PERIODIC SOLUTIONS OF INHOMOGENEOUS SCHRÖDINGER FLOWS INTO 2-SPHERE

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Abstract. In this paper, we consider the so called generalized inhomogeneous Schrödinger flows from a closed Riemann surface $M$ into the standard 2-sphere $S^2$ associated with the energy functional given by

$$E_{f, P}(u) = \int_M \left( \frac{1}{2} |\nabla u|^2 + P(u^3) \right) dV_g.$$ 

We showed the existence of special periodic solutions to the generalized inhomogeneous Schrödinger flows from $M$ with convolution symmetry (especially $M = S^2$) into $S^2$ when the function $f$ and $P$ satisfy certain conditions respectively. Especially, we show that the inhomogeneous Heisenberg spin chain system from a closed Riemann surface with convolution symmetry admits some special periodic solutions if the coupling function $f$ satisfies some suitable conditions. We also prove that there exist an infinite number of special periodic solutions to the Landau-Lifshitz system with an external magnetic field from $S^2$ into $S^2$.

1. Introduction. Let $(M, g_M)$ be a Riemannian manifold and $(N, J, h)$ be a Kähler manifold. For a smooth map $u : M \to N$ the energy is defined by

$$E(u) = \frac{1}{2} \int_M |\nabla u|^2 dV_{g_M}.$$ 

In local charts, we have

$$|\nabla u(x)|^2 = g_M^{ij}(x) \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\beta}{\partial x^j} h_{\alpha\beta}(u(x)).$$ 

The tension field of $u$ can be written as

$$\tau(u)^\alpha = \Delta_{g_M} u^\alpha + \Gamma_{\beta\gamma}^\alpha(u) \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\gamma}{\partial x^j} g_M^{ij}(x),$$

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where $\Delta_{g_M}$ is the Laplace-Beltrami operator on $(M, g_M)$, $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of the Riemannian connection on $(N, h)$. The following flow

$$\frac{\partial u}{\partial t} = J(u)\tau(u)$$

is called the Schrödinger flow on maps from $M$ into $N$ (see [5]). In some cases, the flow is equivalent to a nonlinear Schrödinger equation (see [3]). One has gotten some results on the existence of solutions, precisely we refer to [2, 5, 6, 16, 17, 24, 25].

In this paper, we consider a generalized Schrödinger flow

$$\frac{\partial u}{\partial t} = J(u)\tau_{f,P}(u),$$

here

$$\tau_{f,P}(u) = f\tau(u) + \nabla f \cdot \nabla u - \nabla P(u).$$

It is easy to see that the above flow preserves that following energy

$$E_{f,P}(u) = \int_M \left( \frac{1}{2}f|\nabla u|^2 + P(u) \right) dV_{g_M},$$

where $f$ is a smooth real function on $M$ and $P$ is a real function on $N$.

Let $S^2 \subset \mathbb{R}^3$ be the unit 2-sphere. Since $S^2$ is a Kähler manifold and

$$u \times : T_u S^2 \to T_u S^2 \quad (1)$$

is the standard complex structure on $S^2$, where $\times$ denotes the cross product in $\mathbb{R}^3$, then, the generalized Schrödinger flow from $M$ into $S^2$ can be written as the following equation

$$u_t = u \times \tau_{f,P}(u). \quad (2)$$

These flows are also closely related to mechanics and physics ([9, 13, 19]). For $f \equiv 1$ and $P \equiv 0$, the Schrödinger flow into $S^2$ is just the well-known Heisenberg spin chain system (also called ferromagnetic spin chain system or Landau-Lifshitz equations [19]) which arises in the description of the magnetization of a ferromagnet. In this case, the tension field can be expressed by

$$\tau(u) = \Delta_{g_M} u + |\nabla u|^2 u,$$

hence, the Landau-Lifshitz system can be written simply as

$$u_t = u \times \Delta_{g_M} u. \quad (3)$$

In the case the coupling function $f \neq 1$ and $P \equiv 0$, the flow (2) is just so-called inhomogeneous Heisenberg spin chain system, which also attracted one’s attention for long time. We refer to [17], [23] and references therein.

In the case $f \equiv 1$ and $P(u) = \frac{1}{2}(1 - u_3^2)$, the flow (2) still stems from physics and is called Landau-Lifshitz system with an external magnetic field([19]). Gustafson and Shatah discussed the existence and stability problem for the special periodic solutions to the Landau-Lifshitz system from $\mathbb{R}^2$ into $S^2$ in [9].

Recently, in [7, 24] Ding and Yin studied the existence of some special periodic solutions to the flow on maps from an oriented closed Riemann surface $M$, which has an isometric convolution $T : M \to M$ (i.e. $T$ is an isometry and $T^2 = I$) such that the fixed point set of $T$ consists of a finite number of embedded closed geodesics, into $S^2$. When $M = S^2$, they employed the Sacks-Uhlenbeck’s perturbation method, Palais’s symmetric criticality principle and so called odd extension to approach this problem and showed the following theorem.
Theorem A ([Ding-Yin]) The Schrödinger flow on maps from $S^2$ into $S^2$ admits for any $\lambda > 0$ an infinite number of inequivalent periodic solutions with period $\frac{2\pi}{\lambda}$.

Inspired by [7] we will consider the existence of the special periodic solutions to the generalized Schrödinger flows from an oriented closed Riemann surface, which has a certain symmetry of above isometric convolution, into $S^2$. We also need to assume that the $P(y)$ is a function of the third variable $y_3$ of $y$, i.e., $P(y) \equiv P(y_3)$. Then, in this case the energy functional is of the form

$$E_{f,P}(u) = \int_{M} \left( \frac{1}{2} f|\nabla u|^2 + P(u_3) \right) dV_M. \tag{4}$$

In this paper, we restrict us to the case the domain manifold $M$ is an oriented closed Riemann surface. Our task is to show the existence of some special periodic solutions to the generalized Schrödinger flow from $M$ with convolution symmetry (or $M = S^2$) into $S^2$ when $f$ and $P$ satisfy some reasonable conditions respectively.

In the present situation, we will encounter some new difficulties which come from the coupling function $f$ and the polynomial $P$. Especially, $f$ makes it harder than [7] to construct the supersolution and the subsolution of the related elliptic problem from $S^2$ into $S^2$. Fortunately, we can find considerably general sufficient conditions on the positive smooth function $f$ such that we may distinguish the different solutions corresponding different topological degree (see the subsection 3.2).

In order to state our main results, we choose the spherical polar coordinates $(r, \theta)$ centered at the north pole with metric $g = dr^2 + \sin^2 r d\theta^2$ on $M = S^2 \subset \mathbb{R}^3$. Then $f$ can be expressed as $f(r, \theta)$. We say that $f$ is radial symmetric if $f = f(r)$. We always assume that $P(u_3) = \sum_{i=0}^p a_i u_3^i$ is a polynomial with respect to $u_3$. Theorem 1.1. Assume $f$ is a positive radial symmetric function on $S^2$ and $f(r) = f(\pi - r)$. For any constant $\lambda > 0$ and any positive integer $d$, suppose that the coefficient $a_i$'s of the polynomial $P(u_3)$ satisfy:

1. $b_1 < -\sum_{i=0}^p 2^{i-1}|b_i|$, where we denote $\sum_{i=0}^p a_i(x - 1)^i - \lambda(x - 1) =: \sum_{i=0}^p b_i x^i$;
2. $\sum_{i=1}^p i a_i x^{i-1} - \lambda < 0$, $\forall x \in [-1,1]$;

and that the coupling function $f$ satisfies:

3. $\omega(f) := \max_{\Theta} f - \min_{\Theta} f < \delta$, where

$$\delta = \left( \exp \left( \frac{16\pi \sum_{i=1}^p i a_i + \lambda}{-\pi \sum_{i=1}^p 2^{i-1}|b_i|} \right) \frac{29d \min f}{\pi \sum_{i=1}^p i a_i + \lambda} \right)^{-1} \cdot \frac{29d \min f}{(4\pi + 29)d};$$

4. $f_r = \frac{\partial f}{\partial r} \leq K \sin r$, where $K = \min_{x \in [-1,1]} (\lambda - \sum_{i=1}^p i a_i x^{i-1}) > 0$.

Then the generalized Schrödinger flow (2) from $S^2$ into $S^2$ admits $d$ inequivalent periodic solutions of period $\frac{2\pi}{\lambda}$.

For the well-known inhomogeneous Heisenberg spin chain system, i.e., $f \not\equiv 1$ and $P(u_3) \equiv 0$, as a corollary we obtain the following which generalized Theorem A due to Ding and Yin ([7]):

Theorem 1.2. Suppose $f$ is a positive radial symmetric function which satisfies $f(r) = f(\pi - r)$ and $P(u_3) \equiv 0$. For any constant $\lambda > 0$ and any positive integer $d$, if

$$\omega(f) < \left( \exp \left( \frac{16\pi \lambda + 29d \min f}{\pi \lambda} \right) - 1 \right)^{-1} \cdot \frac{16\pi \lambda + 29d \min f}{(4\pi + 29)d},$$

then the generalized Schrödinger flow (2) from $S^2$ into $S^2$ admits $d$ inequivalent periodic solutions of period $\frac{2\pi}{\lambda}$. 
and
\[ f_r \leq \frac{\lambda \sin r}{d}, \]
then the inhomogeneous Schrödinger flow (2) from \( S^2 \) into \( S^2 \) admits \( d \) inequivalent periodic solutions with period \( \frac{2\pi}{\lambda} \).

For the Landau-Lifshitz model stated in [9], i.e., \( f \equiv 1 \) and \( P(u_3) = \frac{1}{2}(1 - u_3^2) \), as a corollary we have the following:

**Theorem 1.3.** For any constant \( \lambda > 1 \), if \( f \equiv 1 \) and \( P(u_3) = \frac{1}{2}(1 - u_3^2) \), then the generalized Schrödinger flow (2) from \( S^2 \) into \( S^2 \), i.e., the Landau-Lifshitz system, admits an infinite number of inequivalent periodic solutions with period \( \frac{2\pi}{\lambda} \).

We also have the following:

**Corollary 1.** For any constant \( \lambda > 0 \), if \( f \equiv 1 \) and \( P \) satisfies the conditions (1) and (2) in Theorem 1.1, then the generalized Schrödinger flow (2) from \( S^2 \) into \( S^2 \) admits an infinite number of inequivalent periodic solutions with period \( \frac{2\pi}{\lambda} \).

The rest of the paper is organized as follows. In the section 2 we discuss the variational reductions of the generalized Schrödinger flows. We prove our main theorem and corollaries in the section 3. We will show that when \( M \) has a convolution symmetry which splits \( M \) into two parts \( M_+ \) and \( M_- \), the reduced variational problem can be further reduced to one on \( M_+ \) or \( M_- \) with Dirichlet conditions. Employing the Sacks-Uhlenbeck’s perturbation method [22] we can show the direct minimization may lead to a nontrivial solution provided the infimum of the functional \( I \) satisfies an inequality (Lemma 3.1). By what we call the odd extension the original variational problem also be solved (The approach is similar with [7]). Then, in Theorem 3.1 we prove the key inequality holds true under some assumptions on \( f \), the parameter \( \lambda \) and the polynomial \( P \), hence the solution existence follows. In the last section we remark something on the stationary solution related to the flows.

2. The reduced variational problems for \( N = S^2 \). In this section, following the reduction method in [7] we will reduce equation (2) to an elliptic one by making use of the Killing vector field of the target manifold \( S^2 \). The solutions of the elliptic equation correspond to global solutions of (2). Usually we consider \( S^2 \) as a unit sphere in \( \mathbb{R}^3 \) centered at the origin. Then the rotation (anti-clockwise) around \( x_3 \)-axis produces a family of holomorphic isometries \( S_t \), which is periodic with period \( 2\pi \), i.e. \( S_t \equiv S_{t+2\pi} \). Precisely, we have

**Lemma 2.1.** Let \( S_t \) be the one-parameter group of rotations (holomorphic isometries) around \( x_3 \)-axis with \( S_0 = I \) (the identity map) and the corresponding holomorphic Killing vector field on \( S^2 \) denoted by \( V \). Then \( u(t) = S_t g \) with \( g : M \to S^2 \) is a solution to (2) if and only if \( g \) is a solution to the equation

\[ \tau_{f,P}(g) = -J(g)V(g). \] (5)

**Proof.** We know the Euler-Lagrange operator of \( E_{f,P} \) can be written by
\[ \tau_{f,P}(u) = \tau_f(u) - \nabla P(u_3), \]
where
\[ \tau_f(u) = f \tau(u) + \nabla f \cdot \nabla u = f(\Delta u + |\nabla u|^2 u) + \nabla f \cdot \nabla u, \]
and
\[ \nabla P(u_3) = \sum_{i=1}^{p} i a_i u_3^{i-1}(e_3 - u_3 u), \]
here \(e_3 = (0, 0, 1)\) is the north pole of \(S^2\). Since \(S_t\) is an one-parameter subgroup of the rotation group \(SO(2)\) on \(S^2\) which leaves the direction \(e_3\) fixed, then, for any \(t \in \mathbb{R}\) we can see easily that there holds
\[ e_3 - \langle e_3, S_t \circ g \rangle (S_t \circ g) = dS_t(e_3 - \langle e_3, g \rangle g), \]
then
\[ \nabla P(\langle S_t \circ g, e_3 \rangle) = dS_t \nabla P(\langle g, e_3 \rangle). \]

On the other hand,
\[ f \tau(S_t \circ g) + \nabla f \cdot \nabla(S_t \circ g) = f dS_t (\tau(g)) + \nabla f \cdot dS_t (\nabla g) = dS_t (f \tau(g)) + dS_t (\nabla f \cdot \nabla g), \]
then we have
\[ \tau_{f,p}(u) = \tau_{f,p}(S_t \circ g) = dS_t (\tau_{f,p}(g)). \quad (6) \]
Furthermore, we have
\[ u_t = \frac{\partial}{\partial t}(S_t \circ g) = V(S_t \circ g) = dS_t (V(g)), \]
The last equality is implied by the group property \(S_s \circ S_t = S_{s+t}\). In fact, differentiating the identity with respect to \(s\) at \(s = 0\) leads to
\[ dS_t (V) = \frac{d(S_t \circ S_s)}{ds} \big|_{s=0} = \frac{d(S_s \circ S_t)}{ds} \big|_{s=0} = V(S_t). \]
Hence, we obtain the following
\[ J(u)u_t = J(S_t \circ g) dS_t (V(g)) = dS_t (J(g)V(g)). \quad (7) \]
Here we used the fact that \(S_t\) is holomorphic, hence \(J \circ dS_t = dS_t \circ J\).
Combining (6)-(7), we get
\[ 0 = \tau_{f,p}(u) + J(u)u_t = dS_t (\tau_{f,p}(g) + J(g)V(g)). \]
Since \(dS_t\) is an isomorphism on \(S^2\), then we must have
\[ \tau_{f,p}(g) = -J(g)V(g), \]
we have proved the lemma. \(\square\)

Generally, it is difficult to solve the above equation (5) if it is not of a variational structure. In fact, in [1] Chen and Jost have ever considered the existence of the following
\[ \tau(u) = W(u) \]
where \(W\) is a smooth vector field defined on the target manifold \(N\). They established some existence results on the Dirichlet problems of the equation in the case the domain manifold \(M\) is a compact manifold with boundary and \(N\) is a compact manifold with nonpositive curvature.
Remark 1. It is well-known that $S^2$ is a Kähler-Einstein manifold with constant curvature $1$ and that $JV$ is the gradient of the first eigenfunction $F(u) = u_3$ (see [12]). We denote $JV$ by $\nabla F$ with the form

$$\nabla F(z) = e_3 - z_3 z.$$  

(8)

As $S_{\lambda t + 2\pi} = S_{\lambda t}$, then $S_{\lambda t}$ is of period $\frac{2\pi}{\lambda}$, so is the solutions of Schrödinger flow $S_{\lambda t} \circ g$. We have $\nabla F_\lambda = \lambda(e_3 - u_3 u)$, where $F_\lambda = \lambda u_3$. So the equation (5) becomes

$$\tau_{f, P}(g) = -\nabla F_\lambda(g),$$  

(9)

the Euler-Lagrange equation of the Energy Functional

$$I(v) = E_{f, P}(v) - \int_M F_\lambda(v) dV_g.$$  

(10)

For $P(u_3) = \sum_{i=0}^p a_i u_3^i$, it is easy to see that

$$\nabla P(z_3) = \sum_{i=1}^p i a_i (z_3)^{i-1}(e_3 - z_3).$$

By Reduction Lemma 2.1, the reduced equation can be given by

$$f(\Delta_g v + |\nabla v|^2_g v) + \nabla f \cdot \nabla v = (\sum_{i=1}^p i a_i (v_3)^{i-1} - \lambda)(e_3 - v_3),$$  

(11)

which is the Euler-Lagrange equation of the energy functional

$$I(v) = E_f(v) + \int_M (P(v_3) - \lambda v_3) dV_g.$$  

(12)

We are interested in the nontrivial solutions. Notice that if $P(v_3) = \lambda v_3$, the solutions are just the inhomogeneous harmonic maps which correspond to stationary ($t$-independent) solutions of inhomogeneous Schrödinger flows. Even if $P(v_3) \neq \lambda v_3$, (11) may have trivial solutions which maps $M$ into a single point so that both sides of (11) vanish. There are at least two such points on $S^2$, the north pole $\mathcal{N}$ and the south pole $\mathcal{S}$. Notice that (13) is replaced by the north pole $\mathcal{N}$.
And then if we have a smooth solution \( g_+ : M_+ \to S^2 \) of (11) which satisfy the boundary value condition (13), we can define the odd extension as

\[
g(x) = \begin{cases} 
g_+(x), & x \in M_+, \\S_\pi \circ g_+(T(x)), & x \in M_-.
\end{cases}
\]

**Lemma 2.2.** Assume \( f \circ T = f \). Then, \( g \) is a smooth solution of (11) on \( M \).

**Proof.** Since \( T : M \to M \) and \( S_\pi : S^2 \to S^2 \) are isometries, it is easy to see that \( g_-(x) = S_\pi \circ g_+(T(x)) \) is a solution to (11) on \( S_\pi \) satisfying the same boundary condition (13). On the boundary \( C \), condition (13) implies that tangent maps for both \( g_+ \) and \( g_- \) vanish along the tangent directions of \( C \), so \( g \) is \( C^1 \) along tangent directions. Then we need to check on the normal directions. Let \( n \) be the normal unit vector at a point \( y \in C \). We have

\[
dg_-(y)n = dS_\pi \circ dg_+(y) \circ dT(y)n.
\]

Since \( T \) is an isometric convolution and \( y \) is a fixed point of \( T \), we have \( dT(y)n = -n \).

On the other hand, since \( g_+(y) = S \in S^2 \) is fixed under the rotation \( S_\pi \) and \( dS_\pi = -I \) on \( TS \), we get that \( dg_-(y)n = dg_+(y)n \) i.e. \( g \) is \( C^1 \) along the normal directions too. On the other hand, \( f = f \circ T \) implies that \( df(y)n = 0 \).

Hence, it is easy to see that \( g \) is a smooth solution to (11) on \( M_+ \cup M_- \). Indeed, to verify that \( g \) is a weak solution to (11) in the sense of distribution it suffices to verify \( g \) is a distributional solution on any neighbourhood \( \Omega \subset M \) which satisfies that \( \Omega \cap C \neq \emptyset \). Since we have known that \( g \) is a smooth solution on \( M_+ \) and \( M_- \) respectively, for any \( \phi \in C^\infty_0(\Omega, g^{-1}TS^2) \), we have

\[
\int_{\Omega \cap M_+} \left[ f(\Delta g + |\nabla g|^2g) + \nabla f \cdot \nabla g - (\sum_{i=1}^p \lambda (g_3 g_3)^{i-1} - \lambda)(e_3 - g_3 g) \right] \phi dx = 0,
\]

and

\[
\int_{\Omega \cap M_-} \left[ f(\Delta g + |\nabla g|^2g) + \nabla f \cdot \nabla g - (\sum_{i=1}^p \lambda (g_3 g_3)^{i-1} - \lambda)(e_3 - g_3 g) \right] \phi dx = 0.
\]

Therefore, in view of the fact on \( C \)

\[
\frac{\partial g_+}{\partial n} f = \frac{\partial g_-}{\partial n} f,
\]

by integrating by parts we infer from the above integral equalities

\[
-\int_\Omega f \nabla g \cdot \nabla \phi dx + \int_\Omega \left[ f |\nabla g|^2g - (\sum_{i=1}^p \lambda (g_3 g_3)^{i-1} - \lambda)(e_3 - g_3 g) \right] \phi dx = 0.
\]

So, \( g \) is a weak solutions on \( M \). As \( g \) is \( C^1 \) smooth on \( M \), by the standard elliptic theory we know that \( g \) is a smooth solution of (11) on \( M \). \( \Box \)

3. The existence of special periodic solutions into \( S^2 \).

3.1. **An existence result.** For any positive integer \( d \), we define the function space

\[
X_d = \{ v \in H^1 \cap C^0(M_+, S^2) : v(C) = \{ S \}, \text{degree}(v) = d \}.
\]

The degree of \( v \) is considered like this: by the boundary condition (13), we treat \( M_+^2 \) as a closed manifold \( M_+^2 \), and \( v \) can be viewed as a continuous map from \( M_+^2 \) into
Then the degree is the standard degree for mappings between closed manifolds. It is known that (cf. [20])

\[ E(v) = \frac{1}{2} \int_{M^+_g} |\nabla v|^2 dV_g \geq 4\pi d. \] (15)

Then we have

**Lemma 3.1.** Let \( \lambda > 0 \) and

\[ m_d = \inf \{ I(v) : v \in X_d \}. \]

If

\[ m_d < 4\pi d \min f + (P(-1) + \lambda) \text{Vol}(M_+), \] (16)

then equation (11) has a nontrivial solution on \( M \).

**Proof.** Using the odd extension, we need only consider the case on \( M_+ \) with the boundary condition as (13). Following the perturbation method of Sacks-Uhlenbeck [22]. The perturbation functional is chosen as

\[ I_\alpha(v) = E_\alpha^f(v) + \int_{M_+} (P(v_3) - \lambda v_3) dV, \]

here \( \alpha > 1 \) and

\[ E_\alpha^f(v) = \frac{1}{2} \int_{M_+} f (|\nabla v|^2 + 1)^{\alpha - 1} dV. \]

For \( \alpha > 1 \), we define

\[ X^{\alpha}_d = \{ v \in W^{1,2\alpha}(M_+, S^2) : v(C) = S, \text{ degree}(v) = d \}. \]

As in [22], \( I_\alpha \) has a smooth minimizer \( g_\alpha \) in \( X^{\alpha}_d \). The problem is to study what may happen when \( \alpha \) goes to 1. Since \( 2\alpha > 2 \), the Sobolev embedding theory ensures that the integral

\[ \int_{M_+} (P(v_3) - \lambda v_3) dV \]

will not make any trouble to the convergence. So, the blow-up analysis of Sacks and Uhlenbeck (see [11, 22]) can be used here because \( f \) is smooth and positive. A similar argument as in [4, 11] says that, as \( \alpha \to 1 \), there may exist a finite number of points \((p_1, \cdots, p_k)\) in \( M_+ \) or on the boundary \( C \) satisfying \( g_\alpha \to g_1 \) in \( C^\infty(M_+ \setminus \{p_1, \cdots, p_k\}) \).

We have

\[ \lim_{\alpha \to 1} m^\alpha_d = \lim_{\alpha \to 1} I_\alpha(g_\alpha) = \lim_{\alpha \to 1} E_\alpha^f(g_\alpha) + \lim_{\alpha \to 1} \int_{M_+} (P((g_\alpha)_3) - \lambda(g_\alpha)_3) dV. \]

Since \( g_\alpha \) is weakly convergent to \( g_1 \) in \( H^{1,2} \), we have

\[ \lim_{\alpha \to 1} \int_{M_+} (g_\alpha)_3^2 dV = \int_{M_+} (g_1)_3^2 dV. \]

And since the degree of \( g_\alpha \) is \( d \), we get

\[ E_\alpha^f(g_\alpha) \geq E_f(g_\alpha) \geq E(g_\alpha) \min f \geq 4\pi d \min f. \]
Thus if \( g_1 \equiv S \) is constant, i.e. \( (g_1) \equiv -1 \) then
\[
\lim_{\alpha \to 1} m_d^\alpha \geq 4\pi d \min_f + (P(-1) + \lambda)Vol(M_+).
\]
On the other hand, using the fact that \( \cup_{n>1} X_d^\alpha \) is dense in \( X_d \), so
\[
m_d = \lim_{\alpha \to 1} m_d^\alpha \geq 4\pi d \min_f + \lambda Vol(M_+).
\]
It contradicts the assumption (16). So (16) gives us a nontrivial solution.

By the above discussion we know that \( f_1 \) is a smooth solution of (11) with finite many isolated singularities. If \( p_i \) is an interior singularity of \( g_1 \), the following Lemma 3.2 claims that \( p_i \) is removable as \( f \) is smooth and positive. If \( p_i \) is on the boundary of \( M_+ \), due to the special boundary condition, we can employ the odd extension method to extend the domain to a punctured disk around \( p_i \) and use Lemma 3.2 to remove the singularity. Hence \( g_1 \) is a smooth and the odd extension of \( g_1 \) is a smooth solution on entire \( M \). \( \square \)

**Lemma 3.2. (Removability of isolated singularities) Let \( D \) be the unit disk in \( \mathbb{R}^2 \), \( N \) be a closed Riemannian manifold. If \( u : D \setminus \{0\} \to N \) is a smooth map with finite energy and satisfies the following equation:
\[
\tau(u) = \beta \nabla u + g
\]
where \( \beta \in C^0(D) \) and \( g \in L^p(D, TN) \) for some \( p > 2 \), then \( u \) can be extended to the disk \( D \) as a \( W^{2,p} \) map. Moreover, if \( \beta \) and \( g \) is smooth, \( u \) can be extended smoothly.

The proof of the lemma was given in [11]. In the present case, we may apply it since \( f \) is smooth and \( g \) is smooth and bounded.

Next, we need to prove the following

**Theorem 3.3.** Let \( M \) be an oriented closed Riemann surface with convolution symmetry and \( f \) be a positive real function satisfying convolution symmetry, i.e. \( f \circ T = f \). Suppose that for \( \lambda \neq 0 \) the coefficients \( a_i \) of the polynomial \( P(\cdot) \) satisfy (1) in Theorem 1.1. Then there exist \( \delta > 0 \) such that, if \( \omega(f) = \max f - \min f < \delta \), (11) has at least one solution on \( M \).

**Proof.** Lemma 3.1 tells us that it is sufficient to verify (16).

Following [7] we will construct a test function in \( X_d \) to verify (16). Pick a point \( p \) away from the boundary \( C \), and choose an isothermal coordinate system \( (x, y) \) centered at \( p \) on a neighborhood \( D \) away from \( C \). For simplicity we assume \( D = \{(x, y) : |x^2 + y^2| < 1\} \). The metric \( g \) has the following form in \( D \)
\[
g = \phi(x, y)(dx^2 + dy^2) = \phi(r, \theta)(dr^2 + r^2 d\theta^2)
\]
with \( 0 < a \leq \phi(r, \theta) \leq b \)

We will define the test map \( u : M_+ \to S^2 \) which maps \( M_+ \setminus D \) into \( S \) and satisfies
\begin{enumerate}
  \item \( u(\partial D) = S \);
  \item \( u(D) \) covers \( S \) \( d \) times \( (d \geq 1) \);
  \item \( I(u) = E_{f,p}(u) - \lambda \int_D u_3 dV < 4\pi d \min f + (P(-1) + \lambda)Vol(D) \).
\end{enumerate}

We choose the map \( u : D \to S^2 \) so that it is rotationally symmetric and has the form
\[
u(r, \theta) = (\sin h(r) \cos d\theta, \sin h(r) \sin d\theta, \cos h(r)).
\]
We have

\[ E_f(u) \geq E(u) \min f = E(h) \min f = \pi \min f \int_0^1 (h_r^2 + \frac{d^2 \sin^2 h}{r^2}) r dr, \]

\[ \int_D u_3 dV = \int_0^1 \cos h(r) \varphi(r) r dr, \]

here

\[ 2\pi a \leq \varphi(r) = \int_{S^1} \phi(r, \theta) d\theta \leq 2\pi b. \]

Notice that \( E(u) \) is conformal invariant, so \( E(u) = E(h) \) is independent of \( \phi \).

Next we choose a family of test function \( h_c(r) \) with \( c > 0 \) as

\[ h_c(r) = 2 \arctan \frac{r^d}{c} + 2 \arctan(cr^{2d}). \]

It is easy to see that \( h_c(1) = \pi, \) so the corresponding \( u_c \) satisfies i) and ii).

We claim that if \( f \) satisfies

\[ \omega(f) \leq \left( \frac{\exp \left( \frac{8(\sum_{i=1}^p |a_i| + \lambda)Vol(D) + 29d \min f}{-\pi a(b_1 + \sum_{i=2}^{2^p-1} |b_i|)} \right)}{8(\sum_{i=1}^p |a_i| + \lambda)Vol(D) + 29d \min f} - 1 \right)^{-1}, \]

and the coefficient \( a_i \) of \( P(\cdot) \) satisfies \( b_1 < -\sum_{i=2}^{2^p-1} |b_i|, \) then there exists

\[ c^2 = \left( \frac{\exp \left( \frac{8(\sum_{i=1}^p |a_i| + \lambda)Vol(D) + 29d \min f}{-\pi a(b_1 + \sum_{i=2}^{2^p-1} |b_i|)} \right)}{8(\sum_{i=1}^p |a_i| + \lambda)Vol(D) + 29d \min f} - 1 \right)^{-1}, \]

such that

\[ I(h_c) < 4d \pi \min f + (P(-1) + \lambda)Vol(D). \]

In order to prove the above assertion we need to estimate the following integrals with respect to small parameter \( c \). By a direct computation we have

\[ (h_c)_r = \frac{2cd^d-1}{c^2 + r^{2d}} + \frac{4cd^{2d-1}}{1 + c^2 r^{2d}} \]

and

\[ (h_c)_r^2 r = \frac{4c^2 d^2 r^{2d-1}}{(c^2 + r^{2d})^2} + \frac{16c^2 d^4 r^{4d-1}}{(1 + c^2 r^{2d})^2} + \frac{16c^2 d^2 r^{2d}}{(c^2 + r^{2d})^2} = A_1 + B_1 + C_1. \]

For the first term of the above equality, we have

\[ \int_0^1 A_1 dr = 2dc^2 \int_0^1 \frac{d^2}{(c^2 + r^{2d})^2} = 2d - 2d \frac{c^2}{1 + c^2} < 2d, \]

and it is easy to see that

\[ \int_0^1 B_1 dr < \int_0^1 16c^2 d^2 r^{4d-1} dr = 4dc^2. \]

As for \( C_1 \), because

\[ \int_0^1 \frac{r^{3d-1}}{c^2 + r^{2d}} dr = \frac{1}{d} \int_0^1 \frac{r^{2d} dr}{c^2 + r^{2d}} = \frac{1}{d} \int_0^1 \frac{t^2}{c^2 + t^2} dt = \frac{1}{d} (1 - c \arctan \frac{1}{c}), \]
we get
\[ \int_0^1 C_1 \, d\rho < \frac{16c^2 d^2}{d} \left( 1 - c \arctan \frac{1}{c} \right) < 16dc^2. \]

Hence, we have
\[ \int_0^1 (h_c)^2 r \, d\rho < 2d + 20dc^2. \quad (19) \]

To estimate other integral terms in \( E(h_c) \), we need to make the following calculations.

\[ \sin h_c = \frac{2c^2 r^{2d}}{c^2 + r^{2d}} \left( 1 - \frac{c^2 - r^{2d}}{c^2 + r^{2d}} \right) + \frac{2c^{2d} r^{2d}}{1 + c^2 r^{2d}}, \]
\[ \sin^2 \frac{h_c}{r} = A_2 \left( \frac{1 - c^2 r^{4d}}{1 + c^2 r^{4d}} \right)^2 + B_2 \left( \frac{1}{1 + c^2 r^{4d}} \right)^2 + C_2 \left( \frac{1 - c^2 r^{4d}}{(1 + c^2 r^{4d})^2} \right), \]
where
\[ A_2 = \frac{4c^2 r^{2d-1}}{(c^2 + r^{2d})^2}, \]
\[ B_2 = r c^2 r^{4d-1} \frac{(c^2 - r^{2d})}{(c^2 + r^{2d})^2}, \]
\[ C_2 = \frac{8c^2 r^{3d-1}}{(c^2 + r^{2d})^2}. \]

We need only to calculate
\[ d^2 \int_0^1 A_2 \, d\rho = \int_0^1 \frac{2dc^2}{(c^2 + r^{2d})^2} \, dt = 2d - \frac{2dc^2}{1 + c^2} < 2d, \]
\[ d^2 \int_0^1 B_2 \, d\rho = \int_0^1 \frac{2dc^2 r^{2d}}{(c^2 + r^{2d})^2} \, dr = 2dc^2 \int_0^1 \frac{t(c^2 - t)^2}{(c^2 + t)^2} \, dt \leq 2dc^2 \int_0^1 t \, dt = dc^2. \]

By substitute \( r^d = ct \), we get
\[ d^2 \int_0^1 C_2 \, d\rho = 8dc^3 \int_0^{1/c} \frac{t^2(1 - t^2)}{(1 + t^2)^2} \, dt < 8dc^3 \int_0^{1/c} 1 \, dt = 8dc^2. \]

Then we have
\[ \int_0^1 d^2 \sin \frac{h_c}{r} \, d\rho < 2d + 9dc^2. \quad (20) \]

Combining (19)-(20), we get
\[ E(h_c) < 4d\pi + 29dc^2. \]

To estimate
\[ \int_D \cos(h_c) \, dV, \]
we calculate
\[
\cos h_c = \frac{c^2 - r^{2d}}{c^2 + r^{2d} + c^2 r^{4d}} - \frac{2c r^d}{c^2 + r^{2d} + c^2 r^{4d}} - \frac{2c r^{3d}}{c^2 + r^{2d} + c^2 r^{4d}}.
\]
Then,
\[
\int_D \cos(h_c) dV = \int_0^1 \left( -1 + \frac{2c^2}{c^2 + r^{2d} + c^2 r^{4d}} - \frac{2c r^d}{c^2 + r^{2d} + c^2 r^{4d}} - \frac{2c r^{3d}}{c^2 + r^{2d} + c^2 r^{4d}} \right) \phi(r) r dr
\]
\[
= -Vol(D) + \int_0^1 \left( \frac{2c^2}{c^2 + r^{2d} + c^2 r^{4d}} - \frac{2c r^d}{c^2 + r^{2d} + c^2 r^{4d}} - \frac{2c r^{3d}}{c^2 + r^{2d} + c^2 r^{4d}} \right) \phi(r) r dr,
\]
One can check that
\[
\frac{2c^2 r^{4d}}{c^2 + r^{2d} + c^2 r^{4d}} < 2c^2,
\]
\[
\frac{2c^2}{c^2 + r^{2d} + c^2 r^{4d}} < 4c^2,
\]
\[
\frac{4c^2 r^{3d}}{c^2 + r^{2d} + c^2 r^{4d}} < 4c^2,
\]
and
\[
\int_0^1 \frac{2c^2}{c^2 + r^{2d}} \phi(r) r dr \geq 4\pi a c^2 \int_0^1 \frac{r dr}{c^2 + r^2} = 2\pi a c^2 \log(1 + \frac{1}{c^2}).
\]
Then we obtain
\[
\int_D \cos(h_c) dV \geq -Vol(D) + 2\pi a c^2 \log(1 + \frac{1}{c^2}) - 8c^2 Vol(D).
\]
For the term
\[
\int_0^1 (P(\cos h_c) - \lambda \cos h_c) \phi(r) r dr,
\]
we denote
\[
\cos(h_c) = -1 + \frac{2c^2}{c^2 + r^{2d} + c^2 r^{4d}} - \frac{2c r^d}{c^2 + r^{2d} + c^2 r^{4d}} - \frac{2c r^{3d}}{c^2 + r^{2d} + c^2 r^{4d}} - \frac{4c^2 r^{3d}}{c^2 + r^{2d} + c^2 r^{4d}}
\]
\[
= A + B,
\]
where
\[
A = -1 + \frac{2c^2}{c^2 + r^{2d}}.
\]
\[
B = \frac{2c^2 r^{4d}}{c^2 + r^{2d} + c^2 r^{4d}} - \frac{2c^2}{c^2 + r^{2d} + c^2 r^{4d}} - \frac{4c^2 r^{3d}}{c^2 + r^{2d} + c^2 r^{4d}} + \frac{4c^2 r^{3d}}{c^2 + r^{2d} + c^2 r^{4d}}.
\]
Notice that
\[
|B| < 8c^2,
\]
and
\[ |(P(x) - \lambda x)| \leq \sum_{i=1}^{p}|a_i| + \lambda \]
on \([-1, 1]\). The mean value theorem says that
\[ P(A + B) - \lambda(A + B) \leq P(A) - \lambda(A) + 8c^2(\sum_{i=1}^{p}|a_i| + \lambda). \]
Then we infer
\[
\int_0^1 (P(\cos h_c) - \lambda \cos h_c) \varphi(r)r dr
\]
\[
\leq \int_0^1 P(-1 + \frac{2c^2}{c^2 + r^2}) \varphi(r)r dr
\]
\[
-\lambda \int_0^1 (-1 + \frac{2c^2}{c^2 + r^2}) \varphi(r)r dr + 8c^2(\sum_{i=1}^{p}|a_i| + \lambda)Vol(D)
\]
\[
= \int_0^1 \sum_{k=0}^{p} b_k(\frac{2c^2}{c^2 + r^2})^k \varphi(r)r dr + 8c^2(\sum_{i=1}^{p}|a_i| + \lambda)Vol(D)
\]
\[
\leq (P(-1) + \lambda)Vol(D)
\]
\[
+ \int_0^1 \sum_{k=0}^{p} b_k(\frac{2c^2}{c^2 + r^2})^k \varphi(r)r dr + 8c^2(\sum_{i=1}^{p}|a_i| + \lambda)Vol(D)
\]
\[
\leq (P(-1) + \lambda)Vol(D)
\]
\[
+ (b_1 + \sum_{i=2}^{p} 2^{i-1}|b_i|)2\pi ac^2 log(1 + \frac{1}{c^2}) + 8c^2(\sum_{i=1}^{p}|a_i| + \lambda)Vol(D),
\]
where the last inequality follows from
\[
\left| \frac{2c^2}{c^2 + r^2} \right| \leq 2.
\]
By the assumption \(a_i\) satisfies the condition (1) in Theorem 1.1, i.e.,
\[ b_1 + \sum_{i=2}^{p} 2^{i-1}|b_i| < 0. \]
Combining the last inequality with the estimates on the integral terms of \(E_{f, p}(h_c)\)
we obtain the following
\[
E_{f, p}(h_c) - \lambda \int_D \cos(h_c)dV < (4\pi + 29dc^2) \max f + (P(-1) + \lambda)Vol(D)
\]
\[
+ (b_1 + \sum_{i=2}^{p} 2^{i-1}|b_i|)2\pi ac^2 log(1 + \frac{1}{c^2})
\]
\[
+ 8c^2(\sum_{i=1}^{p}|a_i| + \lambda)Vol(D)
\]
\[
= 4\pi \min f + 29dc^2 \min f + (4\pi + 29)d\omega(f)
\]
\[
+ (P(-1) + \lambda)Vol(D)
\]
\[
+ (b_1 + \sum_{i=2}^{p} 2^{i-1}|b_i|)2\pi ac^2 log(1 + \frac{1}{c^2})
\]
\[
+ 8c^2(\sum_{i=1}^{p}|a_i| + \lambda)Vol(D).
\]
From the above inequality we can easily see that, to make the following inequality hold true
\[
E_{f, p}(h_c) - \lambda \int_D \cos(h_c)dV \leq 4\pi \min f + (P(-1) + \lambda)Vol(D),
\]
it is sufficient that $\omega(f)$ and $c$ satisfy simultaneously

$$(4\pi + 29)d\omega(f) + (b_1 + \Sigma_{i=2}^{p}2^{i-1}|b_i|)\pi ac^2 \log(1 + \frac{1}{c^2}) \leq 0$$

and

$$8c^2(\Sigma_{i=1}^{p}|a_i| + \lambda)Volf(D) + 29dc^2 \min f + (b_1 + \Sigma_{i=2}^{p}2^{i-1}|b_i|)\pi ac^2 \log(1 + \frac{1}{c^2}) = 0.$$  

From the last equality we obtain

$$\log(1 + \frac{1}{c^2}) = \frac{8(\Sigma_{i=1}^{p}|a_i| + \lambda)Volf(D) + 29d \min f}{-\pi a(b_1 + \Sigma_{i=2}^{p}2^{i-1}|b_i|)},$$

i.e.

$$c^2 = \left(\exp\left(\frac{8(\Sigma_{i=1}^{p}|a_i| + \lambda)Volf(D) + 29d \min f}{-\pi a(b_1 + \Sigma_{i=2}^{p}2^{i-1}|b_i|)} - 1\right)\right)^{-1}.$$  

From the above inequality (21), it follows

$$\omega(f) \leq \frac{-\pi a(b_1 + \Sigma_{i=2}^{p}2^{i-1}|b_i|)c^2 \log(1 + \frac{1}{c^2})}{(4\pi + 29)d}.$$  

Substituting (22) and (23) into the last inequality, we derive

$$\omega(f) \leq \frac{-\pi a(b_1 + \Sigma_{i=2}^{p}2^{i-1}|b_i|)c^2 \log(1 + \frac{1}{c^2})}{(4\pi + 29)d} = \left(\exp\left(\frac{8(\Sigma_{i=1}^{p}|a_i| + \lambda)Volf(D) + 29d \min f}{-\pi a(b_1 + \Sigma_{i=2}^{p}2^{i-1}|b_i|)} - 1\right)\right)^{-1}.$$  

Now, we set

$$\delta = \left(\frac{8(\Sigma_{i=1}^{p}|a_i| + \lambda)Volf(D) + 29d \min f}{-\pi a(b_1 + \Sigma_{i=2}^{p}2^{i-1}|b_i|)} - 1\right)^{-1}.$$  

then condition (3) is satisfied and we have finished the proof.

**Remark 2.** From the proof of Theorem 3.1, it is easy to see that $\delta$ depends only on $\min f$, the metrics on $D$, $d$, $P(\cdot)$ and $\lambda$. For the case $M = S^2$, the parameter $c$ depends only on $\min f$, $d$, $a_i$ and $\lambda$.

3.2. **Existence of several special periodic solutions on $S^2$.** Although we have had an existence result, we did not know whether there are blow-ups or not, the weak limit may be in a different homotopy class. In this subsection, we use the rotational symmetry of the domain manifold $S^2$ to ensure that we obtain different solutions for different $d$’s.

In the spherical polar coordinates centered at the north pole, the $S^1$-action as a rotation around $x_3$-axis is represented as

$$S_t(r, \theta) = (r, \theta + t).$$

A map $u : S^2_d \rightarrow S^2$ is called $S^1$-invariant if it satisfies for some $d \neq 0$

$$u = S_{-dt} \circ u \circ S_t, \; \forall \; t.$$  

(25)
Since \( S \) is fixed under the \( S^1 \)-action, it must be mapped to either \( S \) or \( N \) of the target \( S^2 \). we still use the Banach manifold 

\[
X^\alpha_d = \{ v \in W^{1,2\alpha}(S^2_+, S^2_+) : v(C) = \{ S \}, degree(v) = d \}.
\]

The sub-manifold of \( S^1 \)-invariant maps is 

\[
Z^\alpha_d = \{ u \in X^\alpha_d : u = S_{-dt} \circ u \circ St, \ \forall \ t \in \mathbb{R} \}.
\]

It is not hard to see that (25) implies that (24) since 

\[
\int \left[ \langle \nabla u \rangle^2 - \lambda^2 u^3 \right] d\sigma = \int \left[ \langle \nabla u \rangle^2 - \lambda^2 u^3 \right] d\sigma.
\]

Moreover, we have 

\[
I(\bar{u}) \leq I(u),
\]

when we minimize \( I_\alpha \) in \( \Sigma^\alpha_d \), we assume \( l(r) \equiv 0 \) since it will decrease \( E^\alpha_f \) by (27). Moreover, we have

**Lemma 3.4.** The function \( I_\alpha(u) \) on \( X^\alpha_d \) has a critical point in \( \Sigma \subset X^\alpha_d \) with \( l \equiv 0 \) and 

\[
0 \leq h(r) \leq \pi, \ r \in [0, \pi/2].
\]

**Proof.** Let \( u_n \) be a minimizing sequence for \( I_\alpha \) in \( \Sigma^\alpha_d \) with the form (26) where \( l_n = 0 \). We show that we can modify \( h_n \) into the form (28). In fact, for any \( u \) and corresponding \( h \), we can define 

\[
\bar{h}(r) = \begin{cases} 
  h(r), & h(r) < \pi, \\
  2\pi - h(r), & h(r) \geq \pi.
\end{cases}
\]

Check that the corresponding \( \bar{u} \) satisfies \( E^\alpha_f(\bar{u}) = E^\alpha_f(u) \) by (27). We also have 

\[
P(\bar{u}_3) = P(u_3) \quad \text{and} \quad \int_{S^2_+} \bar{u}_3dV = \int_{S^2_+} u_3dV \text{ by } \cos(2\pi - h) = \cos h.
\]

Then \( I(\bar{u}) = I(u) \) similarly, changing \( h_n \) into \( \bar{h}_n \) will not change the value of \( I_\alpha \). So we can assume \( h_n \leq \pi \). Next we may denote \( h = h_n \) and define 

\[
\bar{h} = \begin{cases} 
  h, & h \geq 0, \\
  0, & h < 0.
\end{cases}
\]

Since \( \lambda > 0 \), replacing \( h_n \) by \( \bar{h}_n \) will decrease \( I_\alpha \), so we can assume that \( h_n \) satisfies (28).

It is standard that we may assume that \( u_n \) converge weakly in \( W^{1,2\alpha} \) and uniformly in \( C^0 \) to some \( u_{\alpha} \). It follows that the corresponding \( h_\alpha \) satisfies (28). Moreover, since \( E^\alpha_f(u) \) is weakly lower-semi-continuous and \( P(u_3) \) is weakly continuous, \( u_\alpha \) is a critical point of \( I_\alpha \) in \( \Sigma^\alpha_d \). Finally, we note that \( I_\alpha \) is invariant under the \( S^1 \)-action on \( X^\alpha_d \), and \( \Sigma^\alpha_d \) is just the submanifold of fixed points of this action, by Palais’ Principle of symmetric criticality [14], \( u_\alpha \) is also a critical point of \( I_\alpha \) on \( X^\alpha_d \).
The arguments in the proofs of Lemma 3.2 and Theorem 3.1 can be used to show
that as α → 1, u_α converges weakly in H^1 to some u_1 which is a critical point of I
with the form (26) (l = 0), except h_1(0) ≠ 0. If h_1(0) ≠ 0, we must have h_1(0) = π.
Lemma 3.1 ensures us that u_1 is nontrivial, so h_1 is not identically π. In this case,
we will construct a “super solution” by u_1, then we can get a critical point u of I
with the form (26) that u ∈ X^1_β ⊂ X^1_α = X_δ.

Notice that if u is a critical point of I with the form (26), then the function h
satisfies the following Euler-Lagrange equation

\[ f(h_{rr} + \frac{\cos r}{\sin r} h_r - \frac{d^2 h \sin h}{\sin^2 r}) + f_r h_r = (-\sum_{i=1}^p i a_i \cos^{i-1} h + \lambda) \sin h. \] (29)

Assume h_1 is a solution of (29) and satisfies

\[ 0 ≤ h_1 ≤ \pi, \quad h_1(0) = \pi, \quad h_1(\pi/2) = \pi, \]

we will construct a weak super-solution [20] with h(0) = 0 as follow.

**Definition 3.5.** w(r) ∈ C^1(0, π/2) is called a weak super-solution of equation (29),
if ∀φ ≥ 0 in C^∞_φ(0, π), we have

\[ \int_0^{\pi/2} -f \sin r \phi_r h_r + \phi(-f \frac{d^2 \sin h}{\sin r} + \sin r \sin h(\sum_{i=1}^p i a_i \cos^{i-1} h - \lambda)) \, dr ≤ 0. \]

We must have h_1(r) > 0 for r ∈ (0, π/2], or otherwise h_1(r_0) = 0 at some
point r_0 ∈ (0, π/2]. h_1 achieves its minimum at r_0, so that (h_1)_r(r_0) = 0, by the
uniqueness of the solution of regular ODEs, h_1 ≡ 0, it is a contradiction.

Now we can construct the weak super-solution. Set

\[ \phi_c(r) = 2 \arctan[c(\tan \frac{r}{2})^d]. \]

This function is strictly increasing on r ∈ [0, π/2] with \( \phi_c(0) = 0 \) and \( \phi_c(\pi/2) < \pi. \) And it satisfies

\[ (\phi_c)_{rr} + \frac{\cos r}{\sin r} (\phi_c)_r - \frac{d^2 \sin \phi_c \cos \phi_c}{\sin^2 r} = 0. \]

In fact, the map corresponding \( \phi_c \) is just the S^1-invariant harmonic map of degree
d from S^2 to S^2. We will use this later. By continuity of h_1, there exist a constant
δ_1 so that h_1 > δ_1. For c small enough, h_1 > δ_1 > \phi_c. Since u_1 is nontrivial, h_1 is
not constant and must be smaller than π at some point. Then for c large enough,
there exists r ∈ (0, π/2) that h_1(r) < \phi_c(r). So there must be a c_0 in between such
that h_1(s) = \phi_{c_0}(s) at some s ∈ (0, π/2). Let s be the first point of intersection
between h_1 and \( \phi_{c_0} \), thus we have

\[ \frac{d(h_1(s))}{dr} ≤ \frac{d(\phi_{c_0}(s))}{dr}. \]

Then we denote

\[ w(r) = \begin{cases} \phi_{c_0}(r), & r < s, \\ h(r), & r ≥ s. \end{cases} \]

Notice that w is continuous at s and smooth elsewhere. If f satisfies

\[ f_r ≤ \frac{(-\sum_{i=1}^p i a_i \cos^{i-1}(\phi_{c_0}(r)) + \lambda) \sin \phi_{c_0}(r)}{\phi_{c_0}(r)_r} \]
on \([0, s]\), we can check that \(w\) is a super-solution on \([0, \pi/2]\) \(\{s\}\) because \(\phi_{co}\) and \(h_1\) are super-solutions on \([0, s]\) and \((s, \pi/2]\) respectively. So we have
\[
f(w_{rr} + \frac{\cos r}{\sin r} w_r - \frac{d^2}{\sin^2 r} \sin w \cos w) + f_r w_r + (\sum_{i=1}^{p} i a_i \cos^{i-1} w - \lambda) \sin w \leq 0
\]
on \([0, \pi/2]\) \(\{s\}\). Next, for any \(\phi \geq 0\) belonging to \(C_0^\infty([0, \pi/2])\), integrating by part in \([0, s]\), we have
\[
\int_0^s \left( f \sin r \phi w_{rr} + \phi \left( -f \frac{d^2}{\sin r} \sin w \cos w + \sin r \sin w(\sum_{i=1}^{p} i a_i \cos^{i-1} w - \lambda) \right) \right) dr \leq 0,
\]
then
\[
f(s) \sin s \phi(s) w_r(s) + \int_s^\pi \left( -f \sin r \phi_r w_r + \phi \left( -f \frac{d^2}{\sin r} \sin w \cos w + \sin r \sin w(\sum_{i=1}^{p} i a_i \cos^{i-1} w - \lambda) \right) \right) dr \leq 0.
\]
On \((s, \pi/2]\),
\[
\int_s^{\pi/2} \left\{ f \sin r \phi w_{rr} + \phi \left( -f \frac{d^2}{\sin r} \sin w \cos w + \sin r \sin w(\sum_{i=1}^{p} i a_i \cos^{i-1} w - \lambda) \right) \right\} dr \leq 0,
\]
i.e.
\[
-f(s) \sin s \phi(s) w_r(s) + \int_s^{\pi/2} \left( -f \sin r \phi_r w_r + \phi \left( -f \frac{d^2}{\sin r} \sin w \cos w + \sin r \sin w(\sum_{i=1}^{p} i a_i \cos^{i-1} w - \lambda) \right) \right) dr \leq 0.
\]
Moreover, noting
\[
\frac{d(h_1(s))}{dr} \leq \frac{d(\phi_{co}(s))}{dr},
\]
and combining the integral inequalities (29) on \([0, s]\) and (30) on \([s, \pi/2]\), we can see that \(w\) is a super-solution on the interval \([0, \pi/2]\).

Since \(w\) is a weak super solution and \(v = 0\) is a trivial sub-solution, and \(w > 0 = v\) on \((0, \pi/2]\), using Perron’s method directly ([20, 10]), one can show the existence of \(h\) which is a solution of (29) with \(h(0) = 0\).

The remaining is that under the assumptions in Theorem 1.1 we need to verify whether \(f_r\) satisfies the inequality appeared in the above discussion or not, and for different \(d\) whether the corresponding \(u_d\) differs from each other or not.

Fortunately, by a simple computation we have
\[
\sin \phi_{co}(r) = \frac{2c_0(\tan \frac{r}{2})^d}{1 + c_0^2(\tan \frac{r}{2})^{2d}} = \sin r, \quad \phi_{co}(r) = \frac{2c_0(\tan \frac{r}{2})^d}{1 + c_0^2(\tan \frac{r}{2})^{2d}} = \sin r.
\]
By the assumption of Theorem 1.1, we have
\[
f_r \leq g(r) = \frac{K \sin r}{d},
\]
where
\[
K = \min_{x \in [-1, 1]} \left\{ (\lambda - \sum_{i=1}^{p} i a_i x^{i-1}) > 0 \right\}.
\]
So, it ensures that the following inequality holds true
\[
fr \leq \frac{-\sum_{i=0}^{p}ia_i \cos^{i-1} \left( \phi_{co}(r) + \lambda \right) \sin \phi_{co}(r)}{\phi_{co}(r)} = \left( -\sum_{i=1}^{p}ia_i \cos^{i-1}(\phi_{co}(r)) + \lambda \right) \frac{\sin r}{d},
\]
since \( \phi_{co}(r) \in [0, \pi] \).

Notice that \( S^2_+ \) is isometric to \((D^2, \rho(|x|)ds^2)\), where \( ds^2 \) denotes the Euclidean metric of \( \mathbb{R}^2 \) and
\[
\rho(|x|) = \frac{4}{(1 + |x|^2)^{2}}.
\]
Thus, in Theorem 3.1 we have \( Vol(D) = 2\pi \) and \( a = 1 \), and then \( \delta \) can be written explicitly by
\[
\delta = \left( \exp \left( \frac{8(\sum_{i=1}^{p}i|a_i| + \lambda)Vol(D) + 29d \min f}{-\pi a(b_1 + \sum_{i=2}^{p}2^{i-1}|b_i|)} \right) - 1 \right)^{-1}
\]
\[
\times \frac{(4\pi + 29)d}{16\pi(\sum_{i=1}^{p}i|a_i| + \lambda) + 29d \min f},
\]
\[
\forall \lambda > 0, a_i satisfy conditions (1) and (2), and \( f \) satisfies \( \omega(f) < \delta \) and \( f_r \leq g(r) \) on \([0, s]\), we find a solution on \( S^2_+ \) of (29), by using the Reduction Lemma and the method of odd extension we get the special periodic solution from \( S^2 \) to \( S^2 \).

Moreover, notice that
\[
\left( \exp \left( \frac{16\pi(\sum_{i=1}^{p}i|a_i| + \lambda) + 29d \min f}{-\pi(b_1 + \sum_{i=2}^{p}2^{i-1}|b_i|)} \right) - 1 \right)^{-1}
\]
is decreasing with respect to \( d \), and
\[
\frac{16\pi(\sum_{i=1}^{p}i|a_i| + \lambda) + 29d \min f}{(4\pi + 29)d} = \frac{16\pi(\sum_{i=1}^{p}i|a_i| + \lambda) + 29 \min f}{4\pi + 29}
\]
is also decreasing with respect to the mapping degree \( d \), thus we know that \( \delta \) is decreasing with respect to \( d \). It is obvious that
\[
g(r) = \frac{K \sin r}{d}
\]
is decreasing with respect to \( d \) too. Therefore, for each positive integer \( d < d \), we can find a solution \( u_d \). As the solutions are of \( S^1 \)-invariant form (26), different \( d \) corresponds to different \( u \). Thus we have found \( d \) different periodic solutions and finish the proof of Theorem 1.1.

**Remark 3.** It is easy to see that for any fixed \( \lambda \) there exist many polynomials \( \lambda \) satisfy the conditions (1) and (2) in Theorem 1.1. It is clear that if \( (a_1 - \lambda) \) is small enough then the conditions is met. For example, for \( P(x) = 1 + a_1x + x^2 + x^3 \), to make (1) and (2) hold true, we only need \( a_1 < \lambda - 9 \).

**Remark 4.** In Theorem 1.1, while \( P \) satisfies some suitable conditions, the restriction on \( f \) is quite loose. It is easy to check that if \( a_1 \to -\infty \) or \( \lambda \to \infty \), we have \( \delta \to \infty \) and \( g(r)|_{r=\pi} \to \infty \). So, for sufficiently large \( -a_1 \) or sufficiently large \( \lambda \), we only need \( f \) to satisfy \( f_r|_{r=0} = f_r|_{r=\pi} = 0 \) besides smoothness, positivity and
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symmetry. For radial function \( f \), it is obvious that its gradient vanishes at the poles.

Next, we give the proofs of Theorem 1.2, Theorem 1.3 and Corollary 1:

The Proof of Theorem 1.2. Since \( P \equiv 0 \), we have \( a_i = 0 \). Then \( b_1 = -\lambda \) and \( b_i = 0 \) with \( i \geq 2 \). So, from the proof of Theorem 1.1 we can see easily

\[
\delta = \left( \frac{\exp \left( \frac{16\pi \left( \sum_{i=1}^{p} i |a_i| + \lambda \right) + 29d \min f}{\pi(b_1 + \sum_{i=2}^{p} 2^{i-1} |b_i|)} \right) - 1}{4\pi + 29d} \right)^{-1},
\]

and

\[
g(r) = \frac{K \sin r}{d} = \frac{\lambda \sin r}{d}.
\]

Thus we complete the proof of Theorem 1.2.

The Proof of Theorem 1.3. For the case \( P(u_3) = \frac{1}{2}(1 - u_3^2) \), it is easy to verify \( \lambda > 1 \) implies that (1) and (2) hold true. Since \( f \equiv 1 \), according to Theorem 1.1 it follows that (2) admits solutions from \( S^2 \) into \( S^2 \) and for different \( d \)'s we obtain different solutions. Then we get an infinite number of inequivalent periodic solutions, and finish the proof.

The Proof of Corollary 1. As \( f \equiv 1 \) and \( P \) satisfies the conditions (1) and (2), for every integer \( d > 0 \) Theorem 1.1 says (2) admits a corresponding \( S^1 \)-invariant periodic solution \( u_d \) and \( u_d \) is different from each other. Thus we get an infinite number of different solutions, and finish the proof.

4. Some remarks on stationary solutions. We know that the periodic solutions to the Schrödinger flows contain a class of special solutions, i.e. the stationary solutions (harmonic maps). Obviously, the stationary solutions of the flow (2) is just a harmonic map \( u \) from \( M \) into \( S^2 \) with a prescribed potential \( P(u) \), which satisfies the following

\[\tau(u) - \nabla P(u) = 0.\]

Especially, for the two dimensional Landau-Lifshitz system in [19] the above elliptic system can be written by

\[\Delta u + |\nabla u|^2 u + u_3(c_3 - u_3 u) = 0, \quad (32)\]

since the energy functional for this case is of the form

\[E_P(u) = \frac{1}{2} \int_{\partial M} \{ |\nabla u|^2 + (1 - u_3^2) \} dV_M = E(u) + \frac{1}{2} \int_{\partial M} (1 - u_3^2) dV_M.\]

Let’s recall that in [18] Peng and Wang have discussed the existence for such harmonic maps from \( S^2 \) into \( S^2 \) with a symmetric potential. Concretely, they obtained that if the potential \( P : S^2 \to \mathbb{R} \) is symmetric with respect to a plane containing the origin point of \( \mathbb{R}^3 \) and the restriction of \( P \) on the fixed point set with respect to the symmetry (i.e. the intersecting circle of \( S^2 \) and the plane) is constant, then the existence problem of the harmonic maps of degree 1 with the symmetric potential \( P \) is solvable. It follows that there exist solution maps from \( S^2 \) into \( S^2 \) of degree 1 to (32) as \( P(u) = \frac{1}{2} \int_{\partial M} (1 - u_3^2) \) is symmetric with respect to the plane \( (x_1, x_2, 0) \).
and $P(u)$ is constant on the circle $S^1 \equiv \{(x_1, x_2, 0) : x_1^2 + x_2^2 = 1\}$. More generally, if $P(u_3) = P(-u_3)$, then there exists a harmonic map $S^2 \to S^2$ of degree 1 with the potential $P$. It is natural to ask the following two problems:

**Question 1.** Can we prove that there exists a stationary solution from $S^2 \to S^2$ of degree $n \neq 1$ of the above Landau-Lifshitz system?

**Question 2.** If the domain manifold is an oriented closed Riemann surface with convolution symmetry and $P$ is invariant under the convolution transformation, can we show that (32) admits a solution?

As for the stationary solutions to inhomogeneous Schrödinger flows, we may also consider some similar questions with the above questions. For instance, we may discuss the existence of $f$-harmonic maps from an oriented compact Riemann surface $M$ (or $\mathbb{R}^2$) into a compact Riemannian (or Kähler) manifold.

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