IS PLA LARGE?

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ABSTRACT. We examine the class of functions representable by an analytic sum

\[ f(t) = \sum_{n \geq 0} c(n)e^{int} \]

converging almost everywhere. We show that it is dense but that it is first category and has zero Wiener measure.

1. INTRODUCTION

We say that a function \( f \) on the circle \( \mathbb{T} \) belongs to PLA (pointwise limits of analytic sums) if it can be decomposed to a trigonometric series with positive frequencies (1) converging almost everywhere (a.e.)

An important observation is that the representation (1) is unique. It follows from the Abel summation and Privalov uniqueness theorems. An analogy with the Riemannian theory suggests that the coefficients could be recovered by Fourier formulas, provided that \( f \) is integrable (in other words, \( \text{PLA} \cap L^1 \subset H^1 \), the Hardy space).

This was disproved in our note [KO03], where we constructed a series (1) converging outside some compact \( K \) of zero measure to a bounded function \( f \), but which is not \( f \)'s Fourier series. Later we proved [KO04, KO] that a function \( f \) which admits such a non-classic representation can be smooth, even \( C^\infty \), and characterized precisely the maximal possible smoothness in terms of the rate of decrease of the Fourier coefficients.

The following density theorem is a simple consequence of these results:

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Theorem 1. PLA is dense in the space $C(\mathbb{T})$. Moreover, it is dense in the spaces of smooth functions $C^k(\mathbb{T})$ for every $k = 1, 2, \ldots$ in their respective norms.

The approach taken in [KO03, KO04, KO] is complex-analytic, and information is derived by examining the related function $F(z) = \sum c(n)z^n$ in the disk $\{|z| < 1\}$. In this paper we present a purely real-analytic construction, and use it to prove the following density result: any measurable function can be carried into PLA by a uniformly small perturbation.

Theorem 2. Let $f \in L^0(\mathbb{T})$, $\epsilon > 0$. Then there is a decomposition

$$f = g + h, \quad g \in \text{PLA}, \quad \|h\|_{C(\mathbb{T})} < \epsilon. \tag{2}$$

For a stronger version, see theorem 2' below.

This seems like a good place to compare PLA with its “classic” part, the Hardy spaces. Theorems 1 and 2 should be contrasted against the fact that the Hardy space $H^2$ is closed in $L^2$ and has infinite co-dimension. Another interesting fact is that PLA functions may exhibit jump discontinuities (again, in $H^1$ this is impossible) — this is a corollary from theorem 2. Finally it is worth to note that PLA contains non-constant real functions. This comes from our approach in [KO03] where a singular inner function $I(z)$ in the unit disc was constructed such that

$$f(t) := \frac{1}{I(e^{it})} \in \text{PLA}.$$  

Hence $f + \overline{f} = f + 1/f$ gives the required example.

On the other hand we prove that PLA is rather thin in the sense of Wiener measure and Baire category

Theorem 3. PLA $\cap C(\mathbb{T})$ is a set of first category in $C(\mathbb{T})$.

Theorem 4. The Wiener measure of PLA $\cap C(\mathbb{T})$ is zero.

2. SMALL PERTURBATIONS AND PLA

2.1. Density of PLA. First we deduce theorem 1. According to [KO], there exists a $C^\infty$ function $f \in \text{PLA}$ such that

$$\hat{f}(l) \neq 0 \text{ for some } l < 0.$$  

In fact the last inequality holds for infinitely many negative $l$-s. Otherwise if $L$ is the smallest such number then

$$\sum_{n=L}^{\infty} c(n-L)e^{int} - \sum_{n=0}^{\infty} \hat{f}(n-L)e^{int}$$
(c(n) from (1)) is a non-trivial analytic sum converging to zero a.e., contradicting Privalov’s uniqueness theorem. Hence for any s < 0, by multiplying with an appropriate exponential we can get an \( f_s \in \text{PLA} \cap C^\infty \) with \( \hat{f}_s(s) = 1 \).

Next, for an arbitrary \( N \) consider the discrete convolution with \( e^{ist} \),

\[
F_{N,s}(t) := \frac{1}{2\pi} \sum_{j=0}^{N-1} f_s(t - 2\pi j/N) e^{is(2\pi j/N)}.
\]

Since PLA is a translation invariant linear space, \( F_{N,s} \in \text{PLA} \) and

\[
\lim_{n \to \infty} F_{N,s} \to f \ast e^{ist} = e^{ist}
\]

in the \( C^k \) norm for any \( k = 1, 2, \ldots \). Therefore any exponential \( e^{ist} \) admits an approximation by PLA functions, and theorem 1 follows.

2.2. **Lemmas.** We will use the technique from [KO01] (where one may find additional references and historical comments). By \( L^0(\mathbb{T}) \) we denote the space of measurable functions \( f : \mathbb{T} \to \mathbb{C} \) endowed with the distance function

\[
\rho(f, g) := \inf \{ \epsilon : m\{|f - g| > \epsilon\} < \epsilon \}
\]

where \( m \) is the normalized Lebesgue measure on \( \mathbb{T} \). For a trigonometric polynomial

\[
P(t) = \sum c(n)e^{int}
\]

we use the following notations

- \( \text{spec } P = \{ n \in \mathbb{Z} : c(n) \neq 0 \} \)
- \( P^*(t) = \sup_{t < m} \left| \sum_{n=t}^{m} c(n)e^{int} \right| \)
- \( P_{[r]}(t) = P(rt), r \in \mathbb{N} \).

For two trigonometric polynomials \( g \) and \( h \) consider the following “special product”

\[
P = gh_{[r]} \text{ where } r > 3 \deg g.
\]

Then the following is true (see (10) in [KO01]; compare also with [O85] sec. 1.1 and [K96] lemma 15):

\[
P^*(t) \leq |g(t)| \cdot ||h^*||_\infty + 2g^*(t)||h||_\infty,
\]

where \( \hat{h} \) is the Fourier transform and \( || \cdot ||_p \) is the \( l^p \) or \( L^p \) norm, according to context.

**Lemma 1.** For any \( \epsilon > 0 \) there exists a polynomial \( h \) with \( \text{spec } h \subset [1, \infty[ \), \( \rho(h, 1) < \epsilon \) and \( ||\hat{h}||_\infty < \epsilon \).

The proof can be found in [KO01] lemma 4.1.
Lemma 2. Given a segment $I \subset \mathbb{T}$ and $\delta > 0$ there is a trigonometric polynomial $P$ such that

(i) $\text{spec } P \subset [0, \infty)$,
(ii) $\rho(P, 1_I) < \delta$, $1_I$ being the indicator function, and
(iii) $P^*(t) < \delta$ outside of $I_\delta$, a $\delta$-neighborhood of $I$.

Proof. Using lemma 1 find a polynomial $h$ such that $\rho(h, 1) < \frac{1}{3}\delta$ and $\|\hat{h}\|_\infty < \frac{1}{24}\delta^2$. Next approximate $1_I$ by a trigonometric polynomial $g$ such that

$$|g(t)| < \frac{\delta}{2 \|h^*\|_\infty} \quad t \notin I_\delta \quad \rho(g, 1_I) < \frac{1}{3}\delta \quad \|g^*\|_\infty < \frac{6}{\delta}.$$ 
Finding $g$ can be done by interpolating $1_I$ by a trapezoid function and then taking a sufficiently large partial sum of the Fourier expansion. Estimating $g^*$ can be done, for example, by noting that a trapezoid function is a difference of two triangular functions $T$ and for each $\|T^*\|_\infty \leq \|\hat{T}\|_1 = \|T\|_\infty \leq 3/\delta$. Set $P := gh_\|_\|$. Then (iii) implies (i) and $\text{spec } h \subset [1, \infty]$ implies (i) provided $r$ is large enough. □

A direct consequence:

Lemma 3. Given $\delta$ and a step function $\varphi$ which is 0 on a set $U$, there is a polynomial $P$ with (i) above such that

(iv) $\rho(P, \varphi) < \delta$,
(v) $\rho(P^*, 0) < \delta$ on $U$ (by which we mean $\rho(P^* \cdot 1_U, 0) < \delta$).

Lemma 4. Given $\delta$, $\alpha$ and a step function $\psi$, $|\psi| < \alpha$ on $U$ there are polynomials $P$ and $Q$ such that

(iii) $\|Q\|_\infty < \alpha$

are satisfied.

Proof. Fix $Q$ with (vii) and $\rho(Q, \psi) < \frac{1}{3}\delta$ on $U$. Define a step function $\varphi$ which is 0 on $U$ and $\rho(\varphi, \psi - Q) < \frac{2}{3}\delta$. Now apply lemma 3 and get the result. □

2.3. Proof of theorem 2. Let $f$ and $\varepsilon > 0$ be given. We may assume $\rho(f, 0) < \frac{1}{\varepsilon^2}$. Define inductively sequences of trigonometric polynomials $\{P_k\}, \{Q_k\}$ satisfying the conditions

(i) $\rho\left(f, \sum_{k \leq n} P_k + Q_k\right) < 4^{-n}$
(ii) $\text{spec } P_n \subset [0, \infty[\$
(iii) $\|Q_n\|_\infty < \varepsilon 2^{-n}$. 


Start with $P_0 = Q_0 = 0$. Suppose $P_k$ and $Q_k$ are defined with the requirements above for $k < n$. Set $f_n := f - \sum_{k<n}P_k + Q_k$. Approximate $f_n$ by a step-function $S_n$ such that

$$\rho(S_n, f_n) < 4^{-n}.$$  

We get

$$\rho(S_n, 0) < 4^{-n+2}. \quad (4)$$

Denote $U_n = \{t : |S_n(t)| < \epsilon 2^{-n}\}$. Apply lemma 4 for $\psi = S_n$, $U = U_n$, $a = \epsilon 2^{-n}$ and $\delta = 4^{-n}$. We get polynomials $P_n$ and $Q_n$ such that (i)-(iii) are fulfilled for $k = n$. Notice that if $\epsilon 2^{-n} < 4^{-n+1}$ then (4) implies that $mU_n < 4^{-n+2}$ and then condition (v) of lemma 4 gives $\rho(P_n^*, 0) < 4^{-n}$ on $U$. These two together give

$$\rho(P_n^*, 0) < 2^{-n} \quad \text{for } n \text{ sufficiently large.}$$

This means that the series $\sum P_n$ converges a.e. and it defines a function $g \in \text{PLA}$. Hence denoting $h := \sum Q_n$ we finish the proof. 

Remark 1. Since theorem 2 is stronger than the first part of theorem 1, one might wonder whether the second part admits an equivalent improvement. In fact it does not, namely, there exists a function $f \in C(T)$ which does not admit any decomposition $f = g + h$ with $g \in \text{PLA}$ and $h \in C^1$. We plan to exhibit this example in a subsequent paper.

2.4. Representations by “almost analytic” series. D. E. Menshov proved that any $f \in L^0(T)$ can be decomposed to an a.e. converging trigonometric series

$$f(t) = \sum_{n \in \mathbb{Z}} c(n)e^{int} \quad (5)$$

(see [B64, K96, KO01]). The above technique gives the following, “almost-analytic” version:

**Theorem 2’.** Any $f \in L^0(T)$ can be decomposed in an almost everywhere convergent series (5) with the “negative” part $f_-$ converging uniformly on $T$. Further, the negative part can be taken to have arbitrarily small $U(T)$ norm.

We remind that the $U$-norm of a function $F$ is defined by

$$\|F\|_{U(T)} := \sup_{N \geq 0} \left\| \sum_{n=\infty}^{N} \hat{F}(n)e^{int} \right\|_\infty.$$
We shall only sketch the proof of theorem 2'. It requires the technique of separating the measure error and the uniform error. First we will need

**Lemma 5.** For any $\gamma > 0$ and $\delta > 0$ there exists a trigonometric polynomial $h$ satisfying

- $\hat{h}(0) = 0$, $||\hat{h}||_\infty < \delta$
- $m\{t : |h(t) - 1| > \delta\} < C\gamma$
- $||h^*||_\infty \leq 1/\gamma$.

This is lemma 2.1 from [KO01], which is the “non-analytic counterpart” of lemma 1. Next we state a replacement for lemma 4:

**Lemma 4’.** Given $\gamma$, $\delta$, $a$ and a step function $\psi$, $|\psi| < a$ on $U$ there are polynomials $P$ and $Q$ such that

- (vi) $m\{t : |P + Q - \psi| > \delta\} < C\gamma$
- (vii) $||Q||_U < a/\gamma$.

The construction of $P$ and $Q$ is generally similar, but one has to take $Q_{\text{lemma 4'}} = Q_{\text{lemma 4}}h_{[r]}$ where $h$ comes from lemma 5 with the same $\gamma$ and a sufficiently small $\delta$. Notice that there is no price to pay in lemma 5 for decreasing $\delta$. The proof of theorem 2 then applies mutatis mutandis.

**Remark 2.** Notice that if one replaces convergence almost everywhere by convergence in measure then any function $f \in L^0$ admits an analytic representation (1). This was proved in [KO01]. However, here the representation is not unique.

### 3. Category and Measure

Here we prove theorems 3 and 4.

#### 3.1. Relatives.

We will use

**Definition.** (see [O86]) For two functions $f$ and $g$ we say that $g$ is a relative of $f$ if there is a compact $K$ of positive measure on the circle, and an absolutely continuous homeomorphism $h : \mathbb{T} \to \mathbb{T}$ such that

$$g(t) = f(h(t)) \quad \forall t \in K.$$  

In this paper the notion of relatives will be used through the following lemma. Denote $C_A := H^\infty \cap C$ i.e. the space of continuous boundary values of analytic functions on the disk.

**Lemma 6.** If $f \in \text{PLA}$ then it has a relative $g \in C_A$. 
Indeed, let \( f = \sum_{n \geq 0} c(n) e^{int} \). Consider the analytic extension
\[
F(z) = \sum_{n \geq 0} c(n) z^n \quad z \in \mathbb{D}.
\]

According to Abel’s summation theorem \( F \) has non-tangential boundary values equal to \( f(t) \) at the point \( z = e^{it} \) for almost every \( t \). Fix a compact \( K, mK > 0 \), where this limit is uniform. Consider the so-called Privalov domain \( P = P_K \) i.e. the subset of \( \mathbb{D} \) created by removing, for every arc \( I \) from the complement of \( K \), a disk \( D_K \) orthogonal to \( \partial \mathbb{D} \) at the end points of \( I \). If \( I \) is larger than a half circle, remove \( \mathbb{D} \setminus D_I \) instead of \( D_I \) so in both cases you remove the component containing \( I \). Let \( H \) be the Riemannian mapping of the closed unit disc onto \( P \). It is well known that \( H \) belongs to \( C_A \) and its boundary values define an absolutely continuous homeomorphism of \( \mathbb{T} \) onto \( \partial P \). It is easy to see that \( g(t) := F(H(e^{it})) \) is a relative of \( f \). □

So, theorem [3] will follow from

**Lemma 7.** The set of functions with relatives in \( C_A \) is first category in \( C(\mathbb{T}) \).

For the proof of this lemma we need

**Lemma 8.** Given numbers \( \delta, M > 0 \) one can define a positive \( \epsilon(\delta, M) \) such that for any function \( g \) in \( L^2(\mathbb{T}) \), satisfying
\[
||g||_2 < M \quad |g(t) - c_1| < \epsilon \text{ on } E_1 \quad |c_1 - c_2| > \delta \quad |g(t) - c_2| < \epsilon \text{ on } E_2
\]
where \( E_j \) are two disjoint sets of measure \( > \delta \) and \( c_j \) are two constants, one gets that \( g \) is not in \( H^2 \).

**Proof.** This is basically a consequence of Jensen’s inequality. Assume \( g \) admits an extension inside the disc as an \( H^2 \) function. Consider the subharmonic function \( G(z) := \log |g(z) - c_1| \). Then
\[
G(0) \leq \int_{\partial \mathbb{D}} G = \int_{E_1} + \int_{\partial \mathbb{D} \setminus E_1} \leq \delta \log \epsilon + \int_{\partial \mathbb{D}} |g(z) - c_1| \leq \\
\leq \delta \log \epsilon + M + \frac{M}{\sqrt{\delta}} + \epsilon.
\]

so taking \( \epsilon \) sufficiently small we would get \( |g(0) - c_1| < \frac{\epsilon}{2} \delta \). Repeating the same argument for \( |g(0) - c_2| \) leads to a contradiction. □

**Proof of lemma [7]** Denote (for small positive \( s \) and \( r \), and a large \( M \)) by \( F(s, r, M) \) the class of all functions \( f \in C(\mathbb{T}) \) satisfying that there exists
A $h \in \text{hom}(\mathbb{T})$ such that for any measurable $A \subset \mathbb{T}$, $m(A) < s$ implies $m(h(A)) < r/2$,

- A compact $K \subset \mathbb{T}$, $m(K) > r$,

- $g \in H^\infty$, $\|g\|_2 < M$ such that $f \circ h^{-1} = g$ on $K$.

Claim. $\mathcal{F} = \mathcal{F}(s, r, M)$ is nowhere dense in $C(\mathbb{T})$.

Take $f \in C(\mathbb{T})$ and a number $a > 0$. Approximate $f$ uniformly (with error $< a$) by a step-function $q$. Denote by $n$ the number of intervals $I$ of constancy of $g$. We may assume that $1/n < s$ and that all the numbers $\{q(I)\}$ are different. Denote

$$
\delta := \min \left\{ \frac{r}{2n}, \min_{I \neq I'} |q(I) - q(I')| \right\}
$$

Find $\epsilon(\delta, M)$ from lemma 8. The claim will be proved once we show that any $u$ with $\|u - q\|_2 < \epsilon$ is not in $\mathcal{F}$.

Indeed, suppose there are $h$, $K$ and $g$ as in the definition of $\mathcal{F}$. One can see easily that for at least two intervals $I = I_1, I_2$

$$
m(h(I) \cap K) > \delta.
$$

So, the function $g = u \circ h^{-1}$ satisfies all conditions of lemma 8 therefore it can not belong to $H^\infty$. This contradiction proves the claim, and lemma 7 and theorem 3 follow.

Remark 3. The following conjecture (which would emphasize sharpness of Theorems 2, 2') looks likely: one can construct a function $f \in C(\mathbb{T})$ such that all its relatives $g$ satisfy the condition that $\{\hat{g}(n)\} \notin l^1(\mathbb{Z}_-)$ or even stronger, $\notin l^p(\mathbb{Z}_-$), $p < 2$. Notice that for $l_p(\mathbb{Z})$ this is true, see [O86].

3.2. Proof of theorem 4 This follows quite easily from the Fourier representation of Brownian motion (see e.g. [K85 §16.3]). For simplicity, we will prove it for the complex-valued Brownian bridge, i.e. complex Brownian motion $W$ on $[0, 2\pi]$ conditioned to satisfy $W(0) = W(2\pi)$. As is well known, $n\widehat{W}(n)$ are independent standard Gaussians (one may take this as the definition of the Brownian bridge). We will only use the fact that $\widehat{W}(-1)$ is independent from the other variables. In other words we can write (as measure spaces) $C(\mathbb{T}) = \Omega \times \mathbb{C}$ where $\Omega$ is the space of functions satisfying $\hat{f}(-1) = 0$ equipped with the measure of Brownian bridge conditioned to satisfy this, and $\mathbb{C}$ equipped with the Gaussian measure. Since $e^{-it}$ is not in PLA, for any $f \in \Omega$ there can be no more than one value $x(f) \in \mathbb{C}$ for which $f + xe^{-it} \in \text{PLA}$. Since the measure of a single
point is always zero, we get by Fubini’s theorem that the measure of PLA is zero.

Fubini’s theorem requires that \( \text{PLA} \cap C(\mathbb{T}) \) be measurable. In fact it is Borel. To show this, endow PLA with the distance function \( d(f,g) = \rho((f - g)^*, 0) \) where \( f^* \) is defined as for polynomials, i.e.

\[
f^*(t) = \sup_{l < m} \left| \sum_{n=l}^{m} c(n)e^{int} \right| \quad f(t) = \sum_{n=0}^{\infty} c(n)e^{int}.
\]

It is easy to see that \( d \) makes PLA into a separable complete metric space. Souslin’s theorem \([K66, \S 39, IV]\) states that a one-to-one continuous map of a (Borel set in a) Polish space is Borel. Using this for the identity map from PLA to \( L^0(\mathbb{T}) \) shows that PLA is a Borel subset of \( L^0(\mathbb{T}) \). The restriction of a Borel set in \( L^0 \) to \( C \) is Borel so we get that \( \text{PLA} \cap C \) is Borel in \( C \) and theorem 4 is proved.

We remark that the same argument can be used to prove theorem 3. One needs only use a version of Fubini’s theorem for categories, see \([O80, \text{theorem 15.4}]\).

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