The trivial solution of the gravitational energy-momentum tensor problem

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Abstract

In the literature one often finds the claim that there is no such thing as an energy-momentum tensor for the gravitational field, and consequently, that the total energy-momentum conservation can only be defined in terms of a gravitational energy-momentum pseudo-tensor. Nevertheless, by relaxing the assumption that gravitational energy-momentum tensor should only depend on first derivatives of the metric, the Einstein equation leads to a trivial result that gravitational energy-momentum tensor is essentially the Einstein tensor. We discuss various peculiarities of such a definition of energy-momentum and argue that all these peculiarities have a sensible physical interpretation.

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1 Introduction

In general relativity, the matter energy-momentum tensor $T^{\mu\nu}$ satisfies the covariant conservation law

$$\nabla_\mu T^{\mu\nu} = 0.$$  \hfill (1)

Unlike the local covariant conservation $\nabla_\mu j^\mu = 0$ of a vector $j^\mu$, the local covariant conservation (1) of a tensor, in general, does not lead to a global conservation of matter energy. If $n^\mu$ is the unit vector normal to a spacelike hypersurface $\Sigma$, the global matter energy $\int_\Sigma d^3x \sqrt{|g^{(3)}|} n^\mu n^\nu T_{\mu\nu}$, in general, depends on $\Sigma$. (An exception is a spacetime with a symmetry characterized by a timelike Killing vector $\xi^\mu$, because then one can introduce the local energy-momentum vector $p^\mu = T^{\mu\nu} \xi_\nu$. In this case (1) implies the local vector conservation $\nabla_\mu p^\mu = 0$, which follows from the facts that (i) $T^{\mu\nu}$ is a symmetric tensor and (ii) the Killing vector $\xi^\mu$ obeys $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$.)
The fact that (1) does not imply global conservation of matter energy has a simple physical interpretation: the energy-momentum of matter can be exchanged with the energy-momentum of the gravitational field. But this suggests that there should be an energy-momentum tensor $t^{\mu \nu}$ of the gravitational field itself, such that the total energy-momentum tensor

$$T^\mu_\nu = T^\mu_\nu + t^{\mu \nu}$$

is conserved in the ordinary sense

$$\partial_\mu T^\mu_\nu = 0.$$ 

If so, then one can introduce the global 4-momentum

$$P^\mu_\text{tot} = \int d^3 x \, T^\mu_0,$$

which, due to (3), obeys the global conservation

$$\frac{dP^\mu_\text{tot}}{dx^0} = 0.$$

Nevertheless, in general-relativity textbooks one often finds the claim that such a gravitational energy-momentum tensor $t^{\mu \nu}$ does not exist [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. According to the mentioned textbooks, the best one can do is to construct a pseudo-tensor quantity $t^{\mu \nu}$ which does not transform as a tensor under general coordinate transformations. The pseudo-tensor $t^{\mu \nu}$ can be chosen in many inequivalent ways [15], while the most popular choice is the one by Landau and Lifshitz [6].

Contrary to this widely accepted claim that the gravitational energy-momentum tensor $t^{\mu \nu}$ does not exist, in this paper we point out that it does. Moreover, it turns out to be trivial to construct it, if one is willing to relax one common assumption – that $t^{\mu \nu}$ should be constructed from the metric $g_{\mu \nu}$ and its first derivatives $\partial_\alpha g_{\mu \nu}$. By allowing $t^{\mu \nu}$ to depend also on the second derivatives $\partial_\alpha \partial_\beta g_{\mu \nu}$, we find the trivial solution to the problem of constructing the gravitational energy-momentum tensor $t^{\mu \nu}$; the appropriate tensor $t^{\mu \nu}$ turns out to be proportional to the Einstein tensor $G^{\mu \nu}$.

Indeed, such a definition of energy-momentum has also been proposed a long time ago by Lorentz [18] and Levi-Civita [19]. However, textbooks rarely mention the possibility of such a definition of energy-momentum, and when they do, they dismiss it as inadequate [4, 16, 20]. In this paper we reexamine various peculiarities of such a definition of energy-momentum and argue that these peculiarities are not a valid reason to dismiss it.

## 2 The gravitational energy-momentum tensor

Let us start from the Einstein equation

$$G^{\mu \nu} = 8 \pi G_N T^{\mu \nu},$$

where $G_N$ is the Newton constant.
where $G_N$ is the Newton constant and $G^{\mu\nu}$ is the Einstein tensor

$$G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R. \quad (7)$$

Eq. (6) can also be written as

$$T^{\mu\nu} - \frac{1}{8\pi G_N} G^{\mu\nu} = 0. \quad (8)$$

Applying the derivative $\partial_\mu$ on both sides of (8), one gets

$$\partial_\mu \left( T^{\mu\nu} - \frac{1}{8\pi G_N} G^{\mu\nu} \right) = 0. \quad (9)$$

But this is precisely the ordinary conservation equation (3), provided that in (2) one makes the identification

$$t^{\mu\nu} \equiv -\frac{1}{8\pi G_N} G^{\mu\nu}. \quad (10)$$

Hence, the tensor (10) can naturally be interpreted as the energy-momentum tensor of the gravitational field. It depends on the metric $g_{\mu\nu}$ and its first and second derivatives $\partial_\alpha g_{\mu\nu}$ and $\partial_\alpha \partial_\beta g_{\mu\nu}$, respectively.

The matter energy-momentum tensor $T^{\mu\nu}$ usually depends on matter fields and their first derivatives, but not on second derivatives of the matter fields. Nevertheless, there is no any physical reason why it should be the case for all energy-momentum tensors. Therefore we do not see any physical problem with the fact that the gravitational energy-momentum tensor (10) depends on the second derivatives of the gravitational field $g_{\mu\nu}$.

That (10) is the natural energy-momentum tensor for the gravitational field can also be seen from the total action

$$S_{\text{tot}} = S_{\text{grav}} + S_{\text{matter}}, \quad (11)$$

where $S_{\text{matter}}$ is the matter action and $S_{\text{grav}}$ is the pure gravity action

$$S_{\text{grav}} = \frac{1}{16\pi G_N} \int d^4x \sqrt{|g|} R. \quad (12)$$

The matter energy-momentum tensor is defined as (see e.g. [21])

$$T_{\mu\nu} = \frac{-2}{\sqrt{|g|}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}. \quad (13)$$

Likewise, by defining the gravitational energy-momentum tensor as

$$t_{\mu\nu} = \frac{-2}{\sqrt{|g|}} \frac{\delta S_{\text{grav}}}{\delta g^{\mu\nu}}, \quad (14)$$
one recovers (10). In the same spirit, one can define the total energy momentum tensor as
\[ T^{\mu\nu}_{\text{tot}} = -\frac{2}{\sqrt{|g|}} \frac{\delta S_{\text{tot}}}{\delta g^{\mu\nu}}, \] (15)
which leads to
\[ T^{\mu\nu}_{\text{tot}} = T^{\mu\nu} - \frac{1}{8\pi G_N} G^{\mu\nu}. \] (16)
In this way the Einstein equation (8) can be interpreted as a constraint that the total energy-momentum tensor must vanish.

The vanishing of the total energy-momentum tensor is the main source of the critique of (16) in the literature [4, 16, 20]. Essentially, it is claimed that a concept of a vanishing energy-momentum is useless. While we agree that a vanishing energy-momentum is less useful than energy-momentum which can take different values in different physical situations, we do not accept that it is totally useless. In particular, vanishing of the total energy-momentum tensor can also be viewed as a covariant version of the Hamiltonian constraint \( H_{\text{tot}} = 0 \) appearing in the canonical formulation of gravity [8, 15, 22]. In the quantum theory, the vanishing of the total Hamiltonian has a very deep physical consequence, leading to the famous problem of time in quantum gravity [23, 24]. To note at least one possible use of it, let us only mention that it might be a key to the solution of the black-hole information paradox [25].

Zee [16] makes a further critique of a vanishing total energy-momentum by comparing it with the Newton equation written as \( F - ma = 0 \), which one might attempt to interpret as the claim that “the total force vanishes”. While there is some point in such a comparison, in our opinion it misses the deeper geometrical message of the Einstein equation, which expresses the fact that general relativity is \textit{diffeomorphism invariant}. In particular, it means that \( \mu 0 \) and \( 0\nu \) components of the Einstein equation (6) are not really analogous to the Newton equation, but are non-dynamical constraint equations not containing second time derivatives. In this sense, a better analogue is a classical particle with an action invariant under reparameterizations of the time coordinate, leading to the vanishing total Hamiltonian (see e.g. [26, 27]).

Another related unappealing feature of such a definition of the gravitational energy-momentum \( t^{\mu\nu} \) is the fact that it vanishes at all points at which \( T^{\mu\nu} \) vanishes. In particular, it means that the gravitational wave propagating through a spacetime without matter carries zero energy-momentum. But this result should not be surprising, given that the very definition of the gravitational wave is non-covariant in essence. Namely, the definition of gravitational waves rests on a non-covariant split of the metric \( g_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu} \), where \( \gamma_{\mu\nu} \) is an arbitrary background metric (usually chosen to be the Minkowski metric \( \eta_{\mu\nu} \)) and \( h_{\mu\nu} \) is a disturbance, the propagation of which is identified with the gravitational wave. Neither \( \gamma_{\mu\nu} \) nor \( h_{\mu\nu} \) transforms as a tensor under general coordinate transformations. Thus the fact that the covariant energy-momentum tensor of a gravitational wave can vanish reflects the fact that the gravitational wave itself is not a covariant object.

Note also that Einstein equation (8) and positivity of the matter energy-density \( T^{00} \) imply that gravitational energy-density \( t^{00} \) is negative at points at which matter
is present. This negativity of gravitational energy reflects the attractive nature of gravity when it acts on matter.

A possible reason for worry is also the fact that the left-hand side of (9) is not a tensor, owing to the fact that the ordinary derivative $\partial_\mu$ is not a covariant object. In most cases that would be a problem, but here it is not a problem because (9) is valid in all coordinate frames. This is a consequence of the fact that the bracket in (9) vanishes itself due to (8), so that the vanishing of the derivative of the bracket is rather trivial.

To conclude, we believe that there are good arguments for accepting the gravitational energy-momentum tensor (10) as physically viable, despite of some peculiarities associated with it.

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