Lepton mass and mixing in a Neutrino Mass Model
based on $S_4$ flavor symmetry

V. V. VIEN$^{(1),(2)}$

(1) Institute of Research and Development, Duy Tan University,
182 Nguyen Van Linh, Da Nang City, Vietnam,
(2) Department of Physics, Tay Nguyen University,
567 Le Duan, Buon Ma Thoat, DakLak, Vietnam
wvienk16@gmail.com

We study a neutrino mass model based on $S_4$ flavor symmetry which accommodates lepton mass, mixing with non-zero $\theta_{13}$ and CP violation phase. The spontaneous symmetry breaking in the model is imposed to obtain the realistic neutrino mass and mixing pattern at the tree-level with renormalizable interactions. Indeed, the neutrinos get small masses from one SU(2)$_L$ doublet and two SU(2)$_L$ singlets in which one being in 2 and the two others in 3 under $S_4$ with both the breakings $S_4 \rightarrow S_3$ and $S_4 \rightarrow Z_3$ are taken place in charged lepton sector and $S_4 \rightarrow K$ in neutrino sector. The model also gives a remarkable prediction of Dirac CP violation $\delta_{CP} = \frac{\pi}{2}$ or $-\frac{\pi}{2}$ in the both normal and inverted spectrum which is still missing in the neutrino mixing matrix. The relation between lepton mixing angles is also represented.

Keywords: Neutrino mass and mixing; Models beyond the standard model; Non-standard-model neutrinos, right-handed neutrinos, discrete symmetries.

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1. Introduction

The Standard Model (SM) is one of the most successful theories in the elementary particle physics, however, it leaves some unresolved issues that have been empirically verified, such as the fermion masses and mixing, the mass hierarchies problem and the CP-violating phases. It is obvious that the SM must be extended. Theoretically, there are several proposals for explanation of smallness of neutrino mass and large lepton mixing such as the Neutrino Minimal Standard Model$^{1,2}$, Two-Higgs-doublet model$^3$, the scotogenic model$^4$, the Georgi-Glashow model$^5$, $SO(10)$ grand unification$^6$, the texture zero models$^7$, $3\times3$ models$^8$ and so on.

$^a$Depending on the particle content, there exist models which generate an active neutrino mass at 1-loop$^9$, 2-loop$^{10,11,12}$, or 3-loop$^{13,14}$ level, but Ma’s scotogenic model seems to be the simplest extension.

$^b$For some other scenarios of this type of model, the reader can see in Ref. $^{20}$. 
Among the possible extensions of SM, probably the simplest one obtained by adding right-handed neutrinos to its original structure which has been studied in Refs. [1–7]. However, these extensions do not provide a natural explanation for large mass splitting between neutrinos and the lepton mixing was not explicitly explained [27].

There are five well-known patterns of lepton mixing [28], however, the Tribimaximal one proposed by Harrison-Perkins-Scott (HPS) [29–32] seems to be the most popular and can be considered as a leading order approximation for the recent neutrino experimental data. In fact, the absolute values of the entries of the lepton mixing matrix $U_{P M N S}$ are given in Ref. [33]

$$|U_{P M N S}| = \begin{pmatrix}
0.801 \rightarrow 0.845 & 0.514 \rightarrow 0.580 & 0.137 \rightarrow 0.158 \\
0.225 \rightarrow 0.517 & 0.441 \rightarrow 0.699 & 0.614 \rightarrow 0.793 \\
0.246 \rightarrow 0.529 & 0.464 \rightarrow 0.713 & 0.590 \rightarrow 0.776
\end{pmatrix}. \tag{2}$$

The best fit values of neutrino mass squared differences and the leptonic mixing angles given in Ref. [33] as shown in Tabs. 1 and 2.

### Table 1. The experimental values of neutrino mass squared splittings and leptonic mixing parameters, taken from Ref. [33] for normal hierarchy.

| Best fit ±1σ | 3σ range |
|--------------|---------|
| $\Delta m_{21}^2 [10^{-5}\text{eV}^2]$ | $7.50^{+0.19}_{-0.17}$ | $7.02 \rightarrow 8.09$ |
| $\Delta m_{31}^2 [10^{-3}\text{eV}^2]$ | $2.457^{+0.047}_{-0.047}$ | $2.317 \rightarrow 2.607$ |
| $\sin^2 \theta_{12}$ | $0.304^{+0.013}_{-0.012}$ | $0.270 \rightarrow 0.344$ |
| $\sin^2 \theta_{23}$ | $0.452^{+0.052}_{-0.052}$ | $0.382 \rightarrow 0.643$ |
| $\sin^2 \theta_{13}$ | $0.0218^{+0.0010}_{-0.0010}$ | $0.0186 \rightarrow 0.0250$ |
| $\delta [\text{c}^2]$ | $306^{+39}_{-70}$ | $0 \rightarrow 360$ |

### Table 2. The experimental values of neutrino mass squared splittings and leptonic mixing parameters, taken from Ref. [33] for inverted hierarchy.

| Best fit ±1σ | 3σ range |
|--------------|---------|
| $\Delta m_{21}^2 [10^{-5}\text{eV}^2]$ | $7.50^{+0.19}_{-0.17}$ | $7.02 \rightarrow 8.09$ |
| $\Delta m_{31}^2 [10^{-3}\text{eV}^2]$ | $-2.449^{+0.048}_{-0.047}$ | $-2.590 \rightarrow -2.307$ |
| $\sin^2 \theta_{12}$ | $0.304^{+0.013}_{-0.012}$ | $0.270 \rightarrow 0.344$ |
| $\sin^2 \theta_{23}$ | $0.579^{+0.0011}_{-0.0011}$ | $0.389 \rightarrow 0.644$ |
| $\sin^2 \theta_{13}$ | $0.0219^{+0.0010}_{-0.0010}$ | $0.0188 \rightarrow 0.0251$ |
| $\delta [\text{c}^2]$ | $254^{+62}_{-62}$ | $0 \rightarrow 360$ |
The large lepton mixing angles given in Tabs. 1, 2 are completely different from the quark mixing ones defined by the Cabibbo-Kobayashi-Maskawa (CKM) matrix. This has stimulated works on flavor symmetries and non-Abelian discrete symmetries, which are considered to be the most attractive candidate to formulate dynamical principles that can lead to the flavor mixing patterns for quarks and leptons. There are various recent models based on the non-Abelian discrete symmetries, see for example $A_4^{36-54}$, $S_3^{55-95}$, $S_4^{96-124}$, $D_4^{125-135}$, $T_7^{146-150}$. However, in all these papers, the fermion masses and mixings generated from non-renormalizable interactions or at loop level but not at tree-level.

In this work, we investigate another choice with $S_4$ group, the permutation group of four objects, which is also the symmetry group of a cube. It has 24 elements divided into 5 conjugacy classes, with 1, 1', 2, 3, and 3' as its 5 irreducible representations. A brief of the theory of $S_4$ group is given in.

We note that $S_4$ has not been considered before in this kind of the model in this scenario. This model is different from our previous works because the 3-3-1 models (based on $SU(3)_C \otimes SU(3)_L \otimes U(1)_Y \otimes U(1)_X$) itself is an extension of the SM.

The rest of this work is organized as follows. In Sec. 2 we present the necessary elements of the model and introduce necessary Higgs fields responsible for the lepton masses. Sec. 3 is devoted for the quark mass and mixing at tree level. We summarize our results and make conclusions in the section 4. Appendix A briefly provides the theory of $S_4$ group with its Clebsch-Gordan coefficients. Appendix B, Appendix C and Appendix D provide the breakings of $S_4$ by $3$, $3'$ and $2$, respectively.

2. Lepton mass and mixing

The symmetry group of the model under consideration is

$$G = SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \otimes U(1)_X \otimes S_4$$

where the electroweak sector of the SM is supplemented by an auxiliary symmetry $U(1)_X$ plus a $S_4$ flavour symmetry whereas the strong interaction one is retained. The reason for adding the auxiliary symmetry $U(1)_X$ was discussed fully in. The lepton content of the model, under $[SU(2)_L, U(1)_Y, U(1)_X, S_4]$, is summarized in Tab. 3.

| Fields | $\psi_{1,2,3L}$ | $l_{1(2,3)R}$ | $\nu_R$ | $\phi$ | $\phi'$ | $\varphi$ | $\chi$ | $\zeta$ |
|--------|----------------|----------------|--------|-------|--------|---------|-------|-------|
| SU(2)$_L$ | 2 | 1 | 1 | 2 | 2 | 2 | 1 | 1 |
| U(1)$_Y$ | $-1$ | $-2$ | 0 | 1 | 1 | 1 | 0 | 0 |
| U(1)$_X$ | $1$ | $1$ | 0 | 0 | 0 | $-1$ | 0 | 0 |
| $S_4$ | $3$ | $1(2)$ | $3$ | $3$ | $3'$ | $1$ | $3$ | $2$ |

In this scenario, fermion masses and mixing angles are generated from renormalizable Yukawa interactions and at tree-level.
After electroweak breaking, the mass Lagrangian for the charge d leptons becomes

\[ -\mathcal{L}_l = h_1(\bar{\psi}_L\phi)_L^1l_R^1 + h_2(\bar{\psi}_L\phi)_L^2l_R^2 + h_3(\bar{\psi}_L\phi')_L^3l_R^3 + H.c. \]  

(4)

Theoretically, a possibility that the Tribimaximal mixing matrix \(U_{HPS}\) can be decomposed into only two independent rotations may provide a hint for some underlying structure in the lepton sector,

\[ U_{HPS} = \begin{pmatrix} \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ i & \omega & \omega^2 \\ i & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \cong U_L^T U_\nu, \]  

(5)

where \(\omega = \exp(2\pi i/3) = -1/2 + i\sqrt{3}/2\).

All possible breakings of \(S_4\) group under triplets \(3\) and \(3'\) are presented in appendices \(\text{Appendix B}\) and \(\text{Appendix C}\), respectively. To obtain charged-lepton mixing satisfying \(\text{(5)},\) in this work we argue that both the breakings \(S_4 \rightarrow S_3\) and \(S_4 \rightarrow Z_3\) are taken place in charged lepton sector. The breaking \(S_4 \rightarrow S_3\) can be achieved by a \(SU(2)_L\) doublet \(\phi\) with the third alignment given in \(\text{Appendix B}\) i.e., \(\langle \phi \rangle = (\langle \phi_1 \rangle, \langle \phi_1 \rangle, \langle \phi_1 \rangle)\) under \(S_4\), where

\[ \langle \phi_1 \rangle = (0 \ v)^T, \]  

(6)

and the breaking \(S_4 \rightarrow Z_3\) can be achieved by another \(SU(2)_L\) doublet \(\phi'\) with the third alignment given in \(\text{Appendix C}\) i.e., \(\langle \phi' \rangle = (\langle \phi'_1 \rangle, \langle \phi'_1 \rangle, \langle \phi'_1 \rangle)\) under \(S_4\), where

\[ \langle \phi'_1 \rangle = (0 \ v')^T. \]  

(7)

After electroweak breaking, the mass Lagrangian for the charged leptons becomes

\[ -\mathcal{L}_l^{mass} = (\bar{l}_{1L}, \bar{l}_{2L}, \bar{l}_{3L})M_l(l_{1R}, l_{2R}, l_{3R})^T + H.c., \]  

(8)

where

\[ M_l = \begin{pmatrix} h_1 v & h_2 v - h_3 v' & h_2 v + h_3 v' \\ h_2 v - h_3 v' & (h_2 v - h_3 v')\omega & (h_2 v + h_3 v')\omega^2 \\ h_2 v + h_3 v' & (h_2 v - h_3 v')\omega^2 & (h_2 v + h_3 v')\omega \end{pmatrix}. \]  

(9)

The mass matrix \(M_l\) in Eq. \(\text{(9)}\) is diagonalized by \(U_L^T M_l U_R = \text{diag}(m_e, m_\mu, m_\tau)\), with

\[ m_e = \sqrt{3} h_1 v, \quad m_\mu = \sqrt{3}(h_2 v - h_3 v'), \quad m_\tau = \sqrt{3}(h_2 v + h_3 v'), \]  

(10)
and 2 under SU(2)\_L doublet φ'. This is the reason why φ' was additional introduced to φ in lepton sector.

Now, by combining Eq. (10) with the experimental values for masses of the charged leptons given in Ref. 165,

\[
m_e \simeq 0.51099 \text{ MeV}, \quad m_\mu = 105.65837 \text{ MeV}, \quad m_\tau = 1776.82 \text{ MeV} \quad (12)
\]

It follows that \( h_1 \ll h_2, h_3 \) and \( h_2 \simeq h_3 \) if \( v' \simeq v \). On the other hand, if we suppose that \( v \sim 100 \text{ GeV} \) then

\[
h_1 \sim 10^{-6}, \quad h_2 \sim h_3 \sim 10^{-3}, \quad (13)
\]

i.e, in the model under consideration, the hierarchy between the masses for charged-leptons can be achieved if there exists a hierarchy between Yukawa couplings \( h_1 \) and \( h_{2,3} \) in charged-lepton sector as given in Eq. (13).

The neutrino masses arise from the couplings of \( \bar{\psi}_L \nu_R \) and \( \bar{\nu}_R \nu_R \) to scalars, where \( \bar{\psi}_L \nu_R \) transforms as 2 under SU(2)\_L and \( 1 \oplus 2 \oplus 3 \oplus 3' \) under \( S_4 \); \( \bar{\nu}_R \nu_R \) transform as 1 under SU(2)\_L and \( 1 \oplus 2 \oplus 3 \oplus 3' \) under \( S_4 \). Note that under \( S_4 \) symmetry, each tensor product \( 3 \oplus 3 \oplus 3 \) contains one invariant\(^4\). On the other hand, \( 2 \oplus 2 = 1 \oplus 3 \) and \( 3 \oplus 3 = 1 \oplus 3 \oplus 5 \) under SU(2)\_L. For the known SU(2)\_L scalar doublets, only two available interactions \( (\bar{\psi}_L \phi)\_3 \nu_R, (\bar{\nu}_L \phi')\_3 \nu_R \), but explicitly suppressed because of the \( U(1)_X \) symmetry. We therefore additionally introduce one SU(2)\_L doublet \( (\varphi) \) and two SU(2)\_L singlets \( (\chi, \zeta) \), respectively, put in \( 1, 3 \) and \( 2 \) under \( S_4 \) as given in Tab. 6.

It is need to note that \( \varphi \) contributes to the Dirac mass matrix in the neutrino sector and \( \chi \) contributes to the Majorana mass matrix of the right-handed neutrinos. We also note that the \( U(1)_X \) symmetry forbids the Yukawa terms of the form \( (\bar{\psi}_L \phi)\_3 \nu_R \) and yield the expected results in neutrino sector, and this is interesting feature of \( X \)-symmetry.

All possible breakings of \( S_4 \) group under triplet \( 3 \) and doublet \( 2 \) are given in appendices Appendix B and Appendix D, respectively. To obtain a realistic neutrino spectrum, i.e, resulting the non-zero \( \theta_{13} \) and CP violation, in this work, we argue

\[
U_L = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad U_R = 1.
\]
that the breaking $S_4 \to K$ must be taken place in neutrino sector. This can be achieved within each case below.

(1) A SU(2)$_L$ doublet $\chi$ put in 3 under $S_4$ with the VEV is chosen by

$$\langle \chi_1 \rangle = v_\chi, \quad \langle \chi_2 \rangle = \langle \chi_3 \rangle = 0.$$  \hfill (14)

(2) Another SU(2)$_L$ doublet $\zeta$ put in 2 under $S_4$ with the VEV given by

$$\langle \zeta \rangle = (\langle \zeta_1 \rangle, \langle \zeta_2 \rangle), \quad \langle \zeta_i \rangle = v_{\zeta_i} \quad (i = 1, 2).$$  \hfill (15)

The Yukawa Lagrangian invariant under $G$ symmetry in neutrino sector reads:

$$- L_\nu = \frac{x^2}{2} (\bar{\psi}_L \tilde{\phi} \nu_R)^3 + \frac{y^2}{2} (\bar{\nu}_c R \chi)^3 s \nu_R + M^2 \bar{\nu}_c R \nu_R + \frac{z^2}{2} (\bar{\nu}_c R \zeta)^3 s \nu_R + H.c.,$$  \hfill (16)

where $M$ is the bare Majorana mass for the right-handed neutrino.

After electroweak breaking, the mass Lagrangian for the neutrinos is given by

$$- L_\nu^\text{mass} = \frac{1}{2} \bar{\chi}_L M_\nu \chi_L + H.c.,$$  \hfill (17)

where

$$\chi_L \equiv (\nu_L \nu_R)^T, \quad M_\nu \equiv \begin{pmatrix} 0 & M_D \\ M_D^T & M_R \end{pmatrix},$$  \hfill (18)

$$\nu_L = (\nu_1 L \nu_2 L \nu_3 L)^T, \quad \nu_R = (\nu_1 R \nu_2 R \nu_3 R)^T,$$

and the mass matrices $M_D, M_R$ are then obtained by

$$M_D = m_D \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_R = \begin{pmatrix} M + M_1 + M_2 & 0 & 0 \\ 0 & M + \omega M_1 + \omega^2 M_2 & M' \\ 0 & M' & M + \omega^2 M_1 + \omega M_2 \end{pmatrix},$$

$$M' = yv_\chi, \quad m_D = xv_\varphi, \quad M_i = zv_\chi, \quad (i = 1, 2),$$ \hfill (19)

with $v_\varphi = \langle \varphi \rangle$, and $M_D$ is the Dirac neutrino mass matrix, $M_R$ is the right-handed Majorana neutrino mass matrix.

The effective neutrino mass matrix, in the framework of seesaw mechanism, is given by

$$M_{\text{eff}} = -M_D^T M_R^{-1} M_D = \begin{pmatrix} A & 0 & 0 \\ 0 & B_1 & C \\ 0 & C & B_2 \end{pmatrix},$$  \hfill (20)

where

$$A = -\frac{m_D^2}{M + M_1 + M_2}, \quad B_{1,2} = \frac{m_D^2 [-2M + M_1 + M_2 \pm i\sqrt{3}(M_1 - M_2)]}{2\mathfrak{R}},$$

$$C = \frac{m_D^2 M'}{2\mathfrak{R}}, \quad 2\mathfrak{R} = M^2 + M_1^2 + M_2^2 - MM_1 - MM_2 - M_1 M_2 - M^2.$$  \hfill (21)
The matrix $M_{\text{eff}}$ in (20) can be diagonalized as follows $U_\nu^T M_{\text{eff}} U_\nu = \text{diag}(m_1, m_2, m_3)$, with

$$m_1 = \frac{1}{2} \left( B_1 + B_2 + \sqrt{(B_1 + B_2)^2 + 4C^2} \right), \quad m_2 = A,$$

$$m_3 = \frac{1}{2} \left( B_1 + B_2 - \sqrt{(B_1 + B_2)^2 + 4C^2} \right),$$

and

$$U_\nu = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{K + 1}} & 0 & \frac{K}{\sqrt{K + 1}} \\ -\frac{K}{\sqrt{K + 1}} & 0 & -\frac{1}{\sqrt{K + 1}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix},$$

$$K = \frac{B_1 - B_2 - \sqrt{(B_1 - B_2)^2 + 4C^2}}{2C}.$$  \hspace{1cm} (24)

The lepton mixing matrix, obtained from the matrices $U_\nu$ and $U_L$ in Eqs. (11) and (23), is expressed as

$$U = U_L^T U_\nu = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 - K & 1 & 1 + K \\ \frac{1 - K}{\sqrt{(1 + K)^2 + 1}} & \frac{1}{\sqrt{(1 + K)^2 + 1}} & \frac{1}{\sqrt{(1 + K)^2 + 1}} \\ \frac{1}{\sqrt{K + 1}} & \frac{K}{\sqrt{K + 1}} & -\frac{1}{\sqrt{K + 1}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix},$$

where $K$ is defined in Eq. (24).

In the standard parametrization, the lepton mixing matrix can be parametrized as

$$U_{PMNS} = \begin{pmatrix} c_{12}c_{13} & -s_{12}c_{13} & -s_{13}e^{-i\delta} \\ s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} + s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} + c_{12}c_{23}s_{13}e^{i\delta} & c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \times P,$$  \hspace{1cm} (26)

where $P = \text{diag}(1, e^{i\alpha}, e^{i\beta})$, and $c_{ij} = \cos \theta_{ij}$, $s_{ij} = \sin \theta_{ij}$ with $\theta_{12}$, $\theta_{23}$ and $\theta_{13}$ being the solar, atmospheric and reactor angles, respectively, and $\delta = [0, 2\pi]$ is the Dirac CP violation phase while $\alpha$ and $\beta$ are two Majorana CP violation phases.

Comparing the lepton mixing matrix in Eq. (24) to the standard parametrization in Eq. (26), one obtains $\alpha = 0, \beta = \pi/2$, and

$$s_{13}e^{-i\delta} = \frac{-1 - K}{\sqrt{3} \sqrt{K^2 + 1}},$$

$$t_{12} = \frac{\sqrt{K^2 + 1}}{K - 1},$$

$$t_{23} = -\frac{1}{K + \omega}.$$  \hspace{1cm} (29)

Substituting $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ into Eq. (29) yields:

$$\text{Re}K = t_{23}^2 - 4t_{23} + 1 = \frac{1 - t_{23}^2}{2(t_{23}^2 - t_{23} + 1)}, \quad \text{Im}K = \frac{\sqrt{3}}{2} t_{23}^2 - t_{23} + 1.$$  \hspace{1cm} (30)
It is easily to see that $|K| = \sqrt{(ImK)^2 + (ReK)^2} = 1$. Combining Eq. (27) and Eq. (28) we obtain:

$$e^{-i\delta} = \frac{1}{\sqrt{3}s_{13}t_{12}} \frac{1 + K}{1 - K}.$$  

or

$$-i \frac{t_{23} - 1}{s_{13}t_{12}(t_{23} + 1)} = \cos \delta - i \sin \delta.$$  

(31)

By equating the real and imaginary parts of the equation (31), we get

$$\cos \delta = 0, \quad \sin \delta = \frac{t_{23} - 1}{s_{13}t_{12}(t_{23} + 1)}.$$  

(32)

Since $\cos \delta = 0$ so that $\sin \delta$ must be equal to $\pm 1$, it is then $\delta = \frac{\pi}{2}$ or $\delta = -\frac{\pi}{2}$. The value of the Jarlskog invariant $J_{CP}$ which determines the magnitude of CP violation in neutrino oscillations is determined

$$J_{CP} = \frac{1}{8} \cos \theta_{13} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \sin \delta.$$  

(33)

Once $\theta_{12}$, $\theta_{23}$ and $\theta_{13}$ have been determined experimentally, the size of $J_{CP}$ depends essentially only on the magnitude of the currently not well determined value of the Dirac phase $\delta$. Thus, our model predicts the maximal Dirac CP violating phase which is the same as in Refs. [166, 167] but the difference comes from $\theta_{23}$. Namely, in Refs. [166, 167] $\theta_{23} = \pi/4$ but in our model $\theta_{23} \neq \pi/4$ which is more consistent with the recent experimental data given in Tabs. 1, 2 and this is one of the most striking prediction of the model under consideration.

At present, the precise evaluation of $\theta_{23}$ is still an open problem while $\theta_{12}$ and $\theta_{13}$ are now very constrained [163]. From Eq. (32), as will see below, our model can provide constraints on $\theta_{23}$ from $\theta_{12}$ and $\theta_{13}$ which satisfy the data given in Ref. [163].

(i) In the case $\delta = \frac{\pi}{2}$, from (32) we have the relation among three Euler’s angles as follows:

$$t_{23} = \frac{1 + s_{13}t_{12}}{1 - s_{13}t_{12}},$$  

(34)

or

$$s_{23}^2 = \frac{(1 - s_{12}^2) \left( 1 + \sqrt{s_{13}^2 s_{13}^2} \right)^2}{2[1 + s_{12}^2(s_{13}^2 - 1)]}.$$  

(35)

In Fig. 1 we have plotted the values of $s_{23}^2$ as a function of $s_{12}^2$ and $s_{13}^2$ with $s_{12}^2 \in (0.270, 0.344)$, $s_{13}^2 \in (0.0186, 0.0250)$ given in Ref. [33] in the case $\delta = \frac{\pi}{2}$ at the 3$\sigma$ level.

Taking the new data $s_{12}^2 = 0.30 (\theta_{12} = 33.46^\circ)$ and $s_{13}^2 = 0.0245 (\theta_{13} = 9.00^\circ)$ we obtain $s_{23}^2 = 0.6014$, i.e, $\theta_{23} = 50.8507^\circ$ which is larger than 45$^\circ$, and

$$K = -0.938924 - 0.344125i, \quad (|K| = 1).$$  

(36)
Fig. 1. $s_{23}^2$ as a function of $s_{12}^2$ and $s_{13}^2$ with $s_{12}^2 \in (0.270,0.344)$, $s_{13}^2 \in (0.0186,0.0250)$ in the case $\delta = \frac{\pi}{2}$ at the $3\sigma$ level.

The lepton mixing matrix in (25) then takes the form

$$U \simeq \begin{pmatrix} 0.82841 & 0.57735 & -0.147252 \\ -0.53546 & 0.57735 & -0.78743 \\ -0.29295 & 0.57735 & 0.64742 \end{pmatrix},$$

(37)

which is consistent with constraint in Eq. (2).

Combining (24) and the values of $K$ in (36), we obtain the relation

$$B_1 = B_2 - (2.75481 \times 10^{-7} + 0.68825i)C.$$

(38)

(ii) Similar to the case with $\delta = \frac{\pi}{2}$, in the case $\delta = -\frac{\pi}{2}$, we find the followings relation:

$$s_{23}^2 = \frac{(1 - s_{12}^2) \left(-1 + \sqrt{\frac{s_{12}^2}{s_{12}^2 - 1}}\right)^2}{2(1 + s_{12}^2 s_{13}^2 - 1)}.$$

(39)

In Fig. 2, we have plotted the values of $s_{23}^2$ as a function of $s_{12}^2$ and $s_{13}^2$ with $s_{12}^2 \in (0.270,0.344)$, $s_{13}^2 \in (0.0186,0.0250)$ given in Ref. [33] in the case $\delta = -\frac{\pi}{2}$ at the $3\sigma$ level.

If $s_{12}^2 = 0.30$ and $s_{13}^2 = 0.0245$ we obtain $s_{23}^2 = 0.39860 (\theta_{23} = 39.15^\circ)$, and

$$K = -0.938924 + 0.344125i, \quad (|K| = 1).$$

(40)

In this case the lepton mixing matrix in (25) takes the form:

$$U \simeq \begin{pmatrix} 0.82967 & 0.57735 & -0.14725 \\ -0.53546 & 0.57735 & -0.78743 \\ -0.29295 & 0.57735 & 0.64742 \end{pmatrix},$$

(41)

The relation between $B_{1,2}$ and $C$ is determined as follows

$$B_1 = B_2 - (2.75481 \times 10^{-7} - 0.68825i)C.$$

(42)
Fig. 2. $s_{23}^2$ as a function of $s_{12}^2$ and $s_{13}^2$ with $s_{12}^2 \in (0.270, 0.344)$, $s_{13}^2 \in (0.0186, 0.0250)$ in the case $\delta = -\frac{\pi}{2}$ at the $3\sigma$ level.

2.1. Normal case ($\Delta m_{23}^2 > 0$)

In this case, substituting $B_1$ from (38) into (22) and taking the two experimental data on squared mass differences of neutrinos given in Ref. 33, $\Delta m_{21}^2 = 7.50 \times 10^{-5} \text{eV}^2$ and $\Delta m_{31}^2 = 2.457 \times 10^{-3} \text{eV}^2$, we get a solution $^i\text{E.1}$ (in [eV]) as shown in Appendix E. Using the upper bound on the absolute value of neutrino mass Refs. 168–170 we can restrict the values of $A$, $A \leq 0.6 \text{eV}$. However, in the case in (E.1), $|A| \in (0.00867, 0.02) \text{eV}$ can reach the normal neutrino mass hierarchy which is depicted in Fig. 3h.

In the model under consideration, the effective neutrino mass from tritium beta decay $m_\beta = \sqrt{\sum_{i=1}^{3}|U_{ei}|^2m_i^2}$ and the neutrino mass obtained from neutrinoless double-beta decays $m_{\beta\beta} = |\sum_{i=1}^{3}U_{ei}^2m_i|$ are depicted in Fig. 4. We also note that in the normal spectrum, $|m_1| \approx |m_2| < |m_3|$, so $m_1 \equiv m_{\text{light}}$ is the lightest neutrino mass.

To get explicit values of the model parameters, we assume $A = 10^{-2} \text{eV}$, which is safely small.$^8$ Then the other neutrino masses are explicitly given as

$$m_1 = -5.00 \times 10^{-3} \text{eV}, m_2 = 10^{-2} \text{eV}, m_3 \simeq -4.982 \times 10^{-2} \text{eV}, \quad (43)$$
$$m_{\beta\beta} = 1.88866 \times 10^{-3} \text{eV}, m_\beta = 1.02156 \times 10^{-2} \text{eV}, \quad (44)$$
$$|m_1| + |m_2| + |m_3| = 6.48197 \times 10^{-2} \text{eV}, \quad (45)$$

$^8$The system of equations has two solutions but they have the same absolute values of $m_{1,2,3}$; the unique difference is the sign of them. So, here we only consider in detail the case in Eq. (E.1).

$^i$The expressions (E.1), (22) and (38) show that $m_i$ ($i = 1, 2, 3$) depends only on one parameter $A \equiv m_2$ so we consider $m_{1,3}$ as functions of $m_2$. However, to have an explicit hierarchy on neutrino masses $m_2$ should be included in the figures.

$^i$The precise value of the mass of neutrinos is still an open question, however, it lies in the range of a few eV.
Fig. 3. \( |m_{1,2,3}| \) as functions of \( A \) in the normal hierarchy with a) \( A \in (-0.02, -0.00867) \) eV and b) \( A \in (0.00867, 0.02) \) eV.

Fig. 4. \( m_\beta, m_{\beta\beta} \) and \( |m_{\text{light}}| \) as functions of \( A \) in the normal hierarchy with a) \( A \in (-0.02, -0.00867) \) eV and b) \( A \in (0.0087, 0.02) \) eV.

and

\[
B_{1,2} = -(2.74098 \pm 0.821343i) \times 10^{-2} \text{ eV},
\]

\[
C = (2.38676 - 1.28329i) \times 10^{-2} \text{ eV} \simeq 2.38676 \times 10^{-2} \text{ eV}. \tag{46}
\]

Furthermore, combining Eqs. \((21)\) and \((46)\) we get a solution:

\[
M' = (2.39395 - 1.28716 \times 10^{-7}i)M, \quad m_D = (-0.158066 + 3.21117 \times 10^{-17}i)\sqrt{M},
\]

\[
M_{1,2} = (-2.22488 \pm 2.15951 \times 10^{-7}i)M. \tag{47}
\]

\(^1\)This system of equations has two solutions, however, these solutions differ only by the sign of \( m_D \) (or the sign of \( m_{1,2,3} \)) which has no effect on the neutrino oscillation experiments.
and
\[ x = (-0.158066 + 3.21117 \times 10^{-17}i)\sqrt{M}/v_\varphi \simeq -0.158066\sqrt{M}/v_\varphi, \]
\[ y = (2.39395 - 1.28716 \times 10^{-7}i)M/v_\chi \simeq 2.39395M/v_\chi, \]
\[ z = (-2.22488 + 2.15951 \times 10^{-7}i)M/v_\zeta \simeq -2.22488M/v_1, \]
\[ \varepsilon_2 = (0.572442 + 1.52625 \times 10^{-7}v_\zeta) \simeq 0.572442v_\zeta. \]  

Eq. (49) shows that \( v_\zeta_1 \) and \( v_\zeta_2 \) are different from each other but in the same order of magnitude. The solution in Eq. (43) constitutes the normal spectrum and consistent with the constraints on the absolute value of the neutrino masses. Similarly, in the case \( \delta = -\frac{\pi}{2} \), the numerical fit of all parameters to lepton mass and mixing data is summarized in Tab. 4.

Table 4. The observables and parameters of the model in the case \( \delta = -\frac{\pi}{2} \).

| Observables | Data fit 3\( \sigma \) range from Ref. 33 | The values of the model parameters |
|-------------|-----------------------------------------|-----------------------------------|
| \( \theta_{12}^\circ \) | 31.29 \( \rightarrow \) 35.91 | 33.46 |
| \( \theta_{23}^\circ \) | 38.2 \( \rightarrow \) 53.3 | 39.15 |
| \( \theta_{13}^\circ \) | 7.87 \( \rightarrow \) 9.11 | 9.0 |
| \( \Delta m_{21}^2 \) | (7.02 \( \rightarrow \) 8.09) \( \times \) 10\(^{-5}\) eV\(^2\) | 7.50 |
| \( \Delta m_{32}^2 \) | (2.317 \( \rightarrow \) 2.607) \( \times \) 10\(^{-3}\) eV\(^2\) | 2.457 |
| \( m_1 \) [eV] | \( \rightarrow \) | 5 \( \times \) 10\(^{-3}\) |
| \( m_2 \) [eV] | \( \rightarrow \) | 10\(^{-2}\) |
| \( m_3 \) [eV] | \( \rightarrow \) | 5.05668 \( \times \) 10\(^{-2}\) |
| \( \sum m_i \) [eV] | \( \rightarrow \) | 4.55668 \( \times \) 10\(^{-2}\) |
| \( m_{\beta 3} \) [eV] | \( \rightarrow \) | 1.20486 \( \times \) 10\(^{-3}\) |
| \( m_{\beta 3} \) [eV] | \( \rightarrow \) | 1.02949 \( \times \) 10\(^{-3}\) |
| \( A \) [eV] | \( \rightarrow \) | 10\(^{-2}\) |
| \( B_{1,2} \) [eV] | \( \rightarrow \) | \(-2.77834 \pm 0.835034i\) \( \times \) 10\(^{-2}\) |
| \( C \) [eV] | \( \rightarrow \) | 2.42654 \( \times \) 10\(^{-2}\) |

The parameters \( x, y, z \) are given as follows:
\[ x \simeq -0.158066\sqrt{M}/v_\varphi, \quad y \simeq 2.40384M/v_\chi, \quad z \simeq 1.27474M/v_1, \]
\[ \varepsilon_2 \simeq 1.74932v_\zeta. \]

2.2. Inverted case (\( \Delta m_{32}^2 < 0 \))

By taking the two experimental data on squared mass differences of neutrinos for the inverted hierarchy given in Ref. 33 \( \Delta m_{21}^2 = 7.50 \times 10^{-5}\) eV\(^2\) and \( \Delta m_{31}^2 = -2.449 \times 10^{-3}\) eV\(^2\), we obtain the relation\(^4\) between \( m_{1,3} \) and \( m_2 = A \) as shown in Fig. 5.

\(^1\)In the case \( v_\zeta_1 = v_\zeta_2 \), i.e., \( M_1 = M_2 \), the lepton mixing matrix \( U_{lep} \) in Eq. (25) becomes an exact tri-bimaximal mixing which can be considered as a good approximation for the recent neutrino experimental data. Hence, the condition \( v_\zeta_1 \neq v_\zeta_2 \) is necessary to reach the realistic neutrino spectrum, and the relation\(^4\) is satisfy this condition.

\(^2\)We only consider here one solution with \( \delta = \frac{\pi}{2} \).
\( m_3 \equiv m_{\text{light}}^I \) is the lightest neutrino mass, and the effective neutrino mass from tritium beta decay and the neutrino mass obtained from neutrinoless double-beta decays are plotted in Fig. 6.

With \( A = 5.1 \times 10^{-2} \text{ eV} \), we get explicit values of the model parameters as follows:

\[
 m_1 \simeq 5.026 \times 10^{-2} \text{ eV}, \quad m_2 = 5.1 \times 10^{-2} \text{ eV}, \quad m_3 \simeq 8.775 \times 10^{-3} \text{ eV},
\]
\[
 m_{\beta \beta}^I \simeq 5.1786 \times 10^{-2} \text{ eV}, \quad m_\beta^I \simeq 5.1063 \times 10^{-2} \text{ eV}, \quad \sum = \simeq 0.11003 \text{ eV}.
\]

In the inverted spectrum, \( m_3 \sim m_2 < m_1 \) hence \( m_3 \) can be considered as the lightest neutrino mass.
and

\[ B_{1,2} = (2.95171 \mp 0.760222i) \times 10^{-2} \text{eV}, \quad C \simeq 2.20914 \times 10^{-2} \text{eV}. \] (54)

Now, combining (21) and (54) yields \[ M' = -0.979204 M, \quad m_D = 0.139816 \sqrt{M}, \]
\[ M_1 = -0.1138 M, \quad M_2 = -0.502898 M, \] (55)

and
\[ x = 0.139816 \sqrt{M}/v_\phi, \quad y = -0.979204 M/v_\chi, \quad z = -0.1138 M/v_\zeta, \] (56)
\[ v_\zeta_2 = 4.41914 v_\zeta_1. \] (57)

Eq. (57) shows that \(v_\zeta_1\) and \(v_\zeta_2\) are different from each other but in the same order of magnitude.

3. Quark mass

The quarks content of the model under \([SU(2)_L, U(1)_Y, U(1)_X, S_4]\) symmetries, respectively, given in Tab. 5, where \(i = 1, 2, 3\) is a family index of three lepton families, which are in order defined as the components of the \(3\) representations under \(S_4\).

| Fields | \(Q_{iL}\) | \(u_{1R}\) | \(u_{2,3R}\) | \(d_{1R}\) | \(d_{2,3R}\) |
|--------|------------|------------|------------|------------|------------|
| SU(2)_L | 2          | 1          | 1          | 1          | 1          |
| U(1)_Y | 1/3        | 4/3        | 4/3        | -2/3       | -2/3       |
| U(1)_X | 0          | 0          | 0          | 0          | 0          |
| \(S_4\) | 3          | 1          | 2          | 1          | 2          |

The Yukawa interactions are \[ L_q = h_1^u (\bar{Q}_{1L} \tilde{\phi})_{12} u_{1R} + h^n (\bar{Q}_{1L} \phi)_{22} u_{2R} + h^n (\bar{Q}_{1L} \phi')_{22} u_{3R} + h^n (\bar{Q}_{1L} \phi')_{22} u_{3R} + H.c. \] (58)

With the VEV alignments of \(\phi\) and \(\phi'\) as given in Eqs. (50) and (57), the mass Lagrangian of quarks reads
\[ L_q^{mass} = (\bar{u}_{1L}, \bar{u}_{2L}, \bar{u}_{3L}) M_u (u_{1R}, u_{2R}, u_{3R})^T + (\bar{d}_{1L}, \bar{d}_{2L}, \bar{d}_{3L}) M_d (d_{1R}, d_{2R}, d_{3R})^T + H.c. \] (59)

*a* This system of equations has two solutions, however, these solutions differ only by the sign of \(m_D\) which has no effect in the neutrino oscillation experiments.

*b* Here, \(\phi = i \sigma_2 \phi^* = \begin{pmatrix} \phi_0^R \\ -\phi_0^\dagger \end{pmatrix} \sim [2, -1, 0, 3], \) and \(\phi' \sim [2, -1, 0, 3']\).
where the mass matrices for up- and down-quarks are, respectively, obtained as follows

\[
M_u = \begin{pmatrix}
h_u^u v & h_u^u v - h_u^u v' & h_u^u v + h_u^u v' \\
h_u^d v (h_u^u v - h_u^u v') & (h_u^u v + h_u^u v') \omega^2 \\
h_u^d v (h_u^u v - h_u^u v') \omega & (h_u^u v + h_u^u v') \omega
\end{pmatrix},
\]

(60)

\[
M_d = \begin{pmatrix}
h_d^u v & h_d^u v - h_d^u v' & h_d^u v + h_d^u v' \\
h_d^d v (h_d^u v - h_d^u v') & (h_d^u v + h_d^u v') \omega^2 \\
h_d^d v (h_d^u v - h_d^u v') \omega & (h_d^u v + h_d^u v') \omega
\end{pmatrix}.
\]

(61)

The structure of the up- and down-quark mass matrices in Eqs. (60) and (61) are similar to those in Ref. [17], i.e., in the model under consideration there is no CP violation in the quark sector. The matrices \( M_u \) and \( M_d \) in Eqs. (60), (61) are, respectively, diagonalized as

\[
U_{L}^{d+} M_u U_{R}^{d} = \text{diag} \left( \sqrt{3} h_u^u v, \sqrt{3} (h_u^u v - h_u^u v'), \sqrt{3} (h_u^u v + h_u^u v') \right)
\]

\[
eq \text{diag} \left( m_u, m_c, m_t \right),
\]

(62)

\[
U_{L}^{d+} M_d U_{R}^{d} = \text{diag} \left( \sqrt{3} h_d^u v, \sqrt{3} (h_d^u v - h_d^u v'), \sqrt{3} (h_d^u v + h_d^u v') \right)
\]

\[
eq \text{diag} \left( m_d, m_s, m_b \right),
\]

(63)

where \( U_{L}^{d} = U_{R}^{d} = U_L \), with \( U_L \) given in (11), are the unitary matrices, which couple the left-handed up- and down-quarks to those in the mass bases, respectively, and \( U_{L}^{u} = U_{R}^{u} = 1 \). Therefore, in this case, we get the quark mixing matrix

\[
U_{CKM} = U_{L}^{d} U_{L}^{u} = 1.
\]

(64)

This is the common property for some models based on discrete symmetry groups[15] and can be seen as an important result of the paper since the experimental quark mixing matrix is close to the unit matrix. A small permutations such as a violation of \( S_3 \) symmetry due to unnormal Yukawa interactions will possibly providing the desirable quark mixing pattern[165]. A detailed study on this problem is out of the scope of this work and should be skip.

In similarity to the charged leptons, the masses of pairs \((c, t)\) and \((s, b)\) quarks are also separated by the \( \phi \) scalar. The up and down quark masses are

\[
m_u = \sqrt{3} h_1^u v, \quad m_c = \sqrt{3} (h_1^u v - h_1^u v'), \quad m_t = \sqrt{3} (h_1^u v + h_1^u v'), \quad m_d = \sqrt{3} h_1^d v, \quad m_s = \sqrt{3} (h_1^d v - h_1^d v'), \quad m_b = \sqrt{3} (h_1^d v + h_1^d v'),
\]

(65)

The current mass values for the quarks are given by[165]

\[
m_u = 230^{+0.7}_{-0.5} \text{ MeV}, \quad m_c = 1.275 \pm 0.025 \text{ GeV}, \quad m_t = 173.21 \pm 0.51 \pm 0.71 \text{ GeV},
\]

\[
m_d = 4.8^{+0.5}_{-0.3} \text{ MeV}, \quad m_s = 95 \pm 5 \text{ MeV}, \quad m_b = 4.18 \pm 0.03 \text{ GeV}.
\]

(66)
With the help of Eqs. (62), (63) and (66) we obtain the followings relations:

\[
\begin{align*}
    h_u &= \frac{5.03695 \times 10^{10}}{v}, \\
    h_d &= \frac{1.23409 \times 10^9}{v}, \\
    h_u' &= \frac{4.96334 \times 10^{10}}{v'}, \\
    h_d' &= \frac{1.17924 \times 10^9}{v'}, \\
    h_u' &= \frac{1.32791 \times 10^6}{v}, \\
    h_d' &= \frac{2.77128 \times 10^6}{v},
\end{align*}
\]

or

\[
\begin{align*}
    h_u / h_d &\simeq 40, \\
    h_u' / h_d &\simeq 42, \\
    h_d'/h_1 &\simeq 2,
\end{align*}
\]

\[
\begin{align*}
    h_u / h_1 &\simeq 3.8 \times 10^4, \\
    h_d'/h_1 &\simeq 4.5 \times 10^2,
\end{align*}
\]

i.e, \( h_1 \) and \( h_1' \) are in the same order but \( h_u (h_u') \) is one magnitude order larger than \( h_d (h_d') \). On the other hand, in the case \( |v| \sim |v'| \) we get \( h_u / h_1 \simeq 3.7 \times 10^4 \), \( h_d / h_1' \simeq 4.2 \times 10^2 \).

To get explicit values of the Yukawa couplings in the quark sector, we assume \( v' \sim v \sim 100 \text{ Gev} \) then

\[
\begin{align*}
    h_u &= 0.503695, \\
    h_u' &= 0.496334, \\
    h_u &= 1.32791 \times 10^{-5}, \\
    h_d &= 1.23409 \times 10^{-2}, \\
    h_d' &= 1.17924 \times 10^{-2}, \\
    h_d' &= 2.77128 \times 10^{-5}.
\end{align*}
\]

We note that, the quarks mixing matrix in Eq. (67) has no predictive power for quarks mixing but their masses are consistent with the recent experimental data.

4. Conclusions

We have proposed a neutrino mass model based on \( S_4 \) flavor symmetry which accommodates lepton mass, mixing with non-zero \( \theta_{13} \) and CP violation phase, and the quark mixing matrix is unity at tree level. The realistic neutrino mass and mixing pattern obtained at the tree-level with renormalizable interactions by one \( SU(2)_L \) doublet and two \( SU(2)_L \) singlets in which one being in \( \mathbf{2} \) and the two others in \( \mathbf{3} \) under \( S_4 \) if both the breakings \( S_4 \rightarrow S_3 \) and \( S_4 \rightarrow Z_3 \) are taken place in charged lepton sector and the breaking \( S_4 \rightarrow K \) taken place in neutrino sector. The model also gives a remarkable prediction of Dirac CP violation \( \delta_{CP} \simeq \frac{\pi}{2} \) or \( -\frac{\pi}{2} \) in the both normal and inverted spectrum.

Appendix A. \( S_4 \) group and Clebsch-Gordan coefficients

For convenience, we will refer to some properties of \( S_4 \). \( S_4 \) has 24 elements divided into 5 conjugacy classes, with \( \mathbf{1}, \mathbf{1}', \mathbf{2}, \mathbf{2}', \mathbf{3} \) and \( \mathbf{3}' \) as its 5 irreducible representations. Any element of \( S_4 \) can be formed by multiplication of the generators \( S \) and \( T \) obeying the relations \( S^4 = T^3 = 1, ST^2S = T \). In this paper, we work in the basis
where $3, 3'$ are real representations whereas $2$ is complex. One possible choice of generators is given as follows

\[ \mathbf{1} : S = 1, \quad T = 1 \]
\[ \mathbf{1}' : S = -1, \quad T = 1 \]
\[ \mathbf{2} : S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} \]
\[ \mathbf{3} : S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \]

(A.1)

where $\omega = e^{2\pi i/3} = -1/2 + i\sqrt{3}/2$. All the group multiplication rules of $S_4$ as given below.

\[ \mathbf{1} \otimes \mathbf{1} = \mathbf{1}(11), \quad \mathbf{1}' \otimes \mathbf{1}' = \mathbf{1}(11), \quad \mathbf{1} \otimes \mathbf{1}' = \mathbf{1}'(11), \]  
\[ \mathbf{1} \otimes \mathbf{2} = \mathbf{2}(11, 12), \quad \mathbf{1}' \otimes \mathbf{2} = \mathbf{2}(11, -12), \]
\[ \mathbf{1} \otimes \mathbf{2}' = \mathbf{2}'(11, 12, 13), \quad \mathbf{1}' \otimes \mathbf{2}' = \mathbf{2}'(11, 12, 13), \]
\[ \mathbf{2} \otimes \mathbf{2} = \mathbf{1}(12 + 21) \oplus \mathbf{1}'(12 - 21) \oplus \mathbf{2}(22, 11), \]
\[ \mathbf{2} \otimes \mathbf{3} = \mathbf{3}((1 + 2)1, \omega(1 + \omega)2, \omega^2(1 + \omega^2)23) \]
\[ \oplus \mathbf{3}'((1 - 2)1, \omega(1 - \omega)2, \omega^2(1 - \omega^2)23) \]
\[ \mathbf{2} \otimes \mathbf{3}' = \mathbf{3}'((1 + 2)1, \omega(1 + \omega)2, \omega^2(1 + \omega^2)23) \]
\[ \oplus \mathbf{3}'((1 - 2)1, \omega(1 - \omega)2, \omega^2(1 - \omega^2)23) \]
\[ \mathbf{3} \otimes \mathbf{2} = \mathbf{1}(11 + 22 + 33) \oplus \mathbf{2}(11 + \omega^222 + \omega33, 11 + \omega22 + \omega^233) \]
\[ \oplus \mathbf{3}((23 + 32, 31 + 13, 12 + 21) \oplus \mathbf{3}'(23 - 32, 31 - 13, 12 - 21), \]
\[ \mathbf{3}' \otimes \mathbf{2}' = \mathbf{1}'(11 + 22 + 33) \oplus \mathbf{2}'(11 + \omega^222 + \omega33, 11 + \omega22 + \omega^233) \]
\[ \oplus \mathbf{3}'((23 + 32, 31 + 13, 12 + 21) \oplus \mathbf{3}'(23 - 32, 31 - 13, 12 - 21), \]
\[ \mathbf{3} \otimes \mathbf{3}' = \mathbf{1}'(11 + 22 + 33) \oplus \mathbf{2}'(11 + \omega^222 + \omega33, 11 - \omega22 - \omega^233) \]
\[ \oplus \mathbf{3}'((23 + 32, 31 + 13, 12 + 21) \oplus \mathbf{3}'(23 - 32, 31 - 13, 12 - 21). \]

(A.2) to (A.11)

where the subscripts $s$ and $a$ respectively refer to their symmetric and anti-symmetric product combinations as explicitly pointed out. In the Eqs. (A.2) to (A.11) we have used the notation $\mathbf{3}(1, 2, 3)$ which means some $\mathbf{3}$ multiplet such as $x = (x_1, x_2, x_3) \sim \mathbf{3}$ or $y = (y_1, y_2, y_3) \sim \mathbf{3}$ etc. and so on. Moreover, the numbered multiplets such as $\ldots, i_1, i_2, \ldots$ mean $\ldots, i_1, j_1, \ldots$ where $x_i$ and $y_j$ are the multiplet components of different representations $x$ and $y$, respectively.
The rules to conjugate the representations $1, 1', 2, 3$, and $3'$ are given by

$$2^*(1^*, 2^*) = 2(2^*, 1^*), \quad 1^*(1^*) = 1(1^*), \quad 1''(1^*) = 1'(1^*), \quad (A.12)$$
$$2^*(1^*, 2^*, 3^*) = 2(1^*, 2^*, 3^*), \quad 2''(1^*, 2^*, 3^*) = 2'(1^*, 2^*, 3^*), \quad (A.13)$$

where, for example, $2^*(1^*, 2^*)$ denotes some $2^*$ multiplet of the form $(x_1^*, x_2^*) \sim 2^*$.

**Appendix B. The breakings of $S_4$ by triplet 3**

For triplets $\mathbf{2}$ we have the followings alignments:

1. The first alignment: $\langle \phi_1 \rangle \neq \langle \phi_2 \rangle \neq \langle \phi_3 \rangle$ then $S_4$ is broken into $\{1\} \equiv \{\text{identity}\}$, i.e. $S_4$ is completely broken.
2. The second alignment: $0 \neq \langle \phi_1 \rangle \neq \langle \phi_2 \rangle = \langle \phi_3 \rangle \neq 0$ or $0 \neq \langle \phi_1 \rangle = \langle \phi_2 \rangle \neq \langle \phi_3 \rangle \neq 0$ then $S_4$ is broken into $Z_2$ which consists of the elements $\{1, TSTS^2\}$ or $\{1, TSS^2\}$ or $\{1, S^2TS\}$, respectively.
3. The third alignment: $\langle \phi_1 \rangle = \langle \phi_2 \rangle \neq \langle \phi_3 \rangle \neq 0$ then $S_4$ is broken into $S_3$ which consists of the elements $\{1, TSTS^2, STS^2, S^2T\}$.
4. The fourth alignment: $0 = \langle \phi_2 \rangle \neq \langle \phi_1 \rangle = \langle \phi_3 \rangle \neq 0$ or $0 = \langle \phi_1 \rangle \neq \langle \phi_2 \rangle = \langle \phi_3 \rangle \neq 0$ then $S_4$ is broken into $Z_2$ which consists of the elements $\{1, TSTS^2\}$ or $\{1, TSS^2\}$ or $\{1, S^2T\}$, respectively.
5. The fifth alignment: $0 = \langle \phi_2 \rangle \neq \langle \phi_1 \rangle \neq \langle \phi_3 \rangle \neq 0$ or $0 = \langle \phi_1 \rangle \neq \langle \phi_2 \rangle \neq \langle \phi_3 \rangle \neq 0$ or $0 \neq \langle \phi_1 \rangle \neq \langle \phi_2 \rangle \neq \langle \phi_3 \rangle = 0$ then $S_4$ is completely broken.
6. The sixth alignment: $0 \neq \langle \phi_1 \rangle \neq \langle \phi_2 \rangle = \langle \phi_3 \rangle = 0$ or $0 \neq \langle \phi_2 \rangle \neq \langle \phi_3 \rangle = \langle \phi_1 \rangle = 0$ or $0 \neq \langle \phi_3 \rangle \neq \langle \phi_1 \rangle = \langle \phi_2 \rangle = 0$ then $S_4$ is broken into Klein four group $K$ which consists of the elements $\{1, S^2, TSTS^2, TST\}$ or $\{1, TS^2T^2, STS^2, T^2S\}$ or $\{1, T^2ST^2, ST^2, S^2T\}$, respectively.

**Appendix C. The breakings of $S_4$ by triplet $3'$**

For triplets $\mathbf{2'}$ we have the followings alignments:

1. The first alignment: $\langle \phi'_1 \rangle \neq \langle \phi'_2 \rangle \neq \langle \phi'_3 \rangle$ then $S_4$ is broken into $\{1\} \equiv \{\text{identity}\}$, i.e. $S_4$ is completely broken.
2. The second alignment: $0 \neq \langle \phi'_1 \rangle \neq \langle \phi'_2 \rangle = \langle \phi'_3 \rangle \neq 0$ or $0 \neq \langle \phi'_1 \rangle = \langle \phi'_2 \rangle \neq \langle \phi'_3 \rangle \neq 0$ or $0 \neq \langle \phi'_1 \rangle = \langle \phi'_2 \rangle \neq \langle \phi'_3 \rangle = 0$ then $S_4$ is broken into $\{1\} \equiv \{\text{identity}\}$, i.e. $S_4$ is completely broken.
3. The third alignment: $\langle \phi'_1 \rangle = \langle \phi'_2 \rangle = \langle \phi'_3 \rangle \neq 0$ then $S_4$ is broken into $Z_3$ that consists of the elements $\{1, T, T^2\}$.
4. The fourth alignment: $0 = \langle \phi'_2 \rangle \neq \langle \phi'_1 \rangle = \langle \phi'_3 \rangle \neq 0$ or $0 = \langle \phi'_1 \rangle \neq \langle \phi'_2 \rangle = \langle \phi'_3 \rangle \neq 0$ or $0 = \langle \phi'_3 \rangle \neq \langle \phi'_1 \rangle = \langle \phi'_2 \rangle = 0$ then $S_4$ is broken into $Z_2$ which consists of the elements $\{1, T^2S\}$ or $\{1, TST\}$ or $\{1, ST^2\}$, respectively.
5. The fifth alignment: $0 = \langle \phi'_1 \rangle \neq \langle \phi'_2 \rangle \neq \langle \phi'_3 \rangle \neq 0$ or $0 = \langle \phi'_1 \rangle \neq \langle \phi'_2 \rangle \neq \langle \phi'_3 \rangle \neq 0$ or $0 \neq \langle \phi'_1 \rangle \neq \langle \phi'_2 \rangle \neq \langle \phi'_3 \rangle = 0$ then $S_4$ is completely broken.
The sixth alignment: \( 0 \neq \langle \phi' \rangle_1 \neq \langle \phi' \rangle_2 = \langle \phi' \rangle_3 = 0 \) or \( 0 \neq \langle \phi' \rangle_1 \neq \langle \phi' \rangle_2 \neq \langle \phi' \rangle_3 = 0 \) or \( 0 \neq \langle \phi' \rangle_1 \neq \langle \phi' \rangle_2 \neq \langle \phi' \rangle_3 \neq 0 \) then \( S_4 \) is broken into a four-element subgroup generated by a four-cycle, which consisting of the elements \( \{1, S, S^2, S^3\} \) or \( \{1, TST^2, ST, TS^2T^2\} \) or \( \{1, TS, T^2ST, T^2S^2T\} \), respectively.

Appendix D. The breakings of \( S_4 \) by doublet 2

1. The first alignment: \( \langle \zeta_1 \rangle = \langle \zeta_2 \rangle \) then \( S_4 \) is broken into an eight-element subgroup, which is isomorphic to \( D_4 \).
2. The second alignment: \( \langle \zeta_1 \rangle \neq 0 = \langle \zeta_2 \rangle \) or \( \langle \zeta_1 \rangle = 0 \neq \langle \zeta_2 \rangle \) then \( S_4 \) is broken into \( A_4 \) consisting of the identity and the even permutations of four objects.
3. The third alignment: \( \langle \zeta_1 \rangle \neq \langle \zeta_2 \rangle \neq 0 \) then \( S_4 \) is broken into a four-element subgroup consisting of the identity and three double transitions, which is isomorphic to Klein four group \( K \).

Appendix E. The solution with \( \delta = \frac{\pi}{2} \) in the normal case

By substituting \( B_1 \) from (38) into (22) and taking the two experimental data on squared mass differences of neutrinos given in Ref. [33] \( \Delta m^2_{21} = 7.50 \times 10^{-5} \text{eV}^2 \) and \( \Delta m^2_{31} = 2.457 \times 10^{-3} \text{eV}^2 \), we get a solution (in \( [\text{eV}] \)) as follows:

\[
C = 0.5\sqrt{\alpha - 2\beta},
\]

\[
B_2 = -0.5\sqrt{4A^2 - 0.0003 + (1.37741 \times 10^{-7} + 0.34412i)C} - 0.5\sqrt{(3.52631 + 3.792 \times 10^{-7}i)C^2},
\]

where

\[
\alpha = (0.0026169 - 2.81407 \times 10^{-10}i) + (2.26866 - 2.43959 \times 10^{-7}i)A^2, \quad (E.2)
\]

\[
\beta = -2.2987 \times 10^{-7} + 4.94378 \times 10^{-14}i + (0.00296843 - 6.38415 \times 10^{-10}i)A^2 + (1.2867 - 2.7673 \times 10^{-7}i)A^4. \quad (E.3)
\]

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