Quantum Monodromy
and Semi–Classical Trace Formulae

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1. Introduction

Trace formulæ provide one of the most elegant descriptions of the classical-quantum correspondence. One side of a formula is given by a trace of a quantum object, typically derived from a quantum Hamiltonian, and the other side is described in terms of closed orbits of the corresponding classical Hamiltonian. In algebraic situations, such as the original Selberg trace formula, the identities are exact, while in general they hold only in semi-classical or high-energy limits. We refer to a recent survey [14] for an introduction and references.

In this paper we present an intermediate trace formula in which the original trace is expressed in terms of traces of quantum monodromy operators directly related to the classical dynamics. The usual trace formulæ follow and in addition this approach allows handling effective Hamiltonians.

Let \( P = (1/i)h\partial_z \) be the semi-classical differentiation operator on the circle, \( x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}, 0 < h < 1 \). The classical Poisson formula can be written as follows: if \( \hat{f} \in C^\infty_c(\mathbb{R}) \) then

\[
\text{tr} f(P/h) = \frac{1}{2\pi i} \sum_{|k| \leq N} \int_{\mathbb{R}} f(z/h) \left( e^{2\pi iz/h} \right)^k \frac{d}{dz} \left( e^{2\pi iz/h} \right) dz,
\]

where \( N \) depends on the support of \( \hat{f} \), and we think of \( M(z,h) = e^{2\pi iz/h} : \mathbb{C} \to \mathbb{C} \) as the monodromy operator for the solutions of \( P - z \). It acts on functions in one dimension lower (zero dimension here), identified geometrically with the functions on the transversal to the closed curve (\( S^1 \) here), and analytically with \( \ker(P - z) \) (\( \mathbb{C} \) here).

Now let \( P \) be a semi-classical, self-adjoint, principal type operator, with symbol \( p \) (for instance \( P = -h^2\Delta + V(x), p = \xi^2 + V(x) \)), and let \( \gamma \subset p^{-1}(0) \) be a closed primitive orbit of the Hamilton flow of \( p \). We can define the monodromy operator, \( M(z,h) \) for \( P - z \) along \( \gamma \), acting on functions in one dimension lower, that is, on functions on the transversal to \( \gamma \) in the base. We then have

**Theorem 1.** Suppose that there exists a neighbourhood of \( \gamma, \Omega \), satisfying the condition

\[
m \in \Omega \text{ and } \exp tH_p(m) = m, \quad p(m) = 0, \quad 0 < |t| \leq TN \implies m \in \gamma,
\]

where \( T \) is the primitive period of \( \gamma \). If \( \hat{f} \in C^\infty_c(\mathbb{R}), \supp \hat{f} \subset (-NT + C, NT - C) \setminus \{0\}, C = C(p) \geq 0, \chi \in C^\infty_c(\mathbb{R}), \) and \( A \in \Psi^{0,0}_h(X) \) is a microlocal cut-off to a sufficiently small neighbourhood of \( \gamma \), then

\[
\text{tr} \left( f(P/h)\chi(P)A \right) = \frac{1}{2\pi i} \sum_{-N-1}^{N-1} \text{tr} \int_{\mathbb{R}} f(z/h)M(z,h)^k \frac{d}{dz} M(z,h)\chi(z)dz + O(h^\infty),
\]

where \( M(z,h) \) is the semi-classical monodromy operator associated to \( \gamma \).

The dynamical assumption on the operator means that in a neighbourhood of \( \gamma \) there are no other closed orbits of period less than \( TN \), on the energy surface \( p = 0 \). We avoid a neighbourhood of \( 0 \) in the support of \( \hat{f} \) to avoid the dependence on the microlocal cut-off \( A \).
The monodromy operator quantizes the Poincaré map for $\gamma$ and its geometric analysis gives the now standard trace formula of Selberg, Gutzwiller and Duistermaat-Guillemin (see [1] for a recent proof and a historical discussion, and Sect.7 for a derivation based on Theorem 1). The term $k = -1$ corresponds to the contributions from “not moving at all” and the other terms to contributions from going $|k + 1|$ times around $\gamma$, in the positive direction when $k \geq 0$, and in the negative direction, when $k < -1$. For non-degenerate orbits we analyse the traces on monodromy operators in Sect.7 and recover the usual semi-classical trace formulæ in our general setting – see Theorem 3.

Theorem 1 is a special case of the more general Theorem 2 presented in Sect.6. Motivated by effective Hamiltonians in which the spectral parameter appears non-linearly, we give there a trace formula for a family $P(z)$ with the special case corresponding to $P - z$. For an example of a use of effective Hamiltonians in an interesting physical situation we refer to [7]. The effective Hamiltonian described there comes from the “Peierls substitution”, and the celebrated “Onsager rule” is a consequence of a calculation of traces.

The point of view taken here is purely semi-classical but when translated to the special case of $C^\infty$-singularities/high energy regime, it is close to that of Marvizi-Melrose [10] and Popov [12]. In those works the trace of the wave group was reduced to the study of a trace of an operator quantizing the Poincaré map. In [12] it was used to determine contributions of degenerate orbits and our formula could be used for that as well.

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2. Outline of the proof

To present the idea of our proof we use it to derive the classical Poisson summation formula (1.1). The left hand side there can be written using the usual functional calculus based on Cauchy’s formula:

$$
\text{tr} f \left( \frac{P}{\hbar} \right) = \frac{1}{2\pi i} \text{tr} \int \frac{z}{h} (P - z)^{-1} dz, \quad \Gamma = \Gamma_+ - \Gamma_-, \quad \Gamma \pm = \mathbb{R} \pm i\mathbb{R},
$$

where we take the positive orientation of $\mathbb{R}$ and $R > 0$ is an arbitrary constant. We make an assumption on the support of the Fourier transform on $f$:

$$
\text{supp} \hat{f} \subset (-2\pi N, 2\pi N).
$$

We would like to replace $(P - z)^{-1}$ by an effective Hamiltonian which measures the obstruction to the solvability of $(P - z)u = f$. For that we introduce a Grushin problem (see for instance [6] for applications of this method in spectral problems, and for references):

$$
P(z) \overset{\text{def}}{=} \begin{pmatrix}
P - z & R_-(z) \\
R_+(z) & 0
\end{pmatrix} : H^1(S^1) \times \mathbb{C} \longrightarrow L^2(S^1) \times \mathbb{C},
$$

where $R_\pm(z)$ should be chosen so that $P(z)$ is invertible. If we put

$$
R_+ u \overset{\text{def}}{=} u(0),
$$

then we can locally solve

$$
\begin{cases}
(P - z)u = 0 \\
R_+ u = v
\end{cases},
$$

by putting

$$
u = I_+(z)v = \exp(ixx/\hbar)v, \quad -\epsilon < x < 2\pi - 2\epsilon.
$$
This is the forward solution, and we can also define the backward one by
\[ u = I_- (z)v = \exp(izx/h)v, \quad -2\pi + 2\epsilon < x < \epsilon. \]

The monodromy operator \( M(z, h) : \mathbb{C} \to \mathbb{C} \), can be defined by
\[
(2.4) \quad I_+(z)v(\pi) = I_-(z)M(z, h)v(\pi),
\]
and we immediately see that
\[
M(z, h) = \exp \frac{2\pi iz}{h}.
\]
We use \( I_\pm(z) \) and the point \( \pi \) to work with objects defined on \( S^1 \) rather than on its cover: a more intuitive definition of \( M(z, h) \) can be given by looking at a value of the solution after going around the circle but that has some technical disadvantages.

Let \( \chi \in C^\infty(S^1, [0, 1]) \) have the properties
\[
\chi(x) \equiv 1, \quad -\epsilon < x < \pi + \epsilon, \quad \chi(x) \equiv 0, \quad -\pi + 2\epsilon < x < -2\epsilon,
\]
and put
\[
E_+(z) = \chi I_+(z) + (1 - \chi) I_-(z).
\]
We see that
\[
(P - z)E_+ = [P, \chi]I_+(z) - [P, \chi]I_-(z) = [P, \chi]_+I_+(z) - [P, \chi]_-I_-(z),
\]
where \([P, \chi]_-\) denotes the part of the commutator supported near \( \pi \). This can be simplified using (2.4):
\[
(P - z)E_+ + [P, \chi]_-I_-(z)(I - M(z, h)) = 0,
\]
which suggests putting
\[
R_-(z) = [P, \chi]_-I_-(z),
\]
so that the problem
\[
\begin{cases}
(P - z)u + R_-(z)u_- = 0 \\
R_+(z)u = v
\end{cases}
\]
has a solution:
\[
\begin{cases}
u = E_+(z)v \\
u_- = E_{-+}(z)v
\end{cases},
\]
with \( E_{-+}(z) = I - M(z, h) \).

One can show\(^1\) that with this choice of \( R_+(z) \), (2.3) is invertible and then
\[
\mathcal{P}(z)^{-1} = \mathcal{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix},
\]
where all the entries are holomorphic in \( z \), and \( E_+(z) \), \( E_{-+}(z) \), are as above. The operator \( E_{-+}(z) \) is the effective Hamiltonian in the sense that its invertibility controls the existence of the resolvent:
\[
(2.5) \quad (P - z)^{-1} = E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z).
\]
Inserting this in (2.1) and using the holomorphy of \( E(z) \) gives
\[
\text{tr} f \left( \frac{P}{h} \right) = -\frac{1}{2\pi i} \int f \left( \frac{z}{h} \right) \text{tr} E_+(z)E_{-+}(z)^{-1}E_-(z)dz = -\frac{1}{2\pi i} \int f \left( \frac{z}{h} \right) \text{tr} E_-(z)E_+(z)E_{-+}(z)^{-1}dz,
\]

\(^1\)In this situation it is quite easy but it will be done in greater generality in Sect.5.
where we used the cyclicity of the trace. Differentiating $\mathcal{E}(z)P(z) = Id$ shows
\[ E_-(z)E_+(z) = \partial_z E_-(z) + E_-(z)\partial_z E_-(z), \]
which inserted in the previous identity gives
\[ \text{tr} f \left( \frac{P}{h} \right) = -\frac{1}{2\pi i} \int_{\Gamma} f \left( \frac{z}{h} \right) \text{tr} \partial_z E_-(z)E_-(z)^{-1} dz, \]
where we eliminated the other term using countour deformation.

We then define the
\[ \chi(z) = \text{tr} f \left( \frac{z}{h} \right) \partial_z E_-(z)E_-(z)^{-1} \]
for different points on $\Gamma$, and this gives (1.1).

To construct the monodromy operator we fix two different points on $\gamma$, $m_0, m_1$ (corresponding
to 0 and $\pi$ in the example), and their disjoint neighbourhoods, $W_+$ and $W_-$ respectively. We then consider local kernels of $P - z$ near $m_0$ and $m_1$ (that is, sets of distributions satisfying $(P - z)u = 0$ near $m_i$'s), $\ker_{m_{j}}(P - z), j = 0, 1$, with elements microlocally defined in $W_{\pm}$. and the forward and backward solutions:
\[ I_{\pm}(z) : \ker_{m_{j}}(P - z) \rightarrow \ker_{m_{j}}(P - z). \]
We then define the quantum monodromy operator, $\mathcal{M}(z)$ by
\[ \mathcal{M}(z) = I_+(z), \quad \mathcal{M}(z) : \ker_{m_{j}}(P - z) \rightarrow \ker_{m_{j}}(P - z). \]
The operator $P$ is assumed to be self-adjoint with respect to some inner product $(\cdot, \cdot)$, and we define the quantum flux norm on $\ker_{m_{j}}(P - z)$ as follows:\footnote{See [6] for an earlier mathematical development of this basic quantum mechanical idea.}: let $\chi$ be a microlocal cut-off function,
with basic properties of the function $\chi$ in the example. Roughly speaking $\chi$ should supported near $\gamma$ and be equal to one near the part of $\gamma$ between $W_+$ and $W_-$. We denote by $[P, \chi]_{W_+}$ the part of the commutator supported in $W_+$, and put

$$\langle u, v \rangle_{QF} \stackrel{\text{def}}{=} \langle ([h/i] P, \chi)_{W_+}, u, v \rangle, \quad u, v \in \ker_{m_0}(P - z).$$

It is easy to check that this norm is independent of the choice of $\chi$ – see the proof of Lemma 4.4. This independence leads to the unitarity of $\mathcal{M}(z)$:

$$\mathcal{M}(z) u, \mathcal{M}(z) u_{QF} = \langle u, u \rangle_{QF}, \quad u \in \ker_{m_0}(P - z).$$

For practical reasons we identify $\ker_{m_0}(P - z)$ with $\mathcal{D}'(\mathbb{R}^{n-1})$, microlocally near $(0, 0)$, and choose the identification so that the corresponding monodromy map is unitary (microlocally near $(0, 0)$ where $(0, 0)$ corresponds to the closed orbit intersecting a transversal identified with $T^*\mathbb{R}^{n-1}$). This gives

$$M(z, h) : \mathcal{D}'(\mathbb{R}^{n-1}) \to \mathcal{D}'(\mathbb{R}^{n-1}),$$

microlocally defined near $(0, 0)$ (see Sect.3 for a precise definition of this notion) and unitary there. This is the operator appearing in Theorem 1 and it shares many properties with its simple version $\exp(2\pi i z/h)$ appearing for $S^1$.

As shown in the example of the Poisson formula, traces can be expressed in terms of traces of effective Hamiltonians ($E_{-\pm}(z)$ there). Hence in our final formula, we replace $P - z$ by a more general operator $P(z)$, for which we do not demand holomorphy $z$ but only that $P(z)$ is self-adjoint for $z$ real and that it is an almost analytic family of operators. In Theorem 2 we will compute the trace of

$$-\frac{1}{i} \text{tr} \int f(z/h) \partial_z [\tilde{\chi}(z) \partial_z P(z) P(z)^{-1}] \mathcal{L}(dz),$$

which for $P(z) = P - z$ reduces to (2.6).

The only prerequisite to reading the paper is the basic calculus of semi-classical pseudodifferential operators (see [3]). In Sect.3 we review various aspects of semi-classical microlocal analysis needed here. In Sect.4 we define the quantum time and quantum monodromy. Then in Sect.5 we follow the procedure described for $S^1$ to solve a Grushin problem allowing us to represent $P(z)^{-1}$ near a closed orbit. That is applied in the proof of the trace formula in Sect.6, and in Sect.7 we derive the more standard trace formula in the case of a non-degenerate orbit.

3. Semi-classical operators and their almost analytic extensions

Let $X$ be a compact $C^\infty$ manifold. We introduce the usual class of semi-classical symbols on $X$:

$$S^{m,k}(T^*X) = \{ a \in C^\infty(T^*X \times (0, 1]) : |\partial^\alpha_x \partial^\beta_{\xi} a(x, \xi; h)| \leq C_{\alpha, \beta} h^{-m} |\xi|^{m-k+|\beta|} \},$$

and the class corresponding pseudodifferential operators, $\Psi_{h}^{m,k}(X)$, with the quantization and symbol maps:

$$\text{Op}_h^w : S^{m,k}(T^*X) \to \Psi_{h}^{m,k}(X)$$

$$\sigma_h : \Psi_{h}^{m,k}(X) \to S^{m,k}(T^*X)/S^{m-1,k-1}(T^*X),$$

with both maps surjective, and the usual properties

$$\sigma_h(A \circ B) = \sigma_h(A)\sigma_h(B),$$

$$0 \to \Psi^{m-1,k-1}(X) \to \Psi^{m,k}(X) \xrightarrow{\sigma_h} S^{m,k}(T^*X)/S^{m-1,k-1}(T^*X) \to 0.$$
a short exact sequence, and

\[ \sigma_h \circ \text{Op}_h^w : S^{m,k}(T^*X) \to S^{m,k}(T^*X)/S^{m-1,k-1}(T^*X), \]

the natural projection map. The class of operators and the quantization map are defined locally using the definition on \( \mathbb{R}^n \):

\[ \text{Op}_h^w(a)(u(x)) = \frac{1}{(2\pi h)^n} \int \int a \left( \frac{x + y}{2}, \xi \right) e^{i(x-y,\xi)/h} u(y) dy \xi, \]

and we refer to [3] or [13] for a detailed discussion. We remark only that unlike the invariantly defined symbol map, \( \sigma_h \), the quantization map \( \text{Op}_h^w \) can be chosen in many different ways.

In this paper we consider pseudo-differential operators as acting on half-densities and consequently the symbols will also be considered as half-densities – see [8, Sect.18.1] for a general introduction and the Appendix to this paper for a semi-classical discussion. For notational simplicity we suppress the half-density notation. The only result we will need here is that in Weyl quantization, the symbol is well defined up to terms of order \( O(h^2) \) – see Appendix.

For \( a \in S^{m,k}(T^*X) \) we define

\[ \text{ess-sup}_{h} a \subset T^*X \cup S^*X, \quad S^*X \overset{\text{def}}{=} (T^*X \setminus 0)/\mathbb{R}_+, \]

where the usual \( \mathbb{R}_+ \) action is given by multiplication on the fibers: \((x, \xi) \mapsto (x, t\xi)\), as

\[ \text{ess-sup}_{h} a = C((x, \xi) \in T^*X : \exists \varepsilon > 0 \partial_x^\alpha \partial_\xi^\beta a(x', \xi') = O(h^\varepsilon), \quad d(x, x') + |\xi - \xi'| < \varepsilon \}
\]

\[ \cup C((x, \xi) \in T^*X \setminus 0 : \exists \varepsilon > 0 \partial_x^\alpha \partial_\xi^\beta a(x', \xi') = O(h^\varepsilon|\xi'|^{-\varepsilon}),
\]

\[ d(x, x') + 1/|\xi'| + |\xi'|/|\xi| - \xi'/|\xi'| < \varepsilon / \mathbb{R}_+ \]

For \( A \in \Psi_{h}^{m,k}(X) \), then define

\[ WF_h(A) = \text{ess-sup}_{h} a, \quad A = \text{Op}_h^w(a), \]

noting that, as usual, the definition does not depend on the choice of \( \text{Op}_h^w \). For

\[ u \in C^\infty((0,1]_h; D'(X)), \quad \exists N_0, \quad h^{-N_0} u \quad \text{is bounded in } D'(X), \]

we define

\[ WF_h(u) = C((x, \xi) : \exists A \in \Psi_{h}^{0,0}(X) \sigma_h(A)(x, \xi) \neq 0, \quad Au \in h^\infty C^\infty((0,1]_h; C^\infty(X))) \].

When \( u \) is not necessarily smooth we can give a definition analogous to that of \( \text{ess-sup}_{h} a \). Since in this note we will be concerned with a purely semi-classical theory and deal only with compact subsets of \( T^*X \) this definition is sufficient for our purposes (for more general definitions of wave front set which include this usual semi-classical definition, see [11]).

To discuss almost analytic continuation of semi-classical pseudodifferential operators let us first recall the scalar case. For \( f \in C^\infty(\mathbb{R}) \), an almost analytic extension of \( f \) is \( \tilde{f} \in C^\infty(\mathbb{C}) \) such that locally uniformly

\[ \tilde{\partial}_z \tilde{f}(z) = O(|\Im z|^{\infty}), \quad \tilde{f}|_{\Re z} = f. \]

The almost analytic extensions were introduced by Hörmander and are unique up to \( O(|\Im z|^{\infty}) \) terms (see [3, Sect.8] and references given there).

Suppose now that

\[ A(x) \in C^\infty(\mathbb{R}^2; \Psi_{h}^{m,k}(X)) \]
is a smooth family of pseudodifferential operators. We can then find \( a(x) \in C^\infty(\mathbb{R}_x; S^{m,k}(T^*X)) \) such that \( A(x) = \text{Op}_h^u(a(x)) \). We then define the \textit{almost analytic extension of the family} \( A(x) \) as 
\[
\tilde{A}(z) = \text{Op}_h^u(\tilde{a}(z)),
\]
where \( \tilde{a}(z) \in C^\infty(\mathbb{C}_z; S^{m,k}(T^*X)) \) is an almost analytic extension of \( a(x) \). To justify this definition we need the following easy

**Lemma 3.1.** If \( a(x) \in C^\infty(\mathbb{R}_x; S^{m,k}(T^*X)) \) then there exists an almost analytic extension of \( a(x) \) satisfying
\[
\tilde{a}(z) \in C^\infty(\mathbb{C}_z; S^{m,k}(T^*X)), \quad \partial_x^\alpha \partial_z^\beta \tilde{a}(z)(x, \xi; h) = O(|\text{Im } z|^{k-|\beta|}).
\]

We will also need certain aspects of the theory of semi-classical Fourier Integral Operators. Rather than review the full theory we will consider a special class, to which the general calculus reduces in local situations. Thus let \( A(t) \) be a smooth family of pseudodifferential operators, \( A(t) = \text{Op}_h^u(a(t)), a(t) \in C^\infty([-1,1]; S^{0,-}\infty(T^*X)) \), such that for all \( t \), \( \text{WF}(A(t)) \in T^*X \). We then define a family of operators
\[
U(t) : L^2(X) \to L^2(X), \quad hD_t U(t) + U(t) A(t) = 0, \quad U(0) = U_0 \in \Psi^{0,0}_h(X).
\]

This is an example of a family of \( h \)-Fourier Integral Operators, \( U(t) \), associated to canonical transformations \( \kappa(t) \), generated by the Hamilton vector fields \( H_{a_0(t)} \), where the real valued \( a_0(t) \) is the \( h \)-principal symbol of \( A(t) \),
\[
\frac{d}{dt}\kappa(t)(x, \xi) = (\kappa(t))_{\cdot}(H_{a_0(t)}(x, \xi)), \quad \kappa(0)(x, \xi) = (x, \xi), \quad (x, \xi) \in T^*X.
\]

All that we will need in this note is the \textit{Egorov theorem} which can be proved directly from this definition: when \( U_0 \) in (3.2) is elliptic (that is \( |\sigma(U_0)| > c > 0 \) on \( T^*X \), then for \( B \in \Psi^{m,k}_h(X) \)
\[
\sigma(V(t) B U(t)) = (\kappa(t))^* \sigma(B),
\]
where the approximate inverse is constructed by taking
\[
hD_t V(t) - A(t) V(t) = 0, \quad V(0) = V_0, \quad V_0 U_0 - I, \quad U_0 V_0 - I \in \Psi^{-\infty, -\infty}_h(T^*X),
\]
the existence of \( V_0 \) being guaranteed by the ellipticity of \( U_0 \). The proof of (3.3) follows from writing \( B(t) = V(t) B U(t) \), so that, in view of the properties of \( V(t) \),
\[
hD_t V(t) - A(t) V(t) = 0, \quad V(0) = V_0, \quad \text{WF} \Psi^{-\infty, -\infty}_h(X), \quad B(0) = B_0.
\]
Since the symbol of the commutator is given by \( (h/i){H}_{a_0(t)} \sigma(B(t)) \), (3.3) follows directly from the definition of \( \kappa(t) \).

If \( U = U(1) \), say, and the graph of \( \kappa(1) \) is denoted by \( C \), we conform to the usual notation and write
\[
U \in \mathcal{H}_h^1(X \times X; C'), \quad C' = \{(x, \xi; y, -\eta) : (x, \xi) = \kappa(y, \eta)\},
\]
which means that \( U \) is an \( h \)-Fourier Integral Operator associated to the canonical graphs \( C \). Locally all \( h \)-Fourier Integral Operators associated to canonical graphs are of the form \( U(1) \) thanks to the following well known

**Lemma 3.2.** Suppose that \( U_1, U_2 \) are open neighbourhoods of \((0,0) \in T^*\mathbb{R}^n \), and \( \kappa : U_1 \to U_2 \) is a canonical transformation satisfying \( \kappa(0,0) = (0,0) \). Then there exists a smooth family of canonical transformations \( \kappa_t : U_1 \to U_2, 0 \leq t \leq 1, \) satisfying \( \kappa_0 = \text{id}, \kappa_1 = \kappa, \kappa_t(0,0) = (0,0) \).
Proof. Since the symplectic group, $Sp(n, \mathbb{R})$, is connected we can first deform $\kappa$ so that $d\kappa(0,0) = Id$. Hence, near $(0,0)$, $(x(\kappa(y, \eta)), \xi(\kappa(y, \eta))); y, \eta) \mapsto (x, \eta)$ is surjective, and on the graph of $\kappa$, $y$ and $\eta$ can be regarded as functions of $x$ and $\eta$. Since the symplectic forms, $-d(\langle y, d\eta \rangle)$ $d(\langle \xi, dx \rangle)$ are equal, their difference can be written locally as a differential:

$$\langle y, d\eta \rangle + \langle \xi, dx \rangle = d\phi, \quad \phi = \phi(x, \eta), \quad d\phi(0,0) = 0,$$

so that $\kappa : (\phi'_0(x, \eta), \eta) \mapsto (x, \phi'_0(x, \eta))$. We could now take as our family

$$\kappa_t : (t\phi'_0(x, \eta) + (1-t)x, \eta) \mapsto (x, t\phi'_0(x, \eta) + (1-t)\eta).$$

The two steps can be connected smoothly by making the deformations flat at their junction. □

The almost analytic continuation of a family of $h$-Fourier Integral Operators defined by (3.2) is obtained by means of the following

**Lemma 3.3.** Suppose that $U(t)$ is defined by (3.2) and that $\hat{A}(z)$ is an almost analytic continuation of the family $A(t)$, as given by Lemma 3.1. Let $\hat{U}(z) = \hat{U}(t + is)$ be the solution of

$$\frac{1}{i} hD_z \hat{U}(t + is) + \hat{U}(t + is)\hat{A}(t + is) = 0, \quad \hat{U}(t + is)_{|s=0} = U(t).$$

Then for $|\text{Im } z| \leq h \log h^{-L}$ we have

$$\|\hat{U}(z)\|_{L^2 \rightarrow L^2} \leq C \exp(C|\text{Im } z|/h) \quad (3.5)$$

$$\|\hat{D}_z \hat{U}(z)\|_{L^2 \rightarrow L^2} = O(|\text{Im } z|^\infty) \quad (3.6)$$

$$hD_z \hat{U}(z) = \hat{A}(z)\hat{U}(z) + O_{L^2 \rightarrow L^2}(|\text{Im } z|^\infty). \quad (3.7)$$

**Proof.** To see (3.5) we write

$$h \frac{d}{ds} \|\hat{U}(t + is)v\|^2 = 2 \text{Re}(\hat{U}(t + is)\hat{A}(t + is)v, \hat{U}(t + is)v) \leq C\|\hat{U}(t + is)v\|^2, \quad s > 0.$$

Let us now take $v$ with $\|v\| = 1$ so that, by integration,

$$\|\hat{U}(t + is)v\|^2 \leq \|\hat{U}(t)v\|^2 + C \int_0^s \|\hat{U}(t + i\sigma)v\|^2 d\sigma.$$

Since this holds for every $v$ with $\|v\| = 1$ we can replace the left hand side of the inequality by $\|\hat{U}(t + is)v\|^2$, and the standard Gronwall inequality argument shows that

$$\|\hat{U}(t + is)v\|^2 \leq C e^{Cs/\hbar},$$

which is the desired bound. Putting $\hat{V}(t + is) = \hat{D}_z \hat{U}(t + is)$ we have

$$h\partial_x \hat{V}(t + is) = \hat{U}(t + is)\partial_z \hat{A}(t + is) + \hat{V}(t + is)\hat{A}(t + is) = \hat{V}(t + is)\hat{A}(t + is) + O(s^{\infty}),$$

$$|s| < h \log h^{-L}, \quad \hat{V}(t + is)_{|s=0} = 0,$$

where the initial condition came from the equation on the real axis: $hD_z \hat{U}(t + is)_{|s=0} = U(t)A(t)$. As in the argument for (3.5), this implies (3.6) and (3.7).

Our definitions of pseudo-differential operators and of (the special class of) $h$-Fourier Integral Operators were global. It is useful and natural to consider the operators and their properties microlocally. We consider classes of *tempered* operators:

$$T : C^\infty(X) \rightarrow C^\infty(X),$$

where $C^\infty(X)$ is the class of smooth functions on $X$.
and for any semi-norms $\| \cdot \|_1$ and $\| \cdot \|_2$ on $C^\infty(X)$ there exists $M_0$ such that
\[\| Tu \|_1 = O(h^{-M_0}) \| u \|_2.\]

For open sets, $V \subset T^* X$, $U \subset T^* X$, the operators defined microlocally near $V \times U$ are given by equivalence classes of tempered operators given by the relation
\[T \sim T' \iff A(T - T') B = O(h^\infty) : D'(X) \to C^\infty(X),\]
for any $A, B \in \Psi^0_h(X)$ such that
\[WF(A) \subset \tilde{V}, \quad WF(B) \subset \tilde{U},\]
\[\tilde{V} \Subset \tilde{V} \Subset T^* X, \quad \tilde{U} \Subset \tilde{U} \Subset T^* X, \quad \tilde{U}, \tilde{V} \text{ open}.\]

The equivalence class $T$, $h$-Fourier Integral Operator associated to a local canonical graph $C$ if, again for any $A$ and $B$ above
\[ATB \in L^0(X \times X; \tilde{C}',)\]
where $C$ needs to be defined only near $U \times V$.

We say that $P = Q$ microlocally near $U \times V$ if $APB - AQB = O_{L^2 \to L^2}(h^\infty)$, where because of the assumed pre-compactness of $U$ and $V$ the $L^2$ norms can be replaced by any other norms. For operator identities this will be the meaning of equality of operators in this paper, with $U, V$ specified (or clear from the context). Similarly, we say that $B = T^{-1} \text{microlocally near } V \times V$, if $BT = I$ microlocally near $U \times U$, and $TB = I$ microlocally near $V \times V$. More generally, we could say that $P = Q$ microlocally on $W \subset T^* X \times T^* X$ (or, say, $P$ is microlocally defined there), if for any $U, V, U \times V \Subset W$, $P = Q$ microlocally in $U \times V$. We should stress that “microlocally” is always meant in this semi-classical sense in our paper.

In this terminology we have a characterization of local $h$-Fourier Integral Operators, which is essentially the converse of Egorov’s theorem:

**Lemma 3.4.** Suppose that $U = O(1) : L^2(X) \to L^2(X)$, and that for every $A \in \Psi^0_h(X)$ we have
\[AU = UB, \quad B \in \Psi^0_h(X), \quad \sigma(B) = \kappa^* \sigma(A),\]
microlocally near $(m_0, m_0)$ where $\kappa : T^* X \to T^* X$ is a symplectomorphism, defined locally near $m_0$, $\kappa(m_0) = m_0$. Then
\[U \in L^0_h(X \times X; C'), \quad \text{microlocally near } (m_0, m_0),\]
\[C' = \{(x, \xi; y, -\eta) : (y, \eta) = \kappa(x, \xi)\}.\]

**Proof.** From Lemma 3.2 we know that there exists a family of local symplectomorphisms, $\kappa_t$, satisfying $\kappa_t(m_0) = m_0$, and $\kappa_1 = \kappa$, $\kappa_0 = id$. Since we are working locally, there exists a function $a(t)$, such that $\kappa_t$ is generated by its Hamilton vector field $H_{a(t)}$. Let us now consider
\[hD_t U(t) = U(t) A(t), \quad U(1) = U, \quad 0 \leq t \leq 1.\]

The same arguments as the one used in the proof of (3.3) shows that $U(0)$ satisfies
\[\{U(0), A\} = O(h), \quad \text{for any } A \in \Psi^0_h(X).\]

In fact, we take $V(t)$ with $V(0) = Id$ microlocally near $(m_0, m_0)$, so that
\[AU(t)V(t) = U(t)V(t)(V(t)^{-1}BV(t)) = U(t)V(t)A + O(h),\]
where we used Egorov’s theorem and the assumption that $\sigma(B) = \kappa^* \sigma(A)$. Putting $t = 0$ gives (3.9). By Beals’s Lemma [3, Prop.8.3] we conclude that $U(0) \in \Psi^0_h(X)$, and hence $U$ is a microlocally defined $h$-Fourier Integral Operator associated to $\kappa$. □
If the open sets $U$ or $V$ in (3.8) are small enough, so that they can be identified with neighbourhoods of points in $T^* \mathbb{R}^n$, we can use that identification to state that $T$ is microlocally defined near, say, $(m, (0, 0))$, $m \in T^* X$, $(0, 0) \in T^* \mathbb{R}^n$. An example useful here is given in the next proposition.

By Darboux’s theorem we know that if $p$ is a function with a non-vanishing differential then there exists a local canonical transformation $\kappa$ such that $\kappa^* p = \xi_1$ where $\xi_1$ is part of a coordinate system in which the symplectic form is the canonical one $d(\xi, dx)$. The quantum version is given in

**Proposition 3.5.** Suppose that $P \in \Psi_{h,k}^0(X)$ is a semi-classical real principal type operator: $p = \sigma(P)$ is real, independent of $h$, and

$$p = 0 \implies dp \neq 0.$$  

For any $m_0 \in \Sigma_p \triangleq \{m \in T^* X : p(m) = 0\}$ there exists a local canonical transformation $\kappa: T^* X \to T^* \mathbb{R}^n$ defined near $((0, 0), m_0)$, and an $h$-Fourier Integral Operator, $T$, associated to its graph, such that

$$\kappa^* \xi_1 = p,$$

$$TP = hD_{x_1} T, \text{ microlocally near } ((0, 0), m_0),$$

$$T^{-1} \text{ exists microlocally near } (m_0, (0, 0)).$$

For the reader’s convenience we outline a self-contained proof of this semi-classical analogue of the standard $C^\infty$ result [8, Proposition 26.1.3’].

**Proof.** By assumption $dp(m_0) \neq 0$, and consequently Darboux’s theorem gives $\kappa$ with the desired properties. Lemma 3.2 then gives us a family of symplectic transformations $\kappa_t$. If $T_0 = U(1)$, where $U(1)$ was defined using the family $\kappa_t$, then (3.3) shows that $T_0 P - hD_{x_1} = E \in \Psi_{1,0}^{-1.0}$ microlocally near $(0,0)$. Hence we look for $A$ such that $hD_{x_1} + E = AhD_{x_1}A^{-1}$, microlocally near $(0,0)$. That is the same as solving

$$[hD_{x_1}, A] + EA = 0,$$

Since the principal symbol of $P$ is independent of $h$, same is true for the principal symbol of $E$, $e$. Hence we can find $A \in S^{0,0}(T^* \mathbb{R}^n)$, independent of $h$, $a(0,0) \neq 0$, and such that

$$\frac{1}{i}\{\xi_1, a\} + ea = 0$$

near $(0,0)$. Choosing $A_0$ with the principal symbol $a$ we can now find $A_j \in \Psi_{h}^{-j,0}(T^* \mathbb{R}^n)$ so that

$$[hD_{x_1}, A_0 + A_1 + \cdots + A_N] + E(A_0 + A_1 + \cdots + A_N) \in \Psi_{h}^{-N,0}(T^* \mathbb{R}^n).$$

We then put $A \sim A_1 + A_2 + \cdots + A_N + \cdots$ which is elliptic near $(0,0)$, and finally $T = A^{-1}T_0$. \hfill $\Box$

Using the proposition we can transplant objects related to $P$ to the much easier to study objects related to $hD_{x_1}$. In particular, we can microlocally define

$$\ker_{m_0}(P) \triangleq T^{-1}(\ker(hD_{x_1})), \quad \ker(hD_{x_1})) = \{u \in D'(\mathbb{R}^n) : hD_{x_1}u = 0\}.$$  

Since $\ker(hD_{x_1}))$ can be identified with $D'(\mathbb{R}^{n-1})$ we can also identify $\ker_{m_0}(P)$ with $D'(\mathbb{R}^{n-1})$, microlocally near $(m_0, (0,0))$:

$$K : D'(\mathbb{R}^{n-1}) \longrightarrow \ker_{m_0}(P), \quad K = T^{-1} \pi^*, \quad \pi : x \mapsto (x_2, \cdots x_n).$$
4. Quantum Time and Quantum Monodromy

Let \( P(z) \in C^\infty(I_z; \Psi^0_{\hbar}(X)) \), \( I = (-a, a) \subset \mathbb{R} \), be a smooth family of real principal type operators, with principal symbols, \( p(z) \), independent of \( \hbar \). We will assume that

\[
\Sigma_{p(z)} \overset{\text{def}}{=} \{ m \in T^*X : p(z)(m) = 0 \} \subset T^*X, \quad \text{for } z \in I
\]

\( P(z) \) is formally self-adjoint for \( z \in I \).

We assume that \( m_0(z) \) is a smooth family of periodic points of \( H_{p(z)} \), with the minimal periods \( T(z) \) also smooth in \( z \), and the orbits \( \gamma(z) \):

\[
\exp(T(z)H_{p(z)})(m_0(z)) = m_0(z), \quad \gamma \overset{\text{def}}{=} \{ \exp(tH_{p(z)})(m_0(z)) : 0 \leq t \leq T(z) \}.
\]

When no confusion is likely to arise we may drop the dependence on \( z \) in the notation.

Let \( \Omega \) be a neighbourhood of \( \gamma(0) \) in \( T^*X \),

\[
\Omega \simeq \gamma(0) \times \mathbb{B}_{2^n-1}(0, \epsilon),
\]

and we assume that for \( z \in I \), the orbits \( \gamma(z) \) are also contained in \( \Omega \). We now introduce a covering space of this tubular neighbourhood

\[
\check{\Omega} \simeq \mathbb{R} \times \mathbb{B}_{2^n-1}(0, \epsilon), \quad \pi : \check{\Omega} \rightarrow \Omega,
\]

with the lift of \( p(z) \) denoted by \( \check{p}(z) \), and we will use the same notation for other objects.

We start with the following

**Lemma 4.1.** The tubular neighbourhood, \( \Omega \), of \( \gamma(0) \), can be chosen small enough, so that the cover \( \check{\Omega} \) contains no closed orbits of \( H_{\check{p}(z)} \), \( z \in [-\delta, \delta] \subset I \), for some small \( \delta > 0 \).

**Proof.** Let \( m \mapsto \check{t}(m) \) be a smooth function on \( \check{\Omega} \) with the property that \( \check{t}(\exp(tH_{\check{p}(0)}) = t \), and that \( \check{d} = \pi^* d\check{t} \), where \( d\check{t} \) is a well defined one form in \( \Omega \). Then \( H_{\check{p}(0)}\check{t} > 0 \) on the lift of \( \gamma \), and by shrinking \( \check{\Omega} \) if necessary we conclude that \( H_{\check{p}(0)}\check{t} > 0 \) on \( \check{\Omega} \). By the periodicity and and a compactness argument we conclude that this holds for 0 replaced by \( z \in [-\delta, \delta] \). Hence there are no closed orbits of \( H_{\check{p}(z)} \) in \( \check{\Omega} \). \qed

We will now replace \( \check{\Omega} \) by a finite part: \( \check{\Omega} \simeq [-L, L] \times \mathbb{B}_{2^n-1}(0, \epsilon), \quad L \gg T \).

A classical time function, \( \check{q}(z) \in C^\infty(\check{\Omega}; \mathbb{R}) \), on \( \check{\Omega} \) is defined as a solution of

\[
\partial_\check{t} \check{p}(z) = -\{\check{p}(z), \check{q}(z)\}.
\]

In view of Lemma 4.1 this equation can be solved (strictly speaking that may involve shrinking \( \check{\Omega} \) further depending on the initial data, but for simplicity of exposition we will ignore this point), and we can in particular consider solutions satisfying \( \check{q}(\check{m}_0(z)) = 0 \). In a neighbourhood of \( m_0 = m_0(0) \in \Omega \) we can define \( q(z) \in C^\infty \) such that

\[
\check{q}(z) = \pi^* q(z), \quad \text{near } \check{m}_0, \quad q(z)(m_0(z)) = 0 \quad \text{\pi : } \check{\Omega} \rightarrow \Omega.
\]

We clearly have \( \partial_\check{t} p = \{p, q\} \) near \( m_0 \). This defines the local classical time near \( m_0 \). We also define the first return classical time near \( m_0 \) by demanding that

\[
\check{q}(z) = \pi^* q_\circ(z), \quad \text{near } \exp(T(0)H_{\check{p}(0)}\check{m}_0), \quad q_\circ(z)(m_0(z)) = \check{q}(\exp(T(z)H_{\check{p}(z)}(\check{m}_0(z)))).
\]
An iteration procedure similar to the one recalled in the proof of Proposition 3.5 gives the quantum analogues microlocally defined near $m_0$:

$$\partial_t P(z) = -\frac{i}{\hbar} [P(z), Q(z)], \quad \sigma(Q(z)) = q(z),$$

(4.2)

$$\partial_t Q(z) = -\frac{i}{\hbar} [P(z), Q(z)], \quad \sigma(Q(z)) = q(z),$$

(4.3)

Replacing $Q(z)$ by $(Q(z) + Q(z)^*)/2$, we can assume that $Q(z)$ is formally self-adjoint. We clearly have

$$Q(z) - Q(z) : \ker_{m_0} P(z) \longrightarrow \ker_{m_0} P(z).$$

Then $Q(z)$ is the quantum time near $m_0$, and $Q_{\circ}$ is the first return quantum time near $m_0$. See the proof of Lemma 7.4 for further discussion of these objects in the classical context.

For $(z, w)$ near $(0, 0)$, and microlocally near $m_0$, we can solve the following system of equations

$$\begin{align*}
(hD_z - Q(z)L) U(z, w) &\overset{\text{def}}{=} hD_z U(z, w) - Q(z) U(z, w) = 0 \\
(hD_w + Q(w)R) U(z, w) &\overset{\text{def}}{=} hD_w U(z, w) + U(z, w) Q(w) U(z, w) = 0
\end{align*}$$

(4.4)

with the initial condition $U(0, 0) = \text{Id}$, and with $U(z, w)$ bounded on $L^2$ (microlocally near $(m_0, m_0)$): the solvability of the system follows from the fact that

$$[hD_z - Q(z)_L, hD_w + Q(w)_R] = 0.$$

We easily check that (as always, microlocally)

$$U(z, z) = \text{Id}, \quad U(z, w)U(w, v) = U(z, v),$$

and that $U(z, w)$ is unitary. In fact,

$$hD_z(U(z, z)) = Q(z) U(z, z) - U(z, z) Q(z) = -[U(z, z), Q(z)], \quad U(0, 0) = \text{Id},$$

and $U(z, z) = \text{Id}$ is the unique solution. The other property is derived similarly:

$$hD_w(U(z, w)U(w, v)) = -U(z, w) Q(w) U(w, v) + U(z, w) Q(w) U(w, v) = 0,$$

$$U(z, w) U(w, v)|_{w=z} = U(z, v).$$

By varying $m_0$ along the orbit of $H_{p(0)}$, and by extending $Q(z)$ maximally forward (+) and backward (−), we can define semi-global versions of $U(z, w)$:

$$U_+(z, w) \text{ microlocally on a neighbourhood of the diagonal over} \{\exp tH_p(m_0) : -\epsilon < t < T(0) - 2\epsilon\},$$

$$U_-(z, w) \text{ microlocally on a neighbourhood of the diagonal over} \{\exp tH_p(m_0) : -\epsilon < t < T(0) - 2\epsilon\}.$$

The operators have the following intertwining property:

**Proposition 4.2.** Microlocally near the diagonal over

$$\{\exp tH_p(m_0) : -\epsilon < \pm t < T(0) - 2\epsilon\},$$

and for $z, w$ close to 0, we have

$$P(z) U_{\pm}(z, w) = U_{\pm}(z, w) P(w).$$
Proof. We define \( P^\sharp(w) = U(w, z)P(z)U(z, w) \) and differentiate with respect to \( w \):

\[
\begin{align*}
\hbar D_w P^\sharp(w) &= Q(w)U(w, z)P(z)U(z, w) - U(w, z)P(z)U(z, w)Q(w) = -[P^\sharp(w), Q(w)], \\
P^\sharp(w)|_{w=z} &= P(z),
\end{align*}
\]

that is, \( P^\sharp(w) \) satisfies (4.2) and consequently \( P^\sharp(w) = P(w) \).

By replacing the local quantum time, \( Q(z) \), by the first return quantum time, \( Q_C(z) \) (see (4.2), (4.3)), we also define \( U_C(z, w) \),

\[
U_C(z, z) = 1d, \quad U_C(z, w)U_C(w, v) = U_C(z, v) \text{ microlocally near } m_0.
\]

This definition will be useful when we study the quantum monodromy operator. To introduce it, we first define the forward and backward propagators:

\[
I_{\pm}(z) : \ker_{m_0(z)}(P(z)) \rightarrow \mathcal{D}'(X)
\]

(4.6)

\[
P(z)I_{\pm}(z) = 0, \quad \text{microlocally near } m_0(z),
\]

That the operators \( I_{\pm}(z) \) are microlocally well defined follows from Proposition 3.5, and “microlocally” is meant via the identification of \( \ker_{m_0(z)}(P(z)) \) with \( \mathcal{D}'(\mathbb{R}^{n-1}) \) as in (3.11). We fix \( m_0 = m_0(0) \) as in the definition of \( \Omega \) above, and define

\[
W_+ = \text{ a neighbourhood of } m_0 \text{ in } T^*X,
\]

(4.7)

\[
W_- = \text{ a neighbourhood of } \exp((T(0)/2)H_p(0))m_0 \text{ in } T^*X,
\]

\[
W_- \subset \bigcup_{|t+T(0)/2|<\epsilon} \exp th_p(0)(W_+),
\]

noting that for \( z \) small enough, we can replace \( m_0, T(0), p(0), \) by \( m_0(z), T(z), p(z) \) in this definition. This shows that \( I_-(z) \) maps \( \ker_{m_0(z)}(P(z)) \) onto \( \ker_{\exp((T(z)/2)H_p(z))(m_0(z))}(P(z)) \), microlocally near \( W_- \times W_+ \). This means that the left microlocal inverse exists and we can give the following

**Definition.** The (absolute) quantum monodromy operator

\[
\mathcal{M}(z) : \ker_{m_0(z)}(P(z)) \rightarrow \ker_{m_0(z)}(P(z)),
\]

is microlocally defined near \( W_+ \) by

\[
I_+(z) f = I_-(z)\mathcal{M}(z)f, \quad f \in \ker_{m_0(z)}(P(z)), \quad \text{microlocally near } W_-.
\]

The quantum monodromy operator,

\[
\mathcal{M}(z) : \mathcal{D}'(\mathbb{R}^{n-1}) \rightarrow \mathcal{D}'(\mathbb{R}^{n-1}),
\]

is microlocally defined near \( (0,0) \in T^*\mathbb{R}^{n-1} \), by

\[
M(z) = K(z)^{-1}\mathcal{M}(z)K(z),
\]

(4.9)

where \( K(z) \) is as in (3.11).

The basic properties are given in
Proposition 4.3. Let $U(z, w)$ and $U_\circ(z, w)$ be given by (4.2), (4.3), and (4.4). Then the following diagram commutes (microlocally near $m_0$):

$$
\begin{array}{ccc}
\ker_{m_0(w)}(P(w)) & \xrightarrow{M(w)} & \ker_{m_0(w)}(P(w)) \\
U(z, w) \downarrow & & \downarrow U_\circ(z, w) \\
\ker_{m_0(z)}(P(z)) & \xrightarrow{M(z)} & \ker_{m_0(z)}(P(z))
\end{array}
$$

Choosing $K(z)$, so that $K(z) = U(z, w)K(w)$, we also have

$$
hD_z M(z) = [K(z)^{-1}(Q_\circ(z) - Q(z))K(z)]M(z),
$$

where we recall that by (4.2) and (4.3), $Q_\circ(z) - Q(z) : \ker_{\tilde{m}_0(z)}(\tilde{P}(z)) \rightarrow \ker_{\tilde{m}_0(z)}(\tilde{P}(z))$, and hence $K(z)^{-1}$ is well defined.

Proof. We need to show that $U_\circ(z, w)M(w) = M(z)U(z, w)$, and since $U_\circ$ is naturally defined using the covering space, we will translate this into a statement there. We can microlocally define $\tilde{P}(w)$ on $\tilde{\Omega}$ and then,

$$
\tilde{I}_+(w) : \ker_{\tilde{m}_0(w)}(\tilde{P}(w)) \rightarrow \ker(\tilde{P}(w)),
$$

and we define

$$
\tilde{M}(w) : \ker_{\tilde{m}_0(w)}(\tilde{P}(w)) \rightarrow \ker_{\exp(T(w)H_{\tilde{p}(w)}(\tilde{m}_0(w))}(\tilde{P}(w)),$$

by restricting $\tilde{I}_+(w)$ to a neighbourhood of $\exp(T(w)H_{\tilde{p}(w)})(\tilde{m}_0(w))$. Since for $\pi : \tilde{\Omega} \rightarrow \Omega$, we microlocally have

$$
\pi_* : \ker_{\exp(T(w)H_{\tilde{p}(w)}(\tilde{m}_0(w))}(\tilde{P}(w)) \rightarrow \ker_{m_0(w)}P(w),
$$

$$
\pi_*\tilde{M}(w)\pi^* = M(w).
$$

Using the quantized version of $\tilde{q}$ in (4.1), we also define $\tilde{U}(z, w)$, so that $\tilde{U}(z, w)\tilde{P}(w) = \tilde{P}(z)\tilde{U}(z, w)$. In particular we have

$$
\tilde{U}(z, w)\tilde{I}_+(w) = \tilde{I}_+(z)\tilde{U}(z, w).
$$

Restricting (microlocally) to a neighbourhood of

$$
(\exp(T(z)H_{\tilde{p}(z)})(\tilde{m}_0(z)), \tilde{m}_0(z)) \in \tilde{\Omega} \times \tilde{\Omega},
$$

and projecting to $\Omega \times \Omega$, we obtain

$$
U_\circ(z, w)M(w) = M(z)U(z, w).
$$

To see (4.11) we first note that differentiation of $K(z) = U(z, w)K(w)$ and the definition of $U(z, w)$ gives

$$
hD_z K(z) = Q(z)K(z).
$$

We then use the commutative diagram to see that

$$
K(z)M(z) = U_\circ(z, w)M(w)U(w, z)K(z).
$$

Differentiating this with respect to $z$ and using the previous equation gives

$$
K(z)hD_z M(z) = (Q_\circ(z) - Q(z))K(z)M(z).
$$

We then recall that by (4.2) and (4.3), $Q(z) - Q_\circ(z) : \ker_{\tilde{m}_0(z)}(\tilde{P}(z)) \rightarrow \ker_{\tilde{m}_0(z)}(\tilde{P}(z))$, and hence $K(z)^{-1}$ can be applied to both sides. □
We can define the Poincaré map for \( \gamma \) with primitive period \( T \):
\[
C : T^*\mathbb{R}^{n-1} \to T^*\mathbb{R}^{n-1},
\]
defined near \((0,0)\), \(C(0,0) = (0,0)\), as follows: for a neighborhood of \( m_0 \in \gamma \), \(U_0, U_0/\exp(tH_p)\) can be identified with a neighborhood of \((0,0)\) in \(T^*\mathbb{R}^{n-1}\) (using the local identification of \( p \) with \( \xi_1 \), as in the proof of Proposition 3.5), with \([m_0]\) corresponding to \((0,0)\). The Poincaré map is then given by
\[
C : \kappa^{-1}([m]) \mapsto \kappa^{-1}([\exp(TH_p)m]),
\]
(4.12)
[\(m\) \(\in U_0/\exp(tH_p)\), \(\kappa : T^*\mathbb{R}^{n-1} \to U_0/\exp(tH_p)\).]

It will always be understood that \(\kappa\) chosen here is the symplectic transformation corresponding to \(K = K(z)\) in (3.11) and (4.9).

To study quantum properties of the monodromy operator it is convenient to introduce \(\chi \in C_{\infty}^0(T^*X)\) satisfying
\[
\chi \equiv \begin{cases} 1 & \text{near } \{ \exp(tH_p(0)(m_0)) : \epsilon < t < T(0)/2 - \epsilon \} \\ 0 & \text{near } \{ \exp(tH_p(0)(m_0)) : \epsilon < -t < T(0)/2 - \epsilon \} \end{cases}
\]
(4.13)
\(\Omega \cap \{ m : \chi(m) \neq 1 \} \cap \{ m : \chi(m) \neq 0 \} \subset W_+ \cup W_-\),
where \(W_\pm\) are as in (4.7), and \(\Omega\) is a small neighbourhood of \(\gamma\). If \(\rho_{\pm} \equiv 1\) microlocally near \(W_\pm\), and \(\rho_{\pm} \equiv 0\) near \(W_{\mp}\), we define
\[
[P,\chi]_{W_{\pm}} = \rho_{\pm}[P,\chi],
\]
where we use the same notation for \(\chi\) and \(\text{Op}_\hbar(\chi)\). We then have the basic property of the quantum flux (see [6]):

**Lemma 4.4.** Let \(K(z)\) be in (3.11). Then
\[
U(z) \overset{\text{def}}{=} K(z)^*[(i/h)P(z),\chi]_{W_+}K(z) : \mathcal{D}'(\mathbb{R}^{n-1}) \to \mathcal{D}'(\mathbb{R}^{n-1})
\]
is microlocally positive near \((0,0)\) in \(T^*\mathbb{R}^{n-1}\) and independent of \(\chi\) with the properties (4.13).

If we replace \(K(z)\) by \(K(z)U(z)^{-\frac{1}{2}}\) then
\[
(4.14) \quad K(z)^*[(i/h)P(z),\chi]_{W_+}K(z) = I d \text{ microlocally near } (0,0) \text{ in } T^*\mathbb{R}^{n-1}.
\]

**Proof.** We note that if \(P(z)u = 0\) near \(W_+\), and \(\tilde{\chi}\) is another function satisfying (4.13), then
\[
K(z)^*[(i/h)P(z),\chi - \tilde{\chi}]u = K(z)^*P(z)(\chi - \tilde{\chi})u - K(z)^*P(z)(\chi - \tilde{\chi})u = 0,
\]
since \(P(z)u = 0\), and \(K(z)^*P(z) = (P(z)K(z))^* = 0\). The positivity also comes from expanding the commutator and using Proposition 3.5:
\[
\langle \tilde{K}(z)^*[(i/h)\partial_x,\chi]_{W_+}\tilde{K}(z)u, u \rangle = \langle \partial_x, \chi \rho_{\pm} \tilde{K}(z)u, \tilde{K}(z)u \rangle \geq \langle \tilde{\rho}u, \tilde{\rho}u \rangle,
\]
where \(\tilde{K}(z)\) is the composition of \(K(z)\) and \(T\) of Proposition 3.5, and \(\tilde{\rho} \equiv 1\) in a neighbourhood of \((0,0)\) in \(T^*\mathbb{R}^{n-1}\) (we again use the same notation for the function and its quantization).

From now on, our choice of \(K(z)\) in (3.11) is made so that (4.14) holds. We only need to check that we still have
\[
K(z) = U(z,w)K(w).
\]
In fact, we have in general, in the microlocal sense,
\[
K(z)^*[(i/h)P(z),\chi]_{W_+}K(z) = K(w)^*U(w,z)[(i/h)P(z),\chi]_{W_+}U(z,w)K(w) = K(w)^*[(i/h)P(w),\chi]_{W_+}K(w),
\]
and the last expression is unchanged if we replace \( \bar{\chi} \) by \( \chi \) (the quantum flux property used before). We also used the unitarity of \( U(z, w) \).

With this choice of \( K(z) \) we have the following important and well known

**Proposition 4.5.** The monodromy operator, \( M(z) \), defined by (4.9) with \( K(z) \) satisfying (4.14) is microlocally unitary:

\[
M(z)^* = M(z)^{-1} \quad \text{microlocally near } (0, 0) \in T^*\mathbb{R}^{n-1},
\]

and it is an \( h \)-Fourier Integral Operator:

\[
M(z) \in I^0(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; C(z)'),
\]

where \( C(z) \) is the Poincaré map (4.12).

**Proof.** We need to show that for \( v \in \mathcal{D}'(\mathbb{R}^{n-1}) \) with \( WF_h(v) \) in a neighbourhood of \((0, 0)\), we have

\[
(M(z)v, M(z)v) = \langle v, v \rangle + O(h^\infty)\|v\|^2.
\]

If we put \( u = K(z)v \), use (4.14), and the definition of \( M(z) \), (4.9), then the left hand side of (4.15) becomes:

\[
\langle K(z)^*[P(z), \chi]w, M(z)u, K(z)^{-1}M(z)u \rangle = \langle [P(z), \chi]w, M(z)u, M(z)u \rangle.
\]

As in the proof of Lemma 4.4, we see that for \( 0 < t < T(0)/2 + \epsilon \), the right hand side of the previous expression is equal to (modulo \( O(h^\infty) \))

\[
\langle (i/h)P(z), (\exp(tH_{p(0)})^\ast \chi)[\exp(-tH_{p(0)})w, I_{-}(z)M(z)u, I_{-}(z)M(z)u] \rangle,
\]

which corresponds to moving the support of \( \chi \) in the direction opposite to the flow of \( H_{p(0)} \), and simultaneously moving \( W_+ \) so that (4.13) holds.

Similarly, for \( -T(0)/2 - \epsilon < t < 0 \), the right hand side of (4.15) is equal to

\[
\langle (i/h)P(z), (\exp(-tH_{p(0)})^\ast \chi)[\exp(tH_{p(0)})w, I_{+}(z)u, I_{+}(z)u] \rangle.
\]

For \( t \sim T(0)/2 \), \( \exp(\pm tH_{p(0)})(W_+) \subset W_- \), and hence

\[
\langle (i/h)P(z), (\exp(tH_{p(0)})^\ast \chi)[\exp(tH_{p(0)})w, I_{+}(z)u, I_{+}(z)u] =
\]

\[
\langle (i/h)P(z), (\exp(tH_{p(0)})^\ast \chi)[\exp(-tH_{p(0)})w, U_{-}(z)M(z)u, I_{-}(z)M(z)u] \rangle, \quad t \sim T(0)/2,
\]

from the definition of \( M(z) \), (4.8). But this shows (4.15) proving the first part of the proposition.

To see the second part we use use Lemma 3.4, and the obvious conjugation properties of the solution in the model case discussed in Proposition 3.5: going around the closed orbit we obtain that the underlying symplectomorphism is given by the Poincaré map. \( \square \)

So far we have discussed only the case of \( z \in \mathbb{R} \). We can now consider almost analytic extensions of the operators \( Q(z) \), \( Q_\infty(z) \), \( U_\pm(z, w) \), \( I_\pm(z) \), and \( M(z) \). For that we consider a complex neighbourhood of \( I \subset \mathbb{R} \):

\[
I_{h, L} = \{ z : \Re z \in I, \ |\Im z| \leq Lh \log(1/h) \}.
\]

The families of pseudo-differential operators \( P(z) \), \( Q(z) \), and \( Q_\infty(z) \) have almost analytic extensions given by Lemma 3.1, and we use the same notation for them. We then use Lemma 3.3 and (4.4) to extend \( U(z, w), U_\infty(z, w) \), and \( U_\pm(z, w) \) to \( (z, w) \in I_{h, L} \times I_{h, L} \). We then have

\[
P(z)U_\bullet(z, w) = U_\bullet(z, w)P(w), \quad (z, w) \in I_{h, L} \times I_{h, L},
\]
microlocally (that is, in particular modulo $O(h^\infty)$). Indeed, for $x,w \in I \times I$, and $|y| \leq Lh \log(1/h)$, we have, as in the proof of Lemma 3.3,
\[
\partial_y [P(x + iy)U(x + iy, w) - U(x + iy, w)P(w)] = O(y^\infty),
\]
\[
[P(x + iy)U(x + iy, w) - U(x + iy)P(w)]|_{y=0} = 0.
\]
Hence we can define
\[
I_\pm(z) = U_\pm(z, w)I_\pm(w), \quad (z, w) \in I_{0,1} \times I,
\]
so that $P(\zeta)I_\pm(\zeta) = 0$.

To define an almost analytic extension of $M(\zeta)$ we first almost analytically extend the pseudodifferential operator $K(\zeta)^{-1}(Q(\zeta) - Q(\zeta))K(\zeta)$, and then use $(4.11)$ and Lemma 3.2. In particular, Proposition 4.5 gives,
\[
M(\zeta)^{-1} = M(\zeta)^*, \quad |\text{Im } \zeta| \leq Lh \log(1/h).
\]

5. GRUSHIN PROBLEM NEAR A CLOSED Trajectory

As in the previous section we assume that $P(\zeta)$ is self-adjoint for $\zeta \in \mathbb{R}$, and denote by the same symbol the almost analytic continuation of $P(\zeta)$. Although the inverse of $P(\zeta)$ does not normally exist near $\gamma = \gamma(0)$ for all $\zeta \in I$ we will describe $P(\zeta)^{-1}$ in terms of the inverse of a microlocal effective Hamiltonian $E_{\zeta^+}(\zeta) = I - M(\zeta)$. We will do it first for $\zeta$ real and then use the extensions of operators $U_{\pm}(\zeta, w)$ described at the end of the last section to transplant the results to complex values of $\zeta$.

To do that we follow the now standard Grushin reduction [6], and consider the system
\[
\mathcal{P}(\zeta) = \begin{pmatrix}
(i/h)P(\zeta) & R_-(\zeta) \\
R_+(\zeta) & 0
\end{pmatrix} : \mathcal{D}'(X) \times \mathcal{D}'(\mathbb{R}^{n-1}) \longrightarrow \mathcal{D}'(X) \times \mathcal{D}'(\mathbb{R}^{n-1}),
\]
defined microlocally near $\gamma \times (0,0)$, and where the operators $R_{\pm}$ need to be suitably chosen.

We will successively build the operator $\mathcal{P}(\zeta)$ and its inverse. We start by putting
\[
R_+(\zeta) = K(\zeta)^*[(i/h)P(\zeta), \chi]|_{W_+},
\]
and $u$ with $Pu = 0$ near $W_+$, $R_+(\zeta)u$, is its Cauchy data. Hence $u = K(\zeta)v$ provides a local solution to the microlocal Cauchy problem:
\[
P(\zeta)u = 0 \quad \text{and} \quad R_+(\zeta)u = v
\]
To obtain a global Cauchy problem we need to introduce $R_-(\zeta)$. To do that we define
\[
K_f(\zeta) = I_+(\zeta)K(\zeta), \quad K_b(\zeta) = I_-(\zeta)K(\zeta),
\]
where the operators $I_{\pm}(\zeta)$ are defined in $(4.6)$. We recall the definition of the monodromy operator:
\[
K_f(\zeta) = K_b(\zeta)M(\zeta) \text{ microlocally near } W_- \times (0,0).
\]
We can build a solution of $(5.3)$ in $\Omega \setminus W_-$ by putting
\[
E_+(\zeta)v = \chi K_f(\zeta)v + (1 - \chi)K_b(\zeta)v,
\]
so that in particular, $E_+(\zeta)v = K(\zeta)v$, in $W_+$, and consequently
\[
R_+(\zeta)E_+(\zeta) = Id \text{ microlocally near } (0,0) \in T^*\mathbb{R}^{n-1}.
Applying the operator, and using (5.4) we obtain
\[
\frac{i}{\hbar} P(z) E_z v = \left[ (i/\hbar) P(z), \chi \right]_{W_+} K_f(z) v - \left[ (i/\hbar) P(z), \chi \right]_{W_-} K_b(z) v
\]
\[
= \left[ (i/\hbar) P(z), \chi \right]_{W_-} K_b(z)(M(z) - I)v.
\]
Hence we obtain a globally (near $\gamma$) solvable Cauchy problem by putting
\[
(5.7) \begin{cases} 
\frac{i}{\hbar} P(z) u + R_-(z) u_- = 0 \\
R_+(z) u = v
\end{cases},
\]
with
\[
R_-(z) = \left[ (i/\hbar) P(z), \chi \right]_{W_-} K_b(z).
\]
The problem (5.7) is solved by putting
\[
(5.8) \begin{cases} 
u = E_+(z) v, & u_- = E_{-+}(z) v, \quad E_{-+}(z) \overset{\text{def}}{=} I - M(z), \\
u_+ = E_-(-z) v,
\end{cases}
\]
where $E_+(z)$ was given by (5.5).

The definitions (5.2) and (5.8) give $P(z)$ in (5.1). If the microlocal inverse, $E(z)$, exists, it is necessarily given by
\[
(5.10) E(z) = \left( \begin{array}{cc} E(z) & E_+(z) \\
E_-(z) & E_{-+}(z) \end{array} \right)
\]
where $E_+(z)$ and $E_{-+}(z)$ have already been constructed.

It remains to find $E(z)$, $E_-(z)$, and to show that the resulting operator $E(z)$ is the right and left microlocal inverse of $P(z)$. For the right inverse, this means solving
\[
(5.11) \begin{cases} 
\frac{i}{\hbar} P(z) u + R_-(z) u_- = v \\
R_+(z) u = v_+
\end{cases}.
\]

We first introduce the forward and backward fundamental solutions of $(i/\hbar) P(z)$:
\[
L_f(z) \quad \text{microlocally defined near } (\Omega \times \epsilon, \Omega)_+,
\]
\[
L_b(z) \quad \text{microlocally defined near } (\Omega \times \epsilon, \Omega)_-,
\]
where $(\Omega \times \epsilon, \Omega)_\pm$ is given by
\[
(\Omega \times \epsilon, \Omega)_\pm \overset{\text{def}}{=} \left( \bigcup_{m \in \Omega} \{(\exp(t H_{\rho(0)}) m, m) \} \right) \cap \Omega \times \Omega : -\epsilon < t < T(0) - 2\epsilon.
\]

To do that we use Proposition 3.5 and the corresponding local forward and backward fundamental solutions:
\[
L^0_f v(x) = \int_{-\infty}^{x_1} v(t, x') dt,
\]
\[
L^0_b v(x) = -\int_{x_1}^{\infty} v(t, x') dt,
\]
$v \in \mathcal{E}'(\mathbb{R}^n)$.

We will now try to build an approximate solution of $(i/\hbar) P(z) u = v$ using $L_\bullet(z)$. For that let us put
\[
\tilde{u} = L_f(z)(1 - \chi)v.
\]
Let us also define $\chi_b, \chi_f$ satisfying (4.13) and in addition,
\[
\chi_b = 1 \text{ on } \text{supp } \chi \cap W_+, \quad \chi = 1 \text{ on } \text{supp } \chi_f \cap W_+.
\]
We now put
\[ \tilde{u} = L_f(z)(1 - \chi)v, \]
where we can think of \( \tilde{u} \) as being microlocally defined on the covering space of \( \Omega, \tilde{\Omega} \) (see the proof of Proposition 4.3). Hence, \( P(z)\tilde{u} = 0 \) to the right of the support of \( 1 - \chi \) (in the direction of the flow), in particular on the support of \( \chi_f \). Hence, to the right of the support of \( 1 - \chi \),
\[ \tilde{u} = K(z)K(z)^*[(i/h)P(z), \chi_f]_{W_+} \tilde{u} \]
\[ = K(z)K(z)^*[i/h)P(z), \chi_f]_{W_+} L_f(z)(1 - \chi)v. \]

If we use the notation from the proof of Proposition 4.3 and put \( \tilde{K}_f(z) = \tilde{I}_f(z)K(z) \), then in the forward direction of propagation past the support of \( 1 - \chi \), we have in \( \tilde{\Omega} \),
\[ (5.12) \]
\[ \tilde{u} = \tilde{K}_f(z)K(z)^*[i/h)P(z), \chi_f]_{W_+} L_f(z)(1 - \chi)v. \]

Similarly, if \( \tilde{u} = L_b(z)v \) then, left to the support of \( \chi \), we have \( P(z)\tilde{u} = 0 \), and we can extend \( \tilde{u} \) further left, microlocally in \( \tilde{\Omega} \):
\[ (5.13) \]
\[ \tilde{u} = \tilde{K}_b(z)K(z)^*[i/h)P(z), \chi_b]_{W_+} L_b(z)v. \]

We can think of \( \tilde{u} \) and \( \tilde{u} \) as multivalued in \( \Omega \) and we will define, near \( W_- \),
\[ L_{ff}v = \text{second branch of } \tilde{u} \text{ near } W_-; \]
\[ L_{bb}v = \text{second branch of } \tilde{u} \text{ near } W_- \]

With this notation we put
\[ (5.14) \]
\[ u_0 = E_0(z)v \equiv \begin{cases} L_b(z)v + L_f(z)(1 - \chi)v & \text{outside } W_- \\ L_b(z)v + (1 - \chi)L_{bb}(z)v + L_f(z)(1 - \chi)v + \chi L_{ff}(1 - \chi)v & \text{in } W_- \end{cases} \]

An application of \( (i/h)P(z) \) gives
\[ (i/h)P(z)u_0 = v - [(i/h)P(z), \chi]_{W_-} L_{bb}(z)v + [(i/h)P(z), \chi]_{W_-} L_{ff}(z)(1 - \chi)v, \]
and using (5.12) and (5.13) (where we now drop the hat as we are taking the second branch of \( \tilde{u} \) and \( \tilde{u} \), and the definition of \( M(z) \), we get
\[ \frac{i}{h} P(z)u_0 = v - [(i/h)P(z), \chi]_{W_-} K_b(z)K(z)^*[(i/h)P(z), \chi_b]_{W_+} L_b(z)v \]
\[ + [(i/h)P(z), \chi]_{W_-} K_f(z)K(z)^*[i/h)P(z), \chi_f]_{W_+} L_f(z)(1 - \chi)v \]
\[ = v - [(i/h)P(z), \chi]_{W_-} K_b(z) \{K(z)^*[i/h)P(z), \chi_b]_{W_+} L_b(z)v \}
\[ - M(z)K(z)^*[(i/h)P(z), \chi_f]_{W_+} L_f(z)(1 - \chi)v \}. \]

In other terms,
\[ (5.15) \]
\[ \frac{i}{h} P(z)E_0(z)v + R_-(z)E_0_-(z)v = v, \]
where we defined \( E_0(z) \) by (5.14) and
\[ (5.16) \]
\[ E_{0,-}(z) = K(z)^*[i/h)P(z), \chi_b]_{W_+} L_b(z)v - M(z)K(z)^*[i/h)P(z), \chi_f]_{W_+} L_f(z)(1 - \chi)v. \]

If we now put
\[ E(z) \overset{\text{def}}{=} E_0(z) - E_+ (z)R_+(z)E_0(z), \quad E_-(z) \overset{\text{def}}{=} E_0_-(z)v - E_+ (z)R_+(z)E_0(z), \]
then \( E(z) \) given by (5.10) is a right microlocal inverse of \( P(z) \).
To show that it is also a left inverse, we observe that
\[
P(z)^* = \begin{pmatrix} -\frac{i}{h}P(z) & R_+(z)^* \\ R_-^\dagger(z) & 0 \end{pmatrix} : D'(\mathbb{R}^{n-1}) \to D'(\mathbb{R}^{n-1}),
\]
is microlocally defined in the same region as \(P(z)\) and is essentially of the same form but with \(W_+\) replaced by \(W_-\) and \(\chi\) by \(1 - \chi\):
\[
R_+(z)^* = [(i/h)P(z), \chi]_{W_+} K(z),
R_-^\dagger(z) = K^b(z)^* [(i/h)P(z), \chi]_{W_-}.
\]
To see this we first note that
\[
K^b(z)^* [(i/h)P(z), \chi]_{W_-} K^b(z) = -Id.
\]
In fact, as in the proof of Proposition 4.5, (4.14) is invariant under the change of \(\chi\) and \(W_\pm\), as long as (4.13) hold. In particular, for \(0 < t < T(0) - \epsilon\,
\[
K^b(z)^* [(i/h)P(z), (\exp(tH_{p(0)})^* \chi)]_{\exp(-tH_{p(0)})W_+} K^b(z) = Id.
\]
For \(t \sim T(0)/2\), \(W_+\) is moved to \(W_-\), and \((\exp(-tH_{p(0)})^* \chi)\) satisfies the properties of \(1 - \chi\).
Hence, using the independence of \(\chi\),
\[
K^b(z)^* [(i/h)P(z), (1 - \chi)]_{W_-} K^b(z) = Id_{D'(\mathbb{R}^{n-1})}, \quad \text{microlocally near } (0,0) \in T^*\mathbb{R}^{n-1}.
\]
If we now replace \(K(z)\) by \(K^b(z)\), then \(K(z)\) plays the rôle of \(K^b(z)\), and this proves that \(R_+(z)^*\) is the same as \(-R_-^\dagger(z)\) with \(W_+\) and \(W_-\) switched and \(\chi\) replaced by \(1 - \chi\).
Hence, a similar argument to the one used for the construction of \(E(z)\) shows that \(P(z)^*\) has a right inverse,
\[
F(z)^* = \begin{pmatrix} F(z) & F_+(z) \\ F_-(z) & F_-^\dagger(z) \end{pmatrix}^*.
\]
Then \(F(z)^*\) is a left inverse of \(P(z)\), and the usual argument \((F(z)^* P(z) E(z) = E(z)\), microlocally) shows that it is equal to our right inverse.

**Remark.** By constructing part of the left inverse directly we can arrive at a simplier expression for \(E_-(z)\):
\[
E_-(z) = -\left(M(z)K_f(z)^* \chi + K^b(z)^*(1 - \chi)\right),
\]
and it is useful to have it. To obtain it we will directly solve the problem
\[
\begin{align*}
E_-(z)(i/h)P(z) + E_+(z)R_+(z) &= 0 \\
E_-(z)R_-(z) &= Id_{D'(\mathbb{R}^{n-1})}
\end{align*}
\]
Motivated by the structure of \(E_+(z)\) and the fact that \(R_-(z)\) is close to being an adjoint of \(R_+(z)\) (if it were, then \(E_-(z)\) would simply be the adjoint of \(E_+(z)\)), we put
\[
E_-(z) = -\left(M(z)K_f(z)^* \chi + K^b(z)^*(1 - \chi)\right).
\]
We now compute
\[
-E_-(z)R_-(z) = (M(z)K_f(z)^* \chi + K^b(z)^*(1 - \chi))[\chi_{W_+} K^b(z)] = K^b(z)^* [(i/h)P(z), \chi]_{W_-} K^b(z).
\]
To analyze the last expression, we note that \(K(z)\), in the definition of \(K^b(z)\) was chosen, in Lemma 4.4, so that \(K(z)^* [(i/h)P(z), \chi]_{W_+} K(z) = Id\). As in the proof of Proposition 4.5, this is invariant under the change of \(\chi\) and \(W_\pm\), as long as (4.13) hold: for \(0 < t < T(0) - \epsilon\,
\[
K^b(z)^* [(i/h)P(z), (\exp(tH_{p(0)})^* \chi)]_{\exp(-tH_{p(0)})W_+} K^b(z) = Id.
\]
For $t \sim T(0)/2$, $W_+$ is moved to $W_-$, and $(\exp(-tH_{p(0)})^*\chi$ satisfies the properties of $1 - \chi$. Hence, using the independence of $\chi$,

$$K_\theta(z)^*[(i/h)P(z), (1 - \chi)]_{W_+}K_\theta(z) = Id_{\mathcal{D}'(\mathbb{R}^{n-1})}, \quad \text{microlocally near } (0, 0) \in T^*\mathbb{R}^{n-1}.$$ 

This shows that $\tilde{E}_-(z)R_-(z) = Id$ and we need to verify the first identity in (5.18). For that we use $K_\theta(z)^*P(z) = 0$, $M(z)K_f(z)^* = K_\theta(z)^*$, near $(0, 0) \times W_- \subset T^*\mathbb{R}^{n-1} \times T^*X$, to obtain

$$-\tilde{E}_-(z)(i/h)P(z) = (M(z)K_f(z)^*\chi + K_\theta(z)^*(1 - \chi))(i/h)P(z)$$
$$= M(z)K_f(z)^*[\chi, (i/h)P(z)]_{W_+} - K_\theta(z)^*[\chi, (i/h)P(z)]_{W_+}$$
$$= K(z)^*[(i/h)P(z), \chi]_{W_+} - M(z)K(z)^*[(i/h)P(z), \chi]_{W_+}$$
$$= (1 - M(z))R_+(z) = E_-(z)R_+(z),$$

and that establishes (5.18), so $\tilde{E}_-(z) = E_-(z)$ and we have (5.17).

So far we considered only the case of $z \in \mathbb{R}$, and $P(z) = P(z)^*$. Arguing as at the end of Sect.4, we see that all the operators occurring in the construction of $\mathcal{P}(z)$ and $\mathcal{E}(z)$ have almost analytic extensions to $|\text{Im} z| < Lh \log(1/h)$ for any $L$. It follows that the extension of $\mathcal{E}(z)$ is a microlocal inverse of the extension of $\mathcal{P}(z)$ modulo $|\text{Im} z|^\infty$, which in this neighbourhood of the real axis is $O(h^\infty)$, that is, it remains a microlocal inverse. The bounds on the continuation of $\mathcal{E}(z)$ follow from (3.5). This gives

**Proposition 5.1.** Let $P(z)$ be an almost analytic extension of the self-adjoint family of operators $P(z) \in \mathcal{C}^\infty(I; \Psi^{0,k}(X))$, such that

The flow of $H_p$ has a closed orbit $\gamma$,

on which $p = \sigma(P(0)) = 0$ and $dp \neq 0$.

Then, there exist operators $R_+(z)$, defined in $|\text{Im} z| \leq Lh \log(1/h)$, such that

$$\mathcal{P}(z) = \begin{pmatrix}
(i/h)P(z) & R_-(z) \\
R_+(z) & 0
\end{pmatrix} : \mathcal{D}'(X) \times \mathcal{D}'(\mathbb{R}^{n-1}) \rightarrow \mathcal{D}'(X) \times \mathcal{D}'(\mathbb{R}^{n-1}),$$

defined microlocally near $\gamma \times (0, 0)$, has a microlocal inverse there:

$$\mathcal{E}(z) = \begin{pmatrix}
E(z) & E_+ (z) \\
E_- (z) & E_- (z)
\end{pmatrix} = O(e^{C|\text{Im} z|/h}) : L^2(X) \times L^2(\mathbb{R}^{n-1}) \rightarrow L^2(X) \times L^2(\mathbb{R}^{n-1}),$$

and $E_- (z) = I - M(z)$, where $M(z)$ is the quantum monodromy operator defined by (4.9).

**Remark.** The constant $C$ in the estimate of the norm of $\mathcal{E}(z)$ could be described more explicitly if stronger conditions on $P(z)$ were made. If we assumed (6.1) then $C$ could be related to $C_p$ in (6.11).

6. Proof of the trace formula

We can now prove the main result of the paper. We strengthen our assumptions further here by demanding that $P(z)$ is a smooth family of operators, self-adjoint for the real values of the parameter, and elliptic off the real axis.
Theorem 2. Let \( P(z) \in \mathcal{C}^\infty(I; \Psi_h^0(X)) \), \( I = (-a,a) \subset \mathbb{R} \), be a family of self-adjoint, principal type operators, such that \( \Sigma_z = \{ m : \sigma(P(z)) = 0 \} \subset T^*X \) is compact. We assume that
\[
\sigma(\partial_z P(z)) \leq -C < 0, \text{ near } \Sigma_z.
\]
(6.1)
We also assume that for \( z \) near 0, the Hamilton vector field, \( H_{p(z)} \), \( p(z) = \sigma(P(z)) \), has a simple closed orbit \( \gamma(z) \subset \Sigma_0 \) with period \( T(z) \), and that \( \gamma(z) \) has a neighbourhood \( \Omega \) such that
\[
(6.2) \quad m \in \Omega \text{ and } \exp tH_{p(z)}(m) = m, \quad p(m) = 0, \quad 0 < |t| \leq T(z)N + \epsilon, \quad z \in I, \Rightarrow m \in \gamma(z),
\]
where \( T(z) \) is the period of \( \gamma(z) \), assumed to depend smoothly on \( z \). Let \( A \in \Psi_h^{0,0}(X) \) be a microlocal cut-off to a sufficiently small neighbourhood of \( \gamma(0) \).

Then if \( P(z) \) is an almost analytic extension of \( P(z) \), \( z \in \mathbb{R} \), \( \chi \in \mathcal{C}_c^\infty(I) \), \( \tilde{\chi} \in \mathcal{C}_c^\infty(\mathbb{C}) \), its almost analytic extension, \( f \in \mathcal{C}^\infty(\mathbb{R}) \), and supp \( f \subset (-N(C_p - \epsilon) + C, N(C_p - \epsilon) - C) \setminus \{0\} \), we have,
\[
\frac{1}{\pi} \operatorname{tr} \oint f(z/h)\partial_z \left[ \tilde{\chi}(z) \partial_z P(z) P(z)^{-1} \right] A\mathcal{L}(dz) =
\]
\[
-\frac{1}{2\pi i} \sum_{N-1}^{N+1} \operatorname{tr} \int f(z/h)^{M(z,h)k} \frac{d}{dz} M(z,h)\chi(z)dz + O(h^\infty),
\]
where \( M(z,h) \) is the quantum monodromy operator defined in (4.9) with \( K(z) \) satisfying (4.14). The constant \( C_p > 0 \), in the condition on \( f \) depends on \( p(z) \) only and is given in (6.11).

We observe that the left hand side of (6.3) is independent of the choice of the almost analytic extension of \( \chi \): if \( \tilde{\chi}^2 \) is another extension then, then \( \tilde{\chi}^2 - \tilde{\chi}^2 = O(\vert \text{Im } z \vert^\infty) \). In view of Lemma 6.1 below,
\[
(\tilde{\chi}(z) - \tilde{\chi}^2(z)) \partial_z P(z) P(z)^{-1}
\]
is smooth in \( z \), and \( O(\vert \text{Im } z \vert^\infty) \). By Green’s formula and holomorphy of \( f \), the corresponding integral vanishes.

As described in Sect.2 Theorem 1 is an immediate consequence of Theorem 2.

Before proceeding with a proof we remark that we can assume that
\[
P(z) \in \Psi_h^{0,0}(X),
\]
since \( P(z) \) can be multiplied by an \( z \)-independent elliptic \( B \in \Psi_h^{0,-k}(X) \), without changing (6.3).

We start with a lemma which justifies taking the traces in (6.3):

Lemma 6.1. Under the assumptions of Theorem 2, \( P(z)^{-1} \) exists in \( U \setminus I \), where \( U \) is a complex neighbourhood of \( J \Subset I \), and
\[
\|P(z)^{-1}\| \leq C|\text{Im } z|^{-1}, \quad 0 < |\text{Im } z| \leq 1/C.
\]
(6.4)
Proof. Let \( \psi = \psi^m(x, hD; z) \) be a microlocal cut-off to a small neighbourhood of \( \Sigma_z \). Let us put \( v = P(z)u \), so that (semi-classical) elliptic regularity gives
\[
\|(1 - \psi)u\| \leq C\|v\| + O(h^\infty)\|u\|.
\]
For complex values of \( z \) we write
\[
P(z) = P(\text{Re } z) + \text{Im } z Q(z),
\]
where \( P(\text{Re } z) \) is self-adjoint and \( \sigma(Q(z)) > 1/C > 0 \) near \( \Sigma_z \). This shows that
\[
\text{Im}(P(z)\psi u, \psi u) = \text{Im} \text{ Re}(Q(z)\psi u, \psi u) \geq \text{Im} z \left( \|\psi u\|^2/C - O(h^\infty)\|u\|^2 \right),
\]
where we used the semi-classical Gårding inequality.
We also write
\[ \text{Im}(P(z)u, u) - \text{Im}(P(z)\psi u, \psi u) = \text{Im}(z ((Q(z)u, u) - (Q(z)\psi u, \psi u)) = \text{Im} z \mathcal{O}(1) \| (1 - \psi) u \| u \| = \text{Im} z \mathcal{O}(1) \left( \| v \| u \| + \mathcal{O}(h^{\infty}) \| u \|^2 \right), \]
where we used elliptic regularity (6.4) in the last estimate. Then, applying (6.5),
\[ \| u \| v \| \geq \text{Im}(P(z)\psi u, \psi u) - \text{Im} z \mathcal{O}(1) \left( \| v \| u \| + \mathcal{O}(h^{\infty}) \| u \|^2 \right) \geq \text{Im} z \left( \| \psi u \|^2 / C - \mathcal{O}(1) \| v \| u \| - \mathcal{O}(h^{\infty}) \| u \|^2 \right). \]
For small \( \text{Im} z \) the term \( \| v \| u \| \) on the left hand side can be absorbed in the right hand side, and by adding \( \text{Im} z \| (1 - \psi) u \|^2 \) to both sides we obtain
\[ \text{Im} z \| u \|^2 / C \leq \| u \| v \| + \mathcal{O}(h^{\infty}) \text{Im} z \| u \|^2, \]
and that gives
\[ \| u \| \leq \frac{C}{\text{Im} z} \| v \|, \]
proving the estimate for \( P(z)^{-1} \).

**Proof of Theorem:** Using Proposition 5.1 we can formally write
\[ P(z)^{-1} A = E(z) A - E_{+}(z)E_{-}(z)^{-1}E_{-}(z) A, \quad E_{-}(z) = I - M(z), \]
microlocally near \( \Omega \), and for \( 0 < |\text{Im} z| \leq L h \log(1/h) \), with any \( L \). To apply this formal expression rigorously, we rewrite the left hand side of (6.3) as
\[
\frac{1}{\pi} \text{tr} \int f(z/h) \partial \chi(z) \partial_z P(z) P(z)^{-1} A \mathcal{L}(dz) = \frac{1}{\pi} \sum_{\pm} \text{tr} \int_{\mathbb{C}_{\pm}} f(z/h) \partial \chi(z) \partial_z P(z) P(z)^{-1} A \mathcal{L}(dz). \tag{6.6}
\]
Then, motivated by the formal Neumann series expansion of \( (I - M(z))^{-1} \) we define
\[
T_{N}^{+}(z) \overset{\text{def}}{=} E(z) A - E_{+}(z) \sum_{k=0}^{N} M(z)^k E_{-}(z) A, \tag{6.7}
\]
so that
\[
P(z)^{-1} A = T_{N}^{+}(z) + P(z)^{-1} R_{-}(z) M(z)^{N+1} E_{-}(z) A, \tag{6.8}
\]
microlocally near \( \gamma \) and for \( 0 < |\text{Im} z| \leq L h \log(1/h) \), for any \( L \). In fact, from \( \mathcal{P}(z) \mathcal{E}(z) = \text{Id} \), and \( E_{\pm}(z) = I - M(z) \), we have
\[
P(z)E_{+}(z) = -R_{-}(z)(I - M(z)), \quad P(z)E(z) = I - R_{-}(z)E_{-}(z), \tag{6.9}
\]
and hence
\[
P(z)T_{N}^{+}(z) = P(z)(E(z) A - E_{+}(z) \sum_{k=0}^{N} M(z)^k E_{-}(z) A)
= A - R_{-}(z)E_{-}(z) A + R_{-}(z)(I - M(z)) \sum_{k=0}^{N} M(z)^k E_{-}(z) A
= A - R_{-}(z)M(z)^{N+1} E_{-}(z) A,
\]
which gives (6.8).
To use this in (6.6) we need to have the support of the almost analytic extension of the cut-off function $\chi$ to be contained in the region where $|\text{Im} z| \leq Lh \log(1/h)$. To do that we follow the method of [3, Sect.12] by fixing an almost analytic extension of $\chi, \chi^\#, \text{ and then putting}

$$
\tilde{\chi} = \tilde{\chi}_{L,h} = \chi^\# \psi_{L,h}, \quad \psi_{L,h}(z) = \psi \left( \frac{\text{Im} z}{Lh \log(1/h)} \right), \quad \psi(t) = \begin{cases} 1, & |t| < 1/2 \\ 0, & |t| > 1 \end{cases}.
$$

By the remark after the statement of Theorem 2, (6.3) is independent of the choice of $\tilde{\chi}$ and hence we can use $\tilde{\chi}_{L,h}$. Since now $\mathcal{O}(|\text{Im} z|^{\infty}) = \mathcal{O}(h^{\infty})$, the almost analyticity of $P(z)$ also shows that the left hand side of (6.3) can be rewritten as

$$(6.9) \quad \frac{1}{\pi} \text{tr} \int f(z/h) \bar{\partial}_z \tilde{\chi}(z) \partial_z P(z) P(z)^{-1} A \mathcal{L}(dz),$$

and this is what we will use from now on.

We claim that with the choice of $\tilde{\chi}$ above

$$(6.10) \quad \text{tr} \int_{\mathbb{C}^+} f(z/h) \bar{\partial}_z \tilde{\chi}(z) \partial_z P(z) P(z)^{-1} A \mathcal{L}(dz) = \text{tr} \int_{\mathbb{C}^+} f(z/h) \bar{\partial}_z \tilde{\chi}(z) \partial_z P(z) T_N^+(z) \mathcal{L}(dz) + \mathcal{O}(h^{L/C}),$$

where $C$ is fixed depending on $N$ and supp $\hat{f}$.

To show this we first need the following

**Lemma 6.2.** *The almost analytic continuation of the monodromy operator satisfies, for $z$ sufficiently close to 0, and for any $L$,

$$
\|M(z)\| \leq e^{-(C_p-\epsilon)|\text{Im} z|/h} + \mathcal{O}(h^{\infty}), \quad 0 < |\text{Im} z| < Lh \log(1/h),
$$

$$(6.11) \quad \|M(z)^{-1}\| \leq e^{(C_p-\epsilon)|\text{Im} z|/h} + \mathcal{O}(h^{\infty}), \quad -Lh \log(1/h) < |\text{Im} z| < 0,
$$

where $C_p$ is positive thanks to (6.1).***

**Proof.** We use the differential equation (4.11) and observe that for $z$ real, and $m_0(z) \in \gamma(z)$,

$$
\sigma(K(z)^{-1}(Q(z) - Q(z))K(z))(0,0) = -\int_0^{T(z)} \sigma(\partial_2 P(z))(\exp(tH_{p(0)})(m_0(z)))dt.
$$

Hence, writing $z = x + iy, \quad 0 < y < Lh \log(1/h)$, and $B(z) = K(z)^{-1}(Q(z) - Q(z))K(z)$, we have, for $v \in \mathcal{D}'(\mathbb{R}^{n-1})$, with $WF(v)$ close to $0, 0$,

$$
h \frac{d}{dy} (\|M(z)v\|^2) = h \frac{d}{dy} (\langle M(z)v, M(z)v \rangle

= \text{i}h(\partial_z - \partial_2)(\langle M(z)v, M(z)v \rangle

= -\langle (B(z)M(z)v, M(z)v) - \langle M(z)v, B(z)M(z)v \rangle + \mathcal{O}(|\text{Im} z|^{\infty})\|v\|^2

= -\langle (B(z) + B(z)^*)M(z)v, M(z)v \rangle + \mathcal{O}(|\text{Im} z|^{\infty})\|v\|^2.
$$

The Gårding inequality now shows that for $x$ small enough,

$$
h \frac{d}{dy} (\|M(z)v\|^2) \leq - (C_p - \epsilon)\|M(z)v\|^2 + \mathcal{O}(y^{\infty})\|v\|^2.
$$

Since by Proposition 4.5, $\|M(x)v\|^2 = \|v\|^2(1 + \mathcal{O}(h^{\infty}))$, the lemma follows. \qed
Proof of (6.10): By (6.6) and (6.8) we need to estimate

\[
\text{tr} \int_{\mathbb{C}_+} f(z/h) \tilde{\partial}_z \tilde{\chi}_{L,h}(z) \, \partial_z P(z) \, P(z)^{-1} \, R_-(z) M(z)^{N+1} E_-(z) \, A \, L(dz),
\]

where by Lemmas 6.1 and 6.2 we have

\[
\|M(z)^{N+1}\| \leq \|M(z)\|^{N+1} \leq e^{-(C_r-\epsilon)(N+1) \frac{\log z}{h}}, \quad 0 \leq \text{Im } z \leq L \log(1/h),
\]

\[
\|P(z)^{-1}\| \leq 1/|\text{Im } z|.
\]

All the operators coming from \(\mathcal{P}(z)\) and \(\mathcal{E}(z)\) are bounded by \(\exp(C|\text{Im } z|/h)\), and if \(\text{supp } \tilde{f} \subset [-b+C, b-C]\), then

\[
|f(z/h)| \leq C e^{(b-C) \frac{\text{Im } z}{h}}.
\]

Using the definition of \(\tilde{\chi}_{L,h}\), the above estimates, and the characteristic function

\[
\rho_{L,h}(t) = \text{Id}_{Lh \log(1/h)/2 \leq t \leq Lh \log(1/h)},
\]

we can bound (6.12) by a constant times

\[
h^{-n} \int \left( |\tilde{\partial}_z \tilde{\chi}_{L,h}(z)| + (L \log(1/h))^{-1} \rho_{L,h}(\text{Im } z) \right) \left| \text{Im } z \right|^{-1} e^{\frac{\text{Im } z}{h} (b-(C_r-\epsilon)(N+1))} L(dz) \leq
\]

\[
Ch^{-n} \int_{0 \leq \text{Im } z < L \log(1/h)} \left| \text{Im } z \right|^{-1} e^{\frac{\text{Im } z}{h} (b-(C_r-\epsilon)(N+1))} L(dz)
\]

\[
+ Ch^{-n} (Lh \log(1/h))^{-2} L^{(C_r-\epsilon)(N+1)-b} \leq C_X h^{L/C-n},
\]

where \(C > 0\) is fixed. \(\square\)

With (6.10) established, we have to study the leading term on its right hand side which we rewrite using the definition (6.7) and the cyclicity of the trace:

\[
\frac{1}{\pi} \text{tr} \int_{\mathbb{C}_+} \tilde{\partial}_z(\tilde{\chi}_{L,h}) \partial_z P(z) E(z) A \ f(z/h) L(dz) -
\]

\[
\frac{1}{\pi} \text{tr} \int_{\mathbb{C}_+} \tilde{\partial}_z(\tilde{\chi}_{L,h}) \left( \sum_{k=0}^{N} M(z)^k E_-(z) A \partial_z P(z) E_+(z) \right) f(z/h) L(dz).
\]

Since all the operators are almost analytic (in particular \(\tilde{\partial}_z E_\bullet(z) = \mathcal{O}(h^\infty)\) on the support of \(\tilde{\chi}_{L,h}\) and \(f(z/h)\) is holomorphic, we can apply Green’s formula and reduce the first integral to an integral over the real axis:

\[
\frac{1}{2\pi i} \text{tr} \int_{\mathbb{R}} \chi(z) f(z/h) \, \partial_z P(z) E(z) Adz.
\]

To analyze the second term (with integration still over \(\mathbb{C}_+\)) we see that the explicit expressions \(E_-(z)\) and \(E_+(z)\), (5.5) and (5.17), show that

\[E_-(z) A \partial_z P(z) E_+(z) = A_1(z) M(z) + A_2(z), \quad A_j(z) \in C^\infty(I_{L}; \Psi^{0,-\infty}(\mathbb{R}^{-n-1})).\]

To analyze the contributions to the trace we need the following simple

**Lemma 6.3.** Suppose that \(A \in \Psi^{0,-\infty}_h(\mathbb{R}^{-n-1})\) and \(U \in I^0_h(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; C^\alpha)\), satisfy

\[m \in WF_h(A) \implies (m, m) \notin C\).

Then, for \(\chi \in C^\infty_c(\mathbb{R}^{n-1})\),

\[\text{tr} \chi AU = \mathcal{O}(h^\infty).\]
Proof. It is clear that if \( \chi_1, \chi_2 \in \mathcal{C}_c^\infty(T^*\mathbb{R}^{n-1}) \) have disjoint supports then \( \text{tr} \chi_1 A \chi_2 = O(\hbar^\infty) \), where we denoted the quantizations by the same symbols. Using the hypothesis, we can write \( \chi A U \) as a sum of negligible terms \( (O(\hbar^\infty)) \), and of terms of that form. \( \square \)

The assumption \((6.2)\) implies that the main contribution (modulo \( O(\hbar^\infty) \) as usual) to the trace of

\[
M(z)^k E_+(z) A \partial_z P(z) E_+(z), \quad k > 0,
\]
comes from an arbitrarily small neighbourhood of the fixed point, \((0,0) \in T^*\mathbb{R}^{n-1}, \text{of } C(z)\), the canonical relation of \( M(z) \). We can therefore replace \( A \) by 1 and introduce a microlocal cut-off, \( \rho^w \), to a neighbourhood of \((0,0)\):

\[
\text{tr} M(z)^k E_+(z) A \partial P(z) E_+(z) = M(z)^k E_+(z) \partial P(z) E_+(z) \rho^w + O(\hbar^\infty), \quad k > 0.
\]

For \( k = 0 \) the same discussion is valid for the contribution of \( M(z) A_1(z) \) in \( E_+(z) \partial P(z) E_+(z) \), but for the pseudo-differential contribution, \( A_2(z) \), we need to use the support assumption on \( \hat{f} \): 0 \( \notin \) supp \( \hat{f} \). We write

\[
\text{tr} \int_{\mathcal{C}_+} f(z/\hbar) \partial_z \chi(z) A_2(z) \rho^w L(dz) = \text{tr} \int_{\mathbb{R}} f(z/\hbar) \chi(z) A_2(z) \rho^w dz = O(\hbar^\infty),
\]

by the standard argument: put \( g(z) = \text{tr} \ A_2(z) \rho^w \), so that by Plancherel’s theorem

\[
2\pi \int_{\mathbb{R}} f \left( \frac{z}{\hbar} \right) g(z) dz = \hbar \int_{\mathbb{R}} \hat{f}(\hbar \zeta) \hat{g}(-\zeta) d\zeta
\]

\[
= \hbar^{M+1} \int_{\mathbb{R}} \hat{f}(\hbar \zeta)/(\hbar \zeta)^M \hat{g}(-\zeta) d\zeta
\]

\[
= O(\hbar^{M+1}).
\]

Hence the second term in \((6.13)\) becomes

\[
-\frac{1}{\pi} \text{tr} \int_{\mathcal{C}_+} \partial_z \chi(z) \left( \sum_{k=0}^N M(z)^k E_-(z) \partial_z P(z) E_+(z) \right) \rho^w f(z/\hbar) L(dz),
\]

where \( \rho^w \) is a microlocal cut-off to a neighbourhood of \((0,0)\).

We recall that when \( R_\pm \) are independent of \( z \), the following standard formula holds:

\[
\partial_z M(z) = -\partial_z E_+(z) = E_-(z) \partial_z P(z) E_+(z),
\]

as is easily seen from \( \partial_z E_+ = -E_+ \partial_z P E_+ \). In the general case, the same argument gives

\[
\partial_z M(z) = -\partial_z E_+(z)
\]

\[
= E_-(z) \partial_z P(z) E_+(z) + E_+(z) \partial_z R_+(z) E_+(z) + E_-(z) \partial_z R_-(z) E_-(z).
\]

Inserting this we obtain the following expression for \((6.15)\):

\[
-\frac{1}{\pi} \text{tr} \int_{\mathcal{C}_+} f(z/\hbar) \partial_z \chi(z) \sum_{k=0}^N M(z)^k \partial_z M(z) \rho^w L(dz)
\]

\[
+ \frac{1}{\pi} \text{tr} \int_{\mathcal{C}_+} f(z/\hbar) \partial_z \chi(z) \sum_{k=0}^N M(z)^k (1 - M(z))^{-1} \partial_z R_+(z) E_+(z) \rho^w L(dz)
\]

\[
+ \frac{1}{\pi} \text{tr} \int_{\mathcal{C}_+} f(z/\hbar) \partial_z \chi(z) \sum_{k=0}^N M(z)^k E_-(z) \partial_z R_-(z) (1 - M(z))^{-1} \rho^w M(z)^k L(dz)
\]

\[
= J_1 + J_2 + J_3
\]
By Green’s formula

\[ J_1 = -\frac{1}{2\pi i} \sum_{k=0}^{N} \text{tr} \int_{\mathbb{R}} f(z/h) \chi(z) M(z)^k \partial_z M(z) \rho^w \, dx, \]

which is a term appearing in (6.3). We want to show that the remaining two terms, \( J_2, J_3 \), are negligible.

To see this we need

**Lemma 6.4.** We have

\[ \partial_z R_+(z) E_+(z), \quad E_-(z) \partial_z R_-(z) \in \mathcal{C}^\infty(I_x; \Psi_h^{1, -\infty}(X)). \]

**Proof.** From the definitions (5.2), (5.5), and from (4.14) we see that

\[ \partial_z R_+(z) E_+(z) = \partial_z (K^*(z)((i/h) P(z), \chi|_{W_z}) K(z) = -K^*(z)((i/h) P(z), \chi|_{W_z}) \partial_z K(z). \]

From the proof of Proposition 4.3 we recall that

\[ hD_z K(z) = -Q(z) K(z), \]

and hence

\[ \partial_z R_+(z) E_+(z) = (i/h)^{K^*(z)((i/h) P(z), \chi|_{W_z}) Q(z) K(z). \]

This expression is microlocal near \((0, 0)\) and as far as \(K(z)\) is concerned microlocal near \((m_0, 0, 0))\). Hence we can use a model given in Proposition 3.5: \(P(z) = hD_{z_1}\) (the microlocal \(z\)-dependent conjugation will not affect the uniform pseudo-differential behaviour), and

\[ K(z)u(x_1, x') = \frac{1}{(2\pi h)^{n-1}} \int e^{i((y', y') - \phi_z(x', y'))/h} a_z(x', y') u(y') dy' d\eta', \]

where we used local representation of the \(h\)-Fourier Integral Operators (see the proof of Proposition 7.3 below for the derivation of a local representation). After composing the operators, and applying the stationary phase method we arrive at the following expression for the kernel of \(\partial_z R_+(z) E_+(z)\):

\[ \frac{1}{(2\pi h)^{n-1}} \int e^{i((y', y') - \phi_z(x', y'))/h} A_z(y', x', \eta') d\eta', \quad A_z \in S^{1, -\infty}, \]

which by a standard “Kuranishi trick” argument (see the appendix) shows that we get a smooth \(z\)-dependent family of pseudo-differential operators. \(\square\)

In \(J_2\) we can replace \(\sum_{k=0}^{N} M(z)^k(1 - M(z))\) by \(1 - M(z)^{N+1}\). As in the proof of (6.10), we show that the term corresponding to \(M(z)^{N+1}\) is negligible. The remaining term is transformed to an integral over \(\mathbb{R}\):

\[ \frac{1}{2\pi i} \int_{\mathbb{R}} f(z/h) \chi(z) \text{tr}(\partial_z R_+(z) E_+(z) \rho^w) \, dx, \quad 0 \notin \text{supp} \hat{f} \]

which is negligible by Lemma 6.4 and (6.14). Similar arguments then apply to \(J_3\).

To summarize, we have shown that

\[ \text{tr} \int_{\mathbb{C}_+} f(z/h) \tilde{\chi}(z) \partial_z \tilde{\chi}(z) \tilde{\chi}(z) P(z) P(z)^{-1} \tilde{\chi}(z) \tilde{\chi}(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \chi(z) f(z/h) \partial_z P(z) E(z) \, Adz - \frac{1}{2\pi i} \sum_{k=0}^{N} \text{tr} \int_{\mathbb{R}} f(z/h) \chi(z) M(z)^k \partial_z M(z) \rho^w \, dx + \mathcal{O}(h^\infty), \]
The same arguments apply and Green’s formula gives

\[ T_N(z) \overset{\text{def}}{=} E(z)A + E_+(z) \sum_{k=1}^{N} (M(z)^{-1})^k E_-(z)A. \]

The same arguments apply and Green’s formula gives

\[ -\frac{1}{2\pi i} \text{tr} \int_{\mathbb{R}} \chi(z)f(z/h)E(z)Adz - \frac{1}{2\pi i} \text{tr} \int_{\mathbb{R}} \chi(z)f(z/h) \sum_{k=1}^{N+1} (M(z)^{-k}) \partial_k M(z)dz + O(h^\infty). \]

When we now add the contributions from the integrations over \( \mathbb{C}_\pm \) we see that the integrals involving \( E(z) \) cancel and the remaining terms give (6.3)

\[ \square \]

7. Trace formula for non-degenerate closed trajectories

We say that a closed trajectory \( \gamma(z) \) of \( P(z) \) is of \( N \)-fold non-degenerate if

\[ \det(I - (dC(z)_{m_0(z)})^k) \neq 0, \quad 0 \neq |k| \leq N, \]

where \( C(z) \) is the Poincaré map of \( \gamma(z) \), (4.12). When this holds our theorem translates into the standard semi-classical trace formula, generalizing the Gutzwiller, Balian-Bloch, and Duistermaat-Guillemin trace formulae.

We start by a general discussion of traces of Fourier Integral Operators.

**Lemma 7.1.** Suppose that \( B \), microlocally defined near \( (0,0) \in T^*\mathbb{R}^n \) is given by

\[ Bu(x) = \frac{1}{(2\pi h)^n} \int \int e^{i(\phi(x,\eta)-\eta\gamma)/h} b(x,\eta,h)u(y)dyd\eta \]

where \( \phi(x,\eta) \) is defined near \( (0,0) \), \( a \) is a classical symbol of order 0, supported near \( (0,0) \), \( \phi'(0,0) = 0 \), and \( \phi''_{x\eta}(0,0) \neq 0 \). The corresponding canonical transformation is given by

\[ \kappa : (\phi'_x(x,\eta),\eta) \mapsto (x,\phi'_x(x,\eta)). \]

It is defined between two neighbourhoods of \( (0,0) \) and we assume that \( (0,0) \) is its only fixed point there, and

\[ \det(d\kappa(0,0) - 1) \neq 0. \]

Under these assumptions

\[ \text{tr} B = i^{\frac{1}{2}s} \left( \frac{b_0(0,0) + O(h)}{|\det \phi''_{x\eta} \det(d\kappa(0,0) - 1)|^\frac{1}{2}} \right), \quad s = \text{sgn} \left( \begin{array}{cc} \phi''_{xx} & 1 \\ \phi''_{x\eta} & \phi''_{\eta\eta} - 1 \end{array} \right), \]

where the signature of a symmetric matrix \( A \), \( \text{sgn} A \), is the difference between the number of positive and negative eigenvalues.

**Proof.** The fact that (7.3) defines a smooth map is equivalent to the assumption that

\[ \det \phi''_{x\eta} \neq 0. \]

Here and in the following, second derivatives of \( \phi \) are computed at \( (0,0) \) if nothing else is specified. The differential, \( d\kappa(0,0) \), is the map \( (\delta_x,\delta_\eta) \mapsto (\delta_x,\delta_\xi) \), where

\[ \delta_y = \phi''_{x\eta} \delta_x + \phi''_{x\eta} \delta_\eta \]
\[ \delta_\xi = \phi''_{x\eta} \delta_\eta + \phi''_{x\eta} \delta_x. \]
Here we can express $\delta_x$ and $\delta_z$ in terms of $\delta_y$, $\delta_\eta$, and it follows that $d\kappa(0,0)$ is given by the matrix:

$$d\kappa(0,0) = \left( \begin{array}{ccc} (\phi''_{yx})^{-1} & -\phi''_{yx} & -\phi''_{y\eta} \\ \phi''_{zx} & \phi''_{x\eta} & \phi''_{x\eta} \\ \phi''_{z\eta} & \phi''_{z\eta} \end{array} \right) .$$

We find the following factorization:

$$d\kappa(0,0) - 1 = \left( \begin{array}{ccc} -(\phi''_{yx})^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \left( \begin{array}{ccc} 1 & -\phi''_{yx}(\phi''_{yx})^{-1} & -\phi''_{yx} \\ 0 & \phi''_{x\eta} & \phi''_{x\eta} - 1 \\ 0 & \phi''_{z\eta} - 1 & \phi''_{z\eta} \end{array} \right) .$$

In particular,

$$\det(d\kappa(0,0) - 1) = \frac{1}{\det \phi''_{yx}} \det \left( \begin{array}{ccc} \phi''_{yx} & \phi''_{yx} & \phi''_{yx} - 1 \\ \phi''_{x\eta} - 1 & \phi''_{x\eta} & \phi''_{x\eta} \\ \phi''_{z\eta} & \phi''_{z\eta} - 1 & \phi''_{z\eta} \end{array} \right) .$$

Here

$$\det B = \left( \begin{array}{ccc} \det \frac{1}{i} \left( \begin{array}{ccc} \phi''_{yx} & \phi''_{yx} & \phi''_{yx} - 1 \\ \phi''_{x\eta} - 1 & \phi''_{x\eta} & \phi''_{x\eta} \\ \phi''_{z\eta} & \phi''_{z\eta} - 1 & \phi''_{z\eta} \end{array} \right) \right)^{-1/2} \mathcal{O}(h) \right),$$

Here we choose the branch of the square root of the determinant on the set of non-degenerate symmetric matrices with non-negative real part which is equal to 1 for the identity. Using (7.9), we get (7.6).

To give a geometric meaning to the signature $s$ appearing in (7.5) in terms of a Maslov index we first recall the definition of the Hörmander-Kashiwara index of a Lagrangian triple: let $\lambda_1, \lambda_2, \lambda_3$ be Lagrangian planes in a symplectic vector space $(V, \omega)$, and put

$$s(\lambda_1, \lambda_2, \lambda_3) = \text{sgn} \, Q(\lambda_1, \lambda_2, \lambda_3),$$

where $Q(\lambda_1, \lambda_2, \lambda_3)$ is a quadratic form on $\lambda_1 \oplus \lambda_2 \oplus \lambda_3$ given by

$$Q(\lambda_1, \lambda_2, \lambda_3)(v_1 \oplus v_2 \oplus v_3) = \omega(v_1, v_2) + \omega(v_2, v_3) + \omega(v_3, v_1),$$

see [9] for a comprehensive introduction. Here we only mention that if $\lambda_i$'s are mutually transversal, then $s(\lambda_1, \lambda_2, \lambda_3)$ is the only symplectic invariant of such three Lagrangian planes. It is antisymmetric and satisfies the cocycle condition.

We then have

**Lemma 7.2.** Let $V = T^* \mathbb{R}^n \times T^* \mathbb{R}^n$ with the symplectic form $\omega = \omega_1 - \omega_2$, where $\omega_1$ and $\omega_2$ are the canonical forms on the factors. In the notation of Lemma 7.1, let $\Gamma_{dc}$ be the graph of $d\kappa(0,0)$, $\Delta \subset T^* \mathbb{R}^n \times T^* \mathbb{R}^n$ be the diagonal, and $M = \{0\} \oplus \mathbb{R}^n \oplus \mathbb{R}^n \oplus \{0\} \subset T^* \mathbb{R}^n \times T^* \mathbb{R}^n$. Then

$$s(\Gamma_{dc}, \Delta, M) = -\text{sgn} \left( \begin{array}{ccc} \phi''_{yx} & 0 & 0 \\ 0 & \phi''_{y\eta} & 0 \\ 0 & 0 & \phi''_{y\eta} \end{array} \right) .$$

**Proof.** Let us write

$$\phi''(0,0) = \left( \begin{array}{cc} \alpha & \beta \\ 0 & \gamma \end{array} \right) .$$

Then $\Gamma_{dc} = \{(x, \alpha x + \beta \eta; \beta^* x + \gamma \eta, \eta) : (x, \eta) \in \mathbb{R}^n \times \mathbb{R}^n\}$. Since $\Delta$ and $M$ are transversal, [9, Lemma 1.5.4] says that

$$s(\Gamma_{dc}, \Delta, M) = -\text{sgn} \omega(\pi \cdot \cdot \cdot ) |\Gamma_{dc}| ,$$
where \( \pi : T^*\mathbb{R}^n \times T^*\mathbb{R}^n \to M \) is the projection along \( \Delta : \pi(x, \xi; y, \eta) = (0, \xi - \eta; y - x, 0) \). In the \((x, \eta)\) coordinates on \( \Gamma_{de} \), \( \omega(\pi \bullet, \bullet)|_{\Gamma_{de}} \) is then given by \( \{ \Omega \bullet, \bullet \} \), where

\[
\Omega = \left( \begin{array}{ccc} \alpha & \beta - 1 \\ \beta^t - 1 & \gamma - 1 \end{array} \right),
\]

which proves the lemma.

As is well known, and as will be seen in the proof of the next proposition, any locally defined Fourier Integral Operator can be represented by (7.2). To compute its trace in terms of invariantly defined objects we also have to recall the definition of the Maslov index of a curve of linear symplectomorphisms – see [2] for more details and references.

Thus let \( \Gamma(t) \subset T^*\mathbb{R}^n \times T^*\mathbb{R}^n \), \( a \leq t \leq b \), be a curve of graphs of symplectomorphisms. Choose a subdivision \( a = t_0 < t_1 < \cdots < t_k = b \), such that, for all \( j = 1, \ldots, k \), there is a Lagrangian subspace \( M^j \) transversal to \( \Gamma(t) \) and the diagonal, \( \Delta \), for \( t \in [t_{j-1}, t_j] \). We now follow [2] and define the Maslov index of a curve of linear symplectomorphisms as

\[
\mu \overset{\text{def}}{=} \frac{1}{2} \sum_{j=1}^k \left( s(\Gamma(t_{j-1}), \Delta, M^j) - s(\Gamma(t_j, \Delta, M^j)) \right).
\]

It is independent of the choice of the transversal Lagrangians, \( M^j \), and of the subdivision.

We can now prove

**Proposition 7.3.** Suppose that \( U(t) \) is a family of Fourier Integral Operators defined using a family of pseudodifferential operators, \( A(t) \in \Psi^{0,0}(T^*X) \), as in (3.2):

\[
h D_t U(t) + A(t) U(t) = 0, \quad U(0) = U_0 \in \Psi^{0,0}(T^*X).
\]

Let us also assume that \( \alpha_t = \sigma(A(t)) \), the Weyl symbol (with a possible dependence on \( h \) in the subprincipal symbol part) of \( A(t) \), is real and generates a family of canonical transformations:

\[
\frac{d}{dt} \kappa_t(x, \xi) = (\kappa_t)_*(H_{\alpha_t}(x, \xi)), \quad \kappa_0(x, \xi) = (x, \xi), \quad (x, \xi) \in T^*X
\]

\[
\kappa_t(0, 0) = (0, 0).
\]

If

\[
\det(1 - d\kappa_T(0, 0)) \neq 0,
\]

then

\[
\text{tr} \, U(T) = i^{\mu(T)} \frac{(1 + \mathcal{O}(h)) \cdot e^{-i \int_0^T \alpha_t(0, 0) \, dt / h}}{|\det(d\kappa_T(0, 0) - 1)|^{1/2}} \sigma(U_0)(0, 0),
\]

where \( \mu(T) \) is the Maslov index of the curve of linear symplectic transformations \( d\kappa_t(0, 0), 0 \leq t \leq T \).

**Proof.** Let us first assume that for \( 0 \leq t \leq T \)

\[
(\kappa_t(y, \eta); (y, \eta)) \mapsto (x(\kappa_t(y, \eta)), x) \text{ is surjective near } (0, 0).
\]

We follow the presentation from [6, Appendix a]. Let \( a_t \) be the Weyl-symbol of \( A_t \) defined modulo \( \mathcal{O}(h^2) \) (if there is a subprincipal symbol we include it in the principal one and obtain an \( h \) dependent symbol). Consequently the influence of the subprincipal symbol will be accounted for.
as an $\mathcal{O}(h)$-dependence in the canonical transformation $\kappa_t$. Let $\kappa_t$ be the canonical transformation generated by $H_{\alpha_t}$ as described in the statement of the proposition. We can then view $\kappa_t$ as the canonical transformation associated to $U(t)$ (defined modulo $\mathcal{O}(h^2)$) and we claim that

$$U(t)u(x) = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}(\phi(t; x, \eta) - u \eta)} b(t, x, \eta; h) u(y) dy d\eta,$$

where

$$\partial_t \phi(t, x, \eta) + a_t(x, \partial_x \phi) = 0, \quad \phi(0, x, \eta) = x \cdot \eta,$$

so that $\kappa_t : (\partial_t \phi, \eta) \mapsto (x, \partial_x \phi)$. The amplitude $b$ has to satisfy

$$\begin{align*}
(hD_t + a_t^w(x, hD))(e^{i\phi(t, x, \eta)/h} b(t, x, \eta; h)) &= 0, \\
(\partial_t \phi + hD_t + e^{-i\phi/h} a_t^w(x, hD)e^{i\phi/h})(b) &= 0.
\end{align*}$$

Here the Weyl symbol of $e^{-i\phi/h} a_t^w e^{i\phi/h}$ is $q_t(x, \xi) = a_t(x, \phi_x^t + \xi) + \mathcal{O}(h^2)$, and using that $\partial_t \phi = -a_t(x, \partial_x \phi)$, we get

$$(hD_t + \text{Op}(f_t(x, \xi)))b = \mathcal{O}(h^2),$$

where $f_t(x, \xi) = a_t(x, \phi_x^t + \xi) - a_t(x, \phi_x^t)$ (and $\eta$ is just a parameter). This gives

$$hD_t + \frac{1}{2} \sum_{j=1}^n ((\partial_{x_j} a_t) hD_{x_j} + hD_{x_j} \circ (\partial_{x_j} a_t)) b_0 = 0,$$

for the leading part of $b$. With $\nu_t = \sum (\partial_{x_j} a_t) \partial_{x_j}$, this can also be written

$$(\partial_t + \nu_t + \frac{1}{2} \text{div} \nu_t)b_0 = 0,$$

or

$$\begin{align*}
(\partial_t + L_\nu)(b_0(t, x, \eta)(dx_1 \wedge ... \wedge dx_n)^{1/2}) &= 0,
\end{align*}$$

where $\mathcal{L}_\nu$ denotes the Lie derivative. If we consider $b_0(t, x, \eta)(dx_1 \wedge ... \wedge dx_n)^{1/2}$ as a half-density on $\Lambda_{\phi_{\nu\eta}} = \{(x, \partial_x \phi_{\nu\eta}(x, \eta))\}$, then (7.19) means that

$$\kappa_t^*(b_0(t, x, \eta)(dx_1 \wedge ... \wedge dx_n)^{1/2}|_{\Lambda_{\phi_{\nu\eta}}} = (dx_1 \wedge ... \wedge dx_n)^{1/2}|_{\Lambda_{\phi_0\eta}}.$$  

From (7.7) it follows that the restriction of the differential of $\kappa_t$ to $T\Lambda_{\phi_{\nu\eta}}$ followed by the $x$-space projection is given by

$$\delta_y \mapsto (\phi''_{y \eta x})^{-1}(\delta_y),$$

so (7.20) says that

$$\frac{b_0(t, x, \eta)}{(\phi''_{y \eta x})^{1/2}} = 1.$$

We note that $\det \phi''_{y \eta x} > 0$ for $0 \leq t \leq T$.

From (7.16) and $d\phi_t(0, 0) = 0$ (since $\kappa_t(0, 0) = (0, 0)$) we see that

$$\phi_T(0, 0) = -\int_0^T a_t(0, 0) dt.$$
Applying Lemma 7.1 and Lemma 7.2 we obtain (7.13): under the assumption (7.14) we only need one transversal Lagrangian in (7.11), and we can take $M$ from Lemma 7.2. Then

$$\mu(T) = \frac{1}{2} (s(\Gamma_{\text{dco}}, \Delta, M) - s(\Gamma_{\text{dco}}, \Delta, M))$$
$$= \frac{1}{2} (s(\Delta, \Delta, M) - s(\Gamma_{\text{dco}}, \Delta, M))$$
$$= -\frac{1}{2} s(\Gamma_{\text{dco}}, \Delta, M) = \frac{1}{2} \text{sgn} \left( \begin{array}{cc} \phi_{xx}' & -1 \\ \phi_{xy}' & -1 \end{array} \right).$$

In the case (7.14) does not hold for $0 \leq t \leq T$, we have to choose different coordinates in which (7.14) holds for $t_{j-1} \leq t \leq t_j$. That gives corresponding Lagrangians $M^j$ (defined as $M$ was) and the phase shifts add up precisely to give (7.11). In fact, we can conjugate $U(t)$ by an $h$-Fourier Integral Operator (so without affecting the trace), so that for $t_1 - \delta < t < t_2 + \delta$ the resulting operator is given by

$$\tilde{U}(t)u(x) = \frac{\iota'}{(2\pi h)^n} \int e^{\frac{\iota}{h} (\tilde{\phi}(t, x, y) - y \cdot \eta)} \tilde{h}(t, x, \eta; h) u(y) dy dh,$$

where we can arrange that $\tilde{h} > 0$, and that (7.12) holds with $T = t_1, t_2$. We then use Lemma 7.1 and the geometric discussion above to compute the trace:

$$\text{tr} U(t_1) = \iota' i^{-\frac{1}{2}} \left( 1 + O(h) \right) e^{-i \int_{t_1}^{t_2} dt / h} \frac{1}{|\det(\partial_{\xi_1}(0, 0) - 1)|^{\frac{1}{2}}} \sigma(U_0)(0, 0),$$

where

$$\tilde{s} = -\text{sgn} \left( \begin{array}{cc} \phi_{xx}' & -1 \\ \phi_{xy}' & -1 \end{array} \right).$$

As in Lemma 7.2 we interpret $\tilde{s}$ as

$$\tilde{s} = s(\Gamma_{\text{dco}}, \Delta, M^2),$$

where $M_2$ was chosen in new coordinates. Comparing it with the previous expression for the trace (where we put $T = t_1$ and $M = M^1$), we see that

$$\nu = -s(\Gamma_{\text{dco}}, \Delta, M^1) + s(\Gamma_{\text{dco}}, \Delta, M^2).$$

We now use $\tilde{U}(t)$ to compute the trace at $t = t_2$ which in view of the expression for $\nu$, and the fact that $s(\Gamma_{\text{dco}}, \Delta, M^1) = 0$ is

$$\text{tr} U(t_2) = i^{\frac{1}{2}} (s(\Gamma_{\text{dco}}, \Delta, M^1) - s(\Gamma_{\text{dco}}, \Delta, M^1) + s(\Gamma_{\text{dco}}, \Delta, M^2) - s(\Gamma_{\text{dco}}, \Delta, M^2)) \times$$

$$\frac{(1 + O(h)) e^{-i \int_{t_1}^{t_2} dt / h} \sigma(U_0)(0, 0)}{|\det(\partial_{\xi_1}(0, 0) - 1)|^{\frac{1}{2}}},$$

and comparing with (7.11) see that the power of $i$ is given by the Maslov index for the curve $d\xi(0, 0), 0 \leq t \leq t_2$. We can continue in the same way which gives us the final index $\mu(T)$.

We now want to evaluate the trace of $M(z, h)^k M'(z, h)$, and for this we need to identify the Maslov factor and the phase. For this we recall the definition of the classical action:

$$(7.22) \quad I(z) \overset{\text{def}}{=} \int_{\gamma(z)} \xi dx.$$

The well known relation with the periods is given in
Lemma 7.4. Let $q(z)$ and $q_\zeta(z)$ be the local time and the first return local time defined in (4.2) and (4.3). Then

$$
(q_\zeta(z) - q(z))|_{\gamma(z)} = - \int_0^{T(z)} \sigma(\partial_2 P(z))(\exp(tH_p(z))(m_0(z)))dt,
$$

$$
(q_\zeta(z) - q(z))|_{\gamma(z)} = \frac{dI}{dz}(z).
$$

Proof. The first identity follows directly from the definition and was already used in the proof of Lemma 6.2.

Since $\partial_z p(z) \neq 0$, we can write $p(z) = c(z)(z-\tilde{p})$. Hence on $p(z) = 0$, the equations for $q$ and $q_\zeta$ are

$$
H_p q = -1, \quad H_p q_\zeta = -1,
$$

and consequently $(q_\zeta(z) - q(z))|_{\gamma(z)}$ is the period of $\gamma(z)$, thought of as an orbit of $H_p$ on $\tilde{p} = z$.

We now introduce an isotropic submanifold, $\Gamma$, of $T^*(X \times \mathbb{R})$, where the new variable (on $\mathbb{R}$) is denoted by $\zeta$ with $z$ its dual variable:

$$
\Gamma = \{(m; (\zeta, z)) \in T^*(M \times \mathbb{R}) : m \in \gamma(z), \ z = q(z)(m), z_0 \leq z \leq z_1\}.
$$

The symplectic form $d\xi \wedge dx + dz \wedge d\zeta$ vanishes on $\Gamma$, and hence we obtain from Stokes’s theorem:

$$
I(z_1) - I(z_0) = \int_{\gamma(z_1)} \xi dx - \int_{\gamma(z_2)} \xi dx
$$

$$
= \int_{z_0}^{z_1} (q(z)(m_0(z)) - q_\zeta(z)(m_0(z))) dz = \int_{z_0}^{z_1} T(z)dz,
$$

which proves the lemma.

Using this lemma we will be able to identify the phase in the trace of the monodromy operator. For that let $T(z)$ be the quantum time appearing in (4.11):

$$
T(z) = K(z)^{-1}(Q(z) - Q_\zeta(z))K(z),
$$

so that that formula becomes

$$
hD_z M(z) = T(z)M(z).
$$

This and Proposition 7.3 show that the phase factor in $\text{tr} M(z,h)^k T(z)$ satisfies

$$
J_k'(z) = k(q_\zeta(z) - q(z))|_{\gamma(z)}.
$$

In fact, for any family of $P(z,h)$ satisfying the assumptions of Proposition 5.1, we can associate to $M(z,h)^k$ (not necessarily satisfying the non-degeneracy condition) a phase factor, $J_k(z)$ which has to satisfy

$$
J_k(z) = kI(z) + C_k,
$$

We want to show that $C_k = 0$. For that we note that if we put $P_\epsilon(z,h) = P(z,h/\epsilon)$ then the corresponding $J_k(z)$ is given by $J_k(z)\epsilon$. On the other hand, the action corresponding to $P_\epsilon$ is $kI(z)\epsilon$. Since we can consider $P_\epsilon$ as another deformation of our operator we must have

$$
\forall \epsilon > 0, \quad J_k(z)\epsilon = kI(z)\epsilon + C_k,
$$

and for that we need that $C_k = 0$.

To obtain the Maslov factor we need to find a family of symplectic transformations interpolating between the identity and the Poincaré map. For that let us fix $z$ and supress dependence on $z$ in the subsequent formulæ. We want to define a family $M(t) : \mathcal{D}'(\mathbb{R}^{n-1}) \rightarrow \mathcal{D}'(\mathbb{R}^{n-1})$, of
\( h \)-Fourier Integral Operators such that \( M(0) = Id \) and \( M(T) = M \). To do this we modify the definition of \( I_+ \) in (4.6) to

\[
I_+(t) : \ker_{m_0} P \longrightarrow \ker_{\exp tH_{m_0}}(P)
\]

We also generalize the definition of \( K \) to

\[
K(t) : \mathcal{D}'(\mathbb{R}^{n-1}) \longrightarrow \ker_{\exp tH_{m_0}}(P),
\]

defined using Proposition 3.5 as in (3.11). We can now define

\[
M(t) \overset{\text{def}}{=} K(t)^{-1}I_+(t)K(0) : \mathcal{D}'(\mathbb{R}^{n-1}) \longrightarrow \mathcal{D}'(\mathbb{R}^{n-1}),
\]

microlocally near \((0,0)\). This family has desired properties and quantizes a curve of local symplectomorphism \( \kappa_t \):

\[
\kappa_t = (\Phi_t)^{-1}\Phi_0 : T^*\mathbb{R}^{n-1} \longrightarrow T^*\mathbb{R}^{n-1}, \quad \kappa_t(0,0) = (0,0),
\]

where \( \Phi_t \) symplectically identifies a neighbourhood of \((0,0)\) in \( T^*\mathbb{R}^{n-1} \) with a submanifold of \( p = 0, S_t \), transversal to \( \gamma \) at \( \exp tH_p(m_0) \), and \( \Psi_t : S_0 \longrightarrow S_t \) is the restriction of the flow \( \exp(sH_p) \) to \( S_t \). The construction above allows an arbitrary choice of \( S_t \) and \( \Phi_t \).

We can summarize this discussion in

**Proposition 7.5.** Suppose that the orbit \( \gamma(z) \) is primitive and \( N \)-fold non-degenerate in the sense that (7.1) holds. Let \( I(z) \) be the classical actions defined by (7.22), and \( T(z) \) the periods of \( \gamma(z) \).

If \( t, 0 \leq t \leq T(z) \) parametrizes \( \gamma(z) \), let \( S_t \) be a family of submanifolds of \( p(z) = 0 \), transversal to \( \gamma(z) \) at \( t, \Phi_t \) a symplectic identification of \( S_t \) with a neighbourhood of \((0,0)\) in \( T^*\mathbb{R}^{n-1} \), and \( \Psi_t : S_0 \rightarrow S_t \) the restriction of the flow to \( S_t \). Then for \( 0 < |k| \leq N \),

\[
\text{tr} M(z,h)^{k-1}hD_z M(z,h) = \frac{e^{ikI(z)}e^{\nu_k(z)z^2}q_\gamma(z-q(z))|\gamma(z)|}{|dC_m(z)|^k - 1} (1 + \mathcal{O}(h)),
\]

where \( \nu_k(z) \) is the Maslov index of the curve of linear symplectic transformations:

\[
d(\Phi_t^{-1}\Psi_0(0,0)), \quad 0 \leq t \leq kT(z).
\]

**Remark.** The Maslov index \( \nu_k(z) \) is a locally constant function of \( z \); it does not change as long as (7.1) holds. Its value may depend on the non-unique choices of the identifications \( \Phi_t \), and the transversals \( S_t \). Since \( \exp(\nu_k I/2) \) is determined uniquely (as it appears in the trace!), \( \nu \) is determined only modulo 4. In the case when \( \gamma(z) \rightarrow \pi(\gamma(z)) \) is a diffeomorphism, with \( \pi : T^*X \rightarrow X \), the natural projection, a choice of transversals submanifolds in the base gives natural \( S_t \)'s in \( p = 0 \subset T^*X \). Thus in the case of the geodesic flow on a Riemannian manifold \( \nu \) is the index of a closed geodesic.

The usual semi-classical trace formula for non-degenerate orbits follows from Theorem 1 and the following

**Proposition 7.6.** Suppose that the assumptions of Theorem 1 are satisfied, \( M(z) \) is the quantum monodromy defined in (4.9), and in addition the closed orbit \( \gamma = \gamma(0) \) is \( N \)-fold non-degenerate (7.1). Then, for \( k \neq 0, |k| \leq N \), and \( g \in C^\infty_c(\mathbb{R}) \), we have

\[
\frac{1}{2\pi i} \text{tr} \int_{\mathbb{R}} \hat{g}(z/h) M(z,h)^{k-1} \frac{d}{dz} M(z,h)\chi(z)dz = \frac{e^{ikS_\gamma/h + i\nu_{\gamma,k}z^2}T_\gamma g(kT_\gamma)}{|\det((dC_\gamma)^k - I)|^2} + \mathcal{O}(h),
\]

where \( T_\gamma \) is the primitive period of \( \gamma \), \( dC_\gamma \) is the linear Poincaré map, \( S_\gamma \), the classical action of \( \gamma \), and \( \nu_{\gamma,k} \) the Maslov index of \( k\gamma \).
Proof. Since $P$ is assumed to be self-adjoint, the subprincipal symbol of $P$ is zero. Let $\kappa_{z}$ be the Poincaré map and assume that $(d\kappa_{0}(0,0))^{k} - 1$ is non-degenerate. Let $T(z)$ be the quantum time appearing in (7.23) above. Using the cyclicity of the trace, we can write the left hand side of (7.25) as

$$\frac{1}{2\pi} \text{tr} \int_{\mathbb{R}} \chi(z) \hat{g}(\frac{z}{\hbar}) M(z) T(z) \frac{dz}{\hbar},$$

The leading symbol of $T(z)$ at the fixed point is the period $T(z) = dI(z)/dz$, where $I(z)$ is the action along the closed trajectory. Then to leading order, (7.26) becomes

$$\frac{i^{\mu}}{2\pi} \int_{\mathbb{R}} \chi(z) \hat{g}(\frac{z}{\hbar}) \frac{e^{ikI(z)/\hbar}}{|\det((d\kappa_{0}(0,0))^{k} - 1)|^{\frac{1}{2}}} T(z) \frac{dz}{\hbar},$$

where $\mu$ is the Maslov index. Write $E = z/\hbar$, so that

$$\frac{I(z)}{\hbar} = \frac{I(0)}{\hbar} + I'(0) E + \mathcal{O}(\hbar).$$

Then (7.27) becomes, again to leading order,

$$\frac{\mu^{\nu}}{2\pi} \int \hat{g}(E) e^{ikI(0)/\hbar} \frac{E}{|\det((d\kappa_{0}(0,0))^{k} - 1)|^{\frac{1}{2}}}.$$ 

The usual Gutzwiller trace formula for a more general class of operators is given in

Theorem 3. Suppose that the assumptions of Theorem 1 hold and that in addition $\gamma$ is an $N$-fold non-degenerate orbit in the sense that (7.1) holds. Then in the notation of Proposition 7.6 we have

$$\text{tr} f(P/\hbar) \chi(P) A = \frac{1}{2\pi} \sum_{k=-N}^{N} \frac{e^{ikS_{\gamma}/\hbar + ik\omega_{\gamma} z^{2} \frac{\gamma}{\hbar} T_{\gamma} \hat{f}(-kT_{\gamma})}}{|\det((d\kappa_{0}(0,0))^{k} - I)|^{\frac{1}{2}}} + \mathcal{O}(\hbar).$$

Appendix

In the classical treatment of pseudo-differential operators, the subprincipal symbol is invariant under coordinate changes when the pseudo-differential operators are considered as acting on half-densities. This invariance is particularly nice in the Weyl calculus, where the subprincipal symbol is contained in the leading symbol – see [8, Sect.18.5].

For the reader’s convenience we present here a self-contained discussion of the analogous result in the semiclassical setting.

We use the informal notation for sections of the half-density bundles:

$$u \in C^{\infty}(X, \Omega^{\frac{k}{2}}_{X}) \iff u = u(x) |dx|^{\frac{1}{2}},$$

$$a \in S^{0,0}(T^{*}X, \Omega^{\frac{k}{2}}_{T^{*}X}) \iff a = a(x, \xi) |dx|^{\frac{1}{2}} |d\xi|^{\frac{1}{2}},$$

which captures the transformation laws under changes of coordinates:

$$u(x) |dx|^{\frac{1}{2}} = \tilde{u}(\tilde{x}) |d\tilde{x}|^{\frac{1}{2}}, \quad \tilde{x} = \kappa(x) \iff \tilde{u}(\kappa(x)) |d\kappa'(x)|^{\frac{1}{2}} = u(x),$$

where for a linear tranformation $A$ we denote its determinant by $|A|$.

We observe that the half-density sections over $T^{*}X$ are identified with functions if we consider symplectic changes of variables, and in particular

$$(x, \xi) \mapsto (\tilde{x}, \tilde{\xi}) = (\kappa(x), (\kappa'(x))^{\frac{1}{2}} \xi).$$

(A.1)
As stated after (3.1) in this paper we considered pseudodifferential operators acting on half-densities:

$$\Psi^{m,k}_h(X) = \Psi^{m,k}_h(X, \Omega^\frac{1}{2}_X),$$

with distributional kernels given by

$$(A.2) \quad K_a(x,y)|dx|^\frac{1}{2}|dy|^\frac{1}{2} = \frac{1}{(2\pi h)^n} \int \hat{a} \left( \frac{x+y}{2}, \xi \right) e^{i(x-y,\xi)/h} dx d\xi |dx|^\frac{1}{2} |dy|^\frac{1}{2}.$$

We will show that

$$(A.3) \quad K_a(x,y)|dx|^\frac{1}{2}|dy|^\frac{1}{2} = K_a(\tilde{x}, \tilde{y})|d\tilde{x}|^{\frac{1}{2}}|d\tilde{y}|^{\frac{1}{2}} \quad \text{with} \quad \tilde{a}(\tilde{x}, \tilde{\xi}) = a(x, \xi) + \mathcal{O}(h^2(\xi)^{-2}),$$

where $(\tilde{x}, \tilde{\xi})$ is given by (A.1).

To establish (A.3) we start with its right hand side using the coordinates $\tilde{x} = \kappa(x)$ and $\tilde{y} = \kappa(y)$:

$$\frac{1}{(2\pi h)^n} \int \tilde{a} \left( \frac{\kappa(x) + \kappa(y)}{2}, \tilde{\xi} \right) e^{i(\kappa(x) - \kappa(y),\tilde{\xi})/h} d\tilde{x} d\tilde{\xi} |d\tilde{x}|^{\frac{1}{2}} |d\tilde{y}|^{\frac{1}{2}}.$$

We now apply the “Kuranishi trick” and for that write

$$\kappa(x) - \kappa(y) = F(x,y)(x-y), \quad F(x,y) = \kappa' \left( \frac{x+y}{2} \right) + \mathcal{O}((x-y)^2),$$

$$(A.4) \quad \kappa(x) + \kappa(y) = \kappa \left( \frac{x+y}{2} \right) + \mathcal{O}((x-y)^2).$$

We put $\xi = F(x,y)^{\frac{1}{2}} \tilde{\xi}$ and rewrite the expression above as

$$\frac{1}{(2\pi h)^n} \int \left( \tilde{a} \left( \kappa \left( \frac{x+y}{2} \right), (K(x,y))^{-1} \xi \right) + \mathcal{O}(x-y)^2 \right) e^{i(x-y,\xi)/h} |K(x,y)|^{-1}$$

$$\quad \cdot d\xi |\kappa'(x)|^{\frac{1}{2}} |\kappa'(y)|^{\frac{1}{2}} |dx|^{\frac{1}{2}} |dy|^{\frac{1}{2}} =$$

$$\frac{1}{(2\pi h)^n} \int \left( \tilde{a} \left( \kappa \left( \frac{(x+y)/2}{2} \right), (\kappa' ((x+y)/2))^{t^{-1}} \xi \right) + \mathcal{O}(x-y)^2 \right) e^{i(x-y,\xi)/h} |\kappa'((x+y)/2)|^{-1}$$

$$\quad \cdot d\xi |\kappa'(x)|^{\frac{1}{2}} |\kappa'(y)|^{\frac{1}{2}} |dx|^{\frac{1}{2}} |dy|^{\frac{1}{2}},$$

and the terms $\mathcal{O}((x-y)^2)$ will influence the symbol only modulo $\mathcal{O}((\xi)^{-2}h^2)$ (by integration by parts based on $(x-y) \exp((x-y,\xi)/h) = hD_x \exp((x-y,\xi)/h)$), and hence can be neglected.

We now observe that

$$|\kappa'((x+y)/2)|^2 = |\kappa'(y)||\kappa'(x)| + \mathcal{O}((x-y)^2),$$

and consequently

$$K_{\tilde{a}}(\tilde{x}, \tilde{y}) =$$

$$\frac{1}{(2\pi h)^n} \int \left( \tilde{a} \left( \kappa \left( \frac{(x+y)/2}{2} \right), (\kappa' ((x+y)/2))^{t^{-1}} \xi \right) + \mathcal{O}(h^2(\xi)^{-2}) \right) e^{i(x-y,\xi)/h} |d\xi| |dx|^{\frac{1}{2}} |dy|^{\frac{1}{2}},$$

which is the same as $K_a(x,y)|dx|^\frac{1}{2}|dy|^\frac{1}{2}$, if

$$a(x, \xi) = \tilde{a} \left( \kappa(x), (\kappa'(x)^t)^{-1} \xi \right) + \mathcal{O}(h^2(\xi)^{-2}).$$

This proves (A.3) completing the appendix.
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