Cosmological models in conformal representations of Jordan theory

R. M. Avakyan, E. V. Chubaryan, G.H. Harutyunyan, A. V. Hovsepyan, A. S. Kotanjyan

Abstract. This paper is devoted to the investigation of conformally-related variants of the modified tensor-scalar Jordan theory on the example of determination of the comparative characteristics of the model Universe in "Einstein" and "proper" frames. Within the framework of this model, we consider the possibility for the accelerated expansion of the Universe at the recent epoch.

1. Introduction

The role of conformal transformations in gravitational theories has been discussed by H. Weyl [1], W. Pauli [2], A.Z. Petrov [3], R. Dicke [4], etc (for a recent review see [5]). In the Jordan-Brans-Dicke (JBD) theory the specificity of the scalar field is particularly evident in the conformal representations, where the scalar field is separated from the metric part of the gravitational field, and as it turns out, it may play a definite role in the phenomenon of acceleration of the expanding Universe. In the absence of matter fields, JBD theory is conformally-invariant [6]. Under certain conformal transformations it takes a standard "Einstein" form. In this case, the energy-momentum tensor for minimally or conformally coupled scalar field is added to the energy-momentum tensor of matter. There are clear advantages to have this conformal relation: one can use solutions already obtained for one representation to generate solutions of another, equivalent, representation.

The present paper is devoted to the theoretical study of the evolution of the Universe within the framework of various variants of JBD theory. We consider the possibility for the accelerated expansion of the Universe at the recent epoch. By taking into account that in General Relativity the arguments in favor of the cosmological constant are convincing enough, we introduce a similar quantity in JBD theory, assuming that the corresponding field is a scalar one. However, the latter is non-dynamical and is governed by the gravitational scalar.

2. The cosmological problem with the cosmological scalar \( \varphi = \Lambda y/y_0 \)

The modified version of the Jordan theory, in addition to the metric tensor, contains a scalar field \( y(x) \) in its gravitational sector. The corresponding action functional has the form [7]

\[
W = \frac{1}{c} \int \left\{ -\frac{y}{2c} \left[ R + 2\Lambda \frac{y}{y_0} - \xi g^{\mu \nu} \frac{y_{\nu} y_{\mu}}{y^2} \right] \right\} \sqrt{-g} d^4x
\]  

(1)
where $k = 8\pi/c^2$, $y_0$ is a constant, $\zeta$ is the dimensionless parameter of the theory and we use the notation $y_\mu = \partial_\mu y$. After the conformal transformation $\tilde{g}_{\mu\nu} = g_{\mu\nu}y/y_0$, the action (1) takes the form \[8\]

$$\tilde{W} = \frac{1}{c} \int \left[ -\frac{y_0}{16\pi} \left( \tilde{R} + 2\Lambda \right) + \frac{1}{2} \tilde{g}^{\mu\nu} \Phi_\mu \Phi_\nu \right] \sqrt{-g} d^4x,$$

were the new scalar field $\Phi$ is defined by the relation

$$\Phi_\mu = \frac{y_\mu}{y} \sqrt{\frac{(3 + 2\zeta) y_0}{16\pi}}.$$

The action (2) coincides with the Einstein action in the presence of a minimally-coupled scalar field ("Einstein" frame of the Jordan theory) [8]. The corresponding cosmological problem, described by the Friedmann-Robertson-Walker (FRW) line element

$$dS^2 = dt^2 - a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)],$$

has been discussed in [11]. In the present paper we compare the results obtained for the model with a minimally coupled scalar field with the model in the presence of a self-consistent scalar field in the case of action (1) and for the metric

$$dS^2 = d\tau^2 - R^2(\tau)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)].$$

The time coordinates and the scale factors in two different representations are related by

$$R(\tau) = (y/y_0)^{-1} a(t), \quad d\tau = (y/y_0)^{-1/2} dt.$$  \hspace{1cm} (6)

As in the paper [9], we solve the standard cosmological problem with the notations $H = \dot{R}/R$, $\dot{\psi} = \dot{y}/y$, $q = RR/R^2$, where the dot stands for the derivative with respect to $\tau$. The set of field equations reads

$$\frac{1}{yR^3} \frac{d(yR^3)}{d\tau} = \frac{2}{3 + 2\zeta} \left( \varphi - y \frac{d\varphi}{dy} \right),$$

$$2\dot{H} + 3H^2 - \psi H + \zeta\psi^2/2 - \varphi = 0,$$

$$3H^2 - \zeta\psi^2/2 + 3\psi H - \varphi = 0.$$  \hspace{1cm} (9)

By taking into account that in the problem under consideration $\varphi = \Lambda y/y_0$, the equation (7) is reduced to $\ddot{y}/y = -3\psi H$, or equivalently

$$\ddot{\psi} + \psi^2 + 3H\psi = 0.$$  \hspace{1cm} (10)

From here, for the evolution of the expanding Universe ($H > 0$) we get

$$\dot{y}/y < 0 \Rightarrow \dot{\psi}/\psi^2 < -1.$$  \hspace{1cm} (11)

Further, subtracting (9) from (8), we get

$$2\dot{H} + \zeta\psi^2 - 4\psi H = 0.$$  \hspace{1cm} (12)
Defining a new function \( \dot{y} = z(y) \), the latter is reduced to the Euler equation [10]:

\[
y^2 z'' - 2z' y - 3\zeta z = 0,
\]

with the solution [11]

\[
\psi = \frac{\dot{y}}{y} = c_1 \left( \frac{y}{y_0} \right)^{\frac{1+\sigma}{2}} + c_2 \left( \frac{y}{y_0} \right)^{\frac{1-\sigma}{2}}, \quad \sigma = \sqrt{3(3+2\zeta)}.
\]

After the substitution of (14) into (7) and (9), we obtain the relation between the constants of integration \( c_1 \) and \( c_2 \):

\[
c_1 c_2 = -\Lambda / (3 + 2\zeta),
\]

from which it follows that \( c_1 \) and \( c_2 \) have opposite signs, when \( \Lambda > 0 \), and one of these constants is equal to zero, when \( \Lambda = 0 \).

By taking into account that

\[
\frac{\ddot{y}}{y}/\frac{\dot{y}}{y} = c_1 \left( \frac{3 + \sigma}{2} \right) y^{\frac{1+\sigma}{2}} + c_2 \left( \frac{3 - \sigma}{2} \right) y^{\frac{1-\sigma}{2}} < 0,
\]

we conclude that \( c_1 < 0, c_2 > 0 \). These constants are determined from the condition \( y = y_0 \) at \( t = 0 \):

\[
\psi_0 = -|c_1| + c_2, \quad |c_1| c_2 = \Lambda / (3 + 2\zeta).
\]

As a result one finds the following relations

\[
|c_1| = -\psi_0 / 2 + \sqrt{\psi_0^2 / 4 + \Lambda / (3 + 2\zeta)}, \quad c_2 = \psi_0 / 2 + \sqrt{\psi_0^2 / 4 + \Lambda / (3 + 2\zeta)}.
\]

By taking into account that our ultimate aim is to compare the obtained results with the corresponding problem for the action (2), let us present (14) in the form

\[
\psi = c_2 \left( \frac{y}{y_0} \right)^{\frac{1-\sigma}{2}} \left[ 1 - (|c_1| / c_2) (y/y_0)^\sigma \right].
\]

After the integration of this we find the following expression

\[
\left( \frac{y}{y_0} \right)^{\sigma/2} = \frac{1}{\alpha} \tanh \left( \frac{\sigma}{2} \frac{\Lambda}{2\zeta + 3} t + \delta \right) = \frac{1}{\alpha} \tanh \left( \frac{3}{2} H_0 \sqrt{\frac{0}{\Omega \Lambda}} t + \delta \right).
\]

Here we have introduced the notations

\[
\alpha = \frac{|c_1|}{c_2} < 1, \quad e^{2\delta} = \frac{1 + \alpha}{1 - \alpha}, \quad \frac{0}{\Omega \Lambda} = \frac{\Lambda}{3H_0^2}.
\]

From (10) one can see that

\[
\dot{\psi}/\psi + \ddot{y}/y + 3\dot{R}/R = 0,
\]

from which for the function \( R/R_0 \) one obtains

\[
\left( \frac{R}{R_0} \right)^3 = \frac{\psi_0 y_0}{\psi \bar{y}}.
\]

For the numerical calculations of the parameters \( H, \Omega \Lambda, \Omega y, q \) it is convenient to use the following combination of equations (10) and (12):

\[
\frac{d}{dt} \left( \frac{\psi}{\psi} \right) - \frac{\dot{\psi}}{\psi} - \frac{4 + 3\zeta}{2} \psi^2 = 0.
\]
With the definition $z(\psi) = \dot{\psi}/\psi$, the equation (24) is reduced to a known class of differential equations (see for example, [10]):

$$zz' - z - (4 + 3\zeta)\psi/2 = 0.$$  

Taking $z = u\psi$, the variables are separated and we find

$$\psi^2 = 4(\sigma - 1 + 2u)\frac{1-\sigma}{\sigma} (\sigma + 1 - 2u)^{-\frac{1-\sigma}{\sigma}}.$$  

(25)

In order to find the Einsteinian limit ($\zeta \to \infty$), we present (25) in the form

$$\left(\frac{\psi}{2}\right)^{2\sigma} = \frac{\sigma - 1}{\sigma + 1}\frac{1}{(\sigma^2 - 1)^\sigma} \left(1 + \frac{|u|}{1 - \sigma}\right)^{1-\sigma} \left(1 + \frac{|u|}{1 + \sigma}\right)^{-1-\sigma},$$  

(26)

from which it follows that $\lim_{\sigma \to \infty} \psi = 1/\sigma \to 0$. As a result, the solution of the equation (26) can be written as

$$u = \frac{\ddot{\psi}}{\psi^2} = \frac{1}{2} - \sigma \frac{1 + \alpha (y/y_0)^\sigma}{2} 1 - \frac{\alpha (y/y_0)^\sigma}{2}.$$  

(27)

### 3. Determination of the integral parameters

It is easy to see that all relevant functions are expressed in terms of the ratio $\psi/H$, which can be directly determined from (10):

$$H/\psi = -(1 + u)/3.$$  

(28)

Then, from (12), for the deceleration parameter one finds

$$q = 1 + \frac{H^2}{H_0^2} = 1 + \frac{2\psi}{H} - \frac{\zeta \psi^2}{2 H^2}.$$  

(29)

From (9) we have

$$1 - \zeta \frac{\psi^2}{6 H^2} + \frac{\psi}{H} = \frac{\varphi}{3 H^2} = \Omega_\Lambda,$$  

(30)

where $\Omega_\Lambda$ is the effective energy contribution of the cosmological scalar. It should be noted that (30) allows us to obtain $\psi_0/H_0$ for $y = y_0$:

$$\frac{\psi_0}{H_0} = \frac{\Lambda}{3 H_0^2} = 1 - \frac{\zeta \psi_0^2}{6 H_0^2} + \frac{\psi_0}{H_0},$$  

(31)

which is needed for the calculation of $\alpha = |c_1|/c_2$ from (18). For the evaluation of $H/H_0$ we use (28):

$$\frac{H}{H_0} = -\psi \frac{1 + u}{H_0} \frac{3}{3},$$

and the behavior of $R/R_0$ is determined from (23).

By taking into account that we consider the late stages of the Universe expansion, the expression (20) for large positive values of the argument can be presented in the form

$$(y/y_0)^{\sigma/2} \approx (1 - 2e^{-2x})/\alpha,$$  

(32)

where $x = (3/2) H_0 (\sqrt{\Omega_0} - 3/2) t$ and we determine the relation between conformally related time coordinates by (6). As a result one gets $\tau = \alpha^{1/\sigma} t$.

In figures 1-4 we have plotted the time evolution of the functions $H/H_0$, $R/R_0$, $q$, $\Omega_y$, $\Omega_\Lambda$ for the values of the parameter $\zeta = 50$ and $\zeta = 1000$.  


Figure 1. Time dependence of the Hubble parameter $H/H_0$. The time is measured in units $5 \cdot 10^9$ years. Dashed and full curves correspond to values $\zeta = 50$ and $\zeta = 1000$, respectively.

Figure 2. Time dependence of the scale factor $R/R_0$ for $\zeta = 50$ (dashed curve) and $\zeta = 1000$ (full curve).

Figure 3. Time dependence of the "deceleration" parameter $q$ for $\zeta = 50$ (dashed curve) and $\zeta = 1000$ (full curve).
**Figure 4.** Time dependence of the energy contribution of the scalar field, $\Omega_y$, and of the field, corresponding to cosmological scalar, $\Omega_\Lambda$. Dashed and full curves correspond to $\zeta = 50$ and $\zeta = 1000$, respectively.

4. **Nonminimally coupled scalar field in the presence of $\varphi = \Lambda$**

In this section we consider the problem described by the action functional [7]- [13]

$$W = \int \left\{ -y \left[ (R + 2\Lambda) + \zeta g^{\mu\nu} \nabla_\mu y \nabla_\nu y \right] \right\} \sqrt{-g} d^4 x. \quad (33)$$

Taking the line element in the FRW form (4), we introduce the cosmological functions

$$H = \frac{\dot{a}}{a}, \quad \psi = \frac{\dot{y}}{y}, \quad q = \frac{a\ddot{a}}{a}, \quad \Omega_\Lambda = \frac{\Lambda}{3H^2}. \quad (34)$$

The system of equations in convenient combinations is the following:

$$3H^2 - \psi^2 \zeta / 2 + 3\psi H = \Lambda \quad (35)$$

$$3H\psi - \zeta \psi^2 - 3qH^2 = -\frac{2\Lambda \zeta}{3 + 2\zeta} \quad (36)$$

$$\frac{d(ya^3)}{dt} = \frac{2\Lambda}{3 + 2\zeta} ya^3. \quad (37)$$

In order to solve the equation (37) it is reasonable to calculate the derivative of the expression

$$\frac{d(ya^3)}{dt} = ya^3 (\psi + 3H). \quad (38)$$

By taking into account (37), we can see that

$$\dot{\psi} + \psi^2 + 3H\psi = \frac{2\Lambda}{3 + 2\zeta}. \quad (39)$$

Now, combining the equations (35) and (36), we obtain

$$3\dot{H} + 9H^2 + 3H\psi = 2\Lambda \frac{3 + 4\zeta}{3 + 2\zeta}. \quad (40)$$
and, as a result,

\[
d\frac{d^2(ya^3)}{dt^2} - 2\Lambda ya^3 \left(\frac{4}{3 + 2\zeta}\right) = 0.
\]  

(41)

The solution of the equation (41) can be written in the form

\[
ya^3 = c_1 \cosh At + c_2 \sinh At = c \cosh (At + \alpha), \text{ for } c_1 > c_2,
\]  

(42)

\[
ya^3 = c_1 \cosh At + c_2 \sinh At = c \sinh (At + \alpha), \text{ for } c_1 < c_2,
\]  

(43)

where \(A^2 = 2\Lambda \frac{(3 + 4\zeta)}{(3 + 2\zeta)}\). In the case (43), \(ya^3 = c \sinh (At + \alpha)\), the integration constants are determined from the following relations

\[
\psi_0 + 3H_0 = A \coth \alpha
\]  

(44)

\[
y_0a_0^3 = c \sinh (At + \alpha).
\]  

(45)

From the equation (37) we have

\[
\dot{y}a^3 = y_0a_0^3 + \frac{2\Lambda c}{A(3 + 2\zeta)} \left[\cosh (At + \alpha) - \cosh \alpha\right].
\]  

(46)

From here it follows that

\[
\psi = \frac{\psi_0 + H_0 \sqrt{\frac{6\Omega}{(3 + 3\zeta)(3 + 2\zeta)}} \left[\sinh At + \coth \alpha (\cosh At - 1)\right]}{\cosh At + \coth \alpha \sinh At}.
\]  

(47)

Now, by using equations (38), (44) and (47), we get

\[
\frac{H}{H_0} = \sqrt{\frac{6\Omega}{(3 + 3\zeta)(3 + 2\zeta)}} \frac{[(\zeta + 1)(\sinh At + \coth \alpha (\cosh At - 1)) + 1]}{\cosh At + \coth \alpha \sinh At}.
\]  

(48)

At the initial time \(t = 0\) from (35) one can find the ratio \(\psi_0 / H_0\):

\[
\frac{\psi_0}{H_0} = \frac{3}{\zeta} \left(1 \pm \sqrt{1 + \frac{2\zeta}{3} \left(1 - \frac{\zeta}{\Omega}\right)}\right).
\]  

(49)

From (44) and (49) it is easy to find the parameter \(\alpha\):

\[
\coth \alpha = \frac{1}{\zeta \sqrt{\frac{6\Omega}{2\Omega\zeta}}} \sqrt{\frac{3 + 2\zeta}{4 + 3\zeta}} \left(1 \pm \sqrt{1 + \frac{2\zeta}{3} \left(1 - \frac{\zeta}{\Omega}\right) + \zeta}\right).
\]  

(50)

The expression for the scale factor \(a(t)\) is obtained by integrating (48):

\[
\frac{a}{a_0} = \left[\cosh At + \coth \alpha \sinh At\right]^{\frac{1 + \zeta}{1 + 2\zeta}} \left[\sqrt{\coth^2 \alpha - 1} - \sqrt{\coth^2 \alpha + 1} + 1\right]^{\frac{1 - (\zeta + 1)}{1 - 1/2}} \frac{1}{\sqrt{\coth^2 \alpha - 1}} \cosh \alpha.
\]  

(51)
The deceleration parameter $q$ can be found combining the equations (36) and (35):

$$q = \frac{6\Omega_{\Lambda}}{3 + 2\zeta} \left( \frac{H_0}{H} \right)^2 - \frac{\psi}{H} - 2. \quad (52)$$

The energy contribution of the scalar field at an arbitrary time is determined from the relation

$$\Omega_y = 1 - \Omega_{\Lambda} = 1 - \frac{6\Omega_{\Lambda}}{3 + 2\zeta} \left( \frac{H_0}{H} \right)^2 = \frac{\zeta}{6} \left( \frac{\psi}{H} \right)^2 - \frac{\psi}{H}. \quad (53)$$

where

$$\Omega_{\Lambda} = 1 - \frac{6\Omega_{\Lambda}}{3 + 2\zeta} \left( \frac{H_0}{H} \right)^2 = \frac{\zeta}{6} \left( \frac{\psi}{H} \right)^2 - \frac{\psi}{H}. \quad (54)$$

We will not consider the case of (42), as in the problem under consideration tanh $\alpha$ and coth $\alpha$ take the same numerical value (50) which is greater than one.

In figures 5-8 we have displayed the time dependence of the functions $H(t)/H_0$, $q(t)$, $\Omega_y(t)$ and $a(t)/a_0$. As before, the time is measured in units of $t'$ defined as:

$$At = H_0 \sqrt{\frac{0}{6\Omega_{\Lambda}} \frac{4 + 3\zeta}{3 + 2\zeta}} = t' \sqrt{\frac{2(4 + 3\zeta)}{3(3 + 2\zeta)}}, \quad (55)$$

(in the Einsteinian limit one has $At = t'$).

![Figure 5](image-url)

**Figure 5.** Time dependence of $H(t')/H_0$ for $\zeta = 50$ (upper curve in the left sector), $\zeta = 500$ and $\zeta = 1000$ (upper and lower curves in the converging pair).

5. Conclusion

As a result, for the time dependence of the parameters in the model of the Universe similar qualitative features are obtained as in [8]. The quantitative differences are related to the form of the potential energy for the scalar field with general conclusion confirming the essential dependence of the influence of scalar field from the interaction parameter $\zeta$ [12]. On the example of the Universe acceleration (see figure 3) it is clearly seen that for large positive values of $\zeta$ (of the order $10^3$) the acceleration practically coincides with that for General Relativity. For smaller values of $\zeta$, the influence of the scalar field on the expansion rate is stronger and this leads to the corresponding decrease of $q$. 


Figure 6. Time dependence of the scale factor $a(t')/a_0$ for the values $\zeta = 50$ (dashed curve), $\zeta = 500$ (full curve in the converging pair), $\zeta = 1000$ (dashed curve in the converging pair).

Figure 7. Time dependence of $q(t')$ (in units $5 \cdot 10^9$ years) for $\zeta = 500$ and for $\Omega_\Lambda = 0.8$ (dashed curve), $\Omega_\Lambda = 0.75$ (full curve), $\Omega_\Lambda = 0.6$ (dotted curve).

Figure 8. Time dependence of the energy contributions $\Omega_y(t)$ (dashed curve) and $\Omega_\Lambda(t)$ (full curve) for $\zeta = 500$. 
For nonminimally coupled scalar field in the presence of $\varphi = \Lambda$ we consider the proper representation of the Jordan modified theory, taking into account the presence of a cosmological scalar $\varphi = \Lambda$. The results obtained for different values of the dimensionless parameter $\zeta$ (50, 500, 1000) are presented in figures 5-8. These results show that the behavior of the curves for $\zeta = 500$ and $\zeta = 1000$ is practically indistinguishable, in contrast to the case $\zeta = 50$. This confirms the proximity of the Jordan theory and General Relativity for large values of $\zeta$ [14]. The results described above show that the role of scalar fields in variants with small $\zeta$ is different from that in General Relativity, providing a possibility to adjust the deceleration parameter $q$ in comparing with the observational data. Figure 7 shows the time dependence of the deceleration parameter for different values of the energy contribution of the cosmological scalar. Figure 8 displays the energy contributions of the scalar field and the cosmological scalar typical for all versions of the theory. In comparing the obtained results with other models for a flat universe within tensor-scalar theory [15] a well-defined tendency of development is observed: the energy contributions of ordinary matter and the scalar field steadily converge to zero, whereas the contribution due to the presence of the cosmological parameter grows.

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References
[1] Weyl H 1923 Raum, Zeit, Materie (Berlin: Springer-Verlag)
[2] Pauly W 1981 Theory of Relativity (New-York: Dover Publications)
[3] Petrov A Z 1966 New Methods in General Relativity Theory (Moscow: Nauka).
[4] Dicke V 1962 Phys. Rev. 125 2163.
[5] Clifton T Ferreira P G Padilla A and Skordis C 2012 Phys. Rept. 513 1
[6] Jordan P 1955 Gewerkraft und Weltall (Braunschweig).
[7] Avakyan R M Harutyunyan G H and Papoyan V V 2005 Astrophysics 48 455.
[8] Abrahamyan L Avakyan R M Harutyunyan G H Hovsepyan A V and Chubaryan E V 2013 Astrophysics 56 471.
[9] Avakyan R M Harutyunyan G H Hovsepyan A V and Kotanjyan A S 2013 Astrophysics 56 617.
[10] Kamke E 1976 Handbook on Ordinary Differential Equations (Moscow: Nauka).
[11] Avakyan R M Harutyunyan G H 2005 Astrophysics 48 633.
[12] Harutyunyan G H and Papoyan V V 2002 Physics of Particle and Nuclei 33 (7) 114.
[13] Avakyan R M Harutyunyan G H and Papoyan V V 2008 Astrophysics 51 151.
[14] Will C M 1993 Theory and Experiment in Gravitational Physics, (Cambridge: Cambridge University Press)
[15] Avakyan R M Harutyunyan G H and Ovsepyan A V 2010 Astrophysics 53 317.