How to Transfer between Arbitrary $n$-Qubit Quantum States by Coherent Control and Simplest Switchable Noise on a Single Qubit

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We explore reachable sets of open $n$-qubit quantum systems, the coherent parts of which are under full unitary control and that have just one qubit whose Markovian noise amplitude can be modulated in time such as to provide an additional degree of incoherent control. In particular, adding bang-bang control of amplitude damping noise (non-unital) allows the dynamic system to act transitively on the entire set of density operators. This means one can transform any initial quantum state into any desired target state. Adding switchable bit-flip noise (unital), on the other hand, suffices to explore all states majorised by the initial state. We have extended our open-loop optimal control algorithm (DYNAMO package) by such degrees of incoherent control so that these unprecedented reachable sets can systematically be exploited in experiments. As illustrated for an ion trap experimental setting, open-loop control with noise switching can accomplish all state transfers one can get by the more complicated measurement-based closed-loop feedback schemes.

Recently, dissipation has been exploited for quantum state engineering [1, 2] so that evolution under constant noise leads to long-lived fixed-point states. Lloyd and Viola [3] showed that closed-loop feedback from one resettable ancilla qubit suffices to simulate any quantum dynamics of open systems. Both concepts were used to combine coherent dynamics with optical pumping on an ancilla qubit for dissipative preparation of entangled states [4] or quantum maps [5]. — Besides, full control over the Kraus operators [6] or the environment [7] allows for interconverting arbitrary quantum states.

Manipulating quantum systems with high precision is paramount to exploring their properties for pioneering experiments, and also in view of new technologies [8]. Therefore it is most desirable to extend the current toolbox of optimal control [9] by systematically incorporating dissipative control parameters.

Here we prove that it suffices to include as a new control parameter a single bang-bang switchable Markovian noise amplitude for one qubit (no ancilla) into an otherwise noiseless and coherently controllable network to increase the power of the dynamic system such that any target state can be reached from any initial state. Thus we extend our optimal-control platform DYNAMO [10] by controls over Markovian noise sources. To illustrate possible applications, we demonstrate the initialisation step [11] of quantum computing, i.e. the transfer from the thermal state to the pure state $|00...0\rangle$ (as well as the opposite process), the interconversion of random pairs of mixed states, and finally the noise-driven generation of maximally entangled states.

**Theory.** We consider the quantum Markovian master equation of an $n$-qubit system as a bilinear control system ($\Sigma$):

$$\dot{\rho}(t) = -(i\hat{H}_u + \Gamma)\rho(t) \quad \text{and} \quad \rho(0) = \rho_0 \quad (1)$$

with $H_u := H_0 + \sum_j u_j(t)H_j$ comprising the free-evolution Hamiltonian $H_0$, the control Hamiltonians $H_j$ switched by piecewise constant control amplitudes $u_j(t) \in \mathbb{R}$ and $H_u$ as the corresponding commutator superoperator. Take $\Gamma$ to be of Lindblad form

$$\Gamma(\rho) := -\sum_\ell \gamma_\ell(t)(V_\ell\rho V_\ell^\dagger - \frac{1}{2}(V_\ell^\dagger V_\ell\rho + \rho V_\ell^\dagger V_\ell)) \quad (2)$$

where now $\gamma_\ell(t) \in [0, \gamma_\ell]$ with $\gamma_\ell > 0$ will be used as additional piecewise constant control parameters.

In the sequel we will consider mostly systems with a single Lindblad generator in the non-unital case it is the Lindblad generator for amplitude damping, $V_\alpha$, while in the unital case it is the one for bit flip, $V_b$, defined as

$$V_\alpha := 1^{\otimes n-1}_{2} \otimes |0\rangle \langle 1| \quad \text{and} \quad V_b := 1^{\otimes n-1}_{2} \otimes \sigma_x/2 \quad (3).$$

Here we follow the Lie-algebraic setting along the lines of [12, 13]. As in $12$, we say the control system on $n$ qubits meets the condition for (weak) Hamiltonian controllability if the Lie closure under commutation is

$$\langle iH_0, iH_j | j = 1, \ldots, m \rangle_{\text{Lie}} = su(N) \quad \text{with} \quad N := 2^n \quad (4).$$

Now the reachable set $\text{reach}_\Sigma(\rho_0)$ is defined as the set of all states $\rho(T)$ with $T \geq 0$ that can be reached from $\rho_0$ following the dynamics of ($\Sigma$). If Eqn (1) holds, without relaxation one can steer from any initial state $\rho_0$ to any other state $\rho_{\text{target}}$ with the same eigenvalues. In other words, for $\gamma = 0$ the control system ($\Sigma$) acts transitively on the unitary orbit $U(\rho_0) := \{U\rho_0U^\dagger | U \in SU(N)\}$ of the respective initial state $\rho_0$. This holds for any $\rho_0$ in the set of all density operators, termed $\text{pos}_1$ henceforth.

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Under coherent control and constant noise ($\gamma > 0$ non-switchable) it is difficult to give precise reachable sets for general $n$-qubit systems that satisfy Eqn. 1 only upon including the system Hamiltonian ($H_0$). Based on seminal work by Uhrmann [17, 18], majorisation criteria that are powerful if $H_0$ is not needed to meet Eqn. 1 now just give upper bounds to reachable sets by inclusions. But with increasing number of qubits $n$, these inclusions become increasingly inaccurate and have to be replaced by Lie-semigroup methods as described in [22].

In the presence of bang-bang switchable relaxation on a single qubit in an $n$-qubit system, here we show that the situation improves significantly and one obtains the following results, both proven in the Supplement 22:

**Theorem 1.** Let $\Sigma_a$ be an $n$-qubit bilinear control system as in Eqn. 1 satisfying Eqn. 1 for $\gamma = 0$. Suppose the $n^{\text{th}}$ qubit (say) undergoes (non-unital) amplitude-damping relaxation, the noise amplitude of which can be switched in time between two values as $\gamma(t) \in \{0, \gamma_a\}$ with $\gamma_a > 0$. If the free evolution Hamiltonian $H_0$ is diagonal (e.g., Ising-ZZ type), and if there are no further sources of decoherence, then the system $\Sigma_a$ acts transitively on the set of all density operators $\rho_{a}$:

$$\text{reach}_{\Sigma_a}(\rho_0) = \rho_{a} = 0 \text{ for all } \rho_0 \in \rho_{a},$$

(5)

where the closure is understood as the limit $\gamma_a T \to \infty$.

**Theorem 2.** Let $\Sigma_b$ be an $n$-qubit bilinear control system as in Eqn. 1 satisfying Eqn. 1 ($\gamma = 0$) now with the $n^{\text{th}}$ qubit (say) undergoing (unital) bit-flip relaxation with switchable noise amplitude $\gamma(t) \in \{0, \gamma_a\}$. If the free evolution Hamiltonian $H_0$ is diagonal (e.g., Ising-ZZ type), and if there are no further sources of decoherence, then in the limit $\gamma_a T \to \infty$ the reachable set to $\Sigma_b$ explores all density operators majorised by the initial state $\rho_0$, i.e.

$$\text{reach}_{\Sigma_b}(\rho_0) = \{\rho \in \rho_{a} | \rho \prec \rho_0\} \text{ for any } \rho_0 \in \rho_{a}. \text{(6)}$$

The conditions for the drift Hamiltonian $H_0$ in the theorems above can be relaxed. The details are given in the Supplement 22 along with the proofs.

The scenarios of Eqn. 4 can be generalised to the Lindblad generator $V_0 := \left( \begin{array}{cc} 0 & (1-\theta) \\ \theta & 0 \end{array} \right)$ with $\theta \in [0, 1]$. If $\theta \neq 1/2$ the noise qubit has a unique fixed point $\rho_{\infty}(\theta)$. Comparing this to the canonical density operator of a qubit with energy level splitting $\Delta$, relaxation by $V_0$ gives the same fixed point as equilibrating the system via the noisy qubit with a local heat bath of inverse temperature

$$\beta := \frac{1}{k_B T} = \frac{2}{\Delta} \text{artanh}(\delta(\theta)) \text{ with } \delta(\theta) := \frac{\tilde{\theta}^2 - \theta^2}{\theta^2 + \theta^2}$$

using the shorthand $\tilde{\theta} := 1 - \theta$. As limiting cases, pure amplitude damping is brought about by a bath with zero temperature (i.e. $\theta = 0$), while pure bit-flip corresponds to the high-temperature limit $T \to \infty$ (i.e. $\theta \to \frac{1}{2}$).

While a single-qubit system with unitary control and bang-bang switchable noise generator $V_0$ can clearly be steered (asymptotically) to any state with purity less or equal to the larger of the purities of $\rho_0$ and $\rho_{\infty}(\theta)$, the situation for $n \geq 2$ qubits is more involved: using coherent control, relaxation of a diagonal initial state can be limited to a single pair of eigenvalues at a time if all the remaining ones can be arranged in pairs $(\rho_{ii}, \rho_{jj})$, each satisfying

$$\theta^2/\tilde{\theta}^2 \leq \rho_{ii}/\rho_{jj} \leq \tilde{\theta}^2/\theta^2. \text{(7)}$$

Yet Eqn. 7 poses no restriction in the important task of cooling: starting from the maximally mixed state, optimal control protocols with period-wise relaxation by $V_0$ interspersed with unitary permutation of diagonal density operator elements is more general than the partner-pairing approach [24] to *algorithmic cooling* with bias $\delta(\theta)$ and $0 \leq \theta < 1/2$. This type of algorithmic cooling proceeds also just on the diagonal elements of the density operator, but it involves no transfers limited to a single pair of eigenvalues (details in the Supplement 22).

Exploring Model Systems. To challenge our extended optimal control algorithm, we first consider two examples of state transfer where the target states are on the boundary of the respective reachable sets of the initial states, in other words, they can only be reached asymptotically ($\gamma_a T \to \infty$). To illustrate Theorems 1 and 2 we then demonstrate noise-driven transfer (i) between random pairs of states under controlled amplitude damping noise and (ii) between random pairs of states satisfying $\rho_{\text{wargt}} \prec \rho_0$ under controlled bit-flip noise in Examples 3 and 4. We finish with entanglement generation in an experimental trapped ion system in Example 5.

In Examples 1–4, our system is an $n$-qubit chain with uniform Ising-ZZ nearest-neighbour couplings given by $H_0 := \pi J \sum_k \frac{1}{2} \sigma_z^{(k)} \sigma_z^{(k+1)}$ and piecewise constant $x$ and $y$ controls (that need not be bounded) on each qubit locally, so the control systems satisfy Eqn. 1. We add controllable noise (either amplitude-damping or bit flip) with amplitude $\gamma(t) \in \{0, \gamma_a\}$ acting on each terminal qubit. In all the examples we set $\gamma_a = 5J$.

**Example 1.** Here, as for initialising a quantum computer [11], the task is to turn the thermal initial state $\rho_{\text{th}} := \frac{1}{2} \mathbb{1}$ into the pure target state $|00...0\rangle$ by unitary control and controlled amplitude damping. For $n$ qubits, the task can be accomplished in an $n$-step protocol: let the noise act on each qubit $q$ for the time $\tau_q$ to populate the state $|0\rangle/\sqrt{2}$, and permute the qubits between the steps. A linear chain requires $\sum_{q=1}^n (q-1) = \frac{n(n+1)}{2}$ nearest-neighbour swaps. Since all the intermediate states are diagonal, the swaps can be replaced with $i$-swaps, each taking a time of $\frac{1}{\tau_q}$ under the Ising-ZZ coupling. The residual Frobenius error $\delta_F$ is minimised when all the $\tau_q$ are equal, giving

$$\delta_F^2(\epsilon) = 1 - 2(1-\frac{1}{\tau_q})^n + (1-\epsilon + \frac{1}{\tau_q} - 2)^n, \text{(8)}$$

where $\epsilon := e^{-\gamma_a T\tau_q/n}$ and $T_n := \sum_{q=1}^n \tau_q$. Linearizing this expression and adding the time for the $i$-swaps, the total
shown in the Supplement [23, D].

Example 2. In turn, consider ‘erasing’ the pure initial state $|0\ldots0\rangle$ to the thermal state $\rho_{th} = \frac{1}{2} I$ to the zero-state $\rho_{0000}$ in a 3-qubit Ising-ZZ chain with controlled amplitude-damping noise on qubit one as in Example 1. (a) Quality versus total sequence duration $T$. The dashed line gives the upper bound from Eqn. (8), and the dots (red circles for averages) individual optimization runs with random initial states. Noise amplitudes were initialised in three distinct blocks of equal duration to help the optimisation towards an economic solution. (b) Evolution of the eigenvalues under controlled bit-flip noise (dashed). In both cases (a) shows the median of 9 optimisation runs for each of the eight random state pairs. Representative examples of evolution of the eigenvalues for an amplitude-damping transfer (b) and for a bit-flip transfer (c). In the former case, a typical feature is the initial zeroing of the smaller half of the eigenvalues while the larger half are re-distributed among themselves. Only at the very end are the smaller eigenvalues resurrected.

Example 3. We illustrate transitivity under controlled amplitude damping on one qubit plus general unitary control by transfers between pairs of random 3-qubit density operators. Fig. 2(a) shows the algorithm to converge well to $\delta_F = 10^{-3}$. As shown in Fig. 2(b), the best sequence zero to the smaller half of the eigenvalues as soon as possible just to revive them in the very end after the larger half has been balanced among themselves.

Example 4. Similarly allowing for controlled bit-flip noise on one qubit plus general unitary control, we address the transfer between arbitrary pairs of 3-qubit density operators with $\rho_{\text{target}} \prec \rho_0$. Fig. 2(a) again illustrates the good convergence of the algorithm. Many of these profiles exhibit “terraces” which could indicate local quasi-optima. The unital case may be harder to optimise in general: (1) the majorisation condition entails that a suboptimal transfer made early in the sequence cannot be outbalanced later in the control sequence; it can only be mended in a next iteration; (2) the necessity for simultaneous decoupling (like Trotterisation in the proof of Thm. 2) adds to the hardness of the optimisation.

Example 5. The final example addresses entanglement generation in a system similar to the one in [4]. It consists of four trapped ion qubits coherently controlled by lasers. On top of individual local $z$-controls $(u_{z1}, \ldots, u_{z4})$ on each qubit, one can pulse on all the qubits simultaneously by the joint $x$- and $y$-controls $F_v := \frac{1}{2} \sum_{j=1}^{4} \sigma_{v,j}$ with $v = x, y$ as well as by the quadratic terms $F_2 := (F_v)^2$. All the control amplitudes are expressed as multiples of an interaction strength $\alpha$. In contrast to [4] where the protocol resorts to an ancilla qubit to be added (following [2]) for a measurement-based circuit on the $4 + 1$ system, here we do without the ancilla qubit by making just the terminal qubit subject to controlled amplitude-damping noise with strength $\gamma_{a1}$, to drive the system.
Figure 3. State transfer from the thermal state to the four-qubit GHZ state in the ion-trap system of Example 5 similar to 3. By controlled ‘pumping’ (amplitude damping) on one qubit, one can do without closed-loop measurement-based circuits involving an additional ancilla qubit as required in 3, 4. Our sequence (a) drives the system to the state (c), which differs from the target state $|\text{GHZ}_4\rangle$ by an error of $\delta \rho \approx 5 \times 10^{-3}$. The time evolution of the eigenvalues (b) illustrates parallel action on all the eigenvalues under the sequence.

from the thermal initial state $\rho_{th} := \frac{1}{2} |0000\rangle \langle 0000| + |1111\rangle \langle 1111|$ to the pure entangled target state $|\text{GHZ}_4\rangle = \frac{1}{2} (|0000\rangle + |1111\rangle)$. As shown in Fig. 3, the optimised controls use the noise with maximal amplitude over its entire duration interrupted just by two short periods of purely unitary control.

Discussion. By unitary controllability, we may diagonalise the initial and the target states. So transferring a diagonal initial state into a diagonal target state can be considered as the normal form of the state-transfer problem. It can be treated analytically, because it is easy to separate dissipation-driven changes of eigenvalues from unitary coherent actions of permuting eigenvalues and decoupling drift Hamiltonians. Now the difference between optimising amplitude-damping non-unital transfer (as in Thm. 1) and bit-flip unital transfer (as in Thm. 2) becomes evident: In the non-unital case, transitive action on the set of all density operators clearly helps to escape from suboptimal intermediate control sequences during the optimisation. Yet in the unital case, the majorisation condition $\rho_{\text{target}} \prec \rho(t) \prec \rho_0$ for all $0 \leq t \leq T$ and the boundary conditions $\rho(0) = \rho_0$, $\rho(T) = \rho_{\text{target}}$ (at worst for $\gamma, T \to \infty$) explain potential algorithmic traps: one may easily arrive at an intermediate state $\rho_m(t) \prec \rho_0$ that comes closer to the target state, but will never reach it as it fails to meet the reachability condition $\rho_{\text{target}} \prec \rho_m(t)$; see the Supplement 23, 24 for how to avoid this.

One may contrast our method with the closed-loop control method in 3; originally designed for quantum-map synthesis using projective measurement of a coupled resettable ancilla qubit plus full unitary control to enact arbitrary quantum operations (including state transfers), with Markovian evolution as the infinitesimal limit. Applied to state transfer, the present method instead relies on a switchable local Markovian noise source and requires no measurement nor an ancilla 23.

Conclusions and Outlook. We have proven that by adding as a new control parameter bang-bang switchable Markovian noise on just one system qubit, an otherwise coherently controllable $n$-qubit network can explore unprecedented reachable sets: in the case of amplitude-damping noise (or any noise process in its unitary equivalence class, with compatible drift) one can convert any initial state $\rho_0$ into any target state $\rho_{\text{target}}$, while under switchable bit-flip noise (or any noise process unitarily equivalent) one can transfer any $\rho_0$ into any target $\rho_{\text{target}} \prec \rho_0$ majorised by the initial state. These results have been further generalised and compared to equilibrating the system with a finite-temperature bath.

To our knowledge, this is the first time these features have been systematically explored as open-loop control problems and solved in a minimal setting by coherent local controls and bang-bang modulation of a single local noise source that is exactly Markovian. For state transfer, our open-loop protocol ensures full state controllability, so here it is as powerful as the closed-loop measurement-based feedback scheme in 3 (see the Supplement 23, 24). Thus it may serve to simplify many experimental implementations.

We have extended our optimal-control platform DY-NAMO 10 by controls over Markovian noise amplitudes. Our method was also shown to supersede algorithmic cooling 24. As possible applications, we demonstrated the initialisation step of quantum computing (i.e. the transfer from the thermal state to the pure zero-state 11), the erasure, and the interconversion of random pairs of mixed states, as well as the noise-controlled generation of maximally entangled states.

We anticipate that our approach of switching or even modulating the amplitudes of standard Markovian noise processes as additional open-loop control parameters (in an otherwise coherently controllable system) will pave the way to many experimental applications. For instance, bit flips may be induced by external random processes and amplitude damping may be mimicked by pumping. If needed to facilitate experimental implementation, our algorithm can be specialised such as to separate dissipative and unitary evolution. Otherwise, the algorithm parallels coherent and incoherent controls to an extent usually going beyond analytical tractability.

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[25] The Supplement explains how for state transfer, Markovian open-loop controllability already implies full state controllability (including transfers by non-Markovian processes), while for the lift to Kraus-map controllability it remains an open question, whether closed-loop feedback control is not only sufficient (as established in [3]), but also necessary in the sense that it could not be replaced by open-loop unitary control plus control over local Markovian noise.
[26] A T-transform is a convex combination $\lambda I + (1 - \lambda)Q$, where Q is a pair transposition matrix and $\lambda \in [0, 1]$.
[27] $R_{b}(t)$ of Eqn. (A5) covers $\lambda \in [\frac{1}{2}, 1]$, while $\lambda \in [0, \frac{1}{2}]$ is obtained by unitarily swapping the elements before applying $R_{b}(t); \lambda = \frac{1}{2}$ is obtained in the limit $\epsilon \to 0$.
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[41] For simplicity, we only consider the extreme case of non-unital maps (such as amplitude damping) allowing for pure-state fixed points and postpone the generalised cases parameterised by $\theta$ in Appendix B till the very end.
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Appendix A: Proofs of the Main Theorems and Generalisations

**Theorem 1.** Let $\Sigma_n$ be an $n$-qubit bilinear control system as in Eqn. (1) satisfying Eqn. (4) for $\gamma = 0$. Suppose the $n^{th}$ qubit (say) undergoes (non-unital) amplitude-damping relaxation, the noise amplitude of which can be switched in time between two values as $\gamma(t) \in \{0, \gamma_s\}$ with $\gamma_s > 0$. If the free evolution Hamiltonian $H_0$ is diagonal (e.g., Ising-ZZ type), and if there are no further sources of decoherence, then the system $\Sigma_n$ acts transitively on the set of all density operators $\text{pos}_1$:

$$\text{reach}_{\Sigma_n}(\rho_0) = \text{pos}_1 \quad \text{for all } \rho_0 \in \text{pos}_1,$$

where the closure is understood as the limit $\gamma_s T \to \infty$.

**Proof.** We keep the proofs largely constructive. By unitary controllability, $\rho_0$ may be chosen diagonal as $\rho_0 = \text{diag}(r_0)$. Since a diagonal $\rho_0$ commutes with a diagonal free evolution Hamiltonian $H_0$, the evolution under noise and coupling remains purely diagonal, following

$$r(t) = \left[ I_2^{\otimes (n-1)} \otimes \begin{pmatrix} 1 & 1 - \epsilon \\ 0 & \epsilon \end{pmatrix} \right] r_0 =: R_\epsilon(t) r_0,$$

where $\epsilon := e^{-\gamma_s t}$ and $R_\epsilon(t)$ is by construction a stochastic matrix. With the noise switched off, full unitary control includes arbitrary permutations of the diagonal elements. Any of the pairwise relaxative transfers between diagonal elements $\rho_{ii}$ and $\rho_{jj}$ (with $i \neq j$) lasting a total time of $\tau$ can be neutralised by permuting $\rho_{ii}$ and $\rho_{jj}$ after a time

$$\tau_{ij} := \frac{1}{\gamma_s} \ln \left( \frac{\rho_{ii} e^{\gamma_s \tau} + \rho_{jj}}{\rho_{ii} + \rho_{jj}} \right),$$

and letting the system evolve under noise again for the remaining time $\tau - \tau_{ij}$. Thus with $2^{n-1} - 1$ such switches all but one desired transfer can be neutralised. As $\rho(t)$ remains diagonal under all permutations, relaxative and coupling processes, one can obtain any state of the form

$$\rho(t) = \text{diag}(\ldots, [\rho_{ii} + \rho_{jj} \cdot (1 - e^{-\gamma_s t})]_{ii}, \ldots, [\rho_{jj} \cdot e^{-\gamma_s t}]_{jj}, \ldots).$$

Sequences of such transfers between single pairs of eigenvalues $\rho_{ii}$ and $\rho_{jj}$ and their permutations then generate (for $\gamma_s T \to \infty$) the entire set of all diagonal density operators $\Delta \subset \text{pos}_1$. By unitary controllability one gets all the unitary orbits $\mathcal{U}(\Delta) = \text{pos}_1$. Hence the result. \hfill \Box

**Theorem 2.** Let $\Sigma_n$ be an $n$-qubit bilinear control system as in Eqn. (1) satisfying Eqn. (4) ($\gamma = 0$) now with the $n^{th}$ qubit (say) undergoing (unital) bit-flip relaxation with switchable noise amplitude $\gamma(t) \in \{0, \gamma_s\}$. If the free evolution Hamiltonian $H_0$ is diagonal (e.g., Ising-ZZ type), and if there are no further sources of decoherence, then in the limit $\gamma_s T \to \infty$ the reachable set to $\Sigma_n$ explores all density operators majorised by the initial state $\rho_0$, i.e.

$$\text{reach}_{\Sigma_n}(\rho_0) = \{ \rho \in \text{pos}_1 \mid \rho \prec \rho_0 \} \quad \text{for any } \rho_0 \in \text{pos}_1.$$

**Proof.** Again consider the initial state $\rho_0 := \text{diag}(r_0)$. The evolution under the noise remains diagonal following

$$r(t) = \left[ I_2^{\otimes (n-1)} \otimes \frac{1}{\gamma_s} \begin{pmatrix} 1 + \epsilon & (1 - \epsilon) \\ (1 - \epsilon) & (1 + \epsilon) \end{pmatrix} \right] r_0 =: R_\epsilon(t) r_0,$$

where $\epsilon := e^{-\gamma_s t}$ and $R_\epsilon(t)$ is doubly stochastic. In order to limit the relaxative averaging to the first two eigenvalues, first conjugate $\rho_0$ with the unitary

$$U_{12} := I_2 \oplus \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \otimes 2^{n-1}-1,$$

to obtain $\rho_0' := U_{12} \rho_0 U_{12}^\dagger$. Then the relaxation acts as a $T$-transform [20] on the first two eigenvalues of $\rho_0'$, while leaving the remaining ones invariant.
Yet the protected subspaces have to be decoupled from the free evolution Hamiltonian $H_0$ assumed diagonal. Any such $H_0$ decomposes as $H_0 =: H_{0,1} \otimes 1 + H_{0,2} \otimes \sigma_z$, where $H_{0,1}$ and $H_{0,2}$ are diagonal. The term with $H_{0,1}$ commutes with $\rho_0'$ and can thus be neglected. The other term can be sign-inverted using $\pi$-pulses in the $x$-direction on the noisy qubit (generated by $H_{1x}$),

$$e^{i\hat{H}_{1x}t} e^{-i(\Gamma+i\hat{H}_{0,2} \otimes \sigma_z)t} e^{-i\pi \hat{H}_{1x}t} = e^{-i(\Gamma-i\hat{H}_{0,2} \otimes \sigma_z)t},$$

which also leave the bit-flip noise generator invariant. Thus $H_0$ may be fully decoupled in the Trotter limit

$$\lim_{k \to \infty} (e^{-\frac{i}{T}(\Gamma+i\hat{H}_{0,2} \otimes \sigma_z)} e^{-\frac{i}{2T}(\Gamma-i\hat{H}_{0,2} \otimes \sigma_z)})^k = e^{-i\Gamma T}.$$

By combining permutations of diagonal elements with selective pairwise averaging by relaxation, any $T$-transform of $\rho_0$ can be obtained in the limit $\gamma_s T \to \infty$:

$$\rho(t) = \text{diag} \left( \ldots, \frac{1}{2}[\rho_{ii} + \rho_{jj} + (\rho_{ii} - \rho_{jj}) \cdot e^{-\frac{2}{T}t}]_{ii}, \ldots, \frac{1}{2}[\rho_{ii} + \rho_{jj} + (\rho_{jj} - \rho_{ii}) \cdot e^{-\frac{2}{T}t}]_{jj}, \ldots \right).$$

Now recall that a vector $y \in \mathbb{R}^N$ majorises a vector $x \in \mathbb{R}^N$, $x \prec y$, if and only if there is a doubly stochastic matrix $D$ with $x = Dy$, where $D$ is a product of at most $N-1$ such $T$-transforms (e.g., Thm. B.6 in [28] or Thm. II.1.10 in [29]). Actually, by the work of Hardy, Littlewood, and Pólya [30] this sequence of $T$-transforms is constructive [28, p32] as will be made use of later. Thus in the limit $\gamma_s T \to \infty$ all diagonal vectors $r \prec r_0$ can be reached and hence by unitary controllability all the states $\rho \prec \rho_0$.

Finally, to see that one cannot go beyond the states majorised by the initial state, observe that controlled unitary dynamics combined with bit-flip relaxation is still completely positive, trace-preserving and unital. Thus it takes the generalised form of a doubly-stochastic linear map $\Phi$ in the sense of Thm. 7.1 in [31], which for any hermitian matrix $A$ ensures $\Phi(A) \prec A$. Hence the (closure of the) reachable set is indeed confined to $\rho \prec \rho_0$.

The conditions for the drift Hamiltonian $H_0$ in theorems above can be relaxed to the following generalisations. Any free evolution Hamiltonian $H_0$ may be diagonalised by a unitary transformation: $H_0 = U H_0^{\text{diag}} U^\dagger$. The same transformation $U$, when applied to the Lindblad generator $V$, yields a new Lindblad generator $V' := UVU^\dagger$. If $V$ satisfies Theorem 1 or 2 with any diagonal free evolution Hamiltonian, then $V'$ will satisfy them with $H_0$. Degenerate eigenvalues of $H_0$ yield some freedom in choosing $U$ which, together with arbitrary permutations, can be used to make $V'$ simpler to implement (e.g. local).

Moreover, the theorems above are stated under very mild conditions. So

1. the theorems hold a fortiori if the noise amplitude is not only a bang-bang control $\gamma(t) \in \{0, \gamma_s\}$, but may vary in time within the entire interval $\gamma(t) \in [0, \gamma_s]$; the use of this will demonstrated in more complicated systems elsewhere;
2. likewise, if several qubits come with switchable noise of the same type (unital or non-unital), then the (closures of the) reachable sets themselves do not alter, yet the control problems can be solved more efficiently;
3. needless to say, a single switchable non-unital noise process (equivalent to amplitude damping) on top of unital ones suffices to make the system act transitively;
4. for systems with non-unital switchable noise (equivalent to amplitude damping), the (closure of the) reachable set under non-Markovian conditions cannot grow, since it already encompasses the entire set of density operators (see Sec. E)—yet again the control problems may become easier to solve efficiently;
5. likewise in the unital case, the reachable set does not grow under non-Markovian conditions, since the Markovian scenario already explores all interconversions obeying the majorisation condition (see also Sec. E);
6. the same arguments hold for a coded logical subspace that is unitarily fully controllable and coupled to a single physical qubit undergoing switchable noise.
Appendix B: Generalisation of Noise Generators and Their Relation to Coupling to Finite-Temperature Baths

1. Generalised Lindblad Terms

The noise scenarios of the previous theorems can be generalised by using the Lindblad generator \( V_\theta := \begin{pmatrix} 0 & 1-	heta \\ \theta & 0 \end{pmatrix} \) with \( \theta \in [0, 1] \). Using the short-hand \( \overline{\theta} := 1 - \theta \), the Lindbladian and its exponential turn into

\[
\Gamma(\theta) = -\begin{pmatrix} -\theta^2 & 0 & 0 & \overline{\theta}^2 \\ 0 & \overline{\theta} - \frac{1}{2} & \overline{\theta} & 0 \\ 0 & \overline{\theta} & \overline{\theta} - \frac{1}{2} & 0 \\ \theta^2 & 0 & 0 & -\overline{\theta}^2 \end{pmatrix}, \quad \text{and} \quad \exp(-\gamma_* t \Gamma(\theta)) = \begin{pmatrix} c_\theta (\overline{\theta}^2 + \theta^2 \varepsilon_\theta) & 0 & 0 & c_\theta \overline{\theta}^2 (1 - \varepsilon_\theta) \\ 0 & \varepsilon_\theta' \cosh(\gamma_* t \overline{\theta}) & \varepsilon_\theta' \sinh(\gamma_* t \overline{\theta}) & 0 \\ 0 & \varepsilon_\theta' \sinh(\gamma_* t \overline{\theta}) & \varepsilon_\theta' \cosh(\gamma_* t \overline{\theta}) & 0 \\ c_\theta \overline{\theta}^2 (1 - \varepsilon_\theta) & 0 & 0 & c_\theta (\overline{\theta}^2 + \theta^2 \varepsilon_\theta) \end{pmatrix}
\]

with \( c_\theta := \frac{1}{\overline{\theta}^2 + \theta^2} \), \( \varepsilon_\theta := e^{-\gamma_* t/c_\theta} \) and \( \varepsilon_\theta' := e^{\gamma_* t(\overline{\theta}^2 - 1/2)} \) as further short-hands. Choosing the initial state diagonal, the action on a diagonal vector of an \( n \)-qubit state takes the following form that can be decomposed into a (scaled) convex sum of a pure amplitude-damping part and a pure bit-flip part (cp. Eqs. [A2], [A6])

\[
R_\theta(t) = 1^{2 \otimes (n-1)}_2 \otimes \left[ c_\theta \left( \frac{(\overline{\theta}^2 + \theta^2 \varepsilon_\theta)}{(\overline{\theta}^2 (1 - \varepsilon_\theta))(\overline{\theta}^2 + \theta^2 \varepsilon_\theta))} \right) \right]
\]

In order to limit the entire dissipative action over some fixed time \( \tau \) to the first two eigenvalues (as in Thm. 1), one may switch again as in Eqn. [A3] after a time

\[
\tau_{ij}(\theta) := \frac{c_\theta}{\gamma_*} \ln \left( \frac{e^{\gamma_* t/c_\theta} (\rho_{ii} - \theta^2 \rho_{jj}) + (\overline{\theta}^2 \rho_{jj} - \theta^2 \rho_{ii})}{(\theta^2 - \overline{\theta}^2)(\rho_{ii} + \rho_{jj})} \right).
\]

This is meaningful as long as \( 0 \leq \tau_{ij}(\theta) \leq \tau \), which corresponds to the condition

\[
\frac{\theta^2}{\overline{\theta}^2} \leq \rho_{ii}/\rho_{jj} \leq \frac{\overline{\theta}^2}{\theta^2}.
\]

If \( \theta \neq 1/2 \) the noise qubit has a unique fixed point,

\[
\rho_\infty(\theta) = c_\theta \begin{pmatrix} \overline{\theta}^2 & 0 \\ 0 & \theta^2 \end{pmatrix}.
\]

Comparing this with the canonical density operator of a qubit with energy level splitting \( \Delta \) at inverse temperature \( \beta := \frac{1}{k_B T} \),

\[
\rho_\beta := \frac{1}{2 \cosh(\beta \Delta/2)} \begin{pmatrix} e^{\beta \Delta/2} & 0 \\ 0 & e^{-\beta \Delta/2} \end{pmatrix},
\]

we can see that the parameter \( \theta \) corresponds to the inverse temperature

\[
\beta(\theta) = \frac{2}{\Delta} \arctanh(\delta(\theta)) \quad \text{with} \quad \delta(\theta) := \frac{\overline{\theta}^2 - \theta^2}{\overline{\theta}^2 + \theta^2}.
\]

Thus (for \( t \to \infty \)) the relaxation by the single Lindblad generator \( V_\theta \) shares the canonical fixed point with equilibrating the system via the noisy qubit with a local bath of temperature \( \beta(\theta) \). As limiting cases, pure amplitude damping is brought about by a bath with zero temperature \( T_\theta = 0 \) (i.e. \( \theta = 0 \)), while pure bit-flip shares the canonical fixed point with the high-temperature limit \( T_\theta \to \infty \) (i.e., \( \theta \to \frac{1}{2} \)), see also Sec. [B2] for the relation to heat baths.
Whereas in a single-qubit system with unitary control and bang-bang switchable noise generator \( V_\theta \) it is straightforward to see that one can (asymptotically) reach all states with purity less or equal to the larger of the purities of the initial state \( \rho_0 \) and \( \rho_\infty(\theta) \),

\[
\text{reach}_{1 \text{qubit}, \Sigma_\theta}(\rho_0) = \{ \rho \mid \rho < \rho_0 \} \cup \{ \rho' \mid \rho' < \rho_\infty(\theta) \},
\]

the situation for \( n \geq 2 \) qubits is more involved: relaxation of a diagonal state can only be limited to a single pair of eigenvalues, if all the remaining ones can be arranged in pairs each satisfying Eqn. \([16]\).

However, Eqn. \([16]\) poses no restriction in an important special case, i.e. the task of cooling: starting from the maximally mixed state, optimal control protocols with period-wise relaxation by \( V_\theta \) interspersed with unitary permutation of diagonal density operator elements clearly include the partner-pairing approach \([22]\) to algorithmic cooling with bias \( \delta(\theta) \) defined in Eqn. \([16]\), as long as \( 0 \leq \theta < 1/2 \). Note that this type of algorithmic cooling proceeds also just on the diagonal elements of the density operator, but it involves no transfers limited to a single pair of eigenvalues. Let \( \rho_\delta \) define the state(s) with highest asymptotic purity achievable by partner-pairing algorithmic cooling with bias \( \delta \). As the pairing algorithm is just a special case of unitary evolutions plus relaxation brought about by \( V_\theta \), one arrives at

\[
\text{reach}(\rho_0) \supseteq \text{reach}(\rho_\delta) \quad \text{for any} \quad \rho_0,
\]

because any state \( \rho_0 \) can clearly be made diagonal to evolve into a fixed-point state obeying Eqn. \([16]\), from whence the purest state \( \rho_\delta \) can be reached by partner-pairing cooling.

To see this in more detail, note that a (diagonal) density operator \( \rho_\theta \) of an \( n \)-qubit system is in equilibrium with a bath of inverse temperature \( \beta(\theta) \) coupled to its terminal qubit, if the pairs of consecutive eigenvalues satisfy

\[
\frac{\rho_{ii}}{\rho_{i+1,i+1}} = \frac{\theta^2}{\bar{\theta}^2} \quad \text{for all odd } i < 2^n.
\]

Hence (for \( \theta \neq 1/2 \)) such a \( \rho_\theta \) is indeed a fixed point under uncontrolled drift, i.e. relaxation by \( V_\theta \) and evolution under a diagonal Hamiltonian \( H_\theta \) thus extending Eqn. \([16]\) to \( n \) qubits. Now, if (say) the first pair of eigenvalues is inverted by a selective \( \pi \) pulse (which can readily be realized by unitary controls with relaxation switched off), a subsequent evolution under the drift term only affects the first pair of eigenvalues as

\[
\frac{1}{\theta^2 + \bar{\theta}^2} \left[ \frac{(\theta^2 + \bar{\theta}^2)\varepsilon_\theta}{\theta^2(1 - \varepsilon_\theta)} \right] \left( \frac{\varepsilon_\theta \theta^2}{c_\theta \theta^2} \right) = \frac{1}{\theta^2 + \bar{\theta}^2} \left[ \frac{(\theta^2 + \bar{\theta}^2)\varepsilon_\theta}{\theta^2 - \varepsilon_\theta(\theta^2 - \bar{\theta}^2)} \right] \left( \frac{c_\theta \theta^2}{\varepsilon_\theta} \right).
\]

In other words, the evolution then acts as a \( T \)-transform on the first eigenvalue pair. Since the switching condition Eqn. \([16]\) is fulfilled at any time, all \( T \)-transformations with \( \varepsilon_\theta \in [0, 1] \) on the first pair of eigenvalues can be obtained and preserved during transformations on subsequent eigenvalue pairs.

Hence from any diagonal fixed-point state \( \rho_\theta \) (including \( \rho_\delta \) as a special case), those other diagonal states (and their unitary orbits) can be reached by pairwise \( T \)-transforms as long as the remaining eigenvalues can be arranged such as to fulfill the stopping condition Eqn. \([16]\). Suffice this to elucidate why for \( n \geq 2 \) a fully detailed determination of the asymptotic reachable set in the case of unitary control plus a single switchable \( V_\theta \) on one qubit appears involved and will thus be treated elsewhere.

2. Bosonic or Fermionic Heat Baths

Following the lines of \([32\, Ch. 3.4.2]\), a qubit with the Hamiltonian

\[
H = -\hbar \omega_0 \frac{\sigma_\pi}{2}
\]

coupled to a bosonic (or fermionic) heat bath with inverse temperature \( \beta \) via a coupling of the form \( \sigma_\pi \otimes (a + a^\dagger) \) is described in the Born-Markov approximation by a Lindblad equation (of the form of Eqs. \([1], \[2]\)) with two dissipator terms, \( \Gamma_{\sigma^-} \) and \( \Gamma_{\sigma^+} \),

\[
\dot{\rho} = -i[H + \gamma (1 \pm n(\omega_0)) \Gamma_{\sigma^-} + \gamma n(\omega_0) \Gamma_{\sigma^+}] \rho,
\]
where \( n(\omega) := 1/(e^{\beta \hbar \omega} - 1) \) is the Planck (or Fermi) distribution and \( \gamma \) is the relaxation rate constant. In the vec-superoperator representation this yields a Liouvillian which takes the form

\[
\Gamma_T \propto \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & -\frac{1}{2} & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} + n(\omega_0) \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 0 \\
1 & 0 & 0 & -1
\end{pmatrix}. \tag{B15}
\]

In the zero-temperature limit, \( n(\omega_0) \to 0 \) and only the \( \Gamma_{\sigma^-} \) term remains. In the limit \( T \to \infty \), we have \( n(\omega_0) \to \infty \) for bosons and \( n(\omega_0) \to 0 \) for fermions, and one thus obtains in both cases

\[
\lim_{T \to \infty} \Gamma_T \propto \Gamma_{\{\sigma^+, \sigma^-\}} = - \begin{pmatrix}
-1 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
1 & 0 & 0 & -1
\end{pmatrix}. \tag{B16}
\]

This is equivalent to dissipation under the two Lindblad operators \({\{\sigma_x, \sigma_y\}}\) since \( \Gamma_{\{\sigma_x, \sigma_y\}} = 2 \Gamma_{\{\sigma^+, \sigma^-\}} \). In contrast, \( \sigma_x \) as the only Lindblad operator (generating bit-flip noise) gives

\[
\Gamma_{\{\sigma_x\}} = - \begin{pmatrix}
-1 & 0 & 0 & 1 \\
0 & -1 & 1 & 0 \\
0 & 1 & -1 & 0 \\
1 & 0 & 0 & -1
\end{pmatrix} \tag{B17}
\]

in agreement with Eqn. (B11). Note, however, that the propagators \( e^{-t\Gamma_x} \) generated by \( \Gamma_{\{\sigma^+, \sigma^-\}} \) vs \( \Gamma_{\{\sigma_x\}} \) only act on diagonal density operators (and those with purely imaginary coherence terms \( \rho_{12} = \rho_{21}^* \)) in an indistinguishable way. However, the relaxation of the real parts of the coherence terms \( \rho_{12} = \rho_{21}^* \) differs: \( \Gamma_{\sigma_x} \) leaves them invariant, see also Eqn. (B2), while \( \Gamma_{\{\sigma^+, \sigma^-\}} \) does not. In other words, \( \Gamma_{\sigma_x} \) does have nontrivial invariant subspaces used in Eqn. (A7), while \( \Gamma_\infty \) does not. Also for non-zero temperatures, there is no direct equivalence between relaxation under \( \Gamma_T \) and \( \Gamma(\theta) \).

**Appendix C: GRAPE Extended by Noise Controls**

In state transfer problems the fidelity error function used in [110] is valid if the purity remains constant or the target state is pure. In contrast to closed systems, in open ones these conditions need not hold. Thus here we use a full Frobenius-norm distance-based error function instead: \( \delta_F^2 := \|X_{M:0} - X_{\text{target}}\|_F^2 \), where \( X_{k:0} = X_k \cdots X_1 \text{vec}(\rho_0) \) is the vectorised state after time slice \( k \), \( X_k = e^{-\Delta t L_k} \) is the propagator for time slice \( k \) in the Liouville space, and \( L_k := iH_u(t_k) + \Gamma(t_k) \). The gradient of the error is obtained as

\[
\frac{\partial \delta_F^2}{\partial u_j(t_k)} = 2 \text{Re} \text{tr} \left( (X_{M:0} - X_{\text{target}}) \frac{\partial X_{M:0}}{\partial u_j(t_k)} \right), \tag{C1}
\]

where

\[
\frac{\partial X_{M:0}}{\partial u_j(t_k)} = X_M \cdots X_{k+1} \frac{\partial X_k}{\partial u_j(t_k)} X_{k-1} \cdots X_1 \text{vec}(\rho_0). \tag{C2}
\]

Exact expressions for the derivatives of \( X_k \) [110] require \( L_k \) to be normal, which does not hold in the general case of open systems of interest here. Instead we may use, e.g., the finite-difference method to compute the partial derivatives as

\[
\frac{\partial X_k}{\partial u_j(t_k)} = \lim_{s \to 0} \frac{\exp(-\Delta t(L_k + isH_j)) - \exp(-\Delta t L_k)}{s}, \tag{C3}
\]

\[
\frac{\partial X_k}{\partial \gamma_j(t_k)} = \lim_{s \to 0} \frac{\exp(-\Delta t(L_k + s\Gamma V_j)) - \exp(-\Delta t L_k)}{s}. \tag{C4}
\]

The optimal value of \( s \) is obtained as a tradeoff between the precision of the gradient and numerical accuracy, which starts to deteriorate when \( s \) becomes very small.
The work of Hardy, Littlewood, and Pólya [30] (HLP) provides a constructive scheme ensuring the majorisation condition $\rho_{\text{target}} \prec \rho(t) \prec \rho_0$ for all $0 \leq t \leq T$ to be fulfilled for all intermediate steps. Let the initial and the target state be given as diagonal vectors with the eigenvalues of the respective density operator in descending order, so $\rho_0 = \text{diag}(y_1, y_2, \ldots, y_N)$ and $\rho_{\text{target}} = \text{diag}(x_1, x_2, \ldots, x_N)$. Following [30, p321], fix $j$ to be the largest index $\lesssim \sum_{n=1}^{m-1} \frac{1}{n} = 1$.
satisfying with random initial sequences. (b) Evolution of the eigenvalues under the controls of the best of the line as the upper bound from Eqn. (D2). Dots (red circles for averages) denote individual numerical optimal-control runs with random initial sequences. (c) shows that the noise is always maximised, and the unitary actions generated by \((u_{x\nu}, u_{y\nu})\) are fully parallelised with it.

such that \(x_j < y_j\) and let \(k > j\) be the smallest index with \(x_k > y_k\). Define \(\delta := \min\{(y_j - x_j), (x_k - y_k)\}\) and \(\lambda := 1 - \delta/(y_j - y_k)\). This suffices to construct

\[
y' := \lambda y + (1 - \lambda)Q_{jk} y
\]

satisfying \(x < y' < y\). Here the pair-permutation \(Q_{jk}\) interchanges the coordinates \(y_k\) and \(y_j\) in \(y\). So \(y'\) is a T-transform of \(y\), and Ref. [28] shows that by \(N - 1\) successive steps of \(T\)-transforming and sorting, \(y\) is converted into \(x\). Now the \(T\)-transforms \(\lambda I + (1 - \lambda)Q_{jk}\) can actually be brought about by switching on the bit-flip noise according to Eqn. (A10) for a time interval of duration

\[
\tau_{jk} := -\frac{2}{\gamma_*} \ln|1 - 2\lambda|.
\]

With these stipulations one obtains an iterative analytical scheme for transferring any \(\rho_0\) by unitary control and switchable bit-flip noise on a terminal qubit into any \(\rho_{\text{target}}\) satisfying the reachability condition \(\rho_{\text{target}} < \rho_0\).

**Scheme for Transferring Any \(n\)-Qubit Initial State \(\rho_0\) into Any Target State \(\rho_{\text{target}} < \rho_0\) by Unitary Control and Switchable Bit-Flip Noise on Terminal Qubit:**

1. Switch off noise to \(\gamma = 0\), diagonalise target \(U_x \rho_{\text{target}} U_x^\dagger =: \text{diag}(x)\) to obtain diagonal vector in descending order \(x = (x_1, x_2, \ldots, x_N)\); keep \(U_x\);
2. Apply unitary evolution to diagonalise \(\rho_0\) and set \(\tilde{\rho}_0 := \text{diag}(y)\);
3. Determine index pair \((j, k)\) by the HLP scheme;
4. Apply unitary evolution to permute entries \((y_1, y_j)\) and \((y_2, y_k)\) of \(y\), so \(\text{diag}(y) = \text{diag}(y_j, y_k, \ldots)\);
5. Apply unitary evolution \(U_{12}\) of Eqn. (A7) to turn \(\rho_y = \text{diag}(y)\) into protected state;
6. Switch on bit-flip noise on terminal qubit \(\gamma(t) = \gamma_*\) for duration \(\tau_{jk}\) of Eqn. (22) (while decoupling as in Eqn. (A9)) to obtain \(\rho_y\);
7. To undo step (5), apply inverse unitary evolution \(U_{12}^\dagger\) to re-diagonalise \(\rho_y\) and obtain next iteration of diagonal vector \(y = y'\) and \(\rho_y = \text{diag}(y)\);
8. Go to (2) and terminate after \(N - 1\) loops (\(N := 2^n\));
9. Apply inverse unitary evolution \(U_x^\dagger\) from step (0) to take final \(\rho_y\) to \(U_x^\dagger \rho_y U_x \simeq \rho_{\text{target}}\).

Note that the general HLP scheme need not always be time-optimal: E.g., a model calculation shows that just the dissipative intervals for transferring \(\text{diag}(1, 2, 3, \ldots, 8)/36\) into \(1_8/8\) under a bit-flip relaxation-rate constant \(\gamma_* = 5J\) and achieving the target with \(\delta_F = 9.95 \cdot 10^{-5}\) sum up to \(T_{\text{relax}} = 12/J\) in the HLP-scheme, while a greedy alternative can make it within \(T_{\text{relax}} = 6.4/J\) and a residual error of \(\delta_F = 6.04 \cdot 10^{-5}\).
Appendix F: Outlook on the Relation to Extended Notions of Controllability in Open Quantum Systems

The current results also pave the way to an outlook on controllability aspects of open quantum systems on a more general scale, since they are much more intricate than in the case of closed systems [3, 6, 13, 16, 21, 22, 32, 33].

Here we have taken profit from the fact that like in closed systems (where pure-state controllability is strictly weaker than full unitary controllability [50, 51]), in open quantum systems Markovian state transfer appears less demanding than the operator lift to the most general scenario of arbitrary quantum map generation (including non-Markovian ones) first connected to closed-loop feedback control in [3]. Therefore in view of experimental implementation, the question arises how far one can get with open-loop control including noise modulation and whether the border to closed-loop feedback control is drawn by Markovianity.

Due to their divisibility properties [38, 39] that allow for an exponential construction (of the connected component) as Lie semigroup [10], Markovian quantum maps are a well-defined special case of the more general completely positive trace-preserving (CPTP) semigroup of Kraus maps, which clearly comprise non-Markovian ones, too. While some controllability properties of general Kraus-map generation have been studied in [3, 41], a full account of controllability notions in open systems should also encompass state-transfer to give the following major scenarios:

1. Markovian state-transfer controllability (MSC),
2. Markovian map controllability (MMC),
3. general (Kraus-map mediated) state controllability (KSC) (including the infinite-time limit of ‘dynamic state controllability’ (DSC) [6, 7]),
4. general Kraus-map controllability (KMC) [3, 6].

Writing ‘⊆’ and ‘⊂’ in some abuse of language for ‘weaker than’ and ‘strictly weaker than’, one obviously has at least MSC ⊆ KSC and MMC ⊆ KMC, while DSC ⊆ KMC was already noted in the context of control directly over the Kraus operators [6]. In pursuing control over environmental degrees of freedom, Pechen [7, 10] also proposed a scheme, where both coherent plus incoherent light (the latter with an extensive series of spectral densities depending on ratios over the difference of eigenvalues of the density operators to be transferred) were shown to suffice for interconverting arbitrary states with non-degenerate eigenvalues in their density-operator representations.

Yet the situation outlined above is more subtle, since unital and non-unital cases may differ. In this work, we have embarked on unital and non-unital Markovian state controllability, MSC [41].

Somewhat surprisingly, in the non-unital case (equivalent to amplitude damping), the utterly mild conditions of unitary controllability plus bang-bang switchable noise amplitude on one single internal qubit (no ancilla) suffice for acting transitively on the set of all density operators (Theorem 1). Hence these features fulfill the maximal condition KSC already. In other words, for cases of non-unital noise equivalent to amplitude damping (henceforth indexed by ‘nu’), KSCnu implies KSC. Moreover, under the reasonable assumption that the mild conditions in Theorem 1 are in fact the weakest for controlling Markovian state transfer MSCnu in our context, Theorem 1 shows that MSCnu implies KSC via KSCnu. So in the (extreme) non-unital cases, there is no difference between Markovian and non-Markovian state controllability. — On the other hand in order to compare non-unital with unital processes, taking Theorems 1 and 2 together proves MSCu ⊂ MSCnu, since the former is restricted by the majorisation condition of Theorem 2.

Similarly, in the unital case (equivalent to bit-flip), the mild conditions of unitary controllability plus bang-bang switchable noise amplitude on one single internal qubit suffice for achieving all state transfers obeying majorisation (Theorem 2). Hence again they fulfill the maximal condition KSCu at the same time. This is because state transfer under every unital CPTP Kraus map (be it Markovian or non-Markovian) has to meet the majorisation condition; so we get KSCu. On the other hand, the majorisation condition itself imposes the restriction KSCu ⊂ KSC. Again, under the reasonable assumption that the mild conditions in Theorem 2 are in fact the weakest for controlling Markovian state transfer MSCu in our context, Theorem 2 shows that MSCu implies KSCu. Thus also in the unital case, there is no difference between Markovian and non-Markovian state controllability.

The results on these two cases, i.e. non-unital and unital (in the light of Appendix B seen as the limits θ = 0 and θ = 2, respectively), can therefore be summarized as follows:

**Corollary 1.** In the two scenarios of Theorem 1 (non-unital) and 4 (unital), Markovian state controllability already implies Kraus-map mediated state controllability and one finds

\[
\text{MSCu} \implies \text{KSCu} \implies \text{KSC}
\]

**(F1)**
However, whether $\text{MSC}_\theta \implies \text{KSC}_\theta$ also holds in the generalisation of Appendix B, where $\theta$ can range over the entire interval $\theta \in [0, \frac{1}{2}]$ (with $\theta = 0$ giving the limiting cases $\text{MSC}_{nu}, \text{KSC}_{nu}$ and $\theta = \frac{1}{2}$ yielding $\text{MSC}_{u}, \text{KSC}_{u}$), currently remains an open question.

This has an important consequence for experimental implementation of state transfer in open quantum systems: On a general scale in $n$-qubit systems, unitary control plus measurement-based closed-loop feedback from one resettable ancilla (as, e.g., in Ref. [4] following [3]) can be replaced by unitary control plus open-loop bang-bang switchable non-unital noise (equivalent to amplitude damping) on a single internal qubit. This is because both scenarios are sufficient to ensure Markovian and non-Markovian state controllability $\text{KSC}$. Example 5 in the main part illustrates this general simplifying feature.

Yet some questions with regard to the operator lift to map synthesis remain open: Assessing a demarcation between MMC and KMC (and their unital versus non-unital variants) seems to require different proof techniques than used here. In a follow-up study we will therefore further develop our lines of assessing the differential geometry of Lie semigroups in terms and their Lie wedges [16, 22] to this end, since judging upon Markovianity on the level of Kraus maps is known to be more intricate [39, 42]. More precisely, time-dependent Markovian channels come with a general form of a Lie wedge in contrast to time-independent Markovian channels, whose generators form the special structure of a Lie semialgebra (i.e. a Lie wedge closed under Baker-Campbell-Hausdorff multiplication). In [16], we have therefore drawn a detailed connection between these differential properties of Lie semigroups and the different notions of divisibility studied as a defining property of Markovianity in the seminal work [38].

Again, these distinctions will decide on simplest experimental implementations in the sense that measurement-based closed-loop feedback control may be required for non-Markovian maps in KMC, while open-loop noise-extended control may suffice for Markovian maps in MMC. More precisely, closed-loop feedback control was already shown to be sufficient for KMC in [3] (which was the aim that work set out for), yet it remains to be seen whether it is also necessary, and in particular, if it is necessary for the (supposedly) weaker notion MMC. If it turns out not to be necessary, then measurement-based closed-loop feedback control on a system extended by one resettable ancilla [3, 5] would be not be stronger than our open-loop scenario of full unitary control extended by (non-unital) noise modulation not only in the case of state transfer, but also in quantum-map synthesis with direct bearing on the simplification of quantum simulation experiments [5].