Geometry and control of the nonholonomic integrator: 
An electrodynamics analogy

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Abstract—We consider some generalizations of the classical nonholonomic integrator and give a geometric approach to characterize controllability for these systems. We use Stokes’ theorem and results from complex analysis to obtain necessary and sufficient conditions for controllability of these systems. Furthermore, we show that optimal trajectories of certain minimum energy optimal control problems defined on these systems can be identified with the trajectory of a charged particle in an electromagnetic field.

I. INTRODUCTION
In this article, we give a new geometric characterization of controllability for a generalized model of the nonholonomic integrator and study some minimum energy optimal control problems on these models. The relationship between optimal control problems for example, minimum energy problems on nonlinear systems and geometric problems on Riemannian manifolds (such as geodesics) is well known in the control literature due to seminal works of Brockett [11] and the references therein. The celebrated prototype of a nonlinear control system to understand these connections is the nonholonomic integrator or the Brockett integrator. In this article, we explore further into this prototypical example and its variants and show how some of the optimal control problems are analogous to classical electrodynamics problems such as force acting on a particle in an electromagnetic field. The optimal state transfer of the nonholonomic integrator and general nonholonomic systems using sinusoids was demonstrated in the works of Murray and Sastry [2] with applications in motion planning. Motion planning has since been an active research area as can be seen in the works of [9], [5], [6], [7], [8], [9], [10], [11], [12] and the references therein. We used orthogonal polynomials such as Legendre and Chebyshev polynomials for steering the nonholonomic integrator in [13] and showed that these orthogonal polynomials can serve as optimal inputs for appropriate cost functions using Sturm-Liouville theory. The following nonlinear control system

\[ \dot{x}_1 = u_1, \dot{x}_2 = u_2, \dot{x}_3 = -x_2 u_1 + x_1 u_2 \]  

is known as the nonholonomic integrator. Notice that the dynamics in the third state co-ordinate is actually a differential 1–form in \( \mathbb{R}^3 \) or \( \mathbb{R}^2 \). These type of systems arise in robotics and motion planning [13], [16], [10], [11], [12] and the references therein. The following more general form

\[ \dot{x}_1 = u_1, \dot{x}_2 = u_2, \dot{x}_3 = f_1(x_1, x_2) u_1 + f_2(x_1, x_2) u_2 \]  

was considered in [16], [10], [11], [12]. Borrowing these ideas, we consider [2] and its various generalizations in this article. These models are important because they provide a canonical form a wider class of nonholonomic control systems and the more general nonholonomic systems can be better understood by studying these specific nonholonomic systems.

We use notions such as the curl of a vector field from multi-variable calculus to give necessary and sufficient conditions for controllability of [2] and its generalizations. We show that minimum energy optimal control problems for these models can be identified with the classical electrodynamics problem of a particle in an electromagnetic field. We then give another characterization of controllability using holomorphic functions from complex analysis.

Organization: In the next section, we give some preliminaries to be used in this paper. In Section II, we obtain a necessary and sufficient condition of controllability for generalizations of the nonholonomic integrator using the curl operator. Then, in Section III, we explore the relationship between some minimum energy optimal control problems on the nonholonomic integrator and classical electrodynamics. In Section IV, we study controllability of the general nonholonomic integrator using tools from complex analysis.

Notation: The scalars and scalar valued functions are denoted by small-face letters, vectors and vector valued functions are denoted by bold-face letters and matrices and matrix valued functions are denoted by capital letters. The gradient operator on a scalar valued function \( \phi \) is denoted as \( \nabla \phi \), the curl operator on a vector field \( f \) in \( \mathbb{R}^3 \) or \( \mathbb{R}^2 \) is denoted by \( \nabla \times f \) and the divergence operator is denoted by \( \nabla \cdot f \). The closed loop integral over a closed curve \( \gamma \) is denoted by \( \oint \gamma \), and the surface integral over a surface \( S \) is denoted by \( \int_S f \). The line element on a manifold is denoted by \( ds \).

II. PRELIMINARIES
We refer to Equation (1) as the nonholonomic integrator on \( \mathbb{R}^2 \) for reasons which will become clear later. Notice that if \( x_3(0) = x_3(1) \) and \( x_2(0) = x_2(1) \), then the variable \( x_3 \) measures the area formed the projection of the state trajectory on the first two state components were defined as in (1) and dynamics on the remaining state components were defined using different 1–forms. These type of systems arise in robotics and motion planning (13, 16, 10, 11, 12) and the references therein. The following general form

\[ \dot{x}_1 = u_1, \dot{x}_2 = u_2, \dot{x}_3 = f_1(x_1, x_2) u_1 + f_2(x_1, x_2) u_2 \]  

of the general nonholonomic integrator using tools from complex analysis.

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We refer to model considered in Equation (2) as the general nonholonomic integrator or the general nonholonomic integrator on $\mathbb{R}^2$ associated with a vector field $f = (f_1, f_2)$ on $\mathbb{R}^2$.

The following system is refereed as the generalized nonholonomic integrator on $\mathbb{R}^m$.

$$\dot{x}_i = u_i, \quad i = 1, \ldots, m,$$

$$\dot{x}_{ij} = x_i u_j - x_j u_i, \quad i < j = 1, \ldots, m. \quad (3)$$

Suppose $x_i(0) = x_i(T) = 0$, $\forall i = 1, \ldots, m$. Then, the co-ordinates $x_{ij}$ measure the area of the closed curve obtained by the projection of the state trajectory onto $x_i - x_j$ plane.

We also consider the following form of the nonholonomic integrator on $\mathbb{R}^3$ in the sequel

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = u_3,$$

$$\dot{x}_4 = f_1(x_1, x_2, x_3)u_1 + f_2(x_1, x_2, x_3)u_2 + f_3(x_1, x_2, x_3)u_3. \quad (4)$$

Moreover, we also consider the following generalization of (3)

$$\dot{x}_i = u_i, \quad i = 1, \ldots, m,$$

$$\dot{x}_{ij} = f_i(x_i, x_j)u_i + f_j(x_i, x_j)u_j, \quad i < j = 1, \ldots, m. \quad (5)$$

We refer the reader to [14] and [13] for more generalizations of the nonholonomic integrator. As far as steering of the classical nonholonomic integrator and generalized nonholonomic integrator is concerned, [2] gave a steering algorithm using sinusoids which also holds for a wider class of nonholonomic systems such as the ones defined above.

We briefly mention the following example from [13] which gives optimal sinusoidal inputs for the nonholonomic integrator when the cost function is the minimum input energy function.

**Example 2.1 ([2]):** For the system defined by (1), we want to find the minimum energy input to drive the state from the origin to a specified point $(0, 0, a)$ from $t = 0$ to $t = 1$. The cost function is $J = \int_0^1 (u_1^2 + u_2^2)dt$ subject to the system dynamics. Using system equations to eliminate $u_1$ and $u_2$, we obtain the cost function $\int_0^1 (\dot{x}_1^2 + \dot{x}_2^2)dt$ subject to $\dot{x}_3 - x_1 \dot{x}_2 + x_2 \dot{x}_1 = 0$. Therefore, the augmented cost function is

$$J_a = \int_0^1 (\dot{x}_1^2 + \dot{x}_2^2 + p(t)(\dot{x}_3 - x_1 \dot{x}_2 + x_2 \dot{x}_1))dt.$$

Applying the first order necessary conditions from calculus of variations, we obtain $p(t) = c$ and

$$\dot{x}_1 + c \dot{x}_2 = 0$$

$$\dot{x}_2 - c \dot{x}_1 = 0.$$

Now using $\dot{x}_1 = u_1$ and $\dot{x}_2 = u_2$, we have the following first order ode

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

We need to find $u(0)$ and $c$ using initial and final conditions. Let’s write $u = H u$ for first order equations in $u_1, u_2$. Hence, $u(t) = e^{Ht} u(0)$. Note that $e^{Ht}$ is orthogonal, hence, the norm of $\|u(t)\| = \|u(0)\|$ remains constant for all time. From the terminal conditions, it follows that $c = 2n\pi$ where $n = 0, \pm1, \pm2, \ldots$. Suppose $a > 0$, then the cost is minimum when $n = 1$ and $\|u\| = 2\pi a$ with the direction of $u$ being arbitrary.

For an arbitrary terminal time $T$, it turns out that $cT = 2n\pi$. Thus, for $n = 1, c = \frac{2\pi}{T}$ and

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \cos \frac{2\pi}{T}t & -\sin \frac{2\pi}{T}t \\ \sin \frac{2\pi}{T}t & \cos \frac{2\pi}{T}t \end{bmatrix} \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix}.$$ 

Let $u_i(0) = \sqrt{\frac{2\pi}{T}}$, $i = 1, 2$. Therefore, with sinusoidal inputs of appropriate frequencies, one can always steer the system from the origin to any point $(0, 0, a)$ in time $T$. The frequencies are chosen depending upon the terminal time $T$ so that for $x_1$ and $x_2$, we are integrate the sinusoids over the full period.

**Holomorphic functions and Cauchy’s integral formula:** A function $F : \mathbb{C} \to \mathbb{C}$ is called holomorphic if it is complex differentiable at each point in $\mathbb{C}$. Let $z = x_1 + i x_2 \in \mathbb{C}$ and $F(z) = F_1(x_1, x_2) + i F_2(x_1, x_2)$. Then, for a holomorphic function $F$, its real and imaginary parts $F_1, F_2$ satisfy Cauchy-Riemann equations given by (17)

$$\frac{\partial F_1}{\partial x_1} = \frac{\partial F_2}{\partial x_2}, \quad \frac{\partial F_2}{\partial x_1} = -\frac{\partial F_1}{\partial x_2}. \quad (6)$$

Let $U \subseteq \mathbb{C}$ be an open subset and $F$ be a holomorphic function on $U$. Let $\gamma \subset U$ be a closed curve. Then, Cauchy’s integral theorem says that $\oint_{\gamma} F(z)dz = 0$. Let $a$ be a point in the interior of the curve $\gamma$. Then, Cauchy’s integral formula says that (17)

$$F(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{F(z)}{z-a} dz \quad (7)$$

which can be proved using Cauchy’s integral theorem.

**III. CHARACTERIZATION OF CONTROLLABILITY USING THE CURL OPERATOR**

Consider the system (2). By Green’s theorem, $x_3$ measures $\int \int \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} dx_1 dx_2$ over the area enclosed by the loop obtained by the projection of the state trajectory on $\mathbb{R}^2$. We can measure the divergence or the curl of a vector field using the $x_3$ coordinate. In specific, one can define a vector field on $\mathbb{R}^2$. Suppose the projection of the state trajectory on $\mathbb{R}^2$ forms a loop. The $x_3$ coordinate measures the curl of the vector field. We now show how it is related to controllability.

The following theorem gives necessary and sufficient conditions for controllability of (4) in terms of the geometry of the underlying vector field $f$. Notice that one only needs to check arbitrary state transfer of the state variable $x_4$.

**Theorem 3.1:** Consider system (4) and let $f = (f_1, f_2, f_3)$ be a continuously differentiable vector field on $\mathbb{R}^3$. The following are equivalent

1) The system (4) is controllable.
2) There exists a closed loop $\gamma \subseteq \mathbb{R}^3$ such that the line integral $\oint_{\gamma} f \cdot dx \neq 0$.
3) $\nabla \times f \neq 0$ on $\mathbb{R}^3$.

**Proof:** (1) $\Rightarrow$ (2) Suppose $\oint_{\gamma} f \cdot dx = 0$ for every closed loop $\gamma \subseteq \mathbb{R}^3$, then one cannot do a state transfer from the origin to $(0, 0, a)$ hence, the system is uncontrollable. (2) $\Rightarrow$ (3) follows from Stokes’ theorem. Suppose (3) is satisfied. Let $S$ be a some two dimensional surface in $\mathbb{R}^3$ with the boundary $\gamma$ such that $\nabla \times f \neq 0$ on $S$ and the surface integral $\int_{S} (\nabla \times f) \cdot dS \neq 0$. Since
$x_1, x_2, x_3$ are controllable, one can choose $u_i$ ($i = 1, 2, 3$) such that the projection of the state trajectory on $\mathbb{R}^3$ is given by $\gamma$. Now since $\iint_{\mathbb{S}^2}(\nabla \times f)\cdot dS \neq 0$, using Stokes’ theorem, $x_3$ can also be steered which proves controllability.

Corollary 3.2: Consider system \([4]\) and let $f = (f_1, f_2)$ be a continuously differentiable vector field on $\mathbb{R}^2$. The following are equivalent

1. The system \([4]\) is controllable.
2. There exists a closed loop $\gamma \in \mathbb{R}^2$ such that the line integral $\int f_x f_y \cdot dx \neq 0$.
3. $\nabla \times f \neq 0$ on $\mathbb{R}^2$.

Proof: Follows from the proof of Theorem \([3\)].

Example 3.3: Consider the following system

$$x_1 = u_1, \quad x_2 = u_2, \quad x_3 = x_2 u_1 - x_1 u_2$$

from \([10]\), \([11]\), \([12]\) which describes the motion of a planar rigid body with two oscillators. Clearly since $\nabla \times f \neq 0$, the system is controllable. We now give an explicit steering of the system from $(0,0,0)$ at $t = 0$ to $(0,0,a)$ at $t = 1$, where $a > 0$. Suppose $u_1 = c_1 \cos(2\pi t)$ and $u_2 = c_2 \sin(2\pi t)$. Therefore, $x_1(t) = \frac{c_1}{2\pi} \sin(2\pi t)$ and $x_2(t) = \frac{c_2}{2\pi} (1 - \cos(2\pi t))$. Now $x_3(t)$ is given by

$$x_3(t) = \int_0^t \frac{c_1^2}{4\pi^2} (1 - \cos(2\pi \tau))^2 \cos(2\pi \tau) - \frac{c_1^2}{4\pi^2} \sin^2(2\pi \tau) d\tau \sin(2\pi \tau)$$

$$\Rightarrow x_3(1) = \int_0^1 \frac{c_1^2}{4\pi^2} (1 - \cos(2\pi \tau))^2 \cos(2\pi \tau) - \frac{c_1^2}{4\pi^2} \sin^2(2\pi \tau) d\tau$$

$$\Rightarrow x_3(1) = a = \frac{c_1^2}{4\pi^2}.$$  

Thus, for appropriate choices of $c_1, c_2$, we can steer the system from $(0,0,0)$ to $(0,0,a)$.

Remark 3.4: Notice that if $\nabla \times f = 0$, then the system is uncontrollable. Thus, if $f$ is a gradient vector field i.e. $f = \nabla \phi$ for some potential function $\phi$, then \([2]\) and \([4]\) are uncontrollable.

Consider the system defined by \([2]\). Let $f = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \nabla \phi$. Then, $\nabla \times f \neq 0$ in general. Therefore, one can construct controllable systems using a scalar potential function. Similarly, for systems defined by \([4]\), we can construct controllable systems using $f = H \nabla \phi$ where

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ or } H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and so on. There are other choices of $H$ as well which ensure that $\nabla \times (H \nabla \phi) \neq 0$ apart from the ones given above.

Corollary 3.5: Consider the system defined by \([5]\) and for $1 \leq i < j \leq n$, let $F_{ij} = (f_i(x_i, x_j), f_j(x_i, x_j))$ be continuously differentiable vector fields on $\mathbb{R}^2$. The following are equivalent

1. The system \([5]\) is controllable.
2. For each $1 \leq i < j \leq m$, there exists a closed loop $\gamma \in \mathbb{R}^2$ such that the line integral $\int f_x F_{ij} \cdot dx \neq 0$.
3. $\nabla \times F_{ij} \neq 0$ on $\mathbb{R}^2$.

Proof: Follows from the proof of Theorem \([3\]) and the previous corollary.

Remark 3.6: Notice that since $\mathbb{R}^3$ and $\mathbb{R}^2$ are simply connected, $\nabla \times f = 0 \iff f = \nabla \phi$ for some scalar function $\phi$. Therefore, \([4]\) is uncontrollable $\iff f$ is a gradient vector field.

Example 3.7: Consider a system

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = \frac{x_1}{x_1^2 + x_2^2} u_1 + \frac{x_2}{x_1^2 + x_2^2} u_2$$

defined over $\mathbb{R}^3 \setminus \{(0,0,x_3)\}$. Notice that $\nabla \times f = 0$ over $\mathbb{R}^3 \setminus \{(0,0,x_3)\}$ and the system is uncontrollable. Since the state space is not simply connected, although $\nabla \times f = 0$, $f$ is not a gradient vector field.

Suppose we want to steer \([2]\) from the origin to $(0,0,a)$. The $x_3$ coordinate is given by

$$x_3(t) = \int_0^t (f_1 u_1 + f_2 u_2) dt.$$  

Choose $u_1, u_2$ as orthogonal polynomials so that $x_1(1) = x_2(1) = 0$.

$$x_3(1) = \int_0^1 (f_1 u_1 + f_2 u_2) dt.$$  

We need to choose orthogonal $u_1, u_2$ such that the above integral is nonzero. To steer from the origin to $(a,b,c)$, choose constant inputs to steer the state to some point say $(a,b,d)$. Then use orthogonal polynomials to steer along $x_3$ without affecting $x_1, x_2$.

One can similarly steer the nonholonomic integrator \([4]\) on $\mathbb{R}^3$. (Notice that in the case above, choosing $u_1 = -f_2, u_2 = f_1$, the motion can be constrained to $x_1 - x_2$ plane.)

Example 3.8: Consider the system given by \([2]\) where $f_1(x_1,x_2) = x_1^2 - x_2^2$ and $f_2(x_1,x_2) = 2x_1 x_2$. It follows that $\nabla \times f = 4x_2 \neq 0$. Therefore, the system is controllable by Theorem \([3\]).

Suppose we want to steer the system from the origin at $t = 0$ to $(0,0,a)$ at $t = 1$ where $a > 0$. Suppose $u_1 = c_1 \cos(2\pi t)$ and $u_2 = c_2 \sin(2\pi t)$. Therefore, $x_1(t) = \frac{c_1}{2\pi} \sin(2\pi t)$ and $x_2(t) = \frac{c_1^2}{2\pi} (1 - \cos(2\pi t))$. Now $x_3(t)$ is given by

$$x_3(t) = \int_0^t \left( \frac{c_1^2}{4\pi^2} \sin^2(2\pi \tau) - \frac{c_1^2}{4\pi^2} (1 - \cos(2\pi \tau))^2 \right) c_1 \cos(2\pi \tau)$$

$$\Rightarrow x_3(1) = \int_0^1 \left( \frac{c_1^2}{4\pi^2} \sin^2(2\pi \tau) - \frac{c_1^2}{4\pi^2} (1 - \cos(2\pi \tau))^2 \right) c_1 \cos(2\pi \tau)$$

$$\Rightarrow x_3(1) = \int_0^1 \left( \frac{c_1^2}{4\pi^2} (1 - \cos(2\pi \tau))^2 \cos(2\pi \tau) + \frac{c_1^2}{2\pi} \sin^2(2\pi \tau) \right) dt$$

$$\Rightarrow x_3(1) = \int_0^1 \left( \frac{c_1^2}{4\pi^2} (1 - \cos(2\pi \tau))^2 \cos(2\pi \tau) + \frac{c_1^2}{2\pi} \sin^2(2\pi \tau) \right) dt$$

$$\Rightarrow x_3(1) = \frac{c_1^2}{4\pi^2} a = \frac{c_1^2}{4\pi^2}.$$  

Thus, for appropriate choices of $c_1, c_2$, we can steer the system from $(0,0,0)$ to $(0,0,a)$.

IV. OPTIMAL CONTROL ON THE GENERAL NONHOLONOMIC INTEGRATOR AND CLASSICAL ELECTRODYNAMICS

Consider the minimum energy control problem for the system \([2]\).

Applying Euler-Lagrange equations on the augmented Lagrangian $L = \dot{x}_1^2 + \dot{x}_2^2 + \lambda (\dot{x}_3 - f_1 \dot{x}_1 - f_2 \dot{x}_2)$, we have

$$\frac{d}{dt}(2\dot{x}_1 - \lambda f_1) = -\lambda \frac{\partial f_1}{\partial x_1} x_1 + \frac{\partial f_2}{\partial x_1} x_2$$

$$\Rightarrow 2\dot{x}_1 - \lambda \frac{\partial f_1}{\partial x_1} x_1 + \frac{\partial f_2}{\partial x_1} x_2 = 0$$

$$\frac{d}{dt}(2\dot{x}_2 - \lambda f_2) = -\lambda \frac{\partial f_1}{\partial x_2} x_1 + \frac{\partial f_2}{\partial x_2} x_2$$

$$\Rightarrow 2\dot{x}_2 - \lambda \frac{\partial f_1}{\partial x_2} x_1 + \frac{\partial f_2}{\partial x_2} x_2 = 0.$$
Therefore, (substituting λ for λ/2)

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$ \hspace{1cm} (12)

Notice that $-\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1}$ gives the curl of the vector field $(f_1, f_2)$ on $\mathbb{R}^2$. One can choose appropriate form for $x_3$ so that one obtains the divergence instead.

**Remark 4.1:** It follows from (12) and results from (2) that if the curl of the vector field $f = (f_1, f_2)$ on $\mathbb{R}^2$ is constant, then the optimal inputs for (2) are given by sinusoids for a state transfer from the origin to $(0,0,a)$.

Rewriting (12) as

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \lambda \begin{bmatrix} -\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} & \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where $f = (f_1, f_2)$, $\lambda$ is a constant. Notice that $(\nabla \times f) = (0,0,\lambda)$ where the closed loop integral is over the closed curve obtained by projection of the state trajectory in $\mathbb{R}^3$ on to $\mathbb{R}^3$. This can be interpreted as the work done by the vector field $f$ along the curve. Thus, $x_4$ measures the work done by $f$ along the projected curve. Furthermore, by Stokes’ theorem, $x_4$ also measures the flux of the magnetic field $B = \nabla \times f$ passing through any surface whose boundary is given by the closed curve obtained above by the projection of the state trajectory from the origin to $(0,0,a)$ on $\mathbb{R}^3$.

**Remark 4.3:** Consider a revised Lagrangian $L = x_1^2 + x_2^2 + x_3^2 - \lambda(f_1\dot{x}_1 + f_2\dot{x}_2 + f_3\dot{x}_3)$ where we have used dynamics in $x_4$ to conclude that the Lagrange multiplier $\lambda$ is constant. Notice that for a particle in a magnetic field $B = \nabla \times f$ moving with velocity $\dot{x}$, the Lagrangian is given by $\frac{1}{2}m\dot{x}^2 + qx.A$. In the case of the nonholonomic integrator on $\mathbb{R}^3$, $A = f$. Therefore, the revised Lagrangian can be identified with the Lagrangian for electrodynamics. Thus, the optimal control problem is also a classical mechanics problem. Now for the revised Lagrangian, the Hamiltonian is preserved and we have Hamiltonian dynamics.

A. Incorporating a drift term in the nonholonomic integrator and its relation to the force on a particle in an electromagnetic field

Consider the following system with a drift

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2, x_3)u_1 + f_2(x_1, x_2, x_3)u_2 + f_3(x_1, x_2, x_3)u_3 \\ g(x_1, x_2, x_3)u_1 + f_1(x_1, x_2, x_3)u_2 + f_2(x_1, x_2, x_3)u_3 \end{bmatrix}$$ \hspace{1cm} (19)

and the minimum energy control problem of minimizing $\int_0^1 (u_1^2 + u_2^2 + u_3^2)dt$ on (4). The augmented Lagrangian is $L = x_1^2 + x_2^2 + x_3^2 + \lambda((x_4 - g - f_1\dot{x}_1 - f_2\dot{x}_2 - f_3\dot{x}_3))$. Using Euler-Lagrange equations, one obtains

$$\begin{align*}
2\dot{x}_1 &= -\lambda \frac{\partial g}{\partial x_1} + \lambda \frac{\partial f_1}{\partial x_2} x_2 + \lambda \left( \frac{\partial f_1}{\partial x_3} - \frac{\partial f_2}{\partial x_3} \right) x_3 \\
2\dot{x}_2 &= -\lambda \frac{\partial f_1}{\partial x_2} x_1 + \lambda \frac{\partial f_1}{\partial x_3} x_3 + \lambda \left( -\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_1} \right) x_1 \\
2\dot{x}_3 &= -\lambda \frac{\partial f_1}{\partial x_3} x_1 + \lambda \frac{\partial f_1}{\partial x_1} x_1 + \lambda \left( -\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_2} \right) x_2
\end{align*}$$ \hspace{1cm} (15-17)

and $\lambda$ is a constant. Substituting $\lambda$ for $\lambda/2$, the above equations can be written in the matrix form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \lambda \begin{bmatrix} 0 & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_1} \\ -\frac{\partial f_1}{\partial x_2} & 0 & -\frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_1} \\ -\frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_1} & 0 & -\frac{\partial f_1}{\partial x_2} \\ -\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

where $x_4$ is a constant. Notice that (19) is controllable $\iff \nabla \times f \neq 0$ using similar arguments used in Theorem 4.1. If $\nabla \times f = 0$, then $x_4$ cannot be steered arbitrarily.

**Remark 4.4:** One can show that (19) is controllable $\iff \nabla \times f \neq 0$ using similar arguments used in Theorem 4.1. If $\nabla \times f = 0$, then $x_4$ cannot be steered arbitrarily.

**Remark 4.5:** Consider a 4-vector potential $(\phi/c, A)$ in analogy with the 4-vector potential $(\phi/c, A)$ in electrodynamics. The
Lorenz gauge condition is given by $\frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} + \nabla \mathbf{A} = 0$. Thus, $\phi/c = g$ and $\mathbf{A} = \mathbf{f}$. Since only $\nabla \times \mathbf{f}$ decides the controllability of the system, one can ignore the curl free part in the Helmholtz decomposition and $\mathbf{f}$ can be assumed to be solenoidal. Therefore, $\nabla \mathbf{f} = 0$. Moreover, $\frac{\partial \mathbf{f}}{\partial t} = 0$ which implies that Lorenz gauge conditions are satisfied in the above case as well. If $g$ has an explicit time dependence, then $\nabla \mathbf{f} \neq 0$ but we do not consider this case here.

Note that in electrodynamics, $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = \frac{\partial \mathbf{A}}{\partial t} - \nabla \phi$. Now for the control systems considered above, if $g = 0$ and $\mathbf{f}$ has no explicit time dependence, then $\mathbf{E} = 0$ and $\mathbf{B} = \nabla \times \mathbf{f}$. When $g \neq 0$ or if $\mathbf{f}$ is time dependent, then $\mathbf{E} \neq 0$. Notice that controllability of $(2)$, $(4)$ and $(19)$ can be related to the presence of a magnetic field. In the absence of magnetic fields i.e., $\nabla \times \mathbf{f} = 0$, the system becomes uncontrollable.

B. State dependent cost function and electrodynamics analogy

Consider the following optimal control problem on $(4)$ involving a quadratic cost on the first three components of the state

$$ J = \int_0^1 (x_1^2 + x_2^2 + x_3^2 + u_1^2 + u_2^2 + u_3^2) \, dt $$

The augmented Lagrangian is $L = x_1^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + \lambda (x_4 - f_1 x_1 - f_2 x_2 - f_3 x_3)$. Using Euler-Lagrange equations, one obtains

$$ 2\dot{x} = 2x + \lambda (\nabla \times \mathbf{f}) \times \dot{x} $$

where $x \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$. One could consider a state dependent term $g(x_1, x_2, x_3) > 0$ instead of the quadratic term $x_1^2 + x_2^2 + x_3^2$ in the cost function $(24)$ to obtain

$$ 2\dot{x} = \nabla g(x) + \lambda (\nabla \times \mathbf{f}) \times x. $$

Thus, the electrodynamics analogy can be obtained for the general nonholonomic integrator with a drift term or for the general nonholonomic integrator with a state dependent term $g(x_1, x_2, x_3) > 0$ in the cost function as shown in Equation $(24)$ where $g(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$. It is clear that all these optimal control problems on general versions of nonholonomic integrator can be identified with solving an electrodynamics problem.

C. Optimal control of a planar rigid body with two oscillators

We now demonstrate with an example, how to reduce the minimum energy optimal control problem (which is also related to finding the trajectory of a particle in a magnetic field) to solving an elliptic integral.

**Example 4.6:** Consider the following system which describes the motion of a planar rigid body with two oscillators. (Refer Example $(3.3)$).

$$ x_1 = u_1, \quad x_2 = u_2, \quad x_3 = x_2^2 u_1 - x_1^2 u_2. $$

Here, we want to optimize the following cost function given by

$$ J = \int_0^1 (u_1^2 + u_2^2) \, dt $$

subject to the constraints that $(x_1(0), x_2(0), x_3(0)) = (0, 0, 0)$ and $(x_1(1), x_2(1), x_3(1)) = (0, 0, c)$. Here, $f = (x_2^2, -x_1^2)$ and the equations of motion are

$$ 2\dot{x} = \lambda (\nabla \times \mathbf{f}) \times \dot{x} $$

(27)

$$ \dot{x}_1 = \lambda (x_1 + x_2) x_2 $$

(28)

$$ \dot{x}_2 = -\lambda (x_1 + x_2) x_1 $$

(29)

Now, we define a change variables as follows $y = x_1 - x_2$, $z = x_1 + x_2$, then equations of motion reduce to

$$ \ddot{y} = \lambda z \ddot{z} $$

(30)

$$ \ddot{z} = -\lambda \dot{y} $$

(31)

$$ \dot{y}^2 + \dot{z}^2 = 0 $$

(32)

$$ \dot{y}^2 + \dot{z}^2 = r^2 $$

(33)

where $r$ is a constant. Thus, integrating Equation $(30)$ and then substituting the result into Equation $(33)$, we obtain

$$ \dot{y} = \lambda \frac{\dot{z}}{r} $$

(34)

$$ \dot{z} = r^2 - (\lambda \frac{\dot{z}}{r})^2 $$

(35)

$$ dt = \frac{\sqrt{(r + c)^2 + (r - c)^2}}{\sqrt{(r + c)^2 + (r - c)^2} \sin \theta} \, \cos \theta \, d\theta $$

(36)

$$ dt = \frac{2 \, d\theta}{(r + c) \sqrt{1 + \kappa^2 \sin^2 \theta}} $$

(37)

$$ dt = \frac{2 \, d\theta}{(r + c) \sqrt{1 + \kappa^2 \sin^2 \theta}} $$

(38)

Now let $\kappa = \sqrt{\frac{r - c}{r + c}}$, we have,

$$ dt = \frac{2 \, d\theta}{(r + c) \sqrt{1 + \kappa^2 \sin^2 \theta}} $$

(39)

$$ \kappa = \sqrt{\frac{r - c}{r + c}} $$

Let $u = \frac{\pi}{2} - \theta$, then,

$$ \sqrt{(r + c) \kappa} \, du = - \frac{du}{\sqrt{1 + \kappa^2 (1 - \sin^2 u)}} = - \frac{du}{\sqrt{1 + \kappa^2 (1 - \sin^2 u)}} $$

(40)

$$ \sqrt{(r + c) \kappa} \, du = \frac{du}{\sqrt{1 + \kappa^2 (1 - \sin^2 u)}} = \frac{du}{\sqrt{1 + \kappa^2 (1 - \sin^2 u)}} $$

(41)

$$ \sqrt{(r + c) \kappa^2} \, dt + b = F(u \frac{\kappa^2}{\kappa^2 + 1}) $$

(42)

where, in the Equation $(42)$, $F(\psi | \kappa^2)$ represents the incomplete elliptic integral of the first kind. Thus, the optimal control problem reduces to solving the elliptic integral. For further details on elliptic integrals, we refer the reader to $(20)$. 
Consider the complex function \( F(x_1 + ix_2) = f_2(x_1, x_2) + if_1(x_1, x_2) \) where \( f_1(x_1, x_2) \) and \( f_2(x_1, x_2) \) are both real valued functions corresponding to the vector field \( \mathbf{f} = (f_1, f_2) \). Let \( z = x_1 + ix_2 \) and \( u_C = u_1 + iu_2 \). Let \( \gamma \) be a closed curve in \( \mathbb{C} \). Then,

\[
\oint F \, u_C \, dt = \oint ((f_2u_1 - f_1u_2) + if_1u_1 + f_2u_2) \, dt = \oint ((f_2dx_1 - f_1dx_2) + if_1dx_1 + f_2dx_2). \quad (43)
\]

The function \( F \) is defined in such a way that the imaginary part of \( F \, u_C \) can be identified with the curl of \( f \) (since \( \oint f_2dx_1 - f_1dx_2 = \int S \nabla \times f \, dS \)), \( S \) being the area enclosed by the closed curve \( \gamma \) whereas; the imaginary part of the line integral can be identified with the curl of \( f \) (since \( \oint f_2dx_1 + f_2dx_2 = \int S \nabla \times f \, dS \)).

**Lemma 5.1:** Consider \( \mathbb{C} \) and let \( F(x_1 + ix_2) = f_2(x_1, x_2) + if_1(x_1, x_2) \). If \( F \) is holomorphic, then \( \mathbb{C} \) is uncontrolled.

**Proof:** The proof follows from Cauchy’s integral theorem since for holomorphic functions, the integral over a closed loop in the complex plane is zero. Thus, the \( x_3 \) co-ordinate is uncontrollable.

\[
\oint F \, u_C \, dt = 0
\]

**Lemma 5.2:** Consider \( \mathbb{C} \) and let \( F(x_1 + ix_2) = f_2(x_1, x_2) + if_1(x_1, x_2) \) such that \( F \) is not holomorphic. Let \( \gamma \) be a closed loop in the complex plane enclosing the origin such that \( F \) has a pole in the region enclosed by \( \gamma \) and suppose \( \gamma \) has a nonzero winding number. If the residue of \( F \) at the point inside \( \gamma \) is zero, then \( \mathbb{C} \) is controllable.

**Proof:** Since \( x_1, x_2 \) are controllable, one can choose \( u_1, u_2 \) such that the projection of the state trajectory on the complex plane is given by \( \gamma \). Notice that \( x_3 = f_1 u_1 + f_2 u_2 \) and by the residue theorem from complex analysis, \( x_3 \) can be steered if the residue of \( F \) at the point inside \( \gamma \) is nonzero real number.

**Example 5.3:** Consider the classical nonholonomic integrator with \( f_1 = -x_2 \) and \( f_2 = x_1 \). Therefore, \( F = f_2 + if_1 = x_1 - ix_2 \). In fact, \( F(z) = \bar{z} \) is the complex conjugate of \( z \). It can be easily checked that the Cauchy-Riemann equations are not satisfied and the function \( F \) is not holomorphic. This agrees with the fact that the classical nonholonomic integrator is controllable. Let \( \gamma \) be the unit circle centered at \( z = 0 \). Notice that

\[
2\pi i = \oint F \, u_C = \oint (x_1u_1 + x_2u_2) + i(x_1u_2 - x_2u_1) \, dt.
\]

This implies that the purely imaginary part of \( F \, u_C \) which captures the dynamics of \( x_3 \) variable of the nonholonomic integrator is controllable. However, \( \oint (x_1u_1 + x_2u_2) = 0 \) for every closed curve \( \gamma \). Therefore, if \( x_3 = (x_1u_1 + x_2u_2) \), then the system is not controllable as the closed loop complex integral is always purely imaginary. This can also be verified using the fact that for \( x_3 = x_1u_1 + x_2u_2 \), \( F = (x_1, x_2) \) and \( \nabla \times F = 0 \) which implies uncontrollability. Thus, uncontrollability in this case is a consequence of the curl of \( f \) being zero and from complex analytic viewpoint, it follows from the residue theorem.

**Example 5.4:** Suppose \( F = \frac{1}{z} = \frac{x_1 - iy_2}{x_1^2 + y_2^2} \) which is not holomorphic and has a pole at the origin. The integral along the closed loop (which is the unit circle) using Cauchy’s integral formula is given by \( 2\pi i \). Consider a system

\[
\begin{align*}
\dot{x}_1 &= u_1, \\
\dot{x}_2 &= u_2, \\
\dot{x}_3 &= \frac{x_1}{x_1^2 + y_2^2} u_2 - \frac{x_2}{x_1^2 + y_2^2} u_1
\end{align*}
\]

defined over \( \mathbb{R}^3 \setminus \{0,0,x_3\} \). Complexify this system using \( z = x_1 + iy_2 \) and \( u_C = u_1 + iu_2 \). Suppose \( F = \frac{1}{z} = \frac{x_1 - iy_2}{x_1^2 + y_2^2} = f_2 + if_1 \), therefore, \( F \, u_C = \frac{x_1 u_1 + x_2 u_2}{x_1^2 + y_2^2} + i(\frac{-x_2 u_1 + x_1 u_2}{x_1^2 + y_2^2}) \). Let \( \gamma \) be any closed loop in \( \mathbb{C} \), which encloses the origin and let \( T_p \) be the time period of curve traversal by the chosen inputs \( (u_1, u_2) \). Let \( \dot{z} = u_C \) and \( \dot{x}_3 = \text{Im}(F \, u_C) \). It follows that

\[
x_3(nT_p) - x_3(0) = \oint (\frac{x_1}{x_1^2 + y_2^2} u_2 - \frac{x_2}{x_1^2 + y_2^2} u_1) \, dt = 2\pi \text{Im}(F \, u_C) = 0
\]

and the system is uncontrollable since the real part of the residue is zero and the dynamics in \( x_3 \) is given by \( \text{Re}(F \, u_C) \).

Now consider a nonholonomic system on the complex plane defined as

\[
\begin{align*}
\dot{z} &= u_C, \\
\dot{w} &= F \, u_C
\end{align*}
\]

where \( z_1 = x_1 + iy_2, u_C = u_1 + iu_2, F = f_2 + if_1, w = w_1 + iw_2 \) and \( x_1, x_2, u_1, u_2, w_1, w_2 \) are real variables whereas \( f_1, f_2 \) are functions of real variables \( x_1, x_2 \). In the four dimensional real vector space, this system is represented as follows

\[
\begin{align*}
\dot{x}_1 &= u_1, \\
\dot{x}_2 &= u_2, \\
\dot{w}_1 &= f_2 u_1 - f_1 u_2, \\
\dot{w}_2 &= f_1 u_1 + f_2 u_2.
\end{align*}
\]

We now demonstrate how to control a family of complex control systems defined above where the complex function \( F = f_2 + if_1 \) is not holomorphic.

**Example 5.5:** Consider a family of nonholomorphic functions \( F(z) = (\bar{z})^n, n \in \mathbb{N} \) and \( n > 1 \). Consider the nonholonomic system given by

\[
\begin{align*}
\dot{z} &= u_C, \\
\dot{w} &= F \, u_C
\end{align*}
\]

where \( z, w \) are defined in \( \mathbb{C} \). Let \( \gamma_a \) be the unit circle centered at \( z = a \); then we have (by substituting \( \bar{v} = z - a \) and using the residue theorem)

\[
\oint F \, u_C = \oint (\bar{z})^n \, dz = \oint_{\gamma_a} (\bar{v} + \bar{a})^n \, dv = \oint_{\gamma} \frac{1}{\bar{v} + \bar{a}}^n \, dv = 2n\pi i(\bar{a}^{n-1}).
\]

Thus, the dynamics of \( w_2 \) can be controlled choosing \( a = 1 \) and appropriate \( u_1 \) and \( u_2 \). Note that \( w_1 \) remains unaffected since the real part of the closed loop integral considered above is zero. Now to control \( w_1 \), we choose a different complex point say \( a = e^{\frac{i\pi}{2}} \).

Then, \( \bar{a}^{n-1} = e^{-i(n-1)\frac{\pi}{2}} = -e^{\frac{i\pi}{2}} \). Notice that \( e^{\frac{i\pi}{2}} \) has both real and imaginary part for all \( n \geq 2 \), and for \( n = 2 \) it has only the purely
imaginary part. Furthermore, the residues obtained above, when considered as real vectors form a two dimensional real subspace. Thus, the dynamics given by $\dot{w}_1 = \text{Re}\{2\pi \cdot u_C\}$ is controllable and the system with both $w_1$ and $w_2$ is also controllable, as we have two independent real directions associated with the two residues. Now we propose the following algorithm to steer the system from the origin to $(0, 0, a, b)$, where $a, b \neq 0$.

1) Choose the inputs $u_1 = 2\pi \sin(2\pi t)$, $u_2(t) = 2\pi \cos(2\pi t)$, then $x_1 = 1 - \cos(2\pi t)$, $x_2 = \sin(2\pi t)$. This realizes the curve $\gamma_1$ in $x_1 - x_2$ plane and by the discussion above, $w_2(1) = 2n\pi$ and $w_1(1) = 0$.

2) At $t = 1$, change the inputs to $u_1 = \frac{\pi}{n} \sin\left(\frac{\pi}{n}\right)$, $u_2 = -\frac{\pi}{n} \cos\left(\frac{\pi}{n}\right)$, then $x_1(t) = -\cos\left(\frac{\pi}{n} t\right) + \cos\left(\frac{\pi}{n}\right)$, $x_2(t) = -\sin\left(\frac{\pi}{n} t\right) + \sin\left(\frac{\pi}{n}\right)$, this realizes the curve $\gamma_2$, where $a = e^{i\pi n}$ in $x_1 - x_2$ plane and by the above discussion at $t = 1 + 2n$, this $\gamma_2$ curve is looped around once, and $w_1(1 + 2n) = 2n\pi \sin\left(\frac{\pi}{n}\right)$ and $w_2(1 + 2n) = 2n\pi - 2n\pi \cos\left(\frac{\pi}{n}\right)$.

3) Thus, the steering of this system from the origin to $(0, 0, a, b)$ can be done by scaling the inputs and scaling the time taken to traverse the curves $\gamma_1$, such that $(a, b) = c_1(0, 2n\pi) + c_2(2n\pi \sin\left(\frac{\pi}{n}\right), -2n\pi \cos\left(\frac{\pi}{n}\right))$ where $c_1$ and $c_2$ are scaling coefficients of $u_1$ and $u_2$ respectively.

**Remark 5.6:** For a control system given by (44), for any continuous inputs $u_1$ and $u_2$, we cannot restrict the dynamics of (44) to the $x_1 - x_2$ plane. This can be justified as follows. Consider the dynamics of coordinates $w_1$ and $w_2$, we have

$$\dot{w}_1 = f_1 u_1 + f_2 u_2 = (f_1, f_2) \cdot (u_1, u_2)$$

$$\dot{w}_2 = f_1 u_2 - f_2 u_1 = (-f_2, f_1) \cdot (u_1, u_2)$$

To restrict ourselves to $x_1 - x_2$ we need to have $\dot{w}_1 = \dot{w}_2 = 0$. Since $(f_1, f_2)$ and $(-f_2, f_1)$ are orthogonal vectors, any $(u_1, u_2)$ cannot be both non-zero and be perpendicular to both of these vectors.

**Remark 5.7:** The complex analytic results mentioned in this section also hold for nonholonomic systems given by (5) by considering pairwise systems on the complex plane $\mathbb{C}$ for all $i, j$ pairs $1 \leq i < j \leq n$.

**VI. CONCLUSIONS**

We considered generalizations of the classical nonholonomic integrator to define some specific nonholonomic systems using the notion of vector fields. We obtained necessary and sufficient conditions for controllability of these systems using geometric concepts such as the curl of a vector field. In specific, we showed that controllability is equivalent to the curl of the underlying vector field being nonzero. We also considered minimum energy optimal control problems on these general nonholonomic integrators and showed that the optimal trajectories are same as the trajectory of a charged particle in a magnetic field. We also considered a specific system with a drift term and showed that the optimal trajectories are given by a charged particle in an electromagnetic field. We then included a specific state dependent cost function term in the Lagrangian and showed that optimal trajectories are again given by the trajectory of a particle in an electromagnetic field. We then gave a complex analytic viewpoint to nonholonomic integrator and its generalizations and use properties such as holomorphicity, Cauchy’s integral theorem and the residue theorem from complex analysis to characterize controllability.

The future work involves extending these ideas for more general nonholonomic systems.

**REFERENCES**

[1] R. W. Brockett, “Control theory and singular Riemannian geometry,” in New Directions in Applied Mathematics, pp. 11–27, 1981.

[2] R. Murray and S. Sastry, “Nonholonomic Motion Planning: Steering Using Sinusoids,” IEEE Transactions on Automatic Control, vol. 38, no. 5, pp. 700–716, 1993.

[3] S. Sastry, Nonlinear Systems: Analysis, Stability and Control. Springer, 1999.

[4] S. M. LaValle, Planning algorithms. Cambridge University Press, 2006.

[5] M. Belabbas and S. Liu, “New Method for Motion Planning for Nonholonomic Systems using Partial Differential Equations,” in IEEE, ACC, pp. 4189–4194, 2017.

[6] S. Liu, Y. Fan, and M.-A. Belabbas, “Geometric Motion Planning for Affine Control Systems with Indefinite Boundary Conditions and Free Terminal Time,” arXiv:2001.04540v1, pp. 1–7, 2020.

[7] S. Liu, Y. Fan, and M.-A. Belabbas, “Affine Geometric Heat Flow and Motion Planning for Dynamic Systems,” IFAC PapersOnline, vol. 52, no. 16, pp. 168–173, 2019.

[8] H. C. Henninger and J. D. Biggs, “Optimal under-actuated kinematic motion planning on the $e$–group,” Automatica, vol. 90, pp. 185–195, 2018.

[9] J. D. Biggs and N. Horri, “Optimal geometric motion planning for a spin-stabilized spacecraft,” Systems & Control Letters, vol. 61, pp. 609–616, 2012.

[10] R. Yang, P. Krishnaprasad, and W. Dayawansa, “Optimal control of a rigid body with two oscillators,” in In W. F. Shadwick, P.S. Krishnaprasad, & T.S. Ratiu (Eds.), Mechanics day, fields institute communications, pp. 233–260, 1996.

[11] J. Carinena, J. Clement-Gallardo, and A. Ramos, “Motion on lie groups and its applications in control theory,” Reports on Mathematical Physics, vol. 51, pp. 159–170, 2003.

[12] A. Zuyev and V. Grushkovskaya, “Motion planning for control-affine systems satisfying low-order controllability conditions,” International Journal of Control, vol. 90, no. 11, pp. 2517–2537, 2017.

[13] P. Shvaramakrishna and A. S. A. Dilip, “Steering nonholonomic integrator using orthogonal polynomials,” arXiv:2006.01379, pp. 1–10, 2020.

[14] R. W. Brockett and L. Dai, “Non-holonomic Kinematics and the Role of Elliptic Functions in Constructive Controllability,” Nonholonomic Motion Planning, edited by Z. Li, J.F. Canny, pp. 11–27, 1993.

[15] R. Murray, Z. Li, and S. Sastry, A Mathematical Introduction to Robotic Manipulation. CRC Press, 1994.

[16] L. Gurvits and Z. Li, “Smooth Time-Periodic Feedback Solutions for Nonholonomic Motion Planning,” Nonholonomic Motion Planning, edited by Z. Li, J.F. Canny, pp. 11–27, 1993.

[17] G. Strang, Introduction to Applied Mathematics. Wellesley-Cambridge Press, 1986.

[18] V. I. Arnold, Mathematical Methods of Classical Mechanics. Springer, second ed., 1989.

[19] D. J. Griffiths, Introduction to Electrodynamics. Pearson, fourth ed., 2012.

[20] P. F. Byrd and M. D. Friedman, Handbook of elliptic integrals for engineers and physicists, vol. 67 of Grundlehren der mathematischen Wissenschaften. Springer, 2013.