ON THE IMAGE IN THE TORUS OF SPARSE POINTS ON DILATING ANALYTIC CURVES

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Abstract. It is known that the image in \( \mathbb{R}^2/\mathbb{Z}^2 \) of a circle of radius \( \rho \) in the plane becomes equidistributed as \( \rho \to \infty \). We consider the following sparse version of this phenomenon. Starting from a sequence of radii \( \{\rho_n\}_{n=1}^{\infty} \) which diverges to \( \infty \) and an angle \( \omega \in \mathbb{R}/\mathbb{Z} \), we consider the projection to \( \mathbb{R}^2/\mathbb{Z}^2 \) of the \( n \)-th roots of unity rotated by angle \( \omega \) and dilated by a factor of \( \rho_n \). We prove that if \( \rho_n \) is bounded polynomially in \( n \), then the image of these sparse collections becomes equidistributed, and moreover, if \( \rho_n \) grows arbitrarily fast, then we show that equidistribution holds for almost all \( \omega \). Interestingly, we found that for any angle there is a sequence of radii growing to \( \infty \) faster than any polynomial for which equidistribution fails dramatically. In greater generality, we prove this type of results for dilations of varying analytic curves in \( \mathbb{R}^d \). A novel component of the proof is the use of the theory of \( \alpha \)-minimal structures to control exponential sums.

1. Introduction

To put our work in context we first note the following general problem. Consider a Lie group \( G \), let \( \Gamma \leq G \) be a lattice and let \( \pi : G \to G/\Gamma \) be the natural projection. Assume that \( \gamma_T : [0,1] \to G, T \in \mathbb{R}_{>0} \) is a family of curves which expand as \( T \to \infty \). Then a natural question that arises is what are the weak-* limits of the probability measures on \( G/\Gamma \) given by \( \mu_T(f) \coloneqq \int_0^1 f(\pi(\gamma_T(s)))\,ds \) for \( f \in C_c(G/\Gamma) \), as \( T \to \infty \).

The above question was extensively studied in recent years, see for example the following (not complete) list \cite{Ran84, Sha09a, Sha09b, BF09, Yan16, KSS18, Kha20, Yan20}.

In this paper we consider in the Euclidean setting a natural discrete analogue of the above. Namely, we consider sequences of expanding curves in \( \mathbb{R}^d \) and we prove statements concerning the distribution of the projection to the \( d \)-torus \( \mathbb{T}^d \coloneqq \mathbb{R}^d/\mathbb{Z}^d \) of probability counting measures supported on discrete subsets of those curves. We note that problems similar in flavor to the ones studied in this paper were considered in the setting of hyperbolic surfaces for translations of horocycles in \cite{MS03} and more recently in \cite{BSY20}.

The main results of our paper are described roughly as follows. We show that there is a certain threshold for the sparsity of the sampled discrete points so that if it is not crossed, then their images in the \( d \)-torus become equidistributed (Theorem \ref{thm:equidist}) and if this threshold is crossed, then the mentioned equidistribution might fail (Theorems \ref{thm:equidist-fail} and \ref{thm:equidist-fail-generic}). Nevertheless, we have found that when there is no restriction on the level of sparsity, the equidistribution is still generic with respect to a certain perturbation (Theorem \ref{thm:equidist-generic}).

1.1. Preliminaries to the main results. In the following we discuss the types of curves which we study in the paper and introduce some notations and conventions.

Using the data of an analytic function \( \phi : [0,1]^m \times [0,1] \to \mathbb{R}^d \), a sequence \( \{\rho_n\}_{n=1}^{\infty} \subseteq \mathbb{R} \) which diverges to \( \infty \) and a sequence \( \{x_n\}_{n=1}^{\infty} \subseteq [0,1]^m \), we consider the curves

\[
\gamma_n(t) \coloneqq \rho_n \phi(x_n, t), \quad t \in [0,1].
\]

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Our main goal will be to describe the weak-* limits of sequences of probability counting measures on \( T^d \equiv \mathbb{R}^d / \mathbb{Z}^d \) of the form

\[
\mu_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{\pi(\gamma_n(k/n))},
\]

where \( \pi : \mathbb{R}^d \to \mathbb{R}^d / \mathbb{Z}^d \) denotes the natural projection.

We will say that \( \{\mu_n\}_{n=1}^{\infty} \) equidistribute if

\[
\lim_{n \to \infty} \mu_n(f) = \int_{T^d} f \, dx, \quad \forall f \in C(T^d),
\]

where \( dx \) denotes the normalized Haar measure.

**Remark.** In all cases we consider, the sequence of continuous curves equidistributes, namely

\[
\lim_{n \to \infty} \int_0^1 f(\pi(\gamma_n(t))) dt = \int_{T^d} f \, dx, \quad \forall f \in C(T^d).
\]

In qualitative terms, when the points \( \{\gamma_n(k/n)\}_{k=1}^{n} \) don’t become sparse on the continuous curves \( \gamma_n \) as \( n \to \infty \), then one may deduce the equidistribution of (1.2) from the equidistribution of the continuous curves, but when the points \( \{\gamma_n(k/n)\}_{k=1}^{n} \) become very sparse on the continuous curves \( \gamma_n \) as \( n \to \infty \), then it is no longer possible to use the continuous equidistribution to study the discrete one. We note that the sparsity of the discrete points \( \{\gamma_n(\frac{k}{n})\}_{k=1}^{n} \) can be measured by the rate of divergence of ratio \( \rho_n/n \) to \( \infty \) as \( n \to \infty \).

We now describe a rationality property of curves which will be used to give conditions for equidistribution. Given \( j \in \mathbb{N} \) and a smooth curve \( \gamma : [0, 1] \to \mathbb{R}^d \), we define

\[
\gamma^{(j)}(t) \equiv (\gamma^{(j)}_1(t), \ldots, \gamma^{(j)}_d(t)), \quad t \in [0, 1],
\]

where \( \gamma^{(j)}_i \) is the \( j \)-th derivative of \( \gamma_i \).

**Definition 1.1.** We say that a smooth curve \( \gamma : [0, 1] \to \mathbb{R}^d \) is rationally non-degenerate of order \( \kappa \in \mathbb{N} \cup \{\infty\} \), if

\[
\kappa = \sup \left\{ j \in \mathbb{N} \mid \langle h, \gamma^{(j)}(t) \rangle \text{ is a non-zero function in } t, \forall h \in \mathbb{Z}^d \setminus \{0\} \right\},
\]

and we say that a family of smooth curves \( \phi : [0, 1]^m \times [0, 1] \to \mathbb{R}^d \) is rationally non-degenerate of order \( \kappa \in \mathbb{N} \cup \{\infty\} \) if

\[
\kappa = \inf \{ \text{order of } \phi(x, \cdot) \mid x \in [0, 1]^m \}.
\]

We will abbreviate the term “rationally non-degenerate” by RND throughout the text.

### 1.2. Equidistribution for polynomial sparsity.

The first main result we would like to discuss concerns the conditions for the equidistribution of the measures of the form (1.2). Our theorem states that the higher the non-degeneracy of the curves is, the sparser we can sample the curves to obtain equidistribution. If the order of non-degeneracy is \( \infty \), then \( \rho_n \) is allowed to grow at an arbitrary fixed polynomial rate to assure that the sequence of measures (1.2) equidistributes.

**Theorem 1.2.** Let \( \phi : [0, 1]^m \times [0, 1] \to \mathbb{R}^d \) be a family of RND analytic curves of order \( \kappa \), with \( 2 \leq \kappa \leq \infty \). Assume that \( \rho_n \to \infty \) such that:

- \( \rho_n = o(n^\kappa) \) if \( \kappa < \infty \),
- \( \{\rho_n\}_{n=1}^{\infty} \) grows polynomially if \( \kappa = \infty \), namely, there exists \( l \in \mathbb{N} \) such that \( \rho_n = o(n^l) \),

and let \( \{x_n\}_{n=1}^{\infty} \subseteq [0, 1]^m \). Then, the sequence of measures given by (1.2) for the curves (1.1) equidistributes.
Remark. It is an artifact of our proof that Theorem 1.2 is stated for \( \kappa \geq 2 \) and not for all \( \kappa \in \mathbb{N} \). It seems to us that Theorem 1.2 with \( \kappa = 1 \) is also true, yet, since for \( \rho_n = o(n) \) the points \( \{ \rho_n \phi(x_n, k/n) \}_{k=1}^n \) don’t get sparse on the curves \( \rho_n \phi(x_n, t) \), we didn’t make the effort to include a proof.

We find the following examples to be noteworthy.

Example. Let \( \psi : [0,1]^m \to \text{GL}_2(\mathbb{R}) \) be an analytic map, namely, the entries of the matrices \( \psi(x) \) are analytic functions. Then the family of ellipses
\[
\phi(x,t) = (\cos(2\pi t), \sin(2\pi t)) \cdot \psi(x),
\]
is a RND family of analytic curves of order \( \infty \) to which we may apply Theorem 1.2.

Example. The following example shows that the way expanding curves are sampled can have dramatic effects on the equidistribution of (1.2).

Let \( \alpha \) be an irrational number and consider the following two parameterizations of the line segment \( \{ (t,\alpha t) \mid t \in [0,1] \} \),
\[
\gamma_1(t) = (t,\alpha t), \quad \gamma_2(t) = \left( \sin \left( \frac{\pi t}{2} \right), \alpha \sin \left( \frac{\pi t}{2} \right) \right), \quad t \in [0,1].
\]
The important distinction between the two curves is that \( \gamma_1 \) is RND of order 1 and \( \gamma_2 \) is RND of order \( \infty \). Let \( \rho_n = n^\kappa \) for an arbitrary \( \kappa \in \mathbb{N} \), then \( \{ \pi (\rho_n \gamma_1(k/n)) \}_{k=1}^n \) will not equidistribute since the first coordinate is zero modulo one. On the other-hand, Theorem 1.2 implies that the points \( \{ \pi (\rho_n \gamma_2(k/n)) \}_{k=1}^n \) will equidistribute.

1.3. Counter examples. Our goal in the following is to discuss the possibility of failure of equidistribution when the conditions on \( \{ \rho_n \}_{n=1}^\infty \) given in Theorem 1.2 are not met.

1.3.1. The case of RND curves of finite order. We now consider dilations of a single curve
\[
\gamma_n(t) = \rho_n \gamma(t),
\]
where \( \gamma : [0,1] \to \mathbb{R}^d \) is analytic. The content of Theorem 1.3 below is to show that for a RND curve of order \( \kappa < \infty \), the condition \( \rho_n = o(n^\kappa) \) of Theorem 1.2 is rather sharp.

Theorem 1.3. Assume that \( \gamma : [0,1] \to \mathbb{R}^d \) is a RND analytic curve of order \( \kappa < \infty \). Then there exists a sequence \( \{ \rho_n \}_{n=1}^\infty \) which satisfies \( n^\kappa \leq \rho_n \leq n^{(\kappa+1)/2} \), \( \forall n \in \mathbb{N} \), so that \( \{ \mu_n \}_{n=1}^\infty \) given by (1.2) for the curves \( \gamma_n = \rho_n \gamma \) will not equidistribute.

Example. There are RND analytic curves of order \( \kappa \) such that \( \{ \mu_n \}_{n=1}^\infty \) will not equidistribute for \( \rho_n = n^\kappa \). Indeed, consider
\[
\gamma(t) = (t^\kappa, t^{\kappa+1}),
\]
then the first coordinate of \( n^\kappa \gamma(\frac{1}{n^\kappa}) \) is zero modulo one.

1.3.2. The case of RND curves of order \( \infty \). If we consider an arbitrary family of curves \( \phi : [0,1]^m \times [0,1] \to \mathbb{R}^d \), then the following is true.

Theorem 1.4. Let \( \phi : [0,1]^m \times [0,1] \to \mathbb{R}^d \) be a family of curves and fix \( \{ x_n \}_{n=1}^\infty \subseteq [0,1]^m \). Then there exists a sequence \( \{ \rho_n \}_{n=1}^\infty \) diverging to \( \infty \) with
\[
\rho_n \ll \left( 3.5 \right)^d \quad , \quad \forall n \in \mathbb{N},
\]
for which \( \{ \mu_n \}_{n=1}^\infty \) given by (1.2) for the curves \( \gamma_n = \rho_n \phi(x_n, \cdot) \) will not equidistribute.

If \( \phi : [0,1]^m \times [0,1] \to \mathbb{R}^d \) is a family of RND analytic curves of order \( \infty \), then by Theorem 1.2 it is necessary that the sequence \( \{ \rho_n \}_{n=1}^\infty \) given in Lemma 1.4 satisfies for all \( \kappa \in \mathbb{N} \) that
\[
\lim_{n \to \infty} \frac{n^\kappa}{\rho_n} = 0.
\]

Namely, for RND curves of order \( \infty \), the “bad dilations” necessarily exceed polynomial growth, yet there are “bad dilations” which can be bounded exponentially.
1.4. Equidistribution beyond polynomial growth. To illustrate the content of our last main result (Theorem 1.3) we now discuss a particular example.

Consider for \( n \in \mathbb{N} \) the rotated \( n \)’th roots of unity on the unit circle

\[
    r^{(n)}_{k,\omega} \defeq \left( \sin \left( \frac{2\pi k}{n} + \omega \right), \cos \left( \frac{2\pi k}{n} + \omega \right) \right), \quad k = 1, \ldots, n, \ \omega \in \mathbb{R}/\mathbb{Z}.
\]

We denote by \( \{\rho_n\}_{n=1}^{\infty} \subseteq \mathbb{R} \) a sequence diverging to \( \infty \) and we consider the sequence of measures

\[
    \mu_{n,\omega} \defeq \frac{1}{n} \sum_{k=1}^{n} \delta_{\pi(\rho_n r^{(n)}_{k,\omega})}
\]

for \( n \in \mathbb{N}, \ \omega \in \mathbb{R}/\mathbb{Z} \).

Consider the case that \( \rho_n \) is bounded polynomially in \( n \). Since for all \( \omega \in \mathbb{R}/\mathbb{Z} \) the curve

\[
    \gamma_{\omega}(t) = (\sin(2\pi t + \omega), \cos(2\pi t + \omega)), \quad t \in [0, 1]
\]

is RND analytic curve of order \( \infty \), we obtain by Theorem 1.2 that for all \( \omega \in \mathbb{R}/\mathbb{Z} \) the sequence of measures \( \{\mu_{n,\omega}\}_{n=1}^{\infty} \) equidistributes as \( n \to \infty \).

Theorem 1.3 below sheds light on the case that Theorem 1.4 states that for any fixed \( \omega \) that \( \{\rho_n\}_{n=1}^{\infty} \) is RND analytic curve of order \( \infty \). Since for all \( \omega \in \mathbb{R}/\mathbb{Z} \) the sequence of measures \( \{\mu_{n,\omega}\}_{n=1}^{\infty} \) equidistributes as \( n \to \infty \).

Almost all rotations. We say that a family of closed curves \( \varphi : [0, 1]^m \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^d \) is RND analytic of order \( \infty \), if the lift of \( \varphi \) defined by

\[
    \phi(x, t) = \varphi(x, t + \mathbb{Z}), \quad (x, t) \in [0, 1]^m \times [0, 1],
\]

is RND analytic family of curves of order \( \infty \). Let \( \{x_n\}_{n=1}^{\infty} \subseteq [0, 1]^m \), \( \{\rho_n\}_{n=1}^{\infty} \subseteq \mathbb{R} \) and \( \omega \in [0, 1] \). We denote

\[
    \gamma_{\omega,n}(t) \defeq \rho_n \varphi(x_n, t + \omega + \mathbb{Z}), \quad t \in [0, 1],
\]

and for \( \gamma_{\omega,n} \) we define the measure \( \mu_{\omega,n} \) as in (1.2).

Theorem 1.5. Assume that \( \varphi : [0, 1]^m \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^d \) is an analytic family of RND curves of order \( \infty \). Fix \( \{x_n\}_{n=1}^{\infty} \subseteq [0, 1]^m \) and \( \{\rho_n\}_{n=1}^{\infty} \subseteq \mathbb{R} \) such that \( \rho_n \to \infty \). Then, \( \{\mu_{\omega,n}\}_{n=1}^{\infty} \) equidistributes, for almost all \( \omega \in [0, 1] \).

1.5. Proof ideas and organization of the paper. To prove equidistribution we estimate exponential sums. In these estimates, it is crucial to control the sub-level sets of the amplitude function (the one that appears as the argument of the exponential). This control is reflected both in bounding the number of connected components and the measure of the sub-level sets as illustrated in Proposition 2.1. This connects our discussion to the theory of o-minimal structures, which allows to prove such estimates in impressive generality by an argument which we find to be elegant (see Section 2). As far as we know, this work is novel in its use of o-minimality to control exponential sums, yet we note that the use of o-minimality to control exponential integrals already appears in [PS19].

The structure of the paper is as follows:

- In Section 2 we establish the mentioned properties of sub-level sets, and in Appendix A we discuss the facts which we need from the theory of o-minimal structures.
- In Sections 3 and 4 we prove Theorems 1.2 and 1.3 respectively.
- In Section 5 we discuss counter examples for equidistribution (Theorems 1.4 and 1.5).
Notational conventions.
- For $n \in \mathbb{N}$, we set $[n] \overset{\text{def}}{=} \{1, 2, ..., n\}$.
- $e(x) \overset{\text{def}}{=} e^{2\pi i x}$.
- Vectors will be denoted by bold letters, namely $\mathbf{x}$ will stand for a $n$-tuple, and by small letters, as $x_j$, we denote the coordinates of $\mathbf{x}$.
- We denote for $x, y \in \mathbb{R}^d$ by $\langle x, y \rangle$ the usual Euclidean inner product, and by $\|x\|$ the Euclidean norm.
- We denote for $x, y \in \mathbb{R}^d$ by $|S|$ its Lebesgue volume and for any set $S$ we denote by $\#S$ the number of elements in $S$.
- We will use the notations $\ll, o(\cdot), O(\cdot)$, as in the book [IK04] (see introduction in [IK04]).

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2. Preliminaries

Let $N \in \mathbb{N}$ and assume that $F : [0, 1]^N \times [0, 1] \to \mathbb{R}$ is a non-constant analytic function. Let
\begin{equation}
\Sigma \overset{\text{def}}{=} \left\{ x \in [0, 1]^N \mid F(x, t) = 0, \; \forall t \in [0, 1] \right\},
\end{equation}
and note that $\Sigma$ is a proper closed subset of $[0, 1]^N$. We denote by $\text{dist}(x, \Sigma)$ the usual Euclidean distance between $x \in \mathbb{R}^N$ and $\Sigma$. Then, for all $\epsilon > 0$ small enough, the subset
\begin{equation}
\Sigma_\epsilon \overset{\text{def}}{=} \left\{ x \in [0, 1]^N \mid \text{dist}(x, \Sigma) \geq \epsilon \right\},
\end{equation}
is non-empty, and the function $F(x, \cdot)$ is non-zero for all $x \in \Sigma_\epsilon$. When $\Sigma = \emptyset$ we take the convention that $\Sigma_\epsilon = [0, 1]^N$ for all $\epsilon > 0$.

We define for $\delta > 0$ and $x \in [0, 1]^N$ the set
\begin{equation}
F_{x, \delta} \overset{\text{def}}{=} \{ t \in [0, 1] \mid |F(x, t)| \geq \delta \}.
\end{equation}

Proposition 2.1. Assume that $F : [0, 1]^N \times [0, 1] \to \mathbb{R}$ is a non-constant analytic function. Then, there exists $\alpha > 0$ such that for all $\epsilon \in (0, 1)$ and $x \in \Sigma_\epsilon$ it holds that $F_{x, \epsilon^\alpha}$ is a union of $O(1)$ closed intervals, say
\begin{equation}
F_{x, \epsilon^\alpha} = I_1 \cup \ldots \cup I_m, \; m = O(1),
\end{equation}
and
\begin{equation}
1 - |F_{x, \epsilon^\alpha}| \ll \epsilon.
\end{equation}

The proof of Proposition 2.1 involves familiarity with the theory of o-minimal structures. We refer the reader not familiar with o-minimal theory to Appendix [A] where we discuss all the required details needed for the proof below.

We denote by $\mathbb{R}_{an}$ the o-minimal structure expanding the real field generated by the restricted analytic functions (see Section [A.2]).
By the uniform bounds on fibers property of o-minimal structures, there exists $M = M(F) > 0$ such that $F_{x,\delta}$ is a union of at most $M$ intervals (see Theorem A.7 and observe that $F_{x,\delta} \overset{\text{def}}{=} \{(x, \delta, t) \in [0, 1]^N \times \mathbb{R}_{>0} \times [0, 1] \mid (F(x, t))^{t^2 - \delta^2} \geq 0\}$ is definable in $\mathbb{R}_{an}$), which proves (2.3).

The bound (2.4) is more special, namely, it is not necessarily true in an arbitrary o-minimal structure. The essential ingredient we will use below in the proof of (2.4) is that $\mathbb{R}_{an}$ is polynomially bounded (see Definition A.9).

We consider

(2.5)

$A \overset{\text{def}}{=} \{(\epsilon, \delta) \in (0, 1) \mid \forall x \in \Sigma_{\epsilon}, \forall \xi \in [0, 1] \text{ it holds that } (\xi - \frac{\epsilon}{2}, \xi + \frac{\epsilon}{2}) \cap F_{x,\delta} \neq \emptyset\}$.

**Lemma 2.2.** For all $(\epsilon, \delta) \in A$ and all $x \in \Sigma_{\epsilon}$ it holds

$$1 - |F_{x,\delta}| \leq \epsilon(M + 1),$$

where $M$ is a uniform bound on the number of intervals comprising $F_{x,\delta}$.

**Proof.** Assume not, namely assume that there exists $(\epsilon, \delta) \in A$ and $x \in \Sigma_{\epsilon}$ such that

$$1 - |F_{x,\delta}| > \epsilon(M + 1).$$

Since $[0, 1] \setminus F_{x,\delta}$ consists of at most $m + 1 \leq M + 1$ intervals, there exists one of them, say

$$[0, 1] \setminus F_{x,\delta} \supseteq I_0,$$

with length $l > \frac{\epsilon(M + 1)}{m + 1} \geq \epsilon$. Then for the center point of $I_0$, say $\xi_0 \in I_0$, we have

$$I_0 \supseteq \left(\xi_0 - \frac{\epsilon}{2}, \xi_0 + \frac{\epsilon}{2}\right).$$

Namely, there exists $\xi_0 \in [0, 1]$ such that $(\xi_0 - \frac{\epsilon}{2}, \xi_0 + \frac{\epsilon}{2}) \cap F_{x,\delta} = \emptyset$, which is a contradiction since $(\epsilon, \delta) \in A$. \hfill \Box

**Lemma 2.3.** For all $\epsilon > 0$ such that $\Sigma_{\epsilon} \neq \emptyset$ there exists $\delta > 0$ such that $(\epsilon, \delta) \in A$.

**Proof.** Fix an arbitrary $\epsilon > 0$ with $\Sigma_{\epsilon} \neq \emptyset$. Assume (for contradiction) that the statement of the lemma is not true. Then, for all $\delta \in (0, 1]$, $\exists x_{\delta} \in \Sigma_{\epsilon}, \exists \xi_{\delta} \in [0, 1]$, such that $(\xi_{\delta} - \frac{\epsilon}{2}, \xi_{\delta} + \frac{\epsilon}{2}) \subseteq [0, 1] \setminus F_{x_{\delta},\delta}$, namely

(2.6)  

$$|F(x_{\delta}, t)| < \delta, \forall t \in \left(\xi_{\delta} - \frac{\epsilon}{2}, \xi_{\delta} + \frac{\epsilon}{2}\right).$$

By compactness of $\Sigma_{\epsilon}$ and $[0, 1]$ we obtain a sequence $\{\delta_n\}_{n=1}^{\infty}$ such that $\delta_n \rightarrow 0$, $x_{\delta_n} \rightarrow x_0 \in \Sigma_{\epsilon}$ and $\xi_{\delta_n} \rightarrow \xi_0 \in [0, 1]$, and by (2.6) we deduce that there is a neighborhood of $\xi_0$ in which $F(x_0, \cdot)$ is analytic, it vanishes on $[0, 1]$, which is a contradiction as $x_0 \in \Sigma_{\epsilon}$. \hfill \Box

**Proof of Proposition 2.1, bound (2.4).** By Lemma 2.3 there exists $\epsilon_0 > 0$ such that $(0, \epsilon_0) \subseteq \pi_{2,1}(A)$, where $\pi_{2,1} : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the projection to the first coordinate.

By Lemma A.6 we get that $A$ is definable in $\mathbb{R}_{an}$, and by the definable choice theorem (see Theorem A.8), there is a definable function $\delta(\cdot) : (0, \epsilon_0) \rightarrow (0, 1)$ whose graph is in $A$. Since $\mathbb{R}_{an}$ is polynomially bounded, we obtain by Corollary A.10 that there is an $\alpha > 0$ such that $\delta(\epsilon) \geq \epsilon^\alpha$ for all $\epsilon \in (0, \epsilon_0)$ for some $0 < \epsilon_0 < \epsilon_0$. This completes the proof since

$$1 - |F_{x,\epsilon}^\alpha| \leq 1 - |F_{x,\delta(\epsilon)}| \overset{\text{Lemma 2.2}}{\leq} (M + 1)\epsilon, \forall \epsilon \in (0, \epsilon_0).$$

\hfill \Box
3. Proof of theorem 1.2

Given \( \phi : [0,1]^m \times [0,1] \to \mathbb{R}^d \), \( x \in [0,1]^m \), \( \rho \in \mathbb{R}_{>0} \) and \( h \in \mathbb{Z}^d \setminus \{0\} \), we define

\[
(3.1) \quad f_{n,x,\rho}(t) \overset{\text{def}}{=} \rho \left( h, \phi \left( \frac{x + t}{n} \right) \right), \quad t \in [0,n].
\]

We shall always assume that \( \phi \) is analytic. The main result of this section is the following.

**Proposition 3.1.** Let \( 2 \leq l \in \mathbb{N} \) and \( h \in \mathbb{Z}^d \setminus \{0\} \) be such that for all \( x \in [0,1]^m \) the function \( F_l(x,t) \overset{\text{def}}{=} \frac{\partial^l}{\partial t^l} (h, \phi (x,t)) \) is non-zero in \( t \). Assume that \( \{\delta_n\}_{n=1}^{\infty} \), \( \{\eta_n\}_{n=1}^{\infty} \subseteq [0,1) \) are such that \( \lim_{n \to \infty} n^{\delta_n} = \lim_{n \to \infty} n^{\eta_n} = \infty \). Then there exists a sequence \( \{E_n\}_{n=1}^{\infty} \) converging to zero such that for all \( x \in [0,1]^m \) and \( \rho \in [n^{\delta_n}, n^{1-\eta_n}] \) it holds

\[
(3.2) \quad \frac{1}{n} \sum_{k=1}^{n} e \left( f_{n,x,\rho}(k) \right) \ll E_n,
\]

where the implied constant is independent of \( x \in [0,1]^m \) and \( \rho \in [n^{\delta_n}, n^{1-\eta_n}] \).

**Proof that Proposition 3.1 yields Theorem 1.2.** Let \( \phi : [0,1]^m \times [0,1] \to \mathbb{R}^d \) be a RND analytic family of order \( \kappa \in \mathbb{N} \) \( \cup \{\infty\} \). Then by the definition of RND (see Definition 1.1), it holds for all \( h \in \mathbb{Z}^d \setminus \{0\} \) and for all \( x \in [0,1]^m \) that \( \frac{\partial^l}{\partial t^l} (h, \phi (x,t)) \) is non-zero in \( t \) for all \( l \leq \kappa \). Fix \( N \ni \rho \leq \kappa \) and let \( \rho_n \to \infty \) such that \( \rho_n = o(n^l) \). Then there exist sequences \( \{\delta_n\}_{n=1}^{\infty} \), \( \{\eta_n\}_{n=1}^{\infty} \subseteq [0,1) \) that satisfy \( \lim_{n \to \infty} n^{\delta_n} = \lim_{n \to \infty} n^{\eta_n} = \infty \) for which

\[
n^{\delta_n} \leq \rho_n \leq n^{1-\eta_n},
\]

for all large enough \( n \). Using the notation (1.1) we rewrite (3.1) by

\[
f_{n,x,\rho}(t) = \left( h, \gamma_n \left( \frac{t}{n} \right) \right),
\]

where \( x_n \in [0,1]^m \). By Weyl’s equidistribution criterion (see e.g. [IK04, Chapter 21]) we deduce that Proposition 3.1 implies the equidistribution of \( \{\mu_n\}_{n=1}^{\infty} \) (defined in (1.2)). \( \Box \)

Our main tool in the proof of Proposition 3.1 will be the following Van der Corput estimate which we borrow from [IK04, Chapter 8, Theorem 8.20].

**Theorem 3.2.** For all \( 2 \leq j \in \mathbb{N} \) there exists a constant \( \kappa_j > 0 \) such that \( \forall g \in C^j(I) \) where \( I = [a,b] \) with \( b-a \geq 1 \) that satisfy

\[
(3.3) \quad \eta \leq \frac{\partial^j}{\partial t^j} g(t) \leq \sigma \eta, \quad \forall t \in I,
\]

with \( \eta > 0 \) and \( \sigma \geq 1 \), it holds

\[
\sum_{k \in \mathbb{Z}} e(g(k)) \leq \kappa_j \left( \sigma^{2^j-1} \eta^j |I| + \eta^{-\tau_j} |I|^{1-2^j} \right),
\]

where \( \tau_j = (2^j - 2)^{-1} \).

Before applying Theorem 3.2 to the exponential sums of our interest (see Lemma 3.3), we need the following observation which follows from Section 2.

Fix \( 2 \leq l \in \mathbb{N} \) and \( h \in \mathbb{Z}^d \setminus \{0\} \) such that for all \( x \in [0,1]^m \) the analytic function \( \frac{\partial^l}{\partial t^l} (h, \phi (x,t)) \) is non-zero in \( t \). Let \( N \ni j \leq l \) and observe that \( \frac{\partial^j}{\partial t^j} (h, \phi (x,t)) \) is also non-zero in \( t \) for all \( x \in [0,1]^m \). The latter is equivalent to that \( \Sigma^{(j)} \) (defined in (2.1)) is empty for the analytic function \( F_j(x,t) \overset{\text{def}}{=} \frac{\partial^j}{\partial t^j} (h, \phi (x,t)) \), for all \( j \leq l \). Then, by Proposition 2.1 we deduce that there exist \( M_j, \alpha_j > 0 \) such that for all \( x \in [0,1]^m \) and \( \epsilon \in (0,1) \) it holds that

\[
\mathcal{F}_{x,\epsilon,\alpha_j}^{(j)} \overset{\text{def}}{=} \left\{ t \in [0,1] \mid \left| \frac{\partial^j}{\partial t^j} (h, \phi (x,t)) \right| \geq \epsilon \alpha_j \right\},
\]
is a union of at most $M_j$ intervals, say

\[(3.3) \quad \mathcal{F}_{x,\epsilon}^{(j)} = I_{1,x}^{(j)} \cup \ldots \cup I_{m_x,x}^{(j)}, \quad m_x \leq M_j,\]

and

\[(3.4) \quad 1 - \left| \mathcal{F}_{x,\epsilon}^{(j)} \right| \ll \epsilon.\]

**Lemma 3.3.** Assume that $2 \leq j \leq l$ and let $n \in \mathbb{N}$. Then for all $x \in [0,1]^m$ and $\epsilon \in (0,1)$ we have

\[(3.5) \quad e^{\alpha_j(\tau_j - 2^{-j})} \left( \frac{\rho}{n^j} \right)^{\tau_j} + e^{-\tau_j \alpha_j} \left( \frac{\rho}{n^j} \right)^{-\tau_j} n^{-2^{-j}} + \frac{1}{n} + \epsilon.\]

where $\tau_j = (2^j - 2)^{-1}$, and the implied constant is independent of $x, n$ and $\epsilon$.

**Proof.** Fix $2 \leq j \leq l$. First, by using (3.3) and (3.4), we obtain that

\[\# \left\{ 1 \leq k \leq n \mid \frac{k}{n} \not\in \mathcal{F}_{x,\epsilon}^{(j)} \right\} \ll n \epsilon,\]

which implies by the trivial estimate that

\[\frac{1}{n} \sum_{1 \leq k \leq n, \frac{k}{n} \not\in \mathcal{F}_{x,\epsilon}^{(j)}} e(f_{n,x}(k)) \ll \epsilon.\]

Next, let

\[c_j \overset{\text{def}}{=} \sup \left\{ \frac{\partial^j}{\partial t^j} \left( h, \phi(x,t) \right) \mid x \in [0,1]^m, \quad t \in [0,1] \right\},\]

and note that by the chain rule,

\[\frac{d^j}{dt^j} f_{n,x}(t) = \frac{\rho}{n^j} \frac{\partial^j}{\partial t^j} \left( h, \phi\left(x,\frac{t}{n}\right) \right).\]

Therefore, for all $t \in nI_{i,x}^{(j)}$ (appearing in (3.3))

\[(3.6) \quad \frac{\rho}{n^j} e^{\alpha_j} \leq \left| \frac{d^j}{dt^j} f_{n,x}(t) \right| \leq \frac{\rho}{n^j} c_j.\]

We denote

\[\eta = \frac{\rho}{n^j} e^{\alpha_j},\]

\[\sigma = c_j \epsilon^{-\alpha_j},\]

and we rewrite (3.6) by

\[\eta \leq \left| \frac{d^j}{dt^j} f_{n,x}(t) \right| \leq \sigma \eta, \quad \forall t \in nI_{i,x}^{(j)}.\]

Assume $|nI_{i,x}^{(j)}| \geq 1$, then by Theorem 3.2

\[(3.7) \quad \frac{1}{n} \sum_{1 \leq k \leq n, k \in nI_{i,x}^{(j)}} e(f_{n,x}(k)) \ll e^{\alpha_j(\tau_j - 2^{-j})} \left( \frac{\rho}{n^j} \right)^{\tau_j} + e^{-\tau_j \alpha_j} \left( \frac{\rho}{n^j} \right)^{-\tau_j} n^{-2^{-j}}.\]

Together with the trivial estimate on the intervals $nI_{i,x}^{(j)}$ with $|nI_{i,x}^{(j)}| < 1$, we find that
\[
\frac{1}{n} \sum_{1 \leq k \leq n, \frac{k}{n} \in F_n^{(j)}} e(f_{n,x,p}(k)) \\
\ll \epsilon^{\alpha_j (\tau_j - 2^{-j})} \left( \frac{\rho}{n^j} \right)^{\tau_j} + \epsilon^{\tau_j \alpha_j} \left( \frac{\rho}{n^j} \right)^{-\tau_j} n^{-2^{-j}} + \frac{1}{n}.
\]

Proof of Proposition 3.1. Fix \(2 \leq l \in \mathbb{N}\) and let \(h \in \mathbb{Z}^d \setminus \{0\}\) such that for all \(x \in [0,1]^d\) the function \(\frac{\partial^l f}{\partial^l y}(h, \phi(x, t))\) is non-zero in \(t\). Let \(\{\delta_n\}_{n=1}^\infty, \{\eta_n\}_{n=1}^\infty \subseteq [0,1)\) be such that \(\lim_{n \to \infty} n^{\delta_n} = \lim_{n \to \infty} n^{\eta_n} = \infty\) and assume that

\[n^{\delta_n} \leq \rho \leq n^{1-\eta_n}.
\]

We pick \(\lambda \in \mathbb{R}\) such that

\[\rho = n^\lambda,
\]

so that by (3.8)

\[\delta_n \leq \lambda \leq l - \eta_n.
\]

According to \(\lambda \in [\delta_n, l - \eta_n]\), we define

\[j_n(\lambda) \overset{\text{def}}{=} \begin{cases} 2 & \delta_n \leq \lambda \leq 1, \\
\lceil \lambda \rceil + 1 & 1 < \lambda < l - 1, \\
l & l - 1 \leq \lambda \leq l - \eta_n,
\end{cases}
\]

and

\[\nu_n(\lambda) \overset{\text{def}}{=} \frac{1}{2} \min \left\{ \frac{j_n(\lambda) - \lambda}{\alpha_{j_n(\lambda)} (2^{-j_n(\lambda)} \frac{2^{2-j_n(\lambda)}}{\tau_{j_n(\lambda)}} - 1)}, \frac{1}{\alpha_{j_n(\lambda)}} \left( \frac{2^{2-j_n(\lambda)}}{\tau_{j_n(\lambda)}} - (j_n(\lambda) - \lambda) \right) \right\},
\]

where \(\tau_j\) is defined in Theorem 3.2 and \(\alpha_j\) is given in Lemma 3.3. We would like to plug in

\[\epsilon_n(\lambda) \overset{\text{def}}{=} n^{-\nu_n(\lambda)}
\]

into the estimate (3.3). For that to be useful, we would like first to verify that there exist \(\epsilon_n\) such that

\[\epsilon_n(\lambda) \leq \epsilon_n,
\]

and \(\epsilon_n \to 0\) (this, by Lemma 3.3 will yield estimate (3.17) below). We verify this by estimating from below each of the terms appearing in the minimum of (3.11). It will be useful to note that

\[\frac{2^{2-j}}{\tau_j} = 4 - 2^{3-j}.
\]

- The term \(\frac{j_n(\lambda) - \lambda}{\alpha_{j_n(\lambda)} (2^{-j_n(\lambda)} \frac{2^{2-j_n(\lambda)}}{\tau_{j_n(\lambda)}} - 1)}\): An inspection of (3.10) implies that,

\[\eta_n \leq j_n(\lambda) - \lambda, \quad \forall \lambda \in [\delta_n, l - \eta_n],
\]

and since \(j_n(\lambda) \geq 2\), we deduce by (3.13) that \(\frac{2^{2-j_n(\lambda)}}{\tau_{j_n(\lambda)}} - 1 \leq 3\). Hence

\[\frac{j_n(\lambda) - \lambda}{\alpha_{j_n(\lambda)} (2^{-j_n(\lambda)} \frac{2^{2-j_n(\lambda)}}{\tau_{j_n(\lambda)}} - 1)} \geq \frac{\eta_n}{3\alpha_{j_n(\lambda)}}.
\]
• The term \( \frac{1}{\alpha_{j_n}(\lambda)} \left( \frac{2^{j_n(\lambda)}}{\tau_{j_n}(\lambda)} - (j_n(\lambda) - \lambda) \right) \): First assume that \( \lambda > 1 \). Then, \( j_n(\lambda) \geq 3 \), and as a consequence (see (3.13))

\[
\frac{2^{j_n(\lambda)}}{\tau_{j_n}(\lambda)} \geq 3.
\]

Moreover, \( j_n(\lambda) - \lambda \leq 2 \). Hence

\[
\frac{1}{\alpha_{j_n}(\lambda)} \left( \frac{2^{j_n(\lambda)}}{\tau_{j_n}(\lambda)} - (j_n(\lambda) - \lambda) \right) \geq \frac{1}{\alpha_{j_n}(\lambda)}.
\]

Next, assume \( \lambda \leq 1 \). Then \( j(\lambda) = 2 \), and we have

\[
j_n(\lambda) - \lambda \leq 2 - \delta_n,
\]

whence

\[
\frac{1}{\alpha_{j_n}(\lambda)} \left( \frac{2^{j_n(\lambda)}}{\tau_{j_n}(\lambda)} - (j_n(\lambda) - \lambda) \right) \geq \frac{\delta_n}{\alpha_2}.
\]

We denote \( \alpha = \max\{\alpha_j\}_{j=1}^1 \) and \( \nu_n = \frac{1}{2} \min \{ \frac{1}{\alpha}, \frac{\delta_n}{\alpha_2} \} \). Then we conclude from (3.11), (3.14), (3.15) and (3.16) that \( \nu_n(\lambda) \geq \nu_n \). Importantly, we note that

\[
\epsilon_n(\lambda) = \frac{1}{n^{\nu_n(\lambda)}} \leq \frac{1}{n^{\nu_n}} = \epsilon_n.
\]

Recall that \( n^{-\eta_n} \to 0 \) and \( n^{-\delta_n} \to 0 \), hence \( \epsilon_n \to 0 \). Now, by plugging in (3.9) and (3.12) into (3.10), we get that

\[
\frac{1}{n} \sum_{k=1}^{n} e(f_{n,x,\nu}(k)) \ll n^{T_{1,n}(\lambda)} + n^{T_{2,n}(\lambda)} + \frac{1}{n} + \epsilon_n,
\]

where

\[
T_{1,n}(\lambda) \equiv \alpha_{j_n}(\lambda) \nu_n(\lambda) \left( \frac{2^{j_n(\lambda)}}{\tau_{j_n}(\lambda)} - \tau_{j_n}(\lambda) (j_n(\lambda) - \lambda) \right),
\]

\[
T_{2,n}(\lambda) \equiv \tau_{j_n}(\lambda) \left( \alpha_{j_n}(\lambda) \nu_n(\lambda) + j_n(\lambda) - 2^{j_n(\lambda)} \right).
\]

To finish the proof it remains to show that there exist sequences \( \{T_{1,n}\}_{n=1}^\infty \), \( \{T_{2,n}\}_{n=1}^\infty \) such that \( T_{i,n}(\lambda) \leq T_{i,n} \) for \( i = 1, 2 \), and such that \( n^{T_{1,n}} \to 0 \) and \( n^{T_{2,n}} \to 0 \).

• The term \( T_{1,n}(\lambda) \): By definition of \( \nu_n(\lambda) \) we have

\[
\nu_n(\lambda) \leq \frac{1}{2} \frac{j_n(\lambda) - \lambda}{\alpha_{j_n}(\lambda) \left( \frac{2^{j_n(\lambda)}}{\tau_{j_n}(\lambda)} - 1 \right)},
\]

hence,

\[
T_{1,n}(\lambda) \leq \alpha_{j_n}(\lambda) \left( \frac{1}{2} \right) \frac{j_n(\lambda) - \lambda}{\alpha_{j_n}(\lambda) \left( \frac{2^{j_n(\lambda)}}{\tau_{j_n}(\lambda)} - 1 \right)} \left( \frac{2^{j_n(\lambda)}}{\tau_{j_n}(\lambda)} - \tau_{j_n}(\lambda) (j_n(\lambda) - \lambda) \right)
\]

\[
= - \frac{1}{2} \tau_{j_n}(\lambda) (j_n(\lambda) - \lambda).
\]

An inspection of (3.10) shows that \( \eta_n \leq j_n(\lambda) - \lambda \), which combined with (3.11) gives

\[
T_{1,n}(\lambda) \leq \frac{1}{2} \tau_{j_n}(\lambda) \eta_n.
\]

We define

\[
T_{1,n} = - \frac{1}{2} \min \{ \tau_1 \}_{i=2}^n \eta_n,
\]

then \( T_{1,n}(\lambda) \leq T_{1,n} \) and as \( n^{-\eta_n} \to 0 \), we obtain that \( n^{T_{1,n}} \to 0 \).
• The term $T_{2,n}(\lambda)$: By our definition of $\nu_n(\lambda)$ we have

$$\nu_n(\lambda) \leq \frac{1}{2} \alpha_j(\lambda) \left( \frac{2^{2-j_n(\lambda)}}{\tau_j(\lambda)} - (j_n(\lambda) - \lambda) \right),$$

hence

(3.19)

$$T_{2,n}(\lambda) \leq \tau_j(\lambda) \left( \frac{1}{2} \frac{1}{\alpha_j(\lambda)} \left( \frac{2^{2-j_n(\lambda)}}{\tau_j(\lambda)} - (j_n(\lambda) - \lambda) \right) + j_n(\lambda) - \lambda \right) - 2^{2-j_n(\lambda)}$$

$$= \frac{\tau_j(\lambda)}{2} \left( \frac{2^{2-j_n(\lambda)}}{\tau_j(\lambda)} - (j_n(\lambda) - \lambda) \right).$$

By (3.15) and (3.16) we deduce from (3.19) that

$$T_{2,n}(\lambda) \leq \frac{1}{2} \tau_j(\lambda) \alpha_j(\lambda) \min \left\{ \frac{1}{\alpha_j(\lambda)}, \frac{\delta_n}{\alpha_2} \right\}$$

Define

$$T_{2,n} = \frac{1}{2} \min \{ \tau_i \alpha_i \} \frac{\delta_n}{\max \{ \alpha_i \} \alpha_2},$$

then $T_{2,n}(\lambda) \leq T_{2,n}$ and since $n^{-\delta_n} \to 0$, we find that $n^{T_{2,n}} \to 0$. 

\(\square\)

4. PROOF OF THEOREM 1.5

For $\varphi : [0, 1]^m \times \mathbb{R} / \mathbb{Z} \to \mathbb{R}^d$, $\{x_n\}_{n=1}^{\infty} \subseteq [0, 1]^m$, $h \in \mathbb{Z}^d \setminus \{0\}$ and $\{\rho_n\}_{n=1}^{\infty} \subseteq \mathbb{R}_{>0}$, we define the function

$$S_n(\omega) \overset{\text{def}}{=} \frac{1}{n} \sum_{k=1}^{n} e \left( \left\langle h, \rho_n \varphi \left( x_n, \frac{k}{n} + \omega + \mathbb{Z} \right) \right\rangle \right), \ \omega \in [0, 1].$$

The following proposition is the main result of this section.

Proposition 4.1. Assume that $\varphi : [0, 1]^m \times \mathbb{R} / \mathbb{Z} \to \mathbb{R}^d$ is a family of RND analytic curves of order $\infty$. Let $\{x_n\}_{n=1}^{\infty} \subseteq [0, 1]^m$, $h \in \mathbb{Z}^d \setminus \{0\}$ and $\rho_n \to \infty$ be arbitrary. Then, for almost every $\omega \in [0, 1]$,

$$\lim_{n \to \infty} S_n(\omega) = 0.$$

We now show that Proposition 4.1 implies Theorem 1.5. Since a countable intersection of full measure sets is of full measure, it follows from Proposition 4.1 that $\lim_{n \to \infty} S_n(\omega) = 0$ for all $h \in \mathbb{Z}^d \setminus \{0\}$, for almost every $\omega \in [0, 1]$. Hence Theorem 1.5 follows by Weyl’s equidistribution criterion.

To prove Proposition 4.1 we will use the Borel-Cantelli lemma with the following estimate of fourth moments which we prove in Section 4.1.

Proposition 4.2. Assume that $\varphi : [0, 1]^m \times \mathbb{R} / \mathbb{Z} \to \mathbb{R}^d$ is a family of RND analytic curves of order $\infty$ and fix $h \in \mathbb{Z}^d \setminus \{0\}$. For $x \in [0, 1]^m$, $\rho > 0$ and $n \in \mathbb{N}$ let

$$S_n(x, \rho, \omega) = \frac{1}{n} \sum_{k=1}^{n} e \left( \left\langle h, \rho \varphi \left( x, \frac{k}{n} + \omega + \mathbb{Z} \right) \right\rangle \right), \ \omega \in [0, 1].$$

Then, there exists $\tau > 0$ such that for $n \in \mathbb{N}$, $x \in [0, 1]^m$ and $\rho \geq n^\tau$, it holds

(4.1)

$$\int_{0}^{1} |S_n(x, \rho, \omega)|^4 d\omega \ll \frac{1}{n^2},$$

where the implied constant is independent of the parameters $n$, $x$ and $\rho$.

We now explain how the statement of Proposition 4.2 implies Proposition 4.1.
Proof of Proposition 4.2. Let \( \varphi : [0, 1]^m \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^d \) be a family of RND analytic curves of order \( \infty \), \( \{x_n\}_{n=1}^\infty \subseteq [0, 1]^m \), \( h \in \mathbb{Z}^d \setminus \{0\} \) and \( \rho_n \to \infty \) be arbitrary. Let \( \tau \) be the exponent stated to exist in Proposition 4.2.

We partition the sequence \( \{\rho_n\}_{n=1}^\infty \) into two subsequences,

\[
S_+ \overset{\text{def}}{=} \{ n \in \mathbb{N} \mid \rho_n > n^\tau \}, \quad S_- \overset{\text{def}}{=} \{ n \in \mathbb{N} \mid \rho_n \leq n^\tau \}.
\]

If \( n \in S_+ \), then

\[
\left| \left\{ \omega \in [0, 1] \mid |S_n(\omega)| \geq n^{-1/8} \right\} \right| \leq \frac{\int_0^1 |S_n(\omega)|^4 \, d\omega}{n^{-1/2}} \lesssim \frac{1}{n^{3/2}},
\]

and this is summable. Therefore, if \( S_+ \) is infinite, the Borel-Cantelli lemma shows

\[
\left| \left\{ \omega \in [0, 1] \mid \exists N > 0 \text{ such that } |S_n(\omega)| < n^{-1/8}, \forall n \in S_+, n \geq N \right\} \right| = 1,
\]

whence,

\[
\left| \left\{ \omega \in [0, 1] \mid \lim_{S_- \ni n \to \infty} S_n(\omega) = 0 \right\} \right| = 1.
\]

Next, assume without loss of generality that \( S_- \) is infinite (otherwise the proof is done by the above). We note that by the assumption on \( \varphi \), for all \( \omega \in [0, 1] \), the family

\[
\phi_\omega(x, t) \overset{\text{def}}{=} \varphi(x, t + \omega + Z), \quad (x, t) \in [0, 1]^m \times [0, 1],
\]

is a RND analytic family of curves of order \( \infty \). Hence, Theorem 1.2 implies that for all \( \omega \in [0, 1] \) it holds

\[
\lim_{S_- \ni n \to \infty} S_n(\omega) = 0.
\]

\( \square \)

4.1. Proof of Proposition 4.2. For the rest of the section we let \( \varphi : [0, 1]^m \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^d \) be a family of RND analytic curves of order \( \infty \) and we fix \( h \in \mathbb{Z}^d \setminus \{0\} \).

Let us denote for \( k \in [n]^4 \) and \( x \in [0, 1]^m \)

\[
f_{x, n, k}(\omega) \overset{\text{def}}{=} \left\langle h, \sum_{i=1}^4 (-1)^{i+1} \varphi \left( x, \frac{k_i}{n} + \omega + Z \right) \right\rangle, \quad \omega \in [0, 1],
\]

then using the above notation we get that

\[
\int_0^1 |S_n(x, \rho, \omega)|^4 \, d\omega = \int_0^1 \left( S_n(x, \rho, \omega) \overline{S_n(x, \rho, \omega)} \right)^2 \, d\omega = \frac{1}{n^2} \sum_{k \in [n]^4} \int_0^1 e(\rho f_{x, n, k}(\omega)) \, d\omega
\]

Next, we recall the following well known estimate (see e.g. [IK04, Chapter 8, Lemma 8.10]) which we will use below in the proof Lemma 4.3.

**Lemma 4.3.** There exists an absolute constant \( c > 0 \) such that \( \forall f \in C^2(I) \) where \( I = [a, b] \) that satisfy

\[
\left| \frac{d^2}{d\omega^2} f(\omega) \right| \geq \lambda, \quad \forall \omega \in I,
\]

it holds

\[
\int_I e(f(\omega)) \, d\omega \leq \frac{c}{\sqrt{\lambda}}.
\]
We consider the following analytic function
(4.5)  
\[ \Phi(x, y, \omega) \overset{\text{def}}{=} \left\langle h, \sum_{i=1}^{4} (-1)^{i+1} \frac{\partial^2}{\partial \omega^2} \varphi(x_i, y_i + \omega + Z) \right\rangle, \quad (x, y, \omega) \in [0, 1]^m \times [0, 1]^4 \times [0, 1], \]
which isn’t constant (since \( \varphi \) is RND of order \( \infty \)), and we note that it satisfies
(4.6)  
\[ \Phi \left( x, \frac{1}{n} k, \omega \right) = \frac{d^2}{d\omega^2} f_{x, n, k}(\omega). \]

We denote (as in Section 2)
\[ \Sigma = \left\{ (x, y) \in [0, 1]^m \times [0, 1]^4 \mid \Phi(x, y, t) = 0, \forall t \in [0, 1] \right\}, \]
and for \( n \in \mathbb{N} \) (see (2.2)),
\[ \Sigma_{n-2} = \left\{ (x, y) \in [0, 1]^m \times [0, 1]^4 \mid \text{dist}((x, y), \Sigma) \geq n^{-2} \right\}. \]

**Lemma 4.4.** There exists \( \tau > 0 \) with the following property: for all \( n \in \mathbb{N} \), \( \rho \geq n^\tau \), \( x \in [0, 1]^m \) and \( k \in [n]^4 \) such that \( (x, \frac{1}{n} k) \in \Sigma_{n-2} \) it holds
\[ \int_0^1 e(\rho f_{x, n, k}(\omega)) d\omega \ll \frac{1}{n^2}, \]
where the implied constant is independent of the parameters \( x, n \) and \( k \) (depends on \( \Phi \) only).

**Proof.** We apply Proposition 2.1 for the function \( \Phi \) to get \( \alpha = \alpha(\Phi), M = M(\Phi) > 0 \) such that for \( (x, \frac{1}{n} k) \in \Sigma_{n-2} \) it holds (see (4.6))
\[ F_{(x, \frac{1}{n})}, n^{-2\alpha} \overset{\text{def}}{=} \left\{ \omega \in [0, 1] \mid \left| \frac{d^2}{d\omega^2} f_{x, n, k}(\omega) \right| \geq n^{-2\alpha} \right\}, \]
is a union of at most \( M \) intervals and
(4.7)  
\[ 1 - \left| F_{(x, \frac{1}{n})}, n^{-2\alpha} \right| \ll \frac{1}{n^2}, \]
where the implied constant depends on \( \Phi \) only. We define \( \tau = 4 + 2\alpha \) and we deduce that for \( \rho \geq n^\tau \) it holds
\[ \left| \frac{d^2}{d\omega^2} f_{x, n, k}(\omega) \right| \geq n^4, \forall \omega \in F_{(x, \frac{1}{n})}, n^{-2\alpha}. \]

By applying Lemma 4.3 on each of the intervals composing \( F_{(x, \frac{1}{n})}, n^{-2\alpha} \), we obtain that
(4.8)  
\[ \int_{F_{(x, \frac{1}{n})}, n^{-2\alpha}} e(\rho f_{x, n, k}(\omega)) d\omega \ll \frac{1}{n^2}. \]
By (4.7) and (4.8) the proof is complete. \( \square \)

In order to show (4.1) it remains to prove
(4.9)  
\[ \# \left\{ k \in [n]^4 \mid \left( x, \frac{1}{n} k \right) \in [0, 1]^m \setminus \Sigma_{n-2} \right\} \ll n^2, \]
uniformly in \( x \in [0, 1]^m \). Indeed, (4.1) will follow by applying the estimate of Lemma 4.4 on the terms of (4.3) for \( k \in [n]^4 \) such that \( (x, \frac{1}{n} k) \in \Sigma_{n-2} \) and by applying the trivial estimate on the terms of (4.3) for \( k \in [n]^4 \) such that \( (x, \frac{1}{n} k) \in [0, 1]^m \setminus \Sigma_{n-2} \).

To prove (4.9) the following lemma is needed.
Lemma 4.5. Consider the following parallelograms

\[(4.10) \quad H_1 \overset{\text{def}}{=} \{(a, a, b, b) \mid a, b \in [0, 1]\}, \quad H_2 \overset{\text{def}}{=} \{(a, b, b, a) \mid a, b \in [0, 1]\},\]

and for \((l_1, l_2) \in \mathbb{Z}^2\) let

\[(4.11) \quad 1 \overset{\text{def}}{=} (l_1 + l_2, 0, l_2 - l_1, 0).\]

Then there exists \(j_0 \in \mathbb{N}\) such that for all \(x \in [0, 1]^m\) it holds

\[(4.12) \quad \{y \in [0, 1]^4 \mid (x, y) \in \Sigma\} \subseteq \bigcup_{i=1}^{2} \bigcup_{\|l\|_\infty \leq j_0} \left( H_i + \frac{1}{j_0} l \right).\]

Proof of Lemma 4.5. Recall that \(\varphi\) is a family of RND analytic curves of order \(\infty\), hence for any \(x \in [0, 1]^m\), we have that

\[
\psi(x, \omega) \overset{\text{def}}{=} \frac{\partial^2}{\partial \omega^2} \langle h, \varphi(x, \omega) \rangle, \quad \omega \in \mathbb{R}/\mathbb{Z},
\]

is a non-constant smooth function. Since the Fourier series of a smooth function on \(\mathbb{R}/\mathbb{Z}\) converges uniformly to the function (see e.g. [EW17]) and since \(\psi(x, \cdot)\) is not constant, it follows that there exists \(j \in \mathbb{Z} \setminus \{0\}\) such that

\[
\hat{\psi}(x, j) = \int_{\mathbb{R}/\mathbb{Z}} \psi(x, \omega)e(-j\omega)d\omega \neq 0.
\]

Since \(\psi(x, \cdot)\) is real, we have that \(\hat{\psi}(x, j)\) is the complex conjugate of \(\hat{\psi}(x, -j)\), so for convenience we may assume that \(j \in \mathbb{N}\).

We denote

\[
j_0(x) = \min\{j \in \mathbb{N} \mid \hat{\psi}(x, j) \neq 0\},
\]

and in the following we show that \(j_0(x)\) is bounded in \(x \in [0, 1]^m\). Assume for contradiction that there exists a sequence \(\{x_i\}_{i=1}^{\infty} \subseteq [0, 1]^m\) such that \(j_0(x_i) \to \infty\). By compactness, we may assume without loss of generality that \(x_i \to x_0 \in [0, 1]^m\). By continuity, it follows that \(\hat{\psi}(x_0, j) = 0\) for all \(j \in \mathbb{N}\), which implies in turn that \(\psi(x_0, \cdot)\) is constant, which is a contradiction. Whence we conclude that \(j_0(x)\) is bounded.

For \(\alpha \in \mathbb{R}/\mathbb{Z}\) we denote

\[
\tau_\alpha \psi(x, \omega) = \psi(x, \omega + \alpha),
\]

and observe that

\[
\tau_\alpha \hat{\psi}(x, j) = e(j\alpha) \hat{\psi}(x, j).
\]

We rewrite the function \(\Phi\) (defined in (4.5)) as

\[
\Phi(x, y, \omega) = \sum_{i=1}^{4} (-1)^{i+1} \psi(x, \omega + y_i + \mathbb{Z}),
\]

and by the above we conclude that the \(j_0(x)\)'th Fourier coefficient of \(\Phi(x, y, \cdot)\) is

\[(4.13) \quad \hat{\psi}(x, j_0(x)) \left( e(j_0(x)y_1) - e(j_0(x)y_2) + e(j_0(x)y_3) - e(j_0(x)y_4) \right).
\]

By (4.13) we deduce that

\[
\{y \in [0, 1]^4 \mid (x, y) \in \Sigma\} \subseteq \{y \in [0, 1]^4 \mid e(j_0(x)y_1) - e(j_0(x)y_2) + e(j_0(x)y_3) - e(j_0(x)y_4) = 0\}.
\]

Now, we recall the identity

\[
e(a) + e(b) = 2 \cos(\pi(a - b))e\left(\frac{a + b}{2}\right), \quad a, b \in \mathbb{R},
\]

which yields

\[
e(j_0(x)y_1) - e(j_0(x)y_2) + e(j_0(x)y_3) - e(j_0(x)y_4) = 0 \iff \left\{ \begin{array}{l}
\cos(\pi j_0(x)(y_1 - y_3)) = \cos(\pi j_0(x)(y_2 - y_4)) \quad , \\
e(j_0(x)^{2\pi + 2\pi}) = e(j_0(x)^{2\pi + 2\pi}) \quad .\end{array} \right.
\]
there exist $l_1, l_2 \in \mathbb{Z}$ such that

\begin{equation}
(4.14) \quad \begin{cases}
y_1 - y_3 = y_2 - y_4 + \frac{2l_1}{j_0(x)}, \text{ or } y_1 - y_3 = -(y_2 - y_4) + \frac{2l_1}{j_0(x)}, \\
y_1 + y_3 = y_2 + y_4 + \frac{2l_2}{j_0(x)}.
\end{cases}
\end{equation}

For any fixed $l_1, l_2, l_1', l_2' \in \mathbb{Z}$, the solutions to (4.14) in $y \in [0,1]^4$ are included in

\begin{equation}
(4.15) \quad \left\{ H_1 + \left( \frac{l_1 + l_2}{j_0(x)}, 0, \frac{l_2 - l_1}{j_0(x)}, 0 \right) \right\} \cup \left\{ H_2 + \left( \frac{l_1' + l_2'}{j_0(x)}, 0, \frac{l_2' - l_1'}{j_0(x)}, 0 \right) \right\},
\end{equation}

where $H_i$ is defined in (4.10). We also deduce that $\|l\|_\infty, \|l'\|_\infty \leq j_0(x)$, since otherwise (4.15) will not intersect $[0,1]^4$. Finally if $N_0 \in \mathbb{N}$ is a bound for $j_0(x)$, then (4.12) follows with $j_0 \overset{\text{def}}{=} \text{lcm}(1, \ldots, N_0)$.

**Proof that (4.19) holds.** We denote the finite union of parallelograms

\begin{equation}
(4.16) \quad H \overset{\text{def}}{=} \bigcup_{i=1}^2 \bigcup_{\|l\|_\infty \leq j_0} \left( H_i + \frac{1}{j_0} \right)
\end{equation}

satisfying by Lemma 1.5 that for all $x \in [0,1]^m$ it holds

\begin{equation}
(4.17) \quad \left\{ y \in [0,1]^4 \mid (x, y) \in \Sigma \right\} \subseteq H.
\end{equation}

We claim that for all $x \in [0,1]^m$ it holds

\begin{equation}
(4.18) \quad \left\{ y \in [0,1]^4 \mid \text{dist}((x, y), \Sigma) < \frac{1}{n^2} \right\} \subseteq \left\{ y \in [0,1]^4 \mid \text{dist}(y, H) < \frac{1}{n^2} \right\}.
\end{equation}

Indeed, let $y_0 \in [0,1]^4$ such that $\text{dist}((x, y_0), \Sigma) < \frac{1}{n^2}$. Then there exists $(a, b) \in \Sigma$ such that

$$\sqrt{\|x - a\|^2 + \|y_0 - b\|^2} < \frac{1}{n^2},$$

which in turn implies that

$$\|y_0 - b\| < \frac{1}{n^2}.$$

By (4.17) we have that $b \in H$, which shows that $y_0 \in \left\{ y \in [0,1]^4 \mid \text{dist}(y, H) < \frac{1}{n^2} \right\}$.

We now conclude that

$$\# \left\{ \left. k \in [n]^4 \mid (x, \frac{1}{n}k) \in [0,1]^m \right. \right\} = \# \left( \left\{ y \in [0,1]^4 \mid \text{dist}((x, y), \Sigma) < \frac{1}{n^2} \right\} \cap \frac{1}{n} \mathbb{Z}^4 \right)$$

\[ \leq \# \left( \left\{ y \in [0,1]^4 \mid \text{dist}(y, H) < \frac{1}{n^2} \right\} \cap \frac{1}{n} \mathbb{Z}^4 \right) \]

\[ \leq \sum_{i=1}^2 \sum_{\|l\|_\infty \leq j_0} \# \left( \left\{ y \in [0,1]^4 \mid \text{dist}(y, H_i + \frac{1}{j_0}) < \frac{1}{n^2} \right\} \cap \frac{1}{n} \mathbb{Z}^4 \right) \]

Therefore, to prove the estimate $\# \left\{ k \in [n]^4 \mid (x, \frac{1}{n}k) \in [0,1]^m \right\} \ll n^2$ uniformly in $x \in [0,1]^m$, it is sufficient to verify that

\begin{equation}
(4.19) \quad \# \left( \left\{ y \in [0,1]^4 \mid \text{dist}(y, H_i + v) < \frac{1}{n^2} \right\} \cap \frac{1}{n} \mathbb{Z}^4 \right) \ll n^2,
\end{equation}
uniformly in \(i \in \{1, 2\}\) and \(\mathbf{v} \in [0, 1]^4\). To prove (4.19) we note that for \(n \in \mathbb{N}\) there is a cover
\[
\bigg\{ \mathbf{y} \in [0, 1]^4 \mid \text{dist} (\mathbf{y}, H_i + \mathbf{v}) < \frac{1}{n^2} \bigg\} \subseteq S_1(n) \cup \ldots \cup S_{m(n)}(n),
\]
where \(m(n) \ll n^2\) uniformly in \(\mathbf{v}\), such that
\[
(4.20) \quad \text{Euclidean diameter of } S_j(n) \leq \frac{1}{n} \cdot \frac{1}{n}.
\]
By (4.20), each set \(S_j(n)\) can contain at most one rational vector \(\frac{1}{n} \mathbf{k}\) where \(\mathbf{k} \in [n]^4\), which shows (4.19).

5. Counter examples

Our main tool to prove Theorems 1.3 and 1.4 is the well known Diirichlet’s simultaneous approximation theorem, which we recall now.

**Theorem 5.1.** For any \(M \in \mathbb{N}\) and \(\mathbf{x} \in \mathbb{R}^N\), there exists \(\mathbf{p} \in \mathbb{Z}^N\) and \(q \in \{1, \ldots, M\}\) such that
\[
\|q\mathbf{x} - \mathbf{p}\|_\infty \leq \frac{1}{M^{1/N}}.
\]

**Proof of Theorem 5.1.** Assume that \(\gamma\) is RND analytic curve of order \(\kappa \in \mathbb{N}\). Then, there exists \(\mathbf{h} \in \mathbb{Z}^d \setminus \{0\}\) such that
\[
(5.1) \quad \langle \mathbf{h}, \gamma^{(\kappa+1)}(t) \rangle = 0, \; \forall t \in [0, 1] .
\]
Then (5.1) implies that \(\langle \mathbf{h}, \gamma(t) \rangle\) is a polynomial of degree \(\kappa\), say
\[
(5.2) \quad \langle \mathbf{h}, \gamma(t) \rangle = a_\kappa t^\kappa + \ldots + a_0, \; t \in [0, 1].
\]
By Dirichlet’s theorem for each \(n \in \mathbb{N}, M = n^{\kappa+1}\) and the points
\[
(a_\kappa, a_{\kappa-1}, \ldots, a_1) \in \mathbb{R}^\kappa,
\]
we find that there exists \(\tilde{\rho}_n \in \mathbb{N}\) such that \(\tilde{\rho}_n \leq n^{\kappa+1}\), and \(\mathbf{p} = (p_1, \ldots, p_\kappa)\) where \(p_i \in \mathbb{Z}\), such that
\[
(5.3) \quad |\tilde{\rho}_n a_i - p_i| \leq \frac{1}{n^{\kappa+1}}, \; \forall i \in \{1, \ldots, \kappa\}.
\]
We observe that
\[
(5.4) \quad |n^i j^{\kappa-i} \tilde{\rho}_n a_{\kappa-i} - n^i j^{\kappa-i} p_{\kappa-i}| = n^i j^{\kappa-i} |\tilde{\rho}_n a_{\kappa-i} - p_{\kappa-i}| \leq \frac{1}{n^{\kappa}}.
\]
By (5.3) and (5.4) we deduce for each \(j \in \{1, \ldots, n\}\) that
\[
|\left( \langle \mathbf{h}, (n^\kappa \tilde{\rho}_n) \gamma \left( \frac{j}{n} \right) \rangle - n^\kappa \tilde{\rho}_n a_0 \right) - \sum_{i=1}^{\kappa} n^i j^{\kappa-i} p_{\kappa-i}| \leq \frac{\kappa}{n^{\kappa}}.
\]
We denote
\[
\delta_{n,j} \overset{\text{def}}{=} \left( \langle \mathbf{h}, (n^\kappa \tilde{\rho}_n) \gamma \left( \frac{j}{n} \right) \rangle - n^\kappa \tilde{\rho}_n a_0 \right) - \sum_{i=1}^{\kappa} n^i j^{\kappa-i} p_{\kappa-i},
\]
\(1): \) Such a sequence of covers is obtained by partitioning the parallelogram \((H_i + \mathbf{v}) \cap [0, 1]^4\) into \(\ll n^3\) parallelograms \(P_l(n)\) of side length \(\ll \frac{1}{n^{1/3}}\) and defining \(S_l(n) \overset{\text{def}}{=} P_l(n) \cup D_{\frac{1}{n^{1/3}}} \), where \(D_r \overset{\text{def}}{=} \{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 \mid \sqrt{\alpha_1^2 + \alpha_2^2} \leq r \}\) for an orthonormal basis \(\{\mathbf{u}_1, \mathbf{u}_2\}\) of \(H_i^\perp\) the plane orthogonal to \(H_i\).
and we conclude that \( \lim_{n \to \infty} \delta_{n,j} = 0 \) uniformly in \( j \), which implies in turn that
\[
(5.5) \quad \sum_{j=1}^{n} e \left\langle h, n^n \tilde{\rho}_n \gamma \left( \frac{j}{n} \right) \right\rangle = e(n^n \tilde{\rho}_n a_0) \sum_{j=1}^{n} e(\delta_{n,j}) \sim n.
\]
We define the sequence \( \rho_n \overset{\text{def}}{=} n^n \tilde{\rho}_n \), and we note that \( n^n \leq \rho_n \leq n^{n+1} \). Finally, by (5.5), we get that \( \left\{ \frac{1}{n} \sum_{j=1}^{n} e \left( \left\langle h, n^n \rho_n \gamma \left( \frac{j}{n} \right) \right\rangle \right) \right\}_{n=1}^{\infty} \) will not converge to zero. \( \square \)

**Proof of Theorem 1.4.** Let \( \phi : [0,1]^m \times [0,1] \to \mathbb{R}^d \) be a family curves, \( \{x_n\}_{n=1}^{\infty} \subseteq [0,1]^m \).

Then by Dirichlet’s theorem for each \( n \in \mathbb{N} \), \( M = 3^{dn} \) and the points
\[
(\log n) (\phi(x_n,1/n), \phi(x_n,2/n), \ldots, \phi(x_n,1)) \in \mathbb{R}^{dn},
\]
there exists \( \tilde{\rho}_n \in \mathbb{N} \) such that \( \tilde{\rho}_n \leq 3^{dn} \), and \( p = (p_1, \ldots, p_n) \) where \( p_i \in \mathbb{Z}^d \), such that
\[
(5.6) \quad \left| \tilde{\rho}_n \left( (\log n) \phi \left( x_n, \frac{j}{n} \right) \right) - p_j \right|_\infty \leq \frac{1}{3^n} \quad \forall j \in \{1, \ldots, n\}.
\]
Denote \( \rho_n \overset{\text{def}}{=} \tilde{\rho}_n \log n \), and note that \( \rho_n \to \infty \) and \( \rho_n \leq (3.5)^{dn} \) for all large enough \( n \). Finally, let \( \gamma_n \overset{\text{def}}{=} \rho_n \phi(x_n, \cdot) \), and observe by (5.6) that \( \{\pi(\gamma_n(j/n))\}_{j=1}^{n} \) is contained in a strict subset of \( \mathbb{R}^d/\mathbb{Z}^d \) of measure \( \left( \frac{2}{3} \right)^d \) for all \( n \in \mathbb{N} \), so that \( \{\mu_n\}_{n=1}^{\infty} \) will not equidistribute. \( \square \)

**Appendix A. Basic notions in the theory of o-minimal structures.**

In order to make our paper self contained we discuss some basic notions in the theory of o-minimal structures which we use in the proof of Proposition 2.1. For more details on o-minimal structures we refer to the book [vdD98].

**Definition A.1.** A structure \( \mathcal{A} \) on the real field \( \mathbb{R} \) is a sequence \( \mathcal{A} \overset{\text{def}}{=} (\mathcal{A}_d)_{d=1}^{\infty} \) where \( \mathcal{A}_d \) is a subset of the power set of \( \mathbb{R}^d \), satisfying the following requirements for all \( d, m \in \mathbb{N} \):

1. \( \mathcal{A}_d \) is a Boolean algebra, namely \( \emptyset \in \mathcal{A}_d \) and \( \mathcal{A}_d \) is closed under the operation of taking a complement or by performing a finite union.

2. The diagonals \( \Delta_{i,j} \overset{\text{def}}{=} \{(x_1, \ldots, x_d) \mid x_i = x_j \} \) for all \( 1 \leq i < j \leq d \) belong to \( \mathcal{A}_d \).

3. For all \( A \in \mathcal{A}_d \) and \( B \in \mathcal{A}_m \) it holds that \( A \times B \in \mathcal{A}_{d+m} \) and \( B \times A \in \mathcal{A}_{d+m} \).

4. For all \( A \in \mathcal{A}_{d+m} \) it holds that \( \pi_{d+m,d}(A) \in \mathcal{A}_d \), where \( \pi_{d+m,d} : \mathbb{R}^{d+m} \to \mathbb{R}^d \) denotes the projection to the first \( d \) coordinates.

5. The graphs of addition and multiplication are in \( \mathcal{A}_3 \).

**Definition A.2.** Let \( \mathcal{A} \) be a structure on the real field. If any set that belong to \( \mathcal{A}_1 \) is comprised of a finite union of points or intervals, then we say that \( \mathcal{A} \) is an o-minimal structure.

**Definition A.3.** Fix a structure on the real field \( \mathcal{A} = (\mathcal{A}_d)_{d=1}^{\infty} \).

We say that \( A \subseteq \mathbb{R}^d \) is definable in \( \mathcal{A} \) if \( A \in \mathcal{A}_d \), we say that \( f : A' \to \mathbb{R}^m \) for \( A \in \mathcal{A}_d \) is definable in \( \mathcal{A} \) if the graph of \( f \) is in \( \mathcal{A}_{d+m} \), and we say that a constant \( c \in \mathbb{R} \) is definable in \( \mathcal{A} \) if \( \{c\} \in \mathcal{A}_1 \).

**Lemma A.4.** The following hold for any structure \( \mathcal{A} = (\mathcal{A}_d)_{d=1}^{\infty} \) on the real field:

1. Let \( \sigma : \{1, \ldots, d\} \to \{1, \ldots, d\} \) be a permutation. Then \( A \in \mathcal{A}_d \) is definable if and only if
   \[
   A^{\sigma} \overset{\text{def}}{=} \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid (x_{\sigma(1)}, \ldots, x_{\sigma(d)}) \in A\}
   \]
   is definable.

2. The definition given here is equivalent to the definition given in [vdD98] Section 2]. We note that the notion of a structure is more general, see e.g. [vdD98].
(2) For a definable function \( f : \mathbb{R}^d \to \mathbb{R} \) and a definable set \( A \in \mathcal{A}_d \), it holds that
\[
\{ x \in A \mid f(x) = 0 \}, \{ x \in A \mid f(x) > 0 \}
\]
are definable.

(3) If \( f : \mathbb{R}^m \to \mathbb{R} \) is definable, then the restriction of \( f \) to a definable set \( A \in \mathcal{A}_m \) is a definable function.

(4) If \( f : \mathbb{R}^m \to \mathbb{R}^n \) and \( g : \mathbb{R}^n \to \mathbb{R}^k \) are definable, then so is the composition \( g \circ f \).

(5) If \( f, g : \mathbb{R}^m \to \mathbb{R} \) are definable, then \( f + g, f - g, f \cdot g \) are definable, and \( f/g \) defined in the domain \( \{ x \in \mathbb{R}^m \mid g(x) \neq 0 \} \) is definable.

(6) If \( c_1, \ldots, c_n \in \mathbb{R} \) are definable constants and \( f : \mathbb{R}^{m+n} \to \mathbb{R} \) is definable, then so is \( g(x_1, \ldots, x_m) \overset{\text{def}}{=} f(x_1, \ldots, x_m, c_1, \ldots, c_n) \).

Proof. The proof of the lemma relies only on Definition A.1, and follows in a rather straightforward manner. In the following we only prove (2), and leave the rest for the reader.

We first show that 0 and \( \mathbb{R}_{>0} \) are definable. We have
\[
\{(x, x, 2x) \mid x \in \mathbb{R}\} = \{(x, y, x + y) \mid x, y \in \mathbb{R}\} \cap \Delta_{1,2},
\]
and
\[
\{0\} = \pi_{3,1} (\{(x, x, 2x) \mid x \in \mathbb{R}\} \cap \Delta_{2,3}),
\]
which shows that 0 is definable. To show that \( \mathbb{R}_{>0} \) is definable, we observe that
\[
\{(x, x, x^2) \mid x \in \mathbb{R}\} = \{(x, y, xy) \mid x \in \mathbb{R}\} \cap \Delta_{1,2},
\]
and
\[
\mathbb{R}_{>0} = \pi_{4,1} ((\mathbb{R} \times \{(x, x, x^2) \mid x \in \mathbb{R}\}) \cap \Delta_{1,3}) \setminus \{0\}.
\]

Let \( f : \mathbb{R}^d \to \mathbb{R} \) be definable, let \( A \in \mathcal{A}_d \), and denote the graph of \( f \) by \( \Gamma(f) \). Then
\[
\{ x \in A \mid f(x) > 0 \} = \pi_{d+1,d} (\Gamma(f) \cap (A \times \mathbb{R}_{>0})),
\]
and
\[
\{ x \in A \mid f(x) = 0 \} = \pi_{d+1,d} (\Gamma(f) \cap (A \times \{0\})).
\]
are definable.

A.1. Formulae. A convenient way to describe sets in a structure on the real field is to use formulae which are defined as follows.

Definition A.5. An atomic formula in a structure on the real field \( \mathcal{A} = (\mathcal{A}_d)_{d=1}^{\infty} \) is any of the following expressions
\begin{itemize}
  \item \((x_1, \ldots, x_d) \in A \) is an atomic formula, where \( A \in \mathcal{A}_d \),
  \item \( f(x_1, \ldots, x_d) > 0 \) and \( f(x_1, \ldots, x_d) = 0 \) are atomic formulae, where \( f : A' \to \mathbb{R} \) is a definable function.
\end{itemize}

A formula is defined inductively by the following
\begin{itemize}
  \item An atomic formula is a formula
  \item If \( \phi(x_1, \ldots, x_d) \) and \( \psi(x_1, \ldots, x_d) \) are formulae, then \( \phi \land \psi, \phi \lor \psi, \neg \phi \) and \( \phi \Rightarrow \psi \) are formulae.
  \item If \( A \in \mathcal{A}_m \), and \( \phi(x_1, \ldots, x_n, y_1, \ldots, y_m) \) is a formula, then
  \[ \exists (y_1, \ldots, y_m) \in A(\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)) \]
  and
  \[ \forall (y_1, \ldots, y_m) \in A(\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)) \]
  are formulae.
\end{itemize}

A.1.1. Description of sets via formulae. We fix a structure on the real field \( \mathcal{A} = (\mathcal{A}_d)_{d=1}^{\infty} \), and in the following we denote by \( A \) a definable set and we denote by \( f \) a definable function \( f : A' \to \mathbb{R} \) for \( A' \in \mathcal{A}_d \).

Atomic formulae define sets by the following obvious manner:
\[
\begin{align*}
(x_1, \ldots, x_d) \in A & \quad \text{defines} \quad A, \\
 f(x_1, \ldots, x_d) > 0 & \quad \text{defines} \quad \{ x \in A' \mid f(x) > 0 \}, \\
 f(x_1, \ldots, x_d) = 0 & \quad \text{defines} \quad \{ x \in A' \mid f(x) = 0 \}.
\end{align*}
\]
Now assume that the formulae \( \phi(x, y) \) and \( \psi(x, y) \) define \( \Phi \subseteq \mathbb{R}^n \times \mathbb{R}^m \) and \( \Psi \subseteq \mathbb{R}^n \times \mathbb{R}^m \) correspondingly. Then the following formulae define sets by interpreting logical symbols with Boolean operations and projections in the following standard way:

\[
\begin{align*}
\phi(x, y) \land \psi(x, y) & \text{ defines } \Phi \cap \Psi, \\
\phi(x, y) \lor \psi(x, y) & \text{ defines } \Phi \cup \Psi, \\
\neg \phi(x, y) & \text{ defines } \mathbb{R}^{m+n} \setminus \Phi, \\
\forall y \in A(\phi(x, y)) & \text{ defines } \pi_{m+n,n}(\Phi \cap (\mathbb{R}^n \times A)), \\
\forall y \in A(\psi(x, y)) & \text{ defines the same set as } \neg \exists y \in A(\neg \phi(x, y)), \\
\phi(x, y) \Rightarrow \psi(x, y) & \text{ defines the same set as } \neg \phi(x, y) \lor \psi(x, y).
\end{align*}
\]

By Definition \( \text{A.1} \) and Lemma \( \text{A.4} \) we deduce that formulae yield definable sets.

A.2. The o-minimal structure of restricted analytic functions. We obtain a partial order on all structures expanding the real field by declaring that for two structures \( \mathcal{A} \) and \( \mathcal{A}' \) it holds that \( \mathcal{A}' \leq \mathcal{A} \) if and only if \( \mathcal{A}' \subseteq \mathcal{A} \) for all \( d \in \mathbb{N} \).

In this paper we are only interested in the structure \( \mathbb{R}_{an} \) known as the field of real numbers with restricted analytic functions, which is defined to be the smallest structure on the real field in which all real numbers are definable and all functions \( f : \mathbb{R}^d \to \mathbb{R} \) which are real analytic in \([0, 1]^d\) and vanish in \( \mathbb{R}^d \setminus [0, 1]^d \) (namely, \( f \) is the restriction of an analytic function in a neighborhood of \([0, 1]^d\)) are definable.

In \( \text{vdD86} \) it was shown that \( \mathbb{R}_{an} \) is an o-minimal structure.

Lemma A.6. Let \( F : [0, 1]^N \times [0, 1] \to \mathbb{R} \) be an analytic function. Then the set \( A \) defined in Section 2 is definable in \( \mathbb{R}_{an} \).

Proof. In order to prove the claim we give an explicit formula which defines

\[
A \overset{\text{def}}{=} \left\{ (\epsilon, \delta) \in [0, 1]^2 \mid \forall \xi \in \Sigma, \forall \epsilon \in [0, 1] \text{ it holds that } (\xi - \frac{\epsilon}{2} , \xi + \frac{\epsilon}{2}) \cap F_{\delta, \delta} \neq \emptyset \right\},
\]

where \( \Sigma \overset{\text{def}}{=} \{ x \in [0, 1]^N \mid d(x, \Sigma) \geq \epsilon \} \) and \( d(\cdot, \Sigma) : \mathbb{R}^N \to \mathbb{R} \) is the Euclidean distance function from the closed set \( \Sigma \overset{\text{def}}{=} \{ x \in [0, 1]^N \mid F(x, \cdot) = 0 \} \).

We now show that \( d(\cdot, \Sigma) \) is definable in \( \mathbb{R}_{an} \). Indeed, we note that the set \( \Sigma \) is definable since it is defined by

\[
\forall t \in [0, 1](F(x, t) = 0).
\]

Therefore we get that the following is a formula in \( \mathbb{R}_{an} \)

\[
(x, y) \in \mathbb{R}^N \times \mathbb{R}_{\geq 0} \land \left( \forall \epsilon > 0, \exists \xi \in \Sigma(y^2 \leq \|x - \xi\|^2 \leq (y + \epsilon)^2) \right),
\]

which defines the graph of \( d(\cdot, \Sigma) : \mathbb{R}^N \to \mathbb{R} \), showing that \( d(\cdot, \Sigma) \) is definable.

We may now deduce that the following is a formula in \( \mathbb{R}_{an} \)

\[
\phi(\epsilon, \delta, x, \xi) \overset{\text{def}}{=} (x, \xi, (\delta, \epsilon)) \in [0, 1]^N \times [0, 1] \times (0, 1]^2 \land (d(x, \Sigma) - \epsilon > 0 \\
\Rightarrow \exists t \in [0, 1]((\xi - \frac{\epsilon}{2} \leq t \leq \xi + \frac{\epsilon}{2}) \land F(x, t)^2 - \delta^2 \geq 0)).
\]

Since \( A \) is defined by

\[
(\epsilon, \delta) \in (0, 1]^2 \land \forall (x, \xi) \in [0, 1]^N \times [0, 1] \phi(\epsilon, \delta, x, \xi)
\]

we get that \( A \) is definable.

A.3. Needed results from the theory of o-minimal structures. The following two properties hold for an arbitrary o-minimal structure \( \mathcal{A} = (\mathcal{A}_d)_{d=1}^\infty \).

Theorem A.7. \( \text{vdD98} \) Chapter 3, Corollary 3.6] Let \( A \subseteq \mathcal{A}_{m+n} \). Then there exists \( N \in \mathbb{N} \) such for all \( x \in \mathbb{R}^m \) the set \( A_x \overset{\text{def}}{=} \{ y \in \mathbb{R}^n \mid (x, y) \in A \} \) has at most \( N \) connected components.

Theorem A.8. \( \text{vdD98} \) Chapter 6, Proposition 1.2] Let \( A \subseteq \mathcal{A}_{m+n} \). Then there is a definable map \( f : \pi_{m+n,n}(A) \to \mathbb{R}^{m+n} \) such that the graph of \( f \) is contained in \( A \).
An important property possessed by \( \mathbb{R}_{an} \) is the property of polynomial boundedness (see [vdD86]) which we define below.

**Definition A.9.** A structure \( \mathcal{A} = (\mathcal{A}_d)_{d=1}^{\infty} \) is polynomially bounded if for every definable \( f : \mathbb{R} \to \mathbb{R} \) there exists \( m \in \mathbb{N} \) such that \( f(t) = O(t^m) \) as \( t \to \infty \).

We have the following straightforward corollary (for a more general version see [vdDM96, 4.13])

**Corollary A.10.** Let \( f : (0, a) \to \mathbb{R} \) be strictly positive and bounded function definable in \( \mathbb{R}_{an} \). Then there exist \( 0 < c, \kappa \) and \( 0 < \epsilon < a \) such that \( ct^\kappa \leq f(t) \) for all \( t \in (0, \epsilon) \).

**Proof.** For convenience we extend \( f \) to the real line by defining \( f(1/t) = 1 \) for all \( t /\in (0, a) \) (which also gives a function definable in \( \mathbb{R}_{an} \)). Since \( f \) is non-vanishing, we may consider the function \( g : \mathbb{R} \to \mathbb{R} \) defined by \( g(s) = f(1/s) \), (which is definable since \( 1/s \) for \( s \neq 0 \) is definable, and composition of definable functions gives a definable function).

Since \( \mathbb{R}_{an} \) is polynomially bounded, there is \( c' > 0 \) such that \( 1/(1/s) \leq c's^m \) for all \( s \) large enough. Therefore, for any \( t \) close enough to 0 from the right we have that

\[
f(t) \geq \frac{1}{c'} t^m.
\]

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