QUASI-ISOMETRY INVARIANTS FROM DECORATED TREES OF CYLINDERS OF TWO-ENDED JSJ DECOMPOSITIONS

CHRISTOPHER H. CASHEN

ABSTRACT. We construct quasi-isometry invariants of a one-ended finitely presented group by considering the tree of cylinders of a two-ended JSJ decomposition of the group. When the group satisfies additional quasi-isometric rigidity hypotheses we construct finer invariants by also considering relative amounts of stretching across edges of the tree of cylinders.

1. Introduction

Gromov proposed a program of classifying finitely generated groups up to the geometric equivalence relation of quasi-isometry [9].

Stallings’s Theorem [23, 24] shows that the existence of a splitting of a group $G$ as a graph of groups over a finite subgroup is a quasi-isometry invariant. Finitely presented groups admit a maximal such splitting [6]. A theorem of Papasoglu and Whyte [20] says that the collection of quasi-isometry types of one-ended vertex groups of a maximal decomposition of an infinite-ended, finitely presented group is a complete quasi-isometry invariant. Underlying this theorem is the fact that the way in which one-ended vertex groups are arranged in the maximal graph of groups decomposition does not change the quasi-isometry type of the whole group. This is because all the edge groups are finite, so all vertex groups coarsely intersect in bounded subsets, regardless of adjacency in the graph of groups decomposition.

To attempt an analogous study of one-ended groups, we must study splittings over infinite subgroups. Papasoglu [19] showed that the existence of splittings of a finitely presented one-ended group over two-ended subgroups is quasi-isometry invariant, provided that the group is not commensurable to a surface group. Moreover, such a group admits a maximal decomposition as a graph of groups over two-ended subgroups, known as a JSJ decomposition, and Papasoglu’s results say that quasi-isometries respect the JSJ decomposition. In particular, the quasi-isometry types of the essential vertex groups of the JSJ decomposition are invariant under quasi-isometries. However, the quasi-isometry types of the vertex groups alone can not give complete quasi-isometry invariants, because now not all vertex groups have the same coarse intersections. Vertices that are adjacent in the Bass-Serre tree $T$ of the decomposition have vertex groups that intersect in two-ended subgroups, while the coarse intersection of non-adjacent vertices may or may not be bounded.

Date: March 13, 2014.
2010 Mathematics Subject Classification. 20E05, 20E06, 20E08, 20F65, 20F67.
Key words and phrases. Quasi-isometry, JSJ decomposition.

This work supported by the European Research Council (ERC) grant of Goulana ARZHANTSEVA, grant agreement #259527.
Guirardel and Levitt [11] show how to rearrange \( T \) into a different tree \( \text{Cyl}(T) \) called the tree of cylinders of \( T \). The essential vertices of \( T \) will still be vertices in \( \text{Cyl}(T) \). The tree of cylinders has the additional property that if \( v \) and \( v' \) are vertices in \( T \) that also appear as vertices in \( \text{Cyl}(T) \), then the coarse intersection of the corresponding vertex groups is two-ended if \( v \) and \( v' \) are at distance 2 in \( \text{Cyl}(T) \) and bounded otherwise. It follows from Papasoglu’s results, see Theorem 2.5, that a quasi-isometry between finitely presented one-ended groups induces an isomorphism between their trees of cylinders, which, moreover, preserves the quasi-isometry types of vertex stabilizers.

Our first result, Theorem 4.1, gives a concise way to encode the structure of the tree of cylinders as a quasi-isometry invariant of the group. This gives a finer quasi-isometry invariant than just quasi-isometry types of the essential vertices.

Our second result, Theorem 6.4, imposes the additional hypotheses that the group is hyperbolic and the vertex groups of the JSJ decomposition are hyperbolic and relatively quasi-isometrically rigid, see Definition 5.2. In this case we can measure the amount of stretch across and edge of the JSJ decomposition, and we use this stretch parameter to give finer versions of the quasi-isometry invariants of Theorem 4.1.

Acknowledgements. We thank Mladen Bestvina and Gilbert Levitt for interesting conversations related to this work. In particular, the idea that a version of Leighton’s Theorem would provide a concise description of our structure invariants is due to Bestvina.

2. Preliminaries

We assume familiarity with standard concepts such as Cayley graphs, ends of spaces, and (Gromov) hyperbolic geometry. See [3] for background.

2.1. Coarse Geometry. Let \((X,d_X)\) and \((Y,d_Y)\) be proper geodesic metric spaces. Subsets of \( X \) are \textit{coarsely equivalent} if they are bounded Hausdorff distance from one another. A subset \( A \) is \textit{coarsely contained} in \( B \) if \( A \) is coarsely equivalent to a subset of \( B \). Two maps \( \phi \) and \( \phi' \) from \( X \) to \( Y \) are \textit{bounded distance} from each other if \( \sup_{x \in X} d_Y(\phi(x),\phi'(x)) < \infty \).

A map \( \phi: X \to Y \) is a \textit{coarse embedding} if there exist unbounded, non-decreasing functions \( \rho_0 \) and \( \rho_1 \) such that for all \( x \) and \( x' \) in \( X \) we have \( \rho_0(d_X(x,x')) \leq d_Y(\phi(x),\phi(x')) \leq \rho_1(d_X(x,x')) \). A coarse embedding is a \textit{quasi-isometric embedding} if \( \rho_0 \) and \( \rho_1 \) are linear. It is a \textit{quasi-isometry} if, in addition, \( \phi(X) \) is coarsely equivalent to \( Y \).

Let \( \mathcal{QI}(X,Y) \) denote the set of quasi-isometries from \( X \) to \( Y \) modulo bounded distance. If \( \phi \in \mathcal{QI}(X,Y) \), let \( \overline{\phi} \in \mathcal{QI}(Y,X) \) be the \textit{quasi-isometric inverse} of \( \phi \) such that \( \overline{\phi} \circ \phi \) and \( \phi \circ \overline{\phi} \) are bounded distance from the identity maps on \( X \) and \( Y \), respectively. Let \( \mathcal{QI}(X) = \mathcal{QI}(X,X) \) be the group of quasi-isometries of \( X \).

Quasi-isometries respect coarse equivalence of subsets. If \( \mathcal{P} \) is a set of coarse equivalence classes of subsets of \( X \), and \( \mathcal{P}' \) is a set of coarse equivalence classes of subsets of \( Y \), let \( \mathcal{QI}((X,\mathcal{P}),(Y,\mathcal{P}')) \) be the subset of \( \mathcal{QI}(X,Y) \) consisting of quasi-isometries that induce bijections between \( \mathcal{P} \) and \( \mathcal{P}' \). Similarly, \( \mathcal{QI}((X,\mathcal{P})) = \mathcal{QI}((X,\mathcal{P}),(X,\mathcal{P})) \) is a subgroup of \( \mathcal{QI}(X) \).

Let \( [X] \) denote the set of geodesic metric spaces quasi-isometric to \( X \). If \( \mathcal{P} \) is a set of coarse equivalence classes of subsets of \( X \), let \( [(X,\mathcal{P})] \) denote the set of
pairs \( (Y, P') \) where \( Y \) is a geodesic metric space and \( P' \) is a collection of coarse equivalence classes of subsets of \( Y \) such that there exists a quasi-isometry from \( X \) to \( Y \) that induces a bijection from \( P \) to \( P' \). We call \( \|X\| \) the quasi-isometry type of \( X \) and \( \|(X, P)\| \) the relative quasi-isometry type of \( (X, P) \).

2.2. Graphs of Groups. Let \( \Gamma \) be a finite oriented graph. If \( e \) is an edge, let \( \iota(e) \) be its initial vertex, and let \( \tau(e) \) be its terminal vertex. Let \( e \) be the edge \( e \) with opposite orientation, so that \( \tau(e) = \iota(e) \).

A graph of groups \( \Gamma = (\Gamma, \{G_\gamma\}, \{\phi_e\}) \) consists of a finite directed graph \( \Gamma \), groups \( G_\gamma \) for each vertex and edge \( \gamma \) in \( \Gamma \), and injections \( \phi_e : G_e \hookrightarrow G_{\tau(e)} \) for each edge \( e \). We require \( G_e = G_{\bar{e}} \) for each edge \( e \).

A graph of groups \( \Gamma \) has an associate fundamental group \( G \) obtained by amalgamating the vertex groups \( G_v \) over the edge groups \cite{22}. We say that \( \Gamma \) is a graph of groups decomposition of \( G \).

The Bass-Serre tree \( T \) of \( \Gamma \) is the tree on which \( G \) acts without edge inversions, such that \( G \backslash T = \Gamma \) and the stabilizer of a vertex \( \tilde{v} \) in \( T \) is a conjugate in \( G \) of the group \( G_v \), where \( v \) is the image of \( \tilde{v} \) in \( \Gamma = G \backslash T \).

We require that a graph of groups decomposition does not have a valence one vertex such that the incident edge map is surjective. This is equivalent to requiring that the Bass-Serre tree does not have a proper subtree that is invariant under the group action.

**Definition 2.1.** Given a vertex group \( G_v \) of \( \Gamma \), the peripheral structure coming from incident edge groups, \( P_v \), is the set of distinct coarse equivalence classes of \( G_v \)-conjugates of the images of the edge injections \( \phi_e : G_e \hookrightarrow G_v \) for edges \( e \) with \( \tau(e) = v \).

We will be interested in graphs of groups in which the edge groups are two-ended. Recall that a finitely generated, two-ended group is virtually cyclic, that is, it has an infinite cyclic subgroup of finite index. Commensurability of edge stabilizers defines an equivalence relation on the edge of the Bass-Serre tree \( T \) of such a splitting. The equivalence classes of edges are called cylinders. Every cylinder is a subtree of \( T \) \cite[Lemma 4.2]{11}. It follows that we get another tree \( \text{Cyl}(T) \), called the tree of cylinders of \( T \), by taking the dual tree to the covering of \( T \) by cylinders.

2.3. JSJ Decompositions. In this paper we are interested in two-ended JSJ decompositions of finitely presented, one-ended groups. Equivalent descriptions of such decompositions appear in Dunwoody and Sageev \cite{7}, Fujiwara and Papasoglu \cite{8}, and Guirardel and Levitt \cite{10}.

Following Papasoglu \cite{19}, we will give a more geometric description of JSJ decompositions. First, we need some terminology.

If \( L \) is a path connected subset of a geodesic metric space \( (X, d_X) \), let \( d_L \) denote the induced length metric on \( L \). A quasi-line in \( X \) is a path connected subset \( L \) such that \( (L, d_L) \) is quasi-isometric to \( \mathbb{R} \) and the inclusion map of \( L \) into \( X \) is a coarse embedding.

A quasi-line \( L \) is separating if its complement has at least two essential components, that is, components that are not contained in any finite neighborhood of \( L \). In particular, if \( G \) splits over two-ended subgroup then the two-ended subgroup is bounded distance from a separating quasi-line. Separating quasi-lines cross if each travels arbitrarily deeply into two different essential complementary components of the other.
Let $G_v$ be a vertex group in a graph of groups decomposition. Let $\mathcal{P}_v$ be the peripheral structure on $G_v$ coming from incident edge groups. Let $\Sigma$ be a hyperbolic pair of pants. Let $\mathcal{P}_{\partial \Sigma}$ be the peripheral structure on the universal cover $\tilde{\Sigma}$ of $\Sigma$ consisting of the coarse equivalence classes of the components of the preimages of the boundary curves.

**Definition 2.2.** We say $v$ is **hanging** if $(G_v, \mathcal{P}_v)$ is quasi-isometric to $(\tilde{\Sigma}, \mathcal{P}_{\partial \Sigma})$. We say $v$ is **rigid** if it is not two-ended, not hanging, and does not split over a two-ended subgroup relative to its incident edge groups.

**Definition 2.3.** Let $G$ be a finitely presented one-ended group that is not commensurable to a surface group. A **JSJ decomposition** of $G$ is a (possibly trivial) graph of groups decomposition $\Gamma$ with two-ended edge groups satisfying the following conditions:

(a) Every vertex group is two-ended, hanging, or rigid.
(b) Every cylinder in the Bass-Serre tree $T$ of $\Gamma$ that contains exactly two hanging vertices also contains a rigid vertex.

This definition is equivalent to those cited above. The essential facts are that:

1. Hanging vertices contain crossing pairs of separating quasi-lines.
2. Every pair of crossing separating quasi-lines is coarsely contained in a conjugate of a hanging vertex group.
3. A separating quasi-line that is not crossed by any other separating quasi-line is coarsely equivalent to a conjugate of an edge group.
4. Every edge group is coarsely equivalent to a separating quasi-line that is not crossed by any other separating quasi-line.

**Remark.** Condition (b) implies that the hanging vertex groups are maximal hanging.

**Remark.** The case that a vertex group is the fundamental group of a pair of pants and the incident edge groups glue on to the boundary curves is called ‘rigid’ in the usual JSJ terminology because there are no splittings of the pair of pants group relative to the boundary subgroups. Algebraically, such a vertex behaves like our rigid vertices, but geometrically this is a hanging vertex.

Note that these properties do not in general define a unique graph of groups decomposition, but rather a deformation space of decompositions [10]. However, the hanging and rigid vertex groups are elliptic in every JSJ decomposition. Furthermore, all JSJ decompositions are in the same deformation space, and the tree of cylinders of a decomposition depends only on the deformation space [11, Theorem 1], up to $G$–equivariant isomorphism, so there is a unique JSJ tree of cylinders.

**Definition 2.4.** Let $G$ be a finitely presented one-ended group. The **JSJ tree of cylinders** $\text{Cyl}(G)$ of $G$ is the tree of cylinders of the Bass-Serre tree of any JSJ decomposition of $G$.

If $\Gamma$ is a JSJ decomposition of $G$ and $T$ is its Bass-Serre tree, then rigid and hanging vertex groups are contained in more than one cylinder. As a result, $\text{Cyl}(G)$ is bipartite, with one part consisting of vertices $V_C$, corresponding to cylinders of $T$ and the other part consisting of vertices $V_H$ and $V_R$ corresponding to hanging and rigid vertices, respectively, of $T$. 
Since quasi-isometries coarsely preserve quasi-lines, and preserve the crossing and separating properties of quasi-lines, Papasoglu concludes quasi-isometry invariance of JSJ decompositions:

**Theorem 2.5** (cf [19, Theorem 7.1]). Let \(G\) and \(G'\) be finitely presented one-ended groups. Suppose \(\psi: G \to G'\) is a quasi-isometry. Then there is a constant \(C\) such that \(\psi\) induces an isomorphism \(\psi^*: \text{Cyl}(G) \to \text{Cyl}(G')\) that preserves vertex type — cylinder, hanging, or rigid — and for \(v \in \text{Cyl}(G)\) takes \(G_v\) to within distance \(C\) of \(G'_{\psi^*(v)}\).

**Proof.** Two-ended subgroups of \(G\) are bounded distance from each other if and only if they are commensurable, so cylinders are coarse equivalence classes of edge and two-ended vertex stabilizers. These are exactly the coarse equivalence classes of separating quasi-lines that are not crossed by other separating quasi-lines, so quasi-isometries induce a bijection between cylinders. The remaining statements are from [19, Theorem 7.1]. \(\square\)

**Corollary 2.6.** If \(v\) is a rigid or hanging vertex in \(\text{Cyl}(G)\) then \(\psi_v = \pi_{\psi^*(v)} \circ \psi|_{G_v} \in \text{QI}((G_v, P_v), (G'_{\psi^*(v)}, P_{\psi^*(v)}))\) where \(\pi_{\psi^*(v)}\) takes the image of \(\psi|_{G_v}\) to \(G'_{\psi^*(v)}\) by closest point projection.

**Remark.** \(\pi_{\psi^*(v)}\) is coarsely well defined since \(\psi(G_v)\) is within distance \(C\) of \(G'_{\psi^*(v)}\).

### 3. Structure Invariants for Decorated Trees

Let \(G\) be a group. Let \(T\) be a simplicial tree upon which \(G\) acts cocompactly and without inversions. Let \(\delta: T \to D\) be a \(G\)-invariant map from the vertices of \(T\) to a finite set of ‘decorations’. Decorations \(\delta: T \to D\) and \(\delta': T \to D'\) are equivalent if there exists a bijection \(\psi: D \to D'\) such that \(\delta' = \psi \circ \delta\).

A **structure invariant** is obtained from a \(G\)-invariant decoration \(\delta: T \to D\) by iteratively refining the decorations as follows. Set \(D_0 = D\) and \(\delta_0 = \delta\). Beginning with \(i = 0\), for each \(v \in T\) define:

\[
f_{v,i}: D_1 \to \mathbb{N} \cup \{0, \infty\}: d \mapsto \#\{w \in \delta_i^{-1}(d) \mid w \text{ is adjacent to } v\}
\]

Define \(\delta_{i+1}(v) = (\delta_0(v), f_{v,i-1})\), and define \(D_{i+1}\) to be the image of \(\delta_{i+1}\) in \(D_0 \times (D_i \to \mathbb{N} \cup \{0, \infty\})\).

The function \(i \mapsto \#D_i\) is non-decreasing, and if there is some \(s\) such that \(\#D_s = \#D_{s+1}\) then the refinement is **stable**: \(D_j\) is equivalent to \(D_s\) for all \(j \geq s\). Since for all \(i\) the size of \(D_i\) is bounded above by the number of vertices of \(G\setminus T\), the refinement stabilizes in finitely many steps. Let \(\delta_s: T \to D_s\) be a stable decoration. Then \(\delta_s(v) = \delta_s(v')\) implies \(f_{v,s} = f_{v',s}\). Let \(\pi_0: D_s \to D_0\) be projection to the first coordinate. Choose an ordering of \(D\) and let \(D[j]\) denote the \((j+1)\)-st decoration. Similarly, for each \(0 \leq j < \#D_s\) choose an ordering of \(\pi_0^{-1}(D[j])\). Order \(D_s\) lexicographically, and let \(D_s[i]\) denote the \((i+1)\)-st decoration.

**Definition 3.1.** The **structure invariant** \(S(T, \delta, D)\) is the \(\#D_s \times \#D_s\) matrix whose \(j, k\)-entry is the number of vertices in \(\delta_k^{-1}(D_s[j])\) adjacent to each vertex of \(\delta_j^{-1}(D_s[i])\), together with the decoration of the rows and columns by \(D\) that sends the \((i+1)\)-st row or column to \(\pi_0(D_s[i])\).
$S(T, \delta, D)$ is a block matrix with a block consisting of entries with the same row and column decorations. The structure invariant is well defined up to permuting the $D$–blocks and permuting rows and columns within $D$–blocks, i.e., up to the choice of orderings of $D$ and the $\pi_0^{-1}(D[j])$.

**Proposition 3.2.** Let $\delta: T \to D$ be a $G$–invariant decoration of a cocompact $G$–tree. Let $\delta': T' \to D$ be a $G'$–invariant decoration of a cocompact $G'$–tree. There exists a decoration-preserving isomorphism $\phi: T \to T'$ if and only if $S(T, \delta, D) = S(T', \delta', D)$, up to permuting rows and columns within $D$–blocks.

**Proof.** It is clear that decoration-preserving isomorphic trees have the same structure invariants, up to choosing the orderings of the decorations. For the converse, assume that we have reordered within $D$–blocks so that $S(T, \delta, D) = S(T', \delta', D) = S$. Let $\delta_s: T \to D_s$ and $\delta'_s: T' \to D_s$ be the stable decorations of $T$ and $T'$. Choose an arbitrary basepoint $v \in T$. Define $\chi(v)$ by choosing an arbitrary vertex in $(\delta'_s)^{-1} \circ \delta_s(v)$. Extend $\chi$ to adjacent vertices by choosing arbitrary bijections between $\delta_s^{-1}(D_s[k]) \cap N_1(v)$ and $(\delta'_s)^{-1}(D_s[k]) \cap N_1(\chi(v))$ for each $0 \leq k < \#D_s$. Such a bijection exists because the cardinality of these sets is the $j,k$–entry of $S$, where $\delta_s(v) = D_s[j]$. Continue in this way to extend $\chi$ to all of $T$. By construction $\delta_s = \delta'_s \circ \chi$, so $\chi$ is decoration-preserving:

$$\delta = \pi_0 \circ \delta_s = \pi_0 \circ \delta'_s \circ \chi = \delta' \circ \chi$$

□

**Remark.** When $T$ is the universal cover of a finite graph $\Gamma$ and the initial set of decorations is trivial then the structure invariant we have defined is better known as the degree refinement of $\Gamma$. The lemma says that two graphs have the same degree refinement if and only if they have isomorphic universal covers. A theorem of Leighton [14] says that such graphs in fact have a common finite cover.

**Observation.** We get a quasi-isometry invariant of a group $G$ by taking the structure invariant of a cocompact $QI(G)$–tree with a $QI(G)$–equivariant coloring.

This is a rather basic observation, but it does not seem to have appeared in the literature in this generality.

Behrstock and Neumann [1, 2] have used special cases of this type of invariant, in a different guise, to classify fundamental groups of some families of compact irreducible 3–manifolds of zero Euler characteristic. In both papers the tree is Bass-Serre tree for the geometric decomposition of such a 3–manifold along tori and Klein bottles, which is the higher dimensional antecedent of the JSJ decompositions considered in this paper. When the geometric decomposition has only Seifert fibered pieces the vertices are decorated by the quasi-isometry type of the universal cover of the corresponding Seifert fibered manifold. There are only two possible quasi-isometry types, according to whether or not the Seifert fibered piece has boundary. Every vertex in the Bass-Serre tree has infinite valence, so each entry of the structure invariant is either 0 or $\infty$.

Behrstock and Neumann [1] state their result in terms of bi-similarity classes of bi-colored graphs. They show that each bi-similarity class is represented by a unique minimal graph, and that two such 3–manifolds are quasi-isometric if and only if the bi-colored Bass-Serre tree of their geometric decompositions have the
same representative minimal graph. Their minimal bi-colored graphs carry exactly
the same information as the structure invariant of the decorated Bass-Serre tree.
One can construct their graph by taking the vertex set to be the stable decoration
set $D_s$ and connecting vertex $D_s[j]$ to vertex $D_s[k]$ by an edge if and only if the
$j,k$–entry of $S$ is $\infty$. The vertices of the graph are ‘bi-colored’ by the projection
$\pi_0: D_s \to D$. Conversely, $S$ can be recovered by replacing each edge in the graph
by infinitely many edges, lifting the bi-coloring to the universal covering tree, and
calculating the structure invariant.

The second paper [2] extends their results to cases where the decomposition
involves some hyperbolic pieces. The decorations there are more complex.

4. Quasi-isometry Invariants from Decorated JSJ Trees of Cylinders

Combining Proposition 3.2 with Theorem 2.5 and Corollary 2.6 proves:

**Theorem 4.1.** If $G$ is a finitely presented one-ended group not commensurable
to a surface group, then the structure invariant for the JSJ tree of cylinders is a
quasi-isometry invariant of $G$, with respect to any of the following decorations:

1. Vertex type: rigid, hanging, or cylinder.
2. Vertex type and, if $v$ is rigid, $[G_v]$.
3. Vertex type and, if $v$ is rigid, $[[G_v, P_v]]$.

**Theorem 4.2.** If $G$ is hyperbolic and the JSJ decomposition of $G$ has no rigid
vertices then the invariant of Theorem 4.1 is a complete quasi-isometry invariant.

**Proof.** Hyperbolicity implies that the cylinders are finite. The result then follows
by an easy adaptation of the argument of Behrstock and Neumann in [1]. The
torsion-free case is written up in the thesis of William Malone [15].

4.1. Example. Let $\Gamma$ be the graph of groups in Figure 1, with infinite cyclic edge
groups and vertex groups $G_{v_1} = \langle x_1, x_2, x_3 \rangle \cong \mathbb{Z}^3$ and $G_{v_2} = \langle y_1, y_2, y_3, y_4 \rangle \cong \mathbb{Z}^4$.
Edge labels indicate the image of a chosen generator of the edge group. The quotient
of the JSJ tree of cylinders is shown in Figure 2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{The graph of groups $\Gamma$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{The graph of cylinders of $\Gamma$.}
\end{figure}

Take the initial decoration from Theorem 4.1 (2), where $c$ represents ‘cylinder’
and $r$ represents ‘rigid’:

$$\delta_0: c_1, c_2 \mapsto c$$

$$v_1 \mapsto (r, [\mathbb{Z}^3])$$

$$v_2 \mapsto (r, [\mathbb{Z}^4])$$
Cylinders in the orbit of \( c_1 \) are not adjacent to any \( \mathbb{Z}^4 \) vertices, while cylinders in the orbit of \( c_2 \) are adjacent to infinitely many \( \mathbb{Z}^4 \) vertices. Thus, the first refinement distinguishes \( c_1 \) from \( c_2 \), as \( f_{c_2}((r, \mathbb{Z}^4)) = \infty \) while \( f_{c_1}((r, \mathbb{Z}^4)) = 0 \).

We have distinguished all the vertices in the quotient graph, so this refinement is stable, and the structure invariant is:

\[
\begin{array}{ccc}
\text{c} & (r, \mathbb{Z}^3) & (r, \mathbb{Z}^4) \\
(r, \mathbb{Z}^3) & 1 & 1 & 0 & 0 \\
(r, \mathbb{Z}^4) & 0 & 1 & 0 & 0
\end{array}
\]

Let \( \Gamma' \) be the graph of groups in Figure 3, with infinite cyclic edge groups and vertex groups \( G'_{v_1} = \langle x_1, x_2, x_3 \rangle \cong \mathbb{Z}^3 \) and \( G'_{v_2} = \langle y_1, y_2, y_3, y_4 \rangle \cong \mathbb{Z}^4 \). The quotient of the JSJ tree of cylinders is shown in Figure 4.

**Figure 3.** The graph of groups \( \Gamma' \).

\[
G'_{v_1} \xrightarrow{x_2} \xrightarrow{x_1} G'_{v_2}
\]

**Figure 4.** The graph of cylinders of \( \Gamma' \).

The initial decoration distinguishes all vertices in the graph of cylinders, so it is already stable. The structure invariant is:

\[
\begin{array}{ccc}
\text{c} & (r, \mathbb{Z}^3) & (r, \mathbb{Z}^4) \\
(r, \mathbb{Z}^3) & 2 & 0 & 0 \\
(r, \mathbb{Z}^4) & 1 & 0 & 0
\end{array}
\]

Since we have different structure invariants, the fundamental groups of \( \Gamma \) and \( \Gamma' \) are not quasi-isometric.

### 5. Relative Rigidity

Let \( \mathcal{I}(X) \) denote the quotient of Isom(\( X \)) in \( Q\mathcal{I}(X) \). Let \( i: \mathcal{I}(X) \to \text{Isom}(X) \) denote a map that selects an isometry from each equivalence class in \( \mathcal{I}(X) \). For \( g \in G \), let \( L_g \in \text{Isom}(G) \) be left multiplication. Let \( \mathcal{H} \) be a finite collection of two-ended subgroups of \( G \), and let \( \mathcal{P} \) the collection of distinct coarse equivalence classes of conjugates of elements of \( \mathcal{H} \).

**Definition 5.1.** We say \( G \) is weakly quasi-isometrically rigid relative to \( \mathcal{P} \), or \( (G, \mathcal{P}) \) is weakly quasi-isometrically rigid, if there is a proper geodesic metric space \( X \) and a quasi-isometry \( \mu: G \to X \) such that \( \mu \mathcal{Q}\mathcal{I}(G, \mathcal{P})\mu = \mathcal{I}(X, \mu(\mathcal{P})) \) and the distance between \( \mu \circ L_g \circ \mu \) and an isometry of \( X \) is bounded independent of \( g \in G \).
Definition 5.2. We say \( G \) is quasi-isometrically rigid relative to \( P \), or \((G, P)\) is quasi-isometrically rigid, if \( G \) is weakly quasi-isometrically rigid relative to \( P \) and, in addition, if \( \mu': G \to X \) is another quasi-isometry satisfying the definition of weak quasi-isometric rigidity, then \( \mu' \circ \mu \in \mathcal{I}(X) \).

Lemma 5.3. If \( G \) virtually splits over a zero or two-ended group relative to \( H \) then \((G, P)\) is not weakly quasi-isometrically rigid.

Proof. If \( G \) virtually splits it is not hard to show that the multiplicative constants of \( \mathcal{QI}(G, P) \) are unbounded. However, they must be bounded if \( \mathcal{QI}(G, P) \) is quasi-isometrically conjugate to an isometry group. \( \square \)

If \( X \) is an irreducible symmetric space other than real or complex hyperbolic space, then all quasi-isometries are already bounded distance from isometries [18, 13], so \((G, P)\) is quasi-isometrically rigid for every \( G \) quasi-isometric to \( X \) and every \( P \). The peripheral structure plays no role in this case.

If \( X \) is a real or complex hyperbolic space of dimension at least 3 and \( G \) is quasi-isometric to \( X \) then \( G \) is quasi-isometrically rigid relative to any non-empty peripheral structure, by a theorem of Schwartz [21].

If \( X' \) is the 3-valent tree, \( G \) is quasi-isometric to \( X' \), and \( G \) does not virtually split over 0 or 2-ended subgroups relative to \( H \), then \((G, P)\) is quasi-isometrically rigid [5, 4]. In this case the model space \( X \) depends on \( P \).

If \( X = \mathbb{H}^2 \), \( \phi: G \to X \) is a quasi-isometry, and \( G \) does not virtually split over 2-ended subgroups relative to \( H \), then \((G, P)\) is weakly quasi-isometrically rigid. To see this, note that \( \phi \) induces a cobounded quasi-action of \( G \) on \( X \). Such a quasi-action is quasi-isometrically conjugate to an isometric action on \( X \) [16]. A result of Kapovich and Kleiner [12] then shows that \( G \) has finite index in \( \mathcal{QI}(G, P) \). But this means that \( \mathcal{QI}(G, P) \) quasi-acts on \( X \), and the claim follows by conjugating this quasi-action to an isometric action.

Question 1. If \( G \) is a hyperbolic group that is not quasi-isometric to \( \mathbb{H}^2 \) and does not virtually split over a zero or two-ended subgroup relative to \( H \), is \((G, P)\) quasi-isometrically rigid?

6. AN ENHANCED DECORATION FOR HYPERBOLIC GROUPS WITH QUASI-ISOMETRICALLY RIGID VERTICES

For the rest of the paper we suppose \( G \) is a one-ended hyperbolic group with a JSJ decomposition \( \Gamma \) such that every rigid vertex group \( G_v \) is hyperbolic and weakly quasi-isometrically rigid relative to the peripheral structure \( P_v \) coming from incident edge groups.

Lemma 6.1. We may assume that \( \text{Cyl}(G) \) is the Bass-Serre tree \( T \) of \( \Gamma \).

Proof. Let \( T \) be the Bass-Serre tree of \( \Gamma \). Since \( G \) and the vertex groups of \( \Gamma \) are hyperbolic, the cylinders in \( T \) are finite. Then [11, Proposition 5.2] says that \( \text{Cyl}(T) \) and \( T \) are in the same deformation space, which means that there is a finite sequence of moves that transforms one to the other. These moves result in moves on \( \Gamma \) which do not change the rigid and hanging vertex group or the peripheral structures on these subgroups. Replace \( \Gamma \) by this new JSJ decomposition. \( \square \)

Our goal is to find a decoration of \( T \) that gives finer quasi-isometry invariants than Theorem 4.1. This new decoration will incorporate information about the
edges of $\mathcal{T}$. For compatibility with Section 3, it is convenient to subdivide edges of $\mathcal{T}$ so that we only decorate vertices. Let $\Gamma^*$ be the graph of groups decomposition of $G$ obtained from $\Gamma$ by subdividing each edge and taking the stabilizer of each new vertex to be equal to the stabilizer of its edge. Let $\mathcal{T}^*$ be the Bass-Serre tree, which is just $\mathcal{T}$ with subdivided edges. $\mathcal{T}^* = V_C \amalg V_R \amalg V_H \amalg V_S$, where $V_*$ are, respectively, the cylinder vertices, (weakly) rigid vertices, hanging vertices, and subdivided-edge vertices. Vertices in $V_C$, $V_R$ and $V_H$ are only adjacent to vertices in $V_S$, and each vertex in $V_S$ is adjacent to one vertex in $V_C$ and one in either $V_R$ or $V_H$.

Let $q: \mathcal{T}^* \to \Gamma^*$ be the quotient map of the $G$–action.

### 6.1. A Model Space for $G$.

Now we construct a proper geodesic metric space $X$ quasi-isometric to $G$ as a coarse Bass-Serre complex, following [17]. For each vertex $v \in \mathcal{T}^*$ we will choose a model vertex space $X_v$ and a quasi-isometry $\mu_v : X_v \to X_v$. For each edge $e \in \mathcal{T}^*$ we will choose an edge space $X_e = \mathbb{R} \times [0, 1]$ and a quasi-isometry $\mu_e : G_e \to X_e$. Then if $v$ is the terminus of $e$ we attach $\mathbb{R} \times \{1\} \subset X_e$ to $X_v$ by a map $\phi_e$ such that $\phi_e \circ \mu_e$ and $\mu_v \circ \phi_e$ agree to within bounded error.

For $v \in V_C \amalg V_S$ define $X_v = \mathbb{R}$.

Fix a hyperbolic pair of pants $\Sigma$ and for $h \in V_H$, let $X_h = \Sigma$, where

$$\mu_h : (G_h, \mathcal{P}_h) \to (\Sigma, \mathcal{P}_\partial \Sigma)$$

is a quasi-isometry as in Definition 2.2.

For each vertex $r \in \Gamma^*$ such that $(G_r, \mathcal{P}_r)$ is weakly rigid, choose a model space and quasi-isometry $\mu_r : G_r \to X_r$ as in Definition 5.1. We can, and do, choose these quasi-isometries so that if $\psi \in \mathcal{I}(G_r, (\mathcal{P}_r), (G_{r'}, (\mathcal{P}_{r'})))$ then $X_r = X_{r'}$ and $\mu_r(\mathcal{P}_r) = \mu_{r'}(\mathcal{P}_{r'})$, so $\mu_r \circ \psi \circ \mu_r^{-1} \in \mathcal{I}(X_r, \mu_r(\mathcal{P}_r))$. For $r \in V_R$, let $X_r$ be a copy of $X_{a(r)}$.

**Lemma 6.2.** If $\phi_0$ and $\phi_1$ are isometries of $X_r$ at bounded distance from one another, then the distance between them is bounded in terms of the hyperbolicity constant of $X_r$.

**Proof.** Since $X_r$ is quasi-isometric to a non-elementary hyperbolic group, for every $x \in X$ one can choose three distinct points $\xi_0$, $\xi_1$, and $\xi_2$ in the Gromov boundary of $X$ such that $x$ is in the coarse center of the ideal geodesic triangle with vertices $\xi_0$, $\xi_1$, and $\xi_2$. Then for $i \in \{0, 1\}$, $\phi_i(x)$ is in the coarse center of an ideal geodesic triangle with vertices $\partial \phi_i(\xi_0)$, $\partial \phi_i(\xi_1)$, and $\partial \phi_i(\xi_2)$. Since $\phi_0$ and $\phi_1$ are at bounded distance from one another, they extend to the same homeomorphism of the Gromov boundary, so $\partial \phi_0(\xi_j) = \partial \phi_1(\xi_j)$ for all $j$. Thus, the two ideal triangles, and hence their coarse centers, are bounded distance from each other, with the bound depending only on the hyperbolicity constant of $X_r$. \[\Box\]

For each $v \in V_S \amalg V_C$, the stabilizer $G_v$ is virtually cyclic, so we can choose an element $z_v \in G_v$ such that $\langle z_v \rangle < G_v$ is an infinite cyclic subgroup of minimal index. Suppose that $e$ is an edge in $\mathcal{T}^*$ connecting $s \in V_S$ to $c \in V_C$. We identify $G_e$ with $G_s$. Replacing, $z_v$ with its inverse, if necessary, there are positive integers $a$ and $b$ such that $\phi_v(z_v^a) = z_v^b$. The ratio $b/a$ is independent of the particular choices of minimal index cyclic subgroups. Attacx $\mathbb{R} \times \{0\} \subset X_e$ to $\mathbb{R} = X_e$ by the identity map on $\mathbb{R}$. Attach $\mathbb{R} \times \{1\} \subset X_e$ to $\mathbb{R} = X_e$ by multiplication by $b/a$. 

For each edge $e \in T^*$ connecting $s \in V_S$ to $q \in V_Q$, attach $\mathbb{R} \times \{0\} \subset X_e$ to $\mathbb{R} = X_s$ by the identity map on $\mathbb{R}$. Attach $\mathbb{R} \times \{1\} \subset X_e$ to $X_q$ by identifying $\{n\} \times \{1\} \in X_t$ to $\phi_c(z^n_t) \in G_q = X_q$, and interpolating along geodesic subsegments.

For each edge $e \in T^*$ connecting $s \in V_S$ to $r \in V_R$, attach $\mathbb{R} \times \{0\} \subset X_e$ to $\mathbb{R} = X_s$ by the identity map on $\mathbb{R}$. $\phi_c(G_s)$ is a maximal virtually cyclic subgroup of $G_r$ that is conjugate to one of the elements of $H$. Thus, there is an element $P_e \in \mathcal{P}_r$ corresponding to $e$. Recall that $P_e$ is a coarse equivalence class of subset of $G_r$. Let $g_s = \phi_c(z^s_s)$. A quasi-axis $\gamma$ of $g_s < G_s$ is contained in $P_e$. $\mu_r(\gamma) \subset X_r$ is therefore a quasi-geodesic contained in $\mu_r(P_e)$. Since $X_r$ is hyperbolic, $\mu_r(P_e)$ contains a geodesic $\gamma'$. Let $\pi: X_r \to \gamma'$ send each point of $X_r$ to a closest point of $\gamma'$. There may not be a unique closest point, but there is a bound on the diameter of possible candidates in terms of the hyperbolicity constant of $X_r$. Choose a basepoint $y_0 \in \gamma$ and let $x_0 = \pi(\mu_r(y_0))$. The distance from $\mu_r \circ L_{g^n_s} \circ \overline{p}(x_0)$ to $\pi(\mu_r \circ L_{g^n_s} \circ \overline{p}(x_0))$ is bounded, independent of $n$. Thus,

$$d(x_0, \pi(\mu_r \circ L_{g^n_s} \circ \overline{p}(x_0))) = d(x_0, \mu_r \circ L_{g^n_s} \circ \overline{p}(x_0)) = d(x_0, (i(\mu_r \circ L_{g^n_s} \circ \overline{p}))(x_0)) = d(x_0, (i(\mu_r \circ L_{g^n_s} \circ \overline{p}))(x_0)) = n \cdot \left( \lim_{m \to \infty} \frac{1}{m} \cdot d(x_0, (i(\mu_r \circ L_{g^n_s} \circ \overline{p}))(x_0)) \right) + e_4$$

In each case $e_j$ is an error, independent of $n$. The second equality is by definition of weak rigidity and Lemma 6.2. The third equality is again by Lemma 6.2, because $(i(\mu_r \circ L_{g^n_s} \circ \overline{p}))^n$ is an isometry that is bounded distance from the identity map $i(\mu_r \circ L_{g^n_s} \circ \overline{p})$.

Now, we can attach $\mathbb{R} \times \{1\} \subset X_e$ to $\gamma' \subset X_r$ by a map that stretches distance by $\lim_{m \to \infty} \frac{1}{m} \cdot d(x_0, (i(\mu_r \circ L_{g^n_s} \circ \overline{p})))^m(x_0))$, and this map will coarsely commute with $\mu_r \circ \phi_c \circ P_e$.

### 6.2. The Enhanced Decoration.

For $c \in V_C$, let $\text{Rig}(c)$ denote the vertices $s \in V_S$ adjacent to $c$ whose other neighbor is in $V_R$.

Let $s \in \text{Rig}(c)$. Let $e$ be the edge from $s$ to $c$, and let $e'$ be the edge from $s$ to $r \in V_R$. Recall that the map attaching $\mathbb{R} \times \{1\} \subset X_e$ to $X_r$ stretches distance by $\lim_{m \to \infty} \frac{1}{m} \cdot d(x_0, (i(\mu_r \circ L_{g^n_s} \circ \overline{p})))^m(x_0))$, and that map attaching $\mathbb{R} \times \{1\} \subset X_e$ to $X_r$ stretches distance by $\frac{1}{m} \cdot d(x_0, (i(\mu_r \circ L_{g^n_s} \circ \overline{p})))^m(x_0))$.

Define the stretch across $s$, $\text{Str}(s)$, to be $\frac{1}{m} \cdot \lim_{m \to \infty} \frac{1}{m} \cdot d(x_0, (i(\mu_r \circ L_{g^n_s} \circ \overline{p})))^m(x_0))$.

For $s \in \text{Rig}(c)$, define the relative stretch:

$$\text{relStr}(s) = \frac{\text{Str}(s)}{\min_{s' \in \text{Rig}(c)} \text{Str}(s')}$$

**Definition 6.3.** Decorate $T^*$ as in Theorem 4.1 (3), except that for $s \in V_s$ such that $s$ is adjacent to a vertex of $V_R$, the decoration on $s$ additionally includes the number $\text{relStr}(s)$.

**Theorem 6.4.** Let $G$, $\Gamma$, and $T$ and $G'$, $\Gamma'$, and $T'$ be groups with JSJ decompositions and Bass-Serre trees as defined above. Let $\delta: T^* \to \mathcal{D}$ and $\delta': T'^* \to \mathcal{D}'$ be the decorations of Definition 6.3. Build geometric models $X$ and $X'$ as in Section 6.1, with the additional requirement that if there exist weakly rigid vertices
$r \in \Gamma$ and $r' \in \Gamma'$ with $\| (G_r, \mathcal{P}_r) \| = \| (G'_r, \mathcal{P}'_r) \|$ then we choose $X_r = X_{r'}$ and $\mu_r(\mathcal{P}_r) = \mu_r(\mathcal{P}'_r)$. If $G$ and $G'$ are quasi-isometric, then the structure invariants $S(T^*, \delta, \mathcal{D})$ and $S(T'^*, \delta, \mathcal{D})$ are equivalent.

**Remark.** If all vertices of $V_R$ are actually quasi-isometrically rigid, not just weakly rigid, then $S(T^*, \delta, \mathcal{D})$ gives a quasi-isometry invariant of $G$. Otherwise, the values of the relative stretches depend also on the choices of the weakly rigid model spaces. We can still use the structure invariants to distinguish two given groups $G$ and $G'$, provided we choose the same weakly rigid model vertex spaces for $G$ and $G'$.

**Proof.** Suppose $\psi: G \to G'$ is a quasi-isometry. Then there is a quasi-isometry $\mu' \circ \psi \circ \bar{\mu}: X \to X'$ and an induced isomorphism $\psi^*: \mathcal{T}^* \to \mathcal{T}'^*$. We must show that for $s_0$ in $V_S$ of $\mathcal{T}^*$ adjacent to vertices $c \in V_C$ and $r_0 \in V_R$, we have $\text{relStr}(s_0) = \text{relStr}(\psi^*(s_0))$.

If $s_0$ is the only vertex in $\text{Rig}(c)$ then $\text{relStr}(s_0) = 1$ and $\psi^*(s_0)$ is the only vertex in $\text{Rig}(\psi^*(c))$, so $\text{relStr}(\psi^*(s_0)) = 1$. Otherwise, let $s_1$ be a vertex of $\text{Rig}(c)$ distinct from $s_0$, and let $r_1$ be the unique vertex of $V_R$ adjacent to $s_1$. Let $e_i$ be the edge connecting $s_i$ to $r_i$. Choose a pair of points $x_0$ and $y_0$ in $X_{r_0} \cap X_{e_0}$. There is a unique point $x_c \in X_c$ that is distance 2 from $x_0$. Similarly, there is a unique point $x_1 \in X_{r_1}$ that is distance 4 from $x_0$, and unique points $y_c \in X_c$ and $y_1 \in X_{r_1}$ at distance 2 and 4 from $y_0$, respectively.

\[
\frac{d_{X_{r_0}}(x_0, y_0)}{d_{X_{r_1}}(x_1, y_1)} = \frac{\text{Str}(s_0) \cdot d_{X_c}(x_c, y_c)}{\text{Str}(s_1) \cdot d_{X_c}(x_c, y_c)} = \frac{\text{relStr}(s_0)}{\text{relStr}(s_1)}
\]

$\psi$ induces a quasi-isometry $\psi_{r_0} \in \text{QI}((G_{r_0}, \mathcal{P}_{r_0}), (G'_{r_0}, \mathcal{P}'_{r_0}))$. Thus:

\[
\mu'_{\psi^*(r_0)} \circ \psi_{r_0} \circ \bar{\mu}_{r_0} \in \text{QI}((X_{r_0}, \mu_{r_0}(\mathcal{P}_{r_0})), (X'_{r_0}, \mu'_{\psi^*(r_0)}(\mathcal{P}'_{\psi^*(r_0)}))
\]

However, we required that $(X'_{\psi^*(r_0)}, \mu'_{\psi^*(r_0)}(\mathcal{P}'_{\psi^*(r_0)})) = (X_{r_0}, \mu_{r_0}(\mathcal{P}_{r_0}))$, so, by weak quasi-isometric rigidity, $\mu'_{\psi^*(r_0)} \circ \psi_{r_0} \circ \bar{\mu}_{r_0}$ is bounded distance from an isometry of $X_{r_0}$. The same is true for $r_1$.

$\psi$ sends $X_{r_0} \cap X_{e_0}$ to within bounded distance of $X'_{\psi^*(r_0)} \cap X'_{\psi^*(e_0)}$ so let $p_0$ be a point of $X'_{\psi^*(r_0)} \cap X'_{\psi^*(e_0)}$ close to $\psi(x_0)$. Let $p_1$ be the closest point of $X'_{\psi^*(r_1)} \cap X'_{\psi^*(e_1)}$ to $p_0$. Now, $d(p_0, p_1) = 4$, and $d(x_0, x_1) = 4$, so $d(p_1, \psi(x_1))$ is bounded in terms of the quasi-isometry constants of $\psi$, independent of $x_0$. Similarly, let $q_0$ be a point of $X'_{\psi^*(r_0)} \cap X'_{\psi^*(e_0)}$ close to $\psi(y_0)$ and $q_1 \in X'_{\psi^*(r_1)} \cap X'_{\psi^*(e_1)}$ closest to $q_0$.

Since $\mu'_{\psi^*(r_0)} \circ \psi_{r_0} \circ \bar{\mu}_{r_0}$ is bounded distance from an isometry, $d_{X'_{\psi^*(r_0)}}(p_0, q_0) = d_{X_{r_0}}(x_0, y_0)$ up to bounded error that is independent of $x_0$ and $y_0$. Similarly, $d_{X'_{\psi^*(r_1)}}(p_1, q_1) = d_{X_{r_1}}(x_1, y_1)$ up to bounded error. We make these bounded errors negligible by taking $x_0$ and $y_0$ to be far apart:

\[
\frac{\text{relStr}(s_0)}{\text{relStr}(s_1)} = \frac{d_{X_{r_0}}(x_0, y_0)}{d_{X_{r_1}}(x_1, y_1)} = \lim_{d(x_0, y_0) \to \infty} \frac{d_{X_{r_0}}(x_0, y_0)}{d_{X_{r_1}}(x_1, y_1)}
\]

\[
= \lim_{d(p_0, q_0) \to \infty} \frac{d_{X'_{\psi^*(r_0)}}(p_0, q_0)}{d_{X'_{\psi^*(r_1)}}(p_1, q_1)} = \frac{\text{relStr}(\psi^*(s_0))}{\text{relStr}(\psi^*(s_1))}
\]
So, $\psi^*$ preserves the ratios of relative stretches. However, we have normalized so that there exists an $s_1 \in \text{Rig}(c)$ such that $\text{relStr}(s_1) = 1$ is minimal, so $\psi^*$ must actually preserve the values of relStr.

6.3. Example. Let $G_i = \langle a, b, t \mid tu_i t = v_i \rangle$, where $u_i$ and $v_i$ are words in $\langle a, b \rangle$ given below. In each case $G_i$ should be thought of as an HNN extension of $\langle a, b \rangle$ over $\mathbb{Z}$ with stable letter $t$. Subdividing the edge gives a JSJ decomposition $\varGamma_i$ of $G_i$ whose Bass-Serre $\mathcal{T}_i$ is equal to its tree of cylinders.

Let $u_0 = a, v_0 = abab^2, u_1 = ab, v_1 = a^2b^2, u_2 = ab^2$, and $v_2 = a^2b$.

The methods of [5] show that for each $i$, the pair $\{u_i, v_i\}$ is Whitehead minimal, with Whitehead graph equal to the complete graph on 4 vertices, so $\langle a, b \rangle$ is quasi-isometrically rigid relative to the peripheral structure $\mathcal{P}_i$ coming from incident edge groups, and the rigid model space is just the Cayley graph for $\langle a, b \rangle$ with respect to $\{a, b\}$, ie, the 4-valent tree. Furthermore, $QI(\langle a, b \rangle, \mathcal{P}_1)$ is transitive on $\mathcal{P}_1$, and $[\langle (a, b), \mathcal{P}_1 \rangle] = [\langle (a, b), \mathcal{P}_2 \rangle] = [\langle (a, b), \mathcal{P}_3 \rangle]$. Therefore, Theorem 4.1 can not distinguish these three groups.

In $\mathcal{T}_i$, $V_C$ is a single orbit, and each vertex $c \in V_C$ is adjacent to exactly two vertices $s$ and $s'$ in $V_S$. These are adjacent to rigid vertices $r$ and $r'$, respectively. Let us assume that $s$ corresponds to an edge that attaches to $X_r$ along the image of a conjugate of $\langle u_i \rangle$ and $s'$ corresponds to an edge that attaches to $X_r$ along the image of a conjugate of $\langle u_i \rangle$. The stabilizer of $s$ and $s'$ are equal to the stabilizer of $c$, which is infinite cyclic. This, together with the fact that the rigid model space is the Cayley tree for $\langle a, b \rangle$ means that $\text{Str}(s)$ is the word length of $v_i$ in $\langle a, b \rangle$, and $\text{Str}(s')$ is the word length of $u_i$ in $\langle a, b \rangle$. Thus, $\text{relStr}(s') = 1$ and $\text{relStr}(s) = \frac{|v_i|}{|u_i|} = 5, 2, \text{ or } 1$, as $i = 0, 1 \text{ or } 2$, respectively. By Theorem 6.4, no one of these groups is quasi-isometric to the other.

Whether our enhanced decoration gives a complete quasi-isometry invariant, or whether further enhancements can achieve this, are interesting questions that we plan to explore in future work. This requires a fuller understanding of the relative quasi-isometry groups $QI(G_v, \mathcal{P}_v)$.

It would also be interesting to remove the assumption of hyperbolicity of $G$, which would allow the cylinder spaces $X_c$ to be quasi-isometric to Baumslag-Solitar groups.

References

[1] Jason A. Behrstock and Walter D. Neumann, Quasi-isometric classification of graph manifold groups, Duke Math. J. 141 (2008), no. 2, 217–240, doi:10.1215/S0012-7094-08-14121-3. MR 2376814 (2009c:20070)

[2] Jason A. Behrstock and Walter D. Neumann, Quasi-isometric classification of non-geometric 3-manifold groups, J. Reine Angew. Math. 669 (2012), 101–120, doi:10.1515/CRELLE.2011.143. MR 2980453

[3] Martin R. Bridson and André Haefliger, Metric spaces of non-positive curvature, Grundlehren der mathematischen Wissenschaften, vol. 319, Springer, Berlin, 1999. MR 1744486 (2000k:53038)

[4] Christopher H. Cashen, Splitting line patterns in free groups, preprint, 2010, arXiv:1006.2492.

[5] Christopher H. Cashen and Nataša Macura, Line patterns in free groups, Geom. Topol. 15 (2011), no. 3, 1419–1475, doi:10.2140/gt.2011.15.1419. MR 2825316

[6] Martin J. Dunwoody, The accessibility of finitely presented groups, Invent. Math. 81 (1985), no. 3, 449–457, doi:10.1007/BF01388581. MR 807066 (87d:20037)
[7] Martin J. Dunwoody and Michah E. Sageev, *JSJ-splittings for finitely presented groups over slender groups*, Invent. Math. **135** (1999), no. 1, 25–44, doi:10.1007/s002220050278. MR 1664694 (2000b:20050)

[8] Koji Fujiwara and Panos Papasoglu, *JSJ-decompositions of finitely presented groups and complexes of groups*, Geom. Funct. Anal. **16** (2006), 70–125.

[9] Mikhael Gromov, *Infinite groups as geometric objects*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983) (Warsaw), PWN, 1984, pp. 385–392. MR 804694 (87c:57033)

[10] Vincent Guirardel and Gilbert Levitt, *JSJ decompositions: definitions, existence, uniqueness. I: The JSJ deformation space*, preprint, 2009, arXiv:0911.3173.

[11] Vincent Guirardel and Gilbert Levitt, *Trees of cylinders and canonical splittings*, Geom. Topol. **15** (2011), no. 2, 977–1012, doi:10.2140/gt.2011.15.977. MR 2821568 (2012k:20052)

[12] Michael Kapovich and Bruce Kleiner, *Hyperbolic groups with low-dimensional boundary*, Ann. Sci. École Norm. Sup. (4) **33** (2000), no. 5, 647–669, doi:10.1016/S0012-9593(00)01049-1. MR 1834498 (2002j:20077)

[13] Bruce Kleiner and Bernhard Leeb, *Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings*, Inst. Hautes Études Sci. Publ. Math. (1997), no. 86, 115–197 (1998). MR 1608566 (98m:53068)

[14] Frank Thomson Leighton, *Finite common coverings of graphs*, J. Combin. Theory Ser. B **33** (1982), no. 3, 231–238, doi:10.1016/0095-8956(82)90042-9. MR 693362 (85a:05068)

[15] William Malone, *Topics in geometric group theory*, Ph.D. thesis, University of Utah, Salt Lake City, UT, 2010.

[16] Vladimir Markovic, *Quasisymmetric groups*, J. Amer. Math. Soc. **19** (2006), no. 3, 673–715, doi:10.1090/S0894-0347-06-00518-2. MR 2220103 (2007c:37057)

[17] Vincent Guirardel and Gilbert Levitt, *Hyperbolic groups with low-dimensional boundary*, Ann. Sci. École Norm. Sup. (4) **33** (2000), no. 5, 647–669, doi:10.1016/S0012-9593(00)01049-1. MR 1834498 (2002j:20077)

[18] Pierre Pansu, *Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un*, Ann. of Math. (2) **129** (1989), no. 1, 1–60, doi:10.2307/1971484. MR 979599 (90m:53058)

[19] Paneos Papasoglu, *Quasi-isometry invariance of group splittings*, Ann. of Math. (2) **161** (2005), no. 2, 759–830, doi:10.4007/annals.2005.161.759. MR 2153400 (2006d:20076)

[20] Panos Papasoglu and Kevin Whyte, *Quasi-isometries between groups with infinitely many ends*, Comment. Math. Helv. **77** (2002), no. 1, 133–144, doi:10.1007/s00014-002-8334-2. MR 1898396 (2003c:20049)

[21] Richard Evan Schwartz, *Symmetric patterns of geodesics and automorphisms of surface groups*, Invent. Math. **128** (1997), no. 1, 177–199, doi:10.1007/s002220050139. MR 1437498 (98d:58143)

[22] Jean-Pierre Serre, *Trees*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation. MR 1954121 (2003m:20032)

[23] John R. Stallings, *On torsion-free groups with infinitely many ends*, Ann. of Math. (2) **88** (1968), 312–334. MR 0228573 (37 #4153)

[24] John R. Stallings, *Group theory and three-dimensional manifolds*, Yale University Press, New Haven, Conn., 1971, A James K. Whittemore Lecture in Mathematics given at Yale University, 1969, Yale Mathematical Monographs, 4. MR 0415622 (54 #3705)

**Fakultät für Mathematik**
**Universität Wien**
1090 Vienna, Austria
**E-mail address:** christopher.cashen@univie.ac.at
**URL:** http://www.mat.univie.ac.at/~cashen