Solutions for the Klein-Gordon and Dirac equations on the lattice based on Chebyshev polynomials

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The main goal of this paper is to adopt a multivector calculus scheme to study finite difference discretizations of Klein-Gordon and Dirac equations for which Chebyshev polynomials of the first kind may be used to represent a set of solutions. The development of a well-adapted discrete Clifford calculus framework based on spinor fields allows us to represent, using solely projection based arguments, the solutions for the discretized Dirac equations from the knowledge of the solutions of the discretized Klein-Gordon equation. Implications of those findings on the interpretation of the lattice fermion doubling problem is briefly discussed.

Keywords: Chebyshev polynomials; discrete Dirac operators; lattice fermion doubling; spinor fields.

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I. INTRODUCTION

I.1. State of art

For a variety of reasons, the study of equations from relativistic wave mechanics through the incorporation of a fermionic lattice structure in the discrete space-time, plays an important role far beyond the design of non-perturbative methods in Quantum Electrodynamics (QED) and Quantum Chromodynamics (QCD) (cf. [21, Chapters 4 & 5]). Such kind of lattice structure, also used in the study Higgs and Yukawa models (cf. [21, Chapter 6]), was widely popularized during the last decade from its crucial role on the representation of Ising models as local conformal structures of spin-type (cf. [6, 19]).

Historically it was D. Bohm (cf. [2]) one of the firsts that recognizes such need and E.A.B. Cole [7] one amongst many that present the former routes addressing to this topic. With the seminal works of K. Wilson [27] and Kogut-Susskind [18], involving lattice regularizations of Dirac equations, it was realized that there is a spectrum degeneracy phenomenon provided by the replication of fermionic states in the massless limit \( m \to 0 \) for the resulting discretized equations, the so well-know lattice fermion doubling problem.

Years later, J. M. Rabin (cf. [23]) explained with some detail that such gap is indeed a direct consequence of the non-trivial topology of the lattice momentum space supplied by a cut-off. Such kind of topology, isomorphic to a \( n \)-dimensional torus \( \mathbb{R}^n/h\mathbb{Z}^n \), corresponds in the momentum space to the restriction of all momenta to the cube \( Q_h = \left[ -\pi h, \pi h \right]^n \), the so-called Brillouin zone (cf. [15]). Moreover, it was explained in detail that this gap is only filled by the chiral breaking of the symmetries provided by the continuum model.

The proper mathematical foundation of lattice fermion doubling, traced in depth by Nielsen–Ninomiya in [22] and proved afterwards by D. Friedan in [14], shows that through lattice regularizations of fermionic fields the existence of doublers always hold when one impose translational invariance, hermiticity and locality constraints. That is, the doubling of solutions underlying discretized models always arise under this set of constraints (cf. [14, Sections 2 & 3]).

Driven by the combination of these ideas with Beccher-Joos’s insights [1] on Dirac-Kähler fermions, other kinds of lattice formulations such as the approaches obtained by J. Vaz [25] and Kanamori–Kawamoto [17] were obtained. The methods employed on both formulations were essentially build up from Dimakis–Müller-Hoissen approach on noncommutative differential calculus over discrete sets (cf. [10]).

In a different context, mainly driven by the need of obtaining factorizations for discrete Laplacians on combinatorial surfaces, I’ve introduced in collaboration with U. Kähler and F. Sommen (cf. [11]) finite difference approximations for the Dirac operator. In the spirit of multivector calculus, it was explained on my PhD dissertation [12] that such kind of discretization is interrelated with the Dirac-Kähler formalism considered in [1, 17, 25].

Interestingly enough (but not yet fully adopted or known in the community) combination of tools from finite difference potentials (cf. [4, 16]) and interpolation theory (cf. [5, 15]) arising in this context may also be useful in the modelling of problems of quantum field theory over the phase space \( h\mathbb{Z}^n \times \left[ -\pi h, \pi h \right]^n \) (see, for instance, [20]).
I.2. Outline of the paper

In this paper one will show the feasibility of discrete Clifford calculus in the exact representation of the solutions of some discretized equations from wave mechanics. To this end, a consistent multivector calculus scheme through the lattice \( h\mathbb{Z}^n \) will be introduced in Section III with the aim of investigate in Section III the solutions of the discretized time-harmonic Klein-Gordon equation

\[
\Delta_h f(x) = m^2 f(x)
\]

for a given finite difference approximation \( \Delta_h \) of the Laplace operator \( \Delta = \sum_{j=1}^{n} \partial^2_{x_j} \), and moreover, the solutions of a discretized Dirac equation from the knowledge of the solutions of (1). Such characterization corresponds in the paper to Proposition III.1 and Corollary III.1.

The problems of foremost interest treated in Section III also involve the hypercomplex extension of the Chebyshev polynomials of the first kind

\[
T_k(\lambda) = \frac{1}{2} \left( \lambda + \sqrt{\lambda^2 - 1} \right)^k + \frac{1}{2} \left( \lambda - \sqrt{\lambda^2 - 1} \right)^k
\]

whose hypergeometric series representation is given by

\[
T_k(\lambda) = \frac{1}{2} F_1 \left( -k; \frac{1}{2}; \frac{1}{\lambda^2} \right).
\]

The choice of this kind of polynomials, also ubiquitous in E.A.B. Cole’s former approach (cf. [2, p. 650]), was motivated from its wide range of applications far beyond the computation of discrete cosine transforms (cf. [3]).

Following the same train of thought of Borstnik-Nielsen’s seminal paper on multivector calculus (cf. [3]), one will look further in Section III to the emboid a Clifford algebra with signature \((0, n)\) onto the algebra \( \mathcal{A}_h \) of all real-valued lattice functions in \( h\mathbb{Z}^n \).

To this end, one will formulate the wedge (\( \wedge \)) and the dot (\( \cdot \)) product between a Clifford generator and a multivector function on the lattice by taking into account two kind of displacement actions on the axis of \( n \)-dimensional Euclidean space. The Clifford algebra over \( \mathbb{R}^{p+q} \) with signature \((q, p)\) corresponds to a real associative algebra with identity 1, containing \( \mathbb{R} \) and \( \mathbb{R}^{p+q} \) as subspaces, and in which the basis elements \( e_1, e_2, \ldots, e_p, e_{p+1}, \ldots, e_{p+q} \) satisfy the following graded anti-commuting relations

\[
e_j e_k + e_k e_j = -2\delta_{jk}, \quad j, k = 1, 2, \ldots, p
\]

\[
e_j e_{k+q} + e_{k+q} e_j = 0, \quad j, k = 1, 2, \ldots, p
\]

\[
e_{j+q} e_{k+q} + e_{k+q} e_{j+q} = 2\delta_{jk}, \quad j, k = 1, 2, \ldots, p.
\]

Based on the above constraints, a basis \( e_J \) for \( \mathcal{C}_{q,p} \) then consists on \( e_J = e_{j_1} e_{j_2} \ldots e_{j_r} \), where \( J = \{ j_1, j_2, \ldots, j_r \} \) is a partially ordered subset of \( \{1, 2, \ldots, p, p+1, \ldots, p+q\} \) with cardinality \(|J| = r\) so that \( 0 \leq r \leq p+q \). For \( J = \emptyset \) (empty set) one will use the convention \( e_\emptyset = 1 \).

Then, any Clifford number \( a \in \mathcal{C}_{q,p} \) may thus be expressed as

\[
a = \sum_{r=0}^{p+q} \sum_{|J|=r} a_{J} e_J.
\]

Herewith, it is important to notice that \( \mathcal{C}_{q,p} \) is a universal algebra of dimension \( 2^{p+q} \) linear isomorphic to the exterior algebra \( \Lambda^* (\mathbb{R}^{p+q}) \) (cf. [23, Chapter 2]).

In such way, the elements of \( \mathbb{R} \) in \( \mathcal{C}_{q,p} \) are represented as \( a = a e_\emptyset \) whereas the vectors of \( \mathbb{R}^{p+q} \) corresponds in \( \mathcal{C}_{q,p} \) to the ansatz \( z = \sum_{j=1}^{p} x_j e_j + y_j e_{j+q} \). Notice also

\[
abla
\]

will avoids a priori problems related with associativity and distributivity on the product between a Clifford basis and a multivector function with membership in \( \Lambda^* \mathcal{A}_h \).

The major challenging here against [2] will be the introduction of a unitary action over the resulting multivector space through a unitary local action \( \chi_h(x) \) on \( h\mathbb{Z}^n \), closely related with a pseudoscalar representation within the Clifford algebra of signature \((n, n)\). As one will see in Section IV such local action incorporates the Kogut-Susskind fermion regularization.

The intriguing aspect besides this staggered based formulation is that such lattice actions will also endow projection operators that will provide, as in [2], a direct sum decomposition involving exterior algebras of chiral and achiral type, similar to spinor spaces \( \frac{1}{2} (1 \pm \gamma) \mathcal{C}_{0,n} \) that appear on the direct sum decomposition

\[
\mathcal{C}_{n,n} = \frac{1}{2} (1 + \gamma) \mathcal{C}_{0,n} \oplus \frac{1}{2} (1 - \gamma) \mathcal{C}_{0,n}.
\]

so that \( \gamma \) is a pseudoscalar of \( \mathcal{C}_{n,n} \) satisfying \( \gamma^2 = 1 \) (cf. [2, Sections IV. \& V.]).

II. MULTIVECTOR CALCULUS ON THE LATTICE

II.1. Real Clifford algebras

Let \( \mathbb{R}^{p+q} \) be the standard \((p + q)\)-dimensional Euclidean space. The Clifford algebra over \( \mathbb{R}^{p+q} \) with signature \((q, p)\) corresponds to a real associative algebra with identity 1, containing \( \mathbb{R} \) and \( \mathbb{R}^{p+q} \) as subspaces, and in which the basis elements \( e_1, e_2, \ldots, e_p, e_{p+1}, \ldots, e_{p+q} \) satisfy the following graded anti-commuting relations

\[
e_j e_k + e_k e_j = -2\delta_{jk}, \quad j, k = 1, 2, \ldots, p
\]

\[
e_j e_{k+q} + e_{k+q} e_j = 0, \quad j, k = 1, 2, \ldots, p
\]

\[
e_{j+q} e_{k+q} + e_{k+q} e_{j+q} = 2\delta_{jk}, \quad j, k = 1, 2, \ldots, p.
\]
that \( x = \sum_{j=1}^{p} x_j e_j \) corresponds to the \( C\ell_{0,p} \)-valued representation of the \( p \)-tuple \((x_1, x_2, \ldots, x_p)\) of \( \mathbb{R}^p \) whereas \( y = \sum_{j=1}^{q} y_j e_{q+j} \) corresponds to \( C\ell_{0,q} \)-valued representation of the \( q \)-tuple \((y_1, y_2, \ldots, y_q)\) of \( \mathbb{R}^q \).

There are essentially three automorphisms that leave the multivector structure of \( C\ell_{q,p} \) invariant. They are defined as follows:

- **The main involution** \( a \mapsto a' \) is defined recursively via
  \[
  (ab)' = a'b', \\
  (a_j e_j)' = a_j e_j e_{j+q}, \\
  e_j' = -e_j \\
  \text{and} \\
  e_{j+q}' = e_{j+q}.
  \]
  \hspace{1cm} (4)

- **The reversion** \( a \mapsto a^* \) is defined recursively via
  \[
  (ab)^* = b^*a^*, \\
  (a_j e_j)^* = a_j e_j e_{j+q}, \\
  e_j^* = e_j \\
  \text{and} \\
  e_{j+q}^* = e_{j+q}.
  \]
  \hspace{1cm} (5)

- **The \( \dagger \)-conjugation** \( a \mapsto a^\dagger \) is defined recursively via
  \[
  (ab)^\dagger = b^\dagger a^\dagger, \\
  (a_j e_j)^\dagger = a_j e_j e_{j+q}, \\
  e_j^\dagger = -e_j \\
  \text{and} \\
  e_{j+q}^\dagger = e_{j+q}.
  \]
  \hspace{1cm} (6)

Notice that the \( \dagger \)-conjugation may be rewritten as a composition between main involution and reversion, that is \((a')^* = (a')^\dagger = a^\dagger \) holds for every \( a \in C\ell_{q,p} \) whereas in case when \( a \) belongs to \( C\ell_{0,p} \) the Clifford number \( a^\dagger \) coincides with the standard conjugation \( \overline{a} \) over \( \mathbb{C} \). So, any Clifford vector \( x = \sum_{j=1}^{p} x_j e_j \) of \( C\ell_{0,p} \) satisfies \( x^\dagger = x \) and \( x^\dagger = x^* = -x \).

Let us now turn our attention to the description of the real Clifford algebra \( C\ell_{n,n} \) as a canonical realization of the algebra of endomorphisms \( \text{End}(C\ell_{0,n}) \).

First, recall that in terms of the main involution automorphism, the Clifford product \( e_j a \) may be decomposed as

\[
e_j a = \frac{1}{2} (e_j a + (a e_j)^*) + \frac{1}{2} (e_j a - (a e_j)^*)
\]

The symmetric and skew-symmetric part of the above summand thus give rise to the multivector counterparts for the dot (\( \bullet \)) and wedge (\( \wedge \)) product. Indeed, introducing \( e_j \bullet a \) and \( e_j \wedge a \) as

\[
e_j \bullet a = -\frac{1}{2} (e_j a - a'e_j) \text{ and } e_j \wedge a = -\frac{1}{2} (e_j a + a'e_j)
\]

one obtains \( e_j a = -e_j \bullet a + e_j \wedge a \).

Next, let us identify the generators of \( C\ell_{n,n} \) as left and right endomorphisms via the canonical correspondences

\[
e_j : a \mapsto e_j a \text{ and } e_{j+n} : a \mapsto a' e_j.
\]

From the combination of (7) with (8) it is clear that the set of relations \( e_j (e_j a) = e_j^2 a = -a \) and \( (a' e_j) e_j = e_j^2 a = a \) reveals the canonical isomorphism between \( \text{End}(C\ell_{0,n}) \) and \( C\ell_{n,n} \) provided by (7).

The dot and wedge products defined above suggest the introduction of the set of operators \( e_j^\dagger : a \mapsto e_j \bullet a \) and \( e_j^- : a \mapsto e_j \wedge a \), the so-called Witt basis. From (7) \( e_j^\dagger = e_j \bullet (\cdot) \) and \( e_j^- = e_j \wedge (\cdot) \) may be rewritten as

\[
e_j^\dagger = \frac{1}{2} (e_j a - e_j a') \text{ and } e_j^- = \frac{1}{2} (e_j a + e_j a').
\]

Therefore, the set \( \{e_j^\dagger, e_j^- : j = 1, 2, \ldots, n\} \) also forms a basis for \( \text{End}(C\ell_{0,n}) \). The remaining set of graded anti-commuting relations are given by

\[
e_j e_k e_j^- e_k^- = 0, \\
e_j e_k e_k^- e_j^- = 0, \\
e_j^- e_k^- e_k e_j = \delta_{jk}.
\]

Conversely, any generator of \( C\ell_{n,n} \) may be rewritten as a linear combination involving the basis elements of \( \text{End}(C\ell_{0,n}) \). The remaining linear combinations are given by

\[
e_j = e_j^- - e_j^\dagger \text{ and } e_{j+n} = e_j^- + e_j^\dagger.
\]

### II.2. Discrete multivector functions

From now, one will adopt the multivector representation \( x = \sum_{j=1}^{n} x_j e_j \) when one refers to the \( n \)-tuple \((x_1, x_2, \ldots, x_n)\) of \( \mathbb{R}^n \) and the displacements \( x \pm h e_j \) along the \( x_j \)-axis when one refer to forward/backward shifts \((x_1, x_2, \ldots, x_j \pm h, \ldots, x_n)\) over the lattice \( h\mathbb{Z}^n \) with mesh width \( h > 0 \). Any \( C\ell_{0,n} \)-valued function \( f(x) \) may thus be represented as

\[
f(x) = \sum_{r=0}^{n} \sum_{|\mathbf{J}|=r} f_{\mathbf{J}}(x) e_{\mathbf{J}}, \text{ with } x \in h\mathbb{Z}^n.
\]

The algebra containing all the lattice functions \( f_{\mathbf{J}}(x) \) will be denoted by \( \Lambda h \) whereas the linear space containing the summands of the form \( \sum_{|\mathbf{J}|=r} f_{\mathbf{J}}(x) e_{\mathbf{J}} \) will be denoted by \( \Lambda^* A_h \). Clearly, one has \( \Lambda^0 A_h = A_h \) and graded direct sum decomposition

\[
\Lambda^* A_h = \bigoplus_{r=0}^{n} \Lambda^r A_h.
\]

With the aim of fill the lack of commutativity over the lattice \( h\mathbb{Z}^n \) one will associate to the basis elements \( e_j^\dagger = e_j \bullet (\cdot) \) and \( e_j^- = e_j \wedge (\cdot) \) provided by (8) the following noncommutative actions over \( \Lambda^* A_h \):

\[
e_j^\dagger f(x) = f(x - h e_j)' e_j^\dagger, \\
e_j^- f(x) = f(x + h e_j)' e_j^-.
\]

It is clear from (9) that the action of each \( e_j^\dagger e_j^- \) resp. \( e_j^- e_j^\dagger \) on \( f(x) \) are commutative. Also, from (7) one can
see that the set of operators $e_j^\pm e_j^\pm$, resp. $e_j^- e_j^+$, mutually commute and satisfy, for each $j = 1, 2, \ldots, n$, the set of idempotent relations $e_j^\pm (e_j^\pm e_j^\pm f(x)) = e_j^\pm e_j^\pm f(x)$ resp. $e_j^- e_j^+ (e_j^- e_j^+ f(x)) = e_j^- e_j^+ f(x)$. Moreover, from (8) and (9) the mappings
\[
e_j^+ e_j^- : f(x) \mapsto e_j \cdot (e_j \wedge f(x))
\]
\[
e_j^- e_j^+ : f(x) \mapsto e_j \wedge (e_j \cdot f(x))
\]
have the following $Cl_{n,n}$ representation:
\[
e_j^+ e_j^- = \frac{1}{2} (1 + e_j + e_j^n e_j) \quad \text{and} \quad e_j^- e_j^+ = \frac{1}{2} (1 - e_j + e_j^n e_j).
\]

Here one would like to observe that from (9) the set of graded commutator mappings on $End(\Lambda^* A_h)$, defined as
\[
[e_j^+, e_j^-] : f(x) \mapsto e_j \cdot (e_j \wedge f(x)) - e_j \wedge (e_j \cdot f(x))
\]
correspond in $Cl_{n,n}$ to the bivectors $e_j + e_j^n e_j$. Moreover, they mutually commute and satisfy, for each $j = 1, 2, \ldots, n$, the set of unitary relations $[e_j^+, e_j^-] (e_j^\pm, e_j^\pm f(x)) = f(x)$. That allows us to obtain two projection operators as idempotents of $End(\Lambda^* A_h)$.

To this end, let us introduce the following local action on $h\mathbb{Z}^n$:
\[
\chi_h(x) = \prod_{j=1}^n (-1)^{x_j} [e_j^+, e_j^-].
\]

A short computation shows that $\chi_h(x)^2 = \prod_{j=1}^n (-1)^{x_j} [e_j^+, e_j^-]^2 = 1$, and thus, the unitary relation $\chi_h(x) (\chi_h(x) f(x)) = f(x)$. Therefore, elements of the form $\frac{1}{2} (1 \pm \chi_h(x)) \in End(\Lambda^* A_h)$ are also idempotent, that is
\[
\frac{1}{2} (1 \pm \chi_h(x)) \left( \frac{1}{2} (f(x) \pm \chi_h(x) f(x)) \right) = \frac{1}{2} (f(x) \pm \chi_h(x) f(x)).
\]

This allows us to introduce the multivector spaces $\Lambda^*_+ A_h$, defined as follows:
\[
\Lambda^*_+ A_h = \left\{ \frac{1}{2} (f(x) + \chi_h(x) f(x)) : f(x) \in \Lambda^* A_h \right\}
\]
\[
\Lambda^- A_h = \left\{ \frac{1}{2} (f(x) - \chi_h(x) f(x)) : f(x) \in \Lambda^* A_h \right\}.
\]

In concrete, for any lattice function $f_{\pm}(x)$ with membership in $\Lambda^*_+ A_h \oplus \Lambda^- A_h$ there exist two multivector functions $u(x), v(x) \in \Lambda^* A_h$ such that
\[
f_{\pm}(x) = \frac{1}{2} (u(x) + \chi_h(x) u(x)) + \frac{1}{2} (v(x) - \chi_h(x) v(x)).
\]

The uniqueness of $u(x)$ and $v(x)$ is thus assured by the null relations $(1 - \chi_h(x)) (u(x) + \chi_h(x) u(x)) = (1 + \chi_h(x)) (v(x) - \chi_h(x) v(x)) = 0$. Indeed, by letting act $\frac{1}{2} (1 + \chi_h(x))$ and $\frac{1}{2} (1 - \chi_h(x))$ on both sides of (12) one gets
\[
\frac{1}{2} (f_{\pm}(x) + \chi_h(x) f_{\pm}(x)) = \frac{1}{2} (u(x) + \chi_h(x) u(x))
\]
\[
\frac{1}{2} (f_{\pm}(x) - \chi_h(x) f_{\pm}(x)) = \frac{1}{2} (v(x) - \chi_h(x) v(x)).
\]

One will use the subscript notations $f_{\pm}(x)$ and $f_{\pm}(x)$ to denote the multivector functions of the form $f_{\pm}(x) = \frac{1}{2} (f(x) \pm \chi_h(x) f(x))$. The bold notations $\gamma_{\pm}(x), g(x), \ldots, u(x)$ and so on will be only used when one refer to a multivector function belonging to $\Lambda^* A_h$.

**Remark II.1** In dimension $n = 4$, the direct sum $\Lambda^*_+ A_h \oplus \Lambda^- A_h$ looks like a Dirac-like spinor structure based on the homogeneous representation of the special unitary group $SU(2)$ modulo the idempotents $\frac{1}{2} (1 + \chi_h(x))$ and $\frac{1}{2} (1 - \chi_h(x))$ give rise to two independent irreducible representations of the special orthogonal group $SO(3)$.

When one takes the multivector extension of $C^*$ as the noncommutative ring of quaternions $\mathbb{H}$, from the isomorphism $Cl_{0,3} \cong \mathbb{H} \oplus \mathbb{H}$ it follows therefore that $\Lambda^*_+ A_h \oplus \Lambda^- A_h$ may also be represented, up to the permutation sign $(-1)^{\frac{x_j}{2}}$ in terms of Dirac matrices $\gamma_j$, with $j = 0, 1, 2, 3$ (cf. [2]), Subsections 2.2 & 2.3).

One refer the reader to [3, Subsection IV.H.] on which such discussion involving a $SU(2) \times SU(2)$ representation for the Lorentz group was taken in the context of Weyl bispinors.

### II.3. Discrete Dirac operators

After defining the spaces of multivector functions through the last subsection, one move now to the construction of finite difference discretizations for the Dirac operator in intertwining with the formulations [12] and [11]. In order to proceed one define, for each $j = 1, 2, \ldots, n$, the forward/backward finite difference operators $\partial^j_h / \partial^n_h$ by the coordinate formulæ
\[
\partial^j_h f(x) = \frac{f(x + h e_j) - f(x)}{h},
\]
\[
\partial^n_h f(x) = \frac{f(x) - f(x - h e_j)}{h}.
\]

Take into account the forward/backward finite difference Dirac operators $D^j_h = \sum_{j=1}^n e_j \partial^j_h$ already considered in [13], one could associate to each lattice function $f(x)$ the multivector actions
\[
\partial^j_h f(x) = D^j_h \star f(x) \quad \text{and} \quad \partial^n_h f(x) = D^n_h \wedge f(x).
\]

By means of the Witt basis $e_j^\pm$ defined in [3], the actions $\partial^j_h = D^j_h \star (-)$ and $\partial^n_h = D^n_h \wedge (-)$ on $\Lambda^* A_h$ resp.
$\Lambda^*_+A_h \oplus \Lambda^-_+A_h$ correspond to
\[
\partial^+_h = \sum_{j=1}^{n} e^+_j \partial^{+j}_h \quad \text{and} \quad \partial^-_h = \sum_{j=1}^{n} e^-_j \partial^{-j}_h. \tag{15}
\]

It is clear from \([11]\) that $\partial^{+j}_h$ and $\partial^{-j}_h$ are interrelated by the shift operators $S^{+j}_h f(x) = f(x \pm he_j)$, i.e.,
\[
S^{-j}_h (\partial^{+j}_h f(x)) = \partial^{+j}_h (S^{+j}_h f(x)) = \partial^{+j}_h f(x)
\]
\[
S^{+j}_h (\partial^{-j}_h f(x)) = \partial^{-j}_h (S^{-j}_h f(x)) = \partial^{-j}_h f(x). \tag{16}
\]

Moreover, for two Clifford-vector-valued functions $f(x)$ and $g(x)$ the action of each $\partial^{+j}_h/\partial^{-j}_h$ on $f(x)g(x)$ gives rise to the set of product rules
\[
\partial^{+j}_h (f(x)g(x)) = \partial^{+j}_h f(x) \ g(x) + f(x + he_j) \partial^{+j}_h g(x)
\]
\[
\partial^{-j}_h (f(x)g(x)) = \partial^{-j}_h f(x) \ g(x) + f(x - he_j) \partial^{-j}_h g(x). \tag{17}
\]

Now let us examine the actions of $\partial^+_h$ and $\partial^-_h$ on Clifford-vector-valued lattice functions. First, recall that the Grassmannian identities $e^+_j e^+_k + e^-_k e^-_j = 0$ provided by \([9]\) lead to the nilpotent relations
\[
\partial^+_h (\partial^+_h f(x)) = \partial^+_h (\partial^-_h f(x)) = 0.
\]

Based on \([10]\), one can also prove the following lemma, corresponding to generalized Leibniz rules at the level of discrete multivector calculus.

**Lemma II.1** For two lattice functions $f(x)$ and $g(x)$ with membership in $\Lambda^*_+A_h$, one have the generalized Leibniz rules
\[
\partial^+_h (f(x)g(x)) = (\partial^+_h f(x))g(x) + f(x')(\partial^+_h g(x))
\]
\[
\partial^-_h (f(x)g(x)) = (\partial^-_h f(x))g(x) + f(x')(\partial^-_h g(x)).
\]

**Proof II.1** Starting from the endomorphism representation provided by \([14]\), the summand(s) splitting
\[
\partial^+_h (f(x)g(x)) = \sum_{j=1}^{n} e^+_j \partial^{+j}_h f(x)g(x) + \]
\[
+ \sum_{j=1}^{n} e^+_j f(x \pm he_j) (\partial^{+j}_h g(x))
\]
follows straightforwardly from direct application of the product rules \([14]\) and from linearity arguments.

The summand(s) $\sum_{j=1}^{n} e^+_j \partial^{+j}_h f(x)g(x)$ equals to $(\partial^+_h f)(x)g(x)$ whereas from noncommutative constraints \([10]\)
\[
\sum_{j=1}^{n} e^+_j f(x \pm he_j) (\partial^{+j}_h g(x)) = \sum_{j=1}^{n} f(x)e^+_j (\partial^{+j}_h g(x)) =
\]
\[
f(x')(\partial^+_h g(x)).
\]

This results into $\partial^+_h (f(x)g(x)) = (\partial^+_h f)(x)g(x) + f(x')(\partial^+_h g(x))$, as desired.

The above properties altogether shows in turn that, for a given algebra $A_h$ of real-valued functions over the lattice $hZ^n$ with mesh width $h > 0$, the pair $(\partial^+_h, \Lambda^*_+A_h)$ encodes a universal differential calculus over a hypercubic lattice (cf. \([10, 17, 23]\)) whereas $\partial^-_h$ plays the role of the codifferential operator. Indeed, one has the raising and lowering properties, $\partial^+_h : \Lambda^*_+A_h \mapsto \Lambda^{r+1}_+A_h$ and $\partial^-_h : \Lambda^*_+A_h \mapsto \Lambda^{r-1}_+A_h$, respectively.

Having in mind the Dirac-Kähler formalism over differential forms (cf. \([21]\) Subsection 4.3.3), one can introduce the finite difference Dirac operator $D_h$ over $\Lambda^*_+A_h \oplus \Lambda^-_+A_h$ as $D_h = \partial^-_h - \partial^+_h$.

It is easy to see from the splittings $e_j = e^-_j - e^+_j$ and $e_j+n = e^-_j + e^+_j$ provided by \([8]\) that $D_h$ admits the following $Cl_{n,n}$ based representation
\[
D_h = \sum_{j=1}^{n} e_j + \frac{\partial^-_h - \partial^+_h}{2}. \tag{18}
\]

The above formula, corresponding to the decomposition of $D_h$ through $Cl_{n,n}$ into a symmetric plus a skew-symmetric part roughly shows that $D_h$ also gives rise to a finite difference discretization of $D = \sum_{j=1}^{n} e_j \partial x_j$ (cf. \([11]\) for which the symmetric part corresponds to the central finite difference Dirac operator $\frac{1}{2} (D^-_h + D^+_h)$ and the skew-symmetric part equals to $\frac{1}{2} \Box_h$, where
\[
\Box_h = -\sum_{j=1}^{n} e_{j+n} \partial^-_h \partial^+_h
\]
denotes the $Cl_{n,n}$-valued extension of the lattice d’Alembert operators $-\partial^-_h \partial^+_h = \frac{1}{h} (\partial^-_h - \partial^+_h)$ (cf. \([21]\) Subsection 1.5.1]).

There many basic properties regarding the multivector operators \([13]\) and \([14]\) that could be formulating only in terms of the graded commuting relations \([9]\) and \([3]\), respectively. The next proposition, involving the factorization of the star Laplacian
\[
\Delta_h f(x) = \sum_{j=1}^{n} f(x + he_j) + f(x - he_j) - 2f(x) \frac{h^2}{h^2} \tag{19}
\]
will be of special interest on the subsequent section.

**Proposition II.1** For the finite difference discretization of $\Delta_h$ defined by equation \([16]\) it holds
\[
\partial^+_h \left( \partial^-_h f(x) \right) + \partial^-_h \left( \partial^+_h f(x) \right) = \Delta_h f(x)
\]
\[
D_h \left( D_h f(x) \right) = -\Delta_h f(x). \tag{16}
\]

**Proof II.2** Recall that from \([14]\) one has, for each $j = 1, 2, \ldots, n$, the factorization relations
\[
\partial^+_h \left( \partial^-_h f(x) \right) = \frac{f(x + he_j) + f(x - he_j) - 2f(x)}{h^2}.
\]
On the other hand, since from \( (16) \) the forward and backwards \( \partial_{h,j}^+ / \partial_{h,j}^- \) mutually commute, it follows from the duality relations \( e_k^+ e_k^- = e_k^- e_k^+ = \delta_{kk} \) provided by \( (12) \) that the set of identities

\[
\Delta_h f(x) = \sum_{j,k=1}^{2} \left( e_k^+ e_k^- + e_k^- e_k^+ \right) \partial_{h,j}^+ (\partial_{h,k}^- f(x)) = \partial_{h,j}^+ (\partial_{h,k}^- f(x)) + \partial_{h,k}^- (\partial_{h,j}^+ f(x)).
\]

Similarity, the combination of \( (5) \) with the finite difference property

\[
\partial_{h,j}^+ \partial_{h,k}^- = \left( \frac{(\partial_{h,j}^- + \partial_{h,k}^+)}{2} \right)^2 \left( \frac{(\partial_{h,j}^- - \partial_{h,k}^+)}{2} \right)^2
\]

lead to

\[
-\Delta_h f(x) = \sum_{j=1}^{2} e_j^2 \left( \frac{(\partial_{h,j}^- + \partial_{h,j}^+)}{2} \right)^2 f(x) + e_j^2 \left( \frac{(\partial_{h,j}^- - \partial_{h,j}^+)}{2} \right)^2 f(x) = -D_h (D_h f(x)).
\]

**Remark II.2** The statement \( D_h (D_h f(x)) = -\Delta_h f(x) \) also yields as direct consequence of the combination of the formula \( \partial_{h,j}^+ (\partial_{h,k}^- f(x)) + \partial_{h,k}^- (\partial_{h,j}^+ f(x)) = -D_h (D_h f(x)) = \Delta_h f(x) \) with Lemma \( II.2 \).

### III. KLEIN-GORDON AND DIRAC EQUATIONS ON THE LATTICE

#### III.1. The factorization approach

After the preliminary construction provided in Section \( II \) one now enter in the heart of the matter. Our first task consists in the exact formulation of an exact discretized model for the Dirac equation. Unlike the Dirac-Kähler formulations \( I, \ I1, \ I3 \) one will incorporate the local unitary action \( \chi_h(x) \) on \( \Lambda_+ A_h \oplus \Lambda_- A_h \) to ensure the factorization of the discretized Klein-Gordon operator \( -\Delta_h + m^2 \) provided by the equation \( (11) \) in terms of a discretized Dirac-field operator. Such action is close to the Dirac-Hestenes spinor field action (cf. \( [21] \) Subsection 3.7).

To this end, one consider for a multivector function \( f_+ (x) \) with membership in \( \Lambda_+ A_h \oplus \Lambda_- A_h \), the following discretized Dirac equation on \( hZ^n \) for a free particle with mass \( m \):

\[
D_h \ f_+ (x) = m \ \chi_h(x) \ f_+(x).
\]  

(20)

From the substitution provided by \( (12) \) on both sides of \( (20) \), it can be easily concluded, based on the idempotency relation \( \chi_h(x)(\chi_h(x)f(x)) = f(x) \), the following equivalent formulation as a coupled system of equations

\[
\begin{cases}
D_h \ f_+(x) = m \ f_+(x) \\
D_h \ f_-(x) = -m \ f_-(x)
\end{cases}
\]

(21)

so that the solutions \( f_+(x) \) and \( f_-(x) \) of \( (21) \) are the chiral and the achiral components of the spinor vector-field \( f_+(x) \).

Considering simply the multivector finite difference operator \( D_h - m \chi_h(x) \), the solutions of \( (20) \) could be described in terms of the linear space \( \ker (D_h - m \chi_h(x)) \) whereas the linear spaces \( \frac{1}{2} (1 \pm \chi_h(x)) \ker (D_h - m \chi_h(x)) \) of chiral/achiral type contain the solutions of \( (21) \). In addition, the \( Cl_{n,n} \)-valued representation underlying \( D_h \) and \( \chi_h(x) \) allows us to derive the subsequent set of results in a reliable way.

**Proposition III.1** When acting on the multivector space \( \Lambda_+ A_h \oplus \Lambda_- A_h \), the discretized Dirac-field operator \( D_h - m \chi_h(x) \) satisfies

\[
(D_h - m \chi_h(x))^2 = -\Delta_h + m^2.
\]

Moreover, one has

\[
\ker (D_h - m \chi_h(x)) = (D_h - m \chi_h(x)) \left[ \ker (-\Delta_h + m^2) \right].
\]

**Proof III.1** From the direct computation

\[
(D_h - m \chi_h(x))^2 = D_h^2 - mD_h \chi_h(x) - m\chi_h(x) D_h + m^2 \chi_h(x)^2
\]

one easily recognizes the star Laplacian splitting \( (D_h)^2 = -\Delta_h \) (Lemma \( II.1 \)) and the unitary relation \( \chi_h(x)^2 = 1 \) as well.

On the other hand, from \( (3) \) one observe that the set anti-commuting relations \( e_j (e_j e_j) = \text{null} \) and \( e_j (e_j e_j) = \text{null} \) for each \( j = 1, 2, \ldots, n \). This results into the following set of graded anti-commuting relations

\[
e_j \chi_h(x) = -\chi_h(x) e_j \quad \text{and} \quad e_j e_j \chi_h(x) = -\chi_h(x) e_j e_j,
\]

and moreover, into the basic identity \( D_h \chi_h(x) = -\chi_h(x) D_h \). Whence

\[
(D_h - m \chi_h(x))^2 = -\Delta_h + m^2.
\]

For the proof of \( \ker (D_h - m \chi_h(x)) = (D_h - m \chi_h(x)) \left[ \ker (-\Delta_h + m^2) \right] \) one recall first that from the factorization \( (D_h - m \chi_h(x))^2 = -\Delta_h + m^2 \), each solution of \( (20) \) is also a solution of the discretized Klein-Gordon equation \( (7) \), and thus, the linear space \( \ker (D_h - m \chi_h(x)) \) is a subspace of \( \ker (-\Delta_h + m^2) \).

Therefore, one gets the inclusion

\[
\ker (D_h - m \chi_h(x)) \subset (D_h - m \chi_h(x)) \left[ \ker (-\Delta_h + m^2) \right].
\]

Conversely, let \( f_-(x) \) be a multivector function with membership in \( (D_h - m \chi_h(x)) \left[ \ker (-\Delta_h + m^2) \right] \), that is

\[
f_-(x) = D_h g(x) - m \chi_h(x) g(x),
\]

with \( \Delta_h g(x) = m^2 g(x) \).
By applying $D_h - m\chi_h(x)$ to $f_+(x)$, it follows then
\[ D_h f_+(x) - m\chi_h(x) f_+(x) = -\Delta_h g(x) + m^2 g(x) = 0. \]

Thus $f_+(x) \in \ker (D_h - m\chi_h(x))$ and whence $(D_h - m\chi_h(x)) \ker (-\Delta_h + m^2) \subset \ker (D_h - m\chi_h(x))$.

**Corollary III.1** Let $g(x) \in \Lambda_+^\omega A_h$ be a solution of the discretized Klein-Gordon equation
\[ \Delta_h g(x) = m^2 g(x). \]

Then, the solutions of the coupled system (21) are given by
\[ f_+(x) = \frac{1}{2} (1 + \chi_h(x)) (D_h g(x) - mg(x)) \]
\[ f_-(x) = \frac{1}{2} (1 - \chi_h(x)) (D_h g(x) - mg(x)). \]

**Remark III.1** The above corollary is also valid for solutions of the discretized Klein-Gordon equation (1) with fine, for each $T$ the Clifford-vector-valued polynomial $T_{\alpha}(x, y; a)$ by the multi-variable formula
\[ T_{\alpha}(x, y; a) = \prod_{j=1}^n 2F1 \left( -x_j, y_j; 1, \frac{1}{2}, 1 - \lambda; \chi_h(x) \right) a. \]

Here and elsewhere $2F1$ denotes the hypergeometric series expansion defined from the Pochhammer symbol $(\lambda)_s = \frac{\Gamma(\lambda+s)}{\Gamma(\lambda)}$ by the formula
\[ 2F1 (b_1, b_2; c_1; t) = \sum_{s=0}^{\infty} \frac{(b_1)_s (b_2)_s}{(c_1)_s} \frac{t^s}{s!}. \]

**Remark III.2** A striking aspect besides this construction is that the term $\frac{1}{2} \sum_{\alpha} \chi_h(x)$ within each $2F1$ representation is indeed an element of $End(\Lambda_+^n A_h)$ so that each $2F1 \left( -x_j, y_j; 1, \frac{1}{2}, \frac{1}{2}; \chi_h(x) \right) a$ shall be understood as an operational action. This means that the right-hand side of (22) is in fact a $n-$composition formula.

Let us now take a close look to the multi-variable representation of (22) in terms of the classical Chebyshev polynomials of the first kind $T_k(\lambda)$. For $\alpha = 0$ (the zero $n-$vector), the symmetric identity $2F1 \left( -k, k; \frac{1}{2}, \frac{1}{2}; \frac{1}{2} \right) = 2F1 (k, -k; \frac{1}{2}, \frac{1-\lambda}{2})$ allows us to extend the Chebyshev polynomials also for negative integers. That is, putting $k = \frac{1+\lambda}{h}$ on the right-hand side of (22) one always have the identity $T_{[x,j]}(y_j) = 2F1 \left( -\frac{x_j}{h}, \frac{x_j}{h}, 1; \frac{1}{2}, \frac{1}{2} \right) x_j$.

So one has
\[ T_{\alpha}(x, y; a) = \prod_{j=1}^n \left( y_j + \sqrt{y_j^2 - 1} \right) + (y_j - \sqrt{y_j^2 - 1})^s \]
\[ \times \prod_{j=1}^n \left( y_j + \sqrt{y_j^2 - 1} \right) + (y_j - \sqrt{y_j^2 - 1})^s. \]

For $\alpha \neq 0$, the idempotency of $\frac{1+\chi_h(x)}{2}$ allows us to represent, for suitable values of $y$, the hypercomplex polynomials (22) as elements with membership in $\Lambda_+^n A_h$ or in $\Lambda_+^\omega A_h$. So one starts with the binomial identity
\[ \left( \frac{1 - \lambda}{2} + \mu \frac{1 + \chi_h(x)}{2} \right)^s = \sum_{r=0}^{s} \left( \begin{array}{c} s \\ r \end{array} \right) \left( \frac{1 - \lambda}{2} \right)^{s-r} \mu^r \frac{1 + \chi_h(x)}{2}. \]

Application of the above formula results, after a straightforward computation based on linearity arguments, into the hypergeometric identity
\[ 2F1 \left( -k, k; \frac{1 - \lambda}{2} + \mu \frac{1 + \chi_h(x)}{2} \right) = \]
\[ = \frac{1 + \chi_h(x)}{2} 2F1 \left( -k, k; \frac{1 - \lambda}{2} + \mu \right). \]

Recalling (22) one finds for the substitutions $\lambda = y_j + 2\alpha_j$, $\mu = \pm \alpha_j$, and $k = \frac{1}{h}$, the set of identities
\[ 2F1 \left( -x_j, y_j; 1, \frac{1}{2}, \frac{1}{2}; \chi_h(x) \right) = \]
\[ = \frac{1}{2} \left( T_{[x, j]}(y_j + 2\alpha_j) \pm \chi_h(x) T_{[x, j]}(y_j + 2\alpha_j) \right). \]

After some straightforward manipulations based again on the idempotency of $\frac{1+\chi_h(x)}{2}$, one thus gets, for $\alpha \neq 0$, the set of relations
\[ T_{\alpha}(x, y; a) = \]
\[ = \frac{1}{2} \left( T_{\alpha}^{(0)}(x, y, 2\alpha; a) + \chi_h(x) T_{\alpha}^{(0)}(x, y, 2\alpha; a) \right. \]
\[ \left. T_{\alpha}^{(-\alpha)}(x, y; a) = \right] \]
\[ = \frac{1}{2} \left( T_{\alpha}^{(0)}(x, y, 2\alpha; a) - \chi_h(x) T_{\alpha}^{(0)}(x, y, 2\alpha; a) \right). \]

that naturally result into the following lemma.

**Lemma III.1** For each $x \in h\mathbb{Z}^n$ and $y, \alpha \in \mathbb{R}^n$, and $a \in C\mathbb{L}_0^n$. Then we have the following:
1. The Clifford-vector-valued polynomials of the form $T_{\alpha}^{(0)}(x, y, 2\alpha; a)$ belong to $\Lambda_+ A_h$.
2. For $\alpha \neq 0$, the Clifford-vector-valued polynomials of the form $T_{\alpha}^{(0)}(x, y; a)$ resp. $T_{\alpha}^{(-\alpha)}(x, y; a)$ belong to $\Lambda_+ A_h$ resp. $\Lambda_+ A_h$. 

The above description shows the polynomials \( T_h^{(\alpha)}(x, y; a) \) and \( T_h^{(-\alpha)}(x, y; a) \) could be obtained by projecting the multi-variable \( Cl_{0,n} \)-valued Chebyshev polynomials \( T_h^{(0)}(x, y + 2\alpha; a) \) on the multivector spaces \( \Lambda_x^* A_h \) and \( \Lambda_y^* A_h \), respectively. From this interrelationship the next proposition is rather obvious.

**Proposition III.2** For each \( j = 1, 2, \ldots, n \) the hypercomplex polynomials \( T_h^{(\alpha)}(x, y; a) \) defined in (24) satisfy the set of three-term recurrence relations

\[
T_h^{(\pm \alpha)}(x + h e_j, y; a) + T_h^{(\pm \alpha)}(x - h e_j, y; a) = (2y_j + 4\alpha_j)T_h^{(\pm \alpha)}(x, y; a).
\]

**Proof III.2** From the set of relations (24), it is sufficient to show that the three-term recurrence formula holds for every \( T_h^{(0)}(x + h e_j, y + 2\alpha; a) \).

First, recall that for each \( \gamma \in Cl_{0,n} \) satisfying \( \gamma^2 = 1 \), the auxiliary function of the form \( G_h(t, x; \gamma) = (\lambda + \gamma \sqrt{\lambda^2 - 1})^m \) satisfy the recurrence formulae

\[
G_h(t + h, x; \gamma) = \left(\lambda + \gamma \sqrt{\lambda^2 - 1}\right) G_h(t, x; \gamma)
\]

\[
G_h(t - h, x; \gamma) = \frac{G_h(t, x; \gamma)}{\lambda + \gamma \sqrt{\lambda^2 - 1}}
\]

This gives rise to

\[
G_h(t + h, x; \gamma) + G_h(t - h, x; \gamma) = \frac{(\lambda + \gamma \sqrt{\lambda^2 - 1})^2 + 1}{\lambda + \gamma \sqrt{\lambda^2 - 1}} G_h(t, x; \gamma)
\]

\[
= \frac{2\lambda^2 + 2\gamma \lambda \sqrt{\lambda^2 - 1}}{\lambda + \gamma \sqrt{\lambda^2 - 1}} G_h(t, x; \gamma)
\]

\[
= 2\alpha G_h(t, x; \gamma).
\]

Since the Chebyshev polynomials \( T_{x, j}^{\alpha}(y_j + 2\alpha) \) could be rewritten as

\[
T_{x, j}^{\alpha}(y_j + 2\alpha) = \frac{1}{2} (G_h(x_j, y_j + 2\alpha; 1) + G_h(x_j, y_j + 2\alpha; -1))
\]

from the substitutions \( \lambda = y_j + 2\alpha \) one then have

\[
T_{x, j}^{\alpha}(y_j + 2\alpha) + T_{x, j}^{\alpha}(y_j + 2\alpha) = (2y_j + 4\alpha_j)T_{x, j}^{\alpha}(y_j + 2\alpha)
\]

so that, for each \( j = 1, 2, \ldots, n \), the set of three-term recurrence formulae

\[
T_h^{(0)}(x + h e_j, y + 2\alpha; a) + T_h^{(0)}(x - h e_j, y + 2\alpha; a) = (2y_j + 4\alpha_j) T_h^{(0)}(x, y + 2\alpha; a).
\]

follow naturally in the view of the multi-variable representation (23).

Now one are in conditions to write down the solutions of the discretized Klein-Gordon equation (11) in terms of the hypercomplex counterpart of the Chebyshev polynomials provided by (22) and moreover, the spinor vector-field components of the Dirac equation (20).

First, recall that based on the coordinate formula (19) one can rewrite equation (11) as

\[
\sum_{j=1}^{n} f(x + h e_j) + f(x - h e_j) = ((m h)^2 + 2\alpha) f(x).
\]

Furthermore, Proposition II.2 tells us that the constraint \( \sum_{j=1}^{n} (2y_j + 4\alpha_j) = (m h)^2 + 2\alpha \) is a necessary condition for \( T_h^{(0)}(x, y + 2\alpha; a) \), and moreover, \( T_h^{(\pm \alpha)}(x, y; a) \) being solutions of (25).

In the view of Corollary III.1 if one chooses \( y \) and \( \alpha \) in such way that \( \sum_{j=1}^{n} y_j = (m h)^2 \) and \( \sum_{j=1}^{n} \alpha_j = \frac{\alpha}{2} \), one can compute the solutions of (21) from

\[
g(x) = T_h^{(\alpha)}(x, y; a) + T_h^{(-\alpha)}(x, y; a).
\]

A short computation based on projection arguments and on the intertwining property \( D_{h \chi}(x) = -\chi_h(x)D_h \) even shows that the solutions \( f_+(x) \) and \( f_-(x) \) of (21) are thus given by

\[
f_+(x) = D_h T_h^{(-\alpha)}(x, y; a) - m T_h^{(\alpha)}(x, y; a)
\]

\[
f_-(x) = D_h T_h^{(\alpha)}(x, y; a) - m T_h^{(-\alpha)}(x, y; a).
\]

**IV. SOME REMARKS ON LATTICE FERMION DOUBLING**

In relativistic quantum mechanics the Hamiltonian \( -\Delta + m^2 \) encoded on the time-harmonic Klein-Gordon equation \( \Delta f(x) = m^2 f(x) \) could be derived by quantizing the paravector representations \( \zeta^+ = \xi_0 - i \xi \) and \( \zeta^- = -\xi_0 + i \xi \) of the complexified Clifford algebra \( Cl_\mathbb{C} \otimes Cl_{0,n} \) for which \( \xi_0 \in \mathbb{R}, i^2 = -1 \) and \( \xi = \sum_{j=1}^{n} \xi_j e_j \in Cl_{0,n} \).

Namely, such quantization scheme could be formulated in terms of the energy term \( E = m^2 \) and the Dirac operator \( D = \sum_{j=1}^{n} e_j \partial_{x_j} \) from the prime quantization rules

\[
\xi_0 \mapsto i E \quad \text{and} \quad \xi_j \mapsto i \partial_{x_j}.
\]

Thus, the substitution \( \xi_0^2 = \sum_{j=1}^{n} \xi_j^2 \) (assuming the energy normalization in terms of the speed of light) gives rise to the quadratic equations \( \zeta^+ \zeta^- = \zeta^- \zeta^+ = \sum_{j=1}^{n} \xi_j^2 - m^2 \) that in turn yield the factorization relations, analogous to the ones obtained in [8]:

\[
(D - i m)(D + i m) = (D + i m)(D - i m) = -\Delta + m^2.
\]

Therefore, the solutions of the Klein-Gordon equation in **continuum** could be formulated on the momentum space through the energy-momentum relation \( \sum_{j=1}^{n} \xi_j^2 = m^2 \).
When one replaces the $D$ by the central difference Dirac operator $\frac{1}{2} \left( D_{h/2}^- + D_{h/2}^+ \right)$ one can mimicking the above quantization procedure also for the derivation of the discretized Klein-Gordon equation. That is, based on coordinate action of $\frac{1}{2} \left( D_{h/2}^- + D_{h/2}^+ \right)$ on $f(x)$ given by

$$\frac{1}{2} \left( D_{h/2}^- f(x) + D_{h/2}^+ f(x) \right) = \sum_{j=1}^{n} e_j \left( x + \frac{h}{2} e_j \right) - f \left( x - \frac{h}{2} e_j \right)$$

and from the Taylor series representation $S_{h/2}^{\pm j} = \exp \left( \pm \frac{h}{2} \partial_{e_j} \right)$ carrying the shift operators $S_{h/2}^{\pm j} f(x) = f \left( x \pm \frac{h}{2} e_j \right)$ (cf. [13, Subsection 2.2]) it follows that the factorization of the discretized Klein-Gordon operator $-\Delta_h + m^2$ provided by the set of relations

$$-\Delta_h + m^2 = \left( \frac{1}{2} D_{h/2}^- + \frac{1}{2} D_{h/2}^+ + \frac{im}{2} \right) \left( \frac{1}{2} D_{h/2}^- + \frac{1}{2} D_{h/2}^+ - \frac{im}{2} \right)$$

encode the quantization correspondence

$$\sum_{j=1}^{n} \frac{2}{h} \sin \left( \frac{h \xi_j}{2} \right) e_j \rightarrow \frac{1}{2} \left( D_{h/2}^- + D_{h/2}^+ \right)$$

and the following energy-momentum relation on $h\mathbb{Z}^n$ as well:

$$\sum_{j=1}^{n} \frac{4}{h^2} \sin^2 \left( \frac{h \xi_j}{2} \right) = m^2.$$ 

On the flavor of finite difference potentials (cf. [4, 16]), such kind of quantization on the lattice that results into the aforementioned energy-momentum relation was already considered, in a hidden way, when the authors obtained integral representations involving Green’s-type functions based on the fact that the restriction of the continuous Fourier transform to $Q_h = \left[ -\frac{\pi}{h}, \frac{\pi}{h} \right]^n$ gives an inverse for the discrete Fourier transform over $h\mathbb{Z}^n$ (see also [21, Section 1.5] and [13, Subsection 4.2] for further analogies and comparisons).

The drawback besides this quantization scheme is twofold: for the factorization of the discretized Klein-Gordon equation (11) one needs to consider, in addition to the lattice $\frac{2}{h} \mathbb{Z}^n$ with mesh width $\frac{1}{2} > 0$ and the left-hand side of the above summand has 2n + 1 zeros inside the Brillouin zone Qh = $\left[ -\frac{\pi}{h}, \frac{\pi}{h} \right]^n$, and so, such discretization yields spectrum degeneracy for $\frac{1}{2} \left( D_{h/2}^- + D_{h/2}^+ \right)$ in the massless limit $m \rightarrow 0$, as already discussed in Subsection 4.1.

When one formulates the discretized Dirac equation $D_h f(x) = m\chi_h(x) f(x)$ from the finite difference discretization $D_h = \frac{1}{2} \left( D_{h/2}^- + D_{h/2}^+ \right) + \frac{h}{2} \Box_h$ provided by [13], one assures that its momentum representation provided by the prime quantization rules $\xi_j \mapsto i\partial_x$ and $-i\partial_x \mapsto x_j$ over $C^\infty_{\text{n,n}}$:

$$\sum_{j=1}^{n} \frac{1}{h} \sin(h \xi_j) e_j \hat{f}(\xi) + \sin^2 \left( \frac{h \xi_j}{2} \right) e_j \mapsto \hat{f}(\xi) = m \prod_{j=1}^{n} e_{j+n} \cos \left( \frac{\pi}{h} \xi_j \right) \hat{f}(\xi) \quad (27)$$

behaves like $\frac{1}{2} \left( D_{h/2}^- + D_{h/2}^+ \right)$ in the neighborhood of $\xi_j = 0$ and $\xi_j = \pm \frac{\pi}{hn}$ and to avoid gaps in a neighborhood of $\xi_j = \pm \frac{\pi}{hn}$. Contrary to Kanamori-Kawamoto’s approach (cf. [17, Subsection 6]), the mass term was treated as a local spinor field potential $m \chi_h(x)$ that produces, as it was explained in Subsection 4.2 a commutative action over the lattice $h\mathbb{Z}^n$. The connection with staggered fermions of Kogut-Susskind type was recognized from the shift operators

$$\cos \left( \frac{\pi}{h} \partial_{\xi_j} \right) \hat{f}(\xi) \mapsto \hat{f}(\xi + \frac{\pi}{h} e_j) + \hat{f}(\xi - \frac{\pi}{h} e_j).$$

as representations of the permutation term $(-1)^{\frac{m^2}{h^2}}$ on the momentum space (cf. [21, Subsection 4.3]). This rotation symmetry action on $h\mathbb{Z}^n$ is thus essential to assure that to the corners of the Brillouin zone $Q_h = \left[ -\frac{\pi}{h}, \frac{\pi}{h} \right]^n$ (that is, the doublers) goes to a single point in the limit $h \rightarrow 0$.

At the level of the phase space coordinates $(x, \xi) \in h\mathbb{Z}^n \times Q_h$, one can further introduce the local transformation actions

$$\hat{f}(\xi) \mapsto T_{h}^{(\pm \epsilon)}(x, y; a) \hat{f}(\xi), \hat{f}(\xi) \mapsto T_{h}^{(-\epsilon)}(x, y; a) \hat{f}(\xi)$$

underlying the multivector representations $y = \sum_{j=1}^{n} 2 \sin^2 \left( \frac{h \xi_j}{2} \right) e_j$ and $e = \sum_{j=1}^{n} e_j$ of the n–tuples $2 \sin^2 \left( \frac{h \xi}{2} \right), 2 \sin^2 \left( \frac{h \xi}{2} \right), \ldots, 2 \sin^2 \left( \frac{h \xi}{2} \right)$ and $(\frac{1}{2}, \frac{1}{2},\ldots, \frac{1}{2})$, respectively.

Essentially, this local transformations provide families of solutions for the discretized Klein-Gordon equation as well as they assure the existence of two conserved spinorial flows of chiral and achiral type, respectively, for the momentum representation (27). From Corollary 4.1 one can also see that the action of the Dirac-field operator $D_h - m\chi_h(x)$ on such transformations produce solutions for the Dirac equation (20) over the phase space $h\mathbb{Z}^n \times \left[ -\frac{\pi}{h}, \frac{\pi}{h} \right]^n$. Moreover, the solutions of (21), written in $\Lambda^+_n \mathcal{A}_h \otimes \Lambda^+_n \mathcal{A}_h$ as

$$\hat{f}_+(x, \xi) = D_h \left( T_{h}^{(-\epsilon)}(x, y; a) \hat{f}(\xi) \right) - m T_{h}^{(\epsilon)}(x, y; a) \hat{f}(\xi),$$

$$\hat{f}_-(x, \xi) = D_h \left( T_{h}^{(\epsilon)}(x, y; a) \hat{f}(\xi) \right) - m T_{h}^{(-\epsilon)}(x, y; a) \hat{f}(\xi).$$

provide the required spinor vector-field components of (20).
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