SOME CONSTRUCTIONS OF ALMOST PARA-HYPERHERMITIAN STRUCTURES ON MANIFOLDS AND TANGENT BUNDLES

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Abstract. In this paper we give some examples of almost para-hyperhermitian structures on the tangent bundle of an almost product manifold, on the product manifold $M \times \mathbb{R}$, where $M$ is a manifold endowed with a mixed 3-structure and on the circle bundle over a manifold with a mixed 3-structure.

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1. Introduction

The para-hyperhermitian structures arise in a natural way in theoretical physics, both in string theory and integrable systems ([10], [12], [13], [15], [25]). This kind of structures have been intensively studied with different names by many authors (see [2], [14], [18], [19], [22], [27] and more).

In this paper we construct some classes of manifolds endowed with almost para-hyperhermitian structures. The paper is organized as follows. In section 2 we recall the definition and fundamental properties of almost para-hyperhermitian manifolds.

In Section 3 we give an almost para-hyperhermitian structure on the tangent bundle of an almost para-hermitian manifold and study its integrability.

The concept of mixed 3-structure has been introduced in [17]. In section 4 we study the manifolds endowed with such structures and prove that $M$ is a manifold with a mixed 3-structure, then $M = M \times \mathbb{R}$ can be endowed with an almost para-hyperhermitian structure. We construct also an almost para-hyperhermitian structures on a principal circle bundle over a smooth manifold endowed with a metric mixed 3-structure.

2. Preliminaries on almost para-hyperhermitian manifolds

An almost product structure on a smooth manifold $M$ is a tensor field $P$ of type (1,1) on $M$, $P \neq \pm Id$, such that:

(1) $P^2 = Id$.

An almost complex structure on a smooth manifold $M$ is a tensor field $J$ of type (1,1) on $M$ such that:

(2) $J^2 = -Id$.

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An almost para-hypercomplex structure on a smooth manifold $M$ is a triple $H = (J_α)_{α = 1, 2, 3}$, where $J_1$ is an almost complex structure on $M$ and $J_2, J_3$ are almost product structures on $M$, satisfying:

$$J_2 J_1 = -J_1 J_2 = J_3.$$  

In this case $(M, H)$ is said to be an almost para-hypercomplex manifold.

It is easy to see that any almost para-hypercomplex manifold is of dimension $4n$, for a non-negative integer $n$.

A semi-Riemannian metric $g$ on $(M, H)$ is said to be para-hyperhermitian if it satisfies:

$$g(J_α X, J_α Y) = ε_α g(X, Y), \forall α = 1, 2, 3$$

for all vector fields $X, Y$ on $M$, where $ε_1 = 1, ε_2 = ε_3 = -1$.

Moreover, $(M, g, H)$ is said to be an almost para-hyperhermitian manifold. It is clear that the signature of $g$ is $(2n, 2n)$ because, at each point of $M$, there is a local pseudo-orthonormal frame field, called adapted frame, given in the following way:

$$\{E_1, ..., E_m, J_1 E_1, ..., J_1 E_m, J_2 E_1, ..., J_2 E_m, J_3 E_1, ..., J_3 E_m\}.$$  

If $h$ is an arbitrary semi-Riemannian metric on an almost para-hypercomplex manifold $(M, H)$, then we can define always a para-hyperhermitian metric $g$ on $M$ by:

$$g(X, Y) = \frac{1}{4}[h(X, Y) + \sum_{α=1}^{3} ε_α h(J_α X, J_α Y)]$$

for all vector fields $X, Y$ on $M$.

An almost para-hypercomplex manifold $(M, H)$ is said to be a para-hypercomplex manifold if each $J_α$, $α = 1, 2, 3$, is integrable, that is, if the corresponding Nijenhuis tensors:

$$N_α(X, Y) = [J_α X, J_α Y] - J_α[X, J_α Y] - J_α[J_α X, Y] - ε_α[X, Y]$$

$α = 1, 2, 3$, vanish for all vector fields $X, Y$ on $M$. In this case $H$ is said to be a para-hypercomplex structure on $M$.

We remark that if two of the structures $J_1, J_2, J_3$ are integrable, then the third structure is also integrable because we have:

$$2N_α(X, Y) = N_β(J_γ X, J_γ Y) + N_γ(J_β X, J_β Y) - J_β N_γ(Y, J_β X, Y) - J_β N_γ(X, J_β Y) - J_γ N_β(Y, J_γ X, Y) - J_γ N_β(X, J_γ Y) + ε_α ε_β N_γ(X, Y) + ε_α ε_γ N_β(X, Y) + ε_β ε_γ N_α(X, Y)$$

for any even permutation $(α, β, γ)$ of $(1, 2, 3)$, where $ε_1 = 1, ε_2 = ε_3 = -1$.

An almost hermitian paracuaternionic manifold is a triple $(M, σ, g)$, where $M$ is a smooth manifold, $σ$ is a rank 3-subbundle of $End(TM)$ which is locally spanned by an almost para-hypercomplex structure $H = (J_α)_{α = 1, 2, 3}$ and $g$ is a para-hyperhermitian metric in respect with $H$.

3. **AN ALMOST PARA-HYPERHERMITIAN STRUCTURE ON THE TANGENT BUNDLE OF AN ALMOST PARA-HERMITIAN MANIFOLD**

An almost para-hermitian structure on a differentiable manifold $M$ is a pair $(P, g)$, where $P$ is an almost product structure on $M$ and $g$ is a semi-Riemannian metric on $M$ satisfying:

$$g(PX, PY) = -g(X, Y),$$
for all vector fields $X,Y$ on $M$.

In this case, $(M,P,g)$ is said to be an almost para-hermitian manifold. It is easy to see that the dimension of $M$ is even. Moreover, if $\nabla P = 0$ then $(M,P,g)$ is said to be a para-Kähler manifold.

We remark that this kind of manifolds appeared for the first time in [22].

Now, let $(M,P,g)$ be an almost para-hermitian manifold and $T M$ be the tangent bundle, endowed with the Sasakian metric:

$$G(X,Y) = (g(KX, KY) + g(\pi_* X, \pi_* Y)) \circ \pi$$

for all vector fields $X,Y$ on $T M$, where $\pi$ is the natural projection of $T M$ onto $M$ and $K$ is the connection map (see [11]).

We remark that if $X \in \Gamma(T M)$, then there exists exactly one vector field on $T M$ called the "horizontal lift" (resp. "vertical lift") of $X$ such that for all $t \in T M$:

$$\pi_* X^h_t = X^v_{\pi(t)}, \quad \pi_* X^v_t = 0_{\pi(t)}, \quad K X^h_t = 0_{\pi(t)}, \quad K X^v_t = X_{\pi(t)}.$$

**Remark 3.1.** It is immediately that for any affine connection $\nabla$ on $M$ we have:

$$[X^h, Y^h] = [X,Y]^h - (R(X,Y)t)^v, \quad [X^v, Y^v] = 0,$$

$$[X^h, Y^v] = (\nabla_X Y)^v, \quad [X^v, Y^h] = -(\nabla_Y X)^v,$$

for all $X,Y \in T_t(TM)$, $t \in T M$, where $R$ is the curvature tensor of $\nabla$ on $M$.

**Theorem 3.2.** Let $(M,P,g)$ be an almost para-hermitian manifold. Then $T M$ admits an almost para-hypercomplex structure $H$ which is para-hyperhermitian in respect to $G$.

**Proof.** We define three tensor fields $J_1, J_2, J_3$ on $T M$ by the equalities:

\begin{align*}
J_1 X^h &= X^v, & J_1 X^v &= -X^h, \\
J_2 X^h &= (PX)^v, & J_2 X^v &= (PX)^h, \\
J_3 X^h &= (PX)^h, & J_3 X^v &= -(PX)^v.
\end{align*}

We can easily see that we have:

$$J_1^2 = -J_2^2 = -J_3^2 = -Id,$$

$$J_2 J_1 = -J_1 J_2 = J_3$$

and

$$G(J_1 X, J_1 Y) = -G(J_2 X, J_2 Y) = -G(J_3 X, J_3 Y) = G(X,Y).$$

**Corollary 3.3.** The tangent bundle of the pseudosphere $S^6_3$ admits an almost para-hyperhermitian structure.

**Proof.** The assertion is clear from above Theorem because the pseudosphere $S^6_3$ can be endowed with an almost para-hermitian structure (see [3]).

**Corollary 3.4.** Let $(M,g)$ be a semi-Riemannian manifold and $T^* M$ its cotangent bundle. Then the tangent bundle of the cotangent bundle $TT^* M$ admits an almost para-hyperhermitian structure.

**Proof.** The assertion is clear from Theorem 3.2 because the cotangent bundle $T^* M$ can be endowed with an almost para-hermitian structure (see [26]).
Theorem 3.5. Let \((M, P, g)\) be an almost para-hermitian manifold. Then the almost para-hypercomplex structure \(H = (J_{\alpha})_{\alpha=1,\ldots,3}\) on \(TM\) given by Theorem [3.2] is integrable if and only if \((M, P)\) is a flat para-Kähler manifold.

Proof. Using Remark 3.1 we deduce that the Nijenhuis tensor of \(J_1\) is given by:

\[
N_1(X^h, Y^h) = [J_1X^h, J_1Y^h] - J_1[X^h, J_1Y^h] - J_1[J_1X^h, Y^h] - [X^h, Y^h]
\]

\[
= J_1(\nabla_Y X)^v - J_1(\nabla_X Y)^v - [X^h, Y^h] + (R(X, Y)t)^v
\]

(12)

\[
N_1(X^v, Y^v) = [J_1X^v, J_1Y^v] - J_1[X^v, J_1Y^v] - J_1[J_1X^v, Y^v] - [X^v, Y^v]
\]

\[
= (\nabla_Y X)^v - (R(X, Y)t)^v - J_1(\nabla_X Y)^v + J_1(\nabla_X Y)^v
\]

(13)

\[
N_1(X^v, Y^h) = [J_1X^v, J_1Y^h] - J_1[X^v, J_1Y^h] - J_1[J_1X^v, Y^h] - [X^v, Y^h]
\]

\[
= - (\nabla_Y X)^v + J_1([X^h, Y^h] + (R(X, Y)t)^v) + (\nabla_X Y)^v
\]

(14)

\[
N_1(X^h, Y^v) = [J_1X^h, J_1Y^v] - J_1[X^h, J_1Y^v] - J_1[J_1X^h, Y^v] - [X^h, Y^v]
\]

\[
= (R(X, Y)t)^h,
\]

(15)

for all \(X, Y \in T_1(TM), t \in TM\).

Similarly as above, we deduce:

\[
N_2(X^h, Y^h) = (P(\nabla_Y P)X - P(\nabla_X P)Y)^h - (R(X, Y)t)^v,
\]

(16)

\[
N_2(X^v, Y^v) = ((\nabla_P X)^v - (\nabla_P Y)^v + (R(P, Y)t)^v,
\]

(17)

\[
N_2(X^h, Y^v) = -(P(\nabla_P X)^v + (\nabla_P Y)^v + (PR(X, Y)t)^h,
\]

(18)

\[
N_2(X^v, Y^h) = ((\nabla_P X)^v + P(\nabla_Y P)X)^v + (PR(X, Y)t)^h
\]

(19)

and

\[
N_3(X^h, Y^h) = ((\nabla_P X)^v + (\nabla_P Y)^v + (PR(X, Y)t)^h
\]

(20)

\[
N_3(X^v, Y^v) = 0,
\]

(21)

\[
N_3(X^h, Y^v) = - (PR(X, Y)t)^h,
\]

(22)

\[
N_3(X^v, Y^h) = ((\nabla_P X)^v - (\nabla_P Y)^v)^v.
\]

(23)

The proof is now complete from (12)-(23). \(\square\)

Corollary 3.6. Let \((M, P, g)\) be an almost para-hermitian manifold and \(H = (J_{\alpha})_{\alpha=1,\ldots,3}\) the almost para-hypercomplex structure on \(TM\) given by Theorem [3.2]. If \(J_2\) is integrable, then \(H\) is a para-hypercomplex structure on \(TM\).

Proof. The proof is clear from Theorem [3.5] \(\square\)
4. Almost para-hyperhermitian structures and manifolds endowed with mixed 3-structures

**Definition 4.1.** Let $M$ be a differentiable manifold equipped with a triple $(\phi, \xi, \eta)$, where $\phi$ is a field of endomorphisms of the tangent spaces, $\xi$ is a vector field and $\eta$ is a 1-form on $M$. If we have:

$$\phi^2 = -\epsilon I + \eta \otimes \xi, \; \eta(\xi) = \epsilon$$

then we say that:

i. $(\phi, \xi, \eta)$ is an almost contact structure on $M$, if $\epsilon = 1$ (cf. [5]).

ii. $(\phi, \xi, \eta)$ is a Lorentzian almost paracontact structure on $M$, if $\epsilon = -1$ (cf. [24]).

We remark that many authors also include in the above definition the conditions that $\phi \xi = 0$ and $\eta \circ \phi = 0$, although these are deducible from the conditions (24) (see [5]).

**Definition 4.2.** ([17]) Let $M$ be a differentiable manifold which admits an almost contact structure $(\phi_1, \xi_1, \eta_1)$ and two Lorentzian almost paracontact structures $(\phi_2, \xi_2, \eta_2)$ and $(\phi_3, \xi_3, \eta_3)$, satisfying the following conditions:

i. $\eta_\alpha(\xi_\beta) = 0, \forall \alpha \neq \beta$;

ii. $\phi_\alpha(\xi_\beta) = -\phi_\beta(\xi_\alpha) = \epsilon_\gamma \xi_\gamma$;

iii. $\eta_\alpha \circ \phi_\beta = -\eta_\beta \circ \phi_\alpha = \epsilon_\gamma \eta_\gamma$;

iv. $\phi_\alpha \phi_\beta - \eta_\beta \otimes \xi_\alpha = -\phi_\beta \phi_\alpha + \eta_\alpha \otimes \xi_\beta = \epsilon_\gamma \phi_\gamma$,

where in (25), (26) and (27), $(\alpha, \beta, \gamma)$ is an even permutation of $(1,2,3)$ and $\epsilon_1 = 1, \epsilon_2 = \epsilon_3 = -1$.

Then the manifold $M$ is said to have a mixed 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,3}$.

**Definition 4.3.** If a manifold $M$ with a mixed 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,3}$ admits a semi-Riemannian metric $g$ such that:

$$g(\phi_\alpha X, \phi_\alpha Y) = \epsilon_\alpha g(X, Y) - \eta_\alpha(X)\eta_\alpha(Y),$$

and

$$g(X, \xi_\alpha) = \eta_\alpha(X)$$

for all $X, Y \in \Gamma(TM)$ and $\alpha = 1,2,3$, then we say that $M$ has a metric mixed 3-structure and $g$ is called a compatible metric.

Moreover, if $(\phi_1, \xi_1, \eta_1, g)$ is a Sasakian structure, i.e.(see [5]):

$$(\nabla_X \phi_1)Y = g(X, Y)\xi_1 - \eta_1(Y)X$$

and $(\phi_2, \xi_2, \eta_2, g)$, $(\phi_3, \xi_3, \eta_3, g)$ are Lorentzian para-Sasakian structures, i.e.(see [24]):

$$(\nabla_X \phi_2)Y = g(\phi_2 X, \phi_2 Y)\xi_2 + \eta_2(Y)\phi_2^2 X,$$

$$(\nabla_X \phi_3)Y = g(\phi_3 X, \phi_3 Y)\xi_3 + \eta_3(Y)\phi_3^2 X,$$

where $\nabla$ is the Levi-Civita connection of $g$, then $((\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,3}, g)$ is said to be a mixed Sasakian 3-structure on $M$. 
Proposition 4.4. Any manifold \( M \) with a mixed 3-structure \((\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,3}\) admits a compatible semi-Riemannian metric.

Proof. A such type of metric can be constructed from any semi-Riemannian metric \( f \) on \( M \), in four steps:

i. First we define the metric \( u \) by:
\[
u(X, Y) = f(\phi_1^2X, \phi_1^2Y) + \eta_1(X)\eta_1(Y)
\]
and we can see that \( u(X, \xi_1) = \eta_1(X) \).

ii. We define now a new metric \( v \) by:
\[
v(X, Y) = u(\phi_2^2X, \phi_2^2Y) - \eta_2(X)\eta_2(Y)
\]
and we have that \( v(X, \xi_2) = \eta_2(X) \), for \( \alpha = 1, 2 \).

iii. We define a new metric \( h \) by:
\[
h(X, Y) = v(\phi_3^2X, \phi_3^2Y) - \eta_3(X)\eta_3(Y)
\]
and we can easily see that \( h(X, \xi_3) = \eta_3(X) \), for \( \alpha = 1, 2, 3 \).

iv. Finally we define the metric \( g \) by:
\[
g(X, Y) = \frac{1}{4}[h(X, Y) + \sum_{\alpha=1}^{3} \epsilon_\alpha [h(\phi_\alpha X, \phi_\alpha Y) + \eta_\alpha(X)\eta_\alpha(Y)]
\]
and a straightforward verification shows that \( g \) provides a compatible semi-Riemannian metric on \((M, (\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,3})\). \(\Box\)

Remark 4.5. If \((M^{4n+3}, (\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,3}, g)\) is a manifold with a metric mixed 3-structure, then it is easy to see that the signature of \( g \) is \((2n + 1, 2n + 2)\) because one can check that, at each point of \( M \), there always exists a pseudo-orthonormal frame field given in the following way:
\[
\{(E_i, \phi_1 E_i, \phi_2 E_i, \phi_3 E_i)_{i=1,n}, \xi_1, \xi_2, \xi_3\}.
\]

Example 4.6. 1. It is easy to see that if we define \((\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,3}\) in \( \mathbb{R}^3 \) by their matrices:
\[
\phi_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \phi_3 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
\[
\xi_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},
\]
\[
\eta_1 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} -1 & 0 & 0 \end{pmatrix}, \quad \eta_3 = \begin{pmatrix} 0 & 0 & -1 \end{pmatrix},
\]
then \((\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,3}\) is a mixed 3-structure on \( \mathbb{R}^3 \).

We define now \((\phi'_\alpha, \xi'_\alpha, \eta'_\alpha)_{\alpha=1,3}\) in \( \mathbb{R}^{4n+3} \) by:
\[
\phi'_\alpha = \begin{pmatrix} \phi_\alpha & 0 \\ 0 & J_\alpha \end{pmatrix}, \quad \xi'_\alpha = \begin{pmatrix} \xi_\alpha \\ 0 \end{pmatrix}, \quad \eta'_\alpha = \begin{pmatrix} \eta_\alpha & 0 \end{pmatrix},
\]
for \( \alpha = 1, 2, 3 \), where \( J_1 \) is the almost complex structure on \( \mathbb{R}^{4n} \) given by:
\[
J_1((x_i)_{i=1,4n}) = (-x_2, x_1, -x_4, x_3, \ldots, -x_{4n-2}, x_{4n-3}, -x_{4n}, x_{4n-1}),
\]
and \( J_2, J_3 \) are almost product structures on \( \mathbb{R}^{4n} \) defined by:
\[
J_2((x_i)_{i=1,4n}) = (-x_{4n-1}, x_{4n}, -x_{4n-3}, x_{4n-2}, \ldots, -x_3, x_4, -x_1, x_2),
\]
and
\[
J_3((x_i)_{i=1,4n}) = (x_{4n-1}, -x_{4n}, x_{4n-3}, x_{4n-2}, \ldots, x_3, -x_4, x_1, x_2).
\]
(33) \( J_3((x_1)_{i=1..4n}) = (x_{4n}, x_{4n-1}, x_{4n-2}, x_{4n-3}, \ldots, x_4, x_3, x_2, x_1) \).

Since \( J_2J_1 = -J_1J_2 = J_3 \), it is easily checked that \( (\phi'_\alpha, \xi'_\alpha, \eta'_\alpha)_{\alpha=1..3} \) is a mixed 3-structure on \( \mathbb{R}^{4n+3} \).

2. Let \( (\overline{M}, \overline{g}) \) be a \((m + 2)\)-dimensional semi-Riemannian manifold with index \( q \in \{1, 2, \ldots, m + 1\} \) and let \((M, g)\) be a hypersurface of \( \overline{M} \), with \( g = \overline{g}|_M \). We say that \( M \) is a lightlike hypersurface of \( \overline{M} \) if \( g \) is of constant rank \( m \) (see [4]).

We consider the vector bundle \( TM^\perp \) whose fibres are defined by:

\[
T_pM^\perp = \{ Y_p \in T_pM, \overline{\nabla}_p(X, Y)_p = 0, \forall X_p \in T_pM \}, \forall p \in M.
\]

If \( S(TM) \) is the complementary distribution of \( TM^\perp \) in \( TM \), which is called the screen distribution, then there exists a unique vector bundle \( ltr(TM) \) of rank 1 over \( M \) so that for any non-zero section \( \xi \) of \( TM^\perp \) on a coordinate neighborhood \( U \subset M \), there exists a unique section \( N \) of \( ltr(TM) \) on \( U \) satisfying:

\[
\overline{g}(N, \xi) = 1, \quad \overline{g}(N, N) = \overline{g}(W, W) = 0, \quad \forall W \in \Gamma(S(TM)|_U)
\]

(see [4]).

In a lightlike hypersurface \( M \) of an almost hermitian paraquaternionic manifold \((\overline{M}, \overline{\xi}, \overline{\sigma})\) such that \( \xi \) and \( N \) are globally defined on \( M \), there is a mixed 3-structure (see [17]).

3. The unit sphere \( S^{4n+3}_{2n+1} \) is the canonical example of manifold with a mixed Sasakian 3-structure. This structure is obtained by taking \( S^{4n+3}_{2n+1} \) as hypersurface of \((\mathbb{R}^{4n+4}_{2n+2}, \overline{g})\). It is easy to see that on the tangent spaces \( T_pS^{4n+3}_{2n+1}, p \in S^{4n+3}_{2n+1} \), the induced metric \( g \) is of signature \((2n + 1, 2n + 2)\).

If \((J_\alpha)_{\alpha=1..3}\) is the canonical paraquaternionic structure on the \( \mathbb{R}^{4n+4}_{2n+2} \) and \( N \) is the unit normal vector field to the sphere, we can define three vector fields on \( S^{4n+3}_{2n+1} \) by:

\[
\xi_\alpha = -J_\alpha N, \quad \alpha = 1, 2, 3.
\]

If \( X \) is a tangent vector to the sphere then \( J_\alpha X \) uniquely decomposes onto the part tangent to the sphere and the part parallel to \( N \). Denote this decomposition by:

\[
J_\alpha X = \phi_\alpha X + \eta_\alpha(X)N.
\]

This defines the 1-forms \( \eta_\alpha \) and the tensor fields \( \phi_\alpha \) on \( S^{4n+3}_{2n+1} \), where \( \alpha = 1, 2, 3 \).

Now we can easily see that \( (\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1..3} \) is a mixed Sasakian 3-structure on \( S^{4n+3}_{2n+1} \).

4. Since we can recognize the unit sphere \( S^{4n+3}_{2n+1} \) as the projective space \( P^{4n+3}_{2n+1}(\mathbb{R}) \), by identifying antipodal points, we have also that \( P^{4n+3}_{2n+1}(\mathbb{R}) \) admits a mixed Sasakian 3-structure.

**Theorem 4.7.** If \( M \) is a manifold with a mixed 3-structure \((\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1..3}\) then \( \overline{M} = M \times I \) can be endowed with an almost para-hyperhermitian structure, where \( I \) is \( \mathbb{R} \) or some open interval in \( \mathbb{R} \).

**Proof.** We define three tensor fields \( J_1, J_2, J_3 \) on \( \overline{M} \) by the equalities:

\[
J_\alpha = \begin{pmatrix}
\phi_\alpha & \xi_\alpha \\
-f_\alpha \eta_\alpha & 0
\end{pmatrix}, \quad \alpha = 1, 2, 3.
\]

where \( f \) is a positive function on \( I \).
By a straightforward computation one can check:
\[
J_1^2 = -J_2^2 = -J_3^2 = -\text{Id},
\]
\[
J_2J_1 = -J_1J_2 = J_3
\]
and the proof is complete because we can construct now a para-hyperhermitian metric on \(\mathcal{M} = M \times \mathbb{R}\) from any arbitrary semi-Riemannian metric.

**Corollary 4.8.** \(S^{4n+3}_{2n+1} \times 1\) and \(P^{4n+3}_2(\mathbb{R}) \times 1\) can be endowed with almost para-hyperhermitian structures, where \(1\) is \(\mathbb{R}\) or some open interval in \(\mathbb{R}\).

**Proof.** From Example 4.6 we have that \(S^{4n+3}_{2n+1}\) and \(P^{4n+3}_2(\mathbb{R})\) are mixed 3-Sasakian manifolds and, consequently, each structure \(J_{\alpha}\) given by (34) is integrable.

**Corollary 4.9.** Let \(M\) be a manifold endowed with a mixed 3-structure \((\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,3}\). Then the dimension of \(M\) is \(4n + 3\), where \(n\) is a non-negative integer.

**Proof.** The assertion is clear from Theorem 4.7.

**Remark 4.10.** If \(M^{4n+3}\) is a manifold endowed with a mixed 3-Sasakian structure \((\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,3}, g)\), then we can define a para-hyper-Kähler structure \(\{J_\alpha\}_{\alpha=1,3}\) on the cone \((C(M), \mathcal{g}) = (M \times \mathbb{R}_+, dr^2 + r^2 g)\), by:
\[
\begin{cases}
J_\alpha X = \phi_\alpha X - \eta_\alpha(X) \Phi \\
J_\alpha \Phi = \xi_\alpha
\end{cases}
\]
for any vector field \(X \in \Gamma(TM)\) and \(\alpha = 1, 2, 3\), where \(\Phi = r \partial_r\) is the Euler field on \(C(M)\).

Moreover, conversely, if a cone \((C(M), \mathcal{g}) = (M \times \mathbb{R}_+, dr^2 + r^2 g)\) admits a para-hyper-Kähler structure \(\{J_\alpha\}_{\alpha=1,3}\), then we can identify \(M\) with \(M \times \{1\}\) and we have a mixed 3-Sasakian structure \((\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,3}, g)\) on \(M\) given by:
\[
\xi_\alpha = J_\alpha(\partial_r), \quad \phi_\alpha X = \nabla_X \xi_\alpha, \quad \eta_\alpha(X) = g(\xi_\alpha, X),
\]
for any vector field \(X \in \Gamma(TM)\) and \(\alpha = 1, 2, 3\).

Finally, since a para-hyper-Kähler manifold is Ricci-flat, we conclude that \(M\) is an Einstein space with Einstein constant \(\lambda = 4n + 2\) (see [7]).

**Remark 4.11.** Let \(P = P(M, \pi, S^1)\) be a principal circle bundle over a smooth manifold \(M^{4n+3}\) with a metric mixed 3-structure \((\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,3}, g)\).

Let \(g\) be a semi-Riemannian metric on \(P\) such that \(\pi : (P, g) \rightarrow (M, g)\) is a semi-Riemannian submersion. Putting \(\mathcal{V}_x = \text{Ker} \pi_* x\), for any \(x \in M\), we obtain an integrable distribution \(\mathcal{V}\), which is called vertical distribution and corresponds to the foliation of \(M\) determined by the fibres of \(\pi\). The complementary distribution \(\mathcal{H}\) of \(\mathcal{V}\), determined by the semi-Riemannian metric \(g\), is called horizontal distribution. We have now the decomposition:
\[
TP = \mathcal{H} \oplus \mathcal{V}.
\]

We recall that the sections of \(\mathcal{V}\), respectively \(\mathcal{H}\), are called the vertical vector fields, respectively horizontal vector fields. An horizontal vector field \(X^h\) on \(P\) is said to be basic if \(X^h\) is \(\pi\)-related to a vector field \(X\) on \(M\). It is clearly that every vector field \(X\) on \(M\) has a unique horizontal lift \(X^h\) to \(P\) and \(X^h\) is basic.

**Theorem 4.12.** There is an almost para-hyperhermitian structure on any principal circle bundle \(P = P(M, \pi, S^1)\) over a manifold \(M\) with a metric mixed 3-structure \((\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,3}, g)\).
Proof. Making use of the metric mixed 3-structure on $M$ we can define three tensor fields $J_1, J_2, J_3$ on $P$ by the equalities:

$$
\begin{cases}
J_\alpha X^h = (\phi_\alpha X)^h + \eta_\alpha (X)\Theta \\
J_\alpha \Theta = -(\xi_\alpha)^h
\end{cases}
$$

for $\alpha = 1, 2, 3$, where $\Theta$ is the nowhere vanishing vertical vector field on $P$ which generates the $S^1$ action on $P$.

Now, we can easily see that we have:

$$
J_1^2 = -J_2^2 = -J_3^2 = -\text{Id},
J_2 J_1 = -J_1 J_2 = J_3
$$

and the proof is now complete since we can construct a para-hyperhermitian metric from any arbitrary semi-Riemannian metric on $P$.

□

Corollary 4.13. $F_{2n+1} \times S^1$, $S^2_{4n+3} \times S^1$ and $P_{2n+1}(\mathbb{R}) \times S^1$ can be endowed with almost para-hyperhermitian structures.

Proof. The assertion is clear from Example 4.10 and Theorem 4.12. □

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