QUANTUM COSMOLOGY FOR A QUADRATIC THEORY OF GRAVITY

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ABSTRACT

For pure fourth order ($\mathcal{L} \propto R^2$) quantum cosmology the Wheeler-DeWitt equation is solved exactly for the closed homogeneous and isotropic model. It is shown that by imposing as boundary condition that $\Psi = 0$ at the origin of the universe the wave functions behave as suggested by Vilenkin.

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1 Introduction

Non-linear modifications of the Einstein-Hilbert action, have a long history [1], and they are of interest, among others, for the following reasons: first, because the effective gravitational action predicted by closed bosonic, heterotic and supersymmetric strings give the usual Einstein term plus a correction quadratic in the curvatures [2]. Second, these theories can be renormalized when quantized [3]. Third, pure gravity inflationary models emerge on adding an $R^2$ term to the usual gravitational Lagrangian [4]. On the other hand, some non-linear Lagrangians can be chosen with the property that the field equations for the metric are second order, these are the so-called Lovelock actions [5] which can be regarded as formed by the dimensional continuation of the Euler characteristics of lower dimensions [6,7,8]. These Gauss-Bonnet terms seem to be of importance for the quantization of these theories.

In this paper we consider a toy quantum model for the theories mentioned. The action will be constructed only with the square of the Ricci scalar and the corresponding Wheeler-De Witt equation for the closed isotropic cosmological model will be solved by separation of variables. The Vilenkin conditions will result after choosing the vanishing of the wave function at the beginning
of the universe, as first proposed by DeWitt [9]. Previous work considering non-linear gravitational Lagrangians has been made by Hawking and Luttrell [10], Boulware [11], Buchbinder and Lyanhovich [12], and Vilenkin [13] among others.

2 Wheeler DeWitt Equation

The $R + \gamma R^2$ theory of gravity, at the classical level is related to Einstein’s theory of gravity with a scalar field as the source of gravity by a conformal transformation [14], a similar relation exists for the scalar-tensor theories of gravitation [15]. This relation has been exploited by Kasper [16] to obtain approximate solutions to the pure quantum cosmology of fourth order.

The Theory studied by Kasper for the spatially closed Robertson-Walker metric,

\[
ds^2 = -c^2 dt^2 + a^2(t) \left( \frac{dr^2}{1 - r^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right),
\]

where $a(t)$ is the expansion factor of the universe, leads to the following form of the Wheeler -DeWitt equation in fourth order quantum cosmology:
\[
\left[ \frac{\partial^2}{\partial \beta^2} - \frac{\partial^2}{\partial \alpha \partial \beta} - 12e^{6\alpha+3\beta} + 144e^{4\alpha+2\beta} \right] \psi(\alpha, \beta) = 0. \tag{2}
\]

where \( \alpha = \ln a, \beta = \ln R \) and \( R \) is the scalar curvature; therefore this model is restricted to \( R > 0 \), since the above equation was obtained using the signature -+++, classical anti deSitter space is excluded. Notice that the lapse function in that paper is taken equal to 1 in contrast to the paper of Hawking and Luttrell where it is proportional to the scale factor. Therefore the time coordinates are different in the two cases.

In order to solve the above equation it is useful to introduce the following change of independent variables,

\[
x = e^{2\alpha + \beta} = a^2 R, \quad y = \alpha + \beta = \ln(aR), \tag{3}
\]

the new variables extend over the half plane \( x > 0 \) (since \( R > 0 \)). The resulting differential equation is now

\[
\left[ x \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x \partial y} + \frac{\partial}{\partial x} + 12x^2 - 144x \right] \psi(x, y) = 0. \tag{4}
\]
We look for a solution by separation of variables,

$$\psi(x, y) = F(x)G(y).$$

(5)

After substitution into the differential equation we obtain

$$G' = \nu G,$$

(6)

$$xF'' + (\nu + 1)F' + [-144x + 12x^2]F = 0,$$

(7)

where \( \nu \) is the separation constant.

The solution for \( G \) is trivial, \( G = G_0 e^{\nu y} \). The solution for \( F \) is a little more complicated, if \( \nu \neq \pm 1 \) it can be obtained as an infinite series. But, if \( \nu = \pm 1 \) it is possible to obtain closed form solutions.

2.1 \( \nu = -1 \) Solution

In this case Eq.(7) reduces to
\[ F'' + [-144 + 12x]F = 0 \] (8)

This equation can be transformed into a Bessel equation of order $1/3$ in the variable $u = -12^{4/3} + 12^{1/3}x, F = P(u),$

\[ P'' + uP = 0. \] (9)

The solution can be written as follows

\[ F = F_1 Ai(-u) + F_2 Bi(-u) \] (10)

here Ai and Bi are the Airy functions and $F_1, F_2$ are constants that will be fixed by the boundary conditions. The complete state function of the universe is

\[ \psi(x, y) = e^{-y}[c_1 Ai(-u) + c_2 Bi(-u)] \] (11)

Taking the limit of this solution when $x$ is large results into the WKB solution obtained in reference [16]
\[ F \rightarrow (-u)^{-1/4}e^{\pm \frac{2}{3}u^{3/2}} = (-12^{1/3}x)^{-1/4}e^{\pm \frac{2}{3}(-12^{1/3}x)^{3/2}} \]  

The particular solution of this section is also important when looking at the asymptotic solution \((x \to \infty)\) in the case of arbitrary value for \(\nu\) as will be shown in the next section.

### 2.2 Boundary condition and WKB approximation

If we restore the original variables \(a\) and \(R\) in Eq.(11) we have

\[ \psi(a, R) = \frac{c_1 Ai[-12^{1/3}(a^2 R - 12)] + c_2 Bi[-12^{1/3}(a^2 R - 12)]}{aR} \]  

at \(a = 0\) the values of the Airy functions in the numerator of the wave function of the universe are finite in contrast with the denominator that vanishes. Therefore the only sensible thing to do is choosing the constants \(c_1\) and \(c_2\) so that the whole numerator vanishes at \(a = 0\), and that corresponds to the following choice,
With this choice the numerator in Eq. (13) dominates the behaviour at the beginning of the universe and the wave function vanishes at $a = 0$.

We recall here that the vanishing of the wave function of the universe at $a = 0$ is the boundary condition which Bryce DeWitt [9] argued must be imposed on it, because it has the effect of keeping the wavepackets away from the singularity. On the other hand if we consider Eq. (8) and apply the WKB method to it, taking into account the right boundary conditions for the case where the potential goes to infinity, that is the case here because our coordinate $x$ is defined only for positive values, we obtain,

$$
\psi(x)_{WKB} = \frac{\text{Sinh}\left[\int_0^x \sqrt{(12(12-x))}dx\right]}{[12(12-x)]^{1/4}},
$$

(15)

Now the WKB solution vanishes at $a = 0$ ($x=0$). The behaviour of the wave function that we have obtained is similar to the hydrogenic wave functions when the angular momentum $l \neq 0$ in the sense that they vanish at the origin of the radial coordinate.
2.3 $\nu = 1$ Solution

If in Eq. (7) we eliminate the first derivative term, to put the equation into its normal form by means of the transformation

$$F(x) = x^{-(\nu+1)/2}g(x).$$  \hfill (16)

the differential equation is now

$$g'' + [-12(12 - x) + \frac{1 - \nu^2}{4x^2}]g = 0,$$  \hfill (17)

and we notice that for $\nu = 1$ the differential equation for $g(x)$ is the same as Eq.(8). Therefore the complete wave function of the universe is

$$\psi(x, y) = e^{+y} \left[ c_1 Ai(-u) + c_2 Bi(-u) \right] \frac{1}{x}.$$  \hfill (18)

Here we also take as the boundary condition the vanishing of the wave function at $a = 0$, that means that the constants $c_i$ are given by Eq. (14).
3 Series Solutions

Before looking at the series solution to Eq. (6) for $\nu \neq \pm 1$, it is useful to consider the asymptotic limit $x \to \infty$. For that purpose, we look once more at the normal form of the differential equation

$$u'' + [-12(12 - x) + \frac{1 - \nu^2}{4x^2}]u = 0 \quad (19)$$

and it is clear that in the asymptotic region ($x \to \infty$) this equation reduces to the equation solved exactly in the previous section.

In the case $\nu \neq \pm 1$ we can use Frobenius method assuming a solution for Eq.(6) of the form

$$F(x) = |x|^\alpha \sum_{n=0}^{\infty} a_n x^n. \quad (20)$$

The indicial equation is given by

$$\alpha(\alpha + \nu) = 0, \quad (21)$$

and the solutions are
$\alpha_1 = 0, \alpha_2 = -\nu.$ \hfill (22)

The recurrence relations are

\begin{align*}
a_n(n + \alpha - \alpha_1)(n + \alpha - \alpha_2) &= 144a_{n-2} - 12a_{n-3} \hfill (23) \\
a_n(n + \alpha)(n + \alpha + \nu) &= 144a_{n-2} - 12a_{n-3}. \hfill (24)
\end{align*}

From the above equations the series for both solutions is completely determined in case $\nu$ is not an integer; if that is the case the second solution can be obtained by the standard procedure used in Frobenius method.

## 4 Final remarks

The simplified model used in this paper has allowed us to exactly solve, for the first time, the quantum cosmology of fourth order for a closed Friedmann
model. The analogous problem for spatially flat model was solved by Reuter [17]. Actions close related to Chern-Simmons are of particular interest at the present time; we consider our present work as an attempt in the program of studying the quantum cosmology of these more general Lagrangians of topological origin.

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