Boundedness of Pseudodifferential Operators on Banach Function Spaces

Alexei Yu. Karlovich

To Professor Antônio Ferreira dos Santos

Abstract. We show that if the Hardy-Littlewood maximal operator is bounded on a separable Banach function space $X(\mathbb{R}^n)$ and on its associate space $X'(\mathbb{R}^n)$, then a pseudodifferential operator $\text{Op}(a)$ is bounded on $X(\mathbb{R}^n)$ whenever the symbol $a$ belongs to the Hörmander class $S^{n(\rho^{-1})}_{\rho,\delta}$ with $0 < \rho \leq 1$, $0 \leq \delta < 1$ or to the the Miyachi class $S^{n(\rho^{-1})}_{\rho,\delta}(\kappa,n)$ with $0 \leq \delta \leq \rho \leq 1$, $0 \leq \delta < 1$, and $\kappa > 0$. This result is applied to the case of variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$.

1. Introduction

We denote the usual operators of first order partial differentiation on $\mathbb{R}^n$ by $\partial_{x_j} := \partial/\partial x_j$. For every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ with non-negative integers $\alpha_j$, we write $\partial^\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$. Further, put $|\alpha| := \alpha_1 + \cdots + \alpha_n$, and for each vector $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, define $\xi^\alpha := \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$. Let $\langle \cdot, \cdot \rangle$ stand for the scalar product in $\mathbb{R}^n$ and $|\xi| := \sqrt{\langle \xi, \xi \rangle}$ for $\xi \in \mathbb{R}^n$.

Let $C_0^\infty(\mathbb{R}^n)$ denote the set of all infinitely differentiable functions with compact support. Recall that, given $u \in C_0^\infty(\mathbb{R}^n)$, a pseudodifferential operator $\text{Op}(a)$ is formally defined by the formula

$$\text{(Op}(a)u)(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} a(x,\xi)u(y)e^{i\langle x-y,\xi \rangle} dy,$$

where the symbol $a$ is assumed to be bounded in both the spatial variable $x$ and the frequency variable $\xi$, and satisfies certain regularity conditions.

An example of symbols one might consider is the Hörmander class $S^m_{\rho,\delta}$ introduced in [16] and consisting of $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ with

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|} \quad (x,\xi \in \mathbb{R}^n),$$

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where
\[ m \in \mathbb{R}, \quad 0 \leq \delta, \rho \leq 1 \]
and the positive constants \( C_{\alpha,\beta} \) depend only on \( \alpha \) and \( \beta \). Along with the Hörmander class \( S^m_{\rho,\delta} \), we will consider the generalized Hörmander class \( S^m_{\rho,\delta}(\mathcal{Z}, \mathcal{Z}') \) introduced by Miyachi [29]. We will call \( S^m_{\rho,\delta}(\mathcal{Z}, \mathcal{Z}') \) the Miyachi class of symbols. Its quite technical definition is postponed to Subsection 2.1. Here we only note that symbols in the Miyachi classes may lie beyond \( C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \) (that is, they are non-smooth, in general).

Let \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \). For a cube \( Q \subset \mathbb{R}^n \), put
\[ f_Q := \frac{1}{|Q|} \int_Q f(x) \, dx. \]
Here, and throughout, cubes will be assumed to have their sides parallel to the coordinate axes and \( |Q| \) will denote the volume of \( Q \). The Fefferman-Stein sharp maximal operator \( f \mapsto f^\# \) is defined by
\[ f^\#(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx \quad (x \in \mathbb{R}^n), \]
where the supremum is taken over all cubes \( Q \) containing \( x \). Let \( 1 \leq q < \infty \). Given \( f \in L^q_{\text{loc}}(\mathbb{R}^n) \), the \( q \)-th maximal operator is defined by
\[ (M_q f)(x) := \sup_{Q \ni x} \left( \frac{1}{|Q|} \int_Q |f(y)|^q \, dy \right)^{1/q} \quad (x \in \mathbb{R}^n), \]
where the supremum is taken over all cubes \( Q \) containing \( x \). For \( q = 1 \) this is the usual Hardy-Littlewood maximal operator, which will be denoted by \( M \).

The boundedness of pseudodifferential operators with smooth and non-smooth symbols on the classical Lebesgue spaces \( L^p(\mathbb{R}^n) \) was studied by many authors. We refer to the monographs by Coifman and Meyer [8], Kumano-go [21], Journé [18], Taylor [37], Stein [36], Hörmander [17], Abels [1] and also to the papers by Miyachi [29] and Ashino, Nagase, and Vaillancourt [5] for corresponding results and further references.

Miller [27] proved the boundedness of pseudodifferential operators with symbols \( a \in S^0_{1,0} \) on the weighted Lebesgue spaces \( L^p(\mathbb{R}^n, w) \) with \( 1 < p < \infty \) and Muckenhoupt weights \( w \in A_p(\mathbb{R}^n) \). One of the key ingredients in his proof was the pointwise estimate
\[ (\text{Op}(a)f)^\#(x) \leq C_q(M_qf)(x) \quad (x \in \mathbb{R}^n), \quad (1) \]
where \( q \in (1, \infty) \) and \( C_q > 0 \) is independent of \( f \in C^\infty_0(\mathbb{R}^n) \). Another ingredients are the Fefferman-Stein inequality (see e.g. [15] Theorem 5) and self-improving properties of Muckenhoupt weights. Further, estimate (1) and the boundedness results for \( \text{Op}(a) \) on \( L^p(\mathbb{R}^n, w) \) with \( p \in (1, \infty) \) and \( w \in A_p(\mathbb{R}^n) \) were extended to other classes of smooth and non-smooth symbols. We refer, for instance, to the works by Nishigaki [33], Yabuta [38, 39, 40, 41], Miyachi and Yabuta [30], Álvarez and Hounie [2], Álvarez, Hounie, and Pérez [3], Michalowski, Rule, and Staubach [26] and the references therein.
Rabinovich and Samko [35, Theorem 5.1] proved the boundedness of pseudodifferential operators with symbols $a \in S_{1,0}^0$ on so-called variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ (see Subsection 3.1). Their proof did not rely on (1). Instead, they obtained another (more precise) pointwise estimate for $(\text{Op}(a)f)^\#(x)$ in the spirit of [4]. Recently the author and Spitkovsky [19, Theorem 1.2] proved the boundedness of $\text{Op}(a)$ on variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ for the symbols $a \in S_{\rho,\delta}^m(\rho - 1)$ with $0 < \rho \leq 1$ and $0 \leq \delta < 1$. That proof relies on (1) (obtained in [26]), on the Fefferman-Stein inequality for variable Lebesgue spaces, and on a certain self-improving property of the Hardy-Littlewood maximal function on $L^{p(\cdot)}(\mathbb{R}^n)$.

The aim of the present paper is to extend the results of [19, 35] to the case of so-called Banach function spaces. Our proof is based on estimate (1), on the Fefferman-Stein inequality for Banach function spaces proved recently by Lerner [23], and on a self-improving property of the Hardy-Littlewood maximal function on Banach function spaces proved by Lerner and Pérez [25]. Note that our results are true for all symbols classes admitting estimate (1).

We choose here the classical Hörmander classes $S_{\rho,\delta}^m$ of smooth symbols and the Miyachi classes $S_{\rho,\delta}^m(\rho',\kappa')$ of non-smooth symbols just as an illustration of the fact that the assumptions on smoothness of symbols imposed in [19, 35] can be essentially relaxed.

The set of all Lebesgue measurable complex-valued functions on $\mathbb{R}^n$ is denoted by $M$. Let $M^+$ be the subset of functions in $M$ whose values lie in $[0, \infty]$. The characteristic function of a measurable set $E \subset \mathbb{R}^n$ is denoted by $\chi_E$ and the Lebesgue measure of $E$ is denoted by $|E|$. The set of all Lebesgue measurable complex-valued functions on $\mathbb{R}^n$ is denoted by $M$. Let $M^+$ be the subset of functions in $M$ whose values lie in $[0, \infty]$. The characteristic function of a measurable set $E \subset \mathbb{R}^n$ is denoted by $\chi_E$ and the Lebesgue measure of $E$ is denoted by $|E|$.

**Definition 1.1 ([6, Chap. 1, Definition 1.1]).** A mapping $\rho : M^+ \to [0, \infty]$ is called a Banach function norm if, for all functions $f, g, f_n (n \in \mathbb{N})$ in $M^+$, for all constants $a \geq 0$, and for all measurable subsets $E$ of $\mathbb{R}^n$, the following properties hold:

(A1) \[ \rho(f) = 0 \iff f = 0 \text{ a.e.}, \quad \rho(af) = a\rho(f), \quad \rho(f + g) \leq \rho(f) + \rho(g), \]

(A2) \[ 0 \leq g \leq f \text{ a.e.} \implies \rho(g) \leq \rho(f) \quad (\text{the lattice property}), \]

(A3) \[ 0 \leq f_n \uparrow f \text{ a.e.} \implies \rho(f_n) \uparrow \rho(f) \quad (\text{the Fatou property}), \]

(A4) \[ |E| < \infty \implies \rho(\chi_E) < \infty, \]

(A5) \[ |E| < \infty \implies \int_E f(x) \, dx \leq C_E \rho(f) \]

with $C_E \in (0, \infty)$ which may depend on $E$ and $\rho$ but is independent of $f$.

When functions differing only on a set of measure zero are identified, the set $X(\mathbb{R}^n)$ of all functions $f \in M$ for which $\rho(|f|) < \infty$ is called a Banach function space. For each $f \in X(\mathbb{R}^n)$, the norm of $f$ is defined by

\[ \|f\|_{X(\mathbb{R}^n)} := \rho(|f|). \]

The set $X(\mathbb{R}^n)$ under the natural linear space operations and under this norm becomes a Banach space (see [6, Chap. 1, Theorems 1.4 and 1.6]).
If $\rho$ is a Banach function norm, its associate norm $\rho'$ is defined on $\mathcal{M}^+$ by
\[
\rho'(g) := \sup \left\{ \int_{\mathbb{R}^n} f(x)g(x)\,dx : f \in \mathcal{M}^+, \rho(f) \leq 1 \right\}, \quad g \in \mathcal{M}^+.
\]
It is a Banach function norm itself [6, Chap. 1, Theorem 2.2]. The Banach function space $X'(\mathbb{R}^n)$ determined by the Banach function norm $\rho'$ is called the associate space (Köthe dual) of $X(\mathbb{R}^n)$. The Lebesgue space $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, are the the archetypical example of Banach function spaces. Other classical examples of Banach function spaces are Orlicz spaces, rearrangement-invariant spaces, and variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$.

Note that we do not assume that $X(\mathbb{R}^n)$ is rearrangement-invariant (see [6, Chap. 2]). Therefore, we are not allowed to use the interpolation theory to study the boundedness of $\text{Op}(a)$ on $X(\mathbb{R}^n)$.

**Theorem 1.2 (Main result).** Let $X(\mathbb{R}^n)$ be a separable Banach function space such that the Hardy-Littlewood maximal operator $M$ is bounded on $X(\mathbb{R}^n)$ and on its associate space $X'(\mathbb{R}^n)$. If $a$ belongs to one of the following symbol classes:

(a) the Hörmander class $S_{\rho,\delta}^{n(\rho-1)}$ with $0 < \rho \leq 1$ and $0 \leq \delta < 1$;
(b) the Miyachi class $S_{\rho,\delta}^{n(\rho-1)}(\varkappa, n)$ with $0 \leq \delta \leq \rho \leq 1$, $0 \leq \delta < 1$, and $\varkappa > 0$;

then $\text{Op}(a)$ extends to a bounded operator on $X(\mathbb{R}^n)$.

The paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.2. First, we collect the main ingredients. We give the precise definition of the Miyachi class $S_{\rho,\delta}^{m(\rho-1)}(\varkappa, \varkappa')$ in Subsection 2.1. The Fefferman-Stein inequality for Banach function spaces is stated in Section 2.3. A certain self-improving property of the Hardy-Littlewood maximal operator on Banach function spaces is discussed in Subsection 2.4. Precise assumptions on our symbols guaranteeing (1) are stated in Subsection 2.5. Finally, we assemble these ingredients in Subsection 2.6 and prove Theorem 1.2.

In Section 3 we apply Theorem 1.2 to the case of variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$. In Subsection 3.1 we recall the definition and some basic properties of variable Lebesgue spaces. In Subsection 3.2 we discuss the boundedness of the Hardy-Littlewood maximal operator on $L^{p(\cdot)}(\mathbb{R}^n)$. In particular, we recall that $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ if and only if $M$ is bounded on its associate space. This allows us to simplify little bit the formulation of Theorem 1.2 for $L^{p(\cdot)}(\mathbb{R}^n)$ in Subsection 3.3.

## 2. Proof of the main result

### 2.1. The Miyachi class

The following class of symbols was introduced by Miyachi [29] (see also [28, 30]). If $h \in \mathbb{R}^n$ and $f$ is a function on $\mathbb{R}^n$, then the first and the second
differences are denoted by
\[
\Delta_x(h)f(x) := f(x + h) - f(x), \\
\Delta_x^2(h)f(x) := f(x + 2h) - 2f(x + h) + f(x).
\]

Let
\[
m \in \mathbb{R}, \quad 0 \leq \delta, \rho \leq 1, \quad \varkappa > 0, \quad \varkappa' > 0.
\]

Let \( k \) and \( k' \) be nonnegative integers satisfying
\[
k < \varkappa \leq k + 1, \quad k' < \varkappa' \leq k' + 1.
\]

The Miyachi class \( S_{\rho,\delta}^m(\varkappa, \varkappa') \) consists of all functions \( a \) on \( \mathbb{R}^n \times \mathbb{R}^n \) such that the derivatives \( \partial_x^\beta \partial_\xi^\alpha a(x, \xi) \) exist in the classical sense for \( |\beta| \leq k \) and \( |\alpha| \leq k' \) and the following four conditions are fulfilled:

(i) if \(|\beta| \leq k \) and \(|\alpha| \leq k' \), then
\[
|\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \leq A(1 + |\xi|)^{m+|\beta|+\delta-\rho|\alpha|};
\]

(ii) if \(|\beta| = k \) and \(|\alpha| \leq k' \), \( h \in \mathbb{R}^n \), and \(|h| \leq (1 + |\xi|)^{-\delta} \), then
\[
|\Delta_x^2(h) \partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \leq A(1 + |\xi|)^{m+|\beta|+\delta-\rho|\alpha|}|h|^{\varkappa-k};
\]

(iii) if \(|\beta| \leq k \) and \(|\alpha| = k' \), \( \eta \in \mathbb{R}^n \) and \(|\eta| \leq (1 + |\xi|)^{\rho}/4 \), then
\[
|\Delta_\xi^2(\eta) \partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \leq A(1 + |\xi|)^{m+|\beta|+\delta-\rho|\alpha|+\varkappa' - k'}|\eta|^{\varkappa'-k'};
\]

(iv) if \(|\beta| = k \) and \(|\alpha| = k' \), \( h, \eta \in \mathbb{R}^n \), and \(|h| \leq (1 + |\xi|)^{-\delta}, |\eta| \leq (1 + |\xi|)^{\rho}/4 \), then
\[
|\Delta_x^2(h) \Delta_\xi^2(\eta) \partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \leq A(1 + |\xi|)^{m+|\beta|+\delta-\rho|\alpha|+\varkappa' - k'}|h|^{\varkappa-k}|\eta|^{\varkappa'-k'}.
\]

Here the constant \( A \) is independent of the multi-indices \( \alpha, \beta \) and the variables \( x, \xi, h, \eta \in \mathbb{R}^n \). The smallest such constant is denoted by \( \|a\|_{m,\rho,\delta,\varkappa,\varkappa'} \).

It is not difficult to see that if \( \varkappa_2 \leq \varkappa_1 \) and \( \varkappa'_2 \leq \varkappa'_1 \), then
\[
S_{\rho,\delta}^m(\varkappa_1, \varkappa'_1) \subset S_{\rho,\delta}^m(\varkappa_2, \varkappa'_2)
\]
and
\[
\|a\|_{m,\rho,\delta,\varkappa_2,\varkappa'_2} \leq \text{const} \|a\|_{m,\rho,\delta,\varkappa_1,\varkappa'_1}.
\]

If \( \varkappa \) (resp. \( \varkappa' \)) is not integer, then \( \Delta_x^2(h) \) (resp. \( \Delta_\xi^2(\eta) \)) can be replaced by \( \Delta_x^1(h) \) (resp. \( \Delta_\xi^1(\eta) \)). It should also be remarked that the assumptions \(|h| \leq (1 + |\xi|)^{-\delta} \) and \(|\eta| \leq (1 + |\xi|)^{\rho}/4 \) can be replaced by \( h \in \mathbb{R}^n \) and \(|\eta| \leq (1 + |\xi|)/4 \) if one modifies the constant \( A \).

2.2. Density of smooth compactly supported functions

**Lemma 2.1.** The set \( C_0^\infty(\mathbb{R}^n) \) is dense in a separable Banach function space \( X(\mathbb{R}^n) \).

The proof is standard. For details, see [20, Lemma 2.10(b)], where this fact is proved for \( n = 1 \). The proof for arbitrary \( n \) is a minor modification of that one.
2.3. The Fefferman-Stein inequality for Banach function spaces

Let $S_0(\mathbb{R}^n)$ be the space of all measurable functions $f$ on $\mathbb{R}^n$ such that
$$|\{x \in \mathbb{R}^n : |f(x)| > \lambda\}| < \infty$$
for any $\lambda > 0$. Chebyshev’s inequality
$$|\{x \in \mathbb{R}^n : |f(x)| > \lambda\}| \leq \frac{1}{\lambda^q} \int_{\mathbb{R}^n} |f(x)|^q \, dx$$
holds for every $q \in (0, \infty)$ and $\lambda > 0$. In particular, it implies that
$$\bigcup_{q \in (0, \infty)} L^q(\mathbb{R}^n) \subset S_0(\mathbb{R}^n).$$

It is obvious that $f^#$ is pointwise dominated by $Mf$. Hence, by Axiom (A2),
$$\|f^#\|_{X(\mathbb{R}^n)} \leq \text{const}\|f\|_{X(\mathbb{R}^n)} \quad \text{for } f \in X(\mathbb{R}^n)$$
whenever $M$ is bounded on $X(\mathbb{R}^n)$. The converse inequality for Lebesgue spaces $L^p(\mathbb{R}^n)$, $1 < p < \infty$, was proved by Fefferman and Stein (see [15, Theorem 5] and also [36, Chap. IV, Section 2.2]). The following extension of the Fefferman-Stein inequality to Banach function spaces was proved in [23, Corollary 4.2].

**Theorem 2.2 (Lerner).** Let $M$ be bounded on a Banach function space $X(\mathbb{R}^n)$. Then $M$ is bounded on its associate space $X'(\mathbb{R}^n)$ if and only if there exists a constant $C_\# > 0$ such that, for all $f \in S_0(\mathbb{R}^n)$,
$$\|f\|_{X(\mathbb{R}^n)} \leq C_\# \|f^#\|_{X(\mathbb{R}^n)}.$$  

2.4. Self-improving property of maximal operators on Banach function spaces

If $1 < q < \infty$, then from the Hölder inequality one can immediately get that
$$\langle Mf \rangle(x) \leq \langle M_qf \rangle(x) \quad (x \in \mathbb{R}^n).$$
Thus, the boundedness of any $M_q$, $1 < q < \infty$, on a Banach function space $X(\mathbb{R}^n)$ immediately implies the boundedness of $M$. A partial converse of this fact, called a self-improving property of the Hardy-Littlewood maximal operator, is also true. It was proved in [25, Corollary 1.3] (see also [24] for another proof) in a more general setting of quasi-Banach function spaces.

**Theorem 2.3 (Lerner-Pérez).** Let $X(\mathbb{R}^n)$ be a Banach function space. Then $M$ is bounded on $X(\mathbb{R}^n)$ if and only if $M_q$ is bounded on $X(\mathbb{R}^n)$ for some $q \in (1, \infty)$.

2.5. The crucial pointwise estimate

**Theorem 2.4.** If $a$ belongs to one of the following symbol classes:

(a) the Hömander class $S^{n(\rho-1)}_{\rho,\delta}$ with $0 < \rho \leq 1$ and $0 \leq \delta < 1$;

(b) the Miyachi class $S^{n(\rho-1)}_{\rho,\delta}(\varpi,n)$ with $0 \leq \delta \leq \rho \leq 1$, $0 \leq \delta < 1$, and $\varpi > 0$;
then for every \( q \in (1, \infty) \) there exists a constant \( C_q > 0 \) such that
\[
(Op(a)f)^\#(x) \leq C_q(M_qf)(x) \quad (x \in \mathbb{R}^n)
\]
for all \( f \in C_0^\infty(\mathbb{R}^n) \).

Part (a) was recently proved by Michalowski, Rule, and Staubach \[26,\ Theorem 3.3\]. Their estimate generalizes the pointwise estimate by Miller \[27,\ Theorem 2.8\] for \( a \in S_{0,0}^1 \) and by Álvarez and Hounie \[2,\ Theorem 4.1\] for \( a \in S_{m,\rho}^\sigma \) with the parameters satisfying \( 0 < \delta \leq \rho \leq 1/2 \) and \( m \leq n(\rho - 1) \).

Part (b) follows from the estimate by Miyachi and Yabuta \[30,\ Theorem 2.4\].

**Corollary 2.5.** If the conditions of Theorem 2.4 are fulfilled, then \( Op(a)f \in S_0(\mathbb{R}^n) \) for every \( f \in C_0^\infty(\mathbb{R}^n) \).

**Proof.** By using the well-known \( L^p \)-estimates for the sharp maximal function (see \[15,\ Theorem 5\]) and for the maximal function \( M_q \), one can show that if (2) holds for all \( f \in C_0^\infty(\mathbb{R}^n) \), then \( Op(a) \) extends to a bounded operator on \( L^p(\mathbb{R}^n) \) for \( q < p < \infty \). In particular, this implies that \( Op(a)f \in S_0(\mathbb{R}^n) \) for every function \( f \in C_0^\infty(\mathbb{R}^n) \). \( \square \)

**2.6. Proof of Theorem 1.2**

The presented proof is an adaptation of the proof of \[19,\ Theorem 1.2\]. Its idea goes back to Miller \[27\]. Suppose \( f \in C_0^\infty(\mathbb{R}^n) \). Then \( Op(a)f \in S_0(\mathbb{R}^n) \) in view of Corollary 2.5. By Lerner’s theorem (Theorem 2.2), there exists a constant \( C_\# > 0 \) such that
\[
\| Op(a)f \|_{X(\mathbb{R}^n)} \leq C_\# \| (Op(a)f)^\# \|_{X(\mathbb{R}^n)}
\]
Further, by the crucial pointwise estimate (Theorem 2.4), for every \( q \in (1, \infty) \), there is a constant \( C_q > 0 \) such that
\[
(Op(a)f)^\#(x) \leq C_q(M_qf)(x) \quad (x \in \mathbb{R}^n)
\]
Hence, by Axioms (A1) and (A2),
\[
\|(Op(a)f)^\#\|_{X(\mathbb{R}^n)} \leq C_q \| M_qf \|_{X(\mathbb{R}^n)}.
\]
On the other hand, since \( M \) is bounded on \( X(\mathbb{R}^n) \), by the Lerner-Pérez theorem (Theorem 2.3), there is a constant exponent \( q_0 \in (1, \infty) \) and a constant \( C_{q_0}' > 0 \) such that
\[
\| M_{q_0}f \|_{X(\mathbb{R}^n)} \leq C_{q_0}' \| f \|_{X(\mathbb{R}^n)}
\]
Thus, combining (3)–(5), we arrive at
\[
\| Op(a)f \|_{X(\mathbb{R}^n)} \leq C_\# C_{q_0} C_{q_0}' \| f \|_{X(\mathbb{R}^n)}
\]
for all \( f \in C_0^\infty(\mathbb{R}^n) \). It remains to recall that, in view of Lemma 2.1, \( C_0^\infty(\mathbb{R}^n) \) is dense in \( X(\mathbb{R}^n) \) whenever \( X(\mathbb{R}^n) \) is separable. Thus, \( Op(a) \) extends to a bounded operator on the whole space \( X(\mathbb{R}^n) \) by continuity. \( \square \)
3. Pseudodifferential operators on variable Lebesgue spaces

3.1. Variable Lebesgue spaces

Let $p : \mathbb{R}^n \to [1, \infty]$ be a measurable a.e. finite function. By $L^{p(\cdot)}(\mathbb{R}^n)$ we denote the set of all complex-valued functions $f$ on $\mathbb{R}^n$ such that

$$I_{p(\cdot)}(f/\lambda) := \int_{\mathbb{R}^n} |f(x)/\lambda|^{p(x)} dx < \infty$$

for some $\lambda > 0$. This set becomes a Banach function space when equipped with the norm

$$\|f\|_{p(\cdot)} = \inf \{\lambda > 0 : I_{p(\cdot)}(f/\lambda) \leq 1\}.$$

It is easy to see that if $p$ is constant, then $L^{p(\cdot)}(\mathbb{R}^n)$ is nothing but the standard Lebesgue space $L^p(\mathbb{R}^n)$. The space $L^{p(\cdot)}(\mathbb{R}^n)$ is referred to as a variable Lebesgue space.

We will always suppose that

$$1 < p_- := \text{ess inf}_{x \in \mathbb{R}^n} p(x), \quad \text{ess sup}_{x \in \mathbb{R}^n} p(x) =: p_+ < \infty. \quad (6)$$

Under these conditions, the space $L^{p(\cdot)}(\mathbb{R}^n)$ is separable and reflexive, and its associate space is isomorphic to $L^{p'(\cdot)}(\mathbb{R}^n)$, where

$$1/p(x) + 1/p'(x) = 1 \quad (x \in \mathbb{R}^n)$$

(see e.g. [9, Chap. 2] or [14, Chap. 3]).

3.2. The Hardy-Littlewood maximal function on variable Lebesgue spaces

By $\mathcal{M}(\mathbb{R}^n)$ denote the set of all measurable functions $p : \mathbb{R}^n \to [1, \infty]$ such that (6) holds and the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Assume that (6) is fulfilled. Diening [12] proved that if $p$ satisfies

$$|p(x) - p(y)| \leq \frac{c}{\log(e + 1/|x - y|)} \quad (x, y \in \mathbb{R}^n) \quad (7)$$

for some $c > 0$ independent of $x, y \in \mathbb{R}^n$ and $p$ is constant outside some ball, then $p \in \mathcal{M}(\mathbb{R}^n)$. Further, the behavior of $p$ at infinity was relaxed by Cruz-Uribe, Fiorenza, and Neugebauer [10] [11], where it was shown that if $p$ satisfies (7) and there exists a $p_\infty > 1$ such that

$$|p(x) - p_\infty| \leq \frac{c}{\log(e + |x|)} \quad (x \in \mathbb{R}^n) \quad (8)$$

with $c > 0$ independent of $x \in \mathbb{R}^n$, then $p \in \mathcal{M}(\mathbb{R}^n)$. Following [14, Section 4.1], we will say that if conditions (7)–(8) are fulfilled, then $p$ is globally log-Hölder continuous. The class of all globally log-Hölder continuous exponents will be denoted by $P^{\log}(\mathbb{R}^n)$.

Conditions (7) and (8) are optimal for the boundedness of $M$ in the pointwise sense; the corresponding examples are contained in [34] and [10].
However, neither (7) nor (8) is necessary for \( p \in \mathcal{M}(\mathbb{R}^n) \). Nekvinda [31] proved that if \( p \) satisfies (6)–(7) and

\[
\int_{\mathbb{R}^n} |p(x) - p_{\infty}|^{1/(p(x)-p_{\infty})} \, dx < \infty \tag{9}
\]

for some \( p_{\infty} > 1 \) and \( c > 0 \), then \( p \in \mathcal{M}(\mathbb{R}^n) \). One can show that (8) implies (9), but the converse, in general, is not true. The corresponding example is constructed in [7]. Nekvinda further relaxed condition (9) in [32]. Lerner [22] (see also [14, Example 5.1.8]) showed that there exist discontinuous at zero or/and at infinity exponents, which nevertheless belong to \( \mathcal{M}(\mathbb{R}^n) \). Thus, the class of exponents in \( \mathcal{P}^{\text{log}}(\mathbb{R}^n) \) satisfying (6) is a proper subset of the class \( \mathcal{M}(\mathbb{R}^n) \).

We will need the following remarkable result proved in [13, Theorem 8.1] (see also [14, Theorem 5.7.2]).

**Theorem 3.1 (Diening).** We have \( p \in \mathcal{M}(\mathbb{R}^n) \) if and only if \( p' \in \mathcal{M}(\mathbb{R}^n) \).

We refer to the recent monographs [9, 14] for further discussions concerning the class \( \mathcal{M}(\mathbb{R}^n) \).

### 3.3. Boundedness of pseudodifferential operators on variable Lebesgue spaces

Combining Theorem 1.2 and Theorem 3.1, we immediately arrive at the following.

**Theorem 3.2.** Suppose \( p \in \mathcal{M}(\mathbb{R}^n) \). If \( a \) belongs to one of the following symbol classes:

(a) the Hörmander class \( S_n^{\rho,0} \) with \( 0 < \rho \leq 1 \) and \( 0 \leq \delta < 1 \);

(b) the Miyachi class \( S_n^{\rho,0}(\kappa, n) \) with \( 0 \leq \delta \leq \rho \leq 1 \), \( 0 \leq \delta < 1 \), and \( \kappa > 0 \);

then \( \text{Op}(a) \) extends to a bounded operator on \( L^{p(.)}(\mathbb{R}^n) \).

Part (a) of the above theorem was obtained by the author and Spitkovsky in [19, Theorem 1.2]. Part (b) is new.

**Corollary 3.3 (Rabinovich-Samko).** Let \( p \in \mathcal{P}^{\text{log}}(\mathbb{R}^n) \) satisfy (6). If \( a \in S_1^{0,0} \), then \( \text{Op}(a) \) extends to a bounded operator on \( L^{p(.)}(\mathbb{R}^n) \).

**Proof.** This statement immediately follows from part (a) of the previous result because \( \mathcal{P}^{\text{log}}(\mathbb{R}^n) \) is a (proper!) subset of \( \mathcal{M}(\mathbb{R}^n) \). \( \square \)

This result was proved in [35, Theorem 5.1].

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Alexei Yu. Karlovich
Departamento de Matemática
Faculdade de Ciências e Tecnologia
Universidade Nova de Lisboa
Quinta da Torre
2829–516 Caparica
Portugal

e-mail: oyk@fct.unl.pt