HOLOMORPHIC FACTORIZATION OF DETERMINANTS OF LAPLACIANS ON RIEMANN SURFACES AND A HIGHER GENUS GENERALIZATION OF KRONECKER’S FIRST LIMIT FORMULA

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Abstract. For a family of compact Riemann surfaces $X_t$ of genus $g > 1$, parameterized by the Schottky space $\mathfrak{S}_g$, we define a natural basis of $H^0(X_t, \omega^n_{X_t})$ which varies holomorphically with $t$ and generalizes the basis of normalized abelian differentials of the first kind for $n = 1$. We introduce a holomorphic function $F(n)$ on $\mathfrak{S}_g$ which generalizes the classical product $\prod_{m=1}^{\infty} (1 - q^m)^2$ for $n = 1$ and $g = 1$. We prove the holomorphic factorization formula

$$\frac{\det' \Delta_n}{\det N_n} = c_{g,n} \exp \left\{ -\frac{6n^2 - 6n + 1}{12\pi} S \right\} |F(n)|^2,$$

where $\det' \Delta_n$ is the zeta-function regularized determinant of the Laplace operator $\Delta_n$ in the hyperbolic metric acting on $n$-differentials, $N_n$ is the Gram matrix of the natural basis with respect to the inner product given by the hyperbolic metric, $S$ is the classical Liouville action — a Kähler potential of the Weil-Petersson metric on $\mathfrak{S}_g$ — and $c_{g,n}$ is a constant depending only on $g$ and $n$. The factorization formula reduces to Kronecker’s first limit formula when $n = 1$ and $g = 1$, and to Zograf’s factorization formula for $n = 1$ and $g > 1$.

1. Introduction

Let $s$ and $\tau$ be complex numbers with $\text{Re} \, s > 1$, $\text{Im} \, \tau > 0$, and define

$$E(\tau, s) = \sum_{(m,n) \in \mathbb{Z}^2} \frac{(\text{Im} \, \tau)^s}{|m + n\tau|^{2s}}.$$

This series was introduced by Kronecker in 1863; see [Wei76]. It admits meromorphic continuation to the entire $s$-plane with a single simple pole at $s = 1$, and satisfies the functional equation

$$\pi^{-s} \Gamma(s) E(\tau, s) = \pi^{s-1} \Gamma(1-s) E(\tau, 1-s),$$  

(1.1)
where $\Gamma(s)$ is Euler’s gamma-function. Kronecker’s first limit formula asserts that

\[(1.2) \quad E(\tau, s) = \frac{\pi}{s - 1} - \pi \log \left\{ \frac{4 \text{Im} \tau |\eta(\tau)|^4}{\exp(2\Gamma'(1))} \right\} + O(s - 1)\]

as $s \to 1$, where $\eta(\tau)$ is the Dedekind eta-function:

\[\eta(\tau) = q^{1/24} \prod_{m=1}^{\infty} (1 - q^m), \quad q = e^{2\pi i \tau}.\]

See [Wei76] and [Lan87] for the proof, and for applications to number theory.

Equation (1.2) admits an interpretation in terms of the spectral geometry of the elliptic curve $E_\tau \simeq L \setminus \mathbb{C}$, $L = \mathbb{Z} + \mathbb{Z} \tau$, which goes back to [RS73]. Namely, assign to $E_\tau$ the flat metric $\frac{1}{\text{Im} \tau} |dz|^2$, in which the area of $E_\tau$ is 1. Let

\[\Delta_0(\tau) = -\text{Im} \tau \frac{\partial^2}{\partial z \partial \bar{z}}\]

be the Laplace operator in this metric on $E_\tau$, acting on functions. Its eigenvalues are

\[\lambda_\ell = \frac{\pi^2 |\ell|^2}{\text{Im} \tau}, \quad \ell \in L.\]

Its determinant is defined by zeta function regularization: the function $\zeta(\tau, s) = \sum_{\ell \neq 0} \lambda_\ell^{-s}$, defined initially for Re $s > 1$, admits meromorphic continuation to the entire $s$-plane, and one defines

\[\text{det}' \Delta_0(\tau) = \exp \left\{ -\frac{\partial}{\partial s} \bigg|_{s=0} \zeta(\tau, s) \right\},\]

where the prime indicates omission of zero eigenvalues. Since $\zeta(\tau, s) = \pi^{-2s} E(\tau, s)$, it follows from (1.1) and (1.2) that

\[(1.3) \quad \text{det}' \Delta_0(\tau) = 4 \text{Im} \tau |\eta(\tau)|^4.\]

This formula has been used in string theory for the one-loop computation in the perturbative approach of Polyakov (see, e.g., [D'H99] and references therein).

We restate (1.3) in a form convenient for generalization to higher genus. Consider the Schottky uniformization of the elliptic curve: $E_\tau \simeq \Gamma \setminus \mathbb{C}^*$, where $\Gamma$ is the cyclic group generated by the dilation $w \mapsto qw$, with fundamental region $D = \{ w \in \mathbb{C}^* : |q| < |w| \leq 1 \}$. The push-forward of the Euclidean metric $(\text{Im} \tau)^{-1} |dz|^2$ by the map $w = e^{2\pi i z}$ takes the form $\rho(w)|dw|^2$, where $\rho(w) = (4\pi^2 \text{Im} \tau |w|^2)^{-1}$. Setting

\[S(\tau) = \frac{i}{2} \int_D \left\{ \frac{\partial \log \rho}{\partial w} \right\}^2 dw \wedge d\bar{w} = 4\pi^2 \text{Im} \tau,\]
we can rewrite (1.3) as

\[
\left(\frac{\det' \Delta_0(\tau)}{\text{Im } \tau}\right) = 4 \exp \left\{-\frac{1}{12\pi} S(\tau)\right\} |F(q)|^2,
\]

where

\[
F(q) = \prod_{m=1}^{\infty} (1 - q^m)^2.
\]

Note that \(\det' \Delta_0(\tau)\) depends only on the isomorphism class of \(E_\tau\), which in turn depends only on \(q\), and that \(\text{Im } \tau\) also depends only on \(q\). Hence (1.4) is an equality of functions on \(\{q \in \mathbb{C} : 0 < |q| < 1\}\).

In this paper we extend (1.4) and (1.5) from elliptic curves to compact Riemann surfaces of genus \(g>1\), and from functions to \(n\)-differentials (sections of the \(n\)-th power of the canonical bundle). To formulate the main result, which may be interpreted as a higher genus generalization of Kronecker’s first limit formula, we first recall some basic facts about uniformization of Riemann surfaces and about Teichmüller and Schottky spaces (see Section 2 for more detail). Each compact Riemann surface \(X\) of genus \(g>1\) carries a unique hyperbolic metric (a Hermitian metric of constant negative curvature \(-1\), with respect to which one can define the Laplace operator \(\Delta_0(X)\) acting on functions on \(X\), its zeta function (analogous to \(\zeta(\tau,s)\) defined above), and its regularized determinant \(\det' \Delta_0(X)\). The Riemann moduli space is the set \(\mathcal{M}_g\) of isomorphism classes of compact Riemann surfaces of genus \(g>1\); it carries a natural structure of a complex orbifold of dimension \(3g-3\). This generalizes the space \(\text{PSL}(2,\mathbb{Z})\backslash \{\tau \in \mathbb{C} : \text{Im } \tau > 0\}\) of isomorphism classes of elliptic curves. The determinant \(\det' \Delta_0\) is a real-analytic function on \(\mathcal{M}_g\).

Now suppose that the Riemann surface \(X\) is marked, i.e., has a distinguished canonical system of generators \(\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\) of the fundamental group \(\pi_1(X,x_0)\), \(x_0 \in X\). With respect to this marking we may define a normalized basis \(\varphi_1, \ldots, \varphi_g\) of the space of holomorphic 1-forms — abelian differentials of the first kind — by the requirement \(\int_{\alpha_k} \varphi_j = \delta_{jk}\); then the period matrix \(\tau\) is defined by \(\tau_{jk} = \int_{\beta_k} \varphi_j\). It satisfies \(\text{Im } \tau_{jk} = \langle \varphi_j, \varphi_k \rangle = \frac{1}{2} \int_X \varphi_j \wedge \overline{\varphi_k}\) by the Riemann bilinear relations. The Teichmüller space \(\mathcal{T}_g\) is the set of isomorphism classes of marked Riemann surfaces of genus \(g\); it is the universal cover of \(\mathcal{M}_g\), and it carries a natural structure of a complex manifold of dimension \(3g-3\) with respect to which the entries of \(\tau\) are holomorphic functions. For \(g>1\), the Teichmüller space generalizes the upper half-plane \(\{\tau \in \mathbb{C} : \text{Im } \tau > 0\}\), and \(\det \text{Im } \tau\) will play the role of the factor \(\text{Im } \tau\) appearing in (1.4).

In fact, \(\det \text{Im } \tau\) is a well defined function on the Schottky space \(\mathcal{S}_g\), which is an intermediate cover of \(\mathcal{M}_g\) \(\mathcal{T}_g \rightarrow \mathcal{S}_g \rightarrow \mathcal{M}_g\) defined as follows. A marked Schottky group is a discrete subgroup \(\Gamma\) of the group of linear fractional transformations \(\text{PSL}(2,\mathbb{C})\), with distinguished free generators \(L_1, \ldots, L_g\) satisfying the following condition: there exist \(2g\) smooth
Jordan curves $C_r$, $r = \pm 1, \ldots, \pm g$, which form the oriented boundary of a domain $D \subset \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, such that $L_r C_r = -C_{-r}$, $r = 1, \ldots, g$. If $\Omega$ is the union of images of $D$ under $\Gamma$, then $\Gamma \setminus \Omega$ is a compact Riemann surface of genus $g$. According to the classical retrosection theorem, every compact Riemann surface may be realized in this manner; if it is marked, the condition $C_k$ homotopic to $\alpha_k$ for each $k > 0$ fixes the marked group up to overall conjugation in $\text{PSL}(2, \mathbb{C})$. The overall conjugation may be fixed by a normalization condition — see section 2.1. The Schottky space $\mathcal{S}_g$ is the space of marked normalized Schottky groups with $g$ generators. It is a complex manifold of dimension $3g - 3$, covering $M_g$ and with universal cover $\mathcal{T}_g$, and $\det \text{Im} \tau$ is a well defined function on it \cite{Zog89}. The Schottky space $\mathcal{S}_g$ generalizes the space $\{q \in \mathbb{C} : 0 < |q| < 1\}$ discussed above.

Like the Teichmüller space $\mathcal{T}_g$, the Schottky space $\mathcal{S}_g$ carries a natural Kähler metric, the Weil-Petersson metric. Its global Kähler potential can be explicitly constructed as follows. Let $\rho(z) dz^2$ be the hyperbolic metric on $\Omega$ — the pull-back of the hyperbolic metric on $X \simeq \Gamma \setminus \Omega$. Following \cite{ZT87b}, set

$$
S = \frac{i}{2} \int_D \left( \frac{\partial \log \rho}{\partial z} \right)^2 dz \wedge d\bar{z} + \frac{i}{2} \sum_{k=2}^g \int_{C_k} \left( \log \rho - \frac{1}{2} \log |L'_k|^2 \right) \left( \frac{L''_k}{L'_k} dz - \frac{L''_k}{L'_k} d\bar{z} \right) + 4\pi \sum_{k=2}^g \log |c(L_k)|^2,
$$

(1.6)

where for $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma$, we denote $c(\gamma) = c$. The function $S : \mathcal{S}_g \to \mathbb{R}$ is called the classical Liouville action (see \cite{ZT87b} and \cite{TT03} for details and motivation). According to \cite{ZT87b}, the function $-S$ is a Kähler potential of the Weil-Petersson metric on $\mathcal{S}_g$, i.e.,

$$
\partial \bar{\partial} S = 2i\omega_{WP},
$$

(1.7)

where $\partial$ and $\bar{\partial}$ are, respectively, the $(1,0)$ and $(0,1)$ components of the deRham differential $d$ on $\mathcal{S}_g$, and $\omega_{WP}$ is the symplectic form of the Weil-Petersson metric. For $g > 1$, the function $S$ on $\mathcal{S}_g$ will play the role of the function $S(\tau) = -2\pi \log|q|$ on $\{q \in \mathbb{C} : 0 < |q| < 1\}$ appearing in (1.4).

Now we can formulate the following remarkable generalization of (1.4) and (1.5) to higher genus Riemann surfaces.

**Theorem 1** (P. Zograf). Let $g > 1$, and let $\det' \Delta_0$, $\text{Im} \tau$ and $S$ be the functions on the Schottky space $\mathcal{S}_g$ defined above. Then there exists a holomorphic function $F : \mathcal{S}_g \to \mathbb{C}$ such that

$$
\frac{\det' \Delta_0}{\det \text{Im} \tau} = c_g \exp \left\{ -\frac{1}{12\pi} S \right\} |F|^2,
$$

(1.8)
where $c_g$ is a constant depending only on $g$. For points in $S_g$ corresponding to Schottky groups $\Gamma$ with exponent of convergence $\delta < 1$, the function $F$ is given by the following absolutely convergent product:

$$F = \prod_{\{\gamma\}} \prod_{m=0}^{\infty} \left( 1 - q^{1+m}_{\gamma} \right),$$

where $q_{\gamma}$ is the multiplier of $\gamma \in \Gamma$, and $\{\gamma\}$ runs over all distinct primitive conjugacy classes in $\Gamma$, excluding the identity.

See section 2.1 for the definition of $\delta$, $q_{\gamma}$, and primitive $\gamma$. The factorization formula (1.8) was proved in [Zog89], and the representation (1.9) was discovered later [Zog97]. We will refer to (1.8) together with (1.9) as the Zograf factorization formula, or simply Zograf’s formula. Note that when $g = 1$, the theorem still holds provided that $\Delta_0$ and $S$ are defined as in the discussion of elliptic curves above. In this case, (1.8) becomes (1.5).

Associated to the Riemann surface $X$ is the Selberg zeta function

$$Z(s) = \prod_{\{\gamma\}} \prod_{m=0}^{\infty} \left( 1 - q^{s+m}_{\gamma} \right),$$

where $\{\gamma\}$ runs over all distinct nontrivial primitive conjugacy classes in a Fuchsian group uniformizing $X$. Defined initially for $\text{Re} \ s > 1$, the Selberg zeta function admits analytic continuation to the entire $s$-plane, and, according to [DP86] and [Sar87],

$$\det' \Delta_0 = e^{c_0(2g-2)} Z'(1)$$

for some constant $c_0$. Hence Zograf’s formula gives a factorization of $Z'(1)$, considered as a function on $S_g$.

To motivate the extension from functions to $n$-differentials on $X$, we first describe a geometric interpretation of Zograf’s formula, in the context of the Quillen metric and the local index theorem for families. We write $\omega_X$ for the holomorphic cotangent bundle of $X$, and call a smooth section of $\omega_X^n$ an $n$-differential. Let $\mathcal{M}_g = \mathcal{M}_{g,1}$ be the universal curve — the moduli space of compact Riemann surfaces of genus $g > 1$ with one marked point — and let $p : \mathcal{M}_g \to \mathcal{M}_g$ be the corresponding forgetful map. Denote by $T_V.\mathcal{M}_g$ the vertical holomorphic tangent bundle of the fibration $p$, and for each positive integer $n$, denote by $\Lambda_n$ the direct image bundle $p_n(T_V.\mathcal{M}_g^{-n})$ over $\mathcal{M}_g$. Then the fibre of $\Lambda_n$ over a point $t \in \mathcal{M}_g$ is isomorphic to the vector space $H^0(X_t, \omega_{X_t}^n)$ of holomorphic $n$-differentials on the Riemann surface $X_t = p^{-1}(t)$. Let $\lambda_n = \det \Lambda_n$ be the corresponding determinant line bundle over $\mathcal{M}_g$. The hyperbolic metric on the fibres of $p$ defines a natural Hermitian metric on $\Lambda_n$ and on hence on $\lambda_n$. The Quillen metric
on $\lambda_n$ is defined by
\[
\|\varphi\|_{Q,n}^2 = \frac{\|\varphi\|_n^2}{\det' \Delta_n} = \frac{\det N_n}{\det' \Delta_n},
\]
where $\|\cdot\|_n$ is the Hermitian metric mentioned above, $\varphi = \varphi_1 \wedge \cdots \wedge \varphi_d$ is a local holomorphic section of $\lambda_n$ at $t \in \mathcal{M}_g$, $[N_n]_{jk} = \langle \varphi_j, \varphi_k \rangle$ is the Gram matrix of the basis $\varphi_1, \ldots, \varphi_d$ of $H^0(X_t, \omega^n_{X_t})$, and $\Delta_n$ is the Laplace operator in the hyperbolic metric on $X_t$ acting on $n$-differentials. The Quillen metric has the remarkable property that the Chern form of the Hermitian line bundle $(\lambda_n, \|\cdot\|_{Q,n})$ over $\mathcal{M}_g$ is proportional to the Weil-Petersson symplectic form $\omega WP$:
\[
\partial \bar{\partial} \log \frac{\det N_n}{\det' \Delta_n} = \frac{6n^2 - 6n + 1}{6\pi i} \omega WP.
\]
This is the local index theorem for families (see [BK86, BJ86, ZT87a]).

Theorem 1 together with (1.7) constitute a refinement of (1.11) in the case $n = 1$. Let $\varphi = \varphi_1 \wedge \cdots \wedge \varphi_g$ be the local holomorphic section of $\lambda_1$ determined by the normalized basis $\varphi_1, \ldots, \varphi_g$ of abelian differentials of the first kind on $X_t$. Then Theorem 1 provides (by means of the function $F$) an isometry between the line bundle $\lambda_1$ with the Quillen metric, and the line bundle over $\mathcal{M}_g$ canonically determined by carrying the Hermitian metric $\exp \left\{ \frac{1}{12\pi} S \right\}$ (see Section 3 in [Zog89] for details). (We have used the fact that $\det' \Delta_n = \det' \Delta_{1-n}$, see e.g. [ZT87a].) Expressed differently, Zograf’s factorization formula is a “$\partial \bar{\partial}$ antiderivative” of (1.11).

Based on (1.11), it is natural to expect an analogue of Theorem 1 to hold for all positive integer $n$. However, there are two principal differences between the cases $n = 1$ and $n > 1$.

First, for $n = 1$ there is a canonical choice of a lattice of maximal rank in $H^0(X, \omega_X)$ provided by the dual to $H_1(X, \mathbb{Z})$, which gives rise to the classical normalized basis of abelian differentials described above. Topology does not fix such a lattice in $H^0(X, \omega^n_X)$ when $n > 1$. Nevertheless, using Schottky uniformization and corresponding Eichler cohomology groups, we construct a natural basis of $H^0(X_t, \omega^n_{X_t})$ which is canonical up to a choice of basis in a space of polynomials, varies holomorphically with $t \in \mathcal{S}_g$, and reduces to the classical normalized basis of abelian differentials of the first kind when $n = 1$.

Second, for $n = 1$ the holomorphic quadratic differential on $X = X_t$ which corresponds to the $(1, 0)$ form $\partial \log \det' \Delta_0$ at $t \in \mathcal{S}_g$ is given by a local expression in terms of the Green’s function of $\partial_1$. However, for $n > 1$ the corresponding local expression is not holomorphic, and a holomorphic projection must be applied to obtain $\partial \log \det' \Delta_n$, which makes the entire expression non-local. Still, we prove that up to a known “holomorphic anomaly”, (which gives rise to the factor involving the classical Liouville action
between the line bundle \( \lambda_{S_{Z^{2}}} \) gives a factorization of \( \det \) is a constant depending only on \( g \). For the holomorphic line bundle over \( g > n > m \), considered as functions on \( \mathcal{S}_{g} \), then exists a holomorphic function \( F(n) : \mathcal{S}_{g} \to \mathbb{C} \) such that

\[
\frac{\det ' \Delta_{n}}{\det N_{n}} = c_{g,n} \exp \left\{ \frac{-6n^2 - 6n + 1}{12\pi} S \right\} |F(n)|^2,
\]

where \( c_{g,n} \) is a constant depending only on \( g \) and \( n \). The function \( F(n) \) is given by the following absolutely convergent product:

\[
F(n) = (1 - q_{L_{1}}) \cdots (1 - q_{L_{2}})^2 (1 - q_{L_{3}}) \prod_{\{\gamma\}} \prod_{m=0}^{\infty} (1 - q_{\gamma}^{n+m}),
\]

where \( q_{\gamma} \) is the multiplier of \( \gamma \in \Gamma_{t} \), \( \{\gamma\} \) runs over all distinct primitive conjugacy classes in the marked normalized Schottky group \( \Gamma_{t} \), excluding the identity, and \( L_{1}, \ldots, L_{g} \) are the free generators fixing the marking of \( \Gamma_{t} \).

See section 2.11 for the definitions of \( q_{\gamma} \) and primitive \( \gamma \), and for the normalization of the marked Schottky group. For \( n > 1 \) and \( g > 1 \), we have \( \det ' \Delta_{n} = C_{g,n} Z(n) \), where \( Z(s) \) is the Selberg zeta function (1.10) and \( C_{g,n} \) is a constant depending only on \( g \) and \( n \) ([DPS86, Sar87]), so that Theorem 2 gives a factorization of \( Z(n) \) for integers \( n > 1 \), considered as functions on \( \mathcal{S}_{g} \). As in the case of Zograf’s formula, the function \( F(n) \) defines an isometry between the line bundle \( \lambda_{n} \) over \( \mathcal{M}_{g} \) equipped with the Quillen metric, and the holomorphic line bundle over \( \mathcal{M}_{g} \) determined by the Hermitian metric \( \exp \{ \frac{\omega - 6n + 1}{12} S \} \). Theorem 2 together with (1.11), immediately implies the local families index theorem (1.13), of which it may be considered the “\( \partial \bar{\partial} \) antiderivative”.

Theorem 2. Let \( g > n > 1 \), and let \( \det ' \Delta_{n} \) and \( S \) be the functions on Schottky space \( \mathcal{S}_{g} \) defined above. Let \( p : \mathcal{J}_{g} \to \mathcal{S}_{g} \) be the universal Schottky curve, let \( T_{V} \mathcal{J}_{g} \) be the vertical tangent bundle, and let \( \varphi_{1}, \ldots, \varphi_{a_{n}} \) be the family of global holomorphic sections of \( p_{*}(T_{V} \mathcal{J}_{g}^{-n}) \) (the “natural basis” for \( n \)-differentials) defined in Section 7 below, forming a basis for each fibre. For \( t \in \mathcal{S}_{g} \) let \( \left[ N_{n}\right]_{jk}(t) = \langle \varphi_{j}(t), \varphi_{k}(t) \rangle \), where the inner product is induced from the hyperbolic metric on the compact Riemann surface \( X \simeq \Gamma \backslash \Omega \).

Thus we arrive at the main result of the paper.

\[
T_{n}(z) = \lim_{z' \to z} \left( n \frac{\partial}{\partial z'} - (1 - n) \frac{\partial}{\partial z} \right) \left( K_{n}(z, z') - \frac{1}{\pi} \frac{1}{z - z'} \right), \quad z \in \Omega,
\]

where \( K_{n} \) is the Green’s function for the \( \partial_{n} \)-operator. The advantage of this representation is that, although \( T_{n} \) fails to be holomorphic, \( \frac{\partial T_{n}}{\partial z} \) can be characterized explicitly, and the projection can be avoided by means of a contour integration. In this we make rigorous the heuristic outline given in [Mar87], where \( T_{n} \) arises as the “stress-energy tensor of Faddeev-Popov ghosts” (or “\( b \) and \( c \) fields of spins \( n \) and \( 1 - n \)” on the Riemann surface \( X \simeq \Gamma \backslash \Omega \).

See section 2.1 for the definitions of \( q \), and for the normalization of the marked Schottky group. For \( n > 1 \) and \( g > 1 \), we have \( \det ' \Delta_{n} = C_{g,n} Z(n) \), where \( Z(s) \) is the Selberg zeta function (1.10) and \( C_{g,n} \) is a constant depending only on \( g \) and \( n \) ([DPS86, Sar87]), so that Theorem 2 gives a factorization of \( Z(n) \) for integers \( n > 1 \), considered as functions on \( \mathcal{S}_{g} \). As in the case of Zograf’s formula, the function \( F(n) \) defines an isometry between the line bundle \( \lambda_{n} \) over \( \mathcal{M}_{g} \) equipped with the Quillen metric, and the holomorphic line bundle over \( \mathcal{M}_{g} \) determined by the Hermitian metric \( \exp \{ \frac{\omega - 6n + 1}{12} S \} \). Theorem 2 together with (1.11), immediately implies the local families index theorem (1.13), of which it may be considered the “\( \partial \bar{\partial} \) antiderivative”.

\[
\frac{\det ' \Delta_{n}}{\det N_{n}} = c_{g,n} \exp \left\{ \frac{-6n^2 - 6n + 1}{12\pi} S \right\} |F(n)|^2,
\]
Heuristically, the function $F(n)$ on $\mathcal{G}_g$ can be interpreted as a holomorphic determinant $\det' \bar{\partial}_n(t)$ of the family of $\bar{\partial}_n$-operators on Riemann surfaces $X_t$, $t \in \mathcal{G}_g$, in accordance with arguments in \cite{Kni89}. We note in passing that the functions $F(1)$ and $F(2)$ enter the “Polyakov measure for the $D = 26$ theory of closed bosonic strings” \cite{BK86 Kni89 D'H99}.

The content of the paper is the following. In Section 2 we collect the facts we will need on Kleinian groups, Green’s functions, Teichmüller and Schottky spaces, and the classical Liouville action. In Section 3 we express the Green’s function of $\bar{\partial}_n$ in terms of Poincaré series, thus completing the outline given in \cite{Mar87}. Section 4 describes our choice of a natural, holomorphically varying basis of $H^0(X_t, \omega^X_{X_t})$. Finally in Section 5 we prove Theorem 2. For $n = 1$, our proof is essentially the argument of \cite{Zog97}, which establishes Theorem 1 for those Schottky groups with exponent of convergence $\delta < 1$. (For the first part of Theorem 1 when $\delta \geq 1$, we refer to \cite{Zog89}.)

The results of this paper may be extended to the case where the $n$-differentials on $X$ are twisted by a character of the Schottky group, or equivalently, a unitary character of $\pi_1(X)$, generalizing Kronecker’s second limit formula. In this case, comparison with known bosonization results yields a product formula for theta functions in genus $g > 1$, generalizing the Jacobi triple product formula when $g = 1$. We intend to return to this in a sequel to this paper.

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## 2. Necessary basic facts

Here we fix notation, and recall the basic definitions and known results we will need.

### 2.1. Kleinian groups. \cite{Ber75}

By definition, a *Kleinian group* is a discrete subgroup $\Gamma$ of the group of Möbius transformations $\text{PSL}(2, \mathbb{C})$ which acts properly discontinuously on some non-empty open subset of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The largest such subset $\Omega \subset \hat{\mathbb{C}}$ is called the *ordinary set* of $\Gamma$ and its complement is called the *limit set* of $\Gamma$.

For integers $n$ and $m$, an automorphic form of type $(n, m)$ for $\Gamma$ is a function $f : \Omega \to \hat{\mathbb{C}}$ such that

$$f(z) = f(\gamma z) \gamma'(z)^n \bar{\gamma}'(z)^m$$

for all $z \in \Omega$, $\gamma \in \Gamma$.

We write the space of smooth forms of type $(n, m)$ as $A^{n,m}(\Omega, \Gamma)$ (abbreviating $A^{n,0} = A^n$), and the space of holomorphic forms of type $(n, 0)$ as
A function group is a Kleinian group which leaves some connected component \( \Omega_0 \subseteq \Omega \) invariant, and a uniformization of a Riemann surface \( X \) is a function group \( \Gamma \) with invariant component \( \Omega_0 \subseteq \Omega \) such that \( X \simeq \Gamma/\Omega_0 \). Since \( \Omega_0 \) is invariant, we can define the restrictions \( A^{n,m}(\Omega_0, \Gamma) \) and \( \mathcal{H}^n(\Omega_0, \Gamma) \).

The exponent of convergence of a Kleinian group \( \Gamma \) is the infimum of \( \delta \in \mathbb{R} \) such that the series \( \sum_{\gamma \in \Gamma} |\gamma'(z)|^\delta \) converges for all \( z \in \Omega \). For all Kleinian groups, \( \delta < 2 \).

A Kleinian group \( \Gamma \) is called a Fuchsian group if it leaves some Euclidean disc invariant; we will assume the disc has been conjugated to the upper half-plane \( \mathbb{H} = \{ z = x + iy \in \mathbb{C} : y > 0 \} \), so that \( \Gamma \subset \text{PSL}(2, \mathbb{R}) \).

A Kleinian group \( \Gamma \) is called a Schottky group if it is generated by \( L_1, \ldots, L_g \) satisfying the following condition: there exist \( 2g \) smooth Jordan curves \( \hat{C}_r \), \( r = \pm 1, \ldots, \pm g \), which form the oriented boundary of a domain \( D \subset \hat{\mathbb{C}} \), such that \( L_r \hat{C}_r = -\hat{C}_{-r} \), \( r = 1, \ldots, g \) (the negative sign indicating opposite orientation). The domain \( D \) is a fundamental region for \( \Gamma \). A Schottky group is a function group, and a free group on generators \( L_1, \ldots, L_g \). Each nontrivial element \( \gamma \) of \( \Gamma \) is loxodromic: there exists a unique number \( q_\gamma \in \mathbb{C} \) (the multiplier) such that \( 0 < |q_\gamma| < 1 \) and \( \gamma \) is conjugate in \( \text{PSL}(2, \mathbb{C}) \) to \( z \mapsto q_\gamma z \), that is,

\[
\frac{\gamma z - a_\gamma}{\gamma z - b_\gamma} = q_\gamma \frac{z - a_\gamma}{z - b_\gamma}
\]

for some \( a_\gamma, b_\gamma \in \hat{\mathbb{C}} \) (respectively, the attracting and repelling fixed points). A marked Schottky group is a Schottky group together with an ordered set of free generators \( L_1, \ldots, L_g \); it is normalized if \( a_{L_1} = 0, b_{L_1} = \infty, \) and \( a_{L_2} = 1 \).

It will be convenient to define \( L_{-r} := L_r^{-1} \), so that \( L_r \hat{C}_r = -\hat{C}_{-r} \) is true for all \( r \in \{ \pm 1, \ldots, \pm g \} \). We abbreviate \( a_r := a_{L_r}, b_r := b_{L_r} \) and \( q_r := q_{L_r} \). Denote by \( D_r \) the connected component of \( \hat{\mathbb{C}} - \hat{C}_r \) containing \( b_r \), for \( r = \pm 1, \ldots, \pm g \), so that \( -\hat{C}_r \) is the oriented boundary of \( D_r \) and \( L_s^s(D) \subseteq D_{-r} \) for \( s > 0 \). Since \( \Gamma \) is free, every nontrivial \( \gamma \in \Gamma \) has a unique expression as a reduced word, \( \gamma = L_{r_1}^{s_1} \cdots L_{r_m}^{s_m} \), for some \( r_j \in \{ \pm 1, \ldots, \pm g \} \), \( s_j > 0, j = 1, \ldots, m \), where \( |r_j| \neq |r_{j+1}| \) for \( j = 1, \ldots, m-1 \).

We collect some facts we will need about the action of a Schottky group on \( \hat{\mathbb{C}} \) below.

**Lemma 2.1.** Let \( \Gamma \) be a marked Schottky group. With notation as above, the following statements hold.

(i) For all \( r \neq j \) and \( s > 0 \), \( L_s^s(D_j) \subset D_{-r} \).

(ii) Let \( \gamma = L_{r_1}^{s_1} \cdots L_{r_m}^{s_m} \in \Gamma \) be a reduced word. Then \( a_\gamma \in D_{-r_1} \) and \( b_\gamma \in D_{r_m} \).

(iii) Let \( \gamma = L_{r_1}^{s_1} \cdots L_{r_m}^{s_m} \in \Gamma \) be a reduced word. Then

\[
\gamma^{-1}(a_r) \in \begin{cases} D_{r_m} & \text{if } \gamma \neq L_r \text{ for all } s > 0, \\ D_{-r} = D_{-r_m} & \text{if } \gamma = L_r \text{ for some } s > 0. \end{cases}
\]
\[ \omega - n \text{ class, and for any integers } \]  
\[ \text{Riemann surface of genus } 2. \]

The operators \( \text{primitive } \gamma \) and \( s > 0. \) Then this shows that \( a \mathcal{O} \) bundle \( m \). The operators \( \Delta \) space of smooth differential forms of type \( p,q \) on the line bundle \( \omega \) on \( X \). For future use, we mention that an element \( \gamma \) be the holomorphic cotangent bundle of \( X \). The metric determines \( \bar{\partial} \) \( n \) component \( \Omega \) 0. When \( z \) \( \partial \mathcal{O} \) \( \mathcal{O} \) closure of \( X \) carries a unique hyperbolic metric (a \( \partial \mathcal{O} \) \( \mathcal{O} \) \( \partial \mathcal{O} \) \( \gamma \) 1. \( X \) holds a unique hyperbolic metric (a \( \partial \mathcal{O} \) the inner \( \mathcal{O} \) 0 \( \mathcal{O} \) \( \partial \mathcal{O} \) \( \gamma \) 0 \( \mathcal{O} \) \( \partial \mathcal{O} \) \( \gamma \) \( \partial \mathcal{O} \) \( \mathcal{O} \). Let \( H \) \( \partial \mathcal{O} \) \( \mathcal{O} \) \( \partial \mathcal{O} \) \( \mathcal{O} \) \( \partial \mathcal{O} \) \( \mathcal{O} \) \( \partial \mathcal{O} \).

\[ \langle \varphi, \psi \rangle = \iint_D \varphi \bar{\psi} d^{1-n-m}d^2z, \]

on \( \mathcal{O}^n.m(X) \), where \( D \) is a fundamental region for \( \Gamma \) in \( \Omega_0 \), and \( d^2z = \frac{i}{2}dz \wedge d\bar{z} \) is the Euclidean area form on \( \Omega_0 \). The metric and complex structure determine a connection

\[ D = \partial_n \oplus \bar{\partial}_n : \mathcal{O}^{0,0}(X,\omega^n_X) \to \mathcal{O}^{1,0}(X,\omega^n_X) \oplus \mathcal{O}^{0,1}(X,\omega^n_X) \]
on the line bundle \( \omega^n_X \), given locally by

\[ \bar{\partial}_n = \frac{\partial}{\partial z} \quad \text{and} \quad \partial_n = \rho^{n} \frac{\partial}{\partial z} \rho^{-n}. \]
The metric determines \( \bar{\partial} \) Laplacians \( \Delta_n = \Delta_n^{0,0} = \bar{\partial}_n \partial_n \) and \( \Delta_{n,1} = \Delta_{n,1}^{0,1} = \bar{\partial}_n \bar{\partial}_n^{*} \) acting on vector spaces \( \mathcal{O}^n(X) \) and \( \mathcal{O}^{n,1}(X) \) respectively, where \( \bar{\partial}_n = -\rho^{-1} \partial_n \) is the adjoint of \( \partial_n \) with respect to (2.1).

Let \( \mathcal{H}^{n,m}(X) \) be the \( \mathcal{O}^2 \) closure of \( \mathcal{O}^{n,m}(X) \) with respect to the inner product (2.1). The operators \( \Delta_n \) and \( \Delta_{n,1} \) are self-adjoint and non-negative.
and have pure discrete spectrum in the Hilbert spaces $\mathcal{H}^n(X)$ and $\mathcal{H}^{n,1}(X)$. The corresponding eigenvalues $0 \leq \lambda_0 \leq \lambda_1 \leq \cdots$ of $\Delta_n$ (the non-zero eigenvalues of $\Delta_n$ and $\Delta_{n,1}$ coincide) have finite multiplicity and accumulate only at infinity. The determinant of $\Delta_n$ is defined by zeta regularization: 

$$
\zeta_n(s) = \sum_{\lambda_k > 0} \lambda_k^{-s},
$$

defined initially for Re $s > 1$, has a meromorphic continuation to the entire $s$-plane \[MP49\], and by definition \[RS71\], \[RS73\].

$$
\det \Delta_n = e^{-\zeta'_n(0)}.
$$

The non-zero spectrum of $\Delta_{1-n}$ is identical to that of $\Delta_{n,1}$ (see, e.g., \[ZT87a\]), so that $\det \Delta_n = \det \Delta_{1-n}$. Hence without loss of generality we will usually assume $n \geq 1$.

Denote by $I_n$ and $P_n$, respectively, the identity operator in $\mathcal{H}^n(X)$, and the orthogonal projection operator from $\mathcal{H}^n(X)$ onto $\mathcal{H}^{n,1}(X) = \ker \partial_n = \ker \Delta_n$. The Green’s operators for $\partial_n$ and $\Delta_n$ for $n \geq 1$ are the unique operators $K_n : \mathcal{H}^{n,1}(X) \to \mathcal{H}^n(X)$ and $G_n : \mathcal{H}^n(X) \to \mathcal{H}^n(X)$ respectively, such that:

**GF1.** $K_n \partial_n = G_n \Delta_n = I_n - P_n$.

**GF2.** $K_n|_{\ker \partial_n} = 0$ and $G_n|_{\ker \Delta_n} = 0$.

They are related by $K_n = G_n \partial_n$. Now, let $X \simeq \Gamma \backslash \Omega_0$ for some function group $\Gamma$ and invariant component $\Omega_0$. The Green’s functions for $\partial_n$ and $\Delta_n$ are the unique automorphic forms in two variables $K_n(z, z')$ and $G_n(z, z')$ respectively, smooth for $z' \neq \gamma z$, $z, z' \in \Omega_0$ and $\gamma \in \Gamma$, satisfying

$$(K_n \psi)(z) = \int\int_D K_n(z, z')\psi(z') \, d^2z' \quad \text{for all} \quad \psi \in A^{n,1}(\Omega_0, \Gamma)$$

and

$$(G_n \psi)(z) = \int\int_D G_n(z, z')\psi(z') \, d^2z' \quad \text{for all} \quad \psi \in A^n(\Omega_0, \Gamma).$$

The form $K_n(z, z')$ is of type $(n, 0)$ in $z$ and type $(1 - n, 0)$ in $z'$, and the form $G_n(z, z')$ is of type $(n, 0)$ in $z$ and type $(1 - n, 1)$ in $z'$. Both forms are holomorphic in $z$. The relation $K_n = G_n \partial_n$ implies

$$
K_n(z, z') = - (\partial'_{-n})^* G_n(z, z') = \rho(z')^{-n} \frac{\partial}{\partial z} (\rho(z')^{n-1} G_n(z, z')).
$$

**Remark 1.** Our convention differs from \[ZT87a\], where the Green’s function $\tilde{G}_n(z, z')$ is defined by $(G_n \psi)(z) = \langle \tilde{G}_n(z, \cdot), \psi \rangle$. The two are related by $G_n(z, z') = \rho(z')^{-n} \tilde{G}_n(z, z')$.

The Green’s function $Q_n(z, z')$ for $\Delta_n$ on the upper half plane $\mathbb{H}$ is uniquely determined by the following properties:

1. $Q_n(z, z')$ is smooth for $z \neq z'$;
2. $Q_n(\gamma z, \gamma' z') \gamma'(z)^n \gamma'(z')^{1-n} \gamma'(z') = Q_n(z, z')$ for all $\gamma \in \text{PSL}(2, \mathbb{R})$ and $z \neq z'$;

3. $Q_n(z, z') = -\frac{1}{2}(\text{Im } z')^{-2} \log |z - z'|^2 + O(1)$ as $z \to z'$;

4. $\Delta_n Q_n(z, z') = 0$ for $z \neq z'$;

and an additional growth condition as $z \to \partial \mathbb{H}$ (see [Hej76]). The terminology is justified since if $X \simeq \Gamma \setminus \mathbb{H}$ for a Fuchsian group $\Gamma$, then

$$G_n(z, z') = \sum_{\gamma \in \Gamma} Q_n(z, \gamma z') \gamma'(z')^{1-n} \gamma'(z').$$

Correspondingly, the Green’s function $R_n(z, z')$ for $\partial_n$ on $\mathbb{H}$ is $R_n(z, z') = -(\bar{\partial}_{1-n}^*) Q_n(z, z')$, and from the defining properties of $Q_n(z, z')$ we derive

$$R_n(z, z') = \frac{1}{\pi} \cdot \frac{1}{z - z'} \left( \frac{\bar{z} - z'}{\bar{z} - z} \right)^{2n-1}.$$

### 2.3. Teichmüller and Schottky spaces.

[Ber72] [Ber75] [Hej75] A marked Riemann surface is a compact Riemann surface $X$ of genus $g > 1$, equipped with (up to an inner automorphism of $\pi_1(X, x_0)$) a canonical system of generators $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ of $\pi_1(X, x_0)$, i.e., a system with the single relation $\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \cdots \alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1} = 1$. Marked Riemann surfaces will be denoted by $[X] = (X; \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g)$. Let $T_g$ be the Teichmüller space of marked Riemann surfaces of genus $g > 1$.

For a marked Riemann surface $[X]$, let $\mathcal{N}$ be the smallest normal subgroup in $\pi_1(X, x_0)$ containing $\alpha_1, \ldots, \alpha_g$. By the classical retrosection theorem, there exists a Schottky group $\Gamma \simeq \pi_1(X, x_0)/\mathcal{N}$ with ordinary set $\Omega$ such that $X \simeq \Gamma \setminus \Omega$. The group $\Gamma$ is unique if we require it to be normalized; we will always assume that $\Gamma$ is normalized and marked by generators $L_1, \ldots, L_g$ corresponding to the cosets $\beta_1 \mathcal{N}, \ldots, \beta_g \mathcal{N}$. The correspondence

$$[X] \mapsto (a_3, \ldots, a_g, b_2, \ldots, b_g, a_1, \ldots, a_g)$$

defines a complex-analytic map $\Psi : T_g \to \mathbb{C}^{3g-3}$. Its image $\mathcal{S}_g = \Psi(T_g)$ is a domain in $\mathbb{C}^{3g-3}$, called the Schottky space, and $\Psi$ is a covering map onto $\mathcal{S}_g$. The correspondence $t \mapsto \Gamma_t \setminus \Omega_t$ defines a complex-analytic covering map $\mathcal{S}_g \to \mathcal{M}_g$.

Equivalently, the Schottky space $\mathcal{S}_g$ may be defined as the set of marked, normalized Schottky groups of rank $g > 1$, with a complex structure described as follows. For every $t \in \mathcal{S}_g$, let $X_t \simeq \Gamma_t \setminus \Omega_t$ be the corresponding Riemann surface, and let $\bar{\partial}_t(t)$ and $\Delta_{\Omega_t}(t)$ be as defined in section 2.4 for the surface $X_t$. Then the holomorphic tangent space $T_t \mathcal{S}_g$ is naturally isomorphic to $\mathcal{H}^{-1,1}(\Omega_t, \Gamma_t) = \ker \Delta_{\Omega_t}^0(t) \subset \mathcal{A}^{-1,1}(\Omega_t, \Gamma_t)$ — the space of harmonic Beltrami differentials — while the holomorphic cotangent space $T^*_t \mathcal{S}_g$ is naturally isomorphic to $\mathcal{H}^2(\Omega_t, \Gamma_t) = \ker \bar{\partial}_t(t) \subset \mathcal{A}^2(\Omega_t, \Gamma_t)$ — the space of holomorphic quadratic differentials. For $\mu \in \mathcal{H}^{-1,1}(\Omega_t, \Gamma_t)$ and
$q \in \mathcal{H}^2(\Omega_t, \Gamma_t)$, the pairing is given by

$$(\mu, q) = \int_{D_t} \mu \, q \, d^2z,$$

where $D_t$ is a fundamental region for $\Gamma_t$. The inner product $\langle \cdot, \cdot \rangle$ on harmonic $(-1, 1)$-differentials defines a Hermitian metric on the Schottky space $\mathcal{S}_g$. This metric is Kähler, and coincides with the projection onto $\mathcal{S}_g$ of the Weil-Petersson metric on $\mathcal{F}_g$ (see [Ahl61]). We will call it the Weil-Petersson metric on $\mathcal{S}_g$ and will denote its symplectic form by $\omega_{WP}$.

In this definition of $\mathcal{S}_g$, one defines complex coordinates for a neighbourhood of $t \in \mathcal{S}_g$, called Bers coordinates, as follows. Given $\mu = \mathcal{H}^{-1,1}(\Omega_t, \Gamma_t)$ satisfying $\|\mu\|_{\infty} = \sup_{z \in \Omega_t} |\mu(z)| < 1$, there exists a unique homeomorphism $f^\mu : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ fixing $0, 1, \infty$ and satisfying the Beltrami equation

$$\frac{\partial f^\mu}{\partial z} = \mu \frac{\partial f^\mu}{\partial \bar{z}}.$$ 

Set $\Gamma^\mu = f^\mu \circ \Gamma \circ (f^\mu)^{-1}$, $\Omega^\mu = f^\mu(\Omega)$, and $X^\mu = \Gamma^\mu \backslash \Omega^\mu$. Choosing a basis $\mu_1, \ldots, \mu_{3g-3}$ for $\mathcal{H}^{-1,1}(\Omega_t, \Gamma_t)$ gives $\mu = \varepsilon_1 \mu_1 + \cdots + \varepsilon_{3g-3} \mu_{3g-3}$, where $\varepsilon_i \in \mathbb{C}$. The correspondence $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{3g-3}) \mapsto \Psi([X^\mu])$ introduces complex coordinates in a neighborhood of $t \in \mathcal{S}_g$; the corresponding complex structure agrees with that given by the first definition, considering $\mathcal{S}_g$ as a domain in $\mathbb{C}^{3g-3}$. In terms of Bers coordinates,

$$\omega_{WP} \left( \frac{\partial}{\partial \varepsilon_k}, \frac{\partial}{\partial \varepsilon_l} \right) = \frac{i}{2} \langle \mu_k, \mu_l \rangle \quad \text{at } t \in \mathcal{S}_g.$$

The Schottky universal curve is a fibration $p : \mathcal{S}_g \rightarrow \mathcal{S}_g$ with fibre $\pi^{-1}(t) = X_t \simeq \Gamma_t \backslash \Omega_t$ for $t \in \mathcal{S}_g$. Let $T_V \mathcal{I}_g \rightarrow \mathcal{S}_g$ be the holomorphic vertical tangent bundle — the holomorphic line bundle over $\mathcal{S}_g$ consisting of vectors in the holomorphic tangent space $T \mathcal{S}_g$ that are tangent to the fibres $X_t = \pi^{-1}(t)$. A family $\varphi^\varepsilon$ of $(n, m)$-differentials on Riemann surfaces $X^{\varepsilon^\mu}$ is defined as a smooth section of the line bundle

$$(T_V \mathcal{I}_g)^{-n} \otimes (T_V \mathcal{I}_g)^{-m} \rightarrow \mathcal{S}_g.$$ 

The hyperbolic metric $\rho$ gives rise to a family of $(1, 1)$-differentials and defines a natural Hermitian metric on the line bundle $T_V \mathcal{I}_g \rightarrow \mathcal{S}_g$, whose restriction to each fibre coincides with the hyperbolic metric. It also defines a Hermitian metric in the bundle $(T_V \mathcal{I}_g)^{-n} \rightarrow \mathcal{S}_g$, and in the direct image bundle $\Lambda_n = p_\ast((T_V \mathcal{I}_g)^{-n}) \rightarrow \mathcal{S}_g$. The fibre of $\Lambda_n$ over $t \in \mathcal{S}_g$ is the vector space $\mathcal{H}^n(\Omega_t, \Gamma_t)$, and the corresponding Hermitian metric is given by (2.1). The pullback of an $(n, m)$-differential $\varphi^\varepsilon$ over $X^{\varepsilon^\mu}$ is an $(n, m)$-differential over $X = X^0$, defined by

$$f^{\varepsilon^\mu}_{z^k}(\varphi^\varepsilon) = \varphi^\varepsilon \circ f^\varepsilon^\mu (f_{z^k}^{\varepsilon^\mu})^n (f_{z^m}^{\varepsilon^\mu})^m,$$

where $f^{\varepsilon^\mu} : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ is the corresponding solution of Beltrami equation. The Lie derivatives of the family $\varphi^\varepsilon$ in the directions $\mu$ and $\bar{\mu}$, where $\mu \in \mathbb{C}$,
The functions $\delta_\mu \varphi = \frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon=0} f_{s}^{\varepsilon\mu}(\varphi^\varepsilon) \in A^{n,m}(X)$ and $\bar{\delta}_\mu \varphi = \frac{\partial}{\partial \bar{\varepsilon}} \bigg|_{\varepsilon=0} f_{s}^{\varepsilon\mu}(\varphi^\varepsilon) \in A^{n,m}(X)$.

Every smooth function $\varphi$ on $S_g$ is naturally identified with a family of $(0,0)$-differentials, constant along the fibres of $p$, which we will continue to denote by $\varphi$. In this case the Lie derivative coincides with the usual directional derivative,

$$\delta_\mu \varphi = \partial \varphi(\mu) \quad \text{and} \quad \bar{\delta}_\mu \varphi = \bar{\partial} \varphi(\mu),$$

where $\partial$ and $\bar{\partial}$ are the $(1,0)$ and $(0,1)$ components, respectively, of the deRham differential $d$ on the complex manifold $S_g$. Similarly, for a family of linear operators $A_\varepsilon : A^{k,l}(X^{\varepsilon\mu}) \to A^{m,n}(X^{\varepsilon\mu})$ we define the Lie derivatives by

$$\delta_\mu A = \frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon=0} \left( f_{s}^{\varepsilon\mu} A^\varepsilon (f_{s}^{\varepsilon\mu})^{-1} \right)$$

and $\bar{\delta}_\mu A = \frac{\partial}{\partial \bar{\varepsilon}} \bigg|_{\varepsilon=0} \left( f_{s}^{\varepsilon\mu} A^\varepsilon (f_{s}^{\varepsilon\mu})^{-1} \right)$,

so that

$$\delta_\mu (A(\varphi)) = \delta_\mu A(\varphi) + A(\delta_\mu \varphi) \quad \text{and} \quad \bar{\delta}_\mu (A(\varphi)) = \bar{\delta}_\mu A(\varphi) + A(\bar{\delta}_\mu \varphi).$$

Now we present some variational formulas we will need. For $\mu \in H^{-1,1}(\Omega, \Gamma)$ define

$$F_\mu = \frac{\partial}{\partial \varepsilon} f_{s}^{\varepsilon\mu} \bigg|_{\varepsilon=0} \quad \text{and} \quad \Phi_\mu = \frac{\partial}{\partial \bar{\varepsilon}} f_{s}^{\varepsilon\mu} \bigg|_{\varepsilon=0}.$$  

Then [Ahl61]

$$\frac{\partial F_\mu}{\partial \varepsilon} = \mu \quad \text{and} \quad \Phi_\mu = 0,$$

and $\chi_\mu[\gamma] = \frac{F_\mu \circ \gamma}{\gamma'} - F_\mu$ is a polynomial of order $\leq 2$ every $\gamma \in \Gamma$. (For groups other than Schottky, function $\Phi_\mu$ is holomorphic on $\Omega$ but not necessarily zero.) Note that the normalization of $f_{s}^{\varepsilon\mu}$ implies that $F_\mu(0) = F_\mu(1) = F_\mu(\infty) = 0$, and hence $\chi_\mu[L_1](0) = 0$, $\chi_\mu[L_1](\infty) = 0$, and $\chi_\mu[L_2](1) = 0$. (Here $F_\mu(\infty) = 0$ means $F_\mu(z) = o(|z|^2)$ as $z \to \infty$, and similarly for $\chi_\mu[L_1]$.) Another classical result of Ahlfors [Ahl61] is that for the family $\rho$ of $(1,1)$-differentials given by the hyperbolic metric,

$$\delta_\mu \rho = 0 \quad \text{and} \quad \bar{\delta}_\mu \rho = 0.$$

From this one finds (see, e.g., [ZT87a]),

$$\delta_\mu \bar{n} = -\mu n \quad \text{and} \quad \delta_\mu n = 0,$$

and hence

$$\delta_\mu \Delta_n = \rho^{-1} \mu n_{n+1} \partial_n.$$
If $\varphi$ is a smooth family of holomorphic automorphic forms of type $(n,0)$, then differentiating $\bar{\partial}_n \varphi = 0$ one gets

$$\bar{\partial}_n (\delta_\mu \varphi) = \mu \partial_n \varphi \quad \text{and} \quad \bar{\partial}_n (\bar{\delta}_\mu \varphi) = 0,$$

where the last equation follows from $\bar{\delta}_\mu \bar{\partial}_n = 0$. Finally, for $t \in \mathfrak{S}_g$ let $\gamma_t \in \Gamma_t$ be a group element corresponding to a fixed element $[\gamma]$ under the isomorphism $\Gamma_t \simeq \pi_1(X,x_0)/\mathcal{N}$. Then the multipliers $q_\gamma$ give rise to a holomorphic function $q_\gamma : \mathfrak{S}_g \to \mathbb{C}$. Identifying $T^*_t \mathfrak{S}_g \simeq H^2(\Omega_t, \Gamma_t)$, we have (see e.g. [Zog89])

$$\partial q_\gamma = -\frac{g_\gamma}{\pi} \sum_{\sigma \in \langle \gamma \rangle \backslash \Gamma} \frac{(a_\gamma - b_\gamma)^2}{(\sigma z - a_\gamma)^2 (\sigma z - b_\gamma)^2} \sigma'(z)^2,$$

where the sum runs over the set of left cosets in $\Gamma$ of the cyclic subgroup generated by $\gamma$.

### 2.4. Classical Liouville action.

The Schottky space $\mathfrak{S}_g$ is a domain of holomorphy [Hej75], so that the Weil-Petersson metric on $\mathfrak{S}_g$ has a globally defined Kähler potential. Here we present the potential for the Weil-Petersson metric constructed in [ZT87b]. It is given by the “classical Liouville action” — the critical value of the “Liouville action functional” on the family of Riemann surfaces parameterized by the Schottky space $\mathfrak{S}_g$ — and has the additional property of establishing a relation between Fuchsian and Schottky uniformizations.

Namely for $t \in \mathfrak{S}_g$ set $X = X_t$; for convenience, we omit the subscript $t$ here and write $X \simeq \Gamma \backslash \Omega$, etc. Let $\rho(z) |dz|^2$ be the hyperbolic metric on $\Omega$, pulled back from the hyperbolic metric on $X \simeq \Gamma \backslash \Omega$. Let $D$ be a fundamental region for the marked Schottky group $\Gamma$ (see section 2.1). Set

$$S = \int \int_D \left( \frac{\partial \log \rho}{\partial z} \right)^2 + \rho \quad d^2 z$$

$$+ \frac{i}{2} \sum_{k=2}^{g} \oint_{C_k} \left( \log \rho - \frac{1}{2} \log |L_k|^2 \right) \left( \frac{L_k^*}{L_k} \ dz - \frac{\bar{L}_k}{L_k} \ d\bar{z} \right)$$

$$+ 4\pi \sum_{k=2}^{g} \log |c(L_k)|^2,$$

where for $\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma$, we denote $c(\gamma) = c$. This definition does not depend on a particular choice of the fundamental region $D$. The values $S_t$ for $t \in \mathfrak{S}_g$ define a smooth function $S : \mathfrak{S}_g \to \mathbb{R}$, called the classical Liouville action (see [ZT87b] for motivation and details, and [TT03] for a cohomological interpretation). The function $S$ is invariant with respect to transformations of $\mathfrak{S}_g$ corresponding to permutations of the generators of the marked Schottky group [Zog89]. For a holomorphic function $f$ with
\[ f' \neq 0, \text{ the } \text{Schwarzian derivative of } f \text{ is} \]
\[ S(f) = \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2. \]

For \( X \simeq \Gamma \setminus \Omega \) let \( J : \mathbb{H} \to \Omega \) be the universal covering of \( \Omega \) and set
\[ \vartheta = 2 S(J^{-1}). \]

Though the mapping \( J \) is not one-to-one, it follows from the properties of \( J \) and \( S \) that \( \vartheta \) is a well-defined element of \( \mathcal{H}^2(\Omega, \Gamma) \) \([ZT87b]\). Correspondingly, the smooth family \( \vartheta_t \) of holomorphic quadratic differentials on \( X_t \) gives rise to a \((1,0)\)-form \( \vartheta \) on \( \mathfrak{S}_g \).

**Proposition 2.2.** The function \( S : \mathfrak{S}_g \to \mathbb{R} \) has the following properties.

(i) \( \partial S = \vartheta; \)

(ii) \( \partial \overline{\partial} S = 2i \omega_{\mathcal{V}}. \)

**Proof.** See \([ZT87b]\) (and \([TT03]\) for generalization to Kleinian groups of class A). \( \square \)

**3. Poincaré series and the Green’s function of \( \bar{\partial}_n \)**

Let \( X \simeq \Gamma \setminus \Omega_0 \) for some function group \( \Gamma \) and invariant component \( \Omega_0 \), and let \( n \) be a positive integer. In this section we define a meromorphic Poincaré series \( \tilde{K}_n(z,z') \) and a smooth kernel \( K^0_n(z,z') \) associated with the subspace \( \mathcal{H}^n(\Omega_0, \Gamma) = \ker \bar{\partial}_n \), such that for \( n > 1 \) the Green’s function \( K_n(z,z') \) of \( \bar{\partial}_n \) is given by \( K_n = \tilde{K}_n + K^0_n \). (There is a slight modification when \( n = 1 \).) This completes the outline sketched in \([Mar87]\).

For convenience, assume that \( \infty \) is in the limit set of \( \Gamma \). For \( n > 1 \), fix points \( A_1, \ldots, A_{2n-1} \) in the limit set of \( \Gamma \), such that
\[ \forall j \exists \text{ at most } n - 1 \text{ distinct } k \text{ such that } A_k = A_j. \]

If \( n = 1 \), fix a single point \( A_1 \) in the ordinary set of \( \Gamma \). Then for \( n \geq 1 \) and \( z, z' \in \Omega_0 \) with \( z' \neq \gamma z \) for all \( \gamma \in \Gamma \), define \([Ber67]\)
\[ \tilde{K}_n(z,z') = \frac{1}{\pi} \sum_{\gamma \in \Gamma} \frac{1}{\gamma z - z'} \left( \prod_{j=1}^{2n-1} \frac{z' - A_j}{\gamma z - A_j} \right) \gamma'(z)^n, \]
with the natural conventions if \( A_j = \infty \) for one or more \( j \).

**Lemma 3.1.** Let \( \Gamma \) and \( \tilde{K}_n \) be defined as above.

(i) Suppose \( n > 1 \). For \( z, z' \in \Omega_0 \) with \( z' \neq \gamma z \) for all \( \gamma \in \Gamma \), the series \( \tilde{K}_n(z,z') \) converges absolutely and uniformly on compact subsets. It defines a meromorphic function on \( \Omega_0 \times \Omega_0 \) with only simple poles, at \( z' = \gamma z, \gamma \in \Gamma \).

(ii) Suppose that \( \Gamma \) has exponent of convergence \( \delta < 1 \). Then for \( z, z' \in \Omega_0 \) with \( z' \neq \gamma z \) and \( z \neq \gamma A_1 \) for all \( \gamma \in \Gamma \), the series \( \tilde{K}_1(z,z') \) converges absolutely and uniformly on compact subsets. It defines a
A direct computation shows that for any \( F \) cocycle is a map \( \chi: \Gamma \rightarrow \Pi \) with coefficients in \( \Pi \) and denote by \( \rho(z)|dz|^2 \) is the hyperbolic metric on \( \Omega_0 \). This was proved in [Ber71] using Ahlfors’ estimates for \( \rho \), under the assumption that \( A_1, \ldots, A_{2n-1} \) are distinct points in the limit set. Exactly the same proof works when some of the \( A_j \) coincide, provided they satisfy condition (3.1). Because \( A_1 \) is in the ordinary set for \( n = 1 \), (ii) follows immediately from the definition of \( \delta \). □

Let \( \Pi_{2n-2} \) be the vector space of polynomials of degree \( \leq 2n-2 \), considered as a right \( \Gamma \)-module with the \( \gamma \in \Gamma \) acting on \( p \in \Pi_{2n-2} \) by
\[
\gamma \ast p = p \circ \gamma \cdot (\gamma')^{1-n},
\]
and denote by \( Z^1(\Gamma, \Pi_{2n-2}) \) the vector space of 1-cocycles for the group \( \Gamma \) with coefficients in \( \Pi_{2n-2} \) — the \textit{Eichler cocycles} [Ber67]. Explicitly, a cocycle is a map \( \chi: \Gamma \rightarrow \Pi_{2n-2} \) satisfying
\[
\chi(\gamma z_1 \gamma_2) = \gamma_2 \ast \chi(\gamma_1) + \chi(\gamma_2) \quad \text{for all} \quad \gamma_1, \gamma_2 \in \Gamma.
\]
A direct computation shows that for any \( \gamma \in \Gamma \),
\[
\hat{K}_n(\gamma z, z') \gamma'(z)^n = \hat{K}_n(z, z') \quad \text{and} \quad \hat{K}_n(z, \gamma z') \gamma'(z')^{1-n} = \hat{K}_n(z, z') + \chi_{\hat{K}}(\gamma)(z, z'),
\]
where \( \chi_{\hat{K}}(z, \cdot) \in Z^1(\Gamma, \Pi_{2n-2}) \) for every \( z \in \Omega_0 \), and \( \chi_{\hat{K}}(\gamma)(\cdot, z') \in \mathcal{H}^n(\Omega_0, \Gamma) \) for every \( \gamma \in \Gamma \) and \( z' \in \mathbb{C} \).

Now, let \( \varphi_1, \ldots, \varphi_d \) be a basis for \( \mathcal{H}^n(\Omega_0, \Gamma) \), where \( d = (2n-1)(g-1) \) (\( n > 1 \)), or \( d = g \) (\( n = 1 \)). Define \textit{potentials} \( F_k \) (there should be no confusion with \( F_\mu \) defined in section 2.3) of the automorphic forms \( \varphi_k \) by
\[
F_k(z) = -\frac{1}{\pi} \iint_{\Omega_0} \frac{\rho(\zeta)^{1-n} \overline{\varphi_k}(\zeta)}{\zeta - z} \prod_{j=1}^{2n-1} \frac{z - A_j}{\zeta - A_j} d^2\zeta
\]
(3.3)
\[
= -\int_{D_0} \rho(\zeta)^{1-n} \overline{\varphi_k}(\zeta) \hat{K}_n(\zeta, z) d^2z
\]
\[
= -\langle \hat{K}_n(\cdot, z), \varphi_k \rangle,
\]
where \( \rho(\zeta) \) is the hyperbolic metric on \( \Omega_0 \). Note that though \( \hat{K}_n(\cdot, z) \) is not in \( \mathcal{F}^n(\Omega_0, \Gamma) \), the inner product given by (2.1) is still well-defined. The
function $F_k$ on $\Omega_0$ has the property

$$(3.4) \quad \frac{\partial F_k}{\partial \bar{z}} = \rho^{1-n} \bar{\varphi}_k.$$  

Let $[N_n]_{jk} = \langle \varphi_j, \varphi_k \rangle$ be the Gram matrix of the basis $\varphi_1, \ldots, \varphi_d$ with respect to the inner product (2.1), and let $N_n^{jk} = [N_n^{-1}]_{jk}$ be the inverse matrix. For $z, z' \in \Omega_0$ set

$$(3.5) \quad K_0^n(z, z') = \sum_{j=1}^{d} \sum_{k=1}^{d} N_n^{kj} \varphi_j(z) F_k(z').$$

It follows from (3.4) that

$$(3.6) \quad \frac{\partial K_0^n}{\partial \bar{z}'}(z, z') = P_n(z, z')$$

is the integral kernel of the orthogonal projection $P_n : H^n(\Omega_0, \Gamma) \to H^n(\Omega_0, \Gamma)$. For any $\gamma \in \Gamma$ we have

$$K_0^n(\gamma z, z') \gamma'(z)^n = K_0^n(z, z')$$

$$K_0^n(z, \gamma z') \gamma'(z)^{1-n} = K_0^n(z, z') - \sum_{j=1}^{d} \sum_{k=1}^{d} N_n^{kj} \varphi_j(z) (\chi_{\hat{K}}[\gamma](\cdot, z'), \varphi_k)$$

$$= K_0^n(z, z') - \chi_{\hat{K}}[\gamma](z, z'),$$

since $\chi_{\hat{K}}[\gamma](\cdot, z') \in H^n(\Omega_0, \Gamma)$. Hence $\hat{K}_n + K_0^n$ is an automorphic form of type $(n, 0)$ in $z$ and type $(1-n, 0)$ in $z'$.

**Proposition 3.2.** Let $\Gamma$, $\hat{K}_n$ and $K_0^n$ be defined as above, and let $K_n$ be the Green’s function for $\partial_n$ on $\Gamma \setminus \Omega_0$ defined in section 2.2. Then:

(i) for $n > 1$ and $z, z' \in \Omega_0$,

$$K_n(z, z') = \hat{K}_n(z, z') + K_0^n(z, z');$$

(ii) if $\delta < 1$, then for $z, z' \in \Omega_0$,

$$K_1(z, z') - K_1(z, A_1) = \hat{K}_1(z, z') + K_0^1(z, z').$$

**Proof.** First we verify condition GF1, i.e., show that for any $\varphi \in A^n(\Omega_0, \Gamma)$,

$$\iint_{D_0} (\hat{K}_n + K_0^n)(z, z') (\partial_n \varphi)(z') d^2 z' = \varphi(z) - (P_n \varphi)(z),$$

where $D_0$ is a fundamental region for $\Gamma$ in $\Omega_0$. We have

$$\iint_{D_0} (\hat{K}_n + K_0^n)(z, z') (\partial_n \varphi)(z') d^2 z' = I_1 - I_2,$$
where
\[
I_1 = \lim_{\varepsilon \to 0} \int \int_{D_0 \backslash \{|z' - z| \leq \varepsilon\}} \bar{\partial}'_n \left( (\tilde{K}_n + K_0^n)(z, z') \varphi(z') \right) d^2 z',
\]
\[
I_2 = \lim_{\varepsilon \to 0} \int \int_{D_0 \backslash \{|z' - z| \leq \varepsilon\}} \bar{\partial}'_n \left( (\tilde{K}_n + K_0^n)(z, z') \right) \varphi(z') d^2 z'.
\]
By Stokes' theorem, \( I_1 \) is a sum of an integral over the boundary of \( D_0 \), which vanishes since \( (\hat{K}_n + K_0^n)(z, z') \varphi(z') \) is a \((1, 0)\)-differential in \( z' \), and a boundary term around the singularity \( z' = z \), so that
\[
I_1 = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \oint_{|z' - z| = \varepsilon} \left( \frac{\varphi(z')}{z' - z} + O(1) \right) d z' = \varphi(z).
\]
Since \( \tilde{K}_n(z, z') \) is holomorphic in \( z' \) for \( z' \neq z \), using (3.6) we get
\[
I_2 = (P_n \varphi)(z).
\]
Since condition GF2 is vacuous for \( n > 1 \), the above establishes (i) in that case. When \( n = 1 \), the above argument shows that the operators \( K_1 \) and \( \hat{K}_1 + K_0^1 \) agree on \( \text{Im} \bar{\partial}_1 \), that is,
\[
K_1(z, z') = \hat{K}_1(z, z') + K_0^1(z, z') + \psi(z)
\]
for some \( \psi \in \mathcal{H}^1(\Omega_0, \Gamma) \). Setting \( z' = A_1 \) evaluates \( \psi \) and yields (ii).

Remark 2. It follows that
\[
\frac{\partial K_1}{\partial z'}(z, z') = \frac{\partial \hat{K}_1}{\partial z'}(z, z') + \frac{\partial K_0^1}{\partial z'}(z, z'),
\]
which is Fay’s formula relating Bergmann and Schiffer kernels on a compact Riemann surface [Fay77]. This was used in the proof of the local families index theorem [11.11] in the case \( n = 1 \) given in [ZT87a], and was the starting point for the proof of Zograf’s factorization formula [L97] in [Zog97].

4. Natural basis for \( H^0(\mathcal{G}_g, \Lambda_n) \)

It was proved by Kra [Kra84] that the direct image vector bundle
\[
\Lambda_n = p_*(T_V \mathcal{G}_g)^{-n} \to \mathcal{G}_g
\]
is holomorphically trivial, i.e., there exist \( \varphi_1, \ldots, \varphi_d \in H^0(\mathcal{G}_g, \Lambda_n) \) such that for each \( t \in \mathcal{G}_g \), the holomorphic \( n \)-differentials \( \varphi_1(t), \ldots, \varphi_d(t) \) on \( X_t \) form a basis of the fibre \( \mathcal{H}^n(X_t) \). For \( n = 1 \), the abelian differentials \( \varphi_1(t), \ldots, \varphi_g(t) \) on the Riemann surface \( X_t \) with the classical normalization
\[
\int_{\alpha_k} \varphi_j = \delta_{jk}
\]
form such a basis, since every \( t \in \mathcal{G}_g \) uniquely determines the \( \alpha \)-cycles on the Riemann surface \( X_t = \Gamma_t \backslash \Omega_t \) (see [Zog89]). Here we construct a natural
basis of the global sections of $\Lambda_n$ for $n > 1$, which reduces to the former when $n = 1$.

Let $\Gamma$ be normalized, marked Schottky group with distinguished system of generators $L_1, \ldots, L_g$. For $n > 1$, a cocycle $\chi \in Z^1(\Gamma, \Pi_{2n-2})$ is called normalized if
\[
\frac{\partial^r \chi[L_1]}{\partial z^r}(z) = 0, \quad 0 \leq r \leq n - 2, \quad \chi[L_1](z) = o(|z|^n) \text{ as } z \to \infty,
\]
and $\chi[L_2](1) = 0$. Every cocycle $\chi \in Z^1(\Gamma, \Pi_0) = Z^1(\Gamma, \mathbb{C})$ is called normalized by definition. Let $\tilde{\Pi}^1(\Gamma, \Pi_{2n-2})$ be the vector space of normalized Eichler cocycles. Since any cocycle may be normalized by adding a coboundary $b \in B^1(\Gamma, \Pi_{2n-2})$ — a cocycle $b[\gamma] = \gamma_* p - p$ for some $p \in \Pi_{2n-2}$ — and every normalized $b \in B^1(\Gamma, \Pi_{2n-2})$ is identically zero, we have an isomorphism
\[
H^1(\Gamma, \Pi_{2n-2}) := Z^1(\Gamma, \Pi_{2n-2})/B^1(\Gamma, \Pi_{2n-2}) \simeq \tilde{\Pi}^1(\Gamma, \Pi_{2n-2}).
\]
Let $\Pi^g_{2n-2} = \Pi_{2n-2} \times \cdots \times \Pi_{2n-2}$, and define
\[
\Pi^g_{2n-2} = \{(p_1, \ldots, p_g) \in \Pi^g_{2n-2} : p_1(z) = cz^{n-1}, \quad p_2(1) = 0\}.
\]

Since the group $\Gamma$ is free, the mapping from $\tilde{\Pi}^1(\Gamma, \Pi_{2n-2})$ to $\Pi^g_{2n-2}$ given by
\[
\chi \mapsto (\chi[L_1], \ldots, \chi[L_g])
\]
is an isomorphism. Fix a basis of $\Pi^g_{2n-2}$; this fixes a basis
\[
\xi_1, \ldots, \xi_d \in \tilde{\Pi}^1(\Gamma, \Pi_{2n-2}) \simeq H^1(\Gamma, \Pi_{2n-2}).
\]
This basis depends only on $\Gamma$ as an abstract group — that is, $\xi_k[\gamma]$ depends only on the reduced word $L_{t_1}^\pm \cdots L_{t_m}^\pm$ representing $\gamma$. Thus we have defined a basis of $H^1(\Gamma, \Pi_{2n-2})$ simultaneously for all normalized marked Schottky groups $\Gamma_t, \quad t \in \mathcal{S}_g$.

Now we define a basis for $\mathcal{H}^n(\Omega, \Gamma)$ corresponding to the basis $\xi_1, \ldots, \xi_d$ of $\tilde{\Pi}^1(\Gamma, \Pi_{2n-2})$ associated with a fixed basis of $\Pi^g_{2n-2}$. For this purpose we use the Bers map $\beta^* : \mathcal{H}^n(\Omega, \Gamma) \to H^1(\Gamma, \Pi_{2n-2})$, where $\chi = \beta^*(\varphi)$ is defined by
\[
\chi[\gamma] = F \circ \gamma \cdot (\gamma')^{1-n} - F,
\]
with $F$ a potential of the holomorphic $n$-differential $\varphi$ given by (3.3). The potential $F$ depends on the points $A_1, \ldots, A_{2n-1}$ in the limit set of $\Gamma$; a different choice of normalization points adds a coboundary to $\chi$. We will always choose the normalization points to be $0, \ldots, 0, 1, \infty, \ldots, \infty$. With this normalization, we get a mapping
\[
\tilde{\beta}^* : \mathcal{H}^n(\Omega, \Gamma) \to \tilde{\Pi}^1(\Gamma, \Pi_{2n-2}).
\]
Since the Bers mapping $\beta^*$ is injective, $\tilde{\beta}^*$ is also; and the vector spaces $\mathcal{H}^n(\Omega, \Gamma)$ and $\tilde{Z}^1(\Gamma, \Pi_{2n-2})$ have the same dimension $d$, so $\tilde{\beta}^*$ is a complex anti-linear isomorphism. Define a basis $\psi_1, \ldots, \psi_d$ of $\mathcal{H}^n(\Omega, \Gamma)$ by

$$\tilde{\beta}^*(\psi_k) = \xi_k,$$

and let $\varphi_1, \ldots, \varphi_d$ be the dual basis of $\mathcal{H}^n(\Omega, \Gamma)$ with respect to the inner product (2.1):

$$\langle \varphi_j, \psi_k \rangle = \delta_{jk}.$$

**Lemma 4.1.** The holomorphic $n$-differentials $\varphi_1(t), \ldots, \varphi_d(t) \in \mathcal{H}^n(X_t)$, constructed above for every point $t \in \mathcal{S}_g$, define global holomorphic sections $\varphi_1, \ldots, \varphi_d$ of the bundle $\Lambda_n$ over $\mathcal{S}_g$.

**Proof.** It follows from the construction that the $\varphi_j$ are smooth global sections of $\Lambda_n$; we must show they are holomorphic. Fix $t \in \mathcal{S}_g$ and abbreviate $\varphi_j(t) = \varphi_j, \Gamma_t = \Gamma, \text{etc.}$ Let $\mu \in \mathcal{H}^{-1,1}(\Omega, \Gamma)$ represent a tangent vector at $t$. It follows from (2.3) that $\bar{\partial}^n(\bar{\delta}^j \varphi_j) = 0$, i.e., $\bar{\delta}^j \varphi_j \in \mathcal{H}^n(\Omega, \Gamma)$. But by the definition of $\xi_k$ and Stokes' theorem,

$$\delta_{jk} = \langle \varphi_j, \psi_k \rangle = \iint_D \varphi_j \frac{\partial F_k}{\partial z} \, d^2z = -\frac{1}{2i} \sum_{r=1}^g \oint_{C_r} \varphi_j \xi_k [L_r] \, dz. \tag{4.1}$$

Since $\xi_k[L^\varepsilon]$ do not depend explicitly on $\varepsilon$ and $\Phi_\mu = 0$, we have $\bar{\delta}^j \xi_k[L] = 0$, so

$$0 = -\frac{1}{2i} \sum_{r=1}^g \oint_{C_r} (\bar{\delta}^j \varphi_j) \xi_k[L_r] \, dz = \langle \bar{\delta}^j \varphi_j, \psi_k \rangle$$

for each $k$, and we conclude $\bar{\delta}^j \varphi_j = 0$. \hfill $\Box$

**Remark 3.** It is necessary to take the dual basis $\varphi_j$ because the $\psi_k$ are not holomorphic sections of the bundle $\Lambda_n \to \mathcal{S}_g$. This is related to the fact that the Bers mapping $\beta^*$ is complex anti-linear.

We say that the sections $\varphi_1, \ldots, \varphi_d$ form a natural basis of $H^0(\mathcal{S}_g, \Lambda_n)$ corresponding to the basis $\xi_1, \ldots, \xi_d$ of $\tilde{Z}^1(\Gamma, \Pi_{2n-2})$ associated with a fixed basis of $\overline{\Pi}_{2n-2}$ (for brevity, a natural basis). Note that for $n = 1$, if we make the choice

$$\xi_k[L_r] = -2i\delta_{kr},$$

we recover the classical normalized basis of abelian differentials; we add this condition to the definition of natural basis when $n = 1$.

The vector bundle $\Lambda_n \to \mathcal{S}_g$ has a Hermitian metric defined by the inner product (2.1) on the fibres $\mathcal{H}^n(\Omega_t, \Gamma_t), \ t \in \mathcal{S}_g$, which induces a Hermitian metric $\| \cdot \|_n^2$ on its determinant line bundle $\lambda_n = \wedge^d \Lambda_n$. The natural basis gives a global holomorphic section $\varphi = \varphi_1 \wedge \cdots \wedge \varphi_d$ of $\lambda_n$, with

$$\| \varphi \|_n^2 = \det N_n,$$
where \([N_n]_{jk} = \langle \varphi_j, \varphi_k \rangle\). The metric and complex structure define a connection on \(\lambda_n\), which in the holomorphic frame given by \(\varphi\) is \(d + \partial \log \det N_n\), where \(d = \partial + \bar{\partial}\) is the deRham operator on \(\mathcal{S}_g\).

When \(n = 1\), the connection \((1, 0)\) form on \(\mathcal{S}_g\) can be found explicitly. By the Riemann bilinear relations, \(N_1 = \text{Im} \tau\), and we have Rauch’s formula [Rau65]

\[
\partial \tau_{jk}(\mu) = -2i \int_D \varphi_j \varphi_k \mu \, d^2 z
\]

for \(\mu \in \mathcal{H}^{-1,1}(\Omega, \Gamma)\), from which we obtain

\[
\partial \log \det N_1(\mu) = -\int\int_D \sum_{j=1}^d \sum_{k=1}^d N_1^{kj} \varphi_j \varphi_k \mu \, d^2 z,
\]

where \(N_1^{jk} = [N^{-1}]_{jk}\).

There is an analog of (4.2) for the natural basis when \(n > 1\). Namely, let

\[
T_0^n(z) = \left( n \frac{\partial}{\partial z'} - (1 - n) \frac{\partial}{\partial z} \right) K_0^n(z, z') \bigg|_{z' = z},
\]

where \(K_0^n\) is given by (3.5), and define

\[
\varkappa_n[\gamma] = T_0^n \circ \gamma \cdot (\gamma')^2 - T_0^n
\]

for each \(\gamma \in \Gamma\). Then we have the following.

**Proposition 4.2.** Let \(\varphi_1, \ldots, \varphi_d\) be a natural basis of \(H^0(\mathcal{S}_g, \Lambda_n)\) as constructed above. Fix \(t \in \mathcal{S}_g\) and abbreviate \(\varphi_j(t) = \varphi_j\), \(\Gamma_1 = \Gamma\), etc. Let \(N_n, T_n, \varkappa_n\) be defined as above, and recall the notation for the marked normalized Schottky group \(\Gamma\) fixed in section 2.1. Then for \(\mu \in \mathcal{H}^{-1,1}(\Omega, \Gamma) \simeq T_t \mathcal{S}_g\) with potential \(F_\mu\), we have

\[
\partial \log \det N_n(\mu) = \int\int_D T_0^n \mu \, d^2 z + \frac{1}{2i} \sum_{r=1}^g \oint_{C_r} \varkappa_n[L_r] F_\mu \, dz.
\]

**Proof.** Using holomorphy of the family \(\varphi_j\), Stokes’ theorem, \(\psi_j = \sum_{k=1}^d N_n^{kj} \varphi_k\) and (2.3), we have

\[
\partial \log \det N_n(\mu) = \sum_{j=1}^d \sum_{k=1}^d N_n^{kj} \langle \delta_\mu \varphi_j, \varphi_k \rangle = \sum_{j=1}^d \sum_{k=1}^d N_n^{kj} \int\int_D (\delta_\mu \varphi_j) \frac{\partial F_k}{\partial z} \, d^2 z
\]

\[
= - \int\int_D (\partial_\mu K^n_0) \Delta \mu \, d^2 z - \frac{1}{2i} \sum_{r=1}^g \oint_{C_r} \sum_{j=1}^d (\delta_\mu \varphi_j) \xi_j[L_r] \, dz,
\]
where $\Delta$ stands for the restriction on the diagonal $z' = z$. This implies
\[
\partial \log \det N_n(\mu) = \int \int_D T^n_0 \mu \, d^2 z - n \int \int_D \partial_1 (K^n_0|_{\Delta}) \mu \, d^2 z
\]
\[- \frac{1}{2i} \sum_{r=1}^g \oint_{C_r} \sum_{j=1}^d (\delta_{\mu} \varphi_j) \xi_j[L_r] \, d\bar{z},
\]
since $T^n_0 = - (\partial_n K^n_0)|_{\Delta} + n\partial_1 (K^n_0|_{\Delta})$. Using Stokes’ theorem again and $\partial_{-1} \mu = 0$, we obtain
\[
\int \int_D \partial_1 (K^n_0|_{\Delta}) \mu \, d^2 z = \frac{1}{2i} \sum_{r=1}^g \oint_{C_r} \sum_{j=1}^d \varphi_j \xi_j[L_r] \mu \, d\bar{z}.
\]
Hence we must show that
\[
(4.6) \quad \sum_{r=1}^g \oint_{C_r} \varpi[L_r] F_{\mu} \, d\bar{z} = - \sum_{r=1}^g \oint_{C_r} \sum_{j=1}^d (\delta_{\mu} \varphi_j) \xi_j[L_r] \, d\bar{z} + n\varphi_j \xi_j[L_r] \mu \, d\bar{z}.
\]
It follows from (4.1) that
\[
\sum_{r=1}^g \oint_{C_r} (\delta_{\mu} \varphi_j) \xi_k[L_r] \, d\bar{z} + \varphi_j (\delta_{\mu} \xi_k[L_r]) \, d\bar{z} + \varphi_j \xi_k[L_r] \mu \, d\bar{z} = 0,
\]
and we have
\[
\delta_{\mu} \xi_k[L_r] = \frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon = 0} \xi_k[L_r] \circ f_{\varepsilon}^\mu \cdot (f_{\varepsilon}^\mu)^{1-n}
= \frac{\partial \xi_k[L_r]}{\partial z} F_{\mu} + (1-n) \xi_k[L_r] \frac{\partial F_{\mu}}{\partial z},
\]
since, by construction, $\xi_k^\varepsilon[L_z]$ does not depend explicitly on $\varepsilon$. Using the identity
\[
0 = \oint_{C_r} d(\varphi_j \xi_k[L_r] F_{\mu})
= \oint_{C_r} \frac{\partial}{\partial z} (\varphi_j \xi_k[L_r] F_{\mu}) \, d\bar{z} + \varphi_j \xi_k[L_r] \mu \, d\bar{z},
\]
we obtain
\[
- \sum_{r=1}^g \oint_{C_r} (\delta_{\mu} \varphi_j) \xi_k[L_r] \, d\bar{z} + n\varphi_j \xi_k[L_r] \mu \, d\bar{z}
= \sum_{r=1}^g \oint_{C_r} \left( n\varphi_j \frac{\partial \xi_k[L_r]}{\partial z} - (1-n) \frac{\partial \varphi_j}{\partial z} \xi_k[L_r] \right) F_{\mu} \, d\bar{z}.
\]
Now, a straightforward computation shows that
\[
\varpi_n[\gamma] = \sum_{j=1}^d \left( n\varphi_j \frac{\partial \xi_j[\gamma]}{\partial z} - (1-n) \frac{\partial \varphi_j}{\partial z} \xi_j[\gamma] \right),
\]
which establishes (4.6) and completes the proof.

To show the agreement of (4.5) with (4.2) when \( n = 1 \), it suffices to observe that for this case, the properties of the potential \( F_k \) of the basis element \( \varphi_k \) imply that

\[
F_k(z) = \int_{A_1}^z \varphi_k(\zeta) \, d\zeta - \int_{A_1}^z \varphi_k(\zeta) \, d\zeta.
\]

5. Proof of Theorems 1 and 2

Since the functions \( \det \Delta_n, \det N_n \) and \( S \) on the Schottky space \( \mathfrak{S}_g \) are real-valued and the function \( F(n) \) on \( \mathfrak{S}_g \) is holomorphic, to prove Theorems 1 and 2 it sufficient to show that

\[
\partial \log \det \Delta_n - \partial \log F(n) = \partial \log \det N_n - \frac{6n^2 - 6n + 1}{12\pi} \partial S
\]

at all points in \( \mathfrak{S}_g \). The \((1,0)\) forms on \( \mathfrak{S}_g \) appearing on the right hand side of (5.1) are given by Propositions 2.2 and 4.2. Here we complete the proof by computing the \((1,0)\) forms on the left hand side.

5.1. Computation of \( \partial \log \det \Delta_n \).

Let \( X \) be a compact Riemann surface, with \( X \approx \Gamma \setminus \Omega_0 \) for some function group \( \Gamma \) with invariant component \( \Omega_0 \), and let \( \rho(z) |dz|^2 \) be the hyperbolic metric on \( \Omega_0 \). Define

\[
T_n(z) = \lim_{z' \to z} \left( n \frac{\partial}{\partial z'} - (1 - n) \frac{\partial}{\partial z} \right) \left( K_n(z, z') - \frac{1}{\pi} \frac{1}{z - z'} \right),
\]

where \( K_n \) is the Green’s function for \( \bar{\partial} n \) on \( \Gamma \setminus \Omega_0 \) defined in section 2.2. When \( \Omega_0 = \mathbb{H} \), we will denote \( T_n = T_{n}^{\text{Fuchs}} \). It easy to see that \( T_{n}^{\text{Fuchs}} \in A^2(\mathbb{H}, \Gamma) \). Indeed, it follows from (2.2) that

\[
\left( n \frac{\partial}{\partial z'} - (1 - n) \frac{\partial}{\partial z} \right) R_n(z, z') = \frac{1}{\pi} \frac{1}{(z - z')^2} + O(z - z')
\]

as \( z' \to z \), so that

\[
T_{n}^{\text{Fuchs}}(z) = \lim_{z' \to z} \left( n \frac{\partial}{\partial z'} - (1 - n) \frac{\partial}{\partial z} \right) \left( K_n(z, z') - R_n(z, z') \right).
\]

It follows from property 2 in section 2.2 that \( (K_n - R_n)|_{\Delta} \) is a \((1,0)\) form, and the identity

\[
T_{n}^{\text{Fuchs}} = - \left( \bar{\partial} (K_n - R_n) \right) |_{\Delta} + n \partial_1 ((K_n - R_n)|_{\Delta})
\]

proves the claim. Here \( \Delta \) stands for the restriction on the diagonal \( z' = z \).

Lemma 5.1. Let \( X \approx \Gamma \setminus \Omega_0 \) for a function group \( \Gamma \) with invariant component \( \Omega_0 \), let \( J : \mathbb{H} \to \Omega_0 \) be the holomorphic covering map of \( \Omega_0 \) by \( \mathbb{H} \), and let \( T_n \) and \( T_{n}^{\text{Fuchs}} \) be defined as above. Then on \( \Omega_0 \),

\[
T_n = T_{n}^{\text{Fuchs}} \circ J^{-1} \cdot ((J^{-1})')^2 + \frac{6n^2 - 6n + 1}{12\pi} S(J^{-1}),
\]

where \( S \) denotes the Schwarzian derivative \( \overset{\circ}{2} \). In particular, \( T_n \in A^2(\Omega_0, \Gamma) \).
Proof. Note that while $J^{-1}$ is multiple-valued, the right side is a well-defined element of $\mathcal{A}^2(\Omega_0, \Gamma)$. The equality follows from the identity
\[
\lim_{z' \to z} \left( n \frac{\partial}{\partial z} - (1 - n) \frac{\partial}{\partial z'} \right) \left( \frac{J'(z)^n J'(z')^{1-n}}{J(z) - J(z')} - \frac{1}{z - z'} \right) = \frac{6n^2 - 6n + 1}{6} \mathcal{S}(J),
\]
which is verified by direct computation. This is the classical result when $n = 1$. □

Remark 4. In conformal field theory, this result is known as the statement that “$b\text{-}c$ system with spins $n$ and $1 - n$ has central charge $6n^2 - 6n + 1$” (see, e.g., [D’H99] and references therein).

Proposition 5.2. Let $\det \Delta_n$ be the function on the Schottky space $\mathcal{S}_g$ defined in section 2.2, and let $\vartheta$ be the $(1, 0)$ form on $\mathcal{S}_g$ defined in section 2.4. For each $t \in \mathcal{S}_g$, abbreviate $T_n = T_n(t)$, $\Omega = \Omega_t$, $\Gamma = \Gamma_t$, etc. Then for $\mu \in \mathcal{H}^{-1,1}(\Omega, \Gamma) \simeq T_1 \mathcal{S}_g$,
\[
\partial \log \det \Delta_n(\mu) = \iint_D T_n \mu \, d^2 z - \frac{6n^2 - 6n + 1}{12\pi} \vartheta(\mu).
\]

Proof. Set $\mu^{\text{Fuchs}} = \mu \circ J \frac{T'}{J'}$. It follows from Lemma 5.1 that it is sufficient to prove
\[
\partial \log \det \Delta_n(\mu) = \iint_D T_n^{\text{Fuchs}} \mu^{\text{Fuchs}} \, d^2 z,
\]
where $D \subset \mathbb{H}$ is a fundamental region for a Fuchsian group uniformizing the Riemann surface $X \simeq \Gamma \backslash \Omega$. Using the identity (5.3) and $\partial^{-1} \mu = 0$, this reduces to the statement
\[
\partial \log \det \Delta_n(\mu) = -\iint_D (\partial_n(K_n - R_n)) \big|_{\Delta} \mu \, d^2 z,
\]
which is Theorem 1 in [ZT87a]. □

5.2. Computation of $\partial \log F(n)$. Let $\Gamma$ be a marked, normalized Schottky group. For positive integer $n$ define
\[
F_0(n) = \prod_{\{\gamma\}} \prod_{m=0}^{\infty} \left( 1 - q_{\gamma}^{n+m} \right),
\]
where $\{\gamma\}$ runs over all distinct primitive conjugacy classes in $\Gamma$, omitting the identity, and $q_{\gamma}$ is the multiplier of $\gamma$ — see section 2.1. The product converges absolutely if and only if the series $\sum_{\{\gamma\}} \sum_{m=0}^{\infty} |q_{\gamma}|^{m+n}$ converges. One shows that this series converges provided that the multiplier series $\sum_{[\gamma]} |q_{\gamma}|^{n}$ converges, where $[\gamma]$ runs over all distinct conjugacy classes (not necessarily primitive) in $\Gamma$. By a theorem of B"user [Bus96], for a Schottky group $\Gamma$ with exponent of convergence $\delta$, the latter series converges provided $n > \delta$. It
is known that $\delta < 2$, hence for $n > 1$ the product $F_0(n)$ converges absolutely for all Schottky groups $\Gamma$, and the product $F_0(1)$ converges absolutely provided that $\delta < 1$. Now we define

\begin{equation}
F(n) = \begin{cases} 
F_0(1) & \text{if } n = 1, \\
(1 - q_1)^2 \cdots (1 - q_1^{n-1})^2 (1 - q_2^{n-1}) F_0(n) & \text{if } n > 1.
\end{cases}
\end{equation}

For $n \geq 2$ the expression $F(n)$ defines a holomorphic function on $\mathcal{S}_g$. For $n = 1$ the function $F = F(1)$ is defined on the open subset of $\mathcal{S}_g$ characterized by $\delta < 1$.

\textbf{Remark 5.} The product $\prod_{\{\gamma\}} (1 - q_1^s)$ was briefly described in [Bow79], where it was asserted that with the values of $q_1^s$ chosen appropriately, the product is defined for all $Re s > \delta$ and has an analytic continuation to the entire $s$-plane. To our knowledge these results have not yet been proved. The function $|F_0(n)|^2$ coincides with a product of Ruelle type zeta functions $R_\rho(s)$ associated to the hyperbolic 3-manifold $X^3$ defined by $\Gamma$, considered in [Fri86]:

$$|F_0(n)|^2 = Z_n(s),$$

where $Z_n(s) = \prod_{m=0}^{\infty} R_{\rho_n+m}(s + m)$, and $\rho_{n+m}$ is the representation of $\pi_1(X^3)$ on $O(2)$ taking a closed geodesic with twist parameter $\theta$ to a rotation of angle $(n + m)\theta$.

Set

\begin{equation}
\hat{T}_n(z) = \lim_{z' \to z} \left( n \frac{\partial}{\partial z'} - (1 - n) \frac{\partial}{\partial z} \right) \left( \hat{K}_n(z, z') - \frac{1}{\pi} \frac{1}{z - z'} \right),
\end{equation}

where $\hat{K}_n$ is the Poincaré series (3.2). We have

\begin{equation}
T_n = \hat{T}_n + T_n^0,
\end{equation}

where $T_n^0$ and $T_n$ are defined in (4.3) and (5.2) respectively. Since $T_n \in \mathcal{A}^2(\Omega, \Gamma)$, we have for $\gamma \in \Gamma$,

$$\hat{T}_n \circ \gamma \cdot (\gamma)^2 - \hat{T}_n = -\varpi_n[\gamma],$$

where $\varpi_n[\gamma]$ is given by (1.4).

\textbf{Proposition 5.3.} Let $F(n) : \mathcal{S}_g \to \mathbb{C}$ be defined by (5.4) and (5.5). Fix $t \in \mathcal{S}_g$ and abbreviate $\Gamma_t = \Gamma$, etc. Let $\hat{T}_n$ and $\varpi_n$ be defined by (5.6) and (4.4) respectively, corresponding to $X_t = X = \Gamma \backslash \Omega$, and recall the notation for the marked normalized Schottky group $\Gamma$ fixed in section 2.1. For $\mu \in \mathcal{H}^{-1,1}(\Omega, \Gamma) \simeq T_1 \mathcal{S}_g$ with potential $F_\mu$, the $(1,0)$ form $\partial \log F(n)$ satisfies

$$\partial \log F(n)(\mu) = \int_D \hat{T}_n \mu \, d^2z - \frac{1}{2i} \sum_{r=1}^{9} \oint_{C_r} \varpi[L_r] F_\mu \, dz.$$
Proof. For $\gamma \in \Gamma$, $\gamma \neq \text{id}$, and $z \in \Omega$, we introduce the abbreviations

$$A_\gamma(z) = \frac{1}{\pi} \left( nq \gamma^{-1} + (1 - n)q \gamma \right) \frac{\gamma'(z)}{(\gamma z - z)^2},$$

$$B_\gamma(z) = \lim_{z' \to z} \frac{1}{\pi} \left( n \frac{\partial}{\partial z'} - (1 - n) \frac{\partial}{\partial z} \right) \frac{1}{\gamma z - z} \left( \prod_{j=1}^{2n-1} \frac{z' - A_j}{\gamma z - A_j} \right)^n \gamma'(z),$$

and split the computation into three steps.

**Step 1.** Claim that the right hand side can be written as

$$\int \int \hat{T}_n \mu d^2z - \frac{1}{2i} \sum_{r=1}^{g} \oint_{C_r} \omega[L_r] F_\mu \, dz = -\frac{1}{2i} \sum_{\gamma \in \Gamma} \sum_{\gamma \neq \text{id}} \oint_{C_r} B_\gamma \chi_\mu[L_r] \, dz.$$

We have

$$\int \int \hat{T}_n \mu d^2z = \int \int \partial(\hat{T}_n F_\mu) d^2z = \frac{1}{2i} \sum_{r=1}^{g} \left( \oint_{C_r} \hat{T}_n F_\mu \, dz \right)$$

$$= -\frac{1}{2i} \sum_{r=1}^{g} \oint_{C_r} \left( (\hat{T}_n - \omega_n[L_r])(F_\mu + \chi_\mu[L_r]) - \hat{T}_n F_\mu \right) \, dz$$

$$= -\frac{1}{2i} \sum_{r=1}^{g} \oint_{C_r} \hat{T}_n \circ L_r(L'_r)^2 \chi_\mu[L_r] \, dz + \frac{1}{2i} \sum_{r=1}^{g} \oint_{C_r} \omega_n[L_r] F_\mu \, dz.$$

But for any Eichler cocycle, $\chi[\gamma^{-1}] = -\chi[\gamma] \circ \gamma^{-1}$, so we have

$$\oint_{C_r} \hat{T}_n \circ L_r(L'_r)^2 \chi_\mu[L_r] \, dz = \oint_{C_{-r}} \hat{T}_n \chi_\mu[L_{-r}] \, dz.$$

This, together with $\hat{T}_n(z) = \sum_{\gamma \in \Gamma \setminus \{\text{id}\}} B_\gamma(z)$, converging absolutely and uniformly on compact subsets of $\Omega$, establishes (5.8). Note that the non-automorphy of $\hat{T}_n$ necessitates the use of the integral over $C_{-r}$ rather than $C_r$.

**Step 2.** Computation of $\partial \log F_0(n)$. Claim that

$$\partial \log F_0(n)(\mu) = \frac{1}{2i} \sum_{\gamma \in \Gamma} \sum_{\gamma \neq \text{id}} \oint_{C_r} A_\gamma \chi_\mu[L_{-r}] \, dz.$$
Indeed, using the expression \( \log F_0(n) = -\sum_{m=1}^{\infty} \sum_{\gamma} \frac{1}{m} \frac{q^{mn}}{1 - q^{m}} \) and the series (2.4), we get

\[
\partial \log F_0(n) = \frac{1}{\pi} \sum_{\gamma} \sum_{m=1}^{\infty} \frac{nq^{m(n-1)} + (1 - n)q^{mn}}{1 - q^{m}} \cdot \frac{(a_{\gamma} - b_{\gamma})^2}{(1 - q^{m})^2 (\sigma z - a_{\gamma})^2 (\sigma z - b_{\gamma})} \sigma'(z)^2
\]

\[
= \frac{1}{\pi} \sum_{\gamma} \sum_{m=1}^{\infty} \frac{nq^{m-1} + (1 - n)q^{m-1}}{(\sigma - 1)z - z} \sigma'(z)^2
\]

\[
= \sum_{\gamma \in \Gamma} A_\gamma(z),
\]

where we have identified \( T^*_t \mathcal{G}_g \simeq \mathcal{H}^2(\Omega, \Gamma) \). The convergence is absolute and uniform on compact subsets of \( \Omega \). Since \( \partial \log F_0(n) \), unlike \( \bar{T}_n \), is automorphic, applying Stokes’ theorem as in step 1 gives (5.9).

**Step 3.** When \( n = 1 \), we have \( \pi[\gamma] = 0 \) and \( A_\gamma(z) = B_\gamma(z) \), so the proposition is proved. For the case \( n > 1 \) we use the assumption that the normalization points \( A_1, \ldots, A_{2n-1} \) are 0, \( \infty, \ldots, \infty \) (see Section 4), and show that

\[
(5.10) \quad \partial \left( \log \prod_{j=1}^{n-1} (1 - q_1^{j})^2 (1 - q_2^{n-1}) \right)(\mu)
\]

\[
= \frac{1}{2i} \sum_{\gamma \in \Gamma} \sum_{\gamma \neq \text{id}} \int_{C_{-r}} (A_\gamma - B_\gamma) \chi_{\mu} [L_{-r}] \, dz.
\]

We first compute the right hand side of (5.10). Suppose \( \gamma \neq L_1^m, L_{-1}^m \) or \( L_2^m \) for any \( m > 0 \). Direct computation verifies that \( (A_\gamma - B_\gamma)(z) \chi_{\mu} [L_{-r}](z) \) is regular at \( \infty \), with poles at \( b_\gamma \), \( \gamma^{-1}(0) \), \( \gamma^{-1}(1) \) and \( \gamma^{-1}(\infty) \). By part (iii) of Lemma 2.4 all these poles are in a single domain \( D_{r_m} \) bounded by \( C_{r_m} \) for \( \gamma = L_{r_1}^{s_1} \cdots L_{r_m}^{s_m} \), so that every integral in (5.10) is zero. Thus the computation reduces to the cases when \( \gamma = L_1^m, L_{-1}^m \) or \( L_2^m \) for \( m > 0 \). For \( \gamma = L_1^m, m > 0 \), using Lemma 2.4 again we see that \( 0 \in D_{-1} \) and \( \gamma^{-1}(1), \infty \in D_1 \). By an elementary computation, using the identity

\[
\sum_{m=1}^{\infty} \frac{nq^{mn} + (1 - n)q^{m+1}}{1 - q^m} = \sum_{m=n}^{\infty} \frac{mq^m}{1 - q^m}, \ |q| < 1,
\]
and the normalization $\chi_\mu[L_{-1}](z) = az$, we get
\[
\frac{1}{2i} \sum_{m=1}^\infty \oint_{C_{-1}} (A_{L_{-1}}^m - B_{L_{-1}}^m)(z) \chi_\mu[L_{-1}](z) \, dz = a \sum_{j=1}^{n-1} \frac{j_q^1}{1 - q_1^j}.
\]
When $\gamma = L_{-1}^m, m > 0$, we have $\gamma^{-1}(1), 0 \in D_{-1}$ and $\infty \in D_1$. Changing $z \mapsto 1/z$ we get as before,
\[
\frac{1}{2i} \sum_{m=1}^\infty \oint_{C_{-1}} (A_{L_{-1}}^m - B_{L_{-1}}^m)(z) \chi_\mu[L_{-1}](z) \, dz = a \sum_{j=1}^{n-1} \frac{j_q^1}{1 - q_1^j}.
\]
For $\gamma = L_{2}^m$ we have by Lemma 2.1 that $1 \in D_{-2}$ and $b_2, \gamma^{-1}(0), \gamma^{-1}(\infty) \in D_2$. By an elementary computation, using the normalization $\chi_\mu[L_{-2}](z) = b(z - 1) + c(z - 1)^2$, we get
\[
\frac{1}{2i} \sum_{m=1}^\infty \oint_{C_{-2}} (A_{L_{-2}}^m - B_{L_{-2}}^m)(z) \chi_\mu[L_{-2}](z) \, dz = b(n-1) - q_2^{n-1}.
\]
To compute the left hand side of (5.10), we use (2.21) and the identity
\[
\sum_{r=1}^{\infty} \oint_{C_r} \oint_{\gamma(L)} \frac{\gamma'(z)^2}{(\gamma z - a)^2(\gamma z - b)^2} \chi_\mu[L_{-r}](z) \, dz = \oint_C \frac{\chi_\mu[L](z)}{(z - a)^2(z - b)^2} \, dz,
\]
where $a = a_L, b = b_L$ and circles $C$ and $C' = -L(C)$ form the boundary for a fundamental domain of $\langle L \rangle$ in $\mathbb{C} \setminus \{a, b\}$. (It readily follows from Stokes’ theorem and automorphy properties of the sum $\sum_{\gamma \in (L)} \Gamma$, see [Kra83]). This computation establishes (5.10) and completes the proof of the proposition.

\[\square\]

Theorem 2 now follows from (5.7) and Propositions 2.1, 2.2, 5.2 and 5.3 in the case $n > 1$. For $n = 1$, this also gives a proof of Zograf’s formula — Theorem 1 — for Schottky groups with $\delta < 1$. For the remainder of Theorem 1 we refer to [Zog89].

Remark 6. Note that the functions $\det' \Delta_n, F_0(n)$ and $S$ on $\mathcal{G}_g$ are invariant with respect to the transformations of $\mathcal{G}_g$ which correspond to permutations of the generators $L_1, \ldots, L_g$, whereas the function $\det N_n$ is not. Consequently Theorem 2 implies that the extra factors in the definition of $F(n)$ guarantee that the product $\det N_n |F(n)|^2$ is invariant with respect to these transformations. This can be also verified by a direct computation.

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