General approach to the study of vacuum space-times with an isometry

Francesc Fayos†§ and Carlos F. Sopuerta‡§
† Departament de Física Aplicada, UPC, E-08028 Barcelona, Spain
‡ Relativity and Cosmology Group, School of Computer Science and Mathematics, Mercantile House, Hampshire Terrace, PO1 2EG Portsmouth, United Kingdom
E-mail: labfm@ffn.ub.es, carlos.sopuerta@port.ac.uk

Abstract. In vacuum space-times the exterior derivative of a Killing vector field is a 2-form (named here as the Papapetrou field) that satisfies Maxwell’s equations without electromagnetic sources. In this paper, using the algebraic structure of the Papapetrou field, we will set up a new formalism for the study of vacuum space-times with an isometry, which is suitable to investigate the connections between the isometry and the Petrov type of the space-time. This approach has some advantages, among them, it leads to a new classification of these space-times and the integrability conditions provide expressions that determine completely the Weyl curvature. These facts make the formalism useful for application to any problem or situation with an isometry and requiring the knowledge of the curvature.

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1. Introduction

The search of exact solutions of the Einstein field equations has been one of the most active fields of research in general relativity. Despite of the non-linearity of the Einstein equations and its invariance under general changes of coordinates, a wide variety of exact solutions have been found (see [1]). Although we do not have exact solutions describing the gravitational field in complex realistic systems, the development of techniques for the search of exact solutions as well as the study of some particular ones have been of crucial importance for the understanding of general relativity and for some developments within its framework, like perturbation theory, numerical methods, etc. There are lots of techniques and methods to find exact solutions of Einstein’s equations. Among them, we want to emphasize two: First, the imposition of symmetries (Killing symmetries, conformal Killing symmetries, etc.) on the space-time metric and second, the imposition of a special algebraic structure for the space-time (described by the algebraic structure of the Weyl tensor). It is worth to note that many important techniques have been

§ Also at the Laboratori de Física Matemàtica, Societat Catalana de Física, I.E.C., Barcelona, Spain
developed starting from these two types of simplifications of the Einstein equations. However, despite of the number of works dedicated to these techniques and their applications, as far as we know there are few studies establishing connections between the existence of symmetries and the particular algebraic structure of the space-time, even in the case of Killing symmetries (see [1], chapter 33). In this paper we will set up a new formalism to study solutions of Einstein’s equations with a Killing vector field (KVF hereafter) which, as we will see, is suitable to study such connections as well as other important related subjects. Such a formalism is well-motivated taking into account that there is a great variety of physical situations with a symmetry described by a KVF.

The starting point is the fact, firstly noticed by Papapetrou [2], that the exterior derivative of a KVF is a 2-form satisfying Maxwell’s equations, with the KVF playing the role of the electromagnetic potential and satisfying the covariant Lorentz gauge. This 2-form, which we will call the Papapetrou field associated with the KVF (sometimes also called the Killing 2-form or the Killing bivector [3, 4]), has been used and applied to several subjects, like the search and study of exact solutions of Einstein’s field equations (see, e.g., [5, 6]), or the study of black holes in the presence of external electromagnetic fields (see [7]). Other applications were described in [8], where a systematic study was made. Specifically, general covariant expressions for the principal null direction(s) of the Papapetrou field in terms of quantities associated with the Killing were found, and moreover, some differential properties of the principal null direction(s) were studied and the conditions for a principal direction to be aligned with a multiple principal direction of an algebraically special vacuum space-time were given.

The existence of the Papapetrou field provides a classification of the vacuum space-times having a KVF. This classification will consider whether the Papapetrou field is regular (two different null principal directions) or singular (only one null principal direction), the Petrov type, and the possible different alignments of the principal directions of the Papapetrou field with those of the space-time (the principal directions of the Weyl tensor). This classification will lead us to consider a new approach based on the well-known Newman-Penrose (NP hereafter) formalism [9] and on the structure and properties of the Papapetrou field. The first important point is that we will write all the equations involved using a NP basis adapted to the principal direction(s) of the Papapetrou field. And the second important point is that we will extend the usual framework of the NP formalism by adding new variables and their corresponding equations. The first set of new variables will be the components of the KVF in the adapted basis chosen, and the equations that we will use for them are the equations defining the Papapetrou field combined with the Killing equations. To close the system we will need to add two new variables: The eigenvalues in the regular case, or a complex function in the singular case. The equations for these two variables are just the Maxwell equations of simplicity we will restrict ourselves to vacuum space-times, but the main ideas on which our formalism is based can be extended to space-times with other types of energy-momentum content.
equations for the Papapetrou field. All these ingredients together provide a framework that incorporates explicitly the existence of a KVF and that allows to control in a clear and transparent way the algebraic structure of the space-time and the possible alignments of the principal direction(s) of the Papapetrou field with those of the Weyl tensor.

Using this framework, we have studied the integrability conditions for some variables and the compatibility conditions of other sets of equations. As a result of this study we have obtained explicit expressions for the components of the Weyl tensor in terms of the components of the KVF, some spin coefficients, and the two additional functions associated with the Papapetrou field. Therefore, it is not necessary to solve the second Bianchi identities for them, instead these identities will provide more equations for the spin coefficients (apart from the NP equations). As we will see, other integrability or compatibility conditions will be also reduced to equations for the spin coefficients. To sum up, this formalism provide expressions for the Weyl tensor (which depend only on spin coefficients and variables associated with the KVF) and the main problem is reduced to study the set of equations for the spin coefficients (the NP equations and the integrability and compatibility conditions). These characteristics of the formalism make it very appropriate for any situation or problem requiring the knowledge of the curvature (the Weyl tensor) and its properties.

The plan of the paper is as follows: In section 2 we review some useful material on the algebraic structures of the Papapetrou field and the Weyl tensor, and we will propose a new classification for vacuum space-times with a KVF. In section 3 we introduce the formalism for the case of vacuum space-times. In section 4 we study in full generality the integrability and compatibility conditions for some sets of equations and describe a general scheme to proceed further. In section 5 we apply our formalism to two particular examples. First, we will study completely the case in which the Papapetrou field associated with the KVF is singular. And in the second example, we will examine the case of Petrov type III vacuum space-times, specifically we will study the possibility of alignment of the multiple principal direction of the space-time with one of the Papapetrou field. We will finish with some comments and remarks in section 6. In Appendix A we give some formulae for the Kundt class of Petrov types N and III vacuum metrics that have been used in the examples of section 5. Through this paper we will follow the notation and conventions of [1] unless otherwise stated.

2. Some important facts and ideas

In order to introduce a new formalism to deal with vacuum space-times with a KVF, as the one described in the introduction, it will be crucial to consider algebraic structures associated (in a local way) with the KVF and with the space-time. In the case of the space-time the algebraic structure mostly used and studied in the literature is the algebraic structure of the Weyl tensor, whose classification (nowadays called Petrov classification [10, 11, 12]) can be found, for instance, in [1, 13, 14, 15]. On the other
hand, in the case of a KVF, the algebraic structure to be associated with it will be the algebraic structure of its exterior derivative. Since the exterior derivative of a KVF is a 2-form, the algebraic classification will be identical to that of the electromagnetic field. In what follows we will review these algebraic structures and introduce a new classification for space-times with a KVF, establishing some notation and introducing some formulae that will be used along this paper.

One of the starting points in the development of this work is the fact, already recognized by Papapetrou [2], that a KVF $\xi$ can always be seen as the vector potential of an electromagnetic field, the Papapetrou field associated with $\xi$, defined as follows

$$F \equiv d\xi,$$

(1)

where $d$ denotes the exterior derivative. Using the Killing equations

$$\xi_{a;b} + \xi_{b;a} = 0,$$

(2)

where a semicolon means covariant differentiation, and the Ricci identities for $\xi$

$$\xi_{a;bc} = R_{abcd}\xi^d,$$

(3)

where $R_{abcd}$ denotes the Riemann tensor, we can show that $F_{ab} (= \xi_{b;a} - \xi_{a;b} = 2\xi_{b;a})$ satisfies the Maxwell equations

$$F_{[ab;c]} = 0, \quad F_{ab}^{;b} = J^a,$$

(4)

where $J^a$ is the conserved current given by

$$J^a \equiv 2R_b^a\xi^b \Rightarrow J^a_{;a} = 0,$$

where $R_{ab} \equiv R_c{}^{ac}\xi_b$ is the Ricci tensor. It follows directly from these expressions that if the KVF is an eigenvector of the Ricci tensor with zero eigenvalue, then $F_{ab}$ satisfies Maxwell’s equations in the absence of electromagnetic charge and current distributions. Obviously, this is in particular true for vacuum space-times. Finally, as a consequence of the Killing equations (2), $\xi$ satisfies the covariant version of the Lorentz gauge

$$\xi^a_{;a} = 0.$$

From now on we will consider that the algebraic structure of the Papapetrou field $F_{ab}$ (1), a 2-form, is the algebraic structure associated with the KVF $\xi$. In an arbitrary NP basis $\{k, \ell, m, \bar{m}\}$, a basis made up of two real null vectors ($k, \ell$) and a complex null vector $m$ and its complex conjugate $\bar{m}$ (a bar means complex conjugation) such that

$$k \cdot \ell = -m \cdot \bar{m} = -1, \quad k \cdot m = \ell \cdot \bar{m} = 0,$$

an arbitrary 2-form $F_{ab}$ ($F_{[ab]} = F_{ab}$) can be written in the following form

$$\tilde{F}_{ab} = \Phi_0U_{ab} + \Phi_1W_{ab} + \Phi_2V_{ab},$$

where the tilde denotes the self-dual operation

$$\tilde{F}_{ab} = F_{ab} + i * F_{ab} \quad (*F_{ab} \equiv \frac{1}{2} \eta_{ab}{}^{cd} F_{cd}),$$
denotes the dual operation and $\eta_{abcd}$ the volume 4-form of the space-time. Finally, the complex 2-forms $U, V, \text{ and } W$ are given by

$$U_{ab} \equiv -2\ell_{[a}m_{b]} \, , \quad V_{ab} \equiv 2k_{[a}m_{b]} \, , \quad W_{ab} \equiv 2m_{[a}m_{b]} - 2k_{[a}\ell_{b]} \, .$$

Therefore, the complex scalars $\Phi_0, \Phi_1, \text{ and } \Phi_2$ are

$$\Phi_0 = F_{ab}^a m^b, \quad \Phi_1 = \frac{1}{2} F_{ab}^a (k^b + m^a m^b), \quad \Phi_2 = F_{ab}^a \bar{m}^a \ell^b \, .$$

The algebraic classification of a 2-form consists of two differentiated cases:

(i) The regular case ($\tilde{F}_{ab} \tilde{F}_{ab} \neq 0$). In this case we can pick a NP basis so that the self-dual 2-form $\tilde{F}$ can be written as follows

$$\tilde{F}_{ab} = \Phi_1 W_{ab} \, ,$$

where $\Phi_1$ coincide with $-(\phi + i \beta)$, being $(\phi, -\phi)$ and $(\beta, -\beta)$ the real eigenvalues of $F_{ab}$ and $*F_{ab}$ respectively. Moreover, the base vectors $\ell$ and $k$ are the corresponding eigenvectors. In this adapted basis, the 2+2 characteristic structure of regular 2-forms, sometimes called the Maxwellian structure, appears explicitly: $(k, \ell)$ span the 2-planes of the principal directions, and $(m, \bar{m})$ the orthogonal ones. In [8], a covariant way of obtaining them for Papapetrou fields was given.

(ii) The singular case ($\tilde{F}_{ab} \tilde{F}_{ab} = 0$). Now we can choose the NP basis so that $\tilde{F}$ can be cast in the form

$$\tilde{F}_{ab} = \phi V_{ab} \, , \quad \phi \equiv \Phi_2 \, ,$$

being $k$ the only principal direction.

Apart from these two cases, we have the situation in which $F_{ab} = 0$. Obviously, in this case there are no preferred principal null directions and the KVF is a constant vector field ($\xi_{ab} = 0$). For vacuum spacetimes we have two different cases depending on whether the KVF is null or non-null. When the KVF is null the metric corresponds to the plane-fronted gravitational waves, also called pp waves (see, e.g., [1]). On the other hand, when the KVF is non-null the spacetime is Minkowski. Therefore, since the case $F_{ab}$ is well understood we will not consider it in what follows.

In both cases we have some freedom in the choice of the NP basis. First we have the freedom given by the following transformations:

$$k \rightarrow k' = Fk, \quad \ell \rightarrow \ell' = F^{-1}\ell \, ,$$

$$m \rightarrow m' = e^{2iC}m \, ,$$

where $F$ and $C$ are arbitrary real functions. This exhausts the freedom in the regular case, but in the singular case $\ell$ is not fixed at all, and then, we can choose its direction. This additional freedom is described by the following transformation:

$$\ell \rightarrow \ell' = \ell + E m + \bar{E} \bar{m} + \bar{E} E k \, ,$$

Here we have changed the notation with respect to our previous paper [8], where the eigenvalues where called $(\alpha, \beta)$, in order to avoid confusion with the spin coefficients with the same name.
where $E$ is an arbitrary complex function. Finally, it is important to note that whereas in the regular case the quantities $(\phi, \beta)$ are invariant under the transformations (8,9), the quantity $\phi$ in the singular case it is not invariant. In fact, we can use these transformations to choose its value in an arbitrary way.

In the case of the space-time, the algebraic structure usually considered is the algebraic structure of the Weyl tensor, which in vacuum coincides with the Riemann tensor. As we have said before, the algebraic classification of this tensor is the Petrov classification. There are different ways of dealing with this classification (see [1]), but the most convenient one for our purposes is based on the expression for the self-dual Weyl tensor

$$\tilde{C}_{abcd} = C_{abcd} + i \ast C_{abcd}, \quad (\ast C_{abcd} \equiv \frac{1}{2} \eta_{ab}^{ef} C_{efcd}), \quad (11)$$

in an arbitrary NP basis (see, e.g., [1])

$$\frac{1}{2} \tilde{C}_{abcd} = \Psi_0 U_{ab} U_{cd} + \Psi_1 (U_{ab} W_{cd} + W_{ab} U_{cd}) + \Psi_2 (V_{ab} U_{cd} + U_{ab} V_{cd} + W_{ab} W_{cd}) + \Psi_3 (V_{ab} W_{cd} + W_{ab} V_{cd}) + \Psi_4 V_{ab} V_{cd}, \quad (12)$$

Taking into account the definitions (9) and (12), the complex components of the Weyl tensor $\Psi_A (A = 0, \ldots, 4)$ are given by

$$\Psi_0 = \frac{1}{2} \tilde{C}_{abcd} k^a m^b k^c m^d, \quad \Psi_1 = \frac{1}{2} \tilde{C}_{abcd} k^a \ell^b k^c \ell^d, \quad \Psi_2 = \frac{1}{2} \tilde{C}_{abcd} k^a \ell^b k^c \ell^d, \quad \Psi_3 = \frac{1}{2} \tilde{C}_{abcd} k^a \ell^b \ell^c m^d, \quad \Psi_4 = \frac{1}{2} \tilde{C}_{abcd} \ell^a m^b \ell^c \ell^d. \quad (13)$$

Then, we can distinguish five algebraic types (called I, II, III, D and N) according with the number of roots, and their multiplicity, of the polynomical equation

$$\Psi_0 + 4E\Psi_1 + 6E^2\Psi_2 + 4E^3\Psi_3 + E^4\Psi_4 = 0,$$

for the complex variable $E$. We will not enter here in the details (see, e.g., [1, 13, 14, 15]), the important point is that the complex scalars $\Psi_A (A = 0, \ldots, 4)$ contain all the information about the algebraic structure of the space-time. We can also include one more case, the Petrov type O, which corresponds to conformally-flat space-times, i.e., to a vanishing Weyl tensor ($\Psi_A = 0$).

Using the algebraic structures we have just described we can introduce a new classification of the vacuum space-times having at least one KVF, or more precisely, of the pairs $\{(V_4, g), \xi\}$, which includes the fact that there are space-times with more than one KVF. Then, we will classify these pairs according to the following properties:

- The algebraic type of the Papapetrou associated with $\xi$: Regular $(\phi + i \beta \neq 0)$ or singular $(\phi + i \beta = 0)$.
- The algebraic type of the space-time $(V_4, g)$, or equivalently, of the Weyl tensor of $g$: I, II, III, D, N, or O.
- The degree of alignment of the principal directions of the Papapetrou field with those of the Weyl tensor. For instance, in the case of a singular Papapetrou field and a type N space-time there would be only two cases, the case in which the unique principal directions of these objects are aligned and the case in which they are not.
We can have a more refined classification by considering also some intrinsic differential properties, specifically, the differential properties of the principal directions both of the Papapetrou field and the Weyl tensor: Whether they are geodesic or not, the shear, the expansion, and the rotation.

3. A formalism for vacuum space-times with an isometry

The classification presented above raises some questions, as for example in which cases fall the known exact solutions, or which restrictions impose each particular case of the classification, whether or not there are empty cases, and also, whether this classification can help in the search of new solutions. In order to answer these questions and to study other important related issues we are going to introduce a new formalism. For the sake of simplicity we will only consider here the case of vacuum space-times \((V_4, g)\) \((R_{ab} = 0)\) possessing a non-null KVF \(\xi\), i.e., its norm \(N = \xi^a \xi_a \neq 0\).

The case of vacuum space-times with a null Killing vector was considered in our previous work \[8\]. Taking into account the characteristics of our classification, this formalism will be an extension of the well-known NP formalism \[9\], which provides a clear and elegant way of controlling the algebraic structure of the space-time. As in other schemes where a particular basis is used, the equations and variables in the NP formalism can be organized in the following way (see \[13, 14\] for details): Instead of considering the components of the metric tensor as variables we use the components of a NP basis \((z^a) = (k^b, \ell^b, m^b, \bar{m}^b)\) with respect to a coordinate system \(\{x^a\}\). The equations for them are the expressions defining the connection associated with such basis, which can be found by applying the commutators of the NP basis vectors [see equations (44-47) bellow] to the coordinate system \(\{x^a\}\). The next set of variables are the components of this connection, which in the NP formalism are described by the spin coefficients, which are 12 complex scalars \((\kappa, \sigma, \rho, \epsilon, \nu, \lambda, \mu, \gamma, \tau, \pi, \alpha, \beta)\). The equations for the spin coefficients (and also integrability conditions for the previous equations) are the so-called NP equations [see, e.g., \[1\], equations (7.28)-(7.45), for the sake of brevity we will not list these equations here], which are simply the expressions for the Riemann tensor components in terms of the complex connection. In the case of vacuum space-times, the components of the Riemann tensor are just the components of Weyl tensor, which are considered also as new variables. In the NP formalism they are described by the five complex scalars \(\Psi_A\) \((A = 0, \ldots, 4)\) defined by equations \([13, 14]\). The equations for these complex scalars come from the second Bianchi identities [see \[1\], equations (7.61)-(7.71)], we will not list these equations here], which at the same time are the integrability conditions for the NP equations. With these equations we get a closed system of equations for the whole set of variables. The literature is plenty of

\[\text{Note that there is a misprint in equation (7.63) of [1], instead of } \ldots - \Delta \Psi_4 \ldots \text{ it should read } \ldots - D \Psi_4 \ldots.\]
examples in which the integration of these equations, in a great variety of situations, has led to the discovery of new exact solutions of the Einstein equations (see [1] and references therein).

Now, we will extend this formalism. The first step will be to include new variables related to the KVF and the corresponding equations for them. To that end, we start by writing an arbitrary KVF in a NP basis \( \{k, l, m, \bar{m}\} \)

\[
\xi = -\xi_k k - \xi_l l + \xi_m m + \xi_{\bar{m}} \bar{m} ,
\]

where \( \xi_k, \xi_l, \xi_m, \) and \( \xi_{\bar{m}} \) denote the components of \( \xi \) in the NP basis, defined by

\[
\xi_k \equiv k^a \xi_a , \quad \xi_l \equiv l^a \xi_a , \quad \xi_m \equiv m^a \xi_a , \quad \xi_{\bar{m}} \equiv \bar{m}^a \xi_a .
\]

Taking into account that \( \xi \) is a real vector field, \( \xi_k \) and \( \xi_l \) are real scalars, and \( \xi_m \) and \( \xi_{\bar{m}} \) are complex ones and related by

\[
\xi_m = \bar{\xi}_{\bar{m}} .
\]

Hence, we only need to consider \( \xi_m \) and its complex conjugate \( \bar{\xi}_{\bar{m}} \). On the other hand, the norm of the KVF, \( N \), can be written in terms of \( \xi_k, \xi_l, \) and \( \xi_m \) as follows

\[
N = -2\xi_k \xi_l + 2\xi_m \bar{\xi}_{\bar{m}} .
\]

As is obvious these new variables must satisfy the Killing equations

\[
\mathcal{L}(\xi)_{ab} = \xi_{a;b} + \xi_{b;a} = 0 ,
\]

where \( \mathcal{L}(\xi) \) denotes Lie differentiation along \( \xi \). Using the expression (15) for the KVF, we can project the Killing equations onto the NP basis \( \{k, l, m, \bar{m}\} \). Then, we obtain the following equations for the components of the Killing

\[
\begin{align*}
D\xi_k & \equiv (\epsilon + \bar{\epsilon})\xi_k + \bar{\kappa}\xi_m + \kappa \bar{\xi}_m = 0 , \\
\Delta\xi_l & \equiv (\gamma + \bar{\gamma})\xi_l - \nu \xi_m - \bar{\nu}\bar{\xi}_m = 0 , \\
\Delta\xi_k & \equiv D\xi_l - (\gamma + \bar{\gamma})\xi_k + (\epsilon + \bar{\epsilon})\xi_l + (\bar{\tau} - \tau)\xi_m + (\tau - \bar{\tau})\bar{\xi}_m = 0 , \\
D\xi_m & \equiv \delta\xi_k - (\bar{\rho} - \rho - \bar{\epsilon})\xi_k + \kappa\xi_l + (\bar{\rho} - \rho + \epsilon)\xi_m + \sigma\bar{\xi}_m = 0 , \\
\Delta\xi_m & \equiv \delta\xi_l - \bar{\nu}\xi_k + (\tau + \alpha + \beta)\xi_l - (\mu + \gamma - \bar{\gamma})\xi_m + \bar{\lambda}\bar{\xi}_m = 0 , \\
\delta\xi_m & \equiv (\alpha - \beta)\xi_m - \bar{\lambda}\xi_k + \sigma\xi_l = 0 , \\
\bar{\delta}\bar{\xi}_m & \equiv (\alpha - \beta)\bar{\xi}_m - (\alpha - \beta)\xi_m - (\mu + \bar{\mu})\xi_k + (\rho + \bar{\rho})\xi_l = 0 ,
\end{align*}
\]

where \( D, \Delta, \delta \) and \( \bar{\delta} \) denote the directional derivatives along the NP basis vectors, defined as follows

\[
D \equiv k^a \partial_a , \quad \Delta \equiv l^a \partial_a , \quad \delta \equiv m^a \partial_a , \quad \bar{\delta} \equiv \bar{m}^a \partial_a .
\]

Equations (18)-(24) are completely equivalent to the Killing equations (17). The usual way of dealing with the integrability conditions for the Killing equations (17) is to differentiate them repeatedly. As is well-known, these integrability conditions are
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The equations for the regular case follow by putting $\phi = 0$, and to obtain the equations for the singular case we have to put $\phi = \beta = 0$. As is clear, the case $\phi = \beta = \phi = 0$ corresponds to the case of a constant KVF. Furthermore, we can check that these equations contain the Killing equations (18-24). On the other hand, to write these
equations we have used a NP basis adapted to the Papapetrou field. Apart from the fact that this has allowed us to write the equations in the simplest form, there is another important advantage related with the classification put forward in section 2. It is quite simple to implement the idea of alignment of a principal direction of the Papapetrou field with one of the Weyl tensor. For instance, by taking \( \Psi_0 = 0 \) we impose the principal direction of the Papapetrou field \( k \) to be aligned with a principal direction of the space-time.

Apart from the remarkable fact that in the set of equations (28-37) the directional derivatives of the components of the KVF are uncoupled, it is important to note that we have expressions for all the directional derivatives of all the components of the KVF. On the other hand, these partial differential equations are linear in the components of the KVF and inhomogeneous (\( \xi_a = 0 \) is not a solution). The last point is due to the appearance of \( \phi + i \beta \) in the regular case and \( \phi \) in the singular one. The last step in our development will be to consider these quantities as new variables and to complete our description by adding the corresponding equations for them. It turns out that the equations for \( \phi + i \beta \) and \( \phi \) are just the Maxwell equations for the Papapetrou field, equations (4), written in the NP basis in which it takes its canonical form [equations (6,7)]. The explicit form of these equations in the regular case is

\[
D(\phi + i \beta) = 2\rho(\phi + i \beta),
\]

\[
\Delta(\phi + i \beta) = -2\mu(\phi + i \beta),
\]

\[
\delta(\phi + i \beta) = 2\tau(\phi + i \beta),
\]

\[
\bar{\delta}(\phi + i \beta) = -2\pi(\phi + i \beta),
\]

and in the singular case we get

\[
D\phi = (\rho - 2\epsilon)\phi,
\]

\[
\delta\phi = (\tau - 2\beta)\phi,
\]

and

\[
\kappa = \sigma = 0,
\]

which means that the only principal direction, \( k \), is geodesic and shear-free. As we can see, these equations close the system of equations for the variables \( (\xi_k, \xi_l, \xi_m) \) and \( \phi + i \beta \) or \( \phi \).

To sum up, we have set up a formalism for vacuum space-times possessing a KVF, which is an extension of the NP formalism, by considering the algebraic structure associated with the KVF. Firstly, this extension consists in written all the equations involved in a NP basis adapted to the algebraic structure of the Papapetrou field. A covariant procedure to get such a NP basis was given in our previous paper [8]. Here, we have to remember that it is fixed up the transformations (8,9) in the regular case and the transformations (8,9,10) in the singular case. Secondly, we have introduced the
following new variables: (i) the components of a KVF \((\xi_k, \xi_l, \xi_m)\) in the adapted NP basis and, (ii) a complex variable associated with the Papapetrou field. In the regular case it is made up of the eigenvalues, \(\phi + i \beta\), whereas in the singular case it is the complex function \(\phi\). These new variables, together with the corresponding NP basis, determine completely the Papapetrou field of \(\xi\). The equations for the components of the KVF are \((28-37)\), whereas the equations for \((\phi, \beta)\) are \((38-41)\), and the equations for \(\phi\) are \((12-13)\).

4. Study of the integrability conditions

In this section we will exploit the formalism we have just introduced. The best point to start with are the equations for the components of the KVF \((28-37)\) because they are uncoupled and we know all the possible derivatives. Then, it is straightforward to study their integrability conditions. To that end, we have to use the commutators of the directional derivatives \((25)\), which are given by the following expressions

\[
\triangle D - D \triangle = (\gamma + \bar{\gamma})D + (\epsilon + \bar{\epsilon})\Delta - (\tau + \bar{\tau})\delta - (\pi + \bar{\pi})\delta, \tag{44}
\]

\[
\delta D - D \delta = (\bar{\alpha} + \beta - \bar{\pi})D + \kappa \Delta - \sigma \delta - (\bar{\rho} + \epsilon + \bar{\epsilon})\delta, \tag{45}
\]

\[
\delta \Delta - \Delta \delta = -\bar{\nu}D + (\tau - \bar{\alpha} - \beta)\Delta + \bar{\lambda} \delta + (\mu - \gamma + \bar{\gamma})\delta, \tag{46}
\]

\[
\bar{\delta} \delta - \delta \bar{\delta} = (\bar{\mu} - \mu)D + (\bar{\rho} - \rho)\Delta - (\bar{\alpha} - \beta)\bar{\delta} - (\bar{\beta} - \alpha)\delta. \tag{47}
\]

Then, the integrability conditions are obtained by applying these commutators to the components of the KVF, \((\xi_k, \xi_l, \xi_m)\). In this process, derivatives of the spin coefficients and derivatives of the quantities \(\phi + i \beta\) and \(\phi\) will appear. We can use the NP equations [equations (7.28-7.45) in \([1]\)] and the Maxwell equations [equations (38-43)] to eliminate some of them. We have carried out this study and we have found that the integrability conditions are local algebraic relationships involving the complex scalars \(\Psi_A\), the spin coefficients, the components of the KVF, and the quantities \(\phi + i \beta\) and \(\phi\). This contrasts with the usual treatment where, as we have explained before, the integrability conditions involve derivatives of the curvature [see equations (26)]. In the general case, the explicit expressions for the integrability conditions of the equations \((28-37)\) are

\[
2(\Psi_0 \xi_m - \Psi_1 \xi_k) = \kappa (\phi + i \beta), \tag{48}
\]

\[
2(\Psi_0 \xi_l - \Psi_1 \xi_m) = \sigma (\phi + i \beta), \tag{49}
\]

\[
2(\Psi_1 \xi_m - \Psi_2 \xi_k) = \rho (\phi + i \beta) - \kappa \phi + \bar{\kappa} \bar{\phi}, \tag{50}
\]

\[
2(\Psi_1 \xi_l - \Psi_2 \xi_m) = \tau (\phi + i \beta) - \sigma \phi, \tag{51}
\]

\[
2(\Psi_3 \xi_k - \Psi_2 \xi_m) = \pi (\phi + i \beta) + \rho \phi + \bar{\rho} \bar{\phi}, \tag{52}
\]

\[
2(\Psi_3 \xi_m - \Psi_2 \xi_l) = \mu (\phi + i \beta) + \tau \phi, \tag{53}
\]

\[
2(\Psi_4 \xi_k - \Psi_3 \xi_m) = \lambda (\phi + i \beta) + \bar{\delta} \phi + 2 \alpha \phi. \tag{54}
\]
As before, the equations for the regular case are obtained by taking \( \phi + i \beta = 0 \). We can see these expressions as equations for the complex scalars of the Weyl tensor, \( \Psi_A \). Then, analyzing their structure we realize that they can be grouped into four pairs of equations: (58) for \( (\Psi_0, \Psi_1) \), (59) for \( (\Psi_1, \Psi_2) \), (60) for \( (\Psi_2, \Psi_3) \), and (54) for \( (\Psi_3, \Psi_4) \). The important point is that for each pair of equations, considered as equations for the two different complex scalars \( \Psi_A \) that appear, its determinant is proportional (with a non-zero proportional factor) to the quantity \(-2\xi_l\xi_l + 2\xi_m\xi_m\), that is to say, to the norm of the KVF (16), which we have assumed to be non-zero. Hence, we can solve these four pair of equations for the corresponding Weyl complex scalars. Then, we get one expression for the scalars \( \Psi_0 \) and \( \Psi_4 \), and two expressions for the scalars \( \Psi_1, \Psi_2, \) and \( \Psi_3 \):

\[
\Psi_0 = \frac{\phi + i \beta}{N} (\kappa \xi_m - \sigma \xi_k),
\]

\[
\Psi_1 = \frac{\phi + i \beta}{N} (\kappa \xi_l - \sigma \xi_m) = \frac{\phi + i \beta}{N} (\rho \xi_k - \tau \xi_k) + \frac{\phi}{N} (\sigma \xi_k - \kappa \xi_m) + \frac{\delta \xi_k}{N},
\]

\[
\Psi_2 = \frac{\phi + i \beta}{N} (\rho \xi_l - \tau \xi_m) + \frac{\phi}{N} (\sigma \xi_m - \kappa \xi_l) + \frac{\delta \xi_l}{N}
\]

\[
\Psi_3 = \frac{\phi + i \beta}{N} (\mu \xi_m - \pi \xi_l) + \frac{\phi}{N} (\tau \xi_m - \rho \xi_l) + \frac{\delta \xi_l}{N}
\]

\[
\Psi_4 = \frac{\phi + i \beta}{N} (\mu \xi_l - \pi \xi_l) + \frac{\phi}{N} (\tau \xi_l - \rho \xi_m) + \frac{\delta \xi_m}{N}.
\]

These expressions constitute the first important achievement of the proposed formalism. They determine completely the components of Weyl tensor in terms of spin coefficients and quantities constructed from the KVF, and the dependence on these quantities is algebraic. Therefore, they save us to solve the second Bianchi identities, which are the equations for the complex scalars \( \Psi_A \). Instead of that, we only have to substitute the expressions we have just obtained in the second Bianchi identities to obtain a set of consistency relations. Their form will be explained later. Now, it is important to remark the usefulness of the equations (57), first because they can be applied to the question we mention before of establishing connections between Killing symmetries and the algebraic structure of the space-time, and second, because they can applied to any problem involving a KVF and which requires to solve the second Bianchi identities.

The following step is to see how we can exploit this information. The first step is to profit the fact that we have two for expressions for \( \Psi_1, \Psi_2, \) and \( \Psi_3 \). From them we
obtain three relationships between the components of the KVF, spin coefficients, and the quantities $\phi + i \beta$ or $\phi$. The explicit form of these relations is

\[
(-\tau \xi_k - \kappa \xi_l + \rho \xi_m + \sigma \bar{\xi}_m)(\phi + i \beta) = (\kappa \xi_m - \sigma \xi_k)\phi - \bar{\kappa}\xi_m \bar{\phi},
\]

(61)

\[
(\mu \xi_k - \rho \xi_l - \pi \xi_m + \tau \bar{\xi}_m)(\phi + i \beta) = (-\tau \xi_l - \kappa \xi_l + \rho \xi_m + \sigma \bar{\xi}_m)\phi + (\bar{\kappa}\xi_l + \bar{\sigma}\xi_m)\bar{\phi},
\]

(62)

\[
(\nu \xi_k + \pi \xi_l - \lambda \xi_m - \mu \bar{\xi}_m)(\phi + i \beta) = -(2\gamma \xi_k + \rho \xi_l - 2\alpha \xi_m - \tau \bar{\xi}_m)\phi - \bar{\sigma}\xi_l \bar{\phi} - \xi_k \Delta \phi + \xi_m \bar{\delta} \phi.
\]

(63)

In the regular case ($\phi = 0$ and $\phi + i \beta \neq 0$) these relations tell us that the following three complex vector fields must be orthogonal to the KVF $\xi$:

\[
X_1 = -\tau k - \kappa l + \rho m + \sigma \bar{m},
\]

\[
X_2 = \mu k - \rho l - \pi m + \tau \bar{m},
\]

\[
X_3 = \nu k + \pi l - \lambda m - \mu \bar{m}.
\]

It turns out that $X_1$ and $X_3$ are connection 1-forms in the complex vectorial formalism of Cahen, Debever and Defrise [13] (see, e.g., [13]). Moreover, it is possible to see that $X_2$ can be written as the exterior derivative of a complex function, namely

\[
X_2 = \frac{1}{2} d \log(\phi + i \beta).
\]

The consequences for the singular case will be treated in the next section, where all the possible metrics and KVFs with a singular Papapetrou field will be determined.

Now, in the same way that we have dealt with the components of the KVF, we will study the integrability conditions for the Maxwell equations, that is to say, the integrability conditions for $\phi + i \beta$ in the regular case and for $\phi$ in the singular case. In the regular case the situation is like in the case of the components of the KVF because we know explicitly all the directional derivatives of $\phi + i \beta$. Then, applying the commutators ([14-17]) to the equations ([18-11]) and using the NP equations we can write the integrability conditions in the regular case (with $\phi + i \beta \neq 0$) as follows

\[
\delta \tau = (\alpha - \beta) \tau + (\alpha - \beta) \tau + \rho \bar{\mu} - \bar{\rho} \mu,
\]

(64)

\[
\nabla \kappa - \bar{\delta} \pi = \kappa (2\mu - \bar{\mu} + 3\gamma + \bar{\gamma}) + \sigma (\bar{\beta} - 3\alpha - 2\tau - \bar{\tau}) - 2\Psi_1,
\]

(65)

\[
D \pi + \bar{\delta} \rho = \rho (\alpha + \bar{\beta}) - \pi (\epsilon - \bar{\epsilon}) - \bar{\kappa} \mu - \bar{\sigma} \tau,
\]

(66)

\[
\nabla \tau + \delta \mu = -\mu (\alpha + \beta) + \alpha (\gamma - \bar{\gamma}) + \rho \bar{\nu} + \lambda \pi,
\]

(67)

\[
\nabla \pi - \delta \lambda = -\pi (\mu + \gamma - \bar{\gamma}) - \lambda (\alpha - \beta + \tau + (\rho - 2\rho) \nu) - \mu \tau + \Psi_3.
\]

(68)

As we can see, these equations only involve spin coefficients, in the same as the NP equations, therefore we have got more equations for the spin coefficients. Obviously, these integrability conditions can have other alternative but equivalent forms by using the NP equations. In the singular case the situation is a little bit different because we only know two directional derivatives of $\phi$, given by equations ([12,13]). We can study
their compatibility by applying the commutator \((39)\) to \(\phi\). The result is that they are always compatible.

We finish with some comments on the consistency of the expressions for the Weyl complex scalars \((56-60)\) with respect to the second Bianchi identities in the regular case (the singular case will be completely solved in the next section). As we have said before, we must insert these expressions in the second Bianchi identities to see what conditions they impose. It is clear that when we introduce the \(\Psi_A\) in the second Bianchi identities we get expressions containing directional derivatives of the spin coefficients, of the components of the KVF, and of the eigenvalues \(\phi\) and \(\beta\). Since we know all the directional derivatives of the last two sets of variables, they are given in \((28-37)\) and \((38-41)\), they can be eliminated. Therefore, the consistency conditions imposed by the second Bianchi identities can be seen as differential equations for the spin coefficients. Due to the size of these equations we will not write their explicit expressions here. The conclusion of this discussion is that all the integrability, compatibility or consistency conditions can be reduce to differential equations for the spin coefficients. To sum up, the equations for the spin coefficients that we would get come from: (i) The NP equations (see, e.g., \([1]\)). (ii) The integrability conditions for the Maxwell equations [equations \((64-68)\)]. (iii) The equations we would get by introducing the expressions \((56-60)\) for the complex scalars \(\Psi_A\) in the second Bianchi identities. The next step in this study would be to look at the compatibility conditions for the whole set of differential equations for the spin coefficients. In the next section we study two examples in which ways of how to proceed further are shown.

5. Two illustrative examples

In order to show how the formalism we have just described works, in this section we will study two particular cases of the classification given in section 2: (i) Singular Papapetrou fields in vacuum space-times. (ii) Regular Papapetrou fields in Petrov type III vacuum space-times.

5.1. Singular Papapetrou fields

In what follows we will study completely the singular case, i.e., we will determine all the vacuum metrics admitting a KVF whose associated Papapetrou field is singular. We will also determine the possible KVFs for each case. The singular case is characterized by the following relations

\[ \phi + i \beta = 0 \quad \text{and} \quad \phi \neq 0. \]

We start with the analysis of the integrability conditions for the components of the KVF, equations \((28-37)\), which are the expressions for the Weyl complex scalars \(\Psi_A\), equations \((56-60)\). From \((56)\) and \((57)\) we have

\[ \Psi_0 = \Psi_1 = 0, \]

(69)
and therefore, the space-time must be algebraically special (this result was already found in \[8\], theorem 3). Then, the Goldberg-Sachs theorem \[20\] tells us that \( k \) must be geodesic and shearfree (see also \[4, 21, 22\])

\[
\kappa = \sigma = 0 .
\] (70)

Another way of arriving at this result is to use the Mariot-Robinson theorem \[23, 24\], which tells us that a vacuum space-time containing a singular 2-form solution of Maxwell’s equations (without electromagnetic sources) must be algebraically special, and the principal direction of this 2-form must be geodesic and shear-free, that is, equations (69, 70) must hold in that case. Finally, we showed before that (70) is also a consequence of the Maxwell equations for the Papapetrou field. On the other hand, using this result we deduce, from equation (58), that

\[
\Psi_2 = 0 ,
\]

and therefore, taking into account the Petrov classification, we have shown the following result: “The algebraic type of any vacuum space-time possessing a KVF such that its associated Papapetrou field is singular must be III, or N, or O. Moreover, the principal direction of the Papapetrou field must be aligned with the multiple principal direction of the space-time.”

Following with the study, the consequences of the equations (61-63) for the singular case are

\[
\tau \xi_k - \rho \xi_m = 0 ,
\] (71)

\[
(\rho \xi_l - \tau \bar{\xi}_m)\phi + \xi_k(\Delta \phi + 2 \gamma \phi) - \xi_m(\bar{\delta} \phi + 2 \alpha \phi) = 0 .
\] (72)

At this point, we are going to use the freedom in the choice of the NP basis in order to simplify the problem. To that end, it is crucial to consider two differentiated cases depending on whether \( \rho \) is zero or not.

5.1.1. Case \( \rho \neq 0 \). In this case we can use the freedom given by the transformation (10) to have

\[
\tau = 0 .
\]

Looking at the equations giving the change of the spin coefficients (see, e.g., \[1\]) this can be achieved by choosing the function \( E \) in (10) equals to \(-\tau/\rho\). After using this transformation, equation (71) implies that

\[
\xi_m = 0 .
\] (73)

Moreover, we can use the freedom (8) to have

\[
\xi_l = s \xi_k , \quad s^2 = 1 .
\] (74)

Then, taking into account that now

\[
N = -2s \xi_k^2 ,
\] (75)
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the KVF will be timelike (spacelike) when \( s = 1 \) (\( s = -1 \)). From the equations for the components of the KVF \((28-37)\), and using \((73,74)\), we extract the following consequences

\[
\begin{align*}
\epsilon + \bar{\epsilon} &= \gamma + \bar{\gamma} = 0, \\
\pi &= \lambda = \rho - s\bar{\mu} = 0, \\
\phi &= 4s(\alpha + \bar{\beta})\xi_k = 2\nu\xi_k \implies \nu = 2s(\alpha + \bar{\beta}),
\end{align*}
\]

where we have used that \( \xi_k \neq 0 \) [otherwise, from \((75)\), \( N = 0 \)]. The first part of equation \((76)\) tells us that \( k \) is affinely parametrized. Using the remaining freedom in the choice of the NP basis, described by the transformation \((1)\), we can have

\[\epsilon = \gamma = 0,\]

and therefore, since \( \kappa = \epsilon = \pi = 0 \), the NP basis is parallelly transported along \( k \).

Introducing \((77)\) into \((52)\) and taking into account that \( \xi_k \) cannot be zero, we obtain an expression for \( \Psi_3 \)

\[\Psi_3 = 2s\rho(\alpha + \bar{\beta}).\]  

Moreover, if we introduce \((77)\) into the Maxwell equation \((43)\) and use the following equation

\[\delta(\alpha + \bar{\beta}) = 0,\]

which comes from the NP equation \((7.41)\) in [1] and the expression for \( \nu \) \((77)\), we get the condition

\[(\alpha + \bar{\beta})(\bar{\alpha} + 3\beta)\xi_k = 0.\]  

Therefore, there are two possibilities. The first one is \( \alpha + \bar{\beta} = 0 \), which implies, through \((78)\) and \((54)\), that the space-time must be Minkowski and \( \phi = 0 \). Moreover, we can see that these conditions imply \( \rho = 0 \) against our initial assumptions. Then, we must follow the other possibility in \((79)\), i.e., we have to take

\[\alpha = -3\bar{\beta}.\]

At this point, the only independent quantities are: \( \rho, \beta, \xi_k, \) and \( \Psi_4 \). The rest are identically zero or can be expressed in terms of them. From the expressions that we have obtained and from the NP equations, the equations for \( \beta \) can be written as follows

\[
\begin{align*}
D\beta &= \bar{\rho}\beta, \\
\triangle\beta &= -s\bar{\rho}\beta, \\
\delta\beta &= -\frac{1}{4}s\Psi_4, \\
\bar{\delta}\beta &= 0.
\end{align*}
\]

Now, let us study the compatibility of the equations \((80)\) and \((81)\). To that end, we have to consider the Bianchi identity \((7.63)\) in [1]. In our case this equation reads

\[D\Psi_4 = 2\rho(12s\bar{\beta}^2 + \Psi_4) - 4s\bar{\beta}\bar{\delta}\rho.\]
Then, applying the commutator (45) to $\beta$, the compatibility condition we get is
\[ \bar{\rho} \beta^2 = 0. \]
Since we assumed that $\rho \neq 0$, the only possibility is $\beta = 0$, but this, as before, implies that $\alpha = \phi = \Psi_3 = \Psi_4 = 0$, and this implies $\rho = 0$, therefore, we have reached again a contradiction. Therefore, the conclusion is that the case $\rho \neq 0$ is empty.

5.1.2. Case $\rho = 0$. In this case, $\kappa = \sigma = \rho = 0$ and hence, the metric belongs to the Kundt class [23, 26, 27]. As we can see in reference [1] (Chapter 27, section 27.5.1), there are only two possible families of solutions for the Petrov types III and N. In both of them the line element has the form
\[ ds^2 = 2d\zeta d\bar{\zeta} - 2du(dv + Wd\zeta + \bar{W}d\bar{\zeta} + Hdu), \tag{82} \]
where the metric functions are given by:

**Family 1:**

\[ W = W^o(u, \zeta), \quad H = \frac{1}{2}(W_{,\bar{\zeta}} + \bar{W}_{,\zeta})v + H^o, \tag{83} \]

\[ H^o = H^o(u, \zeta, \bar{\zeta}) \quad \text{and} \quad H^o_{,\zeta\bar{\zeta}} - \text{Re}(W_{,\bar{\zeta}}^2 + \bar{W}W_{,\zeta} + W_{,\zeta\bar{\zeta}} + W_{,\zeta u}) = 0. \tag{84} \]

**Family 2:**

\[ W = -\frac{2v}{\zeta + \bar{\zeta}} + W^o(u, \zeta), \quad H = -\frac{v^2}{(\zeta + \bar{\zeta})^2} + \frac{W^o + \bar{W}^o}{\zeta + \bar{\zeta}}v + H^o, \tag{85} \]

\[ H^o = H^o(u, \zeta, \bar{\zeta}) \quad \text{and} \quad (\zeta + \bar{\zeta}) \left( \frac{H^o + \bar{W}^o}{\zeta + \bar{\zeta}} \right)_{,\zeta\bar{\zeta}} = W^o_{,\zeta} \bar{W}^o_{,\zeta}. \tag{86} \]

The NP basis can be constructed as follows: $k$ is aligned with the principal direction of the space-time, so we can take it to be
\[ k = -du, \quad \bar{k} = \frac{\partial}{\partial v}. \tag{87} \]
Using the freedom (10) we can choose $\ell$ to be
\[ \ell = -(Hdu + dv + Wd\zeta + \bar{W}d\bar{\zeta}), \quad \bar{\ell} = \frac{\partial}{\partial u} - H \frac{\partial}{\partial v}, \tag{88} \]
and finally, we can take $m$ as follows
\[ m = -d\zeta, \quad \bar{m} = W \frac{\partial}{\partial u} - \frac{\partial}{\partial \zeta}. \tag{89} \]
Then, using the explicit expressions given in (83-86) for each family we can study whether or not there are solutions and to determine them. To that end we are going to consider the Petrov types III and N (the only possible ones) separately.

In the case of Petrov type III solutions ($\Psi_3 \neq 0$), from equation (52) we have
\[ \xi_k = 0. \]
Taking into account this fact, we have studied the consequences of the equations (28-37) using the expressions given in the Appendix A for the spin coefficients and the scalars $\Psi_3$ and $\Psi_4$ in each family. In this study we have assumed, according to the hypothesis that define this subcase, that $\phi \neq 0$ and $\Psi_3 \neq 0$. After some calculations we have found that these equations are incompatible. Therefore, the conclusion is that there is not any Petrov type III vacuum solution with a KVF having a singular Papapetrou field. Hence, only Petrov N and O vacuum space-times can have a KVF with a singular Papapetrou field.

Now, let us study the case of Petrov type N solutions, characterized by $\Psi_3 = 0$. In this situation, equation (54) implies that
\[ \tau = 0. \]
This condition is very important because it is invariant under any of the freedoms we have in the choice of the NP basis [transformations (8,9,10)]. If we look now at the expressions given in Appendix A for the spin coefficients, we can see that $\tau \neq 0$ for the second family of solutions. Therefore, we can only find solutions in the first family. In this family the condition $\Psi_3 = 0$ implies
\[ W^\rho_{\tilde{\zeta} \tilde{\zeta}} = 0, \]
and using the remaining freedom in the choice of the coordinates (see [1]) we can have $W^\rho = 0$. The resulting line-element corresponds to a $pp$ wave. All the particular classes of these space-times admitting KVFs, apart from the null KVF $\partial/\partial v$, were studied and classified by Ehlers and Kundt [28] (see also [1]). We have studied the consequences of the equations (28-37) and we have found that the solutions allowed belong to the following two classes:

**Class 1:** The only metric function, $H$, can be written as follows
\[ H = f(\zeta) + \bar{f}(\bar{\zeta}), \quad (90) \]
where $f$ is an arbitrary complex function of $\zeta$. This case was already studied in [8]. The KVF is given by
\[ \xi = -dv - 2H du \iff \xi_k = -1, \quad \xi_l = -H, \quad \xi_m = 0, \quad (91) \]
and hence, $N = -2H$. Therefore, the KVF can be either timelike or spacelike. Its associated Papapetrou field is given by
\[ F = 2du \wedge (H_\zeta d\zeta + H_{\bar{\zeta}} d\bar{\zeta}) \implies \phi = 2H_\zeta. \]

**Class 2:** Now, the metric function $H$ has the following form
\[ H = f(u, \zeta) + \bar{f}(u, \bar{\zeta}), \quad (92) \]
where $A$ is an arbitrary complex function of $u$. In this case we have
\[ \xi_k = 0, \quad \xi_m = \xi_m(u), \quad \xi_l = \xi_m \zeta + \bar{\xi}_m \bar{\zeta}, \]
with
\[ A(u) = A(u) \zeta^2. \]
therefore, the KVF is spacelike and is given by
\[ \vec{\xi} = - (\xi_m' \zeta + \bar{\xi}_m' \bar{\zeta}) \frac{\partial}{\partial v} - \xi_m \frac{\partial}{\partial \zeta} - \xi_m \frac{\partial}{\partial \bar{\zeta}}, \quad \xi = \left( \xi_m' \zeta + \bar{\xi}_m' \bar{\zeta} \right) du - \bar{\xi}_m d \bar{\zeta} - \xi_m d \zeta, \] (93)
where \( ' \equiv d/du \). From this expressions, the Papapetrou field is
\[ F = -2 du \wedge \left( \xi_m' d \zeta + \bar{\xi}_m' d \bar{\zeta} \right) \implies \phi = -2\xi_m'. \]

In both classes of solutions the principal direction is \( \partial / \partial v \) [see equation (87)], that is, it is parallel to the null KVF (the normal to the wave fronts) and also to the principal direction of the space-time.

To complete the study of the singular case we have to consider the Petrov type O. Since we are considering vacuum space-times the only possibility is the Minkowski space-time
\[ ds^2 = -2 du dv + 2 d\zeta d \bar{\zeta}. \] (94)

A very convenient NP basis is the following
\[ \vec{k} = \frac{\partial}{\partial v}, \quad \vec{l} = \frac{\partial}{\partial u}, \quad \vec{m} = \frac{\partial}{\partial \zeta}, \quad \vec{\bar{m}} = \frac{\partial}{\partial \bar{\zeta}}. \]

All the spin coefficients associated with this basis vanish and then it is simple to find all the possible KVFs. The most general KVF having a singular Papapetrou field is given by
\[ \xi_k = C_1, \quad \xi_l = \frac{1}{2}(\bar{\phi} \zeta + \phi \bar{\zeta} + C_2), \quad \xi_m = -\frac{1}{2}(\bar{\phi} u + C_3), \]
\[ \bar{\xi} = -\frac{1}{2}(\bar{\phi} \zeta + \phi \bar{\zeta} + C_2) \frac{\partial}{\partial \bar{v}} - C_1 \frac{\partial}{\partial u} - \frac{1}{2}(\bar{\phi} u + C_3) \frac{\partial}{\partial \zeta} - \frac{1}{2}(\bar{\phi} u + C_3) \frac{\partial}{\partial \bar{\zeta}}, \] (95)
where \( C_1, C_2 \) are arbitrary real constants and \( C_3 \) and \( \phi \) are arbitrary complex constants. Then, the Papapetrou field associated with this KVF has the following form
\[ F = -du \wedge (\bar{\phi} d \zeta + \phi d \bar{\zeta}). \]

With this we have finished the study of the singular case, that is to say, we have found all the possible vacuum metrics possessing a KVF with a singular Papapetrou field. To sum up, we have showed the following result: “The only vacuum space-times admitting a KVF whose associated Papapetrou field is singular are the classes of pp waves given by (90) and (92) and the Minkowski space-time (94).” The corresponding KVFs are given in (91), (93) and (95) respectively.

5.2. Regular Papapetrou fields in Petrov type III vacuum space-times

The previous study covers a branch of the classification put forward in section 4. In what follows, we will study another particular case of this classification in which the space-time is Petrov type III and the Papapetrou field is regular. Specifically, we are going to show the following statement: “In any Petrov type III vacuum space-time with a Killing vector field whose Papapetrou field is regular, the multiple principal direction
of the space-time cannot be aligned with either of the two principal directions of the Papapetrou field.”

This means that the algebraic structure of the Papapetrou field cannot be completely adapted to that of the Weyl tensor contrary to what happens, for instance, in the case of the Kerr metric [29], where the two principal directions of the Papapetrou field associated with the timelike KVF coincide with the two principal multiple directions of the space-time (see, e.g., [8]). More precisely, the statement above implies that the only possibility of alignment is between the single direction of the space-time and one of the principal directions of the Papapetrou field. In other words, the cases of the classification corresponding to single and double alignment with the principal multiple direction of the space-time are empty.

The idea to prove the statement is to assume that one of the principal directions of the Papapetrou field, say \( k \), is aligned with the multiple principal direction of the space-time, and then, to reach a contradiction. As is clear, the consequences of the alignment are

\[
\Psi_0 = \Psi_1 = \Psi_2 = 0 .
\]

In this situation, the consequences of the integrability conditions (48-55) [or equivalently, expressions (56-60)] are: First, \( k \) is geodesic and shear-free (this is also a consequence of the Goldberg-Sachs theorem [20])

\[
\kappa = \sigma = 0 . \tag{96}
\]

\( k \) is expansion- and rotation-free

\[
\rho = 0 , \tag{97}
\]

the spin coefficient \( \tau \) also vanishes

\[
\tau = 0 , \tag{98}
\]

and the following remaining conditions

\[
2\Psi_3 \xi_k = \pi(\vartheta + i \beta) , \tag{99}
\]

\[
2(\Psi_3 \xi_l - \Psi_4 \xi_m) = -\nu(\vartheta + i \beta) , \tag{100}
\]

\[
2\Psi_3 \xi_m = \mu(\vartheta + i \beta) , \tag{101}
\]

\[
2(\Psi_3 \bar{\xi}_m - \Psi_4 \xi_k) = -\lambda(\vartheta + i \beta) . \tag{102}
\]

From (96,97) we deduce that the metric must belong to the Kundt class of solutions [23]. And in particular, since we are considering Petrov type III vacuum space-times, it must belong to the families given in the previous subsection [Equations (82-86) and Appendix A]. Moreover, we can use the remaining freedom in the choice of the NP basis, given by transformations (8,9), to have

\[
\epsilon = \alpha = \beta = 0 .
\]
The next step will be to study the stability of the relationships (99-102) under the action of the directional derivatives $D, \triangle, \delta,$ and $\bar{\delta}$. That is to say, we will apply these operators to them and we will use the second Bianchi identities and the NP equations to substitute the directional derivatives of $(\Psi_3, \Psi_4)$ and of the spin coefficients respectively. Finally, we will check whether or not we get new conditions. To that end, we will assume that $\Psi_3 \neq 0$, otherwise the space-time would be of Petrov type N.

Applying the four directional derivatives to (99) we get expressions for the four directional derivatives of $\pi$. Then, applying $D$ to (100), using the expression we have got for $\triangle \pi$, and the equations (99-102), we arrive at the following interesting equation (always assuming that $\alpha + i \beta \neq 0$)

$$\pi \delta \Psi_4 - \mu D \Psi_4 + \Psi_3^2 = 0.$$  

This relation shows that “in Petrov III vacuum space-times with a KVF we cannot have a double alignment of the principal directions of the Weyl tensor with those of the Papapetrou field.” Otherwise, it would imply $\Psi_4 = 0$, and this would imply that $\Psi_3 = 0$, which would lead to the Minkowski space-time, in contradiction with our assumptions.

We could follow further the study of the integrability conditions but since we know that the metric must belong to the families given in (82-86), we can use these expressions, like in the previous subsection, to determine whether or not there are solutions and in the case that there were, to find their exact form. However, there is an important difference with respect to the singular case: Now we do not have the freedom in the choice of the NP basis given by (10) (the direction determined by $\ell$ is fixed because it corresponds now to one of the principal directions of the Papapetrou field), and therefore we cannot fix the NP basis like in (82-86). The only exception is $k$ because it must be aligned with the principal direction of the space-time and hence, it must coincide with (87).

On the other hand, since condition (98) holds we can use the same argument as in the singular case, namely, independently of the directions of $\ell$ and $m$, the condition (98) will still hold (see, e.g., [1] for the formulae for the transformation of the spin coefficients). Therefore, taking into account that in the second family $\tau$ cannot vanish (see Appendix A) we deduce that we can only find solutions in the first family.

For this study we can take a NP basis consisting of (87) and

$$\ell' = \ell + E m + E \bar{m} + E \bar{E} k, \quad m' = e^{2iC} m,$$

where $\ell$ and $m$ are given in (88-89), and $E$ and $C$ are complex and real functions respectively that must be determined. In this situation, we have studied the equations for the components of the KVF (28-37) and we have found that they have not any solution for the Petrov type III. Therefore, this completes the proof of the statement given at the beginning of this subsection.

6. Remarks and conclusions

In this paper we have set up a formalism to study vacuum space-times with a KVF. This formalism exploits the fact that we can associate an algebraic structure with the
KVF through its exterior derivative, the Papapetrou field. Introducing new variables related with the Papapetrou field, and writing all the equations with respect to a NP basis adapted to its algebraic structure, we have obtained a new framework in which we can study the connections between the existence of Killing symmetries and the algebraic structure of the space-time, a subject scarcely studied in the literature. Moreover, this formalism provides, in a natural way, a classification of the space-times with a KVF. The cases in which there are alignments of the principal direction(s) of the Papapetrou field with those of the space-time are, a priori, the most simple to be dealt with this formalism. In this sense, in this paper we have seen that in the case of singular Papapetrou fields there is always alignment and that the class of space-times is very limited. In the case of regular Papapetrou fields we have studied Petrov type III vacuum space-times arriving at the conclusion that there is no possible alignment of the multiple principal direction of the space-time with some of the two principal directions of the Papapetrou field. In contrast with this situation, there are other vacuum space-times in which we can find alignments. An interesting example is the case of the Kerr metric in which the two multiple principal directions of the space-time are aligned with those of the Papapetrou field (see, e.g., [8]). Other cases with alignment have been studied in [30].

On the other hand, it is remarkable the fact that the study of the integrability conditions for the components of the KVF leads directly to explicit expressions for the components of the Weyl tensor in terms of the connection (spin coefficients) and components of the Papapetrou field [(\(\phi\), \(\beta\) or \(\phi\)]. Then, we do not need to solve the second Bianchi identities, but to study the conditions that they impose on the spin coefficients. In conclusion, this formalism is suitable for the study of any problem or situation in which the knowledge of the Weyl complex scalars is required. Some topics in which this formalism may be helpful are: Search and study of exact solutions, perturbations of black holes preserving a symmetry (e.g., axisymmetric or stationary perturbations), the question of the equivalence of metrics, the construction of numerical algorithms, etc.

Finally, it is worthwhile to discuss the possible extensions of this formalism. In this sense it is important to note that the assumption that the space-time would have a vanishing energy-momentum tensor is not fundamental for the development of the formalism. Therefore, a possible way of extending it is to consider other types of energy-momentum content. Other kind of extension would be to consider the present scheme applied to other symmetries, as for instance those generated by conformal KVFs.

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Appendix A. Explicit expressions for the spin coefficients

In this appendix we consider the NP basis \( \{k, \ell, m, \bar{m}\} \) given in equations (87-89), associated with the subclass of Kundt metrics for vacuum and Petrov types III and N, which are determined by expressions (82-86). Using these expressions we can compute the spin coefficients and the non-zero Weyl complex scalars \( \Psi_3 \) and \( \Psi_4 \). After some calculations, the result is

**Family 1:**

\[
\begin{align*}
\kappa &= \epsilon = \sigma = \rho = \tau = \pi = \alpha = \beta = \lambda = 0, \\
\gamma &= \frac{1}{2}W_\zeta, \\
\mu &= \frac{1}{2}(W_\zeta - \bar{W}_\zeta), \\
\nu &= -H^o_\zeta + \bar{W}_{,u} + \frac{1}{2}(\bar{W}_\zeta + W_\zeta)\bar{W} - \frac{1}{2}vW_\zeta\zeta, \\
\Psi_3 &= -\frac{1}{2}W_\zeta\zeta, \\
\Psi_4 &= H^o_\zeta\zeta - \bar{W}W_\zeta\zeta + \frac{1}{2}vW_\zeta\zeta\zeta.
\end{align*}
\]

**Family 2:**

\[
\begin{align*}
\kappa &= \sigma = \rho = \epsilon = \lambda = 0, \\
\tau &= -\pi = 2\alpha = 2\beta = \frac{1}{\zeta + \bar{\zeta}}, \\
\gamma &= -\frac{v}{(\zeta + \bar{\zeta})^2} + \frac{\bar{W}^o}{\zeta + \bar{\zeta}}, \\
\mu &= -\frac{W^o - \bar{W}^o}{\zeta + \bar{\zeta}}, \\
\nu &= -H^o_\zeta + \bar{W}_{,u}^o + \frac{2H^o - v\bar{W}^o_\zeta + \bar{W}^o_\zeta + W^o\bar{W}^o_\zeta}{\zeta + \bar{\zeta}} + \frac{v}{(\zeta + \bar{\zeta})^2}W^o - \bar{W}^o, \\
\Psi_3 &= -\frac{\bar{W}^o_\zeta}{\zeta + \bar{\zeta}}, \\
\Psi_4 &= H^o_\zeta\zeta - \bar{W}^o_\zeta + \frac{v\bar{W}^o_\zeta - 2H^o \zeta - (W^o + 3\bar{W}^o)\bar{W}^o_\zeta}{\zeta + \bar{\zeta}} + 2\frac{H^o + v\bar{W}^o_\zeta + W^o\bar{W}^o}{(\zeta + \bar{\zeta})^2}.
\end{align*}
\]

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