THE PLANCHEREL FORMULA FOR COUNTABLE GROUPS

BACHIR BEKKA

Abstract. We discuss a Plancherel formula for countable groups, which provides a canonical decomposition of the regular representation of such a group $\Gamma$ into a direct integral of factor representations. Our main result gives a precise description of this decomposition in terms of the Plancherel formula of the FC-center $\Gamma_{fc}$ of $\Gamma$ (that is, the normal subgroup of $\Gamma$ consisting of elements with a finite conjugacy class); this description involves the action of an appropriate totally disconnected compact group of automorphisms of $\Gamma_{fc}$. As an application, we determine the Plancherel formula for linear groups. In an appendix, we use the Plancherel formula to provide a unified proof for Thoma’s and Kaniuth’s theorems which respectively characterize countable groups which are of type I and those whose regular representation is of type II.

1. Introduction

Given a second countable locally compact group $G$, a fundamental object to study is its unitary dual space $\hat{G}$, that is, the set irreducible unitary representations of $G$ up to unitary equivalence. The space $\hat{G}$ carries a natural Borel structure, called the Mackey Borel structure (see [Mac57, §6] or [Dix77, §18.6]). A classification of $\hat{G}$ is considered as being possible only if $\hat{G}$ is a standard Borel space; according to Glimm’s celebrated theorem (Gli61), this is the case if and only if $G$ is of type I in the following sense.

Recall that a von Neumann algebra is a self-adjoint subalgebra of $L(H)$ which is closed for the weak operator topology of $L(H)$, where $H$ is a Hilbert space. A von Neumann algebra is a factor if its center only consists of the scalar operators.

Let $\pi$ be a unitary representation of $G$ in a Hilbert space $H$ (as we will only consider representations which are unitary, we will often drop the adjective “unitary”). The von Neumann subalgebra of $L(H)$

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generated by $\pi(G)$ coincides with the bicommutant $\pi(G)''$ of $\pi(G)$ in $L(H)$; we say that $\pi$ is a factor representation if $\pi(G)''$ is a factor.

**Definition 1.** The group $G$ is of **type I** if, for every factor representation $\pi$ of $G$, the factor $\pi(G)''$ is of type I, that is, $\pi(G)''$ is isomorphic to the von Neumann algebra $L(K)$ for some Hilbert space $K$; equivalently, the Hilbert space $\mathcal{H}$ of $\pi$ can written as tensor product $K \otimes K'$ of Hilbert spaces in such a way that $\pi$ is equivalent to $\sigma \otimes I_{K'}$ for an irreducible representation $\sigma$ of $G$ on $K$.

Important classes of groups are known to be of type I, such as semi-simple or nilpotent Lie groups. A major problem in harmonic analysis is to decompose the left regular representation $\lambda_G$ on $L^2(G, \mu_G)$ for a Haar measure $\mu_G$ as a direct integral of irreducible representations. When $G$ is of type I and unimodular, this is the content of the classical Plancherel theorem: there exist a unique measure $\mu$ on $\hat{G}$ and a unitary isomorphism between $L^2(G, \mu_G)$ and the direct integral of Hilbert spaces $\int_{\hat{G}} (\mathcal{H}_{\pi} \otimes \overline{\mathcal{H}_{\pi}}) d\mu(\pi)$ which transforms $\lambda_G$ into $\int_{\hat{G}} (\pi \otimes \overline{\pi}) d\mu(x)$, where $\overline{\mathcal{H}_{\pi}}$ is the conjugate of the Hilbert space $\mathcal{H}_{\pi}$ of $\pi$; in particular, we have a Plancherel formula

$$\|f\|_2^2 = \int_{\hat{G}} \text{Tr}(\pi(f)^* \pi(f)) d\mu(\pi)$$

for all $f \in L^1(G, \mu_G) \cap L^2(G, \mu_G)$, where $\|f\|_2$ is the $L^2$-norm of $f$, $\pi(f)$ is the value at $f$ of the natural extension of $\pi$ to a representation of $L^1(G, \mu_G)$, and $\text{Tr}$ denotes the standard trace on $L(\mathcal{H}_{\pi})$; for all this, see [Dix77, 18.8.1].

When $G$ is not type I, $\lambda_G$ usually admits several integral decompositions into irreducible representations and it is not possible to single out a canonical one among them. However, when $G$ is unimodular, $\lambda_G$ does admit a canonical integral decomposition into factor representations; this the content of a Plancherel theorem which we will discuss in the case of a discrete group (see Theorem A).

Let $\Gamma$ be a countable group. As discussed below (see Theorem E), $\Gamma$ is usually not of type I. In order to state the Plancherel theorem for $\Gamma$, we need to replace the dual space $\hat{G}$ by the consideration of Thoma’s dual space $\text{Ch}(\Gamma)$ which we now introduce.

Recall that a function $t : \Gamma \to \mathbb{C}$ is of positive type if the complex-valued matrix $(t(\gamma_i^{-1} \gamma_j))_{1 \leq i, j \leq n}$ is positive semi-definite for any $\gamma_1, \ldots, \gamma_n$ in $\Gamma$.

A function of positive type $t$ on $\Gamma$ which is constant on conjugacy classes and normalized (that is, $t(e) = 1$) will be called a **trace** on $G$.

The set of traces on $\Gamma$ will be denoted by $\text{Tr}(\Gamma)$.
Let \( t \in \text{Tr}(\Gamma) \) and \((\pi_t, \mathcal{H}_t, \xi_t)\) be the associated GNS triple (see [BHV08, C.4]). Then \( \tau_t : \pi(\Gamma)'' \to \mathbb{C} \), defined by \( \tau_t(T) = \langle T\xi_t | \xi_t \rangle \) is a trace on the von Neumann algebra \( \pi_t(\Gamma)'' \), that is, \( \tau_t(T^*T) \geq 0 \) and \( \tau_t(TS) = \tau_t(ST) \) for all \( T,S \in \pi_t(\Gamma)'' \); moreover, \( \tau_t \) is faithful in the sense that \( \tau_t(T^*T) > 0 \) for every \( T \in \pi_t(\Gamma)'' \), \( T \neq 0 \). Observe that \( \tau_t(\pi(f)) = t(f) \) for \( f \in \mathbb{C}[\Gamma] \), where \( t \) denotes the linear extension of \( t \) to the group algebra \( \mathbb{C}[\Gamma] \).

The set \( \text{Tr}(\Gamma) \) is a convex subset of the unit ball of \( \ell^\infty(\Gamma) \) which is compact in the topology of pointwise convergence. An extreme point of \( \text{Tr}(\Gamma) \) is called a \textbf{character} of \( \Gamma \); we will refer to \( \text{Ch}(\Gamma) \) as Thoma's \textbf{dual space}.

Since \( \Gamma \) is countable, \( \text{Tr}(\Gamma) \) is a compact metrizable space and \( \text{Ch}(\Gamma) \) is easily seen to be a \( G_\delta \) subset of \( \text{Tr}(\Gamma) \). So, in contrast to \( \hat{\Gamma} \), Thoma's dual space \( \text{Ch}(\Gamma) \) is always a standard Borel space.

An important fact is that \( \text{Tr}(\Gamma) \) is a simplex (see [Tho64, Satz 1] or [Sak71, 3.1.18]); by Choquet theory, this implies that every \( \tau \in \text{Tr}(\Gamma) \) can be represented as integral \( \tau = \int_{\text{Ch}(\Gamma)} td\mu(t) \) for a unique probability measure \( \mu \) on \( \text{Ch}(\Gamma) \).

As we now explain, the set of characters of \( \Gamma \) parametrizes the factor representations of finite type of \( \Gamma \), up to quasi-equivalence; for more details, see [Dix77, §17.3] or [BH, §11.C].

Recall first that two representations \( \pi_1 \) and \( \pi_2 \) of \( \Gamma \) are quasi-equivalent if there exists an isomorphism \( \Phi : \pi_1(\Gamma)'' \to \pi_2(\Gamma)'' \) of von Neumann algebras such that \( \Phi(\pi_1(\gamma)) = \pi_2(\gamma) \) for every \( \gamma \in \Gamma \).

Let \( t \in \text{Ch}(\Gamma) \) and \( \pi_t \) the associated GNS representation. Then \( \pi_t(\Gamma)'' \) is a factor of finite type. Conversely, let \( \pi \) be a representation of \( \Gamma \) such that \( \pi(\Gamma)'' \) is a factor of finite type and let \( \tau \) be the unique normalized trace on \( \pi(\Gamma)'' \). Then \( t := \tau \circ \pi \) belongs to \( \text{Ch}(\Gamma) \) and only depends on the quasi-equivalence class \([\pi]\) of \( \pi \).

The map \( t \to [\pi_t] \) is a bijection between \( \text{Ch}(\Gamma) \) and the set of quasi-equivalence classes of factor representations of finite type of \( \Gamma \).

The following result is a version for countable groups of a Plancherel theorem due to Mautner [Mau50] and Segal [Seg50] which holds more generally for any unimodular second countable locally compact group; its proof is easier in our setting and will be given in Section 3 for the convenience of the reader.

\textbf{Theorem A. (Plancherel theorem for countable groups)} Let \( \Gamma \) be a countable group. There exists a probability measure \( \mu \) on \( \text{Ch}(\Gamma) \), a measurable field of representations \( t \mapsto (\pi_t, \mathcal{H}_t) \) of \( \Gamma \) on the standard Borel space \( \text{Ch}(\Gamma) \), and an isomorphism of Hilbert spaces between \( \ell^2(\Gamma) \).
and \( \int_{Ch(\Gamma)} \mathcal{H}_t d\mu(t) \) which transforms \( \lambda_\Gamma \) into \( \int_{Ch(\Gamma)} \pi_t d\mu(t) \) and has the following properties:

(i) \( \pi_t \) is quasi-equivalent to the GNS representation associated to \( t \); in particular, the \( \pi_t \)'s are mutually disjoint factor representations of finite type, for \( \mu \)-almost every \( t \in Ch(\Gamma) \);

(ii) the von Neumann algebra \( L(\Gamma) := \lambda_\Gamma(\Gamma)'' \) is mapped onto the direct integral \( \int_{Ch(\Gamma)} \pi_t(\Gamma)'d\nu(t) \) of factors;

(iii) for every \( f \in C[\Gamma] \), the following Plancherel formula holds:

\[
\|f\|_2^2 = \int_{Ch(\Gamma)} \tau_t(\pi_t(f)^* \pi_t(f))d\mu(t) = \int_{Ch(\Gamma)} t(f^* * f)d\mu(t).
\]

The measure \( \mu \) is the unique probability measure on \( Ch(\Gamma) \) such that the Plancherel formula above holds.

The probability measure \( \mu \) on \( Ch(\Gamma) \) from Theorem \( A \) is called the **Plancherel measure** of \( \Gamma \).

**Remark 2.** The Plancherel measure gives rise to what seems to be an interesting dynamical system on \( Ch(\Gamma) \) involving the group \( Aut(\Gamma) \) of automorphisms of \( \Gamma \). We will equip \( Aut(\Gamma) \) with the topology of pointwise convergence on \( \Gamma \), for which it is a totally disconnected topological group. The natural action of \( Aut(\Gamma) \) on \( Ch(\Gamma) \), given by \( t^g(\gamma) = t(g^{-1}(\gamma)) \) for \( g \in Aut(\Gamma) \) and \( t \in Ch(\Gamma) \), is clearly continuous.

Since the induced action of \( Aut(\Gamma) \) on \( \ell^2(\Gamma) \) is isometric, the following fact is an immediate consequence of the uniqueness of the Plancherel measure \( \mu \) of \( \Gamma \):

the action of \( Aut(\Gamma) \) on \( Ch(\Gamma) \) preserves \( \mu \).

For example, when \( \Gamma = \mathbb{Z}^d \), Thoma’s dual \( Ch(\Gamma) \) is the torus \( T^d \), the Plancherel measure \( \mu \) is the normalized Lebesgue measure on \( T^d \) which is indeed preserved by the group \( Aut(\mathbb{Z}^d) = GL_d(\mathbb{Z}) \). Dynamical systems of the form \( (\Lambda, Ch(\Gamma), \mu) \) for a subgroup \( \Lambda \) of \( Aut(\Gamma) \) may be viewed as generalizations of this example.

We denote by \( \Gamma_{fc} \) the FC-centre of \( \Gamma \), that is, the normal subgroup of elements in \( \Gamma \) with a finite conjugacy class. It turns out (see Remark \( 6 \)) that \( t = 0 \) on \( \Gamma \\setminus \Gamma_{fc} \) for \( \mu \)-almost every \( t \in Ch(\Gamma) \). In particular, when \( \Gamma \) is ICC, that is, when \( \Gamma_{fc} = \{e\} \), the regular representation \( \lambda_\Gamma \) is factorial (see also Corollary \( 5 \)) so that the Plancherel formula is vacuous in this case.

In fact, as we now see, the Plancherel measure of \( \Gamma \) is entirely determined by the Plancherel measure of \( \Gamma_{fc} \). Roughly speaking, we will see that the Plancherel measure of \( \Gamma \) is the image of the Plancherel
measure of $\Gamma_c$ under the quotient map $\text{Ch}(\Gamma_c) \rightarrow \text{Ch}(\Gamma_c)/K\Gamma$, for a compact group $K\Gamma$ which we now define.

Let $K\Gamma$ be the closure in $\text{Aut}(\Gamma_c)$ of the subgroup $\text{Ad}(\Gamma)|_{\Gamma_c}$ given by conjugation with elements from $\Gamma$. Since every $\Gamma$-conjugation class in $\Gamma_c$ is finite, $K\Gamma$ is a compact group. By a general fact about actions of compact groups on Borel spaces (see Corollary 2.1.21 and Appendix in [Zim84]), the quotient space $\text{Ch}(\Gamma_c)/K\Gamma$ is a standard Borel space.

Given a function $t : H \rightarrow \mathbb{C}$ of positive type on a subgroup $H$ of a group $\Gamma$, we denote by $\hat{t}$ the extension of $t$ to $\Gamma$ given by $\hat{t} = 0$ outside $H$. Observe that $\hat{t}$ is of positive type on $\Gamma$ (see for instance [BH, 1.F.10]).

Here is our main result.

**Theorem B. (Plancherel measure: reduction to the FC-center)**

Let $\Gamma$ be a countable group. Let $\nu$ be the Plancherel measure of $\Gamma_c$ and $\lambda_{\Gamma_c} = \int_{\text{Ch}(\Gamma_c)} \pi_t d\nu(t)$ the integral decomposition of the regular representation of $\Gamma_c$ as in Theorem [A]. Let $\hat{\nu}$ be the image of $\nu$ under the quotient map $\text{Ch}(\Gamma_c) \rightarrow \text{Ch}(\Gamma_c)/K\Gamma$.

(i) For every $K\Gamma$-orbit $O$ in $\text{Ch}(\Gamma_c)$, let $m_O$ be the unique normalized $K\Gamma$-invariant probability measure on $O$ and let $\pi_O := \int_O \pi_t m_O(t)$. Then the induced representation $\hat{\pi}_O := \text{Ind}_{\Gamma_c}^{\Gamma}\pi_O$ is factorial for $\hat{\nu}$-almost every $O$ and we have a direct integral decomposition of the von Neumann algebra $L(\Gamma)$ into factors

$$L(\Gamma) = \int_{\text{Ch}(\Gamma_c)/K\Gamma} \hat{\pi}_O(\Gamma)^{\prime\prime} d\hat{\nu}(O).$$

(ii) The Plancherel measure of $\Gamma$ is the image $\Phi_*(\nu)$ of $\nu$ under the map

$$\Phi : \text{Ch}(\Gamma_c) \rightarrow \text{Tr}(\Gamma), \quad t \mapsto \int_{K\Gamma} \hat{t}^g dm(g),$$

where $m$ is the normalized Haar measure on $K\Gamma$.

It is worth mentioning that the support of $\mu$ was determined in [Tho67]. For an expression of the map $\Phi$ as in Theorem [B]ii without reference to the group $K\Gamma$, see Remark [7].

As we next see, the Plancherel measure on $\Gamma_c$ can be explicitly described in the case of a linear group $\Gamma$. We first need to discuss the Plancherel formula for a so-called **central group**, that is, a central extension of a finite group.

Let $\Lambda$ be a central group. Then $\Lambda$ is of type I (see Theorem [E]). In fact, $\hat{\Lambda}$ can be described as follows; let $r : \hat{\Lambda} \rightarrow \hat{Z}(\Lambda)$ be the restriction map, where $Z(\Lambda)$ is the center of $\Lambda$. Then, for $\chi \in \hat{Z}(\Lambda)$, every
\[ \pi \in r^{-1}(\chi) \] is equivalent to a subrepresentation of the finite dimensional representation \( \text{Ind}^A_{Z(\Lambda)} \chi \), by a generalized Frobenius reciprocity theorem (see [Mac52, Theorem 8.2]); in particular, \( r^{-1}(\chi) \) is finite. The Plancherel measure \( \nu \) on \( \text{Ch}(\Lambda) \) is, in principle, easy to determine: we identify every \( \pi \in \widehat{\Lambda} \) with its normalized character given by \( x \mapsto \frac{1}{\dim \pi} \text{Tr} \pi(x) \); for every Borel subset \( A \) of \( \text{Ch}(\Lambda) \), we have

\[
\nu(A) = \frac{1}{\#(Z(\Lambda))} \sum_{\pi \in r^{-1}(\chi)} \frac{\#(A \cap r^{-1}(\chi))}{(\dim \pi)^2} d\chi,
\]

where \( d\chi \) is the normalized Haar measure on the abelian compact group \( Z(\Lambda) \).

**Corollary C. (The Plancherel measure for linear groups)** Let \( \Gamma \) be a countable linear group.

(i) \( \Gamma_{fc} \) is a central group;

(ii) \( K_{\Gamma} \) coincides with \( \text{Ad}(\Gamma)|_{\Gamma_{fc}} \) and is a finite group;

(iii) the Plancherel measure of \( \Gamma \) is the image of the Plancherel measure of \( \Gamma_{fc} \) under the map \( \Phi : \text{Ch}(\Gamma_{fc}) \to \text{Tr}(\Gamma) \) given by

\[
\Phi(t) = \frac{1}{\#\text{Ad}(\Gamma)|_{\Gamma_{fc}}} \sum_{s \in \text{Ad}(\Gamma)|_{\Gamma_{fc}}} t^s.
\]

When the Zariski closure of the linear group \( \Gamma \) is connected, the Plancherel measure of \( \Gamma \) has a particularly simple form.

**Corollary D. (The Plancherel measure for linear groups-bis)** Let \( G \) be a connected linear algebraic group over a field \( k \) and let \( \Gamma \) be a countable Zariski dense subgroup of \( G \). The Plancherel measure of \( \Gamma \) is the image of the normalized Haar measure \( d\chi \) on \( Z(\Gamma) \) under the map

\[
\widehat{Z(\Gamma)} \to \text{Tr}(\Gamma), \quad \chi \mapsto \tilde{\chi}
\]

and the Plancherel formula is given for every \( f \in C[\Gamma] \) by

\[
\|f\|_2^2 = \int_{\widehat{Z(\Gamma)}} \mathcal{F}((f^* \ast f)|_{Z(\Gamma)})(\chi) d\chi,
\]

where \( \mathcal{F} \) is the Fourier transform on the abelian group \( Z(\Gamma) \).

The previous conclusion holds in the following two cases:

(i) \( k \) is a countable field of characteristic 0 and \( \Gamma = G(k) \) is the group of \( k \)-rational points in \( G \);

(ii) \( k \) is a local field (that is, a non discrete locally compact field), \( G \) has no proper \( k \)-subgroup \( H \) such that \( (G/H)(k) \) is compact, and \( \Gamma \) is a lattice in \( G(k) \).
Corollary D generalizes the Plancherel theorem obtained in [CPJ94, Theorem 4] for $\Gamma = G(Q)$ and in [PJ95, Theorem 3.6] for $\Gamma = G(Z)$, in the case where $G$ is a unipotent linear algebraic group over $\mathbb{Q}$; indeed, $G$ is connected (since the exponential map identifies $G$ with its Lie algebra, as affine varieties) and these two results follow from (i) and (ii) respectively.

In an appendix to this article, we use Theorem A to give a unified proof of Thoma’s and Kaniuth’s results ([Tho64], [Tho68], [Kan69]) as stated in the following theorem. For a group $\Gamma$, we denote by $[\Gamma, \Gamma]$ its commutator subgroup. Recall that $\Gamma$ is said to be virtually abelian if it contains an abelian subgroup of finite index.

The regular representation $\lambda_\Gamma$ is of type I (or type II) if the von Neumann algebra $L(\Gamma)$ is of type I (or type II); equivalently (see Corollaire 2 in [Dix69, Chap. II, §3, 5]), if $\pi_t(\Gamma)'''$ is a finite dimensional factor (or a factor of type II) for $\mu$-almost every $t \in \operatorname{Ch}(\Gamma)$ in the Plancherel decomposition $\lambda_\Gamma = \int_{\operatorname{Ch}(\Gamma)} \pi_t d\mu(t)$ from Theorem A.

**Theorem E (Thoma, Kaniuth).** Let $\Gamma$ be a countable group. The following properties are equivalent:

(i) $\Gamma$ is type I;
(ii) $\Gamma$ is virtually abelian;
(iii) the regular representation $\lambda_\Gamma$ is of type I;
(iv) every irreducible unitary representation of $\Gamma$ is finite dimensional;
(iv') there exists an integer $n \geq 1$ such that every irreducible unitary representation of $\Gamma$ has dimension $\leq n$.

Moreover, the following properties are equivalent:

(v) $\lambda_\Gamma$ is of type II;
(vi) either $[\Gamma : \Gamma_{fc}] = \infty$ or $[\Gamma : \Gamma_{fc}] < \infty$ and $[\Gamma, \Gamma]$ is infinite.

Our proof of Theorem E is not completely new as it uses several crucial ideas from [Tho64] and especially from [Kan69] (compare with the remarks on p.336 after Lemma in [Kan69]); however, we felt it could be useful to have a short and common treatment of both results in the literature. Observe that the equivalence between (i) and (iii) above does not carry over to non discrete groups (see [Mac61]).

**Remark 3.** Theorem E holds also for non countable discrete groups. Write such a group $\Gamma$ as $\Gamma = \bigcup_j H_j$ for a directed net of countable subgroups $H_j$. If $L(\Gamma)'''$ is not of type II (or is of type I), then $L(H_j)$ is not of type II (or is of type I) for $j$ large enough. This is the crucial tool for the extension of the proof of Theorem E to $\Gamma$; for more details, proofs of Satz 1 and Satz 2 in [Kan69].
This paper is organized as follows. In Section 2, we recall the well-known description of the center of the von Neumann algebra of a discrete group. Sections 3 and 4 contain the proofs of Theorems A and B. In Section 5, we prove Corollaries C and D. Section 6 is devoted to the explicit computation of the Plancherel formula for a few examples of countable groups. Appendix A contains the proof of Theorem E.

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2. On the center of the group von Neumann algebra

Let $\Gamma$ be a countable group. We will often use the following well-known description of the center $Z = \lambda(\Gamma)^{\prime\prime} \cap \lambda(\Gamma)'$ of $L(\Gamma) = \lambda(\Gamma)^{\prime\prime}$.

Observe that $\lambda(\Gamma)^{\prime\prime}$ is a von Neumann subalgebra of $L(\Gamma)$, for every subgroup $H$ of $\Gamma$. For $h \in \Gamma_{fc}$, we set

$$T_{[h]} := \lambda(1_{[h]}) = \sum_{x \in [h]} \lambda_{\Gamma}(x) \in \lambda(\Gamma_{fc})^{\prime\prime},$$

where $[h]$ denotes the $\Gamma$-conjugacy class of $h$.

**Lemma 4.** The center $Z$ of $L(\Gamma) = \lambda_{\Gamma}(\Gamma)^{\prime\prime}$ coincides with the closure of the linear span of $\{T_{[h]} \mid h \in \Gamma_{fc}\}$, for the strong operator topology; in particular, $Z$ is contained in $\lambda_{\Gamma}(\Gamma_{fc})^{\prime\prime}$.

**Proof.** It is clear that $T_{[h]} \in Z$ for every $h \in \Gamma_{fc}$. Observe also that the linear span of $\{T_{[h]} \mid h \in \Gamma_{fc}\}$ is a unital selfadjoint algebra; indeed, $T_{[h^{-1}]} = T_{[h]}^{\ast}$ for every $h \in \Gamma_{fc}$ and $\{1_{[h]} \mid h \in \Gamma_{fc}\}$ is a vector space basis of the algebra $C[\Gamma_{fc}]^{\ast}$ of $\Gamma$-invariant functions in $C[\Gamma_{fc}]$.

Let $T \in Z$. We have to show that $T \in \{T_{[h]} \mid h \in \Gamma_{fc}\}^{\prime\prime}$. For every $\gamma \in \Gamma$, we have

$$\lambda_{\Gamma}(\gamma)\rho_{\Gamma}(\gamma)(T_{[\delta_{e}]}(\gamma) = (\lambda_{\Gamma}(\gamma)\rho_{\Gamma}(\gamma)(T_{\delta_{e}})) = (T\lambda_{\Gamma}(\gamma)\rho_{\Gamma}(\gamma))\delta_{e} = T_{\delta_{e}},$$

and this shows that $f := T_{\delta_{e}}$, which is a function in $\ell^{2}(\Gamma)$, is invariant under conjugation by $\gamma$. The support of $f$ is therefore contained in $\Gamma_{fc}$.

Write $f = \sum_{[h] \in \mathcal{C}} c_{[h]}1_{[h]}$ for a sequence $(c_{[h]}_{[h] \in \mathcal{C}})$ of complex numbers with $\sum_{[h] \in \mathcal{C}} \#([h]|c_{[h]}|^{2} < \infty$, where $\mathcal{C}$ is a set of representatives for the $\Gamma$-conjugacy classes in $\Gamma_{fc}$. Let $\rho_{\Gamma}$ be the right regular representation of $\Gamma$. Since $T \in \lambda_{\Gamma}(\Gamma)^{\prime\prime}$ and $\rho_{\Gamma}(\Gamma) \subset \lambda_{\Gamma}(\Gamma)'$, we have, for every $x \in \Gamma$,

$$T(\delta_{x}) = T\rho_{\Gamma}(x)(\delta_{e}) = \rho_{\Gamma}(x)(f) = \rho_{\Gamma}(x) \left( \sum_{[h] \in \mathcal{C}} c_{[h]}1_{[h]} \right) = \sum_{[h] \in \mathcal{C}} c_{[h]}T_{[h]}(\delta_{x}),$$
where the last sum is convergent in $\ell^2(\Gamma)$. We also have $T^*(\delta_x) = \sum_{[h] \in C} c_{[h]} T_{[h]^{-1}}(\delta_x)$.

Let $S \in \{T_{[h]} \mid h \in \Gamma_{fc}\}'. For every $x, y \in \Gamma$, we have

$$\langle ST(\delta_x) \mid \delta_y \rangle = \langle S \left( \sum_{[h] \in C} c_{[h]} T_{[h]}(\delta_x) \right) \mid \delta_y \rangle = \langle \sum_{[h] \in C} c_{[h]} ST_{[h]}(\delta_x) \mid \delta_y \rangle$$

$$= \sum_{[h] \in C} c_{[h]} \langle T_{[h]} S(\delta_x) \mid \delta_y \rangle = \langle S(\delta_x) \mid \sum_{[h] \in C} c_{[h]} T_{[h]^{-1}}(\delta_y) \rangle$$

$$= \langle S(\delta_x) \mid T^*(\delta_y) \rangle = \langle TS(\delta_x) \mid \delta_y \rangle$$

and it follows that $ST = TS$. \qed

The following well-known corollary shows that the Plancherel measure is the Dirac measure at $\delta_e$ in the case where $\Gamma$ is ICC group, that is, when $\Gamma_{fc} = \{e\}$.

**Corollary 5.** Assume that $\Gamma$ is ICC. Then $L(\Gamma) = \lambda_{\Gamma}(\Gamma)''$ is a factor.

### 3. Proof of Theorem A

Consider a direct integral decomposition $\int_X \pi_x d\mu(x)$ of $\lambda_{\Gamma}$ associated to the centre $Z$ of $L(\Gamma) = \lambda_{\Gamma}(\Gamma)''$ (see [8.3.2] [Dix77]); so, $X$ is a standard Borel space equipped with a probability measure $\mu$ and $(\pi_x, H_x)_{x \in X}$ is measurable field of representations of $\Gamma$ over $X$, such that there exists an isomorphism of Hilbert spaces

$$U : \ell^2(\Gamma) \rightarrow \int_X \oplus H_x d\mu(x)$$

which transforms $\lambda_{\Gamma}$ into $\int_X \pi_x d\mu(x)$ and for which $UZU^{-1}$ is the algebra of diagonal operators on $\int_X \oplus H_x d\mu(x)$. (Recall that a diagonal operator on $\int_X \oplus H_x d\mu(x)$ is an operator of the form $\int_X \varphi(x) I_{H_x} d\mu(x)$ for an essentially bounded measurable function $\varphi : X \rightarrow C$.)

Then, upon disregarding a subset of $X$ of $\mu$-measure 0, the following holds (see [8.4.1] [Dix77]):

1. $\pi_x$ is a factor representation for every $x \in X$;
2. $\pi_x$ and $\pi_y$ are disjoint for every $x, y \in X$ with $x \neq y$;
3. we have $U \lambda_{\Gamma}(\Gamma)'' U^{-1} = \int_X \pi_x(\Gamma)'' d\mu(x)$.

Let $\rho_\Gamma$ be the right regular representation of $\Gamma$. Let $\gamma \in \Gamma$. Then $U \rho_\Gamma(\gamma) U^{-1}$ commutes with every diagonalisable operator on $\int_X \oplus H_x d\mu(x)$, since $\rho_\Gamma(\gamma) \in L(\Gamma)'$. It follows (see [Dix69] Chap. II, §2, No 5, Théorème 1]) that $U \rho_\Gamma(\gamma) U^{-1}$ is a decomposable operator, that is, there exists a measurable field of unitary operators $x \mapsto \sigma_x(\gamma)$ such that
\[ U \rho_\Gamma(\gamma) U^{-1} = \int_X \sigma_x(\gamma) \, d\mu(x). \]
So, we have a measurable field \( x \mapsto \sigma_x \) of representations of \( \Gamma \) in \( \int_X \mathcal{H}_x \, d\mu(x) \) such that

\[
U \rho_\Gamma(\gamma) U^{-1} = \int_X \sigma_x(\gamma) \, d\mu(x) \quad \text{for all } \gamma \in \Gamma.
\]

Let \( (\xi_x)_{x \in X} \in \int_X \mathcal{H}_x \, d\mu(x) \) be the image of \( \delta_e \in \ell^2(\Gamma) \) under \( U \). We claim that \( \xi_x \) is a cyclic vector for \( \pi_x \) and \( \sigma_x \), for \( \mu \)-almost every \( x \in X \).

Indeed, since \( \delta_e \in \ell^2(\Gamma) \) is a cyclic vector for both \( \lambda_\Gamma \) and \( \rho_\Gamma \),

\[
\{(\pi_x(\gamma)\xi_x)_{x \in X} \mid \gamma \in \Gamma\} \quad \text{and} \quad \{(\sigma_x(\gamma)\xi_x)_{x \in X} \mid \gamma \in \Gamma\}
\]

are countable total subsets of \( \int_X \mathcal{H}_x \, d\mu(x) \) and the claim follows from a general fact about direct integral of Hilbert spaces (see Proposition 8 in Chap. II, §1 of [Dix69]).

Since \( \lambda_\Gamma(\gamma) \delta_e = \rho_\Gamma(\gamma^{-1}) \delta_e \) for every \( \gamma \in \Gamma \) and since \( \Gamma \) is countable, upon neglecting a subset of \( X \) of \( \mu \)-measure 0, we can assume that

\[
\begin{align*}
(4) & \quad \pi_x(\gamma)\xi_x = \sigma_x(\gamma^{-1})\xi_x; \\
(5) & \quad \pi_x(\gamma)\sigma_x(\gamma') = \sigma_x(\gamma')\pi_x(\gamma); \\
(6) & \quad \xi_x \text{ is a cyclic vector for both } \pi_x \text{ and } \sigma_x,
\end{align*}
\]

for all \( x \in X \) and all \( \gamma, \gamma' \in \Gamma \).

Let \( x \in X \) and let \( \varphi_x \) be the function of positive type on \( \Gamma \) defined by

\[
\varphi_x(\gamma) = \langle \pi_x(\gamma)\xi_x \mid \xi_x \rangle \quad \text{for every } \gamma \in \Gamma.
\]

We claim that \( \varphi_x \in \text{Ch}(\Gamma) \). Indeed, using (4) and (5), we have, for every \( \gamma_1, \gamma_2 \in \Gamma \),

\[
\begin{align*}
\varphi_x(\gamma_2\gamma_1\gamma_2^{-1}) &= \langle \pi_x(\gamma_2\gamma_1\gamma_2^{-1})\xi_x \mid \xi_x \rangle = \langle \pi_x(\gamma_2\gamma_1)\sigma_x(\gamma_2)\xi_x \mid \xi_x \rangle \\
&= \langle \sigma_x(\gamma_2)\pi_x(\gamma_2\gamma_1)\xi_x \mid \xi_x \rangle = \langle \pi_x(\gamma_1)\xi_x \mid \sigma_x(\gamma_2^{-1})\sigma_x(\gamma_2)\xi_x \rangle \\
&= \langle \pi_x(\gamma_1)\xi_x \mid \xi_x \rangle = \varphi_x(\gamma_1).
\end{align*}
\]

So, \( \varphi_x \) is conjugation invariant and hence \( \varphi_x \in \text{Tr}(\Gamma) \). Moreover, \( \varphi_x \) is an extreme point in \( \text{Tr}(\Gamma) \), since \( \pi_x \) is factorial and \( \xi_x \) is a cyclic vector for \( \pi_x \).

Finally, since \( U : \ell^2(\Gamma) \to \int_X \mathcal{H}_x \, d\mu(x) \) is an isometry, we have for every \( f \in C[\Gamma] \),

\[
\|f\|^2 = f^* f(e) = \langle \lambda_\Gamma(f^* f) \delta_e \mid \delta_e \rangle = \|\lambda_\Gamma(f)\delta_e\|^2 = \|U(\lambda_\Gamma(f)\delta_e)\|^2 \\
= \int_X \|\pi_x(f)\xi_x\|^2 \, d\mu(x) = \int_X \varphi_x(f^* f) \, d\mu(x).
\]

The measurable map \( \Phi : X \to \text{Ch}(\Gamma) \) given by \( \Phi(x) = \varphi_x \) is injective, since \( \pi_x \) and \( \pi_y \) are disjoint by (2) and hence \( \varphi_x \neq \varphi_y \) for \( x, y \in X \) with
\( x \neq y \). It follows that \( \Phi(X) \) is a Borel subset of \( \text{Ch}(\Gamma) \) and that \( \Phi \) is a Borel isomorphism between \( X \) and \( \Phi(X) \) (see [Mac57, Theorem 3.2]).

Pushing forward \( \mu \) to \( \text{Ch}(\Gamma) \) by \( \Phi \), we can therefore assume without loss of generality that \( X = \text{Ch}(\Gamma) \) and that \( \mu \) is a probability measure on \( \text{Ch}(\Gamma) \). With this identification, it is clear that Items (i), (ii) and (iii) of Theorem A are satisfied and that the Plancherel formula holds.

It remains to show the uniqueness of \( \mu \). Let \( \nu \) any probability measure on \( \text{Ch}(\Gamma) \) such that the Plancherel formula. By polarization, we have then \( \delta_e = \int_{\text{Ch}(\Gamma)} \nu(t) \), which is an integral decomposition of \( \delta_e \in \text{Tr}(\Gamma) \) over extreme points of the convex set \( \text{Tr}(\Gamma) \). The uniqueness of such a decomposition implies that \( \nu = \mu \).

**Remark 6.** (i) For \( \mu \)-almost every \( t \in \text{Ch}(\Gamma) \), we have \( t = 0 \) on \( \Gamma \setminus \Gamma_{fc} \).

Indeed, let \( \gamma \notin \Gamma_{fc} \). Then \( \langle \lambda_\Gamma(\gamma) \lambda_\Gamma(h) \delta_e | \delta_e \rangle = 0 \) for every \( h \in \Gamma_{fc} \) and hence

\[
(\ast) \quad \langle \lambda_\Gamma(\gamma)T \delta_e | \delta_e \rangle = 0 \quad \text{for all} \quad T \in \lambda_\Gamma(\Gamma_{fc})'.
\]

With the notation as in the proof above, let \( E \) be a Borel subset of \( X \). Then \( T_E := U^{-1}P_E U \) is a projection in \( \mathcal{Z} \), where \( P_E \) is the diagonal operator \( \int_X 1_E(x) I_{H_x} d\mu(x) \). It follows from Lemma 4 and (\ast) that

\[
\int_E \varphi_\gamma(x) d\mu(x) = \langle T_E \lambda_\Gamma(\gamma) \delta_e | \delta_e \rangle = \langle \lambda_\Gamma(\gamma) T_E \delta_e | \delta_e \rangle = 0.
\]

Since this holds for every Borel subset \( E \) of \( X \), this implies that \( \varphi_\gamma(x) = 0 \) for \( \mu \)-almost every \( x \in X \).

As \( \Gamma \) is countable, for \( \mu \)-almost every \( x \in X \), we have \( \varphi_\gamma(x) = 0 \) for every \( \gamma \notin \Gamma_{fc} \).

(ii) Let \( \lambda_\Gamma = \int_{\text{Ch}(\Gamma)} \pi_t d\mu(t) \), \( \rho_\Gamma = \int_{\text{Ch}(\Gamma)} \sigma_t d\mu(t) \), and \( \delta_e = (\xi_t)_{t \in \text{Ch}(\Gamma)} \) be the decompositions as above. For \( \mu \)-almost every \( t \in \text{Ch}(\Gamma) \), the linear map

\[
\pi_t(\Gamma)' \to \mathcal{H}_t, \quad T \mapsto T \xi_t
\]

is injective. Indeed, this follows from the fact that \( \xi_t \) is cyclic for \( \sigma_t \) and that \( \sigma_t(\Gamma) \subset \pi_t(\Gamma)' \).

### 4. Proof of Theorem B

Set \( N := \Gamma_{fc} \) and \( X := \text{Ch}(N) \). Consider the direct integral decomposition \( \int_X \pi_t d\nu(t) \) of \( \lambda_N \) into factor representations \( (\pi_t, K_t) \) of \( N \) with corresponding traces \( t \in X \), as in Theorem A.

Let \( K_\Gamma \) be the compact group which is the closure in \( \text{Aut}(N) \) of \( \text{Ad}(\Gamma)|_N \). Since the quotient space \( X/K_\Gamma \) is a standard Borel space, there exists a Borel section \( s : X/K_\Gamma \to X \) for the projection map \( X \to X/K_\Gamma \). Set \( \Omega := s(X/K_\Gamma) \). Then \( \Omega \) is a Borel transversal for
The Plancherel measure $\nu$ can accordingly be decomposed over $\Omega$: we have

$$\nu(f) = \int_{\Omega} \int_{\mathcal{O}_\omega} f(t)dm_\omega(t)d\nu'(\omega)$$

for every bounded measurable function $f$ on $\text{Ch}(\Gamma_k)$, where $m_\omega$ be the unique normalized $K_\Gamma$-invariant probability measure on the $K_\Gamma$-orbit $\mathcal{O}_\omega$ of $\omega$ and $\nu'$ is the image of $\nu$ under $s$.

Let $\omega \in \Omega$ and set

$$\pi_{\mathcal{O}_\omega} := \int_{\mathcal{O}_\omega} \pi t dm_\omega(t),$$

which is a unitary representation of $N$ on the Hilbert space

$$K_\omega := \int_{\mathcal{O}_\omega}^{\oplus} K_{\mathcal{O}_\omega}.$$

For $g \in \text{Aut}(N)$, let $\pi^g_{\mathcal{O}_\omega}$ be the conjugate representation of $N$ on $K_\omega$ given by $\pi^g_{\mathcal{O}_\omega}(h) = \pi_{\mathcal{O}_\omega}(g(h))$ for $h \in N$.

**Step 1** There exists a unitary representation $U_\omega : g \mapsto U_{\omega, g}$ of $K_\Gamma$ on $K_\omega$ such that

$$U_{\omega, g}\pi_{\mathcal{O}_\omega}(h)U_{\omega, g}^{-1} = \pi_{\mathcal{O}_\omega}(g(h)) \quad \text{for all} \quad g \in K_\Gamma, h \in N;$$

in particular, $\pi^g_{\mathcal{O}_\omega}$ is equivalent to $\pi_{\mathcal{O}_\omega}$ for every $g \in K_\Gamma$.

Indeed, observe that the representations $\pi_t$ for $t \in \mathcal{O}_\omega$ are conjugate to each other (up to equivalence) and may therefore be considered as defined on the same Hilbert space.

Let $g \in K_\Gamma$. Then $\pi^g_{\mathcal{O}_\omega}$ is equivalent to $\int_{\mathcal{O}_\omega}^{\oplus} \pi t^g dm_\omega(t)$. Define a linear operator $U_{\omega, g} : K_\omega \to K_\omega$ by

$$U_{\omega, g}((\xi_t)_{t \in \mathcal{O}_\omega}) = (\xi_{gt})_{t \in \mathcal{O}_\omega} \quad \text{for all} \quad (\xi_t)_{t \in \mathcal{O}_\omega} \in K_\omega.$$

Then $U_{\omega, g}$ is an isometry, by $K_\Gamma$-invariance of the measure $m_\omega$. It is readily checked that $U_\omega$ intertwines $\pi_{\mathcal{O}_\omega}$ and $\pi^g_{\mathcal{O}_\omega}$ and that $U_\omega$ is a homomorphism. To show that $U_\omega$ is a representation of $K_\Gamma$, it remains to prove that $g \mapsto U_{\omega, g}\xi$ is continuous for every $\xi \in K_\omega$.

For this, observe that $K_\omega$ can be identified with the Hilbert space $L^2(\mathcal{O}_\omega, m_\omega) \otimes K$, where $K$ is the common Hilbert space of the $\pi_t$'s for $t \in \mathcal{O}_\omega$; under this identification, $U_\omega$ corresponds to $\kappa \otimes I_K$, where $\kappa$ is the Koopman representation of $K_\Gamma$ on $L^2(\mathcal{O}_\omega, m_\omega)$ associated to the action $K_\Gamma \curvearrowright \mathcal{O}_\omega$ (for the fact that $\kappa$ is indeed a representation of $K_\Gamma$, see [BHV08 A.6]) and the claim follows.
Next, let
\[ \tilde{\pi}_\omega := \text{Ind}_N^\Gamma \pi_\omega \]
be the representation of \( \Gamma \) induced by \( \pi_\omega \).

We recall how \( \tilde{\pi}_\omega \) can be realized on \( \ell^2(R, K_\omega) = \ell^2(R) \otimes K_\omega \), where \( R \subseteq \Gamma \) is a set of representatives for the cosets of \( N \) with \( e \in R \). For every \( \gamma \in \Gamma \) and \( r \in R \), let \( c(r, \gamma) \in N \) and \( r \cdot \gamma \in R \) be such that \( r \gamma = c(r, \gamma) r \cdot \gamma \). Then \( \tilde{\pi}_\omega \) is given on \( \ell^2(R, K_\omega) \) by
\[
(\tilde{\pi}_\omega(\gamma)F)(r) = \pi_\omega(c(r, \gamma))(F(r \cdot \gamma)) \quad \text{for all } F \in \ell^2(R, K_\omega).
\]

**Step 2** We claim that there exists a unitary map
\[ \tilde{U}_\omega : \ell^2(R, K_\omega) \to \ell^2(R, K_\omega) \]
which intertwines the representation \( I_{\ell^2(R)} \otimes \pi_\omega \) and the restriction \( \tilde{\pi}_\omega|_N \) of \( \tilde{\pi}_\omega \) to \( N \); moreover, \( \omega \mapsto \tilde{U}_\omega \) is a measurable field of unitary operators on \( \Omega \).

Indeed, we have an orthogonal decomposition
\[
\ell^2(R, K_\omega) = \bigoplus_{r \in R} (\delta_r \otimes K_\omega)
\]
into \( \tilde{\pi}_\omega(N) \)-invariant; moreover, the action of \( N \) on every copy \( \delta_r \otimes K_\omega \) is given by \( \pi_\omega c(r, \gamma) \). For every \( r \in R \), the unitary operator \( U_{\omega,r} : K_\omega \to K_\omega \) from Step 1 intertwines \( \pi_\omega \) and \( \pi_\omega^r \). In view of the explicit formula of \( U_{\omega,r} \), the field \( \omega \mapsto U_{\omega,r} \) is measurable on \( \Omega \).

Define a unitary operator \( \tilde{U}_\omega : \ell^2(R, K_\omega) \to \ell^2(R, K_\omega) \) by
\[
\tilde{U}_\omega(\delta_r \otimes \xi) = \delta_r \otimes U_{\omega,r}(\xi) \quad \text{for all } \xi \in K_\omega.
\]
Then \( \tilde{U}_\omega \) intertwines \( I_{\ell^2(R)} \otimes \pi_\omega \) and \( \tilde{\pi}_\omega|_N \); moreover, \( \omega \mapsto \tilde{U}_\omega \) is a measurable field on \( \Omega \).

Observe that the representation \( \lambda_N \) is equivalent to \( \int_\Omega \pi_\omega d\nu(\omega) \).
Since \( \lambda_\Gamma \) is equivalent to \( \text{Ind}_N^\Gamma \lambda_N \), it follows that \( \lambda_\Gamma \) is equivalent to \( \int_\Omega \tilde{\pi}_\omega d\nu(\omega) \).

In the sequel, we will identify the representations \( \lambda_N \) on \( \ell^2(N) \) and \( \lambda_\Gamma \) on \( \ell^2(\Gamma) \) with respectively the representations
\[
\int_\Omega \pi_\omega d\nu(\omega) \quad \text{on } \mathcal{K} := \int_\Omega K_\omega d\nu(\omega)
\]
and
\[
\int_\Omega \tilde{\pi}_\omega d\nu(\omega) \quad \text{on } \mathcal{H} := \int_\Omega \ell^2(R, K_\omega) d\nu(\omega).
\]
Step 3 The representations $\widehat{\pi_\omega}$ are factorial and are mutually disjoint, outside a subset of $\Omega$ of $\nu$-measure 0.

To show this, it suffices to prove (see [Dix77, 8.4.1]) that the algebra $\mathcal{D}$ of diagonal operators in $\mathcal{L}(\mathcal{H})$ coincides with the center $\mathcal{Z}$ of $\lambda_\Gamma(\Gamma)'$.

Let us first prove that $\mathcal{D} \subset \mathcal{Z}$. For this, we only have to prove that $\mathcal{D} \subset \lambda_\Gamma(\Gamma)'$, since it is clear that $\mathcal{D} \subset \lambda_\Gamma(\Gamma)''$.

By Step 2, for every $\omega \in \Omega$, there exists a measurable field $\omega \to \widetilde{U}_\omega$ of unitary operators $\widetilde{U}_\omega : \ell^2(R, K_\omega) \to \ell^2(R, K_\omega)$ intertwining $I_{\ell^2(R)} \otimes \pi_\omega$ and $\pi_\omega|_N$. So,

$\widetilde{U} := \int_{\Omega}^{\oplus} \widetilde{U}_\omega d\nu(\omega)$

is a unitary operator on $\mathcal{H}$ which intertwines $I_{\ell^2(R)} \otimes \lambda_N$ and $\lambda_\Gamma|_N$; it is obvious that $\widetilde{U}$ commutes with the diagonal operators on $\mathcal{H}$.

Let $\varphi : \Omega \to \mathbb{C}$ be a measurable essential bounded function on $\Omega$. By Theorem A.ii, the corresponding diagonal operator $T = \int_{\Omega}^{\oplus} \varphi(\omega) I_{K_\omega} d\nu(\omega)$ on $K$ belongs to $\lambda_N(N)^\prime$.

For the corresponding diagonal operator

$\widetilde{T} = \int_{\Omega}^{\oplus} \varphi(\omega) I_{\ell^2(R, K_\omega)} d\nu(\omega)$

on $\mathcal{H}$, we have $\widetilde{T} = I_{\ell^2(R)} \otimes T$. So, $\widetilde{T}$ belongs to $(I_{\ell^2(R)} \otimes \lambda_N)(N)^\prime$.

Since $\widetilde{U}$ commutes with $\widetilde{T}$ and intertwines $I_{\ell^2(R)} \otimes \lambda_N$ and $\lambda_\Gamma|_N$, it follows that

$\widetilde{T} = \widetilde{U} (I_{\ell^2(R)} \otimes T) \widetilde{U}^{-1} \in \lambda_\Gamma(N)^\prime \subset \lambda_\Gamma(\Gamma)^\prime$.

So, we have shown that $\mathcal{D} \subset \mathcal{Z}$. Observe that this implies (see Théorème 1 in Chap. II, §3 of [Dix69]) that $L(\Gamma)$ is the direct integral $\int_{\Omega}^{\oplus} \pi_\omega(\Gamma)^\prime d\nu(\omega)$ and that $\mathcal{Z}$ is the direct integral $\int_{\Omega}^{\oplus} \mathcal{Z}_\omega d\nu(\omega)$, where $\mathcal{Z}_\omega$ is the center of $\pi_\omega(\Gamma)^\prime$.

Let $\widehat{T} \in \mathcal{Z}$. Then $\widehat{T} \in \lambda_\Gamma(N)^\prime$, by Lemma 4. So, $\widehat{T} = \int_{\Omega}^{\oplus} T_\omega d\nu(\omega)$, where $T_\omega$ belongs to the center of $\pi_\omega(\Gamma)^\prime$, for $\nu$-almost every $\omega$. Since $\pi_\omega(\Gamma)^\prime$ is equivalent to $I_{\ell^2(R)} \otimes \pi_\omega$ and since $\pi_\omega\Gamma$ and hence $I_{\ell^2(R)} \otimes \pi_\omega$ is a factor representation, it follows that $T_\omega$ is a scalar operator, for $\nu$-almost every $\omega$. So, $\widehat{T} \in \mathcal{D}$.

As a result, we have a decomposition

$\lambda_\Gamma = \int_{\Omega}^{\oplus} \pi_\omega d\nu(\omega)$
of \( \lambda \) as a direct integral of pairwise disjoint factor representations. By the argument of the proof of Theorem A, it follows that, for \( \dot{\nu} \)-almost every \( \omega \in X \), there exists a cyclic unit vector \( \xi_\omega \in \ell^2(R, K_\omega) \) for \( \pi_{\omega} \) so that

\[
\varphi_\omega := \langle \pi_{\omega} (\cdot) \xi_\omega | \xi_\omega \rangle
\]

belongs to \( \text{Ch}(\Gamma) \). In particular,

\[
\hat{M}_\omega := \pi_{\omega}(\Gamma)''
\]

is a factor of type II\(_1\) and its normalized trace is the extension of \( \varphi_\omega \) to \( \hat{M}_\omega \), which we again denote by \( \varphi_\omega \). Our next goal is to determine \( \varphi_\omega \) in terms of the character \( \omega \in \text{Ch}(N) \).

Fix \( \omega \in \Omega \) such that \( \hat{M}_\omega \) is a factor. We identity the Hilbert space \( K_\omega \) of \( \pi_{\omega} \) with the subspace \( \delta_e \otimes K_\omega \) and so \( \pi_{\omega} \) with a subrepresentation of the restriction of \( \pi_{\omega} \) to \( N \).

Let \( \eta_\omega \in K_\omega \) be a cyclic vector for \( \pi_{\omega} \) such that

\[
\omega = \langle \pi_{\omega} (\cdot) \eta_\omega | \eta_\omega \rangle.
\]

For \( g \in K_\Gamma \), define a normal state \( \psi_{\omega,g} \) on \( \hat{M}_\omega \) by the formula

\[
\psi_{\omega,g}(\hat{T}) = \langle \hat{T} U_{g,\omega}^{-1} \eta_\omega | U_{g,\omega}^{-1} \eta_\omega \rangle \quad \text{for all } \hat{T} \in \hat{M}_\omega,
\]

where \( U_{g,\omega} \) is the unitary operator on \( K_\omega \) from Step 1.

Consider the linear functional \( \psi_\omega : \hat{M}_\omega \to \mathbb{C} \) given by

\[
\psi_\omega(\hat{T}) = \int_{K_\Gamma} \psi_{\omega,g}(\hat{T})dm(g) \quad \text{for all } \hat{T} \in \hat{M}_\omega,
\]

where \( m \) is the normalized Haar measure on \( K_\Gamma \).

**Step 4** We claim that \( \psi_\omega \) is a normal state on \( \hat{M}_\omega \).

Indeed, it is clear that \( \psi_\omega \) is a state on \( \hat{M}_\omega \). Let \( (\hat{T}_n) \) be an increasing sequence of positive operators in \( \hat{M}_\omega \) with \( \hat{T} = \sup_n \hat{T}_n \in \hat{M}_\omega \).

For every \( g \in K_\Gamma \), the sequence \( (\psi_{\omega,g}(\hat{T}_n)) \) is increasing and its limit is \( \psi_{\omega,g}(\hat{T}) \). It follows from the monotone convergence theorem that

\[
\lim_n \psi_\omega(T_n) = \lim_n \int_{K_\Gamma} \psi_{\omega,g}(T_n)dm(g) = \int_{K_\Gamma} \psi_{\omega,g}(T)dm(g) = \psi_\omega(T).
\]

So, \( \psi_\omega \) is normal, as \( \hat{M}_\omega \) acts on a separable Hilbert space.

**Step 5** We claim that, writing \( \gamma \) instead of \( \pi_{\omega}(\gamma) \) for \( \gamma \in \Gamma \), we have

\[
\psi_\omega(\gamma) = \begin{cases} 
\int_{K_\Gamma} \omega^g(\gamma)dm(g) & \text{if } \gamma \in N \\
0 & \text{if } \gamma \notin N.
\end{cases}
\]
Moreover, $\psi_\omega$ coincides with the trace $\varphi_\omega$ on $\widetilde{\mathcal{M}}_\omega$ from above. Indeed, let $\gamma \in N$. Since

$$U_{\omega,g} \pi_\omega(\gamma) U_{\omega,g}^{-1} = \pi_\omega(g(\gamma)),$$

we have

$$\psi_\omega(\gamma) = \int_{K_\Gamma} \langle \pi_\omega(g(\gamma)) \eta_\omega | \eta_\omega \rangle dm(g) = \int_{K_\Gamma} \langle \pi_\omega(g(\gamma)) \eta_\omega | \eta_\omega \rangle dm(g)$$

$$= \int_{K_\Gamma} \omega^g(\gamma) dm(g).$$

Let $\gamma \in \Gamma \setminus N$. Then, by the usual properties of an induced representation, $\pi_\omega(\gamma)(K_\omega)$ is orthogonal to $K_\omega$. It follows that

$$\psi_{\omega,g}(\gamma) = \langle \pi_\omega(\gamma) U_{g,\omega}^{-1} \eta_\omega | U_{g,\omega}^{-1} \eta_\omega \rangle = 0$$

and hence $\psi_\omega(\gamma) = 0$.

In particular, this shows that $\psi_\omega$ is a $\Gamma$-invariant state on $\widetilde{\mathcal{M}}_\omega$; since $\psi_\omega$ is normal (Step 4), it follows that $\psi_\omega$ is a trace on $\mathcal{M}_\omega$. As $\mathcal{M}_\omega$ is a factor (see Step 3), the fact that $\tilde{\psi}_\omega = \varphi_\omega$ follows from the uniqueness of normal traces on factors (see Corollaire p. 92 and Corollaire 2 p.83 in [Dix69, Chap. I, §6]).

**Step 6** The Plancherel measure $\mu$ on $\Gamma$ is the image of $\nu$ under the map $\Phi$ as in the statement of Theorem B.ii.

Indeed, for $f \in C[\Gamma]$, we have by Step 5

$$\|f\|_2^2 = \int_{\Omega} \|\pi_\omega(f)\|^2 d\hat{\nu}(\omega) = \int_{\Omega} \psi_\omega(f^* f) d\hat{\nu}(\omega)$$

$$= \int_{\Omega} \int_{K_\Gamma} \omega^g((f^* f)|_N) dm(g) d\hat{\nu}(\omega)$$

$$= \int_{\Omega} \pi_\omega(t) \int_{K_\Gamma} \omega^g((f^* f)|_N) dm(g) d\nu(t)$$

$$= \int_{\text{Ch}(\Gamma)} \Phi(t)(f^* f) d\nu(t)$$

and the claim follows.

**Remark 7.** The map $\Phi$ in Theorem B.ii can be described without reference to the group $K_\Gamma$ as follows. For $t \in \text{Ch}(\Gamma_k)$ and $\gamma \in \Gamma$, we
have

$$
\Phi(t)(\gamma) = \begin{cases} 
\frac{1}{\#[\gamma]} \sum_{x \in [\gamma]} t(x) & \text{if } \gamma \in \Gamma_{fc} \\
0 & \text{if } \gamma \notin \Gamma_{fc}
\end{cases},
$$

where $[\gamma]$ denotes the $\Gamma$-conjugacy class of $\gamma$. Indeed, it suffices to consider the case where $\gamma \in \Gamma_{fc}$. The stabilizer $K_0$ of $\gamma$ in $K\Gamma$ is an open and hence cofinite subgroup of $K\Gamma$; in particular, the $K\Gamma$-orbit of $\gamma$ coincides with the $\Ad(\Gamma)$-orbit of $\gamma$ and so $\{\Ad(x) \mid x \in [\gamma]\}$ is a system of representatives for $K\Gamma/K_0$. Let $m_0$ be the normalized Haar measure on $K_0$. The normalized Haar measure $m$ on $K\Gamma$ is then given by

$$
m(f) = \frac{1}{[\gamma]} \sum_{x \in [\gamma]} \int_{K_0} f(\Ad(x)g) dm_0(g) \text{ for every continuous function } f \text{ on } K\Gamma.
$$

It follows that

$$
\Phi(t)(\gamma) = \int_{K\Gamma} t(g(\gamma)) dm(g) = \frac{1}{[\gamma]} \sum_{x \in [\gamma]} t(x).
$$

5. Proofs of Corollary [C] and Corollary [D]

Let $\Gamma$ be a countable linear group. So, $\Gamma$ is a subgroup of $GL_n(k)$ for a field $k$, which may be assumed to be algebraically closed. Let $G$ be the closure of $\Gamma$ in the Zariski topology of $GL_n(k)$ and let $G_0$ be the irreducible component of $G$. As is well-known, $G_0$ has finite index in $G$ and hence $\Gamma_0 := G_0 \cap \Gamma$ is a normal subgroup of finite index in $\Gamma$.

Let $\gamma \in \Gamma_{fc}$. On the one hand, the centralizer $\Gamma_\gamma$ of $\gamma$ in $\Gamma$ is a subgroup of finite index of $\Gamma$; therefore, the irreducible component of the Zariski closure of $\Gamma_\gamma$ coincides with $G_0$. On the other hand, the centralizer $G_\gamma$ of $\gamma$ in $G$ is clearly a Zariski-closed subgroup of $G$. It follows that the irreducible component of $G_\gamma$ contains (in fact coincides with) $G_0$ and hence

$$
\Gamma_0 = G_0 \cap \Gamma \subset G_\gamma \cap \Gamma = \Gamma_\gamma.
$$

As a consequence, we see that $\Gamma_0$ acts trivially on $\Gamma_{fc}$ and hence $\Ad(\Gamma)|_{\Gamma_{fc}}$ is a finite group. In particular, $\Gamma_0 \cap \Gamma_{fc}$ is contained in the center $Z(\Gamma_{fc})$ of $\Gamma_{fc}$; so $Z(\Gamma_{fc})$ has finite index in $\Gamma_{fc}$ which is therefore a central group. This proves Items (i) and (ii) of Corollary [C]. Item (iii) follows from Theorem [B].ii.

Assume now that $G$ is connected, that is $G = G_0$. Then $\Gamma = \Gamma_0$ acts trivially on $\Gamma_{fc}$ and so $\Gamma_{fc}$ coincides with the center $Z(\Gamma)$ of $\Gamma$. This proves the first part of Corollary [D].

It remains to prove that the assumption $G = G_0$ is satisfied in Cases (i) and (ii) of Corollary [D].
(i) Let $G$ be a connected linear algebraic group over a countable field $k$ of characteristic 0. Then $\Gamma = G(k)$ is Zariski dense in $G$, by [Ros57, Corollary p.44]).

(ii) Let $G$ be a connected linear algebraic group $G$ over a local field $k$. Assume that $G$ has no proper $k$-subgroup $H$ such that $(G/H)(k)$ is compact. Then every lattice $\Gamma$ in $G(k)$ is Zariski dense in $G$, by [Sha99, Corollary 1.2].

6. The Plancherel Formula for some countable groups

6.1. Restricted direct product of finite groups. Let $(G_n)_{n \geq 1}$ be a sequence of finite groups. Let $\Gamma = \prod_{n \geq 1}^{\prime} G_n$ be the restricted direct product of the $G_n$'s, that is, $\Gamma$ consists of the sequences $(g_n)_{n \geq 1}$ with $g_n \in G_n$ for all $n$ and $g_n \neq e$ for at most finitely many $n$. It is clear that $\Gamma$ is an FC-group.

Set $X_n := \text{Ch}(G_n)$ for $n \geq 1$ and let $X = \prod_{n \geq 1} X_n$ be the cartesian product equipped with the product topology, where each $X_n$ carries the discrete topology. Define a map $\Phi : X \to \text{Tr}(\Gamma)$ by

$$\Phi((t_n)_{n \geq 1})(((g_n)_{n \geq 1}) = \prod_{n \geq 1} t_n(g_n)$$

for all $(t_n)_{n \geq 1} \in X, (g_n)_{n \geq 1} \in \Gamma$ (observe that this product is well-defined, since $g_n = e$ and hence $t_n(g_n) = 1$ for almost every $n \geq 1$). Then $\Phi(X) = \text{Ch}(\Gamma)$ and $\Phi : X \to \text{Ch}(\Gamma)$ is a homeomorphism (see [Mau51, Lemma 7.1]).

For every $n \geq 1$, let $\nu_n$ be the measure on $X_n$ given by

$$\nu_n(\{t\}) = \frac{d_t^2}{\#G_n}$$

for all $t \in X_n,$

where $d_t$ is the dimension of the irreducible representation of $G_n$ with $t$ as character; observe that $\nu_n$ is a probability measure, since $\sum_{t \in X_n} d_t^2 = \#G_n$.

Let $\nu = \otimes_{n \geq 1} \nu_n$ be the product measure on the Borel subsets of $X$. The Plancherel measure on $\Gamma$ is the image of $\nu$ under $\Phi$ (see Equation (5.6) in [Mau51]). The regular representation $\lambda_\Gamma$ is of type II if and only if infinitely many $G_n$’s are non abelian (see loc.cit., Theorem 1 or Theorem [E] below).

6.2. Infinite dimensional Heisenberg group. Let $F_p$ be the field of order $p$ for an odd prime $p$ and let $V = \bigoplus_{i \in \mathbb{N}} F_p$ be a vector space over $F_p$ of countable infinite dimension. Denote by $\omega$ the symplectic form on $V \oplus V$ given by

$$\omega((x,y), (x', y')) = \sum_{i \in \mathbb{N}} (x_iy_i' - y_ix_i')$$

for $(x,y), (x', y') \in V \oplus V.$
The “infinite dimensional” Heisenberg group over $\mathbb{F}_p$ is the group $\Gamma$ with underlying set $V \oplus V \oplus \mathbb{F}_p$ and with multiplication defined by
\[(x, y, z)(x', y', z') = (x + x', y + y', z + z' + \omega(((x, y), (x', y'))))\]
for $(x, y, z), (x', y', z') \in \Gamma$.

The group $\Gamma$ is an FC-group; since $p \geq 3$, its center $Z$ coincides with $[\Gamma, \Gamma]$ and consists of the elements of the form $(0, 0, z)$ for $z \in \mathbb{F}_p$. Observe that $\Gamma$ is not virtually abelian.

Let $z_0$ be a generator for the cyclic group $Z$ of order $p$. The unitary dual $\hat{Z}$ consists of the characters defined by $\chi_{\omega}(z_0^j) = \omega^j$ for $j \in \{0, 1, \ldots, p - 1\}$ and $\omega \in C_p$, where $C_p$ is the group of $p$-th roots of unity in $C$.

For $\omega \in C_p$, the subspace
\[H_\omega = \{f \in \ell^2(\Gamma) \mid f(z_0x) = \omega f(x) \text{ for every } x \in \Gamma\}\]
is left and right translation invariant and we have an orthogonal decomposition of $\ell^2(\Gamma) = \bigoplus_{\omega \in C_p} H_\omega$. The orthogonal projection $P_\omega$ on $H_\omega$ belongs to the center of $L(\Gamma)$ and is given by
\[P_\omega(f)(x) = \frac{1}{p} \sum_{i=0}^{p-1} \omega^{-i} f(z_0^i x) \quad \text{for all } f \in \ell^2(\Gamma), x \in \Gamma.\]

One checks that $\|P_\omega(\delta_e)\|_2^2 = 1/p$.

Let $\pi_\omega$ be the restriction of $\lambda_\Gamma$ to $H_\omega$. Observe that $H_1$ can be identified with $\ell^2(\Gamma/Z)$ and $\pi_1$ with $\lambda_{\Gamma/Z}$.

For $\omega \neq 1$, the representation $\pi_\omega$ is factorial of type II$_1$ and the corresponding character is $\tilde{\chi}_\omega$ (for more details, see the proof of Theorem 7.D.4 in [BH]).

The integral decomposition of $\delta_e$ is
\[\delta_e = \frac{1}{p} \int_{\Gamma/Z} \chi d\nu(\chi) + \frac{1}{p} \sum_{\omega \in C_p \setminus \{1\}} \tilde{\chi}_\omega,\]
with the corresponding Plancherel formula given for every $f \in C[\Gamma]$ by
\[\|f\|_2^2 = \frac{1}{p} \int_{\Gamma/Z} |F(P_1(f))(\chi)|^2 d\nu(\chi) + \frac{1}{p} \sum_{\omega \in C_p \setminus \{1\}} \sum_{j=0}^{p-1} (f^* \ast f)(z_0^j) \omega^j,\]
where $\nu$ is the normalized Haar measure of the compact abelian group $\Gamma/Z$ and $F$ the Fourier transform. In particular, $\lambda_{\Gamma}(\Gamma)''$ is a direct sum of an abelian von Neumann algebra and $p - 1$ factors of type II$_1$. For a more general result, see [Kap51, Theorem 2].
6.3. An example involving $SL_d(Z)$. Let $\Lambda = SL_d(Z)$ for an odd integer $d$. Fix a prime $p$ and for $n \geq 1$, let $G_n = SL_d(Z/p^nZ)$, viewed as (finite) quotient of $\Lambda$. Let $\Gamma$ be the semi-direct product $\Lambda \rtimes \prod_{n \geq 1}' G_n$, where $\Lambda$ acts diagonally in the natural way on the restricted direct product $G := \prod_{n \geq 1}' G_n$ of the $G_n$’s.

Since $\Lambda$ is an ICC-group, it is clear that $\Gamma_{fc} = G$. The group $K_\Gamma$ as in Theorem B can be identified with the projective limit of the groups $G_n$’s, that is, with $SL_d(Z_p)$, where $Z_p$ is the ring of $p$-adic integers. Since $\Lambda$ acts trivially on Ch($G$), the same is true for the action of $K_\Gamma$ on Ch($G$).

Let $\lambda_G = \int^{\oplus}_{\text{Ch}(G)} \pi_t d\nu(t)$ be the Plancherel decomposition of $\lambda_G$ (see Example 6.1). It follows from Theorem B that the Plancherel decomposition of $\lambda_\Gamma$ is

$$\lambda_\Gamma = \int^{\oplus}_{\text{Ch}(G)} \text{Ind}_{G}^{\Gamma} \pi_t d\nu(t).$$

Appendix A. Proof of Theorem E

A.1. Easy implications. The implications $(iv') \Rightarrow (iv)$ and $(i) \Rightarrow (iii)$ are obvious; if $(iv)$ holds then $\Gamma$ is a so-called CCR group and so $(i)$ holds, by a general fact (see [Dix77 5.5.2]).

We are going to show that $(ii) \Rightarrow (iv')$, Assume that $\Gamma$ contains an abelian normal subgroup $N$ of finite index. Let $(\pi, \mathcal{H})$ be an irreducible representation of $\Gamma$.

Denote by $\mathcal{B}$ the set of Borel subsets of the dual group $\hat{N}$ and by $\text{Proj}(\mathcal{H})$ the set of orthogonal projections in $L(\mathcal{H})$. Let $E : \mathcal{B}(\hat{N}) \to \text{Proj}(\mathcal{H})$ be the projection-valued measure on $\hat{N}$ associated with the restriction $\pi|_N$ by the SNAG Theorem (see [BHV08 D.3.1]); so, we have

$$\pi(n) = \int_{\hat{G}} \chi(n) dE(\chi) \quad \text{for all} \ n \in N.$$

The dual action of $\Gamma$ on $\hat{N}$, given by $\chi^\gamma(n) = \chi(\gamma^{-1}n\gamma)$ for $\chi \in \hat{N}$ and $\gamma \in \Gamma$, factorizes through $\Gamma/N$. Moreover, the following covariance relation holds

$$\pi(\gamma) E(B) \pi(\gamma^{-1}) = E(B^\gamma) \quad \text{for all} \ B \in \mathcal{B}(\hat{N}),$$

where $B^\gamma = \{\chi^\gamma \mid \chi \in B\}$.

Let $S \in \mathcal{B}(\hat{N})$ be the support of $E$, that is, $S$ is the complement of the largest open subset $U$ of $\hat{N}$ with $E(U) = 0$. We claim that $S$ consists of a single $\Gamma$-orbit.

Indeed, let $\chi_0 \in S$ and let $(U_n)_{n \geq 1}$ be a sequence of open neighbourhoods of $\chi$ with $\bigcap_{n \geq 1} U_n = \{\chi_0\}$. Fix $n \geq 1$. The set $U_n^\Gamma$ is $\Gamma$-
invariant and hence $E(U_n^T) \in \pi(\Gamma)'$, by the covariance relation. Since $\pi$ is irreducible and $E(U_n) \neq 0$, we have therefore $E(U_n^T) = I_H$. By the usual properties of a projection-valued measure, this implies that
\[ E(\chi_0^T) = E(\bigcap_{n \geq 1} U_n^T) = I_H, \]
and the claim is proved.

Since $S$ is finite, we have $H = \bigoplus_{\chi \in S} H_\chi$, where
\[ H_\chi := \{ \xi \in H \mid \pi(n)\xi = \chi(n)\xi \quad \text{for all} \quad n \in N \}; \]
moreover, since $N$ is a normal subgroup, we have $\pi(\gamma)H_\chi = H_\chi \gamma$ for every $\chi \in S$ and every $\gamma \in \Gamma$.

Let $H$ be the stabilizer of $\chi_0$ and let $T \subset \Gamma$ be a set of representatives of the right $T$-cosets of $H$. Then $H_\chi$ is invariant under $\pi(H)$ and we have
\[ H = \bigoplus_{t \in T} H_\chi = \bigoplus_{t \in T} \pi(t)H_\chi. \]
This shows that $\pi$ is equivalent to the induced representation $\text{Ind}^\Gamma_H \sigma$, where $\sigma$ is the subrepresentation of $\pi|_H$ defined on $H_\chi$.

We claim that $\text{Ind}^\Gamma_H \sigma$ is contained in $\text{Ind}^\Gamma_N \chi_0$. Indeed, as is well-known (see [BHV08, E.2.5]), $\text{Ind}^H_N(\sigma|_N)$ is equivalent to the tensor product representation $\sigma \otimes \lambda_{H/N}$, where $\lambda_{H/N}$ is the quasi-regular representation on $\ell^2(\Gamma/H)$. Since $H/N$ is finite, $1_H$ is contained in $\lambda_{H/N}$ and therefore $\sigma$ is contained in $\text{Ind}^H_N(\sigma|_N)$. Notice that $\sigma|_N$ is a multiple $n\chi_0$ of $\chi_0$, for some cardinal $n$. We conclude that $\pi = \text{Ind}^\Gamma_H \sigma$ is contained in $\text{Ind}^\Gamma_H(\text{Ind}^H_N n\chi_0) = n \text{Ind}^\Gamma_N \chi_0$. Since $\pi$ is irreducible, it follows that $\pi$ is contained in $\text{Ind}^\Gamma_N \chi_0$.

Now, $\text{Ind}^\Gamma_N \chi_0$ has dimension $[\Gamma : N]$; hence, $\dim \pi \leq [\Gamma : N]$ and so $(iv')$ holds.

**A.2. Proof of the other implications.** We have to give the proof of the implication $(iii) \Rightarrow (i)$ and the equivalence $(v) \iff (iv)$.

In the sequel, $\Gamma$ will be a countable group and $\lambda_\Gamma = \int_{\text{Ch}(\Gamma)} \pi_t d\mu(t)$ the direct integral decomposition given by the Plancherel Theorem A. Recall (see Section 3) that, if we write $\tilde{\delta}_e = \int_{\text{Ch}(\Gamma)} \xi_t d\mu(t)$, then $\xi_t$ is a cyclic vector in the Hilbert space $H_t$ of $\pi_t$ and $t = \langle \pi_t(\cdot)\xi_t \mid \xi_t \rangle$, for $\mu$-almost every $t \in \text{Ch}(\Gamma)$.

**A.2.1. Case where the FC-centre of $\Gamma$ has infinite index.** We assume that $[\Gamma : \Gamma_{fc}]$ is infinite; we claim that $\lambda_\Gamma$ is of type II.

By Remark 3, i, there exists a subset $X$ of $\text{Ch}(\Gamma)$ with $\mu(X) = 1$ such that $t = 0$ outside $\Gamma_{fc}$ for every $t \in X$. 

Let \( t \in X \). Then the factor \( \pi_t(\Gamma)'' \) is infinite dimensional. Indeed, since \([\Gamma: \Gamma_{fc}]\) is infinite, we can find a sequence \((\gamma_n)_{n \geq 1}\) in \( \Gamma \) with \( \gamma_m^{-1}\gamma_n \notin \Gamma_{fc} \) for every \( m, n \) with \( m \neq n \). Then

\[
\langle \pi_t(\gamma_n)\xi_t \mid \pi_t(\gamma_m)\xi_t \rangle = \langle \pi_t(\gamma_m^{-1}\gamma_n)\xi_t \mid \xi_t \rangle = t(\gamma_m^{-1}\gamma_n) = 0,
\]

for \( m \neq n \); so, \((\pi_t(\gamma_n)\xi_t)_{n \geq 1}\) is an orthonormal sequence in \( H_t \). This implies that \((\pi_t(\gamma_n))_{n \geq 1}\) is a linearly independent sequence in \( \pi_t(\Gamma)'' \) and the claim is proved.

Observe that we have proved, in particular, that \( \Gamma \) is not of type I.

A.2.2. Reduction to FC-groups. Let \( \text{Ch}(\Gamma)_{fd} \) be the set of \( t \in \text{Ch}(\Gamma) \) for which \( \pi_t(\Gamma)'' \) is finite dimensional. Then

\[
\text{Ch}(\Gamma)_{fd} = \bigcup_{n \geq 1} \text{Ch}(\Gamma)_n,
\]

where \( \text{Ch}(\Gamma)_n \) is the set of \( t \) such that \( \dim \pi_t(\Gamma)'' = n \). We claim that \( \text{Ch}(\Gamma)_n \) and hence \( \text{Ch}(\Gamma)_{fd} \) is a measurable subset of \( \text{Ch}(\Gamma) \).

Indeed, let \( \mathcal{F} \) be collection of finite subsets of \( \Gamma \). For every \( F \in \mathcal{F} \), let \( C_F \) be the set of \( t \in \text{Ch}(\Gamma) \) such that the family \((\pi_t(\gamma))_{\gamma \in F}\) is linearly independent, equivalently (see Remark 6.ii), such that \((\pi_t(\gamma)\xi_t)_{\gamma \in F}\) is linearly independent. Since \( t = \langle \pi_t(\cdot)\xi_t \mid \xi_t \rangle \), it follows that

\[
C_F := \{ t \in \text{Ch}(\Gamma) \mid \det(t(\gamma^{-1}\gamma')) \neq 0 \quad \text{for all} \quad (\gamma, \gamma') \in F \times F \}
\]

and this shows that \( C_F \) is measurable. Since

\[
\text{Ch}(\Gamma)_n = \bigcup_{F \in \mathcal{F}, \#F = n} C_F \setminus \left( \bigcup_{F' \in \mathcal{F}, \#F' > n} C_{F'} \right)
\]

and \( \mathcal{F} \) is countable, it follows that \( \text{Ch}(\Gamma)_n \) is measurable.

Assume now that \([\Gamma: \Gamma_{fc}]\) is finite. Observe that, for a cyclic representation \( \pi \) of \( \Gamma_{fc} \), the induced representation \( \text{Ind}_{\Gamma_{fc}}^\Gamma \pi \) is cyclic and so \((\text{Ind}_{\Gamma_{fc}}^\Gamma \pi)(\Gamma)'' \) is finite dimensional if and only if \( \pi(\Gamma_{fc})'' \) is finite dimensional.

let \( \nu \) be the Plancherel measure of \( \Gamma_{fc} \). It follows from Theorem \( B \) that \( \mu(\text{Ch}(\Gamma)_{fd}) = \nu(\text{Ch}(\Gamma_{fc})_{fd}) \); in particular, we have \( \mu(\text{Ch}(\Gamma)_{fd}) = 0 \) (or \( \mu(\text{Ch}(\Gamma)_{fd}) = 1 \)) if and only if \( \nu(\text{Ch}(\Gamma_{fc})_{fd}) = 0 \) (or \( \nu(\text{Ch}(\Gamma_{fc})_{fd}) = 1 \), that is, \( \lambda_\Gamma \) is of type I (or of type II) if and only if \( \lambda_{\Gamma_{fc}} \) is of type I (or of type II).

Observe also that \( \Gamma \) is virtually abelian if and only if \( \Gamma_{fc} \) is virtually abelian. As a consequence, we see that it suffices to prove the implication \((iii) \Rightarrow (i)\) and the equivalence \((v) \leftrightarrow (iv)\) in the case where \( \Gamma = \Gamma_{fc} \).
We will need the following lemma of independent interest, which is valid for an arbitrary countable group \( \Gamma \).

Let \( r : \text{Ch}(\Gamma) \to \text{Tr}([\Gamma, \Gamma]) \) be the restriction map. We will identify \( \text{Ch}(\Gamma/[[\Gamma, \Gamma]]) \) with the set \( \{ s \in \text{Ch}(\Gamma) \mid r(s) = 1_{[\Gamma, \Gamma]} \} \), that is, with the set of unitary characters of \( \Gamma \). Observe that, for every \( s \in \text{Ch}(\Gamma/[[\Gamma, \Gamma]]) \) and \( t \in \text{Ch}(\Gamma) \), we have \( st \in \text{Ch}(\Gamma) \).

**Lemma 8.** Let \( \Gamma \) be a countable group and \( t, t' \in \text{Ch}(\Gamma) \) be such that \( r(t) = r(t') \). Then there exists \( s \in \text{Ch}(\Gamma/[[\Gamma, \Gamma]]) \) such that \( t' = st \).

**Proof.** The integral decomposition of \( 1_{[\Gamma, \Gamma]} \in \text{Tr}(\Gamma) \) into characters is given by
\[
1_{[\Gamma, \Gamma]} = \int_{\text{Ch}(\Gamma/[[\Gamma, \Gamma]])} s d\nu(s),
\]
where \( \nu \) is the Haar measure of \( \Gamma/[[\Gamma, \Gamma]] \). By assumption, we have \( t1_{[\Gamma, \Gamma]} = t'1_{[\Gamma, \Gamma]} \) and hence
\[
t1_{[\Gamma, \Gamma]} = \int_{\text{Ch}(\Gamma/[[\Gamma, \Gamma]])} t s d\nu(s) = \int_{\text{Ch}(\Gamma/[[\Gamma, \Gamma]])} t' s d\nu(s).
\]
By uniqueness of integral decomposition, it follows that the images \( \nu_t \) and \( \nu_{t'} \) of \( \nu \) under the maps \( \text{Ch}(\Gamma/[[\Gamma, \Gamma]]) \to \text{Ch}(\Gamma) \) given respectively by multiplication with \( t \) and \( t' \) coincide. In particular, the supports of \( \nu_t \) and \( \nu_{t'} \) are the same, that is, \( t \text{Ch}(\Gamma/[[\Gamma, \Gamma]]) = t' \text{Ch}(\Gamma/[[\Gamma, \Gamma]]) \) and the claim follows. \( \square \)

We assume from now on that \( \Gamma = \Gamma_{fc} \).

**Step 1** We claim that the regular representation \( \lambda_\Gamma \) is of type II if and only if \([\Gamma, \Gamma] \) is finite.

We have to show that \( \mu(\text{Ch}(\Gamma)_{id}) > 0 \) if and only if \([\Gamma, \Gamma] \) is finite.

Assume first that \([\Gamma, \Gamma] \) is finite. The representation \( \lambda_{\Gamma/[[\Gamma, \Gamma]]} \), lifted to \( \Gamma \), is a subrepresentation of \( \lambda_\Gamma \), since \( \ell^2([\Gamma, \Gamma]) \) can be viewed in an obvious way as \( \Gamma \)-invariant subspace of \( \ell^2(\Gamma) \). As \( \Gamma/[[\Gamma, \Gamma]] \) is abelian, \( \lambda_{\Gamma/[[\Gamma, \Gamma]]} \) is of type I and so \( \mu(\text{Ch}(\Gamma)_{id}) > 0 \).

Conversely, assume that \( \mu(\text{Ch}(\Gamma)_{id}) > 0 \). Since \( \Gamma \) is an FC-group, it suffices to show that \( \Gamma \) has a subgroup of finite index with finite commutator subgroup (see \[\text{Neu55} \text{ Lemma 4.1}] \).

As \( \mu(\text{Ch}(\Gamma)_{id}) > 0 \) and \( \text{Ch}(\Gamma)_{id} = \bigcup_{n \geq 1} \text{Ch}(\Gamma)_n \), we have \( \mu(\text{Ch}(\Gamma)_n) > 0 \) for some \( n \geq 1 \). It follows from (*) that there exists \( F \in \mathcal{F} \) with \( |F| = n \) such that \( \mu(C_F \cap \text{Ch}(\Gamma)_n) > 0 \).

Let \( \Lambda \) be the subgroup of \( \Gamma \) generated by \( F \). Since \( \Gamma \) is an FC-group and \( \Lambda \) is finitely generated, the centralizer \( H := \text{Cent}_\Gamma(\Lambda) \) of \( \Lambda \) in \( \Gamma \) has finite index.
Let $t \in C_F \cap \text{Ch}(\Gamma)_{\mu}$ and $\gamma_0 \in H$. On the one hand, since $(\pi_t(\gamma))_{\gamma \in F}$ is a basis of the vector space $\pi_t(\Gamma)$, we have $\pi_t(\Lambda)^{\prime\prime} = \pi_t(\Gamma)^{\prime\prime}$. On the other hand, as $\gamma_0$ centralizes $\Lambda$, we have $\pi_t(\gamma_0) \in \pi_t(\Lambda)^{\prime\prime}$. Hence, $\pi_t(\gamma_0)$ belongs to the center $\pi_t(\Gamma)^{\prime\prime} \cap \pi_t(\Gamma)$ of the factor $\pi_t(\Gamma)^{\prime\prime}$ and so $\pi_t(\gamma_0)$ is a scalar multiple of $I_{\mathcal{H}_t}$. It follows in particular that $\pi_t$ is trivial on $[H, H]$. As a result, the subrepresentation $\int_{C_F \cap \text{Ch}(\Gamma)_{\mu}} \pi_t \, d\mu(t)$ of $\lambda_{\Gamma}$ is trivial on $[H, H]$. Since the matrix coefficients of $\lambda_{\Gamma}$ vanish at infinity, it follows that $[H, H]$ is finite and the claim is proved.

In view of what we have shown so far, we may and will assume from now on that $[\Gamma, \Gamma]$ is finite and that $\lambda_{\Gamma}$ is of type I. We are going to show that $\Gamma$ is a virtually abelian (in fact, a central) group and this will finish the proof of Theorem E.

Set $N := [\Gamma, \Gamma]$ and let $r : \text{Ch}(\Gamma) \to \text{Tr}(N)$ be the restriction map.

**Step 2** We claim that there exist finitely many functions $s_1, \ldots, s_m$ in $\text{Tr}(N)$ such that $r(t) \in \{s_1, \ldots, s_m\}$, for $\mu$-almost every $t \in \text{Ch}(\Gamma)$.

Indeed, since $\lambda_{\Gamma}$ is of type I, there exists a subset $X$ of $\text{Ch}(\Gamma)$ with $\mu(X) = 1$ such that $\pi_t(\Gamma)^{\prime\prime}$ is finite dimensional for every $t \in X$.

Let $t \in X$. The Hilbert space $H_t$ of $\pi_t$ is finite dimensional and $\pi_t$ is a (finite) multiple of an irreducible representation $\sigma_t$ of $\Gamma$. As $\pi_t$ and $\sigma_t$ have the same normalized character, we may assume that $\pi_t$ is irreducible.

Let $K$ be an irreducible $N$-invariant subspace of $H_t$ and let $\rho$ be the corresponding equivalence class of representation of $N$. For $g \in \Gamma$, the subspace $\pi_t(g)K$ is $N$-invariant with $\rho^g$ as corresponding representation of $N$. Since $\pi_t$ is irreducible, we have $H_t = \bigoplus_{g \in \Gamma} \pi_t(g)K$.

Let $L$ be the stabilizer of $\rho$; observe that $L$ has finite index in $\Gamma$, since $L$ contains the centralizer of $N$ and $\Gamma$ is an FC-group. Let $g_1, \ldots, g_r$ be a set of representatives for the coset space $\Gamma/L$. Then $H_t = \bigoplus_{j=1}^r \pi_t(g_j)K_{\rho}$, where $K_{\rho}$ is the sum of all $N$-invariant subspaces of $H_t$ with corresponding representation equivalent to $\rho$. The normalized trace of the representation of $N$ on $\pi_t(g_j)K_{\rho}$ is $\chi_{\rho}^{g_j}$, where $\chi_{\rho}$ is the normalized character of $\rho$. It follows that, for every $g \in N$, we have

$$t(g) = \frac{1}{r} \sum_{j=1}^r \chi_{\rho}^{g_j}(g).$$

Since $[\Gamma, \Gamma]$ is finite, $[\Gamma, \Gamma]$ has only finitely many equivalence classes of irreducible representations and the claim follows.
**Step 3** We claim that the center $Z(\Gamma)$ has finite index in $\Gamma$.

Indeed, by Step 2, there exists a subset $X$ of $\text{Ch}(\Gamma)$ with $\mu(X) = 1$ and finitely many $t_1, \ldots, t_m \in X$ such that $r(t) \in \{r(t_1), \ldots, r(t_m)\}$ and such that $\mathcal{H}_t$ is finite dimensional for every $t \in X$.

It follows from Lemma 8 that, for every $t \in X$, there exists $s \in \text{Ch}(\Gamma/\Gamma, \Gamma)$ such that $t = st_i$ for some $i \in \{1, \ldots, m\}$ and hence $\dim \pi_t(\Gamma)'' = \dim \pi_{t_i}(\Gamma)''$. As a result, we can find a finitely generated normal subgroup $M$ of $\Gamma$ such that $\dim \pi_t(\Gamma)'' = \dim \pi_t(M)''$ for every $t \in X$.

Since the centralizer $C$ of $M$ in $\Gamma$ has finite index, it suffices to show that $C$ contains $Z(\Gamma)$.

For $g \in C$, $\gamma \in \Gamma$ and $x \in M$, we have $t(x^{-1}g\gamma^{-1}) = t(x^{-1}\gamma)$, that is,

$$\langle \pi_t(g\gamma^{-1})\xi_t | \pi_t(x)\xi_t \rangle = \langle \pi_t(\gamma)\xi_t | \pi_t(x)\xi_t \rangle.$$ 

Since $\dim \pi_t(\Gamma)'' = \dim \pi_t(M)''$, the linear span of $\pi_t(M)\xi_t$ is dense in $\mathcal{H}_t$ and this implies that $\pi_t(g\gamma^{-1})\xi_t = \pi_t(\gamma)\xi_t$ for all $g \in C$, $\gamma \in \Gamma$, and $t \in X$. It follows that $\lambda_t(g\gamma^{-1})\delta_e = \lambda_t(\gamma)\delta_e$ and hence $g\gamma^{-1} = \gamma$ for all $g \in C$ and $\gamma \in \Gamma$; so, $C \subset Z(\Gamma)$.

**References**

[BHV08] B. Bekka, P. de la Harpe, and A. Valette, *Kazhdan's property (T)*, New Mathematical Monographs, vol. 11, Cambridge University Press, Cambridge, 2008. 13, 12, 20, 21

[BH] B. Bekka and P. de la Harpe, *Unitary representations of groups, duals, and characters*, Graduate Studies in Mathematics, American Mathematical Society, Providence, RI. 3, 5, 19

[CPJ94] L. Corwin and C. Pfeffer Johnston, *On factor representations of discrete rational nilpotent groups and the Plancherel formula*, Pacific J. Math. 162 (1994), no. 2, 261–275. 7

[Dix77] J. Dixmier, *$C^*$-algebras*, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. 1, 2, 3, 9, 14, 20

[Dix69] ________, *Les algèbres d’opérateurs dans l’espace hilbertien (algèbres de von Neumann)*, Gauthier-Villars Éditeur, Paris, 1969 (French). Deuxième édition, revue et augmentée; Cahiers Scientifiques, Fasc. XXV. 1, 2, 10, 13, 16

[Gli61] J. Glimm, *Type I $C^*$-algebras*, Ann. of Math. (2) 73 (1961), 572–612. 1

[Kan69] E. Kaniuth, *Der Typ der regulären Darstellung diskreter Gruppen*, Math. Ann. 182 (1969), 334–339 (German). 7

[Kap51] I. Kaplansky, *Group algebras in the large*, Tohoku Math. J. (2) 3 (1951), 249–256. 19

[Mac57] G.W. Mackey, *Borel structure in groups and their duals*, Trans. Amer. Math. Soc. 85 (1957), 134–165. 11

[Mac52] G. W. Mackey, *Induced representations of locally compact groups. I*, Ann. of Math. (2) 55 (1952), 101–139. 10
[Mac61] G.W. Mackey, *Induced representations and normal subgroups*, Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960), Jerusalem Academic Press, Jerusalem; Pergamon, Oxford, 1961, pp. 319–326.

[Mau50] F. I. Mautner, *The structure of the regular representation of certain discrete groups*, Duke Math. J. **17** (1950), 437–441.

[Mau51] __________, *The regular representation of a restricted direct product of finite groups*, Trans. Amer. Math. Soc. **70** (1951), 531–548.

[Neu55] B. H. Neumann, *Groups with finite classes of conjugate subgroups*, Math. Z. **63** (1955), 76–96.

[PJ95] C. Pfeffer Johnston, *On a Plancherel formula for certain discrete, finitely generated, torsion-free nilpotent groups*, Pacific J. Math. **167** (1995), no. 2, 313–326.

[Ros57] M. Rosenlicht, *Some rationality questions on algebraic groups*, Ann. Mat. Pura Appl. (4) **43** (1957), 25–50.

[Sak71] S. Sakai, *C*-algebras and *W*-algebras*, Springer-Verlag, New York-Heidelberg, 1971. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 60. MR0442701

[Seg50] I. E. Segal, *An extension of Plancherel’s formula to separable unimodular groups*, Ann. of Math. (2) **52** (1950), 272–292.

[Sha99] Y. Shalom, *Invariant measures for algebraic actions, Zariski dense subgroups and Kazhdan’s property (T)*, Trans. Amer. Math. Soc. **351** (1999), no. 8, 3387–3412.

[Tho64] E. Thoma, *Über unitäre Darstellungen abzählbarer, diskreter Gruppen*, Math. Ann. **153** (1964), 111–138 (German).

[Tho68] __________, *Eine Charakterisierung diskreter Gruppen vom Typ I*, Invent. Math. **6** (1968), 190–196.

[Tho67] __________, *Über das reguläre Mass im dualen Raum diskreter Gruppen*, Math. Z. **100** (1967), 257–271 (German).

[Zim84] R. J. Zimmer, *Ergodic theory and semisimple groups*, Monographs in Mathematics, vol. 81, Birkhäuser Verlag, Basel, 1984.

BACHIR BEKKA, UNIV RENNES, CNRS, IRMAR–UMR 6625, CAMPUS BEAULIEU, F–35042 RENNES CEDEX, FRANCE

E-mail address: bachir.bekka@univ-rennes1.fr