\textbf{\textit{\alpha-CONTINUITY PROPERTIES OF THE SYMMETRIC \alpha-STABLE PROCESS}}

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\textbf{Abstract.} Let $D$ be a domain of finite Lebesgue measure in $\mathbb{R}^d$ and let $X^D_t$ be the symmetric $\alpha$-stable process killed upon exiting $D$. Each element of the set $\{\lambda_i^\alpha\}_{i=1}^{\infty}$ of eigenvalues associated to $X^D_t$, regarded as a function of $\alpha \in (0, 2)$, is right continuous. In addition, if $D$ is Lipschitz and bounded, then each $\lambda_i^\alpha$ is continuous in $\alpha$ and the set of associated eigenfunctions is precompact. We also prove that if $D$ is a domain of finite Lebesgue measure, then for all $0 < \alpha < \beta \leq 2$ and $i \geq 1$,

$$\lambda_i^\alpha \leq \left[\lambda_i^\beta\right]^{\alpha/\beta}.$$  

Previously, this bound had been known only for $\beta = 2$ and $\alpha$ rational.

\section{1. Introduction}

Let $X_t$ be a $d$-dimensional symmetric $\alpha$-stable process of order $\alpha \in (0, 2]$. The process $X_t$ has stationary independent increments and its transition density $p^\alpha(t, z, w) = f_t^\alpha(z - w)$ is determined by its Fourier transform

$$\exp(-t|z|^\alpha) = \int_{\mathbb{R}^d} e^{iz \cdot w} f_t^\alpha(w) dw.$$  

These processes have right continuous sample paths and their transition densities satisfy the scaling property

$$p^\alpha(t, x, y) = t^{-d/\alpha} p^\alpha(1, t^{-1/\alpha} x, t^{-1/\alpha} y).$$  

When $\alpha = 2$ the process $X_t$ is a $d$-dimensional Brownian motion running at twice the usual speed. The non-local operator associated to $X_t$ is $(-\Delta)^{\alpha/2}$ where $\Delta$ is the Laplace operator in $\mathbb{R}^d$.

Let $D$ be a domain in $\mathbb{R}^d$ and let $X^D_t$ be the symmetric $\alpha$-stable process killed upon leaving $D$. We write $p^\alpha_D(t, x, y)$ for the transition density of $X^D_t$ and $H_\alpha$ for its associated non-local self-adjoint positive

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operator. It is well known that if $D$ has finite Lebesgue measure then the spectrum of $H_\alpha$ is discrete. Let

$$0 < \lambda_1^\alpha(D) < \lambda_2^\alpha(D) \leq \lambda_3^\alpha(D) \leq \cdots$$

be the eigenvalues of $H_\alpha$, and let

$$\varphi_1^\alpha, \varphi_2^\alpha, \varphi_3^\alpha, \cdots$$

be the corresponding sequence of eigenfunctions.

Several authors have studied properties of the eigenvalues and eigenfunctions of $H_\alpha$. One common theme has been to extend results on Brownian motion ($\alpha = 2$) to analogous results for symmetric $\alpha$-stable processes. For example, R. M. Blumenthal and R. K. Getoor [7] have shown Weyl’s asymptotic law holds: if $D$ is a bounded open set and $N(\lambda)$ is the number of eigenvalues less than or equal to $\lambda$, then there exists a constant $C_d$, depending only on $d$, such that

$$N(\lambda) \approx C_d \frac{m(D)}{\Gamma(d+1)} \lambda^d \text{ as } \lambda \to \infty,$$

provided $m(\partial D) = 0$, where $m$ is Lebesgue measure.

If $D \subseteq \mathbb{R}^d$ is a domain, define the inner radius $R_D$ to be the supremum of the radii of all balls contained in $D$. R. Bañuelos et al [6] and P. Méndez-Hernández [17] have shown if $D$ is a convex domain with finite inner radius $R_D$ and $I_D$ is the interval $(-R_D, R_D)$, then

$$\lambda_1^\alpha(I_D) \leq \lambda_1^\alpha(D).$$

Moreover, if $D \subseteq \mathbb{R}^d$ has finite volume and $D^*$ is a ball in $\mathbb{R}^d$ with the same volume as $D$, then it was proved in [6] that the Faber-Krahn inequality holds:

$$\lambda_1^\alpha(D^*) \leq \lambda_1^\alpha(D).$$

Another line of inquiry taken by those authors was to consider the eigenvalues as a function of the index $\alpha$. For instance, if $D$ is a convex domain with finite inner radius $R_D$, then

$$\frac{2^\alpha \Gamma(1 + \frac{\alpha}{2}) \Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{1}{2}) R_D^\alpha} \leq \lambda_1^\alpha(D) \leq \lambda_1^\alpha(B_{R_D}),$$

where $B_{R_D}$ is a ball in $\mathbb{R}^d$ of radius $R_D$. They also proved that if $D \subseteq \mathbb{R}^d$ has finite volume, then

$$\lambda_1^\alpha(D) \leq [\mu_1(D)]^{\alpha/2}, \quad (1.1)$$

where $\mu_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on $D$. 
For the Cauchy process, i.e. \( \alpha = 1 \), and bounded Lipshitz domains, R. Bañuelos and T. Kulczycki \[5\] extended (1.1) to
\[
(1.2) \quad \lambda_1^i(D) \leq [\mu_i(D)]^{1/2}, \quad i = 1, 2, \ldots,
\]
where
\[
0 < \mu_1(D) < \mu_2(D) \leq \cdots
\]
are all the Dirichlet eigenvalues of \(-\Delta\) on \(D\). Their proof of (1.2) is based on a variational formula for \(\lambda_1^i(D)\) they developed from a connection with the Steklov problem for the Laplacian. They also obtained many detailed properties of the eigenfunctions \(\varphi_1^1\) for the Cauchy process.

By finding a connection with the symmetric stable process with rational index \(\alpha\) and PDEs of order higher than 2, R. D. DeBlassie \[12\] derived a variational formula for the eigenvalues which led to the following extension of (1.1) and (1.2):
\[
(1.3) \quad \lambda_1^\alpha(D) \leq [\mu_i(D)]^{\alpha/2}, \quad i = 1, 2, \ldots,
\]
for all rational \(\alpha \in (0, 2)\) and certain bounded domains \(D \subseteq \mathbb{R}^d\). The class of admissible domains includes convex polyhedra, Lipschitz domains with sufficiently small Lipschitz constant and \(C^1\) domains.

In this article, we study the eigenvalues and eigenfunctions regarded as functions of the index \(\alpha\). Our first result concerns continuity of the eigenvalues.

**Theorem 1.1.** Let \(D\) be a domain of finite Lebesgue measure. Then, as a function of \(\alpha \in (0, 2)\), \(\lambda_1^\alpha\) is right continuous for each positive integer \(i\).

In order to prove Theorem 1.1 we need the following interesting monotonicity property extending (1.3) above.

**Theorem 1.2.** Let \(D\) be a domain of finite Lebesgue measure in \(\mathbb{R}^n\). If \(0 < \alpha < \beta \leq 2\), then for all positive integers \(i\),
\[
[\lambda_1^\alpha(D)]^{1/\alpha} \leq [\lambda_i^\beta(D)]^{1/\beta}.
\]

By requiring more regularity of \(\partial D\), we can prove the following extension of Theorem 1.1.

**Theorem 1.3.** Let \(D\) be a bounded Lipschitz domain. Then, as a function of \(\alpha \in (0, 2)\), \(\lambda_1^\alpha\) is continuous for each positive integer \(i\).

We will obtain Theorem 1.3 from the following result that we believe is of independent interest.
Theorem 1.4. Let $D$ be a bounded Lipschitz domain. If $\alpha_m$ converges to $\alpha \in (0, 2)$ then for each positive integer $i$, $\{\varphi_{i,m}^\alpha : m \geq 1\}$ is precompact in $C(D)$ equipped with the sup norm. Moreover, if $\lambda_{i,m}^\alpha$ converges to $\lambda$, then any limit point of $\{\varphi_{i,m}^\alpha : m \geq 1\}$ is an eigenfunction of $H_\alpha$ and $\lambda$ is the corresponding eigenvalue.

As a corollary of the proof of the last theorem, we obtain continuity of the first eigenfunction as a function of $\alpha$.

Theorem 1.5. If $D$ is a bounded Lipschitz domain and $\alpha_m$ converges to $\alpha \in (0, 2)$, then $\varphi_{1,m}^\alpha$ converges uniformly to $\varphi_1^\alpha$ on $D$.

The article is organized as follows. In section 2 we present some results needed in the proof of Theorem 1.1. We establish Theorem 1.1 in section 3 by proving upper semicontinuity and right lower semicontinuity of the eigenvalues via Dirichlet forms. In section 4 we prove Theorem 1.2 using an extension of an operator inequality from [12]. Lower semicontinuity of the eigenvalues, for Lipschitz domains, is proved in section 5 using Theorem 1.4. This will yield Theorems 1.3 and 1.5. Section 6 deals with certain weak convergence results needed to prove Theorem 1.4. Finally, in section 7 we prove Theorem 1.4.

2. Preliminary results

Throughout this section we will assume the domain $D$ has finite Lebesgue measure. We denote by $C^\infty_c(D)$ the set of $C^\infty$ functions with compact support in $D$. The inner product and the norm in $L^2(D)$ will be denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|_2$, respectively.

For any domain $D \subseteq \mathbb{R}^d$, we define $\tau_D$ to be the first exit time of $X_t$ from $D$, i.e.,

$$\tau_D = \inf\{t > 0 : X_t \notin D\}.$$ 

Let

$$\mathcal{F}_\alpha = \left\{ \varphi \in L^2(\mathbb{R}^d) : \int \int \frac{[\varphi(y) - \varphi(x)]^2}{|y - x|^{d+\alpha}} \, dydx < \infty \right\}.$$ 

The Dirichlet form $(\mathcal{E}_\alpha, \mathcal{F}_\alpha)$ associated to $X_t$ is given by

$$\mathcal{E}_\alpha(\psi, \varphi) = A(d, \alpha) \int \int \frac{[\psi(y) - \psi(x)] [\varphi(y) - \varphi(x)]}{|y - x|^{d+\alpha}} \, dydx,$$

for all $\psi, \varphi \in \mathcal{F}_\alpha$, where
\[ A(d, \alpha) = \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{2^\alpha \pi^{d/2} \Gamma\left(\frac{\alpha}{2}\right)}. \]

It is well known that the Dirichlet form corresponding to \( X_t^D \) is given by \( (\mathcal{E}_\alpha, \mathcal{F}_{\alpha,D}) \), where
\[ \mathcal{F}_{\alpha,D} = \{ u \in \mathcal{F}_\alpha : \text{a quasi continuous version of } u \text{ is 0 quasi everywhere in } D^c \}. \]

Recall that for all \( \psi, \varphi \) in the domain of \( H_\alpha \) we have
\[ \mathcal{E}_\alpha(\psi, \varphi) = \langle \psi, H_\alpha \varphi \rangle. \]

As seen in Theorem 4.4.3 of [14], \( \mathcal{F}_{\alpha,D} \) is the closure of \( C_\infty^c(D) \) in \( \mathcal{F}_\alpha \) with respect to the norm
\[ \| \varphi \|_\alpha = \sqrt{\mathcal{E}_\alpha(\varphi, \varphi)} + \| \varphi \|_2. \]

**Lemma 2.1.** Let \( \varphi, \psi \in C_\infty^\infty(D) \). Then the function
\[ \mathcal{E}_\alpha(\varphi, \psi) : (0, 2) \to \mathbb{R}, \]

is continuous on \((0, 2)\).

**Proof.** Let \( \varphi, \psi \in C_\infty^\infty(D) \), and let \( \beta \in (\alpha - \delta, \alpha + \delta) \), where \( \delta = \frac{1}{2} \min \{2 - \alpha, \alpha\} \). Then there exists a constant \( C > 0 \), depending only on \( \varphi \) and \( \psi \), such that
\[ \left| \frac{\psi(y) - \psi(x)}{|y - x|^{d+\beta}} \right| \left| \frac{\varphi(y) - \varphi(x)}{|y - x|^{d+\beta-2}} \right| \leq \frac{C}{|y - x|^{d+\beta-2}} \leq C \max \left\{ \frac{1}{|y - x|^{d+\alpha-\delta-2}}, \frac{1}{|y - x|^{d+\alpha+\delta-2}} \right\}. \]

Since \( D \) has finite measure a simple computation using polar coordinates shows
\[ \max \left\{ \frac{1}{|y - x|^{d+\alpha-\delta-2}}, \frac{1}{|y - x|^{d+\alpha+\delta-2}} \right\}, \]

is integrable in \( D \times D \). The result immediately follows from the dominated convergence theorem. \( \square \)

We end this section with some basic estimates on \( L^2 \) norms to be used in the next section. Suppose \( k \) is a positive integer, \( 0 < \epsilon < 1 \), and \( \varphi_1, \ldots, \varphi_k \in L^2(D) \) satisfy
\[ |\langle \varphi_i, \varphi_j \rangle| < \frac{\epsilon}{4k^2}, \quad i \neq j, \]
\[ \left( 1 - \frac{\epsilon}{4k^2} \right) < \| \varphi_i \|_2^2 < \left( 1 + \frac{\epsilon}{4k^2} \right). \]
for all \(1 \leq i, j \leq k\). If \(\psi = \sum_{i=1}^{k} a_i \varphi_i\), with \(\|\psi\|_2 = 1\), then we now show

\[
\frac{1}{1 + \epsilon/2} \leq \sum_{i=1}^{k} a_i^2 \leq \frac{1}{1 - \epsilon/2},
\]
and \(\varphi_1, \ldots, \varphi_k\) are linearly independent.

For the proof, note we have

\[
1 = \langle \psi, \psi \rangle = \sum_{i=1}^{k} a_i^2 \|\varphi_i\|_2^2 + 2 \sum_{i=1}^{k} \sum_{j > i} a_i a_j \langle \varphi_i, \varphi_j \rangle \geq \sum_{i=1}^{k} a_i^2 \left( 1 - \frac{\epsilon}{4k^2} \right) - 2 \sum_{i=1}^{k} \sum_{j > i} |a_i| |a_j| \frac{\epsilon}{4k^2} \geq \sum_{i=1}^{k} a_i^2 \left( 1 - \frac{\epsilon}{4k^2} \right) - (k^2 - k) \sum_{i=1}^{k} a_i^2 \frac{\epsilon}{4k^2} \geq (1 - \epsilon/2) \sum_{i=1}^{k} a_i^2,
\]
and we conclude

\[
\sum_{i=1}^{k} a_i^2 \leq \frac{1}{1 - \epsilon/2}.
\]

Similar computations give the remaining assertions.

3. **Proof of Theorem 1.1**

We will use the following well known result, see [10].

**Theorem 3.1.** Let \(H\) be a non-negative self-adjoint unbounded operator with discrete spectrum \(\{\lambda_i\}_{i=1}^{\infty}\), and domain \(\text{Dom}(H)\). Then for \(i \geq 1\)

\[
\lambda_i = \inf \{ \lambda(L) : L \subseteq \text{Dom}(H), \text{dim}(L) = i \},
\]
where

\[
\lambda(L) = \sup \{ \langle Hf, f \rangle : f \in L, \|f\|_2 = 1 \},
\]
and \(L\) is a vector subspace of \(\text{Dom}(H)\) of dimension \(i\).

We will prove the right continuity of the \(k\)-th eigenvalue in several steps.
Proposition 3.2. Let $D$ be a domain of finite Lebesgue measure. Then for all $k \geq 1$
\[ \limsup_{\beta \to \alpha} \lambda_k^\beta(D) \leq \lambda_k^\alpha(D). \]

Proof. Let $0 < \epsilon < 1$ and $k \geq 1$. Recall $C_c^\infty(D)$ is dense in $\text{Dom}(H_\alpha)$ under the norm $\| \cdot \|_\alpha$. Then for all $\alpha \in (0,2)$, there exist $\varphi_1, \ldots, \varphi_k \in C_c^\infty(D)$ such that
\begin{equation}
|\langle \varphi_1^\alpha, \varphi_j^\alpha \rangle - \langle \varphi_i^\alpha, \varphi_j^\alpha \rangle| < \frac{\epsilon}{8k^2},
\end{equation}
and
\begin{equation}
|\mathcal{E}_\alpha(\varphi_i^\alpha, \varphi_j^\alpha) - \mathcal{E}_\alpha(\varphi_i^\alpha, \varphi_j^\alpha)| < \frac{\epsilon}{8k^2},
\end{equation}
for all $1 \leq i, j \leq k$.

Thanks to Lemma 2.1 there exists $\eta_0$ such that for all $\beta \in (\alpha - \eta_0, \alpha + \eta_0)$
\begin{equation}
|\mathcal{E}_\alpha(\varphi_i^\alpha, \varphi_j^\alpha) - \mathcal{E}_\beta(\varphi_i^\alpha, \varphi_j^\alpha)| < \frac{\epsilon}{8k^2}.
\end{equation}

Notice (3.3) implies
\[ |\langle \varphi_i^\alpha, \varphi_j^\alpha \rangle| < \frac{\epsilon}{8k^2}, \quad i \neq j, \]
and
\[ 1 - \frac{\epsilon}{8k^2} < \| \varphi_i \|_2^2 < 1 + \frac{\epsilon}{8k^2}, \]
for all $1 \leq i, j \leq k$. Then by the comments at the end of section 2, we know $\varphi_1, \ldots, \varphi_k$ are linearly independent.

Theorem 3.1 implies
\[ \lambda_k^\beta(D) \leq \lambda(\mathcal{L}_k), \]
where $\mathcal{L}_k = \text{span}\{\varphi_1, \ldots, \varphi_k\}$ and
\[ \lambda(\mathcal{L}_k) = \sup \{ \langle H_\beta f, f \rangle : f \in \mathcal{L}_k, \| f \|_2 = 1 \}. \]

Take $\psi = \sum_{i=1}^k a_i \varphi_i \in \mathcal{L}_k$ such that
\begin{equation}
\lambda(\mathcal{L}_k) \leq \mathcal{E}_\beta(\psi, \psi) + \epsilon/4,
\end{equation}
and
\[ \| \psi \|_2 = 1. \]

Thanks to (2.1), with $\epsilon$ there replaced by $\epsilon/2$, we have
\[ \sum_{i=1}^k a_i^2 \leq 2. \]
Then since
\[ |E_\beta(\psi, \psi) - E_\alpha(\psi, \psi)| \leq \sum_{i=1}^{k} \sum_{j=1}^{k} |a_i a_j| |E_\beta(\varphi_i, \varphi_j) - E_\alpha(\varphi_i, \varphi_j)|, \]

(3.5) implies
\[ |E_\beta(\psi, \psi) - E_\alpha(\psi, \psi)| < \frac{\epsilon}{4}. \]

Thus
\[ \lambda_\beta^k(D) \leq E_\alpha(\psi, \psi) + \frac{\epsilon}{2}. \]

Consider \( \psi_0 = \sum_{i=1}^{k} a_i \varphi_i^\alpha \). By (2.1) we have
\[ \frac{1}{1 + \epsilon/4} \leq \|\psi_0\|_2^2 = \sum_{i=1}^{k} a_i^2 \leq \frac{1}{1 - \epsilon/4}. \]

Following the argument used to obtain (3.7), one easily proves (3.4) implies
\[ |E_\alpha(\psi_0, \psi_0) - E_\alpha(\psi, \psi)| < \frac{\epsilon}{4}. \]

Hence
\[ \lambda_\beta^k(D) \leq E_\alpha(\psi, \psi) + \epsilon/2 \leq E_\alpha(\psi_0, \psi_0) + 3\epsilon/4 = \sum_{i=1}^{k} a_i^2 \lambda_i^\alpha(D) + 3\epsilon/4 \leq \lambda_k^\alpha(D) \sum_{i=1}^{k} a_i^2 + 3\epsilon/4 \leq \frac{1}{1 - \epsilon/4} \lambda_k^\alpha(D) + 3\epsilon/4, \]

and the result immediately follows.

**Proposition 3.3.** Let \( D \) be a domain of finite Lebesgue measure. If \( D_n \subset \subset D \) is a sequence of \( C^\infty \) domains increasing to \( D \), then for all \( k \geq 1 \),
\[ \lim_{n \to \infty} \lambda_k^\alpha(D_n) = \lambda_k^\alpha(D). \]

**Proof.** By domain monotonicity,
\[ \lambda_k^\alpha(D) \leq \lambda_k^\alpha(D_n). \]
Hence it suffices to show

\[(3.9) \limsup_{n \to \infty} \lambda^\alpha_k(D_n) \leq \lambda^\alpha_k(D).\]

To this end, let \(\eta > 0\). Following the arguments presented above, we can prove there exist \(k\) linearly independent functions \(\varphi_1, \ldots, \varphi_k \in C_c^\infty(D)\) such that

\[\lambda^\alpha_k(D) \geq \lambda_\alpha(L_k) - \eta,\]

where \(L_k\) is the vector space generated by \(\{\varphi_1, \ldots, \varphi_k\}\) and

\[\lambda_\alpha(L_k) = \sup \{ \langle H_\alpha f, f \rangle : f \in L_k, \|f\|_2 = 1 \} .\]

Then there exists \(n_0\) such that the supports of \(\varphi_1, \ldots, \varphi_k\) are contained in \(D_{n_0}\). Consequently for large \(n\), Theorem 3.3 implies

\[\lambda^\alpha_k(D) \geq \lambda_\alpha(L_k) - \eta \geq \lambda^\alpha_k(D_n) - \eta.\]

Hence upon letting \(n \to \infty\) and \(\eta \to 0\), we get (3.9).

\[\square\]

**Proposition 3.4.** Let \(D\) be a domain of finite Lebesgue measure. Then for all \(k \geq 1\)

\[\liminf_{\beta \to \alpha^+} \lambda^\beta_k(D) \geq \lambda^\alpha_k(D).\]

**Proof.** Let \(D\) be a domain with finite Lebesgue measure, and let \(D_n \subset \subset D\) be a sequence of bounded \(C^\infty\) domains increasing to \(D\). Such a sequence can be constructed using the regularized distance function—see page 171 in [19].

By Theorem 1.2

\[\lambda^\alpha_k(D_n) \leq \left[ \lambda^\alpha_{k+\epsilon}(D_n) \right]^{\alpha/(\alpha+\epsilon)}.\]

There is no danger of circular reasoning here because the proof of Theorem 1.2 given in the next section is independent of Theorem 1.1. Now let \(n \to \infty\) and appeal to Proposition 3.3 to get

\[\lambda^\alpha_k(D) \leq \left[ \lambda^\alpha_{k+\epsilon}(D) \right]^{\alpha/(\alpha+\epsilon)}.\]

Upon letting \(\epsilon \to 0\), we get the desired \(\liminf\) behavior.

\[\square\]

Combining Propositions 3.2 and 3.4 we get Theorem 1.1.
4. Proof of Theorem 1.2

The first result we need is the following extension of Theorem 1.3 in [12]. It says the operator $e^{-(-\Delta)^{\alpha/2}t}$ dominates $e^{-H_\alpha t}$ on $L^2(D)$.

**Theorem 4.1.** Let $D$ be a bounded smooth domain. If $0 < \alpha \leq 2$ then for all $\psi \in L^2(D)$,

$$\langle \psi, e^{-(-\Delta)^{\alpha/2}t}\psi \rangle \geq \langle \psi, e^{-H_\alpha t}\psi \rangle.
$$

**Proof.** The proof of the case given in section 6 of [12] goes through with the following changes. All lemma and equation labels are from that article. In the proof of Lemma 6.1, it is enough to use the bound from Lemma 3.2 in place of (5.3).

The second expression in the scaling relation (6.2) should be replaced by

$$b_\ell = a_\ell/M^\alpha$$

and any subsequent appearance of $a_\ell/M^2$ should be replaced by $a_\ell/M^\alpha$.

In the proof of Lemma 6.2 it is not necessary to appeal to the Weyl Asymptotic formula. It is enough that $a_\ell \to \infty$ as $\ell \to \infty$ and then one needs only examine the function $F(x) = x^\alpha e^{-Bx}$ instead of $x^{2n/d}e^{-Bx^{2/d}}$.

The only other change is at the end of section 6 where it is shown $g(1/\delta) \to 0$ as $\delta \to 0$. This time use the bound

$$p^\alpha(t,x,y) \leq \frac{c_{\alpha,d}}{t^{d/\alpha}}.
$$

□

With the aid of this Theorem, we can now prove Theorem 1.2 for bounded smooth domains.

**Theorem 4.2.** Let $D$ be a bounded smooth domain in $\mathbb{R}^n$. If $0 < \alpha < \beta \leq 2$, then for all positive integers $i$,

$$\left[\lambda^\alpha_i(D)\right]^{1/\alpha} < \left[\lambda^\beta_i(D)\right]^{1/\beta}.
$$

**Proof.** Let $0 < \alpha < \beta \leq 2$. Denote by $\sigma_t$ the stable subordinator of index $\frac{\alpha}{\beta}$ with

$$E[e^{-s\sigma_t}] = e^{-ts^{\alpha/\beta}}, \text{ for all } s > 0.
$$

It is well known that for $\psi \in L^2(D)$,

$$E\left[\langle \psi, e^{-(-\Delta)^{\beta/2}\sigma_t}\psi \rangle\right] = \langle \psi, e^{-(-\Delta)^{\alpha/2}t}\psi \rangle,$$

and

$$E\left[\langle \psi, e^{-H_\beta\sigma_t}\psi \rangle\right] = \langle \psi, e^{-(H_\alpha)^{\alpha/\beta}t}\psi \rangle.$$
Then taking \( \psi = \sum_{j=1}^{i} a_j \varphi_j^\beta \) with \( \|\psi\|_2 = \sum_{j=1}^{i} a_j^2 = 1 \), Theorem 4.1 implies

\[
\langle \psi, e^{-(\Delta)^{\alpha/2}} \psi \rangle \geq \sum_{j,k=1}^{i} a_j a_k \langle \varphi_j^\beta, e^{-(H_\beta)^{\alpha/\beta}} \varphi_k^\beta \rangle
\]

\[
= \sum_{j,k=1}^{i} a_j a_k E \left[ \langle \varphi_j^\beta, e^{-(H_\beta)} \varphi_k^\beta \rangle \right]
\]

\[
= \sum_{j,k=1}^{i} a_j a_k E \left[ \delta_{jk} e^{-\sigma_i \lambda_j^\beta(D)} \right]
\]

\[
= \sum_{j=1}^{i} a_j^2 e^{-t[\lambda_j^\beta(D)]^{\alpha/\beta}}
\]

\[
\geq \sum_{j=1}^{i} a_j^2 e^{-t[\lambda_j^\beta(D)]^{\alpha/\beta}}
\]

\[
= e^{-t[\lambda_i^\beta(D)]^{\alpha/\beta}}.
\]

On the other hand, Theorem 1 in [1] states that any set of zero \( \beta \)-Riesz capacity also has zero \( \alpha \)-Riesz capacity, i.e., \( F_{\beta,D} \subseteq F_{\alpha,D} \). Thus, by the formula

\[
\mathcal{E}_\alpha(\psi, \psi) = \lim_{t \to \infty} \frac{1}{t} \langle \psi, \left( 1 - e^{-t(\Delta)^{\alpha/2}} \right) \psi \rangle,
\]

we conclude

\[
(4.1) \quad \mathcal{E}_\alpha(\psi, \psi) \leq \left[ \lambda_i^\beta(D) \right]^{\alpha/\beta},
\]

for all \( i \geq 1 \).

The desired bound follows from (4.1) and Theorem 3.1.

Now we can prove Theorem 1.2. Let \( D \subseteq \mathbb{R}^d \) be a domain of finite Lebesgue measure. Suppose \( D_n \subset \subset D \) is a sequence of bounded \( C^\infty \) domains increasing to \( D \).

Thanks to Theorem 1.2 if \( 0 < \alpha < \beta \leq 2 \) then for all positive integers \( n \) and \( i \),
\[ \left[ \frac{\lambda_i^\alpha(D_n)}{\alpha} \right]^{1/\alpha} \leq \left[ \frac{\lambda_i^\beta(D_n)}{\beta} \right]^{1/\beta}. \]

Letting \( n \to \infty \), Proposition 3.3 implies
\[ \left[ \frac{\lambda_i^\alpha(D)}{\alpha} \right]^{1/\alpha} \leq \left[ \frac{\lambda_i^\beta(D)}{\beta} \right]^{1/\beta}, \]
as desired. \( \square \)

5. Proof of Theorems 1.3 and 1.5

We now show how Theorem 1.4 implies Theorem 1.3. In order to simplify the notation, throughout this section we will write \( \lambda_k^\alpha \) for \( \lambda_k^\alpha(D) \) and \( \mu_k \) for \( \mu_k(D) \).

**Proof of Theorem 1.3.** We proceed by induction on \( i \). For \( i = 1 \), let \( \{\alpha_m\}_{m=1}^\infty \) be a sequence converging to \( \alpha \) in \((0,2)\). Consider any subsequence \( \beta_r = \alpha_m \). Theorem 1.2 implies the sequence \( \{\lambda_{1m}^\alpha\}_{m=1}^\infty \) is bounded, and so there is a subsequence \( \gamma_{t_r} = \beta_{t_r} \) such that \( \lambda_{1t_r}^\alpha \) converges as \( t \to \infty \), say to \( \lambda \). Thanks to Theorem 1.4 we can choose a subsequence \( \eta_{t_r} = \gamma_{t_r} \) such that \( \phi_{1t_r}^\beta \) converges uniformly to \( \phi \) an eigenfunction of \( H_\alpha \) with eigenvalue \( \lambda \). Since \( \phi_{1t_r}^\beta \) is nonnegative, so is \( \phi \). But the only nonnegative eigenfunction of \( H_\alpha \) is \( \phi_{1}^\alpha \). Hence we have shown any subsequence of \( \lambda_{1m}^\alpha \) contains a further subsequence converging to \( \lambda_1^\alpha \). We conclude that
\[ \lim_{m \to \infty} \lambda_{1m}^\alpha = \lambda_1^\alpha. \]
Note this also proves Theorem 1.5.

Next assume the theorem is true for \( j \leq i \). We verify it is true for \( j = i + 1 \). We will show
\[ \liminf_{\beta \to \alpha} \lambda_{i+1}^\beta \geq \lambda_{i+1}^\alpha. \] (5.1)
Combined with the lim sup behavior from Proposition 3.2, we conclude the desired result
\[ \lim_{\beta \to \alpha} \lambda_{i+1}^\beta = \lambda_{i+1}^\alpha. \]

To get the lim inf behavior, by way of contradiction, assume \( \lambda = \liminf_{\beta \to \alpha} \lambda_{i+1}^\beta < \lambda_{i+1}^\alpha \). Let \( \{\alpha_m\}_{m=1}^\infty \) be a sequence converging to \( \alpha \) with\( \beta_r = \alpha_{m_r} \) such that:
\[ \lim_{m \to \infty} \lambda_{i+1}^{\alpha_{m_r}} = \lambda. \] (5.2)
By the induction hypothesis, \( \lambda_{j}^{\alpha_{m_r}} \) converges to \( \lambda_{j}^\alpha \) for \( j \leq i \). Then Theorem 1.4 implies we can choose a subsequence \( \beta_r = \alpha_{m_r} \) such that:
For each \( j, 1 \leq j \leq i \), \( \lambda^\beta_j \) converges to \( \lambda^\alpha_j \), and \( \varphi^\beta_j \) converges uniformly to an eigenfunction \( \varphi_j \) of \( H_\alpha \) with corresponding eigenvalue \( \lambda^\alpha_j \).

The limit \( \lambda \) from (12) is an eigenvalue of \( H_\alpha \), and \( \varphi^{\beta + 1}_i \) converges uniformly to an eigenfunction \( \varphi_{i+1} \) of \( H_\alpha \) with eigenvalue \( \lambda^\alpha \).

Since \( \lambda^\alpha \) is an eigenvalue strictly less than \( \lambda^\alpha_{i+1} \), we can choose positive integers \( \ell \) and \( m \) such that \( \ell \leq m \leq i \), \( \lambda^\alpha_m = \lambda^\alpha \) and \( \lambda^\alpha_{m+1} < \lambda^\alpha_\ell = \cdots = \lambda^\alpha_1 \leq \lambda^\alpha_{i+1} \).

In particular, if \( E \) is the eigenspace corresponding to \( \lambda^\alpha = \lambda^\alpha_m \), \( \text{dim}(E) = m - \ell + 1 \).

On the other hand, the uniform convergence implies for \( j_1, j_2 \in \{1, \ldots, i + 1\} \)
\[
\delta_{j_1 j_2} = \int_D \varphi^\beta_{j_1} \varphi^\beta_{j_2} \, dx \text{ converges to } \int_D \varphi_{j_1} \varphi_{j_2} \, dx.
\]

Thus \( \{\varphi_1, \ldots, \varphi_{i+1}\} \) is an orthonormal set, and so \( \{\varphi_\ell, \ldots, \varphi_m\} \cup \{\varphi_{i+1}\} \) is an orthonormal subset of \( E \). This forces \( \text{dim}(E) \geq m - \ell + 2 \), which contradicts (12). We conclude (5.1) holds.

6. Weak Convergence Results

Let \( \mathbb{D}[0, \infty) \) be the space of right continuous functions \( \omega: [0, \infty) \to \mathbb{R}^d \) with left limits. That is, \( \omega(t^+) = \lim_{s \to t^+} \omega(s) = \omega(t) \) and \( \omega(t^-) = \lim_{s \to t^-} \omega(s) \) exists. The usual convention is \( \omega(0^-) := \omega(0) \). Let \( X_t(\omega) = \omega(t) \) be the coordinate process and let \( \mathcal{F}_t \) be the \( \sigma \)-algebra generated by the cylindrical sets. We equip \( \mathbb{D}[0, \infty) \) with the Skorohod topology. Our main reference is Chapter 3 in Ethier and Kurtz [13]. Let \( P^\alpha_x \) denote the law on \( \mathbb{D}[0, \infty) \) of the symmetric \( \alpha \)-stable process started at \( x \); the corresponding expectation will be denote by \( E^\alpha_x \).

**Lemma 6.1.** If \( (x_n, \alpha_n) \) converges to \( (x, \alpha) \) in \( \mathbb{R}^n \times (0, 2) \), then \( P^\alpha_{x_n} \) converges weakly to \( P^\alpha_x \) in \( \mathbb{D}[0, \infty) \).

**Proof.** Using characteristic functions it is easy to show the corresponding finite dimensional distributions converge. Thus by Theorem 7.8 on page 131 in [13], it suffices to show \( \{P^\alpha_{x_n}: n \geq 1\} \) is tight. For this, note for \( \beta = \alpha \) or \( \alpha_n \) and \( y = x \) or \( x_n \), \( P^\beta_y \) solves the martingale problem:

a) \( P^\beta_y(X_0 = y) = 1 \)
b) for each \( f \in C^2_{\beta}(\mathbb{R}^d) \),
\[
f(X_t) - f(X_0) - \int_0^t \mathcal{L}_\beta f(X_s) ds
\]
is a \( P^\beta_y \)-martingale, where \( C^2_{\beta}(\mathbb{R}^d) \) is the space of functions with bounded continuous derivatives up to and including order 2 and

\[
\mathcal{L}_\beta f(x) = A(d, \alpha) \int_{\mathbb{R}^d \setminus \{x\}} \frac{f(y) - f(x) - \nabla f(x) \cdot (y - x) I(|y - x| < 1)}{|y - x|^{d+\alpha}} dy,
\]

see section 2 of [3]. It is easy to show for any \( f \in C^2_{\beta}(\mathbb{R}^d) \) there exists \( C_f > 0 \) independent of \( \alpha \) and \( x \) such that \( f(X_t) - f(X_0) - C_f t \) is a \( P^{\alpha_n}_x \)-supermartingale. Then by Proposition 3.2 in [2], \( \{P^{\alpha_n}_x: n \geq 1\} \) is tight on \( \mathbb{D}[0,t] \) for all \( t \). Even though that result is stated in one dimension and \( x_n \equiv x \), it is easy to check the proof works in higher dimensions with \( x_n \) converging to \( x \). \( \square \)

The next step is to show for each \( T > 0 \) the distribution of \( X_{T \wedge \tau_D} \) under \( P^{\alpha_n}_x \) converges to that under \( P^\alpha_x \) as \( (x_n, \alpha_n) \) converges to \( (x, \alpha) \). To this end, define

\[
\mathcal{A}_D = \{ \omega \in \mathbb{D}[0,\infty): d(X[0,\tau_D(\omega) - r], D^c) > 0 \text{ for all rational } r < \tau_D(\omega) \},
\]

and

\[
\mathcal{C}_D = \{ \omega \in \mathbb{D}[0,\infty): X(\tau_D(\omega)) \in \overline{D^c} \} \cap \mathcal{A}_D.
\]

Here \( X[0,t] = \{X_s: 0 \leq s \leq t\} \) and \( d(A,B) \) is the distance between \( A \) and \( B \).

**Lemma 6.2.** For open \( D \subseteq \mathbb{R}^d \), \( \tau_D \) is continuous on \( \mathcal{C}_D \).

**Proof.** Let \( \omega \in \mathcal{C}_D \) and suppose \( \omega_n \) converges to \( \omega \) in \( \mathbb{D}[0,\infty] \). We will show \( \tau_D(\omega_n) \) converges to \( \tau_D(\omega) \). Let

\[
\Lambda' = \{ \lambda: [0,\infty) \to [0,\infty) \mid \lambda \text{ is strictly increasing and surjective} \}
\]

Proposition 5.3 (a) and (c), on page 119, in [13] implies that for each \( T > 0 \) there exist \( \lambda_n \in \Lambda' \) such that

\[
\lim_{n \to \infty} \sup_{0 \leq t \leq T} |\lambda_n(t) - t| = 0,
\]

\[
\lim_{n \to \infty} \sup_{0 \leq t \leq T} |\omega_n(t) - \omega(\lambda_n(t))| = 0.
\]

First we show

\[
\liminf_{n \to \infty} \tau_D(\omega_n) \geq \tau_D(\omega).
\]
Let $\delta \in (0, \tau_D(\omega)/2)$ be rational and set

$$
\varepsilon = d(\omega[0, \tau_D(\omega) - \delta], D^c).
$$

Since $\omega \in C_D$ we have $\varepsilon > 0$. Using $T = \tau_D(\omega)$ in (6.1)–(6.2), there exists $N$ such that for $n \geq N$,

$$
\begin{aligned}
\{ t - \delta < \lambda_n(t) < t + \delta & \quad \text{for all } t \leq T, \\
\sup_{0 \leq t \leq T} |\omega_n(t) - \omega(\lambda_n(t))| & < \frac{\varepsilon}{2}.
\end{aligned}
$$

In particular, for all $t \leq \tau_D(\omega) - 2\delta$ and $n \geq N$

$$
\lambda_n(t) < t + \delta \leq \tau_D(\omega) - \delta < T.
$$

Thus $\omega(\lambda_n(t)) \in D$ and $d(\omega(\lambda_n(t)), D^c) \geq \varepsilon$. Therefore

$$
\omega_n(t) \in B(\omega(\lambda_n(t)), \varepsilon/2) \subseteq D,
$$

for all $t \leq \tau_D(\omega) - 2\delta$ and $n \geq N$. This implies $\tau_D(\omega_n) > \tau_D(\omega) - 2\delta$ for $n \geq N$. Take the lim inf as $n \to \infty$ and then let $\delta \to 0$ to get (6.3).

To finish, we show

$$
\limsup_{n \to \infty} \tau_D(\omega_n) \leq \tau_D(\omega).
$$

Given that $\omega \in C_D$ and $\omega$ is right continuous, we can choose $\delta > 0$ such that

$$
\varepsilon := d(\omega[\tau_D(\omega), \tau_D(\omega) + 2\delta], D) > 0.
$$

Using $T = \tau_D(\omega) + 2\delta$ in (6.1)–(6.2) we can choose $N$ such that for $n \geq N$, (6.4) holds for this choice of $\delta, \varepsilon$ and $T$. In particular, for $n \geq N$,

$$
\tau_D(\omega) < \lambda_n(\tau_D(\omega) + \delta) < \tau_D(\omega) + 2\delta,
$$

and

$$
|\omega_n(\tau_D(\omega) + \delta) - \omega(\lambda_n(\tau_D(\omega) + \delta))| < \frac{\varepsilon}{2}.
$$

Together these imply

$$
d(\omega_n(\tau_D(\omega) + \delta), D) > 0, \quad n \geq N,
$$

which in turn yields

$$
\tau_D(\omega_n) \leq \tau_D(\omega) + \delta, \quad n \geq N.
$$

Taking the lim sup as $n \to \infty$ and then letting $\delta \to 0$ yields (6.5). □

**Lemma 6.3.** If $D$ is a bounded domain that satisfies an exterior cone condition, or if $D$ is a cone. Then for all $x \in D$ and $0 < \alpha < 2$,

$$
P^\alpha_x(C_D \cap \{ X(\tau_D^-) \in D \}) = 1.
$$
Proof. If $D$ is bounded and satisfies a uniform exterior cone condition, it is known that

$$P^\alpha_x( X(\tau_D) \in \partial D ) = 0,$$

see Lemma 6 in [9]. If $D$ is a cone, we can apply Lemma 6 in [9] to $D \cap B_M(0)$ and letting $M \to \infty$, we get (6.6).

The proof of Theorem 2 in [15] implies

$$P^\alpha_x( X(\tau_D) \in \partial D, X(\tau_D) \in E ) = 0, \quad E \subseteq \overline{E} \subseteq \overline{D}^c,$$

(see the lines before the footnote on page 89). Combined with (6.6),

$$P^\alpha_x( X(\tau_D) \in \overline{D}^c, X(\tau_D) \in D ) = 1.$$

Thus to prove the lemma we need to show

$$P^\alpha_x( d(X[0, \tau_D - r], D^c ) > 0 \text{ for all rational } r < \tau_D ) = 1.$$

Let

$$D_n = \left\{ x \in D : d(x, D^c) > \frac{1}{n} \right\}$$

and observe $\tau_{D_n} \leq \tau_D$ increases to some limit $L \leq \tau_D$. By quasi-left continuity, $X(\tau_{D_n}) \to X(L)$ almost surely. One easily sees $X(L) \notin D$, i.e., $\tau_D \leq L$. Hence $\tau_D = L$, and the increasing limit of $\tau_{D_n}$ is $\tau_D$.

If for some rational $r < \tau_D$ we have

$$d(X[0, \tau_D - r], D^c ) = 0$$

then for some sequence $s_n \leq \tau_D - r$,

$$d( X_{s_n}, D^c ) \to 0.$$

It is no loss to assume $s_n$ converges, say to $s$. Choose $N$ such that for all $n \geq N$

$$\tau_D - r < \tau_{D_n} \leq \tau_D.$$

Given $n \geq N$, choose $M_n$ such that for all $m \geq M_n$,

$$d( X_{s_m}, D^c ) < \frac{1}{2n}.$$

Then for such $m$, $X_{s_m} \in D_n^c$, which forces

$$\tau_{D_n} \leq s_m \leq \tau_D - r.$$

Let $m \to \infty$ to get $\tau_{D_n} \leq s \leq \tau_D - r$, then let $n \to \infty$ to get $\tau_D = \lim_{n \to \infty} \tau_{D_n} \leq \tau_D - r$; contradiction. Thus (6.8) holds. \[\square\]

**Lemma 6.4.** Let $D$ be a bounded domain that satisfies an uniform exterior cone condition, and let $f$ be a bounded continuous function on $\mathbb{R}^d$. If $(x_n, \alpha_n)$ converges to $(x, \alpha)$ in $D \times (0, 2)$, then for each $T > 0$,

$$E^n_{x_n} [ f(X_{T \wedge \tau_D}) ] \to E^x_{\alpha} [ f(X_{T \wedge \tau_D}) ].$$
Proof. Let

\[ C_D(T) = C_D \cap \{ X(\tau_D) \in D \} \cap \{ \tau_D \neq T \} \cap \{ \lim_{s \to T} X_s = X_T \}. \]

Recall that the symmetric \( \alpha \)-stable process has no fixed discontinuities.
Then by the eigenfunction expansion of \( P_\alpha^x(\tau_D > t) \),

\[ P_\alpha^x(\tau_D \neq T, \lim_{s \to T} X_s = X_T) = 1. \]

Thanks to Lemma 6.3,

\[ P_\alpha^x(C_D(T)) = 1. \]

If we can show \( \omega \in C_D(T) \rightarrow \omega(T \wedge \tau_D) \) is continuous, then by an extension of the continuous mapping theorem (Theorem 5.1 in [4]), the desired conclusion will follow.

Let \( \omega \in C_D(T) \) and suppose \( \omega_n \) converges to \( \omega \) in \( D[0, \infty) \). Define

\[ t_n = T \wedge \tau_D(\omega_n) \quad t = T \wedge \tau_D(\omega). \]

By Lemma 6.2, \( \lim_{n \to \infty} t_n = t \). Applying Proposition 6.5 (a) on page 125 in [13],

\[ \lim_{n \to \infty} |\omega_n(t_n) - \omega(t)| = 0. \]

If \( |\omega_n(t_n) - \omega(t)| \) converges to 0, then clearly

\[ \lim_{n \to \infty} \omega_n(T \wedge \tau_D(\omega_n)) = \omega(T \wedge \tau_D(\omega)), \]

as desired.

On the other hand, if

\[ (6.9) \quad \lim_{n \to \infty} |\omega_n(t_n) - \omega(t^-)| = 0, \]

then we distinguish two cases: \( T > \tau_D(\omega) \) and \( T < \tau_D(\omega) \) (recall \( \omega \in C_D(T) \) implies \( \tau_D(\omega) \neq T \)).

Let us first assume \( T > \tau_D(\omega) \). Since \( \tau_D(\omega_n) \) converges to \( \tau_D(\omega) \) by Lemma 6.2, \( t_n = \tau_D(\omega_n) \) for large \( n \). Hence by (6.9)

\[ \lim_{n \to \infty} \omega_n(\tau_D(\omega_n)) = \lim_{n \to \infty} \omega_n(t_n) = \omega(t^-) = \omega(\tau_D(\omega)^-). \]

Notice if \( \lim_{n \to \infty} \omega_n(\tau_D(\omega_n)) = y \) exists, then \( y \in D^c \). But then \( \omega \in C_D(T) \) implies \( y = \omega(\tau_D(\omega)^-) \in D \); contradiction. Thus \( T > \tau_D(\omega) \) is not possible.

Finally, if \( T < \tau_D(\omega) \), then by Lemma 5.2

\[ T < \tau_D(\omega). \]
for \( n \) large. Since \( \omega \in \mathcal{C}_D(T) \), (6.9) becomes
\[
\omega_n(T) \to \omega(T^-) = \omega(T).
\]
We conclude
\[
\lim_{n \to \infty} \omega_n(T \land \tau_D(\omega_n)) = \lim_{n \to \infty} \omega_n(T) = \omega(T^-) = \omega(T) = \omega(T \land \tau_D(\omega)).
\]
In any event, we get the desired continuity. \( \square \)

7. Proof of Theorem 1.4

We will need the following lemma; it is formula (2.7) in [5]. Though the authors do not mention the statement concerning continuity in \( \alpha \), it is possible to trace back through the literature they cite to see the statement holds.

**Lemma 7.1.** If \( D \subseteq \mathbb{R}^d \) is a bounded Lipschitz domain, then for some positive continuous functions \( C(\alpha) \) and \( \beta(\alpha) \),
\[
E^\alpha_x(\tau_D) \leq C(\alpha) \delta_D^\beta(\alpha)(x), \quad \text{for all } x \in D.
\]
The next result immediately follows.

**Corollary 7.2.** Given a bounded Lipschitz domain \( D \) and compact \( K \times [a, b] \subseteq \overline{D} \times (0, 2) \),
\[
\sup \{ E^\alpha_x[\tau_D] : (x, \alpha) \in K \times [a, b] \} < \infty.
\]
Corollary 7.2 will allow us to get equicontinuity of the eigenfunctions near \( \partial D \). For the interior of \( D \) we need the following Krylov–Safanov type of theorem. Let
\[
G^\alpha_0 g(x) = E^\alpha_x \left[ \int_0^{\tau_D} g(X_t) dt \right]
\]
be the 0-resolvent of the killed symmetric \( \alpha \)-stable process in \( D \).

**Lemma 7.3.** Suppose \( g \) is bounded with support in \( \overline{D} \). Then for each \( x \in D \) there exist positive continuous functions \( C(\alpha) \) and \( \beta(\alpha) \), independent of \( g \), such that for all \( y \in D \)
\[
| G^\alpha_0 g(x) - G^\alpha_0 g(y) | \leq C(\alpha) \left[ \sup |G^\alpha g| + \sup |g| \right] |x - y|^\beta(\alpha).
\]
Proof. This theorem is essentially due to Bass and Levin (see their Proposition 4.2 on page 387). While they consider the 0-resolvent
\[ S_0g(x) = E_x \left[ \int_0^\infty g(X_t)dt \right], \]
their proof also works for the killed resolvent because their crucial formula
\[ S_0g(y) = E_y \left[ \int_{\tau_{B(x,r)}} g(X_t)dt \right] + E_y \left[ S_0g(X_{\tau_{B(x,r)}}) \right] \]
holds when \( S_0 \) is replaced by \( G_0^\alpha \) and \( E_y \) is replaced by \( E_\alpha y \), where \( r > 0 \) is such that \( B(x,r) \subset D \). Since we are restricted to \( D \) instead of \( \mathbb{R}^d \), the numbers \( C(\alpha) \) and \( \beta(\alpha) \) depend on \( x \), in contrast to the case treated by Bass and Levin. Moreover, it is a simple matter to go through their proof and see the numbers \( C(\alpha) \) and \( \beta(\alpha) \) can be chosen to depend continuously on \( \alpha \). □

Corollary 7.4. Assume \( D \) is bounded and Lipschitz. Then for each \( x \in D \) and \([a, b] \subseteq (0, 2)\), there exist positive \( C \) and \( r \) such that
\[ |G_0^\alpha g(x) - G_0^\alpha g(y)| \leq C|x - y|^r \sup |g| \]
for all \( y \in D \), \( \alpha \in [a, b] \) and bounded \( g \) with support in \( \overline{D} \).

Proof. By Corollary 7.2
\[ \sup |G^\alpha g| \leq \sup |g| \cdot \sup E_\alpha x(\tau_D) \]
\[ \leq \sup |g| \cdot C \]
where \( C \) is independent of \( \alpha \in [a, b] \) and \( g \). The result follows from this and the continuity of \( C(\alpha) \) and \( \beta(\alpha) \) from Theorem 7.3. □

At last we can prove Theorem 1.4. It is well known that
\[ 0 \leq p^\alpha_D(t, x, y) \leq p^\alpha(t, x, y). \]
Moreover,
\[ p^\alpha(t, x, y) \leq C(\alpha)t^{-d/\alpha} \]
where \( C(\alpha) \) is continuous in \( \alpha \) (see (2.1) in [8]).

Let \( \{\alpha_m\}_{m=1}^\infty \) be a sequence converging to \( \alpha \). Recall that for all \( \beta \in (0, 2) \), \( \{\varphi_m^\beta\}_{m=1}^\infty \) is an orthonormal set. Then thanks to the symmetry
of the heat kernel and \( \varphi_i^\beta(x) = e^{\lambda_i^\beta t} \int_D p_D^\beta(t, x, y) \varphi_i^\beta(y) dy \)

\[
\leq e^{\lambda_i^\beta t} \sqrt{\int_D \left[ p_D^\beta(t, x, y) \right]^2 dy} 
= e^{\lambda_i^\beta t} p_D^\beta(2t, x, x) 
\leq e^{\lambda_i^\beta t} \sqrt{C(\beta)} \left(2t\right)^{\alpha/\beta}.
\]

In particular, taking \( t = 1 \) and using Theorem 1.2,

\[
(7.3) \quad \sup_{x \in D, m \geq 1} \varphi_i^{\alpha_m}(x) \leq \sup_{m \geq 1} e^{\lambda_i^{\alpha_m}} \sqrt{C(\alpha_m)} \frac{2d/\alpha_m}{2d/\alpha_m} < \infty.
\]

Thus for each \( i \geq 1 \), the sequence \( \{\varphi_i^{\alpha_m}\}_{m=1}^\infty \) is uniformly bounded. Next we show the sequence \( \{\varphi_i^{\alpha_m}\}_{m=1}^\infty \) is pointwise equicontinuous on \( D \).

Indeed, since (7.4)

\[
\varphi_i^\beta = \lambda_i^\beta G_0^\beta \varphi_i^\beta,
\]

Corollary 7.4 implies that for each \( x \in D \) there exist \( C \) and \( r \) such that

\[
|\varphi_i^{\alpha_m}(x) - \varphi_i^{\alpha_m}(y)| = \lambda_i^{\alpha_m} |G_0^{\alpha_m} \varphi_i^{\alpha_m}(x) - G_0^{\alpha_m} \varphi_i^{\alpha_m}(y)| 
\leq C \left[ \sup_{m \geq 1} [\mu_i]^{\alpha_m/2} \right] \left[ \sup_{u \in D, m \geq 1} |\varphi_i^{\alpha_m}(u)| \right] |x - y|^r
\]

for all \( m \geq 1 \) and \( y \in D \). Thanks to (7.3), we get the desired equicontinuity for \( x \in D \).

As for \( x \in \partial D \), first notice (7.4) and Lemma 7.1 imply there are \( r \) and \( C \) independent of \( m \) such that for each \( z \in D \)

\[
|\varphi_i^{\alpha_m}(z)| \leq \left[ \sup_{m \geq 1} [\mu_i]^{\alpha_m/2} \right] \left[ \sup_{y \in D, m \geq 1} |\varphi_i^{\alpha_m}(y)| \right] E_z^{\alpha_m}(\tau_D) 
\leq C \left[ \delta_D(z) \right]^r.
\]

Thus \( \varphi_i^{\alpha_m} \) is continuous on \( \overline{D} \) with boundary value 0. Hence if \( x \in \partial D \) then

\[
|\varphi_i^{\alpha_m}(x) - \varphi_i^{\alpha_m}(y)| = |\varphi_i^{\alpha_m}(y)| 
\leq C \left[ \delta_D(y) \right]^r 
\leq C|x - y|^r.
\]
By Ascoli’s Theorem, the sequence $\{\varphi_{\alpha}^{\alpha_m}\}_{m=1}^{\infty}$ is precompact in $C(D)$. 

Next assume $\{\lambda_{\alpha}^{\alpha_m}\}_{m=1}^{\infty}$ converges to $\lambda$. We show any limit point $\varphi$ of the sequence $\{\varphi_{\alpha}^{\alpha_m}\}_{m=1}^{\infty}$ is an eigenfunction of $H_{\alpha}$ and the corresponding eigenvalue is $\lambda$. Choose a subsequence $\beta_r = \alpha_{m_r}$ such that, as $r \to \infty$, $\varphi_{i}^{\beta_r}$ converges uniformly to $\varphi$ on $\overline{D}$. Since $\varphi_{i}^{\beta_r}$ and $\varphi$ are 0 on $\partial D$, we can extend them to all of $\mathbb{R}^d$ by taking them to be 0 outside $D$. Then

$$
E_x^\beta_r \left[ \varphi_{i}^{\beta_r}(X_{t\wedge \tau_D}) \right] = E_x^\beta_r \left[ \varphi_{i}^{\beta_r}(X_t) I_{\tau_D > t} \right]
$$

and $\varphi_{i}^{\beta_r}$ converges to $\varphi$ uniformly on $\mathbb{R}^d$. Thus we have

$$
e^{-\lambda^\beta_r t} \varphi_{i}^{\beta_r}(x) = \int_D p_D^\beta(t, x, y) \varphi_{i}^{\beta_r}(y) dy
$$

(7.5)

$$
e^{-\lambda t} \varphi(x) = E_x^\alpha \left[ \varphi(X_{t\wedge \tau_D}) \right] = E_x^\alpha \left[ \varphi(X_t) I_{\tau_D > t} \right]
$$

Lemma 6.4 and the uniform convergence of $\varphi_{i}^{\alpha_m}$ to $\varphi$ imply

$$
\lim_{r \to \infty} E_x^\beta_r \left[ \varphi_{i}^{\beta_r}(X_{t\wedge \tau_D}) - \varphi(X_{t\wedge \tau_D}) \right] + E_x^\beta_r \left[ \varphi(X_{t\wedge \tau_D}) \right] = E_x^\alpha \left[ \varphi(X_{t\wedge \tau_D}) \right].
$$

Since the left hand side of (7.5) converges to $e^{-\lambda t} \varphi(x)$, we conclude

$$
e^{-\lambda t} \varphi(x) = E_x^\alpha \left[ \varphi(X_{t\wedge \tau_D}) \right] = E_x^\alpha \left[ \varphi(X_t) I_{\tau_D > t} \right] = \int_D p_D^\alpha(t, x, y) \varphi(y) dy.
$$

Hence $\varphi$ is an eigenfunction of $H_{\alpha}$, and the corresponding eigenvalue is $\lambda$. 

□

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