GLOBAL BIFURCATIONS AND A PRIORI BOUNDS OF POSITIVE SOLUTIONS FOR COUPLED NONLINEAR SCHRÖDINGER SYSTEMS

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Abstract. In this paper, we consider the following coupled elliptic system
\[
\begin{cases}
-\Delta u + \lambda_1 u = \mu_1 |u|^2 u + \beta uv^2 - \gamma v & \text{in } \mathbb{R}^N, \\
-\Delta v + \lambda_2 v = \mu_2 |v|^2 v + \beta uv^2 - \gamma u & \text{in } \mathbb{R}^N, \\
u(x), v(x) \to 0 & \text{as } |x| \to +\infty.
\end{cases}
\]
Under symmetric assumptions \( \lambda_1 = \lambda_2, \mu_1 = \mu_2 \), we determine the number of \( \gamma \)-bifurcations for each \( \beta \in (-1, +\infty) \), and study the behavior of global \( \gamma \)-bifurcation branches in \([-1, 0] \times H^1 \times H^1 \). Moreover, several results for \( \gamma = 0 \), such as priori bounds, are of independent interests, which are improvements of corresponding theorems in [6] and [35].

1. Introduction. Consider the following doubly coupled nonlinear Schrödinger system
\[
\begin{align*}
-i \frac{\partial}{\partial t} \Phi &= \Delta \Phi + \mu_1 |\Phi|^2 \Phi + \beta |\Phi|^2 \Psi - \gamma \Psi & \text{for } t > 0, \quad x \in \mathbb{R}^N, \\
-i \frac{\partial}{\partial t} \Psi &= \Delta \Psi + \mu_2 |\Psi|^2 \Psi + \beta |\Phi|^2 \Psi - \gamma \Phi & \text{for } t > 0, \quad x \in \mathbb{R}^N, \\
\Phi(t, x) &\to 0, \quad \Psi(t, x) \to 0 & \text{as } |x| \to +\infty, \quad t > 0,
\end{align*}
\]
where \( N = 2, 3, \mu_1 \) and \( \mu_2 \) are positive constants, \( \gamma \) and \( \beta \) are linear and nonlinear coupling constants, respectively. The problem (1) has many applications in physics, especially in nonlinear optics, see [1, 3, 8, 15, 24, 25, 26, 31] and references therein. The solutions \( \Phi \) and \( \Psi \) denote the first and the second components of the beam in Kerr-like photorefractive media [1]. The positive constant \( \mu_j \) is for self-focusing
in the $j$-th component of the beam, $j = 1, 2$. The nonlinear coupling constant $\beta$ is the interaction between the two components of the beam. The interaction is attractive if $\beta > 0$, and repulsive if $\beta < 0$. The linear coupling is generated either by a twist applied to the fiber in the case of two linear polarization, or by an elliptic deformation of the fibers core in the case of circular polarizations. Problem (1) also arises in the dynamics of the Bose-Einstein condensates with two degrees of freedom (see [19, 27, 24, 34, 30] and references therein for more details). Physically, $\Phi$ and $\Psi$ are the corresponding condensate amplitudes, $\mu_j$ and $\beta$ are the intraspecies and interspecies scattering lengths. When $\beta < 0$, the interactions are repulsive [34]; in contrast, when $\beta > 0$, they are attractive. The linear coupling constant $\gamma$ denotes the strength of the radio-frequency or electric coupling (see [17]).

Consider solitary wave solutions of system (1), i.e. solutions in the form $\Phi(t, x) = e^{i\lambda_1 t}u(x)$, $\Psi(t, x) = e^{i\lambda_2 t}v(x)$, then system (1) is transformed into the following elliptic system

$$
\begin{cases}
-\Delta u + \lambda_1 u = \mu_1 u^3 + \beta uv^2 - \gamma v, \\
-\Delta v + \lambda_2 v = \mu_2 v^3 + \beta vu^2 - \gamma u,
\end{cases}
$$

where $\lambda_j > 0, \beta, \gamma$ are constants. A solution $(u, v)$ is called nontrivial if $u \neq 0$ and $v \neq 0$; a solution $(u, v)$ is semi-trivial if $(u, v)$ is of type $(u, 0)$ or $(0, v)$. We call a nontrivial solution $(u, v)$ positive if $u > 0$ and $v > 0$.

When system (2) is only nonlinearly coupled, i.e. $\gamma = 0$, extensive research has been done regarding the existence, multiplicity and asymptotic behavior of nontrivial solutions to system (2). We refer to [4, 7, 11, 13, 14, 22, 28] and references therein. When system (2) is only linearly coupled, i.e. $\gamma \neq 0$, $\beta = 0$, Ambrosetti et al. [5] studied the existence and asymptotic behavior of the multi-bump solutions of system (2). When $\gamma \beta \neq 0$, to our best knowledge, only a few interesting results have been obtained in [9, 24, 33]. In particular, the second and third author of the current paper obtained some existence results for system (2) in [33] by using variational methods and bifurcation techniques.

In spite of these work, there are still many questions left unanswered. In this paper, we consider the following three topics:

(1) when $\gamma \neq 0$ and $\lambda_1 = \lambda_2$, $\mu_1 = \mu_2$, we consider the number of $\gamma$-bifurcations (see definition below) and describe the behavior of global bifurcation branches (Section 2);

(2) when $\gamma = 0$, consider five dimensional bifurcation in $(\mu_1, \mu_2, \lambda_1, \lambda_2, \beta)$ (Section 3);

(3) when $\gamma = 0$, $\lambda_1 = \lambda_2$, refine the results in [6] and [35] (Section 4).

For convenience, we introduce some notations first. Denote

$$
H^1 (\mathbb{R}^N) = \left\{ u \in L^2 (\mathbb{R}^N) : |\nabla u| \in L^2 (\mathbb{R}^N) \right\},
$$

$$
H^1_1 (\mathbb{R}^N) = \left\{ u \in H^1 (\mathbb{R}^N) : u \text{ is radially symmetric} \right\}
$$

and $X := H^1_1 (\mathbb{R}^N) \times H^1_1 (\mathbb{R}^N)$ is a Hilbert space with inner product

$$
\langle \vec{u}, \vec{v} \rangle_b = \langle \nabla u_1, \nabla v_1 \rangle_{L^2} + \lambda_1 \langle u_1, v_1 \rangle_{L^2} + \langle \nabla u_2, \nabla v_2 \rangle_{L^2} + \lambda_2 \langle u_2, v_2 \rangle_{L^2}, \forall \vec{u}, \vec{v} \in X.
$$

Then the associated norms is

$$
\| \vec{u} \|_b = \sqrt{|u_1|^2 + |u_2|^2},
$$
where
\[ \|u_1\| = \sqrt{\langle \nabla u_1, \nabla u_1 \rangle_{L^2} + \lambda_1 \langle u_1, u_1 \rangle_{L^2}}, \quad \|u_2\| = \sqrt{\langle \nabla u_2, \nabla u_2 \rangle_{L^2} + \lambda_2 \langle u_2, u_2 \rangle_{L^2}}. \]

Denote by \( w_{\lambda_j, \mu_j} \) the non-degenerate positive radial solution of
\[
\begin{align*}
-\Delta w + \lambda_j w &= \mu_j w^3 \quad \text{in } \mathbb{R}^N, \\
w(0) &= \max_{x \in \mathbb{R}^N} w(x), \quad w \in H^1_\text{r}(\mathbb{R}^N). 
\end{align*}
\] (3)

Note that, by the classical bootstrap argument, solutions of system (2) which are in \( X \) also in \( C^2(\mathbb{R}^N) \times C^2(\mathbb{R}^N) \).

2. \( \gamma \)-bifurcation of fully symmetric system. Consider system (2) under symmetric assumptions \( \lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1 \), i.e.
\[
\begin{align*}
-\Delta u + u &= u^3 + \beta uv^2 - \gamma v, \\
-\Delta v + v &= v^3 + \beta vu^2 - \gamma u, \\
(u, v) &\in X.
\end{align*}
\] (4)

Denote \( w = w_{1,1} \). It has been shown in [33] that problem (4) has a synchronized solution branch in \( \mathbb{R}^2 \times X \),
\[
\mathcal{T}_w = \left\{ (\gamma, \beta, \sqrt{\frac{1+\gamma}{1+\beta}} w(\sqrt{1+\gamma} x), \sqrt{\frac{1+\gamma}{1+\beta}} w(\sqrt{1+\gamma} x)) : \beta > -1, \gamma > -1 \right\}.
\]

Let \( \mathcal{T}_w|_\beta \) be the restriction of \( \mathcal{T}_w \) for fixed \( \beta \), which is therefore only parameterized in \( \gamma \). Bifurcations with respect to \( \mathcal{T}_w|_\beta \) will be called \( \gamma \)-bifurcations. For fixed \( \beta \in (-1, +\infty) \), finitely many \( \gamma \)-bifurcation points were found with respect to \( \mathcal{T}_w|_\beta \). Moreover, there is a global bifurcation branch emanating from each bifurcation point. In this section, we shall give more information about these global \( \gamma \)-bifurcation branches, where the word “global” is understood by taking \( (-1,0] \times X \) as the whole space. Precisely, we consider the following two interesting problems left in [33]:

1. How many \( \gamma \)-bifurcation points are there with respect to \( \mathcal{T}_w|_\beta \) for a fixed \( \beta \in (-1, +\infty) \)?

2. How does each global \( \gamma \)-bifurcation branch behave in terms of Rabinowitz’s bifurcation theorem [29]?

The bifurcation parameters depend on the following eigenvalue problem
\[
\begin{align*}
-\Delta \phi + \phi &= \eta w^2 \phi, \\
\phi &\in H^1_\text{r}(\mathbb{R}^N). 
\end{align*}
\] (5)

It is well known (see [6]) that problem (5) possesses a sequence of eigenvalues
\[ \eta_1 = 1 < \eta_2 < \eta_3 < \cdots \]
with \( \eta_k \to +\infty \) as \( k \to +\infty \). Let
\[
f(\beta) = \frac{3 - \beta}{1 + \beta}.
\]
It is easy to see that \( f \) is a strictly decreasing function in \((-1, +\infty)\) and \( \lim_{\beta \to (-1)^+} f(\beta) = +\infty \). So we can denote \( \beta_k := f^{-1}(\eta_k) \) for any \( k \geq 1 \). In particular, we can see \( \beta_1 > \beta_2 > \beta_3 > \cdots \) with \( \beta_k \to (-1)^+ \) as \( k \to +\infty \).

To save notations, we refer \( \gamma \)-bifurcations simply as bifurcation in the following theorem.

**Theorem 2.1.** For any fixed \( \beta \in (-1, +\infty) \), one has:

(a) If \( \beta \in [\beta_{k+1}, \beta_k), k \geq 1 \), system (2) has exactly \( k \) bifurcation points \( \{ (\gamma_l, u_{\gamma_l}, v_{\gamma_l}) \}^k \) with respect to \( T_w|_\beta \), where \(-1 < \gamma_1 < \ldots < \gamma_k < 0\). Moreover, for each bifurcation point \( (\gamma_l, u_{\gamma_l}, v_{\gamma_l}) \), \( 1 \leq l \leq k \), there is a global bifurcation branch \( S^\beta \subseteq (-1, 0) \times X \) of positive solutions which does not return to the synchronized branch, such that \( S^\beta \) satisfies one of the following three properties

(i) meets infinity at \( \gamma = -1 \),
(ii) meets infinity at \( \gamma = 0 \),
(iii) meets \( \{0\} \times X \).

Moreover, for any \( (\gamma, u, v) \in S^\beta \setminus \{ (\gamma_l, u_{\gamma_l}, v_{\gamma_l}) \} \), the difference \( u - v \) has precisely \( l - 1 \) simple zeroes;

(b) If \( \beta \in [1, +\infty) \), system (2) has no bifurcation point in \((-1, 0)\) with respect to \( T_w|_\beta \).

**Remark 1.** An important difference lies between the \( \gamma \)-bifurcations and \( \beta \)-bifurcations is that the space dimension \( N \) now affects the behavior of synchronized solution branch \( T_w \). Precisely, using the Pohozaev identity, we have

\[
\|w_{\gamma}\|_{H^1}^2 = \|\nabla w\|_2^2(1 + \gamma)^2 - \frac{N}{2} + \|w\|_2^2(1 + \gamma)^{1 - \frac{N}{2}},
\]

where \( w_\gamma(x) = \sqrt{1 + \bar{\gamma}}w(\sqrt{1 + \bar{\gamma}}x) \), thus for any \( (u, v) \in T_w \),

\[
\|u, v\|_X^2 = \frac{\|\nabla w\|_2^2(1 + \gamma)^{2 - \frac{N}{2}} + \|w\|_2^2(1 + \gamma)^{1 - \frac{N}{2}}}{1 + \beta}.
\]

Figure 1 illustrates the possible global bifurcation branches at a bifurcation point \( (\gamma_l, u_l, v_l) \in T_w \), \( 1 \leq l \leq k \). Each bifurcation branch either blows up at a \( \gamma = -1 \) or \( \gamma = 0 \), or reaches to \( \{0\} \times X \). The dashed lines represent the three possible situations of bifurcation branch and only one of them could happen.
It has been shown in [33] that system (2) is invariant under the following transformation:

$$\sigma : \mathbb{R}^2 \times X \to \mathbb{R}^2 \times X, \ \sigma(\gamma, \beta, u, v) = (-\gamma, \beta, u, -v).$$

Under this $\sigma$-invariance, one can easily get the symmetric bifurcation results about opposite sign solutions for $\gamma \in [0, 1]$.

To get precise descriptions of the global $\gamma$-bifurcations, we need a few lemmas. First, it is well known that the following eigenvalue problem

$$\begin{cases}
- \Delta \phi + \frac{1}{\gamma^2} \phi = \eta w^2 \phi, \\
\phi \in H^1_0(\mathbb{R}^N)
\end{cases}$$

has a sequence of simple eigenvalues

$$\eta_1(\gamma) < \eta_2(\gamma) < \cdots < \eta_j(\gamma) < \cdots, \text{ and } \eta_j(\gamma) \to +\infty \text{ as } j \to +\infty.$$

Moreover, the eigenfunction $\phi_j$ corresponding to $\eta_j(\gamma)$ has precisely $k - 1$ simple zeroes (see Theorem XIII.7.53 and Corollary 7.56. of [18], or [12]). It has been shown in [33, Lemma 3.1] that $\eta_j(\gamma), j \geq 1$, is a decreasing and continuous function of $\gamma \in (-1, +\infty)$. Unfortunately, there is a little gap in the proof of continuity. Here we fill this gap with proper modifications. We would like to point out that the continuity and monotonicity of $\eta_j(\gamma)$ are only needed, both in [33] and the current paper, for $\gamma \in (-1, 1)$. Therefore, we may assume in the following $-1 < \gamma_1 < \gamma_2 < 1$. Let $C(\gamma) := (1 - \gamma)/(1 + \gamma)$, then $C(\gamma) \to +\infty$ as $\gamma \to (1)^-$ and $C(\gamma) \to 0$ as $\gamma \to 1^-$. In addition, $C(\gamma)$ is strictly decreasing on the interval $(-1, 1)$.

Recall the variational characterization of $\eta_j(\gamma)$,

$$\eta_j(\gamma) = \sup E_{j-1} \inf \{J(\phi, \gamma) : \phi \in E_j \},$$

where $E_j$ denotes a $j$-dimensional subspace of $H^1_0(\mathbb{R}^N)$, $E_j^\perp$ denotes the orthogonal space of $E_j$ in $H^1_0(\mathbb{R}^N)$, and

$$J(\phi; \gamma) := \frac{\int_{\mathbb{R}^N} (|\nabla \phi|^2 + C(\gamma)\phi^2) \, dx}{\int_{\mathbb{R}^N} \phi^2 \, dx}.$$

**Lemma 2.2.** For each $j \geq 1$, the eigenvalue $\eta_j(\gamma)$ is a continuous and strictly decreasing function of $\gamma$ in the interval $(-1, 1)$. Moreover, for any $j \geq 1$,

$$\eta_j(\gamma) \to +\infty \text{ as } \gamma \to (1)^+.$$

**Proof.** Monotonicity. Let $-1 < \gamma_1 < \gamma_2 < 1$ and $E_{j-1}^*$ be the $j - 1$-dimensional subspace of $H^1_0(\mathbb{R}^N)$ associated to $\eta_j(\gamma_2)$, i.e.

$$\eta_j(\gamma_2) = \inf \{J(\phi; \gamma_2) : \phi \in (E_{j-1}^*)^+\}.$$

In the case $j = 1$, denote $(E_0^*)^+ = H^1_0(\mathbb{R}^N)$. By the characterization of $\eta_j$,

$$\begin{align*}
\eta_j(\gamma_1) - \eta_j(\gamma_2) &\geq \inf_{\phi \in (E_{j-1}^*)^+} J(\phi, \gamma_1) - \inf_{\phi \in (E_{j-1}^*)^+} J(\phi, \gamma_2) \\
&\geq \inf_{\phi \in (E_{j-1}^*)^+} (J(\phi, \gamma_1) - J(\phi, \gamma_2)) \\
&= (C(\gamma_1) - C(\gamma_2)) \inf_{\phi \in (E_{j-1}^*)^+} \frac{\int_{\mathbb{R}^N} \phi^2 \, dx}{\int_{\mathbb{R}^N} \phi^2 \, dx}.
\end{align*}$$
Therefore the monotonicity of $C(\gamma)$ implies that $\eta_j(\gamma)$ is decreasing, for any $j \geq 1$. **Continuity in $(-1, 1)$.** For any $j \geq 1$ and $-1 < \gamma_1 < \gamma_2 < 1$,

$$\eta_j (\gamma_1) - \eta_j (\gamma_2) = \sup_{E_{j-1}} \inf_{\phi \in E_{j-1}} J(\phi; \gamma_1) - \sup_{E_{j-1}} \inf_{\phi \in E_{j-1}} J(\phi; \gamma_2) \leq \sup_{E_{j-1}} \left( \inf_{\phi \in E_{j-1}} J(\phi; \gamma_1) - \inf_{\phi \in E_{j-1}} J(\phi; \gamma_2) \right) \leq (C(\gamma_1) - C(\gamma_2)) \sup_{E_{j-1}} \frac{\int_{\mathbb{R}^N} (\phi_j^2)^2 \, dx}{\int_{\mathbb{R}^N} w(\phi_j^2)^2 \, dx} \leq \left[ \frac{C(\gamma_1) - 1}{C(\gamma_2) - 1} \right] \sup_{E_{j-1}} \frac{\int_{\mathbb{R}^N} |\nabla \phi_j|^2 \, dx + C(\gamma_2) \int_{\mathbb{R}^N} \phi_j^2 \, dx}{\int_{\mathbb{R}^N} w(\phi_j^2)^2 \, dx} = \left[ \frac{C(\gamma_1) - 1}{C(\gamma_2) - 1} \right] \eta_j (\gamma_2),$$

where $\phi_j^{\gamma_2}$ is a minimizer of $J(\phi; \gamma_2)$ in $E_{j-1}^\perp$.

Now for any $\gamma_0 \in (-1, 1)$, $j \geq 1$, we see from (7) and (8) that

$$0 \leq \lim_{\gamma \to \gamma_0} (\eta_j (\gamma) - \eta_j (\gamma_0)) \leq \lim_{\gamma \to \gamma_0} \left( \frac{C(\gamma)}{C(\gamma_0)} - 1 \right) \eta_j (\gamma_0) = 0 \quad (9)$$

and also

$$0 \leq \lim_{\gamma \to \gamma_0} (\eta_j (\gamma_0) - \eta_j (\gamma)) \leq \lim_{\gamma \to \gamma_0} \left( \frac{C(\gamma_0)}{C(\gamma)} - 1 \right) \eta_j (\gamma) \leq \lim_{\gamma \to \gamma_0} \left( \frac{C(\gamma_0)}{C(\gamma)} - 1 \right) \eta_j (\gamma_0) = 0 \quad (10)$$

Combine (9) and (10),

$$\lim_{\gamma \to \gamma_0} \eta_j (\gamma) = \eta_j (\gamma_0).$$

Hence $\eta_j (\gamma)$ is a continuous and strictly decreasing function of $\gamma$.

For the right limit of $\eta_j$ at $-1$, we fix any $E_{j-1}$, then

$$\eta_j (\gamma) = \sup_{E_{j-1}} \inf_{\phi \in E_{j-1}^\perp} J(\phi, \gamma) \geq \inf_{\phi \in E_{j-1}^\perp} J(\phi, \gamma) = \inf_{\phi \in E_{j-1}^\perp} \frac{\int_{\mathbb{R}^N} |\nabla \phi|^2 \, dx + C(\gamma) \int_{\mathbb{R}^N} \phi^2 \, dx}{\int_{\mathbb{R}^N} w(\phi^2)^2 \, dx} \geq \frac{C(\gamma)}{\|w\|^2_{L^\infty}}.$$

Therefore $\eta_j (\gamma) \to +\infty$ as $\gamma \to (-1)^+$. \qed

It has been shown in [33] that system (2) has no positive solution if $\lambda_1 = \lambda_2 = 1$, $\gamma = -1$ and $\beta > 0$. Here we improve this result by allowing $\beta > -1$.

**Lemma 2.3.** System (4) has no positive solution if $\gamma = -1$ and $\beta > -1$.

**Proof.** Assume for contradiction that $(u, v)$ is a positive solution of system (4). Adding the two equations together, one sees that

$$-\Delta (u + v) = -\Delta (u + v) + (1 + \gamma)(u + v) = u^3 + v^3 + \beta u^2 v + \beta v^2 u.$$

If $\beta \in (-1, 1]$,

$$u^3 + v^3 + \beta u^2 v + \beta v^2 u = (u + v) (u^2 + (\beta - 1) uv + v^2) \geq \frac{1 + \beta}{2} (u + v) (u^2 + v^2) \geq \frac{1 + \beta}{4} (u + v)^3.$$
On the other hand, if $\beta > 1$,

$$u^3 + v^3 + \beta u^2 v + \beta v^2 u = (u + v) (u^2 + (\beta - 1)uv + v^2) \geq \frac{1}{2} (u + v)^3.$$ 

Thus for any $\beta > -1$,

$$u^3 + v^3 + \beta u^2 v + \beta v^2 u \geq \gamma_0 (u + v)^3,$$

where $\gamma_0 = \min \left\{(1 + \beta)/4, 1/2\right\}$. Now setting $U = u + v$, we get that

$$-\Delta U \geq \gamma_0 U^3.$$

So the Liouville type theorem [16, Theorem 2.3] implies $U \equiv 0$. This is a contradiction.

The proof of Theorem 2.1 requires the following a priori bounds on positive solutions $(\gamma, u, v)$, provided $\gamma$ is bounded.

**Lemma 2.4.** Given a compact set $B \subset (-1, +\infty)$ and fixed $\beta > -1$, there exists a constant $C = C(\beta)$ such that for any positive solution $(\gamma, u, v)$ of system (4) with $\gamma \in B$ we have

$$\|u\|_{L^\infty(\mathbb{R}^N)}, \|v\|_{L^\infty(\mathbb{R}^N)} \leq C.$$ 

**Proof.** We follow a blow up procedure introduced by Gidas and Spruck [21]. Since the method is standard, we only sketch the argument. Assuming for contradiction that there is a sequence of solutions $(\gamma_n, u_n, v_n)$ to system (4) such that

$$\max_{x \in \mathbb{R}^N} u_n(x) + \max_{x \in \mathbb{R}^N} v_n(x) \to +\infty \quad \text{as} \quad n \to +\infty.$$

Without loss of generality, we may assume that

$$M_n := u_n(0) = \max_{x \in \mathbb{R}^N} u_n(x) \geq \max_{x \in \mathbb{R}^N} v_n(x).$$

Set $x = y/M_n$ and define $U_n, V_n : \mathbb{R}^N \to \mathbb{R}$ by

$$U_n(y) = \frac{u_n \left( \frac{y}{M_n} \right)}{M_n}, \quad V_n(y) = \frac{v_n \left( \frac{y}{M_n} \right)}{M_n}.$$ 

Then

$$\max_{y \in \mathbb{R}^N} V_n(y) \leq \max_{y \in \mathbb{R}^N} U_n(y) = 1$$

and $(U_n, V_n)$ solves the rescaled problem

$$\begin{cases}
-\Delta U_n = U_n^3 + \beta U_n V_n^2 - \frac{U_n}{M_n^2} - \frac{2\gamma_n V_n}{M_n^2}, \\
-\Delta V_n = V_n^3 + \beta V_n U_n^2 - \frac{V_n}{M_n^2} - \frac{2\gamma_n U_n}{M_n^2}.
\end{cases}$$

Noting that $\gamma_n$ is bounded, passing to a subsequence if necessary, we see that $(U_n, V_n) \to (U_0, V_0) \in L^2_{\text{loc}}(\mathbb{R}^N)$ as $n \to +\infty$, which is a nontrivial and nonnegative bounded radial solution of

$$\begin{cases}
-\Delta u = u^3 + \beta uv^2 \quad \text{in} \quad \mathbb{R}^N, \\
-\Delta v = v^3 + \beta vu^2 \quad \text{in} \quad \mathbb{R}^N.
\end{cases}$$

It follows from [16, Theorem 2.1] that $(U_0, V_0) = (0, 0)$, which contradicts $U_0(0) = 1$.

Now we derive a uniform decaying rate of positive solutions to system (4) in $X$. 

Lemma 2.5. For any fixed $\beta \in (-1, +\infty)$, $\gamma \in (-1, 0]$ and given positive solution $(u, v)$ of (4), there exists $M > 0$ such that $u, v \leq Mr^{-(N-1)/2}e^{-r(1+\gamma)/2}$ and $|u'| \leq Mr^{1-N}e^{-r(1+\gamma)/4}$ for $r$ large enough.

Proof. Let $(u, v)$ be a positive solution of (4), then $U = u + v$ satisfies

$$-U'' - \frac{N-1}{r}U' + (1 + \gamma)U = U(u^2 + (\beta - 1)uv + v^2).$$ (11)

Let $\zeta = r^{(N-1)/2}U$ and denote by

$$\alpha = \frac{2}{1+\gamma}, \quad p(r) = \frac{(N-1)(N-3)}{4r^2}, \quad q(r) = 1 + \gamma - u^2(r) - (\beta - 1)u(r)v(r) - v^2(r)$$

then it follows that

$$(\zeta^\alpha)'' = \alpha(\alpha - 1)\zeta^{\alpha-2}\zeta^2 + \alpha (q(r) + p(r))\zeta^\alpha.$$ Since $u(r), v(r), p(r) \to 0$ as $r \to +\infty$, there exists $R_1 > 1$ such that $q(r) + p(r) \geq (1 + \gamma)/2$ for $r \geq R_1$, which implies

$$(\zeta^\alpha)'' \geq \zeta^\alpha, \quad \text{for any } r \geq R_1.$$ Thus $Q = e^{-r}((\zeta^\alpha)' + \zeta^\alpha)$ is non-decreasing in $(R_1, +\infty)$. Case 1. If $Q$ remains non-positive in $(R_1, +\infty)$, then by an argument similar to that of [32, Theorem 3], we have $(e^r\zeta^\alpha)' \leq 0$ and $\zeta^\alpha = O(e^{-r})$ in this interval. Thus there exists $M > 0$ such that $U \leq Mr^{-(N-1)/2}e^{-r(1+\gamma)/2}$ for $r \geq R_1$.

Case 2. If there exists $R_2 \geq R_1$ such that $Q \geq 2\delta > 0$ in $(R_2, +\infty)$, then $(\zeta^\alpha)' + \zeta^\alpha$ is not integrable near $+\infty$. Let $\tilde{\zeta} = \zeta^2$, by some simple calculations, we can show that

$$(\tilde{\zeta})^{2-1}\left(\frac{\alpha}{2}(\tilde{\zeta}') + \tilde{\zeta}\right) = (\zeta^\alpha)' + \zeta^\alpha.$$ Since $U(r) \to 0$ as $r \to +\infty$, there exists $R_3 \geq R_2$ such that $U(r) \leq 1$ for any $r \geq R_3$. Thus

$$r^{\frac{(N-1)(\alpha-2)}{2}}\left(\frac{\alpha}{2}(\tilde{\zeta}') + \tilde{\zeta}\right) \geq r^{\frac{(N-1)(\alpha-2)}{2}}U^{\alpha-2}\left(\frac{\alpha}{2}(\tilde{\zeta}') + \tilde{\zeta}\right) = (\zeta^\alpha)' + \zeta^\alpha \geq 2\delta e^{r}$$

for any $r \in (R_3, +\infty)$. It follows that $\alpha(\tilde{\zeta})'/2 + \tilde{\zeta}$ is not integrable near $+\infty$. On the other hand, since $\tilde{\zeta} = r^{N-1}U^2$,

$$\int_{R_3}^{\infty} \left(\frac{\alpha}{2}(\tilde{\zeta}') + \tilde{\zeta}\right) \frac{dr}{r} \leq \left(1 + \frac{(N-1)\alpha}{2R_3}\right)\int_{R_3}^{\infty} r^{N-1}U^2 dr + \alpha \int_{R_3}^{\infty} r^{N-1}U U' < +\infty$$

due to the fact that $U \in H^1_1(\mathbb{R}^N)$. A contradiction. Thus Case 2 does not occur.

Therefore, there exists $M > 0$ such that $U(r) \leq Mr^{(1-N)/2}e^{-r(1+\gamma)/2}$ for large enough $r$. Since $u, v$ are nonnegative, the exponential decay of $u$ and $v$ follows.

To show the exponential decay of $u'$ and $v'$, we first note that $h(r) := e^r\zeta^\alpha$ is decreasing in $(R_1, +\infty)$ since $(e^r\zeta^\alpha)' \leq 0$ for $r \geq R_1$. It follows that

$$U = \frac{h^\frac{1}{2}(r)}{r^{(N-1)/2}e^{r(1+\gamma)/2}}$$

is decreasing in $(R_1, +\infty)$. Hence $r^{N-1}U'(r) \leq 0$ for any $r \geq R_1$. By (11), we find that

$$-(r^{N-1}U')' = r^{N-1}[U(u^2 + (\beta - 1)uv + v^2) - (1 + \gamma)U].$$ (12)
This combining with the exponential decay of $U$ implies that $r^{N-1}U'$ is increasing in $(R_4, +\infty)$ for some constant $R_4 \geq R_1$. Then
\[
\lim_{r \to +\infty} r^{N-1}U' =: l_\infty \in (-\infty, 0].
\]
If $l_\infty < 0$, there holds
\[
\frac{2l_\infty}{r^{N-1}} \leq U'(r) \leq \frac{l_\infty}{2r^{N-1}} \text{ for all sufficiently large } r.
\]
When $N = 3$, integrating this over $[r, +\infty)$, we get
\[
\frac{2l_\infty}{N r^N} \leq U(r) \leq \frac{l_\infty}{2N r^N},
\]
which contradicts the exponential decay of $U$. Hence, $l_\infty = 0$. When $N = 2$, one has $U' \notin L^1(0, +\infty)$, which contradicts $U \in L^\infty(0, +\infty)$. So, we still get $l_\infty = 0$.

Integrating (11) from $r$ to $+\infty$,
\[
r^{N-1}U'(r) = \int_r^{+\infty} s^{N-1}[U(u^2 + (\beta - 1)uv + v^2) - (1 + \gamma)U] ds.
\]
It follows that
\[
r^{N-1}|U'(r)| \leq M' \int_r^{+\infty} s^{(N-1)/2} e^{-s(1+\gamma)/2} ds \leq M'' e^{-r(1+\gamma)/4}
\]
for some positive constants $M'$ and $M''$, i.e.
\[
|u'(r) + v'(r)| \leq M'' r^{1-N} e^{-r(1+\gamma)/4}
\]
for $r$ large enough. The proof is completed. \(\square\)

Note that $M$ depends on $(u, v)$ in Lemma 2.5. If $\gamma \in [\overline{\gamma}, \overline{\gamma}] \subset (-1, 0)$ for $\overline{\gamma}, \overline{\gamma} \in (-1, 0)$ fixed, we can choose $M$ in Lemma 2.5 uniformly with respect to $(u, v) \in S$ where
\[
S := \{(u, v) \in X : (u, v) \text{ is a positive solution of system (4), } \gamma \in [\overline{\gamma}, \overline{\gamma}]\}.
\]

**Lemma 2.6.** For any fixed $\beta \in (-1, +\infty)$, there exists $M > 0$ such that $u, v \leq M r^{-(N-1)/2} e^{-r(1+\gamma)/2}$ and $|u' + v'| \leq M r^{1-N} e^{-r(1+\gamma)/4}$ for all $(u, v) \in S$ and $r$ large enough.

**Proof.** Suppose on the contrary that there exists a sequence $\{u_k, v_k\} \subset S$ and a sequence of numbers $\{\gamma_k\} \subset [\overline{\gamma}, \overline{\gamma}]$ such that
\[
u_k(r_k) > kr_k^{-(N-1)/2} e^{-r_k(1+\gamma)/2} \text{ or } v_k(r_k) > kr_k^{-(N-1)/2} e^{-r_k(1+\gamma)/2}
\]
for some $r_k \geq 0$ and $\lim_{k \to +\infty} \gamma_k = \gamma_0 \in [\overline{\gamma}, \overline{\gamma}]$. Clearly, $(u_k, v_k)$ satisfies
\[
\begin{cases}
  u''_k + \frac{N-1}{r} u'_k + g(u_k, v_k) = 0, \\
  v''_k + \frac{N-1}{r} v'_k + f(u_k, v_k) = 0,
\end{cases}
\]
where
\[
g(u_k, v_k) = u_k^3 + \beta u_k v_k^2 - \gamma_k u_k - u_k, \quad f(u_k, v_k) = v_k^3 + \beta v_k u_k^2 - \gamma_k u_k - v_k.
\]
According to Lemma 2.4, $\|u_k\|_{L^\infty(\mathbb{R}^N)}$ and $\|v_k\|_{L^\infty(\mathbb{R}^N)}$ are bounded.

It is well known that $u_k$ and $v_k$ are monotone decreasing on $[0, \infty)$ for $\beta \geq 0$. On the other hand, if $\beta \in (-1, 0)$, we cannot conclude the monotonicity of $u_k$ and $v_k$ as above, since the system is no longer cooperative. Whereas, we can employ the
idea of “moving planes” method to show that \( u_k \) and \( v_k \) are monotone decreasing for \( r \) sufficiently large.

**Claim.** There exists \( \lambda^* > 0 \) such that \( u_k \) and \( v_k \) are monotone decreasing in \([\lambda^*, +\infty)\) for any \( k \in \mathbb{N} \).

We first extend the definition domain of \( u_k \) and \( v_k \) into \( \mathbb{R} \) such that they are even, which are still denoted by \( u_k \) and \( v_k \) for convenience. Clearly, they still satisfy (13) in \( \mathbb{R} \setminus \{0\} \).

Set \( u_k^\lambda(r) = u_k(r^\lambda) \), \( v_k^\lambda(r) = v_k(r^\lambda) \), \( U_k^\lambda(r) = u_k^\lambda(r) - u_k(r) \) and \( V_k^\lambda(r) = v_k^\lambda(r) - v_k(r) \) with \( r^\lambda = 2\lambda - r \) denoting the reflection of \( r \in (\lambda, +\infty) \) with respect to \( r = \lambda \). Then, it is enough to show that \( U_k^\lambda(r) \geq 0 \) and \( V_k^\lambda(r) \geq 0 \) in \((\lambda, +\infty)\) for all \( \lambda \geq \lambda^* \).

Without loss of generality, we assume for contradiction that for all \( \lambda > 0 \) there exist \( k_0 \in \mathbb{N} \) and \( r_0 \in (\lambda^*, +\infty) \) such that \( U_{k_0}^\lambda(r_0) < 0 \). Clearly, we have that

\[
\frac{\partial g}{\partial v}(0, 0) = -1 = \frac{\partial f}{\partial v}(0, 0), \quad \frac{\partial g}{\partial u}(0, 0) = -\gamma_{k_0} = \frac{\partial f}{\partial u}(0, 0).
\]

So, we can choose \( \varepsilon_0 > 0 \) small enough such that

\[
\frac{\partial g}{\partial u}(u, v) < 0, \quad \frac{\partial f}{\partial u}(u, v) < 0, \quad \frac{\partial g}{\partial v}(u, v) > 0, \quad \frac{\partial f}{\partial v}(u, v) > 0
\]

if \(|u| + |v| < \varepsilon_0\). Since \( \lim_{r \to +\infty} u_{k_0}(r) = 0 = \lim_{r \to +\infty} v_{k_0}(r) \) and \( \lim_{r \to +\infty} u_{k_0}^\lambda(r) = 0 = \lim_{r \to +\infty} v_{k_0}^\lambda(r) \) for any fixed \( \lambda > 0 \), we can take \( \bar{\lambda} > 0 \) such that max \( \{u_{k_0}, u_{k_0}^\lambda\} \) + max \( \{v_{k_0}, v_{k_0}^\lambda\} < \varepsilon_0 \) when \( r > \bar{\lambda} \). Obviously, one has that \( U_{k_0}^\lambda(\lambda) = 0 \) and \( U_{k_0}^\lambda(r) \) tends to zero at infinity. Hence, \( U_{k_0}^\lambda \) attains its infimum in \((\lambda^*, +\infty)\). So, for any fixed \( \lambda \geq \bar{\lambda} \), there exists \( r_0 \in (\lambda^*, +\infty) \) such that

\[
U_{k_0}^\lambda(r_0) = \min_{r \in (\lambda^*, +\infty)} U_{k_0}^\lambda(r) < 0.
\]

It follows that \( \left(U_{k_0}^\lambda\right)^{''}(r_0) \geq 0 \) and \( \left(U_{k_0}^\lambda\right)^{'}(r_0) = 0 \).

Note that \( u_{k_0}^\lambda(r) \) and \( v_{k_0}^\lambda(r) \) still satisfy system (4). Then, \( U_{k_0}^\lambda \) and \( V_{k_0}^\lambda \) satisfy

\[
\begin{cases}
\left(U_{k_0}^\lambda\right)^{''} + \frac{N-1}{r} \left(U_{k_0}^\lambda\right)^{'} + g \left(u_{k_0}^\lambda, v_{k_0}^\lambda\right) - g \left(u_{k_0}, v_{k_0}\right) = 0, \\
\left(V_{k_0}^\lambda\right)^{''} + \frac{N-1}{r} \left(V_{k_0}^\lambda\right)^{'} + f \left(u_{k_0}^\lambda, v_{k_0}^\lambda\right) - f \left(u_{k_0}, v_{k_0}\right) = 0.
\end{cases}
\]

By the Taylor expansion, we have that

\[
\begin{cases}
\left(U_{k_0}^\lambda\right)^{''} + \frac{N-1}{r} \left(U_{k_0}^\lambda\right)^{'} + \frac{\partial g}{\partial v}(\xi_1(r, \lambda), v_{k_0}) U_{k_0}^\lambda + \frac{\partial g}{\partial u}(u_{k_0}^\lambda, \eta_1(r, \lambda)) V_{k_0}^\lambda = 0, \\
\left(V_{k_0}^\lambda\right)^{''} + \frac{N-1}{r} \left(V_{k_0}^\lambda\right)^{'} + \frac{\partial f}{\partial v}(\xi_2(r, \lambda), v_{k_0}^\lambda) U_{k_0}^\lambda + \frac{\partial f}{\partial u}(u_{k_0}, \eta_2(r, \lambda)) V_{k_0}^\lambda = 0,
\end{cases}
\]

where

\[
\xi_i(r, \lambda) \in \left(\min \left\{u_{k_0}, u_{k_0}^\lambda\right\}, \max \left\{u_{k_0}, u_{k_0}^\lambda\right\}\right), \\
\eta_i(r, \lambda) \in \left(\min \left\{v_{k_0}, v_{k_0}^\lambda\right\}, \max \left\{v_{k_0}, v_{k_0}^\lambda\right\}\right)
\]

for \( i = 1, 2 \).

Thus, for fixed \( \lambda \geq \bar{\lambda} \), we obtain that

\[
\frac{\partial g}{\partial u}(\xi_1(r_0, \lambda), v_{k_0}(r_0)) U_{k_0}^\lambda(r_0) \leq -\frac{\partial g}{\partial v}(u_{k_0}(r_0), \eta_1(r_0, \lambda)) V_{k_0}^\lambda(r_0).
\]  
(14)

Since the left side of (14) is positive, we must have \( V_{k_0}^\lambda(r_0) < 0 \). Take \( r_1 \in (\lambda, +\infty) \) such that

\[
V_{k_0}^\lambda(r_1) = \min_{r \in (\lambda, +\infty)} V_{k_0}^\lambda(r) < 0.
\]
Repeating the above argument, we can show that $U_{k_0}^\lambda (r_1) < 0$ and
\[
\frac{\partial f}{\partial u} (u_{k_0} (r_1), \eta_2 (r_1, \lambda)) V_{k_0}^\lambda (r_1) \leq -\frac{\partial f}{\partial u} (\xi_2 (r_1, \lambda), v_{k_0}^\lambda (r_1)) U_{k_0}^\lambda (r_1). \tag{15}
\]
For $\lambda \geq \overline{\lambda}$, set
\[
\alpha (\lambda) = \frac{\partial q}{\partial u} (\xi_1 (r_0, \lambda), v_{k_0} (r_0)), \quad \theta (\lambda) = \frac{\partial q}{\partial v} (u_{k_0}^\lambda (r_0), \eta_1 (r_0, \lambda)), \\
\psi (\lambda) = \frac{\partial f}{\partial u} (\xi_2 (r_1, \lambda), v_{k_0}^\lambda (r_1)), \quad \delta (\lambda) = \frac{\partial f}{\partial v} (u_{k_0} (r_1), \eta_2 (r_1, \lambda))
\]
and $\phi (\lambda) = \alpha (\lambda) \delta (\lambda) - \theta (\lambda) \psi (\lambda)$. Since $U_{k_0}^\lambda$ and $V_{k_0}^\lambda$ are negative at $r_0$ and $r_1$, we have
\[
\begin{align*}
u_{k_0}^\lambda (r_0) &< u_{k_0} (r_0), \quad \xi_1 (r_0, \lambda) \leq u_{k_0} (r_0), \quad \eta_1 (r_0, \lambda) \leq v_{k_0} (r_0), \\
v_{k_0}^\lambda (r_1) &< v_{k_0} (r_1), \quad \xi_2 (r_1, \lambda) \leq u_{k_0} (r_1), \quad \eta_2 (r_1, \lambda) \leq v_{k_0} (r_1).
\end{align*}
\]
These inequalities imply that
\[
\lim_{\lambda \to +\infty} \phi (\lambda) = \det \begin{pmatrix} -1 & -\gamma_{k_0} \\ -\gamma_{k_0} & -1 \end{pmatrix} = 1 - \gamma_{k_0}^2 > 0.
\]
Take $\lambda \geq \overline{\lambda}$ large enough such that $\phi (\lambda) > 0$. By (14) and (15),
\[
U_{k_0}^\lambda (r_0) \geq -\frac{\theta (\lambda)}{\alpha (\lambda)} V_{k_0}^\lambda (r_0) \geq -\frac{\theta (\lambda)}{\alpha (\lambda)} V_{k_0}^\lambda (r_1) \geq \frac{\theta (\lambda) \psi (\lambda)}{\alpha (\lambda) \delta (\lambda)} U_{k_0}^\lambda (r_1) \geq \frac{\theta (\lambda) \psi (\lambda)}{\alpha (\lambda) \delta (\lambda)} U_{k_0}^\lambda (r_0) > U_{k_0}^\lambda (r_0),
\]
which is a contradiction. Therefore the claim holds.

By the standard elliptic regularity argument, up to subsequences, $(u_k, v_k) \to (u_0, v_0)$ in $C^2_{\text{loc}} (\mathbb{R}^N)$ where $(u_0, v_0)$ is a nonnegative solution of
\[
\begin{align*}
-\Delta u_0 - \frac{N-1}{r} u_0' + u_0 &= u_0^3 + \beta u_0 v_0^2 - \gamma_0 v_0, \\
-\Delta v_0 - \frac{N-1}{r} v_0' + v_0 &= v_0^3 + \beta v_0 u_0^2 - \gamma_0 u_0.
\end{align*}
\tag{16}
\]
Set
\[
E_k (r) = \frac{1}{2} \left( (u_k')^2 + (v_k')^2 \right) + \frac{1}{4} \left( u_k^4 + v_k^4 \right) - \frac{1}{2} \left( u_k^2 + v_k^2 \right) + \frac{\beta}{2} u_k^2 v_k^2 - \gamma_k u_k v_k, \\
E_0 (r) = \frac{1}{2} \left( (u_0')^2 + (v_0')^2 \right) + \frac{1}{4} \left( u_0^4 + v_0^4 \right) - \frac{1}{2} \left( u_0^2 + v_0^2 \right) + \frac{\beta}{2} u_0^2 v_0^2 - \gamma_0 u_0 v_0.
\]
Then, we have that
\[
E_k' (r) = (u_k'' + u_k') - \beta u_k v_k' - \gamma_k v_k u_k') u_k' + (v_k'' + v_k') - \beta v_k u_k' - \gamma_k u_k v_k' v_k' = -\frac{N-1}{r} \left( (u_k')^2 + (v_k')^2 \right) \leq 0,
\]
i.e. $E_k (r)$ is decreasing. By Lemma 2.5, we know that $\lim_{r \to +\infty} E_k (r) = 0$, then $E_k (r) \geq 0$ for $r \geq 0$. Since $(u_k, v_k) \to (u_0, v_0)$ in $C^2_{\text{loc}} (\mathbb{R}^N)$, we have $E_k \to E_0$ in $C^1_{\text{loc}} (\mathbb{R}^N)$ and $E_0 (r) \geq 0$ for $r \geq 0$.

It follows from the claim that $u_0$ and $v_0$ are non-increasing in $[\lambda^*, +\infty)$ and bounded. So,
\[
\lim_{r \to +\infty} u_0 (r) =: u_{0, \infty} \geq 0 \quad \text{and} \quad \lim_{r \to +\infty} v_0 (r) =: v_{0, \infty} \geq 0.
\]
We claim that \( u_{0,\infty} = v_{0,\infty} = 0 \). If not, then \( u_{0,\infty} > 0 \) or \( v_{0,\infty} > 0 \). Since \( E_0, u_0 \) and \( v_0 \) are bounded in \([0, +\infty)\), \( u'_0 \) and \( v'_0 \) are also bounded. Letting \( r \to +\infty \) in (16), we obtain that

\[
\begin{aligned}
\left\{ \begin{array}{l}
u \in \mathbb{R}^+,  \\
u \in \mathbb{R}^+
\end{array} \right.
\end{aligned}
\]

\[
\begin{aligned}
u_{0,\infty} (\nu_{0,\infty}^2 + \beta \nu_{0,\infty}^2 - 1) - \gamma v_{0,\infty} = 0,
\end{aligned}
\]

\[
\begin{aligned}
u_{0,\infty} (\beta \nu_{0,\infty}^2 + \nu_{0,\infty}^2 - 1) - \gamma u_{0,\infty} = 0.
\end{aligned}
\]

(17)

Now by some elementary computations, we have that

\[
\lim_{r \to +\infty} E_0(r) = \frac{1}{4} (u_{0,\infty}^4 + v_{0,\infty}^4) - \frac{1}{2} (u_{0,\infty}^2 + v_{0,\infty}^2) + \frac{\beta}{2} u_{0,\infty} v_{0,\infty}^2 - \gamma u_{0,\infty} v_{0,\infty}
\]

\[
= -\frac{1}{4} (u_{0,\infty}^4 + v_{0,\infty}^4) - \frac{1}{2} u_{0,\infty}^2 v_{0,\infty}^2 < 0,
\]

which contradicts with the fact that \( E_0(r) \geq 0 \) for \( r \geq 0 \).

By Lemma 2.5, there exists \( M_0 > 0 \) such that \( u_0, v_0 \leq M_0 e^{-(N-1)/2} e^{-r(1+\gamma)/2} \) for \( r \) large enough. Set \( q_k(r) = 1 + \gamma_k = u_k^2(r) - (\beta - 1) u_k(r) v_k(r) - v_k^2(r) \). By \((u_k, v_k) \to (u_0, v_0)\) in \(C^2_{\text{loc}}(\mathbb{R}^N)\), there exist \( r_* > 0 \) and \( k_* \in \mathbb{N} \) such that \( q_k(r) + p(r) \geq (1 + \gamma)/2 \) for any \( r \geq r_* \) and \( k \geq k_* \), where \( p(r) \) is the same as that of Lemma 2.5. Then as that of Lemma 2.5, we can show that there exists \( M > 0 \) which is independent on \((u_k, v_k)\) such that \( u_k, v_k \leq M r^{-\frac{N-1}{2}} e^{-r(1+\gamma)/2} \) for \( r \geq r_* \) and \( k \geq k_* \), which is a contradiction. Finally, by an argument similar to that of Lemma 2.5, we can obtain the exponential decay of \(|u'_k + v'_k|\).

With Lemma 2.4 and Lemma 2.6 in hand, we obtain the following uniform boundedness of nonnegative solutions of system (2) in \(X\).

**Lemma 2.7.** Given \( \beta \in (-1, +\infty) \) fixed. Then the set of nonnegative solutions of system (2) is uniformly bounded in \(X\) for all \( \gamma \in [\overline{\gamma}, \overline{\gamma}] \).

**Proof.** Suppose on the contrary that there exists a nonnegative solution sequence \( \{ (\gamma_n, u_n, v_n) \} \) of system (4) with \( \gamma_n \in [\overline{\gamma}, \overline{\gamma}] \), such that \( \|u_n\| + \|v_n\| \to +\infty \) and \( \gamma_n \to \gamma^* \in [\gamma, \overline{\gamma}] \) as \( n \to +\infty \). Note that when \( \gamma^* = \overline{\gamma} \) or \( \gamma^* = \gamma \), the limit should be understood as one-side limit.

From Lemma 2.6, we see that there exist \( R > 0 \) and \( M > 0 \) such that

\[
u_n, v_n \leq M r^{-\frac{N-1}{2}} e^{-r(1+\gamma)/2}
\]

for any \( r \geq R \) and any \( n \in \mathbb{N} \). Invoking Lemma 2.4, we have that

\[
\int_{\mathbb{R}^N} u_n^2 dx = \int_{B_{R_0}(0)} u_n^2 dx + \int_{B_{R_0}^c(0)} u_n^2 dx \leq C \left( 1 + \int_{R}^{+\infty} e^{-r(1+\gamma)} dr \right)
\]

\[
\leq C \left( 1 + \frac{1}{1 + \gamma} e^{-R(1+\gamma)} - \frac{1}{1 + \gamma} \lim_{r \to +\infty} e^{-r(1+\gamma)} \right)
\]

for some positive constant \( C \), where \( B_{R_0}^c(0) \) is the complementary set of \( B_{R_0}(0) \) in \(\mathbb{R}^N\). Since \( \gamma \in (-1, \overline{\gamma}] \), one has that

\[
\lim_{r \to +\infty} e^{-r(1+\gamma)} = 0.
\]

It follows that \( \|u_n\|_{L^2} \) is uniformly bounded. Similarly, we have that \( \|v_n\|_{L^2} \) is uniformly bounded. So we must have \( \|\nabla u_n\|_{L^2} \to +\infty \) or \( \|\nabla v_n\|_{L^2} \to +\infty \) as \( n \to +\infty \). Without loss of generality, we assume that \( \|\nabla u_n\|_{L^2} \to +\infty \) as \( n \to +\infty \).
Multiplying the equation for $u$ in system $(2)$ by $u_n/\|\nabla u_n\|_{L^2}^2$ and integrating on $\mathbb{R}^N$, then using Lemma 2.4, we obtain that

$$
1 = \int_{\mathbb{R}^N} u_n^2 dx + \beta \int_{\mathbb{R}^N} v_n^2 dx - \|u_n\|_{L^2}^2 - \gamma_n \int_{\mathbb{R}^N} u_n v_n dx
$$

$$
\leq \frac{C \int_{\mathbb{R}^N} u_n^2 dx}{\|\nabla u_n\|_{L^2}^2} + \|\beta\| \frac{C \int_{\mathbb{R}^N} v_n^2 dx}{\|\nabla u_n\|_{L^2}^2} - \|u_n\|_{L^2}^2
$$

$$
- \gamma_n \left( \int_{\mathbb{R}^N} u_n^2 dx \right)^{1/2} \left( \int_{\mathbb{R}^N} v_n^2 dx \right)^{1/2} \xrightarrow{n \to +\infty} 0
$$
as $n \to +\infty$, which is a contradiction. \hfill \Box

To guarantee the positiveness of bifurcation solutions, we consider the following modified system of $(2)$

$$
\begin{aligned}
-\Delta u + u &= u_+^3 + \beta u_+^2 v - \gamma v, \\
-\Delta v + v &= v_+^3 + \beta v_+^2 u - \gamma u,
\end{aligned}
$$

where $u_+ = \max\{u, 0\}$ and $v_+ = \max\{v, 0\}$. For any $\gamma \in (-1, 0)$ and $\beta \in \mathbb{R}$, system $(18)$ has only nonnegative solutions. To see this fact, let $u_- = \min\{u, 0\}$ and $v_- = \min\{v, 0\}$. Then multiplying the first equation with $u_-$, the second with $v_-$ and integrating, we obtain that

$$
\|u_-\|^2 + \int_{\mathbb{R}^N} \gamma u_- v_- \, dx \leq \|u_-\|^2 + \int_{\mathbb{R}^N} \gamma u_- v \, dx = 0,
$$

$$
\|v_-\|^2 + \int_{\mathbb{R}^N} \gamma u_- v_- \, dx \leq \|u_-\|^2 + \int_{\mathbb{R}^N} \gamma u_- v_- \, dx = 0,
$$

where the facts $\gamma u_- \geq 0$, $u_- \leq v$, $\gamma v_- \geq 0$ and $u_- \leq u$ in $\mathbb{R}^N$ are used. Adding the above two inequalities together, we have that

$$
0 \leq (1 + \gamma) \left( \|u_-\|^2 + \|v_-\|^2 \right) \leq \|u_-\|^2 + \|v_-\|^2 + 2 \int_{\mathbb{R}^N} \gamma u_- v_- \, dx \leq 0.
$$

It follows that $u_- \equiv 0$ and $v_- \equiv 0$. Therefore, $u \geq 0$ and $v \geq 0$ in $\mathbb{R}^N$. Since $(18)$ does not have semi-trivial solution, for any $\gamma \in (-1, 0)$ and $\beta \in \mathbb{R}$, the strong maximum principle implies that every nontrivial solution of system $(18)$ is a positive solution of system $(2)$.

To prove Theorem 2.1, we present a Rabinowitz’s type global bifurcation result. Let $E$ be a real Banach space with the norm $\|\cdot\|$, $\mathcal{O}$ be an open subset of $\mathbb{R} \times E$ and $\text{pr}_E(\mathcal{O})$ be the projection of $\mathcal{O}$ on $E$. Consider the following operator equation

$$
u = \lambda Lu + H(\lambda, u) := G(\lambda, u),
$$

where $L : \text{pr}_E(\mathcal{O}) \to \text{pr}_E(\mathcal{O})$ is a compact linear operator and $H : \mathcal{O} \to E$ is compact with $H(0, ||u||)$ at $u = 0$ uniformly on bounded $\lambda$ intervals in $\mathcal{O}$. Denote

$$\mathcal{S} := \{ (\lambda, u) : (\lambda, u) \text{ satisfies equation } (19) \text{ and } u \neq 0 \}.$$

Let $\text{pr}_E(\mathcal{O})$ be the projection of $\mathcal{O}$ on $\mathbb{R}$ and $r(L)$ be the characteristic value set of $L$. Thus the Leray-Schauder degree $\deg(I - G(\lambda, u), B_r(0), 0)$ is well defined for arbitrary $r$-ball $B_r(0)$ in $\mathcal{O}$ and $\lambda \notin r(L)$. Applying similar arguments in the proof of [29, Lemma 1.2 and Theorem 1.3] with obvious changes, we obtain the following result.
Lemma 2.8. If $\mu \in pr_\mathbb{R}(\mathcal{O})$ is a characteristic value of $L$ such that the Leray-Schauder degree $\deg(I - G(\lambda, u), B_r(0))$ changes when $\lambda$ passes $\mu$, then $\mathcal{S}$ possesses a maximal subcontinuum $\mathcal{C}_\mu \subset \mathcal{O}$ such that $(\mu, 0) \in \mathcal{C}_\mu$ and one of the following three properties is satisfied by $\mathcal{C}_\mu$:

(i) $\mathcal{C}_\mu$ is unbounded in $\mathcal{O}$,
(ii) meets $\partial \mathcal{O}$,
(iii) meets $(\overline{\mathcal{O}}, 0)$, where $\overline{\mathcal{O}} \in pr_\mathbb{R}(\mathcal{O})$ is another characteristic value of $L$.

If $\mathcal{O}$ is bounded, Lemma 2.8 is just Corollary 1.12 of [29]. So Lemma 2.8 can be seen the complement of Corollary 1.12 of [29].

Proof of Theorem 2.1. The local bifurcations of system (4) has been established in [33], therefore we shall concentrate on the number of bifurcations with respect to $T_{w}\beta$ and also the behavior of global bifurcations.

(a) It has proved in [33] that $\gamma$ is a parameter of local bifurcation if and only if

$$f(\beta) = \eta_l(\gamma)$$

for some $j \geq 1$. Fix $k \geq 1$ and $\beta \in [\beta_k, \beta_{k+1})$. This is equivalent to $f^{-1}(\eta_k) \leq \beta < f^{-1}(\eta_{k+1})$. But $f^{-1}(\eta_l(0)) = 1$, which implies $\beta \leq 1$. Since $f$ is strictly decreasing in $(-1, +\infty)$, there holds

$$f(\beta_k) < f(\beta) \leq f(\beta_{k+1}),$$

i.e. $\eta_k(0) < f(\beta) \leq \eta_{k+1}(0)$. By Lemma 2.2 there exists a unique number $\gamma_l \in (-1, 0)$ for each $l \in \{1, 2, \ldots, k\}$ such that $\eta_l(\gamma_l) = f(\beta)$.

From now on, we fix $\beta \in [\beta_{k+1}, \beta_k]$ and $l \in \{1, 2, \ldots, k\}$. Define the functional $I_\gamma^+: X \to \mathbb{R}$ of (18) by

$$I_\gamma^+(u, v) = \frac{1}{2} (\|u\|^2 + \|v\|^2) + \gamma \int_{\mathbb{R}^N} uv \, dx - \frac{1}{4} \int_{\mathbb{R}^N} (u_+^4 + v_+^4) \, dx - \frac{\beta}{2} \int_{\mathbb{R}^N} u_+^2 v_+^2 \, dx.$$

Let $(u_\gamma, v_\gamma)$ be a critical point of $I_\gamma^+$ on $T_{w}\beta$. By arguments similar to [6, Lemma 3.3] with obvious changes, we get

$$\deg(\nabla I_\gamma^+(u_\gamma, v_\gamma)) = (-1)^{m(\gamma)},$$

where $m(\gamma)$ is the index of the quadratic form $D^2 I_\gamma^+(u_\gamma, v_\gamma)$. Since every eigenvalue of problem (6) has multiplicity one, so by (20), $\deg(\nabla I_\gamma^+(u_\gamma, v_\gamma))$ changes when $\gamma$ crosses a value $\gamma_l$. Now it follows from Lemma 2.8 by taking $\mathcal{O} = (-1, 0) \times X$ that there exists a global $\gamma$-bifurcation branch $S_\beta^\gamma$ occurs at $(\gamma_l, u_{\gamma_l}, v_{\gamma_l})$. According to [33], the kernel space of linearized system of (4) is spaced by $\{((\phi_l, -\phi_l))\}$, where $\phi_l$ is the eigenfunction corresponding to $\eta_l(\gamma_l)$ of (6). Note that $\phi_l$ has $l - 1$ simple zeros. Now for $(\gamma, u, v) \in S_\beta^\gamma$ near the bifurcation point $(\gamma_l, u_{\gamma_l}, v_{\gamma_l})$, it follows from [29, Lemma 1.24] that

$$u = u_{\gamma_l} + (\gamma - \gamma_l) \phi_l + o(\gamma - \gamma_l), \quad v = v_{\gamma_l} - (\gamma - \gamma_l) \phi_l + o(\gamma - \gamma_l).$$

That is to say, $S_\beta^\gamma$ is curve near the bifurcation point $(\gamma_l, u_{\gamma_l}, v_{\gamma_l})$ in space $X$ and then also in $C^1(\mathbb{R}^N) \times C^1(\mathbb{R}^N)$ by bootstrap the perturbation term $o(\gamma - \gamma_l)$. Therefore

$$u - v = 2(\gamma - \gamma_l) \phi_l + o(\gamma - \gamma_l).$$

has precisely $l - 1$ simple zeroes provided $\gamma$ is close to $\gamma_l$. Let $S_j$ denote the set of functions in $H^1_r(\mathbb{R}^N)$ also in $C^1_{loc}(\mathbb{R}^N)$ by bootstrap argument which have exactly $j - 1$ interior simple zeroes for any $j \in \mathbb{N}$. So $u - v \in S_l$ provided $\gamma$ is close to $\gamma_l$.

We claim that $u - v \in S_l$ for any $(\gamma, u, v) \in S_\beta^\gamma$. 


Suppose on the contrary that there exists $\gamma, u, v \in S^\beta_l$ such that $u - v \notin S_l$. Then there exists $(\gamma^*, u^*, v^*) \in S^\beta_l$ such that $h := u_* - v_*$ has a double zero because $S_l$ is an open set (see [29]). Without loss of generality, we may assume that $(\gamma^*, u^*, v^*)$ be the first such kind of point as $\gamma$ moves away from $\gamma_l$ along $S_l$. Then one can see that $h$ satisfies the following equation

$$-h''(r) - \frac{N - 1}{r} h' + (1 - \gamma^*) h = (u^2 + (1 - \beta) u_* v_* + v^2) h(r).$$

Then by an argument similar to [6, Theorem 2.3], we can easily show that $h(r) \equiv 0$. Thus $S^\beta_l$ returns to the synchronized branch at $\gamma = \gamma_*$. Since $(0, u_0, v_0)$ is not a bifurcation point of system (2) with respect to $T_{w|\beta}$, we have $\gamma \neq \gamma_0$. It follows that $\gamma_0 = \gamma_m$ for some $m \in \{1, 2, \ldots, k\}$ \{l\}. Reasoning as the above, $u - v$ has precisely $m - 1$ simple zeroes provided $\gamma$ is close to $\gamma_m$ for any $(\gamma, u, v) \in S^\beta_l$. So there must exist a point $(\gamma^*, u^*, v^*) \in S^\beta_l$ with $\gamma^* \in (\gamma_l, \gamma_m) \cup (\gamma_m, \gamma_l)$ such that $u^* - v^*$ has a double zero. Similarly, one has $u^* - v^* \equiv 0$. This contradicts the fact that $(\gamma^*, u^*, v^*)$ be the first such kind of point.

So we obtain that $u - v$ has precisely $l - 1$ simple zeroes for any $(\gamma, u, v) \in S^\beta_l \setminus \{\gamma_l, u_{\gamma_l}, v_{\gamma_l}\}$ and $S^\beta_l$ does not return to the synchronized branch for any $1 \leq l \leq k$. It follows that $S^\beta_l \cap (T_{w|\beta} = (\gamma_l, u_{\gamma_l}, v_{\gamma_l})$, i.e. the third alternative of Lemma 2.8 cannot occur. Moreover, by Lemma 2.3, no bifurcation branch can reach to the boundary $\{-1\} \times X$. On the other hand, if $S^\beta_l$ meets the boundary $\{0\} \times X$, then the corresponding solution $(u, v)$ cannot be semitrivial, due to the nodal property of $u - v$ and $u, v$ are nonnegative. At last, by Lemma 2.7, the remaining possibilities of Lemma 2.8 then give us part (a).

(b) For any $\beta \in [1, +\infty)$, one has that $f(\beta) \leq f(1) = \lambda_1 = 1$. Then the continuity and monotonicity of $\eta_j(\gamma)$ $(j \geq 1)$ implies that there does not exist $\gamma \in (-1, 0)$ such that $\eta_j(\gamma) = f(\beta)$. So system (2) has no bifurcation point in $(-1, 0)$ with respect to $T_{w|\beta}$. \hfill $\square$

Remark 2. Note that for fixed $\beta_0 \in (0, +\infty)$ and $l \in \mathbb{N}$, if there exist $\gamma_0 \geq 0$ such that $f(\beta_0) = \eta_1(\gamma_0)$, then using the same arguments as Theorem 2.1, we can obtain that $(\gamma_0, u_{\gamma_0}, v_{\gamma_0})$ is a bifurcation point of system (2) with respect to $T_{w|\beta}$. However, as it is explained in [33, Remark 3.2], the above condition is not always satisfied. So we only consider the case $\gamma \in (-1, 0]$.

3. Five dimensional bifurcation when $\gamma = 0$. In this section, we investigate the bifurcation phenomenon of system (2) with $\gamma = 0$, i.e.,

$$\begin{cases}
-\Delta u + \lambda_1 u = \mu_1 u^3 + \beta uv^2 & \text{in } \mathbb{R}^N, \\
-\Delta v + \lambda_2 v = \mu_2 v^3 + \beta vu^2 & \text{in } \mathbb{R}^N, \\
u(x), v(x) \to 0 & \text{as } |x| \to +\infty.
\end{cases} \tag{21}
$$

For any $b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \beta) \in \mathbb{R}_+^4 \times [0, +\infty)$ with $\mathbb{R}_+ = (0, +\infty)$, problem (21) has the following three solutions

$$U_0 = (0, 0), \ U_{b,1} = (w_{\lambda_1, \mu_1}, 0), \ U_{b,2} = (0, w_{\lambda_2, \mu_2}),$$

where $w_{\lambda, \mu}$ denote the unique positive solution of (3). We shall use a global multi-parameter bifurcation theorem by P. M. Fitzpatrick et al. [20] to improve the corresponding results of [8]. As pointed out in [20, Remark 2.4], Theorem 2.4
of [20] improves the corresponding result of Alexander and Antman [2, Theorem 2.2] because its assumptions are somewhat weaker, and the global structure of bifurcation branch is given. We can obtain the global structure of bifurcation branch by directly applying Theorem 2.4 of [20] rather than study it via Čech cohomology as [8].

By an argument similar to that of [22, Corollary 2.4] with obvious changes, we have the following compactness result.

**Lemma 3.1.** If \( B \subset (\mathbb{R}^4_+ \times [0, +\infty)) \) is compact, then the set
\[
S_B := \{ (u, v) : (u, v) \text{ is radially symmetric positive solution of problem (21) with } b \in B \}
\]
is compact in \( X \).

Now we describe the five dimensional bifurcation branches. In fact, any positive solutions of system (21) with \( \beta > 0 \) must be radially symmetric and decreasing (see [10]). To find positive solutions, we confine the problem to the nonnegative cone \( \mathbb{P} := \{ \overrightarrow{u} = (u, v) \in X : u \geq 0, v \geq 0 \} \).

Then we shall find solutions in \( \mathbb{R}^5_+ \times \mathbb{P} \) via bifurcation technique. In order to state the main results, it is convenient to introduce some notations. Let
\[
\mathcal{T}_j = \{ (b, U_{b,j}) : b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \beta) \in \mathbb{R}^5_+ \}, \ j = 1, 2
\]
be the set of trivial solutions and
\[
\mathcal{S} = \{ (b, \overrightarrow{u}) : (b, \overrightarrow{u}) \in \mathbb{R}^5_+ \times \mathbb{P} \text{ solves system (21), } u > 0, v > 0 \}.
\]
We also need the function \( \xi : \mathbb{R}^4_+ \to \mathbb{R}^4_+ \) defined by
\[
\xi(s) := \inf_{\phi \in H^1_1(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} (|\nabla \phi|^2 + s \phi^2) \, dx}{\int_{\mathbb{R}^N} w^2 \phi^2 \, dx},
\]
where \( w = w_{1,1} \) is the unique positive ground state solution of (3). Without loss of generality, we may assume that \( \mu_1 \leq \mu_2 \).

The main result of this section is the following theorem.

**Theorem 3.2.** There exist connected sets \( \mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{S} \) with
\[
\mathcal{C}_1 \cap \mathcal{T}_1 = \{ (b, U_{b,1}) : b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \mu_1 \xi(\lambda_2/\lambda_1)) \},
\]
\[
\mathcal{C}_2 \cap \mathcal{T}_2 = \{ (b, U_{b,2}) : b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \mu_2 \xi(\lambda_1/\lambda_2)) \}.
\]
These sets have topological dimension at least 5 at every point and are bounded on any compact set of \( \mathbb{R}^5_+ \times [0, +\infty) \). Moreover, one of the following two properties is also satisfied by \( \mathcal{C}_j, \ j = 1, 2 \):
(a) \( \mathcal{C}_j \) is unbounded;
(b) \( \mathcal{C}_j \cap \partial (\mathbb{R}^5_+ \times \mathbb{P}) \neq \emptyset \).

The existence of connected sets \( \mathcal{C}_j \subset \mathcal{S} \) emanating from \( b_j, \ j = 1, 2 \) in fact has been proved in [8], where
\[
b_1 = (\lambda_1, \lambda_2, \mu_1, \mu_2, \mu_1 \xi(\lambda_2/\lambda_1)), \ b_2 = (\lambda_1, \lambda_2, \mu_1, \mu_2, \mu_2 \xi(\lambda_1/\lambda_2)).
If \(\lambda_1 = \lambda_2\) and \(\mu_1 \neq \mu_2\), Bartsch-Wang-Wei also showed that \(C_1 \cup C_2\) covers \((B_1^- \cap B_2^+) \cup (B_1^+ \cap B_2^-)\), where

\[
\begin{align*}
B_1^- &= \{ b = (\lambda_1, \lambda_1, \mu_1, \mu_2, \beta) \in \mathbb{R}_+^5 : \beta < \mu_1 \xi(1) \}, \\
B_1^+ &= \{ b = (\lambda_1, \lambda_1, \mu_1, \mu_2, \beta) \in \mathbb{R}_+^5 : \beta > \mu_1 \xi(1) \}, \\
B_2^- &= \{ b = (\lambda_1, \lambda_1, \mu_1, \mu_2, \beta) \in \mathbb{R}_+^5 : \beta < \mu_2 \xi(1) \}, \\
B_2^+ &= \{ b = (\lambda_1, \lambda_1, \mu_1, \mu_2, \beta) \in \mathbb{R}_+^5 : \beta > \mu_2 \xi(1) \}.
\end{align*}
\]

Although the conditions \(\lambda_1 = \lambda_2\) and \(\mu_1 \neq \mu_2\) are not written out explicitly in [8, Theorem 1.1], they are used in their argument (see [8, P. 361]) in order to get \(C_1 \cup C_2\) covers \((B_1^- \cap B_2^+) \cup (B_1^+ \cap B_2^-)\). While, we do not need these conditions in Theorem 3.2. So Theorem 3.2 improves the corresponding ones of [8] in this sense.

The energy functional associated with system (21) is

\[
E_b(u,v) = \frac{1}{2} \int_{\mathbb{R}^N} \left| \nabla u \right|^2 + \left| \nabla v \right|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} (\lambda_1 u^2 + \lambda_2 v^2) \, dx - \frac{\mu_1}{4} \int_{\mathbb{R}^N} u^4 \, dx - \frac{\mu_2}{4} \int_{\mathbb{R}^N} v^4 \, dx - \frac{\beta}{2} \int_{\mathbb{R}^N} u^2 v^2 \, dx
\]

for \(b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \beta) \in \mathbb{R}_+^5\) and \(\overline{u} = (u,v) \in X\). As a consequence of the Sobolev embedding theorem, \(E_b : X \to \mathbb{R}\) is well-defined and is a \(C^2\)-functional. The gradient of \(E_b\) with respect to \((\cdot, \cdot)_b\) can be computed as

\[
\nabla_b E(\overline{u}) = -\nabla \Lambda (\overline{u}) - f_b(\overline{u}) = \overline{u} - A_b(\overline{u}),
\]

where \(\Lambda = \text{diag}(\lambda_1, \lambda_2)\) and \(f_b(\overline{u}) = (\mu_1 u^3 + \beta uv^2, \mu_2 v^3 + \beta vu^2)\). By the compact embedding of \(H_0^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)\) for \(2 < q < 2^*\) the map

\[
A_b : \mathbb{R}_+^5 \times X \to X, \quad A_b(\overline{u}) = A_b(\overline{u})
\]

is completely continuous. It has been shown that in [8] the only possible bifurcation point on \(T_j\) is \(b_j\). Since \(P \subset X\) is closed and convex, there exists a retraction \(r : X \subset P\). To find the fixed point of \(A_b\) in \(P\) is equivalent to find \(\overline{u} \in X\) such that

\[
\overline{u} - A_b(r(\overline{u})) = 0, \quad b \in \mathbb{R}_+^5.
\]

**Proof of Theorem 3.2.** Taking \(\mathcal{O} = \mathbb{R}_+^5 \times \bar{X}\) with \(\bar{X} = \{ \overline{u} - U_{b,1} : \overline{u} \in X \}\), then \(\mathcal{O}\) is an open subset of \(\mathbb{R}^5 \times \bar{X}\). Define \(F : \mathcal{O} \to \bar{X}\) by

\[
F(b, \overline{u}) = A_b(r(U_{b,1} + \overline{u})) - U_{b,1}.
\]

Then \(F\) is a completely continuous mapping satisfying \(F(b,0) = 0\) whenever \((b,0) \in \mathcal{O}\). We refer to \(\{ \mathbb{R}^5 \times \{0\} \} \cap \mathcal{O}\) as the trivial solutions. Clearly, \(F(b, \overline{u}) = \overline{u}\) is equivalent to \(\overline{u} := U_{b,1} + \overline{u} \in X\) solving \(A_b(r(\overline{u})) = \overline{u}\).

For any fixed \(b^* = (\lambda_1, \lambda_2, \mu_1, \mu_2, \beta^*) \in \mathbb{R}^5\) with \(\beta^* = \mu_1 \xi(\lambda_2^*/\lambda_1^*)\), let \(h : \mathbb{R}^5 \to \mathbb{R}^4\) be defined by

\[
h(\lambda_1, \lambda_2, \mu_1, \mu_2, \beta) = (\lambda_1, \lambda_2, \mu_1, \mu_2) - (\lambda_1^*, \lambda_2^*, \mu_1^*, \mu_2^*)
\]

Clearly, \(h\) is continuously differentiable and has \(0\) as a regular value. Let \(\Gamma = h^{-1}(0)\), then \(b^* \in \Gamma\). For any \(\varepsilon > 0\), taking

\[
\tilde{b} = (\lambda_1^*, \lambda_2^*, \mu_1^*, \mu_2^*, \beta^* - \varepsilon), \quad \tilde{b} = (\lambda_1^*, \lambda_2^*, \mu_1^*, \mu_2^*, \beta^* + \varepsilon).
\]
It is easy to see that \( b \) and \( \overline{b} \) lie in the same component of \( \Gamma \). Choosing \( \varepsilon \) small enough, we can assume that neither \( b \) nor \( \overline{b} \) are \( \Gamma \) bifurcation points of \( \mathcal{F} := I - F \). Lemma 2.4 of [8] shows that

\[
\text{ind} \left( \mathcal{F}_b, 0 \right) \neq \text{ind} \left( \mathcal{F}_{\overline{b}}, 0 \right).
\]

Now, all the assumptions of Theorem 2.4 of [20] are satisfied. So there exists a closed connected subset, \( \tilde{C}_1 \), of \( \{(b, \overline{v}^i) : (b, \overline{v}^i) \in \mathcal{O}, \overline{v}^i \neq 0, \mathcal{F} (b, \overline{v}^i) = 0\} \), whose dimension at each point is at least 5, and \( \tilde{C}_1 \cap \Gamma = b^* \). Moreover, one of the following three properties is also satisfied by \( \tilde{C}_1 \):

(a) \( \tilde{C}_1 \) is unbounded,
(b) \( \tilde{C}_1 \cap \partial \mathcal{O} \neq \emptyset \),
(c) \( \tilde{C}_1 \cap \{\Gamma \setminus \{b^*\}\} \neq \emptyset \).

Lemma 2.2 of [8] implies that the third alternate does not occur. Let

\[
C_1 = \left\{(b, U_{b,1} + \overline{v}^i) : (b, \overline{v}^i) \in \tilde{C}_1\right\}.
\]

Then \( C_1 \) bifurcates from \( T_1 \) at \( b^* \) and satisfies the alternatives (a) or (b). From Lemma 3.1, we can easily get that if \( B \subset \mathbb{R}_{+}^4 \times [0, +\infty) \) is compact then there exists a uniform bound \( R > 0 \) such that \( C_1 \cap (B \times \mathbb{P}) \subset B \times B_R(0) \).

Similarly, we can get the set \( C_2 \) emanating from \( T_2 \) at \( b_2 \) and satisfying the desired conclusions.

4. **A few results comparing with Bartsch-Dancer-Wang's work when \( \gamma = 0 \)**

Note that the results of Theorem 3.2 imply \( C_j \) covers \( B_j^- \) or \( B_j^+ \) for any fixed \( \lambda_1, \lambda_2, \mu_1 \) and \( \mu_2 \). If \( \lambda_1, \lambda_2, \mu_1 \) and \( \mu_2 \) are fixed, \( \beta \) is the bifurcation parameter. Furthermore, if we allow \( \lambda_1 = \lambda_2 \), we can obtain the following theorem regarding the global structure of \( C_j \).

**Theorem 4.1.** If \( \lambda_1, \lambda_2, \mu_1 \) and \( \mu_2 \) are fixed constants with \( \lambda_1 = \lambda_2 \), then

\[
prC_1 = \{(\lambda_1, \lambda_1, \mu_1, \mu_2) \times [0, \mu_1) \}, \quad prC_2 = \{(\lambda_1, \lambda_1, \mu_1, \mu_2) \times (\mu_2, +\infty)\},
\]

where \( prC_j \) denotes the projection of \( C_j \) on \( \mathbb{R}_{+}^4 \times [0, +\infty), j = 1, 2 \).

We see from Theorem 4.1 that \( C_1 \cup C_2 \) covers \( (B_1^- \cap B_2^+) \cup (B_1^+ \cap B_2^-) \). Note that here we still do not require \( \mu_1 \neq \mu_2 \). So Theorem 4.1 improves the corresponding ones of [8] in the case of \( \lambda_1 = \lambda_2 \). To prove Theorem 4.1, we first show the following theorem.

**Theorem 4.2.** Suppose that \( \lambda_1 = \lambda_2 = \lambda > 0 \) and \( \mu_1 = \mu_2 = \mu > 0 \). Then all the positive solutions of system (21) at \( \beta = \mu \) have the following form

\[
u = \sqrt{\frac{\lambda}{\mu}} \cos \theta \cdot w \left( \sqrt{\lambda x} \right), \quad \nu = \sqrt{\frac{\lambda}{\mu}} \sin \theta \cdot w \left( \sqrt{\lambda x} \right), \quad \theta \in \left( 0, \frac{\pi}{2} \right).
\]

Theorem 4.2 has been proved by Wei-Yao [35] in the case of \( N = 1 \). From Theorem 3.2, we can see that \( C_1 \cup C_2 \) cannot blow up in \( \mathbb{R}_{+}^4 \times [0, +\infty) \). Moreover, for given \( \lambda = (\lambda_1, \lambda_2, \mu_1, \mu_2) \in \mathbb{R}_{+}^4 \) with \( \lambda_1 = \lambda_2 = \lambda > 0 \), [35, Theorem 1.1 and Theorem 1.3] and [11, Theorem 1.1] show that system (21) has a unique positive solution for \( \beta \in [0, \beta_0) \cup (\mu_1 - \beta^0, \mu_1) \cup (\mu_2, +\infty) \) for some \( \beta_0, \beta^0 > 0 \) small enough. It is just the following synchronized solution

\[
(u_0, v_0) = \left( \frac{\lambda (\mu_2 - \beta)}{\mu_1 \mu_2 - \beta^2} w \left( \sqrt{\lambda x} \right), \frac{\lambda (\mu_1 - \beta)}{\mu_1 \mu_2 - \beta^2} w \left( \sqrt{\lambda x} \right) \right).
\]
Combine these facts with Theorems 4.1–4.2, we obtain the following corollary.

**Corollary 1.** Given \( b = (\lambda_1, \lambda_2, \mu_1, \mu_2) \in \mathbb{R}_+^4 \) with \( \lambda_1 = \lambda_2 \) and \( \mu_1 \leq \mu_2 \), problem (21) has

(i) at least one positive solution for any \( \beta \in [\beta_0, \mu_1 - \beta^0] \) with some \( \beta_0, \beta^0 > 0 \) small enough which lies on \( S_0 \);

(ii) a unique positive solution \((u_0, v_0)\) for any \( \beta \in [0, \beta_0) \cup (\mu_1 - \beta^0, \mu_1) \cup (\mu_2, +\infty) \).

Moreover, if \( \mu_1 = \mu_2 = \mu > 0 \), then system (21) has infinitely many positive solutions at \( \beta = \mu \) which must be in the form of (22).

See Figure 2 for illustrations of Corollary 1.

![Figure 2](image-url)

**Figure 2.** Schematic diagrams of \( \beta \)-bifurcation branches

From the results of [6], we know that \( S_0 \) can extend to \( \beta < 0 \) if \( \lambda_1 = \lambda_2 \). In fact, \( S_0 \) at least contains the following synchronized solution branch

\[
T_0^0 = \{ (\beta, u_0, v_0) : \beta \in (-\sqrt{\mu_1 \mu_2}, \mu_1) \}.
\]

When \( \lambda_1 = \lambda_2 = 1 \), \( \mu_1 \leq \mu_2 \), Bartsch-Dancer-Wang [6] gave detailed descriptions about local and global bifurcations of positive solutions to system (21) with respect to \( T_0^0 \). Let

\[
g : (-\sqrt{\mu_1 \mu_2}, \mu_1) \to (1, +\infty), \quad g(\beta) = \frac{3\mu_1 \mu_2 - 2\beta (\mu_1 + \mu_2) + \beta^2}{\mu_1 \mu_2 - \beta^2}
\]

and \( \eta_k (k \in \mathbb{N}) \) be the \( k \)th eigenvalue of (5). Then \( \beta_k = g^{-1}(\eta_k) (k \geq 2) \) is shown to be a bifurcation parameter (see Lemma 3.1 and Lemma 3.2 of [6]). Noting Theorems 2.1, 2.3 of [6] and Corollary 1, we can immediately get the following corollary (see Figure 3).

**Corollary 2.** Suppose that \( \lambda_1 = \lambda_2 = 1 \), \( \mu_1 = \mu_2 = 1 \). Then for each integer \( k \geq 2 \) there exists a component \( S_k \) of positive solutions of problem (21) emanating at \( (\beta_k, u_{\beta_k}, v_{\beta_k}) \in T_0^0 \) and there are no other bifurcation points along \( T_0^0 \) with \( \beta < 1 \). The projection \( \text{pr} : S_k \to \mathbb{R} \) onto the parameter space satisfies \((-\infty, \beta_k) \subset \text{pr}(S_k) \subset (-\infty, \beta_2) \). For any \((\beta, u, v) \in S_k \) the difference \( u - v \) has precisely \( k - 1 \) simple zeroes. Moreover, problem (21) has infinitely many positive solutions at \( \beta = 1 \) which must be in the form of (22) with \( \lambda = \mu = 1 \).

The difference between Corollary 2 and [6, Theorem 2.3] is the conclusion at \( \beta = 1 \). In [6], Bartsch-Dancer-Wang showed that at the point \( \beta = 1 \) the bifurcating solutions are explicitly given by (22) with \( \lambda = \mu = 1 \) and there are no further
solutions of problem (21) near \((1, w, 0)\) or \((1, 0, w)\). While, we can see from Corollary 2 that problem (21) with \(\beta = 1\) only has positive solutions that are explicitly given by (22) with \(\lambda = \mu = 1\). So Corollary 2 improves the corresponding results of [6].

Proof of Theorem 4.2. Assume that \((u, v)\) be a positive solution of system (21) at \(\beta = \mu\). Then we multiply the first equation in problem (21) by \(u\), the second equation by \(u^2 / (v + \varepsilon)\) for any \(\varepsilon > 0\) small enough, and integrate resulting equations over \(\mathbb{R}^N\). This yields

\[
\int_{\mathbb{R}^N} \left( |\nabla u|^2 + \frac{u^2}{(v + \varepsilon)^2} |\nabla v|^2 - 2 \frac{u}{v + \varepsilon} \nabla u \nabla v \right) \, dx = \int_{\mathbb{R}^N} \Gamma_\varepsilon(x) \, dx,
\]

where

\[
\Gamma_\varepsilon = \mu u^4 \left( 1 - \frac{v}{v + \varepsilon} \right) + \mu u^2 v^2 \left( 1 - \frac{v}{v + \varepsilon} \right) + \lambda u^2 \left( \frac{v}{v + \varepsilon} - 1 \right).
\]

In particular, we take \(\varepsilon = 1/n\). Then we have that

\[
\Gamma_{1/n} \leq \mu u^4 + \mu u^2 v^2 =: F.
\]

Since the embedding of \(H_1^1(\mathbb{R}^N) \hookrightarrow L^4(\mathbb{R}^N)\) is compact, one has that

\[
\int_{\mathbb{R}^N} F(x) \, dx = \mu \int_{\mathbb{R}^N} u^4 \, dx + \mu \int_{\mathbb{R}^N} u^2 v^2 \, dx
\]

\[
\leq \mu \int_{\mathbb{R}^N} u^4 \, dx + \mu \left( \int_{\mathbb{R}^N} u^4 \, dx \right)^{1/2} \left( \int_{\mathbb{R}^N} v^4 \, dx \right)^{1/2} < +\infty.
\]

So \(F\) is an integrable function defined on \(\mathbb{R}^N\). On applying the Dominated Convergence Theorem we find that

\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} \Gamma_{1/n}(x) \, dx = \int_{\mathbb{R}^N} \lim_{n \to +\infty} \Gamma_{1/n}(x) \, dx = 0.
\]

Furthermore, by the Fatou’s Lemma, we obtain that

\[
0 \leq \int_{\mathbb{R}^N} \left( |\nabla u|^2 + \frac{u^2}{v + \varepsilon} |\nabla v|^2 - 2 \frac{u}{v + \varepsilon} \nabla u \nabla v \right) \, dx
\]

\[
= \int_{\mathbb{R}^N} \liminf_{n \to +\infty} \left( |\nabla u|^2 + \frac{u^2}{(v + 1/n)^2} |\nabla v|^2 - 2 \frac{u}{v + 1/n} \nabla u \nabla v \right) \, dx
\]

\[\text{Figure 3. Global bifurcation diagram in the symmetric case } \mu_1 = \mu_2 = 1.\]
\[
\leq \liminf_{n \to +\infty} \int_{\mathbb{R}^n} \left( \nabla u^2 + \frac{u^2}{(v + 1/n)^2} |\nabla v|^2 - 2\frac{u}{v + 1/n} \nabla u \nabla v \right) dx
\]

= \liminf_{n \to +\infty} \int_{\mathbb{R}^n} \Gamma_{1/n}(x) \, dx = 0.

It follows that \( u = cv \) for some positive constant \( c \). Substituting \( u = cv \) into system (21), we can get

\[-\Delta v + \lambda v = \mu (1 + c^2) v^3.\]

Consequently we have that

\[v = \sqrt{\frac{\lambda}{\mu}} \sqrt{\frac{1}{1 + c^2}} w\left(\sqrt{\lambda} x\right), \quad u = c \sqrt{\frac{\lambda}{\mu}} \sqrt{\frac{1}{1 + c^2}} w\left(\sqrt{\lambda} x\right).\]

Clearly, there exists \( \theta \in (0, \pi/2) \) such that \( c\sqrt{1/(1 + c^2)} = \cos \theta \). Then one has \( \sqrt{1/(1 + c^2)} = \sin \theta \).

\[\square\]

**Proof of Theorem 4.1.** If \( \mu_1 \neq \mu_2 \), it has been shown in [8] that system (21) does not have any positive solution if \( \beta \in [\mu_1, \mu_2] \). Thus, \( \mathbb{R}^4_+ \times (0, \mu_1) \subset \text{pr} \mathcal{C}_1 \subset \mathbb{R}^4_+ \times [0, \mu_1] \) and \( \text{pr} \mathcal{C}_2 = \mathbb{R}^4_+ \times (\mu_2, +\infty) \). It is well known that system (21) has unique positive solution for \( \beta = 0 \). So one has \( \text{pr} \mathcal{C}_1 = \mathbb{R}^4_+ \times [0, \mu_1] \).

If \( \mu_1 = \mu_2 := \mu \), Theorem 4.2 shows that system (21) has infinitely many positive solutions at \( \beta = \mu \) and it does not have other positive solution besides (22). From [35, Theorem 1.1, Theorem 1.3], the positive solution of (21) is unique for \( \beta > \mu \). So \( \mathcal{C}_2 \) is just the synchronized solution branch. Therefore, we have \( \text{pr} \mathcal{C}_1 = \mathbb{R}^4_+ \times [0, \mu) \) and \( \text{pr} \mathcal{C}_2 = \mathbb{R}^4_+ \times (\mu, +\infty) \).

\[\square\]

**Proof of Corollary 1.** For given \( b = (\lambda_1, \lambda_2, \mu_1, \mu_2) \in \mathbb{R}^4_+ \), let \( \mathcal{S}_{j-1} \) denote the section of \( \mathcal{C}_j \) over \( \{(\lambda_1, \lambda_2, \mu_1, \mu_2)\} \times \mathbb{R}_+ \), which may not be connected. From Remark 2.3 of [2], we know that \( \mathcal{S}_{j-1} \) has a component which satisfies one of the alternatives listed in Theorem 3.2, which is still denoted by \( \mathcal{S}_{j-1} \). The existence results are known for \( \beta \in [\beta_0, \mu_1 - \beta^0] \).

We first consider the case of \( \mu_1 < \mu_2 \). In this case, it is well known that problem (21) does not have a nontrivial solution with nonnegative component if \( \beta \in [\mu_1, \mu_2] \). So \( \mathcal{S}_1 \) cannot go to right in the direction of \( \beta \). From Lemma 3.1, we can see that \( \mathcal{C}_1 \) cannot blow up on any bounded set of \( \beta \). It follows that \( \mathcal{S}_1 \) cannot blow up on any bounded set of \( \beta \). So \( \mathcal{S}_1 \) must meet \( \{(0) \times X\} \). Similarly, \( \mathcal{S}_2 \) cannot go to left in the direction of \( \beta \), so it must coincide with \( \{(\beta, u_0, v_0) : \beta > \mu_2\} \)

We now consider the case of \( \mu_1 = \mu_2 \). From Theorem 4.2, we have that (21) has infinitely many positive solutions at \( \beta = \mu \) which must be the form of (22). So \( \mathcal{S}_1 \) still cannot go to right in the direction of \( \beta \). Then reasoning as above, we can obtain the desired conclusions.

\[\square\]

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