EDGE-TRANSITIVE BI-CAYLEY GRAPHS

MARSTON CONDER, JIN-XIN ZHOU, YAN-QUAN FENG, AND MI-MI ZHANG

Abstract. A graph Γ admitting a group H of automorphisms acting semi-regularly on the vertices with exactly two orbits is called a bi-Cayley graph over H. Such a graph Γ is called normal if H is normal in the full automorphism group of Γ, and normal edge-transitive if the normaliser of H in the full automorphism group of Γ is transitive on the edges of Γ. In this paper, we give a characterisation of normal edge-transitive bi-Cayley graphs, and in particular, we give a detailed description of 2-arc-transitive normal bi-Cayley graphs. Using this, we investigate three classes of bi-Cayley graphs, namely those over abelian groups, dihedral groups and metacyclic p-groups. We find that under certain conditions, ‘normal edge-transitive’ is the same as ‘normal’ for graphs in these three classes. As a by-product, we obtain a complete classification of all connected trivalent edge-transitive graphs of girth at most 6, and answer some open questions from the literature about 2-arc-transitive, half-arc-transitive and semisymmetric graphs.

1. Introduction

In this paper we describe an investigation of edge-transitive bi-Cayley graphs, in which we show that in the ‘normal’ case, such graphs can be arc-transitive, half-arc-transitive, or semisymmetric, and we exhibit some specific families of examples of each kind. As a by-product of this investigation, we answer two open questions about edge-transitive graphs posed in 2001 by Marušič and Potočnik [42]. Before proceeding, we give some background to this topic, and set some notation.

First, if G is a group acting on a set Ω, then the stabiliser in G of a point α ∈ Ω is the subgroup Ga = \{g ∈ G | gα = α\} of G. The group G is said to be semi-regular on Ω if Ga = 1 for every α ∈ Ω, and regular on Ω if G is transitive and semi-regular on Ω.

A graph Γ is called a Cayley graph if it admits a group G of automorphisms acting regularly on its vertex-set V(Γ). In that case, Γ is isomorphic to the graph Cay(G, S) with vertex-set G and edge-set \{g, xg \mid g ∈ G, x ∈ S\}, where S is the subset of elements of G taking the identity element to one of its neighbours (see [5 Lemma 16.3]): and then the automorphism group of Γ contains a subgroup R(G) = \{R(g) \mid g ∈ G\}, where R(g) is right multiplication (the permutation of G given by R(g) : x ↦ xg for x ∈ G), for each g ∈ G.

If, instead, we require the graph Γ to admit a group H of automorphisms acting semi-regularly on V(Γ) with two orbits, then we call Γ a bi-Cayley graph (for H). In this case, H acts regularly on each of its two orbits on V(Γ), and the two corresponding induced subgraphs are Cayley graphs for H. In particular, we may label the vertices of these two subgraphs with elements of two copies H0 and H1 of H, and find that there are subsets R, L and S of H (with |R| = |L|) such that the edges of those two induced subgraphs are of the form \{h0, (xh)0\} with h0 ∈ H0 and x ∈ R, and \{h1, (yh)1\} with h1 ∈ H1 and y ∈ L, while all remaining edges are of the form \{h0, (zh)1\} with z ∈ S and where h0 and h1 are the elements of H0 and H1 that represent a given h ∈ H. This gives a concrete realisation of Γ in terms of H, R, L and S.

Conversely, if H is any group, and R, L and S are subsets of H with |R| = |L| such that 1H ∉ R = R−1 and 1H ∉ L = L−1, then the graph Γ with vertex set being the union H0 ∪ H1 of two copies of H and with edges of the form \{h0, (xh)0\}, \{h1, (yh)1\} and \{h0, (zh)1\} with x ∈ R, y ∈ L and z ∈ S, and h0 ∈ H0 and h1 ∈ H1 representing a given h ∈ H, is a bi-Cayley graph for H. Indeed H acts as a semi-regular group

2000 Mathematics Subject Classification. 05C25, 20B25.
Key words and phrases. bi-Cayley graph, edge-transitive, semisymmetric, half-arc-transitive.

The second author was partially supported by the National Natural Science Foundation of China (11271012) and the Fundamental Research Funds for the Central Universities (2015JBM110).

The third author was partially supported by the National Natural Science Foundation of China (11231008, 11571035).
of automorphisms by right multiplication, with \( H_0 \) and \( H_1 \) as its orbits on vertices. We denote this graph by \( \text{BiCay}(H, R, L, S) \), and denote the group of automorphisms induced by \( H \) on the graph as \( R(H) \).

Bi-Cayley graphs, which have sometimes been called semi-Cayley graphs, form a class of graphs that has been studied extensively — as in \( [8, 10, 15, 20, 21, 24, 25, 26, 28, 33, 35, 36, 37, 38, 39, 40, 45, 53] \). Note that some authors label the vertices of a bi-Cayley graph for a group \( H \) with ordered pairs \( (h, i) \) for \( h \in H \) and \( i \in \{0, 1\} \), while we are using \( h_i \) to denote \( (h, i) \).

Various well known graphs can be constructed as bi-Cayley graphs. For example, the famous Petersen graph is a bi-Cayley graph over a cyclic group of order 5, and the Gray graph \( [5] \) (which is the smallest trivalent semisymmetric graph), is a bi-Cayley graph over a metacyclic group of order 27. Similarly, in Bouwer’s answer \( [9] \) to Tutte’s question \( [49] \) about the existence of half-arc-transitive graphs with even valency, the smallest graph in his family is also a bi-Cayley graph over a non-abelian metacyclic group of order 27. We note that all of these interesting small bi-Cayley graphs are edge-transitive. This motivated us to investigate the class of all edge-transitive bi-Cayley graphs.

Although we know that a bi-Cayley graph \( \Gamma = \text{BiCay}(H, R, L, S) \) has a ‘large’ group of automorphisms, with just two vertex-orbits, it is difficult in general to decide whether a bi-Cayley graph is edge-transitive. It helps to consider the subgroup \( R(H) \) of the automorphism group \( \text{Aut}(\Gamma) \) induced by \( H \) on \( \Gamma \), and also its normaliser \( N_{\text{Aut}(\Gamma)}(R(H)) \). The latter subgroup was characterised in \( [53] \) (see Proposition 2.2 in the next section), making it possible to determine whether or not \( N_{\text{Aut}(\Gamma)}(R(H)) \) is transitive on the edges of \( \Gamma \). A bi-Cayley graph \( \Gamma = \text{BiCay}(H, R, L, S) \) is called normal if \( R(H) \) is normal in \( \text{Aut}(\Gamma) \), and normal edge-transitive if \( N_{\text{Aut}(\Gamma)}(R(H)) \) is transitive on the edge-set of \( \Gamma \).

Similarly, we say that a bi-Cayley graph \( \Gamma = \text{BiCay}(H, R, L, S) \) is normal locally arc-transitive if the stabilizer \( (N_{\text{Aut}(\Gamma)}(R(H)))_{i_0} \) of the ‘right’ identity vertex \( i_0 \) in \( N_{\text{Aut}(\Gamma)}(R(H)) \) is transitive on the neighbourhood of \( i_0 \) in \( \Gamma \), and normal half-arc-transitive if \( N_{\text{Aut}(\Gamma)}(R(H)) \) is transitive on the vertex-set and edge-set of \( \Gamma \) but intransitive on the arc-set of \( \Gamma \).

These notions generalise the corresponding ones for Cayley graphs, which have also been studied extensively. For example, a Cayley graph \( \Gamma = \text{Cay}(G, S) \) is normal edge-transitive if \( N_{\text{Aut}(\Gamma)}(R(G)) \) is transitive on the edges of \( \Gamma \). The study of such graphs was initiated in \( [16] \) by Praeger, and they play an important role in the study of edge-transitive graphs (as in \( [30] \), for example).

In this paper, we focus attention on normal edge-transitive bi-Cayley graphs.

Our motivation comes partly from the work of Li \( [29] \) on bi-normal Cayley graphs. Let \( \Gamma = \text{Cay}(G, S) \) be a Cayley graph on a group \( G \), and let \( R(G) \leq X \leq \text{Aut}(\Gamma) \). Then \( \Gamma \) is said to be \( X \)-bi-normal if the maximal normal subgroup \( \bigcap_{s \in X} R(G)^s \) of \( X \) contained in \( R(G) \) has index 2 in \( R(G) \), and if \( X = \text{Aut}(\Gamma) \), then \( \Gamma \) is called a \( X \)-bi-normal Cayley graph. Note that by definition, a connected bi-normal Cayley graph \( \Gamma = \text{Cay}(G, S) \) is a normal bi-Cayley graph over \( H = \bigcap_{s \in X} R(G)^s \).

In \( [29] \) Question 1.2(a)], Li asked whether there exist 3-arc-transitive bi-normal Cayley graphs, and in \( [29] \) Problem 1.3(b)], he asked for a good description of 2-arc-transitive bi-normal Cayley graphs. As we will see later, \( R = L = \emptyset \) for every connected normal edge-transitive bi-Cayley graph \( \text{BiCay}(H, R, L, S) \), and hence our first theorem below provides answers to these two questions.

**Theorem 1.1.** Let \( \Gamma \) be a connected bi-Cayley graph \( \text{BiCay}(H, \emptyset, \emptyset, S) \). Then \( N_{\text{Aut}(\Gamma)}(R(H)) \) acts transitively on the 2-arcs of \( \Gamma \) if and only if the following three conditions hold:

(a) there exists an automorphism \( \alpha \) of \( H \) such that \( S^\alpha = S^{-1} \),

(b) the setwise stabiliser of \( S \setminus \{1\} \) in \( \text{Aut}(H) \) is transitive on \( S \setminus \{1\} \), and

(c) there exists \( s \in S \setminus \{1\} \) and an automorphism \( \beta \) of \( H \) such that \( S^\beta = s^{-1}S \).

Furthermore, \( N_{\text{Aut}(\Gamma)}(R(H)) \) is not transitive on the 3-arcs of \( \Gamma \).

Next, it is natural to consider the classification of edge-transitive graphs into three distinct types: arc-transitive graphs, half-arc-transitive graphs (which are vertex- but not arc-transitive), and semisymmetric graphs (which are edge- but not vertex-transitive). We show that there are normal edge-transitive bi-Cayley graphs of each type:
Theorem 1.2. If $\Gamma$ is a normal edge-transitive bi-Cayley graph, then $\Gamma$ can be either arc-transitive, half-arc-transitive or semisymmetric. Furthermore, infinitely many examples exist in each case.

Note that this contrasts with the situation for Cayley graphs, which are vertex-transitive and therefore cannot be semisymmetric. To prove Theorem 1.2, we need only prove the assertion about the existence of the graphs in each case, and to do that, we consider bi-Cayley graphs over abelian groups, dihedral groups and metacyclic $p$-groups. Any such graph may be called bi-abelian graph, or a bi-dihedrant, or a bi-$p$-metacirculant, respectively. In the bi-abelian case, we have the following:

Proposition 1.3. Every connected bi-Cayley graph $\Gamma = BiCay(H,R,L,S)$ over an abelian group is vertex-transitive. Moreover, if $\Gamma$ is half-arc-transitive, then $R \cup L$ is non-empty and does not contain any involution, $|R| = |L|$ is even, $|S| > 2$, and the valency of $\Gamma$ is at least 6.

We then use this to study trivalent edge-transitive graphs with small girth, motivated by the work of Conder and Nedela [12] and Kutnar and Marušič [27] on classification of connected trivalent arc-transitive graphs of small girth. We obtain the following generalisation, to all connected trivalent edge-transitive graphs of girth 6 (noting that a trivalent edge-transitive graph is either arc-transitive or semisymmetric).

Theorem 1.4. All connected trivalent edge-transitive graphs of girth at most 6 are known, and in particular, every connected trivalent semisymmetric graph has girth at least 8.

Trivalent semisymmetric graphs of girth 8 are known to exist, and include the Gray graph; see [13].

Our semisymmetric normal edge-transitive bi-Cayley graphs are constructed from bi-dihedrants. The motivation for us to consider semisymmetric bi-dihedrants is the work of Marušič and Potočnik [42] on worthy semisymmetric tetracirculants. A graph is called tetracirculant if its automorphism group contains a cyclic semi-regular subgroup with four orbits, and a graph is said to be worthy if no two of its vertices have exactly the same set of neighbours. Marušič and Potočnik proposed two questions (Problems 4.3 and 4.9 in [42]) regarding the existence of worthy semisymmetric tetracirculants. Every bi-dihedrant is a tetracirculant, and an unworthy graph cannot be edge-regular, so our next theorem provides a positive answer to each of those.

Theorem 1.5. Every connected semisymmetric bi-dihedrant has valency at least 6, and examples of semisymmetric bi-dihedrants of valency $2k$ exist for each odd integer $k \geq 3$. In particular, there exists a family of edge-regular semisymmetric bi-dihedrants of valency 6.

Next, a bi-$p$-metacirculant is a bi-Cayley graph over a metacyclic $p$-group. (A group $G$ is metacyclic if it has a normal subgroup $N$ such that both $N$ and $G/N$ are cyclic.) For example, the smallest graph in a family of edge-transitive graphs constructed by Bouwer [9] is a tetravalent half-arc-transitive bi-Cayley graph over a metacyclic group of order 27. This motivated us to consider tetravalent half-arc-transitive bi-$p$-metacirculants in general, leading to our final theorem below.

Theorem 1.6. If $\Gamma$ is a tetravalent vertex- and edge-transitive bi-Cayley graph over a non-abelian metacyclic $p$-group $H$, for some odd prime $p$, and $R(H)$ is a Sylow $p$-subgroup of $Aut(\Gamma)$, then $R(H)$ is normal in $Aut(\Gamma)$, and so $\Gamma$ is a normal bi-Cayley graph. Indeed there exist such tetravalent half-arc-transitive bi-$p$-metacirculants, for every odd prime $p$.

Using this theorem, a complete classification of tetravalent half-arc-transitive bi-$p$-metacirculants will be given in [51]. Also when proving Theorem 1.6 we were led to study a general question posed in 2008 by Marušič and Šparl [43, p.368] about metacirculants, and we give a positive answer to their question (and point out an error in a paper by Li, Song and Wang [32] that claimed to do the same thing).

2. Preliminaries

Throughout this paper, groups are assumed to be finite, and graphs are assumed to be finite, connected, simple and undirected. For the group-theoretic and graph-theoretic terminology not defined here, we refer the reader to [6, 50]. We proceed by introducing some basic concepts and terminology.
2.1. Definitions and notation. For a finite, simple and undirected graph $\Gamma$, we use $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$ and $\text{Aut}(\Gamma)$ to denote the vertex-set, edge-set, arc-set and full automorphism group of $\Gamma$, respectively.

We let $d(u,v)$ be the distance between vertices $u$ and $v$ in $\Gamma$, and let $D$ be the diameter of $\Gamma$, which is the largest distance between two vertices in $\Gamma$. For any vertex $v$ of $\Gamma$, we let $\Gamma(v)$ be the neighbourhood of $v$, and more generally, we define $\Gamma_i(v) = \{ u \mid d(u,v) = i \}$ (the set of vertices at distance $i$ from $v$), for $1 \leq i \leq D$.

For any subset $B$ of $V(\Gamma)$, the subgraph of $\Gamma$ induced by $B$ is denoted by $\Gamma[B]$, and the neighbourhood of $B$ in $\Gamma$ is defined as $\Gamma_B = \bigcup_{v \in B} (\Gamma(v)) \setminus B$.

For each non-negative integer $s$, an $s$-arc in $\Gamma$ is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_s, v_s)$ of vertices of $\Gamma$ such that $v_{i-1}$ is adjacent to $v_i$ for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$, or in other words, such that any two consecutive $v_j$ are adjacent and any three consecutive $v_j$ are distinct. In particular, a 0-arc is a vertex, and a 1-arc is usually called an arc. The graph $\Gamma$ is said to be $s$-arc-transitive if $\text{Aut}(\Gamma)$ is transitive on the set of all $s$-arcs in $\Gamma$. In particular, 0-arc-transitive means vertex-transitive, and 1-arc-transitive means arc-transitive, or symmetric. Similarly, $\Gamma$ is $s$-arc-regular if $\text{Aut}(\Gamma)$ acts regularly (sharply-transitively) on the set of all $s$-arcs in $\Gamma$. In particular, $\Gamma$ is 1-arc-regular, or simply arc-regular, if given any two arcs of $\Gamma$, there exists a unique automorphism of $\Gamma$ taking one arc to the other.

A graph $\Gamma$ is edge-transitive or edge-regular if $\text{Aut}(\Gamma)$ acts transitively or regularly on $E(\Gamma)$, respectively, and $\Gamma$ is semisymmetric if $\Gamma$ has constant valency and is edge- but not vertex-transitive, while $\Gamma$ is half-arc-transitive if $\Gamma$ is vertex-transitive and edge-transitive but not arc-transitive.

We use $C_n$ to denote the multiplicative cyclic group of order $n$, and $\mathbb{Z}_n$ for the ring of integers mod $n$, and $\mathbb{Z}_n^*$ for the multiplicative group of units mod $n$ (the elements coprime to $n$ in $\mathbb{Z}_n$). Also we use $D_n$, $A_n$ and $S_n$ respectively for the dihedral, alternating and symmetric groups of degree $n$. For two groups $M$ and $N$, we use $N \rtimes M$ to denote a semi-direct product of $N$ by $M$ (with kernel $N$ and complement $M$). Finally, for a subgroup $H$ of a group $G$, we denote by $C_G(H)$ and $N_G(H)$ respectively the centraliser and normaliser of $H$ in $G$, and for a permutation group $G$ on a set $\Omega$, we let $G(\Delta)$ and $G_\Delta$ respectively be the pointwise and setwise stabiliser of a subset $\Delta$ of $\Omega$.

2.2. Basic properties of bi-Cayley graphs. In this subsection, we let $\Gamma$ be a connected bi-Cayley graph $\text{BiCay}(H,R,L,S)$ over a group $H$. It is easy to prove some basic properties of such a $\Gamma$, as in [55]:

**Proposition 2.1.** The following hold for any connected bi-Cayley graph $\text{BiCay}(H,R,L,S)$:

(a) $H$ is generated by $R \cup L \cup S$;
(b) $S$ can be chosen to contain the identity of $H$ (up to graph isomorphism);
(c) $\text{BiCay}(H,R,L,S) \cong \text{BiCay}(H,R',L',S')$ for every automorphism $\alpha$ of $H$; and
(d) $\text{BiCay}(H,R,L,S) \cong \text{BiCay}(H,L,R,S^{-1})$.

We will say the triple $(R,L,S)$ of subsets of $H$ is a bi-Cayley triple if $R = R^{-1}$, $L = L^{-1}$ and $1 \in S$, and we will say that the two bi-Cayley triples $(R,L,S)$ and $(R',L',S')$ for the same group $H$ are equivalent, and write $(R,L,S) \sim (R',L',S')$, if either $(R',L',S') = (L,R,S^{-1})$, or $(R',L',S') = (R,L,S)$. Note that by parts (c) and (d) of Proposition 2.1, the bi-Cayley graphs for any two equivalent bi-Cayley triples are isomorphic.

Now we consider the automorphisms of the bi-Cayley graph $\Gamma = \text{BiCay}(H,R,L,S)$. Recall that $H$ acts as a semi-groupular automorphism of $V(\Gamma)$ by right multiplication, with $H_0$ and $H_1$ as its orbits on vertices. Indeed each $g \in H$ induces an automorphism $R(g)$ of $\Gamma$ given by $h^R(g) = (hg)_i$ for $i \in \{0,1\}$ and $h \in H$, and then $R(H) = \{ R(g) \mid g \in H \} \leq \text{Aut}(\Gamma)$. Next, for any automorphism $\alpha$ of $H$ and any elements $x,y,g \in H$, we may define two permutations $\delta_{\alpha,x,y}$ and $\sigma_{\alpha,y}$ on $V(\Gamma) = H_0 \cup H_1$ as follows:

\[
\begin{align*}
\delta_{\alpha,x,y} & : h_0 \mapsto (xh^\alpha)_1 \quad \text{and} \quad h_1 \mapsto (yh^\alpha)_0, \quad \text{for each } h \in H,
\sigma_{\alpha,y} & : h_0 \mapsto (h^\alpha)_0 \quad \text{and} \quad h_1 \mapsto (gh^\alpha)_1, \quad \text{for each } h \in H,
\end{align*}
\]

and then define

\[
I = \{ \delta_{\alpha,x,y} \mid R^\alpha = x^{-1}Lx, \; L^\alpha = y^{-1}Ry \; \text{and} \; S^\alpha = y^{-1}S^{-1}x \},
\]

and

\[
F = \{ \sigma_{\alpha,y} \mid R^\alpha = R, \; L^\alpha = g^{-1}Lg \; \text{and} \; S^\alpha = g^{-1}S \}.
\]

With the above notation, the proposition below is easy to prove.
Proposition 2.2. [55 Theorem 1.1] Let $\Gamma = \text{BiCay}(H, R, L, S)$ be a connected bi-Cayley graph over the group $H$. Then $N_{\text{Aut}(\Gamma)}(R(H)) = R(H) \rtimes F$ if $I = \emptyset$, and $N_{\text{Aut}(\Gamma)}(R(H)) = R(H)(F, \delta_{x,y})$ if $I \neq \emptyset$ and $\delta_{x,y} \in I$. Moreover, for every $\delta_{x,y} \in I$ the following hold:

(a) $(R(H), \delta_{x,y})$ acts transitively on $V(\Gamma)$, and
(b) if $\alpha$ has order 2 and $x = y = 1$, then $\Gamma$ is isomorphic to the Cayley graph $\text{Cay}(H \rtimes \langle \alpha \rangle, R \cup \alpha S)$.

3. General theory on normal edge-transitive Cayley graphs

3.1. General properties. We begin this section with the following lemma, which shows that every normal edge-transitive bi-Cayley graph is bipartite.

Lemma 3.1. Let $\Gamma = \text{BiCay}(H, R, L, S)$ be a connected normal edge-transitive bi-Cayley graph over the group $H$. Then $R = L = \emptyset$, and hence $\Gamma$ is bipartite, with the two orbits of $R(H)$ on $V(\Gamma)$ as its parts.

Proof. Let $X = N_{\text{Aut}(\Gamma)}(R(H))$, which is edge-transitive on $\Gamma$. Now suppose $R \neq \emptyset$. Then $\{1_0, r_0\}$ is an edge of $\Gamma$, for some $r \in R$, and by edge-transitivity of $X$ on $\Gamma$, we know that some element $g \in X$ takes $\{1_0, 1_1\}$ to $\{1_0, r_0\}$. Also $H_0$ is an orbit of $R(H)$, and $R(H)$ is normal in $X$, and therefore $H_0$ is a block of imprimitivity for $X$ on $V(\Gamma)$. But then since $1_0^g \in \{1_0, r_0\}$ we have $H_0^g = H_0$, while on the other hand, since $1_0^g \in \{1_0, r_0\}$ we have $H_0^g = H_0$, and therefore $H_0 = H_1$, contradiction. Thus $R = \emptyset$. Similarly, $L = \emptyset$, and the rest follows easily. □

The next lemma shows that a normal bi-Cayley graph cannot be 3-arc-transitive.

Lemma 3.2. Let $\Gamma = \text{BiCay}(H, \emptyset, \emptyset, S)$ be a connected bi-Cayley graph, and let $X = N_{\text{Aut}(\Gamma)}(R(H))$. Then $X_{1_01_1} = \langle \sigma_{a,1} \mid \alpha \in \text{Aut}(H, S \setminus \{1\}) \rangle$, and hence $X$ does not act transitively on the 3-arcs of $\Gamma$.

Proof. By Proposition 2.2 we know that $X_{1_0} = \langle \sigma_{a,s} \mid \alpha \in \text{Aut}(H), s \in S, S^0 = s^{-1}S \rangle$, and hence that $X_{1_01_1} = \langle \sigma_{a,1} \mid \alpha \in \text{Aut}(H), S^0 = S \rangle = \langle \sigma_{a,1} \mid \alpha \in \text{Aut}(H, S \setminus \{1\}) \rangle$. Next, if $X$ acts transitively on the 3-arcs of $\Gamma$, then $X_{1_01_01_1}$ acts transitively on $\Gamma(1_1) \setminus \{1_0\}$, for some $s \in S \setminus \{1\}$. But for any $\sigma_{a,1} \in X_{1_01_01_1}$, we have $s_1 = s \alpha^{-1} = 1 \cdot (s^0)^{-1} = (s^0)_1$, and so $s^0 = s$, and then $(s^{-1})_0 = (s^{-1})_0 = (s^{-1})_0$, and this contradicts the transitivity of $X_{1_01_01_1}$ on $\Gamma(1_1) \setminus \{1_0\}$. □

3.2. Normal locally arc-transitive bi-Cayley graphs. The following proposition gives a characterisation of normal locally arc-transitive bi-Cayley graphs.

Proposition 3.3. Let $\Gamma = \text{BiCay}(H, \emptyset, \emptyset, S)$ be a connected bi-Cayley graph, and let $X = N_{\text{Aut}(\Gamma)}(R(H))$. Then $\Gamma$ is normal locally arc-transitive if and only if $\Gamma(1_0) = \{s_1 : s \in S\}$ is an orbit of the subgroup $F = \langle \sigma_{a,g} \mid S^0 = g^{-1}S \rangle$. Moreover, if $\Gamma$ is normal locally arc-transitive, then

(a) $X$ acts transitively on the arcs of $\Gamma$ if and only if there exists $\alpha \in \text{Aut}(H)$ such that $S^0 = S^{-1}$;
(b) $X$ acts semisymmetically on $\Gamma$ if and only if there exists no $\alpha \in \text{Aut}(H)$ such that $S^0 = S^{-1}$.

Proof. The first assertion is obvious, so now suppose that $\Gamma$ is normal locally arc-transitive. Then all we have to do is prove that $X$ acts transitively on the arcs of $\Gamma$ if and only if $S^0 = S^{-1}$ for some $\alpha \in \text{Aut}(H)$. If $X$ acts transitively on the arcs of $\Gamma$, then there exists $\delta_{x,y} \in I$ such that $(1_0, 1_1) \delta_{x,y} = (1_1, 1_0)$, and then $1_1 = 1_0 \delta_{x,y} = (x \cdot 1^0)_1 = x_1$ and $1_0 = 1_1 \delta_{x,y} = (y \cdot 1^0)_0 = y_0$, and it follows that $x = y = 1$, so $\delta_{x,y} \in I$, and therefore $S^0 = S^{-1}$ (by (2.2)). Conversely, if $S^0 = S^{-1}$ for some $\alpha \in \text{Aut}(H)$, then $\delta_{x,y}$ takes $s_0$ to $(s^{-1})_1$, and it follows that $X$ is vertex-transitive on $\Gamma$, and therefore arc-transitive on $\Gamma$. □

Recall that if a connected Cayley graph $\Gamma = \text{Cay}(G, S)$ on a group $G$ is bi-normal and arc-transitive, then $\Gamma$ is a normal arc-transitive bi-Cayley graph over $H = \bigcap_{x \in X} R(G)^x$. Conversely, by Propositions 2.3 and 2.2, every arc-transitive normal bi-Cayley graph must be a Cayley graph. On the other hand, an arc-transitive normal bi-Cayley graph is not necessarily a bi-normal Cayley graph.

We can now prove our first main theorem.

Proof of Theorem 1.1. For necessity, suppose that $X = N_{\text{Aut}(\Gamma)}(R(H))$ acts transitively on the 2-arcs of $\Gamma$. Then $X$ acts transitively on the arcs of $\Gamma$, and so (a) holds, by Proposition 2.3. Also $X_{1_01_1}$
acts transitively on $\Gamma(1_0) \setminus \{1_1\}$, and since $X_{1_0 s_1} = (\sigma_{a,1} \mid \alpha \in \text{Aut}(H,S))$ by Lemma 3.2 it follows that for all $s, t \in S \setminus \{1\}$, there exists $\sigma_{a,1} \in X_{1_0 s_1}$ such that $t_1 = s^{t}_{\sigma_{a,1}} = (1 \cdot s^a)_1 = (s^a)_1$, and so $t = s^a$.

Thus $\text{Aut}(H,S \setminus \{1\})$ is transitive on $S \setminus \{1\}$, proving (b). Furthermore, for any $s \in S \setminus \{1\}$ there exists $\sigma_{a,s} \in X_{1_0}$ such that $1^{\sigma_{a,s}} = s_1$, and then since $\sigma_{a,s} \in X_{1_0} = \Gamma$, we must have $S^\alpha = s^{-1}S$, so (c) holds.

For sufficiency, suppose the conditions (a) to (c) hold. Then by (a) and Proposition 2.2 we find that $X$ is vertex-transitive on $\Gamma$, and by (b) and (c), we find that $X_{1_0}$ is 2-transitive on $\Gamma(1_0)$. Thus $X$ acts transitively on the 2-arcs of $\Gamma$.

3.3. Normal half-arc-transitive bi-Cayley graphs.

**Proposition 3.4.** Let $\Gamma = \text{BiCay}(H,\emptyset,\emptyset,S)$ be a connected bi-Cayley graph, and let $X = \text{N}_{\text{Aut}(\Gamma)}(R(H))$.

Then $X$ acts transitively on the vertices and edges but not on the arcs of $\Gamma$ if and only if

(a) $X_{1_0}$ has exactly two orbits on $\Gamma(1_0)$ of equal size, say $O_1 = 1_1^{X_{1_0}}, O_2 = x_1^{X_{1_0}}$, and

(b) there exists $\alpha \in \text{Aut}(H)$ and $S^\alpha = S^{-1}x$.

**Proof.** Suppose $X$ acts transitively on the vertices and edges but not on the arcs of $\Gamma$. Then for any $s \in S$, we have $X_{1_0 s_1} = X_{1_0 s_1}$, so $|X : X_{1_0 s_1}| = |X : X_{1_0 s_1}| = |E(\Gamma)| = |V(G)||\Gamma(1_0)|/2$, and therefore $|X_{1_0} : X_{1_0 s_1}| = |\Gamma(1_0)|/2$. Thus $X_{1_0}$ has exactly two orbits on $\Gamma(1_0)$ of equal size, and (a) holds.

Now choose $x$ so that $1_1$ and $x_1$ lie in different orbits of $X_{1_0}$ on $\Gamma(1_0)$. Then there exists $\delta_{\alpha,a,b} \in X$ such that $\{1_0, 1_1\}^{\delta_{\alpha,a,b}} = \{1_0, x_1\}$, and then $1_0^{\delta_{\alpha,a,b}} = x_1$ and $1_1^{\delta_{\alpha,a,b}} = 1_1$, so that $a = x$ and $b = 1$. In this case $\alpha \in \text{Aut}(H)$, and $S^\alpha = S^{-1}x$, so (b) holds.

Conversely, suppose (a) and (b) hold. Then $X$ is vertex-transitive on $\Gamma$, by (b), but not arc-transitive on $\Gamma$, by (a). Next, for any edge $\{h_0, g_1\}$, we have $\{h_0, g_1\}^{R(h^{-1})} = \{1_0, (gh)^{-1}\}$, and by (a) it follows that $\{gh\}_1 \in O_1$ or $O_2$. If $\{gh\}_1 \in O_1$, then clearly $\{h_0, g_1\}$ lies in the same orbit of $X$ as $\{1_0, 1_1\}$. On the other hand, if $\{gh\}_1 \in O_2$, then there exists $\sigma \in X_{1_0}$ such that $(gh)_1^\sigma = x_1$, and then by (b) it follows that $\delta_{\alpha,x,1}$ is an automorphism of $\Gamma$ with $\{1_0, 1_1\}^{\delta_{\alpha,x,1}} = \{x_0, 1_1\}$, and then $\{1_0, (gh)_1\}^{\delta_{\alpha,x,1}} = \{1_0, 1_1\}$, so again, $\{h_0, g_1\}$ lies in the same orbit of $X$ as $\{1_0, 1_1\}$. Thus $X$ is edge-transitive on $\Gamma$.

Here we remark that in contrast to normal arc-transitive bi-Cayley graphs, normal half-arc-transitive bi-Cayley graphs may be non-Cayley; see Section 7.

4. EDGE-TRANSITIVE BI-ABELIAN GRAPHS

In this section, we prove Proposition 4.1. We do this in two steps.

**Proposition 4.1.** Let $\Gamma = \text{BiCay}(H,R,L,S)$ be a connected bi-Cayley graph over an abelian group $H$.

Then the following hold:

(a) If $R = L = \emptyset$, then $R(H) \rtimes \langle \delta_{0,1,1} \rangle$ is regular on $V(\Gamma)$, where $\alpha$ is the automorphism of $H$ that maps every element of $H$ to its inverse.

(b) If $\Gamma$ is edge-transitive, then $\Gamma$ is vertex-transitive.

**Proof.** Suppose $R = L = \emptyset$. Since $H$ is abelian, there exists an automorphism $\alpha$ of $H$ such that $\alpha$ maps every element of $H$ to its inverse, and in particular, $S^\alpha = S^{-1}$. It then follows from Proposition 2.2 that $\delta_{0,1,1}$ is an automorphism of $\Gamma$ of order 2 interchanging $H_0$ and $H_1$, and $R(H) \rtimes \langle \delta_{0,1,1} \rangle$ is regular on $V(\Gamma)$, which proves (a). Next, for (b), suppose $\Gamma$ is edge- but not vertex-transitive. Then $\Gamma$ is semisymmetric, and hence bipartite, with its parts being the two orbits of $\text{Aut}(\Gamma)$ on $V(\Gamma)$. It follows that $H_0$ and $H_1$ are two partition sets of $\Gamma$, and so $R = L = \emptyset$, but then by (a), $\Gamma$ is vertex-transitive after all, contradiction. Thus $\Gamma$ is vertex-transitive.

**Proposition 4.2.** Let $\Gamma = \text{BiCay}(H,R,L,S)$ be a connected half-arc-transitive bi-Cayley graph over an abelian group $H$.

Then the following hold:

(a) $R \cup L$ is non-empty and contains no involution.

(b) $|R| = |L|$ is even, and $|S| > 2$.

(c) $\Gamma$ has valency 6 or more.
Proof. Again let $\alpha$ be the automorphism of $\Gamma$ that takes every element of $H$ to its inverse. If $R = L = \emptyset$, then by Proposition 1.3, $\delta_{a,1,1} \in \text{Aut}(\Gamma)$ and $(1,0,1)^{\delta_{a,1,1}} = (1,1,0)$, which implies that $\Gamma$ is arc-transitive, contradiction. Hence $R \cup L$ is non-empty. Also if $R$ contains an involution $h$, then $\{1,0,0\} \in E(\Gamma)$ and $(1,0,0)^{R(h)} = (h,0,0)$, which again implies that $\Gamma$ is arc-transitive, contradiction. Similarly, $L$ does not contain an involution. This proves (a). Moreover, because $R = R^{-1}$ and $L = L^{-1}$, it also implies that $|R| = |L|$ is even. Next, since $\Gamma$ is half-arc-transitive, the valency $|\Gamma(1)| = |R| + |S|$ is even, and so $|S|$ is even, and therefore $|S| \geq 2$. Also by half-arc-transitivity, $\text{Aut}(\Gamma)$ has two orbits on $\Gamma(1)$ of equal size, say $B_1$ and $B_2$, with $r_0$ and $(r_0)^{-1}$ being in different orbits, for any $r \in R$. It follows that half of the elements of $R$ are contained $B_1$, and half are in $B_2$. But now suppose $|S| = 2$, say $S = \{1,s\}$. Then since $|B_1| = |B_2| = (|R| + |S|)/2 = |R|/2 + 1$, the other two neighbours $1$ and $s_1$ of $1_0$ must lie in different orbits of $\text{Aut}(\Gamma)_{1_0}$. On the other hand, it is easy to check that $\sigma_{a,s} \in \text{Aut}(\Gamma_{1_0})$ takes $x_1$ to $s_1$, contradiction. Hence $|S| > 2$, which proves (b). Finally, this implies that the valency $|R| + |S|$ of $\Gamma$ is at least $2 + 4 = 6$, proving (c). \hfill \Box

In fact, 6 is the minimum valency of all connected half-arc-transitive bi-Cayley graphs over an abelian group, and is achieved by the following example:

Example 4.3. Let $\Gamma = \text{BiCay}(H,R,L,S)$, where $H = \langle a \rangle = C_{28}$, and $R = \{a,a^{-1}\}$, $L = \{a^{13},a^{-13}\}$ and $S = \{1,a,a^6,a^{19}\}$. Then $\Gamma$ has valency 6, and an easy computation using Magma \cite{Magma} shows that $\Gamma$ is half-arc-transitive, with $\text{Aut}(\Gamma) \cong (C_7 \times Q_8) \rtimes C_3$, where $Q_8$ is the quaternion group.

To complete this section, we give two easy corollaries of the above two propositions.

Corollary 4.4. No connected bi-Cayley graph over an abelian group is semisymmetric.

Corollary 4.5. Let $\Gamma = \text{BiCay}(H,\emptyset,\emptyset,S)$ be a connected trivalent edge-transitive bi-Cayley graph over a cyclic group $H \cong C_n$. Then $n = 2$ or $4$, or $n$ is a divisor of $k^2 + k + 1$ for some $k \in \mathbb{Z}_n^*$. Furthermore, if $n \geq 13$ then $\Gamma$ is 1-arc-regular.

Proof. By Proposition 1.3 we know that $\Gamma$ is an arc-transitive Cayley graph over the group $R(H) \rtimes \langle \delta_{a,1,1} \rangle$, which is dihedral of order $2n$. It then follows from a theorem in \cite{H4} on Cayley graphs over dihedral groups that $n = 2$ or 4, or $n$ divides $k^2 + k + 1$ for some $k \in \mathbb{Z}_n^*$. If $n = 2$ or 4, then $\Gamma$ is isomorphic to the complete graph $K_4$ or the cube graph $Q_3$, while if $n = 3$ or 7 then $\Gamma$ is isomorphic to the complete bipartite graph $K_{3,3}$ or the Heawood graph, and in all other cases (with $k \geq 3$ and $n \geq 13$), $\Gamma$ is 1-arc-regular. \hfill \Box

5. Trivalent edge-transitive graphs with girth at most 6

The aim of this section is to give a classification of all connected trivalent edge-transitive graphs with girth 6, and to prove Theorem 1.3.4. We achieve this in two stages, the first being a special case, and the second the general case.

5.1. Trivalent normal edge-transitive bi-abelian graphs. We begin by defining a family of connected trivalent edge-transitive bi-abelian graphs.

Let $n$ and $m$ be any two positive integers with $nm^2 \geq 3$. If $n = 1$ take $\lambda = 0$, while if $n > 1$ take $\lambda \in \mathbb{Z}_n^*$ such that $\lambda^2 - \lambda + 1 \equiv 0 \pmod{n}$. Now define

$$\Gamma_{m,n,\lambda} = \text{BiCay}(H,\emptyset,\emptyset,\{1,x,xy\})$$

where $H = \mathcal{H}_{m,n} = \langle x \rangle \times \langle y \rangle \cong C_{nm} \times C_m$.

Lemma 5.1. Let $X = N_A(\mathcal{H}_{m,n})$, where $A = \text{Aut}(\Gamma_{m,n,\lambda})$. Then:

(a) if $n \leq 3$, then $X$ acts transitively on the 2-arcs of $\Gamma_{m,n,\lambda}$, while

(b) if $n > 3$, then $X$ acts transitively on the arcs but not on the 2-arcs of $\Gamma_{m,n,\lambda}$.

Moreover, if $nm^2 > 4$ then $\Gamma_{m,n,\lambda}$ has girth 6.

Proof. First, there exists an automorphism $\alpha$ of $\mathcal{H}_{m,n}$ that takes $(x,y)$ to $(x^{\lambda^0}y,x^{-(\lambda^2-\lambda+1)}y^{-\lambda})$. To see this, note that $x = (x^{\lambda^0}y)^{-\lambda} \cdot (x^{-(\lambda^2-\lambda+1)}y^{-\lambda})^{-1}$, so that $x^{\lambda^0}y$ and $x^{-1}(\lambda^2-\lambda+1)y^{-\lambda}$ generate $\mathcal{H}_{m,n}$. Next, $\lambda-1 \in \mathbb{Z}_n^*$ since $\lambda(\lambda-1) \equiv -1 \pmod{n}$, and so $x^{\lambda^0}y$ has order $m_1$ dividing $m$. It follows that $x^{(\lambda^0-1)m_1} = 1 = y^{m_1}$, so that $mn$ divides $m_1n(\lambda-1)$ and $m$ divides $m_1n$, and then
\( \frac{m}{n} \) divides \( \lambda - 1 \) and \( n \), and hence divides \( \text{GCD}(\lambda - 1, n) = 1 \), so \( m = m_1 \). Thus \( x^{\lambda-1}y \) has order \( mn \).

Similarly, \( x^{-(\lambda^2 - \lambda + 1)}y^x \) has order \( k \) for some \( k \) dividing \( m \) (since \( \lambda^2 - \lambda + 1 \equiv 0 \mod n \)), and then because \( y^{k\lambda} = 1 = x^{k(\lambda^2 - \lambda + 1)} \), we find that \( \frac{m}{x} | \lambda \) and \( \frac{m}{x} | \frac{m}{x} | \lambda^2 - \lambda + 1 \), and therefore \( \frac{m}{x} = 1 \), which gives \( m = k \), and thus \( x^{-(\lambda^2 - \lambda + 1)}y^x \) has order \( m \).

Note also that \( \{1, x, x^y\}^\alpha = \{1, x^{\lambda-1}y, x^{\lambda^2 - \lambda + 1}y - y^{\lambda-1}x\} = \{1, x^{\lambda-1}y, x^{\lambda^2 - \lambda + 1}y - x\} = x^{\lambda - 1}\{1, x, x^y\} \), so that \( \alpha \) acts like left multiplication by \( x^{-1} \) on the set \( S = \{1, x, x^y\} \). It follows that \( \sigma_{\alpha,x} \) is an automorphism of \( \Gamma_{m,n,\lambda} \) that fixes \( 1 \) only, and its girth is at most 3. Consequently, suppose \( X \) acts transitively on the 2-ars of \( \Gamma_{m,n,\lambda} \). Then \( \sigma_{\alpha,x} \) fixes the vertex \( 1 \) and induces a 3-cycle on its neighbours, so \( \Gamma_{m,n,\lambda} \) is arc-transitive.

Moreover, there exists an automorphism of \( \tilde{H} \) that inverts every element, since \( H \) is abelian, and hence by Proposition 3.3 we find that \( \Gamma_{m,n,\lambda} \) is arc-transitive.

Next, if \( n \leq 3 \), then since \( n \) divides \( \lambda^2 - \lambda + 1 \), we have \( n = 1 \) or \( n = 3 \) and 3, moreover, if \( n = 1 \) then \( \lambda = 0 \), while if \( n = 3 \) then \( \lambda = 2 \). In both cases it is easy to check that there is an automorphism \( \beta \) of \( \mathcal{H}_{m,n} \) taking \( (x, y) \) to \( (x^{\lambda y}, x^{\lambda y-\lambda}y^-\lambda) \). This automorphism swaps \( x \) with \( x^{\lambda}y \), and so by Theorem 1.1 the group \( X \) acts transitively on the 2-ars of \( \mathcal{H}_{m,n,\lambda} \). Conversely, suppose \( X \) acts transitively on the 2-ars of \( \mathcal{H}_{m,n,\lambda} \). Then there exists \( \beta \in \mathcal{A}(\mathcal{H}_{m,n}) \) such that \( \beta \) swaps \( x \) with \( x^{\lambda y} \), so \( \beta \) swaps \( m \lambda \), and it follows that \( \lambda^2 \equiv 1 \mod n \). Then since \( \lambda^2 - \lambda + 1 \equiv 0 \mod n \), we find that \( \lambda \equiv 2 \mod n \), and so \( 0 \equiv \lambda^2 - \lambda + 1 \equiv 4 \mod n \), which implies that \( n \leq 3 \).

Finally, we consider the girth of \( \mathcal{H}_{m,n,\lambda} \), which is even, since \( \mathcal{H}_{m,n,\lambda} \) is bipartite. In all cases, \( \mathcal{H}_{m,n,\lambda} \) contains a 6-cycle, namely \( (1, 0, x, (x^{\lambda y}-y^{\lambda}1), (x^{\lambda}-y^{1}1), (x^{\lambda}-y^{1}1), 1) \), and so its girth is at most 6.

On the other hand, if the girth is at most 4, then it is one of the two connected trivalent arc-transitive graphs of girth 4, namely the complete bipartite graph \( K_{3,3} \) or the cube graph \( Q_3 \) (see [14]). These are the graphs that occur in the cases \( (m, n, \lambda) = (1, 3, 2) \) and \( (2, 1, 0) \) respectively, and are also the only cases with order at most 8, and hence with \( mn \leq 2 \). See also [22] Lemma 4.1.

Proposition 5.2. Let \( \Gamma = \text{BiC Cay}(H, \emptyset, \emptyset, S) \) be a connected trivalent normal edge-transitive bi-Cayley graph over an abelian group \( H \). Then \( \Gamma \cong \Gamma_{m,n,\lambda} \) for some \( m, n, \lambda \).

Proof. Let \( X = N_{\text{Aut}(\Gamma)}(R(H)) \). Then by Proposition 3.3 \( \Gamma(1_0) \) is an orbit of \( X_{1_0} \), so there exists \( \alpha \in \text{Aut}(H) \) and \( a \in H \) such that \( \sigma_{\alpha,a} \) cyclically permutes the three neighbours of \( 1_0 \) in \( \Gamma \), and it follows that \( S = \{1, a, aa^{-1}\} \). Now let \( b = aa^{-1} \). Then \( \sigma_{\alpha,a} \) induces the 3-cycle \((1, a_1, b_1) \) on \( \Gamma(1_0) \), so \( ab\alpha = 1 \), which gives \( b\alpha = a^{-1} \). Hence in particular, \( a \) and \( b \) have the same order. Also by connectedness of \( \Gamma \), we have \( H = (a, b) \). Next let \( n = |(a) \cap (b)| = n \) and \( m = |(a) \cap (b)| = |(b) : (a) \cap (b)| \). Then we find that \( |(a)| = |(b)| = nm \), and \( (a) \cap (b) = (a^{\lambda m}) = (b^{\lambda m}) \), and it follows that \( b^m = a^{\lambda m} \) for some \( \lambda \in \mathbb{Z}_n \) when \( n > 1 \), or with \( \lambda = 0 \) when \( n = 1 \). Moreover, we have

\[
a^{-m} = (b^{\alpha m})^m = (b^m)^\alpha = (a^{\lambda m})^\alpha = (a^{-1})^{\lambda m} = a^{-\lambda m} = a^{-\lambda m} a^{\lambda m} = a^{\lambda m} (a^{-\lambda} + \lambda^2),
\]

and therefore \( \lambda^2 - \lambda + 1 \equiv 0 \mod n \) (because \( a \) has order \( mn \)). Finally, letting \( x = a \) and \( y = a^{-\lambda} \), we have \( H = (a, b) = (x) \times (y) \), with \( S = \{1, a, b\} = \{1, x, x^{\lambda y}\} \), and thus \( \Gamma \cong \Gamma(m, n, \lambda) \). \( \square \)

5.2. Trivalent edge-transitive graphs with small girth. In this subsection, we use Proposition 4.1 to study trivalent edge-transitive graphs with girth at most 6, and prove Theorem 1.4. This is partially motivated by the work in [14] and [27] on the classification of trivalent arc-transitive graphs of small girth. A natural question is whether there exists a trivalent semisymmetric graph of girth at most 6. We will show that the answer is negative. First, we prove the following:

Lemma 5.3. Let \( \Gamma \) be a connected trivalent edge-transitive graph of girth 6, and let \( A = \text{Aut}(\Gamma) \). If \( c \) is the number of 6-cycles passing through an edge in \( \Gamma \), then \( c = 2, 4, 6, \) or \( 8 \). Moreover,

\begin{enumerate}[(a)]
\item if \( c = 2 \), then \( A \cong C_6 \) for every vertex \( v \) of \( \Gamma \), or \( A \cong S_3 \) for every vertex \( v \) of \( \Gamma \), while
\item if \( c = 3 \), then \( \Gamma \) is isomorphic to the Heawood graph, the Pappus graph, the generalised Petersen graph \( P(10, 3) \), or the generalised Petersen graph \( P(8, 3) \), with \( c = 8, 4, 6 \) or \( 4 \) respectively.
\end{enumerate}

Proof. Let \( u \) be any vertex of \( \Gamma \). Since every 6-cycle passing through \( u \) uses two of the three edges incident with \( u \), and every edge lies in c 6-cycles, the number of 6-cycles through \( u \) is \( b = 3c/2 \). In particular, this
is independent of $u$, and $c$ is even. Also because $\Gamma$ has valency 3 and girth 6, it is easy to see that there are at most eight 6-cycles passing through an edge of $\Gamma$, and so $c = 2, 4, 6$ or 8 (and $b = 3, 6, 9$ or 12).

Similarly, if $x$ is a vertex at distance 3 from $u$, then there are at most three 6-cycles passing through both $u$ and $x$, and $|\Gamma(x) \cap \Gamma_2(u)|$ is at most 3. Moreover, if $|\Gamma(x) \cap \Gamma_2(u)| = 3$, then we know from [54, Lemma 4.6] that $\Gamma$ is isomorphic to the Heawood graph or the generalised Petersen graph $P(8, 3)$. For these two graphs, we have $c = 8$ and 6 respectively, and from now on, we may suppose that $|\Gamma(x) \cap \Gamma_2(u)| \leq 2$ for every vertex $x \in \Gamma_3(u)$. In particular, since there are $2|\Gamma_2(u)| = 12$ edges between $\Gamma_2(u)$ and $\Gamma_3(u)$, under the latter assumption we find that $b \leq 6$, and so $c \leq 4$.

Now suppose $c = 4$. Then $b = 6$, and also $|\Gamma_3(u)| = 6$, with every vertex in $\Gamma_3(u)$ adjacent to two of the vertices in $\Gamma_2(u)$, so $|\Gamma(x) \cap \Gamma_2(u)| = 2$ for every $x \in \Gamma_3(u)$. Note that this holds for every vertex $u$.

Next, let $y \in \Gamma_4(u)$. If we choose $v \in \Gamma(u)$ such that $y \in \Gamma_4(u) \cap \Gamma_3(v)$, so that $v$ is a neighbour of $u$ on some path of length 4 from $u$ to $y$, then $|\Gamma(y) \cap \Gamma_2(v)| = 2$, and in particular, $|\Gamma(y) \cap \Gamma_3(u)| \geq 2$. Then since each vertex in $\Gamma_3(u)$ is adjacent to just one vertex in $\Gamma_4(u)$, while each vertex in $\Gamma_4(u)$ is adjacent to two vertices in $\Gamma_3(u)$, it follows that $|\Gamma_4(u)| \leq 3$.

Next, if $|\Gamma(y) \cap \Gamma_3(u)| = 3$ then $|\Gamma_4(u)| = 2$, with both vertices in $\Gamma_4(u)$ having three neighbours in $\Gamma_3(u)$, so $\Gamma$ has diameter 4, with $|V(\Gamma)| = 1 + |\Gamma(u)| + |\Gamma_2(u)| + |\Gamma_4(u)| + |\Gamma_4(u)| = 1 + 3 + 6 + 6 + 2 = 18$. Then by what we know about edge-transitive trivalent graphs of small order from [12, 13], we find that $\Gamma$ is isomorphic to the Pappus graph. On the other hand, if $|\Gamma(y) \cap \Gamma_3(u)| = 2$ for all $y \in \Gamma_4(u)$, then there are at most three edges from $\Gamma_4(u)$ to $\Gamma_3(u)$, so $|\Gamma_3(u)| \leq 3$. But also if $z \in \Gamma_5(u)$ then the same argument as above shows that $|\Gamma(z) \cap \Gamma_4(u)| \geq 2$, and it follows that $|\Gamma_3(u)| = 1$ and $|\Gamma(z) \cap \Gamma_4(u)| = 3$. Hence in this case $\Gamma$ has diameter 5, with $|V(\Gamma)| = 1 + |\Gamma(u)| + |\Gamma_2(u)| + |\Gamma_3(u)| + |\Gamma_4(u)| + |\Gamma_5(u)| = 1 + 3 + 6 + 6 + 3 + 1 = 20$, and then from [12] we find that $\Gamma$ is isomorphic to the generalised Petersen graph $P(10, 3)$. (Note: the only other edge-transitive trivalent graph of order 20 is the dodecahedral graph, which has girth 5.)

Finally, suppose $c = 2$. Then $b = 3$, and by edge-transitivity, each of the three neighbours of $u$ lies in two of the three 6-cycles passing through $u$, and each 2-arc of the form $(v, u, w)$ lies in exactly one of them. The same holds at any neighbour of $u$, and it follows that each of six vertices in $\Gamma_2(u)$ lies in exactly one of the three 6-cycles passing through $u$. Also just three of the vertices in $\Gamma_3(u)$ lie on these cycles, and are then adjacent to two vertices in $\Gamma_2(u)$, while all other vertices in $\Gamma_3(u)$ are adjacent to a single vertex in $\Gamma_4(u)$. Since there are $2|\Gamma_2(u)| = 12$ edges between $\Gamma_2(u)$ and $\Gamma_3(u)$, we find that $|\Gamma_3(u)| = 6/2 + 6 = 9$, and the induced subgraph on $\{u\} \cup \Gamma(u) \cup \Gamma_2(u) \cup \Gamma_3(u)$ is as shown in Figure 1.

![Figure 1. Local subgraph of $\Gamma$ in the case of $c = 2$.](image)

Now from Figure 1 (or the argument leading to it) we see that any automorphism in $A = \operatorname{Aut}(\Gamma)$ that fixes $u$ and each of its three neighbours must fix all the vertices of each of the three 6-cycles passing through $u$, and hence fixes every vertex of $\Gamma_2(u)$. Hence $A_{\{u, \Gamma(u)\}} = A_{\{u, \Gamma(u)\} \cup \Gamma_2(u)}$ for every $u \in V(\Gamma)$. By connectedness and induction, it follows that $A_{\{u, \Gamma(u)\}}$ fixes every vertex of $\Gamma$, and is therefore trivial. In particular, $A_u$ acts faithfully on $\Gamma(u)$, and so by edge-transitivity, $A_u \cong C_3$ or $S_3$. Moreover, if $w$ is any neighbour of $u$, then $|A_u| = 3|A_{uw}| = |A_w|$, and thus either $A_v \cong C_3$ for all $v \in V(\Gamma)$, or $A_v \cong S_3$ for all $v \in V(\Gamma)$.

The above lemma helps us to find all trivalent edge-transitive graphs of girth 6, as follows.

**Proposition 5.4.** Let $\Gamma$ be a connected trivalent edge-transitive graph of girth 6. Then either $\Gamma \cong \Gamma_{m,n,\lambda}$ with $nm^2 > 9$ (as defined in [54]), or $\Gamma$ is isomorphic to the Heawood graph, the Pappus graph, the
generalised Petersen graph $P(8, 3)$, or the generalised Petersen graph $P(10, 3)$. In particular, in all cases, the graph $\Gamma$ is arc-transitive.

Proof. Let $A = \text{Aut}(\Gamma)$. First we show that $\Gamma$ is bipartite. If $\Gamma$ is arc-transitive, then this follows from [13] Corollary 6.3 or from what was proved for girth 6 in [14], while if $\Gamma$ is not arc-transitive, then $\Gamma$ is semisymmetric and hence bipartite. Moreover, $A = \text{Aut}(\Gamma)$ acts transitively on each part of $\Gamma$.

Next, let $\{u, v\} \in E(\Gamma)$, and take $B = \langle A_u, A_v \rangle$. Then $B$ is edge- but not vertex-transitive on $\Gamma$, and by edge-transitivity, we have $|A_u : A_{uv}| = 3 = |A_v : A_{uv}|$. If there are more than two 6-cycles passing through $\{u, v\}$ in $\Gamma$, then by Lemma 5.3(b), we know that $\Gamma$ is isomorphic to the Heawood graph, the Pappus graph, $P(8,3)$ or $P(10,3)$, all of which are arc-transitive.

From now on, we will suppose that there are exactly two 6-cycles passing through $\{u, v\}$, and hence (by Lemma 5.3(a)) that $A_u \cong A_v \cong C_3$ or $S_3$.

Under this assumption, it follows that if $\Gamma$ is arc-transitive, then $A$ is a quotient of one of the Djoković–Miller amalgams $1'$ or $2'$ and $2''$ from [10] used in [12] and [14], while if $\Gamma$ is semisymmetric then $A$ is a quotient of one of the Goldschmidt amalgams $G_1$ and $G_1^3$ from [23] used in [13]. In particular, $A$ can be obtained from one of those amalgams by adding extra relations to their defining presentations, to force a circuit of length 6 in the resulting graph.

When $\Gamma$ is arc-transitive, we find from similar ideas as carried out in [14] that for girth 6 the amalgam $2''$ can be eliminated, and furthermore, if $A_u \cong A_v \cong C_3$ then $A$ is a quotient of the ordinary $(2,3,6)$ triangle group $\langle a, h \mid a^2 = h^3 = (ha)^6 = 1 \rangle$, with the image of $h$ and generating $A_u$, and the image of $h^a$ generating $A_v$, while if $A_u \cong A_v \cong S_3$ then $A$ is a quotient of either the extended $(2,3,6)$ triangle group $\langle a, h, p \mid a^2 = h^3 = (ha)^6 = p^2 = (ap)^2 = (hp)^2 = 1 \rangle$, with the images of $h$ and $p$ and generating $A_u$, and the images of $h^a$ and $p^a (= p)$ generating $A_v$. In both cases, the elements $h$ and $k = h^a$ satisfy the relations $h^3 = k^3 = (hk)^3 = 1$, and hence generate a subgroup of $A$ of index 2 or 4 that acts transitively on each part of $\Gamma$.

On the other hand, when $\Gamma$ is semisymmetric, we can perform the same kind of analysis as was carried out in [14] for the case of girth 6.

If $A_u \cong A_v \cong C_3$ then $A$ is a quotient of the free product $C_3 * C_3 = \langle h, k \mid h^3 = k^3 = 1 \rangle$, with the images of $h$ and $k$ generating $A_u$ and $A_v$, and having girth 6 implies that some further relation $w = 1$ is satisfied, where $w = w(h, k)$ is a word of length 6 in the generators $h$ and $k$. Without loss of generality, $w = h^{r_1}k^{s_1}h^{r_2}k^{s_2}h^{r_3}k^{s_3}$ with $r_1 = \pm 1$ and $s_1 = \pm 1$ for $1 \leq i \leq 3$, and then an easy computation using MAGMA [7] shows that every such relation forces the quotient to have order at most 24, except in the cases where $w = (hk)^3$, $(hk^{-1})^3$, $(h^{-1}k)^3$ or $(h^{-1}k^{-1})^3$. But we know from [13] that a semisymmetric trivalent graph has order at least 54 and hence at least 81 edges, so $|A| \geq 24$. Also we can replace each of $h$ and $k$ by its inverse, and so we may conclude that $w = (hk)^3$, and again we have elements $h$ and $k$ satisfying the relations $h^3 = k^3 = (hk)^3 = 1$.

Similarly, if $A_u \cong A_v \cong S_3$ then $A$ is a quotient of $\langle h, k, p \mid h^3 = k^3 = p^2 = (hp)^2 = (kp)^2 = 1 \rangle$, with $A_u$ and $A_v$ being the images of $(h, p)$ and $(k, p)$, and in this case girth 6 implies that some relation of the form $w = 1$ or $w = p$ is satisfied, where $w$ is as above. Here the analogous MAGMA computation shows that there are only four such relations that produce a quotient of order greater than 60, namely $(hk)^3 = 1$ and the others obtainable by replacing $h$ and/or $k$ by their inverses.

Hence in all cases, whether $\Gamma$ is arc-transitive or semisymmetric, $A = \text{Aut}(\Gamma)$ contains two elements $h$ and $k$ that fix the vertices $u$ and $v$ and induce 3-cycles on $\Gamma(u)$ and $\Gamma(v)$, respectively, and satisfy the relations $h^3 = k^3 = (hk)^3 = 1$. Also it is clear that the subgroup $L$ generated by $h$ and $k$ is edge-transitive, with two orbits on vertices of $\Gamma$ (namely the two parts of $\Gamma$), and has index at most 2 in $B = \langle A_u, A_v \rangle$; indeed $B = \langle h, k \rangle$ or $\langle h, k, p \rangle$ in each of the above cases.

Now let $J$ be the subgroup of $L = \langle h, k \rangle$ generated by $x = hkh$ and $y = h^{-1}k$. Then $J$ is normal in $L$, because

\[ x^h = kh^2 = kh^{-1} = y^{-1} \quad \text{and} \quad y^h = k^{-1}h = k^2h = y^{-1}x, \]

while

\[ x^k = k^{-1}hkhk = k^{-1}(hk)^2 = k^{-1}(hk)^{-1} = k^{-2}h^{-1} = kh^{-1} = y^{-1} \quad \text{and} \quad y^k = k^{-1}h = k^2h = y^{-1}x, \]
and it follows that also \( J \) is abelian, since \( y^x = y^{hkh} = (y^{-1}x)^{kh} = (x^{-1}y^{-1})^h = (x^{-1})^h = y \).

Moreover, \( Jh = Jk \), since \( hk^{-1} = y \in J \), and therefore \( L/J = ⟨Jh, Jk⟩ = ⟨Jh⟩ \), and then since \( h \) has order 3 it follows that \( |L/J| = 1 \) or 3. On the other hand, if \( L = J \) then \( L \) is abelian so \( h \) commutes with \( k \) and therefore the edge-transitive group \( L = ⟨h, k⟩ \) has order at most 9, which is impossible since \( Γ \) has girth 6. Hence \( |L : J| = 3 \). In particular, \( h \notin J \) and \( k \notin J \), so \( J \) is complementary to each of \( L_u = ⟨h⟩ \) and \( L_v = ⟨k⟩ \) in \( L \), and it follows that \( J \) acts semi-regularly on \( V(Γ) \), with two orbits, namely the two parts of \( Γ \). Thus \( Γ \) is a bi-Cayley graph over the abelian group \( J \).

But furthermore, we can show that \( J \) is normal in \( A = \text{Aut}(Γ) \), in all cases. For if \( Γ \) is semisymmetric and \( A_u ≅ A_v ≅ C_3 \), then \( A = ⟨h, k⟩ = L \), while if \( Γ \) is semisymmetric and \( A_u ≅ A_v ≅ S_3 \), then \( A = ⟨h, k, p⟩ \) for some \( p \) satisfying \( p^2 = ⟨hp⟩^2 = (kp)^2 = 1 \), and then
\[
x^p = (hk)^p = h^{-1}k^{-1}h^{-1} = x^{-1} \quad \text{and} \quad y^p = (hk)^p = h^{-1}k = h^{-1}k^{-2} = x^{-1}y.
\]

On the other hand, if \( Γ \) is arc-transitive and \( A_u ≅ A_v ≅ C_3 \), then \( A = ⟨h, a⟩ \) for some involution \( a \) conjugating \( h \) to \( k \), and then
\[
x^a = (hk)^a = khk = h^{-1}k^{-1}h^{-1} = x^{-1} \quad \text{and} \quad y^a = (hk^{-1})^a = kh^{-1} = y^{-1},
\]
while if \( Γ \) is arc-transitive and \( A_u ≅ A_v ≅ S_3 \), then \( A = ⟨h, a, p⟩ \) for some \( a \) as above, and some \( p \) satisfying \( p^2 = (ap)^2 = (hp)^2 = 1 \), and then \( h^p = h^{-1} \) while \( k^p = (ah)^p = ah^{-1}a = k^{-1} \), and so again \( x^p = x^{-1} \) and \( y^p = x^{-1}y \). Thus \( Γ \) is a normal bi-Cayley graph over the abelian group \( J \).

We can now apply Proposition 5.2 which tells us that \( Γ \) is isomorphic to \( Γ_{m,n,λ} \) for some \( m, n, λ \) with \( λ^2 = λ + 1 \equiv 0 \mod n \). Hence in particular, \( Γ \) is arc-transitive.

Finally, if \( mn^2 < 9 \) then it is easy to see that \((n, m) = (3, 1), (7, 1), (1, 2), \) or \((1, 3) \). Also if \((n, m) = (3, 1) \) or \((1, 2) \), then \( Γ_{m,n,λ} ≅ K_{3,3} \) or \( Q_3 \), each of which has girth 4, while if \((n, m) = (7, 1) \) or \((1, 3) \), then \( Γ_{m,n,λ} \) is isomorphic to the Heawood graph or Pappus graph, which do not satisfy our assumption on the number of 6-cycles through an edge. Thus \( mn^2 > 9 \) completing the proof.

We can now prove Theorem 1.4 showing that all trivalent edge-transitive graphs of girth at most 6 are known, and are arc-transitive.

Proof of Theorem 1.4 The arc-transitive trivalent graphs of girth less than 6 are known (see [13] or [27]), and by [22] Lemma 4.1, there is no semisymmetric trivalent graph of girth less than 6, and Proposition 5.2 gives all trivalent edge-transitive graphs of girth exactly 6, with none being semisymmetric.

6. EDGE-TRANSITIVE BI-DIHEDRANTS

In this section, we investigate edge-transitive bi-dihedrants (that is, bi-Cayley graphs over dihedral groups). We will show there are no semisymmetric bi-dihedrants of valency at most 5, and on the other hand, by considering normal edge-transitive bi-dihedrants, that there exist semisymmetric bi-dihedrants of valency \( 2k \) for every odd integer \( k > 1 \). Also we give a characterisation of 6-valent edge-regular semisymmetric bi-Cayley graphs over a dihedral group \( D_n \) of odd degree \( n \). In turn, this enables us to answer two questions proposed in 2001 by Marušič and Potočnik on semisymmetric tetracirculants [42].

6.1. The smallest valency of semisymmetric bi-dihedrants. The aim of this subsection is to prove the following theorem.

Theorem 6.1. Let \( Γ = \text{BiCay}(H, R, L, S) \) be a connected semisymmetric bi-Cayley graph over a dihedral group \( H = ⟨a, b⟩ \mid a^3 = b^2 = (ab)^3 = 1 ⟩ ≅ D_n \) (for some \( n ≥ 3 \)). Then the valency of \( Γ \) is at least 6.

Proof. First, \( Γ \) is bipartite, and its two parts are the orbits of both \( A = \text{Aut}(Γ) \) and its subgroup of \( H \) on \( V(Γ) \). It follows that \( R = L = 0 \). By Proposition 2.1(b), we may assume that \( 1 ∈ S \), and since \( Γ \) is not vertex-transitive, we find by Proposition 2.2 that there is no automorphism of \( H \) mapping \( S \) to \( S^{-1} \).

It follows that \( S \setminus \{1\} \) cannot consist entirely of involutions, and so \( S \) must contain at least one element of order greater than 2. On the other hand, as \( Γ \) is connected, \( S \) must contain at least one element of the form \( h = a^i \) (with \( i ∈ Z_n \)), and then by replacing \( b \) by \( ba^i \) if necessary, we may suppose that \( i = 0 \) and hence that \( S \) contains \( b \). If, however, this is the only involution in \( S \), then all other elements of \( S \) are powers of \( s \), and so the automorphism of \( H \) taking \( (a, b) \) to \( (a^{-1}, b) \) inverts every element of \( S \), so takes \( S \) to
$S^{-1}$, which is impossible. Similarly, if $S$ contains just one other involution of the form $ba^j$ (with $j \in \mathbb{Z}_n$), then the automorphism of $H$ taking $(a, b)$ to $(a^{-1}, ba^j)$ swaps the involutions $b$ and $ba^j$ and inverts every other element of $S$, and so takes $S$ to $S^{-1}$, which again is impossible. Hence $S$ contains at least three involutions, as well as an element of order greater than 2. In particular, the valency $|S|$ of $\Gamma$ is at least 5.

To complete the proof, we need only show that $|S|$ cannot be 5. So we assume the contrary. Then we know that $S = \{1, b, ba^i, ba^j, a_k^i, a_k^j\}$ where $0 < i < j < n$, and $a_k^i$ is not an involution. In particular, we cannot have $i \equiv -i \equiv k \mod n$, or $j \equiv -j \equiv k \mod n$, or $i - j \equiv j - i \equiv k \mod n$. Also the fact that no automorphism of $H$ taking $(a, b)$ to $(a^{-1}, b)$ or $(a^{-1}, ba^j)$ or $(a^{-1}, ba^i)$ is allowed to take $S$ to $S^{-1}$ implies that $i \not\equiv -j \mod n$, and $j \not\equiv 2i \mod n$, and $i \not\equiv 2j \mod n$.

We proceed by considering the number of cycles of length 4 through a given edge, which by edge-transitivity of $\Gamma$ must be a constant.

Up to reversal, there are either three or six 4-cycles through the edge $\{1_0, 1_1\}$, namely the three of the form $(1_0, 1_1, x_0, x_1)$ for $x \in \{b, ba^i, ba^j\}$, plus

\[(1_0, 1_1, (ba^i)_0, (a^k)_0, (a^k)_0, b_1) \quad \text{if} \quad k \equiv i \mod n,
\]
or \[(1_0, 1_1, (ba^i)_0, (a^k)_0, (a^k)_0, b_1) \quad \text{if} \quad k \equiv j \mod n,
\]
or \[(1_0, 1_1, (ba^i)_0, (a^k)_0, (a^k)_0, b_1) \quad \text{if} \quad k \equiv -i \mod n,
\]
or \[(1_0, 1_1, (ba^j)_0, (a^k)_0, (a^k)_0, b_1) \quad \text{if} \quad k \equiv j \mod n,
\]
or \[(1_0, 1_1, (ba^j)_0, (a^k)_0, (a^k)_0, b_1) \quad \text{if} \quad k \equiv -j \mod n,
\]
or \[(1_0, 1_1, (ba^j)_0, (a^k)_0, (a^k)_0, b_1) \quad \text{if} \quad k \equiv i \mod n.
\]

Note that no two of the above six congruences involving $k$ can occur simultaneously, by the restrictions we have on $i$, $j$, and $k$. Hence up to reversal, the number of 4-cycles through any given edge is 3 or 6.

Next, up to reversal the 4-cycles through $\{1_0, b_1\}$ are $(1_0, b_1, b_0, b_1)$ and $(1_0, b_1, (ba^i)_0, (a^k)_0)$, plus

\[(1_0, b_1, (ba^i)_0, (a^k)_0, (a^k)_0, 1_1) \quad \text{if} \quad k \equiv i \mod n,
\]
or \[(1_0, b_1, (ba^i)_0, (a^k)_0, (a^k)_0, 1_1) \quad \text{if} \quad k \equiv j \mod n,
\]
or \[(1_0, b_1, (ba^i)_0, (a^k)_0, (a^k)_0, 1_1) \quad \text{if} \quad k \equiv -i \mod n,
\]
or \[(1_0, b_1, (ba^j)_0, (a^k)_0, (a^k)_0, 1_1) \quad \text{if} \quad k \equiv j \mod n,
\]
or \[(1_0, b_1, (ba^j)_0, (a^k)_0, (a^k)_0, 1_1) \quad \text{if} \quad k \equiv -j \mod n,
\]
or \[(1_0, b_1, (ba^j)_0, (a^k)_0, (a^k)_0, 1_1) \quad \text{if} \quad 2i \equiv 0 \mod n,
\]
or \[(1_0, b_1, (ba^j)_0, (a^k)_0, (a^k)_0, 1_1) \quad \text{if} \quad 2j \equiv 0 \mod n.
\]

It follows that the number of 4-cycles through $\{1_0, b_1\}$ is not 3 or 6, unless $0 \equiv 2i \equiv 2j \mod n$ and $k \not\equiv \pm(i - j) \mod n$.

But now suppose $2i \equiv 0 \mod n$ and $k \not\equiv \pm(i - j) \mod n$. Then $n$ is even, and $i \equiv \frac{n}{2} \mod n$, and up to reversal the 4-cycles through $\{1_0, (ba^j)_1\}$ are $(1_0, (ba^j)_1, (ba^j)_0, 1_1)$ and $(1_0, (ba^j)_1, (ba^j)_0, (a^k)_0)$, plus

\[(1_0, (ba^j)_1, (ba^j)_0, 1_1) \quad \text{if} \quad k \equiv j \mod n,
\]
or \[(1_0, (ba^j)_1, (ba^j)_0, (a^k)_0) \quad \text{if} \quad k \equiv -j \mod n.
\]

Hence the number of 4-cycles through the edge $\{1_0, (ba^j)_1\}$ is 2 or 5, contradiction. The same holds when the roles of $i$ and $j$ are reversed, and so this completes the proof.

6.2. A class of normal edge-transitive bi-dihedrants. In this subsection, we construct a class of normal edge-transitive bi-Cayley graphs over dihedral groups of degree 5 or more, and thereby prove there exists a semisymmetric bi-dihedral of valency $2k$ for every odd integer $k \geq 3$.

Example 6.2. Let $n$ and $k$ be integers with $n \geq 5$ and $k \geq 2$, such that there exists an element $\lambda$ of order $2k$ in $\mathbb{Z}_n$ such that

\[1 + \lambda^2 + \lambda^4 + \cdots + \lambda^{2(k-2)} + \lambda^{2(k-1)} \equiv 0 \mod n.
\]

Now let $H$ the dihedral group $D_n = \langle a, b \mid a^n = b^2 = (ab)^2 = 1 \rangle$ of degree $n$, and for each $i \in \mathbb{Z}_k$, let $c_i = 1 + \lambda^2 + \lambda^4 + \cdots + \lambda^{2(i-1)} + \lambda^{2i} \quad \text{and} \quad d_i = \lambda c_i = \lambda + \lambda^3 + \lambda^5 + \cdots + \lambda^{2i-1} + \lambda^{2i+1},$
and then define $\Gamma(n, \lambda, 2k)$ as the 2k-valent bi-Cayley graph $\text{BiCay}(H, \emptyset, \emptyset, S)$ over $H$, where

$$S = S(n, \lambda, 2k) = \{a^c : i \in \mathbb{Z}_k\} \cup \{ba^k : i \in \mathbb{Z}_k\}.$$ 

It is easy to see that $\Gamma(n, \lambda, 2k)$ contains the $2n$-cycles $(1_i, a_1, a_0, (a^2)_1, (a^2)_0, \ldots, (a^{n-1})_1, (a^{n-1})_0, 1_1)$ and $(b_0, b_1, (ba)_0, (ba)_1, (ba^2)_0, (ba^2)_1, \ldots, (ba^{n-1})_0, (ba^{n-1})_1)$, and the edge $(1_0, b_1)$, so $\Gamma(n, \lambda, 2k)$ is connected.

Also it is easy to see that $|S| = 2k$, and $c_{k-1} \equiv d_{k-1} \equiv 0 \mod n$, and $1 + \lambda d_i \equiv c_{i+1} \mod n$ for all $i \in \mathbb{Z}_k$. Next let $\alpha$ be the automorphism of $H$ that takes $(a, b)$ to $(a^k, ba)$. Then $S^\alpha = bS$, and $\sigma_{\alpha, b}$ is an automorphism of $\Gamma(n, \lambda, 2k)$ that fixes the vertex $1_0$ and cyclically permutes the 2k neighbours of $1_0$; indeed $\sigma_{\alpha, b}$ takes $(a^c)_1$ to $(ba^{c+1})_1$, and $(ba^d)_1$ to $(b^2a^{1+\lambda d})_1 = (a^{c+1})_1$, for all $i \in \mathbb{Z}_k$. Hence in particular, this shows that $\Gamma(n, \lambda, 2k)$ is normal edge-transitive.

The following natural problem arises.

**Problem A** Determine which of the graphs $\Gamma(n, \lambda, 2k)$ are semisymmetric.

We will give some partial answers to this problem, in the situation where $\lambda^k \equiv -1 \mod n$.

**Proposition 6.3.** If $k$ is even and $\lambda^k \equiv -1 \mod n$, then $\Gamma(n, \lambda, 2k)$ is arc-transitive.

**Proof.** Let $\beta$ be the automorphism of $H$ taking $(a, b)$ to $(a^{-1}, ba^\ell)$ where $\ell = d_{(k-2)/2} = \lambda + \lambda^3 + \ldots + \lambda^{k-1}$. Clearly $\beta$ has order 2. Also $(a^c)^\beta = a^{-c} \in S^{-1}$ for all $i \in \mathbb{Z}_k$, while $(ba^d)^\beta = ba^\ell - d_i$ for all $i \in \mathbb{Z}_k$, and because $\lambda^k \equiv -1 \mod n$ we find that if $0 < 2i + 1 < k$ then

$$\ell - d_i \equiv (\lambda + \lambda^3 + \ldots + \lambda^{k-1}) - (\lambda^2 + \lambda^3 + \ldots + \lambda^{2i-1})$$

$$\equiv \lambda^{2i+1} + \lambda^{2i+3} + \ldots + \lambda^{k-1}$$

$$\equiv \lambda^{2i+1} + \lambda^{2i+3} + \ldots + \lambda^{k-1} + (\lambda + \lambda^k + \lambda^{k+1})$$

$$\equiv \lambda + \lambda^3 + \ldots + \lambda^{2i-1} + \lambda^k + \lambda^{k+1} + \ldots + \lambda^{k-1}$$

$$\equiv d_{(k+2)/2} \mod n,$$

while if $2i + 1 > k$ then

$$\ell - d_i \equiv (\lambda + \lambda^3 + \ldots + \lambda^{k-1}) - (\lambda + \lambda^3 + \ldots + \lambda^{2i-1})$$

$$\equiv -\lambda^{k+1} - \lambda^{k+3} + \ldots + \lambda^{2i-1}$$

$$\equiv \lambda^{k+1} + \lambda^{k+3} + \ldots + \lambda^{2i-1} \equiv d_{(2i-k)/2} \mod n,$$

and so $(ba^d)^\beta = ba^\ell - d_i = ba^d_{(2i+k)/2} = (ba^d_{(2i+k)/2})^{-1} \in S^{-1}$ for all $i \in \mathbb{Z}_k$. Hence the automorphism $\beta$ takes the set $S = S(n, \lambda, 2k)$ to $S^{-1}$, and so by Proposition 2.2 we find that $\Gamma(n, \lambda, 2k)$ is vertex-transitive, and therefore arc-transitive. \qed

**Proposition 6.4.** If $k$ is odd and $\lambda^k \equiv -1 \mod n$, then $\Gamma(n, \lambda, 2k)$ is semisymmetric.

**Proof.** First, let $\ell = d_{(k-3)/2} = \lambda + \lambda^3 + \ldots + \lambda^{k-2}$. Then since $k$ is odd and $\lambda^k \equiv -1 \mod n$, we have

$$0 \equiv 1 + \lambda^2 + \ldots + \lambda^{2(k-1)} \equiv c_{(k-1)/2} + \lambda^k d_{(k-3)/2} \equiv c_{(k-1)/2} - d_{(k-3)/2} \mod n,$$

so $1 + \lambda \ell \equiv c_{(k-1)/2} \equiv d_{(k-3)/2} \equiv \ell \mod n$, and therefore $(ba^d)^{\sigma_{\alpha, b}} = (ba^d)^{\alpha} = (ba^{1+\lambda^d})_0 = (ba^d)_0$. Hence $(ba^d)_0$ is fixed by $\sigma_{\alpha, b}$, which cyclically permutes the 2k neighbours of $1_0$, and it follows that those 2k neighbours of $1_0$ are also the 2k neighbours of $(ba^d)_0$.

Before proceeding, we note that since $1 + \lambda \equiv \ell \mod n$, we also have

$$2\ell \equiv 1 + \ell + \lambda \ell \equiv 1 + (\lambda + \lambda^3 + \ldots + \lambda^{k-2}) + (\lambda^2 + \lambda^4 + \ldots + \lambda^{k-1}) \equiv 1 + \lambda + \lambda^2 + \lambda^3 + \ldots + \lambda^{k-2} + \lambda^{k-1},$$

and therefore $2\ell(1 - \lambda) \equiv (1 + \lambda + \lambda^2 + \lambda^3 + \ldots + \lambda^{k-2} + \lambda^{k-1})(1 - \lambda) \equiv 1 - \lambda^k \equiv 2 \mod n$.

Next, let $B$ be the set of all vertices of $\Gamma$ having the same neighbourhood as $1_0$. Note that $B$ contains both $1_0$ and $(ba^d)_0$, and therefore $|B| \geq 2$. We claim that $B$ is a block of imprimitivity for $Aut(\Gamma)$ on $V(\Gamma)$. To see this, note that if $\sigma \in Aut(\Gamma)$ then all vertices of $B^\sigma$ must have the same neighbourhood (since the same holds for vertices in $B$), and hence if $B \cap B^\sigma \neq \emptyset$, then every vertex in $B^\sigma$ has the same neighbourhood as $1_0$, so $B^\sigma \subseteq B$ and this gives $B^\sigma = B$. Hence in particular, $B$ is a block of imprimitivity for $R(H) \leq Aut(\Gamma)$ on $H_0$, and so $H_B = \{ x \in H | x_0 \in B \}$ is a subgroup of $H$, with order $|H_B| = |B|$ because $R(H)$ acts regularly on $H_0$. But also $B \subseteq \Gamma(1_1)$, and so $B$ is a block of imprimitivity.
for Aut(Γ)_{10} on Γ(11) as well. Hence |B| divides |Γ(11)| = 2k. Moreover, B contains (ba^k)_{0}, so H_B contains the involution ba^k, and hence H_B is dihedral, of order |H_B| = |B| = 2j for some j dividing k.

Now suppose that Γ is vertex-transitive. Then some automorphism θ of Γ takes 1_0 to 1_1, and it follows that C = B^θ is the set of all vertices of Γ having the same neighbourhood as 1_1, and C is a block of imprimitivity for Aut(Γ)_{10} on Γ(10), and the subgroup H_C = \{ y ∈ H \mid y_1 ∈ C \} of H is dihedral, of order |C| = |B| = 2j. In particular, since the automorphism σ_{α,b} fixes 1_0 and cyclically permutes the 2k neighbours of 1_0, the block C is preserved by \( σ_{α,b}^{2k/|C|} = σ_{α,b}^{k/j} \), and hence also by σ_{α,b}^2. Accordingly, C contains the image of 1_1 under σ_{α,b}^k, namely (ba^{(k-3)/2})_{1} = (ba^k)_{1}, and therefore H_C contains ba^k.

On the other hand, consider the automorphism \( τ = σ_{α,b}R(b) \). This takes h_1 to (bb^2b)_{1} for all h ∈ H, and so its effect on H_1 is the same as the permutation induced by the automorphism ψ of H that takes a to ba^0b = ba^{λ}b = a^{−λ}b and b to bb^2b = b(ba)b = ba^{−1}. In particular, τ fixes 1_1, and so τ preserves C (the set of all vertices of Γ having the same neighbourhood as 1_1), and ψ preserves H_C. It follows that H_C contains (ba^k)^ψ = ba^{−1−λ} = ba^{−ℓ}, and hence also (ba^{−ℓ})^1ba^{−ℓ} = a^{2ℓ}, and hence also \( (a^{2ℓ})^{1−λ} = a^{2(1−λ)} = a^2 \), because \((1−λ)2ℓ ≡ 2 \mod n\). But H_C has order 2j, which divides 2k and hence divides \( |Z^*| = φ(n) \), and so \( |H_C ∩ (a)| = j ≤ k ≤ φ(n)/2 < n/2 \). Thus a^2 cannot lie in H_C, and which is a contradiction. □

Remarks: We believe that the hypothesis \( λ^k ≡ −1 \mod n \) in Proposition 6.4 is not actually required, and that Proposition 6.4 can be extended to all cases other than those covered by Proposition 6.3.

In other words, we believe that \( Γ(n, λ, 2k) \) is arc-transitive if and only if k is even and \( λ^k ≡ −1 \mod n \). Furthermore, we believe that if \( λ^k \not≡ −1 \mod n \), then the graph \( Γ = Γ(n, λ, 2k) \) is not just semisymmetric, but edge-regular (or equivalently, the stabiliser in Aut(Γ) of any edge is trivial). This certainly holds in all cases where \( n ≤ 300 \), such as \((n, λ, k) = (21, 2, 3), (68, 9, 4) \) or \((35, 2, 6)\), as shown by a computation using MAGMA [7]. In the next subsection, we will prove it holds whenever \( k = 3 \).

6.3. The case \( k = 3 \). By Theorem 6.1 every semisymmetric bi-dihedral has valency at least 6, and therefore 3 is the smallest possible value of k of interest in this section. Also by Proposition 6.4, we know that when \( k = 3 \) the graph \( Γ(n, λ, 6) \) is semisymmetric if \( λ^3 ≡ −1 \mod n \). In this subsection, we prove that \( Γ(n, λ, 6) \) is edge-regular (and therefore semisymmetric) whenever \( λ^3 \not≡ −1 \mod n \), and this gives a complete solution for Problem A in the case \( k = 3 \).

Theorem 6.5. The graph \( Γ(n, λ, 2k) \) is semisymmetric whenever \( k = 3 \), and moreover, if \( k = 3 \) and \( λ^3 \not≡ −1 \mod n \), then \( Γ(n, λ, 6) \) is edge-regular, with cyclic vertex-stabiliser.

Proof. Let \( Γ = Γ(n, λ, 2k) \) and \( A = Aut(Γ) \). We know from Proposition 6.4 that \( Γ \) is semisymmetric whenever \( λ^3 ≡ −1 \mod n \), and hence in what follows, we will assume that \( λ^3 \not≡ −1 \mod n \).

The smallest value of n for which this happens is 21, with \( λ = ±2 \) or \( ±10 \) (in \( Z_{21} \)), and the next smallest n is 39, with \( λ = ±4 \) or \( ±10 \) (in \( Z_{39} \)). Note that n must be odd, for otherwise λ would be odd but then \( c_{k−1} = c_2 = 1 + λ^2 + λ^4 \), could not be 0 \mod n.

By considering 4- and 6-cycles that contain the edge \((1_0, 1_1)\), we will prove that the stabiliser \( A_{1_0,1_1} \) of the arc \((1_0, 1_1)\) is trivial, and then that the stabiliser \( A_{1_0,1_1} \) of the edge \((1_0, 1_1)\) is trivial, so that \( Γ \) is semisymmetric, and edge-regular.

But first, we will set some notation for later use. Define
\[
H_{0c} = \{(a^0)_i : i ∈ Z_n\}, \quad H_{0d} = \{(ba^0)_i : i ∈ Z_n\}, \quad H_{1c} = \{(a^1)_i : i ∈ Z_n\} \quad \text{and} \quad H_{1d} = \{(ba^1)_i : i ∈ Z_n\}.
\]

These form a partition of V(Γ) into four subsets of size n, with \( H_0 = H_{0c} ∪ H_{0d} \) and \( H_1 = H_{1c} ∪ H_{1d} \), and they are blocks of imprimitivity for \( R(H) × ⟨σ_{α,b}⟩ \) on \( V(Γ) \), with \( σ_{α,b} \) preserving each of \( H_0 \) and \( H_{1d} \), and interchanging \( H_{1c} \) with \( H_{1d} \). In fact both \( R(a) \) and \( σ_{α,b}^2 \) preserve these four subsets, while all of \( R(b), σ_{α,b} \) and \( σ_{α,b}R(b) \) do not, and it follows that the kernel of the action of \( R(H) × ⟨σ_{α,b}⟩ \) on \( \{H_{0c}, H_{0d}, H_{1c}, H_{1d}\} \) is the index 4 subgroup \( M = ⟨R(a), σ_{α,b}⟩ \).

Also define
\[
Δ_0 = \{(a^{−c})_i : i ∈ Z_3\}, \quad Φ_0 = \{(ba^{d})_i : i ∈ Z_3\}, \quad Δ_1 = \{(a^{c})_i : i ∈ Z_3\} \quad \text{and} \quad Φ_1 = \{(ba^{d})_i : i ∈ Z_3\},
\]
so that \( Δ_0 ∪ Φ_0 = G(1_1) \) and \( Δ_1 ∪ Φ_1 = G(1_0) \), the neighbourhoods of the vertices \( 1_1 \) and \( 1_0 \).
Next, the assumption that $\lambda^3 \not\equiv -1 \mod n$ implies that no power of $\lambda$ is congruent to $-1 \mod n$, and so the set $\{-\lambda^j : 1 \leq j \leq 5\}$ is disjoint from $\{\lambda^j : 1 \leq j \leq 5\}$. It follows that the set $\Gamma(1_0)$ of vertices at distance 2 from $1_0$ is the union of the following three disjoint sets:

- $U_1 = \{(a^{ci-cj})_0 : i, j \in \mathbb{Z}_3, i \neq j\} = \{(a^{ci})_0 : \ell \in \{\pm 1, \pm \lambda^2, \pm \lambda^4\}\}$,
- $U_2 = \{(ba^{ci+dj}_0 : i, j \in \mathbb{Z}_3\} = \{(ba^{ci})_0 : \ell \in \{0, 1, 1 + \lambda, -\lambda^4, -\lambda^5, 1 - \lambda^5, \lambda - \lambda^4, -\lambda^4 - \lambda^5\}\}$,
- $U_3 = \{(a^{ci-dj})_0 : i, j \in \mathbb{Z}_3, i \neq j\} = \{(a^{ci})_0 : \ell \in \{\pm \lambda, \pm \lambda^3, \pm \lambda^5\}\}$.

Note that each of the vertices in $U_1$ and $U_3$ has only one common neighbour with $1_0$, while each of the vertices in $U_2$ has two common neighbours with $1_0$, and hence up to reversal there are only nine 4-cycles containing $1_0$, namely $(1_0, (a^{ci})_1, (ba^{ci+dj}_0), (ba^{ci+dj}_1))$ for each pair $(i, j) \in \mathbb{Z}_3 \times \mathbb{Z}_3$. It follows that the stabiliser $A_{1_0}$ of the vertex $1_0$ must preserve the set $U_2$. Moreover, the arc-stabiliser $A_{1_0,1_1}$ must permute the three edges $\{(ba^{ci})_0, (ba^{ci})_1\}$ among themselves, as these are the only edges between vertices of $\Gamma(1_1)$ and $\Gamma(1_0)$ that lie in 4-cycles through $1_0$ and $1_1$.

Therefore $A_{1_0,1_1}$ preserves the subset $\Phi_1$ of $\Gamma(1_0)$ and its complement $\Delta_1 \setminus \{1_1\} = \{(a^{ci})_1 : i \in \{0, 1\}\}$ in $\Gamma(1_0) \setminus \{1_1\}$, as well as the subset $\Phi_0$ of $\Gamma(1_1)$ and its complement $\Delta_0 \setminus \{1_0\} = \{(a^{ci})_0 : i \in \{0, 1\}\}$ in $\Gamma(1_1) \setminus \{1_0\}$, and it also sets up a pairing between the subsets $\Phi_0$ and $\Phi_1$.

The situation is illustrated in Figure 2.

![Figure 2. The ball of radius 2 centered at the vertex 1_0 in Γ(n, λ, 6)](image)

On the other hand, the vertex $1_0$ lies in many 6-cycles, but it is not difficult to check that if $n \geq 39$ then a 3-arc of the form $(u_1, 1_0, 1_1, u_0)$ with $u_1 = (a^{ci})_1 \in \Delta_1 \setminus \{1_1\}$ and $u_0 = (a^{ci})_0 \in \Delta_0 \setminus \{1_0\}$ lies in no 6-cycle when $i = j$, and in just one 6-cycle $((a^{ci})_1, 1_0, 1_1, (a^{ci})_0, (a^{ci})_1, (a^{ci})_0)$ when $i \neq j$. Also in the smallest cases, where $n = 21$, this 3-arc lies in three 6-cycles when $i = j$, but only one 6-cycle when $i \neq j$. It follows that the arc-stabiliser $A_{1_0,1_1}$ must preserve the set of two edges of the form $\{(a^{ci})_0, (a^{ci})_1\}$ for $i \in \{0, 1\}$, and this sets up a pairing between the subsets $\Delta_0 \setminus \{1_0\}$ and $\Delta_1 \setminus \{1_1\}$.

Thus we have a pairing between the five vertices of $\Gamma(1_1) \setminus \{1_0\}$ and the five vertices of $\Gamma(1_0) \setminus \{1_1\}$, given by $(ba^{ci})_0 \leftrightarrow (ba^{ci})_1$ for $i \in \mathbb{Z}_3$ and $(a^{ci})_0 \leftrightarrow (a^{ci})_1$ for $i \in \{0, 1\}$, such that $A_{1_0,1_1}$ permutes the corresponding edges among themselves. By edge-transitivity, a similar thing holds for every edge in $\Gamma$. 
It now follows that $A_{10}$ acts faithfully on $\Gamma(1_0)$. For suppose $g$ is an automorphism in $A_{10}$ that fixes every neighbour of $1_0$ in $\Gamma$. Then $g$ fixes $1_1$, and also fixes the partner in $\Gamma(1_1)$ of every other vertex in $\Gamma(1_0)$, and so fixes all of $\Gamma(1_1)$. By edge-transitivity, the same argument applies to other neighbours of $1_0$, and it follows that $g$ fixes every vertex at distance 2 from $1_0$. Then by induction and connectedness, we find that $g$ fixes all vertices of $\Gamma$, and hence $g$ is trivial.

Also the action of $A_{10}$ is imprimitive on $\Gamma(1_{00})$, with two blocks $\Delta_1$ and $\Phi_1 = \Delta_1^{\sigma_{a,b}}$ of size 3, because if $u_1 \in \Delta_1 \cap \Delta_2$ for some $g \in A_{10}$, then $g$ must preserve the set of all vertices of $\Gamma(1_{00})$ that lie in 4-cycles containing the edge $\{1_0, u_1\}$, namely $\Phi_1 = \{(ba^{i})_1 : i \in \mathbb{Z}_3\}$, and hence also $\Delta_{01}^{g} = \Delta_1$. Thus $A_{10}$ is isomorphic to a subgroup of the wreath product $S_3 \wr C_2$, so $A_{10}$ is a $(2, 3)$-group, of order dividing 72.

Next, let $A^*$ be the subgroup of $A$ preserving the parts $H_{00}$ and $H_1$ of $\Gamma$, so that $A^*$ has index 1 or 2 in $A$. Then $A^*$ contains $R(H)$ and $A_{10}$, and since $R(H)$ acts regularly on each part of $\Gamma$, it follows that $A^*$ is the complement of $R(H)A_{10}$ of these two subgroups. If $p$ is any prime divisor of $|A^*|$ such that $p > 3$, then $p$ cannot divide $|A_{10}|$ and therefore $p$ divides $|R(H)| = |H| = 2n$, so $p$ divides $n$, and then since $\lambda^2 + \lambda^2 + 1 \equiv 0 \pmod{n}$, we find that $p \neq 5$, so $p \geq 7$. Moreover, for every such $p$, the dihedral subgroup $R(H)$ of order $2n$ has a unique Sylow $p$-subgroup $P$, which is then also a Sylow $p$-subgroup of $A^*$ and is normal in the subgroup $R(H) \langle \sigma_{a,b} \rangle$ of order 12 in $A^*$. Hence the index of its normaliser in $A^*$ divides $|A_{10}| : |\langle \sigma_{a,b} \rangle|$, which divides $72/6 = 12$, and because $p \geq 7$, it follows that $P$ is normal in $A^*$. The product of all such Sylow subgroups is therefore (a cyclic) normal Hall $(2, 3)$-subgroup $N$ of $A^*$.

We can use this fact to prove that $A_{10}$ preserves the set $H_{0c}$ (of all vertices of the form $(a^j)_0$), and that $A_{10,1}$ preserves the set $H_{1c}$ (of all vertices of the form $(a^j)_1$). Before doing that, observe that $\sigma_{a,b}$ preserves $H_{0c}$, and that $A_{10} = \langle A_{10,1}, \sigma_{a,b} \rangle$, since $\sigma_{a,b}$ fixes $1_0$ and acts regularly on $\Gamma(1_0)$. Hence all we need to do is prove that $A_{10,1}$ preserves both $H_{0c}$ and $H_{1c}$.

If $n$ is coprime to 3, then the normal subgroup $N$ of $A^*$ must be the cyclic subgroup generated by $R(a)$, and the four sets above are its orbits, which are therefore blocks of imprimitivity for $A^*$ on $V(\Gamma)$, and so $A_{10,1}$ preserves both $H_{0c}$ and $H_{1c}$.

On the other hand, suppose $n \equiv 0 \pmod{3}$. Then $n \not\equiv 0 \pmod{9}$ (since $\lambda^2 + \lambda^2 + 1 \equiv 0 \pmod{n}$), and it follows that $n = 3|N|$, and $N$ is the cyclic subgroup generated by $R(a)^3$, and $H_{0c} = 1_N^N \cup a_N^N \cup (a^{-1})_N^N$. Now if $x \in A_{10,1}$, then $(1_N^N)^x = (1_N^N)^{x_1} = 1_N^N \subseteq H_{0c}$, and by our earlier observations, $x$ preserves the set $\Delta_1 \setminus \{1_1\} = \{(a^{\lambda i})_1, (a^{\alpha i})_1\}$, and hence also preserves the set $U_1 = \{(a^\ell)_1 : \ell \in \{1, \pm \lambda^2, \pm \lambda^4\}\}$ of certain vertices at distance 2 from $1_0$. It follows that
\[
(a^N_0 \cup (a^{-1})_0^N)^x = a^N_0 \cup (a^{-1})_0^N = a^N_0 \cup (a^{-1})_0^N \subseteq U_1 \subseteq H_{0c},
\]
and thus $A_{10,1}$ preserves $H_{0c}$. Similarly, every $x \in A_{10,1}$ must preserve the set $\Phi_1 = \{(ba^{i})_1 : i \in \mathbb{Z}_3\}$, and hence also preserves the set $U_3 = \{(a^\ell)_1 : \ell \in \{1, \pm \lambda, \pm \lambda^3, \pm \lambda^5\}\}$, and since $\lambda$ is a unit mod $n$ we have $\lambda \equiv \pm 1 \pmod{3}$, and so $a^N_0 \cup (a^{-1})_1^N = (a^\lambda)^N \cup (a^{-\lambda})_1^N$. It follows that
\[
(a^N_1 \cup (a^{-1})_1^N)^x = (a^\lambda)_1^N \cup (a^{-\lambda})_1^N \subseteq U_3 \subseteq H_{1c},
\]
and hence also $(1_1^N)^x = (1_1^N)^{x_1} = 1_1^N \subseteq H_{1c}$, we find that $A_{10,1}$ preserves $H_{1c}$, as required.

Now let $X$ be the subgraph of $\Gamma$ induced on all vertices in $H_{0c} \cup H_{1c}$ and let $Y$ be the subgraph induced on all vertices in $H_{0c} \cup H_{1d}$. Then each of $X$ and $Y$ is a trivalent bi-Cayley graph over the cyclic group $\langle a \rangle \cong C_n$; indeed clearly $X$ is the graph BiCay$(\langle a \rangle, \emptyset, \emptyset, \{a^i : i \in \mathbb{Z}_3\})$, while $Y$ is isomorphic to $\text{BiCay}(\langle a \rangle, 0, 0, \{a^i : i \in \mathbb{Z}_3\})$. Also the subgroup of $A$ generated by $R(a)$ and $\sigma_{a,b}$ acts transitively on the edges of both subgraphs. Hence each is a connected edge-transitive bi-Cayley graph over the cyclic group $\langle a \rangle \cong C_n$, and so by Proposition 7.4 each of them is arc-transitive, and then since $n \geq 21$, it follows from Corollary 4.6 that each of $X$ and $Y$ is 1-arc-regular.

We saw above that the arc-stabiliser $A_{10,1}$ preserves $H_{0c}$ and $H_{1c}$, and it follows that $A_{10,1}$ preserves $H_{0d}$ and $H_{1d}$ as well. Hence $A_{10,1}$ induces a group of automorphisms of each of $X$ and $Y$. Then since $X$ is arc-regular, and $A_{10,1}$ fixes the arc $(1_0, 1_1)$ of $X$, we find that $A_{10,1}$ acts trivially on $X$. In particular, $A_{10,1}$ fixes all the vertices $(a^i)_0$ that are common to $X$ and $Y$, including the vertices in $\Delta_1 = \{(a^i)_1 : i \in \mathbb{Z}_3\}$, and it follows that $A_{10,1}$ fixes all the vertices in $\Phi_1 = \{(ba^{i})_1 : i \in \mathbb{Z}_3\}$, each of which lies in a unique 4-cycle with $1_0$ and a given vertex of $\Delta_1$. Indeed $A_{10,1}$ acts trivially on $Y$. Hence the subgroup $A_{10,1}$ of $A_{10}$ acts trivially on $\Delta_1 \cup \Phi_1 = \Gamma(1_0)$, and so $A_{10,1}$ itself is trivial.
In particular, every automorphism of $\Gamma$ is uniquely determined by its effect on $1_0$ and $1_1$. Also we now find that $A_{1\theta} = \langle A_{1\theta 1}, \sigma_{\alpha, b} \rangle$, which has order 6, so $A^* = R(H)A_{1\theta} = R(H) \times \langle \sigma_{\alpha, b} \rangle$, which has order $12n$. Consequently the index 4 normal subgroup $M = \langle R(a), \sigma_{\alpha, b}^2 \rangle$ of $A^*$ has odd order $3n$, and is therefore a characteristic subgroup of of $A^*$. Moreover, since $R(a)^{\sigma_{\alpha, b}} = \sigma_{\alpha, b}^{-1}R(a)\sigma_{\alpha, b}$ takes $1_0 \mapsto (a^\lambda)_0$, we know that $R(a)^{\sigma_{\alpha, b}} = R(a^\lambda) = R(a)^\lambda$, and so conjugation by $\sigma_{\alpha, b}^2$ induces an automorphism of $\langle R(a) \rangle$ of order 3, centralising the 3-part of $\langle R(a) \rangle$ because this has order at most 3. It follows that $J = \langle R(a) \rangle$ is the only cyclic normal subgroup of order $n$ in $M$, and $J$ is therefore characteristic in both $M$ and $A^*$.

We complete the proof by using these facts to show that the edge-stabiliser $A_{1\theta 1}$ is trivial. If that is not the case (so that $\Gamma$ is arc-transitive), then the latter subgroup has order 2, and is generated by an automorphism $\theta$ that swaps the vertices $1_0$ and $1_1$. Moreover, from our earlier observations, we know that $\theta$ must preserve the subsets $\Delta_1 \setminus \{1\} = \{(a^c_i) : i \in \{0, 1\}\}$ and $\Delta_0 \setminus \{1\} = \{(a^{-c}_i) : i \in \{0, 1\}\}$, and so $\theta$ swaps $a_1$ with $(a^{-c})_0$ where $c = c_0 \equiv 1$ or $c = c_1 \equiv 1 + \lambda^2 \mod n$. It then follows that also $\theta$ swaps $a_0$ with $(a^c)_0$. Similarly, $\theta$ swaps $\Phi_0 = \{(b^{d}_a)_0 : a \in \mathbb{Z}_3 \}$ with $\Phi_1 = \{(b^{d_1}) : i \in \mathbb{Z}_3 \}$, in a way that preserves the pairing $(ba^{d_0})_0 \leftrightarrow (ba^{d_1})_1$ for $i \in \mathbb{Z}_3$. In particular, $\theta$ swaps $b_0$ with $(ba^{d_1})_1$ for some $d \in \{d_0, d_1, d_2\}$, and then also swaps $b_1$ with $(ba^{d_1})_0$.

Now consider what happens to the automorphisms $\langle R(a), R(b) \rangle$ and $\sigma_{\alpha, b}$ under conjugation by $\theta$. Clearly $\theta$ normalises $A^*$ hence normalises the characteristic subgroups $J$ and $M$ of $A^*$. Also the fact that $A_{1\theta 1}$ is trivial implies that every automorphism is uniquely determined by its effect on $1_0$ and $1_1$. First $R(b)^\theta$ takes $1_0$ to $(ba^{d_0})_0$, and $1_1$ to $(ba^{d_1})_1$, and therefore $R(b)^\theta = R(ba) = R(b)R(a)^d$. Next, $R(a)^\theta$ lies in $J^\theta = J = \langle R(a) \rangle$, and as $R(a)^\theta$ takes $1_0$ to $(a^{1^\theta}) = (a^{-c})_0$, we have $R(a)^\theta = R(a)^{-c}$. Similarly, $\sigma_{\alpha, b}$ fixes $1_1$ and takes $1_0$ to $(b_1)^\theta = (ba^{d_0})_0$, and it follows that $\sigma_{\alpha, b}^\theta = (\sigma_{\alpha, b}(R(b)))^j$ for some $j$, because $\sigma_{\alpha, b}R(b)$ fixes $1_1$ and induces the 6-cycle $(1_0, b_0, (a^{-1})_0, (ba^{d_0})_0, (a^{-(1+\lambda^2)})_0, (ba^{\lambda+\lambda^2})_0)$ on $\Gamma(1_1)$. In fact $j = \pm 1$, since $\sigma_{\alpha, b}$ and $\sigma_{\alpha, b}R(b)$ have order 6, and then it follows that $d = d_2 = 0$ or $d = d_1 = \lambda + \lambda^3$. Hence the automorphism of $A^* = R(H) \times \langle \sigma_{\alpha, b} \rangle$ induced by $\theta$ takes $\langle R(a), R(b), \sigma_{\alpha, b} \rangle$ to $\langle R(a)^{-c}, R(b)R(a)^d, (\sigma_{\alpha, b}R(b))^{\pm 1} \rangle$.

In particular, $\theta$ conjugates the involution $\sigma_{\alpha, b}^3$ to $\langle \sigma_{\alpha, b}R(b) \rangle^3$, and hence $\theta$ swaps $(ba^{\lambda})_0$ with $(ba^{\lambda})_1$. Also $R(a) = R(a)^\theta = R(a)^(-c)^2 = 1$ and so $c^2 \equiv (-c^2) \equiv 1 \mod n$, and this eliminates the possibility that $c = c_1 = 1 + \lambda^2$, because $(1 + \lambda^2)^2 = 1 + 2\lambda^2 + \lambda^4 = \lambda^2 \not\equiv 1 \mod n$. Thus $c = 1$, and so $R(a)^\theta = R(a)^{-1}$. Finally, since $R(ba^{\lambda})^\theta$ takes $1_0$ to $((ba^{\lambda})_0)^\theta = (ba^{\lambda})_0$, we find that $\theta$ centralises $R(ba^{\lambda})$, and so $R(ba^{\lambda}) = R(ba^{\lambda})^\theta = (R(b)R(a)^d)^\theta = R(b)R(a)^d$, which gives $2\lambda \equiv d \equiv \lambda$ or $\lambda + \lambda^3 \mod n$. Both of these cases are impossible, however, since $\lambda \not\equiv 0$ and $\lambda^2 \not\equiv 1 \mod n$.

Hence no such $\theta$ exists, and therefore $\Gamma$ is semisymmetric and edge-regular. 

The graphs $G(n, \lambda, \kappa)$ with $k = 6$ investigated above provide the answers to two open questions.

For non-empty subsets $S_{00}$, $S_{01}$, $S_{10}$ and $S_{11}$ of $\mathbb{Z}_n$, define $X = T(S_{00}, S_{01}, S_{10}, S_{11})$ as the graph with vertex set $\mathbb{Z}_n \times \mathbb{Z}_2 \times \mathbb{Z}_2$, and edges all pairs of the form $\{(x, 0, i), (y, 1, j)\}$ where $i, j \in \mathbb{Z}_2$ and $y - x \in S_{ij}$. The translation mapping $t \mapsto t+1$ on $\mathbb{Z}_n$ clearly induces a semi-regular automorphism of $X$ of order $n$, with four cycles on $V(X)$. Any graph admitting a semi-regular automorphism $\pi$ with four cycles on vertices is called an $(\pi, \tau)$-tetracirculant. Such graphs were considered in a 2001 paper [42] by Marušič and Potočnik, who showed that every semisymmetric $(\pi, \tau)$-tetracirculant is isomorphic some $T(S_{00}, S_{01}, S_{10}, S_{11})$. Also if $S_{00} = S_{01} = R$ and $S_{10} = S_{11} = T$, then $T(R, R, T, T)$ is called a generalised Folkman tetracirculant. (See [42] for the definition of generalised Folkman graph.)

In their 2001 paper, Marušič and Potočnik [42] posed the two questions below.

**Problem B [42] Problem 4.3** Is there a semisymmetric tetracirculant which is not a generalised Folkman tetracirculant?

**Problem C [42] Problem 4.9** Is there a semisymmetric $(\pi, \tau)$-tetracirculant $\Gamma$ such that the four orbits of $\langle \pi \rangle$ are blocks of imprimitivity of $\text{Aut}(\Gamma)$, but $\Gamma$ is not a generalised Folkman tetracirculant?

We can now give the answer “Yes” to both questions.

For suppose $X = T(R, R, T, T)$ is a semisymmetric generalised Folkman tetracirculant. Then the vertices $(0, 0, 0)$ and $(0, 0, 1)$ have exactly the same neighbours in $X$, namely all the vertices $(y, 1, j)$ with
j ∈ Z_2 and y ∈ R ∪ T, and so there exists an automorphism of X that swaps (0, 0, 0) with (0, 0, 1) and fixes all others. In particular, this implies that X can not be edge-regular.

But now every bi-dihedral BiCay(D_n, ø, ø, S) is a tetracirculant, admitting the natural cyclic subgroup R(C_n) as a group of automorphism with four orbits. In particular, every bi-dihedral Γ = Γ(n, λ, µ, ν) considered above with λ^3 ≠ 1 mod n is a semisymmetric (n, π)-tetracirculant, and furthermore, the four orbits H_{0c}, H_{0d}, H_{1c}, H_{1d} of the semi-regular cyclic subgroup ⟨R(a)⟩ of its automorphism group Aut(Γ) are blocks of imprimitivity for R(H) × (σ_{a,b}) = A^* = Aut(Γ). On the other hand, by Theorem 6.5, every such Γ is edge-regular, and so Γ cannot be a semisymmetric generalised Folkman tetracirculant.

6.4. Proof of Theorem 1.5. Combining Theorem 6.1, Proposition 6.4 and Theorem 6.5 together gives a proof of Theorem 1.5. Note that the graphs in Theorem 1.5 are worthy, because every unworthy graph is not edge-regular, by the same argument as given in the penultimate paragraph of the previous subsection.

6.5. Edge-regular bi-dihedrants of valency 6. Here we give a classification of all 6-valent edge-regular semisymmetric bi-Cayley graphs over a dihedral group D_n of odd degree n. We begin as follows:

Lemma 6.6. If Γ is a connected 6-valent bi-Cayley graph over the dihedral group H = D_n of order 2n, where n is odd, and Γ is both semisymmetric and edge-regular, then R(H) is normal in A = Aut(Γ), and the stabiliser in A of the vertex Γ is cyclic of order 6.

Proof. Let a and b be generators for H satisfying the usual relations a^n = b^2 = 1 and bab = a^{-1}, and let J be the subgroup of A = Aut(Γ) generated by (a). Then the orbits of J are the four subsets H_{0c}, H_{0d}, H_{1c}, H_{1d} defined in the proof of Theorem 6.5. Also note that Γ is bipartite, and since Γ is 6-valent and edge-regular, we have |A_{1_0}| = |A_{1_1}| = 6 and A = R(H)A_{1_0}, so |A| = 12n.

We begin by proving that if J = ⟨R(a)⟩ is a normal subgroup of A, then J is self-centralising in A. Note that conjugation of J from A to Aut(J), which is abelian since J is cyclic, and J is contained in the kernel C = C_A(J), so A/C is abelian, of order dividing |A|/|J| = 12. On the other hand, the involution b does not centralise a, so |A/C| is even, and therefore |C| = n, 2n, 3n or 6n.

Suppose |C| = 3n or 6n. Then |A/C| = 4 or 2, so every Sylow 3-subgroup of A is contained in C. Let P be any one of them, and take M = JP. This contains J as a central subgroup of order n, so |M| = 3n, which is odd, and therefore M preserves the bipartition of Γ. Also M/J is the only subgroup of C/J of order 3 (since |C/J| = 6 or 3), so M is normal in A. Moreover, since both J ∩ M_{1_0} and J ∩ M_{1_1} are trivial, we find that M = J ∩ M_{1_0} = J ∩ M_{1_1} ≅ C_n × C_3, and hence M is abelian. Now let N = ⟨M_{1_0}, M_{1_1}⟩. Then N is abelian, and isomorphic to C_3 × C_3 since Γ is edge-regular, and also N is characteristic in M and hence normal in A, and therefore N contains the stabiliser of every vertex of Γ. It follows that every orbit of N on V(Γ) has length 3. In particular, the neighbourhood Γ(1_0) of the vertex 1_0 is the union of two such orbits, namely Δ_1 = 1_0^1 and Φ_1 = x_1^N for some x ∈ H \ lcm(a). This, however, implies that if h_0 is any other vertex of Δ_0 = 1_0^3, then every edge incident with h_0 lies in the same orbit under N as {1_0, 1_1} or {1_0, x_1}, so 1_0 and h_0 have exactly the same neighbours, and therefore Γ is unwarthy, so cannot be edge-regular. (Indeed Γ ≅ K_{6,6}) Thus |C| ≤ 2n.

Next, suppose |C| = 2n. Then C is generated by J and an involution that centralises J, so C is cyclic. It follows that C cannot act regularly on both H_0 and H_1, for otherwise Γ would be a bi-Cayley graph over C_{2n}, and by Proposition 6.1 it would be vertex-transitive and hence arc-transitive, so not edge-regular. Without loss of generality, C does not regularly on H_0, and then the vertex-stabiliser C_{1_0} is a non-trivial characteristic subgroup of C and therefore normal in A, so fixes every vertex of H_0. It follows that C_{1_0} is semi-regular on H_1, with orbits of length 2. But then the two vertices in each orbit of C_{1_0} on H_1 have exactly the same neighbours, and therefore Γ cannot be edge-regular, another contradiction.

Hence the only possibility is that |C| = n, in which case J = C = C_A(J).

We now proceed to use similar arguments to show that R(H) ∼ A. To do this, we let K be the core of R(H) in A. Now since |A : R(H)| = |A_{1_0}| = 6, we see that A/K is isomorphic to a subgroup of S_6, and then R(H)/K is isomorphic to a subgroup of S_5, but also R(H)/K is a quotient of the dihedral group R(H) ∼ D_n of twice odd order, so the only possibilities are {1}, C_2, D_3 and D_5.

If R(H)/K ∼ C_2, then since R(H) is dihedral of twice odd order, we have K = J = ⟨R(a)⟩ ∼ C_n, and so A/K = A/J = A/C_A(J), which is abelian, but then R(H)/K ∼ A/K, so R(H) ∼ A, contradiction.
Also if $R(H)/K \cong D_3$, then $A/K$ has order 60 and is a product of $D_3$ and $A_{10} \cong C_6$, and hence is solvable. Some elementary group theory then shows that $A/K$ has a normal Sylow 5-subgroup, but then since $|A : J| = 12$ this Sylow 5-subgroup must be $J/K$, and so $J \triangleleft A$, contradiction.

Next, suppose that $R(H)/K \cong D_3$. Then clearly $K = \langle R(a^3) \rangle$, and $A/K$ has order 36 and is a product of $D_3$ and $A_{10} \cong C_6$. In particular, $J = \langle R(a) \rangle \leq C_A(K)$, but also $J$ is not normal in $A$, for otherwise $K = \text{core}_A(R(H))$ would contain $J$, and so $J \not\leq C_A(K)$, and therefore $|C_A(K)|$ is a proper multiple of $n$, dividing $|A| = 12n$. If $|K| = 1$ then $R(H) \cong R(H)/K \cong D_3$ and it follows that $2n = |H| = 6$ and $\Gamma \cong K_{6,6}$, again contrary to the assumption that $\Gamma$ is edge-regular. Thus $|K| > 1$. Also $|K| = |R(H)/6 = n/3|$, so $|K|$ is odd, and therefore $K = \langle R(a^3) \rangle$ cannot be centralised by $R(b)$, which in turn implies that $|C_A(K)|$ divides $|A|/2 = 6n$. On the other hand, $|C_A(K)| \neq 2n$, for otherwise $J$ would be a normal Hall 2'-subgroup of $C_A(K)$, and hence characteristic in $C_A(K)$ and normal in $A$. Thus $|C_A(K)| = 3n$ or $6n$. Moreover, $|C_A(K)/K|$ is either $3n/(n/3) = 9$ or $6n/(n/3) = 18$.

Now let $P$ be a Sylow 3-subgroup of $C_A(K)$, and let $M = KP$. Then $M/K = KP/K$ is a Sylow 3-subgroup of $C_A(K)/K$, so $|M/K| = 9$, and hence is normal in $C_A(K)/K$, and therefore characteristic in $C_A(K)/K$, and hence normal in $A/K$. (In fact $M/K \cong C_3 \times C_3$, because $A$ is the product of the complementary subgroups $R(H)$ and $A_{10}$, with $|A_{10}| = 6$.) Thus $M$ is a normal subgroup of $A$, of order $9|K| = 3n$. Also $J/K$ (of order 3) must be contained in $M/K$, so $|R(a)| = J \leq M$. Consequently, just as before, we find that every orbit of $M$ on $V(\Gamma)$ is one of the four orbits $H_{0e}$, $H_{0d}$, $H_{1e}$ and $H_{1d}$ of $J$. Also the induced subgraph $X$ on $H_{0e} \cup H_{1e}$ is a 3-valent bi-Cayley graph over $\langle a \rangle \cong C_n$, on which $M$ acts edge-transitively, with $M_{10} \cong M_{11} \cong C_3$.

Also we note that $X$ is connected. Suppose for convenience of $X$, with parts $C_0 = V(C) \cap H_0$ and $C_1 = V(C) \cap H_1$. Then $|C_0| = |C_1|$, since $C$ is 3-valent. Also each $C_i$ is a block of imprimitivity for the action of $A$ on $V(\Gamma)$. Next let $x$ and $y$ be involutory automorphisms in $A_{10} \setminus M_{10}$ and $A_{11} \setminus M_{11}$, respectively. Then $C_0^x = C_0$ and $C_1^x \subseteq H_{0d}$, while $C_0^y = C_1$ and $C_1^y \subseteq H_{0d}$. The induced subgraphs on $C_0 \cup C_1^y = (C_0 \cup C_1)^y$ and $C_0^y \cup C_1 = (C_0 \cup C_1)^y$ are isomorphic to $C$, and contain edges from $H_{0b}$ to $H_{1d}$ and from $H_{0d}$ to $H_{1e}$. Hence the induced subgraph on $C_0 \cup C_1^y \cup C_1^y$ contains edges from $H_{1d}$ to $H_{0d}$. Hence the induced subgraph on $C_0 \cup C_1 \cup C_1^y \cup C_1^y$ is connected and 6-valent, so must be $C$, and therefore $C = X$.

This means we can apply Corollary 2.14 to $X$, and conclude that $n$ divides $t^2 + t + 1$ for some $t \in \mathbb{Z}/n$, and hence that $n$ is not divisible by 9. It follows that the 3-part of $|C_A(K)|$ is 9, so $|P| = 9$, and therefore $P$ is abelian. Hence also $M = KP$ is abelian, and in particular, $M$ centralises $J$. Again, however, this implies that the stabiliser of every vertex of $\Gamma$ is a subgroup of $N = \langle M_{10}, M_{11} \rangle \cong C_3 \times C_3$, and so $\Gamma$ is unwholesome, which contradicts edge-regularity.

Thus $R(H)/K$ is not isomorphic to $D_3$, so must be trivial, and we find $R(H) = K \cap A$, as required.

Finally, since $J = \langle R(a) \rangle$ is characteristic in $R(H)$, it follows that $J \triangleleft A$, and then $A/J = A/C_A(J)$, which is abelian, and so $A_{10} \cong A/R(H)$ is abelian, of order 6, and therefore cyclic. \hfill \Box

We can now give and prove the main theorem in this subsection.

**Theorem 6.7.** Let $\Gamma = \text{BiCay}(H, \emptyset, \emptyset, S)$ be a connected 6-valent bi-Cayley graph over the dihedral group $H = D_n$ of order 2n. Then $\Gamma$ is semisymmetric and edge-regular if and only if $\Gamma \cong \Gamma(n, \lambda, 6)$ for some integer $\lambda$ satisfying $\lambda^6 \equiv 1 \mod n$ and $1 + \lambda^2 + \lambda^4 \equiv 0 \mod n$ but $\lambda^3 \neq -1 \mod n$.

**Proof.** First, if $\Gamma \cong \Gamma(n, \lambda, 6)$ where $\lambda^3 \neq -1 \mod n$, then by Theorem 6.6 we know that $\Gamma$ is edge-regular, so it remains to prove the converse. So suppose $\Gamma$ is semisymmetric and edge-regular, and let $A = \text{Aut}(\Gamma)$.

By Proposition 2.11 up to graph isomorphism we may assume that $S$ generates $H$, and contains the identity element of $H$, so that $1 \in \{ s_1 : s \in S \} = \Gamma(1)$. Also by Lemma 6.6, we know that $R(H) \leq A$ and $A_{10} \cong C_n$, and so by Proposition 2.11 we may take $A_{10} = \langle \sigma_{a,n} \rangle$ for some $a \in \text{Aut}(H)$ and $v \in H$.

Now $\sigma_{a,n}$ takes $1$ to $v_1$, so $\sigma_{a,n}^i$ takes $1$ to $(v_0^i, v_0^{i-1} \ldots v_0^0)$ for all $i \geq 1$, and as $\sigma_{a,n}$ has order 6, it follows that $S = \{ 1 \} \cup \{ v_0^i v_0^{i-1} \ldots v_0^0 : 1 \leq i \leq 5 \}$. In particular, since these elements have to generate $H$, we find that $v$ cannot lie in the maximal cyclic subgroup $C_n$ of $H = D_n$, so must be an involution. Next, let $u$ be any generator of the subgroup $C_n$, and suppose that $\alpha$ takes $u$ to $u^3$, and $v$ to $vu^3$, where $\lambda \in \mathbb{Z}_n$ and $j \in \mathbb{Z}_n$. Then it is easy to see that

$$S = \{ 1, v, vu^\alpha, vu^\alpha v^\alpha, vu^\alpha v^\alpha v^\alpha, vu^\alpha v^\alpha v^\alpha v^\alpha v^4 \} = \{ 1, v, vu^3, u^{j+1+\lambda^2}, vu^{j+\lambda^3} \},$$
and also that $1_1 = \sigma_{\lambda, \alpha}^1 = (u(\varphi(\lambda + \lambda^3))_1 = (\varphi(\lambda + \lambda^3))_1$, so $\varphi(\lambda + \lambda^3) = 1$. Moreover, since $S$ generates $H$, we see that $j$ must be a unit mod $n$, so $\varphi$ has order $n$, and therefore $1 + \lambda^3 = 0$ mod $n$. It follows that $1 - \lambda^6 = (1 - \lambda^3)(1 + \lambda^3 + \lambda^4)$ $\equiv 0$ mod $n$, and so $\lambda^6 = 1$ mod $n$. Also $\lambda^3 \neq 1$ mod $n$, for otherwise $0 \equiv 1 + \lambda^3 + \lambda^4 \equiv 3$ mod $n$, which implies $n = 3$, but then $\Gamma \cong K_5,6$, which is arc-transitive.

We can now take $a = u^1$ and $b = v$ as our canonical generators for $H = D_n$, and with these we have $S = \{1, b, a, ba, a^2 + \lambda^3, ba + \lambda^3\} = S(n, \lambda, 6)$. Also $\lambda^3 \neq 1$ mod $n$, for otherwise $1_0$ and $(ba^\lambda)_0$ have the same neighbours, and so $\Gamma$ is unworthy and hence cannot be edge-regular. Similarly $\lambda^2 \neq 1$ mod $n$, for otherwise $1_0$ and $(ba^\lambda)_0$ have the same neighbours, and again $\Gamma$ cannot be edge-regular.

Thus $\lambda^6 = 1$ mod $n$ and $1 + \lambda^3 + \lambda^4 = 0$ mod $n$ but $\lambda^3 \neq 1$ mod $n$, and $\Gamma \cong (\lambda, \lambda, 6)$, as required. □

7. Tetravalent half-arc-transitive bi-$p$-metacirculants

In this final section, we consider tetravalent half-arc-transitive graphs that are constructible as normal edge-transitive bi-Cayley graphs over metacyclic $p$-groups. One motivation for this comes from some work of Bouwer [9] in 1970. Bouwer confirmed Tutte’s question [49] about the existence of half-arc-transitive graphs with even valency at least 4, and the smallest graph in his family is a bi-Cayley graph over a nonabelian metacyclic group of order 27. Another motivation comes from the literature on half-arc-transitive metacirculants of prime-power order. An $(m, n)$-metacirculant is a graph $\Gamma$ of order $mn$ which admits an automorphism $\sigma$ of order $n$ such that $\langle \sigma \rangle$ is semi-regular on $V(\Gamma)$, and an automorphism $\tau$ normalising $\langle \sigma \rangle$ such that $\tau$ has a cycle of size $m$ on $V(\Gamma)$ and cyclically permutes the $m$ orbits of $\langle \sigma \rangle$.

Metacirculant graphs were introduced by Alspach and Parsons [1], and have many interesting and important properties. A graph is called a weak metacirculant if it admits a metacyclic group of automorphisms acting transitively on vertices. It is easy to see that every metacirculant is a weak metacirculant, and in 2008, Marušić and Šparl [45, p.368] asked whether the converse is true or false.

In a recent paper, Li, Song and Wang [32] claimed to prove that the converse is false, in a theorem stating that every non-split metacyclic $p$-group with $p$ an odd prime acts transitively on the vertices of some half-arc-transitive 4-valent graph that is a weak metacirculant but not a metacirculant. Unfortunately they made a mistake in the first paragraph of their proof of Theorem 1.3 in [32], and that theorem is incorrect, as we will see from Theorem 7.2 below.

Nevertheless it is still true that not every weak metacirculant is a circulant. In fact, the 6-valent bi-Cayley graph on the cyclic group $C_2^3$ that we gave in Example 4.3 is a half-arc-transitive graph of order 56 that is a weak metacirculant (with the subgroup $C_2 \times Q_8$ of its automorphism group being also a non-split extension of $C_4$ by $C_{14}$), but not a metacirculant — as can be confirmed by an easy computation, with the help of Magma [7] if necessary. Two other examples of order 800 have also been found very recently by Šparl and Antončič [2], in the census of all 4-valent half-arc-transitive graphs up to order 1000 created by Potočnik, Spiga and Verret [47]. An infinite family of 6-valent examples (generalising Example 4.3) will be constructed in [55], using the methods developed in the current paper.

For the remainder of this section, we let $p$ be an odd prime. Also we need some additional background. If $G$ is a metacyclic group, then every subgroup $H$ of $G$ is also metacyclic (for if $M$ is a normal cyclic subgroup of $G$ such that $G/M$ is cyclic, then $H \cap M$ is a cyclic normal subgroup of $H$, and similarly $H/(H \cap M) \cong HM/M$ is cyclic). For any group $G$, the unique minimal normal subgroup $N$ of $G$ such that $G/N$ is a $p$-group is denoted by $O_p(G)$. Also if $G$ has a normal $p'$-subgroup $C$ such that $G = PC$ for some Sylow $p$-subgroup $P$ of $G$, then $C$ is called a normal $p$-complement in $G$.

Now let $G$ be any finite group having a nonabelian metacyclic Sylow $p$-subgroup $P$. Then by a theorem of Sasaki [48, Proposition 2.1], we find that $N_G(P) \cap O_p(G) = O_p(N_G(P))$, and moreover, if $N_G(P)$ has a normal $p$-complement, then so does $G$. Then by another theorem of Sasaki [48, Proposition 2.2] and a theorem of Lindenberg [31], on automorphisms of split and non-split metacyclic $p$-groups (respectively), we obtain the following:

**Proposition 7.1.** Let $G$ be a finite group having a nonabelian metacyclic Sylow $p$-subgroup $P$. If $P$ is non-split, then $G$ has a normal $p$-complement. On the other hand, if $P$ is split, and therefore a semidirect product $K \rtimes Q$ of cyclic $p$-groups, then either $G$ has a normal $p$-complement, or $P$ has an automorphism $\beta$ such that $P \cap O_p(G) = P \cap O_p(N_G(P)) = K^{\beta}$. 

We use Proposition 7.1 to study tetravalent half-arc-transitive metacirculants of prime-power order.

**Theorem 7.2.** Let $\Gamma$ be a connected 4-valent half-arc-transitive graph of order $p^n$ for some odd prime $p$. Then $\Gamma$ is a weak metacirculant if and only if $\Gamma$ is a metacirculant.

*Proof.* Clearly we need only prove necessity. So let $G$ be a metacyclic subgroup of $A = \text{Aut}(\Gamma)$ that acts transitively on $V(\Gamma)$, and let $P$ be a Sylow $p$-subgroup of $G$. Then $P$ is metacyclic. On the other hand, since $\Gamma$ is half-arc-transitive and 4-valent, the stabiliser $A_v$ of any vertex $v \in V(\Gamma)$ is a 2-group, and hence $p$ cannot divide $|A_v|$. Then because $|P| = p^n = |V(\Gamma)|$, we find that $P$ is regular on $V(\Gamma)$, and therefore $\Gamma$ is an edge-transitive Cayley graph for $P$. In particular, $P$ is non-abelian, for otherwise the inversion automorphism of $P$ gives an arc-reversing automorphism of $\Gamma$, which is impossible since $\Gamma$ is half-arc-transitive. Moreover, $P$ is a Sylow subgroup of $A$, complemented by the Sylow 2-subgroup $A_v$. On the other hand, $A_v$ is not normal in $A$, for otherwise $A_v$ would fix every vertex of $\Gamma$, so $A_v$ would be trivial, but then $\Gamma$ could not be edge-transitive. Thus $A_v$ has no non-trivial normal 2-subgroup. It follows from Proposition 7.1 (applied to $A$ rather than $G$), that $P$ is a split metacyclic group, and the Cayley graph $\Gamma$ for $P$ is a metacirculant. \hfill $\Box$

As well as contradicting Theorem 1.3 in [32], the above proof shows that a tetravalent half-arc-transitive weak metacirculant of odd prime-power order is a Cayley graph for a split metacyclic $p$-group. Hence we may call a tetravalent half-arc-transitive Cayley graph for a metacyclic $p$-group a *$p$-metacirculant*. Analogously, we define a *bi-$p$-metacirculant* to be a bi-Cayley graph over a metacyclic $p$-group.

The next theorem shows that most tetravalent vertex- and edge-transitive bi-Cayley graphs over non-abelian metacyclic $p$-groups are normal. To prove it, we need the concept of a quotient graph. If $G$ is a group of automorphisms of a graph $\Gamma$, and $N$ is a normal subgroup of $G$, then the quotient graph of $\Gamma$ relative to $N$ is defined as the graph $\Gamma_N$ whose vertices are the orbits of $N$ on $V(\Gamma)$, and with two orbits adjacent if there exists an edge in $\Gamma$ between vertices in those two orbits.

**Theorem 7.3.** Let $\Gamma$ be a connected tetravalent bi-Cayley graph over a non-abelian metacyclic $p$-group $H$, where $p$ is an odd prime, and suppose $R(H)$ is a Sylow subgroup of a subgroup $G$ of $\text{Aut}(\Gamma)$ that acts transitively on both the vertices and the edges of $\Gamma$. Then $H$ is a split metacyclic group, and $R(H)$ is normal in $G$. Moreover, if $p > 3$ then $R(H)$ is normal in $\text{Aut}(\Gamma)$, and so $\Gamma$ is a normal bi-Cayley graph.

*Proof.* We begin by noting that $|V(\Gamma)| = 2 |H| = 2p^n$, where $n \geq 3$ because $H$ is non-abelian. Also $\Gamma$ is vertex-transitive and edge-transitive (by the hypothesis on $G$), and therefore $\Gamma$ is either arc-transitive or half-arc-transitive. Then since the valency of $\Gamma$ is 4, the stabiliser $A_v$ of any vertex $v \in \Gamma$ is a group of order $2^c3$ or $2^c$ for some $c$, and accordingly $|A| = |V(\Gamma)||A_v| = 2^{c+1}p^n$ or $2^{c+1}3p^n$, depending on whether or not $\Gamma$ is 2-arc-transitive. In the latter case, $A$ is a $\{2, p\}$-group, and therefore soluble (by Burnside’s $p^aq^b$ theorem). The analogous property holds for the subgroup $G$ of $A$, and so either $G$ acts transitively on the 2-arcs of $\Gamma$, or $G$ is a $\{2, p\}$-group, and therefore soluble.

Now suppose $G$ has a normal $p$-complement, say $Q$. Then the product $QG_v$ is a $p'$-group, which must also be complementary to $R(H)$, so $|QG_v| = |G : R(H)| = |Q|$ and therefore $Q \leq H$. Then since $\Gamma$ is 4-valent and $G$-edge-transitive and $G$-vertex-transitive, the quotient graph $\Gamma_Q$ is either 1-valent or 2-valent, and hence is a cycle or $K_2$, so $\text{Aut}(\Gamma_Q)$ is cyclic or dihedral. But $Q$ is the kernel of the action of $G$ on $V(\Gamma_Q)$, so $R(H) \cong R(H)/Q \leq G/Q \leq \text{Aut}(\Gamma_Q)$, and hence this cannot happen.

Thus $G$ has no normal $p$-complement, and so by Proposition 7.1 (again applied to $G$), it follows that $H$ is a split metacyclic group.

Next, we show that $R(H) \triangleleft G$. This is a little complicated, so we assume there is a counter-example, and proceed in several steps to show that cannot happen.

**Step 1.** We prove that $G$ has no non-trivial normal 2-subgroup.

Suppose $G$ has a non-trivial normal 2-subgroup $N$. Then the quotient graph $\Gamma_N$ has valency 2 or 4. If its valency is 2, then $\Gamma_N$ is a cycle of order $p^2$, but then $R(H)$ cannot act faithfully on $\Gamma_N$ (because the $p$-subgroup $R(H)$ is non-cyclic), and so $\Gamma_N$ must have valency 4. Now if $v$ is any vertex of $\Gamma$, then $v$ and its four neighbours lie in five different orbits of $N$, so $N_v$ must fix each neighbour of $v$. By connectedness,
$N_e$ fixes every vertex of $\Gamma$, and so $N_e$ is trivial, and therefore $N$ is semi-regular on $\Gamma$. In particular, $|N|$ must divide $|V(\Gamma)| = 2p^n$, and so $|N| = 2$. Also the order of $\Gamma_N$ is $p^n$, and $N$ is the kernel of the action of $G$ on $\Gamma_N$, so $\overline{G} = G/N$ is a group of automorphisms of $\Gamma_N$.

Next, let $g$ be the involutory generator of $N$. Then $g$ cannot preserve the orbits $H_0$ and $H_1$ of $R(H)$, since each has odd size $p^n$, and it follows that $(R(H), g) = N \times R(H)$ is transitive on vertices. Moreover, the orbits of $N$ form a system of imprimitivity for $G$ on $V(\Gamma)$, and it follows that no orbit of $N$ can be contained in $H_0$ or $H_1$, for the same reason. In turn, this implies that $R(H)$ acts transitively and hence regularly on the vertices of $\Gamma_N$, so $\Gamma_N$ is a Cayley graph for $R(H) \cong H$.

In particular, since its valency 4 is less than $2p$, it follows from \cite[Corollary 1.2]{31} that $\Gamma_N$ is a normal Cayley graph, with $R(H) = R(H)N/N$ normal in $\text{Aut}(\Gamma_N)$. Hence $R(H)N/N$ is also normal in $G/N$, and so $R(H)N$ is normal in $G$. But $R(H)$ has index 2 in $R(H)N$, and is therefore a normal Sylow $p$-subgroup of $R(H)N$, so is characteristic in $R(H)N$, and hence $R(H)$ is normal in $G$, contradiction.

**Step 2.** We show that every minimal normal subgroup of $G$ is a $p$-group.

Here we make use of \cite[Lemma 3.1]{31}, which shows that if $J$ is an arc-transitive group of automorphisms of a tetravalent connected graph of order $2p^m$ where $m > 1$ (and $p$ is prime), then every minimal normal subgroup of $J$ is solvable. In particular, this is true for $G$, if $G$ acts transitively on the arcs of $\Gamma$. On the other hand, if $G$ does not act arc-transitively on $\Gamma$, then by our earlier observations, $G$ is a $\{2, p\}$-group, and so $G$ itself is solvable. Hence in both cases, a minimal normal subgroup of $G$ is solvable, and therefore an elementary abelian group. But this cannot be a 2-group (by step 1), and cannot be a 3-group (otherwise it would be generated by an element of order 3 fixing a vertex), and thus every minimal normal subgroup of $G$ is a $p$-group.

**Step 3.** Let $M = O_p(G)$ be the largest normal $p$-subgroup of $G$, and consider $C_G(M)$ and $G/M$.

Let $C = C_G(M)$. Then conjugation of $M$ by $G$ makes $G/C$ isomorphic to a subgroup of $\text{Aut}(M)$.

Now suppose that $G/M$ has a normal 2-subgroup contained in $CM/M$, say $L/M$. Then $M \leq L \leq CM$, but since $C \leq CM$, and $M$ is a $p$-group, every 2-subgroup of $CM$ is contained in $C$, and so every Sylow 2-subgroup of $L$ is contained in $C = C_G(M)$, and therefore $L = M \times Q$, where $Q$ is a Sylow 2-subgroup of $L$. But now this makes $Q$ characteristic in $L$ and hence normal in $G$, which is impossible since $G$ has no non-trivial normal 2-subgroup. Thus $G/M$ has no such subgroup.

Next let $\text{Aut}^g(M) = \{ \alpha \in \text{Aut}(M) \mid g^\alpha \Phi(M) = g\Phi(M), \forall g \in M \}$, where $\Phi(M)$ is the Frattini subgroup of $M$. Then $\text{Aut}^g(M)$ is a normal $p$-subgroup of $\text{Aut}(M)$, with $\text{Aut}(M)/\text{Aut}^g(M) \leq \text{Aut}(M)/\Phi(M)$; see, for example, \cite[pp. 81–83]{13}. Also let $K$ be the subgroup of $G$ containing $C$ for which $K/C = (G/C) \cap \text{Aut}^g(M)$. Then $K/C$ is a normal $p$-subgroup of $G/C$, and $G/K \leq \text{Aut}(M/\Phi(M))$.

Now suppose for the moment that $K$ is a $p$-group. Then $K \leq M \leq R(H)$, because $M = O_p(G)$. Also $M \neq R(H)$, for otherwise $R(H) \leq G$, and hence the index of each of $K$ and $M$ in $R(H)$ is divisible by $p$. On the other hand, $R(H)$ is metacyclic, and therefore $M$ is metacyclic, or possibly cyclic. But if $M$ is cyclic, then $\Phi(M) \cong C_p$ and so $G/K \leq \text{Aut}(M/\Phi(M)) \cong C_{p-1}$, and which implies that $K$ is a Sylow $p$-subgroup of $G$, and so $R(H) = K = M \leq G$, contradiction. Hence $M$ is a non-cyclic metacyclic $p$-group. It follows that $M/\Phi(M) \cong C_p \times C_p$, so $G/K \leq \text{Aut}(M/\Phi(M)) \cong \text{Aut}(C_p \times C_p) \cong \text{GL}(2, p)$, and then since $|\text{GL}(2, p)| = (p^2 - 1)(p^2 - p)$ is not divisible by $p^2$, we find that $|R(H) : K| = p$, and so $K = M$. Thus we have shown that if $K$ is a $p$-group, then $K = M$ and $G/M$ is isomorphic to a subgroup of $\text{GL}(2, p)$. In particular, this happens if $C \leq M$ (for then $C$ is a $p$-group and hence so is $K$).

**Step 4.** We show that $G$ is 2-arc-transitive on $\Gamma$.

Suppose that $G$ does not act transitively on the 2-arcs of $\Gamma$. Then $G$ is a $\{2, p\}$-group. This implies that $CM = M$, for otherwise if $L/M$ were a minimal normal subgroup of $G/M$ contained in $CM/M$, then by the maximality of $M$ as a normal $p$-subgroup of $G$, we would find that $L/M$ is a normal 2-group of $G/M$, which is impossible by what we showed at the beginning of Step 3. Thus $C \leq M$, and hence also $K = M$ and $G/M \leq \text{GL}(2, p)$, by what we showed at the end of Step 3. Then since $G/M$ is a $\{2, p\}$-group (but not a 2-group), it follows that the image of $G/M$ in $\text{PGL}(2, p) = \text{GL}(2, p)/\text{Z}(\text{GL}(2, p))$ is isomorphic to a subgroup of $C_p \times C_{p-1}$, and hence has a cyclic normal subgroup of order $p$. In turn,
since $Z(\text{GL}(2,p)) \cong \mathbb{Z}_p^* \cong C_{p-1}$, this implies that $G/M$ has a cyclic normal Sylow $p$-subgroup of order $p$. But that must be $R(H)/M$, and so once again $R(H) \lhd G$, contradiction.

**Step 5.** By considering the quotient graph $\Gamma_M$, we show that $|R(H) : M| = p$.

Since $M$ is a proper subgroup of $R(H)$, we know that $M$ has at least $2p$ orbits on $V(\Gamma)$. Also by the 2-arc-transitivity of $G$ on $\Gamma$, we know that $M$ is the kernel of the action of $G$ on $V(\Gamma_M)$, and $G/M$ is a 2-arc-transitive group of automorphisms of $\Gamma_M$.

Now suppose that $|R(H)/M| > p$. Then $|V(\Gamma_M)| = 2p^2$, and so we can use Lemma 3.1 again, to conclude that $G/M$ has a minimal normal subgroup $N/M$ that is soluble. This cannot be a $p$-group or a 3-group, for the same reasons as before, and so must be a 2-group. Also the number of orbits of $N/M$ of $V(\Gamma_M)$ is $|R(H)/M| > p > 2$, and hence the same is true for the number of orbits of $N$ on $V(\Gamma)$, and as $G$ is 2-arc-transitive on $\Gamma$, the quotient graph $\Gamma_N$ has valency 4. Accordingly, just as in step 1, we find that $N$ acts semi-regularly on $V(\Gamma)$, and $|N| = 2|M|$. Moreover, $R(H)N/N$ acts regularly on $V(\Gamma_N)$, and so $\Gamma_N$ is a tetrahedral Cayley graph for $R(H)N/N$. Also $R(H)\cap N = M$, and therefore $R(H)N/N \cong R(H)/(R(H)\cap N) \cong R(H)/M$, which is a metacyclic $p$-group. If $R(H)/M$ is abelian, then by Corollary 1.3 we find that $G_N$ is a normal Cayley graph, with $R(H)N/N \lhd G/N$, and therefore $R(H) \lhd G$, contradiction. On the other hand, if $R(H)/M$ is non-abelian, then once again by Corollary 1.2 we have $R(H)N/N \leq G/N$, contradiction. Thus $|R(H)/M| = p$.

**Step 6.** This is the last step, in which we show that $|R(H) : M| \neq p$.

As $G$ is 2-arc-transitive on $\Gamma$, and $|R(H)/M| = p$, again we find that $M$ is the kernel of the action of $G$ on $V(\Gamma_M)$, and that $G/M$ is a 2-arc-transitive group of automorphisms of $\Gamma_M$, but this time $|V(\Gamma)| = 2|R(H)/M| = 2p$, and so $\Gamma_M$ is one of the symmetric graphs of order $2p$ classified in [11]. Indeed from Theorem 4.2 and Table 1 in [11], we can see that there are only three possibilities, as follows:

- $p = 5$ and $A_5 \leq G/M \leq S_5 \times C_2$,
- $p = 7$ and $\text{PSL}(2, 7) \leq G/M \leq \text{PGL}(2, 7)$,
- $p = 13$ and $\text{PSL}(3, 3) \leq G/M \leq \text{PSL}(3, 3).C_2$.

Now if $C \leq M$, then $G/M \not\leq \text{GL}(2,p)$, which implies that $A_5 \not\leq \text{GL}(2,5)$, or $\text{PSL}(2, 7) \not\leq \text{GL}(2,7)$, or $\text{PSL}(3, 3) \not\leq \text{GL}(2,13)$, respectively, but none of these is possible (as can be shown with the help of Magma if necessary), and it follows that $M < CM$.

Again let $L/M$ be a minimal normal subgroup of $G/M$ contained in $CM/M$. By step 3 we know that $L/M$ cannot be a 2-group, and so $L/M$ must be $A_5$, $\text{PSL}(2, 7)$ or $\text{PSL}(3, 3)$. In particular, $L/M$ is non-abelian simple, and then because $L/M = (L/M)' = L/M$, we find that $L'/M = L$.

If $L' \neq L$, then $M \not\leq L'$ and $L' \cap M < M$, with $L'/L' \cap M \cong L'/M \cong L/M$, and therefore $|L'| = |L/M||L'/M|$. Also in each of the three cases listed above, the $p$-part of $|L/M|$ is $p$, and it follows that every Sylow $p$-subgroup of $L'$ has order $p|L'/M| < p|M| = |R(H)|$. Thus $|L'/M| < |R(H)|/p$, and we find that $L'$ has at least $p$ orbits on $V(\Gamma)$. But also $L'$ is characteristic in $L$ and hence normal in $G$, and so the 2-arc-transitivity of $G$ on $\Gamma$ implies that $L'$ is semi-regular on $V(\Gamma)$, and so $|L'|$ divides $|V(\Gamma)| = 2|R(H)| = 2p^n$. This makes $L'$ a $\{2, p\}$-group, and therefore $L'/M$ is soluble, which is impossible because $L/M$ is non-abelian simple. Hence $L = L'$, so $L$ is perfect.

Next, if $M$ is abelian, then $M \leq C_G(M) = C$ and so we may suppose that $L \leq C$, in which case $M \leq L \leq C$ and therefore $M \leq Z(L)$. Since also $L = L'$, it follows that $M$ is isomorphic to a subgroup of the Schur multiplier of the simple group $L/M$. The Schur multipliers of $A_5$, $\text{PSL}(2, 7)$ and $\text{PSL}(3, 3)$ are all cyclic (of orders 2, 2 and 1), however, while $M$ is not (since $M/\Phi(M) \cong C_p \times C_p$), contradiction. Hence $M$ is non-abelian.

To complete this step, we consider the subgroup $C_L(M)$, which is normal in $L$. The quotient $L/C_L(M)$ is isomorphic to a subgroup of $\text{Aut}(M)$, which is soluble by Lemmas 2.4, 2.6, 2.7, and so $L/C_L(M)$ is soluble. But $L$ itself is not soluble, and it follows that $C_L(M)$ cannot be soluble. Next, because $L/M$ is simple and $C_L(M)/M \lhd L/M$, we know that $C_L(M)M = L$ or $M$, but the latter cannot occur since $M$ is soluble. Thus $C_L(M)M = L$, and so $C_L(M)/C(L(M) \cap M) \cong C_L(M)/M = L/M$.

Also $C_L(M) \cap M \neq M$ since $M$ is non-abelian, and so just as we did above for $L'$, we find that every Sylow $p$-subgroup of $C_L(M)$ has order $p|C_L(M) \cap M| < p|M| = |R(H)|$, and hence $C_L(M)$ has at least $p$
orbits on $V(\Gamma)$. But also $C_L(M) = L \cap C_G(M) \triangleleft G$, and so $C_L(M)$ is semi-regular on $V(\Gamma)$. In particular, $|C_M(L)|$ divides $|V(\Gamma)| = 2|R(H)| = 2p^n$, and so $C_L(M)$ is soluble, contradiction.

This final contradiction eliminates any possibility of a counter-example, and therefore $R(H) \triangleleft G$.

Finally, because $\Gamma$ is 4-valent, the stabiliser $A_9$ of any vertex is a $\{2, 3\}$-group, and then since $R(H)$ acts regularly on each part of $\Gamma$, it follows that the index $[A : R(H)]$ is of the form $2^3 p^3$. Hence if $p > 3$, then $R(H)$ is a Sylow $p$-subgroup of $A$, and so we can take $G = A$, and find that $R(H) \not\triangleleft A$, so that the bi-Cayley graph $\Gamma$ is normal. □

We remark that $R(H)$ is not always normal in $\text{Aut}(\Gamma)$ when $p = 3$. A counter-example is the bi-Cayley graph $\text{BiCay}(H, \emptyset, \emptyset, S)$, where $H$ is the metacyclic group $\langle a, b \mid a^9 = b^7 = 1, b^{-1}ab = a^4 \rangle \cong C_9 \times C_7$, and $S = \{1, a, ab, a^4b^2\}$. In fact, a computation using Magma \cite{Magma} shows that this graph of order 54 is 2-arc-transitive, with automorphism group of order 1296, but is not normal as a bi-Cayley graph.

This example provided the idea for construction of the families of tetravalent half-arc-transitive bi-p-metacirculants appearing in the next two lemmas. The members of both families are constructed from metacyclic groups of the form $C_p \times C_p$ with presentation $\langle a, b \mid a^{p^2} = b^p = 1, b^{-1}ab = a^{1+p}\rangle$, for odd primes $p$. Note that for every such $p$, the centre of a group of this form is the cyclic subgroup of order $p$ generated by $a^p$, and the elements of order dividing $p$ form the index $p$ subgroup generated by $a^p$ and $b$.

**Lemma 7.4.** For any odd prime $p$, let $H$ be the metacyclic group $\langle a, b \mid a^{p^2} = b^p = 1, b^{-1}ab = a^{1+p}\rangle$ of order $p^3$, and then let $\mathcal{G}_p = \text{BiCay}(H, \emptyset, \emptyset, S)$ where $S = \{1, a^2, a^pb^2, a^2-pb^2\}$. Then $\mathcal{G}_p$ is a 4-valent edge-regular half-arc-transitive bi-$p$-$p$-metacirculant over $C_p \rtimes_{1+p} C_p$, and is also a Cayley graph, for all $p$.

**Proof.** First, it is easy to see that $H$ has an automorphism $\alpha$ taking $a$ to $a^{-1}$, and $b$ to $b$, and then $S^\alpha = \{1, a^2, a^{-p}b^2, a^{2-p}b^2\} = a^{-2}S$. By Proposition 2.2 it follows that $\sigma_{a, a^2}$ is an automorphism of $\mathcal{G}_p$ that fixes $1_0$, and interchanges $1_1$ with $(a^2_1)$, and $(a^{-2}b^2_1)$ with $(a^2b^2_1)$. Next, $a^{-(p+1)}$ has order $p^2$, since $p + 1$ is coprime to $p^2$, and $a^p$ centralises $b$ (indeed $Z(H) = \langle a^p \rangle$), so $a^p b$ has order $p$, and it follows that $a' = a^{-(p+1)}$ and $b' = a^p b$ satisfy the same relations as $a$ and $b$. Hence there exists an automorphism $\beta$ of $H$ that takes $a$ to $a^{-(p+1)}$ and $b$ to $a^p b$, and then $S^{\beta} = \{1, a^{2(1+p)}, a^p b^2, a^{2-p}b^2\} = S^{-1}(a^2b^2)$, so by Proposition 2.2 we find that $\delta_{a'a^p b^2, 1}$ is an automorphism of $\mathcal{G}_p$ that takes $(1_0, 1_1)$ to $((a^2b^2_1), 1_0)$.

In particular, $\delta_{a^a a^p b^2, 1}$ takes a vertex of $H_0$ to a vertex of $H_1$, so $(R(H), \delta_{a^a a^p b^2, 1})$ is transitive on the vertices of $\mathcal{G}_p$. Similarly, the orbit of the arc $(1_0, 1_1)$ under $\langle \sigma_{a, a^2}, \delta_{a^a a^p b^2, 1} \rangle$ includes $(1_0, (a^2_1))$ and also $(\langle a^2b^2_1\rangle, 1_0)$ and $(\langle a^{-2}b^2_1\rangle, 1_0)$, and so $G = \langle R(H), \sigma_{a, a^2}, \delta_{a^a a^p b^2, 1} \rangle$ acts transitively on the edges of $\mathcal{G}_p$.

For $p = 3$, an easy computation with MAGMA \cite{Magma} shows that $\text{Aut}(\mathcal{G}_3) = G$, which has order $108 (= 4p^3)$, and hence $\mathcal{G}_3$ is half-arc-transitive. Hence we may suppose that $p > 3$. Then also because $\mathcal{G}_p$ is 4-valent, the stabiliser $G_v$ of any vertex is a $\{2, 3\}$-group, and it follows that $R(H)$ is a Sylow $p$-subgroup of $G$, and hence $R(H)$ is normal in Aut($\mathcal{G}_p$), by Theorem 7.3.

Now suppose $\mathcal{G}_p$ is arc-transitive. Then since $R(H) \triangleleft \text{Aut}(\mathcal{G}_p)$, also $\mathcal{G}_p$ is normal locally arc-transitive, and so by Proposition 3.3 some automorphism $\gamma$ of $H$ takes $S$ to $S^{-1}$. To consider this possibility, note that the non-trivial elements $a^2, a^p b^2$ and $a^{2-p}b^2$ in $S$ have orders $p^2$, $p$ and $p^2$, respectively, and their inverses are $a^{-2}, a^{-p}b^{-2}$ and $(a^{2-p}b^2)^{-1} = b^{-2}a^{-2} = a^{(p-2)(1+p)}b^{-2} = a^{(p-2)(1+2p)}b^{-2} = a^{(p-2)(1-3p)}b^{-2}$. Hence $\gamma$ takes $(a^2, a^p b^2)$ to either $(a^2, a^{-1-2p}b^{-2})$ or $(a^{2-3p}b^{-2})$, and then $\gamma$ takes $a^p b^2 = a^{2(p-1)}a^{2-3p}b^{-2}$ to either $a^{2(1-p)}a^{-3p}b^{-2} = a^{-5p}b^{-2}$ or $a^{2(1-p)}a^{-2} = a^{-2p}$, a contradiction in both cases. Hence $\mathcal{G}_p$ is not arc-transitive, and is therefore half-arc-transitive.

Next, if the stabiliser in $\mathcal{G}_p$ of the edge $(1_0, 1_1)$ is non-trivial, then it must contain $\sigma_{a, 1}$ for some non-trivial automorphism $\gamma$ of $H$ that preserves $S$, and then $\gamma$ has to swap $a^2$ with $a^{2-p}b^2$, but in that case $\gamma$ takes $a^p b^2 = a^{2(p-1)}a^{2-3p}b^{-2}$ to $(a^{2-p}b^2)^{-2}a^2$, which is of the form $a^2 b^{-2}$ for some $\xi$, and therefore $\gamma$ cannot fix $a^p b^2$. Hence $\mathcal{G}_p$ is edge-regular as well.

Finally, recall that the subgroup $J = \langle R(H), \delta_{a^a a^p b^2, 1} \rangle$ acts transitively on the vertices of $\mathcal{G}_p$. It is also easy to see that $\langle \delta_{a^a a^p b^2, 1} \rangle^2 = R(a^p b^2) \in R(H)$, and so $\langle \delta_{a^a a^p b^2, 1} \rangle^2 = 1$, and then because $R(H) \triangleleft A$ it follows that $J = \langle R(H), \langle \delta_{a^a a^p b^2, 1} \rangle^p \rangle \cong H \rtimes C_2$, of order $2p^3$, and therefore $J$ acts regularly on $\mathcal{G}_p$. Thus $\mathcal{G}_p$ is a Cayley graph for $H \rtimes C_2$. □
The second family can be handled in a similar way, but it differs from the first one at a few points.

**Lemma 7.5.** For any prime $p$ congruent to 1 modulo 4, let $H$ be the same metacyclic group of order $p^3$ as used in Lemma 7.4, namely $\langle a, b \mid a^p = b^2 = 1, b^{-1}ab = a^{1+p}, \rangle$, let $s = \frac{1 + \lambda + \sqrt{\lambda^2 - 4}}{2}$ where $\lambda$ is a square root of $-1$ in $\mathbb{Z}_{p^2}$, and let $H_p = \text{BiCay}(H, \emptyset, \emptyset, T)$ where $T = \{1, a, a^2b^2, a^{1-s}b^2\}$. Then $H_p$ is a 4-valent half-arc-transitive bi-$p$-metacirculant over $C_{p^2} \rtimes_1 C_p$, but is not a Cayley graph, for all $p$.

**Proof.** Let $\alpha$ be the automorphism of $H$ taking $(a, b)$ to $(a^{-1}, b)$. Then $T^\alpha = \{1, a^{-1}, a^{-s}b^2, a^{1-s}b^2\} = a^{-1}T$, and so by Proposition 2.2 we find that $\sigma_{\alpha, a}$ is an automorphism of $H_p$ that fixes $1_0$ and interchanges $1_1$ with $a_1$, and $(a^2b^2)_1$ with $(a^{1-s}b^2)_1$. Next, we note that $s(1 - \lambda - \lambda p) \equiv 1 - \lambda p \ mod \ p^2$, since

$$2s(1 - \lambda - \lambda p) \equiv (1 + \lambda - \lambda p)(1 - \lambda - \lambda p) \equiv (1 - \lambda^2) - 2\lambda p \equiv 2(1 - \lambda p) \ mod \ p^2.$$

Accordingly, we find that $a^{s(1 - \lambda - \lambda p)}b^2 = a^{-s}b^2$, which has order $p$, and then further, that $a^{(p+1)}$ and $a^{s(1 - \lambda - \lambda p)}b^2$ satisfy the same defining relations as the alternative generators $a$ and $b^2$ for $H$, namely $a^2 = (b^2)^p = 1$ and $b^{-1}ab = a^{1+s}2p$. This implies the existence of an automorphism $\beta$ that takes $a$ to $a^{p+1}$, and $b^2$ to $a^{-s}b^2$, respectively. The effect of this automorphism $\beta$ on the other two non-trivial elements of $S$ is given by

$$(a^2b^2)\beta = a^{2s(p+1)}a^{s(1 - \lambda - \lambda p)}b^2 = a^{-s}b^2 \quad \text{and}$$

$$(a^{1-s}b^2)\beta = a^{(s - \lambda)(p+1)}a^{-\lambda}b^2 = a^{\lambda + 1 - \lambda p - s}b^2 = a^{2s - b^2} = a^{s}b^2,$$

with the latter occurring since the displayed congruence above gives $s\lambda(p+1) \equiv s - 1 + \lambda p \ mod \ p^2$. This now implies that $T^\beta = \{1, a^{p+1}, a^{-s}b^2, a^2b^2\} = T^{-1}a^{-s}b^2$, once it is noted that

$$(a^{1-s}b^2)^{-1}a^2b^2 = b^{-2}a^{2s-1}b^2 = a^{2s - (1+2p)} = a^{(\lambda - \lambda p)}(1+2p) = a^{\lambda - \lambda p + 2\lambda p} = a^{\lambda(p+1)}.$$

Hence by Proposition 2.2 we have an automorphism $\delta_{\beta, a, b^2, 1}$ of $H_p$ that takes $(1_0, 1_1)$ to $(a^2b^2)_1, (1_0)$.

In particular, $\delta_{\beta, a, b^2, 1}$ takes a vertex of $H_0$ to a vertex of $H_1$, so $\langle R(H), \delta_{\beta, a, b^2, 1} \rangle$ is transitive on the vertices of $H_p$, and the orbit of the arc $(1_0, 1_1)$ under $\langle \sigma_{\alpha, a}, \delta_{\beta, a, b^2, 1} \rangle$ includes $(1_0, a_1)$ and also $(a^2b^2)_1, 1_0)$ and $((a^{1-s}b^2)_1, 1_0)$, and therefore $G = \langle R(H), \sigma_{\alpha, a}, \delta_{\beta, a, b^2, 1} \rangle$ acts transitively on the edges of $H_p$. Also $p$ is at least 5, and $H_p$ is 4-valent, so the stabiliser $G_u$ of any vertex is a $\{2, 3\}$-group, and it follows that $R(H)$ is a Sylow $p$-subgroup of $G$, and then by Theorem 7.3 that $R(H)$ is normal in $\text{Aut}(H_p)$.

Now suppose $H_p$ is arc-transitive. Then since $R(H) \lhd \text{Aut}(H_p)$, also $H_p$ is normal locally arc-transitive, and hence by Proposition 3.3 there exists an automorphism $\gamma$ of $H$ that takes $S$ to $S^{-1}$. This time the non-trivial elements $u = a, v = a^2b^2$ and $w = a^{1-s}b^2$ in $S$ (which all have order $p^2$) satisfy $vw^{-1} = a^{2s-1}$, and it is a relatively straightforward exercise to prove that no permutation of the inverses of those elements satisfies the analogous relation. In fact, each of $v^{-1}wu^{2s-1}$ and $w^{-1}vu^{2s-1}$ is a non-trivial power of $a$, while in the other four cases, the relevant product lies outside $\langle a \rangle$. Hence $H_p$ is not arc-transitive, and is therefore half-arc-transitive.

Next, if the stabiliser in $H_p$ of the edge $(1_0, 1_1)$ is non-trivial, then it must contain $\sigma_{\gamma, a}$ for some non-trivial automorphism $\gamma$ of $H$ that preserves $S$. But a similar exercise to the one above shows that no non-trivial permutation of the elements $u, v, w$ defined above satisfies the analogue of the relation $vw^{-1} = a^{2s-1}$; in fact $vw^{-1}u^{-2s}$ is a non-trivial power of $a$, while in the relevant four cases, the relevant product lies outside $\langle a \rangle$. Hence no such $\gamma$ exists, and therefore $H_p$ is edge-regular.

Finally, let $\delta$ be the automorphism $\delta_{\beta, a, b^2, 1}$ of $H_p$ referred to earlier. It is easy to check that $\delta^2$ takes $1_0$ to $(a^{s-1}b^2)_0$, and $1_1$ to $(a^2b^2)_1$, so that $\delta^2$ has the same effect on as $H_p$, as $\sigma_{\alpha, a}R(a^{s-1}b^2)$. Since $R(H)$ is normal in $A$ with index 4 but does not contain $\sigma_{\alpha, a}$, it follows that the quotient $A/R(H)$ is cyclic of order 4, generated by the image of $\delta$. In particular, $A$ has a unique subgroup of index 2 (and order $2p^3$), namely $\langle R(H), \delta^2 \rangle = \langle R(H), \sigma_{\alpha, a} \rangle$. This subgroup preserves the two parts $H_0$ and $H_1$ of $H_p$, however, so does not act transitively on vertices, and therefore $H_p$ cannot be a Cayley graph.

**Proof of Theorem 1.6.** The first part of this theorem was proved in Theorem 7.3 and the rest follows from Lemmas 7.4 and 7.5.
8. Proof of Theorem 1.2

Theorem 1.2 asserts that every normal edge-transitive bi-Cayley graph $\Gamma$ is either arc-transitive, half-arc-transitive or semisymmetric, and that examples of each kind exist. The first part is easy to see: if $\Gamma$ is not vertex-transitive, then it is semisymmetric, while if it is vertex-transitive but not arc-transitive, then it is half-arc-transitive. The second part follows from Theorem 5.4, Proposition 6.4 and Lemmas 7.4 and 7.5, which show the existence of infinitely many examples in each case.

References

1. B. Alspach, T.D. Parsons, A Construction for vertex-transitive graphs, Canad. J. Math. 34 (1982), 307–318.
2. I. Antonić, P. Šparl, Classification of quartic half-arc-transitive weak metacirculants of girth at most 4, Discrete Math. 339 (2016), 931–945.
3. A. Araujo, I. Kovács, K. Kutnar, L. Martínez, D. Marušič, Partial sum quadruples and bi-Abelian digraphs, J. Combin. Theory Ser. A 119 (2012), 1811–1831.
4. Y.G. Baik, Y.-Q. Feng, H.S. Sim, M.Y. Xu, On the normality of Cayley graphs of abelian groups, Algebra Colloq. 5 (1998), 297–304.
5. N. Biggs, Algebraic Graph Theory, 2nd ed., Cambridge University Press, Cambridge, 1993.
6. J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Elsevier North Holland, New York, 1976.
7. W. Bosma, J. Cannon, C. Playoust, The MAGMA algebra system I: The user language, J. Symbolic Comput. 24 (1997), 235–265.
8. I.Z. Bouwer, An edge but not vertex transitive cubic graph, Bull. Can. Math. Soc. 11 (1968), 533–535.
9. I.Z. Bouwer, Vertex and edge transitive but not 1-transitive graphs, Canad. Math. Bull. 13 (1970), 231–237.
10. L. Cao, J. Meng, Super-connected and hyper-connected cubic bi-Cayley graphs, Chin. Quart. J. of Math. 24 (2009), 53–57.
11. Y. Cheng, J. Oxley, On weakly symmetric graphs of order twice a prime, J. Combin. Theory Ser. B 42 (1987), 196–211.
12. M. Conder, P. Dobcsányi, Trivalent symmetric graphs on up to 768 vertices, J. Combin. Math. Combin. Comput. 40 (2002), 41–63.
13. M. Conder, A. Malnič, D. Marušič, P. Potočnik, A census of semisymmetric cubic graphs on up to 768 vertices, J. Algebr. Comb. 23 (2006), 255–294.
14. M. Conder, R. Nedela, Symmetric cubic graphs of small girth, J. Combin. Theory Ser. B 97 (2007), 757–768.
15. M.J. de Resmini, D. Jungnickel, Strongly regular semi-Cayley graphs, J. Algebr. Comb. 1 (1992), 171–195.
16. D.Z. Džoković, G.L. Miller, Regular groups of automorphisms of cubic graphs, J. Combin. Theory Ser. B 29 (1980), 195–230.
17. S.F. Du, M.Y. Xu, A classification of semisymmetric graphs of order 2pq, Comm. in Algebra 28 (2000), 2685–2715.
18. Y.Q. Feng, R. Nedela, Symmetric cubic graphs of girth at most 7, Acta Univ. M. Belii Math. 13 (2006), 33–55.
19. J. Folkman, Regular line-symmetric graphs, J. Combin. Theory 3 (1967), 215–232.
20. X. Gao, W. Liu, Y. Luo, On the extendability of certain semi-Cayley graphs of finite abelian groups, Discrete Math. 311 (2011), 1978–1987.
21. X. Gao, Y. Luo, The spectrum of semi-Cayley graphs over abelian groups, Linear Algebra Appl. 432 (2010), 2084–2100.
22. M. Giudici, Maximal subgroup of almost simple groups with socle PSL$(2, q)$, arXiv: math. GR/0703685v1, 23 Mar 2007.
23. D. Goldschmidt, Automorphisms of trivalent graphs, Ann. Math. 111 (1980), 377–406.
24. M. Hladík, D. Marušič, T. Pisanski, Cyclic Haar graphs, Discrete Math. 244 (2002), 137–152.
25. I. Kovács, B. Kuzman, A. Malnič, S. Wilson, Characterization of edge-transitive 4-valent bicirculants, J. Graph Theory 69 (2012), 441–463.
26. I. Kovács, A. Malnič, D. Marušič, Š. Miklavič, One-matching bi-Cayley graphs over abelian groups, European J. Combin. 30 (2009), 602–616.
27. K. Kutnar, D. Marušič, A complete classification of cubic symmetric graphs of girth 6, J. Combin. Theory Ser. B, 99 (2009), 162–164.
28. K.H. Leung, S.L. Ma, Partial difference triples, J. Algebr. Comb. 2 (1993), 397–409.
29. C.H. Li, Finite s-arc transitive Cayley graphs and flag transitive projective planes, Proc. Amer. Math. Soc. 133 (2004), 31–41.
30. C.H. Li, On finite edge transitive graphs and rotary maps, J. Combin. Theory Ser. B 98 (2008), 1063–1067.
31. C.H. Li, H.S. Sim, Automorphisms of Cayley graphs of metacyclic groups of prime-power order, J. Austral. Math. Soc., 70 (2001), 223–231.
32. C.H. Li, S.J. Song, D.J. Wang, A characterization of metacyclics, J. Combin. Theory Ser. A 120 (2013), 39–48.
33. X. Liang, J. Meng, Connectivity of bi-Cayley graphs, Ars Combin. 88 (2008), 27–32.
34. W. Lindenber, Eine Klasse von p-Gruppen, deren Automorphismengruppen p-Gruppen sind, Period. Math. Hungar. 5 (1974), 171–183.
35. Z.P. Lu, C.Q. Wang, M.Y. Xu, Semisymmetric cubic graphs constructed from bi-Cayley graphs of $A_n$, Ars Combin. 80 (2006), 177–187.
36. Y. Luo, X. Gao, On the extendability of Bi-Cayley graphs of finite abelian groups, Discrete Math. 309 (2009), 5943–5949.
37. A. Malnič, D. Marušić, P. Šparl, On strongly regular bicirculants, European J. Combin. 28 (2007), 891–900.
38. L. Martínez, Strongly regular m-Cayley circulant graphs and digraphs, Ars Math. Contemp. 8 (2015), 195–213.
39. L. Martínez, A. Araluze, New tools for the construction of directed strongly regular graphs: Difference digraphs and partial sum families, J. Combin. Theory Ser. B 100 (2010), 720–728.
40. D. Marušić, Strongly regular bicirculants and tricirculants, Ars Combin. 25 (1988), 11–15.
41. D. Marušić, T. Pisanski, Symmetries of hexagonal molecular graphs on the torus, Croat. Chem. Acta 73 (2000), 969–981.
42. D. Marušić, P. Potočnik, Semisymmetry of generalized Folkman graphs, European J. Combin. 22 (2001), 333–349.
43. D. Marušić, P. Šparl, On quartic half-arc-transitive metacirculants, J. Algebr. Comb. 28 (2008), 365–395.
44. F. Menegazzo, Automorphisms of p-groups with cyclic commutator subgroup, Rend. Sem. Mat. Univ. Padova 90 (1993), 81–101.
45. T. Pisanski, A classification of cubic bicirculants, Discrete Math. 307 (2007), 567–578.
46. C.E. Praeger, Finite normal edge-transitive Cayley graphs, Bull. Austral. Math. Soc. 60 (1999), 207–220.
47. P. Potočnik, P. Spiga, G. Verret, A census of 4-valent half-arc-transitive graphs and arc-transitive digraphs of valence two, Ars Math. Contemp. 8 (2015), 133–148.
48. H. Sasaki, The mod p cohomology algebras of finite groups with metacyclic Sylow p-subgroups, J. Algebra 192 (1997), 713–733.
49. W.T. Tutte, Connectivity in graphs, University of Toronto Press, Toronto, 1966.
50. H. Wielandt, Finite Permutation Groups, Academic Press, New York, 1964.
51. M.-M. Zhang, J.-X. Zhou, Tetravalent half-arc-transitive bi-metacirculants, in preparation.
52. J.-X. Zhou, Super restricted edge-connectivity of regular edge-transitive graphs, Discrete Appl. Math. 160 (2012), 1248–1252.
53. J.-X. Zhou, F.-Q. Feng, Tetravalent s-transitive graphs of order twice a prime power, J. Aust. Math. Soc. 88 (2010), 277–288.
54. J.-X. Zhou, F.-Q. Feng, Super-cyclically edge-connected regular graphs, J. Comb. Optim. 26 (2013), 393–411.
55. J.-X. Zhou, Y.-Q. Feng, The automorphisms of bi-Cayley graphs, J. Combin. Theory Ser. B 116 (2016), 504–532.
56. J.-X. Zhou, M.-M. Zhang, On half-arc-transitive bi-circulants, submitted.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF AUCKLAND, PRIVATE BAG 92019, AUCKLAND 1142, NEW ZEALAND
E-mail address: m.conder@auckland.ac.nz

DEPARTMENT OF MATHEMATICS, BEIJING JIAOTONG UNIVERSITY, BEIJING 100044, P.R. CHINA
E-mail address: jxzhou@bjtu.edu.cn

DEPARTMENT OF MATHEMATICS, BEIJING JIAOTONG UNIVERSITY, BEIJING 100044, P.R. CHINA
E-mail address: yqfeng@bjtu.edu.cn

DEPARTMENT OF MATHEMATICS, BEIJING JIAOTONG UNIVERSITY, BEIJING 100044, P.R. CHINA
E-mail address: 14118412@bjtu.edu.cn