Color Superconductivity in Dense QCD and Structure of Cooper Pairs∗

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The two-flavor color superconductivity is examined over a wide range of baryon density with a single model. To study the structural change of Cooper pairs, quark correlation in the color superconductor is calculated both in the momentum space and in the coordinate space. At extremely high baryon density (\(\sim O(10^3 \rho_0)\)), our model becomes equivalent to the usual perturbative QCD treatment and the gap is shown to have a sharp peak near the Fermi surface due to the weak-coupling nature of QCD. On the other hand, the gap is a smooth function of the momentum at lower densities (\(\sim O(10\rho_0)\)) due to strong color magnetic and electric interactions. The size of the Cooper pair is shown to become comparable to the averaged inter-quark distance at low densities, which indicates a crossover from BCS to BEC (Bose-Einstein condensation) of tightly bound Cooper pairs may take place at low density.

I. INTRODUCTION

Because of the asymptotic freedom and the Debye screening in QCD, deconfined quark matter is expected to be realized for baryon densities much larger than the normal nuclear matter density \(\rho_0\). This weak-coupling fermionic matter is, however, qualitatively different from a perturbed free Fermi gas system, if the temperature is low enough. This is because any attractive quark-quark interaction in the cold quark matter causes an instability of the Fermi surface due to the formation of Cooper pairs and leads to the color superconducting phase [2, 3, 4].

Current understanding on the color superconductivity has been based on two different theoretical approaches. One is an analysis by the Schwinger-Dyson equation with perturbative one gluon exchange, which is valid in weak-coupling limit at asymptotically high densities [5]. The dominant contribution to the formation of Cooper pairs comes from the collinear scattering through the long range magnetic gluon exchange [6]. In such a weak-coupling regime, formation of Cooper pairs takes place only in a small region near the Fermi surface. The other approach is the mean-field approximation with the QCD inspired 4-Fermi model introduced to study lower density regions [3, 7]. In this approach, magnitude of the gap becomes as large as 100 MeV and is almost constant in the vicinity of Fermi surface.

The main purpose of this talk is to discuss the superconducting gap over a wide range of baryon density with a single model and to make a bridge between high and low density regimes [8]. To this aim, we make an extensive analysis on the structural change in spatial-momentum dependence of a Cooper pair in 2-flavor color superconductivity at zero temperature. The momentum dependence of the gap, diffuseness of the Fermi surface, quark-quark correlations in the superconductor are the characteristic quantities reflecting the departure from the weak-coupling picture. In particular, we show that the spatial size of Cooper pairs which is calculated from the quark-quark correlations, indicates a clear deviation from the weak-coupling BCS theory. More complete and detailed analysis was done in Ref. [8].

* Invited talk at the Joint CSSM/JHF Workshop on Physics at Japan Hadron Facility (March 14-21, Adelaide, 2002).
II. GAP EQUATION WITH SPATIAL MOMENTUM DEPENDENCE

Using the standard Nambu-Gor’kov formalism with a two component Dirac spinor $\Psi = (\psi, \bar{\psi}^\dagger)$, the quark self-energy $\Sigma(k_\rho)$ with the Minkowski 4-momentum $k_\rho$ satisfies

$$
\Sigma(k_\rho) = -i \int \frac{d^4q}{(2\pi)^4} g^2 \Gamma^\rho_\mu S(q_\rho) \Gamma^\alpha_\mu D_{\mu\nu}^a(q_\rho - k_\rho),
$$

(1)

where $D_{\mu\nu}^a$ is the gluon propagator in medium, $S(q_\rho)$ is the full quark propagator, and $\Gamma^\rho_\mu$ is the quark-gluon vertex. We study the gap function in the flavor anti-symmetric, color anti-symmetric and $J = 0^+$ channel (the most attractive channel within the one-gluon exchange model) [12]:

$$
\Delta(k_\rho) = (\lambda_2 \tau_2 C\gamma_5) \left( \Delta_+(k_\rho) \Lambda_+(\hat{k}) + \Delta_-(k_\rho) \Lambda_-(\hat{k}) \right),
$$

(2)

where $\tau_2$ is the Pauli matrix acting on the flavor space, $\lambda_2$ is a color anti-symmetric Gell-Mann matrix, and $C$ is the charge conjugation. $\Lambda_\pm(k) \equiv (1 \pm \hat{k} \cdot \alpha)/2$ is the projector on positive (+) and negative (−) energy quarks.

For the vertex in Eq. (1), we use $\Gamma^\rho_\mu = \text{diag.}(\gamma_\rho \lambda^a/2, - (\gamma_\rho \lambda^a/2)^T)$. For $g^2$ in Eq. (1), we use a momentum dependent coupling $g^2(q, k)$ in the “improved ladder approximation” [3] ($\beta_0 = (11N_c - 2N_f)/3 = 29/3$):

$$
g^2(q,k) = \frac{16\pi^2}{\beta_0} \ln((p_{\text{max}}^2 + p_c^2)/\Lambda^2), \quad p_{\text{max}} = \max(q, k),
$$

(3)

where $p_c^2$ plays a role of a phenomenological infrared regulator which prevents the coupling constant from being too large at low momentum $q, k \sim \Lambda_{\text{QCD}}$. At high momentum, $g^2$ shows the same logarithmic behavior as the usual running coupling with $\Lambda$ identified with $\Lambda_{\text{QCD}}$. The quark propagator in the improved ladder approximation in the vacuum is known to have a high momentum behavior consistent with that expected from the renormalization group and the operator product expansion. We adopt $\Lambda = 400$ MeV and $p_c^2 = 1.5 \Lambda^2$ which are determined to reproduce the low energy meson properties for $N_f = 2$ vacuum [10].

The gluon propagator in Eq. (1) in the Landau gauge reads,

$$
D_{\mu\nu}(k_\rho; k_0 < |k|) = - \frac{P^T_{\mu\nu}}{k^2 + iM^2|k_0|/|k|} - \frac{P^L_{\mu\nu}}{k^2 + m_D^2},
$$

(4)

where $P^T_{\mu\nu}$ are the transverse and longitudinal projectors. The longitudinal part of the propagator (the electric part) has a static screening by the Debye mass $m_D^2 = (N_f/2\pi^2)g^2\mu^2$, while the transverse part (the magnetic part) has dynamical screening by the Landau damping $M^2 = (\pi/4)m_D^2$. This form is a quasi-static approximation of the full gluon propagator in the sense that only the leading frequency dependence is considered [11].

A. Full gap equation

The gap equation under the approximations shown above is obtained from the 2-1 element of the Schwinger-Dyson (matrix) equation [11] which reads

$$
\Delta_\pm(k) = \int_0^\infty dq \, V_\pm(q,k; \epsilon_+^q, \epsilon_+^k) \frac{\Delta_+(q)}{\sqrt{E_+(q)^2 + |\Delta_+(q)|^2}}
$$

$$
+ \int_0^\infty dq \, V_\mp(q,k; \epsilon_-^q, \epsilon_-^k) \frac{\Delta_-(q)}{\sqrt{E_-(q)^2 + |\Delta_-(q)|^2}}.
$$

(5)

Here we used a simplified notation: $\Delta_\pm(\epsilon_+^k, k) \rightarrow \Delta_\pm(k)$ with $E_\pm(q) \equiv q \mp \mu$ and $\epsilon_\pm^k$ being the quasi-particle energy as a solution of $(\epsilon_\pm^k)^2 = E_\pm^2(q) + \Delta_\pm^2(\epsilon_\pm^q, q)$. The explicit form of $V_\pm$ is given in Ref. [3].

At extremely high density, the Cooper pairing is expected to take place only near the Fermi surface due to weak coupling property. In this case, we can safely neglect the antiquark-pole contribution for calculating $\Delta_+$. Furthermore, one may replace the momentum dependent coupling constant [12] by that on the Fermi surface $g^2(\mu, \mu)$. On the other hand, at low densities, sizable diffusion of the Fermi surface occurs and the weak-coupling approximation is not justified, and we need to solve the coupled gap equations Eq. (1) numerically.
B. Occupation number, correlation function and coherence length

To clarify the structural change of the color superconductor from high to low densities, it is useful to examine the following physical quantities.

[I] The quark and antiquark occupation number which is related to the diagonal (1-1) element of the quark propagator, $\langle \psi_{j}^{1\mu}(t, y)\bar{\psi}_{j}^{\mu}(t, x)\rangle_{\text{super}}$, in the Nambu-Gor'kov formalism:

$$n_{\pm}^{1,2}(q) = \frac{1}{2} \left( 1 - \frac{E_{\pm}(q)}{\sqrt{E_{\pm}(q)^2 + |\Delta_{\pm}(q)|^2}} \right), \quad n_{\pm}^{3}(q) = \theta(\mu - q), \quad n_{\pm}^{3}(q) = 0,$$

where the superscripts (1, 2 and 3) stand for color indices. Since the third axis in the color space is chosen to break the color symmetry, quarks with the third color remains ungapped.

[III] The quark and antiquark correlation functions in momentum space $\tilde{\varphi}_{\pm}(q)$ and in coordinate space $\varphi_{\pm}(r)$: They reflect the internal structure of the Cooper pairs in color superconductor. These correlations are defined through the off-diagonal (1-2) element of the quark propagator, $\langle \psi_{j}^{1\mu}(t, x)\bar{\psi}_{j}^{\mu}(t, y)\rangle_{\text{super}}$:

$$\tilde{\varphi}_{\pm}(q) = \frac{\Delta_{\pm}(q)}{2\sqrt{E_{\pm}(q)^2 + |\Delta_{\pm}(q)|^2}}, \quad \varphi_{\pm}(r) = N \int d^3q \frac{e^{iqr}}{(2\pi)^3} \varphi_{\pm}(q),$$

where $N$ is a normalization constant determined by $\int d^3r |\varphi_{\pm}(r)|^2 = 1$.

The coherence length $\xi_{c}$ characterizing the typical size of a Cooper pair: It is defined simply as a root mean square radius of $\varphi_{\pm}(r)$:

$$\xi_{c}^2 = \frac{\int d^3r \varphi_{\pm}^2(r)}{\int d^3r |\varphi_{\pm}(r)|^2} = \frac{\int_{0}^{\infty} dk k^2 |d\varphi_{\pm}(k)/dk|^2}{\int_{0}^{\infty} dk k^2 |\varphi_{\pm}(k)|^2}.$$

It can be shown that the quark correlation $\varphi_{\pm}(r)$ in the weak-coupling limit behaves as

$$\varphi_{\pm}(r \to \infty) \propto \sin(\mu r) \left( \frac{\mu r}{(\mu r)^{3/2}} \right) \cdot e^{-r/(\pi \xi_{p})},$$

where the Pippard length is given by $\xi_{p} = (\pi \Delta_{\pm}(\mu))^{-1}$.

In a typical type-I superconductor in metals, the Pippard length is of the semi-macroscopic order $\xi_{p} \sim 10^{-4}$ cm, whereas inverse Fermi momentum is of the microscopic order $k_{F}^{-1} \sim 10^{-8}$ cm. Besides, there is another scale $\omega_{D}$, the Debye cutoff, which limits the range of attractive interactions to be just around the Fermi surface $|k - k_{F}| < \omega_{D}$. The inverse of the Debye cutoff is in between the two scales above: $\omega_{D}^{-1} \sim 10^{-6}$ cm. Therefore there is a clear scale hierarchy, $\Delta \ll \omega_{D} \ll k_{F}$. On the other hand, since there is no intrinsic scale $\omega_{D}$ in QCD, scale hierarchy at extremely high density simply reads $\Delta \sim \mu e^{-c/\mu} \ll k_{F} \sim \mu$. At lower densities, however, such scale separation becomes questionable for $g$ is not small.

III. MOMENTUM DEPENDENT GAP FROM HIGH TO LOW DENSITIES

In Fig. 1(a), we show the gap $\Delta_{\pm}(k)$ as a solution of the full gap equation for a wide range of densities. Since the actual position of the Fermi surface moves as we vary the density, we use $k/\mu$ as a horizontal axis in the figure which helps us to understand the change of global behavior. The figure shows that the sharp peak at high density gradually gets broadened and simultaneously the magnitude of the gap increases as we decrease the density.

The characteristic features at high density are (i) there is a sharp peak at the Fermi surface, and (ii) the gap decays rapidly but is nonzero for momentum far away from the Fermi surface. The property (i) is similar to the standard BCS superconductivity but (ii) is not, due to the absence of intrinsic ultraviolet Debye-cutoff of the gluonic interaction in QCD. As for the magnitude of the gap at high density, if one estimates the several contributions to the kernel $V_{c}$ in the gap equation separately, one finds that the color-electric interaction enhances the gap considerably. This may be understood as follows: In the coordinate space, the Debye-screened electric interaction is a Yukawa potential. Such a short-range interaction alone can form only a loosely bound Cooper pairs and a very small gap. However, if the magnetic and electric interactions coexist, small size Cooper pairs are formed primarily by the long-range magnetic interaction. Then, even the short-range electric interaction becomes effective to generate further attraction between the quarks.
The characteristic features at low density are that (iii) the sharp peak at the Fermi surface disappears, and (iv) all the contributions neglected in the weak-coupling limit are actually not negligible. A close look at the effects of each contributions to the gap leads us to a conclusion that the color superconductivity at low density is not a phenomenon just around the Fermi surface. This is confirmed by computing the occupation number following eq. (6). The result shows that the Fermi surface is diffuse substantially at low density.

In Fig. 1(b), the gap at the Fermi surface $\Delta_+(\mu)$ is shown as a function of the chemical potential. It decreases monotonically as $\mu$ increases as is shown on the figure, but turns into an increase at much higher density $\mu > 10^6$ MeV. An analytic solution based on the weak coupling approximation (includes only the magnetic gluons, fixed coupling, and ignores the anti-quark pole contribution) is also shown in Fig. 1(b). The magnitude of the analytic solution is normalized to the numerical solution at the highest density $\mu = 2^{12}\Lambda \approx 1.6 \times 10^6$ MeV. At high density, $\mu$-dependence of the numerical result is in good agreement with the analytic form which has a parametric dependence $\Delta_+(\mu) \propto g^{-5/2} \mu \exp(-3\pi^2/\sqrt{2g})$ with $g^2 = g^2(\mu, \mu)$. On the other hand, the difference of the two curves at low density implies the failure of the weak-coupling approximation.

### IV. CORRELATION FUNCTION AND COHERENCE LENGTH

One of the advantages of treating the momentum dependent gap is that we are able to calculate the correlation function which physically corresponds to the “wavefunction” of the Cooper pair. Such correlations have been first studied in Ref. [12] in the context of the color superconductivity (but with much simpler model than eq. (5), and only at lower density).

Figure 2(a) shows the coherence length $\xi_c$ of a quark Cooper pair defined as the root mean square radius of the correlation function [see Eq. (8)]. The size of a Cooper pair becomes smaller as we go to lower densities. This tendency is understood by the behavior of the Pippard length $\xi_p = 1/\pi \Delta_+(\mu)$ (which gives a rough estimate of the coherence length) together with the behavior of $\Delta_+(\mu)$ shown in Fig. 1(b).

Also, the Cooper pair becomes smaller as density increases beyond $\mu = 2^{12}\Lambda$. However, it does not necessarily imply the existence of tightly bound Cooper pairs. In fact, the size of a Cooper pair makes sense only in comparison to the typical length scale of the system, namely the averaged inter-quark distance $d_q$ for free quarks defined as

$$d_q = \left( \frac{\pi^2}{2} \right)^{1/3} \frac{1}{\mu}.$$  

As we go to higher densities, the ratio $\xi_c/d_q$ increases monotonically as shown in Fig. 2(b). Namely loosely bound
Cooper pairs similar to the BCS superconductivity in metals are formed at extremely high densities. (Recall that the typical ratio in superconductivity is $k_F/\Delta \sim 10^4$)

At the lowest density in Fig. 2, the size of the Cooper pair is less than 4 fm and the ratio $\xi_c/d_q$ is less than 10. This is not similar to the usual BCS system. The transition from $\xi_c/d_q \gg 1$ to $\xi_c/d_q \sim 1$ as $\mu$ decreases is analogous to the crossover from the BCS-type superconductor to the Bose-Einstein condensation (BEC) of tightly bound Cooper pairs [13]. Our result here suggests that the quark matter possibly realized in the core of neutron stars may be rather like the BEC of tightly bound Cooper pairs.

For better understanding of the internal structure of the quark Cooper pair, let us consider the correlation function in the coordinate space. Fig. 3 shows the spatial correlation of a Cooper pair at various chemical potentials normalized as $\int d^3r |\varphi_+(r)|^2 = 1$.

At high density, most of the quarks participating in forming a Cooper pair have the Fermi momentum $k_F = \mu$ giving a sharp peak in the momentum space correlation. In the coordinate space, this corresponds to an oscillatory distribution with a wavelength $\lambda = 1/\mu$ without much structure near the origin. (The oscillation is also evident
from the factor \( \sin(\mu r) \) in the approximate correlation function Eq. (3). At lower densities, accumulation of the correlation near the origin in the coordinate space is much more prominent in Fig. 3. This implies a localized Cooper pair composed of quarks with various momentum.

V. SUMMARY AND DISCUSSION

We have studied the spatial-momentum dependence of a superconducting gap and the structure of the Cooper pairs in two-flavor color superconductivity, using a single model for a very wide region of density. Nontrivial momentum dependence of the gap manifests itself at low densities, where relatively large QCD coupling allows the Cooper pairing to take place in a wide region around the Fermi surface. Note that our results can be easily extended to \( N_f = 3 \) case. Our results imply that the quark matter which might exist in the core of neutron stars or in the quark stars could be rather different from that expected from the weak-coupling BCS picture and could be more like a BEC of tightly bound Cooper pairs. Study of the finite \( T \) phase transition of this strong coupling system is currently under investigation.

Acknowledgments

T.H. would like to thank A. W. Thomas, A. G. Williams, D. Leinweber and the members of the local organizing committee for giving him an opportunity to give a talk at the Joint CSSM/JHF Workshop on Physics at Japan Hadron Facility (March 14-21, Adelaide, 2002). This research was supported in part by the National Science Foundation under Grant No. PHY99-07949.