Shaping for Construction D Lattices

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Abstract—In this paper we propose a practical and general method for shaping a coding lattice \( \Lambda_c \) obtained by the construction D. The proposal assumes only that the coarse/shaping lattice \( \Lambda_s \) is an integer lattice. To achieve this, a new way to encode and index a construction-D Voronoi lattice code is presented which has linear complexity in the block length. We test the performance of our proposal using a construction-D lattice obtained from an extended BCH code, and shaping is performed by a Gosset Lattice \( (E_8) \). This results in a shaping gain of 0.54 dB, which is close to the expected gain of the \( E_8 \), 0.65 dB.

Index Terms—Shaping. Construction-D, Lattices, Voronoi constellations, Gosset Lattice, \( E_8 \), BCH code

I. INTRODUCTION

Lattices are a natural tool for transmitting digital information over the additive white Gaussian noise (AWGN). In fact, an optimal block code for a high-SNR band-limited AWGN channel consists of a dense packing of signal points that lie inside a hyper-sphere, and lattices provide a structured way to find such dense packings \[1\]. Theoretical results that prove that lattices achieve capacity can be seen in \[2\]–\[4\].

Good transmission strategies using lattice codes is divided in two main steps: first, find a high dimensional lattice with a dense packing of points in an \( n \)-dimensional space. This is known as the coding lattice. Second, find a method to select only the points of the coding lattice inside a specific region of the \( n \)-space, which optimally, should approximate to an hyper-sphere. This operation is called shaping, and ensures power constraints and that a finite number of points can be transmitted.

For the first step, many authors proposed good lattice designs, based on error-correcting codes, which consist in translating to the Euclidean space the techniques used for designing effective iteratively decodable error-correcting codes over finite fields \[5\]. Codes like turbo codes, low-density parity-check (LDPC) codes, polar codes and BCH codes, were recently proposed to construct lattice codes that have the potential to achieve maximum coding gain when dimension grows. \[6\]–\[9\].

In contrast to coding, shaping is less exploited in the literature. Usually, the power constraint is achieved using a hypercube for shaping, which can be implemented with very low complexity. Recently an efficiently scheme was proposed in \[1\] for shaping any lattice obtained via construction-A. In this paper we generalize this method for any lattice obtained via multilevel construction-D. We know from \[10\] that the Construction D achieves higher coding gains than construction-A and also that multilevel codes can construct large modulations from simple single-level codes.

In summary, we propose a method for constructing Voronoi constellations with a pair of nested lattices: a coding lattice, obtained via construction D, and a shaping lattice, which is an integer lattice satisfying some simple constraints that will be described in the paper. The goal is to only transmit the points of the coding lattice inside the Voronoi region of shaping lattice. This is done by finding a set of coset representatives of the quotient group defined by these lattices and then performing a quantization operation. We propose an efficient way to find those representatives for a construction-D lattice, allowing practical implementation of shaping operation. This is the main contribution of this paper. To the best of authors’ knowledge, this is the first practical scheme for shaping construction-D lattices in the literature.

To test our proposal, we implement a construction-D lattice obtained via extended BCH code, since this lattice outperforms LDPC and turbo-code lattices in small dimensions \[6\]. Shaping is performed by copies of the \( E_8 \) lattice, which has the greatest packing density in dimension eight and is the best known quantizer in this same dimension \[11\]. The Voronoi code obtained will be called Gosset constellation or \( E_8 \) constellation. We will show that our proposal results in a shaping gain of 0.54 dB, which is close to the theoretical shaping gain of \( E_8 \), 0.65 dB \[1\].

This paper is organized as follows. Section II recalls general lattice definitions, lattices obtained from error correct codes - specifically a general description of construction-D lattice - and figures of merit for lattices when applied to AWGN channel. In Section III we present theorem 1, which will allow to perform shaping. Based in this theorem, a new way to encode and index construction-D lattices with linear complexity in the block length is described. Section IV uses the theoretical results from previous sections to construct the coding lattice based on extended BCH codes and the shaping lattice based in scaled copies of \( E_8 \) lattice. The construction proposed in section IV was implemented and the results are shown in section V. Section VI concludes this work, with our achievements and some remarks.

II. PRELIMINARIES

The first part of this section recalls some lattice definitions which will be used in this paper. The second part recalls the definitions of construction D lattices. For a complete lattice review, see \[11\]–\[13\].

The following notation is adopted: boldface uppercase letters, such as \( G \), denote matrices, and its components are
lowercase of the same letter, for instance $g_{i,j}$. The $n \times n$ identity matrix is $I_n$. Boldface lowercase letters, such as $\mathbf{x}$, are column vectors, and the components are lowercase of the same letter $x_i$. The operation $[.]^t$ denotes transpose, so $\mathbf{x} = [x_1, x_2, ..., x_n]^t$. The set $\mathbb{F}_q$ denotes a finite field with $q$ elements. The set of real numbers is $\mathbb{R}$ and the set of integers is $\mathbb{Z}$.

A. Lattice Definitions

A lattice $\Lambda$ is a discrete additive subgroup of $\mathbb{R}^n$. Mathematically, a lattice $\Lambda$ can be defined as the set of all $\mathbf{x}$, satisfying

$$\Lambda = \{ \mathbf{x} = \mathbf{G} \cdot \mathbf{b} : \mathbf{b} \in \mathbb{Z}^n \} \quad (1)$$

The $n \times n$ matrix $\mathbf{G} = [\mathbf{g}_1 \ \mathbf{g}_2 \ ... \ \mathbf{g}_n]$ is a lattice generator matrix, whose columns, known as lattice generator vectors, are given by $\mathbf{g}_i$. The shortest-distance quantization operator finds the closest point of the lattice $\Lambda$ to any point $y \in \mathbb{R}^n$:

$$Q_{\Lambda}(y) = \arg\min_{\mathbf{x} \in \Lambda} ||y - \mathbf{x}||^2 \quad (2)$$

The Voronoi region of the lattice, $\mathcal{V}$, is the set of points that are closer to the origin than to any other point of the lattice $\Lambda$:

$$\mathcal{V}(\Lambda) = \{ y \in \mathbb{R}^n \mid Q_{\Lambda}(y) = \mathbf{0} \} \quad (3)$$

The volume of the lattice is defined as:

$$\text{Vol}(\Lambda) = |\det(\mathbf{G})| \quad (4)$$

where $\det(\mathbf{G})$ is the volume of fundamental parallelotope formed by the lattice generator vectors. Although a lattice has infinite different basis, it can be shown that the volume does not depend on the particular choice of the generator matrix $\mathbf{G}$. It can also be shown that this is also equal to the volume of the Voronoi region: $|\det(\mathbf{G})| = \text{Vol}(\mathcal{V}(\Lambda))$.

Two lattices are nested if $\Lambda_s \subseteq \Lambda_c$. Equivalently the generator matrix of $\Lambda_s$ can be related to that of $\Lambda_c$ by an integer matrix $\mathbf{M}$ according to [14]:

$$\mathbf{G}_s = \mathbf{M} \cdot \mathbf{G}_c \quad (5)$$

For any $\mathbf{x} \in \Lambda_c$, the set $\mathbf{x} + \Lambda_s$ is the coset of $\Lambda_s$ in $\Lambda_c$ containing $\mathbf{x}$ and $\mathbf{x}$ is called coset leader. In other words the set $\mathbf{x} + \Lambda_s$ is the lattice $\Lambda_s$ shifted by the vector $\mathbf{x} \in \Lambda_c$. Note that two elements $\mathbf{x}_1$ and $\mathbf{x}_2 \in \Lambda_c$ generate the same coset, as long as $\mathbf{x}_1 - \mathbf{x}_2 \in \Lambda_s$. In other words, different coset leaders can generate the same coset.

Given a nested lattice pair we define the quotient group $\Lambda_c/\Lambda_s$ as the set of all coset, i.e.,

$$\frac{\Lambda_c}{\Lambda_s} = \{ \mathbf{x} + \Lambda_s : \mathbf{x} \in \Lambda_c \} \quad (6)$$

The number of distinct cosets $\mathcal{M}$ is the quotient group cardinally [13] p. 21):

$$\mathcal{M} = \left| \frac{\Lambda_c}{\Lambda_s} \right| = \frac{\text{Vol}(\Lambda_s)}{\text{Vol}(\Lambda_c)} = |\det(\mathbf{M})| \quad (7)$$

A Voronoi constellation or lattice code $\mathcal{C}$ is the set of points of a translation of $\Lambda_c$ inside the Voronoi region of $\Lambda_s$: $\mathcal{C} = (\Lambda_c + \mathbf{d}) \cap \mathcal{V}(\Lambda_s)$. In this case, $\Lambda_s$ is called shaping lattice, $\Lambda_c$ is called coding lattice and $\mathcal{V}(\Lambda)$ is called the shaping region. The translation $\mathbf{d}$ is applied to ensure that the final constellation has zero average, and thus minimal power and does not change fundamental lattice parameters. Thus the average Voronoi constellation power per dimension $P(C)$ satisfies:

$$P(C) = P(\Lambda_c) + P(\mathbf{d}) \quad (8)$$

where, $P(\Lambda_c)$ is the average power per dimension of the lattice $\Lambda_c$ constrained by Voronoi region of $\Lambda_s$ and $P(\mathbf{d}) = \frac{1}{n}||\mathbf{d}||^2$.

Note that each point of $\Lambda_s$ inside the shaping region $\mathcal{V}(\Lambda_s)$ is associated to a different coset. Thus, choosing a point in the Voronoi constellation $\mathcal{C}$ can be seen as choosing first a coset in the quotient group, characterized by a coset leader $\mathbf{x}$ - it can be any point of the coset - and then finding the coset leader of this coset that is inside the shaping region $\mathcal{V}(\Lambda_s)$. This latter step is performed by modulo-$\Lambda_s$ operation:

$$\mathbf{x}' = \mathbf{x} - Q_{\Lambda_s}(\mathbf{x}) \quad (9)$$

In consequence, $\mathbf{x}' \in \mathcal{C}$. Note that this is the same as finding the element of the coset $\mathbf{x} + \Lambda_s$ with minimum Euclidean norm, because each different point of a coset is mapped in the same unique point of the set $\mathcal{C}$.

The information rate of any Voronoi constellation is defined as

$$R = \frac{\log_2 \mathcal{M}}{n} \quad (10)$$

The signal-to-noise ratio of the AWGN channel is given by

$$\text{SNR} = \frac{P}{\sigma^2} = 2R \frac{E_b}{N_0} = 2 \frac{E_s}{N_0} \quad (11)$$

where $P = \mathbb{E}[x^2]$ is the transmitted power, $\sigma^2 = N_0/2$ is the noise variance, $E_b$ is the energy per bit and $E_s$ is the energy per symbol.

A standard measure of lattice performance is the so called volume-to-noise ratio, defined as

$$\text{VNR} = \frac{\text{Vol}(\Lambda)^{\frac{2}{n}}}{2\pi \sigma^2} \quad (12)$$

For a given family of lattices with increasing dimension, a necessary condition for its error probability to vanish as the lattice dimension increases is that $\text{VNR} > 1$. This is also known as the Poltyrev limit. Families of lattices that have vanishing error probability are said to be good for coding or AWGN-good or Poltyrev-limit-achieving.

The power savings achieved by a given shaping region, when compared to a standard PAM constellation, is captured by the shaping gain. To define it, let $\mathcal{S}$ be a compact bounded region of the $n$-space, which will be used for shaping. Let $\text{Vol}(\mathcal{S})$ be its volume, and $P(\mathcal{S})$ be its average power per
dimension, computed assuming that the transmitted points are continuously and uniformly distributed in $\mathbb{R}^q$. The shaping gain $\gamma_s(S)$ is defined as $[1]$.

$$
\gamma_s(S) = \frac{\text{Vol}(S)}{12P(S)}
$$

As we will be working with nested lattices, $S$ will be the Voronoi region of shaping lattice $\Lambda_s$.

**B. Lattice from codes: construction-D**

An important way to generate lattices is from error-correcting codes. This can be done in many ways $[11]$. In this paper, we focus on the so called construction-D lattices, which are generated by a set of nested codes $C_i$ in $\mathbb{F}_q^n$.

**Definition 1.** Let $g_1, g_2, \ldots, g_n$ be a basis for $\mathbb{F}_q^n$. For $a \geq 1$, let $C_0 \subseteq C_1 \subseteq \cdots \subseteq C_a = \mathbb{F}_q^n$ be a sequence of nested linear block code over $\mathbb{F}_q^n$. If $g_1, g_2, \ldots, g_n$, span $C_i$, for $i \in \{0, 1, \cdots, a - 1\}$, then a construction-D lattice $\Lambda_D$, is the set

$$
\Lambda_D = \{ c_0 + q \cdot c_1 + \cdots + q^i \cdot c_i + \cdots + q^{a-1} \cdot c_{a-1} + q^a \cdot z \} \quad (14)
$$

where $c_i \in C_i$, $i = \{0, \cdots, a - 1\}$, $z \in \mathbb{Z}^n$. Each code $C_i$ has rate $R = k_i/n$ and minimum distance $d_i$. Because the codes are nested $R_0 \leq R_1 \leq \cdots \leq R_a$ and $d_0 \geq d_1 \geq \cdots \geq d_a$. Also, the basis vectors for code $C_i$, with generator vectors $g_1, g_2, \ldots, g_n$, are a subset of those for code $C_{i+1}$ with generator vectors $g_1, g_2, \ldots, g_{k_{i+1}}$. Since $C_a = \mathbb{F}_q^n$, $R_a = 1$, $k_a = n$ and $d_a = 1$.

Let $G_i = [g_1, \cdots, g_n]$ be a generator matrix of code $C_i$ and $u_i$ a message vector with length $k_i$, with $i = \{0, \cdots, a - 1\}$. Then, each codeword $c_i$ is given by $G_i \cdot u_i$, where the operations here, and only here, are performed over $\mathbb{F}_q$. Thus, using (14) any point $x \in \Lambda_D$ can be written as

$$
x = c_0 + q^1 c_1 + \cdots + q^i c_i + \cdots + q^{a-1} c_{a-1} + q^a z
$$

Note that the sums in the equation above are performed in the regular algebra of $\mathbb{R}^n$. An $n \times n$ generator matrix $G$ of this lattice as $[1]$ is a matrix of the form $[6]$:

$$
G = G_a \cdot D,
$$

where $D$ is a diagonal matrix with diagonal entries

$$
d_{ii} = q^i \text{ for } k_j \leq i < k_{j+1}.
$$

The volume of a construction-D lattice is given by $[6]$

$$
\text{Vol}(\Lambda_D) = q^{an - \sum_{i=0}^{a-1} k_i} \quad (18)
$$

For a construction-D lattice with $q = 2$ and nested linear binary codes $C_i$, let the minimum distance of code $C_i$ be $d_i \geq 4^{a-1}/\gamma$ for $i = \{0, 1, \cdots, a - 1\}$, where $\gamma \in \{1, 2\}$. Then a Construction D lattice has squared minimum distance satisfying $[15]$.

$$
d_{\text{min}}^2 \geq 4^a / \gamma
$$

Note that the construction A is a particular case of the construction D, obtained when we set $\alpha = 1$. As a result, construction A yields lattices that are generated by a single code in $\mathbb{F}_q^n$ and it is not a multilevel construction as the construction D.

**III. ENCODING AND INDEXING CONSTRUCTION-D LATTICES**

Encoding is the process of mapping the information to be transmitted, represented by a vector of integers, to the codewords of $C$. Indexing is the inverse operation, mapping codewords of $C$ to information integers $[14]$.

In the first part of this section we present a simple way to find a coset leader of all the different cosets of the quotient group $\Lambda_s/\Lambda_c$ when $\Lambda_c$ is a construction-D lattice and $\Lambda_s \subseteq q^a \mathbb{Z}^n$. To that end, we will prove a theorem that generalizes lemma 1 of $[5]$, which was derived for construction-A lattices. This is the main contribution of this paper and to the best of our knowledge results in the first practical scheme for shaping construction D lattices. In the second part of this section, based on the encoding operation, we propose a way to recover the transmitted information vector.

The following theorem characterizes the quotient group $\Lambda_s/\Lambda_c$ by producing an explicit set of coset leaders when $\Lambda_c$ is a construction-D lattice and $\Lambda_s$ is any integer lattice satisfying $\Lambda_s \subseteq q^a \mathbb{Z}^n$. It allows us to create a Voronoi constellation because each coset leader can then be mapped to a different point inside the Voronoi region of $\Lambda_s$ when modulo-$\Lambda_s$ operation as in (9) is performed.

**Theorem 1:** Let $\Gamma \subseteq \mathbb{Z}^n$ be any integer lattice. Let $\Lambda_s = q^a \Gamma \subseteq q^a \mathbb{Z}^n$, and let $\Lambda_c = C_0 + qC_1 + \cdots + q^{a-1} C_{a-1} + q^a \mathbb{Z}^n$ be a construction-D lattice. Let the generator matrix in Column-style Hermite normal form $[13]$ of $\Gamma$ be

$$
T = \begin{pmatrix}
t_{1,1} & t_{1,2} & \cdots & t_{1,n} \\
0 & t_{2,2} & \cdots & t_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & t_{n,n}
\end{pmatrix} \in \mathbb{Z}^{n \times n}
$$

Define a set $S$ as the Cartesian product $S = \{0, \ldots, t_{1,1} - 1\} \times \{0, \ldots, t_{2,2} - 1\} \times \cdots \times \{0, \ldots, t_{n,n} - 1\}$.

A complete set of coset leaders of the quotient group $\Lambda_s/\Lambda_c$ is given by

$$
\sum_{n=0}^{a-1} q^i C_i + q^a S = \left\{ \sum_{n=0}^{a-1} q^i c_i + q^a s \mid c_i \in C_i, s \in S \right\}
$$

1 This is known as the continuous approximation. Even though the transmitted points in $C$ are discrete, this approximation is good for large lattice constellations, and makes some computations much easier.
Proof: This proof follows the same structure of the proof of lemma 1 in [5] with the necessary modifications.

First we prove that each element of \((c_0, \ldots, c_{a-1}, s) \in A = C_0 \times \cdots \times C_{a-1} \times S\) generates a different point in \(B = \{\sum_{n=0}^{a-1} q^n C_i + q^n S\}\). To that end, we prove that

\[
|\sum_{n=0}^{a-1} q^n C_i + q^n S| = |S| \prod_{n=0}^{a-1} |C_i|.
\]  

(22)

Let \(x = \sum_{n=0}^{a-1} q^n c_i + q^n s\) and \(y = \sum_{n=0}^{a-1} q^n d_i + q^n t\). Suppose that \(x = y\) for \(c_i, d_i \in C_i\) and \(s, t \in S\). If we apply modulo-\(q\) operation in both sides, it is easy to see that \(c_0 \equiv d_0 \mod q\) and so \(c_0 = d_0\). If we apply modulo-\(q^2\) operation in both sides by the same argument \(c_0 + q c_1 = d_0 + q d_1\), but \(c_0 = d_0\), consequently \(c_1 = d_1\). Moving recursively upwards with modulo-\(q^i\) operation to \(i = 3, 4, \ldots, a\) we conclude that \(c_i = d_i\) for every \(i\), and \(s = t\). Thus, we have a bijection between the sets.

Now we prove that \(A\) and \(B\) have the same cardinality, that is, \(|\sum_{n=0}^{a-1} q^n C_i + q^n S| = |A_c/A_s|\). Given \(\Omega\) and by the triangularity of \(\Omega\) and the definition of \(S\), the quotient group carnality is

\[
|A_c/A_s| = Vol (q^n \det \Gamma) = \frac{q^n |S|}{q^n - \sum_{i=0}^{a-1} k_i} = \frac{|S| \prod_{n=0}^{a-1} |C_i|}{\prod_{n=0}^{a-1} |C_i|} = \prod_{n=0}^{a-1} |C_i|.
\]  

(23)

Now, we prove that any two elements of \(A\) belong to different cosets, or equivalently given two elements \(x\) and \(y\) of \(B\), if they belong to the same coset, then \(x = y\). Because of the definition \(\Omega\), if \(x\) and \(y\) are in the same coset, then \(x - y\) is in \(A_s \subseteq q^n Z^n\). This is true if \(\sum_{i=0}^{a-1} q^n c_i - \sum_{i=0}^{a-1} d_i = 0\) which holds only if \(c_i = d_i\) for all \(i\) in the sum, because this in the only way to obtain 0 which is the only element that satisfies the condition \(x - y \subseteq q^n Z^n\). Thus \(x\) and \(y\) are in the same coset only if \(c_i = d_i\) for \(i = \{0, \ldots, a - 1\}\) and \(x - y = q^n (s - t) \in A_s = q^n \Gamma\), or equivalently \(s - t \in \Gamma\).

To conclude the proof, we prove that \(s - t \in \Gamma\). By definition \(\Gamma\), this is true if \((s - t) = T z\) for \(z \in Z^n\). Let \(U = \{u_{i,j}\} = T^{-1}\) be the inverse of \(T\); it is upper triangular and \(u_{i,i} = t_{i,i}^{-1}\) for every \(i\). Also, we can write \(z = U(s - t)\). This relation implies that, if \(s - t \in \Gamma\),

\[
\sum_{j=0}^{n} u_{i,j} (s_j - t_j) \in Z \text{ for every } i.
\]  

(24)

This is a triangular system of equations. Starting at \(i = n\), the condition \(24\) is \(t_{n,1}^{-1} (s_n - t_n) \in Z\), which implies \(s_n - t_n = 0\) because \(t_{ii}\) is an integer by the definition of the Hermite normal form and if \(t_{ii} = 1\), \(s_i = t_i = 0\) because of the definition of the set \(S\). Using the equality \(s_n = t_n\) in the case \(i = n - 1\), we obtain that \(s_{n-1} = t_{n-1}\) too. Moving recursively backwards to \(i = n - 2, n - 3, \ldots, 1\) we conclude that \(s_i = t_i\) for every \(i\), i.e., \(s = t\) and, as wanted, \(x = y\).

Corollary 1: Let \(C \subseteq F_q^n\). Let \(\Gamma \subseteq Z^n\) be any integer lattice. Let \(A_s = q \Gamma \subseteq q \Gamma^n\) and \(A_c = C + q \Gamma^n\) a construction-A lattice. Let the generator matrix in Hermite normal form \(T\) of \(\Gamma\) be as in \(22\). Then a complete set of coset leaders of the quotient group \(A_c/A_s\) is given by

\[
C + q \Gamma = \{c + q s \mid c \in C, s \in S\}
\]  

(25)

Proof: It is sufficient to set \(a = 1\) in \(21\) in order to obtain a construction-A via construction D.

Remark 1: Note that a cubic shaping is obtained when \(T = I_n\). In this case the set \(S\) encodes no information because \(S = \{0\} \times \{0\} \times \cdots \times \{0\}\). Also, in this case we have that

\[
\frac{A_c}{q \Gamma^n} = \{\sum_{n=0}^{a-1} q^n c_i \mid c_i \in C_i\}
\]  

(26)

If we set \(a = 1\) in order to obtain a construction-A lattice, we obtain the relation

\[
\frac{A_c}{q \Gamma^n} = \{c \mid c \in C\}
\]  

(27)

which is easily proved by the isomorphism between \(A_c/q \Gamma^n\) and \(C\).

A. Encoding

This section describes a method to encode message vectors \(m\) to Voronoi constellation points \(x\).

As suggested by Theorem 1, to have a coset leader we need to choose a codeword \(c_i\) of each code \(C_i\) and an element of the set \(S\). Each codeword is obtained by encoding a message vector \(u_i \in F_q^{k_i}\). Then, each message \(m\) is chosen from the set \(\mathcal{M} = F_q^{k_0} \times F_q^{k_1} \times \cdots \times F_q^{k_{a-1}} \times S\). An element of this set is of the form \(m = (u_0, u_1, \ldots, u_{a-1}, s)\), i.e., each message has \(\sum_{i=0}^{a-1} k_i + n\) coordinates.

In summary, the encoding procedure is done as follows:

1) Pick an element of the set \(\mathcal{M}\).
2) Encode each element \(u_i\) in order to obtain \(c_i\).
3) Find the coset leader by computing

\[
x = \sum_{i=0}^{n-1} q^i c_i + q^n s
\]  

(24)

This is a triangular system of equations. Starting at \(i = n\), the condition \(24\) is \(t_{n,1}^{-1} (s_n - t_n) \in Z\), which implies \(s_n - t_n = 0\) because \(t_{ii}\) is an integer by the definition of the Hermite normal form and if \(t_{ii} = 1\), \(s_i = t_i = 0\) because of the definition of the set \(S\). Using the equality \(s_n = t_n\) in the case \(i = n - 1\), we obtain that \(s_{n-1} = t_{n-1}\) too. Moving recursively backwards to \(i = n - 2, n - 3, \ldots, 1\) we conclude that \(s_i = t_i\) for every \(i\), i.e., \(s = t\) and, as wanted, \(x = y\).

Note that our encoding complexity is dominated by the quantization. In this paper we use a sphere decoder algorithm as in [16] to perform this operation, which has polynomial complexity with the dimension [16]. Despite this complexity
we still can perform this encoding procedure because we can create a high-dimensional shaping lattice with direct sums of small-dimensional lattices. These lattices retain the shaping gain of the small-dimension lattices, and the modulo operation is simpler, as it can be performed directly on the small-dimension lattices. [17].

B. Indexing

This section describes the inverse operation of encoding, called indexing. It obtains the message \( m = (u_0, u_1, \ldots, u_{a-1}, s) \) given a Voronoi constellation point \( x' \) obtained as in the last section after the modulo-\( \Lambda_s \) operation. The indexing procedure is done as follows:

1) Denote \( r_j = \sum_{i=0}^{j} q^i c_i \), for \( j = 1, \ldots, a-1 \). So \( r_0 = c_0, r_1 = c_0 + q c_1, \ldots, r_{a-1} = \sum_{i=1}^{a-1} q^i c_i \).

The point \( x' \) can be written as \( x' = r_{a-1} + q^0 s - Q_{\Lambda_s}(x) \), where \( x \) is a coset leader obtained at step 3 of the encoding procedure. Applying \( \text{mod } q \) operation to \( x' \) we obtain \( r_0 = c_0 \) and we obtain \( c_0 \). That is because \( Q_{\Lambda_s}(x) \in \mathbb{Z}^q \) by definition.

Applying \( \text{mod } q \) operation to \( x' \) we obtain \( r_1 = c_0 + q c_1 \), as we already have \( c_0 \). \( c_1 \) is obtained as \( c_1 = r_1 - c_0 \).

Generalizing, let \( r_{-1} = 0 \), starting by \( i = 1 \) moving upwards to \( i = 2, \ldots, a-1 \), any codeword \( c_i \) is obtained as

- First, obtain \( r_i \) by performing \( \text{mod } q^{i+1} \) operation:
  \[
  r_i = x' \mod q^{i+1}
  \]
- Second, obtain \( c_i \) by
  \[
  c_i = \frac{r_i - r_{i-1}}{q^i}
  \]

2) Obtain \( u_i \) by indexing, or decoding, each \( c_i \in C_i \). This step depends on code choice and how it was encoded.

3) Since we know each \( c_i \in C_i \) and \( x' \) we can recover

\[
  v = \frac{x' - r_{a-1}}{q^a} = s - \frac{1}{q^a} Q_{\Lambda_s}(x) = s - p \in \mathbb{Z}^n
\]

Note that \( v \) can be written as \( v = s - \mathbf{T} z \) for some unknown \( z \in \mathbb{Z}^n \) because \( p = \frac{1}{q^a} Q_{\Lambda_s}(x) \in \Gamma \).

4) Finally, use \( v \) and \( \mathbf{T} \) to find \( s \) by performing step 4.5 and 4.6 described in section III-B of [5].

IV. LATTICE DESIGN

In this section we describe the coding and shaping lattices design. The first part of this section describes the construction-D lattice using nested BCH codes proposed in [6]. Other codes for construction D, as turbo lattices, polar code lattices and LDPC code lattices can be found in [7], [8] and [9] respectively.

We also describe how to design the construction-D shaping lattice based on Theorem 1. We also briefly describe a well-known technique to construct a high dimensional shaping lattice with a low dimensional one. [5], [13], [17]. This results in a shaping strategy that retains the low-dimensional lattice shaping gain, while avoiding the complexity of performing high-dimensional sphere decoding.

A. Coding Lattice: Extended BCH Code Lattice

This section describes the design of a two-level nested extended BCH code lattice where each codeword is obtained from a BCH code with an additional parity check bit [18]. Thus, \( a = 2 \) and \( q = 2 \).

We start by fixing the dimension of the BCH code in \( n = 128 \). This allows a practical algorithm implementation of the code and is sufficient to apply and show the shaping scheme proposed in this paper.

Now we select a pair of nested BCH codes which maximizes the minimum distance of the construction-D lattice given in [19]. To that end we set \( \gamma = 1 \), which implies \( d_{\text{min}} = 4 \) and \( d_{\text{min}} = 16 \). Thus, we select \( C_0 \) and \( C_1 \) as \( (128, 78, 16) \) and \( (128, 78, 16) \).

The nested constraint (\( \Lambda_s \subseteq \Lambda_r \)) must also be satisfied. For this purpose, the generator polynomial \( g_i(x) \) of code \( C_i \) must satisfy \( g_j(x) = g(x) g_i(x) \) for \( j < i \), for minimal polynomials of appropriate degrees. In our case with \( a = 2 \) we have \( g_0(x) = g(x) g_1(x) \). Finally, with codewords \( c_0 \in C_0 \) and \( c_1 \in C_1 \) we construct the coding lattice as in [15],

\[
  \Lambda_c = \{ c_0 + 2 \cdot c_1 + 4 \cdot s \},
\]

where \( s \in S \).

B. Shaping Lattice: Gosset (E8) Lattice

In this section we propose the construction of the shaping lattice with a low dimensional lattice: the Gosset Lattice (E8) which has the greatest packing density in dimension 8 not only among lattices but for any packing [11], [13]. Using the continuous approximation [13], this lattice provides nominal shaping gain of 0.65 dB [1].

The Gosset lattice \( E_8 \) can be used to shape any \( n \)-dimensional coding lattice by simply taking the elements of the \( n \)-dimensional vector \( x \) in blocks of eight elements, and performing shaping with \( E_8 \) in each block, as in step 4 of the encoding procedure. The advantage is that quantizing to \( E_8 \) is feasible. The disadvantage is that the shaping gain is limited to eight dimensions.

In fact, it is always possible to obtain an \( n \)-dimensional shaping lattice from an \( n' \)-dimensional shaping lattice, with \( n' < n \), as long as \( l = n/n' \in \mathbb{Z} \). For this purpose, consider the \( n \)-dimensional shaping lattice in theorem 1 and \( \Lambda_q = q^a \Gamma \). We want to construct \( \Gamma \) as the sum of copies of a low-dimensional lattice \( E_{8} ' \) for that to end, the generator matrix of \( \Gamma \) is written as [12],

\[
  \mathbf{T} = \begin{pmatrix}
  \alpha \mathbf{G_s} & 0 & \cdots & 0 \\
  0 & \alpha \mathbf{G_s} & \cdots & 0 \\
  \vdots & \ddots & \ddots & \vdots \\
  0 & \cdots & 0 & \alpha \mathbf{G_s}
  \end{pmatrix} \in \mathbb{Z}^{n \times n}
\]

where \( \mathbf{T} \) is the matrix described in theorem 1 and \( \mathbf{G_s} \) is the low dimensional shaping lattice generator matrix in Column-style Hermite normal form.
Note that \( \mathbf{G}_s \) is scaled by \( \alpha \) for two reasons. First, to make \( \mathbf{T} \) an integer matrix and consequently \( \Gamma \) an integer lattice as required in theorem 1. Second to change the information rate by expanding the shaping region and consequently putting more points inside it.

Since we are using an extended BCH code with dimension \( n = 128 \), this strategy can be used with the Gosset lattice \( E_8 \), since 128 is a multiple of 8. In this case, the shaping lattice is

\[
\Lambda_s = q^0 \Gamma = 4 \Gamma
\]

(30)

Note that \( \Lambda_s \) and \( \Lambda_c \) are nested and satisfy the requirements of theorem 1, because,

\[
\Lambda_s = q^0 \Gamma = 4 \Gamma \subseteq q^0 \mathbb{Z}^n = 4 \mathbb{Z}^n \subseteq \Lambda_c
\]

(31)

With this configuration, we can calculate the information rate \( R \) of our transmission system. Using equation (7), (18) and (30),

\[
M = \frac{\text{Vol}(\Lambda_s)}{\text{Vol}(\Lambda_c)} = \frac{q^n \text{Vol} \Gamma}{q^n s \sum_{i=0}^{n-1} k_i} = \alpha^n (\det \mathbf{G}_s)^i q^{\sum_{i=0}^{n-1} k_i}
\]

(32)

Let the code rate of our code be,

\[
R_c = \frac{\sum_{i=0}^{n-1} k_i}{an}
\]

(33)

Finally, using (10) and (33),

\[
R = \frac{n \log_2 \alpha + t \log_2 (\det \mathbf{G}_s) + \sum_{i=0}^{n-1} k_i \log_2 q}{n} + \log_2 \alpha + \frac{1}{n} \log_2 (\det \mathbf{G}_s) + a R_c \log_2 q \text{ bits/dim}
\]

(34)

In our case, as we use \( a = 2 \) and \( q = 2 \) we have,

\[
R_c = \frac{k_0 + k_1}{2n}
\]

(35)

and,

\[
R = \log_2 \alpha + \frac{1}{n} \log_2 (\det \mathbf{G}_s) + 2R_c \text{ bits/dim}
\]

(36)

V. SIMULATION RESULTS

In this section, we show the simulation results for our proposed lattice constellation, obtained from a construction-D with 2 nested extended BCH code with dimension 128. In the first part, the performance over the power-unconstrained AWGN channel is shown. This measures the performance for a system without any power restriction, so no shaping needs to be employed. In the second part we show the performance over a power-constrained AWGN channel which is obtained when we restrict the points to be transmitted inside the shaping lattice Voronoi region. We compare the performance of two shaping strategies: lattice shaping with copies of the \( E_8 \) lattice, and cubic shaping with the cubic lattice, which is known to have no shaping gain.

After transmission, the received signal \( y \in \mathbb{R}^n \) is a noisy version of \( x \in \Lambda_c \) and can be written as,

\[
y = x + w
\]

(37)

However, before decoding the channel output, we multiply \( y \in \mathbb{R}^n \) by a constant \( c \),

\[
c = \frac{\text{SNR}}{1 + \text{SNR}} = \frac{P}{P + \sigma^2}
\]

(38)

which is known as Wiener coefficient. The use of this factor is important for achieving the capacity using lattices, as explained in \([19]\). Thus, what we actually decode is \( cy \) with \( c \) as in \( (38) \).

In order to find the point \( x' \in \Lambda_c \) closest to \( cy \), soft-input soft-output decoding of binary codes is used. In our case, ordered statistics decoding (OSD) algorithm with order-l reprocessing is used as described in \([20]\). As described in \([6]\), for our construction, order-4 reprocessing for code \( C_0 \) and order-1 reprocessing for code \( C_1 \) yield a performance close to maximum likelihood decoding. However, we use order-3 reprocessing for \( C_0 \) decoding for two reasons, first because it achieves only slightly higher error-rate performance than that of order-4 (see \([6]\)), and second because of the reduced computational complexity.

A. Performance over Power-Unconstrained AWGN Channel

In this section we assume an infinite constellation. This means that we do not have any power constraint in our system and we can transmit any lattice point. An AWGN with noise power \( \sigma^2 \) is added in each transmitted point and OSD-decoding algorithm is used. The unconstrained scenario performance is analysed by the VNR parameter defined in \([12]\) and is a measure of the performance of the coding lattice.

![Figure 1](image)

Figure 1. SER and WER performance of 2-level extended BCH code lattices with dimension \( n = 128 \) over AWGN channel without power constraint.

Figure 1 shows the symbol-error rate (SER) and word-error-rate (WER) as a function of VNR of our construction-D lattice. We define WER as the probability of detecting a lattice point different than the one that was transmitted, and SER is the probability of error in any one of the \( n \) coordinates.

For \( n = 128 \) dimensions, a construction-D lattice with extended BCH code outperforms turbo lattices \([7]\) and LDLC.
lattices [21]. For more results in this type of construction see [6].

B. Performance over Power-Constrained AWGN Channel

In this section we assume a finite constellation, limiting the points to be transmitted with the shaping operation presented in theorem 1. To the best of our knowledge, lattice shaping has never been performed for construction-D lattices.

The average power per dimension can be obtained by the continuous approximation as in (13). From [1], it is known that $\gamma_5(5) = 1.112$ when $S$ is the $E_8$ Voronoi region. As our shaping lattice is as in (30) and $\alpha = 2$, we have,

$$\text{Vol} S^{2/n} = \text{Vol}(4\Gamma)^{2/n} = 4^n (\det T)^{2/n} = (8^n)^{2/n} = 64$$

and the average power is,

$$P(S) = \frac{64}{12 \times 1.112} = 4.7962$$

As mentioned in section I-A the average value of the constellation, $d$, is subtracted from all the symbols before transmission, to ensure minimal energy consumption. We estimated $d$ taking the sample mean of 20,000 randomly generated symbols. The largest absolute value of the elements of this estimate was 0.07231, which is negligible compared to the entries of the lattice constellation. Therefore, $P(d) \approx 0$ in (8) and we ignored this subtraction and transmitted the lattice constellation directly. The resulting average power per dimension, obtained as the average power of the same 20,000 randomly generated symbols, is,

$$P(S) = 4.6382$$

this is the average power per dimension which was used to obtain the $E_8$ performance in figure 2.

Example 2: Information Rate and Average Power per dimension of Hyper-Cube Constellation

A column style Hermite generator matrix for a cubic lattice is $T = I_n$. As before, $R_c = (78 + 120)/(2 \times 128) = 0.77$ is the code rate of our extended 2-level BCH code. In order to have the same information rate as in example 1, we set $\alpha = 2$. Using (36), the code rate is,

$$R = \log_2 2 + \frac{1}{128} \times \log_2 1 + 2 \times 0.77 = 2.54 \text{ bits/dim}$$

The shaping lattice is $\Lambda_s = 8I_n$, yielding hyper-cubing shaping. The shaping operation in this case is trivial. First, we reduce $\mod 8$ every point transmitted, yielding the possible symbols $\{0, 1, 2, 3, 4, 5, 6, 7\}$. In order to put points inside Voronoi region of $\Lambda_s$ we then subtract 4 from every coordinate obtaining the hyper-cube constellation with possible values $\{-4, -3, -2, -1, 0, 1, 2, 3\}$. To have the minimum average power, the average of all the possible transmitted points must be zero. To achieve this, we sum 0.5 to all points (this makes the constellation symmetric) to obtain the zero average hyper-cube constellation, with possible values $\{\pm 0.5, \pm 1.5, \pm 2.5, \pm 3.5\}$. This is equivalent to translating the coding lattice with a vector $d$ as in (8) with all entries equal to 0.5.

Since all coordinates are equally likely in this case, the average power equal is trivial to compute:
\[ P(S) = 5.25 \]

Figure 2 shows the performance of our system when shaping is performed with the lattices in examples 1 and 2. We consider two-dimensional passband transmission, which means that we have a complex Gosset \((E_8)\) constellation and a 64-QAM constellation. In comparison with the hyper-cube shaping, a gain of 0.54 dB is obtained for WER of \(10^{-3}\) when using the \(E_8\) to construct the shaping lattice. This value is close to 0.65 dB which verifies the theoretical shaping gain for \(E_8\) lattice [1].

VI. CONCLUSIONS

We presented a general method to construct Voronoi constellations for any construction-D lattice. This is possible because of theorem 1, which is the main result of the paper, enabling a new way to encode and index with linear complexity in block length. We tested our shaping strategy with a coding lattice built from an extended BCH code with dimension 128. This outperforms LDPC codes and Turbo codes in similar dimension. However, order-1 reprocessing decoding algorithm of BCH code is not linear in block length and this fact limit our BCH dimension to 128. In order to approach capacity, other codes with linear complexity decoding algorithm could be used, e.g. LDPC codes. Those codes, allied with the linear complexity of the proposed encoding and indexing operation, have the potential to approach Shannon’s capacity.

The proposed shaping strategy uses the sphere decoding for shaping. This is computationally feasible for dimensions 32 [16]. The largest achievable shaping gain in dimension 32 is almost 1.2 dB, which is close to 1.53 dB, which is the maximum possible shaping gain in any dimension (Fig. 3 [1]). In other words, using our proposal it is possible to garner most of the benefits of shaping.

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