On The Total Edge Irregularity Strength of Some Copies of Books Graphs

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Abstract. Let \( G = (V(G), E(G)) \) be a graph and \( k \) be a positive integer. A total \( k \)-labeling of \( G \) is a map \( f : V(G) \cup E(G) \rightarrow \{1,2,\cdots,k\} \). The edge weight \( uv \) under the labeling \( f \) is denoted by \( w_f(uv) \) and defined by \( w_f(uv) = f(u) + f(uv) + f(v) \). A total \( k \)-labeling of \( G \) is called edge irregular if there are no two edges with the same weight. The total edge irregularity strength of \( G \), denoted by \( tes(G) \), is the minimum \( k \) such that \( G \) has an edge irregular total \( k \)-labeling. The labeling was introduced by Bača, Jendrol, Miller, and Ryan in 2007. In this paper, we determine the total edge irregularity strength of some copies of book graphs.

1. Introduction

Let \( G = (V,E) \) be a graph and \( k \) be a positive integer. A total \( k \)-labeling of \( G \) is a map \( f : V(G) \cup E(G) \rightarrow \{1,2,\cdots,k\} \).

In 2007, Bača et al. introduced a kind of total \( k \)-labeling of graph, namely an edge irregular total \( k \)-labeling. [1] The definition of the labeling is given in Definition 1.1.

Definition 1.1. [1] Let \( G = (V,E) \) be a graph. For an integer \( k \), a total \( k \)-labeling \( f : V \cup E \rightarrow \{1,2,\cdots,k\} \) is called an edge irregular total \( k \)-labelling of \( G \) if every two distinct edges \( e = uv \) and \( f = wx \) in \( E \) satisfy \( w_f(e) \neq w_f(f) \), where \( w_f(e) = f(u) + f(e) + f(v) \).

The notation \( w_f(e) \) is called by the weight of edge \( e \) under the labeling \( f \).

Definition 1.2. [1] The minimum \( k \) for which a graph \( G \) has an edge irregular total \( k \)-labeling, denoted by \( tes(G) \), is called the total edge irregularity strength of \( G \).

In the paper [1], Bača et al gave a lower bound and an upper bound on \( tes(G) \) for arbitrary graph \( G \). The bounds are given in Theorem 1.1.

Theorem 1.1 [1] Let \( G = (V,E) \) be a graph with vertex set \( V \) and a non-empty edge set \( E \). Then

\[
\left\lfloor \frac{|E| + 2}{3} \right\rfloor \leq tes(G) \leq |E|.
\]
Bača et al. [1] also gave the exact value of $tes(G)$ for $G$ are paths, cycles, and friendships. [1]

Path with order $n$ is a connected graph with 2 vertices of degree 1 and $n - 2$ vertices of degree 2. Cycle with order $n$ is a connected graph with every vertex has degree 2. The friendship graph $F_n$ may be visualised as $n$ triangles sharing a common vertex (but otherwise independent). The exact values of $tes$ of path, cycle, and friendship are given in Theorem 1.2, Theorem 1.3, and Theorem 1.4, respectively.

Theorem 1.2 [1] Let $P_n$ be a path with $n$ vertices. Then
$$tes(P_n) = \left\lceil \frac{n+1}{3} \right\rceil.$$ 

Theorem 1.3 [1] Let $C_n$ be a cycle with $n$ vertices. Then
$$tes(C_n) = \left\lceil \frac{n+2}{3} \right\rceil.$$ 

Theorem 1.4 [1] $tes(F_n) = \left\lceil \frac{3n+2}{3} \right\rceil$. 

Ivančo and S. Jendrol’ [2] posed a conjecture that for arbitrary graph $G \neq K_5$,
$$tes(G) = \max \left\{ \left\lfloor \frac{|E(G)| + 2}{3} \right\rfloor, \left\lfloor \frac{\Delta(G) + 1}{2} \right\rfloor \right\}.$$ 

In [3], Ramdani, et al. gave an upper bound on the total edge irregularity strength of disjoint union of graphs as follows.

Theorem 1.5 [3] The total edge irregularity strength of disjoint union of graphs $G_1, G_2, \cdots, G_m, \ m \geq 2$, is
$$tes\left( \bigcup_{i=1}^{m} G_i \right) \leq \sum_{i=1}^{m} tes(G_i) - \left\lfloor \frac{m-1}{2} \right\rfloor.$$ 

Siddiqui et al., in [5], considered the total edge irregularity strength of the disjoint union sun graphs.

A sun graph $M_n$ is defined as the graph obtained by adding a pendent edge to every vertex of the cycle $C_n$. [5]

The union of graphs $G_1$ and $G_2$ with disjoint vertex sets $V_1$ and $V_2$ and edge sets $E_1$ and $E_2$, is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$.

Lemma 1.1. [5] Let $n \geq 3$. Then $tes(2M_n) = \left\lceil \frac{4n+2}{3} \right\rceil$.

Theorem 1.6. [5] Let $p, n \geq 3$ be two integers. Then the total edge irregularity strength of the disjoint union of $p$ isomorphic sun graphs is
$$\left\lceil \frac{2(pn+1)}{3} \right\rceil.$$ 

Lemma 1.2. [5] $tes(M_{n_1} \cup M_{n_2}) = \left\lceil \frac{2(n_1+n_2+1)}{3} \right\rceil$.

Lemma 1.3. [5] $tes(M_{n_1} \cup M_{n_2} \cup M_{n_3}) = \left\lceil \frac{2(n_1+n_2+n_3+1)}{3} \right\rceil$. 

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Theorem 1.7. [5] Let \( p \geq 4 \). Then the total edge irregularity strength of the disjoint union of \( p \) consecutive non isomorphic sun graphs is
\[
\left\lceil \frac{2(\sum_{j=1}^{p} n_j + 1)}{3} \right\rceil.
\]

In [4], Nurdin et al. determined the total edge irregular strength of the corona product of paths with some graphs. The corona product of a graph \( G \) with a graph \( H \), denoted by \( G \odot H \), is a graph obtained by taking one copy of a \( n \)-vertex graph \( G \) and \( n \) copies \( H_1, H_2, \ldots, H_n \) of \( H \), and then joining the \( i \)-th vertex of \( G \) to every vertex in \( H_i \).

The value of \( tes(P_m \odot P_n) \) and \( tes(P_m \odot C_n) \) are given in Theorem 1.8 and Theorem 1.9, respectively.

Theorem 1.8 [4]. For any integer \( m, n \geq 2 \),
\[
tes(P_m \odot P_n) = \left\lceil \frac{2mn + 1}{3} \right\rceil.
\]

Theorem 1.9 [4]. For any integer \( m \geq 2 \) and \( n \geq 3 \),
\[
tes(P_m \odot C_n) = \left\lceil \frac{(2n + 1)m + 1}{3} \right\rceil.
\]

Ahmad et al., in [6], determined the total edge irregularity strength of disjoint union of friendship graphs. The results are given in theorem and corollary as follows.

Theorem 1.10 [6] Let \( F_{n,j} \) be a friendship graph with \( n_j \) triangles, \( n_j \geq 3 \) and \( 1 \leq j \leq m, m \geq 2 \). Let \( G \equiv \bigcup_{j=1}^{m} F_{n,j} \) be a disjoint union of friendship graphs \( F_{n,j} \). Then
\[
tes\left( \bigcup_{j=1}^{m} F_{n,j} \right) = 1 + \sum_{j=1}^{m} n_j.
\]

Corollary 1.1. [6] Let \( F_n \) be a friendship graph with \( n \) triangles, \( n \geq 3 \) and let \( mF_n \) be the disjoint union of \( m \) copies of \( F_n, m \geq 2 \). Then
\[
tes(mF_n) = mn + 1.
\]

In [7], Indriati et al. gave the total edge irregularity strength of generalized web graph. A generalized web graph \( W_{n,m} \), \( n \geq 3, m \geq 2, \), is a graph obtained by joining all vertices \( v_{i,m}, 1 \leq i \leq n \), of the generalized prism \( P_m^n \) to a further vertex \( w \), called the centre. Thus \( W_{n,m} \) contains \( mn + 1 \) vertices and \( 2mn \) edges. The \( tes(W_{n,m}) \) is given in Theorem 1.11.

Theorem 1.11 [7] Let \( W_{n,m} \), \( n \geq 3, m \geq 2 \), be the generalized web graph. Then,
\[
tes(W_{n,m}) = \left\lceil \frac{2mn + 2}{3} \right\rceil.
\]

The helm graph \( H_n \) is the graph obtained from a wheel by attaching a pendant edge at each vertex of the \( n \)-cycle. The flower graph \( FL_n \) is the graph obtained from a helm by joining each pendant vertex to the central vertex of the helm. In [8], Jeyanthi and Sudha prove that \( tes(FL_n) = \left\lceil \frac{4n+2}{3} \right\rceil \).

Siddiqui et al., in [9], determined the total edge irregularity strength of disjoint union. The result is as follows.
Theorem 1.12 [9] Let \( m, n \geq 3 \) be two integers. Then, the total edge irregularity strength of a disjoint union \( mH_n \) of \( m \) copies of a helm graph \( H_n \) is \( mn + 1 \).

2. Main Result

In this paper, we determine the total edge irregularity strength of some copies of book graphs. Book graphs is the Cartesian product of path \( P_2 \) and star \( S_n \).

The Cartesian product \( G \Box H \) of graphs \( G \) and \( H \) is a graph such that the vertex set of \( G \Box H \) is the Cartesian product \( V(G) \times V(H) \) and any two vertices \((u, u_0)\) and \((v, v_0)\) are adjacent in \( G \Box H \) if and only if \( u = v \) and \( u_0 \) is adjacent with \( v_0 \) in \( H \), or \( u_0 = v_0 \) and \( u \) is adjacent with \( v \) in \( G \).

The total edge irregularity strength of some copies of book graphs is given in Theorem 2.1.

**Theorem 2.1** Let \( P_2 \Box S_n \) be the book graph, which is the Cartesian product of path \( P_2 \) and star \( S_n \). Let \( m(P_2 \Box S_n) \) be the \( m \) copies of \( P_2 \Box S_n \). Then for \( n \geq 2 \) and \( m \geq 1 \),

\[
tes(m(P_2 \Box S_n)) = nm + \left\lfloor \frac{m+2}{3} \right\rfloor.
\]

**Proof.** Let the vertex set of the \( j \)-th copy of \( P_2 \Box S_n \) is

\[
\{u_{i,j} \mid i = 0,1,2,\ldots,n\} \cup \{v_{i,j} \mid i = 0,1,2,\ldots,n\}
\]

and the edge set is

\[
\{u_{0,j}u_{i,j} \mid i = 1,2,\ldots,n\} \cup \{v_{0,j}v_{i,j} \mid i = 1,2,\ldots,n\} \cup \{u_{i,j}v_{i,j} \mid i = 0,1,2,\ldots,n\}
\]

for every \( j = 1,2,\ldots,m \).

Define an edge irregular total \( \left(nm + \left\lfloor \frac{m+2}{3} \right\rfloor\right) \) - labeling \( f \) of \( m(P_2 \Box S_n) \) as follows.

\[
f(u_{i,j}) = n(j-1) + \left\lfloor \frac{j+1}{3} \right\rfloor \text{ for } 0 \leq i \leq n \text{ and } 1 \leq j \leq m;
\]

\[
f(v_{i,j}) = nj + \left\lfloor \frac{j+2}{3} \right\rfloor \text{ for } 0 \leq i \leq n \text{ and } 1 \leq j \leq m;
\]

\[
f(u_{0,j}u_{i,j}) = (n+1)(j-1) - 2\left\lfloor \frac{j-2}{3} \right\rfloor + i \text{ for } 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq m;
\]

\[
f(v_{0,j}v_{i,j}) = (n+1)(j-1) - 2\left\lfloor \frac{j-1}{3} \right\rfloor + i + 1 \text{ for } 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq m;
\]

\[
f(u_{i,j}v_{i,j}) = (n+1)(j-1) - \left\lfloor \frac{2j-2}{3} \right\rfloor + i \text{ for } 0 \leq i \leq n \text{ and } 1 \leq j \leq m.
\]

From the labeling \( f \), we have the weight of edges of \( m(P_2 \Box S_n) \) as follows.

\[
w_f(u_{0,j}u_{i,j}) = (3n+1)(j-1) + i + 2 \text{ for } 1 \leq i \leq n \text{ and } 1 \leq j \leq m;
\]

\[
w_f(u_{i,j}v_{i,j}) = (3n+1)(j-1) + n + i + 3 \text{ for } 0 \leq i \leq n \text{ and } 1 \leq j \leq m;
\]

\[
w_f(v_{0,j}v_{(i+1),j}) = (3n+1)(j-1) + 2n + i + 3 \text{ for } 1 \leq i \leq n \text{ and } 1 \leq j \leq m.
\]
From the weight formula above, can be seen that there are no two edges with the same weight. So, \( f \) is a total edge irregular \( \left( nm + \left\lceil \frac{m+2}{3} \right\rceil \right) \) – labeling of \( m(P_2 \square S_n) \). So that, we have an inequality as follows.

\[
tes(m(P_2 \square S_n)) \leq \left( nm + \left\lceil \frac{m+2}{3} \right\rceil \right).
\tag{2.1}
\]

On the other hand, the number of edges of \( m(P_2 \square S_n) \) is \( m(3n+1) \). So, by using Theorem 1.1, we have

\[
tes(m(P_2 \square S_n)) \geq \left\lceil \frac{m(3n+1) + 2}{3} \right\rceil
\]
\[
= \left\lceil \frac{3nm + m + 2}{3} \right\rceil
\]
\[
= nm + \left\lceil \frac{m + 2}{3} \right\rceil
\]

So that, we have an inequality (2.2)

\[
tes(m(P_2 \square S_n)) \geq nm + \left\lceil \frac{m + 2}{3} \right\rceil.
\tag{2.2}
\]

From inequality 2.1 and 2.2, we have an equality as follows.

\[
tes(m(P_2 \square S_n)) = nm + \left\lceil \frac{m + 2}{3} \right\rceil.
\]

Figure 2.1 gives an illustration of the notating of vertices in \( m(P_2 \square S_n) \) for \( m = 2 \) and \( n = 4 \).

\[
\text{Figure 2.1. The book graph } 2(P_2 \square S_4).
\]

In the next figure, can be seen the edge irregular total \( \left( (2)(4) + \left\lceil \frac{2+2}{3} \right\rceil = 10 \right) \) – labeling \( f \) of \( 2(P_2 \square S_4) \).
Figure 2.2. The total edge irregular 10-labeling $f$ of $2(P_2 \Box S_4)$.

The weight of each edge of $2(P_2 \Box S_4)$, under the labeling in Figure 2.2, can be seen in the Figure 2.3.

Figure 2.3. The weight of edges of $2(P_2 \Box S_4)$ under the labeling $f$ in Figure 2.2.

It can be seen that under the labeling $f$, there are no two edges in $2(P_2 \Box S_4)$ with the same weight.

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