The Petrov type D isolated null surfaces

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Abstract

Generic black holes in vacuum-de Sitter/anti-de Sitter spacetimes are studied in quasi-local framework, where the relevant properties are captured in the intrinsic geometry of the null surface (the horizon). Imposing the quasi-local notion of stationarity (null symmetry of the metric up to second order at the horizon only) we perform the complete classification of all the so-called special Petrov types of these surfaces defined by the properties (structure of principal null direction) of the Weyl tensor at the surface. The only possible types are: II, D and O. In particular, all the geometries of type O are identified. The condition distinguishing type D horizons, taking the form of a second order differential equation on certain complex invariant constructed from the Gaussian curvature and the rotation scalar, is shown to be an integrability condition for the so-called near horizon geometry equation. The emergence of the near horizon geometry in this context is equivalent to the hypersurface orthogonality of both double principal null directions. We further formulate a no-hair theorem for the Petrov type D axisymmetric null surfaces of topologically spherical sections, showing that the space of solutions is uniquely parametrized by the horizon area and angular momentum.

Keywords: isolated null surfaces, Weyl tensor, Petrov type D, no-hair theorem, black holes

1. Introduction

The theory of isolated null surfaces is a part of the local approach to black holes (BH) [1–3]. Intrinsic geometry of those surfaces, that is the induced metric and the induced covariant derivative, has physically relevant features of the geometry of stationary BH horizons. The theory is applicable to cosmological horizons and to the null boundaries of the conformally
compactified asymptotically flat spacetimes [4]. Moreover, it may be applied to the black hole holograph construction of spacetimes about Killing horizons [5, 6]. There are analogies between the properties of isolated null surfaces and the properties of BHs. Isolated null surfaces admit their mechanics, an analog of the BH “thermodynamics” [2], the rigidity theorem [7], and the uniqueness theorems [8, 9]. The key difference between the BHs and the isolated null surfaces lies in the degrees of freedom. While the families of stationary BH solutions are finite dimensional, the intrinsic geometry of isolated null surfaces has local degrees of freedom. That makes their theory far more general. The current paper develops the observation that at each non-extremal isolated null surface the spacetime Weyl tensor can be determined by the intrinsic geometry via the Einstein equations and some stationarity assumption. We find all the possible Petrov types of isolated null surface and focus on the case of the Petrov type D. We derive and investigate the Petrov type D equation on data defined on a 2-slice of an isolated null surface: a Riemannian metric $g_{AB}$ and a co-vector $\omega^A$ modulo gradient. Moreover, we find all the Petrov type O geometries. We show, that the Petrov type D equation is also an integrability condition for the so-called near horizon geometry equation [10], first discovered and studied as the extremal isolated horizon equation [1, 8]. We find the geometric consequences of the emergence of the near horizon geometry equation in that new context. It is non-twisting of the double principal null directions of the Weyl tensor. We also characterize type D isolated null surfaces that have spherical section and are axisymmetric. A detailed derivation of those solutions is contained in the accompanying paper [11]. The result is a generalization of the earlier geometric result [9] to the case of non-zero cosmological constant.

Physically, the results presented in this paper are relevant for the issue of the local, geometric characterization of the horizons contained in the Kerr–(anti) de Sitter spacetimes. They answer the question of what local properties single out the Kerr–(anti) de Sitter horizons in a much larger class of all the isolated null surfaces. The characterization of the axisymmetric solutions we formulate can be considered as a local no hair theorem.

Geometrically, the Petrov type D equation is a new equation of mathematical physics, that seems to lead to new interesting structures that can be considered in a 2D Riemannian geometry. Also, as an integrability condition for the near horizon geometry equation, the Petrov type D equation may be useful for solving the problem of dimensionality of the space of solutions to the near horizon geometry equation.

2. Isolated null surfaces

2.1. Notation and convention

Throughout this paper we consider a 4D spacetime that consist of a manifold $M$ and a metric tensor $g_{\mu\nu}$ of the signature $-+++$. By $\nabla_\mu$ we denote the torsion free covariant derivative in $M$, corresponding to $g_{\mu\nu}$ via

$$\nabla_\alpha g_{\mu\nu} = 0.$$  

About the metric tensor we assume the vacuum Einstein equations with a cosmological constant $\Lambda$,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0,$$

where $G_{\mu\nu}$ is the Einstein tensor.

In $M$ we study a 3D null surface

$$H \subset M.$$
We assume that \( H \) contains a slice that intersects each null curve in \( H \) exactly once. In other words, \( H \) has the topology

\[
H = S \times \mathbb{R},
\]

where \( S \) is the 2D space of the null curves in \( H \) assumed to be connected.

Throughout this paper we use the following (abstract) index notation [12]:

- Indices of the spacetime tensors are denoted by lower Greek letters: \( \alpha, \beta, \gamma, \ldots = 1, 2, 3, 4 \)
- Tensors defined in 3D space \( H \) carry indexes denoted by lower Latin letters: \( a, b, c, \ldots = 1, 2, 3 \)
- Capital Latin letters \( A, B, C, \ldots = 1, 2 \) are used as the indexes of tensors defined in \( S \).

Non-expanding and isolated null surfaces in four dimensions can be dealt with by either the covariant framework or using the Newman–Penrose (NP) formalism [13]. Both of the approaches are used in [1]: the covariant one in the main part, and the NP one in appendix B of that work. In the current paper we use the covariant framework as an official language. Of course, in some derivations we apply adapted null frames. Then, locally in the paper, we introduce necessary elements of the NP formalism explicitly on a current basis. Our notation is consistent with that of [1] and [13], except that as in [1], the vectors \( k, \ell \) introduced in [13] correspond to vectors \( \ell, n \) in our work.

2.2. Non-expanding shear-free null surfaces

The spacetime metric tensor \( g_{\mu\nu} \) induces a degenerate metric tensor \( g_{ab} \) in the null surface \( H \). The degeneracy means, that at every point \( x \in H \) there is a vector \( \ell \in T_xH \) such that

\[
\ell^a g_{ab} = 0.
\]

The distribution of the degenerate direction is a 1D sub-bundle \( L \subset T(H) \). Every section \( \ell \in \Gamma(L) \) of \( L \) is a non-trivial vector field tangent to \( H \) and orthogonal to \( H \), including itself, at every point. The integral curves tangent to \( L \) are null at every point, they foliate \( H \) and each of them is geodesic in the spacetime \( M \). We refer to them as null generators of \( H \).

**Assumption 0.** For every pair \( X \) and \( Y \) of vector fields tangent to \( H \), the spacetime vector field \( X^\alpha \nabla_\alpha Y^\mu \) is also tangent to \( H \),

\[
X, Y \in \Gamma(T(H)) \Rightarrow \nabla X Y \in \Gamma(T(H)).
\]

In other words, the spacetime covariant derivative \( \nabla_\mu \) preserves the tangent bundle \( T(H) \) and endows \( H \) with a covariant derivative \( \nabla_a \) via the restriction.

An equivalent assumption would be the vanishing of the extrinsic curvature of \( H \), that is

\[
X^\alpha Y^\beta \nabla_\alpha \ell_\beta = 0
\]

for every pair \( X, Y \in \Gamma(T(H)) \) and every \( \ell \in \Gamma(L) \).

**Definition 0.** The pair \( (g_{ab}, \nabla_a) \) is called the intrinsic geometry of \( H \).

The derivative \( \nabla_a \) in \( H \) is torsion free and satisfies the pseudo metricity condition

\[
\nabla_c g_{ab} = 0.
\]

It follows from those properties of \( \nabla_a \) and from the degeneracy of \( g_{ab} \) that for every \( \ell \in \Gamma(L) \), the Lie derivative of \( g_{ab} \) vanishes [14].
\[ \mathcal{L}_\ell g_{ab} = 0. \]  \hspace{1cm} (7)

That property could be yet another equivalent version of assumption 0.

If \( g_{ab} \) were a degenerate metric tensor in a general null surface, we would decompose \( \mathcal{L}_\ell g_{ab} \) uniquely into the expansion \( \theta \) and shear \( \sigma_{ab} \)

\[ \frac{1}{2} \mathcal{L}_\ell g_{ab} = \frac{1}{2} \theta g_{ab} + \sigma_{ab}, \]  \hspace{1cm} (8)

where \( \sigma_{ab} \) is traceless in each of the 2D fibers of \( T(H)/L \). Hence, our surface \( H \) (7) is non-expanding and shear-free (NE-SF). The opposite is also true: if a null surface is non-expanding and shear-free then the spacetime covariant derivative \( \nabla_\alpha \) preserves the bundle \( T(H) \) [1].

**Remark.** We have assumed that surface \( H \) is both, non-expanding and shear-free. However, whenever the equation (1) are satisfied, the vanishing of the expansion \( \theta \) implies the vanishing of the shear \( \sigma_{ab} \) (8). This is a standard application of the Raychaudhury equation that is satisfied by a geodesic \( \ell \in \Gamma(L) \) (that is such that \( \ell^a \nabla_a \ell = 0 \)), namely

\[ F^a(\theta) = -\frac{1}{2} \theta^2 - ||\sigma||^2 - R_{\mu\nu\ell\ell}^{\mu\nu}. \]  \hspace{1cm} (9)

The second term is the norm of the shear in the 2D space of vectors orthogonal to \( L \) modulo \( L \), and the third vanishes owing to the vacuum Einstein equation (1) [1]. For that reason "non-expanding" sufficiently well characterizes the properties of \( H \).

### 2.3. Ingredients of NE-SF null surface geometry \( (g_{ab}, \nabla_a) \)

We continue the discussion of the intrinsic geometry \( (g_{ab}, \nabla_a) \) of a non-expanding and shear-free null surface \( H \). Now we will use the natural projection

\[ \Pi : H \to S \]  \hspace{1cm} (10)

onto the space \( S \) of the null curves in \( H \). It follows from (3) and (7) that on \( S \) there is a uniquely defined Riemannian metric tensor \( g_{AB} \), such that the degenerate metric \( g_{ab} \) is the pullback of \( \Pi^* g_{AB} \).

\[ g_{ab} = \Pi^*_{\alpha \beta} \Lambda^B_\alpha \Lambda^A_\beta g_{AB}. \]  \hspace{1cm} (11)

In this way, the degenerate metric tensor \( g_{ab} \) in \( H \) is all determined by the Riemannian metric \( g_{AB} \) defined on the 2D manifold \( S \). The metric \( g_{AB} \) determines the unique torsion free covariant derivative \( \nabla_A \) in \( S \) that satisfies the metricity condition

\[ \nabla_A g_{BC} = 0. \]

The action of the covariant derivative \( \nabla_a \) in \( H \) on a co-vector field \( W \in \Gamma(T^*(H)) \) orthogonal to the fibers of \( L \), that is such that for every \( \ell \in \Gamma(L) \)

\[ \ell^a W_a = 0, \]

is determined by \( g_{ab} \) [4]. Specifically

\[ \nabla_a W_b = \partial_a W_b + \frac{1}{2} \mathcal{L}_{\hat{W}} g_{ab}, \]

where \( \hat{W}^a \) is any vector field on \( H \) such that

\[ \hat{W}^a g_{ab} = W_b. \]
We can think of that part of $\nabla_a$ as coinciding (vaguely speaking) with the covariant derivative $\nabla_A$ on $S$. The remaining part of $\nabla_a$ is independent of $g_{ab}$. It can be defined by introducing on $H$ any co-vector field $n_a \in \Gamma(T^*(H))$ that is nowhere orthogonal to $L$, and considering the tensor

$$S_{ab} := \nabla_a n_b.$$  \hfill (12)

We may choose

$$n_a := -\nabla_a \nu$$

and a function $\nu : H \to \mathbb{R}$ adapted to a given non-vanishing vector field $\ell \in \Gamma(L)$, such that

$$\ell^a \partial_a \nu = 1.$$  

Then,

$$S_{ab} = S_{(ab)},$$

and

$$\ell^a n_a = -1, \quad \text{and} \quad \mathcal{L}_{\ell} n_a = 0.$$  

The one form

$$\omega_a^{(\ell)} := \ell^b S_{ab}$$

depends only on $\ell$ and may be defined by the equality

$$\nabla_a \ell^b = \omega_a^{(\ell)} \ell^b.$$  \hfill (13)

The function

$$\kappa^{(\ell)} = \omega_a^{(\ell)} \ell^a,$$  \hfill (14)

is a self-acceleration of $\ell$,

$$\ell^a \nabla_a \ell^b = \kappa^{(\ell)} \ell^b.$$  \hfill (15)

While the rotation 1-form potential transforms with rescaling $\ell$,

$$\omega_a^{(f\ell)} = \omega_a^{(\ell)} + \partial_a \ln f,$$  \hfill (16)

its external derivative

$$\Omega_{ab} = \partial_b \omega_a^{(\ell)} - \partial_a \omega_b^{(\ell)}$$  \hfill (17)

is invariant of the intrinsic geometry $(g_{ab}, \nabla_a)$.

**Definition 1.** The 1-form $\omega_a^{(\ell)}$ (13) is called rotation 1-form potential, the function $\kappa^{(\ell)}$ (14) is called surface gravity, the 2-form $\Omega$ (17) is called the rotation 2-form.

The word *rotation* may be understood just as a part of the definition. However, when the angular momentum of a NE-SF null surface is well defined, then it is indeed given by an integral of an expression proportional to $\Omega$ [1].

2.4. The vacuum constraints on the intrinsic geometry of the NE-SF null surfaces

The vacuum Einstein equation (1) with the cosmological constant $\Lambda$ imply constraints on the intrinsic geometry $(g_{ab}, \nabla_a)$ of a NE-SF null surface $H$. They were derived, analyzed, and solved in [1].
One of the constraints is called the 0th Low, and it is satisfied by the surface gravity $\kappa^{(\ell)}$ and the rotation 1-form potential $\omega^{(\ell)}$ of an arbitrary non-vanishing $\ell \in \Gamma(L)$, namely

$$\partial_a \kappa^{(\ell)} = \mathcal{L}_\ell \omega^{(\ell)}_a.$$  \hfill (18)

This constraint has an immediate consequence for the rotation 2-form $\Omega_{ab}$. To calculate $\Omega_{ab}$, we can choose any null vector field in $H$, in particular, we may use $\ell_o \in \Gamma(L)$ such that

$$\ell_o^a \nabla_a \ell_o^b = 0.$$  

Then, the 0th Law implies

$$0 = \partial_a \kappa^{(\ell_o)} = \mathcal{L}_\ell \omega^{(\ell_o)}_a.$$  \hfill (19)

Then it is easy to see, that

$$\mathcal{L}_\ell \Omega_{ab} = 0 \quad \text{for} \quad \ell = \ell_o,$$  \hfill (20)

holds for $\ell = \ell_o$, and even for every $\ell^a \in L(H)$.

The remaining constraints on $(g_{ab}, \nabla_a)$ can be formulated in terms of a given $\ell \in \Gamma(L)$ and the function $v: H \to \mathbb{R}$. Every value $v_1$ of the function $v$ defines naturally a section of $(10)$,

$$s_{v_1}: S \to H.$$  

We use the section to pullback to $S$ the co-vectors defined on $H$. To start with, the pullbacks

$$s^*_{v_1} g_{ab} = g_{AB}, \quad s^*_{v_1} \Omega_{ab} = \Omega_{AB},$$  \hfill (21)

are independent of the value $v_1$ of the function $v$, of the choice of a function $v$ itself and of the null vector field $\ell \in \Gamma(L)$. Without the lack of generality one can rescale every $\ell \in \Gamma(L)$ so that

$$\kappa^{(\ell)} = \text{const.}.$$  

Hence, we restrict ourselves to that case. Next, consider the pullback

$$\omega_A := s^*_{v_1} \omega^A \ell^\ell.$$  \hfill (22)

The result depends on the choice of $\ell$ and also on the choice of $v$, unless $\kappa^{(\ell)} = 0$. However, given those choices, $\omega_A$ is independent of the value $v_1$. Of course, the two pullbacks $\omega_A$ and $\Omega_{AB}$ are related with each other, namely

$$d \omega_{AB} = \Omega_{AB}.$$  

Next, we promote the gradient $\partial_a v$ to the one form $n_a \in \Gamma(T^*(H))$,

$$n_a := -\partial_a v,$$  \hfill (23)

invoke the corresponding tensor $S_{ab}$ (12) and pull it back onto $S$

$$S_{AB}(v) := s^*_{v_1} S_{AB}$$  

(we have dropped the suffix 1 at $v_1$). The result is value of $v$ dependent. The dependence is another constraint implied by (1), namely

$$\frac{d}{dv} S_{AB}(v) = -\kappa^{(\ell)} S_{AB}(v) + \nabla_{(A} \omega_{B)} + \omega_A \omega_B - \frac{1}{2} R_{AB} + \frac{1}{2} \Lambda g_{AB},$$  \hfill (25)

where $R_{AB}$ is the Ricci tensor of the metric $g_{AB}$.

In conclusion, the intrinsic geometry $(g_{ab}, \nabla_a)$ of a NE-SF null surface in a vacuum spacetime with the cosmological constant $\Lambda$ can be determined by choosing $\ell \in \Gamma(L)$ (in other words: fixing a scale and origin of the geodesic parameter $v$) such that
\[ \kappa^{(\ell)} = \text{const.}, \]

a section

\[ s_{\nu}: S \to H, \]

and pullbacks by \( s_{\nu}^* \) onto \( S \): \( g_{AB}, \omega_A \) (22), and \( S_{AB} \) (24). The constraints (3), (7), (18) and (25) provide the degenerate metric tensor \( g_{ab} \) and the covariant derivative \( \nabla_a \) on \( H \).

2.5. Vacuum isolated null surfaces

Consider a null NE-SF surface \( H \) of intrinsic geometry \((g_{ab}, \nabla_a)\) that admits \( \ell \in \Gamma(L) \) non-vanishing on a dense subset of \( H \) and such that

\[ [\mathcal{L}_\ell, \nabla_a] = 0. \quad (26) \]

Notice, that while the condition (7) is preserved by every transformation \( \ell^a \mapsto f \ell^a \) with arbitrary function \( f \) defined on \( H \), the condition (26) is generically invariant only for \( f = f_0 = \text{const.} \).

The equalities (13) and (26) imply

\[ \mathcal{L}_\ell \omega_a^{(\ell)} = 0, \quad (27) \]

hence the 0th Law (18) takes the following form

\[ \partial_a \kappa^{(\ell)} = 0, \quad (28) \]

meaning that the surface gravity \( \kappa^{(\ell)} \) is constant along (every connected part) of the null surface \( H \). A rescaling of a null symmetry generator \( \ell \) by a constant results in rescaling the surface gravity

\[ \kappa^{(f \ell)} = f \kappa^{(\ell)}. \quad (29) \]

Therefore, generically there are two cases: either

\[ \kappa^{(\ell)} \neq 0 \quad (30) \]

or

\[ \kappa^{(\ell)} = 0. \quad (31) \]

**Definition 2.** A null surface \( H \) (2) is isolated whenever it is NE-SF and its intrinsic geometry \((g_{ab}, \nabla_a)\) admits \( \ell \in \Gamma(L) \), non-vanishing on \( H \) and such that

\[ [\mathcal{L}_\ell, \nabla_a] = 0. \quad (32) \]

An isolated null surface is non-extremal if the surface gravity \( \kappa^{(\ell)} \) does not vanish, and is called extremal if the surface gravity \( \kappa^{(\ell)} \) vanishes.

There exist exceptional cases, though, of isolated null surfaces that admit two or more dimensional null symmetry group [1]. In those cases, an isolated surface is non-extremal with respect to one \( \ell \in \Gamma(L) \) and extremal with respect to another \( \ell_0 \in \Gamma(L) \). Because of those special cases, while considering isolated surfaces we also indicate the vector \( \ell \) (up to constant rescaling) that generates a null symmetry (26).

In the non-extremal case the constraint (25) allows to express the tensor \( S_{AB} \) by \( g_{AB} \) and \( \omega_A \). Indeed, it follows from (26), that
\[ \frac{d}{dv} S_{AB} = 0, \]

hence the constraint (25) reads

\[ S_{AB} = \frac{1}{\kappa(\ell)} \left( \nabla(A\omega_B) + \omega_A \omega_B - \frac{1}{2} R_{AB} + \frac{1}{2} \Lambda g_{AB} \right). \] (33)

In conclusion, in the non-extremal case of isolated null surface \( H \) and the infinitesimal generator \( \ell \in \Gamma(L) \) of the symmetry, the intrinsic geometry \((g_{ab}, \nabla a)\) is determined by the degenerate metric \( g_{ab} \), the rotation 1-form potential \( \omega^a(\ell) \), and the value of the self acceleration \( \kappa(\ell) \). That data is free modulo the constraints (3), (7), (27) and (28) and the signature +++ of \( g_{ab} \) projected to \( T(H)/L \).

The rotation 1-form potential \( \omega^a(\ell) \) and the degenerate metric tensor \( g_{ab} \) on \( H \) can be reconstructed given a section \( s_{v_1} : S \to H \) and data on \( S \): \( g_{AB} \) and \( \omega_A \) (22). The 1-form \( \omega_A \) depends on the section. A transformation

\[ v' = v - f, \quad \ell^a \partial_a f = 0 \]

(where \( f : S \to \mathbb{R} \) can be an arbitrary function) induces the respective change of \( \omega^a(\ell) \), in particular

\[ \omega'_A = \omega_A + \kappa(\ell) \partial_A f. \] (34)

In the extremal isolated horizon case, on the other hand, data freely defined on \( S \) is different. The equation (25) with \( \kappa(\ell) = 0 \) becomes a constraint on \( g_{AB} \) and \( \omega_A \) (22), namely [1]

\[ \nabla(A\omega_B) + \omega_A \omega_B - \frac{1}{2} R_{AB} + \frac{1}{2} \Lambda g_{AB} = 0. \] (35)

Notice, that in the extremal case the pull back (22) is independent of the section \( s_{v_1} \).

**Definition 3.** Given a 2D surface \( S \) endowed with a Riemannian metric tensor \( g_{AB} \) and a 1-form \( \omega_A \), the vacuum extremal isolated horizon equation with a cosmological constant \( \Lambda \) is the equation (35).

Solutions to that equation lead to a classification of extremal (degenerate) Killing horizons. Therefore, the study of its properties and solutions attracted interest of mathematical relativists [1, 8, 15, 16]. Secondly, every solution to that equation determines an exact solution to the Einstein vacuum equations foliated by bifurcated Killing horizons whose common part is an extremal Killing horizon [17, 18]. Those spacetimes are called Near Horizon Geometries and describe neighborhoods of extremal isolated Killing horizons [10]. Because of that the equation itself is (most) often called a near horizon geometry equation. There are still open questions about that equation, for example existence of non-axisymmetric solutions on topological \( S^2 \). We will see below, that somewhat surprisingly, our study of non-extremal isolated horizon will lead us to the new results on the extremal isolated horizon equation.

### 3. The Weyl tensor of vacuum non-extremal isolated null surfaces with a cosmological constant

In this section we consider a 3D, non-extremal isolated null surface \( H \) introduced above, endowed with an intrinsic geometry \((g_{ab}, \nabla a)\) and a null symmetry generator \( \ell \in \Gamma(L) \). The surface \( H \) is contained in a spacetime \( M \) whose metric tensor \( g_{\mu\nu} \) satisfies the vacuum Einstein equation (1) with a (possibly zero) cosmological constant \( \Lambda \).
There exists an extension of the vector field $\ell^a$ to a vector field $t^\mu$ defined in a neighborhood of $H$ in $M$ such that

$$t^\mu |_H = \ell, \quad \text{and} \quad \mathcal{L}_t g_{\mu\nu} |_H = 0.$$  

(36)

(We construct an example of $t$ below in section 3.3.) Henceforth, we assume throughout the paper, that every $t$ like that is also a symmetry of the spacetime Weyl tensor $C^{\alpha\beta\gamma\delta}$ at $H$:

**Assumption 1.** Every vector field $t$ that satisfies (36) satisfies also

$$\mathcal{L}_t C^{\alpha\beta\gamma\delta} |_H = 0.$$  

(37)

That assumption, which we will refer to as the assumption of stationarity to the second order, allows to attribute all the spacetime Weyl tensor components to the intrinsic geometry of $H$.

### 3.1. The complex invariant of non-extremal isolated null surface

Consider the data introduced in section 2.5 and defined on the 2D space $S$ of the null geodesics in $H$, that is: a Riemannian metric tensor $g_{AB}$ and a differential 1-form $\omega_A$. There are two scalar invariants that can be constructed from them. One is the Gaussian curvature $K$ (half of the Ricci scalar) of the metric tensor. The second one characterizes the rotation 2-form $\Omega_{AB}$ in terms of the area 2-form $\eta_{AB}$,

$$\Omega_{AB} = : \mathcal{O} \eta_{AB}. \quad (38)$$

The key role in our paper will be played by a suitable complex valued combination of the real valued invariants.

**Definition 4.** The complex invariant of isolated null surface $H$ is the complex valued function $\Psi$ defined on the space $S$ of the null geodesics in $H$ by the following formula

$$\Psi := -\frac{1}{2} (K + i\mathcal{O}). \quad (39)$$

Both, $K$ and $\mathcal{O}$ are derived from the freely given $g_{AB}$ and $\omega_A$, respectively. In particular, the rotation scalar $\mathcal{O}$, can be expressed by an *a priori* unconstrained function $U : S \rightarrow \mathbb{R}$,

$$\mathcal{O} = -\nabla^A \nabla_A U. \quad (40)$$

### 3.2. Spacetime null frame adapted to $H$

We will use the Newman–Penrose decomposition of the spacetime Weyl tensor in terms of a spacetime null frame $(e_1^\mu, e_2^\mu = e_1^\mu, e_3^\mu, e_4^\mu)$ and its dual $(e_1^\mu, ..., e_4^\mu)$, such that

$$g_{\mu\nu} = e_1^\mu e_2^\nu + e_1^\nu e_2^\mu - e_3^\mu e_4^\nu + e_3^\nu e_4^\mu.$$  

The frame is assumed to be defined in a neighborhood of $H$ in $M$, and to be adapted to the isolated null surface $H$ in a way described below. To begin with, we set

$$e_4 |_H = \ell, \quad (41)$$

hence the frame is well defined at every $x \in H$ such that $\ell \neq 0$. Then, the complex valued vector field $e_1$ is by definition of a null frame orthogonal to $\ell$ at $H$, hence, it is tangent to $H$. We choose it such that
where
\[ e^1_a \big|_{\mu} =: m^a. \]
This choice is possible due to (7). Next
\[ e_2 = e_1. \]
(44)
The outstanding vector field
\[ e^3_\mu =: n^\mu \]
is transversal to \( H \).
We restrict further the ambiguities in the vector fields \( m^a \) and \( n^\mu \) by using the 1-form \( n_a \) (23) and assuming about \( e^4_\mu \), that its pullback \( e^4_a \) to \( H \) coincides with \( n_a \),
\[ e^4_a \big|_{\mu} = n_a. \]
(45)
With those frames \((e^1_\mu, ..., e^4_\mu)\) and the dual \((e^1_\mu, ..., e^4_\mu)\) we will apply the Newman–Penrose formalism [13] and use the consistent notation.

**Definition 5.** A null frame \((e^1_\mu, ..., e^4_\mu)\) is called adapted to a given isolated null surface \( H \) and the null symmetry generator \( \ell \) if it satisfies the conditions (41)–(45).

On the other hand, our null spacetime 4-frame provides a 3-frame tangent to \( H \): \((m^a, \bar{m}^a, \ell^a)\). The corresponding dual 3-co-frame \((m_a, \bar{m}_a, -n_a)\) coincides with the pullback to \( H \) of \((e^1_\mu, e^2_\mu, e^4_\mu)\). Due to (42) the co-frame is Lie dragged by \( \ell \),
\[ \mathcal{L}_\ell m_a = 0 = \mathcal{L}_\ell n_a. \]
(46)
In that frame,
\[ g_{ab} = m_a \bar{m}_b + m_b \bar{m}_a. \]
The components of the intrinsic covariant derivative \( \nabla_a \) on \( H \) are expressed below in terms of the Newman–Penrose coefficients corresponding to the spacetime frame \((e^1_\mu, ..., e^4_\mu)\) introduced above:

- \[ \nabla_a \ell^b = (\alpha + \bar{\beta}) m_a + (\bar{\alpha} + \beta) \bar{m}_a - (\epsilon + \bar{\epsilon}) n_a \ell^b, \]
- \[ m^a \nabla_a n_b = \lambda m_a + \mu m_a - \pi n_a; \]
- \[ m^a \nabla_a m_b = -(\alpha - \beta) m_a + (\bar{\alpha} - \beta) \bar{m}_a + (\epsilon - \bar{\epsilon}) n_a = -\bar{m}^a \nabla_a m_b, \]

where additional identities hold on \( H \) [1]:

- \[ \epsilon = \bar{\epsilon}, \]
- \[ \bar{\mu} = \mu, \]
- \[ \pi = \alpha + \beta, \]
- \[ \kappa(\ell) = \epsilon + \bar{\epsilon}. \]

Owing to the symmetry (26), all the coefficients are constant along the null geodesics in \( H \). Consistently with the Newman–Penrose formalism let us denote by \( D \), the differential operator \( e^4_\mu \partial_\mu \) that at \( H \) coincides with \( \ell^a \partial_a \).

---

3 Vectors \( k, \ell \) introduced in [13] correspond to vectors \( \ell, n \) in our work.
\[ D := \ell^a \partial_a. \]  
(51)  

Hence, the Newman–Penrose coefficients of \( \nabla_a \) satisfy  
\[ D\alpha = D\beta = D\lambda = D\mu = D\epsilon = 0, \]  
(52)  

and still  
\[ \kappa^{(\ell)} = \text{const.} \neq 0 \]  
according to the earlier considerations.  

In that co-frame the rotation 1-form potential takes the following form  
\[ \omega_a^{(\ell)} = (\alpha + \beta)m_a + (\bar{\alpha} + \bar{\beta})\bar{m}_a - \kappa^{(\ell)}n_a. \]  
(53)  

It follows that the push forward \( \Pi_* m \) onto \( S \) is a uniquely defined vector field on (a neighborhood in) \( S \),  
\[ m^A := \Pi_* A^a m^a, \]  
and equips \( S \) with a null frame for the metric \( g_{AB} \),  
\[ g_{AB} = m_A \bar{m}_B + \bar{m}_A m_B. \]  

Using that 2-frame \( (m^a, \bar{m}^a) \) and the dual 2-co-frame \( (\bar{m}_A, m_A) \), and the Newman–Penrose coefficients we express the pullbacks \( (22), (24) \) and \( (33) \),  
\[ \omega_A = (\alpha + \beta) m_A + (\bar{\alpha} + \bar{\beta}) \bar{m}_A, \]  
(54)  
\[ S_{AB} = \mu (m_A \bar{m}_B + \bar{m}_A m_B) + \lambda m_A m_B + \bar{\lambda} \bar{m}_A \bar{m}_B, \]  
(55)  

where still \( S_{AB} \) is determined by \( g_{AB} \) and \( \omega_A \) by the equality \( (33) \), hence the functions \( \lambda \) and \( \mu \) can be expressed by \( \alpha, \beta \) and the differential operators \( \delta \) and \( \bar{\delta} \), defined consistently with the Newman–Penrose formalism  
\[ \delta := m^A \partial_A. \]  
(56)  

3.3. The Weyl tensor  
The spacetime Weyl tensor \( C^\mu_{\alpha\beta\gamma} \) in the null frame formalism is expressed by five complex valued Newman–Penrose components \( \Psi_0, ..., \Psi_4 \),  
\[ \Psi_0 = C_{4141}, \quad \Psi_1 = C_{4341}, \quad \Psi_2 = C_{4123}, \quad \Psi_3 = C_{3432}, \quad \Psi_4 = C_{3232}. \]  
(57)  

We now use a null frame adapted to \( H \) (see the previous subsection) and consider \( \Psi_0, ..., \Psi_4 \) on \( H \). It turns out, that the first 4 components are automatically constant along the null generators of \( H \), that is, that  
\[ D\Psi_I = 0, \quad I = 0, 1, 2, 3. \]  
(58)  

The general reason for \( (58) \) is, that \( \Psi_0, ..., \Psi_3 \) at \( H \) are determined by the intrinsic geometry \( (g_{ab}, \nabla_a) \) and the adapted null frame, while \( \ell \) is an infinitesimal symmetry of both, the frame and the intrinsic geometry. We will provide explicit expressions for \( \Psi_0, ..., \Psi_3 \) below.  

The component \( \Psi_4 \), on the other hand, is a subject to an evolution equation along the null generators of \( H \) with initial value arbitrarily set at any fixed transversal section \( \nu = \nu_1 \). Specifically, one of the components of the tensorial Bianchi identity
\[ \nabla_{\alpha} C_{\beta\gamma\delta}^{\alpha} = 0 \]  \hspace{1cm} (59)

takes the following form
\[ 0 = D\Psi_4 - 3\lambda\Psi_2 - 2(2\pi + \alpha)\Psi_3 + 2\kappa^{(\ell)}\Psi_4. \]  \hspace{1cm} (60)

However, if we additionally assume that also for \( \Psi_4 \),
\[ D\Psi_4 = 0 \]  \hspace{1cm} (61)
is true, then, for non-zero \( \kappa^{(\ell)} \) the last coefficient \( \Psi_4 \) of the Weyl tensor becomes determined by the intrinsic geometry \( (g_{ab}, \nabla_a) \) of \( H \) as well.

**Assumption 1'.** We are assuming that the Weyl tensor component \( \Psi_4 \) in a null frame adapted to \( H \) is constant along the null geodesics in \( H \).

Technical assumption 1' follows from the geometrically formulated assumption 1 and is actually the only reason we needed assumption 1. Let us demonstrate that assumptions 1 implies 1'. We have specified at \( H \) the vector field \( n^\mu \) that is the \( e_3^\mu \) element of the adapted null frame. It is the normalized to \( \ell^\mu n_\mu = -1 \) null vector orthogonal to the space-like foliation of \( H \) defined by the constancy surfaces of the function \( v \). Let us extend \( n^\mu \) to a neighborhood of \( H \) by assuming
\[ n^\mu \nabla_\mu n^\nu = 0. \]
Next, let us define in a neighborhood of \( H \) a vector field \( t^\mu \) such that
\[ t^\mu |_n = \ell^\mu, \quad \text{and} \quad L_n t = 0. \]
By the construction, the resulting vector field \( t \) satisfies
\[ L_te_1^\mu |_n = L_te_2^\mu |_n = L_te_3^\mu |_n = L_te_4^\mu |_n = 0 \]
hence, also
\[ L_t g_{\mu\nu} |_n = 0. \]
Now, according to assumption 1,
\[ L_t C_{\beta\gamma\delta}^{\alpha} |_n = 0. \]  \hspace{1cm} (62)
In particular, that implies that assumption 1' is satisfied. It is also easy to show that assumptions 1' implies 1.

Now, all the Weyl tensor components \( \Psi_0, ..., \Psi_4 \) can be expressed by \( g_{ab}, \nabla_a \) and their derivatives with respect to \( n^\mu \) and \( m^\mu \). We derive them below and discuss their properties.

The components \( \Psi_0 \) and and \( \Psi_1 \) vanish identically, due to the vanishing of the expansion and shear of the vector field \( \ell \),
\[ \Psi_0 = \Psi_1 = 0. \]  \hspace{1cm} (63)
The Weyl tensor component \( \Psi_2 \) is determined by the components of the intrinsic covariant derivative \( \nabla_a \) on \( H \), namely
\[ \Psi_2 = \delta\beta - \delta\alpha + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \Lambda/6. \]  \hspace{1cm} (64)
Not by accident, is it closely related to the invariant (39), namely [1]
\[ \Psi_2 = \Psi + \frac{\Lambda}{6}. \]  \hspace{1cm} (65)
and obviously is invariant with respect to the allowed transformations of the adapted null frame \((e_1, ..., e_4)\).

The general Newman–Penrose expression for \(\Psi_3\) is
\[
\Psi_3 = \delta \mu - \delta \lambda + \mu (\alpha + \beta) + \lambda (\alpha - 3 \beta).
\] (66)

But we remember, that \(\lambda\) and \(\mu\) can be expressed by \(\alpha, \beta, \kappa(\ell)\) via (33), (55) and (49). Specifically,
\[
\lambda = \frac{1}{\kappa(\ell)} (\delta \pi + \pi (\pi + \alpha - \beta)),
\] (67)
\[
\mu = \frac{1}{2 \kappa(\ell)} \left( \nabla^A \omega_A + 2 \pi \bar{\pi} - K + \Lambda \right),
\] (68)
where the divergence of \(\omega_A\) in terms of Newman–Penrose coefficients reads
\[
\nabla^A \omega_A = \delta \pi + \bar{\delta} \bar{\pi} - (\alpha - \bar{\beta}) \bar{\pi} - (\bar{\alpha} - \beta) \pi.
\] (69)

With the help of the last two equalities, \(\Psi_3\) can be expressed directly by \(\Psi_2\), namely
\[
\Psi_3 = \frac{1}{\kappa(\ell)} (\bar{\delta} + 3 \alpha + 3 \beta) \Psi_2.
\] (70)

One can easily see that either by inspection or by applying the Bianchi identities (59). The latter imply
\[
0 = D \Psi_3 - \bar{\delta} \Psi_2 + \kappa(\ell) \Psi_3 - 3 \pi \Psi_2.
\]

The last Newman–Penrose component \(\Psi_4\) of the Weyl tensor at \(H\) by the assumption (61) becomes
\[
\Psi_4 = \frac{1}{2 \kappa(\ell)} (\bar{\delta} \Psi_3 - 3 \lambda \Psi_2 + 2 (2 \pi + \alpha) \Psi_3).
\] (71)

### 3.4. The possible Petrov types

The conclusion from section 3.3 is, that assuming the vacuum Einstein equations, knowing the value of the cosmological constant \(\Lambda\), and making assumption I (or I'), we can attribute the Petrov type of the Weyl tensor at a point \(x \in H\) to the intrinsic geometry \((g_{ab}, \nabla_a)\). The meaning of equation (63) is that the vector \(\ell\) is parallel to a double principal null direction of the Weyl tensor. Hence, the Petrov type may be II, D, III, N or O. A necessary condition for the type III or \(N\) is the vanishing of the component \(\Psi_2\) calculated above in equation (65). Notice, however, that it follows from the formulae (70) and (71) that for every open subset of \(H\),
\[
\Psi_2 = 0 \Rightarrow \Psi_3 = \Psi_4 = 0.
\]

In the consequence, if the Petrov type of the Weyl tensor is constant at \(H\), the only possibilities are: II, D or O. If \(\Psi_2 = 0\) at \(H\), then the Petrov type is necessarily O. It happens when
\[
K = \frac{1}{3} \Lambda, \quad \text{and} \quad \Omega_{AB} = 0.
\] (72)

Concluding:

**Theorem 0.** Suppose \(H\) is a 3D non-extremal isolated null surface in a 4D spacetime such that the vacuum Einstein equation (1) with cosmological constant \(\Lambda\) and the assumption I on
stationarity to the second order are satisfied. If the Petrov type of the spacetime Weyl tensor is constant on \( H \), then the Petrov type is one of the following: \( \text{II, D, or O} \). In particular, the necessary conditions and sufficient for the Petrov type to be \( \text{O} \) on all of \( H \) are (72).

If at some point \( x \in H \)
\[
\Psi_2(x) \neq 0,
\]
then the Weyl tensor at \( x \) is either of the Petrov type \( \text{II} \) if
\[
2\Psi_2^3(x) - 3\Psi_2(x)\Psi_4(x) \neq 0,
\]
or of the Petrov type \( \text{D} \) when
\[
2\Psi_2^3(x) - 3\Psi_2(x)\Psi_4(x) = 0.
\]
The (non-)vanishing of \( 2\Psi_2^3(x) - 3\Psi_2(x)\Psi_4(x) \) is independent of the ambiguities remaining in our choice of the null frame \((e_1, e_2, e_3, e_4)\).

3.5. A remark on the Petrov type of the crossover section

In this paper we consider isolated null surfaces such that the infinitesimal generator \( \ell \) of the null symmetry vanishes nowhere. In the non-extremal case, if the integral lines of \( \ell \) can be extended sufficiently far, there is a limit in the spacetime \( M \) as we go along each null geodesic in \( H \) in which
\[
\ell \to 0.
\]
If
\[
\kappa^{(\ell)} > 0,
\]
then the limit is in the past, and vice versa. The limiting points form a space-like 2-surface \( S_0 \subset M \), that bounds \( H \) from the past or from the future. Let us call it a crossover surface. On the other hand, the Weyl tensor components
\[
\Psi_3 = C_{\mu\rho\alpha\delta} n^\mu \ell^\rho n^\alpha \bar{m}^\delta = \text{const.,} \quad \Psi_4 = C_{\mu\rho\alpha\delta} n^\mu \bar{m}^\rho n^\alpha \bar{m}^\delta = \text{const.}
\]
along each line, hence they extend to the limit. The scaling properties of those components as
\[
e_4' = \frac{\ell}{u}, \quad e_3' = un
\]
are
\[
\Psi_3' = u\Psi_3, \quad \Psi_4' = u^2\Psi_4.
\]
So, if we adjust the function \( u \) on \( H \) such that \( u \to 0 \) when we approach \( S_0 \) while \( e_4' \) and \( e_3' \) are both finite in the limit, then
\[
\Psi_3' = 0, \quad \Psi_4' = 0
\]
at the crossover surface. Hence, the null vector \( e_3' \) orthogonal to the crossover surface is a double principal null direction of the Weyl tensor. That makes the Petrov type of the Weyl tensor \( \text{D} \) at the crossover surface.

In a similar way it is easy to prove that \( S_{\text{ab}}' \to 0 \) at the crossover surface \( S_0 \), that makes the orthogonal null vector \( e_4' \) non-expanding and shear free at \( S_0 \).
The above concluded properties of the crossover boundary of $H$ are true for every isolated null surface extendable sufficiently far. They do not cause any constraints on the intrinsic geometry $(g_{ab}, \nabla_a)$.

4. The Petrov type D isolated null surfaces

4.1. Known examples of the Petrov type D isolated null surfaces

Explicit examples of the Petrov type D non-extremal isolated null surfaces are provided by the exact solutions to the vacuum Einstein’s equations with a cosmological constant (positive, negative or zero) that contain non-extremal Killing horizons. A well known family are the non-extremal vacuum black hole solutions: Schwarzschild and Kerr ($\Lambda = 0$), Schwarzschild–de Sitter and Kerr–de Sitter ($\Lambda > 0$), Schwarzschild–anti de Sitter, Kerr–anti de Sitter ($\Lambda < 0$). Their event horizons, inner horizons (Cauchy horizons) and cosmological horizons are the vacuum isolated null surfaces of the Petrov type D. Another class of known type D spacetimes that contain non-extremal isolated horizons are the axi-symmetric spacetimes foliated by two transversal to each other families of the Killing horizons $[17, 18]$. They are also known as near horizon geometries $[10]$. Each of those Killing horizons is simultaneously extremal and non-extremal, and of the Petrov type D. In the case of $\Lambda = 0$, it was proven $[9]$ that every axi-symmetric Petrov type D non-extremal isolated horizon is one of the examples listed above. In the current paper we formulate the generalization of that theorem to $\Lambda \neq 0$, (for detailed proof see $[11]$).

4.2. The Petrov type D equation

In this subsection we study the conditions for the Petrov type D formulated above. The condition is imposed on the following data freely defined on a 2-manifold $S$, the space of the null generators of an isolated null surface $H$:

- $g_{AB}$—a Riemannian metric tensor,
- $\Omega_{AB}$—an exact (rotation) 2-form, represented by a function $U (40)$,
- $\Lambda$—an arbitrary value of the cosmological constant.

The surface gravity

$$\kappa^{(f)} \neq 0$$

takes arbitrarily fixed value and disappears from the considerations.

The Petrov type D condition will be expressed by the invariant (39)

$$\Psi = \frac{1}{2} ( -K + i \Delta U ).$$

(73)

The metric tensor is represented by a complex null 2-co-frame $(\bar{m}_A dx^A, m_A dx^A)$,

$$g_{AB} = m_A \bar{m}_B + \bar{m}_A m_B.$$  

The rotation 2-form is defined in terms of the rotation 1-form potential

$$\omega_A = (\alpha + \beta) m_A + (\bar{\alpha} + \bar{\beta}) \bar{m}_A$$

$$\Omega_{AB} = \partial_A \omega_B - \partial_B \omega_A.$$
The definition of the Newman–Penrose coefficients $\alpha, \beta : S \to \mathbb{C}$ is completed by the formula for the commutator of the tangent 2-frame $(m^A, \bar{m}^A)$ dual to the coframe,

$$[\delta, \bar{\delta}] = (\beta - \alpha)\delta - (\bar{\beta} - \bar{\alpha})\bar{\delta},$$

(74)

where the vector field $m^A$ is identified with the operator

$$\delta = m^A \partial_A.$$

In terms of $g_{AB}$ and $\Omega_{AB}$, the following functions $\Psi_2, \Psi_3$ and $\Psi_4$ are defined on $S$

$$\Psi_2 = \Psi + \frac{\Lambda}{6},$$

(75)

$$\Psi_3 = \frac{1}{\kappa(\ell)} \left( \delta + 3(\alpha + \bar{\beta}) \right) \Psi_2,$$

(76)

$$\Psi_4 = \frac{1}{2\kappa(\ell)} \left( \delta \Psi_3 - 3\lambda \Psi_2 + 2(3\alpha + 2\bar{\beta})\Psi_3 \right),$$

(77)

with the function $\lambda$ defined in (67) and (49).

The Petrov type D condition reads

$$\Psi_2(x) \neq 0,$$

(78)

$$2\Psi_2^2(x) = 3\Psi_2(x)\Psi_4(x).$$

(79)

After plugging in the formulae (75)–(77) we obtain an equation on $\Psi_2$

$$4(\delta \Psi_2)^2 - 3(\alpha - \bar{\beta})\delta \Psi_2 - 3\Psi_2 \delta \Psi_2 = 0.$$  

(80)

Remarkably, the equation can be written in an equivalent compact form

$$(\bar{\delta} + \alpha - \bar{\beta})\delta (\Psi_2(x))^{-\frac{1}{2}} = 0.$$  

(81)

After using (65), the final equation on $(g_{AB}$ and $\Omega_{AB}$ in terms of) the complex invariant $\Psi$ reads

$$(\bar{\delta} + \alpha - \bar{\beta})\delta \left( \Psi(x) + \frac{\Lambda}{6} \right)^{-\frac{1}{2}} = 0.$$  

(82)

This is our Petrov type D equation.

The operators featuring in (82) are well known in the GHP formalism [13]. Using the so-called edth$'$ operator, the equation reads just

$$\bar{\delta}' \delta' (\Psi_2(x))^{-\frac{1}{2}} = 0.$$  

(83)

Notice, that the only dependence on the 1-form $\omega$ is the function $\nabla^A \nabla_A U$ present in $\Psi$ that depends on $\Omega$ only. The vector field $m^A$ (and the corresponding operator $\delta$) is defined modulo the local rotations

$$m^A = e^{i\varphi} m^A,$$

(84)

with a real valued function $\varphi$. The equation (82) must be invariant with respect to them. We can see the invariance after writing the equation in terms of the covariant derivative $\nabla_A$ of the metric tensor $g_{AB}$, namely
\[
(\delta\delta - (\bar{m}^B \nabla_B \bar{m}^A) \partial_A) \left( (\Psi(x) + \frac{\Lambda}{6})^{-\frac{i}{4}} \right) = 0, \tag{85}
\]
that may be written in a more suggestive way,
\[
\bar{m}^A m^B \nabla_A \nabla_B \left( (\Psi(x) + \frac{\Lambda}{6})^{-\frac{i}{4}} \right) = 0, \tag{86}
\]
explicitly invariant with respect to (84).

In summary, we have proved the following:

**Theorem 1.** Suppose \( H \) is a 3D non-extremal isolated null surface in a 4D spacetime such that the vacuum Einstein equation (1) with cosmological constant \( \Lambda \) and the assumption 1 on stationarity to the second order (or assumption \( \Gamma \)) are satisfied. Then, the necessary and sufficient condition for the spacetime Weyl tensor to be of the Petrov type D at each point of the null geodesic \( x \in S \) is, that the invariant \( \Psi(x) \) satisfies the following two conditions:
\[
\Psi(x) \neq -\frac{\Lambda}{6}, \tag{87}
\]
and,
\[
\bar{m}^A m^B \nabla_A \nabla_B \left( (\Psi(x) + \frac{\Lambda}{6})^{-\frac{i}{4}} \right) = 0. \tag{88}
\]

A remark is in order. To interpret the Petrov type D equation without using any specific null frame, one can use the concept of the anti-holomorphic covariant derivative. For every tensor field \( T \) defined on \( S \), consider the following operation
\[
\nabla^{(0,1)} T := \nabla_{\bar{m}} T \otimes m_A dx^A,
\]
that turns \( T \) into a new tensor. This is the anti-holomorphic covariant derivative. It is invariant with respect to the transformations (84). With the anti-holomorphic covariant derivative, the equation (86) reads
\[
\left( \nabla^{(0,1)} \right)^2 \left( (\Psi(x) + \frac{\Lambda}{6})^{-\frac{i}{4}} \right) = 0. \tag{89}
\]

5. **The Petrov type D equation and the near horizon geometry equation**

5.1. **The Petrov type D equation as an integrability condition for the near horizon geometry equation**

Consider a 2D manifold \( S \), a metric tensor \( g_{AB} \), a differential 1-form \( \omega_A \) and a constant \( \Lambda \). Suppose they satisfy the vacuum extremal isolated horizon equation with a cosmological constant \( \Lambda \)
\[
\nabla_{(A} \omega_{B)} + \omega_A \omega_B - \frac{1}{2} R_{AB} + \frac{1}{2} \Lambda g_{AB} = 0. \tag{90}
\]
This equation introduced and studied in [1, 8] is better known as near horizon geometry equation [10].
Our observation is that the corresponding 2-form
\[ \Omega := d\omega, \]
together with \( g_{AB} \) and \( \Lambda \) satisfy the Petrov type D equation (86) of the previous subsection. The proof is easy, given that the formulae are already at our disposal. Use the given 2-manifold \( S \) endowed with \( g_{AB}, \omega_A \) and \( \Lambda \) to construct on \( S \times \mathbb{R} \) the intrinsic geometry of a vacuum non-expanding isolated null surface of the cosmological constant \( \Lambda \). Then, the near horizon geometry equation (90) means that the tensor \( S_{AB} \) (24) and (55) vanishes identically. Indeed,
\[ S_{AB} = \frac{1}{\kappa^{(2)}} \left( \nabla_A \omega_B + \omega_A \omega_B - \frac{1}{2} R_{AB} + \frac{1}{2} \Lambda g_{AB} \right) = 0. \] (91)
That is the components of \( S_{AB} \) vanish, namely
\[ \lambda = \mu = 0. \]
In turn,
\[ \Psi_3 = \tilde{\delta} \mu - \delta \lambda + \mu (\alpha + \tilde{\beta}) + \lambda (\tilde{\alpha} - 3 \tilde{\beta}) = 0, \] (92)
and so does \( \Psi_4 \),
\[ \Psi_4 = \frac{1}{2\kappa^{(2)}} \left( \tilde{\delta} \Psi_3 - 3 \lambda \Psi_2 + 2 (3 \alpha + 2 \tilde{\beta}) \Psi_3 \right) = 0. \] (93)
Hence, the second Petrov type D condition (79) is satisfied. The catch is, that the first condition (78) may be not true in some \( x \in S \). The reason is, that in general nothing prevents the invariant \( \Psi \) (39) of \( (g_{AB}, \omega_A) \) from satisfying
\[ \Psi(x) + \frac{\Lambda}{6} = 0 \]
at some \( x \in S \).

We have thus demonstrated a proof of the following theorem:

**Theorem 2.** Suppose a differential 1-form \( \omega_A \) and a Riemannian metric tensor
\[ g_{AB} = m_A \bar{m}_B + \bar{m}_A m_B, \]
both defined on a 2D manifold \( S \) satisfy the following (near horizon geometry) equation:
\[ \nabla_A \omega_B + \omega_A \omega_B - \frac{1}{2} R_{AB} + \frac{1}{2} \Lambda g_{AB} = 0, \] (94)
with a constant \( \Lambda \). Then, the invariant \( \Psi \) defined by (39) with
\[ \Omega_{AB} := \partial_A \omega_B - \partial_B \omega_A \]
satisfies the equation
\[ m^A m^B \nabla_A \nabla_B \left( \Psi(x) + \frac{\Lambda}{6} \right)^{-\frac{1}{2}} = 0 \] (95)
at every \( x \in S \) such that
\[ \Psi(x) + \frac{\Lambda}{6} \neq 0. \] (96)
Remark. If $S = S_2$ (topologically) and $(g_{AB}, \omega_A)$ satisfy the NHG equation (90), then, by the global argument given in [8] (for $\Lambda = 0$ but it can be easily generalized to $\Lambda \neq 0$), the inequality (96) is satisfied at every $x \in S$, unless $\Psi^+ \Lambda = 0$ identically on $S$. The same can be shown with bit more effort for arbitrary orientable compact $S$ [19].

In conclusion, theorem 2 may also be used as an integrability condition for the near horizon geometry equation to investigate the space of solutions.

Remark. For every extremal isolated null surface $H$ contained in a vacuum spacetime, the corresponding $g_{AB}$ and $\omega_A$ defined on a cross section $S$ of $H$ satisfy the hypothesis of theorem 2. Hence, they also satisfy the conclusion, that is the type D equation whenever (96). Nonetheless, the Petrov type of the spacetime Weyl tensor at $H$ may be and generically is different than that of D. Indeed, theorem 1 concerns non-extremal isolated null surface only. The technical reason is that when $\kappa^{(\ell)} = 0$, then an independent of the variable $v$ tensor $S_{AB}$ decouples from $g_{AB}$ and $\omega$, and actually can be arbitrary at a given cross section (25). The same is true about $\Psi_4$ (60). The arbitrariness of $S_{AB}$ on $S$ passes to $\Psi_3$ (see (55 ) and (66)).

5.2. Non-twisting of the second principal null direction of the Weyl tensor

At each point of a Petrov type D non-extremal isolated null surface there are two double principal directions of the Weyl tensor. The first one is orthogonal to $H$ and coincides with the direction of the null symmetry generator $\ell$. The second one, however, is generically twisting. In that generic case we can not choose $e_3$ in our adapted null frame to be pointing in the second double-principal direction because in the definition of an adapted null frame we have assumed that both the null vectors $e_3$ and $e_4$ are non-twisting at $H$. In this subsection, however, a second principal null direction that will emerge below will also be hyper-surface orthogonal due to the extremal isolated horizon (near horizon geometry) equation (90).

Suppose that there is a section of a non-extremal isolated null surface $H$

$$s : S \rightarrow H$$

such that the 1-form

$$\omega_A = s^* \omega_A^{(\ell)}$$

satisfies the equation (90). Then, the direct meaning of this equation is that the transversal to $H$ and orthogonal to $s(S) \subset H$ null vector field $n^\mu$ is non-expanding and shear-free. We can also normalize $n^\mu$ such that

$$n_\mu \ell^\mu = -1$$

and choose a spacetime null frame adapted to $H$, such that

$$e_3^\mu |_n = n^\mu.$$  

Notice, that if $\ell$ were vanishing at $s(S)$ we could not do that. As we showed in the previous subsection, in this frame

$$\Psi_3 |_n = \Psi_4 |_n = 0.$$  

Therefore, the isolated null surface $H$ is of the Petrov type D. Moreover, in that case the transversal to $H$ principal null direction of the Weyl tensor is orthogonal to the 2-surface $s(S) \subset H$, hence it is hyper-surface orthogonal. Applying to the slice $s(S)$ the symmetry of $H$ generated
by $\ell$, we obtain a foliation of $H$ whose leaves are 2-sections of the same properties as $s(S)$, hence all $H$ is of the Petrov type D.

An opposite implication exists but is a bit more complicated. Suppose that $H$ is an isolated null surface of the Petrov type D and that the transversal to $H$ principal null direction of the Weyl tensor is hyper-surface orthogonal. Then, the hyper-surfaces are 2D, space-like and foliate $H$. Let $s : S \to H$ be a section such that $s(S)$ is a leaf of the foliation. We choose in $H$ an adapted null frame $(e_1^\mu, ..., e_4^\mu)$ such that $e_3^\mu$ is orthogonal to $s(S)$. It follows from the definition of a double principal null direction of the Weyl tensor, that in this frame

$$\Psi_4 = \Psi_3 = 0.$$ 

The vanishing of $\Psi_3$ and $\Psi_4$ on $s(S)$ implies first, that due to (71)

$$0 = \Psi_4 = - \frac{3}{2\kappa(\ell)} \lambda \Psi_2,$$

(97)

and since $\Psi_2$ does not vanish for the type D, the shear of the vector field $n^\mu$ vanishes,

$$\lambda = 0.$$ 

(98)

That furnishes the traceless part of the equation (90). Next, using the equation (70) with $\lambda = 0$,

$$0 = \Psi_3 = \delta \mu + \mu (\alpha + \beta).$$

(99)

Now, either

$$\mu = 0,$$

(which in turn implies the trace part of (90)), or the pullback $\omega$ of the rotation 1-form potential (53) becomes a pure gradient

$$\omega_A = \partial_A \ln \mu.$$ 

(100)

As a consequence, the invariant rotation 2-form

$$\Omega_{AB} = 0,$$

(101)

which makes $H$ non-rotating.

There is a subtlety about the section orthogonal to the assumed principal null direction of the Weyl tensor. For $S$ that is not simply connected it may happen, that the section is not continuous. However, the pullback $(s^* \omega^{(\ell)})_A$ is continuous and differentiable. That follows from the invariance of the foliation and of $\omega_s^{(\ell)}$ on the flow of the vector field $\ell^a$.

We can conclude our findings in the following way:

**Theorem 3.** Suppose $H$ is a 3D non-extremal isolated null surface in a 4D spacetime such that the vacuum Einstein equation (1) with cosmological constant $\Lambda$ and assumption 1 on stationarity to the second order (or assumption 1’ ) are satisfied. Let $\ell^a$ be the generator of the null symmetry of $H$. Let $s : S \to H$ be a section of (10). Then,

(a) The null direction transversal to $H$ and orthogonal to $s(S)$ is non-expanding and shear-free if and only if the 1-form

$$\omega_A := s^* \omega_s^{(\ell)}$$

satisfies the near horizon geometry equation (90).

(i) If the null direction transversal to $H$ and orthogonal to $s(S)$ is non-expanding and shear-free, then at every $x \in s(S)$ it is a double principal direction of the Weyl tensor or the Weyl tensor vanishes at $x$. 

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(ii) Suppose the rotation scalar 2-form $\Omega_{AB}$ does not identically vanish on $S$. If the null direction orthogonal to $s(S)$ and transversal to $H$ is a double principal null direction of the Weyl tensor, then it is non-expanding and shear free.

(iii) The symmetry of $H$ generated by $\ell^a$ spreads the slice $s(S)$ to a foliation of $H$ by slices of the same geometric properties.

Incidentally, in the outstanding non-rotating case, the condition (100) imposed on $\mu$ already determined via the Bianchi identities by $\omega$ becomes a constraint. However, non-trivial solutions do exist: an example is a spherically symmetric section of the Schwarzschild horizon.

5.3. Isolated null surfaces that are simultaneously non-extremal and extremal

As we mentioned above it is possible that an isolated null surface admits two null symmetry generators, one non-extremal and another one extremal. We study this special case now. We will argue now, that it is necessarily of the Petrov type D.

Consider a non-expanding isolated null surface $H$ of a symmetry generator $\ell$ and intrinsic geometry $(h, \nabla)$. Suppose there exists another null vector field tangent to $H$, $\ell_o$ say, non-vanishing on a dense subset of $H$ and such that

$$\kappa(\ell_o) = 0, \quad [\mathcal{L}_\ell, \nabla_a] = 0.$$  \hspace{1cm} (102)

The symmetry generators are related to each other by

$$\ell = f_1 \ell_o, \quad df_1 \neq 0.$$  \hspace{1cm} (103)

On $H$ we have two rotation 1-form potentials: $\omega^{(\ell)}$ and $\omega^{(\ell_o)}$. They are related to each other by

$$\omega^{(\ell)} = \omega^{(\ell_o)} + d\ln f_1.$$  \hspace{1cm} (104)

For every section $s : S \rightarrow H$, we have two pullback 1-forms:

$$\omega = s^* \omega^{(\ell)}$$

and

$$\omega^{(o)} = s^* \omega^{(\ell_o)} = \omega - d\ln^* f_1.$$  \hspace{1cm} (105)

In the latter one, $\omega^{(o)}$ is independent of the choice of the section $s$ (because $\kappa^{(\ell_o)} = 0$), and it satisfies the extremal isolated horizon equation

$$\nabla (A \omega^{(o)}_A + \omega^{(o)}_A \omega^{(o)}_B) - \frac{1}{2} R_{AB} + \frac{1}{2} \Lambda g_{AB} = 0.$$  \hspace{1cm} (106)

There exists however a (local) section $s$ such that

$$\ell^o |_{S} = \ell_o.$$  \hspace{1cm} (107)

Indeed, there is the function $u : H \rightarrow \mathbb{R}$ such that [1]

$$Du = \kappa^{(\ell)} u, \quad \ell_o^a \partial_a u = f_2, \quad Df_2 = 0.$$  \hspace{1cm} (108)

The desired section $s$ is defined by the condition

$$u |_{s(S)} = \frac{f_2}{\kappa^{(\ell)}}.$$  \hspace{1cm} (109)

For this section $s : S \rightarrow H$ the pullbacks of $\omega^{(\ell_o)}$ and $\omega^{(\ell)}$ coincide,

$$\omega^{(o)} = \omega.$$
hence, the equation (90) holds for \( \omega \) as well. It implies (see theorem 2), that the Weyl tensor at \( H \) is of the Petrov type D.

The opposite is also true. Given a non-extremal isolated horizon \( H \) of the symmetry generator \( \ell \), suppose there is a section \( s : S \rightarrow H \), such that

\[
\omega = s^* \omega(\ell)
\]

satisfies the extremal isolated null surface equation (90). Consider the null vector field \( \ell_o \) tangent to \( H \) defined uniquely by the following two conditions

\[
\kappa(\ell_o) = 0, \quad \text{and} \quad \ell_o|_{S(S)} = \ell.
\]

(107)

It vanishes nowhere, and it follows from (90) that [18]

\[
[L_{\ell_o}, \nabla_a] = 0,
\]

(108)
everywhere on \( H \). Hence, \( \ell_o \) makes \( H \) an extremal isolated null surface. That is, \( H \) has two null symmetries: the non-extremal one \( \ell \) and the extremal one \( \ell_o \).

We can conclude this subsection with the following theorem:

**Theorem 4.** Suppose \( H \) is a 3D non-extremal isolated null surface in a 4D spacetime such that the vacuum Einstein equation (1) with cosmological constant \( \Lambda \) and assumption 1 on stationarity to the second order (or assumption 1') are satisfied; let \( \ell \) be the generator of the null symmetry of \( H \).

(i) Suppose there exists a null vector field \( \ell_o \) tangent to \( H \) and non-vanishing on \( H \), such that

\[
\kappa(\ell_o) = 0, \quad \text{and} \quad [L_{\ell_o}, \nabla_a] = 0.
\]

(109)

Then, at every point \( x \in H \) the spacetime Weyl tensor is either of the Petrov type D or of the Petrov type O.

(ii) Suppose, there is a section \( s : S \rightarrow H \) such that the corresponding \( \omega \) (22) satisfies the extremal isolated null surface equation (90). Then, there is a nowhere vanishing function \( f \) defined on \( H \) such that the vector field

\[
\ell_o := f \ell
\]

satisfies

\[
\kappa(\ell_o) = 0, \quad \text{and} \quad [L_{\ell_o}, \nabla_a] = 0.
\]

(110)

6. No hair theorem for the Petrov type D axisymmetric isolated horizons

In this subsection we consider an axisymmetric non-extremal isolated horizon \( H \) such that the space \( S \) of the null generators is diffeomorphic to a 2D sphere. We outline the results that will be derived in detail in the accompanying paper [11]. Let \( \Phi^a \in \Gamma(T(H)) \) be the generator of the 1D group of rotations of \( H \), that preserve the geometry \( (g_{ab}, \nabla_a) \) invariant. That is

\[
\mathcal{L}_{\Phi} g_{ab} = 0, \quad [\mathcal{L}_{\Phi}, \nabla_a] = 0.
\]

In the consequence, the vector field \( \Phi \) Lie drags the rotation 2-form invariant,

\[
\mathcal{L}_{\Phi} \Omega = 0.
\]
The projection $\Pi : H \rightarrow S$ (10) pushes forward the vector field $\Phi^A$ onto a uniquely defined vector field $\Phi^A$ on $S$, $\Phi^A \in \Gamma(T(S))$. It becomes a Killing vector of the metric tensor $g_{AB}$, and Lie draggers the pulled back rotation 2-form

$$\mathcal{L}_\Phi \Omega_{AB} = 0.$$

Therefore, the problem of finding all the Petrov type D isolated null surfaces in this class amounts to solving the Petrov type D equation (86) on a 2-sphere assuming the axial symmetry of the unknowns: a metric $g_{AB}$ and an exact 2-form $\Omega_{AB}$.

To characterize axisymmetric solutions to (86) it is convenient to introduce on $S$ a function $\chi$ defined by the area element,

$$\Phi^A \epsilon_{AB} = \partial_B \chi.$$

It turns out, that every axisymmetric complex valued function $\Psi$ that solves the Petrov type D equation (86) has a form

$$\Psi = \frac{1}{(a_1 \chi + a_2)^2} - \frac{\Lambda}{6},$$

(111)

where $a_1, a_2 \in \mathbb{C}$ are constant.

On the 2-sphere, the complex valued invariant $\Psi$ satisfies a global topological constraint following from its relation (39) with the Gaussian curvature $K$ (the Gauss–Bonnet theorem) and the exact 2-form $\Omega$ (the Stokes’ theorem), namely

$$\int_S \Psi \epsilon = -2\pi.$$

(112)

Still, an axisymmetric metric tensor $g_{AB}$ and rotation 2-form $\Omega_{AB}$, respectively integrated out of $\Psi$ via (39) have to be regular at the poles. A detailed analysis shows [11] (for the $\Lambda \neq 0$ case), that the resulting family of solutions is 2D. Moreover, the two parameters can be chosen to be the area $A$,

$$A = \int_S \epsilon,$$

and the angular momentum [2]. The general formula for the angular momentum of an axisymmetric isolated horizon reads

$$J = -\frac{1}{4\pi} \int_S \chi \text{Im} \Psi \epsilon.$$

(113)

In conclusion, the result reads:

**Theorem 5 (No-hair).** On a topological 2-sphere $S$, every axisymmetric solution $g_{AB}$ and $\Omega_{AB}$ to the Petrov type D equation (86) with a cosmological constant $\Lambda$ is uniquely determined by a pair of numbers: the area $A$, and the angular momentum $J$. For every pair $(A, J)$ such that:

- **for positive cosmological constant**: $J \in (-\infty, \infty)$ for $A \in (0, \frac{12\pi}{\Lambda})$ and $|J| \in \left[0, \frac{A}{8\pi} \sqrt{\frac{\Lambda}{12\pi} - 1} \right]$ for $A \in \left(\frac{12\pi}{\Lambda}, \infty\right)$;
- **for negative cosmological constant**: $J \in (-\infty, \infty)$ and $A \in (0, \infty)$;

$^4$The case in which $A = \frac{12\pi}{\Lambda}$ has been excluded.
there is a unique solution. Every solution gives rise to a non-extremal vacuum isolated horizon with the cosmological constant $\Lambda$ and of the Petrov type D.

A large class of those isolated horizons are embeddable in Schwarzschild/Kerr ($\Lambda = 0$) or Schwarzschild/Kerr (anti) de Sitter (or just de Sitter) spacetimes. There exist also exceptional cases, which admit another null symmetry that is extremal. Those are embeddable in the near (extremal) horizon spacetimes.

The spacetime characterization of the Kerr metric is available in the literature and it provides the uniqueness properties [20, 21]. The results outlined in this section were derived for $\Lambda = 0$ in [9]. For the generalization to the $\Lambda \neq 0$ case see the accompanying paper [11].

7. Summary

The results of this paper are collected in the theorems 0–5. The vacuum Einstein equations (EE) and assumption 1 on stationarity to the second order at the non-extremal isolated surface $H$ makes the spacetime Weyl tensor $C$ thereon determined by the intrinsic geometry. In that way properties of $C$ at $H$ become properties of the intrinsic geometry. The components of $C$ are constrained at $H$ in such a way, that the Petrov type is either II or D or O. The Petrov types III and N can emerge only at measure zero subsets of $H$ (theorem 0). $H$ can be the Petrov type O only if it is non-rotating and the Gaussian curvature of a spacelike section equals $\frac{1}{\Lambda}$. The condition for the Petrov type D takes the form of a complex differential equation on certain complex invariant of the intrinsic geometry constructed from the Gaussian curvature and the rotation scalar invariant (theorem 1). In general, the transversal double principal direction $n'$ of the Weyl tensor is twisting. There are, however, special cases such that $n'$ is orthogonal to a foliation of $H$ with spacelike slices. Then, the rotation 1-form potential pulled back to any of those slices satisfies the near horizon geometry equation (theorem 3). That case has one more geometric characterization, namely $H$ is both, non-extremal and extremal (theorem 4). We also formulate a no-hair theorem valid for the Petrov type D axisymmetric null surfaces of topologically spherical cross-sections (isolated horizons). The intrinsic geometry of each of them is completely determined by two parameters: the area $A$ and the angular momentum $J$ (theorem 5). $A$ and $J$ take arbitrary values (of course, $A > 0$). A detailed derivation is presented in the accompanying paper [11]. A large class of those isolated horizons is embeddable in Kerr(Schwarzshild)-(anti) de Sitter spacetimes provided $n'$ is twisting. In the hypersurface orthogonal case $H$ can be embedded in a near extremal horizon limit spacetime [22]. This work is the first one of the series of papers on the type D isolated null surfaces [11, 23, 24]. The derivation of axisymmetric solutions to the type D equation on topological sphere—the proof of theorem 5 above—is presented in [11]. The general solution of the type D equation on genus $>0$ sections of isolated horizons is found in [24]. The subset of solutions to the type D equation that is closed with respect to the map $(g, \omega) \mapsto (g, -\omega)$ is studied in [23]. It is shown that each of those solutions has an extra symmetry, that makes it axisymmetric in the case of topological sphere. That last result was motivated by a new work by Cole et al [25] on the Killing spinor characteristic data. According to that work, in the $\Lambda = 0$ case, our type D equation is the consistency condition for the Killing spinors characteristic data. The Killing spinor generates the axial symmetry.
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