Treatment of multi freedom constraints in geometrically nonlinear stability analysis of truss structures using penalty function method

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Abstract. This paper focus on the treating the multi freedom constraints in geometrically nonlinear buckling and post buckling analysis of truss structure using finite element method. The solution of geometrically nonlinear buckling problem of truss structure having multi freedom constraints is required to incorporate the boundary constraints to the nonlinear master stiffness equation. This work proposes approach to impose the multi freedom constraints and construct stiffness equation for solving the nonlinear buckling problems of truss structures using Penalty function method. The nonlinear balanced equations for buckling analysis of truss structures are developed by minimization of the augmented potential energy function of system by incorporating the multi freedom constraints and changing the constrained problem to the non-constrained problem using Penalty objective function. Using the arc length method, this research proposes the incremental-iterative algorithm for solving the nonlinear balanced equations. Based on proposed algorithm, the calculation program for determining equilibrium paths is written. The numerical test results show the efficiency of proposed method in analysing buckling and post buckling behaviour of truss structures.

1. Introduction

The stability of truss structures is a phenomenon associated with buckling and post buckling behaviour. Investigating buckling and post buckling behaviour is very important since post buckling means loss the stability of structure associated with large displacement and may be lead to destruction the structure. In recent years, numerous research work addressed the buckling and post-buckling behavior of truss structures. The computational formulation and solution of nonlinear problem of truss structures involving nonlinearities is widely studied in many literature based on analytical and numerical approach [1-7]. The finite element method is considered as the most popular and efficient method used to formulate and solve the geometrically nonlinear problem of truss structures. Using finite element method, the buckling analysis of truss considering large displacement and rotation leads to solving the nonlinear incremental equations of truss. In the cases of truss structures having multi freedom constraints, the boundary constraint considerably increases the difficulty in constructing in solving nonlinear incremental balanced equations of system due to necessary to modify the system of master stiffness equation for imposing multi freedom constraints and getting equation solver. The implementing multi freedom constraints is done by changing the assembled master stiffness equations to produce a modified system equation based on the master stiffness equation. In many applications of the finite element method, the penalty augmentation method and Lagrange multiplier adjunction
method are better suited to general implementations, whether linear or nonlinear [8-9]. This research employed the penalty function method for imposing boundary constraints and constructing stiffness equation for solving the nonlinear buckling problem because of its straightforward computer implementation. For solving geometrically nonlinear buckling and post buckling problem of truss system, the research proposes to employ arc length method due to its efficient to predict the proper response and follow the nonlinear equilibrium path through limit. Therefore, a new incremental-iterative algorithm for solving geometrically nonlinear buckling and post buckling problem of truss system and calculation program are established. The numerical test results shows the efficiency of proposed method in analysing buckling and post buckling behaviour of truss structures.

2. Penalty function method for treatment of linear multi freedom constraints

Linear multi freedom constraint is defined as type of the constraints where displacement components are dependent on each other and constraints can be expressed as linear functional equations. When the functional equation involves interaction of nodal displacements at multiple nodes, the multi freedom constraint is said to be multi point constraints. Accounting for multi freedom constraints is done by changing the assembled master equations to produce a modified system of equations.

For imposing these boundary constraints the penalty function method is easiest, and perhaps earliest method which was still employed in solving finite element problem [10-11].

Consider the nth degree of freedom system.

The global vector of nodal displacement unknowns of system is 

\[ u = [u_1, u_2, \ldots, u_n]^T, \quad u \in \mathbb{R}^n \]

The augmented potential energy of the unconstrained finite element model is 

\[ \Pi(u) \]

Equation of linear multi freedom constraints is expressed as 

\[ g_k(u) = 0, \quad k = 1, m \]

and it can be written in matrix form

\[ g(u) = Au - b = 0 \quad (1) \]

Where: 

\[ g(u) = \{g_1(u), g_2(u), \ldots, g_m(u)\}^T; \quad A \equiv A_{m \times n}, b \equiv b_{m \times 1} \]

The penalty augmented system of equations can be obtained by minimization of the augmented potential energy function by incorporating the constraints (1) as follows

\[ \min \left\{ \Pi(u) : g(u) = 0, \quad u \in \mathbb{R}^n \right\} \quad \text{or} \quad \min \left\{ \Pi(u) : Au - b = 0, \quad u \in \mathbb{R}^n \right\} \]

The augmented potential energy can be computed as the sum of strain energy of system \( U(u) \) and external work \( (-u^T \cdot P) \)

\[ \Pi(u) = U(u) - u^T \cdot P \quad (3) \]

Where \( P = \{P_1, P_2, \ldots, P_n\}^T \) is global nodal force vector.

The concept of the penalty function method [4, 5], for imposing multi freedom constraints, is to change from the constrained optimization problem to non-constrained optimization problem using the penalty objective function \( Q(u, w) \)

\[ Q(u, w) = U(u) - u^T \cdot P + \frac{1}{2} w (Au - b)^T (Au - b) \quad (4) \]

Using penalty function method, each multi freedom constraint is viewed as the presence of a fictitious elastic structural element parameterized by a numerical weight \( w \).

Solving the non-constrained minimization problem \( \min \left\{ Q(u, w) : u \in \mathbb{R}^n \right\} \) by taking the derivative of \( Q(u, w) \) with respect to \( u \) and setting equal to zero as below
\[
\frac{\partial U(u)}{\partial u} - P + w.A^T.(Au - b) = 0
\]  
(5)

Eq. (5) can be written as

\[
\frac{\partial U(u)}{\partial u} + w.A^T.(Au - b) = P
\]  
(6)

In geometrically nonlinear analysis, the function \( \frac{\partial U(u)}{\partial u} \) is nonlinear function with respect to \( u \), therefore the system of equations (6) is nonlinear.

3. Problem solving and algorithm

For solving the nonlinear system of equations (6) using finite element method, it is necessary to divide total load into many steps \( \Delta P \) and construct incremental equation [12-15].

The incremental equation can be constructed by utilizing Taylor series formula for a short of \( \delta u \) to expand function of (6) around of variable point, keeping only linear term in \( \delta u \), expressing as

\[
\frac{\partial U(u)}{\partial u} + w.A^T.(Au - b) + \left\{ \frac{\partial}{\partial u} \left( \frac{\partial U(u)}{\partial u} \right) + w.A^T.A \right\} \delta u = P + \Delta P
\]  
(7)

By transporting components of equation (7), getting

\[
\left\{ \frac{\partial}{\partial u} \left( \frac{\partial U(u)}{\partial u} \right) + w.A^T.A \right\} \delta u = P + \Delta P - \left\{ \frac{\partial U(u)}{\partial u} + w.A^T.(Au - b) \right\}
\]  
(8)

With:

\( \delta u = \{ \delta u_1, \delta u_2, ..., \delta u_n \}^T \) is vector of nodal incremental displacements;

\[
\frac{\partial}{\partial u} \left( \frac{\partial U(u)}{\partial u} \right) = \begin{bmatrix} \frac{\partial^2 U(u)}{\partial u_i \partial u_j} \end{bmatrix}_{i,j=1...n} \quad \text{is the Hessian matrix of the } U(u).
\]

Incremental equation (8) can be written in compact form as

\[
\bar{K}(u) \delta u = P + \Delta P - q(u)
\]  
(9)

Where: \( \bar{K}(u) = \left[ K(u) + w.A^T.A \right], q(u) = \left\{ \frac{\partial U(u)}{\partial u} + w.A^T.(Au - b) \right\} \)

\[
K(u) = \frac{\partial}{\partial u} \left( \frac{\partial U(u)}{\partial u} \right) = \begin{bmatrix} \frac{\partial^2 U(u)}{\partial u_i \partial u_j} \end{bmatrix}, (i, j = 1...n) \quad \text{is tangent stiffness matrix of structures}
\]

This research employs spherical arc length method [16,17], the most effective in solving non-linear problem under consideration exhibits one or more critical points, to establish the algorithm for solving nonlinear incremental equations (9) to investigate geometrically buckling and post buckling behaviour of truss structures. Based on incremental-iterative algorithm using arc length method the calculation program has been written.

4. Numerical example

4.1. Example formulation

Investigate the buckling and post buckling behaviour of the truss structures (is shown in Figure 1) considering large displacement and rotation), having linear multi freedom constraints. All truss bars
made of the same material and had the same geometrical. The geometric parameters, material parameters are given as follows: \( A = 4cm^2; E = 2.1\times 10^4 kN / cm^2 \)

Equation of linear multi freedom and multi point constraints is expressed as

Multi freedom constraint: \( u_1 = 0; u_4 + u_6 = 0; u_8 = 87 \)

Multi point constraint: \( u_1 - u_5 - 1000[1 - \cos(5°)] = 0 \)

Writing equations of all multi freedom and multi point constraints in matrix form

\[
g(u) = Au - b = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_8 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -87 \\ 1000[1 - \cos(5°)] \end{bmatrix} = 0
\]

For investigating the convergence speed of the proposed method, the problem was solved with different options of weight values \( w \), as following

\( w_0 = 10^2, w_1 = 10^3, w_2 = 10^4, \quad w_3 = 10^5, \quad w_4 = 10^6, \quad w_5 = 10^9 \)

4.2. Numerical results

The calculating results of numerical example nodal displacement and internal forces in different cases with weight values. For investigating the efficient of proposed method, the load – displacement

Figure 1. Examined truss system having multi freedom and multi point constraints.
equilibrium path $P - u_4$ in cases with different weight values is shown in Figures 2 - 3, the load-
internal force equilibrium path $P - N_1, N_2, N_3, N_4, N_5$ in case with the weight value $w_5$.

![Figure 2](image1.png)

**Figure 2.** Load – displacement equilibrium path $P - u_4$ in cases with the weight values $w_0, w_1, w_2, w_3, w_4, w_5$.

![Figure 3](image2.png)

**Figure 3.** Load-internal force equilibrium path $P - N_1, N_2, N_3, N_4, N_5$ in case with the weight value $w_5$.

### 4.3. Comments

Analysing the results shown in paragraph 4.2., the following comments can be given:

- The calculation results of geometrically nonlinear buckling and post buckling analysis of truss
  structures is convergence when the weight values is increasing;
Six Limit Points can be found in the load – displacement equilibrium path $P-u_4$, including four Load Limit Points and two Displacement Limit Points. It can be stated that the proposed algorithm is very effective for finding Limit Points in post buckling analysis.

5. Conclusions
The penalty function method can be effectively used for incorporating the multi freedom constraints and producing the nonlinear solving equations of truss system.

The proposed method and algorithm has remarkable advantage in analysing geometrically nonlinear buckling and post buckling behaviour of truss structures having complicated equilibrium path with many Limit Points.

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