Abstract

In this paper we initiate an approach that deals with the problem of calculating average properties of messages traveling on networks, by employing concepts and methods that are used for the study of the many-body problem in the field of physics. We set up a framework that simplifies enormously the problem and, through a concrete example, we show how it can be applied to a broad class of networks and protocols.

1 Introduction

The technological advances of the last two decades have made possible the construction of large networks both on a global scale (e.g. Internet) and on a local scale (e.g. Local Area Networks or LANs, massively parallel machines). As the number of the constituent nodes and the number of communication messages increase, it is interesting – and probably useful – to ponder what the effective properties of a given network will be, under a certain ‘load’ of messages. For, since the messages can originate from any node, at any time, and be sent to any other node, it is reasonable to expect that one message during its travel from its origin to its destination will be influenced by the existence of the other messages traveling on the same network. Hence, although a message needs a given amount of time to travel between two arbitrary nodes on an empty network, it will need a longer time to travel the same distance on a loaded network. In this paper, we shall present a quite general approach that deals with issues related to this problem. The specific model, presented herein, is an abstraction of real communication networks (e.g. optical networks) and real problems on massively parallel machines (e.g. the hot-potato or deflection routing). This generality of the model naturally allows several extensions of itself that will be considered at a later time.

To begin with, consider a number \(N\) of nodes that are distributed on a D-dimensional surface, that are connected to each other by a number \(N_c\) of links (physical or virtual), and that continuously communicate with each other through these links. The nodes are enumerated by an integer number \(i\), and their location in physical space is given by the coordinates of each node \(\mathbf{r}^i = (x_1^i, x_2^i, \ldots, x_D^i)\). Thus, the links can be considered as D-dimensional vectors \(\mathbf{V}_{ij}\) whose components are given by the usual vector formulas, if the
coordinates of the two connected nodes $i$ and $j$ are known. An immediate consequence of
this definition is that the vector-matrix $\vec{V}_{ij}$ is antisymmetric (i.e. $\vec{V}_{ij} = -\vec{V}_{ji}$). The speed
of communication $c$ (along any of the $i = 1, 2, \ldots, D$ dimensions), among the nodes, is a
characteristic of the network and is considered to be known. Each link can carry only one
message at a time, and each node can send or receive messages but it is not allowed to
store-and-forward a message. Therefore, the number ($M_t$) of messages that exist at any
time on this network cannot, obviously, exceed the number of the links. We shall call the
ratio of the number of messages to the number of links ($M_t/N_c$) the *coefficient of saturation* or
simply the *saturation* ($s$) of our network. Moreover, each node can send only a number ($M'$) of messages that is not larger than the number ($N_c'$) of its connections with the other
nodes. At any instant of time, the $M_t$ messages are distributed randomly among the $N$
nodes, and each one of them is sent on a different distance which is also considered to be
random. Each node $i$ is aware of the following three things:

1. Its global location ($r^i$),
2. its connections ($\vec{V}_{ij}$), and
3. the algorithm that describes what should be done when a message is received. Here-
   after, we will refer to this latter algorithm as the *protocol* of our network.

Let a message be sent from the node-origin $i$ to the node-destination $j$. If the nodes are
not connected directly, then the message will have to reach its destination via a number
of other nodes that will enable this connection. As far as the operation of this network is
concerned, we will consider that each message carries the following two things:

1. The coordinates of its final node-destination ($\vec{x}^j$), and
2. its priority ($p$).

This information will be read at every node ($k$) that intervenes during its travel from
the initial node-origin to the final node-destination. Hence, with this information and the
protocol of the network each node is able to decide which of its connections ($\vec{V}_{kl}$) will be
used for each message that arrives, and each message will eventually arrive at its final node-
destination (except, perhaps, for some pathological situations that are not of statistical
significance).

It should now be clear that this network is a good model for variants of hot-potato
routing that can be used by parallel machines such as the HEP multiprocessor (Smith, 1981),
massively-parallel machines such as the Caltech Mosaic C (Seitz, 1992), and by high-speed
communication networks (Maxemchuk, 1989). It should also be appropriate for optical
networks (Acampora & Shah, 1991; Szymanski, 1990; Zhang & Acampora, 1991), where
avoiding storage of a message is highly desirable since optical transmission and switching
rates are much faster than the inter-conversion between the optical and electronic forms of
a message. Despite the practical importance of hot-potato routing and of optical networks,
the problem of understanding what the general properties of these systems will be and
the issue of optimizing them is largely open. The abstract construction, that we described
above, may help us to resolve some of the problems that arise in practice, because it refers to continuous or dynamic routing; in contrast to batch routing for which much is known (see, for example, Borodin & Hopcroft, 1985; Kaklamanis et al., 1993; Borodin et al., 1995). Furthermore it contains locally the attribute of greediness, in the sense that the path that is followed is adaptive to the current load of the network and is not predetermined. For such a dynamic construction, we can define states of “equilibrium” where the number of transmitted messages equals, on the average, the number of received messages.

The new approach is essentially an application of ideas and methods that are found in the study of many-body problems. The many-body problem is not associated with a specific branch of physics. Various forms of it range from solid state to nuclear physics and from quantum field theory to the problem of turbulence. The basic idea is that there are a number of elementary ‘entities’ that they interact with each other and that this interaction is sufficiently strong to influence the dynamical evolution of the system. In our case, the elementary ‘entities’ are the messages that travel through the links of the network. However, the dynamics of the many-body systems in physics is usually known and it is given by the evolution equation (e.g. the Schrödinger equation), whereas in the many-message system the motion of each message is dictated by the protocol, which is not an explicit evolution equation. Thus, one should attribute probabilities to the free propagation and the scattering process that are based on the qualitative features of the network and the protocol. This is not hard for some simple cases (see section 3), whereas for more complicated cases we can always simulate the local dynamics of the protocol and deduce its scattering properties numerically. At any rate, once these probabilities have been found, the final result can be evaluated to an arbitrary order.

In section 2 we will introduce the idea of the propagator for the messages. The calculation of the latter gives us the probability of a message to travel a certain distance at a certain time. In section 3 we perform explicit calculations for a specific network topology and protocol. Finally, in section 4 we discuss our results and suggest what other concepts, used in the case of the physical many-body problem, could be useful in this context.

2 Messages on Networks as a Many-Body Problem

In this section we will first introduce the idea of the propagators for our model-network. We will consider only “equilibrium” states although our formulation does allow the treatment of cases “far from equilibrium.” We shall also content ourselves with the study of spatially homogeneous networks; that is, networks for which the node-origin of a message does not enter the calculation explicitly. Hence, we can always refer to messages originating at $r = 0$ and $t = 0$. Then we will introduce the decisive role of the protocol in the dynamics of the many-message problem. As we have already mentioned, if a message is going to interact with another message, it will do so at the nodes. The protocol describes completely this interaction, but somewhat differently than the common cases treated in the physical many-body problems. For in the case of, say, a many-electron system we know the rules of the interaction in the form of a differential equation whereas in the case of the many-message problem we do not have such knowledge.
2.1 Message propagators

Let \( P(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2) \) be the probability density that if a message is sent from the node at the point \( \mathbf{r}_1 \) and at time \( t_1 \), then it will be received at the point \( \mathbf{r}_2 \) and at the later time \( t_2 \). The saturation does not appear explicitly in this notation but we do consider \( P \) as a function of the saturation \( s \) (see below). In the sequel, we will consider those protocols that are compatible with the following scattering mechanism:

1. There is a probability \( P_s(i) \) that the message will be scattered by the node \( (i) \) and \( 1 - P_s(i) \) that it will take the same path as in the non-interacting case. The \( P_s(i) \) turns out to be a function of the saturation (in some simple cases this quantity completely specifies it; see the next section), of the coordinates, and possibly of time (in cases where the load is not uniform in space nor steady in time).

2. The probability distribution of message-paths after scattering at \((i)\) should be independent of each other, and the joint probability for the whole path should be the product of the probabilities for each part of the path. That is, \( P(\mathbf{r}_1 \to \mathbf{r}_i), (\mathbf{r}_i \to \mathbf{r}_2) = P_o(\mathbf{r}_i, \mathbf{r}_1, t_i - t_1) \) and \( P_o(\mathbf{r}_2, \mathbf{r}_i, t_2 - t_i) \).

Hence, the total probability \( P(\mathbf{r}_2, \mathbf{r}_1, t_2 - t_1) \) will be the sum of the probabilities for all the possible paths. However, unlike the analogous cases of its physical counterparts, in our model case the instant of time \( t_i \) is not independent of the node location; since all messages propagate with the same speed. This means that only a subset of the possible paths is allowed to contribute for a given distance and a given time separation. In particular, the probability that a message will travel from \( \mathbf{r}_i \) to \( \mathbf{r}_j \) is equal to zero unless

\[
t_j - t_i \geq (\Pi_N(\mathbf{r}_j - \mathbf{r}_i))/c .
\]

Furthermore, and for the same reason, the discreteness of the network implies the discreteness of the time variable. Thus, we finally get the following series for the total probability

\[
P(\mathbf{r}_2, \mathbf{r}_1; t_2, t_1) = P_o(\mathbf{r}_2, \mathbf{r}_1; t_2, t_1) + \sum_i \sum_i P_o(\mathbf{r}_i, \mathbf{r}_1; t_n, t_1) P_s(i) P_o(\mathbf{r}_2, \mathbf{r}_i; t_2, t_n)
\]
where the times \( t_n, t_m, \ldots, \in [t_2, t_1] \) refer to the time of the scattering and must satisfy the above mentioned distance-time inequality (3).

Note that the above paragraphs show that Feynman diagrams can be used to accomplish the infinite summations involved to arbitrary order. Nevertheless, we will not use this ‘language’, because it would require the introduction of more techniques and concepts than we really need for the purposes of the present paper. Instead we will simply invoke a simple notation, that is still compact, and write \( P_{ij}^{0} \) for \( P_{o}(r_j, r_i; t_j, t_i) \), \( P_{i} \) for \( P_{s}(i) \), and similarly for the other quantities. For instance, equation (3) will now read

\[
P_{21}^{21} = P_{21}^{21} + \sum_{i} P_{o}^{1i} P_{o}^{2i} + \sum_{i \neq j} P_{o}^{1j} P_{o}^{ij} P_{i}^{2i} + \cdots,
\]

where the summation over the time variable is implied by the corresponding node.

### 2.2 The Role of The Protocol

In the previous sections we have emphasized that if a message interacts with another message, it will do so at the nodes. The rules of this interaction are determined completely by the protocol. Thus we have here a case quite distinct from its natural counterpart, in the sense that we are able to specify completely the local dynamics of ‘collisions.’ This is a propitious feature particular to the many-message system and it rises the certainty that the effective properties of networks will be amenable not only to a calculation but also to an optimization (at least, in principle), by carefully designing an appropriate protocol or an appropriate network for a given protocol.

It turns out that, as in the case of a gas, a crucial idea is that the joint distribution for two messages can be written as the product of the two individual probabilities, i.e. it is assumed that there are no correlations among the messages. This is easily implemented by choosing randomly the first message to be sent, from a number of messages that arrived simultaneously at the specific node. The Internet is a good candidate for applying such a hypothesis, but other types of networks that have a significant saturation may also profit from incorporating such a principle in their protocol.

The hypothesis of random messages is not necessary though, since local ‘scattering’ experiments can be performed numerically in order to completely determine the nature of the collisions for an arbitrary protocol. In particular, this numerical simulation of the collision process should perform a study of an isolated node that is receiving instantaneously two (for a binary collision), three, four, etc. messages. Then the protocol will determine the probability that a message will be scattered, and consequently the calculation of the total probability can be made.
3 A Network Application

What we have presented as a general formulation, we shall now apply it to a specific case that is of practical importance. In particular, we will give here a network application of our model for which, under certain assumptions, we can calculate explicitly the propagator. To begin with, let $D = 2$ and consider an $N \times N$ orthogonal lattice with a uniform spacing along each direction. That is, each $x_i, y_i$ ($i = 1, 2, \ldots, N$) can take only values that are integer multiples of the grid spacing $\Delta$. The speed of transmission for all the messages and all directions is the same and equal to $c$. Moreover, we let all messages have the same priority $p$.

To complete our setting, let the protocol be given by the following rules:

(i) Check for messages.
   If only one message exists then
   go to step (ii)
   else
   choose randomly one of them and follow step (ii).

(ii) Form the vector that joins the local node with the final node-destination of the particular message.
   If this vector is the null vector then
   the message has arrived at its final destination.
   Else
   project it onto the local connection vector-matrix,
   choose the unoccupied link with the largest projection,
   and send the message.

On such a network, we consider the case where messages are continuously sent and received in such a manner that a steady state is approximately reached. What is the propagator for such a system? In other words, what is the probability that a message will reach its destination after a certain time has elapsed? What is the average time that a message needs to travel a given distance $R$, for a specific saturation level $(s)$? What is the uncertainty or variance for this quantity? Of course, the answer to the first question encompasses the answers to the other two questions, but our point is that the calculation of such practical quantities can be carried out in detail.

3.1 Explicit Calculation of the Message Propagator

The starting points are the equation (4) and the general characteristics of the network, as they have been described in the first part of this section. That is, in order to evaluate $P^{21}$, we will evaluate the free propagator $P_o$ and the scattering probability $P^i$ for this network, and then we will substitute them in equation (4).
We consider the form of the free propagator for this particular network to be given by

\[ P_o(r; t) = H(t^*) \sqrt{(1 - s)} \left(1 - \frac{r}{2N}\right)^s \exp[-(1 - s)\alpha t^*], \tag{5} \]

where \( t^* \) stands for \((t - \Pi_N r/c)\), \( r \) is the projection \((\Pi_N r)\), \( \alpha \) is a constant, and \( H \) denotes the step Heaviside function. The exact value of \( \alpha \) can be chosen at the end of the calculation to be such that normalizes the total probability. In this heuristic formula we take into account both the geometric features of the network and its load (given here by the saturation \( s \)). In equation (5), we assume that the free propagator depends nonlinearly on both the saturation and the size of the network. Although other choices can be appropriate as well, there is not a general method that evaluates the free propagator. For a specific network and a specific load, which nonetheless can be functions of space and time, the free propagator should be chosen so that it satisfies some general criteria. In our example, for instance, the probability peaks at the time that corresponds to an uninterrupted travel but then falls exponentially. Moreover, as \( s \) goes to zero the probability goes to the unit value, only for \( t = \Pi_N r/c \) (as it should), whereas \( P_o \) goes to zero as \( s \) goes to one.

The calculation of the scattering probability is based also on the general features of the network (loss of memory, locality, greediness). However, we have already remarked that the current approach allows a local numerical study of the scattering process for an arbitrary protocol, and if not all the conditions enforced in this example are met then the numerical study should be preferred. Once these calculations have been made, and the local ‘dynamics’ of the system is fully determined, we can obtain results regarding the global behaviour of our system.

The calculation of the scattering probability involves the calculation of the probability that two or more messages will be at the same node, at the same time (let us call this the event \( I \)), and the probability that more than one messages will ask for the same link (let us call this the event \( II \)). The random selection of a message (see the protocol above) turns out to be an advantage in the current theory. It is the local randomness of the network that allows us to consider certain events as independent, and thereby to calculate the probability of the combined events as the product of the probabilities of each event separately. Thus, the next step is to express explicitly the scattering probability in terms of the probability that a message will ask for a certain link, and the probability that a message exists on a link. It should be clear that, due to the geometric symmetry and the uniform load of the messages on the network, the former is equal to 0.25. Whereas, the latter probability is, by definition, equal to the saturation \((s)\) of the system. Let us denote by \( M_i \) \((i = 1, 2, 3, 4)\) the event that message \( i \) asks for a particular link, then the scattering probability \( P^i \) should be the product of \( Pr\{I\} \) and \( Pr\{II\} \), where

\[ Pr\{I\} = 1 - (1 - s)^4 - 4s(1 - s)^3 \tag{6} \]

and

\[ Pr\{II\} = Pr\{M_1\} (Pr\{M_2\} + Pr\{M_3\} + Pr\{M_4\} - Pr\{M_2\} Pr\{M_3\} Pr\{M_4\}) + Pr\{M_2\} Pr\{M_3\} Pr\{M_4\} \tag{7} \]
The first of the above equations states that the probability of having two or more messages is equal to unity minus the probability of having no message, minus the probability to have only one message. Since we can have up to four messages (for this particular network) and we know the probability that one message will occupy a link (this is equivalent to knowing the saturation) we use the binomial distribution and obtain the final result, i.e. equation (6). The second of these equations is just a number. We have written the expression explicitly, in order to reveal the underlying assumption of statistical independence among the events $M_i$ ($i = 1, 2, 3, 4$). However, $Pr\{M_1\} = Pr\{M_2\} = Pr\{M_3\} = Pr\{M_4\} = 0.25$ and consequently we obtain $Pr\{II\} = 0.14453125$. The final conclusion is that

$$P^i = 0.14453125 (1 - (1 - s)^4 - 4s(1 - s)^3), \quad (8)$$

where $s$ denotes the saturation of the system.

We are now ready to calculate explicitly to an arbitrary order the propagator as given by equation (3). We have

$$P^{21} = P^{21}_o + \sum_i P^{i1}_o P^i P^{2i}_o$$

$$+ \sum_{i \neq j} P^{j1}_o P^j P^{ij} P^i P^{2i}_o$$

$$+ \sum_{i \neq j \neq k} P^{k1}_o P^k P^{jk} P^j P^{ij} P^i P^{2i}_o$$

$$+ \cdots. \quad (9)$$

Nevertheless, equation (3) indicates that $P^i$ is independent of $i$ and therefore can be taken outside the summation symbols. Furthermore, we will assume that each free propagator in each sum is equal to $P_o(r_{21}/n; t)$, where $n$ equals the order of the summation. That is, if we consider the first sum over a single node then $n = 1$, if we consider the second sum over two nodes then $n = 2$, etc. This assumption allows us to take into account all the orders of the summation. Hence, the summed products can be factored out and give a single sum. In fact, it is not hard to show that the final result is

$$P^{21} = H(t_\ast) \exp(-\alpha(1 - s)t_\ast) \sum_{n=1}^{N^2} (1 - s)^{n/2} \left( \frac{P^i N}{n} \right)^{n-1} \left( 1 - \frac{r}{2nN} \right)^{n s}. \quad (10)$$

The above sum converges extremely rapidly for given values of $s$ and $N$. According to (14) as the saturation of the network increases the probability that a message will travel a certain distance at a certain time becomes a weak function of time. On the other hand, as the saturation becomes negligible the effect of multiple scattering vanishes.

4 Discussion

We have presented a new framework for the study of problems related to the effective properties of networks, when many messages are communicated at the same time through their links. Although we have given a result obtained somewhat heuristically for a simple problem, it should be clear that real-world problems, of the types mentioned in the introduction,
can be tackled by the same method. The summations in (4) need not be simplified and the calculation can be done numerically up to several orders. Numerical evaluation of these terms is to be preferred, since it allows greater flexibility and speed. It is our belief that analytical results (such as the one given in section 3) are important to reveal the qualitative behaviour of the system, however numerical investigations should be superior in accuracy; especially for moderate sized networks. Such numerical experiments are currently undertaken and will be presented elsewhere. These numerical experiments will imitate the local behaviour of the dynamics and obtain the probability of free propagation and the scattering probability.

Unlike its physical counterpart, the local dynamics of the many-message system is not universal. Messages interact with each other according to a specific protocol, which we are able to change at will. Thus each protocol should be studied separately for a specific network. The protocol need not even be the same for the whole network. In that case, a partition of the network may be necessary in order to handle correctly the interactions.

For networks in a state of equilibrium, such as the one of our example, other methods and concepts of statistical mechanics can also be of importance in describing their collective properties. The notion of entropy, energy, temperature, and other thermodynamic concepts can be applied in these cases. For example, it may turn out that certain networks have a critical “temperature” above which the performance of the network degrades considerably.

We believe that the ideas presented will be developed further and provide a systematic way of dealing with large networks. The ultimate goal of these interdisciplinary study is the optimization of existing networks by altering the protocol and the selection of the best network structure for a given class of protocols.

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References

[1] Acampora, A. S. & Shah, S. I. A. 1991 Multihop lightwave networks: a comparison of store-and-forward and hot-potato routing. In INFOCOM, 10–19. IEEE.

[2] Borodin, A. & Hopcroft, J. 1985 Routing, merging, and sorting on parallel models of computation. Journal of Computer and System Sciences 30, 130–145.

[3] Borodin, A., Rabani, Y. & Schieber, B. 1995 Deterministic many-to-many hot potato routing. Technical Report RC 20107, IBM Watson Research Report.

[4] Kaklamanis, C., Krizanc, D. & Rao, S. 1993 Hot-potato routing on processor arrays. In Proceedings of the 5th Symposium on Parallel Algorithms and Architectures, 273–282. ACM.
[5] Maxemchuk, N. F. 1989 Comparison of deflection and store-and-forward techniques in the manhattan street and shuffle exchange networks. In *INFOCOM*, 800–809. IEEE.

[6] Seitz, C. 1992 The Caltech Mosaic C: an experimental, fine-grain multicomputer. In *Proceedings of the 4th Symposium on Parallel Algorithms and Architectures*. Keynote speech. San Diego. ACM

[7] Smith, B. 1981 Architecture and applications of the HEP multiprocessor computer system. In *Proceedings of (SPIE) Real Time Signal Processing IV*, 241–248.

[8] Szymanski, T. 1990 An analysis of hot potato routing in a fiber optic packet switched hypercube. In *INFOCOM*, 918–926. IEEE.

[9] Wiegel, F. W. 1986 *Introduction to Path-Integral Methods in Physics and Polymer Science*. Philadelphia: World Scientific.

[10] Zhang, Z. & Acampora, A. S. 1991 Performance analysis of multihop lightwave networks with hot potato routing and distance age priorities. In *INFOCOM*, 1012–1021. IEEE.