Low Dimensional Supersymmetries in SUSY Chern-Simons Systems and Geometrical Implications

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May 22, 2014

Abstract

We study in detail the underlying graded geometric structure of abelian $N = 2$ supersymmetric Chern-Simons theory in (2 + 1)-dimensions. This structure is an attribute of the hidden unbroken one dimensional $N = 2$ supersymmetries that the system also possesses. We establish the result that the geometric structures corresponding to the bosonic and to the fermionic sectors are equivalent fibre bundles over the (2 + 1)-dimensional manifold. Moreover, we find a geometrical answer to the question why some and not all of the fermionic sections are related to a $N = 2$ supersymmetric algebra. Our findings are useful for the quantum theory of Chern-Simons vortices.

Introduction

Chern-Simons terms [1–3] are important terms that can be consistently be added to (2+1)-dimensional Lagrangians of gauge field theories [4, 5]. These terms provide particularly interesting properties to the these theories, altering the long distance behavior and modifying the solitonic solutions [1–3].

The supersymmetric extensions of (2 + 1)-dimensional gauged Chern-Simons models and particularly the ones with global $N = 2$ spacetime supersymmetry, have the interesting attribute which is the fact that the zero modes of fermions and bosons are directly related to the zero modes of the bosonic field fluctuations, with the latter describing massless bosonic modes around the vortices (see [5], and references therein).

A complete quantum theory of supersymmetric Chern-Simons vortices is still lacking, hence providing information related to such theories may shed light to various aspects of this quantum theory. Particularly, finding underlying hidden symmetries or inherent patterns that can be classified according to an already known symmetry, may reduce the complexity of a generally difficult problem.

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In this context, in a recent paper [6] we presented an appealing property of the zero modes of Chern-Simons models with global $N = 2$ spacetime supersymmetry. Specifically, we established fact that the fermionic and bosonic zero modes of $N = 2$ Abelian Chern-Simons gauge models in (2+1)-dimensions are related to two $N = 2$, $d = 1$ supersymmetric quantum algebras [7,13,15,18,20,26]. We focused in the limiting case $\kappa = 0$, with $\kappa$ the coupling of the Chern-Simons term. The latter theory is nothing else but a supersymmetric extension of the Landau-Ginzburg model. As we demonstrated in [6], the two $N = 2$, $d = 1$, supersymmetries combine to form an $N = 4$ extended supersymmetry, with non zero central charge [27,38]. We argued that this result is a direct consequence of the $N = 2$ global spacetime supersymmetry, possessed by the initial system. To this end, in this paper we shall investigate the geometric implications that each $N = 2$, $d = 1$, supersymmetry generates and in addition we shall see that there is a geometric reason behind the fact that the one dimensional supersymmetries combine to form $N = 4$ extended supersymmetry. In particular, as we shall evince, the supersymmetric quantum mechanics (SUSY QM hereafter) algebras provide a grading on the Hilbert space of solutions. In turn this grading provides the system with an interesting underlying geometric structure. As we shall see, the grading of the unbroken $N = 2$ SUSY QM algebra makes the (2 + 1)-dimensional space a graded manifold, denoted $(\mathbf{X}, \mathcal{A})$, with body $\mathbf{X}$ and structure sheaf $\mathcal{A}$. This graded manifold can be used to construct composite fibre bundles over the space $\mathbf{X}$. Regarding the fermionic sector, the supersymmetric structure is only for some sections of the spin bundle over $\mathbf{X}$. With the help of the graded manifold we can explain how this result occurs, since the covariant differential of the $U(1)$-twisted spin bundle over $\mathbf{X}$ is reducible to a covariant differential of a composite fibre bundle over $\mathbf{X}$, a bundle that contains the graded bundle in its structure. This reducibility is actually done in terms of the corresponding connections of the composite fibre bundles. In the bosonic case, the covariant differential of the corresponding composite fibre bundle is projectable to the one of the total $U(1)$-twisted fibre bundle, the sections of which are the bosonic fluctuations. The fact that the fermionic and bosonic zero modes of $N = 2$ Abelian gauge models in (2 + 1)-dimensions are directly related, has an immediate impact on the underlying geometric structure. Particularly we shall establish the result that the composite fibre bundles for fermions and bosons are equivalent, which means that there is some isomorphism between these. This in turn can be used to establish the fact that the bundles belong to the same K-theoretic equivalence class. This K-theoretic equivalence is actually the reason that the two $N = 2$, $d = 1$ supersymmetries form an $N = 4$, $d = 1$ supersymmetry.

This paper is organized as follows. In section 1 we review the general theoretical framework of abelian $N = 2$ supersymmetric Chern-Simons theory in (2 + 1)-dimensions and briefly present the result of [6], that there is an unbroken $N = 2$ SUSY QM algebra underlying both the fermionic and bosonic sector of the theory. In section 2 we describe in detail the additional geometric structure over the (2 + 1)-dimensional space $\mathbf{X}$, which is implied by the unbroken $N = 2$ SUSY QM algebra and discuss the features of the fermionic and bosonic geometric structures over $\mathbf{X}$. We also discuss why these structures are equivalent, a feature that must be a result of the dimensionality of the space $\mathbf{X}$ and also due to the global $N = 2$ supersymmetry.
1 Supersymmetric Abelian Chern-Simons Theory

In order to make the article self-contained, we briefly review the abelian $N = 2$ supersymmetric Chern-Simons theory in (2 + 1)-dimensions and present the basic result of [6]. As we already noted, one feature of the Abelian Chern-Simons supersymmetric model under study is that, within the $N = 2$ supersymmetry framework, the fermionic zero modes are directly related to the bosonic zero modes. This fact has particularly interesting consequences, in reference to the underlying one dimensional supersymmetries and also for the geometrical structures, as we shall demonstrate in a later section. Let us briefly present the models that we will use in the following. Adopting the notation of references [5, 6], the Lagrangian for the $N = 2$ spacetime supersymmetric model is,

$$
L = -\frac{1}{4}F_{\mu\nu} \epsilon^{\mu\nu\lambda} F_{\mu\nu} A_\lambda - |D_\mu \phi|^2 - \frac{1}{2}(\partial_\mu N)^2
$$

where $D_\mu = \partial_\mu - ieA_\mu$ and the gamma matrices satisfy the relation $\gamma^\mu \gamma^\nu = -\eta^\mu\nu - \epsilon^{\mu\nu\lambda} \gamma_\lambda$. The fields $\psi, \chi$ are two component spinors, with $\psi$ being a Weyl charged fermionic field and $\chi$ a neutral complex Weyl two component spinor.

1.1 Zero Modes of the Fermionic Sector

Since the zero modes of the fermionic sector of the $N = 2$ Lagrangian in the background of self-dual vortices are very important in what follows, so let us see how we can find these. The fermionic equations of motion corresponding to Lagrangian (1), are given by the following equations:

$$
\gamma^i D_i \psi + i\epsilon^0 A^0 \psi - \sqrt{2}\epsilon \phi \chi = 0 \quad (2)
$$

where $D_\mu = \partial_\mu - ieA_\mu$ and the gamma matrices satisfy the relation $\gamma^\mu \gamma^\nu = -\eta^\mu\nu - ie^{\mu\nu\lambda} \gamma_\lambda$. The fields $\psi, \chi$ are two component spinors, with $\psi$ being a Weyl charged fermionic field and $\chi$ a neutral complex Weyl two component spinor.

By making the following conventions:

$$
\gamma^0 = \sigma_3, \quad \gamma^1 = i\sigma_2, \quad \gamma^2 = i\sigma_1, \quad \psi = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}, \quad \chi = \begin{pmatrix} \chi_\uparrow \\ \chi_\downarrow \end{pmatrix}
$$

and moreover assuming positive values of the flux, we obtain the equations of motion for the fermions:

$$
(D_1 + iD_2)\psi_\downarrow - \sqrt{2}\epsilon \phi \chi_\uparrow = 0 \quad (4)
$$

$$
(\partial_1 - i\partial_2)\chi_\uparrow + i\epsilon A^0 \psi_\downarrow = 0
$$

We shall explore the theory for the limiting case $\kappa = 0$ of the Chern-Simons parameter $\kappa$. Note that, when the Chern Simons coupling $\kappa$ becomes zero, the $N = 2$ Lagrangian reduce to the $N = 2$ Abelian Higgs model.
1.2 Fluctuating Bosonic Zero Modes of the $N = 2$ Supersymmetric System

In addition to the fermionic zero modes, the zero modes of the bosonic fields fluctuations are very necessary to reveal the underlying geometric and symmetric structure of the system. Let us give a brief account on how to find these. In order to do this, we focus our interest on the bosonic part of the Lagrangian [1]. The theory has two ground states, namely the non-symmetric one, with $|\phi| = v, N = 0$ and a symmetric one with $\phi = 0, N = \frac{ev^2}{\kappa}$. Solutions of the topological soliton type exist in non-symmetric phase which has the following asymptotic behavior [5]:

$$\lim_{r \to \infty} N(r) \to 0, \quad \lim_{r \to \infty} |\phi(r)| \to v$$

(5)

and additionally a quantized flux $\Phi = \pm \frac{2\pi n}{e}$. In the symmetric Higgs phase, $\phi = 0, N = ev^2/\kappa$, non-topological solutions exist with the following asymptotic behavior:

$$\lim_{r \to \infty} N(r) \to \frac{ev^2}{\kappa} + \text{const.} \frac{1}{r^2}, \quad \lim_{r \to \infty} |\phi(r)| \to \text{const.} \frac{1}{r^a}$$

(6)

Furthermore, all static solutions must satisfy the Gauss law:

$$\partial_i F_{i0} + \kappa F_{12} - i e (|\phi|^2 + \kappa N - ev^2) = 0$$

(7)

Integrating over the whole space we obtain a static configuration of magnetic flux $\Phi = \int d^2x F_{12}$ which has a total electric charge, $Q = -e\Phi$. The energy of the configuration is bounded from below by the relation $E \geq ev^2|\Phi|$, and is saturated if the configurations satisfy the self-duality equations:

$$(D_1 \mp iD_2)\phi = 0$$

$$F_{12} \pm (e|\phi|^2 + \kappa N - ev^2) = 0$$

$$A^0 \mp N = 0$$

$$\partial_i F_{i0} + \kappa F_{12} - i e (|\phi|^2 D^0 \phi - D^0 \phi^* \phi) = 0$$

(8)

The equations of the zero modes fluctuations are obtained by varying the self-duality equations (8) around the static classical vortex configuration, and can be cast in the following form:

$$(D_1 + iD_2)\delta \phi - i e \phi (\delta A_1 + i\delta A_2) = 0$$

$$\partial_1 \delta A_2 - \partial_2 \delta A_1 + e (\phi^* \delta \phi + \phi \delta \phi^*) + k \delta A^0 = 0$$

(9)

We will set $\kappa = 0$ in order to describe the Landau-Ginzburg vortex situation, as we did in the fermionic case. In that case, one can consistently set $A^0 = N = 0$, just as we did in the fermionic case.
1.3 $N = 2$ Supersymmetric Quantum Mechanics Algebra in the Fermionic Sector

The fermionic equations of motion (4) for $\kappa = 0$ can be cast as,

\[
(D_1 + iD_2)\psi_\downarrow - \sqrt{2}e\phi\chi_\uparrow = 0 \\
(\partial_1 - i\partial_2)\chi_\uparrow - \sqrt{2}e\phi^*\psi_\downarrow = 0 \\
(D_1 - iD_2)\psi_\uparrow + \sqrt{2}e\phi\chi_\downarrow = 0 \\
(\partial_1 + i\partial_2)\chi_\downarrow + \sqrt{2}e\phi^*\psi_\uparrow = 0
\] (10)

The last two equations of relation (10) have no solutions describing localized fermions, but only have some trivial solutions [5]. However, the first two equations of relation (10), have $2n$ normalized solutions, with $n$ the vorticity number [5]. Thereby, we can form the operator $\mathcal{D}_{LG}$, corresponding to the first two equations of (10),

\[
\mathcal{D}_{LG} = \begin{pmatrix} D_1 + iD_2 & -\sqrt{2}e\phi \\ -\sqrt{2}e\phi^* & \partial_1 - i\partial_2 \end{pmatrix}
\] (11)

acting on the vector:

\[
|\Psi_{LG}\rangle = \begin{pmatrix} \psi_\downarrow \\ \chi_\uparrow \end{pmatrix}.
\] (12)

Consequently, the first two equations of (10) can be cast as:

\[
\mathcal{D}_{LG}|\Psi_{LG}\rangle = 0
\] (13)

The solutions of the above equation are the zero modes of the operator $\mathcal{D}_{LG}$. Recalling that the first two equations of (10) have $2n$ normalized solutions, we can easily state that:

\[
\dim \ker \mathcal{D}_{LG} = 2n
\] (14)

Furthermore, the adjoint of the operator $\mathcal{D}_{LG}$, namely $\mathcal{D}^\dagger_{LG}$, is equal to:

\[
\mathcal{D}^\dagger_{LG} = \begin{pmatrix} D_1 - iD_2 & \sqrt{2}e\phi \\ \sqrt{2}e\phi^* & \partial_1 + i\partial_2 \end{pmatrix}
\] (15)

and acts on the vector:

\[
|\Psi'_{LG}\rangle = \begin{pmatrix} \psi_\uparrow \\ \chi_\downarrow \end{pmatrix}.
\] (16)

The zero modes of the adjoint operator $\mathcal{D}^\dagger_{LG}$, correspond to the solutions of the last two equations of (10), with the obvious replacement $e \rightarrow -e$. Obviously, since the last pair of equations of relation (10) have no normalized solutions, the corresponding kernel of the adjoint operator is null, that is:

\[
\dim \ker \mathcal{D}^\dagger_{LG} = 0
\] (17)

The normalization condition for the solutions of (10) is crucial for our analysis, since only for such solutions the operator $\mathcal{D}_{LG}$ is Fredholm, a result that can be verified by (14) and
The fermionic system in the self-dual vortices background, possesses an unbroken $N = 2, d = 1$ supersymmetry. Indeed, we can form the supercharges and the quantum Hamiltonian of this $N = 2, d = 1$ SUSY algebra in terms of the operator $D_{LG}$, and these are equal to:

\[
Q_{LG} = \begin{pmatrix} 0 & D_{LG} \\ 0 & 0 \end{pmatrix}, \quad Q_{LG}^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & D_{LG}^\dagger \end{pmatrix}, \quad H_{LG} = \begin{pmatrix} D_{LG}D_{LG}^\dagger & 0 \\ 0 & D_{LG}^\dagger D_{LG} \end{pmatrix}
\] (18)

These three elements of the algebra, satisfy the $d = 1$ SUSY QM algebra:

\[
\{Q_{LG}, Q_{LG}^\dagger\} = H_{LG}, \quad Q_{LG}^2 = 0, \quad Q_{LG}^\dagger^2 = 0
\] (19)

The Hilbert space of the supersymmetric quantum mechanical system, $H_{LG}$ is a $Z_2$ graded vector space, with the grading provided by the operator $W$, an involution operator known as the Witten parity. This operator commutes with the total Hamiltonian and anti-commutes with the supercharges,

\[
[W, H_{LG}] = 0, \quad \{W, Q_{LG}\} = \{W, Q_{LG}^\dagger\} = 0
\] (20)

Moreover, the Witten parity $W$, satisfies the following identity,

\[
W^2 = 1
\] (21)

As we already mentioned, the Witten parity $W$, spans the total Hilbert space into subspaces which are classified according to their $Z_2$ parity. Hence the total Hilbert space of the quantum system can be written as:

\[
H = H^+ \oplus H^-
\] (22)

with the vectors that belong to the two subspaces $H^\pm$, classified according to their Witten parity, to even and odd parity states, that is:

\[
H^\pm = \mathcal{P}^\pm H = \{|\psi\rangle : \mathcal{W}|\psi\rangle = \pm|\psi\rangle\}
\] (23)

Furthermore, the corresponding Hamiltonians of the $Z_2$ graded spaces are:

\[
H_+ = D_{LG}D_{LG}^\dagger, \quad H_- = D_{LG}^\dagger D_{LG}
\] (24)

The operator $W$, in the case at hand, can be represented in the following matrix form:

\[
W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\] (25)

The eigenstates of $\mathcal{P}^\pm$ which are denoted $|\psi^\pm\rangle$, satisfy the following relation:

\[
\mathcal{P}^\pm|\psi^\pm\rangle = \pm|\psi^\pm\rangle
\] (26)
Hence, we call them positive and negative parity eigenstates, with “parity” referring to the $P^\pm$ operator, which is nothing else but the Witten parity operator. Using the representation (25) for the Witten parity operator, the parity eigenstates can be represented by the vectors,

$$|\psi^+\rangle = \begin{pmatrix} |\phi^+\rangle \\ 0 \end{pmatrix}, \quad |\psi^-\rangle = \begin{pmatrix} 0 \\ |\phi^-\rangle \end{pmatrix}$$

(27)

with $|\phi^\pm\rangle \in \mathcal{H}^\pm$. Turning back to the fermionic system at hand, it is easy to write the fermionic states of the system in terms of the SUSY QM algebra. We already wrote down the supercharges, namely relation (18), and then it is easy to verify that:

$$|\Psi_{LG}\rangle = |\phi^-\rangle = \begin{pmatrix} \psi_\downarrow \\ \chi_\uparrow \end{pmatrix}, \quad |\Psi'_{LG}\rangle = |\phi^+\rangle = \begin{pmatrix} \psi_\uparrow \\ \chi_\downarrow \end{pmatrix}$$

(28)

Therefore, the corresponding even and odd parity SUSY QM states are the following:

$$|\psi^+\rangle = \begin{pmatrix} |\Psi'_{LG}\rangle \\ 0 \end{pmatrix}, \quad |\psi^-\rangle = \begin{pmatrix} 0 \\ |\Psi_{LG}\rangle \end{pmatrix}$$

(29)

on which, the Hamiltonian and the supercharges act. Supersymmetry is unbroken if the Witten index is a non-zero integer. The Witten index for Fredholm operators is equal to:

$$\Delta = n_- - n_+$$

(30)

with $n_\pm$ the number of zero modes of $\mathcal{H}_\pm$ in the subspace $\mathcal{H}^\pm$, with the constraint that these are finitely many.

When the Witten index is zero and also if $n_+ = n_- = 0$, then supersymmetry is broken. However, if $n_+ = n_- \neq 0$ the system has still an unbroken supersymmetry.

The Witten index is connected to the Fredholm index of the operator $D_{LG}$, as follows:

$$\Delta = \dim \ker \mathcal{H}_- - \dim \ker \mathcal{H}_+ = \dim \ker D_{LG} D_{LG}^\dagger - \dim \ker D_{LG}^\dagger D_{LG} = \dim \ker D_{LG} - \dim \ker D_{LG}^\dagger$$

(31)

Using equations (14) and (17), the Witten index is equal to:

$$\Delta = -2n$$

(32)

Thereupon, the fermionic system in the self-dual Landau-Ginzburg vortices background with $N = 2$ spacetime supersymmetry, has an $N = 2$, $d = 1$ unbroken supersymmetry. We could argue that this result could stem from the fact that the initial system has an unbroken $N = 2$ spacetime supersymmetry, so the Hilbert space of the zero modes states also has a remnant $N = 2$, $d = 1$ supersymmetric quantum algebra. However, this is not true since global spacetime supersymmetry in $d > 1$ dimensions and supersymmetric quantum mechanics, that is $d = 1$ supersymmetry, are not the same. The SUSY QM supercharges do not generate transformations between fermions and bosons and also these supercharges classify the Hilbert space of quantum states according to the group $\mathbb{Z}_2$. So we can state that the supersymmetric quantum mechanics algebra is not similar to a global spacetime supersymmetry, but nevertheless the SUSY QM algebra is a graded algebra.
1.4 $N = 2$ Supersymmetric Quantum Mechanics Algebra of the Bosonic Fluctuations

We now turn our focus to the bosonic zero modes equations. For $\kappa = 0$ and $A^0 = N = 0$ these can be cast in the following form:

\begin{align*}
(D_1 + iD_2)\delta \phi - ie\phi(\delta A_1 + i\delta A_2) &= 0 \quad (33) \\
(\partial_1 - \partial_2)(\delta A_1 + i\delta A_2) + 2ie\phi^*\delta \phi &= 0
\end{align*}

Interestingly enough, the above equations become identical to equations (38), if we substitute:

$$
\psi_\downarrow = \delta \phi, \quad \chi_\uparrow = \frac{i}{\sqrt{2}}(\delta A_1 + i\delta A_2) \quad (34)
$$

As we shall see, this will play a crucial role when we will address the equivalence of the geometrical structures corresponding to fermions and bosons. Due to this fact, it is easy to conclude that the number of the zero modes corresponding to equation (33), are equal to the total number of zero modes corresponding to the first two equations of relation (10), that is $2n$. As in the fermionic case, a $N = 2$ SUSY quantum mechanical algebra underlies the bosonic system as well. The structure of the algebra is the same as the fermionic one, with the difference that the corresponding Hilbert space vectors are different. Indeed, equations (33) can be written in the following form:

$$
\mathcal{D}'_{LG}|\Phi_{LG}\rangle = 0 \quad (35)
$$

where, the operator $\mathcal{D}'_{LG}$ is:

$$
\mathcal{D}'_{LG} = \left( \begin{array}{cc}
D_1 + iD_2 & -\sqrt{2}e\phi \\
-\sqrt{2}e\phi^* & \partial_1 - i\partial_2
\end{array} \right) \quad (36)
$$

and acts on the vector:

$$
|\Phi_{LG}\rangle = \left( \begin{array}{c}
\delta \phi \\
\frac{i}{\sqrt{2}}(\delta A_1 + i\delta A_2)
\end{array} \right) \quad (37)
$$

Hence we arrive to the conclusion that:

$$
\dim \ker \mathcal{D}'_{LG} = 2n \quad (38)
$$

Furthermore, as in the fermionic case, the adjoint $\mathcal{D}'_{LG}^\dagger$ has null "ker".

$$
\dim \ker \mathcal{D}'_{LG}^\dagger = 0 \quad (39)
$$

The operators $\mathcal{D}_{LG}^\dagger$ and $\mathcal{D}'_{LG}^\dagger$ are Fredholm as well, and thereby any which operator constructed from these, is also Fredholm. The supercharges and the Hamiltonian, that constitute the $N = 2$, $d = 1$ algebra in the bosonic case, are:

$$
\mathcal{Q}_{LG} = \left( \begin{array}{cc}
0 & \mathcal{D}'_{LG} \\
0 & 0
\end{array} \right), \quad \mathcal{Q}'_{LG} = \left( \begin{array}{cc}
0 & 0 \\
\mathcal{D}'_{LG}^\dagger & 0
\end{array} \right), \quad \mathcal{H}'_{LG} = \left( \begin{array}{cc}
\mathcal{D}_{LG} \mathcal{D}_{LG}^\dagger & 0 \\
0 & \mathcal{D}_{LG}^\dagger \mathcal{D}_{LG}
\end{array} \right) \quad (40)
$$
These three elements of the algebra, also satisfy the $d = 1$ SUSY algebra:

$$\{ Q'_{LG}, Q'^\dagger_{LG} \} = H'_{LG}, Q^2_{LG} = 0, \quad Q'^\dagger_{LG} Q^2_{LG} = 0 \quad (41)$$

Supersymmetry is unbroken, since the corresponding Witten index $\Delta'$ is a non-zero integer, in this case too. Indeed:

$$\Delta' = -2n \quad (42)$$

We found that the bosonic fluctuations in the self-dual Landau-Ginzburg vortex background, are related to an $N = 2$ SUSY quantum mechanics algebra, which is identical to the fermionic SUSY quantum mechanics algebra, that we came across earlier. We now turn our focus to find what impact has this SUSY QM algebra on the geometric structures that can be constructed over the $(2 + 1)$-dimensional space.

2 Geometrical Implications of the $N = 2$ SUSY QM Algebras

The existence of underlying $N = 2$, $d = 1$ supersymmetric quantum algebras in the fermionic and bosonic systems we presented in the previous sections, has some geometrical implications on the geometric spaces over $X$, to which spaces the fermions and bosons are sections of the corresponding fibre bundles.

It is convenient to discuss in short what is our aim, why we are motivated to search to find such extra geometric structures over $X$ and also what are our main results. The motivation to search for extra underlying geometrical structure comes from the fact that both in the fermionic and bosonic sector, there exists an underlying graded Hilbert vector space, given by vectors of the form (28) and also (37). Since the fermions and bosons are sections of some total fibre bundles, it is obvious that some of these sections belong to another affine vector bundle that has some sort of an inherent graded structure. We shall construct such a structure in the following. Moreover, in reference to fermions, not all sections of the total spin bundle, belong to this graded manifold. This can be verified by looking relation (10). Only the first two of these equations are associated to an unbroken $N = 2$ SUSY QM algebra. Therefore, this suggests that the connection of the total spin manifold, is reducible to some other connection which is related to the underlying graded manifold. Finally, the fact that the operators (11) and (36) for fermions and bosons are equal, strongly suggests some equivalence relation between the corresponding composite graded fiber structures. In the following subsections we shall extensively address these issues.

2.1 Geometric Structures for the Fermionic Sector

We denote $X$ the $(2 + 1)$-dimensional spacetime upon which the model we described in this paper is built on. We first study the fermionic sector. The fermions are sections of the $U(1)$–twisted fibre bundle $P \times S \otimes U(1)$, where $S$ is the representation of the Spin group $Spin(3)$, which in three dimensions is irreducible, and $P$, the double cover of
the principal $SO(3)$ bundle on the tangent manifold $TX$. Some of these spinors belong to the vector space of the supersymmetric quantum algebra. The graded vector space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, that some of the sections of the fibre bundle $P \times S \otimes U(1)$ belong, implies a new structure on the manifold $X$ and particularly $X$ is up-lifted to a graded manifold $(X, \mathcal{A})$ (for the issues of connections on manifolds see for example [39–42], while for connections on graded manifolds see [39,43]). Indeed, the existence of the $N = 2$ SUSY QM algebra, $\mathcal{W}, \mathcal{Q}_{LG}, \mathcal{Q}_{LG}^\dagger$ and especially the involution $\mathcal{W}$, generates the $\mathbb{Z}_2$ graded vector space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. The subset $\mathcal{H}^+$ contains $\mathcal{W}$-even vectors and $\mathcal{W}$-odd vectors. This grading in turn is an additional algebraic structure on the $(2+1)$-dimensional manifold $X$. As we already mentioned, $X$ is up-lifted to a graded manifold $(X, \mathcal{A})$, with $\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$ an $\mathbb{Z}_2$ graded algebra. Particularly, $\mathcal{A}$ is a sheaf of $\mathbb{Z}_2$-graded commutative $\mathbb{R}$-algebras of total rank $m$ ($m = 2$ for our case). Moreover, the sheaf $\mathcal{A}$ underlies the vector space $\mathcal{H}$ and this sheaf makes the space $\mathcal{H}$ an $\mathbb{Z}_2$-graded $\mathcal{A}$-module. Indeed, this can be verified from the fact that:

$$A_+ \cdot M_+ \subset M_+, \quad A_+ \cdot M_- \subset M_-, \quad A_- \cdot M_+ \subset M_-, \quad A_- \cdot M_- \subset M_+$$  \hspace{1cm} (43)

The sheaf $\mathcal{A}$ contains the endomorphism $W$ (the involution of the SUSY quantum algebra), $W : \mathcal{H} \to \mathcal{H}$, with $W^2 = I$, which provides the $\mathbb{Z}_2$-grading on $\mathcal{H}$, i.e.:

$$W \mathcal{H}^\pm = \pm 1$$  \hspace{1cm} (44)

Hence, $\text{End}(\mathcal{H}) \subseteq \mathcal{A}$. The sheaf $\mathcal{A}$ is called a structure sheaf of the graded manifold $(X, \mathcal{A})$, while $X$ is called the body of $(X, \mathcal{A})$. In the following we shall mainly be interested on the connections of this graded manifold, hence it worths noting that given an open neighborhood $U$ of $x \in X$, we have that locally (recall that $m = 2$): 

$$\mathcal{A}(U) = C^\infty(U) \otimes \wedge R^m$$  \hspace{1cm} (45)

Hence the structure sheaf $\mathcal{A}$ is isomorphic to the sheaf $C^\infty(U) \otimes \wedge R^m$ of some exterior vector bundle $\wedge \mathcal{H}_E^* = U \times \wedge R^m$, with $\mathcal{H}_E$ an affine vector bundle with fiber the vector space $\mathcal{H}$. Then actually, the structure sheaf $\mathcal{A} = C^\infty(U) \otimes \wedge \mathcal{H}$, is isomorphic to the sheaf of sections of the exterior vector bundle $\wedge \mathcal{H}_E^* = R \oplus (\oplus_{k=1}^m \wedge^k) \mathcal{H}_E^*$. The fibre bundle $\mathcal{H}_E$ is very important for the definition of a graded connection. The connections of the graded manifold $(X, \mathcal{A})$ constitute an affine space modeled on the linear space of sections of the vector bundle $TX^* \otimes \wedge \mathcal{H}_E^* \otimes \mathcal{H}_E$. Note that, a connection preserves the grading of the vector space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. The sections of the bundle $TX^* \otimes \wedge \mathcal{H}_E^* \otimes \mathcal{H}_E$ are operators that belong to the sheaf $\mathcal{A}$ with $\mathcal{H}$-valued forms as elements in some specific representation, the dimension of which depends on the rank of the sheaf. In the case at hand, these are $2 \times 2$ matrices with $\mathcal{H}$-valued forms as matrix elements. The graded connection we just described will be used in the following as an auxiliary object, since we are not interested in this connection, but in the composition of this with another connection we shall describe.

Recall that the Dirac covariant differential corresponding to the connection of the total bundle, $P \times S \otimes U(1)$, which we denote $\gamma_s$, results to the equations (10). The fact that some
of these equations are associated to an unbroken $N = 2$ SUSY QM algebra, implies that the connection $\gamma_s$ is reducible to another connection $\gamma_C$, related in some way to the graded manifold $(X, A)$. Actually this reducibility implies that the total covariant differential of the fibre bundle $P \times S \otimes U(1)$, reduces to a simpler one, which when integral sections of the fibre bundle $P \times S \otimes U(1)$ are taken into account, results to the first two equations of relation (10). In this article, we shall predominantly be interested for integral sections of the fibre bundles that we will use. The non-zero modes are also interesting since these provide information for the quantum theory of supersymmetric Chern-Simons vortices, as we saw in [6]. The $N = 2$ SUSY QM algebra implies an underlying geometric structure that is described by the following diagram:

$$
\begin{array}{cccc}
X & \overset{\gamma_s}{\rightarrow} & S \times P \otimes U(1) \\
& & \downarrow \gamma_E \\
& & \gamma_{SE} \\
& & TX^* \otimes \wedge H_{\mathcal{E}}^* \otimes H_{\mathcal{E}}
\end{array}
$$

The arrows do not show the direction of the projective maps of the corresponding fibre bundles, but the directions of the connections of the corresponding fibre bundles. The connections appearing on the arrows are defined to be morphisms of the following fibre maps:

$$
\gamma_s : P \times S \otimes U(1) \rightarrow J^1(P \times S \otimes U(1)), \quad (\text{Bundle map, } \pi_s : P \times S \otimes U(1) \rightarrow X) \\
\gamma_E : TX^* \otimes \wedge H_{\mathcal{E}}^* \otimes H_{\mathcal{E}} \rightarrow J^1(TX^* \otimes \wedge H_{\mathcal{E}}^* \otimes H_{\mathcal{E}}) \quad (\text{Bundle map, } \pi_E : TX^* \otimes \wedge H_{\mathcal{E}}^* \otimes H_{\mathcal{E}} \rightarrow X) \\
\gamma_{SE} : (P \times S \otimes U(1))_G \rightarrow J^1((P \times S \otimes U(1))_G) \quad (\text{Bundle map, } \pi_{SE} : P \times S \otimes U(1) \rightarrow TX^* \otimes \wedge H_{\mathcal{E}}^* \otimes H_{\mathcal{E}})
$$

with $J^1Y_i$ the jet bundle of the corresponding bundle $Y_i$. The underlying geometrical structure is a composite fibre bundle over the manifold $X$, which is:

$$
P \times S \otimes U(1) \xrightarrow{\pi_{SE}} TX^* \otimes \wedge H_{\mathcal{E}}^* \otimes H_{\mathcal{E}} \xrightarrow{\pi_E} X \quad (47)
$$

We can define the composite connection corresponding to the composite fibre bundle (47) as follows:

$$
\gamma_C = \gamma_{SE} \circ \gamma_E \quad (48)
$$

which is nothing else than the composition of the connections $\gamma_{SE}$ and $\gamma_E$. In order such a connection to exist, there must be a canonical map between the jet bundles of the fibre bundles that constitute the composite fibre bundle. The composite connections have a direct impact on the first order differential operators and the corresponding covariant
differentials. It is convenient here to remember the definition of the first order differential and of the covariant differential corresponding to some connection. For a general fibre bundle \( Y \to X \), a section \( s_Y : X \to Y \), and a connection \( \gamma_Y : Y \to J^1 Y \), the first order differential is:

\[
D_{\gamma_Y} : J^1 Y \to TX^* \otimes VY
\] (49)

with \( VY \) the vertical subbundle of \( Y \), which in the case of \( Y \) is a vector bundle \( VY = Y \times Y \). The covariant differential corresponding to \( \gamma_Y \), denoted \( \nabla_{\gamma_Y} \), is:

\[
\nabla_{\gamma_Y} = D_{\gamma_Y} \circ J^1 s_Y : X \to TX^* \otimes VY
\] (50)

Hence, the total covariant differential of the bundle \( P \times S \otimes U(1) \to X \), is equal to:

\[
\nabla_{\gamma_s} = D_{\gamma_s} \circ J^1 s_E : X \to TX^* \otimes V(P \times S \otimes U(1)) \equiv TX^* \otimes P \times S \otimes U(1)
\] (51)

Note that this covariant differential, when it acts on integral sections, results to the set of equations (10). Before we proceed, let us discuss something very crucial for the proceeding analysis. Let \( s_E \) and \( s_{SE} \) be the sections of the following bundles:

\[
s_E : X \to TX^* \otimes \wedge H^* \otimes H_E
\]

\[
s_{SE} : TX^* \otimes \wedge H^* \otimes H_E \to P \times S \otimes U(1)
\] (52)

We denote \( Y_h \), the restriction \( Y_h = s_E^*(P \times S \otimes U(1)) \) of the fibre bundle \( \pi_{SE} : P \times S \otimes U(1) \to TX^* \otimes \wedge H^* \otimes H_E \), to the submanifold \( s_E(X) \subset P \times S \otimes U(1) \), through the inclusion

\[
i_h : Y_h \hookrightarrow P \times S \otimes U(1)
\] (53)

Given the sections \( s_E \) and \( s_{SE} \), their composition is \( s_C = s_{SE} \circ s_E \), which is a section of the fibre bundle \( \pi_a : P \times S \otimes U(1) \to X \), with \( s_C(X) \subset P \times S \otimes U(1) \). The definition of the covariant differential for the composite bundle, denoted as \( \nabla^{\gamma_C} \), corresponding to the connection (48) follows easily:

\[
\nabla^{\gamma_C} = D_{\gamma_C} \circ J^1 s_C : X \to TX^* \otimes VY_h \equiv TX^* \otimes Y_h
\] (54)

The covariant differential when applied to integral sections of \( Y_h \) (which are a subset of the integral sections of \( P \times S \otimes U(1) \)), gives rise to the first two equations of relation (10). Therefore, we observe that the covariant differential, \( \nabla^{\gamma_s} \) of the total bundle \( P \times S \otimes U(1) \) is reducible to the covariant differential \( \nabla^{\gamma_C} \). Formally, this implies that the connection \( \gamma_s \) is reducible to \( \gamma_C \) (but in any case not projectable), so the following diagram is commuting:

\[
\begin{array}{ccc}
P \times S \otimes U(1) & \xrightarrow{\gamma_s} & J^1 Y_h \\
i_h & & J^1 i_h \\
Y_h & \xrightarrow{\gamma_C} & J^1 P \times S \otimes U(1)
\end{array}
\]

where the inclusion map \( i_h \) is the one of relation (53) and \( J^1 i_h \) the jet prolongation of this inclusion map.
2.2 Geometric Structures for the Bosonic Sector

The same arguments we employed for the fermionic sector, also hold for the bosonic sector. In this case the geometric structure implied by the $N = 2$ SUSY QM underlying the bosonic sector, is represented by the following diagram:

\[ X \xrightarrow{\gamma_B} TX^* \times \mathcal{C} \otimes U(1) \]

\[ \xrightarrow{\gamma_E} TX^* \otimes \wedge \mathcal{H}_\xi^* \otimes \mathcal{H}_\xi \]

As in the fermionic sector, $X$ is the $(2 + 1)$-dimensional manifold, while the $U(1)$-twisted fibre bundle $TX^* \times \mathcal{C} \otimes U(1)$ has sections that are actually the bosonic field variations $\delta\phi, \delta A_i, i = 1, 2$. Hence, we have a composite fibre bundle

\[ TX^* \times \mathcal{C} \otimes U(1) \xrightarrow{\pi_{BE}} TX^* \otimes \wedge \mathcal{H}_\xi^* \otimes \mathcal{H}_\xi \xrightarrow{\pi_E} X \] (55)

The connections of this composite fibre bundle are defined similarly to those of the fermionic sector, and are of the following form:

\[ \gamma_B : TX^* \times \mathcal{C} \otimes U(1) \rightarrow J^1(TX^* \times \mathcal{C} \otimes U(1)), \] (56)

\[ \gamma_E : TX^* \otimes \wedge \mathcal{H}_\xi^* \otimes \mathcal{H}_\xi \rightarrow J^1(TX^* \otimes \wedge \mathcal{H}_\xi^* \otimes \mathcal{H}_\xi) \]

\[ \gamma_{BE} : (TX^* \times \mathcal{C} \otimes U(1))_G \rightarrow J^1((TX^* \times \mathcal{C} \otimes U(1))_G) \]

The covariant differential corresponding to the connection $\gamma_B$, which we denote $\nabla^{\gamma_B}$, when it acts to integral sections of the fibre bundle $\pi_B : TX^* \times \mathcal{C} \otimes U(1) \rightarrow X$, yields the differential equations (33). In addition, the covariant differential corresponding to the composite connection $\gamma_F = \gamma_{BE} \circ \gamma_E$, which we denote $\nabla^{\gamma_F}$, when it acts to integral composite sections of the composite fibre bundle, yields the same set of equations. This can only be true if the connection $\gamma_B$ is projectable over $\gamma_F$. Note that the covariant differential $\nabla^{\gamma_B}$ is a map of the form:

\[ \nabla^{\gamma_B} = D_{\gamma_B} \circ J^1s_B : X \rightarrow TX^* \otimes V(TX^* \times \mathcal{C} \otimes U(1)) \] (57)

with $s_B$ the corresponding integral section, while the covariant differential $\nabla^{\gamma_F}$ is of the form:

\[ \nabla^{\gamma_F} = D_{\gamma_F} \circ J^1s_F : X \rightarrow TX^* \otimes VY' \] (58)
In the above relation, $Y'$ is some subbundle of the fibre bundle $TX^* \times C \otimes U(1)$. The fact that $\gamma_B$ is projectable over (but not reducible) $\gamma_F$, means that the following diagram is commuting:

$$
\begin{array}{ccc}
TX^* \times C \otimes U(1) & \xrightarrow{\gamma_B} & J^1(TX^* \times C \otimes U(1)) \\
\downarrow \pi_{bs} & & \downarrow J^1 \pi_{bs} \\
Y' & \xrightarrow{\gamma_F} & J^1 Y'
\end{array}
$$

with $\pi_{bs}$, the projection $\pi_{bs} : TX^* \times C \otimes U(1) \to Y'$.

### 2.3 Bundle Isomorphisms Between Fermionic and Bosonic Sectors and K-theoretic Arguments

Recall relations (11) and (36). The operators $D_{LG}$ and $D'_{LG}$, apart from the fact that they act in a subset of fermionic sections and bosonic sections respectively, they are identical. This is very important from a geometric aspect, in view of the results we presented in the previous two subsections. Clearly, such an equality is by far not accidental. Recall that these operators are actually the covariant differentials of the corresponding composite fibre bundles we saw earlier. Hence the covariant differentials $\nabla^{\gamma_F}$ and $\nabla^{\gamma_B}$ are actually equal. This fact clearly signals some sort of equivalence between the fibre bundle $Y_h$ and $Y'$, which can be quantified by the existence of an isomorphism $\Phi$, so that the following diagram is commuting:

$$
\begin{array}{ccc}
Y' & \xrightarrow{\Phi} & Y_h \\
\downarrow \pi_b & & \downarrow \pi_s \\
X & & \end{array}
$$

with $\pi_b$ and $\pi_s$ the projections from the total bundle space to the base space $X$ of the fibre bundles $Y' \to X$ and $Y_h \to X$ respectively. We can establish that conclusion by thinking as follows: Since $\nabla^{\gamma_F} \equiv \nabla^{\gamma_B}$, this implies some isomorphism between the fibre bundles $TX^* \otimes YY'$ and $TX^* \otimes YY_h$, which isomorphism induces an isomorphism between the vertical subbundles $VY'$ and $VY_h$. In turn, this isomorphism induces the isomorphism between the spaces $Y'$ and $Y_h$ (recall that $Y'$ and $Y_h$ are affine vector bundles).

One could argue that this bundle equivalence between $Y'$ and $Y_h$, could be extended to some sort of K-theoretic equivalence. Indeed, following the same line of argument as above, the bundles $Y_h \oplus I^n$ and $Y' \oplus I^n$, with $I^n$ some trivial bundle over $X$, are stably equivalent, that is:

$$
Y_h \oplus I^n \approx Y' \oplus I^n
$$

(59)

Hence they belong to the same equivalence class in $K(X)$, that is $[Y'] = [Y_h]$. 

14
Concluding Remarks

In this paper we studied an abelian $N = 2$ supersymmetric extension of the Landau-Ginzburg model. Both the bosonic and the fermionic sectors have a common underlying unbroken $N = 2$ SUSY QM algebra. This structure provides both sectors with an additional geometric structure over the $(2+1)$-dimensional space $X$. Particularly, this geometric structure is based on a composite bundle over the space $X$, with a graded manifold over $X$, being a basic ingredient of the composite bundle. We saw that this geometric structure underlies both the fermionic and the bosonic sector, with the difference that in the bosonic sector, the covariant differential of the corresponding total bundle $\nabla^B$ is projectable to $\nabla^F$ corresponding to the total bundle space that is related to the graded manifold. In the fermionic case, the covariant differential $\nabla^F$ is reducible to $\nabla^C$ corresponding to some subbundle $Y_h$ of the initial total twisted spin bundle.

Due to the fact that the zero modes of the fermions and bosons are related, the geometric structures corresponding to bosons and fermions are equivalent. In particular, we found a direct correlation between the fermionic and bosonic bundles, in terms of an isomorphism. This result is a consequence of the global $N = 2$ supersymmetry that the system possesses and owing to the fact that the space $X$ is $(2 + 1)$-dimensional. It is intriguing that, apart from the $N = 2$ supermanifold we can construct over $X$, we also found that $X$ is a $\mathbb{Z}_2$-graded manifold, with the grading being provided by the involution $W$, which is the Witten parity of the SUSY QM system. It would be interesting to try to find if there is any direct correlation between the supermanifold corresponding to global $N = 2$ supersymmetry and that of the graded manifold (which is not a supermanifold). This task is highly motivated by the fact that every graded manifold defines a DeWitt supermanifold. However, we must be cautious because the SUSY QM in the bosonic sector is related to the bosonic fluctuations. This would require analysis of some corresponding jet module, something that is interesting but also out of the scope of this paper.

Finally, in order to support our arguments for the constructed geometric structures, we used only integral sections of the corresponding fibre bundles. The integral sections are very useful for the quantum theory of Chern-Simons vortices. This is due to the fact that the bosonic ones correspond to collective coordinates describing the vortices positions and slow velocity kinematics. Moreover, the fermionic ones represent the degeneracy of the solitonic states. Hence, we believe that this extra geometric structure of the SUSY Chern-Simons model, is relevant for the complete quantum theory of Chern-Simons vortices.

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