Plate models and hidden informations.

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Abstract. A plate or a shell structure is mainly characterized by the fact that the thickness is small compared to the other dimensions. This allows some kinematical and constitutive approximations which can be justified in a mathematical framework as far as the energy of the model is concerned. But in most cases, the three dimensional local behaviour is not represented by the plate or the shell model. For instance, interlayer stresses in a multilayered plate can be singular and lead to a delamination. But they are not contained in a standard plate model. The local waves trapped in one of these layers are not represented in these structural models. The goal of this paper is to give few possibilities to overcome these difficulties.

1. Introduction

A large number of papers and books have been devoted to the plate theory. But the most important contribution is certainly the one given by Kirchhoff and Love [17], [12]. It is recalled hereafter in a first part. The ideas based on a more accurate approximation of the transverse shear strains introduced by Mindlin [18] and Reissner [21] have also contributed to meaningful improvements of the models, in particular for the numerical aspects. Concerning the validation of these models, the natural question was to derive an upper bound of the error between the three dimensional solution and the one of the plate model. This was first obtained using an asymptotic method. The idea of the asymptotic strategy is due to Friedrichs and Dressler [9] but the mathematical framework which enables one to derive a full justification came later (see the books by P.G. Ciarlet [4] and its very complete bibliography which resumes the various contributions and [7] which focus on linearly elastic plates). The main error bounds are recalled in the second section of this paper. Nevertheless, the boundary-layer which plays an important role in the delamination of multilayered plates contains strange but precious informations. The goal of the third fourth and fifth sections is to explain how these informations can be derived from the plate model.

This paper, which is an extension of the Kirchhoff-Love’s method to the boundary-layer effect, corresponds to a twisting free boundary condition on a part of the lateral edge of the plate.

2. The almost usual Kirchhoff-Love’s model

Let us start with the notations. The plate that we consider occupies in space the open set \( \Omega^\varepsilon \), the small parameter \( \varepsilon \) denoting half the thickness. The middle surface is \( \omega \) with a boundary \( \gamma \) and the other geometrical notations are explicited on figure 1. The lateral boundary is \( \Gamma^\varepsilon = \Gamma_0^\varepsilon \cup \Gamma_1^\varepsilon \). The plate is clamped on \( \Gamma_0^\varepsilon = \gamma_0 \times ] - \varepsilon, \varepsilon [ \) and the twisting is free on \( \Gamma_1^\varepsilon = \gamma_1 \times ] - \varepsilon, \varepsilon [ \). The displacement field is \( u = \{ u_i \} \) and the linearized strain field is: \( \gamma(u) = \{ \gamma_{ij}(u) \} \) where
\( \gamma_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i) \) where \( i, j \in \{1, 2, 3\} \) and the notation \( \partial_i \) is the partial derivative with respect to the coordinate \( x_i \). The 2\textsuperscript{nd} Piola-Kirchhoff’s stress is denoted by \( \sigma = \{\sigma_{ij}\} \). The

\[ \sigma = R\gamma(u), \quad \text{or else} \quad \forall i, j, k, l \in \{1, 2, 3\}, \quad \sigma_{ij} = R_{ijkl}\gamma_{kl}(u), \quad \text{where} \quad R_{ijkl} = R_{klij} = R_{jikl}, \quad (1) \]

and it is commonly admitted that the materials used in a plate are monoclinic with respect to the thickness direction so that:

\[ \forall \alpha, \beta, \lambda \in \{1, 2\}, \quad R_{\alpha\beta\lambda 3} = 0. \]

This property enables to decouple the inplane stresses and the transverse shear stresses which is a basic point in plate and shell models. Furthermore the coefficient \( S_{ijkl} \) are assumed to be constant with respect to the in plan coordinates \( x_\alpha \) (even constant) and piecewise constant with respect to \( x_3 \) . It is convenient to use the compliance \( S = R^{-1} \) instead of the stiffness \( R \). Therefore, the constitutive relationship can also be written:

\[ S_{ijkl}\sigma_{kl} = \gamma_{ij}(u) \quad \text{and} \quad S_{\alpha\beta\lambda 3} = 0. \quad (2) \]

Furthermore, the ellipticity-continuity condition is always satisfied:

\[ \exists c_0 > 0, \quad c_1 > 0, \quad \text{s.t.} \quad \forall \tau = \{\tau_{ij}\}, \quad c_0 \sum_{i,j=1,3} |\tau_{ij}|^2 \leq S_{ijkl}\tau_{ij}\tau_{kl} \leq c_1 \sum_{i,j=1,3} |\tau_{ij}|^2 \quad (3) \]

Let us now introduce the principle of virtual work. First of all the admissible space for the displacements is defined by (see figure 1 for a geometrical representation, \( r \) being a rotation along \( \gamma_1 \)):

\[ V = \{v = \{v_i\}, \quad v_i \in H^1(\Omega^e), \quad v_i = 0 \text{ on } \Gamma_0^e, \quad v_3 = 0, \quad v_\alpha b_\alpha = x_3 r \text{ on } \Gamma_1^e \} \quad (4) \]

Thus, the principle of virtual work for a dynamical model without external load but with initial conditions for instance, can be stated as follows (the upper double dot is the second order time derivative):

\[ \forall v \in V, \quad \int_{\Omega^e} q\ddot{u}_i v_i + \int_{\Omega^e} \sigma_{ij}\gamma_{ij}(v) = 0 \quad (5) \]

\[ u(x, 0) = u_d(x) \quad \dot{u}(x, 0) = u_v(x). \quad (6) \]

The set of equations (2), (5) and (6) is the three dimensional dynamical model for the plate that we consider in this paper (with the boundary conditions chosen). A classical result based on Korn’s inequality [8] enables one to prove the classical following statement.

Figure 1. The 3D plate and the boundary conditions (left) an example (right)
Theorem 1 Let \( ud \in V \) and \( uv \in [L^2(\Omega^e)]^3 \). The coefficients \( S_{ijkl} \) of the compliance tensor are supposed to satisfy (3). For any \( T > 0 \), there exists a unique solution \( u \in C^0([0, T]; V) \cap C^1([0, T]; [L^2(\Omega^e)]^3) \) to (2)-(5)-(6). If the initial data are smooth enough additional regularity can also be obtained by derivating in time the model or/and using the classical regularity results for elliptic operator (see P. Grisvard [11]).

Our goal now is to derive a surface model which would be a nice approximation of the three dimensional one. But a cornerstone is also to be able to derive hidden informations from the plate solution using few mathematical analysis between the 3D and the 2D. Concerning the error estimate, the basic trick is to use Prager and Synge’s inequality [20] following the initial idea of P. Ladevèze [14] for static problems and with some additional and important mathematical modifications concerning the dynamical aspect and the boundary-layer effects. The strategy that we develop in this paper is mathematically similar to the one presented in [4] and [7]. But in our opinion, it is more efficient for practical applications and a better mechanical understanding of what happens in a thin structure.

3. The Kirchhoff-Love’s theory
Let us introduce \textit{a priori} assumptions which can be fully justified by an asymptotic method based on the smallness of the thickness of the plate [4] [7]. There are two sets of such hypothesis: one concerning an approximation of the kinematics, the other concerns the energy.

3.1. The Kirchhoff-Love’s kinematical assumption
Due to the smallness of the thickness compared to the other dimension of a plate, one can suggest to neglect the necking strain \( \gamma_{33}(u) \) but also (less justified but acceptable) the transverse shear strain \( \gamma_{\alpha 3}(u) \). Hence one introduces a subspace of \( V \) defined by:

\[
V_{KL} = \{ v = \{ v_i \} \in V, \partial_3 v_3 = 0, \partial_\alpha v_3 + \partial_3 v_\alpha = 0, \alpha = 1, 2 \},
\]  

and this space can be identified to \( V_t \times V_3 \) defined by:

\[
\begin{align*}
V_t &= \{ v = \{ v_\alpha \}, v_\alpha \in H^1(\omega), v_\alpha = 0 \text{ on } \gamma_0 \} \\
V_3 &= \{ v \in H^2(\omega), v = 0 \text{ on } \gamma_0 \cup \gamma_1 \}.
\end{align*}
\]  

In fact, there is an isomorphism between \( V^{KL} \) and \( V_t \times V_3 \) defined by:

\[
\begin{align*}
v &= (v_\alpha, v_3) \in V_t \times V_3 \rightarrow j_{KL}(v) = (\{ v_\alpha - x_3 \partial_3 v_3 \}, v_3) \in V^{KL}.
\end{align*}
\]  

The kinematical Kirchhoff-Love’s hypothesis can be stated as follows:

\textit{Hypothesis 1 (kinematics)} The displacement \( u \) solution to three dimensional model can be approximated by a function \( u^{KL} \) in the space \( V^{KL} \).

3.2. The implicit Koiter’s assumption
In a different formulation W.T. Koiter [13] underlined that the Kirchhoff-Love’s kinematical hypothesis was not sufficient for defining a consistent plate or shell model. Hence he stated that the elastic energy (or free energy) was only dependent on the inplane stresses (or strains). This is an assumption. But in a mathematical framework this sentence is not sufficiently precise. And one controversy was that Koiter omitted to mention explicitly the possibility of transverse and necking stresses. This is why it appeared necessary to reformulate his statement in a more precise way as follows.

3
Hypothesis 2 (energy) The elastic energy due to the transverse shear and the necking stresses are negligible compared to the one of the in plan stresses.

This second hypothesis is a little bit difficult to handle completely and it is at the origin of many misunderstandings in plate and shell models. In fact, it contains two informations: the first one is that in the expression of the complementary energy:

$$\sigma \in [L^2(\Omega^e)]_s^9 \rightarrow E(\sigma) = \int_{\Omega^e} S_{ijkl} \sigma_{ij} \sigma_{kl},$$

the contribution due to transverse shear stresses $$\sigma_{\alpha 3}$$ and the necking stress $$\sigma_{33}$$ can be neglected.

Hence, one sets:

$$E(\sigma) \simeq \int_{\Omega^e} S_{\alpha\beta\lambda\mu} \sigma_{\alpha\beta} \sigma_{\lambda\mu}.$$ 

This hypothesis doesn’t mean that $$\sigma_{\alpha 3} = \sigma_{33} = 0$$, but only that their energy can be neglected. In fact, the transverse and the necking stresses will be defined from the equilibrium equations as functions which are only distributions in many cases. Therefore a regularity condition must be be satisfied in order to ensure that their energy is negligible compared to the one of the in-plane stresses $$\sigma_{\alpha\beta}$$.

3.3. The Kirchhoff-Love’s model

The plate’s model that we are dealing with in this paper is defined as follows:

$$\begin{align*}
\text{Find } (u^{KL}, \sigma^{KL}) \in V^{KL} \times [L^2(\Omega^e)]_s^9, \text{ (s for symmetry), s.t.:} \\
\forall \tau \in [L^2(\Omega^e)]_s^9, \int_{\Omega^e} S_{\alpha\beta\lambda\mu}^{KL} \sigma_{\alpha\beta} \tau_{\lambda\mu} = \int_{\Omega^e} \gamma_{\alpha\beta}(u^{KL}) \tau_{\alpha\beta}, \\
\forall v \in V^{KL}, \int_{\Omega^e} \sigma_{ij}^{KL} \gamma_{ij}(v) + \int_{\Omega^e} \varrho \hat{u} \cdot v = 0, \\
u^{KL}(0) = P^1_{KL}(u_d), \quad v^{KL}(0) = P^0_{KL}(u_v),
\end{align*}$$

(10)

where $$P^1_{KL}$$ (resp. $$P^0_{KL}$$) is the euclidian projection from $$V$$ onto $$V_{KL}$$ (resp. $$[L^2(\Omega^e)]^3$$). Let us point out that the space for the virtual displacement field is $$V^{KL}$$ and not $$V$$. This is a cornerstone in plate modeling. Let us give the first basic result (see for instance H. Brezis [3] or G. Duvaut-J.L. Lions [8]).

Theorem 2 The model (10) has a unique solution $$u^{KL}_d \in C^0([0,T]; V) \cap C^1([0,T]; [L^2(\omega)^2])$$ and $$u^{KL}_v \in C^0([0,T]; V) \cap C^0([0,T]; [L^2(\omega)^3])$$ as soon as $$u_d \in V$$ and $$u_v \in [L^2(\Omega^e)]^3$$. This solution can be characterized as follows:

$$\begin{align*}
\text{first of all let us set:} \\
R^{2m}_{\alpha\beta\lambda\mu} = \int_{-\varepsilon}^{\varepsilon} [S_{\alpha\beta\lambda\mu}]^{-1}, \quad R^{2f}_{\alpha\beta\lambda\mu} = \int_{-\varepsilon}^{\varepsilon} x^2_{3} [S_{\alpha\beta\lambda\mu}]^{-1}, \quad q_{a} = \int_{-\varepsilon}^{\varepsilon} \varrho, \quad I_{3} = \int_{\omega} x^2_{3} \varrho,
\end{align*}$$

(11)

and $$\nabla$$ is the inplane gradient operator.
The displacements $w_{KL}(t) \in V_{1} \quad w_{3}(t) \in V_{3}$ are solution of:

$$\forall \nu_{\alpha} \in V_{1}, \int_{\omega} R_{\alpha \beta \lambda \mu}^{2m} \gamma_{\alpha \beta}(w_{KL}) \gamma_{\lambda \mu}(\nu_{\alpha}) + \int_{\omega} \partial_{\alpha} \overline{w}_{\alpha} \nu_{\alpha} = 0,$$

$$\forall v_{3} \in V_{3}, \int_{\omega} R_{\alpha \beta \lambda \mu}^{2f} \partial_{\alpha \beta} w_{KL} \partial_{\lambda \mu} v_{3} + \int_{\omega} I_{3} \nabla \overline{w}_{KL}; \nabla v_{3} = 0. \quad (12)$$

Remark 1 It is convenient to introduce the in plan stress resultant and the bending moment by:

$$\begin{align*}
\alpha \beta = & \int_{-\varepsilon}^{\varepsilon} \sigma_{\alpha \beta} = R_{\alpha \beta \lambda \mu}^{2m} \gamma_{\lambda \mu}(\nu_{\alpha}), \\
\alpha \beta = & \int_{-\varepsilon}^{\varepsilon} x_{3} \sigma_{\lambda \mu} = -R_{\alpha \beta \lambda \mu}^{2f} \partial_{\lambda \mu} u_{3}.
\end{align*} \quad (13)$$

Therefore, the two dimensional equilibrium equations are respectively for the membrane and the bending behaviour:

$$\begin{align*}
\forall \nu_{\alpha} \in V_{1}, \int_{\omega} n_{\alpha \beta} \partial_{\alpha} \nu_{\beta} + \int_{\omega} \partial_{\alpha} \overline{w}_{\alpha} \nu_{\alpha} = 0, \\
\forall v_{3} \in V_{3}, - \int_{\omega} m_{\alpha \beta} \partial_{\alpha \beta} v_{3} + \int_{\omega} \partial_{\alpha} \overline{w}_{3} v_{3} + \int_{\omega} I_{3} \nabla \overline{w}_{3}; \nabla v_{3} = 0. \quad (14)
\end{align*}$$

Finally, the in plan stress $\sigma_{KL}^{KL}$ is given by the following expression (the upper index $^{-1}$ denotes the inverse):

$$\sigma_{KL} = [S_{\alpha \beta \lambda \mu}]^{-1} \gamma_{\lambda \mu} (w_{KL}) = [S_{\alpha \beta \lambda \mu}]^{-1} [R_{\lambda \mu \eta \xi}^{2m}]^{-1} n_{KL}^{KL} - x_{3}[S_{\alpha \beta \lambda \mu}]^{-1} [R_{\lambda \mu \eta \xi}^{2f}]^{-1} m_{KL}^{KL}. \quad (15)$$

For instance, in a multilayered plate, this in plan stress is piecewise linear through the thickness of the plate. Another important point concerns the boundary conditions on $\Gamma_{b}$. In the three dimensional model, the free twisting movement implies that one has $\sigma_{\alpha \beta a_{3} b_{3}} = 0$. From integrations by parts on the equations (14) and because of the boundary conditions satisfied by the displacements in $V_{KL}$, one has (see for instance [7]):

$$m_{a_{3} b_{3}} = 0, \partial_{a}(m_{a_{3} b_{3} a_{3} b_{3}} + \partial_{a} n_{a_{3} b_{3}} b_{3} = 0. \quad (16)$$

Hence, the 3D torsion-free condition is not satisfied by the plate solution. This phenomenon is at the origin of a torsion boundary-layer that we describe in the following.

4. The error bound

In order to obtain a validation of the plate model it is useful to construct a 3D stress field which satisfy exactly the principle of virtual work (PPW). This strategy has been first observed by P. Ladevèze [14], but it is also necessary that this 3D stress field could be connected to an admissible displacement field belonging to the space $V$. This is the most technical part which was overcome in [7] using a space interpolation method due to J.L. Lions [15]. The point is that no matter the method you are using, (asymptotic expansion or a priori assumptions) the error bound strategy is the same: i) define a 3D stress field which satisfies the PPW, ii) apply
an interpolation result. Let us sketch the method. First of all, let us consider the 3D in plan
equilibrium equation in which $\sigma_{\alpha\beta}^{KL}$ and $u_{\alpha}^{KL}$ are known (but not yet the transverse shear stress $\sigma_{\alpha3}^{KL}$):

$$\forall v_t \in V_t, \quad \int_{\omega} \sigma_{\alpha\beta}^{KL} \partial_{\beta} v_{\alpha} = -\{ \int_{\Omega^{s}} \varrho^{i} \sigma_{\alpha\beta}^{KL} v_{\alpha} + \int_{\Omega^{s}} \sigma_{\alpha\beta}^{KL} \partial_{\beta} v_{\alpha} \}. \quad (17)$$

One can make explicit the local conditions contained in this weak formulation:

$$\begin{align*}
\partial_{\beta} \sigma_{\alpha3}^{KL} &= \varrho(u_{\alpha3}^{KL} - x_{3} \partial_{\alpha} u_{3}^{KL}) - \partial_{\beta} \sigma_{\alpha3}^{KL} , \\
\sigma_{\alpha3} &= 0 \text{ for } x_{3} = \pm \varepsilon .
\end{align*} \quad (18)$$

One can compute $\sigma_{\alpha3}^{KL}$ explicitly from these relations (the two boundary conditions are compatible because of (14)):

$$\sigma_{\alpha3}^{KL} = u_{3}^{KL} \int_{-\varepsilon}^{x_{3}} g - \partial_{\alpha} u_{3}^{KL} \int_{-\varepsilon}^{x_{3}} x_{3} g - \int_{-\varepsilon}^{x_{3}} \partial_{\beta} \sigma_{\alpha3}^{KL} . \quad (19)$$

Under classical regularity assumptions one can claim that $\sigma_{\alpha3}^{KL} \in C^{0}([0,T]; L^{2}(\Omega^{s})).$ But there is a compatibility relation which should be satisfied by the Kirchhoff-Love’s solution in order to ensure that the equation (17) is fully satisfied. It is obtained by an integration by part on the right hand side of (17) and is due to the fact that $v_{\alpha} a_{\alpha} \neq 0$ on the boundary $\Gamma_{1}$. It is the twisting free edge condition:

$$\sigma_{\alpha\beta} a_{\alpha} b_{\beta} = 0 \text{ and unfortunately } \sigma_{\alpha3}^{KL} a_{\alpha} b_{\beta} \neq 0 . \quad (20)$$

This is the starting point of the boundary-layer theory for the thin structure and it is discussed in the next section. For the time being, let us go on with the computation of the necking stress $\sigma_{33}^{KL}$. From (14) one should have:

$$\forall v_{3}, \quad (0,v_{3}) \in V_{t} , \int_{\Omega^{s}} \sigma_{33}^{KL} \partial_{3} v_{3} = -\{ \int_{\Omega^{s}} \sigma_{33}^{KL} \partial_{3} v_{3} + \int_{\Omega^{s}} \varrho u_{3}^{KL} v_{3} \}. \quad (21)$$

The solution is:

$$\sigma_{33}^{KL} = u_{3}^{KL} \int_{-\varepsilon}^{x_{3}} g - \int_{-\varepsilon}^{x_{3}} \partial_{3} \sigma_{33}^{KL} . \quad (22)$$

If the plate solution is smooth enough ($u_{3}^{KL} \in C^{2}([0,T]; H^{3}(\omega))$ and $u_{3} \in C^{2}([0,T]; H^{4}(\omega))$), one can claim that both $\sigma_{33}^{KL}$ and $\sigma_{33}^{KL} \in C^{0}([0,T]; L^{2}(\Omega^{s}))$. Because the stress field $\sigma_{33}^{KL}$ coupled with the displacement field $u_{3}^{KL}$ doesn’t satisfy exactly the PPW (due to (20)). Therefore one can construct a boundary-layer model which takes into account an additional term ($\chi_{ij}^{BL}; \chi_{ij}^{BL}$) (displacement and stress field) such that $\sigma_{ij}^{KL} + \chi_{ij}^{BL}$ would be close to satisfy the PPW.

4.1. The boundary-layer

First of all, only the torsion effect is involved. Therefore the local model which is parametrized by the curvilinear coordinate along the boundary $\gamma_{1}$ is set over a semi-infinite strip $S^{t}$ as shown on figure 2. The torsion displacement the magnitude of which is denoted here by $\varphi$, is transverse to the strip. It is solution to the following equations where the shear modulus in each layer - assumed to be the same in $s \xi$ and $s x_{3}$ ($R_{33x3} = R_{33s\xi}$) for sake of simplicity- is denoted by $G$ and $\nabla_{i}(\text{resp. } \text{div}_{i})$ is the gradient (resp. the divergence) with respect to the coordinates ($\xi, x_{3}$) as shown on figure 2:

$$\begin{cases}
-\text{div}_{i}(G \nabla_{i} \varphi) = 0 \text{ in } S^{t} , \\
\frac{\partial \varphi}{\partial x_{3}} = 0 \text{ for } x_{3} = \pm \varepsilon , \quad G \frac{\partial \varphi}{\partial \xi} = -\sigma_{\alpha3}^{KL} a_{\alpha} b_{\beta} \text{ for } \xi = 0 , \lim_{\xi \rightarrow \infty} \varphi = 0 .
\end{cases} \quad (23)$$
The solution of this model has several remarkable properties that are discussed in the following sections. The existence and uniqueness of a solution is classical. It is obtained analytically from a Fourier series by (one has $A_0 = 0$ because the average of $G \frac{\partial \varphi}{\partial \xi}(0, x_3)$ is zero):

$$
\varphi(\xi, x_3) = \sum_{k=1,\infty}^{\infty} A_n e^{-\frac{n\pi\xi}{2\varepsilon}} \cos\left(\frac{n\pi x_3}{2\varepsilon}\right),
A_n = -\frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \sigma_{\alpha\beta}^{KL} a_\alpha b_\beta G \sin\left(\frac{n\pi x_3}{2\varepsilon}\right).
$$

In fact the previous series converges in the space $H^1(S^\varepsilon)$. The corrector term is then defined as follows in the local frame $(a, b, e_3)$ (see figure 1) where $\eta$ is a truncation function equal to 1 in the vicinity of $\xi = 0$:

$$
\Phi_i = \varphi a_i \eta(\xi), \ a_3 = 0, \ \chi_{ij} = \eta(\xi) R_{ijkl}(\partial_k \Phi_l + \partial_l \Phi_k)/2.
$$

But the point is to describe the local behaviour of this solution in the neighbourhood of a point like $A_k$. A singular behaviour appears due to the discontinuity of the coefficient $G$ from one layer to the other. Thus one considers the following model the solution of which describes the one of $\varphi$ in the neighbourhood of the boundary between the two layers (see point $A_k$ on figure 3 for the notations). Therefore another basis of harmonic functions can be used.
One can write in a semi-infinite strip \( H^\varepsilon = [0, \infty[ - h, h[ \):

\[
\varphi(s, \xi, x_3) = \frac{A_0}{2} + \sum_{n \geq 1} \left[ C_n \cos\left(\frac{n\pi x_3}{h}\right) + D_n \sin\left(\frac{n\pi x_3}{h}\right) \right] e^{-\frac{n\pi \xi}{h}},
\]

(26)

the coefficients \( C_n \) and \( D_n \) are defined in order to satisfy the boundary condition at \( \xi = 0 \) by:

\[
C_n = \frac{1}{n\pi} \int_{-h}^{h} G \sigma_{\alpha\beta}^{KL} a_{\alpha} b_{\beta} \sin\left(\frac{n\pi x_3}{h}\right), \quad D_n = -\frac{1}{n\pi} \int_{-h}^{h} G \sigma_{\alpha\beta}^{KL} a_{\alpha} b_{\beta} \cos\left(\frac{n\pi x_3}{h}\right).
\]

(27)

In fact, only the series with the sine functions leads to a singularity. Let us set:

\[
\chi(x_3) = \begin{cases} 
1 & \text{if } x_3 > x_3^{A_k}, \\
-1 & \text{if } x_3 < x_3^{A_k},
\end{cases}
\]

One can split \(-\frac{\sigma_{\alpha\beta}^{KL} a_{\alpha} b_{\beta}}{G}\) as follows (where \( g \) is continuous on \([-h, h[\)):

\[
-\frac{\sigma_{\alpha\beta}^{KL} a_{\alpha} b_{\beta}}{G} = \kappa \chi(x_3) + g(x_3),
\]

(28)

where \( \kappa \) is a coefficient which takes into account the discontinuity of the material properties (details are given in J.L. Davet and Ph. D [6]). The contribution of the first term on the right handside of (28), to the series representing \( \varphi \) can be explicited analytically and the second term is smoother (derivatives are \( C_0 \)). A simple computation leads to (let us set \( x_3^\delta = x_3 - x_3^{A_k} \)):

\[
\frac{\partial \varphi^\delta}{\partial x_3^\delta}(s, \xi, x_3^\delta) = \frac{2 \kappa}{\pi} \log \left(\frac{1 + 2 \cos(\pi x_3^\delta) e^{-\xi \pi/h} + e^{-2 \xi \pi/h}}{1 - 2 \cos(\pi x_3^\delta) e^{-\xi \pi/h} + e^{-2 \xi \pi/h}}\right).
\]

(29)

This singular function represented on figure 3. It is a singular peeling stress which can be at the origin of a delamination.

4.2. The mathematical result

The basic point in plate or shell theory is to justify that the the 3D and 2D solutions are closed enough. But this can be obtained in an energy norm (excepted if the two solutions are very smooth in order to be able to apply the implicit function theorem for linear or nonlinear models [5]), and no pointwise error bounds have been obtained for singular behaviours. Therefore, all the conclusions drawn from the plate model should be justified from an energetical point of view. The fundamental result is the following one and its proof is quite standard from the classical methods used in variational methods and from the results given in [4] and [7].

**Theorem 3** Let us introduce the error variables by:

\[
\begin{align*}
\overline{\sigma}_{ij} & = \sigma_{ij}^\varepsilon - \sigma_{ij}^{KL} - \chi_{ij}, \\
\overline{u}_i & = u_i^\varepsilon - u_i^{KL} - \Phi_i.
\end{align*}
\]

(30)

There exists two constant \( c_0 \) and \( c_1 \) which are independent on \( \varepsilon \) and can be computed from the plate solution, such that:

\[
\forall t > 0, \quad \frac{1}{2} \int_{\Omega^\varepsilon} g |\overline{\nu}(t)|^2 + \frac{1}{2} \int_{\Omega^\varepsilon} \overline{\sigma}_{ij} \gamma_{ij}(\overline{\nu})(t) \leq (E(0) + c_0 \varepsilon) e^{c_1 t}.
\]
Hence the error is in $\sqrt{\varepsilon}$ in the energy norm as soon as the error on the initial conditions (term $E(0)$) is small enough. The energy norm is:

$$||u(t)||_E = \sqrt{\frac{1}{2} \int_{\Omega^e} \varrho|\dot{\varepsilon}|^2 + \frac{1}{2} \int_{\Omega^e} \sigma_{ij} \gamma_{ij}(\varepsilon^e)}.$$

It is possible to upgrade Theorem 3 in different ways by adding new terms. This can be done in two different ways. One consists in using the asymptotic method based on the fact the thickness of the plate compared to its other dimensions, is a small parameter. This is explained in [4] and [7]. Another one consists in enlarging the approximation spaces for both the displacements and the stress fields. This is the case of the Reissner and Mindlin models [18][21].

5. Few aspects contained in the TBL

The energy contained in the boundary-layer model is meaningful because of Theorem 3. The logarithmic singularity leads an energy term (the hidden regularity of $m_{\alpha\beta}$ and $n_{\alpha\beta}$ (see J.L. Lions [16]) enables one to claim this property. A numerical simulation of this effect is represented on figure 4 for different time in the bending case of the plate.

![Figure 4](image_url)

**Figure 4.** The singular stress (continuous) in the vicinity of the singular point $A_k$ at two different time steps along the free edge ($\xi = 0$) and the loading applied (piecewise linear)

5.1. Delamination in the boundary-layer

Due to the singular peeling stress in the boundary-layer, a debonding between two layers can occur. It is called a delamination. As far as its length is small enough, one can justify the use of the plate model which doesn’t take into account the presence of this small crack. But the boundary-layer model does. And the energy release rate which is the opposite of the energy derivative with respect to crack length, is an energy stable quantity. Therefore it can be evaluated from the solution of the boundary-layer term. Let us consider that a small (length $\approx \varepsilon$) delamination crack has appeared at the interface between two layers. The local behavior of $\varphi$ near the crack tip can be computed following the standard methods used in fracture mechanics (see H.D. Bui [2]). In fact $K$ is the so-called stress intensity factor and $\theta$ represents a virtual movement of the crack tip (parallel to it). In fact its support is included in a neighbourhood $B^c$ of the crack tip and the energy release rate only depends on its value at this point. The advantage of this second expression is that it is perfectly stable from the energetical point of view, $(D_t \theta$ is the Jacobian matrix of $\theta$, $(.,.)$ is the scalar product).
Analytical computation:

\[
\begin{align*}
\varphi &= \frac{K}{G} \sqrt{r} \sin\left(\frac{\xi}{2}\right) + \varphi^R(\xi, x_3), \\
K^2 &= -\frac{8}{\pi(G^+ + G^-)} \frac{\partial E}{\partial l},
\end{align*}
\]

where the energy release rate is defined by:

\[
-\frac{\partial E}{\partial l} = \frac{1}{2} \int_{B^*} |\dot{\varphi}|^2 - G|\nabla_t \varphi|^2 \text{div}_{t}(\theta)
+ \int_{B^*} G(D_t \theta \nabla_t \varphi, \nabla_t \varphi)
\]

Analytical expression of the singularity at the crack tip and the energy release rate.

5.2. Love-Stoneley’s local waves
The boundary-layer model can also be (for highest frequencies) modeled by a dynamical model (wave equation). The dynamical term can be neglected for low frequency terms, but can include meaningful energy contributions for hight frequencies. Even if the pate is vibrating at a low frequency, the discontinuity of material (sandwich structure) can be at the origin of a spill over process on highest frequencies because of the local waves. For instance, the reflection of incident waves on a crack is more efficient than on an interface where a part of the energy is swallowed by the Love-Stoneley waves mainly localized in the weakest part of the sandwich plate. Because of the simplicity of the wave operator for the twisting movement, all the eigenmodes are known analytically [10]. Let us show one of them on figure 5 which is computed for the half upper part of the infinite strip without crack. The model consists in finding \(\varphi\) such that:

\[
\dot{\varphi} - G \frac{\partial^2 \varphi}{\partial \xi^2} - \frac{\partial}{\partial x_3} (G \frac{\partial \varphi}{\partial x_3}) = 0,
\]

\[
G \frac{\partial \varphi}{\partial \xi}(t, 0, x_3) = -\sigma_{a_b} a_{a_b} b_j, \quad G \frac{\partial \varphi}{\partial x_3} = 0 \text{ for } x_3 = \pm \varepsilon,
\]

for instance: \(\frac{\partial \varphi}{\partial \xi} = 0\) at \(\xi = L >> \varepsilon\) and initial conditions.

Figure 5. Love-Stoneley local wave in the twisting model (half strip and the coordinates \(\xi\) varies from left to right

\[
\begin{align*}
\dot{\varphi} - G \frac{\partial^2 \varphi}{\partial \xi^2} - \frac{\partial}{\partial x_3} (G \frac{\partial \varphi}{\partial x_3}) &= 0, \\
G \frac{\partial \varphi}{\partial \xi}(t, 0, x_3) &= -\sigma_{a_b} a_{a_b} b_j, \quad G \frac{\partial \varphi}{\partial x_3} = 0 \text{ for } x_3 = \pm \varepsilon,
\end{align*}
\]
The numerical simulation (figure 6) shows that the energy measured on the hard part is much smaller without a crack (left) than when there is one (right). The loadin case is the one defined in the boundary layer from the plate solution in dynamics.

Figure 6. Density of total energy: left without crack (important absorption) and right (weak absorption) with a crack at the upper interface. The loading case is the same as on figure 4.

6. Conclusion
This paper gives few ideas concerning plate theory and focus on two important points:

- how to construct a plate (or a shell) model which is accurate when the thickness is small,
- how to derive additional informations using a boundary layer model.

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