Morse theory and infinite families of harmonic maps between spheres

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Abstract. Existence of an infinite sequence of harmonic maps between spheres of certain dimensions was proven by Bizoń and Chmaj. This sequence shares many features of the Bartnik-McKinnon sequence of solutions to the Einstein-Yang-Mills equations as well as sequences of solutions that have arisen in other physical models. We apply Morse theory methods to prove existence of the harmonic map sequence and to prove certain index and convergence properties of this sequence. In addition, we generalize the result of Bizoń and Chmaj to produce infinite sequences of harmonic maps not previously known. The key features “responsible” for the existence and properties of the sequence are thereby seen to be the presence of a reflection (Z2) symmetry and the existence of a singular harmonic map of infinite index which is invariant under this symmetry.

§0 Introduction

A countably infinite sequence, \{hi\}, of harmonic maps from an \((m + 1)\)-dimensional sphere, \(S^{m+1}\), into itself for \(2 \leq m \leq 5\) was discovered and analyzed by Bizoń [4] and Bizoń and Chmaj [1]. In these references, existence of this sequence was proven via a shooting argument, and analytic arguments and/or numerical evidence also was presented that this sequence satisfies the following properties: (1) As \(i \to \infty\), we have \(h_i \to h_\infty\) pointwise except at the poles, where \(h_\infty\) denotes a singular harmonic map which maps all of \(S^{m+1}\) into the equator of \(S^{m+1}\). (2) The sequence of energies, \(\{E_i\}\), is monotone increasing and converges to the energy, \(E_\infty\), of \(h_\infty\). (3) The index of \(h_i\) is \(i\).

The above properties bear a remarkable resemblance to the properties of the Bartnik-McKinnon sequence and the colored black hole sequences of Einstein-Yang-Mills theory (see Volkov and Gal’tsov [14] for a review). Here one considers static, spherically symmetric, asymptotically flat, nonsingular solutions of the Einstein-Yang-Mills equations where the static Killing field either remains strictly timelike everywhere (the Bartnik-McKinnon case) or becomes null on a regular event horizon (the colored black hole case). The Bartnik-McKinnon solutions are labeled by a positive integer, \(i\), whereas the colored black holes are

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labeled by \(i\) and a positive real number, \(r_0\), corresponding to the radius of the event horizon. As \(i \to \infty\), the Bartnik-McKinnon sequence converges (in the sense described in section 3.1 of [14]) to the extreme Reissner-Nordstrom solution with unit magnetic charge. At fixed \(r_0\), the colored black hole sequence converges (in the sense described in section 4.1 of [14]) to a unit magnetically charged Reissner-Nordstrom solution. The mass of both the Bartnik-McKinnon and colored black hole sequences increases monotonically with \(i\) and converges to the mass of the limiting Reissner-Nordstrom solution. Finally, numerical evidence indicates that—if one suitably restricts the function space so that only “even parity” perturbations (involving only variables that are nonzero in the background) are considered—the index of both the \(i\)th Bartnik-McKinnon and the \(i\)th colored black hole solution is \(i\). Sequences of solutions with similar properties have also been found in a number of other models, in particular, static, spherically symmetric solutions in Yang-Mills-dilaton theory [1] and self-similar wave maps from Minkowski spacetime into \(S^3\) (or, equivalently, harmonic maps from the hyperboloid, \(H^3\), into \(S^3\)) [3].

The fact that very similar sequences of solutions exist in the quite different contexts of harmonic map theory and Einstein-Yang-Mills theory suggests that there should be an explanation of the existence and properties of these sequences of solutions that depends only on some general properties of the equations, not on their detailed form. An early attempt to provide such an explanation (made prior to the discovery of the harmonic map sequence) was given in [13]. In that reference, it was proposed that a key property related to the existence of the Bartnik-McKinnon and colored black hole sequences is the presence of a symmetry that ensures a “degeneracy” among solutions, i.e., that if a solution of a given mass exists, then other solutions (obtained by action of the symmetry on this solution) of the same mass also must exist. The following heuristic argument for the existence of the Bartnik-McKinnon sequence was given. (A similar argument also was given for the colored black hole sequence.) Suppose that on the phase space, \(\Gamma\), appropriate to the problem there exists a “mass flow” vector field (satisfying suitable smoothness properties and invariant under the symmetries), such that the mass monotonically decreases along the integral curves of this vector field, and such that these integral curves always asymptotically approach a critical point of mass (corresponding to a stationary solution of the Einstein-Yang-Mills equations [13]). By the positive mass theorem, the global minimum of mass is flat spacetime with a pure gauge Yang-Mills field, as well as copies of this solution under the symmetry. However, in addition to these global minima of mass (and, possibly, other local minima of mass), there should exist other critical points of mass: the integral curves of the mass flow vector field should not be able to bifurcate discontinuously between the different local minima, so there should exist points of phase space that do not flow to any local minimum. Consequently, these must flow to critical points of nonzero index. Indeed, if the set of points, \(\Gamma_1\), that do not flow to a local minimum of mass has the structure of a hypersurface of co-dimension 1, then the critical point of minimum mass within \(\Gamma_1\) would have index 1, corresponding to the first Bartnik-McKinnon solution. The argument can now be repeated, replacing \(\Gamma\) by \(\Gamma_1\), to argue for existence of a critical point of index 1 within \(\Gamma_1\) and, hence, index 2 within the original phase space \(\Gamma\). It was proposed in [13] that this would account for the second Bartnik-McKinnon solution; continued iteration of this argument should generate the entire Bartnik-McKinnon sequence. This argument would naturally account for the fact that the index of the \(i\)th Bartnik-McKinnon solution is \(i\), as well as for the fact that the masses in the sequence increase monotonically.
However, in addition to its heuristic nature and its major gaps concerning the existence of a suitable mass flow vector field, the manifold nature of $\Gamma_i$, etc., the argument given in [13] suffers from the following three serious deficiencies: First, the relevant symmetry was identified in [13] as being the “large gauge transformations” of the Yang-Mills field. However, no analog of this symmetry exists in the harmonic map problem, so if a common explanation is sought, this could not be the relevant symmetry in the Einstein-Yang-Mills context. Second, no argument was given as to why $\Gamma_1$ (or any of the higher $\Gamma_i$) should be connected or—if not connected—why the symmetry should map any connected component of $\Gamma_1$ into itself. If the symmetry fails to map a connected component of $\Gamma_1$ into itself, no bifurcation of the mass flow on $\Gamma_1$ need occur, and the above argument for additional critical points breaks down. Note that this difficulty would leave intact the argument for a critical point of index 1 (i.e., the first Bartnik-McKinnon solution) since $\Gamma$ is connected, but if the relevant symmetry is taken to be the large gauge transformations, there is no reason to expect that any higher members of the Bartnik-McKinnon sequence need exist. Third, the argument given in [13] does not account for any of the convergence properties of the Bartnik-McKinnon sequence as $i \to \infty$.

In this paper, we will give a Morse theoretic proof of the existence and properties of a generalization of the sequence of harmonic maps between spheres found in [2] and [4]. By doing so, we will—in the context of the harmonic map problem rather than the Einstein-Yang-Mills problem—in effect, cure all of the deficiencies as well as close all of the gaps in the general argument sketched in [13]. In our proofs, the relevant symmetry on the space of maps between spheres will be seen to be a $\mathbb{Z}_2$ symmetry corresponding to composing a given map between the spheres with the reflection isometry about the equatorial plane of the image sphere. It will be seen that an additional fact playing a crucial role in our proofs is the existence of a harmonic map (namely, $h_\infty$) which is invariant under this symmetry and which has infinite index. At a heuristic level, the presence of $h_\infty$ ensures that the heat flow has the appropriate bifurcation properties to obtain an infinite sequence of solutions. However, a number of technical difficulties would arise if we attempted to use heat flow arguments in our proofs. The heat flow for maps into a sphere can develop singularities in finite time, which makes it difficult to regard it as a flow on the space of maps. As is typical in Morse theoretic arguments on infinite-dimensional spaces, we replace the heat flow by an energy-decreasing flow which is defined in a more ad hoc manner, but has the two advantages that it is defined for all time and its flow lines converge to limiting maps as time tends to infinity. It then becomes possible to apply the essential observation of Morse theory, which is that nontrivial topology in the space of maps forces the existence of critical points for the energy which are not minima. More precisely, one can apply a minimax argument for each homology class in the configuration space: the minimum of the collection of numbers $E$ such that the chosen homology class can be realized in the space of maps with energy no larger than $E$ is a critical value. However, the space of maps in our case is parametrized by real-valued functions on the real line, so has no nontrivial topology. We avoid this problem by (1) showing that all critical points have energy bounded by the energy of the unique singular critical point (corresponding to a map which collapses the domain sphere to the equator in the target sphere) and (2) exploiting the $\mathbb{Z}_2$ symmetry. This leads to a configuration space with nontrivial homology classes in infinitely many dimensions. These homology classes are the essential explanation for the existence of the infinite sequence of harmonic maps.
Several technical complications arise in the course of the proof which may obscure the main ideas, so we will outline a naive version of the argument here. The first step is to reduce the harmonic map equation to an ordinary differential equation by prescribing the map along codimension one slices in the domain sphere. This leads to the condition that the map along such slices must be an eigenmap, i.e. a harmonic map with constant energy density. By an appropriate choice of coordinates, we can then reduce the problem to that of finding critical points of an energy functional $E$ on a Hilbert space $H$ of functions $h : \mathbb{R} \rightarrow \mathbb{R}$. There is an involution of $H$ given by multiplication by $-1$ which preserves $E$. It has a unique fixed point, corresponding to the function $h_\infty$ which is identically zero, and corresponding to a singular harmonic map which collapses the domain to the equator of the target. We can show that all other critical points of $E$ correspond to smooth harmonic maps with energy strictly less than $E(h_\infty)$. Under suitable conditions, $h_\infty$ has infinite index as a critical point of $E$. Ideally, we would exploit this as follows. Consider the punctured Hilbert space $H^* = H - \{h_\infty\}$, and divide by the involution to obtain a space $\overline{H}^*$ with the homotopy type of $\mathbb{R}P^\infty$. In particular, the homology of $\overline{H}^*$ is $\mathbb{Z}_2[x]$, where $x$ is a class of degree 1. In the cases where $h_\infty$ has infinite index, we can realize all of the homology of $\overline{H}^*$ in the portion consisting of functions with energy less than $E(h_\infty)$. If Morse theory could be applied naively, it would tell us that there must be an infinite sequence of critical points, with at least one critical point of index $k$ for every $k \geq 0$. The idea of finding harmonic maps between spheres by reducing to an ordinary differential equation is an old one, first used by Smith \cite{Smith1, Smith2} and used more recently by Ding \cite{Ding}, Eells-Ratto \cite{Eells-Ratto} and Pettinati-Ratto \cite{Pettinati-Ratto}. To the best of our knowledge, most of the work done in this direction has focused on showing that harmonic maps exist, without trying to show that they exist in profusion. The work of Bizoń and Bizoń-Chmaj appears to be the first in this direction.

It should be noted that the static, spherically symmetric Einstein-Yang-Mills equations also possess a $\mathbb{Z}_2$ symmetry given by the transformation $w \rightarrow -w$ in the notation of \cite{Bizoń-Chmaj} (see eqs.(2.50)-(2.53) of that reference). In addition, the Reissner-Nordstrom solutions with unit magnetic charge are invariant under this symmetry and have infinite index. This suggests that it might be possible to give a similar Morse theoretic proof of the existence and properties of the Bartnik-McKinnon and colored black hole sequences; the main problem would be to show that the relevant functional satisfies the Palais-Smale condition. However, we shall not pursue this issue here.

The contents of this paper are as follows. In §1, we describe the basic framework for studying special classes of harmonic maps which can be interpreted as solutions of ordinary differential equations. We establish certain basic properties of these solutions, including the bound on the energy mentioned above. In §2, we study the index of the singular map which collapses the domain sphere onto the equator of the target, and show that it is infinite under certain circumstances. In the final section, we apply Morse theory for functions on compact convex sets to establish the existence of an infinite sequence of harmonic maps.

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§1 Preliminaries

Let $S^n$ be the n-dimensional sphere with the Riemannian metric induced from its identification with the unit sphere in Euclidean space. We identify the $(n+1)$-sphere with north and south poles removed with

$(-\frac{\pi}{2}, \frac{\pi}{2}) \times S^n$

with the metric

$\, d\theta^2 + \cos^2 \theta \, d\psi^2,$

where $\theta$ is the coordinate along $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $d\psi^2$ is the round metric on $S^n$. If we fix a map $F : S^m \to S^n$, then a function $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ with $f(\pm \frac{\pi}{2}) = \pm \frac{\pi}{2}$ gives a map $\tilde{f} : S^{m+1} \to S^{n+1}$ via $\tilde{f}(\theta, \psi) = (f(\theta), F(\psi))$.

We are interested in harmonic maps which can be written in this form with $m \geq 2$. In order for this to be reasonable, we must impose a condition on $F$.

**Definition 1.1.** $F : S^m \to S^n$ is an eigenmap if it is harmonic and the energy density $|dF|^2 = \omega$ is constant as a function on $S^m$. In this situation, $\omega$ is called the eigenvalue of the map.

One of the simplest examples of an eigenmap is the identity map $S^m \to S^m$; the corresponding eigenvalue is $m$. Other examples are given by the Hopf maps $S^3 \to S^2$, $S^7 \to S^4$ and $S^{15} \to S^8$; these have eigenvalues 8, 16 and 32, respectively. Any eigenmap $S^m \to S^n$ is obtained from a collection of $n+1$ eigenfunctions $\xi_1, \ldots, \xi_{n+1}$ of the Laplacian on $S^m$ satisfying $\sum \xi_i^2 = 1$. There is no general classification of eigenmaps between spheres, but a number of examples and partial results are known. For example, there are eigenmaps produced by what is known as the Hopf construction, generalizing the classical Hopf maps. There are also eigenmaps associated with harmonic eiconals. These are harmonic polynomials on $\mathbb{R}^{n+1}$ whose gradients have unit length along the unit sphere; the gradients then give harmonic maps $S^n \to S^m$. Consult Eells-Ratto [4], Chapter VIII for more information.

Once we assume that $F$ is an eigenmap, the condition for $\tilde{f}$ to be a harmonic map reduces to a differential equation for $f$:

$$f'' - m \tan \theta f' + \frac{\omega}{2} \sec^2 \theta \sin 2f = 0.$$  

This is the Euler-Lagrange equation for the following energy functional:

$$J(f) = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [(f')^2 + \frac{\omega \cos^2 f}{\cos^2 \theta}] \cos^n \theta \, d\theta,$$

which gives the energy of $\tilde{f}$ in the usual sense. It will often be useful to make the change of variables $x = \log(\tan \frac{1}{2}(\theta + \frac{\pi}{2}))$ and to set $h(x) = f(2 \tan^{-1}(e^x) - \frac{\pi}{2})$. Then the energy becomes

$$E(h) = \frac{1}{2} \int_{-\infty}^{\infty} [(h')^2 + \omega \cos^2 h] \text{sech}^{m-1} x \, dx,$$

while the Euler-Lagrange equation becomes

$$h'' - (m - 1) \tanh x h' + \frac{\omega}{2} \sin 2h = 0.$$
We will denote the map of spheres associated to \( h \) by \( \tilde{h} \). If \( v : \mathbb{R} \to \mathbb{R} \) is \( C^2 \) with compact support, then the formula for the second variation of \( E \) is

\[
\frac{d^2}{dt^2} E(h + tv)|_{t=0} = \int_{-\infty}^{\infty} \left[ (v')^2 - \omega \cos 2hv^2 \right] \text{sech}^{m-1} x \, dx
\]

(7)

\[
\quad = -\int_{-\infty}^{\infty} \left[ (v' \text{sech}^{m-1} x)' + \omega \cos 2hv \text{sech}^{m-1} x \right] v \, dx
\]

\[
\quad = -\int_{-\infty}^{\infty} \left[ v'' - (m-1) \tanh xx' + \omega \cos 2hv \text{sech}^{m-1} x \right] v \, dx.
\]

The first of these integrals will be abbreviated as \( Q(v, v) \).

Define a weighted Sobolev space \( H \) to be the completion of the space of smooth functions \( h : \mathbb{R} \to \mathbb{R} \) satisfying

\[
\int_{-\infty}^{\infty} \left[ (h')^2 + h^2 \right] \text{sech}^{m-1} x \, dx < \infty
\]

(8)

with respect to the norm \( \| h \|_H^2 \) defined by the integral above. Define \( E : H \to \mathbb{R} \) by the same formula as in the previous paragraph. \( E \) is easily seen to be a smooth function on \( H \), with critical points given by functions satisfying the Euler-Lagrange equation given above.

We are interested in critical points of \( E \) lying in the closed convex set \( C \subset H \) given by

\[
C = \{ h \in H : |h(x)| \leq \frac{\pi}{2}, x \in \mathbb{R} \}.
\]

(9)

The fact that this set is closed follows from the Sobolev embedding theorem for ordinary Sobolev spaces on a finite interval. We begin by proving a few simple properties of these critical points. Most, but not all, of this can be found in [2][4]. In fact, we will need to study critical points of functionals which are perturbations of the energy, so define

\[
E_\nu(h) = \frac{1}{2} \int_{-\infty}^{\infty} \left[ (h')^2 + \omega(1 + \nu) \cos^2 \text{sech}^{m-1} x \, dx,
\]

(10)

where \( \nu \) is a \( C^2 \) function on \( \mathbb{R} \) with \( |\nu| < 1 \) and compact support. The Euler-Lagrange equation for this functional is

\[
h'' - (m-1) \tanh xx' + \frac{\omega}{2}(1 + \nu) \sin 2h = 0.
\]

(11)

Given a solution \( h \in C \) of this equation, define

\[
W(x) = \frac{1}{2} (h')^2 + \frac{\omega}{2}(1 + \nu) \sin^2 h.
\]

(12)

We calculate that

\[
\frac{dW}{dx} = h'' h' + \omega(1 + \nu) \sin h \cos h h' + \frac{\omega}{2} \nu' \sin^2 h
\]

\[
\quad = (h'' + \frac{\omega}{2}(1 + \nu) \sin 2h) h' + \frac{\omega}{2} \nu' \sin^2 h
\]

\[
\quad = (m-1) \tanh xx(h')^2 + \frac{\omega}{2} \nu' \sin^2 h.
\]

(13)

Thus, \( W \) is increasing when \( x \gg 0 \), and decreasing when \( x \ll 0 \). If \( W(x_0) \geq \frac{\omega}{2} \) for some \( x_0 \gg 0 \), then either \( h \) is constant with value \( \pm \frac{\pi}{2} \) or \( h'(x) > \epsilon > 0 \) for all \( x > x_0 \). The latter is impossible, since \( h \in C \). A similar argument applies if \( x_0 \ll 0 \). Hence, \( W(x) \leq \frac{\omega}{2} \) for all \( x \) sufficiently far from zero. As a consequence, the limits

\[
\lim_{x \to \pm \infty} W(x) = L_{\pm}
\]

(14)

both exist, and

\[
\lim_{x \to \pm \infty} W'(x) = 0.
\]

(15)
The latter implies that \( h' \) approaches zero as \( x \) tends to \( \pm \infty \). Taken together, the fact that both \( W \) and \( h' \) have limits at infinity implies that

\[
\lim_{x \to \pm \infty} \sin^2 h = \frac{2L_{\pm}}{\omega},
\]

and therefore \( h \) itself has limits at \( \pm \infty \). The Euler-Lagrange equation now implies that \( h'' \) also has limits at \( \pm \infty \), and since \( \lim_{x \to \pm \infty} h'(x) = 0 \), that limit must be zero. The Euler-Lagrange equation then implies that

\[
\lim_{x \to \pm \infty} \sin 2h = 0,
\]

which means that \( h \) approaches 0 or \( \pm \frac{\pi}{2} \) as \( x \) approaches \( \pm \infty \). If the limit is zero at either extreme, then \( W \) approaches zero as well. Since it is increasing when \( x \gg 0 \) and decreasing when \( x \ll 0 \), and nonnegative everywhere, this implies that it must be zero for \( x \gg 0 \) or \( x \ll 0 \). The latter implies that \( h = 0 \) on an open set, so is zero everywhere. Thus, unless \( h = h_{\infty} \),

\[
\lim_{x \to \pm \infty} h(x) = \pm \frac{\pi}{2}.
\]

This implies that the corresponding map \( \tilde{h} \) between spheres is continuous at the poles of \( S^{m+1} \). The Euler-Lagrange equation for \( E_\nu \) corresponds to harmonic map equation in a neighborhood of each pole, so by regularity of continuous solutions of the harmonic map equation, \( \tilde{h} \) must be a smooth map.

It is also possible to compare the perturbed energy of \( h \) with that of \( h_{\infty} \). When \( m \leq 1 \), \( h_{\infty} \) has infinite energy, so from here on we will assume \( m \geq 2 \). Integrating by parts, and using the fact that \( h' \) tends to zero at \( \pm \infty \), we find that

\[
E_\nu(h) = \frac{1}{2} \int_{-\infty}^{\infty} h\left[-h'sech^{m-1}x\right]' + \omega(1+\nu) \cos^2 hsech^{m-1}xdx
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} \left[h \sin 2h + \cos^2 h\right] \omega(1+\nu) sech^{m-1}xdx.
\]

The function \( \frac{1}{2}h \sin 2h + \cos^2 h \) is bounded by 1 when \( h \in C \), and is identically equal to 1 only if \( h = h_{\infty} \). Thus, \( E_\nu(h) \leq E_\nu(h_{\infty}) \), with equality only if \( h = h_{\infty} \).

We summarize this in the following

**Proposition 1.2.** Assume that \( m \geq 2 \). Any critical point \( h \in C \) of \( E_\nu \) is either the singular map \( h_{\infty} \) or satisfies

1. \( \lim_{x \to \infty} h(x) = \pm \frac{\pi}{2} \), \( \lim_{x \to -\infty} h(x) = \pm \frac{\pi}{2} \)
2. \( E_\nu(h) < E_\nu(h_{\infty}) \)
3. The index and nullity of the Hessian of \( E_\nu \) at \( h \) are finite.

**Proof:** The only remaining issue is the proof of (3). The critical points of \( E_\nu \) correspond to harmonic maps with a potential function from \( S^{m+1} \) to \( S^{n+1} \). As such, they satisfy a quasilinear system of elliptic PDE which, in a neighborhood of each pole, agrees with the usual harmonic map equation. One can calculate the Hessian of the perturbed energy functional on the space of all maps \( S^{m+1} \to S^{n+1} \); it corresponds to an elliptic operator which, aside from a zeroth order term deriving from the potential, agrees with the corresponding operator for the usual energy. This is an operator of Laplacian type, so the assertions about the index and nullity of the Hessian follow easily. QED

It is of interest to consider functions satisfying certain symmetry conditions: either

\[
h(-x) = h(x)
\]
or
\[ h(-x) = -h(x). \] (21)

If \( h \) satisfies condition 1.2.1 in the previous Proposition, then (20) implies that the corresponding map between spheres is homotopically trivial, while (21) implies that the map between spheres is in the homotopy class of the suspension of \( F \). We will let \( H^+ \) be the set of functions in \( H \) satisfying (20), and \( H^- \) the set of functions in \( H \) satisfying (21); similarly, \( C^+ = C \cap H^+ \) and \( C^- = C \cap H^- \). Notice that, if \( \nu(-x) = \nu(x) \) (as we shall assume henceforth), then the lefthand side of (11) satisfies (20) if \( h \) does, and it satisfies (21) if \( h \) does. This implies that any critical point of \( E_\nu \), regarded as a function on \( H^+ \) or \( H^- \), is actually a critical point of \( E_\nu \) regarded as a function on \( H \).

\section{The index of the singular map}

In this section, we will show that the index of the singular map corresponding to \( h_\infty \) is infinite in certain cases. The Hessian of \( E_\nu \) at \( h \) is given by
\[
Q_\nu(v, v) = \int_{-\infty}^{\infty} [(v')^2 - \omega(1 + \nu) \cos (2h)v^2] \text{sech}^{m-1} x dx
\] (22)

for any \( v \in H \). Let
\[
w = v \text{sech}^{m-1} x. \] (23)

Notice that if \( v \in H \), then
\[
\int_{-\infty}^{\infty} [(w')^2 + w^2] dx
\]
(24)\[
= \int_{-\infty}^{\infty} [(v')^2 + v^2] \text{sech}^{m-1} x dx
\]
\[
+ \int_{-\infty}^{\infty} \left[ \left( \frac{m-1}{2} \right)^2 \text{sech}^{m-1} x \tanh^2 x v^2 - (m-1) \text{sech}^{m-1} x \tanh x v v' \right] dx
\]
\[
= \int_{-\infty}^{\infty} [(v')^2 + v^2] \text{sech}^{m-1} x dx - \int_{-\infty}^{\infty} \left[ \frac{m-1}{2} \text{sech}^{m-1} x \tanh x v \cdot v' \right] dx
\]
\[
+ \int_{-\infty}^{\infty} \left[ \frac{m-1}{2} \text{sech}^{m-1} x \tanh x \right] v^2 + \left( \frac{m-1}{4} \right) \text{sech}^{m-1} x \tanh^2 x v^2 ] dx
\]

The middle term in the last expression vanishes, while the third is comparable to
\[
\int_{-\infty}^{\infty} v^2 \text{sech}^{m-1} x dx,
\]
so \( v \in H \) if and only if \( w \in L^2_1 \), the usual Sobolev space of \( L^2 \) functions on \( \mathbb{R} \) whose first derivatives are also in \( L^2 \). A calculation shows that the integral defining the Hessian becomes
\[
\int_{-\infty}^{\infty} [(w' + \frac{m-1}{2} \tanh x w)^2 - \omega(1 + \nu) \cos (2h)w^2] dx
\]
(26)
\[
= \int_{-\infty}^{\infty} [(w')^2 + (m-1) \tanh x w w' + \frac{(m-1)^2}{4} \tanh^2 x - \omega(1 + \nu) \cos (2h)] w^2] dx
\]
Thus, if \( (m-1)^2/4 < \omega \), then there are finite-dimensional subspaces of \( H \) of arbitrarily large dimension on which the Hessian of \( E_\omega \) at \( h_\infty \) is negative definite.

**Proof:** Under the assumption that \( (m-1)^2/4 < \omega \), we can find an \( \epsilon > 0 \) and a \( K > 0 \) such that

\[
V(x) = \left[ \frac{(m-1)^2}{4} - \left( \frac{(m-1)^2}{4} + \frac{m-1}{2} \right) \text{sech}^2 x - \omega(1 + \nu) \right] < -\epsilon
\]

whenever \( |x| > K \).

Choose \( a, c > 0 \) and consider the piecewise linear test function

\[
F(x) = \begin{cases} 
0, & x \not\in [c, c + 2a] \\
-2a - x, & x \in [c, c + a] \\
c + 2a - x, & x \in (\omega + a, c + 2a].
\end{cases}
\]

Then \( F \) is in the Sobolev space \( L^2 \), and it is easy to calculate that

\[
Q_\nu(F, F) = 2a + \int_{-\infty}^{\infty} V(x) F(x)^2 dx.
\]

Choosing \( c > K \) (so that \( V(x) < -\epsilon \) throughout the support of \( F \)), we find that

\[
Q_\nu(F, F) < 2a - \epsilon \int_{-\infty}^{\infty} F(x)^2 dx = 2a - \epsilon \frac{2a^3}{3}.
\]

Thus, if \( a^2 > 3\epsilon^{-1} \), then \( Q_\nu(F, F) < 0 \). Given some \( a \) satisfying this condition, for any positive integer \( i \), define \( F_i \) as above with \( c = K + 2ai \). Then \( Q_\nu \) is negative definite on any subspace of \( L^2 \) generated by any finite collection of the \( F_i \). **QED**

The identity map satisfies the hypothesis of this result for \( 2 \leq m \leq 5 \). The Hopf maps for which it holds are \( S^3 \to S^2 \) and \( S^7 \to S^4 \). Among other maps produced by the Hopf construction, there are maps \( S^5 \to S^3 \) and \( S^9 \to S^8 \) which satisfy the hypothesis. There are also maps associated to harmonic eiconals to which the result applies. For example, there are harmonic eiconals of polynomial degree 3 on \( \mathbb{R}^5 \), \( \mathbb{R}^8 \), \( \mathbb{R}^{14} \) and \( \mathbb{R}^{26} \), corresponding to harmonic self-maps of \( S^4 \), \( S^7 \), \( S^{13} \) and \( S^{25} \) with Brouwer degrees 0, 2, 2 and 2 and eigenvalues 18, 27, 45 and 81, respectively. The first three satisfy the hypothesis.

Notice that \( h_\infty \in H^+, H^- \). It can be shown that the index of \( h_\infty \) as a critical point of \( E_\nu \) on either of these spaces continues to be infinite under the same hypothesis on \( m \) and \( \omega \). This follows by a small modification of the argument above, where the piecewise linear function \( F \) given there is replaced by \( F(x) + F(-x) \) in the case of \( H^+ \), and \( F(x) - F(-x) \) in the case of \( H^- \).
§3 Morse theory applied

It is now possible to apply the elements of Morse theory on convex sets to the $E_\nu$. As basic references on this subject, we take Chang [5] and Struwe [12]. We first recall the notion of a critical point of a function on a convex set ([5], Definition 6.4 or [12], II,1.3).

Definition 3.1. $h_0 \in C$ is a C-critical point of $E_\nu$ if

\begin{equation}
(32) \quad dE_\nu(h_0)(h - h_0) \geq 0
\end{equation}

whenever $h \in C$. Equivalently, let

\begin{equation}
(33) \quad g_\nu(h_0) = \inf dE_\nu(h_0)(h - h_0),
\end{equation}

where $h$ ranges over elements of $C$ with $\| h - h_0 \|_H < 1$. Then $h_0$ is a C-critical point for $E_\nu$ iff $g_\nu(h_0) = 0$.

It should be noted that $g_\nu$ is a continuous function on $C$. We can define the notions of $C^+$-critical and $C^-$-critical points of $E_\nu$ on $C^+ = C \cap H^+$ and $C^- = C \cap H^-$; it is simply necessary to let $h$ range over elements of $C\pm$ rather than elements of $C$. It is not hard to see that any $C\pm$-critical point is also $C$-critical, since the differential of $E_\nu$ at $h$ satisfies the same symmetry condition as $h$.

We now compare the C-critical points with ordinary ones.

Lemma 3.2. Any C-critical point $h_0$ of $E_\nu$ is a critical point for $E_\nu$ as a function on $H$.

Proof: If $h_0$ is not identically equal to $\pm \pi$, then there is a nonempty open set of $\mathbb{R}$ on which it takes values in $(-\pi/2, \pi/2)$. On this open set, we can test $h_0$ by smooth variations with compact support; when these are sufficiently small, we do not leave $C$. Hence, $h_0$ satisfies the Euler-Lagrange equation for $E_\nu$ on the open set where it is not equal to $\pm \pi$.

Near any point $x_0 \in \mathbb{R}$ where $h(x_0) = \pm \pi/2$, $h$ only satisfies a variational inequality. The condition given in the definition above implies that

\begin{equation}
(34) \quad \int_{-\infty}^{\infty} [h'/\omega(1 + \nu) \sin 2h]\sech^{m-1} x \, dx \geq 0
\end{equation}

for any smooth $v$ with compact support which is nonpositive near $h^{-1}(\pi/2)$ and nonnegative near $h^{-1}(-\pi/2)$. This implies that, as a distribution,

\begin{equation}
(35) \quad h'' - (m - 1) \tanh x h' + \frac{\omega}{2} (1 + \nu) \sin 2h = \mu_+ - \mu_-,
\end{equation}

where $\mu_+, \mu_-$ are positive Radon measures supported on $h^{-1}(\pi/2), h^{-1}(-\pi/2)$, respectively. $\mu_+, \mu_-$ are the distributional derivatives of two monotone functions $F_+, F_-$ which are locally constant outside of $h^{-1}(\pi/2), h^{-1}(-\pi/2)$, respectively. This implies that

\begin{equation}
(36) \quad h'(x) = C + \int_{0}^{x} [(m - 1) \tanh th' - \frac{\omega}{2} (1 + \nu) \sin 2h] \, dt + F_+ - F_-.
\end{equation}

Thus, $h'$ is continuous, except possibly for jump discontinuities on $h^{-1}(\pi/2), h^{-1}(-\pi/2)$, with upward jumps on the former and downward jumps on the latter.

If $x_0 \in h^{-1}(\pi/2)$, then $x < x_0$ implies

\begin{equation}
(37) \quad \frac{h(x) - h(x_0)}{x - x_0} \geq 0,
\end{equation}

\end{eqnarray*}
while $x > x_0$ implies

$$h(x) - h(x_0) \leq 0.$$  

The only way this can be compatible with the fact that $h'$ is only allowed upward jumps on $h^{-1}(\frac{x}{2})$ is if $h'$ is continuous at $x_0$ with $h'(x_0) = 0$. A similar argument applies if $x_0 \in h^{-1}(\frac{x}{2})$. Hence, $h'$ is continuous on $\mathbb{R}$.

Choose a maximal interval $I$ contained in $h^{-1}(\frac{x}{2}, \frac{x}{2})$. If $I$ is not all of $\mathbb{R}$, then there is some endpoint $x_0$ contained in either $h^{-1}(\frac{x}{2})$ or $h^{-1}(\frac{x}{2})$. On $I$, $h$ coincides with a smooth solution of the Euler-Lagrange equation for $E_\nu$, and extends to $x_0$ as a $C^1$ function with $h(x_0) = \pm \frac{x}{2}$ and $h'(x_0) = 0$. But the only solution with this property is constant, contradicting the assumption that $h$ does not attain $\pm \frac{x}{2}$ as values in $I$. Hence, $I = \mathbb{R}$, and $h$ satisfies the Euler-Lagrange equation everywhere. \textbf{QED}

In order to apply Morse theory, the following result is needed.

\textbf{Proposition 3.3.} If $m > 1$, the functional $E_\nu : C \to \mathbb{R}$ satisfies the Palais-Smale condition, i.e. if $(h_i) \subset C$ is a sequence with $E_\nu(h_i)$ uniformly bounded and

$$\lim_{i \to \infty} g_\nu(h_i) = 0,$$

then a subsequence of the $h_i$ converges strongly to a critical point of $E_\nu$ in $C$.

\textbf{Proof:} $E_\nu$ is a smooth function on $H$. The fact that $E_\nu(h_i)$ is uniformly bounded implies that

$$\int_{-\infty}^{\infty} (h'_i)^2 \text{sech}^{m-1} dx$$

is uniformly bounded. Since $h_i \in C$ and $m > 1$, this implies that $\| h_i \|_H$ is uniformly bounded so, by passing to a subsequence if necessary, we can assume that $h_i$ converges weakly in $H$ to some $h \in C$. By Rellich’s Lemma, the restriction of $(h_i)$ to any bounded interval $[-a, a]$ is precompact in $L^2([-a, a])$. Set $h_{0,i} = h_i$. Then, for each positive integer $k$, we can choose a subsequence $(h_{k,i})$ of $(h_{k-1,i})$ so that $h_{k,i}$ converges in $L^2([-k, k])$ as $i \to \infty$. Replacing $(h_i)$ by the diagonal subsequence, we may assume that $(h_i)$ converges in $L^2$ on any bounded interval. On the other hand, if $k$ is large enough, then

$$\int_{|x| \geq k} |h_i|^2 \text{sech}^{m-1} dx$$

is as small as we like, since $|h_i| \leq \frac{x}{2}$. This implies that

$$\int_{-\infty}^{\infty} |h_i - h_j|^2 \text{sech}^{m-1} dx \to 0$$

as $i, j \to \infty$.

We can write

$$dE_\nu(h_i)(h_i - h_j) - dE_\nu(h_j)(h_i - h_j)$$

$$= 2 \int_{-\infty}^{\infty} [(h'_i - h'_j)^2 - \omega(1 + \nu)(\sin h_i \cos h_i - \sin h_j \cos h_j)(h_i - h_j)] \text{sech}^{m-1} dx.$$
The second term is bounded in absolute value by
\[ 2\omega \int_{-\infty}^{\infty} (1 + \nu)|h_i - h_j|^2 \text{sech}^{m-1} x dx, \]
so tends to zero as \( i, j \to \infty \). The fact that \( g(h_i) \) tends to zero implies that the expression in (45) is bounded by arbitrarily small positive numbers for \( i, j \gg 0 \), which implies that
\[ \int_{-\infty}^{\infty} (h'_i - h'_j)^2 \text{sech}^{m-1} x dx \]
tends to zero as \( i, j \to \infty \). This implies that the subsequence converges to \( h \) in \( H \). QED

Recall that a function \( F : M \to \mathbb{R} \) on a Hilbert manifold is said to be a Morse function if its critical points are isolated and have nondegenerate Hessians. Define
\[ \tilde{H} = \{ h \in H | h \neq k\pi, h \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \}. \]

Theorem 1.1 of [8] implies that, for a generic set of \( \nu \) in the space of compactly supported \( C^2 \) functions on the line, \( E_\nu \) is a Morse function on \( \tilde{H} \). In particular, we can choose a sequence \( \nu_j \) converging uniformly to zero so that \( E_j = E_{\nu_j} \) is a Morse function on \( \tilde{H} \). When \( |\nu| < 1 \), it is straightforward to see that \( h \equiv \frac{\pi}{2} + k\pi \) gives a global minimum for \( E_\nu \) on \( H \) and that the Hessian is positive definite, so for \( j \gg 0 \), each \( E_{\nu_j} \) is a Morse function on the enlarged Hilbert manifold given by
\[ \overline{H} = \{ h \in H | h \neq k\pi, k \in \mathbb{Z} \}. \]
Note that \( h_\infty \) is a critical point for any \( E_\nu \), and by the result in the previous section, has infinite index iff it has infinite index as a critical point for \( E \).

It will be important for our purposes to use the symmetry \( E_\nu(h) = E_\nu(-h) \). To exploit this, we will work on the space \( \tilde{C} = (C - \{ h_\infty \})/\pm \). \( \tilde{C} \) is a locally convex set in the sense of [8], Definition 6.2, and each \( E_\nu \) descends to a smooth function on \( \tilde{C} \) satisfying the Palais-Smale condition on the subset of \( \tilde{C} \) where \( E_\nu < E_\nu(h_\infty) \). The basic deformation lemma of Morse theory, adapted to our context, is the following.

**Lemma 3.4.** Fix \( \nu, \lambda < E_\nu(h_\infty) \) and \( \tau > 0 \). Let \( C_\lambda = E_\nu^{-1}((-\infty, \lambda)) \), and \( \tilde{C}_\lambda = C_\lambda/\pm \). Let \( K_\lambda \) be the set of critical points of \( E_\nu \) in \( \tilde{C} \) with \( E_\nu(h) = \lambda \), and let \( N \) be an open neighborhood of \( K_\lambda \) in \( \tilde{C} \). Then there exist \( \epsilon \in (0, \tau) \) and a continuous map \( \Phi : [0, 1] \times \tilde{C} \to \tilde{C} \) such that
1. \( \Phi(t, h) = h \) if either \( t = 0 \), \( |E_\nu(h) - \lambda| < \tau \) or \( h \) is a critical point of \( E_\nu \);
2. \( E_\nu(\Phi(t, h)) \) is nonincreasing as a function of \( t \) for any \( h \);
3. \( \Phi(1, \tilde{C}_\lambda + \epsilon - N) \subset \tilde{C}_\lambda - \epsilon \); and
4. \( \Phi(1, \tilde{C}_\lambda + \epsilon) \subset \tilde{C}_\lambda - \epsilon \cup N \).

**Proof:** This is essentially [12], II,1.9. The only difference is that Struwe works there with an actual convex set rather than the kind of quotient we are dealing with. Thus, one has to ensure that the construction of the map \( \Phi \) is invariant under \( \pm \), which is straightforward. Compare [8], Theorem 3.3 and §6.2. QED

The other result from Morse theory we will need is concerned with the way the topology of \( \tilde{C}_\lambda \) changes as \( \lambda \) passes a critical value of \( E_\nu \). Before we can apply this, we need to verify the assumption in [12], II, 3.3. Let \( h \in \tilde{C} - \{ h_\infty \} \) be a critical point for \( E_\nu \), and let \( Q_h \) be the Hessian of \( E_\nu \) at \( h \). As mentioned previously, \( H \) decomposes into a direct sum
\[ H = H_+ \oplus H_0 \oplus H_- , \]
corresponding to the subspaces on which \( Q_h \) is positive-definite, zero and negative-definite, and the latter two subspaces are finite-dimensional. In fact, the dimension of \( H_0 \) is at most 1. That the dimension is at most two follows from the fact that the relevant differential equation has a 2-dimensional space of solutions locally; that it is at most 1 follows from the fact that the two points at infinity fall into the limit point case of Weyl’s classification of singular points for a second order differential equation. The assumption we need in order to apply the theory of \([12]\) is verified by the following.

**Lemma 3.5.** If \( h \in C - \{h_\infty\} \) is a critical point of \( E_\nu \), then there is an open neighborhood \( U \) of \( 0 \in H_- \) such that \( h + U \subset C \).

**Proof:** For any \( v, w \in H \),

\[
Q_h(v, w) = \int_{-\infty}^{\infty} [v'(x) - \omega(1 + \nu) \cos 2hvw] \sech^{m-1} x \, dx
\]

Fixing \( v \) and letting \( w \) vary over \( H \), we obtain a bounded linear functional on \( H \), so there is a bounded linear operator \( A : H \to H \) such that

\[
Q_h(v, w) = \langle Av, w \rangle_H.
\]

\( A \) is a symmetric operator, and \( H_- \) is a direct sum of the eigenspaces for \( A \) corresponding to negative eigenvalues.

Suppose that \( Av = \lambda v \) with \( \lambda < 0 \). Then the fact that \( Q_h(v, w) = \lambda \langle v, w \rangle_H \) for all smooth compactly supported \( w \) implies that

\[
(v' \sech^{m-1} x)' = (\lambda - 1)^{-1} [\lambda + \omega(1 + \nu) \cos 2h] \sech^{m-1} x.
\]

When \( x \) is sufficiently large in absolute value, \( \lambda + \omega(1 + \nu) \cos 2h \) is negative. This implies that \( v' \) is increasing when \( v \) is positive, and is decreasing when \( v \) is negative. Hence, \( v \) cannot be zero when \( x \) is large in absolute value. We can choose a basis \( v_1, \ldots, v_N \) for \( H_- \) consisting of eigenfunctions corresponding to eigenvalues \( \lambda_1, \ldots, \lambda_N \), satisfying the condition that \( v_i > 0 \) for \( x \gg 0 \).

Consider \( v = \sum \epsilon_i v_i \) and suppose that \( \lim_{x \to \infty} h(x) = \frac{\pi}{2} \). We need to show that \( h + v \leq \frac{\pi}{2} \) for all sufficiently small \( \epsilon_i \). This is true on any bounded interval, so we need only show it is so when \( x \gg 0 \). Let \( g = \frac{\pi}{2} - h \), \( w = \sum \epsilon v_i \) and define the following Wronskian-like quantity:

\[
W(x) = \langle g'(x)w(x) - g(x)w'(x) \rangle \sech^{m-1} x.
\]

Then

\[
W''(x) = (g'(x) \sech^{m-1} x) w(x) - g(x) (w'(x) \sech^{m-1} x)'
\]

\[
= -\frac{\nu}{2} \sin 2h \sech^{m-1} x \, w(x) - g(x) \sum \epsilon_i (\lambda_i - 1)^{-1} [\lambda_i + \omega(1 + \nu) \cos 2h] v_i \sech^{m-1} x.
\]

This is negative when \( x \gg 0 \). On the other hand, \( \lim_{x \to \infty} W(x) = 0 \), so \( W(x) > 0 \) for \( x \gg 0 \). This implies that \( (w/g)' < 0 \). Hence, \( (w/g) < C \) for some \( C > 0 \) and \( x \gg 0 \), so the required condition holds when \( \epsilon_i < C^{-1} \) for each \( i \). A similar argument applies when \( \lim_{x \to \infty} h(x) = -\frac{\pi}{2} \) or \( x \) tends to \( -\infty \). QED

With this in place, we can state the second result from Morse theory.

**Theorem 3.6.** Suppose \( \lambda \) is a critical value of \( E_\nu : \tilde{C} \to \mathbb{R} \) with \( \lambda < E_j(h_\infty) \), where, as defined above, \( E_j = E_{\nu_j} \). There are finitely many critical points \( p_1, \ldots, p_N \) in \( E_j^{-1}(\lambda) \). If the indices of these critical points are are \( i_1, \ldots, i_N \), respectively, then, for any sufficiently small
$\epsilon > 0$, $\tilde{C}_{\lambda + \epsilon}$ is homotopy equivalent to $C_{\lambda - \epsilon}$ with disks of dimensions $i_1, \ldots, i_N$ attached along their boundaries. (If $i_k = 0$, then we add a point to $C_{\lambda - \epsilon}$ as a disjoint component.)

**Proof:** The fact that there are only finitely many critical points follows from the fact that each critical point of $E_j$ (except possibly $h_{\infty}$) is isolated together with the Palais-Smale condition. The rest is II, Theorem 3.6 in [12]. QED

We are now in a position to prove our main result. Define the extended index of a critical point $h$ of $E_j$ to be the sum of the dimensions of $H_-$ and $H_0$. Since $\dim H_0 \leq 1$, the extended index of $h$ is either $i$ or $i + 1$, where $i$ is the index of $h$.

**Theorem 3.7.** Suppose that $F : S^m \to S^n$ is an eigenmap with eigenvalue $\omega$, $m > 1$, and

$$\frac{(m - 1)^2}{4} < \omega.$$  

There is an infinite sequence $(h_k) \subset \tilde{C}$ of critical points for $E$ such that

1. the extended index of $h_k$ is at least $k$, whereas the index of $h_k$ is at most $k$, and
2. the $h_k$ converge strongly to $h_{\infty}$.

It should be remembered that each $h_k$ corresponds to a pair of harmonic maps $S^{m+1} \to S^{n+1}$. If $F$ is the identity map, then the eigenvalue is $m$, and the assumption reduces to $2 \leq m \leq 5$, which gives back the results of [2] and [4].

**Proof:** $C$ is a contractible space, as is $C - \{h_{\infty}\}$. This implies that $\tilde{C}$ has the homotopy type of the classifying space for $\mathbb{Z}_2$, i.e. that of an infinite-dimensional real projective space. It follows that the cohomology ring of $\tilde{C}$ is the polynomial ring $\mathbb{Z}_2[x]$, where $x$ has degree 1. Since $h_{\infty}$ has infinite index as a critical point of $E_j$ when $j \gg 0$, we can choose, for any positive integer $k$, a $(k + 1)$-dimensional subspace $V$ of $H$ on which the Hessian of $E_j$ at $h_{\infty}$ is negative definite. Let $S$ be a small sphere in $V$ centered at the origin; when $S$ is sufficiently small, $E_j$ takes values strictly less than $E_j(h_{\infty})$ on $h_{\infty} + S$. This implies that the nontrivial homology class in $\tilde{C}$ of degree $k$ exists already in some $\tilde{C}_\lambda$ with $\lambda < E_j(h_{\infty})$. As described in the previous Theorem, the homotopy type of $\tilde{C}_\lambda$ is obtained by taking a point (corresponding to the unique global minimum of $E_j$ in $\tilde{C}$) and attaching disks of dimensions determined by the indices of critical points with energies less than $\lambda$. The only way to create a homology class of degree $k$ is by attaching a disk of dimension $k$. Hence, $E_j$ must have at least one critical point of every possible index, for each $j \gg 0$.

Now choose a critical point $f_j$ of $E_j$ of index $k$ for each $j \gg 0$. We will show that $(f_j)$ satisfies the hypothesis of the Palais-Smale condition for $E$. We know that

$$E_j(f_j) < E_j(h_{\infty}) = \frac{\omega}{2} \int_{-\infty}^{\infty} (1 + \nu_j) \sech^{m-1} x \, dx \leq \omega \int_{-\infty}^{\infty} \sech^{m-1} x \, dx,$$

so is bounded independent of $j$. On the other hand, the fact that $\nu_j$ converges uniformly to zero implies that

$$E_j(f_j) = \frac{1}{2} \int_{-\infty}^{\infty} [(f_j')^2 + \omega(1 + \nu_j) \cos^2 f_j] \sech^{m-1} x \, dx$$

$$\geq \frac{1}{4} \int_{-\infty}^{\infty} [(f_j')^2 + \omega \cos^2 f_j] \sech^{m-1} x \, dx = \frac{1}{2} E(f_j)$$
when $j \gg 0$. Thus, $E$ is uniformly bounded on the sequence $(f_j)$. On the other hand, $f_j$ satisfies the Euler-Lagrange equation
\begin{equation}
  f_j'' - (m - 1) \tanh xf_j' + \frac{\omega}{2} (1 + \nu_j) \sin 2f_j = 0,
\end{equation}
which implies that
\begin{equation}
  dE(f_j)(v) = -2 \int_{-\infty}^{\infty} [f_j'' - (m - 1) \tanh xf_j' + \frac{\omega}{2} \sin 2f_j] \tanh^{-1} x d x
\end{equation}
\begin{equation}
  = -\omega \int_{-\infty}^{\infty} \nu_j v \sin 2f_j \sech^{-1} x d x.
\end{equation}
The integral on the last line is bounded in absolute value by
\begin{equation}
  \omega \| \nu_j \|_{C^0} \int_{-\infty}^{\infty} |v| \sech^{-1} x d x.
\end{equation}
This tends to zero as $j \to \infty$, so $(f_j)$ satisfies the Palais-Smale condition. We can thus choose a subsequence which converges in $H$ to some critical point $h_k$ of $E$ in $C$.

We need to show that $h_k \neq h_\infty$, and that its extended index is at least $k$. Define $c_k$ to be the infimum of all $\lambda$ such that the nontrivial homology class of degree $k$ in $\tilde{C}$ can be represented by a cycle in $E^{-1}((-\infty, \lambda))$. From the fact that the class can be represented as described above by an embedding of a real projective space of dimension $k$ in $\tilde{C}$ along which the energy is everywhere less than $E(\infty)$, it follows that $c_k < E(\infty)$. The fact that
\begin{equation}
  |E_j(h) - E(h)| = \left| \frac{1}{2} \int_{-\infty}^{\infty} \nu_j \cos h \sech^{-1} x d x \right| \leq C \sup_{x \in \mathbb{R}} |\nu_j(x)|
\end{equation}
implies that $E_j$ converges uniformly to $E$ on $H$. Thus, for any $\epsilon \in (0, \frac{1}{4} (E(\infty) - c_k))$, $j \gg 0$ implies that
\begin{equation}
  E^{-1}((-\infty, c_k + \frac{\epsilon}{2})) \subset E_j^{-1}((-\infty, c_k + \epsilon))
\end{equation}
and $c_k + \epsilon < E_j(\infty)$. But this means that $E_j^{-1}((-\infty, c_k + \epsilon))$ must contain some critical point of index $k$. We can therefore assume that $E_j(f_j) < c_k + \epsilon$, which would imply that $E(h_k) < c_k + \epsilon$. This shows that $h_k \neq h_\infty$.

To see that the extended index of $h_k$ is at least $k$, we can look at the difference between the Hessians of $E_j$ and $E$ at $f_j$ and $h_k$, respectively. We find
\begin{equation}
  D^2E_j(f_j)(v, w) - D^2E(h_k)(v, w) = \frac{\omega}{2} \int_{-\infty}^{\infty} \left[ \cos 2h_k - (1 + \nu_j) \cos 2f_j \right] v \sech^{-1} x d x.
\end{equation}
This tends to zero as $j \to \infty$, uniformly in $v, w$ as they range over any bounded set in $H$. This implies that the Hessian of $E_j$ at $f_j$ converges to that of $E$ at $h_k$. The extended index is upper semicontinuous on the space of continuous quadratic forms on $H$, so the extended index of $h_k$ is at least $k$. Similarly, the index is lower semicontinuous, so the index cannot be greater than $k$.

Finally, the sequence of $h_k$ satisfies the hypothesis of the Palais-Smale condition, so converges to some critical point of $E$. By an argument similar to the one just given, the limit must have infinite index, so the limiting critical point must be $h_\infty$. QED

The analogous argument can be carried out for $C^+$ and $C^-$. This leads to the following conclusion.
Theorem 3.8. Suppose that $F : S^m \rightarrow S^n$ is an eigenmap with eigenvalue $\omega$, $m > 1$, and

$$\frac{(m - 1)^2}{4} < \omega.$$  

There are infinite sequences of critical points for $E$ in $C^+$ and $C^-$, each of which converges strongly to $h_\infty$.

These are the generalizations of the infinite sequences of degree 0 and degree 1 harmonic maps found in [2] and [4]. It is of interest to ask which homotopy classes of maps between spheres can be represented as suspensions of eigenmaps of spheres. As mentioned previously, the Hopf maps $S^3 \rightarrow S^2$ and $S^7 \rightarrow S^4$ are eigenmaps and satisfy the hypothesis of Theorem 2.1. Therefore, the homotopy classes of their suspensions contain infinitely many harmonic representatives. In the case of the map $S^3 \rightarrow S^2$, we obtain a map representing the nontrivial class in $\pi_4(S^3) = \mathbb{Z}_2$. In the case of $S^7 \rightarrow S^4$, we obtain a generator of $\pi_8(S^5) = \mathbb{Z}_{24}$. The maps $S^5 \rightarrow S^4$ and $S^9 \rightarrow S^8$ mentioned in §2 produce infinite families of harmonic maps of the form $S^6 \rightarrow S^5$ and $S^{10} \rightarrow S^9$. The relevant homotopy groups are again isomorphic to $\mathbb{Z}_2$, but we do not know whether the suspensions of the two original maps represent the nontrivial class. The maps associated to the cubic harmonic eiconals on $\mathbb{R}^8$ and $\mathbb{R}^{14}$ give infinite sequences of harmonic self-maps of degrees 0 and 2 defined on $S^8$ and $S^{14}$.

As we have already mentioned, there are other settings where similar ideas may apply. We will briefly summarize the characteristics of the problem discussed here which make the argument above possible.

1. The configuration space is contractible, being in this case a Hilbert space.
2. The energy functional satisfies the Palais-Smale condition.
3. There is a reflection symmetry of the configuration space preserving the energy functional. There is a unique fixed point for this symmetry, corresponding to a critical point for the energy.
4. The index of the fixed point is infinite.
5. All critical points with energy less than that of the fixed point have finite index.
6. Possibly after small perturbations of the energy, the critical points with energy less than that of the fixed point are nondegenerate.

Of course, there are variations of these conditions which may be treated along similar lines.

References

[1] P. Bizoń. Saddle-point solutions in Yang-Mills-dilaton theory. Phys. Rev., D47:1656–1663, 1993.
[2] P. Bizoń. Harmonic maps between three-spheres. Proc. Roy. Soc. London Ser. A, 451(1943):779–793, 1995.
[3] P. Bizoń. Equivariant self-similar wave maps from Minkowski spacetime into the 3-sphere. math-ph/9910026.
[4] P. Bizoń and T. Chmaj. Harmonic maps between spheres. Proc. Roy. Soc. London Ser. A, 453(1957):403–415, 1997.
[5] K.-c. Chang. Infinite-dimensional Morse theory and multiple solution problems. Birkhäuser Boston Inc., Boston, MA, 1993.
[6] W. Y. Ding. Symmetric harmonic maps between spheres. Comm. Math. Phys., 118(4):641–649, 1988.
[7] J. Eells and A. Ratto. Harmonic maps and minimal immersions with symmetries. Methods of ordinary differential equations applied to elliptic variational problems. Princeton University Press, Princeton, NJ, 1993.
[8] D. Motreanu. Generic existence of Morse functions on infinite-dimensional Riemannian manifolds and applications. In *Global differential geometry and global analysis (Berlin, 1990)*, pages 175–184. Springer, Berlin, 1991.

[9] V. Pettinati and A. Ratto. Existence and nonexistence results for harmonic maps between spheres. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 17(2):273–282, 1990.

[10] R. T. Smith. Harmonic mappings of spheres. *Bull. Amer. Math. Soc.*, 78:593–596, 1972.

[11] R. T. Smith. Harmonic mappings of spheres. *Amer. J. Math.*, 97:364–385, 1975.

[12] M. Struwe. *Plateau’s problem and the calculus of variations*. Princeton University Press, Princeton, NJ, 1988.

[13] D. Sudarsky and R. M. Wald. Extrema of mass, stationarity, and staticity, and solutions to the Einstein-Yang-Mills equations. *Phys. Rev. D (3)*, 46(4):1453–1474, 1992.

[14] M. Volkov and D. Gal’tsov. Gravitating non-abelian solitons and black holes with Yang-Mills fields. *Physics Reports*, 319:1–83, 1999.