On the stability of orthogonally Jensen additive and quadratic functional equation∗

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Abstract

We consider the stability of the orthogonal Jensen additive and quadratic equations in $F$-spaces, through applying and extending the approach to the proof of a 2010 result of W.Frčner and J.Sikorska, we presenting a new method to get the stability. Moreover, we work in a more general and natural condition than considered before by other authors.

Keywords: stability, orthogonality, Jensen additive mapping, Jensen quadratic mapping, $F$-space

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1 Introduction and preliminaries

Assume that $X$ is a real inner product space and $f : X \to \mathbb{R}$ is a solution of the orthogonal Cauchy functional equation $f(x + y) = f(x) + f(y), \langle x, y \rangle = 0$. By the Pythagorean theorem $f(x) = \|x\|^2$ is a solution of the conditional equation. Of course, this function does not satisfy the additivity equation everywhere. Thus orthogonal Cauchy equation is not equivalent to the classic Cauchy equation on the whole inner product space. This phenomenon may show the significance of study of orthogonal Cauchy equation.

In the recent decades, stability of functional equations have been investigated by many mathematicians (see [1]). The first author treating the stability of the Cauchy equation was D.H. Hyers [2] by proving that if $f$ is a mapping from a normed space $X$ into a Banach space satisfying $\|f(x + y) - f(x) - f(y)\| \leq \epsilon$ for some $\epsilon > 0$, then there is a unique additive mapping $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \epsilon$.

R. Ger and J. Sikorska [3] investigated the orthogonal stability of the Cauchy functional equation $f(x + y) = f(x) + f(y)$, namely, they showed that if $f$ is a function from an orthogonality space $X$ into a real Banach space $Y$ and $\|f(x + y) - f(x) - f(y)\| \leq \epsilon$ for all $x, y \in X$ with $x \perp y$ and some $\epsilon > 0$, then there exists exactly one orthogonally additive mapping $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \frac{16}{3} \epsilon$ for all $x \in X$.

The first author treating the stability of the quadratic equation was F. Skof [4] by proving that if $f$ is a mapping from a normed space $X$ into a Banach space $Y$ satisfying $\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \epsilon$ for some $\epsilon > 0$, then there is a unique quadratic function $g : X \to Y$ such that...

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\[ \|f(x) - g(x)\| \leq \frac{7}{5}. \] P. W. Cholewa extended Skof's theorem by replacing \( X \) by an abelian group. Skof's result was later generalized by S. Czerwik in the spirit of Hyers-Ulam. The stability problem of functional equations has been extensively investigated by some mathematicians (see [7, 8, 9, 10, 11, 12, 13, 14]).

The orthogonally quadratic equation

\[ f(x + y) + f(x - y) = 2f(x) + 2f(y), x \perp y \]

was first investigated by F. Vajzović when \( X \) is a Hilbert space, \( Y \) is the scalar field, \( f \) is continuous and \( \perp \) means the Hilbert space orthogonality. Later, H. Driljević, M. Fochi, M. Moslehian, and G. Szabó generalized this result.

The notion of orthogonality goes a long way back in time and various extensions have been introduced over the last decades. In particular, proposing the notion of orthogonality in normed linear spaces has been the object of extensive efforts of many mathematicians.

Let us recall the standard definition of orthogonality by Rätz:

**Definition 1.1.** Let \( X \) be a real linear space with \( \dim X \geq 2 \) and let \( \perp \) be a binary relation on \( X \) such that

1. \( x \perp 0 \) and \( 0 \perp x \) for all \( x \in X \);
2. if \( x, y \in X \setminus \{0\} \) and \( x \perp y \), then \( x \) and \( y \) are linearly independent;
3. if \( x, y \in X \) and \( x \perp y \), then for all \( \alpha, \beta \in \mathbb{R} \) we have \( \alpha x \perp \beta y \);
4. for any two-dimensional subspace \( P \) of \( X \) and for every \( x \in P \), \( \lambda \in [0, \infty) \), there exists \( y \in P \) such that \( x \perp y \) and \( x + y \perp \lambda x - y \).

An ordered pair \((X, \perp)\) is called an orthogonality space.

In 2010, Fechner and Sikorska studied the stability of orthogonality and proposed the definition of orthogonality as follows.

**Definition 1.2.** Let \( X \) be an Abelian group and let \( \perp \) be a binary relation defined on \( X \) with the properties:

1. if \( x, y \in X \) and \( x \perp y \), then \( x \perp -y, -x \perp y \) and \( 2x \perp 2y \);
2. for every \( x \in X \), there exists a \( y \in X \) such that \( x \perp y \) and \( x + y \perp x - y \).

It’s worth noting that every orthogonal space satisfies these conditions as well as any normed linear space with the isosceles orthogonality, but Pythagorean orthogonality no longer satisfies these conditions.

Although various studies on stability have been successfully conducted, there are not many corresponding stability results due to the non-linear structure of the infinite-dimensional \( F \)-space. The nonlinear structure of \( F \)-space plays an important role in functional analysis and other mathematical fields. The \( L^p([0, 1]) \) for \( 0 < p < 1 \) equipped with the metric \( d(f, g) = \int \|f(x) - g(x)\|^pdx \) is an example of an \( F \)-space but not a Banach space. Besides these, for \( F \)-spaces, and we recommend readers to read the literature [23, 24].

**Definition 1.3.** Consider \( X \) be a linear space. A non-negative valued function \( \|\cdot\| \) achieves an \( F \)-norm if satisfies the following conditions:

1. \( \|x\| = 0 \) if and only if \( x = 0 \);
2. \( \|\lambda x\| = \|x\| \) for all \( \lambda, |\lambda| = 1 \);
3. \( \|x + y\| \leq \|x\| + \|y\| \) for all \( x, y \in X \);
4. \( \|\lambda_n x\| \to 0 \) provided \( \lambda_n \to 0 \);
5. \( \|\lambda_n x_n\| \to 0 \) provided \( x_n \to 0 \);
6. \( \|\lambda_n x_n\| \to 0 \) provided \( \lambda_n \to 0, x_n \to 0 \).
Then \((X, \| \cdot \|)\) is called an \(F^\ast\)-space. An \(F\)-space is a complete \(F^\ast\)-space.

An \(F\)-norm is called \(\beta\)-homogeneous \((\beta > 0)\) if \(\|tx\| = |t|^\beta \|x\|\) for all \(x \in X\) and all \(t \in C\) (see \([25, 26]\)).

If a quasi-norm is \(p\)-subadditive, then it is called \(p\)-norm \((0 < p < 1)\). In other words, if it satisfies
\[
\|x + y\|^p \leq \|x\|^p + \|y\|^p, \quad x, y \in X.
\]

We note that the \(p\)-subadditive quasi-norm \(\| \cdot \|\) induces an \(F\)-norm. We refer the reader to \([23]\) and \([27]\) for background on it.

**Definition 1.4.** \([28]\) A quasi–norm on \(\| \cdot \|\) on vector space \(X\) over a field \(K(\mathbb{R})\) is a map \(X \rightarrow [0, \infty)\) with the following properties:

1. \(\|x\| = 0\) if and only if \(x = 0\);
2. \(\|ax\| = |a|\|x\|\), \(a \in \mathbb{R}, x \in X\);
3. \(\|x + y\| \leq C(\|x\| + \|y\|), \quad x, y \in X\),

where \(C \geq 1\) is a constant independent of \(x, y \in X\). The smallest \(C\) for which (3) holds is called the quasi-norm constant of \((X, \| \cdot \|)\).

It is worth to note that the well-known Aoki–Rolewicz theorem \([23]\) in nonlocally convex theory, that is, which asserts that for some \(0 < p \leq 1\), every quasi-norm admits an equivalent \(p\)-norm.

Various more results for the stability of functional equations in quasi-Banach spaces can be seen in \([29, 30]\). However, the results are more interesting and meaningful when orthogonality is taken into account.

Let \(X\) be an orthogonality space and \(Y\) a real Banach space. A mapping \(f : X \rightarrow Y\) is called orthogonally Jensen additive if it satisfies the so-called orthogonally Jensen additive functional equation
\[
2f\left(\frac{x + y}{2}\right) = f(x) + f(y)
\]
for all \(x, y \in X\) with \(x \perp y\). A mapping \(f : X \rightarrow Y\) is called orthogonally Jensen quadratic if it satisfies the so-called orthogonally Jensen quadratic functional equation
\[
2f\left(\frac{x + y}{2}\right) + 2f\left(\frac{x - y}{2}\right) = f(x) + f(y)
\]
for all \(x, y \in X\) with \(x \perp y\).

In this paper, we apply some ideas and extend the results from \([22]\) to prove the stability of the orthogonally Jensen additive functional equation (1.1) and of the orthogonally Jensen quadratic functional equation (1.2) in \(F\)-spaces and quasi-Banach spaces.

## 2 Stability of the Jensen additive functional equations

Applying some ideas from \([22]\), we deal with the conditional stability problem for (1.1).

First, we give the following lemma which is important for our main results in this paper.

**Lemma 2.1.** Let \(X\) be an Abelian group, and \(Y\) be a \(\beta\)-homogeneous \(F\)-space. For \(\varepsilon \geq 0\), assume \(f : X \rightarrow Y\) be a mapping such that for all \(x, y \in X\) and a constant \(C > 0\) one has
\[
\|f(2x) - \frac{3}{8}f(4x) + \frac{1}{8}f(-4x)\| \leq C. \quad (2.1)
\]

Let
\[
h(x, n) = \left\| f(2x) - \frac{2^n + 1}{2 \cdot 4^n} f\left(2^{n+1}x\right) + \frac{2^n - 1}{2 \cdot 4^n} f\left(-2^{n+1}x\right) \right\|
\]
and
\[ g_n(x) = \frac{2^n + 1}{2 \cdot 4^n} f(2^n x) - \frac{2^n - 1}{2 \cdot 4^n} f(-2^n x), \quad n \in \mathbb{N}. \]

Then we have that
\[(1) \ |h(x, n + 1) - h(x, n)| \leq C \left[ \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^\beta + \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^\beta \right] \quad \text{and moreover, we have}
\]
\[ h(x, n) \leq C \left( \sum_{n=1}^\infty \left[ \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^\beta + \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^\beta \right] + 1 \right) \quad (2.2) \]

for all \( n \in \mathbb{N}. \)

(2) \( (g_n(x))_{n \in \mathbb{N}} \) is a Cauchy sequence for every \( x \in X \). Hence, the mapping \( g : X \to Y \) can be defined as:
\[ g(x) := \lim_{n \to \infty} g_n(x) \]

and then we have
\[ \|f(2x) - g(2x)\| \leq C \left( \sum_{n=1}^\infty \left[ \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^\beta + \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^\beta \right] + 1 \right) \]

for all \( x \in X. \)

Proof. First, with the help of (2.1), through a simple estimate we obtain
\[
\left\| f(2x) - \frac{2^n + 1}{2 \cdot 4^n} f(2^{n+1} x) + \frac{2^n - 1}{2 \cdot 4^n} f(-2^{n+1} x) \right\|
\leq \left\| f(2x) - \frac{2^n + 1}{2 \cdot 4^n} f(2^n x) + \frac{2^n - 1}{2 \cdot 4^n} f(-2^n x) \right\|
+ \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^\beta \left\| f(2^n+1 x) - \frac{3}{8} f(2^{n+2} x) + \frac{1}{8} f(-2^{n+2} x) \right\|
+ \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^\beta \left\| f(-2^{n+1} x) - \frac{3}{8} f(-2^{n+2} x) + \frac{1}{8} f(2^{n+2} x) \right\|
\leq \left\| f(2x) - \frac{2^n + 1}{2 \cdot 4^n} f(2^n x) + \frac{2^n - 1}{2 \cdot 4^n} f(-2^n x) \right\|
+ C \left[ \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^\beta + \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^\beta \right]
\]

which implies
\[
\left\| f(2x) - \frac{2^n + 1}{2 \cdot 4^n} f(2^{n+1} x) + \frac{2^n - 1}{2 \cdot 4^n} f(-2^{n+1} x) \right\|
- \left\| f(2x) - \frac{2^n + 1}{2 \cdot 4^n} f(2^n x) + \frac{2^n - 1}{2 \cdot 4^n} f(-2^n x) \right\|
\leq C \left[ \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^\beta + \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^\beta \right]
\]

Now we let
\[ h(x, n) = \left\| f(2x) - \frac{2^n + 1}{2 \cdot 4^n} f(2^{n+1} x) + \frac{2^n - 1}{2 \cdot 4^n} f(-2^{n+1} x) \right\|, \]
so we have that
\[ |h(x, n + 1) - h(x, n)| \leq C \left[ \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^\beta + \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^\beta \right], \]
and then
\[
h(x, n) = \sum_{i=2}^{n} (h(x, i) - h(x, i - 1)) + h(x, 1) 
\leq C \left( \sum_{n=1}^{\infty} \left[ \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^\beta + \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^\beta \right] + 1 \right).
\]
This means that
\[
\begin{align*}
\|f(2x) - 2^n f(2^n x) + 2^n f(-2^n x)\| &\leq C \left( \sum_{n=1}^{\infty} \left[ \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^\beta + \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^\beta \right] + 1 \right), 
\quad x \in X.
\end{align*}
\]
The next step is to prove that for each \( x \in X \) the sequence
\[ g_n(x) := \frac{2^n + 1}{2 \cdot 4^n} f(2^n x) - \frac{2^n - 1}{2 \cdot 4^n} f(-2^n x), \quad n \in \mathbb{N} \]
is convergent in \( Y \). Since \( Y \) is complete, it suffices to show that \( (g_n(x))_{n \in \mathbb{N}} \) is a Cauchy sequence for every \( x \in X \). Applying estimate (2.1) twice then we have
\[
\|g_n(x) - g_{n+1}(x)\| = \left\| \frac{2^n + 1}{2 \cdot 4^n} \left( f(2^n x) - \frac{3}{8} f(2^{n+1} x) + \frac{1}{8} f(-2^{n+1} x) \right) - \frac{2^n - 1}{2 \cdot 4^n} \left( f(-2^n x) - \frac{3}{8} f(-2^{n+1} x) + \frac{1}{8} f(2^{n+1} x) \right) \right\| 
\leq C \left[ \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^\beta + \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^\beta \right]
\]
for each \( n \in \mathbb{N} \), which gives us that \( (g_n(x))_{n \in \mathbb{N}} \) is a Cauchy sequence.

Hence, the mapping \( g : X \to Y \) can be defined as:
\[ g(x) := \lim_{n \to \infty} g_n(x) \]
for all \( x \in X \). Combining with (2.1) we have
\[
\|f(2x) - g(2x)\| \leq C \left( \sum_{n=1}^{\infty} \left[ \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^\beta + \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^\beta \right] + 1 \right)
\]
with \( x \in X \). \( \square \)

Next, we would like to show the main result of this section: In this section, let \( X \) be an Abelian group and let \( \perp \) be a binary relation defined on \( X \) with the properties:
(a) for all \( x \in X \), \( 0 \perp x \) or \( x \perp 0 \);
(b) if \( x, y \in X \) and \( x \perp y \), then \( -x \perp -y \) and \( 2x \perp 2y \).
Theorem 2.1. Let $X$ be an Abelian group, and $Y$ be a $\beta$-homogeneous $F$-space. For $\varepsilon \geq 0$, assume $f : X \to Y$ be a mapping such that for all $x, y \in X$ one has

$$x \perp y \implies \|2f \left( \frac{x + y}{2} \right) - f(x) - f(y)\| \leq \varepsilon$$

(2.3)

and

$$\|f(x) + f(-x)\| \leq \varepsilon.$$  

(2.4)

Then there exists a mapping $g : X \to Y$ such that

$$x \perp y \implies 2g \left( \frac{x + y}{2} \right) = g(x) + g(y)$$

(2.5)

and

$$\|f(x) - g(x)\| \leq \left( \sum_{n=1}^{\infty} \left[ \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^\beta + \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^\beta \right] + 1 \right) \cdot \frac{1 + 2^\beta + 4^\beta + 8^\beta + 16^\beta}{8^\beta} \cdot \varepsilon$$

(2.6)

for all $x \in 2X = \{ 2x : x \in X \}$. Moreover, the mapping $g$ is unique on the set $2X$.

Proof. It is not hard to get $\|f(0)\| \leq \frac{1}{2^\beta} \varepsilon$ with the help of (2.4). Then for all $x \in X$, by (a), we have $0 \perp x$ or $x \perp 0$, so combined with (2.3) we can write

$$\|2f \left( \frac{x}{2} \right) - f(x)\| \leq \left( \frac{1}{2^\beta} + 1 \right) \varepsilon$$

and then we have

$$\|2f(x) - f(2x)\| \leq \left( \frac{1}{2^\beta} + 1 \right) \varepsilon.$$

Then it is easy to get the following inequation:

$$\|3f(4x) - 8f(2x) - f(-4x)\|$$

$$= \|4 \left[ f(4x) - 2f(x) \right] + 16 \left[ f(2x) - 2f(x) \right] + \left[ f(-4x) + f(4x) \right] \|$$

$$\leq 4^\beta \cdot \left( \frac{1}{2^\beta} + 1 \right) \varepsilon + 16^\beta \cdot \left( \frac{1}{2^\beta} + 1 \right) \varepsilon + \varepsilon$$

$$= (1 + 2^\beta + 4^\beta + 8^\beta + 16^\beta) \varepsilon.$$

This proves that

$$\left\| f(2x) - \frac{3}{8}f(4x) + \frac{1}{8}f(-4x) \right\| \leq \left( 1 + 2^\beta + 4^\beta + 8^\beta + 16^\beta \right) \cdot \frac{\varepsilon}{8^\beta}, \quad x \in X.$$  

(2.7)

According to Lemma 2.1, we can easily get

$$\left\| f(2x) - \frac{2^n + 1}{2 \cdot 4^n} f \left( \frac{2^n + 1}{2} \cdot x \right) + \frac{2^n - 1}{2 \cdot 4^n} f \left( \frac{2^n - 1}{2} \cdot x \right) \right\|$$

$$\leq \left( \sum_{n=1}^{\infty} \left[ \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^\beta + \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^\beta \right] + 1 \right) \cdot \frac{1 + 2^\beta + 4^\beta + 8^\beta + 16^\beta}{8^\beta} \cdot \varepsilon$$

(2.8)

for $x \in X$ and $n \in \mathbb{N}$. Moreover, for each $x \in X$ the sequence

$$g_n(x) := \frac{2^n + 1}{2 \cdot 4^n} f \left( \frac{2^n + 1}{2} \cdot x \right) - \frac{2^n - 1}{2 \cdot 4^n} f \left( \frac{2^n - 1}{2} \cdot x \right), \quad n \in \mathbb{N}$$

is convergent in $Y$. 


Hence, the mapping \( g : X \rightarrow Y \) can be defined as:
\[
g(x) := \lim_{n \rightarrow \infty} g_n(x)
\]
for all \( x \in X \). Combining with (2.8) we have
\[
\|f(2x) - g(2x)\| \\
\leq \left( \sum_{n=1}^{\infty} \left[ \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^\beta + \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^\beta \right] + 1 \right) \cdot \frac{1 + 2\beta + 4\beta + 8\beta + 16\beta}{8\beta} \cdot \varepsilon
\]
with \( x \in X \).

In order to prove that \( g \) is orthogonally additive observe first that for \( x, y \in X \) such that \( x \perp y \) and \( n \in \mathbb{N}, n > 1 \) we have
\[
\frac{2^n + 1}{2 \cdot 4^n} \cdot 2f \left( \frac{2^n(x + y)}{2} \right) - \frac{2^n - 1}{2 \cdot 4^n} \cdot 2f \left( \frac{-2^n(x + y)}{2} \right) \\
- \frac{2^n + 1}{2 \cdot 4^n} f(2^n x) + \frac{2^n - 1}{2 \cdot 4^n} f(-2^n x) - \frac{2^n + 1}{2 \cdot 4^n} f(2^n y) + \frac{2^n - 1}{2 \cdot 4^n} f(-2^n y)
\]
\[
\leq \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^\beta \left\| 2f \left( \frac{2^n(x + y)}{2} \right) - f(2^n x) - f(2^n y) \right\| \\
+ \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^\beta \left\| 2f \left( \frac{2^n(-x - y)}{2} \right) - f(-2^n x) - f(-2^n y) \right\|
\]
\[
\leq \left[ \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^\beta + \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^\beta \right] \cdot \varepsilon.
\]
Moreover, letting \( n \rightarrow \infty \), we get (2.5).

Now, we show the uniqueness of \( g \). Assuming \( g' \) as another mapping satisfying (2.5) and (3.1) that yields:
\[
\|g(x) - g'(x)\| \\
\leq \|g(x) - f(x)\| + \|g'(x) - f(x)\| \\
\leq 2 \left( \sum_{n=1}^{\infty} \left[ \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^\beta + \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^\beta \right] + 1 \right) \cdot \frac{1 + 2\beta + 4\beta + 8\beta + 16\beta}{8\beta} \cdot \varepsilon
\]
for all \( x \in 2X \).

On the other hand, the mapping \( g - g' \) satisfies (2.5) and thus, in particular, (2.8) with \( \varepsilon = 0 \). By applying (2.8) to \( g - g' \) we see that
\[
g(2x) - g'(2x) = \frac{2^n + 1}{2 \cdot 4^n} \left[ g \left( 2^{n+1} x \right) - g' \left( 2^{n+1} x \right) \right] - \frac{2^n - 1}{2 \cdot 4^n} \left[ g \left( -2^{n+1} x \right) - g' \left( -2^{n+1} x \right) \right]
\]
and therefore
\[
\|g(2x) - g'(2x)\| \\
\leq \left(\frac{2^n + 1}{2 \cdot 4^n}\right)^\beta \|g(2^{n+1}x) - g'(2^{n+1}x)\| + \left(\frac{2^n - 1}{2 \cdot 4^n}\right)^\beta \|g(-2^{n+1}x) - g'(-2^{n+1}x)\|
\]
\[
\leq \left[\left(\frac{2^n + 1}{2 \cdot 4^n}\right)^\beta + \left(\frac{2^n - 1}{2 \cdot 4^n}\right)^\beta\right] \cdot \frac{\sum_{n=1}^{\infty} \left[\left(\frac{2^n + 1}{2 \cdot 4^n}\right)^\beta + \left(\frac{2^n - 1}{2 \cdot 4^n}\right)^\beta\right] + 1}{1 + 2^\beta + 4^\beta + 8^\beta + 16^\beta} \cdot \varepsilon
\]
for \(x \in X\).
Combining the both inequalities, we can easily get the thesis. □

By the same method, we can also obtain the stability result for different target spaces as the following corollary, where the space \(Y\) is equipped with quasi-norm.

**Corollary 2.1.** Let \(X\) be an Abelian group, and \(Y\) be a quasi-Banach space. For \(\varepsilon \geq 0\), assume \(f : X \to Y\) be a mapping such that for all \(x, y \in X\) one has
\[
x \perp y \implies \|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\| \leq \varepsilon
\]
and
\[
\|f(x) + f(-x)\| \leq \varepsilon.
\]

Then there exists a mapping \(g : X \to Y\) such that
\[
x \perp y \implies 2g\left(\frac{x+y}{2}\right) = g(x) + g(y)
\]
and
\[
\|f(x) - g(x)\| \leq \left(\sum_{n=1}^{\infty} \left[\left(\frac{2^n + 1}{2 \cdot 4^n}\right)^p + \left(\frac{2^n - 1}{2 \cdot 4^n}\right)^p\right] + 1\right)^\frac{1}{p} \cdot \frac{(1 + 2^p + 4^p + 8^p + 16^p)\cdot \varepsilon}{8^\beta}
\]
for all \(x \in 2X = \{2x : x \in X\}\). Moreover, the mapping \(g\) is unique on the set \(2X\).

**Proof.** Let \(\| \cdot \|_p = \| \cdot \|^p\), then it is obviously that \((Y, \| \cdot \|_p)\) is \(p\)-homogeneous, we obtain
\[
x \perp y \implies \|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\|_p \leq \varepsilon^p
\]
and
\[
\|f(x) + f(-x)\|_p \leq \varepsilon^p.
\]
According to Theorem 2.1, we obtain that there exists a mapping \(g : X \to Y\) such that
\[
x \perp y \implies 2g\left(\frac{x+y}{2}\right) = g(x) + g(y)
\]
and
\[
\|f(x) - g(x)\|_p \leq \left(\sum_{n=1}^{\infty} \left[\left(\frac{2^n + 1}{2 \cdot 4^n}\right)^\beta + \left(\frac{2^n - 1}{2 \cdot 4^n}\right)^\beta\right] + 1\right) \cdot \frac{1 + 2^\beta + 4^\beta + 8^\beta + 16^\beta}{8^\beta} \cdot \varepsilon^p
\]
for all \(x \in 2X = \{2x : x \in X\}\). Moreover, the mapping \(g\) is unique on the set \(2X\) and the claim follows. □
3 Stability of the orthogonally Jensen quadratic functional equation

Applying and extending some ideas from [22], we deal with the conditional stability problem for (1.2).

The main result of this section: In this section, let \( X \) be an Abelian group and let \( \perp \) be a binary relation defined on \( X \) with the properties:

\((a')\) for all \( x \in X \), \( 0 \perp x \) and \( x \perp 0 \);
\((b')\) if \( x, y \in X \) and \( x \perp y \), then \( -x \perp -y \) and \( 2x \perp 2y \).

**Theorem 3.1.** Let \( X \) be an Abelian group, and \( Y \) be a \( \beta \)-homogeneous \( F \)-space. For \( \varepsilon \geq 0 \), assume \( f : X \rightarrow Y \) be a mapping such that for all \( x, y \in X \) one has

\[ x \perp y \implies \left\| 2f \left( \frac{x+y}{2} \right) + 2f \left( \frac{x-y}{2} \right) - f(x) - f(y) \right\| \leq \varepsilon \]  

(3.1)

Then there exists a mapping \( g : X \rightarrow Y \) such that

\[ x \perp y \implies 2g \left( \frac{x+y}{2} \right) + 2g \left( \frac{x-y}{2} \right) = g(x) + g(y) \]  

(3.2)

and

\[ \left\| f(x) - g(x) \right\| \leq \left( \sum_{n=1}^{\infty} \left[ \left( \frac{2^n+1}{2 \cdot 4^n} \right)^{\beta} + \left( \frac{2^n-1}{2 \cdot 4^n} \right)^{\beta} \right] + 1 \right) \cdot \frac{1 + 2\beta + 2^{1-\beta} + 2^{1-2\beta}}{8\beta} \cdot \varepsilon \]  

(3.3)

for all \( x \in 2X = \{ 2x : x \in X \} \). Moreover, the mapping \( g \) is unique on the set \( 2X \).

**Proof.** For all \( x \in X \). By \((a')\), we have \( 0 \perp 0 \), \( 0 \perp x \) and \( x \perp 0 \), so according to \( 0 \perp 0 \) and (3.1) we have

\[ \left\| 2f(0) + 2f(0) - f(0) - f(0) \right\| \leq \varepsilon \]

and then we have

\[ \left\| f(0) \right\| \leq \frac{\varepsilon}{2^\beta}. \]  

(3.4)

According to \( 0 \perp x \), we can write

\[ \left\| 2f \left( \frac{x}{2} \right) + 2f \left( -\frac{x}{2} \right) - f(0) - f(x) \right\| \leq \varepsilon \]

it is easy to get

\[ \left\| 2f \left( \frac{x}{2} \right) + 2f \left( -\frac{x}{2} \right) - f(x) \right\| \leq \left( \frac{1}{2^\beta} + 1 \right)\varepsilon \]  

(3.5)

Since \( x \perp 0 \), we have

\[ \left\| 2f \left( \frac{x}{2} \right) + 2f \left( \frac{x}{2} \right) - f(x) - f(0) \right\| \leq \varepsilon \]

combined with (3.4), it is obvious that

\[ \left\| 4f \left( \frac{x}{2} \right) - f(x) \right\| \leq \left( \frac{1}{2^\beta} + 1 \right)\varepsilon \]  

(3.6)

On the other hand, by (3.5) and (3.6), we have

\[ \left\| 2f \left( \frac{x}{2} \right) + 2f \left( -\frac{x}{2} \right) - 4f \left( \frac{x}{2} \right) \right\| \leq 2\left( \frac{1}{2^\beta} + 1 \right)\varepsilon \]
so we have
\[ \| f \left( \frac{x}{2} \right) - f \left( -\frac{x}{2} \right) \| \leq 2^{1-\beta} (\frac{1}{2\beta} + 1) \varepsilon \]
this means that
\[ \| f(x) - f(-x) \| \leq 2^{1-\beta} (\frac{1}{2\beta} + 1) \varepsilon. \]
Then it is easy to get
\[
\begin{align*}
\| 3f(4x) - 8f(2x) - f(-4x) \| &= \| 2f(4x) - 8f(2x) + f(4x) - f(-4x) \| \\
&\leq 2^{\beta} \left( \frac{1}{2\beta} + 1 \right) \varepsilon + 2^{1-\beta} \left( \frac{1}{2\beta} + 1 \right) \varepsilon \\
&= (1 + 2^{\beta} + 2^{1-\beta} + 2^{1-2\beta}) \varepsilon.
\end{align*}
\]
This proves that
\[
\begin{align*}
\| f(2x) - \frac{3}{8} f(4x) + \frac{1}{8} f(-4x) \| &\leq \frac{1 + 2^{\beta} + 2^{1-\beta} + 2^{1-2\beta}}{8^{\beta}} \varepsilon, \quad x \in X. \tag{3.7}
\end{align*}
\]
According to Lemma 2.1, we can easily get
\[
\begin{align*}
\| f(2x) - \frac{2^n + 1}{2 \cdot 4^n} f(2^{n+1}x) + \frac{2^n - 1}{2 \cdot 4^n} f(-2^{n+1}x) \| &\leq \sum_{n=1}^{\infty} \left[ \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^\beta + \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^\beta \right] + 1 \cdot \frac{1 + 2^{\beta} + 2^{1-\beta} + 2^{1-2\beta}}{8^{\beta}} \varepsilon. \tag{3.8}
\end{align*}
\]
for \( x \in X \) and \( n \in \mathbb{N} \). Moreover, for each \( x \in X \) the sequence
\[ g_n(x) := \frac{2^n + 1}{2 \cdot 4^n} f(2^n x) - \frac{2^n - 1}{2 \cdot 4^n} f(-2^n x), \quad n \in \mathbb{N} \]
is convergent in \( Y \).
Hence, the mapping \( g : X \to Y \) can be defined as:
\[ g(x) := \lim_{n \to \infty} g_n(x) \]
for all \( x \in X \). Combining with (3.8) we have
\[
\| f(2x) - g(2x) \| \leq \sum_{n=1}^{\infty} \left[ \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^\beta + \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^\beta \right] + 1 \cdot \frac{1 + 2^{\beta} + 2^{1-\beta} + 2^{1-2\beta}}{8^{\beta}} \varepsilon, \quad x \in X.
\]
In order to prove that \( g \) is orthogonally additive observe first that for \( x, y \in X \) such that \( x \perp y \)
and $n \in N, n > 1$ we have

$$\left\| 2g_n \left( \frac{x+y}{2} \right) + 2g_n \left( \frac{x-y}{2} \right) - g_n(x) - g_n(y) \right\|$$

$$= \left\| \frac{2^n + 1}{2 \cdot 4^n} \cdot 2f \left( \frac{2^n(x+y)}{2} \right) - \frac{2^n - 1}{2 \cdot 4^n} \cdot 2f \left( \frac{2^n(x+y)}{2} \right) \right. - \left. \frac{2^n + 1}{2 \cdot 4^n} \cdot 2f \left( \frac{2^n(x-y)}{2} \right) - \frac{2^n - 1}{2 \cdot 4^n} \cdot 2f \left( \frac{2^n(x-y)}{2} \right) \right\|$$

$$= \left\| \frac{2^n + 1}{2 \cdot 4^n} \cdot \left[ 2f \left( \frac{2^n(x+y)}{2} \right) + 2f \left( \frac{2^n(x-y)}{2} \right) - f (2^n x) - f (2^n y) \right] \right. - \left. \frac{2^n - 1}{2 \cdot 4^n} \cdot \left[ 2f \left( \frac{2^n(-x+y)}{2} \right) + 2f \left( \frac{2^n(-x-y)}{2} \right) - f (-2^n x) - f (-2^n y) \right] \right\|$$

$$\leq \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^\beta \left\| 2f \left( \frac{2^n(x+y)}{2} \right) + 2f \left( \frac{2^n(x-y)}{2} \right) - f (2^n x) - f (2^n y) \right\| + \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^\beta \left\| 2f \left( \frac{2^n(-x+y)}{2} \right) + 2f \left( \frac{2^n(-x-y)}{2} \right) - f (-2^n x) - f (-2^n y) \right\|$$

$$\leq \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^\beta + \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^\beta \varepsilon.$$ 

Moreover, letting $n \to \infty$, we get $\|g\| = 0$. 

Now, we show the uniqueness of $g$. Assuming $g'$ as another mapping satisfying (3.2) and (3.3) that yields:

$$\|g(x) - g'(x)\|$$

$$\leq \|g(x) - f(x)\| + \|g'(x) - f(x)\|$$

$$\leq 2 \left( \sum_{n=1}^{\infty} \left[ \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^\beta + \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^\beta \right] + 1 \right) \cdot \frac{1 + 2^\beta + 2^{1-\beta} + 2^{1-2\beta}}{8^\beta} \varepsilon$$

for all $x \in 2X$.

On the other hand, the mapping $g - g'$ satisfies (3.2) and thus, in particular, (3.1) with $\varepsilon = 0$. By applying (3.3) to $g - g'$ we see that

$$g(2x) - g'(2x) = \frac{2^n + 1}{2 \cdot 4^n} \left[ g \left( 2^{n+1} x \right) - g' \left( 2^{n+1} x \right) \right]$$

and therefore

$$\|g(2x) - g'(2x)\|$$

$$\leq \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^\beta \|g \left( 2^{n+1} x \right) - g' \left( 2^{n+1} x \right)\| + \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^\beta \|g \left( -2^{n+1} x \right) - g' \left( -2^{n+1} x \right)\|$$

$$\leq \left[ \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^\beta + \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^\beta \right] \cdot 2 \left( \sum_{n=1}^{\infty} \left[ \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^\beta + \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^\beta \right] + 1 \right) \cdot \frac{1 + 2^\beta + 2^{1-\beta} + 2^{1-2\beta}}{8^\beta} \varepsilon$$

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for \( x \in X \).

Combining the both inequalities, we can easily get the thesis. \( \square \)

By the same method, we can also obtain the stability result for different target spaces as the following corollary, where the space \( Y \) is equipped with quasi-norm.

**Corollary 3.1.** Let \( X \) be an Abelian group, and \( Y \) be a quasi-Banach space. For \( \varepsilon \geq 0 \), assume \( f : X \to Y \) be a mapping such that for all \( x, y \in X \) one has

\[
x \perp y \implies \|2f \left( \frac{x + y}{2} \right) + 2f \left( \frac{x - y}{2} \right) - f(x) - f(y)\| \leq \varepsilon
\]

Then there exists a mapping \( g : X \to Y \) such that

\[
x \perp y \implies 2g \left( \frac{x + y}{2} \right) + 2g \left( \frac{x - y}{2} \right) = g(x) + g(y)
\]

and

\[
\|f(x) - g(x)\| \leq \left( \sum_{n=1}^{\infty} \left[ \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^p + \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^p \right] + 1 \right)^{\frac{1}{p}} \cdot \frac{1 + 2^p + 2^{1-p} + 2^{1-2p}}{8} \cdot \varepsilon
\]

for all \( x \in 2X = \{ 2x : x \in X \} \). Moreover, the mapping \( g \) is unique on the set \( 2X \).

**Proof.** Let \( \| \cdot \|_p = \| \cdot \|^p \), then it is obviously that \((Y, \| \cdot \|_p)\) is \( p \)-homogeneous, we obtain

\[
x \perp y \implies \|2f \left( \frac{x + y}{2} \right) + 2f \left( \frac{x - y}{2} \right) - f(x) - f(y)\| \leq \varepsilon^p.
\]

According to Theorem 2.1 we obtain that there exists a mapping \( g : X \to Y \) such that

\[
x \perp y \implies 2g \left( \frac{x + y}{2} \right) + 2g \left( \frac{x - y}{2} \right) = g(x) + g(y)
\]

and

\[
\|f(x) - g(x)\|_p \leq \left( \sum_{n=1}^{\infty} \left[ \left( \frac{2^n + 1}{2 \cdot 4^n} \right)^p + \left( \frac{2^n - 1}{2 \cdot 4^n} \right)^p \right] + 1 \right)^{\frac{1}{p}} \cdot \frac{1 + 2^p + 2^{1-p} + 2^{1-2p}}{8^p} \cdot \varepsilon^p
\]

for all \( x \in 2X = \{ 2x : x \in X \} \). Moreover, the mapping \( g \) is unique on the set \( 2X \) and the claim follows. \( \square \)

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