$\eta$-glueball mixing from $N_f = 2$ lattice QCD

Xiangyu Jiang,1,2,∗ Wei Sun,1,† Feiyu Chen,1,2 Ying Chen,1,2,‡ Ming Gong,1,2 Zhaofeng Liu,1,2,3 and Renqiang Zhang1,2

1Institute of High Energy Physics, Chinese Academy of Sciences, Beijing 100049, People’s Republic of China
2School of Physical Sciences, University of Chinese Academy of Sciences, Beijing 100049, People’s Republic of China
3Center for High Energy Physics, Peking University, Beijing 100871, People’s Republic of China

We perform the first lattice study on the mixing of the isoscalar pseudoscalar meson $\eta$ and the pseudoscalar glueball $G$ in the $N_f = 2$ QCD at the pion mass $m_\pi \approx 350$ MeV. The $\eta$ mass is determined to be $m_\eta = 714(16)$ MeV. Through the Witten-Veneziano relation, this value can be matched to a mass value of $\sim 981$ MeV for the SU(3) counterpart of $\eta$. Based on a large gauge ensemble, the $\eta - G$ mixing energy and the mixing angle are determined to be $|z| = 107(15)(2)$ MeV and $|\theta| = 3.46(46)^\circ$ from the $\eta - G$ correlators that are calculated using the distillation method. We conclude that the $\eta - G$ mixing is tiny and the topology induced interaction contributes most of $\eta$ mass owing to the QCD U(1) anomaly.

I. INTRODUCTION

Chiral symmetry is an intrinsic symmetry of quantum chromodynamics (QCD) in the massless limit of quarks and is spontaneously broken due to the nonzero quark condensate. The spontaneous symmetry breaking (SSB) of the SU(3) × SU(3) into the flavor SU(3) generates eight Goldstone particles sorted into the pseudoscalar fields. Using the physical mass of an additional Goldstone particle, namely, a light pseudoscalar glueball in the $N_f = 2$ SU(3) sector of the chiral symmetry, it breaks into U(1) that corresponds to the baryon number conservation expects the existence of an additional Goldstone particle, namely, a light flavor singlet pseudoscalar meson. The $\eta$ meson is predominantly a flavor singlet but is too massive to be taken as a candidate for this Goldstone particle. The $\eta'$ meson is its large production rate in the $J/\psi$ radiative decays [5], which are abundant in gluons and are expected to favor the production of glueballs. This observation along with the original mechanism of $\eta'$ mass manifests the strong coupling of $\eta'$ to gluons and thereby prompts the conjecture that $\eta'$ may mix substantially with pseudoscalar glueballs since they have the same quantum number. The KLOE Collaboration analyzed the processes $\phi \to \gamma \eta$ and $\phi \to \gamma \eta'$ and found that the $\eta'$-glueball mixing might be required and the mixing angle can be as large as $(22 \pm 3)^\circ$ [6]. In contrast, another phenomenological analysis of the KLOE result gave the mixing angle $(12 \pm 13)^\circ$ which is consistent with zero within the large error [7]. A phenomenological analysis of the processes $\psi(\psi')$ decaying into a pseudoscalar and a vector final states obtained the $\eta'$-glueball mixing angle to be around $9^\circ$ by considering the $\eta$-$\eta'$-glueball mixing model [8]. Obviously, the determined mixing angle varies in a fairly large range. Based on the $\eta'$-glueball mixing picture, there have been theoretical discussions on the possibility of $\eta(1405)$ as a pseudoscalar glueball candidate [8–12]. However, the quenched lattice QCD studies [13–15] predict that the mass of the pseudoscalar glueball is around 2.4–2.6 GeV, which is confirmed by lattice simulations with dynamical quarks [16–19]. This raised a question on $\eta(1405)$ as a glueball candidate because of its much lighter mass. On the other hand, the strong hint for $\eta(1405)$ to be a glueball candidate is the observation that there exist three isoscalar pseudoscalar mesons $\eta(1295)$, $\eta(1405)$ and $\eta(1475)$ such that one of them is surplus according to the quark model. If $\eta(1405)$ and $\eta(1475)$ are the same state [20], there is no need for a pseudoscalar glueball state in this mass region. Some mixing model studies also favor the pseudoscalar glueball to have a mass heavier than 2 GeV [21, 22].

In this work, we will investigate the possible mixing of isoscalar pseudoscalar meson and the pseudoscalar glueball in $N_f = 2$ lattice QCD. The isoscalar pseudoscalar meson is named $\eta$ throughout this work, which is the SU(2) counterpart of the SU(3) flavor singlet (approximately $\eta'$) in the $N_f = 3$ case. We have

∗ jiangxiangyu@ihep.ac.cn
† sunwei@ihep.ac.cn
‡ cheny@ihep.ac.cn
generated a large ensemble of gauge configurations with \( N_f = 2 \) degenerated \( u, d \) quarks at a pion mass \( m_\pi \approx 350 \) MeV, so we can make a precise determination of \( \eta \) mass. The calculation of \( \eta' \) mass (and the \( \eta' - \eta \) mixing) has been performed in several \( N_f = 2 + 1 \) lattice QCD studies, whose results are in agreement with the physical value after the chiral extrapolation [23–26]. A systematic and comprehensive lattice study on the properties of \( \eta \) and \( \eta' \) is presented in Ref. [27], where the masses, the decay constants and the gluonic matrix elements of \( \eta \) and \( \eta' \) have been derived to a high precision. There are also many studies on the \( \eta \) mass from \( N_f = 2 \) lattice QCD [19, 28–30]. According to the Witten-Veneziano mechanism (WV), in the \( N_f = 2 \) case, pion mass and \( \eta \) mass are related as \( m_\eta^2 = m_\pi^2 + m_0^2 \), where \( m_0^2 \) is the correction from the topology induced interaction and is proportional to \( N_f \). As a check of WV, we would like to take a look at this relationship and use the obtained \( m_0^2 \) to predict the \( \eta' \) mass in the physical case (a pioneering work following this way in the quenched approximation can be found in Ref. [31]). After that, we calculate the correlation functions of the \( \eta \) operator and the glueball operator, from which the mixing angle can be extracted. The strategy of the study is similar to that used in the \( \eta_c \)-glueball mixing [32]. As an exploratory study, we tentatively treat the pseudoscalar glueball as a stable particle and ignore its resonance nature in the presence of light sea quarks. Obviously, the numeric task involves the calculation of the annihilation diagrams of \( u, d \) quarks, so we adopt the distillation method [33] which enables the gauge covariant smearing of quark fields and the all-to-all quark propagators (perambulators) simultaneously. Since we also have the perambulators of the valence charm quark, we also calculate the \( \eta-\eta_c \) correlation functions and explore their properties.

This paper is organized as follows: In Sec. II we describe the lattice setup, operator construction and formulation of correlation functions. Section III gives the theoretical formalism of the meson-glueball mixing, where the data analyses and the results can be found. The discussion and summary are given in Sec. IV, and the preliminary results from \( \eta-\eta_c \) correlation functions are presented here.

II. NUMERICAL DETAILS

A. Lattice setup

We generate gauge configurations with \( N_f = 2 \) degenerate \( u, d \) on an \( L^3 \times T = 16^3 \times 128 \) anisotropic lattice. We use the tadpole-improved Symanzik’s gauge action for anisotropic lattices [13, 15] and the tadpole-improved anisotropic clover fermion action [19, 34]. The parameters in the action are tuned to give the anisotropy \( \xi = a_s/a_t \approx 5.3 \), where \( a_t \) and \( a_s \) are the temporal and spatial lattice spacings, respectively. The anisotropy \( \xi \approx 5.3 \) is checked by the dispersion relation of the pseudoscalar \( \pi \) (along with that of \( \eta \) calculated using the distillation method, see Sec. III), which takes the continuum form

\[
E^2_X(p) a_t^2 = m^2_X a_t^2 + \frac{1}{\xi^2} |p|^2 a_s^2,
\]

where \( X \) refers to a specific hadron state and \( p \) is the spatial momentum \( p = \frac{2\pi}{a_T} \hat{n} \) on the lattice, and \( \hat{n} \) is the mode of spatial momentum. Figure 1 shows the energies obtained from the correlation functions of \( \pi, \eta \) and \( \rho \) at different spatial momentum modes up to \( \vec{n} = (1, 2, 2) \). The data points of \( \pi \) and \( \eta \) fall on straight lines perfectly and can be well described by Eq. (1) with \( \xi = 5.365(5) \) and 5.34(3), respectively (illustrated as shaded lines in the figure). The fit to energies in the \( \rho \) channel using Eq. (1) gives \( \xi = 5.58(1) \) which deviates from 5.3 drastically. This can be definitely attributed to the influence of nearby P-wave \( \pi\pi \) scattering states that have the same center-of-mass momentum as \( \rho \). So we do not use \( \rho \) meson to check the \( \xi \) value.

There are subtleties in the determination of the lattice spacing for our lattice setup. We intend to generate gauge configurations at a pion mass \( m_\pi \sim 300 \) MeV. As the first step, we calculate the static potential

\[
V(r) = V_0 - \frac{e_c}{r} + \sigma r
\]

through Wilson loops to determine \( e_c \) and the string tension \( \sigma a_s^2 \) (in lattice units). Then the spatial lattice spacing \( a_s \) is determined in the unit of the Sommer’s scale parameter \( r_0 \) according to the condition \( \frac{dV}{dr} |_{r=r_0} = $
Even though the reason is still unknown for this relation, the masses of \( \eta \), \( \pi \), \( \rho \), and \( \omega \) mesons \([5]\) are collected in their respective systematics \([5]\\). Since our quark mass is substantially far from the chiral limit, it is more reasonable to use the value extrapolated from the physical pion state. The second error when listing our physical values is due to the uncertainty of \( \Delta m^2 \), which is 2 \( \text{GeV}^2 \), where the second error is due to the uncertainty of \( \Delta m^2 \). On the other hand, the mass of \( \eta \) is not a physical state, the mass squared difference \( m_{\eta}^2 - m_{\eta}^2 \approx 0.570 \text{ GeV}^2 \) also satisfies the empirical relation in Eq. (4). In this study, the dimensionless masses of \( \pi \) and \( \rho \) is determined to be \( m_{\pi} a_t = 0.5055(13) \) and \( m_{\rho} a_t = 0.1204(20) \). In this sense, we assume the relation of Eq. (4) is somewhat general for heavy-light mesons and use it to set the scale parameter \( a_t \). Of course, one should take caution to use this relation since \( \rho \) is experimentally a wide resonance and decays into \( P \)-wave \( \pi \pi \) states by 99\%. On our lattice, the lowest \( P \)-wave \( \pi \pi \) energy threshold in the rest frame of \( \pi \pi \) is 2\( E_{\pi}(p) a_t \approx 0.1795 \) with \( \xi \approx 5.3 \), which is substantially higher than \( m_{\rho} a_t \). This means \( \rho \) in its rest frame is a stable particle, such that the mass value \( m_{\rho} \) is reliable. In practice, we make the least squares fitting to the mass squared differences over the \( m_{\eta}, m_{\eta} \), \( m_{\rho} \), and \( m_{\omega} \) systems where \( n \) refers to the \( u, d \) quarks, and get the value \( \Delta m^2 = 0.568(8) \text{ GeV}^2 \), which serves as an input to the lattice scale parameter \( a_t^{-1} = 6.894(51) \text{ GeV} \) and the corresponding spatial lattice spacing \( a_s \approx 0.1517(11) \text{ fm} \). We emphasize that the errors quoted in the values of \( a_t \) and \( a_s \) include only the statistical errors of \( m_{\pi} a_t \) and \( m_{\rho} a_t \), as well as the uncertainty of \( \Delta m^2 \).

Accordingly, the \( u, d \) mass parameter in this study gives \( m_{\pi} = 348.5(1.0)(2.6) \text{ MeV} \) and \( m_{\rho} = 830.5(6.3)(6.1) \text{ MeV} \), where the second error is due to the uncertainty of \( a_t^{-1} \). For most of this paper, we will not show the second error when listing our physical values. This means that most of the errors below are just statistical errors. We will bring the \( a_t^{-1} \) error back to physical values in conclusion. Using the \( m_{\pi} \), we have \( m_{\pi} L_s \approx 3.9 \) for this lattice setup, which warrants the small finite volume effects. The number of configurations of our gauge ensemble is 6991, which is crucial for the glueball-relevant studies. The details of the gauge ensemble are given in Table II.

For the valence charm quark, we adopt the clover fermion action in Ref. [37], and the charm quark mass parameter is tuned to give \( (m_{q_c} + 3m_{J/\psi})/4 = 3.069 \text{ GeV} \). With the tuned charm quark mass parameter, we generate the perambulators of charm quark on our ensemble, from which the masses of \( \eta_c \) and \( J/\psi \) are derived precisely to be \( m_{\eta_c} = 2.9750(3) \text{ GeV} \) and \( m_{J/\psi} = 3.0988(4) \text{ GeV} \). The according 1\( S \)-hyperfine splitting is \( \Delta m_{HF} = m_{J/\psi} - m_{\eta_c} = 123.8(5) \text{ MeV} \). We also check the dispersion relation in Eq. (1) for \( \eta_c \) and \( J/\psi \) up to the momentum mode \( \vec{n} = (1, 2, 2) \). As shown in Fig. 2, the dispersion relation is almost perfectly satisfied with \( \xi = 5.341(2) \) and \( 5.307(5) \) for \( \eta_c \) and \( J/\psi \), respectively.

As a further check of our scale setting scheme, we also calculate the masses of \( D \) and \( D^* \) and get \( m_D = 1.882(1) \text{ GeV} \) and \( m_{D^*} = 2.023(1) \text{ GeV} \) on a fraction of configurations of our ensemble. The hyperfine splitting

---

**Table I.** Experimental values of the masses of the pseudoscalar (P) and vector mesons (V) of quark configurations \( n\bar{n}, n\bar{s}, n\bar{c}, s\bar{c}, n\bar{b} \) and \( s\bar{b} \) \([5]\). Here \( n \) refers to the light \( u, d \) quarks. The right most column lists the \( m_P^2 - m_S^2 \) \((\text{GeV}^2)\). The hyperfine splitting is derived precisely to be \( m_{\eta} = 0.686(4) \text{ GeV} \) from lattice QCD by the HPQCD collaboration \([35]\). Even though \( \eta \) is not a physical state, the mass squared difference \( m_{\eta}^2 - m_{\eta}^2 \approx 0.570 \text{ GeV}^2 \) also satisfies the empirical relation in Eq. (4).

| \( g\bar{q} \) | \( m_P \) (GeV) | \( m_S \) (GeV) | \( m_P^2 - m_S^2 \) (GeV\(^2\)) |
|------------|-------------|-------------|-----------------|
| \( n\bar{n} \) | 0.775 | 0.140 | 0.581 |
| \( n\bar{s} \) | 0.896 | 0.494 | 0.559 |
| \( s\bar{s} \) | 1.020 | 0.686 \([35]\) | 0.570 |
| \( n\bar{c} \) | 2.010 | 1.870 | 0.543 |
| \( s\bar{c} \) | 2.112 | 1.968 | 0.588 |
| \( n\bar{b} \) | 5.325 | 5.279 | 0.481 |
| \( s\bar{b} \) | 5.415 | 5.367 | 0.523 |

---

**Table II.** Parameters of the gauge ensemble.

| \( L^3 \times T \) | \( \beta \) | \( a_0^{-1} \) (GeV) | \( \xi \) | \( m_\pi \) (MeV) | \( N_{\text{ch}} \) |
|---------------|-----------|-------------------|---------|-----------------|-------|
| \( 16^3 \times 128 \) | 2.0 | 6.894(51) | \( \sim 5.3 \) | 348.5(1.0) | 6991 |
The dispersion relations of $\eta_c$ and $J/\psi$. The data points are lattice results and the shaded lines illustrate the fit results using Eq. (1). The fitted results of $\xi$ are $5.341(2)$, $5.307(5)$ for $\eta_c$ and $J/\psi$, respectively.

**TABLE III.** The masses of $J/\psi$, $\eta_c$, $D$ and $D^*$. The PDG value of $m_{D^{*+}}$ is the average of masses of $D(2400)^0$ and $D(2400)^+$.

| $X$ | $\eta_c$ (GeV) | $J/\psi$ | $D$ | $D^*$ |
|---|---|---|---|---|
| $m_X$ (GeV) | 2.975(3) | 3.098(4) | 1.882(1) | 2.023(1) |
| PDG [5] | 2.983 | 3.097 | $\sim 1.867$ | $\sim 2.008$ |
| $\xi$ | $5.341(2)$ | $5.307(5)$ | $5.32(2)$ | $5.31(3)$ |

$\Delta_{\text{HFS}}(D) = m_{D^*} - m_D = 0.141(2)$ GeV almost reproduces the experimental values $m_{D^{*0}} - m_{D^0} = 0.1420(7)$ GeV and $m_{D^{*+}} - m_{D^+} = 0.14060(7)$ GeV [5]. This manifests that our tuning of charm quark mass works well. The dispersion relation Eq. (1) is also checked to be correct for $D$ and $D^*$ with $\xi = 5.32(2)$ and $5.31(3)$, respectively. The figure is similar to Fig. 1 and Fig. 2 and is omitted here to save space.

Table III collects the results of $\eta_c$, $J/\psi$, $D$ and $D^*$ mesons. Along with the values of $\xi$ derived from the dispersion relations of $\pi$ and $\eta$, we can see that the values of $\xi$ for different mesons are in agreement with the value $\xi = 5.30$ during the parameter tuning and are consistent with each other within 1%.

**B. Operator construction and distillation method**

The principal goal of this work is to investigate the possible mixing of the pseudoscalar glueball and the pseudoscalar $q\bar{q}$ meson, therefore the quark annihilation diagrams should be taken care of. For this to be done, we adopt the distillation method [33] which provides a smearing scheme for quark fields (Laplacian Heaviside smearing) and the calculation strategy of the all-to-all propagators of the smeared quark fields that are distilled to be perambulators in the Laplacian Heaviside subspace of the spatial Laplacian operator $-\nabla^2$. Since we plan to investigate the $\eta - \eta_c$ correlation functions as well, we calculate the perambulators of $u, d$ and $c$ quarks on our large gauge ensemble in the Laplacian Heaviside space spanned by the $N = 70$ eigenvectors of $-\nabla^2$ operator with the lowest eigenvalues. For the pseudoscalar glueball operator, we adopt the strategy in Refs. [14, 15] to get the optimized Hermitian operator $O_G(t) = O_G^T(t)$ coupling mainly to the ground state glueball based on different prototypes of Wilson loops and gauge link smearing schemes (see Appendix for the details).

For the isoscalar $\eta$, the interpolation field can be defined as

$$O_\Gamma = \frac{1}{\sqrt{2}} \left[ \bar{q}(s)\Gamma u(s) + \bar{d}(s)\Gamma d(s) \right].$$

where $\Gamma$ refers to $\gamma_5$ or $\gamma_4\gamma_5$, $u(s)$ and $d(s)$ are Laplacian Heaviside smeared $u, d$ quark fields. Thus the correlation function of $O_\Gamma$ can be expressed as

$$C_{TT}(t) = \frac{1}{T} \sum_{t_1 = 1}^T \sum_{x, y} \langle O_\Gamma(x, t + t_s)O_\Gamma^T(y, t_s) \rangle,$$

with $C_T(t)$ and $D_T(t)$ being the contributions from the connected and disconnected diagrams, respectively. We also consider the following correlation functions

$$C_{GG}(t) = \frac{1}{T} \sum_{t_1 = 1}^T \langle O_G(t + t_s)O_G(t_s) \rangle,$$

$$C_{GT}(t) = \frac{1}{T} \sum_{t_1 = 1}^T \sum_{x} \langle O_G(t + t_s)O_\Gamma^T(x, t_s) \rangle,$$

$$C_{TG}(t) = \frac{1}{T} \sum_{t_1 = 1}^T \sum_{x} \langle O_\Gamma(x, t + t_s)O_G(t_s) \rangle,$$

$$C_{TT}(t) = \frac{1}{T} \sum_{t_1 = 1}^T \sum_{x, y} \langle O_\Gamma(x, t + t_s)O_\Gamma^T(y, t_s) \rangle$$

where the $\mp$ sign comes from the Hermiticity of $O_\Gamma$, it takes minus sign for $\Gamma = \gamma_5$ (anti-Hermitian) and plus sign for $\gamma_4\gamma_5\gamma_5$ (Hermitian). The operator $O_{\Gamma_c} = \bar{c}(s)\Gamma_c c(s)$ where $\Gamma_c = \gamma_5, \gamma_4\gamma_5\gamma_5$ is also defined in terms of the Laplacian Heaviside smeared charm quark field $\bar{c}(s)$. Obviously, all of these correlation functions except for $C_{GG}(t)$ are contributed by quark annihilation diagrams and can be dealt with conveniently in the framework of the distillation method.

**C. $\eta$ mass as a further calibration**

We calculate two types of correlation functions for $\eta$, namely $C_{\gamma_5\gamma_5}(t)$ and $C_{(\gamma_4\gamma_5)\gamma_5}(t)$. We do observe
the finite volume artifact that $C_{\gamma_5\gamma_5}(t)$ approaches to a nonzero constant when $t$ is large, as shown in Fig. 3. It has been argued that this constant term comes from the topology of QCD vacuum and can be approximately expressed as $a^2(\chi_{\text{top}} + Q^2/V)/T$ where $a$ is the lattice spacing (in the isotropic case), $\chi_{\text{top}}$ is the topological susceptibility, $Q$ is the topological charge, $V$ is the spatial volume and $T$ is the temporal extension of the lattice [30, 38, 39]. This can be understood from the anomalous axial vector current relation that the function contains a nonzero constant with a large error when $t$ is large, as shown in Fig. 3.

As usual, the $t$-behavior of these correlation functions can be seen more clearly through their effective mass functions defined as

$$m_{\text{eff}}(t) = \ln \frac{C_{TT}(t)}{C_{TT}(t+1)}$$

(8)

where $\Gamma$ refers to $\gamma_5$ or $\gamma_4\gamma_5$. Figure 4 shows these effective mass functions. Benefited from the large statistics of our gauge ensemble, the effective mass plateau starts from $t \sim 10$, and the signal-to-noise ratio keeps good for $t$ beyond 20 in the case $\Gamma = \gamma_4\gamma_5$. By a combined fit of $C_{\gamma_5\gamma_5}(t)$ and $C_{(\gamma_4\gamma_5)(\gamma_4\gamma_5)}(t)$ using two mass terms, we obtain the best-fit mass of $\eta$ to be

$$m_\eta = 714.1(5.8) \text{ MeV}.$$  

(9)

where the error is obtained through the jackknife analysis. Considering the uncontrolled systematic uncertainties in our calculation (the continuum extrapolation and the chiral extrapolation have not been performed), this value is compatible with other determinations [28–30]. According to WV, the large mass of the flavor singlet pseudoscalar meson has a direct connection with the topology of QCD vacuum. In the $N_f = 2$ case, to the leading order of chiral perturbation theory, $m_\eta$ is related to $m_\pi$

$$m_\eta^2 = m_\pi^2 + m_0^2,$$

(10)

where the parameter $m_0^2$ is defined in terms of the topological susceptibility $\chi_{\text{top}}$ and the pion decay constant $f_\pi$, namely

$$m_0^2 = m_\eta^2 - m_\pi^2 \approx 4N_f f_\pi^2 \chi_{\text{top}}.$$  

(11)

Usually, $\chi_{\text{top}}$ is for the pure gauge case and is expected to be independent of flavor number $N_f$. Using the values of $m_\pi = 348.5(1.0)$ MeV and $m_\eta$ in Eq. (9), $m_0^2$ can be derived to be $m_0^2 = 0.3885(77)$ GeV$^2$. According to the leading order chiral perturbation theory for $N_f = 2$, the $m_\pi$ dependence of $f_\pi$ is [40]

$$f_\pi = \sqrt{2F} \left[ 1 + \xi \left( \ln \frac{m_{\pi}^{\text{phys}}}{m_{\pi}^2} + t_4 \right) + O(\xi^2) \right]$$

(12)

where $m_{\pi}^{\text{phys}} = 134.98$ MeV is the physical pion mass, $\xi$ is approximated by $m_{\pi}^2/(4\pi F)^2$, the pion decay constant in the chiral limit $F \approx 86(1)$ MeV and the low energy constant $t_4 = 4.40(28)$ are taken from FLAG 2019 [41]. For our $m_\pi = 348.5(1.0)$ MeV, we have

$$\frac{f_\pi}{f_{\pi}^{\text{phys}}} = 1.17(1)^{+0.03}_{-0.02}$$

(13)

where the first error comes from the error of $F$ and the second one is due to the uncertainty of $t_4$. Using this value and $f_{\pi}^{\text{phys}} = 130.2(8)$ MeV [41–44], $\chi_{\text{top}}^{1/4}$ is estimated to be $\chi_{\text{top}}^{1/4} = 183(3)$ MeV, which is very close to the phenomenological value 180 MeV and the lattice value 185.3(5.6) MeV [4]. In the physical $N_f = 3$ case at physical pion mass, the GMOR relation implies...
the mass of the singlet counterpart of the pseudoscalar octet should be \( m_1^2 = (2m_K^2 + m_B^2)/3 = 0.170 \text{ GeV}^2 \). Thus using the topological susceptibility we obtained, we can estimate the mass of the flavor singlet pseudoscalar meson to be

\[
m_{\eta_1}^2 = m_1^2 + \frac{12\chi_{\text{top}}}{f_\pi^2} \approx 0.961 \text{ GeV}^2,
\]

which corresponds to \( m_{\eta_1} \approx 0.981(29) \text{ GeV} \) and is not far from the experimental value \( m_{\eta'} = 0.958 \text{ GeV} \). The GMOR relation also implies \( m_{\eta_8}^2 = (4m_K^2 - m_B^2)/3 \approx 0.321 \text{ GeV}^2 \). This confirms again that the Witten-Veneziano mechanism for the flavor singlet pseudoscalar mass works fairly well both for the SU(2) and SU(3) flavor symmetry.

III. TOWARD THE \( \eta \)-GLUEBALL MIXING

A. Theoretical consideration

In order to investigate the possible mixing between the pseudoscalar glueball and \( \eta \), we must parametrize the correlation functions in Eq. (7). We adopt the following theoretical logic. As usual in the lattice study, the correlation function \( C_{XY}(t) \) of operator \( O_X \) and \( O_Y \) can be parametrized as

\[
C_{XY} \approx \sum_{n \neq 0} \left[ \langle 0 | O_X | n \rangle \langle n | O_Y^t(0) | 0 \rangle \left( e^{-E_n t} \pm e^{-E_n (T-t)} \right) \right]
\]

where the \( \pm \) sign is for the same and opposite Hermiticities of \( O_X \) and \( O_Y \), respectively, and \( |n\rangle \) are the eigenstates of the lattice Hamiltonian \( \hat{H} \) defined as \( \hat{H}|n\rangle = E_n |n\rangle \) with \( E_n \) being the corresponding eigenenergies. For a given quantum number, \( |n\rangle \)'s establish an orthogonal and complete set, namely, \( \sum_n \langle n | n \rangle = 1 \) with the normalization condition \( \langle n | n \rangle = \delta_{mn} \). In principle, \( \hat{H} \) only exists heuristically, so we do not know the exact particle configurations of these eigenstates simply from the correlation function in Eq. (15). As far as the flavor singlet pseudoscalar channel is concerned in the \( N_f = 2 \) QCD theory, each of the state \( |n\rangle \) should be a specific admixture of bare \( \eta \) states and bare glueballs if they exist theoretically (here we ignore the multihadron states temporarily) and can be taken as the states in the eigenstate set \( \{|\alpha_n\rangle, n = 1, 2, \ldots\} \) of the free Hamiltonian \( \hat{H}_0 \). Since we are working in a unitary lattice framework for \( u, d \) quarks, in principle this state set is orthogonal and complete with the normalization condition \( \langle \alpha_m | \alpha_n \rangle = \delta_{mn} \). Now we introduce the interaction Hamiltonian \( \hat{H}_I \) to account for the dynamics of the possible mixing, such that \( |n\rangle \) of \( \hat{H} = \hat{H}_0 + \hat{H}_I \) can be expanded in terms of \( \{|\alpha_m\rangle\} \) as

\[
|n\rangle = \sum_m C_{nm} |\alpha_m\rangle
\]

with \( \sum_{m} |C_{nm}|^2 = 1 \). In this sense, one can say that \( |n\rangle \) is an admixture of states \( |\alpha_m\rangle \) whose fractions are \( |C_{nm}|^2 \), respectively. Furthermore, if \( \hat{H}_I \) is small relative to \( \hat{H}_0 \), then to the lowest order of the perturbation theory, one has

\[
|n\rangle = |\alpha_n\rangle + \sum_{m \neq n} \frac{\langle \alpha_m | \hat{H}_I | \alpha_n \rangle}{E_n^{(0)} - E_m^{(0)}} |\alpha_m\rangle
\]

\[
E_n = E_n^{(0)} + \sum_{m \neq n} \frac{|\langle \alpha_m | \hat{H}_I | \alpha_n \rangle|^2}{E_n^{(0)} - E_m^{(0)}}
\]

where \( E_n^{(0)} \) is the eigenenergy of \( |\alpha_n\rangle \) and is ordered from low to high.

The experimentally observed isoscalar pseudoscalars are \( \eta, \eta', \eta(1295), \eta(1405/1475), \text{ etc.} \), which are identified as \( I = 0 \) members of different \( q\bar{q} \) SU(3) nonets of different radial quantum numbers. As for the SU(2) case of this study, since there is only one isoscalar in each isospin quartet, the spectrum can be simplified largely. Because the smeared quark field suppresses the contribution of excited states in the correlation functions of \( \eta \), as a simple approximation, we can truncate the spectrum of \( \eta \) state to be \( \eta \) and \( \eta^* \) with \( \eta^* \) taking account into all the excited \( \eta \) states. On the other hand, quenched lattice QCD predicted the mass of the lowest pseudoscalar glueball is around 2.4–2.6 GeV. This seems to be confirmed by the correlation function \( C_{GG}(t) \) of \( G \) and \( G^* \) in the state basis \( \{|\alpha_i\rangle, i = 1, 2, \ldots\} \) with \( G^* \) standing for all the excited states of the pseudoscalar glueball. Finally, we have the following state basis

\[
|\alpha_i\rangle = |\eta\rangle, |\eta^*\rangle, \ldots, |G\rangle, |G^*\rangle, \ldots,
\]

With this state basis, the free Hamiltonian \( \hat{H}_0 = \text{diag} \{m_\eta, m_{\eta'}, m_G, m_{G^*}\} \) is diagonal with the matrix elements being the bare masses of the basis states, respectively, and being ordered from low to high. Theoretically, \( |\eta\rangle \) and \( |\eta^*\rangle \) are orthogonal, so do the states \( |G\rangle \) and \( |G^*\rangle \). Thus the interaction Hamiltonian \( \hat{H}_I \) can be expressed as

\[
\hat{H}_I = \begin{pmatrix}
0 & 0 & x_1 & y_1 \\
0 & 0 & x_2 & y_2 \\
x_1 & x_2 & 0 & 0 \\
y_1 & y_2 & 0 & 0
\end{pmatrix},
\]

where \( x_i, y_i \) are called mixing energies sometimes. Accordingly, we have the following state expansion of \( |n\rangle \)

\[
|1\rangle \approx |\eta\rangle + \frac{x_1}{m_\eta - m_G} |G\rangle + \frac{y_1}{m_\eta - m_{G^*}} |G^*\rangle
\]

\[
|2\rangle \approx |\eta^*\rangle + \frac{x_2}{m_{\eta'} - m_G} |G\rangle + \frac{y_2}{m_{\eta'} - m_{G^*}} |G^*\rangle
\]

\[
|3\rangle \approx |G\rangle + \frac{x_1}{m_G - m_\eta} |\eta\rangle + \frac{x_2}{m_G - m_{\eta'}} |\eta^*\rangle
\]

\[
|4\rangle \approx |G^*\rangle + \frac{y_1}{m_{G^*} - m_\eta} |\eta\rangle + \frac{y_2}{m_{G^*} - m_{\eta'}} |\eta^*\rangle.
\]
FIG. 5. The correlation function of \( \eta - G \) with \( \Gamma = \gamma_5 \), called \( C_{G\gamma_5} \). The value of this correlator tends to zero at \( t = 0 \) within errors, and the error seems to be a constant independent of \( t \). The correlator approaches a positive (negative) constant when \( t < \frac{T}{2} (t > \frac{T}{2}) \) which might be due to the contribution from topology.

B. The \( \Gamma = \gamma_5 \) case

Now we explore the possibility of glueball-\( \eta \) mixing through the correlation function \( C_{G\gamma_5}(t) \). Let us take a look at the \( t \)-dependence of \( C_{G\gamma_5}(t) \) shown in Fig. 5. We have the following observations:

1. \( C_{G\gamma_5}(t) \) is anti-symmetric with respect to \( t = T/2 \) and tends to 0 at \( t = 0 \). This is understood because \( O_G \) is hermitian by construction (and even under the time reversal transformation \( T \)) while \( \gamma_5 \) is anti-Hermitian and \( T \)-odd. At \( t = 0 \), since the product \( O_G(0)\gamma_5(0) \) is \( T \)-odd, its vacuum expectation value \( \langle O_G\gamma_5(0) \rangle \) certainly vanishes.

2. \( C_{G\gamma_5}(t) \) approaches a positive (negative) constant when \( t < \frac{T}{2} (t > \frac{T}{2}) \). This may be due to the constant contribution from the topology similar to the case of \( C_{\gamma_5\gamma_5}(t) \) discussed in Sec. II C. Since \( C_{G\gamma_5}(t) \) is now \( T \)-odd, so does the topology contribution.

3. Even though its central value is smooth, the error of \( C_{G\gamma_5}(t) \) is almost constant throughout the time range.

In order to find a function form to describe the time behavior of \( C_{G\gamma_5}(t) \), we take the following approximations

\[
O_G^i(0) \approx \sum_{i \neq 0} \sqrt{Z_G} \ket{G_i}
\]

\[
O_{\gamma_5}^i(0) \approx \sum_{i \neq 0} \sqrt{Z_{\gamma_5}} \ket{\eta_i},
\]

(21)

by the assumptions \( \langle 0|O_G|\eta_i \rangle \approx 0 \) and \( \langle 0|O_{\gamma_5}|G_i \rangle \approx 0 \) similar to those adopted in the \( \eta - \eta' \) mixing studies [23, 24].

Consequently, we have the following coupling matrix elements of the operators \( O_G \) and \( O_{\gamma_5} \),

\[
\begin{align*}
\langle 0|O_G|1 \rangle &= \frac{x_1 \sqrt{Z_G}}{m_\eta - m_G} + \frac{y_1 \sqrt{Z_G}}{m_\eta - m_G^*} \\
\langle 0|O_G|2 \rangle &= \frac{x_2 \sqrt{Z_G}}{m_\eta - m_G} + \frac{y_2 \sqrt{Z_G}}{m_\eta - m_G^*} \\
\langle 0|O_G|3 \rangle &= \sqrt{Z_G} \\
\langle 0|O_G|4 \rangle &= \sqrt{Z_G^*}
\end{align*}
\]

(22)

and

\[
\begin{align*}
\langle 0|O_{\gamma_5}|1 \rangle &= \frac{\sqrt{Z_{\gamma_5}}}{T} \\
\langle 0|O_{\gamma_5}|2 \rangle &= \frac{\sqrt{Z_{\gamma_5}}}{T} \\
\langle 0|O_{\gamma_5}|3 \rangle &= -\frac{x_1 \sqrt{Z_{\gamma_5}}}{m_\eta - m_G} + \frac{x_2 \sqrt{Z_{\gamma_5}}}{m_\eta - m_G^*} \\
\langle 0|O_{\gamma_5}|4 \rangle &= -\frac{y_1 \sqrt{Z_{\gamma_5}}}{m_\eta - m_G} - \frac{y_2 \sqrt{Z_{\gamma_5}}}{m_\eta - m_G^*}
\end{align*}
\]

(23)

As an exploratory study and for the simplicity of the future data analysis, we temporarily neglect \( \eta^* \) contributes to \( C_{G\gamma_5}(t) \). That is to say, we take the further approximation \( O_{\gamma_5}^i(0) \approx \sqrt{Z_{\gamma_5}} \ket{\eta} \). Thus after inserting Eqs. (20), (21), (22), (23) to Eq. (15) and ignore the terms relevant to \( |\eta^*\rangle \), we have the approximate expression of \( C_{G\gamma_5}(t) \) as

\[
C_{G\gamma_5}(t) \approx \sqrt{Z_G Z_{\gamma_5}} \frac{x_1}{m_\eta - m_G} (e^{-m_1 t} - e^{-m_2 t}) + \sqrt{Z_G^* Z_{\gamma_5}} \frac{y_1}{m_\eta - m_G} (e^{-m_1 t} - e^{-m_2 t}) - (t \rightarrow (T - t) \text{ terms}).
\]

(24)

The feature of this expression is \( C_{G\gamma_5}(t = 0) = 0 \) and is in accordance with the observation of item (1).

In order to understand the almost constant error of \( C_{G\gamma_5}(t) \), we consider its variance [45]

\[
\delta^2 C_{G\gamma_5}(t) \equiv \langle O_G^2(t)O_{\gamma_5}^2(0) \rangle - C_{G\gamma_5}^2(t).
\]

(25)

The first term on the right-hand side can be viewed as a correlation function of the operator \( O_{\gamma_5}^2 \) and \( O_G^2 \), both of which have the vacuum quantum number 0++ (in the continuum limit) and are expected to have nonzero vacuum expectation values \( \langle O_{\gamma_5}^2 \rangle \neq 0 \) and \( \langle O_G^2 \rangle \neq 0 \). Thus we have

\[
\delta^2 C_{G\gamma_5}(t) \equiv \langle O_G^2(t)O_{\gamma_5}^2(0) \rangle - C_{G\gamma_5}^2(t) + \langle O_G^2 \rangle \langle O_{\gamma_5}^2 \rangle
\]

(26)

where \( \overline{O_G^2(t)} \equiv O_G^2(t) - \langle O_G^2 \rangle \). The almost constant error of \( C_{G\gamma_5}(t) \) implies that the constant term \( \langle O_G^2 \rangle \langle O_{\gamma_5}^2 \rangle \) is large and dominate the variance. This is consistent with the argument in Ref. [45] that the variance of a correlation function is dominated by the possible lowest state, which corresponds to the vacuum state with \( E_{\text{vac}} = \)
0 in our case. This motivates us to consider the temporal derivative of $C_{G\gamma_5}(t)$, namely,

$$\partial_t C_{G\gamma_5}(t) = \frac{1}{2a_t} (C_{G\gamma_5}(t + a_t) - C_{G\gamma_5}(t - a_t)),$$

(27)

such that the constant term in $C_{G\gamma_5}(t)$ and its constant variance can be canceled. This is surely the case. We plot $\partial_t C_{G\gamma_5}(t)$ in Fig. 6, where one can see that $\partial_t C_{G\gamma_5}(t)$ goes to zero when $t$ is large and its relative error is much smaller.

In order for the mixing energies $x_1$ and $y_1$ to be extracted by using Eq. (24), one has to know the parameters $m_1$, $m_3$, $m_4$, $m_\eta$, $m_G$, $m_{G^*}$, $Z_G$, $Z_{G^*}$ and $Z_{\gamma_5,1}$, which, based on the assumptions of Eq. (21), are encoded in the correlation functions $C_{G\gamma_5}(t)$ and $C_{GG}(t)$ as

$$C_{GG}(t) = \sum_i Z_G i \left( e^{-m_{G_i}t} + e^{-m_{G_i}(T-t)} \right)$$

$$C_{G\gamma_5}(t) = \sum_i Z_{\gamma_5,i} \left( e^{-m_{\gamma_5,i}t} + e^{-m_{\gamma_5,i}(T-t)} \right)$$

$$\approx Z_{\gamma_5,1} \left( e^{-m_{\gamma_5,1}t} + e^{-m_{\gamma_5,1}(T-t)} \right)$$

(28)

where we take $i = 1, 2$ for $C_{GG}(t)$ and therefore $G_{1,2}$ refer to $G$ and $G^*$. It should be noted that the second state $|G^*\rangle$ should be considered even though we have built the optimal operator $O_G$ based on a large operator set (see in Appendix), it turns out that there is still a substantial contribution of higher states in $C_{GG}$. To manifest this, we plot the effective mass function $m_{\text{eff}}(t)$ of $C_{GG}(t)$ in Fig. 7 by the definition in Eq. (8), where one can see that the ground state glueball $G$ has not saturated $C_{GG}(t)$ before the signals are undermined by errors. With this observation, we add the second term relevant to $G^*$ to the first equation in Eq. (28) but do not consider its physical meaning.

Since the mixing effect is expected to be small (our final results confirm this), we take the assumptions $m_1 \approx m_\eta$, $m_3 \approx m_G$ and $m_4 \approx m_{G^*}$. In the data analysis procedure, we first rearrange the $N_{\text{conf}}$ measurements into 139 bins with each bin including 50 measurements, and then we perform the one-eliminating jackknife analysis on these data bins. For each time of jackknife resampling, we first extract $m_\eta$, $m_G$, $m_{G^*}$, $Z_G$, $Z_{G^*}$, $Z_{\gamma_5,1}$ and $Z_{\gamma_5,2}$ from $C_{GG}(t)$, $C_{G\gamma_5}(t)$ and $C_{G\gamma_5}(t)$ in the fixed time windows $[t_i, t_h]_{GG} = [1, 14]$, $[t_i, t_h]_{G\gamma_5} = [9, 30]$ and $[t_i, t_h]_{G\gamma_5} = [5, 30]$, as shown in Table IV. Then we feed these parameters to $\partial_t C_{G\gamma_5}(t)$ to determine the parameters $x_1$ and $y_1$. The final results of $m_\eta$, $m_G$ and $|x_1|$ with jackknife errors are obtained to be

$$m_\eta a_t = 0.10358(84),$$

$$m_G a_t = 0.3607(75),$$

$$|x_1| a_t = 0.0155(22).$$

(29)

Note that the definitions in Eqs. (21), (22), (23) are up to a plus or minus sign, we can only determine the absolute value $|x_1|$ of $x_1$. The parameters of the fitting procedure and fitting results are collected in Table IV. The goodness of the fit of Eq. (24) to $\partial_t C_{G\gamma_5}$ using the function is reflected by the $\chi^2 / \text{dof} = 0.96$ in the fitting window $[t_i, t_h]_{G\gamma_5} = [3, 30]$, and is also illustrated in Fig. 6 by the shaded curve. In mean time, making use of the $T$-odd property of $C_{G\gamma_5}(t)$, we average the $t < \frac{T}{2}$ part and $t > \frac{T}{2}$ part and find the errors can be reduced drastically around $t = 0$ except for $C_{G\gamma_5}(t = 0)$, as shown in Fig. 8, where the function of Eq. (24) with fitted parameters is also plotted with shaded curves. It is seen that the function describes the $t$-dependence of $C_{G\gamma_5}(t)$ very well up to an unknown constant term with opposite signs for $t < \frac{T}{2}$ and $t > \frac{T}{2}$.

Since $|x_1|$ is much smaller than $m_G - m_\eta$, to the lowest order of the perturbation theory, we can estimate the
TABLE IV. Ground state mass and mixing angle fitted from operators with $\Gamma = \gamma_5$ and $\Gamma = \gamma_4\gamma_5$ on the ensemble. Values of $m_q$ are the same in both cases because the value is derived from a combined fit to $C_{\gamma_5,\gamma_5}$ and $C_{(\gamma_4\gamma_5),\gamma_4\gamma_5}$. $\chi^2$ are obtained from the fitting results of $C_{\gamma_5,\gamma_5}$.

| $\Gamma$ | $[t_i, t_f]_{\Gamma}$ | $[t_i, t_f]_{\gamma_5}$ | $[t_i, t_f]_{\gamma_4\gamma_5}$ | $\chi^2$/dof | $m_{\mu}\alpha$ | $m_{G}\alpha$ | $|x_1|\alpha$ | $|\theta|$ |
|----------|-----------------|-----------------|-----------------|---------|-------------|-------------|-------------|---------|
| $\gamma_5$ | [9, 30] | [1, 14] | [3, 30] | 0.96 | 0.10358(84) | 0.3607(75) | 0.0155(22) | 3.46(46)$^*$ |
| $\gamma_4\gamma_5$ | [5, 30] | [1, 14] | [0, 30] | 0.15 | 0.10358(84) | 0.3607(75) | 0.0112(55) | 2.5(1.2)$^*$ |

mixing angle $\theta$ of $\eta$ and the ground state glueball $G$ as

$$|\theta| \approx \sin |\theta| \approx \frac{|x_1|}{m_G - m_q} = 3.46(46)^\circ. \quad (30)$$

FIG. 8. $C_{\gamma_5,\gamma_5}$ and used $\gamma$-odd property to fold the data for every gauge configuration. The blue band shows the fitted results. The band doesn’t perfectly match the data point because we just fitted temporal derivative $\partial_t C_{\gamma_5,\gamma_5}$ instead of correlation function $C_{\gamma_5,\gamma_5}$, so we dropped some constant in the fitting result. The $\gamma$-odd property makes the constant negative in $-t$ direction. The x-axis is shifted by 10 to show the near-zero behavior.

C. The $\Gamma = \gamma_4\gamma_5$ case

As a cross-check, we also carry a similar calculation by using the $\Gamma = \gamma_4\gamma_5$ for the interpolation field operator of $\eta$. The corresponding correlation functions $C_{(\gamma_4\gamma_5)^2,(\gamma_4\gamma_5)}(t)$ and $C_{(\gamma_4\gamma_5),(\gamma_4\gamma_5)}(t)$ are calculated using Eq. (7). In contrast to the case of $\Gamma = \gamma_5$, the correlation function $C_{(\gamma_4\gamma_5),(\gamma_4\gamma_5)}(t)$ does not go to zero when $t \to 0$. This is similar to the study of mixing of the pseudoscalar charmonium and the pseudoscalar glueball [32], which can be explained following the same logic that the QCD $U_A(1)$ anomaly may play an important role here. Obviously, the operator $O_{\gamma_4\gamma_5}$ has the same operator structure as the temporal component of the isoscalar axial vector current $j_5^A(x) = \frac{1}{\sqrt{2}} \left[ \bar{u}(x)\gamma_5 u(x) + \bar{d}(x)\gamma_5 d(x) \right]$, which satisfies the anomalous axial vector relation

$$\partial_\mu j_5^A(x) = 2m_q j_5^A(x) + \sqrt{2}q(x), \quad (31)$$

where $j_5^A(x) = \frac{1}{\sqrt{2}} \left[ \bar{u}(x)\gamma_5 u(x) + \bar{d}(x)\gamma_5 d(x) \right]$ is the pseudoscalar density and $q(x) = \frac{2}{\sqrt{2}} e^{\alpha\beta\rho\sigma} G_{\alpha\beta}^g G_{\rho\sigma}^g$ is the anomalous term stemming from the $U_A(1)$ anomaly with $g$ being the strong coupling constant and $G_{\alpha\beta}^g$ being the strength of color fields. Since $j_5^A(x)$ also has the same structure as $O_{\gamma_5,\gamma_5}$, based on the assumption in Eq. (21) we expect

$$m^2_{G_i} f_{G_i} = \langle 0 | \partial_\mu j_5^A(0) | G_i \rangle \approx \sqrt{2} \langle 0 | q(0) | G_i \rangle \quad (32)$$

where $f_{G_i}$ is the decay constant of the pseudoscalar glueball $G_i$. Accordingly we have

$$\langle 0 | j_5^A(0) | G_i(\not{p} = 0) \rangle \approx \sqrt{\frac{2}{m_{G_i}}} \langle 0 | q(0) | G_i(\not{p} = 0) \rangle. \quad (33)$$

FIG. 9. The correlation function of $\eta - G$ with $\Gamma = \gamma_4\gamma_5$, called $C_{\eta - G(\gamma_4\gamma_5)}$. The shaded band illustrates the fitting results using Eq. (34). The fit window starts from $t = 0$ in order to match the near-zero behavior of the correlator. The x-axis is shifted by 5 to show the near-zero behavior.

Previous lattice studies show that pseudoscalar states can be accessed by the operator $q(x)$ [15, 16], thus the nonzero matrix element $\langle 0 | q(0) | G_i \rangle$ implies the coupling $\langle 0 | O_{\gamma_4\gamma_5} | G_i \rangle \neq 0$. If we insist the relation $O_{\gamma_4\gamma_5}^i | 0 \rangle = \sum_{i \neq 0} \sqrt{Z_{G_i}} | G_i \rangle$ still holds, then the correlation function


\[ C_{G(\gamma_5\gamma_5)}(t) \approx Ae^{-m_3t} + \frac{x_1}{m_{G(\gamma_5\gamma_5)}} (e^{-m_1t} - e^{-m_3t}) + \frac{y_1}{m_{G(\gamma_5\gamma_5)}} (e^{-m_1t} - e^{-m_4t}) + (t \to (T - t) \text{ terms}), \]

which is similar to Eq. (24) apart from the first term due to nonzero coupling \(|q(0)|G_i\) (here we ignore the contribution of excited \(\eta\) states). Alongside the correlation function

\[ C_{(\gamma_5\gamma_3)(\gamma_5\gamma_3)}(t) = \sum_i Z_{\gamma_5\gamma_3,i} \left( e^{-m_{\eta_i}t} + e^{-m_{\eta_i}(T-t)} \right) \]

and the first equation of Eq. (28), we carry out a similar jackknife data analysis to the case of \(\Gamma = \gamma_5\) and get the mixing energy \(|x_1|a_t = 0.0112(55)\) and the corresponding mixing angle \(\theta_1 = 2.5(1.2)\)°, which are consistent with the results of \(\Gamma = \gamma_5\) case but have larger errors. The parameters of the fitting procedure and the fit results are also listed in Table IV for comparison. In Fig. 9, the colored curves with error bands are plotted using the fitted parameters. It is seen that the function forms mentioned above also describe the data very well. The fitted results are shown in Table IV and can be compared with the \(\gamma_5\)-case directly. On this ensemble, it is clear that the results of the two cases are compatible with each other within errors.

\section*{IV. DISCUSSION AND SUMMARY}

We generate a large gauge ensemble with \(N_f = 2\) degenerated \(u, d\) dynamical quarks on a \(16^3 \times 128\) anisotropic lattice with the anisotropy \(\xi = a_x/a_t \approx 5.3\). Based on the experimental observation that the differences of mass squares of the heavy-light vector and pseudoscalar mesons are insensitive to quark masses, we propose a new scale setting scheme that the temporal lattice spacing \(a_t\) can be estimated by this quantity. In practice, we use the value \(m_{\pi}^2 - m_{\rho}^2 = 0.568(8)\) GeV\(^2\), which is a least squares fitting of \(n\bar{n}, s\bar{s}, s\bar{s}, c\bar{n}\) and \(c\bar{s}\) mesons with \(n\) referring to the light \(u, d\) quarks. Our \(u, d\) quark mass parameter corresponds to a pion mass \(m_\pi \approx 350\) MeV. We calculate the perambulators of \(u, d\) quarks and the valence charm quark on our ensemble using the distillation method where the quark annihilation diagrams can be calculated efficiently.

We calculate the mass of the isoscalar pseudoscalar meson \(\eta\) to be \(m_\eta = 714(6)(16)\) MeV. The error in the latter bracket is introduced by the estimation of lattice spacing in Sec. II A. This mass value can be matched, through the Witten-Veneziano relation, to the SU(3) flavor singlet pseudoscalar meson mass \(m_\eta' \approx 981\) MeV, which is in good agreement with the \(\eta'\) mass \(m_\eta' = 958\) MeV if the \(\eta - \eta'\) mixing is considered.

The mixing of \(\eta\) and the pseudoscalar glueball is investigated for the first time on the lattice. From the correlation function of the glueball operator and \(\eta\) operator, the mixing energy and the mixing angle are determined to be \(|x_1| = 107(15)(2)\) MeV and \(|\theta| = 3.4(6)\)° given the pseudoscalar glueball mass \(m_G \approx 2.5\) GeV. This mixing angle is so tiny that the mixing effects can be neglected when the properties of \(\eta\) (and also \(\eta'\) in the physical \(N_f = 3\) case) are explored.

With perambulators of light and charm quarks at hand, we also checked correlation functions of \(\eta\) and \(\eta_c\), which are contributed only by annihilation diagrams. The \(\Gamma\) insertion are chosen to be \(\gamma_5\) and \(\gamma_5\gamma_5\). The signal-to-noise ratios of these correlation functions are fairly good, and the effective masses of them defined by Eq. (8) are presented in Fig 10, where the horizontal gray line illustrates the \(\eta\) mass in the lattice unit. Albeit the different behaviors at the small \(t\) range, all the effective masses approach \(m_\eta\) when \(t\) is large. Because of the influence from excited states of \(\eta\) and \(\eta_c\), we cannot fit the curve by the simplification above, and optimized operators for these states are required to get better results. This kind of exploration can be performed in the future.

To summarize, we find that there is little mixing between the flavor singlet pseudoscalar (\(\eta\) is our case) and the pseudoscalar glueball. The topology of the QCD vacuum plays a crucial role in the origin of \(\eta\) mass due to the QCD \(U_A(1)\) anomaly. This is in accordance with the common theoretical considerations [46].
prototype are calculated using smeared gauge links. We loops shown in Fig. 11. The Wilson loops of each glueball are built based on the four prototypes of Wilson glueball, and the CAS ORISE computing environment. (Beijing) and China Spallation Neutron Source (Dongguan), and the HPC clusters at Institute of High Energy Physics (Beijing) and China Spallation Neutron Source (Dongguan), and the CAS ORISE computing environment.

ACKNOWLEDGMENTS

This work is supported by the National Key Research and Development Program of China (No. 2020YFA0406400), the Strategic Priority Research Program of Chinese Academy of Sciences (No. XDB34030302), and the National Natural Science Foundation of China (NNSFC) under Grants No. 11935017, No. 11775229, No. 12075253, No. 12070131001 (CRC 110 by DFG and NNSFC) and No. 12175063. The computations were performed on 110 by DFG and NNSFC) and No. 12175063. The Chroma software system [47] and QUDA library [48, 49] are acknowledged. The computations were performed on the HPC clusters at Institute of High Energy Physics (Beijing) and China Spallation Neutron Source (Dongguan), and the CAS ORISE computing environment.

APPENDIX

A1. Glueball operator construction

The interpolation field operators for the pseudoscalar glueball are built based on the four prototypes of Wilson loops shown in Fig. 11. The Wilson loops of each prototype are calculated using smeared gauge links. We adopt six different schemes to smear gauge links, which are different combinations of single link smearing and double link smearing [14, 15]. The spatial symmetry group on the lattice is the octahedral group $O$ whose irreducible representation $A_1$ corresponds to the spin $J = 0, 4, \ldots$ in the continuum limit. It is expected the glueball of $J = 4$ is higher than the $J = 0$ state in mass, so we can build operators in the $A_1^{+}$ representation to couple with the ground state pseudoscalar glueball. Let $W_{a}(x, t)$ be one prototype of the Wilson loop under a specific smearing scheme, then the $A_1^{+}$ operator in the rest frame of a glueball can be obtained by

$$\phi_\alpha(t) = \sum_x \sum_{R \in O} c^A_\alpha \left[ R \circ W_{a}(x, t) - P R \circ W_{a}(x, t) P^{-1} \right]$$

(36)

where $R \circ W_{a}$ refers to differently oriented Wilson loops after one of the 24 elements of $O (R)$ operated on $W_{a}$, $P$ is spatial reflection operation and $c^A_\alpha$ are the combinatorial coefficients for the $A_1$ representation. Thus we obtain a $A_1^{+}$ operator set $\{\phi_\alpha(t), \alpha = 1, 2, \ldots, 24\}$ based on the four prototypes and six smearing schemes. In order to get an optimized operator $O_G$ that couples most to the ground state glueball, we first calculate the correlation matrix of the operator set,

$$C_{\alpha\beta}(t_1) = \frac{1}{T} \sum_\tau \langle \phi_\alpha(t + \tau) \phi_\beta(\tau) \rangle,$$

(37)

and then solve the generalized eigenvalue problem

$$C_{\alpha\beta}(t_1) w_\beta = \lambda C_{\alpha\beta}(t_0) w_\beta$$

(38)

to get the eigenvector $w_\alpha$ of the largest eigenvalue $\lambda$, which serves as the combinational coefficients of $O_G$, namely,

$$O_G(t) = w_\alpha \phi_\alpha(t).$$

(39)

In practice, we take $t_0 = 1$ and $t_1 = 2$ for the optimized operator coupling more to the ground state pseudoscalar glueball.

[1] J. B. Kogut and L. Susskind, How to Solve the eta \rightarrow 3 \pi Problem by Seizing the Vacuum, Phys. Rev. D 11, 3594 (1975).
[2] E. Witten, Instantons, the Quark Model, and the 1/n Expansion, Nucl. Phys. B 149, 285 (1979).
[3] G. Veneziano, U(1) Without Instantons, Nucl. Phys. B 159, 213 (1979).
[4] K. Cichy, E. Garcia-Ramos, K. Jansen, K. Ottnad, and C. Urbach (ETM), Non-perturbative Test of the Witten-Veneziano Formula from Lattice QCD, JHEP 09, 020, arXiv:1504.07954 [hep-lat].
[5] P. Zyla et al. (Particle Data Group), Review of Particle Physics, PTEP 2020, 083C01 (2020).
[6] F. Ambrosino et al. (KLOE), Measurement of the pseudoscalar mixing angle and eta-prime gluonium content with KLOE detector, Phys. Lett. B 648, 267 (2007), arXiv:hep-ex/0612029.
[7] R. Escribano and J. Nadal, On the gluon content of the eta and eta-prime mesons, JHEP 05, 006, arXiv:hep-ph/0703187.
[8] G. Li, Q. Zhao, and C.-H. Chang, Decays of J/ψ and psi-prime into vector and pseudoscalar meson and the pseudoscalar glueball-anti-q mixing, J. Phys. G 35, 055002 (2008), arXiv:hep-ph/0701020.
[9] H.-Y. Cheng, H.-n. Li, and K.-F. Liu, Pseudoscalar glueball mass from eta - eta-prime - G mixing, Phys. Rev. D 79, 014024 (2009), arXiv:0811.2577 [hep-ph].
A. Chowdhury, A. Harindranath, and J. Maiti, Pseudoscalar glueball in a chiral Lagrangian model with instanton effect, Phys. Rev. D 81, 014003 (2010), arXiv:0903.5032 [hep-ph].

B. A. Li, Chiral field theory of 0+ glueball, Phys. Rev. D 81, 114002 (2010), arXiv:0912.2323 [hep-ph].

Y.-D. Tsai, H.-n. Li, and Q. Zhao, $\eta_c$ mixing effects on charmonium and $B$ meson decays, Phys. Rev. D 85, 034002 (2012), arXiv:1110.6235 [hep-ph].

C. J. Morningstar and M. J. Peardon, Efficient glueball simulations on anisotropic lattices, Phys. Rev. D 56, 4043 (1997), arXiv:hep-lat/9704011.

C. J. Morningstar and M. J. Peardon, The Glueball spectrum from an anisotropic lattice study, Phys. Rev. D 60, 034509 (1999), arXiv:hep-lat/9901004.

Y. Chen et al., Glueball spectrum and matrix elements on anisotropic lattices, Phys. Rev. D 73, 014516 (2006), arXiv:hep-lat/0510074.

A. Chowdhury, A. Harindranath, and J. Maiti, Correlation and localization properties of topological charge density and the pseudoscalar glueball mass in SU(3) lattice Yang-Mills theory, Phys. Rev. D 91, 074507 (2015), arXiv:1409.6459 [hep-lat].

C. M. Richards, A. C. Irving, E. B. Gregory, and C. McNeile (UKQCD), Glueball mass measurements from improved staggered fermion simulations, Phys. Rev. D 82, 034501 (2010), arXiv:2005.2473 [hep-lat].

E. Gregory, A. Irving, B. Lucini, C. McNeile, A. Rago, C. Richards, and E. Rinaldi, Towards the glueball spectrum from unquenched lattice QCD, JHEP 10, 170, arXiv:1208.1858 [hep-lat].

W. Sun, L.-C. Gui, Y. Chen, M. Gong, C. Liu, Y.-B. Liu, Z. Liu, J.-P. Ma, and J.-B. Zhang, Glueball spectrum from $N_f = 2$ lattice QCD study on anisotropic lattices, Chin. Phys. C 42, 093103 (2018), arXiv:1702.08174 [hep-lat].

J.-J. Wu, X.-H. Liu, Q. Zhao, and B.-S. Zou, The Puzzle of anomalously large isospin violations in $\eta(1405)/1475 \to 3\pi$, Phys. Rev. Lett. 108, 081803 (2012), arXiv:1108.3772 [hep-ph].

V. Mathieu and V. Vento, Pseudoscalar glueball and eta - eta-prime mixing, Phys. Rev. D 81, 034004 (2010), arXiv:0910.0212 [hep-ph].

W. Qin, Q. Zhao, and X.-H. Zhong, Revisiting the pseudoscalar meson and glueball mixing and key issues in the search for a pseudoscalar glueball state, Phys. Rev. D 97, 066002 (2018), arXiv:1712.02550 [hep-ph].

N. H. Christ, C. Dawson, T. Iizubuchi, C. Jung, Q. Liu, R. D. Mawhinney, C. T. Sachrajda, A. Soni, and R. Zhou, The $\eta$ and $\eta'$ mesons from Lattice QCD, Phys. Rev. Lett. 105, 241601 (2010), arXiv:1002.2999 [hep-lat].

C. Michael, K. Ottndad, and C. Urbach (ETM), $\eta$ and $\eta'$ mixing from Lattice QCD, Phys. Rev. Lett. 111, 181602 (2013), arXiv:1310.1207 [hep-lat].

H. Fukaya, S. Aoki, G. Cossu, S. Hashimoto, T. Kaneko, and J. Noaki (JLQCD), $\eta$ meson mass from topological charge density correlator in QCD, Phys. Rev. D 92, 111501 (2015), arXiv:1509.00944 [hep-lat].

K. Ottndad and C. Urbach (ETM), Flavor-singlet meson decay constants from $N_f = 2 + 1 + 1$ twisted mass lattice QCD, Phys. Rev. D 97, 054508 (2018), arXiv:1710.07986 [hep-lat].

G. S. Bali, V. Braun, S. Collins, A. Schäfer, and J. Simeth (RQCD), Masses and decay constants of the $\eta$ and $\eta'$ mesons from lattice QCD, JHEP 08, 137, arXiv:2106.05398 [hep-lat].

V. I. Lesk et al. (CP-PACS), Flavor singlet meson mass in the continuum limit in two flavor lattice QCD, Phys. Rev. D 67, 074503 (2003), arXiv:hep-lat/0211040.

K. Hashimoto and T. Iizubuchi, eta-prime meson from two flavor dynamical domain wall fermions, Prog. Theor. Phys. 119, 599 (2008), arXiv:0803.0186 [hep-lat].

P. Dimopoulos et al., Topological susceptibility and $\eta'$ meson mass from $N_f = 2$ lattice QCD at the physical point, Phys. Rev. D 99, 034511 (2019), arXiv:1812.08787 [hep-lat].

Y. Kuramashi, M. Fukugita, H. Minou, M. Okawa, and A. Ukaawa, eta-prime meson mass in lattice QCD, Phys. Rev. Lett. 72, 3448 (1994).

R. Zhang, W. Sun, Y. Chen, M. Gong, L.-C. Gui, and Z. Liu, The glueball content of $\eta c$, Phys. Lett. B 827, 136960 (2022), arXiv:2107.12749 [hep-lat].

M. Peardon, J. Bulava, J. Foley, C. Morningstar, J. Dudek, R. G. Edwards, B. Joo, H.-W. Lin, D. G. Richards, and K. J. Juge (Hadron Spectrum), A Novel quark-field creation operator construction for hadronic physics in lattice QCD, Phys. Rev. D 80, 054506 (2009), arXiv:0905.2160 [hep-lat].

R. G. Edwards, B. Joo, and H.-W. Lin, Tuning for Three-flavors of Anisotropic Clover Fermions with Stout-link Smearing, Phys. Rev. D 78, 054510 (2008), arXiv:0803.3960 [hep-lat].

C. T. H. Davies, E. Follana, I. D. Kendall, G. P. Lepage, and C. McNeile (HPQCD), Precise determination of the lattice spacing in full lattice QCD, Phys. Rev. D 81, 034506 (2010), arXiv:0910.1229 [hep-lat].

A. Abdel-Rehim et al. (ETM), First physics results at the physical pion mass from $N_f = 2$ Wilson twisted mass fermions at maximal twist, Phys. Rev. D 95, 094515 (2017), arXiv:1507.05068 [hep-lat].

G.-Z. Meng et al. (CLQCD), Low-energy $D^*+D^0(D)$ scattering and the Resonance-like Structure $Z^{++}$ (4430), Phys. Rev. D 80, 034503 (2009), arXiv:0905.0752 [hep-lat].

S. Aoki, H. Fukaya, S. Hashimoto, and T. Onogi, Finite volume QCD at fixed topological charge, Phys. Rev. D 76, 054508 (2007), arXiv:0707.0396 [hep-lat].

G. S. Bali, S. Collins, S. Durr, and I. Kanamori, $D_s \to \eta, \eta'$ semileptonic decay form factors with disconnected quark loop contributions, Phys. Rev. D 91, 014503 (2015), arXiv:1406.5449 [hep-lat].

G. Colangelo, J. Gasser, and H. Leutwyler, $\pi\pi$ scattering, Nucl. Phys. B 603, 125 (2001), arXiv:hep-ph/0103088.

S. Aoki et al. (Flavour Lattice Averaging Group), FLAG Review 2019: Flavour Lattice Averaging Group (FLAG), Eur. Phys. J. C 80, 113 (2020), arXiv:1902.08191 [hep-lat].

T. Blum et al. (RBC, UKQCD), Domain wall QCD with physical quark masses, Phys. Rev. D 93, 074505 (2016), arXiv:1411.7017 [hep-lat].

E. Follana, C. T. H. Davies, G. P. Lepage, and J. Shigemitsu (HPQCD, UKQCD), High Precision determination of the $p$, $D$ and $D(s)$ decay constants from lattice QCD, Phys. Rev. Lett. 100, 062002 (2008), arXiv:0706.1726 [hep-lat].

A. Bazavov et al. (MILC), Results for light pseudoscalar mesons, PoS LATTICE2010, 074 (2010), arXiv:1012.0868 [hep-lat].
[45] M. G. Endres, D. B. Kaplan, J.-W. Lee, and A. N. Nicholson, Listening to Noise, PoS LATTICE2011, 017 (2011), arXiv:1112.4023 [hep-lat].

[46] S. D. Bass and P. Moskal, $\eta'$ and $\eta$ mesons with connection to anomalous glue, Rev. Mod. Phys. 91, 015003 (2019), arXiv:1810.12290 [hep-ph].

[47] R. G. Edwards and B. Joo (SciDAC, LHPC, UKQCD), The Chroma software system for lattice QCD, Nucl. Phys. B Proc. Suppl. 140, 832 (2005), arXiv:hep-lat/0409003.

[48] M. A. Clark, R. Babich, K. Barros, R. C. Brower, and C. Rebbi, Solving Lattice QCD systems of equations using mixed precision solvers on GPUs, Comput. Phys. Commun. 181, 1517 (2010), arXiv:0911.3191 [hep-lat].

[49] R. Babich, M. A. Clark, B. Joo, G. Shi, R. C. Brower, and S. Gottlieb, Scaling Lattice QCD beyond 100 GPUs, in SC11 International Conference for High Performance Computing, Networking, Storage and Analysis (2011) arXiv:1109.2935 [hep-lat].