Zigzag Structures of Simple Two-faced Polyhedra

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Abstract

A zigzag in a plane graph is a circuit of edges, such that any two, but no three, consecutive edges belong to the same face. A railroad in a plane graph is a circuit of hexagonal faces, such that any hexagon is adjacent to its neighbors on opposite edges. A graph without a railroad is called tight. We consider the zigzag and railroad structures of general 3-valent plane graph and, especially, of simple two-faced polyhedra, i.e., 3-valent 3-polytopes with only \(a\)-gonal and \(b\)-gonal faces, where \(3 \leq a < b \leq 6\); the main cases are \((a, b) = (3, 6), (4, 6)\) and \((5, 6)\) (the fullerenes).

We completely describe the zigzag structure for the case \((a, b) = (3, 6)\). For the case \((a, b) = (4, 6)\) we describe symmetry groups, classify all tight graphs with simple zigzags and give the upper bound 9 for the number of zigzags in general tight graphs. For the remaining case \((a, b) = (5, 6)\) we give a construction realizing a prescribed zigzag structure.

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1 Introduction: main notions

A graph $G$ consists of a vertex-set $V(G)$ and an edge-set $E(G)$, such that either one or two vertices are assigned to each edge as its ends. A graph is said to be simple if no edge has only one end-vertex and no two edges have identical end-vertices. A plane graph is a particular drawing of a graph in the Euclidean plane using smooth curves that cross each other only at the vertices of the graph. A graph, which has at least one such drawing, is called planar. We only deal with simple connected plane graphs, whose vertex-sets and edge-sets are finite.

By a polyhedron we mean a convex 3-dimensional polytope. The vertices and the edges of a polyhedron form a simple planar 3-connected graph called skeleton. Such a graph has a unique drawing (up to diffeomorphisms of the Euclidean plane), but there exist many distinct polyhedra having this graph as their skeleton; all such polyhedra are of the same combinatorial type. The group of isometries of a polyhedron (so-called point group) is a subgroup of the algebraic symmetry group of the graph. By a theorem of Mani ([Ma71]), there is, for each combinatorial type, at least one polyhedron, for which this inclusion becomes equality. So, we can identify the polyhedron and its graph, as well as the algebraic symmetry group and the point group.

The $v$-vector $v(G) = (\ldots, v_i, \ldots)$ of a polyhedron $G$ enumerates the numbers $v_i$ of vertices of degree $i$. A polyhedron is simple if $v_i = 0$ for $i \neq 3$. The $p$-vector $p(G) = (\ldots, p_i, \ldots)$ of a polyhedron $G$ counts the numbers $p_i$ of faces having $i$ sides. For a connected plane graph $G$, we denote its plane dual graph by $G^*$ and define it on the set of faces of $G$ with two faces being adjacent if they share an edge. Clearly, $v(G^*) = p(G)$ and $p(G^*) = v(G)$.

Let $G$ be a plane graph, such that every of its vertices has at least degree 3. Then all edges, incident to a vertex $x$, can be labeled in counter-clockwise order as $e_1, e_2, \ldots, e_k$, where $k$ is the degree of $x$ in $G$. For any edge $e_i$, $1 \leq i \leq k$, the edges $e_{i+1}$ and $e_{i-1}$ (with $i+1$ and $i-1$ being addition modulo $k$) are called, respectively, the left and the right. A circuit of edges of $G$ is called a zigzag (or a Petrie path [Cox73], geodesic [GrüMo63], left-right path [Sh75]), if, in tracing the circuit, we alternately select, as the next edge, the left neighbor and the right neighbor (see Figure 1). In a 3-valent plane graph, any pair of edges sharing a vertex define a zigzag.
Given an edge, there are two possible directions for extending it to a zigzag. Hence, each edge is covered exactly twice by zigzags. A zigzag $Z$ is called simple if it has no self-intersection. Otherwise, it has at least one edge, say, $e$, in which it self-intersects. Let us choose an orientation on $Z$; call the edge $e$ of type I or type II [Gr"unMo63] if $Z$ traverses it twice in the opposite or in the same direction, respectively (see Figure 1). Clearly, the type of edge does not depend on the choice of orientation. Edges of type I and type II correspond to edges of cocycle and cycle character in [Sh75]. The signature of a zigzag is the pair $(\alpha_1, \alpha_2)$, where $\alpha_1$ and $\alpha_2$ are the numbers of its edges of type I and type II, respectively.

The $z$-vector of a graph $G$ is the vector enumerating the lengths of all its zigzags with their signature as subscript. The simple zigzags are put in the beginning, in increasing order of length, without their signature $(0, 0)$, and separated by a semicolon from others. Self-intersecting zigzags are also ordered by increasing lengths. If there are $m > 1$ zigzags of the same length $l$ and the same signature $(\alpha_1, \alpha_2) \neq (0, 0)$, then we write $l^m_{\alpha_1, \alpha_2}$. For a zigzag $Z$, its intersection vector $\text{Int}(Z) = (\ldots, c_k^m, \ldots)$ is such that $(\ldots, c_k, \ldots)$ is an increasing sequence of sizes $c_k$ of its intersection with all others zigzags, and $m_k$ denote respective multiplicities.

The length of each zigzag is even, because we consecutively take left and right turns. But the length of intersection of two simple zigzags can be odd (see Klein and Dyck maps on the Table II; they both have oriented genus 3 and come as quotients of regular triangulations of the hyperbolic plane having valency 7 and 8, respectively). Returning to our plane case, the number of intersection can be even only.

A graph is called $z$-uniform if all zigzags have the same length and signature, $z$-transitive if its group acts transitively on zigzags. Clearly, $z$-transitivity implies $z$-uniformity. A graph is called $z$-knotted if it has only one zigzag, $z$-balanced if all its zigzags of the same length and same signature, have identical intersection vectors. Clearly, $z$-transitivity implies $z$-balancedness.

Let $Z_1, \ldots, Z_p$ be all zigzags of $G$. On every zigzag choose an orientation; there
are $2^n$ possibilities. Every edge, being covered twice by zigzags (one or two), can be, with respect to the chosen orientation of all zigzags, oriented in the opposite or same direction. We call such an edge of type I or type II with respect to the fixed orientation. If an edge is a self-intersection edge of some zigzag, then the notion of type is independent of the fixed orientation and coincides with the notion of type I and II, which was introduced above.

While for dual graphs $v$- and $p$-vectors are interchanged, the $z$-vector remains the same, except that type I and type II in the signature are interchanged. Other kinds of dualities (for example, interchanging $z$- and $p$-vector, but preserving $v$-vector) are considered in [Li82].

The medial graph $Med(G)$ of $G$ is defined on the set of edges of $G$ with two edges being adjacent if they share a common vertex and if they belong to a common face. The medial graph is 4-valent and $Med(G) = Med(G^*)$. Zigzags of $G$ and $G^*$ correspond to central circuits (see below) of $Med(G)$. Any 4-valent plane graph $H$ is the medial graph for a pair of mutually dual plane graphs: one can assign two colors to the faces of $H$ in the “chess way”, such that no two adjacent faces of $H$ have the same color. Two faces of the same color are said to be adjacent if they share a vertex.

In general, a central circuit of an Eulerian (i.e., having only vertices of even degree) plane graph is a circuit, which is obtained by starting with an edge and continuing at each vertex by the edge opposite to the entering one. A link is one or more circles (components of a link) embedded into the space $\mathbb{R}^3$; a link with one component is called knot. A projection of a link is a drawing of it on the plane with gaps representing underpass and solid line representing overpass. If a drawing of a link has alternating underpasses and overpasses, then it is called alternating; so, we can see it just as a 4-valent plane graph. We will consider only minimal projections, i.e., those without 1-gons (loops). Clearly, each edge belongs to exactly one central circuit and any 4-valent graph without 1-gons can be seen as a minimal projection of an alternating link with components corresponding to its central circuits. Since zigzags in $G$ correspond to central circuits in $Med(G)$, each zigzag corresponds to a component of the corresponding alternating link. The bipartition of edges of a $z$-knotted graph into type I and type II corresponds to a bipartition of the vertices of its medial graph (which is a projection of an alternating knot).
Call a polyhedron *two-faced* if it has only \(a\)- and \(b\)-gonal faces with \(3 \leq a < b \leq 6\). In particular, denote 3-valent two-faced \(n\)-vertex polyhedron by \(3_n\), \(4_n\), \(5_n\) if \((a, b) = (3, 6), (4, 6), (5, 6)\), respectively. Given two circuits \(u_1, \ldots, u_m\) and \(v_1, \ldots, v_m\), an \(m\)-sided prism \(Prism_m\) is formed when every \(u_i\) is joined to \(v_i\) by an edge. Now, an \(m\)-sided antiprism \(APrism_m\) is formed by adding the cycle \(u_1, v_2, u_2, v_3, \ldots, v_m, u_m, v_1, u_1\).

A point group is a finite subgroup of the group \(O(3)\) of isometries of the space \(\mathbb{R}^3\), fixing the origin. The connection with plane graphs come from representing them on the sphere centered at the origin. The list of point groups is splitted into two classes: the infinite families and the sporadic cases. Every point group \(G\) contains a normal subgroup formed by the rotations of \(G\). The group, denoted \(C_m\), is the cyclic group of rotations by angle \(\frac{2\pi}{m}k\) with \(0 \leq k \leq m - 1\) around a fixed axis \(\Delta\). Both groups \(C_{mv}\) and \(C_{mh}\) contains \(C_m\) as normal subgroups of rotations. The group \(C_{mh}\) (respectively, \(C_{mv}\)) is the group, generated by \(C_m\) and a symmetry of plane \(P\), with \(P\) being orthogonal to \(\Delta\) (respectively, containing \(\Delta\)). The group \(D_m\) is the group, generated by \(C_m\) and a rotation by angle \(\pi\), whose axis is orthogonal to \(\Delta\). The point group \(D_{mh}\) (respectively, \(D_{md}\)) is generated by \(C_{mv}\) and a rotation by angle \(\pi\), whose axis is orthogonal to \(\Delta\) and contained in a plane of symmetry (respectively, going between two planes of symmetry). Both, \(D_{mh}\) and \(D_{md}\), contain \(D_m\) as a normal subgroup. If \(N\) is even, one defines the point group \(S_N\) to be the cyclic group generated by the composition of a rotation by angle \(\frac{2\pi}{N}\) with axis \(\Delta\) and a symmetry of plane \(P\) with \(P\) being orthogonal to \(\Delta\). The particular cases \(C_1, C_s\) and \(C_i\) correspond to the trivial group, the plane symmetry group and the central symmetry inversion group. The point groups \(T_d, O_h\) and \(I_h\) are the symmetry group of the Tetrahedron, Cube and Icosahedron; the point groups \(T, O\) and \(I\) are their normal subgroup of rotations. The point group \(T_h\) is formed by all \(f\) and \(-f\) with \(f \in T\). More detailed description of point groups are available, for example, from [RoMa95] and [D1].

In Table 1 zigzag notions are illustrated by the regular and *semiregular* polyhedra (i.e., such that their symmetry group is transitive on vertices). This table is a slightly more precise version (indicating types of self-intersection of zigzags) of Table 1 of [DHL02]. It is easy to see that for self-intersecting zigzags of prisms and an-
| # edges | Polyhedron                  | z-vector | Int. vector |
|---------|----------------------------|----------|-------------|
| 6       | Tetrahedron                | $4^4$    | $2^2$       |
| 12      | Cube                       | $6^4$    | $2^3$       |
| 12      | Octahedron                 | $6^4$    | $2^3$       |
| 30      | Dodecahedron               | $10^6$   | $2^5$       |
| 30      | Icosahedron                | $10^6$   | $2^5$       |
| 24      | Cuboctahedron              | $8^6$    | $2^6$, $0^6$|
| 60      | Icosidodecahedron          | $10^{12}$| $2^{10}$, $0^6$|
| 48      | Rhombicuboctahedron        | $12^8$   | $2^9$, $0^6$|
| 120     | Rhombicicosidodecahedron   | $20^{12}$| $2^{10}$, $0^6$|
| 72      | Truncated Cuboctahedron    | $18^8$   | $6^2$, $2^6$|
| 180     | Truncated Icosidodecahedron| $30^{12}$| $10$, $2^{10}$|
| 18      | Truncated Tetrahedron      | $12^3$   | $6^2$       |
| 36      | Truncated Octahedron       | $12^6$   | $4$, $2^4$  |
| 36      | Truncated Cube             | $18^4$   | $6^2$       |
| 90      | Truncated Icosahedron      | $18^{10}$| $2^9$       |
| 90      | Truncated Dodecahedron     | $30^6$   | $6^2$       |
| 60      | Snub Cube                  | $30^3$, $0^6$ | $8^3$      |
| 150     | Snub Dodecahedron          | $50^3$, $0^6$ | $8^5$      |
| 3m      | Prism$_m$, $m \equiv 0 \pmod{4}$ | $(\frac{3m}{2})^4$ | $\frac{(m)^4}{2}$ |
| 3m      | Prism$_m$, $m \equiv 2 \pmod{4}$ | $(\frac{3m}{2} \cdot 0)^2$ | $\frac{(m)^2}{2}$ |
| 3m      | Prism$_m$, $m \equiv 1, 3 \pmod{4}$ | $(6m_{m,2m})^2$ | $(\frac{m}{2})^3$ |
| 4m      | APrism$_m$, $m \equiv 0 \pmod{3}$ | $(2m)^4$ | $(\frac{m}{3})^3$ |
| 4m      | APrism$_m$, $m \equiv 1, 2 \pmod{3}$ | $2m; 6m_{0,2m}$ | $(\frac{m}{3})^3$ |

| 84      | Klein map                  | $8^{24}$ | $1^8$, $0^{12}$ |
| 48      | Dyck map                   | $6^16$   | $1^6$, $0^9$    |

Table 1: Zigzag structure of Platonic and semiregular polyhedra; also, of Klein and Dyck maps.

tiprisms, the self-intersections of type I, if any, occur on the edges of the two $m$-gons, while those of type II occur on other edges.

The notion of a zigzag was generalized to locally-finite *infinite* planar 3-connected graphs. All such *edge-transitive* graphs without self-intersecting zigzags (i.e., simple circuits and doubly infinite rays are the only zigzags) were classified in [GrSh87]: these are the three regular partitions $(6^3)$, $(3^6)$, $(4^4)$ of the Euclidean plane, the Archimedean partition $(3.6.3.6)$, its dual and several infinities of partitions of the hyperbolic plane. Also the notion of zigzag (Petrie polygon) was extended in [Cox73] p. 223 for $n$-dimensional polytopes and honeycombs.

This paper is based on extensive computation; in particular:

(i) All computations used the GAP computer algebra system [GAP] and the package PlanGraph ([D2]) by the second author.

(ii) The program CPF ([Ha]) by Harmuth was used to generate the graphs of type $3_n$ and $4_n$. In fact, the name $n_x$ for a graph indicates that a graph appears at
x-th position in the output of CPF. Also, CaGe ([CaGe]) was used for graph drawings.

(iii) For graphs of type 5, we used also the face-spiral algorithm notation given in [PoMa95]; a computation by Brinkmann ([Br97]) was used in Table 7.

2 General results for plane graphs

Theorem 2.1 For any planar bipartite graph $G$ there exist an orientation of zigzags, with respect to which each edge has type I.

Proof. We represent the graph $G$ on the sphere. The list of vertices, adjacent to a given vertex, can be organized in counter-clockwise order.

Let the vertex-set $V$ be partitioned into the two subsets $V_1$ and $V_2$ of the bipartition. Fix a zigzag $Z$; it will turn left at vertices, say, $v \in V_1$ and right at vertices $v \in V_2$. It is easy to see that the edges of self-intersection of $Z$ can be only of type I.

Take another zigzag $Z'$, having a common edge $e$ with $Z$. We choose an orientation on $Z'$, such that $e$ is an edge of type I. Then $Z'$ will turn left at vertices $v \in V_1$ and right at vertices $v \in V_2$. Iterating this construction, all zigzags will be oriented and all edges will have type I with respect to this orientation of zigzags. □

In the case of one zigzag, this result was already known ([Sh75]). This theorem is also valid for any bipartite graph, which is embedded in an oriented surface, in view of the well-known topological fact that any two-dimensional orientable manifold admits coherent orientation of its faces.

Proposition 2.2 ([DDF02]) Let $G$ be a plane graph; for any orientation of all zigzags of $G$, we have:

(i) The number of edges of type II, which are incident to any fixed vertex, is even.

(ii) The number of edges of type I, which are incident to any fixed face, is even.

Proof. (i) For each vertex the number of times, that a zigzag enters it, should be equal to the number of times that a zigzag leaves it.

(ii) Passing to the dual graph, the type of edges are interchanged. □
A consequence of the above proposition for any 3-valent plane graph \( G \) (see [DDF02]) is that the set of vertices can be partitioned into two classes, say, class I and class II, where class I stands for vertices, which are incident to three edges of type I, and class II stands for vertices, which are incident to one edge of type I and two edges of type II. If \( n \) denotes the number of vertices of \( G \) and \( n_1 \) denotes the number of its vertices of class I, then the number of edges of type I is \( n_1 + \frac{n}{2} \). Clearly, the number of edges of type II is equal to the number of vertices of class II, i.e., to \( n - n_1 \).

It was conjectured in ([DDF02]) that any 3-valent plane graph, which is \( z \)-knotted, has an odd number of edges of type I. The condition of 3-valency is necessary: for example, the 3-bipyramid (i.e., \( Prism^3 \)) has \( z \)-vector 18\( _{6,3} \).

Clearly, any graph with 1, 2 or 3 zigzags is \( z \)-balanced. See smallest \( z \)-unbalanced 3-valent graph and graphs of type 4\( _n \), 5\( _n \) on Figure 2. We did computations trying to find an example of a \( z \)-unbalanced and \( z \)-uniform 3-valent polyhedron (for \( n \leq 22 \)), graph of type 4\( _n \) (for \( n \leq 250 \)) and 5\( _n \) (for \( n \leq 80 \)); surprisingly, we did not find any such example.

In the rest of this section, we give a local Euler formula for zigzags. Let \( G \) be a plane 3-valent graph. Consider a patch \( A \) in \( G \), which is bounded by \( t \) arcs (i.e., paths of edges) belonging to zigzags (different or coinciding).

We admit also 0-gonal patch \( A \), i.e., just the interior of a simple zigzag. Suppose that the patch \( A \) is regular, i.e., the continuation of any of its bounding arcs (on the zigzag, to which it belongs) lies outside of the patch (see Figure 3). Let \( p'(A) = (..., p'_i, ...) \) be the \( p \)-vector enumerating the faces of \( G \), which are contained in the
There are two types of intersections of arcs on the boundary of a regular patch: either intersection in an edge of the boundary, or intersection in a vertex of the boundary. Let us call these types of intersections *obtuse* and *acute*, respectively (see Figure 3); denote by $t_{ob}$ and $t_{ac}$ the respective number of obtuse and acute intersections. Clearly, $t_{ob} + t_{ac} = t$, where $t$ is the number of arcs forming the patch.

**Theorem 2.3** Let $G$ be a 3-valent plane graph (1-gonal and 2-gonal faces are permitted). If $A$ is a regular patch, then the following equality holds

$$6 - t_{ob} - 2t_{ac} = \sum_{i \geq 1} (6 - i)p'_i.$$  

**Proof.** Let $P(A)$ be the induced plane subgraph of $G$ formed by the patch $A$. The vertices of $P(A)$ have degree 2 or 3; we denote their respective number by $v_2$ and $v_3$. The exterior face is $l$-gonal, where $l$ is the length of the boundary of the patch. Direct enumeration gives the following expressions for the number $|E|$ of edges of $P(A)$

$$|E| = \frac{1}{2}(3v_3 + 2v_2) = \frac{1}{2}(l + \sum_{i \geq 1} ip'_i).$$

Euler’s formula, applied to the plane graph $P(A)$, yields

$$2 = (v_2 + v_3) - |E| + (1 + \sum_{i \geq 1} p'_i).$$

The above two expressions of $|E|$ give

$$1 = -\frac{v_2}{2} + \sum_{i \geq 1} p'_i = (v_2 + v_3) - \frac{l}{2} + \sum_{i \geq 1} (1 - \frac{i}{2})p'_i.$$  

Eliminating $v_3$ yields $3 = v_2 - \frac{l}{2} + \sum_{i \geq 1} (3 - \frac{i}{2})p'_i$.

For example, if the patch $A$ is a 0-gon (i.e., if it is bounded by a simple zigzag), then $l = 2v_2$ and we get

$$3 = \sum_{i \geq 1} (3 - \frac{i}{2})p'_i.$$
Denote by $v'_2$ and $v'_3$ the numbers of vertices on the boundary of $A$, having degree 2 and 3, respectively. Clearly, $l = v'_2 + v'_3$ and $v'_2 = v_2$.

Now we use that $A$ is a regular patch. Specifying acute and obtuse types of arc intersections on the boundary of $A$, we get $v'_2 = v'_3 + t_{ob} + 2t_{ac}$, which gives $6 - t_{ob} - 2t_{ac} = \sum_{i \geq 1} (6 - i)p'_i$.

\[\blacksquare\]

**Corollary 2.4** Let $G$ be a 3-valent plane graph. Let $A$ be a $t$-gonal regular patch with p-vector $(\ldots, p'_i, \ldots)$ in $G$. Then we have:

(i) If $G$ has no $q$-gonal faces with $q > 6$, then $t \leq 6$.

(ii) If $G$ is a two-faced graph $w_n$, $w \in \{3, 4, 5\}$, then

\[6 - t_{ob} - 2t_{ac} = (6 - w)p'_w.\]

**Proof.** (i) By the above theorem

\[6 - t_{ob} - 2t_{ac} = \sum_{1 \leq i \leq 6} (6 - i)p'_i \geq 0,\]

which gives the result. (ii) also follows by direct application of Theorem 2.3. \[\blacksquare\]

### 3 Railroads and pseudo-roads

Let $G$ be a plane graph. Call a railroad of $G$ a circuit of hexagonal faces in $G$, such that each hexagon is adjacent to its neighbors on opposite edges (in [GrimMo63], a simple, i.e., without self-intersections, railroad is called a belt). Clearly, a railroad is bounded by two “parallel” (concentric if the railroad is simple) zigzags. The graph $G$ is called tight if it has no railroads. A railroad in $G$ corresponds to a central circuit in $G^*$, which goes only through 6-valent vertices.

We associate to each railroad a representing plane curve in the following way: in each of its hexagons one connects, by an arc, the midpoints of opposite edges, on which it is adjacent to its two neighbors. The sequence of those arcs can be seen as a Jordan curve in the plane and self-intersections of railroad correspond to self-intersections of the curve.

Those self-intersections can be only double or triple, because a railroad consists of hexagons. If the railroad has no triple self-intersection points, then this curve can be seen as a projection (not minimal, if there are 1-gons) of an alternating knot...
(see, for example, [Kaw96] and [Rol76]) with \( n \) crossings, where \( n \) is the number of self-intersection points. If there are several railroads without triple intersections and triple self-intersections, then the set of plane curves, representing them, can be seen as a projection of an alternating link.

A set of railroads can be seen as a plane graph \( H \) with valency 4 or 6 of its vertices, accordingly to double or triple points of intersection or self-intersection of the curves. Every \( t \)-gonal face \( F \) of \( H \) can be viewed as a regular patch with \( t \) intersections of arcs. In the case, when \( G \) has no \( q \)-gonal faces with \( q > 6 \), Corollary 2.4.(i) implies that \( t \leq 6 \).

A railroad \( R \) is bounded by two zigzags \( Z_1 \) and \( Z_2 \), which have the same length and signature. Each edge of self-intersection of \( Z_1, Z_2 \) corresponds to a self-intersection of \( R \) and so, to an hexagon of self-intersection of \( R \). Since \( Z_1 \) can self-intersect in an edge of type I or type II, there are two types of double self-intersections of railroad. Simple analysis yields two types for triple self-intersections of railroad: either \( Z_1 \) self-intersects in three edges of type I or it self-intersects in one edge of type I and two edges of type II. See Figure 4 for all possibilities.

In order to illustrate the notion of railroads, we give in the remainder of this section two lists: those having, or, respectively, not having, triple self-intersections
of the curves representing railroads in two-faced \(n\)-vertex polyhedra of type \(4_n\) for small \(n\). In Tables 2 and 3, we present the first cases of such curves; the first column being the name of the curve. The smallest graph of type \(4_n\), which is denoted by \(n_x\), \(x\) being the number of the graph in CPF output (see \[Ha\]), is listed in the column “First appearance”. In some cases the curve appears twice in this graph; we mark this situation by putting “twice” in parenthesis. The third and fourth column give the number \(m\) of intersection points and the number \(p_i\) of \(i\)-gons, \(1 \leq i \leq 6\). Columns “Group” and “\(z\)-vector” give the point group and the \(z\)-vector of the corresponding graph of type \(4_n\).

The computation of all railroads in the class of polyhedra \(4_n\), \(n \leq 88\), with at least two self-intersections, all being double self-intersections, produced a list of projections of alternating knots presented in Table 2. We use Rolfsen’s notation, see [Rol76], for knots with at most 10 crossings and those of Thistlethwaite ([Thi]), for other knots. We denote by \(15_y\) and \(18_y\) the 15- and 18-crossing alternating knots, appearing in a graph of type \(4_{80}\), which are \(80_{29}\) and \(80_{28}\), respectively. (The knot \(15_y\) comes from \(APrism_3\) or from knot \(9_{40}\) by inscribing consecutively 3 or 2 triangles, respectively.) All curves, representing railroads with triple self-intersections in the graphs \(4_n\), with \(n \leq 142\) are given in Table 3. The notation \(i – j\), given in first column, means that \(i\) is the number of triple points of the curve and \(j\) is order of appearance (amongst curves with \(i\) triple points) in this table.

**Remark 3.1** Let \(Z\) be a zigzag of signature \((\alpha_1, \alpha_2)\) bounding a railroad \(R\) with \(m_2\) double and \(m_3\) triple self-intersections. Then one has \(m_2 + 3m_3 = \alpha_1 + \alpha_2\).

| Knot | First appearance | \(m\) | \(p_1, \ldots, p_6\) | Group | \(z\)-vector |
|------|------------------|------|----------------------|-------|--------------|
| 0_1  | 20_3             | 0    | 0, 3, 2, 0, 0, 0     | \(D_{3d}\) | 6^{2}; 42_{12}; 0 |
| 3_1  | 26_2             | 3    | 0, 2, 4, 0, 0, 0     | \(D_{3h}\) | 24^{2}; 0, 30_{3}; 0 |
| 4_1  | 64_{17}(twice)   | 4    | 0, 2, 4, 0, 0, 0     | \(D_{2d}\) | 48_{4}; 0 |
| 5_2  | 74_{24}          | 5    | 0, 3, 2, 2, 0, 0     | \(C_2\)  | 18^{2}; 48_{2}; 0, 54_{3}; 0 |
| 6_3  | 74_{21}          | 6    | 0, 2, 4, 2, 0, 0     | \(C_2\)  | 54_{6}; 0, 114_{2}; 0 |
| 7_4  | 80_{30}(twice)   | 7    | 0, 4, 2, 2, 0, 1     | \(D_{3h}\) | 60_{2}; 0 |
| 9_{23}| 62_{16}          | 9    | 0, 2, 6, 2, 0, 1     | \(C_2\)  | 60_{2}; 0, 66_{9}; 0 |
| 9_{40}| 56_{19}          | 9    | 0, 8, 3, 0, 0, 0     | \(D_{6h}\) | 12^{2}; 0, 66_{2}; 0 |
| 11_{32}| 88_{44}         | 11   | 0, 2, 4, 7, 0, 0     | \(C_2\)  | 24^{4}; 84_{9}; 11; 0 |
| 15_{y}| 80_{29}          | 15   | 0, 0, 8, 6, 0, 0     | \(D_{3d}\) | 12^{6}; 64_{2}; 0 |
| 18_{y}| 80_{28}          | 18   | 0, 0, 14, 0, 6, 0    | \(D_3\)  | 24^{2}; 96_{2}; 8; 0 |

Table 2: Alternating knots in railroads of graphs of type \(4_n\), \(n \leq 88\).
Three graphs of type $4_n$ with triple self-intersecting railroads.

Table 3: Curves with triple self-intersections in railroads of graphs of type $4_n$, $n \leq 142$. 

| Curve | First appearance | $m$ | $p_1$, $\ldots$, $p_6$ | Group | $z$ - vector |
|-------|-----------------|-----|------------------------|-------|-------------|
| 1 - 1 | 66_{11} (twice) | 1   | 0, 1, 0, 0, 0, 0       | $D_{3h}$ | $36^1_{30}, 54^1_{32}$ |
| 1 - 2 | 126_{75}       | 7   | 0, 4, 4, 3, 0, 0       | $D_{3h}$ | $18^1_{8}, 90^1_{10}$ |
| 1 - 3 | 108_{53}       | 3   | 1, 2, 3, 0, 0, 0      | $C_5$   | $18^4, 30^1_{30}, 42^1_{10}, 54^1_{10}, 60^2_{0}$ |
| 1 - 4 | 86_{04}        | 4   | 1, 2, 3, 1, 0, 0     | $C_9$   | $60^2_{0}, 138^3_{0}$ |
| 1 - 5 | 114_{36} (twice)| 5   | 0, 4, 3, 0, 1, 0     | $C_{2v}$ | $54^1_{3}, 72^1_{0}$ |
| 1 - 6 | 108_{18} (twice)| 7   | 0, 2, 7, 0, 1, 0     | $C_{2v}$ | $78^1_{0}, 90^1_{0}$ |
| 1 - 7 | 142_{82}       | 5   | 0, 3, 4, 2, 0, 0     | $C_1$   | $24^3, 60^1_{0}, 96^2_{0}, 114^1_{14}$ |
| 1 - 8 | 114_{41}       | 11  | 0, 2, 8, 5, 0, 0     | $C_1$   | $102^2_{0}, 138^2_{0}$ |
| 2 - 1 | 140_{68}       | 2   | 2, 2, 2, 0, 0, 0     | $C_2$   | $24^2, 72^2_{0}, 144^2_{10}$ |
| 2 - 2 | 90_{16}        | 5   | 0, 6, 0, 3, 0, 0     | $D_{3h}$ | $18^4, 54^3_{0}, 72^3_{0}$ |
| 2 - 3 | 122_{21}       | 5   | 2, 2, 3, 2, 0, 0     | $C_2$   | $42^1_{0}, 78^2_{0}, 84^2_{0}$ |
| 2 - 4 | 126_{39}       | 6   | 0, 4, 4, 2, 0, 0     | $C_2$   | $36^2_{0}, 90^2_{0}, 126^2_{10}$ |
| 2 - 5 | 134_{130}      | 6   | 0, 3, 6, 1, 0, 0    | $C_{2v}$ | $36^2, 42^2_{0}, 54^3_{0}, 96^2_{11}$ |
| 2 - 6 | 122_{04}       | 7   | 0, 4, 3, 0, 2, 0     | $C_{2v}$ | $30^1, 54^3_{0}, 96^3_{11}$ |
| 2 - 7 | 124_{100}      | 7   | 0, 3, 6, 2, 0, 0     | $C_2$   | $90^2_{0}, 96^2_{11}$ |
| 2 - 8 | 128_{64}       | 7   | 0, 2, 6, 3, 0, 0     | $C_6$   | $54^3, 96^3_{11}, 138^{17}_{0}$ |
| 2 - 9 | 110_{24}       | 8   | 0, 4, 4, 0, 0, 0     | $C_2$   | $90^2_{12}, 150^{33}_{0}$ |
| 2 - 10| 110_{24}       | 8   | 0, 2, 8, 2, 0, 0     | $C_2$   | $90^2_{12}, 150^{33}_{0}$ |
| 2 - 11| 134_{50}       | 10  | 0, 4, 4, 6, 0, 0     | $C_2$   | $114^3_{14}, 174^{29}_{0}$ |
| 2 - 12| 134_{419}      | 14  | 0, 2, 8, 8, 0, 0     | $C_2$   | $126^2_{5}, 150^{21}_{0}$ |
| 2 - 13| 126_{77}       | 14  | 0, 0, 12, 6, 0, 0    | $D_3$   | $126^2_{5}, 126^{15}_{0}$ |
| 4 - 1 | 72_{17}        | 4   | 0, 6, 4, 0, 0, 0     | $D_3$   | $24, 48^1_{0}, 72^1_{0}$ |
| 6 - 1 | 108_{18}       | 6   | 0, 6, 8, 0, 0, 0    | $D_3$   | $24^3, 36, 108^2_{18}$ |
| 6 - 2 | 135_{102}      | 21  | 0, 1, 20, 6, 2, 0    | $C_2$   | $78^3_{0}, 189^3_{10}$ |
| 6 - 3 | 155_{150}      | 24  | 0, 0, 24, 6, 0, 2    | $D_3$   | $102^2_{0}, 186^{16}_{30}$ |
| 8 - 1 | 144_{121}      | 8   | 0, 6, 12, 0, 0, 0   | $D_2$   | $36, 108^2_{2}, 144^{22}_{10}$ |
Proposition 3.2 Let $G$ be a simple polyhedron with $p$-vector $p = (\ldots, p_i, \ldots)$ and let $G$ be tight. Then the number of zigzags of $G$ is at most
\[
\sum_{i \neq 6} i p_i / 2.
\]

Proof. In fact, each zigzag has a non 6-gonal face on each of its sides, since, otherwise, it would have a railroad on this side; so, the total number of incidences between zigzags and non 6-gons is at least twice the number of zigzags. On the other hand, the number of those incidences is exactly $\sum_{i \neq 6} i p_i$. \hfill \Box

Proposition 3.3 Let $G$ be a graph of type $4_n$, having railroads $R_1, \ldots, R_p$; let $H$ be a plane graph formed by the curves representing those railroads. Then:

(i) Every $t$-gonal face of $H$ with $t = 0, 1$ contains exactly $3-t$ 4-gons of $G$; every 2-gonal face of $H$ contains one or two 4-gons of $G$.

(ii) If $q_t$ is the number of $t$-gonal faces of $H$, then one has the inequality
\[
3 q_0 + 2 q_1 + q_2 \leq 6.
\]

Proof. Every $t$-gonal face $F$ of $H$ can be viewed as a regular patch with $t = t_{ob} + t_{ac}$ (obtuse and acute) intersections of arcs. Let $p'_4$ be the number of 4-gons inside $F$. We will apply Theorem 2.3.

(i) If $t = 0$, then $2p'_4 = 6$ and we are done. If $t = 1$, then
\[
2p'_4 = 6 - t_{ob} - 2t_{ac} \geq 4 \quad \text{and} \quad 2p'_4 = 6 - t_{ob} - 2t_{ac} < 6.
\]

Since $2p'_4$ is even, we get that $2p'_4 = 4$, i.e., every 1-gon contains exactly two 4-gons. If $t = 2$, then
\[
2p'_4 = 6 - t_{ob} - 2t_{ac} \geq 2 \quad \text{and} \quad 2p'_4 = 6 - t_{ob} - 2t_{ac} < 6.
\]

This yields $p'_4 = 1$ or $p'_4 = 2$.

(ii) Any graph of type $4_n$ has six 4-gons; so, the result follows. \hfill \Box

The Euler formula for the $p$-vector of a 3-valent polyhedron, $12 = \sum_t (6 - t) p_t$, is a discrete analog of the Gauss-Bonnet formula, $2\pi(1 - g) = \int_S K(x) dx$, for the Gaussian curvature $K$ of a surface $S$ of genus $g$. So, the $q$-gons can be seen as positively curved, flat, or negatively curved, if $q < 6$, $q = 6$, or $q > 6$, respectively.
Consider a two-faced graph $G$, which is a $3_n$, $4_n$ or $5_n$; it has four 3-gons, six 4-gons, or twelve 5-gons, respectively. Let us call the graph of curvatures of $G$ the following graph (possibly, with loops and multiple edges) having as vertex-set all non 6-gonal faces of $G$. Two vertices, say, non 6-gonal faces $b$ and $c$ of $G$ are called adjacent if there exist a pseudo-road connecting faces $b$ and $c$. A pseudo-road is a sequence of hexagons, say, $a_1, \ldots, a_l$, such that putting $a_0 = b$ and $a_{l+1} = c$, we have that any $a_i, 1 \leq i \leq l$, is adjacent to $a_{i-1}$ and $a_{i+1}$ on opposite edges. Clearly, the graph of curvatures of a graph $w_n, w \in \{3, 4, 5\}$, is regular of degree $w$.

Given an hexagon in a graph of type $3_n$, $4_n$, $5_n$, any pair of opposite edges belongs to a railroad or a pseudo-road; so, we exactly have a triple covering of the set of all hexagons by the set of all railroads and pseudo-roads. Every non 6-gonal face, i.e., $t$-gonal face with $t < 6$, has $t$ adjacencies with the system of pseudo-roads.

For the special case, when our graph is of type $4_n$, any pseudo-road arriving on a 4-gonal face can be extended on the opposite edge. So, the set of such extended pseudo-roads, together with the set of railroads, can be identified with the set of central circuits of the dual of our graph. Hence, extended pseudo-roads can be seen (in the same way as it was done for railroads) as projections of a Jordan curve on the plane with double and, eventually, triple points of self-intersection. Those notions are illustrated in Figure 6. This graph $4_{126}(D_{3h})$ has the following road decomposition: five concentric simple railroads (we indicate only the central, equatorial one), two self-intersecting railroads (they differ only by their opposite position on the sphere; we present only one of them) and all 12 pseudo-roads.

### 4 Two-faced polyhedra

Remind that two-faced polyhedra have only $a$- and $b$-gonal faces with $3 \leq a < b \leq 6$ (see [DeGr99]). Denote by $p_a > 0$ and $p_b \geq 0$ the number of $a$-gonal and $b$-gonal faces.

Any 3-valent $n$-vertex two-faced polyhedron has $3n/2 = (ap_a + bp_b)/2$ edges and satisfies the Euler relation $n - 3\frac{n}{2} + (p_a + p_b) = 2$, i.e., $n = 2(p_a + p_b - 2)$ and $p_a(6 - a) + p_b(6 - b) = 12$. So, $b < 6$ is possible only for 6 simple polyhedra with $(a, b) = (3, 4)(Prism3)$, $(3, 5)(Dürer’s Octahedron)$, $(4, 5)(four$ dual $deltahedra)$. The other cases, namely, $(a, b) = (3, 6), (4, 6), (5, 6)$ are denoted by $3_n, 4_n, 5_n$, where $n$ is
One of two self-intersecting railroads
and the equatorial simple railroad

Figure 6: Road-decomposition of a graph $4_{126}(D_{3h})$.

| $(a, b)$ | Polyhedra                          | Exist if and only if | $p_a$ | $n$          |
|---------|------------------------------------|---------------------|-------|--------------|
| $(5, 6)$ | $5_n$ (fullerenes)                 | $p_6 \in \mathbb{N} - \{1\}$ | $p_5 = 12$ | $n = 20 + 2p_6$ |
| $(4, 6)$ | $4_n$                              | $p_6 \in \mathbb{N} - \{1\}$ | $p_4 = 6$  | $n = 8 + 2p_6$  |
| $(3, 6)$ | $3_n$                              | $p_6/2 \in \mathbb{N} - \{1\}$ | $p_3 = 4$  | $4 + 2p_6$       |
| $(4, 5)$ | 4 dual deltahedra                   | $p_5 = 2, 3, 4, 5$   | $p_4 = 5, 4, 3, 2$ | $n = 10, 12, 14, 16$ |
| $(3, 5)$ | Dürer’s Octahedron                 | $p_5 = 6$           | $p_3 = 2$  | $n = 12$          |
| $(3, 4)$ | $Prism_3$                          | $p_4 = 3$           | $p_3 = 2$  | $n = 6$           |

Table 4: All simple polyhedra with only $a$-gonal and $b$-gonal faces, $3 \leq a < b \leq 6$. The polyhedra $5_n$ are the well known fullerenes (of Organic Chemistry). See Table 4 for all the possibilities. The criterion for the existence of the polyhedra of type $3_n$, $4_n$ and $5_n$ is due to [GrünMo63].

For the types $3_n$, $4_n$ and $5_n$ in Table 4 the case $p_6 = 0$ yields the Tetrahedron, the Cube and the Dodecahedron. Those three polyhedra, together with 4 dual deltahedra and $Prism_3$, are the duals of the eight convex deltahedra. Dürer’s Octahedron is the Cube truncated at two opposite vertices. The four dual deltahedra from the fourth line of Table 4 are: $Prism_5$ (dual 5-bipyramid), dual bisdisphenoid, dual 3-augmented $Prism_3$, dual 2-capped $APrism_4$. Those four dual deltahedra, preceded by the Dürer’s Octahedron and $Prism_3$, are given in Figure 7 with their $z$-vectors and groups.

**Lemma 4.1** Any connected 3-valent plane graph, having only $q$-gonal faces with $3 \leq q \leq 6$, is 2-connected.
Proof. Let $G$ be one such graph and assume that there is one vertex $v$, such that $G - \{v\}$ is disconnected. $G - \{v\}$ has two or three components; clearly, it cannot have three components, since $q \leq 6$. Denote by $C_1$ and $C_2$ the two components and by $e = \{v, v'\}$ the edge linking $C_1$ to $C_2$. Then two edges from $v$ will connect to another vertex $w$ and two edges from $v'$ will connect to another vertex $w'$, since we assume that the faces are incident to at most 6 edges. See below the corresponding drawing.

But the vertex $w$ will disconnect the graph and so, iterating the construction, we obtain an infinite sequence $v_1, \ldots, v_n$ of vertices that disconnect $G$. This contradicts to initial assumption and proves that $G$ is 2-connected.

Denote by $(G_n)_{n \geq 1}$ the 3-valent plane $4(n + 1)$-vertex graph, whose faces are (organized in pairs of adjacent ones) triangles and hexagons. The graph $G_n$ is 2-connected but not 3-connected, its $z$-vector is $4^{n+1}, (4n + 4)^2$; its symmetry group is $D_{2d}$ or $D_{2h}$, if $n$ is even or odd, respectively. The first occurrences are depicted in Figure 8.

**Proposition 4.2** Any 3-valent plane graph having only $q$-gonal faces with $3 \leq q \leq 6$, is 3-connected, except the graphs of the family $(G_n)_{n \geq 1}$.

Proof. Let $G$ be a 3-valent plane graph with $k$-gonal faces, $3 \leq k \leq 6$ and assume that it is not 3-connected. Then there are two vertices, say, $v_1$ and $v_2$, such that...
Figure 9: Some 2-connected graphs

$G - \{v_1, v_2\}$ is disconnected. If $G - \{v_1, v_2\}$ has three components, then since $q \leq 6$, it is of the form depicted in Figure 9.a), which is impossible by the condition $q \geq 3$; so, $G - \{v_1, v_2\}$ has two components, say, $C_1$ and $C_2$.

There are two edges, say, $e_1 = \{v_1, v'_1\}$ and $e_2 = \{v_2, v'_2\}$, that connect $C_1$ to $C_2$. Since $G - \{v_1, v_2\}$ is disconnected, $e_1$ and $e_2$ are both incident to faces $F$ and $F'$. Then, by our assumption on size of faces, we get the following possibilities:

Only first two cases are possible, since, otherwise, we would have a non 2-connected plane graph. See in Figure 9.b),c) the possible continuation of those graphs.

In case b) one obtain an infinite structure; so, we get again a contradiction.

Let us consider now the case c); the two points, $w_1$ and $w_2$, can, either be connected by an edge and we are done, or be non-connected, in which case $w_1$ and $w_2$ disconnect the graph. In the latter case, one can iterate the construction. Since the graph is finite, the construction eventually finish and we get a graph $G_{n_0}$, $n_0 \geq 1$. □

All graphs of type $3_n$, $4_n$, $5_n$ with maximal pair of symmetry ($T$ or $T_d$, $O$ or $O_h$, $I$ or $I_h$, respectively) are known; in fact, the Goldberg-Coxeter construction, denoted $GC_{k,l}$ ([Cox71], [Gold37], [DD03]), implies that any graph of this type $3_n$, $4_n$, $5_n$ and those symmetry are of the form $GC_{k,l}(G_0)$ with $0 \leq l \leq k$ and $G_0$ being Tetrahedron, Cube or Dodecahedron, respectively. Those graphs have $n_0(k^2 + kl + l^2)$ vertices with $n_0 = 4, 8$ or $20$, respectively. Those graphs are tight if and only if $gcd(k, l) = 1$; they are of symmetry $T_d$, $O_h$ or $I_h$ if and only if $l = 0$ or $k = l$.

The polyhedra dual to the 3-valent polyhedra without $q$-gonal faces, $q > 6$, are studied in [Thur98]; they are called there non-negatively curved triangulations (actually, the reference [Sah94] is an application of a preliminary version of [Thur98]).
Table 5: Numbers \(N_3(n)\) of graphs \(3_n\) and \(N_3^t(n)\) of tight \(3_n\) for \(n \leq 512\).

| \(n\) | \(N_3\) | \(N_3^t\) | \(n\) | \(N_3\) | \(N_3^t\) | \(n\) | \(N_3\) | \(N_3^t\) | \(n\) | \(N_3\) | \(N_3^t\) |
|-------|---------|---------|-------|---------|---------|-------|---------|---------|-------|---------|---------|
| 12    | 1       | 1       | 156   | 2       | 3       | 228   | 15      | 2       | 3       | 372   | 15     |
| 16    | 1       | 0       | 88    | 2       | 0       | 232   | 15      | 0       | 232   | 15     |
| 20    | 0       | 1       | 156   | 1       | 3       | 232   | 15      | 0       | 232   | 15     |
| 24    | 1       | 2       | 110   | 6       | 2       | 180   | 15      | 2       | 180   | 15     |
| 28    | 0       | 0       | 88    | 2       | 0       | 232   | 15      | 0       | 232   | 15     |
| 32    | 4       | 0       | 104   | 7       | 0       | 248   | 16      | 0       | 248   | 16     |
| 36    | 3       | 1       | 108   | 8       | 2       | 252   | 20      | 3       | 252   | 20     |
| 40    | 3       | 0       | 112   | 7       | 2       | 256   | 26      | 0       | 256   | 26     |
| 44    | 2       | 2       | 116   | 5       | 5       | 272   | 26      | 0       | 272   | 26     |
| 48    | 7       | 0       | 120   | 3       | 13      | 272   | 26      | 0       | 272   | 26     |
| 52    | 3       | 3       | 124   | 6       | 6       | 272   | 26      | 0       | 272   | 26     |
| 56    | 4       | 0       | 128   | 14      | 0       | 272   | 26      | 0       | 272   | 26     |
| 60    | 5       | 1       | 132   | 9       | 2       | 272   | 26      | 0       | 272   | 26     |
| 64    | 8       | 0       | 136   | 9       | 0       | 272   | 26      | 0       | 272   | 26     |
| 68    | 3       | 3       | 140   | 9       | 3       | 272   | 26      | 0       | 272   | 26     |
| 72    | 7       | 0       | 144   | 14      | 0       | 272   | 26      | 0       | 272   | 26     |
| 76    | 4       | 4       | 148   | 7       | 7       | 272   | 26      | 0       | 272   | 26     |
| 80    | 9       | 0       | 152   | 10      | 0       | 272   | 26      | 0       | 272   | 26     |

Figure 10: A pseudo-road in a graph of type \(3_n\) and a circular railroad.

Theorem 3.4 in [Sah94] implies that \(N_3(n) = O(n^2)\), \(N_4(n) = O(n^3)\) and \(N_5(n) = O(n^9)\), where \(N_w(n)\) denotes the number of polyhedra \(w_n\) for \(w = 3, 4, 5\), respectively.

## 5 Polyhedra \(3_n\)

We present in Table 5 the numbers \(N_3(n)\) of \(3_n\) and the numbers \(N_3^t(n)\) of tight \(3_n\) for \(n \leq 512\).

In [GrüMo63] it was shown that each zigzag in a graph of type \(3_n\) is simple and a general construction of all \(3_n\) was given; we sketch this construction with slightly different notation. If a zigzag contains two edges, say, \(e_1\) and \(e_2\), of a triangle, then its third edge, say, \(e_3\) defines a pseudo-road \(PR\) that finish on another triangle. Denote by \(s\) one fourth of the length of the zigzag; then \(s - 1\) is the number of hexagons of \(PR\). Consider the sequence of concentric simple railroads, which, possibly, goes around the patch depicted in Figure 10; let \(m\) denote the number of zigzags and \(m - 1\)
be the number of corresponding railroads. The same patch (with the same number
$s - 1$ of hexagons between the other two triangles and with the same number $m - 1$
of concentric simple railroads, going around it) occurs on the other side of the sphere.
As in [GrüMo63], we get, by direct computation, the equality $p_6 = 2(sm - 1)$ and
$n = 4sm$, where $p_6$ is the number of hexagons. See Figure (11)c),d) for two examples
with $s = 2$ and $m = 4$.

Take a triangle $T$ and an edge $e$ of this triangle; $e$ belongs to a sequence of
adjacent 6-gons, which is concluded by another triangle $T'$. If one considers the three
edges $e_1, e_2, e_3$, then we obtain three triangles $T_1, T_2, T_3$, respectively, that may be
distinct or not.

**Theorem 5.1** Any graph $3_n$ has exactly one of the following forms:

(i) The Tetrahedron or the Truncated Tetrahedron (the only case, when an
hexagon is adjacent to more than two triangles).

(ii) There are two hexagons, every of which is adjacent to two triangles on
opposite edges; there are one or two such graphs of type $3_n$ depending on $n \equiv 8$
(mod 16) or $n \equiv 0$ (mod 16), respectively. Their symmetry groups are, respectively,
$D_2$ if $n \equiv 8$ (mod 16) and $D_{2h}, D_{2d}$ if $n \equiv 0$ (mod 16).

(iii) There are four hexagons (in two adjacent pairs), every of which is adjacent
to two triangles on non-opposite and non-adjacent edges; there is exactly one such
graph $3_n$ for every $n \geq 16$, $n \equiv 0$ (mod 4).\(^1\) The symmetry group is $D_{2h}$ or $D_{2d}$ if $n$
is even or odd, respectively.

(iv) Each hexagon is adjacent to at most one triangle; there is an one-to-one
correspondence between such graphs $3_n$, having isolated triangles, and IPT fullerenes
$5_n$ (i.e., those having their 12 pentagons organized in 4 triples of mutually adjacent
ones).

**Proof.** For given graph $G$ of type $3_n$, denote by $t(G)$ the maximal number of triangles,
which are adjacent to a hexagon. The only graph of type $3_n$ without hexagons is the
Tetrahedron. The case $t(G) = 4$ corresponds to the 2-connected but not 3-connected
6-vertex plane graph $G_1$ (see Figure 8). The only graph of type $3_n$ with $t(G) = 3$ is

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\(^1\)In the exceptional case $n = 16$, there are only two polyhedra $3_{16}$: the Cube truncated at four
non-adjacent vertices and the Cube truncated at four vertices of two opposite edges. The second
polyhedron is the unique and only example, which is of type (ii) and (iii).
the Truncated Tetrahedron. If \( t(G) = 2 \), then two cases occur: either two triangles are adjacent to an hexagon on opposite edges or they are adjacent to an hexagon on non-opposite and non-adjacent edges.

The first case corresponds to the case \( s = 2 \) of the Grünbaum-Motzkin construction. The divisibility of \( n \) by 8 follows from the formula \( n = 4sm \). At the last step of the Grünbaum-Motzkin construction, namely, after adding railroads, we add a pair of triangles in one of two possible ways. If \( n \equiv 8 \pmod{16} \), then both ways give isomorphic graphs, both of symmetry \( D_2 \), for \( n \geq 8 \) (see Figure 11.b), for this possibility. If \( n \equiv 0 \pmod{16} \), then we get two non-isomorphic graphs, one of symmetry \( D_{2h} \), the other of symmetry \( D_{2d} \) (see Figure 11.c,d) for those two possibilities).

Let our graph be of type (iii), i.e., two hexagons \( H_1, H_2 \) are adjacent to two triangles \( T_1, T_2 \) on non-opposite and non-adjacent edges, as in the picture below.

These two hexagons \( H_1, H_2 \) are adjacent, respectively, to two hexagons \( H'_1, H'_2 \). By replacing the patch \( H_1, H_2, T_1, T_2 \) by two triangles, one gets a graph \( 3n - 4 \), which is also of type (iii). By induction, one gets, for any \( n \equiv 0 \pmod{4} \) with \( n \geq 16 \), an unique graph. A graph of type (iii) has two pairs of hexagons, each pair being adjacent to two triangles; if one does the operation, depicted above, to the two pairs, then the symmetry group remains the same. So, one has a periodicity of order 8 for the symmetry groups and the result on groups follows.

If the graph has \( t(G) = 1 \), then the operation, depicted in picture below, on all triangles realize a one-to-one correspondence.

See Figure 11(a),c),d) for all graphs \( 3n \) having hexagons adjacent to two triangles.

**Proposition 5.2** The graph of curvatures of a \( 3_n \) is one of the three following 4-vertex graphs:
Proof. Take a triangle, say, $T_1$ and an edge $e_1$ of $T_1$. The pseudo-road, which is defined by $T_1$ and $e_1$, establishes an edge between $T_1$ and, say, $T_2$; this pseudo-road is bounded by one zigzag $Z_1$. This zigzag belongs to a sequence of $m$ concentric zigzags $Z_1, Z_2, \ldots, Z_m$. The zigzag $Z_m$ defines a pseudo-road between the remaining triangles $T_3$ and $T_4$, i.e., an edge connecting $T_3$ to $T_4$ in the graph of curvatures. Since any vertex of the graph of curvatures has degree 3, we are done. \qed

Conjecture 5.3 The graph of curvatures of any tight graph of type $3_n$ is as in the first case of Theorem 5.2.

Fix a graph $G$ of type $3_n$. Denote by $T_i$, $1 \leq i \leq 4$, the triangles in $G$, by $(e_i)_{1 \leq i \leq 3}$ the edges of $T_1$, by $PR_i$ the pseudo-road defined by $T_1, e_i$ and by $(4s_i)_{1 \leq i \leq 3}$ the lengths of corresponding zigzags. Without loss of generality, we can suppose that $s_1 \leq s_2 \leq s_3$. Denote by $m_1 - 1, m_2 - 1, m_3 - 1$, respectively, the corresponding number of concentric railroads around each of three patches. Denote by $Z_{i,k}$, where $1 \leq i \leq 3$ and $1 \leq k \leq m_i$, the zigzags of $G$.

Theorem 5.4 For a graph $G$ of type $3_n$, the following properties hold:

(i) Its $z$-vector is $(4s_1)^{m_1}, (4s_2)^{m_2}, (4s_3)^{m_3}$; the number of railroads is $m_1 + m_2 + m_3 - 3$.

(ii) $G$ has at least three zigzags with equality if and only if it is tight.

(iii) If $G$ is tight, then $z(G) = n^3$ (so, each zigzag is a Hamiltonian circuit).

(iv) $G$ is $z$-balanced and $|Z_{i,k} \cap Z_{j,l}| = \begin{cases} 0 & \text{if } i = j, \\ \frac{n}{2m_im_j} & \text{if } i \neq j. \end{cases}$

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Proof. If $Z$ is a fixed zigzag of $G$, then it belongs to a sequence of $m_i$ concentric zigzags of a patch defined by two triangles and a pseudo-road. (i) follow immediately; (ii) is a corollary of (i), since $G$ is tight if and only if all $m_i$ are equal to 1.

Let $G$ be a tight $3_n$; the formula $n = 4sm$ becomes $n = 4s_1 = 4s_2 = 4s_3$. So, $z = n^3$.

Clearly, any two zigzags $Z_{i,k}$ and $Z_{j,l}$ are disjoint if $i = j$. Moreover, the size of the intersection $Z_{i,k} \cap Z_{j,l}$ depends only on $i$ and $j$. Denote by $\beta_{ij}$ the size of the pairwise intersections $|Z_{i,1} \cap Z_{j,1}|$. We obtain the linear system

$$\begin{align*}
4s_1 &= \beta_{13}m_3 + \beta_{12}m_2 \\
4s_2 &= \beta_{23}m_3 + \beta_{12}m_1 \\
4s_3 &= \beta_{23}m_2 + \beta_{13}m_1
\end{align*}$$

which has the unique solution $\beta_{ij} = \frac{n}{2m_im_j}$. If one writes this intersection size as $8s_is_j/n$, then it is easy to see that $G$ is $z$-balanced.

Theorem 5.5 All graphs of type $3_n$ are tight if and only if $\frac{n}{4}$ is prime.

Proof. Assume that $\frac{n}{4}$ is prime; let $G$ be a graph $3_n$ with parameters $s_i, m_i$ defined before Theorem 5.4. If $\frac{n}{4} = s_im_i$, then either $s_i = 1$, or $m_i = 1$. The first case corresponds to a 2-connected but not 3-connected plane graph; so, $m_1 = m_2 = m_3 = 1$ and $G$ has no railroad. The second case $m_i = 1$ means also the absence of a railroad.

Assume that $\frac{n}{4}$ is not prime; if $n = 4pq$ for $q > 1$, then, using the Grünbaum-Motzkin construction, we can construct a graph of type $3_n$ with a system of $q > 1$ concentric zigzags of length $4p$ and so, with at least $q - 1 \geq 1$ railroads.

Remark 5.6 In Table 2, the number of graphs of type $3_n$ for prime $\frac{n}{4}$ is a non-decreasing function.

Theorem 5.7 There exists a tight graph $G$ of type $3_n$ if and only if $\frac{n}{4}$ is odd.

Proof. The unique (for every integer $\frac{n}{4} \geq 4$) graph, defined in Theorem 5.1 (iii), is tight if $\frac{n}{4}$ is odd; we represent this graph in Figure 12(a) for $n = 28$.

Suppose now that $\frac{n}{4}$ is even and that $G$ is tight. We will use the necessary conditions of Theorem 5.4 to get a contradiction.

Consider the patch, formed by a pseudo-road $PR$ between two triangles, say, $T_1$ and $T_2$. Since $G$ is assumed to be tight, there are $\frac{n}{4} - 1$ hexagons in $PR$. Moreover,
Figure 12: The pseudo-road construction.

since there are no railroads, any of the remaining two triangles $T_3$ and $T_4$ will be adjacent to the hexagons of $PR$. Let $i$ be the position of $T_3$ in $PR$. The choice of position for $T_3$ and $T_4$ determines our graph of type $3_n$; so, we need to show that every choice of $i$ leads to a non-tight graph $3_n$. If we find a shorter pseudo-road, then we are done.

Define a pseudo-road $PR'$ by starting with $T_1$ and taking the upper edge if $i$ is even and the lower edge if $i$ is odd (see Figure 12(b)). $PR'$ intersects with $PR$. From the choice of $PR'$, the position $p$ of the first hexagon of intersection, of $PR'$ with $PR$, satisfies $p \equiv 0 \pmod{4}$. Moreover, all positions of hexagons of intersection satisfy this condition. So, $PR'$ is shorter than $\frac{n}{4} - 1$ and the graph is not tight.

The list of all symmetry groups of graphs $3_n$ is known (see [FCS88]); they are (with their first appearance): $D_2(24_2)$, $D_{2h}(16_1)$, $D_{2d}(20_1)$, $T(28_2)$, $T_d(4_1)$ (see [D1] for the corresponding drawings).

For the point groups $D_2$, $D_{2d}$ and $D_{2h}$, we have the following conjecture, which we checked for $n \leq 500$.

Conjecture 5.8 (i) $3_n(D_{2h})$ exists if and only if $\frac{n}{4}$ is even and $n \geq 16$; there are no tight $3_n(D_{2h})$.

(ii) $3_n(D_{2d})$ exists if and only if, either $\frac{n}{4}$ is odd and $n \geq 20$, or $\frac{n}{8}$ is even and $n \geq 24$; the graph defined in Theorem 5.7 (iii) is unique $3_n(D_{2d})$ tight graph if $\frac{n}{4}$ is odd and $n \geq 20$.

(iii) $3_n(D_2)$ exists if and only if $n \geq 24$ and $n \neq 28, 32$; tight $3_n(D_2)$ exists if and only if $\frac{n}{4}$ is odd, $n \geq 44$ and $n \neq 60, 84$; there are $i$ tight $3_n(D_2)$ starting with $\frac{n}{4} = 6i + 5$ for $1 \leq i \leq 9$, except the case $i = 5$ starting with $\frac{n}{4} = 37$. 

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Table 6: Numbers $N_4(n)$ of graphs $4_n$ and $N_4^t(n)$ of tight $4_n$ for $n \leq 260$.

### 6 Polyhedra $4_n$

Since any graph of type $4_n$ is bipartite, we can apply Theorem 4.1. All self-intersections of every zigzag are of type I. Moreover, there exists an orientation of zigzags, with respect to which each edge has type I. We will always work with such orientation. Any self-intersection, double or triple, of railroad is of type I (see Figure 4).

We present in Table 6 the numbers $N_4(n)$ of $4_n$ and the numbers $N_4^t(n)$ of tight $4_n$ for $n \leq 260$.

**Conjecture 6.1**

(i) $z$-kotted $4_n$ exists if and only if $n \geq 30$, $n \equiv 2 \pmod{4}$.

(ii) Tight $4_n$ exists if and only if $n \geq 8$, $n \neq 10, 14$.

(iii) Every tight $4_n$ has at most eight zigzags.

**Theorem 6.2** All symmetry groups of $4_n$ (with their first appearance) are $C_1(40_4)$, $C_s(34_2)$, $C_2(26_1)$, $C_i(140_12)$, $C_{2v}(22_1)$, $C_{2h}(44_1)$, $D_2(24_2)$, $D_3(20_1)$, $D_{2d}(16_1)$, $D_{2h}(20_2)$, $D_{3d}(20_3)$, $D_{3h}(14_1)$, $D_6(84_2)$, $D_{6h}(12_1)$, $O(56_{17})$, $O_h(8_1)$.

**Proof.** Let $G$ be a graph of type $4_n$ and let $r$ be a rotation of $G$ of order $k$. By definition of a graph $4_n$, one has $k = 2, 3, 4$ or 6.

If $k = 6$, then the axis of $r$ goes through two hexagons, say, $H_1$ and $H_2$. Consider around $H_1$ the ring of hexagons adjacent to it; then, after adding $p$ such concentric
rings of hexagons, one will encounter a square and so, by 6-fold symmetry, all squares of $G$. One can then complete, in a unique way, the graph by adding the same number of rings of hexagons on the other side of the sphere. This construction implies the existence of rotations of order two passing through opposite squares and middles of consecutive squares. This implies symmetry $D_6$ or $D_{6h}$.

If $k = 4$, then the axis of $r$ goes through two squares, say, $sq_1$ and $sq_2$. After adding $p$ rings of hexagons around $sq_1$, one finds a square and so, by symmetry, four squares, say, $sq_3$, $sq_4$, $sq_5$, $sq_6$. One can complete the graph in a unique way; from the construction it is clear that there is a 4-fold axis through $sq_3$. So, the group is $O$ or $O_h$.

If $k = 3$, then the axis of $r$ goes, either through one vertex, or one hexagon. After adding $p$ rings of hexagons around this center, one finds three squares; then, adding $q$ rings of hexagons, one finds the three other squares. The patch, formed by those six squares and the $q$ rings of hexagons, has an additional symmetry, formed by exchanging the two triples of squares. So, $G$ has this symmetry too and the symmetry group is $D_3$, $D_{3h}$, or $D_{3d}$.

Assume that $G$ has symmetry $S_4$. The six squares are partitioned in one orbit of two squares, say, $sq_1$ and $sq_2$, and another of four squares. The 2-fold axis goes through $sq_1$ and $sq_2$; moreover, if one consider the four pseudo-roads $PR_i$ from $sq_1$, then all of them stop at $sq_2$. Consider the patch $P$ formed by $PR_1$ and $PR_2$. This patch has two acute angles; so, by Theorem 2.3 it contains exactly one square $sq$. We will show that $sq$ is in the plane, defined by $sq_1$ and $sq_2$, which will prove that there is a plane of symmetry for $G$. If $sq$ is next to $PR_1$ and $PR_2$, then their length is two and the result holds; otherwise, by removing some hexagons, we find a smaller patch with the same property and we conclude by induction.

The above reasonings yield the following list of possibilities: $C_1$, $C_s$, $C_2$, $C_i$, $C_{2v}$, $C_{2h}$, $D_2$, $D_3$, $D_{2d}$, $D_{2h}$, $D_{3d}$, $D_{3h}$, $D_6$, $D_{6h}$, $O$, $O_h$. For every such group, there is a graph of type $4_n$ with $n \leq 140$ having this symmetry; their pictures are available from [D1].

A similar study of point groups for 4-valent plane graphs was done in [DDS].

**Theorem 6.3** Every tight $4_n$ has at most 9 zigzags.

**Proof.** Let $G$ be a tight $4_n$ with zigzags $Z_1$, $Z_2$, ..., $Z_l$. We obtain $2l$ sides, since every
zigzag has two sides. A side \( S \) is called lonely if it is incident to exactly one square \( sq \). We can define a pseudo-road \( PR \), parallel to \( S \), which will begin and finish on \( sq \). On the other side of the pseudo-road, there will be a side \( S' \) of a zigzag \( Z' \), which will be incident two times to \( sq \). Moreover, if \( S' \) is incident exactly two times to \( sq \), then it defines another lonely side \( S'' \) (see Figure 13(b)).

Call \( n_{1a} \) the number of lonely sides in first case and \( n_{1b} \) the number of lonely sides in second case. Also call \( n_2 \) the number of sides incident to exactly two squares (identical or not) and \( n_3 \) the number of sides incident to at least three squares (identical or not). Obviously, \( l = \frac{1}{2}(n_{1a} + n_{1b} + n_2 + n_3) \).

In case a) of lonely side, \( S' \) is incident to at least three squares; so, \( n_{1a} \leq n_3 \), while in case b) \( S' \) is incident exactly two times to \( sq \); so, \( n_{1b} \leq n_2 \). Every square \( sq \) can be incident to 0, 1 or 2 lonely sides; so, one has the inequality \( n_{1a} + \frac{n_{1b}}{2} \leq 6 \). By an enumeration of incidences, one gets \( n_{1a} + n_{1b} + 2n_2 + 3n_3 \leq 4 \times 6 = 24 \). The 4-dimensional polyhedron defined by above inequalities and non-negativity inequalities \( n_{1a}, \ldots, n_3 \geq 0 \) is denoted by \( \mathcal{P} \). We can maximize the linear function \( \frac{1}{2}(n_{1a} + n_{1b} + n_2 + n_3) \) over \( \mathcal{P} \) using the cdd program (see [link]) and obtain the upper bound 9, which is attained for the unique 4-uple \((n_{1a}, n_{1b}, n_2, n_3) = (0, 12, 6, 0)\). \( \Box \)

The following theorem is in sharp contrast with Theorem 5.4(iv) for graphs \( 3_n \) and Theorem 7.1 for graphs \( 5_n \).

**Theorem 6.4** The intersection of every two simple zigzags of a graph of type \( 4_n \), if non-empty, has one of the following forms (and so, its size is 2, 4 or 6).
Proof. Define the graph $H$ as the graph, whose vertices are edges of intersection between simple zigzags $Z$ and $Z'$, with two vertices being adjacent if they are linked by a path belonging to one of $Z$, $Z'$. The graph $H$ is a plane 4-valent graph and $Z$, $Z'$ define two central circuits in $H$. Since $Z$ and $Z'$ are simple, the faces of $H$ are $t$-gons with even $t$.

Applying Theorem 2.3 to a $t$-gonal face $F$ of $H$, we obtain that the number $p'_4$ of 4-gons in $F$ satisfies $6 - t_{ob} - 2t_{ac} = 2p'_4$. So, the numbers $t_{ob}$ and $t_{ac}$ are even, since $t = t_{ob} + t_{ac}$. Also, $6 - t_{ob} - 2t_{ac} \geq 0$. So, $t \leq 6$.

We obtain the following five possibilities for the faces of $H$: 2-gons with two acute angles, 2-gons with two obtuse angles, 4-gons with four obtuse angles, 4-gons with two acute and two obtuse angles, 6-gons with six obtuse angles. 

Take an edge $e$ of a 6-gon in $H$ and consider the sequence (possibly, empty) of adjacent 4-gons of $H$ emanating from this edge. This sequence will stop at a 2-gon or a 6-gon; the case-by-case analysis of angles yields that this sequence has to stop at a 2-gon (see Figure 14.a)).

Take an edge of a 2-gon in $H$ and consider the same construction. If the angles are both obtuse, then the construction is identical and the sequence will terminate at a 2-gon or a 6-gon. If the angles are both acute, then cases b), c) of Figure 14 are possible.

In the first case, all 4-gons contain two obtuse angles and two acute angles; so, the pseudo-road finishes with an edge of two obtuse angles. In the second case, there is a 4-gon, whose angles are all obtuse; this 4-gon is unique in the sequence and its position is arbitrary. Every pair of opposite edges of a 4-gon belongs to a sequence
of 4-gons considered above. So, all angles of a 4-gon are the same, i.e., obtuse. This fact restricts the possibilities of intersections to the three cases of the theorem. \qed

**Corollary 6.5** The only tight \(4_n\), having only simple zigzags, are the Cube and the Truncated Octahedron.

**Proof.** By Theorem 6.4, every two simple zigzags intersect in at most six edges. Since, by Theorem 6.3, there are at most 9 zigzags, we obtain the upper bound \((9 - 1)6 = 48\) on the length of every zigzag. This yields the upper bounds \(\frac{9}{2}48 = 216\) on the number of edges of \(G\) and \(\frac{3}{2}216 = 144\) on the number of its vertices. But, our exhaustive computation in this range of values gave the Cube and the Truncated Octahedron as the only solutions. \qed

### 7 Polyhedra \(5_n\)

Zigzag structure of fullerenes was studied in [DDF02], for which present paper is a follow-up. Here, we add only several observations.

The smallest \(z\)-unbalanced fullerenes \(5_n\) with only simple zigzags are \(5_{108}(D_{2d})\) and \(5_{144}(D_3)\) with \(z\)-vectors \(24^8, 26^4, 28\) and \(28^{12}, 32^3\), respectively. Any \(z\)-uniform or tight \(5_n\), having only simple zigzags, with \(n \leq 200\) is \(z\)-balanced.

Table 7 gives all tight \(5_n\), \(n \leq 200\), with simple zigzags; we conjecture that this list is complete. Apropos, amongst the 9 fullerenes of Table 7 only \(5_{60}(I_h)\), \(5_{88}(T)\), \(5_{140}(I)\) have 12 isolated pentagons, and only \(5_{76}(D_{2d})\) has 6 isolated pairs of adjacent pentagons.

It was conjectured in [DDF02] that for any even \(n \geq 20, n \neq 22\), there exists a tight \(5_n\). Moreover, we could not find a tight graph \(5_n\) with more than 15 zigzags.

| \(n\) | Group | \(z\)-vector | Orb. sizes | Int. vector |
|---|---|---|---|---|
| 20 | \(I_h\) | 10^{10} | 6 | 2^{9} |
| 28 | \(T_d\) | 12^7 | 3,4 | 2^{6} |
| 48 | \(D_3\) | 16^9 | 3,3,3 | 2^{8} |
| 60 | \(I_h\) | 18^{10} | 10 | 2^{9} |
| 60 | \(D_3\) | 18^{10} | 1,3,6 | 2^{9} |
| 76 | \(D_{2d}\) | 22^4, 20^7 | 1,2,4,4 | 4,2^{9} and 2^{10} |
| 88 | \(T\) | 22^{12} | 12 | 2^{11} |
| 92 | \(T_d\) | 24^4, 20^9 | 3,4,6 | 2^{12} and 2^{10}, 0^2 |
| 140 | \(I\) | 28^{15} | 15 | 2^{14} |

Table 7: All tight fullerenes \(5_n\) with \(n \leq 200\), having only simple zigzags.
The list of all symmetry groups of fullerenes $5_n$ is known (see [FeMa95]); they are (with their first appearance):

- $C_{1}(36_3)$, $C_{2}(32_1)$, $C_{4}(56_{314})$, $C_{5}(34_{2})$, $C_{3}(40_{30})$, $D_{2}(28_1)$,
- $S_{4}(44_{82})$, $C_{2v}(30_2)$, $C_{2h}(48_{60})$, $D_{3}(32_6)$, $S_{6}(68_{263})$, $C_{3v}(34_6)$, $C_{3h}(62_{2334})$, $D_{2h}(40_{33})$,
- $D_{2d}(36_6)$, $D_{5}(60_{1794})$, $D_{6}(72_{11144})$, $D_{3h}(26_1)$, $D_{3d}(32_3)$, $T(44_{73})$, $D_{5h}(30_1)$, $D_{5d}(40_1)$,
- $D_{6h}(36_{15})$, $D_{6d}(24_1)$, $T_{d}(28_2)$, $T_{h}(92_{126311})$, $I(140_{x})$, $I_{h}(20_1)$.

There exist fullerenes admitting railroads with triple self-intersection; see, for example, Figure 15.a). More generally, we believe that any curve, which represents a railroad in a graph of type $4_n$, also appears as a curve representing a railroad in some $5_{n'}$.

**Theorem 7.1** For any even number $h \geq 2$, there exists a fullerene $5_n$, with $n = 18h - 8$, having two simple zigzags intersecting in exactly $h$ edges. It has symmetry $T_d$ if $h = 2$ and $D_{2h}$, $D_{2d}$ if $\frac{h}{2}$ is even, odd respectively.

**Proof.** For any even $h \geq 2$ there exist a unique $h$-vertex 4-valent plane graph $H$, whose faces are four 2-gons (in two pairs of adjacent ones) and $\frac{h}{2} - 2$ 4-gons only, and having two simple central circuits (see [DeSt02]). This graph has symmetry $D_{4h}$ for $h = 2, 4$ and, for larger values, symmetry $D_{2h}$, $D_{2d}$ if $\frac{h}{2}$ is even, odd respectively.

We will identify each of the two central circuits of $H$ with simple zigzags, $Z_1$ and $Z_2$, and each vertex with an edge of intersection between them. Every face of $H$ can be seen as a patch in the graph $5_n$ which we will construct, and so, the local Euler formula (2.3) can be applied. Fix a face $F$ of $H$; one can assign to every angle of $F$ an angle (obtuse or acute), so that every 2-gon has one acute and one obtuse angle, while every 4-gon has two obtuse and two acute angles. See below the graph for the first values of $h$.

![Graph for h=4, h=6, and h=8]

So, 2-gonal patches will contain three pentagons, while 4-gonal patches will contain only hexagons. We replace 2- and 4-gonal faces of $H$ by patches depicted below.
The obtained graph has $9(h - 2) + 4$ hexagonal faces, and so, $n = 18h - 8$ vertices. The symmetry group is the same as the one of $H$, except for the first values $h = 2, 4$. See Figure 15(b) for the corresponding graph with $h = 8$. \hfill \Box

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