Belyi parametrisations of elliptic curves and congruence defects

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Abstract: We consider refined integral questions about modular parametrisations of elliptic curves and raise questions about refinements of Belyi’s result in the context of parametrisations of elliptic curves defined over number fields.

Given an elliptic curve $E$ defined over a number field $K$ it follows from Belyi’s theorem (cf. page 71 of [S]) that there is a finite index subgroup $\Gamma$ of $SL_2(\mathbb{Z})$ and a surjective mapping $\mathcal{H}/\Gamma \cup \{\text{cusps}\} := X_\Gamma \rightarrow E$ of Riemann surfaces with $\mathcal{H}$ the upper half-plane. Further given any finite set of points $S$ of $E$ we can choose $\Gamma$ and the mapping so that $S$ is contained in the image of the cusps. This is called a hyperbolic parametrisation of $E$ in Mazur’s article [M].

We will say that such a $\Gamma$ covers $E$. One may choose a $\Gamma$ to have an isomorphism $\mathcal{H}/\Gamma \cup \{\text{cusps}\} := X_\Gamma \simeq E$ of Riemann surfaces. For us the more flexible notion of $\Gamma$ covering $E$ will be more useful.

In a sense the result of Belyi takes in a sense no heed of the properties of the number field $K$ over which $E$ is defined. We can ask for a deepening of Belyi’s result by asking if we can choose $\Gamma$ covering $E$ so that it reflects some of the arithmetical properties of $E$, like its field of definition. It is the goal of this brief note to raise some questions about such a deepening of Belyi’s theorem.

The first clue that such a deepening might exist is the Shimura-Taniyama-Weil conjecture (now a theorem, cf. [BCDT]) that says in the case that $E$ is defined over $\mathbb{Q}$ one can choose a $\Gamma$ that covers $E$ to be a congruence subgroup of $SL_2(\mathbb{Z})$. This is what is called a hyperbolic parametrisation of arithmetic type in [M]. In the appendix to that article it is shown that if $E$ defined over $\mathbb{Q}$ has a hyperbolic parametrisation of arithmetic type then there is an $N$
and a non-constant map $X_0(N) \to E$ defined over $\mathbb{Q}$.

We begin by studying in Section 1 the better-chartered territory of modular parametrisations of elliptic curves where certain issues arise that are easy to see “up to isogeny”, and that we can respond to in certain situations (see Propositions 1 and 2 below) but that we are unable to resolve satisfactorily on the nose. These issues are related to results in [Ri1] controlling kernels of degeneracy maps between Jacobians of modular curves.

Throughout this note we assume the elliptic curves considered do not have CM, as in the CM case the issues we deal with about “Belyi parametrisations” in Section 2 are known (see [Sh1]), and for the considerations of Section 1 this is a simplifying assumption.

1. Modular parametrisations of elliptic curves

Given a positive integer $M$ we consider the modular curve $X_1(M)$ with its canonical model defined over $\mathbb{Q}$ (see [Sh]) in which the 0-cusp is a rational point.

**Definition 1** 1. Let $E$ be an elliptic curve defined over a number field $K$. A $K$-modular parametrisation of $E$ is a non-constant algebraic map $\phi : X_1(M) \to E$, for some positive integer $M$, defined over $K$ that we normalise by requiring that $\phi$ sends the 0-cusp to the origin.

2. Let $E$ be an elliptic curve defined over $\mathbb{Q}$. A $\mathbb{Q}$-modular parametrisation of $E$ is a $K$-modular parametrisation of $E$ for some number field $K$.

**Remark:** Our definitions are inspired by those in [St]. We have relaxed the condition in [St] that the pull-back of a differential of $E$ is a non-zero multiple of the differential on $X_1(N)$ associated to a newform (see [St] for fine conjectures about these constants). This is because later we want to consider “Belyi parametrisations” in which case there is no notion of newforms. The condition that the 0-cusp be sent to the origin is included for ease of comparison with [St].

1.1 $\mathbb{Q}$-modular parametrisations

Let $E$ be a non-CM elliptic curve defined over $\mathbb{Q}$. It is now known that there is a $\mathbb{Q}$-modular parametrisation $X_1(N) \to E$, where $N$ is the conductor of the elliptic curve. This induces by Picard functoriality a map $E \to J_1(N)$: this map that is induced by a $\mathbb{Q}$-modular parametrisation will be referred
to again as a $\mathbb{Q}$-modular parametrisation. (We employ a similar abuse of notation for $\overline{\mathbb{Q}}$-modular parametrisations.) This is justified by noting that starting from a non-trivial homomorphism of abelian varieties $E \to J_1(N)$ defined over $\mathbb{Q}$, we obtain a map $J_1(N) \to E$ by duality and then a map $X_1(N) \to E$ defined over $\mathbb{Q}$ that is the composition $X_1(N) \to J_1(N) \to E$ where $X_1(N) \to J_1(N)$ is with respect to the 0-cusp. This map is non-trivial as the first map induces an isomorphism of cotangent spaces.

If there is a $\mathbb{Q}$-modular parametrisation $\phi : X_1(M) \to E$ for some positive integer $M$, then $N$ divides $M$. For each divisor $d$ of $M/N$ recall that there is a map $\alpha_d^* : J_1(N) \to J_1(M)$ induced by Picard functoriality from the standard degeneracy map $\alpha_d : X_1(M) \to X_1(N)$.

In Theorem 1.9 of [St] it is proved that any modular parametrisation $X_1(M) \to E$ (in the sense of [St]) factors through a modular parametrisation $X_1(N) \to E$. In loc. cit. this is proved using the requirement asked of a modular parametrisation there that the pull-back of a differential of $E$ is the differential of a newform up to scalars (which gives the result “up to isogeny”), and the fact that the natural map $J_1(N) \to J_1(M)$ is injective (which gives the result “on the nose”). We would similarly expect a similar property for the $\mathbb{Q}$-modular parametrisations considered here. Namely we would expect that any $\mathbb{Q}$-modular parametrisation $E \to J_1(M)$, factors through a $\mathbb{Q}$-modular parametrisation $E \to J_1(N)$ via a map $\sum_{d \mid M/N} a_d \alpha_d^* : J_1(N) \to J_1(M)$ with $d$ divisors of $M/N$, and $a_d \in \mathbb{Z}$. But our relaxing one of the conditions of modular parametrisations in [St] introduces certain complications, as we have to contend with more endomorphisms of the Jacobian of the pro-modular curve $\hat{X}$ of [St] than in loc. cit.: more explicitly what we have to control is kernels of sums of certain degeneracy maps $J_1(N) \times \cdots \times J_1(N) \to J_1(M)$. We can prove a universal property at the moment only under the additional assumption that $M$ is squarefree.

**Proposition 1** Let $M$ be a square-free integer. Any $\mathbb{Q}$-modular parametrisation $E \to J_1(M)$ factors through a $\mathbb{Q}$-modular parametrisation $E \to J_1(N)$ via a map $\sum_{d \mid M/N} a_d \alpha_d^* : J_1(N) \to J_1(M)$ with the $a_d$’s integers and $d$ divisors of $M/N$.

**Proof:** The proposition follows “up to isogeny” easily from Eichler-Shimura theory in Chapter 7 of [Sh], Carayol’s theorem of the equality of the geometric and analytic conductor and the Atkin-Lehner theory of newforms: namely the decomposition up to isogeny of the Jacobians $J_1(N), J_1(M)$ into blocks.
formed of simple \(\mathbb{Q}\)-abelian varieties \(A_f\)'s with \(f\)'s running through \(G_{\mathbb{Q}}\) conjugacy classes of newforms, and the determination of the Galois action on the Tate modules of \(A_f\) in terms of the Fourier expansion of the newform \(f\). The blocks arise from packets of oldforms attached to \(f\), which in turn arises from the degeneracy maps \(\alpha_d\) recalled above. For example from this general theory it is clear that the image of \(E\) is contained in the abelian subvariety of \(J_1(M)\) which is the image of \(E' \times \cdots \times E' \subset J_1(N)^{\sigma_0(M/N)}\) (with \(E'\) the optimal curve in the \(\mathbb{Q}\)-isogeny class of \(E\) and \(\sigma_0(M/N)\) the number of divisors of \(M/N\)) in \(J_1(M)\) under the sum of the degeneracy maps \(\sum_{d|MN} \alpha_d^* J_1(N) \rightarrow J_1(M)\). The more refined statement as claimed in the proposition will follow (see Theorem 1.9 of [St]) if we show that the degeneracy map \(\sum_{d|MN} \alpha_d^* J_1(N) \rightarrow J_1(M)\) is injective. This follows from [Ri1] which shows that the degeneracy map \(J_1(N')^2 \rightarrow J_1(N'p)\) is injective if \(p\) is a prime that does not divide \(N'\), for any positive integer \(N'\).

Remarks:

1. By using the method of proof of Theorem 1.6 of [St] one may show that for arbitrary \(M\) a \(\mathbb{Q}\)-modular parametrisation \(\phi : X_1(M) \rightarrow E\) is such that \(\phi^*(\omega_E) = (\sum_{d|MN} a_d \alpha_d^*)(f)\) where \(f\) is the newform associated to \(E\) and \(\omega_E\) a Neron differential of \(E\) and \(a_d \in \mathbb{Z}\).

2. The reason that we have to assume that \(M\) is square-free (or what is really needed to apply [Ri1], the assumption that \((N, M/N) = 1\) and \(M/N\) is square-free) is that in [K1] (where the analog of the result of [Ri1] is proven in the case when \(p|N'\) employing the notation of the proof above) the groups of connected components of kernels of the degeneracy maps considered there are controlled only up to Eisenstein “errors”. If this result could be refined as suggested there (Remark 2.4 on page 641 of [K1]) to show that these groups of connected components are images of Shimura subgroups (see [LO]) we could drop the assumption that \(M\) is squarefree.

1.2 \(\overline{\mathbb{Q}}\)-modular parametrisations

Let \(E\) be a (non-CM) elliptic curve now defined over \(\overline{\mathbb{Q}}\). \(\overline{\mathbb{Q}}\)-modular parametrisations are considered in Ribet’s article [Ri2]. There it is shown that if \(E\) has a \(\overline{\mathbb{Q}}\)-modular parametrisation then \(E\) is a \(\mathbb{Q}\)-curve, i.e., \(E\) is isogenous (over \(\overline{\mathbb{Q}}\)) to all its conjugates.

As a partial converse Ribet showed that any \(\mathbb{Q}\)-curve arises as a factor of a \(GL_2\)-type abelian variety defined over \(\mathbb{Q}\) (see loc. cit.): for this the main
ingredient in [Ri2] is the result of Tate that for the absolute Galois group $G_K$ of a number field $K$, $H^2(G_K, \overline{Q})$ vanishes where the coefficients have trivial $G_K$-action. Conjecturally (for instance Serre’s conjectures imply this) $GL_2$-type abelian varieties over $Q$ are modular, and thus it would follow that all $Q$-curves $E$ have $\overline{Q}$-modular parametrisations. Recent work of Hida (Chapter 5.2 of [H]) and Ellenberg-Skinner, cf. [ES], proves much of this conjecture.

By the result of Tate and the fact that $E$ is non-CM, for a sufficiently small open subgroup $G_K$ of $G_Q$ the $p$-adic representation $G_K \to GL_2(Q_p)$ attached to a $Q$-curve $E$ extends to a representation $G_Q \to \overline{Q}_p GL_2(Q_p)$ that is unique up to twists. Let $E$ have a $\overline{Q}$-modular parametrisation. Let $\rho_E : G_Q \to \overline{Q}_p GL_2(Q_p)$ be a representation that extends the $G_K$-representation afforded by the $p$-adic Tate module of $E$ and that has minimal conductor (say $N$) amongst its twists. Recall that given a newform $f$ and a Dirichlet character $\chi$ we can form its twist $f \otimes \chi$ (which may no longer be a newform). The Galois representations “associated” to these modular forms (that are eigenforms for almost all Hecke operators) are related by twisting by $\chi$: this map (up to a Gauss sum factor) is induced by a certain map of Jacobians of modular curves $R_\chi$ (see [Sh2]). The following proposition is known (for the second part see the appendix of [M]).

**Proposition 2** Let $E$ be a non-CM elliptic curve defined over a number field $K$.

1. If $\phi : X_1(M) \to E$ is a $\overline{Q}$-modular parametrisation then $M$ is divisible by $N$ and the representation $\rho_E$ is attached to a newform $f$. Further there is such a parametrisation with $M = N$. The pull-back of a differential on $E$ under $\phi$ is in the $\overline{Q}$-linear span of images of $f$ under compositions of the degeneracy maps $\alpha^*_M$ and twisting maps $R_\chi$.

2. If $E$ has a $\overline{Q}$-modular parametrisation $X_1(M') \to E$ for some integer $M'$, then it also has a $K$-modular parametrisation $X_1(M) \to E$ for some (possibly different) integer $M$.

**Proof:** The first part follows from what was said earlier, $E$ being non-CM and the Eichler-Shimura theory of the decomposition of the Jacobians of modular curves as under the hypothesis $\rho_E$ is the Eichler-Shimura representation attached to a newform. We now take up the second part. Let $X_1(M') \to E$ be a $\overline{Q}$-modular parametrisation for some integer $M'$. Then as $J_1(M')$ breaks (up to isogeny) over $K$ into the product of abelian varieties $A_f$ attached to conjugacy classes of Hecke eigenforms (which may not
be simple) we deduce from Eicheler-Shimura theory that there is a weight 2 Hecke eigenform $f$ such that the $\wp$-adic $G_K$-representation $\rho_{f,\wp}$ attached to $f$ is isomorphic to the $p$-adic representation $\rho_{E,p}$ associated to $E$ on restriction to an open subgroup of $G_\mathbb{Q}$ with $\wp$ a place above $p$. By the hypothesis on $E$ it follows that this restriction is irreducible and thus the $G_K$-representations $\rho_{f,\wp}$ and $\rho_{E,p}$ are isomorphic up to twisting. Then considering a certain twist $g$ of $f$ we get that $\text{Hom}_{G_K}(\text{Ta}_p(E), \text{Ta}_p(J_1(M)))$ is non-zero for some $M$ and using Falting’s isogeny we get that there is a morphism defined over $K$ from $E \to J_1(M)$. Dualising this map and (pre)composing with the morphism $X_1(M) \to J_1(M)$ with respect to the 0-cusp the proposition follows.

One would want to prove that any $\mathbb{Q}$-modular parametrisation $\phi : X_1(M) \to E$ of $\mathbb{Q}$-curves $E$ induces a map $E \to J_1(M)$ that factors through a $\mathbb{Q}$-modular parametrisation $E \to J_1(N)$ via a morphism $J_1(N) \to J_1(M)$ (recall that $N$ is the least conductor of twists of $G_\mathbb{Q}$-representations which extend the $p$-adic representation attached to $E$ of $G_K$ where $K$ is any field over which $E$ gets defined): this is again easy to prove “up to isogeny”. By considering the pull-back $\phi^*$ on differentials we see that this morphism will be a “$\mathbb{Q}$-linear combination” of compositions of morphisms $\alpha_d^*$s and $R_\chi$’s for some integers $d$ and Dirichlet characters $\chi$.

In the context of Proposition [\[4\] we can prove a weaker universal property without the assumption that $M$ is square-free. We state this result while omitting its proof which is along the lines of Proposition [\[4\] and uses the results of both [K1] and [Ri1] (see also the remarks after Proposition [\[4\]), and some of the considerations in this section. We would hope that it will be possible to prove a cleaner result in the future.

**Proposition 3** Let $K$ be a number field that does not contain any non-trivial abelian extension of $\mathbb{Q}$. Let $E$ be a non-CM elliptic curve over $K$ and $p$ be a “good” prime, i.e., the semisimplification of $\rho_{E,p}$, does not come by restriction from an abelian $G_\mathbb{Q}$-representation. Consider a $K$-modular parametrisation $E \to J_1(M)$ and the induced $G_K$-equivariant map $\text{Ta}_p(E) \to \text{Ta}_p(J_1(M))$ on $p$-adic Tate modules. Then the latter map factors through a $G_K$-equivariant map $\text{Ta}_p(J_1(N)) \to \text{Ta}_p(J_1(M))$.

**Remarks:**

1. This is an easy result for $\mathbb{Q}$-parametrisations as the result is non-trivial only for primes $p$ such that the mod $p$ representation is not irreducible as a
$G_K$-module. The result is again easy up to isogeny, i.e., up to tensoring with $\mathbb{Q}_p$, but integrally does not follow from generalities as in non-trivial cases the $G_K$ representation afforded by the $p$-adic Tate module of $E$ is residually reducible.

2. The condition that $K$ does not contain non-trivial abelian extensions of $\mathbb{Q}$ avoids difficulties arising from the twisting operators $R_\chi$. We cannot do better than this as we do not have exact control of kernels of twisting maps $R_\chi$ (see [K2] for a result in this direction), and even less of kernels of sums of twisting maps and degeneracy maps. We would expect a universal property as in the proposition above to hold without any assumptions on $K$ (i.e., for $\overline{\mathbb{Q}}$-modular parametrisations). This time the “good” $p$ would be those such that the semisimplification of the $G_K$-module $E[p]$ does not come via restriction from an abelian $G_{\mathbb{Q}_{\text{ab}}}$-representation: note that the maps $R_\chi$ are defined over $\mathbb{Q}_{\text{ab}}$.

2. Belyi parametrisations of elliptic curves

Now leaving the highly structured world of modular curves we consider curves that arise from arbitrary finite index subgroups of $SL_2(\mathbb{Z})$. As seen above this is forced on us when considering hyperbolic parametrisations of non-$\mathbb{Q}$ elliptic curves $E$ defined over $\mathbb{Q}$.

Definition 2 Let $E$ be an elliptic curve defined over $\overline{\mathbb{Q}}$.

1. A Belyi parametrisation of $E$ is a non-constant map algebraic map $X_\Gamma \to E$ (defined over $\overline{\mathbb{Q}}$), with $X_\Gamma$ the projective curve (defined over $\overline{\mathbb{Q}}$) associated to a subgroup of finite index $\Gamma$ of $SL_2(\mathbb{Z})$, that we normalise by requiring that the 0-cusp is sent to the origin. We say that such a $\Gamma$ covers $E$.

2. Given a finite index subgroup $\Gamma$ of $SL_2(\mathbb{Z})$ its congruence hull $\Gamma^c$ is the smallest congruence subgroup which contains $\Gamma$: thus $\Gamma^c$ is the intersection of all congruence subgroups of $SL_2(\mathbb{Z})$ which contain $\Gamma$. The congruence defect $cd_\Gamma$ of $\Gamma$ is the index $[\Gamma^c : \Gamma]$.

3. The congruence defect $cd_E$ of $E$ is the smallest congruence defect of a finite index subgroup $\Gamma$ of $SL_2(\mathbb{Z})$ that covers $E$.

By Belyi’s theorem, a Belyi parametrisation of $E$ always exists and thus $cd_E$ is a well-defined invariant associated to the isomorphism class of $E$ (or equivalently to its $j$-invariant): in fact $cd_E$ depends only on the $\overline{\mathbb{Q}}$-isogeny class of $E$. 

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**Question 1** Given a number field $K$ is there a constant $c_K$, such that for all elliptic curves $E$ defined over $K$, $cd_E \leq c_K$?

The STW conjecture answers this affirmatively when $K = \mathbb{Q}$: all elliptic curves that are defined over $\mathbb{Q}$ have congruence defect 1. This is the only piece of evidence for an affirmative answer to the question. We have seen above that one has a (partly conjectural) characterisation of the class of $E$’s with congruence defect 1 as precisely the class of $\mathbb{Q}$-curves. From what we have seen above the appropriate field when considering these questions is:

**Definition 3** The $\mathbb{Q}$-field of an elliptic curve $E$ defined over $\mathbb{Q}$ is the fixed field of the open subgroup of $G_{\mathbb{Q}}$ which fixes the $\mathbb{Q}$-isogeny class of $E$.

Thus the $\mathbb{Q}$-field of $E$ is $\mathbb{Q}$ precisely when $E$ is a $\mathbb{Q}$-curve. A refinement of the question above would be:

**Question 2** Given a number field $K$ is there a constant $c'_K$, such that any elliptic curve with $\mathbb{Q}$-field $K$, $cd_E \leq c'_K$?

**Remarks:**

1. Our questions are naive from the point of view of modular parametrisations (especially from the modern automorphic viewpoint) as say when wanting to parametrise $E$ defined over a totally real number field $K$ of odd degree by quotients of the upper half plane one switches to a quaternion algebra defined over $K$ ramified at all but one infinite place and considers Shimura curves arising from congruence subgroups of this quaternion algebra. Note on the other hand that when the $\mathbb{Q}$-field of $E$ is not totally real one does not have in a direct fashion an “automorphic parametrisation”.

2. It would be of interest to gather computational evidence towards an answer to the questions above. J-P. Serre in an e-mail to the author in response to a message posing the first question above suggested testing out the question in the following situation: Consider $K = \mathbb{Q}(i)$, and a specific elliptic curve over $K$, say one with its $j$-invariant $\frac{1}{n+i}$; $n \geq 2$. Such a curve cannot be uniformized by a congruence subgroup, as since its $j$-invariant is $\frac{1}{n+i}$ it has multiplicative reduction at primes dividing $n+i$ and hence cannot be geometrically isogenous, i.e., isogenous over $\mathbb{Q}$, to its conjugate. What kind of function of $n$ is the congruence defect of such a curve?
3. It will be interesting to see if answers to the questions above have interesting diophantine consequences similar to the rich diophantine consequences of existence of modular parametrisations of elliptic curves.

4. In the enriched situation of modular parametrisations that we considered earlier, by the associated Eichler-Shimura theory, all such parametrisations were “related up to isogeny”. There is no question of expecting such a property of the ensemble of Belyi parametrisations of an elliptic curve $E$ as there is just too much freedom (in for instance the choice of the map $\phi$ above, or in choosing the set $S$ of $E$ which will be in the image of the cusps under such a parametrisation). In other words we cannot expect that there is one Belyi parametrisation $X_\Gamma \to E$ which “generates” all others.

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