Using Maple and GRTensorIII in relativistic spherical models

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Abstract

This article presents some aspects and experience in the use of algebraic manipulation software applied to general relativity. Some years ago certain results were reported using computer algebra platforms, but the growing popularity of graphical platforms such as Maple allows us to approach the problem of the simplifications of many expressions from another point of view. Some simple algebraic programming procedures are presented (in Maple with the GRTensorIII package) to obtain and study material distributions with spherical symmetry and to search for exact solutions of the Einstein field equations. The purpose is to show how useful a computer algebra system can be. All calculations were performed using the GRTensorIII computer algebra package, which runs on Maple 2017, along with several Maple routines that we have used specifically for the simplification of many of the algebraic expressions that are very common in this type of problem.

Keywords: Computer algebra, General relativity, Gravitation theory, Algorithms, Maple, GRTensorIII

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1 Introduction

Computer algebra systems (CAS) have a wide variety of applications in fields that require time-consuming, difficult-to-perform, and error-prone calculations when done manually. They are used more and more frequently and especially when it is necessary to complete a calculation in pages after pages for many hours or perhaps several days. As early as the late 1950s and early 1960s, various programs appeared \[1, 2, 3, 4\] aimed at demonstrate that in the scientific field it is possible to go beyond the purely numerical area and use them to carry out a symbolic calculation. For reasons such as these, various members of the computer algebra family of languages were created, and in fact many were created to perform specific calculations of great complexity in fields such as electronic optics \[5\], celestial mechanics \[6\], quantum electrodynamics \[7, 8, 9, 10, 11, 12\] or general relativity \[13, 14, 15, 16\].

Computer algebra for general relativity (GR) has a long history, beginning almost as early as computer algebra itself in the 1960s.

The first GR program was GEOM, written by J.G. Fletcher \[17\]. Its main ability was to compute the Riemann tensor of a given metric. In 1969, R. A. d’Inverno \[18\] developed ALAM (for Atlas Lisp Algebraic Manipulator) \[19\] and used it to compute the Riemann and Ricci tensors of the Bondi metric. According to \[20\], Bondi and his collaborators took 6 months to complete the original calculations, while the ALAM calculation took 4 minutes and resulted in the discovery of 6 errors in the original paper. Since then, numerous packages have been developed and various investigations have been carried out using manipulative algebraic software (CAS) and in many cases the calculations were of such length that they would have been prohibitively expensive to complete without the aid of a computer \[21\]. Its main advantage is the ability to handle a large number of algebraic calculations and this particularity has allowed advances in fields of theoretical physics such as GR or High Energy Physics (HEP).

Many of these problems have used some free, open source, general purpose software with an emphasis on tensor calculus for GR, such as Java-based REDBERRY \[22, 23\], SAGE \[24\] and SAGEMANIFOLDS written in Python, CADABRA \[25, 26\] developed in C++ and Python. Others, applied specifically for HEP, have been designed based on special algorithms \[27, 28, 12\] and implemented in programs like SCHOONSHIP \[29, 30\], designed by M. Veltman, ASHMEDAI \[31, 32, 33\] by M. Levine, REDUCE \[7, 8, 9, 11\] by A. Hearn, MACSYMA (to become MAXIMA in 1998) \[34, 35, 36\] by J. Moses developed at MIT, or more recent FORM \[37, 38, 39\] by J. Vermaseren. Proprietary software such as MATHEMATICA \[40, 41, 42, 43\] by S. Wolfram or MAPLE \[44, 45, 46, 47, 48, 49, 50, 51, 52\] from B. Char developed at the University of Waterloo. The CAS programs mentioned above constitute only a very small part of the available applications, special purpose and general systems that can be consulted in repositories and lists, maintained and frequently updated \[53, 54, 55, 56, 57, 58\]. Of all the CAS applications mentioned above, we are going to refer in this work to MAPLE.

Maple is a general purpose CAS, initially developed at the University of Waterloo as a result of discussions
on the state of symbolic computing in the 1980s. At the time, large systems such as ALTRAN, CAMAL, REDUCE and MACSYMA, based on the computer technology of the 1960s, they decided to design a new system from scratch, taking advantage of the advances in software engineering available and the lessons of experience. Its basic design features (for example, elementary data structures, input/output, number arithmetic, and elementary simplification) are encoded in a low-level language for efficiency. An important property is that most of the algebraic facilities of the system are implemented using the high-level user language. The basic system, or kernel, is compact and efficient enough to be practical to use for a shared environment or on personal computers with very little main memory. Library functions are loaded into the system as needed, adding features such as polynomial factorization, equation solving, indefinite integration, and matrix manipulation to the system. The modularity of this design allows users to demand available computing resources in proportion to their actual use. It has specialized libraries for elementary and special mathematical functions, it offers support for symbolic and numerical computation with exact results, it can handle a wide set of equation systems, including Diophantine equations, ordinary differential equations (ODE), partial differential equations (PDE), Differential Algebraic Equations (DAE), Delay Differential Algebraic Equations (DDE) and recurrence relations. Initially the kernel of the system was written in macros that could be translated by a locally developed macroprocessor (called Margay) into versions of the kernel in the C programming language for various operating systems; currently only C is used. The GUI was first released with version 5 (Maple V) and continued to be numbered until Maple 18 and then changed to a yearly label. Due to the low demand for main memory to run the kernel and the modular design for many of the possible user applications in a high-level language, it has become one of the main symbolic algebra systems used by many researchers and engineering corporations worldwide. These applications or packages can be programmed by the user and many can be found on the web as GRTensorII or its update GRTensor III. (in short GRTensor).

GRTensor is a computer algebra package for performing calculations in the general area of differential geometry. Its main objective is the computation of tensor components in curved spacetimes specified in terms of a metric or set of basis vectors. The library relies on a series of special commands starting with ”gr” (for example, grcalc, grdisplay, gralter, grdefine, etc.) to deal with a series of (pre) geometric objects, defined as the metric tensor, Ricci tensor and scalar, Einstein tensor, Christoffell symbols, etc. This library of objects can be extended to define new tensors, or use the Newman-Penrose formalism. Although originally designed for use in the field of general relativity, GRTensor is useful in many other fields. There is a version for MATHEMATICA called GRTensorM. The GRTensorII package was originally developed for Maple V and can be run with versions from Maple V Release 3 to Maple 13. The GRTensor III version runs as of Maple 15. All documentation and software are distributed free of charge to help both for research and teaching.
The final objective of this article is to present the calculations, especially in Sec. 2, that have been carried out with an emphasis on the methods, packages and techniques of computational algebra that we use in a spreadsheet developed for the GRTensorIII package running on the Maple 17 platform. Initially, it was written in GRTensorII running on top of Maple 13 and when upgrading GRTensor to version III, it became necessary to upgrade the Maple version as well. The spreadsheet follows the algorithm described in 67, 68 starting from a spherically symmetric perfect fluid distribution of density \( \hat{\rho} \), radial pressure \( \hat{P} \), tangential pressure \( \hat{P}_t \), a flux of unpolarized radiation moving in the radial direction with density \( \hat{\varepsilon} \) and 3\( \hat{\mu} \) the isotropic radiation of the energy density, in a local Minkowskian system. By performing the corresponding coordinate transformations, to a radiative (Bondi) coordinate system, we can construct an Impulse Energy Tensor that must satisfy both the Einstein field equations \( (G_{ab} = \kappa T_{ab}) \) as the conservation equations \( (T^{ab}_{;a} = 0) \). When calculating these equations, it is necessary to simplify them and for this, some procedures and functions have been used, as in 70. In Sec. 3 some comments on the calculations made are presented and in the last section the conclusions and the possibility of extending this procedure to other astrophysical scenarios are detailed.

2 Algorithm structure

As indicated in the previous section, the created spreadsheet basically follows the procedure used in [71] and for the charged case in [72] with the modifications made in 67, 68. In the original sheet, the commands are grouped by sections and each of the sections has a name that suggests the calculation to be carried out. In this build for the article, commands (in red) and output (in blue) from the Maple interpreter will be reproduced. All the commands to perform the calculations of the sheet are present in this work, however, not all the outputs (in blue) were placed and the most extensive ones were edited.

2.1 Login

To start the Maple session we loaded the GRTensorIII libraries, the expression simplification routines were taken from Davies [70] and the directory from where the metrics will be loaded. We start with a reset statement to ensure that you have a new session of the sheet,

\begin{verbatim}
>restart;
>grtw();
\end{verbatim}
2.2 Minkowskian local flat space

It is assumed that the region of the space to be considered is composed of energy density anisotropic material \( \rho \), radial pressure \( P_r \), tangential pressure \( P_t \), radiation isotropic energy density \( 3\mu \) and non-polarized radiation of energy density \( \xi \) propagating in a radial direction. We are going to establish the metric of this region of space.

\[
qload(minkowski);
\]

Calculated \( ds \) for minkowski \( (0.001000 \text{ sec.}) \)

\[
\text{Default spacetime} = \text{minkowski} \\
\text{For the minkowski spacetime:} \\
\text{Coordinates} \\
x(\text{up}) \\
x^a = [t \ x \ y \ z] \\
\text{Line element} \\
ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \\
\text{The Minkowski metric -- plana--}
\]

The covariant components of the metric tensor of the local Minkowski system are calculated and displayed. You can use two \textit{GRTensorIII} instructions: \texttt{grcalc()} to calculate and \texttt{grdisplay()} to show the result. However, \texttt{grcalcd()},

allows you to do the calculation and shows the result, combining the two previous instructions.

\[
g(dn,dn) \]  

Let’s set the values for the contravariant unit vectors in the Minkowski reference system. We will use them to calculate the components of the tensor in the local system

\[
g(dn,dn) \]  

And to establish the Stress-energy tensor of this region, let’s start with: Energy density anisotropic material \( \rho \), Radial pressure \( P \) and Tangential pressure \( P_t \)

\[
g(dn,dn) \]  

Isotropic radiation of energy density \( 3\mu \)

\[
g(dn,dn) \]  

The component corresponding to Polarized radiation

\[
g(dn,dn) \]  

Unpolarized radiation of energy density \( \xi \) propagating in the radial direction

\[
g(dn,dn) \]  

Bringing together the three parts of the Stress-Energy tensor, we get the stress-energy tensor Matter + radiation:

\[
g(dn,dn) \]  

Calculated \( T0(dn,dn) \) for minkowski (0.003000 sec.)
Redefining the radiation and matter variables of the stress-energy tensor:

```plaintext
> grmap(T0(dn, dn), subs, rho_M = rho[0]-3*mu, 'x');
> grmap(T0(dn, dn), subs, P_M = P[0]-mu, 'x');
> grmap(T0(dn, dn), subs, P_t = P[t]-mu, 'x');
> grdisplay(T0(dn,dn));
```

For the minkowski spacetime:

\[
T_{0\alpha\beta} = \begin{pmatrix}
\rho_0 + \xi & -\xi & 0 & 0 \\
-\xi & P_0 + \xi & 0 & 0 \\
0 & 0 & P_t & 0 \\
0 & 0 & 0 & P_t + \mu
\end{pmatrix}
\]  

(4)

### 2.3 Lorentz transformation

Let’s suppose that you have an observer moving in relation to the local system Minkowskian, with a radial velocity \( \omega \). The components of the Stress-Energy tensor in this new system of Lorentz, will be given by the relationship:

\[
\tilde{T}_{\mu\nu} = \Lambda^\alpha_\mu \Lambda^\beta_\nu T_{\alpha\beta}
\]

where \( \Lambda^\alpha_\mu \) is the transformation matrix.

```plaintext
> grdef('umsqrto := 1/sqrt(1-omega^2) ');  
> grdef('Lambda{\a b}:= umsqrto *kdelta{\a $t$}*kdelta{$t b'}
-omega* umsqrto *kdelta{\a $x$}* kdelta{$t b'} -omega* umsqrto *kdelta{\a $t$}* kdelta{$x b'}
+ umsqrto *kdelta{\a $x$}* kdelta{$t b'}+ kdelta{\a $y$}* kdelta{$y b'}+  
 kdelta{\a $z$}* kdelta{$z b'}	; 
```

2 Algorithm structure
> grcalcd(Lambda(up, dn));

\[ CPU \ Time = 0.011 \]

For the minkowski spacetime:

\[ \Lambda(up, dn) \]

\[ \Lambda = \begin{pmatrix}
\frac{1}{\sqrt{1-\omega^2}} & -\frac{\omega}{\sqrt{1-\omega^2}} & 0 & 0 \\
-\frac{\omega}{\sqrt{1-\omega^2}} & \frac{1}{\sqrt{1-\omega^2}} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad (5) \]

Let’s carry out the transformation operation now. To do it, we will establish this transformation as:

> grdef('T1{\ a b} := Lambda[^c \ a] * Lambda[^d \ b] * T0{\ c \ d}');

> grcalcd(T1(dn, dn));

\[ CPU \ Time = 0.012 \]

For the minkowski spacetime:

\[ T1(dn, dn) \]

\[ T1_{\ a \ b} = \begin{pmatrix}
\frac{-\rho_0 + 2\omega\xi + \omega^2 P_0 + \omega^2 \xi}{-1 + \omega^2} & \frac{\omega\rho_0 + 2\omega\xi + \xi + \omega P_0}{-1 + \omega^2} & 0 & 0 \\
\frac{\omega\rho_0 + 2\omega\xi + \xi + \omega P_0}{-1 + \omega^2} & \frac{-\omega^2 \rho_0 + \omega^2 \xi + 2\omega\xi + P_0 + \xi}{-1 + \omega^2} & 0 & 0 \\
0 & 0 & P_t & 0 \\
0 & 0 & 0 & P_t
\end{pmatrix} \quad (6) \]

We obtain the expression of the Stress-energy tensor in the local system, with radial velocity \( \omega \).

### 2.4 Bondi Radiative Coordinate System

As the study we are doing, is related to radiation, then it is logical to assume that we must use a coordinate system according to the theme. Therefore, we are going to use Bondi’s radiative coordinate system as in [73].

> qload(bondi);

Calculated ds for bondi (0.001000 sec.)
Default space time = bondi

For the bondi spacetime:

Coordinates

\[ x^\alpha = [u, r, \theta, \phi] \]

Line element

\[
    ds^2 = \frac{V(u,r)e^{\beta(u,r)}}{r} du^2 + 2e^{2\beta(u,r)} r dr \cdot du - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2
\]

The Bondi metric (Proc. Roy. Soc. A 269, 21) (7)

Changing the expression of the Stress-energy tensor of the local Minkowskiana metric to its structure in the Bondi radiation coordinate system.

\[
    \text{grdef('Vsr:=sqrt(V(u,r)/r)');}
\]
\[
    \text{grdef('Mu\{^a b\}:= exp(beta(u,r))*Vsr*kdelta\{^a u\}*kdelta\{u b\} + exp(beta(u,r))/Vsr*kdelta\{^a r\}*kdelta\{r b\} + exp(beta(u,r))/Vsr*kdelta\{^r r\}*kdelta\{r b\} + r*kdelta\{^a \theta\}*kdelta\{\theta b\} + r*sin(\theta)*kdelta\{^a \phi\}*kdelta\{\phi b\}');}
\]

Defining and showing the matrix of the Minkowski local system transformation to the Lorentz system

\[
    \text{grcalcd(Mu(up, dn));}
\]

Calculated Vsr for bondi (0.002000 sec.)

Calculated grtensor:=kdelta(dn,dn) for bondi (0.002000 sec.)

Calculated grtensor:=kdelta(up,dn) for bondi (0.002000 sec.)

Calculated Mu(up,dn) for bondi (0.011000 sec.)

CPU Time = 0.021

For the bondi spacetime:

\[ Mu (up, dn) \]

\[ M (up, dn) \]
\[ M^a_b = \begin{bmatrix} e^{2\beta(u,r)} \sqrt{V(u,r)} & \frac{e^{2\beta(u,r)}}{r} & 0 & 0 \\ \frac{e^{2\beta(u,r)}}{r} & \sqrt{V(u,r)} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r \sin \theta \end{bmatrix} \] (8)

This matrix of transformation, allows us to obtain the expression of the stress-energy tensor of the local system of Minkowski to the system of radiative coordinates with

\[ \tilde{T}_{ab} = M^\alpha_a M^\beta_b \hat{T}_{\alpha\beta} \] (9)

where \( M^a_b \) is the transformation matrix. As we are performing an operation between two expressions with different metrics, we must specify the space - or metric - corresponding to each term of the multiplication

\[ > \text{gdef('TB<2>{a b}:=Mu}^{c a} *Mu}^{d b}\text{*T1<1>{c d}');} \]

by defining what is the scope of the definition of each term

\[ > \text{grcalcd(1 = minkowski, 2 = bondi, TB(dn, dn));} \]

\[ \text{CPU Time} = 0.202 \]

For the bondi spacetime:

\[ \text{TB} (dn, dn) \]

\[ \text{TB} (dn, dn) \]

\[ TB_{ab} = \begin{bmatrix} -e^{2\beta(u,r)} V(u,r) (\rho_0 + 2\omega + \omega^2 P_0 r + \omega^2 \xi) & -\left(\frac{\omega \rho_0 - \rho_0}{\omega + 1}\right) e^{2\beta(u,r)} & 0 & 0 \\ -\frac{\omega \rho_0 + \omega \rho_0 - \rho_0}{\omega + 1} e^{2\beta(u,r)} & -\left(\frac{\omega P_0 - \rho_0}{\omega + 1}\right) & 0 & 0 \\ 0 & 0 & r^2 P_t & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta P_t \end{bmatrix} \] (10)

2.5 Effective Variables

We are going to define the effective variables in a similar way to the case of [71, 67]. However, here we will denote it as \( \rho_1 \) and \( P_1 \), without a bar at the top:
> grmap(TB(dn, dn), subs, rho[0] = (rho[1]+omega*P[1])/(1-omega), 'x');
> grmap(TB(dn, dn), subs, P[0] = (P[1]+omega*rho[1])/(1-omega), 'x');

Changing the term a bit for radiation

> grmap(TB(dn, dn), subs, xi = (1-omega)*epsilon/(1+omega), 'x');
> gralter(TB(dn, dn), expand, factor);
> grdisplay(TB(dn, dn));

For the bondi spacetime:

\[
TB_{ab} = \begin{bmatrix}
\frac{e^{2\beta(u,r)}V(u,r)(\omega^2\epsilon+\omega^2\rho_1+\omega P_1-2\omega-\omega\rho_1+\epsilon+\rho_1)}{r\omega(1+\omega)^2} & \rho_1 e^{2\beta(u,r)} & 0 & 0 \\
\rho_1 e^{2\beta(u,r)} & \frac{e^{2\beta(u,r)}V(u,r)(P_1+P_0)}{V(u,r)} & 0 & 0 \\
0 & 0 & r^2 P_t & 0 \\
0 & 0 & 0 & r^2 \sin^2 \theta P_t
\end{bmatrix}
\]  

(11)

Using the routines defined in [70], we can obtain an alternate expression for \(TB_{uu}\):

> TBdndnuu := kfactor(hcollect(grcomponent(TB(dn, dn), [u, u]), \{P[1], rho[1], V(u, r)\}, \{r, omega, beta(u, r)\}), V(u, r)*exp(2*beta(u, r))/r);

\[
TB_{dndnuu} = \frac{V(u, r)}{r} e^{2\beta(u,r)} \left( \frac{\omega P_t}{(-1+\omega)^2} + \frac{(-\omega + \omega^2 + 1)\rho_1}{(-1+\omega)^2} + \epsilon \right)
\]  

(12)

or equivalently

\[
TB_{dndnuu} = \frac{V(u, r)}{r} e^{2\beta(u,r)} \left( \frac{\omega (\rho_1 + P_t)}{(1-\omega)^2} + \rho_1 + \epsilon \right)
\]
2.6 Electromagnetic Component

If we are interested in the possibility that the material subject to the study presents an electric charge, it is necessary to include it in the tensor and therefore establish the expression for the Faraday tensor

\[ T_{\alpha\beta} = \frac{1}{4\pi} \left[ F_{\alpha\mu} F_{\beta}^{\mu} + \frac{1}{4} g_{\alpha\beta} (F_{\mu\nu} F^{\mu\nu}) \right] \]  \hspace{1cm} (13)

where \( F^{ab} \) satisfies the equations

\[ \left( \sqrt{g} F^{\alpha\beta} \right)_{,\beta} = 4\pi \sqrt{g} J^{\alpha} \quad F_{[\alpha\beta;\gamma]} = 0 \]

Assuming spherical symmetry, the only non-zero component is \( F \) and we have \( F^{01} \)

\[ (e^{2\beta} r^2 F^{01})_1 = 4\pi e^{2\beta} r^2 J^0 \]
\[ (e^{2\beta} r^2 F^{10})_0 = 4\pi e^{2\beta} r^2 J^1 \]

and when integrating the first:

\[ (e^{2\beta} r^2 F^{01}) = \int_0^r (4\pi e^{2\beta} r^2 J^0) \, dr \equiv Q(u, r) \]

Because of this, we can write

\[ F^{01} = \frac{e^{-2\beta}}{r^2} \cdot Q(u, r) \]

where \( Q(u, r) \), plays the role of electric charge and the electromagnetic component of the stress tensor is

\[ \text{CPU Time} = 0. \]

For the Bondi spacetime:

\[ F(u, up) \]
\[ F(up, up) \]
In terms of $F_{01}$, the electromagnetic component of the stress tensor is

$$F^{ab} = \begin{bmatrix}
0 & F_{01} & 0 & 0 \\
-F_{01} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad (14)$$

The electromagnetic component of the stress tensor is expressed in terms of the electric charge as:

$$\text{CPU Time} = 0.011$$

For the bondi spacetime:

$$\text{Tem}(dn, dn)$$

$$\text{Tem}(dn, dn)$$

$$\text{Tem}_{ab} = \begin{bmatrix}
\frac{1}{8} V(u,r) (e^{2\beta(u,r)})^3 F_{01}^2 & \frac{1}{8} (e^{2\beta(u,r)})^3 F_{01}^2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{8} \pi^2 (e^{2\beta(u,r)})^3 F_{01}^2
\end{bmatrix}, \quad (15)$$

The electromagnetic component of the stress tensor is expressed in terms of the electric charge as:

$$\text{CPU Time} = 0.007$$

$$\text{CPU Time} = 0.007$$
For the bondi spacetime:

\[ \text{Tem}(\text{dn}, \text{dn}) \]
\[ \text{Tem}(\text{dn}, \text{dn}) \]

\[
\text{Tem}_{ab} = \begin{bmatrix}
\frac{1}{8} V(u, r) (e^{2\beta(u, r)})^2 Q^2(u, r) & \frac{1}{8} (e^{2\beta(u, r)})^2 Q^2(u, r) & 0 & 0 \\
\frac{1}{8} V(u, r) (e^{2\beta(u, r)})^2 Q^2(u, r) & 0 & 0 & 0 \\
0 & 0 & \frac{1}{8} Q^2(u, r) & 0 \\
0 & 0 & 0 & \frac{1}{8} \sin(\theta)^2 Q^2(u, r) \\
\end{bmatrix}.
\] (16)

2.7 Stress tensor

The stress tensor total that includes matter + radiation + electric charge is:

\[
> \text{grdef}('T\{a\ b\} := T\{a\ b\} + \text{Tem}\{a\ b\}');
\]

If we want to study neutral or uncharged cases, we can turn OFF the contribution of the electric charge

\[
> \text{#grmap}(\text{T}(\text{dn}, \text{dn}), \text{subs}, Q(u, r) = 0, 'x');
\]

Calculating the covariant components

\[
> \text{grcalc}(\text{T}(\text{dn}, \text{dn}));
\]
\[
> \text{gralter}(\text{T}(\text{dn}, \text{dn}), \text{expand}, \text{factor});
\]
\[
> \text{grdisplay}(\text{T}(\text{dn}, \text{dn}));
\]

For the bondi spacetime:

\[ T(\text{dn}, \text{dn}) \]
\[ T(\text{dn}, \text{dn}) \]
As an example of the use of simplification routines, we can factor the time covariant component

\[
T_{uu} = \frac{1}{8} \left( e^{\beta(u,r)} \right)^2 \left( 8r^4\pi\omega^2 + 8r^4\pi\omega^2\rho_1 + 8r^4\pi\omega P_1 - 16r^4\pi\varepsilon\omega - 8r^4\pi\omega\rho_1 + 8r^4\pi\varepsilon + 8r^4\pi\rho_1 + (Q(u,r))^2 - 2(Q(u,r))^2 \omega + \omega^2(Q(u,r))^2 \right)
\]

\[
T_{ru} = \frac{1}{8} \left( e^{\beta(u,r)} \right)^2 \left( 8r^4\pi\omega + 8r^4\pi\omega + Q(u,r) \right)
\]

\[
T_{ur} = \frac{1}{8} \left( e^{\beta(u,r)} \right)^2 \left( 8r^4\pi\omega + Q(u,r) \right)
\]

\[
T_{rr} = \frac{r}{8} \left( e^{\beta(u,r)} \right)^2 (P_1 + \rho_1)
\]

\[
T_{\theta\theta} = \frac{1}{8} \left( 8r^4\pi\omega + Q(u,r) \right)^2
\]

\[
T_{\phi\phi} = \frac{1}{8} \sin^2(8r^4\pi\omega + Q(u,r)^2)
\]

(17)

As an example of the use of simplification routines, we can factor the time covariant component

\[
> \text{tempTdndnuu} := \text{kfactor(hcollect(grcomponent(T(dn,dn), [u,u])), \{P[1], rho[1], V(u,r)\}, \{r, omega, beta(u,r)\}), V(u,r)*exp(2*beta(u,r))/r);}
\]

\[
\text{tempTdndnuu} := \frac{V(u,r) e^{2\beta(u,r)}}{r} \left( \frac{\omega P_1}{(-1+\omega)^2} + \frac{(-\omega+\omega^2+I)P_1}{(-1+\omega)^2} + \frac{1}{8} \left( 8r^4\pi\omega + Q(u,r)^2 \right) \right)
\]

(18)

The contravariant radial component can be simplified with the help of the previously loaded factoring functions

\[
> \text{tempTupuprr} := \text{kfactor(hcollect(hcollect(grcomponent(T(up,up), [r,r])), \{epsilon, Q(u,r), V(u,r), exp(2*beta(u,r))\}), \{epsilon, P[1], rho[1], Q(u,r)\}), \{epsilon, P[1], rho[1], Q(u,r)\}, \{V(u,r), beta(u,r)\}), exp(-2*beta(u,r))*V(u,r)/r)
\]

\[
\text{tempTupuprr} = \frac{V(u,r) e^{-2\beta(u,r)}}{r} \left( \varepsilon + \frac{(-\omega+\omega^2+I)P_1}{(-1+\omega)^2} - \frac{I}{8} \frac{Q(u,r)^2}{r^4\pi} + \frac{\omega\rho_1}{(-1+\omega)^2} \right)
\]

(19)

that we can simplify as

\[
\text{tempTupuprr} = \frac{V(u,r) e^{-2\beta(u,r)}}{r} \left[ \frac{\omega(\rho_1 + P_I)}{(1-\omega)^2} + \varepsilon + \left( \frac{P_I - Q(u,r)^2}{8\pi r^4} \right) \right]
\]

2.8 Temporary and radial dependency of the Stress Tensor

To calculate the conservation equations, it is necessary to establish that the density \( \rho = \rho(u,r) \), pressure \( P = P(u,r) \), radiation \( \varepsilon = \varepsilon(u,r) \) and tangential pressure \( P_t = P_t(u,r) \) depend on that of the temporal and radial coordinates.
> grmap(T(up, dn), subs, rho[1] = rho(u, r), 'x');
> grmap(T(up, dn), subs, P[1] = P(u, r), 'x');
> grmap(T(up, dn), subs, P[t] = P[t](u, r), 'x');
> grmap(T(up, dn), subs, epsilon = epsilon(u, r), 'x');
> grmap(T(up, dn), subs, omega = omega(u, r), 'x');
> gralter(T(up, dn), simplify);
> grdisplay(T(up, dn));

For the Bondi spacetime:

\[
T(\text{up}, \text{dn})
\]

\[
T(\text{up}, \text{dn})
\]

\[
T^a_{\ b} = \left[\begin{array}{cccc}
\frac{1}{8} \left( \frac{8r^4 \pi \rho(u,r) + Q(u,r)^2}{r^4 \pi} \right), & r \left( \frac{P(u,r) + \rho(u,r)}{V(u,r)} \right), & 0, & 0 \\
\frac{V(u,r)}{r} \left( \frac{\epsilon(u,r) \omega(u,r)^2 + \omega(u,r) P(u,r) - 2 \epsilon(u,r) \omega(u,r) + \omega(u,r) \rho(u,r) + \epsilon(u,r)}{(-1 + \omega(u,r))^2} \right), & -Q(u,r)^2 + \frac{8r^4 \pi P(u,r)}{r^4 \pi}, & 0, & 0 \\
0, & 0, & -\frac{1}{8} \left( \frac{8r^4 \pi P_t(u,r) + Q(u,r)^2}{r^4 \pi} \right), & 0 \\
0, & 0, & 0, & -\frac{1}{8} \left( \frac{8r^4 \pi P_t(u,r) + Q(u,r)^2}{r^4 \pi} \right)
\end{array}\right]
\]

\[(20)\]

### 2.9 Conservation equations (TOV)

These equations are important since the Einstein field equations are composed of two structures: One is the geometry of the $G_{ab}$ system and the other is basically the energy $T_{ab}$. From the expressions that we have taken the geometry is linear in the second derivative and non-linear in the first. However, the impulse energy tensor that we have used is linear. Presumably, this non-linearity, of the geometrical part must have a non-linear version in some expression of the impulse energy tensor. It is therefore necessary to calculate the components of the conservation equation since it is possible that they contain the non-linear expression necessary to correct our choice of the structure of the Stress Tensor.

> grdef('cero{a }:=[0, 0, 0 ]');
> grdef('TOV{b } := T^-{a b;a}= cero{b }');

Calculating the covariant components of the conservation equation:
For the bondi spacetime:

\[ \text{TOV}(dn) \]

\[ \text{TOV}(dn) \]

\[ \text{TOV}_u = \left( \frac{1}{4} r^4 V(u, r) \right) \left( \frac{1}{1 + \omega(u, r)} \right)^3 \left( 8 \left( \omega(u, r)^2 \varepsilon(u, r) + (P(u, r) + \rho(u, r) - 2 \varepsilon(u, r)) \omega(u, r) + \varepsilon(u, r) \right) \right) \cdot V(u, r)^2 r^3 \pi (1 - \omega(u, r)) + \cdots = 0 \] (21)

\[ \text{TOV}_r = \left( \frac{1}{4} r^4 V(u, r) \right) \left( 4 r^5 \pi V(u, r) \left( \frac{\partial}{\partial u} \rho(u, r) \right) + 4 r^5 \pi V(u, r) \left( \frac{\partial}{\partial u} \rho(u, r) \right) - \cdots \cdot \cdot \cdot \right) + 4 r^5 \pi P(u, r) r^4 \pi \left( \frac{\partial}{\partial u} V(u, r) \right) - 4 r^5 \pi \rho(u, r) + \cdots = 0 \] (22)

We can use the simplification routines loaded at the beginning of the sheet to compare the terms observed in the conservation equations. The following simplifications allow us to establish that some of the identities are not really independent:

\[ \text{TOV}_n[0] := \text{hcollect}(\text{simplify}(\exp(2 \beta(u, r)) \cdot \text{lhs} (\text{grcomponent} (\text{TOV}(up), [u])))), \]

\[ \{ P(u, r), \rho(u, r), \text{diff}(P(u, r), r), \text{diff}(Q(u, r), r), P[t](u, r), \} \}

\[ \{ \text{diff}(V(u, r), r), \text{diff}(V(u, r), u), \text{diff}(\beta(u, r), r), \text{diff}(\beta(u, r), u) \} \}; \]

\[ \text{TOV}_n[0] = \left( -\frac{1}{2} \frac{\partial}{\partial u} V(u, r) - \frac{r}{2} \frac{\partial}{\partial u} V(u, r)^2 - \left( \frac{\partial}{\partial r} \beta(u, r) \right) + \frac{2r}{2r} \frac{\partial}{\partial u} \beta(u, r) - \frac{3}{2r} \right) P(u, r) + \]

\[ -\frac{\partial}{\partial r} P(u, r) + \frac{1}{4} Q(u, r) \frac{\partial}{\partial u} Q(u, r) + \left( \frac{1}{2} \frac{\partial}{\partial u} V(u, r) - \frac{r}{2} \frac{\partial}{\partial u} V(u, r)^2 \right) + \]

\[ -\left( \frac{\partial}{\partial u} \beta(u, r) \right) + \frac{2r}{2r} \frac{\partial}{\partial u} \beta(u, r) + \frac{1}{2r} \right) \rho(u, r) + \frac{2P_t(u, r)}{r} + \frac{r}{r} \frac{\partial}{\partial u} P(u, r) + \frac{\partial}{\partial u} \rho(u, r) \] (23)

\[ \text{TOV}_n[1] := \text{hcollect}(\text{hcollect}(\text{simplify}(r \exp(2 \beta(u, r)) \cdot \text{lhs} (\text{grcomponent} (\text{TOV}(up), \ [r]))/V(u, r)), \{V(u, r), \beta(u, r), \text{diff}(V(u, r), r), \text{diff}(V(u, r), u), \text{diff}(} \]
\begin{equation}
\text{TOV}^n_1 := \left( (\omega(u,r))^2 + 1 \right) \frac{\partial \beta(u,r)}{\partial r} + \\
+ \frac{1 - 2r \omega(u,r) + 3 (\omega(u,r))^3}{2} - 3 + 7 \omega(u,r) - 7 (\omega(u,r))^2 - \cdots \\
+ \frac{1 - (\omega(u,r))^2 + 2r (\omega(u,r))^2 - \cdots}{V(u,r)(-1 + \omega(u,r))^2} P(u,r) + \\
- \frac{\omega(u,r) + 2r \omega(u,r)}{V(u,r)(-1 + \omega(u,r))^2} + \frac{1}{2} \frac{\omega(u,r)}{(V(u,r))^2} P(u,r) + \\
+ \left( \frac{1}{4} \frac{\partial Q(u,r)}{\pi r^4} + \frac{1}{V(u,r) \pi r^3} \right) Q(u,r) \cdots
\end{equation}

>\text{TOVn}[2] := \text{hcollect(simplify(r*lhs(grcomponent(TOV(dn), [u]))/V(u, r)), \{P(u, r), rho(u, r), diff(P(u, r), r), diff(Q(u, r), u), diff(V(u, r), r), diff(V(u, r), u), diff(rho(u, r), r), diff(rho(u, r), u), diff(beta(u, r), r), diff(beta(u, r), u), diff(epsilon(u, r), r), diff(omega(u, r), r), epsilon(u, r), omega(u, r), omega(u, r))\});
By comparing term by term, a certain regularity can be observed, and we can verify that the two equations are the same:

\[ \text{TOVn}_2 = \left( \frac{\partial}{\partial r} V(u, r) \right) \omega(u, r) \frac{\omega(u, r)}{(V(u, r)(-1 + \omega(u, r))^3) - \frac{1}{2} \frac{\partial}{\partial r} V(u, r) \frac{\omega(u, r)}{(V(u, r)(-1 + \omega(u, r))^3) + 2 \frac{\partial}{\partial r} \beta(u, r) \omega(u, r) \frac{\omega(u, r)}{(V(u, r)(-1 + \omega(u, r))^3) + \frac{1}{2} \frac{\partial}{\partial r} \beta(u, r) V(u, r)} \right) \]

\[ \text{TOVn}_3 = \left( -\frac{1}{2} \frac{\partial}{\partial r} V(u, r) \right) \omega(u, r) \frac{\omega(u, r)}{(V(u, r)(-1 + \omega(u, r))^3) - \frac{1}{2} \frac{\partial}{\partial r} V(u, r) \frac{\omega(u, r)}{(V(u, r)(-1 + \omega(u, r))^3) + 2 \frac{\partial}{\partial r} \beta(u, r) \omega(u, r) \frac{\omega(u, r)}{(V(u, r)(-1 + \omega(u, r))^3) + \frac{1}{2} \frac{\partial}{\partial r} \beta(u, r) V(u, r)} \right) \]

By comparing term by term, a certain regularity can be observed, and we can verify that the two equations are the same:

\[ \text{TOVn}[3] := \text{hcollect(simplify(lhs(grcomponent(TOV(dn), [r]))), \{P(u, r), rho(u, r), diff(P(u, r), r), diff(Q(u, r), r), P[t](u, r), \{diff(V(u, r), r), diff(V(u, r), u), diff(beta(u, r), r), diff(beta(u, r), u))\})}; \]

\[ \text{TOVn}_3 = \left( -\frac{1}{2} \frac{\partial}{\partial r} V(u, r) \right) \omega(u, r) \frac{\omega(u, r)}{(V(u, r)(-1 + \omega(u, r))^3) - \frac{1}{2} \frac{\partial}{\partial r} V(u, r) \frac{\omega(u, r)}{(V(u, r)(-1 + \omega(u, r))^3) + 2 \frac{\partial}{\partial r} \beta(u, r) \omega(u, r) \frac{\omega(u, r)}{(V(u, r)(-1 + \omega(u, r))^3) + \frac{1}{2} \frac{\partial}{\partial r} \beta(u, r) V(u, r)} \right) \]

\[ \text{TOVn}_3 = \left( -\frac{1}{2} \frac{\partial}{\partial r} V(u, r) \right) \omega(u, r) \frac{\omega(u, r)}{(V(u, r)(-1 + \omega(u, r))^3) - \frac{1}{2} \frac{\partial}{\partial r} V(u, r) \frac{\omega(u, r)}{(V(u, r)(-1 + \omega(u, r))^3) + 2 \frac{\partial}{\partial r} \beta(u, r) \omega(u, r) \frac{\omega(u, r)}{(V(u, r)(-1 + \omega(u, r))^3) + \frac{1}{2} \frac{\partial}{\partial r} \beta(u, r) V(u, r)} \right) \]

\[ \frac{1}{2} \frac{\partial}{\partial r} P(u, r) \frac{Q(u, r)}{V(u, r)} + \frac{1}{2} \frac{\partial}{\partial r} \pi \frac{Q(u, r)}{V(u, r)} + \frac{1}{2} \frac{\partial}{\partial r} \pi \frac{Q(u, r)}{V(u, r)} + \frac{1}{2} \frac{\partial}{\partial r} \pi \frac{Q(u, r)}{V(u, r)} \]

\[ \text{temp03} := \text{collect(simplify(TOVn[0]-TOVn[3]), r)}; \]

\[ \text{temp03} := 0 \]

With the previous expression we can conclude that

\[ e^{2\beta} T_{\alpha\beta} = T_{1,\alpha} = 0 \]

and the same conservation equation is obtained. We have in this way 3 independent conservation equations.
> temp21 := hcollect(hcollect(hcollect(simplify(TOVn[2]-TOVn[1]), \{V(u, r), beta(u, r)\},
{diff(P(u, r), u), diff(V(u, r), r)})}, \{P(u, r), rho(u, r), diff(P(u, r), r), diff(Q(u, r), u)\},
{diff(rho(u, r), u), diff(beta(u, r), r), diff(beta(u, r), u)})}, \{P(u, r), rho(u, r),
P[t](u, r), \{Q(u, r), diff(Q(u, r), r)\}));

$$temp21 := \left( -\frac{1}{2} \frac{\partial V(u, r)}{V(u, r)} - \frac{r \partial V(u, r)}{(V(u, r))^2} - \frac{\partial \beta(u, r)}{\partial r} + 2 \frac{r \beta(u, r)}{V(u, r)} - \frac{3}{2r} \right) P(u, r) + \left( -\frac{1}{2} \frac{\partial V(u, r)}{V(u, r)} \\
- \frac{r \partial V(u, r)}{(V(u, r))^2} - \frac{\partial \beta(u, r)}{\partial r} + 2 \frac{r \beta(u, r)}{V(u, r)} + \frac{1}{2r} \right) \rho(u, r) - \frac{\partial P(u, r)}{\partial r} + \frac{1}{4} \frac{Q(u, r) \frac{\partial Q(u, r)}{\partial r}}{\pi r^4} + \frac{2P_t(u, r)}{r} + \frac{r \left( \frac{\partial P(u, r)}{\partial u} + \frac{\partial \rho(u, r)}{\partial u} \right)}{V(u, r)} \right) \nonumber \quad (28)$$

> temp21m0 := collect(simplify(temp21-TOVn[0]), r);

$$temp21m0 := 0 \quad (29)$$

The difference of the two equations brings us back to the conservation equation of (TOV: Tolman Oppenheimer Volkoff). However, taken separately they are not proporcional to $TOV[0] = TOV[3]$.

### 2.10 Einstein field equations

Taking as input the covariant components of the Einstein field equations:

> grdef('Eins\{a\ b\} := G\{a\ b\} = 8*Pi*T\{a\ b\}');
> grcalc(Eins(dn, dn));
> gralter(Eins(dn, dn), simplify, factor, radsimp);
> grdisplay(Eins(dn, dn));

For the bondi spacetime:

\[
Eins(dn, dn) \\
Eins(dn, dn)
\]
\[
Eins_{uu} = \left( r \frac{\partial}{\partial u} V(u, r) + 2 \left( V(u, r) \right)^2 \frac{\partial}{\partial r} \beta(u, r) - 2 r V(u, r) \frac{\partial}{\partial u} \beta(u, r) - V(u, r) \frac{\partial}{\partial r} V(u, r) + V(u, r) e^{2 \beta(u, r)} \right) r^3 =
\]

\[
= \frac{1}{(-1 + \omega(u, r))^2} \left( V(u, r) e^{2 \beta(u, r)} \left( 8 r^4 \pi e(u, r) (\omega(u, r))^2 + 8 r^4 \pi (\omega(u, r))^2 \rho(u, r) + + 8 r^4 \pi \omega(u, r) P(u, r) - 16 r^4 \pi e(u, r) \omega(u, r) - 8 r^4 \pi (\omega(u, r))^2 \rho(u, r) + + 8 r^4 \pi \rho(u, r) + (Q(u, r))^2 - 2 (Q(u, r))^2 \omega(u, r) + (Q(u, r))^2 (\omega(u, r))^2 \right) \right)
\]

\[
Eins_{ru} = \left( 2 V(u, r) \frac{\partial}{\partial r} \beta(u, r) - \frac{\partial}{\partial r} V(u, r) + e^{2 \beta(u, r)} \right) \frac{e^{2 \beta(u, r)} (8 r^4 \pi \rho(u, r) + (Q(u, r))^2)}{r^3}
\]

\[
Eins_{ur} = \left( 2 V(u, r) \frac{\partial}{\partial u} \beta(u, r) - \frac{\partial}{\partial u} V(u, r) + e^{2 \beta(u, r)} \right) \frac{e^{2 \beta(u, r)} (8 r^4 \pi \rho(u, r) + (Q(u, r))^2)}{r^4}
\]

\[
Eins_{rr} = \left( 4 \frac{\partial}{\partial r} \beta(u, r) \right) = 8 \pi r e^{2 \beta(u, r)} (P(u, r) + \rho(u, r))
\]

\[
Eins_{\theta\theta} = \left( 1 \right) \left( -4 \left( \frac{\partial}{\partial u} \beta(u, r) \right) r^2 + \left( \frac{\partial^2}{\partial u^2} V(u, r) \right) r + 2 \left( \frac{\partial}{\partial r} V(u, r) \right) \left( \frac{\partial}{\partial r} \beta(u, r) \right) \right) + + 2 V(u, r) \left( \frac{\partial^2}{\partial r^2} \beta(u, r) \right) r - 2 V(u, r) \frac{\partial}{\partial r} \beta(u, r) \right) e^{-2 \beta(u, r)} = \frac{8 r^4 P_l(u, r) \pi + (Q(u, r))^2}{r^2}
\]

\[
Eins_{\phi\phi} = \left( \frac{\sin(\theta)}{2} \right) \left( -4 \left( \frac{\partial}{\partial u} \beta(u, r) \right) r^2 + \left( \frac{\partial^2}{\partial u^2} V(u, r) \right) r + 2 \left( \frac{\partial}{\partial r} V(u, r) \right) \left( \frac{\partial}{\partial r} \beta(u, r) \right) \right) + + 2 V(u, r) \left( \frac{\partial^2}{\partial r^2} \beta(u, r) \right) r - 2 V(u, r) \frac{\partial}{\partial r} \beta(u, r) \right) e^{-2 \beta(u, r)} = \frac{\sin(\theta) \left( 8 r^4 P_l(u, r) \pi + (Q(u, r))^2 \right)}{r^2}
\]

Using the simplification routines for the uu component of the Einstein tensor: \(G_{00} = 8\pi T_{00}\)

\[
> \text{lhsEuu} := \text{hcollect}((\text{lhs}(\text{grcomponent}(\text{Eins}\{\text{dn}, \text{dn}\}, \{\text{u}, \text{u}\}))), \{\text{r}, \text{diff}(\text{V}(\text{u}, \text{r}), \text{u}), \text{diff}(\text{beta}(\text{u}, \text{r}), \text{u})\}, \{\text{V}(\text{u}, \text{r}), \text{diff}(\text{V}(\text{u}, \text{r}), \text{r}), \text{diff}(\text{beta}(\text{u}, \text{r}), \text{r})\}))
\]

\[
\text{lhsEuu} = -2 V(u, r) \frac{\partial}{\partial u} \beta(u, r) + \frac{\partial}{\partial u} V(u, r) + + V(u, r) e^{2 \beta(u, r)} + 2 \left( V(u, r) \right)^2 \frac{\partial}{\partial r} \beta(u, r) - V(u, r) \frac{\partial}{\partial r} V(u, r)
\]

\[
(31)
\]

And for the right side of the same component of the Einstein field equation:

\[
> \text{rhsEuu} := \text{kfactor}((\text{hcollect}((\text{hcollect}(\text{rhs}(\text{grcomponent}(\text{Eins}\{\text{dn}, \text{dn}\}, \{\text{u}, \text{u}\})))), \{\text{r}, \text{diff}(\text{V}(\text{u}, \text{r}), \text{u}), \text{diff}(\text{beta}(\text{u}, \text{r}), \text{u})\}, \{\text{V}(\text{u}, \text{r}), \text{diff}(\text{V}(\text{u}, \text{r}), \text{r}), \text{diff}(\text{beta}(\text{u}, \text{r}), \text{r})\}))
\]
\{Q(u, r), V(u, r), \exp(2\beta(u, r)), \epsilon(u, r)\}, \{P(u, r), Q(u, r), \rho(u, r), \\
\epsilon(u, r)\}, \{P(u, r), Q(u, r), \rho(u, r), \epsilon(u, r)\}, \{V(u, r), \beta(u, r)\}, \\\n8\pi \exp(2\beta(u, r)) V(u, r)/r;}

\[
\begin{align*}
\text{rhsEuu} &:= 8\pi e^{2\beta(u, r)} V(u, r) \left( \frac{-\omega(u, r) P(u, r)}{(-1 + \omega(u, r))^2} + \frac{1}{8} \frac{(Q(u, r))^2}{r^2} + \frac{((Q(u, r))^2 - \omega(u, r) + 1) \rho(u, r)}{(-1 + \omega(u, r))^2} + \epsilon(u, r) \right) \quad (32)
\end{align*}
\]

Let’s see now its Mixed components:

> grcalc(Eins(up, dn));
> gralter(Eins(up, dn), expand, factor);
> grdisplay(Eins(up, dn));

For the bondi spacetime:

\[
\begin{align*}
\text{Eins}(up, dn) \\
\text{Eins}(up, dn)
\end{align*}
\]
An interesting expression is the one associated with the field equation (34) and is very similar to the one obtained previously (33), which includes the pressure and is very similar to the one obtained previously (34), which relates density to the functions of the metric.
\[
\text{rhsErr := hcollect(rhs(grcomponent(Eins(up, dn), [r, r])), \{P(u, r), \{V(u, r)\})};
\]

\[
rhsErr := -8 \pi P(u,r) + \frac{(Q(u,r))^2}{r^4}
\]

(35)

Both relate the metrical elements \(- V(u, r) \text{ and } \beta(u, r)\) -- with the physical quantities \(- P(u, r) \text{ and } \rho(u, r)\)-- of pressure and density.

### 2.11 Trace

Let’s calculate the trace of \(G\)

\[
> \text{grdef(‘TG := g\{\text{a} \text{\& b}\}\{\text{a} \text{\& b}\} ‘);} \\
> \text{grcalcd(TG);}
\]

\text{\textit{’CPU Time’ = 0.005}}

For the bondi spacetime:

\[
TG
\]

\[
TG := \frac{1}{e^2 \beta(u,r) r^2} \left( 2 V(u,r) \frac{\partial}{\partial r} \beta(u,r) - 2 \frac{\partial}{\partial r} V(u,r) + 2 e^2 \beta(u,r) + 4 \left( \frac{\partial^2}{\partial u \partial r} \beta(u,r) \right) r^2 \right.
\]

\[
- \left( \frac{\partial^2}{\partial r^2} V(u,r) \right) r - 2 \left( \frac{\partial}{\partial r} V(u,r) \right) \left( \frac{\partial}{\partial r} \beta(u,r) \right) r - 2 V(u,r) \left( \frac{\partial^2}{\partial r^2} \beta(u,r) \right) r \right)
\]

(36)

Very similar to the equation the left side of the angular component, of the field equations \(G_2^2 = 8\pi T_2^2\). Now the trace of the Stress tensor:

\[
> \text{grdef(‘TT := g\{\text{a} \text{\& b}\}\{\text{a} \text{\& b}\} ‘);} \\
> \text{grcalcd(TT);}
\]

\text{\textit{’CPU Time’ = 0.004}}

For the bondi spacetime:

\[
TT
\]

\[
TT = \rho(u,r) - P(u,r) - 2P_t(u,r)
\]

(37)

The contraction of the Field equation \(Traz = g^\mu\nu G_{\mu\nu} = 8\pi g^\mu\nu T_{\mu\nu}\)
This equation of the trace is important since it can be shown that it is equivalent to the equation \((T^\mu_\nu = 0)\), the Tolman-Oppenheimer-Volkoff equation. When comparing the left side of the trace equation with the left side of the angular component \(G^2_2 = 8\pi T^2_2\), of the equations from field, we get

\[
\begin{align*}
\text{difG}_{22}^2 T G &:= e^{-2 \beta(u,r)} \frac{e^{2 \beta(u,r)} - \frac{\partial}{\partial r} V(u,r)}{r^2} \\
\text{difG}_{22}^2 T G &:= -8\pi (-\rho(u,r) + P(u,r) + 2 P_t(u,r))
\end{align*}
\]
\[ T_{0;\lambda}^\lambda = \frac{V}{r^2} \left( 1 + \frac{rV_1}{V} \right) \left[ \varepsilon + \frac{\omega (\rho + P)}{(1 - \omega)^2} \right] + \]
\[ \frac{V}{r} \frac{\partial}{\partial r} \left[ \varepsilon + \frac{\omega (\rho + P)}{(1 - \omega)^2} \right] + \frac{\partial}{\partial u} \left( \rho + \frac{Q^2}{8\pi r^4} \right) = 0 \] (40)

\[ T_{1;\lambda}^\lambda = e^{-2\beta}T_{1;\lambda} = \frac{2}{r^2} \left( \frac{V}{r} \frac{\partial}{\partial r} \left( P - \frac{Q^2}{8\pi r^4} \right) - \left( P_t + \frac{Q^2}{8\pi r^4} \right) \right) = 0 \] (41)

\[ e^{2\beta}T_{1;\lambda}^\lambda = \frac{V}{r^2} \left( 1 + 4r^2 \beta + \frac{rV_1}{V} \right) \left[ \varepsilon + \frac{\omega (\rho + P)}{(1 - \omega)^2} \right] + \frac{V}{2r} \left( 2\beta_1 + \frac{V_1}{V} - \frac{1}{r} \right) (\rho + P) + \]
\[ \frac{2V}{r^2} \left( P - \frac{Q^2}{8\pi r^4} \right) - \left( P_t + \frac{Q^2}{8\pi r^4} \right) \right] + \]
\[ \frac{V}{r} \frac{\partial}{\partial r} \left( P - \frac{Q^2}{8\pi r^4} \right) + \frac{V}{r} \frac{\partial}{\partial r} \left[ \varepsilon + \frac{\omega (\rho + P)}{(1 - \omega)^2} \right] - \frac{\partial}{\partial u} \left( P - \frac{Q^2}{8\pi r^4} \right) = 0 \] (43)

As can be seen in (2.9) and the concordance of these results with those obtained manually, it can allow us to trust that this calculation to be carried out in another coordinate system should have a similar performance. Of course, it is necessary to point out that it is still pending to use the complete equation (31) shown in the article (68). This comparative study between the results obtained so far and solving the complete equation with the terms \( \frac{\partial \varepsilon}{\partial r} \) and \( \frac{\partial \omega}{\partial r} \) will be carried out in a future work to publish.

4 Conclusions

In this article we describe a series of computer procedures used in GR relying on the facilities of an integrated platform such as the Maple package and GRTensorIII. These procedures of computational algebra, numerical and graphic computation can facilitate algebraic calculation both for research purposes and for teaching GR at different levels.

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