SELF-ADJOINT LAPLACIANS ON PARTIALLY AND GENERALIZED HYPERBOLIC ATTRACTORS

SHAYAN ALIKHANLOO¹, MICHAEL HINZ²

ABSTRACT. We construct self-adjoint Laplacians and symmetric Markov semi-groups on partially hyperbolic attractors and on hyperbolic attractors with singularities, endowed with Gibbs u-measures. If the measure has full support, we can also guarantee the existence of an associated symmetric Hunt diffusion process. In the special case of partially hyperbolic diffeomorphisms induced by geodesic flows on manifolds of negative sectional curvature the Laplacians we consider are self-adjoint extensions of well-known classical leafwise Laplacians.

CONTENTS
1. Introduction 1
2. Dirichlet forms and self-adjoint Laplacians 2
3. Regularity and symmetric diffusion processes 7
4. Partially hyperbolic attractors 8
5. Geodesic flows on manifolds with negative curvature 10
6. Hyperbolic attractors with singularities 12
References 16

1. INTRODUCTION

We construct self-adjoint Laplacians on partially hyperbolic attractors, [9, 11, 24, 40, 42, 45, 13, Section 5], and on hyperbolic attractors with singularities, [28, 30, 39, 47, 49, 13, Section 8], endowed with Gibbs u-measures, [13, 39, 42]. In [2] we had already studied self-adjoint Laplacians on uniformly hyperbolic attractors, endowed with SRB-measures, and the present article may be viewed as a continuation of this research.

Here we have two principal goals: The first is to define such Laplacians in situations that are more general than uniform hyperbolicity, but in which the same method can still be applied rather easily. This allows to extend our analysis to many prominent classes of examples, such as geodesic flows on negatively curved manifolds, [4, 6, 10, 12, 16, 18, 22, 27, 43], or attractors of Lorenz, Lozi or Belykh type, [7, 34, 36, 37, 39, 48], or [13, Section 8]. The second principal goal is to point out that in the special case of geodesic flows our construction recovers an analysis that

2010 Mathematics Subject Classification. 31C25, 37D10, 37D30, 37D35, 37D40, 37D45, 47A07, 47B25, 47D07, 60J60.

Key words and phrases. Hyperbolic attractors, Gibbs u-measures, Dirichlet forms, self-adjoint operators, semigroups, diffusions.

¹, ² Research supported by the DFG IRTG 2235: 'Searching for the regular in the irregular: Analysis of singular and random systems'.

arXiv:2105.04470v1 [math.DS] 10 May 2021
had already been established a long time ago, see for instance [52,53]. The extension of this kind of analysis to the attractors of dissipative hyperbolic dynamical systems, started in [2] and continued here, is new. Since we wish to construct Laplacians self-adjoint with respect to Gibbs u-measures, they must locally be superpositions of symmetric Laplacians on local unstable manifolds, endowed with the conditional measures. The main difference to the geodesic flow case is that in general the conditional densities of the Gibbs u-measure may only be Hölder continuous, and this is not sufficient to introduce a 'classical' leafwise Laplacian on functions that are $C^2$ in the unstable directions. We view the situation from a quadratic forms perspective, [1,8,15,19], as it is common in mathematical physics, [44, Section VIII.6], and partial differential equations, [17, Chapter 6]. Based on the stable manifold theorem and the notion of Gibbs u-measure, it is not difficult to construct 'natural' Dirichlet forms, and we regard the unique self-adjoint operators associated with such forms as 'natural' self-adjoint Laplacians on the attractor. The existence of a measure with suitable properties is a main ingredient, see [3] for a related study.

In [2] we had already used the same approach to construct self-adjoint Laplacians on uniformly hyperbolic attractors. Here we distill a simplified abstract version of the basic argument, and this simplification allows to easily apply it to partially hyperbolic attractors and to attractors with singularities. Because compared to [2] subtle details need to be changed or added (such as the definition of rectangles or an additional approximation step), we provide a proof for the main argument, Lemma 2.1. The fact that the present analysis may be seen as a generalization of known results for geodesic flows - which are examples of partially hyperbolic systems - had not been discussed in [2].

We proceed as follows. In Section 2 we introduce a general abstract setup and prove our main results, Theorem 2.1 and Corollary 2.1, on the existence of self-adjoint Laplacians and symmetric Markov semigroups. Section 3 discusses the case when the Gibbs u-measure has full support, in this case we can invoke the theory of regular Dirichlet forms, [19], and guarantee the existence of an associated symmetric Hunt diffusion process, Theorem 3.1. In Section 4 we recall basic definitions on partially hyperbolic maps and attractors and well-known results on the existence of Gibbs u-measures. We then observe that Theorem 2.1, Corollary 2.1 and, in some cases, also Theorem 3.1 apply to partially hyperbolic attractors and yield self-adjoint Laplacians, symmetric semigroups and, in some cases, diffusion processes, Corollary 4.1. In Section 5 we put special emphasis on partially hyperbolic diffeomorphisms induced by geodesic flows, because, as mentioned, this allows to explain that the Laplacians constructed here generalize formerly known special cases, Remarks 5.1 and 5.2. Section 6 contains a discussion of hyperbolic attractors with singularities and Gibbs u-measures on them and the observation that the results from Section 2 also apply to these cases, Corollary 6.1. This discussion of hyperbolic attractors with singularities motivates the notation we employ in Sections 2 and 3.

2. Dirichlet forms and self-adjoint Laplacians

Let $M$ be a smooth Riemannian manifold, $U \subseteq M$ a relatively compact open subset and $r \geq 1$. We assume that there is a sequence $D_1^- \subseteq D_2^- \subseteq \ldots$ of compact
subsets $D^-_i$, all contained in $\mathcal{U}$, and that for each point $z$ in the union
\begin{equation}
D^- = \bigcup_{\ell \geq 1} D^-_\ell
\end{equation}
there is an immersed submanifold $W(z)$ of $U$ of class $C^r$ that contains $z$. We assume further that for any $\ell \geq 1$ the set $D^-_\ell$ can be covered by finitely many subsets $\mathcal{R}_{\ell,i} \subseteq D^-_\ell$, $i = 1, \ldots, n_\ell$, each of which admits a partition $\mathcal{P}_{\ell,i}$ into open subsets $B$ of the submanifolds $W(z)$. As usual we refer to these sets $\mathcal{R}_{\ell,i}$ as rectangles. We finally assume that $\mu$ is a Borel probability measure on $D^-$ such that on each $\mathcal{R}_{\ell,i}$ having positive measure $\mu(\mathcal{R}_{\ell,i}) > 0$ the disintegration identity
\begin{equation}
\mu(E) = \int_{\mathcal{P}_{\ell,i}} \mu_B(E) \mu_{B_\ell,i}(dB), \quad E \subseteq \mathcal{R}_{\ell,i} \text{ Borel},
\end{equation}
holds, where $\mu_{B_{\ell,i}}$ is the pushforward of $\mu$ under the canonical projection onto the elements $B$ of the partition $\mathcal{P}_{\ell,i}$ and for each $B \in \mathcal{P}_{\ell,i}$ the symbol $\mu_B$ denotes the conditional measure on $B$. See for instance [46] or [50].

Now let each of the immersed submanifolds $W(z)$ be endowed with the Riemannian metric inherited from $M$ and let $m_{W(z)}$ denote the corresponding Riemannian volume on $W(z)$. For $\ell$, $i$ and $B \in \mathcal{P}_{\ell,i}$ let $m_B$ denote the restriction to $B$ of $m_{W(z)}$. We say that $\mu$ satisfies the $(AC)$-property if for any rectangle $\mathcal{R}_{\ell,i}$ of positive measure the conditional measures $\mu_B$ are absolutely continuous with respect to $m_B$ for $\mu_{B_{\ell,i}}$-a.e. $B$. We write $\mathcal{M}_{\text{ac}}^{\text{bd}}$ for the set of all Borel probability measures on $D^-$ with the $(AC)$-property and with Radon-Nikodym densities $d\mu_B/dm_B$ that are uniformly bounded and uniformly bounded away from zero.

By $C(D^-)$ we denote the space of continuous functions on $D^-$. For $1 \leq k \leq r$ we write $C^k(M)|_{D^-}$ for the space of restrictions to $D^-$ of functions from $C^k(M)$, clearly a dense subspace of $C(D^-)$ and $C^1(M)|_{D^-}$. We write $C^u(D^-)$ (resp. $C^{u,k}(D^-)$) for the space of Borel functions $\varphi : D^- \to \mathbb{R}$ whose restriction to any immersed submanifold $W(z)$, $z \in D^-$, is a continuous (resp. $C^k$-) function on $W(z)$. It is easily seen that $C(D^-) \subset C^u(D^-)$ and that $C^k(M)|_{D^-} \subset C^{u,k}(D^-)$ for any $1 \leq k \leq r$, [33 Theorem 5.27]. The following facts are straightforward, see [2] Propositions 4.2 and 5.1 for proofs.

**Proposition 2.1.**

(i) For any function $g \in C^1(M)$ and any $x \in D^-$ we have
\[
\left\| \nabla_W g \right\|_{D^-}(x) \leq \left\| \nabla_M g(x) \right\|_{T_x M}.
\]

(ii) For any finite Borel measure $\mu$ on $D^-$ and any $1 \leq k \leq r$ the space $C^k(M)|_{D^-}$ is a dense subspace of $L^2(D^-, \mu)$.

By a quadratic form on $L^2(D^-, \mu)$ we mean a densely defined nonnegative definite symmetric bilinear form on $L^2(D^-, \mu)$, i.e. a pair $(\mathcal{E}, D(\mathcal{E}))$, where $D(\mathcal{E})$ is a dense subspace of $L^2(D^-, \mu)$ and $\mathcal{E}$ is a nonnegative definite and symmetric bilinear form on $D(\mathcal{E})$. We employ the notation $\mathcal{E}(\varphi) := \mathcal{E}(\varphi, \varphi)$. A quadratic form $(\mathcal{E}, D(\mathcal{E}))$ is said to be closed, [44 Section VIII.6], if $D(\mathcal{E})$ is a Hilbert space with respect to the norm
\[
\left\| \varphi \right\|_{D(\mathcal{E})} := \left\{ \mathcal{E}(\varphi) + \left\| \varphi \right\|_{L^2(D^-, \mu)}^2 \right\}^{1/2}, \quad \varphi \in D(\mathcal{E}).
\]
A quadratic form \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) is said to be **closable** if it possesses a closed extension, i.e. there is a closed form \((\mathcal{E}', \mathcal{D}(\mathcal{E}'))\) such that \(\mathcal{D}(\mathcal{E}) \subseteq \mathcal{D}(\mathcal{E}')\) and \(\mathcal{E} = \mathcal{E}'\) on \(\mathcal{D}(\mathcal{E})\). The smallest closed extension of a closable form is referred to as its **closure**.

A closed quadratic form \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) on \(L^2(D^-, \mu)\) is called a **Dirichlet form** if \(\varphi \land 1 \in \mathcal{D}(\mathcal{E})\) for any \(\varphi \in \mathcal{D}(\mathcal{E})\) and \(\mathcal{E}(\varphi \land 1) \leq \mathcal{E}(\varphi)\), see [8] Chapter I, 1.1.1 and 3.3.1, [19] Chapter 1 or [5][15]. A Dirichlet form \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) is called **conservative** if \(1 \in \mathcal{D}(\mathcal{E})\) and \(\mathcal{E}(1) = 0\), [19] p. 49. A Dirichlet form \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) with \(1 \in \mathcal{D}(\mathcal{E})\) is called **local** if \(\mathcal{E}(F(\varphi), G(\varphi)) = 0\) for any \(F, G \in C_0^\infty(\mathbb{R})\) with disjoint supports and any \(\varphi \in \mathcal{D}(\mathcal{E})\), [8] Chapter I, Corollary 5.1.4.

Given \(\varphi \in C^{\infty,1}(D^-)\) we write
\[
\nabla \varphi(z) := \nabla_{W(z)}(\varphi|_{W(z)})(z), \quad z \in D^-,
\]
to define the gradient operator \(\nabla\) on \(C^{\infty,1}(D^-)\). Let
\[
\mathcal{D}_0(\mathcal{E}(\mu)) := \left\{ \varphi \in L^2(D^-, \mu) \cap C^{\infty,1}(D^-) : \int_{D^-} \|\nabla \varphi(z)\|^2_{T_z W(z)} \mu(dz) < +\infty \right\}
\]
and
\[
\mathcal{E}(\mu)(\varphi) := \int_{D^-} \|\nabla \varphi(z)\|^2_{T_z W(z)} \mu(dz), \quad \varphi \in \mathcal{D}_0(\mathcal{E}(\mu)).
\]
We also use the notation
\[
\mathcal{D}_0(\mathcal{E}(\mu)) := \mathcal{D}_0(\mathcal{E}(\mu)) \cap C(D^-).
\]
The following is our main existence result for self-adjoint Laplacians.

**Theorem 2.1.** Suppose that \(\mu \in \mathcal{M}_b^a\).

(i) The quadratic form \((\mathcal{E}(\mu), \mathcal{D}_0(\mathcal{E}(\mu)))\) on \(L^2(D^-, \mu)\) is closable, and its closure \((\mathcal{E}(\mu), \overline{\mathcal{D}}(\mathcal{E}(\mu)))\) is a local conservative Dirichlet form.

(ii) There exists a unique non-positive definite self-adjoint operator \((\mathcal{L}(\mu), \mathcal{D}(\mathcal{L}(\mu)))\) on \(L^2(D^-, \mu)\) such that
\[
\int_{D^-} \mathcal{L}(\mu) u \varphi \, d\mu = -\mathcal{E}(\mu)(u, \varphi)
\]
for all \(u \in \mathcal{D}(\mathcal{L}(\mu))\) and \(\varphi \in \mathcal{D}(\mathcal{E}(\mu))\). In particular, we have
\[
\int_{D^-} \mathcal{L}(\mu) u \, d\mu = 0, \quad u \in \mathcal{D}(\mathcal{L}(\mu)).
\]

(iii) Corresponding statements are true for the quadratic form \((\mathcal{E}(\mu), \mathcal{D}_0(\mathcal{E}(\mu)))\), its closure \((\mathcal{E}(\mu), \mathcal{D}(\mathcal{E}(\mu)))\) and the generator \((\mathcal{L}(\mu), \mathcal{D}(\mathcal{L}(\mu)))\) of the latter. The Dirichlet form \((\mathcal{E}(\mu), \overline{\mathcal{D}}(\mathcal{E}(\mu)))\) is an extension of \((\mathcal{E}(\mu), \mathcal{D}(\mathcal{E}(\mu)))\).

**Remark 2.1.**

(i) The self-adjoint operators \((\mathcal{L}(\mu), \mathcal{D}(\mathcal{L}(\mu)))\) and \((\mathcal{L}(\mu), \mathcal{D}(\mathcal{L}(\mu)))\) may be seen as natural Laplacians on \(D^-\). More precisely, they should be perceived as analogs of Laplacians on weighted manifolds in the sense of [20] Definition 3.17, see Section 5.

(ii) One can run the argument with different choices of a priori domains \((2.4)\), and in general they lead to different closed forms and therefore also to different Laplacians. We concentrate on the closed form \((\mathcal{E}(\mu), \overline{\mathcal{D}}(\mathcal{E}(\mu)))\), because its domain is maximal from a 'transversal' point of view, and on
the closed form \((E^{(u)}, D(E^{(u)}))\), because its domain is well-connected to the topology of \(D^-\). The different possible choices of a priori domains are connected to abstract Dirichlet problems, but this will be discussed elsewhere.

Recall that a strongly continuous semigroup \((P_t)_{t \geq 0}\) on \(L^2(D^-, \mu)\) is said to be symmetric if \(\langle P_t u, v \rangle_{L^2(D^-, \mu)} = \langle u, P_t v \rangle_{L^2(D^-, \mu)}\) for every \(u, v \in L^2(D^-, \mu)\) and all \(t > 0\), and (sub-) Markov if for all \(t > 0\) and every \(u \in L^2(D^-, \mu)\) such that \(0 \leq u \leq 1\) \(\mu\)-a.e. we have \(0 \leq P_t u \leq 1\) \(\mu\)-a.e. See [8, Sections I.1 and I.2] or [19, Section 1.4]. A symmetric Markov semigroup \((P_t)_{t \geq 0}\) on \(L^2(D^-, \mu)\) is conservative if \(P_1 = 1\) for all \(t > 0\) and recurrent if for every nonnegative \(u \in L^1(D^-, \mu)\) we have \(\int_0^\infty P_t u \, dt = 0\) or \(+\infty\) \(\mu\)-a.e. See [19, p. 48/49]. The following statement is a consequence of [19, Lemma 1.3.2 and Theorem 1.4.1] (see also [8, Chapter I, Proposition 3.2.1]) and [19, Theorem 1.6.3 and Lemma 1.6.5], its last claim is immediate from (2.6).

**Corollary 2.1.** Suppose that \(\mu \in M_{bd}^{eq}\). There exists a unique symmetric Markov semigroup \((P_t)_{t \geq 0}\) on \(L^2(D^-, \mu)\) generated by \((E^{(\mu)}, D(E^{(\mu)}))\) as in Theorem 2.1 (ii). It is recurrent and conservative, and \(\mu\) is an invariant measure for \((P_t)_{t \geq 0}\) in the sense that \(\int_{D^-} P_t u \, dm = \int_{D^-} u \, dm\) for all \(u \in L^2(D^-, \mu)\). Corresponding statements are true for the unique symmetric Markov semigroup \((P_t)_{t \geq 0}\) on \(L^2(D^-, \mu)\) generated by \((E^{(\mu)}, D(E^{(\mu)}))\) as in Theorem 2.1 (iii).

The key observation to prove Theorem 2.1 is the following.

**Lemma 2.1.** Suppose that \(\mu \in M_{bd}^{eq}\). Then \((E^{(\mu)}, D_0(E^{(\mu)}))\) is a closable quadratic form on \(L^2(D^-, \mu)\).

With the aid of Proposition 2.1 and Lemma 2.1, Theorem 2.1 now follows easily using [19, Theorem 3.1.1] and [19, Theorem 1.3.1] or [8, Chapter I, Propositions 1.2.2 or 3.2.1]. See [2, Theorem 5.1] and its proof.

We provide a proof of Lemma 2.1. It is very similar to the proof of [2, Lemma 5.1], but the definition of rectangles in the style of [39, 42] used here allows a slight simplification, while the representation of \(D^-\) as a union of the \(D_{1-}\) needs an additional approximation step.

*Proof.* Suppose that \((\varphi_j)_{j=1}^\infty \subseteq D_0(E^{(\mu)})\) is Cauchy w.r.t. the seminorm \((E^{(\mu)})^{1/2}\) and such that \(\lim_j \|\varphi_j\|_{L^2(D^-, \mu)} = 0\). Let \(\varepsilon > 0\). Choose \(j_\varepsilon \geq 1\) large enough so that \(E^{(\mu)}(\varphi_j - \varphi_k)^2 < \varepsilon/2\) for all \(j, k \geq j_\varepsilon\) and choose \(k \geq 1\) large enough such that

\[
\left( \int_{D^- \setminus D_{1-}} \|\nabla \varphi_j(z)\|_{T_{1-} W(z)}^2 \mu(dz) \right)^{1/2} < \frac{\varepsilon}{2}.
\]

Then by the triangle inequality we have

\[
(2.7) \quad \sup_{j \geq j_\varepsilon} \left( \int_{D^- \setminus D_{1-}} \|\nabla \varphi_j(z)\|_{T_{1-} W(z)}^2 \mu(dz) \right)^{1/2} < \varepsilon.
\]

We claim that

\[
(2.8) \quad \lim_j \int_{D_{1-}} \|\nabla \varphi_j(z)\|_{T_{1-} W(z)}^2 \mu(dz) = 0.
\]
If so, then in combination with (2.7) we obtain \( \lim_j \mathcal{E}(\varphi_j) < \varepsilon \), and since \( \varepsilon \) was arbitrary, this shows the closability of \( \mathcal{E}(\varphi) \).

To verify (2.8) let \( R_{\ell,1}, \ldots, R_{\ell,n} \) be a finite cover of \( D_{\ell} \) by rectangles \( R_{\ell,i} \). We may assume they all have positive measure \( \mu \); if not, we can simply omit those rectangles that have measure zero. For each \( i \) the quantity

\[
\int\int_{P_{\ell,i}} \int_B \|\nabla_W(\varphi_j - \varphi_k)(\cdot)\|^2_{T_z W(\cdot)} \mu_B(d\zeta) \mu_{P_{\ell,i}}(dB)
\]

(2.9) can be made arbitrarily small if \( j \) and \( k \) are chosen large enough. Here we have used (2.2). Clearly also

\[
\lim_j \int_{P_{\ell,i}} \int_B (\varphi_j(\zeta))^2 \mu_B(d\zeta) \mu_{P_{\ell,i}}(dB) = \lim_j \int_{R_{\ell,i}} (\varphi_j(z))^2 \mu(dz) = 0.
\]

Each \( B \in \mathcal{P}_{\ell,i} \) is an open subset of some Riemannian manifold \( W = W(z) \), hence itself a Riemannian manifold. Therefore the Dirichlet integral

\[
\psi \mapsto \int_B \|\nabla_W \psi(\zeta)\|^2_{T_z W} \mu_B(d\zeta)
\]

(2.10) can be made arbitrarily small if \( j \) and \( k \) are chosen large enough. Here we have used (2.2). Clearly also

\[
\lim_j \int_{P_{\ell,i}} \int_B (\varphi_j(\zeta))^2 \mu_B(d\zeta) \mu_{P_{\ell,i}}(dB) = \lim_j \int_{R_{\ell,i}} (\varphi_j(z))^2 \mu(dz) = 0.
\]

Each \( B \in \mathcal{P}_{\ell,i} \) is an open subset of some Riemannian manifold \( W = W(z) \), hence itself a Riemannian manifold. Therefore the Dirichlet integral

\[
\psi \mapsto \int_B \|\nabla_W \psi(\zeta)\|^2_{T_z W} \mu_B(d\zeta)
\]

(2.10) can be made arbitrarily small if \( j \) and \( k \) are chosen large enough. Here we have used (2.2). Clearly also

\[
\lim_j \int_{P_{\ell,i}} \int_B (\varphi_j(\zeta))^2 \mu_B(d\zeta) \mu_{P_{\ell,i}}(dB) = \lim_j \int_{R_{\ell,i}} (\varphi_j(z))^2 \mu(dz) = 0.
\]

Each \( B \in \mathcal{P}_{\ell,i} \) is an open subset of some Riemannian manifold \( W = W(z) \), hence itself a Riemannian manifold. Therefore the Dirichlet integral

\[
\psi \mapsto \int_B \|\nabla_W \psi(\zeta)\|^2_{T_z W} \mu_B(d\zeta)
\]

(2.10) can be made arbitrarily small if \( j \) and \( k \) are chosen large enough. Here we have used (2.2). Clearly also

\[
\lim_j \int_{P_{\ell,i}} \int_B (\varphi_j(\zeta))^2 \mu_B(d\zeta) \mu_{P_{\ell,i}}(dB) = \lim_j \int_{R_{\ell,i}} (\varphi_j(z))^2 \mu(dz) = 0.
\]

Each \( B \in \mathcal{P}_{\ell,i} \) is an open subset of some Riemannian manifold \( W = W(z) \), hence itself a Riemannian manifold. Therefore the Dirichlet integral

\[
\psi \mapsto \int_B \|\nabla_W \psi(\zeta)\|^2_{T_z W} \mu_B(d\zeta)
\]

(2.10) can be made arbitrarily small if \( j \) and \( k \) are chosen large enough. Here we have used (2.2). Clearly also

\[
\lim_j \int_{P_{\ell,i}} \int_B (\varphi_j(\zeta))^2 \mu_B(d\zeta) \mu_{P_{\ell,i}}(dB) = \lim_j \int_{R_{\ell,i}} (\varphi_j(z))^2 \mu(dz) = 0.
\]

Each \( B \in \mathcal{P}_{\ell,i} \) is an open subset of some Riemannian manifold \( W = W(z) \), hence itself a Riemannian manifold. Therefore the Dirichlet integral

\[
\psi \mapsto \int_B \|\nabla_W \psi(\zeta)\|^2_{T_z W} \mu_B(d\zeta)
\]

(2.10) can be made arbitrarily small if \( j \) and \( k \) are chosen large enough. Here we have used (2.2). Clearly also

\[
\lim_j \int_{P_{\ell,i}} \int_B (\varphi_j(\zeta))^2 \mu_B(d\zeta) \mu_{P_{\ell,i}}(dB) = \lim_j \int_{R_{\ell,i}} (\varphi_j(z))^2 \mu(dz) = 0.
\]

Each \( B \in \mathcal{P}_{\ell,i} \) is an open subset of some Riemannian manifold \( W = W(z) \), hence itself a Riemannian manifold. Therefore the Dirichlet integral

\[
\psi \mapsto \int_B \|\nabla_W \psi(\zeta)\|^2_{T_z W} \mu_B(d\zeta)
\]

(2.10) can be made arbitrarily small if \( j \) and \( k \) are chosen large enough. Here we have used (2.2). Clearly also

\[
\lim_j \int_{P_{\ell,i}} \int_B (\varphi_j(\zeta))^2 \mu_B(d\zeta) \mu_{P_{\ell,i}}(dB) = \lim_j \int_{R_{\ell,i}} (\varphi_j(z))^2 \mu(dz) = 0.
\]

Each \( B \in \mathcal{P}_{\ell,i} \) is an open subset of some Riemannian manifold \( W = W(z) \), hence itself a Riemannian manifold. Therefore the Dirichlet integral

\[
\psi \mapsto \int_B \|\nabla_W \psi(\zeta)\|^2_{T_z W} \mu_B(d\zeta)
\]

(2.10) can be made arbitrarily small if \( j \) and \( k \) are chosen large enough. Here we have used (2.2). Clearly also

\[
\lim_j \int_{P_{\ell,i}} \int_B (\varphi_j(\zeta))^2 \mu_B(d\zeta) \mu_{P_{\ell,i}}(dB) = \lim_j \int_{R_{\ell,i}} (\varphi_j(z))^2 \mu(dz) = 0.
\]
3. Regularity and symmetric diffusion processes

If $D^-$ is closed (hence compact) and $\text{supp } \mu = D^-$ we write $\Lambda := D^-$. In this case we can employ further results from the theory of regular Dirichlet forms, [19]. A Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(\Lambda, \mu)$ is regular if $\mathcal{D}(\mathcal{E}) \cap C(\Lambda)$ is dense in $\mathcal{D}(\mathcal{E})$ with $|| \cdot ||_{\mathcal{D}(\mathcal{E})}$-norm and dense in $C(\Lambda)$ with the uniform norm. A regular Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(\Lambda, \mu)$ is said to be strongly local if $\mathcal{E}(u, v) = 0$ whenever $u, v \in \mathcal{D}(\mathcal{E}) \cap C(\Lambda)$ are such that $v$ is constant on supp $u$. See [19, p.6].

Statement (i) in the following result is immediate, statement (ii) is a consequence of [19, Theorems 7.2.1 and 7.2.2].

**Theorem 3.1.** Suppose that $\Lambda := D^-$ is closed, $\mu \in \mathcal{M}^{ac}_{bd}$ and supp $\mu = \Lambda$.

(i) The Dirichlet form $(\mathcal{E}^0, \mathcal{D}(\mathcal{E}^0))$ is regular and strongly local.

(ii) There is a $\mu$-symmetric Hunt diffusion process $((X_t)_{t \geq 0}, (\mathbb{P}^x)_{x \in \Lambda \setminus \mathcal{N}})$ on $\Lambda$, with starting points $x$ outside some properly exceptional set $\mathcal{N}$, such that for all bounded Borel functions $u$ on $\Lambda$, any $t > 0$ and $\mu$-a.e. $x \in \Lambda$ we have $P_t u(x) = \mathbb{E}^x [u(X_t)]$.

A strong Markov process is called a diffusion process if its paths are continuous almost surely, [19, Section 4.5]. It is said to be a Hunt process if it satisfies certain specific regularity properties, see [14, Section I.9] or [19, Appendix A.2]. A Borel set $\mathcal{N} \subset \Lambda$ is said to be properly exceptional for a Hunt process $((X_t)_{t \geq 0}, (\mathbb{P}^x)_{x \in \Lambda \setminus \mathcal{N}})$ if it is a $\mu$-null set and $\mathbb{P}^x (X_t \in \mathcal{N} \text{ for some } t \geq 0) = 0$, $x \in \Lambda \setminus \mathcal{N}$, see [19, p. 134 and Theorem 4.1.1]. A Hunt process $((X_t)_{t \geq 0}, (\mathbb{P}^x)_{x \in \Lambda \setminus \mathcal{N}})$ is said to be $\mu$-symmetric if

$$\int_{\Lambda} \mathbb{E}^x [u(X_t)] v(x) \mu(dx) = \int_{\Lambda} u(x) \mathbb{E}^x [v(X_t)] \mu(dx)$$

for any $t > 0$ and every bounded Borel functions $u, v$ on $\Lambda$, see [19, Lemma 4.1.3].

**Remark 3.1.**

(i) The Hunt process $((X_t)_{t \geq 0}, (\mathbb{P}^x)_{x \in \Lambda \setminus \mathcal{N}})$ is unique up to a suitable type of equivalence, see [19, Theorem 4.2.7]. It has infinite life time, [19, Problem 4.5.1]. It may be regarded as a natural analog (in the leafwise sense) of Brownian motion.

(ii) For every bounded Borel $u$ and any $t > 0$ the function $x \mapsto \mathbb{E}^x [u(X_t)]$ is an $(\mathcal{E}^0, \mathcal{D}(\mathcal{E}^0))$-quasi-continuous version of $P_t u$, [19, Theorem 4.2.3].

4. Partially hyperbolic attractors

Let $M$ be a smooth compact Riemannian manifold. A topological attractor for a $C^{r+\alpha}$-diffeomorphism $f : M \to M$ is a compact subset $\Lambda \subseteq M$ for which there is a neighborhood $U$ such that $\overline{f(U)} \subseteq U$ and

$$\Lambda = \bigcap_{n \geq 0} f^n(U).$$

The set $\Lambda$ is $f$-invariant, i.e. $f(\Lambda) = \Lambda$, and it is the largest subset of $U$ with this property. See for instance [10] or [27].

A compact $f$-invariant subset $\Lambda \subseteq M$ is said to be partially hyperbolic if for each $z \in \Lambda$ there exists a continuous $df$-invariant splitting of the tangent space
The subspaces $E$ expanded, but not as sharply as vectors in spaces at $z$. $E$ is ‘dominated’ by the hyperbolic behaviour of the stable and unstable directions.

Given a point $z$ partially hyperbolic set.

We can construct (4.2)

Local stable

Respectively (4.3)

These manifolds $W$ and $V$ are partially hyperbolic set.

Remark 4.1. In the context of partial hyperbolicity the manifolds $V$ and $W$ in (4.2) are also referred to as $\text{local strongly stable}$ and $\text{local strongly unstable}$ manifolds at $z$, respectively. The manifolds $W$ and $W$ in (4.3) are also called the $\text{global strongly stable}$ and $\text{global strongly unstable}$ manifolds at $z$, respectively.

Any partially hyperbolic attractor $\Lambda$ contains the global unstable manifolds of its points,

\[
\Lambda = \bigcup_{z \in \Lambda} W_u(z),
\]

see for instance [24, Theorem 9.1], and the union (4.4) is disjoint.

As usual $B(x, \delta)$ denotes the open ball in $M$ with center $x \in M$ and radius $\delta > 0$. Given $x \in \Lambda$ and $\delta > 0$ let

\[
\mathcal{R}(x, \delta) = \bigcup_{z \in B(x, \delta) \cap \Lambda} V_u(z).
\]
For sufficiently small $\delta$ the set $R(x, \delta)$ admits a measurable partition $\mathcal{P}_{R(x, \delta)}$ into local unstable manifolds $V = V^u(z)$.

Recall that two measures on the same space are said to be equivalent if they are mutually absolutely continuous. An $f$-invariant Borel probability measure $\mu$ on $\Lambda$ is called a Gibbs u-measure if for any $x \in \Lambda$ and (sufficiently small) $\delta > 0$ such that $\mu(R(x, \delta)) > 0$ we have

$$
(4.6) \quad \mu(E) = \int_{\mathcal{P}_{R(x, \delta)}} \mu_V(E) \mu_{P_{\mathcal{R}(x, \delta)}}(dV), \quad E \subseteq \mathcal{R}(x, \delta) \text{ Borel},
$$

where $\mu_{P_{\mathcal{R}(x, \delta)}}$ is the pushforward of $\mu$ under the canonical projection onto the elements $V$ of the partition $\mathcal{P}_{\mathcal{R}(x, \delta)}$, and the conditional measures $\mu_V$ are equivalent to the Riemannian volumes $m_V$ on the manifolds $V$. See for instance [13] Section 5.2.

Gibbs u-measures on partially hyperbolic attractors can be constructed in the same way as SRB-measures on uniformly hyperbolic attractors, [13]. Given a local unstable manifold $V^u(z)$, $z \in \Lambda$, one can consider the sequence $(\mu_n)_n$ of probability measures $\mu_n$ defined by

$$
(4.7) \quad \mu_n := \frac{1}{n} \sum_{i=0}^{n-1} f^i_* m_{V^u(z)},
$$

where $f^i_* m_{V^u(z)}$ is the pushforward of the measure $m_{V^u(z)}$. We quote the following result from [42, Theorem 4]; further details and descriptions can be found in [13, Section 5] and [24, Section 9]. For statement (ii) see [13, Theorem 5.4] or [11].

**Theorem 4.1.** Assume that $f : M \to M$ is a $C^{1+\alpha}$ diffeomorphism and $\Lambda \subseteq M$ is a partially hyperbolic attractor.

(i) There is a Gibbs u-measure $\mu$ on $\Lambda$ with uniformly bounded and Hölder continuous conditional densities $dm_V/dm_V$, that is, $\mu \in \mathcal{M}_{bd}^{ac}$. Any weak limit of $(\mu_n)_n$, as in (4.7), has these properties.

(ii) If for every $z \in \Lambda$ the orbit of the global (strongly) unstable manifold $W^u(z)$ is dense in $\Lambda$, then every Gibbs u-measure $\mu$ has support $\text{supp } \mu = \Lambda$.

Theorem 4.1 makes Theorem 2.1 and its consequences applicable.

**Corollary 4.1.** Let $f$ and $\Lambda$ be as in Theorem 4.1 and let $\mu \in \mathcal{M}_{bd}^{ac}$ be a Gibbs u-measure on $\Lambda$ with uniformly bounded densities.

(i) The quadratic forms $(\mathcal{E}(\mu), D\mathcal{E}(\mu))$ and $(\mathcal{E}(\mu), D\mathcal{E}(\mu))$ as in Theorem 2.1 are local Dirichlet forms on $L^2(\Lambda, \mu)$, their generators are self-adjoint operators on $L^2(\Lambda, \mu)$, and their semigroups are symmetric.

(ii) If for every $z \in \Lambda$ the orbit of the global (strongly) unstable manifold $W^u(z)$ is dense in $\Lambda$, then $(\mathcal{E}(\mu), D\mathcal{E}(\mu))$ is regular and strongly local, and there is an associated $\mu$-symmetric Hunt diffusion process as in Theorem 3.1.

**Proof.** Setting $D^- := \Lambda$, $\ell \geq 1$, we have $D^- = \Lambda$ in (2.1). From (4.5) and the compactness of $\Lambda$ it follows that $\Lambda$ admits a cover by finitely many rectangles $\mathcal{R}(x, \delta)$, each of which is partitioned into local unstable manifolds $V = V(z)$ that are open subsets of the global unstable manifolds $W^u(z)$. Since $\mu \in \mathcal{M}_{bd}^{ac}$ satisfies (4.6), we see that (2.2) holds. \qed

We briefly recall well-known examples for partially hyperbolic diffeomorphisms.
Examples 4.1. One class of examples of partially hyperbolic attractors is generated by direct products \( f \times g : M \times N \to M \times N \), where \( f : M \to M \) is a partially hyperbolic diffeomorphism and \( g : N \to N \) is a diffeomorphism whose dynamical behaviour is less sharp than that of \( f \) in the sense of \cite{40} Section 2.3, Example 3 or \cite{45} Section 2.7]. Then the direct product \( F = f \times g : M \times N \to M \times N \) defined by \( F(x, y) := (f(x), g(y)) \) is a partially hyperbolic diffeomorphism. A particularly simple case arises if \( f \) is an Anosov diffeomorphism and \( g \) is the identity.

Examples 4.2. Suppose \( f : M \to M \) is an Anosov diffeomorphism, \( G \) is a (compact) Lie group \( G \) and \( \theta : M \to G \) is a smooth function. The skew product \( F_\theta : M \times G \to M \times G \) is defined by \( F_\theta(x, y) := (f(x), \theta(x)y), \ x \in M, \ y \in G \), it is a partially hyperbolic diffeomorphism. See for instance \cite{40} Section 2.3, Examples 4 and 5 or \cite{45} Section 2.9].

Examples 4.3. Let \( \phi : \mathbb{R} \times M \to M \) be a flow (generated by a given vector field). By time-\( t \) map we mean the diffeomorphism \( \phi(t, \cdot) : M \to M \). The flow \( \phi \) is said to be partially hyperbolic if its time-1 map \( \phi(1, \cdot) \) is a partially hyperbolic diffeomorphism. A uniformly hyperbolic (or Anosov) flow is a partially hyperbolic flow with one-dimensional central subspace \( E^c(x) = \text{span}\{ \frac{\partial}{\partial t}|_{t=0} \phi(t, x) \}, \ x \in M \). See for instance \cite{27} Definition 17.4.2]. From any Anosov diffeomorphism of a compact Riemannian manifold \( M \) one can construct Anosov flows called suspension flows, see \cite{10} Section 1.11], \cite{27} Section 0.3 or \cite{40} p. 8]. If \( \phi \) is a \( C^{r+\alpha} \) Anosov flow, then the local (strongly) stable and unstable manifolds (4.2) can also be obtained from a continuous time-version of the Stable Manifold Theorem for flows \cite{27} Theorem 17.4.3], \cite{6} Section 7.3.5], and also in \cite{13} a positive real index can be used in place of \( n \) and \( \phi(t, \cdot) \) can replace \( f^n \).

A particular class of examples of Anosov flows is formed by geodesic flows on manifolds of negative sectional curvature. Since for these examples there is an established leafwise analysis to which the theory in Sections 2 and 3 can be compared, we discuss them in a slightly more detailed manner in the next section.

5. Geodesic Flows on Manifolds with Negative Curvature

Let \( M \) be a compact \( C^r \) manifold endowed with a \( C^r \) Riemannian metric, \( r \geq 2 \). Given \( x \in M \) and \( v \in T_x M \), there is a unique geodesic \( \gamma_{x,v} \) such that \( \gamma_{x,v}(0) = x \) and \( \dot{\gamma}_{x,v}(0) = v \). By the geodesic flow on \( M \) we mean the flow \( g : \mathbb{R} \times TM \to TM \) on the tangent bundle \( TM \), defined by

\[ g(t, (x, v)) := (\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t)). \]

Since the length of the tangent vectors are preserved by the geodesic flow, i.e. \( \|\dot{\gamma}_{x,v}(t)\|_{T\gamma_{x,v}(t)} = \|v\|_{T_x M} \), it is usual to consider the restriction of \( g \) to the unit tangent bundle

\[ T^1 M = \{(x, w) \in TM : \|w\|_{T_x M} = 1\}. \]

We assume that \( M \) has negative sectional curvature. Then the \( C^{r-1} \) geodesic flow \( g : \mathbb{R} \times T^1 M \to T^1 M \) is an Anosov flow, \cite{4}, see for instance \cite{12} Theorem 9.4.1] and \cite{27} Sections 17.5 and 17.6.

Examples 5.1. As a guiding ‘example’ consider the Poincaré half plane, that is, \( \mathbb{H} = \{ z \in \mathbb{C} : \text{Im} z > 0 \} \), endowed with the Riemannian metric

\[ \langle v, w \rangle_{T_z \mathbb{H}} = (\text{Im} z)^{-2} \langle v, w \rangle, \quad v, w \in T_z \mathbb{H} = \mathbb{C}, \]
where \( \langle \cdot, \cdot \rangle \) is the usual inner product on \( \mathbb{C} \cong \mathbb{R}^2 \). The manifold \( \mathbb{H} \) itself is not compact, so it does not fully fit our assumptions, but it is the universal cover of any closed hyperbolic surface. It is compactified by adjoining the ideal boundary \( \mathbb{H}(\infty) := \{ z \in \mathbb{C} : \text{Im} \, z = 0 \} \). Its unit tangent bundle \( T^1 \mathbb{H} = \{(z, v) \in \mathbb{H} \times \mathbb{C} : \|v\|_{T_z\mathbb{H}} = 1 \} \) is isomorphic to \( \text{PSL}_2(\mathbb{R}) \), and there is a related simple explicit description of the geodesic flow (5.1), see [6, Section 1.2] or [16, Sections 9.1 and 9.2]. Geodesics are vertical lines or half-circles centered at points on \( \mathbb{H}(\infty) \). Given \( (z, v) \in T^1 \mathbb{H} \), let \( \gamma_{z,v} := \lim_{t \to -\infty} \gamma_{z,v}(t) \), where \( \gamma_{z,v} \) is the geodesic through \( z \) in direction \( v \). The circle \( c(z, v) \) through \( z \) and tangent to \( \mathbb{H}(\infty) \) at \( \gamma_{z,v} \) is called the horocycle through \( z \), and the global strong unstable manifold \( W^u(z, v) \) through \( (z, v) \in T^1 \mathbb{H} \) is formed by all \( (z', v') \in T^1 \mathbb{H} \) such that \( z' \in c(z, v) \) and \( v' \) is the outer normal on \( c(z, v) \) at \( z' \). Further details may also be found in [18, p. 119] or [22, Chapter 3].

Since the time-1 map \( g(1, \cdot) : T^1 M \to T^1 M \) of the geodesic flow (5.1) is a partially hyperbolic diffeomorphism, Theorem 4.1 ensures the existence of a Gibbs \( u \)-measure \( \mu \in M_{loc}^\mu \) on \( T^1 M \). On the other hand the Liouville measure \( \nu \) on \( T^1 M \), defined by

\[
(5.2) \quad \int_{T^1 M} h \, d\nu = \int_M \int_{T^1 \mathbb{H}} h(x, v) \, dm_{S^{n-1}} \, dm_M(x), \quad h \in C(T^1 M),
\]

where \( m_M \) and \( m_{S^{n-1}} \) denote the Riemannian volume on \( M \) and on the sphere \( S^{n-1} = T^1 \mathbb{H} \), respectively, is invariant under \( g(1, \cdot) \), see for instance [43, Appendix C]. But this implies that \( \mu = \nu \). [18, Theorem 7.4.14], and using (5.2) it follows that \( \text{supp} \mu = T^1 M \). It is well known that in the special case of a compact surface of constant negative curvature, this measure also coincides with the Bowen-Margulis measure, i.e. the measure of maximal entropy of the flow, see [38, Chapter 11, Example 2] or [30].

By Corollary 4.1 the forms \( (\mathcal{E}^{(\mu)}, \overline{\mathcal{D}(\mathcal{E}^{(\mu)})}) \) and \( (\mathcal{E}^{(\mu)}, \mathcal{D}(\mathcal{E}^{(\mu)})) \) are local respectively strongly local and regular Dirichlet forms on \( L^2(T^1 M, \mu) \), and in Section 2 we suggested to regard their generators as generalized Laplacians. On the other hand, there is an established analysis involving leafwise Laplacians in the unstable directions, see for instance [52],[54]. We briefly compare \( (\mathcal{E}^{(\mu)}, \mathcal{D}(\mathcal{E}^{(\mu)})) \) to the Laplacians studied in [52],[53].

For any rectangle \( R(x, \delta) \) and any local unstable manifold \( V \) as it appears in the partition (4.5) of this rectangle, let \( \varrho_{V} = d\mu_{V} / dm_{V} \) be the Radon-Nikodym density of the conditional measures \( \mu_{V} \) of \( \mu \) with respect to \( m_{V} \) on \( V \). Assume that \( M \) and \( f \) are of class \( C^\infty \). Then each \( \varrho_{V} \) is a \( C^\infty(V) \)-function. This \( C^\infty \)-regularity was proved in [52, Lemma 2.1], which itself was based on [55, Lemma 2.5]. Setting \( \varrho(z) \equiv \varrho_{V}(z) \) if \( z \) is a point in the partition element \( V \), we can define \( \varrho \) as a \( C^\infty(V) \)-function on \( R(x, \delta) \). On the other hand, a ’classical’ leafwise Laplacian in the unstable directions can be defined by setting

\[
\Delta_{\varphi} := \Delta_{W^u(z)}(\varphi|_{W^u(z)})(z), \quad z \in M,
\]

for any \( \varphi \in C^2(u)(M) \). Here \( \Delta_{W^u(z)} \) denotes the classical Laplace-Beltrami operator on the Riemannian manifold \( W^u(z) \), defined on \( C^2 \)-functions. Now recall that \( \nabla \) defines the leafwise gradient on \( C^1(u)(M) \)-functions as defined in [23] (with \( D^- = M \) and \( W(z) = W^u(z) \)). For functions \( \varphi \in C^2(u)(M) \) we can define

\[
(5.3) \quad \Delta_{\mu, \varphi} := \Delta_{\varphi} + \varrho^{-1}(\nabla \varphi, \nabla \varphi), \quad z \in R(x, \delta),
\]
where we write the shortcut $\langle \nabla \varrho, \nabla \varphi \rangle$ for the function $z \mapsto \langle \nabla \varrho(z), \nabla \varphi(z) \rangle_{T_z W^u(z)}$. Using a smooth partition of unity, definition (5.3) can meaningfully be extended to hold for all $z \in M$. The Laplacian $\Delta_\mu \varphi$ may be seen as a leafwise analog of the Laplacian on weighted manifolds, [20, Definition 3.17]. It had already been studied in [52], see for instance [52, Theorem 1'], where $\mu$ had been shown to be an invariant measure for $\Delta_\mu$. This is fully consistent with our results in the sense that $(\mathcal{L}(\mu), D(\mathcal{L}(\mu)))$ is a self-adjoint extension of $(\Delta_\mu, C^2(M))$ on $L^2(T^1 M, \mu)$, note that we had independently observed the invariance of $\mu$ in (2.6).

Remark 5.1. Observation (5.4) means that for geodesic flows of class $C^\infty$ the theory in this article is simply an $L^2$-version of the smooth theory for the operator $\Delta_\mu$ as considered in [52, 53]. A similar $L^2$-theory for the stable directions, including self-adjoint Laplacians, is studied extensively in [21].

Remark 5.2. We chose the specific example of $C^\infty$-geodesic flows because it is widely known and because for this situation leafwise Laplacians of form (5.3) had been studied in [52]. A definition of leafwise Laplacians as in (5.3) is possible whenever the unstable manifolds are $C^2$ and the conditional densities $C^1$, and this can be guaranteed also for certain more general classes of diffeomorphisms $f : M \to M$, see for instance [32, Remark on p. 534] or [53, p. 168].

For general partially hyperbolic attractors $\Lambda \subseteq M$ a definition of Laplacians $\Delta_\mu$ as in (5.3) seems out of reach: The only regularity information for the conditional densities $\varrho_V$ is their Hölder continuity, too little to give a meaning to (5.3). However, Theorem 2.1 ensures the existence of self-adjoint Laplacians as generators of quadratic forms, and their definition (2.5) and closedness posed no problem. The situation is very similar to the theory of weak solutions in partial differential equations: Divergence form elliptic second order differential operators with bounded measurable coefficients cannot be defined directly as classical operators on a space of $C^2$-functions, but are easily defined as the generators of corresponding quadratic forms. See for instance [17, Chapter 6].

6. Hyperbolic attractors with singularities

We consider a more general class of hyperbolic attractors induced by maps with discontinuities, it had been studied in [39]. A short exposition may also be found in [13, Section 8]. The notation in this section follows [39], up to minor details.

Let $M$ be a smooth Riemannian manifold. Let $U \subseteq M$ be a relatively compact open set and $N \subset U$ a closed subset. Let $f : U \setminus N \to U$ be a $C^{r+\alpha}$-diffeomorphism onto its image. We define

$$N^+ := N \cup \partial U$$

and

$$N^- := \{ y \in U : \text{there are } z \in N^+ \text{ and } z_n \in U \setminus N^+ \text{ with } z_n \to z \text{ and } f(z_n) \to y \}$$

and assume that $f$ is such that

$$\| d^2_z f \| \leq c_1 d(z, N^+)^{-\alpha_1} \quad \text{for any } z \in U \setminus N,$$

$$\| d^2_z f^{-1} \| \leq c_2 d(z, N^-)^{-\alpha_2} \quad \text{for any } z \in f(U \setminus N),$$
where \( c_i > 0, \alpha_i \geq 0, i = 1, 2 \), and \( \| \cdot \| \) denotes the operator norm. A topological attractor with singularities for \( f \) is defined to be the set \( \Lambda := T \) where

\[
D := \bigcap_{n \geq 0} f^n(U^+) \quad \text{and} \quad U^+ := \{ x \in U : f^n(x) \notin N^+, n = 0, 1, 2, \ldots \}.
\]

Given \( z \in M, a > 0 \) and a subspace \( P(z) \subseteq T_zM \), the cone at \( z \) around \( P(z) \) with angle \( \theta \) is the set \( C(z, P(z), \theta) := \{ v \in T_zM : \angle(v, P(z)) \leq \theta \} \). Here we write \( \angle(v, P) := \min_{w \in P} \angle(v, w) \) for any \( P \subseteq T_zM \), and we define \( \angle(P', P) \) for \( P, P' \subseteq T_zM \) in a similar manner.

A topological attractor with singularities \( \Lambda \) is said to be a uniformly hyperbolic attractor with singularities \( \{ \} \) if there exist constants \( c > 0, \lambda \in (0, 1) \), and \( \theta(z) > 0, z \in U \setminus N^+ \), together with subspaces \( P^u(z), P^s(z) \subseteq T_zM, z \in U \setminus N^+ \), of complementary dimension, such that the two families of stable and unstable cones

\[
C^s(z) = C^s(z, P^s(z), \theta(z)) \quad \text{and} \quad C^u(z) = C^u(z, P^u(z), \theta(z))
\]
satisfy the following conditions:

(i) the angles \( \angle(C^s(z), C^u(z)) \), \( z \in U \setminus N^+ \), are uniformly bounded away from zero,

(ii) we have \( df(C^u(z)) \subseteq C^u(f(z)) \) for any \( z \in U \setminus N^+ \) and \( df^{-1}(C^s(z)) \subseteq C^s(f^{-1}(z)) \) for any \( z \in f(U \setminus N^+) \),

(iii) for any \( n \geq 1 \) we have

\[
\| df^n v \|_{T_{f^n(z)}M} \geq \lambda^{-n} \| v \|_{T_zM} \quad \text{for} \quad v \in C^u(z) \quad \text{and} \quad z \in U^+,
\]

\[
\| df^n v \|_{T_{f^{-n}(z)}M} \geq \lambda^{-n} \| v \|_{T_zM} \quad \text{for} \quad v \in C^s(z) \quad \text{and} \quad z \in f^n(U^+).
\]

See [39] Section 1.3 or [13] Section 8.

In the following we assume that \( \Lambda \) is a uniformly hyperbolic attractor with singularities, and we continue to use the above notation. For any \( z \in D \) the subspaces

\[
E^s(z) = \bigcap_{n \geq 0} df^n C^s(f^n(z)) \quad \text{and} \quad E^u(z) = \bigcap_{n \geq 0} df^n C^u(f^n(z))
\]

form a splitting of the tangent space \( T_zM = E^s(z) \oplus E^u(z) \) such that for any \( n \geq 0 \)

\[
\| df^n v \|_{T_{f^n(z)}M} \leq c \lambda^n \| v \|_{T_zM} \quad \text{for} \quad v \in E^s(z);
\]

\[
\| df^n v \|_{T_{f^{-n}(z)}M} \leq c \lambda^n \| v \|_{T_zM} \quad \text{for} \quad v \in E^u(z),
\]

meaning that \( D \) is a uniformly hyperbolic set containd in \( \Lambda \), see [39] p. 128 or [41]. An adapted version of the Stable Manifold Theorem, [39] Proposition 4, guarantees that for sufficiently small \( \varepsilon > 0 \) local stable manifolds \( V^s(z), z \in D^+_\varepsilon \), and local unstable manifolds \( V^u(z), z \in D^-\varepsilon \), exist, where for any \( \ell \geq 1 \) we write

\[
D^+_{\varepsilon, \ell} := \{ z \in \Lambda : d(f^n(z), N^+) \geq \ell^{-1} e^{-\varepsilon n}, n \geq 0 \},
\]

\[
D^-_{\varepsilon, \ell} := \{ z \in \Lambda : d(f^{-n}(z), N^-) \geq \ell^{-1} e^{-\varepsilon n}, n \geq 0 \}
\]

and

\[
D^+_{\varepsilon} := \bigcup_{\ell \geq 1} D^+_{\varepsilon, \ell}, \quad D^-_{\varepsilon} := \bigcup_{\ell \geq 1} D^-_{\varepsilon, \ell}.
\]
It can be shown that \( V^u(z) \subseteq D_\varepsilon^- \) for any \( z \in D_\varepsilon^- \), see [39, Proposition 5]. Analogously to (4.3), the global stable and unstable manifolds are defined as

\[
W^s(z) = \bigcup_{n \geq 0} \hat{f}^{-n}(V^s(f^n(z))), \quad z \in D_\varepsilon^+,
\]

and

\[
W^u(z) = \bigcup_{n \geq 0} \hat{f}^n(V^u(f^{-n}(z))), \quad z \in D_\varepsilon^-,
\]

where we write \( \hat{f}^n(A) := f^n(A \setminus N^+) \) and \( \hat{f}^{-n}(A) := f^{-n}(A \setminus N^-) \), \( A \subseteq \Lambda \).

Now let \( \varepsilon > 0 \) and \( \ell \geq 1 \) be fixed. Given \( x \in D_{\varepsilon,\ell}^- \), we write \( B(z,\delta) \) to denote the open ball in \( U \) centered at \( z \) and with radius \( \delta \), and we write \( B^u(z,\delta) \) for the open ball in \( W^u(z) \) with center \( z \) and radius \( \delta \). By [39, Proposition 7], there are \( r_\ell^{(1)} > r_\ell^{(2)} > r_\ell^{(3)} > 0 \) such that for any \( x \in D_{\varepsilon,\ell}^- \) and any \( z \in B(x,r_\ell^{(3)}) \cap D_{\varepsilon,\ell}^- \) the intersection \( V^u(z) \cap W(x) \) of \( V^u(z) \) and \( W(x) := \exp_x\{v \in E^s(x) : \|v\| \leq r_\ell^{(1)}\} \) is precisely a single point \([z,x]\) and, moreover, \( B^u([z,x],r_\ell^{(2)}) \subseteq V^u(z) \). Given \( x \in D_{\varepsilon,\ell}^- \) and \( \delta \leq r_\ell^{(3)} \) we define the rectangle \( \mathcal{R}_{\varepsilon,\ell}(x,\delta) \) by

\[
\mathcal{R}_{\varepsilon,\ell}(x,\delta) = \bigcup_{z \in B(x,\delta) \cap D_{\varepsilon,\ell}^-} B^u([z,x],r_\ell^{(2)}).
\]

Obviously (6.2) is a partition of \( \mathcal{R}_{\varepsilon,\ell}(x,\delta) \) into the sets \( B = B^u([z,x],r_\ell^{(2)}) \subseteq V^u(z) \).

In the following we consider \( \varepsilon > 0 \) to be fixed and therefore suppress it from notation. That is, we write

\[
D^- := D^-_\varepsilon, \quad D^-_\ell := D^-_{\varepsilon,\ell}, \quad \mathcal{R}_\ell(x,\delta) := \mathcal{R}_{\varepsilon,\ell}(x,\delta)
\]

and so on, this shortcut notation follows [39, p. 129].

We call an \( f \)-invariant Borel probability measure \( \mu \) on \( D^- \) a Gibbs \( u \)-measure if for any \( \ell \geq 1 \), \( x \in D^-_\ell \) and \( \delta \leq r_\ell^{(3)} \) such that \( \mu(\mathcal{R}_\ell(x,\delta)) > 0 \) we have

\[
\mu(E) = \int_{\mathcal{R}_\ell(x,\delta)} \mu_B(E) \, \mu_{\mathcal{P}_{\mathcal{R}_\ell(x,\delta)}}(dB), \quad E \subseteq \mathcal{R}_\ell(x,\delta) \text{ Borel},
\]

with conditional measures \( \mu_B \) equivalent to the Riemannian volumes \( m_B \) on the partition elements \( B \) as in (6.2).

The following existence result for Gibbs \( u \)-measures had been shown in [39, Theorem 1].

**Theorem 6.1.** Let \( \Lambda \) be a uniformly hyperbolic attractor with singularities for the \( C^2 \)-diffeomorphism \( f \) and assume that there are a point \( z \in D^- \) and constants \( c > 0, q > 0, \varepsilon_0 > 0 \) such that for any \( 0 < \varepsilon \leq \varepsilon_0 \), and \( n \geq 0 \)

\[
m_{V^u(z)}(V^u(z) \cap f^{-n}(U(\varepsilon,N^+))) \leq c \varepsilon^q,
\]

where \( U(\varepsilon,N^+) \) is the \( \varepsilon \)-parallel neighborhood of \( N^+ \) in \( M \). Then there is a Gibbs \( u \)-measure \( \mu \in \mathcal{M}_{\text{b}}^u(M) \) with uniformly bounded densities on \( D^- \subseteq \Lambda \), and \( \mu(D^-) = 1 \).

**Remark 6.1.** It follows in particular that \( \mu(N^+) = 0 \).

Theorem 6.1 allows to apply Theorem 2.1 and its consequences.
Corollary 6.1. Let $f$ and $\Lambda$ be as in Theorem 6.1 and let $\mu \in M^\mathrm{ad}_{\Lambda}$ be a Gibbs $u$-measure with uniformly bounded densities on $D^-$. Then $(\mathcal{E}(\mu), \mathcal{D}(\mathcal{E}(\mu)))$ and $(\mathcal{E}(\mu), \mathcal{D}(\mathcal{E}(\mu)))$ as in Theorem 2.1 are local Dirichlet forms on $L^2(D^-, \mu)$, their generators are self-adjoint operators on $L^2(D^-, \mu)$, and their semigroups are symmetric.

Remark 6.2. In general $D^-$ is a proper subset of $\Lambda$, and tangent spaces in the unstable directions are defined only at points of $D^-$. But since $D^-$ has full measure, we have $L^2(\Lambda, \mu) = L^2(D^-, \mu)$, so that the Dirichlet forms constructed in Corollary 6.1 and the associated Laplacians and semigroups may be regarded as objects on $L^2(\Lambda, \mu)$, and in that sense 'on $\Lambda$'.

Proof. The present hypotheses fit into Section 2 with $D^-$ and $D_{-\ell}$, $\ell \geq 1$, as defined here. For each $\ell \geq 1$ finitely many rectangles of type $R_{\ell}(x, \delta)$ cover the compact set $D_{-\ell} \subseteq \Lambda$, and each rectangle admits a partition as in (6.2). The measure $\mu$ satisfies the disintegration identities (2.2) in the form (6.3) and with uniformly bounded conditional densities.

We provide some examples for attractors with singularities.

Examples 6.1. Let $I = (-1, 1)$, $U = I \times I$ and $N = I \times \{0\}$ and let $f : U \setminus N \to U$ be a map of the form $f(x, y) := (g(x, y), h(x, y))$, where $g, h$ are functions given by

\[
g(x, y) = (-B|y|^{\nu_0} + Bx \operatorname{sgn} y|y|^{\nu} + 1) \operatorname{sgn} y,
\]

\[
h(x, y) = ((1 + A)|y|^{\nu_0} - A) \operatorname{sgn} y,
\]

for constants $0 < A < 1$, $0 < B < \frac{1}{7}$, $\nu > 1$, $1/(1 + A) < \nu_0 < 1$. The resulting attractor is the well-known (geometric) Lorenz attractor; it is a uniformly hyperbolic attractor with singularities.

A more common definition of the Lorenz attractor is as the attractor (in ODE sense) for the non-linear system

\[
\dot{x} = -\sigma x + sy, \quad \dot{y} = rx - y - xz \quad \text{and} \quad \dot{z} = xy - bz
\]

for the particular parameters $\sigma = 10$, $b = \frac{8}{3}$ and $r = 28$, illustrated in Figure 1.

Further details and more general classes of Lorenz attractors are discussed in [39, Section 5.2], see also [23, Section 13.3] and [29, Section 2.2].
Examples 6.2. Given $c \in (0, 1)$, let $I = (-c, c)$, $U = I \times I$ and $N = \{0\} \times I$. Define $f: U \setminus N \to U$ by $f(x, y) = (1 + by - a|x|, x)$, where $0 < a < a_0$ and $0 < b < b_0$ for some small $a_0, b_0 > 0$. The map $f$ is called the Lozi map, and the associated attractor is the Lozi attractor [36]. This is the special case of the so-called Lozi-like maps studied in [39][51]. Ergodic and topological properties can be found in [34][37].

Examples 6.3. Let $U = (-1, 1) \times (-1, 1)$ and $N = \{(x, y) : y = kx\} \subset U$. The map $f: U \setminus N \to U$ given by

$$f(x, y) = \begin{cases} \left(\lambda_1(x - 1) + 1, \lambda_2(y - 1) + 1\right) & \text{for } y > kx, \\ \left(\mu_1(x + 1) - 1, \mu_2(y + 1) - 1\right) & \text{for } y < kx, \end{cases}$$

has a hyperbolic attractor with singularity set $N$ whenever $|k| < 1$, $0 < \lambda_1, \mu_1 < 1/2$ and $1 < \lambda_2, \mu_2 < \frac{2}{1+|k|}$. It is called the Belykh attractor. Details can be found in [39, p. 149] and [48].

Remark 6.3. Partially hyperbolic attractors with singularities $\Lambda = \overline{D}$ can similarly be constructed using (6.1), with the difference that $D$ must be a partially hyperbolic set in the sense of Section 4. Stable and unstable manifolds can be constructed analogously, as well as Gibbs u-measures, [39, Theorem 12]. A geometric example of such attractors is provided in [39, Section 5.2, Example 4]. See also [31].

References

[1] S. Albeverio, M. Röckner, Classical Dirichlet forms on topological vector spaces - closability and a Cameron-Martin formula, J. Funct. Anal. 88 (1990), 395–436.
[2] S. Alikhanloo, M. Hinz, SRB-symmetric diffusions on hyperbolic attractors, Preprint (2020), arXiv:2012.05972.
[3] P. Alonso-Ruiz, M. Hinz, A. Teplyaev, R. Trevino, Canonical diffusions on the pattern spaces of aperiodic Delone sets, preprint (2018), arXiv:1801.08956.
[4] D. V. Anosov, Geodesic flows on closed Riemannian manifolds of negative curvature, Trudy Mat. Inst. Steklov. 90 (1967), 3–210.
[5] D. Bakry, I. Gentil, M. Ledoux, Analysis and Geometry of Markov Diffusion Operators, Springer Grundlehren Math. Wiss. vol. 348, Springer, New York, 2014.
[6] L. Barreira, Ya. Pesin, Introduction to Smooth Ergodic Theory, Grad. Studies in Math. Vol. 148, Amer. Math. Soc., Providence, 2013.
[7] V. M. Belykh, Qualitative methods of the theory of nonlinear oscillations in point systems, Gorki University Press (1980).
[8] N. Bouleau, F. Hirsch, Dirichlet Forms and Analysis on Wiener Space, deGruyter Studies in Math. 14, deGruyter, Berlin, 1991.
[9] M. Brin, Ya. Pesin, Partially hyperbolic dynamical systems, Math. USSR Izvestija, 8:1 (1974) 177–218.
[10] M. Brin, G. Stuck, Introduction to Dynamical Systems, Cambridge Univ. Press, Cambridge, 2002.
[11] K. Burns, D. Dolgopyat, Y. Pesin, M. Pollicott, Stable ergodicity for partially hyperbolic attractors with negative central exponents, J. Mod. Dyn. 2(1) (2008), 1–19.
[12] K. Burns, M. Gidea, Differential Geometry and Topology: With a View to Dynamical Systems, Chapman & Hall/CRC Press, Boca Raton, FL, 2005.
[13] V. Climenhaga, S. Luzzatto, Y. Pesin, The geometric approach for constructing Sinaĭ-Ruelle-Bowen measures, J. Stat. Phys. 166 (2017), no. 3-4, 467–493.
[14] R.M. Blumenthal, R.K. Getoor, Markov Processes and Potential Theory, Acad. Press, New York, 1968.
[15] E.B. Davies, Heat Kernels and Spectral Theory, Cambridge Tracts in Math. 92, Cambridge Univ. Press, Cambridge, 1989.
[16] M. Einsiedler, Th. Ward, Ergodic Theory with a View towards Number Theory, Graduate Texts in Math. 299, Springer, London, 2011.
[17] L.C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, Vol. 19, Amer. Math. Soc., Providence, Rhode Island, 1998.

[18] T. Fisher, B. Hasselblatt, *Hyperbolic flows*, Zürich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2019.

[19] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet forms and symmetric Markov processes*, deGruyter, Berlin, New York, 1994.

[20] A. Grigor’yan, *Heat kernel and Analysis on Manifolds*, AMS/IP Stud. in Adv. Math. 47, Amer. Math. Soc., 2009.

[21] U. Hamenstädt, *Harmonic measures for compact negatively curved manifolds*, Acta Math. 178 (1997), 39–107.

[22] B. Hasselblatt, *Ergodic Theory and Negative Curvature*, Springer Lecture Notes series, 2017.

[23] B. Hasselblatt and A. Katok, *A First Course in Dynamics: With a Panorama of Recent Developments*, Cambridge University Press, New York, 2003.

[24] B. Hasselblatt and Y. Pesin, *Partially hyperbolic dynamical systems*, Handbook of dynamical systems, Vol. 1B, Elsevier B. V., 2006, pp. 1–55.

[25] M. Hirsch, C. Pugh, M. Shub, *Invariant manifolds*, Bull. Amer. Math. Soc., 76 (1970), 1015–1019.

[26] M. Hirsch, C. Pugh, M. Shub, *Invariant manifolds*, Lecture Notes in Math., 583, Springer-Verlag, 1977.

[27] A. Katok, B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge Univ. Press, Cambridge, 1995.

[28] A. Katok and J.M. Strelcyn, *Invariant Manifolds, Entropy and Billiards. Smooth Maps with Singularities*, Lecture Notes in Mathematics 1222 Springer: Berlin (1986).

[29] F. Ledrappier, *Harmonic measures and Bowen-Margulis measures*, Israel J. Math. 71 (3) (1990), 275–287.

[30] F. Ledrappier, J.-M. Strelcyn, *A proof of the estimation from below in Pesin’s entropy formula*, Ergod. Th. Dyn. Sys. 2 (1982), 203–219.

[31] F. Ledrappier, L.-S. Young, *The metric entropy of diffeomorphisms. Part I: Characterization of measures satisfying Pesin’s entropy formula*, Ann. Math. 122 (3) (1985), 509–539.

[32] J.M. Lee, *Introduction to Smooth Manifolds*, Grad. Texts in Math. 218, Springer, New York, 2003.

[33] Y. Levy, *Ergodic properties of the Lozi map*, Springer Lecture Notes in Mathematics, 1109, Springer-Verlag, Berlin (1985) 103–116.

[34] R. de la Llave, J. M. Marco, R. Moriyon, *Canonical perturbation theory of Anosov systems and regularity results for the Livsic cohomology equation*, Ann. Math. 123 , (1986), 537–611.

[35] R. Lozi, *Un attracteur étrange du type attracteur de Hénon*, J. Phys., Paris 39, Coll. C5 (1978), 9–10.

[36] M. Misiurewicz, *Strange attractors for the Lozi mappings*, Nonlinear Dynamics, R.G. Helleman, ed. Academic: New York (1980) 348–358.

[37] W. Parry, M. Pollicott, *Zeta Functions and the Periodic Orbit Structure of Hyperbolic Dynamics*, Astérisque (1990), 187–188.

[38] Ya. Pesin, *Dynamical systems with generalized hyperbolic attractors: hyperbolic, ergodic and topological properties*, Ergod. Theory Dyn. Syst. 12(1) (1992), 123–151.

[39] Ya. Pesin, *Lectures on partial hyperbolicity and stable ergodicity*, Zürich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2004.

[40] Ya. Pesin, *Lyapunov characteristic exponents and smooth ergodic theory*, Russ. Math. Surv. 32 (1977), no. 4, 55–114.

[41] Ya. Pesin, Ya. Sinai, *Gibbs measures for partially hyperbolic attractors*, Ergod. Th. Dyn. Sys. 2 (1982), 417–438.

[42] M. Pollicott, *Lectures on ergodic theory and Pesin theory on compact manifolds*, London Mathematical Society Lecture Note Series, vol. 180, Cambridge University Press, Cambridge, 1993.

[43] M. Reed, B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis*, Academic Press, San Diego, 1980.

[44] F. Rodriguez Hertz, M. A. Rodriguez Hertz, and R. Ures, *A survey of partially hyperbolic dynamics*, Fields Institute Comm 51 (2007), 35–88.
[46] V. A. Rokhlin, *On the fundamental ideas of measure theory*, Transl. Amer. Math. Soc., Series 1, 10 (1962), 1–52.

[47] E. A. Sataev, *Invariant measures for hyperbolic maps with singularities*, Russian Math. Surveys 47 (1992), 191–251.

[48] E. A. Sataev, *Ergodic properties of the Belykh map*, J. Math. Sci. 95 (1999), 2564–2575.

[49] J. Schmeling, S. Troubetzkoy, *Dimension and invertibility of hyperbolic endomorphisms with singularities*, Ergod. Th. Dyn. Syst. 18 (1998), 1257–1282.

[50] M. Viana, K. Oliveira, *Foundations of Ergodic Theory*, Cambridge University Press, Cambridge, 2016.

[51] L. S. Young, *Bowen-Ruelle measures for certain piecewise hyperbolic maps*, Trans. Amer. Math. Soc., 287 (1985), 41–48.

[52] C. B. Yue, *Integral formulas for the Laplacian along the unstable foliation and applications to rigidity problems for manifolds of negative curvature*, Ergod. Th. Dyn. Sys. 11 (4) (1991), 803–819.

[53] C. B. Yue, *Brownian motion on Anosov foliations and manifolds of negative curvature*, J. Diff. Geom. 41 (1995), 159–183.

1Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany

Email address: salikhan@math.uni-bielefeld.de

2Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany

Email address: mhinz@math.uni-bielefeld.de