"Bad Metal" Conductivity of Hard Core Bosons

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Two dimensional hard core bosons suffer strong scattering in the high temperature resistive state at half filling. The dynamical conductivity $\sigma(\omega)$ is calculated using non perturbative tools such as continued fractions, series expansions and exact diagonalization. We find a large temperature range with linearly increasing resistivity and broad dynamical conductivity, signaling a breakdown of Boltzmann-Drude quasiparticle transport theory. At zero temperature, a high frequency peak in $\sigma(\omega)$ appears above a "Higgs mass" gap, and corresponds to order parameter magnitude fluctuations. We discuss the apparent similarity between conductivity of hard core bosons and phenomenological characteristics of cuprates, including the universal scaling of Homes et al. (Nature 430, 539 (2004)).

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I. INTRODUCTION

Quantum transport in condensed matter is largely based on the paradigms of Fermi and Bose gases. Boltzmann equation for the conductivity is valid in the weak scattering regime where it yields a Drude form [1, 2],

$$\sigma_{Drude}(T,\omega) = \frac{q^2 n}{m^*} \frac{\tau}{1 - i\omega\tau},$$  \hspace{1cm} (1)

where $T$ is temperature, $\omega$ is frequency, and $q, n, m^*, \tau(T)$ are charge, density, effective mass and scattering time of the constituent quasiparticles.

Interacting bosons in a strong periodic potential may suffer strong enough scattering which invalidates Boltzmann-Drude theory. An example is provided by the two dimensional Hard Core Bosons (HCB) model at half filling. While it is established that the ground state is a bone fide superconductor [3, 4, 5, 6, 7], the resistive (normal) phase involves strongly interacting bosons and vortex pairs. Previous work [8, 9] showed that the lattice (umklapp) scattering dramatically increases vortex mobility. At half filling, it produces an abrupt reversal of the Hall conductivity, and doublet degeneracies associated with each vortex.

HCB models may be experimentally relevant to cold atoms in optical lattices [10, 11], underdoped cuprate superconductors [12, 13, 14], low capacitance Josephson junction arrays [15, 16], and disordered superconducting films [17]. An added advantage of HCB, is that they are described by a quantum spin-half $XY$ model which is amenable to tools of quantum magnetism.

It is the purpose of this paper to compute the conductivity of HCB at half filling. We apply and test a set of non perturbative approaches, including continued fraction representation, series expansions and exact diagonalization. By studying the dynamical structure of the Kubo formula, we can construct well-converging approximants which agree with high accuracy sum rules in a wide regime of temperature. The conductivity at high temperatures is obtained to order of $T^{-3}$. Near the superconducting transition, it is matched with the critical conductivity which was derived by Halperin and Nelson [18]. At zero temperature, the dynamical conductivity is obtained from the relativistic Gross-Pitaevskii field theory, and a variational fit to 12th order moments.

Our key results pertain to the qualitative effects of strong scattering on the conductivity. At high temperatures, the conductivity goes as

$$\sigma(T,\omega) \approx 0.91 \frac{q^2 \tanh(\hbar\omega/(2T))}{\hbar} \exp \left(-\frac{(\omega/\bar{\Omega})^2}{\Omega^2/\bar{\Omega}^2} \right), \hspace{1cm} (2)$$

where $\bar{\Omega}$ is a high frequency scale. Eq. (2), when fit at low frequencies and temperatures, to the form (1), yields an “effective scattering rate” which is equal to $2T$. The resistivity as shown in Fig. 1 exhibits “bad metal” zero
strained creation operators $\tilde{a}_i$, half raising operator with restricted occupation numbers, $n_i \leq 1$. The resistivity’s linear slope defines the proportionality coefficient between the zero temperature superfluid stiffness, and product of $T_s$ and the normal state conductivity near $T_s$. We note that linearly increasing resistivity and an analogous scaling of superfluid stiffness with conductivity called “Homes law”, have been observed in the cuprate family, as depicted in Fig. 7.

In addition, at zero temperature we find a small conductivity peak above a “Higgs mass” gap. This peak is associated with order parameter magnitude fluctuations. These are analogous to coherence oscillations observed in cold atoms during rapid Mott to superfluid quenches. We speculate that perhaps the mid infra-red peak observed in some cuprates at low temperatures, might arise from these magnitude fluctuations. However we emphasize that similarities between HCB and cuprates are only suggestive. This work does not include effects of fermion quasiparticles, magnetic excitations, and inhomogeneities which are experimentally important.

This paper is organized as follows: Section II introduces the HCB model and lists some of the established thermodynamics. Section III discusses the Kubo formula in general, and derives the moment expansion, continued fraction recurrents, and orthogonal polynomials which can be used to evaluate $\sigma$ non perturbatively and generate a high temperature expansion. Section IV derives the particular recurrents for the HCB at half filling. Justification for the rapidly converging harmonic oscillator expansion is provided. Sections IIIIV are quite technical, and could be avoided at first reading. The results of our calculations are provided in the following sections. Section V plots the dynamical conductivity at high temperatures, and explains three approaches which converge to the same curve. Section VI obtains the resistivity as a function of temperature. Section VII obtains the dynamical conductivity and “Higgs mass” at zero temperature. Section VIII summarizes the key results and discusses their possible relevance to the dynamical and DC conductivity of cuprate superconductors.

II. HARD CORE BOSONS

Hard core bosons are defined on lattice sites $i = 1, \ldots, N$ with restricted occupation numbers, $n_i = 0, 1$. The constrained creation operators $\tilde{a}_i^\dagger$ are represented by spin half raising operator

$$\tilde{a}_i^\dagger = S_i^+ = S_i^x + iS_i^y,$$
$$\tilde{a}_i = S_i^- = S_i^x - iS_i^y.$$  

Thus, their commutation relations are

$$[S_i^-, S_j^+] = -\delta_{ij} 2S_i^z,$$
$$[\tilde{a}_i, \tilde{a}_j^\dagger] = 2 \delta_{ij} (1 - 2n_i).$$

A minimal model of HCB hopping with Josephson coupling $J$, coupled to an electromagnetic field $A_{ij}$, is the gauged $S = 1/2$ quantum XY model,

$$H = -2J \sum_{\langle i,j \rangle} \left( e^{i q A_{ij}} \tilde{a}_i^\dagger \tilde{a}_j + \text{H.c.} \right)$$
$$= -2J \sum_{\langle i,j \rangle} \left( e^{i q A_{ij}} S_i^+ S_j^- + \text{H.c.} \right),$$

where $q$ is the boson charge ($= 2e$, for electronic superconductors) and we use units of $\hbar = c = 1$. Here we consider $\langle i, j \rangle$ to be nearest neighbor bonds on the square lattice. In the absence of a chemical potential (i.e. a Zeeman field coupled to $S_i^z$), the Hamiltonian describes a density of half filling (zero magnetization), with half a boson per site.

The uniform current operator, for $A = 0$, is given by

$$J_x = -\frac{2iqJ}{\sqrt{N}} \sum_{\langle i,j \rangle} \left( \tilde{a}_i^\dagger \tilde{a}_j - \tilde{a}_j^\dagger \tilde{a}_i \right)$$
$$= \frac{4qJ}{\sqrt{N}} \sum_{r} \left( S_{r,r+\hat{x}}^z - S_{r+\hat{x},r}^z \right).$$

We note that in one dimension, the current of HCB is a conserved operator since,

$$[H^{1D}, J_x] = 0,$$

and hence the real conductivity trivially vanishes at all finite frequencies and temperatures. In two and higher dimensions, this is not the case: the conductivity has non trivial dynamical structure.

Below we review some established results for the thermodynamic properties of the two dimensional quantum XY model which are relevant to the conductivity.

A. Superfluid stiffness

It is widely believed that at zero temperature $H$ has long range order. Thus, at low temperatures $T \geq 0$, the two dimensional boson superfluid stiffness $\rho_s$, which has units of energy, is finite:

$$\rho_s \equiv q^2 \frac{d^2F(T, n)}{(dA_z)^2} \bigg|_{A=0} > 0,$$

where $F$ is the free energy in the presence of a uniform field $A = A_\hat{x}$. A classical (mean field) approximation at zero temperature yields a non-monotonous density dependence,

$$\rho_s^{cl}(0, n) = 4Jn(1 - n).$$

where $n$ is the mean boson occupation (filling). Half filling $n = 1/2$ is “optimal”, with maximal $\rho_s$. Quantum corrections to $\rho_s^{cl}(0, 1/2)$ enhance it by about 7% [6, 7].
B. BKT Transition

The static order parameter correlations of $\xi$ are described by a renormalized classical XY model. At low temperatures, correlations decay as a power-law in distance, $\rho_s(T)$ decreases with $T$, until it falls discontinuously to zero at $T_{BKT}$, the Berezinskii-Kosterlitz-Thouless (BKT) transition temperature [22, 30, 31]. At half filling, quantum Monte-Carlo (QMC) simulations of [5] have determined $T_{BKT} \approx 1.41J$.

Just below the transition, a universal relation holds:

$$\rho_s(T_{BKT}) = \frac{2}{\pi} T_{BKT}.$$  

(11)

For $T > T_{BKT}$, $\rho_s = 0$, and the correlation length is

$$\xi \sim A \exp \left( \frac{B}{\sqrt{(T - T_{BKT})}} \right),$$  

(12)

where $A = 0.285$ and $B = 1.92$ as determined by QMC [5].

In multilayered systems with weak interlayer coupling [14, 52, 33] $J_c << J$, the 3D transition temperature is higher than $T_{BKT}$ by a factor,

$$T_{c}^{3D} = T_{BKT} \left( 1 + \frac{B^2}{\log^2(0.144J/J_c)} \right),$$  

(13)

C. Boson particle-hole symmetry

The charge conjugation operator

$$C = e^{i\pi \sum_i S_i^z},$$  

(14)

transforms particles into oppositely charged holes:

$$\hat{a}_i \rightarrow \hat{a}_i^\dagger, \quad n_i \rightarrow (1 - n_i), \quad H[nA, n] \rightarrow H[-qA, 1 - n].$$  

(15)

It follows therefore that under reflection about half filling, the Hall conductivity reverses sign, while the superfluid stiffness and longitudinal conductivity are invariant. In the low density limit $n << \frac{1}{2}$, the HCB are effectively unconstrained [34, 35] as seen by [4]. This can be demonstrated by expanding the Holstein-Primakoff bosons representation of spin half,

$$\hat{a}_i = \hat{b}_i^\dagger \sqrt{1 - n_i} \approx \hat{b}_i^\dagger (1 - \frac{1}{2}n_i^2 + O(n_i^2)),$$  

(16)

Truncating the expansion [16], and inserting it into $H$, turns it into an interacting soft bosons model. At low densities, the low excitations of $H$ do not feel the lattice. Thus, the long wavelength properties are well described by the Galilean invariant Gross-Pitaevskii (GP) field theory [10], and the mean field superfluid stiffness goes as

$$\rho_s \sim 4Jn \equiv \frac{\hbar^2}{m_b a^2}n,$$  

(17)

where $m_b, a$ are the boson effective mass and lattice constant, respectively. Similarly, by particle hole transformation [15], $H$ simplifies into a model of weakly interacting holes as $n \rightarrow 1$.

It is around half filling, however, that higher order interactions in [16] and lattice effects become relevant. The Galilean invariant (non-relativistic) GP theory fails to account for important umklapp scattering processes which lead to Hall effect cancellation, and doublet degeneracies of vortex states [8].

III. NON PERTURBATIVE ANALYSIS OF THE KUBO FORMULA

A. Current fluctuations function

Thermodynamic averages are denoted by

$$\langle \cdot \rangle_\beta = \frac{1}{Z} \text{Tr} \left[ e^{-\beta H} (\cdot) \right], \quad Z = \text{Tr} e^{-\beta H},$$  

(18)

where $\beta = 1/T$. The linear response Kubo formula for the longitudinal dynamical conductivity is [36]

$$\sigma(\beta, \omega) = i \frac{\langle \{q^2 K_x \} \rangle_\beta - \Lambda_{xx}(\beta, \omega + i\epsilon)}{\omega + i\epsilon},$$  

(19)

where

$$K_x = -\frac{4J}{N} \sum_\mathbf{r} \left( S_\mathbf{r}^x S_{\mathbf{r}+\mathbf{x}}^x + S_\mathbf{r}^y S_{\mathbf{r}+\mathbf{x}}^y \right),$$  

(20)

is the $x$-kinetic energy, and $\Lambda$ is the retarded current-current response function

$$\Lambda_{xx}(z) = \int_0^\infty dt e^{izt} \langle [J_x(t), J_x(0)]_\beta \rangle,$$  

(21)

$$J_x(t) \equiv e^{iHt} \hat{J}_x e^{-iHt},$$

where the HCB current operator is given by [10]. The (real) conductivity is given by

$$\sigma(\beta, \omega) = q^2 \pi \rho_s(\beta) \delta(\omega) + \frac{\tanh(\beta\omega/2)}{\omega} G''(\beta, \omega),$$  

(22)

where $\{\cdot, \cdot\}$ is an anticommutator. The current fluctuations function $G''(\beta, \omega)$ of HCB at half filling, is our primary object of attention. We cannot rely on a nearly free quasiparticle basis about which to expand Eqs. [15] or [8]. We therefore turn our attention to non perturbative approaches.
B. Moments expansion

It is advantageous in our case, to analyse the dynamical structure of $G''(\omega)$ in the Operator Hilbert Space (OHS). The OHS is a linear space of HCB (spin half) operators, denoted by capital roman letters, $A, B, \ldots$, which are the “hyperstates” of the OHS. In this paper, we use two different inner products. 

(i) The infinite temperature product,

\[ (A, B)_{\infty} = \frac{1}{2N} \text{Tr} (A^\dagger B). \]  

(ii) The zero temperature product,

\[ (A, B)_0 = \langle 0 | A^\dagger B | 0 \rangle, \]  

where $|0\rangle$ is the ground state of the Hamiltonian $H$. It is easy to verify that both definitions obey the Hilbert space conditions for an inner product. Henceforth, we unify the notations, and drop the subscripts $0, \infty$.

“Hyperoperators”, denoted by capital script letters, are linear operators which act on hyperstates of the OHS. The Liouvillian hyperoperator $\mathcal{L}$ is defined by its action on any hyperstate $A$ as,

\[ \mathcal{L} A \equiv [H, A]. \]  

By hermiticity of $H$, and the cyclic property of the trace, the Liouvillian $\mathcal{L}$ is Hermitian for both definitions of the inner products [3,24],

\[ (A, \mathcal{L} B) = (B, \mathcal{L} A)^* . \]  

Therefore, $\mathcal{L}$ has a real eigenspectrum.

The time dependent current operator in the Heisenberg representation $J_x(t)$, is compactly expressed using the evolution hyperoperator,

\[ J_x(t) = e^{i\mathcal{L}t} J_x. \]  

The hyper-resolvent $\mathcal{G}(z)$,

\[ \mathcal{G}(z) = \frac{1}{z - \mathcal{L}}, \]  

is related to the evolution hyper-operator by

\[ e^{i\mathcal{L}t} = \int \frac{dz}{2\pi i} e^{izt} \mathcal{G}(z). \]  

The contour surrounds the spectrum of $\mathcal{L}$, which by Eq. 26 lies on the real axis. The complexified current fluctuations function is

\[ G(\beta, z) = \frac{1}{Z} \text{Tr} \left( e^{-\beta H} J_x \mathcal{G}(z) J_x \right). \]  

Eq. 22 is recovered by its imaginary part on the real axis,

\[ G''(\omega, \omega) = -\frac{1}{Z} \text{ImTr} \left( e^{-\beta H} \left\{ J_x, \frac{1}{\omega - \mathcal{L} + i\epsilon} J_x \right\} \right). \]  

A direct $1/\omega$ expansion of $(\omega - \mathcal{L})^{-1}$ does not yield the required imaginary function [37]. To extract $G''$ one uses complex analysis:

\[ \oint dz z^k G(z) = \oint dz z^k \sum_{n=0}^{\infty} \left( \frac{\mathcal{L}}{z} \right)^n = 2\pi i z^k, \]  

and take the contour around the real axis, to obtain the sum rules for all $k = 0, 2, 4 \ldots \infty$,

\[ \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \omega^k G''(\beta, \omega) = \langle \{ J_x, \mathcal{L}^k J_x \} \rangle_{\beta} = \mu_k(\beta). \]  

All odd-$k$ moments vanish by symmetry of $G''(\omega)$. The moments $\mu_k$ are static (equal time) correlators, which can be evaluated numerically or by series expansions. It is possible, in general, to compute only a finite number of $\mu_k$’s. The remaining task is to derive converging approximants to $G''(\beta, \omega)$ by, in a sense, inverting Eq. 33.

C. The Liouvillian Matrix

In order to invert Eq. 33, we use the structure of the Liouvillian matrix. A tridiagonal matrix representation of the Liouvillian is constructed as follows. We define the root hyperstate by the current operator,

\[ \hat{O}_0 = \frac{1}{\sqrt{(J_x, J_x)}} J_x. \]  

An orthonormal set of hyperstates $\hat{O}_n$ can be generated by sequentially applying $\mathcal{L}$ and orthonormalizing by the Gram-Schmidt procedure,

\[ \hat{O}_{n+1} \equiv c_{n+1} \left( \mathcal{L} \hat{O}_n - \Delta_n \hat{O}_{n-1} \right), \quad n = 1, 2, \ldots \]  

\[ \Delta_n = \left( \hat{O}_n, \mathcal{L} \hat{O}_n \right), \]  

\[ c_{n+1} = \left( |\Delta_n|^2 + \left( \hat{O}_n, \mathcal{L}^2 \hat{O}_n \right) \right)^{-1/2}, \]  

where $\Delta_0 = 0$. Since $J_x$ is hermitian, and $\mathcal{L} J_x^*$ is antihermitian, all $\hat{O}_n$ can be chosen to be hermitian (antihermitian) for even (odd) $n$, since $\Delta_n$ are purely imaginary. Thus it is easy to prove, for both inner products defined in [23] and [24], that $\mathcal{L}$ has no diagonal matrix elements $\langle \hat{O}_n, \mathcal{L} \hat{O}_n \rangle = 0$. Also, it is straightforward to prove by induction that $|\hat{O}_n\rangle$ is an orthonormal set:

\[ \langle \hat{O}_n, \hat{O}_{n'} \rangle = \delta_{nn'}, \quad n, n' = 0, 1, \ldots \infty. \]  

In this basis, $\mathcal{L}$ is given by the Liouvillian matrix,

\[ L_{nn'} \equiv \langle \hat{O}_n | \mathcal{L} | \hat{O}_{n'} \rangle = \begin{pmatrix} 0 & \Delta_1 & 0 & 0 \\ \Delta_1^* & 0 & \Delta_2 & 0 \\ 0 & \Delta_2^* & 0 & \Delta_3 \\ 0 & 0 & \Delta_3^* & \ddots \end{pmatrix}. \]
In both $T = 0, \infty$ limits, the current fluctuations function \( G \) is given by the root expectation value of the resolvent:

$$ G(z) = (J_x, (z - \mathcal{L})^{-1} J_x) = \mu_0 (z - L)^{-1}_{00}. \quad(38) $$

where $\mu_0 = (J_x, J_x)$ is the corresponding zeroth moment. In these limits, all moments of Eq. (38) are root expectation values:

$$ \mu_k = (J_x, \mathcal{L}^k J_x). \quad(39) $$

Using (37), an explicit, and useful relation between recurrents and moments is obtained

$$ \mu_k[\Delta] = \mu_0 (L^k)_{00}; $$

$$ \mu_2 = |\Delta_1|^2; $$

$$ \mu_4 = |\Delta_1|^4 + |\Delta_1|^2|\Delta_2|^2; $$

$$ \mu_6 = |\Delta_1|^6 + 2|\Delta_1|^4|\Delta_2|^2 + |\Delta_1|^2|\Delta_2|^4 + |\Delta_1|^2|\Delta_2|^2|\Delta_3|^2; $$

$$ \vdots = \vdots. \quad(40) $$

D. Continued fraction representation

Inverting $z - L$ in Eq. (38) using elementary algebra yields the continued fraction representation (38):

$$ G(z) = \mu_0 \frac{1}{z - \frac{|\Delta_1|^2}{z - \frac{|\Delta_2|^2}{\vdots}}}. \quad(41) $$

At zero temperature we obtain,

$$ G_{T=0}(z) = 2 \langle 0 | J_x^2 | 0 \rangle \frac{1}{z - \frac{|\Delta_1|^2}{z - \frac{|\Delta_2|^2}{\vdots}}}, \quad(42) $$

where $|0\rangle$ is the ground state of $H$. Similarly, in the high temperature limit, the leading order $G(z)$ is given by

$$ G_{T=\infty}(z) = \frac{2}{2^N} \text{Tr} \left( J_x^2 \right) \frac{1}{z - \frac{|\Delta_1|^2}{z - \frac{|\Delta_2|^2}{\vdots}}}. \quad(43) $$

The infinite list of recurrents fully determines $G''(\omega)$. If only a finite set is computable, extrapolation of $|\Delta_n|^2$ to large $n$ is unavoidable. Some intuition about the relation between recurrents and the imaginary part of the continued fraction function at $z \to \omega + i\epsilon$ is gained by the following examples, for complex functions $F(z)$, where $F'(\omega + i\epsilon) = F'(\omega) - iF''(\omega)$.

(i) A Lorentzian, as given by the Drude form (1), is a somewhat pathological limit: All its even moments except $\mu_0$, are infinite. This form amounts to replacing the denominator’s self energy by a purely imaginary constant,

$$ F(z) = \frac{1}{z - \Sigma_1(z)} $$

$$ \Sigma_1(z) = \frac{|\Delta_1|^2}{z - \frac{\Delta_1}{\gamma}} \Rightarrow \frac{i}{\tau}. \quad(44) $$

(ii) Constant recurrents, $\Delta_n = \Delta$, $n = 1, 2, \ldots$, yield a semicircle imaginary part,

$$ F(z) = \frac{1}{z - \frac{\Delta^2}{\gamma}} $$

$$ F''(\omega) = \sqrt{1 - (\omega/2\Delta)^2}. \quad(45) $$

(iii) Linearly increasing recurrents, $|\Delta_n|^2 = n\Omega^2/2$ define the continued fractions

$$ F(z) = \frac{1}{z - \frac{n\Omega^2}{\gamma}} \quad(46) $$

which is relevant to HCB at high temperatures, as argued in Section VA. \textcolor{red}{In Appendix B it is shown that Im} $F''$ is a Gaussian,

$$ F''(\omega) = \sqrt{\frac{\pi}{\Omega^2}} \exp \left( -\frac{\omega^2}{\Omega^2} \right). \quad(47) $$

Incidentally, the real part of $F'(\omega)$ is the Dawson function \( F' \),

$$ F'(\omega) = \frac{2}{\Omega} \int_0^{\omega/\Omega} e^{t^2} dt. \quad(48) $$

E. Finite temperature corrections

At finite temperatures, the current fluctuations function includes the effects of the thermal density matrix

$$ \rho(\beta) = 2^N e^{-\beta H} / Z(\beta). \quad(49) $$

(The prefactor of $2^N$ is introduced for later convenience). One can write

$$ G(\beta, z) = \langle \{ \rho, J_x \} \rangle, G(z) J_x \rangle \approx $$

$$ \sum_n C_n(\beta) G_n(z), $$

$$ C_n(\beta) = \mu_0 \left( \{ \rho(\beta), \hat{O}_0 \} \right) \hat{O}_n \rangle \approx $$

$$ G_n(z) = \mu_0 (z - L)^{-1}_{00}. \quad(50) $$

Taking $z \to \omega + i\epsilon$ yields an orthogonal polynomial expansion for the current fluctuations function,

$$ G''(\beta, \omega) = \sum_{n=0}^{\infty} C_n(\beta) P_n(\omega) G_0(\omega). \quad(51) $$
The polynomials $P_n$ are orthogonal under the measure defined by $G_0(\omega)$,
\[
\int_\infty^\infty d\omega P_n(\omega)P_m(\omega)G_0(\omega) = \delta_{n,m} \tag{52}
\]
In Appendix we derive explicit expressions for $P_n$ as a function of $G_0$ and the preceeding recurrants $|\Delta_m|^2$, $m = 1, 2, \ldots n$.

IV. KUBO FORMULA FOR HCB AT HALF FILLING

In the previous section we introduced the moments and recurrants of the current fluctuations function $G(z)$. Henceforth, we specialize to HCB at half filling. First we analyze the expected asymptotic behavior of the Liouvillian matrix elements. Second, we construct a variational harmonic oscillator (VHO) basis in which the current fluctuations can be expanded. Third, we generate a high temperature expansion of $G''(\beta, \omega)$.

A. Liouvillian of HCB and Gaussian asymptotics

The Liouvillian hyper-operator $\mathcal{L}$ (25), describes strong scattering in the following sense: When $\mathcal{L}$ acts on a hyperstate composed of $n$ spins,
\[
A_n = \sum_{\alpha_1, \alpha_2, \ldots, \alpha_n} c_{\alpha_1, \alpha_2, \ldots, \alpha_n} S_{\alpha_1}^1 S_{\alpha_2}^2 \ldots S_{\alpha_n}^n,
\]
the number of spins increases or decreases by precisely one, i.e.
\[
\mathcal{L}A_n = A_{n+1} + A_{n-1}. \tag{53}
\]
When $|A_{n+1}| >> |A_{n-1}|$, the primary effect of $\mathcal{L}$ is to proliferate the number of spins. Let us compare the behavior of HCB with weakly interacting bosons. Consider a typical boson liquid Hamiltonian
\[
H^{weak} = a^\dagger H_0 a + \frac{1}{2} ga^\dagger a a^\dagger a. \tag{54}
\]
The action of the respective Liouvillian on a linear operator yields
\[
[H^{weak}, a^\dagger] = a^\dagger H_0 a + ga^\dagger a a^\dagger. \tag{55}
\]
Thus, when $g$ is “smaller” than $H_0$, the primary effect of $\mathcal{L}$ is the first term, which propagates $a^\dagger$, rather than the second term which proliferates it. The root hyperstate $\hat{O}_0 \propto J_x$, is bilinear in spins. By repetitive applications of $\mathcal{L}^n\hat{O}_0$ one obtains clusters of up to $n$ spins. Consider a lattice in two dimensions or higher [41], with coordination number $z > 2$. For a typical $A_n$, there are $nz_{eff}$ available bonds to attach an extra spin to an existing cluster, where $z_{eff} < z$. Therefore, the number of distinct terms in the resulting $A_{n+1}$ is roughly a factor of $nz_{eff}$ more than the number of terms in $A_n$. We thus expect, for such a lattice, most of the weight of the hyperstate $\hat{O}_n$ to consist of $n$ spin operators. Since $\hat{O}_n, \hat{O}_{n-1}$ are normalized, we can crudely estimate the asymptotic $n$ dependence of the recurrants
\[
|\Delta_n|^2 \approx (\hat{O}_n, \mathcal{L}\hat{O}_{n-1})^2 \propto (4J)^2 z_{eff} n, \quad n >> 1. \tag{56}
\]
(The factor $4J$ stems from the commutation relations $[S_i, S_j] = i(\alpha_\beta S_\alpha S_\beta$ and the definition of the Hamiltonian (53). As we have seen in (47), if the asymptotic relation (50) were precise, the continued fraction expansion for $G''$ would lead to a perfect Gaussian of width $4\sqrt{2z_{eff}J}$. However, as demonstrated in Fig. 3 the linearity of $|\Delta_n|^2$ is not precise: When $\mathcal{L}$ acts on $\hat{O}_n$ it generates some small fraction of $n-1$ spin operators, which are not included in the term $\Delta_n\hat{O}_{n-1}$ of Eq. (53). This spoils the exact relation between the index $n$, and the number of spin operators. Nevertheless, the coefficients $C_n(\beta)$ in Eq. (51) are expected to converge rapidly at high temperature. Expanding $C_n(\beta)$ in a high temperature series,
\[
C_n(\beta) = \sum_i C_n^{(i)} \beta^i, \tag{57}
\]
it can be verified that the contribution of any $m$-spin operator $A_m$ to order $\beta^i$ depends on traces such as
\[
\text{Tr} \left[ \{ H^i, J_x \} A_m \right] = 0, \quad m > i + 2. \tag{58}
\]
Therefore, $C_n^{(i)}$ measures the relative weight of $i+2$ spins operators or less, in $\hat{O}_n$. Since, as argued previously, these weights become smaller for large $n$, we can expect for fixed $i$ that
\[
\lim_{n \to \infty} C_n^{(i)} \to 0. \tag{59}
\]
Thus at a finite order $\beta^i$, only a finite number of $C_n^{(i)}$‘s are of substantial magnitude. Phrased differently: In an idealized situation in which $\hat{O}_n$ would contain only products of $(n+2)$-spins, the $C_n$‘s would decay with $\beta$ as
\[
C_n = \frac{1}{Z} \text{Tr} \left( e^{-\beta H} \{ \hat{O}_n, \hat{O}_0 \} \right) \sim \beta^n C_n^{(n)} \beta^{n+2}. \tag{60}
\]
This discussion raises an obvious question: Is there a lattice where Eq. (60) becomes precise? We have preliminary expectations [42], that this would be the case, at least for low orders in $n$, for the Bethe lattice in the limit of large coordination number. The rapid convergence of the harmonic oscillator basis discussed below, indicates that the square lattice at high temperatures is not “far” from the infinite dimensional limit.

B. Variational harmonic oscillator expansion

A high temperature expansion for the conductivity can be generated by choosing a convenient basis to expand
\[ G''(\omega) \] The linear increase of the recurrents, suggest earlier in Eq. \([56]\) implies a Gaussian decay of \(G''(\omega)\) at high frequencies. Therefore it is natural to choose variational harmonic oscillator (VHO) basis such that
\[
\hat{G}''(\beta, \omega) = \sum_{n=0}^{\infty} D_n(\beta) \hat{H}_n(\omega) \psi_0^2(\omega),
\]
where \(\hat{H}_n(\omega) = H_n(\omega/\Omega_v)/\sqrt{2^n n!}\), and
\[
\psi_0^2(\omega) = \frac{1}{\sqrt{\pi \Omega_v^2}} \exp \left( -\frac{\omega^2}{\Omega_v^2} \right).
\]

The function \(\psi_0^2(\omega)\) is the VHO ground state probability density, with a variational frequency scale \(\Omega_v\), and \(\hat{H}_n(\omega)\) are the corresponding (normalized) Hermite polynomials, which constitute an orthogonal set under the measure \(\psi_0^2(\omega)\).

The moments of \(\hat{H}_n(\omega)\psi_0^2(\omega)^2\) are
\[
\lambda_n^k = \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \omega^k H_n(\omega/\Omega_v) \psi_0^2(\omega)
= \sqrt{\frac{2^n}{n!}} 2^{-k} \left( \frac{k/2}{(k-n)/2} \right) \Gamma \left( k + \frac{1}{2} \right),
\]
for \(k \geq n\). By construction, all \(\lambda_n^k\) vanish for \(n > k\).

Using the high temperature expansion of a finite number of moments
\[
\mu_k(\beta) = \sum_{i=0}^{\infty} \mu_k^{(i)} \beta^i, \quad k = 0, 2, \ldots n_{\text{max}},
\]
and similarly,
\[
D_n(\beta) = \sum_{i} D_n^{(i)} \beta^i, \quad n \leq n_{\text{max}}.
\]

We insert \([61]\) into \([63]\) and obtain a finite set of linear equations for each \(D_n^{(i)}\),
\[
\sum_{n=0}^{n_{\text{max}}} \lambda_n^k D_n^{(i)} = \mu_k^{(i)}(\beta), \quad k = 0, 2, \ldots n_{\text{max}}.
\]
Solving for \([66]\) yields the desired coefficients for Eq. \([61]\).

**V. DYNAMICAL CONDUCTIVITY: HIGH TEMPERATURE**

The dynamical conductivity, to leading order in \(\beta\), is
\[
\sigma_{\beta=0} = \frac{\beta}{2} G''(\omega).
\]

We have calculated the infinite temperature current fluctuation \(G''_{T=\infty}\), using three distinct methods. The results of the three approaches show satisfactory agreement, as depicted in Fig. 2: Infinite temperature current fluctuations function \(G''_{T=\infty}(\omega)\). Solid lines (blue online) \(\text{vho}\) depicts the results of the variational harmonic oscillator expansion, and (Green online) \(\text{extr}\) depicts the result of the recurrents extrapolation, shown in Fig. 3. Circles (red online) \(\text{ed}\) depict the exact diagonalization result computed on a \(4 \times 4\) lattice. Inset: Convergence of the variational harmonic oscillator expansion. \(\Delta G''_{\infty}\) are the distances between consecutive approximants, and \(n_{\text{max}}\) is the number of computed moments.

**TABLE I: Low order moments at \(T = \infty\).**

| \(k\) | \(\mu_k^\infty\) |
|------|-----------------|
| 0    | 4               |
| 2    | 64              |
| 4    | 4096            |
| 6    | 544768          |
| 8    | \(1.20906 \times 10^8\) |
| 10   | \(3.96113 \times 10^{10}\) |
| 12   | \(1.75571 \times 10^{13}\) |

These traces were computed numerically on a finite 16 site square lattice with periodic boundary conditions. At \(k \geq 8\) the calculation introduces finite size errors, due to loops of operators which circulate around the system. We eliminated most of the contributions of these loops by averaging over Aharonov-Bohm fluxes through two holes of the torus \([8]\).

The VHO calculation of \(G''\) follows Section \([V\text{B}\text{B}]\). Inverting Eq. \([66]\), for the \(n_{\text{max}}\) lowest moments, we obtain

\[
\mu_k^\infty = \frac{2}{2^N} \text{Tr} \left( J_x k^k J_x \right), \quad k = 0, 2, \ldots 12.
\]
FIG. 3: The recurrents of $G''_\infty(\omega)$ at infinite temperature, and their extrapolation to large index $n$.

$\tilde{G}''_{n_{\text{max}}}$. We obtain the variational frequency

$$\Omega_v = 7.48 J,$$

which minimizes the distance between the two highest order approximants,

$$|\Delta G''|^2(\Omega_v) = \frac{\int_{-\infty}^{\infty} d\omega |\tilde{G}''_{n_{\text{max}}} - \tilde{G}''_{n_{\text{max}}-2}|^2}{\int_{-\infty}^{\infty} d\omega |\tilde{G}''_{n_{\text{max}}-2}|^2}.$$  

We plot $\tilde{G}''$ for $n_{\text{max}} = 12$ as the solid (blue online) curve in Fig. 2. In the inset of Fig. 2 we plot the convergence as a function of $n_{\text{max}}$. The rapid decay of $\Delta G''$ indicates convergence of the VHO expansion at $T = \infty$.

The second calculation extrapolates the recurrents, as shown in Fig. 3. We compute $|\Delta G_n|^2$, $n = 0, 1, \ldots 6$, from the known moments, using (10). The higher order recurrents exhibit an approximate linear increase with $n - 1$ which is consistent with a Gaussian decay of $G''$ at high frequencies as given by the continued fraction example of Eq. (47). We continue the linear slope (see dashed line in Fig. 3) by the extrapolation,

$$|\Delta_{n>6}|^2 \rightarrow z_{\text{eff}}(4J)^2(n-1), \quad z_{\text{eff}} = 3.59,$$

where $z_{\text{eff}}$ is the “effective coordination number”. The approximate function $\tilde{G}''_{n_{\text{str}}}$ is obtained using the continued fraction representation (43), and plotted as the solid (red online) curve in Fig. 2. We notice that the two approaches yield very similar curves.

Last, we compute the infinite temperature current fluctuations function by exact diagonalization (ED) in the Lehmann representation,

$$\tilde{G}_{cd}(\omega) = \frac{2\pi}{2} \sum_{n,m} |\langle n|J_x|m\rangle|^2 \delta(\omega + E_n - E_m).$$

FIG. 4: Dynamical conductivity in the high temperature resistive phase. The function can be fit by Eq. (73).

FIG. 5: The temperature dependent kinetic energy $\langle -K_x \rangle$ (solid, red online), calculated by Refs. [5, 6, 43] and the sum rule for the high temperature expansion of the dynamical conductivity of Fig. 4 (solid, blue online). The vertical dashed line marks the superconducting phase below $T_{BKT}$.

$|\langle n|\rangle$, $E_n$ are the eigenstates and eigenspectrum of $H$ on a $4 \times 4$ lattice with periodic boundary conditions. We expect the finite size effects to be small at infinite temperatures, where the correlation length is much shorter than the lattice size. Indeed, the agreement between the three approaches shown in Fig. 2 supports this expectation.
A. Finite temperature corrections and f-sum rule

Order $\beta^2$ corrections to $G''$ are obtained by a high temperature expansion of the moments $\mu_k^{(2)}$ as defined in Eq. (64). Inverting Eq. (60), the coefficients $D_n^{(2)}$ were computed up to order $n_{\text{max}} = 12$, and inserted into Eq. (65). The resulting temperature dependent dynamical integrated conductivity is plotted in Fig. 4, for several temperatures. The truncation errors in the VHO expansion, and the temperature expansion of the moments $\mu_k^{(2)}$ are relatively small, as the sum rule shows in Eq. (73).

$$\int_{-\infty}^{\infty} d\omega \frac{\tanh(\beta \omega/2)}{\omega} G''(\beta,\omega) = (-q^2 K_x)_{\beta}. \quad (74)$$

The values of $\langle K_x \rangle$ were computed by high temperature series expansion up to 11th order in Ref. [43], and at low temperatures by quantum Monte-Carlo simulations [54]. In Fig. 3 we plot the f-sum rule, Eq. (74), with $G''$ calculated to order $\beta^2$, and the temperature dependent kinetic energy. The corrections grow larger than 10% at $T < 2T_{BKT}$.

We note, however, that numerical satisfaction of Eq. (74) does not, in general, ensure the accuracy of the DC conductivity $\sigma(0)$. In fact, there is an expectation of a cusp at zero frequency arising from high order non-linear coupling between the current and other long lived diffusive modes [44]. The magnitude of this feature is expected to be weak because of particle hole symmetry. From our calculations we can conclude that it is below the resolution provided by the exact diagonalization and the 12th order moment expansion [45].

VI. DC RESISTIVITY: HIGH TEMPERATURE

Based on the calculations of $\sigma(T,\omega)$ of the previous section, the high temperature expansion of the DC resistivity, shown in Fig. 1 is given to order $1/T$ as

$$R(T) = 0.23 R_Q \frac{T}{J} \left(1 - 2.9 (J/T)^2 + \mathcal{O}((J/T)^4)\right). \quad (75)$$

where $R_Q = h/q^2$ is the boson quantum of resistance. In the regime of $T \geq 2T_{BKT}$, the negative corrections of order $T^{-3}$ are relatively small, as the sum rule shows in Eq. (73). As $T_{BKT}$ is approached from above the resistivity drops rapidly. The critical regime was described by Halperin and Nelson [18, 46] (HN) who considered the contribution of unbound vortices to the charge transport coefficients. Using vortex-charge duality, and Einstein’s relation for vortex conductivity, HN derived the critical resistivity as

$$R_{HN} = R_Q \frac{h n_v D_v}{T}, \quad (76)$$

where $D_v$ is vortex diffusion constant, and $n_v$ is the density of free vortices. Estimating that $n_v = \xi(T)^{-2}$ using [12], the critical resistivity is expected to be suppressed toward $T_{BKT}$ as

$$R_{HN}(T) = R_Q \frac{h D_v n_0}{T} A^{-2} \exp \left(-2B \left(\frac{T_{BKT}}{T - T_{BKT}}\right)^{1/2}\right), \quad (77)$$

where $n_0$ is a microscopic density of order $a^{-2}$. The precise diffusion constant for the HCB model is not known, and can be expressed as

$$D_v n_0 = \kappa J/h, \quad (78)$$

where $\kappa$ is an undetermined dimensionless constant. We can estimate the numerical value of $\kappa$ by requiring that $R_{HN}(T)$ matches Eq. (76) at $T \geq 2T_{BKT}$ where the f-sum rule is satisfied to a higher accuracy. We find that matching occurs in the range

$$0.38 < \kappa < 0.51. \quad (79)$$

We can interpret the vortex diffusion constant $D_v = l^2/\tau$, as arising from a scattering time of order $\tau = h/J$ (the inter-site vortex hopping time, according to calculations of Ref. [8] and a short mean free path $l \approx a$. Eqs. (78,79) are consistent with the value $D_v \approx h/m$ which was posited [40] for helium films.

VII. DYNAMICAL CONDUCTIVITY: ZERO TEMPERATURE

At $T = 0$ the conductivity is given by

$$\sigma_0(\omega) = q^2 \pi \rho_s \delta(\omega) + \frac{G''(\omega)}{\omega}. \quad (80)$$

We calculate the current fluctuations $G''$ in two stages. First, we appeal to the relativistic Gross Pitaevskii field theory [48], and obtain the long frequency gap and threshold form of the function. The mass parameter $m$, and the high frequency Gaussian scale $\Omega_0$ are variational fitting parameters. Second, we compute the lowest 12 moments by exact diagonalization of a 20 site cluster. These determine the lowest recurrent parameters $|\Delta n|^2$ which are fit to the variational form.

A. Field theoretical calculation

The relativistic Gross-Pitaevskii (RGP) model describes the long wavelength theory of quantum rotators in the absence of a linear time derivative term [48]. We use it
are described by decoupled phase and magnitude modes, the bubble diagram [49, 50] depicted in Fig. 6, therefore its paramagnetic part, expanded to the same order as Eq. (82), is given by

\[ J_x \to \nabla \to \nabla \]

We introduce a vector potential by shifting \( \nabla \to \nabla + iqA \). The current operator is obtained from \( J_x = \delta S/\delta A_x \), therefore its paramagnetic part, expanded to the same order as Eq. (82), is given by

\[ J_x(x) = q \rho_s (\nabla \phi + 2 \eta \nabla \phi) . \]  

(83)

The lowest order current fluctuations function is given by the bubble diagram [44, 50] depicted in Fig. 6

\[ G_{RGP}(\omega_n) \approx \frac{4q^2}{\beta} \sum_{\nu_n} \frac{d^2k}{(2\pi)^2} \frac{\nu_n}{(c_s^2 \nu_n + k^2)} \]  

(84)

where \( c_s = \sqrt{\rho_s/\Delta^2} \) is the speed of sound. We compute the integral by performing the Matsubara sum, and take \( q \to 0 \) and \( \beta \to \infty \), which yields

\[ G''_{RGP}(\omega) \approx \frac{q^2}{4} \left( \frac{\omega^2 - m^2}{2\omega} \right)^2 \frac{1}{|\omega|} \Theta(|\omega| - m) . \]  

(85)

The dynamical conductivity therefore exhibits finite frequency weight above the mass gap \( m \). We cannot rule out that higher order interactions may produce subgap spectral weight, since magnitude excitations can decay into phason pairs [27].

In addition, the RGP theory Eq. (81) only describes long wave length fluctuations. Therefore, lattice scale and high energy cut-off effects are important for the high frequency tails of \( G'' \). These will be computed directly from the zero temperature Kubo formula.

**B. Variational Fit of Recurrents**

The moments of \( G''_{T=0} \), Eq. (83), are equal to the ground state expectation values,

\[ \mu_k = \langle 0 | \{ J_x, e^{k \cdot J_x} \} | 0 \rangle . \]  

(86)

We compute a set of \( \mu_k, k \leq 12 \) by exact diagonalizations on a \( 5 \times 4 \) lattice. Using Eq. (10) we determine the set of recurrences \( |\Delta_n|^2, n = 1, 2, \ldots 6 \), which are depicted in Fig. 7. Finite size effects are not expected to be large for small \( n \), where \( \Delta_n \) depend mostly on short range correlations. We notice a striking difference between even and odd recurrences, which seem to follow two linearly increasing slopes. This even-odd effect of the recurrences, is an indicator (not a proof) of a gap-like structure [51].
Motivated by the field theoretical calculation, we use the trial function,

$$\tilde{G}_{TR}''(\omega) \propto G''_{RGP}(\omega) \exp(-\omega^2/\Omega_0^2),$$  

where $m$ - the mass gap, and $\Omega_0$ - the high frequency roll off, are variational parameters. We use $\Omega_0$ to determine the trial recurrents. The variational parameters are then determined by a least squares fit between the exact and trial recurrents. Thus we obtain,

$$\Omega_0 \rightarrow 8.6 J,$$
$$m \rightarrow 10.9 J.$$  

The mass gap appears to be similar to the high frequency Gaussian fall-off of the conductivity at high temperatures, as given by the linear slope of the recurrents, Eq. (71).

The resulting dynamical conductivity is depicted in Fig. 8. The figure also includes, for comparison, the results of exact diagonalization on a 16 site cluster, as given, for $\omega > 0$ by

$$\tilde{G}_{ed}''(\omega) = \pi \sum_m |\langle 0|J_2|m\rangle|^2 \delta(\omega + E_0 - E_m).$$  

The oscillations in $\tilde{G}_{ed}''$ are artifacts of finite size gaps in the spectrum. We see that the two curves agree on the position of the central peak and the total spectral weight.

A test of these calculations is provided by the zero temperature $f$-sum rule is,

$$\frac{\hbar}{q^2} \int_{0^+} d\omega \frac{\tilde{G}_{TR}''(\omega)}{\omega} = \langle -K_x \rangle - \rho_s, \quad (90)$$

The left hand side for the varational and exact diagonalization results yields

$$\frac{\hbar}{q^2} \int_{0^+} d\omega \frac{\tilde{G}_{TR}''(\omega)}{\omega} = \left\{ \begin{array}{ll}
0.0148 J & \text{Variational} \\
0.0164 J & \text{ED}.
\end{array} \right. \quad (91)$$

The sum rules are comparable in magnitude to values obtained by QMC [4] for the zero temperature kinetic energy and superfluid stiffness,

$$\langle -K_x \rangle = 1.09765(4) J,$$
$$\rho_s = 1.078(1) J,$$
$$\tilde{\alpha} \sim \langle -K_x \rangle - \rho_s = 0.019(1) J. \quad (92)$$

We notice the very small spectral weight (2%) of the high frequency peak at zero temperature, relative to the condensate weight. This weight is due to the quantum fluctuations of the ground state, and to the non conservation of the current operator in two dimensions. This weight can be ascribed to the magnitude oscillations mode.

**Note:** An important question is whether the mass gap survives corrections to Eq. (52). Within our variational approach, we tried to answer this question by allowing sub-gap spectral weight parameterized by a power-law tail such as

$$\tilde{G}_{TR}''(\omega) = \left( \frac{|\omega|/m} {1 + (|\omega|/m)^\alpha} \right) \exp(-\omega^2/\Omega_0^2). \quad (93)$$

The least squares fit of the recurrents has found that $\alpha$ tends to increase indefinitely. This is consistent, (although not being a proof) with having a true gap at $\omega = m$.

**VIII. DISCUSSION**

**A. Bad Metallicity**

HCB at half filling exhibits "bad metal" characteristics, as demonstrated in Eq. (2) and in Fig. 4.

In contrast to conventional metals and bosonic gases at high temperatures [20, 21, 22], the resistivity of HCB rises approximately linearly, without a sign of saturation at $R \approx \hbar/q^2$. Such behavior signals the breakdown of Boltzmann equation, since the mean free path becomes shorter than interparticle distance [19].

A related quantity is the width of the low frequency conductivity peak, which in metals is called the "Drude peak". Here, Eq. (2) shows that the low frequency temperature dependent peak in $\sigma(\omega)$ is governed mostly by the fluctuations-dissipation factor [52]. If one fits the
width by Eq. (1) in the regime $T_{BKT} < T << \tilde{\Omega}$ obtains,

$$\left(\frac{1}{\tau}\right)^{\text{eff}} = 2T.$$  

We note however, that for our hamiltonian, $T_B$ is not much smaller than $\tilde{\Omega}$. Therefore, separating the two frequency scales can be made larger by additional interactions.

The HCB model relates the asymptotic resistivity slope of the two scales can be made larger by additional interactions. We note however, that above the pseudogap temperature, pair bosons survive above $T_c$. We can operationally define a nominal critical normal state resistivity $R_c$ by $R_c = \frac{dR^\infty}{dT} T_c$. Thus, a HCB version of "Homes law" is obtained:

$$\rho_s(0) = 0.245 \frac{R_Q}{R_c} T_c.$$  

**B. Cuprate Conductivity**

A large linear in $T$ resistivity $\rho_{ab}$ and optical relaxation rate $\tau$ above the in-plane London penetration depth, $\lambda_{bc}$, are widely observed in clean samples, especially near optimal doping. It has been shown that the linear resistivity is not consistent with a proximity to a quantum critical point. It is plausible therefore, that the "bad metal" characteristics of the normal phase of cuprates may be described by lattice bosons in their resistive state. Support to this viewpoint is given by Uemura's empirical scaling law $T_c \propto \rho_{ab}^0(T = 0)$, and the observation of a superfluid density jump in ultrathin underdoped cuprate films. These are consistent with the behavior of a bosonic superfluid, captured by an effective XY model. In undoped cuprates, additional evidence exists that the hole pair bosons survive above $T_c$, up to the pseudogap temperature scale $T^* > T_c$. Thus there have been several theoretical approaches to the superconducting properties of cuprates based on lattice bosons of charge $q = 2e$.

We note however, that above the pseudogap temperature and frequency scale $T^*$, hole pairs completely disintegrate. Thus, above $T^*$, the HCB model cannot be used to obtain the temperature and frequency dependence correctly.

Homes et al. have pointed out, that the superfluid stiffness is generally proportional, with a seemingly universal constant, to the product of "critical conductivity" $\sigma_c^{3D}$, times $T_c$. The "critical conductivity" $\sigma_c$ was experimentally defined by extrapolating $\sigma(\omega, T_c)$ to zero frequency. This empirically universal scaling was interpreted as a sign of "dirty superconductors", where $T_c$ is determined by the disorder driven scattering rate.

Here we promote a different viewpoint, which maintains that Homes law can also be obtained in a disorder free model. The necessary ingredients are strong scattering effects, as described by hard core bosons above $T_{BKT}$. To relate the experimental data of cuprates to the HCB model, we first translate the 3D critical conductivity to a two dimensional critical conductivity:

$$\sigma_c = a_c \sigma_c^{3D},$$

where $a_c \approx 1.5nm$ is the interplane distance. The zero temperature boson superfluid stiffness, can be deduced from the in-plane London penetration depth,

$$\rho_s = a_c \rho_s^{3D} = \frac{\hbar^2 c^2}{16 \pi e^2} \frac{a_c}{a_{ab}},$$

where $c$ is the speed of light, or the optical plasma fre-
quency $\omega_{ps}$:

$$\rho_s = \frac{\hbar^2}{16 \pi e^2} a_s \omega_p^2. \tag{100}$$

In Fig. 9 we plot the data reported in Ref. 24, 63 by translating it into the two dimensional quantities. We find, quite remarkably, that the slope of Eq. (97) lies quite close to the experimental data. We emphasize that the proportionality constant in Eq. (97) is not expected to be universal. A priori, one might expect it to vary with additional interactions and the filling. If the agreement with cuprates’ data is not fortuitous, it would imply that this constant might not be sensitive to moderate variations of $H$.

At very low temperatures, we find that there is a signature of the quantum fluctuations of the order parameter in the optical conductivity at high frequencies. These are magnitude fluctuations characterized by a "Higgs mass" gap at frequency

$$m \approx 10 \rho_s. \tag{101}$$

It is interesting to mention that such a massive magnitude mode, given by the relativistic Gross-Pitaevskii action, appears in the superfluid phase of the strongly interacting Bose Hubbard model at integer filling. It has been associated with the "oscillating superfluidity" experiments of cold atoms in an optical lattice 25, 26, 27. While a clear signature of such a peak in cuprates has not been identified, we speculate that it might be related to the ubiquitous mid infrared peak which has been detected in several compounds at low temperatures 28. We caution however, that at these high frequencies, additional fermionic excitations become increasingly important. In summary, we conclude that some of the "normal state" phenomenology of cuprates, and perhaps other unconventional superconductors, may be described by lattice bosons. However, we emphasize that Eq. (5) oversimplifies these systems, by omitting potentially important ingredients: fermionic excitations, long range Coulomb interactions, interlayer coupling, disorder and inhomogeneities, and of course the HCB conductivity away from half filling. Clearly, these effects need to be accounted for in future work.

**IX. ACKNOWLEDGMENTS**

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**APPENDIX A: ORTHOGONAL POLYNOMIALS AND RECURRENTS**

Iterative use of the matrix inversion formula for $G_n(z)$

$$G_n(z) = \mu_0 \left( z - L \right)^{-1}_{n,0}. \tag{A1}$$

with $L$ defined as in (37) yields, for $z \rightarrow \omega + i\epsilon$

$$C_n^\mu(\omega) = P_n(\omega)G_0(\omega), \tag{A2}$$

where $P_n(\omega)$ is a polynomial that depends on the $|\Delta_m|^2$’s for $m \leq n$, and is given by

$$P_n(\omega) = \prod_{k=1}^{n} |\Delta_k| \det(\omega - L)_{n-1}. \tag{A3}$$

In the above, $(z - L)_{n-1}$ is the upper left $(n-1) \times (n-1)$ sub-matrix of $(z - L)$. The determinants $\det(z - L)_{n-1}$ obey the recursion relation

$$\det(z - L)_{n+1} = z \det(z - L)_n + |\Delta_n|^2 \det(z - L)_{n-1}. \tag{A4}$$

Equation (A4) is a recursion relation for orthogonal polynomials. The polynomials $P_n(\omega)$ defined in Eq. (A3) are therefore orthogonal polynomials under the scalar product defined by

$$\int_{-\infty}^{\infty} d\omega G_0(\omega)P_n(\omega)P_m(\omega) = \delta_{nm}. \tag{A5}$$

The complexity of computation of $C_n(\beta), P_n(\omega)$ depends on obtaining all the low $\Delta_m, m \leq n$. Therefore, the expansion (51) would be useful if it could be truncated at finite $n$, provided that the coefficients $C_n(\beta)$ decay rapidly enough with $n$.

**APPENDIX B: LINEAR RECURRENTS AND THE GAUSSIAN SPECTRAL DENSITY**

Let us consider a sequence of recurrences $|\Delta_n|^2$ given by

$$|\Delta_n|^2 = \frac{1}{2} n \Omega^2 \quad n = 1, 2, \ldots \tag{B1}$$

The continued fraction representation can be solved using the following map. Consider the dimensionless position operator $\hat{x}$ represented by raising and lowering operators of the one dimensional harmonic oscillator

$$x = \frac{1}{\sqrt{2}} (a^\dagger + a). \tag{B2}$$

Since we have $\langle n + 1 | \hat{x} | n \rangle = \sqrt{(n + 1)/2}$, the function $G_n(z)$ of Eq. (50) is equivalent to

$$G_n(z) = \mu_0 \langle n | \frac{1}{z - \Omega \hat{x}} | 0 \rangle. \tag{B3}$$
Using the $x$ representation for the ground state of the harmonic oscillator we have
\[ G_0(\omega) / \mu_0 = -\int_{-\infty}^{\infty} dx \frac{1}{\omega + i \Omega x} \frac{e^{-x^2}}{\sqrt{\pi}} = i \sqrt{\frac{\pi}{\Omega}} e^{-\omega^2/\Omega^2} \left( 1 + \frac{2i}{\pi} \int_0^{\omega^2/\Omega^2} e^{t^2} dt \right). \]
\[ (B4) \]
Likewise, for higher values of $n$ we have
\[ \text{Im} G_n(\omega) / \mu_0 = -\text{Im} \int_{-\infty}^{\infty} dx \frac{N_n H_n(x)}{\omega + i \Omega x} \frac{e^{-x^2}}{\sqrt{\pi}} = \frac{\sqrt{\pi}}{2} e^{-\omega^2/\Omega^2} N_n H_n(\omega/\Omega), \]
\[ (B5) \]
where $N_n = 1/\sqrt{2^n n!}$. 

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