Convergence of radial loop-erased random walk in the natural parametrization

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March 13, 2017

Abstract

In recent work we have shown that loop-erased random walk (LERW) connecting two boundary points of a domain converges to the chordal Schramm-Loewner evolution (SLE$_2$) in the sense of curves parametrized by Minkowski content. In this note we explain how to derive the analogous result for LERW from a boundary point to an interior point, converging towards radial SLE$_2$.

1 Introduction and main results

1.1 Introduction

Let $D$ be a bounded, simply connected domain containing 0 as an interior point. Let $a, b \in \partial D$ and suppose $\partial D$ is analytic near $a, b$. For large integer $N$, let $D_N$ be an approximation of $D$ using the grid $N^{-1}\mathbb{Z}^2$ with $a_N, b_N \in \partial D_N$ approximating $a, b$. In [8, 9] we proved that a loop-erased random walk (LERW) in $D_N$ from $a_N$ to $b_N$ viewed as a continuous curve and parametrized so that each edge is traversed in time a constant times $N^{-5/4}$ converges in the scaling limit to a chordal SLE$_2$ curve in $D$ from $a$ to $b$ parametrized by 5/4-dimensional Minkowski content. In short, LERW converges to SLE in the natural parametrization. The version of LERW studied in [8, 9] is defined by taking a random walk from $a_N$ to $b_N$ conditioned on
staying in \( D_N \) and successively erasing loops as they form to produce a random *chordal* self-avoiding walk. Another natural variant, *radial* LERW, is the loop-erasure of a random walk from an interior point stopped when reaching the boundary. (A third version that we do not directly discuss, is the loop-erasure of a random walk in the whole plane using a limiting procedure, and corresponds to the whole-plane version of SLE\(_2\).) In this note we will continue the work of \cite{8, 9} by proving an analogous result about radial LERW converging to radial SLE\(_2\). The idea of the proof is to use the Markovian coupling of \cite{8, 9} and weight it by the relevant Radon-Nikodym derivatives in order to construct a coupling of the radial processes up to the first time the paths get near the target point. We conclude by giving separate “continuity” estimates for LERW and SLE near the target point in the interior. Another possible strategy would be to redo the work of \cite{8, 9} in the present setting, but this would need the analogue of the sharp one-point estimate of \cite{2} which we currently do not have for the radial version of LERW. We will start by giving a statement of the main result and a sketch of the argument and then discuss the details.

1.2 Notation

In order to state our results we give some notation and describe the set up. We will use some notations of \cite{8, 9} occasionally reminding the reader of their meanings.

Let \( \mathcal{A} \) be the set of bounded, simply connected subsets of \( \mathbb{Z}^2 \) containing 0 as an interior vertex. To each \( A \in \mathcal{A} \) we associate to a simply connected Jordan domain \( D_A \subset \mathbb{C} \) obtained by replacing each vertex with the closed square with axis-parallell sides of length one centered at the vertex, and then taking the interior. We will refer to the domains \( D_A \) as “union of squares” domains; they are in one-to-one correspondence with elements of \( \mathcal{A} \). Suppose boundary edges \( a, b \in \partial_e A \) are given; we write \( a, b \) both for the edge and for its midpoint which is a point in \( \partial D_A \). Let \( \mathcal{W}_{A,0,0} \) be the set of self-avoiding walks (SAW) \( \eta = [\eta_0, \ldots, \eta_k] \) with \( \eta_k = 0, \{\eta_1, \ldots, \eta_{k-1}\} \subset A \) and such that \( a \) is the midpoint of the edge \( [\eta_0, \eta_1] \). (We use the notation \( \eta \) only for SAWs.) We also view \( \eta \) as the continuous curve \( \eta(t) : 0 \leq t \leq k - \frac{1}{2} \) with \( \eta(0) = a \) obtained by traversing the edges of \( \eta \) at unit speed. Throughout the paper, if \( \gamma : [0, t_{\gamma}] \to \mathbb{C} \) is a continuous curve, we also write \( \gamma \) for the curve \( \gamma : [0, \infty) \to \mathbb{C} \) extended in the natural manner as \( \gamma(t) = \gamma(t_{\gamma}), \ t > t_{\gamma} \).

We define a finite measure on \( \mathcal{W}_{A,0,0} \) by

\[
\hat{P}_{A,0,0}(\eta) = p(\eta) \Lambda_\eta(A), \quad \eta \in \mathcal{W}_{A,0,0} \tag{1.1}
\]
where \( \log \Lambda_\eta(A) \) is the random walk loop measure of loops staying in \( A \) that intersect \( \eta \), see [6, Chapter 9], and \( p(\cdot) = 4^{-\text{steps}(\cdot)} \) is the random walk measure on the square grid. Then

\[
\sum_{\eta \in \mathcal{W}_{A,a,0}} \hat{P}_{A,a,0}(\eta) = H_A(0,a),
\]

where \( H_A \) denotes the Poisson kernel for random walk, that is, the probability that simple random walk from 0 exits \( A \) through the edge \( a \). We write

\[
P_{A,a,0}^{\text{rad}}(\eta) = \frac{\hat{P}_{A,a,0}(\eta)}{H_A(0,a)}, \quad \eta \in \mathcal{W}_{A,a,0},
\]

for the probability measure obtained by normalization. This is the probability distribution of radial LERW in \( A \) from \( a \) to 0. \( P_{A,a,0}^{\text{rad}} \) induces a probability measure on parametrized complex curves from \( a \) to 0 in \( D_A \) after parametrizing by arclength. The distributions on other versions of LERW are given by considering different sets of SAWs and normalizing appropriately.

Suppose now that \( D \) is a simply connected domain in \( \mathbb{C} \) containing the origin and assume \( D \) has analytic boundary. For each positive integer \( N \) we define \( D_N = D_{A_N} \) to be the largest simply connected “union of squares” domain containing the origin whose closure is contained in \( N \cdot D \) and we write \( \hat{D} = \hat{D}_N := N^{-1}D_N \) which satisfies \( 0 \in \hat{D} \subset D \). As \( N \to \infty \), the simply connected domain \( \hat{D} \) converges to \( D \) in the Carathéodory sense. Given \( a_N \in \partial A_N \) we write \( \hat{a}_N := N^{-1}a_N \). In order to state the theorem, assume that \( a_N \in \partial D_N \) is a sequence chosen so that \( \hat{a}_N \to a \) as \( N \to \infty \) and for each \( N \), let \( P_{A_N}^{\text{rad}} \) be the probability measure obtained from \( P_{A_N,a_N,0}^{\text{rad}} \) by considering the scaled paths

\[
\eta(t) = \tilde{\eta}_N(t) := N^{-1} \eta(c_* t N^{5/4}), \quad 0 \leq t \leq t_N := \frac{k}{c_* N^{5/4}}. \tag{1.2}
\]

(We use the same notation both for the measure on SAWs and parametrized curves.) Here \( c_* \in (0, \infty) \) is a fixed constant whose value is not known; it is the same constant as the \( c_* \) appearing in [3]. In general, given a SAW \( \eta \), we will write \( \hat{\eta}_N \) for the rescaled path parametrized as in (1.2). Let \( \gamma(t), 0 \leq t \leq t_\gamma \) be radial SLE\(_2\) in \( D \) from \( a \) to 0 parametrized by 5/4-dimensional Minkowski content; we write \( \mu^{\text{rad}} = \mu_{D,a,0}^{\text{rad}} \) for its law.

### 1.3 Statements

With these notations in place the main theorem can be stated as follows.
Theorem 1.1. As $N \to \infty$, the law of $\hat{\eta}$ converges weakly to that of $\gamma$ with respect to the metric $\rho$.

More precisely, for each $\varepsilon > 0$ there exists $N_0 < \infty$ such that if $N > N_0$ then is a coupling of $\gamma$ with distribution $\mu^{rad}$ and $\hat{\eta}$ with distribution $P_{\rho}^{rad}$ such that

$$ P\{\rho(\gamma, \hat{\eta}) > \varepsilon\} < \varepsilon. $$

Here we are using a metric $\rho$ on parametrized curves defined as follows.

Given curves $\gamma^j : [s_j, t_j] \to \mathbb{C}, j = 1, 2$, we let

$$ \rho(\gamma^1, \gamma^2) = \inf \left\{ \sup_{s_1 \leq t \leq t_1} |\alpha(t) - t| + \sup_{s_1 \leq t \leq t_1} |\gamma^1(t) - \gamma^2(\alpha(t))| \right\}, \quad (1.3) $$

where the infimum is taken over increasing homeomorphisms ("reparametrizations") $\alpha : [s_1, t_1] \to [s_2, t_2]$.

Remark. The metric $\rho$ is convenient to work with but it is not the only natural choice. It has the disadvantage that the metric space of parametrized curves $\gamma : [0, t_\gamma] \to \mathbb{C}$ with the metric $\rho$ is not complete. An alternative would be to consider curves $\gamma : [0, t_\gamma] \to \mathbb{C}$ as elements $\gamma = (t, t \mapsto \gamma(t \wedge t)) \in [0, \infty) \times C[0, \infty)$ with a metric

$$ \hat{\rho}\{\gamma^1, \gamma^2\} = |t_1 - t_2| + \sup_{t \in [0, \infty)} |\gamma^1(t \wedge t_1) - \gamma^2(t \wedge t_2)|, $$

if $\gamma^j = (t_j, t \mapsto \gamma^j(t \wedge t_j)), j = 1, 2$.

By a straightforward estimate of the conformal map $f : \hat{D} \to D$ with $f(\hat{a}) = a, f(0) = 0$, one can show that the two curves $\gamma$ and $f \circ \gamma$ are close in the sense of $(1.3)$. See [8, Corollary 7.3] for a proof for the analogous chordal result. Therefore, it suffices to prove the corresponding result where $\gamma$ is replaced with $\hat{\gamma}(t), 0 \leq t \leq t_\gamma$, a radial SLE$_2$ curve from $\hat{a}$ to 0 in $\hat{D}$. The main work is in establishing the following proposition that considers paths stopped before they get too close to the origin.

Proposition 1.2. There exists $c < \infty$ such that for every $r > 0$ and all $N$ sufficiently large, we can define $\hat{\gamma}(t), 0 \leq t \leq t_\gamma$, and $\hat{\eta}(t), 0 \leq t \leq t_N$, on the same probability space $(\Omega, Q)$ so that the following holds.

1. The marginal distribution of $\hat{\gamma}$ is that of radial SLE$_2$ from $\hat{a}$ to 0 in $\hat{D}$ parametrized by $5/4$-dimensional Minkowski content.

2. If $Q_2$ denotes the marginal distribution on $\hat{\eta}$, and $P_{\rho}^{rad}$ the distribution of (scaled) LERW from $\hat{a}$ to 0 in $\hat{D}$ parametrized as in $(1.2)$, then

$$ \|Q_2 - P_{\rho}^{rad}\| \leq cr, $$
where \( \| \cdot \| \) denotes total variation distance.

3. There is a Markovian stopping time \( \tau \) for the coupled pair \((\tilde{\gamma}, \tilde{\eta})\) and an event \( E \) with \( \mathbb{Q}(E) \geq 1 - cr \) on which

\[
\max_{0 \leq t \leq \tau} |\tilde{\gamma}(t) - \tilde{\eta}(t)| \leq r,
\]

and

\[
|\tilde{\gamma}(\tau)| \leq r.
\]

By Markovian stopping time we mean, roughly speaking, that the domain Markov property is valid at the stopping time for each path individually. See Section 4 for a precise statement. Given this proposition, the proof of Theorem 1.1 follows from Proposition 1.3 below. Indeed, using Proposition 1.2 we couple the paths until time \( \tau \) when they are at most at distance \( 2r \) from 0. Given the paths up to this time, we can then extend them independently to paths from \( \tilde{\alpha} \) to 0. Proposition 1.3 shows that the ends of the paths do not increase the \( \rho \)-distance much.

The estimates we give here are not optimal but they more than suffice for our purposes. The notation in the statement is the same as in Proposition 1.2

**Proposition 1.3.** There exists \( c < \infty \) such that on the probability space of Proposition 1.2, except perhaps on an event of probability \( cr^{1/8} \),

\[
t_{\tilde{\gamma}} - \sigma_{r} \leq cr^{1/2};
\]

\[
t_{N}^{*} - \sigma'_{r} \leq cr^{1/2};
\]

\[
\sup_{\sigma_{r} \leq t \leq t_{\tilde{\gamma}}} |\tilde{\gamma}(t)| \leq r^{1/2},
\]

and

\[
\sup_{\sigma'_{r} \leq t \leq t_{N}^{*}} |\tilde{\eta}(t)| \leq r^{1/2},
\]

where

\[
\sigma_{r} = \inf\{t \geq 0 : |\tilde{\gamma}(t)| \leq r\}, \quad \sigma'_{r} = \inf\{t \geq 0 : |\tilde{\eta}(t)| \leq r\}.
\]

**Proof.** See Proposition 3.2 and Proposition 3.7.
Acknowledgements

Lawler was supported by National Science Foundation grant DMS-1513036. Viklund was supported by the Knut and Alice Wallenberg Foundation, the Swedish Research Council, the Gustafsson Foundation, and National Science Foundation grant DMS-1308476. We also wish to thank the Isaac Newton Institute for Mathematical Sciences and Institut Mittag-Leffler where part of this work was carried out.

2 Comparing radial and chordal

We will need to compare measures on paths with different target points – radial and chordal measures, both for LERW and SLE.

2.1 Radial and chordal SLE

We review what is known comparing chordal and radial SLE$\kappa$. (Most of this generalizes to other values of $\kappa$ but for ease we will restrict to $\kappa = 2$.) We will consider the paths in $\tilde{D}$ with $N \geq 10$. (This restriction on $N$ is arbitrary, but we want to avoid some trivialities for small $N$.) All constants in this section will be uniform in $N \geq 10$ but may depend on $D, a, b$. Given a simple curve $\gamma$ from $a$ to $b$ in $D$, we write

$$S_t = S_{D \setminus \gamma(t)}(0; \gamma(t), b)$$

where for a choice of conformal map $F_t: D \setminus \gamma_t \to \mathbb{H}$, $F_t(\gamma(t)) = 0$, $F_t(b) = \infty$, we define

$$S_t(z) = S_{D \setminus \gamma_t}(z; \gamma(t), b) = \sin(\arg F_t(z)).$$

Let $\mu^{\text{rad}}, \mu^{\text{chord}}$ denote the probability measures of radial SLE$\kappa$ from $\tilde{a}$ to 0 and chordal SLE$\kappa$ from $\tilde{a}$ to $\tilde{b}$ in $\tilde{D}$, respectively. We view these as probability measures on curves $\tilde{\gamma}$, parametrized by Minkowski content, stopped at stopping times $\tau$ such that $\tilde{\gamma}(\tau) \in \tilde{D} \setminus \{0\}$, that is, stopped before the paths reach 0 or $\tilde{b}$. Note that the content is a deterministic function of the path, so we could equally well consider the capacity parametrization. Write $h_D(0, b)$ for the Poisson kernel and $h_{\partial D}(a, b)$ for the boundary Poisson kernel. We normalize so that

$$h_{\partial D}(a, b) = S_{D, a, b}^{-2} h_D(0, a) h_D(0, b). \quad (2.1)$$
The radial and chordal laws are absolutely continuous, in fact, if we write \( \tilde{D}_\tau = \tilde{D} \setminus \tilde{\gamma}_\tau \), we have

\[
\frac{d\mu^{\text{rad}}}{d\mu^{\text{chord}}} (\tilde{\gamma}_\tau) = M^{\text{SLE}}_{\tau} := \frac{S_{\tilde{D}_\tau}(0; \tilde{\gamma}(t), \tilde{b})^2}{S_{\tilde{D}}(0, \tilde{a}, \tilde{b})^2} \cdot \frac{h_{\tilde{D}_\tau}(0, \tilde{b})}{h_{\tilde{D}}(0, \tilde{b})}. \tag{2.2}
\]

It is convenient to write the last factor on the right-hand side in coordinates:

\[
\frac{h_{\tilde{D}_\tau}(0, \tilde{b})}{h_{\tilde{D}}(0, \tilde{b})} = \frac{\text{Im} F(0)}{\text{Im} F_\tau(0)} \tag{2.3}
\]

Here \( F : \tilde{D} \rightarrow \mathbb{H}, F(a) = 0, F(b) = \infty \) and \( F_\tau : \tilde{D}_\tau \rightarrow \mathbb{H}, F_\tau(\tilde{\gamma}(\tau)) = 0, F_\tau(\tilde{b}) = \infty \), where these maps have the same normalization, i.e., \( F_\tau \circ F^{-1}(z) = z + o(1) \) as \( z \rightarrow \infty \).

These statements can be made precise by first mapping to the upper half-plane and using the Girsanov theorem; see, e.g., [10]. We state this as a proposition.

**Proposition 2.1.** For every \( r > 0 \) suppose that \( \tau \) is a stopping time such that \( \text{dist}(0, \tilde{\gamma}_\tau) \geq r \) and \( \text{dist}(\tilde{b}, \tilde{\gamma}_\tau) \geq r \). Suppose that \( \tilde{\gamma}(t), 0 \leq t \leq \tau \), has the distribution of chordal SLE\( 2 \) from \( \tilde{a} \) to \( \tilde{b} \) stopped at time \( \tau \) with respect to the measure \( \mu^{\text{chord}} \). If the measure \( \nu \) is defined by the relation

\[
d\nu = M^{\text{SLE}}_{\tau} d\mu^{\text{chord}},
\]

then under \( \nu \), \( \tilde{\gamma}(t), 0 \leq t \leq \tau \), has the distribution of radial SLE\( 2 \) from \( \tilde{a} \) to \( 0 \) stopped at time \( \tau \).

### 2.2 Radial and chordal loop-erased walks

The laws of the LERW aiming towards different points stopped appropriately can be compared in a similar manner as the SLE\( 2 \). Let \( A \in \mathcal{A} \) with \( a, b \in \partial_e A \) given. If \( \eta = [\eta_0, \ldots, \eta_k] \) is a SAW starting at \( a = [\eta_0, \eta_1] \) and otherwise staying in \( A \), we write \( A_\eta = A \setminus \eta \), taking the component of 0 if necessary. Suppose \( a' = [\eta_k, w] \) is an edge such that \( w \in A_\eta \). Then \( a' \) is a boundary edge of \( A_\eta \). We write \( P_{A,a,0}(\eta \oplus a') \) for the probability that the beginning of the LERW from \( a \) to \( 0 \) is given by \( \eta \oplus a' \). Then,

\[
P_{A,a,0}(\eta \oplus a') = 4^{-|\eta|} \Lambda_\eta(A) \frac{H_{A_\eta}(0, a')}{H_A(0, a)},
\]
where $H_{A_\eta}(0, a') = 0$ if $w$ and 0 are not in the same component of $A_\eta$. Let $b$ be a boundary edge of $A$. Then if $P_{A,a,b}$ denotes the probability measure for LERW from $a$ to $b$ in $A$, we have a similar expression,

$$P_{A,a,b}(\eta \oplus a') = 4^{-|\eta|} A_\eta(A) \frac{H_{\partial A_\eta}(a', b)}{H_{A}(a, b)}.$$

We can write this as a “Radon-Nikodym derivative”, in which the loop terms have cancelled:

$$P_{A,a,0}(\eta \oplus a') = 4^{-|\eta|} A_\eta(A) \frac{H_{\partial A_\eta}(a', b)}{H_{A}(0, a)} \frac{H_{\partial A}(a, b)}{H_{\partial A_\eta}(a', b)}$$

provided that the denominator is nonzero. Given $\eta, a'$ as above, we define

$$M_{\text{LERW}}(\eta \oplus a') = M_{\text{LERW}_{A,0,a,b}}(\eta \oplus a') = 4^{-|\eta|} A_\eta(A) \frac{H_{\partial A_\eta}(a', b)}{H_{A}(0, a)} \frac{H_{\partial A}(a, b)}{H_{\partial A_\eta}(a', b)}$$

which naturally defines a martingale with respect to LERW in $A$ from $a$ to $b$.

**Proposition 2.2.** Let $P_{A,a,b}$ be the law of LERW from $a$ to $b$ in $A$. Suppose $\sigma$ is a stopping time such that $H_{\partial A_\eta}(b, a') > 0$, where $\eta = [\eta_0, \ldots, \eta_{\sigma-1}], a' = [\eta_{\sigma-1}, \eta_\sigma]$. If the measure $Q_{A,a,0}$ is defined by the relation

$$dQ_{A,a,0} = M_{\text{LERW}}(\eta \oplus a') dP_{A,a,b},$$

then under $Q_{A,a,0}$, $\eta \oplus a'$ has the distribution of LERW from $a$ to $0$ in $A$ stopped at $\sigma$.

An important fact proved in [3] (Eq. (41)) is that there is a constant $\tilde{c}$ such that

$$H_{\partial A}(a, b) = \tilde{c} H_{A}(0, a) H_{A}(0, b) S_{A,a,b}^{-2} \left[1 + O_\delta(N^{-u})\right] \quad (2.4)$$

if $S_{A,a,b} \geq \delta$. (This also writes the Green’s function next to the boundary as a constant times the boundary Poisson kernel.) Here we are writing $S_{A,a,b}$ for $S_{D,a,b}(0)$ as defined in the previous subsection. If $\text{dist}(\eta, 0) \geq rN$, then we can write

$$M_{\text{LERW}}(\eta \oplus a') = \frac{P_{A,a,b}(\eta \oplus a')}{P_{A,a,b}(\eta \oplus a')} = \frac{S_{A,a,b}^2}{S_{A,a,b}^2} \frac{H_{A}(0, b)}{H_{A}(0, b)} [1 + O_{r,\delta}(N^{-u})], \quad (2.5)$$

provided that the sines are bounded below by $\delta > 0$. This is very similar to [2.2].

Let $I_s$ be the set of SAWs $\eta$ in $A$ such that $\eta \cap C_s \neq \emptyset$. Let $I_s^*$ be the set of SAWs $\eta = [\eta_0, \ldots, \eta_n] \in I_s$ such that $[\eta_0, \ldots, \eta_{n-1}] \notin I_s$. 

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Lemma 2.3. Let $r, \delta > 0$ be given. For all $A, \eta$ with $\eta \in I^*_s$ such that
\[ s \geq rN, \quad \min_{j \leq n} S_{A_j, a_j, b} \geq \delta, \quad A_j = A \setminus \eta[0, \ldots, \eta_j], \quad (2.6) \]
we have
\[ \frac{H_A(0, b)}{H_{A_\eta}(0, b)} = \frac{\text{Im} \ F(0)}{\text{Im} \ F_{\eta}(0)} \left( 1 + \mathcal{O}(r, \delta) \left( N^{-1/20} \right) \right). \]
Here $F_\eta : D_{A_\eta} \to \mathbb{H}$ is the conformal map with $F_\eta(a_\eta) = 0, F_\eta(b) = \infty$ and such that $F_\eta \circ F^{-1}(z) = z + o(1)$ as $z \to \infty$. The error estimate is uniform over $A, \eta$ satisfying (2.6).

Proof. See Section 5. \hfill \square

3 Regularity estimates

This section gives estimates on the content of the SLE and LERW near the target point and estimates for the geometric “sine processes” $S_t$ and $S_{A_j, a_j, b}$.

3.1 SLE estimates

We know that the expected Minkowski content of a chordal SLE$_2$ curve is finite almost surely, at least if the domain is bounded and not too rough. We need a similar estimate for $\mu^{\text{rad}}$ near the bulk point.

Lemma 3.1. There exists $c < \infty$ such that if $\tilde{\gamma}$ is a radial SLE$_2$ path from $\tilde{a}$ to 0, $T_{r, s}$ denotes the $(5/4)$-Minkowski content of $\gamma \cap \{ s < |z| < r \}$, and $T_r = T_{0, r}$, then
\[ E[T_r] \leq c r^{5/4}. \quad (3.1) \]

Proof. We have
\[ E[T_r] = \int_{r \mathbb{D}} G_{D}^{\text{rad}}(z; \tilde{a}, 0) dA(z) \leq c \int_{r \mathbb{D}} G_{D}^{\text{rad}}(z; 1, 0) dA(z), \]
where $G_{D}^{\text{rad}}(z, a, 0)$ denotes the Green’s function for radial SLE$_2$ in $D$ from $a$ to 0. From [1] we have that
\[ G_{D}^{\text{rad}}(z; 1, 0) = c e^{2y(2-d) \sinh y \cosh y |d-2+\beta| \sin(x+iy)^{-\beta}} E_2^{\beta} [g_T(0)^q]. \quad (3.2) \]
Here in the special case $\kappa = 2$, the boundary exponent $\beta = 3, d = 5/4, and q = -3/4$, the cylindrical coordinates $(x, y)$ are defined by $z = e^{-2y+2ix}$, and the expectation is with respect to two-sided radial SLE in $\mathbb{D}$ from 1.
through $z$ with Loewner maps ($g_t$) and with $T$ being the time (in the radial parametrization) at which the path reaches $z$. Since the conformal radius satisfies $g'_T(0)^{-1} \leq 4|z|$ under $P^*_z$, we can see that (3.2) implies

$$G^{rad}_D(z; 1, 0) \leq c|z|^{-3/4},$$

as $|z| \to 0$, and integrating this around 0 gives the stated bound.

**Proposition 3.2.** There exists $c < \infty$ such that the following holds. Let $s \leq 1/4$ and $\sigma_s = \inf\{t : |\tilde{\gamma}(t)| = s\}$. Then, except for an event of probability at most $cs^{1/8}$, we have

$$t_{\gamma} - \sigma_s \leq s^{1/8}$$

and

$$|\tilde{\gamma}(t)| \leq s^{1/2}$$

for $t \geq \sigma_s$.

**Proof.** By the estimates of [4, 5] the probability that $\tilde{\gamma}$ leaves $s^{1/2}\mathbb{D}$ after time $\sigma_s$ is $O(s^{3/4})$. On the event that $\gamma[\sigma_s, t_{\gamma}] \subset s^{1/2}\mathbb{D}$, we have $t_{\gamma} - \sigma_s \leq T_{s^{1/2}}$ where $T_r$ is as in (3.1). Since by Lemma 3.1 $E[T_{s^{1/2}}] \leq cs^{5/8}$, Markov’s inequality implies $P\{T_s \geq s^{1/2}\} \leq cs^{1/8}$. 

Recall that if $f_t : \bar{D} \setminus \tilde{\gamma}_t \to \mathbb{D}$ is the conformal transformation with $f_t(0) = 0$, $f_t(\tilde{\gamma}_t) = 1$, then $\theta_t$ is defined by $f_t(b) = e^{2i\theta_t}$ and $S_t = S(\tilde{\gamma}_t) = \sin \theta_t$; this definition works for any Loewner curve. We will also need the fact that for SLE, $S_t$ is unlikely to get near zero in a finite amount of time.

**Lemma 3.3.** Let $D, a, b$ be given. For every $r > 0$, there exists $\delta_r > 0$ such that if $\gamma$ is a radial SLE$_2$ path from $\tilde{a}$ to $0$ in $\bar{D}$, then

$$P\left\{\min_{0 \leq t \leq \sigma_r} S_t \leq \delta_r\right\} \leq r, \quad (3.3)$$

where $\sigma_r = \inf\{t : |\tilde{\gamma}(t)| \leq r\}$.

**Sketch of proof.** This is standard so we will give a sketch. Consider the process parametrized by the radial parametrization centered at 0: at time $s$ of the radial parametrization, the conformal radius is $e^{-2s/r_D}$. Let $f_t$ be as just above.
The Loewner equation shows that $\theta_t$ satisfies the radial Bessel SDE
\[ d\theta_t = \cot \theta_t \, dt + dW_t, \]
where $W_t$ is a standard Brownian motion. The Schwarz lemma implies that the conformal radius at time $\sigma_r$ at least $r$, and hence $\sigma_r$ corresponds to time at most $-\frac{1}{2} \log r + O(1)$. Hence, this becomes an estimate about the radial Bessel SDE which is standard. Indeed, we could give a more quantitative version of this, but we will not need it.

We will later work with stopping times that correspond to mesoscopic capacity increments, we need to be able to stop the paths slightly later than the stopping time $\sigma_r$, and we will want to say that this time is smaller than $\sigma_r$ given some geometric information. For this, we will use the following deterministic lemma.

**Lemma 3.4.** Let $D, a, b$ be as above. For every $0 < r < 1/4$ and $\delta > 0$ there exists $c > 0$ depending on $D, a, b, r, \delta$ such that the following is true. Suppose $\gamma$ is a simple radial Loewner curve from $a$ to $0$ in $D$. If
\[ \min_{0 \leq t \leq \sigma_r} S_D(0; \gamma(t), b) \geq \delta, \]
then the following estimates hold:

i) $h_{D, \sigma_r}(0, b) \geq c$;

ii) $\text{hcap} [\gamma_{\sigma_r} \setminus \gamma_{2\sigma_r}] \geq c$;

iii) $\text{dist}_I(\gamma_{\sigma_r}, b) \geq c$,

where $\text{dist}_I$ denotes interior distance.

**Proof.** We use the radial parametrization from 0 as above. As before, let $f_t = g_t \circ f_0$, and if we write $\sigma_r$ for the first time $\gamma$ reaches the circle of radius $r$ about 0, we have
\[ \frac{h_{D, \sigma_r}(0, b)}{h_D(0, b)} = |g'_{\sigma_r}(e^{2i\theta_0})|. \]
An easy estimate using the Loewner equation shows that there exists $C_\delta < \infty$ such that if $S_s \geq \delta$ for $0 \leq s \leq \sigma_r$, then
\[ |g'_{\sigma_r}(e^{2i\theta_0})| \geq ce^{-rc_\delta}. \quad (3.4) \]
This gives the first estimate. For the second estimate, recall that halfplane capacity (viewed from \( b \)) can be given as a hitting measure for Brownian bubbles attached at \( b \). More precisely, up to a constant depending on the map to \( \mathbb{H} \) we use to measure capacity, the capacity of \( \gamma[\sigma_{2r}, \sigma_r] \) equals

\[
|f'_0(b)|^2 |g'_{\sigma_{2r}}(\zeta)|^2 \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} P^{\zeta, \varepsilon} (B \cap f_{\sigma_{2r}}(\gamma[\sigma_{2r}, \sigma_r]) \neq \emptyset),
\]

where \( \zeta = e^{2i\theta_0} \) and \( P^{\zeta, \varepsilon} \) denotes the law of a Brownian \( h \)-process \( B \) in \( \mathbb{D} \) from \( (1 - \varepsilon)\zeta \) to \( \zeta \) and we also used the covariance rule. But since \( f_{\sigma_r}(\gamma[\sigma_{2r}, \sigma_r]) \) is a curve in \( \mathbb{D} \) connecting the boundary with a circle of radius at most a constant strictly smaller than 1, the limit in the last display is bounded away from 0. Therefore (3.4) gives the result.

For the last estimate, suppose that \( \varepsilon > 0 \) and let \( \tau \) be the first time \( \gamma \) is at interior distance \( \varepsilon \) from \( b \). Then there is a crosscut \( \beta \) of \( \mathbb{D} \setminus \gamma \) connecting \( \gamma(\tau) \) with \( b \) and \( \text{diam} \beta \leq \varepsilon \). Moreover, \( \beta \) cuts \( \mathbb{D} \setminus \gamma_{\tau} \) into two components, one of which contains 0. By the Beurling estimate, the harmonic measure of \( \beta \) from 0 in \( \mathbb{D} \setminus \gamma_{\tau} \) is at most a constant times \( \varepsilon^{1/2} \) and the same bound (allowing a different constant) holds for \( S_{\mathbb{D}_r}(0; \gamma(\tau), b) \).

\[ \square \]

### 3.2 LERW estimates

We start with an estimate that does not require any smoothness of the boundary of \( A \) and then we use it for the approximations \( \hat{D} \) of the analytic domain \( D \). For \( r > 0 \), let \( C_r = \{ z \in \mathbb{Z}^2 : |z| < r \} \).

**Lemma 3.5.** There exists \( c < \infty \) such that the following holds. Suppose \( A \) is a simply connected finite subset of \( \mathbb{Z}^2 \) containing the origin and \( a \in \partial_e A \). If \( z \in A \), let \( d_z = d_{z,A} = \text{dist}(z, \partial A) \) and let \( d = d_0 \).

- If \( z \in C_{d/4} \), then
  \[
  P^{\text{rad}}_{A,0,a} \{ z \in \eta \} \leq c |z|^{-3/4}.
  \]

- If \( z \in A \setminus C_{d/4} \), then
  \[
  P^{\text{rad}}_{A,0,a} \{ z \in \eta \} \leq c \frac{H_A(z, a)}{H_A(0, a)} G_A(z, 0) d_z^{-3/4}. \tag{3.5}
  \]

**Proof.** Fix \( z \in A \) and let

\[
r = r_z = \frac{1}{4} \cdot \min\{d_z, |z|\},
\]


\[ D_r(z) = C_r + z = \{ w : |w - z| < r \}, \quad D_{2r}(z) = C_{2r} + z = \{ w : |w - z| < 2r \}. \]

Let \( \omega^a, \omega^0 \) be two independent simple random walks starting at \( z \) ending at \( a, 0 \), respectively, and otherwise staying in \( A \). The \((p-)\) measure of the set of such \( \omega^a \) is \( H_A(z, a) \) and the measure of such \( \omega^0 \) is \( G_A(z, 0) \).

- Let \( \tilde{\omega}^0 \) be \( \omega^0 \) with the first edge removed (so that it starts at a nearest neighbor of \( z \)).
- Let \( \eta^a = \text{LE}(\omega^a) \).

Then \( \hat{P}_{A,0,a} \{ z \in \eta \} \) equals the \( p \)-measure of \((\omega^a, \omega^0)\) satisfying

\[
\tilde{\omega}^0 \cap \eta^a = \emptyset.
\]

Let \( \tilde{\eta}^a \) be \( \eta^a \) stopped at the first time that it reaches \( \partial D_r(z) \) and similarly for \( \tilde{\omega}^0 \) using \( \tilde{\omega}^0 \). Then \( \hat{P}_{A,0,a} \{ z \in \eta \} \) is bounded above by the measure of \((\omega^a, \omega^0)\) satisfying

\[
\hat{\omega}^0 \cap \tilde{\eta}^a = \emptyset.
\]

By \cite{[8, Lemma 6.13]}, the normalized probability measure on \( \tilde{\eta}^a \) is comparable to the measure obtained by taking LERW in the larger set \( D_{2r}(z) \) from \( z \) to \( \partial D_{2r}(z) \) (and stopping when exiting \( D_r(z) \)). If we let \( \pi_{2r} \) denote the latter probability, then the measure of \( \omega^a \) that produce \( \tilde{\eta}^a \) is comparable to \( H_A(z, a) \pi_{2r}(\hat{\eta}^a) \). Using \cite{[8, Proposition 6.15]}, we see that that

\[
E_{\pi_{2r}} \left[ \mathbb{P} \{ \tilde{\omega}^0 \cap \tilde{\eta}^a = \emptyset \mid \eta^a \} \right] \leq c r^{-3/4}.
\]

For a given \( \tilde{\omega}^0 \), the measure of possible extensions to walks ending at the origin and avoiding \( \tilde{\eta}^a \) is \( G_{A \setminus \tilde{\eta}^a}(\zeta, 0) \) where \( \zeta \) is the endpoint of \( \tilde{\omega}^0 \) which is a neighbor of \( z \). We now consider two cases.

- If \( z \in A \setminus C_{d/4} \), we bound \( G_{A \setminus \tilde{\eta}^a}(\zeta, 0) \) above by \( G_A(\zeta, 0) \) and use the Harnack inequality to see that this is comparable to \( G_A(z, 0) \).
- If \( z \in C_{d/4} \) we bound \( G_{A \setminus \tilde{\eta}^a}(\zeta, 0) \) above by \( G_{\mathbb{Z}^2 \setminus \tilde{\eta}^a}(\zeta, 0) \). Since \( \tilde{\eta}^a \) is a set of diameter at least \( r \) and \( |\zeta| \geq |z|/2 \), we can use standard arguments to show that

\[
G_{\mathbb{Z}^2 \setminus \tilde{\eta}^a}(\zeta, 0) < c.
\]

In this case, we also see that the Harnack inequality implies that \( H_A(z, a) \propto H_A(0, a) \).
Proposition 3.6. There exists $c = c_D < \infty$ such that for all $(A_n, a_n, b) \in \mathcal{A}_n(D)$,
\[
\sum_{z \in A} \mathbb{P}_{A_n, 0, a_n} \{z \in \eta\} \leq cn^{5/4}.
\]

Proof. We will prove a stronger result. We first see that if $|z| \leq n/2$, then
\[
\mathbb{P}_{A_n, 0, a_n} \{z \in \eta\} \leq c|z|^{-3/4},
\]
and hence for $m \leq n/2$,
\[
\sum_{m \leq |z| \leq m+1} \mathbb{P}_{A_n, 0, a_n} \{z \in \eta\} \leq cm^{1/4}. \tag{3.6}
\]

Let $U_k = \{z \in A : d_z \geq k\}$. Using the smoothness of $D$ and the gambler’s ruin estimate, we see that
\[
G_A(z, 0) \leq \frac{ck}{n}, \quad z \in (\partial U_k) \setminus C_{n/2} \tag{3.7}
\]

Let $S$ be a simple random walk starting at the origin and let $T_k = \inf\{j : S_j \in U_k\}$. Using smoothness of $\partial D$ and gambler’s ruin again, we see that for $w \in \partial U_k$, $H_{U_k}(0, w) \asymp n^{-1}$. Also, the strong Markov property implies that
\[
H_A(0, a) = \sum_{w \in \partial U_k} H_{U_k}(0, w) H_A(w, a).
\]

Hence,
\[
\sum_{w \in \partial U_k} H_A(w, a) \leq cn H_A(0, a).
\]

If $k - 1 \leq d_z < k$, then there exists $w$ within distance two of $z$ with $d_w \geq k$ and hence either $z$ or a nearest neighbor of $z$ is in $\partial U_k$. Using the Harnack inequality, we can then conclude that
\[
\sum_{k-1 \leq d_z < k} H_A(z, a) \leq cn H_A(0, a).
\]

Hence, (3.5) and (3.7) imply that
\[
\sum_{k-1 \leq d_z < k} \mathbb{P}_{A_n, 0, a_n} \{z \in \eta\} \leq ck^{1/4}.
\]

This and (3.6) give the estimate. \qed
Proposition 3.7. There exists $c < \infty$ such that the following holds. Let $A$ be a finite, simply connected domain including the origin, $a \in \partial_e A$, and $n = \text{dist}(0, \partial A)$. Suppose $\eta = [\eta_0, \ldots, \eta_k]$ is a LERW from $a$ to $0$ in $A$, $s \leq 1/2$, and we write
$$\eta = \eta^- \oplus \eta^+$$
where $\eta^- = [\eta_0, \ldots, \eta_m]$ and $m$ is the smallest index with $\eta_m \in C_{s^2 n}$. Then, except for an event of probability at most $c s$,
$$|\eta^+| \leq s^{1/4} n^{5/4}, \quad \eta^+ \subset C_{sn}.$$

Proof. We write $P, E$ for probabilities and expectations under $P_{A,a,0}$. Let $T$ be the total number of points in $C_{sn}$ visited by $\eta$. By Lemma 3.5, we have for $z \in C_{sn}, P\{z \in \eta\} \leq c |z|^{-3/4}$, and hence $E[T] \leq c (sn)^{5/4}$ and
$$P\{T \geq s^{1/4} n^{5/4}\} \leq c s.$$ 

On the event $\{\eta^+ \subset C_{sn}\}$, we have $|\eta^+| \leq T$. Therefore it suffices to show that $P\{\eta^+ \not\subset C_{sn}\} \leq c s$. We will show the stronger fact,
$$P\{\eta^+ \not\subset C_{sn} \mid \eta^-\} \leq c s.$$

Given $\eta^-$, the distribution of $\eta^+$ can be given as follows:

- Take a simple random walk $\omega$ starting at $\eta_m$ and stop it when it reaches the origin. Let $\tilde{\omega}$ be $\omega$ with the first edge removed stopped at the first visit to the origin.

- Condition on the event that $\tilde{\omega} \subset A \setminus \eta^-$. 

- Erase loops from the path.

Hence it suffices to show that
$$P\{\tilde{\omega} \not\subset C_{sn}, \tilde{\omega} \subset A \setminus \eta^- \mid \eta^-\} \leq c s P\{\tilde{\omega} \subset A \setminus \eta^- \mid \eta^-\}.$$ 

Let $q$ be the maximum over $w \in C_{3s^2 n/4} \setminus C_{s^2 n/4}$ of the probability that a random walk starting at $w$ reaches the origin before leaving $A \setminus \eta^m$. Standard estimates show that this comparable to $\log(s^2 n)^{-1}$ and the minimum over $w \in C_{3s^2 n/4} \setminus C_{s^2 n/4}$ is also comparable to this. Let $\omega'$ denote $\tilde{\omega}$ stopped at the first time the path gets distance $s^2 n/2$ from $\eta_m$, and let $z$ denote the endpoint of $\omega'$. By [3, Lemma 6.9],
$$P\{|z| \leq 2s^2 n/3 \mid \omega' \cap \eta^s \neq \emptyset\} \geq c.$$ 

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and hence
\[ P\{\tilde{\omega} \subset A \setminus \eta^- \mid \eta^-\} \geq cq P\{\tilde{\omega} \cap \eta^- \neq \emptyset\}. \]

The Beurling estimate implies that the probability that a walk starting in \( C_{2s^2n} \) reaches \( \partial C_{sn} \) and then returns to \( C_{2s^2n} \) without hitting \( \eta^- \) is \( O(s) \) (it is \( O(\sqrt{s}) \) to reach \( \partial C_{sn} \) and given that, another \( O(\sqrt{s}) \) to return to \( C_{2s^2n} \)). Given that it does this, the probability of reaching the origin before leaving \( A \setminus \eta^s \) is bounded above by \( q \). Hence,

\[ P\{\tilde{\omega} \not\subset C_{sn}, \tilde{\omega} \subset A \setminus \eta^- \mid \eta^-\} \leq csq P\{\tilde{\omega} \cap \eta^- \neq \emptyset\} \leq cP\{\tilde{\omega} \subset A \setminus \eta^- \mid \eta^-\}. \]

We end with the analogue of Lemma 3.3.

**Lemma 3.8.** Let \( D, a, b \) be given. For every \( r > 0 \), there exists \( \delta_r > 0 \) such that if \( \bar{\eta} \) is scaled LERW in \( \bar{D} \) from \( \bar{a} \) to 0, then for all \( N \) sufficiently large,

\[ P\left\{ \min_{0 \leq t \leq \sigma_r} S_{\bar{D} \setminus \bar{\eta}, \bar{b}}(t, \bar{b}) \leq \delta_r \right\} \leq r, \tag{3.8} \]

where \( \sigma_r = \inf\{t : |\bar{\eta}(t)| \leq r\} \).

**Proof.** This follows from Lemma 3.3 and the fact that we know that radial LERW converges to radial SLE\(_2\) in the radial parametrization [7]. \( \square \)

### 4 Coupling: proof of Proposition 1.2

In this section we construct the coupling of Proposition 1.2. Recall the setup: we have a fixed domain \( D \) with analytic boundary with 0 \( \in D \) and with \( a \in \partial D \) given. We assume without loss in generality that \( 1 \leq r_D(0) \leq 2 \). Fix an auxiliary boundary point \( b \in \partial D \) such that if \( F_0 : D \to \mathbb{H} \) is the conformal transformation with \( F_0(a) = 0, F_0(b) = \infty, \Im F_0(0) = 1, |\Re F_0(0)| \leq 1 \). The first three conditions can always be fulfilled, but the last puts a restriction on \( b \) to not be “too close” to \( a \).

For \( N = 1, 2, \ldots \) let \( \bar{D}_N \) be a discrete approximation of \( D \) as above, and we always choose \( \bar{a}_N \in \partial \bar{D}_N \) to correspond to a boundary edge of \( \partial \bar{D}_N \) closest to \( N \cdot a \) and \( \bar{b}_N \) is chosen to approximate \( b \) with the same geometric condition fulfilled: \( F_N : \bar{D}_N \to \mathbb{H}, F_N(\bar{a}_N) = 0, F_N(\bar{b}_N) = \infty \) and \( \Im F_N(0) = 1, |\Re F_N(0)| \leq 1 \). We will write \( \hat{a} = \bar{a}_N, \hat{D} = \bar{D}_N \) etc, and keep the \( N \) implicit in the notation from now on. When we measure capacity of curves in \( \bar{D} \) it is with respect to the map \( F = F_N \).
We will construct our coupling by weighting the coupling of \([8,9]\) by the Radon-Nikodym derivatives discussed in Section 2. Because of this we need to be careful about measurability properties and use the Markovian property of the chordal coupling and we need to consider events on which the Radon-Nikodym derivatives are controlled. Let us review the chordal coupling. Fix \(K < \infty\). In \([8,9]\) we found a sequence \(\varepsilon_N \to 0\) as \(N \to \infty\) (which depends on \(D, a, b, K\)) and for each \(N\) sufficiently large we constructed \((\tilde{\eta}, \tilde{\gamma})\) on the same probability space \((\Omega, \mathcal{F}, \mathbb{P})\) so that the following holds.

- The marginal distribution on \(\tilde{\eta}\) is LERW in \(\tilde{D}\) from \(\tilde{a}\) to \(\tilde{b}\), parametrized as in \([1,2]\). Write \(\mathcal{F}_{n}^{\text{LERW}}\) for the filtration of generated by the LERW up to step \(n\).
  - To be more precise, we start from a probability space on which is defined an infinite sequence of uniform \([0,1]\) random variables \(\{U_j\}\). Then \(\mathcal{F}_{n}^{\text{LERW}}\) can be taken to be the filtration generated by \(U_1, \ldots, U_n\). The probabilities for the at most three possible moves on the \((n+1)\)st step of the LERW are determined by the walk up to time \(n\) (that is, is \(\mathcal{F}_{n}^{\text{LERW}}\)-measurable) and the random variable \(U_{n+1}\) is used to make the choice in the usual manner. We also let \(\tilde{\mathcal{F}}_{n}^{\text{LERW}}\) denote the future sigma algebra generated by \(U_{n+1}, U_{n+2}, \ldots\).

- The marginal distribution on \(\tilde{\gamma}\) is chordal SLE_2 in \(\tilde{D}\) from \(\tilde{a}\) to \(\tilde{b}\), parametrized by \(5/4\)-dimensional Minkowski content. The SLE is generated by a standard Brownian motion \(W_t\) (and then reparametrized) and we write \(\mathcal{F}_{t}^{\text{SLE}}\) for the natural filtration of this Brownian motion, equivalently the filtration of \(\tilde{\gamma}\) run up to capacity \(t\). We let \(\tilde{\mathcal{F}}_{t}^{\text{SLE}} = \sigma\{W_{t+s} - W_t : s \geq 0\}\) be the associated future sigma algebra.

- There exists a universal \(v > 0\) such that if \(h = \lceil N^{-v} \rceil\), then here is a sequence of \(\mathcal{F}_{t}^{\text{LERW}}\)-stopping times \(\{m_k\}\) and a sequence of \(\mathcal{F}_{t}^{\text{SLE}}\)-stopping times \(\{\tau_k\}\) where \(k = 0, 1, \ldots, \lceil 2K/h \rceil\) and an event \(V\) for which the following holds.
  - If \(T_k = c_k^{-1}m_k/N^{5/4}\) and \(t_k = \Theta(\tau_k)\) is the \(5/4\)-dimensional Minkowski content of \(\tilde{\gamma}\) run up to capacity \(\tau_k\), then on the event \(V\) we have
    \[
    \max_{k \in [2K/\delta]} |T_k - t_k| \leq \varepsilon_N.
    \]
Moreover, on $V$,
\[
\max_{k \leq [2K/h]} |\text{hcap} \tilde{\gamma}_k - kh| \leq \varepsilon_N, \quad \max_{k \leq [2K/h]} |\tilde{\eta}(T_k) - \tilde{\gamma}(t_k)| \leq \varepsilon_N.
\]
(The curve $\tilde{\eta}(t)$ is defined by linear interpolation for other times than $T_k$.)

- We have $P(V) \geq 1 - \varepsilon_N$.

The coupling has a Markovian property at the stopping times that can be phrased as follows. There is a filtration $\mathcal{G}_k$ such that
\[
\mathcal{F}^{\text{SLE}}_{\tau_k} \wedge \mathcal{F}^{\text{LERW}}_{m_k} \subset \mathcal{G}_k,
\]
\[
\tilde{\mathcal{F}}^{\text{SLE}}_{\tau_k} \wedge \tilde{\mathcal{F}}^{\text{LERW}}_{m_k} \perp \mathcal{G}_k.
\]
We allow $\mathcal{G}_k$ to contain extra randomness but it must be independent of the future.

We now consider this coupling. For $r > 0$, define the stopping times
\[
\sigma_r^{\text{SLE}} = \inf \{ t \geq 0 : |\tilde{\gamma}(t)| \leq r \}, \quad \sigma_r^{\text{LERW}} = \min \{ j \geq 0 : |\tilde{\eta}_j| \leq r \}.
\]
Given $\delta, \delta' > 0$, define the stopping times
\[
\xi = \inf \{ t \geq 0 : S_t \leq \delta \text{ or } |\tilde{\gamma}(t)| \leq r \},
\]
\[
\xi' = \inf \{ t \geq \xi : \text{hcap}[\tilde{\gamma}_t] = \text{hcap}[\tilde{\gamma}_\xi] + \delta' \}.
\]
Using Lemma 3.3 and Lemma 3.4 we may choose $\delta, \delta', c'$ bounded away from 0 (depending on $r$) such that if $E$ is the event that
\[
\min_{0 \leq t \leq \xi'} S_t \geq \delta, \quad \text{dist}(0, \tilde{\gamma}_{\xi'}) \geq r/2,
\]
then,
\[
\mu^{\text{rad}}(E) \geq 1 - r.
\]
Moreover, if $N$ is sufficiently large, then $h < \delta'$. For the remainder, we only consider $N$ this large.

Let $J$ be the minimum of $[K/h]$ and the first $k$ with $t_k > \xi$. (Recall that $t_k = \Theta(\tau_k)$.) The care in the coupling was taken in order to guarantee that $J$ is a stopping time for the coupled processes. Also, $t_J < \xi'$ since $h < \delta'$. 

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We define the measure $Q^{\text{SLE}}$ on $G_J$ by the relation

$$dQ^{\text{SLE}} = M^{\text{SLE}}_{r_J} dP,$$

and let $Q_1^{\text{SLE}}, Q_2^{\text{SLE}}$ denote the induced marginal measure on $\bar{\gamma}(t), 0 \leq t \leq t_J$ and $\bar{\eta}(t), 0 \leq t \leq T_J$, respectively. By construction $Q_1^{\text{SLE}}$ is the distribution of radial SLE$_2$, parametrized naturally, stopped at half-plane capacity $\tau_J \approx Jh$.

If $K = K_r$ is chosen sufficiently large, except on an event of $Q_1^{\text{SLE}}$-probability $O(r)$,

$$\sigma_r \leq t_J \leq \sigma_r/2,$$

$$\min_{0 \leq t \leq t_J} S_t \geq \delta.$$

(This uses Lemma 3.3 and Lemma 3.4.) Call $E^{\text{SLE}}$ this event that the last two estimates hold. Note that on $E^{\text{SLE}}$, which is $G_J$ measurable, $M^{\text{SLE}}_{r_J}$ is bounded by a constant depending only on $r, \delta$, see Lemma 3.3. We similarly define $Q^{\text{LERW}}$ on $G_J$ by the relation

$$dQ^{\text{LERW}} = M^{\text{LERW}}_{r_J} dP.$$

The paths $\bar{\eta}$ under the marginal $Q_2^{\text{LERW}}$ have the distribution of scaled radial LERW parametrized naturally, stopped at $T_J$. Let $E^{\text{LERW}}$ be the event that

$$\min_{k \leq m_J} S_k^{\text{LERW}} \geq 2\delta, \quad \sigma_r^{\text{LERW}} \leq m_J \leq \sigma_r/2^{\text{LERW}}.$$

Then using Lemma 3.3 we see that

$$Q^{\text{LERW}}(E^{\text{LERW}}) \geq 1 - O(r). \quad (4.1)$$

The $Q^{\text{SLE}}$-marginal on $\bar{\eta}$ is not the LERW distribution, but almost. We will make this precise by estimating the total variation distance between $Q^{\text{SLE}}$ and $Q^{\text{LERW}}$.

**Lemma 4.1.** There are constants $c_{r,\delta}, c < \infty$ such that for $N$ sufficiently large,

$$\|Q^{\text{LERW}} - Q^{\text{SLE}}\| \leq c_{r,\delta}N + cr.$$

**Proof.** Let $U \in G_J$ be an arbitrary event. Then

$$|Q^{\text{LERW}}(U) - Q^{\text{SLE}}(U)| \leq |Q^{\text{LERW}}(U \cap V \cap E^{\text{SLE}}) - Q^{\text{SLE}}(U \cap V \cap E^{\text{SLE}})|$$

$$+ Q^{\text{SLE}}(V^c \cap E^{\text{SLE}}) + Q^{\text{SLE}}((E^{\text{SLE}})^c) + Q^{\text{LERW}}(V^c \cap E^{\text{LERW}}) + Q^{\text{LERW}}((E^{\text{LERW}})^c) + Q^{\text{LERW}}(V \cap (E^{\text{SLE}})^c)$$
We estimate the terms on the right-hand side in order. First, on the event $V \cap E_{\text{SLE}}$, the paths and Loewner chains are close and if $N$ is sufficiently large (depending on $r, \delta$) we can use Lemma 2.3 to see that on this event,

$$|M_{n_j}^{\text{LERW}} - M_{r_j}^{\text{SLE}}| \leq c_{r,\delta} \epsilon N,$$

so

$$|Q_{\text{LERW}}(U \cap V \cap E_{\text{SLE}}) - Q_{\text{SLE}}(U \cap V \cap E_{\text{SLE}})| \leq c_{r,\delta} \epsilon N.$$

Next, on the event $E_{\text{SLE}}$ we have that $M_{r_j}^{\text{SLE}}$ is bounded by a constant depending only on $r, \delta$. So using the fact that $P(V^c) \leq \epsilon N$, we get

$$Q_{\text{SLE}}(V^c \cap E_{\text{SLE}}) \leq c_{r,\delta} \epsilon N.$$

We know that

$$Q_{\text{SLE}}((E_{\text{SLE}})^c) \leq cr.$$

Similarly, using that $M_{n_j}^{\text{LERW}}$ is bounded by $c_{r,\delta}$ on $E_{\text{LERW}}$,

$$Q_{\text{LERW}}(V^c \cap (E_{\text{LERW}})) \leq c_{r,\delta} \epsilon N.$$

Using (4.1) we have

$$Q_{\text{LERW}}((E_{\text{LERW}})^c) \leq cr.$$

And finally, $E_{\text{LERW}} \cap V \subset E_{\text{SLE}} \cap V$, if $N$ is large enough so

$$Q_{\text{LERW}}(V \cap (E_{\text{SLE}})^c) \leq Q_{\text{LERW}}((E_{\text{LERW}})^c) \leq cr.$$

We conclude that

$$\|Q_{\text{LERW}} - Q_{\text{SLE}}\| \leq c_{r,\delta} \epsilon N + cr.$$

Hence if we take $N$ sufficiently large we see that the variation distance is at most $cr$, as claimed. It follows that for such $N$, the marginal of $\tilde{\eta}$ under $Q_{\text{SLE}}$ is within variation distance $cr$ of $Q_{\text{LERW}}$. We take $\tau = m_J$ and $E$ to be the event $E_{\text{SLE}}$. This completes the proof of Proposition 1.2.

5 Proof of Lemma 2.3

We recall the statement.
Lemma. Let $r, \delta > 0$ be given. For all $A, \eta$ with $\eta \in I_\delta^*$ such that
\[ s \geq rN, \quad \min_{j \leq n} S_{A_j, a_j, b} \geq \delta, \quad A_j = A \setminus \eta[0, \ldots, \eta_j], \]
we have
\[ \frac{H_{A}(0, b)}{H_{A_\eta}(0, b)} = \frac{\text{Im } F(0)}{\text{Im } F_\eta(0)} \left( 1 + O_{r, \delta}(N^{-1/20}) \right). \]
Here $F_\eta : D_{A_\eta} \to \mathbb{H}$ is the conformal map with $F_\eta(a_\eta) = 0, F_\eta(b) = \infty$ and such that $F_\eta \circ F^{-1}(z) = z + o(1)$ as $z \to \infty$. The error estimate is uniform over $A, \eta$ satisfying (5.1).

Proof. We start by considering the smaller domain $A_\eta = A \setminus \eta$. By a last exit decomposition we have $4H_{A_\eta}(0, b) = G_{A_\eta}(0, b^*)$. We will write $G_\eta = G_{A_\eta}$. Let $\psi : D_A \to \mathbb{D}, \psi(0) = 0, \psi'(0) > 0$ and $\psi_\eta : D_{A_\eta} \to \mathbb{D}$ with $\psi_\eta(0) = 0, \psi'_\eta(0) > 0$. Write $\zeta = \psi(b)$. It is not hard to show using, e.g., the Loewner equation, that the condition on the sine in (5.1) implies that there is a constant $c_\delta > 0$ such that uniformizing “Loewner map” satisfies $|\psi_\eta \circ \psi^{-1}(\zeta)| \geq c_\delta$. Define
\[ A^* = \{ z \in A : g_\eta(0, z) \geq N^{-1/16} \}, \]
where $g_\eta(z, w)$ is the Green’s function for Brownian motion in the simply connected domain $D_{A_\eta}$. We let
\[ Q = Q(b) = \{ z \in A_\eta : |\psi_\eta(z) - \psi_\eta(b)| \leq c_0 N^{-1/16} \log N \} \cap (A_\eta \setminus A^*), \]
where we will specify $c_0 > 0$ in a moment. Consider simple random walk $S$ and its exit time of $Q$:
\[ \sigma = \min\{ j \geq 0 : S_j \in Q^c \}. \]
We write $Q_{\text{top}} = \partial Q \cap A^*, Q_{\text{sides}} = \partial Q \cap A_\eta$. By Lemma 3.11 of [3] it is possible to choose $c_0 < \infty$ depending only on $r, \delta$ so that
\[ \frac{H_Q(b^*, Q_{\text{sides}})}{H_Q(b^*, Q_{\text{top}})} \leq N^{-2} \tag{5.2} \]
for $N$ sufficiently large and we assume $c_0$ is chosen in this way and that $N$ is large enough that (5.2) holds.

By the strong Markov property,
\[ G_\eta(0, b^*) = \sum_{z \in Q_{\text{top}}} G_\eta(z, 0) P^{b^*} \{ S_\sigma = z \} + \sum_{z \in Q_{\text{sides}}} G_\eta(z, 0) P^{b^*} \{ S_\sigma = z \}. \]
By a rough bound on the Green’s function, there is a constant $c$ such that
\[
\sum_{z \in Q_{\text{top}}} G_{\eta}(z,0) P^{b^*} \{ S_{\sigma} = z \} \leq c H_Q(b^*, Q_{\text{top}}).
\]

When $z \in A^*$ we can apply Theorem 1.2 of [3] to see that
\[
G_{\eta}(0,z) = (2/\pi) g_{\eta}(0,z) [1 + O_{r,\delta}(N^{-1/4})].
\]

Hence,
\[
\sum_{z \in Q_{\text{top}}} G_k(0,z) P^{b^*} \{ S_{\sigma} = z \} \geq c_{r,\delta} N^{-1/16} H_Q(b^*, Q_{\text{top}}).
\]

Consequently, using (5.2)
\[
G_{\eta}(0,b^*) = \sum_{z \in Q_{\text{top}}} G_{\eta}(z,0) P^{b^*} \{ S_{\sigma} = z \} \left[ 1 + O_{r,\delta}(N^{-1}) \right].
\]

Let $\hat{g} = F_\eta \circ F^{-1}$ and $w = F(0)$. Using conformal invariance we have the formula
\[
g_{\eta}(0,z) = \log \left| \frac{\hat{g} \circ F(z) - \hat{g}(w)}{\hat{g} \circ F(z) - \hat{g}(w)} \right| = 2 \frac{\text{Im} F_\eta(0)}{|F(z)|^2 / \text{Im} F(z)} [1 + O((\text{Im} F(z))^{-1})],
\]

where the asymptotic expansion holds for Im $F(z)$ large and we used the normalization of $\hat{g}$ at $\infty$. For $z \in \partial Q$ we have $\text{Im} F(z) \geq C_r N^{1/16}/(\log N)^2$ and so by (5.3)
\[
G_{\eta}(0,b^*) = \sum_{z \in Q_{\text{top}}} G_{\eta}(z,0) P^{b^*} \{ S_{\sigma} = z \} \left[ 1 + O_{r,\delta}(N^{-1/4}) \right]
\]
\[
= (4/\pi) \sum_{z \in Q_{\text{top}}} \frac{\text{Im} F_\eta(0)}{|F(z)|^2 / \text{Im} F(z)} P^{b^*} \{ S_{\sigma} = z \} [1 + O_{r,\delta}(N^{-1/20})].
\]

We now consider the larger domain $A \supset A_\eta$. We keep the sets $Q, Q_{\text{top}}, Q_{\text{top}}$ which are all contained in $A$ or in $\partial A$ for $N$ sufficiently large. We carry out the same argument as for $A_\eta$. The estimates (5.3) and (5.2) hold with $A_\eta$ replaced by $A$; the former using monotonicity of the Green’s function and the latter holds by the assumption (5.1). The conclusion is that
\[
G_A(0,b^*) = \sum_{z \in Q_{\text{top}}} G_A(z,0) P^{b^*} \{ S_{\sigma} = z \} \left[ 1 + O_{r,\delta}(N^{-1/4}) \right]
\]
\[
= (4/\pi) \sum_{z \in Q_{\text{top}}} \frac{\text{Im} F(0)}{|F(z)|^2 / \text{Im} F(z)} P^{b^*} \{ S_{\sigma} = z \} [1 + O_{r,\delta}(N^{-1/20})].
\]

We divide this expression with the one for $G_{\eta}(0,b^*)$ to see that
\[
\frac{G_A(0,b^*)}{G_{\eta}(0,b^*)} = \frac{\text{Im} F(0)}{\text{Im} F_\eta(0)} [1 + O_{r,\delta}(N^{-1/20})],
\]

which, using the last-exit decomposition, is what we wanted to prove. \qed
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