b–VECTORS OF CHORDAL GRAPHS

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ABSTRACT. The b-vector \((b_1, b_2, \ldots, b_d)\) of a graph \(G\) is defined in terms of its clique vector \((c_1, c_2, \ldots, c_d)\) by the equation \(\sum_{i=1}^{d} b_i(x+1)^{i-1} = \sum_{i=1}^{d} c_i x^{i-1}\), where \(d\) is the largest cardinality of a clique in \(G\). We study the relation of the \(b\)-vector of a chordal graph \(G\) with some structural properties of \(G\). In particular, we show that the \(b\)-vector encodes different aspects of the connectivity and clique dominance of \(G\). Furthermore, we relate the \(b\)-vector with the Betti numbers of the Stanley-Reisner ring associated to clique simplicial complex of \(G\).

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1. INTRODUCTION

In this manuscript we study the \(b\)-vector \((b_1, b_2, \ldots, b_d)\) of a chordal graph \(G\) where \(d\) is the largest cardinality of a clique in \(G\). We consider non-complete graphs since in this case we conclude that \(b_i = 1\) for every \(i = 1, \ldots, d\). If \(c_i\) denotes the number of cliques of \(G\) with \(i\) vertices, then the clique vector is given by \(c(G) = (c_1, c_2, \ldots, c_d)\). We note that the \(c\)-vector is the \(f\)-vector of the clique complex of \(G\) shifted by one. This vector is a classical invariant of a graph and has been intensively studied [Fro08, Fro10, HHM+08, Zyk49].

The \(b\)-vector of \(G\) given by \(b(G) = (b_1, b_2, \ldots, b_d)\), is a more recent numerical invariant [HHM+08] defined by the equation

\[
\sum_{i=1}^{d} b_i(x+1)^{i-1} = \sum_{i=1}^{d} c_i x^{i-1}.
\]

Goodarzi [Goo15] showed that the vertex connectivity of \(G\), denoted by \(\kappa\), is encoded in the \(b\)-vector as \(b_i = 1\) for \(i \leq \kappa\) and \(b_{\kappa+1} \neq 1\). We extend this theorem by studying the remaining \(b_i\) and relate them to the number of connected components of \(G, W(G \setminus Y)\), after the deletion of a set \(Y\), with \(i-1\) vertices.

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We also relate \( b_i \) to clique domination (see Definition 2.1). Using this notion, we define the number \( d_i(G) \), which measures how many \( i \)-cliques are necessary to dominate every maximal clique of order at least \( i \) in \( G \). In order to establish the relation between the \( b_i \) and \( d_i(G) \), we need to introduce \( \kappa(G) \), which is defined as the maximum cardinality of the intersection of any pair of maximal cliques in \( G \).

Using these combinatorial invariants of \( G \), we can state the main result regarding the structure of a chordal graph encoded by its \( b \)-vector.

**Theorem 1.1** (See Corollary 5.9 and Theorem 5.13). Let \( G \) be a chordal graph with vertex connectivity \( \kappa \). Then,

(a) \( b_i = \sum_{|Y|=i-1}(W(G-Y) - 1) + 1 \) for \( 1 \leq i \leq \kappa + 1 \), where \( Y \subseteq V(G) \);
(b) \( b_i < \sum_{|Y|=i-1}(W(G-Y) - 1) + 1 \) for \( \kappa + 2 \leq i \leq d \), where \( Y \subseteq V(G) \);
(c) \( b_i \leq d_i(G) \) for every \( i = 1, \ldots, d \);
(d) \( b_i = d_i(G) \) for every \( i > \kappa(G) \);
(e) \( b_i \leq b_j \) for every \( \kappa(G) < j \leq i \).

A key component of parts of the previous theorem is the use of the Betti numbers of the Stanley-Reisner ring associated to \( \Delta(G) \), the simplicial complex of cliques of \( G \). This differs from Goodarzi’s approach, since he only looked at the projective dimension of this ring. In particular, we reinterpret parts of Theorem 1.1 in terms of these Betti numbers using Proposition 4.1 and the fact that \( I_{\Delta(G)} \) has a 2-linear resolution when \( G \) is chordal. This is given by a formula for \( \beta_i(R/I_{\Delta(G)}) \) in terms of the \( b \)-vector (see Proposition 5.12).

**Theorem 1.2.** Let \( G \) be a chordal graph with \( n \) vertices and vertex connectivity \( \kappa \). Let \( \beta_i(R/I_{\Delta(G)}) \) be the \( i \)-th Betti number of the Stanley-Reisner ring \( R/I_{\Delta(G)} \) of \( \Delta(G) \). Then,

(a) \( b_i = \beta_{n-i}(R/I_{\Delta(G)}) + 1 \) for every \( i = 1, \ldots, \kappa + 1 \);
(b) \( b_i < \beta_{n-i}(R/I_{\Delta(G)}) \) for every \( j \geq \kappa + 2 \).

Another key component of the proof of Theorem 1.1 is given by algebraic shifting. In particular, we compare the exterior, \( \Delta(G)^e \), and symmetric, \( \Delta(G)^s \), shiftings of \( \Delta(G) \). As a consequence, we prove that \( I_{\Delta^e} \) and \( I_{\Delta^s} \) have the same graded Betti numbers for every simplicial complex \( \Delta \) such that \( I_{\Delta} \) has a linear resolution. This gives a partial case of a conjecture of Aramova, Herzog and Hibi [AHH00, Conjecture 2.3]. Furthermore, we also prove this conjecture for every \( t \)-skeleton of \( \Delta(G) \), which recover a result by Murai [Mur07].

The final key component for the main result is given by an explicit description a combinatorial shifting for chordal graphs (see Definition 5.4).

2. Background

2.1. Graph terminology. In this manuscript we consider a simple graph \( G = (V(G), E(G)) \) with set of vertices \( V(G) \) and edges \( E(G) \). We also assume that \( G \) is not the complete graph. We now recall some concepts from graph terminology.

The *order* of \( G \) is \( n = |V(G)| \). The *degree* of a vertex \( v \in V(G) \), denoted as \( \deg(v) \), is the number of vertices in \( G \) adjacent to \( v \) and \( N(v) \) denotes the set of neighbors of \( v \) in \( G \). We say that \( G \) is connected if there is a path between any two vertices of \( G \). A *vertex-cut* of \( G \) is a set of vertices whose removal disconnects \( G \). Every graph that is not complete has a vertex-cut. The *vertex connectivity* \( \kappa = \kappa(G) \) of a graph \( G \) is the minimum cardinality of a vertex-cut and a graph is \( k \)-connected if \( k \leq \kappa(G) \). Given a vertex-cut \( Y \) of \( G \), we denote as \( W(G-Y) \) the number of connected components in the graph \( G-Y \).
Since the $b$-vector is defined in terms of its clique vector, we recall all the concepts necessary for our study. A clique is a subset of vertices of a graph such that its induced subgraph is complete and a $i$-clique is a clique of order $i$. The clique vector $c(G)$ of a graph $G$ is a vector $(c_1, c_2, \ldots, c_d)$ in $\mathbb{N}^d$, where $c_i$ is the number of cliques in $G$ with $i$ vertices and $d$ is the largest cardinality of a clique in $G$. A maximum clique of a graph $G$ is a clique such that there is no clique with more vertices and the clique number is the number of vertices in a maximum clique of $G$. A maximal clique in $G$ is a clique which is not contained in any other clique of $G$ and we denote the set of maximal cliques of size $i$ by $\mathcal{C}_i(G)$ for every $1 \leq i \leq d$.

Definition 2.1. We say that a clique $C$ of $G$ dominates a clique $C'$ if $C \subseteq C'$. A dominating $i$–clique of $G$ is a set of cliques of order $i$ in $G$ that dominates all maximal cliques of order at least $i$ in $G$. For every $1 \leq i \leq d$, we take

$$d_i(G) = \min\{|\mathcal{D}| \mid \mathcal{D} \text{ is a dominating } i \text{-clique of } G\}.$$

We say that a dominating $i$–clique $\mathcal{D}$ is minimum if $|\mathcal{D}| = d_i(G)$.

For a simplicial complex $\Delta$ on a set of vertices $V$ and $Y \subseteq V$, we denote as $\Delta_{|V \setminus Y}$ the simplicial subcomplex of $\Delta$ restricted in $Y$ and as $W(\Delta - Y)$ the number of connected components in $\Delta_{|V \setminus Y}$. The $t$–skeleton of $\Delta$ is given by $\Delta^{(t)} = \{\tau \in \Delta \mid \dim(\tau) \leq t\}$, in particular $\Delta^{(1)} = G$. The connectivity of a simplicial complex $\Delta$ is defined as the connectivity of its 1–skeleton. The set of cliques in $G$ forms simplicial complex $\Delta(G)$, known as the clique complex of $G$. Then the well-known $f$-vector of $\Delta(G)$ is exactly the clique vector of $G$.

A graph is chordal if every cycle of length at least 4 has a chord i.e., an edge that is not part of the cycle but connects two vertices of the cycle. Given the clique vector $c(G)$ of a chordal graph $G$, we define the vector $b$–vector of $G$, $b(G) = (b_1, b_2, \ldots, b_d)$, defined as

$$(2.1.1) \quad \sum_{i=1}^{d} b_i(x + 1)^{i-1} = \sum_{i=1}^{d} c_i x^{i-1}.$$ 

We seek to show that as $c(G)$ give us the number the $i$–cliques of $G$, $b(G)$ also give us structural information if $G$ is chordal.

2.2. Free resolutions and Stanley-Reisner rings. In this subsection we consider a polynomial ring $R = K[x_1, \ldots, x_n]$ as a $\mathbb{N}^n$–graded ring with $\deg(x_i) = e_i$ where $e_i$ is the vector with one in the $i$–th entry and zeros elsewhere. We take $m = (x_1, \ldots, x_n)$. Let $M$ be a graded finitely generated $R$–module. Then, there exists a resolution by graded free modules

$$0 \to F_p \xrightarrow{\varphi_{p-1}} F_{i-1} \to \ldots \xrightarrow{\varphi_0} F_0 \to M \to 0,$$

where $\varphi_i$ is represented by a matriz with homogebeous entries in $m$. We can write $F_i = \bigoplus_j R(-j)^{\beta_{i,j}(M)}$, where $R(-\alpha)$ denotes a rank one free module with an homogeneous generator in degree $-\alpha$. By The Hilbert Syzygy Theorem, $p \leq n$. The projective dimension of $M$ is defined by $\text{pd}(M) = p$. By the Auslander-Buchsbaum formula, we have that $\text{depth}(M) = n - \text{pd}(M)$. The numbers $\beta_{i,j}(M)$ are important invariants for $M$, with a vast number of applications in algebra, geometry, and topology. The Castelnuovo-Mumford regularity is defined by $\text{reg}(M) = \max\{j \mid \beta_{i,j}(M) \neq 0 \text{ with } |\alpha| = i + j\}$.

We say that $M$ has $t$–linear resolution if every homogeneous minimal generator of $M$ has degree $t$ and $\beta_{i,i+j}(M) = 0$ for $j \neq t$. 

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We refer Peeva’s book on free resolutions [Pee11] and Eisenbud’s book on syzygies [Eis05] for more details about free resolutions and their geometry.

An ideal generated by monomials is square free, if it is radical. There is a well-known bijection between the squarefree monomial ideals in $R$ and simplicial complexes in $n$.

Given a simplicial complex $\Delta$, the square free monomial associated to $\Delta$ is defined by

$$I_\Delta = (x^\sigma \mid \sigma \notin \Delta).$$

The quotient $R/I_\Delta$ is called the Stanley-Reisner ring of $\Delta$.

Given a square-free monomial ideal $I$, the simplicial complex associated to $I$ is defined by

$$\Delta = \{\sigma \subseteq [n] \mid x^\sigma \notin I\}.$$

We refer to the book by Miller and Sturmfels on combinatorial commutative algebra [MS05] and the survey by Francisco, Mermin and Schweig on Stanley-Reisner theory [FMS14] for more details about these rings.

Hochster’s Formula [Hoc77, Theorem 5.1] is an important result that connects the topology of simplicial complexes to free resolutions, which plays a key role in this manuscript. Let $W = \text{supp}(\alpha)$. Then,

$$\beta_{i+1, \alpha}(R/I_\Delta) = \beta_{i, \alpha}(I_\Delta) = \tilde{H}^{\lfloor |W| - i - 2 \rfloor}((\Delta_{|W} ; K))$$

for every $i \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$.

3. Threshold Graphs

In this section, we focus on the $b$–vector of threshold graphs. This is a key part of our strategy since the $b$–vector determines a unique threshold graph, and therefore, its graph properties. To expand our result to chordal graphs we use results from shifting theory in Subsection 4.2.

3.1. Definition and basic properties. A threshold graph is a graph that can be constructed from a one-vertex graph by the following two operations: addition of one isolated vertex, denoted by $D$, and addition of a vertex connected to all other vertices, denoted by $S$. We set the convention of always start with an $S$–operator. Thus, there is a bijection between threshold graphs and words on the alphabet $\{S, D\}$ beginning with an $S$ reading from left to right.

![Figure 1. Graph corresponding to the word SDSDDS](image)

We consider a vertical line before every $S$ in the word corresponding to the threshold graph $T$, thus breaking the word into subwords. Goodarzi [Goo15] showed that the entry $b_i$ of the $b$–vector $b(G)$ is the length of the $(d-i+1)$–th subword for every $i = 1, \ldots, d$. As a consequence,
there is a bijection between \( b \)-vectors and threshold graphs. For instance, the graph in Figure 3.1, which has the word SDSDDDS, breaks to \(|SD|SDD|S\) and \((b_1, b_2, b_3) = (1, 3, 2)\).

Given a threshold graph \( T \), we recall that \( C_i(T) \) is the set of maximal cliques of size \( i \) in \( G \) and we denote as \( C^*_i \) the \( i \)-clique of \( T \) conformed by all vertices corresponding to all the letters \( S \) in the \((d - j + 1)\)-th subwords, \( 1 \leq j \leq i \). Many properties of a threshold graph can be read off from its word. We collect some simple observations.

**Observation 3.1.** Let \( T \) be a threshold graph with clique number \( d \), then the following hold:

(a) The number of times that \( S \) appears in \( T \) is the clique number of \( T \).
(b) The connectivity of \( T \) is the number of consecutive \( S \) appearing at the end of the word \( T \).
(c) There is only one minimum vertex-cut in \( T \).
(d) For any vertex-cut \( Y \) of \( T \), the graph \( T - Y \) has at most one component with at least 2 vertices.
(e) \( C_i(T) \cup C^*_i \) is the unique minimum dominating \( i \)-clique of \( T \) for every \( 1 \leq i \leq d \).

We point out that there are others characterization of threshold graphs. We recall one that is useful in our study.

**Theorem 3.2** ([MP95]). A graph is threshold if and only if it does not contains the graphs \( G_1, G_2 \) or \( G_3 \) of Figure 2 as an induced subgraph.

\[
\begin{array}{ccc}
\text{G}_1 & | & \text{G}_2 \\
& | & \\
& & \text{G}_3
\end{array}
\]

**Figure 2.**

### 3.2. \( b \)-vectors of threshold graphs

We start this section with a result that establishes a strong relation between the \( b \)-vector and the graphs structure.

**Proposition 3.3.** Let \( T \) be a threshold graph with vertex connectivity \( \kappa \). Then,

(a) \( b_1 = b_2 = \ldots = b_\kappa = 1 \);
(b) \( b_i - 1 = |C_i(T)| \) for every \( \kappa + 1 \leq i \leq d - 1 \) and \( b_d = |C_d(T)| \);
(c) \( b_i = d_i(T) \) for every \( i = 1, \ldots, d \);
(d) \( b_{\kappa+1} = W(T - Y) \) for a minimum vertex-cut \( Y \).

**Proof.**

(a) It follows from Observation 3.1(b).
(b) Let \( \kappa + 1 \leq i \leq d - 1 \). Then, each maximal clique of size \( i \) in \( T \) is forming by a vertex in \( T \) corresponding to a letter \( D \) in the \((d - i + 1)\)-th subword together with the vertices in \( T \) corresponding to all the letters \( S \) in the \((d - j + 1)\)-th subwords for every \( 1 \leq j \leq i - 1 \). Since the \((d - i + 1)\)-th subword has exactly one \( S \) and \( b_i \) corresponds to the cardinality of letters of its \((d - i + 1)\)-th subword, we obtain that \( b_i - 1 = |C_i(T)| \). Finally, if we put the letter \( D \) instead \( S \) in the first position of the word corresponding to \( T \), we obtain the same graph \( T \). Hence, the previous argument holds for \( i = d \) obtaining that \( b_d = |C_d(T)| \).
conclude that contradiction since have that for every

Let be a threshold graph with vertex connectivity . Then,

\[ b_{i+1} < \sum_{|Y|=i} (W(T - Y) - 1) \]

for every .

Proof. We observe that if is not a vertex-cut of , then . Hence the sum \( \sum_{|Y|=i} (W(T - Y) - 1) \) only consider vertex-cuts of . Let denote as the set of vertices in that correspond to the letters in the \((d-i+1)\)-th subword for every \( i = 1, \ldots, d \) (i.e., \( |B_i| = b_i \)). By Proposition 3.3(a), for every \( i = 1, \ldots, \kappa \) the set has only one vertex, say \( x_i \), and by Observation 3.1(c) the only minimum vertex-cut in is the set \( \{x_i\}_{i=1}^\kappa \).

Let be the number of all vertex-cuts of with cardinality , then clearly since \( W(T-Y) - 1 \geq 1 \) for every vertex-cut of with cardinality . Therefore, it is enough to show that \( b_{\kappa+\ell+1} < V_{\kappa+\ell} \) for every \( 1 \leq \ell \leq d - \kappa - 1 \).

Suppose that has order , let be such that its correspond letter is and consider the set \( \bigcup_{j=\kappa+1}^d B_j - \{a\} \) of cardinality \( n - \kappa - 1 \). As \( d \leq n - 1 \) since \( b_{\kappa+1} \geq 2 \), we have that \( \ell < n - \kappa - 1 \). Then, for each \( L \subset \bigcup_{j=\kappa+1}^d B_j - \{a\} \) with cardinality \( |L| = \ell \) we have that \( Y = \{x_j\}_{j=1}^\kappa \cup \{L\} \) is a vertex-cut of with cardinality \( \kappa + \ell \) since vertex is isolate in \( T - Y \). Therefore, there are at least \( \binom{n-\kappa-1}{\ell} \) vertex-cuts of with cardinality \( \kappa + \ell \) concluding that \( \binom{n-\kappa-1}{\ell} \leq V_{\kappa+\ell} \). On the other hand, we notice that \( b_{\kappa+1+\ell} < n - \kappa - 1 \) for every \( 1 \leq \ell \leq d - \kappa - 1 \), otherwise we would have that \( b_j = 1 \) for every \( j = 1, \ldots, \kappa + 1 \), a contradiction since \( b_{\kappa+1} \geq 2 \). Thus, as \( n - \kappa + 1 \leq \binom{n-\kappa-1}{\ell} \) for every \( 1 \leq \ell < n - \kappa - 1 \), we conclude that \( b_{\kappa+1+\ell} < V_{\kappa+\ell} \). \( \square \)

One of the main objectives of this paper is to extend the previous proposition to chordal graphs and to compare the \( b \)-vector of a chordal graph \( G \) with the Betti numbers of \( \Delta(G) \). We end this subsection with a result that further the relation of the \( b \)-vector with connectivity.

Proposition 3.4. Let \( T \) be a threshold graph with vertex connectivity . Then,

\[ b_{i+1} < \sum_{|Y|=i} (W(T - Y) - 1) \]

for every .

Proof. We observe that if \( Y \subseteq V(T) \) is not a vertex-cut of \( T \), then \( W(T - Y) - 1 = 0 \). Hence the sum \( \sum_{|Y|=i} (W(T - Y) - 1) \) only consider vertex-cuts \( Y \) of \( T \). Let denote as the set of vertices in \( T \) that correspond to the letters in the \((d-i+1)\)-th subword for every \( i = 1, \ldots, d \) (i.e., \( |B_i| = b_i \)). By Proposition 3.3(a), for every \( i = 1, \ldots, \kappa \) the set has only one vertex, say \( x_i \), and by Observation 3.1(c) the only minimum vertex-cut in \( T \) is the set \( \{x_i\}_{i=1}^\kappa \).

Let be the number of all vertex-cuts of \( T \) with cardinality \( \kappa + \ell \), then clearly \( V_{\kappa+\ell} \leq \sum_{|Y|=\kappa+\ell} (W(T - Y) - 1) \) since \( W(T - Y) - 1 \geq 1 \) for every vertex-cut \( Y \) of \( T \) with cardinality \( \kappa + \ell \). Therefore, it is enough to show that \( b_{\kappa+1+\ell} < V_{\kappa+\ell} \) for every \( 1 \leq \ell \leq d - \kappa - 1 \).

Suppose that \( T \) has order \( n \), let \( a \in B_{\kappa+1} \) be such that its correspond letter is \( D \) and consider the set \( \bigcup_{j=\kappa+1}^d B_j - \{a\} \) of cardinality \( n - \kappa - 1 \). As \( d \leq n - 1 \) since \( b_{\kappa+1} \geq 2 \), we have that \( \ell < n - \kappa - 1 \). Then, for each \( L \subset \bigcup_{j=\kappa+1}^d B_j - \{a\} \) with cardinality \( |L| = \ell \) we have that \( Y = \{x_j\}_{j=1}^\kappa \cup \{L\} \) is a vertex-cut of \( T \) with cardinality \( \kappa + \ell \) since vertex \( a \) is isolate in \( T - Y \). Therefore, there are at least \( \binom{n-\kappa-1}{\ell} \) vertex-cuts of \( T \) with cardinality \( \kappa + \ell \) concluding that \( \binom{n-\kappa-1}{\ell} \leq V_{\kappa+\ell} \). On the other hand, we notice that \( b_{\kappa+1+\ell} < n - \kappa - 1 \) for every \( 1 \leq \ell \leq d - \kappa - 1 \), otherwise we would have that \( b_j = 1 \) for every \( j = 1, \ldots, \kappa + 1 \), a contradiction since \( b_{\kappa+1} \geq 2 \). Thus, as \( n - \kappa + 1 \leq \binom{n-\kappa-1}{\ell} \) for every \( 1 \leq \ell < n - \kappa - 1 \), we conclude that \( b_{\kappa+1+\ell} < V_{\kappa+\ell} \). \( \square \)
4. Algebraic tools

4.1. Betti numbers and connectivity. The following result has already been pointed out by several authors [Goo15, Theorem 6] [Kat06, Corollary 1.2]. We include a proof in the contexts we need for the sake of completeness.

Proposition 4.1. Let $G$ be a graph of order $n$ and let $\Delta = \Delta(G)$. Then,

$$\beta_{i,i+1}(R/I_\Delta) = \sum_{|Y|=n-i-1} (W(G - Y) - 1),$$

for every $1 \leq i \leq n$, where $Y \subseteq V(G)$. As a consequence, $G$ is $k$-connected if and only if $\beta_{i,i+1}(R/I_\Delta) = 0$ for all $i \geq n - k$. In particular,

$$\kappa(G) = \max\{k \mid \beta_{i,i+1}(R/I_\Delta) = 0 \text{ for all } i \geq n - k\}.$$

Proof. From Hochster’s Formula [Hoc77, Theorem 5.1], we have that

$$\beta_{i,i+1}(R/I_\Delta) = \beta_{i-1,i+1}(I_\Delta) = \sum_{|Y|=i+1} \dim_K \tilde{H}^0(\Delta|_Y; K) = \sum_{|Y|=i+1} (W(G|_Y) - 1) = \sum_{|Y|=n-i-1} (W(G - Y) - 1) = \sum_{|Y|=n-i-1} (W(G - Y) - 1).$$

The rest follows from observing that $\beta_{i,i+1}(Y/I_\Delta) = 0$ if and only if there is no vertex-cut $Y$ such that $|Y| = n - i - 1$. \hfill \Box

The previous result implies the following result, which is already known [Kat06, Goo15].

Corollary 4.2 ([Goo15, Theorem 6]). Let $G$ be a graph and set $\Delta = \Delta(G)$. Then,

$$\text{depth}(R/I_\Delta) \leq \kappa(G) + 1.$$ 

Furthermore, the equality is reached if $G$ is chordal.

4.2. Shifting theory. In order to continue our study of $b$-vectors, we need to recall a few terms and properties of shifting theory for simplicial complex.

Definition 4.3. A simplicial complex $\Delta$ on $[n]$ is shifted if, for $\sigma \in \Delta, i \in \sigma$, and $j \in [n]$ with $j > i$, one has $(\sigma \setminus \{i\}) \cup \{j\} \in \Delta$.

Definition 4.4. A shifting operation on $[n]$ is a map which associates each simplicial complex $\Delta$ on $[n]$ with a simplicial complex $\text{Shift}(\Delta)$ on $[n]$ and which satisfies the following conditions:

(a) $\text{Shift}(\Delta)$ is shifted;
(b) $\text{Shift}(\Delta) = \Delta$ if $\Delta$ is shifted;
(c) $f(\Delta) = f(\text{Shift}(\Delta));$
(d) $\text{Shift}(\overline{\Delta}) \subseteq \text{Shift}(\Delta)$ if $\overline{\Delta} \subseteq \Delta$ and $\overline{\Delta}^{(0)} = \Delta^{(0)}$.

Definition 4.5. We say that a shifting operation $\text{Shift}(\cdot)$ is compatible with Alexander Duality if $\text{Shift}(\Delta^\vee) = \text{Shift}(\Delta)^\vee$.
There are several well-known shifting operations. Among these operations, the exterior shifting $\Delta^e$ and the symmetric shifting $\Delta^s$ stand up. We point out that $\Delta^s$ is only defined in characteristic zero. We omit the definitions as it is not needed for our purposes. However, we refer the interested reader in the book by Herzog and Hibi [HH11, Chapter 11].

We recall a few properties for the exterior and symmetric shifting.

**Theorem 4.6 ([HH11, Theorem 11.4.1]).** Let $\Delta$ be any simplicial complex. The following statements hold:

1. $\text{depth}(R/I_\Delta) = \text{depth}(R/I_{\Delta^e})$.
2. $(\Delta^e)^\vee = (\Delta^\vee)^e$.
3. $\text{reg}(R/I_\Delta) = \text{reg}(R/I_{\Delta^e})$.

If the ground field has characteristic zero, then the same statements hold for the symmetric shifting.

We now mention a conjecture that relates the Betti numbers of algebraic and symmetric shifting.

**Conjecture 4.7 ([AHH00, Conjecture 2.3]).** Let $\Delta$ be a simplicial complex and suppose that $\text{char}(K) = 0$. Then,

$$\beta_{i,j}(I_{\Delta^s}) \leq \beta_{i,j}(I_{\Delta^e}).$$

In order to study the previous conjecture and the $b$-vector of a chordal graph, we need to recall the definition and properties of the $h$-vector of a simplicial complex.

**Definition 4.8.** Let $\Delta$ be a simplicial complex of dimension $d-1$ and let $f(\Delta) = (f_{-1}, \ldots, f_{d-1})$ be the $f$-vector of $\Delta$. The $h$-vector of $\Delta$, $h(\Delta) = (h_0, \ldots, h_d)$, is defined by

$$\sum_{i=0}^{d} h_i t^i = \sum_{i=0}^{d} f_{i-1} t^i (1-t)^{d-i}.$$ 

**Remark 4.9.** We have that

$$h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-i} f_{i-1} \quad \& \quad f_{j-1} = \sum_{i=0}^{j} \binom{d-i}{j-i} h_i.$$

We now recall a result that would imply that the $f$-vector of certain simplicial complexes determine its Betti numbers.

**Theorem 4.10 ([Heg10, Corollary 3.4]).** If $I_\Delta$ has a $t$-linear free resolution, then

$$\beta_{i}(R/I_\Delta) = \sum_{\ell=0}^{t+i} (-1)^{t+i+1} h_{t+i-\ell} \binom{n-d}{\ell}.$$ 

We are able to answer Conjecture 4.7 for ideals with a linear free resolution.

**Proposition 4.11.** Let $\Delta$ be a simplicial complex such that $I_\Delta$ has a $t$-linear free resolution. Then, $\beta_{i,j}(R/I_\Delta) = \beta_{i,j}(R/I_{\Delta^e})$. Furthermore, If $\text{char}(K) = 0$, then

$$\beta_{i,j}(R/I_{\Delta^s}) = \beta_{i,j}(R/I_\Delta) = \beta_{i,j}(R/I_{\Delta^e}).$$

**Proof.** Since $I_\Delta$ has a free $t$-linear resolution, $I_{\Delta^s}$ and $I_{\Delta^e}$ also have a free resolution by Theorem 4.6. We conclude that

$$\beta_{i,j}(R/I_{\Delta^s}) = \beta_{i,j}(R/I_\Delta) = \beta_{i,j}(R/I_{\Delta^e}) = 0.$$
for $j \neq i + t - 1$. In addition, $\beta_i(R/I_{\Delta_{s}}) = \beta_{i,j}(R/I_{\Delta_{s}})$, $\beta_{i,j}(R/I_{\Delta}) = \beta_{i,j}(R/I_{\Delta_{s}})$, and $\beta_{i,j}(R/I_{\Delta_{s}}) = \beta_{i,j}(R/I_{\Delta_{s}})$. Since shifting preserves $f$-vector, it also preserves $h$-vectors. Then, exterior and symmetric shifting preserve total Betti numbers in this case by Theorem 4.10. Hence, 

$$\beta_{i,j}(R/I_{\Delta_{s}}) = \beta_{i,j}(R/I_{\Delta}) = \beta_{i,j}(R/I_{\Delta_{s}}).$$

for $j = i + t - 1$. 

As a consequence of Theorem 4.10 and Proposition 4.1, we prove that shifting preserves numbers related to connectivity for chordal graphs.

**Corollary 4.12.** Let $G$ be a chordal graph with vertex connectivity $\kappa$ and set $\Delta = \Delta(G)$. Then,

$$\sum_{|Y| = \kappa} (W(\Delta - Y) - 1) = \sum_{|Y| = \kappa} (W(\Delta^{e} - Y) - 1),$$

where $Y \subseteq V(G)$. Furthermore, if $\text{char}(K) = 0$, then

$$\sum_{|Y| = \kappa} (W(\Delta - Y) - 1) = \sum_{|Y| = \kappa} (W(\Delta^{s} - Y) - 1).$$

**Proof.** This follows immediately from Propositions 4.1 and 4.11.

We now recall an structural result that characterizes clique complexes of threshold graphs.

**Proposition 4.13** ([Kli07, Theorem 2]). The graph $G$ is threshold if and only if $\Delta(G)$ is shifted.

Thanks to the previous result, we are able to show that the exterior and symmetric shifting yield the same complex for clique complexes of chordal graphs.

**Theorem 4.14.** Let $G$ be a chordal graph. Suppose that $\text{char}(K) = 0$. Then $\Delta(G)^{e} = \Delta(G)^{s}$.

**Proof.** Since $G$ is a chordal graph, $I_{\Delta(G)}$ has a 2–linear resolution. Then, $I_{(\Delta(G))^{e}}$ and $I_{(\Delta(G))^{s}}$ have a 2–linear resolution. Then, $I_{(\Delta(G))^{e}}$ and $I_{(\Delta(G))^{s}}$ are the monomial edge ideals of certain graph. As a consequence, $(\Delta(G))^{e}$ and $\Delta(G)^{s}$ are cliques complexes of certain graphs, say $G_{1}$ and $G_{2}$. By Theorem 4.13, $G_{1}$ and $G_{2}$ are threshold graphs. We obtain that

$$f(\Delta(G_{1})) = f((\Delta(G))^{e}) = f(\Delta) = f((\Delta(G))^{s}) = f(\Delta(G_{2})).$$

Then the clique vector of $G_{1}$ and $G_{2}$ are equal, concluding that $G_{1}$ and $G_{2}$ have the same $b$-vector, so $G_{1} = G_{2}$. Hence $\Delta(G_{1}) = \text{Shift}_{1}(\Delta(G)) = \text{Shift}_{2}(\Delta(G)) = \Delta(G_{2}).$ 

As a consequence of Theorem 4.14, we show Conjecture 4.7 for clique complex of chordal graphs. We point out that the fact that $G^{e} = G^{s}$ was previously known [Mur07].

**Proposition 4.15.** Suppose that $\text{char}(K) = 0$. Then $\Delta(G)^{e} = \Delta(G)^{s}$ for every chordal graph. As a consequence, $(\Delta(G^{(i)})^{e} = (\Delta(G^{(i)})^{s}$ and Conjecture 4.7 holds for $\Delta(G^{(i)})$ for every $i \in \mathbb{N}$.

**Proof.** This is an immediate consequence from Theorem 4.14, and the fact that Shift($\Delta^{(i)}$) = ($\text{Shift}(\Delta)$)$^{(i)}$ for every shifting operation Shift($\cdot$) and $i \in \mathbb{N}$. 

□
5. Chordal graphs

In this section we prove our main results regarding chordal graphs. We start by recalling the definition and properties of a perfect elimination order.

A vertex \( v \) of a graph \( G \) is called simplicial if its adjacency set \( N(v) \) induces a clique. An elimination ordering \( \sigma \) of a graph \( G \) is a bijection \( \sigma : \{1, \ldots, n\} \to V(G) \). Thus, \( \sigma(i) \) is the \( i \)th vertex in the elimination ordering and for \( v \in V(G) \), \( \sigma^{-1}(v) \) gives the position of \( v \) in \( \sigma \). A perfect elimination ordering (PEO) is an elimination ordering \( \sigma = (v_1, v_2, \ldots, v_n) \) where \( v_i \) is a simplicial vertex in the subgraph induced by \( \{v_i, v_{i+1}, \ldots, v_n\} \), \( 1 \leq i \leq n \). Given a PEO \( \sigma \) of a graph \( G \), the monotone adjacency set of \( v \), denoted \( N_\sigma(v) \), is given by

\[
N_\sigma(v) := \{ u \in N(v) \mid \sigma^{-1}(v) < \sigma^{-1}(u) \}
\]

and denote \( n_\sigma(v) := |N_\sigma(v)| \). For a maximal clique \( C \) of \( G \), let define

\[
s(C) := \{ x \in C \mid N_\sigma(x) \not\subseteq C \}.
\]

For any set of vertices \( U = \{u_1, u_2, \ldots, u_r\} \) of \( G \), we say that \( U \) is well ordered if \( \sigma^{-1}(u_i) < \sigma^{-1}(u_j) \) for every \( i, j \in \{1, \ldots, r\} \) with \( i < j \).

As shown below in Theorem 5.1, every nontrivial chordal graph has at least two simplicial vertices.

Theorem 5.1 ([Dir61]). Every chordal graph has a simplicial vertex. If \( G \) is not the complete graph, then it has two nonadjacent simplicial vertices.

We know recall a characterization of chordal graphs related to PEO.

Theorem 5.2 ([FG65]). A graph is chordal if and only if it has a PEO.

The next result shows the existence of a special PEO satisfying properties that are useful for the rest of the paper.

Lemma 5.3. Let \( G \) be a chordal graph of order \( n \), let \( K = \{x_1, x_2, \ldots, x_k\} \) be a maximal clique of \( G \) and let \( C \) be any maximal clique of \( G \). Then there exists a PEO \( \sigma \) of \( G \) such that:

(a) \( \sigma^{-1}(u) < \sigma^{-1}(x) \) for every \( x \in K \) and every \( u \in V(G) - K \), moreover \( \sigma^{-1}(x_i) = n - i + 1 \) for every \( i = 1, \ldots, k \);

(b) \( \sigma^{-1}(u) < \sigma^{-1}(v) \) for every \( u \in C - s(C) \) and every \( v \in s(C) \);

(c) if \( |s(C)| < i \leq |C| \), then there exists a unique vertex \( u \in C - s(C) \) such that \( n_\sigma(u) = i - 1 \);

(d) if \( C' \) is another maximal clique of \( G \) such that \( C' \cap C \neq \emptyset \), then \( \sigma^{-1}(u) < \sigma^{-1}(v) \) for every \( u \in C \setminus C' \) and every \( v \in C \cap C' \) or \( \sigma^{-1}(u) < \sigma^{-1}(v) \) for every \( u \in C' \setminus C \) and every \( v \in C \cap C' \).

Proof. We construct a PEO \( \sigma \) of \( G \) as follows. Let \( \mathcal{C} \) be the set of maximal cliques of \( G \) and suppose that \( |\mathcal{C}| = r \). There exists two nonadjacent simplicial vertices in \( G \) by Theorem 5.1. Then, one of them is a vertex \( u_1 \in V(G) - K \). Suppose that \( u_1 \in C_1 \) for some \( C_1 \in \mathcal{C} \setminus \{K\} \).

Set \( P_1 = \{ u \in C_1 \mid u \) is simplicial in \( G \} \). We note that the graph \( G_2 = G - P_1 \) is chordal and for every \( C \in \mathcal{C} - C_1 \), \( C \) is a maximal clique in \( G_2 \). By Theorem 5.1, there exists a vertex \( u_2 \in V(G_2) - K \) simplicial in \( G_2 \). Suppose that \( u_2 \in C_2 \) for some \( C_2 \in \mathcal{C} \setminus \{K, C_1\} \). Let \( P_2 = \{ u \in C_2 \mid u \) is simplicial in \( G_2 \} \). Proceeding in this way for every \( i = 3, 4, \ldots, r - 1 \), we obtain that the graph \( G_i = G_{i-1} - P_{i-1} \) is chordal. In addition, \( C \) is a maximal clique


in $G_i$ for every $C \in C \setminus \{C_1, C_2, \ldots, C_{i-1}\}$. Then, by Theorem 5.1, there exists a vertex $u_i \in G_i - K$ simplicial in $G_i$. Suppose that $u_i \in C_i$ for some $C_i \in C - \{K, C_1, C_2, \ldots, C_{i-1}\}$. Set $P_i = \{u \in C_i \mid u$ is simplicial in $G_i\}$. We define $\sigma$ as follows. For every $i, j = 1, \ldots, r - 1$ with $i < j$, every $p_i \in P_i$ and every $p_j \in P_j$, we take $\sigma^{-1}(p_i) < \sigma^{-1}(p_j)$ and $\sigma^{-1}(x_i) = n - i + 1$ for every $i = 1, \ldots, k$. Then, $\sigma$ is a PEO of $G$ by construction. We now show that $\sigma$ satisfies the condition (a)-(d).

(a) It follows by the construction of $\sigma$.
(b) We notice that $C = \{C_1, C_2, \ldots, C_r\}$ where $C_r = K$. By construction of $\sigma$, the sets $S_i$ and $s(C_i)$ are a partition of $C_i$ for every $i = 1, \ldots, r$. Suppose that $C = C_i$ for some $i = 1, \ldots, r$. If $i = r$, then $s(C) = \emptyset$. Then, we can assume that $i = 1, \ldots, r - 1$. Let $v \in s(C)$. By definition of the set $s(C)$, $N_\sigma(v) \not\subseteq C$. Therefore, there exists a vertex $z \in N_\sigma(v)$ such that $z \not\in C$. Since $\sigma^{-1}(v) < \sigma^{-1}(z)$ and $z \not\in C$, it follows that $z \in \bigcup_{j=i+1}^r C_j$. Then, by construction of $\sigma$, we have that $\sigma^{-1}(u) < \sigma^{-1}(z)$ for every $u \in S_i = C \setminus s(C)$.

We proceed by contradiction. Suppose that there exits a vertex $u \in C \setminus s(C)$ with $\sigma^{-1}(v) < \sigma^{-1}(u)$. Then, $u \in N_\sigma(v)$. Since $N_\sigma(v)$ is a clique, we have that $uz \in E(G)$. As $z \not\in C$ and $N_\sigma(u) \subseteq C$, we have that $z \not\in N_\sigma(u)$. Since $uz \in E(G)$, we conclude that $u \in N_\sigma(z)$. This means that $\sigma^{-1}(z) < \sigma^{-1}(u)$, a contradiction.

(c) Suppose that $C = \{u_1, u_2, \ldots, u_{|C|}\}$ is well ordered. By (b) it follows that $u_j \not\in s(C)$ for every $j = 1, \ldots, |C| - |s(C)|$. Since $N_\sigma(u_j) \subseteq C$ for every $j = 1, \ldots, |C| - |s(C)|$, we have that $n_\sigma(u_j) = |C| - j$. Therefore, there exists a unique vertex $u \in C \setminus s(C)$ such that $n_\sigma(u) = i - 1$, because $|s(C)| < i \leq |C|$. Let $v \in C \cap C'$. We consider the following cases.

Case (i): $N_\sigma(v) \cap (C' \setminus C) \neq \emptyset$. In this case $N_\sigma(v) \cap (C \setminus C') = \emptyset$, because $N_\sigma(v)$ is a clique. We deduce that $\sigma^{-1}(u) < \sigma^{-1}(v)$ for every $u \in C \setminus C'$, since $u \notin N_\sigma(v)$. Let $v' \in C \cap C'$ be any other vertex different from $v$. If $v' \in N_\sigma(v)$, then $\sigma^{-1}(u) < \sigma^{-1}(v) < \sigma^{-1}(v')$ for every $u \in C \setminus C'$. If $v' \not\in N_\sigma(v)$, then $N_\sigma(v') \cap (C' \setminus C) \neq \emptyset$. Then by a similar argument applied to $v'$, we conclude that $\sigma^{-1}(u) < \sigma^{-1}(v')$ for every $u \in C \setminus C'$.

Case (ii): $N_\sigma(v) \cap (C \setminus C') \neq \emptyset$. The proof is analogous to Case (i).

Case (iii): $N_\sigma(v) \subset (C \cap C')$. In this case, $\sigma^{-1}(u) < \sigma^{-1}(v)$ for every $u \in C \cup C' \setminus (C \cap C')$. Let $v' \in C \cap C'$ be any other vertex different from $v$. If $v' \in N_\sigma(v)$, then $\sigma^{-1}(u) < \sigma^{-1}(v) < \sigma^{-1}(v')$ for every $u \in C \cup C' \setminus (C \cap C')$. If $v' \not\in N_\sigma(v)$, we assume that $N_\sigma(v') \cap (C' \setminus C) \neq \emptyset$. Then by a similar argument to Case (i), we have that $\sigma^{-1}(u) < \sigma^{-1}(v')$ for every $u \in C \setminus C'$. The case when $N_\sigma(v') \cap (C \setminus C') \neq \emptyset$ is analogous. The case $N_\sigma(v') \subset C \cap C'$ implies that $\sigma^{-1}(u) < \sigma^{-1}(v')$ for every $u \in C \cup C' \setminus (C \cap C')$.

We conclude, in any case, that $\sigma^{-1}(u) < \sigma^{-1}(v)$ for every $u \in C \setminus C'$ and every $v \in C \cap C'$ or $\sigma^{-1}(u) < \sigma^{-1}(v)$ for every $u \in C' \setminus C$ and every $v \in C \cap C'$.

\[\Box\]

5.1. A combinatorial shifting of chordal graphs. In this subsection, we define a combinatorial procedure which takes chordal graphs to threshold graphs preserving the clique complex. This procedure results a shifting operation which allows us to have a combinatorial
description of shifting in order to use in our results (for a different study of combinatorial shifting of chordal graphs we refer to Murai’s work on shifting for graphs [Mur07]).

**Definition 5.4.** Let $G$ be a chordal graph with order $n$ and let $K_d = \{x_1, x_2, \ldots, x_d\}$ be a maximum clique of $G$. Let $\sigma$ be a PEO of $G$ that satisfies the conditions of Lemma 5.3 for $G$ and $K_d$ (in particular $\sigma^{-1}(x_i) = n - i + 1$ for every $i = 1, \ldots, d$). For every vertex $u \in V(G) \setminus K_d$ suppose that $N_\sigma(u) = \{u_1, u_2, \ldots, u_{n_\sigma(u)}\}$ is well ordered. We define a map $\alpha_\sigma$ from $E(G)$ to the set of all 2–subsets of $V(G)$ as follows:

(a) $\alpha_\sigma(uv) = uv$ for every edge $uv \in E(K_d)$.
(b) $\alpha_\sigma(uu_i) = ux_i$ for each vertex $u \in V(G) \setminus K_d$ and every $i = 1, \ldots, n_\sigma(u)$.

We define the graph $\alpha_\sigma(G) := (V(G), \alpha_\sigma(E(G)))$.

Figure 3 shows a chordal graph $G$ and the graph $\alpha_\sigma(G) = T$. The labeling in the figure corresponds to a PEO $\sigma$ of $G$ that satisfies the conditions of Lemma 5.3. Vertices 7, 8, 9, 10 corresponds to $K_d = \{x_1 = \sigma(10), x_2 = \sigma(9), x_3 = \sigma(8), x_4 = \sigma(7)\}$. For instance, we observe that if $u \in V(G) \setminus K_d$ is such that $u = \sigma(2)$, as $N_\sigma(u) = \{u_1 = \sigma(3), u_2 = \sigma(4)\}$ then $ux_1$ and $ux_2$ are edges in $T$. On the other hand, we may observe that the PEO $\sigma$ of $G$ in Figure 4 also satisfies the conditions of Lemma 5.3.

![Figure 3](image)

We now show that the previous graph shifting $\alpha_\sigma(G)$ yields the same $b$–vector of $G$. This is a key tool as it yield the same output as exterior or symmetric shifting. However, its construction allows to see that certain combinatorial properties are preserved.

**Theorem 5.5.** Let $G$ be a chordal graph and let $T = \alpha_\sigma(G)$ be as in Definition 5.4. Then $T$ is a threshold graph and both graphs have the same clique vector and $b$–vector.

**Proof.** We note, by the definition of $\alpha_\sigma$, that $T$ is conformed two parts. First, the maximum clique $K_d = \{x_1, x_2, \ldots, x_d\}$, where we may assume that $\sigma^{-1}(x_i) = n - i + 1$ for every $i = 1, \ldots, d$. Second, additional vertices $v$ adjacent to $K_d$ such that if $v \in V(T) \setminus K_d$ with $\deg(v) = i$. Then, $N(v) = \{x_1, x_2, \ldots, x_i\}$.

In order to prove that $T$ is threshold, it is enough by Theorem 3.2 to show that $T$ does not contain graphs $C_4$, $2K_2$ or $P_4$ as an induced subgraph. It is not difficult to see that $T$ does not contain the graphs $C_4$ or $2K_2$ as an induced subgraph. By means of contradiction, suppose that $T$ contains the graph $P_4 = p_1, p_2, p_3, p_4$ as an induced subgraph. By the construction of $T$ we notice that $|P_4 \cap K_d| \geq 2$. If $|P_4 \cap K_d| \geq 3$ we would have a triangle in the induced
subgraph $P_4$ which is a contradiction. Therefore, $|P_4 \cap K_d| = 2$, say $P_4 \cap K_d = \{p_2, p_3\}$. Then, $p_1, p_4 \not\in K_d$. With out loss of generality, we assume that $\deg(p_1) \leq \deg(p_4)$. Then, $N(p_1) \subseteq N(p_4)$ by definition of the graph $T$. We conclude that $p_2p_4 \in E(T)$, a contradiction. Similar arguments holds if another two vertices of $P_4$ are in $K_d$. Therefore the graph $T$ is threshold.

We now show that $G$ and $T$ have the same clique vector, i.e., $c(G) = c(T)$. We note that $|V(G)| = |V(T)|$, and so, $c(G)$ and $c(T)$ coincide in its first coordinate. We now prove that $\alpha$ induces a bijection between the set of $r$–cliques of $G$ and the set of $r$–cliques of $T$ for each $r = 2, 3, \ldots, d$. Let $C_u = \{u_1, u_2, \ldots, u_r\}$ be a well ordered $r$–clique of $G$ not contained in $K_d$. Note that if $u_i \in K_d$ for some $i \in \{1, \ldots, r\}$. Then, $u_j \in K_d$ for every $j > i$, because $\sigma^{-1}(u_i) < \sigma^{-1}(u_j)$ and $\sigma^{-1}(z) < \sigma^{-1}(x)$ for every $z \in V(G) \setminus K_d$ and every $x \in K_d$. Hence, $u_i \not\in K_d$. Consider $N_{\sigma}(u_1) = \{u'_1, u'_2, \ldots, u'_{n_{\sigma}(u_1)}\}$, which is well ordered. As $\{u_2, u_3, \ldots, u_r\} \subseteq N_{\sigma}(u_1)$, there exists $\{j_2, j_3, \ldots, j_r\} \subseteq \{1, \ldots, n_{\sigma}(u_1)\}$ such that $u'_{j_q} = u_q$ for every $q = 2, 3, \ldots, r$. Therefore, the map $\alpha_{\sigma}$ sends the clique $C_u$ in $G$ to a unique clique $\alpha_{\sigma}(C_u) = \{u_1, x_{j_2}, x_{j_3}, \ldots, x_{j_r}\}$ in $T$. Let $C_v$ be any $r$–clique of $G$ such that $\alpha_{\sigma}(C_u) = \alpha_{\sigma}(C_v)$. We now show that $C_u = C_v$. We observe first that $C_u$ is not contained in $K_d$. Otherwise, we would have that $\{u_1, x_{j_2}, x_{j_3}, \ldots, x_{j_r}\} \subseteq K_d$, because the map $\alpha_{\sigma}$ is the identity in $K_d$. This yields a contradiction because $u_i \not\in K_d$.

Consider the well-ordered sets $C_v = \{v_1, v_2, \ldots, v_r\}$ and $N_{\sigma}(v_1) = \{v'_1, v'_2, \ldots, v'_{n_{\sigma}(v_1)}\}$. Then $v_1 \not\in K_d$. Since $\{v_2, v_3, \ldots, v_r\} \subseteq N_{\sigma}(v_1)$, there exists $\{i_2, i_3, \ldots, i_r\} \subseteq \{1, \ldots, n_{\sigma}(v_1)\}$ such that $v'_{i_q} = v_q$ for every $q = 2, 3, \ldots, r$. Therefore, the map $\alpha_{\sigma}$ sends the clique $C_v$ in $G$ to a unique clique $\alpha_{\sigma}(C_v) = \{v_1, x_{i_2}, x_{i_3}, \ldots, x_{i_r}\}$ in $T$. As $\alpha_{\sigma}(C_u) = \alpha_{\sigma}(C_v)$, it follows that $u_1 = v_1$. Since $\sigma$ is a bijection $\sigma : \{1, \ldots, n\} \to V(G)$, we conclude that $i_q = j_q$ for every $q = 2, 3, \ldots, r$. This means $C_u = C_v$.

Finally, we have that $\alpha_{\sigma}(C) \neq \alpha_{\sigma}(C')$ for any distinct $i$–cliques $C$ and $C'$, both contained in $K_d$. Therefore $\alpha_{\sigma}$ induces a bijection between the set of $r$–cliques of $G$ and the set of $r$–cliques of $T$ for each $r = 2, 3, \ldots, d$. We conclude that $c(G) = c(T)$. Hence, $b(G) = b(T)$. 

**Corollary 5.6.** Let $G$ be a chordal graph and let $T = \alpha_{\sigma}(G)$ be as in Definition 5.4. Then $\Delta(T) = \Delta(G)^e$. In particular, $\kappa(G) = \kappa(T)$.

**Proof.** By Theorems 4.13 and 5.5, $\Delta(T)$ is shifted and $T$ is threshold respectively. We also have that $\Delta(G)^e$ is shifted. By Theorem 4.13, there exists a threshold graph $T'$ such that $\Delta(G)^e = \Delta(T')$. Then, $T'$ and $G$ have the same clique vector. By Theorem 5.5, $T$ and $G$ also have the same clique vector. Since $T$ and $T'$ are threshold graphs with the same clique vector, they also have the same $b$–vector. Then, $T = T'$ and $\Delta(T) = \Delta(G)^e$. Since exterior shifting operation preserves depth by Theorem 4.6, we have that $\text{depth}(R/I_{\Delta(G)}) = \text{depth}(R/I_{\Delta(T)})$. Since $G$ and $T$ are chordal graphs, then $I_{\Delta(G)}$ and $I_{\Delta(T)}$ have 2-linear resolutions [Frö90, Theorem 1]. We conclude that $\kappa(G) = \kappa(T)$ by Corollary 4.2. 

**Notation 5.7.** Let us define $\overline{\kappa}(G) := \max\{|C \cap C'| \mid C$ and $C'$ are maximal cliques of $G \}$.

We note that $\kappa(G) \leq \overline{\kappa}(G)$. Next, we show that $d_i(T) \leq d_i(G)$ and the equality holds for $i > \overline{\kappa}(G)$.

**Theorem 5.8.** Let $G$ be a chordal graph and let $T = \alpha_{\sigma}(G)$ as in Definition 5.4. Then,

(a) $d_i(T) \leq d_i(G)$ for every $i = 1, \ldots, d$.
(b) $d_i(T) = d_i(G)$ for every $i > \overline{\kappa}(G)$.
Proof. By Definition 5.4, $K_d = \{x_1, x_2, \ldots, x_d\}$ is a maximum clique of $G$ with $\sigma^{-1}(x_j) = n - j + 1$ for every $j = 1, \ldots, d$.

(a) We note that the claim holds for $i = 1$. Let $\mathcal{D}_i(G)$ be a minimum dominating $i$–clique in $G$ for some $i \in \{2, 3, \ldots, d\}$. Let $C_i \in \mathcal{D}_i(G)$. Suppose that there exits a maximal clique $C$ containing $C_i$ with $i > |s(C)|$. Then, by Lemma 5.3(c), there exists a unique vertex, $u_C$, in $C \setminus s(C)$ such that $n_\sigma(u_C) = i - 1$. We define a map $\varphi$ from $\mathcal{D}_i(G)$ to $\mathcal{D}_i(T)$ as follows. For $C_i \in \mathcal{D}_i(G)$, we set

\[(i) \text{ If every maximal clique } C \text{ of } G \text{ containing } C_i \text{ we have } i \leq |s(C)|, \text{ we set } \varphi(C_i) = \{x_1, x_2, \ldots, x_i\}.
\]

\[(ii) \text{ If there exists a maximal clique } C \text{ containing } C_i \text{ such that } i > |s(C)|, \text{ we set } \varphi(C_i) = \{x_1, x_2, \ldots, x_{i-1}, u_C\}.
\]

First, we show that $\varphi$ is a map from $\mathcal{D}_i(G)$ to $\mathcal{D}_i(T)$. Suppose that (ii) occurs and that there is a maximal clique $C'$, different from $C$, containing $C_i$ and such that $i > |s(C')|$. By Lemma 5.3(c), there exists a unique vertex $u_{C'} \in C' \setminus s(C')$ such that $n_\sigma(u_{C'}) = i - 1$. We prove that $u_C = u_{C'}$. By Lemma 5.3(d), we may assume without loss of generality that $\sigma^{-1}(u) < \sigma^{-1}(v)$ for every $u \in C \setminus C'$ and every $v \in C \cap C'$. Then, by Lemma 5.3(b), we have that $s(C) \subset C \cap C'$. If $C = \{u_1, u_2, \ldots, u_{|C|}\}$ is well ordered, it follows that $u_j \notin s(C)$ for every $j = 1, \ldots, |C| - |s(C)|$. Moreover, since $u_j \notin s(C)$ for every $j = 1, \ldots, |C| - |s(C)|$, we have that $|C| - j \in \{1, \ldots, |C| - |s(C)|\}$. We conclude that $n_\sigma(u_{C\setminus i+1}) = i - 1$. As $u_{C\setminus i+1} \in C \cap C'$, it follows that $u_{C\setminus i+1} = u_C = u_{C'}$, because $u_C$ and $u_{C'}$ are unique in $C$ and $C'$ respectively with the property $n_\sigma(u_C) = n_\sigma(u_{C'}) = i - 1$. Therefore, $\varphi$ is a map from $\mathcal{D}_i(G)$ to $\mathcal{D}_i(T)$.

Second, we show that $\varphi$ is surjective proving that $d_i(T) \leq d_i(G)$ for every $i = 2, 3, \ldots, d$. By Observation 3.1(e), there is only one minimum dominating $i$–clique in $T$ which is formed by $C_i(T)$, the set of maximal $i$–cliques of $T$, and the $i$–clique $\{x_1, x_2, \ldots, x_i\}$. We have that $\{x_1, x_2, \ldots, x_i\} \in \varphi(\mathcal{D}_i(G))$, since there is an $i$–clique $C_i \in \mathcal{D}_i(G)$ such that $C_i \subseteq K_d$. We note that a maximal $i$–clique in $T$ is a clique $\{x_1, x_2, \ldots, x_{i-1}, u\}$ where $u \in V(T) \setminus K_d$. We now show that $\{x_1, x_2, \ldots, x_{i-1}, u\} \in \varphi(\mathcal{D}_i(G))$. By the definition of $\alpha_\sigma$, we have that $n_\sigma(u) = i - 1$. We consider a maximal clique $C$ containing the $i$–clique $u \cup N_\sigma(u)$ in $G$, which means that $u \notin s(C)$. Thus, it follows that $s(C) \subset N_\sigma(u)$ by Lemma 5.3(b). As $|N_\sigma(u)| = i - 1$, we have that $i > |s(C)|$. We conclude that $\varphi(u \cup N_\sigma(u)) = \{x_1, x_2, \ldots, x_{i-1}, u\}$, which proves that $\varphi$ is surjective. Therefore $d_i(T) \leq d_i(G)$ for every $i = 1, \ldots, d$.

(b) Suppose that $\mathcal{D}_i(G) = \{C_1, C_2, \ldots, C_{d_i(G)}\}$. Let $M_j$ be a maximal clique of $G$ containing $C_j$ for every $j = 1, \ldots, d_i(G)$. We have that $C_j$ nor $C_r$ are contained in $M_j \cap M_r$, for every distinct $j, r = 1, \ldots, d_i(G)$, because $i > \tilde{\kappa}(G)$. As a consequence, $M_j \neq M_r$. By definition of $s(M_j)$, we have that $\tilde{\kappa}(G) \geq s(M_j)$ concluding that $i > s(M_j)$. Then, by Lemma 5.3(c), there exists a unique vertex $u_j \in M_j \setminus s(M_j)$ such that $n_\sigma(u_j) = i - 1$ for every $j = 1, \ldots, d_i(G)$.

We now prove that the map is injective for $i > \tilde{\kappa}(G)$. By definition of $\varphi$, we have that $\varphi(C_j) = \{x_1, x_2, \ldots, x_{i-1}, u_j\}$ and $\varphi(C_r) = \{x_1, x_2, \ldots, x_{i-1}, u_r\}$ for every distinct $j, r = 1, \ldots, d_i(G)$. We show that $u_j \neq u_r$, proving that $\varphi(C_j) \neq \varphi(C_r)$. We have $u_j \neq u_r$ if $M_j \cap M_r = \emptyset$. Suppose $M_j \cap M_r \neq \emptyset$. As $M_j \neq M_r$, we may suppose without loss of generality that $\sigma^{-1}(u) < \sigma^{-1}(v)$ for every $u \in M_j \setminus M_r$ and every
Proposition 5.12. \[ N \]
Proof. Since \( i > |M_j \cap M_r| \) and \( n_\sigma(u_j) = i - 1 \), it follows that \( u_j \in M_j \setminus M_r \). Then, \( u_j \neq u_r \), and so, \( \varphi \) is injective. Hence, \( d_i(G) = d_i(T) \) for every \( i > \kappa(G) \).
\[ \square \]

5.2. \textit{b–vectors and betti numbers of chordal graphs}. In this subsection we show our main result regarding the \textit{b–vector} of a chordal graph. In particular, we compute some of the entries of the \textit{b–vector}. We first compute some entries using the combinatorial procedure \( \alpha_\sigma \) introduced in the previous subsection.

Corollary 5.9. Let \( G \) be a chordal graph, then
\begin{enumerate}[(a)]
  \item \( b_i \leq d_i(G) \) for every \( i = 1, \ldots, d \);
  \item \( b_i = d_i(G) \) for every \( i > \kappa(G) \);
  \item \( b_i \leq b_j \) for every \( i, j \) with \( j < i \).
\end{enumerate}

Proof. By Theorem 5.8 and Proposition 3.3(c), we have (a) and (b). Let \( C_{\geq i} \) be the set of all maximal cliques of \( G \) with cardinality at least \( i \) and notice that \( |C_{\geq i}| = d_i(G) \) when \( i \geq \kappa(G) + 1 \). Hence by (b) we have that \( |C_{\geq i}| = b_i \) for every \( i \geq \kappa(G) + 1 \). As \( |C_{\geq i}| \leq |C_{\geq j}| \) for every \( j < i \), we have (c).
\[ \square \]

The next example shows that Theorem 5.8(b) and Corollary 5.9(b) and (c) are best possible.

Example 5.10. For every positive integers \( \kappa, \kappa \) with \( \kappa \leq \kappa \), consider the graph \( G \) conformed by a clique \( K = \{x_1, x_2, \ldots, x_2\} \) of size \( 2\kappa \) together with three additional vertices, \( u_\kappa, u_\kappa \) and \( v_\kappa \) such that \( N(u_\kappa) = \{x_1, x_2, \ldots, x_\kappa\} \), \( N(u_\kappa) = \{x_1, x_2, \ldots, x_\kappa\} \) and \( N(v_\kappa) = \{x_\kappa+1, x_\kappa+2, \ldots, x_2\} \).

We observe that the graph \( G \) of Example 5.10 is chordal, \( \kappa = \kappa(G) \) and \( \kappa = \kappa(G) \). In addition, \( b_i = d_i(T) \) for every \( i = 1, \ldots, d \) where \( T \) is the threshold graph such that \( \alpha_\sigma(G) = T \) by Proposition 3.3(c) and Theorem 5.5. Now, let us consider \( i \leq \kappa \). We first notice that \( b_i = 2 < 3 = d_i(G) \) for \( i = \kappa + 1 \) (which means \( \kappa < \kappa \) since \( i \leq \kappa \)). Second, we notice that \( b_i = 1 < 2 = d_i(G) \) for \( i \neq \kappa + 1 \). In any case \( b_i < d_i(G) \) if \( i \leq \kappa \), showing that Theorem 5.8(b) and Corollary 5.9(b) are best possible. Finally, if \( i = \kappa = \kappa \), we have that \( b_i = 1 < 4 = b_{i+1} \) showing that Corollary 5.9(c) is best possible.

We now study the relation of the \textit{b–vector} of a chordal graph \( G \) with the Betti numbers of the square-free monomial ideal associated to the clique simplicial complex \( \Delta(G) \). We now show that the Betti numbers of \( R/I_{\Delta(G)} \) can be expressed by a formula in terms of its \textit{b–vector} when \( G \) is a chordal graph.

Remark 5.11. The relation between the \textit{c–vector} and the \textit{b–vector} (given indirectly by Formula 2.1.1) can be written explicitly via the formula \( c_i = \sum_{j=0}^{d} \binom{j-1}{i-1} b_j \).

Proposition 5.12. Let \( G \) be a chordal graph, then the Betti numbers of \( R/I_{\Delta(G)} \) can be expressed by a formula in terms of its \textit{b–vector}.

Proof. Since the \textit{f–vector} and the clique vector of \( \Delta(G) \) have the relation \( f_{i-1} = c_i \) for every \( i = 1, \ldots, d \) by Remarks 4.9 and 5.11, the \textit{h–vector} and the \textit{b–vector} of \( \Delta(G) \) have the
following relation

\[(5.2.1) \quad h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{i-j} \left( \sum_{k=j}^{d} \binom{k-1}{k-j} b_k \right)\]

for every \(i = 0, 1, \ldots, d\). Since \(G\) is chordal, \(I_{\Delta(G)}\) has a 2-linear resolution [Frö90, Theorem 1], we have that

\[(5.2.2) \quad \beta_i(R/I_{\Delta(G)}) = \sum_{\ell=0}^{2+i} (-1)^{\ell+i+1} h_{2+i-\ell} \binom{n-d}{\ell} \]

for every \(0 \leq i \leq p\) by Theorem 4.10. Here, \(p\) denotes the projective dimension of the Stanley-Reisner ring \(R\). Therefore, by combining Equations (5.2.1) and (5.2.2), we obtain a formula of the Betti numbers \(\beta_i(R/I_{\Delta(G)})\) in terms of its \(b\)-vector. Precisely

\[
\beta_i(R/I_{\Delta(G)}) = \sum_{\ell=0}^{2+i} (-1)^{\ell+i+1} h_{2+i-\ell} \binom{n-d}{\ell} \left( \sum_{k=j}^{d} \binom{k-1}{k-j} b_k \right) \binom{n-d}{\ell}
\]

for every \(0 \leq i \leq p\).

We now extend a theorem of Goodarzi’s [Goo15, Theorem 1] regarding \(b\)-vector and connectivity of \(G\). In particular, we describe one more entry in terms of connectivity components.

**Theorem 5.13.** Let \(G\) be a chordal graph of order \(n\) and vertex connectivity \(\kappa\). Then,

\[
b_i = \sum_{|Y|=i-1} (W(G - Y) - 1) + 1 = \beta_{n-i}(R/I_{\Delta(G)}) + 1
\]

for every \(i = 1, \ldots, \kappa + 1\) where \(Y \subseteq V(G)\). In addition,

\[
b_i < \sum_{|Y|=i-1} (W(G - Y) - 1) + 1 = \beta_{n-i}(R/I_{\Delta(G)}) + 1
\]

for every \(i = \kappa + 2, \ldots, d\).

**Proof.** By Proposition 4.1, we have that

\[
\beta_{n-i,n-i+1}(R/I_{\Delta(G)}) = \sum_{|Y|=i-1} (W(G - Y) - 1) + 1 = 0
\]

for every \(i = 1, \ldots, \kappa\). Since we already know that \(b_1 = \ldots = b_\kappa = 1\) from Goodarzi’s work [Goo15, Theorem 1], the theorem holds for every \(i = 1, \ldots, \kappa\).

We now focus for the case \(i = \kappa+1\). Let \(T = \alpha_\kappa(G)\) as in Definition 5.4, then \(\Delta(T) = \Delta(G)^e\) and \(\kappa(T) = \kappa(G)\) by Corollary 5.6. We have that \(b_{\kappa+1}(G) = b_{\kappa+1}(T)\) by Theorem 5.5. In
addition,
\[ \sum_{|Y| \geq \kappa} (W(G - Y) - 1) = \beta_{n-n-i+1}(R/I_{\Delta(G)}) \] by Proposition 4.1

\[ = \beta_{n-i}(R/I_{\Delta(G)}) \] because \( I_{\Delta(G)} \) has a 2-linear resolution.

\[ = \beta_{n-i}(R/I_{\Delta(T)}) \] by Proposition 4.11

\[ = \beta_{n-i,n-i+1}(R/I_{\Delta(T)}) \] because \( I_{\Delta(G)} \) has a 2-linear resolution.

\[ = \sum_{|Y| = \kappa} (W(G - Y) - 1) \] by Proposition 4.1.

Then, it suffices to prove our claims for \( T \), which were proved in Propositions 3.3(d) and 3.4 respectively.

\[ \square \]

The \( b \)-vector of \( G \) in Figure 3 is \((1, 4, 3, 2)\) and \( \kappa(T) = \kappa(G) = \bar{\kappa}(T) = \bar{\kappa}(G) = 1 \). We observe that effectively \( b_1 = 1 \) and \( b_i = d_i(G) \) for every \( i \geq 2 \) (Corollary 5.9(b)). Figure 4 shows another chordal graph \( G \) and its graph \( \alpha_{\sigma}(G) = T \). We observe that the \( b \)-vector of \( G \) in Figure 4 is \((1, 2, 3, 2, 2)\) and \( \kappa(T) = \kappa(G) = 1 < 2 = \bar{\kappa}(G) < \bar{\kappa}(T) = 4 \). Hence, \( b_1 = 1 \) and \( b_i = d_i(G) \) for every \( i > 2 = \bar{\kappa}(G) \) (Corollary 5.9(b)). Finally, we observe that edges \((4, 10)\) and \((9, 10)\) of \( T \) are a minimum dominating 2-clique in \( T \) and edges \((3, 5)\), \((4, 5)\) and \((6, 7)\) are a minimum dominating 2-clique in \( G \). Hence, \( b_2 = d_2(T) = 2 < 3 = d_2(G) \). Nevertheless, we know that \( b_2 \) give us another information in \( G \), because \( b_2 = \sum_{|Y| = 1} (W(G - Y) - 1) + 1 = 2 \) by Theorem 5.13.

\[ \sum_{|Y| \geq \kappa} (W(G - Y) - 1) = \beta_{n-n-i+1}(R/I_{\Delta(G)}) \]

\[ = \beta_{n-i}(R/I_{\Delta(G)}) \] because \( I_{\Delta(G)} \) has a 2-linear resolution.

\[ = \beta_{n-i}(R/I_{\Delta(T)}) \] by Proposition 4.11

\[ = \beta_{n-i,n-i+1}(R/I_{\Delta(T)}) \] because \( I_{\Delta(G)} \) has a 2-linear resolution.

\[ = \sum_{|Y| = \kappa} (W(G - Y) - 1) \] by Proposition 4.1.

5.3. Examples. We finish this section by providing examples of pure and matroid complexes arising from chordal and threshold graphs in which we know the value of its \( b \)-vector. A simplicial complex \( \Delta \) is pure if all its facets have the same cardinality and a matroid complex \( \Delta_V \) on a set of vertices \( V \) is a pure simplicial complex such that \( \Delta_{V \setminus S} \) is pure for every \( S \subseteq V \).
Klivans [Kli07] proved that given a graph \( T \), \( \Delta(T) \) is a shifted matroid complex if and only if \( T \) is threshold and its corresponding word is of the form \( SDDDSSSS \). The first example shows that the above is true for pure complexes arising from threshold graphs.

**Example 5.14.** Let \( T \) be a threshold graph and suppose that \( \Delta(T) \) is pure. Note that there is no maximal clique of size \( i \) in \( T \) for \( i \leq d - 1 \). Then, by Proposition 3.3(i)(ii), \( b_i = 1 \) for \( i \leq d - 1 \). Since \( \Delta(T) \) is pure the word corresponding to \( T \) is of the form \( SDDDSSSS \), \( T - S \) is pure for every \( S \subset V(G) \). This is because the corresponding word of \( T - S \) has the same type. Hence, \( \Delta(T) \) is a matroid complex.

**Example 5.15.** Let \( G \) be a chordal graph and suppose that \( \Delta(T) \) is a matroid complex. We show that \( G \) is in fact threshold. For this, it is enough to show that \( G \) does not contain the graphs \( C_4 \), \( 2K_2 \) or \( P_4 \) as an induced subgraph by Theorem 3.2. First, we claim that for every vertex \( v \in V(G) \) and every edge \( u_1u_2 \in E(G) \), we have that \( vu_1 \in E(G) \) or \( vu_2 \in E(G) \). Let \( v, u_1, u_2 \) different vertices in \( G \) such that \( u_1u_2 \in E(G) \) and let \( S = V(G) - \{v, u_1, u_2\} \). As \( \Delta(G) \) is a matroid complex, we have that \( \Delta(G - S) \) is pure which means that \( vu_1 \in E(G) \) or \( vu_2 \in E(G) \). Second, by the previous claim, \( G \) does not contain the graphs \( C_4 \), \( 2K_2 \) or \( P_4 \) as an induced subgraph. We obtain that \( G \) is threshold. Finally, we also notice that \( b_1 = 1 \) for \( i \leq d - 1 \) by Example 5.14.

The last example is when \( G \) is a chordal graph and \( \Delta(G) \) is pure. In this case, the \( b \)-vector satisfies \( 0 < b_1 \leq b_2 \leq \cdots \leq b_d \) [HHM+08, Theorem 1.2]. Hence, from Corollary 5.9(c), we obtain the following.

**Example 5.16.** Let \( G \) be a chordal graph with \( \Delta(G) \) pure. Then, \( b_{\kappa(G)+1} = b_{\kappa(G)+2} = \cdots = b_d \).

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