Weakly Secure Symmetric Multilevel Diversity Coding

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Abstract—Multilevel diversity coding is a classical coding model where multiple mutually independent information messages are encoded, such that different reliability requirements can be afforded to different messages. It is well known that superposition coding, namely separately encoding the independent messages, is optimal for symmetric multilevel diversity coding (SMDC) (Yeung-Zhang 1999). In the current paper, we consider weakly secure SMDC where security constraints are injected on each individual message, and provide a complete characterization of the conditions under which superposition coding is sum-rate optimal. Two joint coding strategies, which lead to rate savings compared to superposition coding, are proposed, where some coding components for one message can be used as the encryption key for another. By applying different variants of Han’s inequality, we show that the lack of opportunity to apply these two coding strategies directly implies the optimality of superposition coding. It is further shown that under a set of particular security constraints, one of the proposed joint coding strategies can be used to construct a code that achieves the optimal rate region.

I. INTRODUCTION

Symmetric multilevel diversity coding (SMDC) was introduced by Roche et al. [1] for applications in distributed data storage and robust network communication. Albanese et al. [2] independently studied the problem of priority encoding transmission (PET), which shares the same mathematical model as SMDC. In a symmetric $L$-level diversity coding system, there are $L$ independent messages $(M_1, M_2, \ldots, M_L)$, where the importance of messages decreases with the subscript $l$. The messages are encoded by $L$ encoders. There are totally $2^L - 1$ decoders, each of which has access to the outputs of a distinct subset of the encoders. A decoder which can access any $\alpha$ encoders, called a Level-$\alpha$ decoder, is required to reconstruct the first $\alpha$ most important messages. The system is symmetric in the sense that the reconstruction requirement of a decoder depends on the set of encoders it can access only via its cardinality.

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It was shown [1], [3] that separately encoding these independent messages, referred to as superposition coding, is optimal in terms of achieving the entire rate region. The characterization of the coding rate region therein involves implicit and uncountably many inequalities, and an explicit characterization of the coding rate region was recently obtained [4]. The problem has also been extended and generalized, e.g., to allow node regeneration [5] and to allow asymmetric decoders [6]. Li et al. [7] studied the multilevel diversity coding problem with at most 3 sources and 4 encoders in a systematic way and obtained the exact rate region of each of the over 7,000 instances with the aid of computation.

The SMDC problem with a strong security guarantee was considered by Balasubramanian et al. [8] and Jiang et al. [9]. In this setting, a security threshold $N$ is given, and the first $N$ messages are degenerate. For the remaining $L - N$ messages $M_\alpha$, $\alpha = N + 1, N + 2, \ldots, L$, in addition to the standard multilevel reconstruction requirement, it is also required that all these messages need to be kept perfectly jointly secure if no more than $N$ encoders are accessible by an eavesdropper. Despite the additional security constraints, it was shown that superposition coding remains to be optimal in terms of both the sum rate [8] and the entire rate region [9].

In this paper we consider a weakly secure setting of the classical SMDC problem, where the security level of each message is specified by a separate security parameter $N_\alpha$. More specifically, for any $\alpha = 1, 2, \ldots, L$, we require the message $M_\alpha$ to be kept perfectly secure if the outputs of no more than $N_\alpha$ encoders are accessible by an eavesdropper. Such a security requirement is “weak” in the sense that the eavesdropper is only prevented from obtaining any information about the individual messages. By comparison, the security requirement of [8]–[10] is strong in that it prevents the eavesdropper to obtain any information about the entire set of messages. The notion of weak security has been considered in various network coding settings [11]–[14] and also channel coding perspectives [15]–[18] in the literature and is generally considered to be more practical for protecting individual messages. For example, when the messages are video sequences, the user should not obtain information about any individual video segment, but obtaining the binary XOR of two video sequences may not be an issue since it will not lead to a meaningfully decodable video sequence. Moreover, a protocol with a weak security constraint can potentially be implemented more efficiently in practical settings and may not require encryption keys. Note that the notion of “weak/strong security” here is different from the asymptotic notion of
weak/strong security in [19]–[22], wherein asymptotic weak security requires vanishing of the information leakage rate and the corresponding strong security requires the vanishing of leaked information content. Another notion of “weak security” is defined in [23] which requires the eavesdropper to be unable to obtain any meaningful information about the source.

On the other hand, the notion of weak security has significantly enriched the collection of secure SMDC problems: Unlike the strongly secure setting where a single security parameter is set for all the messages, for the weakly secure setting, a different security parameter can be set for each message. On the other hand, the notion of weak security has also cast the optimality of superposition coding in much greater doubt, as requiring the messages to be protected only marginally (instead of jointly) significantly opens up the set of feasible coding strategies. The main goals of this paper are: 1) to understand under what configurations of the security parameters \( (N_1, N_2, \ldots, N_L) \) superposition coding remains to be optimal; and 2) to identify optimal coding strategies when superposition coding is suboptimal.

The main message of this paper is that the optimality of superposition coding depends critically on the security parameters \( (N_1, N_2, \ldots, N_L) \). More specifically, we consider a natural joint coding strategy that encodes a pair of messages together by using one of the messages as part of the secret key to secure the other. We term this coding strategy a natural joint coding strategy that encodes a pair of messages together by using one of the messages as part of the secret key to secure the other.

The rest of the paper is organized as follows. We first formulate the problem and state some preliminary results in Section II. In Section III, we state the main results, i) a precise classification of the cases where superposition is sum-rate optimal; ii) the optimal rate region for DS-SMDC.

![Fig. 1: The Weakly Secure SMDC Model](image_url)

In Sections IV and V, we describe the pairwise encoding strategies that reduce coding rates and prove the optimality of superposition under the conditions in i). Section VI is devoted to the proof of the optimal rate region for DS-SMDC. We conclude the paper in Section VII. Some technical proofs can be found in the appendices.

## II. PROBLEM FORMULATION AND PRELIMINARIES

### A. Problem Formulation

Let \( \mathcal{L} \triangleq \{1, 2, \ldots, L\} \), where \( L \geq 2 \). Let \( M_1, M_2, \ldots, M_L \) be a collection of \( L \) mutually independent messages uniformly distributed over the direct product of certain finite sets. For simplicity, we assume the message set to be \( \mathbb{F}_{p_{m_1}} \times \mathbb{F}_{p_{m_2}} \times \cdots \times \mathbb{F}_{p_{m_L}} \), where \( \mathbb{F}_{p_{m}} \) is a finite field of order \( p^{m} \) and \( p \) itself can be an integer power of some prime number. We may also regard \( M_\alpha (\alpha \in \mathcal{L}) \) as \( M_\alpha = (M_{\alpha_1}^1, M_{\alpha_2}^2, \ldots, M_{\alpha_L}^L) \) where \( M_{\alpha_i}^i \in \mathbb{F}_p \) for \( i = 1, 2, \ldots, m_\alpha \).

The weakly secure SMDC problem is depicted in Fig. 1. There are \( L \) encoders, indexed by \( \mathcal{L} \), each of which can access all the \( L \) information messages. There are also \( 2^L - 1 \) decoders. For each \( \mathcal{U} \subseteq \mathcal{L} \) such that \( \mathcal{U} \neq \emptyset \), Decoder-\( \mathcal{U} \) can access the outputs of the subset of encoders indexed by \( \mathcal{U} \). For \( \alpha \in \mathcal{L} \) and any \( \mathcal{U} \) such that \( |\mathcal{U}| = \alpha \), Decoder-\( \mathcal{U} \) can completely recover the first \( \alpha \) messages \( M_1, M_2, \ldots, M_\alpha \). In addition, there is an eavesdropper who has access to the outputs of a subsets \( \mathcal{A} \) of encoders. Let \( \mathcal{N} = (N_1, N_2, \ldots, N_L) \) be \( L \) non-negative integers, where \( N_\alpha < \alpha \) for \( \alpha \in \mathcal{L} \). Weak security requires that each individual message \( M_\alpha \) should be kept perfectly secure from the eavesdropper if \( |\mathcal{A}| \leq N_\alpha \).

Let \( \mathcal{K} \) be the key space. An \((m_1, m_2, \ldots, m_L, R_1, R_2, \ldots, R_L)\) code is formally defined by the encoding functions

\[
E_l : \prod_{i=1}^{L} \mathbb{F}_{p_{m_i}} \times \mathcal{K} \rightarrow \mathbb{F}_{p_{R_l}}, \quad \text{for} \ l \in \mathcal{L} \tag{1}
\]

and decoding functions

\[
D_{\mathcal{U}} : \prod_{l \in \mathcal{U}} \mathbb{F}_{p_{R_l}} \rightarrow \prod_{i=1}^{L} \mathbb{F}_{p_{m_i}}, \quad \text{for} \ \mathcal{U} \subseteq \mathcal{L} \text{ and } \mathcal{U} \neq \emptyset. \tag{2}
\]

Denote the shared key as \( K \) (accessible to all the encoders), which is uniformly distributed in the key space \( \mathcal{K} \). Let \( W_i = E_l(M_1, M_2, \ldots, M_L, K) \) be the output of Encoder-\( l \)
and \( W_{\mathcal{U}} = (W_l : l \in \mathcal{U}) \) for \( \mathcal{U} \subseteq \mathcal{L} \). Define the normalized message rates \( m_l = m_l / \sum_{i=1}^{L} m_i \), from which it follows that \( \sum_i m_i = 1 \). A normalized non-negative rate tuple \( \mathbf{R} = (R_1, R_2, \cdots, R_L) \) is achievable for the normalized message rates \((m_1, \ldots, m_L)\), if for any \( \epsilon > 0 \), there exist an integer \( a \) and an \((am_1, am_2, \ldots, am_L, R_1, R_2, \cdots, R_L)\) code such that

**perfect reconstruction:** \( D_{\alpha}(W_{\mathcal{U}}) = (M_1, M_2, \cdots, M_{|\mathcal{U}|}), \forall \mathcal{U} \subseteq \mathcal{L} \) \( \mathcal{U} \neq \emptyset \),

**perfect secure:** \( H(M_{\alpha}|W_{\mathcal{A}}) = H(M_{\alpha}), \forall \alpha \in \mathcal{L} \) and \( \mathcal{A} \subseteq \mathcal{L} \) s.t. \( |\mathcal{A}| \leq N_\alpha \),

and

**coding rate:** \( R_l + \epsilon \geq \alpha^{-1} R_l, \quad l \in \mathcal{L} \).

The optimal coding rate region \( \mathcal{R} \) is defined as the collection of all achievable rate tuples.

**Remark 1.** Here each message \( M_\alpha \) can be essentially represented in \( m_\alpha \log_2 p \) bits, and each codeword \( W_l \) can be represented in \( R_l \log_2 p \) bits. Thus \( R_l \) can be viewed as the coding rate of encoder \( E_l \), when the definition of the entropy function uses logarithm of base \( p \), which will be adopted from here on. The quantity \( R_l \) is then essentially the normalized \( R_l \).

The minimum achievable normalized sum rate is defined as \( R^*_{\text{sum}} \triangleq \min \sum_{l=1}^{L} R_l \), and one of our main results is a necessary and sufficient condition for superposition coding to be sum-rate optimal. We also study an important case where \( N \) is given by

\[
N_\alpha = \begin{cases} 
\alpha - 1, & \text{for } 1 \leq \alpha \leq r \\
0, & \text{for } r + 1 \leq \alpha \leq L,
\end{cases}
\]

for certain parameters \((L, r)\), where \( r \geq 1 \). We refer to this system as the \((L, r)\) differential-constant secure SMDC (DS-SMDC), where the more important messages (i.e., small \( \alpha \) values) are maximally secure \( (N_\alpha = \alpha - 1) \) and the less important messages do not have any security guarantee at all \( (N_\alpha = 0) \). For the protected messages \((1 \leq \alpha \leq r)\), while the security constraint \( N_\alpha \) grows with the reconstruction requirement \( \alpha \), the difference between \( N_\alpha \) and \( \alpha \) remains to be a constant equal to 1. We refer to this feature as “differential-constant secure”, in contrast to the “level-constant secure” guarantee [8], [9] which requires \( N_\alpha = N \) for all \( \alpha > N \). Denote the optimal coding rate region of the \((L, r)\) DS-SMDC problem by \( \mathcal{R}_{L,r} \), which is the collection of all achievable normalized rate tuples. For \( r = 1 \), the problem reduces to the classical SMDC.

**B. An Achievable Rate Region via Superposition Coding**

Let \( M \) be a message encoded by \( n \) encoders. For any \( 0 \leq c < k \leq n \), the \((c, k, n)\) ramp secret sharing problem [28], also known as the secure symmetrical single-level diversity coding (SS-SSDC) problem in [8], requires that the outputs from any subset of no more than \( c \) encoders provide no information about the message, and the outputs from any subset of \( k \) encoders can completely recover the message. The optimal rate region for this problem can be found in [8], [29], as stated in the following lemma.

**Lemma 1.** The optimal rate region of the \((c, k, n)\) ramp secret sharing problem is the collection of rate tuples \((R_1, R_2, \cdots, R_n)\) such that

\[
\sum_{l=1}^{L} R_l \geq H(M), \quad \forall \mathcal{B} \subseteq \{1, 2, \cdots, n\}, |\mathcal{B}| = k - c.
\]

**Remark 2.** If \( k = c + 1 \), the \((c, k, n)\) ramp secret sharing problem reduces to the \((k, n)\) threshold secret sharing problem and the rate region reduces accordingly.

In light of this result, a natural coding scheme (i.e., superposition coding) for the weakly secure SMDC problem formulated above is to separately encode each message \( M_\alpha \) using an \((N_\alpha, \alpha, L)\) ramp secret sharing code as shown in Fig. 2. The rate region induced by superposition coding provides an inner bound \( \mathcal{R}_{\text{sup}} \) for \( \mathcal{R} \), and by Lemma 1, it can be written as the set of non-negative rate tuples \( \mathbf{R} = (R_1, R_2, \cdots, R_L) \) such that

\[
R_l = \sum_{\alpha=1}^{L} r_l^\alpha, \quad l \in \mathcal{L}
\]

for some \( r_l^\alpha \geq 0, \; l, \alpha \in \mathcal{L}, \) satisfying

\[
\sum_{l \in \mathcal{B}} r_l^\alpha \geq m_\alpha, \quad \forall \mathcal{B} \subseteq \mathcal{L} \text{ s.t. } |\mathcal{B}| = \alpha - N_\alpha.
\]

The induced sum rate provides an upper bound \( \bar{R}_{\text{sum}} \) for \( R^*_{\text{sum}} \), and can be written simply as,

\[
\bar{R}_{\text{sum}} \triangleq \sum_{\alpha=1}^{L} \frac{L m_\alpha}{\alpha - N_\alpha}.
\]

**C. Properties of MDS Code for Secret Sharing**

In this section, we describe in some details two \((n, k)\) maximum distance separable (MDS) codes for ramp secret sharing that achieve the minimum sum rate in Lemma 1, and provide important properties that are instrumental to the joint coding strategy we later propose.

Let \( M = (U_1, U_2, \cdots, U_{k-c}) \) be a length-\((k-c)\) message where each symbol is chosen uniformly and independently from the finite field \( \mathbb{F}_p \). Let \( Z_1, Z_2, \cdots, Z_c \) be independent random keys chosen uniformly from the same finite field \( \mathbb{F}_p \). For \( i = 1, 2, \cdots, k \), define the following length-\(k\) vectors:

\[
f_i = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^T.
\]

Let \( g_1, g_2, \cdots, g_n \) be length-\(k\) vectors with entries from \( \mathbb{F}_p \) such that any \( k \) vectors \( \{h_{j_1}, h_{j_2}, \cdots, h_{j_k}\} \) chosen from the set \( \{f_1, f_2, \cdots, f_k, g_1, g_2, \cdots, g_n\} \) satisfy the full rank condition over \( \mathbb{F}_p \), i.e.,

\[
\text{rank} [h_{j_1}, h_{j_2}, \cdots, h_{j_k}] = k.
\]

It can be shown that as long as \( p \geq n + k \), there exist such vectors \( g_1, g_2, \cdots, g_n \), e.g., it can be chosen as the columns
from a Cauchy matrix. The generator matrices of the two MDS
codes of interest are given, respectively, as
\[
G^{(1)} = [f_{k-c+1} \cdots f_k, g_1, g_2, \cdots, g_{n-c}],
\]
\[
G^{(2)} = [g_1, g_2, \cdots, g_n].
\]

Then the codewords of two MDS codes are, respectively,
\[
[Y_1, Y_2, \cdots, Y_n] = [U_1, \cdots, U_{k-c}, Z_1, \cdots, Z_c]G^{(1)},
\]
\[
[Y_1, Y_2, \cdots, Y_n] = [U_1, \cdots, U_{k-c}, Z_1, \cdots, Z_c]G^{(2)}.
\]

We shall refer these two codes as MDS-A and MDS-B, respectively. By the definition of \( f_{k-c+1}, \cdots, f_k \) in (11), MDS-A has the random keys explicitly as part of the coded message,
\[
[Y_1, Y_2, \cdots, Y_c] = [Z_1, Z_2, \cdots, Z_c].
\]

It is obvious that for both codes, \( M \) and \( Z_1, Z_2, \cdots, Z_c \) can be perfectly recovered from any \( k \) coded symbols.

Since all the coded symbols are linear combinations of the messages and the random keys that are uniformly distributed, we have the following lemma.

**Lemma 2.** Any \( k \) coded symbols of MDS-A and MDS-B are uniformly distributed over \( \mathbb{F}_{p^k} \).

The main difference between the two codes, which is the most relevant to this work, is given in the following two lemmas.

**Lemma 3.** For any integer \( t \) such that \( c \leq t \leq k \), let \( \mathcal{E} \subseteq \{1, 2, \cdots, n\} \) where \( |\mathcal{E}| = t \), and \( \mathcal{A} \subseteq \{1, 2, \cdots, k-c\} \) where \( |\mathcal{A}| = k-t \). The codewords of MDS-A has the following property:
\[
I(Y_{\mathcal{E}}; U_{\mathcal{A}}) = 0,
\]
where \( Y_{\mathcal{E}} \triangleq \{Y_i : i \in \mathcal{E}\} \) and \( U_{\mathcal{A}} \triangleq \{U_i : i \in \mathcal{A}\} \).

**Proof.** We consider the following chain of equality
\[
I(Y_{\mathcal{E}}; U_{\mathcal{A}})
=H(Y_{\mathcal{E}}) - H(Y_{\mathcal{E}}|U_{\mathcal{A}})
=H(Y_{\mathcal{E}}) - H(Y_{\mathcal{E}}|U_{\mathcal{A}}) + H(Y_{\mathcal{E}}|U_{\mathcal{A}}U_{\mathcal{A}'}, Z_1, Z_2, \cdots, Z_c)
=H(Y_{\mathcal{E}}) - H(U_{\mathcal{A}}Z_1, Z_2, \cdots, Z_c|U_{\mathcal{A}})
=H(Y_{\mathcal{E}}) - H(U_{\mathcal{A}}Z_1, Z_2, \cdots, Z_c|Y_{\mathcal{E}}U_{\mathcal{A}})
=H(Y_{\mathcal{E}}) - H(U_{\mathcal{A}}Z_1, Z_2, \cdots, Z_c)
=\frac{t}{k-c}H(M) - \frac{t-c+c}{k-c}H(M)
=0,
\]
where (20) follows from (15), and both (23) and (24) follow from the full rank condition in (12) and the uniform and mutually independent distribution of the messages and the encryption key.

**Remark 3.** For \( t = c \), Lemma 3 reduces to the stated security constraint of parameter \( c \); on the other hand, for \( t > c \) (but \( t \leq k \)), any \( t \) coded symbols reveal no information about any subset of \( k-t \) message symbols.

**Lemma 4.** For any integer \( t \) such that \( 0 \leq t \leq k \), let \( \mathcal{E} \subseteq \{1, 2, \cdots, n\} \) where \( |\mathcal{E}| = t \), and \( \mathcal{A}_1 \subseteq \{1, 2, \cdots, k-c\} \), and \( \mathcal{A}_2 \subseteq \{1, 2, \cdots, c\} \) where \( |\mathcal{A}_1| + |\mathcal{A}_2| = k-t \). The codewords of MDS-B has the following property:
\[
I(Y_{\mathcal{E}}; U_{\mathcal{A}_1}, Z_{\mathcal{A}_2}) = 0,
\]
where \( Y_{\mathcal{E}} \triangleq \{Y_i : i \in \mathcal{E}\} \), \( U_{\mathcal{A}_1} \triangleq \{U_i : i \in \mathcal{A}_1\} \), and \( Z_{\mathcal{A}_2} \triangleq \{Z_i : i \in \mathcal{A}_2\} \).

**Proof.** This is direct from the full-rank condition in (12) and the uniform and mutually independent distribution of the messages and the encryption key.

From the above two lemmas, in contrast to MDS-A, MDS-B has the additional advantage that part of the keys can also be made secure against some \( t \) eavesdroppers. This property becomes important to us in the sequel.
III. MAIN RESULTS

A. Sum-rate Optimality Conditions of Superposition

The main question we seek to answer here is under what condition the equality $R^*_{\text{sum}} = \bar{R}_{\text{sum}}$ will hold, and the following theorem provides the exact answer to this question.

**Theorem 1.** $R^*_{\text{sum}} = \bar{R}_{\text{sum}}$, if and only if for any $\alpha < \beta \in L$ where $m_\alpha, m_\beta > 0$, we have

$$N_\alpha < \alpha \leq N_\beta < \beta, \text{ or } N_\alpha = N_\beta = 0. \quad (27)$$

**Remark 4.** If all $L$ messages are non-degenerate, i.e., all the message entropies are non-zero, the condition in (27) is equivalent to that there exists a $T_s \in \{1, 2, \cdots, L\}$ such that for all $\alpha \in L$,

$$N_\alpha = \begin{cases} 0, & \text{for } \alpha \leq T_s, \\ \alpha - 1, & \text{for } \alpha > T_s. \end{cases} \quad (28)$$

If we do not assume non-degeneration, then the following necessary condition for optimality can be induced from (27): There exists a $T_s \in \{1, 2, \cdots, L\}$ such that for any $\alpha \in L$ satisfying $m_\alpha > 0$,

$$N_\alpha = \begin{cases} 0, & \text{for } \alpha \leq T_s \\ N_\alpha > 0, & \text{for } \alpha > T_s. \end{cases} \quad (29)$$

**Remark 5.** The following are two examples that superposition coding is optimal in terms of achieving the entire rate region and thus Theorem 1 reduces correctly.

- If the threshold in (28) is $T_s = L$, the security constraints are given as

$$N_\alpha = 0, \text{ for all } \alpha \in L, \quad (30)$$

then the problem reduces to the classical SMDC problem without security constraints, where superposition is known to be optimal [3].

- If the threshold in (28) is $T_s = 1$, the security constraint becomes

$$N_\alpha = \alpha - 1, \text{ for all } \alpha \in L, \quad (31)$$

and the problem reduces to the special case of DS-SMDC for $r = L$ in Section III-B.

The following definition will be used in the sequel.

**Definition 1.** For any $\alpha < \beta \in L$, we define two conditions.

- **Condition 1:** $N_\alpha < N_\beta < \alpha$;
- **Condition 2:** $N_\beta \leq N_\alpha$ 

$\text{and } N_\alpha > 0$. \quad (32) \quad (33)

Theorem 1 can be alternatively written in the following form, by taking the complement of the conditions in (27).

**Theorem 1’.** $R^*_{\text{sum}} < \bar{R}_{\text{sum}}$, if and only if there exist $\alpha < \beta \in L$ where $m_\alpha, m_\beta > 0$ such that either **Condition 1** in (32) or **Condition 2** in (33) holds.

We prove Theorem 1 in two parts. In Section IV, we show that superposition is suboptimal under the security constraints in (32) or (33), by providing joint coding strategies that can reduce coding rates. In Section V, the optimality of superposition coding is established by proving that the sum rate is lower bounded by $R_{\text{sum}}$ in (10).

**Remark 6.** Superposition coding is optimal for classical SMDC where there is no security constraints, i.e., suboptimality only happens when there is a security constraint. In view of the suboptimality in Sections IV-A and IV-B, we see intuitively that joint encoding helps only when some message can perform as the secret key of another message.

B. Rate Region of DS-SMDC

When superposition is not optimal, it is generally hard to characterize the coding rate region or even the minimum sum rate, since it is difficult to find the optimal code structures. In this section, we study the $(L, r)$ DS-SMDC problem for which we fully characterize the optimal rate region. The pairwise coding strategy in Section IV-B can be generalized to a multi-message regime, and we obtain a group pairwise coding scheme that achieves the entire rate region of the DS-SMDC problem.

We first present an example that motivates the general group pairwise coding scheme.

**Example 1.** Let $L = 4, (m_1, m_2, m_3, m_4) = (1, 1, 1, 4)$, and $p = 11$. The security constraint for the $(4, 3)$ DS-SMDC problem should be $(N_1, N_2, N_3, N_4) = (0, 1, 2, 0)$. We can follow a naive strategy as illustrated in (34): use generator matrices $G_2$ and $G_3$ generated from MDS-B to encode $M_2$ and $M_3$ separately with encryption keys $Z_2$ and $Z_3^1, Z_3^2$; equally partition $M_4$ into four pieces $M_4^1, M_4^2, M_4^3, M_4^4$.

$$W_1 = (M_2 + Z_2, M_3 + 2Z_3^1 + 9Z_3^2, W_1'), \quad W_2 = (M_2 + 2Z_2, 9M_3 + 8Z_3^1 + 6Z_3^2, W_2'), \quad W_3 = (M_2 + 3Z_2, 6M_3 + 10Z_3^1 + 7Z_3^2, W_3'), \quad W_4 = (M_2 + 4Z_2, 7M_3 + 9Z_3^1 + 7Z_3^2, W_4'); \quad (34)$$

The first part of the group pairwise coding scheme is simply to use $M_4$, specifically $M_4^1, M_4^2, M_4^3$, to replace $Z_2$ and $Z_3^1, Z_3^2$ as secret keys to encrypt $M_2$ and $M_3$, as given in (35).

$$W_1' = (M_2 + M_4^1, \quad M_3 + 2M_4^2 + 9M_4^3, \quad ), \quad W_2' = (M_2 + 2M_4^1, \quad 9M_3 + 8M_4^2 + 6M_4^3, \quad ), \quad W_3' = (M_2 + 3M_4^1, \quad 6M_3 + 10M_4^2 + 7M_4^3, \quad ), \quad W_4' = (M_2 + 4M_4^1, \quad 7M_3 + 9M_4^2 + 7M_4^3, \quad W_4'). \quad (35)$$

The second part of the group pairwise coding scheme simply encodes $M_4^4$ as part of the fourth coded message. Since $M_4^1, M_4^2, M_4^3$ does not need to be separately encoded, rate saving is obtained compared to the naive version. The reconstruction and security requirements of $M_2$ and $M_3$ are immediate from the MDS-B code. The reconstruction requirement of $M_4$ is straightforward since $M_4^1, M_4^2, M_4^3$ is recovered with any three coded symbols.

**Coding scheme for general parameters:**

The group pairwise coding scheme is illustrated in Fig. 3. For each $\alpha \in \{1, 2, \cdots, r\}$, we will use an $(\alpha, L)$-threshold secret sharing scheme to encode $M_\alpha$ and use the last $L - r$ messages $M_{r+1}, \cdots, M_L$ as keys. It is proved in [28] that the
minimum key size for $M_\alpha$ is $(\alpha - 1)m_\alpha$. Thus the total size of keys needed is

$$|K| = \sum_{\alpha=1}^r (\alpha - 1)m_\alpha.$$  \hspace{1cm} (36)

For notational simplicity, we define an auxiliary message $1$ $M_{L+1}$, which is independent with other messages and uniformly distributed over $\mathbb{F}_p^{|m_{L+1}|}$ with

$$m_{L+1} = \left\lfloor \sum_{\alpha=1}^r (\alpha - 1)m_\alpha - \sum_{\alpha=r+1}^L m_\alpha \right\rfloor^+,$$  \hspace{1cm} (37)

where for any $x \in \mathbb{R}$, $[x]^+ \triangleq \max\{0, x\}$. It is easy to check that

$$\sum_{\alpha=r+1}^{L+1} m_\alpha \geq \sum_{\alpha=1}^r (\alpha - 1)m_\alpha.$$  \hspace{1cm} (38)

Thus, there exists a unique $\eta^* \in \{r+1, r+2, \cdots, L+1\}$ such that

$$\sum_{\alpha=r+1}^{\eta^*-1} m_\alpha < \sum_{\alpha=1}^r (\alpha - 1)m_\alpha \leq \sum_{\alpha=r+1}^{\eta^*} m_\alpha.$$  \hspace{1cm} (39)

The parameter $\eta^*$ determines which messages of $M_{r+1}, M_{r+2}, \cdots, M_{L+1}$ will be used as the encryption keys. In light of the definition of $\eta^*$ in (39), denote the first $\frac{\sum_{\alpha=1}^r (\alpha - 1)m_\alpha - \sum_{\alpha=1}^{\eta^*-1} m_\alpha}{m_\eta^*}$ fraction of $M_{\eta^*}$ by $M_{\eta^*}^1$, and the rest by $M_{\eta^*}^2$. Then we use the messages $(M_{r+1}, M_{r+2}, \cdots, M_{\eta^*-1}, M_{\eta^*}^1)$ to replace the keys of $M_1, \cdots, M_r$. The messages $M_{\eta^*}^2, M_{\eta^*+1}, M_{\eta^*+2}, \cdots, M_L$ are separately encoded in the same way as in classical SMDC.

Next, we verify the reconstruction and security constraints.

**Reconstruction:** By the code construction in Section II-C, the reconstruction requirements of all messages $(M_1, M_2, \cdots, M_r)$, $(M_{r+1}, M_{r+2}, \cdots, M_{\eta^*-1}, M_{\eta^*}^1)$, and $(M_{\eta^*}^2, M_{\eta^*+1}, M_{\eta^*+2}, \cdots, M_L)$ are satisfied immediately.

**Security:** The security constraints of $(M_1, M_2, \cdots, M_r)$ is straightforward, and there is no security constraint for $(M_{r+1}, M_{r+2}, \cdots, M_L)$.

**Remark 7:** The first $r$ messages $M_1, M_2, \cdots, M_r$ are encoded separately, and the last $r$ messages $M_{r+1}, M_{r+2}, \cdots, M_L$ are also encoded separately. The reason why we call the coding scheme “group pairwise” is that joint encoding are only performed between the two groups of messages

$$\{M_1, M_2, \cdots, M_r\} \text{ and } \{M_{r+1}, M_{r+2}, \cdots, M_{\eta^*}\}. \hspace{1cm} (40)$$

The group pairwise coding scheme can also be interpreted as superposition coding of the messages $M_1, M_2, \cdots, M_r$, $M_{\eta^*+1}, \cdots, M_L$, where the independent pseudo-messages $M_\alpha^r (1 \leq \alpha \leq L)$ are defined by the message size $m_\alpha^r$ as

$$m_\alpha^r = \begin{cases} 0, & r+1 \leq \alpha \leq \eta^*-1; \\ \sum_{j=r+1}^\alpha m_j - \sum_{j=1}^{r}(j - 1)m_j, & \alpha = \eta^*; \\ m_\alpha, & \eta^* + 1 \leq \alpha \leq L. \end{cases} \hspace{1cm} (41)$$

Then the coding rate region $R_{GP}^{L,r}$ induced by group pairwise coding is the set of $R \geq 0$ such that

$$R_l = \sum_{\alpha=1}^L r_\alpha^l, \text{ for } l \in L, \hspace{1cm} (42)$$

where $r_\alpha^l \geq 0$ and

$$\sum_{l \in B} r_\alpha^l \geq m_\alpha, \text{ for } 1 \leq \alpha \leq r, \hspace{1cm} (43)$$

and

$$\sum_{l \in B} r_\alpha^l \geq m_\alpha^l, \text{ for } \forall B \subseteq L \text{ s.t. } |B| = \alpha, \hspace{1cm} (44)$$

Our main result on DS-SMDC is the following theorem.

**Theorem 2.** $R_{L,r} = R_{GP}^{L,r}$.

**Proof.** The achievability is immediate from the group pairwise coding scheme. The converse is proved through a sophisticated iteration of information inequalities, which can be found in Section VI. \hfill \Box

**Remark 8.** From the group pairwise code design and the converse proof in Section VI, we see that both the group pairwise coding scheme and the converse are compatible with $r = 1$ and $r = L$. Nevertheless, in order to emphasize the specificity of the case $r = L$ and to distinguish superposition and group pairwise joint coding, we discuss the optimality for $r = L$ separately in the following.

1) **Optimality of Superposition Coding for $(L,L)$-DS-SMDC:** For $r = L$, all the messages are protected. We separately encode the $L$ independent messages, where each $M_\alpha$ is encoded using an $(\alpha, L)$ threshold secret sharing scheme. The induced superposition rate region $R_{\text{sup}}$ can be obtained from (8) and (9) by letting $\alpha - N_\alpha = 1$ for all

![Fig. 3: The group pairwise coding scheme.](image-url)
\( \alpha \in \mathcal{L} \). To be specific, \( R_{L,L}^{L} \) is the set of nonnegative rate tuples \( R \) such that
\[
R_l = \sum_{\alpha=1}^{L} r_{\alpha}^l, \text{ for } l \in \mathcal{L}
\]  
(45)
where \( r_{\alpha}^l \geq 0 \), and
\[
r_{\alpha}^l \geq m_{\alpha}, \text{ for } 1 \leq \alpha \leq L.
\]  
(46)
It is easy to eliminate \( r_{\alpha}^l \) \((l, \alpha \in \mathcal{L})\) and obtain the following equivalent characterization of the superposition region,
\[
R_{L,L}^{L} = \{ R : R_l \geq \sum_{\alpha=1}^{L} m_{\alpha}, \text{ for all } l \in \mathcal{L} \}. 
\]  
(47)
The following corollary of Theorem 2 states that superposition coding is optimal for the \((L, L)\) DS-SMDC problem.

**Corollary 2.1.** \( R_{L,L} = R_{L,L}^{L} \).

**Proof.** The proof of the converse part is straightforward, so we omit the details and derive the conclusion directly from Theorem 2. It is easily seen by comparing (42)-(44) and (45)-(46) that \( R_{L,L}^{L} \) reduces to \( R_{L,L} \) for \( r = L \). Thus, by Theorem 2, we have \( R_{L,L} = R_{L,L}^{L} = R_{L,L}^{L,r} \). \( \square \)

### IV. Achievability of Theorem 1: Joint Coding Strategies

In order to prove the necessity part of Theorem 1, we instead prove the sufficiency part of Theorem 1', in the two separate cases given in (32) and (33).

#### A. Low Security Level at Higher Diversity Level

In this section, we provide a joint coding strategy for the case that Condition 1 in (32) holds which provides rate saving, compared to superposition coding. We first discuss a motivating example to illustrate the key insight on how such rate saving is obtained.

**Example 2.** Let \( \mathcal{L} = 3, (\alpha, \beta) = (2, 3), (m_2, m_3) = (2, 2), (N_2, N_3) = (0, 1) \), and \( p = 5 \). Let \( Z_3 \) be an independent random key uniformly chosen from \( \mathbb{F}_p \). Let the two messages be encoded with generator matrices constructed using MDS-A, which induce the coded symbols as shown in Table I(a) through superposition. The important insight is that the coded message of \( M_2 \) can be used as the secret key to encode \( M_3 \), which reduces the coding rate. More precisely, we replace \( Z_3 \) by \( Y_2^1 = Z_2^1 + Z_2^2 \) to serve as the key for \( M_3 \). The coded symbols for this joint coding strategy are shown in Table I(b).

By comparing the two tables, it is seen that the sum rate is reduced since the coded symbol \( Z_3 \) is eliminated. The reconstruction requirements of both \( M_2 \) and \( M_3 \) are straightforward. There is no security requirement on \( M_2 \). For \( M_3 \), it is seen that any one coded symbol \( W_i \) reveals no information about \( M_3 \). For instance, eavesdropping \( W_2 \) gives
\[
H(M_2|W_2) = H(M_3|M_1^1 + 2M_2^1 + Y_2^1, M_2^1 + Y_2^2) \\
= H(M_3) - H(M_1^1 + 2M_2^1 + Y_2^1|M_1^1 + Y_2^1) \\
= H(M_3) - H(M_2^1 + Y_2^1|M_2^1) \\
= H(M_3) - H(M_2^1 + Y_2^1) \\
= H(M_3),
\]  
(52)
where (50) follows from that \( M_1^1 \) is independent of \( M_3, M_1^1 + 2M_2^1 + Y_2^1, Y_2^1 \) and \( M_2^1 \) is independent of \( M_3 + 2M_3 + Y_2^1, Y_2^1 \).

#### Coding strategy for general parameters:

First encode separately \( M_2 \) and \( M_3 \) with generator matrices \( G_\alpha \) and \( G_\beta \) using MDS-A in Section II-C. The coded symbols for superposition coding strategy are as given in Table II(a). The joint coding strategy we propose is then to replace the first \( \theta = \min\{N_\beta, \alpha - N_\beta\} \) encryption key symbols \( \{Z_\beta^2 Z_\beta^3 \cdots Z_\beta^L\} \) by the coded symbols \( \{Y_\beta^1, Y_\beta^1, \cdots, Y_\beta^\theta\} \). The parameter \( \theta \) is strictly positive, which is implied by Condition 1 in (32). Denote the corresponding codewords for \( M_\beta \) thus obtained as \( \{Y_\beta^1, Y_\beta^2, \cdots, Y_\beta^\theta\} \). The joint coding strategy of \( M_\alpha \) and \( M_\beta \) is illustrated in Table II(b) and can be described as follows:
\[
W_i = \begin{cases}
Y_\beta^i, & \text{for } 1 \leq i \leq \theta \\
[Y_\alpha^i, Y_\beta^i], & \text{for } \theta < i \leq L.
\end{cases}
\]  
(53)
By comparing Table II(a) and Table II(b), it can be seen that the coding rate is reduced compared to superposition coding because \( \{Y_\beta^1, Y_\beta^2, \cdots, Y_\beta^\theta\} \) are removed from the codewords, while the rates for all the others are unchanged. Next, we verify the reconstruction and security constraints for the two messages.
Reconstruction: The verification of the reconstruction requirements of both $M_\alpha$ and $M_\beta$ is straightforward.

Security: We consider the security requirements for the two levels separately.

1) Assume we can access $N_\alpha$ coded symbols $W_{B_1}|B_1 = N_\alpha$. Partition $B$ into $B_1$ and $B_2$ such that $B_1 \subseteq \{1, 2, \ldots, \theta\}$ and $B_2 \subseteq \{\theta + 1, \ldots, L\}$. Notice that

$$H(Y_\alpha^{B_2}|M_\alpha, Y_\alpha^{B_1}Y_\beta^{B_2})$$

$$\geq H(Y_\alpha^{B_2}|M_\alpha, Y_\alpha^{B_1}Y_\beta^{B_2})$$

$$= H(Y_\alpha^{B_2}|Y_\alpha^{B_1})$$

$$= H(Y_\alpha^{B_2}),$$

where the second equality follows from the fact that conditioning does not increase entropy, and the last equality follows from Lemma 2 because

$$|B_2| + \theta \leq N_\alpha + \theta$$

$$= N_\alpha + \min\{\alpha - N_\beta, N_\beta\}$$

$$\leq \alpha$$

$$< \beta,$$

where the second inequality follows from $N_\alpha < N_\beta$ which is part of Condition 1 in (32). Since conditioning does not increase entropy, in light of (56), we obtain

$$H(Y_\alpha^{B_2}|M_\alpha, Y_\alpha^{B_1}Y_\beta^{B_2}) = H(Y_\alpha^{B_2}).$$

It follows that

$$I(W_B; M_\alpha)$$

$$= I(W_{B_1}W_{B_2}; M_\alpha)$$

$$= I(Y_\alpha^{B_1}Y_\alpha^{B_2}Y_\beta^{B_2}; M_\alpha)$$

$$= I(Y_\alpha^{B_1}Y_\alpha^{B_2}Y_\beta^{B_2}; M_\beta)$$

$$= I(Y_\alpha^{B_1}Y_\alpha^{B_2}Y_\beta^{B_2}; M_\beta)$$

$$= I(Y_\alpha^{B_2}; M_\alpha)$$

$$= 0,$$

where the last but one equality follows from Lemma 3 and the fact that $|B_1| + |B_2| = N_\alpha$, and (65) follows from (61). Thus indeed $W_B$ reveals nothing about $M_\alpha$.

2) Assume we can access $N_\beta$ coded symbols $W_{B_1}|B_1 = N_\beta$. Partition $B$ into $B_1$ and $B_2$ such that $B_1 \subseteq \{1, 2, \ldots, \theta\}$ and $B_2 \subseteq \{\theta + 1, \ldots, L\}$. We first consider

$$H(Y_\alpha^{B_2}|Y_\alpha^{B_1}Y_\beta^{B_2})$$

$$\geq H(Y_\alpha^{B_2}|Y_\alpha^{B_1}Y_\beta^{B_2}M_\beta)$$

$$\geq H(Y_\alpha^{B_2}|Y_\alpha^{B_1}Y_\beta^{B_2}Z_{\beta}^{\theta+1} \cdots Z_{\beta}^{N_\beta}, M_\beta)$$

$$= H(Y_\alpha^{B_2}|Y_\alpha^{B_1}Y_\beta^{B_2}Z_{\beta}^{\theta+1} \cdots Z_{\beta}^{N_\beta}, M_\beta)$$

$$= H(Y_\alpha^{B_2}|Z_{\beta}^{\theta+1} \cdots Z_{\beta}^{N_\beta}, M_\beta)$$

$$= H(Y_\alpha^{B_2}|Z_{\beta}^{\theta+1} \cdots Z_{\beta}^{N_\beta}, M_\beta)$$

$$= H(Y_\alpha^{B_2}).$$

where both (66) and (67) follow from the fact that conditioning does not increase entropy, (68) follows from that $Y_\beta^{B_2}$ is a function of $(Y_\alpha^{B_1} \cdots Y_\alpha^{\theta}, Z_{\beta}^{\theta+1} \cdots Z_{\beta}^{N_\beta}, M_\beta)$, (70) follows from that $(Z_{\beta}^{\theta+1} \cdots Z_{\beta}^{N_\beta}, M_\beta)$ are independent of $(Y_\alpha^{B_1}, Y_\alpha^{B_2})$, and the last equality follows from Lemma 2, since $|B_2| + \theta \leq \alpha$ which is induced by $\theta \leq \alpha - N_\beta$. Since conditioning does not increase entropy, in light of (71), we obtain

$$H(Y_\alpha^{B_2}|Y_\alpha^{B_1}Y_\beta^{B_2}M_\beta)$$

$$= H(Y_\alpha^{B_2}|Y_\alpha^{B_1}Y_\beta^{B_2}) = H(Y_\alpha^{B_2}).$$

Then we have

$$I(W_B; M_\beta) = I(W_{B_1}W_{B_2}; M_\beta)$$

$$= I(Y_\alpha^{B_1}Y_\alpha^{B_2}Y_\beta^{B_2}; M_\beta)$$

$$= I(Y_\alpha^{B_1}Y_\alpha^{B_2}; M_\beta)$$

$$= I(Y_\alpha^{B_2}; M_\beta)$$

$$= H(Y_\alpha^{B_2}) - H(Y_\alpha^{B_2})$$

$$= 0,$$

where (76) follows from Lemma 3 and the fact that $|B_1| + |B_2| = N_\beta$, and (78) follows from (72). Thus we obtain that $W_B$ reveals nothing about $M_\beta$.

B. Reversed Security Level

We next provide a joint coding strategy for the case that Condition 2 in (33) holds.

Example 3. Let $L = 4, (\alpha, \beta) = (3, 4), (m_3, m_4) = (1, 1), (N_3, N_4) = (2, 1)$, and $p = 11$. We use generator matrix $G_3$ generated using MDS-B to encode $M_3$ separately with encryption keys $Z_1, Z_2$, as given in (80). The joint coding strategy is simply to use $M_4$ to replace $Z_1$ as secret keys to encrypt $M_3$, as given in (81).

$$M_3 + 2Z_1 + 9Z_2, \ 9M_3 + 8Z_1 + 6Z_2,$$

$$6M_3 + 10Z_1 + 7Z_2, \ 7M_3 + 9Z_1 + 7Z_2;$$

$$\rightarrow M_3 + 2M_4 + 9Z_2, \ 9M_3 + 8M_4 + 6Z_2,$$

$$6M_3 + 10M_4 + 7Z_2, \ 7M_3 + 9M_4 + 7Z_2.$$ (81)

Since $M_4$ does not need to be separatedly encoded, rate saving is obtained. The reconstruction and security requirements of $M_3$ are immediate. The reconstruction requirement of $M_4$ is straightforward since everything is recovered with any three coded symbols. The security requirement of $M_4$ can be easily seen that any one coded symbol reveals nothing about $M_4$.

Coding strategy for general parameters:

Next, we present the general coding strategy that $M_\beta$ performs as secret keys for $M_\alpha$ so that we can reduce the coding rates. Let $G_\alpha$ be a generator matrix generated using MDS-B in Section II-C, which can be used to encode $M_\alpha$ separately with encryption keys $(Z_1, Z_2, \ldots, Z_{N_\alpha})$. The joint coding strategy is simply to use $\eta = \min\{N_\alpha, \alpha - N_\beta\}$ symbols of the message $M_\beta$ (i.e., $M_{\beta}^T, M_{\beta}^T, \ldots, M_{\beta}^T$) to replace the encryption keys $(Z_1, Z_2 \cdots Z_\eta)$ for encrypting $M_\alpha$. The parameter $\eta$ is strictly positive, which is implied
by Condition 2 in (33) as well as $\alpha > N_\alpha$. Denote the corresponding coded symbols for $M_\alpha$ after this replacement as $(Y_{\alpha 1}^1, Y_{\alpha 2}^2, \cdots, Y_{\alpha L}^L)$. Since the $\eta$ message symbols of $M_\beta$ do not need to be separately encoded, rate saving is thus obtained.

Next, we verify the reconstruction and security constraints.

**Reconstruction:** By the code construction in Section II-C, both the message $M_\alpha$ and the keys $M_\beta$ can be losslessly recovered from any $\alpha$ coded symbols. Since $\alpha < \beta$, the reconstruction requirements of both $M_\alpha$ and $M_\beta$ are satisfied immediately.

**Security:** The security constraint of $M_\alpha$ is straightforward, and thus let us consider $M_\beta$. For any $B \subseteq \mathcal{L}$ such that $|B| = N_\beta$, let $Y_{\alpha B}^B = (Y_{\alpha i}^i : i \in B)$. By Lemma 4, we have

$$I(Y_{\alpha B}^B; M_\beta) = 0,$$

where $Y_{\alpha 1}^1, Y_{\alpha 2}^2, \cdots, Y_{\alpha L}^L$ are coded symbols of $M_\alpha$ and $Y_{\alpha B}^B = (Y_{\alpha i}^i : i \in B)$ for any $B \subseteq \mathcal{L}$. To see this, observe that if the weakly secure SMDC problem reduces to classical SMDC, (88) is true immediately. Otherwise, by (27), we have $N_L \geq N_\alpha$ for any $\alpha \in \mathcal{L}$. Since we use an $(N_\alpha, \alpha, L)$ ramp secret sharing code to encode $M_\alpha$, any $\alpha$ symbols from the set $\{M_\alpha^1, M_\alpha^2, \cdots, M_\alpha^{N_\alpha}, Y_{\alpha 1}^1, Y_{\alpha 2}^2, \cdots, Y_{\alpha L}^L\}$ can completely recover the whole set. Thus, $(Y_{\alpha B}^B, M_\alpha)$ provide complete information about $Y_{\alpha i}^i$, which verifies (88).

**VI. CONVERSE PROOF OF THEOREM 2**

Before proving Theorem 2, we introduce some terminologies and notations in [3]. Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_L)$ and

$$\mathbb{R}_+^L = \{\lambda : \lambda \neq 0 \text{ and } \lambda_i \in \mathbb{R}, \lambda_i \geq 0 \text{ for } i \in \mathcal{L}\}.$$  

Let $\Omega_\mathcal{L}^\alpha = \{v \in \{0, 1 \}^L : |v| = \alpha\}$, where $|v|$ is the Hamming weight of a vector $v = (v_1, v_2, \cdots, v_L)$. For any $v \in \Omega_\mathcal{L}^\alpha$, let $c_\alpha(v)$ be any nonnegative real number. For any $\lambda \in \mathbb{R}_+^L$ and $\alpha \in \mathcal{L}$, let $f_\alpha(\lambda)$ be the optimal solution to the following optimization problem:

$$f_\alpha(\lambda) \triangleq \max_{v \in \Omega_\mathcal{L}^\alpha} \sum_{v \in \Omega_\mathcal{L}^\alpha} c_\alpha(v)$$

subject to

$$\sum_{v \in \Omega_\mathcal{L}^\alpha} c_\alpha(v) \cdot v \leq \lambda.$$

A set $\{c_\alpha(v) : v \in \Omega_\mathcal{L}^\alpha\}$ is called an $\alpha$-resolution for $\lambda$ if (91) and (92) are satisfied and it will be abbreviated as $\{c_\alpha(\cdot)\}$ if there is no ambiguity. Furthermore, an $\alpha$-resolution is called optimal if it achieves the optimal value $f_\alpha(\lambda)$. In the following proofs, we will take advantage of some lemmas and theorems from [3] and [4], which are enclosed in Appendix A for convenience.

To prove the converse of Theorem 2, we follow the idea of Theorem 2 in [3], i.e., we provide an alternative characterization of the group pairwise region $R_{\mathbb{R}_+^L}$. For simplicity, let $f_{L+1}(\lambda) = 0$ for all $\lambda \in \mathbb{R}_+^L$. For $\eta \in \{r+1, r+2, \cdots, L+1\}$, let

$$g_\eta(\lambda) = \sum_{\alpha=1}^{\eta} f_\lambda(\lambda)m_\alpha + \sum_{\alpha=1}^{r} f_\alpha(\lambda)m_\alpha$$

$$+ f_\lambda(\lambda) \left[ \sum_{\alpha=r+1}^{\eta} m_\alpha - \sum_{\alpha=1}^{r} (\alpha-1)m_\alpha \right].$$

In particular, for $\eta = \eta^*$ which is defined by (39), we have

$$g_\eta^*(\lambda) = \sum_{\alpha=1}^{r} f_\lambda(\lambda)m_\alpha + \sum_{\alpha=r+1}^{L} f_\alpha(\lambda)m_\alpha^*,$$

where $m_\alpha^*$ is defined in (41). From the group pairwise coding scheme in Fig. 3, we have the following intuitions on the coding rates.

i. Superposition of $M_1, M_2, \cdots, M_r$ induces the rate

$$\sum_{l=1}^{L} \lambda_l R_l = \sum_{\alpha=1}^{L} f_\lambda(\lambda)m_\alpha.$$
ii. The messages $M_{r+1}, M_{r+2}, \ldots, M_{r+r}$, perform as keys for $M_1, M_2, \ldots, M_r$. Thus, we do not need extra rates to encode them beyond the rate given in (95).

iii. The other messages $M_{r+1}, M_{r+2}, \ldots, M_L$ will be encoded in the same way as in classical SMDC, i.e., superposition coding. The coding rate is characterized in [3] using the technique of $\alpha$-resolution, which is

$$
\sum_{l=1}^{L} \lambda_l R_l = f_{\eta^r}(\lambda) \left[ \sum_{\alpha=r+1}^{\eta^r} m_\alpha - \sum_{\alpha=1}^{r} (\alpha-1)m_\alpha \right] + \sum_{\alpha=r+1}^{\eta^r} f_\alpha(\lambda)m_\alpha. \tag{96}
$$

Summing up the rates in (95) and (96), we obtain $g_{\eta^r}(\lambda)$ which is the total rate of group pairwise coding. Let $R^*_L$ be the set of all $R \geq 0$ such that

$$\lambda \cdot R \geq g_{\eta^r}(\lambda). \tag{97}
$$

In particular, for $\lambda = (100 \cdots)$ and $\eta^r = L + 1$, the constraint in (97) becomes the single rate bound

$$R_1 \geq \sum_{\alpha=1}^{r} m_\alpha. \tag{98}
$$

For $\lambda = 1$, the constraint in (97) becomes the sum rate bound

$$R^*_\text{sum} = \sum_{\alpha=1}^{r} (L-\alpha+1)m_\alpha + \sum_{\alpha=r+1}^{\eta^r} \frac{Lm_\alpha}{\eta^r} + \sum_{\alpha=\eta^r+1}^{L} \frac{Lm_\alpha}{\alpha}. \tag{99}
$$

For $\eta^r = r + 1$, the constraint becomes

$$\lambda \cdot R \geq \sum_{\alpha=1}^{r} [f_1(\lambda) - (\alpha-1)f_{r+1}(\lambda)] m_\alpha + \sum_{\alpha=r+1}^{\eta^r} f_\alpha(\lambda)m_\alpha. \tag{100}
$$

Inspired by the above intuitions on the group pairwise coding rates, we can alternatively characterize $R^*_L$ in another equivalent form, given in the following theorem.

**Theorem 3.** $R^*_L = R^*_L.$

**Proof.** See Appendix C.

To complete the converse proof of Theorem 2, in light of the fact $R^*_L \subseteq R^*_L$ as well as Theorem 3, we now only need to show $R^*_L \subseteq R^*_L$, i.e., for any $R \in R^*_L$, the following inequality holds

$$\lambda \cdot R \geq g_{\eta^r}(\lambda). \tag{101}
$$

The following lemma provides an alternative representation of $g_{\eta^r}(\lambda)$.

**Lemma 6.** $\max_{\eta=1,2,\ldots,L+r} \{g_{\eta^r}(\lambda)\} = g_{\eta^r}(\lambda).

**Proof.** See Appendix D.

By Lemma 6, it only remains to show that for any $R \in R^*_L$, and $\eta = r + 1, \cdots, L + 1$, the inequality $\lambda \cdot R \geq g_{\eta^r}(\lambda)$ holds. The converse for SMDC in [3] is proved using iterations to extract the entropies $H(M_1), H(M_2), \ldots, H(M_L)$ successively with coefficient $f_1(\lambda), f_2(\lambda), \ldots, f_L(\lambda)$ which have the same form of expression. In the secure setting here, the desired inequality $\lambda \cdot R \geq g_{\eta^r}(\lambda)$ will have two forms of coefficients, i.e., coefficients related to the secure messages and those related to the non-secure messages. The latter is the same as that in [3], but the former is different. For this reason, the iterations in the converse proof in [3] do not apply to the former, i.e., the secure messages. Therefore, we need to derive new iterations to extract the entropies of the secure messages, such that the $r$-th iteration can be connected with the iterations in [3]. Specifically, the main idea of proving $\lambda \cdot R \geq g_{\eta^r}(\lambda)$ is as follows:

i) we extract the entropies $H(M_1), H(M_2), \ldots, H(M_L)$ with proper coefficients in (93) from $\sum_{l=1}^{L} \lambda_l H(W_l)$ successively and iteratively;

ii) when extracting $H(M_\alpha)$ for $\alpha \in \{1,2,\cdots,r\}$, we explicitly design the coefficients of each intermediate term in closed-form so that we can finally connect to the $r$-th iteration of the converse proof in [3];

iii) for $\alpha \geq r + 1$, since there is no security constraints, we simply use the iterations in [3].

One of the main contributions of the converse proof compared with that in [3] is the new technique of explicitly designing the coefficients in closed-form in each iteration for the secure messages. In contrast, in each iteration of the non-secure messages which is simply the iteration in [3], the coefficients in the iteration do not have a closed-form.

Instead of formally proving this inequality here, we provide an example for $(L, r) = (4, 2)$ and $\eta = 3$ to illustrate the main idea, and relegate the formal proof to Appendix E. The connection between this example and the formal proof will be discussed in Remark 11, Remark 12, and Remark 13 in Appendix E. For different $i, j, k \in \{1, 2, 3, 4\}$, we first present two equalities that will be used in the example:

$$H(W_3 | W_1 M_1) = H(M_2 | W_3 M_1) + H(W_1 | W_1 M_1), \tag{102}
$$

$$H(W_2 | W_1 W_3 | M_1) = H(W_1 | W_1 M_3) + H(M_1), \tag{103}
$$

Now we can write the following chain of inequalities without much difficulty:

$$R_1 + R_2 + R_3 + R_4 = H(W_1) + H(W_2) + H(W_3) + H(W_4) \tag{104}
$$

$$= 4H(M_1) + H(W_1 | M_1) + H(W_2 | M_1) + H(W_3 | M_1) + H(W_4 | M_1), \tag{105}
$$

$$= 4H(M_1) + \left\{0H(W_1 | M_1) + \frac{2}{3}H(W_2 | M_1) \right\}_{|S_1} + \left\{H(W_1 | M_1) \right\}_{|S_1}, \tag{106}
$$

$$\geq 4H(M_1) + \frac{8}{3}H(M_2) + \left\{\frac{1}{3}H(W_2 | M_1) + \frac{2}{3}H(W_3 | M_1) + \frac{1}{3}H(W_4 | M_1) \right\}_{|S_1} + \left\{H(W_1 | M_1) \right\}_{|S_1}.
Theorem 2 in [3]: Let \( \{c(v)\} \) be an optimal \( \alpha \)-resolution for \( \lambda \) (c.f. (91), (92)) and \( \tilde{\lambda} = \sum_{v \in \Omega^L} c(v) \cdot v \). An \( \alpha \)-resolution is called \emph{perfect} if the equality in (91) holds, i.e., \( \sum_{v \in \Omega^L} c(v) \cdot v = \lambda \).

**Lemma 2** in [3]: Let \( \{c(v)\} \) be an optimal \( \alpha \)-resolution for \( \lambda \). Then there exists \( 0 \leq l \leq \alpha - 1 \) such that \( \tilde{\lambda}_i - \tilde{\lambda}_i > 0 \) if and only if \( 1 \leq i \leq l \).

**Lemma 4** in [3]:

(i) \( f_\alpha(\lambda) \leq \alpha^{-1} \sum_{i=1}^{L} \lambda_i \).
\( \sum_{v \in \Omega} c(v) = \alpha^{-1} \sum_{i=1}^{L} \lambda_i \) if and only if \( \{c(v)\} \) is a perfect \( \alpha \)-resolution for \( \lambda \). In this case, \( f_\alpha(\lambda) = \alpha^{-1} \sum_{i=1}^{L} \lambda_i \).

Lemma 7 in [3]: For \( \alpha \geq 2 \), \( \lambda \) has a perfect \( \alpha \)-resolution if and only if \( \lambda_1 \leq \lambda_{2} + \cdots + \lambda_{L} \) and only if \( \lambda_1 \leq \lambda_{2} + \cdots + \lambda_{L} \).

Theorem 1 in [4]: \( f_\alpha(\lambda) = \beta \in \{0, 1, \ldots, \alpha-1\} \frac{1}{\alpha-\beta} \sum_{i=\beta+1}^{L} \lambda_i \).

Lemma 1 in [4]: For \( \alpha \geq 2 \), if \( \lambda_1 \leq \lambda_{2} + \cdots + \lambda_{L} \) and only if \( \lambda_1 \leq \lambda_{2} + \cdots + \lambda_{L} \), then \( f_\alpha(\lambda) = \frac{1}{\alpha} \sum_{i=1}^{L} \lambda_i \).

For any permutation \( \omega \) on \( \{1, 2, \ldots, L\} \), denote \( (\lambda_{\omega(1)}, \lambda_{\omega(2)}, \ldots, \lambda_{\omega(L)}) \) by \( \lambda(\omega) \).

Lemma 2 in [4]: \( f_\alpha(\omega(\lambda)) = f_\alpha(\lambda) \) for any \( \alpha \in \lambda \).

Lemma 5 in [4]: Let \( \lambda_1 = (\lambda_{1,1}, \lambda_{1,2}, \ldots, \lambda_{1,L}) \) and \( \lambda_2 = (\lambda_{2,1}, \lambda_{2,2}, \ldots, \lambda_{2,L}) \) be two ordered vectors such that \( \lambda_{1,i} > \lambda_{2,i} \) and \( \lambda_{1,i} > \lambda_{2,i} \) for all \( \lambda_{1} > \lambda_{2} \). For any \( \alpha_0 \in \lambda \), if \( f_{\alpha_0}(\lambda_1) = f_{\alpha_0}(\lambda_2) \), then \( f_{\alpha_0}(\lambda_1) = f_{\alpha_0}(\lambda_2) \) for all \( \alpha \geq \alpha_0 \).

Let \( \lambda_{[i]} \) be the length-\( L \) vector with the first component being 1 and the rest being 0, i.e., \( \lambda_{[i]} = (1, 0, 0, \ldots, 0) \).

Lemma 6 in [4]: If \( \lambda_1 > \sum_{i=2}^{L} \lambda_i \), let \( \lambda' = \left( \sum_{i=2}^{L} \lambda_i, \lambda_2, \lambda_3, \ldots, \lambda_L \right) \). Then for all \( \alpha \in \lambda \),

\[ f_\alpha(\lambda) = \left( \lambda_1 - \sum_{i=2}^{L} \lambda_i \right) f_\alpha(\lambda_{[1]}) + f_\alpha(\lambda') . \]

Lemma 7 in [4]: For any \( \eta \in \{1, 2, \ldots, L-1\} \),

(i) \( \lambda_1 \leq \sum_{i=2}^{L} \lambda_i \), then \( f_\alpha(\lambda) = \frac{1}{\alpha} \sum_{i=1}^{L} \lambda_i \) for \( \alpha = 1, 2, \ldots, \eta+1 \);

(ii) \( \lambda_1 \geq \sum_{i=2}^{L} \lambda_i \), then \( f_\alpha(\lambda) = f_{\alpha-1}(\lambda_2, \lambda_3, \ldots, \lambda_L) \) for \( \alpha = \eta + 1, \eta + 2, \ldots, L \).

Appendix B

Proof of Lemma 5

When condition (27) is satisfied, there must exist a \( T_\alpha \) as defined in (29). For \( \alpha \leq T_\alpha \), since \( N_\alpha = 0 \), we have

\[ \mu_\alpha = \frac{L}{\alpha} \frac{1}{\lambda_{[\alpha]} \sum_{|B_\alpha| = \alpha} H(W_{B_\alpha}|M_{1:0})} . \]

The claim in (86) is exactly inequality (27) in [1] which was proved by applying Han’s inequality.

For any \( \alpha > T_\alpha \), we prove the claim by induction. Firstly, the claim is true for \( \alpha = T_\alpha \). Then we assume the claim is true for \( \alpha = \zeta \) for some \( \zeta \geq T_\alpha \). We now show that it is true for \( \alpha = \varphi \), which is the index of the next non-vanishing message following \( M_{\zeta} \). In light of (86), we only need to show that

\[ \mu_\varphi \geq \frac{L}{\varphi - N_\varphi} H(M_\varphi) + \mu_\varphi . \]

Since now \( \varphi > \zeta > 0 \), the first condition in (27) must hold, i.e.,

\[ N_\varphi < \zeta \leq N_\varphi < \varphi . \]
where (126) follows from the fact that conditioning does not increase entropy, (127) follows from Han’s inequality and the assumption that $\zeta - N_\zeta \leq \varphi - N_\varphi$, and (128) follows from (125).

**Case ii.** $\zeta - N_\zeta > \varphi - N_\varphi$: We derive the iteration as follows,

$$
\mu_\zeta = \frac{L}{\zeta - N_\zeta} \left( \frac{L}{N_\zeta} \right) \left( \frac{L - N_\zeta}{\zeta - N_\zeta} \right) \sum_{(B_1^L, B_2^L) \in \mathcal{B}_\zeta} H(W_{B_1^L} | W_{B_2^L} M_1: \zeta)
\geq \frac{L}{\zeta - N_\zeta} \left( \frac{L}{N_\zeta} \right) \left( \frac{L - N_\zeta}{\zeta - N_\zeta} \right) \sum_{(B_1^L, B_2^L) \in \mathcal{B}_\zeta} \sum_{B_1^L \subseteq \zeta \subseteq L \setminus B_1^L} \sum_{B_2^L \subseteq \varphi \subseteq \zeta \setminus B_1^L} \frac{1}{(\varphi - \zeta)} H(W_{B_1^L} | W_{B_2^L} \cup \zeta M_1: \zeta)
= \frac{L}{\zeta - N_\zeta} \left( \frac{L}{N_\zeta} \right) \left( \frac{L - N_\zeta}{\zeta - N_\zeta} \right) \sum_{D \subseteq \zeta: B_1^L \subseteq D} \sum_{D' = \varphi: B_2^L \subseteq D' \subseteq \zeta - N_\zeta} H(W_{B_1^L} | W_{B_2^L} \cup \zeta M_1: \zeta)
= \frac{L}{\zeta - N_\zeta} \left( \frac{L}{N_\zeta} \right) \left( \frac{L - N_\zeta}{\zeta - N_\zeta} \right) \sum_{D \subseteq \zeta: B_1^L \subseteq D} \sum_{D' = \varphi: B_2^L \subseteq D' \subseteq \zeta - N_\zeta} \frac{1}{(\varphi - \zeta)} H(W_{B_1^L} | W_{B_2^L} \cup \zeta M_1: \zeta)
\geq \frac{L}{\zeta - N_\zeta} \left( \frac{L}{N_\zeta} \right) \left( \frac{L - N_\zeta}{\zeta - N_\zeta} \right) \sum_{D \subseteq \zeta: B_1^L \subseteq D} \sum_{D' = \varphi: B_2^L \subseteq D' \subseteq \zeta - N_\zeta} \frac{1}{(\varphi - \zeta)} H(W_{B_1^L} | W_{B_2^L} \cup \zeta M_1: \zeta)
$$

**APPENDIX C**

**PROOF OF THEOREM 3**

Similar to Lemma 11 in [3], the theorem can be obtained by proving i) $\mathcal{R}_{\text{sp}}^{L_r} \subseteq \mathcal{R}_{L,r}^r$; ii) for any $\lambda \in \mathbb{R}_L^+$, there exists $\mathbf{R} \in \mathcal{R}_{\text{sp}}^{L_r}$ such that $\lambda \cdot \mathbf{R} = g_{\text{sp}}(\lambda)$.

i) We first show that $\mathcal{R}_{\text{sp}}^{L_r} \subseteq \mathcal{R}_{L,r}^r$. For any $1 \leq \alpha \leq r$, let $r^\alpha = (r_1^\alpha, r_2^\alpha, \ldots, r_s^\alpha)$. For any $\lambda \in \mathbb{R}_L^+$ and $\mathbf{R} \in \mathcal{R}_{\text{sp}}^{L_r}$, we have from (43) that

$$
\lambda \cdot r^\alpha \geq f_1(\lambda)m_{\alpha}.
$$

For $r + 1 \leq \alpha \leq L$, let $\{c_\alpha(v)\}$ be an optimal $\alpha$-resolution for $\lambda$, which implies that

$$
\lambda \geq \sum_{v \in \Omega_\alpha^L} c_\alpha(v)v.
$$

Then we have

$$
\lambda \cdot r^\alpha \geq \sum_{v \in \Omega_\alpha^L} c_\alpha(v)v \cdot r^\alpha \geq \sum_{v \in \Omega_\alpha^L} (c_\alpha(v)(v \cdot r^\alpha)) \geq \sum_{v \in \Omega_\alpha^L} (c_\alpha(v)m_{\alpha}^\ast) = \left( \sum_{v \in \Omega_\alpha^L} c_\alpha(v) \right) m_{\alpha}^\ast = f_\alpha(\lambda)m_{\alpha}^\ast
$$

where (135) follows from (134), (137) follows from (44), and (139) follows from the optimality of $\{c_\alpha(v)\}$. Summing up (133) and (139) over $\alpha$, we have

$$
\lambda \cdot \mathbf{R} \geq \sum_{\alpha = 1}^{L} f_1(\lambda)m_{\alpha} + \sum_{\alpha = r + 1}^{L} f_{\alpha}(\lambda)m_{\alpha}^\ast = g_{\text{sp}}(\lambda).
$$

This implies $\mathbf{R} \in \mathcal{R}_{L,r}^r$ and thus $\mathcal{R}_{\text{sp}}^{L_r} \subseteq \mathcal{R}_{L,r}^r$.

ii) We now construct a rate tuple $\mathbf{R}$ for each $\lambda \in \mathbb{R}_L^+$ such that $\mathbf{R} \in \mathcal{R}_{\text{sp}}^{L_r}$ and $\lambda \cdot \mathbf{R} = g_{\text{sp}}(\lambda)$. For $r + 1 \leq \alpha \leq L$, let $\{c_\alpha(v)\}$ be an optimal $\alpha$-resolution for $\lambda$, and let

$$
\tilde{\lambda} = \sum_{v \in \Omega_\alpha^L} c_\alpha(v) \cdot v.
$$

By Lemma 2 in [3], there exists $1 \leq \alpha = \alpha - 1$ such that $\lambda_i > \tilde{\lambda}_i$ if and only if $1 \leq i \leq \alpha$. Let $R_i = \sum_{\alpha = 1}^{L} r_i^\alpha$
for \( l \in \mathcal{L} \). We construct \( R \) by designing the sub-rates \( r_i^\alpha \) as follows.

a) For \( 1 \leq \alpha \leq r \), let 
\[ r_i^\alpha = m_\alpha, \text{ for all } 1 \leq l \leq L. \tag{143} \]

b) For \( r + 1 \leq \alpha \leq \eta^* - 1 \), let 
\[ r_i^\alpha = 0, \text{ for all } 1 \leq l \leq L. \tag{144} \]

c) For \( \eta^* \leq \alpha \leq L \), let 
\[ r_i^\alpha = \begin{cases} 0, & \text{for } 1 \leq l \leq l_\alpha, \\ \frac{m_\alpha^*}{\alpha - l_\alpha}, & \text{for } l_\alpha + 1 \leq l \leq L. \end{cases} \tag{145} \]

We first verify that such a construction implies \( R \in \mathcal{R}_\text{eq}^{L,r} \).

a) For \( 1 \leq \alpha \leq r \), it is obvious that (43) is satisfied.

b) For \( r + 1 \leq \alpha \leq \eta^* - 1 \), since \( m_\alpha^* = 0 \), (44) is satisfied.

c) For \( \eta^* \leq \alpha \leq L \), consider any \( B \subseteq \mathcal{L} \) such that \( |B| = \alpha \). Let \( e_\alpha \) be an \( L \)-vector with the first \( l_\alpha \) components being 0 and the last \( L - l_\alpha \) components being 1. Let \( v_B = (v_1, v_2, \ldots, v_L) \) be such that \( v_i = 1 \) if only if \( i \in B \). Since \( l_i \geq l_\alpha \), we have \( e_\alpha \cdot v_B \geq \alpha - l_\alpha \).

Thus,
\[ \sum_{l \in B} r_i^\alpha = \left( \frac{m_\alpha^*}{\alpha - l_\alpha} e_\alpha \right) \cdot v_B \geq \min_{l \in L} \frac{m_\alpha^*}{\alpha - l_\alpha} \sum_{l \in B} \sum_{\alpha = \eta^*} v_i = \frac{m_\alpha^*}{\alpha - l_\alpha} (\alpha - l_\alpha) \geq m_\alpha^*, \tag{146} \]
\[ \frac{m_\alpha^*}{l_\alpha} (\alpha - l_\alpha) \geq m_\alpha^* \tag{147} \]
\[ = m_\alpha^* \tag{148} \]

Thus, \( R \in \mathcal{R}_\text{eq}^{L,r} \). Now it remains to show that \( \lambda \cdot R = g_{\eta^*}(\lambda) \). We consider the following cases.

a) For \( 1 \leq \alpha \leq r \), it is easy to check that 
\[ \lambda \cdot r^\alpha = f_1(\lambda) m_\alpha. \tag{150} \]

b) For \( r + 1 \leq \alpha \leq \eta^* - 1 \), it is obvious that 
\[ \lambda \cdot r^\alpha = 0. \tag{151} \]

c) For \( \eta^* \leq \alpha \leq L \), only the first \( l_\alpha \) components of \( \lambda - \lambda \) are nonzero. Thus, we have 
\[ \lambda - \sum_{\alpha = \eta^*} c_\alpha(v) \cdot r^\alpha = 0, \tag{152} \]

which implies that 
\[ \lambda \cdot r^\alpha = \sum_{\alpha = \eta^*} c_\alpha(v) \cdot r^\alpha = \sum_{\alpha = \eta^*} c_\alpha(v) \cdot (v \cdot r^\alpha). \tag{153} \]

By Lemma 2 in [3], for any \( v \in \Omega_L^\alpha \) such that \( c_\alpha(v) > 0 \), the first \( l_\alpha \) components are equal to 1, \( \alpha - l_\alpha \) of the other \( L - l_\alpha \) components are equal to 1, and the rest are equal to 0. On the other hand, the first \( l_\alpha \) components of \( r^\alpha \) are equal to zero. Thus, for any \( v \in \Omega_L^\alpha \) such that \( c_\alpha(v) > 0 \), we have 
\[ v \cdot r^\alpha = (\alpha - l_\alpha) \frac{m_\alpha^*}{\alpha - l_\alpha} = m_\alpha^*. \tag{154} \]

Then
\[ \lambda \cdot r^\alpha = \sum_{\alpha = \eta^*} c_\alpha(v)(v \cdot r^\alpha) \tag{155} \]
\[ = \sum_{\alpha = \eta^*} c_\alpha(v) m_\alpha^* \tag{156} \]
\[ = \left( \sum_{\alpha = \eta^*} c_\alpha(v) \right) m_\alpha^* \tag{157} \]
\[ = f_\alpha(\lambda) m_\alpha^*. \tag{158} \]

Summing up (150), (151), and (158) over all \( 1 \leq \alpha \leq L \), we obtain \( \lambda \cdot R = g_{\eta^*}(\lambda) \). Therefore, Theorem 3 is proved.

**APPENDIX D**

**PROOF OF LEMMA 6**

We prove the lemma by proving (i) for \( r + 1 \leq \eta^* \leq L \), 
\[ \sum_{\alpha = r + 1}^{\eta^*} (\alpha - 1)m_\alpha \leq \sum_{\alpha = r + 1}^{\eta^*} (\alpha - 1)m_\alpha \text{ is equivalent to } g_{\eta^*}(\lambda) \geq g_{\eta^*+1}(\lambda) \geq \cdots \geq g_{L+1}(\lambda); \tag{159} \]
(ii) for \( r + 2 \leq \eta^* \leq L + 1 \), 
\[ \sum_{\alpha = r + 1}^{\eta^*} (\alpha - 1)m_\alpha < \sum_{\alpha = r + 1}^{\eta^*} (\alpha - 1)m_\alpha \text{ is equivalent to } g_{\eta^*}(\lambda) > g_{\eta^*+1}(\lambda) \geq \cdots > g_{L+1}(\lambda). \tag{160} \]

(i) For \( \eta^* \leq \eta \leq L \), we have 
\[ g_\eta(\lambda) \geq g_{\eta+1}(\lambda) \tag{159} \]
\[ = \sum_{\alpha = r + 1}^{\eta^*} \sum_{\alpha = 1}^{\eta^*} f_\alpha(\lambda) m_\alpha + f_\eta(\lambda) \left[ \sum_{\alpha = r + 1}^{\eta^*} (\alpha - 1)m_\alpha \right] \tag{161} \]
\[ \geq \sum_{\alpha = r + 2}^{\eta^*} f_\alpha(\lambda) m_\alpha + f_{\eta+1}(\lambda) \left[ \sum_{\alpha = r + 1}^{\eta^*} (\alpha - 1)m_\alpha \right] \tag{162} \]
\[ f_{\eta+1}(\lambda) m_{\eta+1} + f_\eta(\lambda) \left[ \sum_{\alpha = r + 1}^{\eta^*} (\alpha - 1)m_\alpha \right] \tag{163} \]
\[ = \sum_{\alpha = r + 1}^{\eta^*} \sum_{\alpha = 1}^{\eta^*} m_\alpha - \sum_{\alpha = r + 1}^{\eta^*} (\alpha - 1)m_\alpha \tag{164} \]

Thus, we conclude that 
\[ g_{\eta^*}(\lambda) \geq g_{\eta^*+1}(\lambda) \geq \cdots \geq g_{L+1}(\lambda) \tag{165} \]
is equivalent to 
\[ \sum_{\alpha = r + 1}^{\eta^*} (\alpha - 1)m_\alpha \leq \sum_{\alpha = r + 1}^{\eta^*} (\alpha - 1)m_\alpha \tag{166} \]
which is also equivalent to
\[ \sum_{\alpha=1}^{r} (\alpha - 1) \alpha \leq \sum_{\alpha=r+1}^{\eta^*} \alpha. \] (166)

(ii) For \( r + 1 \leq \eta \leq \eta^* \), we have
\[ g_\eta(\lambda) > g_{\eta-1}(\lambda) \] (167)
\[ \Downarrow \]
\[ \sum_{\alpha=r+1}^{L} f_\alpha(\lambda) + f_\eta(\lambda) \left[ \sum_{\alpha=r+1}^{\eta} \alpha - \sum_{\alpha=1}^{r} (\alpha - 1) m_\alpha \right] > \sum_{\alpha=r+1}^{L} f_\alpha(\lambda) m_\alpha + \sum_{\alpha=r+1}^{\eta} \alpha - \sum_{\alpha=1}^{r} (\alpha - 1) m_\alpha \] (168)
\[ \Downarrow \]
\[ (f_\eta(\lambda) - f_\eta(\lambda)) \left[ \sum_{\alpha=r+1}^{\eta} \alpha - \sum_{\alpha=1}^{r} (\alpha - 1) m_\alpha \right] < 0 \] (170)
\[ \Downarrow \]
\[ \sum_{\alpha=r+1}^{\eta^*} \alpha < \sum_{\alpha=1}^{r} (\alpha - 1) m_\alpha. \] (171)

Thus, we conclude that
\[ g_{\eta^*}(\lambda) > g_{\eta^*}(\lambda) > \cdots > g_{r+1}(\lambda) \] (172)
is equivalent to
\[ \sum_{\alpha=r+1}^{\eta^*} \alpha - \sum_{\alpha=1}^{r} (\alpha - 1) m_\alpha \text{ for all } r + 1 \leq \eta \leq \eta^*, \] (173)
which is also equivalent to
\[ \sum_{\alpha=r+1}^{\eta^*} \alpha < \sum_{\alpha=1}^{r} (\alpha - 1) m_\alpha. \] (174)

**Lemma 7.** For any \( \eta = r + 1, r + 2, \cdots, L + 1 \), the rate constraint \( \lambda \cdot R \geq g_\eta(\lambda) \) is redundant in the characterization of \( R_{L,r} \) if
\[ \lambda_1 > \frac{\lambda_2 + \lambda_3 + \cdots + \lambda_L}{\eta - 1}. \] (175)

**Proof.** See Appendix F. \( \square \)

For any \( \eta \in \{r + 1, r + 2, \cdots, L + 1\} \), \( \lambda \) is called an \( \eta \)-considerable coefficient vector if \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_L \) and
\[ \lambda_1 \leq \frac{\lambda_2 + \lambda_3 + \cdots + \lambda_L}{\eta - 1}. \] (176)

Denote the set of all \( \eta \)-considerable coefficient vectors by \( \mathbb{R}^L_\eta \).
Then let
\[ \mathbb{R}^L_{\text{con}} = \bigcup_{\eta=r+1}^{L+1} \mathbb{R}^L_\eta. \] (177)

We have the following property on vectors in \( \mathbb{R}^L_{\text{con}} \), for which a simple proof is given in Appendix G.

**Lemma 8.** For \( \eta = r + 1, r + 2, \cdots, L + 1 \) and \( \lambda \in \mathbb{R}^L_{\text{con}} \), we have \( f_\eta(\lambda) \geq \lambda_1 \).

By Lemma 7, we only need to prove \( \lambda \cdot R \geq g_\eta(\lambda) \) for \( \lambda \in \mathbb{R}^L_{\text{con}} \). Thus, we assume \( \lambda \in \mathbb{R}^L_{\text{con}} \) in the sequel. From Theorem 1 in [4], we can verify that
\[ \sum_{i=1}^{L} \lambda_i \geq f_1(\lambda) \geq \eta f_\eta(\lambda), \] (178)
which implies that
\[ \sum_{i=1}^{L} \lambda_i - (r-1) f_\eta(\lambda) \geq [\eta - (r - 1)] f_\eta(\lambda) > 0. \] (179)

Let \( \xi \in \mathcal{L} \) be the index of \( \lambda \) such that
\[ \sum_{i=1}^{\xi} \lambda_i < \alpha f_\eta(\lambda) \leq \sum_{i=1}^{\xi+1} \lambda_i. \] (180)

For simplicity, let \( \xi_0 = 1 \). From Lemma 8, we can see that
\[ f_\eta(\lambda) \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_L, \] (181)
which implies
\[ \xi_0 \leq \xi_1 < \xi_2 < \cdots < \xi_r. \] (182)
and
\[ \xi_i \geq i. \] (183)

Due to (179), we can subtract \( r - 1 \) of \( f_\eta(\lambda) \) one by one from the sequence \( \lambda_1, \lambda_2, \cdots, \lambda_L \). The subtraction process is illustrated in Fig. 4. For \( \alpha = 1, 2, \cdots, r - 1 \), let \( \gamma_\alpha = (\gamma_1(\alpha), \gamma_2(\alpha), \cdots, \gamma_L(\alpha)) \) be the \( \alpha \)-th subtraction and \( \lambda_\alpha = (\lambda_1(\alpha), \lambda_2(\alpha), \cdots, \lambda_L(\alpha)) \) be the \( \alpha \)-th residue after the first \( \alpha \) subtractions such that
\[ \gamma_i(\alpha) = \begin{cases} \sum_{i=1}^{\xi_0} \lambda_i - (\alpha - 1) f_\eta(\lambda), & \text{if } i = \xi_{\alpha-1} \\ \alpha f_\eta(\lambda) - \sum_{i=1}^{\xi-1} \lambda_i, & \text{if } i = \xi_\alpha \\ \lambda_i, & \text{if } \xi_{\alpha-1} < i < \xi_\alpha \\ 0, & \text{if } i < \xi_{\alpha-1} \text{ or } i > \xi_\alpha. \end{cases} \] (184)
Fig. 4: Illustration of $\gamma_i^{(a)}$

and $\lambda_i^{(a)} = \lambda_i - \sum_{j=1}^{a} \gamma_i^{(j)}$. Thus,

$$\lambda_i^{(a)} = \begin{cases} 
0, & \text{if } i < \xi_a \\
\sum_{j=1}^{a} \lambda_i - \alpha f_\eta(\lambda), & \text{if } i = \xi_a \\
\lambda_i, & \text{if } i > \xi_a
\end{cases} \quad (185)$$

It is easy to check that

$$\sum_{i=\xi_a-1}^{\xi_a} \gamma_i^{(a)} = f_\eta(\lambda) \quad (186)$$

and

$$\gamma_{\xi_a}^{(a)} + \gamma_{\xi_a}^{(a+1)} = \lambda_{\xi_a}. \quad (187)$$

**Remark 11.** In the example at the end of Section VI, the subtraction and residue parameters are the coefficients in (106), which is $X^{(1)} = (0, 0, 0, 0)$, and $\gamma^{(1)} = (1, 1, 1, 0, 0)$.

Let $\lambda^{(r-1)} = \left(\lambda_{r-1}^{(r-1)}, \lambda_{r+1}^{(r-1)}, \ldots, \lambda_L^{(r-1)}\right)$. The following lemma will be used in the converse. The detailed proof of the lemma is given in Appendix H.

**Lemma 9.** $f_{\eta^{-(r-1)}}(\lambda^{(r-1)}) \geq f_\eta(\lambda)$.

By the definition of $f_{\eta^{-(r-1)}}(\lambda^{(r-1)})$ in (90), the value of the objective function $\sum_{v \in \Omega^{\eta^{-(r-1)}}_{L-\xi_{r-1}+1}} c_{\eta^{-(r-1)}}(v)$ lies in the range $[0, f_{\eta^{-(r-1)}}(\lambda^{(r-1)})]$. The inequality in Lemma 9 implies $f_{\eta}(\lambda) \leq [0, f_{\eta^{-(r-1)}}(\lambda^{(r-1)})]$. Thus, there exists an $[\eta - (r-1)]$-resolution $c_{\eta^{-(r-1)}}(v) : v \in \Omega^{\eta^{-(r-1)}}_{L-\xi_{r-1}+1}$ for $\lambda^{(r-1)}$ such that

$$\sum_{v \in \Omega^{\eta^{-(r-1)}}_{L-\xi_{r-1}+1}} c_{\eta^{-(r-1)}}(v) = f_{\eta}(\lambda). \quad (188)$$

For $v \in \Omega^{\eta^{-(r-1)}}_{L-\xi_{r-1}+1}$ such that $c_{\eta^{-(r-1)}}(v) > 0$, let $v = (v_1, v_2, \ldots, v_{L-\xi_{r-1}+1})$ and

$$D_v = \left\{ i \in \{\xi_{r-1}, \xi_{r-1}+1, \ldots, L \} : v_{i-\xi_{r-1}+1} = 1 \right\}. \quad (189)$$

Let $|D| = b_1$. For simplicity, let $D = \{D_1, D_2, \ldots, D_{b_1}\}$.
where the inequality follows from Lemma 8. It is easy to check that for $i \in \{1, 2, \cdots, \xi_{\alpha - 1} - 1\}$,
\[
\sum_{k=\xi_{\alpha - 1}}^{\xi_{\alpha}} \sum_{A_\alpha \in A_{0}^{(\alpha)}} \gamma_{k}^{A_{\alpha}} = \lambda_{i}
\]
(199)
and for $i = \xi_{\alpha - 1}$,
\[
\sum_{k=\xi_{\alpha - 1}}^{\xi_{\alpha}} \sum_{A_\alpha \in A_{0}^{(\alpha)}} \gamma_{k}^{A_{\alpha}} = \gamma_{\xi_{\alpha - 1}}^{(\alpha - 1)}.
\]
(200)
Thus,
\[
\sum_{k=\xi_{\alpha - 1}}^{\xi_{\alpha}} \sum_{A_\alpha \in A_{0}^{(\alpha)}} \gamma_{k}^{A_{\alpha}} = \gamma_{\xi_{\alpha - 1}}^{(\alpha - 1)} + \sum_{A_\alpha \in A_{0}^{(\alpha)}} \gamma_{\xi_{\alpha - 1}}^{A_{\alpha - 1}} = \lambda_{\xi_{\alpha - 1}}.
\]
(201)
where the first equality follows from (200) and (196), and the second equality follows from (187).

For any $k \in \{1, 2, \cdots, \alpha - 1\}$ and $A_\alpha \in A_{0}^{(\alpha)}$, let $A_{0}^{(k)} = \{i_1, i_2, \cdots, i_k\}$ be the set of the first $k$ smallest elements in $A_\alpha$. In particular, $A_{0}^{(\alpha - 1)} = A_\alpha$. Then the condition $\gamma_{i}^{A_{\alpha - 1}} = \sum_{k=\xi_{\alpha}}^{\xi_{\alpha + 1}} \gamma_{k}^{(k)}$ implies that
\[
\gamma_{i}^{A_{\alpha}} = \sum_{A_{\alpha - 1} \in A_{0}^{(\alpha - 1)}} \sum_{k=\xi_{\alpha}}^{\xi_{\alpha + 1}} \gamma_{k}^{A_{\alpha - 1}}.
\]
(202)
For $i \in L$ and $\alpha \in \{1, 2, \cdots, r - 1\}$, we have
\[
\lambda_{i} = \lambda_{i}^{(\alpha)} + \sum_{k=1}^{\alpha} \gamma_{i}^{(k)} = \lambda_{i}^{(\alpha)} + \sum_{k=1}^{\alpha} \sum_{A_{\alpha} \in A_{0}^{(k)}} \gamma_{i}^{A_{\alpha}}.
\]
(203)
In particular, for $\alpha = r - 1$,
\[
\lambda_{i} = \lambda_{i}^{(r - 1)} + \sum_{k=1}^{r - 1} \sum_{A_{\alpha} \in A_{0}^{(k)}} \gamma_{i}^{A_{\alpha}}.
\]
(204)

Let $A^{(r)} = \{\{i\} \cup A_{r - 1} : \gamma_{i}^{A_{r - 1}} > 0 \text{ for } i \in \{\xi_{r - 2}, \xi_{r - 2} + 1, \cdots, \xi_{r - 1}\} \text{ and } A_{r - 1} \in A_{0}^{(r - 1)}\}$. Denote the cardinality of $A^{(r)}$ by $b_{2}$. For simplicity, let $A^{(r)} = \{B_1, B_2, \cdots, B_{b_{2}}\}$. For $j \in \{1, 2, \cdots, b_{2}\}$, (198) implies that
\[
|B_{j}| = r - 1.
\]
(205)
Without loss of generality, let $B_{j} = \{i\} \cup A_{r - 1}$ for some $i \in \{\xi_{r - 2}, \xi_{r - 2} + 1, \cdots, \xi_{r - 1}\}$ and $A_{r - 1} \in A_{0}^{(r - 1)}$. For $k \in \{1, 2, \cdots, r - 1\}$, let
\[
B_{j}^{k} = \begin{cases} A_{k}^{r - 1}, & \text{if } 1 \leq k \leq r - 2 \\ B_{j}, & \text{if } k = r - 1. \end{cases}
\]
(206)

Note that $B_{j}^{k}$ is the set of the first $k$ smallest elements in $B_{j}$. Let $\gamma(B_{j}) = \gamma_{i}^{A_{r - 1}}$ which is the number of $B_{j}$. Then we have
\[
\sum_{j=1}^{b_{2}} \gamma(B_{j}) = \sum_{i=\xi_{r - 2}}^{\xi_{r - 1}} \sum_{A_{r - 1} \in A_{0}^{(r - 1)}} \gamma_{i}^{A_{r - 1}} = \sum_{i=\xi_{r - 2}}^{\xi_{r - 1}} \gamma_{i}^{(r - 1)} = f_{0}(\lambda)
\]
(207)
\[
= \sum_{k=0}^{b_{2}} c(D_{k}),
\]
(208)
where (207) follows from (196), (208) follows from (186), and (209) follows from (190). This implies that we have a one-to-one correspondence between $f_{0}(\lambda)$ of $D_{k}$’s and $f_{0}(\lambda)$ of $B_{j}$’s. The mapping defined by overlap in Fig. 5 is a simple one-to-one correspondence. The inequality in (198) ensures that the number of $D_{k}$’s that contains $\xi_{r - 1}$ is less than or equal to the number of $B_{j}$’s that don’t contain $\xi_{r - 1}$. Thus, there exists a correspondence such that $B_{j} \cap D_{k} = \emptyset$ if $B_{j}$ and $D_{k}$ have overlap in Fig. 5. Without loss of generality, assume the the mapping in Fig. 5 is such a correspondence. Let
\[
O = \{ (j, k) : B_{j} \text{ and } D_{k} \text{ have overlap in Fig. 5} \}.
\]
(210)
Then we have for all $(j, k) \in O$ that
\[
B_{j} \cap D_{k} = \emptyset
\]
(211)
and
\[
|B_{j} \cup D_{k}| = \eta.
\]
(212)
For $k \in \{1, 2, \cdots, b_{1}\}$, let $s_{k} = \sum_{i=1}^{k} c(D_{i})$. For $j \in \{1, 2, \cdots, b_{2}\}$, let $t_{j} = \sum_{i=1}^{j} \gamma(B_{i})$. For $(j, k) \in O$, let $c(B_{j}, D_{k})$ be the length overlap of $B_{j}$ and $D_{k}$ in Fig. 5, which is equal to
\[
c(B_{j}, D_{k}) = \begin{cases} \gamma(B_{j}), & \text{if } s_{k - 1} \leq t_{j - 1} \leq t_{j} \leq s_{k} \\ s_{k} - t_{j - 1}, & \text{if } s_{k - 1} \leq t_{j - 1} \leq s_{k} \leq t_{j} \\ c(D_{k}), & \text{if } t_{j - 1} \leq s_{k - 1} \leq s_{k} \leq t_{j} \\ t_{j} - s_{k - 1}, & \text{if } t_{j - 1} \leq s_{k - 1} \leq t_{j} \leq s_{k} \\ 0, & \text{otherwise}. \end{cases}
\]
(213)
It is easy to check that for $k \in \{1, 2, \cdots, b_{1}\}$,
\[
\sum_{j=1}^{b_{2}} c(B_{j}, D_{k}) = c(D_{k})
\]
(214)
and for $j \in \{1, 2, \cdots, b_{2}\}$,
\[
\sum_{k=1}^{b_{1}} c(B_{j}, D_{k}) = \gamma(B_{j}).
\]
(215)

\begin{center}
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
$D_{1}$ & $D_{2}$ & $D_{3}$ & $D_{4}$ & $D_{5}$ & $D_{6}$ & \cdots & $D_{b_{1}}$
\hline
$B_{1}$ & $B_{2}$ & $B_{3}$ & $B_{4}$ & $B_{5}$ & $B_{6}$ & \cdots & $B_{b_{2}}$
\hline
\end{tabular}
\end{center}
Then we have
\[
\sum_{(j,k) \in O} c(B_j, D_k) = \sum_{k=1}^{b_1} c(D_k) = \sum_{j=1}^{b_2} \gamma(B_j) = f_\eta(\lambda).
\] (216)

The following lemma states the relation between the coefficients \(c(B_j, D_k)\) and \(\lambda\). The detailed proof of the lemma can be found in Appendix I.

**Lemma 10.** For \(i \in \mathcal{L}\), we have
\[
\sum_{(j,k) \in O \cap \mathcal{I}_i \cap D_k} c(B_j, D_k) \leq \lambda_i.
\] (217)

For any \(v \in \Omega^L_i\) and \(v = (v_1, v_2, \ldots, v_L)\), if \(\{i : v_i = 1\} = B_j \cup D_k\) for some \((j,k) \in O\), let \(c_\eta(v) = c(B_j, D_k)\). Otherwise, if there is no \((j,k) \in O\) such that \(\{i : v_i = 1\} = B_j \cup D_k\), let \(c_\eta(v) = 0\). Then by (212) and Lemma 10, we can see that \(c_\eta(v) : v \in \Omega^L_i\) is an optimal \(\eta\)-resolution for \(\lambda\).

For \(i \in \{\xi_{r-1}, \xi_{r-1}+1, \ldots, L\}\) and \(j \in \{1, 2, \ldots, b_2\}\), let
\[
c(\{i\} \cup B_j) = \sum_{k \in \{1, 2, \ldots, \xi_{r-1}\} \cap D_k} c(B_j, D_k).
\] (218)

It is easy to check that
\[
\sum_{j=1}^{b_2} c(\{i\} \cup B_j) = \sum_{(j,k) \in O \cap \mathcal{I}_i} c(B_j, D_k) \leq \lambda_i^{(r-1)}
\] (219)
and
\[
\sum_{i=\xi_{r-1}}^{L} c(\{i\} \cup B_j) = \sum_{i=\xi_{r-1}}^{L} \sum_{k \in \{1, 2, \ldots, \xi_{r-1}\} \cap D_k} c(B_j, D_k)
\]
\[
= \sum_{k=1}^{b_1} c(B_j, D_k)
\]
\[
= \gamma(B_j).
\] (220)
\[
= \gamma(B_j).
\] (221)

**Remark 12.** In the example at the end of Section VI, the parameter \(c(B_j, D_k)\) is the coefficients in (107)-(109), where for example, the coefficient \(\frac{1}{2}H(W_2W_3|W_1M_1M_2)\) in (107) and \(\frac{1}{2}H(W_1W_2W_3|W_1M_1M_2)\) in (108)-(109) is \(c(\{1\}, \{2, 3\}\)). The fact that \(c(B_j, D_k) : (j,k) \in O\) is an optimal \(\eta\)-resolution ensures us to proceed after the \(\eta\)-th iteration in the converse proof. The parameter \(c(\{i\} \cup B_j)\) is the coefficient in (113)-(115), where for example, \(\frac{1}{2}H(W_2|W_1M_{1,2})\) in (115) is \(c(\{i\} \cup B_j)\) for \(i = 2\) and \(B_j = \{1\}\).

Before proving the converse, we introduce two important relations that will be repeated used in the proof. For \(\alpha = 1, 2, \ldots, r-1\), and \(i, j \in \mathcal{L}, B \subseteq \mathcal{L}\) such that \(|B| = \alpha - 1\) and \(i, j \notin B\), we have
\[
H(W_i|W_B|W_{M_{1,\alpha}}) \geq H(W_i|W_jW_B|W_{M_{1,\alpha}})
\]
\[
= H(W_i|W_{(j)\cup B}|W_{M_{1,\alpha}}).
\] (222)

For notational simplicity, let \(\xi_{r-1} = 0\). For \(\alpha = 1, 2, \ldots, r-1\), let
\[
I_\alpha \triangleq \sum_{i=\xi_{r-2}}^{\xi_{r-1}} \sum_{A_{n-1} \in A_0^{\alpha-1}} \gamma_i^{A_{n-1}} H(W_iW_{A_{n-1}}|M_{1,\alpha})
\]
\[
+ \sum_{i=\xi_{r-1}}^{L} \sum_{j=1}^{b_2} c(\{i\} \cup B_j)H(W_i|W_{B_j}^{\alpha-1}|M_{1,\alpha})
\]
\[
+ \sum_{i=1}^{L} \sum_{k=\alpha}^{\gamma_i A_k H(W_i|W_{A_k^{\alpha-1}}|M_{1,\alpha})}.
\] (224)

We have the following lemma which provides an iteration that is useful in the sequel. The proof of the lemma can be found in Appendix I.

**Lemma 11.** \(I_\alpha \geq I_{\alpha+1} + [f_1(\lambda) - \alpha f_\eta(\lambda)] H(M_{\alpha+1})\) for \(\alpha = 1, 2, \ldots, r-1\).

We prove the converse of DS-SMDC (i.e., \(\lambda \cdot \mathbf{R} \geq g_\eta(\lambda)\) for all \(\eta = r+1, r+2, \ldots, L+1\)) as follows.

\[
\lambda \cdot \mathbf{R} = \lambda_1 H(W_1) + \lambda_2 H(W_2) + \cdots + \lambda_r H(W_L)
\]
\[
= (\sum_{i=1}^{L} \lambda_i) H(M_1) + \sum_{i=1}^{L} \lambda_i H(W_i|M_1)
\]
\[
= f_1(\lambda) H(M_1)
\]
\[
+ \sum_{i=1}^{L} \lambda_i^{(r-1)} + \sum_{k=1}^{r-1} \sum_{A_k \in A_k^{\alpha}(\lambda)} H(W_i|M_1)
\]
\[
\geq \sum_{\alpha=1}^{r} [f_1(\lambda) - (\alpha - 1) f_\eta(\lambda)] H(M_{\alpha})
\]
\[
+ \sum_{i=\xi_{r-2}}^{\xi_{r-1}} \sum_{A_{n-1} \in A_0^{\alpha-1}} \gamma_i^{A_{n-1}} H(W_iW_{A_{n-1}}|M_{1,\alpha})
\]
\[
+ \sum_{i=\xi_{r-1}}^{L} \sum_{j=1}^{b_2} c(\{i\} \cup B_j)H(W_i|W_{B_j}^{\alpha-1}|M_{1,\alpha})
\]
\[
= \sum_{\alpha=1}^{r} [f_1(\lambda) - (\alpha - 1) f_\eta(\lambda)] H(M_{\alpha})
\]
\[
+ \sum_{i=\xi_{r-2}}^{\xi_{r-1}} \sum_{A_{n-1} \in A_0^{\alpha-1}} \gamma_i^{A_{n-1}} H(W_iW_{A_{n-1}}|M_{1,\alpha})
\]
\[
+ \sum_{i=\xi_{r-1}}^{L} \sum_{j=1}^{b_2} c(\{i\} \cup B_j)H(W_i|W_{B_j}^{\alpha-1}|M_{1,\alpha})
\]
\[
+ \sum_{(j,k) \in O} c(B_j, D_k)H(W_{D_k}|W_{B_j}|M_{1,\alpha})
\] (229)
Lemma 11 for where (226) follows from (204), (227) follows by applying follows from the fact that in [3], and (229) follows from (215) and

\[ \lambda = \sum_{\alpha \in \Omega_\eta^j} c_n(v) \]

where (226) follows from (204), (227) follows by applying Lemma 11 for \( \alpha = 1, 2, \ldots, r-1 \) successively, (228) follows from the definition of \( B_j \), (232) follows from (216), (233) follows from the fact that \( \{c_n(v) : v \in \Omega_\eta^j\} \) is an optimal \( \eta \)-resolution for \( \lambda \) and the iteration in the converse for SMDC in [3], and (229) follows from (215) and

\[ \geq \sum_{\alpha = 1}^r [f_1(\lambda) - (\alpha - 1)f_\eta(\lambda)] H(M_\alpha) \]

+ \( c(B_j, D_k)H(W_{D_k}W_{B_j}|M_{1:r}) \) (230)

\[ = \sum_{\alpha = 1}^r [f_1(\lambda) - (\alpha - 1)f_\eta(\lambda)] H(M_\alpha) \]

+ \( c(B_j, D_k)H(W_{B_j}D_kM_{r+1}|M_{1:r}) \) (231)

\[ \geq \sum_{\alpha = 1}^r [f_1(\lambda) - (\alpha - 1)f_\eta(\lambda)] H(M_\alpha) \]

+ \( f_\eta(\lambda) \) \( \sum_{\alpha = r + 1}^\eta H(M_\alpha) + \sum_{\alpha = \eta + 1}^L f_\alpha(\lambda)H(M_\alpha) \) (232)

\[ = \sum_{\alpha = 1}^r (f_1(\lambda))H(M_\alpha) + \sum_{\alpha = \eta + 1}^L f_\alpha(\lambda)H(M_\alpha) \]

\[ + f_\eta(\lambda) \left[ \sum_{\alpha = r + 1}^\eta H(M_\alpha) - \sum_{\alpha = 1}^r (\alpha - 1)H(M_\alpha) \right] \]

\[ = \sum_{\alpha = 1}^r f_1(\lambda)m_\alpha + \sum_{\alpha = \eta + 1}^L f_\alpha(\lambda)m_\alpha \]

\[ + f_\eta(\lambda) \left[ \sum_{\alpha = r + 1}^\eta m_\alpha - \sum_{\alpha = 1}^r (\alpha - 1)m_\alpha \right] \] (234)

Letting \( \epsilon \to 0 \), the inequality \( \lambda \cdot R \geq g_\eta(\lambda) \) is proved.

Remark 13. The step-by-step correspondence between the general proof in (225)-(234) and the example in (105)-(110) is as follows:

- The iteration in (227) is the generalization of the step in (107);
- The transform of conditional entropies in (228)-(232) play the same role as (108);
- The application of the \( \alpha \)-resolution technique in (233) is the generalization of that in (109).

APPENDIX F

PROOF OF LEMMA 7

Let \( \lambda' = (\lambda_1', \lambda_2', \ldots, \lambda_L') \), where

\[ \lambda_i' = \lambda_i, \text{ for all } i = 2, 3, \ldots, L \] (241)

and

\[ \lambda_1' = \frac{\lambda_2' + \lambda_3' + \cdots + \lambda_L'}{\eta - 1}. \] (242)

By Lemma 7 in [4], (175) implies that

\[ f_\eta(\lambda) = f_{\eta-1}(\lambda_2, \lambda_3, \ldots, \lambda_L), \] (243)

and similarly, from (242),

\[ f_\eta(\lambda') = f_{\eta-1}(\lambda_2, \lambda_3, \ldots, \lambda_L). \] (244)

Thus, we have

\[ f_\eta(\lambda) = f_\eta(\lambda'), \] (245)

which by Lemma 5 in [4] implies that

\[ f_\alpha(\lambda) = f_\alpha(\lambda'), \text{ for all } \eta \leq \alpha \leq L. \] (246)

The rate constraint \( \lambda' \cdot R \geq g_\eta(\lambda') \) is the following,

\[ \lambda' \cdot R \geq \sum_{\alpha = 1}^r f_1(\lambda')m_\alpha + \sum_{\alpha = \eta + 1}^L f_\alpha(\lambda')m_\alpha \]

\[ + f_\eta(\lambda') \left[ \sum_{\alpha = \eta + 1}^\eta m_\alpha - \sum_{\alpha = 1}^r (\alpha - 1)m_\alpha \right] . \] (247)
This implies

\[ \mathbf{\lambda} \cdot \mathbf{R} = \mathbf{\lambda}' \cdot \mathbf{R} + (\lambda_1 - \lambda_1') \mathbf{R}_1 \]

\[ \geq \sum_{\alpha=1}^{r} f_1(\mathbf{\lambda}) m_\alpha + \sum_{\alpha=r+1}^{L} f_2(\lambda_1') m_\alpha 
+ f_2(\lambda') \left[ \sum_{\alpha=r+1}^{\eta} m_\alpha - \sum_{\alpha=1}^{r} (\alpha - 1) m_\alpha \right] 
+ (\lambda_1 - \lambda_1') \left( \sum_{\alpha=1}^{r} m_\alpha \right) \]

(248)

\[ = r \sum_{\alpha=1}^{r} f_1(\lambda_1) m_\alpha + \sum_{\alpha=r+1}^{L} f_2(\lambda_1) m_\alpha 
+ f_2(\lambda') \left[ \sum_{\alpha=r+1}^{\eta} m_\alpha - \sum_{\alpha=1}^{r} (\alpha - 1) m_\alpha \right], \quad (249) \]

which is exactly the constraint \( \mathbf{\lambda} \cdot \mathbf{R} \geq g_\eta(\mathbf{\lambda}) \). This proves the lemma.

**APPENDIX G**  
**PROOF OF LEMMA 8**

For any \( \mathbf{\lambda} \in \mathbb{R}_+^L \), we have

\[ \lambda_1 \leq \frac{1}{\eta - 1} \sum_{i=1}^{\eta} \lambda_i, \quad (250) \]

which by Lemma 1 in [4] implies that

\[ f_\eta(\mathbf{\lambda}) = \frac{1}{\eta} \sum_{i=1}^{\eta} \lambda_i. \quad (251) \]

It is easy to check that (250) is equivalent to

\[ \lambda_1 \leq \frac{1}{\eta} \sum_{i=1}^{\eta} \lambda_i. \quad (252) \]

Thus, we have \( f_\eta(\mathbf{\lambda}) \geq \lambda_1 \), which proves the lemma.

**APPENDIX H**  
**PROOF OF LEMMA 9**

For \( \alpha = 1, 2, \cdots, r - 1 \), we have

\[ \sum_{i=\xi_{r-1}}^{\xi_{r-1}} \lambda_i^{(\alpha)} = \sum_{i=1}^{r-1} \lambda_i - \alpha f_\eta(\mathbf{\lambda}) \geq (\eta - \alpha) f_\eta(\mathbf{\lambda}). \quad (253) \]

where the inequality follows from (178) and the fact that \( f_1(\mathbf{\lambda}) = \sum_{i=1}^{L} \lambda_i \). In particular, for \( \alpha = r - 1 \),

\[ \sum_{i=\xi_{r-1}}^{\xi_{r-1}} \lambda_i^{(r-1)} \geq [\eta - (r - 1)] f_\eta(\mathbf{\lambda}). \quad (254) \]

Denote the ordered permutation of \( \mathbf{\lambda}^{(r-1)} \) by \( \tilde{\mathbf{\lambda}}^{(r-1)} = (\tilde{\lambda}_{\xi_{r-1}}^{(r-1)} \tilde{\lambda}_{\xi_{r-1}+1}^{(r-1)} \cdots \tilde{\lambda}_{\eta}^{(r-1)}) \). Then from (181), we obtain

\[ \frac{1}{\eta - (r - 1)} \sum_{i=\xi_{r-1}}^{\xi_{r-1}} \tilde{\lambda}_i^{(r-1)} \geq f_\eta(\mathbf{\lambda}) \geq \tilde{\lambda}_{\xi_{r-1}}^{(r-1)} \], \quad (255)

which implies

\[ \tilde{\lambda}_{\xi_{r-1}}^{(r-1)} \leq \frac{1}{\eta - (r - 1)} \sum_{i=\xi_{r-1}+1}^{L} \tilde{\lambda}_i^{(r-1)}. \quad (256) \]

By Lemma 4 and Lemma 7 in [3], this implies that \( \tilde{\mathbf{\lambda}}^{(r-1)} \) has a perfect \([\eta - (r - 1)]\)-resolution (c.f. Appendix A) and

\[ f_{\eta}^{(r-1)}(\tilde{\mathbf{\lambda}}^{(r-1)}) = \frac{1}{\eta - (r - 1)} \sum_{i=\xi_{r-1}}^{L} \tilde{\lambda}_i^{(r-1)}. \quad (257) \]

From Lemma 2 in [4] and (255), this implies that

\[ f_{\eta}^{(r-1)}(\tilde{\mathbf{\lambda}}^{(r-1)}) = f_{\eta}^{(r-1)}(\tilde{\mathbf{\lambda}}^{(r-1)}) \geq f_\eta(\mathbf{\lambda}). \quad (258) \]

This proves the lemma.

**APPENDIX I**  
**PROOF OF LEMMA 10**

Consider the following five cases where the set \( \mathcal{L} \) is partitioned into five subsets.

i. For \( i \in \{1, 2, \cdots, \xi_{r-2} - 1\} \), we have

\[ \sum_{(j,k) \in \mathcal{O}} c(B_j, D_k) = \sum_{i \in B_j} \sum_{k=1}^{b_1} c(B_j, D_k) = \sum_{i \in B_j} \gamma(B_j) \]

\[ = \sum_{k=1}^{\xi_{\xi_1 - 1}} \sum_{A_{\xi_1} \in A_{\xi_1}^{(r-1)}} \gamma_{A_{\xi_1}} \]

\[ = \lambda_{\xi_{r-1}} \], \quad (259)

where (259) follows from (215) and (260) follows from (199).

ii. For \( i = \xi_{r-2} \), it follows from (201) that

\[ \sum_{(j,k) \in \mathcal{O}} c(B_j, D_k) = \sum_{(j,k) \in \mathcal{O}} c(B_j, D_k) = \sum_{A_{\xi_1} \in A_{\xi_1}^{(r-1)}} \gamma_{A_{\xi_1}} \]

\[ = \sum_{k=1}^{\xi_{\xi_1 - 1}} \sum_{A_{\xi_1} \in A_{\xi_1}^{(r-1)}} \gamma_{A_{\xi_1}} \]

\[ = \lambda_{\xi_{r-2}} \]. \quad (261)

iii. For \( i \in \{\xi_{r-2} + 1, \xi_{r-2} + 2, \cdots, \xi_{r-1} - 1\} \), we have

\[ \sum_{(j,k) \in \mathcal{O}} c(B_j, D_k) = \sum_{(j,k) \in \mathcal{O}} c(B_j, D_k) = \sum_{A_{\xi_1} \in A_{\xi_1}^{(r-1)}} \gamma_{A_{\xi_1}} \]

\[ = \gamma_{\xi_{r-1}} \]

\[ = \lambda_{\xi_{r-1}} \]. \quad (262)

where (262) follows from (196) and (263) follows from (184).
iv. For $i = \xi_{r-1}$,
\[
\sum_{(j,k) \in \mathcal{O}} c(B_j, D_k) = \sum_{A_{r-1} \in A_0^{(r-1)}} \gamma_{\xi_{r-1}}^{A_{r-1}} + \sum_{\xi_{r-1} \in D_k} b_2 c(B_j, D_k) = \sum_{A_{r-1} \in A_0^{(r-1)}} \gamma_{\xi_{r-1}}^{A_{r-1}} + \sum_{\xi_{r-1} \in D_k} c(D_k) \quad (264)
\]
\[
\leq \gamma_{\xi_{r-1}}^{A_{r-1}} + \gamma_{\xi_{r-1}}^{(r)} = \lambda_{\xi_{r-1}} \quad (265)
\]
where (264) follows from (214), (265) follows from (196) and the fact that $\{c(D_k) : k = 1, 2, \ldots, b_1\}$ is an $[\eta - (r-1)]$-resolution for $A^{(r-1)}$, and (266) follows from (184).

v. For $i \in \{\xi_{r-1} + 1, \xi_{r-1} + 2, \ldots, L\}$, we have
\[
\sum_{(j,k) \in \mathcal{O}} c(B_j, D_k) = \sum_{i \in D_k} b_2 c(B_j, D_k) = \sum_{i \in D_k} c(D_k) \quad (267)
\]
where (267) follows from (214) and (268) follows from the fact that $\{c(D_k) : k = 1, 2, \ldots, b_1\}$ is an $[\eta - (r-1)]$-resolution for $A^{(r-1)}$.

**APPENDIX J**

**PROOF OF LEMMA 11**

For $\alpha = 1, 2, \cdots, r-1$, we have the following iteration,
\[
I_\alpha = \sum_{i=\xi_{\alpha-1}}^{\xi_{\alpha-1}} \sum_{A_{\alpha-1} \in A_0^{(\alpha-1)}} \gamma_{i}^{A_{\alpha-1}} H(W_i W_{A_{\alpha-1}} | M_{1: \alpha})
\]
\[
+ \sum_{i=\xi_{\alpha-1}}^{L} \sum_{b=1}^{b} c(\{i\} \cup B_j) H(W_i | W_{B_j^{\alpha-1}} M_{1: \alpha})
\]
\[
+ \sum_{i=\xi_{\alpha-1}}^{L} \sum_{k=\alpha+1}^{\alpha+1} \sum_{A_k \in A_0^{(k)}} \gamma_{i}^{A_k} H(W_i | W_{A_k^{\alpha-1}} M_{1: \alpha}) \quad (269)
\]
\[
= \sum_{i=\xi_{\alpha-1}}^{\xi_{\alpha-1}} \sum_{A_{\alpha-1} \in A_0^{(\alpha-1)}} \gamma_{i}^{A_{\alpha-1}} H(W_i W_{A_{\alpha-1}} | M_{1: \alpha})
\]
\[
+ \sum_{i=\xi_{\alpha-1}}^{\xi_{\alpha-1}} \sum_{A_{\alpha-1} \in A_0^{(\alpha-1)}} \gamma_{i}^{A_{\alpha-1}} H(W_i | W_{A_{\alpha-1}} M_{1: \alpha})
\]
\[
+ \sum_{i=\xi_{\alpha-1}}^{\xi_{\alpha-1}} \sum_{j=1}^{b} c(\{i\} \cup B_j) H(W_i | W_{B_j^{\alpha-1}} M_{1: \alpha})
\]
\[
+ \sum_{i=\xi_{\alpha-1}}^{\xi_{\alpha-1}} \sum_{k=\alpha+1}^{\alpha+1} \sum_{A_k \in A_0^{(k)}} \gamma_{i}^{A_k} H(W_i | W_{A_k^{\alpha-1}} M_{1: \alpha}) \quad (270)
\]
\[
= \sum_{i=\xi_{\alpha-1}}^{\xi_{\alpha-1}} \sum_{A_{\alpha-1} \in A_0^{(\alpha-1)}} \gamma_{i}^{A_{\alpha-1}} H(W_i W_{A_{\alpha-1}} | M_{1: \alpha})
\]
\[
+ \sum_{i=\xi_{\alpha-1}}^{\xi_{\alpha-1}} \sum_{A_{\alpha-1} \in A_0^{(\alpha-1)}} \gamma_{i}^{A_{\alpha-1}} H(W_i | W_{A_{\alpha-1}} M_{1: \alpha})
\]
\[
+ \sum_{i=\xi_{\alpha-1}}^{\xi_{\alpha-1}} \sum_{j=1}^{b} c(\{i\} \cup B_j) H(W_i | W_{B_j^{\alpha-1}} M_{1: \alpha})
\]
\[
+ \sum_{i=\xi_{\alpha-1}}^{\xi_{\alpha-1}} \sum_{k=\alpha+1}^{\alpha+1} \sum_{A_k \in A_0^{(k)}} \gamma_{i}^{A_k} H(W_i | W_{A_k^{\alpha-1}} M_{1: \alpha}) \quad (271)
\]
\[
= \sum_{i=\xi_{\alpha-1}}^{\xi_{\alpha-1}} \sum_{A_{\alpha-1} \in A_0^{(\alpha-1)}} \gamma_{i}^{A_{\alpha-1}} H(W_i W_{A_{\alpha-1}} | M_{1: \alpha+1})
\]
\[
+ \sum_{i=\xi_{\alpha-1}}^{\xi_{\alpha-1}} \sum_{j=1}^{b} c(\{i\} \cup B_j) H(W_i | W_{B_j^{\alpha-1}} M_{1: \alpha+1})
\]
\[
+ \sum_{i=\xi_{\alpha-1}}^{\xi_{\alpha-1}} \sum_{k=\alpha+1}^{\alpha+1} \sum_{A_k \in A_0^{(k)}} \gamma_{i}^{A_k} H(W_i | W_{A_k^{\alpha-1}} M_{1: \alpha+1}) \quad (272)
\]
\[
\geq \sum_{i=\xi_{\alpha-1}}^{\xi_{\alpha-1}} \sum_{A_{\alpha-1} \in A_0^{(\alpha-1)}} \gamma_{i}^{A_{\alpha-1}} H(W_i W_{A_{\alpha-1}} | M_{1: \alpha+1})
\]
\[
+ \sum_{i=\xi_{\alpha-1}}^{\xi_{\alpha-1}} \sum_{j=1}^{b} c(\{i\} \cup B_j) H(W_i | W_{B_j^{\alpha-1}} M_{1: \alpha+1})
\]
\[
+ \sum_{i=\xi_{\alpha-1}}^{\xi_{\alpha-1}} \sum_{k=\alpha+1}^{\alpha+1} \sum_{A_k \in A_0^{(k)}} \gamma_{i}^{A_k} H(W_i | W_{A_k^{\alpha-1}} M_{1: \alpha+1}) \quad (273)
\]
\[
= \sum_{i=\xi_{\alpha-1}}^{\xi_{\alpha-1}} \sum_{A_{\alpha-1} \in A_0^{(\alpha-1)}} \gamma_{i}^{A_{\alpha-1}} H(W_i W_{A_{\alpha-1}} | M_{1: \alpha+1})
\]
\[
+ \sum_{i=\xi_{\alpha-1}}^{\xi_{\alpha-1}} \sum_{j=1}^{b} c(\{i\} \cup B_j) H(W_i | W_{B_j^{\alpha-1}} M_{1: \alpha+1})
\]
\[
+ \sum_{i=\xi_{\alpha-1}}^{\xi_{\alpha-1}} \sum_{k=\alpha+1}^{\alpha+1} \sum_{A_k \in A_0^{(k)}} \gamma_{i}^{A_k} H(W_i | W_{A_k^{\alpha-1}} M_{1: \alpha+1}) \quad (274)
\]
\[
= I_{\alpha+1} + [f_1(\lambda) - \alpha f_\eta(\lambda)] H(M_{\alpha+1})
\]
\[
(275)
\]
where (271) follows from
\[
\sum_{i=\xi_{\alpha-1}}^{\xi_{\alpha-1}} \sum_{A_{\alpha-1} \in A_0^{(\alpha-1)}} \gamma_{i}^{A_{\alpha-1}} H(W_i W_{A_{\alpha-1}} | M_{1: \alpha})
\]
\[
= \sum_{i=\xi_{\alpha-1}}^{\xi_{\alpha-1}} \sum_{j=\xi_{\alpha-1}}^{\xi_{\alpha-1}} \sum_{A \in A_0^{(\alpha-1)}} \gamma_{i}^{i \cup A_{\alpha-1}} H(W_{i \cup A_{\alpha-1}} | M_{1: \alpha})
\]
\[
(276)
\]
(273) follows from the fact that conditioning does not increase entropy, and (274) follows from (223).

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