A FOLK QUILLEN MODEL STRUCTURE FOR OPERADS

ITTAY WEISS

Abstract. We establish, by elementary means, the existence of a cofibrantly generated monoidal model structure on the category of operads. By slicing over a suitable operad the classical Rezk model structure on the category of small categories is recovered.

1. Introduction

The aim of the article is to present an elementary construction of a Quillen model structure on the category $\text{Ope}_\Sigma$ of symmetric operads and on the category $\text{Ope}$ of non-symmetric operads where, in each case, every operad is fibrant (and cofibrant). The existence of this model structure was announced without proof at the end of Section 8 of [9]. The same model structure can be deduced from two more recent, and much more sophisticated, results. One is the Cisinski-Moerdijk model structure on simplicial operads [4] and the other result is Giovanni Caviglia’s [3], establishing a model structure on enriched operads. The non-enriched model structure on operads is relatively simple to establish and does not require complicated machinery like Quillen’s small object argument. We thus give an explicit and elementary construction of the model structures, together with explicit generating cofibrations. In that sense the model structures belong to what are commonly referred to as folk model structures. We also note that by slicing either operads category over a suitable operad, one obtains the category $\text{Cat}$ of small categories. When the model structure on operads is transferred to the sliced category one recovers Rezk’s folk model structure on $\text{Cat}$. We also show that the model structure on $\text{Ope}_\Sigma$ is a monoidal model structure when considering the Boardman-Vogt tensor product.

After briefly recalling the main concepts of Quillen model structures, operads, and Rezk’s model structure on $\text{Cat}$, the second section contains the proofs of the main results of this article.

1.1. Quillen model structures. A Quillen model structure on a category $\mathcal{C}$ is a specification of three classes of morphisms, called weak equivalences, fibrations, and cofibrations, such that the following axioms hold.

- $\mathcal{C}$ is small complete and small cocomplete.
- The weak equivalences satisfy the three for two property, namely if $h = g \circ f$ and any two of the three morphisms is a weak equivalence, then so is the third.
- Each of the specified classes of morphisms is closed under retracts.
• Liftings exist. In more detail, consider a commutative square

\[
\begin{array}{ccc}
  a & \rightarrow & c \\
  \downarrow & & \downarrow \\
  b & \rightarrow & d
\end{array}
\]

where the left vertical arrow is a cofibration and the right vertical arrow is a fibration. If any of these morphisms is a weak equivalence, then a diagonal dotted arrow exists, making the diagram commutative.

• Factorizations exist. In more detail, every morphism \( f : a \rightarrow b \) can be factorized as \( f = g \circ h \) where \( g \) is a trivial fibration (i.e., a fibration that is also a weak equivalence) and \( h \) is a cofibration, and \( f \) can also be factorized as \( f = g' \circ h' \) where \( g' \) is a fibration and \( h' \) is a trivial cofibration (i.e., a cofibration that is also a weak equivalence).

We note that typically the verification of the first three axioms is straightforward. It is the reconciliation between the liftings axiom and the factorizations axiom that requires a fine-tuned balance between the three classes of morphisms.

If the category \( C \) is equipped with a monoidal structure, then a Quillen model structure on \( C \) with every object cofibrant is compatible with the tensor product if the following pushout-product axiom is satisfied. For any two morphisms \( F : a \rightarrow b \) and \( G : a' \rightarrow b' \), consider the diagram

\[
\begin{array}{ccc}
  a \otimes a' & \rightarrow & a \otimes b' \\
  \downarrow & \otimes G & \downarrow \\
  b \otimes a' & \rightarrow & c \\
  \downarrow & \otimes G & \downarrow \\
  b \otimes b'
\end{array}
\]

where the square is a pushout, and the corner map \( F \otimes G : c \rightarrow b \otimes b' \) is the induced one from the pushout. The pushout-product axiom states that if both \( F \) and \( G \) are cofibrations, then so is \( F \otimes G \), and, moreover, if any of the given cofibrations is a trivial cofibration, then \( F \otimes G \) is a trivial cofibration. For more details on Quillen model categories see the classical [5], the comprehensive [6], or the recent survey [7].

1.2. Operads. A non-symmetric operad \( \mathcal{P} \), also known as a multicategory, is a collection \( \text{ob}(\mathcal{P}) \) of objects and, for all objects \( p_0, \ldots, p_n, n \geq 0 \), a set \( \mathcal{P}(p_1, \ldots, p_n; p_0) \), also called a hom-set. The elements in each hom set are referred to as morphisms and are also denoted by \( \psi : p_1, \ldots, p_n \rightarrow p_0 \) instead of \( \psi \in \mathcal{P}(p_1, \ldots, p_n; p_0) \). There is further a specified rule for composing morphisms when their domains and codomains suitably match. In more detail, if \( \psi : p_1, \ldots, p_n \rightarrow p_0 \) is a morphism and, for each \( 1 \leq i \leq n \), \( \psi_i : q^1_i, \ldots, q^k_i \rightarrow p_i \) is a morphism, then a morphism \( \psi \circ (\psi_1, \ldots, \psi_n) : q^1_1, \ldots, q^k_1, \ldots, q^1_n, \ldots, q^k_n \rightarrow p_0 \) is designated which is defined to be the composition of the given morphisms. The composition is required to be associative in the obvious sense and there are also identity morphisms,
that is for each object \( p \) there is an identity morphism \( \text{id}_p : p \to p \), which behaves like an identity with respect to the composition. A symmetric operad is an operad where for all \( p_0, \ldots, p_n \) and a permutation \( \sigma \in \Sigma_n \), there is a function \( \sigma^* : \mathcal{P}(p_1, \ldots, p_n; p_0) \to \mathcal{P}(p_{\sigma(1)}, \ldots, p_{\sigma(n)}; p_0) \), and these functions are required to be compatible with the composition operation in the obvious manner.

For both symmetric and non-symmetric operads, the notion of a structure preserving operation between operads is referred to as a functor. A functor \( F : \mathcal{P} \to \mathcal{Q} \) is thus a function \( F : \text{ob}(\mathcal{P}) \to \text{ob}(\mathcal{Q}) \) together with, for all objects \( p_0, \ldots, p_n \), a function \( F : \mathcal{P}(p_1, \ldots, p_n; p_0) \to \mathcal{Q}(Fp_1, \ldots, Fp_n; Fp_0) \), where, of course, the composition and identities are to be respected.

We note that any category \( \mathcal{C} \) gives rise to an operad \( j_!(\mathcal{C}) \) where \( \text{ob}(j_!(\mathcal{C})) = \text{ob}(\mathcal{C}) \). The only morphisms in \( j_!(\mathcal{C}) \) are unary morphisms, given by \( j_!(\mathcal{C})(a; b) = \mathcal{C}(a, b) \), with identities and composition as in \( \mathcal{C} \). Obviously, \( j_!(\mathcal{C}) \) is also a symmetric operad in a unique way. We thus obtain two functors \( j_! : \textbf{Cat} \to \textbf{Op} \) and \( j_! : \textbf{Op} \to \textbf{Cat} \). In the other direction, any operad has an underlying category obtained by neglecting the non-unary morphisms, giving rise to the functors \( j^* : \textbf{Op} \to \textbf{Cat} \) and \( j^* : \textbf{Op}_{\Sigma} \to \textbf{Cat} \). It is easy to see that \( j_! \) is left adjoint to \( j^* \). By means of \( j^* \), the terminology of category theory applies to operads, in particular, a morphism in an operad is an isomorphism if the morphism survives in \( j^*(\mathcal{P}) \) and is an isomorphism there.

Let \( \mathcal{P} \) and \( \mathcal{Q} \) be two symmetric operads. The Boardman-Vogt tensor product of these operads is the symmetric operad \( \mathcal{P} \otimes_{BV} \mathcal{Q} \) with \( \text{ob}(\mathcal{P} \otimes_{BV} \mathcal{Q}) = \text{ob}(\mathcal{P}) \times \text{ob}(\mathcal{Q}) \) in terms of generators and relations as follows. Let \( \mathcal{C} \) be the collection on \( \text{ob}(\mathcal{P}) \times \text{ob}(\mathcal{Q}) \) which contains the following generators. For each \( q \in \text{ob}(\mathcal{Q}) \) and each morphism \( \psi \in \mathcal{P}(p_1, \ldots, p_n; p) \), there is a generator \( \psi \otimes_{bv} q \) in \( \mathcal{C}((p_1, q), \ldots, (p_n, q); (p, q)) \). Similarly, for each \( p \in \text{ob}(\mathcal{P}) \) and a morphism \( \varphi \in \mathcal{Q}(q_1, \ldots, q_m; q) \), there is a generator \( p \otimes_{bv} \varphi \) in \( \mathcal{C}((p, q_1), \ldots, (p, q_m); (p, q)) \).

There are five types of relations among the arrows generated by these generators:

1. \((\psi \otimes_{bv} q) \circ ((\psi_1 \otimes_{bv} q), \ldots, (\psi_n \otimes_{bv} q)) = (\psi \circ (\psi_1, \ldots, \psi_n)) \otimes_{bv} q\)
2. \(\sigma^*(\psi \otimes_{bv} q) = (\sigma^* \psi) \otimes_{bv} q\)
3. \((p \otimes_{bv} \varphi) \circ ((p \otimes_{bv} \varphi_1), \ldots, (p \otimes_{bv} \varphi_m)) = p \otimes_{bv} (\varphi \circ (\varphi_1, \ldots, \varphi_m))\)
4. \(\sigma^*(p \otimes_{bv} \varphi) = p \otimes_{bv} (\sigma^* \varphi)\)
5. \((\psi \otimes_{bv} q) \circ ((p_1 \otimes_{bv} \varphi), \ldots, (p_n \otimes_{bv} \varphi)) = \sigma_{m,n}^* ((p \otimes_{bv} \varphi) \circ ((\psi, q_1), \ldots, (\psi, q_m)))\)

By the relations above we mean every possible choice of morphisms for which the compositions are defined. The relations of type 1 and 2 ensure that for any \( q \in \text{ob}(\mathcal{P}) \), the map \( p \mapsto (p, q) \) naturally extends to a map of operads \( \mathcal{P} \to \mathcal{P} \otimes_{BV} \mathcal{Q} \). Similarly, the relations of type 3 and 4 guarantee that for each \( p \in \text{ob}(\mathcal{P}) \), the map \( q \mapsto (p, q) \) naturally extends to a map of operads \( \mathcal{Q} \to \mathcal{P} \otimes_{BV} \mathcal{Q} \). The relation of type 5 can be pictured as follows. The left hand side is a morphism in the free
operad, represented by the labelled planar tree

while the right hand side is given by the tree

before applying $\sigma^*_{m,n}$, which is the obvious permutation that equates the domains of the two morphisms. With the Boardman-Vogt tensor product, the category $\text{Ope}_\Sigma$ is a closed monoidal category. For an introduction to operads see [8] or, for a source that uses similar notation to the one above (and below), and, in particular, proves the above made claims, see [10].

### 1.3. The folk model structure on $\text{Cat}$

We briefly recount the definition of the Rezk model structure on $\text{Cat}$.

**Theorem 1.1.** The category $\text{Cat}$ of small categories admits a cofibrantly generated cartesian closed Quillen model structure where:

- The weak equivalences are the categorical equivalences.
- The cofibrations are those functors $F : C \to D$ that are injective on objects.
- The fibrations are those functors $F : C \to D$ such that for any $C \in \text{ob}(C)$ and each isomorphism $\psi : FC \to D$ in $D$, there exists an isomorphism $\phi : C \to C'$ for which $F\phi = \psi$.

The model structure is cofibrantly generated and compatible with the cartesian product of categories.

Recently, in [2], conditions are given for the existence of a canonical model structure on the category of small categories enriched in a monoidal category $\mathcal{V}$. The Rezk model structure is recovered by the case $\mathcal{V} = \text{Set}$.

### 2. The model structure on operads

We now present the main result: an elementary presentation of a Quillen model structure on operads.

**Theorem 2.1.** The categories $\text{Ope}$ and $\text{Ope}_\Sigma$ admit a Quillen model structure where:
• The weak equivalences are the operadic equivalences, namely those functors $F : P \to Q$ which are essentially surjective (i.e., such that $j^*(F)$ is essentially surjective) and fully faithful (i.e., where each component function $F : P(p_1, \ldots, p_n; p_0) \to Q(Fp_1, \ldots, Fp_n; Fp_0)$ is a bijection).

• The cofibrations are those functors $F : P \to Q$ that are injective on objects.

• The fibrations are those functors $F : P \to Q$ such that for any $p \in \text{ob}(P)$ and each isomorphism $\psi : Fp \to q$ in $Q$, there exists an isomorphism $\phi : p \to p'$ for which $F\phi = \psi$.

**Proof.** We treat both symmetric and non-symmetric operads together since the symmetric group actions entail no complications. Thus we provide the details for the non-symmetric operads noting at this point that whenever one needs to extend the construction to include symmetric group actions, the extension is the evident one. Notice that a functor $F : P \to Q$ is a fibration (respectively cofibration) if, and only if, $j^*F$ is a fibration (respectively cofibration) in the Rezk model structure. Notice as well that a functor $F : P \to Q$ is a trivial fibration if, and only if, the function $\text{ob}(F) : \text{ob}(P) \to \text{ob}(Q)$ is surjective and $F$ is fully faithful. We now set out to prove the Quillen axioms. Small limits, in both $\text{Ope}$ and $\text{Ope}_\Sigma$ are directly constructed much as they are constructed in the category $\text{Cat}$, posing no difficulties. Showing the existence of small colimits, namely of all small coproducts and all coequalizers, requires some more care. Constructing small coproducts is trivial, but, viewing operads as an extension of categories by means of $j_!$, operads inherit the subtleties of categories. The details of coequalizers of categories can be found in $\text{[1]}$ and, mutatis-mutandis, the same construction gives rise to coequalizers of operads (both symmetric and non-symmetric). The verification of the three for two property and of closure under retracts is routine, and we thus turn now to a detailed proof of the liftings and the factorizations axioms.

Consider a commutative square

\[
\begin{array}{ccc}
P & \xrightarrow{U} & R \\
F \downarrow & & \downarrow G \\
Q & \xrightarrow{H} & S \\
V \uparrow & & \uparrow S \\
\end{array}
\]

where $F$ is a cofibration and $G$ is a fibration. We need to prove the existence of a lift $H$ making the diagram commute, whenever $F$ or $G$ is a weak equivalence. Assume first that $G$ is a weak equivalence. Applying the object functor (that sends an operad $P$ to the set $\text{ob}(P)$) to the lifting diagram we obtain

\[
\begin{array}{ccc}
\text{ob}(P) & \xrightarrow{U} & \text{ob}(R) \\
F \downarrow & & \downarrow G \\
\text{ob}(Q) & \xrightarrow{H} & \text{ob}(S) \\
\end{array}
\]

where $F$ is injective and $G$ is surjective. We can thus find a lift $H$. Let now $\psi \in Q(q_1, \cdots, q_n; q)$, and consider $V(\psi) \in S(Vq_1, \cdots, Vq_n; Vq)$. Since $G$ is fully faithful and $HG = V$ on the level of objects, we obtain that the function

\[G : R(Hq_1, \cdots, Hq_n; Hq) \to S(Vq_1, \cdots, Vq_n; Vq)\]
is an isomorphism. We now define $H(\psi) = G^{-1}(V(\psi))$. It is easily checked that this (uniquely) extends $H$ and makes it into the desired lift.

Assume now that $F$ is a trivial cofibration. We can thus construct a functor $F' : Q \to P$ such that

$$F' \circ F = \text{id}_P$$

together with a natural isomorphism $\alpha : F \circ F' \to \text{id}_Q$ (the theory of natural transformations from a single functor to another functor between operads is almost identical to that of natural transformations between categories). We can moreover choose $\alpha$ such that for each $p \in \text{ob}(P)$, the component at $Fp$ is given by

$$\alpha_{Fp} = \text{id}_{Fp}.$$ 

To define $H : \text{ob}(Q) \to \text{ob}(R)$, let $q \in \text{ob}(Q)$ and consider the object $VFF'q \in \text{ob}(S)$. Since

$$VFF'q = GU'F'q$$

it follows from the definition of fibration that there is an object $H(q)$ and an isomorphism

$$\beta_q : UF'q \to Hq$$

in $R$ such that

$$GHq = Vq$$

and

$$G\beta_q = V\alpha_q.$$ 

We can also choose $\beta$ so as to assure that for every $p \in \text{ob}(P)$

$$HFp = Up$$

and

$$\beta_{Fp} = \text{id}_{Up}.$$ 

Let now $\psi \in Q(q_1, \ldots, q_n; q)$ and define $H(\psi)$ to be the composition of the following composition scheme in $R$:

The resulting $H$ is easily seen to be a functor, and the desired lift.

For the axiom on factorizations, let $F : P \to Q$ be a functor. We first construct a factorization of $F$ into a trivial cofibration followed by a fibration. Construct first the following operad $P'$ with

$$\text{ob}(P') = \{(p, \varphi, q) \in \text{ob}(P) \times Q(Fp, q) \times \text{ob}(Q) \mid \varphi \text{ is an isomorphism}\}$$

and, for objects $(p_1, \varphi_1, q_1), \ldots, (p_n, \varphi_n, q_n), (p, \varphi, q)$, the arrows

$$P'((p_1, \varphi_1, q_1), \ldots, (p_n, \varphi_n, q_n); (p, \varphi, q)) = P(p_1, \ldots, p_n; p)$$
with the obvious operadic structure. If we now define \( G : \mathcal{P} \to \mathcal{P}' \) on objects \( p \in \text{ob}(\mathcal{P}) \) by
\[
G(p) = (p, \text{id}_{Fp}, Fp)
\]
and for any morphism \( \psi \in \mathcal{P}(p_1, \ldots, p_n; p) \) by
\[
G(\psi) = \psi
\]
we evidently get a functor, which is clearly a trivial cofibration. We now define the functor \( H : \mathcal{P}' \to \mathcal{Q} \) on objects \((p, \varphi, q) \in \text{ob}(\mathcal{P}')\) by
\[
H(p, \varphi, q) = q
\]
and on an arrow \( \psi \in \mathcal{P}'((p_1, \varphi_1, q_1), \ldots, (p_n, \varphi_n, q_n); (p, \varphi, q)) \) to be the composition of the composition scheme

\[
\begin{array}{c}
q_1 \\
\vdots \\
q_n
\end{array}
\quad
\begin{array}{c}
Fp_1 \\
F\varphi_1 \\
Fp_n
\end{array}
\quad
\begin{array}{c}
\varphi \\
Fp \\
q
\end{array}
\]

Clearly, \( H \) is a fibration since if \( f : H(p, \varphi, q) \to q' \) is an isomorphism in \( \mathcal{Q} \), then \((p, f\varphi, q')\) is also an object of \( \mathcal{Q} \) and \( \text{id}_p \) is an isomorphism in \( \mathcal{P}' \) from \((p, \varphi, q)\) to \((p, f\varphi, q')\) which, by definition, maps under \( H \) to \( f\varphi \circ F(\text{id}_p) \circ \varphi^{-1} = f \). Since we obviously have that \( F = H \circ G \) we have the desired factorization.

We now proceed to prove that \( F \) can be factored as a composition of a cofibration followed by a trivial fibration. Let \( \mathcal{Q}' \) be the operad with
\[
\text{ob}(\mathcal{Q}') = \text{ob}(\mathcal{P}) \coprod \text{ob}(\mathcal{Q})
\]
and with arrows defined as follows. Given an object \( x \in \text{ob}(\mathcal{Q}') \) let (somewhat ambiguously)
\[
F_x = \begin{cases} 
  x, & \text{if } x \in \text{ob}(\mathcal{Q}), \\
  F_x, & \text{if } x \in \text{ob}(\mathcal{P}).
\end{cases}
\]
Now, for objects \( x_1, \ldots, x_n, x \in \text{ob}(\mathcal{Q}') \) let
\[
\mathcal{Q}'(x_1, \ldots, x_n; x) = \mathcal{Q}(Fx_1, \ldots, Fx_n; Fx).
\]
The operad structure is the evident one. If we now define a functor \( G : \mathcal{P} \to \mathcal{Q}' \) for an object \( p \in \text{ob}(\mathcal{P}) \) and an arrow \( \psi \in \mathcal{P}(p_1, \ldots, p_n; p) \) by
\[
Gp = p
\]
and
\[
G\psi = F\psi,
\]
then we obviously obtain a cofibration. We now define \( H : \mathcal{Q}' \to \mathcal{Q} \) as follows. Given an object \( x \in \text{ob}(\mathcal{Q}') \), if \( x \in \text{ob}(\mathcal{P}) \), then we set \( Hx = Fx \) and if \( x \in \text{ob}(\mathcal{Q}) \), then we set \( Hx = x \) (thus in our slightly ambiguous notation we have that \( Hx = Fx \)). Given an arrow \( \psi \in \mathcal{Q}'(x_1, \ldots, x_n; x) \), defining \( H\psi = \psi \) makes \( H \) into a functor, clearly full and faithful. Moreover \( H \) is a fibration as can easily be seen.
Since obviously $F = H \circ G$, the proof is complete. Note that all operads are both fibrant and cofibrant under this model structure. □

The Boardman-Vogt tensor product is of extreme importance in the theory of operads. It only exists for symmetric operads, since the interchange axiom can not be stated for non-symmetric operads, and without it the resulting tensor product is, in a sense, too large, and not very useful. The model structure on symmetric operads, as we now show, is compatible with the Boardman-Vogt tensor product.

**Theorem 2.2.** The monoidal category $\text{Ope}_\Sigma$ with the Boardman-Vogt tensor product and the model structure defined above is a monoidal model category.

**Proof.** Since all objects are cofibrant we only have to prove that given two cofibrations $F : \mathcal{P} \longrightarrow \mathcal{Q}$ and $G : \mathcal{P}' \longrightarrow \mathcal{Q}'$, the push-out corner map $F \wedge G$ in the diagram

\[
\begin{array}{ccc}
\mathcal{P} \otimes_{BV} \mathcal{P}' & \longrightarrow & \mathcal{P} \otimes_{BV} \mathcal{Q}' \\
\downarrow \mathbb{F} \otimes_{BV} \mathcal{P}' & & \downarrow \mathbb{F} \otimes_{BV} \mathcal{Q}' \\
\mathcal{Q} \otimes_{BV} \mathcal{P}' & \longrightarrow & \mathcal{Q} \otimes_{BV} \mathcal{Q}' \\
\downarrow \mathbb{F} \otimes_{BV} \mathcal{G} & & \downarrow \mathbb{F} \otimes_{BV} \mathcal{G} \\
\mathcal{K} & \longrightarrow & \mathcal{Q} \otimes_{BV} \mathcal{Q}'
\end{array}
\]

is a cofibration which is a trivial cofibration if $F$ is a trivial cofibration. Since $\text{ob}(\mathcal{P} \otimes_{BV} \mathcal{Q}) = \text{ob}(\mathcal{P}) \times \text{ob}(\mathcal{Q})$ and since $\text{ob} : \text{Ope}_\Sigma \rightarrow \text{Set}$ commutes with colimits, applying the functor $\text{ob}$ to the diagram above we obtain the diagram

\[
\begin{array}{ccc}
\text{ob}(\mathcal{P}) \times \text{ob}(\mathcal{P}') & \longrightarrow & \text{ob}(\mathcal{P}) \times \text{ob}(\mathcal{Q}') \\
\downarrow \mathbb{F} \times \mathcal{P}' & & \downarrow \mathbb{H} \\
\text{ob}(\mathcal{Q}) \times \text{ob}(\mathcal{P}') & \longrightarrow & \text{ob}(\mathcal{Q}) \times \text{ob}(\mathcal{Q}') \\
\downarrow \mathbb{F} \times \mathcal{G} & & \downarrow \mathbb{F} \times \mathcal{G} \\
\text{ob}(\mathcal{K}) & \longrightarrow & \text{ob}(\mathcal{Q}) \times \text{ob}(\mathcal{Q}')
\end{array}
\]

which is again a pushout. We are given that $F$ and $G$ are injective, from which it follows that $F \times \mathcal{P}'$ and $\mathcal{P} \times G$ are also injective. It is now easy to verify that $F \wedge G$ is injective as well which proves that the operad map $F \wedge G : \mathcal{K} \rightarrow \mathcal{Q} \otimes_{BV} \mathcal{Q}'$ is a cofibration. Assume now that $F$ in the first diagram is also a weak equivalence, i.e., an operadic equivalence. It is trivial to verify that $F \otimes_{BV} \mathcal{P}'$ is also an equivalence. Thus $F \times \mathcal{P}'$ is a trivial cofibration. Since trivial cofibrations are stable under cobase change it follows that $H$ is a trivial cofibration. Since $F \times \mathcal{Q}'$ is also an equivalence, the three for two property implies that $F \wedge G$ is a trivial cofibration. □
Consider now $\text{Cat}$ with the Rezk model structure and the categories $\text{Ope}$ and $\text{Ope}_\Sigma$ with the model structure given above. Recall the adjunction between categories and operads.

**Lemma 2.3.** The adjunctions $\text{Ope} \stackrel{j_!}{\leftarrow} \text{Cat}$ and $\text{Ope}_\Sigma \stackrel{j_!}{\leftarrow} \text{Cat}$ are Quillen adjunctions.

**Proof.** It is enough to prove that $j_!$ preserves cofibrations and trivial cofibrations. Actually it is trivial to verify the much stronger property that both $j^*$ and $j_!$ preserve fibrations, cofibrations, and weak equivalences. \hfill $\Box$

We end our treatment of the model structure of operads with the following explicit construction of generating cofibrations.

**Theorem 2.4.** The model structures on $\text{Ope}$ and on $\text{Ope}_\Sigma$ are cofibrantly generated.

**Proof.** Again, the symmetric group actions pose no difficulties, and so we give the details in the non-symmetric case; the symmetric case obtained by symmetrizing.

Let $\ast$ be the operad with one object and just one arrow (necessarily the identity, and notice that $\ast$ can be considered as a symmetric or a non-symmetric operad) and let $H$ be the free living isomorphism operad, which has two objects and, besides the necessary identities, just one isomorphism between the two objects. It is a triviality to check that a functor $F : P \to Q$ is a fibration if, and only if, it has the right lifting property with respect to any one of the two possible functors $\ast \to H$.

To characterize the trivial fibrations by right lifting properties we will need to consider several other operads. First of all, it is clear that if a functor $F : P \to Q$ has the right lifting property with respect to $\phi \to \ast$, then $F : \text{ob}(P) \to \text{ob}(Q)$ is surjective (where $\phi$ is the initial operad with no objects). For each $n \geq 1$ consider the operad $Ar_n$ that has $n + 1$ objects $\{0, 1, \ldots, n\}$ and is generated by a single arrow from $(1, \ldots, n)$ to $0$. Thus a functor $Ar_n \to P$ is just a choice of an arrow in $P$ of arity $n$. Let $\partial Ar_n$ be the sub-operad of $Ar_n$ that contains all the objects of $Ar_n$ but only the identity arrows. It now easily follows that if a functor $F : P \to Q$ has the right lifting property with respect to the inclusion $\partial Ar_n \to Ar_n$, then for any objects $p_1, \ldots, p_n, p \in \text{ob}(P)$, the function

$$F : P(p_1, \ldots, p_n; p) \to Q(Fp_1, \ldots, Fp_n; Fp)$$

is surjective. Consider now the operad $PAr_n$ with $n + 1$ objects $\{0, 1, \ldots, n\}$ generated by two distinct arrows from $(1, \ldots, n)$ to $0$ and the obvious map $PAr_n \to Ar_n$ which identifies these two arrows. If a functor $F : P \to Q$ has the right lifting property with respect to $PAr_n \to Ar_n$, then the map

$$F : P(p_1, \ldots, p_n; p) \to Q(Fp_1, \ldots, Fp_n; Fp)$$

is injective. Combining these results we see that if a functor $F : P \to Q$ has the right lifting property with respect to the set of functors

$$\{\phi \to \ast\} \cup \{\partial Ar_n \to Ar_n \mid n \geq 0\} \cup \{PAr_n \to Ar_n \mid n \geq 0\},$$

then $F$ is fully faithful and $F : \text{ob}(P) \to \text{ob}(Q)$ is surjective, which implies that $F$ is a trivial fibration. Finally, since all the functors just mentioned are cofibrations it follows that all trivial fibrations have the right lifting property with respect to
them. This then proves that the trivial fibrations are exactly those functors having the right lifting property with respect to that set.

Finally, we recover the Rezk model structure by exhibiting \( \text{Cat} \) as a slice category of \( \text{Ope} \) as well as of \( \text{Ope}_\Sigma \). Consider again \( \ast \), the operad with one object and only one morphism, necessarily the identity morphism.

**Theorem 2.5.** The slice category \( \text{Ope}/\ast \) (resp. \( \text{Ope}_\Sigma/\ast \)) is isomorphic to \( \text{Cat} \) and slicing the model structure on operads yields the Rezk model structure on categories.

**Proof.** The objects of \( \text{Ope}/\ast \) are functors \( F : \mathcal{P} \to \ast \). Since a functor must preserve arities of morphisms, and \( \ast \) only has one unary morphism, if follows that \( \mathcal{P} \) only has unary morphisms. It is now easy to see that \( F \mapsto j^*(P) \) is an isomorphism of categories between \( \text{Ope}/\ast \) and \( \text{Cat} \). The claim about the model structures is immediate and the argument for symmetric operads is similar. □

**References**

[1] Marek A. Bednarczyk, Andrzej M. Borzyszkowski, and Wieslaw Pawlowski. Generalized congruences—epimorphisms in cat. *Theory Appl. Categ.*, 5:No. 11, 266–280 (electronic), 1999.
[2] C. Berger and I. Moerdijk. On the homotopy theory of enriched categories. *Q. J. Math.*, 64(3):805–846, 2013.
[3] G. Caviglia. A model structure for enriched coloured operads. *arXiv:1401.6983*, 2014.
[4] Denis-Charles Cisinski and Ieke Moerdijk. Dendroidal sets and simplicial operads. *J. Topol.*, 6(3):705–756, 2013.
[5] W. G. Dwyer and J. Spaliński. Homotopy theories and model categories. In *Handbook of algebraic topology*, pages 73–126. North-Holland, Amsterdam, 1995.
[6] Philip S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.
[7] Mark Hovey. Quillen model categories. *J. K-Theory*, 11(3):469–478, 2013.
[8] T. Leinster. *Higher operads, higher categories*, volume 298 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2004.
[9] I. Moerdijk and I. Weiss. Dendroidal sets. *Algebr. Geom. Topol.*, 7:1441–1470, 2007.
[10] Ittay Weiss. From operads to dendroidal sets. In *Mathematical foundations of quantum field theory and perturbative string theory*, volume 83 of *Proc. Sympos. Pure Math.*, pages 31–70. Amer. Math. Soc., Providence, RI, 2011.