Bounds for the Generalized $(\Phi, f)$-Mean Difference

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ABSTRACT

In this paper we establish some bounds for the $(\Phi, f)$-mean difference introduced in the general settings of measurable spaces and Lebesgue integral, which is a two functions generalization of Gini mean difference that has been widely used by economists and sociologists to measure economic inequality.

RESUMEN

En este artículo establecemos algunas cotas para la $(\Phi, f)$-diferencia media introducida en el contexto general de espacios medibles e integral de Lebesgue, que es una generalización a dos funciones de la diferencia media de Gini que ha sido ampliamente utilizada por economistas y sociólogos para medir desigualdad económica.

Keywords and Phrases: Gini mean difference, Mean deviation, Lebesgue integral, Expectation, Jensen’s integral inequality.

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1. Introduction

Let \((\Omega, A, \nu)\) be a measurable space consisting of a set \(\Omega\), a \(\sigma\) -algebra \(A\) of subsets of \(\Omega\) and a countably additive and positive measure \(\nu\) on \(A\) with values in \(\mathbb{R} \cup \{\infty\}\). For a \(\nu\)-measurable function \(w : \Omega \rightarrow \mathbb{R}\), with \(w(x) \geq 0\) for \(\nu\)-a.e. (almost every) \(x \in \Omega\) and \(\int_{\Omega} w(x) \, d\nu(x) = 1\), consider the Lebesgue space

\[
L_w(\Omega, \nu) := \{f : \Omega \rightarrow \mathbb{R}, \text{ } f \text{ is } \nu\text{-measurable and } \int_{\Omega} w(x) |f(x)| \, d\nu(x) < \infty\}.
\]

Let \(I\) be an interval of real numbers and \(\Phi : I \rightarrow \mathbb{R}\) a Lebesgue measurable function on \(I\). For \(f : \Omega \rightarrow I\) a \(\nu\)-measurable function with \(\Phi \circ f \in L_w(\Omega, \nu)\) we define the generalized \((\Phi, f)\)-mean difference \(R_G(\Phi, f; w)\) by

\[
R_G(\Phi, f; w) := \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x) w(y) |(\Phi \circ f)(x) - (\Phi \circ f)(y)| \, d\nu(x) \, d\nu(y) \tag{1.1}
\]

and the generalized \((\Phi, f)\)-mean deviation \(M_D(\Phi, f; w)\) by

\[
M_D(\Phi, f; w) := \int_{\Omega} w(x) |(\Phi \circ f)(x) - E(\Phi, f; w)| \, d\nu(x), \tag{1.2}
\]

where

\[E(\Phi, f; w) := \int_{\Omega} (\Phi \circ f)(y) w(y) \, d\nu(y)\]

the generalized \((\Phi, f)\)-expectation.

If \(\Phi = e\), where \(e(t) = t, t \in \mathbb{R}\) is the identity mapping, then we can consider the particular cases of interest, the generalized \(f\)-mean difference

\[
R_G(f; w) := R_G(e, f; w) = \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x) w(y) |f(x) - f(y)| \, d\nu(x) \, d\nu(y) \tag{1.3}
\]

and the generalized \(f\)-mean deviation

\[
M_D(f; w) := M_D(e, f; w) = \int_{\Omega} w(x) |f(x) - E(f; w)| \, d\nu(x), \tag{1.4}
\]

where \(E(f; w) := \int_{\Omega} f(y) w(y) \, d\nu(y)\) is the generalized \(f\)-expectation.

If \(\Omega = [-\infty, \infty]\) and \(f = e\) then we have the usual mean difference

\[
R_G(w) := R_G(f; w) = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(x) w(y) |x - y| \, dx \, dy \tag{1.5}
\]

and the mean deviation

\[
M_D(w) := M_D(f; w) = \int_{\Omega} w(x) |x - E(w)| \, dx, \tag{1.6}
\]
where \( w : \mathbb{R} \to [0, \infty) \) is a \textit{density function}, this means that \( w \) is integrable on \( \mathbb{R} \) and \( \int_{-\infty}^{\infty} w(t) \, dt = 1 \), and

\[
E(w) := \int_{-\infty}^{\infty} xw(x) \, dx \tag{1.7}
\]
denote the \textit{expectation of} \( w \) provided that the integral exists and is finite.

The mean difference \( R_G(w) \) was proposed by Gini in 1912 [21], after whom it is usually named, but was discussed by Helmert and other German writers in the 1870's (cf. H. A. David [13], see also [26, p. 48]). It has a certain theoretical attraction, being dependent on the spread of the variate-values among themselves and not on the deviations from some central value ([26, p. 48]). Further, its defining integral (1.5) may converge when that of the variance \( \sigma(w) \),

\[
\sigma(w) := \int_{-\infty}^{\infty} (x - E(w))^2 w(x) \, dx, \tag{1.8}
\]
does not. It is, however, more difficult to compute than the standard deviation.

For some recent results concerning integral representations and bounds for \( R_G(w) \) see [5], [6], [8] and [9].

For instance, if \( w : \mathbb{R} \to [0, \infty) \) is a density function we define by

\[
W(x) := \int_{-\infty}^{x} w(t) \, dt, \quad x \in \mathbb{R}
\]
its \textit{cumulative function}. Then we have [5], [6]:

\[
R_G(w) = 2 \text{Cov}(e, W) = \int_{-\infty}^{\infty} (1 - W(y)) W(y) \, dy
\]
\[
= 2 \int_{-\infty}^{\infty} xw(x) W(x) \, dx - E(w)
\]
\[
= 2 \int_{-\infty}^{\infty} (x - E(w)) (W(x) - \gamma) w(x) \, dx
\]
\[
= 2 \int_{-\infty}^{\infty} (x - \delta) \left( W(x) - \frac{1}{2} \right) w(x) \, dx \tag{1.9}
\]
for any \( \gamma, \delta \in \mathbb{R} \) and [6]:

\[
R_G(w) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - y) [W(x) - W(y)] w(x) w(y) \, dx dy. \tag{1.10}
\]

With the above assumptions, we have the bounds [6]:

\[
\frac{1}{2} M_D(w) \leq R_G(w) \leq 2 \sup_{x \in \mathbb{R}} |W(x) - \gamma| M_D(w) \leq M_D(w), \tag{1.11}
\]
for any $\gamma \in [0,1]$, where $W(\cdot)$ is the cumulative distribution of $w$ and $M_D (w)$ is the mean deviation.

Consider the $n$-tuple of real numbers $a = (a_1, ..., a_n)$ and $p = (p_1, ..., p_n)$ a probability distribution, i.e. $p_i \geq 0$ for each $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$, then by taking $\Omega = \{1, ..., n\}$ and the discrete measure, we can consider from (1.1) and (1.2) that (see [7])

$$R_G (a; p) := \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j |\Phi (a_i) - \Phi (a_j)|,$$  \hfill (1.12)

and

$$M_D (a; p) := \frac{1}{2} \sum_{i=1}^{n} p_i \left| \Phi (a_i) - \sum_{j=1}^{n} p_j \Phi (a_j) \right|,$$  \hfill (1.13)

where $a \in I^n := I \times ... \times I$ and $\Phi : I \rightarrow \mathbb{R}$.

The quantity $R_G (a; p)$ has been defined in [7] and some results were obtained.

In the case when $\Phi = e$, then we get the special case of Gini mean difference and mean deviation of an empirical distribution that is particularly important for applications,

$$R_G (a; p) := \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j |a_i - a_j|,$$  \hfill (1.14)

and

$$M_D (a; p) := \frac{1}{2} \sum_{i=1}^{n} p_i \left| a_i - \sum_{j=1}^{n} p_j a_j \right|. \hfill (1.15)$$

The following result incorporates an upper bound for the weighted Gini mean difference [7]:

For any $a \in \mathbb{R}^n$ and any $p$ a probability distribution, we have the inequality:

$$\frac{1}{2} M_D (a; p) \leq R_G (a; p) \leq \inf_{\gamma \in \mathbb{R}} \left[ \sum_{i=1}^{n} p_i |a_i - \gamma| \right] \leq M_D (a; p). \hfill (1.16)$$

The constant $\frac{1}{2}$ in the first inequality in (1.16) is sharp.

For some recent results for discrete Gini mean difference and mean deviation, see [7], [11], [14] and [15].
2. General Bounds

We have:

**Theorem 1.** Let $I$ be an interval of real numbers and $\Phi : I \to \mathbb{R}$ a Lebesgue measurable function on $I$. If $w : \Omega \to \mathbb{R}$ is a $\nu$-measurable function with $w(x) \geq 0$ for $\nu$-a.e. (almost every) $x \in \Omega$ and $\int_\Omega w(x) \, d\nu(x) = 1$ and if $f : \Omega \to I$ is a $\nu$-measurable function with $\Phi \circ f \in L_w(\Omega, \nu)$, then

$$\frac{1}{2} M_D(\Phi, f; w) \leq R_G(\Phi, f; w) \leq I(\Phi, f; w) \leq M_D(\Phi, f; w),$$

(2.1)

where

$$I(\Phi, f; w) := \inf_{\gamma \in \mathbb{R}} \int_\Omega w(x) |(\Phi \circ f)(x) - \gamma| \, d\nu(x).$$

(2.2)

**Demostración.** Using the properties of the integral, we have

$$R_G(\Phi, f; w)$$

$$= \frac{1}{2} \int_\Omega \int_\Omega w(x) w(y) |(\Phi \circ f)(x) - (\Phi \circ f)(y)| \, d\nu(x) \, d\nu(y)$$

$$\geq \frac{1}{2} \int_\Omega w(x) |(\Phi \circ f)(x)\int_\Omega w(y) \, d\nu(y) - \int_\Omega w(y) (\Phi \circ f)(y) \, d\nu(y)| \, d\nu(x)$$

$$= \frac{1}{2} \int_\Omega w(x) |(\Phi \circ f)(x) - \int_\Omega w(y) (\Phi \circ f)(y) \, d\nu(y)| \, d\nu(x)$$

$$= \frac{1}{2} M_D(\Phi, f; w)$$

and the first inequality in (2.1) is proved.

By the triangle inequality for modulus we have

$$|(\Phi \circ f)(x) - (\Phi \circ f)(y)| = |(\Phi \circ f)(x) - \gamma + \gamma - (\Phi \circ f)(y)|$$

$$\leq |(\Phi \circ f)(x) - \gamma| + |(\Phi \circ f)(y) - \gamma|$$

(2.3)

for any $x, y \in \Omega$ and $\gamma \in \mathbb{R}$.
Now, if we multiply \((2.3)\) by \(\frac{1}{2}w(x)w(y)\) and integrate, we get

\[
R_G(\Phi, f; w) = \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x) w(y) |(\Phi \circ f)(x) - (\Phi \circ f)(y)| \, d\nu(x) \, d\nu(y)
\]

\[
\leq \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x) w(y) \left| |(\Phi \circ f)(x) - \gamma + (\Phi \circ f)(y) - \gamma| \right| \, d\nu(x) \, d\nu(y)
\]

\[
= \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x) w(y) |(\Phi \circ f)(x) - \gamma| \, d\nu(x) \, d\nu(y)
\]

\[
+ \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x) w(y) |(\Phi \circ f)(y) - \gamma| \, d\nu(x) \, d\nu(y)
\]

\[
= \frac{1}{2} \int_{\Omega} w(x) |(\Phi \circ f)(x) - \gamma| \, d\nu(x)
\]

\[
+ \frac{1}{2} \int_{\Omega} w(y) |(\Phi \circ f)(y) - \gamma| \, d\nu(y)
\]

(2.4)

for any \(\gamma \in \mathbb{R}\).

Taking the infimum over \(\gamma \in \mathbb{R}\) in (2.4) we get the second part of (2.1).

Since, obviously

\[
I(\Phi, f; w) = \inf_{\gamma \in \mathbb{R}} \int_{\Omega} w(x) |(\Phi \circ f)(x) - \gamma| \, d\nu(x)
\]

\[
\leq \int_{\Omega} w(x) \left| |(\Phi \circ f)(x) - \int_{\Omega} w(y) (\Phi \circ f)(y) \, d\nu(y)| \right| \, d\nu(x)
\]

\[
= M_G(\Phi, f; w),
\]

the last part of (2.1) is thus proved. \(\square\)

By the Cauchy-Bunyakowsky-Schwarz (CBS) inequality, if \((\Phi \circ f)^2 \in L_w(\Omega, \nu)\), then we have

\[
\left[ \int_{\Omega} w(x) \left| (\Phi \circ f)(x) - \int_{\Omega} w(y) (\Phi \circ f)(y) \, d\nu(y) \right| \, d\nu(x) \right]^2
\]

\[
\leq \int_{\Omega} w(x) \left[ (\Phi \circ f)(x) - \int_{\Omega} w(y) (\Phi \circ f)(y) \, d\nu(y) \right]^2 \, d\nu(x)
\]

\[
= \int_{\Omega} w(x) (\Phi \circ f)^2(x) \, d\nu(x)
\]

\[
- 2 \int_{\Omega} w(x) (\Phi \circ f)(x) \, d\nu(x) \int_{\Omega} w(y) (\Phi \circ f)(y) \, d\nu(y)
\]

\[
+ \left[ \int_{\Omega} w(y) (\Phi \circ f)(y) \, d\nu(y) \right]^2 \int_{\Omega} w(x) \, d\nu(x)
\]

\[
= \int_{\Omega} w(x) (\Phi \circ f)^2(x) \, d\nu(x) - \left[ \int_{\Omega} w(x) (\Phi \circ f)(x) \, d\nu(x) \right]^2.
\]
By considering the \emph{generalized} \((\Phi, f)\)-\textit{dispersion}

\[\sigma(\Phi, f; w) := \left( \int_{\Omega} w(x) (\Phi \circ f)^2(x) \, d\nu(x) - \left[ \int_{\Omega} w(x) (\Phi \circ f)(x) \, d\nu(x) \right]^2 \right)^{1/2},\]

then we have

\[M_D(\Phi, f; w) \leq \sigma(\Phi, f; w) \quad (2.5)\]

provided \((\Phi \circ f)^2 \in L_w(\Omega, \nu)\).

If there exists the constants \(m, M\) so that

\[-\infty < m \leq \Phi(t) \leq M < \infty \text{ for almost any } t \in I \quad (2.6)\]

then by the reverse CBS inequality

\[\sigma(\Phi, f; w) \leq \frac{1}{2} (M - m), \quad (2.7)\]

by (2.1) and by (2.5) we can state the following result:

\textbf{Corollary 1.} Let \(I\) be an interval of real numbers and \(\Phi : I \rightarrow \mathbb{R}\) a \textit{Lebesgue measurable function} on \(I\) satisfying the condition (2.6) for some constants \(m, M\). If \(w : \Omega \rightarrow \mathbb{R}\) is a \(\nu\)-measurable function with \(w(x) \geq 0\) for \(\nu\)-a.e. \(x \in \Omega\) and \(\int_{\Omega} w(x) \, d\nu(x) = 1\) and if \(f : \Omega \rightarrow I\) is a \(\nu\)-measurable function with \((\Phi \circ f)^2 \in L_w(\Omega, \nu)\), then we have the chain of inequalities

\[\frac{1}{2} M_D(\Phi, f; w) \leq R_G(\Phi, f; w) \leq I(w) \leq M_D(\Phi, f; w) \leq \sigma(\Phi, f; w) \leq \frac{1}{2} (M - m). \quad (2.8)\]

We observe that, in the discrete case we obtain from (2.1) the inequality (1.16) while for the univariate case with \(\int_{-\infty}^{\infty} w(t) \, dt = 1\) we have

\[\frac{1}{2} M_D(w) \leq R_G(w) \leq I(w) \leq M_D(w) \leq \sigma(\Phi, f; w) \quad (2.9)\]

where

\[I(w) := \inf_{\gamma \in \mathbb{R}} \int_{-\infty}^{\infty} w(x) |x - \gamma| \, dx. \quad (2.10)\]

If \(w\) is supported on the finite interval \([a, b]\), namely \(\int_a^b w(x) \, dx = 1\), then we have the chain of inequalities

\[\frac{1}{2} M_D(w) \leq R_G(w) \leq I(w) \leq M_D(w) \leq \sigma(\Phi, f; w) \leq \frac{1}{2} (M - m). \quad (2.11)\]
3. Bounds for Various Classes of Functions

In the case of functions of bounded variation we have:

**Theorem 2.** Let \( \Phi : [a, b] \to \mathbb{R} \) be a function of bounded variation on the closed interval \([a, b]\). If \( w : \Omega \to \mathbb{R} \) is a \( \nu \)-measurable function with \( w(x) \geq 0 \) for \( \nu \)-a.e. \( x \in \Omega \) and \( \int_{\Omega} w(x) \, d\nu(x) = 1 \) and if \( f : \Omega \to [a, b] \) is a \( \nu \)-measurable function with \( \Phi \circ f \in L^w(\Omega, \nu) \), then

\[
R_G (\Phi, f; w) \leq \frac{1}{2} b \bigvee_a^b (\Phi),
\]

where \( \bigvee_a^b (\Phi) \) is the total variation of \( \Phi \) on \([a, b]\).

**Demostración.** Using the inequality (2.4) we have

\[
R_G (\Phi, f; w) \leq \int_{\Omega} w(x) |(\Phi \circ f)(x) - \gamma| \, d\nu(x)
\]

for any \( \gamma \in \mathbb{R} \).

By the triangle inequality, we have

\[
\left| (\Phi \circ f)(x) - \frac{1}{2} [\Phi(a) + \Phi(b)] \right| \\
\leq \frac{1}{2} |\Phi(a) - \Phi(f(x))| + \frac{1}{2} |\Phi(b) - \Phi(f(x))|
\]

for any \( x \in \Omega \).

Since \( \Phi : [a, b] \to \mathbb{R} \) is of bounded variation and \( d \) is a division of \([a, b]\), namely

\[
d \in D([a, b]) := \{ d := \{ a = t_0 < t_1 < ... < t_n = b \} \},
\]

then

\[
\bigvee_a^b (\Phi) = \sup_{d \in D([a, b])} \sum_{i=0}^{n-1} |\Phi(t_{i+1}) - \Phi(t_i)| < \infty.
\]

Taking the division \( d_0 := \{ a = t_0 < t < t_2 = b \} \) we then have

\[
|\Phi(t) - \Phi(a)| + |\Phi(b) - \Phi(t)| \leq \bigvee_a^b (\Phi)
\]

for any \( t \in [a, b] \) and then

\[
|\Phi(f(x)) - \Phi(a)| + |\Phi(b) - \Phi(f(x))| \leq \bigvee_a^b (\Phi)
\]

for any \( x \in \Omega \).
On making use of (3.3) and (3.4) we get

\[ |(\Phi \circ f)(x) - \frac{1}{2}[\Phi(a) + \Phi(b)]| \leq \frac{1}{2} \sqrt{b - a} \tag{3.5} \]

for any \( x \in \Omega \).

If we multiply (3.5) by \( w(x) \) and integrate, then we obtain

\[ \int_{\Omega} w(x) |(\Phi \circ f)(x) - \frac{1}{2}[\Phi(a) + \Phi(b)]| \leq \frac{1}{2} \sqrt{b - a} \tag{3.6} \]

Finally, by choosing \( \gamma = \frac{1}{2}[\Phi(a) + \Phi(b)] \) in (3.2) and making use of (3.6) we deduce the desired result (3.1).

In the case of absolutely continuous functions we have:

**Theorem 3.** Let \( \Phi : [a, b] \rightarrow \mathbb{R} \) be an absolutely continuous function on the closed interval \([a, b]\). If \( w : \Omega \rightarrow \mathbb{R} \) is a \( \nu \)-measurable function with \( w(x) \geq 0 \) for \( \nu \)-a.e. \( x \in \Omega \) and \( \int_{\Omega} w(x) \, d\nu(x) = 1 \) and if \( f : \Omega \rightarrow [a, b] \) is a \( \nu \)-measurable function with \( \Phi \circ f \in L_w(\Omega, \nu) \), then

\[ R_G(\Phi, f; w) \leq \begin{cases} \|\Phi'\|_{[a,b],\infty} R_G(f; w) & \text{if } \Phi' \in L_\infty([\alpha, \beta]), \\ \frac{1}{2^{1/p}} \|\Phi'\|_{[a,b],p} R_G^1(f; w) & \text{if } \Phi' \in L_p([\alpha, \beta]), \quad \frac{1}{p} + \frac{1}{q} = 1, \end{cases} \tag{3.7} \]

where the Lebesgue norms are defined by

\[ \|g\|_{[\alpha, \beta],p} := \begin{cases} \text{ess}\sup_{t \in [\alpha, \beta]} |g(t)| & \text{if } p = \infty, \\ \left( \int_{\alpha}^{\beta} |g(t)|^p \, dt \right)^{1/p} & \text{if } p \geq 1, \end{cases} \]

and \( L_p([\alpha, \beta]) := \{ g \mid g \text{ measurable and } \|g\|_{[\alpha, \beta],p} < \infty \} \), \( p \in [1, \infty] \).

**Demostración.** Since \( f \) is absolutely continuous, then we have

\[ \Phi(t) - \Phi(s) = \int_{s}^{t} \Phi'(u) \, du \]

for any \( t, s \in [a, b] \).

Using the Hölder integral inequality we have

\[ |\Phi(t) - \Phi(s)| = \left| \int_{s}^{t} \Phi'(u) \, du \right| \leq \begin{cases} \|\Phi'\|_{[a,b],\infty} |t - s| & \text{if } p = \infty, \\ \|\Phi'\|_{[a,b],p} |t - s|^{1/q} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1 \end{cases} \tag{3.8} \]
for any \( t, s \in [a, b] \).

Using (3.8) we then have

\[
\int_\Omega \int_\Omega w(x)w(y) |(\Phi \circ f)(x) - (\Phi \circ f)(y)| \, dv(x) \, dv(y)
\]

\[
\leq \frac{1}{2} \|\Phi\|_{[a,b],p} \int_\Omega \int_\Omega w(x)w(y) |f(x) - f(y)| \, dv(x) \, dv(y)
\]

for any \( x, y \in \Omega \).

Using Jensen’s integral inequality for concave function \( \Psi(t) = t^s, s \in (0,1) \) we have for \( s = \frac{1}{q} < 1 \) that

\[
\int_\Omega \int_\Omega w(x)w(y) |f(x) - f(y)|^{1/q} \, dv(x) \, dv(y)
\]

\[
\leq \left( \int_\Omega \int_\Omega w(x)w(y) |f(x) - f(y)| \, dv(x) \, dv(y) \right)^{1/q},
\]

which implies that

\[
\frac{1}{2} \|\Phi\|_{[a,b],p} \int_\Omega \int_\Omega w(x)w(y) |f(x) - f(y)|^{1/q} \, dv(x) \, dv(y)
\]

\[
\leq \frac{1}{2} \|\Phi\|_{[a,b],p} \left( \int_\Omega \int_\Omega w(x)w(y) |f(x) - f(y)| \, dv(x) \, dv(y) \right)^{1/q}
\]

\[
= \|\Phi\|_{[a,b],p} \left( \frac{1}{2q-1} \int_\Omega \int_\Omega w(x)w(y) |f(x) - f(y)| \, dv(x) \, dv(y) \right)^{1/q}
\]

\[
= \frac{1}{2q-1} \|\Phi\|_{[a,b],p} \left( R_G(f;w) \right)^{1/q}
\]

and the second part of (3.7) is proved.

The function \( \Phi : [a, b] \to \mathbb{R} \) is called of \( r\)-\textit{Hölder type} with the given constants \( r \in (0,1) \) and \( H > 0 \) if

\[
|\Phi(t) - \Phi(s)| \leq H |t - s|^r
\]
for any \( t, s \in [a, b] \).

In the case when \( r = 1 \), namely, there is the constant \( L > 0 \) such that

\[
|\Phi(t) - \Phi(s)| \leq L |t - s|
\]

for any \( t, s \in [a, b] \), the function \( \Phi \) is called \( L \)-Lipschitzian on \([a, b]\).

We have:

**Theorem 4.** Let \( \Phi : [a, b] \to \mathbb{R} \) be a function of \( r \)-Hölder type on the closed interval \([a, b]\). If \( w : \Omega \to \mathbb{R} \) is a \( \nu \)-measurable function with \( w(x) \geq 0 \) for \( \nu \)-a.e. \( x \in \Omega \) and \( \int_\Omega w(x) \, d\nu(x) = 1 \) and if \( f : \Omega \to [a, b] \) is a \( \nu \)-measurable function with \( \Phi \circ f \in L_w(\Omega, \nu) \), then

\[
R_G(\Phi, f; w) \leq \frac{1}{2^{1-r}} H R_G^r(f; w) .
\]

(3.11)

In particular, if \( \Phi \) is \( L \)-Lipschitzian on \([a, b]\), then

\[
R_G(\Phi, f; w) \leq LR_G(f; w) .
\]

(3.12)

Demostración. We have

\[
| (\Phi \circ f)(x) - (\Phi \circ f)(y) | \leq H | f(x) - f(y) |^r
\]

(3.13)

for any \( x, y \in \Omega \).

If we multiply (3.13) by \( \frac{1}{2} w(x) w(y) \) and integrate, then we get

\[
\frac{1}{2} \int_\Omega \int_\Omega w(x) w(y)| (\Phi \circ f)(x) - (\Phi \circ f)(y) | \, d\nu(x) \, d\nu(y)
\leq \frac{1}{2} H \int_\Omega \int_\Omega w(x) w(y) | f(x) - f(y) |^r \, d\nu(x) \, d\nu(y) .
\]

(3.14)

By Jensen’s integral inequality for concave functions we also have

\[
\int_\Omega \int_\Omega w(x) w(y) | f(x) - f(y) |^r \, d\nu(x) \, d\nu(y)
\leq \left( \int_\Omega \int_\Omega w(x) w(y) | f(x) - f(y) | \, d\nu(x) \, d\nu(y) \right)^r .
\]

(3.15)

Therefore, by (3.14) and (3.15) we get

\[
R_G(\Phi, f; w) \leq \frac{1}{2} H \left( \int_\Omega \int_\Omega w(x) w(y) | f(x) - f(y) | \, d\nu(x) \, d\nu(y) \right)^r
= \frac{1}{2^{1-r}} H \left( \frac{1}{2} \int_\Omega \int_\Omega w(x) w(y) | f(x) - f(y) | \, d\nu(x) \, d\nu(y) \right)^r
= \frac{1}{2^{1-r}} H R_G^r(f; w)
\]

and the inequality (3.11) is proved. \( \square \)
We have:

**Theorem 5.** Let \( \Phi, \Psi : [a, b] \to \mathbb{R} \) be continuous functions on \([a, b]\) and differentiable on \((a, b)\) with \(\Psi'(t) \neq 0\) for \(t \in (a, b)\). If \(w : \Omega \to \mathbb{R}\) is a \(\nu\)-measurable function with \(w(x) \geq 0\) for \(\nu\)-a.e. \(x \in \Omega\) and \(\int_{\Omega} w(x) \, d\nu(x) = 1\) and if \(f : \Omega \to [a, b]\) is a \(\nu\)-measurable function with \(\Phi \circ f \in L_w(\Omega, \nu)\), then

\[
\inf_{t \in (a, b)} \left| \frac{\Phi'(t)}{\Psi'(t)} \right| R_G(\Psi, f; w) \leq R_G(\Phi, f; w) \leq \sup_{t \in (a, b)} \left| \frac{\Phi'(t)}{\Psi'(t)} \right| R_G(\Psi, f; w). \quad (3.16)
\]

Demostración. By the Cauchy's mean value theorem, for any \(t, s \in [a, b]\) with \(t \neq s\) there exists a \(\xi\) between \(t\) and \(s\) such that

\[
\frac{\Phi(t) - \Phi(s)}{\Psi(t) - \Psi(s)} = \Phi'\left(\xi\right) / \Psi'\left(\xi\right).
\]

This implies that

\[
\inf_{\tau \in (a, b)} \left| \frac{\Phi'(\tau)}{\Psi'(\tau)} \right| |\Psi(t) - \Psi(s)| \leq |\Phi(t) - \Phi(s)|
\]

\[
\leq \sup_{\tau \in (a, b)} \left| \frac{\Phi'(\tau)}{\Psi'(\tau)} \right| |\Psi(t) - \Psi(s)| \quad (3.17)
\]

for any \(t, s \in [a, b]\).

Therefore, we have

\[
\inf_{\tau \in (a, b)} \left| \frac{\Phi'(\tau)}{\Psi'(\tau)} \right| |\Psi(f(x)) - \Psi(f(y))| \leq |\Phi(f(x)) - \Phi(f(y))|
\]

\[
\leq \sup_{\tau \in (a, b)} \left| \frac{\Phi'(\tau)}{\Psi'(\tau)} \right| |\Psi(f(x)) - \Psi(f(y))| \quad (3.18)
\]

for any \(x, y \in \Omega\).

If we multiply (3.18) by \(w(x) w(y)\) and integrate, we get the desired result (3.16).

**Corollary 2.** Let \( \Phi : [a, b] \to \mathbb{R} \) be a continuous function on \([a, b]\) and differentiable on \((a, b)\). If \(w\) is as in Theorem 5 then we have

\[
\inf_{t \in (a, b)} |\Phi'(t)| R_G(f; w) \leq R_G(\Phi, f; w) \leq \sup_{t \in (a, b)} |\Phi'(t)| R_G(f; w). \quad (3.19)
\]

We also have:

**Theorem 6.** Let \( \Phi : [a, b] \to \mathbb{R} \) be an absolutely continuous function on the closed interval \([a, b]\). If \(w : \Omega \to \mathbb{R}\) is a \(\nu\)-measurable function with \(w(x) \geq 0\) for \(\nu\)-a.e. \(x \in \Omega\) and \(\int_{\Omega} w(x) \, d\nu(x) = 1\)
and if \( f : \Omega \rightarrow [a, b] \) is a \( \nu \)-measurable function with \( \Phi \circ f \in L_w (\Omega, \nu) \), then

\[
R_G (\Phi, f; w) \leq \begin{cases}
\|\Phi'\|_{[a,b], \infty} M (f; w) & \text{if } p = \infty, \\
\|\Phi'\|_{[a,b], p} M^{1/q} (f; w) & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\
\frac{1}{2} (b - a) \|\Phi'\|_{[a,b], \infty} & \text{if } p = \infty, \\
\frac{1}{2} (b - a)^{1/q} \|\Phi'\|_{[a,b], p} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1,
\end{cases}
\tag{3.20}
\]

where \( M (f; w) \) is defined by

\[
M (f; w) := \int_{\Omega} w (x) \left| f (x) - \frac{a + b}{2} \right| \, d\nu (x). \tag{3.21}
\]

**Demostración.** From the inequality (3.8) we have

\[
\left| (\Phi \circ f) (x) - \Phi \left( \frac{a + b}{2} \right) \right| \leq \begin{cases}
\|\Phi'\|_{[a,b], \infty} |f (x) - \frac{a + b}{2}| & \text{if } p = \infty, \\
\|\Phi'\|_{[a,b], p} |f (x) - \frac{a + b}{2}|^{1/q} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1
\end{cases}
\tag{3.22}
\]

for any \( x \in \Omega \).

Now, if we multiply (3.22) by \( w (x) \) and integrate, then we get

\[
\int_{\Omega} w (x) \left| (\Phi \circ f) (x) - \Phi \left( \frac{a + b}{2} \right) \right| \, d\nu (x) \leq \begin{cases}
\|\Phi'\|_{[a,b], \infty} \int_{\Omega} w (x) \left| f (x) - \frac{a + b}{2} \right| \, d\nu (x) & \text{if } p = \infty, \\
\|\Phi'\|_{[a,b], p} \int_{\Omega} w (x) \left| f (x) - \frac{a + b}{2} \right|^{1/q} \, d\nu (x) & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1.
\end{cases}
\tag{3.23}
\]

By Jensen’s integral inequality for concave functions we have

\[
\int_{\Omega} w (x) \left| f (x) - \frac{a + b}{2} \right|^{1/q} \, d\nu (x) \leq \left( \int_{\Omega} w (x) \left| f (x) - \frac{a + b}{2} \right| \, d\nu (x) \right)^{1/q}.
\tag{3.24}
\]

On making use of (3.22), (3.23) and (3.24) we get the first inequality in (3.20).

The last part of (3.20) follows by the fact that

\[
\left| f (x) - \frac{a + b}{2} \right| \leq \frac{1}{2} (b - a)
\]

for any \( x \in \Omega \). ■
4. Bounds for Special Convexity

When some convexity properties for the function $\Phi$ are assumed, then other bounds can be derived as follows.

**Theorem 7.** Let $w : \Omega \to \mathbb{R}$ be a $\nu$-measurable function with $w(x) \geq 0$ for $\nu$-a.e. $x \in \Omega$ and $\int_\Omega w(x) \, d\nu(x) = 1$ and $f : \Omega \to [a, b]$ be a $\nu$-measurable function with $\Phi \circ f \in L_w(\Omega, \nu)$. Assume also that $\Phi : [a, b] \to \mathbb{R}$ is a continuous function on $[a, b]$.

(i) If $|\Phi|$ is concave on $[a, b]$, then

$$R_G(\Phi, f; w) \leq |\Phi(E(f; w))|,$$

(4.1)

(ii) If $|\Phi|$ is convex on $[a, b]$, then

$$R_G(\Phi, f; w) \leq \frac{1}{b - a} \left[ |(b - E(f; w))| \Phi(a) + (E(f; w) - a) \Phi(b) \right].$$

(4.2)

**Demostración.** (i) If $|\Phi|$ is concave on $[a, b]$, then by Jensen’s inequality we have

$$\int_\Omega w(x) |(\Phi \circ f)(x)| \, d\nu(x) \leq |\Phi \left( \int_\Omega w(x) f(x) \, d\nu(x) \right)|.$$  \hspace{1cm} (4.3)

From (4.2) for $\gamma = 0$ we also have

$$R_G(\Phi, f; w) \leq \int_\Omega w(x) |(\Phi \circ f)(x)| \, d\nu(x).$$  \hspace{1cm} (4.4)

This is an inequality of interest in itself.

On utilizing (4.3) and (4.4) we get (4.1).

(ii) Since $|\Phi|$ is convex on $[a, b]$, then for any $t \in [a, b]$ we have

$$|\Phi(t)| = \left| \Phi \left( \frac{(b - t)a + b(t - a)}{b - a} \right) \right| \leq \frac{(b - t)|\Phi(a)| + (t - a)|\Phi(b)|}{b - a}.$$  \hspace{1cm} (4.5)

This implies that

$$|(\Phi \circ f)(x)| \leq \frac{(b - f(x))|\Phi(a)| + (f(x) - a)|\Phi(b)|}{b - a}$$

for any $x \in \Omega$.

If we multiply (4.5) by $w(x)$ and integrate, then we get

$$\int_\Omega w(x) |(\Phi \circ f)(x)| \, d\nu(x) \leq \frac{1}{b - a} \left[ \left( b \int_\Omega w(x) \, d\nu(x) - \int_\Omega w(x) f(x) \, d\nu(x) \right) |\Phi(a)| \\
+ \left( \int_\Omega w(x) f(x) \, d\nu(x) - a \int_\Omega w(x) \, d\nu(x) \right) |\Phi(b)| \right],$$

which, together with (4.4), produces the desired result (4.2).
In order to state other results we need the following definitions:

**Definition 1** ([19]). We say that a function \( f : I \to \mathbb{R} \) belongs to the class \( P(I) \) if it is nonnegative and for all \( x, y \in I \) and \( t \in [0, 1] \) we have

\[
f(tx + (1-t)y) \leq f(x) + f(y).
\]

It is important to note that \( P(I) \) contains all nonnegative monotone, convex and quasi convex functions, i.e. functions satisfying

\[
f(tx + (1-t)y) \leq \max\{f(x), f(y)\}
\]

for all \( x, y \in I \) and \( t \in [0, 1] \).

For some results on \( P \)-functions see [19] and [28] while for quasi convex functions, the reader can consult [18].

**Definition 2** ([3]). Let \( s \) be a real number, \( s \in (0, 1] \). A function \( f : [0, \infty) \to [0, \infty) \) is said to be \( s \)-convex (in the second sense) or Breckner \( s \)-convex if

\[
f(tx + (1-t)y) \leq ts f(x) + (1-t)s f(y)
\]

for all \( x, y \in [0, \infty) \) and \( t \in [0, 1] \).

For some properties of this class of functions see [1], [2], [3], [4], [16], [17], [25], [27] and [29].

**Theorem 8.** Let \( w : \Omega \to \mathbb{R} \) be a \( \nu \)-measurable function with \( w(x) \geq 0 \) for \( \nu \)-a.e. \( x \in \Omega \) and \( \int_\Omega w(x) \, d\nu(x) = 1 \) and \( f : \Omega \to [a, b] \) be a \( \nu \)-measurable function with \( \Phi \circ f \in L^w(\Omega, \nu) \). Assume also that \( \Phi : [a, b] \to \mathbb{R} \) is a continuous function on \([a, b]\).

(i) If \( |\Phi| \) belongs to the class \( P \) on \([a, b]\), then

\[
RG(\Phi, f; w) \leq |\Phi(a)| + \Phi(b); \tag{4.6}
\]

(ii) If \( |\Phi| \) is quasi convex on \([a, b]\), then

\[
RG(\Phi, f; w) \leq \max\{|\Phi(a)|, \Phi(b)|\}; \tag{4.7}
\]

(iii) If \( |\Phi| \) is Breckner \( s \)-convex on \([a, b]\), then

\[
RG(\Phi, f; w) \leq \frac{1}{(b-a)^s} \left[ |\Phi(a)| \int_\Omega w(x)(b-f(x))^s \, d\nu(x) + \Phi(b) \int_\Omega w(x)(f(x)-a)^s \, d\nu(x) \right]
\]

\[
\leq \frac{1}{(b-a)^s} \left[ |\Phi(a)| (b-E(f;w))^s \, d\nu(x) + \Phi(b)(E(f;w)-a)^s \, d\nu(x) \right]. \tag{4.8}
\]
Demostración. (i) Since \(|\Phi|\) belongs to the class \(P\) on \([a, b]\), then for any \(t \in [a, b]\) we have
\[
|\Phi(t)| = \left| \Phi \left( \frac{(b - t) a + b (t - a)}{b - a} \right) \right| \leq |\Phi(a)| + |\Phi(b)|.
\]
This implies that
\[
|\Phi \circ f(x)| \leq |\Phi(a)| + |\Phi(b)|
\] (4.9)
for any \(x \in \Omega\).

If we multiply (4.9) by \(w(x)\) and integrate, then we get
\[
\int_{\Omega} w(x) |\Phi \circ f(x)| \, dv(x) \leq |\Phi(a)| + |\Phi(b)|,
\] (4.10)
which, together with (4.4), produces the desired result (4.6).

(ii) Goes in a similar way.

(iii) By Breckner \(s\)-convexity we have
\[
|\Phi(t)| = \left| \Phi \left( \frac{(b - t) a + b (t - a)}{b - a} \right) \right| \leq \left( \frac{b - t}{b - a} \right)^s |\Phi(a)| + \left( \frac{t - a}{b - a} \right)^s |\Phi(b)|
\]
for any \(t \in [a, b]\).

This implies that
\[
|\Phi \circ f(x)| \leq \frac{1}{(b - a)^s} \left[ (b - f(x))^s |\Phi(a)| + (f(x) - a)^s |\Phi(b)| \right]
\] (4.11)
for any \(x \in \Omega\).

If we multiply (4.11) by \(w(x)\) and integrate, then we get
\[
\int_{\Omega} w(x) |\Phi \circ f(x)| \, dv(x) \leq \frac{1}{(b - a)^s} \left[ |\Phi(a)| \int_{\Omega} w(x) (b - f(x))^s \, dv(x) 
+ |\Phi(b)| \int_{\Omega} w(x) (f(x) - a)^s \, dv(x) \right],
\] (4.12)
which, together with (4.4), produces the first part of (4.8).

The last part follows by Jensen’s integral inequality for concave functions, namely
\[
\int_{\Omega} w(x) (b - f(x))^s \, dv(x) \leq \left( b - \int_{\Omega} w(x) f(x) \, dv(x) \right)^s
\]
and
\[
\int_{\Omega} w(x) (f(x) - a)^s \, dv(x) \leq \left( \int_{\Omega} w(x) f(x) \, dv(x) - a \right)^s,
\]
where \(s \in (0, 1)\).
5. Some Examples

Let \( f : \Omega \to [0, \infty) \) be a \( \nu \)-measurable function and \( w : \Omega \to \mathbb{R} \) a \( \nu \)-measurable function with \( w(x) \geq 0 \) for \( \nu \)-a.e. \( x \in \Omega \) and \( \int_{\Omega} w(x) \, d\nu(x) = 1 \). We define, for the function \( \Phi(t) = t^p, \ p > 0 \), the \textit{generalized (p, f)-mean difference} \( R_G(p, f; w) \) by

\[
R_G(p, f; w) := \frac{1}{2} \int_{\Omega} \int_{\Omega} w(x) w(y) |f^p(x) - f^p(y)| \, d\nu(x) \, d\nu(y) \quad (5.1)
\]

and the \textit{generalized (p, f)-mean deviation} \( M_D(p, f; w) \) by

\[
M_D(p, f; w) := \int_{\Omega} w(x) |f^p(x) - E(p, f; w)| \, d\nu(x), \quad (5.2)
\]

where

\[
E(p, f; w) := \int_{\Omega} f^p(y) w(y) \, d\nu(y) \quad (5.3)
\]

is the \textit{generalized (p, f)-expectation}.

If \( f : \Omega \to [a, b] \subset [0, \infty) \) is a \( \nu \)-measurable function, then by (3.1) we have

\[
R_G(p, f; w) \leq \frac{1}{2} (b^p - a^p). \quad (5.4)
\]

By (3.7) we have

\[
R_G(p, f; w) \leq p \delta_p(a, b) R_G(f; w), \quad (5.5)
\]

where

\[
\delta_p(a, b) := \begin{cases} 
    b^{p-1} & \text{if } p \geq 1, \\
    a^{p-1} & \text{if } p \in (0, 1)
\end{cases}
\]

and

\[
R_G(p, f; w) \leq \frac{p}{2^{1/\alpha}} \left[ \frac{b^{\alpha(p-1)+1} - a^{\alpha(p-1)+1}}{\alpha(p-1)+1} \right]^{1/\alpha} R_G^{1/\beta}(f; w), \quad (5.6)
\]

where \( \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1 \).

From (3.20) we also have

\[
R_G(p, f; w) \leq \begin{cases} 
    \delta_p(a,b) M(f; w), \\
    p \left( \frac{b^{\alpha(p-1)+1} - a^{\alpha(p-1)+1}}{\alpha(p-1)+1} \right)^{1/\alpha} M^{1/\beta}(f; w) & \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1
\end{cases}
\]

\[
\quad \leq \begin{cases} 
    \frac{1}{2} (b-a) \delta_p(a,b), \\
    \frac{1}{2^{1/\alpha}} (b-a)^{1/\beta} p \left( \frac{b^{\alpha(p-1)+1} - a^{\alpha(p-1)+1}}{\alpha(p-1)+1} \right)^{1/\alpha} & \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1
\end{cases}
\]

\[ (5.7)\]
where $M(f; w)$ is defined by (3.21).

If $p \in (0, 1)$, then the function $|\Phi(t)| = t^p$ is concave on $[a, b] \subset [0, \infty)$ and by (4.1) we have

$$R_G(p, f; w) \leq E^p(f; w). \quad (5.8)$$

For $p \geq 1$ the function $|\Phi(t)| = t^p$ is convex on $[a, b] \subset [0, \infty)$ and by (4.2) we have

$$R_G(p, f; w) \leq \frac{1}{b - a} \left( (b - E(f; w)) a^p + (E(f; w) - a b^p) \right). \quad (5.9)$$

Let $f : \Omega \to [0, \infty)$ be a $\nu$-measurable function and $w : \Omega \to \mathbb{R}$ a $\nu$-measurable function with $w(x) \geq 0$ for $\nu$-a.e. $x \in \Omega$ and $\int_{\Omega} w(x) \, d\nu(x) = 1$. We define, for the function $\Phi(t) = \ln t$, the generalized $(\ln, f)$-mean difference $R_G(\ln, f; w)$ by

$$R_G(\ln, f; w) := \frac{1}{2} \int_{\Omega} \left( \int_{\Omega} w(x)(\ln f(x) - \ln f(y)) \, d\nu(x) \, d\nu(y) \right) \quad (5.10)$$

and the generalized $(p, f)$-mean deviation $M_G(\ln, f; w)$ by

$$M_G(\ln, f; w) := \int_{\Omega} w(x)(\ln f(x) - E(\ln, f; w)) \, d\nu(x), \quad (5.11)$$

where

$$E(\ln, f; w) := \int_{\Omega} w(y) \ln f(y) \, d\nu(y) \quad (5.12)$$

is the generalized $(\ln, f)$-expectation.

If $f : \Omega \to [a, b] \subset [0, \infty)$ is a $\nu$-measurable function, then by (3.1) we have

$$R_G(\ln, f; w) \leq \frac{1}{2} (\ln b - \ln a). \quad (5.13)$$

By (3.7) we have

$$R_G(\ln, f; w) \leq \left\{ \begin{array}{l}
\frac{1}{a} R_G(f; w), \\
\frac{1}{2^{1+p}} \left( \frac{b^{p-1} - a^{p-1}}{(p-1) b^{p-1} a^{p-1}} \right)^{1/p} R_G^{1/q}(f; w) \text{ if } p > 1, \frac{1}{p} + \frac{1}{q} = 1.
\end{array} \right. \quad (5.14)$$

By (3.20) we have

$$R_G(\ln, f; w) \leq \left\{ \begin{array}{l}
\frac{1}{a} M(f; w), \\
\left( \frac{b^{p-1} - a^{p-1}}{(p-1) b^{p-1} a^{p-1}} \right)^{1/p} M^{1/q}(f; w) \text{ if } p > 1, \frac{1}{p} + \frac{1}{q} = 1
\end{array} \right. \quad (5.15)$$

$$\leq \left\{ \begin{array}{l}
\frac{1}{2} \left( \frac{b}{a} - 1 \right), \\
\frac{1}{2^{1+q}} (b - a)^{1/q} \left( \frac{b^{p-1} - a^{p-1}}{(p-1) b^{p-1} a^{p-1}} \right)^{1/p} \text{ if } p > 1, \frac{1}{p} + \frac{1}{q} = 1.
\end{array} \right.$$
Now, observe that the function $|\Phi(t)| = |\ln t|$ is convex on $(0, 1)$ and concave on $[1, \infty)$. If $f : \Omega \to [a, b] \subset (0, 1)$ is a $\nu$-measurable function, then by \cite{12} we have
\begin{equation}
R_G(\ln f; w) \leq \frac{1}{b-a} \left[ (b - E(f;w)) |\ln a| + (E(f;w) - a) |\ln b| \right]
\end{equation}
and if $f : \Omega \to [a, b] \subset [1, \infty)$, then by \cite{11} we have
\begin{equation}
R_G(\ln f; w) \leq \ln (E(f;w)).
\end{equation}

The interested reader may state similar bounds for functions $\Phi$ such as $\Phi(t) = \exp t$, $t \in \mathbb{R}$ or $\Phi(t) = t \ln t$, $t > 0$. We omit the details.

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