Stable Manipulation in Voting

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Abstract

We introduce the problem of stable manipulation where the manipulators need to compute if there exist votes for the manipulators which make their preferred alternative win the election even if the manipulators’ knowledge about others’ votes are little inaccurate, that is, manipulation remains successful even under small perturbation of the non-manipulators’ votes. We show that every scoring rule, maximin, Bucklin, and simplified Bucklin voting rules are stably manipulable in polynomial time for single manipulator. In contrast, stable manipulation becomes intractable for the Copeland α voting rule for every α ∈ [0, 1] even for single manipulator. Hence our results show that the well studied single manipulation problem remains polynomial time solvable for scoring rules, maximin, Bucklin, and simplified Bucklin voting rules even if the manipulator is not too sure about the votes of the non-manipulators. However, a little uncertainty about non-manipulators’ votes makes manipulation by single voter intractable for Copeland α voting rule for every α ∈ [0, 1]. However for a constant number of alternatives, we show that the stable manipulation problem is polynomial time solvable for every anonymous and efficient voting rules. Finally we empirically show that the probability that a uniformly random profile is stably manipulable decreases drastically even if manipulator possess little uncertainty about others’ votes.

1 Introduction

Voting has served as a fundamental tool for aggregating preferences of a set of people over a set of alternatives from centuries. A typical voting system consists of a set of alternatives, a set of voters each having a linear order over the set of alternatives as her preference, and a voting rule which selects a set of alternatives as winners depending on the voters’ preferences. However, classical results show that every reasonable voting system with at least 3 alternatives can suffer from manipulation [Gib73, Sat75] — an agent may be able to make her more favored alternative win by misreporting her preference. Bartholdi et al. pioneered the idea of using computational intractability as a barrier to safe guard elections against manipulation [BTT89, BO91]. Indeed, if we have m alternatives and even if the manipulator exactly knows the preferences of all other voters, naively going over all \((m!) - 1\) possible preferences and report the one which results in best outcome for the manipulator is not feasible for any computationally bounded manipulator.

Although the idea of Bartholdi et al. was to use computational intractability as a barrier against manipulation, the computational problem of manipulation turns out to be polynomial time solvable for most of the commonly used voting rules such as the scoring rules, maximin, Copeland, etc. with prominent exception being the single transferable (STV) voting rule. Even for voting rules (STV for example) for which the computational barrier exists against manipulation, it seems that the barrier, in reality, may be substantially weak due to existence of heuristics which work well in practice [FP10, FKKN11, MR15, and references there in].

Motivation: The computational problem of manipulation has mostly been studied in what is called the complete information setting — the manipulator exactly knows the preferences of all other voters. Although, this setting may be the best possible to prove intractability results (if one proves that manipulation is intractable even if the manipulator exactly knows the preferences of all other voters, then
manipulator’s job can only be harder if she does not know some part of others’ preferences), it is hardly practical. Indeed, most applications of voting in AI, voting over a social network for example, involve a large number of voters where the complete information setting is far from reality. This motivates us to study the classical manipulation problem with an incomplete information setting. In our model, for every voter \( v \), the manipulator has a believed preference \( \succ_v \) and an integer \( \delta_v \) denoting the worst case Kendall-Tau (number of pairs which are ranked differently) distance by which the true preference of the voter \( v \) can deviate; low (high respectively) value of \( \delta_v \) corresponds to the manipulator having high (low respectively) confidence on her belief about voter \( v \)'s true preference. Indeed, in many real world election scenarios, the manipulator can form a belief about a voter’s preference based on that voter’s historical data and other activities. However, due to various activities that may have happened since the last election or simply because of the inherent uncertainty in human nature, the voter’s preference may have changed to some extent (quantified as \( \delta_v \)).

1.1 Contribution

In our basic problem, called Stable Manipulation, the input is a set \( A \) of \( m \) alternatives, a set \( V \) of \( n \) voters associated with each of them a believed preference, a distinguished alternative \( c \), and a number \( \ell \) of manipulators. We need to compute if there exists a preference profile for the manipulators which makes \( c \) win the election irrespective of any deviation of every other voter \( v \) from her believed preference \( \succ_v \) by at most \( \delta_v \) under Kendall-Tau distance. We prove that the Stable Manipulation problem is polynomial time solvable for the \( k \)-approval voting rule for any number of manipulators [Theorem 3]. If we have only manipulator (that is \( \ell = 1 \)), then the Stable Manipulation problem is polynomial time solvable for every scoring rule, maximin, Bucklin, and simplified Bucklin voting rules. On the other hand, we show that the Stable Manipulation problem is \( \text{co}-\text{NP} \) – hard for the Copeland\(^\alpha \) voting rule even with one manipulator [Theorem 6]. We summarize our complexity theoretic results in Table 1. Other than that, we show that, for a constant number of alternatives, the stable manipulation problem is polynomial time solvable for every anonymous and efficient voting rule for any number of manipulators [Theorem 7]. Finally we empirically show that the probability that a random profile being stably manipulable drastically decreases even if the manipulator’s uncertainty about others' votes increases slightly for all the common voting rules studied here.

| Voting Rule          | Stable Manipulation                  |
|----------------------|-------------------------------------|
| \( k \)-approval     | \( \mathbb{P} \) for any number of manipulators [Theorem 3] |
| Scoring rules        | \( \mathbb{P} \) for single manipulator [Theorem 1] |
| maximin              | \( \mathbb{P} \) for single manipulator [Theorem 2] \text{NP-hard for } \geq 2 \text{manipulators} [Observation 1] [Faliszewski et al. [FHS08, FHS10]] |
| Copeland\(^\alpha\)   | \textbf{co-NP – Hard} even for single manipulator [Theorem 6] |
| Bucklin              | \( \mathbb{P} \) for single manipulator [Theorem 5] |
| Simplified Bucklin   | \( \mathbb{P} \) for single manipulator [Theorem 4] |

Table 1: Summary of results for Stable Manipulation. Our algorithms work even for the case when manipulators have different \( \delta \) value for different voters. Our hardness work even when the manipulators have the same \( \delta \) value for every voter. Results in bold are proved in this paper.
1.2 Related Work

Initiated by Bartholdi et al. the study of manipulation has been one of the key research focus in computational social choice [BTT89, BO91]. Conitzer et al. showed that, for weighted elections, the coalition manipulation problem which is manipulation by a coalition of voters, is NP-complete even when we have a small constant number of alternatives for most of the commonly used voting rules [CSL07]. Faliszewski and Procaccia exhibited evidence that the computational problem of manipulation may not be computationally challenging on average [FP10]. Mossel and Rácz and Friedgut et al. showed that, for a uniformly random preference profile, reporting a random preference results in a successful manipulation with high probability (1 over some polynomial in the number of voters and the number of alternatives) for any reasonable voting rule [MR15, FKKN11]. We refer to [CW16] for an excellent overview of the computational problem of manipulation. Manipulation comes under a more general class of problems known as election control problems. Election control refers to the phenomenon of influencing the outcome of an election through various means. Other than manipulation, prominent examples of election control problems include bribery, voter deletion, alternative deletion, voter partition, alternative partition, etc. We refer to [FR16] for a comprehensive survey of various kind of election control problems studied in computational social choice.

Limiting manipulators’ access to other voters’ preference profile has been studied before. Conitzer et al. defined the dominating manipulation problem in a bid to model manipulator’s limited information and showed that the commonly used voting rules, except plurality and veto, are resistant to this kind of manipulation [CWX11]. Dey et al. captured the manipulator’s approach to risk into the concept of weak, strong, and opportunistic manipulation and showed that the weak as well as opportunistic manipulations are intractable for all commonly used voting rules except plurality and veto whereas the strong manipulation admits a polynomial time algorithm for most of the common voting rules [DMN18]. Both the above papers model manipulator’s limited information as a partial preference profile (a partial preference for each voter); for every other voter, the manipulator knows the ordering of some of the pairs for sure but does not have any clue about the remaining pairs. On the other hand, our model of manipulator’s limited information cannot be modelled as partial preferences over voters; in our model, intuitively speaking, manipulator’s lack of information is distributed over all pairs of alternatives.

2 Preliminaries

Let us denote the set \{1, 2, \ldots, n\} by \([n]\) for any positive integer \(n\). Let \(A = \{a_1, a_2, \ldots, a_m\}\) be a set of alternatives or alternatives and \(V = \{v_1, v_2, \ldots, v_n\}\) a set of voters. If not mentioned otherwise, we denote the number of alternatives by \(m\) and the number of voters by \(n\). Every voter \(v_i\) has a preference or vote \(\succ_i\) which is a complete order over \(A\). We denote the set of complete orders over \(A\) by \(\mathcal{L}(A)\). We call a tuple of \(n\) preferences \((\succ_1, \succ_2, \ldots, \succ_n)\) \(\in \mathcal{L}(A)^n\) an \(n\)-voter preference profile. An election is defined as a set of alternatives together with a voting profile. Let \(\cup\) denote the disjoint union of sets. A map \(r: \cup_{n,1,\mathcal{A}}(n,1,\mathcal{L}(A)) \rightarrow 2^A \setminus \{\emptyset\}\) is called a voting rule. A voting rule \(r\) is called efficient if the winners under \(r\) can be computed in polynomial time. A voting rule is called anonymous if the set of winners does not depend on the name of the voters. For a voting rule \(r\) and a preference profile \(\succ\) \(\in (\succ_1, \ldots, \succ_n)\), we say a alternative \(x\) wins uniquely if \(r(\succ) = \{x\}\) and \(x\) co-wins if \(x \in r(\succ)\). For a vote \(\succ \in \mathcal{L}(A)\) and two alternatives \(x, y \in A\), we say \(x\) is placed before \(y\) in \(\succ\) if \(x \succ y\); otherwise we say \(x\) is placed after \(y\) in \(\succ\).

A alternative is said to be at the \(i\)th position from the top/left (bottom/right) if there are exactly \((i−1)\) alternatives before (after) it. For any two alternatives \(x, y \in A\) with \(x \neq y\) in an election \(E = (A, \mathcal{P})\), let us define the margin \(\Delta_\mathcal{P}(x, y)\) of \(x\) from \(y\) to be \(|\{i : x \succ_i y\}| - |\{i : y \succ_i x\}|\). Examples of some common voting rules are as follows.

- **Positional scoring rules:** A collection \((\vec{s}_m)_{m \in \mathbb{N}^+}\) of \(m\)-dimensional vectors \(\vec{s}_m = (s_{m,1}, s_{m,2}, \ldots, s_{m,m}) \in \mathbb{N}^m\) with \(s_{m,1} \geq s_{m,2} \geq \ldots \geq s_{m,m}\) and \(s_{m,1} > s_{m,m}\) for every \(m \in \mathbb{N}^+\) naturally defines a voting rule — a alternative gets score \(\alpha_i\) from a vote if it is placed at the \(i\)th position from the bottom, and the score of a alternative is the sum of the scores it receives from all the votes. The winners are the alternatives with maximum score. If \(\alpha_i = 1\) for \(i \in [k]\) and 0 otherwise, then we get the \(k\)-approval voting rule. If \(\alpha_i = m - i\), then we get the Borda rule.

- **Copeland**\(\alpha\): Given \(\alpha \in [0, 1]\), the Copeland\(\alpha\) score of a alternative \(x\) is \(|\{y \neq x : \Delta_\mathcal{P}(x, y) > 0\}|\).
Proposition 1. Observations 1.

If \( i \) manipulation setting only in this short version. For the classical \( M \) easily extend to the unique winner setting. For ease of exposition, we restrict ourselves to the co-winner the problem in the unique winner setting. We remark that all our results, both algorithmic and hardness, in a majority of the preferences.

- **Maximin:** The maximin score of a alternative \( x \) in an election \( E \) is \( \min_{y \neq x} \mathcal{D}_p(x, y) \). The winners are the alternatives with maximum maximin score.

- **Bucklin and simplified Bucklin:** Let \( \ell \) be the minimum integer such that there exists at least one alternative \( x \in A \) whom more than half of the voters place in their top \( \ell \) positions. Then the Bucklin winners are the alternatives who are placed most number of times within top \( \ell \) positions of the votes. The simplified Bucklin winners are the alternatives who appear within the top \( \ell \) positions in a majority of the preferences.

- **Single transferable vote (STV):** In every iteration, the alternative with lowest plurality score (breaking tie using some tie-breaking rule) drops out from the election. The alternative remaining after \( m - 1 \) iterations is the winner.

The Kendall-Tau distance between a pair of preferences \( \succ, \succ' \in \mathcal{L}(A) \), denoted by \( d_{\text{KT}}(\succ, \succ') \), is the number of pairs of alternatives where \( \succ \) and \( \succ' \) differ; that is \( d_{\text{KT}}(\succ, \succ') = |\{(a, b) \in A \times A : a \succ b, a \succ' b \}| \). Alternatively, the Kendall-Tau distance is the number of adjacent swaps needed to convert a preference into another. In this draft, by swaps, we mean only adjacent swaps.

### 2.1 Problem Definition

We now define our problem formally.

**Definition 1 (STABLE MANIPULATION).** Given a set \( A \) of \( m \) alternatives, an \( n \)-voter profile \( \mathcal{P} = \{\succ_i\}_{i \in [n]} \in \mathcal{L}(A)^n \) over \( A \), a distinguished alternative \( c \in A \), a tuple \( (\delta_i)_{i \in [n]} \) of non-negative integers, and the number \( \ell \) (a positive integer) of manipulators, compute if there exists a profile \( (\succ'_1, \ldots, \succ'_{n+\ell}) \in \mathcal{L}(A)^{n+\ell} \) such that, we have \( c \in \tau((\succ'_1, \ldots, \succ'_{n+\ell})) \) for every \( (\succ'_i)_{i \in [n]} \in \mathcal{L}(A)^n \) with \( d_{\text{KT}}(\succ_i, \succ'_i) \leq \delta_i \) for every \( i \in [n] \). We denote an arbitrary instance of **STABLE MANIPULATION** by \( (A, \mathcal{P}, c, (\delta_i)_{i \in [n]}, \ell) \).

The above definition of **STABLE MANIPULATION** requires \( c \) to be a co-winner. One can similarly pose the problem in the unique winner setting. We remark that all our results, both algorithmic and hardness, easily extend to the unique winner setting. For ease of exposition, we restrict ourselves to the co-winner setting only in this short version. For the classical **MANIPULATION** problem, we have \( \delta_i = 0 \) for every \( i \in [n] \). Hence, we have the following observation.

**Observation 1.** If **MANIPULATION** is \( \text{NP-hard} \) for a voting rule \( r \) with \( \ell \) manipulators, then **STABLE MANIPULATION** is also \( \text{NP-hard} \) for \( r \) with \( \ell \) manipulators.

**Proposition 1.** (i) The **STABLE MANIPULATION** problem is \( \text{NP-complete} \) for the maximin and Copeland\(^\alpha \), \( \alpha \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1] \), voting rules even for 2 manipulators.

### 3 Results

We present our theoretical results in this section. We begin with presenting our algorithm for the scoring rules.

**Theorem 1.** There is a polynomial time algorithm for the **STABLE MANIPULATION** problem for every scoring rule if we have only one manipulator.

**Proof.** Let \( \alpha = (\alpha_i)_{i \in [m]} \) be an arbitrary scoring rule and \( (A, \mathcal{P}, c, (\delta_i)_{i \in [n]}, \ell = 1) \) an arbitrary instance of **STABLE MANIPULATION** for \( \alpha \). We iteratively try construct a manipulator’s preference \( \succ_M \) which results in a stable manipulation. Without loss of generality, we place the alternative \( c \) at the first position of \( \succ_M \) in the first iteration. Iteratively, suppose we have already placed alternatives at every position in \{1, \ldots, t - 1\} for some \( 2 \leq t \leq m \) and we next wish to place an alternative at the \( t \)-th position. Let \( A_{t-1} \) be the set of alternatives which are placed within first \( t - 1 \) positions; we obviously have \( c \in A_{t-1} \). We now check if there exists an alternative \( a \in A \setminus A_{t-1} \) which can be placed at the \( i \)-th position “safely”,
then we place the alternative \(a\) at the \(i\)-th position and go to next iteration; otherwise we output that the instance is a **NO** instance. We say that the position \(t\) is “safe” for an alternative \(a \in A \setminus A_{t-1}\) if there does not exist any \(n\)-voter profile \(Q\) such that (i) the Kendall-Tau distance between the \(j\)-th preferences of \(P\) and \(Q\) is at most \(\delta_i\) for every \(j \in [n]\) and (ii) the score of the alternative \(a\) is more than the score of the alternative \(c\) in the profile \((\Omega, \succ_M)\) where \(\succ_M\) is any preference which places the alternatives \(c\) and \(a\) at positions 1 and \(t\) respectively. We next describe how to check, in polynomial time, whether a position \(t \in \{2, \ldots, m\}\) is safe for an alternative \(a\).

Before proceeding further, let us define some notation. For a preference \(\succ \in L(A)\) and an alternative \(a \in A\), let \(\text{rank}(\succ, a)\) be the position of the alternative \(a\) in the preference order \(\succ\). We define \(\text{RS}(\succ, a, k)\) to be the preference \(\succ'\) obtained by shifting the alternative \(a\) to the right by \(\min(k, m - \text{rank}(\succ, a))\) positions. Similarly, we define \(\text{LS}(\succ, a, k)\) to be the preference \(\succ'\) obtained by shifting the alternative \(a\) to the left by \(\min(k, \text{rank}(\succ, a) - 1)\) positions. We use \(S(\succ, a)\) and \(S(\succ, a)\) to denote the score of alternative \(a\) in a preference \(\succ\) and a preference profile \(\succ\). From the given preference profile \(P' = (\succ_{i})_{i \in [n]}\), we construct another preference profile \(Q' = (\succ_{i})_{i \in [n]}\) as follows. For every \(i \in [n]\), let \(i_{i} \in \{0, 1, \ldots, \delta_i\}\) be the integer by which degrading the position of the alternative \(c\) and followed by improving the position of the alternative \(a\) in \(x_i\) is worst possible for \(c\) with respect to \(a\) in \(x_i\). Formally, for \(j \in \{0, 1, \ldots, \delta_i\}\), let \(\Delta(\succ_{i}, a, j, c, \delta_i - j)\) be the decrease of the score of the alternative \(a\) plus the increase in the score of \(c\) if we degrade the position of \(c\) in the preference \(\succ_{i}\) by \(j\) and then we improve the position of \(a\) by \(\delta_i - j\); that is \(\Delta(\succ_{i}, a, j, c, \delta_i - j) = S(\succ_{i}, a) - S(LS(\succ_{i}, c, j)) - S(LS(\succ_{i}, c, j)) + S(LS(\succ_{i}, a, j))\). Then we have \(i_{i} \in \arg \max_{j \in \{0, 1, \ldots, \delta_i\}} \Delta(\succ_{i}, a, j, c, \delta_i - j)\). We define \(\succ_{i}' = \text{LS}(\succ_{i}, c, i_{i})\). Then we have \(\succ_{i}' \succ_S (\succ_{i}, c, \delta_i)\) for all \(i \in [n]\). For an alternative \(a \in A \setminus A_{t-1}\) and position \(t\), we say that the position \(t\) (in the manipulator’s vote) is safe for the alternative \(a\) if \(S(Q', a) + \alpha_1 \geq S(Q', a) + \alpha_t\). This concludes the description of our algorithm. Clearly our algorithm runs in polynomial time. We next prove its correctness.

Suppose the algorithm outputs that the input instance is a **YES** instance. Then we claim that the manipulator’s preference \(\succ_M\) is a successful stable manipulation. Suppose not, then there exist an alternative \(a \in A \setminus \{c\}\) and an \(n\)-voters preference profile \(Q'\) such that (i) the Kendall-Tau distance between the \(j\)-th preferences of \(P\) and \(Q'\) is at most \(\delta_i\) for every \(j \in [n]\) and (ii) the score of the alternative \(a\) is more than the score of the alternative \(c\) in the profile \((Q', \succ_M)\). Suppose the position of the alternative \(a\) in \(\succ_M\) be \(j_a \in \{2, \ldots, m\}\). Then, from the design of the algorithm it follows that \(S(Q', a) + \alpha_1 \geq S(Q', a) + \alpha_t\). From the construction of \(Q', c\) it follows that \(S(Q', c) + \alpha_1 \geq S(Q', a) - S(Q', a)\). Thus we have the following

\[
S(Q', a) - S(Q', c) \leq S(Q', a) - S(Q', a) \leq \alpha_1 - \alpha_t
\]

which implies that \(S(Q', a) + \alpha_1 \geq S(Q', a) + \alpha_t\). However, this contradicts our assumption that the score of the alternative \(a\) is more than the score of the alternative \(c\) in the profile \((Q', \succ_M)\). Hence the instance was indeed a **YES** instance. Now suppose the algorithm outputs that the input instance is a **NO** instance. Now there exists an integer \(t \in \{2, \ldots, m\}\) such that the algorithm does not find any alternative in the \(t\)-th iteration to be placed at position \(t\) safely. We observe that, if a position \(k\) is unsafe for an alternative \(x \in A \setminus \{c\}\), then the position \(k - 1\) is also unsafe for \(x\). Then we have \(m - t + 1\) alternatives, namely the alternatives in the set \(A \setminus A_{t-1}\), who must appear within the rightmost \(m - t\) positions of any manipulator’s preference \(\succ_M\). If \(\succ_M\) results in a successful stable manipulation which is, by pigeon holing principle, impossible. Hence the input instance was indeed a **NO** instance and thus the algorithm is correct.

We next present our polynomial time algorithm for the **Stable Manipulation** problem for the maximin voting rule if we have only one manipulator.

**Theorem 2.** There exists a polynomial time algorithm for the **Stable Manipulation** problem for the maximin voting rule if we have only one manipulator.

**Proof.** Let \((A, \mathcal{P}, c, (\delta_i)_{i \in [n]}, \ell = 1)\) be an arbitrary instance of **Stable Manipulation** for the maximin voting rule. On a high level, our algorithm for the maximin voting rule is similar to our algorithm for scoring rules: we put \(c\) at the first position of the manipulator’s vote in the first iteration, and then iteratively, in the \(t\)-th iteration, if \(A_{t-1}\) is the set of alternatives within the first \(t - 1\) positions, we place an alternative \(a \in A \setminus A_{t-1}\) if it is safe; that is, given the partial preference of the manipulator constructed
so far, placing the alternative \( a \) at the \( t \)-th position does not make the maximin score of \( a \) become more than the maximin score of any \( n \)-voters preference profile \( \Omega \) where the Kendall Tau distance between the \( i \)-th preferences of \( \mathcal{P} \) and \( \Omega \) is not more than \( \delta_i \). The only thing that changes here from the algorithm in Theorem 1 is the algorithm for checking safety which we explain below.

We begin with assuming that the given alternative \( a \) cannot be placed safely at \( t \)-th position given the alternatives at the first \( t-1 \) positions; that is there exists an \( n \)-voters preference profile \( \Omega \) with properties stated above. We first guess an alternative \( b \in A \setminus \{c\} \) (the alternative \( b \) can be the alternative \( a \) itself) such that the maximin score of \( a \) in \( \Omega \) is \( \mathcal{D}_{\Omega}(c,b) \). From the given preference profile \( \mathcal{P} = \{\succ_i\}_{i \in [n]} \), we construct another preference profile \( \mathcal{Q}^a_{\delta_i} = \{\succ_i^{a}\}_{i \in [n]} \) so that (i) the Kendall Tau distance between the \( i \)-th preferences of \( \mathcal{P} \) and \( \mathcal{Q} \) is not more than \( \delta_i \), and (ii) the difference between the maximin score of \( a \) and \( c \) is the maximum possible (the maximin score of \( a \) being higher). For an \( i \in [n] \), the preference \( \succ_i \) can be one of the following types:

**Case I** – it is possible to place the alternative \( c \) on the right of the alternative \( b \) by swapping at most \( \delta_i \) pairs of alternatives: Let \( j_i \) be the minimum number of swaps needed in \( \succ_i \) to place the alternative \( c \) on the right of the alternative \( b \); \( j_i \) is 0 if \( c \) already appears on the right of \( b \). Then we define \( \succ_i^{c} = \text{LS}(\succ_i, c, j_i), a, \delta_i - j_i); \) that is, we first shift \( c \) right to place it immediately after \( b \) and then shift \( a \) left as much as we can.

**Case II** – it is not possible to place the alternative \( c \) on the right of the alternative \( b \) by swapping at most \( \delta_i \) pairs of alternatives: We define \( \succ_i^{c} = \text{LS}(\succ_i, a, \delta_i); \) that is, we shift \( a \) left as much as we can.

This finishes the description of the preference profile \( \mathcal{Q}^a_{\delta_i} \). Let \( \succ_i^{c} \) be any arbitrary completion of the partially constructed preference of the manipulator. We declare the alternative \( a \) to be safe at the \( t \)-th iteration if, for every alternative \( b \in A \setminus \{c\} \), the alternative \( c \) co-wins in the preference profile \( \{\mathcal{Q}^a_{\delta_i}, \succ_i^{c}\} \). This concludes the description of our algorithm. Clearly our algorithm runs in polynomial time. We next prove its correctness.

Suppose the algorithm outputs that the input instance is a YES instance. Then we claim that the manipulator’s preference \( \succ_M \) is a successful stable manipulation. Suppose not, then there exist an alternative \( a \in A \setminus \{c\} \) and an \( n \)-voters preference profile \( \mathcal{R}^a \) such that (i) the Kendall-Tau distance between the \( j \)-th preferences of \( \mathcal{P} \) and \( \mathcal{R}^a \) is at most \( \delta_j \) for every \( j \in [n] \) and (ii) the score of the alternative \( a \) is more than the score of the alternative \( c \) in the profile \( \{\mathcal{R}^a, \succ_M\} \). Suppose the position of the alternative \( a \) in \( \succ_M \) be \( j_a \in \{2, \ldots, m\} \) and \( b \in A \setminus \{c\} \) be an alternative such that the maximin score of the alternative \( c \) is \( \mathcal{D}_{\{\mathcal{R}^a, \succ_M\}}(c,b) \). Then, from the design of the algorithm it follows that the maximin score of \( a \) is more than the maximin score of \( c \) in the profile preference \( \{\mathcal{Q}^a_{\delta_i}, \succ_M\} \). This contradicts our assumption that the algorithm declared that placing the alternative \( a \) at the \( j_a \)-th position in the \( j_a \)-th iteration was safe (since the algorithm has placed the alternative \( a \) at the \( j_a \)-th position in the \( j_a \)-th iteration). Hence, the input instance is indeed a YES instance. Now suppose the algorithm outputs that the input instance is a NO instance. Then, there exists an integer \( t \in \{2, \ldots, m\} \) such that the algorithm finds that it is unsafe for every alternative \( a \in A \setminus A_{t-1} \) to appear before every alternative in \( A \setminus (A_{t-1} \cup \{a\}) \). However, in every possible manipulators preference \( \succ_M \), there exists an alternative \( a \in A \setminus A_{t-1} \) which appears before every alternative in \( A \setminus (A_{t-1} \cup \{a\}) \). Hence the input instance is indeed a NO instance and thus the algorithm is correct.

For the \( k \)-approval voting rule, we are able to reduce the **Stable Manipulation** problem with any number of manipulators to an equivalent maximum flow problem thereby obtaining a polynomial time algorithm.

**Theorem 3.** There exists a polynomial time algorithm for the **Stable Manipulation** problem for the \( k \)-approval voting rule for any number of manipulators and any \( k \).

**Proof.** Let \( (A, \mathcal{P}, c, \{\delta_i\}_{i \in [n]}, t) \) be an arbitrary instance of **Stable Manipulation** for the \( k \)-approval voting rule. We may assume without loss of generality that the alternative \( c \) is placed at the first position in every preference of the manipulators. For every alternative \( a \in A \setminus \{c\} \), we compute the maximum number \( \lambda_a \) of manipulators’ preferences where the alternative \( a \) can appear within the first \( k \) positions in any manipulators’ preference profile which results in successful stable manipulation. Each preference \( \succ_i, i \in [n] \) belongs to exactly one of the following types: (i) simultaneously \( c \) can be placed outside of the first \( k \) positions and \( a \) can be placed within the first \( k \) positions without changing order of more than \( \delta_i \) pairs of alternatives in \( \succ_i \) (ii) \( c \) can be placed outside of the first \( k \) positions without changing order of more than \( \delta_i \) pairs of alternatives and \( a \) can not be placed within the first \( k \) positions without
changing order of more than \( \delta_1 \) pairs of alternatives in \( \succ_i \) (iii) \( c \) can not be placed outside of the first \( k \) positions without changing order of more than \( \delta_1 \) pairs of alternatives and \( a \) can be placed within the first \( k \) positions without changing order of more than \( \delta_1 \) pairs of alternatives in \( \succ_i \) (iv) either \( c \) can be placed outside of the first \( k \) positions or \( a \) can be placed within the first \( k \) positions (but not both) without changing order of more than \( \delta_1 \) pairs of alternatives in \( \succ_i \) (v) both \( c \) can not be placed outside of the first \( k \) positions without changing order of more than \( \delta_1 \) pairs of alternatives and \( a \) can not be placed within the first \( k \) positions without changing order of more than \( \delta_1 \) pairs of alternatives in \( \succ_i \).

Let the number of such preferences respectively be \( n_1, n_2, n_3, n_4, \) and \( n_5 \). Let \( S(\mathcal{P}, x) \) be the \( k \)-approval score of any alternative \( x \in A \) in the profile \( \mathcal{P} \). We define \( \lambda_a = (\ell + n_3 + n_5) - (n_1 + n_3) = \ell + n_5 - n_1 \); that is, loosely speaking, the worst profile for \( c \) with respect to the alternative \( a \) derived from \( \mathcal{P} \) is to push \( c \) outside first \( k \), if possible, and then push \( a \) within first \( k \), if possible. If, for any alternative \( a \in A \setminus \{c\} \), we have \( \lambda_a < 0 \), then the algorithm outputs NO.

We now construct the following flow network \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, s, t, c: E \rightarrow \mathbb{N}_{\geq 1}) \).

\[
\mathcal{V} = \{s, t\} \cup \{u_i : i \in [\ell]\} \cup \{v_a : a \in A \setminus \{c\}\}
\]

\[
\mathcal{E} = \{(s, u_i) : i \in [\ell]\}
\]

\[
\cup \{(u_i, v_a) : i \in [\ell], a \in A \setminus \{c\}\}
\]

\[
\cup \{(v_a, t) : a \in A \setminus \{c\}\}
\]

We now describe the capacities of the edges. The capacity of each outgoing edge from \( s \) is \( k - 1 \). For \( a \in A \setminus \{c\} \), the capacity of the edge \( (v_a, t) \) is \( \lambda_a \). The capacity of every other edge is 1. The algorithm outputs that the instance is a YES instance if and only if there is an \( s \rightarrow t \) flow in \( \mathcal{G} \) of value \( \ell(k - 1) \). This finishes the description of the algorithm. We now prove its correctness.

Suppose the algorithm outputs that the instance is a YES instance. Then \( \mathcal{G} \) has an \( s \rightarrow t \) flow \( f \) of value \( \ell(k - 1) \). We may assume without loss of generality that every edge carries an integral flow in \( f \) since the capacity of every edge is some integer and \( f \) is a maximum flow. We now construct a preference profile \( \mathcal{P}_M \) of \( \ell \) manipulators. The manipulator \( i \) places an alternative \( a \in A \setminus \{c\} \) within first \( k \) positions in \( \mathcal{P}_M \) if and only if the edge \( (u_i, v_a) \) carries 1 unit of flow under \( f \). Since every manipulator places \( c \) at the first position of her preference, and the incoming flow at vertex \( u_i, i \in [\ell] \) is \( k - 1 \) under \( f \), we have described which \( k \) alternatives appear within the first \( k \) position of each manipulator’s preference in \( \mathcal{P}_M \). We claim that the preference profile \( \mathcal{P}_M \) results in a successful stable manipulation. Indeed, for any \( n \)-voters profile \( \mathcal{Q} \) where the Kendall Tau distance between the \( i \)-th preferences of \( \mathcal{P} \) and \( \mathcal{Q} \) is at most \( \delta_i \), we have \( S((\mathcal{Q}, \mathcal{P}_M), c) - S(\mathcal{Q}, a) \geq \lambda_a \) for every alternative \( a \in A \setminus \{c\} \). Since every alternative \( a \in A \setminus \{c\} \) appears within the first \( k \) positions at most \( \lambda_a \) times in \( \mathcal{P}_M \), it follows that \( \mathcal{P}_M \) indeed results in successful stable manipulation. On the other hand, if the algorithm outputs NO, then one of the following two cases happen. In the first case there exists an alternative \( a \in A \setminus \{c\} \) such that \( \lambda_a < 0 \). Consider the preference profile \( \mathcal{Q}^a \) obtained from \( \mathcal{P} \) by simultaneously moving \( c \) outside first \( k \) positions in every preference of type (i), (ii), and (iv) and \( a \) within first \( k \) positions in every preference of type (i) and (iii). We observe that the k-approved score of \( a \) in \( \mathcal{Q}^a \) is more than the k-approved score of \( c \) from \( \mathcal{Q}^a \) plus \( \ell \) and thus the instance is indeed a NO instance. In the second case, suppose the algorithm outputs NO because the maximum flow of \( \mathcal{G} \) is strictly less than \( \ell(k - 1) \). In this case too, for any possible preference profile \( \mathcal{P}_M \) of the manipulators, there exists an alternative \( a \in A \setminus \{c\} \) which appears within first \( k \) positions strictly more than \( \lambda_a \) times. Then, in the profile \( (\mathcal{Q}^a, \mathcal{P}_M) \), the k-approved score of the alternative \( a \) is strictly more than the k-approved score of the alternative \( c \). Hence the instance is indeed a NO instance. \( \square \)

We next show our result for the simplified Bucklin rule.

**Theorem 4.** There exists a polynomial time algorithm for the **STABLE MANIPULATION problem for the simplified Bucklin voting rule if we have only one manipulator.**

**Proof.** Let \( (A, \mathcal{P}, c, (\delta_i)_{i \in [n]}, \ell = 1) \) be an arbitrary instance of **STABLE MANIPULATION** for the simplified Bucklin voting rule. On a high level, our algorithm for the simplified Bucklin voting rule is similar to our algorithm for scoring rules and the maximin voting rule. The only difference being, given a position \( t \in \{2, \ldots, m\} \) and an alternative \( a \in A \setminus \{c\} \), how do we decide if placing the alternative \( a \) at position \( t \) in the manipulator’s vote is safe. We describe this below and skip repeating the other parts since they are exactly similar to the algorithms for the scoring rules and the maximin voting rule.
Let us denote by \( S_k(\mathcal{R}, a) \) the \( k \)-approval score of alternative \( a \) in an \( n \)-voters profile \( \mathcal{R} \). We observe that the alternative \( c \) wins in \( \mathcal{R} \) under the simplified Bucklin rule if and only if, for every other alternative \( a \in \mathcal{A} \setminus \{c\} \) and for every \( k \in \{1, 2, \ldots, m\} \), \( S_k(\mathcal{R}, a) > \frac{n}{2} \implies S_k(\mathcal{R}, c) > \frac{n}{2} \). For any \( k \) let (i) \( n_1 \) be the number of preference \( \succ_{\mathcal{P}} \) where simultaneously \( c \) can be placed outside the first \( k \) positions and \( a \) can be placed within the first \( k \) positions by swapping at most \( \delta_1 \) pairs of alternatives (call these preference type (i)), (ii) \( n_2 \) the number of preference \( \succ_{\mathcal{P}} \) where either \( c \) can be placed outside the first \( k \) positions or \( a \) can be placed within the first \( k \) positions by swapping at most \( \delta_1 \) pairs of alternatives but not both can be done (call these preference type (ii)), (iii) \( n_3 \) the the number of preference \( \succ_{\mathcal{P}} \) where \( c \) can be placed outside the first \( k \) positions and \( a \) can not be placed within the first \( k \) positions by swapping at most \( \delta_1 \) pairs of alternatives (call these preference type (iii)), (iv) \( n_4 \) the number of preference \( \succ_{\mathcal{P}} \) where \( c \) can not be placed outside the first \( k \) positions but \( a \) can be placed within the first \( k \) positions by swapping at most \( \delta_1 \) pairs of alternatives (call these preference type (iv)), (v) \( n_5 \) the number of preference \( \succ_{\mathcal{P}} \) where neither \( c \) can be placed outside the first \( k \) positions nor \( a \) can be placed within the first \( k \) positions by swapping at most \( \delta_1 \) pairs of alternatives (call these preference type (v)). We declare that position \( t \) in the manipulator’s preference is not safe for an alternative \( a \in \mathcal{A} \setminus \{c\} \) if there exists a position \( k \in \{2, \ldots, m\} \) such that there exists an \( n \)-voters preference profile \( \mathcal{Q} \) where (i) the Kendall-Tau distance between the \( i \)-th preferences of \( \mathcal{P} \) and \( \mathcal{Q} \) is at most \( \delta_1 \) for every \( i \in [n] \) and (ii) \( c \) appears within the first \( k \) positions in at most \( \lceil \frac{n}{2} \rceil - 1 \) preferences in \( \mathcal{Q} \) (observe that, including manipulator, we have \( n + 1 \) voters in total) and \( a \) appears within the first \( k \) positions in at least \( \lceil \frac{n}{2} \rceil + 1 \) positions if \( k < t \) and \( \frac{n}{2} \) positions if \( k \geq t \). This happens if and only if there exists an integer \( \ell \in \{0, 1, \ldots, n_2\} \) (\( \ell \) corresponds to the number preferences of type (ii) which are modified to put \( c \) outside the first \( k \) positions) such that we have \( n_2 - \ell + n_4 + n_5 \leq \lceil \frac{n}{2} \rceil - 1 \) and, if \( k \leq t \), then \( n_1 + n_2 - \ell + n_4 \geq \frac{n}{2} + 1 \) and, if \( k > t \), then \( n_1 + n_2 - \ell + n_4 \geq \frac{n}{2} \). This concludes the description of our algorithm. Our algorithm clearly runs in polynomial time. We next argue its correctness. Suppose that the algorithm outputs that the instance is a YES instance. Then we claim that the manipulator’s preference \( \succ_{\mathcal{M}} \) constructed by the algorithm results in a successful stable manipulation. Suppose not, then there exists an \( n \)-voters preference profile \( \mathcal{Q} \) such that (i) the Kendall-Tau distance between the \( i \)-th preferences of \( \mathcal{P} \) and \( \mathcal{Q} \) is at most \( \delta_1 \) and (ii) \( c \) is not a simplified Bucklin winner in \( \langle \mathcal{Q}, \succ_{\mathcal{M}} \rangle \), that is there exists an alternative \( a \in \mathcal{A} \setminus \{c\} \) and a position \( k \in [m] \) such that \( c \) does not appear within the first \( k \) positions in a majority of the preferences whereas \( a \) appears within the first \( k \) positions in a majority of preferences in \( \langle \mathcal{Q}, \succ_{\mathcal{M}} \rangle \). Suppose the alternative \( a \) appears at the \( t \)-th position in \( \succ_{\mathcal{M}} \). Also let the number of preferences of type (ii) where \( c \) is put outside the first \( k \) positions be \( \ell \). Then we consider the profile \( \mathcal{Q}^a \) obtained from \( \mathcal{P} \) where

\( \triangleright \) in preferences of type (i), simultaneously \( c \) is placed outside the first \( k \) positions and \( a \) is placed within the first \( k \) positions in \( \mathcal{Q}^a \).

\( \triangleright \) in \( \ell \) number of preferences of type (ii) and all preference of type (iii), \( c \) is placed outside the first \( k \) positions in \( \mathcal{Q}^a \). In \( n_2 - \ell \) number of preferences of type (ii) and all preference of type (iv), \( a \) is put within the first \( k \) positions in \( \mathcal{Q}^a \).

\( \triangleright \) all preferences of type (v) remain the same in \( \mathcal{P} \) and \( \mathcal{Q}^a \).

It follows that, since \( c \) does not get majority within the first \( k \) positions but \( a \) gets majority within the first \( k \) positions in \( \langle \mathcal{Q}, \succ_{\mathcal{M}} \rangle \), \( c \) does not get majority within the first \( k \) positions but \( a \) gets majority within the first \( k \) positions in \( \langle \mathcal{Q}^a, \succ_{\mathcal{M}} \rangle \). This contradicts our assumption that the algorithm declared the position \( t \) in the manipulator’s preference safe for the alternative \( a \). Hence the input instance is indeed a YES instance.

Now suppose that the algorithm outputs that the instance is a NO instance. For the sake of arriving to a contradiction, let us assume that there exists a manipulator’s preference \( \succ_{\mathcal{M}} \in \mathcal{L}(\mathcal{A}) \) which results in successful stable manipulation. Since our algorithm outputs NO, there exists an iteration \( t \in \{2, \ldots, m\} \) such that, if \( \mathcal{A}_{t-1} \) is the set of alternatives already placed in the first \( t - 1 \) positions by the algorithm, then every alternative \( a \in \mathcal{A} \setminus \mathcal{A}_{t-1} \) was judged unsafe for the position \( t \) in the manipulator’s preference. In this case, there indeed exists an \( n \)-voters profile \( \mathcal{Q}^a \in \mathcal{L}(\mathcal{A})^n \) for every alternative \( a \in \mathcal{A} \setminus \mathcal{A}_{t-1} \) such that (i) the Kendall-Tau distance between the \( i \)-th preferences of \( \mathcal{P} \) and \( \mathcal{Q}^a \) is at most \( \delta_1 \) and (ii) \( c \) appears within the first \( k \) positions in at most \( \lceil \frac{n}{2} \rceil - 1 \) preferences in \( \mathcal{Q}^a \) (observe that, including manipulator, we have \( n + 1 \) voters in total) and \( a \) appears within the first \( k \) positions in at least \( \lceil \frac{n}{2} \rceil + 1 \) positions. if
We declare position $t$ corresponding within the first $T$ preference profile $\tau$ since there are $t$ checked in polynomial time whether $\tau$ results in a successful stable manipulation which is, by pigeon holing principle, impossible. Hence the input instance was indeed a NO instance and thus the algorithm is correct.

The main idea of Theorem 4 can be extended to design a polynomial time algorithm for the Bucklin voting rule.

**Theorem 5.** There exists a polynomial time algorithm for the stable manipulation problem for the Bucklin voting rule if we have only one manipulator.

**Proof.** Let $(A, P, c, (\delta_i)_{i \in [n]}, \ell = 1)$ be an arbitrary instance of Stable Manipulation for the Bucklin voting rule. On a high level, our algorithm for the Bucklin voting rule is similar to our algorithms in Theorem 1, 2 and 4. The only difference being, given a position $t \in \{2, \ldots, m\}$ and an alternative $a \in A \setminus \{c\}$, how do we decide if placing the alternative $a$ at position $t$ in the manipulator’s vote is safe.

We describe this below and skip repeating the other parts.

Let us denote by $S_k(\mathcal R, a)$ the $k$-approval score of alternative $a$ in an $n$-voters profile $\mathcal R$. We observe that the alternative $c$ wins in $\mathcal R$ under the Bucklin rule if and only if, for every other alternative $a \in A \setminus \{c\}$ and for every $k \in \{1, 2, \ldots, m\}$, $S_k(\mathcal R, a) > \frac{k}{2} \implies S_{k-1}(\mathcal R, c) > \frac{k}{2}$ or $S_k(\mathcal R, c) \geq S_k(\mathcal R, a)$. Given a preference $\succ \in \mathcal L(A)$ and an alternative $a \in A \setminus \{c\}$, we define the following set $X = \{x_1, x_2, x_3, x_4\}$ of Boolean variables.

(i) We say that $\succ$ satisfies $x_1$ if and only if the alternative $c$ does not appear within the first $k$ positions in $\succ$.

(ii) We say that $\succ$ satisfies $x_2$ if and only if the alternative $c$ does not appear within the first $k-1$ positions in $\succ$.

(iii) We say that $\succ$ satisfies $x_3$ if and only if the alternative $a$ appears within the first $k-1$ positions in $\succ$.

(iv) We say that $\succ$ satisfies $x_4$ if and only if the alternative $a$ appears within the first $k$ positions in $\succ$.

We define the “type” $T(\succ, a) \subseteq X$ of the preference $\succ$ with respect to an alternative $a \in A \setminus \{c\}$ to be the subset of $X$ satisfied by the preference $\succ$. Before we explain our algorithm for deciding whether a position $t$ is safe for an alternative $a \in A \setminus \{c\}$, we need to define few concepts and notation. Given a preference $\succ \in \mathcal L(A)$, an alternative $a \in A \setminus \{c\}$, and a distance $\delta$, we define the “meta-type” $M(\succ, a, \delta)$ of $\succ$ with respect to $a$ and $\delta$ as the set of all types reachable from $\succ$ within a Kendall-Tau distance of at most $\delta$; that is $M(\succ, a, \delta) = \{Z \subseteq X : \exists Z' \in \mathcal L(A), d_{KT}(\succ, Z') \leq \delta, T(Z', a) = Z\} \subseteq 2^X$. Let the set of all possible meta-types be $M = \{|M_i| : i \in \{\gamma\}\}$. An important observation is that, since $X$ has only 4 elements, only $2^4 = 65536$ (which is a constant) different meta-types are possible. For ease of exposition, let us define $\gamma = 16, \nu = 65536$. For an alternative $a \in A \setminus \{c\}$, let $n_i$ be the number of preferences in $P$ of meta-type $M_i$ and $\lambda_i$ the number of types in the meta-type $M_i$; that is $M_i = \{|T_{i,1}, \ldots, T_{i,\lambda_i}\}$. Another important observation is that, given any preference $\succ \in \mathcal L(A)$, a distance $\delta$, and a type $M \in M_i$, it can be checked in polynomial time whether $M \in M(\succ, a, \delta)$; hence the set $M \in M(\succ, a, \delta)$ can be computed in polynomial time.

We now describe our algorithm for whether $a$ is safe at position $t$. For every $k \in [m]$, we check the following. For a tuple $\tau = (\tau_i)_{i \in [\nu]}$ where $\tau_i = (\ell_i)_{i \in [\lambda_i]}$ such that $\sum_{i=1}^{\nu} \ell_i = n_i$, we define an $n$-voters preference profile $\mathcal Q^\tau$ constructed by converting $\ell_i$ number of preferences of meta-type $M_i$ to preferences of type $T_{i,\ell_i}$. Clearly the Kendall-Tau distance between the $i$-th preferences of $\mathcal P$ and $\mathcal Q^\tau$ is at most $\delta_i$. We declare position $t$ to be safe for the alternative $a$ if and only if, for every $k \in [m]$ and every possible corresponding $\tau$, if $k'$ is the minimum integer such that $c$ gets majority within the first $k'$ positions, then $a$ does not get majority within the first $k' - 1$ positions, and the number of preferences where $a$ appears within the first $k'$ positions is at most the number of preferences where $c$ appears within the first $k'$ positions in $(\mathcal Q^{\tau}, >_M)$ where $>_M$ is any manipulator’s preference where $c$ and $a$ are placed respectively at positions 1 and $t$. This concludes the description of our algorithm. Our algorithm runs in polynomial time since there are $O(n^{16} \mathrm{poly}(m))$ possible tuples $\tau$. We next argue its correctness.
Suppose that the algorithm outputs that the instance is a YES instance. Then we claim that the manipulator's preference $\succ_M$ constructed by the algorithm results in a successful stable manipulation. Suppose not, then there exists an n-voters preference profile $\mathcal{O}$ such that (i) the Kendall-Tau distance between the i-th preferences of $\mathcal{O}$ and $\mathcal{O}^\delta$ is at most $\delta_i$ and (ii) $c$ is not a Bucklin winner in $(\mathcal{O}, \succ_M)$, that is there exists an alternative $a \in A$ at some position $t$ in $\succ_M$ and a position $k$ such that (i) if $k < t$, then $S_k(\mathcal{O}, a) \geq \frac{n}{2} + 1, S_{k-1}(\mathcal{O}, c) \leq \frac{n}{2} - 1$, and $S_k(\mathcal{O}, a) > S_k(\mathcal{O}, c)$, and (ii) if $k \geq t$, then $S_k(\mathcal{O}, a) \geq \frac{n}{2} + 1, S_{k-1}(\mathcal{O}, c) \leq \frac{n}{2} - 1$, and $S_k(\mathcal{O}, a) > S_k(\mathcal{O}, c)$. Let us consider the set $X$ of Boolean variables with respect to the position $k$ and the alternative $a$. Suppose the alternative $a$ appears at the $t$-th position in $\succ_M$. Let us define a tuple $\tau = (\tau_i)_{i \in [n]}$ where $\tau_i = (\ell_i)_{j \in [m]}$ such that $\ell_i$ is the number of preferences of meta-type $M_i$ that are converted into a preferences of type $\ell_j$ in $\mathcal{O}$. Then it follows that (i) if $k < t$, then $S_k(\mathcal{O}^\delta, a) \geq \frac{n}{2} + 1, S_{k-1}(\mathcal{O}^\delta, c) \leq \frac{n}{2} - 1$, and $S_k(\mathcal{O}^\delta, a) > S_k(\mathcal{O}^\delta, c)$, and (ii) if $k \geq t$, then $S_k(\mathcal{O}^\delta, a) \geq \frac{n}{2} + 1, S_{k-1}(\mathcal{O}^\delta, c) \leq \frac{n}{2} - 1$, and $S_k(\mathcal{O}^\delta, a) > S_k(\mathcal{O}^\delta, c)$ which contradicts the fact that the algorithm declared the position $t$ in the manipulator's preference to be safe for the alternative $a$. Hence the instance is indeed a YES instance.

Now suppose that the algorithm outputs that the instance is a NO instance. For the sake of arriving to a contradiction, let us assume that there exists a manipulator's preference $\succ_M \in \mathcal{L}(A)$ which results in successful stable manipulation. Since our algorithm outputs NO, there exists an iteration $t \in \{2, \ldots, m\}$ such that, if $A_{t-1}$ is the set of alternatives already placed in the first $t-1$ positions by the algorithm, then every alternative $a \in A \setminus A_{t-1}$ was judged unsafe for the position $t$ in the manipulator's preference. In this case, there indeed exists an n-voters profile $\mathcal{O}^\delta \in \mathcal{L}(A)^n$ for every alternative $a \in A \setminus A_{t-1}$ such that (i) the Kendall-Tau distance between the i-th preferences of $\mathcal{O}$ and $\mathcal{O}^\delta$ is at most $\delta_i$ and (ii) there exists some position $k$ such that (a) if $k < t$, then $S_k(\mathcal{O}^\delta, a) \geq \frac{n}{2} + 1, S_{k-1}(\mathcal{O}^\delta, c) \leq \frac{n}{2} - 1$, and $S_k(\mathcal{O}^\delta, a) > S_k(\mathcal{O}^\delta, c)$, and (b) if $k \geq t$, then $S_k(\mathcal{O}^\delta, a) \geq \frac{n}{2} + 1, S_{k-1}(\mathcal{O}^\delta, c) \leq \frac{n}{2} - 1$, and $S_k(\mathcal{O}^\delta, a) > S_k(\mathcal{O}^\delta, c)$. We observe that, if a position $k$ is unsafe for an alternative $a \in A \setminus \{c\}$, then the position $k-1$ is also unsafe for $c$. Then we have $m-t+1$ alternatives, namely the alternatives in the set $A \setminus A_{t-1}$, who must appear within the rightmost $m-t$ positions of any manipulator's preference $\succ_M$ if $\succ_M$ results in a successful stable manipulation which is, by pigeon holing principle, impossible. Hence the input instance was indeed a NO instance and thus the algorithm is correct.

Due to Theorem 1, 2, 4 and 5, one may suspect that there may exist a generic algorithm for the Stable Manipulation problem with one manipulator which works for the class responsive and monotone voting rules that Bartholdi et al. defined [BTT89]. Our next result refutes such possibility as we show that the Stable Manipulation problem is coNP-hard for the Copeland voting rule for every $\alpha \in [0, 1]$ which is a responsive and monotone voting rule. We reduce from the X3C problem which is the complement of the classical NP-complete problem X3C. We use the following lemma in our proof of Theorem 6. The X3C and X3C are defined as follows.

**Definition 2** (X3C and X3C). Given an universe $\mathcal{U}$ of 3n elements and a collection $\mathcal{S}$ of $m$ subsets of $\mathcal{U}$ each containing 3 elements, compute if there exists a sub-collection $\mathcal{W} \subseteq \mathcal{S}$ such that (i) $|\mathcal{W}| = n$ and (ii) $\cup_{\mathcal{W} \in \mathcal{S}} = \mathcal{U}$. An instance $(\mathcal{U}, \mathcal{S})$ of X3C is called a YES instance if there indeed exists such a $\mathcal{W}$; otherwise it is called a NO instance. The X3C is the complement problem of X3C: an instance $(\mathcal{U}, \mathcal{S})$ of X3C is a YES instance if and only if $(\mathcal{U}, \mathcal{S})$ is an NO instance of X3C.

Since X3C is NP-complete, it follows that X3C is coNP-hard.

**Lemma 1.** Let $A = \mathcal{B} \cup \Gamma$ be a set of alternatives, and $(Z_{(a,b)})$, $a, b \in \mathcal{B}, a \neq b$ be integers, all with same parity, satisfying $Z_{(a,b)} = -Z_{(b,a)} \forall a, b \in \mathcal{B}$. Let $\delta \geq 0$, a positive integer, be given. Further, suppose that $|\Gamma| > 108$, $\sum_{a, b \in \mathcal{B}, a \neq b} |Z_{(a,b)}|$. Then there exists a preference profile $\mathcal{P} = (\succ_i)_{i \in [m]}$ on the set of alternatives $A$ satisfying

1. $D_{\mathcal{P}}(a, b) = Z_{(a,b)} \forall a, b \in \mathcal{B}, a \neq b$
2. For any two alternatives $a, b \in \mathcal{B}, a \neq b$ and any preference $\succ \in \mathcal{P}$, $|\text{rank}(\succ, a) - \text{rank}(\succ, b)| > \delta$.
3. For any two alternatives $b \in \mathcal{B}, d \in \Gamma$ $D_{\mathcal{P}}(b, d) > 0$ for all profiles $\mathcal{P}' = (\succ_i')_{i \in [m]}$ which satisfy $d_{\text{KT}}(\succ_i, \succ_i') \leq \delta \forall i \in [m]$. The number of preferences in $\mathcal{P}$ is bounded by a polynomial function of $\sum_{a, b \in \mathcal{B}, a \neq b} |Z_{(a,b)}|$ and $\mathcal{P}$ can be constructed in time polynomial in $m + \sum_{a, b \in \mathcal{B}, a \neq b} |Z_{(a,b)}|$.
Proof. Follows from the proof of Lemma 13 in [Dey19].

**Theorem 6.** The Stable Manipulation problem is co-NP-hard for the Copeland voting rule for every $\alpha \in [0,1]$ even if we have only one manipulator and $\delta = 3$ for every preference.

*Proof.* We now prove co-NP hardness. We reduce from $\overline{3}$C. Let $\{U = \{u_i : i \in [3n]\}, S = \{S_j : j \in [m]\}\}$ be an arbitrary instance of $\overline{3}$C. We consider the following instance $\{A, P = (P_1, P_2), c, (\delta_1 = 3)_{i \in [n]}, \ell = 1\}$ of Stable Manipulation.

\[
A = B \cup \Lambda \cup \Gamma \text{ where } \\
B = \{c, x, z\} \cup \{y_{u} : u \in U\}, |\Lambda| = 100n, |\Gamma| = 10m^3n^3 \\
P_1 = \{c \succ \{y_{u} : u \in S\} \succ d_1 \succ d_2 \succ d_3 \succ z \succ d_4 \succ d_5 \succ x \succ \text{others} \} \\
\text{ for some } d_i \in \Gamma, i \in [S]: \forall S \in S
\]

While describing the preferences above, whenever we say ‘others’, the unspecified alternatives are assumed to be arranged in such a way that, for every unspecified alternative $a \in A \setminus \Gamma$, there are at least 6 alternatives from $\Gamma$ in the immediate 6 positions on both left and right of $a$. We also ensure that any alternative in $\Gamma$ appears within top $10mn$ positions at most once in $P_1 \cup P_2$ whereas every alternative in $A \setminus \Gamma$ appears within top $10mn$ position in every preference in $P_1$. This is possible because $|\Gamma| = 10m^3n^3$ and $|A \setminus \Gamma| = 3n + 3$. We now add a preference profile $P_2$ such that we have the following.

\[
\begin{align*}
& \Downarrow D_{P}(c, y_{u}) = D_{P}(x, y_{u}) = 0 \text{ for every } u \in U \\
& \Downarrow D_{P}(z, c) = 0 \\
& \Downarrow D_{P}(y_{u}, z) = 6m \text{ for every } u \in U \\
& \Downarrow D_{P}(z, x) = 2(m - n) - 2
\end{align*}
\]

In $P$, every alternative in $\Lambda$ gets defeated by at least $\frac{1}{2}|\Lambda|$ of the alternatives from $\Lambda$ in pairwise elections by a margin of $6m$. Every alternative in $\{c, x\}$ defeats every alternative in $\Lambda$ in pairwise elections by a margin of $6m$. Every alternative in $\Lambda$ defeats every alternative in $\{y_{u} : u \in U\} \cup \{z\}$ in pairwise elections by a margin of $6m$. Every alternative in $A \setminus \Gamma$ defeats every alternative in $\Gamma$ in pairwise elections by a margin of $6m$. Such a profile $P_2$ (and thus $P$) exists due to Lemma 1 (applied as $B = A \setminus \Gamma$). We now claim that the two instance $s$ are equivalent.

In one direction, suppose the $\overline{3}$C instance is a yes instance. Then there does not exists an exact cover $W \subseteq S$ for $\ell$. We claim that the manipulator’s vote $\succ_M = c \succ \text{others} \succ x$ results in a stable manipulation. To see this, let $Q$ be any $n$-voters profile such that the Kendall-Tau distance between the $t$-th preferences of $P$ and $Q$ is at most 3. We first observe that, since every alternative in $\Lambda \cup \{y_{u} : u \in U\} \cup \{z\}$ gets defeated by at least $\frac{1}{2}|\Lambda|$ of the alternatives from $\Lambda$ and, for every unspecified alternative $a \in A \setminus \Gamma$, there are at least 6 alternatives from $\Gamma$ in the immediate 6 positions on both left and right of $a$ and $\delta = 3$, no alternative in $A \setminus \{c, x\}$ wins in $Q$. Let us define $W$ to be the set of $S \in S$ such that in the corresponding preference in $Q$, $c$ does not appear at the first position. If $|W| > \frac{n}{2}$, the the alternative $z$ defeats $x$ in $(Q, \succ_M)$ and consequently $c$ is a winner in $(Q, \succ_M)$. On the other hand, if $|W| = \frac{n}{2}$, then $W$ is not an exact set cover for $U$, there exists an element $u \in U$ such that $c$ defeats the alternative $y_{u}$ in $(Q, \succ_M)$ and thus $c$ is a winner in $(Q, \succ_M)$.

For the other direction, suppose the $\overline{3}$C instance is a no instance. Let $W \subseteq S$ forms an exact set cover for $\ell$. Let us consider the $n$-voters preference profile obtained from $P$ as: for every $S \in W$, in the corresponding preference, we shift $c$ to right by 3 positions; for every $S \in S \setminus W$, in the corresponding preference, we shift $x$ to left by 3 positions. It follows that, irrespective of the manipulator’s preference $\succ_M \in L(A)$, every alternative in $\{y_{u} : u \in U\}$ defeats $c$ and $x$ defeats $z$. Hence $x$ defeats $1$ more alternative than $c$ and thus the Stable Manipulation instance is a no instance. □

We now present a general result: Stable Manipulation is polynomial time solvable for any polynomial time computable anonymous voting rule if the number of alternatives is $O(1)$.

**Theorem 7.** The Stable Manipulation problem is poly-time solvable for any poly-time computable anonymous voting rule if the number of alternatives is $O(1)$. 

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Proof. Let \( (A, \mathcal{P}) = (\langle i \rangle_{i \in [n]}, c, \{\delta_i\}_{i \in [n]}, \ell) \) be an arbitrary instance of Stable Manipulation. Since the voting rule is anonymous, any preference profile can equivalently be described by the number \( n_\succ_1 \) of times a preference \( \succ \in \mathcal{L}(A) \) appears in the profile. Hence the number of different anonymous preference profiles is \( \binom{n + m! - 1}{m! - 1} = O((n + m! - 1)^{m! - 1}) = O(n^{\ell(1)}) \). We check, for every possible manipulators’ anonymous preference profile \( \succ_M \) (there are only \( \binom{n + m! - 1}{m! - 1} = O((\ell + m! - 1)^{m! - 1}) = O(\ell^{(1)}) \) such preference profiles), if there exists an \( n \)-voters anonymous preference profile \( \Omega \) (by iterating over all possible anonymous preference profiles) where (i) \( c \) does not win in \( (\Omega, \succ_M) \) and (ii) the input preference profile \( \mathcal{P} \) can be modified into the anonymous preference profile \( \Omega \). We reduce the problem in the second condition into a maximum flow problem as follows. In the bipartite graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, s, t, c : E \rightarrow N_{\geq 1}) \) with

\[
\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2, \quad \text{where} \\
\mathcal{V}_1 = \{u_1, u_2, \ldots, u_n\} \\
\mathcal{V}_2 = \{v_\succ \succ \in \mathcal{L}(A)\} \\
\mathcal{E} = \{[u_i, v_\succ] : i \in [n], \succ \in \mathcal{L}(A), d_{KT}(\succ, \succ) \leq \delta_i \} \\
\cup \{[s, u_i] : i \in [n]\} \cup \{[v_\succ, t] : \succ \in \mathcal{L}(A)\}
\]

The capacity of every edge from \( \mathcal{V}_1 \) to \( \mathcal{V}_2 \) and from \( s \) to \( \mathcal{V}_1 \) is 1. If a preference \( \succ \in \mathcal{L}(A) \) appears in \( \Omega \) \( n_\succ \) number of times, we define the capacity of the edge \( (v_\succ, t) \) to be \( n_\succ \). It follows that \( \Omega \) satisfies the second condition if and only if there is a flow of value \( n \) in the above flow network. \( \square \)

4 Experiments

We now empirically study the effect of manipulator’s uncertainty about other voters’ preferences on the number of stably manipulable profiles. To study this, we generate 100 uniformly random preference profiles for various values of \( n \) (the number of voters), \( m \) (the number of alternatives), and \( \delta \) (the Kendall-Tau distance that each vote can change) and for many common voting rules, namely plurality, Borda, Copeland, maximin, Bucklin, and single transferable vote (STV). We use our polynomial time algorithms for the plurality, Borda, and maximin voting rules whereas we use our algorithm in Theorem 7 for other voting rules. We present some of our findings in Figures 3 and 6 deferring all findings to the full version.

![Figure 1: Results for m = 4, n = 4](image1.png)

![Figure 2: Results for m = 5, n = 4](image2.png)

We observe that the probability that a uniformly random preference profile is stably manipulable is almost 0 even if \( \delta = 2 \) for all the above voting rules. We also observe that for the plurality voting rule, this probability is almost 0 even if \( \delta = 1 \). The intuitive reason for this is that, with \( \delta = 1 \), the favorite alternative \( c \) of the manipulator can always be pushed out of the first position in every other vote which foils manipulation. Among voting rules other than plurality, we see that this probability falls most sharply for the STV rule.
Figure 3: Results for $m = 6, n = 4$

Figure 4: Results for $m = 7, n = 4$

Figure 5: Results for $m = 10, n = 30$

Figure 6: Results for $m = 20, n = 30$

Figure 7: Results for $m = 30, n = 30$
5 Conclusion

We have proposed a new model, called stable manipulation, of incorporating manipulator's uncertainty about others' preferences. Intuitively, in our model, unlike existing models in literature, the manipulator's uncertainty is distributed over the whole preference of other voters. We show that the stable manipulation problem usually admits polynomial time algorithm for single manipulator with a prominent exception of Copeland\(^\alpha\) for \(\alpha \in [0, 1]\). We finally show, through simulation, that the probability of a random profile being stably manipulable drops drastically with slight increase of manipulator's uncertainty. Hence restricting information of any voter about the preferences of the others seems quite effective in hindering manipulation.

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