Novel Bounds for the Normalized Laplacian Estrada and Normalized Energy Index of Graphs

Gian Paolo Clemente\textsuperscript{a}, Alessandra Cornaro\textsuperscript{a}

\textsuperscript{a}Department of Mathematics and Econometrics, Catholic University, Milan, Italy
gianpaolo.clemente@unicatt.it, alessandra.cornaro@unicatt.it

Abstract

For a simple and connected graph, several lower and upper bounds of graph invariants expressed in terms of the eigenvalues of the normalized Laplacian matrix have been proposed in literature. In this paper, through a unified approach based on majorization techniques, we provide some novel inequalities depending on additional information on the localization of the eigenvalues of the normalized Laplacian matrix. Some numerical examples show how sharper results can be obtained with respect to those existing in literature.

Keywords: majorization; graphs; normalized Laplacian energy; normalized Laplacian Estrada index; Randić index.
1 Introduction

In literature, several topological indices, related to the structural properties of graphs, have been widely explored. We focus here on the normalized Laplacian Estrada index (see [20] and [21]) and the normalized Laplacian energy index (see [11]), that are based on a particular matrix associated with a graph, called the normalized Laplacian matrix. Properties about the spectrum of this matrix and its relationship to the Randić index have been investigated in several works (see [10], [11], [12] and [17]). In this paper we use a powerful methodology that relies on majorization techniques (see [1], [3], [4] and [6]) in order to localize the graph topological indices we consider. In particular, through this technique, we derive new bounds for these indices taking advantage of additional information on the localization of the eigenvalues of normalized Laplacian matrix. Furthermore, this additional information can be quantified by using numerical approaches developed in [13] and [15] and extended for normalized Laplacian matrix in [14]. Finally, some existing bounds (see [20] and [21]), depending on well-known inequalities on Randić index, have been also improved by using some novel results proposed in [5].

The paper is organized as follows: in Section 2 some preliminaries are given. In Section 3 we provide, through majorization techniques, new bounds for topological indices expressed in terms of the eigenvalues of the normalized Laplacian matrix and we also recover in a straightforward way some results proposed in [21]. The relation between normalized Laplacian Estrada index and Randić index has been used in Section 4 to obtain new inequalities on normalized Laplacian Estrada index. Finally, in Section 5 several numerical results are reported, showing how the proposed bounds are tighter than those given in literature.

2 Notations and Preliminaries

2.1 Basic graph concepts

We consider a simple, connected and undirected graph \( G = (V, E) \) where \( V = \{1, 2, \ldots, n\} \) is the set of vertices and \( E \subseteq V \times V \) the set of edges, \( |E| = m \).

The degree sequence of \( G \) is denoted by \( \pi = (d_1, d_2, \ldots, d_n) \) and it is arranged in non-increasing order \( d_1 \geq d_2 \geq \cdots \geq d_n \), where \( d_i \) is the degree of vertex \( i \).

It is well known that \( \sum_{i=1}^{n} d_i = 2m \) and that if \( G \) is a tree, i.e. a connected graph without
cycles, \( m = n - 1 \).

Let \( A(G) \) be the adjacency matrix of \( G \) and \( D(G) \) be the diagonal matrix of vertex degrees. The matrix \( L(G) = D(G) - A(G) \) is called Laplacian matrix of \( G \), while \( \mathcal{L}(G) = D(G)^{-1/2} L(G) D(G)^{-1/2} \) is known as normalized Laplacian. Let \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \), \( \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \) and \( \gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_n \) be the set of (real) eigenvalues of \( A(G) \), \( L(G) \) and \( \mathcal{L}(G) \) respectively.

We now recall some properties of normalized Laplacian eigenvalues useful for our purpose. For more details we refer the reader to [10], [12] and [17].

**Lemma 1. (see [12])**

Given a connected graph \( G \) of order \( n \geq 2 \), the following properties of the spectrum of \( \mathcal{L}(G) \) hold:

1. \( \sum_{i=1}^{n} \gamma_i = \text{tr}(\mathcal{L}(G)) = n \);

2. \( \sum_{i=1}^{n} \gamma_i^2 = \text{tr}(\mathcal{L}^2(G)) = n + 2 \sum_{(i,j) \in E} \frac{1}{d_id_j} \);

3. \( \frac{n}{n-1} \leq \gamma_1 \leq 2 \). The left inequality is attained if and only if \( G \) is a complete graph, while the right inequality holds when \( G \) is a bipartite graph;

4. \( \gamma_n = 0 \), \( \gamma_{n-1} \neq 0 \) if \( G \) is connected.

### 2.2 Normalized Laplacian indices

The normalized Laplacian Estrada index has been proposed in [21] and it is defined as:

\[
NEE(G) = \sum_{i=1}^{n} e^{(\gamma_i - 1)} = \frac{1}{e} \sum_{i=1}^{n} e^{\gamma_i}.
\]

(1)

In [20], an alternative definition of normalized Laplacian Estrada index has been provided:

\[
\ell EE(G) = \sum_{i=1}^{n} e^{\gamma_i}.
\]

(2)

Notice that \( NEE(G) = \frac{1}{e} \ell EE(G) \), any results derived for \( NEE(G) \) can be trivially re-stated for \( \ell EE(G) \) and viceversa.

Another graph invariant, introduced in [11], is the normalized Laplacian energy index of a graph denoted by:

\[
NE(G) = \sum_{i=1}^{n} |\gamma_i - 1|.
\]

(3)
2.3 Randić index and Majorization techniques

The Randić index is defined as:

$$R(G) = \sum_{(i,j) \in E} \left( \frac{1}{d_i d_j} \right) ,$$

and it can be equivalently expressed as:

$$R(G) = \frac{1}{2} \left( \sum_{(i,j) \in E} \left( \frac{1}{d_i} + \frac{1}{d_j} \right)^2 - \sum_{i=1}^{n} \frac{1}{d_i} \right) .$$

Given a fixed degree sequence $\pi$, let $x \in \mathbb{R}^m$ be the vector whose components are $\frac{1}{d_i} + \frac{1}{d_j}$, with $(i,j) \in E$.

Since $\sum_{i=1}^{m} x_i = \sum_{(i,j) \in E} \left( \frac{1}{d_i} + \frac{1}{d_j} \right) = n$, let $\Sigma_n = \{ x \in \mathbb{R}^m : \sum_{i=1}^{m} x_i = n, x_1 \geq x_2 \geq \cdots \geq x_m \}$. By considering a closed subset $S$ of $\Sigma_n$ whose maximal and minimal elements with respect to the majorization order are $x^*(S)$ and $x_*(S)$, the Randić index can be bounded as follows (see (5) in [5]):

$$L_1 = \frac{\|x_*(S)\|_2^2 - \sum_{i=1}^{n} \frac{1}{d_i}}{2} \leq R(G) \leq \frac{\|x^*(S)\|_2^2 - \sum_{i=1}^{n} \frac{1}{d_i}}{2} = U_1. \quad (4)$$

Inequalities (4) will be used in Section 4 in order to derive new bounds for $N\text{EE}(G)$.

Using the information available on the degree sequence of $G$ and characterizing the set $S$, the minimal and maximal elements $x^*(S)$ and $x_*(S)$ can be easily computed.

In this paper, we focus on a specific case of a graph $G$ with $h$ pendant vertices, whose degree sequence is of the type

$$\pi = (d_1, \cdots, d_{n-h}, 1, \cdots, 1), \quad (5)$$

where $h > 0$ and $n - h \geq 2$ (we do not consider the star graph $S_n$ since it is well-known that $R_1(S_n) = 1$).

It is noteworthy that this method could be applied to other suitable degree sequences.

Pointing out that $\frac{1}{d_{n-h}} + \frac{1}{d_{n-h-1}} < 1 + \frac{1}{d_1}$ holds, we face the set

$$S_1 = \left\{ x \in \mathbb{R}^m : \sum_{i=1}^{m} x_i = n, \quad 1 + \frac{1}{d_1} \leq x_h \leq \cdots \leq x_1 \leq \frac{1}{d_{n-h}} + 1 , \quad \frac{1}{d_1} + \frac{1}{d_2} \leq x_m \leq \cdots \leq x_{h+1} \leq \frac{1}{d_{n-h}} + \frac{1}{d_{n-h-1}} \right\}. \quad (6)$$
For convenience of the reader, we report the expressions of the maximal and minimal elements of $S_1$.

The maximal element is derived by means of Corollary 3 in [5] as follows:

$$x^*(S_1) = \begin{cases} \left[ \begin{array}{c} M_1, \ldots, M_1, \theta, m_1, \ldots, m_1, m_2, \ldots, m_2 \\ k \\ h-k-1 \\ m-h \end{array} \right] & \text{if } n < a^* \\ \left[ \begin{array}{c} M_1, \ldots, M_1, M_2, \ldots, M_2, \theta, m_2, \ldots, m_2 \\ h \end{array} \right] & \text{if } n \geq a^* \end{cases}$$

(7)

where

$$k = \begin{cases} \left\lfloor \frac{n - h(m_1 - m_2) - mm_2}{M_1 - m_1} \right\rfloor & \text{if } n < a^* \\ \left\lfloor \frac{n - h(M_1 - M_2) - mm_2}{M_2 - m_2} \right\rfloor & \text{if } n \geq a^* \end{cases}$$

and $a^* = hM_1 + (m-h)m_2$, $m_1 = 1 + \frac{1}{d_1}$, $m_2 = 1 + \frac{1}{d_2}$, $M_1 = 1 + \frac{1}{d_{n-h}}$, $M_2 = 1 + \frac{1}{d_{n-h-1}}$, and $\theta$ is obtained as the difference between $n$ and the sum of the other components of the vector $x^*(S_1)$.

The minimal element is instead obtained by Corollary 10 in [5] as follows:

$$x_*(S_1) = \begin{cases} \left[ \begin{array}{c} m_1, \ldots, m_1, \frac{n - hm_1}{m-h}, \ldots, \frac{n - hm_1}{m-h} \\ k \\ h \\ m-h \end{array} \right] & \text{if } n < \tilde{a} \\ \left[ \begin{array}{c} n - M_2(m-h), \ldots, n - M_2(m-h) \\ \frac{1}{h} \\ h \\ m-h \end{array} \right] & \text{if } n \geq \tilde{a}, \end{cases}$$

(8)

where $\tilde{a} = hm_1 + (m-h)M_2$ and $m_1, M_2$ have the same meaning of before.

3 Bounds for normalized Laplacian indices via majorization techniques

In this section we provide bounds for normalized Laplacian Estrada index and normalized Laplacian energy index. These descriptors can be expressed in terms of Schur-convex or Schur-concave functions of suitable variables. We briefly recall that Schur-convex (Schur-concave) functions preserve (reverse) the majorization order (see [22] for details).
3.1 Normalized Laplacian Estrada index

Firstly, we focus on $NEE(G)$. Let us consider the set

$$S_0 = \{ \gamma \in \mathbb{R}^{n-1} : \sum_{i=1}^{n-1} \gamma_i = n, \, \gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_{n-2} \geq \gamma_{n-1} \geq 0 \}.$$ 

We can now consider a subset $S_0^1$ of $S_0$:

$$S_0^1 = \{ \gamma \in S_0 : \gamma_1 \geq \alpha \},$$

with $\alpha \geq \frac{n}{n-1}$.

In order to compute the minimal element of $S_0^1$, we apply Corollary 14 in [6] and we obtain:

$$x_*(S_0^1) = \left( \frac{\alpha, n-\alpha, \ldots, n-\alpha}{n-2, \ldots, n-2} \right).$$

By the Schur-convexity of the function $NEE(G)$, we get the following bound:

$$NEE(G) \geq \frac{1}{e} + e^{\alpha-1} + (n - 2)e^{\frac{2\alpha}{n-2}}.$$ 

(9)

Setting $\alpha = \frac{n}{n-1}$, we can easily derive the same result proved in [21], Theorem 3.1:

$$NEE(G) \geq (n - 1) \frac{1}{e} - \frac{1}{e}.$$ 

(10)

Furthermore, by applying a theoretical and numerical methodology (see [7] and [14]), it is possible to compute a different lower bound $\alpha$ for the first eigenvalue of $\gamma_1$ in a fairly straightforward way, that is $\gamma_1 \geq Q$, where

$$Q = \left( n + \sqrt{\frac{b(h^*+1)-n^2}{h^*}} \right) \left( 1 + h^* \right),$$

with $b = n + 2 \sum_{(i,j) \in E} \frac{1}{d_i d_j}$ and $h^* = \left\lfloor \frac{n^2}{b} \right\rfloor$.

It is well-known that, for every connected graph of order $n$:

$$\left( \frac{2}{n} \right) \sum_{(i,j) \in E} \frac{1}{d_i d_j} \geq \frac{1}{n-1},$$

(11)

with inequality attained when $G \cong K_n$ (see [2]). It has been shown in [14] that $Q \geq \frac{n}{n-1}$ and thus we assure that bound (9), by placing $\alpha = Q$, is sharper than (10) (see [5] and [6] for more theoretical details).

6
We can further improve bound (9) by identifying additional information on \( \gamma_2 \). In this case we face the set:

\[
S_0^2 = \{ \gamma \in S_0 : \gamma_1 \geq \alpha , \gamma_2 \geq \beta \}.
\]

Under the assumptions \( \alpha \geq \beta \) and \( \alpha + \beta (n - 2) > n \), by Corollary 14 in [6], the minimal element of \( S_0^2 \) with respect to the majorization order is given by

\[
x_s(S_0^2) = \left( \begin{array}{c}
\alpha, \beta, \frac{n - \alpha - \beta}{n - 3}, \ldots, \frac{n - \alpha - \beta}{n - 3} \end{array} \right)
\]

and we can provide the following bound:

\[
NEE(G) \geq \frac{1}{e} + e^{\alpha - 1} + e^{\beta - 1} + (n - 3)e^{\frac{3 - \alpha - \beta}{n - 3}}.
\]

In [14] the authors found a lower bound \( \beta \) for \( \gamma_2 \), that is \( \gamma_2 \geq R \), where:

\[
R = \frac{n - \sqrt{(n - 2)^2 - 4(n - 1)}}{n - 2}.
\]

They proved that \( R \leq Q \) and numerically showed, for some classes of graphs, that \( Q + R(n - 2) > n \), satisfying the conditions underlying Corollary 14 in [6].

In virtue of these relations it is possible to compute bound (12) that is tighter than (9) with \( \alpha = Q \) and (10) (see [5] and [6] for more theoretical details).

Finally, for bipartite graphs, it is well-known that \( \gamma_1 = 2 \). Hence

\[
S_0^b = \{ \gamma \in \mathbb{R}^{n-2} : \sum_{i=2}^{n-1} \gamma_i = n - 2, \ 2 \geq \gamma_2 \geq \ldots \geq \gamma_{n-2} \geq \gamma_{n-1} \geq 0 \}.
\]

By applying Corollary 14 in [6], we recover the following bound provided in [21], Theorem 3.2:

\[
NEE(G) \geq \frac{1}{e} + e + (n - 2).
\]

Also in this case, we can improve this bound by identifying additional information on \( \gamma_2 \). We face the set:

\[
S_0^{2b} = \{ \gamma \in S_0^b : \gamma_2 \geq \beta \},
\]

under the assumption \( 1 < \beta \leq 2 \). By Corollary 14 in [6], the minimal element of \( S_0^2 \) with respect of majorization order is given by

\[
x_s(S_0^{2b}) = \left( \begin{array}{c}
\beta, \frac{n - 2 - \beta}{n - 3}, \ldots, \frac{n - 2 - \beta}{n - 3} \end{array} \right)
\]
and we can provide the following bound:

\[ \text{NEE}(G) \geq \frac{1}{e} + e^{\beta - 1} + (n - 3)e^{\frac{1}{n-\alpha}}, \]  

(14)

where the lower bound \( \beta = R \) of \( \gamma_2 \) derived in [14] can be also used to compute (14).

In analogy with the results (9) and (12) on \( \text{NEE}(G) \), we can easily derive the following bounds for \( \ell \text{EE}(G) \) for connected non bipartite graphs:

\[ \ell \text{EE}(G) \geq 1 + e^\alpha + (n - 2)e^{\frac{\alpha - \beta}{n-\alpha}}, \]  

(15)

and

\[ \ell \text{EE}(G) \geq e^\alpha + e^\beta + (n - 3)e^{\frac{n - \alpha - \beta}{n-\alpha}}, \]  

(16)

with \( \gamma_1 \geq \alpha, \gamma_2 \geq \beta \) and \( \alpha + \beta(n - 2) > n \).

In Section 5 we will compare these bounds with those proposed in [20] and [21].

### 3.2 Normalized Laplacian energy index

The normalized Laplacian energy index \( \text{NE}(G) \) can be rewritten as a Schur-concave function of the variables \( (\gamma_i - 1)^2, i = 1, \ldots, n \):

\[ \text{NE}(G) = 1 + \sum_{i=1}^{n-1} \sqrt{(\gamma_i - 1)^2}. \]  

(17)

If a lower bound for \( \gamma_1 \) is available, i.e. \( \gamma_1 \geq \alpha \left( \geq \frac{n}{n - 1} \right) \), introducing the new variables \( x_i = (\gamma_i - 1)^2 \) as a function of the eigenvalue \( \gamma_i \) arranged in nonincreasing order, we get:

\[ x_1 \geq k_1 = (\alpha - 1)^2. \]

Let us consider the set

\[ S_{NE} = \{ x \in \mathbb{R}^{n-1} : \sum_{i=1}^{n-1} x_i = 2 \sum_{(i,j)\in E} \frac{1}{d_id_j} - 1, x_1 \geq k_1 \}, \]

where the relation \( \sum_{i=1}^{n-1} x_i = 2 \sum_{(i,j)\in E} \frac{1}{d_id_j} - 1 \) has been obtained by using properties recalled in Lemma 1.

With the same methodology described for \( \text{NEE}(G) \), we can derive the minimal element of \( S_{NE} \) and then the following upper bound:

\[ \text{NE}(G) \leq 1 + \sqrt{k_1} + \sqrt{(n - 2)(n - k_1)}, \]  

(18)
with \( a = 2 \sum_{(i,j) \in E} \frac{1}{d_id_j} - 1 \). This bound could be computed by placing \( k_1 = (Q - 1)^2 \).

Considering also an additional information on \( \gamma_2 \) (i.e. \( \gamma_2 \geq \beta \)), we may face the set:

\[
S^2_{NE} = \{ x \in S_{NE} : x_2 \geq k_2 \}
\]

under the assumptions \( \alpha \geq \beta \) and \( \alpha + \beta(n - 2) > a \).

In this case, by means of the minimal element of \( S^2_{NE} \), we can provide the bound:

\[
NE(G) \leq 1 + \sqrt{k_1} + \sqrt{k_2} + \sqrt{(n - 3)(a - k_1 - k_2)},
\]

where we can place \( k_1 = (Q - 1)^2 \) and \( k_2 = (R - 1)^2 \).

Finally, for bipartite graphs, taking into account that \( \gamma_1 = 2 \), we set:

\[
S^b_{NE} = \{ x \in \mathbb{R}^{n-2} : \frac{1}{2} \sum_{i=2}^{n-1} x_i = 2 \sum_{(i,j) \in E} \frac{1}{d_id_j} - 2 \},
\]

and we derive the bound:

\[
NE(G) \leq 2 + \sqrt{a(n - 2)},
\]

where \( a = 2 \sum_{(i,j) \in E} \frac{1}{d_id_j} - 2 \).

Also in this case, we can improve this result by identifying additional information on \( \gamma_2 \). We face the set:

\[
S^{2b}_{NE} = \{ x \in S^b_{NE} : x_2 \geq k_2 \}
\]

under the assumption \( \frac{a - 2}{n - 2} < \beta \leq 2 \) and we can provide the bound:

\[
NE(G) \leq 2 + \sqrt{k_2} + \sqrt{(n - 3)(a - k_2)},
\]

where the information \( k_2 = (R - 1)^2 \) can be used to compute (21).

### 4 Bounds through Randić Index

In Theorem 3.4 and Theorem 3.5 in [21], the authors provided lower and upper bounds for \( NEE(G) \) of a (bipartite) graph in terms of \( n \) and maximum (or minimum) degree.

This result has been obtained through well-known inequalities on Randić index (see [23]), i.e.

\[
\frac{n}{2d_1} \leq R_{-1}(G) \leq \frac{n}{2d_n}.
\]

Following this idea, we now deduce some bounds for \( NEE(G) \) and its variant \( \ell EE(G) \) by using the methodology based on majorization recalled in Section 2.3. In Section 5.2 we will numerically show that the bounds obtained are tighter than those provided in [20] and [21].
In virtue of (4) and by means of Theorem 3.4 in [21], we easily get the following result for bipartite graph:

**Proposition 1.** Let $G$ be a simple, connected and bipartite graph of order $n$. Then the normalized Laplacian Estrada index of $G$ is bounded as:

$$\frac{1}{e} + e + \sqrt{(n - 2)^2 + 4(L_1 - 1)} \leq \text{NEE}(G) \leq \frac{1}{e} + e + (n - 3) - \sqrt{2(U_1 - 1) + e^{2(U_1 - 1)}}. \quad (22)$$

In the same way as before, by Theorem 3.5 in [21] we have the following bounds for non-bipartite graphs:

**Proposition 2.** Let $G$ be a simple and connected graph of order $n$. Then the normalized Laplacian Estrada index of $G$ is bounded as follows:

$$\sqrt{(n - 1)(1 + (n - 2)e^{\frac{2}{n - 1}}) + 4L_1} \leq \text{NEE}(G) \leq \frac{1}{e} + (n - 1) - \sqrt{2U_1 - 1} + e^{2U_1 - 1}. \quad (23)$$

Notice that, replacing $L_1 = \frac{n}{2d_1}$ and $U_1 = \frac{n}{2d_n}$, we recover the same bounds provided in [21], Theorem 3.4 and 3.5.

Bounds (22) and (23) can be trivially derived for $\ell EE(G)$ by using the proportionality relationship with $\text{NEE}(G)$. For the comparisons provided in Section 5.2, we only report the bound obtained for non-bipartite graph:

$$\sqrt{(n - 1)(e + (n - 2)e^{\frac{n+1}{n-1}}) + 4eL_1} \leq \ell EE(G) \leq 1 + e \left[ (n - 1) - \sqrt{2U_1 - 1} \right] + e^{2U_1}. \quad (24)$$

## 5 Numerical Results

### 5.1 Comparing Bounds derived via majorization techniques

#### 5.1.1 Normalized Laplacian Estrada index

Firstly, we focus on $\text{NEE}(G)$ by comparing for non-bipartite graphs bounds (9) and (12) with (10) proposed in [21]. It has been already analytically proved in Section 3.1 that, when the additional information $\gamma_1 \geq Q$ is considered, bound (10) with $\alpha = Q$ is tighter than (11). We now show how these bounds behave according to different graphs. In particular we analyze two alternative classes of graphs generated by using either the Erdős-Rényi (ER) model $G_{ER}(n, q)$ (see [9], [12], [18] and [19]) or the Watts and Strogatz (WS) model (see [21]). Both models have been generated by using a well-known package of
R (see [16]) and by assuring that the graph obtained is connected. The ER is constructed by connecting nodes randomly such that edges are included with probability \( q \) independent from every other edge. The WS networks have been derived beginning by a simulated \( n \)-node lattice and rewiring each edge at random to a new target node with probability \( p \). As described by [24], we choose a vertex and the edge that connects it to its nearest neighbor in a clockwise sense. With probability \( p \), we reconnect this edge to a vertex chosen uniformly at random over the entire ring, with duplicate edges forbidden; otherwise we leave the edge in place. We repeat this process by moving clockwise around the ring, considering each vertex in turn until one lap is completed. Next, we consider the edges that connect vertices to their second-nearest neighbors clockwise. As before, we randomly rewire each of these edges with probability \( p \) and continue this process, circulating around the ring and proceeding outward to more distant neighbors after each lap, until each edge in the original lattice has been considered once. This construction allows to analyze the behavior of networks between regularity (\( p = 0 \)) and disorder (\( p = 1 \)).

In Table 1 we report the \( NEE(G) \) index and the values of the three mentioned bounds evaluated on non-bipartite graphs generated by using ER model with different number of vertices and with \( q \) equal to 0.5. Relative errors \( r \) measures the absolute value of the difference between the lower bounds and \( NEE(G) \) divided by the value of \( NEE(G) \).

| \( n \) | \( NEE(G) \) | Bound (10) | Bound (9) | Bound (12) | \( r(10) \) | \( r(9) \) | \( r(12) \) |
|---|---|---|---|---|---|---|---|
| 4 | 5.0862 | 4.5547 | 4.6783 | 4.7112 | 10.4488% | 8.0184% | 7.3777% |
| 5 | 6.6073 | 6.1449 | 6.6088 | 6.6265 | 7.2160% | 5.2875% | 4.7463% |
| 6 | 8.3966 | 7.9560 | 7.5345 | 7.5559 | 12.2451% | 11.2235% | 11.0709% |
| 7 | 10.0295 | 9.4331 | 9.4456 | 9.4544 | 5.9460% | 5.8219% | 5.7365% |
| 8 | 11.6328 | 10.5263 | 10.4334 | 10.4391 | 10.3580% | 8.9748% | 8.9036% |
| 9 | 13.2247 | 12.0603 | 11.9674 | 11.9728 | 7.5780% | 7.5019% | 7.4315% |
| 10 | 14.9027 | 13.7266 | 13.6334 | 13.6391 | 7.1768% | 7.0548% | 7.0026% |
| 20 | 29.9226 | 28.9383 | 28.9087 | 28.9134 | 2.5365% | 2.4978% | 2.4712% |
| 30 | 39.9441 | 38.9607 | 38.9311 | 38.9358 | 1.7963% | 1.7940% | 1.7919% |
| 50 | 59.9236 | 58.9392 | 58.9196 | 58.9243 | 0.5225% | 0.5224% | 0.5223% |
| 100 | 109.9001 | 108.9162 | 108.9066 | 108.9113 | 0.5225% | 0.5224% | 0.5223% |

Table 1: Lower bounds for \( NEE(G) \) and relative errors for graphs generated by \( ER(n, 0.5) \) model.

As expected, using bound (12) we observe an improvement with respect to existing bound according to all the analyzed graphs. The improvement is very significant for graphs with a small number of vertices, while it reduces for very large graphs. However, for large graphs formula (10) provided in [21] already gives a very low relative error.

The comparison has been extended in order to test the behaviour of the bounds on alternative graphs generated by using always the ER model with a different probability \( q \). For sake of simplicity we report only the relative errors derived for graphs generated
by using respectively $q = 0.1$ and $q = 0.9$ (see Table 2). In all cases bound (12) assures the best approximation to $\text{NEE}(G)$. We observe a best behaviour of all bounds when $q = 0.9$ because we are moving towards the complete graph. We have indeed that the density of the graphs increases as long as greater probabilities are considered.

| $n$ | $\text{NEE}(G)$ bound (10) | $\text{NEE}(G)$ bound (9) | $\text{NEE}(G)$ bound (12) | $\text{NEE}(G)$ bound (10) | $\text{NEE}(G)$ bound (9) | $\text{NEE}(G)$ bound (12) |
|-----|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| 4   | 5.3414                      | 4.5547                      | 4.5547                      | 0                           | 0                           | 0                           |
| 5   | 6.7503                      | 5.6886                      | 5.6886                      | 2.9935                      | 2.5856                      | 2.5856                      |
| 6   | 7.7271                      | 6.7503                      | 6.7503                      | 6.7503                      | 6.7503                      | 6.7503                      |
| 7   | 8.6097                      | 7.7271                      | 7.7271                      | 7.7271                      | 7.7271                      | 7.7271                      |
| 8   | 9.5638                      | 8.6097                      | 8.6097                      | 8.6097                      | 8.6097                      | 8.6097                      |
| 9   | 10.4988                     | 9.5638                      | 9.5638                      | 9.5638                      | 9.5638                      | 9.5638                      |
| 10  | 11.4394                     | 10.4988                     | 10.4988                     | 10.4988                     | 10.4988                     | 10.4988                     |
| 20  | 20.3947                     | 19.3608                     | 19.3608                     | 19.3608                     | 19.3608                     | 19.3608                     |
| 30  | 30.3853                     | 29.3515                     | 29.3515                     | 29.3515                     | 29.3515                     | 29.3515                     |
| 50  | 50.4424                     | 49.4086                     | 49.4086                     | 49.4086                     | 49.4086                     | 49.4086                     |
| 100 | 100.4293                    | 99.3955                    | 99.3955                    | 99.3955                    | 99.3955                    | 99.3955                    |

Table 2: Lower bounds for $\text{NEE}(G)$ and relative errors for graphs generated respectively by $\text{ER}(n, 0.1)$ and $\text{ER}(n, 0.9)$ models.

Finally, graphs have been simulated by using WS model with different rewiring probabilities $p$. As well-known, intermediate values of $p$ result in small-world networks that share properties of both regular and random graphs. In [24], the authors show that these networks have small mean path lengths and high clustering coefficients. There is indeed a broad interval of $p$ over which the average path is almost as small as random yet the clustering coefficient is significantly greater than random. These small-world networks result from the immediate drop in average path caused by the introduction of few long-range edges. In particular, we analyze the behaviour of bounds in this interval by considering graphs generated with a rewiring probability in the range $p \in (0.01, 0.1)$. At this regard, Table 3 reports bounds evaluated by considering $p = 0.1$. In this case, we observe greater relative errors especially for large graphs. Probably, being these networks very far from complete graphs, bounds tend to assure a weaker approximation. Similar results have been obtained by simulating WS graphs choosing different values of $p$ that belong to the interval.

| $n$ | $\text{NEE}(G)$ bound (10) | $\text{NEE}(G)$ bound (9) | $\text{NEE}(G)$ bound (12) | $\text{NEE}(G)$ bound (10) | $\text{NEE}(G)$ bound (9) | $\text{NEE}(G)$ bound (12) |
|-----|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| 4   | 5.3862                      | 4.8763                      | 4.8763                      | 10.4488                     | 8.0184                     | 7.3717                     |
| 5   | 6.7499                      | 6.2402                      | 6.2402                      | 15.0165                     | 13.6565                    | 12.9815                    |
| 6   | 7.7591                      | 7.1500                      | 7.1500                      | 15.0231                     | 13.6565                    | 12.9815                    |
| 7   | 8.7277                      | 8.3493                      | 8.3493                      | 16.0342                     | 14.6866                    | 13.9815                    |
| 8   | 9.7046                      | 8.8776                      | 8.8776                      | 16.0342                     | 14.6866                    | 13.9815                    |
| 9   | 10.6229                     | 9.3431                      | 9.3431                      | 17.0615                     | 15.9095                    | 15.2271                    |
| 10  | 11.4988                     | 10.4988                     | 10.4988                     | 18.0955                     | 16.9095                    | 15.2271                    |
| 20  | 20.3947                     | 19.9267                     | 19.9267                     | 19.1634                     | 18.9267                    | 17.3134                    |
| 30  | 30.3853                     | 29.9267                     | 29.9267                     | 20.1634                     | 19.9267                    | 18.3134                    |
| 50  | 50.4424                     | 49.9267                     | 49.9267                     | 21.1634                     | 20.9267                    | 19.3134                    |
| 100 | 100.4293                    | 99.9267                    | 99.9267                    | 22.1634                     | 21.9267                    | 20.3134                    |

Table 3: Lower bounds for $\text{NEE}(G)$ and relative errors for graphs generated by $\text{WS}(n, 0.1)$ model.
5.1.2 Normalized Laplacian energy index

We compare here bounds proposed in Section 3.2 for $NE(G)$ with the following upper bounds proposed in [11]:

$$NE(G) \leq 2 \left( \frac{n}{2} \right),$$ (25)

$$NE(G) \leq \sqrt{\frac{15}{28}(n + 1)}.$$ (26)

Table 4 reports main results derived for graphs generated by a $ER(n,0.5)$ model. We observe how both bounds (18) and (19) are tighter than those proposed in [11]. The improvement increases for greater number of vertices.

| n  | $NE(G)$ | bound (25) | bound (26) | bound (18) | bound (19) |
|----|---------|------------|------------|------------|------------|
| 4  | 3.00    | 3.12       | 3.65       |
| 5  | 2.55    | 2.70       | 2.91       |
| 6  | 2.12    | 2.45       | 2.62       |
| 7  | 1.81    | 2.14       | 2.37       |
| 8  | 1.52    | 2.00       | 2.27       |
| 9  | 1.24    | 1.87       | 2.14       |
| 10 | 1.04    | 1.63       | 1.93       |
| 20 | 5.01    | 6.68       | 6.96       |
| 30 | 7.31    | 8.47       | 8.96       |
| 50 | 11.97   | 13.42      | 13.93      |
| 100| 19.24   | 21.16      | 21.50      |
| 50 | 31.75   | 36.24      | 36.99      |
| 100| 63.21   | 70.22      | 70.93      |

Table 4: Upper bounds for $NE(G)$ for graphs generated by $ER(n,0.5)$ model.

Considering instead WS networks, derived as in Section 5.1.1 by assuming a rewiring probability equal to 0.1, we observe in Table 5 greater values of $NE(G)$. In this case, bound (26) gives better results than those observed for ER graphs. However it is confirmed the best approximation when bound (19) is used.

| n  | $NE(G)$ | bound (25) | bound (26) | bound (18) | bound (19) |
|----|---------|------------|------------|------------|------------|
| 4  | 3.00    | 3.12       | 3.65       |
| 5  | 2.55    | 2.70       | 2.91       |
| 6  | 2.12    | 2.45       | 2.62       |
| 7  | 1.81    | 2.14       | 2.37       |
| 8  | 1.52    | 2.00       | 2.27       |
| 9  | 1.24    | 1.87       | 2.14       |
| 10 | 1.04    | 1.63       | 1.93       |
| 20 | 5.01    | 6.68       | 6.96       |
| 30 | 7.31    | 8.47       | 8.96       |
| 50 | 11.97   | 13.42      | 13.93      |
| 100| 19.24   | 21.16      | 21.50      |
| 50 | 31.75   | 36.24      | 36.99      |
| 100| 63.21   | 70.22      | 70.93      |

Table 5: Upper bounds for $NE(G)$ for graphs generated by $WS(n,0.1)$ model.

5.2 Bounds based on Randić Index

We now consider an example based on a specific degree sequence of type (5) in order to explain the details of the procedure used to bound $NEE(G)$ via Randić Index. In the next we will extend the results to several degree sequences of type (5).
Example 1. Let us consider the class $C_{\pi}$ of graphs with the following degree sequence:

$$\pi = (7, 6, 5, 4, 4, 4, 3, 3, 3, 3, 3, 2, 2, 2, 1, 1, 1, 1)$$

We have $n = 20$, $m = 30$ and $h = 4$ pendant nodes. Since $\bar{a} > n$, the minimal element (3) is:

$$x^*(G) = \begin{bmatrix} 8 & 8 & 54 & 54 \\ 7 & 7 & 91 & 91 \\ 4 & 4 & 16 \end{bmatrix}.$$ 

Replacing these values in (4), we find $L_1 = 2.56$, while $\frac{n}{2d_1} = 1.43$.

The bounds for $NEE(G)$ are figured out in Table 6. Furthermore, in order to test how these bounds behave, the exact value of $NEE(G)$ is also needed. Having a huge number of graphs $G \in C_{\pi}$, we randomly generate one million of different graphs belonging to the class $C_{\pi}$. The average value, the minimum and maximum values of the index are also reported in Table 6.

| Reference | Bound  |
|-----------|--------|
| Theorem 3.5 of [21] | 20.12 |
| (23) | 20.23 |
| Min($NEE(G)$) | 20.51 |
| Mean($NEE(G)$) | 23.25 |
| Max($NEE(G)$) | 25.52 |

Table 6: Lower bounds for $NEE(G)$.

Considering instead the upper bound, since $a^* < n$, we compute $k = 12 > h = 4$ and we have that the maximal element (7) is:

$$x^*(G) = \begin{bmatrix} 3 & 3 & 31 & 31 & 31 \\ 2 & 2 & 42 & 42 & 42 \\ 4 & 1 & 8 & 42 & 42 \\ 16 & 91 & 91 & 17 \end{bmatrix},$$ 

leading to $U_1 = 4.96$, while $\frac{n}{2d_n} = 10$.

Upper bounds and values of $NEE(G)$ are summarized in Table 7.

---

1We estimate the total number of graphs with the degree sequence $\pi$ by using the importance sampling algorithm proposed in [8]. Authors show robust results by applying the algorithm with 100,000 trials. In this case we derive a total number of graphs equal to roughly $1.20 \cdot 10^{20}$ with a standard error of $4 \cdot 10^{17}$. However it is noteworthy that also graphs belonging to the same isomorphism class are considered in this value. For the computation of the average values in Tables 6 and 7, we take into account only graphs with a different $NEE(G)$. 

---
Finally, if we know the value of Randić Index, we can directly use it to compute (23). For example, considering a random graph \( G \in C_n \), we obtain \( R_{-1}(G) = 3.0376 \) deriving a better approximation (i.e. \( 20.27 \leq NEE(G) \leq 177.15 \)).

We now evaluate these bounds by randomly generating several degree sequences of type (5). For this aim, \( ER(n, p) \) model has been used to derive different random graphs, where we disregard graphs whose degree sequence does not belong to the set (6). The number of pendant vertices \( h \) varies according to the specific degree sequence obtained. Results have been compared to those analyzed in previous Section 5.1.1. In particular, we report in Table 8 bound (10) and bound (23) \( 1 \) proposed in [21], where bound (23) \( 1 \) has been derived by using in (23) the lower bound \( \frac{n}{2d_1} \) of \( R_{-1}(G) \). These bounds have been compared with bound (9) and bound (12) already analysed in previous section and with bound (23) \( L_1 \) and bound (23) \( R_{-1} \) evaluated by using the first left inequality of (23) and by considering respectively the value of \( L_1 \) or by assuming to know the value of Randić Index \( R_{-1}(G) \).

We further observe that bound (12) based on value of \( Q \) and \( R \) shows the tighter lower bound in all cases by allowing a best approximation respect to bounds based on inequality (23). Furthermore, when inequality (23) is considered, \( L_1 \) leads to a better bound than \( \frac{n}{2d_1} \) used in [21]. Finally, considering the exact value of Randić Index we only get a slight improvement.

### Table 7: Upper bounds for \( NEE(G) \).

| Reference | Bound |
|-----------|-------|
| Theorem 3.5 of [21] | \( 1.7 \times 10^8 \) |
| [23] | 7541.32 |
| \( \text{Min}(NEE(G)) \) | 20.51 |
| \( \text{Mean}(NEE(G)) \) | 23.25 |
| \( \text{Max}(NEE(G)) \) | 25.52 |

### Table 8: Lower bounds for \( NEE(G) \).

| \( n \) | \( m \) | \( d_1 \) | Bound (9) | Bound (12) | Bound (23) \( 1 \) | Bound (23) \( L_1 \) | Bound (23) \( R_{-1} \) |
|--------|--------|--------|-----------|------------|----------------|----------------|----------------|
| 4      | 4      | \( 1.3546 \) | 4.8966    | 4.6466     | 4.7142         | 4.2957         | 4.2440         |
| 5      | 5      | \( 6.1381 \) | 5.0342    | 5.2075     | 5.4602         | 5.6389         | 5.3288         |
| 6      | 6      | \( 7.9224 \) | 6.3479    | 6.2062     | 6.4977         | 6.6989         | 6.2151         |
| 7      | 7      | \( 9.9883 \) | 8.4560    | 8.1182     | 8.5744         | 8.8901         | 8.2693         |
| 8      | 8      | \( 9.9232 \) | 8.1428    | 8.0957     | 8.3688         | 8.5887         | 8.2018         |
| 9      | 9      | \( 1.0470 \) | 9.3333    | 9.7872     | 9.7222         | 9.8843         | 9.2432         |
| 10     | 10     | \( 1.1215 \) | 10.4256   | 10.0100    | 10.1435        | 10.4522        | 10.1071        |
| 15     | 15     | \( 1.3546 \) | 10.4256   | 10.0100    | 10.1435        | 10.4522        | 10.1071        |
| 20     | 20     | \( 1.3546 \) | 20.3941   | 20.1166    | 20.4901        | 20.4646        | 20.2346        |
| 90     | 90     | \( 2.0783 \) | 20.3941   | 20.1166    | 20.4901        | 20.4646        | 20.2346        |
| 100    | 100    | \( 1.1721 \) | 10.3759   | 10.1166    | 10.4901        | 10.4646        | 10.2346        |

On the same graphs upper bounds have been also evaluated by using the right part of
inequality (23). We observe in Table 9 a huge approximation, especially for large graphs, when we apply formula proposed in [21] based on the upper bound \( \frac{n}{2d_n} \) of Randić Index \( R_{-1}(G) \) (see bound (23)). By considering the upper bound based on \( U_1 \) we are able to improve the results, but for large graphs we derive useless bounds in this case too. We have indeed that even when we directly use the value of \( R_{-1}(G) \) we derive bounds significantly larger for graphs with a great number of vertices.

We observe in Table 9 a huge approximation, especially for large graphs, when we apply formula proposed in [21] based on the upper bound \( \frac{n}{2d_n} \) of Randić Index \( R_{-1}(G) \) (see bound (23)). By considering the upper bound based on \( U_1 \) we are able to improve the results, but for large graphs we derive useless bounds in this case too. We have indeed that even when we directly use the value of \( R_{-1}(G) \) we derive bounds significantly larger for graphs with a great number of vertices.

We observe in Table 10 how the proposed bounds significantly improve those in [20].

Considering instead the upper bounds, we compare our results with the following one in [20]:

\[
\ell EE(G) > ne, \quad (27)
\]

\[
\ell EE(G) > 2 + \sqrt{n(n-1)e^2 - 6n + 4}, \quad (28)
\]

\[
\ell EE(G) > \sqrt{n(n-1)e^2 + 4R_{-1}(G) + 5n}. \quad (29)
\]

We observe in Table 10 how the proposed bounds significantly improve those in [20].

Considering instead the upper bounds, we compare our results with the following one in [20]:

\[
\ell EE(G) < e^n + R_{-1}(G) + \frac{n}{2}(3 - n) - 1. \quad (30)
\]

As reported in Table 11 upper bound (30) allows a better approximation than (30). Also in this case, the upper bounds do not show a good behaviour for large graphs.

### Table 9: Upper bounds for \( NEE(G) \).

| \( n \) | \( m \) | \( d_n \) | \( NEE(G) \) | Bound (23) | Bound (25) | Bound (26) | Bound (27) | Bound (29) | Bound (15) |
|---|---|---|---|---|---|---|---|---|
| 4 | 3 | 4.8846 | 4.69 | 4.88 | 4.88 |
| 5 | 3 | 6.2381 | 5.67 | 7.62 | 7.62 |
| 6 | 3 | 6.8223 | 6.35 | 8.18 | 8.18 |
| 7 | 3 | 7.9613 | 10.32 | 8.18 |
| 8 | 6 | 9.2439 | 11.01 | 10.86 |
| 9 | 4 | 10.4376 | 14.15 | 13.19 |
| 10 | 8 | 11.2359 | 21.72 | 10.86 |
| 20 | 10 | 23.2674 | 407.35 | 8.15 |
| 30 | 14 | 30.8047 | 4.42E+06 | 7.62 |
| 50 | 19 | 58.8047 | 1.78E+08 | 7.62 |

### Table 10: Lower bounds for \( \ell EE(G) \).

| \( n \) | \( m \) | \( d_n \) | \( \ell EE(G) \) | Bound (27) | Bound (28) | Bound (29) | Bound (15) | Bound (24) | Bound (16) |
|---|---|---|---|---|---|---|---|---|---|
| 4 | 3 | 13.278 | 10.599 | 12.631 | 12.631 |
| 5 | 3 | 16.957 | 13.333 | 15.223 | 15.223 |
| 6 | 5 | 18.545 | 16.310 | 17.596 | 17.596 |
| 7 | 5 | 21.641 | 19.028 | 18.692 | 18.692 |
| 8 | 6 | 23.122 | 17.315 | 20.326 | 20.326 |
| 9 | 8 | 30.542 | 27.183 | 26.834 | 26.834 |
| 10 | 7 | 63.358 | 54.043 | 54.043 | 54.043 |
| 20 | 15 | 94.687 | 81.548 | 81.548 | 81.548 |
| 50 | 25 | 159.848 | 135.914 | 135.914 | 135.914 |
| 100 | 50 | 318.344 | 271.828 | 271.828 | 271.828 |

### Table 11: Upper bounds for \( \ell EE(G) \).

| \( n \) | \( m \) | \( d_n \) | \( \ell EE(G) \) | Bound (24) | Bound (25) | Bound (26) | Bound (27) | Bound (29) | Bound (15) |
|---|---|---|---|---|---|---|---|---|---|
| 4 | 3 | 13.278 | 10.599 | 12.631 | 12.631 |
| 5 | 3 | 16.957 | 13.333 | 15.223 | 15.223 |
| 6 | 5 | 18.545 | 16.310 | 17.596 | 17.596 |
| 7 | 5 | 21.641 | 19.028 | 18.692 | 18.692 |
| 8 | 6 | 23.122 | 17.315 | 20.326 | 20.326 |
| 9 | 8 | 30.542 | 27.183 | 26.834 | 26.834 |
| 10 | 7 | 63.358 | 54.043 | 54.043 | 54.043 |
| 20 | 15 | 94.687 | 81.548 | 81.548 | 81.548 |
| 50 | 25 | 159.848 | 135.914 | 135.914 | 135.914 |
| 100 | 50 | 318.344 | 271.828 | 271.828 | 271.828 |

### Table 12: Lower bounds for \( \ell EE(G) \).

| \( n \) | \( m \) | \( d_n \) | \( \ell EE(G) \) | Bound (24) | Bound (25) | Bound (26) | Bound (27) | Bound (29) | Bound (15) |
|---|---|---|---|---|---|---|---|---|---|
| 4 | 3 | 13.278 | 10.599 | 12.631 | 12.631 |
| 5 | 3 | 16.957 | 13.333 | 15.223 | 15.223 |
| 6 | 5 | 18.545 | 16.310 | 17.596 | 17.596 |
| 7 | 5 | 21.641 | 19.028 | 18.692 | 18.692 |
| 8 | 6 | 23.122 | 17.315 | 20.326 | 20.326 |
| 9 | 8 | 30.542 | 27.183 | 26.834 | 26.834 |
| 10 | 7 | 63.358 | 54.043 | 54.043 | 54.043 |
| 20 | 15 | 94.687 | 81.548 | 81.548 | 81.548 |
| 50 | 25 | 159.848 | 135.914 | 135.914 | 135.914 |
| 100 | 50 | 318.344 | 271.828 | 271.828 | 271.828 |
Table 11: Upper bounds for $\ell EE(G)$.

6 Conclusions

By using an approach for localizing some relevant graph topological indices based on the optimization of Schur-convex or Schur-concave functions, we derive some new bounds for normalized Laplacian Estrada index and for normalized Laplacian energy index. The proposed bounds can be computed by using additional information on the localization of first and second eigenvalue of normalized Laplacian matrix. A numerical section shows how this approach allows to derive tighter bounds than those provided in the literature. In particular, bound derived directly via majorization technique appear sharper than those depending by the Randić Index. According to the latter ones, it is noteworthy that we analyzed only the results for a specific type of degree sequence, while different bounds could be derived for other suitable degree sequences.

Acknowledgement

The authors are grateful to Monica Bianchi and Anna Torriero for useful advice and suggestions.

Competing interests

The authors declare that they have no competing interests.

Authors’ contribution

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.
References

[1] M. Bianchi, A. Cornaro, J.L. Palacios, and A. Torriero. Bounding the Sum of Powers of Normalized Laplacian Eigenvalues of Graphs through Majorization Methods. MATCH Commun. Math. Comput. Chem, 70(2):707–716, 2013.

[2] M. Bianchi, A. Cornaro, J.L. Palacios, and A. Torriero. Bounds for the Kirchhoff index via majorization techniques. J. Math. Chem., 51(2):569–587, 2013.

[3] M. Bianchi, A. Cornaro, J.L. Palacios, and A. Torriero. New upper and lower bounds for the additive degree-Kirchhoff index. Croatica Chemica Acta, 86(4):363–370, 2013.

[4] M. Bianchi, A. Cornaro, J.L. Palacios, and A. Torriero. New bounds of degree-based topological indices for some classes of c-cyclic graphs. Discrete Appl. Math., 184:62–75, 2015.

[5] M. Bianchi, A. Cornaro, and A. Torriero. A majorization method for localizing graph topological indices. Discrete Appl. Math., 161:2731–2739, 2013.

[6] M. Bianchi, A. Cornaro, and A. Torriero. Majorization under constraints and bounds of the second Zagreb index. Mathematical Inequalities and Applications, 16(2):329–347, 2013.

[7] M. Bianchi and A. Torriero. Some localization theorems using a majorization technique. Journal of Inequalities and Applications, 5:443–446, 2000.

[8] J. Blitzstein and Diaconis P. A sequential importance sampling algorithm for generating random graphs with prescribed degrees. Internet Math., 6(4):489–522, 2010.

[9] B. Bollobás. Random Graphs. Cambridge Univ. Press, London, 2001.

[10] A.E. Brouwer and W.H. Haemers. A lower bound for the Laplacian eigenvalue of a graph - Proof of a conjecture by Guo. Linear Algebra and its Applications, 429:2131–2135, 2008.

[11] M. Cavers, S. Fallat, and S. Kirkland. On the normalized Laplacian energy and general Randić Index $R_{-1}$ of graphs. Linear Algebra and its Applications, 433:172–190, 2010.
[12] F. R. K. Chung, L. Lu, and V. Vu. The spectra of random graphs with given expected degrees. *Proceedings of the National Academy of Sciences*, pages 6313–6318, 2003.

[13] G.P. Clemente and A. Cornaro. Computing Lower Bounds for the Kirchhoff Index Via Majorization Techniques. *MATCH Commun. Math. Comput. Chem.*, 73:175–193, 2015.

[14] G.P. Clemente and A. Cornaro. New Bounds for the Sum of Powers of Normalized Laplacian Eigenvalues of Graphs. *To appear on Ars Mathematica Contemporanea*, 2016.

[15] A. Cornaro and G.P. Clemente. A New Lower Bound for the Kirchhoff Index using a numerical procedure based on Majorization Techniques. *Electronic Notes in Discrete Mathematics*, 41:383–390, 2013.

[16] G. Csardi. Package iGraph. *R Package*, 2014.

[17] D. Cvetković, M. Doob, and H. Sachs. *Spectra of Graphs - Theory and Application*. Academic Press, New York, 1980.

[18] P. Erdős and A. Rényi. On Random Graphs I. *Publicationes Mathematicae*, 6:290–297, 1959.

[19] P. Erdős and A. Rényi. On the evolution of random graphs. *Publications of the Mathematical Institute of the Hungarian Academy of Sciences*, 5:17–61, 1960.

[20] M. Hakimi-Nezhaad, H. Hua, A. Reza Ashrafi, and S. Qian. The normalized laplacian index of graphs. *J. Appl. Math. and Informatics*, 32:227–245, 2014.

[21] J. Li, J.M. Guo, and W.C. Shiu. The Normalized Laplacian Estrada Index of a Graph. *Filomat*, 28-2:365–371, 2014.

[22] A. W. Marshall, I. Olkin, and B. Arnold. *Inequalities: Theory of Majorization and Its Applications*. Springer, 2011.

[23] L. Shi. Bounds on Randić indices. *Discrete Math.*, 309:5238–5241, 2009.

[24] D. J. Watts and S. H. Strogatz. Dynamics of ‘Small World’ Networks. *Nature*, 393:440–444, 1998.