Action principle for the Z4 and BSSN numerical relativity formalisms

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Abstract. A Lagrangian density is provided, that allows to derive the Z4 evolution system from a Palatini-type variation principle. The proposed action includes the supplementary vector \(Z\). The constraint \(Z_{\mu} = 0\) can be imposed just on the initial data, in order to recover true Einstein’s solutions. This opens the door to analogous results for other numerical-relativity formalisms, like BSSN, that can be derived from Z4 by a symmetry-breaking procedure.

The action principle for General Relativity was stated by Hilbert since the very beginning of the theory in the Lagrangian form, although the Hamiltonian formulation had to wait for decades [1, 2]. The reason for this long delay is probably related to the complexity of the Cauchy problem for Einstein’s equations, which becomes manifest in the 3+1 (space plus time) decomposition [2]. The coordinate gauge freedom produces a mismatch between the number of dynamical fields and that of true evolution equations: four of the field equations are indeed (energy-momentum) constraints. This rich structure opens the door to many different approaches.

Using this freedom, by the end of the past century, some hyperbolic extensions of Einstein’s equations were developed with a view on numerical relativity applications [3, 4, 5, 6]. This emergent field is now more mature: there are two main formalisms currently used in numerical simulations. One is BSSN [7, 8], working at the 3+1 level, and the other is the class of generalized harmonic formalisms [9, 10, 11], working at the four-dimensional level. A unifying framework is provided by the Z4 formalism [12], which allows to recover the generalized harmonic one by relating the additional vector field \(Z_{\mu}\) with the harmonic ’gauge sources’ [9]. On the other hand, it allows to recover (a specific version of) BSSN by a symmetry-breaking process in the transition from the four-dimensional to the three-dimensional formulations [13, 14].

There is a growing interest in incorporating the new hyperbolic formulations into the Lagrangian/Hamiltonian framework. An example is the usage of the ’densitized lapse’ [3] as a canonical variable, leading to a modification in the standard form of the canonical evolution equations [15]. Reciprocally, there are very recent attempts of modifying the ADM action [2] in order to incorporate coordinate conditions of the type used in numerical relativity [16, 17], with a view on using symplectic integrators for the time evolution, which could ensure constraint preservation in numerical simulations [18]. On a different context, a well posed evolution formalism developed from a Lagrangian formulation could be a good starting point for Quantum Gravity applications.

In this paper we derive the Z4 formalism from an action principle by introducing a Lagrangian
density which generalizes the Einstein-Hilbert one. This is a crucial step towards the Lagrangian formulation of other numerical-relativity formalisms. We actually consider here the BSSN case, following the symmetry-breaking mechanism described in refs. [13, 14].

1. Generalizing the Einstein-Hilbert action principle

Let us consider the generic action

$$S = \int d^4x \; \mathcal{L}$$

with a Lagrangian density which generalizes the Einstein-Hilbert one by including an extra four-vector $Z_\mu$, namely

$$\mathcal{L} = \sqrt{g} \; g^{\mu \nu} \left[ R_{\mu \nu} + 2 \nabla_\mu Z_\nu \right]$$

(we restrict ourselves to the vacuum case), with the Ricci tensor written in terms of the connection coefficients

$$R_{\mu \nu} = \partial_\mu \Gamma^\rho_{\nu \rho} - \partial_\nu \Gamma^\rho_{\mu \rho} + \Gamma^\rho_{(\mu} \Gamma^\sigma_{\nu) \rho} - \Gamma^\rho_{\sigma \mu} \Gamma^\sigma_{\rho \nu}$$

(round brackets denote symmetrization).

Now let us follow the well-known Palatini approach, by considering independent variations of the metric density $h^{\mu \nu} = \sqrt{g} \; g^{\mu \nu}$, the connection coefficients $\Gamma^\rho_{\mu \nu}$ and the vector $Z_\mu$. From the $h^{\mu \nu}$ variations we get directly the Z4 field equations [12]

$$R_{\mu \nu} + \nabla_\mu Z_\nu + \nabla_\nu Z_\mu = 0,$$

which are currently used in many numerical-relativity developments [14].

From the $\Gamma^\rho_{\mu \nu}$ and the $Z_\mu$ variations we get a coupled set of equations which can be decomposed in a covariant way into the tensor equation

$$\nabla_\rho g^{\mu \nu} = 0,$$

which fixes the connection coefficients in terms of the metric, and the vector condition

$$Z_\mu = 0.$$

Let us note here the different role of the conditions (5) and (6). As there are much more independent connection coefficients than evolution equations in (4), we will consider condition (5) as a constraint enforcing the metric connection 'a posteriori', that is after the variation process. In this way, we will ensure that equations (4) are identical to the original Z4 equations, rather than some affine generalization. For this reason, we will assume a metric connection everywhere in what follows.

The case of condition (6) is different, as the Z4 equations (4) actually provide evolution equations for every component of $Z_\mu$. Then, (6) is a standard primary constraint and we have a choice among different strategies for dealing with it. If we enforce (6) into the Z4 field equations (4), we get nothing but Einstein’s equations. This is not surprising because our Lagrangian obviously reduces to the Einstein-Hilbert one when $Z_\mu$ vanishes. The problem is that the plain Einstein field equations do not lead directly to a well-posed initial data problem. This is why the original harmonic formulation [19, 20, 21] was used instead in the context of the Cauchy problem [22]. For the same reason, other formulations (BSSN [7, 8], generalized harmonic [9, 10, 11], Z4 [12, 13]) are currently considered in numerical relativity.
2. Recovering the Z4 formulation
We can alternatively follow a different strategy. Instead of enforcing (6), we can deal with this condition as an algebraic restriction to be imposed just on the initial data, that is

\[ Z_\mu \mid_{t=0} = 0 \quad (7) \]

In this way, we can keep the Z4 field equations system, which is known to be strongly hyperbolic when supplemented by gauge conditions, like '1+log' or 'freezing shift', suitable for numerical evolution [13, 14]. The consistency of this 'relaxed' approach requires that the constraint (6) should be actually preserved by the Z4 field equations (4). In this way, the solutions obtained from initial data verifying (6) will actually minimize the proposed action (1).

Allowing for the conservation of the Einstein tensor, which is granted after the metric connection enforcement, we derive from (4) the second-order equation, linear-homogeneous in \( Z \)

\[ \nabla_\nu \left[ \nabla^\nu Z_\mu + \nabla^\nu Z_\nu - (\nabla_\nu Z^\nu) g_\nu^{\mu} \right] = 0. \quad (8) \]

It follows that the necessary and sufficient condition for the preservation of the constraint (6) is to impose also its first time-derivative conditions in the initial data, that is

\[ (\partial_0 Z_\mu) \mid_{t=0} = 0. \quad (9) \]

Note that, allowing for (7) and the Z4 field equations, the secondary constraints (9) amount to the standard energy and momentum constraints, which are then to be imposed on the initial data in addition to (7).

This 'relaxed' treatment of the constraints (6) may look unnatural. But it is just the reflection of a common practice numerical relativity ('free evolution' approach), where four of the ten field equations (the energy-momentum constraints) are not enforced during the evolution, being imposed just in the initial data instead. The introduction of the extra four-vector in the Z4 formalism actually provides a simpler implementation of the same idea.

3. Recovering the Z3-BSSN formulation
Note that in all our developments we have preserved general covariance. Our action integral (1) is a true scalar and, in spite of other alternatives, we have avoided the addition of total divergences which could have simplified our developments to some extent, at the price of adding boundary terms. This means that we keep at this point the full coordinate-gauge freedom.

However, as it is well known, the BSSN formulation is not general covariant. It contains just three additional 'contracted-gamma' quantities, associated to the momentum constraint. But the 3+1 splitting between energy and momentum requires a specific choice for the time coordinate, which breaks general covariance. In refs. [13, 14] a symmetry breaking mechanism is proposed, which allows to recover BSSN from the 'Z3 system' [23], which is obtained in turn from the Z4 one by enforcing the energy constraint in the form

\[ Z^0 = 0. \quad (10) \]

We can proceed now like in the preceding section, by the following steps:

- Enforcing both the metric connection condition (5) and (10) in the Z4 field equations.
- Splitting the ten resulting equations into the (energy-related) secondary constraint

\[ \mathcal{E} \equiv \partial_t Z^0 = 0 \quad (11) \]

plus nine evolution equations for the six space metric components and the space vector \( Z_i \).
Note that this splitting is not unique, as any multiple of $E$ (times the metric) can be added to the evolution equations, resulting into a modified evolution system, with a different principal part. In this way we obtain the one-parameter family of (generalized) Z3 evolution systems [13, 14]. For a specific value of the parameter, one can recover a flavor of the BSSN formalism (Z3-BSSN) by means of a conformal decomposition of the spatial tensors and the redefinition of space vector $Z_i$ in terms of the equivalent ‘contracted-Gamma’ quantities, namely

$$\tilde{\Gamma}_i = -\tilde{\gamma}_{ik} \tilde{\gamma}^{jk} + 2 \ Z_i$$

(12)

where $\tilde{\gamma}_{ij}$ is the conformal space metric. The space vector $Z_i$ can be interpreted in this context as the difference between the (contracted) conformal metric connection and the BSSN contracted-Gamma quantities. This provides actually time a three-covariant reformulation of the original BSSN formalism.

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