Modulus Predicting of Viscoelastic Composite Material and Numerical Method of Inverse Laplace Transform

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Abstract. It is studied about the modulus prediction of viscoelastic composite material and numerical method of inverse Laplace transform. According to elastic-viscoelastic correspondence principle, using Laplace transform, the effective modulus of composite materials is achieved by M-T method in the image domain. Then using inverse Laplace transform, the variation rule of modulus with the time and inclusion fraction would be obtained in the time domain.

1. Introduction
Microstructure mechanical analysis methods can establish the overall performance of composite materials which are consisted of elastic matrix and reinforced phase associated with properties of each phase and microstructure parameters. The macroscopic properties of the whole material can be estimated by the properties of each phases.

M-T method based on single inclusion theory is widely used in predicting modulus of composite materials because of its simple model and clear concept [1]. But for composite materials that exhibit viscoelastic behavior, it cannot be applied directly applicable only within the elastic range micromechanics methods. Through the analysis, showing the viscoelastic behavior of the composite material is composed of viscoelastic phase matrix and elastic particle reinforced, so that you can make use of viscoelastic correspondence principle, by Laplace transform domain in use as composite micromechanics forecast effective modulus, and through the inverse Laplace transform, time domain solution can be obtained [2], this paper has been studied in this regard. This forecasting method of effective modulus on viscoelastic composites can be used in viscoelastic composites including solid propellants.

2. MT forecasting method of effective modulus on two-phase composite material with elastic matrix and inclusion
The expressions of bulk modulus, shear modulus and Young's modulus of composites with spherical inclusions were derived using MT method for the spherical inclusions by Wang [2] and Weng[3] as fellow
3. Effective modulus prediction method of two-phase composite body with Viscoelastic matrix

Viscoelastic matrix composite material is filled in the reinforcement, the reinforcement can be regarded as elastic inclusion, but the matrix is viscoelastic, cannot directly use the MT method modulus prediction.

This problem can use elastic-viscoelastic correspondence principle, the viscoelastic problem from time domain transform into image domain to be treated as the elastic problem by Laplace transform [2,4]. The effective modulus of composite materials is achieved by M-T method in the image domain and then the numerical solution of effective modulus in the time domain would be obtained through inverse Laplace transform.

The matrix considered as linear viscoelastic material, the integral form of the constitutive equation is

\[
\sigma_{ik}^0 = \int_{-\infty}^{t} 3K_0 (t - \tau) \left( \frac{d \varepsilon_{ik}^0(\tau)}{d\tau} \right) d\tau
\]

\[
s_{ij}^0 = \int_{-\infty}^{t} 2G_0 (t - \tau) \left( \frac{d \varepsilon_{ij}^0(\tau)}{d\tau} \right) d\tau
\]

\[
\varepsilon_{ik}^0 = \int_{-\infty}^{t} \frac{I_0(t - \tau)}{3} \left( \frac{d \sigma_{ik}^0(\tau)}{d\tau} \right) d\tau
\]

\[
\varepsilon_{ij}^0(\tau) = \int_{-\infty}^{t} \frac{H_0(t - \tau)}{2} \left( \frac{d s_{ij}^0(\tau)}{d\tau} \right) d\tau
\]

Using s as the transformation parameter, the Laplace transform of function \( f(t) \) defined as

\[
\tilde{f}(s) = \int_{0}^{\infty} f(t)e^{-st} dt
\]

\[
s = c + i\omega
\]

Laplace transform of constitutive equation for integral form of linear viscoelastic material is

\[
\tilde{\sigma}_{ik}(s) = 3s\tilde{K}_0(s)\tilde{\varepsilon}_{ik}^0(s)
\]

\[
\tilde{s}_{ij}(s) = 2s\tilde{G}_0(s)\tilde{\varepsilon}_{ij}^0(s)
\]
The matrix material in the Laplace transform space can be seen as an elastic body, corresponding bulk modulus, shear modulus and Young's modulus is defined as

\[
K_0^{\text{TD}}(s) = sK_0(s) = \frac{1}{s\sigma_0(s)}
\]

\[
G_0^{\text{TD}}(s) = sG_0(s) = \frac{1}{s\sigma_0(s)}
\]

\[
E_0^{\text{TD}} = 9K_0^{\text{TD}}C_0^{\text{TD}}/(3K_0^{\text{TD}} + C_0^{\text{TD}})
\]

For easy operation, the Poisson ratio of the matrix can be regarded as constant in the Laplace domain. The bulk modulus and shear modulus of elastic inclusion in image domain and time domain is the same, there are

\[
K_1^{\text{TD}} = K_1, \quad G_1^{\text{TD}} = G_1
\]

Then, effective bulk modulus and shear modulus and Young's modulus of composite material corresponding in the Laplace transform space using MT method obtained as

\[
K^{\text{TD}} = K_0^{\text{TD}} + \frac{f}{K_1^{\text{TD}} - K_0^{\text{TD}}} + \frac{3(1-f)}{3K_0^{\text{TD}} + 4G_0^{\text{TD}}}
\]

\[
G^{\text{TD}} = G_0^{\text{TD}} + \frac{f}{G_1^{\text{TD}} - G_0^{\text{TD}}} + \frac{6(1-f)(K_0^{\text{TD}} + 2G_0^{\text{TD}})}{5G_0^{\text{TD}} (3K_0^{\text{TD}} + 4G_0^{\text{TD}})}
\]

\[
E_{11}^{\text{TD}} = fE_1^{\text{TD}} + (1-f)E_0^{\text{TD}} + \frac{4f(1-f)(v_1 - v_0)^2}{f\frac{1}{K_1^{\text{TD}}} + \frac{1}{K_0^{\text{TD}}} + \frac{1}{G_0^{\text{TD}}}}
\]

Effective modulus of the composites can be obtained by the inverse Laplace transform formula to do. Define the inverse Laplace transform

\[
f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s)e^{st} ds
\]

\[
K(t) = L^{-1}\left[\frac{K^{\text{TD}}}{s}\right]
\]

\[
G(t) = L^{-1}\left[\frac{G^{\text{TD}}}{s}\right]
\]

\[
E_{11}(t) = L^{-1}\left[\frac{E_{11}^{\text{TD}}}{s}\right]
\]
4. Numerical methods of inverse Laplace transform

Peijun Wei [5] have studied the numerical inversion Laplace algorithm which grouped into four categories, and evaluated the characteristics of the various methods. The numerical algorithm based on Foulier Series have a widely adaptation and to be used to tackle the problem of viscoelastic numerical inversion. This numerical algorithm directly from the definition of the Laplace transform, and export the original function \( f(t) \) Fourier series expressions, cosine expression, sine series expression, thus will find \( f(t) \) of the problem into a generalized integral problem.

From the definition of Laplace transform

\[
\tilde{f}(s) = \int_0^{\infty} f(t)e^{-st}dt = \int_0^{\infty} e^{-ct}f(t)[\cos \omega t - i \sin \omega t]dt \quad \text{Re} \tilde{f}(s) = \int_0^{\infty} e^{-ct}f(t) \cos \omega t dt
\]

\[
\text{Im} \tilde{f}(s) = -\int_0^{\infty} e^{-ct}f(t) \sin \omega t dt
\]

Their inverse transformation is

\[
f(t) = f_{i(0)}(t) = \frac{2e^{ct}}{\pi} \int_0^{\infty} \text{Re} \tilde{f}(s) \cos \omega t d\omega \quad (11)
\]

\[
f(t) = f_{i(i)}(t) = -\frac{2e^{ct}}{\pi} \int_0^{\infty} \text{Im} \tilde{f}(s) \sin \omega t d\omega \quad (12)
\]

On the other hand, starting from the definition of inverse Laplace transform

\[
f(t) = \frac{1}{2\pi i} \int_{c-i \infty}^{c+i \infty} \tilde{f}(s)e^{st}ds
\]

\[
e = \frac{1}{2\pi i} \int_{-\infty}^{\infty} [\text{Re} \tilde{f}(s) + i \text{Im} \tilde{f}(s)] \cdot [\cos \omega t + i \sin \omega t]d\omega
\]

\[f(t) \quad \text{is real, we have}
\]

\[
f(t) = \frac{e^{ct}}{\pi} \int_0^{\infty} [\text{Re} \tilde{f}(s) \cos \omega t - \text{Im} \tilde{f}(s) \sin \omega t]d\omega \quad (13)
\]

Here

\[
f_{i(0)}(t) = \frac{2e^{ct}}{T} \left[ \frac{1}{2} \tilde{f}(c) + \sum_{k=1}^{\infty} \text{Re} \tilde{f}(c + k\frac{\pi}{T}) \cos k\frac{\pi t}{T} \right] \quad (14)
\]

\[
f_{i(i)}(t) = -\frac{2e^{ct}}{T} \left[ \sum_{k=1}^{\infty} \text{Im} \tilde{f}(c + k\frac{\pi}{T}) \sin k\frac{\pi t}{T} \right] \quad (15)
\]

\[
f(t) = \frac{e^{ct}}{T} \left[ \frac{1}{2} \tilde{f}(c) + \sum_{k=1}^{\infty} \text{Re} \tilde{f}(c + k\frac{\pi}{T}) \cos k\frac{\pi t}{T} \right.
\]

\[
- \left. \text{Im} \tilde{f}(c + k\frac{\pi}{T}) \sin k\frac{\pi t}{T} \right] \quad (16)
\]

In the formula (14) - (16), \( c \) and \( T \) both are the parameter for the calculation, also known as free parameters, and \( T > t, N \) is the number of stages intercept term, \( i \) is the imaginary unit. In fact, the formula (16) is arithmetic mean of formula (14) and (15), i.e \( f(t) = 0.5[f_{i(0)}(t) + f_{i(i)}(t)] \), it can deal with the case...
of the discontinuous point of the original function. The choice of parameter \( c \) are significantly affect the accuracy of the calculation, in general, there are contrary bias between \( f_{(I)}(t) \) and \( f_{(II)}(t) \), when the parameter \( c \) is a reasonable choice, calculation relatively close; inappropriate choice when the parameter \( c \), two very different results, the calculation results show that the large error \([5,6]\), the use of this feature, the parameters can be identified reasonably selected, whereby the optimization model can be constructed as follows

\[
\min_{c \in \{0, \ldots, c_0\}} |f_{(I)}(t) - f_{(II)}(t)|
\]

(17)

This model is solved to obtain \( c \) and \( f(t) \).

5. Numerical examples

We use Glass/ED-6 composite experimental data\([7]\) to validate our modulus prediction methods. ED-6 resin matrix takes the shape of viscoelasticity, creep compliance is

\[
J(t) = \frac{1}{E_M} + \frac{t}{\eta_M} + \frac{1}{E_r} \left(1 - e^{-E_r/\eta_r t} \right)
\]

(18)

The material constants

\[
\begin{align*}
E_M &= 3.27 \text{GPa}, & E_r &= 1.8 \text{GPa} \\
\eta_M &= 8000 \text{GPa} \cdot h & \eta_r &= 300 \text{GPa} \cdot h \\
\nu_0 &= 0.38
\end{align*}
\]

Glass bead material constants

\[
K_t = 39.43 \text{GPa}, & G_t = 28.35 \text{GPa}
\]

Young's modulus of the matrix can be expressed as a Laplace transform domain

\[
E_o^{TM}(s) = \frac{E_M \eta_M (E_r + \eta_r s)}{E_M E_r + s(E_M \eta_M + E_r \eta_r) + s^2 \eta_M \eta_r}
\]

(19)

Figure 1 shows the creep curve of the 0.25 volume fraction of the composite material, wherein Fig 1 (a) is a graph of creep under hydrostatic stress load, and Fig 1 (b) shows the pure shear creep under load curve. Figure 2 shows the volume fraction of 0.54 when the creep curves composites. Figure 3 shows the Young's modulus and the shear modulus of 0.54 volume fraction of the composite material.
Figure 1. Creep curve (f=0.25)

Figure 2. Creep curve (f=0.54)

Figure 3. Effective modulus of the composite material (f=0.54)
6. Conclusion
Using MT method of micromechanics and elastic-viscoelastic correspondence principle can forecast modulus of composite material with viscoelastic matrix, which effectively solves the MT method micromechanics confined elastomer range of issues, as a new way for modulus forecast of viscoelastic composites. But the method of this paper only for two-phase composite materials are discussed, contains a variety of enhancements for the viscoelastic materials, this method cannot be used directly. Distribution iteration method can forecast modulus for multiphase elastomer composites.

For the viscoelastic composite material which contain many enhancements such as solid propellant must combine the method of this paper with the distribution iteration method to predicate effective modulus, this method is much more complex than modulus forecast of multiphase elastomer composites and two phases of viscoelastic composites, and this is the next step to continue research work.

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