Solving Nonlinear Inverse Problems
Based on the Regularized Modified Gauss–Newton Method

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Abstract—A nonlinear operator equation is investigated in the case when the Hadamard correctness conditions are violated. A two-stage method is proposed for constructing a stable method for solving the equation. It includes modified Tikhonov regularization and a modified iterative Gauss–Newton process for approximating the solution of the regularized equation. The convergence of the iterations and the strong Fejér property of the process are proved. An order optimal estimate for the error of the two-stage method is established in the class of sourcewise representable functions.

Keywords: ill-posed problem, modified Tikhonov method, modified Gauss–Newton method

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1. INTRODUCTION

Consider the inverse problem in the form of a nonlinear ill-posed operator equation
\[ A(u) = f \] (1)
given on two Hilbert spaces \( U, F \). Here, \( A \) is a continuously differentiable operator that has no continuous inverse and the right-hand side \( f \) is specified by its \( \delta \)-approximation \( f_\delta \) such that \( \| f - f_\delta \| \leq \delta \).

To construct a regularized family of approximate solutions, we propose a two-step method in which the equation is first regularized by the modified Lavrentiev–Tikhonov method
\[ A'(u^0)(A(u) - f_\delta) + \alpha B(u - u^0) = 0, \] (2)
and the solution \( u_\alpha \) of Eq. (2) is approximated by applying the modified Newton method
\[ u^{k+1} = u^k - \gamma A'(u^k)^* A(u^k) + \alpha B \] (3)
\[ \times [A'(u^0)^* A(u^0) - f_\delta + \alpha B(u^k - u^0)] \equiv T(u^k), \]
in which the derivative in the step operator \( T(u^k) \) is computed at the fixed point \( u^0 \) in the entire iteration process. That is why iterative process (3) is a modified Gauss–Newton method (GNM) with respect to the operator equation (1). In applications, the operator \( B \) plays the role of an information operator containing important a priori data on the solution (see, e.g., works \cite{1, 2} on thermal sounding of the atmosphere)

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Now we consider the second stage of the method, i.e., analyze the convergence of process (3).

**Theorem 1.** Assume that Eq. (1) is solvable and Eq. (2) is solvable for any \( \alpha, f_0 \). Suppose that the following conditions hold:

(i) \( \| A^t(u^0) \| \leq N_0, \| A(u) - A(u_2) \| \leq N_1 \| u_1 - u_2 \|, \| A^t(u_2) - A^t(u_2') \| \leq N_2 \| u_1 - u_2 \| \)

in a ball \( S(u^0; R) \) containing \( \hat{u}, u_0 \); here, \( \hat{u} \) and \( u_0 \) are solutions of Eqs. (1) and (2), respectively.

(ii) \( B \) is a positive definite operator satisfying the conditions of Lemma 1.

(iii) \( \| u_0 - u_0 \| \leq \rho^0, \rho^0 = \kappa \alpha / (SN_0N_2) \), where \( u_0 \) is the starting point of iterative process (3), which coincides with the trial solution of Eq. (2) and with the point at which the derivative of the operator \( A \) is computed.

Then, for \( \gamma < (\alpha \kappa)/(N_0N_2 + \alpha \| B \|) \), the operator \( T \) implementing process (3) is strongly \( \{u_0\} \)-Fejér in the ball \( S(u_0; r^0) \) and the iterations converge, i.e.,

\[
\lim_{k \to \infty} \| u^k - u_0 \| = 0, \quad (4)
\]

and, for \( \gamma = (\alpha \kappa)/2(N_0N_2 + \alpha \| B \|) \), we have the estimate

\[
\| u^k - u_0 \| \leq q^k \rho^0 \leq q^k \mathcal{F},
\]

where \( \rho^0 \leq \mathcal{F} \) is independent of \( \alpha \).

**Proof.** We introduce the following notation:

\[
D_0 = A^t(u^0)^*A^t(u^0) + \alpha B, \\
F(u) = D_0^{-1}[A^t(u^0)^*A(u) + \alpha B(u - u_0)].
\]

Then

\[
\langle F(u), u - u_\alpha \rangle = \langle F(u) - F(u_\alpha), u - u_\alpha \rangle
\]

\[
= \langle D_0^{-1}[A^t(u^0)^*A(u) - A(u_\alpha)] + \alpha B(u - u_\alpha), u - u_\alpha \rangle
\]

\[
= \langle D_0^{-1}[A^t(u^0)^*A(u) - (A(u) + A^t(u_\alpha)(u - u_\alpha) + \mathcal{E})] + \alpha B(u - u_\alpha), u - u_\alpha \rangle
\]

\[
= \langle D_0^{-1}[A^t(u^0)^*A(u^0) - (A(u) + A^t(u_\alpha)(u - u_\alpha) + \mathcal{E})] + \alpha B(u - u_\alpha), u - u_\alpha \rangle
\]

\[
= \langle D_0^{-1}[A^t(u^0)^*A(u^0) - (A(u) + A^t(u_\alpha)(u - u_\alpha) + \mathcal{E})] + \alpha B(u - u_\alpha), u - u_\alpha \rangle
\]

Taking into account conditions (i) of Theorem 1 and the inequalities

\[
\| D_0^{-1} \| \leq 1/(\kappa \alpha), \quad \| \mathcal{E} \| \leq N_2 \| u - u_\alpha \|^2 / 2,
\]

it follows from (6) that

\[
\| (F(u), u - u_\alpha) \| = \| A^t(u^0)^*A(u) - A^t(u_\alpha) \| \| u - u_\alpha \|^2
\]

\[
= \frac{N_0N_2}{2\kappa \alpha} \| u - u_\alpha \|^2 \
\]

Thus, we obtain the final lower bound

\[
\langle F(u), u - u_\alpha \rangle \geq (1/2) \| u - u_\alpha \|^2.
\]

Additionally, the following upper bound holds:

\[
\| F(u) \|^2 = \| F(u) - F(u_\alpha) \|^2
\]

\[
\leq \frac{(N_0N_1 + \alpha \| B \|^2)}{(\kappa \alpha)^2} \| u - u_\alpha \|^2.
\]

The strong Fejér property of the operator \( T \) means

\[
\| T(u) - u_\alpha \|^2 \leq \| u - u_\alpha \|^2 + \nu \| T(u) - u \|^2 \leq 0
\]

\[
\forall u \in S(u_\alpha; r^0)
\]

for some \( \nu > 0 \), which is equivalent to the inequality

\[
\| F(u) \|^2 \leq \frac{2}{\gamma(1 + \nu)} \langle F(u), u - u_\alpha \rangle.
\]

Combining (8) with (9) yields

\[
\| F(u) \|^2 \leq \frac{2(N_0N_1 + \alpha \| B \|^2)}{(\kappa \alpha)^2} \langle F(u), u - u_\alpha \rangle.
\]

Comparing (11) and (12), we conclude that the operator \( T \) generating iterative process (3) is strongly Fejér for \( \gamma < (\kappa \alpha)/(N_0N_1 + \alpha \| B \|^2) \).
For $\gamma = (\kappa \alpha)/2(N_0N_1 + \alpha \|B\|^2$, the right-hand side of this inequality takes the smallest value, which, for $u = u_k$, implies estimate (5).

3. ERROR ESTIMATION FOR THE TWO-STEP METHOD

**Lemma 2.** Assume that the conditions of Theorem 1 are satisfied. Suppose that Eq. (2) has a solution $u_0$ for any $\alpha > 0$, $f_0 \in F$ and $u_0(\delta)$ belongs to a ball $S(u^0, R)$, where $\alpha(\delta) \to 0$ as $\delta \to 0$. Then the residual of the two-step method satisfies the estimate

$$
\|A'(u^0)(A(v) - f_0)\| \leq [(5N_0N_1 + N_0 \|B\|) R] \alpha(\delta).
$$

Consider the special case $B = I$. To find the error of the approximate solution produced by the two-step method, we need to estimate the error introduced by the regularization method (2). For this purpose, we impose additional conditions on the operator and assume that the solution is sourcewise representable, namely,

(i) assume that $\hat{u}, u_0 \in S(u^0, r^0)$, and there exist a constant $k_0 > 0$ and an element $\phi(u, u^0, w) \in U$ such that, for any $w \in U$ and $u \in S(u^0, r^0)$,

$$
[A'(u) - A'(u^0)]w = A'(u^0)\phi(u, u^0, w),
$$

$$
\|\phi(u, u^0, w)\| \leq k_0 \|u - u^0\| \|w\|;
$$

(ii) the solution is assumed to be sourcewise representable in the class

$$
K = \{\hat{u} : u^0 - \hat{u} = (A'(u^0) A'(u^0))^p v, \|v\| \leq r\},
$$

where $0 < p \leq 1$.

According to Theorem 3.1 from [4] and Lemma 1 from [5], the solution of regularized equation (2) satisfies the estimate

$$
\|u - u_0\| \leq \frac{\max\{1/2, r\}}{1 - \delta^{2/(2p + 1)}} \left(\delta \sqrt{\alpha} + \alpha^p\right),
$$

where $d = k_0 r^0 < 1$.

Equating the terms in parentheses in this inequality, we find the parameter value $\alpha = \delta^{2/(2p + 1)}$ and an estimate for the regularization method (2):

$$
\|u - u_0\| \leq \frac{\max\{1/2, r\}}{1 - \delta^{2/(2p + 1)}} \left(\delta \sqrt{\alpha} + \alpha^p\right),
$$

Combining relations (5) and (13) yields

$$
\|u_k - \hat{u}\| \leq \|u_k - u_0(\delta)\| + \|u_0(\delta) - \hat{u}\| \leq \delta^{2/(2p + 1)} + \delta^{2p/(2p + 1)}.
$$

Equating the terms on the right-hand side of (14), we find an expression for the number of iterations:

$$
k(\delta) = \left[\ln(\delta^{2p/(2p + 1)}/\gamma)/\ln(\delta(\delta))\right].
$$

Substituting $k(\delta)$ into (14) yields an error estimate for the two-step method:

$$
\|u_k - \hat{u}\| \leq 2 \delta^{2p/(2p + 1)}.
$$

Its optimality follows from [6, Lemma 4.2.3, Theorem 4.9.1].

**Remark 1.** In the classical (unmodified) GNM [7, 8], the derivative $A'(u)$ in the step operator is computed at every iteration, while, in the modified GNM (3), it is computed only at the initial point and is stored in the whole iteration process. As a result, the method simplifies significantly in terms of the number of operations, while preserving its optimality in the class of source representable functions.

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**CONFLICT OF INTEREST**

The author declares that he has no conflicts of interest.

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