Symmetric multiple chessboard complexes
and a new theorem of Tverberg type

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Abstract

We prove a new theorem of Tverberg type (Theorem 1.2) which confirms the conjecture of Blagojević, Frick, and Ziegler about the existence of ‘balanced Tverberg partitions’ (Conjecture 6.6 in, Tverberg plus constraints, Bull. London Math. Soc. 46 (2014) 953–967). Among the consequences is a positive answer to the ‘balanced case’ of the problem whether each admissible r-tuple is Tverberg prescribable. The proof relies on the connectivity and shellability properties of multiple chessboard complexes and their symmetric analogues. As revealed by the proof, the conditions in Theorem 1.2 are somewhat weaker than in the original conjecture and we show that the theorem is optimal in the sense that the new (weakened) condition is also necessary.

1 Introduction

Multiple chessboard complexes are studied in their own right [KRW, JVZ] as interesting graph complexes where (in the spirit of [J08]) the analysis of connectivity properties is one of the central themes.

The relevance of general chessboard complexes for theorems of Tverberg type is well known [BMZ, M03, VZ94, ZV92, Z11, Z04]. Perhaps it should not come as a surprise, as anticipated already in [JVZ], that multiple chessboard complexes are not an exception and that they should also quite naturally arise in this context.
In this paper we demonstrate that multiple chessboard complexes and their symmetrized versions, the symmetric multiple chessboard complexes, are indeed natural configuration spaces for the proof of new results of Tverberg type.

Our central new result is Theorem 1.2, originally motivated by the following conjecture.

**Conjecture 1.1 ([BFZ, Conjecture 6.6.])** Let $r \geq 2$ be a prime power, $d \geq 1$, $N \geq (r-1)(d+2)$, and $r(k+1)+s > N+1$ for integers $k \geq 0$ and $0 \leq s < r$. Then, for every continuous map $f : \Delta_N \rightarrow \mathbb{R}^d$, there are $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_N$ such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$, with $\dim \sigma_i \leq k+1$ for $1 \leq i \leq s$ and $\dim \sigma_i \leq k$ for $s < i \leq r$.

Theorem 1.2 establishes the conjecture in full generality. Moreover, it improves the conjecture in the sense that the condition $r(k+1)+s > N+1$ in Conjecture 1.1 is replaced in Theorem 1.2 by a weaker and possibly more natural condition $rk+s \geq (r-1)d$. This condition is indeed weaker since, by assuming both conditions from Conjecture 1.1, we have the inequalities,

$$rk+s > N+1-r \geq (r-1)(d+2)-(r-1) = (r-1)(d+1) > (r-1)d.$$

Observe that the condition $rk+s \geq (r-1)d$ is also necessary. Indeed it expresses the fact that if for a generic affine map $f$ the intersection $\cap_{i=1}^r f(\sigma_i)$ is non-empty then,

$$\text{codim}(f(\sigma_1)) + \cdots + \text{codim}(f(\sigma_r)) = \text{codim}(\cap_{i=1}^r f(\sigma_i)) \leq d.$$

**Theorem 1.2** Let $r \geq 2$ be a prime power, $d \geq 1$, $N \geq (r-1)(d+2)$, and $rk+s \geq (r-1)d$ for integers $k \geq 0$ and $0 \leq s < r$. Then for every continuous map $f : \Delta_N \rightarrow \mathbb{R}^d$, there are $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_N$ such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$, with $\dim \sigma_i \leq k+1$ for $1 \leq i \leq s$ and $\dim \sigma_i \leq k$ for $s < i \leq r$.

Theorem 1.2 is an example of a Tverberg type result where one can prescribe in advance (as in the classical van Kampen - Flores theorem) an upper bound $\dim(\sigma_i) \leq d_i$ on the dimension of simplices in a Tverberg $r$-partition for any map $f : \Delta_N \rightarrow \mathbb{R}^d$. More explicitly it addresses the case where the prescribed bounds $d_i$ are ‘balanced’ in the sense that $|d_i - d_j| \leq 1$ for each $i$ and $j$.

Theorem 1.2 is strong enough to cover the ‘balanced’ case of the general problem whether every admissible $r$-tuple is Tverberg prescribable (Question 6.9. in [BFZ]). An alternative approach to the ‘balanced’ case of this question, based on the equivariant index function and Sarkaria’s inequality, is given in Section 6.1.

As demonstrated in [BFZ] other interesting examples of constrained van Kampen - Flores theorems can be obtained by the elegant technique of ‘Tverberg unavoidable complexes’.

2
2 Symmetric multiple chessboard complexes

Suppose that \( k = (k_i)_{i=1}^n \) and \( p = (p_j)_{j=1}^m \) are two sequences of non-negative integers. The associated multiple chessboard complex

\[
\Delta_{m,n}^{k,p} = \Delta_{m,n}^{k_1,\ldots,k_n;p_1,\ldots,p_m}
\]

is following [KRW] defined as the complex of all rook-placements \( A \subset [m] \times [n] \) such that at most \( k_i \) rooks are allowed to be in the \( i \)-th row (for \( i = 1, \ldots, n \)), and at most \( p_j \) rooks are allowed in the \( j \)-th column (for \( j = 1, \ldots, m \)). As in [JVZ] we pay special attention to the complexes \( \Delta_{m,n}^{k,1} = \Delta_{m,n}^{k_1,\ldots,k_n;1} \) where \( p_1 = \ldots = p_m = 1 \).

Let \( G \subset S_n \) be a subgroup of the symmetric group \( S_n \) acting on the set of rows of the chessboard \([m] \times [n]\). The multiple chessboard complex \( \Delta_{m,n}^{k,1} \) is rarely \( G \)-invariant. Since the \( G \)-invariance of the configuration space is an essential feature of the usual Configuration Space/Test Map - scheme [Z04], it is quite natural to define a symmetric version of \( \Delta_{m,n}^{k,1} \).

**Definition 2.1** The \( G \)-symmetric multiple chessboard complex

\[
\Sigma(\Delta_{m,n}^{k,1}; G) = \Sigma(\Delta_{m,n}^{k_1,\ldots,k_n;1}; G) = \bigcup_{g \in G} \Delta_{m,n}^{k(g(1)),\ldots,k(g(n));1}
\]

is obtained from the multiple chessboard complex \( \Delta_{m,n}^{k_1,\ldots,k_n;1} \) by the symmetrization with respect to \( G \). In the special case \( G = S_n \) we obtain the complex,

\[
\Sigma_{m,n}^{k,1} = \Sigma_{m,n}^{k_1,\ldots,k_n;1} := \Sigma(\Delta_{m,n}^{k_1,\ldots,k_n;1}; S_n)
\]

which is referred to as the symmetric multiple chessboard complex.

**Remark 2.2** If one starts with more general multiple chessboard complexes \( \Delta_{m,n}^{k,p} = \Delta_{m,n}^{k_1,\ldots,k_n;p_1,\ldots,p_m} \) it may be sometimes more natural to symmetrize with respect to the group \( G = H \times Q \) where \( H \) permutes the rows and \( Q \) permutes the columns of the chessboard \([m] \times [n]\). In this paper we do not need these more general complexes.

3 The connectivity of \( \Sigma_{m,n}^{k,1} \)

One of the main results of [JVZ] is a lower bound for the connectivity of the multiple chessboard complex \( \Delta_{m,n}^{k,1} = \Delta_{m,n}^{k_1,\ldots,k_n;1} \).

**Theorem 3.1** ([JVZ Theorem 3.1.]) The generalized chessboard complex \( \Delta_{m,n}^{k_1,\ldots,k_n;1} \) is \( \mu \)-connected where,

\[
\mu = \min\{m - n - 1, k_1 + \cdots + k_n - 2\}
\]

In particular if \( m \geq k_1 + \cdots + k_n + n - 1 \) then \( \Delta_{m,n}^{k_1,\ldots,k_n;1} \) is \((k_1 + \cdots + k_n - 2)\)-connected.
Remark 3.2 As remarked already in [JVZ] the estimate \( \mu \geq m - n - 1 \) in (3) can be significantly improved for small values of \( m \). However, as in [JVZ], we are here mainly interested in the values of \( m \) for which the complex \( \Delta_{m,n}^{k_1,\ldots,k_n;1} \) is \((k_1 + \cdots + k_n - 2)\)-connected since this is precisely the result used in applications to generalized Tverberg theorems. For this reason our working hypothesis will be most of the time the inequality,

\[
m \geq k_1 + \cdots + k_n + n - 1
\]

Motivated by Conjecture 6.6. from [BFZ] (Conjecture 1.1), we focus our attention to the multiple chessboard complex \( K_1 = \Delta_{m,n}^{\nu+1,\nu+1,\nu,\ldots,\nu;1} \) of rook placements allowing at most \( \nu + 1 \) rooks in the first \( s \) rows, and at most \( \nu \) rooks in the remaining \( n - s \) rows. By Theorem 3.1 the complex \( K_1 \) is \( \mu \)-connected where \( \mu = \min\{m - n - 1, \nu n + s - 2\} \). Consequently \( K_1 \) is \( \mu \)-connected where \( \mu = \nu n + s - 2 \), provided the following equivalent of the inequality (4) is satisfied,

\[
m \geq n(\nu + 1) + s - 1
\]

Let us suppose that \( n = p^l \) is a prime power. The abelian group \( G = (\mathbb{Z}/p\mathbb{Z})^l \) acts freely on the rows of the chessboard \([m] \times [n]\). By symmetrization of \( K_1 \) with respect to the group \( G \) we obtain (Definition 2.1) the symmetrized multiple chessboard complex,

\[
\Sigma_G = \Sigma(\Delta_{m,n}^{\nu+1,\nu+1,\nu,\ldots,\nu;1}; G).
\]

More explicitly \( \Sigma = K_1 \cup \cdots \cup K_n \) is the union of \( n \) complexes where each \( K_i \) is obtained from \( K_1 \) by the corresponding permutation \( g \in G \), in particular the simplices of \( \Sigma \) are rook placements on the chessboard \([m] \times [n]\) where up to \( \nu + 1 \) rooks are allowed in some \( s \) rows and \( \nu \) rooks in the remaining \( n - s \) rows. Which combinations are allowed is governed by the action of group \( G \). Note that the group \( G = (\mathbb{Z}/p\mathbb{Z})^l \) acts freely on the complex \( K \).

We find it more convenient to symmetrize with respect to the full symmetry group so we focus our attention to the complex,

\[
\Sigma = \Sigma_{m,n}^{\nu+1,\nu+1,\nu,\ldots,\nu;1} = \Sigma(\Delta_{m,n}^{\nu+1,\nu+1,\nu,\ldots,\nu;1}; \mathfrak{S}_n).
\]

The action of the group \( G = (\mathbb{Z}/p\mathbb{Z})^l \) on the complex \( \Sigma \) is free as before.

We are interested in estimating the connectivity of the complex \( \Sigma = \Sigma_{m,n}^{\nu+1,\nu+1,\nu,\ldots,\nu;1} \). Theorem 3.3 provides such a result and shows that \( \Sigma \) has the same connectivity lower bound as the complex \( K_1 = \Delta_{m,n}^{\nu+1,\nu+1,\nu,\ldots,\nu;1} \).

We offer two approaches to the proof of Theorem 3.3. The first (partial proof of the case \( s \in \{1, n - 1\} \)) is based on the Nerve Lemma and goes along the lines of the proof of Theorem 3.1 given in [JVZ, Section 3]. The second proof is based on the shellability of the complex \( \Sigma \) established in Section 4. We include both proofs since they are independent and use different ideas. Moreover, we believe that a slightly refined version of Theorem 3.1 could be proved to allow the first proof to be extended to the general case of the Theorem 3.3.
Theorem 3.3 Suppose that,
\[ \Sigma = \Sigma_{m,n}^{k_1,\ldots,k_s,k_{s+1},\ldots,k_n;1} = \Sigma_{m,n}^{\nu+1,\ldots,\nu+1,\nu,\ldots,\nu;1} \]
is the symmetric multiple chessboard complex obtained by the \( S_n \)-symmetrization of the multiple chessboard complex \( K_1 = \Delta_{m,n}^{k_1,\ldots,k_s,k_{s+1},\ldots,k_n;1} \) where \( k_1 = \ldots = k_s = \nu + 1 \) and \( k_{s+1} = \ldots = k_n = \nu \). Assume that the following inequality (inequality (5)) is satisfied,
\[ m \geq n(\nu + 1) + s - 1. \] (7)
Then the complex \( \Sigma \) is \( \mu \)-connected where,
\[ \mu = k_1 + \cdots + k_n - 2 = \nu n + s - 2 \] (8)

**Proof 1:** (The case \( s = 1 \) or \( s = n - 1 \)) If \( s = 1 \), then all the complexes \( K_1,\ldots,K_n \) obtained by the symmetrization of \( \Delta_{m,n}^{k_1,\ldots,k_s,k_{s+1},\ldots,k_n;1} \) are \( \mu \)-connected, and the intersection of any \( t \geq 2 \) of them is a generalized chessboard complex \( \Delta_{m,n}^{\nu,1} \), which is at least \( (\mu - 1) \)-connected by Theorem 3.1. The Nerve lemma shows that \( K \) is also \( \mu \)-connected.

If \( s = n - 1 \), then all the complexes \( K_1,\ldots,K_n \) are \( \mu \)-connected, and the intersection of any \( t \geq 2 \) of them is a generalized chessboard complex allowing at most \( \nu + 1 \) rooks in some \( n - t \) rows and at most \( \nu \) rooks in the remaining \( t \) rows. This complex is, by Theorem 3.1, at least \( \min\{m-n-1,(\nu+1)n-t-2\} \)-connected, and the inequality \( \min\{m-n-1,(\nu+1)n-t-2\} \geq \min\{m-n-1,(\nu+1)n-1-2\}-(t-1) = \mu-(t-1) \) completes the proof. \( \Box \)

**Proof 2:** By Theorem 4.2 the complex
\[ \Sigma = \Sigma_{m,n}^{k_1,\ldots,k_s,k_{s+1},\ldots,k_n;1} = \Sigma_{m,n}^{\nu+1,\ldots,\nu+1,\nu,\ldots,\nu;1} \]
is shellable. Since \( \dim(\Sigma) = \nu n + s - 1 \) the complex \( \Sigma \) is \( (\nu n + s - 2) \)-connected and the result follows. \( \Box \)

## 4 Shellability of \( \Sigma_{m,n}^{k_1,\ldots,k_n;1} \)

The following theorem was proved in [JVZ].

**Theorem 4.1** For \( m \geq k_1 + k_2 + \cdots + k_n + n - 1 \) the complex \( \Delta_{m,n}^{k_1,\ldots,k_n;1} \) is shellable.

Here we show that the shelling order for the complex \( \Delta_{m,n}^{k_1,\ldots,k_n;1} \), described in [JVZ Section 4], can be extended to a shelling order on the symmetrization of \( \Delta_{m,n}^{k_1,\ldots,k_n;1} \) with respect to the full symmetric group \( \Sigma_n \).

**Theorem 4.2** Suppose that,
\[ \Sigma = \Sigma_{m,n}^{k_1,\ldots,k_s,k_{s+1},\ldots,k_n;1} = \Sigma_{m,n}^{\nu+1,\ldots,\nu+1,\nu,\ldots,\nu;1} \]
is the symmetric multiple chessboard complex obtained by the \( S_n \)-symmetrization of the multiple chessboard complex \( \Delta_{m,n}^{\nu+1,\ldots,\nu+1,\nu,\ldots,\nu;1} \) where \( k_1 = \ldots = k_s = \nu + 1 \) and \( k_{s+1} = \ldots = k_n = \nu \). Assume the inequality \( m \geq n(\nu + 1) + s - 1 \) (the condition 5) in Section 3. Then the complex \( \Sigma \) is shellable.
**Proof:** The dimension of the simplicial complex $\Sigma$ satisfies the inequality,

$$\dim(\Sigma) + 1 = n\nu + s \leq m - n + 1.$$  

By construction the complex $\Sigma$ is the union,

$$\Sigma = \bigcup \Delta_{a_1, \ldots, a_n:1}^{a_1, \ldots, a_n:1},$$

where $(a_1, \ldots, a_n)$ is an arbitrary permutation of $(k_1, \ldots, k_n) = (\nu + 1, \ldots, \nu + 1, \nu, \ldots, \nu)$. In other words $a_1 + a_2 + \cdots + a_n = n\nu + s \leq m - n + 1$, $a_i \leq \nu + 1$ for each $i$, and at most $s$ of the parameters $a_i$ can be equal $\nu + 1$. Observe that as a consequence of Theorem 1.2 all of the constituent complexes $\Delta_{m,n}^{a_1, \ldots, a_r:1}$ of $\Sigma$ are shellable.

A facet of $\Sigma$ is naturally encoded as an $n$-tuple $(A_1, A_2, \ldots, A_n)$ of disjoint subsets of $[m]$, each of size $\leq \nu + 1$, and at most $s$ of them may have precisely $\nu + 1$ elements. We define the shelling order for the facets of $\Sigma$ by the following construction refining the construction from [JVZ].

$F = (A_1, A_2, \ldots, A_n)$ is a predecessor of $F' = (B_1, B_2, \ldots, B_n)$ if either,

(a) $|A_i| = |B_i|$ for $i = 1, \ldots, i_0 - 1$ and $|A_{i_0}| > |B_{i_0}|$, or

(b) $|A_i| = |B_i| = a_i$ for all $i = 1, 2, \ldots, n$ and in the shelling order of $\Delta_{m,n}^{a_1, \ldots, a_r:1}$ the simplex $F$ is a predecessor $F'$.

Assume that $F = (A_1, A_2, \ldots, A_n)$ precedes $F' = (B_1, B_2, \ldots, B_n)$ in the given order. We are looking for a facet $F''$ that is a predecessor of $F'$ such that for a vertex $v$ of $F'$,

$$F \cap F' \subseteq F'' \cap F' = F' \setminus \{v\}. \quad (9)$$

(a) If $|A_i| = |B_i|$ for $i = 1, \ldots, i_0 - 1$ and $|A_{i_0}| > |B_{i_0}|$, then there exists $j > i_0$ such that $|B_j| > |A_j|$. Hence, there is a vertex $v$ in $B_j$ ($v$ is a square $(x, j)$ in the $j$-th row) that does not belong to $A_j$. Let $B'_0 = B_{i_0} \cup \{x\}$ and $B'_j = B_j \setminus \{x\}$. The facet $F'' = (B_1, \ldots, B'_0, \ldots, B'_j, \ldots)$ is clearly a predecessor of $F'$. Moreover, the facets $F, F'$ and $F''$ clearly satisfy the relation $(9)$.

(b) If $|A_i| = |B_i| = a_i$ for all $i = 1, 2, \ldots, n$ then the existence of $F''$ and $v$ follows from the shelling of the complex $\Delta_{m,n}^{a_1, \ldots, a_r:1}$. $\square$

### 5 Proof of Theorem 1.2

**Proof of Theorem 1.2:** We begin with the observation that the general case of Theorem 1.2 corresponding to the pair of conditions,

$$N \geq (r - 1)(d + 2) \quad rk + s \geq (r - 1)d \quad (10)$$

can be reduced to the case when the second inequality is actually an equality,

$$N \geq (r - 1)(d + 2) \quad rk + s = (r - 1)d \quad (11)$$

Indeed, suppose that for given \(d\) and \(r\) one chooses a pair \((k', s')\) satisfying inequalities (10). Suppose that \((k, s)\) is the unique pair satisfying the equality \(rk + s = (r - 1)d\) and the condition \(0 \leq s < r\). Then either \(k < k'\) or \(k = k'\) and \(s \leq s'\). In both cases the existence of a Tverberg \(r\)-tuple corresponding to the case (11) implies the existence a Tverberg \(r\)-tuple corresponding to the case (10) of the theorem.

Assuming (11) let us consider the simplicial complex,

\[
\Lambda = \Lambda(N, r; k, s) = \bigcup \Delta_{N+1,r}^{k_1,\ldots,k_r;1},
\]

where exactly \(s\) parameters \(k_i\) are equal \(k + 2\) and all other parameters are \(k + 1\). This complex is evidently the configuration space of all possible choices for desired Tverberg partitions with the dimensions constrained as above. A moment’s reflection reveals that \(\Lambda\) is precisely the symmetric multiple chessboard complex \(\Sigma_{m,n}^{\nu+1,\ldots,\nu+1,\nu,\ldots,\nu;1}\) from Theorem 4.2 for the following choice of parameters,

\[
m = N + 1, \quad n = r, \quad \nu = k + 1. \tag{12}
\]

If a desired Tverberg \(r\)-tuple does not exist, there exists a \(G\)-equivariant map

\[F : \Lambda \to \left(\mathbb{R}^d\right)^r_\Delta\]

where \(G = \left(\mathbb{Z}_p\right)^l\) is an elementary abelian group acting freely on \(\Lambda\) and without fixed points on \(\left(\mathbb{R}^d\right)^r_\Delta\). Moreover, there is \(G\)-equivariant homotopy equivalence, \(\left(\mathbb{R}^d\right)^r_\Delta \simeq S^D\) where \(D = (r - 1)(d + 1) - 1\). By a variant of Dold’s theorem [M03, Section 6.2.6] for fixed point free actions in the target space (established by Özaydin, Volovikov, Sarkaria, de Longueville, and other authors) this would lead to a contradiction if we are able to demonstrate that the complex \(\Lambda\) is \(D\)-connected.

By Theorem 3.3 it is sufficient to guarantee the following pair of inequalities,

\[
m \geq n(\nu + 1) + s - 1 \quad \nu n + s - 2 \geq (r - 1)(d + 1) - 1 \tag{13}
\]

for the choice of parameters given in (12). The second inequality in (13) is fulfilled in a tight way as the equality \(kr + s = (r - 1)d\). The first inequality in (13) is equivalent to the inequality,

\[N + 1 \geq r(k + 2) + s - 1 = (r - 1)(d + 2) + 1\]

which is precisely the first inequality in (11). This observation completes the proof of the theorem.

\[\square\]

6 Tverberg A-P problem

Here we briefly discuss the relation of Theorem 1.2 to the problem whether each admissible \(r\)-tuple is Tverberg prescribable. This problem, as formulated in [BFZ] and [B], will be referred to as the Tverberg A-P problem or the Tverberg A-P conjecture.
Definition 6.1 ([BFZ, Definition 6.7]) For \( d \geq 1 \) and \( r \geq 2 \), an \( r \)-tuple \( d = (d_1, \ldots, d_r) \) of integers is admissible if,

\[
[d/2] \leq d_i \leq d \quad \text{and} \quad \sum_{i=1}^{r} (d - d_i) \leq d.
\]

An admissible \( r \)-tuple is Tverberg prescribable if there is an \( N \) such that for every continuous map \( f : \Delta_N \to \mathbb{R}^d \) there is a Tverberg partition \( \{\sigma_1, \ldots, \sigma_r\} \) for \( f \) with \( \dim(\sigma_i) = d_i \).

Question: (Tverberg A-P problem; [BFZ, Question 6.9.]) Is every admissible \( r \)-tuple Tverberg prescribable?

The balanced case of the Tverberg A-P conjecture is settled by Theorem 1.2 if \( r \) is a prime power. Here we offer a different proof based on Sarkaria’s inequality [M03, Proposition 6.2.4]. There is a possibility that a refinement of these methods may lead to other cases of the Tverberg A-P conjecture.

In the Tverberg A-P problem the emphasis is on the existence (rather than the size) of \( N \) which makes a given \( r \)-tuple admissible (Definition 6.1). Nevertheless note that the upper bound on \( N \) provided by Theorem 1.2 is better than the bound that follows from the argument based on Sarkaria’s inequality.

6.1 The balanced case of the A-P conjecture

The balanced case of the Tverberg A-P conjecture is the case when the dimensions \( d_1, \ldots, d_r \) satisfy the condition \( |d_i - d_j| \leq 1 \) for each \( i \) and \( j \). In other words there exist \( 0 \leq s < r \) and \( k \) such that \( d_1 = \ldots = d_s = k + 1 \) and \( d_{s+1} = \ldots = d_r = k \). In this case the second admissibility condition in (14) reduces to the inequality,

\[
 rk + s \geq (r - 1)d
\]

while the first condition is redundant.

Theorem 6.2 Suppose that \( r = p^k \) is a prime power and let \( d = (d_1, \ldots, d_r) \) be a sequence of integers satisfying the condition \( |d_i - d_j| \leq 1 \) for each \( i \) and \( j \). Then if the sequence \( d \) is admissible (Definition 6.1) it is Tverberg prescribable.

Proof: Let us initially carry on the proof in the case when \( r \) is a prime number. For \( \nu = k + 1 \) and \( 1 \leq s < r \) let \( \alpha_s \) be the \( r \)-sequence defined as follows,

\[
\alpha_s(i) = \begin{cases} 
\nu + 1, & i = 1, \ldots, s \\
\nu, & i = s + 1, \ldots, r.
\end{cases}
\]

Moreover, let \( \alpha \) be the constant sequence \( \alpha(i) = \nu + 1 \) for each \( i \).
Suppose that $\Sigma_{m,r}^{1}$ and $\Sigma_{m,r}^{\ast}$ are the associated symmetric multiple chessboard complexes. By applying Sarkaria’s inequality \cite[Proposition 6.2.4]{M03} to the complex $L_0 = \Sigma_{m,r}^{1}$ and its subcomplex $L = \Sigma_{m,r}^{\ast}$ we observe that,

$$\text{Ind}_G(\Sigma_{m,r}^{\ast}) \geq \text{Ind}_G(\Sigma_{m,r}^{1}) - 1. \quad (17)$$

Indeed this is an immediate consequence of the fact that,

$$\dim(\Delta(L_0 \setminus L)) = 0 = \text{Ind}_G(\Delta(L_0 \setminus L)).$$

Let us apply Sarkaria’s inequality again, this time to the complex $L_0 = \Sigma_{m,r}^{\ast}$ and its subcomplex $L = \Sigma_{m,r}^{1}$.

**Lemma 6.3** Let $A = (A_1, \ldots, A_r)$ be a simplex in $L_0 = \Sigma_{m,r}^{\ast}$ and let

$$X_A = \{i \in [r] \mid \alpha_A(i) := |A_i| = \nu + 1\}.$$ 

Then $A \in L_0 \setminus L = \Sigma_{m,r}^{\ast} \setminus \Sigma_{m,r}^{1}$ if and only if,

$$s + 1 \leq |X_A| < r. \quad (18)$$

Let $Q_s^r = \{Z \subset [r] \mid s + 1 \leq |Z| < r\}$. The map $X : L_0 \setminus L \to Q_s^r$ which sends $A$ to $X_A$ is monotone and $G$-equivariant. Moreover, $\dim(Q_s^r) = r - s - 2$. It follows from the monotonicity of the index that,

$$\text{Ind}_G(\Delta(L_0 \setminus L)) \leq \text{Ind}_G(\Delta(Q_s^r)) \leq r - s - 2 \quad (19)$$

and,

$$\text{Ind}_G(\Sigma_{m,r}^{1}) \geq \text{Ind}_G(\Sigma_{m,r}^{\ast}) - r + s + 1. \quad (20)$$

From here and the inequality (17) we finally obtain the inequality,

$$\text{Ind}_G(\Sigma_{m,r}^{1}) \geq \text{Ind}_G(\Sigma_{m,r}^{\ast}) - r + s. \quad (21)$$

Let us assume the inequality,

$$m \geq r(\nu + 1) + r - 1. \quad (22)$$

By \cite[Theorem 3.1]{JVZ} we know that the complex $\Sigma_{m,r}^{1} = \Delta_{m,r}^{1}$ is $\mu$-connected and $(\mu + 1)$-dimensional where $\mu = r(\nu + 1) - 2$. It follows that,

$$\text{Ind}_G(\Sigma_{m,r}^{1}) = \mu + 1 = r(\nu + 1) - 1 \quad (23)$$

and in light of (21),

$$\text{Ind}_G(\Sigma_{m,r}^{1}) \geq \nu s + 1 = rk + s + (r - 1). \quad (24)$$

In order to show that $\alpha_s$ is Tverberg prescribable we should prove that $\text{Ind}_G(\Sigma_{m,r}^{1}) > D$ where $D = (r - 1)(d + 1) - 1$ is the dimension of the target sphere (in the ‘deleted join’ approach to the Tverberg problem \cite{M03}). This inequality is precisely the inequality (15) which completes the proof of the theorem if $r$ is a prime number.

The case of the prime power $r = p^k$ is handled in a similar way. Instead of $\text{Ind}_G$ we use a modified index function $\text{Ind}_G^{\ast}$ described in Proposition 6.4. \hfill \Box
Proposition 6.4 Let \( r = p^k \) be a prime power and let \( G = (\mathbb{Z}_p)^k \) be an elementary abelian \( p \)-group. Let \( \mathcal{C}_G \) be the category of finite, not necessarily free \( G \)-complexes with \( G \)-equivariant maps as morphisms. There exists a sequence \( \{ A_n G \}_{n=0}^{\infty} \) of finite \( G \)-spaces \( \dim(A_n G) = n \) such that the associated index defined on \( \mathcal{C}_G \) by the formula,

\[
\text{Ind}^E_G(X) := \inf\{ n \in \mathbb{N} \mid X \xrightarrow{G} A_n G \}.
\]

has the following properties.

1. \( \text{Ind}^E_G(X) \in \mathbb{N} \cup \{+\infty\} \).
2. \( \text{Ind}^E_G(X) < +\infty \) if \( X \) is a free \( G \)-complex.
3. If \( X \xrightarrow{G} Y \) then \( \text{Ind}^E_G(X) \leq \text{Ind}^E_G(Y) \).
4. \( \text{Ind}^E_G(X \ast Y) \leq \text{Ind}^E_G(X) + \text{Ind}^E_G(Y) + 1 \).
5. If \( X \) is \((n-1)\)-connected then \( \text{Ind}^E_G(X) \geq n \).
6. If \( X \) is free and \( \dim(X) \leq n \) then \( \text{Ind}^E_G(X) \leq n \).
7. (Sarkaria’s inequality) If \( L_0 \) is free and \( L \subseteq L_0 \) is \( G \)-invariant then,
   \[
   \text{Ind}^E_G(L) \geq \text{Ind}^E_G(L_0) - \text{Ind}^E_G(\Delta(L_0 \setminus L)).
   \]
8. If \( n = (r-1)d - 1 \) for some \( d \geq 1 \) then \( A_n G = (S^{r-2})^d = S^{(r-1)d-1} \).
9. \( \text{Ind}^E_G(A_n G) = n \) or equivalently \( A_n G \xrightarrow{G} A_{n+1} G \).
10. If \( r = p \) is a prime then \( \text{Ind}^E_G(X) = \text{Ind}_G(X) \).

Proof: For \( 0 \leq n < r - 2 \) let \( A_n G = E_n G = [r]^{(n+1)} \). If \( n = (r-1)d - 1 \) for some \( d \) then by definition \( A_n G = S^{(r-1)d-1} = (S^{r-2})^d \). If \( n = (r-1)d - 1 + k \) for some \( 0 < k < r - 1 \) let \( A_n G = S^{(r-1)d-1} \ast [r]^k \).

All of the properties are easily verified. Here are some illustrative examples.

(4): Suppose that \( \text{Ind}^E_G(X) = p \) and \( \text{Ind}^E_G(X) = q \) and suppose that \( f : X \xrightarrow{G} A_p G \) and \( f : Y \xrightarrow{G} A_q G \) are \( G \)-equivariant maps. Let \( F : X \ast Y \xrightarrow{G} A_p \ast A_q \) be the induced map. Then (4) follows from the observation that there exists a \( G \)-equivariant map \( A_p G \ast A_q G \xrightarrow{G} A_{p+q+1} G \).

(5): This follows from a result of Volovikov about fixed-point free actions of \( G = (\mathbb{Z}_p)^k \) [MO3, Section 6.2.6] or alternatively from from the ideal-valued index theory [Ziv98, Part II].

(9): Let \( [r]^{(n+1)} = E_n G \). Then \( E_G \xrightarrow{G} A_n G \) and (in light of (5)) \( E_n G \xrightarrow{G} A_{n-1} G \) together imply \( A_n G \xrightarrow{G} A_{n-1} G \). \( \square \)
7 Concluding remarks

It is certainly agreeable to have several proofs of the same result so here we collect some consequences of Theorem [L2] and the connectivity estimates for the symmetric multiple chessboard complexes established in Section 3.

Theorem 6.1 from [BFZ] is a direct consequence of the fact that the complex $\Delta_{d+3,2}^{d/2}$ is $\min\{d + 3 - 2 - 1, 2 \cdot (\frac{d}{2} + 1) - 2\} = d$-connected, which is proved in [JVZ].

Theorem 6.3 from [BFZ] is a direct consequence of the fact that the complex $\Delta_{N+1,r}^{k+1,1}$ is $\min\{(r - 1)(d + 2) + 1 - r - 1, r(k + 1) - 2\}$-connected, which is proved in [JVZ]
and the inequality $r(k + 1) - 2 \geq r\lceil\frac{r-1}{r}d\rceil + r - 2 \geq (r - 1)(d + 1) - 1$.

Similarly Theorems 6.5 and 6.8 from [BFZ], which in part served as a motivation for [BFZ, Conjecture 6.6] (Conjecture 1.1 in our paper) are consequences of our Theorem [L2].

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