PRIMAL-DUAL REGRESSION APPROACH FOR MARKOV DECISION PROCESSES WITH GENERAL STATE AND ACTION SPACE

DENIS BELOMESTNY\textsuperscript{1} AND JOHN SCHOENMAKERS\textsuperscript{2}

Abstract. We develop a regression based primal-dual martingale approach for solving finite time horizon MDPs with general state and action space. As a result, our method allows for the construction of tight upper and lower biased approximations of the value functions, and, provides tight approximations to the optimal policy. In particular, we prove tight error bounds for the estimated duality gap featuring polynomial dependence on the time horizon, and sublinear dependence on the cardinality/dimension of the possibly infinite state and action space. From a computational point of view the proposed method is efficient since, in contrast to usual duality-based methods for optimal control problems in the literature, the Monte Carlo procedures here involved do not require nested simulations.

1. Introduction

Markov decision processes (MDPs) provide a general framework for modeling sequential decision-making under uncertainty. A large number of practical problems from various areas such as economics, finance, and machine learning can be viewed as MDPs. For a classical reference we refer to [18], and for MDPs with application to finance, see [5]. The aim is usually to find an optimal policy that maximizes the expected accumulated rewards (or minimizes the expected accumulated costs). In principle, these Markov decision problems can be solved by a dynamic programming approach; however, in practice, this approach suffers from the so-called “curse of dimensionality” and the “curse of horizon” meaning that the complexity of the program increases exponentially in the dimension of the problem (dimensions of the state and action spaces) and the horizon (at least for problems without discounting). While the curse of dimensionality is known to be unavoidable in the case of general continuous state/action spaces, the possibility of beating the curse of the horizon remains an open issue.

A natural performance metric is given by the value function $V^\pi$ which is the expected total reward of the agent following $\pi$. Unfortunately, even a precise knowledge of $V^\pi$ does not provide reliable information on how far is the policy $\pi$ from the optimal one. To address this issue a popular quality measure is the *regret* of the algorithm which is the difference between the total sum of rewards accumulated when following the optimal policy and the sum of rewards obtained when following the current policy $\pi$. In the setting of finite state- and action space MDPs there is a variety of regret bounds for popular RL algorithms like Q-learning [16], optimistic value iteration [3], and many others. Unfortunately, regret bounds beyond the discrete setup are much less common in the literature. Even more crucial drawback of the regret-based comparison is that regret bounds are typically pessimistic and rely on the unknown quantities of the underlying MDP’s. A simpler, but related, quantity is the *suboptimality gap (policy error)* $\Delta^\pi(x) := V^*(x) - V^\pi(x)$. Since we do not know $V^*$, the suboptimality gap can not be calculated directly. There is a vast amount of literature devoted to theoretical guarantees for $\Delta^\pi(x)$, see e.g. [2], [24], [17] and references therein. However, these bounds share the same drawbacks as the regret bounds. Moreover, known bounds do not apply to the general policy $\pi$ and depend heavily on the particular algorithm which produced it. For instance, in Approximate Policy Iteration (API, [11]) all existing bounds for $\Delta^\pi(x)$ depend on the one-step error induced by the approximation of the action-value function. This one-step error is difficult to quantify since it depends on the unknown smoothness properties of the action-value function. Similarly, in policy gradient methods (see e.g. [23]), there is always an approximation error due to the choice of the family of policies that can be hardly quantified. Though the accuracy
of a suboptimal policy is generally unknown, the lack of theoretical guarantees on a suboptimal policy can be potentially addressed by providing a dual bound, that is, an upper bound (or lower bound) on the optimal expected reward (or cost).

The last decades have seen a high development of duality approaches for optimal stopping and control problems, initiated by the works of [21] and [15] in the context of pricing of American and Bermudan options. Essentially, in the dual approach one minimizes a certain dual martingale representation corresponding to the problem under consideration over a set of martingales or martingale type elements. In general terms, the dual version of an optimal control problem \( V_0^* = \sup_{\alpha} E[R(\alpha)] \) for a reward \( R \) depending on adapted policies \( \alpha \) may be formulated as

\[
V_0^* = \inf_{\text{martingales } M(\alpha)} \mathbb{E}\left[ \sup_{\alpha \text{ in control space}} (R(\alpha) - M(\alpha)) \right].
\]

Thus, in the dual approach one seeks for optimal martingales rather than optimal policies. For optimal stopping problems, [11] showed how to compute martingales using stopping rules via nested Monte Carlo simulations. In [20], the dual representation for optimal stopping (hence American options) was generalized to Markovian control problems. Somewhat later [13] presented a dual representation for quite general control problems in terms of the so-called information relaxation and martingale penalties. On the other hand, the dual representation for optimal stopping was generalized to multiple stopping in [22] and [10]. As a numerical approach to [20], [9] applied regression methods to solve Markov decision problems that can be seen, in a sense, as a generalization of [1]. However, it should be noted that in the convergence analysis of [9], the primal value function estimates showed exponential dependence on the time horizon, and the corresponding dual algorithm was based on nested simulations while its convergence was not analyzed there. Generally speaking, to the best of our knowledge, all error bounds for the primal/dual value function estimates available in the literature so far show exponential dependence on the horizon at least in the case of finite horizon undiscounted optimal control problems, e.g. see also [26].

In this paper, we propose a novel approach to constructing valid dual upper bounds on the optimal value function via simulations and pseudo regression in the case of finite horizon MDPs with general (possibly continuous) state and action spaces. This approach includes the construction of primal value functions via a backwardly structured pseudo regression procedure based on a properly chosen reference distribution (measure). We thus avoid the delicate problem of inverting the empirical covariance matrices. Note that in the context of optimal stopping, a similar primal procedure was proposed in [6], though with accuracy estimates exploding with the number of exercise dates or time horizon. As for the dual part of our algorithm, we avoid nested Monte Carlo simulation that were used in many dual-type methods proposed in the literature so far, see for instance the path-wise optimization approach for MDPs in [13] and [12] for an overview. Instead, for constructing the martingale elements we propose to combine a pointwise pseudo regression approach with a suitable interpolation method such that the martingale property is preserved. Furthermore, we provide a rigorous convergence analysis showing that the error of approximating the true value function via estimated dual value function (duality gap) depends at most polynomially on the time horizon. Moreover, we show that the stochastic part of the error depends sublinearly on the dimension (or cardinality in the finite case) of the state and action spaces. Let us also mention [27] for another approach to avoid nested simulations when estimating the conditional expectations, hence the martingale elements, inside the dual representation. However, [27] left the issue of bounding the duality gap in terms of the error bounds on the primal value functions as an open problem. In this respect, we have solved this problem within the context of the algorithm proposed in this paper.

The paper is organized as follows. The basic setup of the Markov Decision Process and the well-known representations for its maximal expected reward is given in Section 2. Section 3 recalls the dual representation for an MDP from the literature. The primal pseudo regression algorithm for the value functions is described in Section 4 whereas the dual regression algorithm is presented in Section 5. Section 6 and Section 7 are dedicated to the convergence analysis of the primal and dual algorithm, respectively. Appendix A introduces some auxiliary notions needed to formulate an auxiliary result in Appendix B stemming from the theory of empirical processes.
2. Setup and basic properties of the Markov Decision Process

We consider the discrete time finite horizon Markov Decision Process (MDP), given by the tuple

\[ \mathcal{M} = (S, A, (P_h)_{h\in[H]}, (R_h)_{h\in[H]}, F, H), \]

made up by the following items:

- a measurable state space \( (S, S) \) which may be finite or infinite;
- a measurable action space \( (A, A) \) which may be finite or infinite;
- an integer \( H \) which defines the horizon of the problem;
- for each \( h \in [H] \), with \( [H] := \{1, \ldots, H\} \), a time dependent transition function \( P_h : S \times A \to \mathcal{P}(S) \) where \( \mathcal{P}(S) \) is the space of probability measures on \( (S, S) \);
- a time dependent reward function \( R_h : S \times A \to \mathbb{R} \), where \( R_h(x, a) \) is the immediate reward associated with taking action \( a \in A \) in state \( x \in S \) at time step \( h \in [H] \);
- a terminal reward \( F : S \to \mathbb{R} \).

Introduce a filtered probability space \( \Omega := (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[H]}, \mathbb{P}) \) with

\[ (2.1) \quad \Omega := (S \times A)^{\{0,\ldots,H\}}, \quad \mathcal{F} := (S \otimes A)^{\{0,\ldots,H\}}, \quad (\mathcal{F}_t)_{t\in[H]} := ((S \otimes A)^{\{0,\ldots,t\}})_{t\in[H]} \]

For a fixed policy \( \pi = (\pi_0, \ldots, \pi_{H-1}) \) with \( \pi_t : S \to \mathcal{P}(A) \), we consider an adapted controlled process \( (S_t, A_t)_{t=h,\ldots,0} \) on \( \Omega \) satisfying \( S_0 \in S \), \( A_0 \sim \pi_0(S_0) \), and

\[ S_{t+1} \sim P_{t+1}(\cdot | S_t, A_t), \quad A_t \sim \pi_t(S_t), \quad t = 0, \ldots, H-1. \]

The expected reward of this MDP due to the chosen policy \( \pi \) is given by

\[ V^\pi_0(x) := \mathbb{E}_{\pi,x} \left[ \sum_{t=0}^{H-1} R_t(S_t, A_t) + F(S_H) \right], \quad x \in S \]

where \( \mathbb{E}_{\pi,x} \) stands for expectation induced by the policy \( \pi \) and transition kernels \( P_t, t \in [H] \), conditional on the event \( S_0 = x \). The goal of the Markov decision problem is to determine the maximal expected reward:

\[ (2.2) \quad V^*_0 := \sup_{\pi} \mathbb{E}_{\pi,x} \left[ \sum_{t=0}^{H-1} R_t(S_t, A_t) + F(S_H) \right] = \sup_{\pi} V^\pi_0(x_0). \]

Let us introduce for a generic time \( h \in [H] \), the value function due to the policy \( \pi \),

\[ V^\pi_h(x) := \mathbb{E}_{\pi,x} \left[ \sum_{t=h}^{H-1} R_t(S_t, A_t) + F(S_H) \right| S_h = x], \quad x \in S. \]

Furthermore, let

\[ (2.3) \quad V^*_h(x) := \sup_{\pi} V^\pi_h(x) \]

be the optimal value function at \( h \in [H] \). It is well known that under weak conditions, there exists an optimal policy solving \( (2.3) \) which depends on \( S_t \) in a deterministic way. In this case, we shall write \( \pi^*_t(S_t) \) for some mappings \( \pi^*_t : S \to A \). One has the following result, see [18].

**Theorem 1.** Let \( x \in S \) be fixed. It holds \( V^*_H(x) = F(x) \), and

\[ (2.4) \quad V^*_h(x) = \sup_{a \in A} \left( R_h(x, a) + \mathbb{E}_{S_{h+1} \sim P_{h+1}(\cdot | x, a)} \left[ V^*_h(S_{h+1}) \right] \right), \quad h = H-1, \ldots, 0. \]

Furthermore, if \( R_h \) is continuous and the action space is compact, the supremum in \( (2.4) \) is attained at some deterministic optimal action \( a^*_h = \pi^*_h(x) \).

---

1. We further write \( [H] := \{0, 1, \ldots, H\} \) etc.
2. In order to avoid irrelevant measure theoretic technicalities it is assumed that our probability space is supported by discrete time processes, rather than Wiener processes for instance. Nonetheless, it is possible to involve larger probability spaces without essentially affecting the results in this paper.
Let us further introduce recursively $Q^*_h(x, a) = F(x)$, and

$$Q^*_h(x, a) := R_h(x, a) + \mathbb{E}_{S_{h+1} \sim P_{h+1}(|x, a)} \left[ \sup_{a' \in A} Q^*_{h+1}(S_{h+1}, a') \right], \quad h = H - 1, \ldots, 0.$$ 

Then $Q^*_h(x, a)$ is called the optimal state-action function (Q-function) and one thus has

$$V^*_h(x) = \sup_{a \in A} Q^*_h(x, a), \quad \pi^*_h(x) \in \arg \max_{a \in A} Q^*_h(x, a), \quad \text{for } h \in [H].$$

Finally, note that the optimal value function $V^*$ satisfies due to Theorem 1

$$V^*_h(x) = T_h V^*_{h+1}(x), \quad h \in [H],$$

where $T_h V(x) := \sup_{a \in A} \left( R_h(x, a) + P^a_{h+1} V(x) \right)$ with $P^a_{h+1} V(x) := \mathbb{E}_{S_{h+1} \sim P_{h+1}(|x, a)} [V(S_{h+1})].$

### 3. Dual representation

Let us denote by $a_{< t}$ the deterministic vector of actions $a_{< t} = (a_0, \ldots, a_{t-1}) \in A^t$, similarly $a_{\leq t}$ etc., and denote with $S_t \equiv (S_t(a_{\leq t})) \in \{0, \ldots, H\}$ the process defined (in distribution) via

$$S_0 = x, \quad S_{t+1} = S_{t+1}(a_{< t+1}) \sim P_{t+1}(|S_t, a_t), \quad t = 0, \ldots, H - 1.$$ 

Let us also denote by $\Xi$ the class of $H$-tuples $\xi = (\xi_t(\cdot, \cdot), t \in [H])$ consisting of $A^{\leq t} \times F_t$ measurable random variables

$$\xi_t : (a_{< t}, \omega) \in A^t \times \Omega \to \mathbb{R}$$

satisfying

$$\mathbb{E} [\xi_t(a_{< t}, \omega) | F_{t-1}] = 0, \quad \text{for all } (a_{< t}) \in A^t, \quad t \in \{1, \ldots, H\}.$$ 

The next duality theorem, essentially due to [20], may be seen as a generalization of the dual representation theorem for optimal stopping, developed independently in [21] and [15], to Markov decision processes. For a more general dual representations in terms of information relaxation, see [13]. Let us further mention dual representations in the context of multiple stopping developed in [22], [8], and applications to flexible caps studied in [1].

**Theorem 2.** The following statements hold.

(i): For any $\xi \in \Xi$ and any $x \in S$ we have $V^*_{0}^{\text{up}}(x; \xi) \geq V^*_0(x)$ with

$$V^*_{0}^{\text{up}}(x; \xi) := \mathbb{E}_{\pi, x} \left[ \sup_{a_{\geq 0} \in A^H} \left( \sum_{t=0}^{H-1} (R_t(S_t(a_{< t}), a_t) - \xi_{t+1}(a_{< t+1})) + F(S_H(a_{< H})) \right) \right]$$

where as usual we suppress the dependence on $\omega$ for notational simplicity. Hence $V^*_{0}^{\text{up}}(x; \xi)$ is an upper (upper biased) bound for $V^*_0(x)$.

(ii): If we set $\xi^* = (\xi_t^*, t \in [H]) \in \Xi$ with

$$\xi_{t+1}^*(a_{< t+1}) := V^*_{t+1}(S_{t+1}(a_{< t+1})) - \mathbb{E}_{S_{t+1} \sim P_{t+1}(\cdot|S_t(a_{< t}), a_t)} \left[ V^*_{t+1}(S_{t+1}) \right]$$

for $t = 0, \ldots, H - 1$, then, almost surely,

$$V^*_0(x_0) = \sup_{a_{\geq 0} \in A^H} \left( \sum_{t=0}^{H-1} (R_t(S_t(a_{< t}), a_t) - \xi_{t+1}^*(a_{< t+1})) + F(S_H(a_{< H})) \right).$$

**Remark 3.** In Theorem 2 and further below, supremum should be interpreted as essential supremum in case it concerns the supremum over an uncountable family of random variables.

In principle, Theorem 2 may be inferred from [20] or [13]. Nonetheless, also for the convenience of the reader, we here give a concise proof in terms of the present notation and terminology.
Proof. (i) Since for any $\xi \in \Xi$ and policy $\pi$ in $[22]$ one has that
\[
\mathbb{E}_{\pi,x}[\xi_{t+1}(A_{\leq t})] = \mathbb{E}_{\pi,x}[\xi_{t+1}(A_{\leq t}) | F_t] = 0,
\]
for $t = h, ..., H - 1$, it follows that
\[
V^*_t(x) = \sup_{\pi} \mathbb{E}_{\pi,x} \left[ \sum_{t=0}^{H-1} (R_t(S_t, A_t) - \xi^*_{t+1}(A_{\leq t})) + F(S_H(A_{<H})) \right],
\]
from which (3.1) follows immediately.

(ii) We may write for any $a \geq 0 \in A^H$,
\[
\sum_{t=0}^{H-1} (R_t(S_t, A_{<t}) - \xi^*_{t+1}(A_{\leq t})) + F(S_H(A_{<H}))
\]
\[
= \sum_{t=0}^{H-1} R_t(S_t, A_{<t}) + \sum_{t=0}^{H-1} V^*_t(S_{t+1}(A_{\leq t}))
\]
\[
+ \sum_{t=0}^{H-1} E_{S_{t+1} \sim P_t} [V^*_t(S^t_{t+1})] + F(S_H(A_{<H})).
\]
Hence
\[
\sum_{t=0}^{H-1} (R_t(S_t, A_{<t}) - \xi^*_{t+1}(A_{\leq t})) + F(S_H(A_{<H})) = V^*_0(x) + \Delta(x),
\]
with
\[
\Delta(x) := F(S_H(A_{<H})) - V^*_H(S_H(A_{<H})) +
\]
\[
- \sum_{t=0}^{H-1} \left( R_t(S_t, A_{<t}) + E_{S_{t+1} \sim P_t} [V^*_t(S^t_{t+1})] - V^*_t(S_t) \right) \leq 0,
\]
where the latter inequality follows from the Bellman principle, see Theorem [1]. The statement (3.3) now follows by taking the supremum over $a \geq 0 \in A^H$ on the left-hand-side, applying (3.1) and using the sandwich property.

4. PRIMAL REGRESSION ALGORITHM FOR THE VALUE FUNCTION

In Section [3] we will describe regression based martingale methods for computing dual upper bounds based on Theorem [2]. However, these methods require as an input a sequence of (approximate) value functions $V_h$, $h \in [H]$. Below we describe a regression-based algorithm for approximating the value functions $V^*_h$, $h \in [H]$, backwardly in time. In fact, unlike the usual regression, the proposed algorithm is based on a kind of “pseudo” or “quasi” regression procedure with respect to some reference measure $\mu_h$ which is assumed to be such that $P_h(\cdot, a)$ is absolutely continuous w.r.t. $\mu_h$ for any $h \in [H], x \in S$ and $a \in A$. Furthermore, we consider a vector of basis functions
\[
\gamma_K := (\gamma_1, \ldots, \gamma_K)^T, \quad \gamma_k : \mathbb{R} \to \mathbb{R}, \quad k = 1, \ldots, K,
\]
such that the matrix
\[
\Sigma \equiv \Sigma_{h,K} := \mathbb{E}_{X \sim \mu_h} \left[ \gamma_K(X) \gamma_K^T(X) \right]
\]
is analytically known and invertible. This basically means that the choice of basis functions is adapted to the choice of the reference measure. For example, if $\mu_h$ is Gaussian one can choose basis functions to be polynomials or trigonometric polynomials. The algorithm reads then as follows. At $h = H$ we set $V_{H,N}(x) = V_h^*(x) = F(x)$. Suppose that for some $h \in [H]$, the approximations $V_{t,N}$ of $V^*_h$, $h + 1 \leq t \leq H$, are already obtained. We now approximate $V_h^*$ via simulating independent random variables $X_t \equiv X_{h}^t \sim \mu_h, Y_t^a \sim P_{h+1}(\cdot | X_{h}^i, a), a \in A, i = 1, \ldots, N$, and setting
\[
V_{h,N}(x) = T_{h,N}V_{h+1,N}(x) := \sup_a (R_h(x, a) + \tilde{F}_{h+1,N}^a V_{h+1,N}(x))
\]
The use of clipping at level explained as follows. Equation (3.2) and (5.1) imply that, for a particular chosen finite dimensional subspace of $\xi$

By generating a sample

Thus, the quantity $\tilde{V}_{h+1,N}(x)$ aims to approximate the conditional expectation

$x \rightarrow \mathbb{E}_{S' \sim P_{h+1}(\cdot|x,a)}[V_{h+1,N}(S')], \quad a \in A.$

The use of clipping at level $\tilde{L}_{h+1}$ is done to avoid large values of $\beta_{N,a} \gamma_K(x)$. After $H$ steps of the above procedure we obtain the estimates $V_{H,N}, \ldots, V_{0,N}$.

5. Dual regression algorithm

In this section we outline how to construct an upper biased estimate based on Theorem 2 from a given sequence of approximations $V_t$, $t \in [H]$ obtained, for example, as described in Section 4.

Theorem 2(ii) implies that we can restrict our attention to processes $\xi = (\xi_t)_{t \in [H]}$, where the $t + 1$ component of $\xi$ is of the form

$$\xi_{t+1}(a_{\leq t}) = m(S_{t+1}(a_{\leq t}); S_t(a_{<t}), a_t)$$

for a deterministic real valued function $m(\cdot; x, a)$ satisfying

$$\int m(y; x, a) P_{t+1}(dy|x, a) = 0,$$

for all $(x, a) \in S \times A$. Note that the condition (5.2) is time dependent. We shall denote by $\mathcal{M}_{t+1,x,a}$ the set of “martingale” functions $m$ on $S$ that satisfy (5.2) for time $t + 1$, a state $x$, and a control $a$. In this section, we develop an algorithm approximating $\xi^*$ via regression of $V_{t+1}$ on a properly chosen finite dimensional subspace of $\mathcal{M}_{t+1,x,a}$. The idea of approximating $\xi^*$ via regression can be explained as follows. Equations (3.2) and (5.1) imply that, for a particular $t \in [H]$, the component $\xi^{t+1}_{t+1}(a_{\leq t})$ of the random vector $\xi^*$ is given by $\xi^{t+1}_{t+1}(a_{\leq t}) = m^t_{t+1}(S_{t+1}(a_{\leq t}); S_t(a_{<t}), a_t)$, where, for each $(x, a) \in S \times A$, $m_{t+1}^t(\cdot; x, a)$ solves the optimization problem

$$\arg \min_{m \in \mathcal{M}_{t+1,x,a}} \mathbb{E}_{S'_{t+1} \sim P_{t+1}(\cdot|x,a)} \left[ (V_{t+1}^*(S_{t+1}^t) - m(S_{t+1}^t; x, a))^2 \right] =$$

$$\arg \min_{m \in \mathcal{M}_{t+1,x,a}} \text{Var}_{S'_{t+1} \sim P_{t+1}(\cdot|x,a)} \left[ V_{t+1}^*(S_{t+1}^t) - m(S_{t+1}^t; x, a) \right].$$

By generating a sample $Y_1^{x,a}, \ldots, Y_{N}^{x,a}$ from $P_{t+1}(\cdot|x,a)$ we readily obtain a computable approximation of $m_{t+1}^t(\cdot; x, a)$, that is, the solution of (5.3), by

$$\arg \min_{m \in \mathcal{M}_{t+1,x,a}} \left\{ \frac{1}{N} \sum_{i=1}^{N} (V_{t+1}^i(Y_{i}^{x,a}) - m(Y_{i}^{x,a}))^2 \right\},$$

where $\mathcal{M}_{t+1,x,a}$ is some “large enough” finite-dimensional subset of $\mathcal{M}_{t+1,x,a}$.

Let us now discuss possible constructions of the martingale functions $m$ satisfying (5.2). Assume that $S \subseteq \mathbb{R}^d$ and that the conditional distribution $P_{t+1}(\cdot|x, a)$ possesses a smooth density $p_{t+1}(\cdot|x, a)$ with respect to the Lebesgue measure on $\mathbb{R}^d$. Furthermore, assume that $p_{t+1}(\cdot|x, a)$ doesn’t vanish.

\[\text{Actually, for computing } V_0(x_0) \text{ we may replace the above procedure by a standard Monte Carlo simulation when going from } V_1 \text{ to } V_0.\]
on any compact set in \( \mathbb{R}^d \), and that \( p_{t+1}(y|x,a) \to 0 \) for \( |y| \to \infty \). Now consider, for any fixed \((x,a)\), functions of the form

\[
m_{t+1,\phi}(\cdot; x,a) := \langle \nabla \log(p_{t+1}(\cdot|x,a)), \phi \rangle + \text{div}(\phi)
\]

with \( \phi : S \to \mathbb{R}^d \) being a smooth and bounded mapping with bounded derivatives. It is then not difficult to check that

\[
\int_S p_{t+1}(y|x,a) \phi_i(y) \partial_y \log(p_{t+1}(y|x,a)) \, dy = - \int_S p_{t+1}(y|x,a) \partial_y \phi_i(y) \, dy, \quad i = 1, \ldots, d,
\]

and hence \( m_{t+1,\phi} \) satisfies (5.7) for all \((x,a) \in S \times A\). This means that in (5.4), we can take \( \mathcal{M}_{t+1,x,a} = \{ m_{t+1,\phi}(\cdot; x,a) := \phi \in \Phi \} \) where \( \Phi \) is the linear space of mappings \( \mathbb{R}^d \to \mathbb{R}^d \), which are smooth, bounded, and with bounded derivatives. Since \( \phi \to m_{t+1,\phi}(\cdot; x,a) \) is linear in \( \phi \) we moreover have that \( \mathcal{M}_{t+1,x,a} \) is a linear space of real valued functions. So the problem (5.4) can be casted into a standard linear regression problem after choosing a system of basis functions \((m_{t+1,\phi_k}(\cdot; x,a))_{k \in \mathbb{N}} \) due to some basis \((\phi_k)_{k \in \mathbb{N}} \) in \( \Phi \). Needles to say that the problem (5.4) can only be solved on some finite grid, \((x_j,a_k)_{j=1,\ldots,L} \in S \times A\) say, yielding solutions \( \phi_k(\cdot) := \phi(\cdot; x_k,a_k) \) and the corresponding martingale functions \( m_{t+1,\phi_k}(\cdot; x_k,a_k) \). In order to obtain a martingale function \( m_{t+1} \equiv m_{t+1}(\cdot; x,a) \) for a generic pair \((x,a)\) we may apply some suitable interpolation procedure. Loosely speaking, if \((x,a)\) is an interpolation between \((x_k,a_k)\) and \((x_{k'},a_{k'})\) we may interpolate \( \phi(\cdot; x,a) \) between \( \phi_k \) and \( \phi_{k'} \) correspondingly, and set \( m_{t+1} = m_{t+1,\phi}(\cdot; x,a) \). For details regarding suitable interpolation procedures we refer to Section 7.

Let now, for each \( t \in [H] \), and \((x,a) \in S \times A\), the martingale function \( m_{t+1}(\cdot; x,a) \) be an approximate solution of (5.4). Then we can construct an upper bound (upper biased estimate) for \( V_0^*(x_0) \), via a standard Monte Carlo estimate of the expectation

\[
(5.5) \quad V_0^{\text{up}}(x) = \mathbb{E}_{\pi,x} \left[ \sup_{a_{\geq t} \in \mathcal{A}} \left( \sum_{t=0}^{H-1} (R_t(S_t(a_{\geq t}), a_t) - m_{t+1}(S_{t+1}(a_{\leq t}); S_t(a_{\leq t}), a_t)) + F(S_H) \right) \right].
\]

Another way of constructing \( \xi \in \Xi \) is based on the assumption that the chain \( (S_t(a_{\leq t})) \) comes from the system of the so-called random iterative functions:

\[
(5.6) \quad S_t = K_t(S_{t-1}, a_{t-1}, \varepsilon_t), \quad t \in [H],
\]

where \( K_t : S \times A \times \mathbb{E} \to S \), is a measurable map with \( \mathbb{E} \) being a measurable space, and \((\varepsilon_t, t \in [H])\) is an i.i.d. sequence of \( \mathbb{E} \)-valued random variables defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). In this setup we may consider as the underlying probability space \( \Omega := (\mathbb{E} \times A)^{[H]} \) instead of (2.1), with accordingly modified definitions of \( \mathcal{F} \) and \((F_t)\).

Let \( \mathcal{P}_E \) be the distribution of \( \varepsilon_1 \) on \( \mathbb{E} \), and assume that \((\psi_k, k \in \mathbb{N})\) is a system in \( L^2(\mathbb{E}, \mathcal{P}_E) \) satisfying

\[
\int \psi_k(\varepsilon) \, d\mathcal{P}_E = 0, \quad k \in \mathbb{N}.
\]

By then letting

\[
(5.7) \quad \eta_{t+1,K}(x,a) \equiv \eta_{t+1,K}(x,a,\varepsilon_{t+1}) = \sum_{k=1}^K c_k(x,a) \psi_k(\varepsilon_{t+1})
\]

for some natural \( K > 0 \) and “nice” functions \( c_k : S \times A \to \mathbb{R} \), \( k = 1, \ldots, K \), we have that

\[
\xi_{t+1,K}(a_{\leq t}) := \eta_{t+1,K}(S_t(a_{\leq t}), a_t)
\]

is \( \mathcal{F}_{t+1} \)-measurable, and, since \( \int \psi_k(\varepsilon) \, d\mathcal{P}_E(\varepsilon) = 0 \) for \( k \in \mathbb{N} \), it holds that \( \mathbb{E} [\xi_{t+1,K}(a_{\leq t})] \mid \mathcal{F}_t \) = 0. Hence, we have that \( \xi_K = (\xi_{t+1,K}(a_{\leq t}), t \in [H]) \in \Xi \). In this case, we can consider the least-squares problem

\[
(5.8) \quad \inf_{(c_1, \ldots, c_K)} \mathbb{E} \left[ \left( V_{t+1}(Z^{x,a}) - \sum_{k=1}^K c_k \psi_k(\varepsilon_{t+1}) \right)^2 \right], \quad Z^{x,a} \equiv K_{t+1}(x,a,\varepsilon_{t+1}).
\]
for estimating the coefficients in (5.7). Let us further denote $\Sigma_{E,K} := \mathbb{E}_{\varepsilon \sim \mathcal{P}_E} \left[ \psi_K(\varepsilon) \psi_K^T(\varepsilon) \right]$ with $\psi_K(\varepsilon) := [\psi_1(\varepsilon), \ldots, \psi_K(\varepsilon)]^T$. The minimization problem (5.8) is then explicitly solved by

$$\bar{c}_K(x,a) := \Sigma_{E,K}^{-1} \mathbb{E} \left[ V_{t+1}(Z^{x,a}) \psi_K(\varepsilon) \right].$$

In the sequel we assume that $\Sigma_{E,K}$ is known and invertible. This assumption is not particularly restrictive, as we choose the basis $\psi$ ourselves. In order to compute (5.9), we can construct a new sample $U_m(x,a) = V_{t+1}(Z^{x,a})^{-1} \Sigma_{E,K}^{-1} \psi_K(\varepsilon_m)$ with $\varepsilon_m \sim \mathcal{P}_E$, $Z^{x,a}_m = K_{t+1}(x,a,\varepsilon_m)$, $m = 1, \ldots, M$, and estimate its mean $\bar{c}_K(x,a)$ by the empirical mean

$$c_{K,M}(x,a) = [c_{1,M}(x,a), \ldots, c_{K,M}(x,a)]^T := \frac{1}{M} \sum_{m=1}^M U_m(x,a).$$

We so obtain as martingale functions in (5.7),

$$\eta_{t+1,K,M} := c_{K,M}^T(x,a) \psi_K(\varepsilon_{t+1}) = \sum_{k=1}^K c_{k,M}(x,a) \psi_k(\varepsilon_{t+1}).$$

Also note that the problem (5.8) may only numerically be solved on a grid, and a suitable interpolation procedure is required to obtain (5.11) for generic $(x,a) \in S \times A$ (for details see Section 7). Finally, an upper biased upper bound for $V_{0}^\ast(x)$ can be obtained via an independent standard Monte Carlo estimate of the expectation

$$V_{0}^\ast(x) = \mathbb{E}_\pi_x \left[ \sup_{a_0 \mathcal{A}} \left( \sum_{t=0}^{H-1} (R_t(S_t(a_{g,t}),a_t) - \eta_{t+1,K,M}(S_t(a_{\leq t}), a_t)) + F(S_H) \right) \right].$$

In Section 7 we will give a detailed convergence analysis of the dual estimator (5.12). It is anticipated that a similar analysis can be carried out for the dual estimator (5.11), but this analysis is omitted due to space restrictions.

6. Convergence analysis of the primal algorithm

In this section, we carry out the convergence analysis of the primal algorithm designed in Section 4 under some mild assumptions.

Assumption 1. Assume that (5.6) holds. In this case $P^n_{x} f(x) = \mathbb{E}_{\varepsilon \sim \mathcal{P}_E} [f(K_h(x,a,\varepsilon))]$, $(x,a) \in S \times A$. Also assume that the kernels $K_h$ are Lipschitz continuous:

$$|K_h(x,a,\varepsilon) - K_h(x',a',\varepsilon)| \leq L_K \rho((x,a),(x',a')),$$

for some constant $L_K$ not depending on $h$. In (6.1), the metric $\rho \equiv \rho_{S \times A}$ on $S \times A$ is considered to be of the form

$$\rho_{S \times A}((x,a),(x',a')) = \| (\rho_S(x,x'), \rho_A(a,a') ) \|,$$

where $\rho_S$ and $\rho_A$ are suitable metrics on $S$ and $A$, respectively, and $\| (\cdot, \cdot) \|$ is a fixed but arbitrary norm on $\mathbb{R}^2$. In order to avoid an overkill of notation, we will henceforth drop the subscripts $S$, $A$, and $S \times A$, whenever it is clear from the arguments which metric is considered.

Assumption 2. Assume that $\sup_{(x,a) \in S \times A} \{ |R_h(x,a)| \vee |F(x)| \} \leq R_{\max}$ and

$$\sup_{a \in A} \{ |R_h(x,a) - R_h(x',a)| \} \leq L_R \rho(x,x')$$

for some constants $R_{\max}$ and $L_R$ not depending on $h \in [H]$.

We now set

$$\tilde{L}_h := (H - h + 1) R_{\max}, \ h \in [H], \ V^\ast_{0} := \tilde{L}_0 = (H + 1) R_{\max}.$$

Assumption 3. Assume that $|\Sigma_{\gamma,K}^{-1} \gamma_K(x)|_{\infty} \leq \Lambda_K$ for all $x \in S$, $h \in [H]$, and

$$|\gamma_K(x) - \gamma_K(x')| \leq L_{\gamma,K} \rho(x,x')$$

for a constant $L_{\gamma,K} > 0$, where $\| \cdot \|$ denotes the Euclidian norm and $\| \cdot \|_{\infty}$ stands for the $\ell_{\infty}$ norm.
Note that due to (1.1) and (6.2) one has that $|V_{h,N}| \leq \widetilde{L}_h$, $h \in [H]$, and that under Assumptions [1] and [3] one has

$$|T_{h,N}V_{h+1,N}(x) - T_{h,N}V_{h+1,N}(x')| \leq L_R \rho(x, x') + \sup_{a \in A} |\hat{P}_{h+1,N}^a V_{h+1,N}(x) - \hat{P}_{h+1,N}^a V_{h+1,N}(x')|$$

$$\leq L_R \rho(x, x') + \sup_{a \in A} |\beta_{N,a}||\gamma_K(x) - \gamma_K(x')|$$

$$\leq L_R \rho(x, x') + \frac{1}{N} \sum_{n=1}^N \sup_{a \in A} |Z_n^a| |\Sigma_{h,K}\gamma_K(X_n)||\gamma_K(x) - \gamma_K(x')|$$

$$\leq [L_R + V_{\max}^* \Lambda_K \sqrt{K} L_{\gamma,K}] \rho(x, x').$$

Let us denote $L_{V,K} := L_R + V_{\max}^* \Lambda_K \sqrt{K}$. The above estimates imply that $V_{h,N} \in \text{Lip}(L_{V,K})$, and so the function $f(x, a, \varepsilon) := V_{h,N}(K_h(x, a, \varepsilon))$ satisfies

$$|f(x, a, \varepsilon) - f(x', a', \varepsilon)| \leq L_{V,K} L_K \rho((x, a), (x', a'))$$

The next assumption concerns the measures $\mu_1, \ldots, \mu_H$. 

**Assumption 4.** Consider for any $h < l$ the Radon-Nikodym derivative

$$\mathcal{R}_{h,l}(x'|x, \pi) := \frac{P_{h+1}^{x_h} \cdots P_l^{x_l-1}(dx'|x)}{\mu(dx')}$$

where for a generic policy $\pi = (\pi_1, \ldots, \pi_H)$,

$$P_{h+1}^{x_h}(dx'|x, \pi_h(x)).$$

Assume that

$$\mathcal{R}_{\max} := \max_{0 \leq h < l < H, \pi} \left( \int \mu_h(dx) \int \mathcal{R}_{h,l}(x'|x, \pi) \mu(dx') \right)^{1/2} < \infty.$$ 

By the very construction of $V_{h,N}$ from $V_{h+1,N}$, $h \in [H]$, as outlined in Section 4, $V_{h,N}$ may be seen as a random (Lipschitz continuous) function. In particular, for each $x \in S$, $V_{h,N}(x)$ is measurable with respect to the $\sigma$-algebra

$$\mathcal{D}_h^N := \sigma\{Y_{h;N}, \ldots, Y_{H-1;N}\} \text{ with } Y_{h;N} := ((X_{h}^1, \varepsilon_{h}^1), \ldots, (X_{h}^N, \varepsilon_{h}^N))$$

where the pairs $(X_h^i, \varepsilon_h^i) \sim \mu_h \otimes \mathcal{P}_E$ are i.i.d. for $h \in [H]$, $i = 1, \ldots, N$, and Monte Carlo simulated under the measure $P := P_N := (\mu_h \otimes \mathcal{P}_E)^{\otimes H N}$. The following theorem provides an upper bound for the difference between $V_{h,N}$ and $V_h^*$.

**Theorem 4.** Suppose that $\mathbb{E}_{X_{\sim \mu_h}} \left|\gamma_K(X)^2\right| \leq \phi_{\gamma,K}^2$ for all $h \in [H]$. Then for $h \in [H]$,

$$\|V_{h}^* - V_{h,N}(\cdot)\|_{L^2(\mu_h \otimes P)} \leq \mathcal{R}_{\max} \left(H - h\right)\phi_{\gamma,K}\Lambda_K(L_{V,K} L_K \mathcal{I}_D(A) + L_{V,K} L_K \mathcal{D}(A) + V_{\max}^*) \frac{\sqrt{K}}{N} + \sum_{l=h}^{H-1} \mathcal{R}_{K,l},$$

where $\mathcal{R}_{K,l}$ denotes $\leq$ up to an absolute constant, $I_D(A)$ is the metric entropy of $A$, $D(A)$ is the diameter of $A$ as defined in Appendix A, and

$$\mathcal{R}_{K,l} := \sup_{\zeta \in \mathbb{R}^K \times [A]} \mathbb{E}_{X_{\sim \mu_h}} \left[ \sup_{a \in A} \left( \beta_{a,\zeta}^T \gamma_K(X) - P_{h+1}^a V_{h+1,N}(X) \right)^2 \right]^{1/2},$$

where

$$\beta_{a,\zeta} := \arg \sup_{\beta \in \mathbb{R}^K} \mathbb{E}_{X_{\sim \mu_h}} \left[ \left( \beta^T \gamma_K(X) - P_{h+1}^a V_{h+1,N}(X) \right)^2 \right]$$

with

$$V_{h,(\cdot)} := \sup_{a \in A} (R_h(x, a) + T_{l,h+1}^a (\zeta_{a}^T \gamma_K(X))) \text{ for } 0 \leq h < H, \text{ } V_{H,(\cdot)} := F(x).$$
Discussion.

- The quantity \( \mathcal{R}_{K,h} \) is related to the error of approximating the conditional expectation \( P_{h+1}^a V_{h+1,\zeta} \) via a linear combination of the basis functions \( \gamma_1, \ldots, \gamma_K \) in a worst case scenario, that is, for the most unfavorable choice of \( \zeta \). Let us suppose, for illustration, that \( A \) is finite and take some \( h < H - 1 \). One then has

\[
\mathcal{R}_{K,h} \leq \sum_{a \in A} \sup_{\zeta \in \mathbb{R}^{K \times |A|}} \mathbb{E}_{X \sim \mu_h} \left[ \left( \beta_{\gamma}^T \zeta \gamma(X) - P_{h+1}^a V_{h+1,\zeta}(X) \right)^2 \right]^{1/2}
\]

where \( \beta_{\gamma}^T \zeta \gamma(X) \) is the \( L^2(\mu_h) \) projection of \( P_{h+1}^a V_{h+1,\zeta} \) on \( \text{span}(\gamma_K) \) with the corresponding projection error

\[
\mathcal{E}_{K,h}(a, \zeta) := \mathbb{E}_{X \sim \mu_h} \left[ \left( \beta_{\gamma}^T \zeta \gamma(X) - P_{h+1}^a V_{h+1,\zeta}(X) \right)^2 \right]^{1/2}.
\]

Under mild conditions on \( P_{h+1}^a \), (6.7) converges to zero uniformly in \( \zeta \), at a rate depending on the choice of \( \gamma_K \). For example, if the system \( \gamma_1, \gamma_2, \ldots \) is an orthonormal base in \( L^2(\mu_h) \) then \( \max_{a \in A} \sup_{\zeta \in \mathbb{R}^{K \times |A|}} \mathcal{E}_{K,h}(a, \zeta) \lesssim K^{-\beta}, \beta > 0 \), provided that the series

\[
\sum_{K = 1}^{\infty} \sum_{k=1}^{K^2} \mathbb{E}_{X \sim \mu_h}[\gamma_k(X) P_{h+1}^a V_{h+1,\zeta}(X)]^2
\]

is uniformly bounded in \( \zeta \in \mathbb{R}^{K \times |A|} \) and \( a \in A \). Hence, \( \mathcal{R}_{K,h} \lesssim |A| K^{-\beta} \rightarrow 0 \) for \( K \rightarrow \infty \). Note that (6.6) is a worst case estimate, which may be very rough in general.

- Suppose that \( P_{h+1}^a =: P_{h+1} \) does not depend on \( a \in A \), and that \( \gamma_1, \gamma_2, \ldots \) are bounded eigenfunctions (corresponding to nonnegative eigenvalues) of \( P_{h+1} \). Let further \( F(x) = \beta^T \gamma_K(x) \) for some \( \beta \in \mathbb{R}^K \) and \( R_t(x, a) = R_{1,t}(x) R_{2,t}(a) \) with \( R_{1,t}(x) \leq c_t, \gamma_K(x) \geq 0 \), then for \( \tilde{L}_{h+1} \) large enough, \( \mathcal{R}_{K,h} = 0 \) (in this case we may take \( \zeta_a \) independent of \( a \) in the definition of \( V_{h+1,\zeta} \) and only the stochastic part of the error remains:

\[
\|V_h^* - V_h, N\|_{L^2(\mu_h \otimes P)} \lesssim H \mathfrak{R}_{\gamma,K}^{\max} \mathcal{D}(L_{\gamma,K} L_{D}(A) + L_{V,K} L_K D(A) + V_{max}^a) \sqrt{\frac{K}{N}}.
\]

Let us consider the stochastic error (6.8) in more detail for an example where \( A = [0, 1]^d_A \) for some \( d_A \in N \). One then has \( D(A) = \sqrt{d_A} \) and \( \mathcal{D}(A) \lesssim \sqrt{d_A} \). In this example the bound (6.8) depends sub-linearly in \( d_A \). If in addition all basis functions \( \gamma_k \) are uniformly bounded and the infinity matrix norm (i.e. the maximum absolute row sum) of \( \Sigma_{h,K} \) is uniformly bounded from below for all \( K \in N \) and \( h \in [H] \), then \( \mathcal{R}_{h,K} \leq \gamma^{K}/2, \mathcal{D}(A) \lesssim 1, L_{\gamma,K} \leq L_{\gamma,K} H K^{1/2}, V_{max} \leq H \), and the bound in Theorem 4 transforms to

\[
\|V_h^* - V_h, N\|_{L^2(\mu_h \otimes P)} \lesssim \frac{(H - h) H \mathfrak{R}_{\gamma,K}^{\max} \sqrt{d_A} L_{\gamma,K} K^{3/2}}{\sqrt{N}} + \mathfrak{R}_{h,K,l}^{\max} \sum_{l=1}^{H-1} \mathcal{R}_{K,l},
\]

where \( \lesssim \) means inequality up to a constant not depending on \( H, N, K \) and \( \mathcal{D}(A) \). Another relevant situation is the case of finite \( A \). Here \( \mathcal{D}(A) = \sqrt{\log |A|} \) and \( \mathcal{R}(A) = 1 \). Hence (6.9) changes to

\[
\|V_h^* - V_h, N\|_{L^2(\mu_h \otimes P)} \lesssim \frac{(H - h) H \mathfrak{R}_{\gamma,K}^{\max} \sqrt{\log |A|} L_{\gamma,K} K^{3/2}}{\sqrt{N}} + \mathfrak{R}_{h,K,l}^{\max} \sum_{l=1}^{H-1} \mathcal{R}_{K,l}.
\]

Let us point out to a logarithmic dependence of (6.10) on \( |A| \).

- Let us remark on Assumption 4 and discuss the quantity \( \mathfrak{R}_{\gamma,K}^{\max} \). Consider \( S = \mathbb{R}^d \) and assume that the transition kernels are absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R}^d \), that is,

\[
P_{h+1}^a \cdots P_{l}^{\pi_{l-1}}(dy|x) = p_{h+1}^a \cdots p_{l}^{\pi_{l-1}}(y|x) dy.
\]

Further assume that

\[
\sup_{0 \leq h < l \leq H, \pi} p_{h+1}^a \cdots p_{l}^{\pi_{l-1}}(y|x) \leq C e^{-\alpha_H |y-x|^2} \quad \text{for some } C, \alpha_H > 0,
\]
and consider absolutely continuous reference measures $\mu_h(dx) = \mu_h(x) dx$. For the bound (6.4), we then have
\[
\mathfrak{R}_{\text{max}}^2 \leq C \sup_{0 < h < l < H} \int \int \frac{\mu_h(x)}{\mu_l(y)} \left( P_{h+1} \cdots P_{l-1}(y|x) \right)^2 \, dx \, dy \\
\leq C^2 \max_{0 < h < l < H} \int \int \frac{\mu_h(x)}{\mu_l(x+u)} e^{-2\alpha_H u^2} \, dx \, du.
\]
The latter expression can be easily bounded by choosing $\mu_h$ to be Gaussian with an appropriate variance structure depending on $h$. For example, take $d = 1$ and consider
\[
\mu_h(x) = \sqrt{\frac{\alpha_H}{\pi(h+1)}} e^{-\frac{\alpha_H x^2}{h+1}}, \quad h \in [H],
\]
then straightforward calculations yield
\[
\mathfrak{R}_{\text{max}} \leq C \max_{0 < h < l < H} \frac{(l+1)\pi}{\alpha_H \sqrt{2(l-h) - 1}} \leq C \sqrt{H \pi}.
\]
If $\alpha_H$ is polynomial in $H$, then the bound of Theorem 4 also grows polynomially in $H$ as opposed to the most bounds available in the literature. Also note that this bound is obtained under rather general assumptions on the sets $\mathcal{S}$ and $\mathcal{A}$. In particular, we don’t assume that either $\mathcal{S}$ or $\mathcal{A}$ is finite.

7. CONVERGENCE ANALYSIS OF THE DUAL ALGORITHM

7.1. Convergence of martingale functions. For the dual representation (5.12) we construct an $H$-tuple of martingale functions $\tilde{\eta} := (\tilde{\eta}_{l+1,K,M}(x,a), t \in [H])$, see (5.11) as outlined in Section 5, from a given pre-computed $H$-tuple of approximate value functions $(V_{l+1,N}, t \in [H])$ based on sampled data $D_{1}^{N}$ (see (5.8)), and a system of $K_{pr}$ basis functions $\gamma_{K_{pr}}$, as outlined in Section 4.

Let us consider a fixed time $t \in [H]$ and suppress time subscripts where notationally convenient. We fix two (random) grids $\mathcal{S}_{L} := \{x_1, \ldots, x_L\}$ and $\mathcal{A}_{L} := \{a_1, \ldots, a_L\}$ on $\mathcal{S}$ and $\mathcal{A}$, respectively, and obtain values of the coefficient functions $c_{k,M}$ on $\mathcal{S}_{L} \times \mathcal{A}_{L}$ due to $M$ simulations. Next, we construct
\[
\eta_{l+1,K,M}(x,a) \equiv \eta_{l+1,K,M}(x,a,\varepsilon) = \mathbf{c}_{K,M}^{\top}(x,a)\psi(\varepsilon) := \sum_{k=1}^{K} c_{k,M}(x,a)\psi_k(\varepsilon),
\]
for $(x,a) \in \mathcal{S}_{L} \times \mathcal{A}_{L}$. To approximate $\eta_{l+1,K,M}(x,a)$ for $(x,a) \notin \mathcal{S}_{L} \times \mathcal{A}_{L}$, we suggest to use an appropriate interpolation procedure described below, which is particularly useful for our situation where the function to be interpolated is only Lipschitz continuous (due to the presence of the maximum). The optimal central interpolant for a function $f \in \text{Lip}_{\rho}(\mathcal{L})$ on $\mathcal{S} \times \mathcal{A}$ with respect to some metric $\rho$ on $\mathcal{S} \times \mathcal{A}$ is defined as
\[
I[f](x,a) := (H_{l}^{\text{low}}(x,a) + H_{l}^{\text{up}}(x,a))/2,
\]
where
\[
H_{l}^{\text{low}}(x,a) := \max_{(x',a') \in \mathcal{S}_{L} \times \mathcal{A}_{L}} (f(x',a') - \mathcal{L}(f(x,a),(x',a'))),
\]
\[
H_{l}^{\text{up}}(x,a) := \min_{(x',a') \in \mathcal{S}_{L} \times \mathcal{A}_{L}} (f(x',a') + \mathcal{L}(f(x,a),(x',a'))).
\]
Note that $H_{l}^{\text{low}}(x,a) \leq f(x,a) \leq H_{l}^{\text{up}}(x,a)$, $H_{l}^{\text{low}}(x,a), H_{l}^{\text{up}}(x,a) \in \text{Lip}_{\rho}(\mathcal{L})$ and hence $I[f] \in \text{Lip}_{\rho}(\mathcal{L})$. An efficient algorithm to compute the values of the interpolant $I[f]$ without knowing $\mathcal{L}$ in advance can be found in [7]. The so constructed interpolant achieves the bound
\[
||f - I[f]||_{\infty} \leq \mathcal{L}\rho_{\mathcal{L}}(S,A)
\]
\[
:= \mathcal{L} \sup_{(x,a) \in S \times A} \min_{(x',a') \in S \times A} \rho((x,a),(x',a')).
\]
The quantity $\rho_L(S, A)$ is usually called covering radius (also known as the mesh norm or fill radius) of $S_L \times A_L$ with respect to $S \times A$. We set

$$
\tilde{\eta}_{t+1, K, M}(x, a) \equiv \tilde{\eta}_{t+1, K, M}(x, a, \varepsilon) := \sum_{k=1}^{K} \tilde{c}_{k, M}(x, a) \psi_k(\varepsilon) \quad \text{with} \quad \tilde{c}_{k, M} := I[c_{k, M}].
$$

(7.2)

The coefficients $\tilde{c}_{k, M}(x, a)$ in (7.2) are considered as random, which are measurable with respect to $\mathcal{D}_t^N \vee G_t^{M_1}$ with $G_t^{M_1} := \sigma\{\tilde{z}_1, \ldots, \tilde{z}_{t+1}\}$, where $\tilde{z}_m \sim \mathcal{P}_E, m = 1, \ldots, M, t \in [H]$, denote the i.i.d. random drawings used in (1.10). Let us denote the simulation measure (for both primal and dual) with $P \equiv P_{N, M} := P_N \otimes \mathcal{P}_E^{\otimes HM}$ (while slightly abusing notation) with $P_N = (\mu_k \otimes \mathcal{P}_E)^{\otimes HN}$.

Furthermore, denote by $c_K(x, a) = [c_1(x, a), \ldots, c_K(x, a)]^T$ the unique solution of the minimization problem

$$
(7.3) \quad \inf_{c_1, \ldots, c_K} \mathbb{E}_{\varepsilon \sim \mathcal{P}_E} \left[ \left( V_{t+1}^*(K_{t+1}(x, a, \varepsilon)) - \sum_{k=1}^{K} c_k \psi_k(\varepsilon) \right)^2 \right]
$$

for any $(x, a) \in S \times A$, and define $\eta_{t+1, K}(x, a) := c_K^T(x, a) \psi_K(\varepsilon)$. As such, $\eta_{t+1, K}(x, a)$ is the projection of the optimal martingale function $\eta_{t+1}(x, a)$ on span$(\psi_1, \ldots, \psi_K)$.

**Assumption 5.** Assume that $|\Sigma_{E, K}^{-1} \psi_K(\varepsilon)|_{\infty} \leq \Lambda_{E, K}$ for all $\varepsilon \in E$, and that $\mathbb{E}_{\varepsilon \sim \mathcal{P}_E}[|\psi_K(\varepsilon)|^2] \leq \varphi_{E, K}^2$.

The following theorem provides a bound on the difference between the projection $\eta_{t+1, K}(x, a)$ and its estimate (7.2).

**Theorem 5.** Under Assumptions 1, 2, 3 and 5 it holds that

$$
\mathbb{E}_{\varepsilon \sim \mathcal{P}_E} \left[ \sup_{(x, a) \in S \times A} |\eta_{t+1, K}(x, a, \cdot) - \tilde{\eta}_{t+1, K, M}(x, a, \cdot)|^2 \right] \leq \frac{K \Lambda_{E, K}^2 \varphi_{E, K}^2}{\Lambda_{E, K}^2} \sup_{(x, a) \in S \times A} \left\| \frac{dP_{t+1}(\cdot|x, a)}{d\mu_{t+1}(\cdot)} \right\|_{\infty} \|V_{t+1}^* - V_{t+1, N}\|_{L^2(\mu_{t+1} \otimes P)}^2 + K \Lambda_{E, K}^2 \varphi_{E, K}^2 L^2(\mu_{t+1} \otimes P) + K \Lambda_{E, K}^2 \varphi_{E, K}^2 L^2(S, A),
$$

where $\lesssim$ denotes $\leq$ up to a natural constant, the constants $L_{V, K_{t+1}}$, $L_{K}$, and the measure $\mu_{t+1}$ are inferred from the primal procedure in Section 6.

Let us now consider the approximation error

$$
\mathcal{E}_{K,t}^2 := \mathbb{E}_{\varepsilon \sim \mathcal{L}_E} \left[ \sup_{(x, a) \in S \times A} |\eta_{t+1, K}(x, a) - \eta_{t+1}^*(x, a)|^2 \right]
$$

with

$$
\eta_{t+1}^*(x, a) = V_{t+1}^*(K_{t+1}(x, a, \varepsilon)) - \mathbb{E}[V_{t+1}^*(K_{t+1}(x, a, \varepsilon))] \quad (x, a) \in S \times A, \quad t \in [H].
$$

Suppose that one has pointwise

$$
\eta_{t+1}^*(x, a) = \sum_{k=1}^{\infty} c_{k, t+1}^*(x, a) \psi_k(\varepsilon_{t+1}) \quad (x, a) \in S \times A, \quad t \in [H].
$$

If $\|\psi_k\|_{\infty} \leq \psi_k^*$ for all $k \in \mathbb{N}$, then

$$
\mathcal{E}_{K,t}^2 = \mathbb{E} \left[ \sup_{(x, a) \in S \times A} \left( \sum_{k=1}^{\infty} c_{k, t+1}^*(x, a) \psi_k(\varepsilon_t) \right)^2 \right] \leq \sup_{(x, a) \in S \times A} \left( \sum_{k=1}^{\infty} |c_{k, t+1}^*(x, a)|^2 \psi_k^* \right)^2.
$$

If

$$
\sup_{(x, a) \in S \times A} \sum_{k=1}^{\infty} K_1^2 \psi_k^* \leq C < \infty
$$

(7.4)
for some $\beta_0 > 0$, then

\[ \mathcal{E}_{K,t}^2 \leq C^2 K^{-2\beta_0}. \]  

(7.5)

Discussion.

- Let us discuss the quantity $\rho_L(S, A)$. Let $S = [0, 1]^{d_S}$, $A = [0, 1]^{d_A}$ for some $d_S, d_A \in \mathbb{N}$ and let the points $S_L (A_L)$ be uniformly distributed on $S (A)$. Moreover set, $\rho ((x, a), (x', a')) = |x - x'| + |a - a'|$. Then, similarly to [19] it can be shown that

\[ \mathbb{E} \rho_L^p (S \times A) \leq \sqrt{d_S} \left( \frac{p \log L}{L} \right)^{1/d_S} + \sqrt{d_A} \left( \frac{p \log L}{L} \right)^{1/d_A} , \]

(7.6)

where $\leq$ stands for inequality up to a constant not depending on $L$. Using the Markov inequality, we can derive a high probability bound for $\rho_L(S, A)$. Note that if $S$ and $A$ are finite we need not to interpolate and $\rho_L = 0$.

- Assume that all basis functions $(\psi_k)$ are uniformly bounded and that the infinity matrix norm (i.e. the maximum absolute row sum) of the matrix $\Sigma_{E,K}$ is uniformly bounded from below for all $K \in \mathbb{N}$. In this case, $\varrho (\psi, K) \lesssim K^{1/2}$, $\Lambda_{E,K} \lesssim 1$, $L_{V,K,pr} \lesssim L_{\gamma,K,pr} H K_{pr}^{1/2}$. Suppose also that the quantities $\sup_{(x,a) \in S \times A} \| d\mu_{t+1}(x,a) \|_{\infty}$ are uniformly bounded for all $t > 0$. Then using the bound (6.9) and the bound of Theorem 5 we arrive at

\[ \mathbb{E} \rho_p (S \times A) \lesssim D_{t+1}(H, K, K_{pr}, L) + \frac{H K K_{pr}^{1/2} L_{\gamma,K,pr} (\sqrt{d_A} + \sqrt{d_S})}{\sqrt{M}} + \frac{(H - t - 1) H K K_{pr}^{3/2} \mathcal{R}_{\max} L_{\gamma,K,pr} \sqrt{d_A}}{\sqrt{N}}, \]

(7.7)

where $D_{t+1}(H, K, K_{pr}, L)$ denotes the deterministic part of the error reflecting the approximation properties of the systems $\mathcal{R}_{\max}$, $\psi_K$ and the interpolation error due to finite $L$ (see the above discussion for some quantitative estimates). Under the above assumptions, including (7.4), one obtains from (7.5), Theorem 4 and Theorem 5

\[ D_{t+1}(H, K, K_{pr}, L) \lesssim K^{-\beta_0} + H K K_{pr}^{1/2} L_{\gamma,K,pr} \rho_L(S, A) + K \mathcal{R}_{\max} \sum_{l=t+1}^{H-1} \mathcal{R}_{K_{pr},l}, \]

where $\lesssim$ means inequality up to a constant not depending on $H, N, K, K_{pr}$, and $L$. This bound is again polynomial in $H$, provided that $\mathcal{R}_{\max}$ depends polynomially on $H$ (see the discussion after Theorem 1).

7.2. Convergence of upper bounds. Suppose that the estimates $\bar{\eta} = (\bar{\eta}_{t+1}(x,a), t \in [H])$ of the optimal martingale tuple $\eta^* = (\eta^*_t (x,a), t \in [H])$ are constructed based on the sampled data $D_t^{N} \vee G_t^{M} \vee \ldots \vee G_H^{M}$ such that Theorem 5 holds. Consider for $\xi := (\bar{\eta}_{t+1}(S_t(a_{<t}), a_{<t}), a_{<t} \in A_{<t}, t \in [H]),$
Thus, for the Monte Carlo estimate of $V_0^{\text{up}}(x; \xi) - V_0^*(x) = \mathbb{E}_x \left[ \sup_{a \geq 0 \in A^H} \left( \sum_{t=0}^{H-1} (R_t(S_t(a_{<t}), a_t) - \tilde{\eta}_{t+1}(S_t(a_{<t}), a_t)) + F(S_H) \right) \right]
\leq \mathbb{E}_x \left[ \sup_{a \geq 0 \in A^H} \left( \sum_{t=0}^{H-1} (R_t(S_t(a_{<t}), a_t) - \eta_{t+1}^*(S_t(a_{<t}), a_t)) \right) \right]
\leq \sum_{t=0}^{H-1} \mathbb{E}_x \left[ \sup_{(x,a) \in S \times A} |\eta_{t+1}^*(x,a) - \tilde{\eta}_{t+1}(x,a)| \right]^{1/2},

where $\mathbb{E}_x$ denotes the “all-in” expectation, i.e. including the randomness of the pre-simulation, and, the independently simulated trajectories $t \rightarrow S_t(a_{<t})$. Furthermore, similarly,

$$\text{Var} \left[ \sup_{a \geq 0 \in A^H} \left( \sum_{t=0}^{H-1} (R_t(S_t(a_{<t}), a_t) - \tilde{\eta}_{t+1}(S_t(a_{<t}), a_t)) + F(S_H) \right) \right]
= \text{Var} \left[ \sup_{a \geq 0 \in A^H} \left( \sum_{t=0}^{H-1} (R_t(S_t(a_{<t}), a_t) - \tilde{\eta}_{t+1}(S_t(a_{<t}), a_t)) + F(S_H) \right) \right]
\leq \mathbb{E}_x \left[ \left( \sum_{t=0}^{H-1} \sup_{(x,a) \in S \times A} |\eta_{t+1}^*(x,a) - \tilde{\eta}_{t+1}(x,a)| \right)^2 \right].$$

Hence for the standard deviation we get by the triangle inequality,

$$\text{Dev} \left[ \sup_{a \geq 0 \in A^H} \left( \sum_{t=0}^{H-1} (R_t(S_t(a_{<t}), a_t) - \tilde{\eta}_{t+1}(S_t(a_{<t}), a_t)) + F(S_H) \right) \right]
\leq \sum_{t=0}^{H-1} \mathbb{E}_x \left[ \sup_{(x,a) \in S \times A} |\eta_{t+1}^*(x,a) - \tilde{\eta}_{t+1}(x,a)| \right]^{1/2}.\]
where according to Proposition 6, \((\text{component wise applied to the vector function})\) sampled data estimates.

**Proof of Theorem 4.**

\[ \eta \]

\[ \text{Acknowledgments.} \] J.S. gratefully acknowledges financial support from the German science foundation (DFG) via the cluster of excellence MATH+, project AA4-2.

8. Proofs

8.1. Proof of Theorem 4. One-step analysis: Suppose that after \(h\) steps of the algorithm the estimates \(V_{H,N}, \ldots, V_{h+1,N}\) of the value functions \(V^*_H, \ldots, V^*_{h+1}\), respectively, are constructed using sampled data \(D^N_{h+1}\), such that \(\|V_{t,N}\|_\infty \leq \tilde{L}_t \leq V^*_{\max}\) a.s. for all \(t = h+1, \ldots, H\). Denote for \(a \in A\),

\[ \ell^a(\beta) := E \left[ (Z^a - \beta^T \gamma_K(X))^2 \mid D^N_{h+1} \right] \]

\[ Z^a \sim V_{h+1,N}(Y^{a,X}), \quad Y^{a,X} \sim P_{h+1}(\cdot \mid X, a), \quad X \sim \mu_h. \]

The unique minimizer of \(\ell^a(\beta)\) is given by the \(D^N_{h+1}\)-measurable vector

\[ \beta_a = E \left[ Z^a \Sigma^{-1} \gamma_K(X) \mid D^N_{h+1} \right] = E_{X \sim \mu_h} \left[ P^a_{h+1} V_{h+1,N}(X) \Sigma^{-1} \gamma_K(X) \mid D^N_{h+1} \right]. \]

For the estimation of the \(D^N_{h}\)-measurable vector \(\beta_{N,a}\) in (4.3), see (4.1), (4.2), and Assumption 1, it then holds that

\[ E_{\mu_h \otimes P} \left[ \sup_{a \in A} \left( (\beta_{N,a} - \beta_a^T) \gamma_K(X) \right)^2 \mid D^N_{h+1} \right] \]

\[ \leq E_{P} \left[ \sup_{a \in A} |\beta_{N,a} - \beta_a|^2 \mid D^N_{h+1} \right] E_{X \sim \mu_h} \left[ |\gamma_K(X)|^2 \right] \]

\[ \leq \sum_{k=1}^{K} E_{P} \left[ \sup_{a \in A} (\beta_{N,a,k} - \beta_{a,k})^2 \mid D^N_{h+1} \right] E_{X \sim \mu_h} \left[ |\gamma_K(X)|^2 \right], \]

where according to Proposition 6 (component wise applied to the vector function \(f(x, a, \varepsilon) = V_{h+1,N}(K_{h+1}(x, a, \varepsilon)) \Sigma^{-1} \gamma_K(x)\) with \(p = 2\), see (6.3)) one has for \(k = 1, \ldots, K\),

\[ E_{P} \left[ \sup_{a \in A} (\beta_{N,a,k} - \beta_{a,k})^2 \mid D^N_{h+1} \right] \leq \frac{(L_{V,K} L_K I_D(A) + L_{V,K} L_K D(A) + V^*_{\max})^{2} A_{K}^2}{N}. \]

Due to the very structure of \(V_{h+1,N}\) (see (4.1)), we further have

\[ E_{X \sim \mu_h} \left[ \sup_{a \in A} (\beta_a^T \gamma(X) - P^a_{h+1} V_{h+1,N}(X))^2 \mid D^N_{h+1} \right] \leq R^2_{K,h}. \]
and then with (8.1) and (8.2) we have the estimate

\[
(8.3) \quad \mathbb{E}_{\mu_h} \left[ \sup_{a \in A} \left( \Delta_h \left( X \right) - P_{\mu}^{a+1} V_{h+1,N}(X) \right)^2 \right]^{1/2} \leq \mathbb{E}_{\mu_h} \sup_{a \in A} \left( \beta_N^{\mu} \gamma_K(X) - P_{\mu}^{a+1} V_{h+1,N}(X) \right)^2 \leq \mathbb{E}_{\mu_h} \left[ \sup_{a \in A} \left( \beta_N^{\mu} \gamma_K(X) - \beta_a \gamma_K(X) \right)^2 \right]^{1/2} + \mathbb{E}_{X \sim \mu_h} \sup_{a \in A} \left( \beta_a \gamma_K(X) - P_{\mu}^{a+1} V_{h+1,N}(X) \right)^2 \leq 2 \gamma_{\mu,K} \Lambda_K \left( L_{V,K} L_K I_D(A) + L_{V,K} L_K D(A) + V_{\max}^* \right) \sqrt{\frac{K}{N}} + R_{K,h}.
\]

Since the right-hand-side of (8.3) is deterministic, the conditioning on $D_{h+1}^N$ may be dropped and we obtain

\[
(8.4) \quad \mathbb{E}_{\mu_h} \left[ \sup_{a \in A} \left( \Delta_h \left( X \right) - P_{\mu}^{a+1} V_{h+1,N}(X) \right)^2 \right]^{1/2} \leq 2 \gamma_{\mu,K} \Lambda_K \left( L_{V,K} L_K I_D(A) + L_{V,K} L_K D(A) + V_{\max}^* \right) \sqrt{\frac{K}{N}} + R_{K,h}.
\]

**Multi step analysis:** Let us denote for $h \in \{0, \ldots, H\}$

\[
(8.5) \quad \Delta_h^N := \tilde{\Delta}_h^{a+1} V_{h+1,N}(x) - P_{\mu}^{a+1} V_{h+1,N}(x) \quad \text{and} \quad \Delta_h(x) := \sup_{a \in A} |\Delta_h^a(x)|.
\]

Note that

\[
P_{\mu}^{a+1} P_{\mu}^{a'} \left( dx' \right) = \int S P_{\mu}^{a_1} \left( dx' \right) P_{\mu}^{a_2} \left( dx'' \right) P_{\mu}^{a_3} \left( dx''' \right).
\]

We then have

\[
V_h^a(x) - V_h^a(x) = \sup_{a \in A} \left\{ R_h(x, a) + P_{\mu}^{a+1} V_{h+1,N}(x) \right\} - \sup_{a \in A} \left\{ R_h(x, a) + \tilde{\Delta}_h^{a+1} V_{h+1,N}(x) \right\}
= R_h(x, \pi_h^a(x)) + \tilde{\Delta}_h^{a+1} V_{h+1,N}(x) - \sup_{a \in A} \left\{ R_h(x, a) + \tilde{\Delta}_h^{a+1} V_{h+1,N}(x) \right\}
\leq \int \left( V_{h+1}^* - V_{h+1,N} \right) (x') P_{\mu}^{a+1} (dx'|x, \pi_h^a(x)) + \sup_{a \in A} \left\{ R_h(x, a) + \tilde{\Delta}_h^{a+1} V_{h+1,N}(x) \right\} - \sup_{a \in A} \left\{ R_h(x, a) + \tilde{\Delta}_h^{a+1} V_{h+1,N}(x) \right\}
\leq P_{\mu}^{a+1} \left( V_{h+1}^* - V_{h+1,N} \right) (x) + \Delta_h(x)
\]

and analogously,

\[
(8.7) \quad V_h^a(x) - V_h^a(x) \geq P_{\mu}^{a+1} \left[ V_{h+1}^* - V_{h+1,N} \right] (x) - \Delta_h(x).
\]

By iterating (8.6) and (8.7) upwards, and using that $V_{H,N} = V_{H}^f$, we obtain, respectively,

\[
V_h^a(x) - V_h^a(x) \leq \sum_{k=1}^{H-h-1} P_{\mu}^{a+1} \cdots P_{\mu}^{a+k-1} [\Delta_{h+k}](x) + \Delta_h(x), \quad \text{and}
\]

\[
V_h^a(x) - V_h^a(x) \geq - \sum_{k=1}^{H-h-1} P_{\mu}^{a+1} \cdots P_{\mu}^{a+k-1} [\Delta_{h+k}](x) - \Delta_h(x).
\]
We thus have pointwise,

\[ |V^*_h(x) - V_{h,N}(x)| \leq \sum_{k=1}^{H-h-1} \lambda^{\pi_h^{k+1}} P^{\pi_h^{k+1}} \cdots P^{\pi_h^{k+1}} [\Delta_{h+k}](x) \]

which implies

\[ \|V^*_h - V_{h,N}\|_{L^2(\mu_h \otimes P)} \leq 2 \sup_{x,a,\varepsilon} \sum_{k=1}^{H-h-1} \|P^{\pi_h^{k+1}} \cdots P^{\pi_h^{k+1}}[\Delta_{h+k}]\|_{L^2(\mu_h \otimes P)} + \|\Delta_h\|_{L^2(\mu_h \otimes P)}. \]

Hence we have due to Assumption [4]

\[ \|V^*_h - V_{h,N}\|_{L^2(\mu_h \otimes P)} \leq 2R_{\text{max}} \sum_{l=h}^{H-1} \|\Delta_l\|_{L^2(\mu_l \otimes P)} \]

(note that \( R_{\text{max}} \geq 1 \), and then, by the definitions [8.3] and the estimate [8.4], the statement of the theorem follows.

8.2. **Proof of Theorem** [5] For the unique minimizer of (7.3) one has that

\[ c_K(x, a) := \Sigma_{E,K}^1 E[V^*_l(\mathcal{K}_{l+1}(x, a, \varepsilon))\psi_K(\varepsilon)]. \]

Likewise, the unique minimizer of the problem

\[ \inf_{c \in \mathbb{R}^{K}} E_{\varepsilon \sim \pi_e} \left( (V_{l+1,N}(\mathcal{K}_{l+1}(x, a, \varepsilon)) - c^T \psi_K(\varepsilon))^2 \right) \] is given by

\[ c_K(x, a) := \Sigma_{E,K}^1 E_{\varepsilon \sim \pi_e} \left( V_{l+1,N}(\mathcal{K}_{l+1}(x, a, \varepsilon))\psi_K(\varepsilon)|D_{l+1}^N \right). \]

Now let \( c_{K,M}(x, a) \) be the Monte Carlo estimate of \( c_K(x, a) \) as constructed in Section [6], see (5.9) and (5.10). We then have

\[ \mathbb{E}_{\pi_e \otimes P} \left[ \sup_{(x,a) \in S \times A} (c_{K,M} - c_K)^T(x, a)\psi_K(\varepsilon) \right]^2 \leq \mathbb{E}_{\pi_e \otimes P} \left[ \sup_{(x,a) \in S \times A} (c_{K,M} - c_K)^T(x, a) \right]^2 \mathbb{E}_{\varepsilon \sim \pi_e} [\psi_K(\varepsilon)^2] \]

where according to Proposition [5] (applied componentwise with \( p = 2 \) to the vector function \( f(x, a, \varepsilon) = V_{l+1,N}(\mathcal{K}_{l+1}(x, a, \varepsilon))\Sigma_{E,K}^1 \psi_K(\varepsilon) \), see (5.3))

\[ \mathbb{E}_{\pi_e} \left[ \sup_{(x,a) \in S \times A} |(c_{K,M} - c_K)(x, a)|^2 \right] \leq K(LV_{K,M}L_KI_D(S \times A) + L_{V,K,M}L_KD(S \times A) + V_{\text{max}}^*)^2 \Lambda_{E,K}^2. \]

Since for any pair \( (x, a) \in S \times A, \)

\[ |(c_K - c_K)(x, a)|^2 = |\mathbb{E}_{\varepsilon \sim \pi_e} [V^*_l(\mathcal{K}_{l+1}(x, a, \varepsilon)) - V_{l+1,N}(\mathcal{K}_{l+1}(x, a, \varepsilon))\Sigma_{E,K}^1 \psi_K(\varepsilon)|D_{l+1}^N \] \leq \int |V^*_l(\mathcal{K}_{l+1}(x, a, \varepsilon)) - V_{l+1,N}(\mathcal{K}_{l+1}(x, a, \varepsilon))|^2 d\mathbb{P}(\varepsilon) \int |\Sigma_{E,K}^1 \psi_K(\varepsilon)|^2 d\mathbb{P}(\varepsilon) \leq K\Lambda_{E,K} \sup_{(x,a) \in S \times A} \left\| \frac{d\mathbb{P}_0}{d\mu_{l+1}(\cdot)} \right\|_{\infty} \int |V^*_l(y) - V_{l+1,N}(y)|^2 \mu_{l+1}(dy), \]

\[ \int |V^*_l(\mathcal{K}_{l+1}(x, a, \varepsilon)) - V_{l+1,N}(\mathcal{K}_{l+1}(x, a, \varepsilon))|^2 d\mathbb{P}(\varepsilon) \int |\Sigma_{E,K}^1 \psi_K(\varepsilon)|^2 d\mathbb{P}(\varepsilon) \leq K\Lambda_{E,K} \sup_{(x,a) \in S \times A} \left\| \frac{d\mathbb{P}_0}{d\mu_{l+1}(\cdot)} \right\|_{\infty} \int |V^*_l(y) - V_{l+1,N}(y)|^2 \mu_{l+1}(dy), \]
we have

\[
\mathbb{E}_{\varepsilon \sim P_{\varepsilon}} \left[ \max_{(x,a) \in S_L \times A_L} |(c_K - \bar{c}_K)(x,a)|^2 |D_{t+1}^N \right] \\
\leq \max_{(x,a) \in S_L \times A_L} \left| (c_K - \bar{c}_K)(x,a) \right|^2 \mathbb{E}_{\varepsilon \sim P_{\varepsilon}} \left[ \left| \psi_K(\varepsilon) \right|^2 \right] \\
\leq K \varrho_{\psi,K}^2 \Lambda_{E,K}^2 \sup_{(x,a) \in S \times A} \left\| \frac{dP_{t+1}(-|x,a)}{d\mu_{t+1}(-)} \right\|_{\infty} \left\| V_{t+1}^* - V_{t+1,N} \right\|_{L^2(\mu_{t+1})}^2.
\]

Next due to (6.3), we derive for any \( k \in [K] \),

\[
|c_{k,M}(x,a) - c_{k,M}(x',a')| \\
\leq \frac{1}{M} \sum_{m=1}^M \left| V_{t+1,N}(K_{t+1}(x,a,\tilde{\varepsilon}_m)) - V_{t+1,N}(K_{t+1}(x',a,\tilde{\varepsilon}_m)) \right| \left| \sum_{m=1}^M \psi_K(\tilde{\varepsilon}_m) \right|_{\infty} \\
\leq L_{V,K} L_{K,A,K} \rho((x,a),(x',a'))
\]

and so with \( I[c_{K,M}] := (I[c_{1,M}], \ldots, I[c_{K,M}])^\top \) we further have

\[
\mathbb{E}_{P_{\varepsilon} \otimes P_{\mu}} \left[ \sup_{(x,a) \in S \times A} \left| \eta_{t+1,K,M}(x,a) - \tilde{\eta}_{t+1,K,M}(x,a) \right|^2 |D_{t+1}^N \right] \\
= \mathbb{E}_{P_{\varepsilon} \otimes P_{\mu}} \left[ \sup_{(x,a) \in S \times A} \left| (c_{K,M} - I[c_{K,M}])^\top (x,a) \psi_K(\varepsilon_{t+1}) \right|^2 |D_{t+1}^N \right] \\
\leq \varrho_{\psi,K}^2 \mathbb{E}_{P_{\varepsilon}} \left[ \sup_{(x,a) \in S \times A} \left| (c_{K,M} - I[c_{K,M}])^\top (x,a) \right|^2 |D_{t+1}^N \right] \\
\leq \varrho_{\psi,K}^2 \sum_{k=1}^K \mathbb{E}_{P_{\mu}} \left[ \sup_{(x,a) \in S \times A} \left| (c_{k,M} - I[c_{k,M}])^\top (x,a) \right|^2 |D_{t+1}^N \right] \\
\leq K \varrho_{\psi,K}^2 L_{V,K}^2 L_{K,A,K}^2 \rho_{\varepsilon}(S,A),
\]

using (7.1). Finally note that

\[
\eta_{t+1,K} - \tilde{\eta}_{t+1,K,M} = (c_K - \bar{c}_K)^\top \psi_K + (c_K - c_{K,M})^\top \psi_K + \eta_{t+1,K,M} - \tilde{\eta}_{t+1,K,M}
\]

and then the result follows by the triangle inequality, gathering (8.9)–(8.12), and finally taking the unconditional expectation \( \mathbb{E}_{P_{\varepsilon} \otimes P_{\mu}} \).
Appendix A. Some auxiliary notions

The Orlicz 2-norm of a real valued random variable \( \eta \) with respect to the function \( \varphi(x) = e^{x^2} - 1, x \in \mathbb{R} \), is defined by \( \| \eta \|_{\varphi, 2} := \inf \{ t > 0 : \mathbb{E} \left[ \exp \left( \frac{\eta^2}{t^2} \right) \right] \leq 2 \} \). We say that \( \eta \) is sub-Gaussian if \( \| \eta \|_{\varphi, 2} < \infty \). In particular, this implies that for some constants \( C, c > 0 \),

\[
P(|\eta| \geq t) \leq 2 \exp \left( -\frac{c t^2}{\| \eta \|_{\varphi, 2}^2} \right) \quad \text{and} \quad \mathbb{E}[|\eta|^p]^{1/p} \leq C \sqrt{p} \| \eta \|_{\varphi, 2} \quad \text{for all} \quad p \geq 1.
\]

Consider a real valued random process \( (X_t)_{t \in \mathcal{T}} \) on a metric parameter space \( (\mathcal{T}, d) \). We say that the process has sub-Gaussian increments if there exists \( K \geq 0 \) such that

\[
\| X_t - X_s \|_{\varphi, 2} \leq K d(t, s), \quad \forall t, s \in \mathcal{T}.
\]

Let \( (Y, \rho) \) be a metric space and \( X \subseteq Y \). For \( \varepsilon > 0 \), we denote by \( \mathcal{N}(X, \rho, \varepsilon) \) the covering number of the set \( X \) with respect to the metric \( \rho \), that is, the smallest cardinality of a set (or net) of \( \varepsilon \)-balls in the metric \( \rho \) that covers \( X \). Then \( \log \mathcal{N}(X, \rho, \varepsilon) \) is called the metric entropy of \( X \)

\[
I_D(X) := \int_0^{D(X)} \sqrt{\log \mathcal{N}(X, \rho, u)} \, du
\]

with \( D(X) := \text{diam}(X) := \max_{x, x' \in X} \rho(x, x') \), is called the Dudley integral. For example, if \( |X| < \infty \) and \( \rho(x, x') = 1_{\{x \neq x'\}} \) we get \( I_D(X) = \sqrt{\log |X|} \).

Appendix B. Estimation of mean uniformly in parameter

The following proposition holds.

**Proposition 6.** Let \( f \) be a function on \( X \times \Xi \) such that

\[
|f(x, \xi) - f(x', \xi)| \leq L \rho(x, x')
\]

with some constant \( L > 0 \). Furthermore assume that \( \|f\|_\infty \leq F < \infty \) for some \( F > 0 \). Let \( \xi_n, n = 1, \ldots, N \), be i.i.d. sample from a distribution on \( \Xi \). Then we have

\[
\mathbb{E}[f(x, \xi_n) - \mathbb{E}[f(x, \xi_n)]]^{\frac{1}{p}} \leq \frac{LI_D + (LD + F)\sqrt{p}}{\sqrt{N}},
\]

where \( \lesssim \) may be interpreted as \( \leq \) up to a natural constant.

**Proof.** Denote

\[
Z(x) := \frac{1}{\sqrt{N}} \sum_{n=1}^N (f(x, \xi_n) - M_f(x))
\]

with \( M_f(x) = \mathbb{E}[f(x, \xi)] \), that is, \( Z(x) \) is a centered random process on the metric space \( (X, \rho) \).

Below we show that the process \( Z(x) \) has sub-Gaussian increments. In order to show it, let us introduce

\[
Z_n = f(x, \xi_n) - M_f(x) - f(x', \xi_n) + M_f(x')
\]

Under our assumptions we get

\[
\| Z_n \|_{\varphi, 2} \lesssim L \rho(x, x'),
\]

that is, \( Z_n \) is subgaussian for any \( n = 1, \ldots, N \). Since

\[
Z(x) - Z(x') = N^{-1/2} \sum_{n=1}^N Z_n,
\]

is a sum of independent sub-Gaussian r.v, we may apply [25 Proposition 2.6.1 and Eq. (2.16)] to obtain that \( Z(x) \) has sub-Gaussian increments with parameter \( K \approx L \). Fix some \( x_0 \in X \). By the triangular inequality,

\[
\sup_{x \in X} |Z(x)| \leq \sup_{x, x' \in X} |Z(x) - Z(x')| + |Z(x_0)|.
\]

By the Dudley integral inequality, e.g. [25 Theorem 8.1.6], for any \( \delta \in (0, 1) \),

\[
\sup_{x, x' \in X} |Z(x) - Z(x')| \lesssim L \left[ I_D + D \sqrt{\log(2/\delta)} \right]
\]
holds with probability at least $1 - \delta$. Again, under our assumptions, $Z(x_0)$ is a sum of i.i.d. bounded centered random variables with $\psi_2$-norm bounded by $F$. Hence, applying Hoeffding’s inequality, e.g. [25, Theorem 2.6.2.], for any $\delta \in (0, 1)$,

$$|Z(x_0)| \lesssim F \sqrt{\log(1/\delta)}.$$  

□

References

[1] Leif Andersen and Mark Broadie. A Primal-Dual Simulation Algorithm for Pricing Multi-Dimensional American Options. *Management Science*, 50(9):1222–1234, 2004.

[2] András Antos, Rémi Munos, and Csaba Szepesvári. Fitted q-iteration in continuous action-space mdps. 2007.

[3] Mohammad Gheshlaghi Azar, Ian Osband, and Rémi Munos. Minimax regret bounds for reinforcement learning. 70:263–272, 06–11 Aug 2017.

[4] Sven Balder, Antje Mahayni, and John Schoenmakers. Primal-dual linear Monte Carlo algorithm for multiple stopping—an application to flexible caps. *Quant. Finance*, 13(7):1003–1013, 2013.

[5] Nicole Bäuerle and Ulrich Rieder. *Markov decision processes with applications to finance*. Universitext. Berlin: Springer, 2011.

[6] Christian Bayer, Martin Redmann, and John Schoenmakers. Dynamic programming for optimal stopping via pseudo-regression. *Quant. Finance*, 21(1):29–44, 2021.

[7] Gleb Beliakov. Interpolation of Lipschitz functions. *Journal of computational and applied mathematics*, 196(1):20–44, 2006.

[8] Denis Belomestny, Christian Bender, and John Schoenmakers. True upper bounds for bermudan products via non-nested Monte Carlo. *Math. Finance*, 19(1):53–71, 2009.

[9] Denis Belomestny, Anastasia Kolodko, and John Schoenmakers. Regression methods for stochastic control problems and their convergence analysis. *SIAM J. Control Optim.*, 48(5):3562–3588, 2010.

[10] Christian Bender, John Schoenmakers, and Jianing Zhang. Dual representations for general multiple stopping problems. *Math. Finance*, 25(2):339–370, 2015.

[11] Dimitri P. Bertsekas and John N. Tsitsiklis. *Neuro-Dynamic Programming*. Athena Scientific, 1996.

[12] David B Brown, James E Smith, et al. Information relaxations and duality in stochastic dynamic programs: A review and tutorial. *Foundations and Trends® in Optimization*, 5(3):246–339, 2022.

[13] David B. Brown, James E. Smith, and Peng Sun. Information relaxations and duality in stochastic dynamic programs. *Oper. Res.*, 58(4, part 1):785–801, 2010.

[14] Vijay V. Desai, Vivek F Farias, and Ciamac C Moallemi. Bounds for markov decision processes. *Reinforcement learning and approximate dynamic programming for feedback control*, pages 452–473, 2012.

[15] Martin Haugh and Leonid Kogan. Pricing American options: A duality approach. *Oper. Res.*, 52(2):258–270, 2004.

[16] Chi Jin, Zeyuan Allen-Zhu, Sebastien Bubeck, and Michael I Jordan. Is q-learning provably efficient? In S. Bengio, H. Wallach, H. Larochelle, K. Grauman, N. Cesa-Bianchi, and R. Garnett, editors, *Advances in Neural Information Processing Systems*, volume 31. Curran Associates, Inc., 2018.

[17] Bernardo Ávila Pires and Csaba Szepesvári. Policy error bounds for model-based reinforcement learning with factored linear models. In *Conference on Learning Theory*, pages 121–151. PMLR, 2016.

[18] Martin L. Puterman. *Markov decision processes: discrete stochastic dynamic programming*. Wiley Ser. Probab. Math. Stat. New York, NY: John Wiley & Sons, Inc., 1994.

[19] A. Reznikov and E. B. Saff. The covering radius of randomly distributed points on a manifold. *Int. Math. Res. Not. IMRN*, (19):6065–6094, 2016.

[20] L. Rogers. Pathwise stochastic optimal control. *SIAM J. Control and Optimization*, 46:1116–1132, 01 2007.

[21] Leonard C. G. Rogers. Monte Carlo valuation of American options. *Mathematical Finance*, 12(3):271–286, 2002.

[22] J. Schoenmakers. A pure martingale dual for multiple stopping. *Finance Stoch.*, 16:319–334, 2012.

[23] R. S. Sutton and Andrew G. Barto. *Reinforcement Learning: An Introduction*. The MIT Press, second edition, 2018.

[24] Csaba Szepesvári. Algorithms for reinforcement learning. *Synthesis lectures on artificial intelligence and machine learning*, 4(1):1–103, 2010.

[25] Roman Vershynin. *High-dimensional probability*, volume 47 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, 2018. An introduction with applications in data science, With a foreword by Sara van de Geer.

[26] Daniel Z. Zanger. Quantitative error estimates for a least-squares Monte Carlo algorithm for American option pricing. *Finance and Stochastics*, 17(3):503–534, 2013.

[27] Helin Zhu, Fan Ye, and Enlu Zhou. Solving the dual problems of dynamic programs via regression. *IEEE Transactions on Automatic Control*, 63(5):1340–1355, 2017.
1Faculty of Mathematics, Duisburg-Essen University, Thea-Leymann-Str. 9, D-45127 Essen, Germany
   Email address: denis.belomestny@uni-due.de

2Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39, 10117 Berlin, Germany
   Email address: schoenma@wias-berlin.de