MINIMIZING WEAK SOLUTIONS FOR CALABI’S EXTREMAL METRICS ON TORIC MANIFOLDS

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ABSTRACT. In this paper, we discuss a Donaldson’s version of the modified $K$-energy associated to the Calabi’s extremal metrics on toric manifolds and prove the existence of the weak solution for extremal metrics in the sense of convex functions which minimizes the modified $K$-energy.

0. Introduction

The existence of extremal metrics has been recently studied extensively on Kähler manifolds. The goal is to establish a sufficient and necessary condition for the existence of extremal metrics in the sense of Geometric Invariant Theory. There are many important works related to the necessary part ([Ti], [D1], [M1], [M2]). The sufficient part seems more difficult than the necessary part since the existence is related to the solvability of certain fourth-order elliptic equations. On the other hand, an extremal metric can be regarded as a critical point of some geometric energies, such as the Calabi’s energy, the modified $K$-energy, etc. This gives a way to study the existence by using variational method in the sense of Nonlinear Analysis. In this paper, we focus on a class of special Kähler manifolds, namely, toric manifolds and discuss the minimizing weak solution for extremal metrics in the sense of convex functions related to a Donaldson’s version of the modified $K$-energy.

Let $(M, g)$ be a compact Kähler manifold of dimension $n$. Then Kähler form $\omega_g$ of $g$ is given by

$$
\omega_g = \sqrt{-1} \sum_{i,j=1}^{n} g_{ij} dz^i \wedge d\bar{z}^j
$$

in local coordinates $(z_1, ..., z_n)$, where $g_{ij}$ are components of metric $g$. The Kähler class $[\omega_g]$ of $\omega_g$ can be represented by a set of potential functions as follows

$$
\mathcal{M} = \{ \phi \in C^\infty(M) | \omega_\phi = \omega_g + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi > 0 \}.
$$

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According to [Ca], a Kähler metric $\omega_\phi$ in the Kähler class $[\omega_g]$ is called extremal if
\begin{equation}
R(\omega_\phi) = \overline{R} + \theta_X(\omega_\phi)
\end{equation}
for some holomorphic vector field $X$ on $M$, where $R(\omega_\phi)$ is the scalar curvature of $\omega_\phi$, $\overline{R} = \frac{1}{V} \int_M R(\omega_g)\omega^n_g$, $V = \int_M \omega^n_g$ and $\theta_X(\omega_\phi)$ denotes the potential function of $X$ associated to the metric $\omega_\phi$, which is defined by
\[ i_X \omega_\phi = \sqrt{-1} \partial \bar{\partial} \theta_X(\omega_\phi), \int_M \theta_X(\omega_\phi) \omega^n_\phi = 0. \]

By [FM], such an $X$, usually called extremal is uniquely determined by the Futaki invariant $F(\cdot)$. Equation (0.1) can be regarded as an Euler-Lagrange equation of the following modified $K$-energy
\[ \mu(\phi) = -\frac{1}{V} \int_0^1 \int_M \psi_t [R(\omega_{\psi_t}) - \overline{R} - \theta_X(\psi_t)|\omega^n_{\psi_t} \wedge dt, \]
where $\psi_t(0 \leq t \leq 1)$ is a path connecting 0 to $\phi$ in $\mathcal{M}$. In fact, one can show that the functional $\mu(\phi)$ is well-defined, i.e., it is independent of the choice of path $\psi_t$ ([Gua]). Thus $\phi$ is a critical point of $\mu(\cdot)$ iff the corresponding metric $\omega_\phi$ is extremal.

**Definition 0.1.** Let
\[ I(\phi) = \frac{1}{V} \int_M \phi(\omega^n_g - \omega^n_\phi). \]
$\mu(\phi)$ is called proper associated to a subgroup $G$ of the automorphisms group $\text{Aut}(M)$ in Kähler class $[\omega_g]$ if there is a continuous function $p(t) \in \mathbb{R}$ with the property
\[ \lim_{t \to +\infty} p(t) = +\infty, \]
such that
\begin{equation}
\mu(\phi) \geq \inf_{\sigma \in G} p(I(\phi_\sigma)),
\end{equation}
where $\phi_\sigma$ is defined by
\[ \omega_g + \sqrt{-1} \partial \bar{\partial} \phi_\sigma = \sigma^*(\omega_g + \sqrt{-1} \partial \bar{\partial} \phi). \]

There is a natural question: does there exists an extremal metric if the modified $K$-energy $\mu(\phi)$ is proper associated to a reductive subgroup $G$ of $\text{Aut}(M)$? The answer for Kähler-Einstein metrics with positive scalar curvature is positive by Tian ([Ti]). The converse is also true. In this paper, we discuss this question for toric manifolds.
Minimizing Weak Solutions for Calabi’s Extremal Metrics on Toric Manifolds

An n-dimensional toric manifold \( M \) corresponds to a polytope \( P \) in \( \mathbb{R}^n \) which satisfies Delzant’s condition ([Gui]). Let \( G_0 \) be a maximal compact subgroup of torus actions group \( T \) on \( M \). It was showed in [ZZ] that for any \( G_0 \)-invariant \( \phi \),

\[
\mu(\phi) = \frac{2^n n! (2\pi)^n}{V} \mathcal{F}(u)
\]

with

\[
\mathcal{F}(u) = -\int_P \log(\det(D^2u)) dx + \int_{\partial P} u d\sigma - \int_P (\bar{R} + \theta_X) u dx,
\]

where \( u \) is a Legendre function of \( \phi \) which is smooth convex function in \( P \) and can be extended to a continuous function on \( \overline{P} \), and \( d\sigma \) is a natural induced measure from \( dx \). In [D2], Donaldson first obtained (0.3) for the \( K \)-energy. Since Hessian matrix \( (D^2u) \) exists almost everywhere for a convex function \( u \) in \( P \), it is possible to extend \( \mathcal{F}(u) \) to a more general class of convex functions in \( P \).

Let \( \tilde{\mathcal{C}} \) be a set of normalized Legendre functions associated to \( G_0 \)-invariant potential functions on \( M \) (cf. Section 1). Set

\[
\mathcal{C}_* = \{ u \geq 0 \text{ a limit of some sequence of } \{ u_n \} \text{ in } \tilde{\mathcal{C}} \text{ with } \int_{\partial P} u_n d\sigma < C \}.
\]

Then \( \mathcal{C}_* \) is a complete space in the sense of local \( C^0 \)-convergence. Moreover we show that \( \mathcal{F}(u) \) is well-defined in \( \mathcal{C}_* \) although it may be infinity (cf. Section 2). It is easy to see that the Euler-Lagrange equation for \( \mathcal{F}(u) \) in \( \tilde{\mathcal{C}} \) is

\[
-u_{ij}^{ij} = \bar{R} + \theta_X, \quad \text{in } P
\]

where \( \theta_X \) is a potential function of \( X \) which is a linear function in \( P \). We call \( u \) a weak solution of (0.5) in the sense of convex functions if \( \mathcal{F}(u) < \infty \) and \( u \) is a critical point of \( \mathcal{F}(u) \) in \( \mathcal{C}_* \).

The following is our main theorem in this paper.

**Theorem 0.2.** Suppose that \( \mu(\phi) \) is proper for \( G_0 \)-invariant potential functions associated to toric actions group \( T \) on a toric manifold \( M \). Then there exists a weak solution \( u_\infty \) of equation (0.5) for extremal metrics on \( M \) in the sense of convex functions which minimizes \( \mathcal{F}(u) \) in \( \mathcal{C}_* \).

In [ZZ], authors introduced a sufficient condition to verify the properness of \( \mu(\phi) \) on toric manifolds and found some examples which satisfy the condition. Thus according to the above theorem there exists a minimizing weak solution of (0.5) on these toric manifolds. The regularity of minimizing solution of (0.5) is an interested topic. If the minimizing solution \( u_\infty \) in Theorem 2 belongs to \( \tilde{\mathcal{C}} \), then \( u_\infty \) induces an extremal metric on \( M \). We hope to discuss this problem in the future.
The organization of this paper is as follows. In Section 1, we review the modified $K$-energy $\mu(\phi)$ in the sense of Donaldson’s version for convex functions on toric manifolds. In section we will extend $\mathcal{F}(u)$ to more general convex functions. In section 3, we prove the lower semi-continuity of $\mathcal{F}(u)$. In section 4, we prove Theorem 0.2 and discuss some properties about the minimizing weak solution of (0.5).

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1. **Modified K-energy $\mu(\phi)$**

In this section, we recall a Donaldson’s version of the modified K-energy $\mu(\phi)$ on toric manifolds. We assume that $M$ is an $n$-dimensional toric Kähler manifold and $g$ is a $G_0 \cong (S^1)^n$-invariant Kähler metric in the Kähler class, where $G_0$ is a maximal compact subgroup of torus actions group $T$ on $M$. Then under an affine logarithm coordinates system $(w_1, \ldots, w_n)$, its Kähler form $\omega_g$ is determined by a convex function $\psi_0$ on $\mathbb{R}^n$, namely,

$$\omega_g = \sqrt{-1} \partial \bar{\partial} \psi_0$$

is defined on the open dense orbit $T$. Denote $D\psi_0$ to be a gradient map (moment map) associated to $T$. Then the image of $D\psi_0$ is a convex polytope $P$ in $\mathbb{R}^n$. By using the Legendre transformation $y = (D\psi_0)^{-1}(x)$, we see that the function (Legendre function) defined by

$$u_0(x) = \langle y, D\psi_0(y) \rangle - \psi_0(y) = \langle y(x), x \rangle - \psi_0(y(x))$$

is convex on $P$. In general, for any $G_0$-invariant potential function $\phi$ associated to the Kähler class $[\omega_g]$, one gets a convex function $u(x)$ on $P$ by using the above relation while $\psi_0$ is replaced by $\psi_0 + \phi$. Set

$$C = \{u = u_0 + v | u \text{ is a convex function in } P, \ v \in C^\infty(\bar{P})\}.$$

It was showed in [Ab] that functions in $C$ are corresponding to $G_0$-invariant functions in $\mathcal{M}$ (whose set is denoted by $\mathcal{M}_{G_0}$) by one-to-one.

An $n$-dimensional toric manifold $M$ corresponds to a polytope $P$ in $\mathbb{R}^n$ which is described by a common set of some half-spaces,

$$\langle l_i, x \rangle < \lambda_i, \ i = 1, \ldots, d,$$

where $l_i$ are $d$-vectors in $\mathbb{R}^n$ with all components in $\mathbb{Z}$, which satisfy the Delzant condition ([Gui]). Without loss of generality, we may assume that the original point $0$ lies in $P$, so all $\lambda_i > 0$. A special element of $C$ in the sense of $P$ can be constructed as follow ([Gui]),

$$u_P = \sum (-\langle l_i, x \rangle + \lambda_i) \log(-\langle l_i, x \rangle + \lambda_i).$$

(1.1)
The convex function $u_P$ is very useful and we will use it at many places in our paper.

In [D2], Donaldson found a formula for the $K$-energy in the sense of convex functions in $C$. For the modified $K$-energy $\mu(\phi)$, we have ([ZZ]),

**Lemma 1.1.** Let $d\sigma_0$ be the Lebesgue measure on the boundary $\partial P$ and $\nu$ be the outer normal vector field on $\partial P$. Let $d\sigma = \lambda_i^{-1}(\nu, x)d\sigma_0$ on the face $\langle l_i, x \rangle = \lambda_i$ of $P$. Then

\begin{equation}
\mu(\phi) = \frac{2^nn!(2\pi)^n}{V} F(u),
\end{equation}

where

\begin{equation}
F(u) = -\int_P \log(\det(D^2u))dx + \int_{\partial P} u d\sigma - \int_P (\tilde{R} + \theta_X) u dx.
\end{equation}

The functional $F(u)$ is invariant according to the choice of $X$ if $u$ is replaced by adding an affine linear function. For this reason, we normalize $u$ as follows. Let $p \in P$ and set

$\tilde{C} = \{u \in C | \inf_P u = u(p) = 0\}$.

Then for any $u_\phi \in C$ corresponding to $\phi \in M_{G_0}$, one can normalize $u_\phi$ by

$\tilde{u}_\phi = u_\phi - (\langle Du_\phi(p), x - p \rangle + u_\phi(p))$

so that $\tilde{u}_\phi = u_{\tilde{\phi}} \in \tilde{C}$ corresponds to a Kähler potential function $\tilde{\phi} \in M_{G_0}$ which satisfies

$D(\tilde{\phi} + \psi_0)(0) = p$ and $(\tilde{\phi} + \psi_0)(0) = 0$.

In fact, $\tilde{\phi}$ can be uniquely determined by using the affine coordinates transformation $y \to y + y_0$ as follow,

$\tilde{\phi}(y) = (\phi + \psi_0)(y + y_0) - \psi_0(y) - (\phi + \psi_0)(y_0)$.

The following lemma was also proved in [ZZ].

**Lemma 1.2.** There is a constant $C$ independent of $\phi$, such that

\begin{equation}
|\int_P \tilde{u}_\phi dx - I(\tilde{\phi})| \leq C.
\end{equation}

By Lemma 1.2, one can easily get

**Proposition 1.3.** $\mu(\phi)$ is proper for $G_0$-invariant potential functions associated to toric actions $T$ if only if there exists a continuous function $p(t)$ in $\mathbb{R}$ with the property

$\lim_{t \to +\infty} p(t) = +\infty$,

such that

$F(u) \geq p(\int_P u dx), \forall u \in \tilde{C}$.
At the end of this section, we would like to recall a set
\[ C_\infty = \{ u \in C^\infty(P) \cap C(\overline{P}) \mid u \text{ is convex in } P \}. \]
Clearly, \( C \subset C_\infty \). It was proved in [D2]
\[ F(u) > -\infty, \forall u \in C_\infty \]
and
\[ (1.5) \quad \inf_{C_\infty} F(u) = \inf_{\bar{C}} F(u). \]

2. Lower bound of \( F(u) \)

Since \( \bar{C} \) is not a complete space, we need to define a closure set of \( \bar{C} \). We denote by \( P^* \) the union of \( P \) and the open codimension-1 faces. And define
\[ C_* = \{ u \geq 0 \text{ is a limit of some sequence of } \{ u_n \} \text{ in } \bar{C} \text{ with } \int_{\partial P} u_n d\sigma_0 < C \}. \]
Note that for any \( u \in C_* \), by defining \( u \) on the boundary to be
\[ \lim_{t \to 1} u_\infty(tz), \]
then it is a convex function on \( P^* \) with \( \int_{\partial P} u d\sigma_0 < \infty \). On the other hand, \( \int_{\partial P} u_n d\sigma_0 < C \) implies that there is a subsequence of \( u_n \) converging uniformly on any compact set of codimension-1 faces. We denote the limit to be \( \hat{u} \) on any open set of codimension-1 faces. By the convexity of \( u \), it is easy to see that
\[ u|_{\partial P} \leq \hat{u}. \]

Let \( u_0 \in \bar{C} \) and choose \( K_0 > 0 \) so that
\[ \int_{\partial P} u_0 d\sigma_0 < K_0. \]
For any \( K > K_0 \), we denote a subset of convex functions in \( P^* \) by
\[ C_*^K(P) = \{ u \text{ is a convex function in } P^* \text{ with } \int_{\partial P} u d\sigma_0 \leq K \text{ and } \inf_P u = u(0) = 0 \}. \]
Clearly \( u_0 \in C_*^K(P) \). Note that for any sequence \( u_n \) converging locally uniformly to \( u \) in \( C_* \),
\[ \int_{\partial P} u d\sigma_0 = \int_{\partial P} \hat{u} d\sigma_0 = \lim_n \int_{\partial P} u_n d\sigma_0 \leq K. \]
We see that all \( C_*^K(P) \) are complete spaces in sense of local \( C^0 \)-convergence.

We want to extend \( F(u) \) to be defined on \( C_*^K(P) \). Since the linear part
\[ L(u) = \int_{\partial P} u d\sigma - \int_P (\tilde{R} + \theta_X) u dx \]
is well-defined on $C^K_*(P)$, it suffices to define the nonlinear part

$$\int_P \log(\det(D^2u))dx.$$  

Note that $u$ is almost everywhere twice-differentiable since $u$ is convex and so the Hessian matrix $(D^2u)$ exists almost everywhere. We denote Hessian matrix $(\partial^2 u)$ at those twice-differentiable points in $P$. Then one can show that $\det(\partial^2 u)$ corresponds to the regular part $\mu_r[u]$ of Monge-Ampère measure $\mu[u]$ associated to $u$ and so it is a locally integrable function ([TW2]). Thus

$$\int_P \log^+(\det(\partial^2 u))dx$$

is well-defined although the integral may be infinity, where

$$\log^+(\det(\partial^2 u)) = \max(0, \log(\det(\partial^2 u))).$$

But we have

**Proposition 2.1.** Let $u \in C^K_*(P)$. Then

$$\int_P \log^+(\det(\partial^2 u))dx < \infty.$$  

By Proposition 2.1, one sees that

$$\int_P \log(\det(\partial^2 u))dx$$

is integrable. To prove the proposition, we need to regularize $u$ by mollification functions, namely,

$$u_h(x) = h^{-n} \int_P \rho(\frac{x-y}{h})u(y)dy$$

for any small $h > 0$, where $x$ satisfies $h \leq \text{dist}(x, \partial P)$ and $\rho$ is a support function in $B_1(0) \subset \mathbb{R}^n$ with $\int_{B_1(0)} \rho = 1$. A fundamental result is that $(D^2u_h) \rightarrow (\partial^2 u)$ almost everywhere for a convex function $u$ in $P$.

**Lemma 2.2.** Let $u \in C^K_*(P)$ and $u_n$ be a sequence in $C^2(P)$ converging locally uniformly to $u$ with $\det(\partial^2 u_n) \rightarrow \det(\partial^2 u)$ almost everywhere. Suppose that

$$(2.1) \quad \det(\partial^2 u_n), \det(\partial^2 u) \geq \epsilon_0 > 0.$$

Then for any subset $\Omega \subset \subset P$,

$$\int_{\Omega} \log(\det(\partial^2 u))dx = \lim_{n \to \infty} \int_{\Omega} \log(\det(D^2u_n))dx.$$
Proof. Let $\Omega \subset P$ be any Borel subset with $\text{dist}(\Omega, \partial P) \geq \delta > 0$. By using the concavity of log, we have
\[
\int_{\Omega} \log(\det(\partial^2 u)) \, dx \leq |\Omega| \log \left( |\Omega|^{-1} \int_{\Omega} \det(\partial^2 u) \, dx \right) \\
\leq |\Omega| \log \left( |\Omega|^{-1} \frac{\text{osc}(u)}{\delta} \right) \\
= n |\Omega| \frac{\text{osc}(u)}{\delta} - |\Omega| \log(|\Omega|).
\]
Similarly, for $u_n$, it holds
\[
(2.2) \quad \int_{\Omega} \log(\det(\partial^2 u_n)) \, dx \leq n |\Omega| \frac{\text{osc}(u_n)}{\delta} - |\Omega| \log(|\Omega|).
\]
For any $\lambda > 0$, we let $\Omega_{n,\lambda} = \{ x \in \Omega \mid |\log(\det(\partial^2 u_n)) - \log(\det(\partial^2 u))| > \lambda \}$. Then $|\Omega_{n,\lambda}| \to 0$ as $n \to \infty$ since $D^2 u_n \to D^2 u$ almost everywhere. Note
\[
| \int_{\Omega} \log(\det(\partial^2 u_n)) \, dx - \int_{\Omega} \log(\det(\partial^2 u)) \, dx |
\leq | \int_{\Omega_{n,\lambda}} \log(\det(\partial^2 u_n)) \, dx | + | \int_{\Omega_{h,\lambda}} \log(\det(\partial^2 u)) \, dx | \\
+ \int_{\Omega \setminus \Omega_{n,\lambda}} | \log(\det(\partial^2 u_n)) \, dx - \log(\det(\partial^2 u)) \, dx |
\]
It is clear that the first two integrals in the right side of the inequality go to 0 as $n \to \infty$ by (2.2) and the condition $\det(\partial^2 u_n), \det(\partial^2 u) \geq \epsilon_0 > 0$. The last integral also goes to 0 by choosing $\lambda$ small enough. Thus the lemma is proved. \hfill \square

Let $u_P$ be defined as in (1.1). Then

Lemma 2.3. For any $u \in C(\bar{P})$, $\log(\det(\partial^2 u))$ is integrable and
\[
(2.3) \quad \int_{P} \log^+(\det(\partial^2 u)) \leq \int_{P} (u_P)_{i,j}^2 \, dx + \int_{\partial P} u \, d\sigma + C,
\]
where $C = C(u_P)$.

Proof. Without loss of generality, we may assume that $u$ satisfies
\[
\det(\partial^2 u), \, \det(D^2 u_h) \geq 1,
\]
otherwise, we replace $u$ by $u + \frac{|x|^2}{2}$, and $u_h$ by $u_h + \frac{|x|^2}{2}$.

For any $\delta > 0$, we let $P_{\delta}$ be the interior polygon with faces parallel to those of $P$ separated by distance $\delta$. By Lemma 2.2 we have
\[
(2.4) \quad \int_{P_{\delta}} \log(\det(\partial^2 u)) \, dx = \lim_{h \to 0} \int_{P_{\delta}} \log(\det(\partial^2 u_h)) \, dx.
\]
Hence it suffices to prove that $\int_{P_3} \log(\det(\partial^2 u))dx$ is uniformly bounded for $\delta$.

By the convexity of $-\log(\det(\cdot))$ for positive definite matrices, we have
\[
\log(\det(\partial^2 u_h)) \leq \log(\det(D^2 u_P)) + u^i_j(u_h)_{ij} - n.
\]

We need to estimate $\int_{P_3} u^i_j(u_h)_{ij}dx$. Using integration by parts, we obtain
\[
\int_{P_3} u^i_j(u_h)_{ij}dx = \int_{\partial P_3} u^i_j(u_h)_{ij}n_i d\sigma_0 - \int_{\partial P_3} (u_P)^i_j u_h n_i d\sigma_0 + \int_{P_3} (u_P)^i_j u_h n_j d\sigma_0,
\]
where $n$ is the outer normal vector.

Let $x$ be a point of $\partial P_3$ and $\xi = \xi(x)$ be the vector $(\xi^i) = (u^i_j n_j)$. Then $u^i_j(u_h)_{ij} = \nabla_\xi u_h$. Let $y = y(x)$, $\bar{y} = \bar{y}(x)$ and $z = z(x)$ be the intersection points of the ray $\{x + t\xi : t > 0\}$ with the boundary $\partial P_3^{*}$, $\partial P_3^{**}$ and $\partial P$, respectively (see Figure 1 for dimension 2).

It was verified in [D2] that there exist $c_1, c_2 > 0$ such that for any $x \in \partial P_3$, 
\[
|\xi| \leq c_1|y - x| = c_1|\bar{y} - x|; \quad |\bar{y} - x| = |y - x| \leq c_2\delta.
\]

Thus by the convexity of $u_h$, we have
\[
|\nabla_\xi u_h| \leq c_1 \max\{ |u_h(y) - u_h(x)|, |u_h(\bar{y}) - u_h(x)| \}
\]
and
\[ \int_{\partial P} u_{i}^{j}(u_{h})_{ij} n_{j} d\sigma_{0} \]
(2.7) \leq c_{1} \int_{\partial P} \max\{|u_{h}(y) - u_{h}(x)|, |u_{h}(\bar{y}) - u_{h}(x)|\} d\sigma_{0}.

Let \( h \to 0 \). By (2.5), we get
\[
\int_{P} \log(\text{det}(\partial^{2}u)) dx \leq c_{1} \int_{\partial P} \max\{|u(y) - u(x)|, |u(\bar{y}) - u(x)|\} d\sigma_{0}
\]
\[ - \int_{\partial P} (u_{P})_{ij}^{j} u_{n} d\sigma_{0} + \int_{P} (u_{P})_{ij}^{j} u dx \]
\[ \leq c_{1} \int_{\partial P} (|u(y) - u(x)| + |u(\bar{y}) - u(x)|) d\sigma_{0} \]
\[ - \int_{\partial P} (u_{P})_{ij}^{j} u_{n} d\sigma_{0} + \int_{P} (u_{P})_{ij}^{j} u dx. \]

Note that the first integral at the last inequality above goes to 0 as \( \delta \to 0 \) since \( u \) is continuous in \( \bar{P} \), and the second integral converges to \( \int_{\partial P} u d\sigma \) as \( \delta \) goes to 0 since \( (u_{P})_{ij}^{j} n_{i} d\sigma_{\delta} \) converges to \( d\sigma \) on the boundary \( \partial P \) as \( \delta \to 0 \) (\[D2\]). The third integral clearly converges to \( \int_{P} (u_{P})_{ij}^{j} u dx \). Therefore we get (2.3).

Let us recall an adapted co-ordinates system on \( P \). That is, for any point \( X \) on \( \partial P \), there is an affine co-ordinates \( \{x_{1}, ..., x_{n}\} \), such that there is a neighborhood of \( X \) defined by \( p \) inequalities
\[ x_{1}, ..., x_{p} \geq 0, \]
with \( X \) the origin in this system. Then for a \( v \in C \), under an adapted co-ordinates, \( v \) has the form
\[ v = \sum_{i=1}^{p} x_{i} \log x_{i} + w, \]
where \( w \) is a smooth function. Furthermore, by choosing a suitable adapted co-ordinates system one may assume that at \( X = 0 \),
\[ \frac{\partial^{2}w}{\partial x_{i} \partial x_{j}} = \delta_{ij}, i, j > p, \]
and the converse of \( v_{ij}^{j} \) has the following property (\[D3\]),

**Lemma 2.4.**

(2.8) \( (v^{ij}) = \text{diag}(f_{1}x_{1}, ..., f_{p}x_{p}, f_{p+1}, ..., f_{n}) + (\sigma^{ij}), \)
where \( \sigma^{ij} \) is
We need to verify

For simplicity, we just consider the first integral.

\[ \beta \]

Note that here we could not use the continuity of \( \{ E^i \} \)

\text{(2.3)} it suffices to prove

\[ f_i, g_{ij}, \text{ and } h_{ij} \text{ are all smooth functions with } f_i(0) = 1 \text{ and } h_{ij}(0) = 0 \text{ for all } i, j. \]

With this lemma, we are able to deal with functions in \( C_* \).

\textbf{Lemma 2.5.} Assume that \( P \) is a \( n \)-rectangle. Then for any \( u \in C_* \), \( \log^+ (\det (\partial^2 u)) \) is integrable and

\[
\int_P \log^+ (\det (\partial^2 u)) \leq \int_P (u_P)^{ij}_{ij} u dx + \int_{\partial P} u d\sigma + C,
\]

where \( C = C(u_P) \).

\textbf{Proof.} We still use the notations in Lemma 2.3. As in the proof of Lemma 2.3, it suffices to prove

\[
\int_{\partial P} (|u(y) - u(x)| + |u(\bar{y}) - u(x)|) d\sigma_0 \to 0, \text{ as } \delta \to 0.
\]

Note that here we could not use the continuity of \( u \) up to the boundary \( P \).

We need to verify

\[
\int_{\partial P} |u(y) - u(x)| d\sigma_0 \to 0, \int_{\partial P} |u(\bar{y}) - u(x)| d\sigma_0 \to 0.
\]

For simplicity, we just consider the first integral.

Let \( n \)-rectangle \( P \) be bounded with \( (n-1) \)-dimensional faces which are defined by

\[ E^i : x_i = \alpha_i, \quad E'^{n+i} : x_i = \beta_i \text{ for } 0 < i \leq n, \]

where \( \beta_i > \alpha_i \). Then

\[ u_P = \sum (x_i - \alpha_i) \log(x_i - \alpha_i) + \sum (-x_i + \beta_i) \log(-x_i + \beta_i). \]

Let \( \{ E^k_i \}_{i=1}^{2n} \) be the union of corresponding \( (n-1) \)-dimensional faces on \( \partial P \).

We first claim that when \( \delta \) is small enough, \( z \) must lie on \( E^k \) for any \( x \) on \( E^k_i \).

This implies that \( y \) lies on \( E^k_\delta \), and \( \bar{y} \) lies on \( E^k_i \). In fact, for any \( X \) in \( E^k \), we will make the computation in the adapted coordinates \( \{ \tilde{x}_1, ..., \tilde{x}_n \} \) at \( X \).

Note that since \( P \) is a \( n \)-rectangle, the transformation of coordinates just consists of the translations and reflections. Without loss of generality, we assume that \( E^k \) corresponds to \( \{ \tilde{x}_1 = 0 \} \) in an adapted coordinates. Then \( x = (\delta, \tilde{x}_2, ..., \tilde{x}_n) \) and the outer normal vector \( n = (-1, 0, ..., 0) \) at \( x \). Since \( \tilde{x}_2, ..., \tilde{x}_p > 0 \), by Lemma 2.3, we compute the coordinate of \( z \) as follow

\[
\left( 0, \frac{\tilde{x}_2 g_{12} \delta}{f_1 + g_{11} \delta}, ..., \frac{\tilde{x}_p g_{1p} \delta}{f_1 + g_{11} \delta}, \frac{\tilde{x}_{p+1} - g_{1,p+1} \delta}{f_1 + g_{11} \delta}, ..., \frac{x_n - g_{1,n} \delta}{f_1 + g_{11} \delta} \right).
\]
This shows $\tilde{z}_2, ..., \tilde{z}_p > 0$ if $\delta$ is small enough, which means $z$ lies on $E^k$, so the claim is true in the neighborhood of $X$. By using the finite covering of adapted coordinates to $\partial P$, one can get a uniform small $\delta$ so that the claim is true under these adapted coordinates. Hence the claim is verified.

For any point $X$ on an open $p$-dimensional face, it lies on the intersection of $n - p$ faces of dimension $n - 1$, denoted by $E^{k_1}, ..., E^{k_{n-p}}$. One chooses a rectangular-shaped neighborhood $N_{X,\eta}$ in $\bar{P}$ corresponding to a $n-$rectangle $R_{X,\eta}$

$$\begin{cases} 0 < \tilde{x}_i < \eta, \text{ for } 0 < i \leq n - p; \\ -\eta < \tilde{x}_i < \eta, \text{ for } n - p + 1 \leq i \leq n \end{cases}$$

in the adapted coordinates of $X$ with $E^{k_j} \cap N_{X,\eta}$ corresponding the face $\tilde{x}_j = 0$ for any $j \leq n-p$. In particular, for a vertex $X$ of $P$, $N_{X,\eta}$ corresponds to

$$0 < \tilde{x}_i < \eta, \text{ for } 0 < i \leq n$$

in the adapted co-ordinates. Take the union of all $N_{X,\eta}$ with $X$ on the closed $(n - 2)$-dimensional faces and denote it by

$$N_\eta = \bigcup_X N_{X,\eta}$$

(see Figure 2, the dark area gives a description of $N_\eta$ when $n = 2$ and in higher dimension, it is a neighborhood of all $(n - 2)$-dimensional faces).
For a fixed $\eta > 0$, we have
\[\int_{\partial P_\delta} |u(y) - u(x)| \, d\sigma_0 = \int_{N_\eta \cap \partial P_\delta} |u(y) - u(x)| \, d\sigma_0 + \int_{\partial P_\delta \setminus N_\eta} |u(y) - u(x)| \, d\sigma_0\]
(2.10)
as so as $\delta$ is small. Since we have $|y - x| \leq c_2 \delta$ and $|z - x| \leq 2c_2 \delta$, one can suppose $\delta$ to be small enough such that both $y$ and $z$ lie in
\[\begin{cases} N_{2\eta}, & \text{for } x \in N_\eta \cap \partial P_\delta; \\ \bar{P} \setminus N_{2\eta}, & \text{for } x \in \partial P_\delta \setminus N_\eta. \end{cases}\]
Thus letting $\delta \to 0$, we see that the second integral at the right hand of (2.10) goes to zero by the continuity of $u$ in $\bar{P} \setminus N_{2\eta}$. It remains to deal with the first integral. According to the above computation, for any $x = (\delta, \tilde{x}_2, ..., \tilde{x}_n)$ in any codimension−1 face $E^k_\delta$, it must lie in a $N_{X, \eta}$ for some $X \in E^k$. Moreover, in the adapted coordinates, $z$ is a form of
\[z(x) = (0, \tilde{x}_2, ..., \tilde{x}_n) + \delta(0, \phi_2(\tilde{x}), ..., \phi_n(\tilde{x})) ,\]
where $\phi_2(\tilde{x}), ..., \phi_n(\tilde{x})$ are smooth. This implies that $z$ corresponds 1 − 1 to $x$ and $y$ in $N_\eta$ and the measure
\[d\sigma(z) \geq C d\sigma_0(x) .\]
Hence by the convexity of $u$ we obtain
\[\int_{N_\eta \cap \partial P_\delta} |u(y) - u(x)| \, d\sigma_0 \leq C \int_{N_{2\eta} \cap \partial P} u(z) \, d\sigma_0 .\]
The late goes to 0 as $\eta \to 0$ since $u$ is integrable on the boundary $\partial P$. □

Let us begin to prove Proposition 2.1.

**Proof of Proposition 2.1.** For any $X$ on $\partial P$, we choose a polytope-shaped neighborhood $N_X$ in $\bar{P}$ which corresponds to an $n$–rectangle $R_X$ with an adapted coordinates system $\{\tilde{x}_1, ..., \tilde{x}_n\}$ at $X$. Since the determinant of Jacobi matrix associated to the coordinates transformation is equal to 1, we have
\[\int_{N_X} \log(\det(\partial^2_x u)) \, dx = \int_{R_X} \log(\det(\partial^2_{\tilde{x}} u)) \, d\tilde{x} .\]
Applying Lemma 2.5 to the $n$–rectangle $R_X$ with the corresponding choice of $u_{R_X}$ and using the the convexity of $u$, we obtain
\[\int_{R_X} \log^+\det(\partial^2_{\tilde{x}} u) \, d\tilde{x} \leq \int_{R_X} (u_{R_X})_{ij} \tilde{u} \, d\tilde{x} + \int_{\partial R_X} u \, d\bar{\sigma}(\tilde{x}) + C \leq C_1 \int_P u \, dx + C_2 \int_{\partial P} u \, d\sigma(x) + C_3 ,\]
(2.12)
where $C_i = C_i(N_X)$, $i = 1, 2, 3$.

Note that $P \setminus P_\delta$ can be covered by finite neighborhood $\{N_{X_1}, ..., N_{X_m}\}$ as so as $\delta$ is small enough (see Figure 3). On the other hand, since $u \in C(P_\delta)$, by Lemma 2.3, we have

\[
\int_{P_\delta} \log^+(\det(\partial^2 u)) \leq \int_{P_\delta} (u_{P_\delta})_{ij}^2 u dx + \int_{\partial P_\delta} u ds + C
\]

(2.13)

\[
\leq C_1(\delta) \int_P u dx + C_2(\delta) \int_{\partial P} u ds + C_3(\delta),
\]

where $u_{P_\delta}$ is given by

\[
\sum \left(-\langle l_i, x \rangle + \lambda_i - |l_i|\delta\right) \log\left(-\langle l_i, x \rangle + \lambda_i - |l_i|\delta\right).
\]

Hence combining (2.12) and (2.13), we get

\[
\int_P \log^+(\det(\partial^2 u)) \leq \sum \int_{N_{X_i}} \log^+(\det(\partial^2 u)) dx + \int_{P_\delta} \log^+(\det(\partial^2 u)) dx
\]

\[
\leq C'_1 \int_P u dx + C'_2 \int_{\partial P} u ds + C'_3,
\]

where $C'_i$ are independent of $u$. □

**Remark 2.6.** By Lemma 2.3 and 2.5, we also have the following estimate,

\[
\int_P \log(\det(\partial^2 u)) \leq r \int_P (u_P)_{ij}^2 u dx + r \int_{\partial P} u ds + C(u_P, r), \ \forall r > 0,
\]

since we can replace $u_P$ by $r^{-1}u_P$. Hence according to the proof of Proposition 2.1, one sees that for any small $c$ there exist two uniform $C = C(c)$ and $C' = C'(c)$ such that

\[
(2.14) \quad \int_P \log(\det(\partial^2 u)) \leq c \int_{\partial P} u ds + C \int_P u dx + C'.
\]

By Proposition 2.1, we get the following approximation result.
Proposition 2.7. Let $u \in C^K(P)$. Suppose that
\begin{equation}
\int_P \log(\det(\partial^2 u))dx > -\infty.
\end{equation}
Then there exists a sequence $u_n$ in $C$ which locally uniformly converges to $u$ and
\begin{equation}
\int_P \log(\det(\partial^2 u))dx = \lim_{n \to \infty} \int_P \log(\det(\partial^2 u_n))dx.
\end{equation}

Proof. We prove the proposition by the following three steps. Note by (2.15) and Proposition 2.1, we see that $\log(\det(\partial^2 u))$ is integrable.

Step 1. For any $u \in C^K(P)$, there exists a sequence $\{u_n\}$ in $C$, such that (2.16) holds. In fact we choose $u_n(x) = u(r_n x)$, with $r_n \to 1$. By the integrability of $\log(\det(\partial^2 u))$, one sees that (2.16) is true.

Step 2. For any $u \in C_\infty(P)$, there exists a sequence $\{u_n\}$ in $C_\infty$, such that (2.16) holds. To prove this, we extend $u$ to a polytope neighborhood $P_{-\delta}$ of $P$ with each $n-1$-dimensional face parallel to one of $P$. Then the mollification function $u_h$ is well-defined on $P$ for sufficiently small $h$. By the integrability of $\log(\det(\partial^2 u))$, it is easy to see
\[
\int_P \log(\det(\partial^2 u))dx = \lim_{\epsilon \to 0} \int_P \log(\det(\partial^2 [u + \epsilon |x|^2]))dx.
\]
On the other hand, since $P$ can be regarded as a subset of $P_{-\delta}$, by Lemma 2.2, we have
\[
\int_P \log(\det(\partial^2 [u + \epsilon |x|^2]))dx = \lim_{h \to 0} \int_P \log(\det(\partial^2 [u_h + \epsilon |x|^2]))dx.
\]
Thus the above two relations show that sequence $u_h + \epsilon |x|^2$ satisfy (2.16).

Step 3. For any $u \in C_\infty$, there exists a sequence $\{u_n\}$ in $C$, such that (2.16) holds. This was in fact proved in [D2] where a sequence was constructed as follow: Let $\eta_\delta$ be a function defined on an interval $[-\text{diam}(P), \text{diam}(P)]$, which satisfies
\[
\eta_\delta = x \log x, \text{ if } x < \delta,
\]
\[
\eta''_\delta \geq 0, \text{ and }
\]
\[
0 \geq \eta_\delta \geq 2\delta \log \delta.
\]
Let
\[
U_\delta(x) = \sum_i \eta_\delta(\lambda_i - \langle x, l_i \rangle).
\]
Then the sequence $u(r_n x) + U_{\delta_n}(x)$ satisfies (2.16).

Combining the above three steps, we will get a sequence $u_n$ in $C$ such that (2.16) is satisfied.
From the proof of the above proposition we see that the sequence $u_n$ in $\mathcal{C}$ constructed for $u \in \mathcal{C}_K^*(P)$ satisfying (2.15) also have the property,

$$L(u) = \lim_{n \to \infty} L(u_n),$$

where

$$L(u) = \int_{\partial P} u d\sigma - \int_P (\bar{R} + \theta_X) u dx$$

is the linear part of $\mathcal{F}(u)$. Thus we get

**Corollary 2.8.** There exists a $K_0 > 0$ such that for any $K \geq K_0$ it holds

$$\inf_{\mathcal{C}_K^*(P)} \mathcal{F}(\cdot) = \inf_{\mathcal{C}_*} \mathcal{F}(\cdot) = \inf_{\mathcal{C}} \mathcal{F}(\cdot).$$

### 3. Semi-continuity of $\mathcal{F}(u)$

In this section, we discuss the lower semi-continuity of the functional $\mathcal{F}(u)$. First we have

**Lemma 3.1.** Suppose that $u_n \in \mathcal{C}_K^*(P)$ converge locally uniformly to $u \in \mathcal{C}_K^*(P)$ for some $K > 0$, and

(3.1) \[ \int_P \log(\det(\partial^2 u_n)) > -C_0 \]

for some $C_0 > 0$. Then for any $h > 0$,

$$\limsup_{n \to \infty} \int_{P_h} \log(\det(\partial^2 u_n)) \leq \int_{P_h} \log(\det(\partial^2 u)).$$

**Proof.** By (3.1) and Proposition 2.7, it suffices to prove it for $u_n \in \mathcal{C}_2^*(P_h)$. Recall that a convex function on $P$ induces a Monge-Ampere measure $\mu[u]$ through its normal mapping and this is a Radon measure and can be decomposed into a regular part and a singular part as follows,

$$\mu[u] = \mu_r[u] + \mu_s[u].$$

Denote by $S$ the supporting set of $\mu_s[u]$, whose Lebesgue measure is zero. Since $u_n$ converges uniformly to $u$, by the upper semi-continuity of $\mu[u]$, then for any closed subset $F \subset P_h \setminus S$,

(3.2) \[ \limsup_{n \to \infty} \int_F \det(D^2 u_n) dx \leq \int_F \det(D^2 u) dx. \]

For given $\epsilon, \epsilon' > 0$, we let

$$\Omega_k = \{ x \in P_h \setminus S \mid (k-1)\epsilon \leq \log(\det(\partial^2 u)) < k\epsilon \}, k = 0, \pm 1, \pm 2, \ldots.$$
and \( \omega_k \subset \Omega_k \) be a closed set such that \( |\Omega_k \setminus \omega_k| < \frac{\epsilon'}{2^m} \). In particular, we let 
\[ \Omega_{-\infty} = \{ x \in P_h \setminus S \mid \log(\det(\partial^2 u)) = -\infty \} = \{ x \in P_h \setminus S \mid \det(\partial^2 u) = 0 \}, \]
and \( \omega_{-\infty} \subset \Omega_{-\infty} \) be a closed set such that \( |\Omega_{-\infty} \setminus \omega_{-\infty}| < \epsilon' \). Note that \( \Omega_{-\infty} \) is a Lebesgue zero set when \( \int_{P_h} \log(\det(\partial^2 u)) > -\infty \).

First we consider the case that \( \Omega_{-\infty} \) is a Lebesgue zero set. Then for each \( \omega_k \), by convexity of \( \log \) and (3.2), we have
\[
\limsup_{n \to \infty} \frac{1}{|\omega_k|} \int_{\omega_k} \log(\det(D^2 u_n)) \, dx 
\leq \limsup_{n \to \infty} \log \left( \frac{\int_{\omega_k} \det(D^2 u_n) \, dx}{|\omega_k|} \right) 
\leq \limsup_{n \to \infty} \log \left( \frac{\int_{\omega_k} \det(\partial^2 u) \, dx}{|\omega_k|} \right) 
\leq \limsup_{n \to \infty} \log \left( \frac{\int_{\omega_k} e^{k\epsilon} \, dx}{|\omega_k|} \right) 
\leq k\epsilon.
\]

It follows
\[
\limsup_{n \to \infty} \int_{\omega_k} \log(\det(D^2 u_n)) \, dx 
\leq k\epsilon |\omega_k| 
\leq (k - 1)\epsilon |\Omega_k| + \epsilon |\Omega_k| + \frac{|k|\epsilon \epsilon'}{2^{|k|}} 
\leq \int_{\Omega_k} \log(\det(\partial^2 u)) \, dx + \epsilon |\Omega_k| + \frac{|k|\epsilon \epsilon'}{2^{|k|}}.
\]

Hence,
\[
\limsup_{n \to \infty} \int_{\cup \omega_k} \log(\det(\partial^2 u_n)) \, dx 
\leq \int_{\cup \Omega_k} \log(\det(\partial^2 u)) \, dx + \epsilon \cup |\Omega_k| + C\epsilon \epsilon' 
\leq \int_{\cup \Omega_k} \log(\det(\partial^2 u)) \, dx + \epsilon |P_h| + C\epsilon \epsilon'.
\]

Since \( \text{osc} u_n \) are uniformly bounded in \( P_h \), by (2.2), we have
\[
\int_{P_h \setminus \cup \omega_k} \log(\det(\partial^2 u_n)) \, dx \leq C\epsilon'.
\]
Combining the above two inequalities and letting $\epsilon' \to 0$, we get
\[
\limsup_{n \to \infty} \int_{P_h} \log(\det(D^2 u_n)) dx \leq \int_{P_h \setminus S} \log(\det(\partial^2 u)) dx + C\epsilon.
\]
Letting $\epsilon \to 0$ again, we obtain
\[
\limsup_{n \to \infty} \int_{P_h} \log(\det(D^2 u_n)) dx \leq \int_{P_h} \log(\det(\partial^2 u)) dx.
\]
Next we consider the case of $|\Omega_{-\infty}| \neq 0$. In this case, it must hold
\[
\int_{P_h} \log(\det(\partial^2 u)) = -\infty.
\]
By the semi-continuity of $\mu[u]$ one sees
\[
\limsup_{n \to \infty} \frac{\int_{\omega_{-\infty}} \log(\det(D^2 u_n)) dx}{|\omega_{-\infty}|} \leq \limsup_{n \to \infty} \log \left( \frac{\int_{\omega_{-\infty}} \det(D^2 u_n) dx}{|\omega_{-\infty}|} \right) \leq \log \left( \frac{\int_{\omega_{-\infty}} \det(\partial^2 u) dx}{|\omega_{-\infty}|} \right) = -\infty,
\]
which is contradict to the assumption (3.1). Thus $|\Omega_{-\infty}| \neq 0$ is impossible. The proof is finished. □

**Lemma 3.2.** Assume that $P$ is a $n$-rectangle. Suppose that $u_n \in C^*_K(P)$ converges locally uniformly to $u \in C^K(P)$ for some $K > 0$. Then there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and for any $\epsilon > 0$, there exists $\delta_0 > 0$ and a large number $N_0$, such that for any $0 < \delta < \delta_0$ and $n > N_0$, it holds
\[
\int_{P \setminus P_{\delta}} \log(\det(\partial^2 u_n)) < C\epsilon.
\]
for some uniform constant $C$.

**Proof.** Without loss of generality we may assume that
\[
det(\partial^2 u_n) \geq 1.
\]
As in Section 2, we see that there exists a subsequence, still denoted by $u_n$ converging uniformly on any compact subsets of codimension–1 faces to $\hat{u}$ which satisfies
\[
u|_{\partial P} \leq \hat{u}, \quad \int_{\partial P} \hat{u}d\sigma_0 = \lim_{\delta \to 0} \int_{\partial P} u_n d\sigma \leq K.
\]
By using the notations in Section 2, for any \( \epsilon > 0 \), we choose \( \eta > 0 \) small enough, such that

\[
\int_{\nabla_{\epsilon} \cap \partial P} \hat{u}(x) d\sigma < \epsilon.
\]

Let \( \delta' \) and \( \delta \) be two small numbers with \( 0 < \delta' < \delta \). We need to estimate the integral for functions \( \log(\det(D^2 u_n)) \) on \( P_{\delta'} \setminus P_{\delta} \). Let \( u_{n,h} \) be mollification functions of \( u_n \). Then

\[
\int_{P_{\delta'} \setminus P_{\delta}} \log(\det(D^2 u_{n,h})) \leq \int_{P_{\delta'} \setminus P_{\delta}} (\log(\det(D^2 u_P)) - n) + \int_{P_{\delta'} \setminus P_{\delta}} u_{ij}(u_{n,h})_{ij}.
\]

The first integral in the right side of the above inequality can be made arbitrary small as so as \( \delta \) is small. So it suffices to consider the second integral. Note that using the integration by parts, we have

\[
|\int_{\partial P_{\delta'}} \nabla \xi' u_{n,h} d\sigma| \leq C \left( \int_{\partial P_{\delta'}} \max\{|u_{n,h}(\bar{y}') - u_{n,h}(x')|, |u_{n,h}(y') - u_{n,h}(x')|\} d\sigma \right),
\]

and

\[
|\int_{\partial P_{\delta}} \nabla \xi u_{n,h} d\sigma| \leq C \left( \int_{\partial P_{\delta}} \max\{|u_{n,h}(\bar{y}) - u_{n,h}(x)|, |u_{n,h}(y) - u_{n,h}(x)|\} d\sigma \right).
\]
Letting \( h \to 0 \) in both (3.5) and (3.6), then we get
\[
\int_{P_{\delta'} \setminus P_{\delta}} \log(\det(\partial^2 u_n)) \\
\leq \int_{(P_{\delta'} \setminus P_{\delta}) \cap (\bar{P} \setminus N_{\eta})} (u_P)_{ij}^2 u_n \, \sigma_0 - \int_{\partial P_{\delta}} (u_P)_{ij}^2 u_n \, d\sigma_0 \\
+ C \int_{\partial P_{\delta}} (|u_n(\bar{y}) - u_n(x)| + |u_n(y') - u_n(x')|) \, d\sigma_0 \\
(3.7) + C \int_{\partial P_{\delta}} (|u_n(\bar{y}) - u_n(x)| + |u_n(y) - u_n(x)|) \, d\sigma_0.
\]

As in Lemma 2.5, we shall use the adapted coordinates on a union of several \( n \)-rectangles \( N_{\eta} \). By the convergence of \( u_n \) to \( u \), we see that there exists \( N_0 \) such that for \( n > N_0 \) and \( x \in \bar{P} \setminus N_2 \),
\[
|u_n(x) - u(x)| < \epsilon.
\]
Also by the continuity of \( u \) in \( \bar{P} \setminus N_2 \), for any \( x \) in \( \bar{P} \setminus N_{\eta} \), we have
\[
|u(y) - u(x)|, \ |u(\bar{y}) - u(x)|, \ |u(y') - u(x')|, \ |u(\bar{y}') - u(x')| < \epsilon.
\]
Then letting \( \delta \) be small enough, we get the following estimates
\[
(3.8) \int (P_{\delta'} \setminus P_{\delta}) \cap (\bar{P} \setminus N_{\eta}) (u_P)_{ij} u_n \leq C \int_{P_{\delta'} \setminus P_{\delta}} u + C \epsilon \leq C' \epsilon,
\]
\[
(3.9) \int_{\partial P_{\delta} \cap (\bar{P} \setminus N_{\eta})} (|u_n(\bar{y}) - u_n(x)| + |u_n(y') - u_n(x')|) \, d\sigma_0 \leq C \epsilon,
\]
and
\[
(3.10) \int_{\partial P_{\delta} \cap (\bar{P} \setminus N_{\eta})} (|u_n(\bar{y}) - u_n(x)| + |u_n(y) - u_n(x)|) \, d\sigma_0 \leq C \epsilon.
\]
Note that \( (u_P)_{ij} u_n \, d\sigma_0 \) converge to \( d\sigma \) on the boundary \( \partial P \) as \( \delta \) goes to 0. Thus as so as \( \delta \) is small enough we obtain
\[
| \int_{\partial P_{\delta} \cap (\bar{P} \setminus N_{\eta})} (u_P)_{ij} u_n \, d\sigma_0 - \int_{\partial P_{\delta} \cap (\bar{P} \setminus N_{\eta})} (u_P)_{ij} u_n \, d\sigma_0 | \\
\leq | \int_{\partial P_{\delta} \cap (\bar{P} \setminus N_{\eta})} (u_P)_{ij} u_n \, d\sigma_0 - \int_{\partial P_{\delta} \cap (\bar{P} \setminus N_{\eta})} (u_P)_{ij} u_n \, d\sigma_0 | + C \epsilon \\
\leq C \epsilon.
\]
For the integrals on \( N_{\eta} \), we use the convexities of \( u_n \) and estimate
\[
(3.11) \int (P_{\delta'} \setminus P_{\delta}) \cap N_{\eta} (u_P)_{ij} u_n \leq C \int_{\partial P \cap N_{2\eta}} u_n \, d\sigma.
\]
\[
\int_{\partial P \setminus N_{\eta}} (|u_n(y') - u_n(x')| + |u_n(y) - u_n(x)|) d\sigma_0 
\leq C \int_{\partial P \cap N_{2\eta}} u_n d\sigma,
\]
(3.12)
\[
\int_{\partial P \cap N_{\eta}} (|u_n(y) - u_n(x)| + |u_n(y) - u_n(x)|) d\sigma_0 
\leq C \int_{\partial P \cap N_{2\eta}} u_n d\sigma,
\]
(3.13)
\[
\int_{\partial P \cap N_{\eta}} (u_P)^{ij} u_{n;i} d\sigma_0 
\leq \int_{\partial P \cap N_{2\eta}} u_n d\sigma + C\epsilon,
\]
(3.14)
and
\[
\int_{\partial P \cap N_{\eta}} (u_P)^{ij} u_{n;i} d\sigma_0 
\leq \int_{\partial P \cap N_{2\eta}} u_n d\sigma + C\epsilon.
\]
(3.15)
Note that \(u_n\) converges uniformly on any compact subsets of \(\partial P \cap N_{2\eta}\). By choosing \(n\) sufficiently large, we have
\[
\int_{\partial P \cap N_{2\eta}} u_n d\sigma \leq \int_{\partial P \cap N_{2\eta}} \tilde{\sigma} d\sigma + C\epsilon \leq C\epsilon
\]
(3.16)
Therefore combining (3.7)-(3.16), we finally get from (3.4),
\[
\int_{P \setminus \hat{P}_\delta} \log(\det(\partial^2 u_n)) \leq C\epsilon.
\]
By letting \(\delta' \to 0\), we obtain (3.3).

By using the above lemma and the argument in the proof of Proposition 2.1, we can generalize Lemma 3.2 to the general polytope \(P\) which satisfies the delzant’s condition.

**Lemma 3.3.** Let \(P\) be a polytope which satisfies the delzant’s condition. Suppose that \(u_n \in C^K_u(P)\) converges locally uniformly to \(u \in C^K_u(P)\) for some \(K > 0\). Then there exists a subsequence of \(\{u_n\}\), still denoted by \(\{u_n\}\), and for any \(\epsilon > 0\), there exist \(\delta_0 > 0\) and a large number \(N_0\) such that for any \(0 < \delta < \delta_0\) and \(n > N_0\), (3.3) holds.

**Proof.** We also assume that
\[
\det(\partial^2 u_n) \geq 1.
\]
As in the proof Proposition 2.1, for a small enough \(\delta\), we choose a covering of finite polytope-shaped neighborhoods \(\{N_{\delta_i}\}_{i=1}^m\) of \(P \setminus \hat{P}_\delta\) which correspond
to $n$-rectangles $\{R_{X_i}\}_{i=1}^m$ with the adapted coordinates. Then we observe
\[
\int_{P \setminus P_\delta} \log(\det(\partial^2 u_n)) \leq \sum \int_{N_{X_i} \setminus P_\delta} \log(\det(\partial^2 u_n)) \\
\leq \sum \int_{N_{X_i} \setminus (N_{X_i})_{\delta}} \log(\det(\partial^2 u_n)) \\
\leq \sum \int_{R_{X_i} \setminus (R_{X_i})_{\delta'}} \log(\det(\partial^2 u_n))
\]
where $N_{X_i}_{\delta}$ and $(R_{X_i})_{\delta'}$ are the interior polygons corresponding to $N_{X_i}$ and $R_{X_i}$ with faces parallel to those of $R_{X_i}$ separated by distance $\delta$ and $\delta'$. Note that $\delta'$ is less than a scalar multiple of $\delta$ by coordinates transformation. Since $u_n \in C^K(P)$ implies $u_n \in C^{K_i}(R_{X_i})$ for some $K_i > 0$ by the convexity of $u$, the lemma will follow by applying Lemma 3.2 to each $R_{X_i}$.

Now we prove the main result in this section.

**Proposition 3.4.** Suppose that $u_n \in C^K(P)$ converge locally uniformly to $u \in C^K(P)$ for some $K > 0$, and $u_n$ satisfies (3.1). Then
\[
\int_P \log(\det(\partial^2 u)) > -\infty,
\]
and there exists a subsequence of $u_n$ such that
\[
\limsup_{n \to \infty} \int_P \log(\det(\partial^2 u_n)) \leq \int_P \log(\det(\partial^2 u))
\]

**Proof.** On the contrary, we assume
\[
\int_P \log(\det(\partial^2 u)) = -\infty.
\]
Then by Proposition 2.1, we see that for any $M > 0$, there exists $\delta_0 > 0$ such that for any $\delta < \delta_0$,
\[
\int_{P_\delta} \log(\det(\partial^2 u)) < -M.
\]
On the other hand, by Lemma 3.3, there exist $\delta_1 > 0$ and $N_1 > 0$ such that for any $0 < \delta < \delta_1$ and $n > N_1$
\[
\int_{P \setminus P_\delta} \log(\det(\partial^2 u_n)) < C\epsilon.
\]
Then for a fixed $\delta < \min(\delta_0, \delta_1)$, by Lemma 3.1, we see that there exists $N_2 > 0$ such that for $n > N_2$,
\[
\int_{P_\delta} \log(\det(\partial^2 u_n)) \leq \int_{P_\delta} \log(\det(\partial^2 u)) + \epsilon.
\]
Thus by choosing $n > N_0 = \max(N_1, N_2)$ and using (3.19)-(3.21), we get
\[
\int_P \log(\det(\partial^2 u_n)) \leq \int_{P_5} \log(\det(\partial^2 u_n)) + C\epsilon \\
\leq \int_{P_5} \log(\det(\partial^2 u)) + (C + 1)\epsilon \\
\leq -M + (C + 1)\epsilon.
\]

Since $M$ is arbitrary, we derive
\[
\limsup_{n \to \infty} \int_P \log(\det(\partial^2 u_n)) = -\infty.
\]
This is a contradiction. Hence we prove
\[
-\infty < \int_P \log \det(\partial^2 u) < \infty. \tag{3.22}
\]

By (3.22) we see that there exist $\delta_0 > 0$ and $N_3 > 0$ such that for any $0 < \delta < \delta_0$,
\[
\int_{P \setminus P_5} |\log(\det(\partial^2 u))| < \epsilon. \tag{3.23}
\]

Combining (3.20) and (3.21), we get
\[
\int_P \log(\det(\partial^2 u_n)) \leq \int_{P_5} \log(\det(\partial^2 u_n)) + C\epsilon \\
\leq \int_{P_5} \log(\det(\partial^2 u)) + (C + 1)\epsilon \\
\leq \int_P \log(\det(\partial^2 u)) + (C + 2)\epsilon.
\]

Letting $\epsilon \to 0$, we will obtain (3.16) \qed

**Remark 3.5.** Proposition 3.4 implies that the functional $\mathcal{F}(u)$ is lower semi-continuous in $\mathcal{C}_*$ since $L(u)$ is lower semi-continue in $\mathcal{C}_*$.

### 4. Proof of Theorem 0.2

In this section, we prove the existence of minimizing weak solution for the extremal metrics.

**Proposition 4.1.** Suppose that the modified $\mathcal{K}$-energy $\mu(\phi)$ is proper in $\mathcal{M}_{G_0}$ associated to toric actions group $T$ on a toric manifold $M$. Then for any minimizing sequence $u_n$ in $\tilde{\mathcal{C}}$ there exists a subsequence of $u_n$, still denoted by $u_n$, which locally uniformly converge to a convex function $u_\infty$ such that
\[
\mathcal{F}(u_\infty) = \inf_{\tilde{\mathcal{C}}} \mathcal{F}(\cdot).
\]
Moreover, \( u_n|_{\partial P} \to u_\infty|_{\partial P} \) as \( n \to \infty \) almost everywhere.

**Proof.** By the assumption of properness of \( \mu(\phi) \) and Proposition 1.3, one sees that there exist constant \( C_1 \) and \( C_2 > 0 \) such that for any normalized minimizing sequence \( u_n \) in \( \mathcal{C} \),

\[
\int_P u_n \, dx < C_1, \quad -\int_P \log(\det(\partial^2 u_n)) + \int_{\partial P} u_n \, d\sigma < C_2.
\]

On the other hand, by (2.14) in Remark 2.6, we know that for a positive \( c < 1 \) there exist

\[
C = C(c) \quad \text{and} \quad C' = C'(c)
\]

such that

\[
-\int_P \log(\det(\partial^2 u_n)) \geq -c \int_{\partial P} u_n \, d\sigma - C \int_{\partial P} u_n \, dx - C'.
\]

Combining these two inequalities, we get

\[
\int_{\partial P} u_n \, d\sigma \leq \max_{i=1,...,d} \left\{ \frac{1}{|l_i|} \right\} \int_{\partial P} u_n \, d\sigma_0 < K_0
\]

for some \( K_0 > 0 \). This implies \( u_n \in \mathcal{C}_K^*(P) \) for any \( K \geq K_0 \). Thus by the convexities of \( u_n \) there exists a subsequence of \( \{u_n\} \), still denoted by \( \{u_n\} \) which locally uniformly converges to a normalized function \( u_\infty \) in \( \mathcal{C}_*(K_0) \) satisfying

\[
\int_{\partial P} u_\infty \, d\sigma \leq \liminf_{n \to \infty} \int_{\partial P} u_n \, d\sigma < K.
\]

On the other hand, by (4.3) together with (4.1) and (4.2), we also have

\[
\int_P \log(\det(\partial^2 u_\infty)) > -C_0
\]

for some \( C_0 \). Then by Remark 3.5, we get

\[
\mathcal{F}(u_\infty) \leq \liminf_{n \to \infty} \mathcal{F}(u_n) = \inf_{\hat{c}} \mathcal{F}(\cdot).
\]

Hence by Corollary 2.8, we prove

\[
\mathcal{F}(u_\infty) = \inf_{\mathcal{C}_K^*(P)} \mathcal{F}(\cdot) = \inf_{\hat{c}} \mathcal{F}(\cdot).
\]

It remains to prove

\[
u_\infty|_{\partial P} = \hat{u},
\]

where \( \hat{u} = \lim_{n \to \infty} u_n|_{\partial P} \). In fact,

\[
-\int_P \log(\det(\partial^2 u_\infty)) \, dx + \int_{\partial P} \hat{u} \, d\sigma - \int_P u_\infty \, dx
\]

is the minimum of the functional \( \mathcal{F}(\cdot) \) in \( \mathcal{C}_K^*(P) \). This implies

\[
\int_{\partial P} \hat{u} \, d\sigma = \int_{\partial P} u_\infty \, d\sigma.
\]

Thus by the fact \( u_\infty|_{\partial P} \leq \hat{u} \), we conclude that \( u_n|_{\partial P} \to u_\infty|_{\partial P} \) as \( n \to \infty \) almost everywhere.
In the proof of (4.3), we used the estimate (4.2) established in Section 2. In the following we give a direct proof to (4.3). Let \{u_i\} be a minimizing sequence of \(F(u)\) in \(\tilde{C}\). By (1.5), we may assume that for each \(i\) it holds
\[ F(u_i) = \inf_\lambda F(\lambda u_i). \]
Note \(\lambda u_i \in C_\infty\) for any \(\lambda > 0\). We claim
\[ \int_{\partial P} u_i d\sigma = t_i \leq K. \]
Let \(\overline{u}_i = \frac{u_i}{t_i}\) so that
\[ \int_{\partial P} \overline{u}_id\sigma = 1, \text{ for each } i = 1, 2, \ldots \]
Then \(t_i\) is a critical point of function \(F(\lambda u_i)\) for \(\lambda\). So we have
\[ n\text{Vol}(P) t_i = L(\pi_i). \]
Thus the claim will be true if
\[ L(\pi_i) \geq \delta > 0, \text{ for sufficiently large } i. \]
Suppose that the above is not true. Then there exists a subsequence \(\pi_{i_k}\) such that
\[ \lim_{i_k} L(\pi_{i_k}) = 0. \]
It follows
\[ \lim_{i_k} \int_P (\bar{R} + \theta_X) \pi_{i_k} dx = \lim_{i_k} \int_{\partial P} \pi_{i_k} d\sigma = 1. \]
Thus
\[ \int_P \pi_{i_k} dx \geq \delta' > 0, \text{ for sufficiently large } i. \]
On the other hand, by the properness of \(F(u)\), we have
\[ C \geq F(u_i) \geq p(t_i \int_P \pi_i dx), \text{ for each } i = 1, 2, \ldots \]
Hence we get \(t_{i_k} \leq C'\) and
\[ L(\pi_{i_k}) = \frac{n}{t_{i_k}} \geq \frac{n}{C'}. \]
The late is contradict to the assumption (4.5). Therefore the claim is true. (4.4) implies (4.3).

Theorem 0.2 follows from Proposition 4.1. We also have the following corollary.
Corollary 4.2. Let $M$ be a toric Kähler manifold associated to a convex polytope $P$ described by (0.3). Suppose
\begin{equation}
\bar{R} + \theta_X < \frac{n+1}{\lambda_i}, \quad \forall \ i = 1, ..., d.
\end{equation}
Then there exists a minimizing weak solution of equation (0.5) in $M$ in the sense of convex functions.

Proof. It was proved in [ZZ] that under the condition (4.6) we have
\[
F(u) \geq c \int_P u dx - C, \quad \forall u \in \tilde{C}
\]
for some $c, C > 0$. In particular, the modified $K$-energy $\mu(\phi)$ is proper in $\mathcal{M}_{G_0}$ associated to toric actions group $T$. Thus by Proposition 4.1, the corollary is true. \qed

Next we discuss some properties about the minimizing weak solution $u_\infty$.

Proposition 4.3. Suppose that
\[
\theta_X + \bar{R} \geq 0.
\]
Then for any minimizer $u_\infty$ of $F(\cdot)$, the Monge-Ampere measure $\mu[u_\infty]$ has no singular part.

Proof. We use an argument from [TW2] to prove the proposition. Suppose $\mu[u_\infty]$ has non-vanishing singular part $\mu_s[u_\infty]$. Then for any $M > 0$, there must exist a ball $B_r \subset P$ such that
\begin{equation}
\mu_s[u_\infty](B_r) \geq M(\mu_r[u_\infty](B_r) + |B_r|).
\end{equation}
We consider the following Dirichlet problem for Monge-Ampère operator,
\[
\begin{cases}
\mu[v] = M\mu_r[u_\infty] + M \text{ in } B_r, \\
v = u_\infty \text{ on } \partial B_r.
\end{cases}
\]
By the Alexander theorem, the above equation has a unique convex solution $v$. Note
\begin{equation}
\det(\partial^2 v) = M \det(\partial^2 u_\infty) + M, \quad \text{in } B_r.
\end{equation}
By comparison principle, $u_\infty \leq v$ in $B_r$, and the set $E = \{v > u_\infty\}$ is not empty. Define another convex function $\tilde{u}$ by
\[
\begin{cases}
\tilde{u} = u \text{ in } P\backslash E, \\
\tilde{u} = v \text{ in } E.
\end{cases}
\]
We claim $\mathcal{F}(\tilde{u}) < \mathcal{F}(u_\infty)$, so we get a contradiction to the assumption that $u$ is a minimizer. In fact, by choosing $M$ sufficiently large and using (4.8), we have
\[
\mathcal{F}(\tilde{u}) - \mathcal{F}(u_\infty) = -\int_E \log(\det(\partial^2 v))dx + \int_E \log(\det(\partial^2 u_\infty))dx - \int_E (\theta_X + \mathcal{R})(v - u_\infty)dx
\]
\[
\leq -(\log M)|E| - \int_E \log(\det(\partial^2 u_\infty) + 1)dx + \int_E \log(\det(\partial^2 u_\infty))dx
\]
\[
< 0.
\]
The proposition is proved.

Recall $u$ a piecewise linear (PL) function on $P$ if $u$ is a form of
\[
u = \max\{u^1, \ldots, u^r\},
\]
where $u^\lambda = \sum a^\lambda_i x_i + c^\lambda$, $\lambda = 1, \ldots, r$, for some vectors $(a^\lambda_i) \in \mathbb{R}^n$ and some numbers $c^\lambda \in \mathbb{R}$. In particular, when $r = 2$ and $u^2 = 0$, we call $u$ simple PL function with crease $\{u^1 = 0\}$. The following Lemma was proved in [ZZ].

**Lemma 4.4.** Suppose that for each $i = 1, \ldots, d$, it holds
\[
\bar{R} + \theta_X < \frac{n+1}{\lambda_i}, \quad \text{in} \ P.
\]
Then for any PL-function $u$ on $P$, we have
\[
L(u) \geq 0.
\]
Moreover the equality holds if and only if $u$ is an affine linear function.

**Proposition 4.5.** Suppose that for each $i = 1, \ldots, d$, it holds
\[
\bar{R} + \theta_X < \frac{n+1}{\lambda_i}, \quad \text{in} \ P.
\]
Let $u$ be a minimizer of $\mathcal{F}(\cdot)$. Then for any point $x$ in $P$, there exists only one supporting hyperplane at $x$.

**Proof.** Suppose by contrary that there are two distinct supporting hyperplane at $x_0$ in $P$. By adding an affine linear function, we may assume that one of them is zero hyperplane and the other is
\[
\sum_{i=1}^n a_i x_i - x_{n+1} + c = 0.
\]
Then we define a simple PL function $f$ with crease through $x_0$ by
\[
f = \max\{\sum_{i=1}^n a_i x_i + c, 0\}
\]
and write $u$ as

$$u = f + (u - f).$$

It is easy to see that $u - f$ is a convex function $P$. Since $x_0$ lies in $P$, by the lemma above we have $L(f) > 0$. On the other hand,

$$\det(\partial^2 u) = \det(\partial^2 (u - f))$$

almost everywhere. Thus

$$\mathcal{F}(u - f) < \mathcal{F}(u),$$

which is contradict to that $u$ is minimizer. Hence the proposition is true. □

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