MICROSCOPIC SPACE-TIME OF EXTENDED HADRONS : QUANTUM GENERALIZATION AND FIELD EQUATIONS

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Abstract

The microlocal space-time of extended hadrons, considered to be anisotropic is specified here as a special Finsler space. For this space the classical field equation is obtained from a property of the field on the neighbouring points of the autoparallel curve. The quantum field equation has also been derived for the bispinor field of a free lepton in this Finslerian microspace through its quantum generalization below a fundamental length-scale. The bispinor can be decomposed as a direct product of two spinors, one depending on the position coordinates and the other on the directional arguments of the Finsler space. The former one represents the spinor of the macroscopic, an associated Riemannian space-time of the Finsler space, and satisfies the Dirac equation. The directional variable-dependent spinor satisfies a different equation which is solved here. This spinor-part of the bispinor field for a constituent of the hadron can give rise to an additional quantum number for generating the internal symmetry of hadrons. Also, it is seen that in the process of separating the bispinor field and its equation an epoch-dependent mass term arises. Although, this part of the particle-mass has no appreciable contribution in the present era it was very significant for the very early period of the universe after its creation. Finally, the field equations for a particle in an external electromagnetic field for this Finslerian microlocal space-time and its associated Riemannian macrospaces have been found.
I. INTRODUCTION

As early as in 1957, the measurement of electromagnetic form factor of nucleon may be regarded as the first experimental support for the concept of extended structure of the elementary particle (Hofstadter, 1964), although, theoretically it was an older idea. In fact, this concept can be traced back in the electron theory of Lorentz. On the contrary, the elementary particles as the field quanta which are essentially point-like entities originated from the relativistic quantum mechanics in the framework of local field theory which suffers from the well-known divergence difficulties. With an underlying motivation of resolving these difficulties and also for the unified description of elementary particles, Yukawa (1948, 1950) came up with a bilocal field theory which opened the possibility of intrinsic extensions of the elementary particles. Later this simple bilocal model was generalized to multilocal theories. The motivation behind all these theories lies in understanding multifariousness of subatomic particles (whose number went on increasing in the last fifty years) as the ultimate manifestation of their extensions in space and time, the microscopic space-time.

The hadronic matter-extension in the microlocal space-time is also manifested in the composite character of hadrons, and the investigations of such composite pictures of extended hadrons culminated into the theory of quarks by Gell-Mann (1964), and the quark-parton model of Feynman (Bjorken, 1969; Feynman, 1972). So far, no direct observation of free quarks is possible, and consequently the confinement of quarks in the structure of hadrons has to be assumed. Thus, in a sense, these quarks can not be considered as the ordinary particles. Moreover, not only their fractional charges but also their flavour quantum numbers, such as isospin, strangeness, charm, etc. have to be assigned for generating the quantum numbers of hadrons. Even in Weinberg-Salam unified theory of weak and electromagnetic interactions such a phenomenology of assigning individual hypercharges to leptons and quarks has to be adopted. Since the origin of these internal quantum numbers of the constituents is not found, the quark-constituent model of extended hadron-structure remains incomplete. On the other hand, if one considers the extensions of hadron-structure in the microlocal space-time, then there remains the problem of specification of this space-time, which may be different from the macroscopic spaces, such as, the laboratory (Minkowskian) space-time and the large-scale space-time of the universe. Apart from the specification of the microscopic space-time, one has to obtain the field equations for this space-time with an aim of constructing a field theory of hadron interactions. The geometric origin of internal quantum numbers of the constituents is also to be found, making the internal symmetry of hadrons possible.
Another important aspect one has to remember is that the space-time below a length-scale (the Planck scale) is not a meaningful concept. Recently, Adler and Santiago (1999) have modified the uncertainty principle by considering the gravitational interaction of the photon and the particle. From this modified gravitational uncertainty principle it follows that there is an absolute minimum uncertainty in the position of any particle and it is of the order of Planck length. This is also a standard result of superstring theory. Also, the intrinsic limitation to quantum measurements of space-time distances has been obtained (Ng and Van Dam, 1994; and references therein). This uncertainty in space-time measurements does also imply an intrinsic uncertainty of the space-time metric and yields a quantum decoherence for particles heavier than Planck mass. Thus, because of the intrinsic uncertainty of space-time metric, it suffices to give a particle heavier than the Planck mass a classical treatment. Also, the space-time can be defined only as averages over local regions and cannot have any meaning locally. This also indicates that one should treat the space-time as "quantized" below a fundamental length-scale.

In the present article, the microlocal space-time is regarded as a special Finsler space which is anisotropic in nature. In fact, the breaking of discrete space-time symmetries in weak interactions, an anisotropy in the relic background radiation of the universe, and the absence of the effect of cutoff in the spectrum of primary ultra-high energy cosmic protons are all indirect indications of the existence of a local anisotropy in space-time. Consequently, the microlocal space-time should be described by Finsler geometry instead of Riemannian one. Here, we begin, in section 2, with the construction of the classical field equation in the Finsler space. In section 3, a discussion on the quantum generalization of the space-time has been made. In the subsequent section 4, the microlocal space-time has been considered as a quantized special Finsler space; and in section 5, the field equation for this Finsler space of microdomain has been derived. The field for a free lepton is, here, a "bispinor" depending on the position coordinates and the directional variables, and from its field equation it is possible to find the Dirac equations for the spinors of the macrospaces, such as, the usual Minkowski space-time and the background space-time of the universe (the Robertson-Walker space-time). These macrospaces appear as the associated Riemannian spaces of the Finslerian microscopic space. Of course, an alternative approach of deriving the Dirac equations for macrospaces has also been suggested in section 6. In the next section 7, we have found the homogeneous solutions of the directional variable-dependent spinor-part from the separable bispinor field. This spinor can give rise to an additional quantum number for the constituent leptons for generation of the internal symmetry of hadrons. Also, in the process of decomposition of the bispinor an additional mass term depending on the cosmological time appears. The cosmological consequences of this mass term were considered elsewhere (De, 1999;
and the references therein). In section 8, the field equations for a particle in an external electromagnetic field have been obtained for both the microscopic and macroscopic space-times. In the final section 9, some concluding remarks have been made.

II. CLASSICAL FIELD EQUATION

The classical field \( \psi(x, \nu) \) corresponding to a free particle supposed to be a lepton depends on the directional variables \( \nu = (\nu^0, \nu^1, \nu^2, \nu^3) \) of the Finsler space apart from its dependence on the position coordinates \( x = (x^0, x^1, x^2, x^3) \) as this field is an entity of the Finsler space. In fact, all the geometrical objects of any Finsler space, such as the metric tensor, the connection coefficients, depend on the positional as well as directional arguments. The classical field equation for \( \psi(x, \nu) \) can be obtained if one admits the following conjecture (De, 1997) : A property is to be satisfied by \( \psi(x, \nu) \) along the neighbouring points in the Finslerian microdomain on the autoparallel curve (the geodesic ) which is the curve whose tangent vectors result from each other by successive infinitesimal parallel displacement of the type

\[
d\nu^i = -\gamma^i_{hj}(x, \nu)\nu^h dx^j
\]  

where \( \gamma^i_{hj}(x, \nu) \) are the Christoffel symbols of second kind. Now, the property can be stated as the infinitesimal change of \( \psi \) along the autoparallel curve is proportional to the field function itself, that is,

\[
\delta \psi = \psi(\mathbf{x} + d\mathbf{x}, \nu + d\nu) - \psi(\mathbf{x}, \nu) \propto \psi(\mathbf{x}, \nu)
\]

or,

\[
\psi(\mathbf{x} + d\mathbf{x}, \nu + d\nu) - \psi(\mathbf{x}, \nu) = \epsilon(mc/\hbar)\psi(\mathbf{x}, \nu)
\]  

(2)

where the neighbouring points \((\mathbf{x} + d\mathbf{x}, \nu + d\nu)\) and \((\mathbf{x}, \nu)\) lie on the autoparallel curve. Here, the mass term \( m \) appearing in the constant of proportionality is regarded as the inherent mass of the particle and \( \epsilon \) is a real parameter such that \( 0 < \epsilon \leq \ell \), \( \ell \) being a fundamental length.

To the first orders in \( d\mathbf{x} \) and \( d\nu \) we have, from (2),

\[
(d\mathbf{x}^\mu \partial_\mu + d\nu^i \partial_i')\psi(\mathbf{x}, \nu) = \epsilon(mc/\hbar)\psi(\mathbf{x}, \nu)
\]  

(3)

where \( \partial_\mu \equiv \partial/\partial x^\mu \) and \( \partial_i' \equiv \partial/\partial \nu^i \)

Using (1), we arrive at the following equation for \( \psi(\mathbf{x}, \nu) \) :

\[
dx^\mu(\partial_\mu - \gamma^i_{hj}(x, \nu)\nu_h \partial_i')\psi(\mathbf{x}, \nu) = \epsilon(mc/\hbar)\psi(\mathbf{x}, \nu)
\]  

(4)
This equation can also be written in terms of the nonlinear connection \( N^\nu_\mu \) (in local representation) given as
\[
N^\nu_\mu = (1/2) \frac{\partial}{\partial \nu^\mu} (\gamma^\mu_\alpha \nu^\alpha \nu^\beta)
\] (5)

Because of homogeneity property, one finds \( (N^\nu_\mu)_{\nu^\nu} = \gamma^\mu_\alpha \nu^\alpha \nu^\beta \) where \( \nu^\alpha = \frac{dx^\alpha}{ds} \). Consequently, we have
\[
N^\nu_\mu dx^\nu = \gamma^\mu_\alpha \nu^\alpha dx^\beta
\] (6)

Then, the equation (4) becomes
\[
dx^\mu (\partial_\mu - N^\ell_\mu \partial_\ell) \psi(x, \nu) = \epsilon (mc/\hbar) \psi(x, \nu)
\] (7)

or, in terms of covariant \( \frac{\delta}{\delta x^\mu} \equiv \partial_\mu - N^\ell_\mu \partial_\ell \) we have
\[
dx^\mu \frac{\delta \psi(x, \nu)}{\delta x^\mu} = \frac{\epsilon mc}{\hbar} \psi(x, \nu)
\] (8)

Now, as \( dx^\mu \) transforms covariantly like a vector, \( \psi(x, \nu) \) behaves like a scalar under the general coordinate transformations, that is,
\[
\psi'(x', \nu') = \psi(x, \nu)
\] (9)

where \( x'^\mu = x'^\mu(x^\mu) \) and \( \nu'^\mu = X^s_\nu \nu^\nu \) with \( X^s_\nu = \frac{\partial x^s}{\partial x^\nu} \).

It is to be noted that the equation (8) for the classical field function is form-invariant under the general coordinate transformations. Now, if the parameter \( \epsilon \) is identified with the Finslerian arc distance element \( ds \) between the neighbouring points on the autoparallel curve, that is, the geodesic distance element, then we get the classical field equation in the following form:
\[
dx^\mu x^\mu \frac{\delta \psi(x, \nu)}{\delta x^\mu} = \frac{mc}{\hbar} \psi(x, \nu)
\] (10)

The equations (8) and (10) for the field \( \psi(x, \nu) \) are derived from its property on the autoparallel curve at the neighbouring points \( (x^\mu, \nu^\mu) \) and \((x^\mu + dx^\mu, \nu^\mu + dv^\mu)\) for which \( \eta_{\mu\nu} dx^\mu dx^\nu < 0 \). On the contrary, for \( \eta_{\mu\nu} dx^\mu dx^\nu > 0 \), this property should be taken as
\[
\delta \psi = \psi(x + dx, \nu + dv) - \psi(x, \nu) = -\frac{i \epsilon mc}{\hbar} \psi(x, \nu)
\] (11)

which leads to the following equation for \( \psi(x, \nu) \):
\[
dx^\mu \frac{\delta \psi(x, \nu)}{\delta x^\mu} = -\frac{i \epsilon mc}{\hbar} \psi(x, \nu) \text{ for } \eta_{\mu\nu} dx^\mu dx^\nu > 0
\] (12)

Thus, the field equation takes the following form:
\[
idx^\nu \frac{\delta \psi(x, \nu)}{\delta x^\nu} = \frac{\epsilon mc}{\hbar} \sqrt{\theta(dx^2)} \psi(x, \nu)
\] (13)
where $dx^2 = \eta_{\mu\nu}dx^\mu dx^\nu$ and

$$
\theta(z) = \begin{cases} 
1 & \text{for } z \geq 0 \\
-1 & \text{for } z < 0
\end{cases}
$$

(14)

On identification of $\epsilon$ with the arc distance element, the above equation for $\psi(x, \nu)$ becomes

$$
i\nu^\mu \frac{\delta \psi(x, \nu)}{\delta x^\mu} = \frac{mc}{\hbar} \sqrt{\theta(\nu^2)} \psi(x, \nu)
$$

(15)

where

$$\nu^2 = \eta_{\mu\nu}\nu^\mu \nu^\nu$$

### III. QUANTIZED SPACE-TIME

From the classical field functions we now transit to the quantum field or wave functions and their equations on the basis of quantized space-time. Snyder (1947), and Yang (1947) discussed long ago such theories of quantized spacetime. Later, many authors developed the theory [for various references see Blokhintsev (1973), Prugovecki (1984) and Namsrai (1985)]. In the theory of quantized space-time there is no usual conceptual meaning of definite space-time points. In fact, the components of the operators of coordinates are not commuted. Blokhintsev (1973) made a general statement of this problem as follows:

The usual (c - number) coordinates of points $(x^0, x^1, x^2, x^3)$ of space-time, which form a differential manifold $M_4(x)$ (with a certain metric) are changed by linear operators $(\hat{x}^0, \hat{x}^1, \hat{x}^2, \hat{x}^3)$ noncommuting with one another. This leads to the question concerning the measurable numerical coordinates of a point event and the ordering of events in the "operational space" $M_4(\hat{x})$. The reasonable answer to this question can be provided by admitting a mapping of the operational space on a space of eigenvalues of $\hat{x}$ or of functions $f(\hat{x})$ which form a complete set of variables. This set should be sufficient for ordering points in the four-dimensional pseudo-Euclidean space. Toward such a consideration Blokhintsev postulated the space $\mathcal{H}(\phi)$ of eigenfunctions $\phi$ of the complete set at each point of space $M_4(x)$. Out of the several examples considered there the following will be relevant for the present consideration. That was the quantum generalization of the usual Minkowski four-dimensional space-time regarded as a special Finsler space. The length element $ds$ for this case is expressed as

$$ds = N_\mu dx^\mu$$

(16)

where the vector $N_\mu$ is a zero-order form in $dx$. This form is different for time-like, space-like directions and light cone, having three possible values, $N = \pm 1, 0$. 

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Now, the quantum generalization of this Finsler space is achieved through the change of coordinate differentials in (16) by the finite operators

$$\Delta x^\mu = a \gamma^\mu$$

(17)

where $\gamma^\mu$ where $g_m$ are the Dirac matrices and $a$ is a certain length. The operator of interval is taken as follows :

$$\Delta \hat{s} = N_\mu \Delta \hat{x}^\mu \quad \text{for} \quad N^2 = 1 \quad \text{and} \quad N^2 = 0$$

(18a)

$$\Delta \hat{s} = \frac{1}{i} N_\mu \Delta \hat{x}^\mu \quad \text{for} \quad N^2 = -1$$

(18b)

Evidently, from (17) it follows that

$$[\Delta \hat{x}^\mu, \Delta \hat{x}^\nu] = 2ia^2 \hat{I}^{\mu\nu}$$

(19)

where $(\hat{I}^{\mu\nu})$ is the four-dimensional spin operator. The space determined by the relations (17), (18a) and (18b) will be called $\Gamma_4(\hat{x})$-space.

It follows from (19) that the eigenvalues of operators $\Delta \hat{x}^0, \Delta \hat{x}^1, \Delta \hat{x}^2, \Delta \hat{x}^3, \Delta \hat{x}^4$ do not form the complete set. This set can be built out of the eigenvalues of the interval operator $\Delta \hat{s}$ and unit vector $textbf{N}$. By solving the equation for the eigenfunctions $\phi_\lambda$ and eigenvalues of the operator $\hat{\sigma}(N) = \frac{1}{a} \Delta \hat{s}(N)$; that is

$$\hat{\sigma}(N) \phi_\lambda = \lambda \phi_\lambda$$

(20)

it is possible to find out the eigenvalues $\lambda$. They are given as

$$\lambda = \pm \sqrt{N^2} \quad \text{for} \quad N^2 > 0$$

(21a)

$$\lambda = \pm \sqrt{-N^2} \quad \text{for} \quad N^2 < 0$$

(21b)

Thus, the eigenvalues of $\Delta \hat{s}$ are given by

$$\Delta s = \pm a \quad \text{or} \quad 0$$

It follows from (18a) (18b), (19) that the interval operators $\Delta \hat{s}(N')$ and $\Delta \hat{s}(N'')$ for two nonparallel directions $N'$ and $N''$ do not commute :

$$[\Delta \hat{s}(N'), \Delta \hat{s}(N'')] = a^2 \gamma^\mu \gamma^\nu (N' \times N'')_{\mu\nu}$$

(22)

where the symbol $X$ represents the vector product. Hence, each point of the quantized space $\Gamma_4(\hat{x})$ can be crossed only by one (though arbitrary) straight line.
Regarding the ordering of events, Blokhintsev has chosen the sign for the interval in accordance with the concept of time $\tau$ and distance $\ell$. For the time-like interval $\hat{s} = \hat{\tau}, N^2 = 1$, at each point, the rule

$$\lambda = \pm 1, \quad \phi_\lambda \equiv \phi_\pm(\pm N) \quad (23)$$

gives two values of $\tau$, that is, $\tau = \pm a$, whereas for the space-like interval $\hat{s} = \hat{\ell}, N^2 = -1$,

$$\lambda = +1, \quad \phi_\lambda = \phi_+(\pm N) \quad (24)$$

only one sign is admitted, that is, $\ell = a$. Thus, in accordance with this choice, at each point in the space-like direction there can be only one ray($N$), while in time-like direction there can be two rays ($\pm N$). Thereby the ordering of events is determined in the space $\Gamma_4(\hat{x})$. It is realized in the same way as in the Minkowski space with the help of interval $s$ and unit vector $N$. The important difference is that only one line (for $N^2 = 1$) and one ray (for $N^2 = -1$) are admitted at each point. The eigenvalue of interval for $N^2 = 1$ coincides with time $\tau$ in the reference frame, where $N = (1, 0, 0, 0)$ and for $N^2 = -1$ with length $\ell$ in the frame where $N = (0, \tilde{N})$. As to the interval $\Delta s = 0, N^2 = 0$, it defines neither length nor time because at $\Delta s = 0$ the operators $\hat{x}^0$ and $\hat{x}^h(h=1,2,3)$ do not commute with $\Delta s$ in any reference frame. Therefore the seat of points separated by the light cone $N^2 = 0$ is undermined.

Similar quantized space-time has been considered by Namsrai (1985) for the internal space-time $I_4$ of the space-time $R_4 = E_4 + I_4$ where $E_4$ is the external space-time. The coordinates $\hat{x}^\mu \in \hat{R}_4 (\mu = 0, 1, 2, 3)$ are given as

$$\hat{x}^\mu = x^\mu + r^\mu, \quad x^\mu \in E_4 \quad and \quad r^\mu \in I_4$$

The quantization of space-time is realized in two possible ways, namely,

**Case 1:**

$$\hat{x}^\mu = x^\mu + \ell\gamma^\mu \quad (25a)$$

**Case 2:**

$$\hat{x}^\mu = x^\mu + i\ell\gamma^\mu \quad (25b)$$

where $\ell$ is a fundamental length. Thus, $R_4$ is quantized at small distances. For the first and second cases we have, respectively,

$$[\hat{x}^\mu, \hat{x}^\nu] = 2i\ell^2\sigma^{\mu\nu} \quad (26a)$$

and

$$[\hat{x}^\mu, \hat{x}^\nu] = \frac{2}{i}\ell^2\sigma^{\mu\nu} \quad (26b)$$

where

$$\sigma^{\mu\nu} = \frac{1}{2i}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \quad (27)$$

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A mathematical procedure has been prescribed there to provide a passage to the large scale from the small one. This procedure, the averaging of coordinates $r^\mu = \ell \gamma^\mu$ or $i\ell \gamma^\mu$ of the internal space $I_4$, is to trace the $\gamma$ matrices. For example, in the case of first realization,

$$\langle \hat{x}^\mu \rangle_{R_4} = x^\mu, \quad \langle \hat{x}^\mu \hat{x}^\nu \rangle = x^\mu x^\nu + 4\ell^2 g^{\mu\nu}$$

$$\langle s^2 \rangle = \langle \hat{x}^0 \hat{x}^2 \rangle - \langle \hat{x}^2 \rangle = s_0^2 + 16\ell^2$$

where

$$s_0^2 = x_0^2 - \bar{x}^2$$

For the second case,

$$\langle \hat{x}^\mu \hat{x}^\nu \rangle = x^\mu x^\nu - 4\ell^2 g^{\mu\nu}$$

$$\langle \hat{s}^2 \rangle = s_0^2 - 16\ell^2$$

It is to be noted that out of the two realizations we have to choose only one realization for quantization of space-time. In Blokhintsevs approach the quantization corresponds only to the first case of Namsrais approach although there is an ambiguity in the operator of interval $\Delta \hat{s}$ in the former one. In fact, as it is evident from (18a) and (18b) that $\Delta \hat{s}$ differs for $N^2 = 1$ and $N^2 = 0$ from that for $N^2 = -1$.

**IV. MICRODOMAIN AS QUANTIZED FINSLER SPACE**

In De (1997) the microlocal space-time of extended hadrons has been specified as a special Finsler space. The fundamental function $F(x, \nu)$ of this space is given as

$$F^2(x, \nu) = \hat{g}_{ij}(x, \nu)\nu^i \nu^j$$

where

$$\hat{g}_{ij}(x, \nu) = \eta_{ij}g(x)\theta(\nu^2)$$

(28)

Here, $g$ is not, in general, the Finsler metric tensor, but simply represents a homogeneous tensor of degree zero in $\nu$, which is used for the purpose of defining $F$. The Finsler metric can be obtained by using the following formula:

$$g_{ij}(x, \nu) = \frac{1}{2} \frac{\partial^2 F^2(x, \nu)}{\partial \nu^i \partial \nu^j}$$

(29)
The Finsler space introduced here is, in fact, in accord with Riemann’s original suggestion that the positive fourth root of a fourth order differential form might serve as a metric function (Riemann, 1854). That is, we can write

\[ F(x, \nu) = \left\{ g_{\mu \nu \rho \sigma}(x) \nu^\mu \nu^\nu \nu^\rho \nu^\sigma \right\}^{1/4} \]  

(30)

In the present case the tensor field \( g = (g_{\mu \nu \rho \sigma}(x)) \) is taken as

\[ g_{\mu \nu \rho \sigma}(x) = \{g(x)\}^2 \eta_{\mu \nu} \eta_{\rho \sigma} \]  

(31)

Consequently,

\[ F^4(x, \nu) = \{g(x)\}^2 \eta_{\mu \nu} \eta_{\rho \sigma} \nu^\mu \nu^\nu \nu^\rho \nu^\sigma = \{g(x)\nu^2\}^2 \]  

(32)

or,

\[ ds^4 = \{g(x)\eta_{\mu \nu} dx^\mu dx^\nu\}^2 \]  

(33)

From this relation we have two possibilities, namely,

(i) the metric is given as

\[ ds^2 = g(x) \eta_{\mu \nu} dx^\mu dx^\nu \]  

(34)

which corresponds to a Riemannian space with the metric tensor

\[ g_{\mu \nu}(x) = g(x) \eta_{\mu \nu}, \]

(ii) the metric

\[ ds^2 = g(x) \theta(\nu^2) \eta_{\mu \nu} dx^\mu dx^\nu \]  

(35)

which corresponds to the Finsler space with the Fundamental function (28). It is interesting to note that if we insist on the condition of nonnegativeness of \( ds^2 \), then the second possibility from (33) or (30) gives rise to a Finsler geometry and we regard it as the geometry of the microlocal space-time of hadronic matter-extension.

Now, for quantization we write the interval of this Finsler space in the following form:

\[ ds_F = F(x, dx) = L_\mu dx^\mu \]  

(36)

where \( L_\mu \) is a zero-order form in \( \nu^\mu \equiv \frac{dx^\mu}{ds_F} \). In fact, \( L_\mu \) and \( L^\mu \) are given by (for nonnull \( ds_F \))
\[ L^\mu = g(x)\theta(\nu^2)\nu^\mu, \quad \text{and} \quad L_\mu = g(x)\theta(\nu^2)\eta_{\mu\nu}\nu^\nu \]  

(37)

For null \( ds_F \) we can take

\[ L^\mu = kg(x)\theta(\nu^2)\frac{dx^\mu}{dt}, \quad \text{and} \quad L_\mu = kg(x)\theta(\nu^2)\eta_{\mu\nu}\frac{dx^\nu}{dt} \]  

(38)

where \( t \) is a parameter and \( k \) is \((-1) \) p-homogeneous form in the differential \( dx^\mu \) (that is, a positively homogeneous function of degree -1). \( k \) is also supposed to be positive definite for all directions. Then, for nonnull \( ds_F \) we have

\[ L^2 = \eta_{\mu\nu}L^\mu L^\nu = g(x)\theta(\nu^2)F(x, \nu) = g(x)\theta(\nu^2) \]  

(39)

(using (28) and noting that \( F(x, \nu) = F(x, \frac{dx}{ds_F}) = 1 \) For null \( ds_F \),

\[ L^2 = \eta_{\mu\nu}L^\mu L^\nu = k^2 g(x)\theta^2(\nu^2)\eta_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = 0 \]  

(40)

This property of \( L^\mu \) expresses the Finslerian character of the space-time given by the interval \( ds_F \) in (36).

The quantum generalization of this Finsler space is made through the change of coordinate differentials in (36) by the finite operators

\[ \Delta \dot{x}^\mu = \epsilon \sqrt{\theta(\nu^2)}\gamma^\mu(x) \]  

(41)

where \( \gamma^\mu(x) \) \( (\mu = 0, 1, 2, 3) \) are Dirac matrices for the associated curved space (Riemannian) of the finsler space. In fact, the present Finsler space \( F_4 = (M_4, F) \) that describes the microlocal space-time of extended hadrons is a simple type of \((\alpha, \beta) \) - metric Finsler space. The fundamental function \( F \) of this space can be written as

\[ F(x, \nu) = \alpha \sqrt{\theta(\alpha^2)} = \alpha \beta^0 \sqrt{\theta(\alpha^2)} \]

where

\[ \alpha(x, \nu) = \sqrt{g(x)\eta_{ij}\nu^i\nu^j}, \quad g(x) > 0, \]

and \( \beta(x, \nu) \) is a differential 1-form. Obviously, the function \( F \) is \((1) \) p-homogeneous in \( \alpha \) and \( \beta \). The associated space \( R_4 = (M_4, \alpha) \) is Riemannian. This space is conformal to the Minkowski space-time. The matrices \( \gamma^\mu(x) \) are related to the flat space (Dirac) matrices through the vierbein \( V^\mu_\alpha(x) \) as follows:

\[ \gamma^\mu(x) = V^\mu_\alpha(x)\gamma^\alpha \]  

(42)

For the present case, the vierbein fields \( V^\mu_\alpha(x) \) and \( V^\alpha_\mu(x) \) are given as

\[ \gamma^\mu(x) = V^\mu_\alpha(x)\gamma^\alpha \]  

(42)
\[ V_\alpha^\mu(x) = \{g(x)\}^{-1/2} \delta_\alpha^\mu, \quad \text{and} \quad V_\mu^\alpha(x) = \{g(x)\}^{1/2} \delta_\mu^\alpha \] (43)

With the operators (41) we have the operator of interval as

\[ \Delta \hat{s}_F = L_\mu \Delta \hat{x}^\mu \] (44)

Also, from (41), (42), and (43) it follows that

\[ [\Delta \hat{x}^\mu, \Delta \hat{x}^\nu] = 2i \epsilon^{\mu \nu} \sigma^{\mu \nu} \] (45)

Defining the operator \( \hat{\sigma}(L) = \frac{\Delta \hat{s}_F}{\epsilon} \), we have

\[ \hat{\sigma}(L) = \sqrt{\theta(\nu^2)} L_\mu \gamma^\mu(x) = \sqrt{\frac{\theta(\nu^2)}{g(x)}} L_\alpha \gamma^\alpha \] (46)

where \( \gamma^A \) are flat space Dirac matrices. Also, it follows that

\[ \hat{\sigma}^2 = \frac{\theta(\nu^2)}{g(x)} L^2 = 1 \quad \text{for} \quad L^2 \neq 0 \] (47a)

\[ \hat{\sigma}^2 = \frac{\theta(\nu^2)}{g(x)} L^2 = 0 \quad \text{for} \quad L^2 = 0 \] (47b)

Thus, the eigenvalues of \( \hat{\sigma} \) are \( \pm 1 \) and 0. Consequently, those of the interval operator \( \Delta \hat{s}_F \) are \( \pm \epsilon \) and 0. The ordering of events in this space can be determined as in the case of Minkowski (Finsler) space discussed above.

\[\text{V. FIELD EQUATION IN FINSLER SPACE}\]

We shall now derive quantum field (or wave) equation in the Finsler space of microdomain through its quantum generalization. The quantization is admitted in two steps. From the property of the field function as given in (2) and (11) one can write

\[ \delta \psi = dx^\mu \partial_\mu \psi(x, \nu) + dv^\ell \partial_\ell = - \frac{iemc}{\hbar} \sqrt{\theta(\nu^2)} \psi(x, \nu) \] (48)

As the first step, the differentials \( dx^\mu \) are quantized to \( \Delta \hat{x}^\mu \) as given in (41). Consequently, the classical field function \( \psi(x, \nu) \) transforms into a spinor \( \psi_\alpha(x, \nu) \) whose equation becomes

\[ \epsilon \sqrt{\theta(\nu^2)} \gamma_\alpha^\mu(x) \partial_\mu \psi_\beta(x, \nu) + dv^\ell \partial_\ell \psi_\alpha(x, \nu) = - \frac{iemc}{\hbar} \sqrt{\theta(\nu^2)} \psi_\alpha(x, \nu) \] (49)
In the second step, the differentials $dn_l$ are quantized by noting first that since the neighbouring points $(x, \nu)$ and $(x + dx, \nu + d\nu)$ lie on the autoparallel curve of the Finsler space the relation (1) between the differentials $d\nu^\ell$ and $dx^\mu$ must hold. Therefore, the quantized differentials $\Delta \hat{\nu}^\ell$ should be given as

$$\Delta \hat{\nu}^\ell = -\epsilon \sqrt{\theta(\nu^2)} \gamma^\ell_{h\mu}(x, \nu) \nu^h \gamma^\mu(x)$$

and consequently, the field becomes a bispinor (this nomenclature is only formal and is for convenience) $\psi_{\alpha\beta}(x, \nu)$. Then the resulting equation for $\psi_{\alpha\beta}$ in the Finsler space becomes

$$i\hbar \{ \gamma^\mu_{\alpha\beta}(x) \partial_\mu \psi_{\beta\beta}(x, \nu) - \gamma^\mu_{\beta\beta}(x) \gamma^\ell_{h\mu}(x, \nu) \nu^h \partial_\ell \psi_{\alpha\beta}(x, \nu) \} = mc\psi_{\alpha\beta}(x, \nu) \quad (51a)$$

or, in compact form,

$$i\hbar \gamma^\mu(x) (\partial_\mu - \gamma^\ell_{h\mu}(x, \nu) \nu^h \partial_\ell) \psi(x, \nu) = mc\psi(x, \nu) \quad (51b)$$

where it is to be remembered that the first and the second operators on the L.H.S. operate on the first and the second spinor indices of the bispinor $\psi(x, \nu) = \psi_{\alpha\beta}(x, \nu)$ respectively.

**VI. DIRAC EQUATION IN ROBERTSON-WALKER SPACE-TIME**

We shall now find the field equation (for lepton) in the associated Riemannian space of the Finsler space from the above equation of the bispinor field. For this purpose one can first specify the function $g(x)$ in the metric (28) as

$$g(x \equiv F(t) = \exp(\pm b_0 x^0) \text{ or } (b_0 x^0)^n \text{ or } (1 + b_0 x^0)^n \quad (52)$$

where $x^0 = ct$

In this case, the connection coefficients $\gamma^\ell_{h\mu}(x, \nu)$ are separated as

$$\gamma^\ell_{h\mu}(x, \nu) = \zeta(t) \gamma^\ell_{h\mu}(\nu) \quad (53)$$

where

$$2b_0 c \zeta(t) = F'/F(t) \quad (54)$$

In fact, for the present Finsler space it can be easily seen that $\gamma^\ell_{h\mu}$ are independent of the directional arguments. Also, if we calculate $G^\ell_{hk} = \frac{\partial^2 G^\ell}{\partial \nu^h \partial \nu^k}$ with $G^\ell = \frac{1}{2} \gamma^\ell_{ij}(x, \nu) \nu^i \nu^j$
it will be found that $G_{h,k}$ are independent of $\nu$. Such types of Finsler spaces are called affinely-connected spaces or Berwald spaces (Rund, 1959).

Now, the field equation (51a) can be written in the following form:

\[ i\hbar \{ \gamma^\mu(x) \partial_\mu \psi(x, \nu) - \zeta(t) \gamma^k_{\mu h} \nu^h (\psi(x, \nu) \frac{\partial}{\partial t} \gamma^k_{\mu h} (x)) \} = mc \psi(x, \nu) \quad (55) \]

where the bispinor $\psi(x, \nu)$ is represented here as a $4 \times 4$ matrix. The vierbein fields $V^a_\mu$ and the inverse vierbein fields $V^a_\mu$ satisfy

\[ V^a_\mu(x) V^b_\mu(x) = \partial_a^b \quad (56) \]

For the present case, these fields are diagonal and are given by

\[ V^a_\mu(x) = e(t) \partial_a^\mu, \quad V^a_\mu(x) = \frac{1}{e(t)} \partial_a^\mu \quad (57) \]

where $e(t) = \{F(t)\}^{1/2}$. Then, the equation (55) becomes

\[ i\hbar e(t) \{ \gamma^\mu(x) \partial_\mu \psi(x, \nu) - \zeta(t) \psi(x, \nu) \frac{\partial}{\partial t} \gamma^k_{\mu h} \nu^h \} = mc \psi(x, \nu) \quad (58) \]

where $\gamma^\mu$ are now the flat space Dirac matrices.

Let us now decompose $\psi(x, \nu)$ in the following way (De, 1997):

\[ \psi(x, \nu) = \psi_1(x) \times \phi^T(\nu) + \psi_2(x) \times \phi^c(\nu) \quad (59) \]

where the spinors $\psi_1(x)$ and $\psi_2(x)$ are eigenstates of $\gamma^0$ with eigenvalues $+1$ and $-1$ respectively. Also, $\phi(\nu)$ and $\phi^c(\nu)$ satisfy, respectively, the following equations.

\[ i\hbar \gamma^\mu \gamma^k_{\mu h} \nu^h \partial_\nu \phi(\nu) = (Mc - \frac{3i\hbar b_0}{2})\phi(\nu) \quad (60a) \]

\[ i\hbar \gamma^\mu \gamma^k_{\mu h} \nu^h \partial_\nu \phi^c(\nu) = (Mc + \frac{3i\hbar b_0}{2})\phi^c(\nu) \quad (60b) \]

Then, it is easily seen that the field $\psi(x, \nu)$ satisfies the Dirac equation in the "x-space", the associated Riemannian space, which is, in fact, a space-time conformal to the Minkowski flat space. Here, $M$ appears as a constant in the process of separation of the equation (58) and this can be considered as a manifestation of the anisotropic Finslerian character of the microdomain. The equation for $\psi(x, \nu)$ is

\[ i\hbar (\gamma^\mu \partial_\mu + \frac{3b_0}{2} \zeta(t) \gamma^0) \psi(x, \nu) = \frac{c}{e(t)} (m + M\zeta(t)e(t)) \psi(x, \nu) \quad (61) \]

Consequently, the Dirac equation for the Robertson-Walker (RW) space-time can be obtained by a pure-time transformation given as
\[
\frac{dt}{e(t)} = \sqrt{F(t)}dt = dT \quad \text{or,} \quad \int \frac{dt}{e(t)} = T
\]  

(62)

where \( T \) is the cosmological time. With the scale factor \( R(T) \equiv \sqrt{F(t)} \), this transformation can also be written as

\[
t = \int \frac{dT}{R(T)}
\]  

(63)

Here, in the Dirac equation (61) (and also in that for RW space-time) an additional mass term (if \( M \neq 0 \)) appears and it is time-dependent. In fact, the mass of the particle is \( m + M \zeta(t)e(t) \) where \( m \) is the "inherent" mass of the particle as stated earlier. The time-dependent part of the mass, expressed in terms of cosmological time is found to be dominant in the very early universe and has significant effect in that era (De, 1993). It is pointed out here that the "\( \nu \)-part" in the decomposition of \( \psi(x, \nu) \) can give rise to an additional quantum number if it represents the field of the constituent-particle in the hadron configuration (De, 1997). On the other hand, for other cases, time-dependence of the masses is the only physical consequence that is manifested by the \( \nu \)-variable-dependence of the bispinor field. The usual field for the \((x)\)-space can be obtained by an "averaging procedure" such as

\[
\psi(x) = \int \psi(x, \nu)\chi(\nu)d^4\nu
\]  

(64)

where \( \chi(\nu) \) is a (spinor) probability density or a weight function. The field \( \psi(x) \) clearly satisfies the Dirac equation (61) in the \((x)\)-space.

The Dirac equation for the local inertial frame (the Minkowski flat space) can be recovered from (61) by using the vierbein

\[
V^\alpha_\mu(X) = \left( \frac{\partial y^\alpha_x}{\partial x^\mu} \right)_{x=X} \quad (\alpha = 0, 1, 2, 3)
\]  

(65)

which connect the curved space-time with the flat one in normal coordinates \( y^\alpha_x \) (with index \( \alpha \) referring to the local inertial frame) at the point \( X \). In the present case, the index \( \mu \) is associated with the conformal Minkowski space-time which becomes the RW space-time by a pure-time transformation. For fixed \( y^\alpha_x \), the effect of changing \( x^\mu \) is given by

\[
V^\alpha_\mu \rightarrow \frac{\partial x^\nu}{\partial x'^\mu} V^\alpha_\nu
\]  

(66)

On the other hand, \( y^\alpha_x \) may be changed by Lorentz transformation \( \Lambda^\alpha_\beta(X) \), and in this case the vierbeins \( V^\alpha_\mu(X) \) are changed to \( \Lambda^\alpha_\beta(X)V^\beta_\mu(X) \) keeping the metric of the curved space-time invariant.
With vierbeins (57) for the present case, one can obtain from (61) the Dirac equation in local inertial frame in normal coordinate system as

\( i \hbar \gamma^\alpha \partial_\alpha \psi(x, \nu) = mc \psi(x, \nu) \) \hspace{1cm} (67)

if one neglects the extremely small second terms both from the left and right sides of (61). Of course, the mass term \( M\zeta(t)e(t) \) may be retained here, and although it has no appreciable effect on the mass of the particle in the present era its dominance in the very early epoch is very significant (De, 1993, 1999).

Actually, in laboratory space-time (Minkowskian) one can decompose the field as

\( \psi(x, \nu) = \psi(x) X\phi^T(\nu) = \psi(x) \phi^T(\nu) \) \hspace{1cm} (68)

where \( \phi(\nu) \) satisfies the equation

\( i \hbar \gamma^\mu \gamma^\ell h_\mu^h (\nu) \nu^h \partial_\ell \phi(\nu) = Mc \phi(\nu) \) \hspace{1cm} (69)

The decomposition (68) corresponds to that in (59) for the case \( \phi(\nu) = \phi^c(\nu) \) and for flat space-time \( (b_0 \to 0) \). With this decomposition, it is easy to see that (from equation (58)) \( \psi(x) \) satisfies the Dirac equation in flat space-time :

\( i \hbar \gamma^\mu \partial_\mu \psi(x) = mc \psi(x) \) \hspace{1cm} (70)

Here, of course, \( b_0 \to 0 \) (and consequently \( F(t) \to 1 \)) makes \( M \to 0 \), as we shall later see that \( M \) is proportional to \( b_0 \). The bispinor \( \psi(x, \nu) \) which satisfies the Dirac equation (67) in the local inertial frame (the Minkoski space-time) reduces to a spinor through the averaging procedure (64) and this spinor satisfies the Dirac equation in local frame with or without the time-dependent mass term. Thus, the time-dependence of particle masses is the manifestation of the Finslerian character of the space-time whose underlying manifold (the \( \psi(x) \)-space or the associated Riemannian space) is a curved one. In the present case, this space is a RW space-time (obtained through pure-time transformation) which is the background space-time of our universe.

VII. HOMOGENEOUS SOLUTIONS FOR \( \phi(\nu) \) AND \( \phi^c(\nu) \)

We seek a class of solutions for \( \phi(\nu) \) and \( \phi^c(\nu) \) which are homogeneous of degree zero. In fact, the metric tensors of the Finsler space and the fundamental function are homogeneous functions of degree zero and one, respectively, in the directional arguments.
Therefore, one can argue that only this class of homogeneous solutions is physically relevant. For the Finsler space that we are considering, the equations for $\phi(\nu)$ and $\phi^c(\nu)$ for such type of solutions become (from eqn.(60))

$$i\hbar b_0 \sum_{\ell=1}^{3} \gamma^\ell (\nu^\ell \frac{\partial}{\partial \nu^0} + \nu^0 \frac{\partial}{\partial \nu^\ell})\phi(\nu) = -(Mc - \frac{3i\hbar b_0}{2}) \phi(\nu)$$ (71a)

and

$$i\hbar b_0 \sum_{\ell=1}^{3} \gamma^\ell (\nu^\ell \frac{\partial}{\partial \nu^0} + \nu^0 \frac{\partial}{\partial \nu^\ell})\phi^c(\nu) = -(Mc + \frac{3i\hbar b_0}{2}) \phi^c(\nu)$$ (71b)

The general form of the solutions of these equations, which are homogeneous of degree zero in $n$ is given as

$$\phi(\nu) = \{f_1(\frac{\nu^2}{\nu^0}) + i\tilde{\gamma} \cdot \tilde{\nu} f_2(\frac{\nu^2}{\nu^0})\} \omega^b$$ (72)

where $\omega^b$ is a four-component (arbitrary) spinor independent of $\nu$. Here, $f_1$ and $f_2$ are functions of the homogeneous variable (of degree zero in $\nu$) $x = \nu^2/\nu^0$. These functions satisfy the following coupled equations

$$2f'_1(x)(1-x) = ikf_2(x)$$ (73a)

$$(x-3)f_2(x) + 2x(x-1)f'_2(x) = -ikf_1(x)$$ (73b)

where $k$ is a real or complex constant. In fact, it is given by

$$i\hbar b_0 k = -Mc + \frac{3i\hbar b_0}{2}$$ (74)

For $\phi^c(\nu)$ whose form of solution is also given by (72), the constant $k$ is to be replaced by $k'$, and it is given by

$$i\hbar b_0 k' = -Mc - \frac{3i\hbar b_0}{2}$$ (75)

Thus, we have

$$Re\{k\} = 3/2, \quad Re\{k'\} = -3/2, \quad Im\{k\} = Im\{k'\} = Mc/\hbar b_0$$ (76)

For $M = 0$, $k$ and $k'$ are real. For the equation (69), the solution for $\phi(\nu)$ is also of the form (72) but in this case $k$ is purely imaginary, and it is given by $k = iMc/\hbar b_0$.

Now, it is easy to see that $f_1(x)$ and $f_2(x)$ individually satisfy the following second order differential equations
With the substitution \( y = x^{-1/2} \), we get the equation for \( f_1(x) \) from (77a) as

\[
\frac{d^2 f_1}{dy^2} + \frac{k^2 f_1}{(1 - y^2)^2} = 0 \quad (y^2 \neq 1)
\]  

(78)

Again, by the substitution

\[ f_1 = (1 - y^2)^{1/2} \nu \quad (y^2 \neq 1) \]  

(79)

we have the equation for \( \nu \) as

\[
(1 - y^2) \frac{d^2 \nu}{dy^2} - 2y \frac{d\nu}{dy} + \frac{k^2 - 1}{1 - y^2} \nu = 0
\]  

(80)

The solutions for \( \nu \) can be identified for \( y^2 > 1 \) and \( y^2 < 1 \). In fact, for \( y^2 > 1 \), one can make the substitution

\[ f_1 = (y^2 - 1)^{1/2} \nu \]  

(81)

in (78) to arrive at the same-equation (80). For \( y^2 > 1 \), the solution for \( \nu \) is the associated Legendre function of second kind \( Q_\mu(y) \) where \( \mu = (1 - k^2)^{1/2} \). It is to be noted that for \( M = 0 \), we have \( k^2 = k'^2 = 9/4 \) and consequently \( \mu = i\sqrt{5}/2 \).

Now, by using the following integral representation of \( Q^\mu_0(y) \) (Erdlyi, 1953)

\[
Q^\mu_0(y) = (1/2)e^{\pi\mu i}\Gamma(\mu + 1)(y^2 - 1)^{-\mu/2} \int_0^\pi (y + \cos t)^{\mu - 1}\sin t \, dt, \quad \text{for} \quad \text{Re}(\mu + 1) > 0
\]

and with the substitution \( y + \cos t = \xi \) we can have the solution for \( \nu \) as

\[ \nu = Q^\mu_0(y) = (1/2\mu)e^{\pi\mu i}\Gamma(\mu + 1)(y^2 - 1)^{-\mu/2}\{(y + 1)^\mu - (y - 1)^\mu\} \]

Thus, apart from the inessential constant one can take the solution for \( f_1 \) for \( y^2 > 1 \) (that is, for \( y > 1 \) or \( y < -1 \)) as

\[
f_1 = \frac{(y^2 - 1)^{1/2}}{2\mu}\{((y + 1)/y - 1)^{\mu/2} - ((y - 1)/y + 1)^{\mu/2}\}
\]  

(82)

For \( y^2 < 1 \), (that is, for \(-1 < y < 1 \)), it is easy to see that
In alternative forms we can write

\[ f_1 = \frac{(1 - y^2)^{1/2}}{2\mu} \left\{ (1 + y)\mu/2 - (1 - y)^{\mu/2} \right\} \]  

(83)

where \( d = \sqrt{5}/2 \)

Now, \( f_2 \) can be found from (73) and (84), and it is found to be

\[ f_2 = \frac{y\sqrt{y^2 - 1}}{ik} \left[ \cos\{(d/2)\ln\left(\frac{y + 1}{y - 1}\right)\} - (y/d)\sin\{(d/2)\ln\left(\frac{y + 1}{y - 1}\right)\}\right] \]  

for \( y > 1 \) or \( y < -1 \)  

(85a)

\[ f_2 = \frac{y\sqrt{1 - y^2}}{ik} \left[ \cos\{(d/2)\ln\left(\frac{1 + y}{1 - y}\right)\} - (y/d)\sin\{(d/2)\ln\left(\frac{1 + y}{1 - y}\right)\}\right] \]  

for \( -1 < y < 1 \)  

(85b)

Here, \( y \) is given by

\[ y = \nu^0/\sqrt{\nu^2} \]  

(86)

which is a homogeneous function of degree zero in the directional variable \( \nu \). For the case of \( M = 0 \), the solution for \( \phi(\nu) \) from (72) can be written as

\[ \phi(\nu) = F(\nu^0, \bar{\nu})\omega_b \]  

(87)

where

\[ F(\nu^0, \bar{\nu}) = \sqrt{\frac{\nu^0}{\nu^2} - 1}\left[ \frac{1}{(d/2)\ln\nu^0 + \sqrt{\nu^2}} \right] \]  

\[ + \frac{\gamma_j\bar{\nu}}{k\sqrt{\nu^2}} \left( \cos\{(d/2)\ln\nu^0 + \sqrt{\nu^2} - \gamma_j\bar{\nu}\} \right) \nu^0 > \sqrt{\nu^2} \text{ or } \nu^0 < -\sqrt{\nu^2} \]  

(88a)

and

\[ F(\nu^0, \bar{\nu}) = \sqrt{1 - \frac{\nu^0}{\nu^2}} \left[ \frac{1}{(d/2)\ln\nu^0 + \sqrt{\nu^2}} \right] + \frac{\gamma_j\bar{\nu}}{k\sqrt{\nu^2}} \left( \cos\{(d/2)\ln\sqrt{\nu^2} + \nu^0\} \right) \]  

(88b)
\[
\frac{\nu^0}{d\sqrt{\nu^2}} \sin\left(\frac{d/2}{ln\sqrt{\nu^2 + \nu^0}}\right) - \sqrt{\nu^2} < \nu^0 < \sqrt{\nu^2}
\]  

(88b)

The solution for \(\phi^c(\nu)\) can be obtained from (88a) and (88b) by changing \(k = 3/2\) to \(k' = -3/2\). This can also be achieved in the solution by changing \(\nu\) to \(-\nu\) (keeping the constant \(k\) unchanged). That is, the solution for \(\phi^c(\nu)\) is actually given by

\[
\phi^c(\nu) = F(\nu^0, -\nu)\omega^b_c
\]

(89)

where \(\omega^b_c\) is an arbitrary four-component spinor independent of \(\nu\) and may be chosen as

\[
\omega^b_c = i\gamma^2(\omega^b)^* 
\]

(90)

It is to be noted that the solutions for \(\phi(\nu)\) and \(\phi^c(\nu)\) at \(y^2 = 1\), that is, at \(\nu^0 = \pm \sqrt{\nu^2}\) can be obtained from (87), (88a) and (88b) by taking the left and right limits at those points. Obviously, these limits are zero, and consequently, \(\phi(\nu)\) and \(\phi^c(\nu)\) tend to zero at \(\nu^0 = \pm \sqrt{\nu^2}\). This results is also in consistent with the relations. It is easy to see that \(\phi(\nu)\) (and also \(\phi^c(\nu)\)) is finite as \(\nu^2 \to 0\) (that is, \(y \to \infty\)). In fact \(\phi(\nu) \to \omega^b\) as \(\nu^2 \to 0\), that is, as \(\nu \to (\nu^0, 0, 0, 0)\). Also, as \(\nu \to (0, \nu)\), that is, as \(y \to 0\), \(\phi(\nu) \to \frac{\gamma \nu}{k\sqrt{\nu^2}}\omega^b\) and \(\phi^c(\nu) \to -\frac{\gamma \nu}{k\sqrt{\nu^2}}\omega^c_c\).

The following relation also holds good:

\[
\phi^c(\nu) = i\gamma^2 \phi^s(\nu)
\]

(91)

with the following representation of \(\gamma\)-matrices

\[
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \tilde{\gamma} = \begin{pmatrix} 0 & \tilde{\sigma} \\ -\tilde{\sigma} & 0 \end{pmatrix}
\]

(92)

\(\tilde{\sigma} = (\sigma^1, \sigma^2, \sigma^3)\) being Paulis spin matrices. For \(M \neq 0\), that is, for \(Im\{k\} = Im\{k'\} \neq 0\), the solution of (80) is also given by \(\nu = Q^\mu_0(y)\) but in this case \(\mu\) is a complex number. It is given by

\[
\mu = \sqrt{1 - \{(3/2) + id\}^2}
\]

(93)

where \(Im\{k\} = Im\{k'\} = d\).

From (76) we see that the real parts of \(k\) and \(k'\) are fixed but their imaginary parts are arbitrary. One can choose these imaginary parts to be of the same order with their real parts. Presently, we take

\[
Im\{k\} = Im\{k'\} = d = 3/\sqrt{13}
\]

(94)
and therefore
\[ \mu = 1 - \frac{9i}{2\sqrt{13}} \]  (95)

With this changed \( \mu \), the solutions for \( f_1 \) and \( f_2 \) are as above. It is to be noted that this choice of the value for \( d \) can give rise to the finite limits for \( f_1 \) and \( f_2 \) as \( y \to \infty \), and also for \( y \to 0 \). The functions \( f_1 \) and \( f_2 \) at \( y = \pm 1 \) are given, as before, by their limits as \( y \to \pm 1 \) from left as well as from right. It is easy to see that these limits are also finite for this value of \( d \).

With this value of \( d \), we can obtain the following relation from (76):
\[ \frac{3\hbar b_0}{\sqrt{13}} = Mc \quad \text{or,} \quad (0.832)\hbar b_0 = Mc \]  (96)

This relation connects the mass parameter \( M \) with the parameter \( b_0 \) of the space-time. Although, the terms containing \( b_0 \) and \( M \) may be neglected to obtain the usual field equations in local inertial frame of the "\( \mathbf{x} \)-space" (the flat space) but it is seen that the \( \nu \)-dependent spinor \( \phi(\nu) \) (for the case \( M = 0 \)) can give rise to an additional quantum number which can generate the internal symmetry of hadrons (De, 1997).

VIII. EXTERNAL ELECTROMAGNETIC FIELD

The field equation for a particle in an external electromagnetic field can be deduced from a similar property of the fields on the autoparalled curve of the Finsler space as for case of a free field considered above. This property is now expressed by the following relation:
\[ \delta(\psi\chi) = \frac{ie}{hc} \left\{ A_\mu(\mathbf{x},\nu) d\mu \right\} \psi\chi = -\frac{iemc}{h} \sqrt{\theta(\nu^2)} \psi\chi \]  (97)

where \( \chi(\mathbf{x},\nu) \) is a scalar function which may be regarded as a phase factor for the field \( \psi(\mathbf{x},\nu) \). The vector field \( A_\mu(\mathbf{x},\nu) \) represents the external electromagnetic field. From (97), we have
\[ d\mu \left\{ \partial_\mu + \frac{ie}{hc} A_\mu(\mathbf{x},\nu) + \frac{1}{\chi} \partial_\mu \chi \right\} \psi + d\nu \left\{ \partial_\nu + \frac{1}{\chi} \partial_\nu \chi \right\} \psi = -\frac{iemc}{h} \sqrt{\theta(\nu^2)} \psi \]  (98)

As before we can use the relation (1) between \( d\nu^\ell \) and \( d\mu \) to find
\[ d\mu \left\{ \partial_\mu + \frac{ie}{hc} A_\mu(\mathbf{x},\nu) + \frac{1}{\chi} \partial_\mu \chi - \gamma_{h\mu}(\mathbf{x},\nu) \nu^\ell (\partial_\ell + \frac{1}{\chi} \partial_\ell \chi) \right\} \psi = -\frac{iemc}{h} \sqrt{\theta(\nu^2)} \psi \]  (99)
By setting \( \chi = \exp\{\frac{i}{\hbar c} \phi(x, \nu)\} \), this equation (99) can also be written in terms of nonlinear connections \((N^\nu_\mu)\) as follows:

\[
\begin{align*}
dx^\mu \left\{ \partial_\mu + \frac{ie}{\hbar c} (A_\mu(x, \nu) + (1/e) \delta_\mu \phi - (1/e) N^\ell_\mu \partial_\ell \phi) - N^\ell_\mu \partial_\ell \right\} \psi &= - \frac{iemc}{\hbar} \sqrt{\theta(\nu^2)} \psi \quad (100) \\
\end{align*}
\]

or, in terms of "covariant" \( \frac{\delta}{\delta x^\mu} = \partial_\mu - N^\ell_\mu \partial_\ell \), we have

\[
\begin{align*}
dx^\mu \left\{ \frac{\delta}{\delta x^\mu} + \frac{ie}{\hbar c} (A_\mu(x, \nu) + (1/e) \frac{\delta \phi}{\delta x^\mu}) \right\} \psi &= - \frac{iemc}{\hbar} \sqrt{\theta(\nu^2)} \psi \quad (101) \\
\end{align*}
\]

This equation should be regarded as the "classical" field equation. Here, \( A_\mu(x, \nu) + (1/e) \frac{\delta \phi}{\delta x^\mu} \) represents the covariant four-vector, the external electromagnetic field, in the Finsler space. Obviously, \( A_\mu + (1/e) \frac{\delta \phi}{\delta x^\mu} \) is the gauge transformation of the electromagnetic field and manifests the gauge invariance of it in this space.

When the field \( A_\mu(x, \nu) \) and the scalar function \( \chi(x, \nu) \) are separable as follows

\[
A_\mu(x, \nu) = A_\mu(x) - \gamma^k_{\mu h} \nu^h \ddot{A}_k(\nu) \quad (102)
\]

(In fact, we have seen earlier that \( \gamma^k_{\mu h} \) are independent of the directional arguments for the Finsler space we are considering) and

\[
\chi(x, \nu) = \chi(x) \ddot{\chi}(\nu), \quad (103)
\]

we have

\[
\begin{align*}
dx^\mu \left\{ \frac{\delta}{\delta x^\mu} + \frac{ie}{\hbar c} (A_\mu(x) - N^\ell_\mu \ddot{A}_k(\nu) + (1/e) \partial_\mu \phi(x) - (1/e) N^\ell_\mu \partial_\ell \phi(\nu)) \right\} \psi \\
&= - \frac{iemc}{\hbar} \sqrt{\theta(\nu^2)} \psi \quad (104a)
\end{align*}
\]

where \( \phi(x) = (\hbar c/i) \ln \chi(x) \), and \( \ddot{\phi}(\nu) = (\hbar c/i) \ln \ddot{\chi}(\nu) \), that is,

\[
\phi(x, \nu) = \phi(x) + \ddot{\phi}(\nu) \quad (104b)
\]

Also, along the autoparallel curve of the Finsler Space

\[
A_\mu(x, \nu) dx^\mu = A_\mu(x) dx^\mu - N^\ell_\mu \ddot{A}_k(\nu) dx^\mu = A_\mu(x) dx^\mu + \ddot{A}_k(\nu) d\nu^\ell \quad (105)
\]

Now, if we assume

\[
\ddot{A}_k(\nu) + (1/e) \partial_\ell \ddot{\phi}(\nu) = 0, \quad (106)
\]

then the above field equation takes the following form :
\[ dx^\mu \left\{ \frac{\delta}{\delta x^\mu} + \frac{ie}{\hbar c} \bar{A}_\mu(x) \right\} \chi = -\frac{ie mc}{\hbar} \sqrt{\theta(\nu^2)} \psi \]  

(107)

where

\[ \bar{A}_\mu(x) = A_\mu(x) + (1/e) \partial_\mu \phi(x) \]  

(108)

is the usual gauge transformation in the associated curved space (Riemannian) of the Finsler space.

We can now find out quantum field equation starting from (98) and using (104), (105) and (106) on quantum generalization of the Finslerian microdomain in two steps as before. The resulting equation for the bispinor \( \psi_{\alpha \beta}(x, \nu) \) for a particle in an external electromagnetic field \( \bar{A}_\mu(x) \) is found to be

\[
\gamma^{\mu}_{\alpha \alpha'}(x)(ih \partial_\mu - (e/c)\bar{A}_\mu(x))\psi_{\alpha' \beta}(x, \nu) - i\hbar \gamma^\mu_{\beta \beta'}(x)\gamma^\nu_{h\mu}(x, \nu)\nu^h \partial_\ell \psi_{\alpha \beta'}(x, \nu) = mc\psi_{\alpha \beta}(x, \nu) 
\]  

(109)

or, in the following alternative form:

\[
\gamma^\mu(x)(ih \partial_\mu - (e/c)\bar{A}_\mu(x))\psi(x, \nu) - \psi(x, \nu)\gamma^\nu_{h\mu}(x, \nu)\nu^h \partial_\ell \gamma_{\beta \beta'}^\ell(x, \nu)\gamma^{\mu T}(x) = mc\psi(x, \nu) 
\]  

(110)

The field equation for the antiparticle bispinor field \( \psi^c(x, \nu) \) can be obtained from (110) by the transformation \( e \to -e \). The relation between the particle and antiparticle fields is found to be

\[
\psi^c(x, \nu) = i\gamma^2 \psi^*(x, \nu)i\gamma^2 
\]  

(111)

In fact, by taking the complex conjugate of the equation (110) and then by left as well as right multiplications of it by \( i\gamma^2 \), one can arrive at the equation for \( \psi^c(x, \nu) \) if the relation (111) is taken into account.

If we decompose \( \psi(x, \nu) \) as in (68), we have

\[
\psi^c(x, \nu) = i\gamma^2(\psi(x) \times \phi^T(\nu))^*i\gamma^2 \\
= (i\gamma^2 \psi^*(x)) \times (i\gamma^2 \phi^*(\nu))^T 
\]  

(112)

(since \( \gamma^{2T} = \gamma^2 \))

Now, if \( \psi(x) \) represents a particle field in "x-space", then \( \psi^c(x) = i\gamma^2 \psi^*(x) \) is the antiparticle field. Similarly, from (91) it follows that \( i\gamma^2 \phi^*(\nu) = \phi^c(\nu) \), and therefore it follows from (112) that
\[ \psi^c(x, \nu) = \psi^c(x) \times \phi^T(\nu) \] (113)

For the case of decomposition (59), we can similarly arrive at the following decomposition of \( \psi^c(x, \nu) \):

\[ \psi^c(x, \nu) = \psi^c_1(x) \times \phi^T(\nu) + \psi^c_2(x) \times \phi^T(\nu) \] (114)

Here, \( \psi^c_1(x) \) and \( \psi^c_2(x) \) are the eigenstates of \( \gamma^0 \) with eigenvalues -1 and +1 respectively.

Now, from the equation (110), by using the decomposition (59) it is easy to find the following equation for \( \psi(x, \nu) \):

\[
(i\hbar \gamma^\mu \partial_\mu - \frac{e}{c} \gamma^\mu \tilde{A}_\mu(x) + \frac{3i\hbar b_0}{2} \zeta(t) \gamma^0) \psi(x, \nu) = (c/e(t))(m + M\zeta(t)e(t))\psi(x, \nu)
\] (115)

Also, by the averaging procedure (64) we can find the field equation for \( \psi(x) \) for the "x-space" as

\[
\left\{ i\hbar \gamma^\mu \partial_\mu - \frac{e}{c} \gamma^\mu \tilde{A}_\mu(x) + \frac{3i\hbar b_0}{2} \zeta(t) \gamma^0 \right\} \psi(x) = (c/e(t))(m + M\zeta(t)e(t))\psi(x)
\] (116)

The field equation in local inertial frame (the flat Minkowski space-time) can be derived from (116) with the use of the vierbeins \( V^a_\mu(X) \) if one neglects the extremely small terms as before. The equation is

\[
\gamma^\mu(i\hbar \partial_\mu - \frac{e}{c} \tilde{A}_\mu(x))\psi(x) = c(m + M\zeta(t)e(t))\psi(x)
\] (117)

where \( \tilde{A}_\mu(x) = e(t)\tilde{A}_\mu(x) \) (expressed in local coordinates, that is, in the normal coordinates). Of course, \( \tilde{A}_\mu(x) \approx \tilde{A}_\mu(x) \) as \( F(t) \approx 1 \) in the present epoch of the universe.

The equation (117) is the usual field equation in Minkowski space-time because the additional time-dependent mass term \( Mc\zeta(t)e(t) \) is negligible in the present era of the universe, and also it vanishes for the case \( Im\{k\} = Im\{k'\} = 0 \). For non-zero imaginary parts of \( k \) and \( k' \), this time-dependent mass term is significant in the very early era of the universe.

**IX. CONCLUDING REMARKS**

The microlocal space-time of extended hadrons is specified here in accordance with Riemann’s original suggestion of a metric function as the positive fourth root of a fourth
order differential form. The classical field equation has been obtained for this space from an assumed property of the field on the autoparallel curve, and the corresponding quantum field equation for a free lepton has been derived here by making the quantum generalization of this microlocal Finslerian space-time below a fundamental length-scale. In this process of quantization, the field transforms into a bispinor field which can be decomposed as a direct product of the two spinors depending respectively on the position coordinates and the directional arguments. The position coordinate-dependent spinors correspond to the fields for the macroscopic spaces which are the associated Riemannian space-time of the Finslerian microdomain. These spinors satisfy the usual Dirac-equations in those macrospaces. The other spinor depending on the directional variables satisfies a different equation which has been solved here. It is shown elsewhere (De, 1997) that these spinors are responsible for generating additional quantum numbers for the constituents in the hadron-structure. From this Finsler geometric origin of internal quantum numbers of the constituents, the internal symmetry of hadrons was achieved there. In the process of separation of the bispinor field and its equation we have seen that the mass of the particle is \( m + M \zeta(t)e(t) \), where \( M \) is related to the parameter \( bo \) of the space-time by (96). Incorporating this relation (96) into the particle-mass and expressing in terms of the cosmological time \( T \) given in (62) or (63), it is easy to see that the mass of the particle is \( m + \{ (0.832\hbar/c^2)H(T) \} \), where the relations (54) and (57) have also been taken into account. Here \( H(T) \) is the Hubbles function. For a representative particle (muon), the mass of it is \( m(1 + 2\alpha H(T)) \), where \( \alpha = 0.26 \times 10^{-23} \text{sec.} \) It is evident that the particle-mass at the present epoch of the universe is approximately equal to its inherent mass \( m \) to an extremely high degree of accuracy. On the contrary, the epoch-dependent part of the mass had significant contribution around and before the space time \( a \) (after the time-origin of the universe). In De (1993, 1999, 2001, 2002) its cosmological consequences in this very early era of the universe have been discussed. Also, it is to be noted that an "inherent massless" particle can even have mass \( 2\alpha \tilde{m} H(T) \), \( \tilde{m} \) being its mass at the epoch \( \alpha \).

The field equations for a particle in an external electromagnetic field in the Finsler space as well as in the associated Riemannian spaces have been obtained here. In the subsequent papers (De, 2002a,b) we have considered the electromagnetic interaction in the Finsler and its associated spaces. The covariance of the field equations under general coordinate transformations have been discussed there. Also, the field-theoretic and the S-matrix approaches for dealing with the strong interaction of hadrons have been presented.
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