Reverse Mathematics of the uncountability of $\mathbb{R}^*$

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Abstract. In his first set theory paper (1874), Cantor establishes the uncountability of $\mathbb{R}$. We study the latter in Kohlenbach’s higher-order Reverse Mathematics, motivated by the observation that one cannot study concepts like ‘arbitrary mappings from $\mathbb{R}$ to $\mathbb{N}$’ in second-order Reverse Mathematics. Now, it was recently shown that the statement

$\text{NIN} : \text{there is no injection from } [0,1] \text{ to } \mathbb{N}$

is hard to prove in terms of conventional comprehension. In this paper, we show that $\text{NIN}$ is robust by establishing equivalences between $\text{NIN}$ and $\text{NIN}$ restricted to mainstream function classes, like: bounded variation, semi-continuity, and Borel. Thus, the aforementioned hardness of $\text{NIN}$ is not due to the quantification over arbitrary $\mathbb{R} \to \mathbb{N}$-functions in $\text{NIN}$. Finally, we also study $\text{NBI}$, the restriction of $\text{NIN}$ to bijections, and the connection to Cousin’s lemma and Jordan’s decomposition theorem.

1 Introduction and preliminaries

1.1 Aim and motivation

In a nutshell, we study the uncountability of $\mathbb{R}$ from the point of view of Reverse Mathematics. We now explain the aforementioned italicised notions.

First of all, Reverse Mathematics (RM hereafter) is a program in the foundations of mathematics initiated by Friedman ([11,12]) and developed extensively by Simpson and others ([34,35]); an introduction to RM for the ‘mathematician in the street’ is in [36]. In a nutshell, RM seeks to identify the minimum axioms needed to prove theorems of ordinary, i.e. non-set theoretic, mathematics. We assume basic familiarity with RM, including Kohlenbach’s higher-order RM introduced in [18], with more recent results -including our own- in [26,29,31,32].

Now, the biggest difference between ‘classical’ RM and higher-order RM is that the former makes use of $L_2$, the language of second-order arithmetic, while the latter uses $L_{\omega}$, the language of higher-order arithmetic. Thus, higher-order

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objects are only indirectly available via so-called codes or representations in classical RM. In particular, $L_2$ cannot talk about ‘arbitrary mappings from $\mathbb{R}$ to $\mathbb{N}$’. Thus, Simpson (only) proves that the real numbers $\mathbb{R}$ cannot be enumerated as a sequence in classical RM (see [35] II.4.9]). Hence, the higher-order RM of the uncountability of $\mathbb{R}$, discussed next, is a natural (wide-open) topic of study.

Secondly, the uncountability of $\mathbb{R}$ was established in 1874 by Cantor in his first set theory paper [6], which even has its own Wikipedia page, namely [39]. We will study the uncountability of $\mathbb{R}$ in the guise of the following principles:

- **NIN**: there is no injection from $[0, 1]$ to $\mathbb{N}$,
- **NBI**: there is no bijection from $[0, 1]$ to $\mathbb{N}$.

It was established in [29] that **NIN** and **NBI** are hard to prove in terms of (conventional) comprehension, as explained in detail in Remark [1]. One obvious way of downplaying these results is to simply attribute the hardness of **NIN** to the fact that one quantifies over arbitrary third-order objects, namely $\mathbb{R} \to \mathbb{N}$-functions.

In this paper, we establish RM-equivalences involving **NIN** and **NBI**, where some are straightforward (Section 2.1) and others advanced or surprising (Section 2.2). We also study the connection between **NIN** and Cousin’s lemma and Jordan’s decomposition theorem (Section 2.3). In particular, we show that **NIN** is equivalent to the statement that there is no injection from $[0, 1]$ to $\mathbb{Q}$ that enjoys ‘nice’ mainstream properties like bounded variation, semi-continuity, and related notions. Hence, the aforementioned hardness of **NIN** and **NBI** is not due to the latter quantifying over arbitrary third-order functions as exactly the same hardness is observed for mathematically natural subclasses. A recent FOM-discussion initiated by Friedman via [13], brought about this insight, while our results establish that **NIN** is robust in the sense of Montalbán, as follows.

Thirdly, as to the structure of this paper, we introduce some essential axioms and definitions in Section 1.2 while our main results may be found in Section 2. We note that some of our results are proved using **IND$^0$**, a non-trivial fragment of the induction axiom from Section 1.2.1 It is a natural RM-question, posed previously by Hirschfeldt (see [23, §6.1]), whether these extra axioms are needed for the reversal. Neeman provides an example of the necessary use of extra induction in a reversal in[23]. We finish this introductory section with a conceptual remark.

**Remark 1 (Conventional comprehension)** First of all, the goal of RM is to find the minimal axioms that prove a given theorem. In second-order RM,
these minimal axioms are fragments of the comprehension axiom (and related notions), i.e. the statement that the set \{n ∈ N : ϕ(n)\} exists for a certain class of \(L_2\)-formulas. Higher-order RM similarly makes use of ‘comprehension functionals’, i.e. third-order objects that decide formulas in a certain sub-class of \(L_2\). Examples include Kleene’s quantifier \(∃^2\) and the Suslin functional \(S^2\), to be found in Section 1.2.1. We are dealing with conventional comprehension here, i.e. only first- and second-order objects are allowed as parameters.

Secondly, second-order arithmetic \(Z_2\) has two natural higher-order formulations \(Z^2\) and \(Z^{2L}\) based on comprehension functionals, both to be found in Section 1.2.1. The systems \(Z_2, Z^2,\) and \(Z^{2L}\) prove the same second-order sentences by \([15, Cor. 2.6]\). Nonetheless, the system \(Z^2\) cannot prove \(NIN\) or \(NBI\), while \(Z^{2L}\) proves both. Here, \(Z^2\) and \(NIN\) can be formulated in the language of third-order arithmetic, i.e. there is no ‘type mismatch’. The previous negative result is why we (feel obliged/warranted to) say that the principle \(NIN\) is hard to prove in terms of conventional comprehension. Finally, \(NIN\) and \(NBI\) seem to be the weakest natural third-order principles with this hardness property.

1.2 Preliminaries

We introduce axioms and definitions from RM needed below. We refer to \([18, \S 2]\) or \([26, \S 2]\) for Kohlenbach’s base theory \(RCA^0\), and basic definitions like the real numbers \(\mathbb{R}\) in \(RCA^0\). As in second-order RM (see \([35, II.4.4]\)), real numbers are represented by fast-converging Cauchy sequences. To avoid the details of coding real numbers and sets, we often assume the axiom \((∃^2)\) from Section 1.2.1 which can however sometimes be avoided, as discussed in Remark 10.

1.2.1 Some axioms of higher-order arithmetic First of all, the functional \(ϕ\) in \((∃^2)\) is clearly discontinuous at \(f = 11\ldots;\) in fact, \((∃^2)\) is equivalent to the existence of \(F : \mathbb{R} → \mathbb{R}\) such that \(F(x) = 1\) if \(x > 0\), and 0 otherwise (\([18, \S 3]\)).

\[
(∃ϕ^2 ≤ 1)(∀f^1)(∃n)(f(n) = 0) ↔ ϕ(f) = 0.
\]

Related to \((∃^2)\), the functional \(μ^2\) in \((μ^2)\) is also called Feferman’s \(μ\) (\([18]\)).

\[
(∃μ^2)(∀f^1)[(∃n)(f(n) = 0) → [f(μ(f)) = 0 ∧ (∀i < μ(f))(f(i) ≠ 0)] ∧ [(∀n)(f(n) ≠ 0) → μ(f) = 0)].
\]

Intuitively, \(μ^2\) is the least-number-operator, i.e. \(μ(f)\) provides the least \(n \in \mathbb{N}\) such that \(f(n) = 0\), if such number exists. We have \((∃^2) ↔ (μ^2)\) over \(RCA^0\) and \(ACA^0_ω = RCA^0 + (∃^2)\) proves the same \(L_2\)-sentences as \(ACA_0\) by \([15, Theorem 2.5]\). Working in \(ACA^0_ω\), one readily defines a functional \(η : [0,1] → 2^\mathbb{N}\) that converts real numbers to their\(^1\) binary representation.

Secondly, we sometimes need more induction than is available in \(RCA^0_ω\). The connection between ‘finite comprehension’ and induction is well-known from second-order RM (see \([35, X.4.4]\)).

\(^1\) In case there are two binary representations, we choose the one with a tail of zeros.
**Principle 2 (IND₀)** Let \( Y^2 \) satisfy \((∀n ∈ N)(∃ at most one \( f ∈ 2^N \))(Y(f,n) = 0). For \( k ∈ N \), there is \( w^{1^k} \) such that for any \( m ≤ k \), we have
\[
(∃i < |w|)((w(i) ∈ 2^N ∧ Y(w(i), m) = 0)) ↔ (∃f ∈ 2^N)(Y(f, m) = 0).
\]
Thirdly, the Suslin functional \( S^2 \) is defined in \([18]\) as follows:
\[
(∃S^2 ≤ 2^1)(∀f^{1^1})[(∃g^{1^1})(∀n^0)(f(∀n) = 0) ↔ S(f) = 0].
\]
The system \( Π^1_k-CA_0^\omega \equiv RCA_0^\omega + (S^2) \) proves the same \( Π^1_k \)-sentences as \( Π^1_k-CA_0 \) by \([32]\) Theorem 2.2. By definition, the Suslin functional \( S^2 \) can decide whether a \( Σ^1_k \)-formula as in the left-hand side of \((S^2)\) is true or false. We similarly define the functional \( S^2_k \) which decides the truth or falsity of \( Σ^1_k \)-formulas from \( L_2 \); we also define the system \( Π^1_k-CA_0^\omega \) as \( RCA_0^\omega + (S^2_k) \), where \((S^2_k)\) expresses that \( S^2_k \) exists. We note that the operators \( ν_n \) from \([31]\) p. 129] are essentially \( S^2_k \) strengthened to return a witness (if existant) to the \( Σ^1_n \)-formula at hand.
Finally, second-order arithmetic \( Z_2 \) readily follows from \( ∪_k Π^1_k-CA_0^\omega \), or from:
\[
(∃E^{3 ≤ 3})(∀Y^2)[(∃f^{1^1})(Y(f) = 0) ↔ E(Y) = 0],
\]
and we therefore define \( Z_2^{Ω^2} \equiv RCA_0^\omega + (3^3) \) and \( Z_2^ω \equiv ∪_k Π^1_k-CA_0^\omega \), which are conservative over \( Z_2 \) by \([15]\) Cor. 2.6. Despite this close connection, \( Z_2^{Ω^2} \) and \( Z_2^ω \) can behave quite differently, as discussed in Remark\(^*\) The functional from \((3^3)\) is also called \( 3^3 \), and we use the same convention for other functionals.

**1.2.2 Some basic definitions** We introduce the higher-order definitions of ‘set’ and ‘countable’, as can be found in e.g. \([27][29][31]\).

First of all, open sets are represented in second-order RM as countable unions of basic open sets (\([35]\) II.5.6]), and we refer to such sets as ‘RM-open’. By \([35]\) II.7.1], one can effectively convert between RM-open sets and (RM-codes for) continuous characteristic functions. Thus, a natural extension of the notion of ‘open set’ is to allow arbitrary (possibly discontinuous) characteristic functions, as is done in e.g. \([27][31]\]. To make sure (basic) RM-open sets have characteristic functions, we shall always assume \( ACA_0^\omega \) when necessary.

**Definition 3** [Subsets of \( R \)] We let \( Y : R → \{0,1\} \) represent subsets of \( R \) as follows: we write ‘\( x ∈ Y \)’ for ‘\( Y(x) = 1 \)’.

The notion of ‘subset of \( 2^N \) or \( N^N \) now has an obvious definition. Having introduced our notion of set, we now turn to countable sets.

**Definition 4** [Enumerable sets of reals] A set \( A ⊂ R \) is enumerable if there exists a sequence \((x_n)_{n∈N} \) such that \((∀x ∈ R)((x ∈ A) ↔ (∃n ∈ N)(x = x_n)). \)

This definition reflects the RM-notion of ‘countable set’ from \([35]\) V.4.2. Note that given Feferman’s \( μ^2 \), we can remove all elements from a sequence of reals \((x_n)_{n∈N} \) that are not in a given set \( A ⊂ R \).

The definition of ‘countable set of reals’ is now as follows in \( RCA_0^\omega \), while the associated definitions for Baire space are obvious.
Definition 5: [Countable subset of \( \mathbb{R} \)] A set \( A \subseteq \mathbb{R} \) is countable if there exists \( Y : \mathbb{R} \to \mathbb{N} \) such that \( (\forall x, y \in A)(Y(x) = 0 \lor Y(y) = x =_\mathbb{R} y) \). The functional \( Y \) is called injective on \( A \) or an injection on \( A \). If \( Y : \mathbb{R} \to \mathbb{N} \) is also surjective, i.e. \( (\exists n \in \mathbb{N})(\exists x \in A)(Y(x) = n) \), we call \( A \) strongly countable. The functional \( Y \) is then called bijective on \( A \) or a bijection on \( A \).

The first part of Definition 5 is from Kunen’s set theory textbook ([20] p. 63]) and the second part is taken from Hrbacek-Jech’s set theory textbook [14] (where the term ‘countable’ is used instead of ‘strongly countable’). According to Veldman ([38, p. 292]), Brouwer studied set theory based on injections. Hereafter, ‘strongly countable’ and ‘countable’ shall exclusively refer to Definition 5.

Finally, note that the principles \( \text{NIN} \) and \( \text{NBI} \) from Section 1 have now been defined. We have previously studied the RM of \( \text{cocode}_i \) for \( i = 0, 1 \) in [29][31], where the index \( i = 0 \) expresses that a countable set in the unit interval can be enumerated (for \( i = 1 \), we restrict to strongly countable sets).

2 Main results

We establish the results sketched in Section 1. We generally assume (\( \exists^2 \)) from Section 1.2.1 to avoid the technical details involved in the representation of sets and real numbers. Given that \( \text{NIN} \) cannot be proved in \( \mathbb{Z}^2 \) by Remark 1, this seems like a weak assumption.

2.1 Basic robustness results

In this section, we show that \( \text{NIN} \), \( \text{NBI} \), and related principles are relatively robust when it comes to the domain of the mappings therein.

First of all, let \( \text{NIN}^X \) express that there is no injection \( Y : X \to \mathbb{N} \), for \( X \) equal to either the reals \( \mathbb{R} \), Cantor space \( 2^\mathbb{N} \) (also denoted as \( C \)), or Baire space \( \mathbb{N}^\mathbb{N} \).

Theorem 6: The system \( \text{ACA}_0^\omega \) proves \( \text{NIN} \leftrightarrow \text{NIN}^C \leftrightarrow \text{NIN}^\mathbb{N} \leftrightarrow \text{NIN}^\mathbb{R} \).

Proof. First of all, \( \text{NIN} \to \text{NIN}^\mathbb{R} \) and \( \text{NIN}^C \to \text{NIN}^\mathbb{N} \) are trivial, while \( \text{NIN}^\mathbb{R} \to \text{NIN} \) follows by considering the injection \( \frac{1}{2}(1 + \frac{1}{1 + x}) \) from \( \mathbb{R} \) to (0, 1).

Secondly, assume \( \text{NIN} \) and use the usual interval-halving technique (using \( \exists^2 \)) to obtain \( \eta : [0, 1] \to 2^\mathbb{N} \) such that \( \eta(x) \) is the binary representation of \( x \in [0, 1] \), choosing a tail of zeros in the non-unique case. Fix \( Y : 2^\mathbb{N} \to \mathbb{N} \) and define \( Z : [0, 1] \to \mathbb{N} \) as \( Z(x) := Y(\eta(x)) \), which satisfies the axiom of extensionality\(^2\) on \( \mathbb{R} \) by definition. By \( \text{NIN} \), there are \( x, y \in [0, 1] \) with \( x \neq_\mathbb{R} y \) and \( Z(x) = Z(y) \). Clearly, \( \eta(x) \neq \eta(y) \) and \( Y(\eta(x)) = Y(\eta(y)) \), and \( \text{NIN}^C \) follows.

Thirdly, assume \( \text{NIN}^C \), fix \( Z : [0, 1] \to \mathbb{N} \) and let \( (q_n)_{n \in \mathbb{N}} \) be a list of all rational numbers with non-unique binary representation. Define \( Y : 2^\mathbb{N} \to \mathbb{N} \)

\(^2\) Functions \( F : \mathbb{R} \to \mathbb{R} \) are represented by \( \Phi : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N} \) mapping equal reals to equal reals, i.e. extensionality as in \( (\forall x, y \in \mathbb{R})(x =_\mathbb{R} y \to \Phi(x) =_\mathbb{R} \Phi(y)) \) (see [18] p. 289).
as follows: $Y(f) := 3Z(\tau(f))$ in case $\tau(f) := \sum_{n=0}^{\infty} \frac{f(n)}{2^n}$ has a unique binary representation, $Y(f) := 3n + 1$ in case $\tau(f) = q_n$ and $f$ has a tail of zeros, and $Y(f) = 3n + 2$ in case $\tau(f) = q_n$ and $f$ has a tail of ones. By $NIN^C$, there are $f, g \in 2^N$ such that $f \neq_1 g$ and $Y(f) = Y(g)$. Clearly, this is only possible in the first case of the definition of $Z$, i.e. we have $Y(f) = 3Z(\tau(f)) = 3Z(\tau(g)) = Y(g)$. Since also $\tau(f) \neq_2 \tau(g)$, $NIN$ follows and we obtain $NIN \leftrightarrow NIN^C$.

Finally, let $Y : 2^N \to N$ be an injection. For $f \in N^N$, define its graph $X_f := \{(n, f(n)) : n \in N\}$ in $N^2$ and code the latter as a binary sequence $\bar{X}_f$. Note that $f(n) := (\mu m)[(n, m) \in X_f]$ recovers the function $f$ from its graph $X_f$. Modulo this coding, define $Z : N^N \to N$ as $Z(f) := Y(\bar{X}_f)$. By the assumption on $Y$, $Z(f) =_0 Z(g)$ for $f, g \in N^N$ implies $\bar{X}_f =_1 \bar{X}_g$, which implies $f =_1 g$, by the definition of $X_f$. Hence, $\lnot NIN^C \rightarrow \lnot NIN^N$, and we are done. \qed

Similarly, $cocode_0^X$ is the statement that any countable subset of $X$ can be enumerated, while $cocode_1^X$ is the restriction to strongly countable sets.

**Theorem 7 (ACA^0_ω)** For $i = 0, 1$, we have $cocode_i \leftrightarrow cocode_1^{\mathbb{R}} \leftrightarrow cocode_i^{\mathbb{C}}$.

**Proof.** The implication $cocode_i^{\mathbb{R}} \rightarrow cocode_i$ is trivial while the (rescaled) arctangent function is a bijection from $\mathbb{R}$ to $(0, 1)$, which readily yields the reversal.

Now assume $cocode_0^C$ and let $Z : [0, 1] \to N$ be injective on $A \subset [0, 1]$. The functional $Y : 2^N \to N$ defined by $Y(f) := Z(\tau(f))$ is clearly injective on $B := \{\eta(x) : x \in A\}$ where $\eta$ is as in the proof of Theorem[4] Let $(f_n)_{n \in N}$ be a list of all elements in $B$ and note that $(\tau(f_n))_{n \in N}$ is a list of all elements in $A$, i.e. $cocode_0$ follows. Note that if $Z$ is bijective on $A$, then $Y$ is bijective on $B$ by definition, i.e. $cocode_1^C \rightarrow cocode_1$.

Next, assume $cocode_0$, let $Y : 2^N \to N$ be injective on $A \subset 2^N$, and define $Z(x) := Y(\eta(x))$. Then $Z : [0, 1] \to N$ witnesses that $B = \{\tau(f) : f \in A\}$ is countable, and let $(x_n)_{n \in N}$ be an enumeration of $B$. This list is readily converted to a list of all elements in $A$ via $\eta$ and by noting that $\mu^2$ can list all $f \in A$ such that $\tau(f)$ has a non-unique binary representation; we thus have $cocode_0^C$.

We now prove $cocode_0^R \rightarrow cocode_0^C$. Let $Y : 2^N \to \mathbb{N}$ be bijective on $A \subset 2^N$ and let $(f_n)_{n \in N}$ be the list of all $f \in A$ such that $\tau(f)$ has a non-unique binary representation. Now define $D \subset \mathbb{R}$ as: $x \in D$ if either of the following holds:

- $x \in [0, 1]$, $x$ has a unique binary representation, and $\eta(x) \in A$,
- there is $n \in N$ with $x \in (n, n+1, n+2]$ and $x - (n + 1) =_R \tau(f_n)$.

Define $W : \mathbb{R} \to \mathbb{N}$ as $W(x) := Y(\eta(x))$ if $x \in [0, 1]$ and $W(x) := Y(f_n)$ in case $|x| \in (n+1, n+2]$ as in the second case of the definition of $D$. Then $W$ is a bijection on $D$ since $Y$ is a bijection on $A$. The list provided by $cocode_0^R$ for $D$ now readily yields the list required for $A$ as in $cocode_1^C$. \qed

Finally, $NBI^X$ is the statement that there is no bijection from $X$ to $\mathbb{N}$, where $X$ is e.g. $\mathbb{R}$ or $\mathbb{N}^N$. We have the following theorem.

**Theorem 8** The system ACA^0_ω proves $NBI \leftrightarrow NBI^R$ and $NBI \rightarrow NBI^N$. 
Proof. The implication $\text{NBI} \rightarrow \text{NBI}^\mathbb{R}$ is immediate as the (rescaled) tangent function provides a bijection from $(0, 1)$ to $\mathbb{R}$. The inverse of tangent, called $\text{arctangent}$, yields a bijection in the other direction (also with rescaling), i.e. the first equivalence is immediate, as well as $\text{NBI} \leftrightarrow \text{NBI}^{\mathbb{R}^\geq 0}$. We now define a (continuous) bijection from $\mathbb{N}^\mathbb{N}$ to $\mathbb{R}_{\geq 0}$ based on continued fractions. Intuitively, a sequence $(a_n)_{n \in \mathbb{N}}$ of natural numbers is mapped to the real $x \in \mathbb{R}_{\geq 0}$ via the following (generalised) continued fraction:

$$x = a_0 + \frac{1}{1 + \frac{1}{a_1 + \frac{1}{1 + \frac{1}{a_2 + \ddots}}} \quad (\text{CF})$$

The real $x \in \mathbb{R}_{\geq 0}$ in $(\text{CF})$ exists in $\text{ACA}_0^\omega$ in the sense that there is an explicit function $F : (\mathbb{N}^\mathbb{N} \times \mathbb{N}) \rightarrow \mathbb{Q}$ such that $x =_{\mathbb{R}} \lim_{n \rightarrow \infty} F(f)(n)$, where $F(f)(n) \in \mathbb{Q}$ is essentially the continued fraction in $(\text{CF})$ ‘broken off’ after encountering $a_n$. The definition of $F$ can be found in e.g. [22, Ch.1, p. 7-9]. One readily shows that the mapping defined by $(\text{CF})$ is a bijection from $\mathbb{N}^\mathbb{N}$ to $\mathbb{R}_{\geq 0}$ in $\text{ACA}_0^\omega$. \hfill \square

We could prove similar results for a countable set in the unit interval has measure zero, which is intermediate between cocode$_0$ and NIN, which is shown in [21] as an illustration how weak NIN is. Nonetheless, we have the following result.

**Theorem 9 (ACA$_0^\omega$)** A countable set $A \subset [0, 1]$ has weak measure zero.

**Proof.** Fix $A \subset [0, 1]$ and $Y : [0, 1] \rightarrow \mathbb{N}$ injective on $A$. For $\varepsilon > 0$, define $\varepsilon_n := \frac{\varepsilon}{2^n}$, $B := \{(a, b) \in \mathbb{R}^2 : \frac{a+1}{b} \in A \land |b-a| = 2^{-Y(\frac{a+b}{2})}\}$, and $Z((a, b)) := Y(\frac{a+b}{2})$. Clearly, this shows that $A$ has weak measure zero, as required. \hfill \square

We say that a property holds weakly almost everywhere (wae) in case it holds outside a set of weak measure zero as in Footnote 4.

We finish this section with a conceptual remark regarding our base theory.

**Remark 10** We have used ACA$_0^\omega$ as the base theory for the above results, since our notion of ‘set-as-characteristic function’ as in Definition 3 is poorly behaved in the absence of (3^2). One can obtain equivalences over RCA$_0^\omega$, and let us establish NIN$^{\mathbb{N}^\omega}$ → NIN$^\mathbb{N}$ over RCA$_0^\omega$ as an example via the following steps.

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3 For $A \subset \mathbb{R}$, let ‘$A$ has measure zero’ mean that for any $\varepsilon > 0$, there is a sequence of closed intervals $(I_n)_{n \in \mathbb{N}}$ covering $A$ and such that $\varepsilon > \sum_{n=0}^{\infty} |I_n|$ for $J_0 := I_0$ and $J_{i+1} := I_{i+1} \cup \bigcup_{j \leq i} I_j$. This follows from the usual definition as used in mathematics.

4 For $A \subset \mathbb{R}$, let ‘$A$ has weak measure zero’ mean that for any $\varepsilon > 0$, there is a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$, a set $B$ of closed intervals, and $Z : \mathbb{R}^2 \rightarrow \mathbb{N}$ injective on $B$, such that $(\forall a \in A) (\exists (b, c) \in B)(a \in (b, c))$ and $(\forall (b, c) \in B, \forall n \in \mathbb{N})(Z((b, c)) = n \rightarrow |b-c| \leq \varepsilon_n)$ and $\varepsilon \geq \sum_{n=0}^{\infty} \varepsilon_n$. Given cocode$_0$, this is the same as ‘measure zero’.
– Fix any $Y : 2^\mathbb{N} \to \mathbb{N}$, which may or may not be continuous.
– In case $Y$ is continuous, it is immediate that $Y(00\ldots) = Y(00\ldots00\ast11\ldots)$
  for enough instances of 0 on the right.
– In case $Y$ is discontinuous, use the results in [18, §3] to derive $(\exists^2)$ over
  $\mathsf{RCA}_0^\omega$. We can now use the proof of Theorem 6 in
  $\mathsf{ACA}_0^\omega$.

The above proof of course heavily relies on the law of excluded middle.

### 2.2 Advanced robustness results

In this section, we show that $\mathsf{NIN}$ is equivalent to various restrictions involving
notions from mainstream mathematics, like semi-continuity and bounded variation; we first introduce the latter.

First of all, an important weak continuity notion is *semi-continuity*, introduced by Baire in [2] around 1899. By [2] §84, p. 94-95, the notion of quasi-continuity goes back to Volterra; any cliquish function is the sum of two quasi-continuous functions. Moreover, while the limits in the following definition may not exist in $\mathsf{RCA}_0^\omega$, the associated inequalities always make sense.

**Definition 11** [Weak continuity]

– $f : \mathbb{R} \to \mathbb{R}$ is upper semi-continuous if for all $x_0 \in \mathbb{R}$, $f(x_0) \geq \limsup_{x \to x_0} f(x)$.
– $f : \mathbb{R} \to \mathbb{R}$ is lower semi-continuous if for all $x_0 \in \mathbb{R}$, $f(x_0) \leq \liminf_{x \to x_0} f(x)$.
– $f : X \to \mathbb{R}$ is quasi-continuous (resp. cliquish) at $x \in X$ if for any $\varepsilon > 0$ and any open neighbourhood $U$ of $x$, there is a non-empty open ball $G \subset U$ with
  $(\forall y \in G)(|f(x) - f(y)| < \varepsilon)$ (resp. $(\forall y, z \in G)(|f(z) - f(y)| < \varepsilon)$).

Secondly, Jordan introduces the notion of *bounded variation* in [16] around 1881, also studied in second-order RM ([19,25]). Moreover, Jordan proves in [17, §105] that functions of bounded variation are exactly those for which the notion of ‘length of the graph’ makes sense; the latter boasts an even ‘earlier’ history. What is more, Lakatos in [21, p. 148] claims that Jordan did not invent or introduce the notion of bounded variation in [16], but rather discovered it in Dirichlet’s 1829 paper [8].

**Definition 12** [Bounded variation] Any $f : [a, b] \to \mathbb{R}$ has bounded variation on $[a, b]$ if there is $k_0 \in \mathbb{N}$ such that $k_0 \geq \sum_{i=0}^{n} |f(x_i) - f(x_{i+1})|$ for any partition $x_0 = a < x_1 < \cdots < x_{n-1} < x_n = b$.

Functions of bounded variation have only got countably many points of discontinuity (see e.g. [1, Ch. 1]); Dag Normann and the author study this property in higher-order computability theory in [30]. In the latter, we also study regulated functions (called ‘regular’ in [1]), defined as follows (say in $\mathsf{ACA}_0^\omega$).

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5 The notion of arc length was studied for discontinuous regulated functions in 1884 ([33 §1-2]), where it is also claimed to be essentially equivalent to Duhamel’s 1866 approach from [10 Ch. VI]. Around 1833, Dirksen, the PhD supervisor of Jacobi and Heine, provides a definition of arc length that is (very) similar to the modern one (see [9 §2, p. 128]), but with some conceptual problems as discussed in [2] §3.
Definition 13 [Regulated function] A function \( f : [0, 1] \to \mathbb{R} \) is \textit{regulated} if for every \( x_0 \in [0, 1] \), the ‘left’ and ‘right’ limit \( f(x_0-) = \lim_{x \to x_0^-} f(x) \) and \( f(x_0+) = \lim_{x \to x_0^+} f(x) \) exist.

Thirdly, Borel functions are defined in Definition 14, the usual definition of Borel set makes sense in \( \text{ACA}_0^ω \), where \((3^2)\) is used to define countable unions.

Definition 14 [Borel function] Any \( f : [0, 1] \to \mathbb{R} \) is a Borel function in case \( f^{-1}((a, +\infty)) = \{ x \in [0, 1] : f(x) > a \} \) is a Borel set for any \( a \in \mathbb{R} \).

Fourth, recall the induction axiom \( \text{IND}_0 \) from Section 1.2.2. Let \( Y \) be any property such that \( \{ f : [0, 1] \to \mathbb{R} \text{ satisfies } Y \} \) follows from \( \sim f \) has bounded variation on \([0, 1] \) and where this implication can be established over (say) \( \text{ACA}_0^ω \).

Theorem 15 (\( \text{ACA}_0^ω + \text{IND}_0 \)) The following are equivalent to \( \text{NIN} \):

1. \( \text{NIN}_{bv} \): there is no injection from \([0, 1]\) to \( \mathbb{Q} \) that has bounded variation,
2. \( \text{NIN}_Y \): there is no injection from \([0, 1]\) to \( \mathbb{Q} \) that has property \( Y \),
3. \( \text{NIN}_{\text{Riemann}} \): there is no injection from \([0, 1]\) to \( \mathbb{Q} \) that is Riemann integrable,
4. \( \text{NIN}_{\text{Borel}} \): there is no Borel function that is an injection from \([0, 1]\) to \( \mathbb{Q} \),
5. \( \text{NIN}_{\text{reg}} \): there is no injection from \([0, 1]\) to \( \mathbb{Q} \) that is regulated,
6. \( \text{NIN}_{\text{cliq}} \): there is no injection from \([0, 1]\) to \( \mathbb{Q} \) that is cliquish,
7. \( \text{NIN}_{\text{semi}} \): there is no upper semi-continuous injection from \([0, 1]\) to \( \mathbb{Q} \),
8. \( \text{NIN}'_{\text{semi}} \): there is no lower semi-continuous injection from \([0, 1]\) to \( \mathbb{Q} \).

Only the implications involving the final five items require the use of \( \text{IND}_0 \).

Proof. As there is an injection from \( \mathbb{Q} \) to \( \mathbb{N} \) in \( \text{RCA}_0 \), we only need to prove that \( \text{NIN}_{bv} \to \text{NIN} \) over \( \text{ACA}_0^ω \) for the first equivalence. To this end, let \( Y : [0, 1] \to \mathbb{N} \) be an injection and define \( W : [0, 1] \to \mathbb{Q} \) by \( W(x) := \frac{1}{2^{1+\lfloor x \rfloor}} \). Then \( W \) has bounded variation with upper bound 2. Indeed, since \( Y \) is an injection on \([0, 1]\), any sum \( \sum_{i=0}^n |W(x_n) - W(x_{n+1})| \) is at most \( \sum_{i=0}^n \frac{1}{2^{i+1}} \). By \( \text{NIN}_{bv} \), there are \( x, y \in [0, 1] \) with \( x \neq y \) and \( W(x) = W(y) \). This implies the contradiction \( Y(x) = Y(y) \), and \( \text{NIN} \leftrightarrow \text{NIN}_{bv} \) follows. For \( \text{NIN}_{\text{Riemann}} \to \text{NIN} \), the function \( W \) is Riemann integrable following the \( \varepsilon, \delta \)-definition. Indeed, pick \( \varepsilon_0 > 0 \) and find \( k_0 \in \mathbb{N} \) such that \( \frac{1}{2^{k_0}} < \varepsilon_0 \). Since \( Y \) is an injection, if \( P \) is a partition of \([0, 1]\) consisting of \(|P|\)-many points and with mesh \( ||P|| \leq \frac{1}{2^{k_0}} \), it is immediate that the Riemann sum \( S(W, P) \) is smaller than \( \frac{1}{2^{k_0}} \sum_{i=0}^{|P|} \frac{1}{2^{i+k_0}} \), which is at most \( \frac{1}{2^{k_0}} \).

For the implication \( \text{NIN}_{\text{semi}} \to \text{NIN} \), consider the same \( W : [0, 1] \to \mathbb{R} \) and note that \( \limsup_{x \to x_0} W(x) = \liminf_{x \to x_0} W(x) = \lim_{x \to x_0} W(x) \) for any \( x_0 \in [0, 1] \) in case \( Y : [0, 1] \to \mathbb{N} \) is an injection. Hence, \( W(x) \) is upper semi-continuous and \( Z(x) := 1 - W(x) \) is similarly \textit{lower} semi-continuous, since \( \liminf_{x \to x_0} Z(x) = \lim_{x \to x_0} Z(x) = 1 \). The finite sequences provided by \( \text{IND}_0 \) seem essential to establish these semi-continuity claims. One proves \( \text{NIN}_{\text{cliq}} \to \text{NIN} \) in the same way, namely using \( \text{IND}_0 \) to exclude the finitely many ‘too large’ function values. For the implication \( \text{NIN}_{\text{Borel}} \to \text{NIN} \), note that for an injection \( Y : [0, 1] \to \mathbb{N} \) the above function \( W(x) \) is Borel as \( W^{-1}((a, +\infty)) \) for any \( a \in \mathbb{R} \) is either finite or \([0, 1] \), and that these are Borel sets is immediate in \( \text{ACA}_0^ω + \text{IND}_0 \). For
the implication $\text{NIN}_{\text{reg}} \rightarrow \text{NIN}$, consider the same $W : [0, 1] \rightarrow \mathbb{R}$ and note that $W'(0+) = W(1-) = W(x+) = W(x-) = 0$ for $x \in (0, 1)$ in the same way as for the semi-continuity of $W$. Thus, $W$ is regulated and we are done.  

As noted above, a function has bounded variation iff it has finite arc length. The proof of this equivalence ([1] Prop. 3.28]) goes through in $\text{RCA}_0$, i.e. we may replace ‘bounded variation’ by ‘finite arc length’ in the previous theorem.

Fifth, we say that a function has total variation equal to $a \in \mathbb{R}$ in case the supremum over all partitions of $\sum_{i=0}^{n} |f(x_i) - f(x_{i+1})|$ in Def. [12] equals $a$.

**Corollary 16 ($\text{ACA}_0^\omega + \text{IND}_0$) The following are equivalent to $\text{NBI}$:**

- $\text{NBI}_{\text{Riemann}}$: there is no bijection from $[0, 1]$ to $\mathbb{Q}$ that is Riemann integrable,
- $\text{NBI}_{\text{bv}}$: there is no injection from $[0, 1]$ to $\mathbb{Q}$ that has total variation 1,
- $\text{NBI}_{\text{Borel}}$: there is no Borel function that is a bijection from $[0, 1]$ to $\mathbb{Q}$,
- $\text{NBI}_{\text{cliq}}$: there is no bijection from $[0, 1]$ to $\mathbb{Q}$ that is cliquish,
- $\text{NBI}_{\text{semi}}$: there is no upper semi-continuous bijection from $[0, 1]$ to $\mathbb{Q}$,
- $\text{NBI}'_{\text{semi}}$: there is no lower semi-continuous bijection from $[0, 1]$ to $\mathbb{Q}$.

Only the implications involving the final four items require the use of $\text{IND}_0$.

*Proof.* For the first equivalence, $W : [0, 1] \rightarrow \mathbb{R}$ from the proof has total variation exactly 1 in case $Y$ is also surjective. The other equivalences are now immediate by the proof of the theorem.  

As an intermediate conclusion, one readily proves that there are no continuous injections from $\mathbb{R}$ to $\mathbb{Q}$ (say over $\text{ACA}_0^\omega$). However, Theorem [15] and Corollary 16 show that admitting countably many points of discontinuity, one obtains principles that are extremely hard to prove following Remark [1].

Finally, one can greatly generalise Theorem 15 based on Remark 17. Indeed, there are many spaces intermediate between bounded variation and regulated, each of which yields a natural and equivalent restriction of $\text{NIN}$.

**Remark 17 (Intermediate spaces)** The following spaces are intermediate between bounded variation and regulated; all details may be found in [1]. Wiener spaces from mathematical physics are based on $p$-variation, which amounts to replacing $|f(x_i) - f(x_{i+1})|$ by $|f(x_i) - f(x_{i+1})|^p$ in the definition of variation. Young generalises this to $\phi$-variation which instead involves $\phi(|f(x_i) - f(x_{i+1})|)$ for so-called Young functions $\phi$, yielding the Wiener-Young spaces. Perhaps a simpler construct is the Waterman variation, which involves $\lambda_i |f(x_i) - f(x_{i+1})|$ and where $(\lambda_n)_{n \in \mathbb{N}}$ is a sequence of reals with nice properties; in contrast to bounded variation, any continuous function is included in the Waterman space ([1] Prop. 2.23]). Combining ideas from the above, the Schramm variation involves $\phi_i(|f(x_i) - f(x_{i+1})|)$ for a sequence $(\phi_n)_{n \in \mathbb{N}}$ of well-behaved ‘gauge’ functions. As to generality, the union (resp. intersection) of all Schramm spaces yields the space of regulated (resp. bounded variation) functions, while all other aforementioned spaces are Schramm spaces ([1] Prop. 2.43 and 2.46]). In contrast to bounded variation and the Jordan decomposition theorem, these generalised
notions of variation have no known ‘nice’ decomposition theorem. The notion of Korenblum variation does have such a theorem (see [11 Prop. 2.68]) and involves a distortion function acting on the partition, not on the function values.

2.3 Connections to mainstream mathematics

We establish the connection between NIN and two theorems from mainstream mathematics, namely Cousin’s lemma and Jordan’s decomposition theorem.

First of all, our results have significant implications for the RM of Cousin’s lemma. Indeed, as shown in [26], \( \mathbb{Z}_2^\omega \) cannot prove Cousin’s lemma as follows:

\[
(\forall \Psi : \mathbb{R} \to \mathbb{R}^+) (\exists y_0, \ldots, y_k \in [0, 1]) ([0, 1] \subset \bigcup_{i \leq k} B(y_i, \Psi(y_i)), \quad (\text{HBU})
\]

which expresses that the canonical covering \( \bigcup_{x \in [0, 1]} B(x, \Psi(x)) \) has a finite sub-covering, namely given by \( y_0, \ldots, y_k \in [0, 1] \). In [4], it is shown that HBU formulated using second-order codes for Borel functions is provable in \( \text{ATR}_0 \) plus some induction. We now show that this result from [4] is entirely due to the presence of second-order codes. Indeed, by Theorem 18, the restriction of HBU to Borel functions still implies NIN, which is not provable in \( \mathbb{Z}_2^\omega \) by Remark 1. To this end, let \( \text{HBU}_{\text{semi}} \) (resp. \( \text{HBU}_{\text{Borel}} \)) be HBU restricted to \( \Psi : [0, 1] \to \mathbb{R}^+ \) that are upper semi-continuous (resp. Borel) as in Definition 11 (resp. Def. 14).

**Theorem 18 (ACA\(^\omega_0\) + IND\(_0\))** NIN follows from \( \text{HBU}_{\text{semi}} \) and from \( \text{HBU}_{\text{Borel}} \); extra induction is only needed in the first case.

**Proof.** Let \( Y : [0, 1] \to \mathbb{N} \) be an injection and consider \( \Psi(x) := \frac{1}{2^{x+1}+x} \), which is upper semi-continuous and Borel by the proof of Theorem 15. Now consider the uncountable covering \( \bigcup_{x \in [0, 1]} B(x, \frac{1}{2^{x+1}+x}) \) of \([0, 1]\). Since \( Y \) is an injection, we have \( \sum_{i \leq k} |B(x_i, \frac{1}{2^{x_i+1}+x_i})| \leq \frac{1}{2} \) for any finite sequence \( x_0, \ldots, x_k \) of distinct reals in \([0, 1]\). In this light, HBU\(_{\text{semi}}\) and HBU\(_{\text{Borel}}\) are false. We note that the required basic measure theory (for finite sequences of intervals) can be developed in \( \text{RCA}_0 \) ([35, X.1]). \( \square \)

We now show that we can replace ‘Borel’ by ‘Baire class 2’ in Theorem 18 assuming the right (equivalent) definition. Now, Baire classes go back to Baire’s 1899 dissertation ([2]) and a function is ‘Baire class 0’ if it is continuous and ‘Baire class \( n + 1 \)’ if it is the pointwise limit of Baire class \( n \) functions. Baire’s characterisation theorem ([3, p. 127]) expresses that a function is Baire class 1 iff there is a point of continuity of the induced function on each perfect set.

Now let \( \mathcal{B}_2 \) be the class of all \( g : [0, 1] \to \mathbb{R} \) such that \( g = \lim_{n \to \infty} g_n \) on \([0, 1]\) and where for all \( n \in \mathbb{N} \) and perfect \( P \subset [0, 1] \), the restriction \( g_{n+1} \mid P \) has a point of continuity on \( P \). We have the following corollary.

**Corollary 19 (ACA\(^\omega_0\) + IND\(_0\))** We have \( \text{HBU}_{\mathcal{B}_2} \to \text{NIN} \) where the former is the restriction of HBU to \( \Psi : [0, 1] \to \mathbb{R}^+ \) in \( \mathcal{B}_2 \).
Proof. Fix \( A \subset [0,1] \) and \( Y : [0,1] \to \mathbb{N} \) with \( Y \) is injective on \( A \). Define \( \Psi : [0,1] \to \mathbb{R}^+ \) as follows: \( \Psi(x) = \frac{1}{x+1} \) in case \( x \in A \), and 1/8 otherwise. Define \( \Psi_n \) as \( \Psi \) with the condition \( 'Y(x) \leq n+5' \) in the first case. Clearly \( \Psi = \lim_{n\to\infty} \Psi_n \) and \( \Psi \in \mathcal{B}2 \), as \( \Psi_n \) only has at most \( n + 5 \) points of discontinuity (the set of which is not perfect in \( \text{ACA}_0^\omega + \text{IND}_n \)). For a finite sub-covering \( x_0, \ldots, x_k \in [0,1] \) of \( \bigcup_{x \in [0,1]} \mathcal{B}(x, \Psi(x)) \), there must be \( j < k \), with \( x_j \not\in A \). Indeed, the measure of \( \bigcup_{i \leq k} \mathcal{B}(x_i, \Psi(x_i)) \) is otherwise below \( \sum_{n=0}^{k} \frac{1}{x+1} < 1 \), a contradiction as the required basic measure theory can be developed in \( \text{RCA}_0 \) (see [25, X.1]). \( \square \)

Secondly, Jordan proves the following fundamental theorem about functions of bounded variation around 1881 in [16].

**Theorem 20 (Jordan decomposition theorem)** Any \( f : [0,1] \to \mathbb{R} \) of bounded variation is the difference of two non-decreasing functions \( g, h : [0,1] \to \mathbb{R} \).

Formulated using second-order codes, Theorem 20 is provable in \( \text{ACA}_0 \) (see [19, 25]); we now show that the third-order version is hard to prove as in Remark 10.

**Theorem 21 (\( \text{ACA}_0^\omega \))** Each item implies the one below it.

- The Jordan decomposition theorem for the unit interval.
- \( \text{HBU}_{\text{bv}}, \text{i.e. HBU restricted to } \Psi : [0,1] \to \mathbb{R}^+ \text{ of bounded variation.} \)
- \( \text{NIN: there is no injection from } [0,1] \text{ to } \mathbb{N}. \)

Assuming \( \text{IND}_0 \), we may replace the principle \( \text{HBU}_{\text{bv}} \) by the following one:

- For \( f : [0,1] \to \mathbb{R} \) of bounded variation, there is \( x \in [0,1] \) such that \( f \) is continuous (or: quasi-continuous) at \( x \).

**Proof.** The points of discontinuity of a non-decreasing function can be enumerated in \( \text{ACA}_0^\omega \) by [30, Lemma 3.3]. Now assume the Jordan decomposition theorem and fix some \( \Psi : [0,1] \to \mathbb{R}^+ \) of bounded variation. If \( (x_n)_{n \in \mathbb{N}} \) enumerates all the points of discontinuity of \( \Psi \), then the following also covers \([0,1], \)

\[
\bigcup_{q \in \mathbb{Q} \cap [0,1]} \mathcal{B}(q, \Psi(q)) \bigcup \bigcup_{n \in \mathbb{N}} \mathcal{B}(x_n, \Psi(x_n)).
\]

The second-order Heine-Borel theorem (provable in \( \text{WKL}_0 \) by [25, IV.1]) now yields a finite sub-covering, and \( \text{HBU}_{\text{bv}} \) follows. Now assume the latter and suppose \( Y : [0,1] \to \mathbb{N} \) is an injection. Define \( \Psi : [0,1] \to \mathbb{N} \) as \( \Psi(x) := \frac{1}{x+1} \). As in the proof of Corollary 19 any finite sub-covering of \( \bigcup_{x \in [0,1]} \mathcal{B}(x, \Psi(x)) \) must have measure at most 1/2, a contradiction; \( \text{NIN} \) follows and the first part is done.

For the second part of the theorem, we use the first part of the proof, namely that for \( f : [0,1] \to \mathbb{R} \) of bounded variation, the points of discontinuity can be enumerated, say by \( (x_n)_{n \in \mathbb{N}} \). By [25, II.4.9], the unit interval cannot be enumerated, i.e. there is \( y \in [0,1] \) such that \( (\forall n \in \mathbb{N})(x_n \neq y) \). By definition, \( f \) is continuous at \( y \). For the final implication, consider \( \Psi : [0,1] \to \mathbb{R}^+ \) from the first part of the proof. The function \( \Psi \) is everywhere discontinuous in case \( Y \) is an injection; one seems to need \( \text{IND}_0 \) to prove this. Similarly, \( \Psi \) is not quasi-continuous at any \( x \in [0,1] \), and we are done. \( \square \)
In conclusion, basic third-order theorems like Cousin’s lemma and Jordan’s decomposition theorem are ‘hard to prove’ in terms of conventional comprehension following Remark 1. Rather than measuring logical strength in terms of the one-dimensional scale provided by conventional comprehension, we propose an alternative two-dimensional scale, where the first dimension is based on conventional comprehension and the second dimension is based on the neighbourhood function principle NFP (see e.g. [37]). Thus, higher-order RM should seek out the minimal axioms needed to prove a given theorem of third-order arithmetic and these minimal axioms are in general a pair, namely a fragment of conventional comprehension and a fragment of NFP. This two-dimensional picture already exists in set theory where one studies which fragment of ZF and which fragments of AC are needed for proving a given theorem of ZFC. Note that ZF proves NFP as the choice functions in the latter are continuous.

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