NOTES ON THE PHASE STATISTICS
OF THE RIEMANN ZEROS

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ABSTRACT. We numerically investigate, for zeros $\rho = 1/2 + i\gamma$, the statistics of the imaginary part of $\log(\zeta'(1/2 + i\gamma))$, computed by continuous variation along a vertical line from $\sigma = 4$ to $4 + i\gamma$ and then along a horizontal line to $1/2 + i\gamma$.

1. INTRODUCTION

One popular way [5] of visualizing a complex function $w = f(z)$ is to plot $\text{arg}(w)/2\pi$ interpreted as a color at each point $z$ in the domain. This is easily implemented in Mathematica. For the Riemann zeta function the excitement is all near the critical strip: for $1 \ll \text{Re}(s)$, $\zeta(s) \approx 1$ and so the image is monochrome in that region. For $\text{Re}(s) \ll 0$, $\zeta(s) = \pi^{s-1/2} \Gamma((1-s)/2) \Gamma(s/2) \zeta(1-s)$. Again $\zeta(1-s) \approx 1$. For bounded $\sigma$, as $t \to +\infty$, Stirling’s formula shows the argument of the remaining terms is asymptotic to $-t \log(t/2\pi) - t$, which means one sees very regular repeating horizontal bands of color. Meanwhile, near a zero $\rho$ in the critical strip, $\zeta(s) = \zeta'(\rho)(s - \rho) + O(s - \rho)^2$.

Near $\rho$, the image corresponding to the function $s - \rho$ is just the color wheel with all the colors coming together at $s = \rho$. Multiplying by $\zeta'(\rho)$ locally scales the picture by $|\zeta'(\rho)|$ and rotates it by $\text{arg}(\zeta'(\rho))$. See Figure 1 for an image with $7000 \leq t \leq 7010$. (What looks like a double zero is actually the first known example of a Lehmer pair near $t = 7005$.)

Thus the argument of $\zeta'(\rho)$ plays a significant role in the image, inspiring this MathOverflow question. In this paper we begin the numerical investigation of these by examining $5 \cdot 10^6$ zeros with $2.63012 \cdot 10^6 \leq \gamma \leq 4.99238 \cdot 10^6$. 


Starting first with $\zeta(s)$, an unpublished result of Selberg [4, p. 310] implies

**Theorem.** *Suitably normalized, $\arg(\zeta(1/2 + it))$ converges in distribution over fixed ranges to a standard normal variable. More precisely, for $\alpha < \beta$ we have*

\[
\lim_{T \to \infty} \frac{1}{T} \mu(\{ T \leq t \leq 2T \mid \alpha < \frac{\arg(\zeta(1/2 + it))}{\sqrt{\log \log(T)/2}} < \beta \}) = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} \exp(-x^2/2) \, dx,
\]

*where $\mu$ is Lebesgue measure.*
The argument is the imaginary part of $\log(\zeta(1/2 + it))$, computed by continuous variation along a vertical line from, say, $\sigma = 4$ to $4 + it$ and then along a horizontal line to $1/2 + it$. Selberg’s result actually covers the real part of the complex logarithm, $\log |\zeta(1/2 + it)|$ as well.

Returning to $\zeta'$, for the real part of the logarithm we have the following generalization of Selberg’s result, due to Hejhal [1]:

**Theorem.** Assuming the Riemann Hypothesis and a technical condition on the spacing of zeros which is a weak consequence of the Montgomery Pair Correlation Conjecture, then $\log |\zeta'(1/2 + it)|$, suitably normalized, converges in distribution over fixed ranges to a standard normal variable. More precisely, for $\alpha < \beta$ we have

$$\lim_{N \to \infty} \left| \frac{1}{N} \left\{ n : N \leq n \leq 2N, \alpha \leq \log \frac{2\pi \zeta'(1/2 + i\gamma_n)}{\log(\gamma_n/2\pi)} < \beta \right\} \right| = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} \exp(-x^2/2) \, dx$$

Here’s my attempt to make an exposition of Hejhal’s exposition [1, p. 346] of the basic idea behind the proof. First some notation: With

$$\chi(s) = \pi^{s-1/2} \frac{\Gamma((1-s)/2)}{\Gamma(s/2)},$$

define $\phi(s)$ by

$$\chi(s)^{-1/2} = \exp(i\phi(s)),$$

so $\phi$ is real on the critical line. Let $M$ be a large constant, $t$ an auxiliary random variable with $T \leq t \leq 2T$, and $W_t$ the ‘window’

$$W_t = [t - M/ \log T, t + M/ \log T].$$

Let $A(t) = t/2\pi \log (t/2\pi) - t/2\pi$. Let $x = A(u)$, and $\theta(u) = \phi(1/2 + iu)$. Let $P_t(x)$ be the polynomial approximation

$$P_t(x) = \Pi_{\gamma \in W_t} (x - A(\gamma)),$$

and define $\Omega_t(u)$ to be the correction to the approximation so that

$$\zeta(1/2 + iu) = \exp(\Omega_t(u) - i\theta(u))P_t(x).$$

Computing logarithmic derivative (in $u$, being careful with the chain rule) we see

$$i \frac{\zeta'(1/2 + iu)}{\zeta(1/2 + iu)} = \Omega'_t(u) - i\theta'(u) + \frac{P'_t(x)}{P_t(x)} \cdot A'(u).$$
Rearranging gives

\[ i \frac{\zeta'(1/2 + iu)}{A'(u)} = \zeta(1/2 + iu) \cdot \left( \frac{\Omega'_t(u)}{A'(u)} - i \frac{\theta'(u)}{A'(u)} + \frac{P'_t(x)}{P_t(x)} \right). \]

Hejhal makes an estimate (see below) of the term in parenthesis on the right to argue that

\[ \log |\zeta(1/2 + iu)| \quad \text{and} \quad \log \left| \frac{\zeta'(1/2 + iu)}{A'(u)} \right| \]

are (in effect) the same random variable, and so Selberg’s theorem applies.

For this estimate, Hejhal claims he and Bombieri showed previously that the total variation of \( \Omega_t(u) \) on \( W_t \) is \( O_M(1) \) for ‘most’ \( t \).

This means

\[ \int_{W_t} |\Omega'_t(u)| \, du = O_M(1) \]

for ‘most’ \( t \), and so on ‘most’ windows \( W_t, |\Omega'_t(u)| = O_M(\log T) \).

The above was the hard part; \( \theta'(u)/A'(u) \) is elementary. And \( P'_t(x)/P_t(x) = \sum_{\gamma \in W_t} 1/(x - A(\gamma)) \), with average spacing between \( A(\gamma) \) being 1 and the number of terms in the sum \( O_M(1) \). Hejhal argues heuristically that

\[ \log \left| \frac{\Omega'_t(u)}{A'(u)} - i \frac{\theta'(u)}{A'(u)} + \frac{P'_t(x)}{P_t(x)} \right| = O_M(1) \]

except for a subset of small measure. This completes Hejhal’s estimate.

Could this heuristic be extended to the imaginary part of \( \log \zeta'(1/2 + iu) \), defined (again) by continuous variation up the vertical line from 4 to 4 + \( iu \) and along the horizontal line from 4 + \( iu \) to 1/2 + \( iu \)? The challenge is that the estimates above depend on being inside a window of radius \( M/ \log T \). We will see below that the variation along the vertical line is quite regular, so that presents no problem. Along the horizontal line, between the real part 4 and the real part 1/2 + \( M/ \log T \), the zeros of \( \zeta' \) should be infrequent and so the argument of \( \zeta' \) should be changing slowly for ‘most’ values of \( u \).

3. ALGORITHM

The first 10^7 zeros of \( \zeta(s) \) are implemented constants in Mathematica, which can also, of course, easily compute derivatives numerically. Evaluating the argument via continuous variation requires a
little more effort. For a zero $\rho = 1/2 + i\gamma$, the variation along the line from 4 to $4 + i\gamma$ is easy. In this range

$$\zeta'(4 + iy) = -\log(2)2^{-4-iy} - \sum_{n=3}^{\infty} \log(n)n^{-4-iy}.$$ 

The first term has $|\log(2)2^{-4-iy}| = 0.0433217$, while the tail is smaller, bounded by 0.025590, and thus via Rouche theorem $\zeta'(4 + iy)$ winds around the origin as many times as does $-\log(2)2^{-4-iy}$.

Along the horizontal line $s = x + i\gamma$, $1/2 < x \leq 4$, we only need to estimate $\zeta'(s)$ very roughly, to determine when the argument increases by a multiple of $2\pi$. Since we compute at many equally spaced points along the line, directly computing each derivative in Mathematica is wasteful and slow. Instead we make a table of values of $\zeta(s)$ at the equally spaced points, and use a variant of Richardson Interpolation [3, 5.7] of the derivative:

$$(-\zeta(s - 3\Delta x) + 9\zeta(s - 2\Delta x) - 45\zeta(s - \Delta x) + 45\zeta(s + \Delta x) - 9\zeta(s + 2\Delta x) + \zeta(s + 3\Delta x)) / 60\Delta x = \zeta'(s) + O(\Delta x^6).$$

(In fact the next term of the $\Delta x$ series expansion of the left side is $\zeta^{(7)}(s)\Delta x^6/140$.) The derivative $\zeta'(1/2 + i\gamma)$ is computed with the built-in Mathematica implementation.

The step size $\Delta x$ needs to give a sufficiently accurate result even when the horizontal line passes close to a zero of $\zeta'(s)$. Recalling that the zeros of $\zeta'$ in $\text{Re}(s) > 1/2$ tend to be interspersed between the zeros of $\zeta(s)$ on the critical line, we looked for small gaps between

| $k$     | $\Delta$   |
|---------|------------|
| 8546951 | 0.00232317 |
| 5042996 | 0.00296997 |
| 9857600 | 0.00302828 |
| 9675304 | 0.00333645 |
| 7279824 | 0.00337943 |
| 7498518 | 0.00387655 |
| 7060975 | 0.00494052 |

**Table 1.** Small gaps between zeros $\rho_k$ of $\zeta(s)$, $5 \cdot 10^6 \leq k \leq 10^7$.
successive zeros $\rho_k, \rho_{k+1}$ for $5 \cdot 10^6 \leq k \leq 10^7$. With just seven exceptions (see Table 1), the gaps are all greater than 0.005. Based on this we choose for speed a step size $\Delta x$ of 0.0025, accepting that a very small number of phases may be computed incorrectly.

4. DATA

Hiary and Odlyzko [2] have investigated Hejhal’s theorem numerically, for data sets at much larger heights than we consider, and find the convergence rather slow. They also observe a surplus of large values and a deficit of small values. Since we have the data available, for completeness we include in Figure 2 a histogram of values, for $5 \cdot 10^6 \leq k \leq 10^7$, of

$$\frac{\log |2\pi \zeta' (\rho_k) / \log (\gamma_k / 2\pi)|}{\sqrt{\log \log (5 \cdot 10^6)}}.$$ 

Figure 3 is the numerical investigation of the argument, the main goal of the paper. For $5 \cdot 10^6 \leq k \leq 10^7$, the histogram displays

$$\frac{\arg \zeta' (\rho_k) + \pi - \gamma_k \log 2}{\sqrt{\log \log (5 \cdot 10^6)}}.$$

Mathematica computes the mean to be $-0.00043882$ and the standard deviation to be 2.47623. For what it is worth, the third through sixth moments were computed to be 0.00463054, 76.8629, 0.344781, and 1333.96 respectively.

Figure 4 shows both the real and imaginary parts of $\log \zeta' (\rho_k)$. Observe that the apparent surplus of examples with the imaginary part near $\pm \pi$ seems to correlate to the real part being positive and relatively large.

5. SUMMARY

Given the poor fit to a (mean 0) Gaussian for the data in Figure 2, perhaps not much can be conjectured from the data in Figure 3, beyond that there is a distribution for the argument computed by continuous variation. In other words, the naive conjecture that the data are uniform in $(-\pi, \pi]$ appears to be incorrect. We hope this paper inspires others with access to more computing power to investigate further.
Figure 2. $\log|\zeta'(\rho)|$.

Figure 3. $\arg(\zeta'(\rho))$. 
Figure 4. \( \log(\zeta'(\rho)) \).
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