Partial-wave Coulomb $t$-matrices for like-charged particles at ground-state energy

V. F. Kharchenko

Bogolyubov Institute for Theoretical Physics, National Academy of Sciences of Ukraine, UA - 03143, Kyiv, Ukraine
E-mail: vkharchenko@bitp.kiev.ua

Abstract
We study a special case at which the analytical solution of the Lippmann-Schwinger integral equation for the partial wave two-body Coulomb transition matrix for likely charged particles at negative energy is possible. With the use of the Fock’s method of the stereographic projection of the momentum space onto the four-dimensional unit sphere, the analytical expressions for s-, p- and d-wave partial Coulomb transition matrices for repulsively interacting particles at bound-state energy have been derived.

Keywords: partial wave transition matrix, Coulomb interaction, Lippmann-Schwinger equation, Fock method, analytical solution

1. Introduction

The Coulomb transition matrix ($t$-matrix), being a scalar function of the initial and final relative momenta and the energy, provides all information about the system of two interacting charged particles. The analytic properties of the Coulomb $t$-matrix have been discussed in the review [1]. The availability of bound states for systems with oppositely charged particles leads to the appearance of pole singularities of the corresponding Coulomb $t$-matrix at bound-state energies with residues relating with wave functions of the bound states. In the case of likely charged particles the Coulomb $t$-matrix has no energy poles. The analytic properties of the Coulomb $t$-matrix, which in the case of short-range interaction potentials manifest themselves as a singular branch point with a cut along the positive energy axis and related unitarity conditions on and off the energy shell, are more complicated (see review [1]).

The knowledge of the two-body Coulomb transition matrix with the momenta off the energy shell is especially important when studying properties of few-body atomic and nuclear systems containing charged particles with the use of the Faddeev [2,3] and Faddeev-Yakuboskii [4] integral equations. For such systems, Faddeev equations are known to become non-Fredholm even below the decay threshold. The extraction of the main Coulomb singularity and the regularization of three-body equations in this case were proposed by Veselova [5] with the help of the known Gorshkov procedure for two-body systems [6]. The problem of regularization of the integral equations for four-body systems containing charged particles was considered in work [7]. Earlier information relative to the properties of the two-body off-shell Coulomb transition matrix can be found in [1].

Several representations for two-body Coulomb transition matrix are known in the literature [8-16]. Of special interest is the study of the Coulomb transition matrix with the use of the Coulomb system symmetry in the Fock four-dimensional Euclidean space [17]. Earlier, the Fock method was applied in Bratsev-Trifonov’s[10] and Schwinger’s [12] works in order to derive the Coulomb Green’s function in the one-parameter integral form. Expressions for the three-dimensional Coulomb transition matrix with explicitly singled out transferred momentum and energy singularities were obtained in works [15] (for negative energies, $E < 0$) and [16] (for zero and positive energies, $E \geq 0$).

For the first time, a possibility to derive an analytical expression for total wave two-body
Coulomb transition matrices at the ground bound state energy was examined for oppositely charged particles (with the attractive interaction) in the previous work [18]. In this work, on the basis of the Fock method of stereographic projection of the three-dimensional momentum space onto a four-dimensional unit sphere [17], the form of the partial wave Coulomb transition matrices for a system of two likely charged bodies (with the repulsive Coulomb interaction) is analyzed. The consideration begins in Section 2, where the expression obtained earlier in work [14] for the three-dimensional Coulomb transition matrix at the negative energy is used. In Section 3, a general expression for the off-shell partial wave Coulomb $t$-matrix at the negative energy is derived. Section 4 is devoted to the study of the partial wave Coulomb $t$-matrix at the ground bound state energy, and it is shown that a simple analytical expression for partial wave $t$-matrix can be obtained in this case. Explicit analytical expressions for the $s$-, $p$- and $d$-wave components of the Coulomb $t$-matrix are presented. Final remarks and conclusions are made in Section 5.

2. Three-dimensional Coulomb transition matrix at the negative energy with explicitly separated singularities

The three-dimensional Coulomb transition matrix $\langle k|t(E)|k'\rangle$ satisfies the inhomogeneous Lippmann-Schwinger integral equation

$$<k|t(E)|k'> = \langle k|v|k'\rangle + \int \frac{dk''}{(2\pi)^3} \langle k|v|k''\rangle \frac{1}{E-k''^2} <k''|t(E)|k'> .$$

(1)

Here, the free term $\langle k|v|k'\rangle$ is determined by the Coulomb interaction potential $v(r) = q_1 q_2 / r$, where $q_i$ is the charge of the $i$-th particle ($i = 1, 2$), and $r$ is the distance between particles 1 and 2. In the momentum space, this term looks like

$$\langle k|v|k'\rangle = \frac{4\pi q_1 q_2}{|k-k'|^2} .$$

(2)

where $k$ and $k'$ are relative momenta corresponding to the radius-vectors $r$ and $r'$, respectively, in the coordinate space. The kernel of the integral equation (1) is a product of the operator of Coulomb interaction potential (2) and the free Green operator

$$<k|g_0(E)|k'> = \frac{(2\pi)^3 \delta(k-k')}{E-k'^2} ,$$

(3)

where the quantity $E$ is the total energy of the relative motion of particles 1 and 2, and $\mu = m_1 m_2 / (m_1 + m_2)$ is their reduced mass.

In this work, the consideration is confined to the problem of Coulomb scattering of two likely charged particles off the energy shell in case of negative energy

$$E = -\frac{\hbar^2 \kappa^2}{2\mu} .$$

(4)

The consideration is based on the solution on the integral equation (1) for the off-shell three-dimensional Coulomb transition matrix with the explicitly separated singularities in the variables of the transfer momentum and the energy, which was obtained by us early [14]:
The function $c$ is the dimensionless Coulomb parameter, and $\hbar$ is the reduced Planck’s constant. The variable $\omega$ in Eq.(5) stands for the angle between two 4-dimensional unit vectors $e \equiv (e, e_0)$ and $e' \equiv (e', e'_0)$ in the four-dimensional Euclidean space introduced by Fock [17]:

$$e = \frac{2\kappa k}{\kappa^2 + k^2}, \quad e_0 = \frac{\kappa^2 - k^2}{\kappa^2 + k^2}, \quad e' = \frac{2\kappa k'}{\kappa^2 + k'^2}, \quad e'_0 = \frac{\kappa^2 - k'^2}{\kappa^2 + k'^2};$$

$$\cos \theta' = e \cdot e' + e_0 \cdot e'_0. \quad (8)$$

The three-dimensional vectors $k$ and $k'$ lie in a hyperplane, which is a stereographic projection of a sphere with the unit radius. The variable $\omega$ is determined by the relation

$$\sin^2 \frac{\omega}{2} = \frac{\kappa^2 |k - k'|^2}{(\kappa^2 + k^2)(\kappa^2 + k'^2)}, \quad 0 \leq \omega \leq \pi. \quad (9)$$

The function $c(\gamma)$ in Eq.(5) looks like

$$c(\gamma) = \frac{1}{2} \left( 1 - \frac{1}{\pi} \int_0^\pi d\varphi \sin \gamma \varphi \csc \frac{\varphi}{2} \right) \quad (10)$$

or in terms of the digamma functions

$$c(\gamma) = \theta(-\gamma) + \frac{\sin \gamma \pi}{2\pi} \left[ \psi \left( \frac{|\gamma| + 1}{2} \right) - \psi \left( \frac{|\gamma|}{2} \right) - \frac{1}{|\gamma|} \right], \quad (11)$$

where $\psi(x) \equiv d/dx \ln \Gamma(x)$ and $\Gamma(x)$ are the digamma- and gamma-functions [18], and $\theta(x)$ is the Heaviside step function,

$$\theta(x) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x < 0. \end{cases}$$

The first three terms in the square brackets in Eq.(5) contain transferred momentum singularities

$$|k - k'|^{-2}, \quad |k - k'|^{-1} \quad \text{and} \quad \ln \left\{ \kappa |k - k'| / (\kappa^2 + k^2)^{1/2}(\kappa^2 + k'^2)^{1/2} \right\},$$

respectively. The other three terms in Eq.(5) are smooth functions of $|k - k'|$.

The fourth term in expression (5) contains singularities in the energy. They arise only in the case of attractive Coulomb potential (with opposite electric charges, $q_1 q_2 < 0$), when the Coulomb parameter $\gamma$ accepts negative integer values corresponding to the spectrum of bound states of a two-particle system with the energies $E = E_n$,

$$E_n = -\frac{\mu(q_1 q_2)^2}{2\hbar^2 n^2}, \quad n = 1, 2, 3, \ldots.$$
According to EQS. (4) and (6), the corresponding values of the parameter $\kappa$ and the Coulomb parameter $\gamma$ are equal to

$$\kappa_n = \sqrt{-\frac{2\mu E_n}{\hbar^2}} = \mu \frac{|q_1 q_2|}{\hbar^2 n}, \quad \gamma_n = \frac{\mu q_1 q_2}{\hbar^2 \kappa_n} = \frac{|q_1 q_2|}{\hbar^2 n},$$

(13)

respectively. At these points $\gamma = \gamma_n = -n$, so that the function $\cot \gamma \pi$ has pole singularities, and the function $c(\gamma)$ differs from zero: $c(-n) = 1$.

In this case of repulsive Coulomb potential ($\gamma < 0$), the expression for $c(\gamma)$ equals zero at positive values of $\gamma$, $c(n) = 0$, and the fourth term in Eq. (5) is finite and equal to

$$\rho(\gamma) \equiv \frac{2\pi c(\gamma)}{\tan \gamma \pi} \bigg|_{\gamma \to -n} = \rho_n,$$

(14)

where

$$\rho_n = 2nc'(n), \quad c'(n) = -\frac{1}{2\pi} \int_0^\pi d\varphi \cos n \varphi \cot \frac{\varphi}{2}$$

(15)

or, using the function $\beta(x) = \frac{1}{2} \left[ \psi \left( \frac{x + 1}{2} \right) - \psi \left( \frac{x}{2} \right) \right]$, \n
$$\rho_n = (-1)^n \left[ 2n \beta(n) - 1 \right].$$

(16)

The ultimate expression for $\rho_n$ takes the form

$$\rho_n = (-1)^n - 2n \ln 2 - 2n \sum_{m=1}^{n} \frac{(-1)^m}{m} n.$$  \hspace{1cm} (17)

3. Partial wave component of the Coulomb transition matrix at negative energy

Using the partial wave method and expanding the matrix elements of the Coulomb potential and the transition matrix with a negative energy in series in Legendre polynomials $P_l(x)$,

$$\langle k | v | k' \rangle = \sum_{l=0}^{\infty} (2l + 1) v_l(k, k') P_l(\hat{k} \cdot \hat{k}'),$$

$$\langle k | t(E) | k' \rangle = \sum_{l=0}^{\infty} (2l + 1) t_l(k, k'; E) P_l(\hat{k} \cdot \hat{k}'),$$

(18)

where $\hat{k}$ is a unit vector along the vector $k$, and $\hat{k} \cdot \hat{k}' = \cos \theta$, the one-dimensional integral equation for the partial wave component of the transition matrix can be written in the form

$$t_l(k, k'; E) = v_l(k, k') + \int_0^{\infty} \frac{dk'' k''^2}{2\pi^2} v_l(k, k'') \frac{1}{E - \frac{k''^2}{2\mu}} t_l(k'', k'; E).$$

(19)

The inhomogeneous and the kernel of this equation contain a partial wave component of the Coulomb interaction potential

$$v_l(k, k') = \frac{1}{2} \int_0^{\pi} d\theta \sin \theta P_l(\cos \theta) \langle k | v | k' \rangle.$$

(20)
According to definition (18), the partial wave component of the Coulomb transition matrix \( t_l(k, k'; E) \) equals
\[
t_l(k, k'; E) = \frac{1}{2} \int_0^\pi d\theta \sin \theta \, P_l(\cos \theta) \, \langle k \mid t(E) \mid k' \rangle.
\] (21)

Taking into account that expression (5) for the three-dimensional Coulomb transition matrix \( \langle k \mid t(E) \mid k' \rangle \) depends on the angle \( \omega \) between the unit vectors \( e \) and \( e' \) in the four-dimensional Fock space, it is convenient to go in Eq. (21) from the integration over the variables \( k \) describing the relationship between the angles \( \theta \) and \( \omega \). From expression (9) describing the relationship between the angles \( \theta \) and \( \omega \), it follows that
\[
\cos \theta = \frac{\xi}{\eta} - \frac{1}{\eta} \sin^2 \omega = \frac{2\xi - 1 + \cos \omega}{2\eta}, \quad \sin \theta \, d\theta = \frac{1}{2\eta} \sin \omega \, d\omega,
\] (22)
where
\[
\xi = \frac{\kappa^2(k^2 + k'^2)}{(k^2 + \kappa^2)(k'^2 + \kappa^2)} \quad \eta = \frac{2k^2k'k'}{(k^2 + \kappa^2)(k'^2 + \kappa^2)}.
\] (23)

Then the formula (21) can be rewritten as
\[
t_l(k, k'; E) = \frac{1}{4\eta} \int_{\omega_0}^{\omega_\pi} d\omega \sin \omega \, P_l\left(\frac{2\xi - 1 + \cos \omega}{2\eta}\right) \langle k \mid t(E) \mid k' \rangle.
\] (24)

The integration limits in Eq. (24) are determined by the expressions
\[
\omega_0 = 2 \arcsin \sqrt{\frac{\xi - \eta}{\eta}}, \quad \omega_\pi = 2 \arcsin \sqrt{\frac{\xi + \eta}{\eta}},
\] (25)
so that
\[
\cos \omega_0 = 1 - 2\xi + 2\eta, \quad \cos \omega_\pi = 1 - 2\xi - 2\eta, \quad \sin \omega_0 = 2\sqrt{\xi - \eta} \sqrt{1 - \xi + \eta}, \quad \sin \omega_\pi = 2\sqrt{\xi + \eta} \sqrt{1 - \xi - \eta}.
\] (26)

Substituting expression (5) for the three-dimensional Coulomb transition matrix in Eq. (24), we obtain the following formula for the partial wave Coulomb transition matrix \( t_l(k, k'; E) \) at \( E < 0 \):
\[
t_l(k, k'; E) = \frac{\pi q_1 q_2}{k \kappa^2} \int_{\omega_0}^{\omega_\pi} d\omega \, P_l\left(\frac{2\xi - 1 + \cos \omega}{2\eta}\right) \left\{ \cot \frac{\omega}{2} - \pi \gamma \cos \gamma \omega - \gamma \sin \gamma \omega \ln(\sin \frac{\omega}{2}) + 2\pi \gamma c(\gamma) \cot \gamma \pi \sin \gamma \omega \right. \]
\[
\left. + \gamma \cos \gamma \omega x_\gamma(\omega) + 2\gamma^2 \sin \gamma \omega y_\gamma(\omega) \right\},
\] (27)
where
\[
x_\gamma(\omega) = \int_0^\omega d\varphi \sin \gamma \varphi \cot \frac{\varphi}{2}, \quad y_\gamma(\omega) = \int_0^\pi d\varphi \sin \gamma \varphi \ln \left(\sin \frac{\varphi}{2}\right).
\] (28)

The partial wave Coulomb transition matrix \( t_l(k, k'; E) \) is a function of three independent variables \( k, k' \) and \( E \). The quantities \( \xi \) and \( \eta \), as well as the integration limits \( \omega_0 \) and \( \omega_\pi \) in expression (27), also depend on those variables. The quantity \( \kappa \) is connected with the energy \( E \) by the formula (4). By definition (6), the Coulomb parameter \( \gamma \) in expression (27) depends on \( \kappa \) and therefore on the energy \( E \). Note that in expression (27) for the \( l \)-matrix, the Coulomb interaction intensity \( q_1 q_2 \) is contained both in the factor before the integral and owing to the Coulomb parameter \( \gamma \) [see Eq.(6)] in the terms in the curly braces in the integrand.
4. Partial wave Coulomb transition matrices for likely charged particles at the ground bound state energy

The expression (27) for the Coulomb transition matrix contains the double integration over \( \varphi \) and \( \omega \) and is rather complicated. It is easy to see that for separate values of the Coulomb parameter \( \gamma \) (which correspond to certain energy values \( E \)) the integration over \( \varphi \) and \( \omega \) in Eq.(27) can be performed explicitly. In such cases, simple analytical expressions for the partial wave Coulomb \( t \)-matrix can be obtained.

In particular, the integration in the expressions for \( x_\gamma(\omega) \) and \( y_\gamma(\omega) \) (28) are simplified for integer values of the Coulomb parameter \( \gamma \) (Eq. (12)) corresponding to the energy spectrum of bound states of the two-particle system [Eq. (12)] with the energies \( E = E_n \).

Let us consider the form of the of-shell partial wave transition matrix for a repulsive Coulomb potential of interaction between likely charged particles \( (q_1 q_2 > 0) \) at the ground bound state energy \( E = E_n \). In this case, the Coulomb parameter, determined by the expression (13) with \( n = 1 \), is equal to

\[
\gamma = \gamma_1 = 1 ,
\]

the fourth term in the curly braces in Eq.(27), according to (14), is simplified to \( \rho_1 \sin \omega \), and the integration in the fifth and sixth terms is carried out as follows

\[
x_1(\omega) = \int_0^\omega d\varphi \sin \varphi \cot \frac{\varphi}{2} = \omega + \sin \omega ,
\]

\[
y_1(\omega) = \int_0^\omega d\varphi \sin \varphi \ln \left( \sin \frac{\varphi}{2} \right) = \cos^2 \frac{\omega}{2} - 2 \sin^2 \frac{\omega}{2} \ln \left( \sin \frac{\omega}{2} \right) .
\]

As a result, the formula (27) for the partial wave Coulomb transition matrices (with \( l = 0, 1, 2, ... \)) in the case of the repulsive interaction at \( \gamma = 1 \) (which corresponds to \( E = E_1 \)) takes the form

\[
t_l^0(k, k'; -b_1) = \frac{\pi q_1 q_2}{kk'} \int_{\omega_01}^{\omega_1} d\omega \ P_l \left( \frac{2\xi_1 - 1 + \cos \omega}{2\eta_1} \right) \cdot \left\{ \cot \frac{\omega}{2} - \pi \cos \omega + \omega \cos \omega + (\rho_1 - 1) \sin \omega - 2 \sin \omega \ln \left( \sin \frac{\omega}{2} \right) \right\} ,
\]

where, in accordance with Eq. (7),

\[
\rho_1 = 1 - 2 \ln 2 .
\]

The quantities \( \xi_1, \eta_1, \omega_{01} \) and \( \omega_{\pi 1} \) in Eq.(31) are determined by the expressions for \( \xi, \eta, \omega_0 \) and \( \omega_\pi \), in accordance with their definitions (23) and (25), taken at the point \( \kappa = \kappa_1 \),

\[
\xi_1 = \frac{\kappa_1^2 (k^2 + k'^2)}{(k^2 + \kappa_1^2)(k'^2 + \kappa_1^2)} , \quad \eta_1 = \frac{2\kappa_1^2 kk'}{(k^2 + \kappa_1^2)(k'^2 + \kappa_1^2)} ,
\]

\[
\omega_{01} = 2 \arcsin \sqrt{\xi_1 - \eta_1} , \quad \omega_{\pi 1} = 2 \arcsin \sqrt{\xi_1 + \eta_1} ,
\]

Note that the first term in the braces in the general expression (31 for the partial wave Coulomb transition matrix corresponds to the Born approximation:

\[
t_l^{\text{Born}}(k, k'; -b_1) = v_l(k, k') = \frac{2\pi q_1 q_2}{kk'} Q_l \left( \frac{k^2 + k'^2}{2kk'} \right) ,
\]

where \( Q_l(x) \) is the Legendre function of the second kind [18]

\[
Q_l(x) = \frac{1}{2} P_l(x) \ln \left( \frac{x + 1}{x - 1} \right) - W_{l-1}(x) ,
\]
The fourth term in (31), which contains \( \sin \omega \), because of the orthogonality of the Legendre polynomials

\[
\int_{-\pi}^{\pi} d\omega \sin \omega P_l \left( \frac{2\xi_1 - 1 + \cos \omega}{2\eta_1} \right) = 2\eta_1 \int_{0}^{\pi} d\theta \sin \theta P_l(\cos \theta) = 4\eta_1 \delta_{l0} ,
\]

contributes only to the partial s-wave Coulomb \( t \)-matrix.

Integrating over \( \omega \) in the expression (31) in the simplest case with \( l = 0 \), we obtain the following formula for the partial s-wave Coulomb transition matrix for two likely charged particles \( (q_1 q_2 > 0) \):

\[
t'_0(k, k'; E_1) = \frac{\pi q_1 q_2}{kk'} \left\{ 4(\rho_1 - 1)\eta_1 - (2\xi_1 - 1) \ln \left( \frac{\xi_1 + \eta_1}{\xi_1 - \eta_1} \right) - 2\eta_1 \ln \left( \xi^2_1 - \eta^2_1 \right) 
\right.
\]

\[
- \left[ (\pi - \omega_{\pi1}) \sin \omega_{\pi1} - (\pi - \omega_{01}) \sin \omega_{01} \right] .
\]

Note, that in the case of the attractive Coulomb interaction \( (q_1 q_2 < 0) \) the corresponding partial s-wave Coulomb transition matrix, containing \( \cot \gamma \pi \), has the pole singularity at the energy \( E = E_1 \).

Analogously, integrating over \( \omega \) in the expression (31) with \( l = 1 \) and with \( l = 2 \), we obtain the following formulas for the partial p- and d-wave Coulomb transition matrices in the case of the repulsive Coulomb interaction \( (q_1 q_2 > 0) \):

\[
t'_1(k, k'; -b_1) = \frac{\pi q_1 q_2}{kk'} \left\{ -1 - \frac{1}{\eta_1} \left( \xi^2_1 - \xi_1 - \eta^2_1 \right) \ln \left( \frac{\xi_1 + \eta_1}{\xi_1 - \eta_1} \right) 
\right.
\]

\[
+ \frac{1}{8} \left( \omega_{\pi1} - \omega_{01} \right) (2\pi - \omega_{\pi1} - \omega_{01}) 
\]

\[
+ \frac{1}{2} (2\xi_1 - 1) \left[ (\pi - \omega_{\pi1}) \sin \omega_{\pi1} - (\pi - \omega_{01}) \sin \omega_{01} \right]
\]

\[
+ \frac{1}{8} \left[ (\pi - \omega_{\pi1}) \sin 2\omega_{\pi1} - (\pi - \omega_{01}) \sin 2\omega_{01} \right] \}
\]

\[
t'_2(k, k'; -b_1) = \frac{\pi q_1 q_2}{kk'} \left\{ \frac{1}{\eta_1} \left( \xi_1 + \frac{3}{2} \right) - \frac{1}{\eta_1} \left[ \left( \xi^2_1 - \frac{3}{2} \xi_1 - \eta^2_1 + \frac{1}{2} \eta_1 \right) \ln \left( \frac{\xi_1 + \eta_1}{\xi_1 - \eta_1} \right) 
\right.
\right.
\]

\[
+ \frac{3}{16} (2\xi_1 - 1) \left( \omega_{\pi1} - \omega_{01} \right) (2\pi - \omega_{\pi1} - \omega_{01}) 
\]

\[
+ \left( \frac{3}{2} \xi_1 - \frac{3}{2} \xi_1 - \frac{1}{2} \eta_1 + \frac{21}{32} \right) \left[ (\pi - \omega_{\pi1}) \sin \omega_{\pi1} - (\pi - \omega_{01}) \sin \omega_{01} \right]
\]

\[
+ \frac{3}{16} (2\xi_1 - 1) \left[ (\pi - \omega_{\pi1}) \sin 2\omega_{\pi1} - (\pi - \omega_{01}) \sin 2\omega_{01} \right]
\]

\[
+ \frac{1}{32} \left[ (\pi - \omega_{\pi1}) \sin 3\omega_{\pi1} - (\pi - \omega_{01}) \sin 3\omega_{01} \right] \}
\]

(39)

Taking into account the relations

\[
\cos \omega_{\pi1} + \cos \omega_{01} = -2(2\xi_1 - 1) , \quad \cos \omega_{\pi1} - \cos \omega_{01} = -4\eta_1 ,
\]

which follow from Eq.(26), and the relations

\[
(\pi - \omega_{\pi1}) \sin 2\omega_{\pi1} - (\pi - \omega_{01}) \sin 2\omega_{01} = 2(2\xi_1 - 1)A_+ - 4\eta_1 A_+ ,
\]

where

\[
A_+ = \frac{1}{2} \left( \frac{1}{\eta_1} \left( \xi_1 + \frac{3}{2} \right) - \frac{1}{\eta_1} \left( \xi^2_1 - \frac{3}{2} \xi_1 - \eta^2_1 + \frac{1}{2} \eta_1 \right) \ln \left( \frac{\xi_1 + \eta_1}{\xi_1 - \eta_1} \right) + \frac{3}{16} (2\xi_1 - 1) \left( \omega_{\pi1} - \omega_{01} \right) (2\pi - \omega_{\pi1} - \omega_{01}) 
\]

\[
+ \left( \frac{3}{2} \xi_1 - \frac{3}{2} \xi_1 - \frac{1}{2} \eta_1 + \frac{21}{32} \right) \right] (\pi - \omega_{\pi1}) \sin \omega_{\pi1} - (\pi - \omega_{01}) \sin \omega_{01} \}
\]

\[
+ \frac{3}{16} (2\xi_1 - 1) \left[ (\pi - \omega_{\pi1}) \sin 2\omega_{\pi1} - (\pi - \omega_{01}) \sin 2\omega_{01} \right] + \frac{1}{32} \left[ (\pi - \omega_{\pi1}) \sin 3\omega_{\pi1} - (\pi - \omega_{01}) \sin 3\omega_{01} \right] \}
\]
\[ (\pi - \omega_{01}) \sin 3\omega_{01} - (\pi - \omega_{01}) \sin 3\omega_{01} = \left[ 4(2\xi_1 - 1)^2 + 16\eta_1^2 \right] A_- + 16(2\xi_1 - 1)\eta_1 A_+ , \]

where \[ A_\pm \equiv (\pi - \omega_{01}) \sin \omega_{01} \pm (\pi - \omega_{01}) \sin \omega_{01} , \]

the formulas for partial p- and d-wave component \[ t_1(k, k'; -b_1) = \frac{\pi q_1 q_2}{k k'} \left\{ -1 - \frac{1}{\eta_1} \left[ (\xi_1^2 - \xi_1 - \eta_1^2) \ln \left( \frac{\xi_1 + \eta_1}{\xi_1 - \eta_1} \right) \right. \right. \]
\[ + \frac{1}{8} (\omega_{01} - \omega_{01}) (2\pi - \omega_{01} - \omega_{01}) - \frac{1}{4} [(\pi - \omega_{01}) \sin \omega_{01}] \left\} \right. \]
\[ t_2(k, k'; -b_1) = \frac{\pi q_1 q_2}{k k'} \left\{ -1 - \frac{1}{\eta_1} \left[ (\xi_1^2 - \xi_1 + \eta_1^2) \ln \left( \frac{\xi_1 + \eta_1}{\xi_1 - \eta_1} \right) \right. \right. \]
\[ + \frac{1}{8} (\omega_{01} - \omega_{01}) (2\pi - \omega_{01} - \omega_{01}) - \frac{1}{4} [(\pi - \omega_{01}) \sin \omega_{01}] \left\} \right. \]

For comparison, we present the formulas for the corresponding partial p- and d-wave Coulomb transition matrices, which were obtained in the case of attractive interaction \[ (q_1 q_2 < 0) \] [18]:
\[ t_1^d(k, k'; -b_1) = \frac{\pi q_1 q_2}{k k'} \left\{ 4\xi_1 - \frac{3}{\eta_1} \left[ (\xi_1^2 - \xi_1 - \eta_1^2) \ln \left( \frac{\xi_1 + \eta_1}{\xi_1 - \eta_1} \right) \right. \right. \]
\[ - \frac{1}{8} (\omega_{01} - \omega_{01}) (2\pi - \omega_{01} - \omega_{01}) - \frac{1}{4} [(\pi - \omega_{01}) \sin \omega_{01}] \left\} \right. \]
\[ t_2^d(k, k'; -b_1) = \frac{\pi q_1 q_2}{k k'} \left\{ \frac{1}{\eta_1} \left[ (4\xi_1^2 - 5\xi_1 - \frac{8}{3}\eta_1^2 + \frac{3}{2}) \right. \right. \]
\[ - \frac{3}{16} (\omega_{01} - \omega_{01}) (2\pi - \omega_{01} - \omega_{01}) - \frac{1}{4} [(\pi - \omega_{01}) \sin \omega_{01}] \left\} \right. \]

Taking the sign difference for \[ q_1 q_2 \] in the coefficients before the braces in Eqs. (40), (41) and Eqs. (42), (43), we conclude that the expressions for the corresponding partial wave transition matrices in the cases of attractive and repulsive Coulomb interaction differ only in their first terms and in the signs in front of the second terms. Other corresponding terms do not differ among themselves.

5. Discussion and conclusion

The off-shell Coulomb transition matrix is directly connected with the Coulomb Green’s function and includes all information about the system of interacting particles. In the previous
work [18], a possibility to derive an analytical expression for the off-shell Coulomb transition matrix for two particles with the use of the Fock method of stereographic projection of the momentum space onto a four-dimensional unit sphere was studied. In the case of the attractive Coulomb between opposite charges \((q_1 q_2 < 0)\), simple analytical expressions for the partial p-, d- and f-wave transition matrices at the ground bound state energy, i.e. \(E = E_1\), \(t_l^0(k, k'; E_1)\) with \(l = 1, 2\) and 3.

Note that the knowledge of the partial wave Coulomb transition matrix \(t_3(k, k'; E_n)\), the bound state wave function and its derivatives is necessary, in particular, when determining the electric \(2\lambda\)-pole polarizability \(\alpha_\lambda\) \((\lambda = 1, 2, 3, \ldots)\) of a two-particle Coulomb bound system in the state with the energy \(E = E_n\) [20].

In this work, the Fock method is applied in order to derive the partial wave Coulomb transition matrix in the case of repulsive Coulomb interaction (likely charged particles, \(q_1 q_2 > 0\)) at the energy \(E = E_1\). Rather simple analytical expressions are obtained for the partial s-, p- and d-wave transition matrices at the ground bound state energy, i.e. \(t_l^0(k, k'; E_1)\) with \(l = 0, 1\) and 2 [formulas (37), (40) and (41), respectively].

It is of interest, that in the case of particles with likely charges, for which bound states do not exist at all, the simplification of expressions for the partial wave Coulomb \(t\)-matrices takes place at the energies of the discrete spectrum of bound states for oppositely charged particles.

It should be pointed out that the possibility to have a simple analytical form for the partial wave Coulomb \(t\)-matrix is associated with a possibility to carry out analytically the integration over \(\varphi\) and \(\omega\) in the expressions \((28)\) for \(x_\gamma(\omega)\) and \(y_\gamma(\omega)\) and in the expression \((27)\) for \(t_1(k, k'; E)\). In particular, such integration can be done at the energy values that are equal to the energies of the ground and excited bound states in the discrete spectrum \(E_n\), \(n = 1, 2, 3, \ldots\) [the formula \((12)\)]. The procedure can be realized for the partial wave Coulomb matrices \(t_l^0(k, k'; E_n)\) that describe a system with repulsive forces (with likely charged particles, \(q_1 q_2 > 0\)) at all values of \(n\) and \(l\). Analytical expressions for the Coulomb transition matrices \(t_l^0(k, k'; E)\) describing a system with attractive forces (with oppositely charged particles, \(q_1 q_2 < 0\)) can be obtained only at the values \(n\) and \(l\) that do not realize bound states at which the corresponding transition matrix has pole singularity (at each value of \(n\) and the values of the orbital momentum with \(l \leq n - 1\)). The pole singularities arise, for instance, in the partial wave Coulomb transition matrix \(t_0^1(k, k'; E)\) at \(E = E_1\), in the matrices \(t_0^1(k, k'; E)\) and \(t_0^2(k, k'; E)\) at \(E = E_2\), and so forth.

Note that the partial wave Coulomb transition matrix \((27)\) takes the simple analytical form not only at the energy values corresponding to the discrete spectrum of bound states \((12)\) that is equivalently, in accordance with \((13)\), to integer values of the Coulomb parameter \((6)\). A similar simplification can also be obtained for the Coulomb parameter value \(\gamma = \frac{1}{2}\), which is equivalent to the negative energy \(E = 4E_1\).

The present work was partially supported by the National Academy of Sciences of Ukraine (project No. 0112U000054) and by the Program of Fundamental Research of the Department of Physics and Astronomy of NASU (project No. 0112000056).

1. J.C.Y.Chen and A.C.Chen, in Advances in Atomic and Molecular Physics 8, edited by D.B.Bates and I.Estermann (Academic Press, New York, London, 1972) p.p. 71-129.
2. L.D.Faddeev, Sov. Phys. JETP 12(1961)1014-1019.
3. L.D.Faddeev, Mathematical Aspects of the Three-Body Problem in the Quantum Scattering Theory. Isr. Program Sci. Transl., Jerusalem, 1965.
4. O.A.Yakubovsky Sov. J. Nucl. Phys. 5(1967)937-942.
5. A.M.Veselova, Teor. Mat. Fiz. 3(1970)326-331.
6. V.G.Gorshkov, Zh. Eksp. Teor. Fiz. 40(1961)1481-1490.
7. G.Ya.Beil'kin, Vest. Leningrad. Univ. No.13(1978)72.
8. S.Okubo and D.Feldman, Phys. Rev. 117(1960)292-306.
9. E.H.Wichmann and C.H.Woo, J. Math. Phys. 2(1961)178-181.
10. V.F.Bratsev and E.D.Trifonov, Vest. Leningrad. Univ. No.16(1962)36-39.
11. L.Hostler, J. Math. Phys. 5(1964)1235-1240.
12. J.Schwinger, J. Math. Phys. 5(1964)1606-1608.
13. A.M.Perelomov and V.S.Popov, Sov. Phys. JETP 23(1966)118-134.
14. S.A.Shadchin and V.F.Kharchenko, J. Phys. B: At. Mol. Phys. 16(1983)1319-1322.
15. S.A.Storozhenko and S.A.Shadchin, Teor. Mat. Fiz. 76(1988)339-349.
16. H.vanHaeringen, J. Math. Phys. 25(1984)3001-3032.
17. V.A.Fock, Z. Phys. 98(1935)145-154.
18. V.F.Kharchenko, Annals of Physics 374(2016)16-26.
19. I.S.Gradstein, I.M.Ryzhik, Tables of Integrals, Series, and Products (Academic,1980).
20. V.F.Kharchenko, J. Mod. Phys. 4(2013)99-107.