A Precautionary Loss-based Bayesian Algorithm for Zero Failure Analysis of Exponential Model

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Abstract. The aim of this paper is to study the Bayesian reliability analysis of the exponential distribution under a precautionary loss function based on zero failure data. In the Bayesian algorithm, the prior distribution of the parameter is assumed to be Gamma distribution. Bayesian and E-Bayesian estimators of the failure rate and reliability of exponential model are obtained under the precautionary loss function. Finally, a practical example is utilized to validate the effectiveness and robustness of the proposed estimators.

1. Introduction
In the reliability test, all kinds of truncated data are often obtained. In the time truncation test, the "no failure data" is often encountered, especially in the high reliability, it is easier to produce zero-failure data. That is, no product failure was observed at the specified time. Research on reliability of zero failure data is a hot topic in recent years. For the reliability analysis of lifetime model with zero failure data, Fu and Zhang [1] put forward a method of reliability analysis for Weibull distribution, which can make a high confidence assessment for the reliability of product with time-truncated zero failure data. Han [2] introduced the E-Bayesian estimation method to estimate the reliability of Binomial distribution. Xu and Chen [3] studied the interval estimation of failure rate and reliability for exponential distribution using modified Bayesian (M-Bayesian) credible limit method. Jiang et al. [4] introduced a estimating method to construct the interval of failure probability for the Weibull distribution by using the concavity or convexity property of the density function. Jia et al. [5] used an extension of match distribution curve method to compute the point and confidence interval estimators for parameters of Weibull model. More references about statistical analysis about zero-failure data can be found in [6-8].

The study and applications of exponential model have penetrated into various fields, such as reliability analysis, quality control. In the case of zero-failure data, there are some references studied the statistical inference of failure rate and reliability of exponential. But the existing Bayesian reliability analyses for exponential distribution are mainly discussed under the square error loss function. Until recently, some Bayesian estimation of failure rate and reliability under other loss functions such as LINEX loss, entropy loss and bounded loss functions are considered [9,10]. Precaution loss function is a useful loss function, firstly introduced by Norstrom in 1996, which has the following mathematical expression (Norstrom [11]):
Where $\hat{l}$ is an estimator of $l$. Precautionary loss function $L(\hat{l}, \lambda)$ has been used in Bayesian inference for various reliability distributions, such as exponential distribution, Rayleigh distribution and Weibull distribution [12, 13].

Assume that $X$ is the life of a product distributed with exponential distribution whose density function is

$$f(t) = \lambda e^{-\lambda t}, t > 0$$

(2)

Here $\lambda > 0$ is the unknown parameter and it is often called the failure rate. The parameter $R = \exp(-\lambda t)$ is often called the reliability parameter. Assume that there is a time truncation test carried out on the product with an exponential distribution (2), $t_i$ ($i = 1, 2, \ldots, m$) is the censored times under the type-I censored life test, $t_i$ ($i = 1, 2, \ldots, m$) satisfies the condition $t_1 < t_2 < \ldots < t_m$ and $n_i$ is the corresponding sample size of time $t_i$. If there is no failure events in these samples occurring at all during the whole testing process, that is, all the $n_i$ samples have lifetime longer than the corresponding test time $t_i$ for all $i = 1, 2, \ldots, m$. Then we often call such data set $(t_i, n_i), i = 1, 2, \ldots, m$ the zero-failure data set.

This article will study Bayesian and E-Bayesian estimation of failure rate $l$ and reliability $R = \exp(-\lambda t)$ for exponential distribution under the precautionary loss function defined in (1) in case of zero-failure data.

This article is organized as follows: Section 2 introduces some preliminary knowledge about precautionary loss function and prior distribution. Section 3 studies Bayesian and E-Bayesian estimation of failure rate and reliability of exponential distribution. Section 4 discusses the performance of various estimators by a practical example. Finally, conclusions are given in Section 5.

2. Preliminary Knowledge

In the following discussion, we always assume that the $i$-th time truncation test, a total of $X_i$ samples fail, known by the literature [14], $X_i$ is a random variable distributed with the Poisson distribution with the following mathematical expression

$$P_i(X_i = r_i) = \frac{(n_i t_i \lambda)^{r_i}}{(r_i)!} \exp(-n_i t_i \lambda)$$

(3)

Here $r_i = 0, 1, 2, \ldots, n_i, i = 1, 2, \ldots, m$.

Then the likelihood function of parameter $\lambda$ is

$$L(x | \lambda) = \prod_{i=1}^{m} P_i(X_i = r_i)$$

$$= \prod_{i=1}^{m} \frac{(n_i t_i \lambda)^{r_i}}{(r_i)!} \exp(-N \lambda)$$

(4)

where $N = \sum_{i=1}^{m} n_i t_i$. Note that when $r_i = 0$ ($i = 1, 2, \ldots, m$), i.e. in the case of zero-failure data, we have

$$L(0 | \lambda) = \exp(-N \lambda)$$

(5)

Then equation (5) is the likelihood function of $\lambda$ based on zero failure sample data.

**Lemma** Let $\hat{l}$ be an estimator of $l$ and $\pi(\lambda)$ is a prior distribution of $\lambda$. Then under the precautionary loss function (1), the Bayesian estimator of $\lambda$ is given by

$$\hat{\lambda}_B = [E(\lambda^2 | X)]^{0.5}.$$
Provided that \( E(\lambda^2 \mid X) \) exist, and is finite.

**Proof.** Under the precautionary loss function (1), Bayesian risk of the estimator \( \hat{\lambda} \) is
\[
r(\hat{\lambda}) = E[L(\hat{\lambda}, \lambda) \mid X]).
\]
To minimize \( r(\hat{\lambda}) \), we only need
\[
f(\hat{\lambda}) = E(L(\hat{\lambda}, \lambda) \mid X)
\]
almost obtained minimum. Because
\[
f(\hat{\lambda}) = E(\hat{\lambda}^2 - \frac{\hat{\lambda}^2}{\lambda} + \frac{1}{\lambda^2}) \mid X)
\]
Then, the first order derivative of \( f(\lambda) \) is
\[
f'(\lambda) = 1 - \frac{1}{\lambda} E[\lambda^2 \mid X],
\]
Then
\[
f''(\lambda) = \frac{2}{\lambda^3} E[\lambda^2 \mid X] \geq 0
\]
is always true.
Thus the solution of \( f'(\lambda) = 0 \), i.e. \( \hat{\lambda}_y = [E(\lambda^2 \mid X)]^{1/2} \) is the minimum point of \( f(\lambda) \).
Therefore the Bayesian estimator of \( \lambda \) under the precautionary loss function (1) is \( \hat{\lambda}_y = [E(\lambda^2 \mid X)]^{1/2} \).

3. Bayesian and E-Bayesian of failure-rate and Reliability

This section will discuss the Bayesian and E-Bayesian estimation of failure rate and reliability of exponential distribution under precautionary loss function in case of zero-failure data. In the following discussion, the following notations are used to depict the considered problems:

(i) \( t_i \), \( i = 1, 2, \ldots, m \) is the censored times under the type-I censored life test;

(ii) \( n_i \) is the corresponding sample size of time \( t_i \);

(iii) \( N = \sum_{i=1}^{m} n_i \);

(iv) \( (t_i, n_i), i = 1, 2, \ldots, m \) is zero-failure data set (the measurement data).

3.1 Bayesian Estimation

**Theorem 1.** The time truncation test is carried out on the product whose lifetime distributed with exponential distribution (2), Assume that the prior distribution of failure rate \( \lambda \) is Gamma distribution \( \Gamma(a, b) \), that is the density function of \( \lambda \) is
\[
\pi(\lambda; a, b) = \frac{k^a}{\Gamma(a)} \lambda^{a-1} \exp(-b\lambda), \quad \lambda > 0
\]
Then under the precautionary loss function (1),

(i) the Bayesian estimator of failure rate \( \lambda \) is
\[
\hat{\lambda}_B = \frac{a(a + 1)}{b + N}
\]
(ii) the Bayesian estimator of reliability is
\[
\hat{R}_B = \exp(-\hat{\lambda}_B t)
\]
**Proof.** According to equations (5), (7) and Bayes’ Theorem, the posterior density function of failure rate $\lambda$ is

\[
h(\lambda | 0) \propto L(0 | \lambda) \cdot \pi(\lambda) = e^{-N\lambda} \cdot \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-\lambda} = \lambda^{a-1} e^{-(b+N)\lambda}.
\]

(10)

From (10), we know that the posterior distribution of $\lambda$ is also a Gamma distribution $\Gamma(a, \beta + N)$, and the probability density function is as follows:

\[
h(\lambda | 0) = \frac{(b+N)^a}{\Gamma(a)} \lambda^{a-1} e^{-(b+N)\lambda}.
\]

(11)

Then

\[
E[\lambda^2 | 0] = \int_0^\infty \lambda^2 \frac{(b+N)^a}{\Gamma(a)} \lambda^{a-1} e^{-(b+N)\lambda} d\lambda
\]

\[
= \frac{(b+N)^a}{\Gamma(a)} \frac{\Gamma(a+2)}{(b+N)^{a+2}}
\]

\[
= \frac{a(a+1)}{(b+N)^2}
\]

According to Eq.(6), the Bayesian estimator of failure rate $\lambda$ is

\[
\hat{\lambda}_B = \left[ E(\lambda^2 | 0) \right]^{1/2} = \frac{\sqrt{a(a+1)}}{b+N}.
\]

Then we can get the Bayesian estimator of reliability $R$ as follows:

\[
\hat{R}_B = \exp(-\hat{\lambda}_B t).
\]

3.2 E-Bayesian Estimation

In the case of zero-failure data, Han [15] proposed a construction method for determining the prior distribution of failure rate and this method is also called decreasing function method. According to Han [14], the values of hyper parameters $a$ and $b$ should be selected to guarantee that $\pi(\lambda; a, b)$ is a decreasing function of $\lambda$.

The derivative of $\pi(\lambda; a, b)$ is

\[
\frac{d\pi(\lambda; a, b)}{d\lambda} = \frac{b^a}{\Gamma(a)} \lambda^{a-1} \exp(-b\lambda)[(a - 1) - b\lambda],
\]

(12)

Then, we can see that, if $0 < a \leq 1$ and $b > 0$ then $\frac{d\pi(\lambda; a, b)}{d\lambda} < 0$. Thus $\pi(\lambda; a, b)$ is a decreasing function of $\lambda$. Then we can choose the prior distributions of hyper parameters $a$ and $b$ as follows:

\[
\pi_1(a) = U(0,1),
\]

(13)

\[
\pi_2(b) = U(0, C),
\]

(14)

where $C$ is a constant.

**Definition 1** Let $\hat{\lambda}_B(a, b)$ be a Bayesian estimator of parameter $\lambda$, $D = \{(a, b) : 0 < a < 1, 0 < b < C \}$. $C > 0$ is a constant. The function $\pi(a, b)$ is a probability density function of $a$ and $b$ in $D$, then the E-Bayesian estimator of parameter $\lambda$ is defined as follows:

\[
\hat{\lambda}_{EB} = \int_D \hat{\lambda}_B(a, b) \pi(a, b) \, da \, db
\]

(15)
**Theorem 2.** The time truncation test is carried out on the product whose life distributed with an exponential distribution (1), and assume that the prior distribution of failure rate $\lambda$ is Gamma distribution $\Gamma(a, b)$ and the hyper parameters $a$ and $b$ have the prior distribution (13) and (14), respectively, then under the precautionary loss function (1), then

(i) The E-Bayesian estimator of failure rate $\lambda$ is

$$\hat{\lambda}_{EB} = \frac{\ln(C + N) - \ln N}{C} - \sqrt{a(a + 1)}da$$

(ii) The Bayesian estimator of reliability is

$$\hat{R}_{EB} = \exp(-\hat{\lambda}_{EB}t)$$

**Proof.** From Theorem 1, under precautionary loss function and Gamma prior, the Bayesian estimator of $\lambda$ is

$$\hat{\lambda}_B = \frac{\sqrt{a(a + 1)}}{b + N}.$$

Then By Definition 1, the E-Bayesian estimation of failure rate $\lambda$ can be derived as follows:

$$\hat{\lambda}_{EB} = \iint_D \hat{\lambda}(a, b)\pi(a, b)dadb$$

$$= \int_0^C \int_0^C \frac{\sqrt{a(a + 1)}}{b + N}\pi(a)\pi(b)dadb$$

$$= \frac{1}{C} \int_0^C \left[ \int_0^C \frac{\sqrt{a(a + 1)}}{b + N}db \right]dadb$$

$$= \frac{\ln(C + N) - \ln N}{C} - \sqrt{a(a + 1)}da$$

Thus, the E-Bayesian estimator of reliability $\hat{R}$ can be obtained as

$$\hat{R}_{EB} = \exp(-\hat{\lambda}_{EB}t).$$

4. **Examples**

The effectiveness and practicability of the Bayesian and E-Bayesian estimators obtained in this paper is illustrated by an example in [3]. Consider the zero-failure data of type I censored life testing from some engine, and there is no failure in Type-I lifetime test. We only get the zero-failure data given in Table 1 (time unit: hour), which contains 6 sets of 20 data in total. The aim of this example is to estimate the failure rate and reliability when time $T=2000$, that is $R_{2000} = e^{-\hat{\lambda}_{2000}}$.

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 |
|-----|---|---|---|---|---|---|
| $t_i$ | 136 | 282 | 370 | 667 | 1188 | 1335 |
| $n_i$ | 2 | 2 | 3 | 5 | 4 | 4 |

Table 1 Zero failure data of certain model engine

The results are calculated by Theorem 1 and Theorem 2 and shown in Table 2.

| $C$ | 300 | 500 | 800 | 1500 | Range |
|-----|-----|-----|-----|------|-------|
| $\hat{\lambda}_{EB}$ | $5.4135 \times 10^{-5}$ | $5.3792 \times 10^{-5}$ | $5.3287 \times 10^{-5}$ | $5.2958 \times 10^{-5}$ | $0.1177 \times 10^{-5}$ |
| $R_{EB}$ | 0.8974 | 0.8980 | 0.8989 | 0.8995 | 0.0021 |
| $(a, b)$ | (0.5,0.5) | (1.0,0.5) | (1.5,0.5) | (1.5,1.0) |
| $\hat{\lambda}_B$ | $5.6332 \times 10^{-5}$ | $9.1990 \times 10^{-5}$ | $1.2596 \times 10^{-4}$ | $1.2596 \times 10^{-4}$ | $6.9628 \times 10^{-5}$ |
As shown in Table 2, E-Bayesian estimation is robust of failure rate $\lambda$ and $R(000)$ for different value of $C$. But Bayesian estimation has a much dependence with the values of $(a, b)$. Therefore, E-Bayesian estimators can be chosen the estimator of failure rate and reliability of exponential distribution in real application especially in the case of zero-failure data.

5. Conclusions
Reliability testing is a very important element in reliability analysis. It is often carried out with small sample sizes and short duration because of increasing costs and the restriction of time. Therefore, in the time truncation test, zero-failure data is often encountered, especially in high reliability and small sample problems. Bayesian statistical method is a suitable tool in solving the above-mentioned case. Under assuming the gamma distribution as prior distribution of failure rate and under a precautionary loss function, this study derived Bayesian and E-Bayesian estimators of failure rate and reliability of exponential distribution in case of zero-failure data. A practical example shows that the E-Bayesian estimators are more robust than ordinary Bayesian estimators. In the future, we will study the E-Bayesian estimation for other reliability distributions when the testing data is zero-failure data under the precautionary loss function.

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