Integer Echo State Networks: Hyperdimensional Reservoir Computing

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Abstract

We propose an integer approximation of Echo State Networks (ESN) based on the mathematics of hyperdimensional computing. The reservoir of the proposed Integer Echo State Network (intESN) contains only $n$-bits integers and replaces the recurrent matrix multiply with an efficient cyclic shift operation. Such an architecture results in dramatic improvements in memory footprint and computational efficiency, with minimal performance loss. Our architecture naturally supports the usage of the trained reservoir in symbolic processing tasks of analogy making and logical inference.

1 Introduction

This article addresses challenges of the two currently trending research directions: 1.) Binarization of computing architectures for neural information processing [1], [2] and 2.) Neural-symbolic integration [3].

Binarization of filters in convolutional neural networks have demonstrated enormous gains in memory and computational efficiency [4].

Reservoir computing (RC) [5] appeared as an efficient approach for the training process in Recurrent Neural Networks. It is a powerful tool for modeling and predicting dynamic systems both living [6] and technical [7, 8] systems.

Recent results on projecting neural activities of hidden layers of an artificial neural network into a framework for symbolic computations called hyperdimensional computing [9] allow making analogies of kind “What is the Automobile of Air?” directly on image data.

Hyperdimensional computing (HDC) [10] or vector symbolic architectures [11] are frameworks for neural symbolic representation, computation, and analogical reasoning. The distinction from traditional computing is that all entities (objects, phonemes, symbols) are represented by random vectors of very high dimensionality – several thousand dimensions. Complex data structures and analogical reasoning are implemented by simple arithmetical operations (multiplication, addition and permutation) and a well defined similarity metric [10].

We discovered several direct functional similarities between the operations of HDC and of RC. Specifically, these similarities are: 1.) Random projections of the input values onto a reservoir (which in essence is a high-dimensional vector) matches random vector-symbol representations stored in superposition; 2.) The update of the reservoir by a randomly generated reservoir connection matrix matches HDC binding/permutation operation; 3.) The nonlinearity of the reservoir can be matched to
thresholded addition of integers in HDC. We exploited these findings in Integer Echo State Networks (intESN), which perform like ESNs with much less memory and computational power, the major contribution of this paper.

In the proposed architecture, the reservoir of the network contains only \( n \)-bits integers for each neuron, reducing the memory of each neuron from a 32-bit float. The recurrent matrix multiply update is replaced by cyclic shift, which results in the dramatic boosting of the computational efficiency.

The intESN architecture is verified in a set of typical ESN tasks: memorizing of a sequence of symbols; learning a sinusoid; as well as predicting the complex dynamic behavior of Mackey-Glass series. All examples demonstrate satisfactory approximation of conventional ESN performance.

The paper is structured as follows. Background and related work are presented in Section 2. The main contribution - Integer Echo State Networks are described in Section 3. The performance evaluation follows in Section 4. Section 6 presents a discussion and conclusions.

2 Background and related work

There are many practical tasks that require maintaining the history of inputs to be solved efficiently. In the area of artificial neural networks (ANN), such tasks require working memory, which could be implemented by recurrent connections between neurons of a recurrent neural network (RNN). Historically, the task of training RNNs was considered to be much harder than that of feed-forward ANNs (FFNNs) due to vanishing gradient problem [12].

The challenge of training RNNs was addressed from two approaches. First, the vanishing gradient problem can be eliminated through neurons with special memory gates, as it is done in Long Short-Term Memory [13]. Another approach is to reformulate the training process by learning only connections to the last readout layer while keeping the other connections to be fixed. This approach appeared in two similar architectures: Liquid State Machines [14] and Echo State Networks [15]. Both architectures are referred to as reservoir computing (RC) [5]. It is interesting to note, that similar ideas were conceived in the area of FFNNs, which can be seen as an RNN without memory, and are known under the name of Extreme Learning Machines (ELMs) [16]. ELMs are used to solve various machine learning problems including classification, clustering, and regression [17].

Another recent research area is binary reservoir computing with cellular automata (CARC) which started as a interdisciplinary research within three areas: cellular automata, reservoir computing, and hyperdimensional computing. CARC was initially explored in [18] for projecting binarized features into high-dimensional space. Further, in [19] it was applied for modality classification of medical images. The usage of CARC for symbolic reasoning is explored in [20]. The memory characteristics of a reservoir formed by CARC are presented in [21]. Work [22] proposed the usage of coupled cellular automata in CARC.

2.1 Echo State Networks

This subsection briefly describes the conventional ESN. For the detailed tutorial on ESNs diligent readers are referred to [23]. Figure 1 depicts the architecture of the conventional ESN, which includes three layers of neurons. The input layer with \( K \) neurons represents the current value of input signal denoted as \( u(n) \). The output layer (\( L \) neurons) produces the output of the network (denoted as \( y(n) \)) during the operating phase. The reservoir is the hidden layer of the network with \( N \) neurons, with the state of the reservoir at time \( n \) denoted as \( x(n) \).

In general, the connectivity of the ESN is described by four matrices. \( W^{\text{in}} \) describes connections between the input layer neurons and the reservoir, and \( W^{\text{back}} \) does the same for the output layer. Both matrices project the current input and output to the reservoir. The memory in the ESN is due to the recurrent connections between neurons in the reservoir, which are described in the reservoir matrix \( W \). Finally, the matrix of readout connections \( W^{\text{out}} \) transforms the current activity levels in the input layer and reservoir (\( u(n) \) and \( x(n) \), respectively) into the network’s output \( y(n) \).

Note that three matrices (\( W^{\text{in}} \), \( W^{\text{back}} \), and \( W \)) are randomly generated at the network initialization and stay fixed during the network’s lifetime. Thus, the training process is focused on learning the readout matrix \( W^{\text{out}} \). There are no strict restrictions for the generation of projection matrices \( W^{\text{in}} \) and
Figure 1: Architecture of the conventional Echo State Network.

\( W^{\text{back}} \). They are usually randomly drawn from either normal or uniform distributions. The reservoir connection matrix, however, is restricted to possess the echo state property. This property is achieved when the spatial radius of the matrix \( W \) is less or equal than one. Here, the matrix is generated from a Gaussian distribution and then normalized by its maximal eigenvalue.

The update of the network’s reservoir at time \( n \) is described by the following equation:

\[
x(n) = \tanh(Wx(n-1) + W^{\text{in}}u(n) + W^{\text{back}}y(n-1)).
\]

(1)

Note that at each time step neurons in the reservoir apply \( \tanh() \) as the activation function. The nonlinearity prevents the network from exploding by restricting the range of possible values from -1 to 1. The activity in the output layer is calculated as:

\[
\hat{y}(n) = g(W^{\text{out}}[x(n); u(n)]),
\]

(2)

where the semicolon denotes concatenation of two vectors and \( g() \) the activation function of the output neurons, for example, linear or \( \tanh() \).

### 2.1.1 Training process

This paper only considers training with supervised-learning when the network is provided with the ground truth desired output at each update step. The reservoir states \( x(n) \) are collected together with the ground truth \( y(n) \) for each training step. The weights of the output layer connections are acquired by solving the regression problem which minimizes the mean square error between predictions (2) and the ground truth. While this paper does not focus on the readout training task, it should be noted that there are many alternatives reported in the literature including the usage of regression with regularization, online update rules, etc. [23].

### 2.2 Fundamentals of hyperdimensional distributed representations and computing

In a localist representation, which is used in all modern digital computers, a group of bits is needed in its entirety to interpret a representation. In hyperdimensional computing (HDC), all entities (objects, phonemes, symbols) are represented by vectors of very high dimensionality – thousands of bits. The information is spread out in a distributed representation, which contrary to the localist representations, any subset of the bits can be interpreted. Computing with distributed representations utilizes statistical properties of vector spaces with very high dimensionality, which allows for approximate, noise-tolerant, parallel computations. **Item memory** (also referred to as **clean-up memory**) is needed to recover superposed representations assigned to specific concepts. There are several flavors of HDC...
with distributed representations, differentiated by the random distribution of vector symbols, which can be real numbers [24, 25, 26, 27], complex numbers [28], binary numbers [10, 29], or bipolar [25, 30].

We rely on the mathematics of HD computing with bipolar dense distributed representations (BDDR) to develop the intESN, the fundamentals of which are are outlined below.

2.3 Computing with dense bipolar hyperdimensional distributed representations

[10] proposed the use of vectors comprising \( N = 10,000 \) binary elements (HD vectors). The values of each bit of an HD vector are independent equally probable, hence they are called dense distributed representations (DDR). Similarity between two binary DDR-vectors is characterized by Hamming distance, which (for two vectors) measures the number of elements in which they differ. In very high dimensions Hamming distances (normalized by the dimensionality \( N \)) between any arbitrary chosen HD vector and all other vectors in the HD space are concentrated around 0.5. Interested readers are referred to [10] and [31] for comprehensive analysis of probabilistic properties of the hyperdimensional representation space.

The binary dense representations can be equivalently mapped to the case of bipolar representations (BDDR), i.e. where each vector’s component is encoded as “-1” or “+1”. This definition is sometimes more convenient for purely computational reasons. The distance metric for the bipolar case is dot product:

\[
\text{dist} = \mathbf{x}^\top \mathbf{y}
\]

Atomic BDDR vectors are generated randomly and independently, and due to high-dimensions will be near orthogonal with high probability, with similarity (dot product) between such vectors approximately 0. A sequence of vector-symbols can be encoded in the reservoir from using the cyclic shift operation and bundling. The simplest bundling operation is an elementwise summation. When using the elementwise summation, the vector space of the reservoir is no longer bipolar, and from the implementation point of view, it can be practical to bound the values of the reservoir with a threshold value (denoted as \( \kappa \)). The bounding operation is called clipping:

\[
f_\kappa(x) = \begin{cases} 
-\kappa & x \leq -\kappa \\
-\kappa < x < \kappa \\
\kappa & x \geq \kappa \end{cases}
\]

The recovery of atomic vectors from their bundle is performed by finding the most similar vectors stored in the item memory. However, as more vectors are bundled together there is more interference noise and the likelihood of recovering the correct atomic vector declines.

3 Integer Echo State Networks

This section presents the main contribution of the paper – an architecture for Integer Echo State Network. The architecture is illustrated in Figure 2. The intESN is structurally identical to the conventional ESN (see Figure 1) with three layers of neurons: input (\( \mathbf{u}(n) \), \( K \) neurons), output (\( \mathbf{y}(n) \), \( L \) neurons), and reservoir (\( \mathbf{x}(n) \), \( N \) neurons). It is important to note from the beginning that training the readout matrix \( \mathbf{W}_{\text{out}} \) for intESN is the same as for the conventional ESN (Section 2.1.1).

However, other components of the intESN differs from the ESN. First, activations of input and output layers are projected into the reservoir in the form of bipolar dense vectors [26] (denoted as \( \mathbf{u}^{\text{HD}}(n) \) and \( \mathbf{y}^{\text{HD}}(n) \)). For problems where input and output data are described by finite alphabets and each symbol can be treated independently, the mapping to high-dimensional space is achieved by simply assigning a random bipolar vector to each new symbol and storing it in the item memory [10]. In the case with continuous data (e.g., real numbers), we quantized the continuous values into a finite alphabet. The quantization scheme (denoted as \( Q \)) and the granularity of the quantization are problem dependent. Additionally, when there is a need to preserve similarity between quantization levels, distance preserving mapping schemes are applied (see [32, 33]), which can preserve, for example, linear or nonlinear similarity between levels. An example of a discretization and quantization of a
continuous signal as well as its hyperdimensional representation in the item memory is illustrated in Figure 2.

Figure 2: Architecture of the conventional Echo State Network.

Another feature of the intESN is the way the recurrence in the reservoir is implemented. Rather than a matrix multiply, recurrence is implemented via the permutation of the reservoir vector. Note that permutation of a vector can be described in matrix form, which plays the role of $W$ in the intESN. Note that the spatial radius of this matrix equals one. An efficient implementation of permutation can be achieved for a special case – cyclic shift (denoted as $\text{Sh}()$). Figure 2 shows the recurrent connections of neurons in a reservoir with recurrence by cyclic shift of one position. In this case, vector-matrix multiplication $Wx(n)$ is equivalent to $\text{Sh}(x(n), 1)$.

Finally, the intESN uses different nonlinear activation function for reservoir’s neurons, clipping \((4)\). Clipping is characterized by the threshold value $\kappa$ regulating nonlinear behavior of the reservoir and limiting the range of activation values. Note that in the intESN the reservoir is updated only with integer bipolar vectors, and after the clipping the values of neurons are integers in the range between $-\kappa$ and $\kappa$. Thus, each neuron can be represented using only $\log_2(2\kappa + 1)$ bits of memory. For example, when $\kappa = 7$, there are fifteen unique values of a neuron, which can be stored with just four bits.

Summarizing the aforementioned differences, the update of intESN is described as:

$$x(n) = f_\kappa(\text{Sh}(x(n - 1), 1) + u^{\text{HD}}(n) + y^{\text{HD}}(n - 1)). \quad (5)$$

4 Performance evaluation

In this section, the intESN architecture is verified on a set of tasks typical for ESN. First, the memory capacities are compared using the trajectory association task \((28)\), introduced in the area of holographic reduced representations \((34)\). Next, the task of learning a simple dynamic process – a sinusoidal function. Finally, networks are trained to reproduce a complex dynamical system produced by a Mackey-Glass series. In all experiments below, the Moore-Penrose pseudo-inverse was used to learn the readout matrix $W^{\text{out}}$.

4.1 Sequence recall task

The sequence recall task includes two stages: memorization and recall. At the memorization stage, a network continuously stores a sequence of tokens (e.g., letters, phonemes, etc). The number of unique tokens is denoted as $D$ ($D = 27$ in the experiments), and one token is presented as input each timestep. At the recall stage, the network uses the content of its reservoir to retrieve the token stored $d$ steps ago, where $d$ denotes delay. In the experiments, the range of delay varied between 0 and 15.

For ESN, the dictionary of tokens was represented by a one-hot encoding, i.e. the number of input layer neurons was set to the size of the dictionary $K = D = 27$. The same encoding scheme was
Figure 3: The accuracy of the correct decoding of tokens for ESN and intESN for three different values of $N$.

adopted for the output layer, $L = 27$. The input vector was projected to the reservoir by the projection matrix $W^\text{in}$ where each entry was independently generated from the uniform distribution in the range $[-0.1, 0.1]$. The sparsity of the reservoir connection matrix $W$ does not affect the memory of the network. It was fixed to 10%. Nonzero elements of $W$ were generated from the uniform distribution. The spectral radius of the reservoir connection matrix was one.

For intESN, the item memory was populated with $D$ random high-dimensional bipolar vectors. The threshold for the clipping function was set to $\kappa = 3$. The output layer was the same as in ESN with $L = 27$ and one-hot encoding of tokens.

For each value of the delay $d$ a readout matrix $W^\text{out}$ was trained, giving 16 matrices in total. The training sequence presented 2000 random tokens to the network, and only the last 1500 steps were used to compute the readout matrices. The training sequence of tokens delayed by the particular $d$ was used as the ground truth for the activations of the output layer. During the operating phase, both the inclusion of a new token into the reservoir and the recall of the delayed token from the reservoir were simultaneous. Experiments were performed for three different sizes of the reservoir: $N = 100$, $N = 200$, and $N = 300$.

The memory capacity of the network is characterized by the accuracy of the correct decoding of tokens for different values of the delay. Figure 3 depicts the accuracy for both networks ESN (solid lines) and intESN (dashed lines). The capacities of both networks grow with the increased number of neurons in the reservoir. The capacities of ESN and intESN are comparable but feature slightly different behaviors. For all values of $N$ the accuracy of intESN starts to deviate from 100% earlier than that of ESN. On the other hand, ESN features steeper decay and at some point intESN shows higher accuracy. Eventually, all curves converge to the value of the random guess which equals $1/D$.

### 4.2 Learning Sinusoidal Function

Figure 4: Generation of a sinusoidal signal.

The task of learning a sinusoidal function is an example of a learning simple dynamic system with the constant cyclic behavior. The ground truth signal was generated as follows:

$$y(n) = 0.5 \sin(n/4).$$  \hspace{1cm} (6)
In this task, the input layer was not used, i.e. $K = 0$ but the network projected the activations of the output layer back to the reservoir using $W^{\text{back}}$. The output layer had only one neuron ($L = 1$). The reservoir size was fixed to $N = 1000$ neurons. The length of the training sequence was 3000 (first 1000 steps were discarded from the calculation). For ESN, the spectral radius of the reservoir connection matrix was scaled by 0.8. A continuous value of the ground truth signal was fed-in to the ESN during the training.

For intESN, in order to map the input signal to a bipolar vector the quantization was used. The signal was quantized as:

$$y(n)_q = \lfloor 100y(n) \rfloor / 100.$$  \hspace{1cm} (7)

The item memory for the projection of the output layer was populated with bipolar vectors preserving linear (in terms of dot product) similarity between quantization levels \cite{33}. The threshold for the clipping function was set to $\kappa = 3$.

In the operating phase, the network acted as the generator of the signal feeding its previous prediction (at time $n - 1$) back to the reservoir. Figure 4 demonstrates the behavior of intESN during the first 100 prediction steps. The ground truth is depicted by dashed line while the prediction of intESN is illustrated by the shaded area between 10% and 90% percentiles (100 simulations were performed). The figure does not show the performance of the standard ESN as it just follows the ground truth without visible deviations. The intESN clearly follows the values of the ground truth but the deviation from the ground truth is increasing with the number of prediction steps. It is unavoidable for the increasing prediction horizon but, in this scenario, it is additionally accelerated due to the presence of the quantization error at each prediction step. The next subsection will clearly demonstrate effects caused by this process. The error is accumulated because every time when feeding the prediction back to the reservoir of intESN it should be quantized in order to fetch a vector from the item memory.

4.3 Mackey-Glass series prediction

A Mackey-Glass series is generated by the nonlinear time delay differential equation. It is commonly used to assess the predictive power of an RC approach. In this scenario, we followed the preprocessing of data and the parameters of ESN described in \cite{7}. The parameters of intESN (including quantization scheme) were the same as in the subsection above. The length of the training sequence was 3000 (first 1000 steps were discarded from the calculation). Figure 5 depicts results for the first 300 prediction steps. The results were calculated from 100 simulation runs. The figure includes four panels. Each panel depicts the ground truth, the mean value of predictions as well as areas marking percentiles between 10% and 90%. The lower right corner corresponds to intESN while three other panels show performance of the ESN in three different cases related to the quantization of the data.

In these scenarios the ESN was trained to learn the model from the quantized data in order to see to which extent it affects the network. The upper left corner corresponds to the ESN without data quantization. The upper right corner
corresponds to the ESN trained on the quantized data but with no quantization during the operational phase. In such settings, the network closely follows the ground truth for the first 150 steps but then it often explodes. The lower left corner corresponds to the ESN where the data was quantized during both training and prediction. In this scenario, the network was able to produce good prediction just for the first few dozens of steps and then entered the chaotic mode where even the mean value does not reflect the ground truth. These cases demonstrate how the quantization error could affect the predictions especially when it is added at each prediction step. Note that the intESN operated in the same mode as the third ESN. Despite this fact, its performance rather resembles that of the second ESN where the speed deviation of the ground truth is faster. At the same time, the deviation of intESN grows smoothly without a sudden explosion in contrast to the ESN.

5 intESN in the scope of neural-symbolic integration

The complete coverage of the usage of intESN for neural-symbolic integration is a subject for a separate paper. In this section we want to give a taste of such usage. Hyperdimensional computing as proposed by Kanerva and further generalized as vector-symbolic architectures was intended for representing objects, predicates and rules for symbolic manipulation and inference. An example of operation, which can be done with HD vectors is called holistic mapping. It can be used to answer non-trivial queries (e.g., “What is the dollar of Mexico?”) by operating on the whole representation. The first attempt of integration of the reservoir computing with vector symbolic architectures was presented by Yilmaz [20]. Using cellular automata to project the weights of the final hidden layer of the trained network into a hyperdimensional vector, he was able to implement inference of type “What is is a horse of the sky” directly on images [9].

As an illustrative use case for the intESN in the context of analogical reasoning consider an artificial pipeline for functional modeling of cognitive behavior of honeybees from [36]. There hyperdimensional predicates were used in the reinforcement learning context for modeling correspondence between the reward and relationships between objects in a visual scene. A relationship between the objects in a scene with a square above a circle shape would be represented by a hyperdimensional vector constructed as a bundle of role-filler bindings, e.g. \( \text{rel}_{\text{visual}} = \text{role}_{\text{above}} \odot \text{square} + \text{role}_{\text{below}} \oplus \text{circle} \), where “\( \odot \)” denotes operation of binding and “\( + \)” denotes operation of bundling. While encoding discrete variables and shapes is rather straightforward in vector-symbolic architectures, working with continuous signals was not addressed previously. Now with intESN other “senses” represented by continuous signals can naturally be included in the VSA reasoning: \( \text{role}_{\text{tone}} \oplus \text{intESN}_{\text{sinus}} \).

6 Conclusions

In this article we proposed an architecture for integer approximation of the reservoir computing, which is based on the mathematics of hyperdimensional computing. The neurons in the reservoir are described by integers and the update operations include only summation and cyclic shift. Therefore intESN has substantially smaller memory footprint and better computational efficiency compared to the standard Echo State Network with the same number of neurons in the reservoir. The actual number of bits for representing a neuron depends on the clipping threshold \( \kappa \), but can be significantly lower than 32-bit floats in ESN. In all our experiments the results were obtained with \( \kappa = 3 \), which effectively makes it sufficient to represent a neuron with only three bits. We demonstrated that the performance of intESN is comparable to the standard Echo State Network both in terms of memory capacity and potential capabilities for modeling dynamic systems. Further improvements can be made by optimization of the parameters and better quantization schemes for handling continuous values. Naturally, due to the peculiarity of input data projection into the intESN, the accuracy of the network is to a certain degree lower than that of the conventional ESN. This, however, does not undermine the importance of intESNs, which are extremely attractive for memory and power savings, and in the general area of approximate computing, where errors and approximations are becoming acceptable as long as the outcomes have a well-defined statistical behavior.

References

[1] M. Courbariaux, I. Hubara, D. Soudry, R. El-Yaniv, and Y. Bengio. Binarized neural networks: Training neural networks with weights and activations constrained to +1 or -1. arXiv:1602.02830,
[2] I. Hubara, M. Courbariaux, D. Soudry, R. El-Yaniv, and Y. Bengio. Quantized neural networks: Training neural networks with low precision weights and activations. arXiv:1609.07061, pages 1–29, 2016.

[3] S. Bader and P. Hitzler. Dimensions of neural-symbolic integration - a structured survey. arXiv:0511042, pages 1–28, 2005.

[4] M. Rastegari, V. Ordonez, J. Redmon, and A. Farhadi. XNOR-Net: ImageNet Classification Using Binary Convolutional Neural Networks. In European Conference on Computer Vision, volume 9908 of Lecture Notes in Computer Science, pages 525–542, 2016.

[5] M. Lukosevicius and H. Jaeger. Reservoir computing approaches to recurrent neural network training. Computer Science Review, 3(3):127–149, 2009.

[6] F. Triefenbach, A. Jalalvand, B. Schrauwen, and J.-P. Martens. Phoneme recognition with large hierarchical reservoirs. In Advances in Neural Information Processing Systems 23, pages 2307–2315, 2010.

[7] H. Jaeger and H. Haas. Harnessing Nonlinearity: Predicting Chaotic Systems and Saving Energy in Wireless Communication. Science, 304(5667):78–80, 2004.

[8] L. Appeltant and M.C. Soriano and G. Van der Sande and J. Danckaert and S. Massar and J. Dambre and B. Schrauwen and C.R. Mirasso and I. Fischer. Information processing using a single dynamical node as complex system. Nature Communications, 2(1):1–6, 2011.

[9] O. Yilmaz. Analogy making and logical inference on images using cellular automata based hyperdimensional computing. In COCO’15 Proceedings of the 2015th International Conference on Cognitive Computation: Integrating Neural and Symbolic Approaches, volume 1583, pages 19–27, 2015.

[10] P. Kanerva. Hyperdimensional computing: An introduction to computing in distributed representation with high-dimensional random vectors. Cognitive Computation, 1(2):139–159, 2009.

[11] R.W. Gayler. Vector Symbolic Architectures Answer Jackendoff’s Challenges for Cognitive Neuroscience. In , editor, Proceedings of the Joint International Conference on Cognitive Science. ICCS/ASCS, pages 133–138, 2003.

[12] Y. Bengio, P. Simard, and P. Frasconi. Learning long-term dependencies with gradient descent is difficult. Neural Networks, IEEE Transactions on, 5(2):157–166, 1994.

[13] S. Hochreiter and J. Schmidhuber. Long short-term memory. Neural Computation, 9(8):1735–1780, 1997.

[14] W. Maass and T. Natschlager and H. Markram. Real-Time Computing Without Stable States: A New Framework for Neural Computation Based on Perturbations. Neural Computation, 14(11):2531–2560, 2002.

[15] H. Jaeger. Adaptive nonlinear system identification with echo state networks. In Advances in Neural Information Processing Systems 15, pages 593–600, 2003.

[16] G. Huang and Q. Zhu and C. Siew. Extreme learning machine: Theory and applications. Neurocomputing, 70(1-3):489–501, 2006.

[17] G. Huang. What are Extreme Learning Machines? Filling the Gap Between Frank Rosenblatt’s Dream and John von Neumann’s Puzzle. Cognitive Computation, 7:263–278, 2015.

[18] O. Yilmaz. Machine Learning Using Cellular Automata Based Feature Expansion and Reservoir Computing. Journal of Cellular Automata, 10(5-6):435–472, 2015.

[19] D. Kleyko, S. Khan, E. Osipov, and S. P. Yong. Modality Classification of Medical Images with Distributed Representations based on Cellular Automata Reservoir Computing. In IEEE International Symposium on Biomedical Imaging, pages 1–4, 2017.

[20] O. Yilmaz. Symbolic Computation Using Cellular Automata-Based Hyperdimensional Computing. Neural Computation, 27(12):2661–2692, 2015.

[21] S. Nichele and A. Molund. Deep Reservoir Computing Using Cellular Automata. arXiv:1703.02806, pages 1–9, 2017.
[22] N. McDonald. Reservoir computing and extreme learning machines using pairs of cellular automata rules. *arXiv:1703.05807*, pages 1–8, 2017.

[23] M. Lukosevicius. A Practical Guide to Applying Echo State Networks. In *Neural Networks: Tricks of the Trade*, volume 7700 of *Lecture Notes in Computer Science*, pages 659–686, 2012.

[24] T. A. Plate. Holographic reduced representations. *Neural Networks, IEEE Transactions on*, 6(3):623–641, 1995.

[25] S. I. Gallant and T. W. Okaywe. Representing objects, relations, and sequences. *Neural Computation*, 25(8):2038–2078, 2013.

[26] R. W. Gayler. Multiplicative binding, representation operators & analogy. In Gentner, D., Holyoak, K. J., Kokinov, B. N. (Eds.), *Advances in analogy research: Integration of theory and data from the cognitive, computational, and neural sciences*, pages 1–4, New Bulgarian University, Sofia, Bulgaria, 1998.

[27] S. I. Gallant and P. Culliton. Positional binding with distributed representations. In *International Conference on Image, Vision and Computing (ICIVC)*, pages 108–113, 2016.

[28] T. A. Plate. *Holographic Reduced Representations: Distributed Representation for Cognitive Structures*. Stanford: Center for the Study of Language and Information (CSLI), 2003.

[29] D. A. Rachkovskij. Representation and Processing of Structures with Binary Sparse Distributed Codes. *Knowledge and Data Engineering, IEEE Transactions on*, 3(2):261–276, 2001.

[30] A. Rahimi, S. Benatti, P. Kanerva, L. Benini, and J. M. Rabaey. Hyperdimensional biosignal processing: A case study for emg-based hand gesture recognition. In *2016 IEEE International Conference on Rebooting Computing (ICRC)*, pages 1–8, Oct 2016.

[31] P. Kanerva. *Sparse Distributed Memory*. The MIT Press, 1988.

[32] D. A. Rachkovskij, S. V. Slipchenko, E. M. Kussul, and T. N. Baidyk. Sparse Binary Distributed Encoding of Scalars. *Journal of Automation and Information Sciences*, 37(6):12–23, 2005.

[33] D. Widdows and T. Cohen. Reasoning with vectors: A continuous model for fast robust inference. *Logic Journal of the IGPL*, 23(2):141–173, 2015.

[34] Plate T. A. Holographic reduced representations. *Neural Networks, IEEE Transactions on*, 6(3):623–641, 1995.

[35] H. Jaeger. Tutorial on training recurrent neural networks, covering bptt, rtrl, ekf and the echo state network approach. Technical report, Technical Report GMD Report 159, German National Research Center for Information Technology, 2002.

[36] D. Kleyko, E. Osipov, R. W. Gayler, A. I. Khan, and A. G. Dyer. Imitation of honey bees’ concept learning processes using vector symbolic architectures. *Biologically Inspired Cognitive Architectures*, 14:57–72, 2015.