A NOTE FOR EXTENSION OF ALMOST SURE CENTRAL LIMIT THEORY

YU MIAO AND GUANGYU YANG

Abstract. Hörmann (2006) gave an extension of almost sure central limit theorem for bounded Lipschitz 1 function. In this paper, we show that his result of almost sure central limit theorem is also hold for any Lipschitz function under stronger conditions.

1. Introduction

The classical results on the almost sure central limit theorem (ASCLT) dealt with partial sums of random variables. A general pattern is that, if \( X_1, X_2, \ldots \) be a sequence of independent random variables with partial sums \( S_n = X_1 + \cdots + X_n \) satisfying \( (S_n - b_n)/a_n \xrightarrow{L} H \) for some sequences \( a_n > 0, b_n \in \mathbb{R} \) and some distribution function \( H \), then under some mild conditions we have

\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{P}\left\{ \frac{S_k - b_k}{a_k} \leq x \right\} = H(x) \quad \text{a.s.}
\]

for any continuity point \( x \) of \( H \).

Several papers have dealt with logarithmic limit theorems of this kind and the above relation has been extended in various directions. Fahrner and Stadtmüller [2] gave an almost sure version of a maximum limit theorem. Berkes and Horváth [2] obtained a strong approximation for the logarithmic average of sample extremes. Berkes and Csáki [1] showed that not only the central limit theorem, but every weak limit theorem for independent random variables has an analogous almost sure version. For stationary Gaussian sequences with covariance \( r_n \), Csáki and Gonchigdanzan [3] proved an almost sure limit theorem for the maxima of the sequences under the condition \( r_n \log n (\log \log n)^{1+\varepsilon} = O(1) \). For some dependent random variables, Peligrad and Shao [7] and Dudziński [4] obtained corresponding results about the almost sure central limit theorem.

Date: May 26, 2006.

2000 Mathematics Subject Classification. 60F05.

Key words and phrases. Almost sure central limit theorem; Lipschitz function; logarithmic limit theorems.
Recently, Hörmann [6] gave an extension of almost sure central limit theory under some regularity condition as the following form:

$$\lim_{N \to \infty} D_N^{-1} \sum_{k=1}^{N} d_k f \left( \frac{S_k}{a_k} - b_k \right) = \int_{-\infty}^{\infty} f(x) dH(x) \ a.s. \quad (1.1)$$

where $f$ is a bounded Lipschitz 1 function and $D_N = \sum_{k=1}^{N} d_k$, $\{d_k\}_{k \geq 1}$ is a sequence of positive constants. Using his method, we will show that for any Lipschitz function $f$, (1.1) holds under some additional conditions.

At first, we give our main result.

**Theorem 1.1.** Let $X_1, X_2, \ldots$ be independent random variables with partial sums $S_n$ and assume that

1. $(C_1)$ For some numerical sequences $a_n > 0$ and $b_n$, we have $\frac{S_n}{a_n} - b_n \xrightarrow{\mathcal{D}} H$, where $H$ is some (possibly degenerate) distribution function.
2. $(C_2)$ $kd_k = O(1)$ and for some $0 < \alpha < 1$, $d_k k^\alpha$ is eventually non-increasing.
3. $(C_3)$ For some $\rho > 0$, $d_k = O\left(\frac{D_k}{k(\log D_k)^\rho}\right)$.
4. $(C_4)$ There exist positive constants $C, \beta$, such that $a_k/a_l \leq C(k/l)^\beta$ $(1 \leq k \leq l)$. Furthermore, for some $0 < r < \rho$,

$$\mathbb{E}\left|\frac{S_n}{a_n} - b_n\right|^\mu = O(1), \quad \text{for some integer } \mu \geq (2 \lor 4/(\rho - r)). \quad (1.2)$$

Then for any Lipschitz function $f$ on the real line, we have (1.1).

**Remarks 1.2.** Obviously, for any bounded Lipschitz 1 function $f$, under the above assumptions, the equation (1.1) holds, i.e. we can obtain Theorem 1 in [6].

2. **Proof of Theorem 1.1**

In this section, we will give the proof of Theorem 1.1 according to the process of Hörmann in [6].

**Lemma 2.1.** (See Lemma 1 in [6]) Let $(D_N)$ be a summation procedure, then the condition $(C_3)$ of Theorem 1.1 implies that $D_N = o(N^\varepsilon)$ for any $\varepsilon > 0$.

**Lemma 2.2.** Assume that condition $(C_4)$ of Theorem 1.1 is satisfied and $b_n = 0$. Then for every Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ there exists constant $c > 0$ such that

$$\left| \text{Cov} \left( f \left( \frac{S_k}{a_k} \right), f \left( \frac{S_l}{a_l} \right) \right) \right| \leq c(k/l)^\beta \quad (1 \leq k \leq l), \quad (2.1)$$

where $\beta$ is the same as in $(C_4)$. 
Proof. Firstly, we assume $f(0) = 0$. Denoting $\| f \|$ the Lipschitz constant of $f$, we get, by using the independence of $S_k$ and $S_l - S_k$,

$$
\left| \text{Cov} \left( f \left( \frac{S_k}{a_k} \right), f \left( \frac{S_l}{a_l} \right) \right) \right| = \left| \text{Cov} \left( f \left( \frac{S_k}{a_k} \right), f \left( \frac{S_l}{a_l} \right) - f \left( \frac{S_l - S_k}{a_l} \right) \right) \right|
\leq \mathbb{E} \left| f \left( \frac{S_k}{a_k} \right) \right| \mathbb{E} \left| f \left( \frac{S_l}{a_l} - f \left( \frac{S_l - S_k}{a_l} \right) \right) \right|
\leq \| f \|^2 \frac{a_k}{a_l} \mathbb{E} \left[ \frac{S_k^2}{a_k^2} \right] + \| f \|^2 \frac{a_k}{a_l} \mathbb{E} \left[ \frac{S_k}{a_k} \right]^2
\leq 2k \| f \|^2 \mathbb{E} \left[ \frac{S_k^2}{a_k^2} \right] (k/l)^\beta,
$$

where the last inequality is due to condition $(C_3)$. Since the equation (1.2), we can take $c > 0$ such that for any $k \geq 1$, $2k \| f \|^2 \mathbb{E} \left[ \frac{S_k^2}{a_k^2} \right] \leq c$. So (2.1) is obtained.

If $f(0) \neq 0$, we can define a function $g$, such that $g(x) = f(x) - f(0)$, then $g$ is a Lipschitz function and $g(0) = 0$. And noting that,

$$
\text{Cov} \left( f \left( \frac{S_k}{a_k} \right), f \left( \frac{S_l}{a_l} \right) \right) = \text{Cov} \left( g \left( \frac{S_k}{a_k} \right), g \left( \frac{S_l}{a_l} \right) \right)
$$

we complete the proof of the lemma.

\[ \square \]

Remarks 2.3. It is obvious to see that if we replace $\beta$ by any $0 < \beta' < \beta$, the Lemma 2.2 also holds. Hence, without loss of generality, we can assume that $\beta$ is the same as $\alpha$ in condition $(C_2)$ of Theorem 1.1.

Next we will use the following notations,

$$
\xi_l := f \left( \frac{S_l}{a_l} \right) - \mathbb{E} f \left( \frac{S_l}{a_l} \right), \quad \xi_{k,l} := f \left( \frac{S_l - S_k}{a_l} \right) - \mathbb{E} f \left( \frac{S_l - S_k}{a_l} \right).
$$

Lemma 2.4. Assume that condition $(C_4)$ of Theorem 1.1 is satisfied and $b_n = 0$, and define $\xi_l$ and $\xi_{k,l}$ as in (2.2). If $\{d_k, k \geq 1\}$ are arbitrary positive weights, then we have for any $k \leq m \leq n$ and $p \in \mathbb{N}$, $p \leq \mu$,

$$
\mathbb{E} \left[ \sum_{l=m}^n d_l (\xi_l - \xi_{k,l}) \right]^p \leq E_p \left( \sum_{l=m}^n d_l^2 \right)^{p/2},
$$

where $E_p = c_p C^p \| f \|^p \left[ \left( \frac{2^\kappa}{\kappa} \right) \vee \left( 1 + \frac{1}{\kappa} \right) \right]^{p/2}$ and $\kappa = 2\beta$. 

Proof. Let \( Q(l) = Q(k, l) = \xi_l - \xi_{k,l} \), then
\[
\mathbb{E}|Q(l)|^p = \mathbb{E}\left| f\left(\frac{S_l}{a_l}\right) - f\left(\frac{S_l - S_k}{a_l}\right) - \mathbb{E}\left[ f\left(\frac{S_l}{a_l}\right) - f\left(\frac{S_l - S_k}{a_l}\right) \right] \right|^p \\
\leq \|f\|^p(a_k/a_l)^p \mathbb{E}\left( \frac{|S_k|}{a_k} + \mathbb{E}\left( \frac{|S_k|}{a_k} \right) \right)^p \\
\leq C^p \|f\|^p \mathbb{E}\left( \frac{|S_k|}{a_k} + \mathbb{E}\left( \frac{|S_k|}{a_k} \right) \right)^p (k/l)^{p\beta} \\
\leq c_p C^p \|f\|^p (k/l)^{p\beta},
\]
where \( C \) is the same as in condition \((C_4)\) and \( c_p \) is a positive constant such that for all \( k \), \( \mathbb{E}\left( \frac{|S_k|}{a_k} + \mathbb{E}\left( \frac{|S_k|}{a_k} \right) \right)^p \leq c_p \). Thus by the Hölder inequality, we get
\[
\mathbb{E}\left| \sum_{l=1}^n d_l(\xi_l - \xi_{k,l}) \right|^p \leq \sum_{l_1=m}^n \cdots \sum_{l_p=m}^n d_{l_1} \cdots d_{l_p} (\mathbb{E}|Q(l_1)|^p \cdots \mathbb{E}|Q(l_p)|^p)^{1/p} \\
\leq c_p C^p \|f\|^p k^{p\beta} \sum_{l_1=m}^n \cdots \sum_{l_p=m}^n d_{l_1} \cdots d_{l_p} l_1^{-\beta} \cdots l_p^{-\beta} \\
= c_p C^p \|f\|^p k^{p\beta} \left( \sum_{l=m}^n d_l l^{-\beta} \right)^p \\
\leq c_p C^p \|f\|^p m^{p\beta} \left( \sum_{l=m}^n d_l^2 l^{-2\beta-1} \right)^{p/2} \left( \sum_{l=m}^n l^{-2\beta-1} \right)^{p/2}.
\]
For \( m \geq 2 \), it is easy to see that
\[
m^{p\beta} \left( \sum_{l=m}^n l^{-2\beta-1} \right)^{p/2} \leq m^{p\beta} \left( \int_{m-1}^{\infty} l^{-2\beta-1} dl \right)^{p/2} \\
\leq \left( \frac{m}{m-1} \right)^{p\beta} \left( \frac{1}{2\beta} \right)^{p/2} \leq \left( \frac{2^n}{\kappa} \right)^{p/2},
\]
where \( \kappa := 2\beta \). Similarly, we get for \( m = 1 \)
\[
\left( \sum_{l=1}^n l^{-2\beta-1} \right)^{p/2} \leq \left( 1 + \frac{1}{\kappa} \right)^{p/2}.
\]
Hence, we have
\[
\mathbb{E}\left| \sum_{l=m}^n d_l(\xi_l - \xi_{k,l}) \right|^p \leq c_p C^p \|f\|^p \tau(\kappa)^{p/2} \left( \sum_{l=m}^n d_l^2 l \right)^{p/2},
\]
where \( \tau(\kappa) := \left[ \left( \frac{2^n}{\kappa} \right) \lor (1 + \frac{1}{\kappa}) \right] \). This completes the proof of our result. \( \square \)
**Lemma 2.5.** Assume that conditions $(C_2) - (C_4)$ of Theorem 1.1 are satisfied. Further let $b_n = 0$ in condition $(C_4)$ and $f$ be a Lipschitz function. Then for every $p \leq \mu$ and $p \in \mathbb{N}$ we have

$$
\mathbb{E} \left| \sum_{k=1}^{N} d_k \left( f \left( \frac{S_k}{a_k} \right) - E f \left( \frac{S_k}{a_k} \right) \right) \right|^p \leq C_p \left( \sum_{1 \leq k \leq l \leq N} d_k d_l \left( \frac{k}{l} \right)^\beta \right)^{p/2},
$$

where $C_p > 0$ is a constant and $\beta$ is the same as in $(C_4)$.

**Proof.** At first, we set $C_p = (4\gamma)^p$ and

$$ V_{m,n} := \sum_{l=m}^{n} d_l l^{-\beta} \left( \sum_{k=1}^{l} d_k k^\beta \right), \quad (1 \leq m \leq n). $$

For obtaining our result, it is enough to show that the following claim, 

"if the number $\gamma$ is chosen large enough, then

$$
\mathbb{E} \left| \sum_{k=m}^{n} d_k \xi_k \right|^p \leq C_p (V_{m,n})^{p/2}, \quad \text{for all } 1 \leq m \leq n,
$$

where $\xi_k$ is defined as in (2.2)."

We will use induction on $p$ to show (2.4). By Lemma 2.2, we have

$$
\mathbb{E} \left| \sum_{k=m}^{n} d_k \xi_k \right|^2 \leq 2 \sum_{m \leq k \leq l \leq n} d_k d_l \mathbb{E} |\xi_k \xi_l| \leq 2c \sum_{m \leq k \leq l \leq n} d_k d_l (k/l)^\beta \leq 2c V_{m,n}.
$$

Hence if we choose $\gamma$ so large that $(4\gamma)^4 \geq 2c$, then (2.4) holds for $p = 2$.

Assume now that (2.4) is true for $p - 1 \geq 2$. From $kd_k = O(1)$ it follows that there is a positive constant $A$ such that $\sum_{k=1}^{l} d_k k^\beta \geq Al^\beta$. Then we get for $V_{m,n} \leq \gamma$ as the proof of Lemma 2.4, there exists a constant $A_p$ such that

$$
\mathbb{E} \left| \sum_{k=m}^{n} d_k \xi_k \right|^p \leq \sum_{k_1=m}^{n} \cdots \sum_{k_p=m}^{n} d_{k_1} \cdots d_{k_p} (\mathbb{E} |\xi_{k_1}|^p \cdots \mathbb{E} |\xi_{k_p}|^p)^{1/p}
$$

$$
\leq A_p \|f\|^p \sum_{k_1=m}^{n} \cdots \sum_{k_p=m}^{n} d_{k_1} \cdots d_{k_p}
$$

$$
= A_p \|f\|^p \left( \sum_{k=m}^{n} d_k \right)^p
$$

$$
\leq A_p \|f\|^p A^{-p} \left( \sum_{k=m}^{n} d_k k^{-\beta} \left( \sum_{l=1}^{k} d_l l^\beta \right) \right)^p.
$$

Now choose $\gamma$ so large that the $C_p \leq (A_p^{1/p} \|f\|/A)^{p\gamma/p^2}$. In the case of $V_{m,n} \leq \gamma$, we have shown (2.4) is valid.
We now want to show that if for any given $X \geq \gamma$ and the inequality $|2.4|$ holds for $V_{m,n} \leq X$, then it will also hold for $V_{m,n} \leq 3X/2$ and this will show that $|2.4|$ holds for any value of $V_{m,n}$, i.e. complete the induction step.

Assume $V_{m,n} \leq 3X/2$ and set

$$S_1 + S_2 := \sum_{k=m}^{w} d_k \xi_k + \sum_{k=w+1}^{n} d_k \xi_k, \quad T_2 := \sum_{k=w+1}^{n} d_k \xi_{w,k}, \quad (m \leq w \leq n).$$

From the discussion of Lemma 4 in Hörmann, S. [6] (2006), and Lemma 2.1, for a fixed $m$ and $n$ we choose $w$ in such a way that $V_{m,w} \leq X, V_{w+1,n} \leq X$ and $V_{w+1,n}/V_{m,w} = \lambda \in [1/2, 1]$.

From the mean value theorem we get

$$|S_j - T_j| \leq j|S_j - T_j|/(|S_j|^{j-1} + |T_j|^{j-1}) \quad (j \geq 1). \quad (2.5)$$

Since condition $(C_2)$ and Remarks 2.3 there exists a constant $B > 0$ such that for all $l \geq 1$,

$$B \sum_{k=1}^{l} d_k k^\beta \geq l^{1+\beta} d_l.$$ 

This also shows that

$$\sum_{l=m}^{n} ld_l^2 \leq BV_{m,n}, \quad \text{for all } 1 \leq m \leq n.$$ 

By Lemma 2.4 we get for all $j \geq 1$,

$$\mathbb{E}|S_2 - T_2|^{j} \leq F_j (V_{w+1,n})^{j/2},$$

where $F_j = B^{j/2}E_j$ and $E_j$ is the constant in Lemma 2.4.

From the induction hypothesis in the case of $1 \leq j \leq p - 1$ and from the validity of $(2.4)$ for $V_{m,n} \leq X$ in the case of $j = p$, we have

$$\mathbb{E}|S_1|^{j} \leq C_j (V_{m,n})^{j/2}, \quad (1 \leq j \leq p) \quad (2.6)$$

and

$$\mathbb{E}|S_2|^{j} \leq C_j (V_{w+1,n})^{j/2} \leq C_j \lambda^{j/2}(V_{m,n})^{j/2}, \quad (1 \leq j \leq p). \quad (2.7)$$

The remains of the proof are the same as in Lemma 4 in Hörmann, S. [6] (2006), but for completeness, we still give the proof. By $C_r$ inequality, we have

$$\mathbb{E}|T_2|^{j} \leq 2^j C_j \lambda^{j/2}(V_{m,n})^{j/2}, \quad (1 \leq j \leq p). \quad (2.8)$$

Furthermore, from Hölder inequality the following result is easy,

$$\mathbb{E}|S_1|^{j} |S_2 - T_2|^{p-j-1} \leq (\mathbb{E}|S_1|^{p})^{j/p} (\mathbb{E}|S_2 - T_2|^{p})^{1/p} (\mathbb{E}|S_2|^{p})^{(p-j-1)/p} \leq C_p^{(p-1)/p} F_1^{1/p} \lambda^{(p-j)/2}(V_{m,n})^{p/2}. \quad (2.9)$$
The last inequality remains valid, with an extra factor $2^{p-j-1}$ on the right hand side, if $|S_2|^{p-j-1}$ on the left hand side is replaced by $|T_2|^{p-j-1}$. Since $S_1$ and $T_2$ are independent, we get

$$E|S_1 + S_2|^p \leq E|S_1|^p + E|S_2|^p + \sum_{j=1}^{p-1} G_p^j(E|S_1|^j|S_2^{p-j} - T_2^{p-j}| + E|S_1|^jE|T_2|^{p-j}),$$

where $G_p^j$ denote the combination, i.e., $G_p^j = p![(p-j)!]^{-1}$. Substituting (2.5) – (2.9) (using also the analogue of (2.9) with $|T_2|^{p-j-1}$) in the above inequality and get

$$E|S_1 + S_2|^p \leq C_p(V_m,w)^{p/2} \left( 1 + \lambda^{p/2} + C_p^{-1/p} F_p^{1/p} \sum_{j=1}^{p-1} 2^{p-j} G_p^j(p-j)\lambda^{(p-j)/2} + C_p^{-1} \sum_{j=1}^{p-1} 2^{p-j} G_p^j\lambda^{(p-j)/2} C_j C_{p-j} \right).$$

Note that

$$C_p^{-1/p} F_p^{1/p} \leq const \cdot \tau(\kappa)^{1/2} c_p^{1/p}(4\gamma)^{-p}$$

and

$$C_j C_{p-j} / C_p \leq (4\gamma)^{-p}, \hspace{1em} (1 \leq j \leq p-1),$$

thus, by $\lambda \leq 1$, we have

$$C_p^{-1/p} F_p^{1/p} \sum_{j=1}^{p-1} 2^{p-j} G_p^j(p-j)\lambda^{(p-j)/2} \leq const \cdot \tau(\kappa)^{1/2} p c_p^{1/p} \gamma^{-p}$$

and

$$C_p^{-1} \sum_{j=1}^{p-1} 2^{p-j} G_p^j\lambda^{(p-j)/2} C_j C_{p-j} \leq const \cdot \gamma^{-p}.$$

Since $\lambda \geq 1/2$ we have shown that for a large $\gamma$ the relation $E|S_1 + S_2|^p \leq C_p(1 + \lambda)^{p/2}(V_m,w)^{p/2} = C_p(V_m,n)^{p/2}$ is true, i.e., for $V_m,n \leq 3X/2$, (2.4) is valid.

\[ \square \]

**Lemma 2.6.** (See Lemma 5 in [4]) Assume the condition $(C_3)$ of Theorem 1.1 is satisfied. Then for any $\alpha > 0$ and any $\eta < \rho$, we have

$$\sum_{1 \leq k \leq l \leq N} d_k d_l \left( \frac{k}{l} \right)^{\alpha} = O\left( \frac{D_N^2}{(\log D_N)^{\eta}} \right).$$

**Proof of Theorem 1.1.** Without loss of generality, from Lemma 2.5 and Lemma 2.6, we have, for any $\varepsilon > 0$, $p \leq \mu$ and $p \in \mathbb{N},$

$$P\left( \left| \sum_{k=1}^{N} d_k \xi_k \right| > \varepsilon D_N \right) \leq c(p, \varepsilon) (\log D_N)^{-p\eta/2}, \hspace{1em} \text{for} \hspace{1em} N \geq N_0.$$
Since \( \mu \geq (2 \sqrt[4]{4/(\rho - r)}) \) for some \( 0 < r < \rho \), we can choose suitable \( \eta < \rho \) and \( p \) such that \( p\eta > 4 \). By \((C_3)\), we have \( D_{N+1}/D_N \to 1 \), thus we can choose \((N_j)\) such that \( D_{N_j} \sim \exp\{\sqrt{j}\} \). Applying Borel-Cantelli lemma, we get
\[
\lim_{j \to \infty} D^{-1}_{N_j} \sum_{k=1}^{N_j} d_k \xi_k = 0 \quad \text{a.s.}
\]
For \( N_j \leq N \leq N_{j+1} \), we have
\[
D^{-1}_N \sum_{k=1}^{N} |d_k \xi_k| \leq D^{-1}_{N_j} \sum_{k=1}^{N_j} |d_k \xi_k| + 2(D_{N_{j+1}}/D_{N_j} - 1) \quad \text{a.s.}
\]
The convergence of the subsequence implies that whole sequence converges a.s., since \( D_{N_{j+1}}/D_{N_j} \to 1 \). This complete the proof of the theorem. \( \square \)

REFERENCES

[1] Berkes, I., Csáki, E. A universal result in almost sure central limit theory. *Stochastic Process and Applications*, 2001, 94: 105-134.
[2] Berkes, I., Horváth L. The logarithmic average of sample extremes is asymptotically normal. *Stochastic Process and Applications*, 2001, 91(1): 77-98.
[3] Csáki, E., Gonchigdanzan, K. Almost sure limit theorems for the maximum of stationary Gaussian sequences. *Statistics and Probability Letters*, 2002, 58(2): 195-203.
[4] Dudziński, M. A note on the almost sure central limit theorem for some dependent random variables. *Statistics and Probability Letters*, 2003, 61: 31-40.
[5] Fahrner, I., Stadtmüller, U. On almost sure max-limit theorems. *Statistics and Probability Letters*, 1998, 37(3): 229-236.
[6] Hörmann, S. An extension of almost sure central limit theory. *Statistics and Probability Letters*, 2006, 76: 191-202.
[7] Peligrad, M., Shao, Q.M. A note on the almost sure central limit theorem for weakly dependent random variables. *Statistics and Probability Letters*, 1995, 22: 131-136.

DEPARTMENT OF MATHEMATICS AND STATISTICS, WUHAN UNIVERSITY, 430072 HUBEI, CHINA
AND COLLEGE OF MATHEMATICS AND INFORMATION SCIENCE, HENAN NORMAL UNIVERSITY, 453007 HENAN, CHINA.
E-mail address: yumiao728@yahoo.com.cn

DEPARTMENT OF MATHEMATICS AND STATISTICS, WUHAN UNIVERSITY, 430072 HUBEI, CHINA.
E-mail address: study_yang@yahoo.com.cn