A categorical construction of 2-dimensional extended Topological Quantum Field Theory

Vishvajit V S Gautam*
The Institute of Mathematical Sciences
CIT Campus Taramani, Chennai - 600113 INDIA

Abstract

In this paper we propose a naive construction of 2-dimensional extended topological quantum field theories (TQFTs), which can be further generalized to the higher-dimension extended TQFTs.

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1 Introduction

The notion of TQFT is related with the study of path integral for Lagrangian with topological invariance. In some sense a TQFT gives a topological invariant to n-dimensional manifolds. For example 3-dimensional TQFT assigns a 3-manifold $M$ to a numerical invariant $\tau(M)$ (which may be a complex number) such that if $\tau(M) \neq \tau(M')$ for 3-manifolds $M$ and $M'$ then $M$ and $M'$ are not diffeomorphic. A 3-dimensional TQFT also gives numerical invariants of knots, links and ribbon graphs. This theory was introduced by Witten in 1988 to describe a class of quantum field theory whose action is diffeomorphism invariant such as $\tau(M)$ which not only assigns a numerical value to manifold but also preserves the embedded ribbon graphs or any other structure defined over the manifold. Atiyah in 1988 formulated an axiomatic setup for TQFTs. Independently and at about the same time G. Segal formulates a mathematical definition of conformal field theories (CFTs), which is very similarly based on categories and functors.

*E-mail: vishvajit@imsc.res.in; gautamvvs@yahoo.com
Categories play a central role in the mathematical formulation of TQFT. An $n$-dimensional TQFT is defined as a monoidal functor from the category of oriented $n$-cobordisms with disjoint union as tensor product to the category $\mathbf{Vect}$ of finite dimensional vector spaces with usual tensor product of vector spaces. Any modification in cobordism category may lead to a modification in TQFT. This modification can be think as an extended version of TQFT. For example in Chern-Simons Witten TQFT cobordisms are supplied with some extra structures [4].

Similar to the case of TQFTs there are many ways to define extended TQFTs. Kerler and Lyubashenko in [8] introduce a notion of extended TQFTs which involves higher category theory, namely double categories and double functors. Involvement of higher-dimensional algebra indicate that extended TQFTs will preserve extra information. Due to double functority, their extended TQFTs contains both Atiyah notion of a TQFT in dimension three and Segal’s notion of CFT as special cases, though they appear on different categorical levels.

The role of higher-dimensional algebra is clear from the various constructions of extended TQFTs. Baez and Dolan in [2] outline a program in which $n$-dimensional TQFTs are described as $n$-category representation. They described an $n$-dimensional extended TQFT as a weak $n$-functor from the free stable weak $n$-category with duals of one objects to $\mathbf{nHilb}$ the category of $n$-Hilbert spaces, which preserve all levels of duality.

This paper gives a naive categorical construction of a 2-dimensional extended TQFT, which is different from the known constructions of $n$-dimensional extended TQFT for $n = 2$. We use the notion of internal categories $2$-Vector spaces which is in agreement with the 2-Vector spaces defined by Baez and Crans in [1].

2 2- Categories and semistrict monoidal 2-categories

2-categories are the first prototype of higher-dimensional algebra. Importance of symmetric monoidal 2-categories, braided monoidal 2-categories is evident from the recent developments in higher-dimensional algebra, see Baez and Neuchl [3], Day and Street [5], and Kapranov and Voevodsky [7], and reference therein. Semistrict monoidal 2-categories are the base categories for our description of an extended TQFTs. Instead of using the weak version of monoidal 2-categories we will use the semistrict version of monoidal 2-categories, because they are now better understood, and a coherence theorem (cf. [6]) for weak categories says they are equivalent to semistrict ones.
2.1 2-Category

A 2-category $\mathcal{C}$ consists of following data:

- a class $\mathcal{C}_0$ of objects $A$, $B$, $C$, ... which are called $0$-cells;
  
  The objects or 0-cells and arrows or 1-cells form a category, called the underlying category of $\mathcal{C}$ which we also denote by $\mathcal{C}$, with identities $1_A : A \to A$.

- for each pair $A,B$ in $\mathcal{C}_0$, a small category $\mathcal{C}_{1(A,B)}$ whose objects or 1-cells are morphisms $f : A \to B$ etc., and arrows or 2-cells are morphisms of morphisms from $A$ to $B$, which we denote by $\alpha$, $\beta$, $\gamma$...;
  
  A 2-cell pictured as

  $$
  \begin{array}{c}
  \alpha \\
  \downarrow \\
  A \\
  \downarrow \\
  B \\
  \end{array}
  $$

  For any pair 2-cells $\alpha$, $\beta$ in $\mathcal{C}_{1(A,B)}$ the 2-cells composition under which $\mathcal{C}_{1(A,B)}$ form a category is called \textit{vertical composition}.
  
  A 2-cells vertical composition as displayed in:

  $$
  \begin{array}{c}
  \alpha \\
  \downarrow \\
  \beta \\
  \downarrow \\
  \alpha \\
  \downarrow \\
  \alpha \\
  \end{array}
  $$

  The vertical composite $f \Rightarrow h$ is denoted by $\beta \circ \alpha$. Identities of $\mathcal{C}_{1(A,B)}$ are denoted by
Between any pair of 0-cells $A$ and $B$ there are 2-cells $\alpha$ etc. We can compose them under another 2-cells operation known as *horizontal composition*. In this case we have

$$f \circ \alpha \cdot h \cdot f = (\gamma \circ \alpha) \cdot k \cdot g$$

Under this composite $\gamma \circ \alpha : h \cdot f \Rightarrow k \cdot g$ law the 2-cells form a category, with identities

$$1_A$$

Finally the horizontal and the vertical compositions are related with following conditions:
\[(\delta \circ \beta) \circ (\gamma \circ \alpha) = (\delta \circ \gamma) \circ (\beta \circ \alpha)\]

In the situation

we require that horizontal composite of two vertical identities is itself a vertical identity i.e. \(1_f \circ 1_f = 1_f \circ f\).

This structure also provides a horizontal composite of a 2-cell with 1-cell

gives a vertical composite \(h \circ \alpha\). This is same as
A 2-functor $F : \mathcal{C} \to \mathcal{D}$ between 2-categories $\mathcal{C}$ and $\mathcal{D}$ is a triple of functions sending 0-cells, 1-cells, and 2-cells of $\mathcal{C}$ to items of the same types in $\mathcal{D}$ so as to preserve all the categorical structures (domain, codomain, identities, and composites).

A natural example of 2-categories is the category of $\mathbf{2}$-$\mathbf{Top}$ whose objects or 0-cells are topological spaces, 1-cells are continuous maps between spaces, 2-cells are homotopy classes of homotopies between continuous maps.

Another example is the category $\mathbf{Grp}$ whose objects are groups, 1-cells are homomorphism between two groups, and 2-cells $\alpha : f \Rightarrow g$ are the automorphisms of the codomain of $g$.

### 3 semistrict monoidal 2-category

(cf. [6], [7], [3]) A semistrict monoidal 2-category consists of:

1. A 2-category $\mathcal{C}$
2. For any two objects or 0-cells $A$ and $B$ in $\mathcal{C}$, an object $A \otimes B$ in $\mathcal{C}$.
3. The unit object $I \in \mathcal{C}$.
4. 1-cell composite functions $1 \otimes 0, 0 \otimes 1$ and 2-cell composite functions $2 \otimes 0, 0 \otimes 2$ and $\otimes_{1,1'} : 0 \otimes 1' \cong 0 \otimes 1'$ whenever $(0 \otimes 1')(1 \otimes 0) = (1 \otimes 0)(0 \otimes 1')$,

where 0, 1 and 2 stand for the 0-cells, 1-cells and 2-cells respectively.

these composite functions act as follows:
4a. For any 1-cell $f : A \to B$ and any 0-cell $C \in \mathcal{C}$ a 1-cell $f \otimes C : A \otimes B \to A \otimes C$.

4b. For any 1-cell $g : B \to C$ and any 0-cell $A \in \mathcal{C}$ a 1-cell $A \otimes g : A \otimes B \to A \otimes C$.

4c. For any 2-cell $\alpha : f \Rightarrow f'$ and any 0-cell $B \in \mathcal{C}$ a 2-cell $\otimes_f B : A \otimes B \Rightarrow A \otimes C$.

4d. For any 2-cell $\beta : g \Rightarrow g'$ and any 0-cell $A \in \mathcal{C}$ a 2-cell $A \otimes \beta : A \otimes g \Rightarrow A \otimes g'$.

4e. For any two 1-cells $f : A \to B$ and $g : C \to D$ a 2-isomorphism $\otimes_f g : A \otimes g \cong B \otimes g$

\[
\begin{array}{ccc}
A \otimes g & & A \otimes D \\
\downarrow f \otimes C & \otimes_f g & \downarrow \otimes_f g' \\
B \otimes C & \otimes_f g' & B \otimes D
\end{array}
\]

such that the following conditions are satisfied

(i) For any object $A \in \mathcal{C}$ we have 2-functors $A \otimes - : \mathcal{C} \to \mathcal{C}$ and $- \otimes A : \mathcal{C} \to \mathcal{C}$.

(ii) $A \otimes I = A = I \otimes A$ for any object $A$,

$f \otimes I = f = I \otimes f$ for any 1-cell $f$,

$\alpha \otimes I = \alpha = I \otimes \alpha$ for any 2-cell $\alpha$.

(iii) Let $X$ be any object, 1-cell or 2-cell in $\mathcal{C}$, for all $X$, $A$ and $B$ in $\mathcal{C}$ we have

$(A \otimes B) \otimes X = A \otimes (B \otimes X)$, $(A \otimes X) \otimes B = A \otimes (X \otimes B)$, $(X \otimes A) \otimes B = X \otimes (A \otimes B)$.

(iv) For any 1-cells $f : A \to A'$, $g : B \to B'$ and $h : C \to C'$ in $\mathcal{C}$ we have

$\otimes_f A \otimes g, h = A \otimes \otimes_f g, h$, $\otimes_f B, h = \otimes_f, B \otimes h$ and $\otimes_f g \otimes C = \otimes_f g \otimes C$.

(v) For any objects $A$ and $B$ in $\mathcal{C}$ we have $1_A \otimes B = A \otimes 1_B = 1_{A \otimes B}$.  

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(vi) For any 1-cells \( f : A \to A' \), any 1-cells \( g, g' : B \to B' \) and any 2-cell \( \beta : g \Rightarrow g' \) we have 1-cell identities
\[
(A' \otimes g)(f \otimes B) = (f \otimes B')(A \otimes g)
\]
and a 2-cell identity
\[
(A' \otimes \beta) \circ (\otimes_{f,g}) = (\otimes_{f,g'}) \circ (A \otimes \beta).
\]

(vii) For any 1-cells \( g : B \to B' \), any 1-cells \( f, f' : A \to A' \) and any 2-cell \( \alpha : f \Rightarrow f' \) we have 1-cell identities
\[
(A' \otimes g)(f' \otimes B) = (f' \otimes B')(A \otimes g)
\]
and a 2-cell identity
\[
(\alpha \otimes B) \circ (\otimes_{f,g}) = (\otimes_{f',g}) \circ (\alpha \otimes B).
\]

(viii) For any 1-cells \( f : A \to A' \), \( g : B \to B' \) and \( g' : B' \to B'' \) the 2-isomorphism \( \otimes_{f,gg'} \) equals to the pasting of \( \otimes_{f,g} \) and \( \otimes_{f,g'} \) as in the following diagram.

\[
\begin{array}{ccc}
A \otimes B & \rightarrow & A \otimes B' \\
\downarrow \otimes_{f,g} & & \downarrow \otimes_{f,g'} \\
A' \otimes B & \rightarrow & A' \otimes B'
\end{array}
\]

\[
\begin{array}{ccc}
A \otimes B & \rightarrow & A \otimes B'' \\
\downarrow \otimes_{f,g} & & \downarrow \otimes_{f,g'} \\
A' \otimes B' & \rightarrow & A' \otimes B''
\end{array}
\]

Similarly, the 2-isomorphism \( \otimes_{ff',g} \) equals to the pasting of \( \otimes_{f,g} \) and \( \otimes_{f',g} \).

4 Internal Categories

We can define monoid, group, graph, and other structures in a category \( C \). We can also define a category within \( C \) -called a category of objects in \( C \) or an internal category in \( C \). Such an internal category provides a generalize version of
the category $\mathcal{C}$. In what follows we assume category $\mathcal{C}$ is finite complete.

An internal category $\mathcal{C}$ in $\mathcal{C}$ consists of:

- an object $C_0 \in \text{obj}(\mathcal{C})$, called object of objects;
- an object of morphisms $C_1 \in \text{obj}(\mathcal{C})$, called object of arrows;

Together with four maps in $\mathcal{C}$

- source or domain morphism $s : C_1 \rightarrow C_0$ and target or codomain morphism $t : C_1 \rightarrow C_0$;
- an identity arrow $i : C_0 \rightarrow C_1$;
- a composition morphism $\circ : C_1 \times C_0 C_1 \rightarrow C_1$, here composition $\circ$ is defined on the following pullback $(C_1 \times C_0 C_1, p, q)$ of $s$ and $t$:

\[
\begin{array}{ccc}
C_1 \times C_0 C_1 & \rightarrow & C_1 \\
p & \downarrow & t \\
C_1 & \rightarrow & C_0;
\end{array}
\]

This is equal to the following two conditions:

$s \cdot \circ = s \cdot p$, and $t \cdot \circ = s \cdot q$.

These data must satisfy the following commutative conditions, which simply express the usual axiom for a category:

- $s \, i = 1_{C_0} = t \, i$
  specifies domain and codomain of the identity arrows;

- $s \cdot \circ = s \cdot p$, and $t \cdot \circ = t \cdot q$
  assigns the domain and codomain of composite morphisms;

- $\circ \circ (\circ \times C_0 1) = \circ \circ (1 \times C_0 \circ)$
  expresses that associative law for composition in terms of triple pullback;
• $p = \circ \circ (i \times 1)$, and $\circ \circ (1 \times i) = q$
  gives the left and right unit laws for composition of morphisms.

When $\mathcal{C} = \textbf{Set}$ (category of small sets and functions) pullback is the set of
composable pairs $(g, f)$ of arrows.

An internal category in $\textbf{Set}$ is just an ordinary small category which is same
as an object in $\textbf{Cat}$ (category of small categories and functors).

An internal category in $\textbf{Grp}$ (category of groups and homomorphisms) is
a category in which both $C_0$ and $C_1$ are groups, and all the maps $i, s, t$ and
$\circ$ are homomorphisms of groups. We observe that internal category in $\textbf{Grp}$ is
same as a group object in $\textbf{Cat}$.

An internal functor (or functor in $\mathcal{C}$) $F : \mathcal{C} \longrightarrow \mathcal{D}$ between two internal
categories $\mathcal{C}$ and $\mathcal{D}$ of $\mathcal{C}$ consists of :

• morphisms $F_0 : C_0 \longrightarrow D_0$, and $F_1 : C_1 \longrightarrow D_1$ of $\mathcal{C}$;
such that following holds :
  • $F_0 \cdot s = s' \cdot F_1$, and $F_0 \cdot t = t' \cdot F_1$
    preservation of domain and codomain;
  • $F_1 \cdot i = i' \cdot F_0$
    preservation of identity arrows;
  • $F_1 \circ C = \circ D \circ (F_1 \times F_1)$
    preservation of composite arrows.

Using a similar procedure, an internal natural transformation between two
internal functors $F$ and $G$ from $\mathcal{C}$ to $\mathcal{D}$ in $\mathcal{C}$, say $\alpha : F \Rightarrow G$, is a morphism
$\alpha : C_0 \longrightarrow C_1$ which satisfies the following conditions :

• $s \cdot \alpha = F_0$, and $t \cdot \alpha = G_0$
  • $\circ \circ \Delta(\alpha \cdot t \times F_1) = \circ \circ \Delta(G_1 \times \alpha \cdot s)$
    where $\Delta$ is a diagonal morphism.

4.1 2-category of internal categories

Proposition 4.1 Internal categories, internal functors, and internal natural
transformations in $\mathcal{C}$ form a strict 2-category $\mathcal{2C}$. 


We denote by $\textbf{Vect}$, the category of vector spaces over a field $k$ and linear functions.

**Definition 4.2** A 2-vector space is an internal category in $\textbf{Vect}$.

When $\mathcal{C} = \textbf{Grp}$ or $\textbf{Vect}$, we can formulate the definition of an internal categories in $\mathcal{C}$ without the use of the categorical composition $\odot$ in $\mathcal{C}$. (cf. [1]).

**Proposition 4.3** 2-vector spaces, linear functors and linear natural transformations in $\textbf{Vect}$ form a 2-category $2\textbf{Vect}$ the 2-category of 2-vector spaces.

Following the Gray monoid construction of Day and Street [5] we prove

**Proposition 4.4** $2\textbf{Vect}$ forms a semistrict monoidal 2-category.

Note that a discrete 2-vector space category is simply a vector space i.e. an object of $\textbf{Vect}$.

## 5 Extended 2-dimensional TQFTs

An $n$-dimensional TQFT is a symmetric monoidal functor $F : \textbf{Cob}_{n+1} \to \textbf{Vect}$. Here the category $\textbf{Cob}_{n+1}$ has compact oriented $n$-dimensional manifolds as objects and compact oriented cobordisms, which are equivalence classes of $(n + 1)$-manifolds with boundary, between them as morphisms, and it has a monoidal structure (tensor product) given by disjoint union. An example of physicists interest is the 2-category of relative cobordism, $\textbf{Cod}^{rel}_{n+1}$ (cf. [8]) has the underlying category $\textbf{Cob}_{n+1}$ as objects and 1-cells. The 2-cells between two cobordisms are given by $(n + 2)$-dimensional manifolds with boundary satisfying certain conditions. This category can be formulated in a semistrict version of monoidal 2-category.

In $n$-categorical set up, other examples of monoidal 2-categories are

1. category $\textbf{Chcomp}$ has chain complexes as 0-cells, chain maps as 1-cells, and chain homotopies as 2-cells.

2. category $\textbf{n-Cob}$ has 0-manifolds as 0-cells (we assume all manifolds are `compact, smooth, oriented manifolds`), 1-manifolds with corners. i.e. cobordism between 0-manifolds as 1-cells, and 2-manifolds with corners as 2-cells. (cf [9]).

Instead of taking 0-cells as 0-manifolds, one can also start with objects as 1-manifolds with or without corners to get Atiya-Segal-style TQFT.

A 2-dimensional TQFT is a particular case of the above construction. Here the category $\textbf{Cob}_{1+1}$ or $\textbf{Cob}_2$ has compact oriented 1-manifolds as objects and
compact oriented cobordism between them as morphisms.

*Extended TQFTs* constructed by Kerler and Lyubashenko in [8] involves higher category theory, namely double categories and double functors. Their construction of extended version of TQFTs is quite different from the \(n\)-categorical version of extended TQFTs purposed by the Baez and Dolan in [2].

Reshetikhin and Turaev in 1990 constructed a ribbon invariants defined via quantum groups and later in 1991 they succeeded for the first time to construct 3-manifold invariants in a rigorous and mathematically consistent way. In their paper they obtained a projective TQFT in the sense of Atiyah. They use a *semisimple modular category* as input data. Turaev’s monograph [10] fully describe the Reshetikhin and Turaev construction of a TQFT, methods used in [10] show that the construction of a TQFT functor is very complicated and a difficult procedure.

Extended TQFT functor constructed in [8] (which is more complicated then the ordinary TQFT functor of [T]) cannot be considered as a generalized version of Turaev construction of TQFT functor, actually both construction are different because of the different base categories.

Baez and Dolan’s [2] hypothesis for extended TFQTs shows that one needs a different construction of TQFT functors at a different dimensional level, e.g. the TQFT functor which produce 2-dimensional extended TQFTs cannot be (easily) generalized to a 3-dimensional extended TQFT functor. This is because of the base categories (especially the codomain categories). Either they do not have a nice structure in higher dimension or their structure is very complicated, e.g. the enriched \(n\)-categorical version of \(\text{Vect}\) is not very clear in dimension \(n \geq 2\).

For \(n = 2\), one can think 2-vector spaces (in the sense of \(n\)-categories) as a linear space over the category \(\text{Vect}\) of vector spaces. This means that, it is a monoidal category \(\mathcal{V}\) with an external tensor product \(\oplus\) and a functor \(\otimes: \text{Vect} \times \mathcal{V} \to \mathcal{V}\) satisfying various conditions. Involvement of external tensor product over categories suggests that one needs to construct different TQFT functors at different dimensional level. This also suggests that in most of the higher dimensional cases these TQFTs functors will be independent from each other.

In a general situation one can ask “is there any other way to construct an \(n\)-dimensional extended TQFTs such that

- an \((n-1)\)-dimension extended TQFT can be obtained by putting certain constraints on the \(n\)-dimensional extended TQFT functor?
a generalized version of \((n-1)\)-dimensional extended TQFT can be realized.

As an attempt to answer this question, we define 2-dimensional extended TQFTs in the following hypothesis:

**Definition 5.1** A two dimensional extended TQFT is a functor from semistrict monoidal 2-category of \(\mathbf{2-cob}\) to semistrict monoidal 2-category \(\mathbf{2Vect}\) of 2-vector spaces.

Here, internal categories \(\mathbf{2-Vector\ spaces}\) are same as the 2-Vector spaces defined by Baez and Crans in [1].

For the higher dimensional extended TQFTs, one needs to generalize internal categories structure for higher dimensions in such a way that existing base category structures remain preserved, e.g. as in the case of \(\mathbf{2Vect}\), which contains ordinary vector spaces as objects. If we consider \(\mathbf{3Vect}\) to be the category having objects as internal categories of \(\mathbf{2Vect}\) and arrows are internal functors, then under suitable conditions \(\mathbf{3Vect}\) can gives a higher category version of \(\mathbf{2Vect}\) which also contain \(2\)-vector spaces as objects.

Presence of an ordinary vector space in \(\mathbf{2Vect}\) is vital for following two reasons

1. to get a Turaev’s type of TQFT from a 2-dimensional extended TQFT.

2. to generalize Turaev’s type TQFT to 2-dimensional extended TQFT. These two cases can be proved by making a suitable modification or restriction in TQFT functors.

One can also think an \(n\)-categories version of this result in terms of enriched categories by using the Baez’s construction of \(n\)-categories.

Details on the various aspects of 2-dimensional extended TQFT which are discussed here is a subject matter of the next version of this article.

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