A REILLY TYPE INTEGRAL FORMULA AND ITS APPLICATIONS

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Abstract. In this paper, we achieve a Reilly type integral formula associated with the \( \phi \)-Laplacian. As its applications, we obtain Heintze-Karcher and Minkowski type inequalities. Furthermore, almost Schur lemmas are also given. They recover the partial results of Li and Xia in [15]. On the other hand, we also study eigenvalue problem for Wentzell boundary conditions and obtain eigenvalue relationships.

1. Introduction

Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold with the dimension \(n \geq 3\), where \(g\) is the metric. The \(\phi\)-Laplacian associated with \(\phi\) is defined by
\[
\Delta_{\phi} v = e^{\phi} \text{div}(e^{-\phi} \nabla v) = \Delta v - \nabla \phi \nabla v, \quad \forall \, v \in C^\infty(M),
\]
which is symmetric with respect to the \(L^2(M)\) inner product under the weighted measure
\[
d\mu = e^{-\phi} dv_g,
\]
that is,
\[
\int_M \alpha \Delta_{\phi} \beta d\mu = \int_M \beta \Delta_{\phi} \alpha d\mu = - \int_M \nabla \alpha \nabla \beta d\mu, \quad \forall \, \alpha, \beta \in C^\infty_0(M).
\]
Following [1,14,23](or see [8–10,13] and the references therein), the \(m\)-dimensional Bakry-Émery Ricci curvature associated with the above \(\phi\)-Laplacian is given by
\[
\text{Ric}_{\phi,m} = \text{Ric} + \nabla^2 \phi - \frac{1}{m-n} d\phi \otimes d\phi,
\]
where \(m\) is a real constant, and \(m = n\) if and only if \(\phi\) is a constant. Define
\[
\text{Ric}_{\phi} = \text{Ric} + \nabla^2 \phi.
\]
Then \(\text{Ric}_{\phi}\) can be seen as the \(\infty\)-dimensional Bakry-Émery Ricci curvature, that is, \(\text{Ric}_{\phi} := \text{Ric}_{\phi,\infty}\).

For the convenience, we still denote \(\Delta, \nabla\) by the Laplacian operator and gradient operator on \(M\), and \(\overline{\Delta}, \overline{\nabla}\), respectively, by the Laplacian operator
and gradient operator on the boundary $\partial M$. The mean curvature $H$ of $\partial M$ is given by $H = \text{tr}_g(II)$, where $II(X, Y) = g(\nabla_X \nu, Y)$ denotes the second fundamental form of $\partial M$ with $\nu$ the outward unit normal on $\partial M$. For any positive twice differentiable function $V$, we make the following conventions:

$$\widehat{\text{Ric}}_\phi^V = \frac{\Delta \phi}{V} g_{ij} - \frac{1}{V} V_{ij} + \text{Ric}_{\phi, m},$$  \hspace{1cm} (1.2)$$

$$\widehat{\text{Ric}}_\phi^\infty = \frac{\Delta \phi}{V} g_{ij} - \frac{1}{V} V_{ij} + \text{Ric}_\phi,$$ \hspace{1cm} (1.3)$$

$H_\phi = H - \phi \nu$, $II^V = II - (\ln V) \nu \overline{g}$ and $d\sigma$ denotes the measure induced on $\partial M$.

First, we prove the Reilly type formula associated with $\phi$-Laplacian:

**Theorem 1.1.** Let $V$ be a positive twice differentiable function on a given compact Riemannian manifold $M$ with the boundary $\partial M$. For any smooth function $f$, we have the following equality:

$$0 = \int_{\partial M} \left[ -VII^V(V \nabla^V f, V \nabla^V f) - V^3 H_\phi \left( \frac{f}{V} \right)^2 \right.$$

$$- 2V^2 \left( \frac{f}{V} \right) (\nabla \phi - \frac{\Delta \phi}{V} f) \big] d\sigma + \int_M V \left[ (\Delta \phi f - \frac{\Delta \phi}{V} V f)^2 \right.$$

$$- \left| \nabla^2 f - \frac{f}{V} \nabla^2 V \right|^2 - \widehat{\text{Ric}}_\phi^V \left( V \nabla^V f, V \nabla^V f \right) \big] d\mu.$$ \hspace{1cm} (1.4)$$

Denote by $A_{ij} = f_{ij} - \frac{f}{V} V_{ij}$ and $\hat{A}_{ij} = A_{ij} - \frac{\text{tr}(A_{ij})}{n} g_{ij}$ with $\text{tr}(A_{ij}) = \Delta f - \frac{f}{V} \Delta V$. Then, we have $\text{tr}(A_{ij}) = 0$ and

$$|A_{ij}|^2 = |\hat{A}_{ij}|^2 + \frac{1}{n} [\text{tr}(A_{ij})]^2$$

$$= |\hat{A}_{ij}|^2 + \frac{1}{m} [\text{tr}(A_{ij})]^2 + \frac{m-n}{mn} [\text{tr}(A_{ij})]^2$$

$$= |\hat{A}_{ij}|^2 + \frac{1}{m} (\Delta \phi f - \frac{f}{V} \Delta \phi V)^2 + \frac{2}{m} V [\text{tr}(A_{ij})] \nabla \phi \nabla \frac{f}{V}$$

$$- \frac{1}{m} V^2 \left( \nabla \phi \nabla \frac{f}{V} \right)^2 + \frac{m-n}{mn} |\text{tr}(A_{ij})|^2,$$

which gives

$$\left| \nabla^2 f - \frac{f}{V} \nabla^2 V \right|^2 + \text{Ric}_\phi \left( V \nabla^V f, V \nabla^V f \right)$$

$$= |\hat{A}_{ij}|^2 + \frac{1}{m} (\Delta \phi f - \frac{f}{V} \Delta \phi V)^2 + \text{Ric}_{\phi, m} \left( V \nabla^V f, V \nabla^V f \right)$$

$$+ \left( \sqrt{\frac{m-n}{mn}} \text{tr}(A_{ij}) + \sqrt{\frac{n}{m(m-n)}} V \nabla \phi \nabla \frac{f}{V} \right)^2$$

$$\geq \frac{1}{m} (\Delta \phi f - \frac{f}{V} \Delta \phi V)^2 + \text{Ric}_{\phi, m} \left( V \nabla^V f, V \nabla^V f \right)$$ \hspace{1cm} (1.5)$$
provided \( m \in (-\infty, 0) \cup [n, +\infty) \), and equality holds if and only if
\[
\nabla^2 f - \frac{f}{V} \nabla^2 V = \frac{1}{n} \left( \Delta f - \frac{f}{V} \Delta V \right) g
\]  
(1.6)

and
\[
\Delta f - \frac{f}{V} \Delta V + \frac{n}{\sqrt{(m-n)^2}} V \nabla \phi \nabla f = 0.
\]  
(1.7)

Applying (1.5) in (1.4), one obtain the following result immediately:

**Corollary 1.2.** Let \( V \) and \( f \) be as in Theorem 1.1 and \( m \in (-\infty, 0) \cup [n, +\infty) \). Then, we have the following inequalities:
\[
0 \leq \int_{\partial M} \left[ \frac{1}{V} \left( \nabla \nabla V f, \nabla \nabla f \right) - V^3 \phi \left( \frac{\phi}{V}, \frac{f}{V} \right)^2 - 2V^2 \left( \frac{\phi}{V}, \frac{\phi V f - \phi f V}{V} \right) \right] d\sigma + \int_M V \left[ m-1 \left( \Delta \phi f - \frac{\phi V f}{V} \right)^2 - \frac{\hat{\text{Ric}}_V}{\phi, m} \left( \nabla f, \nabla f \right) \right] d\mu,
\]  
(1.8)

where the equality occurs if and only if (1.6) and (1.7) hold.

**Remark 1.1.** In [18] (or see [15, 20]), Qiu and Xia provide a generalization of Reilly’s formula and give some applications. In particular, if \( m = n \), then our (1.4) becomes the formula (1.1) of Li and Xia in [15].

Next, by using the above formula (1.8), we can achieve the following Heintze-Karcher type inequality and Minkowski type inequality:

**Theorem 1.3.** Let \( V \) be a positive twice differentiable function on a given compact Riemannian manifold \( M \) with the boundary \( \partial M \). If \( \hat{\text{Ric}}_V \geq 0 \) and \( H_\phi > 0 \), where \( m \in (-\infty, 0) \cup [n, +\infty) \), then
\[
\int_M V \, d\mu \leq \frac{m-1}{m} \int_{\partial M} V \, d\sigma.
\]  
(1.9)

Moreover, if the equality in (1.9) holds, then \( m = n \) and \( \partial M \) is umbilical.

**Theorem 1.4.** Let \( V \) be a positive twice differentiable function on a given compact Riemannian manifold \( M \) with the boundary \( \partial M \). If \( \hat{\text{Ric}}_V \geq 0 \) and \( II V \geq 0 \), where \( m \in (-\infty, 0) \cup [n, +\infty) \), then
\[
\left( \int_{\partial M} V \, d\sigma \right)^2 \geq \frac{m}{m-1} \int_M V \, d\mu \int_{\partial M} V \, d\sigma.
\]  
(1.10)

Moreover, if the inequality in (1.10) is strict and the equality in (1.11) holds, then we have that \( m = n \), \( \partial M \) is umbilical and the mean curvature \( H \) is constant.
Remark 1.2. If $\partial M = \Sigma \cup (\bigcup_{l=1}^{N} \Sigma_l)$ satisfies the inner boundary condition defined by Definition 2.3 in [15] and we suppose the outermost boundary hypersurface $\Sigma$ is mean convex, the similar results as in Theorem 1.3 can also be achieved. Particularly, when $m = n$, then our Theorem 1.3 and Theorem 1.4 reduce to the Theorem 1.3 and Theorem 1.7 of Li and Xia in [15], respectively.

Let $\mathbb{R}^{n+1}(c)$ be a space form with constant sectional curvature $c$, where $\mathbb{R}^{n+1}(1) = \mathbb{S}^{n+1}(1)$ is a unit sphere if $c = 1$, $\mathbb{R}^{n+1}(-1) = \mathbb{H}^{n+1}(-1)$ is a hyperbolic space if $c = -1$ and $\mathbb{R}^{n+1}(0) = \mathbb{R}^{n+1}$ is an Euclidean space if $c = 0$. Let

$$C_{n,K_1,K_2,\eta_1} = \frac{n - 1}{n} + \frac{(n - 1)K_1 + K_2 + 2\sqrt{(\eta_1 + (n - 1)K_1)K_2}}{\lambda_1},$$

(1.12)

where $\eta_1$ is the first nonzero eigenvalue of problem $\Delta \phi f - \Delta \phi V f = -\eta f V$ on closed manifolds. Next, we will apply Theorem 1.1 to obtain the following almost Schur lemmas on closed hypersurfaces:

**Theorem 1.5.** Let $M$ be an $n$-dimensional closed hypersurface in a space form $\mathbb{R}^{n+1}(c)$ and $V$ be a positive twice differentiable function on $M$. If $\overline{\nabla} V \geq -(n - 1)K_1$, where $K_1$ is a nonnegative constant, then

$$\frac{(n - 1)^2}{n^2} \int_M V(H - \overline{H})^2 d\mu \leq C_{n,K_1,K_2,\eta_1} \int_M V\left(|\bar{\nabla}|^2 - \frac{1}{n}H^2\right) d\mu,$$

(1.13)

where $\overline{H} = \frac{f_M V H d\mu}{\int_M V d\mu}$ and $K_2 = \max(V|\nabla \phi|^2)$ for some nonnegative constant $K_2$.

The $r$-mean curvature $S_r$ on hypersurface $M$ of $\mathbb{R}^{n+1}(c)$ is related to the Newton transformation $P_r$ by

$$\text{tr}(P_r) = (n - r)S_r,$$

where

$$P_{rij} = \frac{1}{r!} \sum_{i_1, \cdots, i_r, j_1, \cdots, j_r} \delta_{i_1 \cdots i_r j_1 \cdots j_r} \cdot h_{i_1 j_1} \cdots h_{i_{r-1}j_{r-1}} h_{i_r j_r}$$

with $\text{div} P_r = 0$. For detail, see [3, 4, 17] and the references therein. Then, we have

**Theorem 1.6.** Let $M$ be an $n$-dimensional closed hypersurface in a space form $\mathbb{R}^{n+1}(c)$ and $V$ be a positive twice differentiable function on $M$. If $\overline{\nabla} V \geq -(n - 1)K_1$, where $K_1$ is a nonnegative constant, then for $2 \leq r \leq n$,

$$\frac{1}{n^2} \int_M V(S_r - \overline{S}_r)^2 d\mu \leq C_{n,K_1,K_2,\eta_1} \int_M V\left(|P_r|^2 - \frac{(n - r)^2}{n}S_r^2\right) d\mu,$$

(1.14)
where $S^V_r = \frac{\int_M V S_r d\mu}{\int_M V d\mu}$ and $K_2 = \max(V|\nabla \phi|^2)$ for some nonnegative constant $K_2$.

On the other hand, by applying Theorem 1.1, we also achieve almost Schur lemmas on closed Riemannian manifolds:

**Theorem 1.7.** Let $M$ be an $n$-dimensional closed Riemannian manifold and $V$ be a positive twice differentiable function on $M$. If $\hat{\text{Ric}}_\phi \geq -(n-1)K_1$, where $K_1$ is a nonnegative constant, then
\[
\frac{(n-2)^2}{4n^2} \int_M V(R - \bar{R})^2 d\mu \leq C_{n,K_1,K_2} \int_M V\left(|\text{Ric}|^2 - \frac{1}{n}R^2\right) d\mu, \tag{1.15}
\]
where $\bar{R} = \frac{\int_M VR d\mu}{\int_M V d\mu}$ and $K_2 = \max(V|\nabla \phi|^2)$ for some nonnegative constant $K_2$.

The $r$-scalar curvature $\sigma_r$ on a Riemannian manifold $M$ is related to the $k$-th Newton tensor $T^{(k)}$ is defined by
\[
T^{(k)}_{ij} = \frac{1}{k!} \sum \delta^{j_1 \cdots j_k} \cdots \delta_{i_1 \cdots i_k} A_{i_1} \cdots A_{i_k},
\]
where $A$ is the well-known Schouten tensor given by
\[
A_{ij} = \frac{1}{n-2} \left(R_{ij} - \frac{1}{2(n-1)} R g_{ij}\right).
\]
Moreover, $\text{tr}_g(T^{(k)}) = (n-k)\sigma_k$. When the metric is locally conformally flat, then we have $\text{div} T^{(k)} = 0$, for example, see [5–7,19]. Therefore, we have

**Theorem 1.8.** Let $M$ be an $n$-dimensional closed locally conformally flat Riemannian manifold and $V$ be a positive twice differentiable function on $M$. If $\hat{\text{Ric}}_V \geq -(n-1)K_1$, where $K_1$ is a nonnegative constant, then for $2 \leq k \leq n$,
\[
\frac{1}{n^2} \int_M V(\sigma_k - \bar{\sigma}_k)^2 d\mu \leq C_{n,K_1,K_2} \int_M V\left(|T^{(k)}|^2 - \frac{(n-k)^2}{n} \sigma_k^2\right) d\mu, \tag{1.16}
\]
where $\bar{\sigma}_k = \frac{\int_M V \sigma_k d\mu}{\int_M V d\mu}$ and $K_2 = \max(V|\nabla \phi|^2)$ for some nonnegative constant $K_2$.

**Remark 1.3.** From our Theorem 1.5 and Theorem 1.7, we can deduce Theorem 1.9 and Theorem 6.1 of Li and Xia in [15]. Moreover, when $V = 1$ and $\phi$ is a constant, our Theorem 1.6 and Theorem 1.8 become Theorem 1.10 and Theorem 1.11 of Cheng in [2], respectively.

In the following, we consider the eigenvalue problem for Wentzell boundary:
\[
\Delta_\phi f - \frac{\Delta_\phi V}{V} f = 0 \text{ on } M, \quad -\beta \left(\Sigma_\phi f - \frac{\Sigma_\phi V}{V} f\right) + V\left(\frac{f}{V}\right)_\nu = \lambda \frac{f}{V} \text{ on } \partial M, \quad (1.17)
\]
where $\beta$ is a given real constant. When $\beta = 0$, the problem (1.17) becomes the following second order Steklov problem:

$$\Delta f - \frac{\Delta f}{V} f = 0 \text{ on } M, \quad V \left( \frac{f}{V} \right) = \nu \frac{f}{V} \text{ on } \partial M. \quad (1.18)$$

Denote by $\lambda_{1,\beta}$ and $p_1$ the first nonzero eigenvalues of (1.17) and (1.18), respectively. Then, we obtain the following:

**Theorem 1.9.** Let $V$ be a positive twice differentiable function on a given compact Riemannian manifold $M$ with the boundary $\partial M$. Suppose that $V II^V \geq c_1$ and $H f \geq c_2$ for two positive constants $c_1, c_2$. Then we have the following:

1. If $\hat{Ric}^V f, m \geq 0$, where $m \in (-\infty, 0) \cup [n, +\infty)$, then the first nonzero eigenvalue $\eta_1$ of the eigenvalue problem $e^\phi \text{div}(e^{-\phi}V^2 \nabla \phi) = -\eta_2$ on the boundary satisfies

$$\eta_1 \geq c_1 c_2, \quad (1.19)$$

with the equality holding if and only if $m = n$.

2. If $\hat{Ric}^V f, m \geq -(m-1)K$ for some nonnegative constant $K$, where $m \in (-\infty, 0) \cup [n, +\infty)$, then the first nonzero eigenvalue $\lambda_{1,\beta}$ of the eigenvalue problem (1.17) satisfy

$$\lambda_{1,\beta} \leq \beta \eta_1 + \frac{1}{2c_2} \left( [2\eta_1 + (m - 1)K] + \sqrt{[2\eta_1 + (m - 1)K]^2 - 4c_1 c_2 \eta_1} \right), \quad (1.20)$$

with the equality holding if and only if $m = n$.

**Theorem 1.10.** Let $V$ be a positive twice differentiable function on a given compact Riemannian manifold $M$ with the boundary $\partial M$. Suppose that $V II^V \geq c_1$ and $H f \geq c_2$ for two positive constants $c_1, c_2$. Then we have the following:

1. If $\hat{Ric}^V f, m \geq -(m-1)K$ for some nonnegative constant $K$, where $m \in (-\infty, 0) \cup [n, +\infty)$, then the first nonzero eigenvalue $p_1$ of the eigenvalue problem (1.18) satisfy

$$p_1 > \frac{c_1 \eta_1}{2\eta_1 + (m - 1)K}. \quad (1.21)$$

2. If $\hat{Ric}^V f, m \geq 0$, where $m \in (-\infty, 0) \cup [n, +\infty)$, then the first nonzero eigenvalue $\lambda_{1,\beta}$ of the eigenvalue problem (1.17) satisfy

$$\lambda_{1,\beta} \geq \frac{c_1}{2} \left( 1 + \beta c_2 + \sqrt{\beta^2 c_2^2 + 2\beta c_2} \right). \quad (1.22)$$

**Remark 1.4.** When $V = 1$, the formula (1.19) is corresponding to Theorem 1.6 of Huang and Ruan [11] and the estimate (1.20) is corresponding to Theorem 1.1 of Wang and Xia in [21]. Our Theorem 1.10 generalizes Theorem 1.3 of Wang and Xia in [21]. Moreover, by letting $K = 0, V = 1$ and $\beta \to 0$
in \([1,20]\), we obtain the estimate (1.4) in Theorem 1.1 of Wang and Xia in \([22]\).

**Acknowledgment.** We would like to thank Dr. Fanqi Zeng for helpful discussions which make the paper more readable.

2. **Proof of results**

2.1. **Proof of Theorem [1.1]** Using the divergence theorem, we have

\[
\int_M V f_{ij}^2 d\mu = \int_{\partial M} \frac{1}{2} V |\nabla f|^2 \nu d\sigma - \int_M e^\phi (e^{-\phi} V f_{ij}) f_i d\mu \\
= \int_{\partial M} \frac{1}{2} V |\nabla f|^2 \nu d\sigma - \int_M (f_{ij} V_j + V f_{ij,j} - V f_{ij} \phi_j) f_i d\mu \\
= \int_{\partial M} \frac{1}{2} V |\nabla f|^2 \nu d\sigma - \int_M \left[ \frac{1}{2} \nabla f \cdot \nabla V_i + V (\Delta_V f_i) f_i + V \text{Ric} \phi (\nabla f, \nabla f) \right] d\mu \\
= \int_{\partial M} \left( \frac{1}{2} V |\nabla f|^2 \nu - \frac{1}{2} |\nabla f|^2 V_i - V (\Delta_V f_i) f_i \right) d\sigma + \int_M \left[ \frac{1}{2} |\nabla f|^2 V_i f_i + V (\Delta_V f) f_i - V \text{Ric} \phi (\nabla f, \nabla f) \right] d\mu, 
\]

(2.1)

\[
-2 \int_M f f_{ij} V_{ij} d\mu = -2 \int_{\partial M} f f_{iu} V_i d\sigma + 2 \int_M [f_{ij} V_i f_j + f (\Delta f_i) V_i \\
+ f \text{Ric} \phi (\nabla f, \nabla f)] d\mu \\
= \int_{\partial M} [2 f (\Delta f) V_i - 2 f f_{iu} V_i + |\nabla f|^2 V_i] d\sigma \\
+ \int_M \left[ - |\nabla f|^2 \Delta_V - 2 f (\Delta f) (\Delta V) \right. \\
- 2 (\Delta f f_i V_i f_i + 2 f \text{Ric} \phi (\nabla f, \nabla f)] d\mu 
\]

(2.2)

and

\[
\int_M \frac{f^2}{V} V_{ij}^2 d\mu = \frac{1}{2} \int_{\partial M} \frac{f^2}{V} |\nabla V|^2 \nu d\sigma - \int_M \left[ \left( \frac{f^2}{V} \right) j V_i V_i \\
+ \frac{f^2}{V} (\Delta_V V_i) f_i + \frac{f^2}{V} \text{Ric} \phi (\nabla V, \nabla V) \right] d\mu \\
= \int_{\partial M} \left[ \frac{1}{2} \frac{f^2}{V} |\nabla V|^2 \nu - \frac{f^2}{V} (\Delta_V V_i) V_i \right] d\sigma \\
+ \int_M \left[ \frac{f^2}{V} (\Delta_V V_i) V_i + (\Delta_V V) \left( \frac{f^2}{V} V_i \right) f_i - \left( \frac{f^2}{V} \right) j V_i V_i \\
- \frac{f^2}{V} \text{Ric} \phi (\nabla V, \nabla V) \right] d\mu. 
\]

(2.3)
Let $A_{ij} = f_{ij} - \frac{f}{V}V_{ij}$. Then we have $A_{ij}^2 = f_{ij}^2 - 2fVf_{ij}V + \frac{f^2}{V^2}V_{ij}^2$ and $\text{tr}(A_{ij}) = \Delta f - \frac{f}{V}\Delta V$. It follows from (2.1)-(2.3) that

$$
\int_M V A_{ij}^2 d\mu = \int_{\partial M} \left[ \frac{1}{2}V|\nabla f|_\nu^2 + \frac{1}{2}V|\nabla f|^2 V_{ij} - V(\Delta f)_V \right] d\sigma - 2f f_{\nu V} V_i + \frac{f^2}{V} |\nabla V|_\nu^2 - \frac{f^2}{V}(\Delta f)_V V_i d\sigma
\]

$$
+ \int_M \left[ - \frac{1}{2}Vf_v^2 \Delta f - (\Delta f)_Vi f_i + (\Delta f)_V \left( \frac{f^2}{V} \right)_i V_i
- \left( \frac{f^2}{V} \right)_j V_{ij} V_i \right] d\mu + \int_M V \left[ (\Delta f - \frac{f}{V}\Delta f)^2 + \text{Ric}_V \left( \nabla \frac{f}{V}, \nabla \frac{f}{V} \right) \right] d\mu,
$$

where we note that $\nabla f - \frac{f}{V}\nabla V = V\nabla \frac{f}{V}$. It is easy to check that

$$
- (\Delta f V)_{g_{ij}} + V_{ij} \left( V\nabla \frac{f}{V}, V\nabla \frac{f}{V} \right) = - (\Delta f V) \left[ |\nabla f|^2 - \left( \frac{f^2}{V} \right)_i V_i \right]
+ V_{ij} f_i f_j - \left( \frac{f^2}{V} \right)_j V_{ij} V_i,
$$

which is equivalent to

$$
(\Delta f V) \left( \frac{f^2}{V} \right)_i V_i - \left( \frac{f^2}{V} \right)_j V_{ij} V_i = \left[ (\Delta f V)_{g_{ij}} + V_{ij} \left( V\nabla \frac{f}{V}, V\nabla \frac{f}{V} \right) \right]
+ (\Delta f V)|\nabla f|^2 - V_{ij} f_i f_j.
$$

Therefore, (2.4) becomes

$$
\int_M V A_{ij}^2 d\mu = \int_{\partial M} \left[ \frac{1}{2}V|\nabla f|_\nu^2 + \frac{1}{2}V|\nabla f|^2 V_{ij} - V(\Delta f)_V \right] d\sigma - 2f f_{\nu V} V_i + \frac{f^2}{V} |\nabla V|_\nu^2 - \frac{f^2}{V}(\Delta f)_V V_i d\sigma
\]

$$
+ \int_M \left[ \frac{1}{2}V|\nabla f|_\nu^2 + |\nabla f|^2 V_{ij} - V(\Delta f)_V \right] d\mu + \int_M V \left[ (\Delta f - \frac{f}{V}\Delta f)^2 - \text{Ric}_V \left( \nabla \frac{f}{V}, \nabla \frac{f}{V} \right) \right] d\mu
$$

\begin{align}
\label{2.6}
= \int_{\partial M} \left[ \frac{1}{2}V|\nabla f|_\nu^2 + |\nabla f|^2 V_{ij} - V(\Delta f)_V \right] d\sigma - 2f f_{\nu V} V_i + \frac{f^2}{V} |\nabla V|_\nu^2 - \frac{f^2}{V}(\Delta f)_V V_i d\sigma
\]

$$
+ \int_M V \left[ (\Delta f - \frac{f}{V}\Delta f)^2 - \text{Ric}_V \left( \nabla \frac{f}{V}, \nabla \frac{f}{V} \right) \right] d\mu.
$$
Using the fact that the boundary $\partial M$ are tangent to $\nu$.

Let $\{e_i\}_{i=1}^n$ be an orthonormal frame field along $\partial M$ such that $\{e_\alpha\}_{\alpha=1}^{n-1}$ are tangent to $\partial M$ and $e_n = \nu$ is normal to $\partial M$. Then we have the following:

$$|\nabla f|_\nu^2 = 2 \sum_{\alpha=1}^{n-1} f_\alpha f_{\alpha\nu} + 2 f_{\nu\nu} f_\nu = 2 \nabla f \nabla f_\nu - 2 II(\nabla f, \nabla f) + 2 f_{\nu\nu} f_\nu,$$

$$|\nabla f|^2 = |\nabla f|_\nu^2 + f_\nu^2,$$

$$\Delta_\phi f = \Delta_\phi f + H_\phi f_\nu + f_{\nu\nu},$$

$$f_{i\nu} V_i = f_{\alpha\nu} V_\alpha + f_{\nu\nu} V_\nu = \nabla V \nabla f_\nu - II(\nabla f, \nabla V) + f_{\nu\nu} V_\nu,$$

$$|\nabla V|_\nu^2 = 2 \sum_{\alpha=1}^{n-1} V_\alpha V_{\alpha\nu} + 2 V_{\nu\nu} V_\nu = 2 \nabla V \nabla V_\nu - 2 II(\nabla V, \nabla V) + 2 V_{\nu\nu} V_\nu,$$

$$\Delta_\phi V = \Delta_\phi V + H_\phi V_\nu + V_{\nu\nu}$$

and

$$f_\nu \nabla f \nabla V = f_\nu \nabla f \nabla V + V_\nu f_\nu^2.$$ 

Therefore, on $\partial M$,

$$\frac{1}{2} V |\nabla f|_\nu^2 + |\nabla f|^2 V_\nu - V(\Delta_\phi f) f_\nu + 2 f(\Delta_\phi f) V_\nu$$

$$- 2 f f_{\nu\nu} V_i + \frac{1}{2} f_\nu^2 |\nabla V|_\nu^2 - \frac{f_\nu^2}{V}(\Delta_\phi V) V_\nu - f_\nu \nabla f \nabla V$$

$$= - V(II - (\ln V)_\nu g)(\nabla f - \frac{f}{V} \nabla V, \nabla f - \frac{f}{V} \nabla V)$$

$$- VH_\phi \left( f_\nu - \frac{f}{V} V_\nu \right)^2 + 2 \frac{f}{V} V_\nu \nabla f \nabla V - \frac{f_\nu^2}{V_\nu} |\nabla V|^2$$

$$+ V \nabla f \nabla f_\nu - V f_\nu \Delta_\phi f + 2 f V_\nu \Delta_\phi f - 2 f \nabla V \nabla f_\nu$$

$$+ \frac{f_\nu^2}{V} V_\nu \nabla V_\nu - \frac{f_\nu^2}{V} V_\nu \Delta_\phi V - f_\nu \nabla f \nabla V.$$ 

Using the fact that the boundary $\partial M$ is closed, then we have

$$\int_{\partial M} \left( V \nabla f \nabla f_\nu - 2 V \nabla f V_\nu + \frac{f_\nu^2}{V} V_\nu \nabla V \right) d\sigma$$

$$= \int_{\partial M} \left( - V f_\nu \nabla f - V f_\nu \Delta_\phi f + 2 f_\nu \nabla f \nabla V + 2 f f_{\nu \nu} \Delta_\phi V$$

$$- V_\nu \nabla f \nabla V - V_\nu \frac{f_\nu^2}{V} \Delta_\phi V \right) d\sigma$$

(2.7)
and
\[
\int_{\partial M} \left[ \frac{1}{2} V |\nabla f|^2 + |\nabla f|^2 V_\nu - V(\Delta_\phi f)f_\nu + 2f(\Delta_\phi f) V_\nu \\
- 2ff_\nu V_i + \frac{1}{2} f_\nu^2 |\nabla V|^2 - f_\nu^2 (\Delta_\phi V) V_\nu - f_\nu \nabla f \nabla V \right] \, d\sigma
\]
\[
= \int_{\partial M} \left[ - V II^V \left( V \nabla \left( \frac{f}{V} \right), V \nabla \left( \frac{f}{V} \right) \right) - V H_\phi \left( f_\nu - (\ln V)\nu f \right)^2 \\
- 2V^2 \left( \frac{f}{V} \right)_\nu \left( \Delta_\phi f - \frac{\Delta_\phi V}{V} f \right) \right] \, d\sigma,
\]
which concludes the proof of Theorem 1.1.

2.2. Proof of Theorem 1.3. We consider the following Dirichlet boundary problem:
\[
\Delta_\phi f - \frac{\Delta_\phi V}{V} f = 1 \text{ on } M, \quad f = 0 \text{ on } \partial M.
\]
Then from (1.8), we obtain
\[
\int_{\partial M} V H_\phi f_\nu^2 \, d\sigma \leq \frac{m-1}{m} \int_M V \, d\mu. \tag{2.10}
\]
On the other hand,
\[
\int_M V \, d\mu = \int_M (V \Delta_\phi f - f \Delta_\phi V) \, d\mu \\
= \int_{\partial M} V f_\nu \, d\sigma \\
\leq \left( \int_{\partial M} \frac{V}{H_\phi} \, d\sigma \right)^{\frac{1}{2}} \left( \int_{\partial M} V H_\phi f_\nu^2 \, d\sigma \right)^{\frac{1}{2}}, \tag{2.11}
\]
which together with (2.10) gives the formula (1.9).

If the equality in (1.9) is attained, then from \( \Delta_\phi f - \frac{\Delta_\phi V}{V} f = 1 \) and (1.7), we have \( \Delta f - \frac{\Delta V}{V} f = \frac{n}{m} \) and \( V \nabla \phi \nabla \left( \frac{f}{V} \right) = -\frac{m-n}{m} \). If \( m > n \), then (1.7) shows that
\[
V \Delta f - f \Delta V + \frac{n}{m-n} V^2 \nabla \phi \nabla \left( \frac{f}{V} \right) = 0,
\]
which is equivalent to
\[
e^{-\frac{n}{m-n} \phi} \left[ e^{\frac{n}{m-n} \phi} V^2 \left( \frac{f}{V} \right) \right]_i = \left[ V^2 \left( \frac{f}{V} \right) \right]_i + \frac{n}{m-n} V^2 \nabla \phi \nabla \left( \frac{f}{V} \right) = 0, \tag{2.12}
\]
where we notice that $V \Delta f = \Delta f - \frac{f}{V} \Delta V - 2 \nabla V \nabla f$. Multiplying both sides of (2.12) with $\frac{f}{V} \text{e}^{\frac{\phi}{m-n}}$ gives

$$0 = \int_M f \left[ \text{e}^{\frac{\phi}{m-n}} V^2 \left( \frac{f}{V} \right) \right]_{\nu} dv_g = - \int_M V^2 \left| \nabla \frac{f}{V} \right|^2 \text{e}^{\frac{\phi}{m-n}} dv_g,$$

and then $f = \theta V$, where $\theta$ is a constant. This contradicts with $\Delta \phi f - \frac{\Delta \phi V f}{V} = 1$. Therefore, we have that if the equality in (1.9) is attained, then $m = n$ and $\phi$ must be constant. Using the similar conclusions as in Theorem 1.3 of [15], we complete the proof.

2.3. Proof of Theorem 1.4. We consider the following Dirichlet boundary problem:

$$\Delta \phi f - \frac{\Delta \phi V f}{V} = 1 \text{ on } M, \quad V \left( \frac{f}{V} \right)_{\nu} = c \text{ on } \partial M,$$

where

$$c = \frac{\int_M V d\mu}{\int_{\partial M} V d\sigma}.$$}

The existence and uniqueness of the solution to above equation is due to Fredholm alternative. Then from (1.8), we obtain

$$0 \leq \int_{\partial M} \left[ -c^2 V H_{\phi} - 2c(V \Delta f - f \Delta \phi V) \right] dv_{\sigma} + \frac{m-1}{m} \int_M V d\mu$$

$$= -c^2 \int_{\partial M} V H_{\phi} d\sigma + \frac{m-1}{m} \int_M V d\mu,$$

which gives (1.11).

Similarly, if the equality in (1.11) is attained the inequality in (1.10) is strict, then we have that $m = n$ and $\phi$ is a constant. Then, according to the arguments as in Theorem 1.7 of [15] finishes the proof.

2.4. Proof of Theorems 1.5-1.8. Firstly, we prove the following

**Proposition 2.1.** Let $V$ be a positive twice differentiable function on an $n$-dimensional closed Riemannian manifold $M$ with $V \overline{\text{Ric}}_V \geq -(n-1)K_1$, where $K_1$ is a nonnegative constant. If the symmetric $(2,0)$-tensor field $T$ defined on $M$ satisfies $\text{div} T = c \nabla (\text{tr} T)$, where $c$ is a constant, then

$$\frac{(nc-1)^2}{n^2} \int_M V (\text{tr} T - \text{tr} T^V)^2 d\mu \leq C_{n,K_1} \int_M V \left| T - \frac{1}{n} (\text{tr} T) g \right|^2 d\mu,$$

where $K_2 = \max(V \left| \nabla \phi \right|^2)$ for some nonnegative constant $K_2$,

$$\text{tr} T^V = \frac{\int_M V \text{tr} T d\mu}{\int_M V d\mu}.$$
and the constant $C_{n,K_1,K_2,\eta_1}$ is given by (1.12). Here $\eta_1$ is the first nonzero eigenvalue of problem $\Delta_\phi f - \frac{\Delta_\phi V}{V} f = -\eta_1 f$.

**Proof.** Let $f$ be a solution to the following problem:

$$\Delta_\phi f - \frac{\Delta_\phi V}{V} f = \text{tr} T - \text{tr} T^V$$
on M.

Denote by $A_{ij} = f_{ij} - \frac{f}{V} V_{ij}$ and $\hat{A}_{ij} = A_{ij} - \frac{\text{tr}(A_{ij})}{n} g_{ij}$ with $\text{tr}(A_{ij}) = \Delta f - \frac{1}{V} \Delta V$. Then, we have $\text{tr}(\hat{A}_{ij}) = 0$ and

$$\int_M V (\text{tr} T - \text{tr} T^V)^2 d\mu = \int_M (\text{tr} T - \text{tr} T^V) (V \Delta_\phi f - f \Delta_\phi V) d\mu = - \int_M (V \nabla f - f \nabla V) \nabla \text{tr} T d\mu. \quad (2.16)$$

Using $\nabla \text{tr} T = \frac{1}{n} \text{div} T$, we have $\frac{n}{n-1} \nabla \text{tr} T = \text{div} \hat{T}$, where $\hat{T}_{ij} = T_{ij} - \frac{1}{n} (\text{tr} T) g_{ij}$. Then (2.16) gives

$$\int_M V (\text{tr} T - \text{tr} T^V)^2 d\mu = - \frac{n}{nc-1} \int_M \hat{T}_{ij} (V f_i - f V_i) d\mu = \frac{n}{nc-1} \int_M V \hat{T}_{ij} B_{ij} d\mu = \frac{n}{nc-1} \int_M V \hat{T}_{ij} \hat{B}_{ij} d\mu \leq \frac{n}{nc-1} \left( \int_M V |\hat{T}_{ij}|^2 d\mu \right)^{\frac{1}{2}} \left( \int_M V |\hat{B}_{ij}|^2 d\mu \right)^{\frac{1}{2}}, \quad (2.17)$$

where $\hat{B}_{ij} = B_{ij} - \frac{1}{n} (\text{tr} B) g_{ij}$ and

$$B_{ij} = A_{ij} - \frac{1}{2} V \left[ \left( \frac{f}{V} \right)_i \phi_j + \phi_i \left( \frac{f}{V} \right)_j \right].$$

It is easy to check that $\text{tr} B = \Delta_\phi f - \frac{\Delta_\phi V}{V} f$. By virtue of the Cauchy inequality, we have

$$|\hat{B}_{ij}|^2 = |B_{ij}|^2 - \frac{1}{n} \left( \Delta_\phi f - \frac{\Delta_\phi V}{V} f \right)^2 \leq (1 + \delta) |A_{ij}|^2 + \left( 1 + \frac{1}{\delta} \right) V^2 |\nabla f|^2 |\nabla \phi|^2 - \frac{1}{n} \left( \Delta_\phi f - \frac{\Delta_\phi V}{V} f \right)^2, \quad (2.18)$$
where $\delta$ is a positive constant to be determined, and then
\[
\int_M V |\tilde{B}_{ij}|^2 \, d\mu \leq (1 + \delta) \int_M V |A_{ij}|^2 \, d\mu + \left(1 + \frac{1}{\delta}\right) \int_M V^3 \left|\nabla \frac{f}{V}\right|^2 \, d\mu
\]
\[
- \frac{1}{n} \int_M V \left(\Delta_\phi f - \frac{\Delta_\phi V f}{V}\right)^2 \, d\mu.
\] (2.19)

On other hand, applying (1.4) on closed manifold $M$, we obtain
\[
\int_M V |A_{ij}|^2 \, d\mu = \int_M V \left|\nabla^2 f - \frac{f}{V}\nabla^2 V\right|^2 \, d\mu
\]
\[
= \int_M \left[V \left(\Delta_\phi f - \frac{f}{V}\Delta_\phi V\right)^2 - V\text{Ric}_\phi \left(V\nabla \frac{f}{V}, V\nabla \frac{f}{V}\right)\right] \, d\mu.
\] (2.20)

Thus, (2.19) becomes
\[
\int_M V |\tilde{B}_{ij}|^2 \, d\mu \leq \left(\frac{n - 1}{n} + \delta\right) \int_M V \left(\Delta_\phi f - \frac{\Delta_\phi V f}{V}\right)^2 \, d\mu + \left(1 + \frac{1}{\delta}\right) K_2
\]
\[
+ (n - 1)(1 + \delta) K_1 \int_M V^2 \left|\nabla \frac{f}{V}\right|^2 \, d\mu.
\] (2.21)

Using the Rayleigh-Reitz principle, we have the first nonzero eigenvalue $\eta_1$ of $\Delta_\phi f - \frac{\Delta_\phi V f}{V}$ can also be characterized by
\[
\eta_1 = \inf_{f \in C^\infty(M)} \frac{\int_M V^2 \left|\nabla \frac{f}{V}\right|^2 \, d\mu}{\int_M V(f)^2 \, d\mu}.
\] (2.22)

Then,
\[
\int_M V^2 \left|\nabla \frac{f}{V}\right|^2 \, d\mu = - \int_M V \left(\frac{f}{V}\right) \left(\Delta_\phi f - \frac{\Delta_\phi V f}{V}\right) \, d\mu
\]
\[
\leq \left[\int_M V \left(\frac{f}{V}\right)^2 \, d\mu\right]^\frac{1}{2} \left[\int_M V \left(\Delta_\phi f - \frac{\Delta_\phi V f}{V}\right)^2 \, d\mu\right]^\frac{1}{2}
\]
\[
\leq \left[\frac{1}{\eta_1} \int_M V^2 \left|\nabla \frac{f}{V}\right|^2 \, d\mu\right]^\frac{1}{2} \left[\int_M V \left(\Delta_\phi f - \frac{\Delta_\phi V f}{V}\right)^2 \, d\mu\right]^\frac{1}{2},
\] (2.23)

which gives
\[
\int_M V^2 \left|\nabla \frac{f}{V}\right|^2 \, d\mu \leq \frac{1}{\eta_1} \int_M V \left(\Delta_\phi f - \frac{\Delta_\phi V f}{V}\right)^2 \, d\mu
\]
\[
= \frac{1}{\eta_1} \int_M V \left(\text{tr} T - \frac{\text{tr} V}{V}\right)^2 \, d\mu.
\] (2.24)
Inserting (2.24) into (2.21), we have
\[ \int_M V|\hat{B}_{ij}|^2 d\mu \leq \frac{1}{\eta_1} \left[ \frac{n-1}{n} \eta_1 + \delta \eta_1 + \left( 1 + \frac{1}{\delta} \right) K_2 ight. \\
+ (n-1)(1+\delta)K_1 \left] \int_M V(\text{tr}T - \text{tr}T^V)^2 d\mu \\
= \frac{1}{\eta_1} \left[ \frac{n-1}{n} \eta_1 + (n-1)K_1 + K_2 + [\eta_1 + (n-1)K_1]\delta \\
+ \frac{K_2}{\delta} \right] \int_M V(\text{tr}T - \text{tr}T^V)^2 d\mu. \tag{2.25} \]

Minimizing the \( \delta \) in (2.25) by taking
\[ \delta = \sqrt{\frac{K_2}{\eta_1 + (n-1)K_1}}, \]
we obtain
\[ \int_M V|\hat{B}_{ij}|^2 d\mu \leq C_{n,K_1,K_2,\eta_1} \int_M V(\text{tr}T - \text{tr}T^V)^2 d\mu. \tag{2.26} \]

Therefore, combining (2.26) with (2.17) concludes the proof of (2.15) and the proof of Proposition 2.1 is finished. \( \square \)

Since the ambient space is a space form, we have the well-known Codazzi equation:
\[ H_{ij,j} = H_i. \]

Hence, we complete the proof of Theorems 1.5 and 1.6 by taking \( c = 1 \) and \( c = 0 \) in the formula (2.15), respectively.

Using the second Bianchi identity, the Ricci curvature \( R_{ij} \) is related to the scalar curvature \( R \) by \( R_{ij,j} = \frac{1}{2} R_{ij} \). Thus, the proof of theorems 1.7 and 1.8 follows by taking \( c = \frac{1}{2} \) and \( c = 0 \) in the formula (2.15), respectively.

2.5. Proof of Theorem 1.9. (1) Let \( f \) be the solution of the following equation:
\[ \Delta_\phi f - \frac{\Delta_\phi V}{V} f = 0 \text{ on } M, \quad f = z \text{ on } \partial M, \]
where \( z \) satisfies \( \Delta_\phi z - \frac{\Delta_\phi V}{V} z = -\eta_1 \frac{z}{V} \), that is, \( z \) satisfies \( e^{\phi}\text{div}(e^{-\phi}V^2\nabla \frac{z}{V}) = -\eta_1 z \). Then from (1.8), we achieve
\[ 0 \geq \int_{\partial M} \left[ c_1 V^2 \left( \nabla \frac{z}{V} \right)^2 + c_2 V^3 \left( \frac{f}{V} \right) \left( \frac{z}{V} \right)^2 - 2\eta_1 V^2 \left( \frac{f}{V} \right) \left( \frac{z}{V} \right) \right] d\sigma. \tag{2.27} \]

Applying
\[ \int_{\partial M} V^2 \left( \nabla \frac{z}{V} \right)^2 d\sigma = -\int_{\partial M} V \frac{z}{V} (V \Delta_\phi z - \frac{z \Delta_\phi V}{V}) d\sigma \]
\[ = \eta_1 \int_{\partial M} V \left( \frac{z}{V} \right)^2 d\sigma \tag{2.28} \]
in (2.27) yields

\[
0 \geq \int_{\partial M} \left[ c_1 \eta_1 V \left( \frac{\bar{z}}{V} \right)^2 + c_2 V^3 \left( \left( \frac{\bar{f}}{\nu} \right)_\nu \right)^2 - 2 \eta_1 V^2 \left( \frac{\bar{f}}{\nu} \right)_\nu \bar{z} \right] d\sigma \\
\geq \left( c_1 \eta_1 - \frac{\eta_1^2}{c_2} \right) \int_{\partial M} V \left( \frac{\bar{z}}{V} \right)^2 d\sigma,
\]

(2.29)

which shows that \( \eta_1 \geq c_1 c_2 \) and the estimate (1.19) follows.

(2) Let \( f \) be the solution of the following equation:

\[
\Delta \phi f - \Delta \phi V f = 0 \text{ on } M, \quad f = z \text{ on } \partial M,
\]

where \( z \) satisfies \( \Delta \phi - \Delta \phi V z = -\eta_1 \bar{z} \), that is, \( z \) satisfies \( e^\phi \text{div}(e^{-\phi} V^2 \nabla \bar{z}) = -\eta_1 z \). Then from (1.8), we also have

\[
\begin{align*}
(m - 1)K \int_{\partial M} V^2 \left( \frac{\bar{f}}{\nu} \right)_\nu \bar{z} d\sigma \\
= (m - 1)K \int_M V^2 \left| \nabla \frac{\bar{f}}{\nu} \right|^2 d\mu \\
\geq \int_{\partial M} \left[ c_1 V^2 \left( \nabla \frac{\bar{z}}{V} \right)^2 + c_2 V^3 \left( \left( \frac{\bar{f}}{\nu} \right)_\nu \right)^2 - 2 \eta_1 V^2 \left( \frac{\bar{f}}{\nu} \right)_\nu \bar{z} \right] d\sigma \\
= \int_{\partial M} \left[ c_1 \eta_1 V \left( \frac{\bar{z}}{V} \right)^2 + c_2 V^3 \left( \left( \frac{\bar{f}}{\nu} \right)_\nu \right)^2 - 2 \eta_1 V^2 \left( \frac{\bar{f}}{\nu} \right)_\nu \bar{z} \right] d\sigma,
\end{align*}
\]

(2.30)

which gives

\[
0 \geq \int_{\partial M} \left[ c_1 \eta_1 V \left( \frac{\bar{z}}{V} \right)^2 + c_2 V^3 \left( \left( \frac{\bar{f}}{\nu} \right)_\nu \right)^2 - [2\eta_1 + (m - 1)K] V^2 \left( \frac{\bar{f}}{\nu} \right)_\nu \bar{z} \right] d\sigma \\
\geq c_2 \int_{\partial M} V^3 \left( \left( \frac{\bar{f}}{\nu} \right)_\nu \right)^2 d\sigma + c_1 \eta_1 \int_{\partial M} V \left( \frac{\bar{z}}{V} \right)^2 d\sigma \\
- [2\eta_1 + (m - 1)K] \left( \int_{\partial M} V \left( \frac{\bar{z}}{V} \right)^2 d\sigma \right)^{\frac{1}{2}} \left( \int_{\partial M} V^3 \left( \left( \frac{\bar{f}}{\nu} \right)_\nu \right)^2 d\sigma \right)^{\frac{1}{2}}.
\]

(2.31)

Therefore, we have proved that

\[
c_2 x^2 - [2\eta_1 + (m - 1)K] x + c_1 \eta_1 \leq 0,
\]

(2.32)

where

\[
x = \left[ \int_{\partial M} V^3 \left( \left( \frac{\bar{f}}{\nu} \right)_\nu \right)^2 d\sigma / \int_{\partial M} V \left( \frac{\bar{z}}{V} \right)^2 d\sigma \right]^{\frac{1}{2}}.
\]

Solving this quadratic inequality with respect to \( x \) gives

\[
x \leq \frac{1}{2c_2} \left( [2\eta_1 + (m - 1)K] + \sqrt{[2\eta_1 + (m - 1)K]^2 - 4c_1 c_2 \eta_1} \right).
\]

(2.33)
On the other hand, from the Rayleigh-Ritz formula (we may refer to [12,21]), we have

\[ \lambda_{1,\beta} \leq \beta \eta_1 + \frac{\int_M V^2 \left| \nabla \frac{f}{V} \right|^2 d\mu}{\int_{\partial M} V(\frac{f}{V})^2 d\sigma} \]

\[ = \beta \eta_1 + \frac{\int_{\partial M} V^2(\frac{f}{V})_\nu \frac{f}{V} d\sigma}{\int_{\partial M} V(\frac{f}{V})^2 d\sigma} \]

\[ \leq \beta \eta_1 + \left[ \int_{\partial M} V^3 \left( \left( \frac{f}{V} \right)_\nu \right)^2 d\sigma / \int_{\partial M} V \left( \frac{z}{V} \right)^2 d\sigma \right]^\frac{1}{2}. \quad (2.34) \]

Hence, the desired estimate (1.20) follows from (2.33) and (2.34).

2.6. Proof of Theorem 1.10

(1) Let \( f \) be the first eigenfunction corresponding to the first eigenvalue \( p_1 \) of the Steklov problem:

\[ \Delta \phi f - \Delta \phi V f = 0 \text{ on } M, \quad V \left( \frac{f}{V} \right)_\nu = p f \text{ on } \partial M. \]

Then from (1.8), we have

\[ (m - 1)Kp_1 \int_{\partial M} V \left( \frac{f}{V} \right)^2 d\sigma \]

\[ = (m - 1)K \int_{\partial M} V^2 \left( \frac{f}{V} \right)_\nu \frac{f}{V} d\sigma \]

\[ = (m - 1)K \int_M V^2 \left| \nabla \frac{f}{V} \right|^2 d\mu \]

\[ \geq \int_{\partial M} \left[ c_1 V^2 \left| \nabla \frac{f}{V} \right|^2 + c_2 V^3 \left( \left( \frac{f}{V} \right)_\nu \right)^2 + 2V^2 \left( \frac{f}{V} \right)_\nu \left( \overline{\nabla} f - \overline{\nabla} V f \right) \right] d\sigma \]

\[ = \int_{\partial M} \left[ c_2 p_1^2 V \left( \frac{f}{V} \right)^2 + (c_1 - 2p_1) V^2 \left| \nabla \frac{f}{V} \right|^2 \right] d\sigma \]

\[ > (c_1 - 2p_1) \int_{\partial M} V^2 \left| \nabla \frac{f}{V} \right|^2 d\sigma. \quad (2.35) \]

Applying the inequality

\[ \eta_1 \int_{\partial M} V \left( \frac{f}{V} \right)^2 d\sigma \leq \int_{\partial M} V^2 \left| \nabla \frac{f}{V} \right|^2 d\sigma, \quad (2.36) \]

in (2.35) gives

\[ 0 > [(c_1 - 2p_1)\eta_1 - (m - 1)Kp_1] \int_{\partial M} V \left( \frac{f}{V} \right)^2 d\sigma. \quad (2.37) \]

Thus, we obtain (1.21).

(2) Let \( f \) be a solution of the following equation:

\[ \Delta \phi f - \frac{\Delta \phi V}{V} f = 0 \text{ on } M, \quad -\beta \left( \overline{\nabla} f - \overline{\nabla} V f \right) + V \left( \frac{f}{V} \right)_\nu = \lambda \frac{f}{V} \text{ on } \partial M, \]
then on $\partial M$, we have

$$
\nabla \phi f - \frac{\Delta \phi f}{V} = \frac{1}{\beta} V \left(\frac{f}{V}\right)_\nu - \frac{\lambda_{1,\beta}}{\beta} f.
$$

Then (1.8) becomes

$$
0 \geq \int_{\partial M} \left[ c_1 V^2 \nabla^2 f \left(\frac{f}{V}\right)_\nu^2 + c_2 V^3 \left(\frac{f}{V}\right)_\nu^2 + 2 V^2 \left(\frac{f}{V}\right)_\nu \left(\nabla \phi f - \frac{\Delta \phi f}{V} \right) \right] d\sigma
$$

$$
= \int_{\partial M} \left[ \left( c_2 + \frac{2}{\beta} \right) V^3 \left(\frac{f}{V}\right)_\nu^2 - \frac{c_1 + 2 \lambda_{1,\beta}}{\beta} V^2 \left(\frac{f}{V}\right)_\nu + \frac{c_1 \lambda_{1,\beta}}{\beta} V f \left(\frac{f}{V}\right)^2 \right] d\sigma.
$$

(2.38)

Inserting the inequality

$$
\left( c_2 + \frac{2}{\beta} \right) V^3 \left(\frac{f}{V}\right)_\nu^2 - \frac{c_1 + 2 \lambda_{1,\beta}}{\beta} V^2 \left(\frac{f}{V}\right)_\nu \geq - \frac{(c_1 + 2 \lambda_{1,\beta})^2}{4 \beta (2 + \beta c_2)} V f \left(\frac{f}{V}\right)^2
$$

into (2.38) yields

$$
0 \geq \left[ \frac{c_1 \lambda_{1,\beta}}{\beta} - \frac{(c_1 + 2 \lambda_{1,\beta})^2}{4 \beta (2 + \beta c_2)} \right] \int_{\partial M} V f \left(\frac{f}{V}\right)^2 d\sigma,
$$

(2.39)

which shows that

$$
4(2 + \beta c_2) c_1 \lambda_{1,\beta} - (c_1 + 2 \lambda_{1,\beta})^2 \leq 0.
$$

(2.40)

Solving this quadratic inequality with respect to $\lambda_{1,\beta}$, we have that either

$$
\lambda_{1,\beta} \geq \frac{c_1}{2} \left( 1 + \beta c_2 + \sqrt{\beta^2 c_2^2 + 2 \beta c_2} \right),
$$

(2.41)

or

$$
\lambda_{1,\beta} \leq \frac{c_1}{2} \left( 1 + \beta c_2 - \sqrt{\beta^2 c_2^2 + 2 \beta c_2} \right).
$$

(2.42)

Note that

$$
0 = - \int_M V \frac{f}{V} \left( \nabla \phi f - \frac{\Delta \phi f}{V} \right) d\mu
$$

$$
= \int_M V^2 \left| \nabla \frac{f}{V} \right|^2 d\mu - \int_{\partial M} V^2 \left(\frac{f}{V}\right)_\nu \frac{f}{V} d\sigma
$$

$$
= \int_M V^2 \left| \nabla \frac{f}{V} \right|^2 d\mu - \lambda_{1,\beta} \int_{\partial M} V \left(\frac{f}{V}\right)^2 d\sigma + \beta \int_{\partial M} V^2 \left| \nabla \frac{f}{V} \right|^2 d\sigma.
$$

(2.43)

By the Rayleigh-Ritz formula (or see (2.34) with $\beta = 0$), we have

$$
\int_M V^2 \left| \nabla \frac{f}{V} \right|^2 d\mu \geq p_1 \int_{\partial M} V \left(\frac{f}{V}\right)^2 d\sigma
$$

(2.44)

and

$$
\int_{\partial M} V^2 \left| \nabla \frac{f}{V} \right|^2 d\sigma \geq \eta_1 \int_{\partial M} V \left(\frac{f}{V}\right)^2 d\sigma.
$$

(2.45)
Inserting (2.44) and (2.45) into (2.43) gives
\[
\lambda_{1,\beta} \geq p_1 + \beta \eta_1 > \frac{c_1}{2} + \beta c_1 c_2,
\]
(2.46)
which shows that (2.42) does not occur. Therefore, we have that (1.22) holds and the proof of Theorem 1.10 is completed.

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