Order parameter model for unstable multilane traffic flow

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We discuss a phenomenological approach to the description of unstable vehicle motion on multilane highways that explains in a simple way the observed sequence of the phase transitions “free flow \(\leftrightarrow\) synchronized mode \(\leftrightarrow\) jam” as well as the hysteresis in these transitions. We introduce a new variable called order parameter that accounts for possible correlations in the vehicle motion at different lanes. So, it is principally due to the “many-body” effects in the car interaction in contrast to such variables as the mean car density and velocity being actually the zeroth and first moments of the “one-particle” distribution function. Therefore, we regard the order parameter as an additional independent state variable of traffic flow. We assume that these correlations are due to a small group of “fast” drivers and taking into account the general properties of the driver behavior formulate a governing equation for the order parameter. In this context we analyze the instability of homogeneous traffic flow manifesting itself in the mentioned above phase transitions and giving rise to the hysteresis in both of them. Besides, the jam is characterized by the vehicle flows at different lanes being independent of one another. We specify a certain simplified model in order to study the general features of the car cluster self-formation under the phase transition “free flow \(\leftrightarrow\) synchronized motion”. In particular, we show that the main local parameters of the developed cluster are determined by the state characteristics of vehicle motion only.

I. INTRODUCTION. MACROSCOPIC MODELS FOR MULTILANE TRAFFIC DYNAMICS

The study of traffic flow formed actually a novel branch of physics since the pioneering works by Lighthill and Whitham [1], Richards [2], and, then, by Prigogine and Herman [3]. It is singled out by the fact that in spite of motivated, i.e. a non-physical individual behavior of moving vehicles (they make up a so-called ensemble of “self-driven particles”, see, e.g., [4–6]), traffic flow exhibits a wide class of critical and self-organization phenomena met in physical systems (for a review see [7–9]). Besides, the methods of statistical physics turn out to be a useful basis for the theoretical description of traffic dynamics [10].

The existence of a new basic phase in vehicle flow on multilane highways called the synchronized motion was recently discovered by Kerner and Rehborn [11], impacting significantly the physics of traffic as a whole. In particular, it turns out that the spontaneous formation of moving jams on highways proceeds mainly through a sequence of two transitions: “free flow \(\rightarrow\) synchronized motion \(\rightarrow\) stop-and-go pattern” [12]. All the three traffic modes are phase states, meaning their ability to persist individually for a long time. Besides, the two transitions exhibit hysteresis [12,13], i.e., for example, the transition from the free flow to the synchronized mode occurs at a higher density and lower velocity than the inverse one. As follows from the experimental data [11,13,14] the phase transition “free flow \(\leftrightarrow\) synchronized mode” is essentially a multilane effect. Recently Kerner [7–9] assumed it to be caused by “Z”-like form of the overtaking probability depending on the car density.

The synchronized mode is characterized by substantial correlations in the car motion along different lanes because of the lane changing maneuvers. So, to describe such phenomena a multilane traffic theory is required. There have been proposed several macroscopic models dealing with multilane traffic flow and based on the gas-kinetic theory [15–20], a compressible fluid model [21] generalizing the approach by Kerner and Konhäuser [22,23], and actually a model [24,25] dealing with the time dependent Ginzburg-Landau equation.

All these models describe traffic flow in terms of the car density \(\rho\), mean velocity \(v\), and, may be, the velocity variance \(\theta\) or ascribe these quantities to vehicle flow at each lane \(\alpha\) individually. In other words, the quantities \(\{\rho, v, \theta\}_\alpha\) are regarded as a complete system of the traffic flow state variables and if they are fixed then all the vehicle flow characteristics should be determined. The given models relate the self-organization phenomena actually to the vehicle flow instability caused by the delay in the driver response to changes in the motion of the nearest cars ahead. In fact, let us briefly consider their simplified version (cf. [26,27]) which, nevertheless, catches the basic features taken into account:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0, \tag{1}
\]

\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{1}{\tau} (v - U). \tag{2}
\]

Here the former term on the right-hand side of Eq. (2),
a so-called pressure term, reflects dispersion effects due to the finite velocity variance $\theta$ of the vehicles and the latter one describes the relaxation of the current velocity within the time $\tau'$ to a certain equilibrium value $U\{\rho, \theta\}$. In particular, for

$$P\{\rho, v, \theta\} = \rho \theta - \frac{\partial v}{\partial x},$$

(3)

where $\eta$ is a “viscosity” coefficient and the velocity variance $\theta$ is treated as a constant, we are confronted with the Kerner-Konhäuser model \[22,23\]. In the present form the relaxation time $\tau'$ characterizes the acceleration capability of the mean vehicle as well as the delay in the driver control over the headway (see, e.g., \[28,30\]). The value of the acceleration time is typically estimated as $\tau' \sim 30 s$ because it is mainly determined by the mean time of vehicle acceleration which is a slower process than the vehicle deceleration or the driver reaction to changes in the headway. The equilibrium velocity $U\{\rho\}$ (here the fixed velocity variance $\theta$ is not directly shown in the argument list) is chosen by drivers keeping in mind the safety, the readiness for risk, and the legal traffic regulations. For homogeneous traffic flow the equilibrium velocity $U\{\rho\} = \bar{\theta}(\rho)$ is regarded as a certain phenomenological function meeting the conditions:

$$\frac{d\bar{\theta}(\rho)}{d\rho} < 0 \quad \text{and} \quad \rho \bar{\theta}(\rho) \to 0 \quad \text{as} \quad \rho \to \rho_0,$$

(4)

where $\rho_0$ is the upper limit of the vehicle density on the road. Since the drivers anticipate risky maneuvers of distant cars also, the dependence $U\{\rho\}$ is nonlocal. In particular, it is reasonable to assume that the driver behavior is mainly governed by the car density $\rho$ at a certain distance “interaction point” $x_a = x + L^*$ rather than at the current one $x$, which gives rise to a new term in Eq. (2) basing on the gas-kinetic theory \[4,14\]. Here, for the sake of simplicity following \[22\], we take this effect into account expanding $\rho(x + L^*)$ and then, $U\{\rho\}$ into the Taylor series and write:

$$U\{\rho\} = \bar{\theta}(\rho) - v_0 L^* \frac{\partial \rho}{\partial x},$$

(5)

where $v_0$ is a certain characteristic velocity of the vehicles. Then linearizing the obtained system of equations with respect to the small perturbations $\delta \rho, \delta v \propto \exp(\gamma t + i k x)$ we obtain that the long-wave instability will occur if (cf. \[22,23,20\])

$$\tau'(\rho_0^2)'^2 > v_0 L^* + \tau' \theta,$$

(6)

in the long-wave limit the instability increment $\text{Re } \gamma$ depends on $k$ as

$$\text{Re } \gamma = k^2 \left[ \tau'(\rho_0^2)'^2 - (v_0 L^* + \tau' \theta) \right],$$

(7)

and the upper boundary $k_{\text{max}}$ of the instability region in the $k$-space is given by the expression:

$$k_{\text{max}}^2 = \frac{\rho}{\tau \eta} \left\{ \left[ \tau'(\rho_0^2)'^2 \right] \left[ (v_0 L^* + \tau' \theta) \right]^{1/2} - 1 \right\}.$$
occurrence of high-flow states in free traffic\(^\ddagger\). The platoons are metastable and their destruction gives rise to the congested vehicle motion \(^5\). In the synchronized mode the weight of the short time-headways is less, however, almost every fourth driver falls below the 1-scc-threshold. In the vicinity of the transition “free flow ↔ synchronized mode” the short time-headways have the greatest weight. In other words, at least near the given phase transition the traffic flow state is to be characterized by two different driver groups which separate from each other in the velocity space and, consequently, in multilane traffic flow there should be another relaxation process distinct from one taken into account by the model (1), (2). In order to move faster than the statistically averaged car a driver should permanently manoeuvring pass by the cars moving ahead. Meeting of several such “fast” drivers seems to cause the platoon formation. Obviously, to drive in such a manner requires additional efforts, so, each driver needs a certain time \(\tau\) to get the decision whether or not to take part in these manoeuvres. Exactly the time \(\tau\) characterizes the relaxation processes in the platoon evolution. It should be noted that the overtaking manoeuvres are not caused by the control over the headway distance and, thus, the corresponding transient processes may be much slower then the driver response to variations in the headway to prevent possible traffic accidents.

The analysis of the obtained optimal-velocity function \(V(\Delta x)\) demonstrates its dependence not only on the headway \(\Delta x\) but also on the local car density. So, in congested flow the drivers supervises the vehicle arrangement or, at least, try to do this in a sufficiently large neighborhood covering several lanes.

Another unexpected fact is that the synchronized mode is mainly distinctive not due to the car velocities at different lanes being equal. In the observed traffic flow various lanes did not exhibit a substantial difference in the car velocity even in the free flow. In agreement with the results obtained by Kerner (1) the synchronized mode is singled out by small correlations between fluctuations in the car flow, velocity and density. There is only a strong correlation between the velocities at different lanes taken at the same time, however, it decreases sufficiently fast as the time difference increases. By contrast, there are strong long-time correlations between the flow and density for the free flow as well as the stop-and-go mode. In these phases the vehicle flow directly depends on the density.

Thereby, the free flow, the synchronized mode and the jammed motion seem to be qualitatively distinct from one another at the microscopic level. So, it is likely that to describe macroscopically traffic phase transitions the set of the state variables \(\{\rho, v, \theta\}_a\) should be completed with an additional parameter (or parameters) reflecting the internal correlations in the car dynamics. In other words, this parameter has to be due to the “many-body” effects in the car interaction in contrast to such \(\textit{external}\) variables as the mean car density and velocity being ac-

\[ u = \frac{\rho_{cl}v_{cl} - \rho_f v_f}{\rho_{cl} - \rho_f}. \] (10)
certain order parameter \( h \) characterizing the correlations, e.g., in the atom arrangement. In the present paper following practically the spirit of the Landau theory we develop a phenomenological approach to the description of the traffic flow instability that ascribes to the vehicle flux an additional internal parameter which will be also called the order parameter \( h \) and allows for the effect of lane changing on the vehicle motion. In this way the free flow and the congested phases become in fact distinctive and solely the conditions of their coexistence as well the dynamics of the transition layer are the subject of specific models.

II. ORDER PARAMETER AND THE INDIVIDUAL DRIVER BEHAVIOR

We describe the vehicle flow on a multilane highway in terms of its characteristics averaged over the road cross-section, namely, by the car density \( \rho \), the mean velocity \( v \), and the order parameter \( h \). The latter is the measure of the correlations in the car motion or, what is equivalent, of the car arrangement regularity forming due to the lane change by the “fast” drivers. Let us discuss the physical meaning of the order parameter \( h \) in detail considering the free flow, synchronized mode and jammed traffic individually (Fig. 2).

A. Physical meaning of the order parameter \( h \) and its governing equation

When vehicles move on a multilane highway without changing the lanes they interact practically with the nearest neighbors ahead only and, so, there should be no internal correlations in the vehicle flow at different lanes. Therefore, although under this condition the traffic flow can exhibit complex behavior, for example, the “stop-and-go” waves can develop, it is actually of one-dimensional nature. In particular, the drivers that would prefer to move faster than then the statistically mean driver will bunch up forming the platoons headed by a relatively slower vehicle. When the cars begin to change lanes for overtaking slow vehicles the car ensembles at different lanes will affect one another. The case of this interaction is due to that a car during a lane change manoeuvre occupies, in a certain sense, two lanes simultaneously, affecting the cars moving behind it at both the lanes. Figure 2 illustrates this interaction for cars 1 and 2 through car 4 changing the lanes. The drivers of both cars 1 and 2 have to regard car 4 as the nearest neighbor and, so, their motion will be correlated during the given manoeuvre and after it during the relaxation time \( \tau' \). In the same way car 1 is affected by car 3 because the motion of car 4 directly depends on the behavior of car 3. The more frequently lane changing is performed, the more correlated traffic flow on a multilane highway. Therefore, it is reasonable to introduce the order parameter \( h \) being the mean density of such car triplets normalized to its maximum possible for the given highway and to regard it as a measure of the multilane correlations in the vehicle flow.

On the other hand the order parameter \( h \) introduced in this way can be regarded as a measure of the vehicle arrangement regularity. Let us consider this question in detail for the free flow, synchronized mode, and jammed traffic individually. In the free flow the feasibility of overtaking makes the vehicle arrangement more regular because of platoon dissipation. So as the order parameter \( h \) grows the free traffic becomes more regular. Nevertheless, in this case the density of the car multilane triplets remains relatively low, \( h \ll 1 \), and the vehicle ensembles should exhibit weak correlations. Whence it follows also that the mean car velocity \( \theta \) is an increasing function of the order parameter \( h \) in the free flow. In the jammed motion (Fig. 2c) leaving current lanes is hampered because of lack of room for the manoeuvres. So the car ensembles at different lanes can be mutually independent in spite of individual complex behavior. In the given case the order parameter must be small too, \( h \ll 1 \), but, in contrast, the car mean velocity should be a decreasing function of \( h \). In fact, for highly dense traffic any lane change of a car requires practically that the neighbor drivers decelerating give place for this manoeuvre.

Figure 3 illustrates the transition “free flow \( \leftrightarrow \) synchronized mode”. As the car density grows in free flow, the “fast” drivers that at first overtake slow vehicles individually begin to gather into platoons headed by more “slow” cars among them but, nevertheless, moving faster than the statistically mean vehicle (Fig. 3a). The platoons are formed by drivers preferring to move as fast as possible keeping short headways without lane changing. Such a state of the traffic flow should be sufficiently inhomogeneous and the vehicle headway distribution has to contain a short headway spike as observed experimentally in [34]. Therefore, even at a sufficiently high car density the free flow should be characterized by weak
multilane correlations and not too great values of the order parameter \( h_f \). The structure of these platoons is also inhomogeneous, they comprise cars whose drivers would prefer to move at different headways (for a fixed velocity) under comfortable conditions, i.e., when the cars moving behind a given car do not jam it or none of the vehicles moving on the neighboring lanes hinders its motion at the given velocity provided it changes the current lane. So, when the density of vehicles attains sufficiently high values and their mean velocity decreases remarkably with respect to the velocity on the empty highway some of the “fast” drivers can decide that there is no reason to move so slowly at such short headways requiring strain. Then they can either overtake the car heading the current platoon by changing lanes individually or leave the platoon and take vacant places (Fig. 3b). The former has to increase the multilane correlations and, in part, to decrease the mean vehicle velocity because the other drivers should give place for this manoeuvres in a sufficiently dense traffic flow. The latter also will decrease the mean vehicle velocity because these places were vacant from the standpoint of sufficiently “fast” drivers only but not from the point of view of the statistically mean ones preferring to keep longer headways in comparison with the platoon headways. Therefore, the statistically mean drivers have to decelerate, decreasing the mean vehicle velocity. The two manoeuvre types make the traffic flow more homogeneous dissipating the platoons and smoothing the headway distribution (Fig. 3 and the low fragment). Besides, the single-vehicle experimental data [33] show that the synchronized mode is singled out by long-distant correlations in the vehicle velocities, whereas the headway fluctuations are correlated only on small scales, which justifies the assumptions of the synchronized mode being a more homogeneous state than the free flow. We think that the given scenario describes the synchronized mode formation which must be characterized by a great value of the order parameter, \( h_s > h_f \), and a lower velocity in comparison with the free flow at the same vehicle density.

In addition, whence it follows that, first, the left boundary of the headway distribution should be approximately the same for both the free flow and the synchronized mode near the phase transition, which corresponds the experimental data [33]. Second, since in this case the transition from the free flow to the synchronized mode leads to the decrease in the mean velocity, the “fast” driver will see no reason to alter their behaviour and to move forming platoons again until the vehicle density decreases and the mean velocity grows enough. It is reasonable to relate this characteristics to the experimentally observed hysteresis in the transition “free flow ↔ synchronized mode” [12–14]. Third, for a car to be able to leave a given platoon the local vehicle arrangement at the neighboring lane should be of special form and when an event of the vehicle rearrangement occurs its following evolution depends also on the particular details of the neighboring car configuration exhibiting substantial fluctuations. Therefore, the synchronized mode can comprise a great amount of local metastable states and correspond to a certain two-dimensional region on the flow-density plane (\( j\rho \)-plane) rather than a line \( j = \vartheta(\rho) \rho \), which matches the experimental data [11] and the modern notion of the synchronized mode nature [3–5]. This feature seems to be similar to that met in physical media with local order, for example, in glasses where phase transitions are characterized by wide range of controlling parameters (temperature, pressure, etc.) rather than their fixed values (see, e.g., [13]).

This uncertainty of the synchronized mode, at least qualitatively, may be regarded as an effect of the internal fluctuations of the order parameter \( h \) and at the first step we will ignore them assuming the order parameter \( h \) to be determined in a unique fashion for fixed values of the vehicle density \( \rho \) and the mean velocity \( v \). Thus for a uniform vehicle flow we write:

\[
\frac{\tau}{dt} dh = -\Phi(h, \rho, v),
\]

where \( \tau \) is the time required of drivers coming to the decision to begin or stop overtaking manoeuvres and the function \( \Phi(h, \rho, v) \) possesses a single stationary point \( h = h(\rho, v) \) being stable and, thus,

\[
\frac{\partial \Phi}{\partial h} > 0.
\]

The latter inequality is assumed to hold for all the values of the order parameter for simplicity. We note that equation (11) also allows for the delay in the driver response to changes on the road. However, in contrast with models similar to (1) and (2), here this effect is not the
origin of the traffic flow instability and, thus, its particular description is not so crucial. Moreover, as discussed in the Introduction, the time \( \tau \) characterizes the delay in the driver decision concerning to the lane changing but not the control over the headway, enabling us to assume \( \tau \gg \tau' \).

The particular value \( h(v, \rho) \) of the order parameter results from the compromise between the danger of accident during changing lanes and the will to move as fast as possible. Obviously, the lower is the mean vehicle velocity \( v \) for a fixed value of \( \rho \), the weaker is the lane changing danger and the stronger is the will to move faster. Besides, the higher is the vehicle density \( \rho \) for a fixed value of \( v \), the stronger is this danger (here the will has no effect at all). These statements enable us to regard the dependence \( h(v, \rho) \) as a decreasing function of both the variables \( v, \rho \) (Fig. 4) and taking into account inequality \((12)\) to write:

\[
\frac{\partial \Phi}{\partial v} > 0, \quad \frac{\partial \Phi}{\partial \rho} > 0, \quad (13)
\]

with the latter inequality stemming from the danger effect only.

Equation \((11)\) describes actually the behavior of the drivers who prefer to move faster than the statistically mean vehicle and whose readiness for risk is greatest. Exactly this group of drivers govern the value of \( h \). There is, however, another characteristics of the driver behavior, it is the mean velocity \( v = \vartheta(h, \rho) \) chosen by the statistically averaged driver taking into account also the danger resulting from the frequent lane changing by the “fast” drivers. This characteristics is actually the same as one discussed in the Introduction but depends also on the order parameter \( h \). So, as a function of \( \rho \) it meets conditions \((1)\). Concerning the dependence of \( \vartheta(h, \rho) \) on \( h \) we can state that generally this function should be increasing for small values of the car density, \( \rho \ll \rho_0 \), because in the given case the lane changing practically makes no danger to traffic and all the drives can overtake vehicle moving at lower speed without risk. By contrast, when the vehicle density is sufficiently high, \( \rho \gtrsim \rho_0 \), the most “impatient” drivers permanently change the lanes for overtaking, making an additional danger to the most part of other drivers. Therefore, in this case the velocity \( \vartheta(h, \rho) \) has to decrease as the order parameter \( h \) increases. For certain intermediate values of the vehicle density, \( \rho \approx \rho_c \), this dependence is to be weak. Fig. 5 shows the velocity \( \vartheta(h, \rho) \) as a function of \( h \) for different values of \( \rho \), where, in addition, we assume the effect of order parameter \( h \in (0,1) \) near the boundary points weak and set

\[
\frac{\partial \vartheta}{\partial h} = 0 \quad \text{at} \quad h = 0 \quad \text{and} \quad h = 1. \quad (14)
\]

We will ignore the delay in the relaxation of the mean velocity to the equilibrium value \( v = \vartheta(h, \rho) \) because the corresponding delay time characterizes the driver control over the headway and should be short, as already discussed above. Then the governing equation \((11)\) for the order parameter \( h \) can be rewritten in the form:

\[
\tau \frac{dh}{dt} = -\phi(h, \rho); \quad \phi(h, \rho) \defeq \Phi[h, \rho, \vartheta(h, \rho)]. \quad (15)
\]

For the steady state uniform vehicle flow the solution of the equation \( \phi(h, \rho) = 0 \) specifies the dependence \( h(\rho) \) of the order parameter on the car density. Let us, now, study its properties and stability.

**B. Nonmonotony of the \( h(\rho) \) dependence and the traffic flow instability**

To study the local characteristics of the right-hand side of Eq. \((15)\) we analyze its partial derivatives

\[
\frac{\partial \phi}{\partial h} = \frac{\partial \Phi}{\partial h} + \frac{\partial \Phi}{\partial \rho} \frac{\partial \vartheta}{\partial h}, \quad (16)
\]

\[
\frac{\partial \phi}{\partial \rho} = \frac{\partial \Phi}{\partial \rho} + \frac{\partial \Phi}{\partial \rho} \frac{\partial \vartheta}{\partial \rho}. \quad (17)
\]

As mentioned above, the value of \( \partial \Phi/\partial \rho \) is solely due to the danger during changing lanes, so this term can be ignored until the vehicle density \( \rho \) becomes sufficiently high. In other words, in a certain region \( \rho < \rho_h \lesssim \rho_0 \) the derivative \( \partial \phi/\partial \rho \sim (\partial \Phi/\partial v)(\partial \vartheta/\partial \rho) < 0 \) by virtue of \((12)\) and \((13)\). So, the local behavior of the function \( h(\rho) \) (meeting the equality \( \partial \phi = 0 \) and, thus, \( dh/d\rho = -(\partial \phi/\partial \rho)(\partial \phi/\partial h)^{-1} \)) depends directly on the sign of the derivative \( \partial \phi/\partial h \), it is increasing or decreasing for \( \partial \phi/\partial h > 0 \) or \( \partial \phi/\partial h < 0 \), respectively.

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FIG. 4. Qualitative sketches of the order parameter \( h \) as a function of the vehicle mean velocity \( v \) and the density \( \rho \) specified by the behavior of individual drivers.

FIG. 5. A qualitative sketch of the mean vehicle velocity vs. the order parameter \( h \) for several fixed values of the vehicle density \( \rho \).
FIG. 6. The region of the traffic flow instability in the $h\rho$-plane and the form of the curve $h(\rho)$ displaying the dependence of the order parameter on the vehicle density. The plot is a qualitative sketch.

For long-wave perturbations $\propto \exp\{ikx\}$ of the car distribution on a highway the density $\rho$ can be treated as a constant at the lower order in $k$. Therefore, according to Eq. (15) the steady state traffic flow is unstable if $\partial \phi / \partial h < 0$.

Due to (12) and (14) the first term on the right-hand side of (10) is dominant in the vicinity of the lines $h = 0$ and $h = 1$, thus, in this region the curve $h(\rho)$ is increasing and the stationary state of the traffic flow is stable. For $\rho < \rho_c$ the value $\partial \eta / \partial h > 0$ (Fig. 5), therefore, the whole region $\{0 < h < 1, 0 < \rho < \rho_c\}$ corresponds to the stable car motion. However, for $\rho > \rho_c$ there can be a region of the order parameter $h$ where the derivative $\partial \phi / \partial h$ changes the sign and the vehicle motion becomes unstable. Indeed, the solution $v = \eta(h, \rho)$ of the equation $\Phi(h, \rho, v) = 0$ can be regarded as the mean vehicle velocity controlled by the “fast” drivers and is decreasing function of $h$ because of $\partial \eta / \partial h = -(\partial \Phi / \partial h)/(\partial \Phi / \partial v)^{-1}$. So, once such “active” drivers become to change the lanes to move faster, they will do this as frequently as possible especially if the mean velocity decreases, which corresponds to a considerable increase in $h$ for a small decrease in $v$. So, it is quite natural to assume that the value of $\partial \eta / \partial h$ for $\rho > \rho_c$ is sufficiently small and

$$
\frac{\partial \phi}{\partial h} = \frac{\partial \Phi}{\partial v} \left( \frac{\partial \theta}{\partial h} - \frac{\partial \eta}{\partial h} \right) < 0. \tag{18}
$$

Under these conditions the instability region does exist, the curve $h(\rho)$ can look like “S” (Fig. 6) and its decreasing branch corresponds to the unstable traffic flow. The lower increasing branch matches the free flow state of the car motion, whereas the upper one should be related to the synchronized mode because it is characterized by the order parameter coming to unity.

C. Hysteresis and the fundamental diagram

The obtained dependence $h(\rho)$ actually describes the first order phase transition in the vehicle motion. Indeed, when increasing the car density exceeds the value $\rho_1$ the free flow becomes absolutely unstable and the synchronized mode forms through a sharp jump of the order parameter. If, however, after that the car density decreases the synchronized mode will persist until the car density attains the value $\rho_2 < \rho_1$. It is a typical hysteresis and the region $(\rho_2, \rho_1)$ corresponds to the metastable phases of traffic flow.

Let us, now, discuss a possible form of the fundamental diagram $j = j(\rho)$ showing the vehicle flux $j = \rho \vartheta(\rho)$ as a function of the car density $\rho$, where, by definition, $\vartheta(\rho) = \vartheta[h(\rho), \rho]$. It should be pointed out that here we confine our consideration to the region of not too large values of the car density, $\rho < \rho_h$, where the transition “free flow ↔ synchronized mode” takes place. The transition “synchronized mode ↔ jammed traffic” will be discussed below. Fig. 7a displays the dependence $\vartheta(h, \rho)$ of the mean vehicle velocity on the density $\rho$ for the fixed limit values of the order parameter $h = 0$ and 1. For a small values of $\rho$ these curves practically coincide with each other. As the vehicle density $\rho$ grows and until it comes close to the critical value $\rho_c$ when the lane change danger becomes substantial, the velocity $\vartheta(1, \rho)$ practically does not depend on $\rho$. So at the point $\rho_c$ at which the curves $\vartheta(1, \rho)$ and $\vartheta(0, \rho)$ meet each other, $\vartheta(1, \rho)$ is to exhibit sufficiently sharp decrease in comparison with the latter one. Therefore, on one hand, the function $j_1(\rho) = \rho \vartheta(1, \rho)$ has to be decreasing for $\rho > \rho_c$. On the other hand, at the point $\rho_c$ for $h \ll 1$ the effect of the lane change danger is not extremely strong, it only makes the lane change ineffective, $\partial \phi / \partial h \approx 0$ (Fig. 6). So it is reasonable to assume the function $j_0(\rho) = \rho \vartheta(0, \rho)$ increasing near the point $\rho_c$. Under the adopted assumptions the relative arrangement of the curves $j_0(\rho)$, $j_1(\rho)$ is demonstrated in Fig. 7b, and Fig. 7c shows the fundamental diagram of traffic flow resulting from Fig. 6 and
I. INTRODUCTION

The previous section has considered uniform traffic flow, so, analyzed actually the individual characteristics of the free flow and the synchronized mode. In the present section we study their coexistence, i.e., the conditions under which a car cluster of finite size forms. This problem, however, requires that the traffic flow model be defined concretely. Therefore, in what follows we will consider a certain simple model which illustrates the characteristic features of the car cluster self-organization without complex mathematical manipulations.

As before, the model under consideration assumes the mean velocity relaxation to be immediate and modifies the governing equation (13) in such a way as to ascribe the order parameter \( h \) to a local car group. In other words, we describe the vehicle flow by the Lighthill–Whitham equation with dissipation (see, e.g., [2] and also Introduction), replace the time derivative in Eq. (15) by the particle derivative, and take into account that the order parameter cannot exhibit substantial variations over scales \( l \sim \theta^{1/2} \tau \lesssim v_0 \tau \) (\( \theta \) is the velocity variance, \( v_0 \) is the typical car velocity in the free flow). Namely, we write:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial [\rho \varrho(h, \rho)]}{\partial x} = D \frac{\partial^2 \rho}{\partial x^2}, \tag{19}
\]

\[
\tau \left[ \frac{\partial h}{\partial t} + \varrho(h, \rho) \frac{\partial h}{\partial x} \right] = \mathcal{L}(h) - \phi(h, \rho) + \xi(x, t). \tag{20}
\]

Let us discuss the meaning of the particular terms of the given model. The Burgers equation (19), as already discussed in Introduction, allows for the fact that drivers govern their motion taking into account not only the behavior of the nearest cars, but the state of traffic flow inside the whole field of their front view of length. The effective diffusivity \( D \) can be estimated as \( D \sim L^* v_0 \), where \( L^* \gg l \) is a front distance looked through by drivers assumed to be much greater that the scale \( l \), so

\[
D \tau \sim l L^* \gg l^2. \tag{21}
\]

The function \( \phi(h, \rho) \) is of the form

\[
\phi(h, \rho) \overset{\text{def}}{=} h(1-h)[\alpha(\rho) - h], \tag{21}
\]

where

\[
a(\rho) = \begin{cases} 
1 & \text{for } \rho < \rho_c, \\
(\rho_c + \Delta - \rho)/\Delta & \text{for } \rho_c < \rho < \rho_c + \Delta, \\
0 & \text{for } \rho > \rho_c + \Delta.
\end{cases}
\]

It describes such a driver behavior that \( h = 0 \) and \( h = 1 \) are the unique stable values of the order parameter for \( \rho < \rho_c \) and \( \rho > \rho_c + \Delta \), respectively, whereas, for \( \rho_c < \rho < \rho_c + \Delta \) the points \( h = 0, h = 1 \) are both locally stable and there is an additional unstable stationary point, namely, \( h = a(\rho) \). The term

\[
\hat{\mathcal{L}}(h) \overset{\text{def}}{=} L^2 \frac{\partial^2 h}{\partial x^2} + \frac{1}{\sqrt{2}} \frac{\partial h}{\partial x}
\]

governs spatial variations in the field \( h(x, t) \) and takes into account that drivers mainly follow the behavior of cars in front of them and cars moving at the rear cannot essentially affect them. The mean car velocity depends on \( h \) and \( \rho \) as

\[
\varrho(h, \rho) \overset{\text{def}}{=} \varrho_{0}(1-h) + [\rho_{c}\varrho_{0} - \nu(\rho - \rho_{c})]h. \tag{24}
\]

The last term on the right-hand side of Eq. (20) characterizes the random fluctuations in the order parameter dynamics:

\[
\langle \xi(x, t) \rangle = 0, \tag{25}
\]

\[
\langle \xi(x, t) \xi(x', t') \rangle = \sigma^2 t \delta(x - x') \delta(t - t'), \tag{26}
\]

where \( \sigma \) is their dimensionless amplitude. Expressions (23) and (24) gives the \( h(\rho) \)-dependence and the fundamental diagram shown in Fig. 8, simplifying the one presented in Fig. [4].

![Fig. 8. The dependence \( h(\rho) \) and the fundamental diagram of traffic flow described by the model (15), (24).](image)

If we ignore the random fluctuations of the order parameter \( h \), i.e., set \( \sigma = 0 \), then the model (15), (24) will give us an artificially long delay (much greater than \( \tau \)) in the order parameter variations from, for example, the
unstable point \( h = 0 \) to the stable point \( h = 1 \). Such a delay can lead to a meaningless great increase of the vehicle density in the free flow without phase transition to congestion. In order to avoid this artifact and to allow for the effect of real fluctuations in the driver behavior we also will assume the amplitude \( \sigma \) to obey the condition \[ (\frac{l}{L^*})^{5/4} \lesssim \sigma \ll 1 \] (\( \sigma \ll 1 \), because, otherwise, the traffic flow dynamics would be totally random). It should be noted that small random variations of the order parameter \( h \) near the points \( h = 0 \), \( h = 1 \) going into the regions \( h < 0 \) and \( h > 1 \), respectively, do not come into conflict with its physical meaning as the measure of the car motion correlations. Indeed, the chosen values \( h = 0 \) and \( h = 1 \) can describe a renormalization of real correlation coefficients \( h = h_1 > 0, h_2 < 1 \).

According to Eq. (24), for the order parameter \( h \) the characteristic scale of its spatial variations is \( l \), so, the layer \( \mathcal{I}_h \) separating the regions where \( h \approx 0 \) and 1 is of thickness about \( l \). Due to inequality (2) the car density on such scales can be treated as constant. Therefore, the transition region \( L^* \) between practically the uniform free flow and the congested phase is of thickness determined mainly by spatial variations of the vehicle density and on such scales the layer \( \mathcal{I}_h \) can be treated as an infinitely thin interface. In addition, the characteristic time scale of the layer \( \mathcal{I}_h \) formation is about \( \tau \), whereas it takes about the time \( \tau_p \sim D/\nu^2 \sim \tau (L^*/l) \gg \tau \) for the layer \( L^* \) to form. Thereby, when analyzing the motion of a wide car clusters we may regard the order parameter distribution \( h(x,t) \) as quasi-stationary for a fixed value of the car density \( \rho \).

Let us, now, consider two possible limits of the layer \( \mathcal{I}_h \) motion under such conditions.

### A. Regular dynamics

In the region \( \rho_c < \rho < \rho_c + \Delta \) until the value of \( a(\rho) \) comes close to the boundaries \( h = 0 \) and \( h = 1 \) the effect of the random fluctuations is ignorable. In this case by virtue of the adopted assumptions the solution of Eq. (24) that describes the layer \( \mathcal{I}_h \) moving at the speed \( u \) is of the form:

\[
h = \frac{1}{2} \left[ 1 + \tanh \left( \frac{x - ut}{\lambda} \right) \right].
\]

Here for the layer \( \mathcal{I}_{\rho_1} \) of the transition “free-flow \( \rightarrow \) synchronized mode” and for the layer \( \mathcal{I}_{\rho_{10}} \) of the opposite transition (Fig. 3)

\[
\lambda_{\rho_1} = \frac{2\sqrt{2}}{\eta_c} \nu, \quad \lambda_{\rho_{10}} = -2\sqrt{2} \eta_c \nu.
\]

![FIG. 9. The distribution of the order parameter and the car density in the vicinity of the layers \( \mathcal{I}_h \) of the transition between the free flow and the synchronized phase as well as the velocity of their motion vs. the local values \( \rho_i \) of the car density.](image)

\[
u_{01} = \nu_0 - \frac{\Delta_v}{2} - \frac{l}{\sqrt{2} \eta_v} \left[ 1 + \eta_v - 2a(\rho_i) \right],
\]

\[
u_{10} = \nu_0 - \frac{\Delta_v}{2} - \frac{l}{\sqrt{2} \eta_v} \left[ 2\eta_v a(\rho_i) - (\eta_v - 1) \right],
\]

where we introduced the quantities:

\[
\Delta_v = \nu(0, \rho_i) - \nu(1, \rho_i),
\]

\[
\eta_v = \frac{\nu_0}{\nu_0} \left[ 1 + \left( \frac{\tau \Delta_v}{2\eta_v} \right)^{1/2} + \frac{\tau \Delta_v}{2\eta_v} \right],
\]

and \( \rho_i \) is the corresponding value of the car density inside the layers \( \mathcal{I}_{\rho_1} \) and \( \mathcal{I}_{\rho_{10}} \).

Expressions (30), (31) describe the regular dynamics of the car cluster formation because the transition, for example, from the free flow to the synchronized phase at a certain point \( x \) is induced by this transition at the nearest points. The dependence of the velocities \( u_{01} \) and \( u_{10} \) on the local car density \( \rho_i \) is illustrated in Fig. 8. The characteristic velocities attained in this type motion can be estimated as

\[
\nu_0 - u \sim \max \left\{ \nu_0 \Delta / \rho_c, \frac{l}{\tau} \right\},
\]

so, under the adopted assumptions the regular dynamics does not allow for the sufficiently fast motion of the layers \( \mathcal{I}_h \) upstream.

### B. Noise-induced dynamics

As the car density \( \rho \) tends to the critical values \( \rho_c \) or \( \rho_c + \Delta \) the value of \( a(\rho) \) comes close to the boundaries \( a(\rho_c) = 1 \) and \( a(\rho_c + \Delta) = 0 \), and the point \( h = 1 \) or \( h = 0 \) becomes unstable, respectively. In this case the effect of the random fluctuations \( \xi(x,t) \) plays a substantial
role. Namely, the phase transition, for example, from the free flow to the synchronized motion (for \( \rho \approx \rho_c + \Delta \)) is caused by the noise \( \xi(x, t) \) and equiprobably takes place at every point of the region wherein \( \rho \approx \rho_c + \Delta \) rather than is localized near the current position of the layer \( \mathcal{L}_{01} \). Under these conditions the motion of the layers \( \mathcal{L}_h \) can be qualitatively characterized by an extremely high velocity in both the directions, which is illustrated in Fig. 9 by dashed lines.

We note that the noise-induced motion, in contrast to the regular dynamics, is to exhibit significant fluctuations in the displacement of the layer \( \mathcal{L}_h \) as well as in its forms. This question is, however, a subject for an individual study.

C. Diffusion limited motion of vehicle clusters

Let us, now, analyze the motion of a sufficiently large cluster that can form on a highway when the initial car density or, what is the same, the average car density \( \bar{\rho} \) belongs to the metastable region, \( \bar{\rho} \in (\rho_c, \rho_c + \Delta) \). The term “sufficiently large” means that the cluster dimension \( L \) is assumed to be much greater than the front distance \( L^* \) looked through by drivers, so, they cannot look round the congestion as a whole. Exactly, in this case a quasi-local description of traffic flow similar to the differential equations (19), (20) is justified.

Converting to the frame \( y = x - ut \) moving at the cluster velocity \( u \), solving Eq. (19) individually for the free flow and the synchronized phase, and treating the layers \( \mathcal{L}_h \) as infinitely thin interfaces we get the following conclusion. Within the frameworks of the given model the car cluster moves upstream sufficiently fast, so, the motion of the layers \( \mathcal{L}_{01} \) and \( \mathcal{L}_{10} \) is governed by the noise \( \xi(x, t) \). In this case the values of the car density at the layers \( \mathcal{L}_{01} \) and \( \mathcal{L}_{10} \) have to be \( \rho_j \approx \rho_c + \Delta \) and \( \rho_j \approx \rho_c \), respectively. Thereby, the cluster velocity \( u \) is mainly determined by the car redistribution governed by the diffusion type processes. The latter feature is the reason why we refer to the cluster dynamics under such conditions as to the diffusion limited motion. The transition region \( \mathcal{L}_{01} \) between practically the uniform free flow state and the cluster contains the exponential increase of the vehicle density inside the free flow phase from the value \( \rho_f \) far from the “interface” \( \mathcal{L}_{01} \) up to \( \rho_j \approx \rho_c + \Delta \) at \( \mathcal{L}_{01} \),

\[
\rho = \rho_f + (\rho_j - \rho_f) \exp\{q_j y\},
\]

where \( q_f = (\bar{\rho}_0 + |u|)/D \sim 1/L^* \) and the frame \( \{y\} \) is attached to the “interface” \( \mathcal{L}_{01} \). The transition region \( \mathcal{L}_{10} \) from the synchronized phase to the uniform free flow is to be localized inside the car cluster. So, it is characterized by the decrease in the vehicle density \( \delta \rho \propto \exp\{q_j y\} \), where \( q_j = (|u| - \nu)/D \), and the vehicle free flow leaving the cluster is uniform at all its points (Fig. 10).

The cluster velocity is directly determined by the motion of the interface \( \mathcal{L}_{01} \). Therefore, assuming also the cluster dimension \( L \) large in comparison with \( L^* \), from Eq. (19) we get the expression of the same form as the Lighthill–Whitham formula (10) relating the cluster velocity \( u \) and the vehicle flux characteristics on the both sides of the layer \( \mathcal{L}_{01} \). Whence it follows that at the first approximation:

\[
u \approx -\nu,
\]

the value \( q_j = 0 \), and the cluster velocity is of the form shown in Fig. 10, under the name “mesocluster”. Assuming the total number of cars on the highway of length \( L_{rd} \) fixed we get the expression for the mesocluster dimension \( L \):

\[
L = 2L_{rd} \frac{\bar{\rho} - \rho_c}{\Delta}.
\]

However, this result is justified only for sufficiently small values of \( (\bar{\rho} - \rho_c)/\Delta \ll 1 \), when the cluster dimension is not too large, \( L_{q_j} \ll 1 \) (nevertheless, \( L \gg L^* \)). Exactly for this reason we refer to such clusters as mesoscopic ones. In order to study the opposite limit, \( L_{q_j} \gg 1 \), we have to take into account that the value \( \rho_f \) is not rigorously equal to \( \rho_c \) but practically is the root \( \rho_f^* > \rho_c \) of the equation \( u_{10}(\rho_f^* - \rho_c) = -\nu \). In this case the Lighthill–Whitham formula (10) gives the expression:

\[
u \approx -\left[\nu + (\bar{\rho}_0 + \nu) \frac{\rho_f^* - \rho_c}{\Delta}\right]
\]

leading to the following estimates of the thickness \( 1/q_j \) of the transition region \( \mathcal{L}_{10} \):

\[
1/q_j \approx \frac{D \Delta}{(\bar{\rho}_0 + \nu)(\rho_f^* - \rho_c)} \sim L^* \frac{\Delta}{(\rho_f^* - \rho_c)}.
\]

The form of such a wide cluster is shown in Fig. 10, its dimension is:

\[
L = L_{rd} \frac{\bar{\rho} - \rho_c}{\Delta}.
\]

and the region of the mean car density corresponding to this limit is specified by the inequality:

\[
\frac{\rho^* - \rho_c}{\Delta} \gg \frac{L^*}{L_{rd}(\rho_f^* - \rho_c)}.
\]

The resulting dependence of the cluster dimension on the mean car density \( \bar{\rho} \) is illustrated in Fig. 10.

FIG. 10. The possible forms of the car clusters and their dimension vs. the mean car density.
IV. PHASE TRANSITION “SYNCHRONIZED MODE ↔ JAM”, BRIEF DISCUSSION

In Sec. IV we have considered the phase transition between the free flow and the synchronized mode. However, according to the experimental data [13] there is an additional phase transition in traffic flow regarded as the transition between the synchronized motion and the jammed “stop-and-go” traffic. This transition occurs at extremely high vehicle densities \( \rho \) coming close to the limit value \( \rho_0 \).

The present section briefly demonstrates that the developed model for the driver behavior also predicts a similar phase transition at high car densities. To avoid possible misunderstandings we, beforehand, point out that the model in its present form cannot describe details of the transition “synchronized mode ↔ jam” because we have not taken into account the delay in the driver response to variations in headway. The latter is responsible for the formation of the “stop-and-go” pattern, so, to describe the jammed traffic on multilane highways we, at least, should combine a governing equation for the order parameter \( h \) and a continuity equation similar to (20), (19) with an equation for the car velocity relaxation similar to (2). This question, however, is worthy of individual study. Besides, the approximations used in Sec. IV to characterize the synchronized mode at the car densities near \( \rho_0 \) do not hold here.

In Sec. IV we have studied the dependence of the order parameter \( h \) on the car density ignoring the first term on the right-hand side of Exp. (17) caused by the dangerous of lane changing. This assumption is justified when the car density is not too high. In extremely dense traffic flow, when the car density exceeds a certain value, \( \rho > \rho_h \approx \rho_0 \), changing lanes becomes sufficiently dangerous and the function \( \Phi(h, v, \rho) \) describing the driver behavior is to depend strongly on the vehicle density in this region. In addition, the vehicle motion becomes sufficiently slow. Under such conditions the former term on the right-hand side of expression (17) should be dominant and, thus, \( \partial \phi / \partial \rho > 0 \). Therefore, the stable vehicle motion corresponding to \( \partial \phi / \partial h > 0 \) matches the decreasing dependence of the order parameter \( h(\rho) \) on the vehicle density \( \rho \) for \( \rho > \rho_h \). So, as the vehicle density \( \rho \) increases the curve \( h(\rho) \) can again go into the instability region (in the \( h\rho \)-plane), which has to give rise to a jump from the synchronized mode with greater values of the order parameter to a new traffic state with its less values (Fig. 11). Obviously, this transition between the two congested phases also exhibit the same hysteresis as one describe in Sec. IV.

We identify the latter traffic state with the jammed vehicle motion. Indeed, in extremely dense traffic lane change is practically depressed, making the car ensembles at different lanes independent of one another. So, in this case vehicle flow has to exhibit weak multilane correlations and we should ascribe to it small values of the order parameter \( h \). It should be noted that the experimental single-vehicle data [8] demonstrates strong correlations of variations in the traffic flux and the car density for both the free flow and the “stop-and-go” motion. By contrast, the synchronized mode is characterized by small values of the cross-covariance between flow, speed, and density. In other words, for the free flow and the “stop-and-go” motion the traffic flux \( j = \partial \rho \) should depends directly on the car density \( \rho \), as it must in the present model if we set \( h = 0 \).

Finalizing the present section we point out that the given model treats the jammed phase as a “faster” vehicle motion then the synchronized mode at the same values of the order parameter. There is no contradiction with the usual view on the synchronized mode as a high flux traffic state. The latter corresponds to the traffic flow at the vehicle densities near the phase transition “free flow ↔ synchronized mode” rather than close to the limit value \( \rho_0 \). Besides, ordinary driver’s experience prompts that a highly dense traffic flow can be blocked at all if one of the cars begin to change the lanes. Nevertheless, in order to describe, at least qualitatively, the real features of the phase transition “synchronized mode ↔ stop-and-go waves” a more sophisticated model is required. The present description only relates it to the instability of the order parameter at high values of the vehicle density.

Besides, the present analysis demonstrates also the nonmonotonic behavior of the order parameter as the car density increases even if we ignore the hysteresis regions and focus our attention the stabel vehicle flow regions only. It should be noted that a similar nonmonotonic dependence of the lane change frequency on the car density as well as the platoon formation has been found in the cellular automaton model for two-lane traffic [14].

V. CLOSING REMARKS

Concluding the paper we recall the key points of the developed model.

We have proposed an original macroscopic approach to the description of multilane traffic flow based on an extended collection of the traffic flow state variables.
Namely, in addition to such characteristics as the car density $\rho$ and mean velocity $v$ being actually the zeroth and first moments of the “one particle” distribution function we introduce a new variable $h$ called the “order parameter”. It stands for the internal correlation in the car motion along different lanes that are due to the lane changing manoeuvres. The order parameter, in fact, allows for the essentially “many-body” effects in the car interaction so it is treated as an independent state variable.

Taking into account the general properties of the driver behavior we have stated a governing equation for the order parameter. Based on current experimental data [11, 13] we have assume the correlations in the car motion on multilane highways to be due to a small group of “fast” drivers, i.e. the drivers who move substantially faster than the statistically mean vehicle continuously overtaking other cars. These “fast” cars, on one hand, increase individually the total rate of vehicle flow but, on the other hand, make the accident danger greater and, thus, cause the statistically mean driver to decrease the velocity. The competition of the two effects depends on the car density and the mean velocity and, as shown, can give rise to the traffic flow instability. It turns out that the resulting dependence of the order parameter on the car density describes in the same way the experimentally observed sequence of phase transitions “free flow $\leftrightarrow$ synchronized motion $\leftrightarrow$ jam” typical for traffic flow on highways [12]. Besides, we have shown that both these transitions should be of the first order type and exhibit hysteresis, matching the experimental data [12, 14].

The synchronized mode is characterized by a large value of the order parameter model from the gas-kinetic theory we note that the appearance of the “fast” driver platoons demonstrates a substantial deviation of the car distribution function from the monotonic quasi-equilibrium form. So, to construct an adequate system of equations dealing with the moments of the distribution function a more sophisticated approximation is required.

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