The Hyers-Ulam stability for nonlinear Volterra integral equations via a generalized Diaz-Margolis’s fixed point theorem

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Abstract: In this work, we prove an existence theorem of the Hyers-Ulam stability for the nonlinear Volterra integral equations which improves and generalizes Castro-Ramos theorem by using some weak conditions.

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1. Introduction and preliminaries

The stability of functional equations was originally raised by Ulam in 1940 and the problem posed by Ulam was the following: ”Under what conditions does there exist an additive mapping near an approximately additive mapping?” (we refer the reader to [1] for details). In 1941, Hyers [2] gave the first answer to Ulam’s question in the case of Banach spaces. Since then a number of generalizations in the study of stability of functional equations have been investigated by several authors; see [3-5] and references therein. A generalized Banach contraction principle in a complete generalized metric space proved by Diaz and Margolis [6] has played an important role in the study of stability of functional equations.

Definition 1.1. [6] Let $X$ be a nonempty set. A function $p : X \times X \to [0, \infty]$ is called a generalized metric on $X$ if the following conditions hold:

(GM1) $p(x, y) = 0$ if and only if $x = y$;

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The pair \((X, p)\) is then called a \textit{generalized metric space}.

We remark that the only one difference of the generalized metric from the usual metric is that the range of the former is permitted to include the infinity.

\textbf{Theorem 1.1. (Diaz and Margolis [6])} Let \((X, p)\) be a complete generalized metric space and \(T : X \to X\) be a selfmap on \(X\). Assume that there exists a nonnegative real number \(\lambda < 1\) such that

\[ p(Tx, Ty) \leq \lambda p(x, y) \quad \text{for all } x, y \in X. \]

Denote \(T^0 = I\), the identity mapping. Then, for a given element \(u \in X\), exactly one of the following assertions is true:

(a) \(p(T^nu, T^{n+1}u) = \infty\) for all \(n \in \mathbb{N} \cup \{0\}\),

(b) there exists a nonnegative integer \(\ell\) such that \(p(T^nu, T^{n+1}u) < \infty\) for all \(n \geq \ell\).

Actually, if the assertion (b) holds, then

(b1) the sequence \(\{T^n u\}_{n \in \mathbb{N} \cup \{0\}}\) is convergent to a fixed point \(\hat{y}\) of \(T\).

(b2) \(\hat{y}\) is the unique fixed point of \(T\) in the set \(S\), where

\[ S = \{x \in X : p(T^\ell u, x) < \infty\}; \]

(b3) \(p(x, \hat{y}) \leq \frac{1}{1-\lambda} p(x, Tx)\) for all \(x \in S\).

\textbf{Definition 1.2. [7, 8]} A function \(\varphi : [0, \infty) \to [0, 1)\) is said to be an \textit{MT-function} (or \textit{R-function}) if \(\limsup_{s \to t^+} \varphi(s) < 1\) for all \(t \in [0, \infty)\).

It is obvious that if \(\varphi : [0, \infty) \to [0, 1)\) is a nondecreasing function or a nonincreasing function, then \(\varphi\) is an \(\mathcal{MT}\)-function. So the set of \(\mathcal{MT}\)-functions is a rich class. In 2012, Du [8] proved the following characterizations of \(\mathcal{MT}\)-functions.

\textbf{Theorem 1.2. [8]} Let \(\varphi : [0, \infty) \to [0, 1)\) be a function. Then the following statements are equivalent.
(a) $\varphi$ is an $MT$-function.

(b) For each $t \in [0, \infty)$, there exist $r_t^{(1)} \in [0, 1)$ and $\varepsilon_t^{(1)} > 0$ such that $\varphi(s) \leq r_t^{(1)}$ for all $s \in (t, t + \varepsilon_t^{(1)})$.

(c) For each $t \in [0, \infty)$, there exist $r_t^{(2)} \in [0, 1)$ and $\varepsilon_t^{(2)} > 0$ such that $\varphi(s) \leq r_t^{(2)}$ for all $s \in [t, t + \varepsilon_t^{(2)}]$.

(d) For each $t \in [0, \infty)$, there exist $r_t^{(3)} \in [0, 1)$ and $\varepsilon_t^{(3)} > 0$ such that $\varphi(s) \leq r_t^{(3)}$ for all $s \in (t, t + \varepsilon_t^{(3)})$.

(e) For each $t \in [0, \infty)$, there exist $r_t^{(4)} \in [0, 1)$ and $\varepsilon_t^{(4)} > 0$ such that $\varphi(s) \leq r_t^{(4)}$ for all $s \in [t, t + \varepsilon_t^{(4)}]$.

(f) For any nonincreasing sequence $\{x_n\}_{n \in \mathbb{N}}$ in $[0, \infty)$, we have $0 \leq \sup_{n \in \mathbb{N}} \varphi(x_n) < 1$.

(g) $\varphi$ is a function of contractive factor; that is, for any strictly decreasing sequence $\{x_n\}_{n \in \mathbb{N}}$ in $[0, \infty)$, we have $0 \leq \sup_{n \in \mathbb{N}} \varphi(x_n) < 1$.

In 2009, Castro and Ramos [9] proved the following existence theorem of the Hyers-Ulam stability for the nonlinear Volterra integral equations.

**Theorem 1.3.** [9, Theorem 5.1] Let $a$ and $b$ be given real numbers with $a < b$ and let $K := b - a$. Assume that there exists a positive constant $L$ such that

$$0 < KL < 1.$$  

Assume that $f : [a, b] \times [a, b] \times \mathbb{C} \to \mathbb{C}$ is a continuous function satisfying

$$|f(x, \tau, y) - f(x, \tau, z)| \leq L |y - z|$$  

for any $x, \tau \in [a, b]$ and $y, z \in \mathbb{C}$.

If there exists a continuous function $y : [a, b] \to \mathbb{C}$ satisfying

$$\left| y(x) - \int_a^x f(x, \tau, y(\tau))d\tau \right| \leq \theta$$

for each $x \in [a, b]$ and some constant $\theta \geq 0$, then there exists a unique continuous function $y_0 : [a, b] \to \mathbb{C}$ such that

$$y_0(x) = \int_a^x f(x, \tau, y_0(\tau))d\tau$$
and
\[ |y(x) - y_0(x)| \leq \frac{\theta}{1 - KL} \]
for all \( x \in [a, b] \).

In this work, we give a generalization of Castro-Ramos theorem by using some weak conditions.

2. Main results

Very recently, Du [10] established the following generalization of Diaz-Margolis’s fixed point theorem.

**Theorem 2.1.** [10] Let \((X, p)\) be a complete generalized metric space and \(T : X \rightarrow X\) be a selfmap on \(X\). Assume that there exists an \(MT\)-function \(\alpha : [0, \infty) \rightarrow [0, 1)\) such that
\[ p(Tx, Ty) \leq \alpha(p(x, y)) p(x, y) \]
for all \(x, y \in X\) with \(p(x, y) < \infty\).

Denote \(T^0 = I\), the identity mapping. Then, for a given element \(u \in X\), exactly one of the following assertions is true:

(a) \(p(T^n u, T^{n+1} u) = \infty\) for all \(n \in \mathbb{N} \cup \{0\}\),

(b) there exists a nonnegative integer \(\ell\) such that \(p(T^n u, T^{n+1} u) < \infty\) for all \(n \geq \ell\).

Actually, if the assertion (b) holds, then

(b1) the sequence \(\{T^n u\}_{n \in \mathbb{N} \cup \{0\}}\) is convergent to a fixed point \(v\) of \(T\).

(b2) \(v\) is the unique fixed point of \(T\) in the set \(\mathcal{L}\), where
\[ \mathcal{L} = \{x \in X : p(T^\ell u, x) < \infty\}; \]

(b3) \(p(x, v) \leq \frac{1}{1 - \alpha(p(x, v))} p(x, Tx)\) for all \(x \in \mathcal{L}\).

In this paper, we prove an existence theorem of the Hyers-Ulam stability for the nonlinear Volterra integral equations which improves and generalizes [9, Theorem 5.1].
Theorem 2.2. Let \( a \) and \( b \) be given real numbers with \( a < b \) and let \( K := b - a \). Assume that \( \varphi : \mathbb{R} \to \mathbb{R} \) is a function which is nondecreasing on \([0, \infty)\) satisfying
\[
\varphi([0, \infty)) \subseteq [0, \delta K^{-1}],
\]
for some constant \( 0 < \delta < 1 \), and \( V : [a, b] \times [a, b] \times \mathbb{C} \to \mathbb{C} \) is a continuous function satisfying
\[
|V(x, \tau, y) - V(x, \tau, z)| \leq \varphi(|y - z|)|y - z| \quad \text{for any } x, \tau \in [a, b] \text{ and } y, z \in \mathbb{C}.
\]
If there exists a continuous function \( y : [a, b] \to \mathbb{C} \) satisfying
\[
\left| y(x) - \int_a^x V(x, \tau, y(\tau))d\tau \right| \leq \theta
\]
for each \( x \in [a, b] \) and some constant \( \theta \geq 0 \), then there exists a unique continuous function \( y_0 : [a, b] \to \mathbb{C} \) such that
\[
y_0(x) = \int_a^x V(x, \tau, y_0(\tau))d\tau
\]
and
\[
|y(x) - y_0(x)| \leq \frac{\theta}{1 - \delta}
\]
for all \( x \in [a, b] \).

**Proof.** Let \( X \) denote the set of all continuous functions from \([a, b]\) to \( \mathbb{C} \). Define a function \( p : X \times X \to [0, \infty] \) by
\[
p(f, g) = \inf\{M \geq 0 : |f(x) - g(x)| \leq M \text{ for all } x \in [a, b]\},
\]
where we adopt the usual convention that \( \inf \emptyset = \infty \). Then \((X, p)\) is a complete generalized metric space. Let \( T : X \to X \) be defined by
\[
(Tf)(x) = \int_a^x V(x, \tau, f(\tau))d\tau
\]
for all \( f \in X \) and \( x \in [a, b] \). It is easy to show \( Tf \in X \) for all \( f \in X \) and this ensures that \( T \) is well defined. Define an \( \mathcal{MT} \)-function \( \alpha : [0, \infty) \to [0, 1) \) by
\[
\alpha(t) = K \varphi(t) \quad \text{for all } t \in [0, \infty).
\]
It is not hard to verify that for all \( f, g \in X \) with \( p(f, g) < \infty \),
\[
p(Tf, Tg) \leq \alpha(p(f, g))p(f, g).
\]
Take \( h \in X \). Since \( p(Tf, f) < \infty \) for all \( f \in X \), we have \( p(Th, h) < \infty \). For any \( f \in X \), since \( f \) and \( h \) are continuous on \([a, b]\), there exists a constant \( c \geq 0 \) such that

\[
|h(x) - f(x)| \leq c \quad \text{for any } x \in [a, b]
\]

which implies \( p(h, f) \leq c < \infty \). Therefore we prove

\[
X = \{ f \in X : p(h, f) < \infty \}.
\]

Applying Theorem 2.1 (b), there exists a unique \( y_0 \in X \) such that \( T^n h \xrightarrow{p} y_0 \) as \( n \to \infty \), \( Ty_0 = y_0 \) and

\[
p(f, y_0) \leq \frac{1}{1 - \alpha(p(f, y_0))}p(f, Tf) \quad \text{for all } f \in X. \tag{2.2}
\]

So

\[
y_0(x) = \int_a^x V(x, \tau, y_0(\tau))d\tau \quad \text{for all } x \in [a, b].
\]

By (2.1), we have

\[
p(y, Ty) \leq \theta. \tag{2.3}
\]

Since \( \alpha(p(y, y_0)) \leq \delta \), by (2.2) and (2.3), we get

\[
p(y, y_0) \leq \frac{1}{1 - \alpha(p(y, y_0))}p(y, Ty) \leq \frac{\theta}{1 - \delta},
\]

which deduce

\[
|y(x) - y_0(x)| \leq \frac{\theta}{1 - \delta} \quad \text{for all } x \in [a, b].
\]

The proof is completed. \( \square \)

**Remark 2.3.** [9, Theorem 5.1] is a special case of Theorem 2.2. Indeed, let \( g : \mathbb{R} \to \mathbb{R} \) be any function. Define \( \varphi : \mathbb{R} \to \mathbb{R} \) by

\[
\varphi(t) = \begin{cases} 
L, & \text{for } t \geq 0, \\
g(t), & \text{otherwise}.
\end{cases}
\]

Put \( \delta := KL \). Then all the conditions of Theorem 2.2 are satisfied and the conclusion of [9, Theorem 5.1] follows from Theorem 2.2.

**References**
[1] S.M. Ulam, A Collection of Mathematical Problems, Interscience Tracts in Pure and Applied Mathematics 8, Interscience Publishers, New York, 1960.

[2] D.H. Hyers, On the stability of the linear functional equation, Proc. Natl. Acad. Sci. USA 27 (1941) 222-224.

[3] V. Lakshmikantham and M.R.M. Rao, Theory of Integro-differential Equations, Stability and Control: Theory, Methods and Applications 1, Gordon and Breach Publ., Philadelphia, 1995.

[4] S.-M. Jung, Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis, in: Springer Optimization and Its Applications, vol. 48, Springer, New York, 2011.

[5] I.A. Rus, Ulam stability of the operatorial equations, in: Functional Equations in Mathematical Analysis, in: Springer Optim. Appl., vol. 52, Springer, New York, 2012, pp. 287-305.

[6] J.B. Diaz and B. Margolis, A fixed point theorem of the alternative, for contractions on a generalized complete metric space, Bull. Amer. Math. Soc. 74 (1968) 305-309.

[7] W.-S. Du, Some new results and generalizations in metric fixed point theory, Nonlinear Anal. 73 (2010) 1439-1446.

[8] W.-S. Du, On coincidence point and fixed point theorems for nonlinear multivalued maps, Topology and its Applications 159 (2012) 49-56.

[9] L.P. Castro and A. Ramos, Hyers-Ulam-Rassias stability for a class of nonlinear Volterra integral equations, Banach J. Math. Anal.(3) (2009), no. 1, 36-43.

[10] W.-S. Du, The generalization of Diaz-Margolis’s fixed point theorem and its application to the stability for generalized Volterra integral equations, submitted.