D1-strings in large RR 3-form flux, quantum Nambu geometry and M5-branes in the C-field

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Abstract
We consider D1-branes in an RR flux background and show that there is a low energy–large flux double-scaling limit where the D1-brane action is dominated by a Chern–Simons–Myers coupling term. As a classical solution to the matrix model, we find a novel quantized geometry characterized by a quantum Nambu 3-bracket. Infinite-dimensional representations of the quantum Nambu geometry are constructed, which demonstrate that the quantum Nambu geometry is intrinsically different from the ordinary Lie algebra-type noncommutative geometry. Matrix models for the II B string, II A string and M-theory in the corresponding backgrounds are constructed. A classical solution of a quantum Nambu geometry in the II A matrix string theory gives rise to an expansion of the fundamental strings into a system of multiple D4-branes and the fluctuation is found to describe an action for a non-Abelian 3-form field strength, which is a natural non-Abelian generalization of the PST action for a single D4-brane. In view of the recent proposals [1, 2] of the M5-branes theory in terms of the D4-branes, we suggest a natural way to include all the KK modes and propose an action for the multiple M5-branes in a constant C-field. The worldvolume of the M5-branes in a C-field is found to be described by a quantum Nambu geometry with self-dual parameters. It is intriguing that our action is naturally formulated in terms of a 1-form gauge field living on a six-dimensional quantum Nambu geometry.

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1. Introduction

It is generally expected that the usual description of spacetime in terms of Riemannian geometry would break down above the Planck energy scale. A possibility is that geometry is quantized and spacetime coordinates become quantum operators. In this case, traditional spacetime concepts such as locality and causality, and even the fundamental nature of spacetime itself,
will have to be re-examined. String theory, as a candidate for a theory of quantum gravity, provides an interesting setup to address some of these questions. One of our motivations is to discover new types of quantum geometries in string theory and to study the physics on these quantum spaces.

There are a number of ways where a quantum geometry could emerge in string theory. One way is to consider open string theory ending on a D-brane with a background $B$-field on it and use the open string to probe the geometry of the D-brane. The obtained form of the noncommutative geometry could be of Moyal type [4–7],

$$[X^\mu, X^\nu] = i\theta^{\mu\nu}, \quad (1.1)$$

or a fuzzy sphere [8, 9]. String effects to arbitrary loops can be included readily [10]. Noncommutative geometry could also arise in matrix models [11, 12] as a classical solution [13–16]. Myers effects [17] could introduce additional terms to the matrix model and lead to new solutions [18–20]. We note that all these quantized geometries are characterized by a commutator and could be referred to as of Lie-algebra type. Inclusion of small fluctuations around these solutions always gives rise to a noncommutative gauge theory [14]. The discovery of a new non Lie-algebra type of quantum geometry in string theory would be interesting. Moreover, one may wonder if the physics of the fluctuations may lead to some new kind of gauge theory. This is another motivation for this work.

This aim has been achieved partially in [21] where it was shown that the consistency between the different descriptions of the M2–M5 intersecting brane system implies that the M5-brane geometry in the presence of a constant 3-form $C$-field takes the form of

$$[X^\mu, X^\nu, X^\lambda] = i\theta^{\mu\nu\lambda}, \quad (1.2)$$

where $\theta$ is a constant and the 3-bracket is given by a Lie 3-bracket. A Lie 3-bracket is multilinear and is antisymmetric under interchange of any pair of its components. Moreover, it satisfies the fundamental identity

$$[[f, g, h], k, l] = [[f, k, l], g, h] + [f, [g, k, l], h] + [f, g, [h, k, l]], \quad (1.3)$$

where $f, g, h, k, l$ are any elements of the algebra. The reason why a Lie 3-bracket appears is because the geometry of the M5-brane was inferred from the boundary dynamics of the open M2-branes which end on it; and the BLG model [22] with boundary was used to describe the open M2-branes. In a quantum theory, it is necessary to understand the relation (1.2) as an operator relation. However, the representation of the Lie 3-algebra relation as transformations on vector spaces or maybe some kind of generalization is still an open question (see [23, 24] for some different approaches). Since the difficulties are mainly due to the insistence of the fundamental identity, it motivates us to look for a 3-bracket geometry of the form (1.2) but with a 3-bracket where the fundamental identity is not required.

1 We remark that very interestingly, nonassociative geometry has also made an appearance in string theory recently [3]. Unlike the example [8] where the nonassociativity is due to a projection of the spectrum, here it seems that the emergence of the nonassociativity is due to the insistence of the use of a geometric language in a non-geometric background. In this paper, we are interested in quantum geometries that are characterized by conventional associative operators. We thank Erik Plauschinn for a discussion on this.

2 In [21], it was also shown that the standard noncommutative geometry (1.1) of D-branes in a constant 2-form $B$-field could be derived similarly by considering the intersecting system of F1 strings and D3 brane in the $B$-field. In this case, the 2-bracket $[,]$ is a Lie 2-bracket and is inherited from the Lie 2-bracket structure of the boundary matrix string theory.

3 One may think that one could easily repeat the analysis of [21] with the open ABJM theory [25] instead and obtain a similar relation (1.2) where $X$’s would be operators. This is however not immediate since the analysis in [21] involves a comparison of the results obtained in different dual descriptions of the M2–M5 intersecting brane system. In particular, a comparison of the information contained in the boundary condition of the open M2-brane system...
We remark that although the fundamental identity plays a crucial role in the BLG model, since it allows gauge transformations to be defined in terms of the 3-bracket and ensures the closure of the supersymmetry algebra, *a priori* there is no reason that a quantum geometry of the 3-bracket form (1.2) should obey the fundamental identity. In particular, if a 3-bracket geometry of the form (1.2) has a different physical origin which is not related to M2-branes, then one would not expect the fundamental identity to be observed. In this paper, we will show that the quantum geometry (1.2) arises as a classical solution of a matrix model of D1-strings in a background of large RR 3-form flux. Here the 3-bracket is given by one which is defined on ordinary operators:

\[
[f, g, h] := fg h + gh f + hfg - fhg - gfh - hgf, \tag{1.4}
\]

where \(f, g, h\) are any three operators and the binary product is the usual operator product. It is easy to see that the fundamental identity is not observed by the 3-bracket (1.4); see [27] for a discussion of this as well as some other algebraic properties of the 3-bracket (1.4). In fact, if one were to give up the fundamental identity, the most natural 3-bracket to consider would be the one given by (1.4) which is the most natural higher order generalization of the commutator. The 3-bracket (1.4) was originally introduced by Nambu [28] as a possible candidate of the quantization of the classical Nambu bracket

\[
\{f, g, h\} := \epsilon^{ijk} \partial_i f \partial_j g \partial_k h. \tag{1.5}
\]

Therefore, we will refer to (1.4) as the quantum Nambu bracket and the geometry (1.2) as the quantum Nambu geometry.

For the standard noncommutative geometry (1.1), a 2-form field strength can be written as

\[
F^{\mu \nu} = -i[X^\mu, X^\nu] \tag{1.5}
\]

when the fluctuation over the noncommutative geometry background is taken into account. What about the fluctuation around the quantum Nambu geometry? It is suggestive to interpret the quantum Nambu bracket of the target space coordinate fields \(X^\mu\) as a 3-form field strength

\[
H^{\mu \nu \lambda} = -i[X^\mu, X^\nu, X^\lambda]. \tag{1.6}
\]

To check this idea, we have to look for a place where a non-Abelian 3-form field strength lives. This leads us to consider the system of multiple D4-branes (where the 3-form field strength would be the Hodge dual of a 2-form field strength) and multiple M5-branes (where the 3-form field strength would be self-dual).

To reach the D4-brane system, we use the D1-strings matrix model to derive the matrix model descriptions for the II B string, II A string and M-theory in a corresponding background of large RR-flux or its uplift to 11 dimensions. For the II A matrix string theory, we find that a classical solution of quantum Nambu geometry is again allowed. We also find that the fluctuation around the solution gives a Lagrangian for a 1-form gauge potential whose form is exactly the same as the dimensional reduced PST action (which describes a single D4-brane) [30] if a quantum Nambu bracket of \(X^\mu\) is identified as (1.6) as a 3-form field strength whose Hodge dual would be the Yang–Mills field strength. Physically, this means that the system of fundamental strings has expanded over the quantum Nambu geometry into a system of multiple D4-branes.

Since a system of multiple D4-branes can be considered as a dimensional reduction of multiple M5-branes on a circle, it has been proposed recently that [1, 2] the instantons on the D4-branes can be identified with the KK modes of the compactified M5-branes, and that by including all the instantons, the D4-branes SYM theory is in fact equal to the M5-branes theory.

and those in the 3-sphere (or 3-ellipsoid in the presence of the C-field) description of the M2-brane spike is needed. However, this is tricky for the ABJM theory since classically only a fuzzy 2-sphere is seen; see [26] for a careful analysis on this issue.
In view of this, we suggest a natural way to include all the KK modes into the D4-branes and propose the action (5.40) for multiple M5-branes in a constant self-dual C-field. Our proposed action is living on a six-dimensional quantum Nambu geometry with the self-dual parameter $\theta^{\mu\nu\lambda}$ and is formulated in terms of a non-Abelian 3-form field strength defined using (1.6). A priori, such an $H^{\mu\nu\lambda}$ may not obey the desired self-duality condition. Nevertheless quite amazingly we find that the self-duality condition emerges naturally from our model. The M5-branes system in a C-field could be reduced to a system of D4-branes in the B-field, and the latter has a worldvolume described by the standard Moyal-type noncommutative geometry. This connection allows us to identify the $\theta^{\mu\nu\lambda}$ parameter of the quantum Nambu geometry as a C-field on the worldvolume of the M5 branes. Therefore, we obtain the result that the worldvolume of the M5-branes in a C-field is described by a quantum Nambu geometry

$$[X^\mu, X^\nu, X^\lambda] = i\theta^{\mu\nu\lambda},$$

with self-dual parameters $\theta^{\mu\nu\lambda} = C^{\mu\nu\lambda}$.

The plan of the paper is as follows. In section 2, we consider D1-branes in a constant RR 3-form flux background. We show that there is a low energy–large flux double-scaling limit such that the D1-brane action is dominated by the RR coupling terms. We then show that the resulting D1-brane matrix model has the quantum Nambu geometry as a classical solution. In section 3, we present some analysis of the mathematical properties of the quantum Nambu geometry. Infinite-dimensional representations are constructed, and we explain how the existence of these representations implies that the quantum Nambu geometry is intrinsically different from the ordinary Lie algebra-type geometry. In section 4, we derive the matrix model descriptions for the II B string, II A string and M-theory in the corresponding backgrounds. In section 5, we argue and propose the action (5.40) as the action for a system of multiple M5-branes in a constant C-field. The worldvolume geometry of the system of M5-branes is argued to be given by a quantum Nambu geometry with self-dual parameters $\theta^{\mu\nu\lambda} = C^{\mu\nu\lambda}$.

In our formulation, the fundamental dynamical variables are a 1-form gauge potential and the 3-form field strength is constructed out of them as a Nambu bracket. We discuss and give comments on this dual formulation. The paper is concluded with some further discussions.

2. Matrix model of D1-strings in large RR 3-form flux

2.1. A II B supergravity background

In the paper [31], an exact II B supergravity background with a constant RR 3-form flux was constructed. The background was constructed by turning on a constant RR 3-form flux in the AdS5 factor of the standard AdS5 $\times S^5$ background. The background has a spacetime which is a direct product

$$\mathcal{M} = \mathcal{M}_5 \times \mathcal{M}_5'$$

and has a nonvanishing dilaton $\Phi$, axion $\chi$, RR potentials $C_2$ and $C_4$ specified by

$$e^{-\Phi} = \chi/(2\sqrt{2}) = \text{constant},$$

$$F_3 = \begin{cases} f_{ijk}, & i, j, k = 1, 2, 3, \\ 0, & \text{otherwise}, \end{cases}$$

$$F_5 = \begin{cases} e_5, & \text{on } \mathcal{M}_5, \\ e_5', & \text{on } \mathcal{M}_5', \\ 0 & \text{otherwise}. \end{cases}$$
In the above, $f$ and $c$ are constants, $\epsilon_5$ and $\epsilon'_5$ are the volume forms on $M_5(\mu = 0, 1, 2, 3, 4)$ and $M_5(\mu = 5, 6, 7, 8, 9)$

$$
\epsilon_{\mu_0...\mu_4} = \sqrt{-g} \epsilon_{\mu_0...\mu_4}, \quad \epsilon'_{\mu_0...\mu_4} = \sqrt{-g} \epsilon_{\mu_0...\mu_4},
$$

(2.5)

and $\epsilon_{\mu_0...\mu_4}$, $\epsilon_{\mu_5...\mu_9}$ are the Levi-Civita symbols: $\epsilon^{01234} = -\epsilon_{01234} = 1, \epsilon^{56789} = \epsilon_{56789} = 1$.

It was found that a consistent background can be constructed if the magnitudes of the RR potentials $C_2$ and $C_4$ are chosen appropriately:

$$
f^2 = \frac{2}{3} c^2.
$$

(2.6)

The metric (in the string frame) takes the form of $\mathbb{R}^3 \times \text{AdS}_2 \times S^5$:

$$
dx^2 = \sum_{i=1}^{3} (dX^i)^2 + R^2 \left( \frac{-dr^2 + dU^2}{U^2} \right) + R^2 d\Omega_5^2,
$$

(2.7)

where

$$
R^2 = 2 e^{-2\Phi} / f^2, \quad R^2 = 80 e^{-2\Phi} / f^2,
$$

(2.8)

and $d\Omega_5^2 = \hat{G}_{ij} \, dx^i dx^j$ is the metric for an $S^5$ of unit radius. Apart from the $\mathbb{R}^3$ part, the spacetime can be understood as a warping of a one-dimensional Minkowski space $M_1$ with a six-dimensional manifold $Y_6$ with a conical singularity at $U = 0$.

For later use, we record the RR 2-form potential

$$
C_2 = f \epsilon_{ijk} X^i dx^j dx^k, \quad i, j, k = 1, 2, 3.
$$

(2.9)

Our convention is $F_{\mu\nu} = \frac{1}{2} \left( \partial_\mu C_{\nu} + \partial_\nu C_{\mu} + \partial_\lambda C_{\mu\nu} \right)$. The expressions for $C_4$ is more complicated. Later we will consider a large $f$ limit for a system of D1-strings in this background. For our purpose, it is enough to note that $C_{\mu_1...\mu_4}$, $(\mu_i = 0, 1, \ldots, 4)$ is proportional to $1/f$ and $C_{\mu_1...\mu_4}$, $(\mu_i = 5, 6, \ldots, 9)$ is proportional to $c R^5 \sim 1/f^4$. We remark that the background is nonsupersymmetric.

2.2. Matrix model of D1-strings in the limit of large $F_3$

Let us consider a system of $N$ parallel D1-branes in this background. The worldvolume action for the D1-branes is given by the non-Abelian Born--Infeld action plus the Chern--Simons term of the Myers type given by [17]

$$
S_{CS} = \mu_1 \int Tr P \left( e^{i\phi \lambda} \sum_n C_n \right) e^{i\int I}.
$$

(2.10)

Here $\mu_1 = 1/(g_s 2\pi a')$, $\lambda = 2\pi a'$ and $X^i = 2\pi a' \Phi^i$. Our background has $B = 0$. With $C_2$ and $C_4$ turned on, the Chern--Simons term reads

$$
S_{CS} = \mu_1 \int Tr \left[ \lambda F \chi + PC_2 + i\lambda^2 F_{i\phi i\phi} C_2 + i\lambda P_{i\phi i\phi} C_3 - \frac{\lambda^3}{2} F_{i\phi i\phi} C_3 \right]
$$

(2.11)

$$
: = S_\chi + S_{C_2} + S_{C_3},
$$

(2.12)

where $S_\chi$, $S_{C_2}$, $S_{C_3}$ denote the terms in $S_{CS}$ that depend on the RR potentials $\chi$, $C_2$ and $C_3$, respectively. Substituting (2.9), we obtain

$$
S_{C_3}/\mu_1 = f \int d^2\sigma \, Tr \left( \frac{1}{2} \epsilon_{ijk} X^i D_\sigma X^j D_\sigma X^k \phi \right) + f \int d^2\sigma \, Tr (iFX^i X^k \epsilon_{ijk})
$$

$$
\equiv f \int d^2\sigma (L_1 + L_2);
$$

(2.13)
here, $\epsilon_0^1 = -\epsilon_0^1 = 1, F = F_0$. From now on we will use $F$ to refer to either the curvature two-form or the component $F_{01}$. It should be clear from the context which is which. Naïvely, if we take a large $F_3$ limit, then the D1-brane action is dominated by $S_{CS}$. This is what we would like to demonstrate now. More precisely, we will show that there is a certain double-scaling limit wherein the dynamics of the system of D1-branes is dominated by the $C_2$ coupling term $S_{C_2}$. To do this, we need to include the non-Abelian Born–Infeld action, examine the large $f$ limit of the equations of motion and keep the parts of the action that contribute in the limit.

The non-Abelian Born–Infeld theory in curved space is however not so well understood. First, for flat space, one may expand the non-Abelian Born–Infeld action in the powers of $F$. However, there is an ambiguity associated with the ordering of $F$ which cannot be fixed with a simple symmetrization procedure [32, 33]. This ambiguity associated with the ordering of $F$ disappears in the Yang–Mills limit. However, in the curved space, there is a new difficulty associated with the incorporation of a curved metric. A natural proposal [34] is to promote the metric to become a matrix $G_{IJ}(X)$ and to incorporate the effect of curved space with the action, for the case of D1-branes reads

$$S_X/\mu_1 := \int d^2\sigma \sqrt{-\det G_{\alpha\beta}} \left( G_{IJ}(X)D_\alpha X^I D_\beta X^J G^{\alpha\beta} + \frac{1}{\alpha'} G_{IJ}(X) G_{KL}(X)[X^I, X^K][X^J, X^L] \right),$$

where a physical gauge $X^\alpha = \sigma^\alpha$ has been taken by making use of the worldvolume diffeomorphism of the non-Abelian Born–Infeld theory. The action (2.14) is highly ambiguous due to the ambiguity in the ordering of $X$ in the metric matrix function $G_{IJ}(X)$. For the case of $p = 0$, it was proposed [34] that the action (2.14) gives the matrix theory in curved space and it was found that a large class (but not all) of the ambiguities could be resolved by requiring that the IR gravitational physics be correctly reproduced. A more general principle is still needed to construct the action unambiguously in general.

Fortunately we will see that these ambiguities will not bother us. Let us assume that in the small $\alpha'$ limit, the system of D1-branes is described by an action of the form (2.14) with $I, J = 2, 3, \ldots, 9$ together with the Chern–Simons coupling (2.11). Note that for our metric, the ambiguity of the action $S_X$ is concentrated entirely in the $S^5$ part. The full D1-strings action is given by

$$S_{D1} := S_X + S_{CS} + S_{YM},$$

where the Yang–Mills term is

$$S_{YM}/\mu_1 = \alpha'^2 \int \sqrt{-\det G_{\alpha\beta}} F_{\alpha\beta} F_{\alpha'\beta'} G^{\alpha\alpha'} G^{\beta\beta'}$$

and the metric is

$$G_{\alpha\beta} = R^2 \sigma^{-2} \eta_{\alpha\beta}, \quad \alpha, \beta = 0, 1,$$

$$G_{ij} = \delta_{ij}, \quad i, j = 2, 3, 4,$$

$$G_{ij'} = R^2 \times \hat{G}_{ij'}(X^k), \quad i', j' = 5, 6, 7, 8, 9,$$

with $\hat{G}_{ij'}$ being the metric for a unit 5-sphere.
It is not difficult to see that

1. the scalars \( X^i \) and \( X'^i \) decouple from each other in the action \( S_{D1} \),
2. the contributions to the equations of motion of \( X^i \) and \( X'^i \) from the various pieces of actions (2.11) and (2.14) are given by

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Contribution of} & S_X & S_{C_2} & i_{B}^b C_4 \\hline
\text{EOM of} X^i & O(1/\alpha') & O(\frac{1}{\alpha'^2}) & 0 \\hline
\text{EOM of} X'^i & O(\frac{1}{\alpha'^3}) & 0 & O(\frac{1}{\alpha'^2}) \\hline
\end{array}
\]

(2.20)

3. the equation of motion of \( X^i \) can be solved with \( X'^i = 0 \).

These are truly independent of the ambiguity of the form of the metric \( g_{ij} \) in the action (2.14).

Therefore, one can set \( X'^i = 0 \) and focus on the sector with only the scalars \( X^i \) and the gauge field activated. Now the action \( S_{C_2} \) is of order \( O(f/\alpha') \) and the piece of action \( i_{B}^b C_4 \) in (2.11) is of order \( O(1/f\alpha'^2) \). Therefore, if we take a double-scaling limit \( \epsilon \to 0 \):

\[
\alpha' \sim \epsilon, \quad f \sim \epsilon^{-a}, \quad a > 0,
\]

such that \( a > 1/2 \), then \( S_{C_2} \) dominates. Moreover, \( S_{YM} \) can be ignored compared to \( S_{C_2} \) if \( a < 2 \). All in all, in the double-scaling limit (2.21) with \( 1/2 < a < 2 \), the low energy action of \( N \) D1-branes in a large \( F_3 \) background is given by

\[
\lim_{\epsilon \to 0} S_{D1} = S_{C_2}.
\]

(2.22)

We remark that the dominance of the system by a topological term is similar to what happened in the discussions of [18], where the effects of a Lorentz force term

\[
L = \frac{\mu_0 H_i}{2} \varepsilon_{ij} \varepsilon_{kl} X^i D_k X^j \quad i = 1, 2,
\]

(2.23)
on the physics of a system of \( N \) D0-branes dissolved in a D2-brane (whose spatial directions are \( i = 1, 2 \)) was studied. There it was found that the equation of motion of \( L \) is solved with any time-independent configuration \( D_i X^i = 0 \) and a specific solution \( [x^i, x'^i] = i\theta \varepsilon^{ij} \), which corresponds to a D2-brane non-vanishing charge density, was considered.

### 2.3. Quantum Nambu geometry as the classical solution

We can now find the equations of motion to the action \( S_{C_2} \); these are

\[
\varepsilon^{\alpha \beta \gamma} \varepsilon_{ijk} [X^i, D_\beta X^j X^k] + \varepsilon^{\alpha \beta \gamma} \varepsilon_{ijk} [D_\beta, X'^i X'^j X'^k] = 0,
\]

(2.24)

\[
\frac{1}{2} \varepsilon_{ijk} D_\alpha X^i D_\beta X^j \varepsilon^{\alpha \beta \gamma} + \varepsilon_{ijk} [F, X^i, X'^j] = 0,
\]

(2.25)

where \([A; B, C] := [B, C]A + A[B, C] + BAC - CAB\) is antisymmetric only in exchange of \(B, C\). This bracket arises since \(\text{Tr}[A, B, C]D = \text{Tr}[D; B, C]A\), in analogy to the relation \(\text{Tr} D[A, B] = \text{Tr}[D, A]B\), which is useful in ordinary Yang–Mills theory.

The first equation is solved with any (covariantly) constant configuration

\[
D_\alpha X^i = 0.
\]

(2.26)

The second equation becomes \(\varepsilon_{ijk} [F, X^i, X'^j] = 0\) and is solved by

\[
F = 0.
\]

(2.27)
Certainly the standard noncommutative geometry

\[ [X^i, X^j] = i \theta^{ij} \]  

(2.28)

is allowed, but there is also a new solution

\[ [X^i, X^j, X^k] = i \theta \epsilon_{ijk} \]  

(2.29)

where \( \theta \) is a constant and the 3-bracket is given by (1.4). We note that the solution (2.29) is not allowed in the standard matrix models [11, 12] where no external \( F_3 \) is turned on.

We remark that the 3-bracket (1.4) was originally introduced by Nambu [28] as a possible candidate of the quantization of the classical Nambu bracket

\[ \{f, g, h\} = \epsilon_{ijk} \partial_i f \partial_j g \partial_k h. \]  

(2.30)

In his paper, Nambu was interested in generalizing the Hamiltonian mechanics to the form (Nambu mechanics)

\[ \frac{df}{dt} = \{H_1, H_2, f\}, \]  

(2.31)

which involves two ‘Hamiltonians’ \( H_1, H_2 \). The concept of a fundamental identity was not considered in his consideration. In fact one can easily check that the fundamental identity is not satisfied for (1.4). The concept of fundamental identity was introduced almost 20 years later by Takhtajan [35] (and by Baryen and Flato independently) as a natural condition for his definition of a Nambu–Poisson manifold, which allows him to formulate the Nambu mechanics in an invariant geometric form similar to that of Hamiltonian mechanics. For example, the fundamental identity implies that the time evolution preserves the Nambu bracket. However, note that for this purpose, a weaker form of the fundamental identity, where two of the elements are fixed: \( k = H_1, l = H_2 \), is sufficient. What we have shown above is that a quantized geometry characterized by the Nambu bracket (1.4) is allowed as a solution to string theory and we will refer to the quantized geometry (2.29) as the quantum Nambu geometry.

3. Analysis of the quantum Nambu geometry

3.1. Representations of the Nambu–Heisenberg commutation relation

An intermediate question to relation (2.29) is that in what sense it characterizes a new quantized geometry. We will address this question in this section.

3.1.1. Finite-dimensional Lie algebraic representations. Let us start with the observation of Nambu [28] that if \( X^i = \alpha l_i \) for a constant \( \alpha \) and \( l_i \) are the generators of the standard \( SU(2) \) algebra

\[ [l_i, l_j] = i \epsilon_{ijk} l_k, \]  

(3.1)

then

\[ [X^i, X^j, X^k] = i \epsilon^{ijk} \alpha^3 C_R, \]  

(3.2)

where \( C_R \) is the quadratic Casimir for the representation \( R \) where \( X^i \) is in. For \( N \times N \) matrices, \( C_N = (N^2 - 1)/4 \) and so if we choose \( \alpha^3 = \theta / C_N \), then we can realize the relation (2.29) with \( N \times N \) matrices. Nambu has also constructed a representation of the relation (2.29) in terms of \( SU(2) \times SU(2) \) representations. In these representations that Nambu constructed, the quantum Nambu bracket is embedded in an underlying Lie algebra (\( SU(2) \) or \( SU(2) \times SU(2) \))

4 As remarked by Takhtajan in [35].
3.1.2. Infinite-dimensional representations. An infinite-dimensional representation of (2.29) has been constructed by Takhtajan [35]. However, his representation is complex as the operators $X^3$ are not represented as Hermitian operators there. As a result, the quantum space is six dimensional. In this subsection, we give two examples of representations where a unitary condition can be imposed and the quantum space is three dimensional. We remark that in the large $N$ limit, there is probably an infinite number of inequivalent representations for the operator relation (2.29). Precisely which representation is to be used is a question that depends on the physics under consideration.

To be concrete, we are interested in constructing representations of the relation

$$[X^1, X^2, X^3] = i \theta,$$  

(3.3)

where $\theta$ is real and there is a certain reality condition which one can impose so that the quantum space (3.3) can be understood as a deformation of a real three-dimensional space.

(1) A representation in terms of $Z, \bar{Z}, X$.

Let us consider Hermitian $X^i$’s and introduce the complex coordinates

$$Z := X^1 + iX^2, \quad \bar{Z} := X^1 - iX^2.$$  

(3.4)

The relation (3.3) can be written in the form

$$[X, Z, \bar{Z}] = 2\theta,$$  

(3.5)

where $X = X^3$. We consider an ansatz for a representation

$$Z(\omega) = f_1(\omega)(\omega + \beta) + f_2(\omega)(\omega - \beta),$$  

(3.6a)

$$\bar{Z}(\omega) = f_3^*(\omega + \beta)(\omega + \beta) + f_4^*(\omega - \beta)(\omega - \beta),$$  

(3.6b)

$$X(\omega) = g(\omega)|\omega\rangle,$$  

(3.6c)

where the state $|\omega\rangle$ is parameterized by a number $\omega$ and $\beta$ is a fixed ‘step’. It is clear that the domain of $\omega$ is one dimensional. Without loss of generality we can take $\beta$ real and $\omega \in \mathbb{R}$. The form of (3.6b) is fixed by (3.6a) by requiring $\bar{Z} = Z^\dagger$. Hermiticity of $X$ requires that $g$ be real. We remark that the introduction of $Z, \bar{Z}$ is motivated by the creation and annihilation operators for the Heisenberg commutation relation. Thus, it would be natural to consider the representation (3.6a–3.6c) with $f_2 = 0$ or $f_3 = 0$. However, this always give a constraint of the form $ZZ + \bar{Z}\bar{Z} = Z(Z)$ for some function $Z$ and so describes at most a two-dimensional space. As a result, we are prompted to try the more general ansatz stated above.

It is easy to obtain

$$[X, Z, \bar{Z}]|\omega\rangle = I_2(\omega)|\omega + 2\beta\rangle + I_{-2}(\omega)|\omega - 2\beta\rangle + I_0(\omega)|\omega\rangle,$$  

(3.7)

where

$$I_2(\omega) = G(\omega)K(\omega), \quad I_{-2}(\omega) = I_2(\omega - 2\beta)^*, \quad I_0(\omega) = F(\omega)(2g(\omega) - g(\omega - \beta)) - F(\omega + \beta)(2g(\omega) - g(\omega + \beta)).$$  

(3.8)

and

$$G(\omega) := g(\omega + 2\beta) + g(\omega) - g(\omega + \beta).$$  

(3.9)
We would like to find functions \( g, f_1, f_2 \) such that
\[
I_2 = I_{-2} = 0 \quad (3.13)
\]
and
\[
I_0 = 2\theta. \quad (3.14)
\]
The first condition can be solved by requiring \( K(\omega) = 0 \) or \( G(\omega) = 0 \). The possibility of \( K = 0 \) is not good since it implies that \( [Z, Z]|\omega\rangle = F(\omega)|\omega\rangle \) and so there is a relation of the form \( [Z, Z] = Z(X) \) for some function \( Z \). This means the relation (3.5) is not intrinsic but reducible to a statement about commutators; this is not what we are after. For this reason, we consider the second possibility
\[
g(\omega + 2\beta) + g(\omega) - g(\omega + \beta) = 0. \quad (3.15)
\]
It is easy to see that it implies a pseudo-periodic condition
\[
g(\omega + 3\beta) = -g(\omega), \quad (3.16)
\]
and it follows that
\[
I_0(\omega) = F(\omega)A(\omega) - F(\omega + \beta)A(\omega - \beta), \quad (3.17)
\]
where
\[
A(\omega) := g(\omega) + g(\omega + \beta). \quad (3.18)
\]
It is \( A(\omega + 3\beta) = -A(\omega), F(\omega + 3\beta) = -F(\omega) \). The condition (3.15) is solved by
\[
g(\omega) = \sin \alpha \omega, \quad \cos \alpha \omega, \quad \text{where} \quad \alpha = \frac{\pi}{3\beta}(6p \pm 1), \quad p \in \mathbb{Z}, \quad (3.19)
\]
or generally a Fourier sum of these modes. For simplicity, let us construct a representation for the simple mode
\[
g(\omega) = \cos \alpha \omega, \quad (3.20)
\]
where \( \alpha \) is as specified in (3.19). Consider the ansatz
\[
F(\omega) = k \sin \left( \alpha \omega - \frac{\alpha \beta}{2} \right). \quad (3.21)
\]
One sees that (3.14) is solved with
\[
k = \frac{2\theta}{\sin \alpha \beta \cos \frac{\alpha \beta}{2}}. \quad (3.22)
\]
This provides a constraint on the two functions \( f_1 \) and \( f_2 \). For example, a simple solution is
\[
|f_1(\omega)|^2 = |f_2(\omega)|^2 = k_0 - \frac{8\theta}{3} \cos \alpha \omega, \quad (3.23)
\]
where \( k_0 > \frac{8\theta}{3} \) is any constant such that the right-hand side above is positive. Without loss of generality, we can take \( \beta = 1 \). The representation space is given by the one-dimensional lattice
\[
\{|\omega + n \rangle : n \in \mathbb{Z} \} \quad (3.24)
\]
and is of countably infinite dimension for each fixed \( \omega \).
(2) A representation with \( Z_3 \) symmetry.

We now demonstrate that there is another way to construct a representation of (2.29) such that the quantum space it represents is three dimensional. In this construction, we assume no reality condition on the fields \( X^i \); thus, so far we have six degrees of freedom. Instead, let us introduce a unitary operator

\[
U|\omega\rangle = |\rho^2 \omega\rangle, \\
U^\dagger|\omega\rangle = |\rho \omega\rangle,
\]

and assuming

\[
X^1|\omega\rangle = (\omega + a)|\omega + 1\rangle,
\]

one obtains

\[
U^\dagger X^1 U|\omega\rangle = (\rho^2 \omega + a)|\omega + \rho\rangle, \\
U^{12} X^1 U^2|\omega\rangle = (\rho \omega + a)|\omega + \rho^2\rangle,
\]

where \( a \in \mathbb{C} \) and \( \rho \) is a cubic root of unity (\( \rho^3 = 1 \)) which is not equal to 1.

Now if the fields \( X^1, X^2 \) and \( X^3 \) are unitarily related to each other by

\[
X^2 = U^\dagger X^1 U, \\
X^3 = U^\dagger X^2 U,
\]

then

\[
X^1|\omega\rangle = (\omega + a)|\omega + 1\rangle, \\
X^2|\omega\rangle = \rho^2 (\omega + a \rho)|\omega + \rho\rangle, \\
X^3|\omega\rangle = \rho (\omega + a \rho^2)|\omega + \rho^2\rangle
\]

and it easy to see that

\[
[X^1, X^2, X^3]|\omega\rangle = 3(a^2 - a)(\rho - \rho^2)|\omega\rangle,
\]

where \( a \in \mathbb{C} \) and \( \rho - \rho^2 \) is pure imaginary. In this representation, the fields \( X^1, X^2, X^3 \) are not Hermitian. They are however related through a unitary transformation, \( U = e^{i\Theta} \), where \( \Theta \) is some Hermitian operator. So in this representation, we have two degrees of freedom from \( X^1 \) and one from \( \Theta \) giving us three real dimensions.

We note that in this representation, the operators \( X^i \) can be constructed as pseudo-differential operators acting on functions \( \langle \omega|\psi\rangle = \psi(\omega) \). Let us start with \( X^1 \) and note that \( \langle \omega + 1|X^1 = (\omega + a)|\omega \rangle \) and so \( X^1\psi(\omega) = \langle \omega|X^1|\psi\rangle = (\omega + a - 1)\psi(\omega - 1) \). Therefore, we obtain

\[
X^1 = (\omega + a - 1) e^{-\frac{\omega}{\rho}}.
\]

Similarly

\[
X^2 = (\rho^2 \omega + a - 1) e^{-\frac{\omega}{\rho^2}}, \\
X^3 = (\rho \omega + a - 1) e^{-\rho^2 \frac{\omega}{\rho^2}},
\]

and for the unitary operator

\[
U = \exp \left[ \ln(\rho) \frac{\partial}{\partial \omega} \right].
\]

The Hermitian conjugate of the unitary operator is

\[
U^\dagger = \exp \left[ \ln(\rho^2) \frac{\partial}{\partial \omega} \right].
\]
In this construction, the representation space is given by the two-dimensional lattice
\[
[|m + nρ⟩ : m, n ∈ \mathbb{Z}]
\]
and is of countably infinite dimension.

In conclusion, we have shown that there are at least two ways to represent \((2.29)\) as a three-dimensional quantum space: either a real representation or having one complex field and introducing a unitary operator relating \(X^2, X^3\). This is in contrast to the representation in \([35]\) where all the fields are complex and not unitarily related.

### 3.2. Integrals

It is an interesting question to construct quantum field theory on the quantum Nambu geometry. An important ingredient that is needed is an invariant integral on the space. Given a general quantum space, sometimes the symmetry is strong enough to determine the integral purely algebraically. For example, this is the case for a compact Lie group and some homogeneous spaces of quantum groups. For the Nambu geometry, this is not the case due to the existence of many inequivalent representations; an integral must be defined using information beyond the algebraic commutation relations. With the representations available, we can use the trace to define an integral. The properties of the integrals as well as the construction of the quantum field theory will be reported elsewhere.

### 4. Matrix theories in the large RR flux background

We would like to perform an expansion around the quantum Nambu geometry and ask what kind of gauge theory would come out. We recall that for the standard Lie algebraic type noncommutative geometry, a 2-form field strength is obtained from the fluctuation over the noncommutative geometry as
\[
F^{\muν} = -i[X^μ, X^ν].
\]
(4.1)

For our quantum Nambu geometry, it is suggestive to interpret the quantum Nambu bracket of the target space coordinate fields \(X^μ\) as a 3-form field strength
\[
H^{μνλ} = -i[X^μ, X^ν, X^λ].
\]
(4.2)

and we would like to check this idea.

Places where a non-Abelian 3-form field strength lives are, for example, multiple D4-branes (where the 3-form field strength would be the Hodge dual to a 2-form field strength) and multiple M5-branes (where the 3-form field strength would be self-dual). To check the idea, we would like to connect to these systems from our D1-branes system. And to do this, let us first derive the Matrix model descriptions for the II B string theory, M-theory and II A string theory in a large flux background using our description (2.13) for the D1-branes.

#### 4.1. II B matrix theory

The II B matrix model can be obtained by a large \(N\) reduction \([36]\) of the D1-string action. Let us first denote the covariant derivative \(iD^α = i∂^α + A^α, α = 0, 1\) as
\[
iD^α = X^α,
\]
(4.3)
and rewrite \(L_1, L_2\) in terms of the \(X^α\)’s:
\[
L_1 = -\frac{1}{2} \text{Tr} \ [X^α, X^β][X^β, X^δ]ε_{αβδ}ε_{ijk},
\]
(4.4)
Although the form of $L_1$ does not look like it, it is not difficult to show that

$$L_1 + L_2 = \frac{1}{40} \text{Tr}[X^a, X^b][X^c, X^d, X^e] \epsilon_{abcde}. \quad (4.6)$$

It is quite remarkable that the D1-branes Chern–Simons coupling to a constant RR $F_3$ flux can be written in such a simple form. The action of $N$ D1-branes in a large $F_3$ limit can thus be written compactly as

$$S_{D1} = \frac{\mu_1 f}{40} \int d^2 \sigma \text{Tr}[X^a, X^b][X^c, X^d, X^e] \epsilon_{abcde},$$

where $a, b, c, d, e = 0, 1, 2, 3, 4$. The large $N$ reduction immediately gives the following $D$-instantonic action (ignoring an unimportant overall numerical constant)

$$S_{IDB} = \frac{f}{g_{sL}} \int d^3 \sigma X^a X^b X^c X^d X^e \epsilon_{abcde}, \quad a, b, c, d, e = 0, 1, 2, 3, 4. \quad (4.8)$$

This gives the matrix model description for the II B string theory in the limit of a large constant RR 3-form flux, and in the sector with $X^a' = 0$, $a' = 5, 6, 7, 8, 9$. In this limit, the Myers term dominates over the standard Yang–Mills term in the IKKT matrix model.

4.2. Matrix model of M-theory

The II B background we considered is invariant under the Killing vector $\partial/\partial x^i$, $i = 2, 3, 4$. Therefore, we can compactify, say $x^i$ on a circle of radius $R_2$ and T-dualize. The corresponding II A background has

**Metric:** $S^1 \times \mathbb{R}^2 \times \text{AdS}_2 \times S^5$, \hspace{1cm} (4.9)

**Constant RR field strength:**

$F_{ij} = F_{2ij}$, $i, j = 3, 4$,

$F_{abcd} = F_{2abcd}$, $a, b, c, d = 0, 1, 3, 4$, \hspace{1cm} (4.10)

$F_{2ab', c'd'e'} = F_{2ab'c'd'e'}$, $a', b', c', d', e' = 5, 6, 7, 8, 9$,

**Constant dilaton:** $e^{\phi} = e^{\phi} \sqrt{\frac{\alpha'}{R_2}}$. \hspace{1cm} (4.11)

Under T-duality, the D1-branes become D2-branes. In the double-scaling limit (2.21), the D2-branes action is given by the T-dual of the D1-branes action (4.7) by applying the usual T-duality rule [11, 37] to the D1-branes action:

$$X^2 = i R_2 D_2,$$ \hspace{1cm} (4.12)

$$\text{Tr} = \int_0^1 \frac{d \sigma_2}{R_2} \text{Tr}. \hspace{1cm} (4.13)$$

We obtain

$$S_{D2} = \frac{f}{g_{sL}} \int d^3 \sigma X^a X^b X^c X^d X^e \epsilon_{abcde}, \hspace{1cm} (4.14)$$

where $a, b, c, d, e = 0, 1, 2, 3, 4$ and we have ignored an unimportant overall numerical constant. Note that since the Chern–Simons coupling is topological, the $R_2$ dependence gets
canceled in (4.14). We note that one can also obtain (4.14) directly from the Chern–Simons coupling of D2-branes in the II A RR flux background (4.10). It is

\[ S_{CS} = \frac{1}{g_s l_s^3} \left[ \int P(C_1) F + \int P(C_3) + \int P(i\lambda_1 i_1 \lambda_2 C_5) \right]. \]  

(4.15)

The \( C_3 \) and \( C_5 \) terms have their origin from the RR 5-form of II B and so they can be ignored in the double-scaling limit (2.21). The \( C_1 \) term then reproduces precisely (4.14).

In addition to D2-branes, the II A side also contains D0, D4, D6 and D8-branes. If we put M-theory on a circle and go to the infinite momentum frame, then only states with positive D0-brane charge are left in the physical description. In general, this includes all the D-branes in the II A theory since by turning on a worldvolume Born–Infeld configuration the D0-branes are charged under the II A theory since by turning on a worldvolume Born–Infeld configuration \( F \wedge \cdots \wedge F \) with \( p/2 \) terms, a Dp-brane is charged under \( C_1 \).

With remarkable insights, BFSS originally proposed that M-theory in flat space in the infinite momentum frame is given by the large \( N \) quantum mechanics of D0-branes [11]. The reason why it is not necessary to include the higher Dp-branes (\( p \) even) is because they can be constructed out of the D0-branes and so they are already included. This is so because in flat space, the worldvolume action for such a system of Dp-branes is given by

\[ S_{YM} = \int d^{p+1} \sigma [X^I, X^J]^2, \]  

(4.16)

where \( X^I = (X^\mu, X^i), \mu = 0, 1, \ldots, p, i = p + 1, \ldots, 9 \) and \( X^\mu = i D^\mu \) and a background with nontrivial \( F^{\alpha \beta} \) is assumed. In this way, one can see that all the higher Dp-brane actions can actually be constructed from the D0-branes and so it is sufficient to include only the D0-branes in the description.

For us, we would like to derive the quantum mechanical description of M-theory in a curved background that corresponds to (4.9)–(4.11) uplifted to 11 dimensions. The 11-dimensional background reads

\[ \text{metric} : ds^2 = e^{-2\phi} ds_{10}^2 + e^{2\phi} (dx^{11} - dx^C)^2, \]  

(4.17)

\[ \text{3-form potential} : C^{(3)} = \frac{1}{6} C_{abc} dx^a dx^b dx^c, \]  

(4.18)

where \( \phi \) is the dilaton in II A theory, \( C_1 \) and \( C_{abc} \) are the RR 1-form potential and RR 3-form potential, respectively, which appear in (4.10).

Let us denote the 11th-dimensional radius by \( R_{11} \). In general, with a suitable worldvolume flux turned on, the higher Dp-branes (\( p \) even) carries D0-brane charges and so in principle should be kept in the infinite momentum frame. However, as in the flat case, it is sufficient to select a subset of degrees of freedom in such a way that all the other degrees of freedom as well as their dynamics could be recovered. Now what is different for our background is that there is a set of non-vanishing RR gauge potentials which lead to explicit Chern–Simons terms in the action of the Dp-branes.

Let us examine this in detail. In the double-scaling limit (2.21), we can ignore the Yang–Mills term and the Chern–Simons coupling to \( C_3 \) and \( C_5 \) (whose origin are both from \( F_5 \) of the II B side) and concentrate on the Chern–Simons coupling of \( C_1 \). Moreover, in the sector where the fields in the sphere directions are set to zero: \( X^\alpha = 0, \alpha = 5, 6, 7, 8, 9 \), the Chern–Simons couplings for D4, D6 and D8-branes are zero. For the D0-branes, we have (ignoring an unimportant overall numerical constant)

\[ S_{D0} = \frac{1}{g_s l_s} \int P(C^{(1)}) = \frac{1}{g_s l_s} \int d^6 \epsilon \epsilon^j X^j D_i X^i, \quad i, j = 3, 4. \]  

(4.19)
Now the action (4.14) is equivalent to its dimensional reduction

$$\frac{f}{g_s l_s} \int \text{d}t \text{Tr} X^a X^b X^c X^d X^e \epsilon_{abcde}, \quad (4.20)$$

since one can always recover $S_{D2}$ by compactifying $X^1, X^2$ and then decompactify using the rules (4.12) and (4.13). Since the action (4.19) can be considered as a special case of (4.20) in a background $[X^1, X^2] = 1$; therefore, we propose that in the large flux limit and in the sector with $X^a = 0, a' = 5, 6, 7, 8, 9$, M-theory in our curved background (4.17), (4.18) is described by the quantum mechanical action

$$S_M = \frac{f}{g_s l_s} \int \text{d}t D_a X^b X^c X^d X^e \epsilon_{bcde}, \quad b, c, d, e = 1, 2, 3, 4. \quad (4.21)$$

Here we have substituted $X^0 = -i D_t$ and we have ignored an unimportant overall numerical constant.

### 4.3. II A matrix string theory

Given the matrix model (4.21) for M-theory, one could follow the procedure of [38] and derive the corresponding II A matrix string theory. To do this, we first rewrite (4.21) in terms of the 11th-dimensional radius

$$R_{11} = g_s l_s, \quad (4.22)$$

and then compactify $X^2$ on a circle of radius $R_2$, and finally perform an 11-2 flip which exchanges the role of the 11th and the second direction of the $T^2$ where our M-theory is compactified on. In practice, this amounts to having instead

$$R_2 = g_s l_s \quad (4.23)$$

and

$$R_{11} = N, \quad (4.24)$$

where a normalization of lightcone momentum $p_+ = 1$ is adopted [38]. In this way, we obtain the matrix string description

$$S_{IIA} = \frac{f}{N} \int \text{d}^3 \sigma \text{Tr} X^a X^b X^c X^d X^e \epsilon_{bcde}, \quad a, b, c, d, e = 0, 1, 2, 3, 4. \quad (4.25)$$

where

$$X^a = i D^a, \quad X^i = \text{scalars}, \quad \alpha = 0, 1, \quad i = 2, 3, 4, \quad (4.26)$$

and we have ignored an unimportant overall numerical constant. We note that the D1-strings action (4.7) and the II A matrix string action (4.25) are indeed the same up to a constant coefficient. This is similar to what was found in [38–40] where the same two-dimensional supersymmetric Yang–Mills theory could have different string interpretations depending on how one associate its parameters with the string theories. This is consistent with the T-duality.

### 5. Multiple D4-branes and M5-branes

#### 5.1. D4-branes in large RR 2-form flux

Let us concentrate on the matrix string theory. Since the matrix string theory (4.25) takes the same form as the original D1-strings action (4.7), it follows immediately that it admits the classical solution

$$[X^a, X^b] = 0, \quad [X^a, X^i] = 0, \quad \alpha = 0, 1, \quad i = 2, 3, 4. \quad (5.1)$$
As before, the commutation relations of $X^i$ among themselves are not constrained. Let us consider the solution $X_{cl}^i = x_i$ of quantum Nambu geometry

$$[x^2, x^3, x^4] = i\theta$$

and consider a fluctuation around it. In the large $N$ limit, we get a set of large $N$ matrices $x_i$. Depending on the representation chosen, they may or may not generate the entire set of $N \times N$ matrices. In general, assume that $x_i$ do not generate the whole set of $N \times N$ matrices. Then, every $N \times N$ matrix can be expressed as a $K \times K$ matrix whose entries are functions of $x_i$ [14]. The expansion of the dynamical variables around the classical solution can thus be parameterized as

$$X^i = x^i 1_{K \times K} + A^i(\sigma, x^i).$$

The action (4.25) becomes

$$S_5 = \frac{f}{N} \int \Sigma_5 \text{tr} X^a X^b X^c X^d \epsilon_{abcde},$$

where $f_{\Sigma_5} = \int d^2 \sigma f_x$ and $f_x$ is an integral on the quantum Nambu geometry which can be constructed from a representation of the geometry. In the large $N$ limit, the trace over large $N$ matrices decompose as usual as $\text{Tr} = f_x \text{tr}.$

We would like to argue that this solution corresponds to a system of $K$ parallel D4-branes. To do this, let us introduce a three-form $H$-field whose components are defined by

$$H^{abc} = - i [X^a, X^b, X^c],$$

$$H^{dce} = - i [X^d, X^e], \quad a, b, c, d, e = 0, 1, 2, 3, 4,$$

where

$$H_{\mu\nu\lambda} := \frac{1}{6\sqrt{-g}} \epsilon_{\mu\nu\lambda\rho\sigma\theta} H_{\rho\sigma\theta}$$

is the Hodge dual of $H_{\mu\nu\lambda}. \quad$Our convention for the Hodge duality operation is $\epsilon_{012345} = 1 = -g^{012345}. \quad$We remark that a similar identification has also been proposed in [21] in the analysis of the M5-brane geometry in a large $C$-field. As a result we obtain

$$S_5 = \int \Sigma_5 t H^{abc} H^{dce} \epsilon_{abcde},$$

where have ignored an unimportant overall constant here.

To see the connection of (5.8) with D4-branes, let us consider the Abelian case. Based on important earlier works [29], PST constructed a covariant action for a self-dual 3-form field strength $H = dB$ living on a single M5-brane [30]:

$$S_{\text{PST}} = \int d^4 \sigma \left[ \sqrt{-g} \frac{1}{4(\partial a)^2} \partial_{\mu} a H^{*\mu\nu\lambda} H_{\nu\lambda\rho} \partial^\rho a + Q(\hat{H}) \right].$$

Here the Greek indices $\mu = 0, 1, \ldots, 5$ and

$$\hat{H}_{\mu\nu} := \frac{1}{\sqrt{(\partial a)^2}} H^{*}_{\mu\nu\lambda} \partial^\lambda a.$$
where
\[ V_{\mu\nu} := -2 \sqrt{\left( \frac{\partial a}{\partial x^7} \right)^2 - g \delta Q \delta \tilde{H}_{\mu\nu}}. \] (5.12)

The equation of motion of the 2-form potential \( B_{\mu\nu} \) is
\[ \epsilon_{\mu\nu\rho\sigma} \left( \frac{\partial a}{\partial x^7} \right) \left( H_{\rho\sigma} \partial^7 a - V_{\rho\sigma} \right) = 0. \] (5.13)

Using the local symmetry (5.11), one can then show that it is equivalent to the self-duality condition
\[ H_{\mu
u\lambda} \partial^\lambda a - V_{\mu\nu} = 0. \] (5.14)

The scalar field \( a \) is introduced to allow six-dimensional covariance and is completely auxiliary due to the symmetry (5.11). If we choose a gauge \( a = x^5 \) and consider the linearized case (i.e. linearized equation of motion) with
\[ Q = -\frac{1}{4} \tilde{H}_{\mu\nu} H^{\mu\nu} \sqrt{-g}, \] (5.15)
then
\[ V_{\mu\nu} = \tilde{H}_{\mu\nu} \sqrt{\left( \frac{\partial a}{\partial x^7} \right)^2} \] (5.16)
and (5.14) becomes the standard self-duality condition
\[ H_{\mu\nu\lambda} = H^{\mu\nu\lambda}. \] (5.17)

In this case, the gauge-fixed PST action reads
\[ S_{\text{PST}} = \frac{1}{16} \int \epsilon^{\mu\nu\sigma} \left( \frac{1}{6} \epsilon_{abcd} H^{abc5} + H^{ab5} H^{cd5} \sqrt{-g} \right), \] (5.18)

where \( a = 0, \ldots, 4 \), etc. See [41] for a detailed discussion of the non-covariant and covariant PST formulations of the M5-brane action. Another equivalent description is to use the superembedding approach [42, 43]. In the next subsection, we will argue how to generalize the PST description for a single M5-brane to the non-Abelian case.

Now if we perform a dimensional reduction on \( x^5 \), the first term \( \epsilon_{abcd} H^{abc5} H^{cd5} \) in the dimensionally reduced gauge-fixed PST action is precisely equal to (5.8). This matching is quite amazing. As for the second term \( H^{ab5} H^{cd5} \), it is identified with the D4-branes’ Yang–Mills Lagrangian \( \sqrt{-g} F_{ab}^5 \) by performing a (Hodge) dualization. However, we have shown above that the Yang–Mills term is negligible in our double-scaling limit; therefore, the \( H^{ab5} H^{cd5} \) term is not seen in the action (5.8). Since a dimensionally reduced M5-brane is simply a D4-brane, this means (5.8), for the Abelian case, does describe a D4-brane in the large RR flux background. For the non-Abelian case, we propose that the action (5.8) describes a sector of multiple D4-branes theory (where \( X^5 = 0 \)) in a large RR 2-form flux background.

We note that since our term (5.8) agrees with the PST action (5.18) only if the map (5.5) is employed, the matching gives us confidence in this identification. We emphasize that the reason that it is possible to write (5.4) in terms of the \( H' \)'s is entirely due to the fact that the D1-branes Chern–Simons action could be combined nicely into the remarkable form (4.7), which is true only for our constant RR-flux in the II B background.

5.2. A proposal for a theory of multiple M5-branes using 1-form gauge field

Recently, it has been argued that [1, 2] the instantons on multiple D4-branes could be identified with the KK modes associated with the compactification of M5-branes on a circle. By including all these modes, it was proposed that the low energy SYM theory of D4-branes is a well-defined
quantum theory and is actually the theory of multiple M5-branes compactified on a circle. Back to our proposed action (5.8) for D4-branes in a large RR flux background, how can we incorporate the higher KK modes in our description? A possible hint is from the identification (5.6). We note that the identification for $H^{de5}$ can be written as

$$H^{de5} = -i[X^d, X^e, X^5]$$

with

$$X^5 = 1.$$  \hspace{1cm} (5.20)

If we think about $X^5$ as a scalar field describing the compactified $X^5$ direction transverse to the D4-brane, then one can understand the relation (5.6) and (5.20) as saying only the zero mode of the M5-branes has been included, i.e. a dimensional reduction to D4-branes. In this picture, it is suggestive to include the higher KK/instantonic modes by promoting $X^5 = 1$ to a general field. The identification (5.5) and (5.19) can be put together as

$$H^{\mu\nu\lambda} = -i[X^\mu, X^\nu, X^\lambda].$$  \hspace{1cm} (5.21)

We would like to propose that it is a different way to write the non-Abelian self-dual 3-form field strength living on M5-branes. In a conventional description, there would be a non-Abelian 2-form potential $B$ and $H = dB + \cdots$, where the $\cdots$ term denotes terms necessarily for the non-Abelianization. Thus, we propose that there is a dual description of the non-Abelian 3-form field strength in terms of the 1-form variables $X = X_\mu d\sigma^\mu$, and the $B$-field and the $X$-field are related, although one can expect the relation to be very complicated. To justify our proposal, one needs to show that $H^{\mu\nu\lambda}$ satisfies the correct equation of motion (i.e. the self-dual equation) and describes the correct number of on-shell degrees of freedom (i.e. 3). We will now construct an appropriate action to try to achieve these goals.

Let us start with the following action:

$$S_{M5} = -\frac{1}{4} \int_{\Sigma_5} \text{tr} \left( \frac{1}{6} \epsilon_{abcdef} H^{abc} H^{de5} + c_2 H^{db5} H_{abc} + c_3 H^{de5} H_{abd5} \right) \sqrt{-g},$$  \hspace{1cm} (5.22)

where $\Sigma_5 = \Sigma_5 \times S^1$ is the worldvolume of the M5-branes and (5.21) is to be used. We will consider a constant metric. The action (5.22) is the most general quadratic action that can be constructed out of the components $H^{abc}$ and $H^{a5b}$ and which is compatible with the $SO(1,4)$ Lorentz symmetry. For a non-Abelian generalization of the PST Lagrangian (5.18), it is expected that $c_2 = 1$ and $c_3 = 0$. Here we have allowed for more general possibility since there is no reason to expect that our action be exactly the same.

Our goal is to construct an action for the non-Abelian 3-form field strength living on a system of M5-branes. Generally, one can turn on a constant $C$-field on the worldvolume of the M5-branes. How could one incorporate a $C$-field in (5.22)? It is useful to recall a similar story for the case of D-branes where it is well known that a constant NSNS $B$-field can be naturally included as a classical solution (corresponds to a non-commutative geometry) of matrix models \[13–16\]. The remarkable feature of this construction is that the different backgrounds that correspond to different $B$-fields arise as different classical solutions of the same degrees of freedom of the underlying matrix model. Therefore, let us follow the same route and consider a reduction of the matrix model to a point. As a result, we get the matrix model

$$S_0 = \frac{1}{4} \text{Tr}(c_1 \epsilon_{abcdef} X^a X^b X^c X^d X^e X^5 + c_2 [X^a, X^b, X^c]^2 + c_3 [X^a, X^b, X^5]^2),$$  \hspace{1cm} (5.23)

where

$$c_1 = 2.$$  \hspace{1cm} (5.24)
and the parameters $c_2$, $c_3$ are to be determined. We will now require that the equation of motion of the matrix model agree with the self-duality condition of $H$. Quite remarkably, this can be achieved with a particular choice of the parameters.

The equations of motion of $S_0$ are
\[ c_1 \epsilon_{abed} X^a X^b X^d X^e + 2c_3 [ [X_a, X_b, X_4], X^e, X^b ]' = 0 \]
and
\[ c_1 \epsilon_{abed} (X^b X^d X^4 X^5 + X^b X^d X^5 X^4 + X^b X^d X^4 X^5 + X^b X^d X^4 X^5 + X^b X^d X^4 X^5) + 6c_2 [[X_a, X_b, X_4], X^b, X^e] + 4c_3 [ [X_a, X_b, X_4], X^b, X^5 ]' = 0. \]

The first equation (5.25) can be written as
\[ \left( \frac{c_1}{6} - c_3 \right) \epsilon_{abed} X^a X^b H^{cde} + 2c_3 \left[ H_{ab5} + \frac{1}{6} \epsilon_{abcd} H^{cde}, X^a, X^b \right]' = 0. \]

Since we want to interpret $H^{\mu \nu \lambda}$ of (5.21) as the non-Abelian field strength on M5-branes, $H^{\mu \nu \lambda}$ must satisfy a Bianchi identity. The most natural gauge covariant version would be
\[ [X^{[\mu}, H^{\nu \lambda]] = 0. \]

We use a convention of $[X^{[\mu}, H^{\nu \lambda]} = [X^\mu, H^{\nu \lambda}] - [X^\nu, H^{\mu \lambda}] + [X^\lambda, H^{\mu \nu}]$. We will comment on its possible origin later. Let us assume this condition holds, particularly
\[ [X^{[\mu}, H^{\nu \lambda]} = 0; \]
then we see that the self-duality condition
\[ H_{ab5} = -\frac{i}{2} \epsilon_{abed} H^{cde} \]
solves (5.25).

Next we turn to (5.26). Using conditions (5.29) and the self-duality condition (5.30), one can show that the LHS of (5.26) can be written as
\[ \left( \frac{c_1}{2} + 18c_2 \right) [H_{a5}, X^5 X^e] + \left( \frac{1}{4} B^{bcde}, X^5 \right) \epsilon_{abed} + \left( \frac{c_1'' - 2c_3}{3} \right) (X^b X^5 H^{cde} - H^{cde} X^b X^5), \]
where
\[ B^{bcde} := c_1' [H^{bcde}, X^5] - 12c_2 [H^{bc5}, X^d] \]
and the constants $c_1'$, $c_1''$ satisfy
\[ c_1' + c_1'' = c_1. \]

To get this, we have split the term proportional to $c_1$ of (5.26) into two terms (with coefficients $c_1'$ and $c_1''$) and used the $c_1'$ term to combine with the $c_2$ term and the $c_1''$ term to combine with the $c_3$ term to arrive at (5.31). We note that the term $B^{bcde}$ is of the form of the Bianchi identity
\[ [X^{[5}, H^{\mu \nu \lambda]] = 0 \]
if $c_1' = 4c_2$. Therefore, the equation of motion (5.26) is satisfied if the coefficients are such that
\[ c_1' = 4c_2, \quad c_1'' = -40c_2, \quad c_3 = -15c_2 \]
and the condition (5.34) is satisfied.

All in all, the equations of motion (5.25) and (5.26) are satisfied if the self-duality condition (5.30) and condition (5.28) are satisfied and if the coefficients $c_i$ are given by
\[ c_2 = -\frac{1}{18} (c_1/2), \quad c_3 = \frac{5}{6} (c_1/2). \]
It is amazing that a set of parameters can be found consistently so that the self-duality condition emerges from a matrix model. This is not guaranteed a priori and provides evidence that the matrix model (5.23) has something to do with a theory of self-dual 3-form field strength.

To get a six-dimensional field theory, we need to consider classical solutions to the equations of motion and incorporate the fluctuations around them to build the six-dimensional theory. An interesting class of solutions which are useful for this purpose is

\[ X^\mu = x^\mu \]

such that

\[ [x^\mu, x^\nu, x^\lambda] = i\theta^{\mu\nu\lambda} 1, \]

(5.37)

where \( \theta^{\mu\nu\lambda} \) are arbitrary constants. Clearly, condition (5.28) is satisfied. Moreover, the self-duality condition is satisfied if the parameter \( \theta^{\mu\nu\lambda} \) is self-dual. Thus, we obtain a six-dimensional quantum Nambu geometry parameterized by the self-dual parameter \( \theta^{\mu\nu\lambda} \).

The fluctuations around the solution can be written as

\[ X^\mu = x^\mu 1_{K \times K} + A^\mu(x), \]

(5.38)

where \( A^\mu \) are \( K \times K \) matrices. The large \( N \) trace becomes

\[ \text{Tr} = \int_x \text{tr}, \]

(5.39)

where \( \int_x \) is determined from the representations of the quantum Nambu geometry (5.37) and our proposal for a theory of \( K \) M5-branes (or more precisely, \( K \) non-Abelian 3-form) is

\[ S_{M5,0} = -\frac{1}{4} \int_x \text{tr} \left( \frac{1}{6} \epsilon_{abcde} H^{abcde} H^{de5} + \left( \alpha - \frac{1}{3} H^{abc} H_{abc} + (1 - \alpha) H^{ab5} H_{ab5} \right) \sqrt{-g} \right). \]

(5.40)

with \( \alpha = 1/6 \). We note that with the self-duality condition, the second and the third term in (5.40) can be summed together and is equal to \( H^{ab5} H_{ab5} \) for any value of \( \alpha \); therefore, the action (5.40) has in fact precisely the same form (including the coefficients) as the non-Abelian generalization of (5.18). However, only for \( \alpha = 1/6 \) can one identify a Bianchi identity and the self-duality condition. The action (5.40) is invariant under the gauge symmetry

\[ A^\mu \rightarrow U^{-1}[x^\mu, U] + U^{-1} A^\mu U, \]

(5.41)

where \( U \) is any \( U(K) \) matrix whose entries are functions of \( x^\mu \). Since \( [x^\mu, U] \) is a linear operation on \( U \) which satisfies the Leibniz rule, we can denote it as

\[ [x^\mu, U] = \partial^\mu U. \]

(5.42)

Generally, the derivative \( \partial^\mu \) depends on the representation being considered and is not the same as the derivative in the coordinate basis.

We remark that our M5-branes system has a quantum Nambu geometry as its worldvolume geometry. What is the physical origin responsible for this quantized spacetime? The emergence of a noncommutative worldvolume on a brane is typically the result of a background gauge potential being turned on in its worldvolume. The fact that the quantization parameter \( \theta^{\mu\nu\lambda} \) is self-dual suggests us to identify it with the self-dual 3-form \( C \)-field on the worldvolume of the M5-branes. This identification is further supported by the fact that if we dimensionally reduced the M5-branes, say on the fifth direction, which amounts to putting \( X^5 = 1 \), then the relation (5.37) reads

\[ [X^\mu, X^\nu, 1] = [X^\mu, X^\nu] = i\theta^{\mu\nu\lambda}. \]

(5.43)

This is the noncommutative geometry over D4-branes with a \( B \)-field whose components are \( B_{\mu\nu} = \theta_{\mu\nu5} \) (we remind the readers that we are considering the linearized limit). Since the \( B \)-field is related to the 11-dimensional \( C \)-field as \( B_{\mu\nu} = C_{\mu\nu5} \), it is correct to identify \( \theta^{\mu\nu\lambda} \) with the constant \( C \)-field \( C^{\mu\nu\lambda} \). All in all, we conclude that the geometry (5.37) is the result of
having a self-dual 3-form $C$-field

$$C_{\mu \nu \lambda} = \theta_{\mu \nu \lambda} \quad (5.44)$$

turned on in the worldvolume of the M5-branes.

In our description, the field strength $H^{\mu \nu \lambda}$ is constructed from the 1-form potentials $A^\mu$ using (5.21) and (5.38). In a conventional description of 3-form field strength, a 2-form potential is used. Our analysis suggests that there may in fact be two equivalent formulations for the theory of multiple M5-branes in a $C$-field, one in terms of a 1-form gauge field as in ours (5.40) and the conventional formulation in terms of a 2-form gauge potential.

Evidence of this can be seen from the counting of the degrees of freedom of our model. Initially we have six fields. If the self-duality equation is in fact the equation of motion of the theory, then the degrees of freedom are reduced to half and we have indeed three degrees of freedom, which is appropriate for a description of a self-dual 3-form field strength. The theory (5.40) would then have all the desirable properties of a theory of non-Abelian self-dual 3-form field strength except that the theory is written manifestly using a 1-form potential as the variables.

A couple of comments on the dual formulation are in order:

1. As noted above, our action (5.40) is equal to the non-Abelian form of the PST action (5.18) when the self-duality condition is satisfied. The agreement of the actions on-shell is a necessary condition for our formulation to be an equivalent description on-shell. Therefore, this agreement provides more support that our proposed action (5.40) indeed provides a dual description of the non-Abelian self-dual 3-form.

2. Our formulation of using a 1-form gauge field $A_\mu$ is supposed to be equivalent to the conventional formulation of using a 2-form gauge potential $B_{\mu \nu}$ only on-shell. As such there can be a relation between the 2-form gauge field and the 1-form gauge field only on-shell. Identifying such a $B$-field from our description is important as it would allow us to couple to self-dual strings.

3. In the conventional formulation, the existence of the tensor gauge symmetry and the self-duality equation is crucial in reducing the 15 components of $B$ to 3. In our formulation, we do get the desired number of degrees of freedom (modulo the issues discussed above) and there is no need of a tensor gauge symmetry. Curiously, in a recent construction of the non-Abelian 3-form theory using a 2-form $B$-field potential [44], it was shown that the tensor gauge symmetry (part of the $G \times G$ symmetry structure constructed there) could be gauge fixed to an ordinary gauge symmetry $G$ (diagonal part of $G \times G$). It is interesting that the gauge-fixed theory has precisely the same gauge symmetry as our proposed description here. This coincidence provides some support to both the description proposed in [44] and the description proposed here.

4. In the above, we have obtained the Bianchi identity and the self-duality condition as a solution of the reduced matrix description. However, to fully justify our proposal, we need to establish that it is the only nontrivial solution. We recall that in the PST action (5.18), one does not get the self-duality condition (5.14) as the equation of motion immediately. To do this, one needs to make crucial use of the symmetry (5.11) which acts on the $B$-field. For our case, it is possible that there is a counterpart of the symmetry (5.11) which acts on the $X$’s; and this symmetry is needed to derive the self-duality equation (and hence the Bianchi identity). It is important to understand whether such a symmetry really exists in our model, and if so, how it acts.

Another possible way to settle the issue is to supersymmetrize our action with (1,0) or (2,0) supersymmetry since supersymmetry would require the 3-form field strength to be self-dual automatically. Supersymmetrization of our system is also needed for describing
M5-branes. In any case, supersymmetry is an important topic and we hope to return to it in future work. See [45] for some recent related works on (2,0) supersymmetry of a non-Abelian self-dual 3-form field strength multiplet.

6. Discussions

In this paper, we have achieved the goal of finding a novel kind of quantum geometry in string theory. The geometry we found is characterized by a quantum Nambu bracket and is intrinsically different from the usual Lie-algebraic-type noncommutative geometry. Starting with an analysis of the D1-branes matrix model, we arrive at the action (5.40) which we proposed to be the theory of the non-Abelian self-dual tensor living on multiple M5-branes. We also found that the worldvolume of the M5-branes in a C-field is described by a quantum Nambu geometry with self-dual parameters $\theta^{\mu \nu \lambda} = C^{\mu \nu \lambda}$. These are the main results of the paper.

It is intriguing that there seems to be a dual formulation of the theory of non-Abelian self-dual 3-form field strength in terms of a 1-form gauge potential. We have discussed various aspects of this dual formulation and how this may be related to the conventional formulation in terms of a 2-form gauge field. As a proposed description for the worldvolume theory of multiple M5-branes, it is interesting to understand how (5.40) could be reduced to the non-abelian Yang–Mills theory of D4-branes. It is also necessary to include supersymmetry in (5.40).

It is an interesting result that the worldvolume of an M5-brane in a C-field background is described by a quantum Nambu geometry with self-dual parameters. One may wonder how to obtain this result by a quantization of an open M2-brane in the presence of a C-field. However, it appears that treating the mixed boundary condition as a constraint and canonically quantizing the system may not be the best way to proceed [46]. It may be possible that a different choice of the quantization variables and a reformulation of the quantization is necessary.

Our proposed action (5.40) for multiple M5-branes in a C-field (5.44) is defined on a quantum Nambu geometry. It is instructive to recall that in the case of D-branes with a B-field, the worldvolume action can be expressed either in terms of a commutative language as a Dirac–Born–Infeld action or in terms of noncommutative geometry, as a much simpler noncommutative Yang–Mills action. This remarkable equivalence is established with the Seiberg–Witten map [7]. In our case, the action for a system of M5-branes with the C-field can in principle be constructed as a non-Abelian generalization of the Abelian PST action (linearized or nonlinear) with the C-field. This action has not been constructed (see however [44] and also [47, 48] for some recent proposals for the case with $C = 0$), but in any case can be expected to be very complicated. Our proposed action (5.40) takes a much simpler form. Like the noncommutative Yang–Mills action, it is supposed to be equal to the full form of a non-Abelian nonlinear PST action with a C-field via some kind of mapping like the Seiberg–Witten map. The understanding of how symmetries are realized in the different models is important in understanding this Seiberg–Witten map.

We have presented a preliminary analysis of the properties of the quantum Nambu geometry. For the usual noncommutative geometry of the Moyal type, the existence of a noncommutative parameter leads to a number of very interesting physical effects [49]. It will be interesting to construct quantum field theory on the quantized Nambu space and to uncover the novel physical effects associated with the existence of the Nambu scale parameter $\theta$. In the dual language, one can try to formulate the operatoric quantum Nambu geometry as an algebra of functions with a $\ast$-product. The construction of the $\ast$-product will be very interesting and will be very helpful in the construction of the quantum field theory.
In the usual noncommutative geometry of the Moyal type, a fluctuation analysis typically leads to a (noncommutative) Yang–Mills gauge theory whose field strength is defined by a commutator. We have performed a small fluctuation analysis around the quantum Nambu geometry describing a system of D4-branes and found that the action can be written in terms of a three-form field strength defined by the quantum Nambu bracket. We have also argued for a dual formulation of the theory of a 3-form field strength whose field strength is defined in terms of the quantum Nambu bracket of a 1-form gauge field. The pattern seems to be that a non-Abelian $N$-form gauge theory is naturally associated with a quantum geometry defined by a quantum $N$-bracket. It will be interesting to develop this further and understand better the properties of the gauge symmetry for these higher-form gauge theories.

A close variant of the quantum Nambu geometry is the geometry defined by the following relation:

$$[X^i, X^j, X^k] = i\lambda \epsilon^{ijk} X^l,$$

which is also defined using the quantum Nambu 3-bracket. In the case of Lie 2-bracket, the fuzzy geometry could be obtained in a matrix model where the Moyal-type noncommutative geometry is a solution by adding a mass term. It would be interesting to see whether (6.1) also arises in string theory and in particular from a modification of our D1-brane matrix model with additional terms. The geometry (6.1) with a Lie 3-bracket has played an important role in the studies of multiple M2-branes [22]. One may wonder if (6.1) for a quantum Nambu bracket may also play some role in the physics of M-branes. It is also interesting to understand better the mathematical properties of (6.1), e.g. its center and representations, and to construct quantum field theory of this kind of quantum space as well.

Finally we emphasize that the quantum Nambu bracket we considered in this paper should not be confused with the quantization of the Nambu–Poisson bracket. The Nambu–Poisson bracket was introduced by Takhtajan [35] and obeys the fundamental identity. Its quantization is a very difficult mathematical problem and so far there is no satisfactory solution to it. It is an interesting question as to whether a quantum Nambu–Poisson bracket, or higher n-ary structures which are constrained by a fundamental identity [50], appears in the description of quantum geometry in string theory. On the other hand, it is very possible that one could generalize the consideration of this paper by considering higher form flux and find a higher $N$-bracket (completely antisymmetrized sum of $N$ elements) quantum geometry in string theory.

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