On positive scalar curvature bordism

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Abstract

Using standard results from higher (secondary) index theory, we prove that the positive scalar curvature bordism groups $\text{Pos}^{\text{spin}}_n(G \times \mathbb{Z})$ are infinite for any $n \geq 1$ and $G$ a group with non-trivial torsion. We construct representatives of each of these classes which are connected and with fundamental group $G \times \mathbb{Z}$. We get the same result for $\text{Pos}^{\text{spin}}_{n+2}(G \times \mathbb{Z})$ if $G$ is finite and contains an element which is not conjugate to its inverse. This generalizes the main result of Kazaras, Ruberman, Saveliev, “On positive scalar curvature cobordism and the conformal Laplacian on end-periodic manifolds” to arbitrary even dimensions and arbitrary groups with torsion.

1 Introduction

The classification of Riemannian metrics of positive scalar curvature (up to suitable natural equivalence relations) is an active object of study in geometry. One popular way to organize this uses the Stolz positive scalar curvature exact sequence (compare [11, Proposition 1.27])

$$\cdots \rightarrow \text{Pos}^{\text{spin}}_n(B\Gamma) \rightarrow \Omega^{\text{spin}}_n(B\Gamma) \rightarrow R^{\text{spin}}_n(B\Gamma) \rightarrow \text{Pos}^{\text{spin}}_{n-1}(B\Gamma) \rightarrow \cdots$$  (1.1)

Here, $\text{Pos}^{\text{spin}}_n(B\Gamma)$ is one of our main objects of study, the group of closed $n$-dimensional spin manifolds $(M, f, g)$ with a reference map $f: M \rightarrow B\Gamma$ and a Riemannian metric $g$ of positive scalar curvature. The equivalence relation is bordism, where the bordisms have to carry the corresponding structure. We use throughout the usual convention that Riemannian metrics on manifolds with boundary (e.g. on a bordism) must have product structure near the boundary.

In this setup, $\Gamma$ is just an arbitrary group. If the starting point is a connected smooth manifold $M$, typically one chooses $\Gamma = \pi_1(M)$. Moreover, $B\Gamma$ is a classifying space for $\Gamma$ and the map $f$ contains essentially the same information as the homomorphism $f_*: \pi_1(M) \rightarrow \Gamma$.

The group $\Omega^{\text{spin}}_n(B\Gamma)$ is the usual bordism group, whereas $R^{\text{spin}}_n(B\Gamma)$ is Stolz’ $R$-group, defined as the set of bordism classes $(W, f, g)$ where $W$ is a compact $(n+1)$-dimensional spin-manifold, possibly with boundary, with a reference map $f: W \rightarrow B\Gamma$, and with a positive scalar curvature metric on the boundary when the latter is non-empty.

Our main goal is to show that the positive scalar curvature bordism groups $\text{Pos}^{\text{spin}}_n(B\Gamma)$ are rich (more precisely, map onto an infinite cyclic group) in new
situations. Because the starting point often is a fixed connected manifold $M$ with fundamental group $\Gamma$, we will show in addition that the infinitely many different non-trivial representatives can be chosen to be connected and with fundamental group $\Gamma$ (mapped bijectively under the reference map to $B\Gamma$), even in dimension 4. Indeed, we are taking up the main result \cite[Theorem 1]{Kazaras} of Kazaras, Ruberman, and Saveliev, which says

1.2 Theorem. Let $n = 4$ or $6$ and $\{1\} \neq G$ be a fundamental group of a 3-dimensional spherical space form if $n = 4$, or more generally a finite group with at least one element not conjugate to its inverse if $n = 6$. Set $\Gamma := G \times \mathbb{Z}$.

Then $\text{Pos}^{\text{spin}}_n(B\Gamma)$ contains infinitely many elements, represented by maps $f : M \to B\Gamma$ where $M$ is connected and $f$ induces an isomorphism in $\pi_1$.

We improve that theorem by allowing more general groups and all even dimensions bigger than 2.

1.3 Theorem. Let $n > 2$ be an even integer. If $n = 4k$ is divisible by 4, let $G$ be an arbitrary finitely presented group which contains a non-trivial torsion element. If $n = 4k + 2$ let $G$ be a finite group such that at least one of its elements is not conjugate to its inverse. Set $\Gamma := G \times \mathbb{Z}$.

Then $\text{Pos}^{\text{spin}}_n(B\Gamma)$ contains infinitely many elements $x_j$, represented by connected manifolds $M_j$ with fundamental group $\Gamma$ as in the Theorem of Kazaras, Ruberman, and Saveliev.

Even better, we have a non-trivial homomorphism $\text{Ind}_\rho : R^{\text{spin}}_n(B\Gamma) \to \mathbb{R}$ and the $x_j$ lift to elements in $R^{\text{spin}}_{n+1}(B\Gamma)$ whose image under $\text{Ind}_\rho$ form an infinite cyclic subgroup of $\mathbb{R}$.

Theorem 1.2 of \cite{Kazaras} is based on the beautiful, but rather complicated and technical index theory for manifolds with periodic ends. The associated bordism invariance is very delicate and uses, for the time being, minimal hypersurface techniques which cause the dimension restriction in Theorem 1.2.

The main point we want to make in this note is that this delicate theory is actually not necessary for the result at hand. Instead, it can be derived from well known (and not too complicated) results in higher index theory, and along the way one gets the more general result. Related techniques and results are developed in \cite{Piazza} which can be used to get even stronger results, compare the discussion below.

2 Secondary higher index

The distinction of bordism classes of metrics of positive scalar curvature in the literature typically relies on secondary index invariants of the Dirac operator, and this is precisely how we prove the main part of Theorem 1.3. Recall that the classical (higher) index of the Dirac operator can be defined for closed manifolds, but also for manifolds with boundary provided the boundary operator is invertible \cite[Section 2.2]{Piazza}. In particular, we have a commutative diagram

$$
\begin{array}{cccc}
\Omega^{\text{spin}}_n(B\Gamma) & \longrightarrow & R^{\text{spin}}_n(B\Gamma) & \\
\downarrow & & \downarrow \text{Ind} & \\
K_n(B\Gamma) & \longrightarrow & K_n(C^*\Gamma) & \\
\end{array}
$$

(2.1)
On positive scalar curvature bordism

Here, $\mu$ is the Baum-Connes assembly map which in this setup is not an isomorphism if $\Gamma$ is not torsion-free.

This diagram is the easy and elementary part of the diagram "mapping the Stolz exact sequence to analysis" developed in [11]. Recall that $\Gamma = G \times \mathbb{Z}$. The diagram (2.1) and the corresponding one for $G$ are closely related via a kind of Künneth theorem. Indeed, we have a transformation, given by product with $S^1$ or its fundamental K-theory class (and using that $BG = BG \times S^1$ with $S^1 = B\mathbb{Z}$)

$$
\begin{align*}
\Omega_{n}^{\text{spin}}(BG) & \longrightarrow R_{n}^{\text{spin}}(BG) & \Omega_{n+1}^{\text{spin}}(BG) & \longrightarrow R_{n+1}^{\text{spin}}(BG) \\
\downarrow & \downarrow \text{Ind} & \downarrow \times[S^1] & \downarrow \text{Ind} \\
K_{n}(BG) & \mu \longrightarrow K_{n}(C^{*}G) & K_{n+1}(BG) & \mu \longrightarrow K_{n+1}(C^{*}G)
\end{align*}
$$

2.3 Lemma. After removing the term $R_{n}^{\text{spin}}(\cdot)$, the map $\times[S^1]$ of (2.2) is an embedding as direct summand of the left diagram into the right. In other words, the maps $\times[S^1]$ are injective and the map $\mu$ is compatible with a suitable direct sum decomposition of the right hand side into the left hand side and a complement.

Proof. This is certainly well known. For the convenience of the reader we give here a detailed proof.

For a multiplicative generalized homology theory $E_{s}$ and for natural transformations of such, in particular for $\Omega_{s}^{\text{spin}} \rightarrow K_{s}$, this is a consequence of the natural Künneth theorem for the special product $X \times S^1$ or equivalently of the suspension isomorphism. The desired splitting of $E_{s}(X) \xrightarrow{\times[S^1]} E_{s+1}(X \times S^1)$ is obtained as the composition

$$E_{s+1}(X \times S^1) \xrightarrow{pr} \tilde{E}_{s+1}(X \times S^1/X \times \ast) \xrightarrow{\sigma^{-1}} \tilde{E}_{s+1}(\Sigma(X_{+})) \xrightarrow{\ast} \tilde{E}_{s}(X_{+}) = E_{s}(X).$$

Here $pr$ is the canonical projection, $X_{+}$ is the disjoint union of $X$ and an additional basepoint, $\Sigma(X_{+})$ its reduced suspension, and $\sigma$ is the suspension isomorphism. The complementary summand is the (injective) image $i_{*}(E_{s+1}(X)) \subset E_{s+1}(X \times S^1)$ for $i : X \leftrightarrow X \times S^1; x \mapsto (x, \ast)$ (or the exterior product with the unit 1), so that we get in particular the direct sum decomposition

$$K_{s+1}(X \times S^1) = K_{s+1}(X) \otimes 1 \oplus K_{s}(X) \otimes [S^1].$$

For the $C^{*}$-algebra K-theory, the Künneth theorem [15, Theorem 4.1] specializes in our situation to the isomorphism (given by external tensor product) $K_{s}(C^{*}G) \otimes K_{s}(C^{*}\mathbb{Z}) \rightarrow K_{s}(C^{*}G \otimes C^{*}\mathbb{Z}) = K_{s}(C^{*}\Gamma)$. Using $K_{s}(C^{*}\mathbb{Z}) = 1 \cdot \mathbb{Z} \oplus [S^1] \cdot \mathbb{Z}$ with unit 1 in $K_{0}$ and $[S^1] \in K_{1}(C^{*}\mathbb{Z})$, the image of $[S^1]$ under the Baum-Connes isomorphism $\mu_{S^1} : K_{s}(S^1) \xrightarrow{\cong} K_{s}(C^{*}\mathbb{Z})$, we obtain the splitting

$$K_{s+1}(C^{*}\Gamma) = K_{s+1}(C^{*}G) \otimes 1 \oplus K_{s}(C^{*}\Gamma) \otimes [S^1].$$

The Baum-Connes map $\mu$ is compatible with external products, proving the rest of the claims. \hfill \square

The strategy is now the following.
(1) We recall a suitable homomorphism \( \text{Ind}_\rho: K_n(C^*G) \to \mathbb{R} \), a delocalized index. “Delocalized” means in particular that \( \text{Ind}_\rho \circ \mu: K_n(BG) \to \mathbb{R} \) is zero.

(2) We will construct an appropriate element \( [W, f, g] \in R_{n}^{\text{spin}}(BG) \) with \( \text{Ind}_\rho(\text{Ind}[W, f, g]) \neq 0 \). It generates an infinite cyclic subgroup of \( R_{n}^{\text{spin}}(BG) \) detected in \( K_n(C^*G) \) and then in \( \mathbb{R} \) as image of the maps \( \text{Ind} \) and \( \text{Ind}_\rho \circ \text{Ind} \), respectively.

(3) By the commutativity of (2.1) and using item (1), no non-trivial element of this infinite cyclic subgroup is in the image of \( \Omega_{n}^{\text{spin}}(BG) \to R_{n}^{\text{spin}}(BG) \), and therefore this infinite cyclic subgroup is mapped injectively to \( \text{Pos}_{n-1}^{\text{spin}}(BG) \).

Finally, we take the Cartesian product with \( S^1 \).

2.4 Corollary. The element \( W \times S^1 \) generates an infinite cyclic subgroup of \( R_{n+1}^{\text{spin}}(B\Gamma) \) detected in \( K_{n+1}(C^*\Gamma) \) as image of the map \( \text{Ind} \).

None of its non-zero elements is in the image of \( \Omega_{n+1}^{\text{spin}}(B\Gamma) \), and so this infinite cyclic group generated by \( W \times S^1 \) injects into \( \text{Pos}_{n}^{\text{spin}}(B\Gamma) \).

Proof. This follows from the commutativity of (2.1) and the fact that the Künneth map (2.2) gives an inclusion as direct summand of the whole map \( K_n(BG) \xrightarrow{\mu} K_n(C^*G) \to K_{n+1}(B\Gamma) \xrightarrow{\mu} K_{n+1}(C^*\Gamma) \).

It remains to construct \( W \) and \( \text{Ind}_\rho \) with the appropriate properties. In all cases, \( W \) will be a null bordism of a disjoint union of lens spaces or lens space bundles, we describe this explicitly later.

Case \( n \equiv 0 \pmod{4} \)

Essentially, this case has been treated in [10] and we refer to this paper for more details. We choose \( \text{Ind}_\rho: K_0(C^*G) \to \mathbb{R} \) to be equal to the difference of the homomorphisms induced by the standard trace and by the trivial representation. Recall that the first one gives the \( L^2 \)-index and it is induced by the canonical trace \( \text{tr}_G: C^*G \to \mathbb{C} \), while the second one gives the ordinary index and is induced by the trivial homomorphism \( G \to \{1\} \).

It is a direct consequence of Atiyah’s \( L^2 \)-index theorem [1] (1.1)] that \( \text{Ind}_\rho \) defined in this way vanishes on the image of \( \mu \).

Small caveat: we have to use a \( C^* \)-completion \( C^*G \) of \( C[G] \) such that the trivial homomorphism extends to this completion. For this, one could use the maximal \( C^* \)-completion or a smaller, more geometric one (based on the direct sum of the regular representation and the trivial representation). The relevant theory is well known, compare e.g. [14].

We have to compute \( \text{Ind}_\rho \) for a compact manifold \( W \) of dimension \( n \) with boundary \( L \) of dimension \( n-1 \), where the boundary is equipped with a metric of positive scalar curvature, and where \( W \) is equipped with a reference map to \( BG \). Again, this is a classical result, an easy special case of the \( (L^2) \)-Atiyah-Patodi-Singer index theorem [13]. This index is the Cheeger-Gromov \( L^2-\rho \)-invariant of the positive scalar curvature metric of the boundary \( L \), where we use the \( G \)-covering pulled back via the map to \( BG \).
Finally, for a cyclic group $C_p$ of order $p$ which we choose to be prime, the $L^2$-$\rho$-invariant is just the Atiyah-Patodi-Singer $\rho$-invariant associated to $\alpha := \frac{1}{p} \rho_{reg} - \rho_1$, the linear combination of the trivial and the regular representation in the complexified representation ring.

For the group $G$ with non-trivial torsion element, we choose an embedding $\iota: C_p \to G$ with induced map $B\iota: BC_p \to BG$. The $L^2$-$\rho$-invariant is compatible with “induction”, meaning that for a map $L \to BC_p \xrightarrow{B\iota} BG$ the $L^2$-$\rho$-invariants with respect to the covering pulled back from $BC_p$ and from $BG$ coincide. In this situation, therefore, $\rho(2)(L) = \rho_\alpha(L)$, see [10, Lemma 2.22].

Finally, we come to the explicit construction of $W$. We choose for the boundary $L$ the disjoint union of an appropriate number $N_L$ of copies of the lens space $L(p; 1, \ldots, 1)$, the quotient of $S^{n-1} \subset \mathbb{C}^n$ by the action of $C_p$ where the generator acts by multiplication with a fixed $p$-th root of unity, equipped with the quotient metric $g_L$ and with map $u: L(p; 1, \ldots, 1) \to BC_p$ inducing an isomorphism on the fundamental group. This space has a spin structure (unique if $p > 2$). The computation of the $\eta$-invariant of the Dirac operator is classical, we get:

\begin{equation}
\rho(2)(L(p, 1, \ldots, 1)) = \rho_\alpha(L) = (-1)^{\frac{n}{2}+1} \sum_{j=1}^{n-1} \frac{1}{p} \frac{1}{|\zeta_j - 1|^{n/2}} \neq 0.
\end{equation}

Here, $\zeta$ is a primitive $p$-th root of unity.

**Proof.** This follows from the equivariant Lefschetz fixed point formula of Donnelly [4, Proposition 4.1] (applied to $S^{n-1}$ bounding an $n$-dimensional hemisphere), as observed in [2, Lemma 2.3], compare also [5, Lemma 2.1] or [6]. Note that the formulas hold in $\mathbb{R}$ and not only in $\mathbb{R}/\mathbb{Z}$ because we have positive scalar curvature and hence no harmonic spinors. We use that for our virtual representation $\rho$ we have $\rho(g) = -1$ if $g \neq e$ and $\rho(e) = 0$.

The group $\Omega^{\text{spin}}_{n-1}(BC_p)$ is finite because $n-1$ is odd, as one deduces readily from the Atiyah-Hirzebruch spectral sequence. Therefore, we find $N_L$ and a spin null bordism $U: W \to BC_p$ of $L := N_L \cdot L(p; 1, \ldots, 1) = \bigsqcup_{i=1}^{N_L} L(p; 1, \ldots, 1)$. Together with the metric $g$ on the boundary this represents the desired element $(W, U, g)$ of $R^{\text{spin}}(BC_p)$.

**2.6 Remark.** In particular, [2, Theorem 0.1] —the non-triviality of $R^{\text{spin}}(BC_p)$— extends to $n = 4$ without any difficulty. Indeed, Botvinnik and Gilkey could have stated the result in this form and their proof would have worked. However, their main focus was on [2, Theorem 0.2] —that the space of metrics of positive scalar curvature on a given manifold has infinitely many component. And this latter result we still can only prove for dimensions $\geq 5$.

**Case** $n \equiv 2 \pmod{4}$

Here, we have nothing new to offer, the construction of $\text{Ind}_\rho$ and $L$ are given in [2]. One uses an appropriate virtual representation $\rho$ of dimension 0 and the associated twisted index (which then vanishes on the image of $\mu$) and a suitable explicitly constructed $L$. For the readers convenience we recall the construction.
Start with the sphere bundle $S$ of the bundle $\eta \oplus \eta \to \mathbb{C}P^1$, where $\eta$ is the tautological complex line bundle over $\mathbb{C}P^1$. Use the diagonal $U(1)$-action to divide by the action of the cyclic subgroup $C_p$ to obtain a manifold $X$. Then set $L_0 := X \times K_r$ where $K$ is the Kummer surface, a 4-dimensional simply connected spin manifold with non-vanishing $\hat{A}$-genus. In [2, Lemma 2.3] a spin structure, positive scalar curvature metric and classifying map to $BC_p$ are constructed for $L_0$. We can now continue exactly as in the case $n \equiv 0 \pmod{4}$, using $L_0$ instead of $L(p, 1, \ldots, 1)$. In [2, Section 3.2] the authors address the gaps in the original proof of [1, Corollary 4.2] for manifolds of dimension $4k + 3$, which we are using here.

3 Positive scalar curvature bordism

To complete the proof of Theorem 1.3 one has to show that we can actually find representatives which are connected and such that the map to $BG$ induces an isomorphism $\pi_1(M) \to \Gamma$. However, this is a general and well-known fact from the Gromov-Lawson surgery construction of positive scalar curvature metrics and is already carried out in [9, Section 5] and [8, Section 9.3]. As we have nothing new to offer here, we don’t repeat this proof.

3.1 Remark. In dimension $n \geq 5$, one can do even better: given an arbitrary
On positive scalar curvature bordism

connected closed spin manifold $M$ with reference map $f: M \to BG$ which is an isomorphism on $\pi_1$ and a manifold $f: M' \to BG$ with positive scalar curvature metric $g'$ bordant to $M \to BG$, one can find a metric $g$ on $M$ of positive scalar curvature such that $[M', f', g'] = [M, f, g] \in \text{Pos}^{\text{spin}}_n(B\Gamma)$. In other words: one can often choose the underlying manifold $M$ representing a class in $\text{Pos}^{\text{spin}}_n(B\Gamma)$.

If $n = 4$, however, this method breaks down. Nonetheless, some trace of this remains true, as shown in [8, Section 9]: in the above situation, one can represent $[M', f', g']$ by $(\tilde{M}, \tilde{f}, g)$ where $\tilde{M}$ is the connected sum of $M$ with an (unspecified) number of copies of $S^2 \times S^2$.

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