Stability of Traveling Oscillating Fronts in Complex Ginzburg Landau Equations

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Abstract. Traveling oscillating fronts (TOFs) are specific waves of the form $U_\star(x,t) = e^{-i\omega t}V_\star(x - ct)$ with a profile $V_\star$ which decays at $-\infty$ but approaches a nonzero limit at $+\infty$. TOFs usually appear in complex Ginzburg Landau equations of the type $U_t = \alpha U_{xx} + G(|U|^2)U$. In this paper we prove a theorem on the asymptotic stability of TOFs, where we allow the initial perturbation to be the sum of an exponentially localized part and a front-like part which approaches a small but nonzero limit at $+\infty$. The underlying assumptions guarantee that the operator, obtained from linearizing about the TOF in a co-moving and co-rotating frame, has essential spectrum touching the imaginary axis in a quadratic fashion and that further isolated eigenvalues are bounded away from the imaginary axis. The basic idea of the proof is to consider the problem in an extended phase space which couples the wave dynamics on the real line to the ODE dynamics at $+\infty$. Using slowly decaying exponential weights, the framework allows to derive appropriate resolvent estimates, semigroup techniques, and Gronwall estimates.

Key words. Traveling oscillating front, nonlinear stability, Ginzburg Landau equation, equivariance, essential spectrum.

AMS subject classification. 35B35, 35B40, 35C07, 35K58, 35Pxx, 35Q56

1. Introduction

In this paper we consider complex-valued semilinear parabolic equations of the form

\begin{equation}
U_t = \alpha U_{xx} + G(|U|^2)U, \quad x \in \mathbb{R}, \ t \geq 0
\end{equation}

with nonlinearity $G : \mathbb{R} \to \mathbb{C}$ and diffusion coefficient $\alpha \in \mathbb{C}$, $\text{Re}\ \alpha > 0$. If the nonlinearity $G$ is a linear resp. a quadratic polynomial over $\mathbb{C}$ then (1.1) leads to the cubic resp. the quintic complex Ginzburg Landau equation. Evolution equations of the form (1.1) admit the propagation of various types of waves which oscillate in time and which either have a front profile or which are periodic in space like wave trains, see [23], [25]. We are interested

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\textsuperscript{3}This work is an extended version of parts of the author’s PhD Thesis [8].
in the stability behavior of a special class of solutions which we call traveling oscillating fronts (TOFs). A TOF is a solution of (1.1) of the form

$$U_\star(x,t) = e^{-i\omega t}V_\star(x-ct)$$

with a profile $V_\star : \mathbb{R} \to \mathbb{C}$ satisfying the asymptotic property

$$\lim_{\xi \to -\infty} V_\star(\xi) = 0, \quad \lim_{\xi \to +\infty} V_\star(\xi) = V_\infty$$

for some $V_\infty \in \mathbb{C}$, $V_\infty \neq 0$. The parameters $\omega, c \in \mathbb{R}$ are called the frequency and the velocity of the TOF.

Figure 1.1 shows a typical TOF obtained by simulating the quintic complex Ginzburg-Landau equation

$$(QCGL) \quad U_t = \alpha U_{xx} + \beta_1 U + \beta_3 |U|^2 U + \beta_5 |U|^4 U, \quad x \in \mathbb{R}, \ t \geq 0$$

with an initial function $U(\cdot,0)$ of sigmoidal shape.

We aim at sufficient conditions under which a TOF is nonlinearly stable with asymptotic phase in suitable function spaces. As initial perturbations we allow functions which can be decomposed into an exponentially localized part and a front-like part which perturbs the limit at $+\infty$. There are two main difficulties to overcome: first, the operator obtained by linearizing about the TOF has essential spectrum touching the imaginary axis at zero in a quadratic way. Second, the perturbation at infinity prevents the use of standard Sobolev
spaces for the linearized operator. The first difficulty will be overcome by exponential weights which shift the essential spectrum to the left, while the second difficulty is handled by analyzing stability in an extended phase space which couples the dynamics on the real line to the dynamics at $+\infty$.

In the following we give a more technical outline of the setting and our basic assumptions, and we provide an overview of the following sections. Our results will be stated for the two-dimensional real-valued system equivalent to (1.1). Setting $U = u_1 + iu_2$, $u_j(x,t) \in \mathbb{R}$, $\alpha = \alpha_1 + i\alpha_2$, $\alpha_j \in \mathbb{R}$ and $G = g_1 + ig_2$ with $g_j : \mathbb{R} \to \mathbb{R}$ the equivalent real-valued parabolic system reads

$$u_t = Au_{xx} + f(u), \quad x \in \mathbb{R}, \ t \geq 0,$$

where

$$A = \begin{pmatrix} \alpha_1 & -\alpha_2 \\ \alpha_2 & \alpha_1 \end{pmatrix}, \quad f(u) = g(|u|^2)u, \quad g(\cdot) = \begin{pmatrix} g_1(\cdot) & -g_2(\cdot) \\ g_2(\cdot) & g_1(\cdot) \end{pmatrix}.$$}

A TOF $U_\ast = u_{\ast,1} + iu_{\ast,2}$ of (1.1) then corresponds to a solution $u_\ast = (u_{\ast,1}, u_{\ast,2})^\top$ of (1.2) of the form

$$u_\ast(x,t) = R_{-\omega t}v_\ast(x - ct), \quad R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},$$

where $R_\theta$ denotes rotation by the angle $\theta \in \mathbb{R}$. The profile $v_* : \mathbb{R} \to \mathbb{R}^2$ satisfies $V_\ast = v_{\ast,1} + iv_{\ast,2}$ and

$$\lim_{\xi \to -\infty} v_\ast(\xi) = 0, \quad \lim_{\xi \to +\infty} v_\ast(\xi) = v_{\infty},$$

where $V_\infty = v_{\infty,1} + iv_{\infty,2}$ and $v_{\infty} \in \mathbb{R}^2$, $v_{\infty} \neq 0$. The vector $v_{\infty}$ is called the asymptotic rest-state and the specific solution $R_{-\omega t}v_{\infty}$ of (1.2) is called a bound state. Figure 1.2 shows a typical profile of a TOF in $(x,u_1,u_2)$-space.

Figure 1.2. Traveling oscillating front.
For the stability analysis it is natural to transform (1.2) into a co-moving and co-rotating frame, i.e. we set \( u(x,t) = R_{-\omega t} v(\xi,t) \), \( \xi = x - ct \) and find that \( v \) solves the equation
\[
(1.5) \quad v_t = Av_{\xi\xi} + cv_{\xi} + S_\omega v + f(v), \quad \xi, t \geq 0, \quad S_\omega := \left( \begin{array}{cc} 0 & -\omega \\ \omega & 0 \end{array} \right).
\]

The time-independent profile \( v_* \) becomes a stationary solution of (1.5), i.e. it solves the ODE
\[
(1.6) \quad 0 = Av_{xx} + cv_x + S_\omega v + f(v), \quad x \in \mathbb{R}.
\]

From the asymptotic property (1.4) one concludes (see Lemma 2.1) that the rest-state satisfies
\[
\lim_{x \to \pm\infty} v_*'(x) = 0, \quad \lim_{x \to \pm\infty} v_*''(x) = 0.
\]

Both assumptions guarantee that we have a stable equilibrium at \( \xi = -\infty \) and a circle of stable equilibria at \( \xi = \infty \) when spatial derivatives are ignored in (1.5). Further conditions will be imposed on the spectrum of the linearized operator
\[
(1.10) \quad Lv = Av_{xx} + cv_x + S_\omega v + Df(v_*)v
\]
in suitable function spaces. In view of (1.7) we expect the linearization \( L \) from (1.10) to have a two-dimensional kernel. In addition, it turns out that the essential spectrum of \( L \), touches the imaginary axis at the origin when considered in the function space \( L^2(\mathbb{R}, \mathbb{R}^2) \). Thus there is no spectral gap between the zero eigenvalue and the remaining spectrum, so that standard approaches to conclude nonlinear stability do not apply; see [12], [14], [22]. We overcome this problem by two devices. First, we impose the following condition
**Assumption 1.3.** (Spectral Condition) The diffusion coefficient $\alpha = \alpha_1 + i\alpha_2$ satisfies

$$\alpha_2 g_2(|v_\infty|^2) + \alpha_1 g_1(|v_\infty|^2) < 0.$$ 

Note that Assumption 1.3 follows from Assumptions 1.1, 1.2 if $\alpha_2 = 0$. Moreover, we will show that Assumption 1.3 guarantees the essential spectrum to have negative quadratic contact with the imaginary axis. Second, we use Lebesgue and Sobolev spaces with exponential weight

$$\eta(x) = e^{\mu \sqrt{x^2 + 1}}, \quad \mu \geq 0.$$ (1.11)

Any sufficiently small $\mu > 0$ will be enough to shift the essential spectrum to the left and allow for a stability result. Weights of this or similar type frequently appear in stability analyses, see e.g. [26], [14, Ch.3.1.1], [11], [13]. We note that the stability statement w.r.t. the $L^\infty$-norm in the second part of [13, Theorem 7.2] comes closest to our results. There a perturbation argument for the case of a positive definite matrix $A$ in (1.5) is employed and the resulting Evans function [1] is analyzed. This leads to more restrictive conditions on the coefficients of the system and on the initial data.

We finish the introduction with a brief outline of the contents of the following sections. In section 2 we complete the basic assumptions 1.1, 1.2, 1.3 by eigenvalue conditions for the operator (1.10) and we state our main results in more technical terms. The approach of the profile towards its rest states is shown to be exponential, and stability with asymptotic phase is stated in weighted $H^1$-spaces. We also explain the main idea of the proof which incorporates the dynamics of (1.5) at $\xi = \infty$ into an extended evolutionary system, see (2.2), (2.3). In Section 3 we discuss in detail the Fredholm properties of the operator $L$ and its extended version in weighted spaces and we derive resolvent estimates. These form the basis for obtaining detailed estimates of the associated (extended) semigroup in Section 4. The subsequent section 5 is devoted to the decomposition of the dynamics into the motion within the underlying two-dimensional symmetry group and within a codimension-two function space. Section 6 then provides sharp estimates for the resulting remainder terms. Then a local existence theorem and a Gronwall estimate complete the proof in Section 7.

Let us finally mention that the techniques of this paper can be used to prove that the general method of freezing ([6], [21], [5]) works successfully for the parabolic equation (1.2) with two underlying symmetries, see [8].

## 2. Assumptions and main results

As a preparation for the subsequent analysis we specify the approach of a TOF towards its rest states.

**Lemma 2.1.** Let $v_\star \in C^2_b(\mathbb{R}, \mathbb{R}^2)$ be the profile of a traveling oscillating front of (1.2) with speed $c > 0$, frequency $\omega \in \mathbb{R}$ and asymptotic rest-state $v_\infty \in \mathbb{R}^2 \setminus \{0\}$. Moreover, suppose $\text{Re} \alpha > 0$ and $g \in C(\mathbb{R}, \mathbb{R}^{2,2})$. Then the following holds:

$$g(|v_\infty|^2) = -S_\omega, \quad \lim_{x \to \pm \infty} v_\star'(x) = 0, \quad \lim_{x \to \pm \infty} v_\star''(x) = 0.$$ 

Using Assumption 1.1 and 1.2 one can conclude that the convergence in Lemma 2.1 of the profile $v_\star$ and its derivatives is exponentially fast.
\textbf{Theorem 2.2.} Let Assumption 1.1 and 1.2 be satisfied and let \( v_* \) be given as in Lemma 2.1. Then \( v_* \in C_0^\infty(\mathbb{R}, \mathbb{R}^2) \) holds and there are constants \( K, \mu_* > 0 \) such that
\[
|v_*(x) - v_\infty| + |v_*'(x)| + |v_*''(x)| + |v_*'''(x)| \leq Ke^{-\mu_* x} \quad \forall x \geq 0,
\]
\[
|v_*(x)| + |v_*'(x)| + |v_*''(x)| + |v_*'''(x)| \leq Ke^{\mu_* x} \quad \forall x \leq 0.
\]

In the Appendix we give the proof of Lemma 2.1 and the main steps of the proof of Theorem 2.2.

With the weight \( \eta \) given by (1.11), let us introduce the weighted \( L^2 \) space
\[
L_\eta^2(\mathbb{R}, \mathbb{R}^n) := \{ v \in L^2(\mathbb{R}, \mathbb{R}^n) : \eta v \in L^2(\mathbb{R}, \mathbb{R}^n) \}, \quad (u, v)_{L_\eta^2} := (\eta u, \eta v)_{L^2}
\]
and the associated weighted Sobolev spaces defined for \( \ell \in \mathbb{N} \) by
\[
H_\eta^k(\mathbb{R}, \mathbb{R}^n) := \{ v \in L_\eta^2(\mathbb{R}, \mathbb{R}^n) \cap H_\eta^2(\mathbb{R}, \mathbb{R}^n) : \partial^k v \in L_\eta^2(\mathbb{R}, \mathbb{R}^n), 1 \leq k \leq \ell \},
\]
\[
\|v\|_{H_\eta^k}^2 := \sum_{k=0}^\ell \|\partial^k v\|_{L_\eta^2}^2.
\]

Let us note that Theorem 2.2 ensures \( v_*^{(j)} \in H_\eta^{3-j}(\mathbb{R}, \mathbb{R}^2) \) for \( 0 \leq \mu < \mu_* \) and \( j = 1, 2, 3 \).

However, the profile \( v_* \) of a TOF does not decay to zero as \( x \to \infty \), and, moreover, we expect the limit \( \rho(t) = \lim_{x \to \infty} v(x,t) \) of a solution of (1.5) to still move with time. Therefore, the idea is to include an ODE for the dynamics of \( \rho(t) \) into the overall system. Formally taking the limit \( x \to \infty \) in (1.5) and assuming \( v_x(x,t), v_{xx}(x,t) \to 0 \) as \( x \to \infty \) we obtain for \( \rho \) the ODE
\[
(2.1) \quad \rho'(t) = S_\omega \rho(t) + f(\rho(t)).
\]

Note that \( v_\infty \) is a stationary solution of (2.1) due to Lemma 2.1, and, by equivariance, there is a whole circle of equilibria \( \{ R_\theta v_\infty : \theta \in S^1 \} \). Next we choose a template function
\[
\hat{\nu}(x) := \frac{1}{2} \tanh(\hat{\mu} x) + \frac{1}{2}, \quad 0 < 2\hat{\mu} \leq \mu_*.
\]

The rate \( \hat{\mu} \) has been chosen such that the approach toward the limits as \( x \to \pm \infty \) is weaker than for the derivatives of the solution in Theorem 2.2. Such a choice is not strictly necessary but will avoid some technicalities in the following. If \( 0 < \mu < 2\hat{\mu} \) we conclude \( v_* - \hat{\nu} v_\infty \in H_\eta^2(\mathbb{R}, \mathbb{R}^2) \) and we also expect the solution \( v \) of (1.5) to satisfy \( v(\cdot, t) - \hat{\nu} \rho(t) \in H_\eta^2(\mathbb{R}, \mathbb{R}^2) \), i.e. to lie in an affine linear space with a time dependent offset given by \( \rho \). Therefore, we introduce the Hilbert space
\[
X_\eta := \{(v, \rho)^\top : v : \mathbb{R} \to \mathbb{R}^2, \rho \in \mathbb{R}^2, v - \hat{\nu} \rho \in L_\eta^2(\mathbb{R}, \mathbb{R}^2) \}
\]
with inner product \( ((u, \rho)^\top, (v, \zeta)^\top)_{X_\eta} = (\rho, \zeta) + (u - \rho \hat{\nu} v - \zeta \hat{\nu} \rho)_{L_\eta^2} \). Similarly, we define the smooth analog
\[
X_\eta^k := \{(v, \rho)^\top \in X_\eta : v \in H_\eta^k, \partial^k v \in L_\eta^2, 1 \leq k \leq \ell \}, \quad \ell \in \mathbb{N}_0
\]
with the norm given by \( \| (v, \rho)^\top \|^2_{X_\eta^k} = \|\rho\|^2 + \|v - \rho \hat{\nu}\|^2_{L_\eta^2} + \sum_{k=1}^\ell \|\partial^k v\|^2_{L_\eta^2} \). We further set
\[
Y_\eta := X_\eta^2 \quad \text{and denote the elements of } X_\eta^2 \text{ by bold letters, for example,}
\]
\[
\mathbf{v} = (v, \rho)^\top, \quad \mathbf{v}_* = (v_*, v_\infty)^\top, \quad \mathbf{v}_0 = (u_0, \rho_0)^\top.
\]
As noted above, Theorem 2.2 implies \( v_* \in v_\infty \hat{v} + H^2_\eta \) and thus \( v_* \in Y_\eta \). Instead of (1.8), we consider the extended Cauchy problem on \( X_\eta \)

(2.2) \( \mathbf{v}_t = \mathcal{F}(\mathbf{v}), \quad \mathbf{v}(0) = v_* + \mathbf{v}_0, \)

where \( \mathcal{F} \) is a semilinear operator given by

(2.3) \( \mathcal{F} : Y_\eta \to X_\eta, \quad \left( \frac{v}{\rho} \right) = \mathbf{v} \mapsto \mathcal{F}(\mathbf{v}) = \begin{pmatrix} A_{xx}v + cv_x + S_\omega v + f(v) \\ S_\omega \rho + f(\rho) \end{pmatrix}. \)

With these settings, \( v_* \) becomes a stationary solution of (2.2), and our task is to prove its nonlinear stability with asymptotic phase. For this purpose, let us extend the group action induced by rotation and translation of elements from \( L^2_\eta \) to \( X_\eta \) as follows:

(2.4) \( a(\gamma) : X_\eta \to X_\eta, \quad \left( \frac{v}{\rho} \right) \mapsto a(\gamma) \left( \frac{v}{\rho} \right) = \begin{pmatrix} R_{-\theta}v(\cdot - \tau) \\ R_{-\theta}\rho \end{pmatrix}, \quad \gamma = (\theta, \tau) \in \mathcal{G} := S^1 \times \mathbb{R}. \)

The operator \( \mathcal{F} \) from (2.3) is then equivariant w.r.t. the group action, i.e. \( \mathcal{F}(a(\gamma)v) = a(\gamma)\mathcal{F}(v) \) for all \( \gamma \in \mathcal{G} \) and \( u \in Y_\eta \). Further a metric on \( \mathcal{G} \) is given by

\[ d_{\mathcal{G}}(\gamma_1, \gamma_2) = |\gamma_1 - \gamma_2|_G, \quad |\gamma|_G := \min_{k \in \mathbb{Z}} |\theta - 2\pi k| + |\tau|, \quad \gamma = (\theta, \tau) \in \mathcal{G}. \]

Finally, we collect the assumptions on the linearized operator \( L : \mathcal{D}(L) = H^2 \subset L^2 \to L^2 \)

from (1.10). The operator \( L \) will turn out to be closed and densely defined. We denote its resolvent set by

\[ \text{res}(L) := \{ s \in \mathbb{C} : sI - L : \mathcal{D}(L) \to L^2 \text{ is bijective} \} \]

and its spectrum by \( \sigma(L) = \mathbb{C} \setminus \text{res}(L) \). The further subdivision of the spectrum into the essential spectrum and the point spectrum varies in the literature (see the five different notions in [9]). We use the following definition (see \( \sigma_{e,4}(L) \) in [9, Ch.I.4,IX.1] or [14, Ch.3] and note the slight deviation from [12],[15]):

(2.5) \( \sigma_{pt}(L) := \{ s \in \sigma(L) : sI - L \text{ is Fredholm of index } 0 \}, \quad \sigma_{ess}(L) := \sigma(L) \setminus \sigma_{pt}(L). \)

When we insert the translates \( v_* (\cdot - \tau) \) from (1.7) into the stationary equation (1.5) and differentiate with respect to \( \tau \) we obtain that the nullspace \( \mathcal{N}(L) \) of \( L \) contains at least \( v_*' \in H^2 \). The following condition requires that there are no (generalized) eigenfunctions and that eigenvalues from the point spectrum lie strictly to the left of the imaginary axis.

**Assumption 2.3 (Eigenvalue Condition).**

There exists \( \beta_E > 0 \) such that \( \text{Re} \ s < -\beta_E \) holds for all \( s \in \sigma_{pt}(L) \). Moreover,

(2.6) \( \dim \mathcal{N}(L) = \dim \mathcal{N}(L^2) = 1. \)

Recall \( v_*' \in \mathcal{N}(L) \) so that (2.6) implies \( \mathcal{N}(L) = \text{span}\{ v_*' \} \). In Theorem 3.5 below we will see that \( L : H^2 \to L^2 \) is not Fredholm, hence 0 belongs to \( \sigma_{ess}(L) \) and not to \( \sigma_{pt}(L) \). For this reason we wrote condition (2.6) explicitly in terms of nullspaces, and \( \text{Re} \ s < -\beta_E \) for \( s \in \sigma_{pt}(L) \) is no contradiction for \( s = 0 \).

Differentiating the equation (1.5) for the stationary continuum (1.7) with respect to the first group variable \( \theta \in S^1 \) produces a second ‘eigenfunction’ \( S_1 v_* \) which, however, does
not belong to $D(L) = H^2$. But this eigenfunction will appear for the extended operator obtained by linearizing $F$ from (2.3) at $v_*$:

\[(2.7) \quad \mathcal{L}_\eta : Y_\eta \to X_\eta, \quad \left(\begin{array}{c} v \\ \rho \end{array}\right) \mapsto \mathcal{L}_\eta \left(\begin{array}{c} v \\ \rho \end{array}\right) = \left(\begin{array}{c} Av_{xx} + cv_x + S_\omega v + Df(v_*)v \\ S_\omega \rho + Df(v_*)\rho \end{array}\right).\]

The subindex $\eta$ indicates that the operator $\mathcal{L}_\eta$ depends on the weight through its domain and range. We further write $\mathcal{L} = \mathcal{L}_1$ in case $\mu = 0, \eta \equiv 1$ and introduce

\[E_\omega := S_\omega + Df(v_\infty)\]

for the second component of the operator. In Section 3 we prove the following result for the point spectrum of the operator $\mathcal{L}_\eta$ defined in (2.7).

**Lemma 2.4.** Let Assumption 1.1, 1.2, 1.3 and 2.3 be satisfied. Then there exists a constant $\mu_1 \in (0, 2\mu)$ such that the following holds for all weight functions (1.11) with $0 < \mu \leq \mu_1$:

i) The eigenvalue $0$ belongs to $\sigma_{pl}(\mathcal{L}_\eta)$ and has geometric and algebraic multiplicity 2, more precisely,

\[(2.8) \quad \mathcal{N}(\mathcal{L}_\eta^2) = \mathcal{N}(\mathcal{L}_\eta) = \text{span}\{\varphi_1, \varphi_2\}, \quad \varphi_1 = (v_*, 0)^\top, \quad \varphi_2 = (S_1 v_*, S_1 v_\infty)^\top.\]

ii) There exists some $\beta_1 = \beta_1(\mu) > 0$ such that all eigenvalues $s \in \sigma_{pl}(\mathcal{L}_\eta) \setminus \{0\}$ satisfy $\Re s < -\beta_1 < 0$.

Now we are in a position to formulate the main result.

**Theorem 2.5.** Let Assumption 1.1, 1.2, 1.3 and 2.3 be satisfied and let $\eta$ be given by (1.11). Then there exists $\mu_0 > 0$ such that for every $\mu \in (0, \mu_0)$ there are constants $\varepsilon_0(\mu), \beta(\mu), K(\mu), C_{\infty}(\mu) > 0$ so that the following statements hold. For all initial perturbations $v_0 \in Y_\eta$ with $\|v_0\|_{X_\eta^1} < \varepsilon_0$ the equation (2.2) has a unique global solution $v \in C((0, \infty), Y_\eta) \cap C^1([0, \infty), X_\eta)$ which can be represented as

\[v(t) = a(\gamma(t))v_* + w(t), \quad t \in [0, \infty)\]

for suitable functions $\gamma \in C^1([0, \infty), G)$ and $w \in C((0, \infty), Y_\eta) \cap C^1([0, \infty), X_\eta)$. Further, there exists an asymptotic phase $\gamma_\infty = \gamma_\infty(v_0) \in G$ such that

\[\|w(t)\|_{X_\eta^1} + |\gamma(t) - \gamma_\infty|_G \leq Ke^{-\beta t} \|v_0\|_{X_\eta^1}, \quad |\gamma_\infty|_G \leq C_\infty \|v_0\|_{X_\eta^1}.\]

This leads to corresponding stability statements for a TOF of the equations (1.8) and (1.2), respectively. For simplicity, we state the result in an informal way under the assumptions of Theorem 2.5 for the extended version of (1.2), i.e.

\[u_t = Au_{xx} + f(u), \quad u(\cdot, 0) = v_* + u_0,\]

\[r_t = f(r), \quad r(0) = v_\infty + \rho_0.\]

Initial perturbations $v_0 = (u_0, \rho_0)$ are assumed to be small in the sense that

\[\|v_0\|_{X_\eta^1} = \|u_0 - \rho_0 \tilde{v}\|_{L^2_\eta}^2 + \|\partial_x u_0\|_{L^2_\eta}^2 + |\rho_0|^2 \leq \varepsilon_0^2.\]

Then the system (2.9) has a unique solution $u = (u, r) \in C((0, \infty), Y_\eta) \cap C^1([0, \infty), X_\eta)$ and there exist functions $(\theta, \tau) \in C^1([0, \infty), S^1 \times \mathbb{R})$ and a value $(\theta_\infty, \tau_\infty) \in S^1 \times \mathbb{R}$ such
that for all $t \geq 0$
$$
\|u(\cdot, t) - R_{-\omega t - \theta(t)} v_\ast (\cdot - ct - \tau(t)) - \dot{\nabla} (r(t) - R_{-\omega t - \theta(t)} v_\ast)\|_{L^2_\eta} + \|\partial_x u(\cdot, t)\|_{L^2_\eta} \\
+ |r(t) - R_{-\omega t - \theta(t)} v_\ast| + |\tau(t) - \tau_\ast| + |\theta(t) - \theta_\ast|
\leq K e^{-\beta t}\|v_0\|_{X^\ast_1}.
$$

Note the detailed expression for the asymptotic behavior of $\lim_{x \to \infty} u(x, t)$ as $t \to \infty$.

3. Spectral analysis of the linearized operator

In this section we study the spectrum of the linear operator $\mathcal{L}_\eta$ from (2.7) and estimate solutions of the resolvent equation

$$
(sI - \mathcal{L}_\eta) v = r, \quad s \in \mathbb{C}, \quad r = (r, \zeta)^\top \in X_\eta.
$$

In the first step we derive resolvent estimates for solutions $v \in Y_\eta$ of (3.1) when $|s|$ is large and $s$ lies in the exterior of some sector opening to the left. The approach is based on energy estimates from [16], [17].

**Lemma 3.1.** Let Assumption 1.1 and 1.2 be satisfied and let $\mu_2 \in (0, 2\tilde{\mu})$. Then there exist constants $\varepsilon_0, R_0, C > 0$ such that the following properties hold for all $0 \leq \mu \leq \mu_2$.

The operator $\mathcal{L}_\eta : Y_\eta \subset X_\eta \to X_\eta$ is closed and densely defined in $X_\eta$. For all

$$
\Omega_\eta := \left\{ s \in \mathbb{C} : |s| \geq R_0, |\arg(s)| \leq \frac{\pi}{2} + \varepsilon_0 \right\}
$$

the equation (3.1) with $v \in Y_\eta$ and $r \in X_\eta$ implies

$$
|s|^2 \|v\|^2_{X_\eta} + \|v_x\|^2_{L^2_\eta} \leq \frac{C}{|s|^2} \|r\|^2_{X_\eta},
$$

$$
|s|^2 \|v\|^2_{X_\eta} + |s| \|v_x\|^2_{L^2_\eta} + \|v_{xx}\|^2_{L^2_\eta} \leq C \|r\|^2_{X_\eta}.
$$

In addition, if $r \in X^1_\eta$ and $v \in X^3_\eta$ then

$$
|s|^2 \|v\|^2_{X^1_\eta} + |s| \|v_x\|^2_{L^2_\eta} + \|v_{xx}\|^2_{L^2_\eta} \leq C \|r\|^2_{X^3_\eta}.
$$

**Proof.** For the proof let us abbreviate $C_\ast := D(f_\ast), C_\infty := D(f_\infty)$ and $(\cdot, \cdot) = (\cdot, \cdot)_{L^2_\eta(\mathbb{R}, \mathbb{R}^2)}$.

From Theorem 2.2 and $\mu \leq \mu_2 < \mu_\ast$ we find for $(v, \rho)^\top \in X_\eta$,

$$
\|C_\ast v - C_\infty \rho \dot{\nabla}\|_{L^2_\eta} \leq \|C_\ast\|_{L^\infty} \|v - \rho \dot{\nabla}\|_{L^2_\eta} + \|(C_\ast - C_\infty) \rho \dot{\nabla}\|_{L^2_\eta} \\
\|(C_\ast - C_\infty) \rho \dot{\nabla}\|_{L^2_\eta} = \|(C_\ast - C_\infty) \rho \dot{\nabla}\|_{L^2_\eta(\mathbb{R}, \mathbb{R})} + \|(C_\ast - C_\infty) \rho \dot{\nabla}\|_{L^2_\eta(\mathbb{R}, \mathbb{R}_+)}
\leq \frac{\varepsilon^2 \mu \|C_\ast - C_\infty\|_{L^\infty}}{2\mu - \mu_\ast} |\rho|^2 + \int_0^\infty \eta^2(x) |Df(v_\ast(x)) - Df(v_\infty)|^2 dx |\rho|^2
\leq K_C |\rho|^2.
$$

From this one infers that the operator $\mathcal{L}_\eta : Y_\eta \to X_\eta$ is bounded. Next, we note that (3.4) implies the closedness of $\mathcal{L}_\eta$. For this purpose, let $\{v_n\}_{n \in \mathbb{N}} \subset Y_\eta$ be given with $v_n \to v$ in $X_\eta$ and $\mathcal{L}_\eta v_n \to w$ in $X_\eta$. Pick $s_0 \in \Omega_\eta$ with $|s_0| \geq 1$. Then (3.4) yields

$$
\|v_n - v_m\|^2_{Y_\eta} \leq |s_0|^2 \|v_n - v_m\|^2_{X_\eta} + |s_0|^2 \|v_{n,x} - v_{m,x}\|^2_{L^2_\eta} + \|v_{n,xx} - v_{m,xx}\|^2_{L^2_\eta}
\leq C_1 |s_0| (v_n - v_m) - \mathcal{L}_\eta v_n - \mathcal{L}_\eta v_m \to 0, \quad n, m \to \infty.
$$
Thus, \( \{v_n\} \in \mathbb{N} \) is a Cauchy sequence in \( Y \), and there is \( \tilde{v} \in Y \) with \( v_n \to \tilde{v} \) in \( Y \). We conclude \( v = \tilde{v} \in Y \) and \( v_n \to v \) in \( Y \). Finally, \( \mathcal{L}_n v = w \) follows from the boundedness of \( \mathcal{L}_n : Y \to X \) and the estimate

\[
\|\mathcal{L}_n v - w\|_{X_n} \leq \|\mathcal{L}_n(v - v_n)\|_{X_n} + \|\mathcal{L}_n v_n - w\|_{X_n} \to 0, \ n \to \infty.
\]

The estimate (3.5) follows by differentiating (3.1) w.r.t. \( x \) and using (3.4). Therefore, it is left to show (3.3) and (3.4). We begin with (3.3). For this purpose, let \( s \in \Omega_0 \) with \( R_0 \) and \( \varepsilon_0 \) still be determined. Take the inner product of (3.1) with \( v \) in \( X_n \) to obtain

\[
(v, r)_{X_n} = (v, (sI - \mathcal{L}_n) v)_{X_n} = \left( \begin{pmatrix} v \\ \rho \end{pmatrix}, \begin{pmatrix} sv - Av_{xx} - cv_x - S_\omega v - C_\nu v \\ s\rho - S_\omega \rho - C_\infty \rho \end{pmatrix} \right)_{X_n}
\]

\[
= \rho^T(sI - \mathcal{L}_n)\rho + (v - \rho \hat{v}, sv - Av_{xx} - cv_x - S_\omega v - C_\nu v - (s\rho - S_\omega \rho - C_\infty \rho)\hat{v})
\]

\[
= s\|v\|^2_{X_n} - \rho^T S_\omega \rho - \rho^T C_\infty \rho
\]

\[
- (v - \rho \hat{v}, Av_{xx})_{L^2} - c(v - \rho \hat{v}, v_x) - (v - \rho \hat{v}, S_\omega (v - \rho \hat{v})) - (v - \rho \hat{v}, C_\nu v - C_\infty \rho \hat{v}).
\]

Integration by parts yields

\[
s\|v\|^2_{X_n} + (v - \rho \hat{v}, Av_{xx})_{L^2}
\]

\[
(3.7)
\]

\[
= \rho^T(S_\omega + C_\infty)\rho - 2(\eta'\eta^{-1}(v - \rho \hat{v}), Av_{xx}) + c(v - \rho \hat{v}, v_x)_{L^2}
\]

\[
+ (v - \rho \hat{v}, S_\omega (v - \rho \hat{v}))_{L^2} + (v - \rho \hat{v}, C_\nu v - C_\infty \rho \hat{v})_{L^2} + (v, r)_{X_n}.
\]

Further, we use Cauchy-Schwarz and Young’s inequality with arbitrary \( \varepsilon_i > 0 \) and (3.6) to obtain the estimates

\[
|c(v - \rho \hat{v}, v_x)| \leq |c| \|v - \rho \hat{v}\|^2_{L^2} + |c| \varepsilon_3 \|v_x\|^2_{L^2},
\]

\[
|((v - \rho \hat{v}, S_\omega (v - \rho \hat{v})))| \leq |\omega| \|v - \rho \hat{v}\|^2_{L^2},
\]

\[
|(v - \rho \hat{v}, C_\nu v - C_\infty \rho \hat{v})| \leq \left\| C_\nu \right\|_{L^\infty} + \frac{1}{4\varepsilon_4} \|v - \rho \hat{v}\|^2_{L^2} + K \varepsilon_4 |\rho|^2.
\]

Take the absolute value in (3.7) and use (3.8)-(3.12) with \( \varepsilon_i = 1 \) to obtain for some \( K_0, K_1 > 0 \)

\[
(3.13)
\]

\[
|s\|v\|^2_{X_n} \leq K_0 \|v\|^2_{L^2} + K_1 \|v\|^2_{X_n} + \|v\|_{X_n} \|r\|_{X_n}.
\]

Next we note that \( (v - \rho \hat{v}, Av_{xx}) = a_1 \|v\|^2_{L^2} - (\rho \hat{v}, Av_{xx}) \) and

\[
|(\rho \hat{v}, Av_{xx})| \leq |A|\|\hat{v}\|_{L^2} \|\hat{v}\|_{L^2} \leq \frac{e^2|A|}{(2\hat{\mu} - \mu)\varepsilon_5} |\rho|^2 + \varepsilon_5 |A| \|v\|^2_{L^2}.
\]

\[
(3.14)
\]
Taking the real part in (3.7) we obtain by using Cauchy-Schwarz, Young’s inequality and (3.9)-(3.12) as well as (3.14) with \( \varepsilon_2 = \varepsilon_5 = \frac{\alpha_1}{\alpha_3 + \varepsilon_4 A}, \varepsilon_3 = \frac{\alpha_1}{4c}, \varepsilon_4 = 1 \) the estimate
\[
\text{Re } s \|v\|_{X_n}^2 + \alpha_1 \|v_x\|_{L^2_\eta}^2 \leq (\varepsilon_5 |A| + \varepsilon_2 |A| + \varepsilon_3 |c|) \|v_x\|_{L^2_\eta}^2 + K_2 \|v\|_{X_n}^2 + \|v\|_{X_n} \|r\|_{X_n} \leq \frac{\alpha_1}{2} \|v_x\|_{L^2_\eta}^2 + K_2 \|v\|_{X_n}^2 + \|v\|_{X_n} \|r\|_{X_n}.
\]
This yields
\[
(3.15) \quad \text{Re } s \|v\|_{X_n}^2 + \frac{\alpha_1}{2} \|v_x\|_{L^2_\eta}^2 \leq K_2 \|v\|_{X_n}^2 + \|v\|_{X_n} \|r\|_{X_n}.
\]
The remaining proof falls naturally into three cases depending on the value of \( s \in \Omega_0 \).

**Case 1:** \( \text{Re } s \geq |\text{Im } s|, \text{Re } s > 0, |s| \geq 2\sqrt{2}K_2 \).
We have \( 0 < \text{Re } s \leq |s| \leq \sqrt{2} \text{Re } s \). Therefore, using (3.15) and Young’s inequality with \( \varepsilon = \frac{\sqrt{2}}{|s|} \), we obtain
\[
\begin{align*}
\frac{|s|}{\sqrt{2}} \|v\|_{X_n}^2 + \frac{\alpha_1}{2} \|v_x\|_{L^2_\eta}^2 &\leq \frac{|s|}{2\sqrt{2}} \|v\|_{X_n}^2 + \|v\|_{X_n} \|r\|_{X_n} \\
&\leq \frac{|s|}{2\sqrt{2}} \|v\|_{X_n}^2 + \frac{|s|}{4\sqrt{2}} \|v\|_{X_n}^2 + \frac{\sqrt{2}}{|s|} \|r\|_{X_n}^2.
\end{align*}
\]
Thus, for a suitable constant \( C \)
\[
|s| \|v\|_{X_n}^2 + \|v_x\|_{L^2_\eta}^2 \leq \frac{C}{|s|} \|r\|_{X_n}^2.
\]

**Case 2:** \( |\text{Im } s| \geq \text{Re } s \geq 0 \).
From (3.15) we have
\[
\|v_x\|_{L^2_\eta} \leq \frac{2}{\alpha_1} \left( K_2 \|v\|_{X_n}^2 + \|v\|_{X_n} \|r\|_{X_n} \right).
\]
Use this in (3.13) and find a constant \( K_3 \) such that
\[
|s| \|v\|_{X_n}^2 \leq K_3 \left( \|v\|_{X_n}^2 + \|v\|_{X_n} \|r\|_{X_n} \right).
\]
Take \( |s| > 2K_3 \) and use Young’s inequality with \( \varepsilon = |s|^{-1} \)
\[
|s| \|v\|_{X_n}^2 \leq \frac{|s|}{2} \|v\|_{X_n}^2 + K_3 \|v\|_{X_n} \|r\|_{X_n} \leq \frac{|s|}{2} \|v\|_{X_n}^2 + \frac{|s|}{4} \|v\|_{X_n}^2 + \frac{K^2}{\varepsilon^2} \|r\|_{X_n}^2,
\]
hence
\[
(3.16) \quad |s| \|v\|_{X_n}^2 \leq \frac{4K^2}{|s|} \|r\|_{X_n}^2.
\]
Using (3.15), (3.16) and taking \( |s| \geq 4K_2 \) yields by Young’s inequality with \( \varepsilon = |s|^{-1} \)
\[
(3.17) \quad \frac{\alpha_1}{2} \|v_x\|_{L^2_\eta}^2 \leq \frac{|s|}{4} \|v\|_{X_n}^2 + \|v\|_{X_n} \|r\|_{X_n} \leq \frac{|s|}{4} \|v\|_{X_n}^2 + \frac{|s|}{4} \|v\|_{X_n}^2 + \frac{1}{|s|} \|r\|_{X_n}^2 + \frac{K_4}{|s|} \|r\|_{X_n}^2.
\]
Combining (3.16) and (3.17) we arrive at the estimate (3.3).

**Case 3:** \( \text{Re } s \leq 0, |\text{Re } s| \leq \varepsilon_0 |\text{Im } s| \). Using (3.13) and (3.15) yields
\[
|\text{Im } s| \|v\|_{X_n}^2 \leq |s| \|v\|_{X_n}^2 \leq K_0 \|v_x\|_{L^2_\eta}^2 + K_1 \|v\|_{X_n}^2 + \|v\|_{X_n} \|r\|_{X_n}.
\]
\[
\leq \frac{2K_0}{\alpha_1} \left( |\Re s| \|v\|_{X_n}^2 + K_2\|v\|_{X_n}^2 + \|v\|_{X_n}\|r\|_{X_n} \right) + K_1\|v\|_{X_n}^2 + \|v\|_{X_n}\|r\|_{X_n}.
\]

Choose \(0 < \varepsilon_0 < \frac{\alpha_1}{4K_0}\), so that \(\frac{2K_0}{\alpha_1}|\Re s| \leq \frac{|\Im s|}{2}\) holds. Then we conclude
\[
|\Im s|\|v\|_{X_n}^2 \leq K_5\left(\|v\|_{X_n}^2 + \|v\|_{X_n}\|r\|_{X_n} \right).
\]

Since \(|s| \leq \sqrt{1 + \varepsilon_0^2}|\Im s|\) we also have
\[
|s|\|v\|_{X_n}^2 \leq K_6\left(\|v\|_{X_n}^2 + \|v\|_{X_n}\|r\|_{X_n} \right).
\]

Now take \(|s| > 2K_6\) and use Young’s inequality with \(\varepsilon = |s|^{-1}\) to find
\[
|s|\|v\|_{X_n}^2 \leq \frac{|s|}{2}\|v\|_{X_n}^2 + K_6\|v\|_{X_n}\|r\|_{X_n} \leq \frac{|s|}{2}\|v\|_{X_n}^2 + \frac{|s|}{4}\|v\|_{X_n}^2 + \frac{K^2_6}{|s|}\|r\|_{X_n}^2,
\]

which yields
\[
(3.18) \quad |s|\|v\|_{X_n}^2 \leq \frac{K_7}{|s|}\|r\|_{X_n}^2.
\]

To complete (3.3), take \(|s| \geq 2K_2\) in (3.15) and use (3.18),
\[
\frac{\alpha_1}{2}\|v_x\|_{L^2_n}^2 \leq |\Re s|\|v\|_{X_n}^2 + \frac{|s|}{2}\|v\|_{X_n}^2 + \|v\|_{X_n}\|r\|_{X_n} \leq |s|\|v\|_{X_n}^2 + \frac{|s|}{2}\|v\|_{X_n}^2 + \frac{1}{2|s|}\|r\|_{X_n}^2
\]
\[
= 2\|v\|_{X_n}^2 + \frac{1}{2|s|}\|r\|_{X_n}^2 \leq \frac{K_8}{|s|}\|r\|_{X_n}^2.
\]

It remains to prove (3.4). The resolvent equation (3.1) implies the following equation in \(X_n\),
\[
\begin{pmatrix}
-v_{xx} \\
0
\end{pmatrix} = \begin{pmatrix}
A^{-1}(-sv + cv + S_\omega v + C\varepsilon + r) \\
A^{-1}(-sp + S_\omega p + C\omega p + \zeta)
\end{pmatrix}.
\]

Thus, with the help of (3.6) we obtain for \(|s| \geq 1\) the estimate
\[
\|v_{xx}\|_{L^2_n}^2 \leq K_9\left(|s|^2\|v\|_{X_n}^2 + \|v_x\|_{L^2_n}^2 + \|v\|_{X_n}\|r\|_{X_n}^2 \right)
\]
\[
\leq 2K_9\left(|s|^2\|v\|_{X_n}^2 + |s|\|v_x\|_{L^2_n}^2 + \|r\|_{X_n}^2 \right).
\]

When combined with (3.3) this proves our assertion. \(\square\)

In the next step we study the Fredholm property of the operator \(L_\eta\) in (2.7). First we consider the operator \(L\) from (1.10) on \(L^2_\eta\) and therefore, as in (2.7), indicate the dependence on the weight \(\eta\) by a subindex. So we introduce
\[
L_\eta : H^2_\eta \to L^2_\eta, \quad v \mapsto Av_{xx} + cv_x + S_\omega v + Df(v)\eta v.
\]

Further, let us transform \(L_\eta\) into unweighted spaces
\[
L_{[\eta]} : H^2 \to L^2, \quad v \mapsto \eta L_\eta(\eta^{-1} v) = Av_{xx} + B(\mu)v_x + C(\mu)v,
\]

(3.19)
where $q(x) = \sqrt{x^2 + 1}$ and

$$B(\mu, x) = cI + \frac{2\mu x}{q(x)} A, \quad C(\mu, x) = S_\omega + Df(v_\ast(x)) - \frac{c\mu x}{q(x)} I + \left(\frac{\mu^2 x^2}{q(x)^2} - \frac{\mu}{q(x)} + \frac{\mu x^2}{q(x)^3}\right) A.$$ 

The limits as $x \to \pm\infty$ of these matrices are given by

$$B_{\pm, \mu} = cI \pm 2\mu A, \quad C_{\pm, \mu} = S_\omega + \mu^2 A + \begin{cases} -c\mu I + Df(v_\ast), & \text{in case } +, \\ c\mu I + Df(0), & \text{in case } - . \end{cases}$$

With these limit matrices we define the piecewise constant operator

$$L_{[\eta], \infty} : H^2 \to L^2, \quad v \mapsto Av_{xx} + (B_{-\mu} \mathbb{I}_{\mathbb{R}_-} + B_{+\mu} \mathbb{I}_{\mathbb{R}_+})v_x + (C_{-\mu} \mathbb{I}_{\mathbb{R}_-} + C_{+\mu} \mathbb{I}_{\mathbb{R}_+})v.$$ 

The following lemma shows that it is sufficient to analyze the Fredholm properties of $L_{[\eta], \infty}$.

**Lemma 3.2.** Let Assumption 1.1 and 1.2 be satisfied, and assume $0 \leq \mu \leq \mu_2$ with $\mu_2$ from Lemma 3.1. Then for each $s \in \mathbb{C}$ the following are equivalent:

i) The operator $sI - L_{[\eta], \infty} : H^2 \to L^2$ is Fredholm of index $k$.

ii) The operator $sI - L_\eta : H^2_\eta \to L^2_\eta$ is Fredholm of index $k$.

iii) The operator $sI - L_\eta : Y_\eta \to X_\eta$ is Fredholm of index $k$.

**Proof.** Both equivalences (i) $\iff$ (ii) and (ii) $\iff$ (iii) follow from the invariance of the Fredholm index under compact perturbations [9, Ch.IX]. For the first equivalence note that the multiplication operator $v \mapsto m(\cdot)v$ is compact from $H^1$ to $L^2$ if $m \in L^\infty$ and $\lim_{x \to \pm\infty} m(x) = 0$ ([4, Lemma 4.1]). This shows that the Fredholm property transfers from $L_{[\eta], \infty}$ to $L_\eta$ (see (3.19)), and thus also to $L_\eta : H^2_\eta \to L^2_\eta$. For the second equivalence use the homeomorphism $T : X_\eta \to L^2_\eta \times \mathbb{R}^2, (v, \rho)^\top \mapsto (v - \rho \hat{v}, \rho)^\top$ and transform $L_\eta$ into the block operator

$$TL_\eta T^{-1} = \begin{pmatrix} L_\eta & \mathcal{K} \\ 0 & E_\omega \end{pmatrix}, \quad \mathcal{K} = \hat{v}_{xx} A + c\hat{v}_x I + \hat{v}(Df(v_\ast) - Df(v_\ast)),$$

Since $\mathcal{K}$ is bounded in $L^2_\eta$ the result follows from the Fredholm bordering lemma ([3, Lemma 2.3]).

The Fredholm property of $sI - L_{[\eta], \infty}$ can be determined from the first order system corresponding to $(sI - L_{[\eta], \infty})v = r$, i.e. $w = (v, v_x)^\top$ and

$$\begin{pmatrix} 0 \\ r \end{pmatrix}^\top = w_x - (M_{-\mu}(s) \mathbb{I}_{\mathbb{R}_-} + M_{+\mu}(s) \mathbb{I}_{\mathbb{R}_+})w,$$

$$M_{\pm, \mu}(s) = \begin{pmatrix} 0 & I \\ A^{-1}(sI - C_{\pm, \mu}) & -A^{-1}B_{\pm, \mu} \end{pmatrix}.$$

We define the $\mu$-dependent Fredholm set by

$$\Omega_F(\mu) = \{ s \in \mathbb{C} : M_{-\mu}(s) \text{ and } M_{+\mu}(s) \text{ are hyperbolic} \}$$

and denote by $m_{\pm, \mu}^s(s, \mu)$ the dimension of the stable subspace of $M_{\pm, \mu}(s)$ for $s \in \Omega_F(\mu)$. Rewriting the eigenvalue problem for $M_{\pm, \mu}(s)$ in $\mathbb{C}^2$ as a quadratic eigenvalue problem in $\mathbb{C}^2$ shows that $\Omega_F(\mu) = \mathbb{C} \setminus \sigma_{\text{disp}, \mu}(L_{\eta})$ holds for the $\mu$-dependent dispersion set,

$$\sigma_{\text{disp}, \mu}(L_{\eta}) = \{ s \in \mathbb{C} : \det(sI - D_{\pm}(\nu, \mu)) = 0 \text{ for some } \nu \in \mathbb{R} \text{ and some sign } \pm \},$$

$$D_{\pm}(\nu, \mu) = -\nu^2 A + i\nu B_{\pm, \mu} + C_{\pm, \mu}. $$
The following Lemma is well-known and appears for example in [14, Lemma 3.1.10], [19], [22, Sec. 3].

**Lemma 3.3.** Let Assumption 1.1 and 1.2 be satisfied, and let $0 \leq \mu \leq \mu_2$ with $\mu_2$ from Lemma 3.1. Then the operator $sI - L_{[\eta],\infty} : H^2 \to L^2$ is a Fredholm operator if and only if $s \in \Omega_F(\mu)$. If $s \in \Omega_F(\mu)$ then the Fredholm index is given by

\[
\text{ind}(sI - L_{[\eta],\infty}) = m^+(s,\mu) - m^-(s,\mu).
\]

(3.22)

**Remark 3.4.** An intuitive argument for the formula (3.22) is as follows. The Fredholm index measures the degrees of freedom of a linear problem minus the number of constraints. In this case there are $m^+(s,\mu)$ forward decaying modes and $m - m^-(s,\mu)$ (\(m = \text{dim}(w) = 4\)) backward decaying modes, adding up to $m^+(s,\mu) + m - m^-(s,\mu)$ degrees of freedom. The condition that these modes fit together at the origin provides $m$ constraints which then leads to the index formula (3.22).

Next we show how the Fredholm index 0 domain extends into the left half plane.

**Theorem 3.5.** Let Assumption 1.1, 1.2 and 1.3 be satisfied and let $\mu_2$ be given by Lemma 3.1. Then there exist constants $\mu_0 \in (0,\mu_2)$ and $\beta, \varepsilon, \kappa > 0$ such that for each $0 \leq \mu \leq \mu_0$ the open set $\Omega_F(\mu)$ has a unique connected component $\Omega_{\infty}(\mu)$ satisfying

\[
\text{S}_{\varepsilon,\beta}(\mu) := \{s \in \mathbb{C} : |\arg(s + \beta \mu)| \leq \frac{\pi}{2} + \varepsilon \mu\} \subset \Omega_{\infty}(\mu), \quad \text{if} \quad \mu > 0;
\]

\[
\text{S}_{\varepsilon,\beta}(0) := \{s \in \mathbb{C} : \text{Re} s \geq -\kappa \min(|\text{Im} s|, \beta)^2 + \varepsilon \min(\beta - |\text{Im} s|, 0)\} \subset \Omega_{\infty}(0), \quad \text{if} \quad \mu = 0.
\]

Moreover, $\Omega_{\infty}(\mu)$ has the properties

i) For all $s \in \Omega_{\infty}(\mu)$ the operator $sI - L_{\eta} : Y_{\eta} \to X_{\eta}$ is Fredholm of index 0.

ii) $\sigma_{\text{ess}}(L_{\eta}) \subseteq \mathbb{C} \setminus \Omega_{\infty}(\mu)$.

Figure 3.2 illustrates the spectral behavior for $\mu = 0$, $L = L_{\eta}$ in case of the quintic Ginzburg Landau equation (QGCGL) with $\alpha = 1$, $\beta_1 = -\frac{3}{8}$, $\beta_3 = 1 + i$, $\beta_5 = -1 + i$. The dispersion set $\sigma_{\text{disp},0}(L)$ consists of 4 parabola-shaped curves. They originate from
purely imaginary eigenvalues of $M_{-0}(s)$ (red) and of $M_{+0}(s)$ (blue). Note that one of the latter curves has quadratic contact with the imaginary axis. The numbers in the connected components of $\mathbb{C} \setminus \sigma_{\text{disp},0}(\mathcal{L})$ denote the Fredholm index as calculated from (3.22). The white components have index 0 with $\Omega_\infty(0)$ being the rightmost component. The essential spectrum $\sigma_{\text{ess}}(\mathcal{L})$ (see (2.5)) is colored green. Every $\mu > 0$ sufficiently small shifts the spectrum to the left (by $\approx -c\mu$) which allows to inscribe a proper sector with tip at $-\beta \mu$ into the unbounded Fredholm 0 component $\Omega_\infty(\mu)$. In case $\mu = 0$ the sector is rounded quadratically for $|\text{Im} s| \leq \beta$; see Figure 3.1.

**Figure 3.2.** Essential spectrum of the linearized operator $\mathcal{L}$ (green) and dispersion set (red/blue) for (QCGL) with $\alpha = 1$, $\beta_1 = -\frac{1}{8}$, $\beta_3 = 1 + i$, $\beta_5 = -1 + i$. The numbers in the connected components indicate the Fredholm index of $sI - \mathcal{L}$, white regions have Fredholm index 0.

**Proof.** We show $\mathcal{S}_{\epsilon,\beta}(\mu) \subseteq \mathbb{C} \setminus \sigma_{\text{disp},0}(\mathcal{L}_\eta) = \Omega_F(\mu)$, so that (3.23) follows by taking $\Omega_\infty(\mu)$ to be the connected component of $\Omega_F(\mu)$ which contains this sector. From (1.3) one finds

\[
D f(v) = g(|v|^2) + 2g'(|v|^2) \begin{pmatrix} v_1^2 & v_1 v_2 \\ v_1 v_2 & v_2^2 \end{pmatrix}.
\]

From (3.20),(3.21) we obtain

\[
D_-(\nu, \mu) = -\nu^2 A + i\nu(cI - 2\mu A) + S + g(0) + \mu^2 A + c\mu I.
\]

The dispersion set contains the two curves of eigenvalues $s(\nu, \mu)$ and $\overline{s(\nu, \mu)}$ of $D_-(\nu, \mu)$ given for $\nu \in \mathbb{R}$ by

\[
s(\nu, \mu) = (\mu^2 - \nu^2)\alpha_1 + 2\nu \mu \alpha_2 + g_1(0) + c\mu + i((\mu^2 - \nu^2)\alpha_2 + \nu c - 2\nu \mu \alpha_1 + \omega + g_2(0)).
\]

An elementary discussion shows that $\text{Re} s(\nu, \mu) \leq g_1(0) < 0$ for all $\nu \in \mathbb{R}$ and $0 \leq \mu \leq \frac{\nu}{|\nu|^2}$. Moreover, for $|\nu|$ large, the values $s(\nu, \mu)$ lie in a sector which has an opening angle with
the negative real axis of at most $\arctan\left(\frac{|\nu_0|}{\alpha_1}\right) < \frac{\pi}{2}$ uniformly for $\mu$ small. Thus there is a sector $S_{\varepsilon,\beta}(\mu)$ as above which has the two curves in its exterior. Next we obtain from (3.20), (3.21), (3.24) and Lemma 2.1

\begin{equation}
D_+(\nu, \mu) = \left(\delta_1(\nu, \mu) + 2\rho_1 - 2\rho_2, \delta_2(\nu, \mu) + 2\rho_2\right), \quad \rho_j = g_j(|\nu_0|^2)v_\infty^2, j = 1, 2,
\end{equation}

\begin{equation}
\delta_1(\nu, \mu) = (\mu^2 - \nu^2)\alpha_1 + 2i\nu\alpha_1 + i\nu - c\mu, \quad \delta_2(\nu, \mu) = (\mu^2 - \nu^2)\alpha_2 + 2i\nu\alpha_2.
\end{equation}

The eigenvalues are

\begin{equation}
s_{\pm}(\nu, \mu) = \delta_1(\nu, \mu) + \rho_1 \pm (\rho_1^2 - 2\delta_2(\nu, \mu)\rho_2 - \delta_2^2(\nu, \mu))^{1/2}.
\end{equation}

Consider first the case $\mu = 0$:

\begin{equation}
s_{\pm}(\nu, 0) = -\alpha_1\nu^2 + \rho_1 \pm i\nu - c\mu \pm R(\nu)^{1/2}, \quad R(\nu) = |\rho|^2 - (\rho_2 - \alpha_2\nu^2)^2.
\end{equation}

For $\nu \to \infty$ we obtain $\Re s_{\pm}(\nu, 0) \sim -\alpha_1\nu^2$ and $\Im s_{\pm}(\nu, 0) \sim \nu - c\mu \pm |\alpha_2|\nu^2$, hence large values lie in a sector opening to the left with angle $< \frac{\pi}{2}$. Further, Assumption 1.2 yields $\Re s_-(\nu, 0) \leq -\alpha_1\nu^2 + \rho_1 \leq \rho_1 < 0$ for all $\nu \in \mathbb{R}$ (independently of the sign front of $R(\nu)$). Next we show that $r(\nu) = \Re s_+(\nu, 0)$ has a unique global maximum at $\nu = 0$, more precisely,

\begin{equation}
r(0) = 0, \quad r''(0) < 0, \quad r'(\nu) \begin{cases}
> 0, & \nu \in (-\infty, 0) \setminus \{\nu_0\}, \\
= 0, & \nu = 0, \\
< 0, & \nu \in (0, \infty) \setminus \{\nu_0\},
\end{cases}
\end{equation}

where $\nu_0 > 0$ is defined by $|\alpha_2|\nu_0^2 = |\rho| + \operatorname{sgn}(\alpha_2)\rho_2 > 0$ for $\alpha_2 \neq 0$ and $\nu_0 = \infty$ in case $\alpha_2 = 0$. Note that $|\nu| > \nu_0$ holds if and only if $R(\nu) < 0$, and $r(\nu) = -\alpha_1\nu^2 + \rho_1$ for $|\nu| = \nu_0$. Moreover, if $\alpha_2 = 0$ then we have $r(\nu) = -\alpha_1\nu^2$. Thus (3.27) holds in both cases. It remains to consider $r'(\nu)$ for $|\nu| < \nu_0$ and $\alpha_2 \neq 0$. We obtain $R(0) = |\rho|^2$; $r(0) = 0$ and

\begin{equation}
r'(\nu) = 2\nu \left(-\alpha_1 + R(\nu)^{-1/2}\alpha_2(\rho_2 - \alpha_2\nu^2)\right).
\end{equation}

This shows $r'(0) = 0$ and $|\rho_1|r''(0) = 2(-\alpha_1|\rho_1| + \alpha_2\rho_2) < 0$ by Assumption 1.3. If $r'$ has a further zero for $|\nu| < \nu_0$ then this implies $\alpha_2(\rho_2 - \alpha_2\nu^2) > 0$, $\operatorname{sgn}(\alpha_2) = \operatorname{sgn}(\rho_2)$, and $|\alpha|^2(\rho_2 - \alpha_2\nu^2)^2 = \alpha_1^2|\rho|^2$. By these sign conditions we have a unique square root given by

\begin{equation}
|\alpha|(\rho_2 - \alpha_2\nu^2) = \operatorname{sgn}(\alpha_2)\alpha_1|\rho|, \quad \text{or} \quad |\alpha||\alpha_2|\nu^2 = |\alpha||\rho_2| - \alpha_1|\rho|.
\end{equation}

But the last equation has no real solution $\nu$ since our Assumptions 1.1-1.3 and $\alpha_2\rho_2 > 0$ imply

\begin{equation}
|\alpha|^2(\rho_2 - \alpha_1|\rho|)^2 = \alpha_2^2\rho_2^2 - \alpha_1^2\rho_1^2 = (\alpha_2\rho_2 - \alpha_1\rho_1)(\alpha_2\rho_2 + \alpha_1\rho_1) < 0.
\end{equation}

Since $r'$ has no further zeros in $(-\nu_0, 0) \cup (0, \nu_0)$ the sign of $r'$ is determined by $r''(0) < 0$. Now consider $s_{\pm}(\nu, \mu)$ for small values $\mu > 0$. For large values of $|\nu|$ the asymptotic behavior is only slightly modified to $\Re s_{\pm}(\nu, \mu) \sim -\alpha_1\nu^2$ and $\Im s_{\pm}(\nu, \mu) \sim \nu(c + 2\mu\alpha_1) \pm |\alpha_2|\nu^2$, and we find a proper sector enclosing the dispersion curves for $|\nu| \geq \nu_1$ uniformly for $0 \leq \mu \leq \mu_1$. The curve $s_-(\nu, 0), |\nu| \leq \nu_1$ is bounded away from the imaginary axis, hence it suffices to consider $s_+(\nu, \mu)$. From (3.25), (3.26) one computes the partial derivatives
3.28, we infer that we can estimate $C[|\nu| |\nu| \leq \frac{C}{2\delta} + C\delta |\nu|^2$ and absorb $C\delta$ into the negative pre-factor of $|\nu|^2$ by taking $\delta$ small and subsequently absorb $\frac{C}{2\delta}$ into the negative pre-factor of $\mu$ by taking $\mu$ small. Thus we can choose $\beta = \frac{\pi}{2}$ and determine $\varepsilon, \nu_2, \mu_2 > 0$ such that

$$|\arg(s_+(\nu, \mu) + \beta \mu)| \geq \frac{\pi}{2} + \varepsilon \mu \quad \text{for all} \quad |\nu| \leq \nu_2, \quad 0 < \mu \leq \mu_2.$$ 

Finally, for $\nu_2 \leq |\nu| \leq \nu_1$, the curve $s_+(\nu, 0)$ is bounded away from the imaginary axis, and our assertion follows by continuity of $s_+$. For $\mu = 0$ the expansion (3.28) ensures $\text{Im } s_+(\nu, 0) = \nu c + \mathcal{O}(|\nu|^2)$ and the existence of some $\beta > 0$ such that

$$\text{Re } s_+(\nu, 0) \leq -2\kappa|\text{Im } s_+(\nu, 0)|^2 \quad \text{for} \quad |\text{Im } s_+(\nu, 0)| \leq \beta, \quad \kappa = \frac{|\rho_1 \alpha_1 + \rho_2 \alpha_2|}{4c^2 |\rho_1|}.$$ 

The inclusion (3.23) then follows for $\varepsilon$ sufficiently small. In addition, we require $\varepsilon \leq \kappa \beta$ which implies

$$S_{\varepsilon, \beta}(0) \subset S_{\varepsilon, T\beta}(0) \quad \text{for} \quad 0 < \beta < \beta.$$ 

For the proof of (i) note that the Fredholm index is constant in $\Omega_\infty(\mu)$. Therefore, it is enough to consider $M_{\pm, \mu}(s)$ for large $s > 0$:

$$\begin{pmatrix} I & 0 \\ 0 & s^{-1/2}I \end{pmatrix} M_{\pm, \mu}(s) \begin{pmatrix} I & 0 \\ 0 & s^{1/2}I \end{pmatrix} = s^{1/2} \begin{pmatrix} 0 & I \\ A^{-1} & 0 \end{pmatrix} + s^{-1/2} \begin{pmatrix} 0 & 0 \\ -A^{-1}C_{\pm, \mu} & -A^{-1}B_{\pm, \mu} \end{pmatrix}.$$ 

The leading matrix $\begin{pmatrix} 0 & I \\ A^{-1} & 0 \end{pmatrix}$ has a two-dimensional stable subspace which belongs to the eigenvalues $-\alpha^{-1}, -\alpha^{-1}$, and a two-dimensional unstable subspace which belongs to $\alpha^{-1}, \alpha^{-1}$. These subspaces are only slightly perturbed for large $s$, and hence we obtain the Fredholm index 0 from Lemma 3.3.

We conclude the proof by noting that assertion (ii) follows from our previous result and the definition of the essential spectrum, cf. (2.5). \hfill \Box

We continue with the

**Proof of Lemma 2.4.** Clearly, Theorem 2.2 ensures that the functions $\varphi_1, \varphi_2$ from (2.8) lie in every $Y_\eta$, where $\eta$ is defined by (1.11) and $0 \leq \mu < 2\tilde{\mu}$, and both are eigenfunctions of $L_\eta$ defined in (2.7). Recall that $E_\omega = S_\omega + Df(v_\infty) = 2 \begin{pmatrix} \rho_1 & 0 \\ \rho_2 & 0 \end{pmatrix}$ (see (3.25) for $\rho_1, \rho_2$) has the eigenvalues 0 and $2\rho_1 < 0$ with eigenvectors $(0, |v_\infty\rangle)^T = S_1v_\infty$ and $(\rho_1, \rho_2)^T$, respectively. Now consider $(v, \zeta)^T \in Y_\eta$ such that $L_\eta(v, \zeta)^T = 0$. From $E_\omega \zeta = 0$ we obtain $\zeta = zS_1v_\infty$ for some $z \in \mathbb{C}$. This shows $(u, 0) := (v, \zeta)^T - z\varphi_2 \in \mathcal{N}(L_\eta)$ and $u = yv_\eta^\prime$ for some $y \in \mathbb{C}$ by Assumption 2.3. Therefore, we obtain $(v, \zeta)^T = z\varphi_2 + y\varphi_1$. Now suppose $L_\eta(v, \zeta)^T = y\varphi_1 + z\varphi_2$ for some $y, z \in \mathbb{C}$. Since $E_\omega$ has only simple eigenvalues we conclude $z = 0$ from the second component and then $y = 0$ from Assumption 2.3. This proves (2.8).

From Theorem 3.5 we infer that $L_\eta : Y_\eta \to X_\eta$ is Fredholm of index 0 and hence also $L_{\eta} : H_2^2 \to L_2^2$ by Lemma 3.2. Assumption 2.3 guarantees that 0 is a simple eigenvalue of $L_{\eta} : H_2^2 \to L_2^2$. Note that we have $v_\eta^\prime \in H_2^2$ and that there is no solution $y \in H_2^2$ of
$L_\eta y = v_\eta'$ since this implies $y \in H^2$ and $y \in \mathcal{N}(L^2) = \mathcal{N}(L)$. Simple eigenvalues are known to be isolated. This may be seen by applying the inverse function theorem to

$$T : H^2_\eta \times \mathbb{R} \to L^2_\eta \times \mathbb{R}, \quad T\left(\frac{v}{s}\right) = \left(\frac{L_\eta v - sv}{(v_\eta', v - v_\eta')_{L^2}}, \frac{v_\eta'}{0}\right).$$

Therefore, there exists some $s_0 = s_0(\mu) > 0$ such that $L_\eta : H^2_\eta \to L^2_\eta$ has no eigenvalues with $|s| \leq s_0$ except $s = 0$.

Finally, we prove assertion (ii) for $\beta_0 = \min(\varepsilon_0, \beta_E, \tilde{\beta}, |\rho_1|)$ where $\beta_E$ is from Assumption 2.3 and $\beta, \tilde{\beta}$ satisfy (3.23), (3.29) as well as $\tilde{\beta} < \min(\beta, \frac{\sqrt{3}}{2}s_0)$. Consider $s \in \sigma_{pt}(L_\eta) \setminus \{0\}$ with $\Re s \geq -\beta_0$ and eigenfunction $(v, \zeta)^\top \in Y_\eta$. From $\sigma(E_\omega) = \{0, 2\rho_1\}$ we obtain $\zeta = 0$ and thus $L_\eta v = sv, v \in H^2_\eta$. We claim that $s \in \mathcal{S}_{s,\tilde{\beta}}(0)$. For $\Re s \geq 0$ this is obvious, while for $0 > \Re s \geq -\beta_0 \geq -\frac{\varepsilon_0}{2}$ this follows from

$$|\Im s| \geq (s_0^2 - (\Re s)^2)^{1/2} \geq \frac{\sqrt{3}}{2}s_0 > \tilde{\beta} + \frac{s_0}{2}, \quad \Re s \geq -\frac{\varepsilon s_0}{2} > -s_0\beta + \varepsilon(\tilde{\beta} - |\Im s|).$$

By Theorem 3.5, $s \in \Omega_\omega(0)$ and $sI - L_\eta : Y_\eta \to X_\eta$ with $\mu = 0$ is Fredholm of index 0. By Lemma 3.2 the same holds for $sI - L : H^2 \to L^2$ and we have shown $s \in \sigma_{pt}(L)$. This contradicts Assumption 2.3 since $\Re s \geq -\beta_E$. \hfill \Box

So far we determined the spectral properties of the linearized operator $L_\eta$ and proved spectral stability of the extended system (2.2) posed on the exponentially weighted spaces $X_\eta$ for positive but small $\mu > 0$. In particular, the essential spectrum of $L_\eta$ is included in the left half plane as well as its point spectrum except the zero eigenvalue which has algebraic and geometric multiplicity 2, i.e. $\mathcal{N}(L_\eta) = \mathcal{N}(L^*_\eta) = \text{span}\{\varphi_1, \varphi_2\}$. Since $L_\eta$ is Fredholm of index 0 the same holds true for the (abstract) adjoint operator $L^*_\eta : D(L^*_\eta) \subset X_\eta \to X_\eta$ and there are two normalized adjoint eigenfunctions $\psi_1, \psi_2 \in D(L^*)$ with

$$\mathcal{N}(L^*_\eta) = \text{span}\{\psi_1, \psi_2\}, \quad (\psi_i, \varphi_j)_{X_\eta} = \delta_{ij}, \quad i, j = 1, 2.$$

We define the map

$$(3.31) \quad P_\eta : X_\eta \to X_\eta, \quad v \mapsto (\psi_1, v)_{X_\eta}\varphi_1 + (\psi_2, v)_{X_\eta}\varphi_2.$$

Then $P_\eta$ is a projection onto $\mathcal{N}(L_\eta)$ and $X_\eta$ can be decomposed into

$$X_\eta = \mathcal{R}(P_\eta) \oplus \mathcal{R}(I - P_\eta) = \mathcal{N}(L_\eta) \oplus \mathcal{N}(L^*_\eta)^\perp.$$

The subspace $\mathcal{N}(L^*_\eta)^\perp$ is invariant under $L_\eta$ and we introduce its intersection with the smooth spaces $X^k_\eta, k = 1, 2$

$$V_\eta := \mathcal{N}(L^*_\eta)^\perp \subset X_\eta, \quad V^k_\eta := V_\eta \cap X^k_\eta.$$

## 4. Semigroup estimates

In the previous section we studied the spectrum of the linearized operator $L_\eta$ on exponentially weighted spaces for positive but small $\mu > 0$ and derived a-priori estimates for the resolvent equation (3.1). Theorem 3.5 shows that there is no essential spectrum in the Fredholm 0 component $\Omega_\omega(\mu)$ and thus also not in the sector $\mathcal{S}_{s,\tilde{\beta}}(\mu)$. When combined with Lemma 3.1 we obtain $\Omega_0 \subset \text{res}(L_\eta)$ for the domain $\Omega_0$ from (3.2). Further, Lemma 2.4 shows that the nonzero point spectrum is bounded away from the imaginary axis. Thus
we conclude from Lemma 3.1 that $\mathcal{L}_\eta$ is a sectorial operator. By the classical semigroup theory, (see [12], [18], [20]) the operator $\mathcal{L}_\eta$ generates an analytic semigroup $\{e^{\mathcal{L}_\eta}t\}_{t \geq 0}$ on $X_\eta$ such that for any $\delta > 0$ there exists a constant $C_\delta$ with $\|e^{\mathcal{L}_\eta}t\|_{X_\eta} \leq C_\delta e^{\delta t}$, $t \geq 0$. Next we avoid the neutral modes of $\{e^{\mathcal{L}_\eta}t\}_{t \geq 0}$ and restrict $\mathcal{L}_\eta$ to $V_\otimes^2$ in order to have exponential decay.

**Theorem 4.1.** Let Assumption 1.1, 1.2, 1.3 and 2.3 be satisfied and let $0 < \mu \leq \min(\mu_0, \mu_1)$ with $\mu_0$ from Theorem 3.5 and $\mu_1$ from Lemma 2.4. Then the linearized operator $\mathcal{L}_\eta : Y_\eta \to X_\eta$ generates an analytic semigroup $\{e^{\mathcal{L}_\eta}t\}_{t \geq 0}$ on $X_\eta$. Moreover, there exist $K = K(\mu) \geq 1$ and $\nu = \nu(\mu) > 0$ such that for all $t \geq 0$ and $w \in V_\otimes^\ell$, $\ell = 0, 1$ the following estimate holds

\[ \|e^{t\mathcal{L}_\eta}w\|_{X_\eta^\ell} \leq Ke^{-\nu t}\|w\|_{X_\eta^\ell}, \]

*Proof.* The first assertion follows by the arguments above. Thus it remains to show the estimate (4.1). For that purpose, we note that the restriction $\mathcal{L}_{V_\chi}$ of $\mathcal{L}_\eta$ to $V_\chi$ is a closed operator on $V_\chi$ with $\mathcal{N}(\mathcal{L}_{V_\chi}) = \{0\}$ and $\mathcal{R}(\mathcal{L}_{V_\chi}) = V_\chi$. Thus $\mathcal{L}_{V_\chi}$ is Fredholm of index 0 and 0 $\notin \sigma(\mathcal{L}_{V_\chi})$. Moreover, the projector $P_\eta$ from (3.31) commutes with $\mathcal{L}_\eta$ which leads to $\text{res}(\mathcal{L}_\eta) \subseteq \text{res}(\mathcal{L}_{V_\chi})$. Therefore, by Theorem 3.5, Lemma 2.4 and Lemma 3.1 we find $\varepsilon = \varepsilon(\mu)$, $\nu = \nu(\mu)$ and a sector $\Sigma_{\varepsilon, \nu} = \{s \in \mathbb{C} : \arg(s + \nu) \leq \pi + \varepsilon\}$ such that $\Sigma_{\varepsilon, \nu} \subseteq \text{res}(\mathcal{L}_{V_\chi})$. Further we can decrease $\varepsilon > 0$ and take $R > 0$ sufficiently large so that $\Sigma_{\varepsilon, \nu} \cap \{|s| \geq R\} \subseteq \Omega_0$. From Lemma 3.1 and the fact that the resolvent is bounded in a compact subset of the resolvent set we then find a constant $C = C(\mu) > 0$ such that for all $w \in V_\otimes^\ell$ and $\ell = 0, 1$ the following holds

\[ \| (sI - \mathcal{L}_{V_\chi})^{-1}w \|_{X_\chi^\ell} = \| (sI - \mathcal{L}_\eta)^{-1}w \|_{X_\eta^\ell} \leq C\|w\|_{X_\chi^\ell}, \quad \forall s \in \Sigma_{\varepsilon, \nu} \cap \{|s| \leq R\}, \]

\[ \| (sI - \mathcal{L}_{V_\chi})^{-1}w \|_{X_\chi^\ell} = \| (sI - \mathcal{L}_\eta)^{-1}w \|_{X_\eta^\ell} \leq \frac{C}{|s|}\|w\|_{X_\chi^\ell}, \quad \forall s \in \Sigma_{\varepsilon, \nu} \cap \{|s| > R\}. \]

Therefore, $\mathcal{L}_{V_\chi}$ is a sectorial operator on $V_\chi$ and the representation of the semigroup

\[ e^{t\mathcal{L}_{V_\chi}} = \int_{\Gamma_{\varepsilon, \nu}} (zI - \mathcal{L}_{V_\chi})^{-1}e^{tz}dz, \quad \Gamma_{\varepsilon, \nu} = \{-\nu + r\exp(\text{sgn}(r)\pi i (\frac{\pi}{2} + \varepsilon)) : r \in \mathbb{R}\} \]

leads in the standard way to the exponential estimate

\[ \|e^{t\mathcal{L}_\eta}w\|_{X_\eta^\ell} = \|e^{t\mathcal{L}_{V_\chi}}w\|_{X_\chi^\ell} \leq Ke^{-\nu t}\|w\|_{X_\chi^\ell}, \quad w \in V_\otimes^\ell, \ell = 0, 1. \]

\[ \square \]

5. Decomposition of the dynamics

The nonlinear operator $\mathcal{F}$ on the right hand side of (2.2) is equivariant w.r.t. the group action $a(\gamma)$ from (2.4) of the group $G = S^1 \times \mathbb{R}$. Every element $\gamma$ of the group can be represented by an angle $\theta$ and a shift $\tau$. The composition $\circ : G \times G \to G$ of two elements $\gamma_1, \gamma_2 \in G$ is given by $\gamma_1 \circ \gamma_2 = (\theta_1 + \theta_2 \mod 2\pi, \tau_1 + \tau_2)$ and the inverse map $\gamma \mapsto \gamma^{-1}$ by $\gamma^{-1} = (-\theta \mod 2\pi, -\tau)$. Both maps are smooth and $G$ is a two dimensional $C^\infty$-manifold.

An atlas of the group $G$ is given by the two (trivial) charts $(U, \chi)$ and $(\tilde{U}, \tilde{\chi})$ defined by

$U = \{\gamma = (\theta \mod 2\pi, \tau) \in G : \theta \in (-\pi, \pi), \tau \in \mathbb{R}\}, \quad \chi : U \to \mathbb{R}^2, \gamma \mapsto \chi(\gamma) = (\theta, \tau),$

$\tilde{U} = \{\gamma = (\theta \mod 2\pi, \tau) \in G : \theta \in (0, 2\pi), \tau \in \mathbb{R}\}, \quad \tilde{\chi} : \tilde{U} \to \mathbb{R}^2, \gamma \mapsto \tilde{\chi}(\gamma) = (\theta, \tau).$

\[ \square \]
We will always work with the chart $\chi$ since the arguments for $\tilde{\chi}$ will be almost identical. Next we show smoothness of the group action $a(\cdot)v$ in $\mathcal{G}$ depending on the regularity of $v$.

**Lemma 5.1.** The group action $a : \mathcal{G} \rightarrow GL[X_\eta]$, $\gamma \mapsto a(\gamma)$ from (2.4) is a homomorphism and $a(\gamma)Y_\eta = Y_\eta \gamma \in \mathcal{G}$. For $v \in X_\eta$ the map $a(\cdot)v : \mathcal{G} \rightarrow X_\eta$ is continuous and for $v \in Y_\eta$ it is continuously differentiable. For $\gamma \in \mathcal{U}$, $\chi(\gamma) = z$ the derivative applied to $h = (h_1, h_2)^T \in \mathbb{R}^2$ is given by

$$(a(\cdot)v \circ \chi^{-1})'(z)h = -h_1a(\gamma)S_1v - h_2a(\gamma)v_x,$$

where $S_1v = (S_1v, S_1\rho)^T$, $v_x = (v_x, 0)^T$ for $v = (v, \rho)^T$.

The proof of Lemma 5.1 is straightforward and will be given in the Appendix. It is based on well known properties of translation and rotation on (weighted) Lebesgue and Sobolev spaces. Next recall the Cauchy problem (2.2) with perturbed initial data

$$u_t = \mathcal{F}(u), \quad u(0) = v_* + v_0.$$ 

We follow the approach in [4], [12] and decompose the dynamics of the solution into a motion along the group orbit $\{a(\gamma)v_* : \gamma \in \mathcal{G}\}$ of the wave and into a perturbation $w$ in the space $V_\eta$. We use local coordinates in $U$ and write the solution $u(t)$ as

$$(5.1) \quad u(t) = a(\gamma(t)))v_* + w(t), \quad \gamma(t) = \chi^{-1}(z(t)) \in \mathcal{U}, w(t) \in V_\eta$$

for $t \geq 0$. Thus $z$ describes the local coordinates of the motion on the group orbit $\mathcal{O}(v_*)$ given by $\gamma$ in the chart $(U, \chi)$ and $w \in V_\eta$ is the difference of the solution to the group orbit in $V_\eta = \mathcal{N}(L^\eta)^\perp$. It turns out that the decomposition is unique as long as the solution stays in a small neighborhood of the group orbit and $\gamma$ stays in $\mathcal{U}$. This will be guaranteed by taking sufficiently small initial perturbations $v_0$. Let $P_\eta$ be the projector onto $\mathcal{N}(L^\eta)$ from (3.31) and recall from (2.8) that $\mathcal{N}(L^\eta)$ is spanned by the eigenfunctions $\varphi_2 = S_1v_*$ and $\varphi_1 = v_{*,x}$. Following [4] we define

$$(5.2) \quad \Pi_\eta : \chi(U) \subset \mathbb{R}^2 \rightarrow \mathcal{N}(L^\eta), \quad z \mapsto P_\eta(a(\gamma^{-1}(z))v_* - v_*).$$

For simplicity of notation we frequently replace $\gamma^{-1}(z)$ by $\gamma$ where $\gamma$ is always taken in our working chart $(U, \chi)$. The next lemma uses $\Pi_\eta$ to show uniqueness of the decomposition (5.1) in a neighborhood of $v_*$. 

**Lemma 5.2.** Let Assumption 1.1, 1.2, 1.3 and 2.3 be satisfied and let $\mu_1$ be given by Lemma 2.4. Then for all $0 < \mu \leq \mu_1$ there is a zero neighborhood $W = W(\mu) \subset \chi(U)$ such that the map $\Pi_\eta : W \rightarrow \mathcal{N}(L^\eta)$ from (5.2) is a local diffeomorphism. Moreover, there is a zero neighborhood $V = V(\mu) \subset \chi(U) \times V$ such that the transformation

$$T_\eta : V \rightarrow X_\eta, \quad (z, w) \mapsto a(\gamma^{-1}(z))v_* - v_* + w$$

is a diffeomorphism with the solution of $T_\eta(z, w) = v$ given by

$$(5.3) \quad z = \Pi^{-1}_\eta(P_\eta v), \quad w = v + v_* - a(\gamma^{-1}(z))v_*.$$

**Proof.** Since $0 < \mu \leq \mu_1$ the projector $P_\eta$ and $\Pi_\eta$ are well defined. By Lemma 5.1 the group action $a$ is continuously differentiable and so is $\Pi_\eta$. Further, $\Pi_\eta(0) = 0$ and its derivative is given by $D\Pi_\eta(0)y = -y_1\varphi_1 - y_2\varphi_2, y \in \mathbb{R}^2$ where $\varphi_1, \varphi_2$. Therefore, $D\Pi_\eta(0)$ is invertible on $\mathcal{N}(L^\eta)$ and the first assertion is a consequence of the inverse function theorem. By the same arguments, $T_\eta$ is continuously differentiable, $T_\eta(0, 0) = 0$ and its derivative is
given by $DT_\eta(0,0) = (D\Pi_\eta(0) \quad I_{V_\eta \to X_\eta}) : \mathbb{R}^2 \times V_\eta \to X_\eta$ which is again invertible. Hence $T_\eta : V \to X_\eta$ is a diffeomorphism on a zero neighborhood $V \subset \chi(U) \times V_\eta$. Finally, applying $P_\eta$ to $T_\eta(z, w) = v$ yields $z = \Pi_\eta^{-1}(P_\eta v)$ while the second equation in (5.3) follows from the definition of $T_\eta$.

Consider a smooth solution $u(t)$, $t \in [0, t_\infty)$ of (2.2) which stays close to the profile of the TOF. In particular, assume that $u(t) - v_\star$, $t \in [0, t_\infty)$ lies in the region where $T_\eta^{-1}$ exists by Lemma 5.2. Then there are unique $z(t) \in \chi(U)$ and $w(t) = (w(t), \zeta(t))^\top \in V_\eta$ for $t \in [0, t_\infty)$ such that

$$u(t) - v_\star = T_\eta(z(t), w(t)) \quad \forall t \in [0, t_\infty),$$

and (5.1) holds. Taking the initial condition from (2.2) into account yields for $t = 0$

$$v_\star + v_0 = u(0) = a(\chi^{-1}(z(0)))v_\star + w(0),$$

which leads to $v_0 = T_\eta(z(0), w(0))$. Therefore, the initial conditions for $z, w$ are given by

$$z(0) = \Pi_\eta^{-1}(P_\eta v_0) =: z_0, \quad w(0) = v_0 + v_\star - a(\chi^{-1}(z(0)))v_\star =: w_0.$$

Now we write the angular and translational components of $z$ explicitly as $z(t) = (\theta(t), \tau(t))$. We insert the decomposition (5.1) into (2.2) and obtain

$$0 = u_t - \mathcal{L}_\eta u = \frac{d}{dt}a(\chi^{-1}(z))v_\star + w_t - a(\gamma) \left( \frac{Av_{\star xx} + cv_{\star x} + S_\omega v_\star}{S_\omega v_\infty} \right) - \left( \frac{Aw_{xx} + cw_x + S_\omega w}{S_\omega \zeta} \right) - \left( \frac{f(R_\theta v_\star (\cdot - \tau) + w)}{f(R_\theta v_\infty + \zeta)} \right).$$

Using the equivariance of $F$ and the derivative of the group action from Lemma 5.1, leads to

$$w_t = \mathcal{L}_\eta w - a(\chi^{-1}(z))\varphi_1 \theta_t - a(\chi^{-1}(z))\varphi_2 \tau_t + r_{[\eta]}(z, w)$$

where the remainder $r_{[\eta]}(z, w)$ is given for $z = (\theta, \tau)$ and $w \in V_\eta$ by

$$r_{[\eta]}(z, w) := \left( \frac{f(R_\theta v_\star (\cdot - \tau) + w)}{f(R_\theta v_\infty + \zeta)} \right) - \left( \frac{f(R_\theta v_\star (\cdot - \tau))}{f(R_\theta v_\infty)} \right) - \left( \frac{Df(v_\star)}{Df(v_\infty)} \right).$$

Let us apply the projector $P_\eta$ to (5.5) and use $w(t) \in V_\eta$, $t \in [0, t_\infty)$ and $P_\eta(w_t - \mathcal{L}_\eta w) = 0$ to obtain the equality

$$0 = P_\eta r_{[\eta]}(z, w) - P_\eta a(\chi^{-1}(z))\varphi_1 \theta_t - P_\eta a(\chi^{-1}(z))\varphi_2 \tau_t.$$

The next lemma shows that equation (5.6) can be written as an explicit ODE for $z = (\theta, \tau)$.

**Lemma 5.3.** Let Assumption 1.1, 1.2, 1.3 and 2.3 be satisfied and let $\mu_1$ be given by Lemma 2.4. Then for all $0 < \mu \leq \mu_1$ the map

$$S_\eta(z) : \mathbb{R}^2 \to \mathcal{N}(\mathcal{L}_\eta), \quad y \mapsto P_\eta a(\chi^{-1}(z))\varphi_1 y_1 + P_\eta a(\chi^{-1}(z))\varphi_2 y_2$$

is continuous, linear and continuously differentiable w.r.t. $z \in (-\pi, \pi) \times \mathbb{R}$. Moreover, there is a zero neighborhood $V = V(\mu) \subset \mathbb{R}^2$ such that $S_\eta(z)^{-1}$ exists for all $z \in V$ and depends continuously on $z$. 
Proof. Since $0 < \mu \leq \mu_1$ the projector $P_\eta$ and the map $S_\eta(z)$ are well defined. Moreover, $S_\eta(z)$ is linear and continuous. Once more the smoothness of the group action, cf. Lemma 5.1, implies that $S_\eta(z)$ is continuously differentiable w.r.t. $z$. Take $w \in \mathcal{N}(\mathcal{L}_\eta) = \text{span}\{\varphi_1, \varphi_2\}$ and recall the adjoint eigenfunctions $\psi_1, \psi_2$ from (3.30). We form the inner products in $X_\eta$ of the equation $S_\eta(z)y = w$, $y \in \mathbb{R}^2$ with the adjoint eigenfunctions:

$$M(z)y = \begin{pmatrix} (\psi_1, w) \\ (\psi_2, w) \end{pmatrix}, \quad M(z) = \begin{pmatrix} (\psi_1, P_\eta a(\chi^{-1}(z))\varphi_1) & (\psi_1, P_\eta a(\chi^{-1}(z))\varphi_2) \\ (\psi_2, P_\eta a(\chi^{-1}(z))\varphi_1) & (\psi_2, P_\eta a(\chi^{-1}(z))\varphi_2) \end{pmatrix}.$$ 

Now $M(0) = I$ and $M(z)$ depends continuously on $z$. Then there exists a zero neighborhood $V \subset \mathbb{R}^2$ such that $M(z)$ is invertible and its inverse depends continuously on $z$. Finally, we obtain for $S_\eta(z)^{-1}$ the representation

$$S_\eta(z)^{-1}w = M(z)^{-1} \begin{pmatrix} (\psi_1, w) \\ (\psi_2, w) \end{pmatrix},$$

which proves our assertion. \qed

As a consequence of Lemma 5.3 we obtain from (5.6) and (5.4) the $z$-equation

$$(5.7) \quad z_t = r^{[z]}(z, w), \quad z(0) = \Pi_\eta^{-1}(P_\eta v_0),$$

where $r^{[z]}$ is given by

$$(5.8) \quad r^{[z]}(z, w) := S_\eta(z)^{-1}P_\eta r^{[f]}(z, w).$$

This equation describes the motion of the solution projected onto the group orbit $O(v_*)$. The last step is to apply the projector $(I - P_\eta)$ to (5.5) and using (5.7) to obtain the equation for the offset $w$ from the group orbit:

$$w_t = \mathcal{L}_\eta w + (I - P_\eta)r^{[f]}(z, w) - (I - P_\eta)(a(\cdot)v_* \circ \chi^{-1})(z)S_\eta(z)^{-1}P_\eta r^{[f]}(z, w)$$

$$=: \mathcal{L}_\eta w + r^{[w]}(z, w)$$

with the remainder $r^{[w]}$ given by

$$(5.9) \quad r^{[w]}(z, w) := \left( (I - P_\eta) - (I - P_\eta)(a(\cdot)v_* \circ \chi^{-1})(z)S_\eta(z)^{-1}P_\eta \right)r^{[f]}(z, w).$$

Finally, the fully transformed system including initial values for $w$ and $z$ reads as

$$(5.10) \quad w_t = \mathcal{L}_\eta w + r^{[w]}(z, w), \quad w(0) = v_0 + v_* - a(\Pi_\eta^{-1}(P_\eta v_0))v_* =: w_0,$$

$$(5.11) \quad z_t = r^{[z]}(z, w), \quad z(0) = \Pi_\eta^{-1}(P_\eta v_0) =: z_0.$$

Reversing the steps leading to (5.10), (5.11) shows that every local solution of this system leads to a solution of (2.2) close to $v_*$ via the transformation (5.1).

6. Estimates of nonlinearities

To study solutions of the system (5.10), (5.11) we need to control the remaining nonlinearities $r^{[w]}, r^{[z]}$ from (5.9) and (5.8). In this section we derive Lipschitz estimates with small Lipschitz constants for the nonlinearities in the space $X^1_\eta$. Of course the estimates will be guaranteed by the smoothness of $f$ from (1.8). In particular, we can assume $f \in C^3$ by Assumption 1.9. However, our choice of the underlying space $X_\eta$ requires somewhat
laborious calculations to derive the estimates. The main work is to take care of the offset which is hidden in the second component of elements in $X_\eta$.

**Lemma 6.1.** Let Assumption 1.1, 1.2, 1.3 and 2.3 be satisfied and let $\mu_1$ be given by Lemma 2.4. Then for every $0 < \mu \leq \mu_1$ there are constant $C = C(\mu) > 0$ and $\delta = \delta(\mu) > 0$ such that for all $z, z_1, z_2 \in B_\delta(0) \subseteq \mathbb{R}^2$ and $w, w_1, w_2 \in B_\delta(0) \subseteq X_\eta^1$ the following holds:

$$i) \| r^{[l]}(z, w_1) - r^{[l]}(z, w_2) \|_{X_\eta^1} \leq C \left( |z| + \max \left\{ \| w_1 \|_{X_\eta^1}, \| w_2 \|_{X_\eta^1} \right\} \right) \| w_1 - w_2 \|_{X_\eta^1},$$

$$ii) \| r^{[l]}(z_1, w) - r^{[l]}(z_2, w) \|_{X_\eta^1} \leq C |z_1 - z_2|,$$

$$iii) \| r^{[w]}(z, w_1) - r^{[w]}(z, w_2) \|_{X_\eta^1} \leq C \left( |z| + \max \left\{ \| w_1 \|_{X_\eta^1}, \| w_2 \|_{X_\eta^1} \right\} \right) \| w_1 - w_2 \|_{X_\eta^1},$$

$$iv) \| r^{[w]}(z_1, w_2) - r^{[w]}(z_2, w_2) \|_{X_\eta^1} \leq C \left( |z_1 - z_2| + \| w_1 - w_2 \|_{X_\eta^1} \right),$$

$$v) \| r^{[z]}(z_1, w_1) - r^{[z]}(z_2, w_2) \| \leq C \left( |z_1 - z_2| + \| w_1 - w_2 \|_{X_\eta^1} \right).$$

**Remark 6.2.** Note that $r^{[l]}(z, 0) = 0$ holds so that the estimates i) and iii) imply linear bounds for the the nonlinearities $r^{[l]}$ and $r^{[w]}$ in $B_\delta(0)$.

**Proof.** Let $\delta$ be so small such that $B_\delta(0) \subseteq \chi(U)$ and $B_\delta(0) \subseteq V$ with $V$ from Lemma 5.3. Then the remainders $r^{[l]}, r^{[w]}, r^{[z]}$ are well defined by Lemmas 2.4, 5.2, and 5.3. Let us set $\gamma = \chi(z) = (\theta, \tau)$ as well as $\gamma_i = \chi(z_i) = (\theta_i, \tau_i), i = 1, 2$. Further we write $w = (w, \zeta)^T$ and $w_i = (w_i, \zeta_i)^T$ for $i = 1, 2$. For the sake of notation we also write $a(\gamma) v = R_{\theta} v (\cdot - \tau)$ for a function $v : \mathbb{R} \to \mathbb{R}^2$.

Throughout the proof, $C = C(\mu)$ denotes a universal constant depending on $\mu$. The smoothness of $f$ and Sobolev embeddings imply

$$\| Df(a(\gamma) v_*) - Df(v_*) \|_{L^\infty} \leq C \| a(\gamma) v_* - v_* \|_{L^\infty}$$

$$\leq C \| a(\gamma) v_* - R_{\theta} v_{\infty} \hat{v} - (v_* - v_{\infty} \hat{v}) \|_{L^\infty} + C |R_{\theta} v_{\infty} - v_{\infty}|$$

$$\leq C |a(\gamma) v_* - R_{\theta} v_{\infty} \hat{v} - (v_* - v_{\infty} \hat{v})\|_{H^1} + C |R_{\theta} v_{\infty} - v_{\infty}|$$

$$\leq C \|a(\chi^{-1}(z)) v_* - v_*\|_{X_\eta^1} \leq C |z|.$$  

(6.1)

The last estimate follows from the smoothness of the group action; see Lemma 5.1. Similarly, we find

$$\| Df(R_{\theta} v_{\infty}) - Df(v_{\infty}) \| \leq C |z|$$

(6.2)

and

$$\| a(\gamma) v_* - R_{\theta} v_{\infty} - (v_* - v_{\infty}) \|_{L^2_{\eta}(\mathbb{R}^+)}$$

$$\leq \| a(\gamma) v_* - R_{\theta} v_{\infty} \hat{v} - (v_* - v_{\infty} \hat{v})\|_{L^2_{\eta}(\mathbb{R}^+)} + |R_{\theta} v_{\infty} - v_{\infty}| \| \hat{v} - 1\|_{L^2_{\eta}(\mathbb{R}^+)}$$

$$\leq C \|a(\chi^{-1}(z)) v_* - v_*\|_{X_\eta^1} \leq C |z|.$$  

(6.3)

By Theorem 2.2 we can also estimate

$$\| v_* - v_{\infty}\|_{L^2_{\eta}(\mathbb{R}^+)} \leq C.$$  

(6.4)

In what follows these estimates will be used frequently. We start with

i). By definition and the triangle inequality we can split the left side of i) into

$$\| r^{[l]}(z, w_1) - r^{[l]}(z, w_2) \|_{X_\eta^1}$$

$$\leq |f(R_{\theta} v_{\infty} + \zeta_1) - f(R_{\theta} v_{\infty} + \zeta_2) - Df(v_{\infty})(\zeta_1 - \zeta_2)|$$
The first term $T_1$ is estimated by

$$T_1 = |f(R_\theta v_\infty + \zeta_1) - f(R_\theta v_\infty + \zeta_2) - Df(v_\infty)(\zeta_1 - \zeta_2)|$$

$$\leq \int_0^1 |Df(R_\theta v_\infty + \zeta_2 + (\zeta_1 - \zeta_2)s) - Df(v_\infty)|ds|\zeta_1 - \zeta_2|$$

$$\leq C \left( \int_0^1 |Df(R_\theta v_\infty + \zeta_2 + (\zeta_1 - \zeta_2)s) - Df(R_\theta v_\infty)|ds + |Df(R_\theta v_\infty) - Df(v_\infty)| \right) |\zeta_1 - \zeta_2|$$

$$\leq C(|z| + \max\{|\zeta_1|, |\zeta_2|\}) |\zeta_1 - \zeta_2| \leq C (|z| + \max\{|w_1|_{H^1}, |w_2|_{H^1}\}) \|w_1 - w_2\|_{H^1}.$$ 

For the second term $T_2$ we have

$$\|f(a(\gamma)v_\ast + w_1) - f(a(\gamma)v_\ast + w_2) - Df(v_\ast)(w_1 - w_2)$$

$$- \hat{v}[f(R_\theta v_\infty + \zeta_1) - f(R_\theta v_\infty + \zeta_2) - Df(v_\infty)(\zeta_1 - \zeta_2)]\|_{L^2_h}$$

$$= \left\| \int_0^1 Df(a(\gamma)v_\ast + w_2 + (w_1 - w_2)s) - Df(v_\ast)ds(w_1 - w_2)$$

$$- \hat{v} \int_0^1 Df(R_\theta v_\infty + \zeta_2 + (\zeta_1 - \zeta_2)s) - Df(v_\infty)ds(\zeta_1 - \zeta_2) \right\|_{L^2_h}$$

$$\leq \left\| \int_0^1 Df(a(\gamma)v_\ast + w_2 + (w_1 - w_2)s) - Df(a(\gamma)v_\ast)ds(w_1 - w_2)$$

$$- \hat{v} \int_0^1 Df(R_\theta v_\infty + \zeta_2 + (\zeta_1 - \zeta_2)s) - Df(R_\theta v_\infty)ds(\zeta_1 - \zeta_2) \right\|_{L^2_h}$$

$$+ \||Df(a(\gamma)v_\ast) - Df(v_\ast)|w_1 - w_2 - \hat{v}[Df(R_\theta v_\infty) - Df(v_\infty)](\zeta_1 - \zeta_2)\|_{L^2_h}$$

$$=: T_4 + T_5.$$ 

$T_5$ is bounded by another two terms

$$T_5 \leq \||Df(a(\gamma)v_\ast) - Df(v_\ast)|w_1 - \hat{v}\zeta_1 - w_2 + \hat{v}\zeta_2\|_{L^2}$$

$$+ \||Df(a(\gamma)v_\ast) - Df(v_\ast) - Df(R_\theta v_\infty) + Df(v_\infty)|(\zeta_1 - \zeta_2)\hat{v}\|_{L^2} =: T_6 + T_7.$$ 

Using (6.1) we have

$$T_6 \leq C|z|\|w_1 - \hat{v}\zeta_1 - w_2 + \hat{v}\zeta_2\|_{L^2} \leq C|z|\|w_1 - w_2\|_{H^1}.$$ 

We bound $T_7$ by two terms, one for the negative and one for the positive half-line:

$$T_7 \leq \||Df(a(\gamma)v_\ast) - Df(v_\ast) - Df(R_\theta v_\infty) + Df(v_\infty)|(\zeta_1 - \zeta_2)\hat{v}\|_{L^2(R_-)}$$

$$+ \||Df(a(\gamma)v_\ast) - Df(v_\ast) - Df(R_\theta v_\infty) + Df(v_\infty)|(\zeta_1 - \zeta_2)\hat{v}\|_{L^2(R_+)} =: T_8 + T_9.$$
Now, (6.1), (6.2) imply
\[ T_8 \leq \| DF(a(\gamma)v_*) - DF(v_*) - DF(R_\theta v_\infty) + DF(v_\infty) \|_{L^\infty} | \zeta_1 - \zeta_2 | \| \hat{\nu} \|_{L^p_0(\mathbb{R})} \]
\[ \leq C |z| |\zeta_1 - \zeta_2| \leq C |z| \| \mathbf{w}_1 - \mathbf{w}_2 \|_{X_1^1}. \]

We use the abbreviations \( \chi_1(s) := v_* + s(a(\gamma)v_* - v_*) \), \( \chi_2(s) := v_\infty + s(R_\theta v_\infty - v_\infty) \), \( s \in [0,1] \) and (6.3), (6.4) to obtain
\[ T_9 = \| [DF(a(\gamma)v_*) - DF(v_*) - DF(R_\theta v_\infty) + DF(v_\infty)](\zeta_1 - \zeta_2) \hat{\nu} \|_{L^p_0(\mathbb{R})} \]
\[ \leq \| \int_0^1 D^2 f(\chi_1(s))[a(\gamma)v_* - v_*, (\zeta_1 - \zeta_2) \hat{\nu}] ds \]
\[ - \int_0^1 D^2 f(\chi_2(s))[R_\theta v_\infty - v_\infty, (\zeta_1 - \zeta_2) \hat{\nu}] ds \|_{L^p_0(\mathbb{R})} \]
\[ \leq \| \int_0^1 D^2 f(\chi_1(s))[a(\gamma)v_* - v_* - R_\theta v_\infty + v_\infty, (\zeta_1 - \zeta_2) \hat{\nu}] ds \|_{L^p_0(\mathbb{R})} \]
\[ + \| \int_0^1 [D^2 f(\chi_1(s)) - D^2 f(\chi_2(s))][R_\theta v_\infty + v_\infty, (\zeta_1 - \zeta_2) \hat{\nu}] ds \|_{L^p_0(\mathbb{R})} \]
\[ \leq C \left( \| a(\gamma)v_* - R_\theta v_\infty - (v_* - v_\infty) \|_{L^p_0(\mathbb{R})} \right) \]
\[ + \| \int_0^1 \chi_1(s) - \chi_2(s) ds \|_{L^p_0(\mathbb{R})} | R_\theta v_\infty - v_\infty | \| \zeta_1 - \zeta_2 | \]
\[ \leq C |z| \| \mathbf{w}_1 - \mathbf{w}_2 \|_{X_1^1}. \]

To estimate \( T_4 \) we use the abbreviations \( w(s) := w_2 + (w_1 - w_2) s \), \( \zeta(s) := \zeta_2 + (\zeta_1 - \zeta_2) s \), \( s \in [0,1] \) and obtain
\[ T_4 = \| \int_0^1 Df(a(\gamma)v_* + w(s)) - Df(a(\gamma)v_*) ds(w_1 - w_2) \]
\[ - \hat{\nu} \int_0^1 Df(R_\theta v_\infty + \zeta(s)) - Df(R_\theta v_\infty) ds(\zeta_1 - \zeta_2) \|_{L^p_0} \]
\[ \leq \| \int_0^1 Df(a(\gamma)v_* + w(s)) - Df(a(\gamma)v_*) ds(w_1 - \zeta_1 \hat{\nu} - w_2 + \zeta_2 \hat{\nu}) \|_{L^p} \]
\[ + \| \int_0^1 Df(a(\gamma)v_* + w(s)) - Df(a(\gamma)v_*) \]
\[ - Df(R_\theta v_\infty + \zeta(s)) + Df(R_\theta v_\infty) ds(\zeta_1 - \zeta_2) \hat{\nu} \|_{L^p} \]
\[ =: T_{10} + T_{11}. \]

Now for every \( s \in [0,1] \) we have
\[ \| DF(a(\gamma)v_* + w(s)) - DF(a(\gamma)v_*) \|_{L^\infty} \leq C \| w_2 + s(w_1 - w_2) \|_{L^\infty} \]
\[ \leq C \max \{ \| w_1 \|_{L^\infty}, \| w_2 \|_{L^\infty} \} \leq C \max \left\{ \| \mathbf{w}_1 \|_{X_1^1}, \| \mathbf{w}_2 \|_{X_1^1} \right\}, \]

where we used that the Sobolev embedding implies for \( i \in \{1,2\} \)
\[ \| w_i \|_{L^\infty} \leq \| w_i - \zeta_i \hat{\nu} \|_{L^\infty} + |\zeta_i| \leq C \| w_i - \zeta_i \hat{\nu} \|_{H^1} + |\zeta_i| \leq C \| \mathbf{w}_i \|_{X_1^1}. \]
Then (6.5) yields
\[
T_{10} \leq \int_0^1 \| Df(a(\gamma)v_* + w(s)) - Df(a(\gamma)v_*) \|_{L^\infty} ds \| w_1 - \zeta_1 \hat{v} - w_2 + \zeta_2 \hat{v} \|_{L^2_0}
\]
\[
\leq C \max \left\{ \| w_1 \|_{X^{1,1}_0}, \| w_2 \|_{X^{1,1}_0} \right\} \| w_1 - w_2 \|_{X^{1,1}_0}.
\]
Further,
\[
T_{11} \leq \left\| \int_0^1 Df(a(\gamma)v_* + w(s)) - Df(a(\gamma)v_*) \right. \\
\left. - Df(R_\theta v_\infty + \zeta(s)) + Df(R_\theta v_\infty) ds(\zeta_1 - \zeta_2) \hat{v} \right\|_{L^2_0(R^+)}
\]
\[
+ \left\| \int_0^1 Df(a(\gamma)v_* + w(s)) - Df(a(\gamma)v_*) \right. \\
\left. - Df(R_\theta v_\infty + \zeta(s)) + Df(R_\theta v_\infty) ds(\zeta_1 - \zeta_2) \hat{v} \right\|_{L^2_0(R^+)}
\]
\[
=: T_{12} + T_{13}.
\]
We write \( \kappa(s) := a(\gamma)v_* + w(s) - R_\theta v_\infty - \zeta(s) \). Then for \( s \in [0, 1] \) there holds
\[
\left\| Df(a(\gamma)v_* + w(s)) - Df(a(\gamma)v_*) - Df(R_\theta v_\infty + \zeta(s)) + Df(R_\theta v_\infty) \right\|_{L^\infty}
\]
\[
= \left\| \int_0^1 D^2 f(R_\theta v_\infty + \zeta(s) + \kappa(s) \tau) [\kappa(s), \cdot] \\
- D^2 f(R_\theta v_\infty + (a(\gamma)v_* - R_\theta v_\infty) \tau) [a(\gamma)v_* - R_\theta v_\infty, \cdot] d\tau \right\|_{L^\infty}
\]
\[
\leq \left\| \int_0^1 D^2 f(R_\theta v_\infty + \zeta(s) + \kappa(s) \tau) [w(s) - \zeta(s), \cdot] d\tau \right\|_{L^\infty}
\]
\[
+ \left\| \int_0^1 (D^2 f(R_\theta v_\infty + \zeta(s) + \kappa(s) \tau) \\
- D^2 f(R_\theta v_\infty + (a(\gamma)v_* - R_\theta v_\infty) \tau) [a(\gamma)v_* - R_\theta v_\infty, \cdot] d\tau \right\|_{L^\infty}
\]
\[
\leq C \| w(s) - \zeta(s) \|_{L^\infty} + C \int_0^1 \| \zeta(s) - (w(s) - \zeta(s)) \tau \|_{L^\infty} d\tau \leq C \max \left\{ \| w_1 \|_{L^\infty}, \| w_2 \|_{L^\infty} \right\}
\]
\[
\leq C \max \left\{ \| w_1 \|_{X^{1,1}_0}, \| w_2 \|_{X^{1,1}_0} \right\} \| \zeta_1 - \zeta_2 \|_1.
\]
where we used \( |\zeta_i| \leq \| w_i \|_{L^\infty}, i = 1, 2 \). So we conclude
\[
T_{12} \leq C \max \left\{ \| w_1 \|_{X^{1,1}_0}, \| w_2 \|_{X^{1,1}_0} \right\} |\zeta_1 - \zeta_2|.
\]
Similarly, for every \( s \in [0, 1] \),
\[
\left\| Df(a(\gamma)v_* + w(s)) - Df(a(\gamma)v_*) - Df(R_\theta v_\infty + \zeta(s)) + Df(R_\theta v_\infty) \right\|_{L^2_0(R^+)}
\]
\[
\leq C \| w(s) - \zeta(s) \|_{L^2_0(R^+)} + C \int_0^1 \| \zeta(s) - (w(s) - \zeta(s)) \tau \|_{L^\infty} d\tau \| a(\gamma)v_* - R_\theta v_\infty \|_{L^2_0(R^+)}
\]
This yields the estimate for $T_{13}$

$$T_{13} \leq C \max \left\{ \| w_1 \|_{X^1}, \| w_2 \|_{X^1} \right\} |\zeta_1 - \zeta_2|.$$

Summarizing, we have shown

$$T_2 = \| f(a(\gamma)v_* + w_1) - f(a(\gamma)v_* + w_2) - Df(v_*)(w_1 - w_2) - \hat{v}(R_0v_\infty + \zeta_1) - f(R_0v_\infty + \zeta_2) - Df(v_\infty)(\zeta_1 - \zeta_2) \|_{L^2_\beta} \leq C \left( |z| + \max \left\{ \| w_1 \|_{X^1}, \| w_2 \|_{X^1} \right\} \right) \| w_1 - w_2 \|_{X^1}.$$
For $I_4$ we have

\[ I_4 \leq C \left( \|a(\gamma)v_* - v_*\|_{L^\infty} + \max\{\|w_1\|_{L^\infty}, \|w_2\|_{L^\infty}\} \right) \|w_1 - w_2\|_{L^\infty} \|v_*\|_{L^2} \]

Hence

\[ T_3 = \left\| \left[ f(a(\gamma)v_* + w) - f(a(\gamma)v_* + w + Df(v_*)(w_1 - w_2) \right] \right\|_{L^2} \]

\[ \leq C \left( |z| + \max\{\|w_1\|_{X^1}, \|w_2\|_{X^1}\} \right) \|w_1 - w_2\|_{X^1}. \]

Finally we have shown

\[ \left\| r^{(r)}(z, w_1) - r^{(r)}(z, w_2) \right\|_{X^1} \leq C \left( |z| + \max\{\|w_1\|_{X^1}, \|w_2\|_{X^1}\} \right) \|w_1 - w_2\|_{X^1}. \]

**ii).** As in i) we frequently use the mean value theorem and the smoothness of $f$ which follows from Assumption 1.1. First, we estimate

\[ \|r^{(r)}(z_1, w) - r^{(r)}(z_2, w)\|_{X^1} = \left\| \left( f(a(\gamma)v_* + w) - f(a(\gamma)v_* + w + f(a(\gamma)v_*) \right) \right\|_{X^1} \]

\[ \leq |f(R_{\theta_1}v_* + \zeta) - f(R_{\theta_2}v_* + \zeta) + f(R_{\theta_1}v_* - f(R_{\theta_2}v_*))| \]

\[ + \|f(a(\gamma)v_* + w) - f(a(\gamma)v_* + w + f(a(\gamma)v_*) + f(R_{\theta_1}v_*) - f(R_{\theta_2}v_*)| L^2 \]

\[ + \left\| \partial_\zeta[f(a(\gamma)v_* + w) - f(a(\gamma)v_* + w)] \right\|_{L^2} + \left\| \partial_\zeta[f(a(\gamma)v_*) - f(a(\gamma)v_*)] \right\|_{L^2} \]

\[ =: J_1 + J_2 + J_3 + J_4 + J_5 + J_6. \]

The smoothness of $f$ implies

\[ J_1 = |f(R_{\theta_1}v_* + \zeta) - f(R_{\theta_2}v_* + \zeta)| \leq C|R_{\theta_1}v_* - R_{\theta_2}v_*| \leq C|z_1 - z_2|. \]

The same holds true for $\zeta = 0$ so that $J_2 \leq C|z_1 - z_2|$. Write $\kappa_1(s) := a(\gamma)v_* + w + (a(\gamma)v_* - a(\gamma)v_*)s$ and $\kappa_2(s) := R_{\theta_2}v_* + \zeta + (R_{\theta_1}v_* - R_{\theta_2}v_*)s$, $s \in [0, 1]$ and obtain for $J_3$,

\[ J_3 = \left\| |f(a(\gamma)v_* + w) - f(a(\gamma)v_* + w) - \hat{\nu}[f(R_{\theta_1}v_* + \zeta) + f(R_{\theta_2}v_* + \zeta)| L^2 \]

\[ = \left\| \int_0^1 Df(a(\gamma)v_* + w + (a(\gamma)v_* - a(\gamma)v_*)s)(a(\gamma)v_* - a(\gamma)v_*)) ds \right\|_{L^2} \]

\[ - \hat{\nu} \int_0^1 Df(R_{\theta_1}v_* + \zeta) + (R_{\theta_1}v_* - R_{\theta_2}v_*)s)(R_{\theta_1}v_* - R_{\theta_2}v_*) ds \right\|_{L^2} \]

\[ \leq \left\| \int_0^1 Df(\kappa_1(s))(a(\gamma)v_* - R_{\theta_1}v_*) - a(\gamma)v_* + R_{\theta_2}v_* \hat{\nu} ds \right\|_{L^2} \]

\[ + \left\| \int_0^1 [Df(\kappa_1(s)) - Df(\kappa_2(s)](R_{\theta_1}v_* - R_{\theta_2}v_*) \hat{\nu} ds \right\|_{L^2} =: J_7 + J_8. \]

We estimate $J_7$ by

\[ J_7 \leq C\|a(\gamma)v_* - R_{\theta_1}v_* \hat{\nu} - a(\gamma)v_* + R_{\theta_2}v_* \hat{\nu}\|_{L^2}. \]
\[
\leq C(a(x^{-1}(z_1))v_* - a(x^{-1}(z_2))v_*) \|x_n
\]
\[
\leq C|z_1 - z_2|
\]
and bound \(J_8\) by two terms
\[
J_8 \leq \left\| \int_0^1 \left[ Df(\kappa_1(s)) - Df(\kappa_2(s)) \right] (R_{\theta_1}v_{\infty} \hat{\nu} - R_{\theta_2}v_{\infty} \hat{\nu}) ds \right\|_{L^2(\mathbb{R}^n)}
\]
\[
+ \left\| \int_0^1 \left[ Df(\kappa_1(s)) - Df(\kappa_2(s)) \right] (R_{\theta_1}v_{\infty} \hat{\nu} - R_{\theta_2}v_{\infty} \hat{\nu}) ds \right\|_{L^2(\mathbb{R}^n)} = J_9 + J_{10}.
\]
Then
\[
J_9 \leq C\| \hat{\nu} \|_{L^2(\mathbb{R}^n)} |R_{\theta_1}v_{\infty} - R_{\theta_2}v_{\infty}| \leq C|z_1 - z_2|
\]
and for \(J_{10}\)
\[
J_{10} \leq C|R_{\theta_1}v_{\infty} - R_{\theta_2}v_{\infty}| \int_0^1 \| \kappa_1(s) - \kappa_2(s) \|_{L^2} ds \leq C|z_1 - z_2|.
\]
Thus we have shown \(J_8 \leq C|z_1 - z_2|\). In particular the estimates hold for \(w = 0, \zeta = 0\). Therefore we also have shown \(J_4 \leq C|z_1 - z_2|\) and it remains to estimate the spatial derivatives \(J_5\) and \(J_6\). We note that for arbitrary \(u \in L^2\) we have by Sobolev embedding
\[
\left\| \left[ Df(a(\gamma_1)v_* + w) - Df(a(\gamma_2)v_* + w) \right] u \right\|_{L^2} \leq C\|a(\gamma_1)v_* - a(\gamma_2)v_*\|_{L^\infty} \left\| u \right\|_{L^2}
\]
\[
\leq C\|u\|_{L^2} \left\| a(\gamma_1)v_* - R_{\theta_1}v_{\infty} \hat{\nu} - a(\gamma_2)v_* + R_{\theta_2}v_{\infty} \hat{\nu}\right\|_{L^\infty} + \left\| R_{\theta_1}v_{\infty} \hat{\nu} - R_{\theta_2}v_{\infty} \hat{\nu}\right\|_{L^\infty}
\]
\[
\leq C\left\| u \right\|_{L^2} |z_1 - z_2|.
\]
This implies
\[
J_5 \leq \left\| [Df(a(\gamma_1)v_* + w) - Df(a(\gamma_2)v_* + w)] w_* \right\|_{L^2}
\]
\[
+ \left\| Df(a(\gamma_1)v_* + w)a(\gamma_1)v_* - Df(a(\gamma_2)v_* + w)a(\gamma_2)v_* \right\|_{L^2}
\]
\[
\leq C\|w_*\|_{L^2} |z_1 - z_2| + \left\| [Df(a(\gamma_1)v_* + w) - Df(a(\gamma_2)v_* + w)] a(\gamma_1)v_* \right\|_{L^2}
\]
\[
+ C\|a(\gamma_1)v_* - a(\gamma_2)v_*\|_{L^2}
\]
\[
\leq C\left\| w_* \right\|_{L^2} + \left\| a(\gamma_1)v_* \right\|_{L^2} |z_1 - z_2| + C\|a(\gamma_1)v_* - a(\gamma_2)v_*\|_{L^2} \leq C|z_1 - z_2|.
\]
In particular the same holds true for \(w = 0\) and we observe \(J_6 \leq C|z_1 - z_2|\). Summarizing, we have shown
\[
\left\| r^{(f)}(z_1, w) - r^{(f)}(z_2, w) \right\|_{X^1} \leq C|z_1 - z_2|.
\]
iii). Since the group action is smooth and since \(P_\eta\) from (3.31) is a projector we have
\[
\left\| \left( (I - P_\eta) - (I - P_\eta)(a(\cdot)v_* \circ \chi^{-1})(z) S_\eta(z)^{-1} P_\eta \right) u \right\|_{X^1} \leq C\|u\|_{X^1} \forall u \in X^1.
\]
Now the claim follows from i).
iv). By the smoothness of the group action and Lemma 5.3 the function \((a(\cdot)v_* \circ \chi^{-1})(z) S_\eta(z)^{-1}\) is continuously differentiable in \(z\). Therefore, we obtain the local Lipschitz estimate
\[
\left\| (a(\cdot)v_* \circ \chi^{-1})(z_1) S_\eta(z_1)^{-1} w - (a(\cdot)v_* \circ \chi^{-1})(z_2) S_\eta(z_2)^{-1} w \right\|_{X^1} \leq C|z_1 - z_2| \|w\|_{X^1}.
\]
Then we use (6.6) and i) to see
\[
\| r^{[w]}(z_1, w) - r^{[w]}(z_2, w) \|_{X_R^1} \\
\leq C \| r^{[f]}(z_1, w) - r^{[f]}(z_2, w) \|_{X_R^1} \\
+ \| (a(\cdot)v_1 \circ \chi^{-1})(z_1)S_{\eta}(z_1)^{-1}P_{\eta}r^{[f]}(z_1, w) - (a(\cdot)v_1 \circ \chi^{-1})(z_2)S_{\eta}(z_2)^{-1}P_{\eta}r^{[f]}(z_1, w) \|_{X_R^1} \\
\leq C |z_1 - z_2|.
\]
Now we obtain using ii) and iii)
\[
\| r^{[w]}(z_1, w_1) - r^{[w]}(z_2, w_2) \|_{X_R^1} \\
\leq \| r^{[w]}(z_1, w_1) - r^{[w]}(z_2, w_1) \|_{X_R^1} + \| r^{[w]}(z_2, w_1) - r^{[w]}(z_2, w_2) \|_{X_R^1} \\
\leq C \left( |z_1 - z_2| + \| w_1 - w_2 \|_{X_R^1} \right).
\]

v). Similar to iv) we have by Lemma 5.3 that $S_{\eta}(z)^{-1}$ is locally Lipschitz w.r.t. $z$. Then we obtain
\[
|r^{[z]}(z_1, w_1) - r^{[z]}(z_2, w_2)| = |S_{\eta}(z_1)^{-1}P_{\eta}r^{[f]}(z_1, w_1) - S_{\eta}(z_2)^{-1}P_{\eta}r^{[f]}(z_2, w_2)| \\
\leq C \| r^{[f]}(z_1, w_1) - r^{[f]}(z_2, w_2) \|_{X_R^1} + \| (S_{\eta}(z_1)^{-1} - S_{\eta}(z_2)^{-1})P_{\eta}r^{[f]}(z_2, w_2) \| \\
\leq C_4 \left( |z_1 - z_2| + \| w_1 - w_2 \|_{X_R^1} \right).
\]

\[
\square
\]

7. Nonlinear stability

In this section we complete the proof of the main Theorem 2.5 according to the following strategy. For sufficiently small initial perturbation $v_0$ in (2.2) we show existence of a local mild solution of the corresponding integral equations of the decomposed system (5.10), (5.11) which reads as

\[
w(t) = e^{t\mathcal{L}_R}w_0 + \int_0^t e^{(t-s)\mathcal{L}_R}r^{[w]}(z(s), w(s))ds,
\]
\[
z(t) = z_0 + \int_0^t r^{[z]}(z(s), w(s))ds.
\]

A Gronwall estimate then shows that the solution exists for all times, that the perturbation $w$ decays exponentially and that $z$ converges to the coordinates of an asymptotic phase. Combining the results with the regularity theory for mild solutions we infer Theorem 2.5 and thus nonlinear stability of traveling oscillating fronts.

Lemma 7.1. Let Assumption 1.1, 1.2, 1.3 and 2.3 be satisfied and let $0 < \mu \leq \min(\mu_0, \mu_1)$ with $\mu_0$ from Theorem 3.5 and $\mu_1$ from Lemma 2.4. Then for every $0 < \varepsilon_1 < \delta$ and $0 < 2K\varepsilon_0 \leq \delta$ with $K$ from Theorem 4.1 and $\delta$ from Lemma 6.1, there exists $t_* = t_*(\varepsilon_0, \varepsilon_1, \mu) > 0$ such that for all initial values $(z_0, w_0) \in \mathbb{R}^2 \times V_R^1$ with
\[
\| w_0 \|_{X_R^1} < \varepsilon_0, \quad |z_0| < \varepsilon_1
\]
there exists a unique solution \((z, w) \in C([0, t_*], \mathbb{R}^2 \times V^1_\eta)\) of (7.1), (7.2) with
\[
\|w(t)\|_{X^1_\eta} \leq 2K\varepsilon_0, \quad |z(t)| \leq 2\varepsilon_1, \quad t \in [0, t_*).
\]
In particular, \(t_*\) is independent of \((z_0, w_0) \in B_{\varepsilon_1}(0) \times B_{\varepsilon_0}(0)\).

**Proof.** Take \(\nu = \nu(\mu) > 0\) from Theorem 4.1, \(C = C(\mu) > 0\) from Lemma 6.1 and let \(t_*\) be so small such that the following conditions are satisfied:
\[
\begin{align*}
(7.3) \quad t_* &< \frac{\varepsilon_1}{2C(\varepsilon_1 + K\varepsilon_0)}, \\
& \leq \frac{2K}{\nu}(1 - e^{-\nu t_*}) < \frac{1}{C}.
\end{align*}
\]

The proof employs a contraction argument in the space \(Z := C([0, t_*], \mathbb{R}^2 \times V^1_\eta)\) equipped with the supremum norm \(\|(z, w)\|_Z := \sup_{t \in [0, t_*]} \{|z(t)| + \|w(t)\|_{X^1_\eta}\}\). Define the map \(\Gamma : Z \to Z\) given by the right hand side of (7.1), (7.2). We show that \(\Gamma\) is a contraction on the closed set
\[
B := \{(z, w) \in Z : \|w(t)\|_{X^1_\eta} \leq 2K\varepsilon_0, \quad |z(t)| \leq 2\varepsilon_1, \quad t \in [0, t_*]\} \subset Z.
\]
Let \((z, w) \in B\). By using the estimates from Theorem 4.1, Lemma 6.1 and (7.3) we obtain for all \(0 \leq t < t_*\)
\[
\begin{align*}
\|e^{t\mathcal{L}_\eta}w_0 + \int_0^t e^{(t-s)\mathcal{L}_\eta}r[w](z(s), w(s))ds\|_{X^1_\eta}
& \leq Ke^{-\nu t}\varepsilon_0 + K\int_0^t e^{-\nu(t-s)}\|r[w](z(s), w(s))\|_{X^1_\eta}ds \\
& \leq Ke^{-\nu t}\varepsilon_0 + KC\int_0^t e^{-\nu(t-s)}\|w(s)\|_{X^1_\eta}ds \\
& \leq Ke^{\varepsilon_0} + \frac{2K^2C\varepsilon_0}{\nu}(1 - e^{-\nu t_*}) \leq 2K\varepsilon_0.
\end{align*}
\]
and
\[
\begin{align*}
|z_0 + \int_0^t r[z](z(s), w(s))ds| & \leq \varepsilon_1 + \int_0^t |r[z](z(s), w(s))| ds \\
& \leq \varepsilon_1 + C\int_0^t |z(s)| + \|w(s)\|_{X^1_\eta} ds \\
& \leq \varepsilon_1 + 2C(\varepsilon_1 + K\varepsilon_0)t_* \leq 2\varepsilon_1.
\end{align*}
\]
Hence \(\Gamma\) maps \(B\) into itself. Further, for \((z_1, w_1), (z_2, w_2) \in B\) and \(0 \leq t < t_*\) we can estimate
\[
\begin{align*}
\|\Gamma(z_1, w_1) - \Gamma(z_2, w_2)\|_Z
& \leq \sup_{t \in [0, t_*]} \left\{ \int_0^t \|r[z](z_1(s), w_1(s)) - r[z](z_2(s), w_2(s))\| ds \\
& \quad + \int_0^t Ke^{-\nu(t-s)}\|r[w](z_1(s), w_1(s)) - r[w](z_2(s), w_2(s))\|_{X^1_\eta} ds \right\} \\
& \leq \left(Ct_* + \frac{KC}{\nu}(1 - e^{-\nu t_*})\right) \|(z_1 - z_2, w_1 - w_2)\|_Z.
\end{align*}
\]
By condition (7.3), the map $\Gamma$ is a contraction on $B$ and the assertion follows from the contraction mapping theorem.

We use the following Gronwall lemma from [4, Lemma 6.3].

**Lemma 7.2.** Suppose $\varepsilon, \nu, C, \tilde{C} > 0$ such that

$$C \geq 1, \quad \varepsilon \leq \frac{\nu}{16CC}$$

and let $\varphi \in C([0, t_\infty), [0, \infty))$ for some $0 < t_\infty \leq \infty$ satisfying

$$\varphi(t) \leq C\varepsilon e^{-\nu t} + \tilde{C} \int_0^t e^{-\nu(t-s)} (\varphi(s)^2 + \varepsilon \varphi(s)) \, ds, \quad \forall t \in [0, t_\infty).$$

Then for all $0 \leq t < t_\infty$ there holds

$$\varphi(t) \leq 2C\varepsilon e^{-\frac{3}{2}\nu t}.$$

Next we prove the stability result for the $(z, w)$-systems (7.1), (7.2) and (5.10), (5.11). The Gronwall estimate ensures that the solution from Lemma 7.1 can not reach the boundary of the region of existence and therefore exists for all times. Moreover, the perturbation $w$ of the TOF decays exponentially. Regularity of the solution will follow by standard results from [2] and [12]. As in [2], we denote by $C^\alpha, \alpha \in (0, 1)$ the space of H"{o}lder continuous functions and by $C^{1,\alpha}$ the space of differentiable functions with H"{o}lder continuous derivative.

**Theorem 7.3.** Let Assumption 1.1, 1.2, 1.3 and 2.3 be satisfied and let $0 < \mu \leq \min(\mu_0, \mu_1)$ with $\mu_0$ from Theorem 3.5 and $\mu_1$ from Lemma 2.4. Then there are $\varepsilon(\mu), \beta(\mu) > 0$ such that for all initial values $(z_0, w_0) \in \mathbb{R}^2 \times V^2_\eta$ with $\|w_0\|_{\mathbb{R}^2 \times X^1_\eta} < \varepsilon$ the following statements hold:

i) There are unique

$$w \in C^\alpha((0, \infty), V^2_\eta) \cap C^{1,\alpha}((0, \infty), V_\eta), \quad z \in C^1([0, \infty), \mathbb{R}^2),$$

for arbitrary $\alpha \in (0, 1)$, satisfying (5.10) in $X_\eta$ and (5.11) in $\mathbb{R}^2$.

ii) There exist $z_\infty = z_\infty(z_0, w_0) \in \mathbb{R}^2$ and $K_0 = K_0(\mu) \geq 1$ such that for all $t \geq 0$

$$\|w(t)\|_{X^1_\eta} + |z(t) - z_\infty| \leq K_0 e^{-\beta t} \|(z_0, w_0)\|_{\mathbb{R}^2 \times X^1_\eta}, \quad |z_\infty| \leq (K_0 + 1) \|(z_0, w_0)\|_{\mathbb{R}^2 \times X^1_\eta}.$$

**Proof.** Recall the constants $K = K(\mu), \nu = \nu(\mu)$ from Theorem 4.1 and $C = C(\mu), \delta = \delta(\mu)$ from Lemma 6.1. We choose $C_0, \varepsilon, \bar{\varepsilon} > 0$ such that $0 < 2K\bar{\varepsilon} < \delta$ and

$$\varepsilon < \min \left( \frac{\delta}{C_0}, \frac{\bar{\varepsilon}}{4K}, \frac{\nu}{16K^2CC_0} \right), \quad C_0 > 2 + \frac{16CK}{3\nu}. \tag{7.4}$$

Let us abbreviate $\xi_0 := \|(z_0, w_0)\|_{\mathbb{R}^2 \times X^1_\eta} < \varepsilon$ and set

$$t_\infty := \sup \left\{ T > 0 : \exists! (z, w) \in C([0, T], \mathbb{R}^2 \times V_\eta) \text{ satisfying (7.1), (7.2) on } [0, T] \right\}.$$

and

$$\|w(t)\|_{X^1_\eta} \leq K\bar{\varepsilon}, \quad |z(t)| \leq C_0\xi_0, \quad t \in [0, T).$$
Then Lemma 7.1 with \( \varepsilon_0 = \bar{\varepsilon} \) and \( \varepsilon_1 = \frac{C_0\bar{\varepsilon}}{2} < \delta \) implies \( t_\infty \geq t_* = t_4(\varepsilon_0, \varepsilon_1, \mu) \) and we denote the unique solution by \( (z, w) \). Using Theorem 4.1 and Lemma 6.1 we estimate for all \( 0 \leq t < t_\infty \)

\[
\|w(t)\|_{X^1_0} \leq \|e^{tL_0}w_0\|_{X^1_0} + \int_0^t \|e^{(t-s)L_0}r^{[w]}(z(s), w(s))\|_{X^1_0} ds
\]

\[
\leq Ke^{-\nu t}\|w_0\|_{X^1_0} + \int_0^t e^{-\nu(t-s)}\|r^{[w]}(z(s), w(s))\|_{X^1_0} ds
\]

\[
\leq Ke^{-\nu t}\|w_0\|_{X^1_0} + KC \int_0^t e^{-\nu(t-s)}(\|z(s)\| + \|w(s)\|_{X^1_0}) \|w(s)\|_{X^1_0} ds
\]

\[
\leq Ke^{-\nu t}\xi_0 + KCC_0 \int_0^t e^{-\nu(t-s)}(\xi_0 + \|w(s)\|_{X^1_0}) \|w(s)\|_{X^1_0} ds.
\]

Then the Gronwall estimate in Lemma 7.2 implies due to (7.4)

\[
\|w(t)\|_{X^1_0} \leq 2Ke^{-\frac{3}{2}\nu t}\xi_0 < 2Ke^{-\frac{3}{2}\nu t}\bar{\varepsilon} < \frac{\bar{\varepsilon}}{2}, \quad t \in [0, t_\infty).
\]

This yields

\[
|z(t)| \leq |z_0| + \int_0^t |r^{[z]}(z(s), w(s))| ds \leq \xi_0 + C \int_0^t \|w(s)\|_{X^1_0} ds
\]

\[
\leq \xi_0 + 2KC\xi_0 \int_0^t e^{-\frac{3}{2}\nu s} ds \leq \xi_0 + \frac{8\nu C \xi_0}{3\nu} < \frac{C_0\xi_0}{2}, \quad t \in [0, t_\infty).
\]

We show that \( t_\infty < \infty \) leads to a contradiction. The estimates (7.5), (7.6) imply

\[
\|w(t_\infty - \frac{1}{2}t_*)\|_{X^1_0} < \frac{\bar{\varepsilon}}{2} = \varepsilon_0, \quad |z(t_\infty - \frac{1}{2}t_*)| < \frac{C_0\xi_0}{2} = \varepsilon_1.
\]

Now we can apply Lemma 7.1 once again to the integral equations (7.1), (7.2) with \( w_0 = w(t_\infty - \frac{1}{2}t_*) \) and \( z_0 = z(t_\infty - \frac{1}{2}t_*) \) and obtain a solution \( (\tilde{z}, \tilde{w}) \) on \( [0, t_*) \) with

\[
\tilde{w}(0) = w(t_\infty - \frac{1}{2}t_*), \quad \|w(t)\|_{X^1_0} \leq K\bar{\varepsilon}, \quad t \in [0, t_*)
\]

\[
\tilde{z}(0) = z(t_\infty - \frac{1}{2}t_*), \quad |z(t)| \leq C_0\xi_0, \quad t \in [0, t_*).
\]

Define

\[
(\tilde{z}, \tilde{w})(t) := \begin{cases} (z, w)(t), & t \in [0, t_\infty - \frac{1}{2}t_*] \\ (\tilde{z}, \tilde{w})(t - t_\infty + \frac{1}{2}t_*) & t \in (t_\infty - \frac{1}{2}t_* , t_\infty + \frac{1}{2}t_*). \end{cases}
\]

Then \( (\tilde{z}, \tilde{w}) \) is a solution on \( [0, t_\infty + \frac{1}{2}t_*) \) with \( \|\tilde{w}(t)\|_{X^1_0} \leq K\bar{\varepsilon} \) and \( |\tilde{z}(t)| \leq C_0\xi_0 \). This contradicts the definition of \( t_\infty \). Hence \( t_\infty = \infty \) and (7.5) holds on \( [0, \infty) \). Further, we see that the integral

\[
z_\infty := z_0 + \int_0^\infty r^{[z]}(z(s), w(s)) ds
\]

exists since

\[
|z(t) - z_\infty| \leq \int_t^\infty |r^{[z]}(z(s), w(s))| ds
\]
\[ \leq C \int_t^\infty \|w(s)\|_{X^1_t} \leq 2KC\xi_0 \int_t^\infty e^{-\frac{2}{3}\nu t} ds = \frac{8KC}{3\nu} e^{-\frac{2}{3}\nu t} \xi_0. \]

Thus the first estimate in ii) is proven with \( K_0 = 2K + \frac{8KC}{3\nu} \) and \( \tilde{\beta} = \frac{2}{3} \nu \). The second estimate is obtained by
\[ |z_\infty| \leq |z(0) - z_\infty| + |z_0| \leq (K_0 + 1)\xi_0. \]

It remains to show the regularity of \((z, w)\). By Lemma 7.1 one infers \( r^{|z|}(z(\cdot), w(\cdot)) \in C((0, \infty), \mathbb{R}^2) \) and thus \( z \in C^1((0, \infty), \mathbb{R}^2) \). Furthermore, let \( r(t) := r^{|w|}(z(t), w(t)) \). Suppose \( 0 \leq s \leq t < \infty \). Then by Lemma 7.1 we find some \( C_r > 0 \) such that
\[ \|r(t) - r(s)\|_{X^1_\eta} = \|r^{|w|}(z(t), w(t)) - r^{|w|}(z(s), w(s))\|_{X^1_\eta} \]
\[ \leq C \left( |z(t) - z(s)| + \|w(t) - w(s)\|_{X^1_\eta} \right) \]
\[ \leq C \left( \int_s^t |r^{|z|}(z(\sigma), w(\sigma))| d\sigma + \int_s^t |r^{|w|}(z(\sigma), w(\sigma))|_{X^1_\eta} d\sigma \right) \]
\[ \leq C \left( \int_s^t \|w(\sigma)\|_{X^1_\eta} d\sigma + C \int_s^t |z(\sigma)| + \|w(\sigma)\|_{X^1_\eta} d\sigma \right) \leq C_r (t - s). \]

This implies \( r \in C^\alpha([0, \infty), X_\eta) \) for every \( \alpha \in (0, 1) \) and for arbitrary \( s > 0 \),
\[ \int_0^s \|r(t)\|_{X^1_\eta} dt = \int_0^s \|r^{|w|}(z(t), w(t))\|_{X^1_\eta} dt \leq C \int_0^s \|w(t)\|_{X^1_\eta} dt < \infty. \]

Now the regularity of \( w \) is a consequence of the well known theory of semilinear parabolic equations and can be concluded, for instance, using [2, Thm. 1.2.1] [12, Thm. 3.2.2].

We conclude with the

Proof of Theorem 2.5. We choose \( \mu_0 \) from Theorem 3.5 and possibly decrease it further such that \( \mu_0 \leq \mu_1 \) with \( \mu_1 \) from Lemma 2.4. We take the sets \( V, W \) from Lemma 5.2 and let \( \delta > 0 \) be so small that the ball \( B_\delta = \{ u \in X_\eta : \|u\|_{X_\eta} \leq \delta \} \) is contained in the image of \( V \) under \( T_\eta \) and its projection \( P_\eta(B_\delta) \) in the image of \( W \) under \( \Pi_\eta \), i.e. \( B_\delta \subset T_\eta(V) \) and \( P_\eta(B_\delta) \subset \Pi_\eta(W) \). Then the inverse maps \( T_\eta^{-1}, \Pi_\eta^{-1} \) exist on \( B_\delta \), respectively \( P_\eta(B_\delta) \), and are diffeomorphic. Moreover, let
\[ C_{\Pi} := \sup_{v \in B_\delta} \frac{\|\Pi_\eta^{-1}(P_\eta v)\|}{\|v\|_{X^1_\eta}} \]
and, since the group action is smooth, we find \( C \geq 1 \) such that
\[ \|a(\chi^{-1}(z_1))v_* - a(\chi^{-1}(z_2))v_*\|_{X^1_\eta} \leq C|z_1 - z_2| \quad \forall z_1, z_2 \in \Pi_\eta^{-1}(P_\eta(B_\delta)). \]

Decrease \( \varepsilon > 0 \) from Theorem 7.3 such that the solution \((z, w)\) of (5.10), (5.11) for initial values smaller than \( \varepsilon \) satisfy \( w(t) \in T_\eta^{-1}(B_\delta) \) and \( z(t) \in \Pi_\eta^{-1}(P_\eta(B_\delta)) \) for all \( t \in [0, \infty) \).

We restrict the size of the initial perturbation \( v_0 \) by the condition
\[ \varepsilon_0 < \min \left( \frac{\varepsilon}{C_{\Pi}(1 + C)}, \frac{\pi}{2K_0 + 1}, \frac{\delta}{2K_0(3C + 1)} \right) \]
with \( K_0 \) from Theorem 7.3. The initial values for the \((z, w)\)-system are defined by
\[ (z_0, w_0) := T_\eta^{-1}(v_0) = \left( \Pi_\eta^{-1}(P_\eta v_0), v_0 + v_* - a(\chi^{-1}(z_0))v_* \right). \]
Then $|z_0| \leq C \|v_0\|_{X_\eta}$ holds and
\[
\| (z_0, w_0) \|_{\mathbb{R}^d \times X_\eta} \leq |z_0| + \| a(\chi^{-1}(z_0)) v_* - v_* \|_{X_\eta} + \| v_0 \|_{X_\eta} \leq C \Pi(1 + C) \varepsilon_0 + \varepsilon_0 < \varepsilon.
\]
Thus, by Theorem 7.3, there are $z \in C^1([0, \infty), \mathbb{R}^2)$ and $w \in C([0, \infty), V^2_\eta \cap C^1((0, \infty), V_\eta))$ such that $(z, w)$ solves (5.10), (5.11) with $z(0) = z_0$, $w(0) = w_0$ and
\[
\| w(t) \|_{X_\eta} \leq K \varepsilon_0, \quad |z(t)| \leq |z(t) - z_\infty| + |z_\infty| \leq (2K_0 + 1) \varepsilon_0 < \pi, \quad t \in [0, \infty).
\]
Hence, $z(t)$ lies in the chart $(U, \chi)$ for all $t \in [0, \infty)$ and we can define $\gamma = \chi^{-1}(z) \in C^1([0, \infty), G)$. Set
\[
u(t) = a(\gamma(t)) v_* + w(t), \quad t \in [0, \infty).
\]
Then $\nu \in C([0, \infty), Y_\eta) \cap C^1([0, \infty), X_\eta)$ and by Lemma 5.2 and the construction of the decomposition in section 5, we conclude $\nu_\eta = F(\nu)$ and $\nu(0) = v_* + v_0$.

With $\gamma_\infty = \chi^{-1}(z_\infty)$ we have by Theorem 7.3,
\[
\| w(t) \|_{X_\eta} + |\gamma(t) - \gamma_\infty|_G = \| w(t) \|_{X_\eta} + |z(t) - z_\infty|
\leq K_0 e^{-\beta t} \| (z_0, w_0) \|_{\mathbb{R}^d \times X_\eta} \leq K e^{-\beta t} \| v_0 \|_{X_\eta},
\]
where $K = C \Pi(1 + C) K_0 + K_0$. We further estimate the asymptotic phase,
\[
|\gamma_\infty|_G \leq |\gamma_0|_G + |\gamma_0 - \gamma_\infty|_G = |z_0| + |z_0 - z_\infty|
\leq C \Pi \| v_0 \|_{X_\eta} + K_0 \| (z_0, w_0) \|_{\mathbb{R}^d \times X_\eta} \leq C_\infty \| v_0 \|_{X_\eta}
\]
with $C_\infty = C \Pi(1 + K_0) + K_0(1 + C \Pi)$. Finally, we show uniqueness of $\nu$. First note
\[
\| \nu(t) - v_* \|_{X_\eta} \leq C |z(t) - z_\infty| + \| w(t) \|_{X_\eta} + C |z_\infty| \leq (3C + 1) K_0 \varepsilon_0 \leq \frac{\delta}{2}.
\]
Assume there is another solution $\tilde{\nu}$ of (2.2) on $[0, T)$ for some $T > 0$. Let
\[
\tau := \sup \{ t \in [0, T) : \| \tilde{\nu} - v_* \|_{X_\eta} \leq \delta \}
\]
Then there is a solution $(\tilde{z}, \tilde{w})$ of (5.10), (5.11) on $[0, \tau)$ such that $T_\eta(\tilde{z}(t), \tilde{w}(t)) = \tilde{\nu}(t) - v_*$. and, therefore, $\tilde{\nu}(t) = a(\tilde{\gamma}(t)) v_* + \tilde{w}(t), \tilde{\gamma}(t) = \chi^{-1}(\tilde{z}(t))$. But since $(z, w)$ is unique we conclude $(\tilde{z}, \tilde{w}) = (z, w)$ and $\nu(t) = \tilde{\nu}(t)$ on $[0, \tau)$. Now assume $\tau < T$. Then we have
\[
\frac{\delta}{2} \geq \| \nu(t) - v_* \|_{X_\eta} = \| \tilde{\nu}(t) - v_* \|_{X_\eta} \quad \text{for all } t \in [0, \tau).
\]
Since the right-hand side converges to $\delta$ as $t \to \tau$, we arrive at a contradiction. \qed

8. Appendix

Consider the differential operator
\[
L_0 u = Au'' + cu',
\]
where $c > 0$ and $A \in \mathbb{R}^{m,m}$ satisfies $\text{Re}(\lambda) > 0$ for all $\lambda \in \sigma(A)$.

**Lemma 8.1** (Limits of solutions). Let $r \in C(\mathbb{R}, \mathbb{R}^m)$ have limits $\lim_{x \to \pm \infty} r(x)$ and let $v \in C^2(\mathbb{R}, \mathbb{R}^m)$ be a bounded solution of $L_0 v = r$. Then the following limits exist and vanish
\[
\lim_{x \to \pm \infty} r(x) = 0 = \lim_{x \to \pm \infty} v'(x) = \lim_{x \to \pm \infty} v''(x).
\]
Proof. Consider first \( x \geq 0 \). Then we can write \( v \) for some \( a_1, a_2 \in \mathbb{R}^n \) as
\[
(8.1) \quad v(x) = Y_1(x)a_1 + Y_2(x)a_2 + v_3(x),
\]
where \( Y_1(x) = I \) and \( Y_2(x) = \exp(-cA^{-1}x) \) form a fundamental system for \( L_0 \) and \( v_3 \) solves \( L_0v_3 = r, \ v_3(0) = v_3'(0) = 0 \), i.e.
\[
(8.2) \quad v_3(x) = \int_0^\infty G(x, \xi)r(\xi)d\xi, \quad G(x, \xi) = \begin{cases} \frac{1}{c}(I - Y_2(x - \xi)), & 0 \leq \xi \leq x, \\ 0, & 0 \leq x < \xi. \end{cases}
\]
By the positivity of \( A \) and \( c \) we have \( |Y_2(x)| \leq C \exp(-bx), x \geq 0 \) for some \( b > 0 \). Since \( v, Y_1, Y_2 \) are bounded on \( \mathbb{R}_+ \), so is \( v_3 \). If \( r_+ = \lim_{x \to \infty} r(x) \neq 0 \) then we have the following lower bound for \( 0 < x_0 < x \)
\[
|v_3(x)| \geq \left| \int_{x_0}^x G(x, \xi)d\xi r_+ \right| - \int_0^{x_0} G(x, \xi)r(\xi)d\xi - \int_{x_0}^x G(x, \xi)(r(\xi) - r_+)d\xi \geq c^{-1}(x - x_0)|r_+| - 2C|A|c^{-1}|r_+| - (1 + C)\|r\|_{L^\infty_{x_0}}(x - x_0)(1 + C)\sup_{\xi \geq x_0}|r(\xi) - r_+|.
\]
The last term can be absorbed into the first term by taking \( x_0 \) large, and the resulting term dominates the middle terms as \( x \to \infty \). Hence \( v_3 \) is unbounded and we arrive at a contradiction. For the derivative we find
\[
v'(x) = Y_2'(x)a_2 - \frac{1}{c} \int_0^x Y_2'(x - \xi)r(\xi)d\xi,
\]
which together with \( r_+ = 0 \) and the exponential decay of \( Y_2' \) yields \( \lim_{x \to \infty} v'(x) = 0 \). Instead of considering \( L_0 \) on \( \mathbb{R}_- \) we reflect domains and consider \( L_0 \) on \( \mathbb{R}_+ \) but now with \( c < 0 \). Formulas (8.1) and (8.2) still hold but with the Green’s function given by
\[
G(x, \xi) = \frac{1}{c} \begin{cases} I - \exp(cA^{-1}\xi), & 0 \leq \xi \leq x, \\ \exp(cA^{-1}(\xi - x)) - \exp(cA^{-1}\xi), & 0 \leq x < \xi. \end{cases}
\]
Note that \( c < 0 \) implies an estimate
\[
|G(x, \xi)| \leq C \begin{cases} 1, & 0 \leq \xi \leq x, \\ \exp(-b(\xi - x)), & 0 \leq x < \xi. \end{cases}
\]
Hence the integral in (8.2) converges and provides a linear upper bound for \( v_3(x) \). Since \( Y_2(x)a_2 \) grows exponentially if \( a_2 \neq 0 \), we obtain \( a_2 = 0 \) from the boundedness of \( v \). As in case \( c > 0 \) we then derive a linear lower bound for \( |v_3(x)| \) if \( r_+ \neq 0 \). In this way, we find again \( r_+ = 0 \) and then \( \lim_{x \to \infty} v'(x) = 0 \) from
\[
v_3'(x) = -A^{-1} \int_x^\infty \exp(cA^{-1}(\xi - x))r(\xi)d\xi.
\]

Proof of Lemma 2.1. The TOF \( v_* \) satisfies
\[
L_0v_* = -S_wv_* - f(v_*) = r,
\]
hence Lemma 8.1 shows \( \lim_{x \to \pm \infty} v'_*(x) = \lim_{x \to \pm \infty} v''_*(x) = 0 \) as well as
\[
0 = \lim_{x \to \infty} r(x) = -S_wv_\infty - f(v_\infty) = -(S_w + g(|v_\infty|^2))v_\infty.
\]
This is the real version of the complex equation \((i \omega + G(|V_\infty|^2))V_\infty = 0\), so that \(S_\omega + g(\|v_\infty\|^2) = 0\) follows.

\[\square\]

**Proof of Theorem 2.2.** The profile \(v_*\) is a solution of (1.6) and \(f \in C^3\) by Assumption 1.1. Therefore \(v_* \in C^3_0(\mathbb{R}, \mathbb{R}^2)\). For the estimate on \(\mathbb{R}_-\) we transform (1.6) into a 4-dimensional first order system with \(w = (w_1, w_2)^T, w_1 = v_*, w_2 = v'_*.\) Then \(w\) solves

\[
(8.3) \quad w' = \mathcal{H}(w), \quad \mathcal{H}(w) = \begin{pmatrix} 0 \\ -A^{-1}(cw_2 + S_\omega w_1 + f(w_1)) \end{pmatrix}
\]

and \(w = (v_*, v'_*)^T \to 0\) as \(x \to -\infty\) (cf. Lemma 2.1). Now zero is an equilibrium of (8.3) with

\[
D\mathcal{H}(0) = \begin{pmatrix} 0 & I_2 \\ -A^{-1}(S_\omega + Df(0)) & -cA^{-1} \end{pmatrix}, \quad Df(0) = \begin{pmatrix} g_1(0) & -g_2(0) \\ g_2(0) & g_1(0) \end{pmatrix}.
\]

One can show that Assumption 1.1 implies zero to be a hyperbolic equilibrium of (8.3) with local stable and unstable manifolds of dimension 2. Since convergence to hyperbolic equilibria is known to be exponentially fast (cf. [24, Theorem 7.6]), we conclude the desired estimate on \(\mathbb{R}_-\).

For the estimate on \(\mathbb{R}_+\) we use an ansatz from [25] with polar coordinates,

\[
(8.4) \quad v_*(x) = r(x) \begin{pmatrix} \cos \phi(x) \\ \sin \phi(x) \end{pmatrix},
\]

and introduce the new variables \(q := \phi\) and \(\kappa := \frac{r'}{r}\). Plugging the ansatz (8.4) into (1.6) then gives the equation for \((r, q, \kappa)\),

\[
(8.5) \quad \begin{pmatrix} \frac{r}{r} \quad q \quad \kappa \end{pmatrix}' = \begin{pmatrix} q^2 - \kappa^2 & -2\kappa q \\ -2\kappa q & -A^{-1}(c\kappa + g_1(|r|^2)) \end{pmatrix} =: \Gamma(r, q, \kappa).
\]

We define \(r_\infty := |v_\infty|\) and \(\phi_\infty := \arg(v_\infty)\). Then we have \(r \to r_\infty\) as \(x \to \infty\), since \(v_* \to v_\infty\) as \(x \to \infty\). In addition, \(v'_* \to 0\) as \(x \to \infty\), by Lemma 2.1, which implies \(r' \to 0\) as \(x \to \infty\).

Therefore we obtain \(\kappa = \frac{r'}{r} \to 0\) as \(x \to \infty\) and further

\[
r' \left(\frac{\cos \phi}{\sin \phi}\right) + rq \left(\frac{-\sin \phi}{\cos \phi}\right) = v'_* \to 0, \quad x \to \infty.
\]

This shows \(q \to 0\). Summarizing we have \((r, \kappa, q) \to (r_\infty, 0, 0)\) as \(x \to \infty\). Now one verifies that \((r_\infty, 0, 0)\) is a hyperbolic equilibrium of (8.5) with stable manifold of dimension equal to 2 and unstable manifold of dimension equal to 1. Again since convergence to hyperbolic equilibria is known to be exponentially fast (cf. [24, Theorem 7.6]), we find \(K_0, \mu_*> 0\) such that for \(x \geq 0\),

\[
|(r', \kappa', q')| = |\Gamma(r, \kappa, q) - \Gamma(r_\infty, 0, 0)| \leq C|(r, \kappa, q) - (r_\infty, 0, 0)| \leq K_0 e^{-\mu_* x}
\]

where we use the fact that \(\Gamma \in C^1\) by Assumption 1.1. Finally we find \(K > 0\) such that

\[
|v_*(x) - v_\infty| + |v'_*(x)| \leq |r(x) \begin{pmatrix} \cos \phi(x) \\ \sin \phi(x) \end{pmatrix} - r_\infty \begin{pmatrix} \cos \phi_\infty \\ \sin \phi_\infty \end{pmatrix}| + |r'(x)| + |r(x)q(x)| 
\]

\[
\leq |r(x) - r_\infty| + |r_\infty||\phi(x) - \phi_\infty| + |r'(x)| + \|r\|_{L_\infty} |g(x)|
\]
\[
\leq |r(x) - r_\infty| + |r_\infty| \int_x^\infty |q(x)| dx + |r'(x)| + \|r\|_{L^\infty} |q(x)| \\
\leq Ke^{-\mu x}
\]
for all \( x \geq 0 \). Since \( f \in C^3 \), the estimates for \( \nu^\prime \) and \( \nu^\prime\prime \) then follow by differentiating (1.5).

Proof of Lemma 5.1. We note that translations on \( L^2_\eta \) are continuous and the estimate \( \|v(x - \tau)\|_{L^2_\eta} \leq e^{\mu |\tau|} \|v\|_{L^2_\eta} \) for all \( v \in L^2_\eta \) holds. Further, if \( v \in H^1_\eta \) it is straightforward to show \( \|v(x - \tau) - v\|_{L^2_\eta} \leq |\tau| e^{\mu |\tau|} \|v_x\|_{L^2_\eta} \) and the same holds true if \( v \) is replaced by the template function \( \hat{\upsilon} \). Using these facts and invariance under rotation of the norms we obtain continuity of the group action on \( X_\eta \) by

\[
\|a(\gamma) v\|_{X_\eta} \leq |\rho| + \|v(x - \tau) - \rho \hat{\upsilon}\|_{L^2_\eta} \leq |\rho| + \|v(x - \tau) - \rho \hat{\upsilon}(x - \tau)\|_{L^2_\eta} + |\rho| \|\hat{\upsilon}(x - \tau) - \hat{\upsilon}\|_{L^2_\eta} \\
\leq |\rho| + e^{\mu |\tau|} (\|v - \rho \hat{\upsilon}\|_{L^2_\eta} + |\rho| |\tau| \|\hat{\upsilon}_x\|_{L^2_\eta}) \leq C \|v\|_{X_\eta}.
\]

Using the continuity of translations on \( L^2_\eta \) once again yields \( \|a(\gamma) v\|_{Y_\eta} \leq C \|v\|_{Y_\eta} \). It is easy to verify the properties \( a(\gamma_1) a(\gamma_2) = a(\gamma_1 \circ \gamma_2) \) and \( a(\gamma)^{-1} = a(\gamma^{-1}) \) so that \( a(\cdot) \in GL[X_\eta] \) is a homomorphism. The continuity of the group action in \( G \) for \( v \in X_\eta \) follows by

\[
\|a(\gamma) v - v\|_{X_\eta}^2 = \|a(\gamma) v - v\|_{X_\eta}^2 + \sum_{a=1}^2 \|R_\theta \partial^a v(x - \tau) - \partial^a v\|_{L^2_\eta}^2 \\
\leq |R_\theta \rho - \rho| + \|R_\theta v(x - \tau) - R_\theta \rho \hat{\upsilon} - (v - \rho \hat{\upsilon})\|_{L^2_\eta} \\
\leq |R_\theta \rho - \rho| + \|R_\theta v(x - \tau) - \rho \hat{\upsilon}\|_{L^2_\eta} + \|R_\theta (v - \rho \hat{\upsilon})\|_{L^2_\eta} \\
\leq |R_\theta - I| \left( |\rho| + \|v - \rho \hat{\upsilon}\|_{L^2_\eta} \right) + \|v(x - \tau) - v\|_{L^2_\eta} \to 0 \quad \text{as} \quad (\theta, \tau) \to 0.
\]

Similarly, for \( v \in Y_\eta \) we have

\[
\|a(\gamma) v - v\|_{Y_\eta}^2 = \|a(\gamma) v - v\|_{X_\eta}^2 + \sum_{a=1}^2 \|R_\theta \partial^a v(x - \tau) - \partial^a v\|_{L^2_\eta}^2 \\
\to 0 \quad \text{as} \quad (\theta, \tau) \to 0.
\]

It is left to show that \( a(\cdot) v \) is of class \( C^1 \) for \( v \in Y_\eta \) and to compute its derivative. For this purpose it suffices to prove the assertion at \( \gamma = 1 \) (0, 0). Let us take \( h = (h_1, h_2) \in \mathbb{R}^2 \) small such that \( \chi^{-1}(h) \in U \). Then

\[
\|a(\chi^{-1}(h)) v - v - h_1 S_1 v + h_2 v_x\|_{X_\eta} \\
\leq \|R_{-h_1} \rho - \rho + h_1 S_1 \rho\|_{L^2_\eta} + \|R_{-h_1} (v - h_2) - \rho \hat{\upsilon}) - (v - \rho \hat{\upsilon}) + h_1 S_1 (v - \rho \hat{\upsilon}) + h_2 v_x\|_{L^2_\eta}.
\]

Since \( \partial_\theta R_\theta \rho |_{\theta=0} = S_1 \rho \), the first term is \( o(|h|) \). The second term is less obvious. We frequently add zero and split into serveral terms

\[
\|R_{-h_1} (v - h_2) - \rho \hat{\upsilon}) - (v - \rho \hat{\upsilon}) + h_1 S_1 (v - \rho \hat{\upsilon}) + h_2 v_x\|_{L^2_\eta} \\
\leq \|R_{-h_1} (v - \rho \hat{\upsilon})(-h_2) - (v - \rho \hat{\upsilon})(-h_2) + h_1 S_1 (v - \rho \hat{\upsilon})(-h_2) + h_1 S_1 (v - \rho \hat{\upsilon})(h_2) \|_{L^2_\eta} \\
+ \|R_{-h_1} \rho \hat{\upsilon}(-h_2) - R_{-h_1} \rho \hat{\upsilon} + (v - \rho \hat{\upsilon})(-h_2) - h_1 S_1 (v - \rho \hat{\upsilon})(h_2) \|_{L^2_\eta} \\
- (v - \rho \hat{\upsilon}) + h_1 S_1 (v - \rho \hat{\upsilon}) + h_2 v_x\|_{L^2_\eta} \\
\leq T_1 + \|R_{-h_1} (v - \rho \hat{\upsilon})(-h_2) - (v - \rho \hat{\upsilon}) + h_2 v_x - \rho \hat{\upsilon} x\|_{L^2_\eta} \\
+ \|h_2 \rho \hat{\upsilon} x + R_{-h_1} \rho \hat{\upsilon}(-h_2) - R_{-h_1} \rho \hat{\upsilon} + h_1 S_1 (v - \rho \hat{\upsilon})(-h_2) + h_1 S_1 (v - \rho \hat{\upsilon})\|_{L^2_\eta} \\
\leq T_1 + T_2 + \|R_{-h_1} [\rho \hat{\upsilon}(-h_2) - \rho \hat{\upsilon} + h_2 \rho \hat{\upsilon} x]\|_{L^2_\eta}
\]
\[ + \|h_2 \rho \hat{v}_x - h_2 R_{-h_1} \rho \hat{v}_x - h_1 S_1 (v - \rho \hat{v}) (- h_2) + h_1 S_1 (v - \rho \hat{v}) 1\|_{L_2}^2 \leq T_1 + T_2 + T_3 + \| - h_1 S_1 (v - \rho \hat{v}) (- h_2) + h_1 S_1 (v - \rho \hat{v}) - h_2 (h_1 S_1 v_x - h_1 S_1 \rho \hat{v}_x) 1\|_{L_2}^2 \]
\[ + \|h_2 \rho \hat{v}_x - h_2 R_{-h_1} \rho \hat{v}_x + h_2 h_1 S_1 v_x - h_2 h_1 S_1 \rho \hat{v}_x 1\|_{L_2}^2 \leq T_1 + T_2 + T_3 + T_4 + T_5 + T_6. \]

Now \( T_1, T_5 = o(|h|) \) holds since rotations are smooth and \( \partial_\theta R_{\theta} \rho_{\theta=0} = S_1 \rho \). Further, \( T_6 = o(|h|) \) is obvious. Finally \( T_2, T_3, T_4 = o(|h|) \) hold, since translations on \( H_1^1 \) are smooth and therefore \( \|v(\cdot - \tau) - v + hv_x\| = o(|h|) \) for \( v \in H_1^1 \). This completes the proof. \( \square \)

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