ON THE ORDER AND THE TYPE OF AN ENTIRE FUNCTION

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(Received May 10, 2022; accepted September 12, 2022)

Abstract. In this short article we present some properties regarding the order and the type of an entire function.

1. Introduction

Let

\[ g(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{C}, \]

be an entire function and

\[ M(r) = M_g(r) := \sup_{|z| \leq r} |g(z)| = \max_{|z|=r} |g(z)|, \quad r > 0, \]

its maximum modulus.

Recall that the order of \( g(z) \) is the quantity

\[ \rho = \rho(g) := \limsup_{r \to \infty} \frac{\ln \ln M(r)}{\ln r} \]

(see [1]). In other words, the order \( \rho \) of \( g(z) \) is the smallest exponent \( \rho' \geq 0 \) such that for any given \( \varepsilon > 0 \) there is a \( r_0 = r_0(\varepsilon) > 0 \) for which

\[ |g(z)| \leq \exp(|z|^\rho' + \varepsilon) \quad \text{whenever} \quad |z| \geq r_0. \]

Clearly, \( 0 \leq \rho \leq \infty. \)

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Key words and phrases: entire function, order, type.

Mathematics Subject Classification: 30D20.

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Let us also recall [1] that if $0 < \rho < \infty$, the quantity

$$
\tau = \tau(g) := \limsup_{r \to \infty} \frac{\ln M(r)}{r^\rho}
$$

is the type (of the order) of $g(z)$. In other words, $\tau$ is the smallest number $\tau' \geq 0$ such that for any given $\varepsilon > 0$ there is a $r_0 = r_0(\varepsilon) > 0$ for which

$$
|g(z)| \leq \exp \left( (\tau' + \varepsilon)|z|^\rho \right) \quad \text{whenever } |z| \geq r_0.
$$

Clearly, $0 \leq \tau \leq \infty$. If $\tau = 0$, we say that $g(z)$ is of minimal type, whereas if $\tau = \infty$, we say that $g(z)$ is of maximal type. In the extreme cases where $\rho = 0$ or $\rho = \infty$ the type is not defined.

It follows easily from (4) and (6) that for any fixed $\zeta \in \mathbb{C}$ the entire functions

$$
g(z) \text{ and } g_\zeta(z) := g(z + \zeta)
$$

have the same order and type.

A well-known fact of complex analysis [1] is that the order $\rho$ and the type $\tau$ of $g(z)$ are given by the formulas

$$
\rho = \limsup_n \frac{n \ln n}{-\ln |a_n|} \quad \text{and} \quad \tau = \frac{1}{e\rho} \limsup_n n|a_n|^\rho/n
$$

respectively, where $a_n, n = 0, 1, \ldots$, are the coefficients of the power series of $g(z)$ as seen in (1).

**Remark 1.1.** (i) Suppose $0 \leq \rho < \infty$. Then (8) implies that for all sufficiently large $n$ we have

$$
\frac{n \ln n}{-\ln |a_n|} \leq \rho + \varepsilon_n
$$

for some sequence $\varepsilon_n$ of positive numbers with $\varepsilon_n \to 0$.

Formula (10) can be written as

$$
\ln \left( |a_n|^{1/n} \right) \leq -\frac{\ln n}{\rho + \varepsilon_n}
$$

and for $\rho' > \rho$ (11) yields

$$
\ln \left( n|a_n|^\rho'/n \right) \leq -\left( \frac{\rho'}{\rho + \varepsilon_n} - 1 \right) \ln n,
$$
which implies (since, eventually, $\rho' > \rho + \varepsilon_n$)

$$\lim_{n} n |a_n|^{\rho'/n} = 0 \quad \text{for } \rho' > \rho. \quad (13)$$

(ii) Suppose $0 < \rho \leq \infty$. Then from (8) we have that there is a subsequence $a_{n_k}$ such that

$$\lim_{k} \frac{n_k \ln n_k}{-\ln |a_{n_k}|} = \rho. \quad (14)$$

It follows that

$$\frac{\ln |a_{n_k}|}{n_k \ln n_k} = -\frac{1}{\rho} + o(1) \iff \ln \left( |a_{n_k}|^{\rho/n_k} \right) = -\frac{1}{\rho} \ln n_k + o(\ln n_k). \quad (15)$$

Thus, for $0 < \rho' < \rho$ formula (15) yields

$$\ln \left( n_k |a_{n_k}|^{\rho'/n_k} \right) = \left( 1 - \frac{\rho'}{\rho} \right) \ln n + o(\ln n_k). \quad (16)$$

Therefore, $\lim_{k} n_k |a_{n_k}|^{\rho'/n_k} = \infty$ and, consequently,

$$\limsup_{n} n |a_n|^{\rho'/n} = \infty \quad \text{for } \rho' \in (0, \rho). \quad (17)$$

Formulas (13) and (17) should be compared with formula (9).

**Remark 1.2.** Let $g(z)$ of (1) be an entire function of order $\rho \in (0, \infty)$, so that its type $\tau$ is defined, with $0 \leq \tau \leq \infty$.

Suppose $0 < \tau < \infty$ and let $a_{n_k}$ be a subsequence for which the $\limsup$ in (9) is attained, i.e.,

$$\tau = \frac{1}{e\rho} \lim_{k} n_k |a_{n_k}|^{\rho/n_k}. \quad (18)$$

Then,

$$|a_{n_k}|^{\rho/n_k} = \frac{e\rho \tau}{n_k} + o\left( \frac{1}{n_k} \right) = \frac{e\rho \tau}{n_k} \left[ 1 + o(1) \right]. \quad (19)$$

Taking logarithms in (19) yields

$$\frac{\rho}{n_k} \ln |a_{n_k}| = \ln(e\rho \tau) - \ln n_k + o(1) \quad (20)$$

or

$$\frac{\rho \ln |a_{n_k}|}{n_k \ln n_k} = -1 + O\left( \frac{1}{\ln n_k} \right). \quad (21)$$
Therefore,

\[ \lim_{k} \frac{n_k \ln n_k}{k - \ln |a_{n_k}|} = \rho, \]

i.e., the lim sup in (8) is also attained for the same subsequence \( a_{n_k} \).

Notice that the converse is not true in general. If \( a_{n_k} \) is a subsequence for which the lim sup in (8) is attained, then the lim sup in (9) may not be attained for \( a_{n_k} \) (e.g., take \( g(z) = \sin z + \cos(2z) \) and \( a_{n_k} = a_{2k+1} \)).

If \( \tau = \infty \) and \( a_{n_k} \) is a subsequence for which the lim sup in (9) is attained, then for any \( M > 0 \) we have

\[ |a_{n_k}|^{\rho/n_k} \geq \frac{M}{n_k} \]

for all sufficiently large \( k \), which implies

\[ \frac{\rho \ln |a_{n_k}|}{n_k \ln n_k} \geq -1 + \frac{\ln M}{\ln n_k} \]

from which it follows that, again, the lim sup in (8) is also attained for the same subsequence \( a_{n_k} \).

Finally, if \( \tau = 0 \), then (9) implies that

\[ \lim_{n} n|a_n|^\rho/n = 0. \]

In this case, though, there may exist subsequences \( a_{n_k} \) for which the lim sup in (8) is not attained. For instance, let \( g(z) = g_e(z) + g_o(z) \), where \( g_e(z) \) is an even entire function of order \( \rho \) and type 0, while \( g_o(z) \) is an odd entire function of order less than \( \rho \). Then \( g(z) \) is of order \( \rho \) and 0 type, but the lim sup in (8) is not attained for the subsequence \( a_{n_k} = a_{2k+1} \).

Another tool that we will need in the sequel is the operator \( (.)^z \) defined as

\[ g^z(z) := \sum_{n=0}^{\infty} |a_n|z^n, \]

when \( g(z) \) is the entire function of (1). Notice that for any \( r > 0 \) we have

\[ \max_{|z| \leq r} |g(z)| \leq g^z(r) = \max_{|z| \leq r} |g^z(z)| \]

(the inequality can be strict). Furthermore, since, in view of (8) and (9), the order and type of \( g(z) \) depend only on the sequence of the absolute values \( \{|a_n|\}_{n \geq 0} \), they remain invariant under \( (.)^z \), i.e.,

\[ \rho(g^z) = \rho(g) \quad \text{and} \quad \tau(g^z) = \tau(g). \]

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Also,

\[(g')^\sharp(z) = \sum_{n=0}^{\infty} |n a_n| z^{n-1} = \sum_{n=0}^{\infty} n |a_n| z^{n-1} = (g^\sharp)'(z),\]

i.e., \((. )^\sharp\) commutes with the derivative operator.

Let us now set

\[a_n(z) := \frac{g^{(n)}(z)}{n!} \quad n = 0, 1, \ldots\]

(so that \(a_n(0) = a_n\)). Then, in view of (7), formulas (8), (9), and (30) yield

\[\rho = \limsup_n \frac{n \ln n}{-\ln |a_n(z)|}\]

and (in the case where \(0 < \rho < \infty\))

\[\tau = \frac{1}{e \rho} \limsup_n n |a_n(z)|^{\rho/n} = \frac{e^{\rho-1}}{\rho} \limsup_n n^{1-\rho} |g^{(n)}(z)|^{\rho/n},\]

respectively, independently of the complex number \(z\). An interesting question here is to inquire into the dependence on \(z\) of the subsequence(s) of \(\{a_n(z)\}\) for which the \(\limsup\) is attained in (31) and (32).

Let us also notice that we can write (31) in the following equivalent form (since \(\lim_n |a_n(z)| = 0\) and, hence, \(-\ln |a_n(z)|\) is eventually positive):

\[e^{-1/\rho} = \limsup_n |a_n(z)|^{\frac{1}{n \ln n}}\]

or, in view of (30) and the fact that \(\lim_n n!^{\frac{1}{n \ln n}} = e\),

\[\theta = \theta(\rho) := e^{1-1/\rho} = \limsup_n |g^{(n)}(z)|^{\frac{1}{n \ln n}}.\]

Clearly, \(\theta = \theta(\rho)\) is smooth and strictly increasing on \([0, +\infty]\), with \(\theta(0) := \theta(0^+) = 0\) and \(\theta(+\infty) = e\).

Also, if

\[\theta^\sharp = \limsup_n \left| (g^\sharp)^{(n)}(z) \right|^{\frac{1}{n \ln n}}, \quad z \in \mathbb{C},\]

then we obviously have \(\theta^\sharp = \theta\) since, as we have seen, \(\rho(g^\sharp) = \rho(g) = \rho\). Thus, if we set

\[m_n(r) := \max_{|z| \leq r} |g^{(n)}(z)|, \quad r > 0,\]

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then, clearly, \( m_n(r) \leq (g^{(n)}z)(r) = (g^z)^{(n)}(r) \) and, therefore

\[
\theta = \limsup_n \left[ m_n(r) \right]^{\frac{1}{\ln n}} = \limsup_n \left[ (g^z)^{(n)}(r) \right]^{\frac{1}{\ln n}}, \quad r > 0.
\]

Likewise, in view of (32), since \( \tau(g^z) = \tau(g) = \tau \) (in the case where \( 0 < \rho < \infty \)) we have

\[
\tau = \frac{e^{\rho-1}}{\rho} \limsup_n n^{1-\rho} \left[ m_n(r) \right]^{\rho/n}
\]

\[
= \frac{e^{\rho-1}}{\rho} \limsup_n n^{1-\rho} \left[ (g^z)^{(n)}(r) \right]^{\rho/n}, \quad r > 0.
\]

Now, let \( \nu = \{n_k\}_{k=1}^{\infty} \) be a sequence of indices, namely a strictly increasing sequence of positive integers. In the present work we are interested in the quantities

\[
\rho_\nu(z) := \limsup_k \frac{n_k \ln n_k}{-\ln |a_{n_k}(z)|}, \quad z \in \mathbb{C}
\]

and

\[
\tau_\nu(z) := \frac{1}{e^\rho} \limsup_k n_k |a_{n_k}(z)|^{\rho/n_k} \quad z \in \mathbb{C}.
\]

Let \( \text{Ran}(\nu) := \{n_1, n_2, \ldots\} \) be the range of \( \nu \) and \( \mathbb{N} := \{1, 2, \ldots\} \) the set of natural numbers. If \( \mathbb{N} \setminus \text{Ran}(\nu) \) is a finite set, then it is clear from (31) and (32) that \( \rho_\nu(z) = \rho \) and (if \( 0 < \rho < \infty \)) \( \tau_\nu(z) = \tau \) for all \( z \in \mathbb{C} \). Therefore, to make things interesting we assume that \( \nu \) is a proper sequence of indices, namely a sequence of indices such that the set \( \mathbb{N} \setminus \text{Ran}(\nu) \) is infinite. In this case it is obvious that there is a unique proper sequence of indices \( \mu = \{m_\ell\}_{\ell=1}^{\infty} \) such that \( \text{Ran}(\mu) \cap \text{Ran}(\nu) = \emptyset \) and \( \text{Ran}(\mu) \cup \text{Ran}(\nu) = \mathbb{N} \). We will call \( \mu \) the complementary sequence of \( \nu \). It is clear that for every \( z \in \mathbb{C} \) we have

\[
\rho = \max \{\rho_\nu(z), \rho_\mu(z)\}
\]

and

\[
\tau = \max \{\tau_\nu(z), \tau_\mu(z)\}.
\]

Let us summarize the main results of the paper: Under the assumption that the sequence of indices \( \nu = \{n_k\}_{k=1}^{\infty} \) satisfies the growth condition \( n_{k+1}/n_k \to 1 \), in Section 2 we show that \( \rho_\nu(z) = \rho \) for almost every \( z \in \mathbb{C} \) and in Section 3 we show that \( \tau_\nu(z) = \tau \) for almost every \( z \in \mathbb{C} \), provided
that $\tau < \infty$; if $\tau = \infty$, then there is a dense $G_\delta$ subset $S$ of $\mathbb{C}$ such that $\tau_\nu(z) = \infty$ for $z \in S$.

These results, apart from being interesting per se, could be used in the study of entire solutions of partial differential equations.

2. Properties of the order

For a proper sequence of indices $\nu = \{n_k\}_{k=1}^\infty$, we set (in the spirit of (34))

\begin{equation}
\theta_\nu(z) := \exp \left(1 - \frac{1}{\rho_\nu(z)}\right) = \limsup_k \left|g^{(n_k)}(z)\right|^{\frac{1}{n_k \ln n_k}}, \quad z \in \mathbb{C}.
\end{equation}

Thus, if $\mu$ is the complementary sequence of $\nu$, then (41) gives

\begin{equation}
0 \leq \theta = \max\{\theta_\nu(z), \theta_\mu(z)\} \leq e.
\end{equation}

We wish to determine how close the quantity $\rho_\nu(z)$ to the order $\rho$ of $g(z)$ is; or, equivalently, how close the quantity $\theta_\nu(z)$ to the constant $\theta$ of (34) is.

Recall that a function $\phi(z)$, defined in a domain $\Omega$ of the complex plane and taking values in $\mathbb{R} \cup \{-\infty\}$, is called subharmonic (in $\Omega$) if it is locally integrable and for any disk $D_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\} \subset \Omega$ we have

\begin{equation}
\phi(z_0) \leq \frac{1}{\pi r^2} \int_{D_r(z_0)} \phi(z) \, dx \, dy
\end{equation}

(here, of course, $z = x + iy = (x, y)$ and the function $\phi(z)$ is subharmonic with respect to the real variables $x$ and $y$). Some authors require (45) to hold for almost every $z_0 \in \Omega$, in order to completely characterize subharmonic functions as functions whose distributional Laplacian is nonnegative [2]. However, such variants of the definition of subharmonicity are nonessential for our analysis.

If $A(z)$ is analytic in a domain $\Omega \subset \mathbb{C}$, then $\ln |A(z)|$ is subharmonic in $\Omega$ (this follows, e.g., from the facts that (i) $\ln |z - z_0|$ is subharmonic and (ii) if $A(z)$ does not vanish in $\Omega$, then $\ln |A(z)|$ is harmonic). Also, since $h(x) = e^{\alpha x}$, $x \in \mathbb{R}$, is convex for any $\alpha > 0$, Jensen’s inequality implies that $|A(z)|^\alpha = e^{\alpha \ln |A(z)|}$ too is subharmonic for any $\alpha > 0$.

**Lemma 2.1.** The function $\theta_\nu(z)$ defined by (43) is subharmonic in $\mathbb{C}$.

**Proof.** Let us set

\begin{equation}
\Phi_n(z) := \sup_{k \geq n} \left|g^{(n_k)}(z)\right|^{\frac{1}{n_k \ln n_k}}, \quad n \geq 2.
\end{equation}
Fix an \( r > 0 \) and restrict \( z \in D_r := D_r(0) = \{ z : |z| \leq r \} \). Then, \(|g^{(n_k)}(z)| \leq (g^*)(n_k)(r)\). Furthermore, as we have seen,
\[
\theta = \limsup_n \left[ \frac{1}{\ln n} \left( g^*(n)(r) \right) \right] \leq e.
\]
It follows that there is an \( M = M(r) \geq 0 \), such that \( \Phi_n(z) \) of (46) is at most \( M \) for all \( n \geq 2 \) and all \( z \in D_r \).

From the discussion preceding Lemma 2.1 we know that \( |g^{(n_k)}(z)|^{\frac{1}{n_k \ln n_k}} \) is subharmonic for any \( k \geq 2 \). It follows easily that \( \Phi_n(z) \) is subharmonic in \( D_r \) for every \( n \geq 2 \) (being finite and the supremum of a sequence of subharmonic functions). Furthermore, it is obvious that \( \Phi_n(z) \) decreases with \( n \) and, in view of (43),
\[
\theta_\nu(z) = \lim_n \Phi_n(z), \quad z \in D_r.
\]
Therefore, by a simple application of the bounded convergence theorem, we can conclude that \( \theta_\nu(z) \) is subharmonic in \( D_r \) and, consequently, since \( r \) is arbitrary, that \( \theta_\nu(z) \) is subharmonic in \( \mathbb{C} \).

**Remark 2.2.** It is a well-known fact (see, e.g., [2]) that a subharmonic function \( \phi(z) \) in a domain \( \Omega \) is equal to an upper semicontinuous function \( \hat{\phi}(z) \) for almost every (a.e.) \( z \in \Omega \). Therefore, Lemma 2.1 implies
\[
\theta_\nu(z) = \hat{\theta}_\nu(z) \quad \text{for a.e.} \quad z \in \mathbb{C},
\]
where \( \hat{\theta}_\nu(z) \) is upper semicontinuous in \( \mathbb{C} \).

**Example 2.3.** Suppose \( n_k = 2k, \, k = 1, 2, \ldots \), and \( g(z) = \sin(\lambda z) \), where \( \lambda \in \mathbb{C} \setminus \{0\} \). Then, \( \rho = 1 \) and \( \tau = |\lambda| \). Furthermore, since
\[
g^{(2k)}(z) = (-1)^k \lambda^{2k} \sin(\lambda z),
\]
we have, in view of (43),
\[
\theta_\nu(z) = \begin{cases} 
1, & \lambda z / \pi \in \mathbb{C} \setminus \mathbb{Z}; \\
0, & \lambda z / \pi \in \mathbb{Z},
\end{cases}
\]
where \( \mathbb{Z} \) is the set of integers. Obviously, \( \theta_\nu(z) \) is subharmonic and it is equal to \( \hat{\theta}_\nu(z) \equiv 1 \) for all except for countably many \( z \in \mathbb{C} \).

We are now ready for the main result of the section.

**Theorem 2.4.** Let \( \nu = \{n_k\}_{k=1}^\infty \) be a proper sequence of indices of subexponential growth, namely
\[
\frac{n_{k+1}}{n_k} \to 1 \quad \text{as} \quad k \to \infty.
\]
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Then, for an entire function \( g(z) \) we have

\[
\theta_{\nu}(z) = \theta \quad \text{for a.e. } z \in \mathbb{C},
\]

where \( \theta \) and \( \theta_{\nu}(z) \) are as in (34) and (43), respectively.

**Proof.** By Lemma 2.1 we have that \( \theta_{\nu}(z) \) is subharmonic in \( \mathbb{C} \). Hence, in view of (45) we must have

\[
\theta_{\nu}(w) \leq \frac{1}{\pi r^2} \int_{D_r(w)} \theta_{\nu}(z) \, dx \, dy \leq \theta
\]

for any \( w \in \mathbb{C} \) and any \( r > 0 \). Therefore, if for some \( w \in \mathbb{C} \) we have that \( \theta_{\nu}(w) = \theta \), then formula (52) implies that \( \theta_{\nu}(z) = \theta \) for a.e. \( z \in D_r(w) \), which in turn implies \( \theta_{\nu}(z) = \theta \) for a.e. \( z \in \mathbb{C} \), since \( r \) is arbitrary.

More generally, let us only assume that the supremum of \( \theta_{\nu}(z) \) on some compact subset of \( \mathbb{C} \) is \( \theta \), namely that there is a sequence \( \{z_n\}_{n=1}^{\infty} \) with \( \lim_n z_n = z^* \in \mathbb{C} \) and \( \lim_n \theta_{\nu}(z_n) = \theta \). We will show that we must again have \( \theta_{\nu}(z) = \theta \) for a.e. \( z \in \mathbb{C} \).

Fix a disk \( D_r(z^*) \) and consider the disks \( D_n := D_{r_n}(z_n) \), \( n = 1, 2, \ldots \), so that \( r_n \) is the largest radius satisfying \( D_n \subset D_r(z^*) \). Using \( w = z_n \) and \( D_r(w) = D_n \) in (52) yields

\[
\theta_{\nu}(z_n) \leq \frac{1}{\pi r_n^2} \int_{D_n} \theta_{\nu}(z) \, dx \, dy \leq \frac{1}{\pi r_n^2} \int_{D_r(z^*)} \theta_{\nu}(z) \, dx \, dy \leq \frac{r^2}{r_n^2} \theta, \quad n \geq 1,
\]

thus, by letting \( n \to \infty \) we obtain

\[
\theta \leq \frac{1}{\pi r^2} \int_{D_r(z^*)} \theta_{\nu}(z) \, dx \, dy \leq \theta,
\]

which tells us that \( \theta_{\nu}(z) = \theta \) for a.e. \( z \in D_r(z^*) \) and, consequently, that \( \theta_{\nu}(z) = \theta \) for a.e. \( z \in \mathbb{C} \).

Finally, we will show that the assumption

\[
\Theta_{\nu}(r) := \sup_{|z| \leq r} \theta_{\nu}(z) < \theta \quad \text{for every } r > 0
\]

leads to a contradiction.

For a given \( r > 0 \) let us assume (55) and fix an \( \varepsilon > 0 \) so that

\[
\Theta_{\nu}(r) + \varepsilon < \theta.
\]

Then, by (43) and (55) we get that for every \( z \in D_r(0) \) there is an integer \( K = K(z) \) such that

\[
\sup_{k \geq K(z)} \left| g^{(n_k)}(z) \right|^{1/n_k} < \Theta_{\nu}(r) + \varepsilon.
\]

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It follows that if we set
\begin{equation}
G_j := \left\{ z \in D_r(0) : \sup_{k \geq j} \left| g^{(n_k)}(z) \right|^{\frac{1}{n_k}} \mathbin{\in} k \mathbin{\in} n_k \Theta_{\nu}(r) + \varepsilon \right\},
\end{equation}
then
\begin{equation}
\bigcup_{j=1}^{\infty} G_j = D_r(0).
\end{equation}
Thus, there is a \( j_0 \) such that the set \( G_{j_0} \) has positive (Lebesgue) measure.

Now, as we have seen in the proof of Lemma 2.1, the function
\begin{equation}
\phi(z) := \sup_{k \geq j_0} \left| g^{(n_k)}(z) \right|^{\frac{1}{n_k}}
\end{equation}
is subharmonic. Thus, as we have mentioned in Remark 2.2 there is an upper semicontinuous function \( \hat{\phi}(z) \) such that \( \phi(z) = \hat{\phi}(z) \) for a.e. \( z \in D_r(0) \). Therefore, the sets
\begin{align*}
G_{j_0} &= \left\{ z \in D_r(0) : \phi(z) < \Theta_{\nu}(r) + \varepsilon \right\} \\
\hat{G} := \left\{ z \in D_r(0) : \hat{\phi}(z) < \Theta_{\nu}(r) + \varepsilon \right\}
\end{align*}
differ by a set of measure 0, i.e., the set \( \hat{G} \setminus G_{j_0} \) has zero (Lebesgue) measure. Furthermore, the upper semicontinuity of \( \hat{\phi}(z) \) implies \cite{2} that \( \hat{G} \) is open (and nonempty since \( G_{j_0} \) has positive measure). Therefore, any open disk \( D_\delta(z_0) \subset \hat{G} \) lies almost entirely in \( G_{j_0} \) in the sense that their symmetric difference has measure 0 (in other words, the area of \( D_\delta(z_0) \cap G_{j_0} \) is equal to the area of \( D_\delta(z_0) \), namely \( \pi \delta^2 \)).

We continue by noticing that the assumption (55) implies that \( \theta_{\nu}(z) < \theta \) for all \( z \in \mathbb{C} \) and, hence, we must have
\begin{equation}
\theta_{\mu}(z) \equiv \theta,
\end{equation}
where \( \mu = \{m_\ell\}_{\ell=1}^{\infty} \) is the complementary sequence of \( \nu \).

Let \( D_\delta(z_0) \) be a disk (with \( \delta > 0 \)) such that \( D_\delta(z_0) \subset \hat{G} \). If \( \Gamma \) is the boundary of \( D_\delta(z_0) \), then due to the previous discussion we can arrange it so that the symmetric difference of \( G_{j_0} \) and \( \Gamma \) has one-dimensional measure 0 (in other words, the “length” (i.e., the one-dimensional measure) of \( \Gamma \cap G_{j_0} \) is equal to the length of \( \Gamma \), namely \( 2\pi \delta \)).

Now, by Cauchy’s integral formula we have
\begin{equation}
g^{(m_\ell)}(z_0) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{g^{(n_k)}(z)}{(z - z_0)^{m_\ell - n_k[\varepsilon] + 1}} \, dz,
\end{equation}
where
\[(63) \quad k[\ell] := \max\{k : n_k < m_\ell\}.\]

Taking absolute values in (62) yields
\[(64) \quad |g^{(m_\ell)}(z_0)| \leq \frac{1}{2\pi \delta^{m_\ell-n_k[\ell]+1}} \oint_{\Gamma} |g^{(n_k[\ell])}(z)| ds,\]

where \(ds\) is the arc-length element of \(\Gamma\).

By (58) we have
\[(65) \quad |g^{(n_k[\ell])}(z)| < [\Theta_\nu(r) + \varepsilon]^{n_k[\ell] \ln n_k[\ell]} \quad \text{for all} \quad k \geq j_0 \quad \text{and a.e.} \quad z \in \Gamma.\]

Thus, by using (65) in (64) we obtain
\[(66) \quad |g^{(m_\ell)}(z_0)| \leq \frac{1}{2\pi \delta^{m_\ell-n_k[\ell]+1}} \oint_{\Gamma} [\Theta_\nu(r) + \varepsilon]^{n_k[\ell] \ln n_k[\ell]} ds
= \frac{[\Theta_\nu(r) + \varepsilon]^{n_k[\ell] \ln n_k[\ell]}}{\delta^{m_\ell-n_k[\ell]}}, \quad k \geq j_0,\]
or
\[(67) \quad |g^{(m_\ell)}(z_0)| \frac{1}{m_\ell \ln m_\ell} \leq \left(\frac{1}{\delta}\right)^{m_\ell-n_k[\ell]} \frac{[\Theta_\nu(r) + \varepsilon]^{n_k[\ell] \ln n_k[\ell]}}{m_\ell \ln m_\ell}, \quad k \geq j_0.\]

Now, it is clear that
\[(68) \quad \frac{m_\ell - n_k[\ell]}{m_\ell \ln m_\ell} \to 0 \quad \text{as} \quad \ell \to \infty.\]

Also, our assumption (50) for \(n_k\) together with the fact that \(n_k[\ell] < m_\ell < n_k[\ell]+1\) imply
\[(69) \quad \frac{n_k[\ell] \ln n_k[\ell]}{m_\ell \ln m_\ell} \to 1 \quad \text{as} \quad \ell \to \infty.\]

Therefore, in view of (68) and (69), formula (67) yields
\[(70) \quad \theta_\mu(z_0) = \limsup_k |g^{(2k+1)}(z_0)| \frac{1}{2k \ln k} \leq \Theta_\nu(r) + \varepsilon < \theta,\]

which contradicts (61). Hence, the assumption (55) is false and we must have \(\sup_{|z| \leq r} \theta_\nu(z) = \theta\) for some \(r > 0\), which, as we have seen earlier in the proof, implies \(\theta_\nu(z) = \theta\) for a.e. \(z \in \mathbb{C}\). \(\square\)
Remark 2.5. Suppose the complementary sequence $\mu$ of $\nu$ is also subexponential. Then, in view of (43), Theorem 2.4 implies immediately that if

$$(71) \quad \mathcal{F}_\nu := \{ z \in \mathbb{C} : \rho_\nu(z) = \rho \} \quad \text{and} \quad \mathcal{F}_\mu := \{ z \in \mathbb{C} : \rho_\mu(z) = \rho \},$$

where $\rho$ is the order of $g(z)$ ($0 \leq \rho \leq \infty$) and the quantities $\rho_\nu(z)$ and $\rho_\mu(z)$ are defined by (39), then both sets $\mathcal{F}_\nu$ and $\mathcal{F}_\mu$ have full measure (and by formula (41) we have $\mathcal{F}_\nu \cup \mathcal{F}_\mu = \mathbb{C}$); in other words, the sets $\mathcal{F}_\nu^c := \mathbb{C} \setminus \mathcal{F}_\nu$ and $\mathcal{F}_\mu^c := \mathbb{C} \setminus \mathcal{F}_\mu$ have Lebesgue measure (i.e. area) zero.

Open Question. Are the sets $\mathcal{F}_\nu^c$ and $\mathcal{F}_\mu^c$ nowhere dense in $\mathbb{C}$? Are they countable?

3. Properties of the type

Now we turn our attention to the type of $g(z)$. Of course, we need to assume that $0 < \rho < \infty$.

Let $\nu = \{ n_k \}_{k=1}^{\infty}$ and $\mu = \{ m_\ell \}_{\ell=1}^{\infty}$ be two complementary sequences of indices. Then, in view of (40) we have

$$\tau_\nu(z) = \frac{1}{e^\rho} \limsup_k n_k |a_{n_k}(z)|^{\rho/n_k}$$

$$= \frac{e^{\rho-1}}{\rho} \limsup_k n_k^{1-\rho} |g^{(n_k)}(z)|^{\rho/n_k}, \quad z \in \mathbb{C},$$

and

$$\tau_\mu(z) := \frac{1}{e^\rho} \limsup_\ell m_\ell |a_{m_\ell}(z)|^{\rho/m_\ell}$$

$$= \frac{e^{\rho-1}}{\rho} \limsup_\ell m_\ell^{1-\rho} |g^{(m_\ell)}(z)|^{\rho/m_\ell}, \quad z \in \mathbb{C},$$

so that, as we have seen in (42), the type $\tau$ of $g(z)$ is the maximum of $\tau_\nu(z)$ and $\tau_\mu(z)$.

The following theorem gives a property of the type of $g(z)$ which is the analog of the property regarding the order of $g(z)$ established in Theorem 2.4.

Theorem 3.1. Let $\tau$ be the type of the entire function $g(z)$, while $\tau_\nu(z)$ is as in formula (72), where $\nu$ is a sequence of indices of subexponential growth.

(i) If $\tau < \infty$, then

$$\tau_\nu(z) = \tau \quad \text{for a.e. } z \in \mathbb{C}. \quad (74)$$

(ii) If $\tau = \infty$, then the set $\{ z \in \mathbb{C} : \tau_\nu(z) = \infty \}$ is a dense $G_\delta$ (therefore uncountable) subset of $\mathbb{C}$.
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Proof. (i) If $\tau < \infty$, we can follow the proof of Lemma 2.1 in order to show that $\tau_{\nu}(z)$ is subharmonic in $\mathbb{C}$. Then, by imitating the proof of Theorem 2.4 we can easily obtain (74).

(ii) Suppose $\tau = \infty$. Then, in view of (32) we have

$$\sigma(z) := \sup_n n^{1-\rho} |g^{(n)}(z)|^{\rho/n} \equiv \infty. \tag{75}$$

Let $\mu = \{m_{\ell}\}_{\ell=1}^{\infty}$ be the complementary sequence of $\nu$. We introduce the quantities

$$\sigma_{\nu}(z) := \sup_{k} (n_k)^{1-\rho} |g^{(n_k)}(z)|^{\rho/n_k}, \quad z \in \mathbb{C}, \tag{76}$$

and

$$\sigma_{\mu}(z) := \sup_{\ell} m_{\ell}^{1-\rho} |g^{(m_{\ell})}(z)|^{\rho/m_{\ell}}, \quad z \in \mathbb{C}, \tag{77}$$

so that

$$\max \{\sigma_{\nu}(z), \sigma_{\mu}(z)\} \equiv \infty. \tag{78}$$

Since $\tau_{\nu}(z) = \infty$ if and only if $\sigma_{\nu}(z) = \infty$, it suffices to prove (ii) for $\sigma_{\nu}(z)$ in place of $\tau_{\nu}(z)$.

Suppose that for some disk $D$ we have

$$\sup_{z \in D} \sigma_{\nu}(z) < \infty. \tag{79}$$

Then, (78) would imply that $\sigma_{\mu}(z) = \infty$ for all $z \in D$, which can be shown to be impossible under (79) by following the approach used in the proof of Theorem 2.4, starting with formula (62). Therefore,

$$\sup_{z \in D} \sigma_{\nu}(z) = \infty \quad \text{for any disk } D. \tag{80}$$

Now, formula (76) implies that $\sigma_{\nu}(z)$ is lower semicontinuous on $\mathbb{C}$ (being the supremum of continuous functions). Hence, the set

$$G_N := \{z \in \mathbb{C} : \sigma_{\nu}(z) > N\} \tag{81}$$

is open. Furthermore, by (80) we have that $G_N$ is dense in $\mathbb{C}$ and, therefore, the set

$$\{z \in \mathbb{C} : \sigma_{\nu}(z) = \infty\} = \bigcap_{N=1}^{\infty} G_N \tag{82}$$

is a dense $G_\delta$ subset of $\mathbb{C}$. \qed
Remark 3.2. Suppose that \( \mu \), too, is subexponential. Then the set
\[
\{ z \in \mathbb{C} : \sigma_\nu(z) = \infty \} \cap \{ z \in \mathbb{C} : \sigma_\mu(z) = \infty \},
\]
being the intersection of two dense \( G_\delta \) sets, is again a dense \( G_\delta \) subset of \( \mathbb{C} \).

Remark 3.3. As we have seen, the functions \( |g^{(n)}(z)|^{|p/n|, \ n = 1, 2, \ldots} \), are subharmonic. It follows that \( \sigma_\nu(z) \) satisfies (45), namely
\[
\sigma_\nu(z_0) \leq \frac{1}{\pi r^2} \int_{D_r(z_0)} \sigma_\nu(z) \, dx \, dy
\]
for any disk \( D_r(z_0) \). Notice, however, that \( \sigma_\nu(z) \) may not be subharmonic, since it might become infinite for some \( z \) or it might not be locally integrable. By using (80) in (84), and arguing as in the beginning of the proof of Theorem 2.4, we can conclude that
\[
\int_D \sigma_\nu(z) \, dx \, dy = \infty \quad \text{for any disk } D.
\]
Actually, with the help of the Poisson integral formula for harmonic functions (and the fact that in any sufficiently smooth domain a subharmonic function is dominated by the harmonic function with the same boundary values) we can get a stronger version of (85), namely
\[
\int_\Gamma \sigma_\nu(z) \, ds = \infty \quad \text{for any circle } \Gamma.
\]
Furthermore, since \( \ln |g^{(n)}(z)| \) is subharmonic, we can work with \( \ln \sigma_\nu(z) \) instead of \( \sigma_\nu(z) \) and conclude that
\[
\int_\Gamma \ln \sigma_\nu(z) \, ds = \infty \quad \text{for any circle } \Gamma.
\]
However, in spite of (87), the question whether \( \tau_\nu(z) = \infty \) for a.e. \( z \in \mathbb{C} \) remains open in the case where \( \tau = \infty \).

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