POSITIVE SOLUTION FOR THE KIRCHHOFF-TYPE EQUATIONS INVOLVING GENERAL SUBCRITICAL GROWTH

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Abstract. In this paper, the existence of a positive solution for the Kirchhoff-type equations in $\mathbb{R}^N$ is proved by using cut-off and monotonicity tricks, which unify and sharply improve the results of Li et al. [Existence of a positive solution to Kirchhoff type problems without compactness conditions, J. Differential Equations 253 (2012) 2285–2294]. Our result cover the case where the nonlinearity satisfies asymptotically linear and superlinear at infinity.

1. Introduction. Considering the following Kirchhoff-type equation

$$
(a + \lambda \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \lambda b \int_{\mathbb{R}^N} u^2 \, dx) (-\Delta u + bu) = g(u), \quad x \in \mathbb{R}^N,
$$

where $N \geq 3$, $a, b$ are positive constants, $\lambda \geq 0$ is a parameter, and the nonlinearity $g$ verifies

$(g_1)$ $g \in C(\mathbb{R}^+, \mathbb{R}^+) \cap [0, \infty)$ and $\lim_{s \to 0^+} \frac{g(s)}{s} = 0$.

$(g_2)$ $\lim_{s \to +\infty} \frac{g(s)}{s^{2^*-1}} = 0$ with $2^* = \frac{2N}{N-2}$.

$(g_3)$ There exists $\zeta > 0$ such that $G(\zeta) := \int_0^\zeta g(t) \, dt > \frac{ab}{2} \zeta^2$.

In recent years, the Kirchhoff-type equations in the whole space $\mathbb{R}^N$ have been studied widely by using the variational methods and various existence results were

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obtained, see for example [1, 3, 4, 5, 8, 9, 10, 11, 13, 14] and the references therein. Especially, in [10], Li et al. investigated equation (1.1) under \((g_1)\) and the following assumptions

\[(g_4) \ |g(s)| \leq C(|s| + |s|^{p-1}) \text{ for all } s \in \mathbb{R}^+ \text{ and some } p \in (2, 2^*).\]

\[(g_5) \ \lim_{s \to +\infty} \frac{g(s)}{s} = +\infty.\]

They obtained the following theorem.

**Theorem A.** Assume that \(N \geq 3\), \(a, b\) are positive constants and \(\lambda \geq 0\) is a parameter. Suppose that the nonlinearity \(g\) satisfies \((g_1), (g_4)\) and \((g_5)\). Then there exists \(\lambda_* > 0\) such that for any \(\lambda \in [0, \lambda_*)\), equation (1.1) has a positive solution.

As well known, owing to the existence of the nonlocal term, usually it demands that the nonlinearity \(g\) satisfies A-R, monotonicity or supercubic growth conditions, which assures the boundedness of any (PS) or Cerami sequence. In Theorem A, the authors did not assume the above conditions. In the present paper, under the weaker assumptions, we consider equation (1.1). Our existence result reads as follows.

**Theorem 1.1.** Assume that \(N \geq 3\), \(a, b\) are positive constants and \(\lambda \geq 0\) is a parameter. Suppose that the nonlinearity \(g\) satisfies \((g_1) - (g_3)\). Then there exists \(\lambda_* > 0\) such that for any \(\lambda \in [0, \lambda_*)\), equation (1.1) has a positive solution.

**Remark 1.** Theorem 1.1 extends Theorem A. Obviously, \((g_1), (g_4)\) and \((g_5)\) are stronger than \((g_1) - (g_3)\). Indeed, by \((g_4)\), one has

\[
\frac{|g(s)|}{|s|^{2^*-1}} \leq C\left(\frac{|s| + |s|^{p-1}}{|s|^{2^*-1}}\right) = C\left(\frac{1}{|s|^{2^*-2}} + \frac{1}{|s|^{2^*-p}}\right) \to 0, \ \text{as} \ s \to +\infty.
\]

Thus \((g_2)\) holds. It follows from \((g_5)\) that there exists \(R > 0\) such that for any \(s > R\), one has

\[
g(s) > 2abs.
\]

Combining \(g(s) \geq 0\) for all \(s \geq 0\), for any \(s \geq \sqrt{2}R\), we have

\[
G(s) = \int_0^R g(t)dt + \int_s^R g(t)dt \geq \int_s^R g(t)dt > ab(s^2 - R^2) \geq \frac{ab}{2} s^2.
\]

Thus \((g_4)\) holds.

Suppose that \(g\) satisfies the following asymptotically linear assumption:

\[
(g_6) \ \lim_{s \to +\infty} \frac{g(s)}{s} = V \in (ab, +\infty).
\]

Then we can deduce that \((g_1)\) and \((g_6)\) are stronger than \((g_1) - (g_3)\). Indeed, by \((g_6)\), we have

\[
\lim_{s \to +\infty} \frac{g(s)}{s^{2^*-1}} = \lim_{s \to +\infty} \frac{g(s)}{s^{8s^{2^*-2}}} = 0.
\]

Thus \((g_2)\) holds. By \((g_6)\), there exists \(R > 0\) such that for any \(s > R\), one has

\[
g(s) > \frac{V + ab}{2} s.
\]
So for any $s \geq \sqrt{\frac{V + ab}{V - ab}} R$, we have

$$G(s) = \int_{0}^{R} g(t)dt + \int_{R}^{s} g(t)dt \geq \int_{R}^{s} g(t)dt > \frac{V + ab}{4}(s^2 - R^2) \geq \frac{ab}{2}s^2,$$

since $g(s) \geq 0$ for all $s \geq 0$. Hence $(g_3)$ holds and then we have the following corollary.

**Corollary 1.** Assume that $N \geq 3$, $a, b$ are positive constants and $\lambda \geq 0$ is a parameter. Suppose that the nonlinearity $g$ satisfies $(g_1)$ and $(g_6)$. Then there exists $\lambda_* > 0$ such that for any $\lambda \in [0, \lambda_*)$, equation (1.1) has a positive solution.

In [10], the authors claimed that it is not clear whether the result in Theorem A still holds for large $\lambda > 0$. In the present paper, we can give an answer. Suppose that the nonlinearity $g$ satisfies

$(g_7)$ $g \in C(\mathbb{R}, \mathbb{R})$ and $\lim_{s \to 0} \frac{g(s)}{s} = 0.$

$(g_8)$ $\limsup_{|s| \to +\infty} \frac{|g(s)|}{|s|^{3^* - 1}} < +\infty.$

Then we have the following the result of nonexistence.

**Theorem 1.2.** Assume that $N \geq 3$, $a, b$ are positive constants and $\lambda > 0$ is a parameter. Suppose that $(g_7)$ and $(g_8)$ hold. Then there exists $\Lambda > 0$ such that for any $\lambda > \Lambda$, equation (1.1) has no nonzero solution.

The paper is organized as follows. Section 2 contains some preliminaries. In Section 3 we give the proof of theorems.

2. Preliminaries. From now on, we will use the following notations.

- $H^1(\mathbb{R}^N)$ denotes the usual Hilbert space endowed with the norm
  $$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + bu^2)dx.$$

- $L^s(\mathbb{R}^N)$ is the usual Lebesgue space endowed with the norm
  $$|u|_s = \left(\int_{\mathbb{R}^N} |u|^s dx\right)^{\frac{1}{s}}, \forall s \in [1, +\infty).$$

- $C, C_i$ denote various positive constants.

Because that we are looking for positive solution, we may assume that $g(s) = 0$ for all $s < 0$. The energy functional for equation (1.1) is defined by

$$I_{\lambda}(u) = \frac{a}{2} \|u\|^2 + \frac{\lambda}{4} \|u\|^4 - \int_{\mathbb{R}^N} G(u)dx.$$

By $(g_1)$ and $(g_2)$, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|G(s)| \leq \frac{\varepsilon}{2}|s|^2 + C_\varepsilon |s|^{2^*}, \text{ for all } s \in \mathbb{R}. \quad (2.1)$$

It is obvious that $I_{\lambda} \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ and has the derivative given by

$$\langle I'_{\lambda}(u), v \rangle = (a + \lambda \|u\|^2)(u, v) - \int_{\mathbb{R}^N} g(u)vdx, \forall u, v \in H^1(\mathbb{R}^N),$$
where
\[(u,v) = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + buv)dx.\]
Since equation (1.1) is autonomous, we can work in
\[H^1_r(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) : u \text{ is radially symmetric}\}.\]
We know that any critical point \(u \in H^1_r(\mathbb{R}^N)\) is, by the principle of symmetric
criticality of Palais (see Theorem 1.28 in [12]), also a critical point on
\[H^1(\mathbb{R}^N).\]
To obtain a bounded Palais-Smale sequence for the functional
\[I_\lambda,\]
following [7] and [10], we use a cut-off function \(\psi \in C^\infty(\mathbb{R}^+, \mathbb{R}^+)\) satisfying
\[
\begin{align*}
\psi(t) &= 1, & t &\in [0, 1] \\
\psi(t) &\in [0, 1], & t &\in [1, 2] \\
\psi(t) &= 0, & t &\in [2, \infty) \\
|\psi'(t)| &\leq 2
\end{align*}
\]
and study the following modified functional
\[I^\psi_\lambda : H^1_r(\mathbb{R}^N) \to \mathbb{R}\]
defined by
\[I^\psi_\lambda(u) = \frac{a}{2} \|u\|^2 + \frac{\lambda}{4} h_T(u)\|u\|^4 - \int_{\mathbb{R}^N} G(u)dx,
\]
where for every \(T > 0\),
\[h_T(u) = \psi\left(\frac{\|u\|^2}{T^2}\right).\]
The functional \(I^\psi_\lambda\) satisfies the geometrical assumptions of the Mountain-pass theorem but we are not able to obtain the boundedness of the Palais-Smale sequences. Hence we use an indirect approach which developed by Jeanjean [6]. We apply the
following Proposition 1 (See Theorem 1.1 and Lemma 2.3 in [6]).

**Proposition 1.** Let \(X\) be a Banach space equipped with a norm \(\|\cdot\|_X\) and let \(J \subset \mathbb{R}^+\) be an interval. We consider a family \(\{\Phi_\mu\}_{\mu \in J}\) of \(C^1\)-functionals on \(X\) of the form
\[\Phi_\mu(u) = A(u) - \mu B(u), \quad \forall \mu \in J,
\]
where \(B(u) \geq 0\) for all \(u \in X\) and such that either \(A(u) \to +\infty\) or \(B(u) \to +\infty\), as \(\|u\|_X \to +\infty\). We assume that there are two points \(v_1, v_2\) in \(X\) such that
\[\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = v_1, \gamma(1) = v_2\},
\]
there hold, \(\forall \mu \in J\)
\[c_\mu = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} \Phi_\mu(\gamma(t)) > \max\{\Phi_\mu(v_1), \Phi_\mu(v_2)\}.
\]
Then, for almost every \(\mu \in J\), there is a sequence for \(\{u_n\} \subset X\), such that
(i) \(\{u_n\}\) is bounded in \(X\),
(ii) \(\Phi_\mu(u_n) \to c_\mu\) and
(iii) \(\Phi'_\mu(u_n) \to 0\) in the dual \(X^*\) of \(X\).
Moreover, the map \(\mu \to c_\mu\) is continuous from the left.

The following Pohozaev equality is important for proving the boundedness of the Palais-Smale sequence. For the proof, please see [10].
Proposition 2. If \( u \) is a nonzero solution of the equation
\[
(a + \lambda h_T(u)\|u\|^2 + \frac{\lambda}{2T^2}\psi'(\frac{\|u\|^2}{T^2})\|u\|^4)(-\Delta u + bu) = \mu g(u), \quad x \in \mathbb{R}^N,
\]
the following Pohozaev equality
\[
\left( a + \lambda h_T(u)\|u\|^2 + \frac{\lambda}{2T^2}\psi'(\frac{\|u\|^2}{T^2})\|u\|^4 \right)\left( \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2dx + \frac{Nb}{2} \int_{\mathbb{R}^N} u^2dx \right)
= \mu N \int_{\mathbb{R}^N} G(u)dx
\]
holds.

In our case, \( X = H^1_s(\mathbb{R}^N) \), \( \Phi_\mu = I_{\lambda,\mu}^T \), \( v_1 = 0 \),
\[
A(u) = \frac{a}{2}\|u\|^2 + \frac{\lambda}{4} h_T(u)\|u\|^4 \quad \text{and} \quad B(u) = \int_{\mathbb{R}^N} G(u)dx.
\]
So the perturbed functional which we study is
\[
I_{\lambda,\mu}^T(u) = \frac{a}{2}\|u\|^2 + \frac{\lambda}{4} h_T(u)\|u\|^4 - \mu \int_{\mathbb{R}^N} G(u)dx
\]
and its derivative is given by
\[
\langle (I_{\lambda,\mu}^T)'(u), v \rangle = a(u,v) + \lambda h_T(u)\|u\|^2(u,v) + \frac{\lambda}{2T^2}\psi'(\frac{\|u\|^2}{T^2})\|u\|^4(u,v)
- \mu \int_{\mathbb{R}^N} g(u)vdx,
\]
for any \( u, v \in H^1_s(\mathbb{R}^N) \).

3. The proof of theorems. As a start, we give the proofs of some lemmas.

Lemma 3.1. Assume that \( N \geq 3 \), \( a, b \) are positive constants and \( \lambda \geq 0 \) is a parameter. Suppose that the nonlinearity \( g \) satisfies \((g_1) - (g_3)\). Then there exist \( \delta \in (0, 1) \) and \( v_2 \in H^1_s(\mathbb{R}^N) \) such that \( I_{\lambda,\mu}^T(v_2) < 0 \) for all \( \mu \in J := [\delta, 1] \).

Proof. Setting \( F(s) = G(s) - \frac{ab}{2}s^2 \), we claim that there exists \( z \in H^1_s(\mathbb{R}^N) \) such that
\[
\int_{\mathbb{R}^N} F(z)dx > 0.
\]
Borrowing the method in [2], it follows from \((g_3)\) that there exists \( \zeta > 0 \) such that \( F(\zeta) > 0 \). For \( R > 1 \), define
\[
w_R(x) = \begin{cases} 
\zeta, & \text{for } |x| \leq R, \\
\zeta(R + 1 - r), & \text{for } r = |x| \in [R, R + 1], \\
0, & \text{for } |x| \geq R + 1.
\end{cases}
\]
Thus \( w_R \in H^1_s(\mathbb{R}^N) \). Letting \( \text{meas}(\cdot) \) denote Lebesgue measure, it is easily checked that
\[
\int_{\mathbb{R}^N} F(w_R)dx \geq F(\zeta)\text{meas}(B_R) - \text{meas}(B_{R+1} - B_R)(\max_{s \in [0, \zeta]} |F(s)|).
\]
Hence there exist two positive constant \( C_1 \) and \( C_2 \) such that
\[
\int_{\mathbb{R}^N} F(w_R)dx \geq C_1 R^N - C_2 R^{N-1}.
\]
For $R > 1$ large enough, this shows that $\int_{R^N} F(w_R) dx > 0$. Letting $z = w_R$, we have proved the claim. Thus there exists $\delta \in (0, 1)$ such that

$$\int_{R^N} (\delta G(z) - \frac{ab}{2} z^2) dx > 0,$$

which implies that there exist $t_0 > 0$ large enough and $v_2(x) := z(\frac{x}{t_0})$ such that

$$\int_{R^N} (\delta G(v_2) - \frac{ab}{2} v_2^2) dx = \int_{R^N} (\delta G(z) - \frac{ab}{2} z^2) dx > \frac{a t_0^{N-2}}{2} \int_{R^N} |\nabla z|^2 dx = \frac{a}{2} \int_{R^N} |\nabla v_2|^2 dx$$

and

$$\|v_2\|^2 = t_0^{N-2} \int_{R^N} |\nabla z|^2 dx + t_0^N \int_{R^N} |\nabla z|^2 dx > 2T^2.$$ 

Then $h_T(v_2) = 0$. Hence for all $\mu \in J := [\delta, 1]$, we have

$$I_{\lambda, \mu}^T(v_2) = \frac{a}{2} \|v_2\|^2 - \mu \int_{R^N} G(v_2) dx$$

$$\leq \frac{a}{2} \|v_2\|^2 - \delta \int_{R^N} G(v_2) dx$$

$$= \frac{a}{2} \int_{R^N} |\nabla v_2|^2 dx - \int_{R^N} (\delta G(v_2) - \frac{ab}{2} v_2^2) dx$$

$$< 0,$$

which completes the proof. □

**Lemma 3.2.** Assume that $N \geq 3$, $a, b$ are positive constants and $\lambda \geq 0$ is a parameter. Suppose that the nonlinearity $g$ satisfies $(g_1) - (g_3)$. Then there exists $\bar{c} > 0$ such that $c_\mu \geq \bar{c}$ for all $\mu \in J$.

**Proof.** For all $\mu \in J$, it follows from (2.1) with $\varepsilon = \frac{ab}{4}$ that

$$I_{\lambda, \mu}^T(u) = \frac{a}{2} \|u\|^2 + \frac{\lambda}{4} b_T(u) \|u\|^4 - \mu \int_{R^N} G(u) dx$$

$$\geq \frac{a}{2} \|u\|^2 - \int_{R^N} G(u) dx \geq C_3 \|u\|^2 - C_4 \|u\|^2^*,$$

which deduces that there exists $\rho > 0$ small enough such that $I_{\lambda, \mu}^T(u) > 0$ for all $0 < \|u\| \leq \rho$ and all $\mu \in J$. In particular, there exists $\bar{c} > 0$ such that $I_{\lambda, \mu}^T(u) \geq \bar{c}$ for all $\|u\| = \rho$ and all $\mu \in J$. For any $\gamma \in \Gamma$, by Lemma 3.1, we know $I_{\lambda, \mu}^T(\gamma(1)) = I_{\lambda, \mu}^T(v_2) < 0$. Thus $\|\gamma(1)\| > \rho$. Combining $\gamma(0) = v_1 = 0$ with the continuity of $\gamma$, there exists $t_\gamma \in (0, 1)$ such that $\|\gamma(t_\gamma)\| = \rho$. Hence we obtain

$$\max_{t \in [0, 1]} I_{\lambda, \mu}^T(\gamma(t)) \geq I_{\lambda, \mu}^T(\gamma(t_\gamma)) \geq \bar{c},$$

which implies $c_\mu \geq \bar{c}$. We complete the proof. □

**Lemma 3.3.** Assume that $N \geq 3$, $a, b$ are positive constants and $\lambda \geq 0$ is a parameter. Suppose that the nonlinearity $g$ satisfies $(g_1) - (g_3)$. Then when $\lambda < \frac{a}{8T^2}$, for each bounded Palais-Smale sequence of $I_{\lambda, \mu}^T$ at the level $c_\mu$, there exists a convergent subsequence in $H^1_{\lambda}(R^N)$. 


Proof. Suppose that \( \{u_n\} \subset H^1_r(\mathbb{R}^N) \) is a bounded Palais-Smale sequence of \( I^T_{\lambda,\mu} \) at the level \( c_\mu \). Up to a subsequence, there exists \( u \in H^1_r(\mathbb{R}^N) \) such that \( u_n \to u \) in \( H^1_r(\mathbb{R}^N) \), \( u_n \to u \) in \( L^{\frac{2N}{N-2}}(\mathbb{R}^N) \) and \( u_n(x) \to u(x) \) a.e. in \( \mathbb{R}^N \). By (g1) and (g2), for any \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) such that
\[
|g(s)| \leq \varepsilon |s| + \varepsilon |s|^{2^* - 1} + C_\varepsilon |s|^\frac{N+1}{N-1},
\]
for all \( s \in \mathbb{R} \).

Thus combining the Hölder and Sobolev inequalities, we have
\[
\left| \int_{\mathbb{R}^N} g(u_n)(u_n - u) \, dx \right|
\leq \varepsilon \int_{\mathbb{R}^N} |u_n| |u_n - u| \, dx + \varepsilon \int_{\mathbb{R}^N} |u_n|^{2^* - 1} |u_n - u| \, dx + C_\varepsilon \int_{\mathbb{R}^N} |u_n|^\frac{N+1}{N-2} |u_n - u| \, dx
\leq \varepsilon |u_n|_2 |u_n - u|_2 + \varepsilon |u_n|^{2^* - 1} |u_n - u|_2 + C_\varepsilon |u_n|_\frac{N+1}{N-2} |u_n - u|_{\frac{N}{N-1}}
\leq C \varepsilon \|u_n\| |u_n - u| + C \varepsilon \|u_n\|^{2^* - 1} |u_n - u| + C \varepsilon \|u_n\|^{\frac{N+1}{N-2}} |u_n - u|_{\frac{N}{N-1}}
\leq C \varepsilon + o(1).
\]

Then we obtain
\[
o(1) = \langle (I^T_{\lambda,\mu})'(u_n), u_n - u \rangle
= a(u_n, u_n - u) + \lambda h_T(u_n) |u_n|^2 (u_n, u_n - u) + \frac{\lambda}{2T^2} \psi'\left(\frac{|u_n|^2}{T^2}\right) \|u_n\|^4(u_n, u_n - u)
- \mu \int_{\mathbb{R}^N} g(u_n)(u_n - u) \, dx
= [a + \lambda h_T(u_n)] |u_n|^2 + \frac{\lambda}{2T^2} \psi'\left(\frac{|u_n|^2}{T^2}\right) \|u_n\|^4(u_n, u_n - u) + o(1).
\]

Since \( \lambda < \frac{a}{\pi^2} \) and
\[
\frac{\lambda}{2T^2} \psi'\left(\frac{|u_n|^2}{T^2}\right) \|u_n\|^4 \leq 4 \lambda T^2 < \frac{a}{2},
\]
(3.2)

one has
\[
0 \leq \frac{a}{2} |(u_n, u_n - u)| \leq [a + \lambda h_T(u_n)] |u_n|^2 + \frac{\lambda}{2T^2} \psi'\left(\frac{|u_n|^2}{T^2}\right) \|u_n\|^4(u_n, u_n - u) = o(1).
\]

Hence \( u_n \to u \) in \( H^1_r(\mathbb{R}^N) \). We complete the proof. \( \square \)

Lemma 3.4. Assume that \( N \geq 3 \), \( a, b \) are positive constants and \( \lambda \geq 0 \) is a parameter. Suppose that the nonlinearity \( g \) satisfies (g1)–(g3). Then when \( \lambda < \frac{a}{\pi^2} \), for almost every \( \mu \in J \), there exists \( u^\mu \in H^1_r(\mathbb{R}^N) \) such that \( (I^T_{\lambda,\mu})'(u^\mu) = 0 \) and \( I^T_{\lambda,\mu}(u^\mu) = c_\mu \).

Proof. By Proposition 1, for almost every \( \mu \in J \), there exists a bounded Palais-Smale sequence \( \{u^\mu_n\} \subset H^1_r(\mathbb{R}^N) \) of \( I^T_{\lambda,\mu} \) at level \( c_\mu \). Combining Lemma 3.3, there exists \( u^\mu \in H^1_r(\mathbb{R}^N) \) such that \( u^\mu_n \to u^\mu \) in \( H^1_r(\mathbb{R}^N) \). Hence \( (I^T_{\lambda,\mu})'(u^\mu) = 0 \) and \( I^T_{\lambda,\mu}(u^\mu) = c_\mu \). We complete the proof. \( \square \)

According to Lemma 3.4, there exists sequences \( \{\mu_n\} \subset J \) and \( \{u_n\} \subset H^1_r(\mathbb{R}^N) \) such that
\[
\mu_n \to 1^-, \quad (I^T_{\lambda,\mu_n})'(u_n) = 0 \quad \text{and} \quad I^T_{\lambda,\mu_n}(u_n) = c_{\mu_n}.
\]
The following lemma shows that \( \|u_n\| \leq T \).

**Lemma 3.5.** Assume that \((I_{\lambda,\mu,n}^T)'(u_n) = 0\) and \(I_{\lambda,\mu,n}^T(u_n) = c_{\mu,n}\) with \( \lambda < \frac{a}{\sqrt{T^2}} \). Then there exist \( T > 0 \) and \( \lambda_* \in (0, \frac{a}{\sqrt{T^2}}) \) such that for any \( \lambda \in (0, \lambda_*), \) up to a subsequence, \( \|u_n\| \leq T \).

**Proof.** It follows from \((I_{\lambda,\mu,n}^T)'(u_n) = 0\) and Proposition 2 that

\[
\left( a + \lambda h_T(u_n)\|u_n\|^2 + \frac{\lambda}{2T^2} \psi'\left( \frac{\|u_n\|^2}{T^2} \right) \|u_n\|^4 \right)
\cdot \left( \frac{N - 2}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \frac{N}{2} \int_{\mathbb{R}^N} u_n^2 dx \right)
= \mu_n N \int_{\mathbb{R}^N} G(u_n) dx.
\]

Combining (3.2) and \(I_{\lambda,\mu,n}^T(u_n) = c_{\mu,n}\), we have

\[
a \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \leq \left( a + \lambda h_T(u_n)\|u_n\|^2 + \frac{\lambda}{2T^2} \psi'\left( \frac{\|u_n\|^2}{T^2} \right) \|u_n\|^4 \right) \int_{\mathbb{R}^N} |\nabla u_n|^2 dx
\]

\[= N c_{\mu,n} + \frac{\lambda N}{4} h_T(u_n)\|u_n\|^4 + \frac{\lambda N}{4T^2} \psi'\left( \frac{\|u_n\|^2}{T^2} \right) \|u_n\|^6. \tag{3.3}\]

According to \( v_2(x) := z(\frac{x}{\sqrt{T}}) \) in Lemma 3.1, define a function \( k : [0, 1] \to H^1_2(\mathbb{R}^N) \) satisfying \( k(0) = 0 \) and \( k(t) = v_2\left( \frac{x}{\sqrt{T}} \right) = z(\frac{x}{\sqrt{T}}) \) for \( t \in (0, 1) \). Then \( k \in \Gamma \) and then

\[
c_{\mu,n} \leq \max_{t \in [0,1]} I_{\lambda,\mu,n}^T(k(t))
\]

\[
\leq \max_{t \in [0,1]} \left\{ \frac{a}{2} \|k(t)\|^2 + \frac{\lambda}{4} h_T(k(t))\|k(t)\|^4 - \delta \int_{\mathbb{R}^N} G(k(t)) dx \right\}
\]

\[
\leq \max_{t \in [0,1]} \left\{ \frac{a(t_0)^N - 2}{2} \int_{\mathbb{R}^N} |\nabla z|^2 dx + \frac{ab(t_0)^N}{2} \int_{\mathbb{R}^N} z^2 dx - \delta(t_0)^N \int_{\mathbb{R}^N} G(z) dx \right\}
\]

\[+ \frac{\lambda}{4} \max_{t \in [0,1]} \left\{ \psi\left( \frac{\|k(t)\|^2}{T^2} \right)\|k(t)\|^4 \right\}
\]

\[
\leq \max_{t \geq 0} \left\{ \frac{atN - 2}{2} \int_{\mathbb{R}^N} |\nabla z|^2 dx + \frac{abtN}{2} \int_{\mathbb{R}^N} z^2 dx - \delta t^N \int_{\mathbb{R}^N} G(z) dx \right\}
\]

\[+ \frac{\lambda}{4} \max_{t \in [0,1]} \left\{ \psi\left( \frac{\|k(t)\|^2}{T^2} \right)\|k(t)\|^4 \right\}
\]

\[:= A_1 + \lambda A_2(T), \tag{3.4}\]

in which we use (3.1) and \( A_1 \) is independent of \( T \). According the definition of \( \psi \), we have

\[
\lambda A_2(T) \leq \lambda T^4, \tag{3.5}\]

\[
\frac{\lambda N}{4} h_T(u_n)\|u_n\|^4 \leq \lambda N T^4 \tag{3.6}\]

and

\[
\frac{\lambda N}{4T^2} \psi\left( \frac{\|u_n\|^2}{T^2} \right)\|u_n\|^6 \leq 4 \lambda NT^4. \tag{3.7}\]
Then from (3.3) – (3.7), one has
\[
\frac{a}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \leq NA_1 + 6\lambda NT^4. \tag{3.8}
\]
On the other hand, it follows from (2.1) with \( \varepsilon = \frac{ab}{4} \), (3.4), (3.5) (3.8) and\( I_{\lambda, \mu_n}^T(u_n) = c_{\mu_n} \) that
\[
\frac{a}{2} \|u_n\|^2 \leq \frac{a}{2} \|u_n\|^2 + \frac{\lambda}{4} h_T(u_n) \|u_n\|^4
\]
\[
= c_{\mu_n} + \mu_n \int_{\mathbb{R}^N} G(u_n) \, dx
\]
\[
\leq A_1 + \lambda T^4 + \frac{ab}{4} \int_{\mathbb{R}^N} u_n^2 \, dx + C \int_{\mathbb{R}^N} u_n^\ast_2 \, dx
\]
\[
\leq A_1 + \lambda T^4 + \frac{ab}{4} \int_{\mathbb{R}^N} u_n^2 \, dx + C \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \right)^{\frac{4}{3}}
\]
\[
\leq A_1 + \lambda T^4 + \frac{ab}{4} \int_{\mathbb{R}^N} u_n^2 \, dx + C(NA_1 + 6\lambda NT^4)^{\frac{4}{3}},
\]
which deduces that
\[
\frac{a}{4} \|u_n\|^2 \leq \frac{a}{2} \|u_n\|^2 + \frac{ab}{4} \int_{\mathbb{R}^N} u_n^2 \, dx
\]
\[
= \frac{a}{2} \|u_n\|^2 - \frac{ab}{4} \int_{\mathbb{R}^N} u_n^2 \, dx
\]
\[
\leq A_1 + \lambda T^4 + C(NA_1 + 6\lambda NT^4)^{\frac{4}{3}}. \tag{3.9}
\]
We suppose by contradiction that there exists no subsequence of \( \{u_n\} \) which is uniformly bounded by \( T \). Without any loss of generality, we can suppose that \( \|u_n\| > T \) for all \( n \). Then combining (3.9), one has
\[
T^2 < \|u_n\|^2 \leq \frac{4A_1}{a} + \frac{4\lambda T^4}{a} + C'(NA_1 + 6\lambda NT^4)^{\frac{4}{3}}. \tag{3.10}
\]
We can find \( T_0 > 0 \) such that
\[
T_0^2 \geq \frac{4A_1}{a} + C'(NA_1)^{\frac{4}{3}} + 1 \tag{3.11}
\]
and \( \lambda_* \in (0, \frac{a}{8T_0^4}) \) such that
\[
\frac{4A_1}{a} + C'(NA_1 + 6\lambda_* NT_0^4)^{\frac{4}{3}} \leq C'(NA_1)^{\frac{4}{3}} + 1. \tag{3.12}
\]
Hence for any \( \lambda \in [0, \lambda_*), (3.10) – (3.12) \) imply that
\[
\frac{4A_1}{a} + C'(NA_1)^{\frac{4}{3}} + 1 < \frac{4A_1}{a} + \frac{4\lambda_* T_0^4}{a} + C'(NA_1 + 6\lambda_* NT_0^4)^{\frac{4}{3}} \leq \frac{4A_1}{a} + C'(NA_1)^{\frac{4}{3}} + 1,
\]
which is a contradiction. We complete the proof. \( \square \)

**Proof of Theorem 1.1.** Let \( T \) and \( \lambda_* \) be as in Lemma 3.5 and fix \( \lambda \in [0, \lambda_*). \) Suppose that \( (I_{\lambda, \mu_n}^T)'(u_n) = 0 \) and \( I_{\lambda, \mu_n}^T(u_n) = c_{\mu_n}. \) By Lemma 3.5, we may assume that \( \|u_n\| \leq T. \) Thus
\[
I_{\lambda, \mu_n}^T(u_n) = \frac{a}{2} \|u_n\|^2 + \frac{\lambda}{4} \|u_n\|^4 - \mu_n \int_{\mathbb{R}^N} G(u_n) \, dx = I_\lambda(u_n) + (1 - \mu_n) \int_{\mathbb{R}^N} G(u_n) \, dx
\]
and

$$\langle (I_{\lambda, \mu_n})'(u_n), v \rangle = a(u_n, v) + \lambda \|u_n\|^2(u_n, v) - \mu_n \int_{\mathbb{R}^N} g(u_n)vdx$$

$$= \langle I_{\lambda}'(u_n), v \rangle + (1 - \mu_n) \int_{\mathbb{R}^N} g(u_n)vdx.$$ 

Since $\mu_n \to 1^-$, $I_{\lambda}(u_n) \to c_1$ and $I_{\lambda}'(u_n) \to 0$ in $H_0^2(\mathbb{R}^N)$. By similar argument as in the proof of Lemma 3.3, there exists $u \in H_0^1(\mathbb{R}^N)$ such that up to a subsequence, $u_n \to u$ in $H_0^1(\mathbb{R}^N)$. Hence $I_{\lambda}(u) = c_1$ and $I_{\lambda}'(u) = 0$. Define $u^\pm := \max\{\pm u, 0\}$, then $u = u^+ - u^-$. Multiplying equation (1.1) by $u^-$, we have

$$(a + \lambda \|u\|^2)\|u^-\|^2 = 0,$$

which deduces that $u^- = 0$ and then $u \geq 0$. By the strongly maximum principle, $u$ is positive. We complete the proof. \qed

Proof of Theorem 1.2. By ($g_7$) and ($g_8$), for any $\varepsilon > 0$, there exists $C_{\varepsilon} > 0$ such that

$$|g(s)| \leq \varepsilon |s| + C_{\varepsilon} |s|^{\min(3, 2^*) - 1}.$$ 

Suppose that equation (1.1) has a nonzero solution $u \in H^1(\mathbb{R}^N)$. Then one has

$$a\|u\|^2 + \lambda \|u\|^4 = \int_{\mathbb{R}^N} g(u)udx$$

$$\leq \varepsilon \int_{\mathbb{R}^N} |u|^2dx + C_{\varepsilon} \int_{\mathbb{R}^N} |u|^{\min(4, 2^*)}dx$$

$$\leq \frac{\varepsilon}{b} \|u\|^2 + C_{\varepsilon} \|u\|^{\min(4, 2^*)}. \quad (3.13)$$

(1) When $N = 3, 4$, choosing $\varepsilon = ab$, (3.13) implies

$$\lambda \|u\|^4 \leq C_1 \|u\|^4.$$ 

If $\lambda > C_1$, we obtain a contradiction.

(2) When $N \geq 5$, choosing $\varepsilon = \frac{ab}{2}$, (3.13) and the Young inequality imply

$$\frac{a}{2} \|u\|^2 + \lambda \|u\|^4 \leq C_2 \|u\|^{2^* - 2}$$

$$\leq \left( \frac{a}{4 - 2^*} \right)^{\frac{1}{2}} \|u\|^2 \left( \frac{4 - 2^*}{a} \right)^{\frac{4 - 2^*}{2}} C_2 \|u\|^{2^{* - 4}}$$

$$\leq \frac{a}{2} \|u\|^2 + \frac{2^* - 2}{2} \left( \frac{4 - 2^*}{a} \right)^{\frac{4 - 2^*}{2^{* - 2}}} C_2^{\frac{2^* - 2}{2}} \|u\|^4.$$ 

If $\lambda > \frac{2^* - 2}{2} \left( \frac{4 - 2^*}{a} \right)^{\frac{4 - 2^*}{2}} C_2^{\frac{2^* - 2}{2}}$, we obtain a contradiction. Define

$$\Lambda = \begin{cases} 
C_1, & N = 3, 4, \\
\frac{2^* - 2}{2} \left( \frac{4 - 2^*}{a} \right)^{\frac{4 - 2^*}{2}} C_2^{\frac{2^* - 2}{2}}, & N \geq 5.
\end{cases}$$

Thus for any $\lambda > \Lambda$, equation (1.1) has no nonzero solution. \qed

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