I. INTRODUCTION

In the paper by Leitner et al. [1] (see also [2, 3]) the matrix element of the transition “neutron → proton” or \( n \rightarrow p \), induced by the charged hadronic vector current \( V_\mu^{(+)}(0) \), has been written in the following form

\[
\langle p(k_p, \sigma_p)|V_\mu^{(+)}(0)|n(k_n, \sigma_n)\rangle = u_p(k_p, \sigma_p)\left(\frac{q_\mu}{q^2}\right) F_1(q^2) + \frac{i\sigma_{\mu\nu}q^\nu}{2m_N} F_2(q^2) u_n(k_n, \sigma_n) =
\]

\[
u_p(k_p, \sigma_p)\left(\gamma_\mu\right) \left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_1(q^2) + \frac{i\sigma_{\mu\nu}q^\nu}{2m_N} F_2(q^2) u_n(k_n, \sigma_n),
\]

where \( u_p(k_p, \sigma_p) \) and \( u_n(k_n, \sigma_n) \) are the Dirac bispinor wave functions of the free proton and neutron in the final and initial states of the transition \( n \rightarrow p \), \( m_N = (m_p + m_n)/2 \) is a nucleon mass or an averaged nucleon mass, expressed in terms of the proton mass \( m_p \) and neutron mass \( m_n \), \( \eta_{\mu\nu} \) is the metric tensor of the Minkowski spacetime, and \( \gamma_\mu \) and \( \sigma_{\mu\nu} = \frac{i}{2}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) \) are the Dirac matrices [4]. Then, \( q = k_p - k_n \) is the momentum transferred, and \( F_1(q^2) \) and \( F_2(q^2) \) are the form factors. The second term in Eq. (1) describes the contribution of the weak magnetism. The right-hand-side (r.h.s.) of Eq. (1) vanishes after multiplication by a momentum transferred \( q^\mu \), i.e.

\[
q^\mu \langle p(k_p, \sigma_p)|V_\mu^{(+)}(0)|n(k_n, \sigma_n)\rangle = 0,
\]

even for \( m_p \neq m_n \). Such a property of the matrix element of the transition \( n \rightarrow p \) testifies conservation of the charged hadronic vector current \( V_\mu^{(+)}(x) \), but only in the sense of the vanishing matrix element \( \langle p(k_p, \sigma_p)|\partial^\mu V_\mu^{(+)}(x)|n(k_n, \sigma_n)\rangle = 0 \). This, of course, should not contradict the hypothesis of conservation of the vector or the CVC hypothesis by Feynman and Gell-Mann [5]. Recently [6] we have shown that the term \(|-q_\mu q^\mu/F_1(q^2)| \) is the contribution of the first class current [7].

This letter is addressed to the analysis of the dynamical nature of the term with the Lorentz structure \( q_\mu q^\mu \). As has been proposed in [6], the vector part of the matrix element of the transition \( n \rightarrow p \), caused by the contributions of the first class current only, should be taken in the following general form

\[
\langle p(k_p, \sigma_p)|V_\mu^{(+)}(0)|n(k_n, \sigma_n)\rangle = \bar{u}_p(k_p, \sigma_p)\left(\gamma_\mu F_1(q^2) + \frac{i\sigma_{\mu\nu}q^\nu}{2m_N} F_2(q^2) + \frac{q_\mu q^\mu}{m_N^2} F_3(q^2)\right) u_n(k_n, \sigma_n).
\]

Below we show that the appearance of the term with the Lorentz structure \( q_\mu q^\mu \) is fully caused by strong low–energy interactions.

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We analyse the Lorentz structure of the matrix elements of the transitions “neutron → proton”, induced by the charged hadronic vector current. We show that the term providing conservation of the charged hadronic vector current in the sense of the vanishing matrix element of the divergence of the charged hadronic vector current of the transitions “neutron → proton” even for different masses of the neutron and proton (see T. Leitner et al., Phys. Rev. C 73, 065502 (2006) and A. M. Ankowski, arXiv:1601.06169 [hep-ph]) has a dynamical origin, related to the G–even first class current contribution. We show that because of invariance of strong low–energy interactions under the G–parity transformations, the G–odd contribution with the Lorentz structure \( q_\mu \), where \( q_\mu \) is a momentum transferred, does not appear in the matrix elements of the “neutron → proton’ transitions.

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The paper is organized as follows. In section II we propose for the analysis of the dynamical nature of the term with the Lorentz structure \( q_\mu q_\nu /q^4 \) to use a strongly coupled \( \pi N \)–system with the linear pion–nucleon pseudoscalar interaction. We show that only the total hadronic isovector vector current, being the sum of the nucleon and mesonic currents, can be locally conserved. In section III we derive the Lorentz structure of the matrix element of the transition \( n \rightarrow p \) using the path–integral technique. In section IV we discuss the obtained results. In Appendix A we calculate the cross sections for the inelastic electron neutrino–nucleon scattering and for the inverse \( \beta \)–decay. In order to illustrate the influence of the contributions of the term with the Lorentz structure \(-q_\mu q_\nu /q^4 F_1(q^2)\) we neglect the contributions of the weak magnetism and recoil of the outgoing nucleon, and the radiative corrections. In the Appendix B we plot the relative contributions of the term \(-q_\mu q_\nu /q^4 F_1(q^2)\). We show that the processes of the inelastic electron neutrino–nucleon scattering and of the inverse \( \beta \)–decay are insensitive to the contributions of the term, responsible for the vanishing of the matrix elements \( \langle p|\partial^\mu V_\mu^{(+)}(0)|n\rangle = \langle n|\partial^\mu V_\mu^{(-)}(0)|p\rangle = 0 \) for different masses of the neutron and proton. In Appendix B we analyse the dynamical nature of the Lorentz structure of the matrix element \( \langle p|A_\mu^{(+)}(0)|n\rangle \) of the transition \( n \rightarrow p \), caused by the charged hadronic axial–vector current. We show that the linear pion–nucleon pseudoscalar interaction, used for the analysis of the dynamical nature of the Lorentz structure of the charged hadronic vector part of the transition \( n \rightarrow p \), allows to reproduce fully the standard Lorentz structure of the axial–vector part of the hadronic \( n \rightarrow p \) transition.

II. HADRONIC VECTOR CURRENT OF STRONGLY COUPLED PION–NUCLEON SYSTEM

As an example of strongly coupled system we consider the \( \pi N \)–system with the simplest linear pseudoscalar interaction \[9\]. The Lagrangian of such a system is given by \[9\]:

\[
\mathcal{L}_{\pi N}(x) = \bar{N}(x)(i\gamma^\mu \partial_\mu - m_N)N(x) + \frac{1}{2} \partial_\mu \bar{\pi}(x) \cdot \partial_\mu \pi(x) - \frac{1}{2} m_\pi^2 \pi^2(x) + g_\pi \bar{N}(x)i\gamma^5 \pi \cdot \bar{\pi}(x)N(x). \tag{4}
\]

Here \( N(x) \) is the nucleon isospin doublet with components \((p(x), n(x))\), where \( p(x) \) and \( n(x) \) are the proton and neutron field operators, \( \bar{\pi}(x) = (\pi^+, \pi^0, \pi^-) \) is the pion field operator, \( m_N \) and \( m_\pi \) are the nucleon and pion masses, \( g_\pi \) is the pion–nucleon coupling constant, \( \gamma^5 \) is the Dirac matrix \[9\], and \( \bar{\pi} = (\tau_1, \tau_2, \tau_3) \) is the Pauli isospin matrix \[9\].

The Lagrangian Eq. (4) is invariant under global isospin transformations \[9\]. This, according to Feynman and Gell–Mann \[9\], leads to the isovector hadronic vector current of the \( \pi N \) system given by

\[
\bar{V}_\mu(x) = \frac{1}{2} \bar{N}(x)\gamma_\mu N(x) + \bar{\pi}(x) \times \partial_\mu \bar{\pi}(x),
\tag{5}
\]

local conservation of which one may check using the equations of motion. The Dirac equation for the nucleon and the Klein–Gordon equation for the pions are given by

\[
(i\gamma^\mu \partial_\mu - m_N + g_\pi i\gamma^5 \bar{\pi} \cdot \bar{\pi}(x))N(x) = 0,
\]

\[
(i\gamma^\mu \partial_\mu - m_\pi^2)\bar{\pi}(x) - g_\pi \bar{N}(x)i\gamma^5 \bar{\pi}(x)N(x) = 0.
\tag{6}
\]

Using the Dirac equation for the nucleon Eq. (6) one may show that the nucleon part of the isovector hadronic vector current Eq. (5) is not conserved

\[
\partial^\mu \left( \frac{1}{2} \bar{N}(x)\gamma_\mu N(x) \right) = \frac{1}{2} \partial^\mu \bar{N}(x)\gamma_\mu \bar{\pi} N(x) + \frac{1}{2} \bar{N}(x)\gamma_\mu \bar{\pi} \partial^\mu N(x) - \bar{\pi}(x) \times g_\pi \bar{N}(x)i\gamma^5 \bar{\pi} N(x).
\tag{7}
\]

Hence, in the strongly coupled \( \pi N \)–system a strong non–conservation of the nucleon part of the isovector hadronic vector current is caused by strong low–energy interactions but not by isospin violation. The divergence of the mesonic part of the isovector hadronic vector current is equal to

\[
\partial^\mu \left( \bar{\pi}(x) \times \partial_\mu \bar{\pi}(x) \right) = \bar{\pi}(x) \times g_\pi \bar{N}(x)i\gamma^5 \bar{\pi} N(x).
\tag{8}
\]

Summing up the contributions Eqs. (7) and (8) we get \( \partial^\mu \bar{V}_\mu(x) = 0 \). This means that in the strongly coupled \( \pi N \)–system only the total hadronic isovector vector current, being the sum of the nucleon and mesonic currents, can be locally conserved.
The charged hadronic vector current responsible for the hadronic $n \rightarrow p$ transition is equal to [3]

$$V_\mu^{(+)}(x) = \tilde{N}(x)\tau^+(\gamma_\mu N(x) + 2\varepsilon^{+bc}\pi^b(x)\partial_\mu\pi^c(x) - \bar{p}(x)\gamma_\mu n(x)) + \sqrt{2}i\left(\pi^0(x)\partial_\mu\pi^-(x) - \pi^-(x)\partial_\mu\pi^0(x)\right),$$

(9)

where $\tau^+ = (\tau^1 + i\tau^2)/2$, $\varepsilon^{+bc} = (\varepsilon^{1bc} + i\varepsilon^{2bc})/2$ and $\varepsilon^{abc}$ is the Levi-Civita isotensor [3]. Now we may calculate the matrix element

$$\langle \mathrm{out}, p(\vec{k}_p, \sigma_p) | V_\mu^{(+)}(0) | \mathrm{in}, n(\vec{k}_n, \sigma_n) \rangle,$$

(10)

where $\langle \mathrm{out}, p(\vec{k}_p, \sigma_p) \rangle$ and $| \mathrm{in}, n(\vec{k}_n, \sigma_n) \rangle$ are the wave functions of the free proton and neutron in the final (i.e. out–state at $t \rightarrow +\infty$) and initial (i.e. in–state at $t \rightarrow -\infty$) states, respectively [3]. Using the relation $\langle \mathrm{out}, p(\vec{k}_p, \sigma_p) \rangle = \langle \mathrm{in}, p(\vec{k}_p, \sigma_p) \rangle | S \rangle$, where $S$ is the $S$–matrix, we rewrite the matrix element Eq. (10) as follows

$$\langle \mathrm{out}, p(\vec{k}_p, \sigma_p) | V_\mu^{(+)}(0) | \mathrm{in}, n(\vec{k}_n, \sigma_n) \rangle = \langle \mathrm{in}, p(\vec{k}_p, \sigma_p) | T \left( \epsilon^i \int d^4x \mathcal{L}_{\pi NN}(x) \right) | \mathrm{out}, n(\vec{k}_n, \sigma_n) \rangle.$$

(11)

Since the transition $n \rightarrow p$ is fully induced by strong low–energy interactions, we define the $S$–matrix only in terms of strong low–energy interactions. For simplicity we propose to use only $\pi NN$–system, a dynamics of which is determined by the Lagrangian Eq. (3). The corresponding $S$–matrix is given by [2]

$$S = T e^i \int d^4x \mathcal{L}_{\pi NN}(x),$$

(12)

where $T$ is a time–ordering operator and $\mathcal{L}_{\pi NN}(x)$ is equal to

$$\mathcal{L}_{\pi NN}(x) = g_\pi \tilde{N}(x)i\gamma^5 \tau \cdot \bar{\pi}(x)N(x).$$

(13)

Plugging Eq. (12) into Eq. (11) we get

$$\langle \mathrm{in}, p(\vec{k}_p, \sigma_p) | V_\mu^{(+)}(0) | \mathrm{in}, n(\vec{k}_n, \sigma_n) \rangle = \langle \mathrm{in}, p(\vec{k}_p, \sigma_p) | T \left( \epsilon^i \int d^4x \mathcal{L}_{\pi NN}(x) V_\mu^{(+)}(0) \right) | \mathrm{in}, n(\vec{k}_n, \sigma_n) \rangle.$$

(14)

The wave functions of the neutron and proton we determine in terms of the operators of creation (annihilation)

$$| \mathrm{in}, n(\vec{k}_n, \sigma_n) \rangle = a_{n, in}^\dagger(\vec{k}_n, \sigma_n) | 0 \rangle,$$

$$| \mathrm{in}, p(\vec{k}_p, \sigma_p) \rangle = | 0 \rangle a_{p, in}^\dagger(\vec{k}_p, \sigma_p).$$

(15)

The operators $(a_{n, in}^\dagger(\vec{k}_n, \sigma_n), a_{p, in}^\dagger(\vec{k}_p, \sigma_p))$ and $(a_{n, in}(\vec{k}_n, \sigma_n), a_{p, in}(\vec{k}_p, \sigma_p))$ obey standard anticommutation relations [4]

$$[a_{n, in}(\vec{k}_n', \sigma_n'), a_{n, in}^\dagger(\vec{k}_n, \sigma_n)] = (2\pi)^3 2\delta_{n, n'} \delta(\vec{k}_n' - \vec{k}_n) \delta_{\sigma_n, \sigma_n'},$$

$$[a_{n, in}(\vec{k}_n', \sigma_n'), a_{n, in}(\vec{k}_n, \sigma_n)] = [a_{n, in}^\dagger(\vec{k}_n', \sigma_n'), a_{n, in}^\dagger(\vec{k}_n, \sigma_n)] = 0,$$

$$[a_{p, in}(\vec{k}_p', \sigma_p'), a_{p, in}^\dagger(\vec{k}_p, \sigma_p)] = (2\pi)^3 2\delta_{n, n'} \delta(\vec{k}_p' - \vec{k}_p) \delta_{\sigma_p, \sigma_p'},$$

$$[a_{p, in}(\vec{k}_p', \sigma_p'), a_{p, in}^\dagger(\vec{k}_p, \sigma_p)] = [a_{p, in}^\dagger(\vec{k}_p', \sigma_p'), a_{p, in}^\dagger(\vec{k}_p, \sigma_p)] = 0.$$

(16)

The vacuum wave function we define as follows $| 0 \rangle = | 0_N \rangle | 0_\sigma \rangle$, where $| 0_N \rangle$ and $| 0_\sigma \rangle$ are the vacuum wave functions of a nucleon and mesons, respectively. Since there are no mesons in the initial and final states of the transition $n \rightarrow p$, the matrix element Eq. (14) we may rewrite as follows

$$\langle 0_N | T \left( \epsilon^i \int d^4x \mathcal{L}_{\pi NN}(x) V_\mu^{(+)}(0) \right) | 0_\sigma \rangle =$$

$$= T \left( \frac{1}{2} \bar{N}(0)\tau^+(\gamma_\mu N(0)) \right) \int D\bar{\pi} \epsilon^i \int d^4x \left( \frac{1}{2} \bar{\partial}_\mu \bar{\pi}(x) \cdot \partial^\mu \bar{\pi}(x) - \frac{1}{2} m_\pi^2 \bar{\pi}^2 + g_\pi \bar{N}(x)i\gamma^5 \tau \cdot \bar{\pi}(x)N(x) \right)$$

$$+ 2\varepsilon^{+bc} \int D\bar{\pi} \pi^b(x) \epsilon^i \int d^4x \left( \frac{1}{2} \bar{\partial}_\mu \bar{\pi}(x) \cdot \partial^\mu \bar{\pi}(x) - \frac{1}{2} m_\pi^2 \bar{\pi}^2 + g_\pi \bar{N}(x)i\gamma^5 \tau \cdot \bar{\pi}(x)N(x) \right) \bigg|_{z=0}.$$

(18)
The integrals are Gaussian. The calculation of the first integral runs as follows. We transcribe it into the form

\[
\int D\pi e^i\int d^4x \left( \frac{i}{2} \partial_\alpha \tilde{\pi}(x) \cdot \partial^\alpha \tilde{\pi}(x) - \frac{1}{2} m_\pi^2 \tilde{\pi}^2 + g_\pi N(x) i\gamma^5 \pi \cdot \tilde{\pi}(x) N(x) \right) = \\
= \int D\pi e^i\int d^4x \left( -\frac{i}{2} \tilde{\pi} \cdot (\square + m_\pi^2 - i0) \tilde{\pi} + g_\pi \bar{N}(x) i\gamma^5 \pi \cdot \tilde{\pi}(x) N(x) \right). 
\]

Then, we make a shift

\[
\tilde{\pi}(x) \rightarrow \tilde{\pi}(x) + \frac{1}{\square + m_\pi^2 - i0} g_\pi \bar{N}(x) i\gamma^5 \pi \cdot N(x) = \tilde{\pi}(x) + \int d^4y \Delta(x-y) g_\pi \bar{N}(y) i\gamma^5 \pi N(y), 
\]

where \(\Delta(x-y)\) is the \(\pi\)-meson propagator. The result of the integration is

\[
\int D\pi e^i\int d^4x \left( \frac{i}{2} \partial_\alpha \tilde{\pi}(x) \cdot \partial^\alpha \tilde{\pi}(x) - \frac{1}{2} m_\pi^2 \tilde{\pi}^2 + g_\pi \bar{N}(x) i\gamma^5 \pi \cdot \tilde{\pi}(x) N(x) \right) = \\
= e^{i\frac{1}{2} g_\pi^2} \int d^4x d^4y \bar{N}(x) i\gamma^5 \pi N(x) \cdot \Delta(x-y) \bar{N}(y) i\gamma^5 \pi N(y).
\]

For the integration of the pionic part of the charge hadronic vector current we use the following procedure. We rewrite the path-integral, given by the second term in the r.h.s. of Eq. (15), with an external source \(\bar{J}(x)\) of the \(\pi\)-meson field:

\[
\varepsilon^{abc} \int D\pi^b(z) \partial_\mu \pi^c(z) e^i\int d^4x \left( \frac{i}{2} \partial_\alpha \tilde{\pi}(x) \cdot \partial^\alpha \tilde{\pi}(x) - \frac{1}{2} m_\pi^2 \tilde{\pi}^2 + g_\pi \bar{N}(x) i\gamma^5 \pi \cdot \tilde{\pi}(x) N(x) \right) \bigg|_{z=0} \rightarrow \\
= \varepsilon^{abc} \int D\pi^b(z) \partial_\mu \pi^c(z) e^i\int d^4x \left( \frac{i}{2} \partial_\alpha \tilde{\pi}(x) \cdot \partial^\alpha \tilde{\pi}(x) - \frac{1}{2} m_\pi^2 \tilde{\pi}^2 + g_\pi \bar{N}(x) i\gamma^5 \pi \cdot \tilde{\pi}(x) N(x) + \bar{J}(x) \tilde{\pi}(x) \right) \bigg|_{z=0} \rightarrow \\
= \varepsilon^{abc} \int D\pi^b(z) \partial_\mu \pi^c(z) e^i\int d^4x \left( -\frac{1}{2} \tilde{\pi}(x) \cdot (\square + m_\pi^2 - i0) \tilde{\pi}(x) + g_\pi \bar{N}(x) i\gamma^5 \pi \cdot \tilde{\pi}(x) N(x) + \bar{J}(x) \tilde{\pi}(x) \right) \bigg|_{z=0}.
\]

Then, the pionic fields in the integrand we replace by functional derivatives with respect to the external source:

\[
\varepsilon^{abc} \int D\pi^b(z) \partial_\mu \pi^c(z) e^i\int d^4x \left( \frac{i}{2} \partial_\alpha \tilde{\pi}(x) \cdot \partial^\alpha \tilde{\pi}(x) - \frac{1}{2} m_\pi^2 \tilde{\pi}^2 + g_\pi \bar{N}(x) i\gamma^5 \pi \cdot \tilde{\pi}(x) N(x) \right) \bigg|_{z=0} \rightarrow \\
\varepsilon^{abc} \left( -i \right) \frac{\delta}{\delta J^b(z)} \frac{\delta}{\delta z^c(z)} \frac{\delta}{\delta J^c(z)} e^i\int d^4x \left( -\frac{1}{2} \tilde{\pi}(x) \cdot (\square + m_\pi^2 - i0) \tilde{\pi}(x) + g_\pi \bar{N}(x) i\gamma^5 \pi \cdot \tilde{\pi}(x) N(x) + \bar{J}(x) \tilde{\pi}(x) \right) \bigg|_{z=0,\bar{J}=0}.
\]

For the calculation of the integral over \(\tilde{\pi}\) we make a change of variables

\[
\tilde{\pi}(x) \rightarrow \tilde{\pi}(x) + \frac{1}{\square + m_\pi^2 - i0} \left( g_\pi \bar{N}(x) i\gamma^5 \pi \cdot N(x) + \bar{J}(x) \right) = \\
= \tilde{\pi}(x) + \int d^4y \Delta(x-y) g_\pi \bar{N}(y) i\gamma^5 \pi N(y) + \bar{J}(y).
\]

As a result, for the integral over \(\tilde{\pi}\) we obtain the following expression

\[
\int D\pi e^i\int d^4x \left( -\frac{1}{2} \tilde{\pi}(x) \cdot (\square + m_\pi^2 - i0) \tilde{\pi}(x) + g_\pi \bar{N}(x) i\gamma^5 \pi \cdot \tilde{\pi}(x) N(x) + \bar{J}(x) \tilde{\pi}(x) \right) = \\
= e^{i\frac{1}{2} g_\pi^2} \int d^4x d^4y \left( g_\pi \bar{N}(x) i\gamma^5 \pi N(x) + \bar{J}(x) \right) \cdot \Delta(x-y) \left( g_\pi \bar{N}(y) i\gamma^5 \pi N(y) + \bar{J}(y) \right).
\]

Plugging Eq. (20) into Eq. (21) and calculating the functional derivatives with respect to external sources we arrive at the expression

\[
\varepsilon^{abc} \int D\pi^b(z) \partial_\mu \pi^c(z) e^i\int d^4x \left( \frac{i}{2} \partial_\alpha \tilde{\pi}(x) \cdot \partial^\alpha \tilde{\pi}(x) - \frac{1}{2} m_\pi^2 \tilde{\pi}^2 + g_\pi \bar{N}(x) i\gamma^5 \pi \cdot \tilde{\pi}(x) N(x) \right) \bigg|_{z=0} = \\
= \varepsilon^{abc} g_\pi^2 \int d^4x \left( \bar{N}(x) i\gamma^5 \pi N(x) \right) \int d^4y \bar{N}(y) i\gamma^5 \pi N(y) \partial_\mu \Delta(-y) \times \frac{1}{2} g_\pi^2 \int d^4x d^4y \left( g_\pi \bar{N}(x) i\gamma^5 \pi N(x') \cdot \Delta(x'-y') \bar{N}(y') i\gamma^5 \pi N(y') \right).
\]

(26)
Thus, after the calculation of the vacuum expectation value Eq.(18), the matrix element Eq.(11) of the transition \( n \rightarrow p \) becomes equal to

\[
\langle \text{out}, p(k_p, \sigma_p) | V^+_{\mu}(0) | \text{in}, n(k_n, \sigma_n) \rangle = N \langle \text{in}, p(k_p, \sigma_p) | T \left( \left( \bar{N}(0) \gamma_{\mu} N(0) + 2 \varepsilon^{+bc} g_\pi^2 \int d^4 x \Delta(x) \bar{N}(x) \gamma^5 \gamma^c N(x) \right) \cdot \Delta(x' - y') \bar{N}(y') \gamma^5 \gamma^c N(y') \right) | \text{in}, n(k_n, \sigma_n) \rangle_n
\]

As the first step towards the analysis of the Lorentz structure of the matrix element of the transition \( n \rightarrow p \), given by Eq.(27), we propose to consider the contributions of order \( g_\pi^2 \). We understand that the value of the coupling constant \( g_\pi \) is sufficiently large. Nevertheless, the Lorentz structure of the matrix element Eq.(27) can be fully understood to order \( g_\pi^2 \).

A. Lorentz structure of the matrix element Eq.(27) to order \( g_\pi^2 \), determined by the mesonic part of the charged hadronic vector current Eq.(9)

To order \( g_\pi^2 \) the contribution of the mesonic part of the charged hadronic vector current is given by the expression

\[
N \langle \text{in}, p(k_p, \sigma_p) | T \left( 2 \varepsilon^{+bc} g_\pi^2 \int d^4 x \Delta(x) \bar{N}(x) \gamma^5 \gamma^c N(x) \right) \cdot \Delta(x') - y' \bar{N}(y') \gamma^5 \gamma^c N(y') \right) | \text{in}, n(k_n, \sigma_n) \rangle_n = \bar{u}_p(k_p, \sigma_p) g_\pi^2 \int d^4 x \Delta(x) e^{ik_p \cdot x} \gamma^5 \int d^4 y (-i) S_F(x - y) \gamma^5 \gamma_\mu \Delta(-y) e^{-ik_n \cdot y} u_n(k_n, \sigma_n)
\]

(28)

where \( S_F(x - y) \) is the nucleon propagator \[9\]. For the derivation of Eq.(28) we have used the relation \( \varepsilon^{+bc} \gamma^c = 2i\tau^+(+) \). In the momentum representation the r.h.s. of Eq.(28) reads

\[
N \langle \text{in}, p(k_p, \sigma_p) | T \left( 2 \varepsilon^{+bc} g_\pi^2 \int d^4 x \Delta(x) \bar{N}(x) \gamma^5 \gamma^c N(x) \right) \cdot \Delta(x') - y' \bar{N}(y') \gamma^5 \gamma^c N(y') \right) | \text{in}, n(k_n, \sigma_n) \rangle_n = \bar{u}_p(k_p, \sigma_p) g_\pi^2 \int d^4 p \gamma_\mu \left( \frac{1}{m_N - \bar{p} - i0} \right) \gamma_5 \left( \frac{1}{m_N - \bar{p} - i0} \right) \gamma_5 \left( \frac{k_p + k_n - 2p}_\mu \right) u_n(k_n, \sigma_n).
\]

(29)

The integral is symmetric with respect to the transformation \( k_p \leftrightarrow k_n \). This means that the momentum integral possesses the following Lorentz structure

\[
\int d^4 p \gamma_\mu \left( \frac{1}{m_N} - \bar{p} - i0 \right) \gamma_5 \left( \frac{1}{m_N} - \bar{p} - i0 \right) \gamma_5 \left( \frac{k_p + k_n - 2p}_\mu \right) u_n(k_n, \sigma_n) = a \gamma_\mu + b P_\mu \bar{P} + c q_\mu \gamma_5 q_\mu,
\]

(30)

which can be confirmed by a direct calculation of the integral, where \( P = (k_p + k_n)/2 \) and \( a, b, c \) are coefficients, which can be determined by a direct calculation of the integral. The symmetry of the integral with respect to the transformation \( k_p \leftrightarrow k_n \) testifies that the term with the Lorentz structure \( q_\mu = (k_p - k_n)_\mu \), which is antisymmetric with respect to the transformation \( k_p \leftrightarrow k_n \), does not appear in the matrix element of the transition \( n \rightarrow p \) in agreement with a suppression of the contributions of the second class currents \[7\].

As a consequence of the relations \( m_N^2 \gg m_\pi^2 \gg q^2 \) one may perform the calculation of the momentum integral Eq.(30) in the heavy baryon approximation \[12\]. Since we are interested in the term with the Lorentz structure \( P_\mu \bar{P} \) only, skipping standard intermediate steps of the calculations for the coefficient \( c \) we obtain the following result

\[
c = \frac{1}{6m_N m_\pi} \arctan \left( \frac{m_N}{m_\pi} \right),
\]

(31)

where we have neglected the contributions of order \( O(1/m_N^2) \). Then, using the Dirac equations \( \bar{u}_p \bar{P} u_n = (m_p + m_n)/2 = m_N \) that does not violate the property of the term with the Lorentz structure \( P_\mu \bar{P} \) to be a contribution of the first class current \[6\] and the Gordon identity \[13\]

\[
\bar{u}_p(k_p, \sigma_p) \frac{(k_p + k_p)_\mu}{2m_N} u_n(k_n, \sigma_n) = \bar{u}_p(k_p, \sigma_p) \gamma_\mu u_n(k_n, \sigma_n) - \bar{u}_p(k_p, \sigma_p) \frac{i \sigma_{\mu\nu} q^\nu}{2m_N} u_n(k_n, \sigma_n)
\]

(32)
we transcribe Eq. (20) into the form

\[
N(\bar{u}_p, \sigma_p) \left( \frac{2 \pi^2}{g_\pi^2} \int d^4 \bar{x} \Delta(x) \bar{N}(x) i \gamma^5 \gamma^\nu N(x) \int d^4 \bar{N}(y) i \gamma^5 \gamma^\nu N(y) \partial_\mu \Delta(-y) \right) n(\bar{k}_n, \sigma_n)_N = \]

\[
= \bar{u}_p(\bar{k}_p, \sigma_p) g_\pi^2 \left( A_N \gamma_\mu + B_N \frac{i \sigma_\mu q^\nu}{2 m_N} + C_N \frac{\sigma_\mu q}{m_N^2} \right) u(\bar{k}_n, \sigma_n), \tag{33}
\]

where \( A_N = a + m_N b \), \( B_N = -m_N b \) and \( C_N = m_N^2 c \).

B. Lorentz structure of the matrix element Eq. (27) to order \( g_\pi^2 \), determined by the nucleon part of the charged hadronic vector current Eq. (9)

To order \( g_\pi^2 \), the dynamical contribution of the nucleon part of the charged hadronic vector current to the matrix element of the transition \( n \rightarrow p \), given by Eq. (27), is determined by the matrix element

\[
N(\bar{u}_p, \sigma_p) \left( \frac{2 \pi^2}{g_\pi^2} \int d^4 \bar{x} d^4 y \bar{N}(x) i \gamma^5 \gamma^\nu N(x) \cdot \Delta(x-y) \bar{N}(y) i \gamma^5 \gamma^\nu N(y) \right) | n(\bar{k}_n, \sigma_n)_N = \]

\[
= \bar{u}_p(\bar{k}_p, \sigma_p) g_\pi^2 \int d^4 \bar{x} d^4 y \gamma_\mu(i) S_F(x-y) \gamma_5 S_F(x-y) i \gamma^5(i) S_F(x-y) \partial_\mu \gamma_5 e^{-ik_n \cdot y} u(\bar{k}_n, \sigma_n) + u(\bar{k}_p, \sigma_p) g_\pi^2 \int d^4 \bar{x} d^4 y e^{ik_n \cdot x} i \gamma^5(-i) S_F(x-y) \gamma_5 \gamma_\mu \gamma_5 e^{-ik_n \cdot y} u(\bar{k}_n, \sigma_n), \tag{34}
\]

where we have used the relations \( \vec{\tau}^2 = 3 \) and \( \vec{\tau} \cdot \tau^{(+)\tau} = -\tau^{(+)\tau} \). The contributions of the first two terms in Eq. (34) can be removed by renormalization of the masses and wave functions of the neutron and proton, respectively.

Thus, a non–trivial contribution comes from the third term only. In the momentum representation it reads

\[
N(\bar{u}_p, \sigma_p) \left( \frac{2 \pi^2}{g_\pi^2} \int d^4 \bar{x} d^4 y \bar{N}(x) i \gamma^5 \gamma^\nu N(x) \cdot \Delta(x-y) \bar{N}(y) i \gamma^5 \gamma^\nu N(y) \right) | n(\bar{k}_n, \sigma_n)_N = \]

\[
= \bar{u}_p(\bar{k}_p, \sigma_p) g_\pi^2 \left( A_N \gamma_\mu + B_N \frac{i \sigma_\mu q^\nu}{2 m_N} + C_N \frac{\sigma_\mu q}{m_N^2} \right) u(\bar{k}_n, \sigma_n). \tag{35}
\]

This integral is also symmetric with respect to the transformation \( k_p \rightarrow k_n \), so it should also have a structure

\[
\int \frac{d^4 p}{\pi^2} \gamma_5 \frac{1}{m_N - k_p + p - i \theta} \gamma_\mu \frac{1}{m_N - k_n + p - i \theta} \frac{1}{m_N^2 - p^2 - i \theta} = a' \gamma_\mu + b' \vec{P} \gamma_\nu + c' q_\mu q_\nu, \tag{36}
\]

where the coefficients \( a' \), \( b' \) and \( c' \) are determined by a direct calculation of the integral. Thus, the contribution of the term with the Lorentz structure \( \gamma_\mu = (k_p - k_n)_\mu \), which is antisymmetric with respect to the transformation \( k_p \rightarrow k_n \), does not appear in the nucleon part of the charged hadronic vector current. A direct calculation of the integral in Eq. (35) gives the following value of the coefficient \( c' \): \( c' = -1/24 m_N^2 \). Using the Dirac equations \( \bar{u}_p \gamma_5 u = m_N \) that do not violate the property of the term with the Lorentz structure \( \gamma_\mu = (k_p - k_n)_\mu \) that gives the r.h.s. of Eq. (38) we transcribe the r.h.s. of Eq. (38) into the form

\[
N(\bar{u}_p, \sigma_p) \left( \frac{2 \pi^2}{g_\pi^2} \int d^4 \bar{x} d^4 y \bar{N}(x) i \gamma^5 \gamma^\nu N(x) \cdot \Delta(x-y) \bar{N}(y) i \gamma^5 \gamma^\nu N(y) \right) | n(\bar{k}_n, \sigma_n)_N = \]

\[
= \bar{u}_p(\bar{k}_p, \sigma_p) g_\pi^2 \left( A_N \gamma_\mu + B_N \frac{i \sigma_\mu q^\nu}{2 m_N} + C_N \frac{\sigma_\mu q}{m_N^2} \right) u(\bar{k}_n, \sigma_n), \tag{37}
\]

where \( A_N = (a' + m_N b')/4 \), \( B_N = -m_N b'/4 \) and \( C_N = m_N^2 c' \).

Summing up the contributions of the nucleon of the mesonic parts of the charged hadronic vector current for the vector part of the matrix element of the transition \( n \rightarrow p \), calculated to order \( g_\pi^2 \), we obtain the expression

\[
\langle \text{out}, p(\bar{k}_p, \sigma_p) | V^\mu_+ \rangle (0) | n(\bar{k}_n, \sigma_n)_N = N(\bar{u}_p, \sigma_p) \left( \frac{2 \pi^2}{g_\pi^2} \int d^4 \bar{x} \Delta(x) \bar{N}(x) i \gamma^5 \gamma^\nu N(x) \cdot \Delta(x-y) \bar{N}(y) i \gamma^5 \gamma^\nu N(y) \right) \]

\[
\times \int d^4 \bar{x} N(y) i \gamma^\nu N(y) \partial_\mu \Delta(-y) \right) e^{i \frac{1}{2} g_\pi^2 \int d^4 \bar{x} d^4 y' \bar{N}(x') i \gamma^5 \gamma^\nu N(x') \cdot \Delta(x' - y') \bar{N}(y') i \gamma^5 \gamma^\nu N(y')} \right) | n(\bar{k}_n, \sigma_n)_N \]

\[
= \bar{u}_p(\bar{k}_p, \sigma_p) \left( 1 + \frac{g_\pi^2}{4 \pi^2} (A_N + A_x) \gamma_\mu + \frac{g_\pi^2}{4 \pi^2} (B_N + B_x) \frac{i \sigma_\mu q^\nu}{2 m_N} + \frac{g_\pi^2}{4 \pi^2} (C_N + C_x) \frac{\sigma_\mu q}{m_N^2} \right) u(\bar{k}_n, \sigma_n). \tag{38}
\]
Thus, we have shown that the matrix element of the transition $n \rightarrow p$, calculated to order $g^2_\pi$, can be expressed in terms of three Lorentz structures $\gamma_\mu, i\sigma_{\mu\nu}q^\nu$ and $q_\mu \bar{q}$, which are induced by the first class current \([7, 8]\). Indeed, the isovector hadronic vector current Eq. (5) has a positive $G$–parity and belongs to the first class current \([3]\).

\[
\bar{V}_\mu(x) \xrightarrow{G} G\bar{V}_\mu(x)G^{-1} = G\left(\bar{N}(x)\frac{1}{2} \tau_5 \gamma_\mu N(x)\right)G^{-1} + G\left(\bar{\pi}(x) \times \partial_\mu \bar{\pi}(x)\right)G^{-1} = N^T(x)C(-\tau_2 \tau_5 \gamma_\mu \tau_2 CN^T(x) + G\bar{\pi}(x)G^{-1} \times G\partial_\mu \bar{\pi}(x)G^{-1} = \bar{V}_\mu(x),
\]

where $T$ is a transposition and $C$ is the charge conjugation matrix \([3]\). For the derivation of the relation Eq. (39), we have used the relations $\gamma_\mu C = -\gamma^5 \mu \tau_2 \tau_5 \gamma_\mu \tau_2 T$ and $N^T(x)N(x) = -N(x)N(x)$ \([3,8]\) and $G\bar{\pi}(x)G^{-1} = -\bar{\pi}(x)$ \([7,8]\).

Since, the coefficient $C_N$ is much smaller than the coefficient $C_\pi$, the contribution of the Lorentz structure $q_\mu \bar{q}$ to the matrix element of the transition $n \rightarrow p$ is practically determined by the mesonic part of the charged hadronic vector current Eq. (19). Of course, the coefficients $A_\pi$ and $A_N$ can depend on the ultra–violet cut–off $\Lambda$. However, such a dependence can be removed by renormalization of the coupling constant $g^2_\pi$ \([12,14]\).

As strong low–energy interactions are invariant under the $G$–parity transformation \([8]\) (see also \([3]\)),

\[
\mathcal{L}_{\pi NN}(x) = g_0 \bar{N}(x)\gamma_5 \tau N(x) \cdot \bar{\pi}(x) \xrightarrow{G} G\left(g_0 \bar{N}(x)\gamma_5 \tau N(x) \cdot \bar{\pi}(x)\right)G^{-1} = N^G(x)\gamma_5 \tau N^G(x) \cdot \bar{\pi}^G(x) = N^T(x)C(-\tau_2 \gamma_5 \tau \tilde{N} \tau_2 CN^T(x) \cdot (-\bar{\pi}(x)) = \mathcal{L}_{\pi NN}(x),
\]

where we have used the relation $C\gamma_5 C = -\gamma^5 \tau_5$ \([3]\), and the terms with the Lorentz structures $\gamma_\mu, i\sigma_{\mu\nu}q^\nu$ and $q_\mu \bar{q}$ possess the positive $G$–parity, i.e. they are the contributions of the first class current \([6]\), the term with the Lorentz structure $q_\mu$, having a negative $G$–parity \([3]\), should not appear in the matrix element of the transition $n \rightarrow p$ Eq. (27) to any order of $g^2_\pi$–expansion. This allows to write \([3]\)

\[
\langle \text{out}, p(\vec{k}_p, \sigma_p)|V_{\mu}^{(+)}(0)|\text{in}, n(\vec{k}_n, \sigma_n)\rangle = \bar{u}_p(\vec{k}_p, \sigma_p) \left(F_1(q^2) \gamma_\mu + F_2(q^2) \frac{i\sigma_{\mu\nu}q^\nu}{2m_N} + F_4(q^2) \frac{q_\mu \bar{q}}{m_N^2}\right) u_n(\vec{k}_n, \sigma_n),
\]

where $F_1(q^2)$, $F_2(q^2)$ and $F_4(q^2)$ are form factors, calculated to all orders of $g^2_\pi$–expansion.

IV. CONCLUSIVE DISCUSSIONS

We have analysed the Lorentz structure of the matrix element of the transition $n \rightarrow p$, caused by the charged hadronic vector current. We have shown that in addition to the standard terms with the Lorentz structure $F_1(q^2) \gamma_\mu$ and $F_2(q^2) i\sigma_{\mu\nu}q^\nu/2m_N$, caused by the contributions of the electric charge distribution and the weak magnetism inside the hadron, one obtains the term with the structure $F_4(q^2) q_\mu \bar{q}/m_N^2$. Using the simplest model of strongly coupled $\pi N$–system with the linear pion–nucleon pseudoscalar interaction we have shown that the contribution of the term with the Lorentz structure $F_4(q^2) q_\mu \bar{q}/m_N^2$ is practically induced by the mesonic part of the hadronic isovector vector current. We have also shown that the term with the Lorentz structure $F_4(q^2) q_\mu/m_N$, caused by the second class current \([7,8]\), cannot be, in principle, induced by strong low–energy interactions invariant under $G$–parity transformations.

A requirement of conservation of the charged hadronic vector current even for different masses of the hadrons in the initial and final states (see \([1,3,13,18]\) and so on) in the sense of the vanishing of the matrix element $\langle p|\partial^\mu V_{\mu}^{(+)}(0)|n\rangle = 0$ of the hadronic $n \rightarrow p$ transition leads to the relation $F_4(q^2) = -(m_N^2/q^2) F_1(q^2)$. Such a relation leads to the appearance of the term $-(q_\mu \bar{q}/q^2) F_1(q^2)$ in the matrix element of the hadronic $n \rightarrow p$ transition.

For simplicity we have restricted our analysis by the simplest theory of $\pi N$ strong interactions described by the Lagrangian Eq. (40) with the linear pseudoscalar $\pi N N$–interaction \([3,12,14]\). However, one may assert that the obtained result, i.e. the existence of the term with the Lorentz structure $q_\mu \bar{q}$ and the suppression of the term with the Lorentz structure $q_\mu$, which are the contributions of the first and second class currents, respectively, should be valid in any theory of strong low–energy interactions \([13,20]\), which are invariant under $G$–transformations \([3]\) (see also \([7]\)). Our assertion is based only on $G$–invariance of such theories. Indeed, it is hardly possible to perform analytical calculations, which are similar to those we have carried out in this paper, within such complicated non–linear theories of meson–nucleon low–energy interactions as \([15,16]\) and Chiral perturbation theory \([20]\).

In Appendix A we have shown that the cross sections for the electron neutrino–neutron scattering and for the inverse $\beta$–decay, calculated in the non–relativistic approximation with respect to the outgoing hadron, are insensitive to the contributions of the term $-(q_\mu \bar{q}/q^2) F_1(q^2)$. That is why one may assert that it is important to search for processes, which are sensitive to the contributions of the term $-(q_\mu \bar{q}/q^2) F_1(q^2)$.
In Appendix B we have analysed the dynamical nature of the Lorentz structure of the matrix element \( \langle p|A^{(+)}_\mu(0)|n \rangle \) of the transition \( n \rightarrow p \), caused by the charged hadronic axial–vector current \( A^{(+)}_\mu(0) \). We have shown that the low–energy pion–nucleon interaction Eq.~[13] allows to reproduce fully the standard Lorentz structure of the axial–part of the hadronic \( n \rightarrow p \) transition [1].

Of course, our results, obtained for the hadronic \( n \rightarrow p \) transition [1], are fully valid for the hadronic \( p \rightarrow n \) transition [17].

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VI. APPENDIX A: CROSS SECTIONS FOR THE INELASTIC ELECTRON NEUTRINO–NEUTRON SCATTERING AND FOR THE INVERSE \( \beta \)-DECAY

In this Appendix we calculate the cross sections for the inelastic scattering \( \nu_e + n \rightarrow p + e^- \) and the inverse \( \beta \)-decay \( \bar{\nu}_e + p \rightarrow n + e^+ \) by taking into account the contributions of the term \(-q_\mu \gamma^\mu/q^2\) responsible for the constraint \( \langle h'|\partial^\mu V^{(\pm)}_\mu(0)|h \rangle = 0 \) even for different masses of incoming \( h \) and outgoing \( h' \) hadrons. Below the contributions of such a term we call the contributions of Exact Conservation of the charged weak hadronic Vector Current or the ECVC effect.

The amplitudes of the inelastic scattering \( \nu_e + n \rightarrow p + e^- \) and the inverse \( \beta \)-decay \( \bar{\nu}_e + p \rightarrow n + e^+ \) we define in the non–relativistic approximation for the outgoing nucleon. They are equal to

\[
M(\nu_e n \rightarrow p e^-) = -\frac{G_F}{\sqrt{2}} V_{ud} \langle p(\vec{k}_p, \sigma_p)|J^{(+)}_\mu(0)|n(\vec{k}_n, \sigma_n)\rangle \left[ \bar{u}_-(\vec{k}_-, \sigma_-)\gamma^\mu(1 - \gamma^5) u_+(\vec{k}_+, \sigma_+) \right]
\]

(A-1)

and

\[
M(\bar{\nu}_e p \rightarrow n e^+) = \frac{G_F}{\sqrt{2}} V_{ud} \langle n(\vec{k}_n, \sigma_n)|J^{(-)}_\mu(0)|p(\vec{k}_p, \sigma_p)\rangle \left[ \bar{v}_+(\vec{k}_+, \sigma_+)\gamma^\mu(1 - \gamma^5) v_-(\vec{k}_-, \sigma_-) \right]
\]

(A-2)

where \( G_F \) and \( V_{ud} \) are the Fermi weak coupling constant and the Cabibbo–Kobayashi–Maskawa (CKM) matrix element [21], \( \bar{u}_-(\vec{k}_-, \sigma_-) \) and \( u_+(\vec{k}_+, \sigma_+, -\frac{1}{2}) \) are the Dirac wave functions of the free electron and electron neutrino with 3–momenta \( \vec{k}_- \) and \( \vec{k}_+ \) and polarizations \( \sigma_\pm = \pm 1 \) and \( -\frac{1}{2} \) respectively, and \( (\gamma_\mu, \gamma^5) \) are the Dirac matrices. Then, \( \bar{v}_+(\vec{k}_+, \sigma_+, +\frac{1}{2}) \) and \( v_-(\vec{k}_-, \sigma_-) \) are the Dirac wave functions of the electron antineutrino and positron with 3–momenta \( \vec{k}_+ \) and \( \vec{k}_- \) and polarizations \( \sigma_\pm = \pm 1 \) and \( +\frac{1}{2} \) respectively. The matrix elements \( \langle p(\vec{k}_p, \sigma_p)|J^{(+)}_\mu(0)|n(\vec{k}_n, \sigma_n)\rangle \) and \( \langle n(\vec{k}_n, \sigma_n)|J^{(-)}_\mu(0)|p(\vec{k}_p, \sigma_p)\rangle \) of the hadronic \( n \rightarrow p \) and \( p \rightarrow n \) transitions we define as follows [1]

\[
\langle p(\vec{k}_p, \sigma_p)|J^{(+)}_\mu(0)|n(\vec{k}_n, \sigma_n)\rangle = \bar{u}_p(\vec{k}_p, \sigma_p) \left( F_1(q^2) \left( \gamma_\mu - \frac{q_\mu q^2}{q^2} \right) + F_A(q^2) \gamma_\mu \gamma^5 \right) u_n(\vec{k}_n, \sigma_n)
\]

(A-3)

and

\[
\langle n(\vec{k}_n, \sigma_n)|J^{(-)}_\mu(0)|p(\vec{k}_p, \sigma_p)\rangle = \bar{u}_n(\vec{k}_n, \sigma_n) \left( F_1(q^2) \left( \gamma_\mu - \frac{q_\mu q^2}{q^2} \right) + F_A(q^2) \gamma_\mu \gamma^5 \right) u_p(\vec{k}_p, \sigma_p)
\]

(A-4)

where \( J^{(\pm)}_\mu(0) = V^{(\pm)}_\mu(0) - A^{(\pm)}_\mu(0) \), \( u_n(\vec{k}_n, \sigma_n) \) and \( u_p(\vec{k}_p, \sigma_p) \) are the Dirac wave functions of the free neutron and proton with 3–momenta and polarizations \( (\vec{k}_n, \sigma_n = \pm 1) \) and \( (\vec{k}_p, \sigma_p = \pm 1) \). Then, \( F_1(q^2) \) and \( F_A(q^2) \) are the vector and axial–vector form factors [1]. The vector parts of the matrix elements Eqs.\[A-3\] and \[A-4\] obey the constraints

\[
\langle p(\vec{k}_p, \sigma_p)|\partial^\mu V^{(+)\mu}(0)|n(\vec{k}_n, \sigma_n)\rangle = \langle n(\vec{k}_n, \sigma_n)|\partial^\mu V^{(-)\mu}(0)|p(\vec{k}_p, \sigma_p)\rangle = 0
\]

(A-5)

even for different masses of the neutron and proton. In the matrix elements Eqs.\[A-3\] and \[A-4\] we have neglected the contributions of the weak magnetism and one–pion exchange [1]. In the approximation, when the squared momentum
transferred $q^2 = (\pm k_p \mp k_n)^2$ is much smaller than the scales $M_F^2$ and $M_A^2$ defining the effective radii of the vector and axial–vector form factors, the matrix elements Eqs. (A-5) and (A-6) can be reduced to the form

$$\langle p(k_p, \sigma_p) | J^{(+)}_\mu (0) | n(k_n, \sigma_n) \rangle = \hat{u}_p(k_p, \sigma_p) \left( \gamma_\mu (1 + \lambda \gamma^5) - \frac{q_{\mu} q}{q^2} \right) u_n(k_n, \sigma_n)$$  \hspace{1cm} (A-6)

and

$$\langle n(k_n, \sigma_n) | J^{(-)}_\mu (0) | p(k_p, \sigma_p) \rangle = \hat{u}_n(k_n, \sigma_n) \left( \gamma_\mu (1 + \lambda \gamma^5) - \frac{q_{\mu} q}{q^2} \right) u_p(k_p, \sigma_p),$$  \hspace{1cm} (A-7)

where $\lambda = -1.2750(9)$ is the axial coupling constant \( \text{[26]} \) \( \text{[see also 22, 23]} \).

In order to illustrate the contribution of the term $-q_{\mu} q / q^2$ responsible for the fulfilment of the constraints Eqs. (A-5), we neglect the contributions of the weak magnetism, recoil and radiative corrections \( \text{[24]} \). Skipping intermediate standard calculations \( \text{[24]} \) we obtain the following cross sections for the inelastic electron neutrino–neutrino scattering $\sigma(E_\nu)$ and the inverse $\beta$-decay $\sigma(E_\nu)$:

$$\sigma(E_\nu) = \sigma_0(E_\nu) + \frac{G_F^2 |V_{ud}|^2}{2\pi} \left[ \frac{2m^2 \Delta}{k_\nu E_\nu} \left[ \ln \left( 1 + \frac{2k_\nu E_\nu}{m^2 - 2E_\nu} \right) - \ln \left( 1 - \frac{2k_\nu E_\nu}{m^2 - 2E_\nu} \right) \right] + \frac{m^2 \Delta^2}{(m^2 - 2E_\nu)^2 - 4k^2 E_\nu^2} \left[ \ln \left( 1 + \frac{2k_\nu E_\nu}{m^2 - 2E_\nu} \right) - \ln \left( 1 - \frac{2k_\nu E_\nu}{m^2 - 2E_\nu} \right) \right] - \frac{4k_\nu E_\nu (m^2 - 2E_\nu)}{(m^2 - 2E_\nu)^2 - 4k^2 E_\nu^2} \right] ,$$  \hspace{1cm} (A-8)

where $E_\nu = E_\nu + \Delta$, $k_\nu = \sqrt{E_\nu^2 - m^2}$ and $\beta_\nu = k_\nu / E_\nu$ are the energy, momentum and velocity of the electron, and

$$\sigma(E_\nu) = \sigma_0(E_\nu) + \frac{G_F^2 |V_{ud}|^2}{2\pi} \left[ \frac{2m^2 \Delta}{k_\nu E_\nu} \left[ \ln \left( 1 + \frac{2k_\nu E_\nu}{m^2 - 2E_\nu} \right) - \ln \left( 1 - \frac{2k_\nu E_\nu}{m^2 - 2E_\nu} \right) \right] + \frac{m^2 \Delta^2}{(m^2 - 2E_\nu)^2 - 4k^2 E_\nu^2} \left[ \ln \left( 1 + \frac{2k_\nu E_\nu}{m^2 - 2E_\nu} \right) - \ln \left( 1 - \frac{2k_\nu E_\nu}{m^2 - 2E_\nu} \right) \right] - \frac{4k_\nu E_\nu (m^2 - 2E_\nu)}{(m^2 - 2E_\nu)^2 - 4k^2 E_\nu^2} \right] ,$$  \hspace{1cm} (A-9)

where $E_+ = E_\nu - \Delta$ and $k_+ = \sqrt{E_+^2 - m^2}$ are the energy and momentum of the positron. The cross sections $\sigma_0(E_\nu)$ and $\sigma_0(E_\nu)$ are given by \( \text{[24]} \)

$$\sigma_0(E_\nu) = (1 + 3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{\pi} k_\nu E_\nu , \quad \sigma_0(E_\nu) = (1 + 3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{\pi} k_+ E_\nu.$$  \hspace{1cm} (A-10)

In the inelastic electron neutrino–neutrino scattering and the inverse $\beta$-decay the energies of neutrino and antineutrino vary in the regions $E_\nu \geq 0$ and $E_\nu \geq (E_\nu)_{\text{thr}} = (m_+ + m_-)^2 / 2m_\nu = 1.8061 \text{ MeV} \text{[24]}$. The terms dependent on $\Delta$ are caused by the ECVC effect. The relative contributions of the ECVC effect to the cross sections under consideration we define as follows

$$R_\nu(E_\nu) = \frac{1}{2} \left\{ \frac{m^2 \Delta}{k_\nu E_\nu} \left[ \ln \left( 1 + \frac{2k_\nu E_\nu}{m^2 - 2E_\nu} \right) - \ln \left( 1 - \frac{2k_\nu E_\nu}{m^2 - 2E_\nu} \right) \right] + \frac{m^2 \Delta^2}{(m^2 - 2E_\nu)^2 - 4k^2 E_\nu^2} \left[ \ln \left( 1 + \frac{2k_\nu E_\nu}{m^2 - 2E_\nu} \right) - \ln \left( 1 - \frac{2k_\nu E_\nu}{m^2 - 2E_\nu} \right) \right] - \frac{4k_\nu E_\nu (m^2 - 2E_\nu)}{(m^2 - 2E_\nu)^2 - 4k^2 E_\nu^2} \right\} ,$$  \hspace{1cm} (A-11)

and

$$R_\nu(E_\nu) = \frac{1}{2} \left\{ \frac{m^2 \Delta}{k_\nu E_\nu} \left[ \ln \left( 1 + \frac{2k_\nu E_\nu}{m^2 - 2E_\nu} \right) - \ln \left( 1 - \frac{2k_\nu E_\nu}{m^2 - 2E_\nu} \right) \right] + \frac{m^2 \Delta^2}{(m^2 - 2E_\nu)^2 - 4k^2 E_\nu^2} \left[ \ln \left( 1 + \frac{2k_\nu E_\nu}{m^2 - 2E_\nu} \right) - \ln \left( 1 - \frac{2k_\nu E_\nu}{m^2 - 2E_\nu} \right) \right] - \frac{4k_\nu E_\nu (m^2 - 2E_\nu)}{(m^2 - 2E_\nu)^2 - 4k^2 E_\nu^2} \right\} .$$  \hspace{1cm} (A-12)
where \( R_x(E_\nu) = \Delta \sigma(E_\nu)/\sigma_0(E_\nu) \), \( R_y(E_\nu) = \Delta \sigma(E_\nu)/\sigma_0(E_\nu) \) with \( \Delta \sigma(E_\nu) = \sigma(E_\nu) - \sigma_0(E_\nu) \) and \( \Delta \sigma(E_\nu) = \sigma(E_\nu) - \sigma_0(E_\nu) \), respectively. The cross sections Eq. (A-8) and Eq. (A-9) are calculated in the laboratory frame in the non-relativistic approximation for outgoing hadrons. Since the most important region of the antineutrino energies for the inverse \( \beta \)-decay is \( 2 \text{MeV} \leq E_\nu \leq 8 \text{MeV} \), in Fig. 1 we plot \( R_x(E_\nu) \) and \( R_y(E_\nu) \) for \( E_\nu \) varying over the regions \( 2 \text{MeV} \leq E_\nu \leq 8 \text{MeV} \) and \( 2 \text{MeV} \leq E_\nu \leq 8 \text{MeV} \), respectively.

Our numerical analysis of the relative contributions of the ECVC effect to the cross sections for the inelastic electron neutrino–neutron scattering and for the inverse \( \beta \)-decay shows that these processes are not sensitive to the ECVC effect. Indeed, the contribution of the ECVC effect to the cross section for the inelastic electron neutrino–neutron scattering is smaller than 0.7% at \( E_\nu \approx 2 \text{MeV} \) and decreases by about two orders of magnitude at \( E_\nu \approx 8 \text{MeV} \). The cross section for the inverse \( \beta \)-decay, applied to the analysis of the deficit of positrons induced by reactor electron antineutrinos \([24, 27]\), should be averaged over the reactor electron antineutrino energy spectrum, which has a maximum at \( E_\nu \approx 4 \text{MeV} \). According to Fig. 1 the contribution of the ECVC effect should decrease the yield of positrons \( Y_{p+} \) by about 0.5%. Since such a contribution is smaller than the experimental error bars \( Y_{p+} = 0.943(23) \), one may argue that the inverse \( \beta \)-decay is insensitive to the contribution of the ECVC effect.

VII. APPENDIX B: THE LORENTZ STRUCTURE OF THE MATRIX ELEMENT OF THE HADRONIC \( n \to p \) TRANSITION, CAUSED BY THE CHARGED HADRONIC AXIAL–VECTOR CURRENT

In this Appendix we analyse the Lorentz structure of the axial–vector part of the hadronic \( n \to p \) transition, induced by the charged hadronic axial–vector current

\[
A^{(+)}(x) = \bar{N}(x)\gamma^+ N(x).
\]

The matrix element of our interest is

\[
\langle \text{out}, p(\vec{k}_p, \sigma_p) | A_{\mu}^{(+)}(0) | \text{in}, n(\vec{k}_n, \sigma_n) \rangle,
\]

where \( \langle \text{out}, p(\vec{k}_p, \sigma_p) \rangle \) and \( | \text{in}, n(\vec{k}_n, \sigma_n) \rangle \) are the wave functions of the free proton and neutron in the final (i.e. out-state at \( t \to +\infty \)) and initial (i.e. in-state at \( t \to -\infty \)) states, respectively \([4]\). Using the relation \( \langle \text{out}, p(\vec{k}_p, \sigma_p) \rangle = \langle p, p(\vec{k}_p, \sigma_p) | S \rangle \), where \( S \) is the S-matrix, we rewrite the matrix element Eq. (B-2) as follows

\[
\langle \text{out}, p(\vec{k}_p, \sigma_p) | A_{\mu}^{(+)}(0) | \text{in}, n(\vec{k}_n, \sigma_n) \rangle = \langle p, p(\vec{k}_p, \sigma_p) | SA_{\mu}^{(+)}(0) | \text{in}, n(\vec{k}_n, \sigma_n) \rangle.
\]

Since the transition \( n \to p \) is fully induced by strong low-energy interactions, we define the S-matrix only in terms of strong low-energy interaction described by the Lagrangian Eq. (4). It is given by (see Eq. (12)). Plugging Eq. (12) into Eq. (B-3) we get

\[
\langle p, p(\vec{k}_p, \sigma_p) | SA_{\mu}^{(+)}(0) | \text{in}, n(\vec{k}_n, \sigma_n) \rangle = \langle p, p(\vec{k}_p, \sigma_p) | T \left( e^{i \int d^4x \mathcal{L}_{\pi NN}(x) A_{\mu}^{(+)}(0)} \right) | \text{in}, n(\vec{k}_n, \sigma_n) \rangle.
\]

After the integration over the pion–fields we arrive at the expression

\[
\langle \text{out}, p(\vec{k}_p, \sigma_p) | A_{\mu}^{(+)}(0) | \text{in}, n(\vec{k}_n, \sigma_n) \rangle = N \langle \text{in}, p(\vec{k}_p, \sigma_p) | T \left( \bar{N}(0)\gamma^+ N(0) \right) \exp \left( i \frac{1}{2} g_\pi^2 \int d^4x' d^4y' \left( \bar{N}(x')i\gamma^5 N(x') \right) \Delta(x' - y') \right) | \text{in}, n(\vec{k}_n, \sigma_n) \rangle.
\]
To order $g^2$ the dynamical contribution of the nucleon part of the charge hadronic axis–vector current to the matrix element of the transition $n \to p$, given by Eq. \[13\], is determined by the matrix element

$$
N \langle n, p(k_p, p) | T \left( \bar N(0) \gamma^\mu \gamma^5 N(0) \right) | n(k_n, \sigma_n) \rangle_N = \frac{1}{2} g^2 \int d^4x d^4y \bar N(x) i\gamma^\mu \gamma^5 \mathcal{P}N(y) \cdot \Delta(x - y) \bar N(y) i\gamma^\mu \gamma^5 \mathcal{P}N(y) \rangle | n(k_n, \sigma_n) \rangle_N =
$$

$$
\left[ \bar u_p(k_p, \sigma_p) \gamma^\mu \gamma^5 u_n(k_n, \sigma_n) \right] 3 \langle 0 | \bar u_p(k_p, \sigma_p) i\gamma^\mu \gamma^5 u_n(k_n, \sigma_n) | 0 \rangle | n(k_n, \sigma_n) \rangle_N
$$

where $q = k_p - k_n$. The contribution proportional to $g^{(i)}(0)$ can be in principle removed by using the normal–ordered form of the four–nucleon operator of interaction. Indeed, replacing $\bar N(x) i\gamma^\mu \gamma^5 \mathcal{P}N(y)$ by $\mathcal{N}(x) i\gamma^\mu \gamma^5 \mathcal{P}N(y)$ : the vacuum expectation value of the operator $\mathcal{N}(x) i\gamma^\mu \gamma^5 \mathcal{P}N(y)$ : is equal to zero. The contribution of the momentum integral of the second term is divergent and proportional to $\mu_q$. As a result, the matrix element Eq. \[15\] we may define as follows

$$
N \langle n, p(k_p, \sigma_p) | T \left( \bar N(0) \gamma^\mu \gamma^5 N(0) \right) | n(k_n, \sigma_n) \rangle_N = \frac{1}{2} g^2 \int d^4x d^4y \bar N(x) i\gamma^\mu \gamma^5 \mathcal{P}N(y) \cdot \Delta(x - y) \bar N(y) i\gamma^\mu \gamma^5 \mathcal{P}N(y) \rangle | n(k_n, \sigma_n) \rangle_N
$$

$$
\left[ \bar u_p(k_p, \sigma_p) \gamma^\mu \gamma^5 u_n(k_n, \sigma_n) \right] \frac{2 g^2}{m_n^2 - q^2 - i0} \int \frac{d^4p}{(2\pi)^4} \frac{1}{2m_N - p - i0} \frac{1}{2m_p - Q - i0} \left\{ \frac{1}{m_N - \mu_q} \frac{1}{m_N - \mu_q - Q - i0} \right\} \frac{1}{m_p - Q^2 - i0}
$$

Thus, for the matrix element Eq. \[15\], calculated to order $g^2$, we obtain the following expression

$$
\langle \text{out}, p(k_p, \sigma_p) | A^{(+)}_\mu (0) | n(k_n, \sigma_n) \rangle = N \langle \text{in}, p(k_p, \sigma_p) | T \left( \bar N(0) \gamma^\mu \gamma^5 N(0) \right) \exp \left( i \frac{1}{2} g^2 \int d^4x d^4y \bar N(x) i\gamma^\mu \gamma^5 \mathcal{P}N(x') \cdot \Delta(x' - y') \bar N(y') i\gamma^\mu \gamma^5 \mathcal{P}N(y') \right) | n(k_n, \sigma_n) \rangle_N =
$$

$$
\left[ \bar u_p(k_p, \sigma_p) \left( 1 + \frac{g^2}{4\pi^2} D_N \right) \gamma^\mu \gamma^5 + \frac{g^2}{4\pi^2} \frac{2m_N q_\mu}{m_n^2 - q^2 - i0} E_N \gamma^5 \right] u_n(k_n, \sigma_n).
$$

Following \[11\] we may argue that the Lorentz structure of the matrix element Eq. \[15\], calculated to order $g^2$, should be valid to all order of the $g^2$–expansion. The latter is, of course, because of invariance of strong low–energy interactions under the $G$–transformations. Thus, the matrix element Eq. \[15\] should have the following Lorentz structure, induced by the first class axis–vector current

$$
\langle \text{out}, p(k_p, \sigma_p) | A^{(+)}_\mu (0) | n(k_n, \sigma_n) \rangle = \bar u_p(k_p, \sigma_p) \left( - F_A(q^2) \gamma^\mu \gamma^5 + \frac{2m_N q_\mu}{m_n^2 - q^2 - i0} F_P(q^2) \gamma^5 \right) u_n(k_n, \sigma_n),
$$

where $F_A(q^2)$ and $F_P(q^2)$ are the axis–vector and pseudoscalar form factors. Taking into account the PCAC hypothesis (or the hypothesis of Partial Conservation of Axial–vector Current) \[28\], \[29\] we may rewrite the r.h.s. of Eq. \[10\] as follows

$$
\langle \text{out}, p(k_p, \sigma_p) | A^{(+)}_\mu (0) | n(k_n, \sigma_n) \rangle = - \bar u_p(k_p, \sigma_p) \left( \gamma^\mu \gamma^5 + \frac{2m_N q_\mu}{m_n^2 - q^2 - i0} \gamma^5 \right) F_A(q^2) u_n(k_n, \sigma_n),
$$

where we have set $F_A(q^2) = - F_P(q^2)$. At $q^2 = 0$ we set $F_A(q^2) = \lambda$, where $\lambda = -1.2750(9)$ is the axial coupling constant \[24\] (see also \[22\], \[23\]). In the chiral limit $m_n \to 0$ the matrix element Eq. \[11\] multiplied by the 4–momentum transferred $q^\mu$ vanishes

$$
\lim_{m_n \to 0} q^\mu \langle \text{out}, p(k_p, \sigma_p) | A^{(+)}_\mu (0) | n(k_n, \sigma_n) \rangle = 0
$$
even for different masses of the neutron and proton, according to the PCAC hypothesis\cite{28,29} pointing out an operator relation \( \partial^\mu \vec{A}_\mu(x) = m_\pi^2 F_\pi \vec{\pi}(x) \), where \( F_\pi \) is the pion–decay constant\cite{21}. In the chiral limit \( m_\pi \to 0 \) we get \( \partial^\mu \vec{A}_\mu(x) = 0 \) agreeing well with Eq. \((B-12)\).