FREE SPACES OVER SOME PROPER METRIC SPACES

A. DALET

ABSTRACT. We prove that the Lipschitz-free space over a countable proper metric space is isometric to a dual space and has the metric approximation property. We also show that the Lipschitz-free space over a proper ultrametric space is isometric to the dual of a space which is isomorphic to $c_0(M)$. 

1. Introduction

For a pointed metric space $(M, d)$, that is a metric space with an origin 0, we denote $\text{Lip}_0(M)$ the space of Lipschitz real-valued functions on $M$ which vanish at 0. Endowed with the norm defined by the Lipschitz constant, this space is a Banach space. Moreover, its unit ball is compact for the pointwise topology, hence it is a dual space.

Let $x \in M$ and define $\delta_x \in \text{Lip}_0(M)^*$ as follows: for $f \in \text{Lip}_0(M)$, $\delta_x(f) = f(x)$. The Lipschitz-free space over $M$, denoted $\mathcal{F}(M)$, is the closed subspace of $\text{Lip}_0(M)^*$ spanned by the $\delta_x$'s: $\mathcal{F}(M) := \text{span}\{\delta_x, \ x \in M\}$. Its dual space is isometrically isomorphic to $\text{Lip}_0(M)$.

Lipschitz-free spaces are considered in [20], where they are called Arens-Eells spaces. The notation we use is due to Godefroy and Kalton [6] where they point out that despite the simplicity of the definition of $\mathcal{F}(M)$, it is not easy to study its linear structure. Although their article was published in 2003, still very little is known about Lipschitz-free spaces. One can check that the Lipschitz-free space over $\mathbb{R}$ is $L_1(\mathbb{R})$, but Naor and Schechtman proved in [17] that $\mathcal{F}(\mathbb{R}^2)$ is not isomorphic to a subspace of any $L_1$. Moreover, Godard [4] proved that $\mathcal{F}(M)$ is isometrically isomorphic to a subspace of an $L_1$-space if and only if $M$ isometrically embeds into an $\mathbb{R}$-tree. We will focus on the notion of approximation property.

A Banach space $X$ has the approximation property (AP in short) if for every positive $\varepsilon$, every $K \subset X$ compact, there exists an operator $T$ on $X$, of finite rank, such that for every $x \in K$, the norm $\|Tx - x\|$ is less than $\varepsilon$.

Let $\lambda \in [1, +\infty)$. The space $X$ has the $\lambda$-bounded approximation property ($\lambda$-BAP) if for every positive $\varepsilon$, every $K \subset X$ compact, there exists an operator $T$ on $X$, of finite rank, such that $\|T\| \leq \lambda$ and for every $x \in K$, the norm $\|Tx - x\|$ is less than $\varepsilon$.

Finally, $X$ has the metric approximation property (MAP) when it has the 1-BAP. Godefroy and Kalton [6] proved that a Banach space has the $\lambda$-BAP if and only if its Lipschitz-free space has the $\lambda$-BAP. Lancien and Pernecká [13] proved that the Lipschitz-free space over a doubling metric space has the BAP and that $\mathcal{F}(\ell_1)$ has a finite-dimensional Schauder decomposition. Hájek and Pernecká improved this last result, in [9] they obtained that $\mathcal{F}(\ell_1)$ has a Schauder basis. However, there

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are not only positive results, Godefroy and Ozawa [7] constructed a compact metric space \( (K, d) \) such that \( \mathcal{F}(K) \) fails the AP. But the author proved in [1] that in the case of countable compact metric spaces, the Lipschitz-free space always has the MAP. In this article, we will prove that the Lipschitz-free space over a countable proper metric space and over a proper ultrametric space is a dual space and has the MAP. More precisely, we show that in the case of a proper ultrametric space, the Lipschitz-free space has an isometric predual which is isomorphic to \( c_0(\mathbb{N}) \).

2. Countable proper metric spaces

A metric space is said to be proper if every closed ball is compact.

For a metric space \((M, d)\), we will denote by \( B(x, r) \) the open ball of center \( x \in M \) and radius \( r > 0 \), and by \( \overline{B}(x, r) \) the closed ball.

The space \( \text{lip}_0(M) \) is the subspace of \( \text{Lip}_0(M) \) of functions \( f \) satisfying:

\[
\forall \varepsilon > 0, \exists \delta > 0 : d(x, y) < \delta \Rightarrow |f(x) - f(y)| \leq \varepsilon d(x, y)
\]

The first result of this section is the following:

Theorem 2.1. Let \( M \) be a countable proper metric space and

\[
S = \left\{ f \in \text{lip}_0(M) : \lim_{r \to +\infty} \sup_{x, y \in \overline{B}(0, r), x \neq y} \frac{f(x) - f(y)}{d(x, y)} = 0 \right\}.
\]

Then, \( \mathcal{F}(M) \) is isometrically isomorphic to \( S^* \).

Before the proof, we need some definitions:

Definition 2.2.

(1) Let \( X \) be a Banach space. A subspace \( F \) of \( X^* \) is called separating if \( x^*(x) = 0 \) for all \( x^* \in F \) implies \( x = 0 \).

(2) For \((M, d)\) a pointed metric space, a subspace \( F \) of \( \text{Lip}_0(M) \) separates points uniformly if there exists a constant \( c \geq 1 \) such that for every \( x, y \in M \), some \( f \in F \) satisfies \( \|f\|_L \leq c \) and \( |f(x) - f(y)| = d(x, y) \).

Definition 2.3. Let \( X \) be a Banach space. We denote \( NA(X) \) the subset of \( X^* \) consisting of all linear forms which attain their norm.

A result of Petunin and Pičko [19] asserts that for a separable Banach space \( X \), if a closed subspace \( F \) of \( X^* \) is separating and is a subset of \( NA(X) \), then \( X \) is isometrically isomorphic to \( F^* \). To use this result we proceed with a few lemmas about the space \( S \).

Lemma 2.4. Let \((M, d)\) be proper pointed metric space. The space \( S \) is a subspace of \( NA(\mathcal{F}(M)) \).

**Proof.** Let \( f \in S \). We may assume that \( f \neq 0 \) and take \( 0 < \varepsilon < \frac{\|f\|_L}{2} \). Since

\[
\lim_{r \to +\infty} \sup_{x, y \in \overline{B}(0, r), x \neq y} \frac{f(x) - f(y)}{d(x, y)} = 0
\]
there exists \( r > 0 \) such that

\[
\sup_{x, y \in \overline{B}(0, r), x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} < \varepsilon.
\]

Thus, \( \|f\|_L = \sup_{x, y \in \overline{B}(0, r), y \neq x} \frac{|f(x) - f(y)|}{d(x, y)} \)

Because \( f \in Lip_0(M) \), the set

\[
\overline{B}_\varepsilon := \{(x, y) \in \overline{B}(0, r)^2, x \neq y, |f(x) - f(y)| \geq \varepsilon d(x, y)\}
\]

is compact and we have

\[
\|f\|_L = \sup_{x, y \in \overline{B}(0, r), y \neq x} \frac{|f(x) - f(y)|}{d(x, y)} = \sup_{(x, y) \in \overline{B}_\varepsilon} \frac{|f(x) - f(y)|}{d(x, y)} = \max_{(x, y) \in \overline{B}_\varepsilon} \frac{|f(x) - f(y)|}{d(x, y)}
\]

Thus, there exist \( x \neq y \) such that \( \|f\|_L = \frac{|f(x) - f(y)|}{d(x, y)} \). With \( \gamma = \frac{1}{d(x,y)}(\delta_x - \delta_y) \), \( \gamma \in F(M) \), we obtain \( \|f\|_L = |f(\gamma)| \), with \( \|\gamma\|_{F(M)} = 1 \) because \( \delta \) is an isometry. Then, \( f \) is norm attaining and \( S \subset NA(F(M)) \).

\[\square\]

**Lemma 2.5.** Let \((M, d)\) be a proper pointed metric space. If \( S \) separates points uniformly, then it is separating.

**Proof.** Using Hahn-Banach theorem it is enough to prove that when \( S \) separates points uniformly, it is weak*-dense in \( Lip_0(M) \).

We will first prove that the condition

\[
\lim_{r \to +\infty} \sup_{x, y \in \overline{B}(0, r), x \neq y} \frac{f(x) - f(y)}{d(x, y)} = 0
\]

is stable under supremum and infimum between two functions.

Let \( f, g \in S \) and \( x \neq y \) in \( M \) such that \( x \) or \( y \) doesn’t belong to \( \overline{B}(0, r) \). We assume that \( f(x) \leq g(x) \), the other case is similar. We need to distinguish two cases:

**• if** \( f(y) \leq g(y) \), then

\[
\frac{\inf(f, g)(x) - \inf(f, g)(y)}{d(x, y)} = \frac{f(x) - f(y)}{d(x, y)}
\]

and

\[
\frac{\sup(f, g)(x) - \sup(f, g)(y)}{d(x, y)} = \frac{g(x) - g(y)}{d(x, y)}
\]

**• if** \( f(y) \geq g(y) \), then

\[
\frac{f(x) - f(y)}{d(x, y)} \leq \frac{\inf(f, g)(x) - \inf(f, g)(y)}{d(x, y)} = \frac{f(x) - g(y)}{d(x, y)} \leq \frac{g(x) - g(y)}{d(x, y)}
\]
and 
\[
\frac{f(x) - f(y)}{d(x, y)} \leq \frac{\sup(f, g)(x) - \sup(f, g)(y)}{d(x, y)} = \frac{g(x) - f(y)}{d(x, y)} \leq \frac{g(x) - g(y)}{d(x, y)}
\]

So we obtain:
\[
\lim_{r \to +\infty} \sup_{x \in \bar{B}(0, r), y \neq x} \frac{\inf(f, g)(x) - \inf(f, g)(y)}{d(x, y)} = 0
\]

and
\[
\lim_{r \to +\infty} \sup_{x \in \bar{B}(0, r), y \neq x} \frac{\sup(f, g)(x) - \sup(f, g)(y)}{d(x, y)} = 0
\]

finally \( \inf(f, g), \sup(f, g) \in S \).

Assume that \( M \) is a proper pointed metric space such that \( S \) separates points uniformly. Mimicking the proof of Lemma 3.2.3 in [20] we obtain the following: there exists \( b \geq 1 \) such that for all \( f \in \text{Lip}_b(M) \), for all \( A \) finite subset of \( M \) containing 0, we can find \( g \in S \) so that \( \|g\|_L \leq b\|f\|_L \) and \( g|_A = f|_A \). Finally, one can deduce that the weak*-closure of \( S \) is \( \text{Lip}_b(M) \), so \( S \) is separating. \( \square \)

Along the proof of Theorem 2.1 we will need a characterization of compact metric spaces which are countable. First define the Cantor-Bendixon derivation. For a compact metric space \((K, d)\) we denote:

- \( M' \) the set of accumulation points of \( M \).
- \( M^{(\alpha)} = (M^{(\alpha-1)})' \), for a successor ordinal \( \alpha \).
- \( M^{(\alpha)} = \bigcap_{\beta<\alpha} M^{(\beta)} \), for a limit ordinal \( \alpha \).

A compact metric space \((K, d)\) is countable if and only if there is a countable ordinal \( \alpha \) such that \( K^{(\alpha)} \) is finite.

**Proof of Theorem 2.1:** Note first that the subspace \( S \) of \( \mathcal{F}(M)^* \) defined previously is closed in \( \mathcal{F}(M)^* \), so it follows from Lemmas 2.4, 2.5 and from Petunîn and Plîcko’s result [19] that we only have to prove that \( S \) separates points uniformly.

Ideas are the same as in the proof of Theorem 2.1 in [11] but for sake of completeness we will give all details. Let \( M \) be a proper countable metric space, \( x, y \in M \) and \( a = d(x, y) \). The ball \( \overline{B}(x, \frac{3a}{2}) \) is compact and countable so there exist a countable ordinal \( \alpha_0, k_1 \in \mathbb{N} \) and \( y_1^1, \ldots, y_{k_1}^1 \in M \) such that
\[
\overline{B}(x, \frac{3a}{2})^{(\alpha_0)} = \{y_1^1, \ldots, y_{k_1}^1\}
\]

We can find \( r_1, s_1, t_1 \) and
\[
u_1^1 < \cdots < \nu_{t_1}^1 \leq \frac{a}{2} < \nu_1^1 < \cdots < \nu_{r_1}^1 < a \leq \nu_1^1 < \cdots < \nu_{s_1}^1 \leq \frac{3a}{2}
\]
such that \( \{d(x, y_i^1), 1 \leq i \leq k_1\} = \{\nu_1^1, \ldots, \nu_{r_1}^1, v_1^1, \ldots, v_{s_1}^1, w_1^1, \ldots, w_{t_1}^1\} \). Now set
\[
u_1 = \min \left( \left\{ \frac{a}{2}, \nu_1^1, \frac{a}{2} - \nu_{r_1}^1, w_1^1 - a, \frac{3a}{2} - w_{t_1}^1 \right\} \cup \{0\} \right)
\]
and define \( \varphi_1 : [0, +\infty) \to [0, +\infty) \) by
we have $C$ and non empty. First if $t \neq 0$ so the equality $C$ Since $C$ we can find $r$ \phi \in \text{lip}(\varphi)$ such that $\|\varphi\|_{L} \leq 2$. With $f(\cdot) = d(\cdot, x)$, we set $C_{1} = f^{-1} \left([0, +\infty) \setminus \left(\bigcup_{i=0}^{r_{1}^{1}} U_{i}^{1} \right) \cup \left(\bigcup_{i=0}^{t_{1}+1} W_{i}^{1} \right)\right)$.

First, if $C_{1}$ is finite or empty, define $h(\cdot) = 2 \left(\varphi_{1} \circ d(\cdot, x) - \varphi_{1} \circ d(0, x)\right)$. Then we have $|h(x) - h(y)| = d(x, y), h(0) = 0$ and $\|h\|_{L} \leq 4$. We need to prove that $h \in \text{lip}_{0}(M)$. Set

$$\delta = \begin{cases} u_{1}/2, & \text{if } C_{1} = \emptyset \\ 1/2 \inf \{(u_{1}, \text{sep}(C_{1})) \cup \{\text{dist}(z, M \setminus C_{1}), z \in D_{1}\}\}, & \text{otherwise} \end{cases}$$

where

$$\text{sep}(C_{1}) = \inf \{d(z, t), z \neq t, z, t \in C_{1}\}$$

and

$$D_{1} = f^{-1} \left([0, +\infty) \setminus \left(\bigcup_{i=0}^{r_{1}^{1}} U_{i}^{1} \right) \cup \left(\bigcup_{i=0}^{t_{1}+1} W_{i}^{1} \right)\right).$$

Since $C_{1}$ is finite we have $\text{sep}(C_{1}) > 0$. Moreover, $\text{dist}(z, M \setminus C_{1}) > 0$ for $z \in D_{1}$ and $D_{1}$ is finite. Thus we deduce that $\delta > 0$.

If follows that every $z \neq t \in M$ such that $d(z, t) \leq \delta$ are not in $D_{1}$ and there exists $0 \leq i \leq r_{1}$ such that $z, t \in f^{-1} \left(U_{i}^{1}\right)$ or $0 \leq i \leq t_{1} + 1$ such that $z, t \in f^{-1} \left(W_{i}^{1}\right)$, so the equality $h(z) = h(t)$ holds, i.e. $h \in \text{lip}_{0}(M)$.

Finally, let us prove that $\lim_{r \to +\infty} \sup_{x \neq y \in \overline{B}(0, r)} \frac{h(x) - h(y)}{d(x, y)} = 0$, that is $h \in S$.

Let $r > 0$ be such that $\overline{B}(x, \frac{3a}{2}) \subset \overline{B}(0, r)$ and $z \notin \overline{B}(0, r)$.

First if $t \notin \overline{B}(x, \frac{3a}{2})$ then $h(z) = h(t)$ and $h(z) - h(t) = 0$.

Secondly if $t \in \overline{B}(x, \frac{3a}{2})$, then

$$\frac{|h(z) - h(t)|}{d(z, t)} = \frac{|h(t)|}{d(z, t)} \leq \frac{d(x, y)}{d(z, t)} \to +\infty 0$$

so $h \in S$.

Assume now that $C_{1}$ is infinite. It is a subset of $\overline{B}(x, \frac{3a}{2})$ thus for every ordinal $\alpha$ we have $C_{1}^{(\alpha)} \subset \overline{B}(x, \frac{3a}{2})^{(\alpha)}$. Moreover, $C_{1} \cap \overline{B}(x, \frac{3a}{2})^{(\alpha)} = \emptyset$ so we have $C_{1}^{(\alpha)} = \emptyset$. Since $C_{1}$ is compact and countable we can find $\alpha_{1} < \alpha_{0}$ such that $C_{1}^{(\alpha_{1})}$ is finite and non empty.

There exist $k_{2} \in \mathbb{N}$ and $y_{2}^{1}, \ldots, y_{k_{2}}^{1} \in C_{1}$ such that $C_{1}^{(\alpha_{1})} = \{y_{2}^{1}, \ldots, y_{k_{2}}^{1}\}$. Then we can find $r_{2}, t_{2} \in \mathbb{N}$ and

$$u_{2}^{1} < \cdots < u_{2}^{k_{2}} < \frac{a}{2} - \frac{u_{1}}{4}, \frac{3a}{2} + \frac{u_{1}}{4} < u_{2}^{1} < \cdots < u_{2}^{k_{2}}$$
such that

\[ \{d(x, y^2) ; 1 \leq i \leq k_2\} = \{u^1_2, \cdots, u^r_2, w^1_2, \cdots, w^{t_2}_2\}. \]

Set

\[ w_2 = \min \left( \left\{ u_1, \left( \frac{a}{1} - \frac{u_2}{2} \right) - u_2^1, w_1^1, \left( \frac{3a}{2} + \frac{u_2}{2} \right) \right\} \right) \]

and define \( \varphi : [0, +\infty) \to [0, +\infty) \) by

\[ \varphi_2(t) = \begin{cases} \varphi_1(t), & t \in \left( \bigcup_{i=0}^{t_1} U_i \right) \text{ or } \left( \bigcup_{i=0}^{t_1} W_i \right) \\ \varphi_1(u_2), & t \in \left( w^2_2 - \frac{w^2_2}{2}, w^2_2 + \frac{w^2_2}{2} \right) := W, 1 \leq i \leq r_2 \\ \varphi_1(w_2), & t \in \left( w^2_2 - \frac{w^2_2}{2}, w^2_2 + \frac{w^2_2}{2} \right) := W, 1 \leq i \leq t_2 \end{cases} \]

and \( \varphi_2 \) is continuous on \([0, +\infty)\) and affine on each interval of

\[ [0, +\infty) \setminus \left( \left( \bigcup_{i=0}^{t_1} U_i \right) \cup \left( \bigcup_{i=0}^{t_1} W_i \right) \right). \]

It is easy to check that \( \|\varphi_2\|_L \leq \frac{8}{3}. \)

Now we set \( C_2 = C_1 \backslash \overline{f^{-1} \left( \left( \bigcup_{i=0}^{t_1} U_i \right) \cup \left( \bigcup_{i=0}^{t_1} W_i \right) \right)} \). First if \( C_2 \) is finite or empty the function \( h(\cdot) = 2 (\varphi_2 \circ d(\cdot, x) - \varphi_2 \circ d(0, x)) \) verifies \( h(0) = 0, |h(x) - h(y)| = d(x, y) \) and \( \|h\|_L \leq \frac{8}{3} \). Moreover, if we set

\[ \delta = \begin{cases} u_2/2, & \text{if } C_2 = 0 \\ 1/2 \min \left\{ u_2, \mathrm{sep}(C_2) \right\} \cup \left\{ \mathrm{dist}(z, M \setminus C_2), \ z \in D_2 \right\} \end{cases} \]

where \( D_2 = C_1 \setminus \overline{f^{-1} \left( \left( \bigcup_{i=0}^{t_2} U_i \right) \cup \left( \bigcup_{i=0}^{t_2} W_i \right) \right)} \), we obtain that \( \delta > 0 \) and when \( z, t \in M \) are such that \( d(z, t) \leq \delta \), then \( h(z) = h(t) \). So finally \( h \) is in \( \text{lip}_0(M) \). The proof of the fact that \( h \) belongs to \( S \) is the same as previously.

If \( C_2 \) is infinite we proceed inductively until we get \( C_n \) finite, which eventually happens because we construct a decreasing sequence of ordinals.

The function \( h \) we finally obtain verifies \( h(0) = 0, |h(y) - h(x)| = d(x, y) \) and

\[ \|h\|_L \leq 2 \prod_{j=1}^{n} \left( 1 + \frac{1}{2^{j-1}} \right) \leq 2 \prod_{j=1}^{+\infty} \left( 1 + \frac{1}{2^{j-1}} \right) := c \]

where \( c \) does not depend on \( x \) and \( y \). Moreover, setting

\[ \delta = \begin{cases} u_n/2, & \text{if } C_n = 0 \\ 1/2 \left( \min \{u_n, \text{sep}(C_n)\} \cup \left\{ \text{dist}(z, M \setminus C_n), \ z \in D_n \right\} \right) \end{cases} \]

we get \( \delta > 0 \) and if \( z, t \in M \) are such that \( d(z, t) \leq \delta \), then \( h(z) = h(t) \), i.e. \( h \in \text{lip}_0(M) \). Finally, \( h \) still verifies

\[ \lim_{r \to +\infty} \sup_{\substack{x \in B(0, r) \ \text{or} \ y \in B(0, r) \ \text{and} \ x \neq y}} \frac{h(x) - h(y)}{d(x, y)} = 0 \]

so we can conclude that \( S \) separates points uniformly and therefore \( F(M) \) is isometrically isomorphic to \( S^* \). \( \square \)

We can now prove the second result of this section:

**Theorem 2.6.** The Lipschitz-free space over a countable proper metric space has the metric approximation property.
Proof: A theorem of A. Grothendieck [8] asserts that if a separable Banach space is isometrically isomorphic to a dual space and has the AP, then it has the MAP. Thus it follows from Theorem 2.1 that it is enough to prove that for $M$ a countable proper metric space, $F(M)$ has the BAP.

We need the following result we can deduce from Lemma 4.2 in [11]:

For any pointed metric space $M$, define for $N \in \mathbb{N}$,

$$A_N = (\overline{B}(0,2^{N+1}) \setminus B(0,2^{-N-1})) \cup \{0\}.$$ 

Then there exists a sequence of operators $S_N : F(M) \to F(A_N)$, of norm less than 72, such that for every $\gamma \in F(M)$ the sequence $(S_N(\gamma))_{N \in \mathbb{N}}$ converges to $\gamma$.

Now let $M$ be a countable and proper metric space. Since every closed ball is compact, the set $A_N$ is countable and compact, for every $N \in \mathbb{N}$. Thus Theorem 3.1 in [1] asserts that $F(A_N)$ has the MAP and since for every $N \in \mathbb{N}$, $F(A_N)$ is separable, there exists $R^N_p : F(A_N) \to F(A_N)$ a sequence of operators of finite-rank, so that for every $\gamma \in F(A_N)$, $\lim_{p \to +\infty} R^N_p \gamma = \gamma$ and $\|R^N_p\| \leq 1$ for every $p \in \mathbb{N}$ (15, see also Theorem 1.e.13 in [15]).

Setting $Q_{N,p} = R^N_p \circ S_N$ we deduce that the range of $Q_{N,p}$ is finite dimensional, $\|Q_{N,p}\| \leq \|R^N_p\| \|S_N\| \leq 72$ and for every $\gamma \in F(M)$,

$$\lim_{N \to +\infty} \lim_{p \to +\infty} R^N_p S_N \gamma = \lim_{N \to +\infty} S_N \gamma = \gamma.$$ 

Thus $F(M)$ has the 72-BAP.

Finally, we can conclude that $F(M)$ has the MAP. 

3. ULTRAMETRIC SPACES

A metric space $(M, d)$ is said to be ultrametric if for every $x, y, z \in M$, we have $d(x, z) \leq \max\{d(x, y), d(y, z)\}$. One can easily prove the following useful properties:

Property 1. For $x, y \in M$ and $r, r' > 0$, if $B(x, r) \cap B(y, r') \neq \emptyset$ and $r \leq r'$ then $B(x, r) \subset B(y, r')$.

Property 2. For $x, y \in M$ and $r > 0$, if $y \in B(x, r)$, then $B(y, r) = B(x, r)$.

Property 3. For $x, y, z \in M$, if $d(x, y) \neq d(y, z)$ then

$$d(x, z) = \max\{d(x, y), d(y, z)\}.$$ 

Property 4. For every $r > 0$ there exists a partition of $M$ in closed balls of radius $r$.

Now let us prove the first result of this section:

Theorem 3.1. The Lipschitz-free space over a proper ultrametric space has the metric approximation property.

Proof: Let $M$ be a proper ultrametric space and $\tau_p$, the topology of pointwise convergence on $Lip_p(M)$. We will construct a sequence $(L_n)_{n \in \mathbb{N}}$ of operators on $Lip_p(M)$, of norm less than 1, such that for every $f \in Lip_p(M)$ the sequence $(L_n f)_{n \in \mathbb{N}}$ converges pointwise to $f$.

Let $n \in \mathbb{N}$. Because $M$ is ultrametric there exists a partition of $\overline{B}(0, n)$ into balls $\overline{B}(x, \frac{1}{n})$. Moreover, the closed ball $\overline{B}(0, n)$ is compact then it is possible to find
We will first compute the norm of $L_n$. Let $f \in Lip_0(M)$ and $x, y \in M$.

- If there exists $i \in \{1, \ldots, k\}$ such that $x, y \in \overline{B}(x_i, \frac{1}{n})$ then clearly:

$$|L_n(f)(x) - L_n(f)(y)| = 0 \leq \|f\|_L d(x, y).$$

- Now assume $x \in \overline{B}(x_i, \frac{1}{n})$ and $y \in \overline{B}(x_j, \frac{1}{n})$ with $i \neq j$.

Remark that because $x \in \overline{B}(x_i, \frac{1}{n})$, we have $\overline{B}(x_i, \frac{1}{n}) = \overline{B}(x, \frac{1}{n})$. Furthermore $y \notin \overline{B}(x_i, \frac{1}{n}) = \overline{B}(x, \frac{1}{n})$, so $d(x, y) > \frac{1}{n}$.

$$|L_n(f)(x) - L_n(f)(y)| = |f(x_i) - f(x_j)| \leq \|f\|_L d(x_i, x_j) \leq \|f\|_L \max\{d(x_i, x), d(x, x)\} = \|f\|_L d(x, x) \leq \|f\|_L \max\{d(x, y), d(y, x)\} = \|f\|_L d(x, y).$$

- Finally, for $x \in \overline{B}(0, n)$ and $y \notin \overline{B}(0, n)$, there exists $i \in \{1, \ldots, k\}$ such that $x \in \overline{B}(x_i, \frac{1}{n})$. Because $x \in \overline{B}(0, n)$, we have $\overline{B}(0, n) = \overline{B}(x, n)$ and since $y \notin \overline{B}(0, n)$, we obtain $d(x, y) > n$. Hence

$$|L_n(f)(x) - L_n(f)(y)| = |f(x_i)| \leq \|f\|_L d(x_i, 0) \leq \|f\|_L \times n \leq \|f\|_L d(x, y).$$

Then $\|L_n(f)\| \leq \|f\|_L$ and $\|L_n\| \leq 1$.

One can easily prove that $L_n$ is $\tau_p - \tau_p$-continuous and that for $f \in Lip_0(M)$, the sequence $(L_n(f))_{n \in \mathbb{N}}$ pointwise converges to $f$. Then it is the adjoint of an operator $R_n : F(M) \to F(M)$ of norm less than 1 such that for every $\gamma \in F(M)$, the sequence $(R_n(\gamma))_{n \in \mathbb{N}}$ weakly-converges to $\gamma$. Finally, for every $n \in \mathbb{N}$ we have $R_n(L(M)) = \text{span}\{\delta_{x_i}, 1 \leq i \leq k\}$, so the operator $R_n$ is of finite rank. Then, because $F(M)$ is separable, using convex combinations and a diagonal argument, we can conclude that $F(M)$ has the MAP \[3\].

It is also possible to prove that the Lipschitz-free space over a proper ultrametric space $M$ is a dual space. We will prove first that in the case of $K$ a compact ultrametric space, $F(K)$ is isometrically isomorphic to $lip_0(K)^*$. We will again use the result of Petunin and Pličko. Note that Theorem 3.3.3 in \[20\] provides an alternative approach.

Before stating the result let us introduce the notion of $\mathbb{R}$-trees and some background about its link with ultrametric spaces:

**Definition 3.2.** A metric space $(T, d)$ is said to be an $\mathbb{R}$-tree when the two following conditions hold:

1. For every $a, b$ in $T$, there exists a unique isometry $\phi : [0, d(a, b)] \to T$ such that $\phi(0) = a$ and $\phi(d(a, b)) = b$.
2. Any continuous and one-to-one mapping $\varphi : [0, 1] \to T$ has the same range as the isometry $\phi$ associated to the points $a = \varphi(0)$ and $b = \varphi(1)$.
Background. P. Buneman proved in [3] that the 4-points property is a characterization of subsets of $\mathbb{R}$-trees, where a metric space $(M, d)$ has the 4-points property if for every $x, y, z$ and $t$ in $M$ we have:

$$d(x, y) + d(z, t) \leq \max \{d(x, z) + d(y, t), d(x, t) + d(y, z)\}.$$ 

In particular any ultrametric space $(M, d)$ has the 4-points property.

It is proved in [16] by Matoušek that, for a subspace $M$ of a tree $T$, it is possible to find a linear extension operator from $\text{Lip}_0(M)$ to $\text{Lip}_0(T)$ which is bounded. In particular $\mathcal{F}(M)$ is complemented in $\mathcal{F}(T)$.

Moreover, Godard proved in [4] that the Lipschitz-free space over an $\mathbb{R}$-tree is an $L_1$-space.

In conclusion if $M$ is a ultrametric space, its Lipschitz-free space is complemented into an $L_1$-space.

Theorem 3.3. If $(K, d)$ is a compact ultrametric space, then $\mathcal{F}(K)$ is isometrically isomorphic to $\text{lip}_0(K)^*$ and $\text{lip}_0(K)$ is isomorphic to $c_0(\mathbb{N})$.

Proof: It is proved in [1] that for a compact metric space $(K, d)$, the space $\text{lip}_0(K)$ is a subset of $\text{NA} (\mathcal{F}(K))$ and it is separating as soon as it separates points uniformly.

Let $(K, d)$ be a compact ultrametric space. To obtain the first part of the result it is enough to prove that $\text{lip}_0(K)$ separates points uniformly. 

Let $x, y \in K$, set $a = d(x, y)$ and define $h : K \to \mathbb{R}$ as follows:

$$\forall z \in K, \ h(z) = d(x, y) \left(1_{B(x, a/2)}(z) - 1_{B(x, a/2)}(0)\right)$$

where $1_{B(x, a/2)}$ is the characteristic function of the open ball $B(x, a/2)$.

Then we have $h(0) = 0$ and $|h(x) - h(y)| = d(x, y)$. We will compute the Lipschitz-constant of $h$:

If $z, t$ are both in $B(x, a/2)$ or both outside $B(x, a/2)$, then

$$|h(z) - h(t)| = 0 \leq 2d(z, t).$$

Take $z \in B(x, a/2)$ and $t \notin B(x, a/2)$, then

$$d(z, t) = \max\{d(x, z), d(x, t)\} = d(x, t) \geq \frac{a}{2} = \frac{d(x, y)}{2} = \frac{|h(z) - h(t)|}{2}.$$ 

Hence the function $h$ is 2-Lipschitz.

To conclude we need to prove that $h \in \text{lip}_0(K)$. We will see that $\delta = \frac{a}{2}$ holds for every $\varepsilon$.

Let $z, t \in K$ such that $d(z, t) < \frac{a}{2}$.

First, if $z \in B(x, a/2)$ then $B(x, a/2) = B(z, a/2)$ and because $d(z, t) < \frac{a}{2}$ we have $t \in B(x, a/2)$ and $h(z) = h(t)$.

Secondly, if $z \notin B(x, a/2)$ then $t$ cannot be in $B(x, a/2)$ and $h(z) = h(t)$.

This proves that $h$ is in $\text{lip}_0(K)$ so this space separates points uniformly and therefore this concludes the proof of the fact that $\mathcal{F}(K)$ is the dual space of $\text{lip}_0(K)$.

A result due to D.R. Lewis and C. Stegall [14] asserts that if a separable dual space is complemented in $L_1$, then it is isomorphic to $\ell_1(\mathbb{N})$. So it follows from the background before the theorem that $\mathcal{F}(K)$ is isomorphic to $\ell_1(\mathbb{N})$.

Finally, Theorem 6.6 in [11] asserts that for a compact metric space $K$, the space $\text{lip}_0(K)$ is isomorphic to a subspace of $c_0(\mathbb{N})$. Moreover, its dual is isomorphic to $\ell_1(\mathbb{N})$, then Corollary 2 in [10] implies that $\text{lip}_0(K)$ is isomorphic to $c_0(\mathbb{N})$. □
More generally for a proper ultrametric space we have the following:

**Theorem 3.4.** Let $M$ be a proper ultrametric space and

$$ S = \left\{ f \in \text{lip}_0(M); \lim_{r \to +\infty} \sup_{x \neq y} \frac{f(x) - f(y)}{d(x, y)} = 0 \right\}. $$

Then $\mathcal{F}(M)$ is isometrically isomorphic to $S^*$ and $S$ is isomorphic to $c_0(\mathbb{N})$.

**Proof:** It is possible to adapt the proof of Theorem 6.6 in [11] to obtain that the space $S$ is isomorphic to a subspace of $c_0(\mathbb{N})$.

**Lemma 3.5.** Let $M$ be a proper metric space. Then for any $\varepsilon > 0$, the space $S$ is $(1+\varepsilon)$-isometric to a subspace of $c_0(\mathbb{N})$.

**Proof:** Assume $\varepsilon < 1$ and consider the space $M \times M$ with the metric:

$$ d((x_1, x_2), (y_1, y_2)) = \max\{d(x_1, y_1), d(x_2, y_2)\}. $$

For every $j \in \mathbb{N}$ and $k \in \mathbb{Z}$ we consider the compact set

$$ C_{j,k} = \{(x_1, x_2) \in M \times M; d(0, x_1) \leq 2^j \text{ and } 2^k \leq d(x_1, x_2) \leq 2^{k+1}\} $$

and $F_{j,k}$ a finite $2^{k-3}\varepsilon$-net of $C_{j,k}$. Then $F := \bigcup_{j \in \mathbb{N}} F_{j,k}$ is countable.

We now define

$$ T : S \to c_0(F) $$

$$ f \mapsto \left( \frac{f(x_1) - f(x_2)}{d(x_1, x_2)} \right)_{(x_1, x_2) \in F}. $$

Justify first that $Tf \in c_0(F)$ for $f \in S$:

Let $\alpha > 0$.

Because $f \in S$, in particular $f \in \text{lip}_0(M)$ and there exists $K \in \mathbb{N}$ such that for every $k \leq -K$, if $d(x_1, x_2) \leq 2^{k+1}$ then $|f(x_1) - f(x_2)| \leq \alpha \frac{d(x_1, x_2)}{d(x_1, x_2)} \leq \alpha$. Thus for every $j \in \mathbb{N}$,

every $k \leq -K$ and $(x_1, x_2) \in C_{j,k}$, we have $\frac{|f(x_1) - f(x_2)|}{d(x_1, x_2)} \leq \alpha$.

Moreover, $\lim_{r \to +\infty} \sup_{x \neq y} \frac{|f(x) - f(y)|}{|d(x, y)|} = 0$, thus there exists $R > 0$ such that

\[ \forall r \geq R, \forall x \notin B(0, r), \forall y \in M, \text{ we have } \frac{|f(x) - f(y)|}{d(x, y)} \leq \alpha. \]

Let $N \in \mathbb{N}$ be such that $2^n \geq 2R, \forall n \geq N$.

If $(x_1, x_2) \in C_{j,k}$ with $j \geq N$ we clearly have $\frac{|f(x_1) - f(x_2)|}{d(x_1, x_2)} \leq \alpha$

Assume now $(x_1, x_2) \in C_{j,k}$ with $k > N$ and $j \leq N$, then

$$ d(0, x_2) \geq d(x_1, x_2) - d(0, x_1) \geq 2^k - R > R $$

that is $x_2 \notin B(0, R)$ and $\frac{|f(x_1) - f(x_2)|}{d(x_1, x_2)} \leq \alpha$.

Finally, we obtain that $Tf \in c_0(F)$, for every $f \in S$. 
Clearly $\|T\| \leq 1$. We will now show that $\|f\|_L \leq (1 + \varepsilon)\|T f\|_{\infty}$.

Let $y_1 \neq y_2 \in M$. There exists $j \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $(y_1, y_2) \in C_{j,k}$ and $(x_1, x_2) \in F_{j,k}$ such that $d((y_1, y_2), (x_1, x_2)) \leq 2^{k-3}\varepsilon$. Then

$$d(y_1, y_2) \geq d(x_1, x_2) - d(x_1, y_1) - d(x_2, y_2) \geq d(x_1, x_2) - 2^{k-2}\varepsilon$$

$$\geq d(x_1, x_2) \left(1 - \frac{\varepsilon}{4}\right).$$

Let $f \in S$,

$$\frac{|f(y_1) - f(y_2)|}{d(y_1, y_2)} \leq \frac{|f(x_1) - f(x_2)|}{d(y_1, y_2)} + \frac{d(x_1, y_1) + d(x_2, y_2)}{d(y_1, y_2)}\|f\|_L$$

$$\leq \frac{|f(x_1) - f(x_2)|}{d(y_1, y_2)} + \frac{\varepsilon}{4}\|f\|_L$$

$$\leq \left(1 - \frac{\varepsilon}{4}\right)^{-1} \frac{|f(x_1) - f(x_2)|}{d(x_1, x_2)} + \frac{\varepsilon}{4}\|f\|_L$$

$$\leq \left(1 - \frac{\varepsilon}{4}\right)^{-1} \|T f\|_{\infty} + \frac{\varepsilon}{4}\|f\|_L$$

Finally, $\|T f\|_{\infty} \leq \|f\|_L \leq (1 + \varepsilon)\|T f\|_{\infty}$ and one can conclude that $S$ is $(1 + \varepsilon)$-isometric to a subspace of $c_0(\mathbb{N})$. \hfill \Box

We now conclude the proof of Theorem 3.3. We previously proved that in the case of a proper metric space, the space $S$ is a subspace of $NA(F(M))$ and it is separating as soon as it separates points uniformly. Therefore in order to use Petunin and Pličko’s result [19] (see also [5]) we only need to prove that, in the case of proper ultrametric space, the space $S$ separates points uniformly.

For given $x, y \in M$, the function $h$ defined as in proof of Theorem 3.3 satisfies $h \in lip_0(M)$, $|h(x) - h(y)| = d(x, y)$ and its Lipschitz constant does not depend on $x$ and $y$.

Let $r > 0$ be such that $B(x, a/2) \subset B(0, r)$, with $a = d(x, y)$. We may and do assume that $d(z, 0) > r$.

First if $t \in \overline{B}(x, \frac{a}{2})$, then

$$\frac{|h(z) - h(t)|}{d(z, t)} = \frac{d(x, y)}{d(z, t)} \rightarrow 0, t \rightarrow \infty.$$

Secondly if $t \notin \overline{B}(x, \frac{a}{2})$, then

$$\frac{|f(z) - f(t)|}{d(z, t)} = 0.$$

Finally, we have $h \in S$, then $S$ separates points uniformly. We can conclude that $S^*$ is isometrically isomorphic to $F(M)$.

The second part of the proof follows the same line than the last part of the proof of Theorem 3.3. \hfill \Box

Remark 3.6. B.R. Kloeckner proved in [12] that the Wasserstein space of a compact ultrametric space is affinely isometric to a convex subset of $\ell_1(\mathbb{N})$. 

FREE SPACES OVER SOME PROPER METRIC SPACES 11
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Laboratoire de Mathématiques de Besançon, CNRS UMR 6623, Université de Franche-Comté, 16 Route de Gray, 25030 Besançon Cedex, FRANCE.
E-mail address: aude.dalet@univ-fcomte.fr