ERROR ANALYSIS OF A BACKWARD EULER POSITIVE PRESERVING STABILIZED SCHEME FOR A CHEMOTAXIS SYSTEM

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Abstract. For a Keller-Segel model for chemotaxis in two spatial dimensions we consider a modification of a positivity preserving fully discrete scheme using a local extremum diminishing flux limiter. We discretize space using piecewise linear finite elements on an quasiuniform triangulation of acute type and time by the backward Euler method. We assume that initial data are sufficiently small in order not to have a blow-up of the solution. Under appropriate assumptions on the regularity of the exact solution and the time step parameter we show existence of the fully discrete approximation and derive error bounds in $L^2$ for the cell density and $H^1$ for the chemical concentration. We also present numerical experiments to illustrate the theoretical results.

1. Introduction

We shall consider a Keller-Segel system of equations of parabolic-parabolic type, where we seek $u = u(x,t)$ and $c = c(x,t)$ for $(x,t) \in \Omega \times [0,T]$, satisfying

$$
\begin{align*}
\begin{cases}
  u_t = \Delta u - \lambda \div (u\nabla c), & \text{in } \Omega \times [0,T], \\
  c_t = \Delta c - c + u, & \text{in } \Omega \times [0,T], \\
  \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial \Omega \times [0,T], \\
  u(\cdot,0) = u^0, & \text{in } \Omega, \\
  c(\cdot,0) = c^0, & \text{in } \Omega,
\end{cases}
\end{align*}
$$

(1.1)

where $\Omega \subset \mathbb{R}^2$ is a convex bounded domain with boundary $\partial \Omega$, $\nu$ is the outer unit normal vector to $\partial \Omega$, $\partial / \partial \nu$ denotes differentiation along $\nu$ on $\partial \Omega$, $\lambda$ is a positive constant and $u^0, c^0 \geq 0$, $u^0 \neq 0$.

The chemotaxis model (1.1) describes the aggregation of slime molds resulting from their chemotactic features, cf. e.g. [18]. The function $u$ is the cell density of cellular slime molds, $c$ is the concentration of the chemical substance secreted by molds themselves, $\epsilon = u - c$ is the ratio of generation or extinction, and $\lambda$ is a chemotactic sensitivity constant.

There exists an extensive mathematical study of chemotaxis models, cf. e.g., [24, 15, 16, 17, 29] and references therein. It is well-known that the solution of (1.1) may blow up in finite time. However, if $\|u_0\|_{L^1(\Omega)} \leq 4\pi \lambda^{-1}$, the solution $(u,c)$ of (1.1) exists for all time and is bounded in $L^\infty$, cf. e.g. [23].

A key feature of the system (1.1) is the conservation of the solution $u$ in $L^1$ norm,

$$
\begin{align*}
\|u(t)\|_{L^1(\Omega)} = \|u^0\|_{L^1(\Omega)}, & \quad 0 \leq t \leq T,
\end{align*}
$$

(1.2)

which is an immediate result of the preservation of non-negativity of $u$,

$$
u^0 \geq 0, \neq 0 \text{ in } \Omega \Rightarrow u > 0 \text{ in } \Omega \times (0,T],
$$

(1.3)

and the conservation of total mass

$$
\int_{\Omega} u(x,t) \, dx = \int_{\Omega} u_0(x) \, dx, \quad 0 \leq t \leq T.
$$

(1.4)

Capturing blowing up solutions numerically is a challenging problem and many numerical methods have been proposed to address this. The main difficulty in constructing suitable numerical schemes is to preserve several essential properties of the Keller-Segel equations such as positivity, mass conservation, and energy dissipation.

Some numerical schemes were developed with positive-preserving conditions, cf. e.g. [14, 9, 8], which depend on a particular spatial discretization and impose CFL restrictions on the time step. Other approaches include, finite-volume based numerical methods, [9, 8], high-order discontinuous Galerkin methods, [12, 13, 20], a flux corrected finite element method, [28], and a novel numerical method based on symmetric reformulation of the chemotaxis system, [21]. For a more detailed review on recent developments of numerical methods for chemotaxis problems, we refer to [10, 30].

In order to maintain the total mass and the non-negativity of the numerical approximations of the system (1.1), Saito in [26] proposed and analyzed a fully discrete method that uses an upwind finite element scheme in space and backward Euler method in time. His proposed finite element scheme made use of Baba and Tabata’s upwind approximation, see [1]. Strehl et al. in [28] proposed a slightly different approach. The stabilization was implemented at a pure algebraic level via algebraic flux correction, see [19]. This stabilization technique can be
applied to high order finite element methods and maintain the mass conservation and the non-negativity of the solution.

In the variational form of (1.1), we seek \( u(\cdot, t) \in H^1 \) and \( c(\cdot, t) \in H^1 \), for \( t \in [0, T] \), such that

\[
\begin{align*}
(u_t, v) + (\nabla u - \lambda u \nabla c, \nabla v) = 0, \quad \forall v \in H^1, \\
(c_t, v) + (\nabla c, \nabla v) + (c - u, v) = 0, \quad \forall v \in H^1,
\end{align*}
\]

where \( (f, g) = \int_{\Omega} f g \, dx \).

In our analysis we consider regular triangulations \( T_h = \{ K \} \) of \( \Omega \), with \( h = \max_{K \in T_h} h_K, h_K = \text{diam}(K) \), and the finite element spaces

\[ S_h := \{ \chi \in C^0 : \chi|_K \in P_1, \forall K \in T_h \}, \]

where \( C^0 = C^0(\Omega) \) denotes the continuous functions on \( \Omega \).

A semidiscrete approximation of the variational problem (1.5) is: Find \( u_h(t) \in S_h \) and \( c_h(t) \in S_h \), for \( t \in [0, T] \), such that

\[
\begin{align*}
(u_{h,t}, \chi) + (\nabla u_h - \lambda u_h \nabla c_h, \nabla \chi) = 0, \quad \forall \chi \in S_h, \quad \text{with} \ u_h(0) = u_0, \\
(c_{h,t}, \chi) + (\nabla c_h, \nabla \chi) + (c_h - u_h, \chi) = 0, \quad \forall \chi \in S_h, \quad \text{with} \ c_h(0) = c_0,
\end{align*}
\]

where \( u_0, c_0 \in S_h \).

We now formulate (1.7) in matrix form. Let \( Z_h = \{ Z_j \}_{j=1}^{N} \) be the set of nodes in \( T_h \) and \( \{ \phi_j \}_{j=1}^{N} \subset \mathcal{S}_h \) the corresponding nodal basis, with \( \phi_j(Z_h) = \delta_{ij} \). Then, we may write

\[
\begin{align*}
u_h(t) = \sum_{j=1}^{N} \alpha_j(t) \phi_j, \quad \text{with} \ u_h^0 = \sum_{j=1}^{N} \alpha_j^0 \phi_j \quad \text{and} \ c_h(t) = \sum_{j=1}^{N} \beta_j(t) \phi_j, \quad \text{with} \ c_h^0 = \sum_{j=1}^{N} \beta_j^0 \phi_j.
\end{align*}
\]

Thus, the semidiscrete problem (1.7) may then be expressed, with \( \alpha = (\alpha_1, \ldots, \alpha_N)^T \) and \( \beta = (\beta_1, \ldots, \beta_N)^T \), as

\[
\begin{align*}
M \alpha'(t) + (S - T \beta) \alpha(t) = 0, \quad \text{for} \ t \in [0, T], \quad \text{with} \ \alpha(0) = \alpha^0, \\
M \beta'(t) + (S + M) \beta(t) = M \alpha(t), \quad \text{for} \ t \in [0, T], \quad \text{with} \ \beta(0) = \beta^0,
\end{align*}
\]

where \( \alpha^0 = (\alpha_1^0, \ldots, \alpha_N^0)^T, \beta^0 = (\beta_1^0, \ldots, \beta_N^0)^T, \ M = (m_{ij}), \ m_{ij} = (\phi_i, \phi_j), \ S = (s_{ij}), \ s_{ij} = (\nabla \phi_i, \nabla \phi_j), \ T = (\tau_{ij}(\beta)) \) and \( \tau_{ij}(\beta) = \lambda \sum_{\ell=1}^{N} \beta_\ell (\phi_j \nabla \phi_\ell, \nabla \phi_i) \), for \( i, j = 1, \ldots, N \).

We will often suppress the index \( \beta \) in the coefficients \( \tau_{ij} = \tau_{ij}(\beta) \) and in \( \mathbf{T} = T \beta \). The matrices \( M \) and \( S \) are both symmetric and positive definite, however \( T \) due to the chemotactical flux \( \lambda u(t) \nabla c(t) \) is not symmetric.

Note that the semidiscrete solutions \( u_h(t), c_h(t) \) of (1.7) are nonnegative if and only if the coefficient vectors \( \alpha(t), \beta(t) \) are nonnegative elementwise. In order to ensure nonnegativity, we may employ the lumped mass method, which results from replacing the mass matrix \( M \) in (1.8)-(1.9) with a diagonal matrix \( M_L \) with elements \( \sum_{j=1}^{N} m_{ij} \).

A sufficient condition for \( \alpha(t) \) to be nonnegative elementwise is that the off diagonal elements of \( S - \mathbf{T} \) are nonpositive. Further, for \( \beta(t) \) to be nonnegative elementwise, it suffices that the off diagonal elements of \( S \) are nonnegative.

Assuming, that \( T_h \) satisfies an acute condition, i.e., all interior angles of a triangle \( K \in T_h \) are less than \( \pi/2 \), we have that \( s_{ij} \leq 0, \) cf. e.g., [11]. Then, in order to ensure that the off diagonal elements of \( S - \mathbf{T} \) are nonpositive we may add an artificial diffusion operator \( \mathbf{D} = D \beta \). This technique is commonly used in conversation laws, cf. e.g., [19] and references therein. This modification of the semidiscrete scheme (1.5) is proposed in [28]. This scheme is often called low-order scheme since we introduce an error which manifests in the order of convergence.

To improve the convergence order of the low-order scheme, Strehl et al. in [28] proposed another scheme, which is called algebraic flux correction scheme or AFC scheme. To derive the AFC scheme we decompose the error, introduced in the low-order scheme by adding the artificial diffusion operator, into internal nodal fluxes. Then we appropriately restore high accuracy in regions where the solution does not violate the non-negativity. There exists various algorithms to limit the internal nodal fluxes. We will consider limiters that will satisfy the discrete maximum principle and linearity preservation on arbitrary meshes, as the one proposed by G. Barrenechea et al. in [4].

Our purpose here, is to analyze fully discrete schemes, for the approximation of (1.1), by discretizing in time the low-order scheme and the AFC scheme using the backward Euler method. We will consider the case where the solution of (1.1) remains bounded for all \( t \geq 0 \), therefore we will assume that \( \|u(t)\|_{L^1} \leq 4 \lambda^{-1} \pi \).

Our analysis of the stabilized schemes is based on the corresponding one employed by Barrenechea et al. in [3]. In order to show existence of the solutions of the nonlinear fully discrete schemes, we employ a fixed point argument and demonstrate that our approximations remain uniformly bounded, provided that \( T_h \) is quasiuniform and our time step parameter \( k \), is such that \( k = O(h^{1+\epsilon}) \), with \( 0 < \epsilon < 1 \).
We shall use standard notation for the Lebesgue and Sobolev spaces, namely we denote \( W^m_2 = W^m_2(\Omega) \), \( H^m = W^m_2 \), \( L^p = L^p(\Omega) \), and with \( \| \cdot \|_{m,p} = \| \cdot \|_{W^m_2} \). \( \| \cdot \|_m = \| \cdot \|_{H^m} \), \( \| \cdot \|_{L^p} = \| \cdot \|_{L^p(\Omega)} \), for \( m \in \mathbb{N} \) and \( p \in [1, \infty] \), the corresponding norms.

The fully discrete schemes we consider approximate \((u^n, c^n)\) by \((U^n, C^n) \in S_h \times S_h\) where \( u^n = u(\cdot, t^n) \), \( c^n = c(\cdot, t^n) \), \( t^n = nk \), \( n = 0, \ldots, N_T \) and \( N_T \in \mathbb{N} \), \( N_T \geq 1 \), \( k = T/N_T \). Assuming that the solutions \((u, c)\) of \((1.1)\) are sufficiently smooth, with \( u \in W^2_p \), with \( p \in (2, \infty) \) and \( c \in W^{2,\infty} \), we derive error estimates of the form

\[
\|U^n - u^n\|_{L^2} \leq C(k + k^{-1/2}h^2 + h^{3/2}|\log h|),
\]

\[
\|C^n - c^n\|_{1} \leq C(k + k^{-1/2}h^2 + h).
\]

The paper is organized as follows: In Section 2 we introduce notation and the semidiscrete low-order scheme and the AFC scheme for the discretization of \((1.1)\). Further, we prove some auxiliary results for the stabilization terms, that we will employ in the analysis that follows and rewrite the low-order and AFC scheme, as general semidiscrete scheme. In Section 3, we discretize the general semidiscrete scheme in time, using the backward Euler method. For a sufficiently smooth solution of \((1.1)\) and \( k = O(h^{1+\epsilon}) \), with \( \epsilon < 1 \), we demonstrate that there exists a unique discrete solution which remains bounded and derive error estimates in \( L^2 \) for the cell density and \( H^1 \) for the chemical concentration. In Section 4, we show that the discrete solution preserves nonnegativity. Finally, in Section 5, we present numerical experiments, illustrating our theoretical results.

## 2. Preliminaries

### 2.1. Mesh assumptions.
We consider a family of regular triangulations \( T_h = \{K\} \) of a convex polygonal domain \( \Omega \subset \mathbb{R}^2 \). We will assume that the family \( T_h \) satisfies the following assumption.

**Assumption 2.1.** Let \( T_h = \{K\} \) be a family of regular triangulations of \( \Omega \) such that any edge of any \( K \) is either a subset of the boundary \( \partial \Omega \) or an edge of another \( K \in T_h \), and in addition

1. \( T_h \) is shape regular, i.e., there exists a constant \( \gamma > 0 \), independent of \( K \) and \( T_h \), such that

\[
\frac{h_K}{\varrho_K} \leq \gamma, \quad \forall K \in T_h,
\]

where \( \varrho_K = \text{diam}(B_K) \), and \( B_K \) is the inscribed ball in \( K \).

2. The family of triangulations \( T_h \) is quasiuniform, i.e., there exists constant \( \varrho > 0 \) such that

\[
\frac{\max_{K \in T_h} h_K}{\min_{K \in T_h} h_K} \leq \varrho, \quad \forall K \in T_h,
\]

3. All interior angles of \( K \in T_h \) are less than \( \pi/2 \).

Let \( N_h := \{i : Z_i \text{ a node of the triangulation } T_h\} \), \( E_h \) be the set of all edges of the triangulation \( T_h \) and \( e_{ij} \in E_h \) denotes an edge of \( T_h \) with endpoints \( Z_i, Z_j \in Z_h \). We denote \( \omega_i \), the collection of triangles with a common edge \( e \in E_h \), see Fig. 2.1, and \( \omega_i, i \in N_h \), the collection of triangles with a common vertex \( Z_i \), i.e. \( \omega_i = \cup_{Z \in E_h} K \), see Fig. 2.1. The sets \( Z_h(\omega) \) and \( E_h(\omega) \) contain the vertices and the edges, respectively, of a subset of \( \omega \subset T_h \) and \( Z^i_h := \{j : Z_j \in Z_h, \text{adjacent to } Z_i\} \). Using the fact that \( T_h \) is shape regular, there exists a constant \( \kappa_\gamma \), independent of \( h \), such that the number of vertices in \( Z^i_h \) is less than \( \kappa_\gamma \), for \( i = 1, \ldots, N \).

Since \( T_h \) satisfies \((2.2)\), we have for all \( \chi \in S_h \), cf., e.g., [6, Chapter 4],

\[
\|\chi\|_{L^\infty} + \|\nabla\chi\|_{L^2} \leq C h^{-1}\|\chi\|_{L^2} \quad \text{and} \quad \|\nabla\chi\|_{L^\infty} \leq C h^{-1}\|\nabla\chi\|_{L^2}.
\]

Further, in our analysis, we will employ the following trace inequality which holds for \( K \in T_h \), cf. e.g., [6, Theorem 1.6.6] together with a homogeneity argument, cf. e.g., [25, Theorem 3.1.2],

\[
\|v\|_{L^1(\partial K)} \leq C_\gamma \|v\|_{H^1(K)},
\]

where \( C_\gamma \) a positive constant that depends on \( \gamma \).

### 2.2. Stabilized semidiscrete methods.
In the sequel we will present two stabilized semidiscrete schemes for the numerical approximation of \((1.1)\), namely the low order scheme and the AFC scheme, which have been proposed in [28].
2.2.1. Low order scheme. The semidiscrete problem (1.7) may be expressed in the matrix form (1.8)-(1.9), where the matrices $M$ and $S$ are symmetric and positive definite. However $T_{\beta}$ is not symmetric, but as we will demonstrate in the sequel, cf. Lemma 3.4, it has a zero-column sum.

For a function $v \in C^0$, let $v = (v_1, \ldots, v_N)^T$ denote the vector with coefficients its nodal values $v_i = v(Z_i)$, $Z_i \in Z_h$, $i = 1, \ldots, N$. We will often express the coefficients $\tau_{ij}$, $i, j = 1, \ldots, N$, of $T_{\beta}$ as functions of an element $\psi \in S_h$, $\tau_{ij} = \tau_{ij}(\psi) = \tau_{ij}(\psi)$, such that $\psi = \sum_j \psi_j \phi_j \in S_h$ and $\psi = (\psi_1, \ldots, \psi_N)^T$. Then the elements of $T_{\beta} = T(\tau_{ij})$, may be expressed equivalently as,

$$
\tau_{ij} = \tau_{ij}(\psi) = \tau_{ij}(\psi) = \lambda(\phi_\ell \nabla \psi, \nabla \phi_\ell) = \lambda \sum_{\ell=1}^{N} \psi_j(\phi_\ell \nabla \phi_\ell, \nabla \phi_\ell).
$$

(2.5)

In order to preserve non-negativity of $\alpha(t)$ and $\beta(t)$, a low order semi-discrete scheme of minimal model has been proposed, cf. e.g. [28], where $M$ is replaced by the corresponding lumped mass matrix $M_L$ and an artificial artificial diffusion operator $D = D_{\beta} = (d_{ij})$ is added to $T$, to eliminate all negative off-diagonal elements of $T$, so that $T + D \geq 0$, elementwise. Thus, assuming that $T_h$ satisfies an acute condition, i.e., all interior angles of a triangle $K \in T_h$ are less than $\pi/2$, gives that $s_{ij} \leq 0$ and hence, the off diagonal elements of $S - T - D$ are nonpositive, $s_{ij} - \tau_{ij} - d_{ij} \leq 0$, $i \neq j$, $i, j = 1, \ldots, N$. However, note that assuming $T_h$ to be acute is not a necessary condition to preserve non-negativity of $\alpha(t)$ or $\beta(t)$.

Also, we will often suppress the index $\beta$ in the coefficients $d_{ij} = d_{ij}(\beta)$, $i, j = 1, \ldots, N$, or express them as functions of an element $\psi \in S_h$, $d_{ij}(\psi) = d_{ij}(\psi)$, such that $\psi = \sum_j \psi_j \phi_j \in S_h$ and $\psi = (\psi_1, \ldots, \psi_N)^T$. Since, we would like our scheme to maintain the mass, $D$ must be symmetric with zero row and column sums, cf. [28], which is true if $D = (d_{ij})_{i,j=1}^{N}$ is defined by

$$
d_{ij} := \max\{-\tau_{ij}, 0, -\tau_{ji}\} = d_{ji} \geq 0, \quad \forall j \neq i \quad \text{and} \quad d_{ii} := -\sum_{j \neq i} d_{ij}.
$$

(2.6)

Thus the resulting system for the approximation of (1.1) is expressed as follows, we seek $\alpha(t), \beta(t) \in \mathbb{R}^N$ such that, for $t \in [0, T]$,

$$
M_L \alpha'(t) + (S - T_{\beta} - D_{\beta}) \alpha(t) = 0, \quad \text{with} \quad \alpha(0) = \alpha^0,
$$

(2.7)

$$
M_L \beta'(t) + (S + M_L) \beta(t) = M_L \alpha(t), \quad \text{with} \quad \beta(0) = \beta^0.
$$

(2.8)

Let for $w \in S_h$, $d_h(w; v) : C^0 \times C^0 \rightarrow \mathbb{R}$, be a bilinear form defined by

$$
d_h(w; v, z) := \sum_{i,j=1}^{N} d_{ij}(w)(v_i - v_{ij}) z_i, \quad \forall v, z \in C^0,
$$

and $(\cdot, \cdot)_h$ be an inner product in $S_h$ that approximates $(\cdot, \cdot)$ and is defined by

$$
(\psi, \chi)_h = \sum_{K \in T_h} \frac{Q_h(K)}{|K|} (\psi, \chi)_h + Q_h(K) (g, \nabla \psi) = \frac{1}{3} \sum_{Z \in Z_h(K)} g(Z) \approx \int_{Z} g(x) dx,
$$

(2.10)

with $Z_h(K)$ the vertices of $K \in T_h$ and $|K|$ the area of $K \in T_h$. Then following [3], the coupled system (2.7)-(2.8) can be rewritten in the following variational formulation: Find $u_h(t), c_h(t) \in S_h$ such that

$$
(u_{h, t}, \chi)_{h} + (\nabla u_h - \lambda u_h \nabla c_h, \nabla \chi) + d_h(c_h; u_h, \chi) = 0, \quad \forall \chi \in S_h,
$$

(2.11)

$$
(c_{h, t}, \chi)_{h} + (\nabla c_h, \nabla \chi) + (c_h - u_h, \chi)_{h} = 0, \quad \forall \chi \in S_h,
$$

(2.12)

with $u_h(0) = u_0 \in S_h$ and $c_h(0) = c_0 \in S_h$.

We can easily see that $(\cdot, \cdot)_h$ induces an equivalent norm to $\| \cdot \|_{L^2}$ on $S_h$. Thus, there exist constants $C, C'$ independent on $h$, such that

$$
C \| \chi \|_h \leq \| \chi \|_{L^2} \leq C' \| \chi \|_h, \quad \text{with} \quad \| \chi \|_h = (\chi, \chi)_h^{1/2}, \quad \forall \chi \in S_h.
$$

(2.13)

2.2.2. Algebraic flux correction scheme. The replacement of the standard FEM discretization (1.8)-(1.9) by the low-order scheme (2.7)-(2.8) ensures nonnegativity but introduces an error which manifests the order of convergence, cf. e.g. [27, 19]. Thus, following Strehl et al. [27], one may “correct” the semidiscrete scheme (2.7)-(2.8) by introducing a flux correction term. Hence, we also consider an algebraic flux correction (AFC) scheme, which involves the decomposition of this error into internodal fluxes, which can be used to restore high accuracy in regions where the solution is well resolved and no modifications of the standard FEM are required. There exists various algorithms to implement an AFC scheme. Here we will follow the one proposed by G. Barrenacpela et. al. in [4].

The AFC scheme is constructed in the following way. Let $f = (f_1, \ldots, f_N)^T$ denote the error of inserting the operator $D_{\beta}$ in (1.8), i.e., $f(\alpha, \beta) = -D_{\beta} \alpha$. Using the zero row sum property of matrix $D_{\beta}$, cf. (2.6), we can
show that the residual admits a conservative decomposition into internodal fluxes,
\begin{equation}
    f_i = \sum_{j \neq i} f_{ij}, \quad f_{ij} = -f_{ij}, \quad i = 1, \ldots, N,
\end{equation}
where the amount of mass transported by the raw anti-diffusive flux \( f_{ij} \) is given by
\begin{equation}
    f_{ij} := f_{ij}(\alpha, \beta) = (\alpha_i - \alpha_j) d_{ij}(\beta), \quad \forall j \neq i, \quad i, j = 1, \ldots, N.
\end{equation}
For the rest of this paper we will call the internodal fluxes as anti-diffusive fluxes. Some of these anti-diffusive fluxes are harmless but others may be responsible for the violation of non-negativity. Such fluxes need to be canceled or limited so as to keep the scheme non-negative. Thus, every anti-diffusive flux \( f_{ij} \) is multiplied by a solution-dependent correction factor \( a_{ij} \in [0, 1] \), to be defined in the sequel, before it is inserted into the equation. Hence, the AFC scheme is the following: We seek \( \alpha(t), \beta(t) \in \mathbb{R}^N \) such that, for \( t \in [0, T] \),
\begin{equation}
    M_L \alpha'(t) + (S - T_{\beta} - D_{\beta}) \alpha(t) = \bar{I}(\alpha(t), \beta(t)), \quad \text{with } \alpha(0) = \alpha^0,
\end{equation}
\begin{equation}
    M_L \beta'(t) + (S + M_L) \beta(t) = M_L \alpha(t), \quad \text{with } \beta(0) = \beta^0,
\end{equation}
where \( \bar{I}(\alpha(t), \beta(t)) = (\bar{I}_1, \ldots, \bar{I}_N)^T \), with
\begin{equation}
    \bar{I}_i := \bar{I}_i(\alpha(t), \beta(t)) = \sum_{j \neq i} a_{ij} f_{ij}, \quad i = 1, \ldots, N,
\end{equation}
and \( a_{ij} \in [0, 1] \) are appropriately defined in view of the anti-diffusive fluxes \( f_{ij} \).

In order to determine the coefficients \( a_{ij} \), one has to fix a set of nonnegative coefficients \( q_i \). In principle the choice of these parameters \( q_i \) can be arbitrary. But efficiency and accuracy can dictate a strategy, which does not depend on the fluxes \( f_{ij} \) but on the type of problem ones tries to solve and the mesh parameters. We will not elaborate more on the choice of \( q_i \), for a more detail presentation we refer to [19] and [3] and the references therein. In the sequel we will employ two particular choices of \( q_i \), cf. Lemma 2.7.

To ensure that the AFC scheme maintains the nonnegativity property, it is sufficient to choose the correction factors \( a_{ij} \) such that the sum of anti-diffusive fluxes is constrained, (cf. e.g., [19]), as follows. Let \( q_i > 0, i \in \mathcal{N}_h \), be given constants that do not depend on \( \alpha \) and
\begin{equation}
    Q_i^+ = q_i (\alpha_i^{\max} - \alpha_i) \quad \text{and} \quad Q_i^- = q_i (\alpha_i^{\min} - \alpha_i), \quad i \in \mathcal{N}_h,
\end{equation}
with \( \alpha_i^{\max}, \alpha_i^{\min} \) the local maximum and local minimum at \( \omega_i \). Then \( a_{ij} \) should satisfy
\begin{equation}
    Q_i^- \leq \sum_{j \neq i} a_{ij} f_{ij} \leq Q_i^+, \quad i \in \mathcal{N}_h.
\end{equation}

Remark 2.1. Note that if all the correction factors \( a_{ij} = 0 \), then the AFC scheme (2.16)-(2.17) reduces to the low-order scheme (2.7)-(2.8).

Remark 2.2. The criterion (2.20) by which the correction factors are chosen, implies that the limiters used in (2.16)-(2.17) guarantee that the scheme is non-negative. In fact, if \( \alpha_i \) is a local maximum in \( \omega_i \), then (2.20) implies the cancellation of all positive fluxes. Similarly, all negative fluxes are canceled if \( \alpha_i \) is a local minimum in \( \omega_i \). In other words, a local maximum cannot increase and a local minimum cannot decrease. As a consequence, \( a_{ij} f_{ij} \) cannot create an undershoot or overshoot at node \( i \).

We shall compute the correction factors \( a_{ij} \) using Algorithm 2.3, which has originally proposed by Kuzmin, cf. [19, Section 4]. Then we have that
\begin{equation}
    Q_i^- \leq R_i^- P_i^- \leq \sum_{j \neq i} a_{ij} f_{ij} \leq R_i^+ P_i^+ \leq Q_i^+, \quad i \in \mathcal{N}_h,
\end{equation}
which implies that (2.20) holds.

Algorithm 2.3 (Computation of correction factors \( a_{ij} \)). Given data:
\begin{enumerate}[1.]
    \item The positive coefficients \( q_i \), such that \( q_i = \mathcal{O}(h) \), \( i \in \mathcal{N}_h \).
    \item The fluxes \( f_{ij} \), \( i \neq j, \ i, j = 1, \ldots, N \).
    \item The coefficients \( \alpha_j, \beta_j, j = 1, \ldots, N \).
\end{enumerate}

Computation of factors \( a_{ij} \):
\begin{enumerate}[1.]
    \item Compute the limited sums \( P_i^\pm := P_i^\pm(\alpha, \beta), \ i \in \mathcal{N}_h \), of positive and negative anti-diffusive fluxes
        \begin{equation}
            P_i^+ = \sum_{j \neq i} \max\{0, f_{ij}\} \quad \text{and} \quad P_i^- = \sum_{j \neq i} \min\{0, f_{ij}\}, \quad i \in \mathcal{N}_h.
        \end{equation}
    \item Retrieve the local extremum diminishing upper and lower bounds \( Q_i^\pm := Q_i^\pm(\alpha), \ i \in \mathcal{N}_h \),
        \begin{equation}
            Q_i^+ = q_i (\alpha_i^{\max} - \alpha_i), \quad Q_i^- = q_i (\alpha_i^{\min} - \alpha_i), \quad i \in \mathcal{N}_h,
        \end{equation}
        where \( \alpha_i^{\max}, \alpha_i^{\min} \) are the local maximum and local minimum in \( \omega_i \),
\end{enumerate}
(3) Compute the coefficients \( \bar{a}_{ij} \), for \( j \neq i \), \( i, j \in \mathcal{N}_h \), which are defined by

\[
(2.21) \quad R_i^+ = \min \left\{ 1, \frac{Q_i^+}{P_i^+} \right\}, \quad R_i^- = \min \left\{ 1, \frac{Q_i^-}{P_i^-} \right\} \quad \text{and} \quad \bar{a}_{ij} = \begin{cases} R_i^+, & \text{if } f_{ij} > 0, \\ 1, & \text{if } f_{ij} = 0, \\ R_i^-, & \text{if } f_{ij} < 0. \end{cases}
\]

Note that if \( P_i^- = 0 \) or \( P_i^+ = 0 \), then we define \( R_i^- = 1 \) or \( R_i^+ = 1 \), respectively.

(4) Finally, the requested coefficients \( a_{ij} \), for \( j \neq i \), \( i, j \in \mathcal{N}_h \) are defined by

\[
a_{ij} := \min \{ \bar{a}_{ij}, \bar{a}_{ji} \}, \quad \text{for } i, j \in \mathcal{N}_h.
\]

Remark 2.4. Following the definition of \( \tau_{ij} \) in (2.5), we may express \( a_{ij} = a_{ij}(v, w) = a_{ij}(v, w) \), with \( v = \sum_j v_j \phi_j, \ w = \sum_j w_j \phi_j \) and \( v = (v_1, \ldots, v_N)^T, \ w = (w_1, \ldots, w_N)^T \in \mathbb{R}^N \).

Remark 2.5. There exist \( \gamma_i \in \mathbb{R}, i \in \mathcal{N}_h \), cf. [4, Section 6], such that

\[
(2.22) \quad v_i - v_i^{\min} \leq \gamma_i(v_i^{\max} - v_i), \quad \forall v \in P_1(\mathbb{R}^2),
\]

for \( v_i = v(\Omega^i) \) and \( v_i^{\max} \) and \( v_i^{\min} \) the local maximum and local minimum, respectively, in \( \omega_i \), where \( \omega_i \) is the union of triangles with \( Z_i \) as a common vertex, see the right patch of Fig. 2.1 and

\[
\gamma_i = \max_{Z_j \in \partial \omega_i} |Z_i - Z_j|/\text{dist}(Z_i, \partial \omega_i^{\text{conv}}),
\]

and \( \omega_i^{\text{conv}} \) is the convex hull of \( \omega_i \). Note that if \( \omega_i \) is symmetric with respect to \( Z_i \), then \( \gamma_i = 1 \), see [4].

Definition 2.6. The limiter \( \bar{a}_{ij} \) defined in (2.21) has the linearity preservation property if

\[
(2.23) \quad \bar{a}_{ij}(v, w) = 1, \quad \text{for } i, j \in \mathcal{N}_h, \text{ and } v \in P_1(\mathbb{R}^2), \ w \in \mathcal{S}_h.
\]

Lemma 2.7. Let the positive coefficients \( q_i, i \in \mathcal{N}_h \), in Algorithm 2.3 be defined by

\[
(2.24) \quad q_i := \gamma_i \sum_{j \neq i} d_{ij}, \quad i \in \mathcal{N}_h,
\]

with \( \gamma_i \) defined in (2.22), then the linearity preservation property (2.23) is satisfied. Further, if there exists \( M > 0 \) such that \( \|\nabla w\|_{L^\infty} \leq M \) for \( w \in \mathcal{S}_h \) and the constants \( q_i, i \in \mathcal{N}_h \), in Algorithm 2.3, are defined by

\[
(2.25) \quad q_i := \gamma_i \frac{m_i}{\nu} \text{ with } \nu \in (0, 1) \text{ and } \nu = O(h^{1+\epsilon}), \ \epsilon \in (0, 1),
\]

with \( m_i \) the diagonal elements of \( M_L \), then there exists \( h_M > 0 \) such that for \( h < h_M, (2.23) \) is satisfied.

Proof. The proof is based on the fact that the linearity preservation is equivalent to have

\[
(2.26) \quad Q_i^+ > P_i^+ \text{ if } f_{ij} > 0 \text{ and } Q_i^- < P_i^- \text{ if } f_{ij} < 0,
\]

see [4, Section 6]. Let \( v \in P_1(\mathbb{R}^2) \) and the positive coefficients \( q_i, i \in \mathcal{N}_h \), in Algorithm 2.3 are defined by (2.24). Following the proof of [4, Theorem 6.1] we can show (2.26) and thus (2.23). Indeed using the definition of \( \gamma_i \) in Remark 2.5 and following [5, Lemma 7] we obtain

\[
P_i^+ = \sum_{j \neq i} \max\{0, f_{ij}\} = \sum_{j \in \mathcal{Z}_h^i, v_i > v_j} \max\{0, d_{ij}(w)(v_i - v_j)\} = \sum_{j \in \mathcal{Z}_h^i, v_i > v_j} d_{ij}(w)(v_i - v_j)
\]

\[
\leq \sum_{j \in \mathcal{Z}_h^i, v_i > v_j} d_{ij}(w)(v_i - v_i^{\min}) \leq \sum_{j \in \mathcal{Z}_h^i} d_{ij}(w)(v_i - v_i^{\min})
\]

\[
\leq \gamma_i(v_i^{\max} - v_i) \sum_{j \in \mathcal{Z}_h^i} d_{ij}(w) = q_i(v_i^{\max} - v_i) = Q_i^+.
\]

Similarly we can show \( Q_i^- < P_i \), for \( i \in \mathcal{N}_h \) and hence (2.23) is satisfied.

Let now the positive coefficients \( q_i, i \in \mathcal{N}_h \), in Algorithm 2.3 be defined by (2.25). For \( w \in \mathcal{S}_h \), in view of the definition of \( d_{ij} = d_{ij}(w) \) in (2.6), (2.5) and the fact that the triangulation \( \mathcal{T}_h \) is shape regular, i.e., (2.1), there exists a constant \( C(\gamma) > 0 \) independent of \( h \), such that

\[
|d_{ij}(w)| \leq |\tau_{ij}(w)| + |\tau_{ij}(w)| \leq \|\nabla w\|_{L^\infty} \sum_{K \in \omega_i} (\|\nabla \phi_i\|_{L^2(K)}\|\phi_j\|_{L^2(K)} + \|\nabla \phi_j\|_{L^2(K)}\|\phi_i\|_{L^2(K)})
\]

\[
\leq C\|\nabla w\|_{L^\infty} \sum_{K \in \omega_i} h_K \leq C(\gamma)h\|\nabla w\|_{L^\infty}.
\]

Using now (2.28) and the fact that the family of triangulations are shape regular, \( m_i = O(h^2) \), there exists \( h_M > 0 \) such that for sufficiently small \( h < h_M \), we get
Hence, the schemes (2.11)–(2.12) and (2.31)–(2.32) can be viewed as the following variational problem: Find $u$ with
\begin{equation}
(2.36)
\end{equation}

Therefore in view of Algorithm 2.3 we get for $v_i > v_j$ that $\bar{v}_{ij} = R_i^{+} = 1$. Similar arguments may be used in case where $\bar{v}_{ij} = R_i^{-} = 1$ and the case where $f_{ij} < 0$. Therefore, (2.26) holds and hence (2.23) is satisfied.

Let us consider now the bilinear form $\tilde{d}_h(s, w; v, \cdot) : C^0 \times C^0 \to \mathbb{R}$, with $s, w \in S_h$, defined by, for $v, z \in C^0$,
\begin{equation}
(2.30)
\end{equation}

Then, employing $\tilde{d}_h$, we can rewrite the AFC scheme (2.16)–(2.17) equivalently in a variational form as: Find $u_h(t) \in S_h$ and $c_h(t) \in S_h$, with $u_h(0) = u_0^h \in S_h$, $c_h(0) = c_0^h \in S_h$, such that
\begin{equation}
(2.31)
\end{equation}
\begin{equation}
(2.32)
\end{equation}

Note that if $a_{ij} \equiv 0$, then $\tilde{d}_h = d_h$ and that $\tilde{d}_h$ and $d_h$ satisfy similar properties. The bilinear forms $d_h$ and $\tilde{d}_h$, introduced in (2.9) and (2.30), can be viewed as $\tilde{d}_h(s, w; v, \cdot) : C^0 \times C^0 \to \mathbb{R}$, with $s, w \in S_h$, where
\begin{equation}
(2.33)
\end{equation}

with $\rho_{ij} = \rho_{ji} \in [0, 1]$. Note that, for $\rho_{ij} = 1$ we have $\tilde{d}_h = d_h$ and for $\rho_{ij} = 1 - a_{ij}$, we get $\tilde{d}_h = \bar{d}_h$. In the sequel, we will derive various error bounds involving the bilinear form $\tilde{d}_h$, which obviously will also hold for either $d_h$ or $\bar{d}_h$. In view of the symmetry of $\rho_{ij}$ and $d_{ij}$, the form (2.33) can be rewritten as
\begin{equation}
(2.34)
\end{equation}

Hence, the schemes (2.11)–(2.12) and (2.31)–(2.32) can be viewed as the following variational problem: Find $u_h(t) \in S_h$ and $c_h(t) \in S_h$ such that
\begin{equation}
(2.35)
\end{equation}
\begin{equation}
(2.36)
\end{equation}

with $u_h(0) = u_0^h \in S_h$ and $c_h(0) = c_0^h \in S_h$.

2.3. Auxiliary results. We consider the $L^2$ projection $P_h : L^2 \to S_h$ and the elliptic projection $R_h : H^1 \to S_h$ defined by
\begin{equation}
(2.37)
\end{equation}
\begin{equation}
(2.38)
\end{equation}

In view of the mesh Assumption 2.1, $P_h$ and $R_h$ satisfy the following bounds, cf. e.g., [6, Chapter 8] and [26].
\begin{equation}
(2.39)
\end{equation}
\begin{equation}
(2.40)
\end{equation}
\begin{equation}
(2.41)
\end{equation}
\begin{equation}
(2.42)
\end{equation}
\begin{equation}
(2.43)
\end{equation}

Thus, for sufficiently smooth solutions $u, c$ of (1.1), there exists $M_0 > 0$, independent of $h$, such that, for $t \in [0, T],$
\begin{equation}
(2.44)
\end{equation}

For the inner product $(\cdot, \cdot)_h$ introduced in (2.10), the following holds.
Lemma 2.8. [7, Lemma 2.3] Let $\varepsilon_h(\chi, \psi) := (\chi, \psi) - (\chi, \psi)_h$. Then,
\[ |\varepsilon_h(\chi, \psi)| \leq C h^{1+j} \| \nabla^i \chi \|_{L^2} \| \nabla^j \psi \|_{L^2}, \quad \forall \chi, \psi \in S_h, \text{ and } i, j = 0, 1, \]
where the constant $C$ is independent of $h$.

Next, we recall various results that will be useful in the analysis that follows. Using the following lemma we have that the bilinear form $d_h$, introduced in (2.9), and hence also $\tilde{d}_h$, defined in (2.30), induces a seminorm on $C^0$.

Lemma 2.9. [3, Lemma 3.1] Consider any $\mu_{ij} = \mu_{ji} \geq 0$ for $i, j = 1, \ldots, N$. Then,
\[ \sum_{i,j=1}^{N} v_i \mu_{ij} (v_i - v_j) = \sum_{i,j=1}^{N} \mu_{ij} (v_i - v_j)^2 \geq 0, \quad \forall v_1, \ldots, v_N \in \mathbb{R}. \]

Therefore, $\overline{d}_h(s, w; \cdot, \cdot) : C^0 \times C^0 \rightarrow \mathbb{R}$, with $s, w \in S_h$, is a non-negative symmetric bilinear form which satisfies the Cauchy-Schwarz’s inequality,
\[ |\overline{d}_h(s, w; v, z)|^2 \leq \overline{d}_h(s, w; v, v) \overline{d}_h(s, w; z, z), \quad \forall v, z \in C^0, \]
and thus induces a seminorm on $C^0$.

Lemma 2.10. There exists a constant $C$ such that for all $w, s, \psi, \chi \in S_h$,
\[ |\overline{d}_h(s, w; \psi, \chi)| \leq C h \| \nabla w \|_{L^\infty} \| \nabla \psi \|_{L^2} \| \nabla \chi \|_{L^2}. \]

Proof. Recall that (2.28) gives
\[ |d_{ij}(w)| \leq C h \| \nabla w \|_{L^\infty}, \quad \forall i, j \in N_h. \]
Similarly we obtain
\[ |d_{ij}(w)| \leq C \| \nabla w \|_{L^2}, \quad \forall i, j \in N_h. \]
Then, in view of (2.34) and the fact that $\rho_{ij} \in [0, 1]$, we obtain
\[ |\overline{d}_h(s, w; \chi, \chi)| \leq C h \| \nabla w \|_{L^\infty} \sum_{K \in T_h} \sum_{z_i, z_j \in \mathcal{Z}_h(K)} (\chi_i - \chi_j)^2 \leq C h \| \nabla w \|_{L^\infty} \sum_{K \in T_h} \| \nabla \chi \|_{L^2(K)}^2 \]
\[ \leq C h \| \nabla w \|_{L^\infty} \| \nabla \chi \|_{L^2}. \]
Therefore, using (2.45) we get the desired result.

Next, using the properties of the correction factors and standard arguments, we can prove two similar estimates to [2, Lemma 2].

Lemma 2.11. There exists a constant $C$ such that for $w, \tilde{w}, \psi, \chi \in S_h$,
\[ |\overline{d}_h(\psi, w; \psi, \chi) - \overline{d}_h(\psi, \tilde{w}; \psi, \chi)| \leq C (\| \nabla (w - \tilde{w}) \|_{L^\infty} + \| \nabla w \|_{L^\infty}) \| \nabla \psi \|_{L^2} \| \nabla \chi \|_{L^2}. \]

Proof. In view of (2.34), we have
\[ \overline{d}_h(\psi, w; \psi, \psi) \quad \overline{d}_h(\psi, \tilde{w}; \psi, \psi) = \sum_{i < j} (d_{ij}(w) \rho_{ij}(\psi, w) - d_{ij}(\tilde{w}) \rho_{ij}(\psi, \tilde{w})(\psi_i - \psi_j)(\chi_i - \chi_j). \]

Also, using (2.6), we get
\[ d_{ij}(\tilde{w}) \rho_{ij}(\psi, \tilde{w}) - d_{ij}(w) \rho_{ij}(\psi, w) = (d_{ij}(\tilde{w}) - d_{ij}(w)) \rho_{ij}(\psi, \tilde{w}) + d_{ij}(w)(\rho_{ij}(\psi, \tilde{w}) - \rho_{ij}(\psi, w)) \]
\[ = \max\{\tau_{ij}(\tilde{w}), 0, -\tau_{ij}(\tilde{w})\} \max\{\tau_{ij}(w), 0, -\tau_{ij}(w)\} \rho_{ij}(\psi, \tilde{w}) \]
\[ + d_{ij}(w) \rho_{ij}(\psi, \tilde{w}) - d_{ij}(w) \rho_{ij}(\psi, w) \]
\[ \leq \| \nabla (w - \tilde{w}) \|_{L^\infty} \leq C \| \nabla (w - \tilde{w}) \|_{L^\infty}. \]
\[ \leq C \| \nabla w \|_{L^\infty} \sum_{K \in \omega_i} h_K \leq C \| \nabla w \|_{L^\infty} \sum_{K \in \omega_i} h_K \leq C (\| \nabla w \|_{L^\infty} + \| \nabla w \|_{L^\infty}), \]
where in the last inequality we have used the fact that the triangulation $T_h$ is shape regular, see (2.1).

Therefore, in view of (2.49) and (2.28), we obtain
\[ |\overline{d}_h(\psi, w; \psi, \chi) - \overline{d}_h(\tilde{w}, \psi; \psi, \chi)| \leq C (\| \nabla (w - \tilde{w}) \|_{L^\infty} + \| \nabla w \|_{L^\infty}) \sum_{i < j} |\psi_i - \psi_j| |\chi_i - \chi_j| \]
\[ \leq C \| \nabla w \|_{L^\infty} + \| \nabla w \|_{L^\infty} \| \nabla \psi \|_{L^2} \| \nabla \chi \|_{L^2}, \]
which gives the desired bound.
An immediate result for the stabilization term (2.9) from Lemma 2.11 is the following.

**Lemma 2.12.** There exists a constant $C$ such that, for $w, \tilde{w}, \psi, \chi \in \mathcal{S}_h$,

$$|d_h(w; \psi, \chi) - d_h(\tilde{w}; \psi, \chi)| \leq C h \|\nabla (w - \tilde{w})\|_{L^\infty} \|\nabla \psi\|_{L^2} \|\nabla \chi\|_{L^2}. $$

**Proof.** In view of (2.9), we have

$$d_h(w; \psi, \chi) - d_h(\tilde{w}; \psi, \chi) = \sum_{i<j} (d_{ij}(w) - d_{ij}(\tilde{w})) (\psi_i - \psi_j)(\chi_i - \chi_j).$$

In view of the proof of Lemma 2.11 and (2.49), we obtain

$$|d_h(w; \psi, \chi) - d_h(\tilde{w}; \psi, \chi)| \leq C h \|\nabla (w - \tilde{w})\|_{L^\infty} \sum_{i<j} |\psi_i - \psi_j| |\chi_i - \chi_j| \leq C h \|\nabla (w - \tilde{w})\|_{L^\infty} \|\nabla \psi\|_{L^2} \|\nabla \chi\|_{L^2},$$

which gives the desired bound.

$\square$

**Lemma 2.13.** Let the triangulation $\mathcal{T}_h$ satisfy Assumption 2.1. Then the correction factor functions $\tilde{a}_{ij}$, defined by Algorithm 2.3, can be written as

$$\tilde{a}_{ij}(\alpha, \beta) = \frac{A_{ij}(\alpha, \beta)}{|\alpha_j - \alpha_i| + B_{ij}(\alpha, \beta)}, \quad i, j \in \mathcal{N}_h,$$

with $\alpha = (\alpha_1, \ldots, \alpha_N)^T$, $\beta = (\beta_1, \ldots, \beta_N)^T \in \mathbb{R}^N$, $\alpha_j \neq \alpha_i$, and $A_{ij}$ and $B_{ij}$ non-negative functions which are continuous functions in $\alpha$, with $\alpha_i \neq \alpha_j$. Further $A_{ij}$ and $B_{ij}$ are Lipschitz-continuous with respect to $\alpha$ in the sets $\{\alpha \in \mathbb{R}^N : \alpha_i > \alpha_j\}$ and $\{\alpha \in \mathbb{R}^N : \alpha_i < \alpha_j\}$,

$$|A_{ij}(\alpha, \beta) - A_{ij}(\tilde{\alpha}, \beta)| \leq \Lambda_{ij}^A(\beta, q_i) \sum_{\ell \in \mathcal{Z}_h(\omega_i)} |\alpha_\ell - \tilde{\alpha}_\ell|$$

and

$$|B_{ij}(\alpha, \beta) - B_{ij}(\tilde{\alpha}, \beta)| \leq \Lambda_{ij}^B(\beta, q_i) \sum_{\ell \in \mathcal{Z}_h(\omega_i)} |\alpha_\ell - \tilde{\alpha}_\ell|,$$

where

$$\Lambda_{ij}^A(\beta, q_i) = \Lambda_{ij}^B(\beta, q_i) = \max_{1 \leq j \leq N} \frac{d_{ij}(\beta)}{d_{ij}(\beta)}.$$

**Proof.** Following [3, Lemma 4.1] we define $A_{ij}$ and $B_{ij}$, for $i \in \mathcal{N}_h$ with $j \in \mathcal{N}_h$ and $d_{ij} > 0$, as

$$A_{ij}(\alpha, \beta) := \frac{1}{d_{ij}(\beta)} \begin{cases} \min\{-P_i^-(\alpha, \beta), -Q_i^-\}, & \text{if } \alpha_i < \alpha_j, \\ \min\{P_i^+(\alpha, \beta), Q_i^+\}, & \text{if } \alpha_i > \alpha_j, \end{cases}$$

and

$$B_{ij}(\alpha, \beta) := \frac{1}{d_{ij}(\beta)} \begin{cases} -\tilde{P}_i^-(\alpha, \beta), & \text{if } \alpha_i < \alpha_j, \\ \tilde{P}_i^+(\alpha, \beta), & \text{if } \alpha_i > \alpha_j, \end{cases}$$

where

$$\tilde{P}_i^+(\alpha, \beta) = \sum_{k=1, k \neq i}^N \max\{0, f_{ik}\}, \quad \tilde{P}_i^-(\alpha, \beta) = \sum_{k=1, k \neq i}^N \min\{0, f_{ik}\}.$$

If $d_{ij} = 0$, then we define $A_{ij} = B_{ij} = 0$.

Following [3, Lemma 3.5], $A_{ij}$ and $B_{ij}$ are continuous functions and are Lipschitz with respect to the first variable $\alpha$, i.e., there exist constants $\Lambda_{ij}^A$ and $\Lambda_{ij}^B$ such that

$$|A_{ij}(\alpha, \beta) - A_{ij}(\tilde{\alpha}, \beta)| \leq \Lambda_{ij}^A(\beta, q_i) \sum_{\ell \in \mathcal{Z}_h(\omega_i)} |\alpha_\ell - \tilde{\alpha}_\ell|$$

and

$$|B_{ij}(\alpha, \beta) - B_{ij}(\tilde{\alpha}, \beta)| \leq \Lambda_{ij}^B(\beta, q_i) \sum_{\ell \in \mathcal{Z}_h(\omega_i)} |\alpha_\ell - \tilde{\alpha}_\ell|.$$

To show (2.56)-(2.57) we will employ the following inequalities from [22, Ineq. (2.3b)-(2.3c)]

$$|\max(a, b) - \max(\tilde{a}, \tilde{b})| \leq |a - \tilde{a}| + |b - \tilde{b}|, \forall a, \tilde{a}, b, \tilde{b} \in \mathbb{R}$$

and

$$|\min(a, b) - \min(\tilde{a}, \tilde{b})| \leq |a - \tilde{a}| + |b - \tilde{b}|, \forall a, \tilde{a}, b, \tilde{b} \in \mathbb{R}.$$
Let us assume $d_{ij}(\beta) > 0$ and that $\alpha_i - \alpha_j$ and $\tilde{\alpha}_i - \tilde{\alpha}_j$ are both positive. The other case for the differences $\alpha_i - \alpha_j < 0$ and $\tilde{\alpha}_i - \tilde{\alpha}_j < 0$ can be treated analogously. In view of (2.54) and using (2.59), we get

$$|A_{ij}(\alpha, \beta) - A_{ij}(\tilde{\alpha}, \beta)| = \frac{1}{d_{ij}(\beta)} |\min\{P_i^+(\alpha, \beta), Q_i^+(\alpha)\} - \min\{P_i^+(\tilde{\alpha}, \beta), Q_i^+(\tilde{\alpha})\}|$$

(2.60)

Then, using the definition of $P_i^+$, given by Algorithm 2.3, we get

$$|P_i^+(\alpha, \beta) - P_i^+(\tilde{\alpha}, \beta)| \leq \sum_{\ell \in \mathbb{Z}_h} |\max\{0, f_{i\ell}(\alpha, \beta)\} - \max\{0, f_{i\ell}(\tilde{\alpha}, \beta)\}|$$

$$\leq \sum_{\ell \in \mathbb{Z}_h, \ell \neq \ell} |\max\{0, f_{i\ell}(\alpha, \beta)\} - \max\{0, f_{i\ell}(\tilde{\alpha}, \beta)\}| + |\max\{0, f_{ij}(\alpha, \beta)\} - \max\{0, f_{ij}(\tilde{\alpha}, \beta)\}|$$

(2.61)

In order to bound $I_1$ we will consider various cases for the the sign of the differences $\alpha_i - \alpha_\ell$, $\tilde{\alpha}_i - \tilde{\alpha}_\ell$, $\ell \in \mathbb{Z}_h, \ell \neq i$. Let $\alpha_i - \alpha_\ell < 0$ and $\tilde{\alpha}_i - \tilde{\alpha}_\ell > 0$, then

$$|\max\{0, f_{i\ell}(\alpha, \beta)\} - \max\{0, f_{i\ell}(\tilde{\alpha}, \beta)\}| \leq d_{i\ell}(\beta)(|\tilde{\alpha}_i - \tilde{\alpha}_\ell| = d_{i\ell}(\beta)(\tilde{\alpha}_i - \tilde{\alpha}_\ell)$$

$$\leq d_{i\ell}(\beta)(|\alpha_i - \alpha_\ell| + |\alpha_\ell - \tilde{\alpha}_\ell|).$$

(2.62)

In a similar way we can show that for $\alpha_i - \alpha_\ell > 0$ and $\tilde{\alpha}_i - \tilde{\alpha}_\ell < 0$, we get

$$\max\{0, f_{i\ell}(\alpha, \beta)\} - \max\{0, f_{i\ell}(\tilde{\alpha}, \beta)\} | \leq d_{i\ell}(\beta)(|\alpha_i - \tilde{\alpha}_i| + |\alpha_\ell - \tilde{\alpha}_\ell|).$$

Also if $\alpha_i - \alpha_\ell < 0$ and $\tilde{\alpha}_i - \tilde{\alpha}_\ell < 0$, then in view of (2.58), we have

$$|\max\{0, f_{ij}(\alpha, \beta)\} - \max\{0, f_{ij}(\tilde{\alpha}, \beta)\}| \leq d_{i\ell}(\beta)(|\alpha_i - \tilde{\alpha}_i| + |\alpha_\ell - \tilde{\alpha}_\ell|).$$

(2.63)

Finally, for $\alpha_i - \alpha_\ell > 0$ and $\tilde{\alpha}_i - \tilde{\alpha}_\ell < 0$, we get

$$|\max\{0, f_{ij}(\alpha, \beta)\} - \max\{0, f_{ij}(\tilde{\alpha}, \beta)\}| = 0.$$}

(2.64)

Therefore, combining the above cases (2.62)–(2.65), and the fact that $d_{i\ell} = 0$ for $\ell \notin \mathbb{Z}_h(\omega)$, we obtain

$$I_1 \leq \sum_{\ell \in \mathbb{Z}_h(\omega)} d_{i\ell}(\beta)(|\alpha_i - \tilde{\alpha}_i| + |\alpha_\ell - \tilde{\alpha}_\ell|).$$

(2.66)

Moreover,

$$I_2 \leq \sum_{j \in \mathbb{Z}_h(\omega)} d_{ij}(\beta)(|\alpha_j - \tilde{\alpha}_j|).$$

(2.67)

Hence, using (2.66) and (2.67) in (2.61), we get

$$|P_i^+(\alpha, \beta) - P_i^+(\tilde{\alpha}, \beta)| \leq C \max_{1 \leq j \leq N} d_{ij}(\beta) \sum_{j \in \mathbb{Z}_h(\omega)} |\alpha_j - \tilde{\alpha}_j|. $$

(2.68)

Next, let $\alpha_i^{\max}$ and $\tilde{\alpha}_i^{\max}$ be the local maximums on patch $\omega_i$ for $\alpha$ and $\tilde{\alpha}$, respectively, i.e., $\alpha_i^{\max} \geq \alpha_j$ and $\tilde{\alpha}_i^{\max} \geq \tilde{\alpha}_j$, $j \in \mathbb{Z}_h(\omega)$, which may occur on different nodes. Therefore, let $\ell, \tilde{\ell} \in \mathbb{Z}_h(\omega)$ such that $\alpha_i^{\max} = \alpha_\ell$ and $\tilde{\alpha}_i^{\max} = \tilde{\alpha}_\ell$, then we obtain

$$|Q_i^+(\alpha) - Q_i^+(\tilde{\alpha})| = q_i[(\alpha_i^{\max} - \alpha_i) - (\tilde{\alpha}_i^{\max} - \tilde{\alpha}_i)] \leq q_i(|\alpha_i^{\max} - \tilde{\alpha}_i^{\max}| - (\alpha_i - \tilde{\alpha}_i)|)$$

(2.69)

We have

$$|\alpha_i^{\max} - \tilde{\alpha}_i^{\max}| = \begin{cases} \alpha_\ell - \tilde{\alpha}_\ell, & \text{if } \alpha_\ell > \tilde{\alpha}_\ell, \text{ and } \ell \neq \tilde{\ell}, \\ \tilde{\alpha}_\ell - \alpha_\ell, & \text{if } \alpha_\ell < \tilde{\alpha}_\ell, \text{ and } \ell \neq \tilde{\ell}, \\ |\alpha_\ell - \tilde{\alpha}_\ell|, & \text{if } \ell = \tilde{\ell}. \end{cases}$$

(2.70)

Therefore, using (2.70) in (2.69) we get

$$|Q_i^+(\alpha) - Q_i^+(\tilde{\alpha})| \leq q_i \sum_{j \in \mathbb{Z}_h(\omega)} |\alpha_j - \tilde{\alpha}_j|.$$
\[
|A_{ij}(\alpha, \beta) - A_{ij}(\bar{\alpha}, \bar{\beta})| \leq C \frac{\max_{1 \leq j \leq N} d_{ij}(\beta) + q_i}{d_{ij}(\beta)} \sum_{j \in \mathbb{Z}_q(\omega_i)} |\alpha_j - \bar{\alpha}_j|.
\]

Similarly, in view of the definition of \(B_{ij}\) we get, for \(d_{ij} > 0\),
\[
|B_{ij}(\alpha, \beta) - B_{ij}(\bar{\alpha}, \bar{\beta})| = \left(\frac{d_{ij}^{-1}(w) \max_{1 \leq j \leq N} d_{ij}(\beta)}{d_{ij}(\beta)} \right) \sum_{j \in \mathbb{Z}_q(\omega_i)} |\alpha_j - \bar{\alpha}_j|.
\]

**Definition 2.14.** Let \(w \in \mathcal{S}_h\), then
\[
A_{ij}(w, q_i) := (d_{ij}^{-1}(w) \max_{1 \leq j \leq N} d_{ij}(w) + q_i) + 1).
\]

**Lemma 2.15.** Let \(\chi, \psi, w \in \mathcal{S}_h\), \(e_{ij} \in \mathcal{E}_h\), \(i, j \in \mathcal{N}_h\), with endpoints \(Z_i, Z_j \in \mathcal{Z}_h\) and \(\bar{\rho}_{ij}(\chi, w) := \rho_{ij}(\chi(w)(\chi_i - \chi_j))\), with \(\rho_{ij}\) given in \((2.33)\). Then there exists \(C > 0\) independent of \(h\) such that \(\bar{\rho}_{ij}\) satisfies,
\[
|\bar{\rho}_{ij}(\chi, w) - \bar{\rho}_{ij}(\psi, w)| \leq C \Lambda_{ij}(w, q_i) \sum_{l \in \mathbb{Z}_q(\omega_i)} |\chi_l - \psi_l|, \quad \forall \chi, \psi, w \in \mathcal{S}_h.
\]

**Proof.** It suffices to consider the case where \(\rho_{ij} = 1 - a_{ij}\). We easily get
\[
|\bar{\rho}_{ij}(\chi, w) - \bar{\rho}_{ij}(\psi, w)| = |(1 - a_{ij}(\chi(w)(\chi_i - \chi_j)) - (1 - a_{ij}(\psi(w)(\psi_i - \psi_j))|
\]
\[
\leq |(\chi_i - \chi_j) - (\psi_i - \psi_j)| + |\Phi_{ij}(\chi(w) - \Phi_{ij}(\psi, w)|,
\]
where \(\Phi_{ij}(\chi, w) := a_{ij}(\chi(w)(\chi_i - \chi_j)), \chi \in \mathcal{S}_h, j \in \mathcal{Z}_h\).

Let \(\chi, \psi \in \mathcal{S}_h\), then following the proof of \([3, \text{Lemma 3.5}]\), we have for \(|\chi_i - \chi_j| \leq \psi_i - \psi_j) \leq 0\) that
\[
|\Phi_{ij}(\chi, w) - \Phi_{ij}(\psi, w)| \leq |\chi_i - \chi_j| + |\psi_i - \psi_j| = |(|\chi_i - \psi_i|) - (\chi_j - \psi_j)| \leq C \sum_{l \in \mathbb{Z}_q(\omega_i)} |\chi_l - \psi_l|.
\]

Further, again in view of the proof of \([3, \text{Lemma 3.5}]\), we have for \(|\chi_i - \chi_j| \leq \psi_i - \psi_j) > 0\) that
\[
\Phi_{ij}(\chi, w) = \Phi_{ij}(\psi, w) = (A_{ij}(\chi, w) - A_{ij}(\psi, w)) \frac{\psi_i - \psi_j}{|\psi_i - \psi_j| + B_{ij}(\psi, w)} + a_{ij}(|\chi_i - \chi_j) (B_{ij}(\psi, w) - B_{ij}(\chi, w))|\psi_i - \psi_j| + (|\chi_i - \chi_j| - (\psi_i - \psi_j))B_{ij}(\psi, w),
\]
where \(\chi = (\chi_1, \ldots, \chi_N)^T\) and \(w = (w_1, \ldots, w_N)^T\). Therefore the desired bound follows from
\[
|\Phi_{ij}(\chi, w) - \Phi_{ij}(\psi, w)| \leq \Lambda_{ij}(\chi, w) - A_{ij}(\psi, w)| + |B_{ij}(\chi, w) - B_{ij}(\psi, w)| + |(|\chi_i - \psi_i| - (\chi_j - \psi_j)|
\]
\[
\leq (\Lambda_{ij} + \Lambda_{ij}^0) \sum_{l \in \mathbb{Z}_q(\omega_i)} |\chi_l - \psi_l| + |(|\chi_i - \psi_i| - (\chi_j - \psi_j)|.
\]

**Remark 2.16.** The Lipschitz constants \(\Lambda_{ij}\), in Lemma 2.15 can be estimated in view of \((2.28)\), as follows,
\[
d_{ij}(w)A_{ij}(w, q_i) = \max_{1 \leq j \leq N} d_{ij}(w) + q_i \leq C(h \norm{\nabla w}_L^\infty + \norm{\psi}_{max}),
\]
where \(\norm{\psi}_{max} = \max_{i \in \mathcal{N}_h} \norm{q_i},\) \(q = (q_1, \ldots, q_N)\) and the coefficients \(q_i, i \in \mathcal{N}_h\), in Algorithm 2.3, are such that \(\text{Lemma 2.7 holds}\).

**Lemma 2.17.** Let \(w \in \mathcal{S}_h\) and \(q_i, i \in \mathcal{N}_h\), the coefficients in Algorithm 2.3 such that \(\text{Lemma 2.7 and 2.15 hold}\). Then for \(v \in H^2\) there exists a constant \(C\) such that for all \(\chi \in \mathcal{S}_h\),
\[
|\bar{d}_h(\psi, w; \chi)| \leq C(h \norm{\nabla w}_L^\infty + \norm{\psi}_{max})(\norm{\nabla(\psi - v)}_L^2 + h^2 \norm{\psi}_L^2)\norm{\nabla \chi}_{L^2}.
\]

**Proof.** We will follow the proof of \([5, \text{Lemma 3}]\). Using \((2.34)\), we have
\[
(2.72) \quad \bar{d}_h(\psi, w; \psi, \chi) = \sum_{i < j} d_{ij}(w)\rho_{ij}(\psi, w)(\psi_i - \psi_j)(\chi_i - \chi_j) = I.
\]

Given \(i \in \mathcal{N}_h\), let \(\hat{v}_i \in \mathcal{P}_1(\omega_i)\) be the unique solution of
\[
(\nabla \hat{v}_i, \nabla \chi)_{L^2(\omega_i)} = (\nabla v, \nabla \chi)_{L^2(\omega_i)}, \quad \forall \chi \in \mathcal{P}_1(\omega_i),
\]
\(
(\hat{v}_i, 1)_{L^2(\omega_i)} = (v, 1)_{L^2(\omega_i)}.
\)
We can extended \(\hat{v}_i\) arbitrarily as a function on \(\mathcal{S}_h\). In view of Lemma 2.7, we have that \(\rho_{ij}(\hat{v}_i) = 0,\) for \(j \in \mathcal{N}_h\). Also, there exists a constant \(C\) independent of \(h\), such that
\[
\norm{\nabla(v - \hat{v}_i)}_{L^2(\omega_i)} \leq Ch \norm{v}_{H^2(\omega_i)}.
\]
Thus $I$ in (2.72) can be rewritten as

$$I = \sum_{i<j} d_{ij}(w) \tilde{\rho}_{ij}(\psi, w)(\chi_i - \chi_j) = \sum_{i<j} d_{ij}(w)(\tilde{\rho}_{ij}(\psi, w) - \tilde{\rho}_{ij}(\hat{v}, w))(\chi_i - \chi_j).$$

Then, we easily get

$$|I| \leq \sum_{i<j} d_{ij}(w)|\tilde{\rho}_{ij}(\psi, w) - \tilde{\rho}_{ij}(\hat{v}, w)||\chi_i - \chi_j|$$

$$\leq \left( \sum_{i<j} d_{ij}(w)^2 |\tilde{\rho}_{ij}(\psi, w) - \tilde{\rho}_{ij}(\hat{v}, w)|^2 \right)^{1/2} \left( \sum_{i,j=1}^N |\chi_i - \chi_j|^2 \right)^{1/2}$$

$$\leq C \left( \sum_{i<j} d_{ij}(w)^2 |\tilde{\rho}_{ij}(\psi, w) - \tilde{\rho}_{ij}(\hat{v}, w)|^2 \right)^{1/2} \|\nabla \chi\|_{L^2}.$$  

In view of Lemma 2.15 and Remark 2.16 we obtain

$$d_{ij}(w)^2|\tilde{\rho}_{ij}(\psi, w) - \tilde{\rho}_{ij}(\hat{v}, w)|^2 \leq C d_{ij}(w)^2 \Lambda_{ij}(w, q_i)^2 \sum_{\ell \in Z_h(\chi)} |\psi_\ell - (\hat{v})_\ell|^2$$

$$\leq C(h \|\nabla w\|_{L^\infty} + \|q\|_{\text{max}})^2 \|\nabla(\psi - \hat{v})\|_{L^2(Z_h(\chi))}^2$$

$$\leq C(h \|\nabla w\|_{L^\infty} + \|q\|_{\text{max}})^2 \|\nabla(\psi - v)\|_{L^2(Z_h(\chi))}^2$$

$$\leq C(h \|\nabla w\|_{L^\infty} + \|q\|_{\text{max}})^2 \|\nabla(\psi - v)\|_{L^2(Z_h(\chi))}^2 + h^2 \|v\|_{H^2(Z_h(\chi))}^2.$$  

Then since the mesh $T_h$ is shape regular, we get the desired result

$$|I| \leq C(h \|\nabla w\|_{L^\infty} + \|q\|_{\text{max}})(\|\nabla(\psi - v)\|_{L^2} + h \|v\|_{L^2})\|\nabla \chi\|_{L^2}. \tag{2.74}$$

\[\square\]

**Lemma 2.18.** Let the bilinear form $\overline{d}_h$ defined in (2.33) and $\tilde{\rho}_{ij}$ defined in Lemma 2.15 and $w, s, v, \chi \in S_h$, then

$$|\overline{d}_h(v, w; v) - \overline{d}_h(s, w; s, \chi)| \leq C(h \|\nabla w\|_{L^\infty} + \|q\|_{\text{max}})\|\nabla(\psi - v)\|_{L^2} \|\nabla \chi\|_{L^2}.$$  

**Proof.** Using (2.34), we have,

$$\overline{d}_h(v, w; v) - \overline{d}_h(s, w; s, \chi) = \sum_{i<j} d_{ij}(w)(\tilde{\rho}_{ij}(v, w) - \tilde{\rho}_{ij}(s, w))(\chi_i - \chi_j),$$

In view of Remark 2.16 and Lemma 2.15, we get

$$|\overline{d}_h(v, w; v) - \overline{d}_h(s, w; s, \chi)| \leq C \sum_{K \in T_h} \sum_{z_i, z_j \in K} |d_{ij}(w)||\tilde{\rho}_{ij}(v, w) - \tilde{\rho}_{ij}(s, w)|(\chi_i - \chi_j)$$

$$\leq C(h \|\nabla w\|_{L^\infty} + \|q\|_{\text{max}})\|\nabla(\psi - s)\|_{L^2} \|\nabla \chi\|_{L^2}.$$

\[\square\]

**Lemma 2.19.** Let $\chi, \psi, w \in S_h, e_{ij} \in E_h, i, j \in N_h$, with endpoints $Z_i, Z_j \in Z_h$, and $\tilde{\rho}_{ij}(\chi, w) := d_{ij}(w)\rho_{ij}(\chi, w)(\chi_i - \chi_j)$, with $\rho_{ij}$ given in (2.33). Then there exists $C > 0$ independent of $h$ such that $\tilde{\rho}_{ij}$ satisfies

$$|\tilde{\rho}_{ij}(\chi, w) - \tilde{\rho}_{ij}(\hat{v}, w)| \leq C h \|\nabla(\psi - \hat{v})\|_{L^\infty(Z_h(\chi))}\|\nabla \chi\|_{L^2(Z_h(\chi))}.$$

**Proof.** It suffices to consider the case where $\rho_{ij} = 1 - a_{ij}$. We easily get

$$|\tilde{\rho}_{ij}(\chi, w) - \tilde{\rho}_{ij}(\hat{v}, w)| = |d_{ij}(w)(1 - a_{ij}(\chi, w))(\chi_i - \chi_j) - d_{ij}(w)(\hat{v})(1 - a_{ij}(\chi, w))(\chi_i - \chi_j)|$$

$$\leq |(d_{ij}(w) - d_{ij}(\hat{v}))(\chi_i - \chi_j)| + |\Phi_{ij}(\chi, w) - \Phi_{ij}(\hat{v}, w)|,$$

where $\Phi_{ij}(\chi, w) := d_{ij}(w)a_{ij}(\chi, w)(\chi_i - \chi_j)$, $\chi \in S_h$, $j \in Z_h$.

Following the proof in Lemma 2.15 we have for $\chi_i - \chi_j > 0$

$$\Phi_{ij}(\chi, w) - \Phi_{ij}(\chi, \hat{w}) = d_{ij}(w)A_{ij}(\chi, w)\frac{\chi_i - \chi_j}{|\chi_i - \chi_j| + B_{ij}(\chi, w)} - d_{ij}(w)A_{ij}(\hat{v}, w)\frac{\chi_i - \chi_j}{|\chi_i - \chi_j| + B_{ij}(\chi, w)}$$

$$= (d_{ij}(w)A_{ij}(\chi, w) - d_{ij}(w)A_{ij}(\chi, \hat{w}))\frac{\chi_i - \chi_j}{|\chi_i - \chi_j| + B_{ij}(\chi, w)}$$

$$+ d_{ij}(w)A_{ij}(\chi, \hat{w})\left(\frac{\chi_i - \chi_j}{|\chi_i - \chi_j| + B_{ij}(\chi, w)} - \frac{\chi_i - \chi_j}{|\chi_i - \chi_j| + B_{ij}(\chi, \hat{w})}\right)$$

$$= I + II.$$  

(2.75)
Let us assume \(d_{ij}(\cdot) > 0, d_{ij}(\hat{\cdot}) > 0\), and \(\chi_i - \chi_j > 0\). The other case for the differences \(\chi_i - \chi_j < 0\) can be treated analogously. In view of (2.54) and using (2.59), we get

\[
|d_{ij}(\cdot)A_{ij}(\chi, w) - d_{ij}(\hat{\cdot})A_{ij}(\chi, \hat{\cdot})| = \min \{ P^+_i(\chi, w), Q^+_i(\chi) \} - \min \{ P^+_i(\chi, \hat{\cdot}), Q^+_i(\chi) \}
\]

\[
\leq |P^+_i(\chi, w) - P^+_i(\chi, \hat{\cdot})|.
\]

(2.77)

Then, using the definition of \(P^+_i\), given by Algorithm 2.3, we get

\[
|P^+_i(\chi, w) - P^+_i(\chi, \hat{\cdot})| \leq \sum_{\ell \in \mathbb{Z}_h} |\max\{0, f_{\ell}(\chi, w)\} - \max\{0, f_{\ell}(\chi, \hat{\cdot})\}|.
\]

(2.78)

In order to bound \(I_1\) we will consider various cases for the the sign of the difference \(\chi_i - \chi_{\ell}, \ell \in \mathbb{Z}_h, \ell \neq i\). Let \(\chi_i - \chi_{\ell} > 0\) then

\[
|\max\{0, f_{\ell}(\chi, w)\} - \max\{0, f_{\ell}(\chi, \hat{\cdot})\}| \leq |d_{\ell}(w) - d_{\ell}(\hat{\cdot})| \leq |\chi_i - \chi_{\ell}|.
\]

(2.79)

For \(\chi_i - \chi_{\ell} \leq 0\) we get

\[
|\max\{0, f_{\ell}(\chi, w)\} - \max\{0, f_{\ell}(\chi, \hat{\cdot})\}| = 0.
\]

(2.80)

Further, since \(\chi_i - \chi_{\ell} > 0\) we have

\[
|\max\{0, f_{\ell}(\chi, w)\} - \max\{0, f_{\ell}(\chi, \hat{\cdot})\}| \leq |d_{\ell}(w) - d_{\ell}(\hat{\cdot})| \leq |\chi_i - \chi_{\ell}|.
\]

(2.81)

Therefore, combining the above cases (2.79)–(2.81), (2.49) and the fact that \(d_{\ell} = 0\) for \(\ell \notin \mathbb{Z}_h(\omega_i)\), we obtain

\[
I_1 + I_2 \leq \sum_{\ell \in \mathbb{Z}_h(\omega_i)} |d_{\ell}(w) - d_{\ell}(\hat{\cdot})| |\chi_i - \chi_{\ell}| \leq C h \|\nabla(w - \hat{\cdot})\|_{L^\infty(\omega_i)} \|\nabla \chi\|_{L^2(\omega_i)}.
\]

Hence, combining (2.77), (2.78) and (2.82) we get

\[
|d_{ij}(w)A_{ij}(\chi, w) - d_{ij}(\hat{\cdot})A_{ij}(\chi, \hat{\cdot})| \leq C h \|\nabla(w - \hat{\cdot})\|_{L^\infty(\omega_i)} \|\nabla \chi\|_{L^2(\omega_i)}.
\]

(2.83)

Therefore,

\[
|I| = |(d_{ij}(w)A_{ij}(\chi, w) - d_{ij}(\hat{\cdot})A_{ij}(\chi, \hat{\cdot}))| \leq \frac{\chi_i - \chi_j}{|\chi_i - \chi_j| + B_{ij}(\chi, w)} \leq C h \|\nabla(w - \hat{\cdot})\|_{L^\infty(\omega_i)} \|\nabla \chi\|_{L^2(\omega_i)}.
\]

(2.84)

We turn now to the estimation of \(II\). We can rewrite \(II\) as,

\[
II = d_{ij}(\hat{\cdot}) \left\{ \frac{A_{ij}(\chi, \hat{\cdot})}{|\chi_i - \chi_j| + B_{ij}(\chi, \hat{\cdot})} \right\} \left( B_{ij}(\chi, \hat{\cdot}) - B_{ij}(\chi, w) \right)
\]

(2.85)

Similarly as we derived (2.83) and in view of the definition of \(B_{ij}\) we get, for \(d_{ij} > 0\),

\[
|d_{ij}(w)B_{ij}(\chi, w) - d_{ij}(\hat{\cdot})B_{ij}(\chi, \hat{\cdot})| = |\tilde{P}^+_i(\chi, w) - \tilde{P}^+_i(\chi, \hat{\cdot})| \leq C h \|\nabla(w - \hat{\cdot})\|_{L^\infty(\omega_i)} \|\nabla \chi\|_{L^2(\omega_i)}.
\]

Therefore

\[
|a_{ij}(x, \hat{\cdot})| \leq C h \|\nabla(w - \hat{\cdot})\|_{L^\infty(\omega_i)} \|\nabla \chi\|_{L^2(\omega_i)}
\]

Further

\[
|a_{ij}(x, \hat{\cdot})| \leq C h \|\nabla(w - \hat{\cdot})\|_{L^\infty(\omega_i)} \|\nabla \chi\|_{L^2(\omega_i)}
\]

(2.87)

Therefore (2.85), (2.86) and (2.87) give

\[
|II| \leq C \|\nabla(w - \hat{\cdot})\|_{L^2(\omega_i)} \|\nabla \chi\|_{L^2(\omega_i)}
\]

Hence combining (2.75), (2.76), (2.84), (2.88) we obtain the desired result. \(\square\)

**Lemma 2.20.** There exists a constant \(C\) such that, for \(w, \hat{w}, \psi, \chi \in \mathcal{S}_h,\)

\[
|\bar{d}_h(\psi, w; \chi) - \bar{d}_h(\psi, \hat{w}; \psi, \chi)\| \leq C h \|\nabla(w - \hat{\cdot})\|_{L^\infty} \|\nabla \psi\|_{L^2} \|\nabla \chi\|_{L^2}.
\]
Proof. Using (2.34), we have,
\[ \bar{d}_h(\psi, w; \psi, \chi) = \sum_{i < j} (\bar{\rho}_{ij}(\psi, w) - \bar{\rho}_{ij}(\psi, \bar{w}))(\chi_i - \chi_j), \]
where \( \bar{\rho}_{ij}(\psi, w) := d_{ij}(w)\rho_{ij}(\psi, w)(\psi_i - \psi_j), \) \( \forall \psi \in \mathcal{S}_h. \) In view of Lemma 2.19, we get
\[ |\bar{d}_h(\psi, w; \psi, \chi) - \bar{d}_h(\psi, \bar{w}; \psi, \chi)| \leq C \sum_{K \in \mathcal{T}_h} \sum_{\gamma \in \mathcal{E}_K} |\bar{\rho}_{ij}(\psi, w) - \bar{\rho}_{ij}(\psi, \bar{w})||\chi_i - \chi_j| \]
\[ \leq C \|v\|_{L^\infty} \|\nabla \psi\|_{L^2} \|\nabla \chi\|_{L^2}. \]

\[ \square \]

Lemma 2.21. Let \( v, w \in \mathcal{S}_h, u \in W_p^1, \) with \( p \in (2, \infty) \) and \( c \in W^2_\infty. \) Then, there exists a constant \( C \) independent of \( h \) such that for \( \chi \in \mathcal{S}_h, \)
\[ \|u \nabla c - v \nabla w, \nabla \chi\| \leq C (h^{3/2} \|\log h\| + \|\nabla w\|_{L^\infty} (h^2 + \|v - R_h u\|_{L^2}) + \|\nabla (w - R_h c)\|_{L^2}) \|\nabla \chi\|_{L^2}. \]

Proof. We easily get the following splitting
\[ (u \nabla c - v \nabla w, \nabla \chi) = (u \nabla (c - R_h c), \nabla \chi) + (u \nabla (R_h c - w), \nabla \chi) \]
\[ + ((u - R_h u) \nabla w, \nabla \chi) + ((R_h u - v) \nabla w, \nabla \chi) = I_1 + \cdots + I_4. \]
Next, we will bound each one of the terms \( I_i, i = 1, 2, 3, 4. \) Note that \( I_1 \) can be rewritten as\[ I_1 = (u \nabla c, \nabla (c - R_h c)), \]
Then applying integration by parts, to get
\[ I_1 = \sum_{K \in \mathcal{T}_h} \int_K (\nabla (u \nabla c)(c - R_h c)) dx - \int_{\partial K} (c - R_h c) u \nabla \chi \cdot n_K ds, \]
where \( n_K \) is the unit normal vector on \( \partial K. \) Employing now the trace inequality (2.4), the error estimates for \( R_h, (2.40), \) and the fact that \( \chi \) is linear on \( K, \) we obtain
\[ |\int_{\partial K} (c - R_h c) u \nabla \chi \cdot n_K ds| \leq C h^{-1/2} \|c - R_h c\|_{L^\infty(K)} \|u\|_{H^1(K)} \|\nabla \chi\|_{L^2(K)} \]
\[ \leq C (c) h^{3/2} \|\log h\| \|u\|_{H^1(K)} \|\nabla \chi\|_{L^2(K)}, \]
where the constant \( C \) does not depend on \( K \in \mathcal{T}_h. \) Hence, summing over all \( K \in \mathcal{T}_h \) and using Cauchy-Schwartz inequality we have
\[ |\sum_{K \in \mathcal{T}_h} \int_{\partial K} (c - R_h c)(u \nabla \chi) \cdot n ds| \leq C (u, c) h^{3/2} \|\log h\| \|\nabla \chi\|_{L^2}. \]
Thus employing the above and (2.40) in (2.89), we get
\[ |I_1| \leq C (u, c) h^{3/2} \|\log h\| \|\nabla \chi\|_{L^2}. \]

Next, using again (2.40), we can easily bound \( I_2, I_3 \) and \( I_4 \) in the following way,
\[ |I_2| \leq \|u\|_{L^\infty} \|\nabla (w - R_h c)\|_{L^2} \|\nabla \chi\|_{L^2}, \]
\[ |I_3| \leq C \|u - R_h u\|_{L^2} \|\nabla w\|_{L^\infty} \|\nabla \chi\|_{L^2} \leq Ch^2 \|\nabla w\|_{L^\infty} \|\nabla \chi\|_{L^2}, \]
\[ |I_4| \leq \|\nabla w\|_{L^\infty} \|v - R_h u\|_{L^2} \|\nabla \chi\|_{L^2}. \]
Therefore, combining the above bounds for \( I_i, i = 1, 2, 3, 4, \) we obtain the desired estimate. \[ \square \]

3. Fully discrete scheme
Let \( N_T \in \mathbb{N}, N_T \geq 1, k = T/N_T \) and \( t^n = nk, n = 0, \ldots, N_T. \) Discretizing in time (2.35)–(2.36) with the backward Euler method we approximate \((u^n, c^n) = (u^k, t^n), c^k, t^n))\) by \((U^n, C^n) \in \mathcal{S}_h \times \mathcal{S}_h, \) for \( n = 0, 1, \ldots, N_T, \) such that,
\[ \bar{d}(U^n, \chi)_h + (\nabla U^n - \mu c^n \nabla C^n, \nabla \chi) + \bar{d}(U^n, C^n, U^n, \chi, h) = 0, \]
\[ \bar{d}(C^n, \chi)_h + (\nabla C^n, \nabla \chi) + (C^n - U^n, \chi)_h = 0, \]
for \( \chi \in \mathcal{S}_h \) and with \( U^0 = u^0_h \in \mathcal{S}_h, C^0 = c^0_h \in \mathcal{S}_h \) and \( \bar{d}U^n = (U^n - U^{n-1})/k. \)
3.1. Error estimates. Given $M > 0$, let $\mathcal{B}_M \subset \mathcal{S}_h \times \mathcal{S}_h$ be such that
\begin{equation}
\mathcal{B}_M := \{ (\chi, \psi) \in \mathcal{S}_h \times \mathcal{S}_h : \|\chi\|_{L^\infty} + \|\nabla \psi\|_{L^\infty} \leq M \}.
\end{equation}
We choose $M = \max(2M_0, 1)$ where $M_0$ depends on the solution of $u, c$ of (1.1) and is defined in (2.44).
In order to show existence of the solution $(U^n, C^n)$ of (3.1)-(3.2), for $n = 0, \ldots, N_T$, we will employ a fixed point argument and show that $(U^n, C^n) \in \mathcal{B}_M$, $n = 0, \ldots, N_T$. Note that in view of (2.44), $(R_n u^0, R_n c^0) \in \mathcal{B}_M$.

In the sequel we will assume that the constants $q_i$, $i \in \mathcal{N}_2$, in Algorithm 2.3 are such that Lemma 2.7 holds and are defined by (2.24). Thus if $(U^n, C^n) \in \mathcal{B}_M$, in view of (2.28), we have
\begin{equation}
\|q\|_{\text{max}} \leq Ch_M.
\end{equation}
Therefore there exists $h_M$ such that for $h \leq h_M$, $\|q\|_{\text{max}}$ can be sufficiently small.

**Remark 3.1.** Note that if Lemma 2.7 holds then there exist small $\delta > 0$ and $h_M > 0$ such that for $h < h_M$, $O(h^{3/2}|\log h| + h\|q\|_{\text{max}}) = O(h^{3/2-\delta})$.

**Theorem 3.2.** Let $(u, c)$ be a unique, sufficiently smooth, solution of (1.1), with $u \in W^2_0$ and $c \in W^2_\infty$. Also, let $n_0 \geq 1$ such that $(U^n, C^n)_{n=0}^{n_0} \in \mathcal{B}_M$, is the unique solution of (3.1)-(3.2) with $U^0 = R_h u^0$ and $C^0 = R_h c^0$. Then for $k, h$ sufficiently small, there exists $C_1 = C_1(M) > 0$, independent of $k, h, n_0$ such that for $n = 0, \ldots, n_0$, we have
\begin{equation}
\|R_h u^n - U^n\|_{L^2} + \|R_h c^n - C^n\|_{L^1} \leq C_1(k + k^{-1/2}h^2 + h^{3/2-\delta}), \quad \text{for small } \delta > 0.
\end{equation}
Proof. Let $\theta^n = U^n - R_h u^n$, $\rho^n = R_h u^n - u^n$, $\rho^n = C^n - R_h c^n$, and $\eta^n = R_h c^n - c^n$, for $n \geq 0$. Then, for $\theta^n$, $n = 1, \ldots, N_T$, we get the following error equation,
\begin{equation}
(\bar{\partial} \theta^n, \chi) + (\nabla \theta^n, \nabla \chi) = -(\omega^n, \chi)_h + (\delta^n, \nabla \chi) + (\rho^n, \chi) - \bar{d}_h(U^n, C^n; U^n, \chi) + \varepsilon_h(P_h u^n, \chi),
\end{equation}
with $\chi \in \mathcal{S}_h$, and
\begin{equation}
\omega^n = (\bar{d}_h R_h u^n - P_h u^n) \quad \text{and} \quad \delta^n = \lambda(u^n \nabla c^n - U^n \nabla C^n).
\end{equation}
Using the error estimations for $R_h$ in (2.40), we easily obtain
\begin{equation}
\|\omega^n\|_{L^2} + \|\rho^n\|_{L^2} \leq C(k + h^2), \quad \text{for } n = 1, \ldots, N_T.
\end{equation}
Also, Lemma 2.21, and the fact that $(U^n, C^n) \in \mathcal{B}_M$, gives
\begin{equation}
(\|\delta^n\|_{\nabla \chi}), \leq C_M(h^{3/2}|\log h| + \|\theta^n\|_{\nabla \chi} + \|\nabla \chi\|_{L^2})\|\nabla \chi\|_{L^2}, \quad \text{for } n = 1, \ldots, n_0,
\end{equation}
where $C_M$ is a general positive constant that depends on $M$.
In addition, the stabilization term on the right hand side may be written as
\begin{align*}
-\bar{d}_h(U^n, C^n; U^n, \chi) &= (-\bar{d}_h(U^n, C^n; U^n, \chi) + \bar{d}_h(R_h u^n, C^n; u^n, \chi)) \\
&- \bar{d}_h(R_h u^n, C^n; R_h u^n, \chi) = I_1 + I_2,
\end{align*}
where for the estimation of $I_1$, we employing Lemma 2.18, (3.4), the fact that $(U^n, C^n) \in \mathcal{B}_M$, to take
\begin{equation}
|I_1| \leq C(h\|\nabla C^n\|_{L^\infty} + \|q\|_{\text{max}}\|\nabla \theta^n\|_{L^2}|\nabla \chi\|_{L^2}) \leq CM h\|\nabla \theta^n\|_{L^2}|\nabla \chi\|_{L^2},
\end{equation}
while for $I_2$, we use Lemma 2.17, (3.4), the fact that $(U^n, C^n) \in \mathcal{B}_M$ and (2.40), we have
\begin{equation}
|I_2| \leq CM h(\|\nabla (R_h u^n - u^n)\|_{L^2}^2 + h^2\|u^n\|_{L^2}^2)^2/2|\nabla \chi\|_{L^2}^2.
\end{equation}
Thus, for this choice,
\begin{equation}
|I_1| \leq CM h\|\nabla \theta^n\|_{L^2}|\nabla \chi\|_{L^2}.
\end{equation}
Also, employing Lemma 2.8 and (2.39), we get
\begin{equation}
|\varepsilon_h(P_h u^n, \chi)| \leq C h^2\|\nabla P_h u^n\|_{L^2}\|\nabla \chi\|_{L^2} \leq Ch^2\|\nabla \chi\|_{L^2}.
\end{equation}
Note now that in view of the symmetry of $(\cdot, \cdot)_h$, we obtain
\begin{equation}
(\bar{d}_h \theta^n, \theta^n)_h = \frac{1}{2k}(\|\theta^n\|_h^2 - \|\theta^{n-1}\|_h^2) + \frac{k}{2}\|\bar{d}_h \theta^n\|_h^2.
\end{equation}
Hence, choosing $\chi = \theta^n$ in (3.6) and using (3.11) we obtain
\begin{equation}
\frac{1}{2k}(\|\theta^n\|_h^2 - \|\theta^{n-1}\|_h^2) + \frac{k}{2}\|\bar{d}_h \theta^n\|_h^2 \\
\leq C(h^2\|\nabla \theta^n\|_{L^2}^2 + \|\theta^n\|_{L^2} + |\delta^n|_{L^2})\|\nabla \theta^n\|_{L^2} + CM h\|\nabla \theta^n\|_{L^2},
\end{equation}
In view of (3.4) there exist \( h_M > 0 \) such that for \( h < h_M \) we can eliminate \( \|\nabla \theta^\alpha\|_{L^2} \). Then \( h < h_M \), since there exists \( h_M \) such that for \( h < h_M \) for \( h \) sufficiently small using (3.11), eliminating \( \|\nabla \theta^\alpha\|_{L^2} \) for \( h \) sufficiently small, employing the estimations (3.7)–(3.10) and Remark 3.1, we get

\[
\|\theta^\alpha\|^2_h \leq \|\theta^{\alpha-1}\|^2_h + C_M k (\|\nabla \zeta^\alpha\|^2_{L^2} + \|\theta^\alpha\|^2_h + E^n),
\]

with \( E^n = O(k^2 + h^{3-2\epsilon}) \), for \( \epsilon > 0 \) small.

Next, in view of (3.2), we get the following error equation for \( \zeta^\alpha \),

\[
(\mathcal{D}\zeta^\alpha, \chi) + (\nabla \zeta^\alpha, \nabla \chi) + (\zeta^\alpha, \chi)_h = -((\tilde{\omega}^\alpha, \chi)_h + (\rho^\alpha, \chi) + \varepsilon_h(\tilde{\omega}^\alpha, \chi),
\]

where

\[
\tilde{\omega}^\alpha = (\partial_R h c^\alpha - P_h c^\alpha_r) + \theta^\alpha, \quad \omega^\alpha = P_h c^\alpha_r + R_h(c^\alpha - u^\alpha).
\]

Thus for sufficiently smooth \( u \) and \( c \) we get

\[
\|\tilde{\omega}^\alpha\|_{L^2} \leq C(k + h^2 + \|\theta^\alpha\|_h) \quad \text{and} \quad |\varepsilon_h(\tilde{\omega}^\alpha, \chi)| \leq C(u, c)h^2 \|\nabla \chi\|_{L^2}.
\]

Let

\[
\|\chi\|^2 = \|\chi\|^2_h + \|\nabla \chi\|^2_{L^2}.
\]

Choosing now \( \chi = \mathcal{D}\zeta^\alpha \) in (3.14), using similar arguments as before and (3.15), we get

\[
\|\zeta^\alpha\|^2 \leq \|\zeta^{\alpha-1}\|^2 + Ck(k^2 + k^{-1}h^4 + \|\theta^\alpha\|_h^2).
\]

Then combining (3.13) and (3.16) we get

\[
\|\theta^\alpha\|^2_h + \|\zeta^\alpha\|^2 \leq \|\theta^{\alpha-1}\|^2_h + \|\zeta^{\alpha-1}\|^2 + C_M k (\|\theta^\alpha\|^2_h + \|\zeta^\alpha\|^2 + \tilde{E}^n),
\]

with \( \tilde{E}^n = O(k^2 + h^{3-2\epsilon} + k^{-1}h^4) \). Next, moving \( \|\theta^\alpha\|^2_h + \|\zeta^\alpha\|^2 \) to the left, we have for \( k \) sufficiently small and \( n = 1, \ldots, n_0 \),

\[
\|\theta^\alpha\|^2_h + \|\zeta^\alpha\|^2 \leq (1 + C_M k)(\|\theta^{\alpha-1}\|^2_h + \|\zeta^{\alpha-1}\|^2) + C_M k \tilde{E}^n.
\]

Hence, since \( \|\theta^0\|_h = \|\zeta^0\| = 0 \), summing over \( n \), we further get, for \( E = \max, \tilde{E}^2 = O(k^2 + h^{3-2\epsilon} + k^{-1}h^4) \)

\[
\|\theta^n\|^2_h + \|\zeta^n\|^2 \leq C(\|\theta^0\|^2_h + \|\zeta^0\|^2) \leq C_M kE \sum_{\ell=0}^{n} (1 + C_M k)^{n-\ell+1} \leq C_1(k^2 + k^{3-2\epsilon} + k^{-1}h^4),
\]

which gives the desired estimate (3.5), with \( C_1 \) depending on \( M \).

Note, that in view of (3.4) and (3.4) in (3.19) we were able to derive a superconvergence error estimate in \( H^1 \)-norm for \( C^n - R_h c^\alpha \) in \( S_h \), \( n = 0, \ldots, n_0 \), which we will employ to show existence of \( (U^n, C^n) \) for \( n = n_0 + 1 \). Also, in view of Theorem 3.2, we can easily derive error bounds for \( U^n - u^\alpha \) and \( C^n - c^\alpha \), \( n = 0, \ldots, n_0 \).

**Theorem 3.3.** Let \( (u, c) \) be a unique, sufficiently smooth, solution of (1.1), with \( u \in W^p_2 \), with \( p \in (2, \infty] \) and \( c \in W^\infty_\infty \). Also, let \( \{U^n, C^n\}_{n=0}^{n_0} \in B_{M, n_0} \geq 1 \) be the unique solution of (3.1)–(3.2) with \( U^0 = R_h u^0 \) and \( C^0 = R_h c^0 \). Then for \( k, h \) sufficiently small, there exists \( C = C(M) > 0 \) independent of \( k, h, n_0 \) such that for \( n = 1, \ldots, n_0 \),

\[
\|U^n - u^n\|_{L^2} \leq C(k + k^{-1/2}h^2 + h^{3-2\epsilon}) \quad \text{and} \quad \|C^n - c^n\|_1 \leq C(k + k^{-1/2}h^2 + h).
\]

**Proof.** Using the error splittings \( U^n - u^\alpha = (U^n - R_h u^\alpha) + (R_h u^\alpha - u^n) \), \( C^n - c^\alpha = (C^n - R_h c^\alpha) + (R_h c^\alpha - c^\alpha) \), Theorem 3.2 and the approximation properties of \( R_h \) in (2.40), we can easily get the desired error estimates. \( \square \)

### 3.2. Existence

Let us assume that the discretization parameters \( k \) and \( h \) are sufficiently small and satisfy \( k = O(h^{1+\epsilon}) \), with \( 0 < \epsilon < 1 \). Then, using a standard contradiction argument, we will show that there exists a solution of the fully discrete scheme (3.1)–(3.2), if \( U^0 = R_h u^1 \), \( C^0 = R_h c^0 \).

We consider the following iteration operator \( \mathcal{G}^\alpha = (\mathcal{G}^{\alpha_1}_1, \mathcal{G}^{\alpha_2}_2) : S_h \times S_h \to S_h \times S_h, (v, w) \to (\mathcal{G}^{\alpha_1}_1 v, \mathcal{G}^{\alpha_2}_2 w) \) defined by

\[
(\mathcal{G}^{\alpha_1}_1 v - U^{n-1}, \chi)_h + k(\nabla \mathcal{G}^{\alpha_1}_1 v, \nabla \chi) + \lambda (\mathcal{G}^{\alpha_1}_1 v, v, \chi) + k d_h (v, \mathcal{G}^{\alpha_1}_1 v, \chi, v, \chi) = \tilde{d}_h (v, w, v, \chi), \quad \forall \chi \in S_h,
\]

\[
(\mathcal{G}^{\alpha_2}_2 w - C^{n-1}, \chi)_h + k(\nabla \mathcal{G}^{\alpha_2}_2 w, \nabla \chi) + k (\mathcal{G}^{\alpha_2}_2 w, v, \chi) = 0, \quad \forall \chi \in S_h,
\]

where, for \( w, s \in S_h \), the bilinear form \( \tilde{d}_h (w, s; v, z) : S_h \times S_h \to \mathbb{R} \), is defined by \( \tilde{d}_h (w, s; v, z) := \tilde{d}_h (w; v, z) - \tilde{d}_h (s; v, z) \), therefore

\[
\tilde{d}_h (w, s; v, z) = \sum_{i,j=1}^N d_{ij}(w)(1 - \rho_{ij}(s))(v_i - v_j)z_i.
\]

In particular, for the low-order scheme we get \( \tilde{d}_h = 0 \), since for this case \( \rho_{ij} = 1 \). Note that, for \( \tilde{d}_h \) Lemmas 2.11 and 2.18 also hold. Obviously, if \( \mathcal{G}^\alpha \) has a fixed point \((v, w)\), then \((U^n, C^n) := (v, w)\) is the solution of the discrete scheme (3.1)–(3.2).
Lemma 3.5. Let \( (3.24) \) where \( v, w \) be a unique, sufficiently smooth, solution of \( G^i v, G^2 w \) where \( 0 < \epsilon < 1 \), there exists a sufficiently large constant \( C_2 \), independent of \( h \), such that \( C_2 > (2C_1 + 1) \) and \((U^{n-1}, C^n-1) \in \hat{B}_n \), where

\[
\hat{B}_n := \{ (\chi, \psi) \in S_h \times S_h : \| \chi - R_h u^n \|_{L^2} + \| \nabla (\psi - R_h c^n) \|_{L^2} \leq C_2 h^{1+\tilde{\epsilon}} \},
\]

where \( \tilde{\epsilon} = \min \{1/2, -\epsilon, (1-\epsilon)/2 \} \) and \( \epsilon \) given by Remark 3.1.

Proof. Using the stability property of \( R_h \), Remark 3.1 and the fact that \( k = O(h^{1+\epsilon}) \), we have

\[
\begin{align*}
&\| U^{n-1} - R_h u^n \|_{L^2} + \| \nabla (C^n-1 - R_h c^n) \|_{L^2} \\
&\quad \leq \| U^{n-1} - R_h u^n \|_{L^2} + \| \nabla (C^n-1 - R_h c^n) \|_{L^2} + k \| R_h \bar{\partial} u^n \|_{L^2} + \| \nabla R_h \bar{\partial} c^n \|_{L^2} \\
&\quad \leq C_1 (k + k^{-1/2} h^2 + h^{3/2-\epsilon}) + C(u, c) k \leq C_2 h^{1+\tilde{\epsilon}}.
\end{align*}
\]

Remark 3.6. In view of the inverse inequalities (2.3), for sufficiently small \( h \) we have that if \( (\chi, \psi) \in \hat{B}_n \) then \( (\chi, \psi) \in B_M \)

Lemma 3.7. Let \( (u, c) \) be a unique, sufficiently smooth, solution of \( (1.1) \), with \( u \in W^2_0 \), and \( p \in (2, \infty] \) and \( c \in W^2_{\infty} \). Also let \((U^{n-1}, C^n-1) \in B_M \) satisfying (3.25) holds and \( (v, w) \in \hat{B}_n \). Then for \( k = O(h^{1+\epsilon}) \) with \( 0 < \epsilon < 1 \), we have \( \{ G^1 v, G^2 w \} \in \hat{B}_n \cap B_M \).

Proof. Let \( p_n = G^1 v - R_h u^n, p_{n-1} = U^{n-1} - R_h u^n, z_n = G^2 w - R_h c^n, \) and \( z_{n-1} = C^n-1 - R_h c^n \). Then, in view of (3.20), \( p_n \) satisfies the error equation

\[
\begin{align*}
&\tilde{\partial} p_n, \chi) + (\nabla p_n, \nabla \chi) + d_h(w; p_n, \chi) = -(w^n, \chi) + (\rho^n, \chi) + (\delta^n, \nabla \chi) \\
&\quad - \epsilon h (p_h u^n, \chi) - d_h(w; R_h u^n, \chi) + \tilde{d}_h(R_h c^n, R_h u^n; R_h u^n, \chi) \\
&\quad + \{ d_h(w, v; v, \chi) - \tilde{d}_h(R_h c^n, R_h u^n; R_h u^n, \chi) \}, \quad \forall \chi \in S_h,
\end{align*}
\]

where

\[
\begin{align*}
\omega^n &= (\tilde{\partial} R_h u^n - P_h u^n), \quad \rho^n = R_h u^n - u^n, \quad \delta^n = \lambda (u^n \nabla c^n - G^1 v \nabla w).
\end{align*}
\]

Using the approximation properties of \( R_h \), (2.40), we easily obtain

\[
\begin{align*}
&\| \omega^n \|_{L^2} + \| \rho^n \|_{L^2} \leq C(k + h^2).
\end{align*}
\]

Next, in view of Lemma 2.21 and the fact that \( (v, w) \in B_M \), cf. Remark 3.6, and Remark 3.1, we have for \( h \) sufficiently small

\[
\begin{align*}
&\| (\delta^n, \nabla \chi) \| \leq C(1 + M \| p_n \|_{L^2} + \| \nabla (w - R_h c^n) \|_{L^2}) \| \nabla \chi \|_{L^2}.
\end{align*}
\]

We can easily rewrite (3.20)–(3.21) in matrix formulation. For this, we introduce the following notation. Let \( \alpha = (\alpha_1, \ldots, \alpha_N)^T \) and \( \beta = (\beta_1, \ldots, \beta_N)^T \), the coefficients, with respect to the basis of \( S_h \), of \( v, w \in S_h \), respectively, and \( \tilde{\alpha}, \tilde{\beta} \in \mathbb{R}^N \) the corresponding vectors for \( G^1 v, G^2 w \), respectively. Then (3.20)–(3.21) can be written as

\[
\begin{align*}
A_1^1 \tilde{\alpha} &= b_{\alpha, \beta} \quad \text{and} \quad A_2^2 \tilde{\beta} = \epsilon^2,
\end{align*}
\]

where

\[
\begin{align*}
A_1^1 &= M_L + k (S - T_\beta - D_\beta), \quad b_{\alpha, \beta} = M_L \alpha^{n-1} + k T^\alpha (\tilde{\alpha}, \tilde{\beta}), \\
A_2^2 &= M_L + k (M_L + S), \quad b_2 = M_L \beta^{n-1} + k M_L \alpha^{n-1}.
\end{align*}
\]

For the matrix \( T_\beta \), introduced in (1.8), we can show that the elements of every column of \( T_\beta \) have zero sum.

Lemma 3.4. Let \( T_h \) satisfy Assumption 2.1, then the matrix \( T_\beta \), introduced in (1.8), has zero-column sum for all \( t \geq 0 \).

Proof. For the proof see Appendix A.
Further, using the definition of $\tilde{d}_h$, see (3.22), we get
\[
-\tilde{d}_h(R_h u^n; R_h c^n; R_h u^n; \chi) = \tilde{d}_h(R_h c^n; R_h u^n; R_h u^n; \chi) - \tilde{d}_h(R_h c^n; R_h u^n; \chi),
\]
and therefore
\[
-\tilde{d}_h(w; R_h u^n, \chi) + \tilde{d}_h(R_h c^n; R_h u^n; R_h u^n; \chi) = (d_h(R_h c^n; R_h u^n; \chi)) - \tilde{d}_h(R_h u^n, R_h c^n; R_h u^n; \chi) = J_1 + J_2,
\]
where $J_1$, $J_2$ may be estimated in view of Lemma 2.12 (with an application of the inverse inequality (2.3)) and the fact that $(v, w) \in \mathcal{B}_n$, we have for $h$ sufficiently small, i.e.,
\[
|J_1| \leq C\|\nabla (w - R_h c^n)\|_{L^2}\|\nabla R_h u^n\|_{L^2}\|\nabla \chi\|_{L^2} \leq CC_2 h^{1+\varepsilon}\|\nabla \chi\|_{L^2}.
\]
Also, using Lemma 2.17, and (3.4) we get
\[
|J_2| \leq C(h\|\nabla R_h c^n\|_{L^\infty} + \|q\|_{\text{max}})(\|\nabla (R_h u^n - u^n)\|_{L^2}^2 + h^2\|u^n\|_{L^2}^{4/2})\|\nabla \chi\|_{L^2}
\leq C h^2\|\nabla \chi\|_{L^2}.
\]
Next, we rewrite the last term in (3.27), such that
\[
\tilde{d}_h(w; v, \chi) - \tilde{d}_h(R_h c^n, R_h u^n; R_h u^n; \chi)
= \left(\tilde{d}_h(w; v, \chi) - \tilde{d}_h(w, R_h u^n; R_h u^n, \chi)\right)
+ \left(\tilde{d}_h(w, R_h u^n; R_h u^n, \chi) - \tilde{d}_h(R_h c^n, R_h u^n; R_h u^n, \chi)\right) = I_1 + I_2.
\]
Note that Lemmas 2.18 and 2.17 are valid also for $\tilde{d}_h$. To estimate the first term, $I_1$, we use Lemma 2.18 together with an inverse estimate, (2.3), the fact that $(v, w) \in \mathcal{B}_n$, and (3.4),
\[
|I_1| \leq C(h\|\nabla w\|_{L^\infty} + \|q\|_{\text{max}})\|\nabla (v - R_h u^n)\|_{L^2}\|\nabla \chi\|_{L^2} \leq Ch^{-1}hM\|v - R_h u^n\|_{L^2}\|\nabla \chi\|_{L^2}
\leq CMC_2 h^{1+\varepsilon}\|\nabla \chi\|_{L^2}.
\]
Next, the second term, $I_2$, can be estimated by using Lemma 2.17 and Remark (3.4), i.e.,
\[
|I_2| \leq CMh(\|\nabla (R_h u^n - u^n)\|_{L^2}^2 + h^2\|u^n\|_{L^2}^{4/2})\|\nabla \chi\|_{L^2} \leq CM h^2\|\nabla \chi\|_{L^2}
\]
Choosing now $\chi = p_n$ in (3.27), employing the corresponding identity as in (3.11) for $p_n$, combining the previous estimations (3.10), (3.28)-(3.33), and eliminating $\|\nabla p_n\|_{L^2}$ we get
\[
\|p_n\|_{H}^2 \leq \|p_n-1\|_{h}^2 + C_M k(\|p_n\|_{H}^2 + k^2 + h^{2+2\varepsilon}).
\]
with $C_M$ denoting a constant that depends on $C_2$ and $M$.

Next, in view of (3.21), we have the following error equation for $z_n$,
\[
(\overline{\partial} z_n, \chi) + (\nabla z_n, \nabla \chi) + (z_n, \chi) = - (\omega^n, \chi) + (\rho^n, \chi) - \varepsilon_h(\tilde{\omega}^n, \chi),
\]
where
\[
\omega^n = u^n - v + (\overline{\partial} R_h c^n - P_h c^n), \quad \text{and} \quad \tilde{\omega}^n = R_h (c^n - u^n) + P_h c^n.
\]
Employing now (2.40) and the fact that $(v, w) \in \mathcal{B}_n$, we obtain
\[
\|\omega^n\|_{L^2} + \|\rho^n\|_{L^2} \leq C(k + h^2) + C_2 h^{1+\varepsilon} \quad \text{and} \quad |\varepsilon_h(\tilde{\omega}^n, \chi)| \leq C h^2\|\nabla \chi\|_{L^2}.
\]
Choosing now $\chi = \overline{\partial} z_n$ in (3.35) and similar arguments as before, we get
\[
\|z_n\|_{L^2} \leq \|z_n-1\|_{H}^2 + C_M k(\|\omega^n\|_{L^2}^2 + h^{2+2\varepsilon}) + C h^4.
\]
Then combining (3.34) and (3.36), we have for $k$ sufficiently small
\[
\|p_n\|_{H}^2 + \|z_n\|_{L^2}^2 \leq (1 + C_M k)(\|p_n-1\|_{H}^2 + \|z_n-1\|_{H}^2) + C_M k E,
\]
with $E = \mathcal{O}(k^2 + h^{2+2\varepsilon} + k^{-1}h^4)$. Then for $k = \mathcal{O}(h^{1+\varepsilon})$ sufficiently small, we obtain
\[
\|p_n\|_{H}^2 + \|z_n\|_{L^2}^2 \leq C_2^{-1} h^{2+2\varepsilon}.\]

Hence, $(G_1^p v, G_2^p w) \in \mathcal{B}_n$. Finally, employing the inverse inequality (2.3), the fact that $(G_1^p v, G_2^p w) \in \mathcal{B}_n$ and (2.43), we get sufficiently small $h$
\[
\|G_1^p v\|_{L^\infty} + \|\nabla G_2^p w\|_{L^\infty} \leq C h^{-1}(\|p_n\|_{L^2} + \|\nabla z_n\|_{L^2}) + \|R_h u^n\|_{L^\infty} + \|\nabla R_h c^n\|_{L^\infty}
\leq C h^2 + M_0 \leq M.
\]

Therefore $(G_1^p v, G_2^p w) \in \mathcal{B}_M$, which concludes the proof. \qed
Theorem 3.8. Let \((u, c)\) be a unique, sufficiently smooth, solution of (1.1), with \(u \in W^2_\infty\) and \(p \in (2, \infty]\) and \(c \in W^2_\infty\). Let the correction factors \(\alpha_{ij}\), for \(i, j = 1, \ldots, N\), be defined as in Lemma 2.13. If \((U^{n-1}, C^{n-1}), (v, w) \in B_M\), such that (3.25) holds. Then for \(h \) sufficiently small, \(k = \mathcal{O}(h^{1+\epsilon})\) with \(\epsilon > 0\), there exists a unique solution \((U^n, C^n) \in B_M\) of the fully-discrete scheme (3.1)–(3.2).

Proof. Obviously, in view of Lemmas 3.5 and 3.7, starting with \((v_0, w_0) = (U^{n-1}, C^{n-1})\) through \(\mathcal{G}^n\), we obtain a sequence of elements \((v_{j+1}, w_{j+1}) = (\mathcal{G}^n_{j+1} v, \mathcal{G}^n_{j+1} w) \in \tilde{B}_n \cap B_M, j \geq 0\).

To show existence and uniqueness of \((U^n, C^n) \in \tilde{B}_n \cap B_M\), it suffices to show that there exists \(0 < L < 1\), such that
\[
||\mathcal{G}^n_v - \mathcal{G}^n_{v}||_{L^2} + ||\mathcal{G}^n_w - \mathcal{G}^n_{w}||_{L^1} \leq L(||v - \bar{v}||_{L^2} + ||w - \bar{w}||_{L^1}), \quad \forall (v, w), (\bar{v}, \bar{w}) \in \tilde{B}_n \cap B_M.
\]

Let \((v, w), (\bar{v}, \bar{w}) \in B_n \cap B_M\), with \(v := v - \bar{v}\) and \(w := w - \bar{w}\). In view of (3.20), we have
\[
\begin{align*}
\langle \mathcal{G}^n_v, \chi \rangle &= k\lambda \langle \mathcal{G}^n_v v \nabla \bar{w}, \nabla \chi \rangle + k \langle \mathcal{G}^n_v \bar{v} \nabla \bar{w}, \nabla \chi \rangle + k \langle d_h(w; \mathcal{G}^n_v \bar{v}), \chi \rangle \\
&\quad + k \langle d_h(w; \mathcal{G}^n_{\bar{v}} v), \chi \rangle \\
&= k \lambda \langle \mathcal{G}^n_v v \nabla \bar{w}, \nabla \chi \rangle - k \langle d_h(w; \mathcal{G}^n_v \bar{v}), \chi \rangle - k \langle d_h(w; \mathcal{G}^n_{\bar{v}} v), \chi \rangle \\
&\quad + k \langle d_h(w; v, v; \chi) \rangle + k \langle d_h(w; \bar{v}, \bar{v}; \chi) \rangle = I_1 + I_2 + I_3.
\end{align*}
\]

Then, in view of the fact that \((\mathcal{G}^n_v \bar{v}, \mathcal{G}^n_{\bar{v}} \bar{w}) \in \tilde{B}_n \cap B_M\) and (2.3)
\[
\begin{align*}
||\nabla \mathcal{G}^n_v \bar{v}||_{L^2} &\leq C ||\nabla (\mathcal{G}^n_v \bar{v} - R_h u^n)||_{L^2} + ||\nabla R_h u^n||_{L^2} \\
&\leq Ch^{-1} ||\mathcal{G}^n_v \bar{v} - R_h u^n||_{L^2} + ||\nabla R_h u^n||_{L^2} \leq Ch^n + M_0 \leq M,
\end{align*}
\]

Similarly, we obtain \(||\nabla v||_{L^2} \leq M\). Using the fact that \((v, w), (\bar{v}, \bar{w})\), \((\mathcal{G}^n_v v, \mathcal{G}^n_{\bar{v}} w) \in \tilde{B}_n \cap B_M\) and Lemmas 2.12, 2.18 and 2.20, (3.4) and (3.3), we get
\[
\begin{align*}
|I_1| &\leq CMk(\|\nabla \bar{w}\|_{L^2} + \|\mathcal{G}^n_v \bar{v}\|_{L^2})\|\nabla \chi\|_{L^2}, \\
|I_2| &\leq Ck(\|\nabla \bar{w}\|_{L^2} + \|\mathcal{G}^n_v \bar{v}\|_{L^2})\|\nabla \chi\|_{L^2} \leq CMk\|\nabla \bar{w}\|_{L^2} \|\nabla \chi\|_{L^2}, \\
|I_3| &\leq CkM(\|\nabla \bar{w}\|_{L^2} + \|\bar{w}\|_{L^2})\|\nabla \chi\|_{L^2},
\end{align*}
\]

where \(C_M\) denotes a general constant that depends on \(M\). Choosing \(\chi = \mathcal{G}^n_v \bar{v}\) in (3.37), using (3.38) and eliminating \(||\nabla \mathcal{G}^n_v \bar{v}\|_{L^2}||\), we get
\[
\|\mathcal{G}^n_v \bar{v}\|_{L^2}^2 \leq CMk(\|\mathcal{G}^n_v \bar{v}\|_{L^2}^2 + \|\bar{w}\|_{L^2}^2 + \|\bar{v}\|_{L^2}^2).
\]

Then for sufficiently small \(k\), we obtain
\[
\|\mathcal{G}^n_v \bar{v}\|_{L^2}^2 \leq CMk(\|\nabla \bar{w}\|_{L^2}^2 + \|\bar{v}\|_{L^2}^2).
\]

Next, in view of (3.21), we get
\[
(\mathcal{G}^n_{\bar{v}} \bar{w}, \chi)_h + k(\nabla \mathcal{G}^n_{\bar{v}} \bar{w}, \nabla \chi) + k(\mathcal{G}^n_{\bar{v}} \bar{w}, \chi) = k(\bar{v}, \chi)
\]

Choosing \(\chi = \mathcal{G}^n_{\bar{v}} \bar{w}\) in (3.40) we get
\[
\|\mathcal{G}^n_{\bar{v}} \bar{w}\|_{L^2}^2 + k\|\mathcal{G}^n_{\bar{v}} \bar{w}\|_{L^2}^2 \leq Ck\|\bar{v}\|_{L^2} \|\mathcal{G}^n_{\bar{v}} \bar{w}\|_{L^2}.
\]

which after eliminating \(||\mathcal{G}^n_{\bar{v}} \bar{w}\|_{L^2}||\) gives
\[
\|\mathcal{G}^n_{\bar{v}} \bar{w}\|_{L^2}^2 \leq Ck\|\bar{v}\|_{L^2}^2.
\]

Thus combining (3.39) and (3.41), we get
\[
\|\mathcal{G}^n_v \bar{v}\|_{L^2}^2 + \|\mathcal{G}^n_{\bar{v}} \bar{w}\|_{L^2}^2 \leq CMk(\|\bar{v}\|_{L^2}^2 + \|\bar{w}\|_{L^2}^2).
\]

Therefore, for \(k\) sufficiently small such that \(C_Mk < 1\), the sequence \((v_j, w_j) \to (U^n, C^n) \in \tilde{B}_n \cap B_M, j \to \infty\) and \((U^n, C^n)\) is the unique solution of the fully-discrete scheme (3.1)–(3.2). \(\square\)

4. Positivity

In this section we will demonstrate that the solution \((U^n, C^n)\) of the the fully discrete scheme (3.1)–(3.2) is nonnegative the initial approximations \((U^n, C^n)\) are non-negative.

The fully discrete scheme (3.1)–(3.2) may be expressed by splitting the bilinear form \(\mathcal{G}_h\) in a similar way as the iteration scheme (3.20)–(3.21),
\[
\begin{align*}
(U^n - U^{n-1}, \chi)_h + k(U^n, \nabla \chi) - \lambda k(U^n \nabla C^n, \nabla \chi) + k d_h(C^n; U^n, \chi) \\
&= k d_h(C^n, U^n; U^n, \chi), \quad \forall \chi \in S_h, \\
(C^n - C^{n-1}, \chi)_h + k(\nabla C^n, \nabla \chi) + k(C^n - U^n, \chi)_h = 0, \quad \forall \chi \in S_h.
\end{align*}
\]
We can easily rewrite (3.1)–(3.2) in matrix formulation. For this, we introduce the following notation. Let $\alpha^n = (\alpha^n_1, \ldots, \alpha^n_N)^T$ and $\beta^n = (\beta^n_1, \ldots, \beta^n_N)^T$, the coefficients, with respect to the basis of $\mathcal{S}_h$, of $U^n, C^n \in S_h$, respectively. Then (3.1)–(3.2) can be written as

\begin{equation}
A^1_{\beta^n} \alpha^n = b_{\alpha^n, \beta^n} \quad \text{and} \quad A^2 \beta^n = b^2,
\end{equation}

where

\begin{equation}
A^1_{\beta^n} = M_L + k (S - T_{\beta^n} - D_{\beta^n}), \quad b_{\alpha^n, \beta^n} = M_L \alpha^{n-1} + k \bar{T}^{n} (\alpha^n, \beta^n),
\end{equation}

\begin{equation}
A^2 = M_L + k (M_L + S), \quad b^2 = M_L \beta^{n-1} + k M_L \alpha^n.
\end{equation}

Next, we will show that for $T^0, C^0 \geq 0$, the solutions $\{U^n, C^n\}^N_{n=0}$ of the discrete scheme (3.1)–(3.2) are non-negative, for small $k$ and $k = \mathcal{O}(h^{1+\epsilon})$ for $0 < \epsilon < 1$.

**Theorem 4.1.** Let the correction factors $\alpha^n_{ij}$, for $i, j = 1, \ldots, N$, be defined as in Lemma 2.13. Then for $U^0, C^0 \geq 0$, the solutions $\{U^n, C^n\}^N_{n=0}$ of the discrete scheme (3.1)–(3.2) are non-negative, for small $k$ and $k = \mathcal{O}(h^{1+\epsilon})$ for $0 < \epsilon < 1$.

**Proof.** Note that in order to show the non-negativity of $\{U^n, C^n\}^N_{n=0}$ it suffices to show that the vectors $\alpha^n, \beta^n$ of the coefficients of $\{U^n, C^n\}^N_{n=0}$ with respect to the basis of $\mathcal{S}_h$, are positive elementwise, i.e., $\alpha^n \geq 0$ and $\beta^n \geq 0$. We will show this by induction. Since $U^0, C^0 \geq 0$, it suffices show that $U^n, C^n \geq 0$ if $U^{n-1}, C^{n-1} \geq 0$. The assumption that $U^{n-1}, C^{n-1} \geq 0$ implies that $\alpha^{n-1}, \beta^{n-1} \geq 0$, respectively. Therefore, in order to show the desired result we will show that $\alpha^n, \beta^n \geq 0$.

The fully discrete scheme (3.1)–(3.2) can equivalently be written in matrix formulation, i.e., (4.1)–(4.2). From the first system, i.e., $A^1_{\beta^n} \alpha^n = b_{\alpha^n, \beta^n}$, we obtain, for $i = 1, \ldots, N$,

$$(m_i + k (s_{ii} - \tau^{n}_{ii} - d^{n}_{ii}) ) \alpha^n_i + k \sum_{j \in \mathcal{Z}_h^{i}} (s_{ij} - \tau^{n}_{ij} - d^{n}_{ij}) \alpha^n_j = m_i \alpha^{n-1}_i + k \sum_{j \in \mathcal{Z}_h^{i}} a_{ij} \alpha^n_j (\alpha^n_i - \alpha^n_j),$$

where $T_{\beta^n} = (\tau^{n}_{ij})_{i,j=1}^N$ and $D_{\beta^n} = (d^{n}_{ij})_{i,j=1}^N$.

Let us assume that $\alpha^n_i = \min_j \alpha_j$. If $\alpha^n_i \geq 0$, then we have the desired result. Therefore we assume that $\alpha^n_i < 0$. In view of the fact that $Q_i^1 = q_i (\alpha^n_{\min} - \alpha_i) = 0$ and (2.20), we get

\begin{equation}
\sum_{j \in \mathcal{Z}_h^i} a_{ij} d_{ij} (\alpha^n_i - \alpha^n_j) = \sum_{j \in \mathcal{Z}_h^i} a_{ij} f_{ij} \geq Q_i - 0 = 0.
\end{equation}

Since $s_{ij} - \tau^{n}_{ij} - d^{n}_{ij} \leq 0$, $i \neq j$ and using the zero row sum property of $S$ and $D_{\beta^n}$, we obtain

$$(m_i - k \tau^{n}_{ii}) \alpha^n_i - k \sum_{j \in \mathcal{Z}_h^i} \tau^{n}_{ij} \alpha^n_i = (m_i + k (s_{ii} - \tau^{n}_{ii} - d^{n}_{ii}) + \sum_{j \in \mathcal{Z}_h^i} (s_{ij} - \tau^{n}_{ij} - d^{n}_{ij})) \alpha^n_i \geq (m_i + k (s_{ii} - \tau^{n}_{ii} - d^{n}_{ii}) \alpha^n_i + k \sum_{j \in \mathcal{Z}_h^i} (s_{ij} - \tau^{n}_{ij} - d^{n}_{ij}) \alpha^n_j = m_i \alpha^{n-1}_i.$$

Therefore $(m_i - k \sum_{K \in \omega_i} \tau^{n}_{ij}) \alpha^n_i \geq 0$. In view of Lemma 3.7 and Theorem 3.8, we have $\| \nabla C^n \|_{L^\infty} \leq M$. Therefore, since $k = \mathcal{O}(h^{1+\epsilon})$, for sufficiently small $h$, there exists constants $c_1, c_2 > 0$ such that

$$(m_i - k \sum_{K \in \omega_i} \tau^{n}_{ij}) \geq \sum_{K \in \omega_i} (c_1 h^2 K - c_2 M h^{1+\epsilon} h_K) > 0.$$ 

Hence $\alpha^n_i \geq 0$, for $j \in \mathcal{N}_h$ with $j \in \mathcal{Z}_h^{i}$ and thus $\alpha^n_i \geq 0$ for $j \in \mathcal{N}_h$.

Finally, since $A^2$ has positive inverse, the non-negativity of $\beta^n$ is a direct consequence of non-negativity of $\alpha^n$.

For the discrete approximation $U^n$ of $u^n$ we can show the following conservation property.

**Lemma 4.2.** Let $\{U^n, C^n\}^N_{n=0}$ be a solution of the fully-discrete scheme (3.1)–(3.2). Then, we have

\begin{equation}
(U^n, 1) = (U^0, 1), \quad \text{for all } n = 1, \ldots, N_T.
\end{equation}

In addition, is conserved in $L^1$-norm for small $k$ and $k = \mathcal{O}(h^{1+\epsilon})$ for $0 < \epsilon < 1$ and $\|q\|_{\max} = \mathcal{O}(h)$, i.e.,

\begin{equation}
\|U^n\|_{L^1} = \|U^0\|_{L^1}, \quad \text{for all } n = 1, \ldots, N_T.
\end{equation}

**Proof.** We can easily see that choosing $\chi = 1 \in \mathcal{S}_h$ in (3.1) we get that

\begin{equation}
(\bar{D} U^n, 1) = 0.
\end{equation}

Thus we can easily obtain the conservation property (4.4). Then combining (4.4) and nonnegativity of $U^n \geq 0$, $n = 0, \ldots, N_T$ for $U_0 \geq 0$, we get the conservation in $L^1$-norm.

\[ \square \]
Figure 5.1. A triangulation of a square domain.

Figure 5.2. Left: Approximation of standard FEM scheme. Right: Approximation of low order scheme. Bottom: Approximation of AFC scheme.

5. Numerical experiments

In this section we present several numerical experiments, illustrating our theoretical results. We consider a uniform mesh $T_h$ of the unit square $\Omega = [0,1]^2$. Each side of $\Omega$ is divided into $M$ intervals of length $h_0 = 1/M$ for $M \in \mathbb{N}$ and we define the triangulation $T_h$ by dividing each small square by its diagonal, see Fig. 5.1. Thus $T_h$ consists of $2M^2$ right-angle triangles with diameter $h = \sqrt{2}h_0$. Obviously $T_h$ satisfies Assumption 2.1. Therefore, the corresponding stiffness matrix $S$ has non-positive off-diagonal elements and positive diagonal elements. In order to illustrate the preservation of positivity of the numerical schemes initially we consider an example where the solution of (1.1) obtains large values in absolute terms, and eventually blows-up at the center of unit square in finite time. The blow-up time for this example has been calculated, cf. e.g., [27] and we provide computations well before the blow-up time.

To construct the approximation at each time level $t_n$ we implement the fixed point iteration scheme (3.23). As a stopping criterion we consider the relative error between two successive solutions of (3.23), in the maximum norm of $\mathbb{R}^N$, with $\text{TOL} = 10^{-8}$. For the computation of the correction factors $a_{ij}$, we use Algorithm 2.3 with $q_i = m_i/k$, where $m_i$, $i = 1,\ldots,N$, are the diagonal elements of $M_L$. Since for every patch $\omega_i$, $i = 1,\ldots,N$, of $T_h$ is symmetric to the node $Z_i$, then $\gamma_i = 1$, $i = 1,\ldots,N$ in Remark 2.5, see [4].

5.1. Blow-up at the center of unit square domain. Choosing for initial conditions in (1.1)

$$u_0 = 1000 e^{-100((x-0.5)^2+(y-0.5)^2)},$$
$$c_0 = 500 e^{-50((x-0.5)^2+(y-0.5)^2)},$$

we can show that the solution $u$ blows-up at a finite time $t^*$, with $t^* \in (8 \times 10^{-5}, 10^{-4}]$, and the peak occurs at the center of $\Omega$, cf. e.g. [27].

Next, we consider a fine square mesh of $\Omega$ with $h_0 = 1/120$ and time step $k = 10^{-5}h_0^{1.01}$, and a set as final time $T$ of our computation $T = 63k < 8 \times 10^{-5}$. We discretize in time (1.7) using the backward Euler method and we observe that at the final time $T$ our discrete approximation of $u$ has negative values along the horizontal line $(x,0.5)$, $x \in [0,1]$ of $\Omega$, cf. Fig. 5.2. However, using the stabilized numerical schemes, of low-order and AFC, given by (3.1)-(3.2), our discrete approximation of $u$ remains positive, cf. Fig. 5.2.
obtained in $L^\infty$.

Next, we consider the following two sets of initial conditions for (1.1). The first one is

$$u_0 = 10 e^{-10((x-0.5)^2+(y-0.5)^2)} + 5,$$

$$c_0 = 0,$$

where note that $\|u_0\|_{L^1} \approx 7.984 < 4\pi$. The second set is

$$u_0 = \sin(\pi x)^2 \cos(\pi y)^2,$$

$$c_0 = 0,$$

where note that $\|u_0\|_{L^1} \approx 0.25 < 4\pi$. Thus, the solution $u$ in both examples does not blow-up.

We consider again the non-stabilized scheme where we discretize in time (1.7) using the backward Euler method and the stabilized schemes given by (3.1)-(3.2). We consider a sequence of triangulations $T_k$ as described above with $h_0 = 1/M$, $M = 10, 20, 40, 80$. The final time is chosen to be $T = 0.01$, and we choose $k = \frac{1}{20} h_0$ for the computation of the error in $H^1$ norm and $k = \frac{1}{320} h_0^2$ for the $L^2$-norm.

Since the exact solution of (1.1) is unknown, the underlying numerical reference solution for each numerical scheme was obtained with $M = 320$ and small time step $k = 10^{-5}$.

In Tables 1 and 2 we present the errors for $u$ for the standard finite element scheme and the stabilized schemes, obtained in $L^2$ and $H^1$ norm and the corresponding approximate order of convergence for the initial conditions (5.2).

In Tables 3 and 4 we present the errors for $u$ for the standard finite element scheme and the stabilized schemes, obtained in $L^2$ and $H^1$ norm and the corresponding approximate order of convergence for the initial conditions (5.3).

| $h_0$ | Stand. FEM | Order | Low Order | AFC Order | Order |
|-------|------------|-------|-----------|-----------|-------|
| 1/10  | 0.04831    | 0.05952 | 0.06108   |           |       |
| 1/20  | 0.01260    | 1.9683  | 0.01667   | 1.8356    | 0.01764 | 1.7919 |
| 1/40  | 0.00316    | 1.9942  | 0.00404   | 2.0453    | 0.00450 | 1.9604 |
| 1/80  | 0.00078    | 2.0234  | 0.00084   | 2.2519    | 0.00103 | 2.1272 |

Table 1. $L^2$-norm error and convergence order for the initial conditions (5.2).

| $h_0$ | Stand. FEM | Order | Low Order | AFC Order | Order |
|-------|------------|-------|-----------|-----------|-------|
| 1/10  | 2.3052     | 2.4347 | 2.4410    |           |       |
| 1/20  | 1.1743     | 0.9731 | 1.3005    | 1.0137    | 1.1757 | 1.0137 |
| 1/40  | 0.5893     | 0.9947 | 0.5967    | 1.0148    | 0.5982 | 1.0151 |
| 1/80  | 0.2893     | 1.0264 | 0.2909    | 1.0365    | 0.2915 | 1.0369 |

Table 2. $H^1$-norm error and convergence order for the initial conditions (5.2).

| $h_0$ | Stand. FEM | Order | Low Order | AFC Order | Order |
|-------|------------|-------|-----------|-----------|-------|
| 1/10  | 0.0081747  | 0.0064746 | 0.0064926 |           |       |
| 1/20  | 0.0022593  | 1.8553  | 0.0019299 | 1.8855    | 0.0019245 | 1.8805 |
| 1/40  | 0.0005752  | 1.9737  | 0.0004902 | 2.0409    | 0.0004902 | 2.0325 |
| 1/80  | 0.0001352  | 2.0914  | 0.0001104 | 2.1825    | 0.0001128 | 2.1681 |

Table 3. $L^2$-norm error and convergence order for the initial conditions (5.3).

| $h_0$ | Stand. FEM | Order | Low Order | AFC Order | Order |
|-------|------------|-------|-----------|-----------|-------|
| 1/10  | 0.25035    | 0.26625 | 0.26634   |           |       |
| 1/20  | 0.13356    | 0.9578  | 0.13297   | 1.1425    | 0.13301 | 1.1426 |
| 1/40  | 0.06753    | 0.9882  | 0.06593   | 1.0696    | 0.06596 | 1.0697 |
| 1/80  | 0.03319    | 1.0239  | 0.03217   | 1.0605    | 0.03218 | 1.0606 |

Table 4. $H^1$-norm error and convergence order for the initial conditions (5.3).
6. CONCLUSIONS

In this paper, we presented the finite element error analysis for the stabilized schemes presented in [28] and [27]. We discretize space using continuous piecewise linear finite elements and time by the backward Euler method. Under assumptions for the triangulation used for the space discretization and the size of the time step \( k \), we showed that the resulting coupled non-linear schemes have a unique solution and also remain non-negative. Further, we showed that the resulting discrete solution remains bounded and derived error estimates in \( L^2 \) and \( H^1 \)-norm in space. Numerical experiments in two dimensions were presented for both the standard FEM and stabilized schemes. In the numerical experiments, we do not observe any significant difference between the low-order scheme and the AFC scheme for the two initial conditions that we have studied.

APPENDIX A. PROOF OF LEMMA 3.4

**Proof.** Since, only the surrounding nodes to \( Z_i \) contributes to the corresponding row of the matrix, let \( Z_i \) be an interior node where the patch which has non-zero support is depicted on the left of the Fig. 2.1. Notice that for \( j = 1, \ldots, 8 \), we have

\[
\tau_{Z_iz_i} = \lambda \int_{\text{supp}(\phi_{Z_i}, \phi_{z_i})} (\nabla c_h \cdot \nabla \phi_{Z_i}) \phi_{z_i} \, dx = \lambda \frac{1}{3} \sum_{K \in \text{supp}(\phi_{Z_i}, \phi_{z_i})} \int_K \nabla c_h \cdot \nabla \phi_{Z_i} \, dx,
\]

where \( c_h \in S_h \) and therefore \( c_h = \sum_{j=1}^{N} \zeta_j \phi_j \). Let \( K_1 \) and \( K_2 \) two triangles such its intersection consists of edge \( e \) with endpoints \( A \) and \( B \), see Fig. 2.1. Let \( \phi_A \) and \( \phi_B \) the basis functions at nodes \( A \) and \( B \), respectively. 

Draganescu et al. [11] have given a closed formula for the computation of following integrals,

\[
\int_{K_1} \nabla \phi_A \cdot \nabla \phi_B \, dx = -\frac{\cot \gamma}{2},
\]

\[
\int_{K_1} |\nabla \phi_A|^2 \, dx = \frac{\sin \alpha}{2 \sin \beta \sin \gamma},
\]

\[
\int_{K_1 \cup K_2} \nabla \phi_A \cdot \nabla \phi_B \, dx = -\frac{\sin(\gamma + \delta)}{2 \sin \gamma \sin \delta}.
\]

To prove the lemma, we need to distinguish the position of node \( Z_i \) in case of internal node and boundary node. Assume that \( Z_i \) is an internal node, with surrounding nodes as depicted in patch \( \Pi_0 \) of Fig. A.1.

Our goal is to prove that

\[
\sum_{j=0}^{8} \tau_{Z_iz_i} = \tau_{Z_iz_i} + \tau_{Z_iz_i} + \cdots + \tau_{Z_iz_i} = 0,
\]

where \( Z_{i0} := Z_i \) and

\[
\tau_{Z_iz_i} = \lambda \sum_{l=1}^{N} \zeta_{Z_i} \int_K \nabla \phi_{Z_i} \cdot \nabla \phi_{Z_i} \, dx, \quad \text{for } j = 0, \ldots, 8.
\]

The sum (A.4), can be written as

\[
\sum_{j=0}^{8} \tau_{Z_iz_i} = \lambda \sum_{l=0}^{8} \zeta_{Z_i} A_l;
\]

with \( A_l, l = 0, \ldots, 8 \) are sum of integrals. We need to prove that \( A_l = 0, l = 0, \ldots, 8 \). We have,
\[ A_0 := \int_{\text{supp}[Z_i]} |\nabla \phi_{Z_i}|^2 \, dx + \sum_{l=1}^8 \int_{K_{i+l}} \nabla \phi_{Z_i} \cdot \nabla \phi_{Z_{i+1}} \, dx, \]
and for \( l = 1, \ldots, 8, \)

\[
A_{l+1} := \int_{K_{i+l}} ((|\nabla \phi_{Z_{i+l}}|^2 + |\nabla \phi_{Z_{i+1}}|^2) |\nabla \phi_{Z_i}| \, dx \\
+ \int_{K_i} \nabla \phi_{Z_{i+1}} \cdot \nabla \phi_{Z_i} \, dx + \int_{K_{i+1}} \nabla \phi_{Z_{i+2}} \cdot \nabla \phi_{Z_{i}} \, dx, \]

where \( A_9 := A_1 \) and with \( Z_{i0} = Z_{i1}, Z_{i10} = Z_{i2}, K_0 = K_1, K_{10} = K_2. \) Using (A.1), (A.2) and (A.3), we get \( A_l = 0, l = 0, \ldots, 8. \) We work similar when \( Z_i \) is a boundary node. \( \square \)

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