Onset of Synchronization in Weighted Complex Networks: the Effect of Weight-Degree Correlation

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By numerical simulations, we investigate the onset of synchronization of networked phase oscillators under two different weighting schemes. In scheme-I, the link weights are correlated to the product of the degrees of the connected nodes, so this kind of networks is named as the weight-degree correlated (WDC) network. In scheme-II, the link weights are randomly assigned to each link regardless of the node degrees, so this kind of networks is named as the weight-degree uncorrelated (WDU) network. Interestingly, it is found that by increasing a parameter that governs the weight distribution, the onset of synchronization in WDC network is monotonically enhanced, while in WDU network there is a reverse in the synchronization performance. We investigate this phenomenon from the viewpoint of gradient network, and explain the contrary roles of coupling gradient on network synchronization: gradient promotes synchronization in WDC network, while deteriorates synchronization in WDU network. The findings highlight the fact that, besides the link weight, the correlation between the weight and node degree is also important to the network dynamics.

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In the past decades, complex networks have become an established framework to understand the behavior of a large variety of complex systems, ranging from biology to social sciences [1–3]. In network studies, one important issue is to investigate the influence of the network topology on the dynamics, e.g., the synchronization behavior of coupled oscillators [4]. For many realistic systems, besides the degree, the weight of the network links also presents the heterogeneous distribution, i.e., the weighted networks. Depending on the correlation between the node degree and the link weight, the weighted systems can be roughly divided into two types: weight-degree correlated (WDC) and weight-degree uncorrelated (WDU) networks [3]. While the collective dynamics taking place on weighted networks have been extensively studied [4], but so far, to the best of our knowledge, the impact of weight-degree correlation on the collective dynamics has not been addressed in literature. In this paper, we consider the effect of weight-degree correlation on the onset of synchronization of a generalized Kuramoto model with fixed total coupling cost. Interestingly, it is found that, by increasing the weight parameter that characterizes the distribution of links weights, the synchronization is monotonically enhanced in WDC network, while the synchronization could be deteriorated firstly and then enhanced in WDU network. Moreover, we explain qualitatively the fundamental mechanism from the viewpoint of gradient network. Our findings may be helpful to the design and optimization of realistic dynamical networks.

I. INTRODUCTION

Due to its broad applications, synchronization of coupled phase oscillators has been extensively studied in the past decades [5, 6]. Recently, with the blooming of network science, this study has been extended to the situation of complex network structures, i.e., the generalized Kuramoto model [7–13]. Different to the traditional studies of regular networks, in the generalized Kuramoto model more attention has been paid to the interplay between the network topology and dynamics. Considering the fact that links in realistic networks are generally of different strength [16–19], more recently the study of onset of synchronization in weighted complex networks has arisen certain interest, where a number of new phenomena have been identified [20–29].

For weighted networks in reality, a typical feature is that the weights of the network links possess a large variation. In particular, it is found that in many realistic networks the link weights roughly follow power-law distributions, say, for instance, the airport network [30], the scientific collaboration network [31], and the mobile communication network [32], to name just a few. In the previous studies of weighted networks, the heterogeneity of link weights has been attributed to the heterogeneous network structure, e.g., the heterogenous degree distribution, and the weight of a link in general is set to be proportional to the product of the degrees of the nodes it connects. For this feature, we call this type of network as the weight-degree correlated (WDC) network, which is often observed in the biological and technological systems, e.g., the E. Coli metabolic network [33] and the airport network [30, 33]. Lately, with the progress of network science, it is also found that some realistic networks, while the link weights also possess a power-law distribution, the link weight and node degree

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are not closely correlated, e.g., the scientific collaboration and email networks [30, 34]. More specifically, in this type of networks the link weights seem to be randomly chosen within a certain range, regardless of the node degrees. For this property, we call this type of network as the weight-degree uncorrelated (WDU) network. Our main task in the present paper is to investigate the dynamical behaviors of these two different types of weighted networks, with the purpose of exploring the effect of weight-degree correlation on the network dynamics.

To study the effect of weight-degree correlation on network dynamics, we will employ the model of networked phase oscillators, i.e., the generalized Kuramoto model, and study the onset of synchronization between the following two weighting schemes. In scheme-I, the coupling weight between a pair of directly connected nodes in the network is set to be proportional to the degrees of these two nodes, so as to generate a WDC-type network. In scheme-II, the weights of the network links also have a power-law distribution, but the weights are randomly distributed among the links, regardless of the node degrees. This gives rise to a WDU-type network. Our main finding is that, by tuning a weight parameter (to be defined later) that governs the heterogeneity of the link weights, the two types of networks have very different synchronization responses. In particular, in WDC-type network, with the increase of the weight parameter, the network synchronization is found to be monotonically enhanced; while in WDU-type network, it is found that the increase of the weight parameter may either suppress or enhance the network synchronization.

We analyze the different synchronization responses of the two networks from the viewpoint of gradient network [25, 33], and get the general conclusion that synchronization is enhanced by coupling gradient in WDC network, while is suppressed in WDU network. These findings highlight that, besides the link weight, the correlation between the link weight and the node degree should be also taken into account when analyzing the network dynamics.

The rest of this paper is organized as follows. In Sec. II we will introduce our model of coupled phase oscillators, as well as the two types of weighting schemes. In Sec. III we will show the transition paths from non-synchronous to synchronous states for the two types of networks, in which the critical coupling strength characterizing onset of synchronization will be defined. In Sec. IV we will study in detail the effect of weight-degree correlation on the network synchronization, in which the reversed synchronization in WDU network will be highlighted. Also, based on the concept of gradient complex network, we will give an analysis to the synchronization performance of the two types of networks. In Sec. V we will give our discussion and conclusion.

II. THE MODEL

In our study, we employ the following two typical models for the network structure: the small-world and scale-free networks. The small-world network is constructed by the method introduced in Ref. [36], in which each node has 2m links and each link is rewired randomly with a probability $p_r$. The scale-free network is generated by the standard BA growth model [37], which has average degree $\langle k \rangle = 2m$, and the node degrees roughly follow a power-law distribution, $P(k) \sim k^{-\gamma}$, with $\gamma = 3$.

For illustration, we set the weights in the two types of networks as follows. For the WDC network, motivated by the empirical observation in scientific collaboration networks [30] and email networks [34], we set the weight, $c_{mn}$, between the pair of directly connected nodes, $m$ and $n$, to be

$$c_{mn} \sim (k_m k_n)^\alpha,$$

where $k_m$ and $k_n$ are the degrees of the nodes $m$ and $n$, respectively, and $\alpha \in R$ is a tunable parameter characterizing the weight distribution, i.e. the weight parameter. For the WDU network, motivated by the empirical observation in scientific collaboration networks [30] and email networks [34], the weight of each link is randomly chosen within the range $[c_{min}, c_{max}]$, while keeping the weight distribution to be having the power-law scaling,

$$P(c) \sim c^{-\beta}.\quad (2)$$

In WDU network, the weight parameter is $\beta$, which, according to the empirical observations, is defined to be positive.

Arranging each node in the network with a phase oscillator and regarding the weight as the coupling strength between the connected nodes, we now can study the dynamical behaviors of the two types of networks constructed above. Our model of coupled phase oscillators reads

$$\dot\phi_m = \omega_m + \varepsilon \sum_{n=1}^{N} c_{mn} \sin(\phi_n - \phi_m),\quad (3)$$

with $N$ the network size, and $m, n = 1, \ldots, N$ the node indices. $\phi_m$ and $\omega_m$ represent, respectively, the phase state and the intrinsic frequency of the $m$th oscillator (node). The weights of the network links are represented by the matrix $\{c_{mn}\}$, with $c_{mn} = 0$ if $m$ and $n$ are not directly connected. $\varepsilon$ is a uniform coupling strength which, for the sake of convenience, is normalized by the node weight intensity $s_m = \sum_{n=1}^{N} c_{mn}$. Our main task is to investigate how the weighting schemes, i.e. weight-degree correlated or non-correlated, will affect the network dynamics.

In next simulations, we fix the network size as $N = 6400$ and the average degree as $\langle k \rangle = 2m = 6$. In constructing the small-world network, we set the rewiring probability $p_r$ as 0.2. For each network example, the intrinsic frequency, $\{\omega_i\}$, of the network oscillators is randomly chosen from the Gaussian distribution $g(\omega) = (2\pi \sigma^2)^{-1/2} \exp(-\omega^2/2\sigma^2)$, which has zero mean value and the unit variance ($\sigma^2 = 1$). Also, the initial states of the oscillators are randomly chosen from the interval $[0, 2\pi]$. With these settings, we then set a value for the uniform coupling strength, $\varepsilon$, and evolve the network according to Eq. (3). The equations are integrated by the fourth-order Runge-Kutta method and the time step is $\Delta t = 0.01$. After discarding a transient process of 2000 time steps, we start to measure the collective behavior of the network, which
Each data is averaged over 20 network runs. is characterized by the network order parameter [9, 13]

\[ R = \langle |\sum_{m=1}^{N} d_m e^{i\phi_m}| \rangle. \]  

Here \( \langle \ldots \rangle \) represents that the result is averaged over a period of 2000 time steps, and \( [\ldots] \) represents the average of 20 network runs. \( d_m \) is the total incoming coupling strength of the node \( n \), which, due to the normalization operation in the model, is unit. By scanning the coupling strength \( \varepsilon \), we can monitor the behavior of \( R \), so as to find the critical coupling strength, \( \varepsilon_c \), where the onset of network synchronization occurs.

III. ONSET OF SYNCHRONIZATION IN WEIGHTED COMPLEX NETWORKS

We start by demonstrating the typical path from non-synchronous to synchronous states in non-weighted complex networks, i.e. the case of \( \alpha = 0 \) in Eq. (1), so as to define the onset of network synchronization. As plot in Fig. 1 (the curve for \( \alpha = 0 \)), when the coupling strength is small, e.g. \( \varepsilon \approx 0 \), the trajectories of the oscillators are almost uncorrelated, and we roughly have \( R \approx O(1/\sqrt{N}) \) for the order parameter. This value of \( R \) keeps almost unchanged till a critical coupling strength, \( \varepsilon_c \), is met. After that, as \( \varepsilon \) increases from \( \varepsilon_c \), the value of \( R \) will be gradually increased and, when \( \varepsilon \) is large enough, it reaches the limit \( R = 1 \) (the state of complete network synchronization). The value \( \varepsilon_c \) is then defined as the onset point for synchronization, which, for the case of non-weighted scale-free network, is about 0.2 (Fig. 1). In numerical simulations, we define the onset of network synchronization as the point where \( R^2 > 2 \times 10^{-2} \). In what follows, we will study how the value of \( \varepsilon_c \) is changed by the weight parameters, \( \alpha \) and \( \beta \), in the two types of networks.

![Figure 1](image1.png)

**FIG. 1:** (Color online) Under the WDC scheme, the variation of the synchronization order parameter, \( R^2 \), as a function of the coupling strength, \( \varepsilon \), for different weight parameters, \( \alpha \). The network is generated by the standard BA model (scale-free network), which has size \( N = 6400 \) and average degree \( \langle k \rangle = 6 \). Onset of synchronization is identified as the coupling strength where \( R^2 \) starts to increase from \( 2 \times 10^{-2} \). The error bars are estimated by the standard deviation. Each data is averaged over 20 network runs.

![Figure 2](image2.png)

**FIG. 2:** (Color online) Under the WDU scheme, the variations of the synchronization order parameter, \( R^2 \), as a function of the coupling strength, \( \varepsilon \), for scale-free networks (a) (c) and small-world networks (b). As a reference, the synchronization of non-weighted network (Binary) is also plotted in each subplot. All the networks have size \( N = 6400 \) and average degree 6. In constructing the WDU network, each link is arranged a weight chosen randomly from the range \([1, 100]\) for (a) and (b), and from the range \([1, 100000]\) for (c). The rewiring probability in generating the small-world networks is \( p_r = 0.2 \). The error bars are estimated by the standard deviation, and each data is averaged over 20 network runs.

We first check the influence of the weight parameter, \( \alpha \), on WDC-type networks. The results are carried for scale-free networks and are plotted in Fig. 1. It is shown that, as the weight parameter increases from -3 to 5, the network synchronization is monotonically enhanced. In particular, the onset of synchronization, i.e. the value of \( \varepsilon_c \), is found
to be gradually shifted to the small values. For instance, when \( \alpha = -3 \), the critical coupling strength is about 7, while this value is decreased to about 0.9 when \( \alpha = 3 \). Another observation in Fig. 1 is that, in the regime of \( \alpha < -1 \), a small change of the weight parameter could induce a significant change to the network synchronization; while in the regime of \( \alpha > -1 \) the similar change in \( \alpha \) has relatively small influence. For instance, by increasing \( \alpha \) from \(-3\) to \(-2\), the critical coupling strength is decreased from 7 to 4.1; while it is almost unchanged when \( \alpha \) is increased from 3 to 5 (see Fig. 1).

We next investigate the influence of the weight parameter on WDU-type networks. In Fig. 2(a) and (c), we plot the variation of the order parameter as a function of the coupling strength for the scale-free network. It is found that, similar to the case of WDC networks, the network synchronization is enhanced. The influence of gradient on the onset of synchronization of coupled phase oscillators has been also investigated \cite{13}, where an important finding is that, besides the gradient strength, the gradient direction also plays a key role in influencing the network synchronization. In particular, in Ref. \cite{13} it is demonstrated that when the gradient direction is pointing from the larger-degree to smaller-degree nodes on each link, the increase of the gradient strength will enhance the synchronization monotonically; while if it is in the opposite direction, the increase of the gradient strength will suppress the synchronization. Noticing of these effects of the weight gradient and gradient direction on the network synchronization, we in the following explore the properties of the gradients in our models of WDC and WDU networks, with a hope to explain the numerical observations in Figs. 1 and 2.

Following the method of Ref. \cite{25}, we construct the gradient networks in our models, as follows. The Eq. \eqref{Eq3} can be rewritten as

\[
\dot{\phi}_m = \omega_m + \varepsilon \sum_{n=1}^{N} g_{mn} \sin(\phi_n - \phi_m),
\]

with \( g_{mn} = c_{mn}/s_m \) the normalized coupling strength, which captures the real coupling strength that node \( m \) receives from node \( n \). The reason for the normalized coupling strength is simply for convenience, since in this way the total network coupling strength will be keeping as constant and is not changing with the weight parameters. That is to say, in our models the change of network synchronization is caused solely by the weight distribution, instead of the change of the coupling cost.

The coupling gradient, \( \Delta g_{mn} \), flowing from node \( n \) to node \( m \) under the condition \( k_n > k_m \) is written as

\[
\Delta g_{mn} = g_{mn} - g_{nm} = \frac{c_{mn}}{s_m} - \frac{c_{mn}}{s_n},
\]

where \( c_{mn} = c_{nm} \) is the weight before the normalization operation, and \( s \) is the node weight intensity defined in Sec. II. Extracting only the gradient element on each of the network links, together with the network nodes, we can construct the gradient network. In our study, the positive direction of the gradient is defined as pointing from the larger-degree to smaller-degree nodes on each link. With this definition, we can measure the gradient at the network level, i.e. the averaged network gradient, which is

\[
\langle \Delta g \rangle = \frac{\sum_{mn,k_n<k_m} \Delta g_{mn}}{M},
\]

where \( \langle \ldots \rangle \) means that the value is averaged over the network links (totally there are \( 2M = N(k) \) links in the network). Please note that the summation in Eq. (7) is under the condition of \( k_m < k_n \), which is based on the direction of coupling gradient defined above. Our next job is to analyze how the tuning of the weight parameter will change the gradient network and, in turn, affect the network synchronization.

IV. THE INFLUENCE OF WEIGHT-DEGREE CORRELATION ON NETWORK SYNCHRONIZATION

In studying the dynamics of directed and weighted complex networks, an effective method is to analyze the properties of the weight gradient \cite{25,26}. For a directed network, the pair of weights on a link in general is different from each other, which causes a gradient (bias) among the connected nodes. The idea behind gradient analysis is to extract the asymmetric part (gradient) on each link, and analyze its functions to the network dynamics separately. In this way, the original network can be virtually regarded as a superposition of two subnetworks: one constituted by the symmetric links and the other one constituted by gradient links. In Ref. \cite{25}, the later is also called the gradient complex network.

Gradient network has been employed in analyzing the network synchronization in literature. In Ref. \cite{25} it is shown that, with the increase of the gradient strength, the complete synchronization in scale-free network can be significantly enhanced. The influence of gradient on the onset of synchronization of coupled phase oscillators has been also investigated \cite{13}, where an important finding is that, besides the gradient strength, the gradient direction also plays a key role in influencing the network synchronization. In particular, in Ref. \cite{13} it is demonstrated that when the gradient direction is pointing from the larger-degree to smaller-degree nodes on each link, the increase of the gradient strength will enhance the synchronization monotonically; while if it is in the opposite direction, the increase of the gradient strength will suppress the synchronization. Noticing of these effects of the weight gradient and gradient direction on the network synchronization, we in the following explore the properties of the gradients in our models of WDC and WDU networks, with a hope to explain the numerical observations in Figs. 1 and 2.

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\[
\langle \Delta g \rangle = \frac{\sum_{mn,k_n<k_m} \Delta g_{mn}}{M},
\]

where \( \langle \ldots \rangle \) means that the value is averaged over the network links (totally there are \( 2M = N(k) \) links in the network). Please note that the summation in Eq. (7) is under the condition of \( k_m < k_n \), which is based on the direction of coupling gradient defined above. Our next job is to analyze how the tuning of the weight parameter will change the gradient network and, in turn, affect the network synchronization.
A. WDC-type Networks

We first analyze the synchronization of WDC network. For this type of weighting scheme, the node weight intensity is
\[ s_m = k_m^{\alpha} \sum_{n \in \mathcal{N}} k_n^{\alpha} = k_m^{1+\alpha} (k_n^{\alpha})_m. \]
According to our definitions above, the coupling gradient between the pair of connected nodes, \( m \) and \( n \), is,
\[ \Delta g_{mn} = \frac{e_{mn}}{s_m} - \frac{e_{nm}}{s_n} = \frac{k_m^{\alpha} k_n^{\alpha}}{k_m^{1+\alpha} (k_n^{\alpha})_m} - \frac{k_m^{\alpha} k_n^{\alpha}}{k_m^{1+\alpha} k_n (k_n^{\alpha})_n}. \]  

For scale-free networks generated by the BA standard model (the model we have adopted for WDC network in Fig. 1), the degrees of the nodes are not correlated, and we can roughly treat \( (k_n^{\alpha})_m \) as a constant, \( (k_n^{\alpha})_m = (1/k_m) \sum k_n^{\alpha} \). Using this approximation, the above equation can be simplified into
\[ \Delta g_{mn} \sim \frac{k_m^{\alpha} k_n^{\alpha}}{k_m^{1+\alpha}} - \frac{k_m^{\alpha} k_n^{\alpha}}{k_m^{1+\alpha} k_n (k_n^{\alpha})_n}. \]  

As shown by the above equation, the coupling gradient arises naturally when one of the conditions, \( \alpha \neq -1 \) or \( k_m \neq k_n \), is satisfied, which is generally met in weighted complex networks.

Separated by the value \( \alpha = -1 \), the function of the gradient at the two sides of the parameter space is completely different. Assuming that \( k_m < k_n \) in Eq. (9), in the region of \( \alpha < -1 \), we statistically have \( s_m > s_n \) and, consequently, \( \Delta g_{mn} < 0 \). Since \( s_m \propto k_m \), the gradient thus is pointing from the smaller-degree to larger-degree nodes. According to the result of Ref. 12, this negative gradient will suppress synchronization and, moreover, in the negative regime with the increase of the gradient strength, \( \langle \Delta g_{mn} \rangle \), the value of \( \varepsilon_c \) will be monotonically decreased. In a similar fashion, in the region of \( \alpha > -1 \), the gradient is pointing from the larger-degree to smaller-degree nodes, i.e., the gradient is positive. In this case, with the increase of the gradient strength, the value of \( \varepsilon_c \) will be also monotonically decreased. Combining the behaviors of \( \varepsilon_c \) in the two regimes, we thus predict a continuous decrease of the critical coupling strength as a function of the weight parameter, \( \alpha \), in the WDC network.

The above predictions are verified by numerical simulations. By the scale-free network used in Fig. 1, we plot the variation of \( \varepsilon_c \) as a function of \( \alpha \) in Fig. 3(a). Clearly, with the increase of \( \alpha \) the network synchronization is found to be monotonically enhanced (a continuous decrease of \( \varepsilon_c \)). In the inset of Fig. 3(a), we also plot the variation of the averaged network gradient, \( \langle \Delta g \rangle \), as a function of \( \alpha \), where a continuous increase of \( \langle \Delta g \rangle \) is evident. The enhanced synchronization by coupling gradient is more clearly shown in Fig. 3(b), in which the value of \( \varepsilon_c \) is found to be monotonically decreased with the increase of the averaged network gradient.

Our findings in WDC network, i.e., the results in Fig. 3 might give indications to the design of realistic networks where synchronization is importantly concerned, e.g., the world-wide airport network 30 and the metabolic network 33. In a practical situation, due to the high cost of maintaining the weight gradient, a tradeoff between the network performance (synchronization) and the weight (coupling) gradient will be necessary. The results of Fig. 3(a) suggest that in the region of \( \alpha < 2 \), the network synchronization can be significantly improved by a small gradient cost (small change of \( \alpha \)); while in the region of \( \alpha > 2 \), to achieve the same performance improvement, the gradient cost could be significantly large. This might be one of the reasons why some realistic networks are found to be possessing weight parameter \( \alpha \approx 0.5 \), e.g., the world-wide airport network 30 and the E. coli metabolic network 33.

B. WDU-type Networks

We next analyze the synchronization of WDU network. As indicated in Fig. 2 in this type of networks with the increase of the weight parameter, \( \beta \), the network synchronization may be either enhanced or suppressed. To have a close look to the relationship between \( \varepsilon_c \) on \( \beta \), we increase \( \beta \) from 0 to 4 gradually, and checking the variation of \( \varepsilon_c \). The numerical results carried on scale-free networks are plotted in Fig. 4(a). In this figure, it is clearly shown that at about \( \beta_c \approx 1.5 \), there exists a maximum value for \( \varepsilon_c \). It is worthwhile to note that the cap-shape variation of \( \varepsilon_c \) may have implication to the function of some realistic networks where network desynchronization
Fig. 2(a), under WDU scheme, the variation of the critical coupling strength, $\varepsilon_c$, as a function of the weight parameter, $\beta$, in (a) and the averaged network gradient, $\langle|\Delta g|\rangle$, in (b). Insets: $\langle|\Delta g|\rangle$ versus $\beta$. Each data is averaged over 20 network runs.

Fig. 4: (Color online) For the same scale-free network as used in Fig. 2(a), under WDU scheme, the variation of the critical coupling strength, $\varepsilon_c$, as a function of the weight parameter, $\beta$. In (a) and the average network gradient, $\langle|\Delta g|\rangle$, in (b). Insets: $\langle|\Delta g|\rangle$ versus $\beta$. Each data is averaged over 20 network runs.

Fig. 5: (Color online) For the same small-world network as used in Fig. 2(b), under WDU scheme, the variation of the critical coupling strength, $\varepsilon_c$, as a function of the weight parameter, $\beta$. In (a) and the average network gradient, $\langle|\Delta g|\rangle$, in (b). Insets: $\langle|\Delta g|\rangle$ versus $\beta$. Each data is averaged over 20 network runs.

Instead of synchronization, is desired. Say, for instance, in the mobile communication network, simultaneous communications may cause the information congestion and should be always avoided. An interesting empirical observation is that, in this weighted network, the weight of the network links follows a roughly pow-law distribution with an exponent $\beta \approx 1.9$. This value is quite close to $\beta_c$ in Fig. 4(a), which might indicate the possible application of weight-degree correlation in affecting network dynamics for this specific network.

We now analyze the synchronization behavior of WDU network from the viewpoint of gradient network. It should be noted that, different to the case of WDC network, in WDU network the direction of the gradient is not uniquely defined. That is, the gradients on the network links may either point from the larger-degree node to the smaller-degree node, or pointing in the opposite direction. As a matter of fact, in WDU network the gradient directions are determined by the weight details, instead of node degrees. That is, the positive and negative gradients are coexisting in a WDU network.

This arises a problem in using the averaged network gradient for the synchronization analysis, since for a large-scale WDU network, due to the random weight arrangement, we roughly have $\langle|\Delta g|\rangle \approx 0$. Regarding of this, in WDU network we use the averaged gradient amplitude, $\langle|\Delta g|\rangle = \frac{\sum_{m,n}|\Delta g_{mn}|}{M}$, to characterize the gradient at the network level. With this adoption, our task now is shifted to exploring the relationship between $\varepsilon_c$ and $\langle|\Delta g|\rangle$.

We first check the variation of $\langle|\Delta g|\rangle$ as a function of $\beta$. The results are plotted in the inset of Fig. 4(a). Different to the scale-free network, it is observed that here the variation of $\langle|\Delta g|\rangle$ has a cap-shape structure. (For WDC network, as $\alpha$ increases from 0, the value of $\langle|\Delta g|\rangle$ is monotonically increased, see Fig. 3(a).) In particular, at the critical parameter $\beta_c$, we also find a maximum value for $\langle|\Delta g|\rangle$. The similar behavior of $\varepsilon_c$ and $\langle|\Delta g|\rangle$ seems to indicate the following: gradient deteriorates synchronization in WDU network. To check this out, we plot in Fig. 4(b) the dependence of $\varepsilon_c$ on $\langle|\Delta g|\rangle$. Now it is clearly seen that, as the network gradient increases, the network synchronization is monotonically suppressed.

The suppressed synchronization by gradient is also found in small-world networks under the WDU scheme. With the small-world network model used in Fig. 2 we calculate the variation of the critical coupling strength as a function of the weight parameter. The results are presented in Fig. 5(a). Again, the maximum value of $\varepsilon_c$ is found at some intermediate value of $\beta$. Similar to the case of scale-free network (Fig. 4), the averaged gradient amplitude, $\langle|\Delta g|\rangle$, also shows a cap-shape structure (inset of Fig. 5(a)), with maximum value of $\langle|\Delta g|\rangle$ appears at $\beta_c$. The relationship between $\varepsilon_c$...
and $\langle |\Delta g| \rangle$ is plotted in Fig. 6(b), where the suppression of synchronization by gradient is evident.

C. Mechanism analysis

Why synchronization is suppressed by gradient in WDU network, while is enhanced in WDC network? To answer this, we compare the structures of the gradient networks generated in the two types of networks, and analyzed their functions to the network synchronization. For a WDC network, according to the study of Ref. [26], the gradient couplings of the network are organized into a spanned tree. More specifically, at the top of the gradient network there locates the largest-degree node of the network, which sending out gradients to its nearest neighbors (the second level). Each node at the second level is then sending out gradients to its own nearest neighbors (the third level), given that it has a larger degree to its neighbors. In this way, all the network nodes will be organized into a one-way coupled tree structure. An important feature of this gradient tree is that, starting from the rooting node (the largest-degree node), every node of the network can be reached by following the gradient couplings. Previous studies on gradient network have shown that [25], for such a spanned gradient tree, the network synchronizability will be monotonically increased with the increase of the gradient strength. From Eqs. (1) and (6), it is straightforward to see that in our model of WDC network, the gradient strength is increasing with the parameter $\alpha$ in a monotonic fashion. We thus attribute the enhanced synchronization in WDC network to the existence of a spanned gradient tree.

To better describe the spanned gradient tree in WDC network, we adopt the simplified network model plotted in Fig. 6(a), which might be also regarded as a motif of a large-scale network. According to our model (the Eq. (3)), under WDC scheme the gradient direction will be pointing from larger-degree node to smaller-degree node on each link. Following this rule, we can construct the gradient network, which is schematically shown in Fig. 6(b). Please note that, to illustrate, here we only show the gradient direction, while neglecting the gradient strength. In Fig. 6(b), it is clearly seen that, leading by node 1, all the network nodes are connected by the gradient couplings in a hierarchical fashion. It is important to note that, while the exact gradient strength of each link is modified by the parameter $\alpha$, the hierarchical structure of the spanned gradient tree is unchanged. In particular, in the gradient network node 4 is always reachable by node 1, by passing the nodes 2 and 3.

We go on to study the gradient couplings in WDU network. To manifest the feature of weight-degree non-correlation, and also for the sake of simplicity, we keep all other weights of the network links the same to those of Fig. 6(b) (WDC network), while changing only the weight between nodes 3 and 4 by a large increment, $\Delta$. Taking into account this single change of the link weight, we now reanalyze the gradient couplings of the network. Since all other weights of the network are unchanged, all the gradients are the same to that of Fig. 6(b), except the one on the link between nodes 2 and 3. According to our model of Eq. (3), the gradient from node 2 to node 3 is $\Delta g_{23} = c_{23}(1/s_3 - 1/s_2)$, with $s$ the weight intensity defined in Sec. II. In WDC network (Fig. 6(b)), because $s_2 > s_3$, we thus have $\Delta g_{23} > 0$. This direction of gradient, however, will be switched in WDU network if $\Delta$ is large enough (Fig. 6(c)). In the WDU network, the gradient from node 2 to node 3 is $\delta g_{23} = c_{23}(1/s_3' - 1/s_2')$, with $s_3' = s_3 + \Delta$. It is not difficult to find that, as long as $\Delta > s_2 - s_3$, the gradient will be pointing from node 3 to node 2. That is, the gradient direction is switched. Once switched, the gradient network will be separated into two sub-trees: one led by node 1 and the other one led by node 3. Previous studies have shown that [26], for such a breaking network, the network synchronizability will be monotonically deteriorated by increasing the gradient strength. Based on this analysis, we attribute the deteriorated synchronization in WDU networks to the breaking of the gradient network.

V. DISCUSSION AND CONCLUSION

The main purpose of the present paper is to highlight the important role of the weight-degree correlation played on the network synchronization. While synchronization in weighted complex network has been extensively studied in literature, to the best of our knowledge, it is the first time that the correlation between the link weight and node degree be investi-
gated. To demonstrate the effect of weight-degree correlation on the network dynamics, we have employed the generalized Kuramoto model as the platform and, following the tradition, adopting the scheme of normalized coupling strength. The main findings of the paper, however, are general and are expected to be observable for other network models and coupling schemes as well, say, for instance, the complete network synchronization.

Motivated by the empirical observations, we have adopted two different weighting schemes for the WDC and WDU networks. The weight parameters, $\alpha$ and $\beta$, thus have different meanings and, in general, should not be directly compared. For this regard, the comparison we have made in the present work is actually on the different dynamical responses of the two types of networks to the change of the weight heterogeneity, which should not be understood as a comparison of the network synchronizability. Direct comparison of the two weighting schemes might be possible for the BA-type scale-free networks, in which we roughly have $\beta = -\gamma/(2\alpha)$. However, to keep this relation, in numerical simulations the network size should be very large (larger than $1 \times 10^6$), which is out of our current computational ability.

While gradient coupling has been identified as important for network synchronization, previous studies have been mainly concentrated on the type of WDC network [28]. As such, a general observation is that, by increasing the gradient strength, the network synchronizability is monotonically increased. This observation, as we show in the present paper, is invalid for the case of WDU network. In particular, in WDU network we find that, instead of enhancement, the increase of the gradient could actually suppress synchronization under some conditions. This finding could be helpful to the design of realistic networks which, depending on the network functionality, may either in favor of network synchronization or network desynchronization. Now, besides the aspect of link weight, to manipulate the network synchronization, we can also adjust the correlation between the link weight and the node degree.

We wish to note that, despite of the recent progresses in the onset of synchronization of networked phase oscillators, at the present stage we are still short of a theory to predict precisely the point of onset of synchronization [10]. This is specially the case when the link weight is of heterogeneous distribution, e.g., the WDC and WDU networks discussed here. While our main findings are based on numerical simulations, it is our believing that these findings are general and stand for other network models and dynamics as well.

In summary, we have investigated the onset of synchronization in weighted complex networks, and explored the influence of weight-degree correlation on the network synchronization. An interesting finding is that, as the weight parameter increases, in WDC network the synchronization is monotonically enhanced, while in WDU network the synchronization could be either enhanced or deteriorated. This finding is obtained from extensive simulations, and is explained from the viewpoint of gradient network. Our study shows that, besides the link weight, the weight-degree correlation is also an important concern of the network dynamics.

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[1] R. Albert and A.-L. Barabási, Rev. Mod. Phys. 74 47 (2002).
[2] M. E. J. Newman, SIAM Rev. 45 167 (2003).
[3] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez and D.-U. Hwang, Phys. Rep. 424 175 (2006).
[4] A. Arenas, A. Diaz-Guilera, J. Kurths, Y. Moreno, and C. Zhou, Phys. Rep. 469, 93 (2008).
[5] Y. Kuramoto, Chemical Oscillations, Waves and Turbulence (Springer-Verlag, Berlin, 1984).
[6] J. Acebron, L. Bonilla, C. Vicente, F. Ritort, and R. Spigler, Rev. Mod. Phys. 77, 137 (2005).
[7] Y. Moreno and A. E. Pacheco, Europhys. Lett. 68, 603 (2004).
[8] T. Ichinomiya, Phys. Rev. E 70, 026116 (2004).
[9] J. G. Restrepo, E. Ott, and B. R. Hunt, Phys. Rev. E 71, 036151 (2005).
[10] J. G. Restrepo, E. Ott, and B. R. Hunt, Chaos 16, 015107 (2005).
[11] J. Gomez-Gardenes, Y. Moreno, and A. Arenas, Phys. Rev. Lett. 98, 034101 (2007).
[12] S. Guan, X. Wang, Y.-C. Lai, and C.-H. Lai, Phys. Rev. E 77, 046211 (2008).
[13] X. Wang, L. Huang, S. Guan, Y.-C. Lai, and C.-H. Lai, Chaos 18, 037117 (2008).
[14] T. Aoki and T. Aoyagi, Phys. Rev. Lett. 102, 034101 (2009).
[15] M. Li, X. Wang, and C.-H. Lai, Chaos 20, 045114 (2010).
[16] S. H. Yook, H. Jeong, A.-L. Barabási, and Y. Tu, Phys. Rev. Lett. 86, 5835 (2001).
[17] M.E.J. Newman, Proc. Natl. Acad. Sci. U.S.A. 98, 404 (2001).
[18] M.E.J. Newman, Phys. Rev. E 64, 016131 (2001).
[19] J. Gomez-Gardenes, G. Zamora-Lopez, Y. Moreno and A. Arenas, PLoS ONE 5, e12313 (2010).
[20] A. E. Motter, C. S. Zhou, and J. Kurths, Europhys. Lett. 69, 334 (2005).
[21] A. E. Motter, C. S. Zhou, and J. Kurths, Phys. Rev. E 71, 016116 (2005).
[22] D.-U. Hwang, M. Chavez, A. Amann, and S. Boccaletti, Phys. Rev. Lett. 94, 138701 (2005).
[23] M. Chavez, D.-U. Hwang, A. Amann, H. G. E. Hentschel, and S. Boccaletti, Phys. Rev. Lett. 94, 218701 (2005).
[24] C.-S. Zhou, A. E. Motter, and J. Kurths, Phys. Rev. Lett. 96, 034101 (2006).
[25] X. Wang, Y.-C. Lai, and C.-H. Lai, Phys. Rev. E 75, 056205 (2007).
[26] X. Wang, C. Zhou, and C.-H. Lai, Phys. Rev. E 77, 056208 (2008).
[27] S. W. Son, B. J. Kim, H. Hong, and H. Jeong, Phys. Rev. Lett. 103, 228702 (2009).
[28] W. X. Wang, L. Huang, Y.-C. Lai and G. Chen, Chaos 19, 013134 (2009).
[29] T. Zhou, M. Zhao, and C. Zhou, New Journal of Physics, 12, 043030 (2010).
[30] A. Barrat, M. Barthelemy, R. Pastor-Satorras, A. Vespignani, Proc. Natl. Acad. Sci. U.S.A. 101, 3747 (2004).
[31] M. Li, J. Wu, D. Wang, T. Zhou, Z. Di, Y. Fan, Physica A 375, 355-364 (2007).
[32] J.-P. Onnela, J. Saramäki, J. Hyvönen, G. Szabó, D. Lazer, K. Kaski, J. Kertész, and A.-L. Barabási, Proc. Nat. Acad. Sci. USA 104, 7332-7336 (2007).
[33] P. J. Macdonald, E. Almaas and A.-L. Barabási, Europhys. Lett. 72, 308 (2005).
[34] Y. Fan, M. Li, P. Zhang, J. Wu, Z. Di, Physica A 378, 583-590 (2007).
[35] Z. Toroczkai and K. E. Bassler, Nature 428, 716 (2004).
[36] D. J. Watts and S. H. Strogatz, Nature 393, 440 (1998).
[37] A.-L. Barabási and R. Albert, Science 286, 509 (1999).
[38] X. Wang, S. Guan, Y.-C. Lai, B. Li, and C.-H. Lai, Europhys. Lett. 88, 28001 (2009).