Toric degenerations of Grassmannians and Schubert varieties from matching field tableaux

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Abstract

We study the combinatorics of Gröbner degenerations of Grassmannians and the Schubert varieties inside them. We provide a family of binomial ideals whose combinatorics is governed by tableaux induced by matching fields in the sense of Sturmfels and Zelevinsky in [SZ93]. We prove that these ideals are all quadratically generated and they yield a SAGBI basis of the Plücker algebra. This leads to a new family of toric degenerations of Grassmannians. Moreover, we apply our results to construct a family of Gröbner degenerations of Schubert varieties inside Grassmannians. We provide a complete characterization of toric ideals among these degenerations in terms of the combinatorics of matching fields, permutations and semi-standard tableaux.

Keywords: Toric degenerations, SAGBI bases, Khovanskii bases, Grassmannians, Schubert varieties, Semi-standard Young tableaux

1. Introduction

Computing toric degenerations of varieties is a powerful tool to extend the deep relationship between combinatorics and toric varieties to more general spaces [And13]. Given a projective variety, a toric degeneration is a flat family whose fiber over zero is a toric variety and all its other fibers are isomorphic to the original variety. Hence, toric degenerations enable us to use the tools developed in toric geometry to study more general varieties.

Toric degenerations of Grassmannians, flag and Schubert varieties have been extensively studied in the literature, see e.g. [GL96, Cal02, FFL17]. The main example of such degenerations is the Gelfand-Cetlin degeneration which is studied with semi-standard tableaux and Gelfand-Cetlin polytopes [ACK18, KM05]. For the Grassmannian, this example has been generalized, see e.g. [SS04, Wit15, RW, BFF+18]. For example, Rietsch and Williams describe a family of toric degenerations of Grassmannians arising from plabic graphs [RW]. Recently, Kaveh and Manon have used tools from tropical geometry to study toric degenerations of ideals in general [KM16]. The main idea is that the initial ideals associated to the top-dimensional facets of tropicalizations of ideals are good candidates for toric degenerations. A similar approach has been taken in studying toric degenerations of Gr(3, n)
in [MS18], and small flag varieties in [BLMM17]. More precisely, tropical Grassmannians defined by Speyer and Sturmfels in [SS04], provide a nice framework for studying toric degenerations of Grassmannians. In [MS18], Mohammadi and Shaw have used this framework together with the theory of matching fields from [SZ93] to show that every coherent matching field has a canonical toric ideal that can be identified as the toric component of the initial ideal associated to a top-dimensional facet of $\text{trop}(\text{Gr}(k,n))$. This, in particular, leads to a family of toric degenerations for $\text{Gr}(3,n)$. Our work extends results from [MS18] to higher-dimensional Grassmannians.

In this work, we are interested in toric degenerations of Grassmannians and Schubert varieties inside them from the point-of-view of algebraic combinatorics. In other words, we have a positive answer to Degeneration Problem, posed by Caldero in [Cal02], for Schubert varieties inside Grassmannians (see [FFL17] and references therein, for other such examples).

Let $\Lambda(k,n)$, or $\Lambda$ when there is no confusion, be a set of $k \times 1$ tableaux of integers corresponding to all $k$-subsets of $[n] = \{1, \ldots, n\}$. By combining tableaux from $\Lambda$ side by side, we can construct larger tableaux. We say a pair of row-wise equal tableaux differ by a swap if they are the same for all but two columns. Moreover, two row-wise equal tableaux are called quadratically equivalent if they can be obtained from each other by a series of swaps. Now, let $R = \mathbb{K}[P]$ denote the polynomial ring with one variable associated to each $k$-subset of $[n]$. For any pair of quadratically equivalent tableaux $T$ and $T'$ one can write a binomial $P_T - P_{T'}$. The ideal $J_{\Lambda} \subset R$ is generated by all such binomials. In other words, two tableaux are quadratically equivalent if and only if their corresponding monomials are equal in $R/J_{\Lambda}$. This ideal is implicitly defined by Sturmfels and Zelevinsky to address questions about determinantal varieties; $\Lambda$ is the matching field defined in [SZ93]. In [FR15], Fink and Rincón provided a link between tropical linear spaces and a specific family so-called coherent matching fields. From a combinatorial point-of-view, it is much more convenient to work with the tableau description of a matching field than the classical definition. In [MS18] Mohammadi and Shaw used the tableaux description of coherent matching fields to study the tropicalization of Grassmannians and provided a necessary combinatorial condition for a top-dimensional facet of $\text{trop}(\text{Gr}(k,n))$ to lead to a toric degeneration of $\text{Gr}(k,n)$. When $J_{\Lambda}$ is quadratically generated, this condition is also sufficient. In particular, for $\text{Gr}(3,n)$, the authors of [MS18] provided a family of matching fields, called 2-block diagonal, whose ideals are quadratically generated and hence, they lead to a family of toric degenerations of $\text{Gr}(3,n)$. Our main result generalizes this to higher-dimensional Grassmannians and shows that, quite surprisingly, this construction leads to many non-isomorphic toric varieties.

In this paper, we study the Plücker ideals of $\text{Gr}(k,n)$, denoted by $G_{k,n}$, and their associated algebras from the point-of-view of Gröbner basis theory and SAGBI basis theory. Namely, we extend the family of block diagonal matching fields defined in [MS18] from $\text{Gr}(3,n)$ to higher-dimensional Grassmannians (see Definition 2.3). Then, we explicitly construct a weight vector for every such matching field and we study its corresponding initial ideal $I_{\Lambda}(G_{k,n})$ (see Definition 2.5). We prove that such ideals are equal to their corresponding matching field ideals $J_{\Lambda}$ and, in particular, they are all toric. We remark that a general matching field rarely gives rise to a toric or even a binomial initial ideal. Moreover, we
prove that the initial ideals in\( \text{in}_w(G_{k,n}) \) are all quadratically generated. Note that describing a minimal generating set of toric ideals or even finding an upper bound for the degree of the generators is a difficult problem. Such questions are usually studied for special families of ideals with combinatorial structures, see e.g. [Whi80, OH99, LM14, DM14].

We describe a minimal generating set of the associated Plücker algebra of in\( \text{in}_w(G_{k,n}) \) in terms of its corresponding matching field tableaux. This is shown by explicitly constructing a SAGBI basis for the Plücker algebra. In combinatorial terms, for each matching field we construct a collection of \( k \times 2 \) tableaux such that every \( k \times 2 \) tableau is row-wise equal to exactly one tableau in the collection. Then, we show that this collection is in bijection with \( k \times 2 \) tableaux in semi-standard form, which provide a SAGBI basis for the Plücker algebra in the Gelfand-Cetlin case. Hence, our results directly generalize the analogous results from the classical Gelfand-Cetlin case, see e.g. [MS05, Theorem 14.11].

The paper is structured as follows: In §2, we fix our notation and introduce our main objects of study. We define the Plücker ideal, block diagonal matching fields and their associated ideals. We also define Schubert varieties and their Gröbner degenerations using matching field ideals (see Definition 2.6). We phrase our goal in studying such degenerations as Question 5.1. In §3, we define tableaux arising from matching fields. Basic properties of tableaux are studied in §3.1. Moreover, in §3.2, we study the matching field tableaux from the point-of-view of algebras, and in Lemmas 3.9 and 3.10, we provide a SAGBI basis for the Plücker algebras associated to block diagonal matching fields. These are the core lemmas needed for our main results. In §4, we study the ideals of block diagonal matching fields from the viewpoint of SAGBI basis theory and Gröbner basis theory. In Theorem 4.1, we show that they are all quadratically generated. Then, we study their associated initial algebras and in Theorem 4.3, we prove that the Plücker variables form a SAGBI basis for the Plücker algebra. Hence, we obtain a family of toric degenerations of Gr\((k,n)\). We then turn our attention to Schubert varieties in §5. Here, our main result is Theorem 5.7 in which we provide a complete characterization of toric ideals arising from block diagonal matching fields for Schubert ideals.

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2. Preliminaries

Throughout we set \([n] := \{1, \ldots, n\}\) and we use \( I_{k,n} \) to denote the collection of subsets of \([n]\) of size \(k\). The symmetric group on \(n\) elements is denoted by \(S_n\). We also fix a field \(\mathbb{K}\) with char\((\mathbb{K}) = 0\). We are mainly interested in the case when \(\mathbb{K} = \mathbb{C}\).
2.1. Flag varieties and Grassmannians

The set of full flags
\[ \{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{K}^n \]
of vector subspaces of \( \mathbb{K}^n \) with \( \dim_k(V_i) = i \) is called the flag variety denoted by \( \text{Fl}_n \), which is naturally embedded in a product of Grassmannians using Plücker variables. Each point in the flag variety can be represented by an \( n \times n \) matrix \( X = (x_{i,j}) \) whose first \( k \) rows generate \( V_k \) which corresponds to a point in the Grassmannian \( \text{Gr}(k,n) \). Therefore, the ideal of \( \text{Fl}_n \), denoted by \( I_n \) is the kernel of the map,

\[ \varphi_n : \mathbb{K}[P_J : \emptyset \neq J \subseteq \{1, \ldots, n\}] \to \mathbb{K}[x_{i,j} : 1 \leq i \leq n-1, 1 \leq j \leq n] \]
sending each Plücker variable \( P_J \) to the determinant of the submatrix of \( X \) with row indices \( 1, \ldots, |J| \) and column indices in \( J \). Similarly, we define the Plücker ideal of \( \text{Gr}(k,n) \), denoted by \( G_{k,n} \), as the kernel of the map \( \varphi_n \), restricted to the ring with variables \( P_J \) with \( |J| = k \).

2.2. Schubert varieties

Let \( \text{SL}(n, \mathbb{C}) \) be the set of \( n \times n \) matrices with determinant 1, and let \( B \) be its subgroup consisting of upper triangular matrices. There is a natural transitive action of \( \text{SL}(n, \mathbb{C}) \) on the flag variety \( \text{Fl}_n \) which identifies \( \text{Fl}_n \) with the quotient group \( \text{SL}(n, \mathbb{C})/B \), since \( B \) is the stabilizer of the standard flag \( 0 \subset \langle e_1 \rangle \subset \cdots \subset \langle e_1, \ldots, e_n \rangle = \mathbb{C}^n \). Given a permutation \( w \in S_n \), \( \sigma_w \) is the \( n \times n \) matrix whose only non-zero entries are 1 in position \((w(i), i)\) for each \( 1 \leq i \leq n \). By the Bruhat decomposition, we can write \( \text{SL}(n, \mathbb{C})/B = \bigsqcup_{w \in S_n} B\sigma_w B/B \).

Given a permutation \( w \), its Schubert variety is

\[ X(w) = \overline{B\sigma_w B/B} \subseteq \text{Fl}_n \]

which is the Zariski closure of the corresponding cell in the Bruhat decomposition. The associated ideal of the Schubert variety \( X(w) \) is \( I(X(w)) = I_n + \langle P_I : I \in S_w \rangle \), where

\[ S_w = \{ I : I \subset [n] \text{ with } I = (i_1, \ldots, i_{|I|}) \leq (w_{i_1}, w_{i_2}, \ldots, w_{i_{|I|}}) \}, \]

and \( w_{i_1} < w_{i_2} < \cdots < w_{i_{|I|}} \) is obtained by ordering the first \( |I| \) entries of \( w \). Here, \( I_n \) is the Plücker ideal defined in §2.1 and \( \leq \) is the component-wise partial order on \( I_{k,n} \).

Similarly, we can study the Schubert varieties inside \( \text{Gr}(k,n) \). The permutations giving rise to distinct Schubert varieties in \( \text{Gr}(k,n) \) are of the form \( w = (w_1, \ldots, w_n) \) where \( w_1 < \cdots < w_k, w_{k+1} < \cdots < w_n \) and \( w_k > w_{k+1} \). Therefore, it is enough to record the permutations of \( S_n \) as \( w = (w_1, \ldots, w_k) \) which will be called a Grassmannian permutation. In this case, we take the subset \( S_{w,k} = S_w \cap \{ I : |I| = k \} \) of \( S_w \).

2.3. Matching fields

Following [MS18], we define matching fields as follows. Some of the most important features of this section are that each matching field induces a canonical toric ideal, and gives rise to a weight vector for the Plücker variables \( P_I \) for \( I \in I_{k,n} \).
Given integers $k$ and $n$, a matching field denoted by $\Lambda(k, n)$, or $\Lambda$ when there is no confusion, is a choice of permutation for each $I \in I_{k,n}$. We think of $I$ as being identified with the Plücker form $P_I$. Given a matching field $\Lambda$ and a subset $I = \{i_1, \ldots, i_k\} \subset [n]$ we consider the set to be ordered by $i_1 < \cdots < i_k$. We think of the permutation $\sigma = \Lambda(I)$ as inducing a new ordering on the elements of $I$, where the position of $i_s$ is determined by the value of $\sigma(s)$. We represent the variable $P_I$ as a $k \times 1$ tableau where the entry of $(\sigma(r), 1)$ is $i_r$.

Let $X = (x_{i,j})$ be a $k \times n$ matrix of indeterminants. To every $k$-subset $I$ of $[n]$ with $\sigma = \Lambda(I)$ we associate the monomial $x_{\Lambda(I)} := x_{\sigma(1)i_1}x_{\sigma(2)i_2} \cdots x_{\sigma(k)i_k}$. The matching field ideal $J_\Lambda$ is defined as the kernel of the monomial map,

$$\phi_\Lambda: \mathbb{K}[P_I] \to \mathbb{K}[x_{ij}] \quad \text{with} \quad P_I \mapsto \text{sgn}(\Lambda(I))x_{\Lambda(I)},$$

where $\text{sgn}(\Lambda(I))$ denotes the signature of the permutation $\Lambda(I)$ for each $I \in I_{k,n}$.

**Definition 2.1.** A matching field $\Lambda$ is coherent if there exists an $n \times n$ matrix $M$ with entries in $\mathbb{R}$ such that for every $I \in I_{k,n}$ the initial of the Plücker form $P_I \in \mathbb{K}[x_{ij}]$ is in $\text{in}_M(P_I) = \phi_\Lambda(P_I)$, where in$_M(P_I)$ is the sum of all terms in $P_I$ of the lowest weight and the weight of a monomial $m$ is the sum of entries in $M$ corresponding to the variables in $m$. In this case, we say that the matrix $M$ induces the matching field $\Lambda$. The weight of each variable $P_I$ is defined as the minimum weight of the terms of the corresponding minor of $M$, and it is called the weight induced by $M$.

**Example 2.2.** Consider the matching field $\Lambda(3, 5)$ which assigns to each subset $I$ the identity permutation. Consider the following matrix:

$$M = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
5 & 4 & 3 & 2 & 1 \\
9 & 7 & 5 & 3 & 1
\end{bmatrix}.$$

The weights induced by $M$ on the variables $P_{123}, P_{124}, \ldots, P_{345}$ are $9, 7, 5, 6, 4, 3, 6, 4, 3, 3$, respectively. Thus, for each $I = \{i, j, k\}$ we have that in$_M(P_I) = x_{i1}x_{j2}x_{k3}$ for $1 \leq i < j < k \leq 5$. Therefore, the matrix $M$ induces $\Lambda(3, 5)$. Below are the tableaux representing $P_I$ for each $I$:

1. $\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 \\
2 & 2 & 2 & 3 & 3 & 4 & 3 & 4 & 4 & 4
\end{array}$

Notice that each initial term in$_M(P_I)$ arises from the leading diagonal. Such matching fields are called diagonal. See, e.g. [SZ93, Example 1.3].

**Definition 2.3.** Given $k, n$ and $0 \leq \ell \leq n$, we define the block diagonal matching field denoted by $B_\ell = (1 \cdots \ell | \ell + 1 \cdots n)$ as the map from $I_{k,n}$ to $S_k$ such that

$$B_\ell(I) = \begin{cases}
\text{id} & \text{if } |I| = 1 \text{ or } |I \cap \{1, \ldots, \ell\}| \neq 1, \\
(12) & \text{otherwise}.
\end{cases}$$

These matching fields are called 2-block diagonal in [MS18]. Note that $\ell = 0$ or $n$ gives rise to the classical diagonal matching field as in Example 2.2.
Example 2.4. Given \( k, n \) and \( 0 \leq \ell \leq n \), the matching field \( B_\ell \) is induced by the matrix:

\[
M_\ell = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\ell & \ell - 1 & \cdots & 1 & n & n - 1 & \cdots & \ell + 1 \\
2n & 2(n - 1) & \cdots & 10 & 8 & 6 & 4 & 2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
n(k - 1) & (n - 1)(k - 1) & \cdots & 5(k - 1) & 4(k - 1) & 3(k - 1) & 2(k - 1) & k - 1
\end{bmatrix},
\]

and hence, it is coherent. We denote \( w_\ell \) for the weight vector induced by \( M_\ell \).

Definition 2.5. Given a block diagonal matching field \( B_\ell \), we denote the initial ideal of \( G_{k,n} \) with respect to \( w_\ell \) by \( \text{in}_{w_\ell}(G_{k,n}) \) and we define it as the ideal generated by polynomials \( \text{in}_{w_\ell}(f) \) for all \( f \in G_{k,n} \), where

\[
\text{in}_{w_\ell}(f) = \sum_{a_j : w_\ell = d} c_{a_j} P^{a_j} \quad \text{for} \quad f = \sum_{i=1}^t c_{a_i} P^{a_i} \quad \text{and} \quad d = \min\{a_i \cdot w_\ell : i = 1, \ldots, t\}.
\]

Definition 2.6. Given a block diagonal matching field \( B_\ell \) and a permutation \( w \in S_n \) we define the ideal

\[
G_{k,n,\ell,w} := \text{in}_{w_\ell}(G_{k,n})|_{P_I = 0} \quad \text{for} \quad I \in S_{w,k}.
\]

This is equivalent to adding the variables \( P_I \), where \( I \in S_{w,k} \), to the initial ideal and then eliminating them. This can be computed using the function \text{eliminate} in Macaulay2 [GS]:

\[
G_{k,n,\ell,w} = \text{eliminate} (\text{in}_{w_\ell}(G_{k,n}) + \langle P_I : I \in S_{w,k} \rangle), \{P_I : I \in S_{w,k}\}).
\]

We say that the variable \( P_I \) vanishes in \( G_{k,n,\ell,w} \) if \( I \in S_{w,k} \).

3. Matching field tableaux

The tableaux arisen in the theory of matching fields in [SZ93, MS18] will be the main tool in the proofs of our main results in §4. Here, we prove important properties about these tableaux. This section, while elementary, is the most technical part of the paper. We provide illustrative examples to make it easier to follow the proofs. The main ingredients needed for our main theorems in §4, about Grassmannians, are Lemmas 3.4 and 3.6 for Theorem 4.1, and Lemmas 3.9, 3.10 and 3.11 for Theorem 4.3. Other results are used to establish these main ingredients.

Throughout this section we fix a block diagonal matching field \( B_\ell = (B_1 | B_2) = (1 \cdots \ell | \ell + 1 \cdots n) \). For each collection \( \mathcal{I} = \{I_1, \ldots, I_t\} \) of non-empty subsets of \( [n] \) we denote by \( T_\mathcal{I} \) or, when there are few columns, by \( T_{I_1 \ldots I_t} \) the tableau with columns \( I_i \). The order of the elements in each column is given by the matching field \( B_\ell \).
\[ T_X \] \[ T_Y \]

\[
\begin{array}{ccccccc}
1 & 1 & 2 & 2 & 3 & 5 & 4 & 5 \\
2 & 2 & 3 & 4 & 4 & 6 & 1 & 3 \\
3 & 4 & 4 & 5 & 6 & 7 & 5 & 6 \\
4 & 6 & 7 & 7 & 8 & 8 & 7 & 8 \\
\end{array}
\]

Figure 1: An example of a tableau for Gr(4, 8) with block diagonal matching field \( B_4 = (1234|5678) \). The columns appearing in the table from left to right are of type 4, 3, 3, 2, 2, 0, 1, 1. The tableau is partitioned into \( T_X \), whose column entries appear in increasing order, on the left and \( T_Y \) on the right.

3.1. Quadratic relations among tableaux

Definition 3.1. Given a tableau \( T \), for each column \( I \in T \) we say that \( I \) is of type \( |I \cap B_1| \).

Importantly, if two tableaux are row-wise equal, then they have the same number of columns of each type. Also the columns in \( T \) whose entries do not appear in increasing order are exactly those of type 1. We distinguish these columns with the following notation.

Let \( X \) be the collection of all \( I \in I_{kn} \) for which \( B_r(I) = id \) and let \( Y \) be the collection of all \( I \in I_{kn} \) for which \( B_r(I) = (12) \). Equivalently, \( Y \) is the collection of all \( I \) of type 1 and \( X \) is the collection of all \( I \) of type 0, 2, 3, \ldots, \( k \). We denote by \( T_X \) the sub-tableau of \( T \) whose columns lie in \( X \) and similarly \( T_Y \) the columns of \( T \) which lie in \( Y \). By convention, we write \( T = [T_X \mid T_Y] \) with \( T_X \) on the left and \( T_Y \) on the right. For example, see Figure 1.

Definition 3.2. Suppose that \( T \) and \( T' \) are two tableaux that are row-wise equal. We say that \( T \) and \( T' \) differ by a swap, or \( T' \) is obtained from \( T \) by a quadratic relation, if \( T \) and \( T' \) are the same for all but two columns. If \( T \) and \( T' \) differ by a sequence of swaps, then we say \( T \) and \( T' \) are quadratically equivalent.

Example 3.3. Consider the diagonal matching field for Gr(3, 6) and the tableaux \( T_1 \) and \( T_3 \) below. Then \( T_1 \) is quadratically equivalent to \( T_3 \) with the following sequence of tableaux,

\[
T_1 = \begin{array}{ccc}
2 & 1 & 3 \\
3 & 2 & 4 \\
4 & 6 & 5 \\
\end{array}, \quad T_2 = \begin{array}{ccc}
2 & 1 & 3 \\
3 & 2 & 4 \\
4 & 5 & 6 \\
\end{array}, \quad T_3 = \begin{array}{ccc}
1 & 2 & 3 \\
2 & 3 & 4 \\
4 & 5 & 6 \\
\end{array}.
\]

We say that a tableau is in semi-standard form if its columns are strictly increasing and its rows are weakly increasing. In this example, \( T_3 \) is in semi-standard form. More generally for the diagonal matching field, if \( T \) is a tableau then there exists a unique semi-standard tableau \( T' \) which is row-wise equal to \( T \). Moreover, \( T' \) is obtained from \( T \) by a sequence of quadratic relations. This follows immediately by applying the following construction to every pair of columns in \( T \). For any two \( k \)-subsets \( I = \{i_1 < i_2 < \cdots < i_k\} \) and \( J = \{j_1 < j_2 < \cdots < j_k\} \), we let \( I' = \{\min\{i_1, j_1\}, \min\{i_2, j_2\}, \ldots, \min\{i_k, j_k\}\} \) and \( J' = \{\max\{i_1, j_1\}, \max\{i_2, j_2\}, \ldots, \max\{i_k, j_k\}\} \). Then \( T_{IJ} \) and \( T_{I'J'} \) are row-wise equal, and \( T_{I'J'} \) is in semi-standard form.
Let $T$ be a tableau. By the above, we may change $T_X$, by a series of quadratic relations, into semi-standard form. Similarly we may transform $T_Y$ so that the entries along each row are weakly increasing and we call this the semi-standard form of $T_Y$. Unless stated otherwise, we assume that any tableau $T$ is written with each part, $T_X$ and $T_Y$, in semi-standard form, see, e.g. Figure 1.

In the following two lemmas, we show how to make row-wise equal tableaux similar, by a sequence of quadratic relations.

**Lemma 3.4.** Suppose that $T$ and $T'$ are row-wise equal tableaux and the leftmost column of each is of type $i$ where $i \in \{2, 3, \ldots, k\}$. Then there exist tableaux $S$ and $S'$ quadratically equivalent to $T$ and $T'$, respectively, such that $S$ and $S'$ contain an identical column of type $i$.

**Proof.** Let $A = (a_1, \ldots, a_k)^{tr}$ be the leftmost column of $T$ and assume that it does not appear as a column in $T'$. Let $B$ be the leftmost column of $T'$ and similarly assume that it does not appear in $T$. Note that $A$ and $B$ have the same type and,

$$B = (a_1, b_2, a_3, \ldots, a_i, b_{i+1}, \ldots, b_k)^{tr}$$

for some $b_{i+1}, \ldots, b_k$.

If $b_2 \neq a_2$ then, without loss of generality, suppose that $a_2 < b_2$. So $a_2$ appears in the second row of $T'$ and we may swap $a_2$ and $b_2$ in $T'$. Thus, we may assume that $b_2 = a_2$.

Suppose $j$ is the smallest index for which $b_j \neq a_j$. Without loss of generality we may assume that $a_j < b_j$. So $a_j$ must appear in the $j^{th}$ row of $T'$. Suppose $a_j$ appears in column $C = (c_1, \ldots, c_k) \in T'$ where $c_j = a_j$. We define $B' = (a_1, \ldots, a_{j-1}, a_j, \ldots, c_k)^{tr}$ and $C' = (c_1, \ldots, c_{j-1}, b_j, \ldots, b_k)^{tr}$. Note that $B'$ and $C'$ are valid tableaux because $a_{j-1} < a_j = c_j$ and $c_{j-1} < c_j = a_j < b_j$. So we may apply the following relation,

$$\begin{array}{c|c}
B & C \\
\hline
a_1 & c_1 \\
\vdots & \vdots \\
a_{j-1} & c_{j-1} \\
b_j & c_j \\
\vdots & \vdots \\
b_k & c_k \\
\end{array} = \begin{array}{c|c}
B' & C' \\
\hline
a_1 & c_1 \\
\vdots & \vdots \\
\end{array}$$

After applying this relation, we have reduced the number of differences in columns $A$ and $B$. So by induction, we can find a sequence of swaps after which the leftmost column of $T$ and $T'$ are equal. \hfill $\Box$

**Example 3.5.** Consider $\text{Gr}(3, 6)$ with block diagonal matching field $B_3 = (123|456)$. Each tableau $T$ below is partitioned with $T_X$ on the left and $T_Y$ on the right, where each part is in semi-standard form. We show that $T_1$ is quadratically equivalent to $T_3$ with the following sequence of quadratic relations. Firstly, we may swap 2 and 3 in the second row of $T_1$ to obtain $T_2$ and then swap 5 and 6 from the first and second columns of $T_2$ to obtain $T_3$.

$$T_1 = \begin{array}{ccc}
1 & 4 & 5 \\
3 & 1 & 2 \\
5 & 6 & 6 \\
\end{array} \quad T_2 = \begin{array}{ccc}
1 & 4 & 5 \\
2 & 1 & 3 \\
5 & 6 & 6 \\
\end{array} \quad T_3 = \begin{array}{ccc}
1 & 4 & 5 \\
2 & 1 & 3 \\
6 & 5 & 6 \\
\end{array}$$
Lemma 3.6. Suppose that $T$ and $T'$ are row-wise equal tableaux and each contain columns of type 0 and 1 only. Then there exist tableaux $S$ and $S'$ quadratically equivalent to $T$ and $T'$, respectively, such that the first two rows of $S$ and $S'$ are identical.

Proof. Since the second row entries of $T_X$ and $T'_X$ lie in $B_2$ and the second row entries of $T_Y$ and $T'_Y$ lie in $B_1$, by semi-standardness of the tableaux, it follows that the second row of $T$ is identical to the second row of $T'$. Note that, if the first row of $T_Y$ is equal to the first row of $T'_Y$, then the first row of $T_X$ is equal to the first row of $T'_X$ and vice versa because $T_X, T'_X, T_Y$ and $T'_Y$ are in semi-standard form. We proceed by induction on the number of differences in the first row of $T$ and $T'$.

Suppose the first row of $T$ is not equal to the first row of $T'$. Let us write the first row of $T_Y$ as $(a_1, a_2, \ldots, a_t)$ for some $t \geq 1$, and similarly write $(b_1, b_2, \ldots, b_t)$ for the first row of $T'_Y$. By assumption there exists $1 \leq i \leq t$ such that $a_i \neq b_i$. Let $i$ be the smallest such index. Let $A = (a_1, \ldots, a_k)^r$ be the column of $T_Y$ whose first entry is $a_i = a_1$ and let $B = (b_1, \ldots, b_k)^r$ be the column of $T'_Y$ whose first entry is $b_i = b_1$. Since $A$ and $B$ are the $i$th column of $T_Y$ and $T'_Y$, respectively, and the second rows are the same, we have that $b_2 = a_2$.

Assume without loss of generality that $a_1 < b_1$. Since the tableaux are semi-standard and row-wise equal, there is a column $C = (c_1, \ldots, c_k)^r$ in $T'_X$ where $c_1 = a_1$. We distinguish the following cases:

Case 1. $b_1 < c_2$. Then we may swap $a_1$ and $b_1$ in $T'$. As a result columns $A$ and $B$ have the same entry in the first and second row and so we have reduced the number of differences in the first row.

Case 2. $b_1 \geq c_2$. Then there is a column $D = (d_1, \ldots, d_k)^r$ in $T_X$ such that $d_2 = c_2$. Now we take cases on $d_1$ and $a_3$.

Case 2.i. $d_1 < a_3$. Then we may swap $d_1$ and $a_1$ in $T$. As a result, columns $C$ and $D$ will have the same entries in the first and second row and so we have reduced the number of differences in the first row.

Case 2.ii. $d_1 \geq a_3$. Then we have $b_1 \geq c_2 > d_1 \geq a_3 > a_1$. In particular $d_1 < b_1$. Hence, $d_1$ appears in the first row for some column in $T'_X$. Call this column $E = (d_1, e_2, \ldots, e_k)^r$, see Figure 2. Since $a_1 < d_1$ and $T'_X$ is in semi-standard form, column $E$ is on the right side of column $C$. Therefore, we may swap the entries $d_1$ and $a_1$ in $T'$. So the first two entries of columns $C$ and $D$ are the same and we have reduced the number of differences in the first row.
Therefore, by induction on the number of differences in the first row of $T$ and $T'$, we can apply quadratic relations to $T$ and $T'$ to ensure the first two rows are identical. \hfill \Box

**Example 3.7.** For Lemma 3.6, Case 1, consider $\text{Gr}(3, 6)$ with the block diagonal matching field $B_\ell = (1|23456)$. Consider the tableaux

\[
T = \begin{array}{ccc}
3 & 2 & 4 \\
4 & 1 & 1 \\
6 & 5 & 5
\end{array}, \quad T' = \begin{array}{ccc}
2 & 3 & 4 \\
4 & 1 & 1 \\
5 & 5 & 6
\end{array}.
\]

We let $A = (2, 1, 5)^{tr}$ in $T$ and $B = (3, 1, 5)^{tr}$ in $T'$. Then, following the proof, let $C = (2, 4, 5)^{tr}$ in $T'$. Since $b_1 = 3 < 4 = c_2$ we may swap 2 and 3 in the first row of $T'$. This makes the first two rows of $T$ and $T'$ identical.

For Lemma 3.6, Case 2.i, consider $\text{Gr}(3, 7)$ with the block diagonal matching field $B_\ell = (1|234567)$. Consider the tableaux $T$ and $T'$ below,

\[
T = \begin{array}{ccc}
3 & 4 & 2 \\
4 & 5 & 1 \\
6 & 7 & 5
\end{array}, \quad T' = \begin{array}{ccc}
2 & 3 & 4 \\
4 & 5 & 1 \\
5 & 6 & 7
\end{array}, \quad \tilde{T} = \begin{array}{ccc}
2 & 4 & 3 \\
4 & 5 & 1 \\
6 & 7 & 5
\end{array}.
\]

Following the proof, since $d_1 = 3 < 5 = a_3$ we may swap $d_1$ and $a_3$ to obtain the tableau $\tilde{T}$ with altered columns $\tilde{D}$ and $\tilde{A}$. Now observe that the first rows of $\tilde{T}$ and $T'$ differ in only two positions, namely the second and third columns.

Finally for Lemma 3.6, Case 2.ii consider $\text{Gr}(3, 8)$ with the block diagonal matching field $B_\ell = (1|2345678)$. Consider the tableaux $T$ and $T'$ below,

\[
T = \begin{array}{ccc}
2 & 4 & 5 \\
3 & 5 & 1 \\
6 & 7 & 4
\end{array}, \quad T' = \begin{array}{ccc}
2 & 3 & 4 \\
3 & 5 & 6 \\
4 & 6 & 7
\end{array}, \quad \tilde{T} = \begin{array}{ccc}
2 & 4 & 3 \\
3 & 5 & 6 \\
4 & 6 & 7
\end{array}.
\]

In this case, we have $d_1 \geq a_3$ so we swap $c_1 = 3$ and $e_1 = 4$ in $T'$ to obtain the tableau $\tilde{T}'$. Notice that the first rows of $T$ and $T'$ differ in last three columns whereas the first rows of $T$ and $\tilde{T}'$ differ only in the third and fourth columns. To make the first two rows identical, we proceed by Case 2.i and swap 3 and 5 in the first row of $\tilde{T}'$.

### 3.2. Plücker algebra of block diagonal matching field ideals.

Let $\mathcal{A}_{k,n}$ be the Plücker algebra for $\text{Gr}(k, n)$. We recall the definition of **SAGBI basis**\(^1\) from [RS90]. In our setting, the set of Plücker forms $\{P_I\}_{I \in \mathcal{I}_{k,n}} \subset \mathbb{K}[x_{ij}]$ is a SAGBI basis for

\[\text{SAGBI stands for Subalgebra Analogue to Gröbner Bases for Ideals. Khovanskii basis from [KM16] generalizes the notion of a SAGBI basis to non-polynomial algebras.}\]
the Plücker algebra $\mathcal{A}_{k,n}$ with respect to the weight vector $w_\ell$ if for all $I \in \mathbf{I}_{k,n}$ the initial form $\text{in}_{w_\ell}(P_I)$ is a monomial and $\text{in}_{w_\ell}(\mathcal{A}_{k,n}) = \mathbb{K}[\text{in}_{w_\ell}(P_I)]_{I \in \mathbf{I}_{k,n}}$. Here, $w_\ell$ is the weight vector induced by the matrix $M_\ell$ from Example 2.4.

We denote $A_\ell$ for the algebra associated to the block diagonal matching field $B_\ell$. The algebra $A_\ell$ has the standard grading and any monomial $P_{I_1}P_{I_2} \cdots P_{I_t} \in A_\ell$ is identified with a tableau of size $k$ by $r$ denoted by $T_{I_1, I_2, \ldots, I_r}$. We show that the subspace $[A_\ell]_2$ of $A_\ell$ spanned by the monomials of degree two has a basis which is in a bijection with the set of semi-standard tableaux with two columns.

In combinatorial terms, for each matching field $B_\ell$ we will construct a collection of two-column tableaux such that every two-column tableau is row-wise equal to exactly one tableau in the collection. Then we will show that this collection is in bijection with two-column tableaux in semi-standard form. We have already seen in Example 3.3 that semi-standard tableaux give a basis for $[A_0]_2$ and so this construction gives a strict generalization.

**Definition 3.8.** Fix $\text{Gr}(k,n)$ and a block diagonal matching field $B_\ell$. We define $T_\ell$ to be the collection of all tableaux $T$ which follow. We partition $T_\ell$ into types each of which is described below. We also define a map $S$ taking each tableau $T \in T_\ell$ to a semi-standard tableau $S(T)$. Note that $S(T)$ does not necessarily lie in $T_\ell$.

We write, $I = \{i_1 < i_2 < \cdots < i_k\}$ and $J = \{j_1 < j_2 < \cdots < j_k\}$.

**Type 1.**

$$T = \begin{array}{cc}
i_1 & j_1 \\
i_2 & j_2 \\
\vdots & \vdots \\
i_k & j_k \\
\end{array} \quad i_1 \leq j_1, i_2 \leq j_2, \ldots, i_k \leq j_k, \quad S(T) = \begin{array}{cc}
i_1 & j_1 \\
i_2 & j_2 \\
\vdots & \vdots \\
i_k & j_k \\
\end{array}.$$

**Type 2.**

$$T = \begin{array}{cc}
i_2 & j_2 \\
i_1 & j_1 \\
i_3 & j_3 \\
\vdots & \vdots \\
i_k & j_k \\
\end{array} \quad i_1 \leq j_1, i_2 \leq j_2, \ldots, i_k \leq j_k, \quad S(T) = \begin{array}{cc}
i_1 & j_1 \\
i_2 & j_2 \\
i_3 & j_3 \\
\vdots & \vdots \\
i_k & j_k \\
\end{array}.$$

**Type 3A.**

$$T = \begin{array}{cc}
i_2 & j_1 \\
i_1 & j_2 \\
i_3 & j_3 \\
\vdots & \vdots \\
i_k & j_k \\
\end{array} \quad i_1 \in B_1, \quad i_2, \ldots, i_k, j_1, \ldots, j_k \in B_2, \quad S(T) = \begin{array}{cc}
i_1 & j_1 \\
i_2 & j_2 \\
i_3 & j_3 \\
\vdots & \vdots \\
i_k & j_k \\
\end{array}.$$
Type 3B(r).

| $i_2$ | $j_1$ |
|-------|-------|
| $i_1$ | $j_2$ |
| $i_3$ | $j_3$ |
| \vdots | \vdots |
| $i_{r-1}$ | $j_{r-1}$ |
| $i_r$ | $j_r$ |
| $i_{r+1}$ | $j_{r+1}$ |
| \vdots | \vdots |
| $i_k$ | $j_k$ |

$T = \begin{array}{cc}
  i_1 & j_1 \\
  i_2 & j_2 \\
  i_3 & j_3 \\
  \vdots & \vdots \\
  i_r & j_r \\
  i_{r+1} & j_{r+1} \\
  \vdots & \vdots \\
  i_k & j_k \\
\end{array}$

$S(T) = \begin{array}{cc}
  i_1 & j_1 \\
  i_2 & j_2 \\
  i_3 & j_3 \\
  \vdots & \vdots \\
  i_r & j_r \\
  i_{r+1} & j_{r+1} \\
  \vdots & \vdots \\
  i_k & j_k \\
\end{array}$

Type 3C(s).

| $i_2$ | $j_1$ |
|-------|-------|
| $i_1$ | $j_2$ |
| $i_3$ | $j_3$ |
| \vdots | \vdots |
| $i_s$ | $j_s$ |
| $i_{s+1}$ | $j_{s+1}$ |
| \vdots | \vdots |
| $i_k$ | $j_k$ |

$T = \begin{array}{cc}
  i_1 & j_1 \\
  i_2 & j_2 \\
  i_3 & j_3 \\
  \vdots & \vdots \\
  i_s & j_s \\
  i_{s+1} & j_{s+1} \\
  \vdots & \vdots \\
  i_k & j_k \\
\end{array}$

$S(T) = \begin{array}{cc}
  i_1 & j_1 \\
  i_2 & j_2 \\
  i_3 & j_3 \\
  \vdots & \vdots \\
  i_s & j_s \\
  i_{s+1} & j_{s+1} \\
  \vdots & \vdots \\
  i_k & j_k \\
\end{array}$

Type 3D(s).

| $i_2$ | $j_1$ |
|-------|-------|
| $i_1$ | $j_2$ |
| $i_3$ | $j_3$ |
| \vdots | \vdots |
| $i_s$ | $j_s$ |
| $i_{s+1}$ | $j_{s+1}$ |
| \vdots | \vdots |
| $i_k$ | $j_k$ |

$T = \begin{array}{cc}
  i_1 & j_1 \\
  i_2 & j_2 \\
  i_3 & j_3 \\
  \vdots & \vdots \\
  i_s & j_s \\
  i_{s+1} & j_{s+1} \\
  \vdots & \vdots \\
  i_k & j_k \\
\end{array}$

$S(T) = \begin{array}{cc}
  i_1 & j_1 \\
  i_2 & j_2 \\
  i_3 & j_3 \\
  \vdots & \vdots \\
  i_s & j_s \\
  i_{s+1} & j_{s+1} \\
  \vdots & \vdots \\
  i_k & j_k \\
\end{array}$

Lemma 3.9. The tableaux $\mathcal{T}_\ell$ span $[A_\ell]_2$.

Proof. Let $I = \{i_1 < i_2 < \cdots < i_k\}$ and $J = \{j_1 < j_2 < \cdots < j_k\}$ be two $k$-subsets of $[n]$. Recall the notation for tableaux in terms of its columns defined on Page 6. We will show that there is a tableau $T_{I,J} \in \mathcal{T}_\ell$ which is row-wise equal to the tableau $T_{I,J}$. We take cases on $I$ and $J$.

Case 1. $B_\ell(I) = B_\ell(J) = id$. The entries of $T_{I,J}$ can be seen in Case 1 of Figure 3. Let,

$I' = \{\min \{i_1, j_1\} < \cdots < \min \{i_k, j_k\}\}, \quad J' = \{\max \{i_1, j_1\} < \cdots < \max \{i_k, j_k\}\}$. 

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Then $T_{rJ'}$ and $T_{IJ}$ are row-wise equal and $T_{rJ'}$ is a tableau of type 1 in $\mathcal{T}_r$.

**Case 2.** $B_t(I) = B_t(J) = (12)$. The entries of $T_{IJ}$ can be seen in Case 2 of Figure 3. Similarly to case 1, let

$$I' = \{\min\{i_1, j_1\} < \cdots < \min\{i_k, j_k\} \}, \quad J' = \{\max\{i_1, j_1\} < \cdots < \max\{i_k, j_k\} \}.$$

Then $T_{rJ'}$ and $T_{IJ}$ are row-wise equal and $T_{rJ'}$ is a tableau of type 2 in $\mathcal{T}_r$.

**Case 3.** $B_t(I) \neq B_t(J)$. Without loss of generality we assume $B_t(I) = (12)$ and $B_t(J) = id$. The entries of $T_{IJ}$ can be seen in Case 3 of Figure 3.

Since $B_t(I) = (12)$ it follows that $i_1 \in B_1$ and $i_2, \ldots, i_k \in B_2$. We now take cases on the type of $J$.

**Case 3a.** The type of $J$ is 0 and $i_2 < j_2$. Let,

$$I' = \{i_1 < \min(i_2, j_1) < \min(i_3, j_3) < \cdots < \min(i_k, j_k) \}, \quad J' = \{\max(i_1, j_1) < \cdots < \max(i_k, j_k) \}.$$

So we have $T_{IJ}$ is row-wise equal to $T_{rJ'}$ and $T_{rJ'}$ is a tableau of type 3A in $\mathcal{T}_r$.

**Case 3b.** The type of $J$ is 0 and $i_2 \geq j_2$. Let $r = \min\{i \geq 2 : j_{i+1} > i_i \}$. Let,

$$I' = \{i_1 < i_2 < i_3 < \cdots < i_r < \min(i_{r+1}, j_{r+1}) < \cdots < \min(i_k, j_k) \}, \quad J' = \{j_1 < j_2 < j_3 < \cdots < j_r < \max(i_{r+1}, j_{r+1}) < \cdots < \max(i_k, j_k) \}.$$

Note that for each $i \in \{3, \ldots, r\}$ we have that $j_i \leq i_{i-1} < i_i$. So $T_{rJ'}$ is a tableau of type 3B(r) in $\mathcal{T}_r$ and $T_{IJ}$ is row-wise equal to $T_{rJ'}$.

**Case 3c.** The type of $J$ is $s \geq 2$ and $i_1 \leq j_1$. Then let,

$$I' = \{i_1 < i_2 < i_3 < \cdots < i_s < \min(i_{s+1}, j_{s+1}) < \cdots < \min(i_k, j_k) \}, \quad J' = \{j_1 < j_2 < j_3 < \cdots < j_s < \max(i_{s+1}, j_{s+1}) < \cdots < \max(i_k, j_k) \}.$$

So we have $T_{IJ}$ is row-wise equal to $T_{rJ'}$ and $T_{rJ'}$ is a tableau of type 3C(s) in $\mathcal{T}_r$.

**Case 3d.** The type of $J$ is $s \geq 2$ and $i_1 > j_1$. Now if $i_1 < j_2$ then we may apply a quadratic relation to swap $i_1$ and $j_2$. So without loss of generality we may assume that $i_1 \geq j_2$. Let,

\begin{figure}[h]
\centering
\begin{tabular}{|c|c|}
\hline
Case 1 & Case 2 & Case 3 \\
\hline
\hline
$i_1$ & $j_1$ & $i_2$ & $j_2$ & $i_2$ & $j_1$ \\
$i_2$ & $j_2$ & $i_1$ & $j_1$ & $i_1$ & $j_2$ \\
$i_3$ & $j_3$ & $i_3$ & $j_3$ & $i_3$ & $j_3$ \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
$i_k$ & $j_k$ & $i_k$ & $j_k$ & $i_k$ & $j_k$ \\
\hline
\end{tabular}
\caption{Tableaux $T_{IJ}$ in Lemma 3.9 for cases 1, 2 and 3.}
\end{figure}
\[ I' = \{ i_1 < i_2 < \cdots < i_s < \text{min}(i_{s+1}, j_{s+1}) < \cdots < \text{min}(i_k, j_k) \}, \]
\[ J' = \{ j_1 < j_2 < \cdots < j_s < \text{max}(i_{s+1}, j_{s+1}) < \cdots < \text{max}(i_k, j_k) \}. \]

So we have \( T_{IJ} \) is row-wise equal to \( T_{I'J'} \) and \( T_{I'J'} \) is a tableau of type 3D(s) in \( \mathcal{T}_I \). And so we have shown that \( \mathcal{T}_I \) spans \([A_I]_2\).

\[ \square \]

**Lemma 3.10.** The tableaux \( \mathcal{T}_I \) are linearly independent in \([A_I]_2\).

**Proof.** Consider the map \( \phi_{B_t} \) from Equation (2.1) in Page 5. Since \( \text{Ker}(\phi_{B_t}) \) is generated by binomials, it suffices to show that if \( T_{IJ} \) and \( T_{I'J'} \) are row-wise equal then they are equal up to reordering the columns.

Recall that the type of a column \( I \) in \( T_{IJ} \) is \([ I \cap B_1 ]\) and if \( T_{IJ} \) and \( T_{I'J'} \) are row-wise equal then their columns must have the same type up to reordering. So we may assume that \([ I \cap B_1 ] = [ I' \cap B_1 ]\) and \( s = [ J \cap B_1 ] = [ J' \cap B_1 ]\). Now we take cases on the type of \( T_{IJ} \) in \( \mathcal{T}_I \).

**Type 1.** In this case, we have that both \( T_{IJ} \) and \( T_{I'J'} \) are in semi-standard form. Since the rows are weakly increasing, it follows that these tableaux must be equal.

**Type 2.** Again we see that \( T_{IJ} \) and \( T_{I'J'} \) are equal since the rows are weakly-increasing.

**Type 3.** We assume that \( B_t(I) = (12) \) and \( B_t(J) = id \). Note that the columns \( J \) and \( J' \) have the same type denoted \( s \). If \( s = 0 \) then \( T_{IJ} \) and \( T_{I'J'} \) are either of type 3A or 3B, see Page 11. Otherwise \( s \geq 2 \) and \( T_{IJ} \) and \( T_{I'J'} \) are either of type 3C or 3D, see Page 12. In both cases we write,
\[ I = \{ i_1 < i_2 < \cdots < i_k \}, \quad J = \{ j_1 < j_2 < \cdots < j_k \}, \]
\[ I' = \{ i'_1 < i'_2 < \cdots < i'_k \}, \quad J' = \{ j'_1 < j'_2 < \cdots < j'_k \}. \]

**Case 1.** Let \( s = 0 \), i.e. \( T_{IJ} \) and \( T_{I'J'} \) are either of type 3A or 3B. Since \( i_1, i'_1 \) are the only elements of \( I \) and \( J \) that belong to \( B_1 \), it follows that \( i_1 = i'_1 \). Hence, by row-wise equality of \( T_{IJ} \) and \( T_{I'J'} \) we have \( j_2 = j'_2 \).

If \( i_2 < j_2 \) then we have that \( T_{IJ} \) is of type 3A. By row-wise equality of \( T_{IJ} \) and \( T_{I'J'} \) we have \( \{ i'_2, j'_2 \} = \{ i_2, j_2 \} \) and so \( i'_2 < j'_2 \). Therefore \( T_{I'J'} \) is also of type 3A. By definition of type 3A, the rows are weakly increasing and so \( T_{IJ} \) and \( T_{I'J'} \) are equal.

If, on the other hand, \( i_2 \geq j_2 \) then \( T_{IJ} \) is of type 3B. Note that \( j'_1 < j'_2 = j_2 \) so \( i_2 \neq j'_1 \). By row-wise equality of \( T_{IJ} \) and \( T_{I'J'} \) we deduce that \( i_2 = i'_2 \) and \( j_1 = j'_1 \). Therefore \( T_{I'J'} \) is also of type 3B. For each \( t \in \{2, \ldots, r\} \) where \( r = \text{min}\{t \geq 2 : j_{t+1} > i_t\} \) we show that \( i_t = i'_t \).

Suppose by contradiction that \( i_t \neq i'_t \) for some \( t \in \{2, \ldots, r\} \). We take \( t \) to be the smallest such value. Note that \( i_2 = i'_2 \) so we may assume \( 3 \leq t \leq r \). By row-wise equality of \( T_{IJ} \) and \( T_{I'J'} \), since \( i'_2 \neq i_2 \), so \( i'_t = j_t \). By definition of \( r \) we have \( j_t \leq i_{t-1} \). By assumption \( t \) was minimal so \( i_{t-1} = i'_{t-1} \). So we have deduced that \( i'_t \leq i'_{t-1} \), a contradiction. Therefore \( i_1 = i'_1, i_2 = i'_2, \ldots, i_r = i'_r \).

Next, we will show that \( j'_r+1 > i'_r \). By row-wise equality of \( T_{IJ} \) and \( T_{I'J'} \), either \( j'_{r+1} = j_{r+1} \) or \( j'_{r+1} = i_{r+1} \). So either \( j'_{r+1} = j_{r+1} > i_r = i'_r \) or \( j'_{r+1} = i_{r+1} > i_r = i'_r \). So we have shown that \( r = \text{min}\{t \geq 2 : j'_t+1 > i'_t\} \). Therefore both \( T_{IJ} \) and \( T_{I'J'} \) are of type 3B(r). Therefore we have \( i_{r+1} \leq j_{r+1}, \ldots, i_k \leq j_k \) and \( i'_r+1 \leq j'_{r+1}, \ldots, i'_k \leq j'_k \). By row-wise equality of \( T_{IJ} \) and \( T_{I'J'} \), it
follows that \( i_{r+1} = i'_{r+1}, \ldots, i_k = i'_k \). And so, we have shown that \( I = I' \), hence \( J = J' \) and the tableaux are equal. This completes the proof for the case where \( T_{IJ} \) and \( T_{J'I'} \) are either of type 3A or 3B.

**Case 2.** Let \( s \geq 2 \), i.e. \( T_{IJ} \) and \( T_{J'I'} \) are either of type 3C or 3D. Consider the first row of the tableaux \( T_{IJ} \), since \( i_2, i'_2 \in B_2 \) and \( j_1, j'_1 \in B_1 \) it follows that \( i_2 = i'_2 \) and \( j_1 = j'_1 \).

If \( i_1 \leq j_1 \) then \( T_{IJ} \) is of type 3C(s) and \( i_1 = i'_1 \) and \( j_2 = j'_2 \). So \( T_{J'I'} \) is also of type 3C(s). Since \( j_3, \ldots, j_s, j'_3, \ldots, j'_s \in B_1 \) and \( i_3, \ldots, i_s, i'_3, \ldots, i'_s \in B_2 \) it follows that \( i_3 = i'_3, \ldots, i_s = i'_s \) and \( j_3 = j'_3, \ldots, j_s = j'_s \). Since \( i_{s+1} \leq j_{s+1}, \ldots, i_k \leq j_k \) and \( i'_{s+1} \leq j'_{s+1}, \ldots, i'_k \leq j'_k \) so by row-wise equality of \( T_{IJ} \) and \( T_{J'I'} \) we deduce that \( i_{s+1} = i'_{s+1}, \ldots, i_k = i'_k \). And so \( T_{IJ} \) and \( T_{J'I'} \) are equal.

If \( i_1 > j_1 \) then \( T_{IJ} \) is of type 3D(s). We deduce that \( i'_1 > j_1 \) because by row-wise equality of \( T_{IJ} \) and \( T_{J'I'} \) either \( i'_1 = j_2 > j_1 \) or \( i'_1 = i_1 > j_1 \). Therefore \( T_{J'I'} \) is also of type 3D(s). By the definition of 3D(s) we have \( i_1 \leq j_2 \) and \( i'_1 \leq j'_2 \). Therefore, \( i_1 = i'_1 \) and \( j_2 = j'_2 \). Since \( j_3, \ldots, j_s, j'_3, \ldots, j'_s \in B_1 \) and \( i_3, \ldots, i_s, i'_3, \ldots, i'_s \in B_2 \) it follows that \( i_3 = i'_3, \ldots, i_s = i'_s \) and \( j_3 = j'_3, \ldots, j_s = j'_s \). Since \( i_{s+1} \leq j_{s+1}, \ldots, i_k \leq j_k \) and \( i'_{s+1} \leq j'_{s+1}, \ldots, i'_k \leq j'_k \), by row-wise equality of \( T_{IJ} \) and \( T_{J'I'} \) we deduce that \( i_{s+1} = i'_{s+1}, \ldots, i_k = i'_k \). And so \( T_{IJ} \) and \( T_{J'I'} \) are equal. Hence, the proof is complete for the case when \( T_{IJ} \) and \( T_{J'I'} \) are either of type 3C or 3D. So we have shown that \( \mathcal{T}_c \) is a linearly independent set.

**Lemma 3.11.** The map \( T \mapsto S(T) \) is a bijection between \( \mathcal{T}_c \) and the set of semi-standard tableaux with two columns and \( k \) rows.

**Proof.** We will show that the inverse to the map \( S \) from Definition 3.8 exists by constructing the map explicitly. Fix a semi-standard tableau,

\[
T = \begin{array}{c|c}
   i_1 & j_1 \\
   i_2 & j_2 \\
   i_3 & j_3 \\
   \vdots & \vdots \\
   i_k & j_k \\
\end{array}
\]

We now take cases on \( i_1, i_2, j_1 \) and \( j_2 \). If we have one of,

- \( i_1, i_2, j_1, j_2 \in B_1 \),
- \( i_1, i_2 \in B_1 \) and \( j_1, j_2 \in B_2 \),
- \( i_1, i_2, j_1, j_2 \in B_2 \),

then we have that \( S^{-1}(T) = T \) is a tableau of type 1.

If \( i_1, j_1 \in B_1 \) and \( i_2, j_2 \in B_2 \), then we have that \( S^{-1}(T) = T' \) is a tableau of type 2, where

\[
T' = \begin{array}{c|c}
   i_2 & j_2 \\
   i_1 & j_1 \\
   i_3 & j_3 \\
   \vdots & \vdots \\
   i_k & j_k \\
\end{array}
\]
If $i_1 \in B_1$ and $j_1, i_2, j_2 \in B_2$, then there are two cases, either $i_2 \leq j_1$ or $i_2 > j_1$.

**Case 1.** $i_2 \leq j_1$. Then $S^{-1}(T) = T_1$ in Figure 4 is a tableau of type $3A$.

**Case 2.** $i_2 > j_1$. Let $r = \min\{t \geq 2 : i_{t+1} > j_t\}$. Then $S^{-1}(T) = T_2$ in Figure 4 is a tableau of type $3B(r)$.

If $i_1, j_1, i_2 \in B_1$ and $j_2 \in B_2$, then there are two cases, either $j_1 < i_2$ or $j_1 \geq i_3$.

**Case 1.** $j_1 < i_2$. Then $S^{-1}(T) = T_3$ in Figure 4 is a tableau of type $3C(s)$, where $s = |I \cap B_1|$.

**Case 2.** $j_1 \geq i_2$. Then $S^{-1}(T) = T_4$ in Figure 4 is a tableau of type $3D(s)$, where $s = |I \cap B_1|$.

So for each $\ell \in \{0, 1, \ldots, n - 1\}$ we have shown that $\mathcal{T}_\ell$ is a basis for $[A_\ell]_2$ and there is a bijection between $\mathcal{T}_\ell$ and semi-standard tableaux with two columns.  

\[\Box\]

### 4. Toric degenerations of Grassmannians

In this section, we state our main results for Grassmannians, where we generalize the results of [MS18] from $\text{Gr}(3, n)$ to higher-dimensional Grassmannians.

**Theorem 4.1.** The ideals of block diagonal matching fields are quadratically generated.

Before giving the proof of Theorem 4.1 we fix our notation. Fix a block diagonal matching field $B_\ell = (1 \cdots \ell | \ell + 1 \cdots n)$. Let $I = \{I_1, \ldots, I_r\} \subseteq 1_{k,n}$ be a non-empty collection of $k$-subsets of $[n]$ and consider the tableau $T_I$. Recall, from §3.1 on Page 7, that $T_I$ is written in the form $[T_X|T_Y]$ where $T_X$ and $T_Y$ are tableaux in semi-standard form. The columns of $T_Y$ are the columns of $T_I$ of type 1 and the remaining columns are contained in $T_X$.

**Proof.** Suppose that $T$ and $T'$ are two row-wise equal tableaux. The statement is equivalent to proving that $T$ and $T'$ differ by a sequence of swaps. We proceed by induction on the number of columns in $T$ and $T'$. We will assume that $T$ and $T'$ contain no identical columns, otherwise we may remove these columns from the subsequent manipulations and apply induction. Note that if $T$ and $T'$ contain two or fewer columns then we are done.

Suppose that the leftmost column of $T$ and $T'$ is of type $i$ where $i \in \{2, 3, \ldots, k\}$. Then by Lemma 3.4 we may perform a sequence of swaps to reduce the number of different columns in $T$ and $T'$.

\[\Box\]
Figure 5: A quadratic relation

Suppose that $T$ and $T'$ contain only columns of type 0 and 1. Then by Lemma 3.6 we may perform a sequence of swaps to make the first two rows of $T$ and $T'$ identical. Let $A = (a_1, a_2, \ldots, a_k)^r$ be the leftmost column of $T_X$ of type 0. Let $B = (b_1, b_2, \ldots, b_k)^r$ be the leftmost column in $T'_X$. By assumption we have that $b_1 = a_1$ and $b_2 = a_2$. Since $A \neq B$, let $i \geq 3$ be the smallest index such that $b_i \neq a_i$. Without loss of generality suppose that $a_i < b_i$. Since $T_X$ and $T'_X$ are in semi-standard form, there is a column $C = (c_1, \ldots, c_k)^r$ in $T'_Y$ such that $c_i = a_i$. Then we may apply the relation in Figure 5.

So we have reduced the number of differences in the leftmost column of $T$ and $T'$. So, by induction on the number of differences in the left column, we can find a sequence of swaps which makes the leftmost column of $T$ and $T'$ equal, which completes the proof. 

Example 4.2. Following the proof above, if the leftmost column of $T$ and $T'$ is of type $i$ where $i \in \{2, 3, \ldots, k\}$ then consider Example 3.5 in §3.1. This example gives a typical manipulation of the tableaux to reduce the number of differences in the leftmost column.

If the leftmost column of $T$ and $T'$ is of type 0 or 1, then Example 3.7 shows how to make the first and second rows of $T$ and $T'$ identical. Continuing this example, we consider the tableaux $T$ and $T'$ below.

Following the notation of the proof of Theorem 4.1, we label the columns with $A, B$ and $C$.

We obtain the tableau $\tilde{T}$ from $T$ by swapping the entries 6,7 in column $B$ with 4,5 in column $C$. As a result the leftmost column of $\tilde{T}$ and $T'$ are identical.

We now turn our attention to one of the main results of our paper.

Theorem 4.3. The Plücker variables $P_I$ form a SAGBI basis for the Plücker algebra of block matching field ideals.
Before giving the proof of Theorem 4.3 we recall from §3.2 on Page 10 that \( \text{in}_{w,\ell}(A^k_n) = \mathbb{K}[\text{in}_{w,\ell}(P_I)]_{I \in I_{k,n}} \) and \( A_{\ell} \) is the graded algebra associated to the block diagonal matching field \( B_{\ell} \). We show that the subspace \( [A_{\ell}]_2 \) of \( A_{\ell} \) spanned by the monomials of degree two has a basis which is in bijection with the set of semi-standard tableaux with two columns.

**Proof.** By Theorem 4.1 it suffices to show that \( \dim([A_{\ell}]_2) = \dim([A_0]_2) \). By a classical result, \([A_0]_2\) has a basis given by the set of semi-standard tableaux with two columns and \( k \) rows. By Lemma 3.9 and 3.10 we have that \([A_{\ell}]_2\) has a basis given by \( T\), from Definition 3.8. And by Lemma 3.11 we have that \( T \mapsto S(T) \) is a bijection, as desired. \( \square \)

As a corollary of the above statements and [Stu96, Theorem 11.4] we have that:

**Corollary 4.4.** Each block diagonal matching field produces a toric degeneration of \( \text{Gr}(k,n) \). Equivalently, the Plücker forms are a SAGBI basis with respect to the weight vectors arising from block diagonal matching fields.

**Remark 4.5.** We remark that this result is a generalization of Corollary 1.5 in [MS18] for \( \text{Gr}(3,n) \). Moreover, our combinatorial description of SAGBI bases of Grassmannians leads to analogous results for flag varieties. Using combinatorial tools of matching field tableaux we have provided a family of toric degenerations of flag varieties and Schubert varieties inside them in [CM19a, CM19b].

Following [MS18], we define the matching field polytope as follows. Given a \( k \times n \) matching field, the **matching field polytope** is the convex hull of the points in \( \mathbb{R}^{k \times n} \) associated to the \( k \)-subsets of \( [n] \). See [MS18, Section 5] for more details. For a coherent matching field \( \Lambda \), this polytope coincides with the polytope of the toric variety defined by the ideal of the matching field \( J_{\Lambda} \). Hence, its combinatorial invariants carry on a lot of information about the variety. For example, the normalized lattice volume of such polytope is equal to the degree of its corresponding variety. Moreover, by comparing the polytopes of different toric varieties arisen in our construction, we can check whether they lead to non-isomorphic varieties.

Notice that up to isomorphism, there are seven polytopes associated to the trop(Gr(3,6)) and four of them can be obtained as polytopes of block diagonal matching fields. See [HJJS09] for further details.

Here, we summarize our computational results on matching field polytopes.

**Remark 4.6.** Using polymake [GJ00] we computed the f-vectors of the polytopes associated to toric ideals of block diagonal matching fields, see Table 5. In particular, Table 5 shows that almost all toric ideals obtained by our construction are non-isomorphic. Note that for some values of \( \ell \), the f-vector is the same but the polytopes are non-isomorphic. For instance, in the case of Gr(3,6) and Gr(4,7) the polytopes associated to the block diagonal matching fields \( B_{\ell} \), for \( \ell = 1 \) and \( \ell = 2 \) have the same f-vectors however, by computing their face lattice, we can show they are non-isomorphic. For some values of \( \ell \), the matching field ideal is trivially isomorphic to the diagonal case, so we list only the f-vectors for the cases \( 0 \leq \ell \leq n - k + 1 \).
5. Toric degeneration of Schubert varieties inside Grassmannians

In this section, we apply our results from §4 for Grassmannians to provide a family of toric degenerations for Schubert varieties. Our aim is to answer the following question which is a reformulation of Degeneration Problem posed by Caldero in [Cal02], in our setting.

Question 5.1. Characterize non-zero toric ideals of type $G_{k,n,t,w}$.

We provide a complete answer to Question 5.1. In particular, we give a complete characterization of toric ideals of type $G_{k,n,t,w}$ from Definition 2.6. We will first distinguish such ideals which are non-zero in Proposition 5.2 and then in Theorem 5.7 we provide a list of combinatorial conditions which lead to toric ideals.

Proposition 5.2. The ideal $G_{k,n,t,w}$ is zero if and only if $w \in Z_{k,n}$, where

$$Z_{k,n} = \{(1,2,\ldots,k-1,i): \ k \leq i \leq n\} \cup \{(1,\ldots,i,\ldots,k,k+1): \ 1 \leq i \leq k-1\}.$$

Proof. To begin we show that $G_{k,n,t,w}$ is zero for each $w \in Z_{k,n}$. We distinguish two cases:

Case 1. Let $w = (1,2,\ldots,k-1,i)$ for some $k \leq i \leq n$. Then the only variables $P_I$ which do not vanish in $G_{k,n,t,w}$ are indexed by:

$$I = \{1,2,\ldots,k-1,j\} \text{ for some } k \leq j \leq i.$$

Suppose by contradiction that $G_{k,n,t,w}$ is non-zero. Then there is a non-trivial relation $P_I P_J = P_{I'} P_{J'}$ in $\text{in}_w(G_{k,n})$ for which $P_I$ and $P_J$ do not vanish. Write $I = \{1,\ldots,k-1,j_1\}$ and $J = \{1,\ldots,k-1,j_2\}$. Note that as multisets $I \cup J$ and $I' \cup J'$ are identical and so $I'$ and $J'$ each contain $1,\ldots,k-1$. So either $I' = I$ or $I' = J$, hence the relation is trivial, a contradiction. Therefore, $G_{k,n,t,w}$ is zero.

Case 2. Let $w = (1,\ldots,i,\ldots,k,k+1)$ for some $1 \leq i \leq k-1$. Then the only variables $P_I$ which do not vanish in $G_{k,n,t,w}$ are indexed by:

$$I = \{1,2,\ldots,\hat{i},\ldots,k+1\} \text{ for some } i \leq j \leq k + 1.$$

Suppose by contradiction that $G_{k,n,t,w}$ is non-zero. Then there is a non-trivial relation $P_I P_J = P_{I'} P_{J'}$ in $\text{in}_w(G_{k,n})$ for which $P_I$ and $P_J$ do not vanish. Write $I = \{1,\ldots,\hat{j}_1,\ldots,k+1\}$ and $J = \{1,\ldots,\hat{j}_2,\ldots,k+1\}$. Note that as multisets $I \cup J$ and $I' \cup J'$ are identical and so $I'$ and $J'$ each contain $1,\ldots,\hat{j}_1,\ldots,\hat{j}_2,\ldots,k+1$. So either $I' = I$ or $I' = J$, hence the relation is trivial, a contradiction. So $G_{k,n,t,w}$ is zero.

Conversely, let $w = (w_1,\ldots,w_k)$ be a Grassmannian permutation not in $Z_{k,n}$. We will show that $G_{k,n,t,w}$ is non-zero. Note that for $k = 1$ the result is trivial. The cases $k = 2$ and $k = 3$ hold by Lemma 5.3 and Lemma 5.4, respectively. So it remains to show the result for $k \geq 4$.

Let $k \geq 4$. We will find a relation $P_I P_J = P_{I'} P_{J'}$ in $\text{in}_w(G_{k,n})$ for which $P_I$ and $P_J$ do not vanish in $G_{k,n,t,w}$. First, note that there exists $1 \leq i \leq k-1$ such that $w_i \neq i$, otherwise
We have verified the result for $Gr(3,4)$, we have that $24\leq 34$ and $1\leq 23$.

Now we show that $I, I', J, J'$ agree on all entries except the final two, it follows that $I,J \leq \{1,2,\ldots,\hat{i},\ldots,\hat{j},\ldots,k+2\}$. Therefore $P_I$ and $P_J$ do not vanish in $G_{k,n,\ell,w}$.  Let,

$$I' = \{1,\ldots,k-2,k-1,k+1\} \text{ and } J' = \{1,\ldots,k-2,k,k+1\}.$$

Now we show that $P_I P_J = P_{I'} P_{J'}$ is a relation in $in_w(G_{k,n})$. We recall that $k \geq 4$. Since each set $I, I', J, J'$ contains one and two, it follows that $B_I(I) = B_I(I') = B_I(J) = B_I(J')$. And so, in the true ordering of the sets $I, I', J, J'$, the final two entries of each ordered set are the largest two elements which are in increasing order. Since $I, I', J, J'$ agree on all entries except the final two, it follows that $P_I P_J = P_{I'} P_{J'}$ is a relation in $in_w(G_{k,n})$.

**Lemma 5.3.** Fix $n$ and let $k = 2$. Let $B = B_\ell$ be a block diagonal matching field. If $w = (w_1, w_2) \notin Z_{2,n}$ is a Grassmannian permutation then $G_{k,n,\ell,w}$ is non-zero.

**Proof.** We write $B'$ for the restriction of $B$ to $\{1,2,3,4\}$. That is, if $B_\ell = (B_1 | B_2)$ then $B'_\ell = (B_1 \cap \{1,2,3,4\} | B_2 \cap \{1,2,3,4\})$ and $\ell' = \min\{4,\ell\}$. Let $v = (2,4)$. Note that $Z_{2,n} = \{(1,2),(1,3),\ldots,(1,n),(2,3)\}$ so if $w \notin Z_{2,n}$ then we have $w_1 \geq 2$ and $w_2 \geq 4$. Therefore $v \leq w$. For each possible $\ell' \in \{0,1,2,3,4\}$ we see from Table 1 that $G_{2,4,\ell',v}$ is non-zero which completes the proof. 

**Lemma 5.4.** Fix $n$ and let $k = 3$. Let $B = B_\ell$ be a block diagonal matching field. If $w = (w_1, w_2, w_3) \notin Z_{3,n}$ is a Grassmannian permutation then $G_{k,n,\ell,w}$ is non-zero.

**Proof.** Denote by $B'$ the block diagonal matching field $B$ restricted to $\{1,\ldots,6\}$. If each entry $w_i \in \{1,\ldots,6\}$, then $w$ is a Grassmannian permutation for $Gr(3,6)$ which is not in $Z_{3,n}$. In Table 2 we have verified the result for $Gr(3,6)$, and so $G_{3,6,\ell',w}$ is non-zero. So there is a relation $P_I P_J = P_{I'} P_{J'}$ such that $P_I$ and $P_J$ do not vanish in $G_{3,6,\ell',w}$. Similarly $P_I$ and $P_J$ do not vanish in $G_{3,n,\ell,w}$, hence $G_{3,n,\ell,w} \neq 0$.

On the other hand, if we do not have each $w_i$ in $\{1,\ldots,6\}$, then we consider $v = (1,3,6) \notin Z_{3,n}$. Since $w \notin Z_{3,n}$ we must have $w_1 \geq 1, w_2 \geq 3$ and by assumption $w_3 > 6$. Therefore $v \leq w$. By the calculation in Table 2 we have that $G_{3,6,\ell',v}$ is non-zero and so there is a relation $P_I P_J = P_{I'} P_{J'}$ such that $P_I$ and $P_J$ do not vanish in $G_{3,6,\ell',v}$. Since $I,J \leq v \leq w$ we have that $P_I$ and $P_J$ do not vanish in $G_{3,n,\ell,w}$. Therefore, $G_{3,n,\ell,w}$ is non-zero.

| Matching fields | Toric permutations |
|-----------------|-------------------|
| Diagonal        | 24 34             |
| (123|4)           | 34                |
| (12|34)          | 24 34             |
| (1|234)          | 34                |
| Zero           | 12 13 14 23       |

Table 1: For each block diagonal matching field $B_\ell$ for $Gr(2,4)$, we list the Grassmannian permutations $w$ for which $G_{2,4,\ell,w}$ is either toric or zero.
Table 2: For each block diagonal matching field \( B_\ell \) for \( \text{Gr}(3,6) \), we list the Grassmannian permutations \( w \) for which \( G_{3,6,\ell,w} \) is toric. The last row of the table represents all permutations leading to the zero ideal.

### Proposition 5.5.

For the diagonal matching field, all non-zero ideals \( G_{k,n,\ell,w} \) are toric.

**Proof.** For diagonal matching field we have \( \ell = 0 \). Suppose \( G_{k,n,\ell,w} \) is non-zero and write \( w = (w_1 < \cdots < w_k) \). Let \( P_I \) and \( P_J \) be variables which do not vanish in \( G_{k,n,\ell,w} \) and suppose that \( I \) and \( J \) are incomparable. Write \( I = \{i_1 < \cdots < i_k\} \) and \( J = \{j_1 < \cdots < j_k\} \). We show that \( P_{I\cap J} \) and \( P_{I\cup J} \) do not vanish in \( G_{k,n,\ell,w} \), where \( I \cap J = \{\min(i_1, j_1) < \cdots < \min(i_k, j_k)\} \) and \( I \cup J = \{\max(i_1, j_1) < \cdots < \max(i_k, j_k)\} \). However, this is immediate since \( P_I \) and \( P_J \) do not vanish. Hence, for every \( t \in [k] \) we have \( i_t \leq w_t \) and \( j_t \leq w_t \). Therefore \( P_I P_J - P_{I\cap J} P_{I\cup J} \) is a relation in \( G_{k,n,\ell,w} \) and in particular the ideal contains no monomials, hence it is toric. \( \square \)

**Remark 5.6.** The ideal \( G_{k,n,\ell,w} \) is known as Hibi ideal [Hib87]. The generators \( P_I P_J - P_{I\cap J} P_{I\cup J} \) can be read from the distributive lattice \( \{I \subseteq [n] : P_I \text{ does not vanish in } G_{k,n,\ell,w}\} \).

### Theorem 5.7.

Let \( w = (w_1, \ldots, w_k) \) be a permutation of \( k \) indices in \([n]\). Then the ideal \( G_{k,n,\ell,w} \) is non-toric if and only if the following hold:

1. \( \ell \neq 0 \)
2. \( w_1 \in \{2, \ldots, n-k\}\setminus\{\ell\} \)
3. \( \{w_2, \ldots, w_k\} \subseteq \{\ell + 1, \ldots, n\}\setminus\{w_1 + 1\} \)

**Proof.** Fix \( \ell \in \{0, 1, \ldots, n-1\} \). If \( \ell = 0 \) then by Proposition 5.5 we have that for any \( w \), \( G_{k,n,\ell,w} \) is either toric or zero. So we may assume that \( \ell > 0 \).

Suppose that \( w \) satisfies the given conditions and write \( B \) for \( B_\ell \). We will construct a relation \( P_I P_J = P_{I'} P_{J'} \) for which only \( P_{I'} \) vanishes in \( G_{k,n,\ell,w} \). For ease of notation let \( u = (w_3, \ldots, w_k) \). Suppose that \( w_1 \in B_1 \). Then consider the following relation among the Plücker variables

\[
P_{w_2,w_1,u} P_{1,w_1+1,u} = P_{w_2,w_1+1,u} P_{1,w_1,u}.
\]

Note that the relation is given with indices ordered according to the matching field \( B \). By assumption, \( 1 < w_1 < w_1 + 1 < w_2 < \cdots < w_k \) so in the above expression the only variable to vanish in \( G_{k,n,\ell,w} \) is \( P_{w_2,w_1+1,u} \) because,

\[
(1, w_1, u) < (1, w_1 + 1, u) < (w_1, w_2, u) = w
\]

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whereas \((w_1 + 1, w_2, u) \notin (w_1, w_2, u)\). Therefore, \(G_{k,n,\ell,w}\) is non-toric as it contains the monomial 
\(P_{w_2,w_1,u}P_{1,w_1+1,u}\). Now suppose that \(w_1 \in B_2\). Then we consider the relation
\[
P_{w_1+1,1,u} P_{w_1,w_2,u} = P_{w_1+1,w_2,u} P_{w_1,1,u}
\]
among the Plücker variables. Similarly, in the above relation the only variable to vanish in 
\(G_{k,n,\ell,w}\) is \(P_{w_1+1,w_2,u}\). So \(G_{k,n,\ell,w}\) contains the monomial 
\(P_{w_1+1,1,u} P_{w_1,w_2,u}\) and hence is non-toric.

For the converse, suppose \(G_{k,n,\ell,w}\) is non-toric. We proceed by induction on \(k\) and \(n\), 
reducing to the cases with \(k = 3\) and \(n = 6\) in which \(w\) has the desired form by direct computation.

Suppose \(k > 3\), we reduce to the case \(k = 3\) as follows. Write \(w = (w_1 < \cdots < w_k)\). 
By assumption \(G_{k,n,\ell,w}\) is non-toric and so contains a monomial \(P_I P_J\) for some subsets 
\(I = \{i_1, \ldots, i_k\}\) and \(J = \{j_1, \ldots, j_k\}\). In \(Gr(k,n)\) this monomial belongs to a relation 
\(P_I P_J = P_I' P_J'\), where at least one of the variables \(P_I, P_J\) vanishes in \(G_{k,n,\ell,w}\). Assume that \(I' = \{i_1', \ldots, i_k'\}\) 
and \(J' = \{j_1', \ldots, j_k'\}\). Now we take cases on \(i_k' \in \{i_k, j_k\}\).

**Case 1.** Let \(i_k' = i_k\). Hence \(j_k' = j_k\). By assumption neither \(P_I\) nor \(P_J\) vanishes in 
\(G_{k,n,\ell,w}\) so \(i_k, j_k \leq w_k\). Therefore \(P_I |_{i_k} P_J |_{j_k}\) does not vanish in 
\(G_{k-1,n,\ell,w'}\) where \(w' = (w_1, \ldots, w_{k-1})\). However in \(G_{k,n,\ell,w}\) we must have that either \(P_I\) or \(P_J\) vanishes so either \(I' \notin w\) or \(J' \notin w\).

Since \(i_k', j_k' \leq w_k\) then either \(I' \setminus i_k' \notin (w_1, \ldots, w_{k-1})\) or \(J' \setminus j_k' \notin (w_1, \ldots, w_{k-1})\). Hence \(P_I P_J\) vanishes in 
\(G_{k-1,n,\ell,w'}\). And so the relation \(P_I |_{i_k} P_J |_{j_k} = P_I |_{i_k'} P_J |_{j_k'}\) gives rise to the monomial 
\(P_I |_{i_k} P_J |_{j_k}\) in \(G_{k-1,n,\ell,w'}\).

**Case 2.** Let \(i_k' = j_k\) and \(i_k' \neq i_k\). Now if we have \(I' \setminus i_k' \neq I' \setminus i_k'\) then we may use the same argument above to show that 
\(P_I |_{i_k} P_J |_{j_k}\) is a monomial in \(G_{k-1,n,\ell,w'}\). Otherwise \(I \setminus i_k = I' \setminus i_k'\) 
and hence \(J \setminus j_k = J' \setminus j_k'\). But then we have, \(I' \leq w\) and \(J' \leq w\) since \(i_k, i_k', j_k, j_k' \leq w_k\). And 
so \(P_I P_J\) does not vanish in \(G_{k,n,\ell,w}\), a contradiction. So we have shown that all permutations 
\(w\) for which \(G_{k,n,\ell,w}\) are non-toric arise from those in \(G_{k-1,n,\ell,w'}\). So we can reduce to the case with 
\(k = 3\).

So we may assume that \(k = 3\). Now, we consider the monomial \(P_I P_J\) contained in 
\(G_{k,n,\ell,w}\). Since we have that \(|I \cup J| \leq 6\), we may reduce to the corresponding case with \(k = 3\) and 
\(n = 6\). See Table 2 for the list of toric ideals among \(G_{3,6,\ell,w}\). \(\square\)

**Example 5.8.** Here, we provide illustrative examples of binomial relations used throughout 
the proof of Theorem 5.7. Let \(n = 9, k = 4, B_4 = (1234)56789\) and \(w = (2,5,8,9)\). We 
consider \(G_{k,n,\ell,w}\), to show that it is non-toric. Consider the relation 
\[
P_{5289}P_{1389} = P_{5389}P_{1289}
\]
in \(G_{k,n,\ell,w}\). Note that the left hand side does not vanish in \(G_{k,n,\ell,w}\) however \((5,3,8,9) \notin (2,5,8,9)\) so the right hand side vanishes. Suppose we are told that \(G_{k,n,\ell,w}\) is non-toric 
with \(n, k, \ell\) and \(w\) as above. And in particular suppose we are given that the ideal contains the monomial 
\(P_{5278}P_{1389}\). Suppose it is obtained from the relation, \(P_{5278}P_{1389} = P_{5389}P_{1278}\) 
where the right hand side vanishes but the left hand side does not. We now consider \(G_{k-1,n,\ell,w'}\), 
where \(w' = (2, 5, 8)\). This is non-toric because it contains the monomial \(P_{527}P_{128}\) arising from 
\(P_{527}P_{138} = P_{538}P_{127}\). To reduce to the case with \(n = 6\) we consider the entries in the indices
Table 3: For small $k, n$ we list the number of toric, zero and non-toric ideals of form $G_{k,n,\ell,w}$.

| Toric | $n$ | Zero | $n$ | Non-Toric | $n$ |
|-------|-----|------|-----|-----------|-----|
|       | $k$ | 4    | 5   | 6         | 7   |
| 2     | 6   | 17   | 34  | 2         | 16  |
| 3     | 23  | 74   |     | 3         | 25  |
| 4     |     | 52   |     | 4         | 36  |
| 5     |     |      |     |           |     |
| 6     |     |      |     |           |     |
| 7     |     |      |     |           |     |
| 8     |     |      |     |           |     |
| 9     |     |      |     |           |     |
| 10    |     |      |     |           |     |
| 11    |     |      |     |           |     |
| 12    |     |      |     |           |     |
| 13    |     |      |     |           |     |
| 14    |     |      |     |           |     |
| 15    |     |      |     |           |     |
| 16    |     |      |     |           |     |
| 17    |     |      |     |           |     |
| 18    |     |      |     |           |     |
| 19    |     |      |     |           |     |
| 20    |     |      |     |           |     |

Table 4: For each $k, n$ we calculate the percentage of pairs $(\ell, w)$ for which $G_{k,n,\ell,w}$ is toric.

| Toric | $n$ |
|-------|-----|
|       | $k$ | 4    | 5   | 6         | 7   | 8   | 9   | 10  | 11  | 12  | 13  | 14  | 15  | 16  | 17  | 18  | 19  | 20  | 21  | 22  |
| 2     | 25  | 34   | 39  | 40        | 40  | 40  | 39  | 39  | 39  | 39  | 39  | 39  | 39  | 39  | 39  | 39  | 39  | 38  |
| 3     | 46  | 62   | 70  | 71        | 71  | 70  | 69  | 68  | 67  | 67  | 66  | 65  | 64  | 64  | 64  | 63  |
| 4     | 58  | 75   | 81  | 83        | 83  | 83  | 82  | 82  | 81  | 80  | 79  | 78  | 78  | 77  | 76  | 75  | 74  |
| 5     | 65  | 83   | 88  | 89        | 89  | 89  | 88  | 87  | 86  | 85  | 84  | 83  | 82  | 82  | 81  |     |
| 6     | 71  | 87   | 92  | 93        | 93  | 92  | 91  | 91  | 90  | 89  | 88  | 88  | 87  | 86  | 86  |     |
| 7     | 74  | 90   | 94  | 95        | 95  | 94  | 94  | 93  | 92  | 91  | 91  | 90  | 90  | 89  |     |
| 8     | 77  | 92   | 96  | 96        | 96  | 95  | 94  | 93  | 93  | 92  | 91  | 90  | 89  |     |
| 9     | 80  | 94   | 97  | 97        | 97  | 97  | 96  | 96  | 95  | 94  | 94  | 94  |     |
| 10    | 82  | 95   | 97  | 98        | 98  | 98  | 97  | 96  | 96  | 95  | 94  | 94  |     |
| 11    | 83  | 96   | 98  | 98        | 98  | 98  | 97  | 97  | 96  | 96  | 95  | 94  |     |
| 12    | 84  | 96   | 98  | 98        | 98  | 98  | 98  | 97  | 97  | 96  | 96  | 95  |     |
| 13    | 86  | 97   | 99  | 99        | 99  | 98  | 98  | 98  | 98  | 97  | 97  | 96  |     |
| 14    | 87  | 97   | 99  | 99        | 99  | 99  | 99  | 98  | 98  | 98  | 98  | 97  |     |
| 15    | 87  | 98   | 99  | 99        | 99  | 99  | 99  | 99  | 99  | 99  | 99  | 98  |     |
| 16    | 88  | 98   | 99  | 99        | 99  | 99  | 99  | 99  | 99  | 99  | 99  | 99  |     |
| 17    | 89  | 98   | 99  | 99        | 99  | 99  | 99  | 99  | 99  | 99  | 99  | 99  |     |
| 18    | 89  | 98   | 99  | 99        | 99  | 99  | 99  | 99  | 99  | 99  | 99  | 99  |     |
| 19    | 90  | 98   | 99  | 99        | 99  | 99  | 99  | 99  | 99  | 99  | 99  | 99  |     |
| 20    |     |      |     |           |     |     |     |     |     |     |     |     |     |     |     |     |     |

of the above relation. These entries are $E = \{1, 2, 3, 5, 7, 8\}$. So, under the order-preserving bijection between $E$ and $\{1, 2, 3, 4, 5, 6\}$, the above is equivalent to looking at $G_{3,6,\ell',w''}$ where $w'' = (2, 4, 6)$ and $\ell' = 3$. Under the bijection, the relation becomes $P_{125}P_{136} = P_{346}P_{125}$. Note that the right hand side vanishes but the left hand side does not. So $w''$ satisfies the criteria of Theorem 5.7 and so, under the bijection, $w'$ and $w$ also satisfy these criteria.

Example 5.9. In Tables 1 and 2 we list all Grassmannian permutations $w$, and all block diagonal matching fields $B_{\ell}$ whose corresponding ideal $G_{2,4,\ell,w}$ and $G_{3,6,\ell,w}$ is either toric or zero. Also, we can explicitly calculate the number and so the percentage of pairs $(\ell, w)$ for which $G_{k,n,\ell,w}$ is toric. See Tables 3 and 4.

Remark 5.10. Fix $k$ and $n$. We can explicitly count the number of distinct pairs $(\ell, w)$ for which $G_{k,n,\ell,w}$ is zero. By Proposition 5.2 there are exactly $n^2$ such pairs. Similarly, we can count the number of pairs $(\ell, w)$ for which $G_{k,n,\ell,w}$ is non-toric. By Theorem 5.7 there are exactly $2\binom{n-1}{k+1}$ such pairs. In total there are $n\binom{n}{k}$ distinct pairs $(\ell, w)$, and so there are $n\binom{n}{k} - 2\binom{n-1}{k+1} - n^2$ pairs which give rise to toric ideals of type $G_{k,n,\ell,w}$ inside Schubert varieties.
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| $(k,n)$ | $\ell$ | The $f$-vector of the toric polytope |
|---------|--------|-------------------------------------|
| (3,6)   | 0,3    | 20 122 372 670 766 571 276 83 14 |
|         | 1      | 20 122 376 690 807 615 302 91 15 |
|         | 2*     | 20 122 376 690 807 615 302 91 15 |
|         | 4      | 20 122 378 701 832 645 322 98 16 |
| (3,7)   | 0      | 35 329 1514 4177 7599 9579 8573 5485 2487 778 159 19 |
|         | 1      | 35 329 1546 4411 8352 10977 10221 6762 3136 986 197 22 |
|         | 2      | 35 329 1548 4424 8388 11032 10271 6789 3144 987 197 22 |
|         | 3      | 35 329 1528 4276 7907 10132 9204 5959 2721 851 172 20 |
|         | 4      | 35 329 1535 4329 8084 10474 9625 6301 2904 913 184 21 |
|         | 5      | 35 329 1555 4483 8606 11495 10893 7336 3458 1100 220 24 |
| (4,7)   | 0,3    | 56 756 4852 18664 48026 87804 118120 119262 |
|         | 1      | 56 756 4997 20102 54503 105309 149704 159294 |
|         | 2      | 56 756 5014 20251 55087 106656 151727 161359 |
|         | 3      | 56 756 4955 19647 52288 98898 137416 142869 |
|         | 4      | 56 756 4932 19458 51605 97483 135591 141406 |
|         | 5      | 56 756 4994 20020 53925 103256 145248 152866 |
|         | 6      | 56 756 5014 20302 55557 108606 156529 169146 |
|         | 7      | 137866 84464 38376 12598 2857 414 33 |
| (3,8)   | 0      | 70 1097 7901 33641 95366 192200 286329 227321 278740 |
|         | 1      | 70 1097 8050 35219 102954 214160 329032 381669 338135 |
|         | 2      | 70 1097 8057 35293 103297 215086 330656 383616 339765 |
|         | 3      | 70 1097 7950 34187 98077 200195 302020 344342 300536 |
|         | 4      | 70 1097 8021 34837 100844 207461 315162 361710 317498 |
|         | 5      | 70 1097 8070 35599 104698 220164 342491 402030 361607 |

Table 5: For each Grt($k,n$) and matching field $B_\ell$ we calculate the $f$-vector of the toric polytope associated to the matching field $B_\ell$. The $f$-vector for the rows (*) are the same as the row above, however these polytopes have non-isomorphic face lattices and so define non-isomorphic toric varieties.