Self-Organised Optimality in Driven Systems with Symmetrical Interactions

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Extremal principles are fundamental in our interpretation of phenomena in nature. One of the best known examples is the second law of thermodynamics, governing most physical and chemical systems and stating the continuous increase of entropy in closed systems. Biological and social systems, however, are usually open and characterised by self-organised structures. Being results of an evolutionary optimisation process, one may conjecture that such systems use resources like energy very efficiently, but there is no proof for this. Recent results on driven systems indicate that systems composed of competing entities tend to reach a state of self-organised optimality associated with minimal interaction or minimal dissipation, respectively. Using concepts from non-equilibrium thermodynamics and game theoretical ideas, we will show that this is universal to an even wider class of systems which, generally speaking, have the ability to reach a state of maximal overall “success”. This principle is expected to be relevant for driven systems in physics, but its main significance concerns biological and social systems, for which only a limited number of quantitative principles are available yet.
Recent simulations of driven multi-particle or multi-agent systems have revealed different kinds of self-organised states that seem to optimise certain aggregate quantities. Hence, in analogy with the concept of “self-organised criticality” for systems driving themselves to a critical state, it is natural to introduce the concept of “self-organised optimality”. As a specific example, one can consider the formation of human trail systems, where it is the discomfort of walking multiplied by the length of the individual ways that is minimised (cf. Figure 1).

Here, we will focus on the dynamics of pedestrian crowds, allowing the intuitive and clear illustration of the non-trivial mechanisms behind self-organised optimality. In crowds of oppositely moving pedestrians, usually lanes of uniform walking direction develop (cf. Figure 2). Obviously, this self-organised collective pattern of motion maximises the average speeds and minimises interactions, since pedestrians would be considerably slowed down by avoidance maneuvers, if the desired walking directions were mixed.

To prove minimal interaction, we will set up continuum equations for the considered systems, but at the same time, we will generalise the problem to arbitrary kinds of populations \( a \) that are composed of identical moving entities \( \alpha \). In the above examples, the different populations correspond to different origin-destination pairs and desired walking directions, respectively. The distribution of the \( N_a \) entities of population \( a \) over the locations \( \mathbf{x} \) of an \( n \)-dimensional space will be represented by the densities \( \rho_a(\mathbf{x}, t) \geq 0 \). The space does not need to be real. It may be an abstract one. In the case of trail formation, for example, a point \( \mathbf{x} \) in space corresponds to a path connecting origin and destination.

Assuming conservation of the number
of entities in each population $a$, we obtain the continuity equations

$$\frac{\partial \rho_a(x,t)}{\partial t} + \nabla \cdot \left[ \rho_a(x,t)V_a(x,t) \right] = 0. \quad (2)$$

Here, $V_a(x,t)$ is the average velocity of entities of population $a$. We will assume that this is given by the gradient

$$V_a(x,t) = \nabla S_a(x,t) \quad (3)$$

of some function $S_a$ of the densities $\rho_b$. The existence of such a function requires the potential condition $\partial V_{ai}(x)/\partial x_j = \partial V_{aj}(x)/\partial x_i$ to be true, but sometimes it can be relaxed. In particular, it is not fulfilled in our pedestrian model. However, assuming homogeneity in $x_1$-direction, we can reduce the problem to the investigation of the one-dimensional dynamics in $x_2$-direction, for which the potential condition is satisfied.

The first terms of a series expansion of $S_a(x,t)$ give

$$S_a(x,t) = S^0_a + \sum_b S_{ab} \rho_b(x,t), \quad (4)$$

but the constants $S^0_a$ do not matter at all. The function $S_a(x,t)$ may be interpreted as the “(expected) success” per unit time for an entity of population $a$ at location $x$, as it is plausible that the entities move into the direction of the greatest increase of success, corresponding to the gradient (3). Positive $S_{ab}$ belong to profitable or attractive interactions between populations $a$ and $b$, whereas competitive or repulsive interactions correspond to negative $S_{ab}$. If an entity of kind $a$ interacts with entities
of kind $b$ at a rate $\nu_{ab}$ and the associated result of the interaction can be quantified by some “payoff” $P_{ab}$, we have the relation $S_{ab} = \nu_{ab}P_{ab}$. However, the above equations differ from the conventional game dynamical equations\cite{12,13} in several respects: 1. We have a topology (like in the game of life)\cite{18,19} but define abstract games for interactive motion in space with the possibility of local agglomeration at a fixed number of entities in each population. 2. The payoff does not depend on the variables that the individual entities can change (i.e. the spatial coordinates $x_i$). 3. Individuals can only improve their success by redistributing themselves in space. 4. The increase of success is not proportional to the difference with respect to the global average of success, but to the local gradient of success in a population.

Now, we will proof that, for symmetric interactions with $S_{ba} = S_{ab}$, the overall success

$$S(t) = \sum_a \int d^nx \rho_a(x,t)S_a(x,t)$$

(5)

is a so-called Lyapunov function which monotonically increases in the course of time, just like some thermodynamic non-equilibrium potentials\cite{22}. Because of (1), we eventually obtain $dS(t)/dt = \sum_a \int d^nx \partial_a \rho_a(x,t)/\partial t \sum_b (S_{ab} + S_{ba})\rho_b(x,t)$. By inserting (2) and (3), and applying (4), we get $dS(t)/dt = -2\sum_a \int d^nx \nabla \cdot [\rho_a(x,t)\nabla S_a(x,t)][S_a(x,t) - S_{0a}]$. Making use of partial integration and the Gaussian integral theorem (for systems with periodic boundary conditions), we finally arrive at

$$\frac{dS(t)}{dt} = 2\sum_a \int d^nx \rho_a(x,t) [\nabla S_a(x,t)]^2 \geq 0.$$  

(6)

This result establishes self-optimisation for symmetrical interactions and can be easily transferred to discrete spaces (see Figure 3). Notice that $S(t)$ is bounded
for any finite system and that \(\text{(8)}\) looks similar to dissipation functions in thermodynamics. If we interpret the function \(-dS(t)/dt\) as a measure of dissipation per unit time in the system, Eq. \(\text{(3)}\) immediately implies that the system approaches a state of minimal dissipation.

According to \(\text{(8)}\), the stationary solution \(\rho_{a}^{\text{st}}(x)\) is characterised by

\[
\rho_{a}^{\text{st}}(x) = 0 \quad \text{or} \quad \nabla S_{a}^{\text{st}}(x) = \sum_{b} S_{ab} \nabla \rho_{b}^{\text{st}}(x) = 0
\]  

(7)

for all \(a\), which is fulfilled by homogeneous or step-wise constant solutions. For the case of two species, a linear stability analysis shows that the homogeneous solution (cf. Figure 3(A)) is unstable if

\[
\rho_{a}^{\text{hom}} S_{aa} + \rho_{b}^{\text{hom}} S_{bb} > 0 \quad \text{or} \quad S_{ab} S_{ba} > S_{aa} S_{bb},
\]  

(8)

where \(\rho_{a}^{\text{hom}} = N_{a}/V\) denotes the homogeneous density and \(V\) the volume of the system. The latter condition in \(\text{(8)}\) is, for example, fulfilled for lane formation by pedestrians, since their interaction rate is proportional to their relative velocity, which is much higher for oppositely moving pedestrians than for pedestrians with the same desired walking direction.

Under the condition \(\text{(8)}\), the stable stationary solution corresponds to complete segregation which, together with \(\text{(5)}\), implies that the system reaches a state of self-organised optimality. Notice that the global optimum is reached by means of short-range interactions \(\text{(3)}\). However, the regions occupied by one population need not be connected (cf. Figure 3(B)). If \(S_{aa} < 0\) for all \(a\), the distributions \(\rho_{a}^{\text{st}}(x)\) tend to be flat, as in the case of lane formation by repulsive pedestrian interactions (Figure 3(B)). Instead, we have agglomeration (locally peaked stationary solutions),
if $S_{aa} > 0$ for all $a$ (Figures 3(C), (D)). The example of trail formation, which is based on attractive interactions between trails, corresponds to Figure 3(C).

Finally, we discuss the influence of noise. Adding diffusion terms $\nabla \cdot [D_a \nabla \rho_a(x, t)]$ to the right-hand side of (2), the relation (4) will not be exactly valid anymore. Thus, optimality will be affected. Moreover, the condition (8) for segregation will become $\rho_{a}^{\text{hom}} S_{aa} + \rho_{b}^{\text{hom}} S_{bb} > 0$ or $(S_{ab} S_{ba} - S_{aa} S_{bb}) \rho_a^{\text{hom}} \rho_b^{\text{hom}} > D_a D_b$. Hence, growing diffusion coefficients will produce more homogeneous equilibrium states, which agrees with intuition.

We call attention to the fact that there is a class of living systems (for which we have given some realistic examples) to which existing methods, notions, and principles of statistical mechanics can be successfully applied. In particular, we have proven that systems, which can be represented as a game between symmetrically interacting populations, approach a stationary state characterised by maximal overall success and minimal dissipation. In other words, as individual entities are trying to optimise their own success, a class of systems tends to reach a state with the highest global success. This non-trivial result is valid only under the conditions discussed above. In contrast to self-organised optimality, self-optimisation without a self-organised state occurs for symmetrical interactions, if condition (8) is not fulfilled (Figure 3(A)). There are also cases of self-organisation without optimality, if (8) is satisfied, but the interactions are not symmetrical. Such a system is exemplified by uni-directional multi-lane traffic of cars and lorries, but even there one observes significantly reduced interactions (lane-changing rates). Furthermore, we point out that the above minimal dissipation principle may be relevant to physical systems like driven granular media, but the most important implications of this generalised thermodynamic concept are expected for biological, social, and economic systems.
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FIG. 1. Schematic representation of the human trail system evolving on an initially homogeneous ground. When the frequency of trail usage is small, the minimal way system (that would connect the four entry points and destinations in the corners by direct ways) cannot be supported in competition with the regeneration of the vegetation. Here, by bundling of trails, the frequency of usage becomes large enough to support the depicted trail system. It corresponds to the optimal compromise between the diagonal ways and the ways along the edges, supplying optimal walking comfort at a minimal detour of 22% for everyone, which is a fair solution.
FIG. 2. Formation of lanes of uniform walking directions in crowds of oppositely moving pedestrians. Blue circles represent pedestrians walking from left to right, red ones represent pedestrians walking into the opposite direction. If $x_\alpha(t)$ denotes the position of pedestrian $\alpha$ at time $t$, $v_\alpha(t) = dx_\alpha(t)/dt$ its velocity, $v_0$ the desired speed, and $e_\alpha$ the desired walking direction, our simplified pedestrian model reads $v_\alpha(t) = v_0 e_\alpha + \sum_{\beta(\neq \alpha)} f_{\alpha\beta}(x_\alpha(t), x_\beta(t))$. Here, $f_{\alpha\beta}$ represents repulsive interactions between pedestrians $\alpha$ and $\beta$, which were assumed to decrease monotonically with their distance $d_{\alpha\beta}(t) = \|x_\alpha(t) - x_\beta(t)\|$. For simplicity, we have specified the interactions as a gradient of a rotationsymmetric potential that depends only on $d_{\alpha\beta}$. Notice that lane formation is not a trivial effect of this model, but it eventually arises due the smaller relative velocity and interaction rate that pedestrians with the same walking direction have. It is clear that lane formation will increase the average velocity in walking direction $E(t) = \langle \langle v_\alpha \cdot e_\alpha \rangle_\alpha \rangle_t/v_0 \leq 1$, which is a measure of “efficiency” or “success”. (Here, $\langle \langle \cdot \rangle_\alpha \rangle_t$ denotes the average over the pedestrians and over time.) Moreover, maximisation of efficiency immediately implies that the system minimises the quantity $\langle \langle -\sum_{\beta(\neq \alpha)} f_{\alpha\beta} \cdot e_\alpha \rangle_\alpha \rangle_t = v_0 - \langle \langle v_\alpha \cdot e_\alpha \rangle_\alpha \rangle_t = v_0(1 - E)$, i.e., the average interaction against the desired direction of motion. (The average interaction perpendicular to it is zero.) For dissipative interactions, minimal dissipation is a direct consequence of minimal interactions.
FIG. 3. Illustration of the various forms of self-optimisation for two different populations: (A) Homogeneous distribution in space, (B) segregation of populations without agglomeration, (C) attractive agglomeration, (D) repulsive agglomeration. In cases (B) to (D), the finally evolving optimal state is related with a self-organised, non-homogeneous state, which corresponds to “self-organised optimality”. The above figures were obtained with a one-dimensional, discrete version of the game-dynamical model defined by equations (2) to (4). We assumed a periodic lattice with $V$ lattice sites $x \in \{1, \ldots, V\}$ and two populations $a \in \{1, 2\}$ with a total of $N = N_1 + N_2 \gg V$ entities (here: $V = 40$ and $N_1 = N_2 = 200$). Furthermore, we applied the following update steps: 1. Calculate the successes $S_a(x, t) = S_a^0 + \sum_b S_{ab} n_b^a(t)/V$, where $n_b^a(t) = \rho_b(x, t)V$ represents the number of entities of population $b$ at site $x$. 2. For each entity $\alpha$, determine a random number $\xi_\alpha$ that is uniformly distributed in the interval $[0, S_{\text{max}}]$ with a large constant $S_{\text{max}}$ (here: $S_{\text{max}} = 20$). 3. Move entity $\alpha$ belonging to population $a$ from site $x$ to site $x+1$, if $[S_a(x+1, t) - S_a(x-1, t)] > \xi_\alpha$, but to site $x-1$, if $[S_a(x-1, t) - S_a(x+1, t)] > \xi_\alpha$. Our simulations started with a random initial distribution of the entities. We applied a random sequential update rule, but a parallel update yields qualitatively the same results. The above figures show the numbers $n_x^1$ and $n_x = (n_x^1 + n_x^2)$ of entities as a function of the lattice site $x$ at time $t = 4000$ and the evolution of the overall success $S(t)$ as a function of time $t$. The fluctuations around the monotonic increase of $S(t)$ are caused by the fluctuations $\xi_\alpha$ and the random sequential update.