POLISH GROUPOIDS AND FUNCTIORAL COMPLEXITY

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Abstract. We introduce and study the notion of functorial Borel complexity for Polish groupoids. Such a notion aims to measure the complexity of classifying the objects of a category in a constructive and functorial way. In the particular case of principal groupoids such a notion coincides with the usual Borel complexity of equivalence relations. Our main result is that on one hand for Polish groupoids with an essentially treeable orbit equivalence relation, functorial Borel complexity coincides with the Borel complexity of the associated orbit equivalence relation. On the other hand, for every countable equivalence relation $E$ that is not treeable there are Polish groupoids with different functorial Borel complexity both having $E$ as orbit equivalence relation. In order to obtain such a conclusion we generalize some fundamental results about the descriptive set theory of Polish group actions to actions of Polish groupoids, answering a question of Arlan Ramsay. These include the Becker-Kechris results on Polishability of Borel $G$-spaces, existence of universal Borel $G$-spaces, and characterization of Borel $G$-spaces with Borel orbit equivalence relations.

1. Introduction

Classification of mathematical structures is one of the main components of modern mathematics. It is safe to say that most results in mathematics can be described as providing an explicit classification of a class of mathematical objects by a certain type of invariants.

In the last 25 years the notion of constructive classification has been given a rigorous formulation in the framework of invariant complexity theory. In this context a classification problem is regarded as an equivalence relation on a standard Borel space (virtually all classification problems in mathematics fit into this category). The concept of constructive classification is formalized by the notion of Borel reduction. A Borel reduction from an equivalence relation $E$ on $X$ to an equivalence relation $E'$ on $X'$ is a Borel function $f : X \to X'$ with the property that, for every $x, y \in X$, $xEy$ if and only if $f(x)E'f(y)$. In other words $f$ is a Borel assignment of complete invariants for $E$ that are equivalence classes of $E'$. The existence of such a function can be interpreted as saying that classifying the objects of $X'$ up to $E'$ is at least as complicated as classifying the objects of $X$ up to $E$. This offers a notion of comparison between the complexity of different classification problems.
Several natural equivalence relations can then be used as benchmarks to measure the complexity of classification problems. Perhaps the most obvious such benchmark is the relation of equality $=_{\mathbb{R}}$ of real numbers. This gives origin to the basic dichotomy smooth vs. nonsmooth: an equivalence relation is smooth if it is Borel reducible to $=_{\mathbb{R}}$. (The real numbers can here be replaced by any other uncountable standard Borel space.) Beyond smoothness the next fundamental benchmark is classifiability by countable structures. Here the test is Borel reducibility to the relation of isomorphism within some class of countable first order structures, such as (ordered) groups, rings, etc. Equivalently one can consider the orbit equivalence relation associated with Borel actions of the Polish group $S_\infty$ of permutations of $\mathbb{N}$. Replacing $S_\infty$ with an arbitrary Polish group yields the notion of an equivalence relation being classifiable by orbits of a Polish group action.

This framework allows one to build a hierarchy between different classification problems. Many efforts have been dedicated to the attempt to draw a picture as complete as possible of classification problems in mathematics and their relative complexity. To this purpose powerful tools such as Hjorth’s theory of turbulence [23] have been developed in order to disprove the existence of Borel reduction between given equivalence relations and to distinguish between the complexity of different classification problems. This can be interpreted as a way to formally exclude the possibility of a full classification of a certain class of objects by means of a given type of invariants. For example the relation of isomorphism of simple separable C*-algebras has been shown to transcend countable structures in [15]; see also [17]. Similar results have been obtained for several other equivalence relations, such as affine homeomorphisms of Choquet simplexes [15], conjugacy of unitary operators on the infinite dimensional separable Hilbert space [31], conjugacy of ergodic measure-preserving transformations of the Lebesgue space [17], conjugacy of homeomorphisms of the unit square [23], conjugacy of irreducible representations of nontype I groups [22] or C*-algebras [12, 32], conjugacy and unitary equivalence of automorphisms of classifiable simple separable C*-algebras [33, 36], isometry of separable Banach spaces [30] and complete order isomorphisms of separable operator systems. Furthermore the relations of isomorphism and Lipschitz isomorphisms of separable Banach spaces, topological isomorphisms of (abelian) Polish groups, uniform homeomorphisms of complete separable metric spaces [16], and the relation of completely bounded isomorphisms of separable operator spaces [2] have been shown to be not classifiable by the orbits of a Polish group action (and in fact to have maximal complexity among analytic equivalence relations). An exhaustive introduction to invariant complexity theory can be found in [19].

Considering how helpful the theory of Borel complexity has been so far in giving us a clear understanding of the relative complexity of classification problems in mathematics, it seems natural to look at refinements to the notion of Borel reducibility that can in some situations better capture the notion of explicit classification from the practice of mathematics. Such a line of research has been suggested in [13], where the results of the present paper have been announced. This is the case for example when the classification problem under consideration concerns a category. In this case it is natural to ask that the classifying map be functorial, and to assign invariants not only to the objects of the category but also to the morphisms. This is precisely what happens in many explicit examples of classification results in mathematics. In fact in many such examples the consideration of invariants of morphisms is essential to the proof. This is particularly the case
in the Elliott classification program of simple C*-algebras, starting from Elliott’s seminal paper of AF algebras \[8\]. Motivated by similar considerations, Elliott has suggested in \[9\] an abstract approach to classification by functors. In this paper we bring Elliott’s theory of functorial classification within the framework of Borel complexity theory. For simplicity we consider only categories where every arrow is invertible, called groupoids. Such categories will be assumed to have a global Borel structure that is at least analytic and makes the set of objects (identified with their identity arrows) a standard Borel space. In the particular case when between any two objects there is at most one arrow (principal groupoids) these are precisely the analytic equivalence relations. One can then consider the natural constructibility requirement for classifying functors, which is being Borel with respect to the given Borel structures. This gives rise to the notion of functorial Borel complexity, which in the particular case of principal groupoids is the usual notion of Borel complexity.

In this article we study such a notion of functorial Borel complexity for groupoids, focusing on the case of Polish groupoids. These are the groupoids where the Borel structure is induced by a topology that makes composition and inversion of arrows continuous and open and has a basis of open sets which are Polish in the relative topology. These include all Polish groups, groupoids associated with Polish group actions, and locally compact groupoids \[12\] Definition 2.2.2]. The latter ones include the holonomy groupoids of foliations and the tangent groupoids of manifolds \[12\] Chapter 2], the groupoids of row-finite directed graphs \[35\], and the localization groupoids of actions of countable inverse semigroups \[12\] Chapter 4]. The main results of the present paper assert that, for Polish groupoids with an essentially treeable equivalence relation, the existence of a Borel reducibility between the groupoids is equivalent to the Borel reducibility of the corresponding orbit equivalence relations. On the other hand, for every countable equivalence relation \(E\) that is not treeable there are two Polish groupoids with orbit equivalence relation \(E\) that have distinct functorial Borel complexity; see section \[8\]. This shows that Borel reducibility of groupoids provides a finer notion of complexity than the usual Borel reducibility of equivalence relations. Having a finer notion of complexity is valuable, because it allows one to further distinguish between the complexity of problems that, in the usual framework, turn out to have the same complexity. An example of this phenomenon occurs in the classification problem for C*-algebras, where it turns out \[10\] \[15\] \[47\] that classifying arbitrary separable C*-algebras is as difficult as classifying the restricted class of C*-algebras that are considered to be well behaved (precisely the amenable simple C*-algebras, or even more restrictively the simple C*-algebras that can be obtained as direct limits of interval algebras).

In order to prove the above mentioned characterization of essentially treeable equivalence relations we will generalize some fundamental results of the theory of actions of Polish groups to actions of Polish groupoids, answering a question of Ramsay from \[15\]. These include the Becker-Kechris results on Polishability of Borel \(G\)-spaces \[8\] Chapter 5], existence of universal Borel \(G\)-spaces \[8\] section 2.6], and characterization of Borel \(G\)-spaces with Borel orbit equivalence relation \[8\] Chapter 7]. The fundamental technique employed is a generalization of the Vaught transform \[50]\ from actions of Polish groups to actions of Polish groupoids. Building upon the results of the present paper, Hjorth’s theory of turbulence is generalized in \[21\] from Polish group actions to Polish groupoids. This is then used therein to obtain an anti-classification result for the class of operator algebras arising from operator algebraic varieties studied in \[7\].
This paper is organized as follows: In section 2 we recall some background notions, introduce the notation to be used in the rest, and state the basic properties of the Vaught transform for actions of Polish groupoids. In section 3 we generalize the local version of Effros’ theorem from Polish group actions to actions of Polish groupoids and infer the Glimm-Effros dichotomy for Polish groupoids and Borel reducibility, refining results from [44]. Section 5 contains the proof of the Polishability result for Borel G-spaces, showing that any Borel G-space is isomorphic to a Polish G-space, where G is a Polish groupoid. A characterization for Borel G-spaces with Borel orbit equivalence relation is obtained as a consequence in section 6. Section 7 considers countable Borel groupoids, i.e. analytic groupoids with only countably many arrows with a given source. It is shown that every such groupoid has a Polish groupoid structure compatible with its Borel structure. In particular all results about Polish groupoids apply to countable Borel groupoids. Finally in section 8 the above mentioned characterization of essentially treeable equivalence relations in terms of Borel reducibility is proved.

This paper includes an appendix written by Anush Tserunyan. In this appendix, it is proved that the Effros Borel structure on the space of closed subsets of a Polish groupoid is standard. We are grateful to Anush Tserunyan for letting us include her result here.

2. Notation and preliminaries

2.1. Descriptive set theory. A Polish space is a separable and completely metrizable topological space. Equivalently a topological space is Polish if it is T_1, regular, second countable, and strong Choquet [30, Theorem 8.18]. A subspace of a Polish space is Polish with respect to the subspace topology if and only if it is a G_δ [30, Theorem 3.11].

A standard Borel space is a space endowed with a σ-algebra which is the σ-algebra of Borel sets with respect to some Polish topology. An analytic space is a space endowed with a countably generated σ-algebra which is the image of a standard Borel space under a Borel function. A subset of a standard Borel space is analytic if it is an analytic space with the relative standard Borel structure. A subset of a standard Borel space is co-analytic if its complement is analytic. It is well known that for a subset of a standard Borel space it is equivalent to being Borel and being both analytic and co-analytic [30 Theorem 14.7]. If X, Y are standard Borel spaces and A is a subset of X × Y, then for x ∈ X the section {y ∈ Y : (x, y) ∈ A} is denoted by A_x. The projection of A onto the first coordinate is {x ∈ X : A_x ≠ ∅}, while the co-projection of A is {x ∈ X : A_x = Y}. The projection of an analytic set is analytic, while the co-projection of a co-analytic set is co-analytic.

If X is a Polish space, then the space of closed subsets of X is denoted by F(X). The Effros Borel structure on F(X) is the σ-algebra generated by the sets \{F ∈ F(X) : F ∩ U ≠ ∅\} for U ⊂ X open. This makes F(X) a standard Borel space [30 section 12.C].

Recall that a subset A of a Polish space X has the Baire property if there is an open subset U of X such that the symmetric difference A △ U is meager [30 Definition 8.21]. It follows from [30 Corollary 29.14] that any analytic subset of X has the Baire property.
A topological space \( X \) is a **Baire space** if every nonempty open subset of \( X \) is not meager. Every completely metrizable topological space is a Baire space; see [30] Theorem 8.4].

If \( X, Y \) are standard Borel spaces, then we say that \( Y \) is **fibered over** \( X \) if there is a Borel surjection \( p : Y \to X \). If \( x \in X \), then the inverse image of \( x \) under \( p \) is called the \( x \)-fiber of \( Y \) and denoted by \( Y_x \). If \( Y_0, Y_1 \) are fibered over \( X \), then the fibered product \( Y_0 \ast Y_1 = \{(y_0, y_1) : p_0(y_0) = p_1(y_1)\} \) is naturally fibered over \( X \). Similarly if \( (Y_n)_{n \in \omega} \) is a sequence of Borel spaces fibered over \( X \) we define \( \ast_{n \in \omega} Y_n = \{(y_n)_{n \in \omega} : p(y_n) = p(y_m) \text{ for } n, m \in \omega\} \), which is again fibered over \( X \). A Borel fibered map from \( Y_0 \) to \( Y_1 \) is a Borel function \( \varphi : Y_0 \to Y_1 \) which sends fibers to fibers, i.e. \( p_1 \circ \varphi = p_0 \).

If \( E \) is an equivalence relation on a standard Borel space \( X \), then a subset \( T \) of \( X \) is a transversal for \( E \) if it intersects any class of \( E \) in exactly one point. A **selector** for \( E \) is a Borel function \( \sigma : X \to X \) such that \( \sigma(x) \in E \) for every \( x \in X \) and \( \sigma(x) = \sigma(y) \) whenever \( x Ey \).

### 2.2. Locally Polish spaces.

**Definition 2.2.1.** A **locally Polish space** is a topological space with a countable basis of open sets which are Polish spaces in the relative topology.

By [30] Theorem 8.18] a locally Polish space is \( T_1 \), second countable, and strong Choquet. Moreover it is a Polish space if and only if it is regular. It follows from [30] Lemma 3.11] that a \( G_\delta \) subspace of a locally Polish space is locally Polish.

Suppose that \( X \) is a locally Polish space. Denote by \( F(X) \) the space of closed subsets of \( X \). The Effros Borel structure on \( F(X) \) is the \( \sigma \)-algebra generated by the sets of the form \( \{ F : F \cap U \neq \emptyset \} \) for \( U \subset X \) open. It is shown in the Appendix that the Effros Borel structure on \( F(X) \) is standard.

One can deduce from this that the Borel \( \sigma \)-algebra of \( X \) is standard. In fact the function \( X \to F(X) \), \( x \mapsto \{ x \} \) is clearly a Borel isomorphism onto the set \( F_1(X) \) of closed subsets of \( F(X) \) containing exactly one element. It is therefore enough to show that \( F_1(X) \) is a Borel subset of \( F(X) \). Fix a countable basis \( A \) of open Polish subsets of \( X \). Suppose also that for every \( U \in A \), a compatible complete metric \( d_U \) on \( U \) is fixed. Observe that \( F_1(X) \) contains precisely the closed subsets \( F \) of \( X \) such that \( F \cap U \neq \emptyset \) for some \( U \in A \) and for every \( U \in A \) such that \( F \cap U = \emptyset \), and for every \( n \in \omega \) there is \( W \in A \) such that \( cl(W) \subset U \), \( diam_U(\text{cl}(W)) < 2^{-n} \) and \( F \cap (X \setminus \text{cl}(W)) = \emptyset \), where \( diam_U(\text{cl}(W)) \) is the diameter of \( W \) with respect to the metric \( d_U \). This shows that \( F_1(X) \) is a Borel subset of \( X \).

### 2.3. The Effros fibered space.

Suppose that \( Z \) is a locally Polish space, \( X \) is a Polish space, and \( p : Z \to X \) is a continuous open surjection. For \( x \in X \) denote by \( Z_x \) the inverse image of \( x \) under \( p \). Define \( F^*(Z) \) to be the space of nonempty subsets of \( Z \) endowed with the Effros Borel structure. Define \( F^*(Z, X) \) to be the Borel subset of closed subsets of \( Z \) contained in \( Z_x \) for some \( x \in X \). The Borel function from \( F^*(Z, X) \) onto \( X \) assigning to an element \( F \) of \( F^*(Z, X) \) the unique \( x \in X \) such that \( F \subset Z_x \) endows \( F^*(Z, X) \) with the structure of fibered Borel space. The obvious embedding of \( F^*(Z_x) \) into \( F^*(Z, X) \) is a Borel isomorphism onto the \( x \)-fiber of \( F^*(Z, X) \).

Consider the set \( \{ \emptyset : x \in X \} \) endowed with the Borel structure obtained from the bijection \( x \leftrightarrow \emptyset_x \). Define \( F(Z, X) \) to be the disjoint union of \( F^*(Z, X) \) with
2.4. Analytic and Borel groupoids. A groupoid $G$ is a small category where every arrow is invertible. The set of objects of $G$ is denoted by $G^0$. We will regard $G^0$ as a subset of $G$ by identifying an object with its identity arrow. A groupoid $G$ is endowed with two canonical maps $s, r : G \to G^0$ (source and range maps) that assign to every arrow $\gamma$ in $G$ its source and range object, respectively. Denote by $G^2$ the set of pairs of composable arrows $G^2 = \{ (\gamma, \rho) : s(\gamma) = r(\rho) \}$. If $A, B$ are subsets of $G$, then $AB$ stands for the set $\{ \gamma \rho : (\gamma, \rho) \in (A \times B) \cap G^2 \}$. In particular if $Y \subseteq X$, then $Y B = \{ \gamma \in B : r(\gamma) \in Y \}$ and $B Y = \{ \gamma \in B : s(\gamma) \in Y \}$. We write $x B$ for $\{ x \} B = r^{-1}[\{ x \}] \cap B$ and $B x$ for $B \{ x \} = s^{-1}[\{ x \}] \cap B$. If $A$ is a set of objects, then the restriction $G|_A$ of $G$ to $A$ (this is called “contraction” in [38,43]) is the groupoid $\{ \gamma \in G : s(\gamma) \in A, r(\gamma) \in A \}$ with set of objects $A$ and operations inherited from $G$.

To every groupoid $G$ one can associate the orbit equivalence relation $E_G$ on $G^0$ defined by $(x, y) \in E_G$ if and only if there is $\gamma \in G$ such that $s(\gamma) = x$ and $r(\gamma) = y$. The function $G \to E_G$, $\gamma \mapsto (r(\gamma), s(\gamma))$ is a continuous surjection. We say that a groupoid is principal when such a map is injective. Thus a principal groupoid is just an equivalence relation on its set of objects. Conversely any equivalence relation can be regarded as a principal groupoid.

The notion of functor between groupoids is the usual notion from category theory. Thus a functor from $G$ to $H$ is a function from $G$ to $H$ such that, for every $\gamma \in G$ and $(\rho_0, \rho_1) \in G^2$ the following hold: $F(s(\gamma)) = s(F(\gamma))$, $F(r(\gamma)) = r(F(\gamma))$, $F(\gamma^{-1}) = F(\gamma)^{-1}$, and $F(\rho_0 \rho_1) = F(\rho_0) F(\rho_1)$.

When $E$ and $E'$ are principal groupoids, then functors from $E$ to $E'$ are in $1:1$ correspondence with homomorphisms from $E$ to $E'$ in the sense of [19] Definition 10.1.3.

**Definition 2.4.1.** An analytic groupoid is a groupoid endowed with an analytic Borel structure making composition and inversion of arrows Borel and such that the set of objects and, for every object $x$, the set of elements with source $x$, are standard Borel spaces with respect to the induced Borel structure. A (standard) Borel groupoid is a groupoid endowed with a standard Borel structure making composition and inversion of arrows Borel and such that the set of objects is a Borel subset.

It is immediate to verify that principal analytic groupoids are precisely analytic equivalence relations on standard Borel spaces. Similarly principal Borel groupoids are precisely the Borel equivalence relations on standard Borel spaces. A functor between analytic groupoids is Borel if it is Borel as a function with respect to the given Borel structures.

2.5. Polish groupoids and Polish groupoid actions.

**Definition 2.5.1.** A topological groupoid is a groupoid endowed with a topology making composition and inversion of arrows continuous.
It is not difficult to see that for a topological groupoid the following conditions are equivalent:

1. Composition of arrows is open.
2. The source map is open.
3. The range map is open.

(See [46 Exercise I.1.8].)

**Definition 2.5.2.** A **Polish groupoid** is a groupoid endowed with a locally Polish topology such that

1. composition and inversion of arrows are continuous and open,
2. the set $G^0$ of objects is a Polish space with the subspace topology,
3. for every $x \in G^0$ the sets $Gx$ and $xG$ are Polish spaces with the subspace topology.

Polish groupoids have been introduced in [44] with the extra assumption that the topology be regular or, equivalently, globally Polish. It is nonetheless noticed in [44] page 362 that one can safely dispense of this additional assumption without invalidating the results proved therein.

Suppose that $G$ is a Polish groupoid and $X$ is a Polish space. A continuous action of $G$ on $X$ is given by a continuous function $p : X \rightarrow G^0$ called anchor map together with a continuous function $(g, x) \mapsto gx$ from $G \times X$ to $G \times X = \{ (\gamma, x) : p(x) = s(\gamma) \}$ to $X$ such that, for all $\gamma, \rho \in G$ and $x \in X$, $\gamma(\rho x) = (\gamma \rho) x$, $p(\gamma x) = r(\gamma)$, and $p(x) x = x$. In such a case we say that $X$ is a Polish $G$-space. Similarly if $X$ is a standard Borel space, then a Borel action of $G$ on $X$ is given by a Borel map $p : X \rightarrow G^0$ together with a Borel map $G \times X \rightarrow X$, $(\gamma, x) \mapsto \gamma x$ satisfying the same conditions as above. In this case $X$ will be called a Borel $G$-space.

Clearly any Polish groupoid acts continuously on its space of objects $G^0$ by setting $p(x) = x$ and $(\gamma, x) \mapsto r(\gamma)$. This will be called the **standard** action of $G$ on $G^0$.

Most of the usual notions for actions of groups, such as orbits, or invariant sets, can be generalized in the obvious way to actions of groupoids. If $X$ is a $G$-space and $x \in X$, then its orbit $\{ \gamma x : s(\gamma) = p(x) \}$ is denoted by $[x]$. The orbit equivalence relation $E_G$ on $X$ is defined by $x E_G y$ if $[x] = [y]$. If $A$ is a subset of $X$, then its saturation $\{ \gamma a : a \in A, \gamma \in Gp(a) \}$ is denoted by $[A]$. An action is called **free** if $\gamma x = \rho x$ implies $\gamma = \rho$ for any $x \in X$ and $\gamma, \rho \in Gp(x)$.

Suppose that $G$ is a Polish groupoid and $X$ is a Borel $G$-space. If $x, y \in G^0$ are in the same orbit define the stabilizer

$$G_x = \{ \gamma \in G : s(\gamma) = p(x) \text{ and } \gamma x = x \}$$

of $x$ and

$$G_{x,y} = \{ \gamma \in G : s(\gamma) = p(x) \text{ and } \gamma x = y \}.$$ 

Observe that by [30] Theorem 9.17 $G_x$ is a closed subgroup of $p(x)Gp(x)$. Therefore $G_{x,y}$ is also closed, since $G_{x,y} = G_{x,x} \rho$ for any $\rho$ such that $s(\rho) = p(x)$ and $\rho x = y$.

Suppose that $X$ and $Y$ are Borel $G$-spaces with anchor maps $p_X$ and $p_Y$. A Borel fibered map from $X$ to $Y$ is a Borel function $\varphi : X \rightarrow Y$ such that $p_Y \circ \varphi = p_X$. A Borel fibered map from $X$ to $Y$ is $G$-equivariant if $\varphi(\gamma x) = \gamma \varphi(x)$ for $x \in X$ and $\gamma \in Gp(x)$. A Borel $G$-**embedding** from $X$ to $Y$ is an injective $G$-equivariant Borel fibered map from $X$ to $Y$. Finally a Borel $G$-**isomorphism** from $X$ to $Y$ is a Borel $G$-embedding which is also onto.
2.6. Some examples of Borel groupoids. In this subsection we show how several natural categories of interest can be endowed (after a suitable parametrization) with the structure of a Borel groupoid.

Let us first consider the category of complete separable metric spaces having surjective isometries as morphisms. This can be endowed with the structure of a Borel groupoid in the following way. Denote by $\mathbb{U}$ the Urysohn universal metric space. (A survey about $\mathbb{U}$ and its remarkable properties can be found in [40].) Let $F(\mathbb{U})$ be the Borel space of closed subsets of $\mathbb{U}$ endowed with the Effros Borel structure. By universality of the Urysohn space, $F(\mathbb{U})$ contains an isometric copy of any separable metric space. Moreover any surjective isometry between closed subsets of $\mathbb{U}$ can be identified with its graph, which is a closed subset of $\mathbb{U} \times \mathbb{U}$. The set CMS of such graphs is easily seen to be a Borel subset of $F(\mathbb{U})$. Moreover a standard computation shows that composition and inversion of arrows are Borel maps. This shows that CMS is a Borel groupoid that can be seen as a parametrization of the category of metric spaces with surjective isometries as arrows.

More generally one can look at the category of separable $\mathcal{L}$-structures in some signature $\mathcal{L}$ of continuous logic. (A complete introduction to continuous logic is [4].) One can identify any $\mathcal{L}$-structure with an $\mathcal{L}$-structure having as support a closed subset of $\mathbb{U}$. In such case the interpretation of a function symbol $f$ can be seen as a closed subset of $\mathbb{U}^{\lfloor f \rfloor + 1}$ where $\lfloor f \rfloor$ denotes the arity of $f$. The interpretation of a relation symbol $B$ can be seen as a closed subset of $\mathbb{U}^{\lfloor B \rfloor} \times \mathbb{R}$ where again $\lfloor B \rfloor$ denotes the arity of $B$. (Here distances and relations are allowed to attain value in the whole real line.) The set Mod($\mathcal{L}$) of such structures can be verified to be a Borel subset of $F(\mathbb{U}) \times \prod_f F(\mathbb{U}) \times \prod_B F(\mathbb{U})$ where $f$ and $B$ range over the function and relation symbols of $\mathcal{L}$. Similar parametrizations of the space of $\mathcal{L}$-structures can be found in [10] and [5]. As before one can regard Mod($\mathcal{L}$) as the Borel groupoid of $\mathcal{L}$-structures. In the particular case when one considers discrete structures one can replace the Urysohn space with $\omega$.

As a particular case of separable structures in a given signature one can consider separable $\mathcal{C}^*$-algebras. (The book [6] is a complete reference for the theory of operator algebras.) The complexity of the classification problem for separable $\mathcal{C}^*$-algebras has recently attracted considerable interest; see [10],[14],[15],[47]. Particularly important classes for the classification program are nuclear and exact $\mathcal{C}^*$-algebras; see [6] section IV.3. Separable exact $\mathcal{C}^*$-algebras are precisely the closed self-adjoint subalgebras of the Cuntz algebra $\mathcal{O}_2$ [34]. Thus the Borel groupoid $\mathcal{C}^*\text{Exact}$ of closed subalgebras of $\mathcal{O}_2$—where a $^*$-isomorphism between closed subalgebras is identified with its graph—can be regarded as a parametrization for the category of exact $\mathcal{C}^*$-algebras having $^*$-isomorphisms as arrows. The category of (simple, unital) nuclear $\mathcal{C}^*$-algebras can be regarded as the restriction of $\mathcal{C}^*\text{Exact}$ to the Borel set of (simple, unital) self-adjoint subalgebras of $\mathcal{O}_2$; see [15] section 7.

We now look at the category of Polish groups with continuous group isomorphisms as arrows. Denote by Iso($\mathbb{U}$) the group of isometries of the Urysohn space endowed with the topology of pointwise convergence. Recall that Iso($\mathbb{U}$) is a universal Polish group [49]; i.e. it contains any other Polish group as closed subgroup. The space $SG(\text{Iso}(\mathbb{U}))$ of closed subgroups of Iso($\mathbb{U}$) endowed with the Effros Borel
structure can be regarded as the standard Borel space of Polish groups. Moreover a continuous isomorphism between closed subgroups of \( \text{Iso}(U) \) can be identified with its graph, which is a closed subgroup of \( \text{Iso}(U) \times \text{Iso}(U) \). It is not difficult to check that the set \( \text{PG} \) of such closed subgroups of \( \text{Iso}(U) \times \text{Iso}(U) \) is a Borel subset of the space \( SG(\text{Iso}(U) \times \text{Iso}(U)) \) of closed subgroups of \( \text{Iso}(U) \times \text{Iso}(U) \) endowed with the Effros Borel structure. (Fix a countable neighborhood basis \( \mathcal{N} \) of the identity in \( \text{Iso}(U) \), and observe that a closed subgroup \( H \) of \( \text{Iso}(U) \times \text{Iso}(U) \) is in \( \text{PG} \) if and only if \( \forall V \in \mathcal{N} \exists U \in \mathcal{N} \) such that \( H \cap (U \times (\text{Iso}(U) \setminus \text{cl}(V))) = \emptyset \) and \( H \cap ((\text{Iso}(U) \setminus \text{cl}(V)) \times U) = \emptyset \).) Moreover a standard calculation shows that composition and inversion of arrows are Borel functions in \( \text{PG} \). (For composition of arrows, observe that if as before \( \mathcal{N} \) is a countable basis of neighborhoods of the identity in \( \text{Iso}(U) \), \( D \) is a dense subset of \( \text{Iso}(U) \), \( A \) and \( B \) are open subsets of \( \text{Iso}(U) \), and \( \varphi, \psi \in \text{PG} \), then \( (\varphi \circ \psi) \cap (A \times B) \neq \emptyset \) if and only if there are \( U, V \in \mathcal{N} \) and \( g, h \in D \) with \( \text{cl}(V)^2 h \subset B \) and \( U g \subset A \) such that \( \psi \cap (U^2 \times (\text{Iso}(U) \setminus \text{cl}(V))) = \emptyset \), \( \varphi \cap (A \times U g) \neq \emptyset \), and \( \psi \cap (U g \times V h) \neq \emptyset \).) This shows that \( \text{PG} \) is a Borel groupoid that can be seen as a parametrization of the category of Polish groups with continuous isomorphisms as arrows.

A similar discussion applies to the category of separable Banach spaces with linear (not necessarily isometric) isomorphisms as arrows. In this case one considers a universal separable Banach space, such as \( C[0, 1] \). One then looks at the standard Borel space of closed subspaces of \( C[0, 1] \) as a set of objects, and the set of closed subspaces of \( C[0, 1] \oplus C[0, 1] \) that code a linear isomorphism between closed subspaces of \( C[0, 1] \) as a set of arrows. The proof that these sets are Borel with respect to the Effros Borel structure is analogous to the case of Polish groups.

### 2.7. The action groupoid

Suppose that \( G \) is a Polish groupoid and \( X \) is a Polish \( G \)-space. Consider the groupoid \( G \times X = \{ (\gamma, x) \in G \times X : s(\gamma) = p(x) \} \), where composition and inversion of arrows are defined by \( (\rho, \gamma)(\gamma, x) = (\rho \gamma, x) \) and \( (\gamma, x)^{-1} = (\gamma^{-1}, \gamma x) \). The set of objects of \( G \times X \) is \( G^0 \times X = \{ (a, x) \in G^0 \times X : p(x) = a \} \). Endow \( G \times X \) with the subspace topology from \( G \times X \). Observe that the function \( X \to G^0 \times X, x \mapsto (p(x), x) \) is a homeomorphism from \( X \) to the set of objects of \( G \times X \). We can therefore identify the latter with \( X \). Under this identification the source of \( (\gamma, x) \) is \( x \) and the range is \( \gamma x \). We claim that \( G \times X \) is a Polish groupoid, called the action groupoid associated with the Polish \( G \)-space \( X \). Clearly the topology is locally Polish, and composition and inversion of arrows are continuous. We need to show that the source map is open. Suppose that \( V \) is an open subset of \( G \), \( U \) is an open subset of \( X \), and \( W \) is the open subset \( \{ (\gamma, x) : \gamma \in V, x \in U \} \) of \( G \times X \). Suppose that \( W \) is nonempty and pick \( (\gamma_0, x_0) \in W \). Thus \( x_0 \in U \) and \( p(x_0) = s(\gamma_0) \in s[V] \). Therefore there is an open subset \( U_0 \) of \( U \) containing \( x_0 \) such that \( p[U_0] \subset s[V] \). We claim now that \( U_0 \) is contained in the image of \( V \) under the source map. In fact if \( x \in U_0 \), then \( p(x) = s(\gamma) \) for some \( \gamma \in V \), and therefore \( x \) is the source of the arrow \( (\gamma, x) \) in \( W \). This concludes the proof of the fact that \( G \times X \) is a Polish groupoid. To summarize we can state the following proposition.

**Proposition 2.7.1.** Suppose that \( G \) is a Polish groupoid, and \( X \) is a Polish \( G \)-space. The action groupoid \( G \times X \) as defined above is a Polish groupoid. Moreover the map \( X \to (G \times X)^0, x \mapsto (p(x), x) \) is a homeomorphism such that, for every \( x, x' \in X \), \( x E_G x' \) if and only if \( (p(x), x) E_{G \times X} (p(x'), x') \).
2.8. **Functorial reducibility.**

**Definition 2.8.1.** Suppose that $G$ and $H$ are analytic groupoids. A *Borel reduction* from $G$ to $H$ is a Borel functor $F$ from $G$ to $H$ such that $xGy \neq \emptyset$ whenever $F(x)HF(y) \neq \emptyset$.

Equivalently a Borel functor $F$ from $G$ to $H$ is a Borel reduction from $G$ to $H$ when the function $G^0 \rightarrow H^0$, $x \mapsto F(x)$ is a Borel reduction from $E_G$ to $E_H$ in the sense of \[19\] Definition 5.1.1.

**Definition 2.8.2.** Suppose that $G$ and $H$ are analytic groupoids. We say that $G$ is *Borel reducible* to $H$—in formulas $G \leq_B H$—if there is a Borel reduction from $G$ to $H$.

The notion of bireducibility is defined accordingly.

**Definition 2.8.3.** Suppose that $G$ and $H$ are analytic groupoids. We say that $G$ is *Borel bireducible* to $H$—in formulas $G \sim_B H$—if $G$ is Borel reducible to $H$ and vice versa.

When $E$ and $E'$ are principal analytic groupoids, then the Borel reductions from $E$ to $E'$ are in 1:1 correspondence with Borel reductions from $E$ to $E'$ in the usual sense of Borel complexity theory; see \[19\] Definition 5.1.1. In particular Definition 2.8.2 generalizes the notion of Borel reducibility from analytic equivalence relations to analytic groupoids.

Similarly as in the case of reducibility for equivalence relations, one can impose further requirements on the reduction map. If $G$ and $H$ are analytic groupoids, we say that $G$ is injectively Borel reducible to $H$—in formulas $G \subseteq_B H$ if there is an injective Borel reduction from $G$ to $H$. When $G$ and $H$ are Polish groupoids, one can also insist that the reduction be continuous rather than Borel. One then obtains the notion of continuous reducibility $\leq_c$ and continuous injective reducibility $\subseteq_c$.

Definition 2.8.2 provides a natural notion of comparison between analytic groupoids. This allows one to build a hierarchy of complexity of analytic groupoids that includes the usual hierarchy of Borel equivalence relations. The *functorial Borel complexity* of an analytic groupoid will denote the position of the given groupoid in such a hierarchy.

2.9. **Category preserving maps.** According to \[41\] Definition A.2] a continuous map $f : X \rightarrow Y$ between Polish spaces is *category preserving* if for any co-meager subset $C$ of $Y$ the inverse image $f^{-1}[C]$ of $C$ under $f$ is a co-meager subset of $X$.

It is not difficult to see that any continuous open map is category preserving \[41\] Proposition A.3].

Category preserving maps satisfy a suitable version of the classical Kuratowski-Ulam theorem for coordinate projections. We will state the particular case of this result for continuous open maps in the following lemma, which is Theorem A.1 in \[41\].

**Lemma 2.9.1.** Suppose that $X$ is second countable space, $Y$ is a Baire space, and $f : X \rightarrow Y$ is an open continuous map such that $f^{-1}\{y\}$ is a Baire space for every $y \in Y$. If $A \subset X$ has the Baire property, then the following statements are equivalent:

1. $A$ is co-meager.
2. $\forall^* y \in Y$, $A \cap f^{-1}\{y\}$ is co-meager in $f^{-1}\{y\}$.
2.10. Vaught transforms. Suppose in the following that $G$ is a Polish groupoid, $\mathcal{A} = \{U_n : n \in \omega\}$ is a basis of Polish open subsets of $G$, and $X$ is a Borel $G$-space.

**Definition 2.10.1.** For $A \subset X$ and $V \subset G$, define the Vaught transforms

$$A^{\Delta V} = \{ x \in X : Vp(x) \neq \emptyset \ \text{and} \ \exists^* \gamma \in Vp(x), \ \gamma x \in A \} \quad \text{and} \quad A^{\ast V} = \{ x \in X : Vp(x) \neq \emptyset \ \text{and} \ \forall^* \gamma \in Vp(x), \ \gamma x \in A \}.$$  

In the particular case when $G$ is a Polish group and $X$ is a Borel $G$-space, this definition coincides with the usual Vaught transform; cf. [19] Definition 3.2.2.

**Lemma 2.10.2.** Assume that $B$ and $A_n$ for $n \in \omega$ are subsets of $X$. If $V$ is an open subset of $G$, then the following hold:

1. $B^{\Delta G}$ and $B^{\ast G}$ are invariant subsets of $X$.
2. $(\cap_n A_n)^{\ast V} = \bigcap_n A_n^{\ast V}.$
3. $(\cup_n A_n)^{\Delta V} = \bigcup_n A_n^{\Delta V}.$
4. $p^{-1} s[V]$ is the disjoint union of $(X \setminus B)^{\ast V}$ and $B^{\Delta V}$.
5. If $B$ is analytic, then $B^{\Delta V} = \bigcup \{ B^{\ast U} : V \supset U \in A \}$ and $B^{\ast V} = \bigcap \{ B^{\Delta U} : V \supset U \in A \}.$

**Lemma 2.10.2** is elementary and can be proved similarly as [19] Proposition 3.2.5. The next lemma can be seen as the analog of [3] Lemma 5.2.5.

**Lemma 2.10.3.** Suppose that $B \subset X$ is analytic and $U \subset G$ is open. If $x \in X$ and $\gamma \in Gp(x)$, then the following statements are equivalent:

1. $\gamma x \in B^{\Delta U}$.
2. $x \in B^{\ast V}$ for some $V \in \mathcal{A}$ such that $V \gamma^{-1} \subset U \gamma(\gamma)$.
3. $x \in B^{\Delta V}$ for some $V \in \mathcal{A}$ such that $V \gamma^{-1} \subset U \gamma(\gamma)$.
4. There are $V, W \in \mathcal{A}$ such that $VW^{-1} \subset U$, $\gamma \in W$, and $x \in B^{\ast V}$.
5. There are $V, W \in \mathcal{A}$ such that $VW^{-1} \subset U$, $\gamma \in W$, and $x \in B^{\Delta V}$.

**Proof.** The implications (5) $\Rightarrow$ (3), (2) $\Rightarrow$ (3), and (4) $\Rightarrow$ (5) are obvious. We present the proof of the nontrivial implications below.

(1) $\Rightarrow$ (2) By hypothesis $U \gamma(\gamma) \neq \emptyset$ and $\exists^* \rho \in U \gamma(\gamma)$ such that $\rho \gamma x \in B$. Therefore $U \gamma \neq \emptyset$ and $\exists^* \rho \in U \gamma$ such that $\rho x \in B$. Since $B$ is analytic and the action is Borel, the set $\{ \rho \in U \gamma : \rho x \in B \}$ is analytic and in particular it has the Baire property. It follows that there is $V \in \mathcal{A}$ such that $Vp(\rho) \neq \emptyset$, $Vp(x) \subset U \gamma$, and $\forall^* \rho \in V, \rho x \in B$. To conclude the proof, observe that $V \gamma^{-1} \subset U \gamma(\gamma)$.

(3) $\Rightarrow$ (1) Observe that $\emptyset \neq Vp(x) \subset U \gamma$. Thus $U \gamma \neq \emptyset$ and $\exists^* \rho \in U \gamma$ such that $\rho x \in B$. Thus $Up(\gamma z) \neq \emptyset$ and $\exists^* \rho \in Up(\gamma x), \rho \gamma x \in B$. This shows that $\gamma x \in B^{\Delta U}$.

(2) $\Rightarrow$ (4) Pick $v \in Vp(x)$ and observe that $v \gamma^{-1} \in U \gamma(\gamma)$. Therefore there are $W, V_0 \in \mathcal{A}$ such that $v \in V_0 \subset V, \gamma \in W$, and $V_0 W^{-1} \subset U$. Moreover since $x \in B^{\ast V}$ and $V_0 \subset V$ we have that $x \in B^{\ast V_0}$.  

If $A$ is a subset of $G \times X$ and $x \in X$, then $A_x$ denotes the $x$-fiber $\{ \gamma \in G : (\gamma, x) \in A \}$ of $A$. The proof of the following lemma is inspired by the proof of the Montgomery-Novikov theorem; see [30] Theorem 16.1.

**Lemma 2.10.4.** If $A$ is a Borel subset of $G \times X$ and $V \subset G$ is open, then

$$\Pi_V^A(A) := \{ x \in X : Vp(x) \neq \emptyset \ \text{and} \ A_x \ \text{is nonmeager in} \ Vp(x) \}$$
is Borel. The same conclusion holds if one replaces “nonmeager” with “co-meager” or “meager”.

**Proof.** Define $E$ to be the class of subsets $A$ of $G \times X$ such that $\Pi_V^\Delta (A)$ is Borel for every nonempty open subset $V$ of $G$. We claim that:

1. $E$ contains the sets of the form $U \times B = \{(\rho, x) \in G \times X : x \in B$, $\rho \in U\}$ for $B \subset X$ Borel and $U \subset G$ open;
2. $E$ is closed by taking countable unions;
3. $E$ is closed by taking complements.

Fix a nonempty open subset $V$ of $G$.

1. If $B \subset X$ is Borel and $U \subset G$ is open, then $\Pi_V^\Delta (U \times B) = B \cap p^{-1} [s [U \cap V]]$.
2. If $A = \bigcup_n A_n$, then $\Pi_V^\Delta (A)$ is the union of $\Pi_V^\Delta (A_n)$ for $n \in \mathbb{N}$.
3. If $A \subset G \times X$, then $\Pi_V^\Delta ((G \times X) \setminus A) = \bigcup_{U_n \subset V} (p^{-1} [s [U_n]] \setminus \Pi_V^\Delta (A))$.

A similar argument shows that the same conclusion holds after replacing “nonmeager” with “meager” or “co-meager”.

**Lemma 2.10.5.** If $A \subset X$ is Borel and $V \subset G$ is open, then $A^\Delta V$ and $A^* V$ are Borel.

**Proof.** Consider the subset $\bar{A} = \{(\rho, x) \in G \times X : \rho x \in A\}$ and observe that $\bar{A}$ is a Borel subset of $G \times X$ such that $A^\Delta V = \Pi_V^\Delta (A)$ and $A^* V = p^{-1} [s [V]] \setminus (X \setminus A)^\Delta V$.

The conclusion now follows from Lemma 2.10.4.

**Lemma 2.10.6.** Assume that $X$ is a Polish $G$-space. If $B \subset X$, $U \subset G$ is open, and $\alpha \in \omega_1$, then the following hold:

1. If $B$ is open, then $B^\Delta U$ is open.
2. If $B$ is $\Sigma^0_\alpha$, then $B^\Delta U$ is $\Sigma^0_\alpha$ relative to $p^{-1} [s [U]]$.
3. If $B$ is $\Pi^0_\alpha$, then $B^* U$ is $\Pi^0_\alpha$ relative to $p^{-1} [s [U]]$.

**Proof.** The proof is analogous to the corresponding one for group actions; see [19, Theorem 3.2.9]. Suppose that $B$ is open, and pick $x \in B^\Delta U$. Thus $Up(x) \neq \emptyset$ and $\exists \rho \in U \rho(x)$ such that $\rho x \in B$. Pick $U_0 \subset U$ open such that $x \in B^{* U_0}$ and $\rho \in U_0 \rho(x)$ such that $\rho x \in B$. Since $B$ is open and the action is continuous there are open subsets $W$ and $V$ containing $x$ and $\rho$ such that $V \subset U_0$, $W \subset B$, and $p [W] \subset s [V]$. We claim that $W \subset B^\Delta U$. In fact if $w \in W$, then $Vp(w) \neq \emptyset$. Moreover since $VW \subset B$, $\exists \rho \in Vp(w)$ such that $\rho w \in B$. This concludes the proof that $B^\Delta U$ is open. The other statements follow via (2), (3), and (4) of Lemma 2.10.2.

Using the Vaught transform it is easy to see that if $X$ is a Borel $G$-space, then the orbit equivalence relation $E^X_G$ is idealistic. (This is well known when $G$ is a Polish group; cf. [19, Proposition 5.4.10].) Recall that an equivalence relation $E$ on a standard Borel space $X$ is idealistic if there is a map $[x]_E \mapsto I_{[x]_E}$ assigning to each equivalence class $[x]_E$ of $E$ an ideal $I_{[x]_E}$ of subsets of $[x]_E$ such that $[x]_E \notin I_{[x]_E}$, and for every Borel subset $A$ of $X \times X$ the set $A_I$ defined by $x \in A_I$ iff $\{y \in [x]_E : (x, y) \in A\} \in I_{[x]_E}$ is Borel; see [19, Definition 5.4.9].

**Proposition 2.10.7.** If $X$ is a Borel $G$-space, then the orbit equivalence relation $E^X_G$ is idealistic.
Assume moreover that $p$ by the choice of the orbit equivalence relation. Pick $\rho \in \text{Gp}(y)$ under $\Phi$ is the set $\{\rho \in \text{Gp}(y) : py \notin S\}$. This shows that $\forall \rho \in \text{Gp}(y)$, $py \notin S$, and hence the definition of $I_C$ does not depend on the choice of $x \in C$. Clearly $C \notin I_C$ since $\text{Gp}(x)$ is a Baire space. It is not difficult to verify that $I_C$ is a $\sigma$-ideal. Suppose that $A \subset X \times X$ is Borel, and consider the set $A_I$ defined by $x \in A_I$ iff $\{y \in [x] : (x, y) \in A\} \subset I[x]$. Observe that $x \in A_I$ iff $\forall \rho \in \text{Gp}(x)$, $(x, \rho x) \notin A$. Consider the action of $G$ on $X \times X$ defined by $p(x, y) = p(y)$ and $\gamma(x, y) = (x, \gamma y)$ for $\gamma \in G \text{p}(x, y) = \text{Gp}(y)$. Observe that

$$x \in A_I \iff \{y \in [x]_E : (x, y) \in A\} \in I[x]_E \iff \forall \rho \in \text{Gp}(x), (x, \rho x) \notin A$$

$$\iff x \in ((X \times X) \setminus A)^*G.$$ 

This shows that $A_I$ is Borel by Lemma 2.10.5.

Let us denote as customary by $E_1$ the tail equivalence relation for sequences in $[0, 1]$. If $E$ is an idealistic Borel equivalence relation, then $E_1$ is not Borel reducible to $E$ by [28] Theorem 4.1. Therefore we obtain from Proposition 2.10.7 the following corollary:

**Corollary 2.10.8.** If $X$ is a Borel $G$-space with Borel orbit equivalence relations, then the orbit equivalence relation $E_1$ is not Borel reducible to $E_X^G$.

**Corollary 2.10.8** holds more generally for arbitrary Borel $G$-spaces, with not necessarily Borel orbit equivalence relations. This was shown by the present author in collaboration with Samuel Coskey, George Elliott, and Ilijas Farah by adapting the proof of [23, Chapter 8].

An equivalence relation $E$ on a standard Borel space $E$ is *smooth* if it is Borel reducible to the relation of equality in some Polish space [19, Definition 5.4.1]. By [19, Theorem 5.4.11] an equivalence relation has a Borel selector precisely when it is smooth and idealistic. Therefore the following corollary follows immediately from Proposition 2.10.7.

**Corollary 2.10.9.** If $X$ is a Polish $G$-space such that $E_X^G$ is smooth, then $E_X^G$ has a Borel selector.

Let us denote as customary by $E_1$ the tail equivalence relation for sequences in $[0, 1]$. If $E$ is an idealistic Borel equivalence relation, then $E_1$ is not Borel reducible to $E$ by [28] Theorem 4.1. Therefore we obtain from Proposition 2.10.7 the following corollary:

**Corollary 2.10.10.** If $G$ and $H$ are Polish groupoids such that $E_G$ and $E_H$ are smooth, then $G \leq_B H$ if and only if $E_G \leq_B E_H$.

2.11. Borel orbits. We now observe that if $G$ is a Polish groupoid, then the orbits of any Polish $G$-space are Borel.

**Proposition 2.11.1.** If $G$ is a Polish groupoid and $X$ is a Polish $G$-space, then the orbit equivalence relation $E_X^G$ is analytic and has Borel classes.

**Proof.** By Proposition 2.10.1 we can consider without loss of generality the case of the standard action of $G$ on its set of objects $G^0$. Fix $x \in G^0$ and consider the right action of $xGx$ on $Gx$ by composition. Observe that $xGx$ is a Polish group and $Gx$ is a right Polish $xGx$-space with closed orbits. Therefore by [19, Proposition 3.4.6] the corresponding orbit equivalence relation $E_{xGx}^G$ has a Borel transversal $T$. The
orbit \([x]\) is the image of \(T\) under the range map \(r\). Since \(r\) is 1:1 on \(T\), it follows that \([x]\) is Borel by \([30]\) Theorem 15.1. Observe now that the orbit equivalence relation \(E_G\) is the image of the standard Borel space \(G\) under the Borel function \(\gamma \mapsto (r(\gamma), s(\gamma))\). This shows that \(E_G\) is analytic. \(\square\)

Similarly as in the case of Polish group actions, a uniform bound on the complexity of the orbits in the Borel hierarchy entails Borelness of the orbit equivalence relation.

**Theorem 2.11.2.** Suppose that \(G\) is a Polish groupoid and \(X\) is a Polish \(G\)-space. The orbit equivalence relation \(E^X_G\) is Borel if and only if there is \(\alpha \in \omega_1\) such that every orbit is \(\Pi_0^\alpha\).

*Proof.* One direction is obvious. For the other one consider for \(\alpha \in \omega_1\) the relation \(E_\alpha\) of \(X\) defined by \((x, y) \in E_\alpha\) iff for every \(G\)-invariant \(\Pi^0_\alpha\) set \(W \subset X\) we have that \(x \in W\) iff \(y \in W\). If every orbit is \(\Pi^0_\alpha\), then \(E^X_G = E_\alpha\). It is thus enough to prove that \(E_\alpha\) is co-analytic for every \(\alpha \in \omega_1\). Consider a universal \(\Pi^0_\alpha\) subset \(U\) of \(\omega^\omega \times X\). Define the action of \(G\) on \(\omega^\omega \times X\) by setting \(p(a, b) = p(b)\) and \(\gamma(a, b) = (a, \gamma b)\). Define now \(T = U^{*G}\) and observe that \(T\) is \(\Pi^0_\alpha\) since \(U\) is \(\Pi^0_\alpha\). Denote by \(T_a\) the section \(\{b \in X : (a, b) \in T\}\) for \(a \in \omega\). Note now that \(b \in T_a\) iff \(b \in (U_a)^{*G}\). This shows that \(T_a\) is a \(G\)-invariant \(\Pi^0_\alpha\) subset of \(X\) for every \(\alpha \in \omega^\omega\). Conversely if \(A\) is a \(G\)-invariant \(\Pi^0_\alpha\) set, then \(A = U_a\) for some \(a \in \omega^\omega\) and hence \(A = A^{*G} = (U_a)^{*G} = T_a\). This shows that \(\{T_a : a \in \omega^\omega\}\) is the collection of all invariant \(\Pi^0_\alpha\) sets. It follows that \((x, y) \in E_\alpha\) iff \(\forall a \in \omega^\omega, (a, x) \in T\). Therefore \(E_\alpha\) is co-analytic. \(\square\)

Theorem 2.11.2 was proved for Polish group actions in \([48]\) sections 3.6 and 3.7.

### 3. Effros’ theorem and the Glimm-Effros dichotomy

#### 3.1. Effros’ theorem.

**Lemma 3.1.1.** Suppose that \(G\) is a Polish groupoid. Consider the standard action of \(G\) on \(G^0\) and the corresponding Vaught transform. If \(A \subset G^0\) is meager, then \(A^{\Delta^G}\) is meager.

*Proof.* The source map \(r : G \to G^0\) is open and, in particular, category preserving; see subsection 2.9. Thus \(r^{-1}[A]\) is a meager subset of \(G\). Therefore, since the source map \(s\) is also open, by Lemma 2.9.1 the set

\[
A^{\Delta^G} = \{x \in X : Gx \cap r^{-1}[A] \text{ is nonmeager}\}
\]

is meager. \(\square\)

**Theorem 3.1.2.** Suppose that \(G\) is a Polish groupoid, \(X\) is a Polish \(G\)-space, and \(x \in X\). Denote by \([x]\) the orbit of \(x\). The following statements are equivalent:

1. \([x]\) is a \(G_\delta\) subset of \(X\).
2. \([x]\) is a Baire space.
3. \([x]\) is nonmeager in itself.

*Proof.* By Proposition 2.11.1 we can assume without loss of generality that \(X = G^0\) and \(G \curvearrowright G^0\) is the standard action. The only nontrivial implication is 3 \(\Rightarrow\) 1. After replacing \(G\) with the restriction of \(G\) to the closure of \([x]\), we can assume that \([x]\) is dense in \(G^0\) and hence nonmeager in \(G^0\). By Proposition 2.11.1 \([x]\) is a Borel subset
of $G^0$ and in particular it has the Baire property. Therefore by [30] Proposition 8.23] the orbit $[x]$ is the union of a meager set $M$ and a $G_δ$ set $U$. One can conclude that $[x] = U^{*G}$ arguing as in [48] Proposition 4.4. Clearly $[x]$ is the union of $U^{*G}$ and $M^{*G}$. By Lemma 3.1.3 $M^{*G}$ is meager and hence, since $[x]$ is nonmeager, $M^{*G} = \emptyset$. Therefore $[x] = U^{*G}$ is $G_δ$ by Lemma 2.10.6

Theorem 2.1 of [44] asserts that it is equivalent for the conditions in Theorem 3.1.2 to hold for all points of $X$.

Suppose that $G$ is a Polish groupoid, $X$ is a Polish $G$-space, and $x \in G^0$. The fiber $Gp(x)$ is a Polish space, and the stabilizer $G_x$ of $x$ is a Polish group acting from the right by composition on $Gp(x)$. One can then consider the quotient space $Gp(x)/G_x$ and the quotient map $π_x : Gp(x) → Gp(x)/G_x$, which is clearly continuous and open. When $G \curvearrowright G^0$ is the standard action of $G$ on its set of objects and $x \in G^0$, then the stabilizer $G_x$ is just $xGx$.

It is not difficult to see that the proof of [44] Theorem 3.2 can be adapted to the context where $G$ is a not necessarily regular Polish groupoid, as observed in [44] page 362]. The following lemma can then be obtained as an immediate consequence.

**Lemma 3.1.3** (Ramsay). Suppose that $G$ is a Polish groupoid and $x \in G^0$. If the orbit $[x]$ of $x$ is $G_δ$, then the map $ϕ_x : Gx/xGx \to [x]$ defined by $ϕ_x(π(γ)) = r(γ)$ is a homeomorphism.

**Corollary 3.1.4.** Suppose that $G$ is a Polish groupoid, $X$ is a Polish $G$-space, and $x \in X$. If the orbit $[x]$ of $x$ is $G_δ$, then the map $ϕ_x : Gp(x)/G_x \to [x]$ defined by $ϕ_x(π(γ)) = γx$ is a homeomorphism.

**Proof.** Consider the action groupoid $G \times X$, and let us identify $X$ with the space of objects of $G \times X$ as in Proposition 2.7.1. Consider the map $ψ$ defined by $Gp(x) \to (G \times X)x$, $γ \mapsto (γ, x)$. Observe that $ψ$ is a continuous map with continuous inverse $(G \times X)x \to Gp(x)$, $(γ, x) \mapsto γ$. Moreover the image of $G_x$ under $ψ$ is precisely $x(G \times X)x$. The proof is then concluded by invoking Lemma 3.1.3.

### 3.2. A Polish topology on quotient spaces.

Suppose in this subsection that $G$ is a Polish groupoid, which is moreover regular. Equivalently the topology of $G$ is (globally) Polish. The following lemma is proved in [44] page 362].

**Lemma 3.2.1** (Ramsay). Suppose that $G$ is a regular Polish groupoid. If $U$ is an open subset of $G$ containing the set of objects $G^0$, then there is an open subset $V$ of $G$ containing the set of objects $G^0$ such that $VV \subset U$.

Fix $x \in G^0$. If $V$ is a neighborhood of $G^0$ in $G$, define the set $A_{V, x} = \{(ρ, γ) ∈ Gx × Gx : ργ^{−1} ∈ V\}$. Observe that if $γ \in Gx$, then the collection of open subsets of the form $Vγ$, where $V$ is an open neighborhood of $r(γ)$ in $G$, is a basis of neighborhoods of $γ$ in $Gx$. It follows from this observation and Lemma 3.2.1 that the collection $U_x = \{A_{V, x} : V$ is a neighborhood of $G^0$ in $G\}$ generates a uniformity compatible with the topology of $Gx$.

Suppose now that $H$ is a closed subgroup of $xGx$, and consider the right action of $H$ on $Gx$ by translation. Denote by $π$ the quotient map $Gx \to Gx/H$. Observe that $π$ is continuous and open. If $V$ is a neighborhood of $G^0$ in $G$ define

$$A_{V, x, H} = \{(π(γ), π(ρ)) ∈ Gx/H × Gx/H : ρhγ^{−1} ∈ V \text{ for some } h ∈ xGx\}.$$
As before the collection \( \mathcal{U}_{x,H} = \{ A_{V,x,H} : V \text{ is a neighborhood of } G^0 \text{ in } G \} \) generates a uniformity compatible with the topology of \( Gx/H \).

**Proposition 3.2.2.** The quotient \( Gx/H \) is a Polish space.

*Proof.* The topology on \( Gx/H \) is induced by a countably generated uniformity, and hence it is metrizable. Since the quotient map \( \pi : Gx \to Gx/H \) is continuous and open, it follows from [19, Theorem 2.2.9] that \( Gx/H \) is Polish. \( \square \)

**Proposition 3.2.3.** Suppose that \( G \) is a regular Polish groupoid and \( x \in G^0 \). Denote by \( \pi \) the quotient map \( \pi : Gx \to Gx/xGx \). The following statements are equivalent:

1. The orbit \( [x] \) of \( x \) is a \( G_\delta \) subset of \( G^0 \).
2. The map \( \phi_x : Gx/xGx \to [x] \) defined by \( \phi_x(\pi(\gamma)) = r(\gamma) \) is a homeomorphism.

*Proof.* The quotient space \( Gx/xGx \) is Polish by Proposition 3.2.2. Therefore if \( \phi_x \) is a homeomorphism, then \( [x] \) is Polish, and hence a \( G_\delta \) subset of \( G^0 \) by [30] Theorem 3.11. The converse implication follows from Lemma 3.1.3. \( \square \)

### 3.3. \( G_\delta \) orbits.

**Lemma 3.3.1.** Suppose that \( G \) is a Polish groupoid and \((U_n)_{n \in \omega}\) is an enumeration of a basis of nonempty open subsets of \( G^0 \). If \( G \) has a dense orbit, then every element of \( \bigcap_n [U_n] \) has dense orbit.

The proof of Lemma 3.3.1 is immediate. Recall that \( [U_n] \) denotes the \( G \)-saturation \( r[s^{-1}[U_n]] \) of \( U_n \).

**Lemma 3.3.2.** Suppose that \( G \) is a Polish groupoid. Define the equivalence relation \( \overline{E} \) on \( G^0 \) by \((x,y) \in \overline{E} \) iff the orbits of \( x \) and \( y \) have the same closure. The equivalence relation \( \overline{E} \) is \( G_\delta \) and contains \( E_G \).

*Proof.* Suppose that \((U_n)_{n \in \omega}\) is an enumeration of a countable open basis of \( G^0 \). We have that \((x,y) \in \overline{E} \) if and only if \( \forall n \in \omega, x \in [U_n] \text{ iff } y \in [U_n] \). It follows that \( \overline{E} \) is \( G_\delta \).

**Lemma 3.3.3.** Suppose that \( G \) is a Polish groupoid such that every orbit of \( G \) is \( G_\delta \). If \( x,y \in G^0 \) are such that \( [x] \neq [y] \) and \( [y] \cap [x] \neq \emptyset \), then \( \overline{[y]} \cap [x] = \emptyset \). Equivalently the quotient space \( G^0/E_G \) is \( T_0 \).

*Proof.* After replacing \( G \) with the restriction of \( G \) to \( [x] \) we can assume that \( \overline{[y]} \subset [x] = X \). Denote by \((U_n)_{n \in \omega}\) an enumeration of a basis of nonempty open subsets of \( G^0 \). By Lemma 3.3.1 every element of \( \bigcap_n [U_n] \) has dense orbit. Since \( [x] \cap [y] = \emptyset \), \( [y] \) is not dense in \( X \) (otherwise it would be co-meager and it would intersect \([x] \)). It follows that, for some \( n \in \omega, y \notin [U_n] \) and hence \([y] \cap U_n = \emptyset \). This shows that \( \overline{[y]} \subset X \setminus U_n \). On the other hand, \( U_n \) is invariant dense open and \([x] \) is co-meager, hence \([x] \subset U_n \). This shows that \( \overline{[y]} \cap [x] = \emptyset \). \( \square \)

**Lemma 3.3.4.** Suppose that \( G \) is a Polish groupoid and \( X \) is a Polish \( G \)-space. If every orbit is \( G_\delta \), then \( E_G^X \) is smooth.
Lemma 3.3.3. We prove the nontrivial implications below.

By [37, Theorem 4.2] it must coincide with the quotient topology. It remains to observe that such a map is Borel. In fact if \( U \) is an open subset of \( G^0 \), then \( \{ x \in G^0 : [x] \cap U \neq \emptyset \} = [U] = r [s^{-1} [U]] \) is open.

Proposition 3.3.5. Suppose that \( G \) is a Polish groupoid and \( X \) is a Polish \( G \)-space. The following statements are equivalent:

1. Every orbit is \( G_\delta \).
2. The orbit equivalence relation \( E^X_G \) is \( G_\delta \).
3. The quotient space \( X/E^X_G \) is \( T_0 \).
4. The quotient topology generates the quotient Borel structure.

Proof. In view of Proposition 2.7.1 we can assume without loss of generality that \( X = G^0 \) and \( G \simeq G^0 \) is the standard action. The implication (2) \( \Rightarrow \) (3) follows from Lemma 3.3.3. We prove the nontrivial implications below.

(1) \( \Rightarrow \) (2) Consider the equivalence relation \( E \) defined as in 3.3.2. Suppose that \( x, y \in X \) are such that \( (x, y) \in \overline{E} \). It follows that \([x] \) and \([y] \) are both dense subsets of \( Y = \overline{[x]} = \overline{[y]} \). Since both the orbit of \( x \) and \( y \) are \( G_\delta \), \([x] \) and \([y] \) are co-meager subsets of \( Y \). It follows that they are not disjoint, and hence \([x] = [y] \). This shows that \( E_G = \overline{E} \) and in particular \( E_G \) is \( G_\delta \).

(3) \( \Rightarrow \) (1) Since the quotient map \( \pi : X \to X/E_G \) is continuous and open, \( X/E_G \) has a countable basis \( \{ U_n : n \in \omega \} \). If \( x \in X \), then

\[
[x] = \bigcap \{ \pi^{-1} [U_n] : n \in \omega, \pi(x) \in U_n \}.
\]

This shows that \([x] \) is \( G_\delta \).

(3) \( \Rightarrow \) (4) The Borel structure generated by the quotient topology is separating and countably generated. By [37, Theorem 4.2] it must coincide with the quotient Borel structure.

(4) \( \Rightarrow \) (3) Observe that the orbits are Borel. Therefore the quotient Borel structure is separating, and hence the quotient topology separates points; i.e. it is \( T_0 \). \( \square \)

The equivalence of the conditions in Proposition 3.3.5 has been proved in [44, Theorem 2.1] under the additional assumption that the orbit equivalence relation is \( F_\sigma \).

3.4. The Glimm-Effros dichotomy. Denote by \( E_0 \) the orbit equivalence relation on \( 2^\omega \) defined by \( (x, y) \in E_0 \) iff \( x(n) = y(n) \) for all but finitely many \( n \in \omega \). Observe that \( E_0 \) can be regarded as the (principal) Polish groupoid associated with the free action of \( \bigoplus_{n \in \omega} \mathbb{Z}/2\mathbb{Z} \) on \( \prod_{n \in \omega} \mathbb{Z}/2\mathbb{Z} \) by translation. The proof of the following result is contained in [44 Section 4]. An exposition of the proof in the case of Polish group actions can be found in [19, Theorem 6.2.1].

Proposition 3.4.1. Suppose that \( G \) is a Polish groupoid. If \( E_G \) is dense and meager in \( G^0 \times G^0 \), then \( E_0 \subseteq_e G \).

Recall that \( E_0 \subseteq_e G \) means that there is an injective continuous functor \( F : E_0 \to G \) such that the restriction of \( F \) to the set of objects is a Borel reduction from \( E_0 \) to \( E_G \); see subsection 2.8. One can then obtain the following consequences.
Proposition 3.4.2. Suppose that $G$ is a Polish groupoid. If $G$ has no $G_{\delta}$ orbits, then $E_0 \subseteq_c G$.

Proof. After replacing $G$ with the restriction of $G$ to a class of the equivalence relation $E$ defined as in Lemma 3.3.2, we can assume that every orbit is dense. By Theorem 3.1.2 every orbit is meager. It follows from Lemma 2.9.1 that $E_G$ is meager. One can now apply Proposition 3.3.1. 

Theorem 3.4.3. Suppose that $G$ is a Polish groupoid. If every $G_{\delta}$ orbit is $F_\sigma$, then either $E_G$ is $G_{\delta}$ or $E_0 \subseteq_c G$.

Proof. Suppose that $E_0 \not\subseteq_c G$. In particular for every $G_{\delta}$ subspace $Y$ of $G^0$, $E_0 \not\subseteq_c G|_Y$. Denote by $E$ the equivalence relation defined as in Lemma 3.3.2. If $y \in X$, define $Y = [y]_{\overline{E}}$ and observe that $Y$ is an $E_G$-invariant $G_{\delta}$ subset of $G^0$. Moreover every $E_G$-orbit contained in $Y$ is dense in $Y$. Since $E_0$ is not continuously reducible to $G$, by Proposition 3.4.2 there is $z \in Y$ such that $[z]_{E_G}$ is a dense $G_{\delta}$ subset of $Y$. In particular $[z]_{E_G}$ is a $G_{\delta}$ subset of $G^0$. Therefore by assumption also $Y \setminus [z]_{E_G}$ is $G_{\delta}$. Since every orbit of $Y$ is dense, $Y \setminus [z]_{E_G}$ must be empty and $[z]_{E_G} = Y = [y]_{\overline{E}}$ is $G_{\delta}$. This shows that every orbit of $G$ is $G_{\delta}$ and hence $E_G$ is $G_{\delta}$ by Proposition 3.3.5.

Corollary 3.4.4. Suppose that $G$ and $H$ are Polish groupoids such that every $G_{\delta}$ orbit is $F_\sigma$. If $E_G$ and $E_H$ are Borel reducible to $E_0$, then $G \leq_B H$ if and only if $E_G \leq_B E_H$.

Proof. Suppose that $E_G \leq_B E_H$. If $E_H$ is smooth, then the conclusion follows from Corollary 2.10.10. If $E_G$ is not smooth, then $G \sim_B H \sim_B E_0$ by Theorem 3.4.3 see Definition 2.8.3.

We can combine Proposition 3.3.5 with Theorem 3.4.3 to get the following result. It was obtained in [14] under the additional assumption that the orbit equivalence relation $E_G$ is $F_\sigma$. The notion of nonatomic and ergodic Borel measure with respect to an equivalence relation can be found in [19] Definition 6.1.5.

Theorem 3.4.5. Suppose that $G$ is a Polish groupoid such that every $G_{\delta}$ orbit is $F_\sigma$. The following statements are equivalent:

1. There is an orbit which is not $G_{\delta}$.
2. $E_0 \subseteq_c G$.
3. $E_0 \leq_B E_G$.
4. There is an $E_G$-nonatomic $E_G$-ergodic Borel probability measure on $G^0$.
5. $E_G$ is not smooth.
6. $E_G$ is not $G_{\delta}$.
7. Some orbit is not open in its closure.

Proof. The implication (1)$\Rightarrow$(2) follows from Theorem 3.4.3. The implication (2)$\Rightarrow$(3) is obvious. For (3)$\Rightarrow$(4), observe that if $f : 2^\omega \to X$ is a Borel reduction from $E_0$ to $E_G$, then the product measure on $2^\omega$, and $\nu$ is the push-forward of $\mu$ under $f$, then $\nu$ is an $E_G$-nonatomic and $E_G$-ergodic Borel probability measure on $G^0$. The implication (4)$\Rightarrow$(5) follows from [19] Proposition 6.1.6. By Lemma 3.3.4 (5) implies (6). The implication (6)$\Rightarrow$(1) is contained in Proposition 3.3.5. Since a set that is open in its closure is $G_{\delta}$, the implication (1)$\Rightarrow$(7) is obvious. Let us show that (7)$\Rightarrow$(1). Suppose that every orbit is $G_{\delta}$, and fix $x \in G^0$. After replacing $G$ with its restriction to the closure of the orbit of $x$, we assume that $x$ has
dense orbit. Therefore \([x]\) is a dense \(G_δ\) in \(x\). Since \([x]\) is by assumption also \(F_σ\), 
\([x] = \bigcup_n F_n\) where the \(F_n\)'s are closed in \(X\). Being \([x]\) nonmeager in \(X\), there is an open subset \(U\) of \(X\) contained in \(F_n\) for some \(n \in \omega\). Hence \([x] = [U]\) is open. □

4. Universal actions

Suppose that \(G\) is a Polish groupoid. The space \(G\) is fibered over the space of objects \(G^0\) via the range map \(r : G \to G^0\). One can then consider the corresponding Effros fibered space \(F(G,G^0)\) of closed subsets of \(G\) contained in \(xG\) for some \(x \in G^0\); see subsection 2.3. Recall that \(F(G,G^0)\) is a standard Borel space fibered over \(G^0\) via the Borel map assigning \(x\) to a closed nonempty subset \(F\) of \(xG\). Moreover \(F(G,G^0)\) has naturally the structure of Borel \(G\)-space given by the map \((\gamma,F) \mapsto \gamma F\) for \(F \subset s(\gamma)G\), where \(\gamma F = \{\gamma ρ : ρ \in F\}\). Similarly the fibered product

\[
\bigstar_{n \in \omega} F(G,G^0) = \{(F_n)_{n \in \omega} \in F(G,G) : \exists x \in G^0 \forall n \in \omega, F_n \subset xG\}
\]

is naturally a Borel \(G\)-space with respect to the coordinate-wise action of \(G\) defined by \(\gamma(F_n)_{n \in \omega} = (\gamma F_n)_{n \in \omega}\). Recall the notion of Borel \(G\)-embedding from subsection 2.5.

**Theorem 4.0.1.** The Borel \(G\)-space \(\bigstar_{n \in \omega} F(G,G^0)\) is a universal Borel \(G\)-space: if \(X\) is any Borel \(G\)-space, then there is a Borel \(G\)-embedding

\[
\varphi : X \to \bigstar_{n \in \omega} F(G,G^0).
\]

The rest of this subsection is dedicated to the proof of Theorem 4.0.1. The following lemma is well known.

**Lemma 4.0.2.** If \(X\) is a Polish space, \(A \subset X\), and \(E(A)\) is the set of \(x \in X\) such that for every neighborhood \(V\) of \(x\), \(V \cap A\) is not meager, then \(E(A)\) is closed in \(X\). Moreover \(A\) has the Baire property iff \(A \triangle E(A)\) is meager.

**Proof.** Clearly \(E(A)\) is closed, and if \(A \triangle E(A)\) is meager, then \(A\) has the Baire property. Observe that if \(A,B \subset X\) are such that \(A \triangle B\) is meager, then \(E(A) = E(B)\). If \(A\) has the Baire property, there is an open subset \(U\) of \(X\) such that \(A \triangle U\) is meager. Thus \(E(A) = E(U)\) is equal to the closure \(\overline{U}\) of \(U\). It follows that \(A \triangle E(A) \subset (A \triangle U) \cup (U \setminus U)\) is meager. □

Suppose that \(X\) is a Borel \(G\)-space. In view of Theorem 5.1.1, we can assume without loss of generality that \(X\) is in fact a Polish \(G\)-space. Fix a countable open basis \(\mathcal{B} = \{B_n : n \in \omega\}\) of nonempty open subsets of \(X\). Assume further that \(\mathcal{A}\) is a countable basis of Polish open subsets of \(G\). Define for \(n \in \omega\) the fibered Borel map \(\varphi_n : X \to F(G,G^0)\) by setting \(\varphi_n(x) = (E(\{γ \in Gp(x) : γx ∈ B_n\}))^{-1}\). Define the Borel fibered map \(\varphi : X \to \bigstar_{n \in \omega} F(G,G^0)\) by \(\varphi(x) = (\varphi_n(x))_{n \in \omega}\).

**Claim.** \(\varphi\) is Borel measurable.

It is enough to show that \(\varphi_n\) is Borel measurable for every \(n \in \omega\). Suppose that \(V \in \mathcal{A}\). We want to show that the set of \(x \in X\) such that \(E(\{γ \in Gp(x) : γx ∈ B_n\})\) meets \(V\) is Borel. Observe that \(E(\{γ \in Gp(x) : γx ∈ B_n\})\) meets \(V\) if and only if there exists \(W \in \mathcal{A}\) such that \(W \subset V\) and the set \(\{γ \in Gp(x) : γx ∈ B_n\}\) is co-meager in \(Wp(x)\). The set of such elements \(x\) of \(X\) is Borel by Lemma 2.10.3.
Proof. \(\varphi\) is \(G\)-equivariant; i.e. \(\varphi(\gamma x) = \gamma \varphi(x)\) for \((\gamma, x) \in G \rtimes X\).

It is enough to show that \(\varphi_n(\gamma x) = \gamma \varphi_n(x)\) for \(n \in \omega\). Observe that
\[
\varphi_n(\gamma x) = (E(\{\rho \in Gr(\gamma) : \rho \gamma x \in B_n\}))^{-1}
\]
and
\[
\varphi_n(x) = (E(\{\tau \in Gp(x) : \tau x \in B_n\}))^{-1}.
\]
We thus have to prove that \(E(\{\rho \in Gr(\gamma) : \rho \gamma x \in B_n\})\) is equal to
\[
E(\{\tau \in Gp(x) : \tau x \in B_n\})^{-1}.
\]
This follows from the fact that \(\tau \mapsto \tau^{-1}\) is a homeomorphism from \(Gp(x)\) to \(Gr(\gamma)\).

Claim. \(\varphi\) is injective.

Assume that \(x, y \in X\) are such that \(\varphi(x) = \varphi(y)\). Thus \(p(x) = p(y)\) and for every \(n \in \omega\), the sets \(\{\gamma \in Gp(x) : \gamma x \in B_n\}\) and \(\{\gamma \in Gp(y) : \gamma y \in B_n\}\) differ by a meager set. Thus \(\forall^* \gamma \in Gp(x), \forall n \in \omega, \gamma x \in B_n\) iff \(\gamma y \in B_n\). Thus for some \(\gamma \in Gp(x), \gamma x \in B_n\) iff \(\gamma y \in B_n\) for all \(n \in \omega\). This implies that \(\gamma x = \gamma y\) and hence \(x = y\).

5. Better topologies

5.1. Polishability of Borel \(G\)-spaces.

Theorem 5.1.1. Suppose that \(G\) is a Polish groupoid. Every Borel \(G\)-space is Borel \(G\)-isomorphic to a Polish \(G\)-space. Equivalently if \(X\) is a Borel \(G\)-space, then there is a Polish topology compatible with the Borel structure of \(G\) that makes the action of \(G\) on \(X\) continuous.

Theorem 5.1.1 answers a question of Ramsay from [45]. The rest of this subsection is dedicated to the proof of Theorem 5.1.1. The analogous statement for actions of Polish groups is proved in a similar way in [3] Theorem 5.2.1. Fix a countable basis \(A\) of Polish open subsets of \(G\). Suppose that \(X\) is a Polish \(G\)-space. We want to define a topology \(t\) on \(X\) such that \(t\) is Polish, the action \(G \rtimes (X, t)\) is continuous; i.e. the anchor map \(p : X \to G^0\) is continuous and \(G \rtimes X \to X, (\gamma, x) \mapsto \gamma x\) is continuous, and \(t\) generates the Borel structure of \(X\).

By Lemma 2.10.5 and \([3\ 5.1.3\ and \ 5.1.4]\) there exists a countable Boolean algebra \(B\) of Borel subsets of \(X\) satisfying the following conditions. (1) For all \(B \in B\) and \(U \in A, B^\Delta U \in B\). (2) The topology \(t'\) generated by the basis \(B\) is Polish.

Observe that the identity function from \(X\) with its original Borel structure to \((X, t')\) is Borel measurable, and hence a Borel isomorphism by \([30\ Theorem 15.1]\). It follows that \(t'\) generates the Borel structure of \(X\). Define \(S\) to be the set \(\{B^\Delta U : B \in B, U \in A\}\) and \(t\) to be the topology on \(X\) having \(S\) as subbasis.

Claim. The action \(G \rtimes (X, t)\) is continuous.

Proof. If \(V \in A,\) then \(p^{-1}[s[V]] = X^\Delta V \in S\). This shows that \(p : X \to G^0\) is \(t\)-continuous. Let us now show that the map \(G \rtimes X \to X\) is \(t\)-continuous. Suppose that \(B \in B, U \in A, (\gamma_0, x_0) \in G \rtimes X\) is such that \(\gamma_0 x_0 \in B^\Delta U\). By Lemma 2.10.3 there are \(W, V \in A\) such that \(V W^{-1} \subseteq U\), \(x_0 \in B^\Delta V\), and \(\gamma_0 \in W\). We claim that \(\gamma x \in B^\Delta U\) for every \(x \in B^\Delta V\) and \(\gamma \in W\). Fix \(x \in B^\Delta V\) and \(\gamma \in W\) and observe that \(V^{-1} \cap VW^{-1} \subseteq U\), and hence it follows from Lemma 2.10.3 that \(\gamma x \in B^\Delta U\). This shows that the action is continuous. \(\square\)
Claim. The space $(X, t)$ is $T_1$.

Proof. Pick distinct points $x, y$ of $X$. If $p(x) \neq p(y)$, then there are disjoint $V, W \in \mathcal{A}$ such that $p(x) \in V$ and $p(y) \in W$. Thus $p^{-1}[V]$ and $p^{-1}[W]$ are open sets separating $x$ and $y$. Suppose that $p(x) = p(y)$. Consider the function $f : Gp(x) \to X \times X$ defined by $f(\gamma) = (\gamma x, \gamma y)$. Observe that $f$ is Borel when $X \times X$ is endowed with the $t' \times t'$ topology. By [30, Theorem 8.38] there is a dense $G_\delta$ subset $Q$ of $Gp(x)$ such that the restriction of $f$ to $Q$ is $(t' \times t')$-continuous. Let $\gamma_0 \in Q$. Since $B$ is a basis for the Polish topology $t'$ on $X$ there are disjoint elements $B, C$ of $B$ such that $\gamma_0 x \in B$ and $\gamma_0 y \in C$. Since $f$ is $(t' \times t')$-continuous on $Q$ there is $U \in \mathcal{A}$ such that $Up(x) \neq \emptyset$ and $f[U p(x) \cap Q] \subset B \times C$. Thus $\forall \gamma \in Up(x), \gamma x \in B$ and $\gamma y \in C$. This shows that $x \in B^{\Delta U}$, $y \in C^{\Delta U}$, $y \notin B^{\Delta U}$, and $x \notin C^{\Delta U}$. This concludes the proof that $(X, t)$ is $T_1$. 

Claim. The space $(X, t)$ is regular.

Proof. Suppose that $B \in \mathcal{B}$ and $U \in \mathcal{A}$. Pick $x_0 \in B^{\Delta U}$. It is enough to show that there is a $t$-open subset $N$ of $B^{\Delta U}$ containing $x_0$ such that the $t$-closure of $N$ is contained in $B^{\Delta V}$. Since $x_0 \in B^{\Delta U}$ by Lemma 2.10.3 there are $W_1, V_1 \in \mathcal{A}$ such that $V_1W_1^{-1} \cup V_1 \subset U$, $p(x_0) \in W_1$, and $x_0 \in B^{\Delta V}$. Since $x_0 \in B^{\Delta V}$, again by Lemma 2.10.3 there are $V_2, W_2 \in \mathcal{A}$ such that $V_2W_2^{-1} \subset V_1$, $p(x_0) \in W_2$, and $x_0 \in B^{\Delta V}$. Define $W \in \mathcal{A}$ such that $p(x_0) \in W \subset W_1^{-1} \cap W_2$. Consider $N = B^{\Delta V} \cap p^{-1}s[W]$ and observe that $N$ is a $t$-open subset of $X$ containing $x_0$. We claim that the closure of $N$ is contained in $B^{\Delta U}$. Define $F = (B^{\Delta V})^\star W$. Define $N \subset F \subset B^{\Delta U}$. Suppose that $x \in X$. If $\gamma \in Wp(x)$ we have that $V_2\gamma^{-1} \subset V_2W^{-1} \subset V_2W_2^{-1} \subset V_1$. Therefore $\gamma x \in B^{\Delta V}$. This being true for every $\gamma \in Wp(x)$, $x \in (B^{\Delta V})^\star W = F$. Suppose now that $x \in F$ and pick $\gamma \in Wp(x)$ such that $\gamma x \in B^{\Delta V}$. We thus have $V_1\gamma \subset V_1W \subset V_1W_1^{-1} \subset U$, which implies by Lemma 2.10.3 that $x = \gamma^{-1}(\gamma x) \in B^{\Delta U}$. This concludes the proof that $N \subset F \subset B^{\Delta U}$. We will now show that the $t$-closure of $N$ is contained in $B^{\Delta U}$. It is enough to show that if $x \notin B^{\Delta U}$, then there is a $t$-open neighborhood of $x$ disjoint from $N$. This is clear if $p(x) \notin s[W]$. Indeed in this case one can fix an open neighborhood $\Omega$ of $p(x)$ disjoint from $s[W]$ and consider the $t$-open neighborhood $p^{-1}[\Omega]$ of $x$. This is disjoint from $N$ since $N \subset p^{-1}s[W]$. Suppose now that $p(x) \in s[W]$. Since $F$ is relatively closed in $p^{-1}s[W]$ and $N \subset F \subset B^{\Delta V} \cap p^{-1}s[W]$, we have that $p^{-1}s[W] \setminus F$ is an open subset of $X$ containing $x$ and disjoint from $N$. This concludes the proof that the closure of $N$ is contained in $B^{\Delta V}$. We have thus found an open neighborhood $N$ of $x$ whose closure is contained in $B^{\Delta V}$. This concludes the proof that $(X, t)$ is regular. 

Claim. The space $(X, t)$ is strong Choquet.

Proof. Define $C$ to be the (countable) set of nonempty finite intersections of elements of $\mathcal{S}$ and observe that $C$ is a basis for $(X, t)$. Fix a well ordering $E$ of the countable set $C \times B \times A$. Let $d'$ be a complete metric on $X$ compatible with the Polish topology $t'$. We want to define a strategy for Player II in the strong Choquet game; see [30, Section 8.D]. Suppose that Player I plays $t$-open sets $N_i$ for $i \in \omega$ and $x_i \in N_i$. At the $i$-th turn Player II will choose an element $(M_i, B_i, U_i)$ of $C \times B \times A$
in such a way that the following properties hold:

1. \(x_i \in M_i\).
2. The \(t\)-closure of \(M_i\) is contained in \(N_i\).
3. The closure of \(U_{i+1}\) in \(U_0\) is contained in \(U_i\).
4. The \(t'\)-closure of \(B_{i+1}\) is contained in \(B_i\).
5. The \(d'\)-diameter of \(B_i\) is less than \(2^{-i}\).
6. The \(d_{U_0}\)-diameter of \(U_i\) is less than \(2^{-i}\) for \(i \geq 1\), where \(d_{U_0}\) is a compatible complete metric on \(U_0\).
7. \(M_i \subset B^\Delta_{U_i}\).

Player II’s strategy is the following: At the \(i\)-th turn pick the \(E\)-least tuple \((M_i, B_i, U_i)\) in \(C \times B \times A\) satisfying properties (1)–(7). We need to show that the set of such tuples is nonempty. Observe that \(x_i \in N_i \subset M_{i-1} \subset B_{i-1}^\Delta\). Thus \(B_{i-1}^\Delta U_{i-1}\) is a countable Boolean algebra and hence there is a countable Boolean algebra \((B, \lor, \land, 0, 1)\) containing \(B_{i-1}^\Delta\). Let \(x_i \in B_{i-1}\) such that (3)–(6) hold. Fix \(x_i \in B_{i-1}\) such that (3)–(5) hold, and \(y_{i+1} \in B_i\) such that \(\gamma x_i \in B_{i+1}\). Consider \(M = B^\Delta_{U_i} \cap N_i\) and observe that \(M\) is a \(t\)-open set containing \(x_i\). Since \((X, t)\) is regular there is \(M_i \in \mathcal{C}\) such that \(x_i \in M_i\) and the closure of \(M_i\) is contained in \(N_i \cap B^\Delta_{U_i}\). This ensures that (1), (2), (7) are satisfied. We now show that this gives a winning strategy for Player II. For every \(i \in \omega\) we have that \(x_i \in M_i \subset B^\Delta_{U_i}\), and hence there is \(\gamma_i \in U_i p(x_i)\) such that \(\gamma_i x_i = y_i \in B_i\). Define \(\gamma\) to be the limit of the sequence \((\gamma_i)_{i \in \omega}\) in \(U_0\) and \(y\) to be the \(t'\)-limit of the sequence \((y_i)_{i \in \omega}\) in \(Y\). Observe that \(p(y) = \lim_j p(y_j) = \lim_j r(\gamma_j) = r(\gamma)\). Define \(x = \gamma^{-1} y \in X\) and observe that \(x\) is the \(t\)-limit of the sequence \((x_i)_{i \in \omega}\). Fix \(i \in \omega\). For \(j > i\) we have that \(x_j \neq N_j \subset M_i\) and hence \(x\) is contained in the \(t\)-closure of \(M_i\), which is in turn contained in \(N_i\). This shows that \(x \in \bigcap_{i \in \omega} N_i\), concluding the proof that Player II has a winning strategy in the strong Choquet game in \((X, t)\).

The proof of Theorem 5.1.1 is finished recalling that a regular \(T_1\) strong Choquet space is Polish [30, Theorem 8.18].

5.2. Finer topologies for Polish \(G\)-spaces.

**Theorem 5.2.1.** Suppose that \(G\) is a Polish groupoid and \((X, \tau)\) is a Polish \(G\)-space. Assume that \(V \subset G\) is an open Polish subset, \(P \subset X\) is \(\Sigma^0_\alpha\) for some \(\alpha \in \omega_1\), and \(Q = P^{\Delta V}\). There is a topology \(t\) on \(X\) such that:

1. \(t\) is a Polish topology;
2. \(t\) is finer than \(\tau\);
3. \(Q\) is \(t\)-open;
4. the action of \(G\) on \((X, t)\) is continuous;
5. \(t\) has a countable basis \(B\) such that for every \(B \in B\) there is \(n \in \omega\) such that \(B\) is \(\Sigma^0_{\alpha+n}\) with respect to \(\tau\).

The analogous statement for actions of Polish groups is proved in a similar way in [3 Theorem 5.1.8]. Let \(A\) be a countable basis of Polish open subsets for \(G\) containing \(V\) and let \(D\) be a countable basis for \((X, \tau)\). By Lemma 2.10.6 and [3 5.1.3, 5.1.4] there is a countable Boolean algebra \(B\) of subsets of \(X\) satisfying the following:

1. for \(B \in B\) and \(U \in A\), \(B^{\Delta V} \subset B\);
2. \(P \in B\);
\( (3) \ \mathcal{D} \subset \mathcal{B}; \)
\( (4) \ \mathcal{B} \) is a basis for a Polish topology \( t'; \)
\( (5) \) for every \( B \in \mathcal{B}, \) there is \( n \in \omega \) such that \( B \) is \( \Sigma_{\alpha+n} \) with respect to \( \tau. \)

Define \( S = \{ B^\Delta V : B \in \mathcal{B}, V \in \mathcal{A} \}, \) and \( S^* = S \cup D. \) Consider the topology \( t \) on \( X \) having \( S^* \) as a subbasis. We claim that \( t \) is a Polish topology finer than \( \tau \) and coarser that \( t' \) making the action continuous. Clearly \( t \) is finer than \( \tau, \) and in particular \( p : (X, t) \to G^0 \) is continuous. The proof that the action is continuous and that \( t \) is a Polish topology is analogous to the proof of Theorem 5.1.1. The following corollary can be obtained from Theorem 5.2.1 together with [3, subsection 5.1.3].

**Corollary 5.2.2.** Suppose that \( G \) is a Polish groupoid, and \( (X, \tau) \) is a Polish \( G \)-space. If \( \mathcal{J} \) is a countable collection of \( G \)-invariant Borel subsets of \( X, \) then there is a Polish topology \( t \) on \( X \) finer than \( \tau \) and making the action continuous such that all elements of \( \mathcal{J} \) are \( t \)-clopen.

### 6. Borel orbit equivalence relations

#### 6.1. A Borel selector for co-sets

Suppose that \( G \) is a Polish groupoid. Denote by \( F(G) \) the (standard) Borel space of closed subsets of \( G \) endowed with the Effros Borel structure; see the Appendix. A similar proof as [30, Theorem 12.13] shows that there is a Borel function \( \sigma : F(G) \setminus \{ \emptyset \} \to G \) such that \( \sigma(A) \in A \) for every nonempty closed subset \( A \) of \( G. \) Denote by \( S(G) \) the Borel space of closed subgroupoids of \( G. \) This is the Borel subset of \( F(G) \) containing the closed subsets \( H \) of \( G \) such that for \( \gamma, \rho \in H, \) \( \gamma^{-1} \in H \) and if \( r(\gamma) = s(\rho), \) then \( \rho \gamma \in H. \) If \( H \in S(G) \) denote by \( \sim_H \) the equivalence relation on \( G \) defined by \( \gamma \sim_H \rho \) iff \( \gamma = \rho h \) for some \( h \in H \) or, equivalently, \( \gamma H = \rho H. \)

**Proposition 6.1.1.** The relation \( \sim \) on \( G \times S(G) \) defined by \( (\gamma, H) \sim (\gamma', H') \) iff \( H = H' \) and \( \gamma H = \gamma' H' \) has a Borel transversal \( T. \)

**Proof.** Define the map \( f \) from \( G \times S(G) \) to \( F(G) \) by \( f(\gamma, H) = \gamma H. \) We claim that \( f \) is Borel. Let us show that if \( U \) is an open subset of \( G, \) then the set \( A_U \) of pairs \( (\gamma, H) \in G \times S(G) \) such that \( \gamma H \cap U \neq \emptyset \) is Borel. Since the set \( \{ (\rho, \gamma, H) \in G \times G \times S(G) : \gamma^{-1} \rho \in H \text{ and } \rho \in U \} \) is Borel, its projection \( A_U \) on the last two coordinates is analytic. We want to show that \( A_U \) is co-analytic. Fix a countable basis of Polish open sets \( \{ U_n : n \in \omega \} \) for \( G. \) Observe that \( (\gamma, H) \in A_U \) if and only if there is \( n \in \omega \) such that \( \gamma U_n \subset U \) and \( U_n \cap H \neq \emptyset. \) It is now enough to show that \( \{ \gamma \in G : \gamma U_n \subset U \} \) is co-analytic. This follows from the fact that

\[
\{ (\gamma, \rho) \in G \times U_n : \text{either } r(\rho) \neq s(\gamma) \text{ or } r(\rho) = s(\gamma) \text{ and } \gamma \rho \in U \}
\]

is a Borel set and its co-projection on the first coordinate is \( \{ \gamma \in G : \gamma U_n \subset U \}. \) If now \( \sigma : F(G) \setminus \{ \emptyset \} \to G \) is a Borel map such that \( \sigma(A) \in A \) for every nonempty closed subset \( A \) of \( G, \) define \( g(\gamma, H) = ((\sigma \circ f)(\gamma, H), H). \) Observe that \( g \) is a Borel selector for \( \sim. \) Therefore the set

\[
T = \{ (\gamma, H) : g(\gamma, H) = (\gamma, H) \}
\]

is a Borel transversal for \( \sim. \)

**Corollary 6.1.2.** If \( G \) is a Polish groupoid and \( X \) is a Borel \( G \)-space, then the orbits are Borel subsets of \( X. \)
Proof. Observe that the stabilizer $G_x$ is a closed subgroup of $p(x)Gp(x)$ by Theorem 9.17. Consider a Borel transversal $T_x$ for the equivalence relation $\sim_{G_x}$. The function $\gamma \mapsto \gamma x$ from $T \cap Gx$ to $X$ is a 1:1 Borel function from $T_x$ onto the orbit of $x$. This shows that the orbit of $x$ is Borel by Theorem 15.1. □

Corollary 6.1.2 also follows from Proposition 2.11.1 and Theorem 5.1.1.

6.2. Borel orbit equivalence relations. Suppose that $G$ is a Polish groupoid and $X$ is a Polish $G$-space. If $x \in X$, then Lemma 2.10.5 and [3, 5.1.3 and 5.1.4] show that there is a sequence $(B_{x,n})_{n \in \omega}$ of Borel subsets of $X$ such that $[x] = B_{x,0}$ and $B(x) = \{B_{x,n} : n \in \omega\}$ is a Boolean algebra that is a basis for a topology $t(x)$ on $X$ making the action continuous, and such that $B^\Delta U \in B(x)$ whenever $B \in B(x)$ and $U \in A$. It is implicit in the proof of Lemma 2.10.5 and [3, 5.1.3, 5.1.4] that, under the additional assumption that the orbit equivalence relation $E^X_G$ is Borel, the dependence of the sequence $(B_{x,n})_{n \in \omega}$ from $x$ is Borel; i.e. the relation $B(y,x,n) \iff y \in B_{x,n}$ is Borel. This concludes the proof of the following lemma; see also [3, Lemma 7.1.3].

**Lemma 6.2.1.** Suppose that $G$ is a Polish groupoid and $X$ is a Polish $G$-space. Assume that $\mathcal{A}$ is a countable basis of Polish open subsets of $G$. If the orbit equivalence relation $E^X_G$ is Borel, then there is a Borel subset $B$ of $X \times X \times X$ such that, letting $B_{x,n} = \{y : (y,x,n) \in B\}$ and $B(x) = \{B_{x,n} : n \in \omega\}$, for every $x \in X$ the following hold: (1) $[x] = B_{x,0}$; (2) $B^\Delta U \in B(x)$ for every $B \in B(x)$ and $U \in A$; (3) $B(x)$ is a Boolean algebra; (4) $B(x)$ is a basis for a Polish topology $t(x)$ on $X$ making $X$ a Polish $G$-space.

The following result provides a characterization of the Borel $G$-spaces with Borel orbit equivalence relation. The analogous result for Polish group actions is [3, Theorem 7.1.2].

**Theorem 6.2.2.** Suppose that $G$ is a Polish groupoid and $X$ is a Borel $G$-space. The following statements are equivalent:

1. The function $X \to F(G)$, $x \mapsto G_x$ is Borel.
2. The function $X \times X \to F(G)$, $(x,y) \mapsto G_{x,y}$ is Borel.
3. The orbit equivalence relation $E^X_G$ is Borel.

Recall that, for $x, y \in G^0$, $G_x$ denotes the (closed) stabilizer $\{\gamma \in Gp(x) : \gamma x = x\}$, while $G_{x,y}$ is the set $\{\gamma \in Gp(x) : \gamma x = y\}$; see subsection 2.5.

**Proof.** By Theorem 5.1.1 we can assume without loss of generality that $X$ is a Polish $G$-space. Fix a countable basis $\mathcal{A} = \{U_n : n \in \omega\}$ of nonempty Polish open subsets of $G$. Denote also by $T \subset G \times S(G)$ a Borel transversal for the relation $(\gamma, H) \sim (\gamma', H')$ if $H = H'$ and $\gamma H = \gamma' H'$ as in Proposition 6.1.1. We prove the nontrivial implications below.

1. $\Rightarrow 2$. Fix a nonempty open subset $U$ of $G$. It is enough to show that the set

$$\{(x,y) \in X \times X : U \cap G_{x,y} \neq \emptyset\}$$

is co-analytic. Observe that $G_{x,y} \cap U \neq \emptyset$ if and only if there is a unique $\gamma \in G$ such that $s(\gamma) = p(x)$, $(\gamma, G_x) \in T$, $\gamma x = y$, and $G_x \cap U \neq \emptyset$. Moreover $G_x \cap U \neq \emptyset$ if and only if there is $n \in \omega$ such that $\gamma U_n \subset U$ and $U_n \cap G_x \neq \emptyset$. Fix $n \in \omega$ and recall that by the proof of Proposition 6.1.1 $\{\gamma \in G : \gamma U_n \subset U\}$ is co-analytic. This concludes the proof that $\{(x,y) \in X \times X : U \cap G_{x,y} \neq \emptyset\}$ is co-analytic.
(1)⇒(3) Observe that \((x, y) \in E_G\) if and only if there is a unique \(\gamma \in G\) such that \((\gamma, G_x) \in T\) and \(\gamma x = y\).

(3)⇒(1) Suppose that \(B, B(x), t(x),\) and \(B_{x,n}\) for \(x \in X\) and \(n \in \omega\) are defined as in Lemma \[6.2.3\]. Observe that the orbit \([x] = B_{x,0}\) is open in \(t(x)\). It follows from Lemma \[3.1.3\] that the map \(G p(x) / G_x \to [x]\), \(\gamma G_x \mapsto \gamma x\) is a \(t(x)\)-homeomorphism. We want to show that for every \(U \in \mathcal{A}\) the set \(\{x \in X : G_x \cap U \neq \emptyset\}\) is Borel. It is enough to show that for every basic nonempty \(U, V\) the set \(\{x \in X : U G_x \cap V \neq \emptyset\}\) is co-analytic. We claim that \(U G_x \cap V \neq \emptyset\) if and only if \(\exists U_m \subseteq U\) such that \(\forall B \in B(x), x \in B^{\Delta U_m}\) implies \(x \in B^{\Delta V}\). In fact suppose that \(U G_x \cap V \neq \emptyset\) and pick \(m\) such that \(U_m \subseteq U\) and \(U_m \gamma \subseteq V\) for some \(\gamma \in G_x\). If \(x \in B^{\Delta U_m}\), then \(x = \gamma^{-1} x \in B^{\Delta U_m} \gamma\) and hence \(x \in B^{\Delta V}\) by Lemma \[2.10.3\]. Conversely suppose that \(U G_x \cap V = \emptyset\) and hence \(\{\gamma x : \gamma \in U\}\) and \(\{\gamma x : \gamma \in V\}\) are disjoint. Fix \(m \in \omega\). Since the map \(G p(x) / G_x \to [x], \gamma G_x \mapsto \gamma x\) is a \(t(x)\)-homeomorphism, the set \(\{\gamma x : \gamma \in U_m\}\) is open in \([x]\). Thus there is \(B \in B(x)\) such that \(x \in B \subseteq \{\gamma x : \gamma \in U_m\}\). Moreover \(\{\gamma \in G p(x) : \gamma x \in B\}\) is an open subset of \(G p(x)\). Therefore there is \(k \in \omega\) such that \(U_k \subseteq U_m\) and \(U_k p(x) \subseteq \{\gamma \in G p(x) : \gamma x \in B\}\). In particular \(x \in B^{\Delta U_k}\) but \(x \notin B^{\Delta V}\). \(\square\)

7. Countable Borel groupoids

7.1. Actions of inverse semigroups on Polish spaces. An inverse semigroup \((T, t)\) is a semigroup \(T\) such that every \(t \in T\) has a semigroup-theoretic inverse \(t^* \in T\). This means that \(t^*\) is the unique element of \(T\) such that \(tt^*t = t\) and \(t^*tt^* = t^*\). If \(T\) is an inverse semigroup, then the set \(E(T)\) of idempotent elements is a commutative subsemigroup of \(T\), and hence a semilattice; see \[12\] Proposition 2.1.1. In particular \(E(T)\) has a natural order defined by \(e \leq f\) iff \(ef = fe = e\). Observe that for every \(t \in T\) the elements \(tt^*\) and \(t^*t\) are idempotent.

Suppose that \(X\) is a Polish space. The semigroup \(\mathcal{H}(X)\) of partial homeomorphisms between open subsets of \(X\) is clearly an inverse semigroup.

Definition 7.1.1. An action \(\theta : T \curvearrowright X\) of a countable inverse semigroup \(T\) on the Polish space \(X\) is a semigroup homomorphism \(\theta : t \mapsto \theta_t\) from \(T\) to \(\mathcal{H}(X)\).

Observe that a semigroup homomorphism between inverse semigroups automatically preserves inverses; see \[12\] Proposition 2.1.1.

7.2. Étale Polish groupoids. Suppose that \(G\) is a Polish groupoid. A subset \(u\) of \(G\) is a bisection if the source and range maps restricted to \(u\) are injective. A bisection of \(G\) is open if it is an open subset of \(G\). It is not difficult to verify that the following conditions are equivalent:

1. The source and range maps of \(G\) are local homeomorphisms from \(G\) to \(G^0\).
2. Composition of arrows in \(G\) is a local homeomorphism from \(G^2\) to \(G\).
3. \(G\) has a countable basis of open bisections.
4. \(G\) has a countable inverse semigroup of open bisections that is the basis for the topology of \(G\).
5. \(G^0\) is an open subset of \(G\).

When these equivalent conditions are satisfied, \(G\) is called an étale Polish groupoid. If \(G\) is an étale Polish groupoid, then in particular for every \(x \in G^0\) the fiber \(Gx\) is a countable discrete subset of \(G\).
7.3. The groupoid of germs. Suppose that $\theta : T \curvearrowright X$ is an action of a countable inverse semigroup on a Polish space. We want to associate to such an action an étale Polish groupoid $G(\theta, T, X)$ that contains all the information about the action. This construction can be found in [11] in the case when $X$ is locally compact.

If $e \in E(T)$ denote by $D_e$ the domain of $\theta_e$. Observe that the domain of $\theta_e$ is $D_{\theta_e}$ and the range of $\theta_e$ is $D_{\theta_e^*}$. Define $\Omega$ to be the subset of $T \times X$ of pairs $(u, x)$ such that $x \in D_{u \cdot u}$. Consider the equivalence relation $\sim$ on $\Omega$ defined by $(u, x) \sim (v, y)$ iff $x = y$ and for some $e \in E(T)$, $ue = ve$ and $x \in D_e$. The equivalence class $[u, x]$ of $(u, x)$ is called the germ of $u$ at $x$. Observe that if $e$ witnesses that $(u, x) \sim (v, x)$, then, after replacing $e$ with $u^*uv^*ve$ we can assume that $e \leq u^*u$ and $e \leq v^*v$. It can be verified as in [11, Proposition 4.7] that if $(u, x)$ and $(v, y)$ are in $\Omega$ and $x = \theta_v(y)$, then $(uv, y) \in \Omega$. Moreover the germ $[uv, y]$ of $uv$ at $y$ depends only on $[u, s]$ and $[v, t]$.

One can then define the groupoid $G(\theta, T, X) = \Omega / \sim$ of germs of the action $T \curvearrowright X$ obtained by setting $G(\theta, T, X)^2 = \{([u, x], [v, y]) : \theta_v(y) = x\}$, $[u, x][v, y] = [uv, y]$, and $[u, x]^{-1} = [u^*, \theta_u(x)]$.

Observe that the map $x \mapsto [e, x]$ from $X$ to $G$, where $e$ is any element of $E(T)$ such that $x \in D_e$ is a well-defined bijection from $X$ to the set of objects $G^0$ of $G$. Identifying $X$ with $G^0$ we have that the source and range maps $s$ and $r$ are defined by $s[u, x] = x$ and $r[u, x] = \theta_u(x)$. We now define the topology of $G(\theta, T, X)$. For $u \in T$ and $U \subset D_{u \cdot u}$ open define $\Theta(u, U) = \{[u, x] : x \in U\}$. It can be verified as in [11] Proposition 4.4, Proposition 4.5, Corollary 4.6, Proposition 4.7, and Proposition 4.8 that the following hold:

1. $G(\theta, T, X)$ is an étale Polish groupoid.
2. The map $x \mapsto [e, x]$ where $e$ is any element of $E(T)$ such that $x \in D_e$ is a homeomorphism from $X$ onto the space of objects of $G(\theta, T, X)$.
3. If $u \in T$ and $U \subset D_{u \cdot u}$, then $\Theta(u, U)$ is an open bisection of $U$, and the map $x \mapsto [u, x]$ is a homeomorphism from $U$ onto $\Theta(u, U)$.
4. If $\mathcal{A}$ is a basis for the topology of $X$, then the collection $\{\theta_\ast(u, A \cap D_{u \cdot u}) : u \in S, A \in \mathcal{A}\}$ is a basis of open bisections for $G(\theta, T, X)$.

7.4. Regularity of the groupoid of germs. The groupoid of germs $G(\theta, T, X)$ for an action $\theta : T \curvearrowright X$ is in general not Hausdorff, even when $X$ is locally compact. Here we isolate a condition that ensures that $G(\theta, T, X)$ is regular.

Define the order $\leq$ on $T$ by setting $u \leq v$ iff $u = vu^*u$. Observe that this extends the order of $E(T)$. Moreover if $u \leq v$, then $u^*u = v^*vu^*v^*v = v^*vu^*u$ and hence $u^*u \leq v^*v$. We say that $T$ is a semilattice if it is a semilattice with respect to the order $\leq$ just defined; i.e. for every pair $u, v$ of elements of $T$ there is a largest element $u \wedge v$ below both $u$ and $v$.

**Proposition 7.4.1.** Suppose that $T$ is a semilattice. If there is a subset $C$ of $T$ such that:

1. for every $u \in T$ and $x \in D_{u \cdot u}$ there is $c \in C$ such that $x \in D_{(u \wedge c)^\ast(u \wedge c)}$, and
2. for every distinct $c, d \in C$, $\Theta(c, D_{c \cdot c}) \cap \Theta(d, D_{d \cdot d}) = \emptyset$,

then the groupoid of germs $G(\theta, T, X)$ is regular.

**Proof.** Suppose that $[u, x]$ is an element of $G(\theta, T, X)$ and $W$ is an open neighborhood of $[u, x]$ in $G(\theta, T, X)$. There is an open subset $U$ of $X$ contained in $D_{u \cdot u}$
such that \([u, x] \in \Theta(u, U) \subset W\). Pick \(c \in C\) such that \(x \in D_{(u \wedge c)^* (u \wedge c)}\) and an open neighborhood \(V\) of \(x\) whose closure \(\overline{V}\) is contained in \(U \cap D_{(u \wedge c)^* (u \wedge c)}\). We claim that \(\Theta(u \wedge c, V)\) is an open neighborhood of \([u, x]\) whose closure is contained in \(W\). To show this it is enough to show that \(\Theta(u \wedge c, \overline{V})\) is closed in \(G(\theta, T, X)\).

Pick \([v, y] \in G(\theta, T, X) \setminus \Theta(u \wedge c, \overline{V})\). If \(y \notin \overline{V}\), then clearly there is an open neighborhood of \([t, y]\) disjoint from \(\Theta(u \wedge c, \overline{V})\). Suppose that \(y \in \overline{V}\). Pick \(d \in C\) such that \(y \in D_{(u \wedge d)^* (u \wedge d)}\). In such case we have that \(\Theta(u \wedge d, D_{(u \wedge d)^* (u \wedge d)}\) is an open neighborhood of \(y\) disjoint from \(\Theta(u \wedge c, \overline{V})\). This concludes the proof. □

7.5. Étale groupoids as groupoids of germs. Suppose that \(G\) is an étale Polish groupoid and \(\Sigma\) is a countable inverse semigroup of open bisections of \(G\). One can define the standard action of \(\Sigma\) on \(G^0\) by setting \(D_e = e\) for every \(e \in E(\Sigma)\), and \(\theta_u : D_u^* u \to D_u u^*\) by \(\theta_u(x) = r(u x)\), where \(u x\) is the only element of \(u\) with source \(x\). The same proof as [11] Proposition 5.4 shows the following fact:

**Proposition 7.5.1.** Suppose that \(\Sigma\) is a countable inverse semigroup of open bisections of \(G\) such that \(\bigcup \Sigma = G\) and for every \(u, v \in \Sigma\), \(u \cap v\) is the union of the elements of \(\Sigma\) contained in \(u \cap v\). Consider the standard action \(\theta : \Sigma \curvearrowright G^0\). The map from \(G(\theta, \Sigma, X)\) to \(G\) assigning to the germ \([u, x]\) of \(u\) at \(x\) the unique element of \(u\) with source \(x\) is well defined, and it is an isomorphism of étale Polish groupoids.

In particular every étale Polish groupoid is isomorphic to the groupoid of germs of an action of an inverse semigroup on a Polish space.

7.6. Borel bisections. We will say that a (standard) Borel groupoid is countable if for every \(x \in G^0\), the set \(G x = s^{-1}(\{x\})\) is countable. Observe that the countable Borel equivalence relations are exactly the principal countable Borel groupoids.

Suppose that \(G\) is a countable Borel groupoid. Observe that the set \(S(G)\) of Borel bisections of \(G\) is an inverse semigroup. The idempotent semilattice \(E(S(G))\) is the Boolean algebra of Borel subsets of \(G^0\). The order \(\leq\) on \(S(G)\) as in subsection 7.4 is defined by \(u \leq v\) iff \(u \subseteq v\). Therefore \(E(S(G))\) is a semilattice with \(u \wedge v = u \cap v\).

The following lemma is an easy consequence of the Luzin-Novikov uniformization theorem [30] Theorem 18.10] applied to the graph of \(s\).

**Lemma 7.6.1.** Suppose that \(X, Z\) are standard Borel spaces and \(s : Z \to X\) is a Borel countable-to-one surjection. There is a countable partition \((P_n)_{n \in \omega}\) of \(Z\) into Borel subsets such that \(s|_{P_n}\) is 1:1 for every \(n \in \omega\).

**Proposition 7.6.2.** If \(G\) is a countable Borel groupoid, then there is a countable partition of \(G\) into Borel bisections. Moreover for every \(n \in \omega\) we have that \(\{x \in G^0 : |G x| = n\}\) is Borel.

**Proof.** The source map \(s : G \to G^0\) satisfies the hypothesis of Lemma 7.6.1. Therefore one can find a countable partition \(\mathcal{H}\) of \(G\) into Borel subsets such that the source map is 1:1 on every element of \(\mathcal{H}\). Define \(\mathcal{C} = \{u \cap v^{-1} : u, v \in \mathcal{H}\}\) and observe that \(\mathcal{C}\) is a countable collection of pairwise disjoint Borel bisections of \(G\).

Observe now that for every \(u \in \mathcal{C}\), \(\{x \in G^0 : \exists \gamma \in u, x = s(\gamma)\} = s[u] = u^{-1} u\) is Borel being the 1:1 image of a Borel set. Moreover \(|G x| = m\) iff \(\exists u_0, \ldots, u_{m-1} \in \mathcal{C}\) pairwise distinct such that \(x \in u_i u_i^{-1}\) for \(i \in m\) and \(\forall w \in \mathcal{C}\) if \(x \in w w^{-1}\), then \(w = u_i\) for some \(i \in m\). □
7.7. A Polish topology on countable Borel groupoids. In this subsection we observe that any countable Borel groupoid is Borel isomorphic to a regular zero-dimensional étale Polish groupoid. Suppose that $G$ is a countable Borel groupoid. Pick a countable partition $C$ of $G$ into full Borel bisections. Denote as before by $S(G)$ the inverse semigroup of Borel bisections of $G$. Consider the smallest inverse subsemigroup $T$ of $S(G)$ containing $C$ with the property that $u \cap v \in T$ whenever $u, v \in T$. This can be seen to exist by taking the intersection of all the inverse subsemigroups of $S(G)$ that contain $C$ and are closed under intersection. One can also construct $T$ starting from $C$ and closing it up under products and intersections. This alternative description shows that $T$ is countable. By Exercise 13.5 there is a zero-dimensional Polish topology $\tau^0$ on $G^0$ generating the Borel structure on $G^0$ such that $u^{-1}u$ is clopen for every $u \in T$. Consider the standard action $\theta$ of $T$ on $(G^0, \tau^0)$ and observe that it satisfies the condition of Proposition 7.4.1. Therefore the associated groupoid of germs $\mathcal{G}(\theta, T, G^0)$ is an étale zero-dimensional regular Polish groupoid. Arguing as in the proof of Proposition 5.4 one can verify that the function $\phi$ from $G$ to $\mathcal{G}(\theta, T, G^0)$ sending $\gamma$ to $[c, s(\gamma)]$ where $c$ is the only element of $C$ such that $\gamma \in C$ is a well-defined Borel isomorphism of countable Borel groupoids.

7.8. Treeable Borel groupoids. Suppose that $G$ is a countable Borel groupoid.

Definition 7.8.1. A Borel graphing $Q$ of $G$ is a Borel subset $Q$ of $G \setminus G^0$ such that $Q = Q^{-1}$ and $\bigcup_{n \in \omega} Q^n = G$, where $Q^0 = G^0$.

Suppose that $Q$ is a graphing of $G$. Define $P^*(Q)$ to be the set of finite nonempty sequences $(\gamma_i)_{i \in n+1}$ in $Q$ such that $r(\gamma_i) = s(\gamma_{i+1})$ and $\gamma_{i+1} \neq \gamma_i^{-1}$ for $i \in n$. For $(\gamma_i)_{i \in n+1}$ in $P^*(Q)$ one can define the product $\gamma_n \gamma_{n-1} \cdots \gamma_0$ in $G$.

Definition 7.8.2. A Borel graphing $Q$ of a countable Borel groupoid $G$ is a treeing if for every $(\gamma_i)_{i \in n+1} \in P^*(Q)$, $\gamma_n \gamma_{n-1} \cdots \gamma_0 \notin G^0$.

A Borel graphing $Q$ of a countable Borel groupoid $G$ is a treeing if and only if for every $\gamma \in G \setminus G^0$ there is exactly one element $(\gamma_i)_{i \in n+1}$ of $P^*(Q)$ such that $\gamma_n \gamma_{n-1} \cdots \gamma_0 = \gamma$. A countable Borel groupoid is treeable when it admits a treeing.

It is not difficult to verify that a principal countable Borel groupoid is treeable precisely when it is treeable as an equivalence relation. A countable groupoid is treeable as a groupoid if and only if it is a free group. In analogy with free groups, if $Q$ is a treeing and $(\gamma_i)_{i \in n+1} \in P^*(Q)$, then we say that $\gamma_n \cdots \gamma_0$ is a reduced word and that the length $l(\gamma_n \cdots \gamma_0)$ of $\gamma_n \cdots \gamma_0$ is $n + 1$.

Proposition 7.8.3. Suppose that $G$ is a countable Borel groupoid. If there is a Borel complete section $A$ for $E_G$ such that $G|_A$ is treeable, then $G$ is treeable.

Proof. Pick a Borel function $f : G^0 \to G$ such that $f(a) = a$ for $a \in A$, $s(f(x)) = x$ and $r(f(x)) \in A$ for $x \in G^0$. Suppose that $Q_A$ is a treeing for $G|_A$. Observe that $Q_A \cup f[\overline{G^0 \setminus A}] \cup f[\overline{G^0 \setminus A}]^{-1}$ is a treeing for $G$. \qed

The rest of this section is dedicated to the proof that a Borel subgroupoid of a treeable countable Borel groupoid is treeable. Recall that a subgroupoid of a groupoid $G$ is a subset $H$ of $G$ with the property that, for any $\gamma, \rho \in H$, $\gamma^{-1} \in H$, and if $r(\rho) = s(\gamma)$, then $\gamma \rho \in H$. A Borel subgroupoid of a standard Borel groupoid $G$ is a subgroupoid $H$ of $G$ that is Borel as a subset of $G$. 

Proposition 7.8.4. Suppose that $G$ is a countable Borel groupoid. Consider a Borel subgroupoid $H$ of $G$. If $G$ is treeable, then $H$ is treeable.

In the particular case when $G$ is a countable group, Proposition 7.8.4 recovers the classical Nielsen-Schreier theorem asserting that a subgroup of a countable free group is free. In the case when $G$ is a principal groupoid, Proposition 7.8.4 recovers Proposition 3.3(iii)]. The rest of this section is dedicated to presenting a proof of Proposition 7.8.4. The strategy of our proof will be a Borel version for groupoids of Schreier’s proof of the Nielsen-Schreier theorem, an exposition of which can be found in [27, Chapter 2].

Proof of Proposition 7.8.4. Denote by $\sim_H$ the equivalence relation on $HG = \{ \gamma \in G : r(\gamma) \in H \}$ defined by setting $\gamma \sim_H \rho$ if and only if $H \gamma = H \rho$. Observe that the $\sim_H$-class of an element $\rho$ of $HG$ is the right coset $H \rho$. We say that a Borel subset $U$ of $G$ has the Schreier property if for any reduced word $\gamma_n \cdots \gamma_0 \in U$ and for any $k \in n + 1$ one has that $\gamma_n \cdots \gamma_k \in U$. A Schreier Borel transversal for $H$ is a Borel transversal for the equivalence relation $\sim_H$ which moreover has the Schreier property.

Suppose that $Q$ is a treeing for $G$. We define a “lexicographical ordering” on each fiber $Gx$. By Proposition 7.6.2 there exists a countable partition $(V_n)_{n \in \omega}$ of $G/G^0$ into Borel bisections. If $\gamma_n \cdots \gamma_0$ and $\gamma_m \cdots \gamma_0$ are reduced words with $s(\gamma_0) = s(\gamma_0') = x$, then set $\gamma_n \cdots \gamma_0 < \gamma_m \cdots \gamma_0'$ if and only if either $n < m$ or $n = m$ and there exists $0 \leq k \leq m$ such that $\gamma_i = \gamma_i'$ for $i \leq k$ and there exists $n \in \omega$ such that $\gamma_k \in V_n$ and $\gamma_k' \notin V_j$ for every $j \in n + 1$. Set also $x < \gamma_n \cdots \gamma_0$ whenever $\gamma_n \cdots \gamma_0$ is a reduced word with $s(\gamma_0) = x$. Observe that $< x$ is a well ordering of $Gx$ with least element $x$. Furthermore the set of pairs $(\gamma, \rho)$ such that $s(\gamma) = s(\rho) = x$ and $\gamma < x \rho$ is a Borel subset of $G \times G$. Now for $\gamma \in HG$ define $\overline{\gamma}$ to be the $< s(\gamma)$-least element of $H \gamma$. Then define $U$ to be the set $\{ \overline{\gamma} : \gamma \in HG \}$. Observe that $U$ is indeed a transversal for $\sim_H$. Furthermore, being a countable-to-one image of a Borel set, $U$ is Borel. It remains to show that $U$ has the Schreier property. Suppose that $\gamma_n+1 \cdots \gamma_0$ is a reduced word such that $\gamma_n+1 \cdots \gamma_1 \notin U$. We want to prove that $\gamma_n+1 \cdots \gamma_0 \notin U$. Let $v$ be the $<s(\gamma)$-least element of $\gamma_n+1 \cdots \gamma_1$. Since by assumption $\gamma_n+1 \cdots \gamma_1 \notin U$, we have that $v < s(\gamma)$ $\gamma_n+1 \cdots \gamma_1$. We claim that $v \gamma_0 < s(\gamma) \gamma_n+1 \cdots \gamma_0$. Indeed if the length of $v$ is less than $n$, then the length of $v \gamma_0$ is less than $n + 1$, and hence $v \gamma_0 < s(\gamma) \gamma_n+1 \cdots \gamma_0$. Suppose that the length of $v$ is equal to $n$. Write $v = \gamma'_n \cdots \gamma_1$. If $\gamma'_1 = \gamma_0^{-1}$, then $v \gamma_0$ has length $n - 1$ and hence $v \gamma_0 < s(\gamma_0) \gamma_n+1 \cdots \gamma_0$. If $\gamma'_1 \neq \gamma_0^{-1}$, then $\gamma'_n \cdots \gamma'_1 \gamma_0$ is a reduced word. Since $\gamma'_n \cdots \gamma'_1 < s(\gamma)$ $\gamma_n \cdots \gamma_1$, we conclude that $\gamma'_n \cdots \gamma'_1 \gamma_0 < s(\gamma) \gamma_n \cdots \gamma_0$. This concludes the proof that $U$ has the Schreier property.

Observe that the function $HG \to U$, $\gamma \mapsto \overline{\gamma}$ by definition satisfies $H \gamma = H \overline{\gamma}$, $\gamma = \overline{\gamma}$ if and only if $\gamma \in U$. Furthermore we have that, for $u \in U$ and $\gamma \in Q$, $Hu = Hu \gamma \gamma^{-1} = H \overline{\gamma \gamma^{-1}}$ and hence $\overline{u \gamma \gamma^{-1}} = u$. Finally observe that $H^0 = U \cap H$.

Define now $A := \{ u \gamma (\overline{u \gamma})^{-1} : u \in U, \gamma \in Q, s(u) = r(\gamma) \} \subset H$. Since $A$ is the countable-to-one image of a Borel set, it is Borel. We claim that $A$ generates $H$, in the sense that every element of $H$ can be written as a product of elements of $A$. To prove such a claim, fix an element $h$ of $H$. We can write $h = \gamma_n \cdots \gamma_0$ where $\gamma_0, \ldots, \gamma_n \in Q$ and $\gamma_0 \cdots \gamma_n$ is a reduced word. Define elements of $U$ inductively as follows. Define $u_0 := r(\gamma_0)$, and $u_{i+1} := \overline{u_i \gamma_i}$ for $i = 1, 2, \ldots, n$. Then define
Let $a_i := u_i\gamma_i u_{i+1}^{-1} = u_i\gamma_i (u_{i+1}^{-1})^{-1} \in A$ for $i = 0, 1, \ldots, n$. Observe that $a_0 \cdots a_n = \gamma_0 \cdots \gamma_n u_{n+1}^{-1} = hu_{n+1}^{-1}$. Thus $u_{n+1} = (a_0 \cdots a_n)^{-1} h \in U \cap H = H^0$. This gives that $a_0 \cdots a_n = h$ and concludes the proof of the claim that $A$ generates $H$. Observe now that $A \cup U = H^0$ and an element $u\gamma (u\gamma)^{-1}$ of $A$ belongs to $U$ if and only if $u\gamma \in U$. We then set $B = A \backslash H^0$ and observe that $B$ is a Borel graphing for $H$.

Now we claim that if $b = u\gamma (u\gamma)^{-1}$ and $b' = \gamma v\rho (\gamma v\rho)^{-1}$ are elements of $B$, then the product $\gamma (u\gamma)^{-1} v\rho$ is equal to a reduced word $\gamma w\rho$ for some $w \in G$, unless $v = \gamma u\gamma$ and $\rho = \gamma^{-1}$, in which case $u = \gamma v\gamma T = v\rho$. Indeed, suppose that $\gamma (u\gamma)^{-1} v\rho$ begins with $\gamma$ and $\gamma v\rho$ ends with $\rho$. We examine now the possibility of cancellation between $(u\gamma)^{-1}$ and $v$. The reduced form of $(u\gamma)^{-1} v\rho$ begins with $\gamma$, because otherwise $v\rho = \rho_0 \cdots \rho_{m-1} \in U$ by the Schreier property of $U$, contradicting the assumption that $v\rho \notin U$. Similarly $\gamma^{-1} \rho$ is a principal groupoid, then $G$ has the lifting property.

It remains to prove that $B$ is not only a graphing but in fact a treeing. Let $b_0 \cdots b_n$ be a product of elements of $B$, where $b_i = u_i\gamma_i (u_{i+1}^{-1})$, $s(b_i) = r(b_{i+1})$ for $i \in n$, $b_{i+1} \neq b_i^{-1}$ for $i \in n$. By the previous observation, there does not exist $i \in n$ such that $\{b_i, b_{i+1}\} = \{u\gamma (u\gamma)^{-1}, v \rho (\gamma v\rho)^{-1}\}$, where $v = \gamma u\gamma$ and $\gamma = \gamma^{-1}$. It thus also follows from the previous observation that the elements $\gamma_0, \ldots, \gamma_n$ of $Q$ appear as letters in the unique way of writing $b_0 \cdots b_n$ as a word with letters from $Q$. In particular, since $Q$ is a treeing for $G$, $b_0 \cdots b_n$ is not an object. This shows that $B$ is a treeing for $H$. \hfill $\square$

8. Functorial Borel complexity and treeable equivalence relations

8.1. The lifting property.

Definition 8.1.1. Suppose that $G$ is a Polish groupoid. We say that $G$ has the lifting property if the following holds: For any Polish groupoid $H$ such that $E_H$ is Borel and any Borel function $f : G^0 \to H^0$ such that $f(x)E_H f(x')$ whenever $x EGX x'$, there is a Borel functor $F : G \to H$ that extends $f$.

Remark 8.1.2. If $E_G$ has the lifting property (as a principal groupoid), then $G$ has the lifting property.

Proposition 8.1.3. A treeable countable Borel groupoid with no elements of order 2 has the lifting property.
Proof. Suppose that \( G \) is a treeable countable Borel groupoid with no elements of order 2, \( H \) is a Polish groupoid such that \( E_H \) is Borel, and \( f : G^0 \to H^0 \) is a Borel function such that \( f(x)E_Gf(x') \) whenever \( xE_Gx' \). Suppose that \( Q \) is a treeing for \( G \). Write \( Q = Q^+ \cup Q^- \) where \( Q^+ = (Q^-)^{-1} \) and \( Q^+ \) and \( Q^- \) are disjoint. Since \( E_H \) is Borel, the map \( (x, y) \mapsto xy \) from \( E_G \) to \( F(H) \setminus \{ \emptyset \} \) is Borel by Theorem \ref{thm:treeing}. Fix a Borel map \( \sigma : F(H) \setminus \{ \emptyset \} \to H \) such that \( \sigma(A) \in A \) for every \( A \in F(H) \setminus \{ \emptyset \} \). Define
\[
\begin{align*}
F(x) &= f(x) \text{ for } x \in G^0, \\
F(\gamma) &= \sigma(f(\tau(\gamma)))Hf(s(\gamma)) \text{ for } \gamma \in Q^+, \\
F(\gamma) &= F(\gamma)^{-1} \text{ for } \gamma \in (Q^+)^{-1}, \text{ and} \\
F(\gamma_0 \cdots \gamma_0) &= F(\gamma_0) \cdots F(\gamma_0) \text{ if } \gamma_0 \cdots \gamma_0 \in G \setminus G^0 \text{ is a reduced word.}
\end{align*}
\]
It is immediate to check that \( F \) is a Borel functor such that \( F|_{G^0} = f \). \( \square \)

**Proposition 8.1.4.** If \( G \) is a Polish groupoid and \( A \subset G^0 \) is a Borel complete section for \( E_G \) such that \( G|_A \) has the lifting property and there is a Borel map \( G^0 \to G, x \mapsto \gamma_x \) such that \( r(\gamma_x) = x \) and \( s(\gamma_x) \in A \) for every \( x \in G^0 \), then \( G \) has the lifting property.

Proof. Without loss of generality we can assume that \( \phi(x) = x \) for \( x \in A \). Define \( a_x := s(\gamma_x) \) for \( x \in X \). Suppose that \( f : G^0 \to H^0 \) is a Borel function such that \( f(x)E_Hf(x') \) whenever \( xE_Hx' \). Since \( G|_A \) has the lifting property there is a Borel functor \( F : G|_A \to H \) such that \( F|_A = f|_A \). Recall that we denote by \( \sigma : F(H) \setminus \{ \emptyset \} \to H \) a Borel function satisfying \( \sigma(C) \in C \) for every nonempty closed subset \( C \) of \( H \). Define the Borel map \( G^0 \to H, x \mapsto h_x \) by
\[
h_x = \sigma(f(x)Hf(a_x)).
\]
Define, for \( \rho \in G \) such that \( s(\rho) = x \) and \( r(\rho) = y \),
\[
F(\rho) := h_yF(\gamma_y^{-1}\rho\gamma_x)h_x^{-1},
\]
and observe that \( F \) is a Borel functor such that \( F|_{G^0} = f \). \( \square \)

Recall that an equivalence relation is called essentially treeable (resp. essentially countable) if it is Borel reducible to a treeable (resp. countable) equivalence relation. Suppose that \( E \) is an equivalence relation on a standard Borel space \( X \). Consider a map \( C \to IC \) that assigns to an equivalence class of \( E \) a \( \sigma \)-ideal of subsets of \( C \). Following [29, page 285], we say that such a map is *Borel* if for every Borel subset \( A \) of \( X \times X \), the set \( A_f \) defined by
\[
x \in A_f \iff \{ y \in [x]_E : (x, y) \in A \} \in I_{[x]_E}
\]
is a Borel subset of \( X \times X \). We say that such a map satisfies the *countable chain condition* (ccc) if every ideal \( I_C \) satisfies the countable chain condition; i.e. any collection of pairwise disjoint subsets of \( C \) that do not belong to \( I \) is countable.

**Lemma 8.1.5.** Suppose that \( E \) is an essentially countable equivalence relation on a standard Borel space \( X \). Then there exists a sequence \( (E_n) \) of smooth equivalence relations on \( X \) such that \( E = \bigcup_n E_n \).

Proof. Suppose that \( F \) is a countable Borel equivalence relation on a standard Borel space \( Y \), and \( f : X \to Y \) is a Borel reduction from \( E \) to \( F \). Let \( f \times f \) be the map \( X \times X \to Y \times Y, (x_0, x_1) \mapsto (f(x_0), f(x_1)) \). By the implication (i)\( \Rightarrow \)(ii) in [29, Theorem 1.5], there exists a sequence \( (F_n) \) of smooth equivalence relations
on $Y$ such that $F = \bigcup_n F_n$. Set $E_n := (f \times f)^{-1}(F_n)$. Observe that, for every $n \in \omega$, $E_n$ is smooth. Furthermore, since $f$ is a Borel reduction from $E$ to $F$, $E = (f \times f)^{-1}(F) = \bigcup_n E_n$. This concludes the proof. $\square$

The proof of Theorem 8.1.6 is similar to the proof of [3, Theorem 8.1].

**Theorem 8.1.6.** Suppose that $G$ is a Polish groupoid. If $E_G$ is essentially treeable, then $E_G$ has the lifting property.

**Proof.** Since $E_G$ is essentially treeable, it is in particular essentially countable. Therefore by Lemma 8.1.5 there exists a sequence $(E_n)$ of smooth equivalence relations on $X$ such that $E_G = \bigcup_n E_n$. Observe now that the assignment $\{x\}_{E_G} \mapsto I_{[x]}_{E_G}$, where $A \in I_{[x]}_{E_G}$ iff $\{\gamma \in xG : s(\gamma) \in A\}$ is meager, is a Borel ccc assignment of $\sigma$-ideals; see subsection 2.10. It follows from the implication (ii)$\Rightarrow$(i) in [20, Theorem 1.5] that there is a countable Borel subset $A$ of $G^0$ meeting every orbit in a countable nonempty set. Thus $(E_G)|_A$ is a treeable equivalence relation. In particular by Proposition 8.1.3 the equivalence relation $(E_G)|_A$ has the lifting property. Therefore $G|_A$ has the lifting property. Since $(E_G)|_A$ is countable one can find a Borel map $p : X \to A$ such that $(x, p(x)) \in E_G$ for every $x \in X$ and $p(x) = x$ for $x \in A$. It follows from Proposition 8.1.4 that $E_G$ has the lifting property. $\square$

**Corollary 8.1.7.** Suppose that $G$ and $H$ are Polish groupoids. If $E_G$ is essentially treeable and $E_H$ is Borel, then $G \leq_B H$ if and only if $E_G \leq_B E_H$.

**Proposition 8.1.8.** Suppose that $G$ is a Polish groupoid. If $E_G$ is essentially countable, then there is an invariant dense $G_\delta$ set $C \subseteq G^0$ such that $(E_G)|_C$ is essentially hyperfinite.

**Proof.** By [24, Theorem 6.2] there is a co-meager and invariant subset $C_0$ of $G^0$ such that $(E_G)|_{C_0}$ is essentially hyperfinite. Pick a dense $G_\delta$ subset $C_1$ of $C_0$ and then define $C = \{x \in X : \forall^* \gamma \in Gx, \gamma x \in C_1\}$. The properties of the Vaught transform together with Lemma 2.9.1 imply that $C$ is an invariant dense $G_\delta$ set contained in $C_0$. In particular $(E_G)|_C$ is essentially hyperfinite. $\square$

**Corollary 8.1.9.** Suppose that $G$ is a Polish groupoid such that $E_G$ is essentially countable. There is an invariant dense $G_\delta$ subset $C$ of $G^0$ with the following property: For any Polish groupoid $H$ such that $E_G \leq_B E_H$ and $E_H$ is Borel, $G|_C \leq_B H$.

8.2. The co-cycle property.

**Definition 8.2.1.** An analytic groupoid $G$ has the co-cycle property if there is a Borel functor $F : E_G \to G$ such that $F(x, x) = x$ for every $x \in G^0$.

It is immediate to verify that a Polish group action $G \acts X$ has the co-cycle property as defined in [24] if and only if the action groupoid $G \times X$ has the co-cycle property as in Definition 8.2.1. The proof of the following proposition is essentially the same of the proof of the implication (ii)$\Rightarrow$(iii) in [26, Theorem 3.7], and it is presented for the convenience of the reader.

**Proposition 8.2.2.** Suppose that $G$ is a countable Borel groupoid and $X$ a Borel $G$-space. If $G \times X$ has the co-cycle property, then there is a free Borel $G$-space $Y$ such that $E_G^Y \sim_B E_G^X$. Moreover if $G$ is treeable, then $E_G^X$ is treeable.
Proof. Since \( G \times X \) has the co-cycle property there is a Borel functor \( F : E^X_G \to G \) such that \( s(F(x,y)) = p(y) \) and \( F(x,y)y = x \). Consider the equivalence relation \( \sim \) on \( G \times X \) defined by \((\gamma, x) \sim (\rho, y) \) iff \((x, y) \in E^X_G \) and \( \gamma F(x, y) = \rho \). Clearly \( \sim \) is Borel. We now show that it has a Borel selector. Observe that the range \( H \) of \( F \) is a Borel subgroupoid of \( G \) (since \( F \) is countable to one). By Proposition 6.1.1 there is a Borel selector \( t : G \to G \) for the equivalence relation \( \gamma \sim_H \gamma' \) iff \( \gamma H = \gamma' H \). Observe that if \((\gamma, x) \sim (\rho, y) \), then \( \gamma H = \rho H \) and hence \((t(\gamma), x_0) \sim (\gamma, x) \). Define \( S(\gamma, x) = (t(\gamma), x_0) \) and observe that \( S \) is a Borel selector for the equivalence relation \( \sim \). Define \( Y \) to be the quotient of \( G \times X \) by \( \sim \). Define now the Borel action of \( G \) on \( Y \) by \( p[\gamma, x] = r(\gamma) \) and \( \rho[\gamma, x] = [\rho \gamma, x] \) for \( \rho \in Gr(\gamma) \). It is easy to verify that such an action is free, and \([\gamma, x] E^X_G [\rho, y] \) iff \( x E^X_G y \). Let us now observe that \( E^X_G \sim_B E^X_G \). If \( q : X \to G \) is a Borel map such that \( s(q(x)) = p(x) \) for every \( x \in X \), then the map \( x \mapsto [q(x), x] \) is a Borel reduction from \( E^X_G \) to \( E^X_G \). Conversely the map \([\gamma, x] \mapsto x^* \) where \((t(\gamma), x^*) = S(\gamma, x) \) is a Borel reduction from \( E^Y_G \) to \( E^X_G \). Suppose finally that \( G \) is treeable with treeing \( Q \). We want to show that \( E^X_G \) is treeable. Since \( E^X_G \sim_B E^Y_G \), it is enough to show that \( E^Y_G \) is treeable. Fix an equivalence class \([\gamma, x]_{E^Y_G} \) of \( E^Y_G \). Observe that the map from \([\gamma, x]_{E^Y_G} \) to \( Gp(x) \) defined by \([\rho, y] \mapsto \rho F(y, x) \) is bijective. One can then consider the tree relation \( R \) on \([\gamma, x]_{E^Y_G} \) obtained as a pull-back of the tree relation on \( Gp(x) \) induced by the treeing \( Q \) of \( G \). Explicitly, this is defined by

\[
[\rho_0, y_0] R [\rho_1, y_1] \iff \rho_0 F(y_0, y_1) \rho'_1 \in Q.
\]

\[\blacksquare\]

Lemma 8.2.3. Suppose that \( G \) is a countable Borel groupoid and \( A \subset G^0 \) is a Borel complete section for \( E_G \). If there is a Borel functor \( \psi : (E_G)|_A \to G \) such that \( \psi(x, x) = x \) for every \( x \in A \), then there exists a Borel functor \( F : E_G \to G \) such that \( F(x, x) = x \) for \( x \in G^0 \) and \( F(x, y) = \psi(x, y) \) for \( (x, y) \in (E_G)|_A \). In particular, \( G \) has the co-cycle property.

Proof. By the Borel selection theorem for countable-to-one maps [30, Theorem 18.10], one can pick a Borel function \( G^0 \to G \), \( x \mapsto \gamma_x \) such that \( \gamma_a = a \) for \( a \in A \), \( s(\gamma_x) \in A \) and \( r(\gamma_x) = x \) for every \( x \in G^0 \). Define \( F : E_G \to G \) by setting

\[
F(x, y) := \gamma_x \psi(s(\gamma_x), s(\gamma_y)) \gamma_y^{-1}.
\]

Observe that \( F \) is Borel, since \( \psi \), the map \( x \mapsto \gamma_x \), and the groupoid operations are Borel. Furthermore

\[
F(x, x) = \gamma_x \psi(s(x), s(x)) \gamma_x^{-1} = \gamma_x^2 \gamma_x^{-1} = x.
\]

If \((x, y), (y, z) \in E_G \), then

\[
F(x, y)F(y, z) = \gamma_x \psi(s(\gamma_x), s(\gamma_y)) \gamma_y^{-1} \gamma_y \psi(s(\gamma_y), s(\gamma_z)) \gamma_z^{-1} = \gamma_x \psi(s(\gamma_x), s(\gamma_y)) \psi(s(\gamma_y), s(\gamma_z)) \gamma_z^{-1} = \gamma_x \psi(s(\gamma_x), s(\gamma_z)) \gamma_z^{-1} = F(x, z).
\]

It is also clear that \( F(x, y) = \psi(x, y) \) for \( x, y \in A \). Therefore \( F \) satisfies the desired properties. \[\blacksquare\]
Lemma 8.2.4. Suppose that $G$ is a countable Borel groupoid. If $E_G \leq_{B} G$, then there is an invariant Borel subset $Y$ of $G^0$ such that $G|_Y$ has the co-cycle property and $(E_G)|_Y \sim_B E_G$.

Proof. Since $E_G \leq_{B} G$ there is a Borel functor $F : E_G \to G$. Since $F$ maps objects to objects, there exists a Borel map $f : G^0 \to G^0$ such that $F(x,x) = f(x)$ for every $x \in G^0$. Since $F$ is a Borel functor and $G$ is a countable Borel groupoid, $F$ and $f$ are countable-to-one. Define $A \subset G^0$ to be the image of $G^0$ under $f$, which is a Borel subset of $G^0$; see [30] Theorem 18.10. By the Borel section theorem for countable-to-one maps [31], Exercise 18.14 there is a Borel function $g : A \to G^0$ such that $(f \circ g)(y) = y$ for every $y \in A$. Define $Y$ to be the union of the orbits of $G$ that meet $A$. Observe that since $F : E_G \to G$ is a Borel functor, $f : G^0 \to Y$ is a Borel reduction from $E_G$ to $(E_G)|_Y$. It follows that $E_G \sim_B E_G|_Y$. The map $(E_G)|_A \to G$, $(x,y) \mapsto F(g(x),g(y))$ together with Lemma 8.2.3 implies that $G|_Y$ has the co-cycle property. □

8.3. Free actions of treeable groupoids. We want to show that if $G$ is a treeable groupoid and $G \acts X$ is a free Borel action of $G$, then the associated orbit equivalence relation is treeable. This will follow from a more general result about $\mathcal{L}$-structured equivalence relations.

Suppose that $\mathcal{L} = \{R_n : n \in \omega\}$ is a countable relational language in first order logic, where $R_n$ has arity $k_n \in \omega$. Suppose that $E$ is a countable Borel equivalence relation on a standard Borel space $X$. According to [26] Definition 2.17 the equivalence relation $E$ is $\mathcal{L}$-structured if there are Borel relations $R_n^E \subset X^{k_n}$ such that, for any $k \in \omega$ and $x_1, \ldots, x_{k_n} \in X$, $x_1, \ldots, x_{k_n}$ belong to the same $E$-class whenever $(x_1, \ldots, x_{k_n}) \in R_n^E$. In particular every $E$-class $[x]$ is the universe of an $\mathcal{L}$-structure $\langle [x], (\{[x]^{k_n} \cap R_n^E\}_{n \in \omega}) \rangle$.

Similarly, if $X$ is a standard Borel space, then a standard Borel bundle $A$ of countable $\mathcal{L}$-structures over $X$ is a standard Borel space $A$ fibered over $X$ via a countable-to-one Borel map $p_A : A \to X$ with fibers $A_x = p_A^{-1}(x)$ for $x \in X$, endowed with Borel subsets $R_n^A \subset A^{k_n}$ such that, for any $k \in \omega$ and $a_1, \ldots, a_{k_n} \in A$, $a_1, \ldots, a_{k_n}$ belong to the same fiber over $X$ whenever $(a_1, \ldots, a_{k_n}) \in R_n^A$. Suppose that $G$ is a Polish groupoid and $A$ is a standard Borel bundle of $\mathcal{L}$-structures over $G^0$. A Borel action $G \acts A$ is a Borel map $(\gamma, a) \mapsto \gamma a$ defined for $(\gamma, a) \in G \times A$ such that $a \in A_s(\gamma)$, and $(a_1, \ldots, a_{k_n}) \in R_n^A$ if and only if $(\gamma a_1, \ldots, \gamma a_{k_n}) \in R_n^A$ for every $n \in \omega$, $\gamma \in G$, and $a_1, \ldots, a_{k_n} \in A_s(\gamma)$. The proof of the following theorem is very similar to the argument at the beginning of section 3.2 in [26], and it is reproduced for the convenience of the reader.

Theorem 8.3.1. Suppose that $G$ is a Polish groupoid such that there is a standard Borel bundle $A$ of countable $\mathcal{L}$-structures and a Borel action $G \acts A$ such that the corresponding orbit equivalence relation $E^A_G$ is smooth, and for every $a \in A$ the stabilizer

$$\text{St}_G(a) = \{\gamma \in Gp_A(a) : \gamma a = a\}$$

is a compact subset of $G$. If $X$ is a standard Borel space, and $G \acts X$ is a free Borel action of $G$ on $X$, then there is an $\mathcal{L}$-structured countable Borel equivalence relation $E$ such that $E \sim_B E^X_G$, and moreover every class of $E$ is isomorphic to some fiber of $A$. 
Proof. By Corollary 2.10.9 there is a Borel selector $t$ for $E^A_G$. Moreover by Theorem 5.1.1 we can assume without loss of generality that the action $G \acts X$ is continuous. Define

$$X \times A = \{(x, a) \in X \times A : a \in A_{p(x)}\}$$

and the action $G \acts (X \times A)$ by $\gamma (x, a) = (\gamma x, \gamma a)$. Observe that such an action is free and in particular the associated orbit equivalence relation $\sim$ is Borel. We now show that $\sim$ has a Borel selector. Suppose that $(x, a) \in X \times A$. Observe that if $(x, a) \sim (y, b)$, then $aE^b_G y$ and hence $t(a) = t(b)$. Therefore $t(a)$ depends only on the $\sim$-class $[x, a]$ of $(x, a)$. Observe that

$$O(x, a) := \{\gamma x : \gamma \in \St([x, a])\} = \{y \in X : (y, t(a)) \in [x, a]\}$$

is a compact subset of $X$. Denote by $\sigma : F(X) \setminus \{\varnothing\} \to X$ a Borel function such that $\sigma(C) \in C$ for every $C \in F(X) \setminus \{\varnothing\}$. Define $S(x, a) = (\sigma(O(x, a)), t(a))$, and observe that $S$ is a Borel selector for $\sim$. Define the standard Borel space $Y = (X \times A)/\sim$ and the countable Borel equivalence relation $E$ on $Y$ by $[x, a]E[y, b]$ if $x \sim y$. We now define for every $E$-class $C = [[x, a]]_E$ an $\mathcal{L}$-structure on $C$. Fix $n \in \omega$ and suppose that $\gamma_i x, a_i$ for $i \in k_n$ are elements of $C$, where $\gamma_i \in Gp(x)$ for $i \in k_n$. Set $\langle \gamma_i x, a_i \rangle_{i \in k_n} \in R^C_n$ if $\gamma^{-1}_i a_i \in R^A_n$. Using the fact that $G$ acts by $\mathcal{L}$-isomorphisms one can verify that this does not depend on the choice of $[x, a] \in C$. Define now $\langle \gamma_i x, a_i \rangle_{i \in k_n} \in R^E_n$ if and only if $[x_0, a_0] E [x_1, a_1] E \cdots E [x_{k_n-1}, a_{k_n-1}]$ and $\langle [x_0, a_0] \rangle_{i \in k_n} \in R^C_n$ where $C = [[x, a]]_E$. This defines Borel relations $R^E_n$ on $E$ that make $E$ $\mathcal{L}$-structured. Moreover the Borel map $f : C \to A$ defined by $f \gamma x, a = \gamma^{-1} a$, where $C$ is the class of $[x, a]$, shows that the $\mathcal{L}$-structure $\langle C, (R_n \cap C^{k_n})_{n \in \omega} \rangle$ is isomorphic to $A_{p(x)}$. Finally we observe now that $E \bar{\sim} E^G_X$.

Let us now consider the particular case of Theorem 8.3.1 when $\mathcal{L}$ is the language with a single binary relation $R$. Assume further that $G$ is a treeable Borel groupoid. A standard Borel bundle of trees over $X$ is a standard Borel bundle $(A_x)_{x \in X}$ of countable $\mathcal{L}$-structures such that $A_x$ is a tree for every $x \in X$.

Corollary 8.3.2. If $G$ is a countable treeable groupoid and $G \acts X$ is a free Borel action, then the orbit equivalence relation $E^G_X$ is treeable.

Proof. Suppose that $Q$ is a treeing of $G$ in the sense of Definition 7.8.2. We define a standard Borel bundle $A$ of trees over $G^0$ as follows. As a set, $A$ is equal to $G$. Furthermore, $A_x = xG$ for every $x \in G^0$. For $x \in G^0$ and $\gamma, \rho \in A_x$ we have that $(\rho_0, \rho_1) \in R^A$ if and only if $\rho^{-1}_1 \rho_0 \in Q$. The fact that $Q$ is a treeing is equivalent to the assertion that $A$ is a standard Borel bundle of trees over $G^0$. Consider the canonical action of $G$ on $A$ by left translation. This is defined by $(\gamma, \rho) \mapsto \gamma \rho$ for $(\gamma, \rho) \in G \times A$ such that $\rho \in A_{\gamma} (\text{i.e. } \gamma, \rho \in G$ and $s(\gamma) = r(\rho)$). Observe that this is indeed a Borel action of $G$ on the standard Borel bundle of trees, since $\rho^{-1}_1 \rho_0 \in Q$ if and only if $(\gamma \rho_1)^{-1} (\gamma \rho_0) \in Q$ for any $\gamma \in G$ and $\rho_0, \rho_1 \in s(\gamma) G = A_{s(\gamma)}$. Furthermore, for every $x \in G^0$ and $\rho \in A_x$ the stabilizer $\{\gamma \in Gr : \gamma \rho = \rho\} = \{x\}$ is trivial. Finally, the orbit equivalence relation $E^G_X$ is the relation $\rho_0, \rho_1 \in E^G_X$ if and only if there exists $\gamma \in Gr(\rho_0)$ such that $\gamma \rho_0 = \rho_1$ if and only if $s(\rho_0) = s(\rho_1)$. Such a relation is clearly smooth, since the map
\( A \to G^0, \rho \mapsto s(\rho) \) is a Borel reduction from \( E_G^A \) to the identity of \( G^0 \). These observations show that the action \( G \acts A \) satisfies the assumptions of Theorem 8.3.1. Therefore applying Theorem 8.3.1 one can deduce that, given any free Borel action \( G \acts X \), the corresponding orbit equivalence relation \( E_G^X \) is treeable. □

8.4. Characterizing treeable equivalence relations. Denote by \( \mathbb{F}_\infty \) the free countable group on infinitely many generators. The following result subsumes [26, Theorem 3.7].

**Theorem 8.4.1.** Suppose that \( E \) is a countable Borel equivalence relation on a standard Borel space \( X \). The following statements are equivalent:

1. \( E \) is treeable.
2. \( E \) has the lifting property.
3. For every countable Borel groupoid \( G \) and Borel action \( G \acts X \) such that \( E_G^X = E \), the groupoid \( G \acts X \) has the co-cycle property.
4. For every Borel action \( \mathbb{F}_\infty \acts X \) such that \( E_{\mathbb{F}_\infty}^X = E \), \( E_{\mathbb{F}_\infty}^X \leq_B \mathbb{F}_\infty \acts X \).
5. For every countable Borel groupoid \( G \) and Borel action \( G \acts X \) such that \( E_G^X = E \), there is a free Borel action \( G \acts Y \) such that \( E_Y^G \sim E_G^X \).
6. For every countable Borel groupoid \( G \) and Borel action \( G \acts X \) such that \( E \subset E_G^X \) there is a free Borel action \( G \acts Y \) such that \( E \sqsubseteq_B E_Y^G \).

**Proof.** The implication (1)⇒(2) follows from Proposition 8.1.3; (2)⇒(3) follows from the fact that if \( E_G^X \) has the lifting property, then \( G \acts X \) has the co-cycle property; (3)⇒(5) follows from Proposition 8.2.2; (4)⇒(1) follows from Lemma 8.2.4 and Proposition 8.2.2; and (6)⇒(1) follows from Corollary 8.3.2 together with the fact that a subrelation of a treeable equivalence relation is treeable [26, Proposition 3.3]; see also subsection 7.8.

We prove the other nontrivial implications below.

(5)⇒(1) Consider an action \( \mathbb{F}_\infty \acts X \) such that \( E = E_{\mathbb{F}_\infty}^X \) and then apply (5) and Corollary 8.3.2.

(2)⇒(6) Since \( E \) has the lifting property, there is a Borel function \( F : E \to G \) such that \( s(F(x,y)) = p(y) \) and \( F(x,y)y = x \) for every \( (x,y) \in E \). Consider on \( G \acts X \) the equivalence relation \( (\gamma,x) \sim \rho \) iff \( xEy \) and \( \rho = \gamma F(x,y) \). Proceeding as in the proof of Proposition 8.2.2 one can show that \( \sim \) has a Borel selector. Thus the quotient \( Y \) of \( G \acts X \) by \( \sim \) is standard. Define the Borel action \( G \acts Y \) by \( p[\gamma,x] = r(\gamma) \) and \( p[\gamma,x] = [p\gamma,x] \). As in the proof of Proposition 8.2.2 one can show that such an action is free. Moreover the map \( x \mapsto [p(x),x] \) is an injective Borel reduction from \( E \) to \( E_Y^G \).

□

The following corollary is an immediate consequence of the implication (4)⇒(1) in Theorem 8.4.1.

**Corollary 8.4.2.** For any nontreeable equivalence relation \( E \) there are countable Borel groupoids \( G \) and \( H \) that have \( E \) as orbit equivalence relation such that \( G \) is not Borel reducible to \( H \). Moreover one can take \( G = E \) and \( H \) to be the action groupoid of an action of \( \mathbb{F}_\infty \).

Corollary 8.4.2 can be interpreted as asserting that functorial Borel complexity provides a finer distinction between the complexity of classification problems in mathematics than the traditional notion of Borel complexity for equivalence relations.
9. Further directions and open problems

Polish groupoids seem to be a broad generalization of Polish groups and Polish group actions. For instance, as shown in section 5, any Borel action of a Polish groupoid is again a Polish groupoid. However, no example is currently known of a Polish groupoid whose orbit equivalence relation is not Borel bireducible with an orbit equivalence relation of a Polish group action.

Problem 9.1. Is there a Polish groupoid $G$ such that the orbit equivalence relation $E_G$ is not Borel bireducible with an orbit equivalence relation of a Polish group action? Can one find such a $G$ for which $E_G$ is moreover Borel?

Problem 9.1 has a tight connection with a conjecture due to Hjorth and Kechris [25, Conjecture 1]. Recall that $E_1$ denotes the relation of tail equivalence for sequences in $[0,1]$. Theorem 4.2 of [28] asserts that $E_1$ is not Borel reducible to the orbit equivalence relation of a Polish group action. The Hjorth-Kechris conjecture asserts that if $E$ is a Borel equivalence relation, then the converse holds; namely, if $E_1$ is not Borel reducible to $E$, then $E$ is Borel bireducible with the orbit equivalence relation of a Polish group action. By Corollary 2.10.8 $E_1$ is not reducible to any Borel orbit equivalence relation of a Polish groupoid. Therefore a groupoid $G$ as in Problem 9.1 such that $E_G$ is moreover Borel would provide a counterexample to the Hjorth-Kechris conjecture.

In the theory of Borel complexity of equivalence relations, a key role is played by equivalence relations that are complete (or universal) for a given class up to Borel reducibility. These equivalence relations provide natural benchmarks of complexity. It would be interesting to analogously find universal elements for natural classes of analytic groupoids.

Problem 9.2. Establish whether the following classes have a universal element up to Borel reducibility: analytic groupoids, Borel groupoids, Polish groupoids, countable groupoids.

In the world of equivalence relations, the phenomenon of universality is widespread. Most natural classes of analytic equivalence relations have universal elements. Moreover such universal elements admit many different descriptions. For example the following equivalence relations are all complete for orbit equivalence relations induced by Polish group actions [10,15,20,40,47,51]:

1. Isomorphism of abelian C*-algebras;
2. Isomorphism of amenable, simple, unital, separable C*-algebras;
3. Isometry of separable Banach spaces;
4. Complete order isomorphism of separable operator systems;
5. Isometry of metric spaces;
6. Conjugacy of isometries of the Urysohn sphere.

As shown in subsections 2.6 and 2.7 the relations (1)–(6) above are naturally the orbit equivalence relation of a Borel groupoid. It would be interesting to know if the functorial Borel complexities of such groupoids are different.

Problem 9.3. Consider the standard Borel groupoids associated with the relations (1)–(6) above. Are these groupoids Borel bireducible?

One can consider a similar problem for the classes of countable first order structures that are Borel complete [15]. Recall that a class of countable first order
structures is Borel complete if the corresponding isomorphism relation is complete for isomorphism relations of countable structures. Borel complete classes include countable trees, countable linear orders, countable fields of a fixed characteristic, and countable groups.

**Problem 9.4.** Are there two Borel complete classes of first order structures such that the corresponding groupoids are not Borel bireducible?

**APPENDIX BY ANUSH TSERUNYAN**

In this appendix we show that if $X$ is a locally Polish space, then the Effros Borel structure on the space $F(X)$ of closed subspaces of $X$ is standard. Recall that, as in Definition 2.2.1, a topological space $X$ is locally Polish if it has a countable basis of open sets which are Polish in the relative topology. If $U$ is an open subset of $X$, we denote by $U^-$ the set $\{ F \in F(X) : F \cap U \neq \emptyset \}$. Define the Effros Borel structure on $F(X)$ to be the Borel structure generated by the sets $U^-$ for $U \subset X$ open.

**Theorem A.** The Borel structure on $F(X)$ is standard.

**Proof.** Suppose that $A = \{ U_n : n \in \omega \}$ is a countable basis of Polish open subnets of $X$. For every $n \in \omega$ denote by $d_n$ a compatible complete metric on $U_n$. Clearly the Effros Borel structure on $F(X)$ is generated by the sets $U^-$ for $U \in A$. Consider the collection $S_A = \{ U^-, X \setminus U^- : U \in A \}$ and the topology $\tau_A$ on $F(X)$ having $S_A$ as subbasis. We will show that the topology $\tau_A$ on $F(X)$ is Polish. Consider the map $c$ from $F(X)$ to $2^\omega$ assigning to $F$ the characteristic function of $\{ n \in \omega : F \cap U_n \neq \emptyset \}$. Clearly $c$ is a $\tau_A$-homeomorphism onto its image. In view of [30, Theorem 3.11], in order to conclude that $(F(X), \tau_A)$ is Polish it is enough to show that the image $Y$ of $c$ is a $G_\delta$ subspace of $2^\omega$. We claim that, for $y \in 2^\omega$, $y \in Y$ if and only if the following conditions hold:

1. For all $n, m$ with $U_n \subseteq U_m$, if $y(n) = 1$, then $y(m) = 1$.
2. For all $n$ and $\varepsilon \in \mathbb{Q}_+$, if $y(n) = 1$, then there is $m$ such that $y(m) = 1$ and for all $i \leq n$ with $U_i \supseteq U_n$, we have $\overline{U}_m^i \subseteq U_n$ and $\text{diam}_i(U_m) < \varepsilon$, where the closure $\overline{U}_m^i$ and diameter $\text{diam}_i(U_m)$ are taken with respect to the metric $d_i$.

Since necessity is obvious, we check that these conditions are sufficient. Let $y \in 2^\omega$ satisfy conditions (i) and (ii), and define the $\tau_A$-closed subset of $X$, $F = \{ x \in X : \forall n \in \omega, x \in U_n \Rightarrow y(n) = 1 \}$. We show that $c(F) = y$. Fix $n \in \omega$ and note that if $y(n) = 0$, then $F \cap U_n = \emptyset$ by definition. So suppose $y(n) = 1$ and we have to find an $x \in F \cap U_n$. Iterating (ii), we get a sequence $(U_{n_i})_{i \in \omega}$ with $n_0 = n$ and such that for all $i \in \omega$, $y(n_i) = 1$, $\overline{U}_{n_{i+1}}^{n_i} \subseteq U_{n_i}$, and $\text{diam}_n(U_{n_i}) \leq 2^{-k}$.

Thus, since the metric $d_n$ on $U_n$ is complete, we get $\{ x \} = \bigcap_i U_{n_i}^n$, for some $x \in U_n$. It remains to show that $x \in F$, but this easily follows from (i). \hfill \Box

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