Abstract

In this Part II, D(10.2), of D(10), we take D(10.1) (arXiv:1302.2054 [math.AG]) as the foundation to define the notion of $Z$-semistable morphisms from general Azumaya nodal curves, of genus $\geq 2$, with a fundamental module to a projective Calabi-Yau 3-fold and show that the moduli stack of such $Z$-semistable morphisms of a fixed type is compact. This gives us a counter moduli stack to D-strings as the moduli stack of stable maps in Gromov-Witten theory to the fundamental string. It serves and prepares for us the basis toward a new invariant of Calabi-Yau 3-fold that captures soft-D-string world-sheet instanton numbers in superstring theory. This note is written hand-in-hand with D(10.1) and is to be read side-by-side with ibidem.

Key words: D-string world-sheet instanton, central charge; Azumaya nodal curve, fundamental module, $Z$-semistable morphism; bubbling $\mathbb{P}^1$-tree, positivity; compactness of moduli.

MSC number 2010: 14D20, 81T30, 14F05, 14N35, 14A22.

Acknowledgements. For this Part II in D(10), we thank D.S. Nagaraj, C.S. Seshadri and Jun Li, Baosen Wu for their works [N-S: II] and [L-W] that influence our thought on the problem. C.-H.L. thanks in addition Baosen Wu for lectures/discussions on [L-W] and literature guide;Montserrat Teixidor i Bigas for conversation on universal vector bundles on nodal curves; Siu-Cheong Lau for discussions on symplectic aspect of D-branes and related open Gromov-Witten theory; Ying-Ing Lee, Chung-Jun Tsai for discussions on special Lagrangian cycles; Murad Alim, Constantin Bachas, Pei-Ming Ho, Albrecht Klemm, Cumrun Vafa for discussions on related stringy issues and literature guide; Eric Sharpe for a communication; Hai-Chau Chang, Heng-Yu Chen, Toshiaki Fujimori, Hui-Wen Lin, Ai-Nung Wang, Chih-Lung Wang and participants of the Geometry Seminar and the String Theory Seminar at National Taiwan University for conversations/discussions on this project, May 2013; Melody Tung Chan, Siu-Cheong Lau, Cumrun Vafa for the enlightening topic courses, spring 2013; Yu-jong Tzeng for instruction on Beamer (LaTeX) for presentation; Chieh-Hsiung Kuan, Silke Prohl, Lanfang Wu for conversations; Department of Mathematics and Department of Physics at National Taiwan University and National Center for Theoretical Sciences - Mathematics Division, Taipei Office, for hospitality. May 2013, while part of the work is in preparation; Alexandra Grot for Bach that accompanies the typing of the notes; Ling-Miao Chou for moral support. S.-T.Y. thanks in addition Department of Mathematics at National Taiwan University for hospitality, fall 2013, while part of the manuscript is in preparation. The project is supported by NSF grants DMS-9803347 and DMS-0074329.
Chien-Hao Liu dedicates D(10.1) and D(10.2) to his uncle Prof. Pin-Hsiung Liu (1925–2004), aunt Ms. Rui-Be Lin, and cousin Master Guo-Guang Shr (Te-Ru Liu) for their hospitality in his first year of college, years of communications with Te-Ru, and lots of cherished memories.
0. Introduction and outline

In a suitable regime of superstring theory, D-branes in a Calabi-Yau space and their most fundamental behaviors can be nicely described mathematically through morphisms from Azumaya spaces with a fundamental module to that Calabi-Yau space. In the earlier work [L-L-S-Y] (D(2): arXiv:0809.2121 [math.AG], with Si Li and Ruifang Song) from the project, we explored this notion for the case of D1-branes (i.e. D-strings) and laid down some basic ingredients toward understanding the notion of D-string world-sheet instantons in this context. In this continuation, D(10), of D(2), we move on to construct a moduli stack of semistable morphisms from Azumaya nodal curves with a fundamental module to a projective Calabi-Yau 3-fold \( Y \). In Part I of the note, D(10.1), we defined the notion of twisted central charge \( Z \) for Fourier-Mukai transforms of dimension 1 and width \([0]\) from nodal curves and the associated stability condition on such transforms and proved that for a given compact stack of nodal curves \( C_M/\mathcal{M} \), the stack \( \mathcal{F}M_{1,[0];Z-ss}^{Y,c} C_M/\mathcal{M} \) of \( Z \)-semistable Fourier-Mukai transforms of dimension 1 and width \([0]\) from nodal curves in the family \( C_M/\mathcal{M} \) to \( Y \) of fixed twisted central charge \( c \) is compact.

In the current Part II (D(10.2)) of D(10), we take D(10.1) as the foundation to define the notion of \( Z \)-semistable morphisms from general Azumaya nodal curves, of genus \( \geq 2 \), with a fundamental module to a projective Calabi-Yau 3-fold (Definition 3.1.1) and show that the moduli stack of such \( Z \)-semistable morphisms of a fixed type is compact (Theorem 4.0). This gives us a counter moduli stack to D-strings as the moduli stack of stable maps in Gromov-Witten theory to the fundamental string. It serves and prepares for us the basis toward a new invariant of Calabi-Yau 3-fold that captures soft-D-string world-sheet instanton numbers in superstring theory. (Cf. Sec. 1, Theme 'Issues on stability conditions for morphisms from general Azumaya nodal curves with a fundamental module'.)

Remark 0.1. [work of Nagaraj-Seshadri and Li-Wu]. This is a remark for experts on the issue of moduli spaces of vector bundles on nodal curves and their degenerations. Our notion of \( Z \)-semistable morphisms from Azumaya nodal curves with a fundamental module to a Calabi-Yau 3-fold is a generalization of the construction from D.S. Nagaraj and C.S. Seshadri of generalized Gieseker moduli spaces in their study of degenerations of moduli spaces of vector bundles on curves ([N-S, II]). Conditions (1), (2), and (3) in Definition 3.1.1 in Sec. 3.1 are indeed a generalization of an abstraction of their related study [N-S, II: Sec. 2]: "Vector bundles over the curves \( X_k \)", where \( X_k \) is a nodal curve from bubbling off a node of a stable curve by attaching a \( \mathbb{P}^1 \)-chain of length \( k \). In this regard, Sec. 2.1 and Sec. 2.2 of this note may be regarded as a counter part to part of [N-S, II: Sec. 2].) On the other hand, with the requirement of compactness of moduli space however constructed in mind, we need something to help demonstrate/control how far a general Fourier-Mukai transform that appears in the reduction is from being in our category. For this we learned from Baosen Wu his work with Jun Li [L-W] the technique of error Hilbert polynomials in their case (turned to error charges in our case). In this regard, Sec. 2.3 and Sec. 4.2 are the parallel to their study of error Hilbert polynomials of sheaves in a degeneration in [L-W: Sec. 3.3 "Numerical criterion" and part of Sec. 5 "Properness of the moduli stacks"].

We acknowledge also the works [Ca], [Gi2], [Kau], [K-L], [M-O-P], [Pa], [P-R], [Sch], [Sun], and [TiB] for related studies that have influenced us in the brewing years since spring 2008.

Convention. Standard notations, terminology, operations, facts in (1) stacks; (2) moduli spaces of sheaves; (3) cohomological techniques in algebraic geometry can be found respectively in (1) [L-MB]; (2) [H-L]; (3) [Gro2], [Ha], [EGAIII].
· All varieties, schemes and their products are over $\mathbb{C}$; a ‘curve’ means a 1-dimensional proper scheme over $\mathbb{C}$.

· The ‘support’ $\text{Supp}(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ on a scheme $Y$ means the scheme-theoretical support of $\mathcal{F}$ unless otherwise noted; $\mathcal{I}_Z$ denotes the ideal sheaf of a subscheme of $Z$ of a scheme $Y$; $l(\mathcal{F})$ denotes the length of a coherent sheaf $\mathcal{F}$ of dimension 0.

· The current note continues the study in [L-L-S-Y] (arXiv:0809.2121 [math.AG], D(2)) and [L-Y3] (arXiv:1302.2054 [math.AG] D(10.1)). A partial review of D-branes and Azumaya noncommutative geometry is given in [L-Y3] (arXiv:1003.1178 [math.SG], D(6)) and [Liu1] (arXiv:1112.4317 [math.AG]) (see also [Liu2] and [Liu3]). Notations and conventions follow these earlier works when applicable.

Outline

0. Introduction.

1. D-string world-sheet instantons, morphisms from Azumaya nodal curves with a fundamental module, Fourier-Mukai transforms, and issues on stability conditions.

2. Preliminaries aiming for a definition of stability conditions for morphisms from general Azumaya nodal curves with a fundamental module.
   2.1 Push-forward and higher direct image of sheaves.
   2.2 A special class of morphisms from Azumaya curves with a fundamental module and positivity of fundamental modules on $\mathbb{P}^1$-trees.
   2.3 Remarks on nonnegative torsion-free sheaves on a $\mathbb{P}^1$-tree.

3. The space of D-string world-sheet instantons: The moduli stack of $Z$-semistable morphisms from Azumaya nodal curves with a fundamental module to a Calabi-Yau 3-fold.
   3.1 The moduli stack of $Z$-semistable morphisms from Azumaya nodal curves with a fundamental module to a Calabi-Yau 3-fold.
   3.2 A natural morphism from $\mathcal{M}^{Z,ss}_{\mathcal{A}^Z([g,r,\chi])}(Y;\beta,c)$ to $\mathcal{F}\mathcal{M}^{1,[0];Z,ss}_{g}(Y;c)$.

4. Compactness of the moduli stack $\mathcal{M}^{Z,ss}_{\mathcal{A}^Z([g,r,\chi])}(Y;\beta,c)$ of $Z$-semistable morphisms.
   4.1 Boundedness of $\mathcal{M}^{Z,ss}_{\mathcal{A}^Z([g,r,\chi])}(Y;\beta,c)$.
   4.2 The error charge of a Fourier-Mukai transform.
   4.3 Completeness of $\mathcal{M}^{Z,ss}_{\mathcal{A}^Z([g,r,\chi])}(Y;\beta,c)$.
1 D-string world-sheet instantons, morphisms from Azumaya nodal curves with a fundamental module, Fourier-Mukai transforms, and issues on stability conditions

We review in this section related notions from [L-L-S-Y] (D(2)) and [L-Y3] (D(10.1)) to bring out terminologies and notations needed. Readers are referred to ibidem for details and references.

Holomorphic D-branes, morphisms from Azumaya schemes with a fundamental module, and Fourier-Mukai transforms

A holomorphic D-brane in superstring theory can be described as a morphism from a scheme with a matrix-type noncommutative structure sheaf (i.e. an Azumaya scheme $(X, O^A_X)$) together with a fundamental $O^A_X$-module $E$ to $(Y, O_Y)$, where $O_Y$ is the structure sheaf of $Y$ in either commutative or noncommutative setting; in notation/symbol,

$$\varphi : (X, O^A_X; E) \rightarrow (Y, O_Y),$$

with a built-in isomorphism $O^A_X \simeq \text{End}_{O_X}(E)$. In true contents, this means a contravariant gluing system of ring-homomorphisms

$$O^A_X \leftarrow O_Y : \varphi^\sharp,$$

which in general does not induce any morphisms directly from $X$ to $Y$. It is through $\varphi^\sharp$ that the $O^A_X$-module $E$ can be pushed forward to an $O_Y$-module, in notation $\varphi_* (E)$, on $Y$.

When the target space $Y$ is a commutative scheme and $E$ is locally free $O_X$-module, then associated to a morphism $\varphi : (X, O^A_X; E) \rightarrow (Y, O_Y)$ is the following diagram

$$O^A_X = \text{End}_{O_X}(E)$$

$$\begin{array}{c}
A_{\varphi} := \text{Im} \varphi^\sharp \leftarrow \varphi^\sharp \rightarrow O_Y \\
\downarrow \text{id} \leftarrow \varphi^\sharp \rightarrow O_Y \\
O_X
\end{array}$$

which defines a subscheme $X_{\varphi} := \text{Spec} A_{\varphi} \subset X \times Y$ together with a coherent sheaf $\tilde{E}_{\varphi}$ supported on $X_{\varphi}$, which is simply the $O^A_X$-module $E$ regarded as an $A_{\varphi}$-module. $\tilde{E}_{\varphi}$ is called the graph of the morphism $\varphi$. It is a coherent sheaf on $X \times Y$ that is flat over $X$, of relative dimension $0$. Conversely, given such a coherent sheaf $\tilde{E}$ on $X \times Y$, a morphism $\varphi_{\tilde{E}} : (X, O^A_X; E) \rightarrow Y$ can be constructed from $\tilde{E}$ by taking

- $E = \text{pr}_1^* \tilde{E}$,
- $O^A_X = \text{End}_{O_X}(E)$, and
- $\varphi_{\tilde{E}}^\sharp : O_Y \rightarrow O^A_X$ is defined by the composition

$$O_Y \xrightarrow{\text{pr}_2^*} O_{X \times Y} \xrightarrow{\iota^\sharp} O_{\text{Supp}(\tilde{E})} \xrightarrow{\iota^\sharp} O^A_X.$$
Here, $X \xleftarrow{pr_1} X \times Y \xrightarrow{pr_2}$ are the projection maps, $\iota: \text{Supp}(\mathcal{E}) \to X \times Y$ is the embedding of the subscheme, and note that $\text{Supp}(\mathcal{E})$ is affine over $X$.

Treating $\mathcal{E}$ as an object in the bounded derived category $D^b(\text{Coh}(X \times Y))$ of coherent sheaves on $X \times Y$, $\mathcal{E}$ defines a Fourier-Mukai transform $\Phi_{\mathcal{E}} : D^b(\text{Coh}(X)) \to D^b(\text{Coh}(Y))$, in short name, a Fourier-Mukai transform from $X$ to $Y$. In this way, the data that specifies a morphism $\varphi: (X, \mathcal{O}^*_X; \mathcal{E}) \to Y$ is matched to a data that specifies a special kind of Fourier-Mukai transform.

**Definition 1.1. [support, dimension, width of Fourier-Mukai transform].** ([L-Y3: Definition 1.1].) For a general $\mathcal{F}^\bullet \in D^b(\text{Coh}(X \times Y))$, we define the (scheme-theoretical) support $\text{Supp}(\mathcal{F}^\bullet)$ to be the (scheme-theoretical) support of $\oplus_i H^i(\mathcal{F}^\bullet)$, the dimension $\dim \mathcal{F}^\bullet$ of $\mathcal{F}^\bullet$ to be the dimension $\dim(\text{Supp}(\mathcal{F}^\bullet))$, and the width of $\mathcal{F}^\bullet$ to be the interval $[i, j]$ such that $H^i(\mathcal{F}^\bullet) \neq 0$, $H^j(\mathcal{F}^\bullet) \neq 0$, and $H^k(\mathcal{F}^\bullet) = 0$, for $k \notin [i, j]$. We’ll denote the width $[i, i]$ by $[i]$.

Thus, for $X$ fixed of pure dimension $d$, the stack of morphisms $(X, \mathcal{O}^*_X; \mathcal{E}) \to Y$ is embedded in the stack of Fourier-Mukai transforms from $X$ to $Y$ of dimension $d$ and width $[0]$; the latter is identical to the stack of $d$-dimensional coherent sheaves on $X \times Y$. Similar statement holds for $X$ not fixed. (Figure 1-1.)

We now specialize to the objects of this subseries D(10): The case of D1-branes (i.e. D-strings) world-sheet instantons.

**D-string world-sheet instantons, central charges, and stability conditions**

In the context of a compactification of Type IIB superstring theory on a Calabi-Yau 3-fold $Y$, an instanton in the effective 4-dimensional, $N = 1$, supersymmetric quantum field theory can be created by “wrapping” a (Euclideanized/Wick-rotated) D-string world-volume on some cycles in $Y$; cf. Figure 1-2. For such instanton to be stable (in the sense of not to decay away by fluctuations as time passes), it is required that this Euclidean D-string world-sheet, together with the Chan-Paton sheaf thereupon, satisfy some stability conditions governed by the central charge of D-branes. In [L-Y3] (D(10.1)), these notions are rephrased/polished (in an appropriate large-radius limit of the Calabi-Yau space with the tension of the fundamental string going to infinity) to the following setting along the line of the D-project:

Let $C$ be a nodal curve with a polarization class $L$ and $Y$ be a projective Calabi-Yau manifold with a complexified Kähler class $B + \sqrt{-1}J$.

**Definition 1.2. [twisted central charge of Fourier-Mukai transform].** ([L-Y3: Definition 2.1.1].) Let $\mathcal{F}$ be a coherent sheaf of dimension 1 on $C \times Y$ and $\Phi_{\mathcal{F}}$ be the Fourier-Mukai transform $\mathcal{F}$ defines. Then, the twisted central charge of $\Phi_{\mathcal{F}}$ associated to the data $(B + \sqrt{-1}J, L)$ is defined to be

$$Z^{B+\sqrt{-1}J,L}(\Phi_{\mathcal{F}}) := Z^{B+\sqrt{-1}J,L}(\mathcal{F}) := \int_{C \times Y} pr_2^* \left( \frac{e^{-(B+\sqrt{-1}J)}}{\sqrt{Id(T_Y)}} \right) \cdot \tau_{C \times Y}(\mathcal{F}),$$

where $\tau_{C \times Y}(\mathcal{F}) := ch(\mathcal{F}) \cdot td(T_{C \times Y})$ is the $\tau$-class of $\mathcal{F}$. 

4
Figure 1-1. The equivalence between a morphism $\varphi$ from an Azumaya scheme with a fundamental module $(X, \mathcal{O}_X^{\mathbb{A}^2} := \text{End}_{\mathcal{O}_X}^{}(\mathcal{E}); \mathcal{E})$ to a scheme $Y$ and a special kind of Fourier-Mukai transform $\mathcal{E} \in \text{Coh}(X \times Y)$ from $X$ to $Y$. 
Lemma 1.3. [twisted central charge: explicit form]. ([L-Y3: Lemma 2.1.2].) Continuing the above notation. Let

\[ \tilde{\beta}(\tilde{F}) := \sum_i d_i[\zeta_i] \in A_1(C \times Y), \]

where \( \zeta_i \) runs through the generic points of \( \text{Supp}(\tilde{F}) \) and \( d_i \) is the dimension of \( \tilde{F}|_{\zeta_i} \) as a \( k_{\zeta_i} \)-vector space. Then,

\[ Z^{B+\sqrt{-1}JL}(\tilde{F}) = (\chi(\tilde{F}) - B \cdot \tilde{\beta}(\tilde{F})) - \sqrt{-1} \left( (J + L) \cdot \tilde{\beta}(\tilde{F}) \right). \]

In particular, for non-zero coherent sheaves on \( C \times Y \) of dimension \( \leq 1 \), \( Z^{B+\sqrt{-1}JL} \) takes its values in the partially completed lower-half complex plane

\[ \mathbb{H}_- := \{ z \in \mathbb{C} \mid \text{either } \text{Im}z < 0 \text{ or } \text{Im}z = 0 \text{ with } \text{Re}z > 0 \}. \]

Definition 1.4. [Z-slope \( \mu^Z \)]. ([L-Y3: Definition 2.1.4].) Continuing Definition 1.2. We define the Z-slope for a non-zero coherent sheaf \( \tilde{F} \) on \( C \times Y \) of dimension \( \leq 1 \) to be

\[ \mu^Z(\tilde{F}) := - \text{Re} \left( \frac{Z^{B+\sqrt{-1}JL}(\tilde{F})}{Z^{B+\sqrt{-1}JL}(\tilde{F})} \right) \bigg/ \text{Im} \left( \frac{Z^{B+\sqrt{-1}JL}(\tilde{F})}{Z^{B+\sqrt{-1}JL}(\tilde{F})} \right) \]

\[ = \left( \chi(\tilde{F}) - B \cdot \tilde{\beta}(\tilde{F}) \right) \bigg/ \left( (J + L) \cdot \tilde{\beta}(\tilde{F}) \right). \]
Definition 1.5. \[Z\text{-semistable, } Z\text{-stable, } Z\text{-unstable, strictly } Z\text{-semistable}.\] (\[\text{[L-Y3: Definition 2.2.1]}\].) Continuing the discussion.

1. A 1-dimensional coherent sheaf \(\tilde{F}\) on \(C \times Y\) is said to be \(Z\text{-semistable}\) (resp. \(Z\text{-stable})\) if \(\tilde{F}\) is pure and \(\mu^Z(F') \leq \langle \mu^Z(F)\rangle\) for any nonzero proper subsheaf \(F' \subset \tilde{F}\). Such \(\tilde{F}\) is called \(Z\text{-unstable}\) if it is not \(Z\text{-semistable, and is called strictly } Z\text{-semistable if it is } Z\text{-semistable but not } Z\text{-stable.}

2. A morphism \(\varphi: (C, O_C^\mathbb{A}^2) := \mathcal{E}nd_{O_C}(E); \mathcal{E}) \rightarrow Y\) is said to be \(Z\text{-semistable}\) (resp. \(Z\text{-stable, } Z\text{-unstable, strictly } Z\text{-semistable})\) if its graph \(\tilde{E}_\varphi \in \text{Coh}_1(C \times Y)\) is \(Z\text{-semistable}\) (resp. \(Z\text{-stable, } Z\text{-unstable, strictly } Z\text{-semistable)\).\)

When the central charge functional \(Z\) is known and fixed either explicitly or implicitly, we may use the terminology: \(semistable, stable, unstable, strictly semistable,\) for simplicity.

Compactness of the moduli stack of Fourier-Mukai transforms from stable curves

Assumption 1.6. \([g \geq 2].\) For the rest of the notes, we assume that \(g \geq 2\) and leave the special cases of \(g = 0\) and \(g = 1\) to separate notes.

Theorem 1.7. \([\mathcal{F}M^1,[0];Z\text{-ss}(Y; c) \text{ compact}].\) (Cf. \([\text{L-Y3: Theorem 3.1}]\). Let \((Y, B + \sqrt{-1}J)\) be a projective Calabi-Yau 3-fold with a fixed complexified K"ahler class, \(\mathcal{M}_g\) be the moduli stack of stable curves of genus \(g\), \(C_{\mathcal{M}_g}/\mathcal{M}_g\) be the associated universal curve over \(\mathcal{M}_g\) with a fixed relative polarization class \(L\). Then the moduli stack \(\mathcal{F}M^1,[0];Z\text{-ss}(Y; c)\) of \(Z^{B+\sqrt{-1}J,L}\text{-semi-stable Fourier-Mukai transforms of dimension 1, width [0], and central charge } c \in \mathbb{H}_-\) from stable curves of genus \(g\) to \(Y\) is compact.

Issues on stability conditions for morphisms from general Azumaya nodal curves with a fundamental module

As in Gromov-Witten theory for stable maps, the domain Azumaya nodal curves with a fundamental module of our intended \(Z\text{-semistable morphisms are not fixed. To have a good mathematical theory of D-string world-sheet instantons, we want our moduli stack of intended } Z\text{-semistable morphisms from Azumaya nodal curves with a fundamental module to } Y\) to be compact. This is a minimal requirement to have a well-defined intersection theory on the moduli stack. Only then may one have a chance for a good-enough tangent-obstruction theory, at least in some important cases, to define \(D\text{-string world-sheet instanton numbers}\) from a purely D-string world-sheet aspect. If achieved, this would give us a counter theory to D-strings as Gromov-Witten theory to the fundamental string.

However, also as in Gromov-Witten theory, degenerations of a naive \(Z\text{-semistable morphism } \varphi\) in our problem may give rise to objects not in our category. For example and in terms of the graph \(\tilde{E}_\varphi \in \text{Coh}_1(C \times Y)\) of \(\varphi, \text{Supp}(\tilde{E}_\varphi)\) may turn to have a vertical component with respect to the projection map \(pr_C: C \times Y \rightarrow C\) and/or \(E = pr_C(\tilde{E}_\varphi)\) may turn to a non-locally-free sheaf on a deformation of \(C\). Just considering morphisms alone, all such bad degenerations/irregularities of morphisms under a deformation can be corrected/absorbed by \(\mathbb{P}^1\) -bubbling trees added to nodal curves, as in Gromov-Witten theory. However, as there is no known universal estimate to relate the complexity of degenerations of intended \(Z\text{-semistable morphisms in our category to a bound on the complexity of the } \mathbb{P}^1\text{-trees needed to absorb the bad degeneration, there is no}
way to select beforehand a large enough polarization class $L$ on nodal curves to guarantee that we never use up its positivity condition to define $Z$-semistability condition on morphisms when extending the polarization class on a nodal curve to its cousin with $\mathbb{P}^1$-bubbling trees. This is the same issue mathematicians already ran into when studying the Gieseker-type compactifications of moduli spaces of vector bundles on stable curves.

Following the lesson learned from these previous studies on moduli spaces of vector bundles on nodal curves ([Gi2], [N-S,II], and e.g. [Kau], [K-L], [Sch], [Sun], [TiB]), the way out of such difficulty in our situation is, in conceptual, nontechnical, and slightly imprecise words, that one needs to separate the underlying domain nodal curve $C$ of a morphism $\varphi : (C, \mathcal{O}_C^{\mathbb{A}^2}; \mathcal{E}) \rightarrow Y$ in our problem into the union of a major nodal subcurve $C_0$ and a minor subcurve $C_1$. $C_0$ in general contributes to the imaginary part of the twisted central charge of $\varphi$ and is where the $Z$-semistability of $\varphi$ should be imposed in the standard way, as given in Definition [L-Y3] now applied only to $\tilde{\mathcal{E}}_{\varphi}|_{C_0}$, while $C_1$ consists only of $\mathbb{P}^1$-bubbling trees, whose contribution to the $Z$-semistability of $\varphi$ has to be imposed by hand (in an as-natural-as-possible way).

The moduli stack $\mathcal{F}_M^{1, (0); Z-ss} (Y; c)$ of $Z_{B+\sqrt{-1}J,L}$-semi-stable Fourier-Mukai transforms of dimension 1, width [0], and central charge $c \in \hat{H}_{-}$ from stable curves of genus $g$ to $Y$ in [L-Y3] D(10.1) was constructed exactly to make such separation of morphisms from over nodal curves to morphisms from over main-versus-minor subcurve of nodal curves precise and, hence, ease the process toward a complete notion/definition of stability conditions for $\varphi$. The compactness of $\mathcal{F}_M^{1, (0); Z-ss} (Y; c)$, cf. Theorem 1.7 above from [L-Y3: Theorem 3.1], justifies that this is a reasonable stack to begin with. What remains to be done constitutes the core of the current note D(10.2): Sec. 2 – Sec. 4.

2 Preliminaries aiming for a definition of stability conditions for morphisms from general Azumaya nodal curves with a fundamental module

To extend the notion of semistable morphisms from Azumaya nodal curves with a fundamental module beyond the domains reviewed in Sec. 1 in such a way that it is natural and that it gives rise to a compact moduli stack, there are two sets of technical issues one has to understand beforehand – one on the behavior of cohomologies under push-pull and the other on the behavior over bubbling $\mathbb{P}^1$-trees. In this section, we explain what paves our path toward Definition [3.1.1] of general semistable morphisms in Sec. [3.1] It is also through this section one can see better why Definition [3.1.1] though conceivably not the only possibility, is very natural.

2.1 Push-forward and higher direct image of sheaves

We address in this subsection related cohomological issues toward the notion of semistable morphisms in our problem.

Lemma 2.1.1. [criterion for pushing forward a flat family to another flat family]. Let $S$ be a separated Noetherian base scheme; $f_S : X'_S \rightarrow X_S$ be an $S$-morphism of separated Noetherian schemes of finite type over $S$; and $\mathcal{F}'_S$ be a quasi-coherent sheaf on $X'_S$ that is flat over $S$. Assume that $R^i f'_S_{*} (\mathcal{F}'_S) = 0$ for all $i > 0$, then $f_S_{*} (\mathcal{F}'_S)$ is a quasi-coherent sheaf on $X_S$ that is flat over $S$. 

8
We remark that in our application, $f_S$ is projective and the same statement holds with ‘quasi-coherent’ replaced by ‘coherent’.

**Proof.** As the statement is local, without loss of generality, one may assume that both $S$ and $X$ are affine with $S = \text{Spec } A_0$, $X = \text{Spec } A$, and $A$ is an $A_0$-algebra. In this case, $f_{S*}(F'_S) = H^0(X'_S, F'_S)^*$, the quasi-coherent sheaf on $X'_S$ associated to the $A$-module $H^0(X'_S, F'_S)$. Similarly, $R^if_{S*}(F'_S) = H^i(X'_S, F'_S)^*$ for $i > 0$. The statement becomes:

- Let $F'_S$ be a quasi-coherent sheaf on $X'_S$ that is flat over $S = \text{Spec } A_0$ with $H^i(X'_S, F'_S) = 0$ for all $i > 0$. Then, $H^0(X'_S, F'_S)$ is a flat $A_0$-module.

Which we will prove.

Let $U = \{U_a\}_a$ be a finite affine open cover of $X'_S$, then $\{H^i(X'_S, F'_S)\}_{i \geq 0}$ is the cohomology associated to the Čech complex

\[
\begin{array}{cccccccc}
C^*(U, F'_S, d) : & C^0(U, F'_S) & \overset{d^0}{\longrightarrow} & C^1(U, F'_S) & \overset{d^1}{\longrightarrow} & C^2(U, F'_S) & \overset{d^2}{\longrightarrow} & \cdots \\
\end{array}
\]

Note that since $F'_S$ is flat over $S$, $C^p(U, F'_S) := \prod_{i_0 < \cdots < i_p} F'_S(U_{i_0} \cdots i_p)$ is a flat $A_0$-module, for all $p \geq 0$. Consider now the following sequence of short exact sequences

\[
\begin{array}{cccccccc}
0 & \rightarrow & H^0(X'_S, F'_S) & \rightarrow & C^0(U, F'_S) & \rightarrow & \text{Im } d^0 & \rightarrow & 0 \\
\| & & \| & & \| & & \| \\
0 & \rightarrow & \text{Ker } d^0 & \rightarrow & C^1(U, F'_S) & \rightarrow & \text{Im } d^1 & \rightarrow & 0 \\
\| & & \| & & \| & & \| \\
0 & \rightarrow & \text{Ker } d^1 & \rightarrow & C^2(U, F'_S) & \rightarrow & \text{Im } d^2 & \rightarrow & 0 \\
\| & & \| & & \| & & \| \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
\end{array}
\]

Here, $\text{Im } d^i = \text{Ker } d^{i+1}$, for all $i > 0$, since $H^i(X'_S, F'_S) = 0$, for all $i > 0$, by assumption. Let $I_0$ be any ideal of $A_0$. Then

\[
\text{Tor}_1^{A_0}(I_0, C^p(U, F'_S)) = 0,
\]

for all $i > 0$ and $p \geq 0$, since $C^p(U, F'_S)$ is a flat $A_0$-module. It follows that

\[
\begin{array}{c}
\text{Tor}_1^{A_0}(I_0, H^0(X'_S, F'_S)) \simeq \text{Tor}_2^{A_0}(I_0, \text{Im } d^0) \\
\| \\
\text{Tor}_2^{A_0}(I_0, \text{Ker } d^1) \simeq \text{Tor}_3^{A_0}(I_0, \text{Im } d^1) \\
\| \\
\cdots \\
\end{array}
\]

Since

\[
\text{Tor}_p^{A_0}(I_0, \text{Im } d^{p-2}) \simeq \text{Tor}_p^{A_0}(I_0, \text{Ker } d^{p-1}) \simeq 0
\]

for $p$ large enough,

\[
\text{Tor}_i^{A_0}(I_0, H^0(X'_S, F'_S)) = 0
\]

for all ideal $I_0$ of $A_0$. This shows that $H^0(X'_S, F'_S)$ is a flat $A_0$-module and hence proves the lemma.

\[\square\]

**Lemma 2.1.2.** [relation on push-forward as a closed condition]. Let $(B, b_0)$ be a smooth pointed curve, $f_B : X'_B \rightarrow X_B$ be a birational projective morphism of Noetherian $B$-schemes whose exceptional locus is of relative dimension 1 under the restriction of $f_B$, $F_B$ be a coherent $\mathcal{O}_{X'_B}$-module that gives a flat family of pure 1-dimensional coherent sheaves on fibers of $X_B$ over $B$, and $f_B^*(F_B) \rightarrow F'_B \rightarrow 0$ be a quotient $\mathcal{O}_{X'_B}$-module that gives a flat family of 1-dimensional coherent sheaf on fibers of $X'_B$ over $B$. Suppose that
Then, the pre-composition with a natural homomorphism $f_{B_0}$

Proof. The pre-composition with a natural homomorphism

$$F_B \rightarrow f_{B_0}^*(F_B) \rightarrow f_{B_0}^*(F'_B)$$

with

$$f_{B_0}^*(F'_B) \simeq F_{B_0} \quad \text{on } X_{B_0}.$$ Here, $(\bullet)_{B - \{b_0}\}}$ and $(\bullet)_{b_0}$ mean the restriction of $(\bullet)$ to over $B - \{b_0\}$ and $b_0$ respectively.

This gives the exact sequence

$$0 \rightarrow F_B \rightarrow f_{B_0}^*(F'_B) \rightarrow Q \rightarrow 0,$$

where $Q$ is supported in an infinitesimal neighborhood of $X_{b_0}$ in $X_B$. This induces a long exact sequence

$$\cdots \rightarrow Tor^1_B(O_{X_{b_0}}, f_{B_0}^*(F'_B)) \rightarrow Tor^1_B(O_{X_{b_0}}, Q) \rightarrow F_{b_0} \rightarrow (f_{B_0}^*(F'_B))|_{b_0} \rightarrow Q_{b_0} \rightarrow 0.$$ This gives the exact sequence

$$0 \rightarrow Q_{b_0} \rightarrow F_{b_0} \rightarrow (f_{B_0}^*(F'_B))|_{b_0} \rightarrow Q_{b_0} \rightarrow 0,$$

since $Tor^1_B(O_{X_{b_0}}, f_{B_0}^*(F'_B)) = 0$ from the flatness of $f_{B_0}^*(F'_B)$ over $B$ and $Tor^1_B(O_{X_{b_0}}, Q) \simeq Q_{b_0}$ via the two-term line-bundle resolution $0 \rightarrow I_{X_{b_0}} \rightarrow O_{X_B} \rightarrow O_{X_{b_0}} \rightarrow 0$ of $O_{X_{b_0}}$. Condition (2) on $F_{b_0}$ implies that $Q_{b_0}$ is 0-dimensional; it follows then from the purity of $F_{b_0}$ that $Q_{b_0}$ (and, hence, $Q$) = 0. This proves that

$$F_B \simeq f_{B_0}^*(F'_B)$$

on $X_B$. In particular, $F_{b_0} \simeq (f_{B_0}^*(F'_B))|_{b_0}$.

It remains to show that the natural homomorphism

$$(f_{B_0}^*(F'_B))|_{b_0} \xrightarrow{\alpha} f_{b_0}^*(F'_B)$$

is an isomorphism. First note that, by the assumptions in the lemma, $Ker(\alpha)$ is 0-dimensional on $X_{b_0}$, which must then vanish since $(f_{B_0}^*(F'_B))|_{b_0} \simeq F_{b_0}$ is pure. Consider now the exact sequence of coherent $O_{X_{b_0}}$-modules

$$0 \rightarrow H \rightarrow F_B \rightarrow F_{b_0} \rightarrow 0,$$

which gives the long exact sequence of coherent $O_{X_{b_0}}$-modules

$$0 \rightarrow f_{B_0}(H) \rightarrow f_{B_0}(F_B) \rightarrow f_{B_0}(F_{b_0}) \rightarrow R^1f_{B_0}(H) \rightarrow R^1f_{B_0}(F_B) \rightarrow \cdots.$$
Assumptions (1) and (2) and the fact that \( \mathcal{F}'_{b_0} \) is supported only on \( X'_b \) imply further that the above long exact sequence can be truncated to give a short exact sequence of coherent \( \mathcal{O}_{X_b} \)-modules

\[
0 \rightarrow (f_{b_0}(\mathcal{F}'_b))_{|b_0} \xrightarrow{\alpha} f_{b_0}(\mathcal{F}'_b) \rightarrow R^1f_{b_0}(\mathcal{H}) \rightarrow 0
\]

with \( R^1f_{b_0}(\mathcal{H}) \) a 0-dimensional coherent sheaf on \( X_{b_0} \).

On the other hand,

\[
\mathcal{H} \simeq \mathcal{F}'_b \otimes \mathcal{O}_{X_b} \mathcal{O}_{X'_b}(X_{b_0})
\]

and, hence, is isomorphic to \( \mathcal{F}'_b \) over an affine neighborhood of \( b_0 \in B \). Since \( R^1f_{b_0}(\mathcal{H}) \) is supported only over \( b_0 \in B \),

\[
R^1f_{b_0}(\mathcal{H}) \simeq R^1f_{b_0}(\mathcal{F}'_b) \simeq 0,
\]

by Assumption (1). It follows that \( \alpha \) is an isomorphism. This concludes the proof.

As we only need to deal with coherent sheaves of relative dimension \( \leq 1 \), we now turn to some criteria naturally occurring in our setting that force the first (and hence all higher) direct image of sheaves to vanish. A basic example is given in the lemma below whose repeating use, together with base changes, is enough to deal with the case of \( \mathbb{P}^1 \)-bubbling from resolving \( A_n \)-singularities on a complex surface from the total space of a complex 1-parameter family of nodal curves. However, what we will actually use for this note is the more powerful/encompassing Lemma 3.2.1 in Sec. 3.2.

**Lemma 2.1.3. [criterion for vanishing first direct-image sheaf].** Let \( f : X' \rightarrow X \) be a projective morphism of Noetherian schemes that fits into the following commutative diagram

\[
\begin{array}{ccc}
X' & \xleftarrow{\subset} & \mathbb{P}_X^1 \\
\downarrow{f} & & \downarrow{pr_X} \\
X & & 
\end{array}
\]

and \( \mathcal{F}' \) be a coherent sheaf on \( X' \) that fits into an exact sequence of the form

\[
\mathcal{O}_{X'}^\oplus k \rightarrow \mathcal{F}' \rightarrow 0,
\]

for some \( k \in \mathbb{Z}_{>0} \). Then,

\[
R^1f_*(\mathcal{F}') = 0.
\]

**Proof.** Since one has the quotient homomorphism \( \mathcal{O}_{\mathbb{P}_X^1} \rightarrow \mathcal{O}_{X'} \) via the embedding \( X' \hookrightarrow \mathbb{P}_X^1 \), one can promote the quotient \( \mathcal{O}_{X'} \)-module \( \mathcal{O}_{\mathbb{P}_X^1}^\oplus k \rightarrow \mathcal{F}' \) to a short exact sequence of coherent \( \mathcal{O}_{\mathbb{P}_X^1} \)-modules:

\[
0 \rightarrow \mathcal{H} \rightarrow \mathcal{O}_{\mathbb{P}_X^1}^\oplus k \rightarrow \mathcal{F}' \rightarrow 0.
\]

This gives a long exact sequence of coherent \( \mathcal{O}_X \)-modules

\[
\cdots \rightarrow R^1pr_{X*}(\mathcal{O}_{\mathbb{P}_X^1}^\oplus k) \rightarrow R^1pr_{X*}(\mathcal{F}') \rightarrow R^2pr_{X*}(\mathcal{H}) \rightarrow \cdots.
\]

Since both \( R^1pr_{X*}(\mathcal{O}_{\mathbb{P}_X^1}^\oplus k) \) and \( R^2pr_{X*}(\mathcal{H}) \) vanish from the known cohomology of projective spaces and the fact that \( pr_X \) has only relative dimension 1, the lemma follows. \( \square \)
Lemma 2.1.4. [vanishing first direct-image sheaf under composition].

Let \( f = h_l \circ \cdots \circ h_1 \) be a composition of morphisms of Noetherian schemes

\[
\begin{array}{cccc}
X_l & \xrightarrow{h_l} & X_{l-1} & \cdots & \xrightarrow{h_2} & X_1 & \xrightarrow{h_1} & X_0 \\
\end{array}
\]

and \( F \) be a coherent sheaf on \( X_l \). Then, \( R^1 f_* (F) = 0 \) if and only if \( R^1 h_i_*( (h_l \circ \cdots \circ h_{i+1})_* (F)) = 0 \) for all \( i = 1, \ldots , l \).

Proof. This is an immediate consequence of the spectral sequence associated to a composition \( f = g \circ h \) of morphisms of Noetherian schemes, which gives the exact sequence

\[
0 \rightarrow R^1 g_* (h_* (F)) \rightarrow R^1 f_* (F) \rightarrow g_* (R^1 h_* (F)) \rightarrow 0.
\]

\[\square\]

2.2 A special class of morphisms from Azumaya curves with a fundamental module and positivity of fundamental modules on \( \mathbb{P}^1 \)-trees

Anticipating the issue of bubbling \( \mathbb{P}^1 \)-trees from removing irregularities of a morphism to haunt us, as in the case of Gromov-Witten theory, we study in this subsection related algebraic geometry for a special class of morphisms from Azumaya \( \mathbb{P}^1 \)-trees with a fundamental module.

A special class of morphisms from Azumaya curves with a fundamental module

Consider a morphism

\[
\varphi : (C, \mathcal{O}^{Az}_C := \text{End}_{\mathcal{O}_X}(\mathcal{E}); \mathcal{E}) \rightarrow Y
\]

from an Azumaya nodal curve with a fundamental module of rank \( r \) to a projective variety \((Y, \mathcal{O}_Y(1))\). Let \( \mathcal{E}_\varphi \), the graph of \( \varphi \), be the associated coherent \( \mathcal{O}_{C \times Y} \)-module on \( C \times Y \) that is flat over \( C \) and of relative dimension 1 (and relative length \( r \)). Suppose \( \varphi \) is a morphism such that the following three special properties hold for \( \mathcal{E}_\varphi \):

1. \( \mathcal{E}_\varphi \) is realizable as a quotient of a trivial bundle on \( C \times Y \)
   \[
   \mathcal{O}^{\oplus k}_{C \times Y} \xrightarrow{\alpha} \mathcal{E}_\varphi;
   \]
2. Its push-forward \( pr_{\leftarrow Y}(\mathcal{E}_\varphi) \) is 0-dimensional, where \( pr_Y : C \times Y \rightarrow Y \) is the projective map;
3. On each irreducible component of \( C \), \( \mathcal{E}_\varphi \) is not the pull-back \( pr_Y^* (\mathcal{F}) \) of some 0-dimensional coherent \( \mathcal{O}_Y \)-module; i.e., \( \varphi \) is not constant on each irreducible component of \( C \).

The decoration/marking \( \alpha \) in Property (1) specifies a morphism

\[
f_{(\varphi, \alpha)} : C \rightarrow Quot_Y(\mathcal{O}^{\oplus k}_Y, r)
\]

from \( C \) to the Quot-scheme \( Quot_Y(\mathcal{O}^{\oplus k}_Y, r) \) of 0-dimensional quotients of \( \mathcal{O}^{\oplus k}_Y \) of length \( r \). To see the implication of Property (2) and Property (3), let us recall how \( Quot_Y(\mathcal{O}^{\oplus k}_Y, r) \) is embedded in a Grassmannian variety, which in turn embeds into a projective space, (e.g., [F-G-I-K-N-V] or [H-L]).
Let \( \mathcal{O}_{\text{Quot}_Y(\mathcal{O}_{\mathcal{V}_Y}^\oplus, r) \times Y}/\text{Quot}_Y(\mathcal{O}_{\mathcal{V}_Y}^\oplus, r) \) be the relative ample line bundle on 
\( (\text{Quot}_Y(\mathcal{O}_{\mathcal{V}_Y}^\oplus, r) \times Y)/\text{Quot}_Y(\mathcal{O}_{\mathcal{V}_Y}^\oplus, r) \) associated to the ample line bundle \( \mathcal{O}_Y(1) \) on \( Y \),

\[
0 \rightarrow \tilde{\mathcal{H}} \rightarrow \mathcal{O}^\oplus_{\text{Quot}_Y(\mathcal{O}_{\mathcal{V}_Y}^\oplus, r) \times Y} \rightarrow \tilde{\mathcal{Q}} \rightarrow 0
\]

be the short sequence of \( \mathcal{O}_{\text{Quot}_Y(\mathcal{O}_{\mathcal{V}_Y}^\oplus, r) \times Y} \)-modules that extends the universal quotient sheaf on \( \text{Quot}_Y(\mathcal{O}_{\mathcal{V}_Y}^\oplus, r) \times Y \), and

\[
pr_1 : \text{Quot}_Y(\mathcal{O}_{\mathcal{V}_Y}^\oplus, r) \times Y \rightarrow \text{Quot}_Y(\mathcal{O}_{\mathcal{V}_Y}^\oplus, r) , \quad pr_2 : \text{Quot}_Y(\mathcal{O}_{\mathcal{V}_Y}^\oplus, r) \times Y \rightarrow Y
\]

be the projection maps. Then, there exists an \( m > 0 \) such that

(a) for all \( m' \geq m \),

\[
R^i pr_{1*}(\tilde{\mathcal{H}}(m')) , \quad R^i pr_{1*}(\mathcal{O}_{\text{Quot}_Y(\mathcal{O}_{\mathcal{V}_Y}^\oplus, r) \times Y}(m')) , \quad R^i pr_{1*}(\tilde{\mathcal{Q}}(m'))
\]

vanish, for all \( i > 0 \),

(b) the push-forward

\[
0 \rightarrow pr_{1*}(\tilde{\mathcal{H}}(m)) \rightarrow pr_{1*}(\mathcal{O}_{\text{Quot}_Y(\mathcal{O}_{\mathcal{V}_Y}^\oplus, r) \times Y}(m)) \rightarrow pr_{1*}(\tilde{\mathcal{Q}}(m)) \rightarrow 0
\]

is an exact sequence of locally free sheaves on \( \text{Quot}_Y(\mathcal{O}_{\mathcal{V}_Y}^\oplus, r) \),

(c) for all \( m' \geq m \), the sequence

\[
pr_{1*}(\tilde{\mathcal{H}}(m)) \otimes pr_{1*}(\mathcal{O}_{\text{Quot}_Y(\mathcal{O}_{\mathcal{V}_Y}^\oplus, r) \times Y}(m' - m)) \rightarrow pr_{1*}(\mathcal{O}_{\text{Quot}_Y(\mathcal{O}_{\mathcal{V}_Y}^\oplus, r) \times Y}(m')) \rightarrow pr_{1*}(\tilde{\mathcal{Q}}(m')) \rightarrow 0
\]

where the first homomorphism is given by multiplication of global sections, is exact.

The quotient sequent sequence

\[
pr_{1*}(\mathcal{O}_{\text{Quot}_Y(\mathcal{O}_{\mathcal{V}_Y}^\oplus, r) \times Y}(m)) \rightarrow pr_{1*}(\tilde{\mathcal{Q}}(m)) \rightarrow 0
\]

\[
\mathcal{O}_{\text{Quot}_Y(\mathcal{O}_{\mathcal{V}_Y}^\oplus, r) \otimes H^0(Y, \mathcal{O}_Y(m))}
\]

specifies then an embedding

\[
\iota : \text{Quot}_Y(\mathcal{O}_{\mathcal{V}_Y}^\oplus, r) \hookrightarrow \text{Gr}_C(H^0(Y, \mathcal{O}_Y(m)), r).
\]

By construction, \( \mathcal{O}_{\text{Quot}_Y(\mathcal{O}_{\mathcal{V}_Y}^\oplus, r) \otimes H^0(Y, \mathcal{O}_Y(m))} \rightarrow pr_{1*}(\tilde{\mathcal{Q}}(m)) \rightarrow 0 \) on \( \text{Quot}_Y(\mathcal{O}_{\mathcal{V}_Y}^\oplus, r) \) is the pull-back of the universal quotient sequence

\[
\mathcal{O}_{\text{Gr}_C(H^0(Y, \mathcal{O}_Y(m))) \otimes H^0(Y, \mathcal{O}_Y(m))} \rightarrow \tilde{\mathcal{Q}} \rightarrow 0
\]

on \( \text{Gr}_C(H^0(Y, \mathcal{O}_Y(m)), r) \) by \( \iota \).

- Note that \( \tilde{\mathcal{Q}} \) is a locally free sheaf on \( \text{Gr}_C(H^0(Y, \mathcal{O}_Y(m)), r) \) of rank \( r \); the quotient homomorphism

\[
\mathcal{O}_{\text{Gr}_C(H^0(Y, \mathcal{O}_Y(m)), r) \otimes \bigwedge^r H^0(Y, \mathcal{O}_Y(m))} \rightarrow \bigwedge^r \tilde{\mathcal{Q}} \quad (\neq \text{det } \tilde{\mathcal{Q}} \neq 0)
\]

defines the Plücker embedding of \( \text{Gr}_C(H^0(Y, \mathcal{O}_Y(m)), r) \) into the projective space \( \mathbb{P}^{N} \), where \( N = \dim H^0(Y, \mathcal{O}_Y(m)) \). In particular, \( \text{det } \tilde{\mathcal{Q}} \) is a very ample line bundle on \( \text{Gr}_C(H^0(Y, \mathcal{O}_Y(m)), r) \).
Back to the study of \( \varphi \) or, equivalently, \( \tilde{\mathcal{E}}_\varphi \). Property (1) of \( \tilde{\mathcal{E}}_\varphi \) initiates the above construction. From the above review, if letting \( \mathcal{O}_{(C \times Y)}/(C(1)) \) be the relative ample line bundle on \( (C \times Y)/C \) associated to the ample line bundle \( \mathcal{O}_Y(1) \) on \( Y \), then

\[
(\iota \circ f_{(\varphi, \alpha)})^*(\tilde{\mathcal{Q}}) \simeq f^*_{(\varphi, \alpha)}(pr_{1*}(\tilde{\mathcal{Q}}(m))) \simeq pr_{C*}(\tilde{\mathcal{E}}_\varphi(m)).
\]

Here, \( pr_C : C \times Y \to C \) is the projection map. In general, \( pr_{C*}(\tilde{\mathcal{E}}_\varphi(m)) \) is not isomorphic to \( pr_{C*}(\tilde{\mathcal{E}}_\varphi) = \mathcal{E} \). However, when Property (2) holds for \( \tilde{\mathcal{E}}_\varphi \), \( \tilde{\mathcal{E}}_\varphi(m') \simeq \mathcal{E} \) for all \( m' \). Thus, in this case,

\[
(\iota \circ f_{(\varphi, \alpha)})^*(\tilde{\mathcal{Q}}) \simeq f^*_{(\varphi, \alpha)}(pr_{1*}(\tilde{\mathcal{Q}}(m))) \simeq \mathcal{E}
\]

and the universal quotient bundle on \( Gr_C(H^0(Y, \mathcal{O}_Y(m)), r) \) is pulled back to a quotient sequence of \( \mathcal{O}_C \)-modules

\[
\mathcal{O}_C \otimes H^0(Y, \mathcal{O}_Y(m)) \longrightarrow \mathcal{E} \longrightarrow 0.
\]

If, in addition, Property (3) also holds, then \( f_{(\varphi, \alpha)} \) is not constant on each irreducible component of \( C \). Since now \( det \mathcal{E} \simeq (\iota \circ f_{(\varphi, \alpha)})^*(det \tilde{\mathcal{Q}}) \) and \( det \tilde{\mathcal{Q}} \) is very ample, \( \mathcal{E} \) has positive degree on each irreducible component of \( C \). In summary:

**Lemma 2.2.1.** [global generation of \( \mathcal{E} \) and positivity of \( deg(\mathcal{E}) \)]. Let \( \varphi : (C, \mathcal{O}^{Az}_{C}; \mathcal{E}) \to Y \) be a morphism as above, whose graph \( \tilde{\mathcal{E}}_\varphi \in \text{Coh}(C \times Y) \) satisfies Properties (1), (2), and (3). Then \( \mathcal{E} \) is globally generated and has positive degree on each irreducible component of \( C \).

When \( C \) is a \( \mathbb{P}^1 \)-tree: positivity of fundamental modules

**Definition 2.2.2.** [\( \mathbb{P}^1 \)-tree and \( \mathbb{P}^1 \)-chain]. A compact (not necessarily connected) nodal curve \( C \) that is simply-connected is called a \( \mathbb{P}^1 \)-tree since in this case all its irreducible components are \( \mathbb{P}^1 \)'s and the dual graph to \( C \) is a tree (i.e. a simply-connected graph with possibly more-than-one connected components ). A \( \mathbb{P}^1 \)-tree that is connected and such that each \( \mathbb{P}^1 \)-component intersects with at most two other \( \mathbb{P}^1 \)-components is called a \( \mathbb{P}^1 \)-chain. The number of \( \mathbb{P}^1 \)-components in the chain is called the length of the chain. For a \( \mathbb{P}^1 \)-chain of length \( > 1 \), there are exactly two \( \mathbb{P}^1 \)-components, each of which intersects with only one other \( \mathbb{P}^1 \)-component. Each is called a \( \mathbb{P}^1 \) at an end of the chain. Cf. Figure 2.2-1.

We now consider the case when \( C \) is a \( \mathbb{P}^1 \)-tree. As in the Gromov-Witten theory for world-sheet instantons created by fundamental strings, such \( \mathbb{P}^1 \)-trees occur from bubbling off of domains to keep the regularity of morphisms in our problem.

Recall the structure theorem of Grothendieck that a locally free sheaf of rank \( r \) on \( \mathbb{P}^1 \) must be of the form \( \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i) \).

**Definition 2.2.3.** [nonnegative/positive/strictly-positive locally-free sheaf on \( \mathbb{P}^1 \)-tree]. A locally free sheaf \( \mathcal{E} \) of rank \( r \) on a \( \mathbb{P}^1 \)-tree \( C \) is called

- *nonnegative* if for each \( \mathbb{P}^1 \)-component of \( C \), \( \mathcal{E}|_{\mathbb{P}^1} \simeq \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i) \) for some non-negative integers \( 0 \leq a_1 \leq \cdots \leq a_r \);

- *positive* if it is nonnegative and each connected component of \( C \) has at least one \( \mathbb{P}^1 \)-component such that \( \mathcal{E}|_{\mathbb{P}^1} \simeq \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i) \) for some non-negative integers \( 0 \leq a_1 \leq \cdots \leq a_r \) with \( a_r > 0 \);
Figure 2-2-1. Two examples of $\mathbb{P}^1$-trees and their dual graph are illustrated. In (a), the $\mathbb{P}^1$-tree has two connected components. In (b), a $\mathbb{P}^1$-chain of length 6 is illustrated. Note that bubbling $\mathbb{P}^1$-trees that arise from resolving/absorbing irregularities of morphisms from a filling (cf. Sec. 4.3) can be complicated.

- $\textit{strictly positive}$ if for each $\mathbb{P}^1$-component of $C$, $E|_{\mathbb{P}^1} \cong \oplus_{i=1}^{r} O_{\mathbb{P}^1}(a_i)$ for some non-negative integers $0 \leq a_1 \leq \cdots \leq a_r$ with $a_r > 0$.

By definition, $\textit{strictly positive} \Rightarrow \textit{positive} \Rightarrow \textit{nonnegative}.$

Lemma 2.2.4. [criterion of strict positivity]. Let $E$ be a locally free sheaf on a $\mathbb{P}^1$-tree $C$. Then $E$ is strictly positive if and only if $E$ is globally generated and of positive degree on each irreducible component.

Proof. That global generatedness and positive degree together imply strict positivity follows immediately from the structure theorem of Grothendieck. For the converse, we postpone till the end of the next theme after the space $H^0(C, E)$ of global sections of $E$ is studied.

Continuing the discussion from the previous theme. Thus, as a consequence of Lemma 2.2.1 and Lemma 2.2.4 one has the following proposition:

Proposition 2.2.5. [necessary condition for $(\mathbb{P}^1$-tree, $E)$ to admit special morphism]. Let $C$ be a $\mathbb{P}^1$-tree and $(C, O^A_C; E)$ be an Azumaya $\mathbb{P}^1$-tree with a fundamental module that admits a special morphism $\varphi$ to $Y$ whose graph $\tilde{E}_\varphi$ satisfies Properties (1), (2), and (3). Then $E$ must be strictly positive on $C$.

This gives us a guide to a part of the stability condition in Sec. 3.1 and is the main reason why Definition 3.1.1 there gives rise to a bounded moduli stack, taking into account the preliminary compactness result in Part I [L-Y3] (D(10.1)) of the work.
$H^0(C,\mathcal{E})$ and sections of $\mathcal{E}$ that vanish at specified points on $C$

Let $C$ be a $\mathbb{P}^1$-tree and $\mathcal{E}$ be a nonnegative locally free sheaf of rank $r$ on $C$.

**Lemma 2.2.6.** [$h^0(\mathcal{E})$]. Let $\{C_{ij} \simeq \mathbb{P}^1 : i, j\}$ be the set of irreducible components of $C$, where $i$ labels the connected components of $C$ and $j$ labels the irreducible components in the $i$-th connected component of $C$. Suppose that $\mathcal{E}|_{C_{ij}} \simeq \oplus_{k=1}^r \mathcal{O}_{\mathbb{P}^1}(a_{ijk})$, $0 \leq a_{ij1} \leq \cdots \leq a_{ijr}$. Then

$$h^0(\mathcal{E}) := \dim H^0(C, \mathcal{E}) = r \cdot |\pi_0(C)| + \sum_{i,j,k} a_{ijk},$$

where $|\pi_0(C)|$ is the number of connected components of $C$. In particular, except the number of connected components of $C$, it is independent of how this collection of nonnegative locally free sheaves on $\mathbb{P}^1$'s are glued to give a nonnegative locally free sheaf $\mathcal{E}$ on a $\mathbb{P}^1$-tree $C$ (i.e., independent of the isomorphism class of the pair $(C, \mathcal{E})$ from gluing except $|\pi_0(C)|$).

**Proof.** Recall that $h^0(\mathcal{O}_{\mathbb{P}^1}(a)) = 1 + a$ for $a \geq 0$. As a consequence of the structure theorem of Grothendieck, any element in a fiber of a nonnegative locally free sheaf $\mathcal{F} \simeq \oplus_{k=1}^r \mathcal{O}_{\mathbb{P}^1}(a_k)$ on $\mathbb{P}^1$ extends to a global section of $\mathcal{F}$. The dimension of the space of such extensions is given by the sum $\sum_{k=1}^r a_k$.

The fact that a tree is simply connected implies that there is no constraint on extending sections to over a new $\mathbb{P}^1$-component when building a nonnegative locally free sheaf on $C$ by gluing a nonnegative locally free sheaf on $\mathbb{P}^1$ one at a time through an isomorphism between paired fibers. For each connected component, the extension-of-global-sections is indifferent to the isomorphism class of the whole sheaf from gluing as well. It follows that, for the $i$-connected component of $C$, the dimension $h^0$ of the space of global sections is given by $r + \sum_{j,k} a_{ijk}$. The lemma follows.

□

**Remark 2.2.7.** [constrained sections and torsions under collapsing $\mathbb{P}^1$-tree]. Let $A \subset C$ be a finite set of $a$-many distinct points on the $\mathbb{P}^1$-tree $C$. Then, the linear space

$$\{s \in H^0(C, \mathcal{E}) \mid s \text{ vanishes at all the points in } A \subset C \}$$

has dimension $\geq h^0(\mathcal{E}) - ra = r(|\pi_0(C)| - a) + \sum_{i,j,k} a_{ijk}$. When $A$ is the set of attached points of a $\mathbb{P}^1$-tree to the rest of a nodal curve, such sections generate torsions when pushing forward a locally free sheaf on the nodal curve by a collapsing morphism that contracts the $\mathbb{P}^1$-tree.

**Finishing the proof of Lemma 2.2.4.** Let $C$ be a $\mathbb{P}^1$-tree and $\mathcal{E}$ be a strictly positive locally-free sheaf on $C$. It’s clear from the definition of strict positivity of $\mathcal{E}$ that $\det \mathcal{E}$ has positive degree on each irreducible component of $C$. Furthermore, as in the proof of Lemma 2.2.6, any element in a fiber of $\mathcal{E}$ extends to a global section of $\mathcal{E}$. It follows that the natural homomorphism of $\mathcal{O}_C$-modules

$$\mathcal{O}_C \otimes H^0(C, \mathcal{E}) \rightarrow \mathcal{E}$$

is surjective. This shows the other direction of the lemma.

□
2.3 Remarks on nonnegative torsion-free sheaves on a $\mathbb{P}^1$-tree.

As our main interest is semistable morphisms from Azumaya nodal curves with a fundamental module, the objects in Sec. 2.2 are well anticipated. However, somewhere along the path toward our goal we need also to understand the basics of nonnegative torsion-free coherent sheaves on a $\mathbb{P}^1$-tree. We devote this subsection to this, with details that are similar to Sec. 2.2 omitted.

Convention 2.3.1. [coherent sheaf of rank $r$]. By convention, a (coherent) sheaf of rank $r$ on a curve $C$ means a coherent $\mathcal{O}_C$-module that has rank $r$ on every irreducible component of $C$.

Discrepancy to flatness and positive sheaves on a $\mathbb{P}^1$-tree

Definition 2.3.2. [discrepancy to flatness]. Let $\mathcal{F}$ be a torsion-free sheaf of rank $r$ on a nodal curve $C$. For $p \in C$, define the discrepancy to flatness of $\mathcal{F}$ at $p$ to be

$$\delta_{\text{flat}}(\mathcal{F}; p) := \dim(\mathcal{F}|_p) - r.$$  

Note that $\mathcal{F}$ is flat at smooth points of $C$. Thus, $\delta_{\text{flat}}(\mathcal{F}; \bullet) = 0$ except possibly at nodes of $C$ and $\mathcal{F}$ is flat if and only of $\delta_{\text{flat}}(\mathcal{F}; \bullet)$ is identically 0. Also, it follows from [Se1: Chapter 8] that $0 \leq \delta_{\text{flat}}(\mathcal{F}; p) \leq r$ for all $p$. Thus, one may define the discrepancy to flatness of $\mathcal{F}$ on $C$ to be

$$\delta_{\text{flat}}(\mathcal{F}) := \sum_{p \in C} \delta_{\text{flat}}(\mathcal{F}; p).$$

It’s clear that a torsion-free sheaf $\mathcal{F}$ of a fixed rank on $C$ is locally free if and only if $\delta_{\text{flat}}(\mathcal{F}) = 0$. This justifies the name.

Definition 2.3.3. [nonnegative/positive/strictly-positive torsion-free sheaf on $\mathbb{P}^1$-tree]. (Cf. Definition 2.2.3) A torsion-free sheaf $\mathcal{F}$ of rank $r$ on a $\mathbb{P}^1$-tree $C$ is called

- nonnegative if for each $\mathbb{P}^1$-component of $C$,

$$\mathcal{F}|_{\mathbb{P}^1}^{\text{torsion-free}} := \mathcal{F}|_{\mathbb{P}^1}/(\mathcal{F}|_{\mathbb{P}^1})_{\text{torsion}} \simeq \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$$

for some non-negative integers $0 \leq a_1 \leq \cdots \leq a_r$, where $(\mathcal{F}|_{\mathbb{P}^1})_{\text{torsion}}$ is the torsion subsheaf of $\mathcal{F}|_{\mathbb{P}^1}$ involved;

- positive if it is nonnegative and each connected component of $C$ has at least one $\mathbb{P}^1$-component such that $(\mathcal{F}|_{\mathbb{P}^1})_{\text{torsion-free}} \simeq \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$ for some non-negative integers $0 \leq a_1 \leq \cdots \leq a_r$, with $a_r > 0$;

- strictly positive if for each $\mathbb{P}^1$-component of $C$, $(\mathcal{F}|_{\mathbb{P}^1})_{\text{torsion-free}} \simeq \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$ for some non-negative integers $0 \leq a_1 \leq \cdots \leq a_r$ with $a_r > 0$.

By definition, strictly positive $\Rightarrow$ positive $\Rightarrow$ nonnegative.

Definition 2.3.4. [weighted graph associated to $(C, \mathcal{F})$]. Let $\mathcal{F}$ be a torsion-free sheaf on a $\mathbb{P}^1$-tree curve $C$ and $\{C_{ij} \cong \mathbb{P}^1 : i, j\}$ be the set of irreducible components of $C$, where $i$ labels the connected components of $C$ and $j$ labels the irreducible components in the $i$-th connected component of $C$. Suppose that $(\mathcal{F}|_{C_{ij}})_{\text{torsion-free}} \cong \bigoplus_{k=1}^r \mathcal{O}_{\mathbb{P}^1}(a_{ijk}), a_{ijk} \leq \cdots \leq a_{ijr}$. Define the weighted graph $\Gamma_{(C, \mathcal{F})}$ associated to $(C, \mathcal{F})$ to be the (unoriented) graph that associates to each $C_{ij}$ a vertex $v_{ij}$, with weight $\{a_{ijk}, \ldots, a_{ijr}\}$, and to each node $p$, say connecting $C_{ij}$ and $C_{ij'}$, an edge $e_{i, jj'}$ connecting $v_{ij}$ and $v_{ij'}$, with weight $\delta_{\text{flat}}(\mathcal{F}; p)$. (By convention, $e_{i, jj'} = e_{i, jj'}$.)
Note that, forgetting the weights, $\Gamma_{(C,F)}$ is simply the dual graph/tree to $C$.

**Definition 2.3.5. [degree].** Continuing the setting in Definition 2.3.4 The degree $\deg(F)$ of a torsion-free sheaf $F$ on a $\mathbb{P}^1$-tree curve $C$ is defined to be $\sum_{i,j,k} a_{ijk}$.

**Lemma 2.3.6. $[h^0(F)]$.** (Cf. Lemma 2.2.6) Continuing the setting in Definition 2.3.4 with the additional assumption that $F$ is nonnegative. Then

$$ h^0(F) := \dim H^0(C, F) = r \cdot |\pi_0(C)| + \deg(F) + \delta_{\text{flat}}(F). $$

where $|\pi_0(C)|$ is the number of connected components of $C$. Again, except the data already encoded in the weighted graph $\Gamma_{(C,F)}$ associated to $(C,F)$, $h^0(F)$ is independent of the isomorphism class of the pair $(C,F)$.

**Proof.** It follows from the classification of the germs of a torsion-free sheaf at a node of a nodal curve, given by C.S. Seshadri in [Se1: Chapter 8], that the torsion-free sheaf $F$ is reconstructible from $(F|_{C_{ij}})^{\text{torsion-free}}$ as follows:

- Let $e$ be an edge in $\Gamma_{(C,F)}$ of weight $\delta_{i,j,j'}$ and connecting vertices $v_{ij}$ and $v_{ij'}$.
- At the level of curves, $e$ means there are a pair of points, $p_{i,j,j'} \in C_{ij}$ and $p_{i,j',j} \in C_{ij'}$, which are glued to give a node $p$ in $C$.
- At the level of sheaves, $e$ means there is a pair of codimension-$\delta$ subspaces,

$$ H_{i,j,j'} \subset (F|_{C_{ij}})^{\text{torsion-free}}|_{p_{i,j,j'}} \quad \text{and} \quad H_{i,j',j} \subset (F|_{C_{ij'}})^{\text{torsion-free}}|_{p_{i,j',j}}, $$

together with projection maps

$$ \pi_{i,j,j'} : (F|_{C_{ij}})^{\text{torsion-free}}|_{p_{i,j,j'}} \to H_{i,j,j'} \quad \text{and} \quad \pi_{i,j',j} : (F|_{C_{ij'}})^{\text{torsion-free}}|_{p_{i,j',j}} \to H_{i,j',j}, $$

and an isomorphism

$$ h_{i,j,j'} : H_{i,j,j'} \to H_{i,j',j}. $$

- The data

$$ \{(H_{i,j,j'}, \pi_{i,j,j'}, H_{i,j',j}, \pi_{i,j',j}; h_{i,j,j'}) \{i,j,j'\} \} $$

corresponds to an edge in $\Gamma_{C,F}$.

recovers the torsion-free sheaf $F$ on $C$ by gluing through the isomorphism $h_*$ of codimension-$\delta$ subspaces in the paired fibers of $(F|_{C_{ij}})^{\text{torsion-free}}$’s at paired points in $C_{ij}$’s with a sheaf-of-local-sections structure through those of $(F|_{C_{ij'}})^{\text{torsion-free}}$’s and the projection maps $\pi_*$. In particular, it specifies the $\mathbb{C}$-vector space structure on the fiber at the node $p$ after gluing as the fibered product

$$ (F|_{C_{ij}})^{\text{torsion-free}}|_{p_{i,j,j'}} \oplus (F|_{C_{ij'}})^{\text{torsion-free}}|_{p_{i,j',j}} $$

$$ : \text{Ker} \left( (-h_{i,j,j'} \circ \pi_{i,j,j'} , \pi_{i,j',j}) : (F|_{C_{ij}})^{\text{torsion-free}}|_{p_{i,j,j'}} \oplus (F|_{C_{ij'}})^{\text{torsion-free}}|_{p_{i,j',j}} \to H_{i,j',j} \right). $$

In terms of this gluing data,

$$ H^0(C, F) \simeq \left\{ (s_{ij})_{ij} \bigg| \begin{array}{c}
    \cdot \ s_{ij} \in H^0(C_{ij}, (F|_{C_{ij}})^{\text{torsion-free}}) \\
    \cdot \ (h_{i,j,j'} \circ \pi_{i,j,j'})(s_{ij}(p_{i,j,j'})) = \pi_{i,j',j}(s_{ij'}(p_{i,j',j})) \\
    \text{for } \{ij, ij'\} \text{ corresponding to an edge in } \Gamma_{(C,F)}
  \end{array} \right\}. $$
Proof. we may assume, without loss of generality, that $h$ and the scheme-theoretical preimage $h_*$ of $h$ is globally generated and the locally-free quotient $(F|_{\mathbb{P}^1})_{\text{torsion-free}}$ of its restriction to each irreducible component $\mathbb{P}^1$ has positive degree.

Lemma 2.3.7. [criterion of strict positivity]. (Cf. Lemma 2.2.4) Let $F$ be a torsion-free sheaf on a $\mathbb{P}^1$-tree $C$. Then $F$ is strictly positive if and only if $F$ is globally generated and the locally-free quotient $(F|_{\mathbb{P}^1})_{\text{torsion-free}}$ of its restriction to each irreducible component $\mathbb{P}^1$ has positive degree.

Proof. We shall adopt the notation from Definition 2.3.4.

Given a strictly positive torsion-free sheaf $F$ on a $\mathbb{P}^1$-tree $C$, by definition the locally-free quotient $(F|_{\mathbb{P}^1})_{\text{torsion-free}}$ of its restriction to each irreducible component $\mathbb{P}^1$ has positive degree. Furthermore, the proof of Lemma 2.3.6 implies that the sequence from the natural evaluation homomorphism $\mathcal{O}_C \times H^0(C, F) \to F \to 0$ is exact.

Conversely, suppose that $\mathcal{O}_C \times H^0(C, F) \to F \to 0$ is exact. Then, so are $\mathcal{O}_{C_{ij}} \times H^0(C, F) \to F|_{C_{ij}} \to 0$. Thus, the composition $\mathcal{O}_{C_{ij}} \times H^0(C, F) \to F|_{C_{ij}} \to (F|_{C_{ij}})_{\text{torsion-free}}$ is surjective. Since, furthermore, $(F|_{C_{ij}})_{\text{torsion-free}}$ has positive degree, $F$ must be strictly positive.

Decrease of $\delta_{\text{flat}}(\bullet)$ when bubbling off a $\mathbb{P}^1$-tree with a positive sheaf

Proposition 2.3.8. [decrease of $\delta_{\text{flat}}(\bullet)$ when bubbling off $\mathbb{P}^1$-tree with positive sheaf].

Let $h : C' \to C$ be a collapsing morphism of nodal curves that contracts a $\mathbb{P}^1$-tree subcurve $C'_u$ of $C'$, $F$ be a torsion-free sheaf on $C$ of rank $r$, and $F'$ be a torsion-free sheaf on $C'$, also of rank $r$, such that

1. $(F'|_{C'_u})_{\text{torsion-free}}$ is positive on $C'_u$;
2. there exists a surjective homomorphism $\beta : h_*(F') \to F$ of $\mathcal{O}_C$-modules.

Then,

$$
\delta_{\text{flat}}(F') - \delta_{\text{flat}}(F) = -\deg((F'|_{C'_u})_{\text{torsion-free}}) < 0.
$$

Proof. Since $F$ is torsion-free and $h_*(F')$ and $F$ have the same rank,

$$
\text{Ker}(\beta) = (h_*(F'))_{\text{torsion}} \quad \text{and} \quad (h_*(F'))_{\text{torsion-free}} \simeq F.
$$

Furthermore, since the restriction $h : C' - C'_u \to C - h(C'_u)$ is an isomorphism, $F'$ is torsion-free, and the scheme-theoretical preimage $h^{-1}(h(C'_u))$ is exactly $C'_u$, $\text{Ker}(\beta)$ is supported at the finite set $h(C'_u)$ of points on $C$. As we are concerned only with the difference

$$
\delta_{\text{flat}}(F') - \delta_{\text{flat}}(F) = \delta_{\text{flat}}(F') - \delta_{\text{flat}}((h_*(F'))_{\text{torsion-free}}),
$$

we may assume, without loss of generality, that $F'$ is locally free on $C' - C'_u$ (and, hence, $F$ and $h_*(F')$ are locally free on $C - h(C'_u)$).

With this additional assumption, let $C' = C'_0 \cup C'_u$ be the decomposition of $C$ into a union of the contracted $\mathbb{P}^1$-tree $C'_0$ and the subcurve $C'_u$ that’s not contracted by $h$. Note that the restriction $h : C'_0 \to C$ is birational, affine, surjective, and of relative dimension 0. Let $p \in h(C'_u)$ and $C'_{u,p} := h^{-1}(p)$, a connected $\mathbb{P}^1$-tree subcurve in $C'_u$. 

19
• Case (a) : $p$ is a smooth point on $C$.

Then, $C_{u,p}'$ has only one contact point $p'$ with $C_0'$. Under $h_*$, $F'|_{C_{u,p}'}$ contributes only to $\text{Ker}(\beta) \subset h_*(\mathcal{F})$ as the torsion subsheaf supported at the smooth point $p$. It has length

$$\deg((F'|_{C_{u,p}'})_{\text{torsion-free}}) + \delta_\text{flat}((F'|_{C_{u,p}'})_{\text{torsion-free}}) + \delta_\text{flat}(\mathcal{F}; p') > 0,$$

as a consequence of Lemma 2.3.6 though irrelevant to $\delta_\text{flat}(\mathcal{F}') - \delta_\text{flat}(\mathcal{F})$.

• Case (b) : $p$ is a node on $C$.

Then, $C_{u,p}'$ has two contact points, $p'_-$ and $p'_+$ with $C_0'$. Under $h_*$, $\tilde{F}'|_{C_{u,p}'}$ may contribute to both $\text{Ker}(\beta)$ and $\delta_\text{flat}(h_*(\mathcal{F})): (h_*(\mathcal{F}'))$.

For the $\text{Ker}(\beta)$ part, again as a consequence of Lemma 2.3.6, $F'|_{C_{u,p}'}$ contributes to $\text{Ker}(\beta)$ as the torsion subsheaf supported at the node $p$ of length $\max \{0, \eta_p - r\}$, where

$$\eta_p := \deg((F'|_{C_{u,p}'})_{\text{torsion-free}}) + \delta_\text{flat}((F'|_{C_{u,p}'})_{\text{torsion-free}}) + \delta_\text{flat}(\mathcal{F}; p')$$

which can be either $\leq r$ or $\geq r$. (By convention, a torsion sheaf of length 0 is the 0-sheaf.)

For the $\delta_\text{flat}((h_*(\mathcal{F}'))_{\text{torsion-free}}; p)$ part, since all local sections in the stalk of $h_*(\mathcal{F}')$ at $p$ contribute to the fiber

$$(h_*(\mathcal{F}'))|_p \simeq \text{Ker}(\beta)|_p + ((h_*(\mathcal{F}'))_{\text{torsion-free}})|_p$$

of $h_*(\mathcal{F})$ at $p$, it follows from Lemma 2.3.6 and the above that

$$\text{dim}((h_*(\mathcal{F}'))_{\text{torsion-free}})|_p = \begin{cases} 
\eta_p + r & \text{if } \eta_p \leq r, \\
2r & \text{if } \eta_p > r.
\end{cases}$$

Note that, when $\eta_p \leq r, \eta_p < \eta_p + r \leq 2r$. Thus, $\mathcal{F}'|_{C_{u,p}'}$ always contributes to $\delta_\text{flat}((h_*(\mathcal{F}'))_{\text{torsion-free}})$:

$$\delta_\text{flat}((h_*(\mathcal{F}'))_{\text{torsion-free}}; p) = \begin{cases} 
\eta_p & \text{if } \eta_p \leq r, \\
r & \text{if } \eta_p > r.
\end{cases}$$

which is always positive.

Summing over the node $p \in h(C'_u) \subset C$ in Case (b) above, one concludes that

$$\delta_\text{flat}(\mathcal{F}') - \delta_\text{flat}(\mathcal{F}) = -\deg((F'|_{C_{u,p}'})_{\text{torsion-free}}) < 0.$$  

This proves the proposition.  

\[\square\]

3 The space of D-string world-sheet instantons: The moduli stack of $Z$-semistable morphisms from Azumaya nodal curves with a fundamental module to a Calabi-Yau 3-fold

With the preparations from [L-L-S-Y] (D(2)), [L-Y3] (D(10.1)), and Sec. 2 of this note, we are now ready to define the notion of $Z$-semistable morphisms of a fixed type from general Azumaya nodal curves with a fundamental module to a Calabi-Yau 3-fold and the stack of such objects, Sec. 3.1. A natural morphism from this stack to the stack $\mathcal{F}\mathcal{M}_g^{1,0;Z-ss}(Y; c)$ of Fourier-Mukai transforms is explained in Sec. 3.2.
3.1 The moduli stack of $Z$-semistable morphisms from Azumaya nodal curves with a fundamental module to a Calabi-Yau 3-fold

We now bring out the main character of the D(10)-series of the project.

$Z$-Semistable morphisms from Azumaya nodal curves with a fundamental module

For the current Sec. 3 and the next Sec. 4, we fix the following data:

- **Domain data:**
  - $\mathcal{M}_g$: the moduli stack of stable curves of genus $g$,
  - $\underline{\text{C}}_{\mathcal{M}_g}/\mathcal{M}_g$: the universal curve over $\mathcal{M}_g$,
  - $[L]$ : a relative positive degree class on $\underline{\text{C}}_{\mathcal{M}_g}/\mathcal{M}_g$.

- **Target data:**
  - $(Y, B + \sqrt{-1}J)$: a projective Calabi-Yau 3-fold with a complexified Kähler class.

**Definition 3.1.1. [Z-(semi)stable morphism].** Let

- $(C', \mathcal{E}')$ be a (connected) nodal curve $C'$ of genus $g$ with a locally free sheaf $\mathcal{E}'$ of rank $r$ and Euler characteristic $\chi$,
- $\varphi : (C', \mathcal{O}_{C'}^{Az} = \text{End}_{O_{C'}}(\mathcal{E}'); \mathcal{E}') \to Y$ be a morphism from an Azumaya nodal curve with a fundamental module to a projective Calabi-Yau 3-fold $Y$ such that $[\varphi_{*}(\mathcal{E}')] = \beta \in A_1(Y)$, (i.e. $\varphi$ is a morphism from an Azumaya nodal curve with a fundamental module to $Y$ of type $(g; r, \chi; \beta)$)
- $\tilde{\mathcal{E}}_{\varphi} \in \text{Coh}(C' \times Y)$ be the graph of $\varphi$, which is a 1-dimensional coherent sheaf on $C' \times Y$ that is flat over $C'$ and of relative dimension 0 and relative length $r$ over $C'$,
- $\rho : C' \to C$ be the collapsing morphism from $C'$ to the stable curve $C$ associated to $C'$ (i.e. $\rho$ stabilizes $C'$ ),
- $Id_Y : Y \to Y$ be the identity map.

We say that $\varphi$ is **$Z$-semistable of type $(g; r, \chi; \beta; c)$** if the following additional conditions hold:

1. $\rho \times Id_Y : (\rho \times Id_Y)^{*}(\tilde{\mathcal{E}}_{\varphi}) =: \tilde{F}$ is a $Z$-semistable Fourier-Mukai transform from $C$ to $Y$ with twisted central charge $Z^{B+\sqrt{-1}[L]}(\tilde{F}) = c$. (Note that $[pr_{Y*}(\tilde{F})] = [pr_{Y*}(\tilde{\mathcal{E}}_{\varphi})] = [\varphi_{*}(\mathcal{E}')] = \beta$.)

2. The natural sequence of homomorphisms of $\mathcal{O}_{C' \times Y}$-modules $(\rho \times Id_Y)^{*}(\tilde{F}) \to \tilde{\mathcal{E}}_{\varphi} \to 0$ is exact.

3. For each $\mathbb{P}^1$-component (denoted by $\mathbb{P}^1$) of the $\mathbb{P}^1$-tree subcurve of $C'$ that is collapsed by $\rho$ to points in $C$, if $\varphi|_{\mathbb{P}^1}$ is a constant morphism, then $\mathbb{P}^1$ has at least three special points (i.e. points where $\mathbb{P}^1$ intersects with other components of $C'$).

We say that $\varphi$ is **$Z$-stable** if Condition (1) above is replaced by:

1. $(\rho \times Id_Y)^{*}(\tilde{\mathcal{E}}_{\varphi}) =: \tilde{F}$ is a $Z$-stable Fourier-Mukai transform from $C$ to $Y$. 

21
When the central charge functional $Z$ is known and fixed either explicitly or implicitly, we may use the terminology semistable morphism, stable morphism for simplicity.

**Remark 3.1.2.** [Conditions (1), (2), (3) in Definition 3.1.1]. The meaning behind Conditions (1), (2), and (3) in Definition 3.1.1 is illuminated below:

1. **Condition (1)** says that, though the notion of (semi)stability of morphisms in our problem cannot be defined completely just using twisted central charge, that for the restriction of the morphism on the “main part” of a general nodal curves remains following the notion of (semi)stability associated to the twisted central charges.

2. **Condition (2)** is a “positivity condition” on the restriction of morphisms to the unstable bubbling $\mathbb{P}^1$-tree of nodal curves. It compensates for the fact that such positivity can be lost and undetectable from the twisted central charge due to “overspreading of the twisted central charge”. This condition says that, while possibly undetectable by twisted central charge, the restriction of a morphism to such $\mathbb{P}^1$-tree can still cost some “positive internal rotating/winding energy”. From this aspect, Condition (2) might be weakened to:

   
   
   \[(2') \text{ The natural sequence of homomorphisms of } \mathcal{O}_{C' \times Y}\text{-modules } (\rho \times Id_Y)^*(\tilde{F}) \to \tilde{E}'_{\phi'} \to 0 \text{ is exact at } 1\text{-dimensional generic points of } \text{Supp}(\tilde{E}'_{\phi}).\]

   
   I.e. requiring only that $\text{Coker}( (\rho \times Id_Y)^*(\tilde{F}) \to \tilde{E}'_{\phi})$ is 0-dimensional, instead of 0.

3. **Condition (3)** reminds one of a similar condition for stable maps in algebraic Gromov-Witten theory ([Ko] and e.g. [Be] for a survey). In our case, even with Conditions (1) and (2), there remains no control of the size of $\mathbb{P}^1$-chains for which the restriction of the morphism is connected-componentwise constant. As such chain takes no twisted central charge and costs no “winding energy’ of any sort, they are completely undetectable and uncontrollable. Fortunately, they are on the other hand the only kind of $\mathbb{P}^1$-trees in a nodal curve such that collapsing the $\mathbb{P}^1$-tree gives rise to a curve that remains nodal. From the above reasoning, to get a bounded family of morphisms, one has no choice but to collapse all such $\mathbb{P}^1$-chains. What’s left is exactly described in Condition (3). (See Sec. 4.3, Theme “Step (d) : Termination of the reduction – Recovery of a regular morphism in our category over $t \in T$” for a more technical and precise discussion.)

**Definition 3.1.3.** [morphism between $Z$-semistable morphisms]. Let

\[
\varphi_1 : (C'_1, \mathcal{O}_{C'_1}^{A^Z_1} : = \text{End}_{\mathcal{O}_{C'}}(\mathcal{E}'_1); \mathcal{E}'_1) \to Y \text{ and } \varphi_2 : (C'_2, \mathcal{O}_{C'_2}^{A^Z_2} := \text{End}_{\mathcal{O}_{C'}}(\mathcal{E}'_2); \mathcal{E}'_2) \to Y
\]

be two $Z$-semistable morphisms from Azumaya nodal curves with a fundamental module to $Y$ of type $(g;r, \chi; \beta, c)$. Then, a morphism from $\varphi_1$ to $\varphi_2$, phrased directly in terms of their graph $\tilde{E}'_{\varphi_1}$ and $\tilde{E}'_{\varphi_2}$ respectively, is a pair $(h', \tilde{h}')$, where

- $h' : C'_1 \to C'_2$ is an isomorphism of nodal curves,
- $\tilde{h}' : (h' \times Id_Y)^*(\tilde{E}'_{\varphi_2}) \to \tilde{E}'_{\varphi_1}$ is an isomorphism of coherent sheaves on $C'_1 \times Y$.

Note that, with the collapsing morphisms $\rho_1 : C'_1 \to C_1$ and $\rho_2 : C'_2 \to C_2$ specified, $h'$ induces a unique isomorphism $h : C_1 \to C_2$ so that the following diagram commutes:

\[
\begin{array}{ccc}
C'_1 & \xrightarrow{h'} & C'_2 \\
\downarrow{\rho_1} & & \downarrow{\rho_2} \\
C_1 & \xrightarrow{h} & C_2
\end{array}
\]
and \( \tilde{h}' \) induces further an isomorphism \( \tilde{h} : (h \times \text{Id}_Y)^* (\tilde{F}_2) \to \tilde{F}_1 \) so that the following diagram commutes

\[
\begin{array}{ccc}
(h \times \text{Id}_Y)^*((p_2 \times \text{Id}_Y)_* (\tilde{E}'_{\varphi_2})) & \sim & (p_1 \times \text{Id}_Y)^* ((h' \times \text{Id}_Y)^* (\tilde{E}'_{\varphi_2})) \\
\downarrow & & \downarrow \\
(h \times \text{Id}_Y)^* (\tilde{F}_2) & \sim & (p_1 \times \text{Id}_Y)^* ((p_1 \times \text{Id}_Y)_* (h')) \\
\end{array}
\]

where the isomorphism \( (h \times \text{Id}_Y)^* ((p_2 \times \text{Id}_Y)_* (\tilde{E}'_{\varphi_2})) \Rightarrow (p_1 \times \text{Id}_Y)^* ((h' \times \text{Id}_Y)^* (\tilde{E}'_{\varphi_2})) \) on the top is the natural homomorphism (isomorphism in the current case) from interchanging the order of push and pull. These induced data of isomorphisms are automatically assumed when in need.

The moduli stack of \( Z \)-semistable morphisms of type \( (g; r, \chi; \beta, c) \)

**Definition 3.1.4. [family of \( Z \)-semistable morphisms].** Let \( S \) be a Noetherian scheme/\( \mathbb{C} \). An \( S \)-family \( \varphi_S \) of \( Z \)-semistable morphisms of type \( (g; r, \chi; \beta, c) \) from Azumaya nodal curves with a fundamental module to the Calabi-Yau 3-fold \( Y \) is given, in terms of their graphs, by the following data:

- \( C'_S/S \) : a flat family of nodal curves of genus \( g \) over \( S \);
- \( \tilde{E}'_S \in \text{Coh} (C'_S \times Y) \) : a coherent sheaf on \( C'_S \times Y \) that is flat over \( C'_S \) and of relative dimension 0 and relative length \( r \) over \( C'_S \) such that
  - as a coherent sheaf on \( C'_S/S \), \( pr_{C'_S} (\tilde{E}'_S) \) is a flat family of locally free sheaves of rank \( r \) and Euler characteristic \( \chi \) on nodal curves over \( S \),
  - for each \( s \in S \), \( [pr_{Y_s} (\tilde{E}'_S)] = \beta \in A_1 (Y) \)

that satisfy the following properties:

1. Let \( \rho_S : C'_S/S \to C/S \) be the collapsing \( S \)-morphism that defines a morphism \( S \to \overline{\mathcal{M}}_g \). Then, \( (\rho_S \times \text{Id}_Y)_* (\tilde{E}'_S) =: \tilde{F}_S \) is a flat family of \( Z \)-semistable Fourier-Mukai transforms from fibers of \( C_S/S \) to \( Y \) with twisted central charge \( Z^{B+\sqrt{-1} [\beta]} (\tilde{F}_s) = c \) for all \( s \in S \).

2. The natural sequence of homomorphisms of \( \mathcal{O}_{C'_S \times Y} \)-modules \( (\rho_S \times \text{Id}_Y)^* (\tilde{F}_S) \to \tilde{E}'_S \to 0 \) is exact.

3. For each \( \mathbb{P}^1 \)-component (denoted by \( \mathbb{P}^1 \)) of the \( \mathbb{P}^1 \)-tree subcurve of \( C'_s, s \in S \), that is collapsed by \( \rho_s \) to points in \( C_s \), if \( \tilde{E}'|_{\mathbb{P}^1 \times Y} \) is isomorphic to \( pr_Y^* (\bullet) \) for some 0-dimensional sheaf \( \bullet \) of length \( r \) on \( Y \) (i.e. \( \varphi_s|_{\mathbb{P}^1} \) is a constant morphism), then \( \mathbb{P}^1 \) has at least three special points in \( C'_s \).

**Definition 3.1.5. [morphism between \( S \)-families of \( Z \)-semistable morphisms].** Let

\[
\varphi_{1,S} : (C'_1, S, \mathcal{O}_{C'_1, S}^\mathbb{A}_S):= \text{End}_{\mathcal{O}_{C'_1, S}} (\mathcal{E}'_1, S)/S \to Y
\]

and

\[
\varphi_{2,S} : (C'_2, S, \mathcal{O}_{C'_2, S}^\mathbb{A}_S):= \text{End}_{\mathcal{O}_{C'_2, S}} (\mathcal{E}'_2, S)/S \to Y
\]

be two \( S \)-families of \( Z \)-semistable morphisms from Azumaya nodal curves with a fundamental module to \( Y \) of type \( (g; r, \chi; \beta, c) \). Then, a morphism from \( \varphi_{1,S} \) to \( \varphi_{2,S} \), phrased directly in terms of their graph \( \mathcal{E}'_{\varphi_{1,S}} \) and \( \mathcal{E}'_{\varphi_{2,S}} \) respectively, is a pair \((h'_S, \tilde{h}'_S)\), where
· $h'_S : C'_{1,S}/S \to C'_{2,S}/S$ is an $S$-isomorphism of nodal curves,

· $\tilde{h}'_S : (h'_S \times \text{Id}_Y)^*(\tilde{\mathcal{E}}'_{\varphi_{2,S}}) \to \tilde{\mathcal{E}}'_{\varphi_{1,S}}$ is an $S$-isomorphism of coherent sheaves on $(C'_{1,S} \times Y)/S$.

As before, with the collapsing morphisms $\rho_{1,S} : C'_{1,S} \to C_{1,S}$ and $\rho_{2,S} : C'_{2,S} \to C_{2,S}$ specified, $h'_S$ induces a unique isomorphism $h_S : C'_1 \to C'_2$ so that the following diagram commutes

$$
\begin{array}{ccc}
C'_{1,S} & \xrightarrow{h'_S} & C'_{2,S} \\
\rho_{1,S} \downarrow & & \downarrow \rho_{2,S} \\
C_{1,S} & \xrightarrow{h_S} & C_{2,S}
\end{array}
$$

and $\tilde{h}'_S$ induces further an isomorphism $\tilde{h}_S : (h_S \times \text{Id}_Y)^*(\tilde{F}_{2,S}) \to \tilde{F}_{1,S}$ so that the following diagram commutes

$$
\begin{array}{ccc}
(h_S \times \text{Id}_Y)^*((\rho_{2,S} \times \text{Id}_Y)_*(\tilde{\mathcal{E}}'_{\varphi_{2,S}})) & \xrightarrow{\sim} & (\rho_{1,S} \times \text{Id}_Y)_*((h'_S \times \text{Id}_Y)^*(\tilde{\mathcal{E}}'_{\varphi_{2,S}})) \\
\downarrow & & \downarrow \\
(h_S \times \text{Id}_Y)^*(\tilde{F}_{2,S}) & \xrightarrow{\tilde{h}_S} & \tilde{F}_{1,S}
\end{array}
$$

**Definition 3.1.6. [stack of $Z$-semistable morphisms].** Define the stack $\mathcal{M}^{Z,ss}_{\text{Az}(g;r,\chi)}(Y;\beta,c)$ of $Z$-semistable morphisms from Azumaya nodal curves with a fundamental module to the Calabi-Yau 3-fold $Y$ of type $(g;r,\chi;\beta,c)$ to be the sheaf of groupoids that associates to each scheme $S$ over $\mathbb{C}$ the groupoid $\mathcal{M}^{Z,ss}_{\text{Az}(g;r,\chi)}(Y;\beta,c)(S)$ whose objects are $S$-families of $Z$-semistable morphisms from Azumaya nodal curves with a fundamental to $Y$ of type $(g;r,\chi;\beta,c)$ in Definition 3.1.4 and whose morphisms are morphisms between $S$-families of $Z$-semistable morphisms in Definition 3.1.5.

**3.2 A natural morphism from $\mathcal{M}^{Z,ss}_{\text{Az}(g;r,\chi)}(Y;\beta,c)$ to $\mathcal{F}M^1_{g,0};Z^{ss}(Y;c)$**

**Lemma 3.2.1. [vanishing of $R^i(\rho \times \text{Id}_Y)_*(\text{graph})$ for semistable morphism].** Let $S$ be an affine base scheme,

$$
\varphi_S : (C'_S, \mathcal{O}^{\mathbb{A}^2}_{C'_S} := \text{End}_{\mathcal{O}_{C'_S}}(\mathcal{E}'_S)/S; \mathcal{E}'_S) \to Y
$$

be an $S$-family of semistable morphisms from Azumaya nodal curves with a fundamental module to $Y$ and $\mathcal{E}'_{\varphi_S} \subset \text{Coh}((C'_S \times Y)/S)$ be its graph. Let $\varphi_S : C'_S \to C_S$ be the built-in contracting homomorphism from nodal curves to stable curves. Then, Condition (2) required of semistable morphisms implies that

$$
R^i(\varphi_S \times \text{Id}_Y)_*(\mathcal{E}'_{\varphi_S}) = 0
$$

in $\text{Coh}((C_S \times Y)/S)$.

**Proof.** Let $\Sigma_S \subset C_S$ be the image of contracted $\mathbb{P}^1$-trees on fibers of $C'/S$ under $\varphi_S$. Then $\Sigma_S$ has relative dimension 0 over $S$. It follows that, after passing to a base change (still denoted by $S$) if necessary, there exists a relative ample Cartier divisor $H_{C_S/S}$ on $C_S/S$ that has no intersection with $\Sigma_S$. Together with any ample Cartier divisor $H_Y$ on $Y$, they define a relative ample line bundle $\mathcal{O}_{(C_S \times Y)/S}(1)$ on $(C_S \times Y)/S$ that has the following property:

$$
R^i(\rho_S \times \text{Id}_Y)_*(\mathcal{E}'_{\varphi} \otimes_{\mathcal{O}_C} \mathcal{O}_{C' \times Y} \otimes (\rho_S \times \text{Id}_Y)^* \mathcal{O}_{(C_S \times Y)/S}(m)) \cong R^i(\rho_S \times \text{Id}_Y)_*(\mathcal{E}'_{\varphi})
$$

24
for all \( m \).

Consider now the push-forward \( \tilde{F}_S := (\rho_S \times \text{Id}_Y)_* (\tilde{E}_\varphi) \) on \( (C_S \times Y)/Y \). Then, there is an \( m > 0 \) such that \( \tilde{F}_S(m) \) is globally generated. Let

\[
\mathcal{O}_{C_S \times Y}^{\oplus k} \to \tilde{F}_S(m) \to 0
\]

be the associated exact sequence of \( \mathcal{O}_{(C_S \times Y)/Y} \)-modules. It follows from Condition (2) and right exactness of tensoring/pulling-back that this induces an exact sequence of \( \mathcal{O}_{C_S \times Y} \)-modules

\[
0; \to \tilde{H}'_S \to \mathcal{O}_{C_S \times Y}^{\oplus k} \to \tilde{E}'_{\varphi} \otimes \mathcal{O}_{C_S \times Y} (\rho_S \times \text{Id}_Y)^*(\mathcal{O}_{(C_S \times Y)/Y}(m)) \to 0,
\]

and its associated long exact sequence of \( \mathcal{O}_{C_S \times Y} \)-modules

\[
\cdots \to R^1(\rho_S \times \text{Id}_Y)_* (\mathcal{O}_{C_S \times Y}^{\oplus k}) \to R^1(\rho_S \times \text{Id}_Y)_* (\tilde{E}'_{\varphi} \otimes \mathcal{O}_{C_S \times Y} (\rho_S \times \text{Id}_Y)^*(\mathcal{O}_{(C_S \times Y)/Y}(m))) \to R^2(\rho_S \times \text{Id}_Y)_* (\tilde{H}'_S) \to \cdots .
\]

Observe that \( R^2(\rho \times \text{Id}_Y)_* (\tilde{H}'_S) = 0 \) since \( \rho_S \times \text{Id}_S : C_S' \times Y \to C_S \times Y \) has relative dimension \( \leq 1 \). In addition, \( R^1(\rho_S \times \text{Id}_Y)_* (\mathcal{O}_{C_S \times Y}^{\oplus k}) = 0 \) as well since \( \rho_S \times \text{Id}_Y \) collapses \( \mathbb{P}^1 \)-trees. It follows that \( R^1(\rho \times \text{Id}_Y)_* (\tilde{E}'_{\varphi} \otimes \mathcal{O}_{C \times Y} (\rho \times \text{Id}_Y)^*(\mathcal{O}_{C \times Y}(m))) = 0 \) and, hence, \( R^1(\rho \times \text{Id}_Y)_* (\tilde{E}_{\varphi}) = 0 \).

So far, this is over the base-changed \( S \) to guarantee the existence of the special \( H_{C_S/S} \) at the beginning of the proof. However, the Theorem on Formal Functions implies that the same vanishing statement must hold over the original \( S \). This proves the lemma.

Continuing the setting in the above lemma. It follows then from Lemma \( \ref{lem:pullback} \) that

\[
\tilde{F}_S := (\rho_S \times \text{Id}_Y)_* (\tilde{E}_\varphi)
\]

is flat over \( S \) and, hence, defines a morphism

\[
S \to \mathcal{F}\mathcal{M}_{g, Z-ss}^{1,[0];} (Y;c).
\]

This shows that the built-in contracting morphisms \( \rho \times \text{Id}_Y \) in Definition \( \ref{def:pullback} \) induce a natural morphism

\[
\Xi : \mathcal{M}_{Z-ss}^{g, \mathcal{A}_g}(Y;c) \to \mathcal{F}\mathcal{M}_{g, Z-ss}^{1,[0]}(Y;c)
\]

from the moduli stack of semistable morphisms from general Azumaya nodal curves with a fundamental module to the moduli stack of Fourier-Mukai transforms from stable curves.

Altogether, one has the following diagram of natural morphisms between moduli stacks:

\[
\begin{array}{ccc}
\mathcal{M}_{Z-ss}^{g, \mathcal{A}_g}(Y;c) & \xrightarrow{\Xi} & \mathcal{F}\mathcal{M}_{g, Z-ss}^{1,[0]}(Y;c) \\
\downarrow \Phi & & \downarrow \\
\mathcal{M}_{\mathcal{A}_g}(Y;c) & \to & \overline{\mathcal{M}}_g \\
\end{array}
\]

Here, \( \Phi : \mathcal{M}_{Z-ss}^{g, \mathcal{A}_g}(Y;c) \to \mathcal{M}_{\mathcal{A}_g}(Y;c) \) is the forgetful morphism

\[
[\varphi : (C', \mathcal{O}_C^\mathcal{A}_g) := \text{End}_{\mathcal{O}_C}(\mathcal{E}'; \mathcal{E}') \to Y] \mapsto [(C', \mathcal{O}_C^\mathcal{A}_g) := \text{End}_{\mathcal{O}_C}(\mathcal{E}'; \mathcal{E}')] = [(C', \mathcal{E}')] ,
\]

\( \mathcal{M}_{\mathcal{A}_g}(Y;c) \to \overline{\mathcal{M}}_g \) comes from the collapsing morphism that stabilizes a nodal curve, and \( \mathcal{F}\mathcal{M}_{g, Z-ss}^{1,[0]}(Y;c) \to \overline{\mathcal{M}}_g \) is the forgetful morphism \( [(C, \tilde{F} \in \text{Coh}(C \times Y))] \to [C] \).
4 Compactness of the moduli stack $\mathcal{M}^{Z-ss}_{A^2(g,r,\chi)}(Y; \beta, c)$ of $Z$-semistable morphisms

We now state the main theorem of the current notes, whose proof takes this whole section:

**Theorem 4.0.** $[\mathcal{M}^{Z-ss}_{A^2(g,r,\chi)}(Y; \beta, c)$ compact]. The stack $\mathcal{M}^{Z-ss}_{A^2(g,r,\chi)}(Y; \beta, c)$ of $Z$-semistable morphisms from Azumaya nodal curves with a fundamental module to the Calabi-Yau 3-fold $Y$ of type $(g; r, \chi; \beta, c)$ is bounded and complete (i.e. compact).

4.1 Boundedness of $\mathcal{M}^{Z-ss}_{A^2(g,r,\chi)}(Y; \beta, c)$

Recall from [L-L-S-Y: Proposition 2.3.1] (D(2)) (with Si Li and Ruifang Song):

**Proposition 4.1.1.** [boundedness of morphisms from bounded family of domains]. Let $(C^+_S, E_S)/S$ be a bounded family of Azumaya nodal curves with a fundamental module of type $(g; r, \chi)$ over a base scheme $S$ (of finite type). Then the stack $\mathcal{M}_{(C^+_S, E_S)/S}(Y; \beta)$ of morphisms of type $(g; r, \chi; \beta)$ from fibers of $(C^+_S, E_S)/S$ to a projective scheme $Y$ is bounded.

It follows that to show that $\mathcal{M}^{Z-ss}_{A^2(g,r,\chi)}(Y; \beta, c)$ is bounded, we only need to show that the image stack of the forgetful morphism $\Phi : \mathcal{M}^{Z-ss}_{A^2(g,r,\chi)}(Y; \beta, c) \to \mathcal{M}_{A^2(g,r,\chi)}$ is bounded. In other words, consider the following commutative diagram of schemes and coherent sheaves thereupon:

$$
\begin{array}{ccc}
\tilde{E}' & \xrightarrow{\rho \times Id_Y} & \tilde{F} \\
C' \times Y & \rightarrow & C \times Y \\
\downarrow p_{C'} & & \downarrow p_C \\
E' & \rightarrow & F \\
\end{array}
$$

Here,

- $\tilde{E}'$ is a coherent $O_{C' \times Y}$-module that gives a $Z$-semistable morphism $[\varphi_{\tilde{E}'}] \in \mathcal{M}^{Z-ss}_{A^2(g,r,\chi)}(Y; \beta, c)$;
- $\rho : C' \to C$ is the contraction of unstable $\mathbb{P}^1$-trees in the nodal curve $C'$ that renders it a stable curve $C$;
- $\tilde{F} = (\rho \times Id_Y)_*(\tilde{E}')$ on $C \times Y$ gives a $Z$-semistable Fourier-Mukai transform $[\tilde{F}] \in \mathcal{FM}_{1;[0]:Z-ss}(Y; c)$;
- $E' = pr_{C'*}(\tilde{E}')$ is locally free on $C'$; and
- $F = pr_{C*}(\tilde{F}) = p_*(E')$ is a coherent $O_C$-module, possibly with torsion.
Then,

**Theorem 4.1.2. [boundedness of stack of Z-semistable morphisms of fixed type].** The substack

\[ \text{Im } \Phi := \left\{ (C', \mathcal{O}_{C'}^A := \mathcal{E}nd_{\mathcal{O}_{C'}}(\mathcal{E}')); \varphi \right\} \]

runs over all \( \mathcal{M}^Z_{\text{Z-ss}}(Y; \beta, c) \) and \( (C', \mathcal{E}') \) arises from the bottom-left corner of the above diagram.

of \( \mathcal{M}^Z_{\text{Z-ss}}(g; r, \chi; \beta, c) \) is bounded and, hence, the moduli stack \( \mathcal{M}^Z_{\text{Z-ss}}(Y; \beta, c) \) of Z-semistable morphisms of type \( (g; r, \chi; \beta, c) \) from Azumaya nodal curves with a fundamental module to the Calabi-Yau 3-fold \( Y \) is bounded.

The proof of the theorem takes the rest of this subsection. For the convenience of referral, we will call the above commutative diagram the *basic diagram of four*.

**Reduction to boundedness of the family of pairs for stable and unstable subcurves.**

Let

\[ C' = C'_0 \cup C'_u \]

be a decomposition of \( C' \) into a union of two subcurves, where \( C'_u \) is the union of all connected \( \mathbb{P}^1 \)-trees that are contracted by \( \rho \), and

\[ C'_u = \left( C'_{u,+} \cup C'_{u,0} \right) \cup C'_{u,(1)} \]

be the further decomposition of \( C'_u \) into the union of three \( \mathbb{P}^1 \) sub-trees such that

- for each \( \mathbb{P}^1 \)-component of \( C'_{u,+} \), \( pr_{Y \ast}(\tilde{\mathcal{E}}'_Y|_{\mathbb{P}^1}) \) is 0-dimensional with \( \varphi_{\tilde{\mathcal{E}}} |_{\mathbb{P}^1} \) a nonconstant morphism;

- for each \( \mathbb{P}^1 \)-component of \( C'_{u,0} \), \( pr_{Y \ast}(\tilde{\mathcal{E}}'_Y|_{\mathbb{P}^1}) \) is 0-dimensional with \( \varphi_{\tilde{\mathcal{E}}} |_{\mathbb{P}^1} \) a constant morphism;

- for each \( \mathbb{P}^1 \)-component of \( C'_{u,(1)} \), \( pr_{Y \ast}(\tilde{\mathcal{E}}'_Y|_{\mathbb{P}^1}) \) is 1-dimensional.

For convenience, denote also the union as subcurves in \( C' \)

\[ C'_{u,(0)} := C'_{u,+} \cup C'_{u,0}. \]

Our goal is to prove the following theorem:

**Theorem 4.1.3. [boundedness for family of pairs for subcurves].** The family of isomorphism classes of all possible pairs \( (C'_0, \mathcal{E}'|_{C'_0}) \) (resp. \( (C'_{u,+}, \mathcal{E}'|_{C'_{u,+}}), (C'_{u,0}, \mathcal{E}'|_{C'_{u,0}}), (C'_{u,(1)}, \mathcal{E}'|_{C'_{u,(1)}}) \)) in the problem is bounded.

Once this is justified, then since the gluing of the four locally free sheaves

\[ (C'_0, \mathcal{E}'|_{C'_0}), (C'_{u,+}, \mathcal{E}'|_{C'_{u,+}}), (C'_{u,0}, \mathcal{E}'|_{C'_{u,0}}), \text{ and } (C'_{u,(1)}, \mathcal{E}'|_{C'_{u,(1)}}) \]
along fibers at attached points contributes only a finite-dimensional moduli to the resulting pair $(C', \mathcal{E}')$ the problem, the resulting family of pairs $(C', \mathcal{E}')$ would then be bounded. This would thus prove Theorem 4.1.2. We have thus reduced the proof of Theorem 4.1.2 to the justification of Theorem 4.1.3.

Before further study in details, note that if letting $H$ be the hyperplane class on $Y$ associated to the very ample line bundle $O_Y(1)$ on $Y$, then, for each $\mathbb{P}^1$-component $\mathbb{P}^1$ of $C_u^{(1)}$, the curve class $[pr_Y_*(\tilde{\mathcal{E}}'_{|\mathbb{P}^1})]$ on $Y$ is effective and hence has a positive intersection number with $H$. Since

$$
\sum_{\mathbb{P}^1 : \mathbb{P}^1\text{-components of } C_u^{(1)}} H \cdot [pr_Y_*(\tilde{\mathcal{E}}'_{|\mathbb{P}^1})] = H \cdot [pr_Y_*(\tilde{\mathcal{E}}'_{|C_u^{(1)}})] < H \cdot \beta,
$$

the number of irreducible components of $C_u^{(1)}$ is uniformly bounded by $H \cdot \beta$. Together with the fact that $C_0'$ is a partial normalization of a subcurve of a stable curve of genus $g$, one has the following lemma:

**Lemma 4.1.4. [boundedness of combinatorial types of $C_0'$ and $C_u^{(1)}$].** The set of combinatorial type (and hence homeomorphism classes) of $C_0'$ and $\mathbb{P}^1$-trees $C_u^{(1)}$ that can occur in our problem is finite, depending only on the type $(g; r, \chi; \beta, c)$ of morphisms considered.

In particular, both numbers of connected components, $|\pi_0(C_0')|$ and $|\pi_0(C_u^{(1)})|$, are uniformly bounded, depending only on $g$ and $\beta$ respectively.

**Boundedness of the family of isomorphisms classes of pairs $(C_{u,+}^{(0)}; \mathcal{E}'|_{C_{u,+}^{(0)}})$**

**(a) Strict positivity of $\mathcal{E}'|_{C_{u,+}^{(0)}}$**

Observe that from the way we define a semistable morphism in Definition 3.1.1 the restriction of $\tilde{\mathcal{E}}'$ to over a $\mathbb{P}^1$-component, denoted simply as $\mathbb{P}^1$, of $C_{u,+}^{(0)}$ satisfies the three properties:

1. $\tilde{\mathcal{E}}'|_{\mathbb{P}^1}$ is realizable as a quotient of a trivial bundle on $C' \times Y$,
2. $pr_Y_*(\tilde{\mathcal{E}}'|_{\mathbb{P}^1})$ is 0-dimensional,
3. $\varphi_{\mathcal{E}'}$ is not constant on this $\mathbb{P}^1$.

in the beginning of Sec. 2.2. In other words, $\varphi_{\mathcal{E}'}|_{C_{u,+}^{(0)}}$ is a special morphism studied in Sec. 2.2. It follows from Proposition 2.2.5 that:

**Lemma 4.1.5. [strict positivity of $\mathcal{E}'|_{C_{u,+}^{(0)}}$].**

$\mathcal{E}'|_{C_{u,+}^{(0)}}$ is a strictly positive locally-free sheaf on $C_{u,+}^{(0)}$.

**(b) $H^0(\mathcal{E}'|_{C_{u,+}^{(0)}})$ and torsions and non-locally-free-ness of $\mathcal{F}$**

We now study how the size of the $\mathbb{P}^1$-tree $C_{u,+}^{(0)}$ influences the push-forward $\mathcal{F} := \rho_*(\mathcal{E}')$ on $C$ in the basic diagram of four.
Consider the decomposition
\[ C_u^{(0)} = C_{u,a}^{(0)} \cup C_{u,b}^{(0)}, \]
where \( C_{u,a}^{(0)} \) is the union of connected components of \( C_u^{(0)} \) each of which contains some connected components of \( C_{u,+}^{(0)} \) and \( C_{u,b}^{(0)} \subset C_u^{(0)}. \) Since \( E'|_{C_{u,0}^{(0)}} \simeq \mathcal{O}^{\oplus r}_{C_{u,-}}, \) one can rephrase Lemma 2.2.6 in the current notations as:

**Lemma 4.1.6.** \([h^0(E'|_{C_{u,a}^{(0)}})]\). (Cf. Lemma 2.2.6) Let \( C_{u,a;ij}^{(0)} \simeq \mathbb{P}^1 \) be the irreducible components of \( C_{u,a}^{(0)}, \) where \( i \) labels the connected components of \( C_{u,a}^{(0)} \) and \( j \) labels the irreducible components in the \( i \)-th connected component of \( C_{u,a}^{(0)} \) that lie in \( C_{u,+}^{(0)}. \) Suppose that \( E'|_{C_{u,0}^{(0)}} \simeq \oplus_{k=1}^r \mathcal{O}_{\mathbb{P}^1}(a_{ijk}). \) Recall from Part (a) that, up to a relabeling, \( 0 \leq a_{ij1} \leq \cdots \leq a_{ijr} \) with \( a_{ijr} > 0. \) Then
\[
h^0(E'|_{C_{u,a}^{(0)}}) := \dim H^0(C_{u,a}^{(0)}, E'|_{C_{u,a}^{(0)}}) = r \cdot |\pi_0(C_{u,a}^{(0)})| + \sum_{i,j,k} a_{ijk},
\]
where \( |\pi_0(C_{u,a}^{(0)})| \) is the number of connected components of \( C_{u,a}^{(0)}. \) In particular, except the number of connected components of \( C_{u,a}^{(0)}, \) it is independent of how this collection of locally free sheaves on \( \mathbb{P}^1 \)'s are glued to give \( E'|_{C_{u,a}^{(0)}} \) on \( C_{u,a}^{(0)}. \)

As a consequence, one has the following lemma:

**Lemma 4.1.7.** \([\text{length of } F_{\text{torsion and bound on }} (C_{u,+}^{(0)}, E'|_{C_{u,+}^{(0)}})].\) Let
\[
n_a := |C_{u,a}^{(0)} \cap (C_0^{(0)} \cup C_0^{(1)})|
\]
be the number of attached points on \( C_{u,a}^{(0)} \) to the rest of \( C'. \) Note that since \( C_{u,a}^{(0)} \cup (C_0^{(0)} \cup C_0^{(1)}) \) is a nodal curve of genus \( \leq g, \) it follows from Lemma 4.1.4 that \( n_a \) is uniformly bounded, depending only on \( (g; r; \chi; \beta, c). \) Let \( F_{\text{torsion}} \) be the torsion subsheaf of \( F = \rho_*(E') \) and \( l(F_{\text{torsion}}) \) be its length. Then,
\[
l(F_{\text{torsion}}) \geq h^0(E'|_{C_{u,a}^{(0)}}) - r n_a.
\]
In particular, if there is a uniform upper bound for \( l(F_{\text{torsion}}) \) then there is a uniform upper bound for the number of irreducible components of \( C_{u,+} \) and a uniform upper bound for \( a_{ijk} \geq 0 \) in Lemma 4.1.6.

**Proof.** Note that \( \rho_+ \) takes
\[
\{s \in H^0(E'|_{C_{u,a}^{(0)+}},) \mid s \text{ vanishes at all the attached points on } C_{u,a}^{(0)} \}
\]
injectively to the torsion subsheaf \( F_{\text{torsion}} \) of \( F. \) Recall Remark 2.2.7 and observe from the explicit formula in Lemma 4.1.6 that
\[
h^0(E'|_{C_{u,a}^{(0)}}) - r n_a \rightarrow \infty
\]
if either the number of irreducible components of \( C_{u,+} \) goes to infinity or some \( a_{ijk} \) in Lemma 4.1.6 goes to infinity except the situation of the connected components \( C'' \) of \( C_{u,a} \) that contain only \( \mathbb{P}^1 \)-chain connected component of \( C_{u,+}^{(0)} \) that has total length \( \leq r, \) with an attached point on
each $\mathbb{P}^1$-end, and with the total degree of $E'|_{\mathbb{P}^1}$'s, where $\mathbb{P}^1$ runs over the $\mathbb{P}^1$-components of $C''$, \[ \leq r \] since collapsing such $C''$ may not create torsions from pushing forward. However, when such collection of $\mathbb{P}^1$-chains in $C''$ do not contribute to $\mathcal{F}_{\text{torsion}}$, they then contribute positively to $\delta_{\text{flat}}(\mathcal{F}_{\text{torsion-free}})$, where $\mathcal{F}_{\text{torsion-free}} := \mathcal{F}/\mathcal{F}_{\text{flat}}$. Since
\[ \delta_{\text{flat}}(\mathcal{F}_{\text{torsion-free}}) \leq r \cdot \text{(number of nodes of } C) \]
and the number of nodes of a stable curve of genus $g$ is uniformly bounded above, the total number of $\mathbb{P}^1$-chains in the union of all such $C''$ is uniformly bounded and, hence, the family of isomorphism classes of $(C'' \cap C_{u,+})', E'|_{C'' \cap C_{u,+}}$ is also bounded. This proves the lemma.

Now $l(\mathcal{F}_{\text{torsion}})$ does have a uniform upper bound since $\mathcal{F}$ comes from the push-forward of the elements in the compact stack $\mathcal{FM}_{[1,0];Z-ss} (Y; c)$ of coherent sheaves. This proves:

**Proposition 4.1.8. [boundedness of $\{(C''(0), C'|_{C''(0)})\}$].** The family of isomorphisms classes of pairs $(C''(0), C'|_{C''(0)})$ is bounded.

**Boundedness of the family of isomorphisms classes of pairs $(C''(0), C'|_{C''(0)})$**

Since $E'|_{C''(0)} \simeq O_{C''(0)}^{\oplus r}$, we only need to show that there is a uniform bound on the combinatorial types of $C''(0)$. Consider the dual graph $\Gamma_{C''(0)}$ of $C''(0)$, which is a tree-subgraph of the dual graph $\Gamma_{C'}$ of $C'$. Since

- the genus of $C'$ is fixed to be $g$,
- the sets of combinatorial types of $C''$, $C''^{(1)}$, and $C''^{(0)}$ respectively are all bounded from Lemma 4.1.4 and the previous theme, and
- each vertex of the tree $\Gamma_{C''(0)}$ has valance at least 3 as a vertex in $\Gamma_{C'}$,

the number of 1-valance vertices of $\Gamma_{C''(0)}$ must be uniformly bounded. This implies that the number of vertices of $\Gamma_{C''(0)}$ whose valance is $\geq 3$ must also be bounded, since
\[ |\{\text{1-valance vertics of a tree } \Gamma\}| \geq |\{\text{vertices of valance } \geq 3 \text{ in } \Gamma\}| + 2. \]

Which implies in turn that the number of 2-valance vertices of $\Gamma_{C''(0)}$ musty also be uniformly bounded since the genus $g(\Gamma_{C'}) \leq g(C') = g$. In conclusion:

**Proposition 4.1.9. [boundedness of $\{(C''(0), C'|_{C''(0)})\}$].** The family of isomorphisms classes of pairs $(C''(0), C'|_{C''(0)})$ is bounded.
Boundedness of the family of isomorphisms classes of pairs \((C'_0, \mathcal{E}'|_{C'_0})\)

(a) **Bound for \(|\chi(\mathcal{E}'|_{C'_0})|\)**

Let \(\mathcal{I}_{C'_u}\) be the ideal sheaf of \(C'_u\) in \(C\). Then one has the short exact sequence of \(\mathcal{O}_C\)-modules

\[
0 \rightarrow \mathcal{I}_{C'_u} \cdot \mathcal{E}' \rightarrow \mathcal{E}' \rightarrow \mathcal{E}'|_{C'_u} \rightarrow 0.
\]

Since \(\mathcal{I}_{C'_u} \cdot \mathcal{E}'\) is supported on \(C'_0\) and \(\rho|_{C'_0} \rightarrow C\) is affine birational of relative dimension 0, one has the short exact sequence

\[
0 \rightarrow \rho_*(\mathcal{I}_{C'_u} \cdot \mathcal{E}') \rightarrow \rho_*(\mathcal{E}') \rightarrow \rho_*(\mathcal{E}'|_{C'_u}) \rightarrow R^1\rho_*(\mathcal{I}_{C'_u} \cdot \mathcal{E}')
\]

and

\[
\chi(\mathcal{I}_{C'_u} \cdot \mathcal{E}') = \chi(\rho_*(\mathcal{I}_{C'_u} \cdot \mathcal{E}')).
\]

Since \(\mathcal{I}_{C'_u}|_{C'_0}\) is the ideal sheaf of the nodes \(C'_0 \cap C'_u\) in \(C'_0\), one has also a short exact sequence of \(\mathcal{O}_{C'_0}\)-modules

\[
0 \rightarrow \mathcal{I}_{C'_u} \cdot \mathcal{E}' \rightarrow \mathcal{E}'|_{C'_0} \rightarrow \mathcal{G}' \rightarrow 0,
\]

where \(\mathcal{G}'\) is 0-dimensional, supported at \(C'_0 \cap C'_u\), and of length \(r|C'_0 \cap C'_u|\), which is uniformly bounded by Lemma \[4.1.4] and Lemma \[4.1.7] .

Since both \(\rho_*(\mathcal{I}_{C'_u} \cdot \mathcal{E}')\) and \(\rho_*(\mathcal{E}'|_{C'_0})\) are torsion free on \(C\), one has a sequence of inclusions of \(\mathcal{O}_C\)-modules

\[
0 \hookrightarrow \rho_*(\mathcal{I}_{C'_u} \cdot \mathcal{E}') \hookrightarrow \mathcal{F}/\mathcal{F}_{tor} \hookrightarrow \rho_*(\mathcal{E}'|_{C'_0}).
\]

From which one deduces that

\[
0 \leq l(\mathcal{F}/(\mathcal{F}_{tor} + \rho_*(\mathcal{I}_{C'_u} \cdot \mathcal{E}'))) \leq l(\rho_*(\mathcal{E}'|_{C'_0})/\rho_*(\mathcal{I}_{C'_u} \cdot \mathcal{E}'))) \leq r|C'_0 \cap C'_u|,
\]

which is, again, uniformly bounded by Lemma \[4.1.4] and Lemma \[4.1.7] . It follows that

\[
|\chi(\mathcal{E}'|_{C'_0}) - \chi(\mathcal{F}/\mathcal{F}_{tor})| = |\chi(\mathcal{I}_{C'_u} \cdot \mathcal{E}') + l(\mathcal{G}') - \chi(\mathcal{F}/\mathcal{F}_{tor})|
\]

\[
= |\chi(\rho_*(\mathcal{I}_{C'_u} \cdot \mathcal{E}'))) + l(\mathcal{G}') - \chi(\mathcal{F}/\mathcal{F}_{tor})|
\]

\[
\leq 2r|C'_0 \cap C'_u|.
\]

Together with Lemma \[4.1.7] and Lemma \[4.1.4] this proves:

**Lemma 4.1.10.** (bound on \(|\chi(\mathcal{E}'|_{C'_0})|\)). There is a uniform upper bound for \(|\chi(\mathcal{E}'|_{C'_0})|\) that depends only on \((g, r, \chi, \beta, c)\).

(b) **Boundedness for the family of pairs \((C'_0, \mathcal{E}'|_{C'_0})\)**

From Part (a), one has an exact sequence of \(\mathcal{O}_C\)-modules

\[
0 \rightarrow \mathcal{F}/\mathcal{F}_{tor} \rightarrow \rho_*(\mathcal{E}'|_{C'_0}) \rightarrow \mathcal{G} \rightarrow 0,
\]

where \(\mathcal{G}\) is a 0-dimensional \(\mathcal{O}_C\)-module of length \(\leq r|C'_0 \cap C'_u|\). This shows that the pairs \((C, \rho_*(\mathcal{E}'|_{C'_0}))\) lie in a bounded family since both \(\mathcal{F}/\mathcal{F}_{tor}\) and \(\mathcal{G}\) do. On the other hand, the pairs \((C'_0, \mathcal{E}'|_{C'_0})\) lie in a bounded family if and only if the pairs \((C, \rho_*(\mathcal{E}'|_{C'_0}))\) lie in a bounded family. This proves:
Proposition 4.1.11. [boundedness of \{(C'_0, \mathcal{E}'|_{C'_0})\}]. The family of isomorphisms classes of pairs \((C'_0, \mathcal{E}'|_{C'_0})\) is bounded.

Boundedness of the family of isomorphisms classes of pairs \((C'^{(1)}_u, \mathcal{E}'|_{C'^{(1)}_u})\)

Let \(C'^{(1)}_{u,i,j} \cong \mathbb{P}^1\) be the irreducible components of \(C'^{(1)}_u\), with \(i\) labeling the connected components of \(C'^{(1)}_u\) and \(j\) labeling the irreducible components in the \(i\)-th connected component of \(C'^{(1)}_u\), and \(\mathcal{E}'|_{C'^{(1)}_{u,i,j}} \cong \bigoplus_{k=1}^r \mathcal{O}_{\mathbb{P}^1}(a_{ijk})\). Recall Lemma 4.1.4 that there is a uniform bound on the number of combinatorial types of the \(\mathbb{P}^1\)-tree \(C'^{(1)}_u\). It follows that to show that the pairs \((C'^{(1)}_u, \mathcal{E}'|_{C'^{(1)}_u})\) lie in a bounded family, we only need to show that the \(a_{ijk}\)'s that can appear in the above decomposition must lie in a bounded interval of \(\mathbb{Z}\) that depends only on \((g; r, \chi; \beta, c)\).

(a) Upper bound for \(a_{ijk}\)

That \(a_{ijk}\) has a uniform upper bound follows from the same argument as in Theme `Boundedness of the family of isomorphisms classes of pairs \((C'^{(0)}_u, \mathcal{E}'|_{C'^{(0)}_u})\)' by comparing \(l((\rho_{*} (\mathcal{E}'|_{C'^{(1)}_u}))_{\text{torsion}}\) and \(l(\mathcal{F}_{\text{torsion}})\).

(b) Lower bound for \(a_{ijk}\)

Let

\[
\iota_0 : C'_0 \hookrightarrow C, \quad \iota^{(0)}_u : C'^{(0)}_u \hookrightarrow C, \quad \iota^{(1)}_u : C'^{(1)}_u \hookrightarrow C
\]

be the inclusion of subcurves. Then the quotient homomorphisms of \(\mathcal{O}_C\)-modules

\[
\mathcal{E}' \rightarrow \mathcal{E}'|_{C'_0}, \quad \mathcal{E}' \rightarrow \mathcal{E}'|_{C'^{(0)}_u}, \quad \mathcal{E}' \rightarrow \mathcal{E}'|_{C'^{(1)}_u}
\]

induce a short exact sequence of \(\mathcal{O}_C\)-modules

\[
0 \rightarrow \mathcal{E}' \rightarrow \iota_0^* (\mathcal{E}'|_{C'_0}) \oplus \iota^{(0)}_u^* (\mathcal{E}'|_{C'^{(0)}_u}) \oplus \iota^{(1)}_u^* (\mathcal{E}'|_{C'^{(1)}_u}) \rightarrow \mathcal{G}' \rightarrow 0,
\]

where \(\mathcal{G}'\) is 0-dimensional of length bounded above by \(r \left( |C'_0 \cap C'^{(0)}_u| + |C'_0 \cap C'^{(1)}_u| + |C'^{(0)}_u \cap C'^{(1)}_u| \right)\), which, in turn, is uniformly bounded above. It follows that

\[
\chi(\mathcal{E}'|_{C'^{(1)}_u}) = \chi(\mathcal{E}') + l(\mathcal{G}') - \chi(\mathcal{E}'|_{C'_0}) - \chi(\mathcal{E}'|_{C'^{(0)}_u})
\]

is uniformly bounded below since \(\chi(\mathcal{E}') = \chi\) is fixed and all \(l(\mathcal{G}')\), \(-\chi(\mathcal{E}'|_{C'_0})\), and \(-\chi(\mathcal{E}'|_{C'^{(0)}_u})\) are uniformly bounded below by the above argument, Proposition 4.1.8, Proposition 4.1.11 and Proposition 4.1.9.

On the other hand, one has the short exact sequence

\[
0 \rightarrow \mathcal{E}'|_{C'^{(1)}_u} \rightarrow \bigoplus_{i,j} \mathcal{E}'|_{C'^{(1)}_{u,i,j}} \rightarrow \mathcal{G}'' \rightarrow 0,
\]

where \(\mathcal{G}''\) is 0-dimensional of length bounded above by \(r \left| (C'^{(1)}_u)_{\text{marked}} \right|\), which, in turn, is uniformly bounded. It follows that

\[
\sum_{i,j,k} (1 + a_{ijk}) = \sum_{i,j} \chi(\bigoplus_{i,j} \mathcal{E}'|_{C'^{(1)}_{u,i,j}}) = \chi(\mathcal{E}'|_{C'^{(1)}_u}) + l(\mathcal{G}'') \geq \chi(\mathcal{E}'|_{C'^{(1)}_u})
\]

is uniformly bounded below. Since the number of terms in the summation \(\sum_{i,j,k}\) is bounded above by \(r(H; \beta)\) and the \(a_{ijk}\)'s are uniformly bounded above by Part (a), \(a_{ijk}\) must be uniformly bounded below as well. Together with Part (a), this proves that:
Proposition 4.1.12. [boundedness of $(C_u^{(1)}, E'_u|_{C_u^{(1)}})]$ The family of isomorphisms classes of pairs $(C_u^{(1)}, E'_u|_{C_u^{(1)}})$ is bounded.

All together, this proves the boundedness of $\text{Im} \Phi$. It follows now from Proposition 4.1.1 that $\mathfrak{M}^Z_{A'(g,r,\chi)}(Y; \beta, c)$ is also bounded. This concludes the proof of Theorem 4.1.2.

4.2 The error charge of a Fourier-Mukai transform.

Similar to the technique used in the work [L-W] of Jun Li and Baosen Wu, we define the notion of error charge that suits our problem in this subsection. It will be used in Sec. 4.3 to show that the bubbling-off procedure, which resolves irregularities of a morphism created by filling-in the central fiber of a 1-dimensional family of morphisms, must terminate in finitely many steps to recover a regular morphism within our category. This proves then the completeness of the stack $\mathfrak{M}^Z_{A'(g,r,\chi)}(Y; \beta, c)$.

The error charge and its basic properties

Definition 4.2.1. [error charge]. Let $C'$ be a nodal curve, $\rho : C' \to C \in \overline{\mathcal{M}}_g$ be the collapsing morphism, and $\tilde{F}'$ be a 1-dimensional coherent $O_{C' \times Y}$-module such that $\text{pr}_{C'Y}(\tilde{F}')$ has rank $r$ on each irreducible component of $C'$. Then, from the existence of a flattening stratification for a coherent sheaf, there exist a finite set $D'$ of points on $C'$ such that

- $\tilde{F}'$ is flat, of relative length $r$, over $U' := C' - D'$;
- as coherent sheaf over $C'$, $\tilde{F}'$ is not flat over points in $D'$.

Define the error charge of $\tilde{F}'$ over $C'$, with respect to the central charge function $Z$, to be the following complex number $\text{Err}^Z_{C'}(\tilde{F}')$:

- For $p' \in C'$, let
  $$\tilde{F}'_{(p')} = \{ s' \in \tilde{F}' \mid \text{Ann}(v) \subset \mathcal{I}_{p'}^k, \text{ for some } k \geq 1 \}.$$

Define
  $$\tilde{F}'_{\text{torsion}/C'} := \sum_{p' \in C'} \tilde{F}'_{(p')}$$

and
  $$\tilde{F}'_{\text{torsion-free}/C'} := \tilde{F}' / \tilde{F}'_{\text{torsion}/C'}.$$ 

Note that $\tilde{F}'_{(p')} = 0$ except for $p' \in D'$ (hence $\sum_{p' \in C'}$ is well-defined) and that $\tilde{F}'_{\text{torsion-free}/C'}$ is now of relative dimension 0 over $C'$; points in $C'$ over which $\tilde{F}'_{\text{torsion-free}/C'}$ is not flat is now a subset of $D' \cap C'_{\text{singular}}$.

- (Cf. Definition 2.3.2) For $p' \in C'$, define the discrepancy to flatness of $\tilde{F}'_{\text{torsion-free}/C'}$ over $p'$ to be
  $$\delta_{\text{flat}/C'}(\tilde{F}'_{\text{torsion-free}/C'}; p') := l(\tilde{F}'_{\text{torsion-free}/C'}|_{p'}) - r = \delta_{\text{flat}}(\text{pr}_{C'Y}(\tilde{F}'_{\text{torsion-free}/C'}), p').$$
The last equality follows from the fact that $\text{Supp}(\tilde{\mathcal{F}}_{\text{torsion-free}/C'})$ is affine, of relative dimension 0 over $C'$. Also, note that $\delta_{\text{flat}/C'}(\tilde{\mathcal{F}}_{\text{torsion-free}/C'; p'}) = 0$ except for a subset of points in $D' \cap C'_\text{sing}$. Thus, we may define the discrepancy to flatness of $\tilde{\mathcal{F}}'_{\text{torsion-free}/C'}$ over $C'$ to be

$$\delta_{\text{flat}/C'}(\tilde{\mathcal{F}}'_{\text{torsion-free}/C'}) := \sum_{p' \in C'} \delta_{\text{flat}/C'}(\tilde{\mathcal{F}}'_{\text{torsion-free}/C'; p'}) .$$

Define the error charge $\text{Err}_{Z, C'}(\tilde{\mathcal{F}}')$ of $\tilde{\mathcal{F}}'/C'$ to be

$$\text{Err}_{Z, C'}(\tilde{\mathcal{F}}') := Z \left( (\rho \times \text{Id}_Y)_*(\tilde{\mathcal{F}}'_{\text{torsion}/C'}) \right) + \delta_{\text{flat}/C'}(\tilde{\mathcal{F}}'_{\text{torsion-free}/C'}) .$$

Note that

$$\text{Err}_{Z, C'}(\tilde{\mathcal{F}}') = \text{Err}_{Z, C'}(\tilde{\mathcal{F}}'_{\text{torsion}/C'}) + \text{Err}_{Z, C'}(\tilde{\mathcal{F}}'_{\text{torsion-free}/C'}) .$$

**Lemma 4.2.2. [vanishing of error charge as criterion of flatness $C'/C$].** Continuing the set-up in Definition 4.2.1. Then, $\tilde{\mathcal{F}}'$ is flat over $C'$ if and only if $\text{Err}_{Z, C'}(\tilde{\mathcal{F}}') = 0$.

**Proof.** Note that the subsheaf $\tilde{\mathcal{F}}'_{(p')}$ of $\tilde{\mathcal{F}}'$ in Definition 4.2.1 can be defined for all $p' \in C'$. With this, the lemma is an immediate consequence of the fact that $\tilde{\mathcal{F}}'$ is flat over $C'$ if and only if $\tilde{\mathcal{F}}'_{(p')} = 0$ and $l(\tilde{\mathcal{F}}'_{(p')}) = r$ for all $p' \in C'$, that $\tilde{\mathcal{F}}'_{(p')}$ is of relative dimension 0 over $C \times Y$ and hence $(\rho \times \text{Id}_Y)_*(\tilde{\mathcal{F}}'_{(p')}) = 0$ if and only if $\tilde{\mathcal{F}}'_{(p')} = 0$, and that $Z(\bullet) = 0$ if and only if $\bullet = 0$.

The following lemma follows from Lemma 1.3 ([L-Y3: Lemma 2.1.2] (D(10.1))):

**Lemma 4.2.3. [definity of $\text{Err}_{Z, C'}(\tilde{\mathcal{F}}')$].** Continuing the set-up in Definition 4.2.1. Then:

1. As a function on $\text{Coh}_1(C' \times Y)$, $\text{Err}_{Z, C'}$ takes its value in a locally-finite rank-2 lattice in $\mathbb{C}$.
2. $-\text{Im} \text{Err}_{Z, C'}(\tilde{\mathcal{F}}') \geq 0$.
3. $-\text{Im} \text{Err}_{Z, C'}(\tilde{\mathcal{F}}') = 0$ if and only $\tilde{\mathcal{F}}'_{(p')}$ is 0-dimensional for all $p' \in C'$.
4. When $-\text{Im} \text{Err}_{Z, C'}(\tilde{\mathcal{F}}'_{(p')}) = 0$, $\text{Err}_{Z, C'}(\tilde{\mathcal{F}}') \in \mathbb{Z}_{\geq 0}$.

**Decrease of the error charge under a bubbling-off of $C'$**

**Definition 4.2.4. [order on $\mathbb{C}$].** Define an order $\prec$ on $\mathbb{C}$ as follows: For $z_1, z_2 \in \mathbb{C}$, we say that $z_1$ *precedes* $z_2$, in notation $z_1 \prec z_2$ (or equivalently $z_2 \succ z_1$), if either $-\text{Im} z_1 < -\text{Im} z_2$ or $-\text{Im} z_1 = -\text{Im} z_2$ and $\text{Re} z_1 < \text{Re} z_2$. When either $z_1 \prec z_2$ or $z_1 = z_2$ holds, one denotes $z_1 \preceq z_2$ (or equivalently $z_2 \succeq z_1$).
Proposition 4.2.5. [decrease of $\text{Err}^Z_C(\bullet)$ under bubbling off]. Continuing the situation in Definition 4.2.1. Let $C''$ be a nodal curve that admits a collapsing morphism

$$h : C'' \rightarrow C'$$

that contracts a connected $\mathbb{P}^1$-tree subcurve $C''_{u_h}$ in $C''$. Note the commutative diagram

$$
\begin{array}{ccc}
C'' & \xrightarrow{h} & C' \\
\downarrow{\rho''} & & \downarrow{\rho'} \\
C & & \\
\end{array}
$$

where $\rho'$ and $\rho''$ are collapsing morphisms that stabilize the nodal curves. Let $\tilde{F}'' \in \text{Coh}_1(C'')$ be a 1-dimensional coherent sheaf on $C'' \times Y$ with the following properties:

1. $\text{pr}_{C''}^*(\tilde{F}'')$ is of rank $r$ on each irreducible component of $C''$.
2. The natural sequence of homomorphisms of $\mathcal{O}_{C'' \times Y}$-modules $(h \times \text{Id}_Y)^*(\tilde{F}') \to \tilde{F}'' \to 0$ is exact.
3. For at least one of the $\mathbb{P}^1$-components of the $\mathbb{P}^1$-tree subcurve of $C''$ that is collapsed by $h$ to points in $C'$, $(\tilde{F}'|_{\mathbb{P}^1})_{\text{torsion-free}/\mathbb{P}^1}$ is not the pull-back of a coherent sheaf on $C' \times Y$ via $h \times \text{Id}_Y$.

Then,

$$\text{Err}^Z_{C''}(\tilde{F}'') \prec \text{Err}^Z_C(\tilde{F}') .$$

The proof takes the rest of this theme, which we now proceed.

Note that under the isomorphism $(h \times \text{Id}_Y)_*(\tilde{F}'') \simeq \tilde{F}'$ from Property (1) of the statement,

$$(h \times \text{Id}_Y)_*(\tilde{F}'')_{\text{torsion}/C''} \subset \tilde{F}'_{\text{torsion}/C'} .$$

In particular, the curve class $[\tilde{F}'_{\text{torsion}/C'}] - [(h \times \text{Id}_Y)_*(\tilde{F}'')_{\text{torsion}/C''}]$ on $C' \times Y$ is effective if it is not zero. Consider the built-in short exact sequence of $\mathcal{O}_{C''}$-modules

$$0 \rightarrow \tilde{F}''_{\text{torsion}/C''} \rightarrow \tilde{F}'' \rightarrow \tilde{F}''_{\text{torsion-free}/C''} \rightarrow 0 .$$

Since $\tilde{F}''_{\text{torsion}/C''}$ has relative dimension 0 over $C' \times Y$ under $h \times \text{Id}_Y$, $R^1(h \times \text{Id}_Y)_*(\tilde{F}'')_{\text{torsion}/C''} = 0$. It follows that one has the following commutative diagram of $\mathcal{O}_{C'' \times Y}$-modules

$$
\begin{array}{ccc}
0 & \xrightarrow{(h \times \text{Id}_Y)_*(\tilde{F}'')_{\text{torsion}/C''}} & (h \times \text{Id}_Y)_*(\tilde{F}'') \xrightarrow{\approx} (h \times \text{Id}_Y)_*(\tilde{F}'')_{\text{torsion-free}/C''} \\
\downarrow{\iota} & & \downarrow{\alpha} \\
0 & \xrightarrow{\iota} & \tilde{F}'_{\text{torsion}/C'} \xrightarrow{\alpha} \tilde{F}'_{\text{torsion-free}/C'} & 0
\end{array}
$$

where both horizontal sequences are exact. This implies that

$$\text{Ker}(\alpha) \simeq \text{Coker}(\iota) .$$

$\alpha$ induces a homomorphism of $\mathcal{O}_{C'}$-modules

$$\text{pr}_{C'\times}(\alpha) : \text{pr}_{C'\times}^*((h \times \text{Id}_Y)_*(\tilde{F}'')_{\text{torsion-free}/C''}) \rightarrow \text{pr}_{C'\times}^*(\tilde{F}'_{\text{torsion-free}/C'}) .$$

35
Since $pr_{C''} \circ (h \times Id_Y) = h \circ pr_{C''}$, $pr_{C''} \alpha$, in turn, defines a homomorphism

$$
\alpha' : h_*(pr_{C''} (\tilde{\mathcal{F}}'_{\text{torsion-free}}/C'')) \rightarrow pr_{C''} (\tilde{\mathcal{F}}'_{\text{torsion-free}}/C').
$$

Note that $Ker(\alpha)$ is either 1-dimensional or 0-dimensional. For the convenience of further discussions, let

$$
C'' = C''_0 \cup C''_u
$$

be the decomposition of $C''$ as a union of the $\mathbb{P}^1$-tree subcurve $C''_u$ that is contracted by $h$ and the remaining subcurve $C''_0$.

**Case (a)**: $Ker(\alpha)$ is 1-dimensional.

This happens exactly when there exists a $\mathbb{P}^1$-component of $C''_u$ such that $\text{Supp}(\tilde{\mathcal{F}}'_{\mathbb{P}^1})$ has an irreducible component that is of relative dimension 1 over both $\mathbb{P}^1$ and $Y$. In this case, $[\tilde{\mathcal{F}}'_{\text{torsion}/C}] - [(h \times Id_Y)_*(\tilde{\mathcal{F}}'_{\text{torsion}/C''})]$ is effective. Thus,

$$
-\text{Im} \text{Err}^{Z}_{C''} (\tilde{\mathcal{F}}') < -\text{Im} \text{Err}^{Z}_{C} (\tilde{\mathcal{F}}')
$$

and, hence, $\text{Err}^{Z}_{C''} (\tilde{\mathcal{F}}'') < \text{Err}^{Z}_{C''} (\tilde{\mathcal{F}}')$, as claimed.

**Case (b)**: $Ker(\alpha)$ is 0-dimensional.

I.e., there exists no $\mathbb{P}^1$-component of $C''_u$ such that $\text{Supp}(\tilde{\mathcal{F}}'_{\mathbb{P}^1})$ has an irreducible component that is of relative dimension 1 over both $\mathbb{P}^1$ and $Y$. Observe first the following lemma:

**Lemma 4.2.6. [positivity of $pr_{C''} (\tilde{\mathcal{F}}'_{\text{torsion-free}}/C'')$ on $C''_u$].** Let

$$
\mathcal{G}'' := \left(pr_{C''} (\tilde{\mathcal{F}}'_{\text{torsion-free}}/C'')|_{C''_u}\right)_{\text{torsion-free}}.
$$

Then, when $Ker(\alpha)$ is 0-dimensional, Properties (2) and (3) in the statement of Proposition 4.2.5 imply that $\mathcal{G}''$ is a positive torsion-free sheaf on the $\mathbb{P}^1$-tree $C''_u$.

**Proof.** To show that $\mathcal{G}''$ is positive is to show that for every $\mathbb{P}^1$-component $\mathbb{P}^1$ of $C''_u$,

$$
(\mathcal{G}''|_{\mathbb{P}^1})_{\text{torsion-free}}
$$

is nonnegative and for the $\mathbb{P}^1$-components of $C''_u$ in Property (3),

$$
(\mathcal{G}''|_{\mathbb{P}^1})_{\text{torsion-free}}
$$

is positive. In this case,

$$
(\mathcal{G}''|_{\mathbb{P}^1})_{\text{torsion-free}} = pr_{C''} \left((\tilde{\mathcal{F}}'_{\mathbb{P}^1})_{\text{torsion-free}}/C''\right)
$$

is locally free and $(\tilde{\mathcal{F}}'_{\mathbb{P}^1})_{\text{torsion-free}}/C'' \subset Coh_1 (\mathbb{P}^1 \times Y)$ defines a special morphism as studied in Sec. 2.2. The same argument that gives Lemma 2.2.1 proves then $(\mathcal{G}''|_{\mathbb{P}^1})_{\text{torsion-free}}$ is positive. This proves the lemma.

In the current case, $R^1pr_{C''} (Ker(\alpha)) = 0$; together with left exactness of push-forward, one has thus:

- **When Ker($\alpha$) is 0-dimensional, the homomorphisms $pr_{C''} (\alpha)$ and, hence,**

$$
\alpha' : h_*(pr_{C''} (\tilde{\mathcal{F}}'_{\text{torsion-free}}/C'')) \rightarrow pr_{C''} (\tilde{\mathcal{F}}'_{\text{torsion-free}}/C'),
$$

**are surjective. Furthermore, Ker($\alpha'$) = pr_{C''}(Ker($\alpha$)).**
Since pr_{C''}(\tilde{F}_{torsion-free/C''}^i) is a torsion-free O_{C''}-module and both h_*(pr_{C''}(\tilde{F}_{torsion-free/C''}^i)) and pr_{C''}(\tilde{F}_{torsion-free/C''}) are of rank r,

\[ Ker(\alpha') = \left( h_*(pr_{C''}(\tilde{F}_{torsion-free/C''}^i)) \right)_{torsion}. \]

**Lemma 4.2.7. [difference of error charges].** When Ker(\alpha) is 0-dimensional, (with the notation from Lemma 4.2.6)

\[ Err_{C''}^Z(\tilde{F}^i) - Err_{C'}^Z(\tilde{F}^i) = -\chi(Ker(\alpha)) - deg(G'') \in \mathbb{Z}_{<0}. \]

**Proof.** When Ker(\alpha) is 0-dimensional, [(\rho'' \times Id_Y)_*(\tilde{F}_{torsion/C''}^i)] = [(\rho' \times Id_Y)_*(\tilde{F}_{torsion/C'}^i)] in A_1(C \times Y). Thus,

\[
Err_{C''}^Z(\tilde{F}^i) - Err_{C'}^Z(\tilde{F}^i) = \chi((\rho'' \times Id_Y)_*(\tilde{F}_{torsion/C''}^i)) - \chi((\rho' \times Id_Y)_*(\tilde{F}_{torsion/C'}^i)) \\
+ \delta_{flat/C''}(\tilde{F}_{torsion-free/C''}^i) - \delta_{flat/C'}(\tilde{F}_{torsion-free/C'}^i). 
\]

Now,

\[
\chi((\rho'' \times Id_Y)_*(\tilde{F}_{torsion/C''}^i)) - \chi((\rho' \times Id_Y)_*(\tilde{F}_{torsion/C'}^i)) = \chi(\tilde{F}_{torsion/C''}^i) - \chi(\tilde{F}_{torsion/C'}^i) \\
= \chi((h \times Id_Y)_*(\tilde{F}_{torsion/C''}^i)) - \chi(\tilde{F}_{torsion/C'}^i) \\
= -\chi(Coker(\iota)) = -\chi(Ker(\alpha)) = -\chi(Ker(\alpha')) \\
= -\chi(\left( h_*(pr_{C''}(\tilde{F}_{torsion-free/C''}^i)) \right)_{torsion}) \leq 0,
\]

while

\[
\delta_{flat/C''}(\tilde{F}_{torsion-free/C''}^i) - \delta_{flat/C'}(\tilde{F}_{torsion-free/C'}^i) = \delta_{flat}(pr_{C''}(\tilde{F}_{torsion-free/C''}^i)) - \delta_{flat}(pr_{C'}(\tilde{F}_{torsion-free/C'}^i)) \\
= -deg(G'') < 0.
\]

Here, the last line follows Proposition 2.3.8 and Lemma 4.2.6. This proves the lemma.

It follows that, when Ker(\alpha) is 0-dimensional, Err_{C''}^Z(\tilde{F}^i) < Err_{C'}^Z(\tilde{F}^i) as claimed as well. This proves Proposition 4.2.5.

### 4.3 Completeness of M^Z_{A^Z(g,r,\chi)}(Y; \beta, c)

We now proceed to prove the completeness of M^Z_{A^Z(g,r,\chi)}(Y; \beta, c) by stepwise reductions of the error charge of the Fourier-Mukai transforms involved.

With the order ≪ on C in Definition 4.2.4 and the properties of the error charge Err_{C'}^Z(\bullet) from Lemma 1.2.3 and Lemma 4.2.2 one has:

- Err_{C'}^Z(\tilde{F}^i) ≪ 0 for all \tilde{F}^i ∈ Coh_1(C' \times Y).
- The equality “≈” holds if and only if \tilde{F}^i is flat over C'.

37
Let $T$ be an affine smooth curve with parameter $t$ and a base-point, denoted by 0 with its ideal sheaf denoted by $m_0 = (t)$, and $U := T - \{0\}$. Consider a $U$-family of $Z$-semistable morphisms from Azumaya nodal curves with a fundamental module to $Y$ of type $(g; r, \chi; \beta, c)$, given by
\[
\mathcal{E}_U' \in \text{Coh}(C'_U \times Y),
\]
where
\[
\bullet \; C'_U \text{ is a } U\text{-family of nodal curves with a built-in collapsing morphism } \rho'_U : C'_U / U \to C_U / U; \quad \text{where } C_U / U \text{ is a } U\text{-family of stable curves of genus } g; \\
\bullet \; \mathcal{E}_U' \text{ is flat, of relative dimension } 0 \text{ and relative length } r, \text{ over } C'_U; \\
\bullet \; \mathcal{F}_U := (\rho'_U \times \text{Id}_Y)^*(\mathcal{E}_U') \in \text{Coh}(C_U \times Y) \text{ is a } U\text{-family of } Z\text{-semistable Fourier-Mukai transforms from stable curves of genus } g \text{ to } Y.
\]
The data is also equipped with a built-in exact sequence of $\mathcal{O}_{C'_U \times Y}$-modules
\[
(\rho'_U \times \text{Id}_Y)^*(\mathcal{F}_U) \longrightarrow \mathcal{E}_U' \longrightarrow 0.
\]
In the discussion below, some procedures may require a finite base change to realize. To avoid overflow of notations, we set the convention that the new base (with the pulled-back family) from a base change will be still denoted by $T$.

**Step (a) : Filling in a Fourier-Mukai transform over $0 \in T$**

Possibly after passing to a base change, one may fill the $U$-family $C'_U$ of nodal curves to a $T$-family of nodal curves with a collapsing morphism $\rho'_T : C'_U \to C_U$ over $T$ that extends the collapsing morphism $\rho'_U : C'_U \to C_U$ over $U$. For the $T$-family $C_T$ of stable curves, it follows from [L-Y3: Sec. 3] (D(10.1)) that one may complete the $U$-family $\mathcal{F}_U$ to a $T$-family
\[
\mathcal{F}_T \in \text{Coh}(C_T \times Y)
\]
of $Z$-semistable Fourier-Mukai transform to stable curves of genus $g$ to $Y$, using the technique of elementary modifications by Stacy Langton [La].

**Step (b) : Filling in a quotient sheaf over $0 \in T$**

It follows from the properness of the Quot-scheme $\text{Quot}_{C'_T \times Y / T}((\rho'_T \times \text{Id}_Y)^*(\mathcal{F}_T), \bullet)$ of quotient sheaves of $(\rho'_T \times \text{Id}_Y)^*(\mathcal{F}_T)$ over $T$ that the $U$-family of quotient sheaves $(\rho'_U \times \text{Id}_Y)^*(\mathcal{F}_U) \rightarrow \mathcal{E}_U' \rightarrow 0$ extends uniquely to a $T$-family of quotient sheaves
\[
(\rho'_T \times \text{Id}_Y)^*(\mathcal{F}_T) \longrightarrow \mathcal{E}_T' \longrightarrow 0,
\]
(with $\mathcal{E}_T'$ flat over $T$), whose restriction to over $0 \in T$ is the quotient sheaf
\[
(\rho'_0 \times \text{Id}_Y)^*(\mathcal{F}_0) \longrightarrow \mathcal{E}_0' \longrightarrow 0.
\]

**Lemma 4.3.1. [vanishing of $R^1(\rho'_0 \times \text{Id}_Y)_*(\mathcal{E}_0')$ on $C_0 \times Y$ and $R^1(\rho'_T \times \text{Id}_Y)_*(\mathcal{E}_T')$ on $C_T \times Y$.]**

\[
R^1(\rho'_0 \times \text{Id}_Y)_*(\mathcal{E}_0') = 0 \quad \text{on } C_0 \times Y \quad \text{and} \quad R^1(\rho'_T \times \text{Id}_Y)_*(\mathcal{E}_T') = 0 \quad \text{on } C_T \times Y.
\]
Proof. These are the consequences of:

- the exact sequences $(\rho_0' \times \text{Id}_Y)^*(\tilde{\mathcal{E}}_0) \to \tilde{\mathcal{E}}'_0 \to 0$ and $(\rho_T' \times \text{Id}_Y)^*(\tilde{\mathcal{F}}_T) \to \tilde{\mathcal{E}}'_T \to 0$ respectively,
- $C'_0 \times Y$ (resp. $C'_T \times Y$) is of relative dimension $\leq 1$ over $C_0 \times Y$ (resp. $C_T \times Y$),
- $R^i((\rho_0' \times \text{Id}_Y)_*(\mathcal{O}_{C'_0 \times Y}) = 0$ (resp. $R^i((\rho_T' \times \text{Id}_Y)_*(\mathcal{O}_{C'_T \times Y}) = 0$).

The details follow the same proof of Lemma 3.2.1.

Consider now the natural homomorphisms $\tilde{\alpha}_0$ and $\tilde{\alpha}_T$ from the compositions

$$\tilde{\mathcal{F}}_0 \to (\rho_0' \times \text{Id}_Y)_* (\rho_0' \times \text{Id}_Y)^*(\tilde{\mathcal{E}}_0) \to (\rho_0' \times \text{Id}_Y)_*(\tilde{\mathcal{E}}'_0)$$

and

$$\tilde{\mathcal{F}}_T \to (\rho_T' \times \text{Id}_Y)_* (\rho_T' \times \text{Id}_Y)^*(\tilde{\mathcal{F}}_T) \to (\rho_T' \times \text{Id}_Y)_*(\tilde{\mathcal{E}}'_T).$$

Lemma 4.3.2. [Coincidence of push-forward on $C_0 \times Y$ and $C_T \times Y$]. The natural homomorphisms

$$\tilde{\alpha}_0 : \tilde{\mathcal{F}}_0 \to (\rho_0' \times \text{Id}_Y)_* (\tilde{\mathcal{E}}'_0) \text{ on } C_0 \times Y \text{ and } \tilde{\alpha}_T : \tilde{\mathcal{F}}_T \to (\rho_T' \times \text{Id}_Y)_* (\tilde{\mathcal{E}}'_T) \text{ on } C_T \times Y$$

are isomorphisms.

Proof. We only need to show that $\tilde{\alpha}_T : \tilde{\mathcal{F}}_T \to (\rho_T' \times \text{Id}_Y)_* (\tilde{\mathcal{E}}'_T)$ is an isomorphism over an open dense complement of a codimension-2 subset of $\text{Supp}(\tilde{\mathcal{F}}_T)$. The lemma then follows from Lemma 4.3.1 and Lemma 2.1.2.

First note that since $\tilde{\mathcal{F}}_T$ is flat over $T$ while $\text{Ker}(\tilde{\alpha}_T) \subset \tilde{\mathcal{F}}_T$ is supported only over $0 \in T$, it must be that $\text{Ker}(\tilde{\alpha}_T) = 0$ and $\tilde{\alpha}_T$ is a monomorphism. Note also that

$$(\rho_T' \times \text{Id}_Y)^*(\tilde{\mathcal{F}}_T)_{\text{flat}/T} = (\rho_T' \times \text{Id}_Y)^*(\tilde{\mathcal{F}}_T) \bigg\{/ \text{torsion sections of } (\rho_T' \times \text{Id}_Y)^*(\tilde{\mathcal{F}}_T) \text{ supported over an infinitesimal neighborhood of } 0 \in T \bigg\}.$$

Since both $\tilde{\mathcal{F}}_T$ and $(\rho_T' \times \text{Id}_Y)_*(\tilde{\mathcal{E}}'_T)$ are flat over $T$, one has the exact sequence

$$((\rho_T' \times \text{Id}_Y)^*(\tilde{\mathcal{F}}_T))_{\text{flat}/T} \xrightarrow{\tilde{\beta}_T'} \tilde{\mathcal{E}}'_T \to 0$$

and $\tilde{\alpha}_T$ can be regarded as the composition

$$\tilde{\mathcal{F}}_T \hookrightarrow (\rho_T' \times \text{Id}_Y)_*(((\rho_T' \times \text{Id}_Y)^*(\tilde{\mathcal{F}}_T))_{\text{flat}/T}) \to (\rho_T' \times \text{Id}_Y)_*(\tilde{\mathcal{E}}'_T).$$

Let $\tilde{e}_T \in (\rho_T' \times \text{Id}_Y)_*(\tilde{\mathcal{E}}'_T)$ be a local section of $(\rho_T' \times \text{Id}_Y)_*(\tilde{\mathcal{E}}'_T)$ that is regular and nowhere-zero over a nonempty open set of $C_0 \times Y$. Then, there is a local section $\tilde{e}'_T \in \tilde{\mathcal{E}}'_T$, regular and nowhere-zero over a nonempty open subset of $C'_0 \times Y$, such that $(\rho_T' \times \text{Id}_Y)_*(\tilde{e}'_T) = \tilde{e}_T$. In turn, there is a local section $\tilde{f}_T \in ((\rho_T' \times \text{Id}_Y)^*(\tilde{\mathcal{F}}_T))_{\text{flat}/T}$, regular and nowhere-zero over a nonempty open subset of $C'_0 \times Y$, such that $\tilde{\beta}_T'(\tilde{f}_T) = \tilde{e}'_T$.

Suppose that $\tilde{e}'_T \notin \tilde{\alpha}_T(\tilde{\mathcal{F}}_T)$. Since $\tilde{\alpha}_U : \tilde{\mathcal{F}}_U \to (\rho_U' \times \text{Id}_Y)_*(\tilde{\mathcal{E}}'_U)$ is an isomorphism, let $\tilde{e}'_U$ be the restriction of $\tilde{e}'_T$ to over $U \subset T$ (and similarly for other local section $(\cdot)_U$), then there exists an $\tilde{f}_U \in \tilde{\mathcal{F}}_U$ such that $\tilde{\alpha}_U(\tilde{f}_U) = \tilde{e}'_U$. From naturality of the construction, $\tilde{\beta}_U(\tilde{f}_U) = \tilde{e}_U$ also holds. Thus, with $\tilde{\mathcal{F}}_T$ as a subsheaf of $(\rho_T' \times \text{Id}_Y)_*(((\rho_T' \times \text{Id}_Y)^*(\tilde{\mathcal{F}}_T))_{\text{flat}/T})$, $\tilde{f}_U$ and $(\rho_T' \times \text{Id}_Y)_*(\tilde{f}_T)|_U$ coincide as local sections of $(\rho_T' \times \text{Id}_Y)_*(((\rho_T' \times \text{Id}_Y)^*(\tilde{\mathcal{F}}_T))_{\text{flat}/T})$ and $\tilde{f}_T$ gives rise to the extension of $\tilde{f}_U$ to $\tilde{f}_T := (\rho_T' \times \text{Id}_Y)_*(\tilde{f}_T) \in \tilde{\mathcal{F}}_T$ with $\tilde{\alpha}_T(\tilde{f}_T) = \tilde{e}_T$.

This proves that $\tilde{\alpha}_T : \tilde{\mathcal{F}}_T \to (\rho_T' \times \text{Id}_Y)_*(\tilde{\mathcal{E}}'_T)$ is an isomorphism over an open dense complement of a codimension-2 subset of $\text{Supp}(\tilde{\mathcal{F}}_T)$ and, hence, the lemma.
If $\text{Err}_{\text{flat}/C_0'}(\tilde{E}_0') = 0$, then go to the last Step (d). Otherwise, move on to the next Step (c) to continue the reduction.

**Step (c) : Turning on the reduction – Bubbling off to absorb/reduce irregularities of the morphism over $0 \in T$**

We first reduce the irregularities of $\varphi_{\tilde{E}_0}$ that come from 1-dimensional subsheaf of $(\tilde{E}_0')_{\text{torsion}/C_0'}$. Once such irregularities are all resolved to obtain a new $\varphi_{\tilde{E}_0'}$, we then reduce the irregularities of the new $\varphi_{\tilde{E}_0'}$ that comes from points on $C_0'$ over which $\tilde{E}_0'$ is not flat.

**Step (c.1) : If $-\text{Im } \text{Err}_{\text{flat}/C_0'}(\tilde{E}_0') > 0$**

Then, there exists a point $p' \in C_0'$ such that $(\tilde{E}_0'_{p'})$ is 1-dimensional. Up to a base change of $T$, let

$$
\begin{array}{ccc}
C_0'' & \xrightarrow{h_T} & C_T' \\
\downarrow & & \downarrow \\
T & & T
\end{array}
$$

be a collapsing morphism of $T$-family of nodal curves from blowing up an appropriate subscheme of $C_T'$ that contains $p'$, now as a point in $C_T'$, such that

- $h_U$ is an isomorphism;
- the $\mathbb{P}^1$-tree subcurve $C_{u_{0'}}$ of $C_0''$ that gets contracted by $h_0$ to $p'$ is connected.

Consider the new $T$-family of 1-dimensional coherent sheaves $\tilde{E}_T'' \in \text{Coh}((C_T' \times Y)/T)$ in the exact sequence over $T$

$$(h_T \times \text{Id}_Y)^* (\tilde{E}_T') \xrightarrow{\tilde{\beta}'_T} \tilde{E}_T'' \rightarrow 0$$

from the Quot-scheme/$T$ completion of the exact sequence over $U$

$$(h_U \times \text{Id}_Y)^* (\tilde{E}_U') \xrightarrow{id} (h_U \times \text{Id}_Y)^* (\tilde{E}_U') \rightarrow 0.$$ 

Note that $\text{Ker}(\tilde{\beta}'_T) = ((h_T \times \text{Id}_Y)^*(\tilde{E}_T'))_{\text{torsion}/T}$ is supported over an infinitesimal neighborhood of $0 \in T$. Then, similar argument in the proof of Lemma 4.3.1 in Step (b) implies that

**Lemma 4.3.3.** [vanishing of $R^i(h_0 \times \text{Id}_Y)_*(\tilde{E}_0'')$ on $C_0' \times Y$ and $R^i(h_T \times \text{Id}_Y)_*(\tilde{E}_T'')$ on $C_T' \times Y$.]

$$R^i(h_0 \times \text{Id}_Y)_*(\tilde{E}_0'') = 0 \text{ on } C_0' \times Y \text{ and } R^i(h_T \times \text{Id}_Y)_*(\tilde{E}_T'') = 0 \text{ on } C_T' \times Y.$$ 

And similar argument in the proof of Lemma 4.3.2 in Step (b) implies that

**Lemma 4.3.4.** [coincidence of push-forward on $C_0' \times Y$ and $C_T' \times Y$.] The natural homomorphisms

$$\tilde{E}_0' \rightarrow (h_0 \times \text{Id}_Y)_*(\tilde{E}_0'') \text{ on } C_0' \times Y \text{ and } \tilde{E}_T' \rightarrow (h_T \times \text{Id}_Y)_*(\tilde{E}_T'') \text{ on } C_T' \times Y$$

from the compositions

$$\tilde{E}_0' \rightarrow (h_0 \times \text{Id}_Y)_*(h_0 \times \text{Id}_Y)^*(\tilde{E}_0') \rightarrow (h_0 \times \text{Id}_Y)_*(\tilde{E}_0'')$$

and

$$\tilde{E}_T' \rightarrow (h_T \times \text{Id}_Y)_*(h_T \times \text{Id}_Y)^*(\tilde{E}_T') \rightarrow (h_T \times \text{Id}_Y)_*(\tilde{E}_T'').$$

are isomorphisms.
By the nature of the blow-up in the construction, one has in addition the following lemma:

**Lemma 4.3.5. [diagram of surjections].** The natural diagram of morphisms from the construction

\[
\begin{array}{ccc}
(Supp (\tilde{E}''_T))_{\text{red}} & \xrightarrow{h_T \times Id_Y} & (Supp (\tilde{E}'_T))_{\text{red}} \\
pr_{C''_T} \downarrow & & \downarrow pr_{C'_T} \\
C''_T & \xrightarrow{h_T} & C'_T
\end{array}
\]

has all the arrows surjections.

It follows that there is a \( \mathbb{P}^1 \)-component of \( C''_{u_0} \) such that \( pr_{Y^*}(\tilde{E}'|_{\mathbb{P}^1}) \) is 1-dimensional and, hence, we are in ‘Case (a) : Ker(\alpha) is 1-dimensional’ of the previous theme. It follows from the discussion there that

\[ 0 \leq -\text{Im } \text{Err}_{C''_0}(\tilde{E}''_0) \leq \text{Im } \text{Err}_{C'_0}(\tilde{E}'_0). \]

If \( \text{Err}_{C''_0}(\tilde{E}''_0) = 0 \), then go to the last Step (d). Else, if \( \text{Err}_{C''_0}(\tilde{E}''_0) \neq 0 \) but \( -\text{Im } \text{Err}_{C''_0}(\tilde{E}''_0) = 0 \), then redenote \( (C''_T, \tilde{E}''_T) \) as \( (C'_T, \tilde{E}'_T) \) and go to the next Step (c.2). Finally, if \( -\text{Im } \text{Err}_{C''_0}(\tilde{E}''_0) \neq 0 \), redenote \( (C''_T, \tilde{E}''_T) \) as \( (C'_T, \tilde{E}'_T) \) and repeat the current Step (c.1).

**Step (c.2) : If \(-\text{Im } \text{Err}_{C''_0}(\tilde{E}''_0) = 0\)**

\( \tilde{E}'_0 \) is now of relative dimension 0 over \( C'_0 \). Let \( p' \in C'_0 \) be such that \( \tilde{E}'_0 \) is not flat over \( p' \). As in Step (c.1), up to a base change of \( T \), let

\[
\begin{array}{ccc}
C''_T & \xrightarrow{h_T} & C'_T \\
\downarrow h_U & & \downarrow \\
T & & T
\end{array}
\]

be a collapsing morphism of \( T \)-family of nodal curves from blowing up an appropriate subscheme of \( C'_T \) that contains \( p' \), now as a point in \( C''_T \), such that

\[
\begin{align*}
\cdot & \text{ } h_U \text{ is an isomorphism;} \\
\cdot & \text{ the } \mathbb{P}^1 \text{-tree subcurve } C''_{u_0} \text{ of } C''_0 \text{ that gets contracted by } h_0 \text{ to } p' \text{ is connected.}
\end{align*}
\]

Consider the new \( T \)-family of 1-dimensional coherent sheaves \( \tilde{E}''_T \in \text{Coh} ((C''_T \times Y)/T) \) in the exact sequence over \( T \)

\[
(h_T \times Id_Y)^*(\tilde{E}'_T) \xrightarrow{\beta''_T} \tilde{E}''_T \rightarrow 0
\]

from the Quot-scheme/T completion of the exact sequence over \( U \)

\[
(h_U \times Id_Y)^*(\tilde{E}'_U) \xrightarrow{\beta'_U} (h_U \times Id_Y)^*(\tilde{E}'_U) \rightarrow 0.
\]

Again, \( \text{Ker}(\beta''_T) = ((h_T \times Id_Y)^*(\tilde{E}'_T))_{\text{torsion}}/T \) is supported over an infinitesimal neighborhood of \( 0 \in T \) and one has the following two lemmas by the same reasoning for the corresponding lemmas in Step (b) and Step (c.1):
Lemma 4.3.6. [vanishing of $R^l(h_0 \times Id_Y)_*(\tilde{E}_0^l)$ on $C_0' \times Y$ and $R^l(h_T \times Id_Y)_*(\tilde{E}_T^l)$ on $C_T' \times Y$].

$$R^l(h_0 \times Id_Y)_*(\tilde{E}_0^l) = 0 \text{ on } C_0' \times Y \quad \text{and} \quad R^l(h_T \times Id_Y)_*(\tilde{E}_T^l) = 0 \text{ on } C_T' \times Y.$$ 

Lemma 4.3.7. [coincidence of push-forward on $C_0' \times Y$ and $C_T' \times Y$]. The natural homomorphisms

$$\tilde{E}_0^l \rightarrow (h_0 \times Id_Y)_*(\tilde{E}_0^l) \text{ on } C_0' \times Y \quad \text{and} \quad \tilde{E}_T^l \rightarrow (h_T \times Id_Y)_*(\tilde{E}_T^l) \text{ on } C_T' \times Y$$

from the compositions

$$\tilde{E}_0^l \rightarrow (h_0 \times Id_Y)_*(h_0 \times Id_Y)^*(\tilde{E}_0^l) \rightarrow (h_0 \times Id_Y)_*(\tilde{E}_0^l)$$

and

$$\tilde{E}_T^l \rightarrow (h_T \times Id_Y)_*(h_T \times Id_Y)^*(\tilde{E}_T^l) \rightarrow (h_T \times Id_Y)_*(\tilde{E}_T^l).$$

are isomorphisms.

Furthermore, one has the following positivity property:

Lemma 4.3.8. [positivity of push-forward to $C_{u_{h0}}'$].

The torsion-free sheaf $(pr_{C_{u_{h0}}'})_{C_{u_{h0}}}$ on the $\mathbb{P}^1$-tree $C_{u_{h0}}'$ is positive.

Proof. Since $pr_{C_{u_{h0}}'}(\tilde{E}_T^l)$ is coherent, there exist an affine neighborhood $U_T'$ of $p'$ in $C_T'$ and a $k > 0$ such that $(pr_{C_{u_{h0}}'}(\tilde{E}_T^l))|U_T'$ is realized as a quotient sheaf

$$O_{U_T'}^{\oplus k} \rightarrow (pr_{C_{u_{h0}}'}(\tilde{E}_T^l))|U_T' \rightarrow 0.$$

The composition of homomorphisms of $O_{U_T' \times Y}$-modules

$$pt_{U_T'}^*(O_{U_T'}^{\oplus k}) \rightarrow pr_{U_T'}^*((pr_{C_{u_{h0}}'}(\tilde{E}_T^l))|U_T') \rightarrow \tilde{E}_T'|U_T'$$

gives a quotient sequence of $O_{U_T' \times Y}$-modules

$$O_{U_T' \times Y}^{\oplus k} \rightarrow \tilde{E}_T'|U_T' \rightarrow 0.$$

Since $\tilde{E}_T'$ is flat over $U_T' - \{p'\}$, of relative length $r$, the above quotient sequence induces a rational map

$$f_T' : U_T' \dashrightarrow Quot_Y(O_Y^{\oplus k}, r)$$

that is regular on $U_T' - \{p'\}$ and has a point of indeterminancy $p'$. Let $U_T'' := h_T^{-1}(U_T') \subset C_T'$. Then, $U_T''$ contains $C_{u_{h0}}'$ and the exact sequence

$$O_{U_T'' \times Y}^{\oplus k} \rightarrow \tilde{E}_T'|U_T'' \rightarrow 0$$

from the composition of homomorphisms of $O_{U_T'' \times Y}$-modules

$$(h_T|U_T'' \times Y)^*(O_{U_T'' \times Y}^{\oplus k}) \rightarrow (h_T|U_T'' \times Y)^*(\tilde{E}_T'|U_T'') \rightarrow \tilde{E}_T'|U_T''$$

defines rational map

$$f_T'' : U_T'' \dashrightarrow Quot_Y(O_Y^{\oplus k}, r)$$
that is regular on \((h_T|_{U''_T})^{-1}(U''_T - \{p'\}) = U''_T - C''_{u_{\rho_0}}\). Furthermore, since \(U''_T\) is smooth on the complement of a codimension-2 subset and \(\text{Quot}_Y(O_Y^{\oplus k}, r)\) is proper, the domain of definition for \(f''_T\) extends over an open dense subset \(V''_0 \subset C''_{u_{\rho_0}}\). The nature of the blow-up construction implies that \(f''_T(V''_0)\) must be 1-dimensional. It follows now by the same argument as that for Lemma 2.2.1 that the torsion-free sheaf \((\text{pr}_{C''_{u_{\rho_0}}}(\tilde{E''}_0)|_{C''_{u_{\rho_0}}})\) torsion-free on the \(\mathbb{P}^1\)-tree \(C''_{u_{\rho_0}}\) is positive.

It follows from Lemma 4.2.7 that, as nonnegative integers,

\[
0 \leq \text{Err}_{C''_0}(\tilde{E''}_0) < \text{Err}_{C''_0}(\tilde{E}_0).
\]

If \(\text{Err}_{\text{flat}/C''_0}(\tilde{E''}_0) = 0\), then go to the next and last Step (d). Otherwise, redenote \((C''_T, \tilde{E''}_T)\) as \((C''_T, \tilde{E}_T)\) and repeat the current Step (c.2).

**Step (d): Termination of the reduction**

- Recovery of a regular morphism in our category over \(t \in T\)

Since

\[
0 \leq \text{Err}_{C''_0}(\tilde{E''}_0) < \text{Err}_{C''_0}(\tilde{E}_0)
\]

by Lemma 4.2.3 and Proposition 4.2.5 and in Step (c) we reduce first \(-\text{Im } \text{Err}_{\cdot}(\cdot)\) until it becomes 0 and then reduce the non-negative-integer \(\text{Err}_{\cdot}(\cdot)\) until it is 0, Step (c) must terminate after finitely many repetitions to give a \(\tilde{E''}_0 \in \text{Coh}_1(C''_0 \times Y)\) with \(\text{Err}_{C''_0}(\tilde{E''}_0) = 0\) since these error charges lie in a locally finite rank-2 lattice in \(\mathbb{C}\).

**Definition 4.3.9. [admissible \(\mathbb{P}^1\)-tree].** Let \(C''\) be a nodal curve. A \(\mathbb{P}^1\)-tree subcurve \(C''_{(1)}\) of \(C''\) is called admissible if \(C''_{(1)}\) satisfies the following condition:

- Let \(C'' = C''_{(0)} \cup C''_{(1)}\) be the decomposition of \(C''\) into the union of \(C''_{(1)}\) and the complementary closed subcurve \(C''_{(0)}\). Then, each connected component of \(C''_{(1)}\) intersects with \(C''_{(0)}\) at only either one or two (smooth) points.

(Note that by definition, connected components of \(C''_{(1)}\) are disjoint from each other in \(C''\).)

**Lemma 4.3.10. [factorization of collapsing morphism].** Let

\[
\begin{array}{ccc}
C''_{T} & \xrightarrow{\rho''_i} & C''_T \\
\downarrow & & \downarrow \\
T & \xrightarrow{\rho''_0} & \mathbb{P}^1
\end{array}
\]

be a \(T\)-morphism between two (flat) \(T\)-families of nodal curves of the same (arithmetic) genus such that \(\rho''_i\) is an isomorphism. (Here, recall that \(0 \in T\) and \(U := T - \{0\}\).) Note that \(\rho''_0\) must be a collapsing morphism that contracts an admissible \(\mathbb{P}^1\)-tree subcurve \(C''_{u_{\rho''_0}}\) of \(C''_0\). Let \(C''_{0;(1)}\) be an admissible \(\mathbb{P}^1\)-tree subcurve of \(C''_0\) that is contained in \(C''_{u_{\rho''_0}}\). Then, up to a base change on \(T\)
with the pulled-back family, there exist collapsing morphisms between $T$-family of nodal curves of the same genus

\[
\begin{array}{ccc}
C''_T & \xrightarrow{h_T} & C''_{0:1} \\
\downarrow \hspace{1cm} f & & \downarrow  \\
T & & T
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
C''_T & \xrightarrow{\rho''_T} & C_T \\
\downarrow \hspace{1cm} f & & \downarrow  \\
T & & T
\end{array}
\]

where $h_T$ contracts exactly $C''_{0:1}$, such that $\rho''_T = \rho''_T \circ h_T$.

**Proof.** Let $H_T$ be an ample Cartier divisor on $C_T$ that is relative very ample on $C_T/T$ and supported in the relative smooth locus of $(C''_T - \rho''_T(C''_{0:1}))/T$. Then the pull-back Cartier divisor $H''_T := \rho''_T^{-1}(H_T)$ is supported on the relative smooth locus of $(C''_T - \rho''_T(C''_{0:1}))/T$ and the tautological quotient sheaf $\mathcal{O}_{C''_T} \otimes_{\mathcal{O}_T} H^0(C''_T, \mathcal{O}_{C''_T}(H''_T)) \to \mathcal{O}_{C''_T}(H''_T)$ from evaluations of global sections determines a morphism $f_T : C''_T \to \text{Proj}(H^0(C''_T, \mathcal{O}_{C''_T}(H''_T)))$ whose image $\text{Im}(f_T)$ is isomorphic to $C_T$ and the morphism $f_T : C''_T \to \text{Im}(f_T)$ recovers $\rho''_T$.

Similarly, up to a base change on $T$, there exists an effective Cartier divisor $\hat{H}''_T$ on $C''_T$, supported in the relative smooth locus of $(C''_T - \hat{H}''_T)/T$. Then $m \hat{H}''_T$, $m$ large enough, determines a collapsing morphism $h''_T$ that contracts exactly $C''_{0:1}$. In turn, up to a base change on $T$, there exists an effective Cartier divisor $H''_T$ on $C''_T$, supported in the relative smooth locus of $(C''_T - h''_T(C''_{0:1}))/T$, and $m H''_T$, $m$ large enough, determines a collapsing morphism $\rho''_T$ that contracts exactly $h''_T(C''_{0:1})$. The lemma follows.

Resume our proof of the completeness of $\mathcal{M}^{Z,ss}_{A^3(g,\tau,\chi)}(Y; \beta, c)$. By construction and with some redenotation, we now have

- a $T$-family of nodal curves $C''_T/T$ with a collapsing $T$-morphism $\rho''_T : C''_T \to C_T$ such that
  - $C''_U/U \simeq C'_U/U$ specified in the beginning;
  - under the above isomorphism, $\rho''_U = \rho_U$.
- a coherent sheaf $\mathcal{E}'_T$ on $C''_T \times Y$ that is flat over $C''_T$, of relative length $r$, with the following properties:
  1. under the $U$-isomorphism $C''_U \simeq C'_U$ above, $\mathcal{E}'_U \simeq \mathcal{E}'_U$.
  2. $\mathcal{F}_T$ gives a (flat) $T$-family of $Z$-semistable Fourier-Mukai transforms from fibers of $C_T/T$ to $Y$.
  3. The natural sequence of homomorphisms $(\rho''_T \times \text{Id}_Y)^*(\mathcal{F}_T) \to \mathcal{E}'_T \to 0$ is exact.

Let $C''_{0:1}$ be the $\mathbb{P}^1$-tree subcurve of $C''_0$ that is collapsed by $\rho''_0$.

**Lemma 4.3.11. [collapsing admissible $\mathbb{P}^1$-tree in $C''_{0:1}$].** Let

- $C''_{0:1}$ be a connected admissible $\mathbb{P}^1$-tree subcurve of $C''_0$ that is contained in $C''_{0:1}$ such that the restriction of $\varphi_{C''_{0:1}}$ to $C''_{0:1}$ is constant, and
where $h_T$ contracts exactly $C''_0$ and $\rho''_T = \rho'''_T \circ h_T$, be the induced factorization of collapsing morphisms from Lemma 4.3.10.

Define

$$\tilde{\mathcal{E}}''_T := (h_T \times \text{Id}_Y)_*(\tilde{\mathcal{E}}''_T).$$

Then, $\tilde{\mathcal{F}}''_T = (\rho''_T \times \text{Id}_Y)_*(\tilde{\mathcal{E}}''_T)$ and the natural sequence $(\rho''_T \times \text{Id}_Y)^*(\tilde{\mathcal{F}}_T) \to \tilde{\mathcal{E}}''_T \to 0$ of homomorphisms is exact.

Proof. This follows from the fact that $H^0(Z; \mathcal{O}_Z) = \mathbb{C}$ for any connected projective variety $Z$ and that $\text{Supp}(\tilde{\mathcal{E}}''_T)$ is affine over $C''_T$.

Thus, after contracting step by step all the admissible $\mathbb{P}^1$-tree subcurves in $C''_0$ that are contained in $C''_0$, and to whose components the restriction of $\varphi_{\mathcal{E}}''_0$ are constant and redenotation, $\tilde{\mathcal{E}}''_T$ now has in addition the following property:

(3) For each $\mathbb{P}^1$-component (denoted by $\mathbb{P}^1$) of the $\mathbb{P}^1$-tree subcurve of $C''_0$ that is collapsed by $\rho''_0$ to points in $C_0$, if $\varphi_{\mathcal{E}}''_0|_{\mathbb{P}^1}$ is a constant morphism, then $\mathbb{P}^1$ has at least three special points.

In other words, we obtains a $T$-family of $Z$-semitstable morphisms from Azumaya nodal curves with a fundamental module to $Y$ of type $(g; r, \chi; \beta, c)$ that, up to a base change, extends the original $U$-family of $Z$-semitstable morphisms of type $(g; r, \chi; \beta, c)$.

This proves the completeness of $\mathcal{M}^{Z-ss}_{A_S(g; r, \chi)}(Y; \beta, c)$.

Together with Sec. 4.1, this proves Theorem 4.0 on the compactness of $\mathcal{M}^{Z-ss}_{A_S(g; r, \chi)}(Y; \beta, c)$. 

45
References

[D-A-L-R] D. Avritzer, H. Lange, and F.A. Ribeiro, Torsion-free sheaves on nodal curves and triples, Bull. Brazilian Math. Soc. 41 (2010), 421 – 447.

[Be] K. Behrend, Algebraic Gromov-Witten invariants, in New trends in algebraic geometry (Warwick, 1996), K. Hulek, F. Catanese, C. Peters, and M. Reid eds., 19 – 70, London Math. Soc. Lecture Note Ser. 264, Cambridge Univ. Press, 1999.

[Ca] L. Caporaso, A compactification of the universal Picard variety over the moduli space of stable curves, J. Amer. Math. Soc. 41 (2010), 589 – 660.

[EGA III] A. Grothendieck, Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I, Inst. Hautes Études Sci. Publ. Math. no. 11, 1961; ibid. II, no. 17, 1963.

[Ei] D. Eisenbud, Commutative algebra with a view toward algebraic geometry, GTM 150, Springer, 1995.

[E-H] D. Eisenbud and J. Harris, The geometry of schemes, GTM 197, Springer, 2000.

[F-G-I-N-V] B. Fantechi, L. Göttsche, L. Illusie, S.L. Kleiman, N. Nitsure, and A. Vistoli, Fundamental Algebraic Geometry, Math. Surv. Mono. 123, Amer. Math. Soc., 2005.

[F-G-I-K-N-V] B. Fantechi, L. Göttsche, L. Illusie, S.L. Kleiman, N. Nitsure, and A. Vistoli, Fundamental Algebraic Geometry, Math. Surv. Mono. 123, Amer. Math. Soc., 2005.

[Gi1] D. Gieseker, On the moduli of vector bundles on an algebraic surface, Ann. Math. 106 (1977), 45 – 60.

[Gi2] ——–, A degeneration of the moduli space of stable bundles, J. Diff. Geom. 19 (1984), 173 – 206.

[Gro1] A. Grothendieck, Sur la classification des fibrés holomorphes sur la sphère de Riemann, Amer. J. Math. 79 (1957), 121 – 138.

[Gro2] ——–, Sur quelques points d’algèbre homologique, Tohoku Math. J. 9 (1957), 119 – 221.

[Gro3] ——–, Techniques de construction et théorèmes d’existence en géométrie algébrique IV: Les schemas de Hilbert, Séminaire Bourbaki, 1960/1961, no. 221.

[G-M] S.I. Gelfand and Y.I. Manin, Methods of homological algebra, Springer, 1996.

[Ha] R. Hartshorne, Algebraic geometry, GTM 52, Springer, 1977.

[H-L] D. Huybrechts, Fourier-Mukai transforms in algebraic geometry, Oxford Univ. Press, 2006.

[H-L] D. Huybrechts and M. Lehn, The geometry of moduli spaces of sheaves, 2nd., Cambridge Univ. Press, 2010.

[H-M] J. Harris and I. Morrison, Moduli of curves, GTM 187, Springer, 1998.

[Kau] I. Kausz, A Gieseker type degeneration of moduli stacks of vector bundles on curves, Transactions Amer. Math. Soc. 357 (2004), 4897 – 4955; arXiv:math/0201197 [math.AG].

[Ko] M. Kontsevich, Enumeration of rational curves via torus actions, The moduli space of curves (Ticino Island, 1994), R. Dijkgraaf, C. Faber and G. van der Geer. eds., 335 – 368, Progress in Math. 129, Birkhäuser, 1995; arXiv:hep-th/9405035.

[K-L] Y.-H. Kiem and J. Li, Vanishing of the top Chern classes of the moduli of vector bundles, J. Diff. Geom. 76 (2007), 45 – 115; arXiv:math/0403033 [math.AG].

[La] S.G. Langton, Valuative criteria for families of vector bundles on algebraic varieties, Ann. Math. 101 (1975), 88–110.

[Liu1] C.-H. Liu, Azumaya noncommutative geometry and D-branes - an origin of the master nature of D-branes, lecture given at the workshop Noncommutative algebraic geometry and D-branes, December 12 – 16, 2011, organized by Charlie Beil, Michael Douglas, and Peng Gao, at Simons Center for Geometry and Physics, Stony Brook University, Stony Brook, NY; arXiv:1112.4317 [math.AG].

[Liu2] ——–, Algebraic and symplectic aspects of D-branes in superstring theory from Azumaya noncommutative geometry - An invitation to Azumaya D-geometry for mathematicians, lecture given at Geometry Seminar, Department of Mathematics at National Taiwan University, May 9, 2013; slides obtainable at https://www.researchgate.net/profile/Chien-Hao_Liu/

[Liu3] ——–, Physics aspects of D-branes from matrix-type noncommutative geometry on D-brane world-volume – An invitation to Azumaya D-geometry for string-theorists, lecture given at String Theory Seminar, Department of Physics at National Taiwan University, May 10, 2013; slides obtainable at https://www.researchgate.net/profile/Chien-Hao_Liu/

[L-MB] G. Laumon and L. Moret-Bailly, Champs algébriques, Ser. Mod. Surveys Math. 39, Springer, 2000.

[L-W] J. Li and B. Wu, Good degeneration of Quot-schemes and coherent systems, arXiv:1110.0390 [math.AG].

[L-Y1] C.-H. Liu and S.-T. Yau, Azumaya-type noncommutative spaces and morphism therefrom: Polchinski’s D-branes in string theory from Grothendieck’s viewpoint, arXiv:0709.1515 [math.AG].
[L-L-S-Y] S. Li, C.-H. Liu, R. Song, S.-T. Yau, *Morphisms from Azumaya prestable curves with a fundamental module to a projective variety: Topological D-strings as a master object for curves*, arXiv:0809.2121 [math.AG]. (D(2))

[L-Y2] C.-H. Liu and S.-T. Yau, *D-branes and Azumaya noncommutative geometry: From Polchinski to Grothendieck*, arXiv:1003.1178 [math.SG]. (D(6))

[L-Y3] ———, *A mathematical theory of D-string world-sheet instantons, I: Compactness of the stack of Z-semistable Fourier-Mukai transforms from a compact family of nodal curves to a projective Calabi-Yau 3-fold*, arXiv:1302.2054 [math.AG]. (D(10.1))

[L-Y4] ———, manuscript in preparation.

[M-O-P] A. Marian, D. Oprea, and R. Pandharipande, *The moduli space of stable quotients*, Geom. Topol. 15 (2011), 1651 – 1706; arXiv:0904.2992 [math.AG].

[N-S] D.S. Nagaraj and C.S. Seshadri, *Degenerations of the moduli spaces of vector bundles on curves I*, Proc. Indian Acad. Sci. (Math. Sci.) 107 (1997), 101 – 137; *II (Generalized Gieseker moduli spaces)*, ibid. 109 (1999), 165 – 201.

[Pa] R. Pandharipande, *A compactification over \(\overline{M}_g\) of the universal moduli space of slope-semistable vector bundles*, J. Amer. Math. Soc. 9 (1996), 425 – 471; arXiv:alg-geom/9502020.

[P-R] M. Popa and M. Roth, *Stable maps and Quot schemes*, Invent. Math. 152 (2003), 625 – 663; arXiv:math/0012221 [math.AG].

[Sch] A. Schmitt, *The Hilbert compactification of the universal moduli space of semistable vector bundles over smooth curves*, J. Diff. Geom. 66 (2004), 169–209; arXiv:math/0312362 [math.AG].

[Se1] C.S. Seshadri, *Fibrés vectoriels sur les courbes algébriques*, Astérisque 96 (1982), Soc. Math. France.

[Se2] ———, *Degenerations of the moduli spaces of vector bundles on curves*, lecture given at School of Algebraic Geometry, Trieste, 1999.

[Sun] X. Sun, *Remarks on Gieseker’s degeneration and its normalization*, Third International Congress of Chinese Mathematicians. Part 1, 177 – 191, AMS/IP Stud. Adv. Math., 42, Amer. Math. Soc., 2008.

[TiB] M. Teixidor i Bigas, *Compactifications of moduli spaces of (semi)stable bundles on singular curves: two points of view*, Collect. Math. 49 (1998), 527 – 548.

[Wu] B. Wu, *Topics in the moduli theory of sheaves*, course Math 265y given at Harvard University, fall 2011.

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