb{C}) \) corresponds to the usual Euclidean norm on \( \mathbb{R}^{(N-1)^2} \).

Set

\[
f_N(W) = \frac{1}{N-1} \sum_{i=1}^{N-1} \text{Tr} \left[ ((w - \gamma) \otimes I_{N-1} - \alpha \otimes W - \beta \otimes A_{N-1})^{-1} (e_{qp} \otimes E_{ii}) \right] \\
\times \text{Tr} \left[ ((w - \gamma) \otimes I_{N-1} - \alpha \otimes W - \beta \otimes A_{N-1})^{-1} (e_{qp'} \otimes E_{ii}) \right],
\]

where for any \((l, r) \in \{1, \ldots, m\}^2\), \( e_{lr} \) denotes the \( m \times m \) matrix where the single nonnull entry equals one and occurs in the row \( l \) and column \( r \).

Using the resolvent identity, for \( H_1, H_2 \in M_{m(N-1)}^{sa}(\mathbb{C}) \),

\[
(w \otimes I_{N-1} - H_1)^{-1} - (w \otimes I_{N-1} - H_2)^{-1} \\
= (w \otimes I_{N-1} - H_1)^{-1} (H_1 - H_2) (w \otimes I_{N-1} - H_2)^{-1},
\] (58)

and \([\text{30]} \text{ Lemma 3.1 (i)}\), one can easily prove that \( f_N \circ \Psi^{-1} \) is Lipschitz with constant \( \| (\Im w)^{-1} \|_3 \). Therefore, according to Lemma \([\text{31]} \)

\[
\mathbb{P} \left( \left| \frac{1}{N-1} \sum_{i=1}^{N-1} (a_i b_i - \mathbb{E}(a_i b_i)) \right| > \varepsilon \right) \leq 2 \exp \left( -CN^{1/2}\| (\Im w)^{-1} \|^{-3}\varepsilon \right).
\]

By Borell-Cantelli lemma, we deduce that, almost surely, when \( N \) goes to infinity, \( \frac{1}{N-1} \sum_{i=1}^{N-1} a_i b_i - \frac{1}{N-1} \sum_{i=1}^{N-1} \mathbb{E}(a_i b_i) \) goes to zero.

Now set

\[
g_N(W) = \text{Tr} \left[ ((w - \gamma) \otimes I_{N-1} - \alpha \otimes W - \beta \otimes A_{N-1})^{-1} (e_{qp} \otimes E_{ii}) \right].
\]
Define also $\tilde{g}_N : \mathbb{R}^{(N-1)^2} \to \mathbb{R}$ by $\tilde{g}_N = g_N \circ \Psi^{-1}$, where $\Psi$ is defined in (37). Note that

$$\|\nabla \tilde{g}_N(\Psi(X))\|^2 = \|\nabla g_N(X)\|_{e}^2.$$ 

Applying Poincaré inequality for $\tilde{g}_N$, we get that

$$\mathbb{E} \left( \left| g_N \left( \frac{W_{N-1}}{\sqrt{N}} \right) - \mathbb{E} \left\{ g_N \left( \frac{W_{N-1}}{\sqrt{N}} \right) \right\} \right|^2 \right) \leq \frac{C}{N} \mathbb{E} \left( \|\nabla g_N(\frac{W_{N-1}}{\sqrt{N}})\|_{e}^2 \right).$$

Note that

$$\|\nabla g_N(X)\|_{e}^2 = \max_{w \in S_1(M_{N-1}(\mathbb{C})_{sa})} \left| \frac{d}{dt} \right|_{t=0} g_N(X + tw) \right|^2,$$

where $S_1(M_{N-1}(\mathbb{C})_{sa})$ denotes the unit sphere of $M_{N-1}(\mathbb{C})$ with respect to $\| \cdot \|_{e}$. Using (38) and (33), it readily follows that, there exists $C > 0$, such that for any $i = 1, \ldots, N - 1$,

$$\mathbb{E} |a_i - \mathbb{E}(a_i)|^2 \leq \frac{C\|(3w)^{-1}\|_{e}^4}{N}$$

and similarly

$$\mathbb{E} |b_i - \mathbb{E}(b_i)|^2 \leq \frac{C\|(3w)^{-1}\|_{e}^4}{N},$$

so that $\frac{1}{N-1} \sum_{i=1}^{N-1} \mathbb{E} \{ (a_i - \mathbb{E}(a_i))(b_i - \mathbb{E}(b_i)) \}$ goes to zero as $N$ goes to infinity. Thus, the proof of (56) is complete.

**Lemma 5.** For any $w \in \mathbb{H}_{m}^{\ast}(\mathbb{C})$, for any $j \in \{1, \ldots, N - 1\}$,

$$\mathbb{E} \{ (R_{N-1}(w - \gamma))_{jj} \} = (\omega_m^{(N)}(w - \gamma) - d_j \beta)^{-1} + O_j^{(u)}(1/\sqrt{N}),$$

where there exists a polynomial $Q$ with nonnegative coefficients such that for any $j \in \{1, \ldots, N - 1\}$ and for any $w \in M_m(\mathbb{C})$ such that $\Im w > 0$,

$$O_j^{(u)}(1/\sqrt{N}) \leq \frac{Q\|(3w)^{-1}\|_{e}}{\sqrt{N}}.$$ 

**Proof.** Denote by $\kappa_3$ the classical third cumulant of $\mu$. According to Corollary 5.5 in [12], for any $j \in \{1, \ldots, N - 1\}$,

$$\mathbb{E} \{ (R_{N-1}(w - \gamma))_{jj} \} = (Y_{N-1}(w))_{jj} + \sum_{i,l=1}^{N-1} \frac{\kappa_3(1 - \sqrt{-1})}{2\sqrt{2}(N-1)\sqrt{N-1}} (Y_{N-1}(w))_{jl} \alpha(Y_{N-1}(w))_{il} \alpha \mathbb{E} \{ (R_{N-1}(w - \gamma))_{jj} \} + O_j^{(u)}(1/N),$$

where $Y_{N-1}(w)$ is a $m \times (N - 1)$ matrix and $O_j^{(u)}(1/N)$ is a function on $w, N, j$, such that
and there exists a polynomial $P$ with nonnegative coefficients such that, for any $w \in M_m(\mathbb{C})$ such that $\Im w > 0$,

$$
\left\| Y_{N-1}(w) - \tilde{Y}_{N-1}(w) \right\| \leq \frac{P(\left\| (\Im w)^{-1} \right\|)}{\sqrt{N}},
$$

with $\tilde{Y}_{N-1}(w) = \left( \omega_{m}^{(N)}(w - \gamma) \otimes I_{N-1} - \beta \otimes A_{N-1} \right)^{-1}$,

- there exists a polynomial $Q$ with nonnegative coefficients such that for any $j \in \{1, \ldots, N-1\}$ and for any $w \in M_m(\mathbb{C})$ such that $\Im w > 0$,

$$
O_j^{(w)}(1/N) \leq \frac{Q(\left\| (\Im w)^{-1} \right\|)}{N},
$$

Now

$$
\left\| \sum_{i,l=1}^{N-1} \frac{k_{3}(1-\sqrt{1})}{2\sqrt{2}(N-1)} Y_{N-1}(w_{ij}) \right\| \leq \frac{C}{\sqrt{N}} \left( \sum_{i=1}^{N-1} \left\| (Y_{N-1})_{ji} \right\|^2 \right)^{1/2} \left( \sum_{i=1}^{N-1} \left\| \mathbb{E} \{(R_{N-1}(w - \gamma))_{ij}\} \right\|^2 \right)^{1/2} \leq \frac{Cm}{\sqrt{N}} \left\| Y_{N-1} \right\| \left\| \mathbb{E} (R_{N-1}(w - \gamma)) \right\| \leq \frac{Cm}{\sqrt{N}^3},
$$

where we used $[12]$ Lemma 8.1, (59) and (33) in the last lines. It readily follows that, for any $j \in \{1, \ldots, N-1\}$,

$$
\mathbb{E} \{(R_{N-1}(w - \gamma))_{jj}\} = \left( \omega_{m}^{(N)}(w - \gamma) \otimes I_{N-1} - \beta \otimes A_{N-1} \right)^{-1}_{jj} + O_j^{(w)}(1/\sqrt{N}),
$$

where there exists a polynomial $Q$ with nonnegative coefficients such that for any $j \in \{1, \ldots, N-1\}$ and for any $w \in M_m(\mathbb{C})$ such that $\Im w > 0$,

$$
O_j^{(w)}(1/\sqrt{N}) \leq \frac{Q(\left\| (\Im w)^{-1} \right\|)}{\sqrt{N}}.
$$

Now, note that there exist two permutation matrices $P$ and $Q$ in $M_{(N-1)m}(\mathbb{C})$ such that, for any matrices $A \in M_m(\mathbb{C})$, $B \in M_{N-1}(\mathbb{C})$, $A \otimes B = P(B \otimes A)Q$. Therefore
\[
\left[ \left( \omega_{m}^{(N-1)}(w - \gamma) \otimes I_{N-1} - \beta \otimes A_{N-1} \right)^{-1} \right]_{pq}
\]

\[
= \text{Tr} \left[ \left( \omega_{m}^{(N-1)}(w - \gamma) \otimes I_{N-1} - \beta \otimes A_{N-1} \right)^{-1} (e_{qp} \otimes E_{jj}) \right]
\]

\[
= \text{Tr} \left[ Q^{-1} \left( I_{N-1} \otimes \omega_{m}^{(N-1)}(w - \gamma) - A_{N-1} \otimes \beta \right)^{-1} P^{-1} P (E_{jj} \otimes e_{qp}) Q \right]
\]

\[
= \text{Tr} \left[ \left( I_{N-1} \otimes \omega_{m}^{(N-1)}(w - \gamma) - A_{N-1} \otimes \beta \right)^{-1} (E_{jj} \otimes e_{qp}) \right]
\]

\[
= \left[ \left( \omega_{m}^{(N-1)}(w - \gamma) - d_{j} \beta \right)^{-1} \right]_{pq}.
\]

Thus,

\[
\mathbb{E} \{(R_{N-1}(w - \gamma))_{jj}\} = \left( \omega_{m}^{(N)}(w - \gamma) - d_{j} \beta \right)^{-1} + O_{j}^{(u)}(1/\sqrt{N}),
\]

where there exists a polynomial \( Q \) with nonnegative coefficients such that for any \( j \in \{1, \ldots, N - 1\} \) and for any \( w \in M_{m}(\mathbb{C}) \) such that \( \Im w > 0 \),

\[
O_{j}^{(u)}(1/\sqrt{N}) \leq \frac{Q(||(\Im w)^{-1}||)}{\sqrt{N}}.
\]

Lemma 5 follows.

\[\Box\]

Note that, using (7) and Lemma 3.1 (i), we have that for any \( w \in H_{m}^{+}(\mathbb{C}) \)

\[
\left\| (\omega_{m}^{(N)}(w) - d_{j} \beta)^{-1} \right\| \leq \left\| (\Im w)^{-1} \right\|, \tag{60}
\]

\[
\left\| (\omega_{m}(w) - d_{i} \beta)^{-1} \right\| \leq \left\| (\Im w)^{-1} \right\|, \tag{61}
\]

and then

\[
\left\| (\omega_{m}^{(N)}(w) - d_{i} \beta)^{-1} - (\omega_{m}(w) - d_{i} \beta)^{-1} \right\|
\]

\[
\leq \left\| (\omega_{m}^{(N)}(w) - d_{i} \beta)^{-1} \left[ \omega_{m}(w) - \omega_{m}^{(N)}(w) \right] (\omega_{m}(w) - d_{i} \beta)^{-1} \right\|
\]

\[
\leq \left\| (\Im w)^{-1} \right\|^2 \left\| \omega_{m}^{(N)}(w) - \omega_{m}(w) \right\|. \tag{62}
\]

Lemma 5 and (56) yield that almost surely for any \( w \in H_{m}^{+}(\mathbb{C}) \),

\[
F_{N}(w) = \frac{1}{N - 1} \sum_{i=1}^{N-1} \left[ \left( (\omega_{m}^{(N)}(w - \gamma) - d_{i} \beta)^{-1} \right]_{pq} (\omega_{m}^{(N)}(w - \gamma) - d_{i} \beta)^{-1} \right]_{p'q'} + o(1)
\]

\[
= \frac{1}{N - 1} \sum_{i=1}^{N-1} \left[ (\omega_{m}(w - \gamma) - d_{i} \beta)^{-1} \right]_{pq} (\omega_{m}(w - \gamma) - d_{i} \beta)^{-1} \right]_{p'q'} + o(1),
\]

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identities \((31), (32)\), one can easily prove that
\[
F_N(w) = \int [(\omega_m^{(N)}(w - \gamma) - t\beta)^{-1}]_{pq} [(\omega_m^{(N)}(w - \gamma) - t\beta)^{-1}]_{p'q'} d\mu_{A_{N-1}}(t) + o(1)
\]
where \(\mu_{A_{N-1}} = \frac{1}{N-1} \sum_{i=1}^{N-1} \delta_{\lambda_i(A_{N-1})}\) is the empirical spectral measure of \(A_{N-1}\). Since \(\mu_{A_{N-1}}\) weakly converges towards \(\mu_{\omega}\), Proposition 6 follows. \(\square\)

**Proposition 7.** When it is defined, let us rewrite
\[
R_{N-1} = \sum_{i,j=1}^{N-1} (R_{N-1})_{ij} \otimes E_{ij},
\]
where \((R_{N-1})_{ij} \in M_m(C)\). We have that, almost surely,
\[
\frac{1}{N-1} \sum_{i,j=1}^{N-1} [(R_{N-1}(\rho \epsilon e_{11} - \gamma))_{ij}]_{pq} [(R_{N-1}(\rho \epsilon e_{11} - \gamma))_{ji}]_{p'q'} 1_{\tilde{\Omega}_{N-1}} \\
\rightarrow_{N \to +\infty} \int [(\omega_m(\rho \epsilon e_{11} - \gamma) - t\beta)^{-1}]_{pq} [(\omega_m(\rho \epsilon e_{11} - \gamma) - t\beta)^{-1}]_{p'q'} d\mu_{\omega}(t) \quad (63)
\]
and
\[
\frac{1}{N-1} \sum_{i,j=1}^{N-1} [(R_{N-1}(\rho \epsilon e_{11} - \gamma))_{ij}]_{pq} [(R_{N-1}(\rho \epsilon e_{11} - \gamma))_{ji}]_{p'q'} 1_{\tilde{\Omega}_{N-1}} \\
\rightarrow_{N \to +\infty} \phi \left\{ \text{Tr}_m \otimes \text{id}_A \left[ R_{\infty}(\rho \epsilon e_{11} - \gamma) (e_{qp} \otimes 1_A) \right] \text{Tr}_m \otimes \text{id}_A \left[ R_{\infty}(\rho \epsilon e_{11} - \gamma) (e_{q'p'} \otimes 1_A) \right] \right\}. \quad (64)
\]

**Proof.** First, with \(w = \rho \epsilon e_{11} - \gamma\), let us rewrite
\[
\frac{1}{N-1} \sum_{i,j=1}^{N-1} [(R_{N-1}(w))_{ij}]_{pq} [(R_{N-1}(w))_{ji}]_{p'q'} \\
= \text{tr}_{N-1} \left\{ \text{Tr}_m \otimes \text{id}_{N-1} \left[ R_{N-1}(w) (e_{qp} \otimes I_{N-1}) \right] \text{Tr}_m \otimes \text{id}_{N-1} \left[ R_{N-1}(w) (e_{q'p'} \otimes I_{N-1}) \right] \right\}.
\]
Thus (64) readily follows from Lemma 4.

Now, For any \(z \in \mathbb{C}\), set \(\epsilon_z = 1\) if \(\Im z \geq 0\) and \(\epsilon_z = -1\) if \(\Im z < 0\). On \(\tilde{\Omega}_{N-1}\), \(F_N\) defined by (64) is well defined at the points \(w = z e_{11}, z e_{11} + i\frac{r}{\epsilon_z}\), for any \(r \in \mathbb{Q} \setminus \{0\}\), \(0 < 1/r < \tau/2\) and any \(z \in \mathbb{C}\) such that \(|z - \rho| < \tau/2\). Using the bounds (36), (57), (59) and the resolvent identities (31), (32), one can easily prove that
\[
\left\| F_N(z e_{11}) 1_{\tilde{\Omega}_{N-1}}, - \int [(\omega_m(z e_{11} - \gamma) - t\beta)^{-1}]_{pq} [(\omega_m(z e_{11} - \gamma) - t\beta)^{-1}]_{p'q'} d\mu_{\omega}(t) \right\|
\leq \frac{1}{r} \left\{ 16/C^3(1 + \|\alpha\|^2) + 1 \right\} + 4/C^2 \epsilon_z \frac{1}{r} 1_{\tilde{\Omega}_{N-1}} \\
+ \left\| F_N(z e_{11} + i\frac{\epsilon_z}{r} I_m) \\
- \int [(\omega_m(z e_{11} + i\frac{\epsilon_z}{r} I_m - \gamma) - t\beta)^{-1}]_{pq} [(\omega_m(z e_{11} + i\frac{\epsilon_z}{r} I_m - \gamma) - t\beta)^{-1}]_{p'q'} d\mu_{\omega}(t) \right\|.
\]
We deduce by letting $N$ go to infinity, using Proposition 6, and then $r$ go to infinity that for any $z \in \mathbb{C}$ such that for $|z - \rho| < \tau/2$, almost surely, $F_N(ze_{11})I_{\hat{\Omega}_{N-1}}$ converges to $\int [\omega_m (ze_{11} - \gamma) - t\beta]^{-1}]p_q d\mu_q(t)$ when $N$ goes to infinity.

Note that using (24), the resolvent identity (31) on $\tilde{\Omega}_{N-1}^1$, and the bound (36), (63) follows from the result for $\rho$ instead of $\rho_N$. The proof of Proposition 7 is complete.

5.3 Basic technical results of negligeability

**Lemma 6.** For any $N$, let $X_N = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}$ be random in $\mathbb{C}^N$ with iid standardized entries ($\mathbb{E}(x_i) = 0$, $\mathbb{E}(|x_i|^2) = 1$, $\mathbb{E}(x_i^2) = 0$) and $\mathbb{E}(|x_i|^4) < \infty$. Let $m$ be a fixed integer number and $\alpha$ be Hermitian $m \times m$ deterministic matrix. Let $B$ be a Hermitian $mN \times mN$ independent matrix such that $\sup_N \|B\| \leq C$. Then

$$\frac{1}{N} (I_m \otimes X_N^*) B (I_m \otimes X_N) - \text{id}_m \otimes \text{tr}_N B = o_P(1).$$

Proof. Let us write $B = \sum_{p,q=1}^m e_{pq} \otimes B^{(pq)}$ where $B^{(pq)}$ are $N \times N$ matrices. Noting that

$$\frac{1}{N} (I_m \otimes X_N^*) B (I_m \otimes X_N) - \text{id}_m \otimes \text{tr}_N B$$

$$= \frac{1}{N} \sum_{p,q=1}^m e_{pq} \left\{ X_N^* B^{(pq)} X_N - \text{tr}_N (B^{(pq)}) \right\},$$

the result readily follows from Lemma 2.7 in [4].

**Lemma 7.** There exists a polynomial $P$ with nonnegative coefficients such that for any $w \in H^+_m(\mathbb{C})$,

$$\text{id}_m \otimes \text{tr}_{N-1} \mathbb{E} [R_{N-1}(w - \gamma)]$$

$$= \text{id}_m \otimes \phi \left( ((w - \gamma) \otimes I - \alpha \otimes x - \beta \otimes a_{N-1})^{-1} \right) + O(1/N),$$

where

$$O(1/N) \leq \frac{P(||(\Re w)^{-1}||)}{N}.$$ 

This result still holds for $w \in M_m(\mathbb{C})$ such that $\Re w < 0$.

Proof. In this proof, any $O(1/N^p)$ denotes a function such that there exists a polynomial $P$ with nonnegative coefficients such that for any $w \in H^+_m(\mathbb{C})$,

$$O(1/N^p) \leq \frac{P(||(\Re w)^{-1}||)}{N^p}.$$ 

According to Theorem 5.7 in [12],

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id_m \otimes tr_{N-1} \mathbb{E} [R_{N-1}(w - \gamma)]
- \text{id}_A \otimes \phi \left( (w - \gamma) \otimes I - \alpha \otimes x - \beta \otimes a_{N-1} \right)^{-1} + E_{N-1}(w) = O \left( \frac{1}{N \sqrt{N}} \right), \quad (65)

where $E_{N-1}(w)$ is given by

$$E_{N-1}(w) = \tilde{G}_{N-1}^{\nu} \cdot \alpha L_{N-1}(w) \alpha_v - \frac{1}{2} \tilde{G}_{N-1}^{\mu} \cdot (\alpha L_{N-1}(w) \alpha_v, \alpha_v L_{N-1}(w) \alpha) - L_{N-1}(w) \quad (66)$$

with

$$\tilde{G}_{N-1}(w) = \text{id}_m \otimes \phi \left( (w - \gamma) \otimes I - \alpha \otimes x - \beta \otimes a_{N-1} \right)^{-1},$$

$$L_{N-1}(w) = \frac{1}{N - 1} \sum_{j=1}^{N-1} (Y_{N-1}(w) \Psi(\omega))_{jj},$$

$\Psi(\omega)$ and $Y_{N-1}(w)$ being defined in Theorem 5.3 [12] and Lemma 5.2 [12] respectively. Set

$$T_N = \frac{1}{(N-1)^2 \sqrt{N-1}} \kappa_3(1 - \sqrt{-1}) \times \sum_{i,j,l=1}^{N-1} (Y_{N-1}(w))_{jl} \mathbb{E} \{ \alpha(R_{N-1}(w))_{ii} \alpha(R_{N-1}(w))_{ll} \alpha(R_{N-1}(w))_{ij} \},$$

where $\kappa_3$ still denotes the third cumulant of $\mu$. Now using [12, Lemme 8.1, (5.5), (8.14)], and (33), one can easily prove that

$$L_{N-1}(w) - T_N = O(1/N).$$

Note moreover that, for any $m \times m$ matrix $B$ with bounded operator norm

$$\text{Tr}_m(BT_N)$$

$$= \frac{1}{(N-1)^2 \sqrt{N-1}} \kappa_3(1 - \sqrt{-1}) \times \sum_{i,j,l=1}^{N-1} \text{Tr}_m \mathbb{E} \{ \alpha(R_{N-1}(w))_{ii} \alpha(R_{N-1}(w))_{ll} \alpha(R_{N-1}(w))_{ij} \} \left( [B \otimes I_{N-1}] Y_{N-1}(w) \right)_{jl}$$

$$= \frac{1}{(N-1)^2 \sqrt{N-1}} \kappa_3(1 - \sqrt{-1}) \times \sum_{i,l=1}^{N-1} \text{Tr}_m \mathbb{E} \{ \alpha(R_{N-1}(w))_{ii} \alpha(R_{N-1}(w))_{ll} \alpha(R_{N-1}(w))_{ij} \} \left( [B \otimes I_{N-1}] Y_{N-1}(w) \right)_{jl},$$

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so that
\[
|\text{Tr}_m(BT_N)| \leq \frac{\sqrt{2} |\kappa_3| |m| \|\alpha\|^2}{(N - 1) \sqrt{N - 1}}
\times \sum_{i, t=1}^{N - 1} \|(\Im w)^{-1}\|^2 \mathbb{E} \left\{ \left( \sum_{i, t=1}^{N - 1} \|[R_{N-1}(w) (B \otimes I_{N-1}) Y_{N-1}(w)]_{it}\|^2 \right)^{1/2} \right\}
\leq \frac{\sqrt{2} |\kappa_3| |m| \|\alpha\|^2 \|(\Im w)^{-1}\|^4 \|B\|}{(N - 1)}
= O(1/N),
\]
so that
\[
T_N = O(1/N)
\]
and therefore, using (35),
\[
E_{N-1} = O(1/N).
\]
Lemma 7 follows. $\square$

**Proposition 8.**
\[
\sqrt{N} \left\{ \text{id}_m \otimes \text{tr}_{N-1} R_{N-1}(\rho_{Ne_{11}} - \gamma) 1_{\tilde{\Omega}_{N-1}} - \text{id}_m \otimes \phi \left( ((\rho_{N e_{11}} - \gamma) \otimes 1_A - \alpha \otimes x - \beta \otimes a_{N-1})^{-1} \right) \right\}
\]
goes to zero in probability.

**Proof.** According to (36), for $N$ large enough, there exists $K > 0$ such that
\[
\|(\rho_{N e_{11}} - \gamma) \otimes 1_A - \alpha \otimes x - \beta \otimes a_{N-1}\| \leq K.
\]
and on $\tilde{\Omega}_{N-1},$
\[
\left\|(\rho_{N e_{11}} - \gamma) \otimes I_{N-1} - \alpha \otimes \frac{W_{N-1}}{\sqrt{N}} - \beta \otimes A_{N-1}\right\| \leq K.
\]
Let $g : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ function with support in $\{C_\epsilon/4 \leq |x| \leq 2K\}$ and such that $g \equiv 1$ on $\{C_\epsilon/2 \leq |x| \leq K\}.$ $f : x \mapsto \frac{g(x)}{x}$ is a $C^\infty$ function with compact support. Note that
\[
\text{id}_A \otimes \phi \left( ((\rho_{N e_{11}} - \gamma) \otimes 1_A - \alpha \otimes x - \beta \otimes a_{N-1})^{-1} \right)
= \text{id}_A \otimes \phi \left( f ((\rho_{N e_{11}} - \gamma) \otimes I - \alpha \otimes x - \beta \otimes a_{N-1}) \right) \quad (67)
\]
and on $\tilde{\Omega}_{N-1},$
\[
R_{N-1}(\rho_{N e_{11}} - \gamma) = f \left( (\rho_{N e_{11}} - \gamma) \otimes I_{N-1} - \alpha \otimes \frac{W_{N-1}}{\sqrt{N}} - \beta \otimes A_{N-1} \right). \quad (68)
\]
According to Lemma 7 for any $z \in \mathbb{C} \setminus \mathbb{R},$
\[
\sqrt{N} \text{id}_m \otimes \text{tr}_{N-1} \mathbb{E} [R_{N-1}(\rho_{N e_{11}} - \gamma - z I_m)]
\]
\[
\sqrt{N} id_A \otimes \phi \left( ((\rho_N e_{11} - \gamma - z I_m) \otimes 1_A - \alpha \otimes x - \beta \otimes a_{N-1})^{-1} \right) + o^{(z)}(1),
\]
where there exist polynomials \( Q_1 \) and \( Q_2 \) with non negative coefficients and \( k \in \mathbb{N} \) such that
\[
\|o^{(z)}(1)\| \leq \frac{Q_1(|3z|)^{-1}}{\sqrt{N}} \leq \frac{1}{\sqrt{N} |3z|^k}.
\]
We recall Helffer-Sjöstrand's representation formula: let \( f \in C^{k+1}(\mathbb{R}) \) with compact support and \( M \) a Hermitian matrix,
\[
f(M) = \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial}F_k(f)(z) (M - z)^{-1} d^2 z
\]
where \( d^2 z \) denotes the Lebesgue measure on \( \mathbb{C} \).
\[
F_k(f)(x + iy) = \sum_{l=0}^{k} \frac{(iy)^l}{l!} f^{(l)}(x) \chi(y)
\]
where \( \chi : \mathbb{R} \to \mathbb{R}^+ \) is a smooth compactly supported function such that \( \chi \equiv 1 \) in a neighborhood of 0, and \( \bar{\partial} = \partial_x + iy \partial_y \).
The function \( F_k(f) \) coincides with \( f \) on the real axis and is an extension to the complex plane. Note that, in a neighborhood of the real axis,
\[
\bar{\partial}F_k(f)(x + iy) = \left( \frac{iy}{k!} \right)^k f^{(k+1)}(x) = O(|y|^k) \text{ as } y \to 0.
\]
Therefore, by Helffer-Sjöstrand functional calculus,
\[
\sqrt{N} id_m \otimes \text{tr}_{N-1} \mathbb{E} \left( f \left( (\rho_N e_{11} - \gamma) \otimes I_{N-1} - \alpha \otimes \frac{W_{N-1}}{\sqrt{N}} - \beta \otimes A_{N-1} \right) \right)
= \frac{1}{\pi} \int_{\mathbb{C} \setminus \mathbb{R}} \bar{\partial}F_k(f)(z) \sqrt{N} id_m \otimes \text{tr}_{N-1} \mathbb{E} \left[ R_{N-1}((\rho_N e_{11} - \gamma - z I_m) \otimes I_{N-1}) \right] d^2 z
\]
and
\[
\sqrt{N} id_A \otimes \phi \left[ f \left( (\rho_N e_{11} - \gamma) \otimes I - \alpha \otimes x - \beta \otimes a_{N-1} \right) \right]
= \frac{1}{\pi} \int_{\mathbb{C} \setminus \mathbb{R}} \bar{\partial}F_k(f)(z) \sqrt{N} id_m \otimes \phi \left( ((\rho_N e_{11} - \gamma - z I_m) \otimes 1_A - \alpha \otimes x - \beta \otimes a_{N-1})^{-1} \right) d^2 z.
\]
Hence, using (69) and (67), we can deduce that
\[
\sqrt{N} id_m \otimes \text{tr}_{N-1} \mathbb{E} \left( f \left( (\rho_N e_{11} - \gamma) \otimes I_{N-1} - \alpha \otimes \frac{W_{N-1}}{\sqrt{N}} - \beta \otimes A_{N-1} \right) \right) + \frac{1}{\pi} \int_{z \in \mathbb{C}, \Im z \neq 0} \partial F_k(f)(z) o^{(z)}(1) d^2 z
\]
with, according to (70) and (73),
\[
\left\| \frac{1}{\pi} \int_{\mathbb{C} \setminus \mathbb{R}} \partial F_k(f)(z) o^{(z)}(1) d^2 z \right\| \leq \frac{C}{\sqrt{N}}.
\]
Thus,
\[
\sqrt{N} \left\{ \text{Id}_m \otimes \text{tr}_{N-1} \left( f \left( (\rho_N e_{11} - \gamma) \otimes I_{N-1} - \alpha \otimes \frac{W_{N-1}}{\sqrt{N}} - \beta \otimes A_{N-1} \right) \right) - \text{id} \otimes \phi \left( ((\rho_N e_{11} - \gamma) \otimes 1_A - \alpha \otimes x - \beta \otimes a_{N-1})^{-1} \right) \right\} \to_{N \to \infty} 0. 
\] (74)

Now, we are going to study the concentration of
\[
\sqrt{N} \text{id}_m \otimes \text{tr}_{N-1} \left( f \left( (\rho_N e_{11} - \gamma) \otimes I_{N-1} - \alpha \otimes \frac{W_{N-1}}{\sqrt{N}} - \beta \otimes A_{N-1} \right) \right)
\]
around its expectation. Define for any \((p, q) \in \{1, \ldots, m\}^2\), \(h_{pq} : M_{N-1}(\mathbb{C}) \to \mathbb{R}\) by
\[
h_{pq}(X) = \frac{1}{N-1} \text{Tr}_m \otimes \text{Tr}_{N-1} \left[ (e_{qp} \otimes I_{N-1}) f \left( (\rho_N e_{11} - \gamma) \otimes I_{N-1} - \alpha \otimes X - \beta \otimes A_{N-1} \right) \right],
\]
so that
\[
\text{id}_m \otimes \text{tr}_{N-1} \left[ f \left( (\rho_N e_{11} - \gamma) \otimes I_{N-1} - \alpha \otimes X - \beta \otimes A_{N-1} \right) \right] = \sum_{p, q=1}^{m} h_{pq} e_{pq}.
\]
Define also \(\tilde{h}_{pq} : \mathbb{R}^{(N-1)^2} \to \mathbb{R}\) by \(\tilde{h}_{pq} = h_{pq} \circ \Psi^{-1}\), where \(\Psi\) is defined in (57). Note that
\[
\left\| \nabla \tilde{h}_{pq}(\Psi(X)) \right\|^2 = \|\text{grad} h_{pq}(X)\|_e^2.
\]
Applying Poincaré inequality for \(\tilde{h}_{pq}\), we get that
\[
\mathbb{E} \left( \left\| h_{pq}(\frac{W_{N-1}}{\sqrt{N}}) - \mathbb{E}(h_{pq}(\frac{W_{N-1}}{\sqrt{N}})) \right\|^2 \right) \leq \frac{C}{N} \mathbb{E} \left( \|\text{grad} h_{pq}(\frac{W_{N-1}}{\sqrt{N}})\|_e^2 \right).
\]

Note that
\[
\|\text{grad} h_{pq}(X)\|_e^2 = \max_{w \in S_1(M_{N-1}(\mathbb{C}))} \left\| \frac{d}{dt} h_{pq}(X + tw)|_{t=0} \right\|^2,
\]
where \(S_1(M_{N-1}(\mathbb{C}))\) denotes the unit sphere of \(M_{N-1}(\mathbb{C})\) with respect to \(\| \cdot \|_e\). For \(w\) in \(S_1(M_{N-1}(\mathbb{C}))\) set
\[
\Delta(t) = f \left( (\rho_N e_{11} - \gamma) \otimes I_{N-1} - \alpha \otimes (X + tw) - \beta \otimes A_{N-1} \right)
- f \left( (\rho_N e_{11} - \gamma) \otimes I_{N-1} - \alpha \otimes X - \beta \otimes A_{N-1} \right)
\]
and
\[
\Delta(t) = \sum_{p', q'=1}^{m} e_{p'q'} \otimes \Delta_{p'q'}(t).
\]
Note that \(\Delta(t) = \Delta(t)^*\) so that \(\Delta_{p'q'}(t) = \Delta_{p'q'}(t)^*\) We have
\[
\left| \frac{d}{dt} h_{pq}(X + tw)|_{t=0} \right|^2 = \left| \lim_{t \to 0} \frac{1}{t} \text{tr}_{N-1} \Delta_{pq}(t) \right|^2.
\]
Now,
\[ |\text{tr}_{N-1} \Delta_{pq}(t)|^2 \leq \text{tr}_{N-1} \Delta_{pq}(t) \Delta_{pq}(t)^*. \]

We have
\[
\begin{align*}
\text{Tr}_m \otimes \text{Tr}_{N-1} \Delta(t)^2 &= \sum_{p',q'=1}^m \text{Tr}_{N-1} \Delta_{p'q'}(t) \Delta_{q'p'}(t) \\
&= \sum_{p',q'=1}^m \text{Tr}_{N-1} \Delta_{pq}(t) \Delta_{p'q'}(t)^*.
\end{align*}
\]

Therefore \( \text{Tr}_{N-1} \Delta_{pq}(t) \Delta_{pq}(t)^* \leq \text{Tr}_m \otimes \text{Tr}_{N-1} \Delta(t)^2 \). Since \( f \) is a Lipschitz function on \( \mathbb{R} \) with Lipschitz constant \( C_L \), its extension on Hermitian matrices is \( C_L \)-Lipschitz with respect to the norm \( \| M \|_2 = (\text{Tr} M^2)^{1/2} \). Thus,
\[
|\text{tr}_{N-1} \Delta_{pq}(t)|^2 \leq \frac{1}{N-1} \text{Tr}_m \otimes \text{Tr}_{N-1} \Delta(t)^2 \leq C^2 \frac{t^2}{N-1} \text{Tr}_m \otimes \text{Tr}_{N-1}(\alpha^2 \otimes w^2) = t^2 \frac{1}{N-1} C^2 \text{Tr}_m \alpha^2.
\]

Therefore,
\[
\sup_{w \in S_1(M_{N-1}(\mathbb{C}))} \left| \frac{d}{dt} \big|_{t=0} h_{pq}(X + tw) \right|^2 \leq \frac{C}{N},
\]
and then
\[
\mathbb{E} \left( \sqrt{N} \left\{ h_{pq} \left( \frac{W_{N-1}}{\sqrt{N}} \right) - \mathbb{E} \left( h_{pq} \left( \frac{W_{N-1}}{\sqrt{N}} \right) \right) \right\} \right) \leq \frac{C}{N}.
\]

It readily follows that
\[
\sqrt{N} \text{id}_m \otimes \text{tr}_{N-1} \left( f \left( (\rho_{N} e_{11} - \gamma) \otimes I_{N-1} - \alpha \otimes \frac{W_{N-1}}{\sqrt{N}} - \beta \otimes A_{N-1} \right) \right) \\
- \sqrt{N} \mathbb{E} \text{id}_m \otimes \text{tr}_{N-1} \left( f \left( (\rho_{N} e_{11} - \gamma) \otimes I_{N-1} - \alpha \otimes \frac{W_{N-1}}{\sqrt{N}} - \beta \otimes A_{N-1} \right) \right)
= o_p(1). \tag{75}
\]

Proposition 8 follows from (68), (74), (75) and (52).

6 Proof of Theorem 3

According to Lemma 1, \( \lambda \) is an eigenvalue of \( M_N \) if and only if
\[
\det \left( \lambda e_{11} \otimes I_N - \gamma \otimes I_N - \alpha \otimes \frac{W_N}{\sqrt{N}} - \beta \otimes A_N \right) = 0
\]
or, since there exist permutation matrices \( K_{Nm} \) and \( K_{mN} \) in \( M_{Nm} \) such that for any \( A \in M_N \) and \( B \in M_m \),
\[
A \otimes B = K_{Nm}(B \otimes A) K_{mN}, \tag{76}
\]
equivalently
\[
\det \left( I_N \otimes (\lambda e_{11} - \gamma) - \frac{W_N}{\sqrt{N}} \otimes \alpha - A_N \otimes \beta \right) = 0.
\]
Thus, \( \lambda \) is an eigenvalue of \( M_N \) if and only if
\[
\exists V \in \mathbb{C}^{Nm} \setminus \{0\}, \left( I_N \otimes (\lambda e_{11} - \gamma) - \frac{W_N}{\sqrt{N}} \otimes \alpha - A_N \otimes \beta \right) V = 0. \quad (77)
\]
Set
\[
V = \sum_{i=1}^{m} V_i \otimes e_i
\]
where \((e_i)_{i=1,...,m}\) is the canonical basis of \( \mathbb{C}^m \) and
\[
V_i = \begin{pmatrix} v_i^{(1)} \in \mathbb{C} \\ v_i^{(2)} \in \mathbb{C}^{N-1} \end{pmatrix}.
\]
(77) can be rewritten
\[
\sum_{i=1}^{m} \begin{bmatrix} v_i^{(1)}(\lambda e_{11} - \gamma)e_i \\ (V_i^{(2)} \otimes (\lambda e_{11} - \gamma)e_i) - \left( \frac{W_N}{\sqrt{N}} v_i^{(1)} + \frac{Y^*}{\sqrt{N}} V_i^{(2)} \right) \alpha e_i - \left( \frac{Y}{\sqrt{N}} v_i^{(1)} + \frac{W_N - 1}{\sqrt{N}} V_i^{(2)} \right) \otimes \alpha e_i - \left( \frac{\theta v_i^{(1)} \beta e_i}{A_{N-1} V_i^{(2)} \otimes \beta e_i} \right) \end{bmatrix} = 0
\]
which leads to the system
\[
\begin{cases}
(\lambda e_{11} - \gamma - \frac{W_{11}}{\sqrt{N}} - \beta \theta) \left( \sum_{i=1}^{m} v_i^{(1)} e_i \right) = \left( \frac{Y}{\sqrt{N}} \otimes \alpha \right) \left( \sum_{i=1}^{m} V_i^{(2)} \otimes e_i \right) \\
I_{N-1} \otimes (\lambda e_{11} - \gamma) - \frac{W_{N-1}}{\sqrt{N}} \otimes \alpha - A_{N-1} \otimes \beta) \left( \sum_{i=1}^{m} V_i^{(2)} \otimes e_i \right) = \left( \frac{Y}{\sqrt{N}} \otimes \alpha \right) \left( \sum_{i=1}^{m} v_i^{(1)} e_i \right)
\end{cases}
\]
Let \( \tau \) be defined by (16). For any \( \lambda \in \mathcal{B}(\rho, \tau) \), according to (15) and (16), we can define on \( \tilde{\Omega}_{N-1} \)
\[
\tilde{R}_{N-1}(\lambda e_{11} - \gamma) = \left( I_{N-1} \otimes (\lambda e_{11} - \gamma) - \frac{W_{N-1}}{\sqrt{N}} \otimes \alpha - A_{N-1} \otimes \beta \right)^{-1}.
\]
The following lines hold on \( \Omega_N \) (defined by (17)).
First, we can deduce from the above system that \( \lambda \in \mathcal{B}(\rho, \tau) \) is an eigenvalue of \( M_N \) if and only if there exists \((v_i^{(1)})_{i=1,...,m} \in \mathbb{C}^m, (V_i^{(2)})_{i=1,...,m} \in \mathbb{C}^{m(N-1)} \), such that:
\[
\sum_{i=1}^{m} v_i^{(1)} e_i \neq 0, \quad (78)
\]
\[
\sum_{i=1}^{m} V_i^{(2)} \otimes e_i = \tilde{R}_{N-1}(\lambda e_{11} - \gamma) \left( \frac{Y}{\sqrt{N}} \otimes \alpha \right) \left( \sum_{i=1}^{m} v_i^{(1)} e_i \right), \quad (79)
\]
\[
(\lambda e_{11} - \gamma - \frac{W_{11}}{\sqrt{N}} - \beta \theta - \frac{1}{N} (Y^* \otimes \alpha) \tilde{R}_{N-1}(\lambda e_{11} - \gamma) (Y \otimes \alpha) \left( \sum_{i=1}^{m} v_i^{(1)} e_i \right) = 0. \quad (80)
\]
Therefore in particular this implies
\[
\det (X_m(N)) = 0, \quad (81)
\]
where
\[ X_m(N) = \lambda(N, \rho) e_{11} - \gamma - \alpha \frac{W_{11}}{\sqrt{N}} - \beta \theta - \frac{1}{N} (Y^* \otimes \alpha) \tilde{R}_{N-1}(\lambda(N, \rho) e_{11} - \gamma) (Y \otimes \alpha), \]
with \( \lambda(N, \rho) \) defined by (18). Now, noticing that
\[ (Y^* \otimes \alpha) \tilde{R}_{N-1}(\lambda(N, \rho) e_{11} - \gamma) (Y \otimes \alpha) = \text{Tr}_{N-1} \otimes \text{id}_m \left[ (E_{11} \otimes I_m) (\tilde{Y}^* \otimes \alpha) \tilde{R}_{N-1}(\lambda(N, \rho) e_{11} - \gamma) (\tilde{Y} \otimes \alpha) \right], \]
where \( \tilde{Y} = (Y|0) \in M_{N-1}(\mathbb{C}) \), and using (76), it is easy to see that
\[ (Y^* \otimes \alpha) \tilde{R}_{N-1}(\lambda(N, \rho) e_{11} - \gamma) (Y \otimes \alpha) = \frac{1}{N} (\alpha \otimes Y^*) R_{N-1}(\lambda(N, \rho) e_{11} - \gamma) (\alpha \otimes Y). \]
Let \( \rho_N \) be as defined by (23). Using the identity
\[ R_{N-1}(\rho_N e_{11} - \gamma) - R_{N-1}(\lambda(N, \rho) e_{11} - \gamma) = (\lambda(N, \rho) - \rho_N) R_{N-1}(\rho_N e_{11} - \gamma) \left( e_{11} \otimes I_{N-1} \right) R_{N-1}(\lambda(N, \rho) e_{11} - \gamma), \]
we have
\[ X_m(N) = H_m(N) + X_m^{(0)}(N), \]
where
\[ X_m^{(0)}(N) = \omega_m^{(N)}(\rho_N e_{11} - \gamma) - \beta \theta, \]
(\( \omega_m^{(N)} \) is defined by (22)),
\[ H_m(N) = (\lambda(N, \rho) - \rho_N) e_{11} - \Delta_1(N) - \Delta_2(N) \]
\[ + (\lambda(N, \rho) - \rho_N) r_1(N) - \alpha \frac{W_{11}}{\sqrt{N}} - (\lambda(N, \rho) - \rho_N)^2 r_2(N) \]
with
\[ r_1(N) = \frac{1}{N} (\alpha \otimes Y^*) R_{N-1}(\rho_N e_{11} - \gamma) I_{\tilde{\Omega}_{N-1}} (e_{11} \otimes I_{N-1}) R_{N-1}(\rho_N e_{11} - \gamma) I_{\tilde{\Omega}_{N-1}} (\alpha \otimes Y), \]
\[ r_2(N) = \frac{1}{N} \alpha \otimes Y^* R_{N-1}(\rho_N e_{11} - \gamma) I_{\tilde{\Omega}_{N-1}} (e_{11} \otimes I_{N-1}) R_{N-1}(\rho_N e_{11} - \gamma) I_{\tilde{\Omega}_{N-1}} (\alpha \otimes Y), \]
\[ \times e_{11} \otimes I_{N-1} R_{N-1}(\lambda(N, \rho) e_{11} - \gamma) I_{\tilde{\Omega}_{N-1}} (\alpha \otimes Y), \]
\[ \Delta_1(N) = \frac{1}{N} \alpha \otimes Y^* R_{N-1}(\rho_N e_{11} - \gamma) I_{\tilde{\Omega}_{N-1}} (\alpha \otimes Y) \]
\[ - \alpha \text{id}_m \otimes \text{tr}_{N-1} \left( \left( R_{N-1}(\rho_N e_{11} - \gamma) I_{\tilde{\Omega}_{N-1}} \right) \right) \alpha, \]
\[ \Delta_2(N) = \alpha \text{id}_m \otimes \text{tr}_{N-1} \left( R_{N-1}(\rho_N e_{11} - \gamma) I_{\tilde{\Omega}_{N-1}} \right) \alpha \]
\[ - \alpha \text{id}_m \otimes \phi \left( (\rho_N e_{11} - \gamma) \otimes 1_A - \alpha \otimes x - \beta \otimes a_{N-1} \right)^{-1} \alpha. \]
First, we have that according to (36) and Lemma 6,

\[ r_1(N) = \alpha \text{id}_m \otimes \text{tr}_{N-1}(R_{N-1}(\rho_N e_{11} - \gamma)) I_{\tilde{\Omega}_{N-1}} (e_{11} \otimes I_{N-1}) R_{N-1}(\rho_N e_{11} - \gamma) I_{\tilde{\Omega}_{N-1}} \alpha = o_P(1). \]

From Lemma 4, almost surely,

\[ \text{id}_m \otimes \text{tr}_{N-1}(R_{N-1}(\rho_N e_{11} - \gamma)) I_{\tilde{\Omega}_{N-1}} (e_{11} \otimes I_{N-1}) R_{N-1}(\rho_N e_{11} - \gamma) I_{\tilde{\Omega}_{N-1}} - \rightarrow_{N \to \infty} \text{id}_m \otimes \phi(R_{\infty}(\rho e_{11} - \gamma) (e_{11} \otimes I_A) R_{\infty}(\rho e_{11} - \gamma)). \]

Therefore,

\[ r_1(N) \xrightarrow{P} N \to \infty \alpha \text{id}_m \otimes \phi(R_{\infty}(\rho e_{11} - \gamma) (e_{11} \otimes I_A) R_{\infty}(\rho e_{11} - \gamma)) \alpha. \quad (82) \]

Now,

\[ \|r_2(N)\| \leq m \|\alpha\|^2 \|R_{N-1}(\rho_N e_{11} - \gamma) I_{\tilde{\Omega}_{N-1}} \|^2 \|R_N(\lambda(N, \rho)e_{11} - \gamma) I_{\tilde{\Omega}_{N-1}} \| \frac{\|Y\|^2_N}{N}. \]

By the law of large numbers,

\[ \frac{\|Y\|^2_N}{N} = \frac{1}{N} \sum_{j=2}^{N} |W_{j1}|^2 = \sigma^2 + o_P(1). \]

Moreover, by Lemma 3 we have

\[ \|R_{N-1}(\rho_N e_{11} - \gamma) I_{\tilde{\Omega}_{N-1}} \| \leq 2/C_\epsilon \text{ and } \|R_N(\lambda(N, \rho)e_{11} - \gamma) I_{\tilde{\Omega}_{N-1}} \| \leq 2/C_\epsilon. \]

Therefore

\[ \|r_2(N)\| = O_P(1). \quad (83) \]

By Lemma 6

\[ \Delta_1(N) = o_P(1). \quad (84) \]

Now, Proposition 5 readily yields

\[ \sqrt{N} \Delta_2(N) = o_P(1). \quad (85) \]

Thus (19), (24), (82), (83), (84) and (85) yield that

\[ H_m(N) = o_P(1). \]

Therefore, according to (81) and (23), with a probability going to one as \( N \) goes to infinity,

\[ 0 = \det X_m(N) = \det(X_m^{(0)}(N) + H_m(N)) = \det(X_m^{(0)}(N)) + \text{Tr}_m B_{X_m^{(0)}}(N) H_m(N) + \epsilon_N = \text{Tr}_m B_{X_m^{(0)}}(N) H_m(N) + \epsilon_N, \]
where

$$B_{X_{m}^{(0)}(N)} = {}^t \text{Com}(X_{m}^{(0)}(N)),$$

$$\epsilon_N = O(||H_m(N)||^2).$$

Thus, using [19], [24], [82], [83], [85] and Proposition 4

$$\sqrt{N} \epsilon_N = o_P(\sqrt{N}(\lambda - \rho_N)) + o_P(1).$$

Hence, with a probability going to one as \(N\) goes to infinity,

$$\sqrt{N}(\lambda(N, \rho) - \rho_N) \left[ Tr_m B_{X_{m}^{(0)}(N)} e_{11} + Tr_m B_{X_{m}^{(0)}(N)} r_1(N) + o_P(1) \right]$$

$$= Tr_m B_{X_{m}^{(0)}(N)} \sqrt{N} \Delta_1(N) + W_{11} Tr_m B_{X_{m}^{(0)}(N)} \alpha + o_P(1).$$

Theorem 3 readily follows from Proposition 4, the independence of \(\Delta_1(N)\) and \(W_{11}\) and the fact that \(\omega_m^{(N)}(\rho e_{11} - \gamma)\) converges towards \(\omega_m(\rho e_{11} - \gamma)\) when \(N\) goes to infinity (see 3)

Lemma 4).

7 Appendix

A probability \(\mu\) satisfies a Poincaré inequality if for any \(C^1\) function \(f : \mathbb{R} \to \mathbb{C}\) such that \(f\) and \(f'\) are in \(L^2(\mu)\),

$$V(f) \leq C_{PI} \int |f'|^2 d\mu,$$

with \(V(f) = \int |f - \int f d\mu|^2 d\mu\).

If the law of a random variable \(X\) satisfies the Poincaré inequality with constant \(C_{PI}\) then, for any fixed \(\alpha \neq 0\), the law of \(\alpha X\) satisfies the Poincaré inequality with constant \(\alpha^2 C_{PI}\).

Assume that probability measures \(\mu_1, \ldots, \mu_M\) on \(\mathbb{R}\) satisfy the Poincaré inequality with constant \(C_{PI}(1), \ldots, C_{PI}(M)\) respectively. Then the product measure \(\mu_1 \otimes \cdots \otimes \mu_M\) on \(\mathbb{R}^M\) satisfies the Poincaré inequality with constant \(C'_{PI} = \max_{i \in \{1, \ldots, M\}} C_{PI}(i)\) in the sense that for any differentiable function \(f\) such that \(f\) and its gradient \(\text{grad} f\) are in \(L^2(\mu_1 \otimes \cdots \otimes \mu_M)\),

$$V(f) \leq C'_{PI} \int ||\text{grad} f||^2 d\mu_1 \otimes \cdots \otimes d\mu_M$$

with \(V(f) = \int |f - \int f d\mu_1 \otimes \cdots \otimes d\mu_M|^2 d\mu_1 \otimes \cdots \otimes d\mu_M\).

Lemma 8. Lemma 4.4.3 and Exercise 4.4.5 in [2] or Chapter 3 in [32]. Let \(\mathbb{P}\) be a probability measure on \(\mathbb{R}^M\) which satisfies a Poincaré inequality with constant \(C_{PI}\). Then there exists \(K_1 > 0\) and \(K_2 > 0\) such that, for any Lipschitz function \(F\) on \(\mathbb{R}^M\) with Lipschitz constant \(|F|_{\text{Lip}}\),

$$\forall \epsilon > 0, \mathbb{P}(|F - \mathbb{E}_\mathbb{P}(F)| > \epsilon) \leq K_1 \exp \left(-\frac{\epsilon}{K_2 \sqrt{C_{PI}|F|_{\text{Lip}}}}\right).$$

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