Approximate Controllability of the System Governed by Double Coupled Semilinear Degenerate Parabolic Equations

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Abstract. This paper concerns the approximate controllability of the initial-boundary problem of double coupled semilinear degenerate parabolic equations. The equations are degenerate at the boundary, and the control function acts in the interior of the spatial domain and acts only on one equation. We overcome the difficulty of the degeneracy of the equations to show that the problem is approximately controllable in $L^2$ by means of a fixed point theorem and some compact estimates. That is to say, for any initial and desired data in $L^2$, one can find a control function in $L^2$ such that the weak solution to the problem approximately reaches the desired data in $L^2$ at the terminal time.

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1 Introduction

In this paper, we investigate the approximate controllability of the following semilinear problem:

$$\frac{\partial u}{\partial t} - \text{div}(a_1(x,t)\nabla u) + g(x,t,u,v) = h(x,t)\chi_{\omega_1}, \quad (x,t) \in Q_T, \quad (1.1)$$

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where $Q_T = \Omega \times (0, T)$, $\Omega$ is a bounded smooth domain in $\mathbb{R}^n$, $T > 0$, $h \in L^2(Q_T)$ is the control function, $\chi$ is the characteristic function, $\omega_1$ and $\omega_2$ are open subsets of $\Omega$ satisfying $\omega_1 \cap \omega_2 \neq \emptyset$, $a_1, a_2 \in C(\overline{Q_T})$ satisfy

$$a_i(x, t) > 0 \quad \text{for} \quad (x, t) \in Q_T, \quad \frac{1}{a_i} \frac{\partial \chi}{\partial t} \in L^\infty(Q_T), \quad i = 1, 2,$$

$u_0, v_0 \in L^2(\Omega)$, $g$ and $q$ are measurable functions in $Q_T \times \mathbb{R} \times \mathbb{R}$ and $Q_T \times \mathbb{R}$, respectively. Furthermore, $g$ and $q$ satisfy

$$g(x, t, \cdot, \cdot) \in C^1(\mathbb{R} \times \mathbb{R}) \quad \text{uniformly for} \quad (x, t) \in Q_T, \quad (1.7)$$

$$\left|\frac{\partial g}{\partial s}(x, t, s, p)\right| + \left|\frac{\partial g}{\partial p}(x, t, s, p)\right| \leq M, \quad (x, t, s, p) \in Q_T \times \mathbb{R} \times \mathbb{R}, \quad (1.8)$$

$$|q(x, t, u) - q(x, t, v)| \leq M|u - v|, \quad (x, t) \in Q_T, \quad u, v \in \mathbb{R}, \quad (1.9)$$

where $M > 0$ is a constant. Since (1.1) and (1.2) may be degenerate at the lateral boundary, the boundary conditions are prescribed not on the whole lateral boundary, but only on $\Sigma_1$ and $\Sigma_2$, which are the nondegenerate and weakly degenerate parts of the lateral boundary. More precisely,

$$\Sigma_i = \left\{ (x, t) \in \partial \Omega \times (0, T) : a_i(x, t) > 0 \right\} \cup \left\{ (x, t) \in \partial \Omega \times (0, T) : a_i(x, t) = 0 \right\},$$

and there exists $0 < \delta < \min\{t, T - t\}$ such that

$$\int_{t - \delta}^{t + \delta} \int_{\{y \in \Omega : |y - x| < \delta\}} \frac{1}{a_i(y, s)} dy ds < +\infty, \quad i = 1, 2.$$

The degenerate equations (1.1) and (1.2) can be used to describe some models from mathematical biology and physics, such as the Lotka-Volterra model and the Keller-Legel model [2, 17]. The degenerate equations (1.1) and (1.2) are double coupled, whose linearized equations are of the form

$$\frac{\partial u}{\partial t} - \text{div}(a_1(x, t) \nabla u) + c_1(x, t) u + c_2(x, t) v = h(x, t) \chi_{\omega_1}, \quad (x, t) \in Q_T, \quad (1.10)$$

$$\frac{\partial v}{\partial t} - \text{div}(a_2(x, t) \nabla v) + c_3(x, t) v = u \chi_{\omega_2}, \quad (x, t) \in Q_T, \quad (1.11)$$
where \(c_i \in L^\infty(Q_T)\) \((i = 1, 2, 3)\). It is noted that \(\omega_1 \cap \omega_2 \neq \emptyset\) is necessary even for the nondegenerate case (see [18]).

In the last forty years, controllability theory has been widely investigated for nondegenerate parabolic equations and there have been a lot of results (see for instance [1, 7, 9–15, 18]). The null controllability and the approximate controllability for heat equations have been proved. The study on the degenerate parabolic equations just began more than ten years ago and a few results have been known. First of all, the following control system governed by a single degenerate parabolic equation has been widely studied

\[
\frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left( x^\alpha \frac{\partial u}{\partial x} \right) + c(x,t)u = h(x,t)\chi_{\omega}, \quad (x,t) \in (0,1) \times (0,T),
\]

\[
u(0,t) = u(1,t) = 0, \quad t \in (0,T), \quad \text{if } 0 < \alpha < 1,
\]

\[
x^\alpha \frac{\partial u}{\partial x}(0,t) = u(1,t) = 0, \quad t \in (0,T), \quad \text{if } \alpha \geq 1,
\]

\[
u(x,0) = u_0(x), \quad x \in (0,1),
\]

where \(c \in L^\infty(Q_T), \alpha > 0\). The Eq. (1.12) is degenerate at the boundary \(x = 0\). It was shown that the problem (1.12)-(1.15) is null controllable if \(0 < \alpha < 2\) [4, 5, 16], while not if \(\alpha \geq 2\) [3]. Although this problem is not null controllable for \(\alpha \geq 2\), it was proved in [3, 19] that it is regional null controllable and approximately controllable, respectively, for each \(\alpha > 0\). Indeed, it was shown in [19] that the problem (1.10), (1.3), (1.5) is approximately controllable in \(L^2(\Omega)\). The null controllability of the linear case for two coupled degenerate equations and the nonlinear case for a single degenerate equation was studied in [6] and [8, 20, 21], respectively. As to the approximate controllability of the coupled degenerate equations, only the linear case was considered. In [22, 23], the linear control system (1.10), (1.11), (1.3)-(1.6) was shown to be approximately controllable, where the controls are constructed by means of its conjugate problem. Such a method should be owed to Lions for nondegenerate equations [14, 15]. Since (1.10) and (1.11) are degenerate, weak solutions with poor regularity should be considered in [22, 23], and the authors had to overcome some technical difficulties to construct the controls.

In this paper, we study the approximate controllability of the semilinear degenerate problem (1.1)-(1.6). Similarly to the nondegenerate case, we prove the approximate controllability of the semilinear problem by using a fixed point argument and the approximate controllability result for the linear problem. The key is to establish many compact estimates to get a fixed point. Since (1.1) and (1.2) are degenerate, classical solutions may not exist and weak solutions with poor regularity should be considered. For such weak solutions, some compact estimates for classical solutions to heat equations are missing. For example, there
is an $L^2$ estimate for the gradient of classical solutions to heat equations, which plays an important role in studying approximate controllability of heat equations, while it fails for weak solutions to the problem (1.1)-(1.6). Therefore, we have to seek techniques to establish desired compact estimates in this paper. By means of the compact estimates and a fixed point theorem, it is shown that the problem (1.1)-(1.6) is approximately controllable in $L^2(\Omega)$. More precisely, for any desired datum $u_d,v_d \in L^2(\Omega)$ and admissible error value $\varepsilon > 0$, it is proved that there exists a control function $h \in L^2(Q_T)$, such that the weak solution $(u,v)$ to the problem (1.1)-(1.6) satisfies

$$\|u(\cdot,T) - u_d(\cdot)\|_{L^2(\Omega)} \leq \varepsilon, \quad \|v(\cdot,T) - v_d(\cdot)\|_{L^2(\Omega)} \leq \varepsilon. \quad (1.16)$$

This paper is organized as follows. In Section 2, we give some compact estimates for weak solutions to the linear problem and prove the well-posedness of the semilinear problem (1.1)-(1.6). In Section 3, we prove the approximate controllability of the problem (1.1)-(1.6) by a fixed point argument and many compact estimates.

## 2 Well-posedness

In this section, we prove the well-posedness of the semilinear problem

$$\begin{align*}
\frac{\partial u}{\partial t} - \text{div}(a_1(x,t)\nabla u) + g(x,t,u,v) &= f(x,t), \quad (x,t) \in Q_T, \\
\frac{\partial v}{\partial t} - \text{div}(a_2(x,t)\nabla v) + q(x,t,v) &= u\chi_{\omega_2}, \quad (x,t) \in Q_T, \\
u(x,t) &= 0, \quad (x,t) \in \Sigma_1, \\
v(x,t) &= 0, \quad (x,t) \in \Sigma_2, \\
u(x,0) &= u_0(x), \quad x \in \Omega, \\
u(x,0) &= v_0(x), \quad x \in \Omega,
\end{align*}$$

where $f \in L^2(Q_T)$ and $u_0,v_0 \in L^2(\Omega)$.

For $i = 1,2$, denote $B_i$ to be the closure of $C^0_0(Q_T)$ with respect to the norm

$$\|w\|_{B_i} = \left(\iint_{Q_T} a_i(x,t) \left( (w(x,t))^2 + |\nabla w(x,t)|^2 \right) \, dx dt \right)^{\frac{1}{2}}, \quad w \in B_i.$$
Definition 2.1. A pair of functions \((u,v)\) is called to be the weak solution of the problem (2.1)-(2.6), if \(u \in L^2(Q_T) \cap B_1, v \in L^2(Q_T) \cap B_2,\) and \((u,v)\) satisfies

\[
\iint_{Q_T} \left( - \frac{\partial \varphi}{\partial t} + a_1 \nabla u \cdot \nabla \varphi + g(x,t,u,v) \varphi \right) \, dx \, dt = \iint_{Q_T} f \varphi \, dx \, dt + \int_{\Omega} u_0(x) \varphi(x,0) \, dx,
\]

\[
\iint_{Q_T} \left( - \frac{\partial \psi}{\partial t} + a_2 \nabla v \cdot \nabla \psi + q(x,t,v) \psi \right) \, dx \, dt = \iint_{Q_T} u \chi_{\omega_2} \psi \, dx \, dt + \int_{\Omega} v_0(x) \psi(x,0) \, dx
\]

for any \(\varphi \in H^1((0,T);L^2(\Omega)) \cap B_1\) and \(\psi \in H^1((0,T);L^2(\Omega)) \cap B_2\) with \(\varphi(\cdot,T) = \psi(\cdot,T) = 0\).

Let us first recall the well-posedness of the linear problem

\[
\begin{align*}
\frac{\partial u}{\partial t} - \text{div}(a_1(x,t) \nabla u) + c_1(x,t) u + c_2(x,t) v &= f(x,t), & (x,t) \in Q_T, \\
\frac{\partial v}{\partial t} - \text{div}(a_2(x,t) \nabla v) + c_3(x,t) v &= u \chi_{\omega_2}, & (x,t) \in Q_T, \\
u(x,t) &= 0, & (x,t) \in \Sigma_1, \\
v(x,t) &= 0, & (x,t) \in \Sigma_2, \\
u(x,0) &= u_0(x), & x \in \Omega, \\
v(x,0) &= v_0(x), & x \in \Omega,
\end{align*}
\]

where \(c_i \in L^\infty(Q_T)\) \((i = 1,2,3)\).

Weak solutions to the problem (2.7)-(2.12) can be defined similarly. As mentioned in [22], the problem (2.7)-(2.12) is well-posed.

Lemma 2.1. For \(c_i \in L^\infty(Q_T)\) with \(\|c_i\|_{L^\infty(Q_T)} \leq M\) \((i = 1,2,3)\), \(f \in L^2(Q_T), u_0,v_0 \in L^2(\Omega)\), the problem (2.7)-(2.12) admits a unique weak solution \((u,v)\). Furthermore, \((u,v)\) satisfies the following properties:

(i) \((u,v)\) satisfies

\[
\|u\|_{L^\infty((0,T);L^2(\Omega))} + \|v\|_{L^\infty((0,T);L^2(\Omega))} + \left\|a_1^{\frac{1}{2}} |\nabla u|\right\|_{L^2(Q_T)} + \left\|a_2^{\frac{1}{2}} |\nabla v|\right\|_{L^2(Q_T)} \leq C \left( \|f\|_{L^2(Q_T)} + \|u_0\|_{L^2(\Omega)} + \|v_0\|_{L^2(\Omega)} \right),
\]

where \(C > 0\) is a constant depending only on \(T\) and \(M\).
Corollary 2.1. Assume that \( \|c_i^{(k)}\|_{L^\infty(Q_T)} (i = 1, 2, 3), \|f^{(k)}\|_{L^2(Q_T)}, \|u_0^{(k)}\|_{L^2(\Omega)} \) and \( \|v_0^{(k)}\|_{L^2(\Omega)} \) are uniformly bounded, and

\[
\begin{align*}
& c_i^{(k)} \rightharpoonup c_i \text{ weakly } * \text{ in } L^\infty(Q_T), \quad i = 1, 2, 3, \quad f^{(k)} \rightharpoonup f \text{ weakly in } L^2(Q_T), \\
& u_0^{(k)} \rightharpoonup u_0 \text{ weakly in } L^2(\Omega), \quad v_0^{(k)} \rightharpoonup v_0 \text{ weakly in } L^2(\Omega) \text{ as } k \to \infty.
\end{align*}
\]
There exists a subsequence of \( \{(u^{(k)}, v^{(k)})\}_{k=1}^{\infty} \), which converges to \((u, v)\) weakly in \(L^2(Q_T) \times L^2(Q_T)\) and strongly in \(L^1(Q_T) \times L^1(Q_T)\), where \((u, v)\) is the solution to the problem (2.7)-(2.12), while \((u^{(k)}, v^{(k)})\) is the solution to the problem (2.7)-(2.12) with \(c_i = c_i^{(k)} (i = 1, 2, 3), f = f^{(k)}, u_0 = u_0^{(k)}, v_0 = v_0^{(k)}\).

**Corollary 2.2.** Assume that \(\|c_i^{(k)}\|_{L^\infty(Q_T)} (i = 1, 2, 3), \|f^{(k)}\|_{L^2(Q_T)}, \|u_0^{(k)}\|_{L^2(\Omega)}\) and \(\|v_0^{(k)}\|_{L^2(\Omega)}\) are uniformly bounded, and

\[
\left( c_i^{(k)} - c_i \right) \rightarrow 0 \text{ weakly * in } L^\infty(Q_T), \quad i = 1, 2, 3, \quad f^{(k)} \rightarrow f \text{ in } L^2(Q_T),
\]

\(u_0^{(k)} \rightarrow u_0 \text{ in } L^2(\Omega), \quad v_0^{(k)} \rightarrow v_0 \text{ in } L^2(\Omega) \text{ as } k \rightarrow \infty.\)

Then \(\{(u^{(k)}, v^{(k)})\}_{k=1}^{\infty}\) converges to \((u, v)\) in \(L^\infty((0, T); L^2(\Omega)) \times L^\infty((0, T); L^2(\Omega))\), where \((u, v)\) is the solution to the problem (2.7)-(2.12), while \((u^{(k)}, v^{(k)})\) is the solution to the problem (2.7)-(2.12) with \(c_i = c_i^{(k)} (i = 1, 2, 3), f = f^{(k)}, u_0 = u_0^{(k)}, v_0 = v_0^{(k)}\).

**Corollary 2.3.** Assume that \(\|c_i^{(k)}\|_{L^\infty(Q_T)} (i = 1, 2, 3), \|f^{(k)}\|_{L^2(Q_T)}, \|u_0^{(k)}\|_{L^2(\Omega)}\) and \(\|v_0^{(k)}\|_{L^2(\Omega)}\) are uniformly bounded, and

\[
c_i^{(k)} \rightarrow c_i \text{ weakly * in } L^\infty(Q_T), \quad i = 1, 2, 3, \quad f^{(k)} \rightarrow f \text{ in } L^2(Q_T),
\]

\(u_0^{(k)} \rightarrow u_0 \text{ in } L^2(\Omega), \quad v_0^{(k)} \rightarrow v_0 \text{ in } L^2(\Omega) \text{ as } k \rightarrow \infty.\)

There exists a subsequence of \(\{(u^{(k)}, v^{(k)})\}_{k=1}^{\infty}\), which converges to \((u, v)\) in \(L^2(Q_T) \times L^2(Q_T)\), where \((u, v)\) is the solution to the problem (2.7)-(2.12), while \((u^{(k)}, v^{(k)})\) is the solution to the problem (2.7)-(2.12) with \(c_i = c_i^{(k)} (i = 1, 2, 3), f = f^{(k)}, u_0 = u_0^{(k)}, v_0 = v_0^{(k)}\).

For any \(w_1, w_2 \in L^1(Q_T)\), we define the functions

\[
\sigma_1[w_1, w_2](x, t) = \int_0^1 \frac{\partial}{\partial s} g(x, t, \lambda w_1(x, t), \lambda w_2(x, t)) d\lambda, \quad (x, t) \in Q_T,
\]

\[
\sigma_2[w_1, w_2](x, t) = \int_0^1 \frac{\partial}{\partial p} g(x, t, \lambda w_1(x, t), \lambda w_2(x, t)) d\lambda, \quad (x, t) \in Q_T,
\]

\[
\sigma_3[w_2](x, t) = \begin{cases} 
\frac{1}{w_2(x, t)} (q(x, t, w_2(x, t)) - q(x, t, 0)), & w_2(x, t) \neq 0, \\
0, & w_2(x, t) = 0, 
\end{cases} \quad (x, t) \in Q_T.
\]
It follows from (1.7)-(1.9) that \( \sigma_i[w_1,w_2], \sigma_2[w_1,w_2], \sigma_3[w_2] \in L^\infty(Q_T) \), and
\[
\begin{align*}
\|\sigma_1[w_1,w_2]\|_{L^\infty(Q_T)} &\leq M, \\
\|\sigma_2[w_1,w_2]\|_{L^\infty(Q_T)} &\leq M, \\
\|\sigma_3[w_2]\|_{L^\infty(Q_T)} &\leq M. \tag{2.14}
\end{align*}
\]
Furthermore, \( \sigma_i \ (i = 1, 2, 3) \) satisfy the following properties, whose proof is the same as [20, Lemma 3.1] and [22, Lemma 3.1].

\begin{lemma}
Assume that \( \{w_1^{(k)}\}_{k=1}^{\infty} \) and \( \{w_2^{(k)}\}_{k=1}^{\infty} \) converge to \( w_1, w_2 \) in \( L^1(Q_T) \), respectively. Then
\[
\begin{align*}
\sigma_i[w_1^{(k)},w_2^{(k)}] &\rightarrow \sigma_i[w_1,w_2], \quad i = 1, 2, \\
\left(\sigma_i[w_1^{(k)},w_2^{(k)}] - \sigma_i[w_1,w_2]\right)^2 &\rightarrow 0, \quad i = 1, 2,
\end{align*}
\]
weakly \( * \) in \( L^\infty(Q_T) \) as \( k \rightarrow \infty \).
\end{lemma}

Below we prove the existence and uniqueness of the weak solution to the problem (2.1)-(2.6).

\begin{theorem}
For any \( f \in L^2(Q_T) \) and \( u_0, v_0 \in L^2(\Omega) \), the problem (2.1)-(2.6) admits a unique weak solution.
\end{theorem}

\begin{proof}
We first prove the existence by the Schauder fixed point theorem. For any \( w_1, w_2 \in L^1(Q_T) \), one gets that \( \sigma_1[w_1,w_2], \sigma_2[w_1,w_2], \sigma_3[w_2] \in L^\infty(Q_T) \) satisfy (2.14).

Owing to Lemma 2.1, the linear problem
\[
\begin{align*}
\frac{\partial u}{\partial t} - \text{div}(a_1(x,t)\nabla u) + \sigma_1[w_1,w_2](x,t)u + \sigma_2[w_1,w_2](x,t)v \\
= f(x,t) - g(x,t,0,0), \quad & (x,t) \in Q_T, \tag{2.15} \\
\frac{\partial v}{\partial t} - \text{div}(a_2(x,t)\nabla v) + \sigma_3[w_2](x,t)v = u\chi_{w_2} - q(x,t,0), \quad & (x,t) \in Q_T, \tag{2.16} \\
u(x,t) = 0, \quad & (x,t) \in \Sigma_1, \tag{2.17} \\
v(x,t) = 0, \quad & (x,t) \in \Sigma_2, \tag{2.18} \\
u(x,0) = u_0(x), \quad & x \in \Omega, \tag{2.19} \\
v(x,0) = v_0(x), \quad & x \in \Omega \tag{2.20}
\end{align*}
\]
admits a unique weak solution \((u,v)\) with \(u \in L^2(Q_T) \cap \mathcal{B}_1\) and \(v \in L^2(Q_T) \cap \mathcal{B}_2\). Define the mapping \(\Lambda\) as follows

\[
\Lambda : L^1(Q_T) \times L^1(Q_T) \to L^1(Q_T) \times L^1(Q_T), \quad (w_1, w_2) \mapsto (u, v),
\]

where \((u,v)\) is the weak solution to the problem (2.15)-(2.20).

Assume that \(\{w_1^{(k)}\}_{k=1}^{\infty}\) and \(\{w_2^{(k)}\}_{k=1}^{\infty}\) converge to \(w_1, w_2\) in \(L^1(Q_T)\), respectively. Then, it follows from Lemma 2.2 and Corollary 2.2 that \(\{\Lambda(w_1^{(k)}, w_2^{(k)})\}_{k=1}^{\infty}\) converges to \(\Lambda(w_1, w_2)\) in \(L^\infty((0,T); L^2(\Omega)) \times L^\infty((0,T); L^2(\Omega))\) and thus in \(L^1(Q_T) \times L^1(Q_T)\). Hence \(\Lambda\) is continuous.

For any \(\{w_1^{(k)}\}_{k=1}^{\infty}, \{w_2^{(k)}\}_{k=1}^{\infty} \subset L^1(Q_T)\), it holds that \(\sigma_1[w_1^{(k)}, w_2^{(k)}], \sigma_2[w_1^{(k)}, w_2^{(k)}]\), \(\sigma_3[w_2^{(k)}] \in L^\infty(Q_T)\) satisfy (2.14). Therefore, there exists a subsequence \(\{k_m\}_{m=1}^{\infty}\) of \(\{k\}_{k=1}^{\infty}\), such that

\[
\left\{ \sigma_1 \left[ w_1^{(k_m)}, w_2^{(k_m)} \right] \right\}_{m=1}^{\infty}, \quad \left\{ \sigma_2 \left[ w_1^{(k_m)}, w_2^{(k_m)} \right] \right\}_{m=1}^{\infty}, \quad \left\{ \sigma_3 \left[ w_2^{(k_m)} \right] \right\}_{m=1}^{\infty}
\]

converge weakly * in \(L^\infty(Q_T)\) as \(m \to \infty\). Thanks to Corollary 2.1, there exists a subsequence \(\{\Lambda(w_1^{(k_m)}, w_2^{(k_m)})\}_{m=1}^{\infty} \subset \{\Lambda(w_1^{(k)}, w_2^{(k)})\}_{k=1}^{\infty}\), which converges in \(L^1(Q_T) \times L^1(Q_T)\). Hence \(\Lambda\) is precompact.

According to the above discussion, the restriction of the mapping \(\Lambda\) on the close and convex hull of the range of \(\Lambda\) satisfies the hypotheses of the Schauder fixed point theorem. So \(\Lambda\) admits a fixed point \((u,v)\) such that \((u,v) = \Lambda(u,v)\) is a weak solution of the problem (2.1)-2.6, where \(u \in L^2(Q_T) \cap \mathcal{B}_1, v \in L^2(Q_T) \cap \mathcal{B}_2\).

Below we prove the uniqueness result. Assume that \((\tilde{u},\tilde{v})\) and \((\check{u},\check{v})\) are two weak solutions to the problem (2.1)-2.6. Set

\[
w_1(x,t) = \tilde{u}(x,t) - \check{u}(x,t), \quad w_2(x,t) = \tilde{v}(x,t) - \check{v}(x,t), \quad (x,t) \in Q_T.
\]

Then \((w_1, w_2)\) is the weak solution to the following linear problem:

\[
\begin{align*}
\frac{\partial w_1}{\partial t} - \text{div}(a_1(x,t) \nabla w_1) + c_1(x,t)w_1 + c_2(x,t)w_2 &= 0, \quad (x,t) \in Q_T, \\
\frac{\partial w_2}{\partial t} - \text{div}(a_2(x,t) \nabla w_2) + c_3(x,t)w_2 &= w_1 \chi w_2, \quad (x,t) \in Q_T, \\
w_1(x,t) &= 0, \quad (x,t) \in \Sigma_1, \\
w_2(x,t) &= 0, \quad (x,t) \in \Sigma_1, \\
w_1(x,0) &= 0, \quad x \in \Omega, \\
w_2(x,0) &= 0, \quad x \in \Omega,
\end{align*}
\]
where for \((x,t) \in Q_T,\)

\[
c_1(x,t) = \int_0^1 \frac{\partial}{\partial s} g(x,t,\lambda \bar{u}(x,t) + (1-\lambda)\tilde{u}(x,t),\lambda \bar{\sigma}(x,t) + (1-\lambda)\tilde{\sigma}(x,t)) \, d\lambda,
\]

\[
c_2(x,t) = \int_0^1 \frac{\partial}{\partial p} g(x,t,\lambda \bar{u}(x,t) + (1-\lambda)\tilde{u}(x,t),\lambda \bar{\sigma}(x,t) + (1-\lambda)\tilde{\sigma}(x,t)) \, d\lambda,
\]

\[
c_3(x,t) = \begin{cases} 
\frac{q(x,t,\tilde{\bar{\sigma}}(x,t)) - q(x,t,\tilde{\sigma}(x,t))}{\bar{\sigma}(x,t) - \tilde{\sigma}(x,t)}, & \bar{\sigma}(x,t) \neq \tilde{\sigma}(x,t), \\
0, & \bar{\sigma}(x,t) = \tilde{\sigma}(x,t).
\end{cases}
\]

It follows from (1.7)-(1.9) that \(c_1, c_2, c_3 \in L^\infty(Q_T).\) Then one gets from Lemma 2.1 that

\[
\|w_1\|_{L^\infty((0,T);L^2(\Omega))} + \|w_2\|_{L^\infty((0,T);L^2(\Omega))} \leq 0,
\]

which implies that

\[
w_1(x,t) = 0, \quad w_2(x,t) = 0, \quad (x,t) \in Q_T.
\]

That is to say, the weak solution to the problem (2.1)-(2.6) is unique. \(\square\)

### 3 Approximate controllability

In this section, we prove the approximate controllability of the semilinear problem (1.1)-(1.6).

First we recall the approximate controllability of the linear problem in [22]. Consider the following linear control system

\[
\begin{align*}
\frac{\partial u}{\partial t} - \text{div}(a_1(x,t) \nabla u) + c_1(x,t)u + c_2(x,t)v &= h(x,t)\chi_{\omega_1}, \quad (x,t) \in Q_T, \\
\frac{\partial v}{\partial t} - \text{div}(a_2(x,t) \nabla v) + c_3(x,t)v &= u\chi_{\omega_2}, \\
&= u(x,t) = 0, \quad (x,t) \in \Sigma_1, \\
v(x,t) &= 0, \quad (x,t) \in \Sigma_2, \\
&= u(x,0) = 0, \quad x \in \Omega, \\
v(x,0) &= 0, \quad x \in \Omega, \\
\|u(\cdot,T) - u_d(\cdot)\|_{L^2(\Omega)} &\leq \varepsilon, \\
\|v(\cdot,T) - v_d(\cdot)\|_{L^2(\Omega)} &\leq \varepsilon,
\end{align*}
\]
where \( c_i \in L^\infty(Q_T) \) \((i = 1,2,3)\), \( \varepsilon > 0 \) is a constant, and \( u_d,v_d \in L^2(\Omega) \). The approximate controllability of the problem (3.1)-(3.8) is related to its conjugate problem

\[
\begin{align*}
-\frac{\partial z}{\partial t} - \text{div}(a_1(x,t) \nabla z) + c_1(x,t)z &= y \chi_{\omega_1}, & (x,t) &\in Q_T, \\
-\frac{\partial y}{\partial t} - \text{div}(a_2(x,t) \nabla y) + c_2(x,t)z + c_3(x,t)y &= 0, & (x,t) &\in Q_T, \\
z(x,t) &= 0, & (x,t) &\in \Sigma_1, \\
y(x,t) &= 0, & (x,t) &\in \Sigma_2, \\
z(x,T) &= z_0(x), & x &\in \Omega, \\
y(x,T) &= y_0(x), & x &\in \Omega.
\end{align*}
\]

Define the mapping

\[
\mathcal{L} : \mathcal{H} \times \mathcal{G} \to L^2(\omega_1 \times (0,T)), \quad ((z_0,y_0),(c_1,c_2,c_3)) \mapsto z_0 \chi_{\omega_1},
\]

where \((z,y)\) is the solution to the problem (3.9)-(3.14), \( \mathcal{H} = L^2(\Omega) \times L^2(\Omega) \) with the norm

\[
\|(z_0,y_0)\|_{\mathcal{H}} = \left( \|z_0\|_{L^2(\Omega)}^2 + \|y_0\|_{L^2(\Omega)}^2 \right)^{1/2}, \quad (z_0,y_0) \in \mathcal{H},
\]

and \( \mathcal{G} = L^\infty(Q_T) \times L^\infty(Q_T) \times L^\infty(Q_T) \) with the norm

\[
\|(c_1,c_2,c_3)\|_{\mathcal{G}} = \|c_1\|_{L^\infty(Q_T)} + \|c_2\|_{L^\infty(Q_T)} + \|c_3\|_{L^\infty(Q_T)}, \quad (c_1,c_2,c_3) \in \mathcal{G}.
\]

Define the functional as follows

\[
J((z_0,y_0),(c_1,c_2,c_3),(u_d,v_d)) = \frac{1}{2} \int_0^T \int_{\omega_1} \|\mathcal{L}((z_0,y_0),(c_1,c_2,c_3))(x,t)\|^2 \, dx \, dt + \varepsilon \|(z_0,y_0)\|_{\mathcal{H}} - (u_d,v_d, (z_0,y_0))_{\mathcal{H}},
\]

where \( (\cdot,\cdot)_{\mathcal{H}} \) is the inner product in \( \mathcal{H} \).

**Lemma 3.1.** The problem (3.9)-(3.14) possesses the property of the unique continuation. That is to say, if

\[
\mathcal{L}((z_0,y_0),(c_1,c_2,c_3)) = 0 \quad \text{a.e. in } \omega_1 \times (0,T),
\]

then \((z,y) = (0,0)\) a.e. in \(Q_T\).
Lemma 3.2. For \((c_1, c_2, c_3) \in \mathcal{G}\) and \((u_d, v_d) \in \mathcal{H}\), the functional \(J(\cdot, (c_1, c_2, c_3), (u_d, v_d))\) achieves its minimum at a unique point \((\hat{z}_0, \hat{y}_0) \in \mathcal{H}\). Furthermore, there exists a control function \(h = z \in L^2(\mathbb{R}_T)\) such that the solution to the problem (3.1)-(3.6) satisfies (3.7)-(3.8), where \((\hat{z}, \hat{y})\) is the weak solution to the conjugate problem (3.9)-(3.14) with \((z_0, y_0) = (\hat{z}_0, \hat{y}_0)\).

Lemmas 3.1 and 3.2 were proved in [22], and the proof is similar to [23, Proposition 2.2, Theorem 2.2]. Such a method should be owed to Lions for heat equations [14,15].

Below we do some preliminaries for the semilinear problem. Define the mapping as follows

\[
\mathcal{M}: \mathcal{G} \times \mathcal{H} \to \mathcal{H}, \quad ((c_1, c_2, c_3), (u_d, v_d)) \mapsto (\hat{z}_0, \hat{y}_0),
\]

where \((\hat{z}_0, \hat{y}_0)\) is the unique minimum point of the functional \(J(\cdot, (c_1, c_2, c_3), (u_d, v_d))\).

Proposition 3.1. Assume that \(B\) is a bounded subset of \(\mathcal{G}, K \subset \mathcal{H}\) is compact. Then \(\mathcal{M}(B \times K)\) is bounded in \(\mathcal{H}\).

Proof. The linearity of \(\mathcal{L}\) and the Hölder inequality imply that \(J\) is a strictly convex and continuous functional. For any \((c_1, c_2, c_3) \in B\) and \((u_d, v_d) \in K\), it holds that

\[
J((0,0), (c_1, c_2, c_3), (u_d, v_d)) = 0.
\]

Therefore, we only need to prove

\[
\inf_{\mathcal{H}} \frac{J((z_0, y_0), (c_1, c_2, c_3), (u_d, v_d))}{\| (z_0, y_0) \|_{\mathcal{H}}} \geq \varepsilon \quad \text{as} \quad \| (z_0, y_0) \|_{\mathcal{H}} \to +\infty. \quad (3.15)
\]

If (3.15) is not true, then there exist three sequences

\[
\left\{ \left( u_d^{(k)}, v_d^{(k)} \right) \right\}_{k=1}^{\infty} \subset K, \quad \left\{ \left( c_1^{(k)}, c_2^{(k)}, c_3^{(k)} \right) \right\}_{k=1}^{\infty} \subset B, \quad \left\{ \left( z_0^{(k)}, y_0^{(k)} \right) \right\}_{k=1}^{\infty} \subset \mathcal{H},
\]

such that

\[
\lim_{k \to \infty} \left\| (z_0^{(k)}, y_0^{(k)}) \right\|_{\mathcal{H}} = +\infty,
\]

\[
\lim_{k \to \infty} \frac{J\left( (z_0^{(k)}, y_0^{(k)}), (c_1^{(k)}, c_2^{(k)}, c_3^{(k)}), (u_d^{(k)}, v_d^{(k)}) \right)}{\left\| (z_0^{(k)}, y_0^{(k)}) \right\|_{\mathcal{H}}} < \varepsilon. \quad (3.16)
\]
It follows from Lemma 3.1 and (3.17) that

\[
\lim_{k \to \infty} \int_0^T \int_{\Omega_1} \mathcal{L} \left( \left( z_0^{(k)}, y_0^{(k)} \right), \left( c_1^{(k)}, c_2^{(k)}, c_3^{(k)} \right) \right) (x,t) \, dx \, dt = 0,
\]

which leads to

\[
\int_0^T \int_{\Omega_1} \mathcal{L} \left( (\bar{z}_0, \bar{y}_0), (c_1, c_2, c_3) \right) (x,t) \, dx \, dt = 0.
\]

(3.17)

It follows from Lemma 3.1 and (3.17) that \((\bar{z}_0, \bar{y}_0) = (0,0)\) a.e. in \(\Omega\). Hence

\[
\lim_{k \to \infty} \frac{J \left( \left( z_0^{(k)}, y_0^{(k)} \right), \left( c_1^{(k)}, c_2^{(k)}, c_3^{(k)} \right), \left( u_d^{(k)}, \nu_d^{(k)} \right) \right)}{\left\| \left( z_0^{(k)}, y_0^{(k)} \right) \right\|_{\mathcal{H}}} \geq \varepsilon - \lim_{k \to \infty} \frac{\left\langle \left( u_d^{(k)}, \nu_d^{(k)} \right), \left( z_0^{(k)}, y_0^{(k)} \right) \right\rangle_{\mathcal{H}}}{\left\| \left( z_0^{(k)}, y_0^{(k)} \right) \right\|_{\mathcal{H}}}
\]
\[
-\varepsilon - \lim_{k \to \infty} \left\langle \left( u_d^{(k)}, v_d^{(k)} \right), \left( \hat{\xi}_0^{(k)}, \hat{\eta}_0^{(k)} \right) \right\rangle_{\mathcal{H}} = \varepsilon,
\]
which contradicts with (3.16). So (3.15) holds.

\[\square\]

**Proposition 3.2.** Assume that \( (u_d^{(k)}, v_d^{(k)})_{k=1}^{\infty} \) converges to \((u_d, v_d)\) in \(\mathcal{H}\) as \(k \to \infty\), \(\|c_i^{(k)}\|_{L^\infty(Q_T)} (i = 1, 2, 3)\) are uniformly bounded, and

\[c_i^{(k)} \rightharpoonup c_i \text{ weakly } * \text{ in } L^\infty(Q_T), \quad k \to \infty, \quad i = 1, 2, 3.\]

Then there exists a subsequence of \(\{\mathcal{M}(\{c_1^{(k)}, c_2^{(k)}, c_3^{(k)}\}, (u_d^{(k)}, v_d^{(k)})\})_{k=1}^{\infty}\) such that it converges to \(\mathcal{M}(c_1, c_2, c_3, (u_d, v_d))\) in \(\mathcal{H}\).

**Proof.** For convenience, denote that

\[
\left( \hat{\xi}_0^{(k)}, \hat{\eta}_0^{(k)} \right) = \mathcal{M} \left( (c_1^{(k)}, c_2^{(k)}, c_3^{(k)}), (u_d^{(k)}, v_d^{(k)}) \right), \quad k = 1, 2, \ldots,
\]

\[
\left( \hat{\xi}_0^{(k)}, \hat{\eta}_0^{(k)} \right) = \mathcal{M} \left( (c_1, c_2, c_3), (u_d, v_d) \right).
\]

It follows from Proposition 3.1 that \(\{(\hat{\xi}_0^{(k)}, \hat{\eta}_0^{(k)})\}_{k=1}^{\infty}\) is bounded in \(\mathcal{H}\). Therefore, there exists a subsequence of \(\{(\hat{\xi}_0^{(k)}, \hat{\eta}_0^{(k)})\}_{k=1}^{\infty}\), denoted by itself for convenience, which converges to \((\hat{\xi}_0, \hat{\eta}_0)\) weakly in \(\mathcal{H}\). By Corollary 2.1, there exists a subsequence of \(\{\mathcal{L}(\{\hat{\xi}_0^{(k)}, \hat{\eta}_0^{(k)}\}, (c_1^{(k)}, c_2^{(k)}, c_3^{(k)}))\}_{k=1}^{\infty}\), denoted by itself for convenience, such that

\[
\mathcal{L} \left( (\hat{\xi}_0^{(k)}, \hat{\eta}_0^{(k)}), (c_1^{(k)}, c_2^{(k)}, c_3^{(k)}) \right) \rightharpoonup \mathcal{L} \left( (\hat{\xi}_0, \hat{\eta}_0), (c_1, c_2, c_3) \right)
\]

weakly in \(L^2(Q_T), \quad k \to \infty\). (3.18)

Therefore, one can get that

\[
\begin{align*}
\lim_{k \to \infty} & \int \frac{1}{2} \int_0^T \int_{\omega_1} \left| \mathcal{L} \left( (\hat{\xi}_0^{(k)}, \hat{\eta}_0^{(k)}), (c_1^{(k)}, c_2^{(k)}, c_3^{(k)}) \right) \right|^2 \, dx \, dt \\
&+ \varepsilon \left\| \left( \hat{\xi}_0^{(k)}, \hat{\eta}_0^{(k)} \right) \right\|_{\mathcal{H}} - \left\langle \left( u_d^{(k)}, v_d^{(k)} \right), \left( \xi_0^{(k)}, \eta_0^{(k)} \right) \right\rangle_{\mathcal{H}} \\
\geq & \int \frac{1}{2} \int_0^T \int_{\omega_1} \left| \mathcal{L} \left( (\hat{\xi}_0, \hat{\eta}_0), (c_1, c_2, c_3) \right) \right|^2 \, dx \, dt \\
&+ \varepsilon \left\| (\hat{\xi}_0, \hat{\eta}_0) \right\|_{\mathcal{H}} - \left\langle (u_d, v_d), (\xi_0, \eta_0) \right\rangle_{\mathcal{H}} \\
= & \int \left( (\hat{\xi}_0, \hat{\eta}_0), (c_1, c_2, c_3), (u_d, v_d) \right). (3.19)
\end{align*}
\]
Moreover, it follows from Corollary 2.3 that
\[
\lim_{k \to \infty} J \left( \left( \hat{z}_0, \hat{y}_0 \right), \left( c_1^{(k)}, c_2^{(k)}, c_3^{(k)} \right), \left( u_d^{(k)}, v_d^{(k)} \right) \right) \\
= \lim_{k \to \infty} \left( \frac{1}{2} \int_0^T \int_{\omega_1} \mathcal{L} \left( \left( \hat{z}_0, \hat{y}_0 \right), \left( c_1^{(k)}, c_2^{(k)}, c_3^{(k)} \right) \right) \right)^2 \, dx \, dt \\
+ \varepsilon \left\| \left( \hat{z}_0, \hat{y}_0 \right) \right\|_{\mathcal{H}} - \left( \left( u_d^{(k)}, v_d^{(k)} \right), \left( \hat{z}_0, \hat{y}_0 \right) \right) \\
= J \left( \left( \hat{z}_0, \hat{y}_0 \right), \left( c_1, c_2, c_3 \right), \left( u_d, v_d \right) \right). \tag{3.20}
\]

It is noted that
\[
J \left( \left( \hat{z}_0, \hat{y}_0 \right), \left( c_1^{(k)}, c_2^{(k)}, c_3^{(k)} \right), \left( u_d^{(k)}, v_d^{(k)} \right) \right) \\
\leq J \left( \left( \hat{z}_0, \hat{y}_0 \right), \left( c_1^{(k)}, c_2^{(k)}, c_3^{(k)} \right), \left( u_d^{(k)}, v_d^{(k)} \right) \right) \\
\leq \lim_{k \to \infty} J \left( \left( \hat{z}_0, \hat{y}_0 \right), \left( c_1^{(k)}, c_2^{(k)}, c_3^{(k)} \right), \left( u_d^{(k)}, v_d^{(k)} \right) \right) \\
= J \left( \left( \hat{z}_0, \hat{y}_0 \right), \left( c_1, c_2, c_3 \right), \left( u_d, v_d \right) \right). \tag{3.21}
\]

Letting \( k \to \infty \) in (3.21) and using (3.19) and (3.20), one gets that
\[
J \left( \left( \hat{z}_0, \hat{y}_0 \right), \left( c_1, c_2, c_3 \right), \left( u_d, v_d \right) \right) \\
\leq \lim_{k \to \infty} J \left( \left( \hat{z}_0, \hat{y}_0 \right), \left( c_1^{(k)}, c_2^{(k)}, c_3^{(k)} \right), \left( u_d^{(k)}, v_d^{(k)} \right) \right) \\
\leq \lim_{k \to \infty} J \left( \left( \hat{z}_0, \hat{y}_0 \right), \left( c_1^{(k)}, c_2^{(k)}, c_3^{(k)} \right), \left( u_d^{(k)}, v_d^{(k)} \right) \right) \\
= J \left( \left( \hat{z}_0, \hat{y}_0 \right), \left( c_1, c_2, c_3 \right), \left( u_d, v_d \right) \right). 
\]

It follows from Lemma 3.1 that \( (\hat{z}_0, \hat{y}_0) = (\hat{z}_0, \hat{y}_0) \), and
\[
\lim_{k \to \infty} J \left( \left( \hat{z}_0^{(k)}, \hat{y}_0^{(k)} \right), \left( c_1^{(k)}, c_2^{(k)}, c_3^{(k)} \right), \left( u_d^{(k)}, v_d^{(k)} \right) \right) \\
= J \left( \left( \hat{z}_0, \hat{y}_0 \right), \left( c_1, c_2, c_3 \right), \left( u_d, v_d \right) \right). \tag{3.22}
\]

Note that \( \{ (\hat{z}_0^{(k)}, \hat{y}_0^{(k)}) \}_{k=1}^{\infty} \) converges weakly to \( (\hat{z}_0, \hat{y}_0) \) in \( \mathcal{H} \). One can prove by using (3.18) and (3.22) that
\[
\lim_{k \to \infty} \left\| (\hat{z}_0^{(k)}, \hat{y}_0^{(k)}) \right\|_{\mathcal{H}} = \left\| (\hat{z}_0, \hat{y}_0) \right\|_{\mathcal{H}},
\]
which yields that \( \{ (\hat{z}_0^{(k)}, \hat{y}_0^{(k)}) \}_{k=1}^{\infty} \) converges to \( (\hat{z}_0, \hat{y}_0) \) in \( \mathcal{H} \). \qed
Now we are ready to prove the approximate controllability of the semilinear problem (1.1)-(1.6) by using the Schauder fixed point theorem. Assume that $u_0, v_0, u_d, v_d \in L^2(\Omega)$. For any $w_1, w_2 \in L^1(Q_T)$, let $\sigma_1[w_1, w_2], \sigma_2[w_1, w_2], \sigma_3[w_2]$ be defined in Section 2. Assume that $(\tilde{u}, \tilde{v})$ is the weak solution to the problem

$$
\frac{\partial \tilde{u}}{\partial t} - \text{div}(a_1(x,t) \nabla \tilde{u}) + \sigma_1[w_1, w_2](x,t)\tilde{u} + \sigma_2[w_1, w_2](x,t)\tilde{v} = -g(x,t,0,0),
$$

$$
(x,t) \in Q_T, (3.23)
$$

$$
\frac{\partial \tilde{v}}{\partial t} - \text{div}(a_2(x,t) \nabla \tilde{v}) + \sigma_3[w_2](x,t)\tilde{v} = \tilde{u} \chi_{\omega_2} - q(x,t,0),
$$

$$
(x,t) \in Q_T, (3.24)
$$

$$
\tilde{u}(x,t) = 0, (x,t) \in \Sigma_1, (3.25)
$$

$$
\tilde{v}(x,t) = 0, (x,t) \in \Sigma_2, (3.26)
$$

$$
\tilde{u}(x,0) = u_0(x), x \in \Omega, (3.27)
$$

$$
\tilde{v}(x,0) = v_0(x), x \in \Omega. (3.28)
$$

Consider the following control system

$$
\frac{\partial \hat{u}}{\partial t} - \text{div}(a_1(x,t) \nabla \hat{u}) + \sigma_1[w_1, w_2](x,t)\hat{u} + \sigma_2[w_1, w_2](x,t)\hat{v} = h \chi_{\omega_1},
$$

$$
(x,t) \in Q_T, (3.29)
$$

$$
\frac{\partial \hat{v}}{\partial t} - \text{div}(a_2(x,t) \nabla \hat{v}) + \sigma_3[w_2](x,t)\hat{v} = \hat{u} \chi_{\omega_2},
$$

$$
(x,t) \in Q_T, (3.30)
$$

$$
\hat{u}(x,t) = 0, (x,t) \in \Sigma_1, (3.31)
$$

$$
\hat{v}(x,t) = 0, (x,t) \in \Sigma_2, (3.32)
$$

$$
\hat{u}(x,0) = 0, x \in \Omega, (3.33)
$$

$$
\hat{v}(x,0) = 0, x \in \Omega, (3.34)
$$

$$
\|\hat{u}(\cdot,T) - (u_d(\cdot) - \tilde{u}(\cdot,T))\|_{L^2(\Omega)} \leq \varepsilon, (3.35)
$$

$$
\|\hat{v}(\cdot,T) - (v_d(\cdot) - \tilde{v}(\cdot,T))\|_{L^2(\Omega)} \leq \varepsilon, (3.36)
$$

It follows from Lemma 3.2 that the control system (3.29)-(3.36) is approximately controllable with a control given by $h = \hat{z} \in L^2(Q_T)$, where $(\hat{z}, \hat{y})$ is the weak solution to the conjugate problem

$$
-\frac{\partial \hat{z}}{\partial t} - \text{div}(a_1(x,t) \nabla \hat{z}) + \sigma_1[w_1, w_2](x,t)\hat{z} = \hat{y} \chi_{\omega_2},
$$

$$
(x,t) \in Q_T, (3.37)
$$

$$
-\frac{\partial \hat{y}}{\partial t} - \text{div}(a_2(x,t) \nabla \hat{y}) + \sigma_2[w_1, w_2](x,t)\hat{z} + \sigma_3[w_2](x,t)\hat{y} = 0,
$$

$$
(x,t) \in Q_T, (3.38)
$$
\[ \begin{align*}
\hat{z}(x,t) &= 0, & (x,t) &\in \Sigma_1, \\
\hat{y}(x,t) &= 0, & (x,t) &\in \Sigma_2, \\
\hat{z}(x,T) &= \hat{z}_0(x), & x &\in \Omega, \\
\hat{y}(x,T) &= \hat{y}_0(x), & x &\in \Omega,
\end{align*} \tag{3.39-3.42} \]

\((\hat{z}_0, \hat{y}_0)\) is the unique minimum point of the functional
\[
J((z_0, y_0), (\sigma_1 [w_1, w_2], \sigma_2 [w_1, w_2], \sigma_3 [w_2]), (u_d(\cdot) - \bar{u}(\cdot, T), v_d(\cdot) - \bar{v}(\cdot, T))).
\]

Let
\[
u(x,t) = \bar{u}(x,t) + \tilde{u}(x,t), \quad v(x,t) = \bar{v}(x,t) + \tilde{v}(x,t), \quad (x,t) \in Q_T.
\]

Then \((u,v)\) is the weak solution to the problem
\[
\begin{align*}
\frac{\partial u}{\partial t} - \text{div}(a_1(x,t) \nabla u) + \sigma_1 [w_1, w_2](x,t) u + \sigma_2 [w_1, w_2](x,t) v &= h \chi_{w_1} - g(x,t,0,0), & (x,t) &\in Q_T, \\
\frac{\partial v}{\partial t} - \text{div}(a_2(x,t) \nabla v) + \sigma_3 [w_2](x,t) v &= u \chi_{w_2} - q(x,t,0), & (x,t) &\in Q_T, \\
u(x,t) = u_0(x), & (x,t) &\in \Sigma_1, \\
v(x,t) = v_0(x), & (x,t) &\in \Sigma_2, \\
u(x,0) = u_0(x), & x &\in \Omega, \\
v(x,0) = v_0(x), & x &\in \Omega,
\end{align*} \tag{3.43-3.48} \]

and satisfies
\[
\|u(\cdot, T) - u_d(\cdot)\|_{L^2(\Omega)} = \|\bar{u}(\cdot, T) - (u_d(\cdot) - \bar{u}(\cdot, T))\|_{L^2(\Omega)} \leq \varepsilon, \\
\|v(\cdot, T) - v_d(\cdot)\|_{L^2(\Omega)} = \|\bar{v}(\cdot, T) - (v_d(\cdot) - \bar{v}(\cdot, T))\|_{L^2(\Omega)} \leq \varepsilon.
\]

For convenience, we denote
\[
\begin{align*}
\tilde{u}(x,t) &= \tilde{u}[w_1, w_2](x,t), & \tilde{v}(x,t) &= \tilde{v}[w_1, w_2](x,t), & (x,t) &\in Q_T, \\
\hat{u}(x,t) &= \hat{u}[w_1, w_2](x,t), & \hat{v}(x,t) &= \hat{v}[w_1, w_2](x,t), & (x,t) &\in Q_T, \\
\hat{z}(x,t) &= \hat{z}[w_1, w_2](x,t), & \hat{y}(x,t) &= \hat{y}[w_1, w_2](x,t), & (x,t) &\in Q_T, \\
\tilde{u}(x,t) &= \tilde{u}[w_1, w_2](x,t), & \tilde{v}(x,t) &= \tilde{v}[w_1, w_2](x,t), & (x,t) &\in Q_T,
\end{align*}
\]

where \((\tilde{u}, \tilde{v}), (\hat{u}, \hat{v}), (\hat{z}, \hat{y})\) and \((u,v)\) are the solutions to the problems (3.23)-(3.28), (3.29)-(3.36), (3.37)-(3.42) and (3.43)-(3.48), respectively.
Theorem 3.1. The semilinear problem (1.1)-(1.6) is approximately controllable. More precisely, for any \( u_0, v_0, u_d, v_d \in L^2(\Omega) \) and \( \varepsilon > 0 \), there exists a control function \( h \in L^2(Q_T) \) such that the weak solution \( (u, v) \) to the problem (1.1)-(1.6) satisfies (1.16).

Proof. Define the mapping

\[
\Lambda : L^1(Q_T) \times L^1(Q_T) \to L^1(Q_T) \times L^1(Q_T), \quad (w_1, w_2) \mapsto (u[w_1, w_2], v[w_1, w_2]).
\]

First let us show that the range of \( \Lambda \) is precompact. Given \( \{w_1^{(k)}\}_{k=1}^{\infty}, \{w_2^{(k)}\}_{k=1}^{\infty} \subseteq L^1(Q_T) \). It follows from (2.14) that there exist subsequences of \( \{\sigma_1[w_1^{(k)}, w_2^{(k)}]\}_{k=1}^{\infty}, \{\sigma_2[w_1^{(k)}, w_2^{(k)}]\}_{k=1}^{\infty} \) and \( \{\sigma_3[w_2^{(k)}]\}_{k=1}^{\infty} \), denoted by themselves for convenience, which converge weakly * in \( L^\infty(Q_T) \). Lemma 2.1 shows that \( \{\hat{z}[w_1^{(k)}, w_2^{(k)}]\}_{k=1}^{\infty} \) is bounded in \( L^2(Q_T) \). Hence there exists a subsequence of \( \{\hat{z}[w_1^{(k)}, w_2^{(k)}]\}_{k=1}^{\infty} \), denoted by itself for convenience, which converges weakly in \( L^2(Q_T) \). Thanks to Corollary 2.1, there exists a subsequence of \( \{(u[w_1^{(k)}, w_2^{(k)}], v[w_1^{(k)}, w_2^{(k)}])\}_{k=1}^{\infty} \), which converges in \( L^1(Q_T) \times L^1(Q_T) \). Hence the range of \( \Lambda \) is precompact.

Second we prove that \( \Lambda \) is continuous. Assume that \( \{w_1^{(k)}\}_{k=1}^{\infty} \) and \( \{w_2^{(k)}\}_{k=1}^{\infty} \) converge to \( w_1, w_2 \) in \( L^1(Q_T) \), respectively. Let us show that \( \{(u[w_1^{(k)}, w_2^{(k)}], v[w_1^{(k)}, w_2^{(k)}])\}_{k=1}^{\infty} \) converges to \( (u[w_1, w_2], v[w_1, w_2]) \) in \( L^1(Q_T) \times L^1(Q_T) \) by the contradiction. Otherwise, there exist \( \{w_1^{(m)}\}_{m=1}^{\infty}, \{w_2^{(m)}\}_{m=1}^{\infty} \) and a positive constant \( \tau > 0 \) such that

\[
\|u[w_1^{(m)}, w_2^{(m)}] - u[w_1, w_2]\|_{L^1(Q_T)} + \|v[w_1^{(m)}, w_2^{(m)}] - v[w_1, w_2]\|_{L^1(Q_T)} \geq \tau, \quad m = 1, 2, \ldots.
\]

Since \( \{w_1^{(m)}\}_{m=1}^{\infty} \) and \( \{w_2^{(m)}\}_{m=1}^{\infty} \) converge to \( w_1, w_2 \) in \( L^1(Q_T) \), respectively, it follows from Lemma 2.2 and Corollary 2.2 that

\[
\left( \hat{u}[w_1^{(m)}, w_2^{(m)}](\cdot, T), \bar{\sigma}[w_1^{(m)}, w_2^{(m)}](\cdot, T) \right) \to \left( \hat{u}[w_1, w_2](\cdot, T), \bar{\sigma}[w_1, w_2](\cdot, T) \right)
\]

in \( \mathcal{H} \), \( m \to \infty \).

Owing to Proposition 3.2, there exists a subsequence of \( \{(\hat{z}_0[w_1^{(m)}, w_2^{(m)}], \hat{y}_0[w_1^{(m)}, w_2^{(m)}])\}_{m=1}^{\infty} \), denoted by itself for convenience, which converges to \( (\hat{z}_0[w_1, w_2], \hat{y}_0[w_1, w_2]) \) in \( \mathcal{H} \). It follows from Corollary 2.2 that \( \{(\hat{z}[w_1^{(m)}, w_2^{(m)}], \hat{y}[w_1^{(m)}, w_2^{(m)}])\}_{m=1}^{\infty} \) converges to \( (\hat{z}[w_1, w_2], \hat{y}[w_1, w_2]) \) in \( L^\infty((0, T); L^2(\Omega)) \times L^\infty((0, T); L^2(\Omega)) \), respectively.
Thus \( \{ (\hat{z}[w_1^{(k_m)},w_2^{(k_m)}],\hat{y}[w_1^{(k_m)},w_2^{(k_m)}]) \}_{m=1}^\infty \) converges to \((\hat{z}[w_1,w_2],\hat{y}[w_1,w_2])\) in \( L^2(Q_T) \times L^2(Q_T) \). Using Corollary 2.2 again, one gets that \( \{ (u[w_1^{(k_m)},w_2^{(k_m)}],v[w_1^{(k_m)},w_2^{(k_m)}]) \}_{m=1}^\infty \) converges to \((u[w_1,w_2],v[w_1,w_2])\) in \( L^1(Q_T) \times L^1(Q_T) \), which contradicts with (3.49). Therefore, \( \Lambda \) is continuous.

Thanks to the above discussion, one gets that the restriction of the mapping \( \Lambda \) to the close and convex hull of the range of \( \Lambda \) satisfies the hypotheses of the Schauder fixed point theorem. Therefore, \( \Lambda \) admits a fixed point \((u,v)\). That is to say, \((u,v) = \Lambda(u,v)\) is a weak solution to the problem (1.1)-(1.6) satisfying (1.16), where \( h \in L^\infty((0,T);L^2(\Omega)) \).

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### References

[1] V. Barbu, Controllability of parabolic and Navier-Stokes equations, Sci. Math. Jpn., 56(1) (2002) 143-211.

[2] H. M. Byrne and M. R. Owen, A new interpretation of the Keller-Segel model based on multiphase modelling, J. Math. Biol., 49(6) (2004) 604-626.

[3] P. Cannarsa, P. Martinez, and J. Vancostenoble, Persistent regional controllability for a class of degenerate parabolic equations, Commun. Pure Appl. Anal., 3(4) (2004) 607-635.

[4] P. Cannarsa, P. Martinez, and J. Vancostenoble, Null controllability of degenerate heat equations, Adv. Differ. Equ., 10(2) (2005) 153-190.

[5] P. Cannarsa, P. Martinez, and J. Vancostenoble, Carleman estimates for a class of degenerate parabolic operators, SIAM J. Control Optim., 47(1) (2008) 1-19.

[6] P. Cannarsa and L. Teresa, Controllability of 1-D coupled degenerate parabolic equations, Electron. J. Differ. Equ., 2009(73) (2009) 1-21.

[7] A. Doubova, E. Fernández-Cara, M. González-Burgos, and E. Zuazua, On the controllability of parabolic systems with a nonlinear term involving the state and the gradient, SIAM J. Control Optim., 41(3) (2002) 798-819.

[8] R. M. Du, C. P. Wang, and Q. Zhou, Approximate controllability of a semilinear system involving a fully nonlinear gradient term, Appl. Math. Optim., 70(1) (2014) 165-183.

[9] C. Fabre, J. P. Puel, and E. Zuazua, Approximate controllability of the semilinear heat equation, Proc. Roy. Soc. Edinburgh Sect. A, 125(1) (1995) 31-61.
[10] E. Fernández-Cara, Null controllability of the semilinear heat equation, ESAIM Control Optim. Calc. Var., 2 (1997) 87-103.

[11] E. Fernández-Cara and E. Zuazua, Null and approximate controllability for weakly blowing up semilinear heat equations, Ann. Inst. H. Poincaré Anal. Non Lineaire. Anal., 17(5) (2000) 583-616.

[12] E. Fernández-Cara and E. Zuazua, The cost of approximate controllability for heat equations: The linear case, Adv. Differ. Equ., 5 (2000) 465-514.

[13] A. V. Fursikov and O. Imanuvilov, Controllability of evolution equations, Lecture Notes Series 34, Seoul National University, 1996.

[14] J. L. Lions, Remarques sur la Contrôlabilité Approchée, Proceedings of “Jornadas Hispano-Francesas sobre Control de Sistemas Distribuidos”, University of Málaga, 1990.

[15] J. L. Lions, Remarks on approximate controllability, J. Anal. Math., 59(1) (1992) 103-116.

[16] P. Martinez and J. Vancostenoble, Carleman estimates for one-dimensional degenerate heat equations, J. Evol. Equ., 6(2) (2006) 325-362.

[17] A. Schiaffino and A. Tesei, Competition systems with Dirichlet boundary conditions, J. Math. Biol., 15(1) (1982) 93-105.

[18] L. Teresa, Insensitizing controls for a semilinear heat equation, Commun. Partial. Differ. Equ., 25 (1-2) (2000) 39-72.

[19] C. P. Wang, Approximate controllability of a class of degenerate systems, Appl. Math. Comput., 203(1) (2008) 447-456.

[20] C. P. Wang, Approximate controllability of a class of semilinear systems with boundary degeneracy, J. Evol. Equ., 10(1) (2010) 163-193.

[21] C. P. Wang and R. M. Du, Approximate controllability of a class of semilinear degenerate systems with convection term, J. Differ. Equ., 254(9) (2013) 3665-3689.

[22] Q. Zhou, F. D. Xu, and D. Gao, Approximate controllability of a class of double coupled linear degenerate parabolic equations, Journal of Jilin University: Science Edition, 58(5) (2020), doi: 10.13413/j.cnki.jdxbb.2020057.

[23] Y. J. Zhu, R. M. Du, and L. Z. Bao, Approximate controllability of a class of coupled degenerate systems, Bound. Value Probl., 127 (2016).