Superconformal vector multiplet self-couplings and generalised Fayet-Iliopoulos terms

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Abstract

As an extension of the recent construction of generalised Fayet-Iliopoulos terms in \( \mathcal{N} = 1 \) supergravity given in [1], we present self-interactions for a vector multiplet coupled to conformal supergravity. They are used to construct new models for spontaneously broken local supersymmetry.

Recently, a one-parameter family of generalised Fayet-Iliopoulos (FI) terms in supergravity were proposed [1], including the one discovered in [2], with the crucial property that no gauged \( R \)-symmetry is required, unlike the standard FI term [3] lifted to supergravity [4,5]. These generalised FI terms make use of composite super-Weyl primary multiplets \( \mathcal{V}(n) \), with \( n \) a real parameter, which are constructed from an Abelian vector multiplet coupled to \( \mathcal{N} = 1 \) conformal supergravity [1]. As usual, the vector multiplet is described using a real scalar prepotential \( V \) defined modulo gauge transformations

\[
\delta_{\lambda} V = \lambda + \bar{\lambda} , \quad \bar{\mathcal{D}}\alpha V = 0 .
\] (1)

The prepotential is chosen to be super-Weyl inert, \( \delta_{\sigma} V = 0 \). It was assumed in [1] that the top component (\( D \)-field) of \( V \) is nowhere vanishing. In terms of the gauge-invariant covariantly chiral field strength [6,7]

\[
W_{\alpha} := -\frac{1}{4}(\bar{\mathcal{D}}^{2} - 4R)\mathcal{D}_{\alpha} V , \quad \bar{\mathcal{D}}\beta W_{\alpha} = 0 ,
\] (2)

this assumption means that the real scalar \( \mathcal{D}W := \mathcal{D}^{\alpha}W_{\alpha} = \bar{\mathcal{D}}\bar{\alpha}W^{\bar{\alpha}} \) is nowhere vanishing [2].

The lowest component of \( \mathcal{D}W \) is proportional to the top component of \( V \), see eq. (19). In the special case \( n = 0 \), the composite \( \mathcal{V} \equiv \mathcal{V}(0) \) is derived as a by-product of the \( \mathcal{N} = 1 \) Goldstino superfield construction proposed in [2], and it reads

\[
\mathcal{V} := -4\frac{W^{2}\bar{W}^{2}}{(\mathcal{D}W)^{3}} , \quad W^{2} := W^{\alpha}W_{\alpha} .
\] (3)

In general, \( \mathcal{V}(n) \) is defined to have the form [1]

\[
\mathcal{V}(n) := \left[\frac{\mathcal{D}^{2}W^{2}\bar{\mathcal{D}}^{2}\bar{W}^{2}}{(\mathcal{D}W)^{4}}\right]^{n} \mathcal{V} .
\] (4)

It has the following properties:

1. \( \mathcal{V}(n) \) satisfies the nilpotency conditions

\[
\mathcal{V}(n)\mathcal{V}(n) = 0 , \quad \mathcal{V}(n)\mathcal{D}_{A}\mathcal{D}_{B}\mathcal{V}(n) = 0 , \quad \mathcal{V}(n)\mathcal{D}_{A}\mathcal{D}_{B}\mathcal{D}_{C}\mathcal{V}(n) = 0 .
\] (5)

\(^{1}\)We make use of the superspace formulation for conformal supergravity described in the Appendix.

\(^{2}\)We follow the notation and conventions of [8].
2. $V_{(n)}$ is gauge invariant, $\delta_V V_{(n)} = 0$.

3. $V_{(n)}$ is super-Weyl inert, $\delta_{\sigma} V_{(n)} = 0$.

The super-Weyl invariance of $V_{(n)}$ follows from the discussion in [10] (see also [11]).

Associated with $V_{(n)}$ is the following generalised FI term $[1]$,

$$J_{\text{FI}}^{(n)}[V; \Upsilon] = \int d^4x d^2\theta d^2\bar{\theta} E \Upsilon V_{(n)} , \quad (6)$$

where $\Upsilon$ is a real scalar with super-Weyl transformation

$$\delta_{\sigma} \Upsilon = (\sigma + \bar{\sigma}) \Upsilon . \quad (7)$$

The composite $V_{(-1)}$ and the associated FI term $J_{\text{FI}}^{(-1)}$ were discovered in [2].

It is $\Upsilon$ which contains information about a specific off-shell supergravity theory. Within the new minimal formulation for $\mathcal{N} = 1$ supergravity [12, 13], $\Upsilon$ can be identified with the corresponding linear compensator

$$\bar{L} = L . \quad (8)$$

In pure old minimal supergravity [7, 16, 17], $\Upsilon$ is given by $\Upsilon = \bar{S}_0 S_0$, where $S_0$ is the chiral compensator, $\bar{D}_a S_0 = 0$, with super-Weyl transformation law $\delta_{\sigma} S_0 = \sigma S_0$. In the presence of chiral matter, however, $\Upsilon$ must be deformed, see below. It should be mentioned that the use of conformal compensators to describe off-shell formulations for supergravity was advocated by many authors including [18, 14, 19, 20].

In the literature there have appeared various applications of the generalised FI terms [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31], including inflationary cosmological models.

It is worth remarking that the right-hand side of (6) can be written in several equivalent forms using the identities

$$V D^2 W^2 = V \frac{D^2 W^2}{(D W)^4} , \quad \mathbb{W}_\alpha := -\frac{1}{4} (D^2 - 4R) D_\alpha V . \quad (9)$$

Actually, it is possible to consider more general FI-like terms of the form

$$J_{\text{FI}}^{(G)}[V; \Upsilon] = \int d^4x d^2\theta d^2\bar{\theta} E \Upsilon V G \left( -\frac{D^2 W^2}{(D W)^2} , -\frac{\bar{D}^2 \bar{W}^2}{(D W)^2} \right) , \quad (10)$$

where $G(z, \bar{z})$ is a real function of one complex variable. The component structure of $J_{\text{FI}}^{(G)}[V; \Upsilon]$ will be discussed below. The original functional (6) corresponds to the choice $G(z, \bar{z}) = (z \bar{z})^n$.

Each of the generalised FI terms (11), including (6), is not superconformal in the sense that the integrand involves a conformal compensator. Quite remarkably, once the condition $D W \neq 0$ is imposed, it is also possible to construct superconformal self-couplings for the vector multiplet. Such superconformal self-couplings are proposed in this note. They are described by super-Weyl invariant functionals of the form

$$S[V] = \frac{1}{2} \int d^4x d^2\theta E W^2 + \int d^4x d^2\theta d^2\bar{\theta} E \frac{W^2 \bar{W}^2}{(D W)^2} \mathcal{H} \left( -\frac{D^2 W^2}{(D W)^2} , -\frac{\bar{D}^2 \bar{W}^2}{(D W)^2} \right) , \quad (11)$$

\[3\] The linear compensator [14] is described by a tensor multiplet [15] such that its field strength $L$ is nowhere vanishing.
where $\mathcal{E}$ is the chiral integration measure \[32, 33\], and $\mathcal{H}(z, \bar{z})$ is a real function of one complex variable. This action is part of the complete supergravity-matter action given by

$$S = S_{\text{SUGRA}} + S[V] - 2\xi \mathbb{J}^{(g)}_{\text{FI}}[V; \Upsilon]$$

(12)

where $S_{\text{SUGRA}}$ denotes an action for supergravity coupled to other matter supermultiplets, for instance

$$S_{\text{SUGRA}} = -3 \int d^4x d^2\theta d^2\bar{\theta} \mathcal{E} S_0 e^{-\frac{1}{3}K(\phi, \bar{\phi})} S_0 + \left\{ \int d^4x d^2\theta d^2\bar{\theta} \mathcal{E} S_0^3 W(\phi) + \text{c.c.} \right\}$$

(13)

which corresponds to the old minimal formulation for $\mathcal{N} = 1$ supergravity. In this case $\Upsilon$ in the generalised FI term in (12) should be $\Upsilon = \bar{S}_0 e^{-\frac{1}{3}K(\phi, \bar{\phi})} S_0$, as was pointed out in \[22, 24\].

In general, the action (11) is highly nonlinear. However its functional form drastically simplifies provided the ordinary gauge field contained in $V$ is chosen to be a flat connection. This means that the gauge freedom (11) may be used to make $V$ a nilpotent superfield obeying the constraints

$$VV = 0, \quad VD_A D_B V = 0, \quad VD_A D_B D_C V = 0.$$  

(14)

Then it can be seen that

$$V_{(n)} = V,$$  

(15)

compare with the analysis in \[9\]. This implies

$$S[V] - 2f \mathbb{J}^{(n)}_{\text{FI}}[V; \Upsilon] = \frac{h}{2} \int d^4x d^2\theta \mathcal{E} W^2 - 2\xi g \int d^4x d^2\theta d^2\bar{\theta} \mathcal{E} TV,$$  

(16)

where we have denoted

$$h := 1 + \frac{1}{2}\mathcal{H}(1, 1) > 0, \quad g := \mathcal{G}(1, 1) \neq 0.$$  

(17)

Modulo an overall numerical factor, (16) is the Goldstino multiplet action proposed in \[9\]. The restriction $h > 0$ is equivalent to the requirement that the Goldstino kinetic term has the correct sign. At the component level, the action (16) is still nonlinear, due to the nilpotency constraints (14). However, the functional form of the action (16) is universal, unlike the complete vector multiplet action in (12), which is a manifestation of the universality of the Volkov-Akulov action \[34, 35\].

To arrive at the Goldstino multiplet action (16), we have made use of the gauge (14) which expresses the fact that the gauge field is switched off. It is actually possible to avoid imposing any gauge condition. In general the gauge prepotential $V$ may be split in two multiplets, one of which contains only a single independent component field – the gauge field itself – while the second part contains the remaining component fields. The point is that the nilpotency conditions (5) allow us to interpret $V_{(n)}$ as a Goldstino superfield of the type proposed in \[9\] provided its $D$-field is nowhere vanishing, which means that $D^2W^2$ is nowhere

\footnote{All the constraints (14) are invariant under local rescalings $V \to e^\rho V$, with the parameter $\rho$ being an arbitrary real scalar superfield. Requiring the action (16) to be stationary under such rescalings gives the constraint $fTV = \frac{1}{4}VDP^a(D^2 - 4R)D_a V$, where $f = \xi g/h$. In conjunction with (14), this constraint defines the irreducible Goldstino multiplet introduced in \[36\].}
vanishing, in addition to the condition $\mathcal{D}W \neq 0$ imposed earlier. Then $\mathbb{V}_{(n)}$ contains only two independent component fields, the Goldstino and $D$-field. We then can introduce a new parametrisation for the gauge prepotential given by

$$V = A_{(n)} + \mathbb{V}_{(n)} .$$  \hspace{1cm} (18)

It is $A_{(n)}$ which varies under the gauge transformation (11), $\delta \lambda A_{(n)} = \lambda + \bar{\lambda}$, while $\mathbb{V}_{(n)}$ is gauge invariant by construction. Modulo purely gauge degrees of freedom, $A_{(n)}$ contains only one independent field, the gauge field.

It is of interest to work out the bosonic sector of the model (12) in the vector multiplet sector. For this purpose we introduce gauge-invariant component fields of the vector multiplet following [11]

$$W_{\alpha} = \psi_{\alpha} , \quad -\frac{1}{2} \mathcal{D}^{\alpha} W_{\alpha} = D , \quad \mathcal{D}_{(\alpha} W_{\beta)} = 2i \hat{F}_{\alpha\beta} = i(\sigma^{ab})_{\alpha\beta} \hat{F}_{ab} ,$$  \hspace{1cm} (19)

where the bar-projection $U|$ of a superfield $U$ means switching off the superspace Grassmann variables, and

$$\hat{F}_{ab} = F_{ab} - \frac{1}{2}(\Psi_{a}\sigma_{b}\bar{\psi} + \psi_{a}\bar{\psi}_{b}) + \frac{1}{2}(\Psi_{b}\sigma_{a}\bar{\psi} + \psi_{a}\bar{\psi}_{b}) ,$$

$$F_{ab} = \nabla_{a} V_{b} - \nabla_{b} V_{a} - T_{ab}^{\ c} V_{c} ,$$  \hspace{1cm} (20)

with $V_{a} = e_{a}^{m}(x) V_{m}(x)$ the gauge one-form, and $\Psi_{a}^{\beta}$ the gravitino. Here $\nabla_{a}$ denotes a spacetime covariant derivative with torsion,

$$[\nabla_{a}, \nabla_{b}] = T_{ab}^{\ c} \nabla_{c} + \frac{1}{2} \mathcal{R}_{abcd} M^{cd} ,$$  \hspace{1cm} (21)

where $\mathcal{R}_{abcd}$ is the curvature tensor and $T_{abc}$ is the torsion tensor. The latter is related to the gravitino by

$$T_{abc} = -\frac{i}{2}(\Psi_{a}\sigma_{c}\bar{\psi}_{b} - \psi_{b}\sigma_{c}\bar{\psi}_{a}) .$$  \hspace{1cm} (22)

For more details, see [8, 11]. We deduce from the above relations that

$$-\frac{1}{4} D^{2} W^{2} = D^{2} - 2F^{2} + \text{fermionic terms} , \quad F^{2} := F^{\alpha\beta} F_{\alpha\beta} .$$  \hspace{1cm} (23)

We conclude that the electromagnetic field should be weak enough to satisfy $D^{2} - 2F^{2} \neq 0$, in addition to the condition $D \neq 0$ discussed above. Direct calculations give the component bosonic Lagrangian

$$\mathcal{L}(F_{ab}, D) = -\frac{1}{2} (F^{2} + \tilde{F}^{2}) + \frac{1}{2} D^{2} \left\{ 1 + \frac{1}{2} \mathcal{H} \left( 1 - \frac{2F^{2}}{D^{2}} , 1 - \frac{2\tilde{F}^{2}}{D^{2}} \right) \right\} \left[ 1 - \frac{2F^{2}}{D^{2}} \right]^{2}$$

$$- \xi D \mathcal{G} \left( 1 - \frac{2F^{2}}{D^{2}} , 1 - \frac{2\tilde{F}^{2}}{D^{2}} \right) \left[ 1 - \frac{2F^{2}}{D^{2}} \right]^{2} \mathcal{Y} .$$  \hspace{1cm} (24)

In order for the supergravity action in (12) to give the correct Einstein-Hilbert gravitational Lagrangian at the component level, one has to impose the super-Weyl gauge $\mathcal{Y} = 1$, see [19, 11] for the technical details.
It is seen that the case \( \mathcal{G}(z, \bar{z}) = (z\bar{z})^{-1} \) is special since the last term becomes linear in \( D \) and independent of the field strength. This simplicity is somewhat misleading since the generalised FI term is nonlinear in the Goldstino for arbitrary \( n \). Of course, this nonlinear fermionic sector disappears in the unitary gauge in which the Goldstino is gauged away. However, we have shown that the model describes spontaneously broken local supersymmetry for any choice of \( \mathcal{G} \), and hence there is nothing unique in the choice \( (25) \) from the conceptual point of view.

As follows from \( (24) \), the auxiliary field \( D \) may be integrated out (at least in perturbation theory) using its equation of motion
\[
\frac{\partial}{\partial D} \mathcal{L}(F_{ab}, D) = 0 ,
\]
leaving a model for nonlinear electrodynamics, \( \mathcal{L}(F_{ab}) \).

Action \( (11) \) is superconformal since it describes the self-interacting vector multiplet coupled to conformal supergravity. In the presence of a conformal compensator, which corresponds to an off-shell supergravity theory, more general couplings exist. Ref. \([11]\) presented a general family of \( U(1) \) duality invariant models for a massless vector multiplet coupled to off-shell supergravity, old minimal or new minimal. Such a theory is described by a super-Weyl invariant action of the form
\[
S_{\text{self-dual}}[V; \Upsilon] = \frac{1}{2} \int d^4 x d^2 \theta \mathcal{E} W^2 + \frac{1}{4} \int d^4 x d^2 \bar{\theta} d^2 \theta E \frac{W^2 \bar{W}^2}{\Upsilon^2} \Lambda(\frac{\omega}{\Upsilon^2}, \frac{\bar{\omega}}{\Upsilon^2}) .
\]

Here \( \omega := \frac{i}{2} \mathcal{D}^2 W^2 \), and \( \Lambda(\omega, \bar{\omega}) \) is a real analytic function satisfying the equation \([37, 38]\)
\[
\text{Im} \left\{ \Gamma - \bar{\omega} \Gamma^2 \right\} = 0 , \quad \Gamma := \frac{\partial(\omega \Lambda)}{\partial \omega} .
\]

These \( U(1) \) duality invariant theories are curved-superspace extensions of the globally supersymmetric systems introduced in \([37, 38]\). The supersymmetric Born-Infeld action coupled to supergravity \([10]\) is obtained by choosing
\[
\Lambda_{\text{SBI}}(\omega, \bar{\omega}) = \frac{k^2}{1 + \frac{1}{2} A + \sqrt{1 + A + \frac{1}{4} B^2}} , \quad A = k^2 (\omega + \bar{\omega}) , \quad B = k^2 (\omega - \bar{\omega}) ,
\]
with \( k \) a coupling constant. It was pointed out by Cecotti and Ferrara \([10]\) that the dynamical system defined by eqs. \((27)\) and \((29)\) is not a unique supersymmetric extension of the Born-Infeld action. One can introduce a two-parameter deformation of \((29)\) obtained by replacing
\[
\omega \rightarrow \omega + \zeta \mathcal{D} W , \quad \zeta = \text{const} ,
\]
with \( \zeta \) a complex parameter. At the component level, the resulting bosonic action coincides with the Born-Infeld one provided the auxiliary field is switched off. However, the freedom to perform shifts \((30)\) is eliminated if one requires the supersymmetric theory to possess \( U(1) \) duality invariance. In other words, no \( \mathcal{D} W \)-dependence is allowed in duality invariant
models. The same condition emerges if the $\mathcal{N} = 1$ vector multiplet is used to describe partial $\mathcal{N} = 2 \to \mathcal{N} = 1$ supersymmetry breaking \[39\].

As pointed out in \[40\], an important property of the standard FI term is that it remains invariant under the second nonlinearly realised supersymmetry of the rigid supersymmetric Born-Infeld action \[39\]. This property implies the supersymmetric Born-Infeld action deformed by a FI term still describes partial $\mathcal{N} = 2 \to \mathcal{N} = 1$ supersymmetry breaking \[40, 41, 42\], and the resulting model is compatible with $U(1)$ duality invariance \[41\]. As for the generalised FI-type terms \[6\], they do not share these fundamental properties.

One may compare \[(11)\] with general $\mathcal{N} = 2$ superconformal actions for a vector multiplet in Minkowski superspace \[43\]

\[
\Gamma = \int d^4x d^4\theta d^4\bar{\theta} \left\{ \frac{c}{2} \ln \mathcal{W} \ln \bar{\mathcal{W}} + \ln \mathcal{W} \Lambda (\bar{\mathcal{W}}^{-2} \ln \mathcal{W}) + \text{c.c.} \\
+ \Sigma (\bar{\mathcal{W}}^{-2} D^4 \ln \mathcal{W}, \mathcal{W}^{-2} \bar{D}^4 \ln \bar{\mathcal{W}}) \right\}, \tag{31}
\]

where $\mathcal{W}$ is a reduced $\mathcal{N} = 2$ chiral superfield constrained by

\[
D^i_i \mathcal{W} = 0, \quad D^{ij} \mathcal{W} = D^{ij} \bar{\mathcal{W}}, \quad D^{ij} := D^{\alpha(i} D^{j)}_{\dot{\alpha}}, \quad \bar{D}^{ij} := \bar{D}^{(i} \bar{D}^{j)}_{\dot{\alpha}}, \tag{32}
\]

which describes the field strength of the vector multiplet. It is assumed in \[(31)\] that the physical complex scalar of the vector multiplet, $\mathcal{W}$, is nowhere vanishing. Unlike the $\mathcal{N} = 1$ case considered in this paper, no assumption is made about the auxiliary iso-triplet, $D^{ij} \mathcal{W}$, since the case of unbroken $\mathcal{N} = 2$ supersymmetry is studied. Because of unbroken supersymmetry, all contributions containing factors of the \textit{primary} superfield $D^{ij} \mathcal{W}$ are omitted due to the fact that this primary operator constitutes the free equation of motion (the functional \[(31)\] is interpreted as a low-energy effective action). The $\mathcal{N} = 1$ analogue of $D^{ij} \mathcal{W}$ is the super-Weyl primary multiplet $\mathcal{D} \mathcal{W}$ we have used above. Since in the $\mathcal{N} = 1$ case we are interested in models for spontaneously broken supersymmetry, which means $\mathcal{D} \mathcal{W}$ is nowhere vanishing, we are no longer allowed to discard terms involving factors of $\mathcal{D} \mathcal{W}$. It is for these reasons that actions of the form \[(11)\] have to be considered.

Let us summarise the main results of this paper. We proposed the new generalised FI terms \[(10)\] which include those constructed earlier \[1, 2\]. We introduced new models for spontaneously broken local supersymmetry \[(12)\] which make use of the novel superconformal vector multiplet self-couplings \[(11)\].

The constructions given in this paper have a natural extension to $\mathcal{N} = 2$ supergravity, which will be described elsewhere.

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\[5\]The action \[(31)\] and constraints \[(32)\] involve $\mathcal{N} = 2$ spinor covariant derivatives $D^i_\alpha$ and $\bar{D}^{\dot{\alpha}}_{\dot{i}}$, with $i = 1, 2$, and the fourth-order operators $D^4$ and $\bar{D}^4$ are defined as $D^4 = (D_\mathcal{W})^2 (\bar{D}_{\bar{\mathcal{W}}})^2$ and $\bar{D}^4 = (D_{\bar{\mathcal{W}}})^2 (\bar{D}_\mathcal{W})^2$.\]
A Conformal supergravity in superspace

In the framework of the vielbein formulation for conformal gravity, the gauge field is a vielbein \( e^a := dx^m e_m^a(x), \) \( e := \det(e_m^a) \neq 0, \) while the metric becomes a composite field defined by \( g_{mn} = e_m^a e_n^b \eta_{ab}, \) with \( \eta_{ab} \) the Minkowski metric. The gauge group of conformal gravity is spanned by general coordinate, local Lorentz and Weyl transformations which act on the torsion-free covariant derivatives

\[
\nabla_a = e_a + \omega_a = e_a^m \partial_m + \frac{1}{2} \omega_{ab} M_{bc} , \quad [\nabla_a, \nabla_b] = \frac{1}{2} R_{ab}^\gamma \sigma_{\gamma cd} M_{cd} , \tag{A.1}
\]

by the rule

\[
\delta \nabla_a = [\xi^b \nabla_b + \frac{1}{2} K^b_{\gamma c} M_{bc}, \nabla_a] + \tau \nabla_a + (\nabla^b \tau) M_{ba} , \tag{A.2}
\]

with the gauge parameters \( \xi^a(x) = \xi^m(x) e_m^a(x), \) \( K^{ab}(x) = -K^{ba}(x) \) and \( \tau(x) \) being completely arbitrary. In (A.1), \( M_{ba} = -\tilde{M}_{ab} \) is the Lorentz generator, \( e_m^a(x) \) the inverse vielbein, \( e_a^m e_m^b = \delta_a^b, \) and \( \omega_{bc}(x) \) the torsion-free Lorentz connection.

In order to describe \( \mathcal{N} = 1 \) conformal supergravity \([14, 15]\) in superspace, the simplest approach is to make use of the Grimm-Wess-Zumino geometry \([16]\), which is at the heart of the Wess-Zumino formulation for old minimal supergravity \([7]\), in conjunction with the super-Weyl transformations discovered by Howe and Tucker \([17]\). The geometry of curved superspace is described by covariant derivatives of the form

\[
\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_a, \bar{\mathcal{D}}^\dot{\alpha}) = E_A^M \partial_M + \frac{1}{2} \Omega_A^{bc} M_{bc} , \tag{A.3}
\]

which obey the graded commutation relations (see \([8]\) for a derivation)

\[
\{\mathcal{D}_a, \mathcal{D}_a\} = -2i \mathcal{D}_{\dot{a}\dot{a}}, \quad \{\mathcal{D}_a, \mathcal{D}_b\} = -4 R M_{\alpha \beta}, \quad \{\mathcal{D}_{\dot{a}}, \mathcal{D}_{\dot{b}}\} = 4 R M_{\dot{a}\dot{b}}, \tag{A.4a}\]

\[
[\mathcal{D}_a, \mathcal{D}_{\dot{b}}] = i \varepsilon_{a\dot{b}} \left( \bar{R} \mathcal{D}_{\dot{b}} + G^{\dot{b}}_{\dot{\gamma}} \mathcal{D}_{\dot{\gamma}} - \mathcal{D}_{\dot{b}} G^{\dot{b}}_{\dot{\gamma}} M_{\dot{\gamma} \dot{\delta}} + 2 \bar{W}^{\dot{b} \dot{\gamma} \dot{\delta}} M_{\dot{\gamma} \dot{\delta}} \right) + i \bar{D}_{\dot{b}} \bar{R} M_{a \beta}, \tag{A.4b}\]

\[
[\mathcal{D}_{\dot{a}}, \mathcal{D}_{\dot{b}}] = -i \varepsilon_{\dot{a}\dot{b}} \left( R \mathcal{D}_{\dot{b}} + G^{\dot{b}}_{\dot{\gamma}} \mathcal{D}_{\dot{\gamma}} - \mathcal{D}^{\dot{b}} G^{\dot{b}}_{\dot{\gamma}} M_{\dot{\gamma} \dot{\delta}} + 2 W_{\dot{b} \dot{\gamma} \dot{\delta}} M_{\dot{\gamma} \dot{\delta}} \right) - i \bar{D}_{\dot{b}} \bar{R} M_{\dot{a} \dot{b}}. \tag{A.4c}\]

Here the torsion tensors \( R, G_a = \tilde{G}_a \) and \( W_{\alpha \beta \gamma} = W_{(\alpha \beta \gamma)} \) satisfy the Bianchi identities:

\[
\bar{D}_{\dot{a}} R = 0 , \quad \bar{D}_{\dot{a}} W_{\alpha \beta \gamma} = 0 , \tag{A.5a}\]

\[
\bar{D}^{\dot{a}} G_{\alpha \dot{\gamma}} = \mathcal{D}_a R, \quad \bar{D}^{\dot{a}} W_{\alpha \beta \gamma} = i \mathcal{D}_{\dot{a}} G_{\beta \dot{\gamma}}. \tag{A.5b}\]

The super-Weyl transformations are

\[
\delta_a \mathcal{D}_a = (\bar{\sigma} - \frac{1}{2} \sigma) \mathcal{D}_a + \mathcal{D}^{\dot{b}} \sigma M_{a \beta}, \tag{A.6a}\]

\[
\delta_{\dot{a}} \mathcal{D}_{\dot{a}} = (\sigma - \frac{1}{2} \bar{\sigma}) \mathcal{D}_{\dot{a}} + (\mathcal{D}^{\dot{b}} \bar{\sigma}) \bar{M}_{\dot{a} \dot{\beta}}, \tag{A.6b}\]

\[
\delta_{\sigma} \mathcal{D}_{\dot{a}\dot{a}} = \frac{1}{2} (\sigma + \bar{\sigma}) \mathcal{D}_{\dot{a}\dot{a}} + \frac{i}{2} \bar{D}_{\dot{a}} \sigma \mathcal{D}_{\dot{a}} + \frac{i}{2} \mathcal{D}_{\dot{a}} \sigma \bar{D}_{\dot{a}} + \mathcal{D}^{\dot{b}} \sigma M_{a \beta} + \mathcal{D}_{\dot{a}} \bar{\sigma} \bar{M}_{\dot{a} \dot{\beta}}. \tag{A.6c}\]
accompanied by the following transformations of the torsion super fields

$$\delta_\sigma R = 2\sigma R + \frac{1}{4}(D^2 - 4R)\bar{\sigma}, \quad (A.7a)$$

$$\delta_\sigma G_{\alpha\dot{\alpha}} = \frac{1}{2}(\sigma + \bar{\sigma})G_{\alpha\dot{\alpha}} + iD_{\alpha\dot{\alpha}}(\sigma - \bar{\sigma}), \quad (A.7b)$$

$$\delta_\sigma W_{\alpha\beta\gamma} = \frac{3}{2}\sigma W_{\alpha\beta\gamma}. \quad (A.7c)$$

Here the super-Weyl parameter $\sigma$ is a covariantly chiral scalar superfield, $\bar{D}_{\dot{\alpha}}\sigma = 0$. The super-Weyl transformations belong to the gauge group of conformal supergravity.

A tensor superfield $\mathcal{T}$ (with its indices suppressed) is said to be super-Weyl primary of weight $(p, q)$ if its super-Weyl transformation law is $\delta_\sigma \mathcal{T} = (p\sigma + q\bar{\sigma})\mathcal{T}$, for some parameters $p$ and $q$.

There exist two other superspace formulations for conformal supergravity \cite{18,19} which have structure groups larger than SL$(2, \mathbb{C})$. These formulations are not used in the present paper, although our results can readily be lifted to U(1) superspace \cite{18} and conformal superspace \cite{19}. At the component level, the latter approach naturally reduces to the super-conformal tensor calculus \cite{25,26,27,28} as demonstrated in \cite{19,29,30}.

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