Online Improper Learning with an Approximation Oracle

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Abstract

We revisit the question of reducing online learning to approximate optimization of the offline problem. In this setting, we give two algorithms with near-optimal performance in the full information setting: they guarantee optimal regret and require only poly-logarithmically many calls to the approximation oracle per iteration. Furthermore, these algorithms apply to the more general improper learning problems. In the bandit setting, our algorithm also significantly improves the best previously known oracle complexity while maintaining the same regret.

1 Introduction

One of the most fundamental and well-studied questions in learning theory is whether one can learn a given problem using an optimization oracle. For online learning in games, it was shown by Kalai and Vempala (2005) that an optimization oracle giving the best decision in hindsight is sufficient for attaining optimal regret.

However, in many non-convex settings, such an optimization oracle is either unavailable or NP-hard to compute. In contrast, in many such cases, efficient approximation algorithms are usually known, and are guaranteed to return a solution within a certain multiplicative factor of the optimum. These include not only combinatorial optimization problems such as MAX CUT, WEIGHTED SET COVER, METRIC TRAVELING SALESMAN PROBLEM, SET PACKING, etc., but also machine learning problems such as LOW RANK MATRIX COMPLETION.

Kakade et al. (2009) considered the question of whether an approximation algorithm is sufficient to obtain vanishing regret compared with an approximation to the best solution in hindsight. They gave an algorithm for this offline-to-online conversion. However, their reduction is inefficient in the number of per-iteration queries to the approximation oracle, which grows linearly with time. Ideally, an efficient reduction should call the oracle only a constant number of times per iteration and guarantee optimal regret at the same time, and this was considered an open question in the literature.

Various authors have improved upon this original offline-to-online reduction under certain cases, as we survey below. Recently, Garber (2017) has made significant progress by giving a more efficient reduction, which improves the number of oracle calls both in the full information and the bandit settings. He explicitly asked whether a near-optimal reduction with only logarithmically many calls per iteration exists.

1.1 Our Results

In this paper we resolve this question on the positive side, and in a more general setting. We give two different algorithms in the full information setting, one based on the online mirror descent (OMD) method.

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and another based on the *continuous multiplicative weight update (CMWU)* algorithm, which give optimal regret and are oracle-efficient. Furthermore, our algorithms apply to more general loss vectors. Our results are summarized in the table below.

### Table 1: Summary of results in the full information setting.

| Algorithm           | $\alpha$-regret in $T$ rounds | oracle complexity per round | loss vectors      |
|---------------------|-------------------------------|----------------------------|-------------------|
| Kakade et al. (2009)| $O(\sqrt{T})$                | $O(T)$                     | general           |
| Garber (2017)       | $O(\sqrt{T})$                | $O(\sqrt{T} \log T)$      | non-negative      |
| Algorithm 1 (OMD)   | $O(\sqrt{T})$                | $O(\log T)$               | PNIP property     |
| Algorithm 5 (CMWU)  | $O(\sqrt{T})$                | $O(\log T)$               | general           |

In addition to these two algorithms, we give an improved bandit algorithm based on OMD: it attains the same $O(T^{2/3})$ regret as in (Kakade et al., 2009; Garber, 2017) with a lower computational cost: our method requires $O(T^{2/3})$ oracle calls over all the $T$ game iterations, as opposed to $O(T)$ in the previous best method.

Besides the improved oracle complexity, our methods have the following additional advantages:

- While the algorithm in (Garber, 2017) requires non-negative loss vectors, our second algorithm, based on CMWU, can work with general loss vectors. Furthermore, our OMD-based algorithm can also work with loss vectors from any convex cone satisfying the *pairwise non-negative inner product (PNIP)* property defined in Definition 4.1 (together with an appropriately chosen regularizer), which is more general than the non-negative orthant.

- Our methods apply to a general online improper learning setting, in which the predictions can be from a potentially different set from the target set to compete against. Previous work considered this different set to be a constant multiple of the target set, which makes sense primarily for combinatorial optimization problems.

However, in many interesting problems, such as **LOW RANK MATRIX COMPLETION**, the natural approximation algorithm returns a matrix of higher rank. This is not in a constant multiple of the set of all low rank matrices, and our additional generality allows us to obtain meaningful results even for this case.

- Our first algorithm is based on the general OMD methodology, and thus allows any strongly convex regularizer. This can give better regret bounds, in terms of the space geometry, compared with the previous algorithm of (Garber, 2017) that is based on online gradient descent and Euclidean regularization. The improvement in regret bounds can be as large as the dimension.

- Our bandit algorithm is based on OMD with a new regularizer that is inspired from the construction of barycentric spanners, and may be of independent interest.

#### 1.2 Our Techniques

The more general one of our algorithms is based on a completely different methodology compared with previous online-to-offline reductions. It is a variant of the *continuous multiplicative weight update (CMWU)* algorithm, or the *continuous hedge* algorithm. Our idea is to apply CMWU over a superset of the target set, and in every iteration the algorithm tries to play the mean of a log-linear distribution. To check feasibility of
this mean, we show how to design a separation-or-decomposition oracle, which either certifies that the mean is infeasible - in this case it provides a separating hyperplane between the mean and the target set and thus gives a more refined superset of the target set, or provides a distribution over feasible points whose average is superior to the mean in terms of the regret. Using this approach, the more oracle calls the algorithm makes, the tighter superset it can obtain, and we show an interesting trade-off between the oracle complexity and the regret bound.

The other algorithm follows the line of Garber (2017). We show how to significantly speed up Garber’s infeasible projection oracle, and to generalize Garber’s algorithm from online gradient descent (OGD) to online mirror descent (OMD).

This additional generality is crucial in our bandit algorithm, where we make use of a novel regularizer in OMD, called the barycentric regularizer, in order to have a low-variance unbiased estimator of the loss vector. This geometric regularization may be of independent interest.

1.3 Related Work

The reduction from online learning to offline approximation algorithms was already considered by Kalai and Vempala (2005). Their scheme, based on the follow-the-perturbed-leader (FTPL) algorithm, requires very strong approximation guarantee from the approximation oracle, namely, a fully polynomial time approximation scheme (FPTAS), and requires an approximation that improves with time. Balcan and Blum (2006) used the same approach in the context of mechanism design.

Kalai and Vempala (2005) also proposed a specialized reduction that works under certain conditions on the approximation oracle, satisfied by some known algorithms for problems such as MAX-CUT. Fujita et al. (2013) further gave more general reductions that apply to problems whose approximation algorithms are based on convex relaxations of mathematical programs. Their scheme is also based on the FTPL method. Recent advancements on black-box online-to-offline reductions were made in (Kakade et al., 2009; Dudík et al., 2016; Garber, 2017). Hazan and Koren (2016) showed that efficient reductions are in general impossible, unless special structure is present. In the settings we consider this special structure is a linear cost function over the space.

Our algorithms fall into one of two templates. The first is the online mirror descent method, which is an adaptive version of the follow-the-regularized-leader (FTRL) algorithm. The second is the continuous multiplicative weight update method, which dates back to Cover’s portfolio selection method (Cover, 1991) and Vovk’s aggregating algorithm (Vovk, 1990). The reader is referred to the books (Cesa-Bianchi and Lugosi, 2006; Shalev-Shwartz, 2012; Hazan, 2016) for details and background on these prediction frameworks. We also make use of polynomial-time algorithms for sampling from log-concave distributions (Lovász and Vempala, 2007).

2 Preliminaries

We use $\|x\|$ to denote the Euclidean norm of a vector $x$. For $x \in \mathbb{R}^d$ and $r > 0$, denote by $B(x, r)$ the Euclidean ball in $\mathbb{R}^d$ of radius $r$ centered at $x$, i.e., $B(x, r) := \{x' \in \mathbb{R}^d : \|x - x'\| \leq r\}$. For $S, S' \subseteq \mathbb{R}^d$, $\beta \in \mathbb{R}$, $y \in \mathbb{R}^d$, and $A \in \mathbb{R}^{d \times d}$, define $S + S' := \{x + x' : x \in S, x' \in S'\}$, $\beta S := \{\beta x : x \in S\}$, $x + S := \{x + y : y \in S\}$, and $AS := \{Ax : x \in S\}$. The convex hull of $S \subseteq \mathbb{R}^d$ is denoted by $\text{CH}(S)$. Denote by $\text{Vol}(S)$ the volume (Lebesgue measure) of a set $S \subseteq \mathbb{R}^d$. Denote by $\Delta^{k-1}$ the probability simplex in $\mathbb{R}^k$, i.e., $\Delta^{k-1} := \{x \in \mathbb{R}^k : x_i \geq 0 \ (\forall i), \sum_i x_i = 1\}$.

A set $C \subseteq \mathbb{R}^d$ is called a cone if for any $\beta \geq 0$ we have $\beta C \subseteq C$. For any $S \subseteq \mathbb{R}^d$, define the dual cone of $S$ as $S^\circ := \{y \in \mathbb{R}^d : x^\top y \geq 0, \ \forall x \in S\}$. $S^\circ$ is always a convex cone, even when $S$ is neither convex nor a cone.
For any closed set \( S \subseteq \mathbb{R}^d \), define \( \Pi_S : \mathbb{R}^d \to S \) to be the projection onto \( S \), namely \( \Pi_S(x) := \arg \min_{x' \in S} \|x' - x\|^2 \). The well-known Pythagorean theorem characterizes an important property of projections onto convex sets:

**Lemma 2.1** (Pythagorean theorem). For any closed convex set \( S \subseteq \mathbb{R}^d \), \( x \in \mathbb{R}^d \) and \( y \in S \), we have \((\Pi_S(x) - x)^	op (\Pi_S(x) - y) \leq 0\), or equivalently, \( \|x - y\|^2 \geq \|\Pi_S(x) - x\|^2 + \|\Pi_S(x) - y\|^2 \).

**Definition 2.2.** A function \( f : A \to \mathbb{R} (A \subseteq \mathbb{R}^d) \) is Legendre if

- \( A \) is convex;
- \( f \) is strictly convex with continuous gradient defined over \( A \)'s interior \( \text{int}(A) \);
- for any sequence \( x_1, x_2, \ldots \in A \) converging to a boundary point of \( A \), \( \lim_{n \to \infty} \|\nabla f(x_n)\| = \infty \).

**Definition 2.3.** For a Legendre function \( \varphi : A \to \mathbb{R} \), the Bregman divergence with respect to \( \varphi \) is defined as \( D_\varphi(x, y) := \varphi(x) - \varphi(y) - \nabla \varphi(y)^	op (x - y) \) (\( \forall x, y \in A \)).

The Pythagorean theorem can be generalized to projections with respect to a Bregman divergence (see e.g. Lemma 11.3 in (Cesa-Bianchi and Lugosi, 2006)):

**Lemma 2.4** (Generalized Pythagorean theorem). For any closed convex set \( S \subseteq \mathbb{R}^d \), \( x \in \mathbb{R}^d \), \( y \in S \), and any Legendre function \( \varphi : \mathbb{R}^d \to \mathbb{R} \), letting \( z = \arg \min_{x' \in S} D_\varphi(x', x) \), we must have \( D_\varphi(y, z) + D_\varphi(z, x) \geq D_\varphi(y, x) \).

**Log-concave distributions.** A distribution over \( \mathbb{R}^d \) with a density function \( f \) is said to be log-concave if \( \log(f) \) is a concave function. For a convex set \( S \) equipped with a membership oracle, there exist polynomial-time algorithms for sampling from any log-concave distribution over \( S \) (Lovász and Vempala, 2007). This can be used to approximately compute the mean of any log-concave distribution.

We have the following classical result which says that every half-space close enough to the mean of a log-concave distribution must contain at least constant probability mass. For simplicity, we only state and prove the result for isotropic (i.e., identity covariance) log-concave distributions, but the result can be easily generalized to allow arbitrary covariance.

**Lemma 2.5.** Consider any isotropic (identity covariance) log-concave distribution \( p \) over \( \mathbb{R}^d \) with mean \( x^* \). Then for any half-space \( H \) such that \( \|x^* - \Pi_H(x^*)\| \leq \frac{1}{2e} \), we have \( \int_H p(x)dx \geq \frac{1}{2e} \).

The proof of Lemma 2.5 is given in Appendix A. As an implication, we have the following lemma regarding mean computation of a log-concave distribution, which is useful in this paper.

**Lemma 2.6.** For any log-concave distribution \( p \) in \( \mathbb{R}^d \) with mean \( x^* \), whose support \( \text{supp}(p) \) is in \( B(0, R) \) (\( R > 0 \)), and any \( \epsilon > 0 \) and \( 0 < \delta < 1 \), it is possible to compute a point \( \tilde{x}^* \) in \( \text{poly} \left(d, \frac{1}{\epsilon}, \log \frac{1}{\delta} \right) \) time such that with probability at least \( 1 - \delta \) we have:

1. \( \|\tilde{x}^* - x^*\| \leq R\epsilon \);
2. for any half space \( H \) containing \( \tilde{x}^* \), \( \int_H p(x)dx \geq \frac{1}{2e} \).

For our purpose in this paper, it always suffices to choose \( \epsilon = \frac{1}{T} \) and \( \delta = \frac{1}{\text{poly}(T)} \) (\( T \) being the total number of rounds) without hurting our regret bounds. Therefore, for ease of presentation, we will assume that we can compute the mean of bounded-supported log-concave distributions exactly.
3 Online Improper Linear Optimization with an Improper Optimization Oracle

Now we describe the problem setting we consider in this paper. Let $\mathcal{K}, \mathcal{K}^* \subseteq B(0, R) (R > 0)$ be two compact subsets of $\mathbb{R}^d$, and let $W \subseteq \mathbb{R}^d$ be a convex cone. Suppose we have an improper linear optimization oracle $\mathcal{O}_{\mathcal{K}, \mathcal{K}^*} : W \to \mathcal{K}$, which given an input $v \in W$ can output a point $\mathcal{O}_{\mathcal{K}, \mathcal{K}^*}(v) \in \mathcal{K}$ such that

$$v^\top \mathcal{O}_{\mathcal{K}, \mathcal{K}^*}(v) \leq \min_{x^* \in \mathcal{K}^*} v^\top x^*.$$ 

In other words, it performs linear optimization over $\mathcal{K}^*$ but is allowed to output a point from a (possibly different) set $\mathcal{K}$. Note that this implicitly requires that $\mathcal{K}$ “dominates” $\mathcal{K}^*$ in all directions in $W$, that is, for all $v \in W$ we must have $\min_{x \in \mathcal{K}} v^\top x \leq \min_{x^* \in \mathcal{K}^*} v^\top x^*$.

**Online improper linear optimization.** Consider a repeated game with $T$ rounds. In round $t$, the player chooses a point $x_t \in \mathcal{K}$ while an adversary chooses a loss vector $f_t \in W \cap B(0, L)$ ($L > 0$), and then the player incurs a loss $f_t^\top x_t$. The goal for the player is to have a cumulative loss that is comparable to that of the best single decision in hindsight.

We assume that the player only has access to the optimization oracle $\mathcal{O}_{\mathcal{K}, \mathcal{K}^*}$. Therefore, it is only fair to compare with the best decision in $\mathcal{K}^*$ in hindsight. The (improper) regret over $T$ rounds is defined as

$$\text{Reg}_{\mathcal{K}, \mathcal{K}^*}(T) := \sum_{t=1}^T f_t^\top x_t - \min_{x^* \in \mathcal{K}^*} \sum_{t=1}^T f_t^\top x^*.$$ 

We sometimes treat $f_t$ as a function on $\mathbb{R}^d$, i.e., $f_t(x) := f_t^\top x$.

**Full information and bandit settings.** We consider both full information and bandit settings. In the full information setting, after the player makes her choice $x_t$ in round $t$, the entire loss vector $f_t$ is revealed to the player; in the bandit setting, only the loss value $f_t(x_t)$ is revealed to the player.

**$\alpha$-regret minimization with an approximation oracle.** The problem of online linear optimization with an approximation oracle considered by Kakade et al. (2009) and Garber (2017) is a special instance in our online improper linear optimization framework. In this problem, the player has access to an approximate linear optimization oracle $\mathcal{O}_K^\alpha$ over $\mathcal{K} (\alpha > 1)$, which given a direction $v \in \mathcal{K}$ as input can output a point $\mathcal{O}_K^\alpha(v) \in \mathcal{K}$ such that

$$v^\top \mathcal{O}_K^\alpha(v) \leq \alpha \min_{x \in \mathcal{K}} v^\top x.$$ 

In this setting we will consider $\mathcal{K} \subseteq \mathbb{R}^d_+$ and $W = \mathbb{R}^d_+$; many combinatorial optimization problems with efficient approximation algorithms fall into this regime. The goal in the online problem is therefore to minimize the $\alpha$-regret, defined as

$$\text{Reg}_K^\alpha(T) := \sum_{t=1}^T f_t^\top x_t - \alpha \min_{x \in \mathcal{K}} \sum_{t=1}^T f_t^\top x.$$ 

To see why this is a special case of online improper linear optimization, note that we can take $\mathcal{K}^* = \alpha \mathcal{K}$ and then the approximation oracle $\mathcal{O}_K^\alpha$ is equivalent to $\mathcal{O}_{\mathcal{K}, \alpha \mathcal{K}}$ and the $\alpha$-regret $\text{Reg}_K^\alpha(T)$ is equal to the improper regret $\text{Reg}_{\mathcal{K}, \alpha \mathcal{K}}(T)$. 

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4 Efficient Online Improper Linear Optimization via Online Mirror Descent

In this section, we give an efficient online improper linear optimization algorithm (in the full information setting) based on online mirror descent (OMD) equipped with a strongly convex regularizer \( \varphi \), which achieves \( O(\sqrt{T}) \) regret when the regularizer \( \varphi \) and the domain of linear loss functions \( W \) satisfy the pairwise non-negative inner product (PNIP) property (Definition 4.1). This property holds for many interesting domains with appropriately chosen regularizers. Notable examples include the non-negative orthant \( \mathbb{R}^d_+ \), the positive semidefinite matrix cone, and the Lorentz cone \( L_{d+1} = \{(x, z) \in \mathbb{R}^d \times \mathbb{R} : \|x\|_2 \leq z\} \).

**Definition 4.1 (Pairwise non-negative inner product).** For a twice-differentiable Legendre function \( \varphi : \mathcal{A} \rightarrow \mathbb{R} \) with domain \( \mathcal{A} \subseteq \mathbb{R}^d \) and a convex cone \( W \subseteq \mathbb{R}^d \), we say \((\varphi, W)\) satisfies the pairwise non-negative inner product (PNIP) property, if for all \( w, w' \in W \) and \( H \in \text{CH}(\mathcal{H}) \), where \( \mathcal{H} = \{\nabla^2 \varphi(x) : x \in \mathcal{A}\} \), it holds that \( w^\top H^{-1} w' \geq 0 \).

**Examples.** \((\varphi, W)\) satisfies the PNIP property if:
- \( \varphi(x) = \frac{1}{2} \|x\|^2 \) (with domain \( \mathbb{R}^d \)) and \( W \subseteq W^\circ \);
- \( \varphi(x) = \sum_{i=1}^d x_i (\log x_i - 1) \) (with domain \( \mathbb{R}^d_+ \)) and \( W = \mathbb{R}^d_+ \);
- \( \varphi(x) = \frac{1}{2} x^\top Q^{-1} x \) (with domain \( \mathbb{R}^d \)), where \( Q = MM^\top, M \in \mathbb{R}^{d \times d} \) is an invertible matrix, and \( W = (M^\top)^{-1} \mathbb{R}^d_+ \). This is useful in our bandit algorithm in Section 6.

4.1 Online Mirror Descent with a Projection-and-Decomposition Oracle

We first show that assuming the availability of a projection-and-decomposition (PAD) oracle (Definition 4.2), we can implement a variant of the OMD algorithm that achieves optimal regret. In Section 4.2, we show how to construct a PAD oracle using the oracle \( O_{\mathcal{K}, \mathcal{K}^*} \). In Section 4.3, we bound the number of oracle calls to \( O_{\mathcal{K}, \mathcal{K}^*} \) in our algorithm.

**Definition 4.2 (Projection-and-decomposition oracle).** A projection-and-decomposition (PAD) oracle onto \( \mathcal{K}^* \), \( \mathcal{PAD}(y, \epsilon, W, \varphi) \), is defined as a procedure that given \( y \in \mathbb{R}^d, \epsilon > 0, \) a convex cone \( W \) and a Legendre function \( \varphi \) produces a tuple \((y', V, p)\), where \( y' \in \mathbb{R}^d, V = (v_1, \ldots, v_k) \in \mathbb{R}^{d \times k} \) and \( p = (p_1, \ldots, p_k)^\top \in \Delta^{k-1} \), such that:

1. \( y' \) is “closer” to \( \mathcal{K}^* \) than \( y \) with respect to the Bregman divergence of \( \varphi \) (and hence is an “infeasible projection”): \( \forall x^* \in \mathcal{K}^*, D_\varphi(x^*, y') \leq D_\varphi(x^*, y) \);

2. \( v_1, \ldots, v_k \in \mathcal{K} \), and \( \sum_{i=1}^k p_i v_i \) is a point that “almost dominates” \( y' \) in all directions in \( W \). In other words, there exists \( c \in W^\circ \) such that \( \| \sum_{i=1}^k p_i v_i + c - y' \| \leq \epsilon \).

The purpose of the PAD oracle is the following. Suppose the OMD algorithm tells us to play a point \( y \). Since \( y \) might not be in the feasible set \( \mathcal{K} \), we can call the PAD oracle to find another point \( y' \) as well as a distribution \( p \) over points \( v_1, \ldots, v_k \in \mathcal{K} \). The first property in Definition 4.2 is sufficient to ensure that playing \( y' \) also gives low regret, and the second property further ensures that we have a distribution of points in \( \mathcal{K} \) that suffers less loss than \( y' \) for every possible loss function so we can play according to that distribution.

Using the PAD oracle, we can apply OMD as in Algorithm 1. Theorem 4.3 gives its regret bound.
Algorithm 1 Online Mirror Descent using a Projection-and-Separation Oracle

\textbf{Input:} Learning rate $\eta > 0$, tolerance $\epsilon > 0$, regularizer $\varphi$, convex cone $W$, time horizon $T \in \mathbb{N}_+$

1: $y_1 \leftarrow \arg\min_{y \in \text{Dom}(\varphi)} \varphi(y)$.
2: \textbf{for} $t = 1$ \textbf{to} $T$ \textbf{do}
3: \hspace{5mm} $(x_t, V, p) \leftarrow \mathcal{PAD}(y_t, \epsilon, W, \varphi)$
4: \hspace{10mm} Play $\tilde{x}_t = v_i$ with probability $p_i$ ($i \in [k]$), where $V = (v_1, \ldots, v_k)$, and observe the loss vector $f_t$
5: \hspace{10mm} $\nabla \varphi(y_{t+1}) \leftarrow \nabla \varphi(x_t) - \eta f_t$
6: \textbf{end for}

\textbf{Theorem 4.3.} Suppose $(\varphi, W)$ satisfies the PNIP property (Definition 4.1). Then for any $\epsilon, \eta > 0$, Algorithm 1 satisfies the following regret guarantee:

$$\forall x^* \in \mathcal{K}^* : \quad \mathbb{E} \left[ \sum_{t=1}^{T} (f_t(\tilde{x}_t) - f_t(x^*)) \right] \leq \frac{1}{\eta} \left( \varphi(x^*) - \varphi(y_1) + \sum_{t=1}^{T} D_\varphi(x_t, y_{t+1}) \right) + \epsilon LT.$$ 

In particular, if $\varphi$ is $\mu$-strongly convex and $A \geq \max_{x^* \in \mathcal{K}^*} (\varphi(x^*) - \varphi(y_1))$, setting $\epsilon = \frac{R}{T}$ and $\eta = \frac{1}{T} \sqrt{\frac{2\mu A}{T^2}}$, we have

$$\forall x^* \in \mathcal{K}^* : \quad \mathbb{E} \left[ \sum_{t=1}^{T} (f_t(\tilde{x}_t) - f_t(x^*)) \right] \leq L \sqrt{\frac{2AT}{\mu}} + LR.$$ 

\textbf{Proof.} First, for any fixed round $t \in [T]$, let $(x_t, V, p)$ be the output of $\mathcal{PAD}(y_t, \epsilon, W, \varphi)$ in this round. We know by the second property of the PAD oracle that there exists $c \in W^\circ$ such that $\|\sum_i p_i v_i + c - x_t\| \leq \epsilon$. Since $\tilde{x}_t$ is equal to $v_i$ with probability $p_i$, letting $\mathfrak{I}_t := \mathbb{E}[\tilde{x}_t] = \sum_i p_i v_i$, we have

$$f_t(\mathfrak{I}_t) - f_t(x_t) = \mathbb{E} [ f_t(\tilde{x}_t) - f_t(x_t) ] = f_t \left( \sum_i p_i v_i - x_t \right) \leq f_t \left( \sum_i p_i v_i - x_t + c \right) \leq \epsilon L. \quad (1)$$

We make use of the following properties of Bregman divergence, which can be verified easily (see e.g. Section 11.2 in (Cesa-Bianchi and Lugosi, 2006)):

$$\forall x, y, z : \quad (x - y) \top (\nabla \varphi(z) - \nabla \varphi(y)) = D_\varphi(x, y) - D_\varphi(x, z) + D_\varphi(y, z). \quad (2)$$

Consider any $x^* \in \mathcal{K}^*$. We have

$$\sum_{t=1}^{T} (f_t(x_t) - f_t(x^*))$$

$$= \sum_{t=1}^{T} \frac{1}{\eta} (\nabla \varphi(x_t) - \nabla \varphi(y_{t+1})) \top (x_t - x^*) \quad \text{(by algorithm definition)}$$

$$= \frac{1}{\eta} \sum_{t=1}^{T} (D_\varphi(x^*, x_t) - D_\varphi(x^*, y_{t+1}) + D_\varphi(x_t, y_{t+1})) \quad \text{(by (2))} \quad (3)$$

$$\leq \frac{1}{\eta} \sum_{t=1}^{T} (D_\varphi(x^*, y_t) - D_\varphi(x^*, y_{t+1}) + D_\varphi(x_t, y_{t+1})) \quad \text{(by property of the PAD oracle)}$$

$$= \frac{1}{\eta} \left( D_\varphi(x^*, y_1) - D_\varphi(x^*, y_{T+1}) + \sum_{t=1}^{T} D_\varphi(x_t, y_{t+1}) \right). \quad \text{(by telescoping)}$$
Combining (1) and (3), we can bound the expected improper regret of Algorithm 1 as
\[
\forall x^* \in K^* : \quad \mathbb{E} \left[ \sum_{t=1}^{T} (f_t(\bar{x}_t) - f_t(x^*)) \right] = \sum_{t=1}^{T} (f_t(\bar{x}_t) - f_t(x^*)) \leq \frac{1}{\eta} \left( D_\varphi(x^*, y_1) - D_\varphi(x^*, y_{T+1}) + \sum_{t=1}^{T} D_\varphi(x_t, y_{t+1}) \right) + \epsilon L T. \tag{4}
\]

By the optimality condition \( \nabla \varphi(y_1)^T (x^* - y_1) \geq 0 \), we have
\[
D_\varphi(x^*, y_1) \leq \varphi(x^*) - \varphi(y_1). \tag{5}
\]

Plugging (5) into (4) and noting \( D_\varphi(x^*, y_{T+1}) \geq 0 \), we finish the proof of the first regret bound.

When \( \varphi \) is \( \mu \)-strongly convex, we have the following well-known property:\footnote{See \url{http://xingyuzhou.org/blog/notes/strong-convexity} for a proof.}
\[
D_\varphi(x, y) \leq \frac{1}{2\mu} \| \nabla \varphi(x) - \nabla \varphi(y) \|^2.
\]

Then by the definition in Algorithm 1 we have
\[
\forall t \in [T] : \quad D_\varphi(x_t, y_{t+1}) \leq \frac{1}{2\mu} \| \nabla \varphi(x_t) - \nabla \varphi(y_{t+1}) \|^2 = \frac{1}{2\mu} \| \eta f_t \|^2 \leq \frac{\eta^2 L^2}{2\mu}. \tag{6}
\]

From the above inequality and the choices of parameters \( \epsilon = \frac{R}{L} \) and \( \eta = \frac{1}{2} \sqrt{\frac{2\mu A}{T}} \), we have
\[
\mathbb{E} \left[ \sum_{t=1}^{T} (f_t(\bar{x}_t) - f_t(x^*)) \right] \leq \frac{A}{\eta} + \frac{\eta L^2 T}{2\mu} + LR \leq L \left( \frac{2AT}{\mu} + LR \right). \tag*{$\square$}
\]

For the problem of \( \alpha \)-regret minimization using an \( \alpha \)-approximation oracle, we have the following regret guarantee, which is an immediate corollary of Theorem 4.3.

**Corollary 4.4.** If \( W \subseteq \mathbb{R}_+^d \), \( K \subseteq B(0, R) \), \( K^* = \alpha K \), \( \varphi(x) = \frac{1}{2} \| x \|^2 \), setting \( \epsilon = \frac{\alpha R}{T} \), \( \eta = \frac{\alpha R}{L \sqrt{T}} \), Algorithm 1 has the following regret guarantee:
\[
\forall x^* \in K^* : \quad \mathbb{E} \left[ \sum_{t=1}^{T} f_t(\bar{x}_t) - \alpha \sum_{t=1}^{T} f_t(x^*) \right] \leq \alpha LR(\sqrt{T} + 1).
\]

### 4.2 Construction of the Projection-and-Decomposition Oracle

Now we show how to construct the PAD oracle using the improper linear optimization oracle \( O_{K, K^*} \). Our construction is given in Algorithm 2.

**Theorem 4.5.** Suppose \( (\varphi, W) \) satisfies the PNIP condition (Definition 4.1) and \( \varphi \) is \( \mu \)-strongly convex.

Then for any \( y \in \mathbb{R}^d \) and \( \epsilon \in (0, R] \), Algorithm 2 must terminate in \( k \leq \frac{4R+2}{\min_{x \in K^*} D_\varphi(x^*, y)} \sqrt{\frac{d \min_{x \in K^*} D_\varphi(x^*, y)}{\epsilon}} \) iterations, and it correctly implements the projection-and-decomposition oracle \( \mathcal{PAD}(y, \epsilon, W, \varphi) \), i.e., its output \( (y', V, p) \) satisfies the two properties in Definition 4.2.

We break the proof of Theorem 4.5 into several lemmas.
Algorithm 2 Projection-and-Decomposition Oracle, \( \mathcal{PAD}(y, \epsilon, W, \varphi) \)

**Input:** Point \( y \in \mathbb{R}^d \), tolerance \( \epsilon > 0 \), convex cone \( W \), regularizer \( \varphi \).

**Output:** \((y', V, p)\), where \( y' \in \mathbb{R}^d \), \( V = (v_1, \ldots, v_k) \in \mathbb{R}^{d \times k} \) for some \( k \) such that \( v_i \in \mathcal{K} (\forall i \in [k]) \), and \( p = (p_1, \ldots, p_k) ^\top \in \Delta^{k-1} \)

1. \( W_1 \leftarrow W \cap B(0, 1) \), \( z_1 \leftarrow y \)
2. \( i \leftarrow 0 \)
3. while \( i < 5d \log \frac{2(R+\|z_{i+1}\|)}{\epsilon} \) do
   4. \( i \leftarrow i + 1 \)
   5. \( w_i \leftarrow \frac{1}{\text{Vol}(W_i)} \)
   6. \( v_i \leftarrow \text{O}_{\mathcal{K}, \mathcal{K}^*}(w_i) \)
   7. \( z_{i+1} \leftarrow \arg\min_{z \in \mathbb{R}^d, w_i ^\top (z - v_i) \geq 0} D_\varphi(z, z_i) \)
   8. \( W_{i+1} \leftarrow W_i \cap \{w \in \mathbb{R}^d : w ^\top (v_i - z_{i+1}) \geq 0\} \)
4. end while
11. Solve \( \min_{p \in \Delta^{k-1}, c \in W^0} \|\sum_{i=1}^k p_i v_i + c - z_{k+1}\| \) to get \( p \)
12. return \( y' = z_{k+1}, V = (v_1, \ldots, v_k), p \)

**Lemma 4.6.** If \((\varphi, W)\) satisfies the PNIP condition (Definition 4.1), then \( z_1, \ldots, z_{k+1} \) computed in Algorithm 2 satisfy \( z_{i+1} - z_i \in W^0 \) for all \( i \in [k] \).

**Proof.** Since we have \( z_{i+1} = \arg\min_{z \in \mathbb{R}^d, w_i ^\top (z - v_i) \geq 0} D_\varphi(z, z_i) \), by the KKT condition, we have
\[
0 = \frac{\partial}{\partial z} \left( D_\varphi(z, z_i) - \lambda w_i ^\top (z - v_i) \right) \bigg|_{z = z_{i+1}} = \nabla \varphi(z_{i+1}) - \nabla \varphi(z_i) - \lambda w_i
\]
for some \( \lambda \geq 0 \). On the other hand, note that \( \nabla \varphi(z_{i+1}) - \nabla \varphi(z_i) = \int_0^1 \nabla^2 \varphi(\gamma z_{i+1} + (1 - \gamma) z_i) \cdot (z_{i+1} - z_i) d\gamma = H(z_{i+1} - z_i) \), for some \( H \in \text{CH} (\mathcal{H}) \), where \( \mathcal{H} = \{\nabla^2 \varphi(x) : x \in \text{Dom} (\varphi)\} \). Therefore, for all \( w \in W \) we have \( w ^\top (z_{i+1} - z_i) = w ^\top H^{-1} H(z_{i+1} - z_i) = \lambda w ^\top H^{-1} w_i \geq 0 \). This means \( z_{i+1} - z_i \in W^0 \).

**Lemma 4.7.** Under the setting of Theorem 4.5, Algorithm 2 terminates in at most
\[
\left[ \begin{array}{c}
4R + 2 \sqrt{\frac{2}{\mu}} \min_{x^* \in \mathcal{K}^*} D_\varphi(x^*, y) \\
5d \log \frac{4R + 2 \sqrt{2P/\mu}}{\epsilon}
\end{array} \right]
\]
iterations.

**Proof.** According to the algorithm, for each \( i \), \( z_{i+1} \) is the Bregman projection of \( z_i \) onto a half-space containing \( \mathcal{K}^* \), since the oracle \( \text{O}_{\mathcal{K}, \mathcal{K}^*} \) ensures \( w_i ^\top v_i \leq w_i ^\top x^* \) for all \( x^* \in \mathcal{K}^* \). Then by the generalized Pythagorean theorem (Lemma 2.4) we know \( D_\varphi(x^*, z_{i+1}) \leq D_\varphi(x^*, z_i) \) for all \( x^* \in \mathcal{K}^* \) and \( i \). Therefore we have \( D_\varphi(x^*, z_i) \leq D_\varphi(x^*, z_{i+1}) = D_\varphi(x^*, y) \) for all \( x^* \in \mathcal{K}^* \) and \( i \).

Let \( P := \min_{x^* \in \mathcal{K}^*} D_\varphi(x^*, y) \). Then there exists \( x^* \in \mathcal{K}^* \) such that \( P = D_\varphi(x^*, y) \geq D_\varphi(x^*, z_i) \geq \frac{\mu}{2} \|x^* - z_i\|^2 \) for all \( i \), where the last inequality is due to the \( \mu \)-strong convexity of \( \varphi \). This implies \( \|z_i\| \leq \|x^*\| + \sqrt{\frac{2P}{\mu}} \leq R + \sqrt{\frac{2P}{\mu}} \) for all \( i \). Therefore, when \( i \geq 5d \log \frac{4R + 2 \sqrt{2P/\mu}}{\epsilon} \), we must have \( i \geq 5d \log \frac{2(\|z_{i+1}\|)}{\epsilon} \), which means the loop must have terminated at this time. This proves the lemma. \( \square \)
Lemma 4.8. Under the setting of Theorem 4.5, for all \( w \in W, \|w\| = 1 \), there exists \( i \in [k] \) such that \( w^\top (v_i - y') \leq \epsilon \).

Proof. We assume for contradiction that there exists a unit vector \( h \in W \) such that \( \min_{i \in [k]} h^\top (v_i - y') > \epsilon \). Note that \( \|v_i - y'\| \leq \|v_i\| + \|y'\| \leq R + \|y'\| \). Letting \( r := \frac{\epsilon}{2(R + \|y'\|)} \), we have

\[
\forall w \in \frac{h}{2} + (W \cap B(0, r)) : \min_{i \in [k]} w^\top (v_i - y') > 0.
\]

Since \( r \leq \frac{1}{2} \) for \( \epsilon \leq R \), we have \( \frac{h}{2} + (W \cap B(0, r)) \subseteq \frac{h}{2} + (W \cap B(0, 1/2)) \subseteq W \cap B(0, 1) = W_1 \).

By the algorithm, we know that for all \( w \in W_1 \setminus W_{k+1} \), there exists \( i \in [k] \) such that \( w^\top (v_i - z_{i+1}) \leq 0 \). Notice that from Lemma 4.6 we know \( z_{j+1} - z_j \in W^0 \) for all \( j \in [k] \). Thus for all \( w \in W_1 \setminus W_{k+1} \) there exists \( i \in [k] \) such that \( w^\top (v_i - y') = w^\top (v_i - z_{k+1}) \leq w^\top (v_i - z_{i+1}) \leq 0 \). In other words, we have

\[
\forall w \in W_1 \setminus W_{k+1} : \min_{i \in [k]} w^\top (v_i - y') \leq 0.
\]

Therefore, we must have \( \frac{h}{2} + (W \cap B(0, r)) \subseteq W_{k+1} \). We also have \( \text{Vol}(W_{i+1}) \leq (1 - 1/(2\epsilon)) \text{Vol}(W_i) \) for each \( i \in [k] \) from Lemma 2.6, since \( W_{i+1} \) is the intersection of \( W_i \) with a half-space that does not contain \( W_i \)'s centroid \( w_i \) in the interior. Then we have

\[
\text{Vol}(W_1) = \text{Vol}(W \cap B(0, 1)) = r^{-d} \text{Vol}(W \cap B(0, r)) \leq r^{-d} \text{Vol}(W_{k+1}) \\
\leq r^{-d}(1 - 1/(2\epsilon))^k \text{Vol}(W_1) < \text{Vol}(W_1),
\]

where the last step is due to \( k \geq 5d \log \frac{1}{\epsilon} = 5d \log \frac{2(R + \|y'\|)}{\epsilon} = 5d \log \frac{2(R + \|z_{k+1}\|)}{\epsilon} \), which is true according to the termination condition of the loop. Therefore we have a contradiction. \( \square \)

We need the following basic property of projection onto a convex cone. The proof is given in Appendix B.

Lemma 4.9. For any closed convex cone \( W \subseteq \mathbb{R}^d \) and any \( x \in \mathbb{R}^d \), we have \( \Pi_W(x) - x \in W^0 \).

The following lemma is a more general version of Lemma 6 in (Garber, 2017).

Lemma 4.10. Given \( v_1, \ldots, v_k \in \mathbb{R}^d, \epsilon \geq 0 \) and a convex cone \( W \subseteq \mathbb{R}^d \), for any \( x \in \mathbb{R}^d \), the following two statements are equivalent:

(A) There exists \( p = (p_1, \ldots, p_k)^\top \in \Delta^{k-1} \) and \( c \in W^0 \) such that \( \|\sum_{i=1}^k p_i v_i + c - x\| \leq \epsilon \).

(B) For all \( w \in W, \|w\| = 1 \), there exists \( i \in [k] \) such that \( w^\top (v_i - x) \leq \epsilon \).

Geometric interpretation of Lemma 4.10. Before proving Lemma 4.10, we discuss its geometric intuition. For simplicity of illustration, we only consider \( \epsilon = 0 \) here. First we look at the case where \( W = \mathbb{R}^d, W^0 = \{0\} \). In this case the lemma simply degenerated to the fact

\[
x \in \text{CH(}\{v_i\}_{i=1}^k\) \iff \text{There is no hyperplane that separates } x \text{ and all } v_i \text{'s.}
\]

In the general case where \( W \subseteq \mathbb{R}^d \) is an arbitrary convex cone, lemma 4.10 becomes

\[
x \in \text{CH(}\{v_i\}_{i=1}^k\) + W^0 \iff \text{There is no direction } w \in W \text{ such that } w^\top x < w^\top v_i \text{ for all } i.
\]

Denote \( F := \text{CH(}\{v_i\}_{i=1}^k\) + W^0 \). For the “\( \Rightarrow \)” side, if \( x \in F \), it is clear that for all \( w \in W \) we must have \( w^\top x \geq w^\top v_i \) for some \( i \). For the “\( \Leftarrow \)” side, if \( x \notin F \), then \( w = \Pi_F(x) - x \) satisfies \( w^\top x < w^\top v_i \) for all \( i \). Moreover it is easy to see \( \Pi_F(x) - x \in W \), which completes the proof. See Figure 1 for a graphic illustration.
We have \(A\)\footnote{Proof of Theorem} where the equality holds only when \(\sum_{i=1}^{k} p_i v_i + c - x \leq \epsilon\), which contradicts the optimality of \((A)\). Thus we have \(B\)\footnote{Proof of Lemma 4.10}. Let \((p^*, c^*) = \arg \min_{p \in \Delta^{k-1}, c \in W^o} \left\| \sum_{i=1}^{k} p_i v_i + c - x \right\|\). Since \(0 \in W\), by the Pythagorean theorem \footnote{Proof of Lemma 2.1}, we have \[
\left\| \sum_{i=1}^{k} p_i^* v_i - x + c^* \right\| \geq \left\| \Pi_W \left( \sum_{i=1}^{k} p_i^* v_i - x + c^* \right) \right\|
\] where the equality holds only when \(\sum_{i=1}^{k} p_i^* v_i - x + c^* \in W\). Now we claim \(\sum_{i=1}^{k} p_i^* v_i - x + c^* \in W\). Otherwise, letting \(c' = c^* + \Pi_W \left( \sum_{i=1}^{k} p_i^* v_i - x + c^* \right) - \left( \sum_{i=1}^{k} p_i^* v_i - x + c^* \right)\), by Lemma 4.9 we have \(c' \in W^o\), and furthermore \[
\left\| \sum_{i=1}^{k} p_i^* v_i - x + c' \right\| = \left\| \Pi_W \left( \sum_{i=1}^{k} p_i^* v_i - x + c^* \right) \right\| < \left\| \sum_{i=1}^{k} p_i^* v_i - x + c^* \right\|,
\] which contradicts the optimality of \((p^*, c^*)\).

Thus we have \(\sum_{i=1}^{k} p_i^* v_i + c^* - x \in W\). Let \(w = \sum_{i=1}^{k} p_i^* v_i + c^* - x\) and \(G = \text{CH}(\{v_1, \ldots, v_k\}) + W^o + \{-x\}\). Then we have \(w = \Pi_G(0)\) by the definition of \((p^*, c^*)\). Since \(G\) is convex and \(v_i - x \in G\) for all \(i \in [k]\), by the Pythagorean theorem \footnote{Proof of Lemma 2.1} we have \(w^\top (v_i - x - w) \geq 0\) for all \(i \in [k]\), which implies \(\|w\|^2 \leq \min_{i \in [k]} \|w^\top (v_i - x)\| \leq \epsilon\|w\|\), i.e., \(\|w\| \leq \epsilon\). Hence we have \((B) \implies (A)\).

Theorem 4.5 is now easy to prove using the above lemmas.

Proof of Theorem 4.5. The upper bound on the number of iterations is proved in Lemma 4.7. In the proof of \footnote{Proof of Theorem 4.7}, we have shown \(D_v(x^*, z_{i+1}) \leq D_v(x^*, z_i)\) for all \(x^* \in K^*\) and \(i\). This implies \(D_v(x^*, y') =\)
$D^*_{new}(x^*, z_{k+1}) \leq D^*_{new}(x^*, z_k) \leq \cdots \leq D^*_{new}(x^*, z_1) = D^*_{new}(x^*, y)$ for all $x^* \in K^*$, which verifies the first property in Definition 4.2. The second property is a direct consequence of combining Lemmas 4.8 and 4.10.

4.3 The Oracle Complexity of Algorithm 1

**Theorem 4.11.** Suppose $(\varphi, W)$ satisfies the PNIP property (Definition 4.1), $\epsilon \in (0, R]$, $\varphi$ is $\mu$-strongly convex and $A$ is an upper bound on $\max_{x^* \in K^*} (\varphi(x^*) - \varphi(y_1))$. Then Algorithm 1 only needs to call $O_{K, K^*}$ for at most $5d \log \left( \frac{4R + 2}{\epsilon} \right)$ times per round.

In particular, setting $\epsilon = \frac{R}{T}$ and $\eta = \frac{1}{T} \sqrt{\frac{2L^2}{\mu}}$, Algorithm 1 only needs to call $O_{K, K^*}$ for at most $5d \log \left( \left( 6\sqrt{T} + 4 \sqrt{\frac{A}{\mu}} + 4 \right) T \right)$ times per round.

**Proof.** According to Theorem 4.5, round $t$ of Algorithm 1 calls $O_{K, K^*}$ for at most $\left[ \frac{4R + 2}{\epsilon} \min_{x^* \in K^*} D^*_{new}(x^*, y_t) \right]$ times. Hence it suffices to obtain an upper bound on $\min_{x^* \in K^*} D^*_{new}(x^*, y_t)$.

According to (4) (substituting $T$ with $t$), we have:

$$\forall t \in [T], \forall x^* \in K^* : \quad D^*_{new}(x^*, y_{t+1}) \leq D^*_{new}(x^*, y_1) + \sum_{j=1}^{t} D^*_{new}(x_j, y_{j+1}) - \eta \sum_{j=1}^{t} (f_j(x_j) - f_j(x^*)) + \epsilon \eta L t.$$

Plug (5) and (6) into the above inequality, we have

$$\forall t \in [T], \forall x^* \in K^* : \quad D^*_{new}(x^*, y_{t+1}) \leq D^*_{new}(x^*, y_1) + \sum_{j=1}^{t} D^*_{new}(x_j, y_{j+1}) - \eta \sum_{j=1}^{t} (f_j(x_j) - f_j(x^*)) + \epsilon \eta L t$$

$$\leq \varphi(x^*) - \varphi(y_1) + \frac{\eta^2 L^2}{2\mu} t - \eta \sum_{j=1}^{t} (f_j(x_j) - f_j(x^*)) + \epsilon \eta L t$$

$$\leq A + \frac{\eta^2 L^2}{2\mu} t + \eta \sum_{j=1}^{t} \|f_j\| \cdot \|x_j - x^*\| + \epsilon \eta L t$$

$$\leq A + \frac{\eta^2 L^2}{2\mu} T + 2\eta L R T + \epsilon \eta L T.$$

For $t = 1$ we also have $D^*_{new}(x^*, y_1) \leq A$. Therefore Algorithm 1 calls $O_{K, K^*}$ for at most

$$\left[ \frac{4R + 2}{\epsilon} \left( A + \left( \frac{\eta^2 L^2}{2\mu} + 2\eta L R + \epsilon \eta L \right) T \right) \right]$$

times per round.
When \( \epsilon = \frac{R}{\mu} \) and \( \eta = \frac{1}{T} \sqrt{\frac{2\mu A}{T}} \), we have

\[
\frac{2}{\mu} \left( A + \left( \frac{\eta^2 L^2}{2\mu} + 2\eta LR + \epsilon \eta L \right) T \right) \leq \frac{4A}{\mu} + 6R \sqrt{\frac{2AT}{\mu}} \leq \left( 2 \sqrt{\frac{A}{\mu}} + 3R \sqrt{T} \right)^2,
\]

so the number of oracle calls per iteration is at most \( 5d \log \left( \left( 6 \sqrt{T} + \frac{4 \sqrt{A}}{\mu} + 4 \right) T \right) \). \( \square \)

## 5 Efficient Online Improper Linear Optimization via Continuous Multiplicative Weight Update (CMWU)

In this section, we design our second online improper linear optimization algorithm (in the full information setting) based on the continuous multiplicative weight update (CMWU) method. Compared with Algorithm 1, the CMWU-based algorithm allows loss vectors to come from a general convex cone \( W \) and does not require the PNIP condition (Definition 4.1).

### 5.1 Separation-or-Decomposition Oracle

We first construct a separation-or-decomposition (SOD) oracle (Algorithm 3) using \( O_{K,K^*} \), which we will use to design the online improper linear optimization algorithm later in this section. Given a point \( x \in B(0, R) \), the SOD oracle either outputs a separating hyperplane between \( x \) and \( K^* \), or outputs a distribution of points in \( K \) which approximately dominates \( x \) in every direction in \( W \). The guarantee of the SOD oracle is summarized in Theorem 5.1.

**Algorithm 3 Separation-or-Decomposition Oracle, \( SOD(x, \epsilon, W) \)**

**Input:** Point \( x \in B(0, R) \), tolerance \( \epsilon > 0 \), convex cone \( W \subseteq \mathbb{R}^d \)

**Output:** Decomposition \( V = (v_1, \ldots, v_k) \in \mathbb{R}^{d \times k} \), \( p = (p_1, \ldots, p_k)^T \in \Delta^{k-1} \), such that \( v_i \in K \) (\forall i \in [k]) and \( \| \sum_{i=1}^k p_i v_i - x + c \| \leq 3 \epsilon \) for some \( c \in W^o \).

Or: Separating hyperplane \((u, b) \in \mathbb{R}^d \times \mathbb{R} \), such that \( \| u \| = 1 \) and \( u^T x \leq b - \epsilon \leq \min_{x^* \in K^*} w^T x^* - \epsilon \).

1. \( k \leftarrow \left\lceil 5d \log \frac{4R}{\epsilon} \right\rceil \)
2. \( W_1 \leftarrow W \cap B(0, 1) \)
3. **for** \( i = 1 \) to \( k \) **do**
   4. \( w_i \leftarrow \frac{\text{Vol}(W_i)}{\text{Vol}(W)} \)
   5. \( v_i \leftarrow O_{K,K^*}(w_i) \)
   6. **if** \( w_i^T x \leq w_i^T v_i - \epsilon \) **then**
      7. **return** Separating hyperplane \( (\frac{w_i}{\|w_i\|}, \frac{w_i^T v_i}{\|w_i\|}) \)
   8. **else**
   9. \( W_{i+1} \leftarrow W_i \cap \{ w \in \mathbb{R}^d : w^T (v_i - x) \geq \epsilon \} \)
10. **end if**
11. **end for**
12. Solve \( \min_{p \in \Delta^{k-1}, c \in W^o} \| \sum_{i=1}^k p_i v_i + c - x \| \) to get \( p \)
13. **return** \( V = (v_1, \ldots, v_k), p \)

**Theorem 5.1.** For any \( x \in B(0, R) \) and \( \epsilon \in (0, 2R] \), the separation-or-decomposition oracle in Algorithm 3, \( SOD(x, \epsilon, W) \), returns one of the two followings, using at most \( k = \left\lceil 5d \log \frac{4R}{\epsilon} \right\rceil \) calls of \( O_{K,K^*} \):
1. A decomposition \( V = (v_1, \ldots, v_k) \in \mathbb{R}^{d \times k}, p = (p_1, \ldots, p_k) \in \Delta^{k-1} \), such that \( v_i \in \mathcal{C} (\forall i \in [k]) \) and \( \| \sum_{i=1}^{k} p_i v_i - x + e \| \leq 3e \) for some \( e \in \mathbb{W}^\circ \).

2. A separating hyperplane \( (w, b) \in \mathbb{R}^d \times \mathbb{R} \), where \( \| w \| = 1 \) and \( w^\top x \leq b - \epsilon \leq \min_{x^* \in \mathcal{K}^*} w^\top x^* - \epsilon \).

The proof of Theorem 5.1 is postponed to Appendix C.

5.2 CMWU with Refining Domains

Now we look at a general online learning setting where the feasible domain is shrinking over time while being a superset of the target domain. Namely, suppose \( \mathcal{K}^* \) is the target domain and \( \mathcal{K}_t \) is the feasible domain in the \( t \)-th round. We assume \( B(0, R) \supseteq \mathcal{K}_0 \supseteq \mathcal{K}_1 \supseteq \mathcal{K}_2 \cdots \supseteq \mathcal{K}_T \supseteq (1 - \gamma)\mathcal{K}^* + \gamma \mathcal{K}_0 \) for some \( \gamma \in (0, 1] \). In round \( t \), the player only knows \( \mathcal{K}_1, \ldots, \mathcal{K}_t \) and does not know \( \mathcal{K}_j \) for all \( j > t \). We can still run CMWU in this setting, using the knowledge of \( \mathcal{K}_j \) at iteration \( t \) - the algorithm is given in Algorithm 4. Theorem 5.2 bounds the regret of Algorithm 4 in this setting.

**Algorithm 4** Continuous Multiplicative Weight Update (CMWU) with Refining Domains

**Input:** Learning rate \( \eta > 0 \), time horizon \( T \in \mathbb{N}_+ \)

1. **for** \( t = 1 \) **to** \( T \) **do**
2. Receive current domain \( \mathcal{K}_t \)
3. Play \( x_t = \frac{\int_{\mathcal{K}_t} e^{-\eta \sum_{i=1}^{t-1} f_i(x)} dx}{\int_{\mathcal{K}_t} e^{-\eta \sum_{i=1}^{t-1} f_i(x)} dx} \)
4. Receive loss vector \( f_t \)
5. **end for**

**Theorem 5.2.** Suppose \( B(0, R) \supseteq \mathcal{K}_0 \supseteq \mathcal{K}_1 \supseteq \mathcal{K}_2 \cdots \supseteq \mathcal{K}_T \supseteq (1 - \gamma)\mathcal{K}^* + \gamma \mathcal{K}_0 \) for \( \gamma \in (0, 1] \). Then for any \( 0 < \eta \leq \frac{1}{LR} \), Algorithm 4 has the following regret guarantee:

\[
\forall x^* \in \mathcal{K}^* : \sum_{t=1}^{T} (f_t(x_t) - f_t(x^*)) \leq \frac{d \log \frac{1}{\eta}}{\eta} + \eta L^2 R^2 T + \gamma L R T - \sum_{t=1}^{T} \delta_t,
\]

where

\[
\delta_t := \log \frac{\int_{\mathcal{K}_{t-1}} e^{-\eta \sum_{i=1}^{t-1} f_i(x)} dx}{\int_{\mathcal{K}_t} e^{-\eta \sum_{i=1}^{t-1} f_i(x)} dx}.
\]

In particular, setting \( \gamma = 1/T \) and \( \eta = \frac{1}{LR} \min \left\{ 1, \sqrt{d \log T} \right\} \), we have

\[
\forall x^* \in \mathcal{K}^* : \sum_{t=1}^{T} (f_t(x_t) - f_t(x^*)) \leq L R \left( 1 + 2 \max \left\{ \sqrt{d T \log T}, d \log T \right\} \right).
\]

**Proof.** We fix any \( x^* \in \mathcal{K}^* \) and denote \( \bar{\mathcal{K}} := (1 - \gamma)x^* + \gamma \mathcal{K}_0 \). Since \( (1 - \gamma)\mathcal{K}^* + \gamma \mathcal{K}_0 \subseteq \mathcal{K}_T \), we have \( \bar{\mathcal{K}} \subseteq \mathcal{K}_T \). We define

\[
z_t(x) := e^{-\eta \sum_{i=1}^{t-1} f_i(x)}, \quad Z_t := \int_{\bar{\mathcal{K}}_t} z_t(x) dx, \quad Z'_t := \int_{\mathcal{K}_{t-1}} z_t(x) dx.
\]
A straightforward calculation gives us:

\[
\log \frac{Z_{T+1}'}{Z_1'} = \log \left( \frac{\int_{K_T} e^{-\eta \sum_{t=1}^T f_t(x)} \, dx}{\int_{K_0} 1 \, dx} \right) \\
\geq \log \left( \frac{\int_{K_0} e^{-\eta \sum_{t=1}^T f_t(x)} \, dx}{\int_{K_0} 1 \, dx} \right) \\
= \log \left( \frac{\int_{K_0} e^{-\eta \sum_{t=1}^T (f_t(x) + \gamma f_t(x))} \, dx}{\int_{K_0} 1 \, dx} \right) \\
\geq \log \left( \frac{\int_{K_0} e^{-\eta \sum_{t=1}^T (f_t(x) + \gamma) \gamma} \, dx}{\int_{K_0} 1 \, dx} \right) \\
= d \log \gamma - \eta \sum_{t=1}^T f_t(x^*) - \eta \gamma \text{LRT}.
\]

On the other hand, we have

\[
\log \frac{Z_{t+1}'}{Z_t} = \log \left( \int_{K_t} \frac{z_t(x) e^{-\eta f_t(x)}}{Z_t} \, dx \right) \\
\leq \log \left( \int_{K_t} \frac{z_t(x) (1 - \eta f_t(x) + (\eta f_t(x))^2)}{Z_t} \, dx \right) \\
\leq \left( \int_{K_t} \frac{z_t(x)}{Z_t} \, dx \right) (1 - \eta f_t(x) + (\eta f_t(x))^2) - 1 \\
\leq \int_{K_t} \frac{z_t(x)}{Z_t} (-\eta f_t(x) + \eta^2 L^2 R^2) \, dx \\
= - \eta f_t \left( \int_{K_t} \frac{z_t(x)}{Z_t} x \, dx \right) + \eta^2 L^2 R^2 \\
= - \eta f_t(x^*) + \eta^2 L^2 R^2,
\]

where the first inequality is due to \( e^a \leq 1 + a + a^2 (\forall a \leq 1) \) and \( |\eta f_t(x)| \leq \eta L R \leq 1 \), the second inequality is due to \( \log a \leq a - 1 \ (\forall a > 0) \), and the third inequality is due to \( |\eta f_t(x)| \leq \eta L R \).

Note that \( \delta_t = \log \frac{Z_t'}{Z_t} \). Combining the two bounds above, we get:

\[
\sum_{t=1}^T (-\eta f_t(x_t) + \eta^2 L^2 R^2) \geq \sum_{t=1}^T \log \frac{Z_{t+1}'}{Z_t} = \log \frac{Z_{T+1}'}{Z_1'} + \sum_{t=1}^T \log \frac{Z_{t+1}'}{Z_t} \\
\geq d \log \gamma - \eta \sum_{t=1}^T f_t(x^*) - \eta \gamma \text{LRT} + \sum_{t=1}^T \delta_t.
\]

In other words,

\[
\sum_{t=1}^T (f_t(x_t) - f_t(x^*)) \leq \frac{d \log \frac{1}{\eta}}{\eta} + \eta L^2 R^2 T + \gamma L R T - \frac{\sum_{t=1}^T \delta_t}{\eta}.
\]

The second regret bound in the theorem follows directly by plugging in the values of \( \gamma \) and \( \eta \) stated in the theorem. Note that \( \delta_t \geq 0 \) for all \( t \in [T] \) since \( K_t \subseteq K_{t-1} \).
5.3 Online Improper Linear Optimization via CMWU

Now we are ready to present our CMWU-based online improper learning algorithm. At a high level, this algorithm is a specialized implementation of Algorithm 4 for the online improper linear optimization problem. The algorithm starts with an initial convex domain \( K_0 \) which is a superset of \( K^* \), and maintains a convex domain \( K_t \) at iteration \( t \). In iteration \( t \), the algorithm first computes the mean \( x_t \) of a log-linear distribution over \( K_t \) using random walk, as in Line 3 of Algorithm 4. Then the algorithm calls the SOD oracle on \( x_t \). If the SOD oracle returns a distribution of points in \( K \), then we can play according to that distribution, since the SOD oracle ensures that the expected loss of this distribution is not much larger than that of \( x_t \). If the SOD oracle returns a separating hyperplane between \( x_t \) and \( K^* \), then the algorithm replaces \( K_t \) with the intersection of the original \( K_t \) and the half-space given by this hyperplane that contains \( K^* \), and repeats the same process for the new \( K_t \) until a decomposition is returned by the SOD oracle. Note that each time \( K_t \) is updated, the mean \( x_t \) of the log-linear distribution is not in the new \( K_t \), which according to Lemma 2.6 implies that a constant probability mass is removed. This allows us to bound the total number of oracle calls. We detail our algorithm in Algorithm 5 and its regret bound in Theorem 5.3.

**Algorithm 5 CMWU for Online Improper Linear Optimization**

| Input: Learning rate \( \eta > 0 \), tolerance \( \gamma > 0 \), initial convex domain \( K_0 \), convex cone \( W \), time horizon \( T \in \mathbb{N}_+ \) |
|---|
| 1: \( K_1 \leftarrow K_0 \) |
| 2: for \( t = 1 \) to \( T \) do |
| 3: \( x_t \leftarrow \frac{\int_{K_t} e^{-\eta \sum_{i=1}^{t-1} f_i(x)} dx}{\int_{K_t} e^{-\eta \sum_{i=1}^{t-1} f_i(x)} dx} \) |
| 4: while SOD\((x_t, 2\gamma R, W)\) returns a separating hyperplane \((w, b) \in \mathbb{R}^d \times \mathbb{R} \) do |
| 5: \( K_t \leftarrow K_t \cap \{x \in \mathbb{R}^d : w^\top x \geq b - 2\gamma R\} \) |
| 6: \( x_t \leftarrow \frac{\int_{K_t} e^{-\eta \sum_{i=1}^{t-1} f_i(x)} dx}{\int_{K_t} e^{-\eta \sum_{i=1}^{t-1} f_i(x)} dx} \) |
| 7: end while |
| 8: Let \((V, p) \in \mathbb{R}^{d \times k} \times \Delta^{k-1}\) be the output of SOD\((x_t, 2\gamma R, W)\) |
| 9: Play \( \tilde{x}_t = v_j \) with probability \( p_i (i = 1, \ldots, k) \), where \( V = (v_1, \ldots, v_k) \) |
| 10: \( K_{t+1} \leftarrow K_t \) |
| 11: Receive loss vector \( f_t \) |
| 12: end for |

**Theorem 5.3.** Suppose that the initial convex domain \( K_0 \) satisfies \( K^* \subseteq K_0 \subseteq B(0, R) \). Then for any \( \gamma \in (0, 1] \) and \( \eta \in (0, \frac{1}{T R}] \), Algorithm 5 satisfies the following regret guarantee:

\[
\forall x^* \in K^*: \quad \mathbb{E} \left[ \sum_{t=1}^{T} (f_t(\tilde{x}_t) - f_t(x^*)) \right] \leq \frac{d \log \frac{1}{\eta}}{\eta} + \eta L^2 R^2 T + 7\gamma L R T - \frac{s}{5\eta},
\]

where \( s = \sum_{t=1}^{T} s_t \), and \( s_t \) is the number of separating hyperplanes returned by the SOD oracle during round \( t \).

In particular, if we set \( \gamma = \frac{1}{T}, \eta = \frac{1}{TR} \min \left\{ 1, \sqrt{\frac{d \log T}{T}} \right\} \), then we have

\[
\forall x^* \in K^*: \quad \mathbb{E} \left[ \sum_{t=1}^{T} (f_t(\tilde{x}_t) - f_t(x^*)) \right] \leq LR \left( 7 + 2 \max \left\{ \sqrt{d T \log T}, d \log T \right\} \right),
\]
and in this case Algorithm 5 calls $O_{K, K^*}$ for $O(dT \log T)$ times in $T$ rounds.

**Proof.** In the proof, we use $\bar{K}_t$ and $\bar{x}_t$ to denote the values of $K_t$ and $x_t$ at the end of iteration $t$ ($\bar{K}_0 = K_0$). We define

$$z_t(x) := e^{-\eta \sum_{i=1}^{t-1} f_i(x)}, \quad Z_t := \int_{K_t} z_t(x) dx, \quad Z'_t := \int_{K_{t-1}} z_t(x) dx, \quad \delta_t := \log \frac{Z'_t}{Z_t}.$$ 

We first prove the following two claims:

(i) For all $t \in \{0, 1, \ldots, T\}$, we have $(1 - \gamma)K^* + \gamma \bar{K}_0 \subseteq \bar{K}_t$.  

(ii) For all $t \in [T]$, we have $\delta_t \geq \frac{s_t}{5}$.  

We use induction to prove (i). It holds for $t = 0$ since $K^* \subseteq K_0 = \bar{K}_0$ and $K_0$ is convex. Suppose it holds for $t - 1$. If $\bar{K}_t = \bar{K}_{t-1}$, then it already holds for $t$. Otherwise, consider the separating hyperplane $(w, b) \in R^d \times R$ obtained in iteration $t$, which is the output of $SOD(x', 2\gamma R, W)$ for some $x'$. By the guarantee of the SOD oracle, we have

$$w^T x' \leq b - 2\gamma R \leq \min_{x \in K^*} w^T x - 2\gamma R.$$  

This implies

$$(1 - \gamma)K^* + \gamma \bar{K}_0 \subseteq (1 - \gamma)K^* + \gamma B(0, R) \subseteq K^* + B(0, 2\gamma R) \subseteq \{x \in \mathbb{R}^d : w^T x \geq b - 2\gamma R\}.$$ 

Note that $\{x \in \mathbb{R}^d : w^T x \geq b - 2\gamma R\}$ is exactly the half-space to intersect with when updating $K_t$. Hence we know that during the execution of the algorithm, $K_t$ is always a superset of $(1 - \gamma)K^* + \gamma \bar{K}_0$. This proves (i).

For (ii), note that each time $K_t$ is updated, the mean of the distribution over $K_t$ with density proportional to $z_t(x)$ is not included in the interior of the new $K_t$. By Lemma 2.6, this implies $\int_{new K_t} z_t(x) dx \leq (1 - \frac{1}{2e}) \int_{old K_t} z_t(x) dx$. Hence we have $-\delta_t = \log \frac{Z'_t}{Z_t} \leq \log (1 - \frac{1}{2e})^{s_t}$, which gives $\delta_t \geq \frac{s_t}{5}$.

Now we show the regret bound. From (i) we know

$$(1 - \gamma)K^* + \gamma \bar{K}_0 \subseteq \bar{K}_T \subseteq \bar{K}_{T-1} \subseteq \cdots \subseteq \bar{K}_1 \subseteq \bar{K}_0 \subseteq B(0, R).$$ 

Therefore, we can apply Theorem 5.2 to get

$$\forall x^* \in K^* : \sum_{t=1}^{T} (f_t(\bar{x}_t) - f_t(x^*)) \leq \frac{d \log \frac{1}{\eta}}{\eta} + \eta L^2 R^2 T + \gamma LRT - \sum_{t=1}^{T} \delta_t$$

$$\leq \frac{d \log \frac{1}{\eta}}{\eta} + \eta L^2 R^2 T + \gamma LRT - \frac{8}{5\eta},$$ 

where the second inequality is due to (ii).

The actual algorithm does not play $\bar{x}_t$, but a random $\tilde{x}_t$. Namely, letting $(V, p) \in \mathbb{R}^{d \times k} \times \Delta^{k-1}$ be the output of $SOD(\bar{x}_t, 2\gamma R, W)$, we have that $\tilde{x}_t$ is equal to $v_i$ with probability $p_i$ ($i = 1, \ldots, k$), where $V = (v_1, \ldots, v_k)$. By Theorem 5.1 we know that there exists $c \in W$ such that $\|\sum_{i=1}^{k} p_i v_i + c - \tilde{x}_t\| \leq 6\gamma R$, which implies (note $f_t \in W \cap B(0, L)$)

$$E[f_t(\tilde{x}_t)] = f_t \left( \sum_{i=1}^{k} p_i v_i \right) \leq f_t \left( \sum_{i=1}^{k} p_i v_i + c \right) \leq f_t(\bar{x}_t) + 6\gamma LR.$$
Therefore we have
\[ \forall x^* \in \mathcal{K}^*: \mathbb{E} \left[ \sum_{t=1}^{T} (f_t(\tilde{x}_t) - f_t(x^*)) \right] = \mathbb{E} \left[ \sum_{t=1}^{T} (f_t(\tilde{x}_t) - f_t(x_t)) \right] + \sum_{t=1}^{T} (f_t(\tilde{x}_t) - f_t(x^*)) \leq 6\gamma LRT + \frac{d \log \frac{1}{\eta}}{\eta} + \eta L^2 R^2 T + \gamma LRT - \frac{s}{5 \eta} \]
\[ = \frac{d \log \frac{1}{\eta}}{\eta} + \eta L^2 R^2 T + 7\gamma LRT - \frac{s}{5 \eta}. \]

Setting \( \gamma = \frac{1}{T} \) and \( \eta = \frac{1}{LR} \min \left\{ 1, \sqrt{\frac{d \log T}{T}} \right\} \), the above bound becomes
\[ \forall x^* \in \mathcal{K}^*: \mathbb{E} \left[ \sum_{t=1}^{T} (f_t(\tilde{x}_t) - f_t(x^*)) \right] \leq LR \left( 7 + 2 \max \left\{ \sqrt{dT \log T}, d \log T \right\} \left( 1 - \frac{s}{10d \log T} \right) \right). \]

We can use the above regret bound to bound the number of oracle calls in Algorithm 5. Since the regret is always lower bounded by \(-2LRT\), the above regret upper bound implies \( s = O(T) \). Therefore Algorithm 5 calls the SOD oracle for \( s + T = O(T) \) times. Note that each implementation of SOD needs to call \( O_{\mathcal{K}, \mathcal{K}^*} \) for \( O \left( d \log \frac{4R}{\beta LR} \right) = O(d \log T) \) times (Theorem 5.1). We conclude that the total number of calls to \( O_{\mathcal{K}, \mathcal{K}^*} \) in Algorithm 5 is at most \( O(d \log T) \).

**Remark.** Intuitively, when the SOD oracle is called in Algorithm 5, between the two outcomes (separation and decomposition) we should prefer decomposition, since this means we can make the play and move on to the next iteration. However, Theorem 5.3 shows an interesting trade-off between oracle complexity and regret: the more oracle calls, the less the regret. This means obtaining separating hyperplanes helps the regret. Interestingly, we obtain our upper bound on the oracle calls by observing that regret can never be lower by \(-2LRT\).

### 6 \( \alpha \)-Regret Minimization in the Bandit Setting

In this section we consider the \( \alpha \)-regret minimization problem in the bandit setting, where \( W = \mathbb{R}_+^d \), \( \mathcal{K} \subseteq \mathbb{R}_+^d \cap B(0, R) \) and \( \mathcal{K}^* = \alpha \mathcal{K} \). Similar to (Kakade et al., 2009), we assume we know a \( \beta \)-barycentric spanner for \( \mathcal{K} \). This concept was first introduced by Awerbuch and Kleinberg (2004).

**Definition 6.1** (Barycentric spanner). A set of \( d \) linearly independent vectors \( \{q_1, \ldots, q_d\} \subseteq \mathbb{R}^d \) is a \( \beta \)-barycentric spanner for a set \( \mathcal{K} \subseteq \mathbb{R}^d \), denoted by \( \beta \text{-BS}(\mathcal{K}) \), if \( \{q_1, \ldots, q_d\} \subseteq \mathcal{K} \) and for all \( x \in \mathcal{K} \), there exist \( \beta_1, \ldots, \beta_d \in [-\beta, \beta] \) such that \( x = \sum_{i=1}^{d} \beta_i q_i \).

Given \( \{q_1, \ldots, q_d\} \) which is a \( \beta \text{-BS}(\mathcal{K}) \), define \( Q := \sum_{i=1}^{d} q_i q_i^\top \) and \( M := (q_1, \ldots, q_d) \in \mathbb{R}^{d \times d} \). Then we have \( Q = MM^\top \) and \( Me_i = q_i, M^{-1} q_i = e_i (\forall i \in [d]) \), where \( e_i \) is the \( i \)-th standard unit vector in \( \mathbb{R}^d \).

**The need for a new regularization.** The bandit algorithm of Garber (2017) additionally requires a certain boundedness property of barycentric spanners, namely:
\[ \max_{i \in [d]} q_i^\top Q^{-2} q_i \leq \chi. \]

However, for certain bounded sets this quantity may be unbounded, such as the two-dimensional axis-aligned rectangle with one axis being of size unity, and the other arbitrarily small. This unboundedness creates
problems with the unbiased estimator of loss vector, whose variance can be as large as certain geometric properties of the decision set. To circumvent this issue, we design a new regularizer called barycentric regularizer, which gives rise to an unbiased estimator coupled with an online mirror descent variant that automatically ensures constant variance.

Similar to (Kakade et al., 2009; Garber, 2017), our bandit algorithm also simulates the full information algorithm with estimated loss vectors. Namely, our algorithm implements Algorithm 1 with a specific barycentric regularizer \( \varphi(x) = \frac{1}{2} x^T Q^{-1} x \). The algorithm is detailed in Algorithm 6, and its regret guarantee is given in Theorem 6.2.

**Algorithm 6 Online Stochastic Mirror Descent with Barycentric Regularization**

**Input:** Learning rate \( \eta > 0 \), tolerance \( \epsilon > 0 \), \( \{q_1, \ldots, q_d\} \) - a \( \beta \)-BS(\( \mathcal{K} \)) for some \( \beta > 0 \), exploration probability \( \gamma \in (0, 1) \), time horizon \( T \in \mathbb{N}_+ \).

1. Instantiate Algorithm 1 with parameters \( \eta, \epsilon, \varphi(x) = \frac{1}{2} x^T Q^{-1} x \), \( W' = (M^T)^{-1} \mathbb{R}_+^d \), and \( T \).

2. for \( t = 1 \) to \( T \) do

3. Receive \( x_t \) (the point to play in round \( t \)) from Algorithm 1

4. \( b_t \leftarrow \begin{cases} 
\text{EXPLORE}, & \text{with probability } \gamma \\
\text{EXPLOIT}, & \text{with probability } 1 - \gamma
\end{cases} \)

5. if \( b_t = \text{EXPLORE} \) then

6. Sample \( i_t \in [d] \) uniformly at random, and play \( q_{i_t} \)

7. Receive loss \( l_t = q_{i_t}^T f_t \)

8. \( \bar{f}_t \leftarrow \frac{1}{d} l_t Q^{-1} q_{i_t} \)

9. else

10. Play \( \tilde{x}_t \) and receive loss \( l_t = \tilde{x}_t^T f_t \)

11. \( \tilde{f}_t \leftarrow 0 \)

12. end if

13. Feed \( \tilde{f}_t \) to Algorithm 1 as the loss vector for round \( t \) (Note that when \( \tilde{f}_t = 0 \), in the next round Algorithm 1 can simply play according to the distribution computed in this round without any oracle calls.)

14. end for

**Theorem 6.2.** Denote by \( z_t \) the point played by Algorithm 6 in round \( t \). Then for any \( \gamma \in (0, 1) \), \( \epsilon \in (0, \alpha R) \) and \( \eta > 0 \), Algorithm 6 satisfies the following regret guarantee:

\[
\forall x^* \in \mathcal{K} : \quad \mathbb{E} \left[ \sum_{t=1}^{T} (f_t(z_t) - \alpha f_t(x^*)) \right] \leq \frac{\alpha^2 \beta^2 d}{2 \eta} + \frac{\eta L^2 R^2 d^2}{2 \gamma} T + 2 \gamma \alpha LRT + \epsilon LT,
\]

and the expected total number of calls to the oracle \( \mathcal{O}^{\alpha} \) in \( T \) rounds is at most

\[
(1 + \gamma T) \left( 1 + 5d \log \frac{4 \alpha R + 2R \sqrt{\frac{\alpha^2 \beta^2 d^2 + \frac{\eta L^2 R^2 d^4}{\gamma} T}{\epsilon}}}{\epsilon} \right).
\]

In particular, setting \( \eta = \frac{\alpha \beta^{1/3}}{LRT^{2/3}}, \epsilon = \frac{R}{T} \) and \( \gamma = \frac{\beta^{2/3} d}{T^{1/3}} \) (assuming \( T > \beta^2 d^3 \) so \( \gamma < 1 \)), we have

\[
\forall x^* \in \mathcal{K} : \quad \mathbb{E} \left[ \sum_{t=1}^{T} (f_t(z_t) - \alpha f_t(x^*)) \right] \leq \alpha LR \left( 3d (\beta T)^{2/3} + 1 \right),
\]

and the expected total number of oracle calls in \( T \) rounds is at most \( O \left( d^2 (\beta T)^{2/3} \log T \right) \).
Proof. Let $x_t, y_t$ and $\tilde{x}_t$ be the same $x_t, y_t$ and $\tilde{x}_t$ appearing in Algorithm 1 during our implementation. We define $\mathcal{F}_t := \mathbb{E}[\tilde{x}_t | y_t]$ similarly to the proof of Theorem 4.3. It is easy to see that $(\varphi, W')$ satisfies the PNIP property (Definition 4.1) and $\tilde{f}_t \in W'$ for all $t \in [T]$, where $W' = (M^T)^{-1} \mathbb{R}_+^d$.

Note that $\nabla \varphi(x) = \nabla^{-1} x$, which implies

$$D_{\varphi}(x_t, y_{t+1}) = \frac{1}{2}(x_t - y_{t+1})^T \nabla^{-1}(x_t - y_{t+1}) = \frac{1}{2}(\nabla \varphi(x_t) - \nabla \varphi(y_{t+1}))^T Q(\nabla \varphi(x_t) - \nabla \varphi(y_{t+1})) = \frac{\eta^2}{2} \tilde{f}_t^T Q \tilde{f}_t.$$  

Using the regret bound (3) in the proof of Theorem 4.3, for any $x^* \in K$ we have

$$\sum_{t=1}^{T} \left( \tilde{f}_t(x_t) - \alpha \tilde{f}_t(x^*) \right) \leq \frac{1}{\eta} \left( D_{\varphi}(\alpha x^*, y_1) - D_{\varphi}(\alpha x^*, y_{T+1}) + \sum_{t=1}^{T} D_{\varphi}(x_t, y_{t+1}) \right) \leq \frac{1}{\eta} \left( \varphi(\alpha x^*) - \min_{y \in \mathbb{R}^d} \varphi(y) + \sum_{t=1}^{T} D_{\varphi}(x_t, y_{t+1}) \right) \leq \frac{1}{\eta} \varphi(\alpha x^*) + \frac{\eta}{2} \sum_{t=1}^{T} \tilde{f}_t^T Q \tilde{f}_t. \tag{7}$$

Since $\{q_i\}_{i=1}^{d}$ is a $\beta$-BS($K$), there exist $\beta_1, \ldots, \beta_d \in [-\beta, \beta]$ such that $x^* = \sum_{i=1}^{d} \beta_i q_i$. Then we have

$$\varphi(\alpha x^*) = \frac{1}{2}(\alpha x^*)^T \nabla^{-1}(\alpha x^*) = \frac{\alpha^2}{2} \|M^{-1} x^*\|^2 = \frac{\alpha^2}{2} \left\| \sum_{i=1}^{d} \beta_i M^{-1} q_i \right\|^2 = \frac{\alpha^2}{2} \left\| \sum_{i=1}^{d} \beta_i e_i \right\|^2 \leq \frac{\alpha^2 \beta^2 d}{2}. \tag{8}$$

We also have

$$\mathbb{E} \left[ \tilde{f}_t^T Q \tilde{f}_t \right] = \gamma \sum_{i=1}^{d} \frac{1}{d} \left( \frac{d}{\gamma} q_i^T f_t \right)^2 q_i^T Q^{-1} Q^{-1} q_i = \frac{L^2 R^2 d}{\gamma} \sum_{i=1}^{d} q_i^T (MM^T)^{-1} q_i = \frac{L^2 R^2 d}{\gamma} \sum_{i=1}^{d} e_i^T e_i = \frac{L^2 R^2 d^2}{\gamma}. \tag{9}$$

Hence by taking expectation on (7) we get

$$\mathbb{E} \left[ \sum_{t=1}^{T} \left( \tilde{f}_t(x_t) - \alpha \tilde{f}_t(x^*) \right) \right] \leq \frac{\alpha^2 \beta^2 d}{2\eta} + \frac{\eta L^2 R^2 d^2}{2\gamma} T. \tag{9}$$

Note that $\mathbb{E}[\tilde{f}_t | x_t] = \gamma \sum_{i=1}^{d} \frac{1}{d \gamma} Q^{-1} q_i q_i^T f_t = Q^{-1} \left( \sum_{i=1}^{d} q_i q_i^T \right) f_t = f_t$. Therefore (8) becomes

$$\mathbb{E} \left[ \sum_{t=1}^{T} \left( f_t(x_t) - \alpha f_t(x^*) \right) \right] \leq \frac{\alpha^2 \beta^2 d}{2\eta} + \frac{\eta L^2 R^2 d^2}{2\gamma} T. \tag{9}$$

Next, by the guarantee of the PAD oracle, for any $t \in [T]$ we know that there exists $c_t \in (W')^\circ$ such that $\| \bar{x}_t + c_t - x_t \| \leq \epsilon$. It is easy to see that $\mathbb{R}_+^d \subseteq W'$, which implies $(W')^\circ \subseteq \mathbb{R}_+^d$, so we know $c_t \in \mathbb{R}_+^d$. Then we have

$$\forall t \in [T] : \quad \mathbb{E} \left[ f_t(x_t) - f_t(x_t) \right] = \mathbb{E} \left[ f_t^T (\bar{x}_t - x_t) \right] \leq \mathbb{E} \left[ f_t^T (\bar{x}_t + c_t - x_t) \right] \leq \epsilon L.$$  

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Thus (9) implies
\[
\mathbb{E} \left[ \sum_{t=1}^{T} (f_t(\bar{x}_t) - \alpha f_t(x^*)) \right] \leq \frac{\alpha^2 \beta^2 d}{2\eta} + \frac{\eta L^2 R^2 d^2}{2\gamma} T + \epsilon LT. \tag{10}
\]

Finally, since the point played in round \(t\), \(z_t\), is equal to \(\bar{x}_t\) (whose expectation is \(\bar{x}_t\)) with probability \(1 - \gamma\), we have
\[
\mathbb{E} \left[ \sum_{t=1}^{T} (f_t(z_t) - \alpha f_t(x^*)) \right] \leq (1 - \gamma) \mathbb{E} \left[ \sum_{t=1}^{T} (f_t(\bar{x}_t) - \alpha f_t(x^*)) \right] + \gamma \cdot 2\alpha LRT
\]
\[
= (1 - \gamma) \mathbb{E} \left[ \sum_{t=1}^{T} (f_t(\bar{x}_t) - \alpha f_t(x^*)) \right] + 2\gamma \alpha LRT
\]
\[
\leq \frac{\alpha^2 \beta^2 d}{2\eta} + \frac{\eta L^2 R^2 d^2}{2\gamma} T + \epsilon LT + 2\gamma \alpha LRT.
\]

**Oracle complexity.** Using Theorem 4.5, we know that when \(b_t = \text{EXPLORE}\), the number of calls to the oracle \(O_K^\phi\) in round \(t\) is at most
\[
5d \log \frac{4\alpha R + 2\sqrt{\frac{2}{\mu} \min_{x^* \in \mathcal{K}} D_\varphi(\alpha x^*, y_t)}}{\epsilon},
\]
where \(\mu\) is the strong convexity parameter of \(\varphi\).

In the above proof of the regret bound, we have ignored the term \(D_\varphi(\alpha x^*, y_{T+1})\) in (7). If we instead keep this term, the regret bound (10) will become
\[
\forall x^* \in \mathcal{K}: \quad \mathbb{E} \left[ \sum_{t=1}^{T} (f_t(\bar{x}_t) - \alpha f_t(x^*)) \right] \leq \frac{\alpha^2 \beta^2 d}{2\eta} + \frac{\eta L^2 R^2 d^2}{2\gamma} T + \epsilon LT - \frac{1}{\eta} \mathbb{E} [D_\varphi(\alpha x^*, y_{T+1})].
\]

In the above inequality, substituting \(T\) with \(t\), we have
\[
\forall t \in [T], \forall x^* \in \mathcal{K}: \quad \mathbb{E} [D_\varphi(\alpha x^*, y_{t+1})] \leq \frac{\alpha^2 \beta^2 d}{2} + \frac{\eta L^2 R^2 d^2}{2\gamma} t + \epsilon \eta L t - \eta \mathbb{E} \left[ \sum_{j=1}^{t} (f_j(\bar{x}_j) - \alpha f_j(x^*)) \right]
\]
\[
\leq \frac{\alpha^2 \beta^2 d}{2} + \frac{\eta L^2 R^2 d^2}{2\gamma} T + \epsilon \eta L T + \eta \cdot 2\alpha LRT
\]
\[
\leq \frac{\alpha^2 \beta^2 d}{2} + \frac{\eta L^2 R^2 d^2}{2\gamma} T + 3\eta \alpha LRT.
\]

The above upper bound is also clearly valid for \(D_\varphi(\alpha x^*, y_1)\).

Since \(\varphi(x) = \frac{1}{2} x^\top Q^{-1} x\) is quadratic, we know that \(\mu = \lambda_{\min}(Q^{-1}) = \frac{1}{\lambda_{\max}(Q)} = \frac{1}{\max_{u \in \mathbb{R}^d, \|u\| = 1} \|u\|} \geq \frac{1}{\sum_{t=1}^{T-1} \|u\|^2} \geq \frac{1}{R^2 d},\) where \(\lambda_{\min}(P)\) and \(\lambda_{\max}(P)\) are respectively the smallest and the largest eigenvalues of a symmetric matrix \(P\).

Note that \(\log(a + \sqrt{x})\) is a concave function in \(x\) for \(a > 0\). By Jensen’s inequality, the expected number of calls to the oracle \(O_K^\phi\) in round \(t\) when \(b_t = \text{EXPLORE}\) is upper bounded by:
\[
\mathbb{E} \left[ 1 + 5d \log \frac{4\alpha R + 2\sqrt{\frac{2}{\mu} \min_{x^* \in \mathcal{K}} D_\varphi(\alpha x^*, y_t)}}{\epsilon} \right]
\]
\[
\leq 1 + \min_{x^* \in \mathcal{K}} \mathbb{E} \left[ 5d \log \frac{4\alpha R + 2\sqrt{\frac{2}{\mu} D_\varphi(\alpha x^*, y_t)}}{\epsilon} \right]
\]
\[
\begin{align*}
\leq 1 + \min_{x^* \in K} 5d \log_{\frac{4\alpha R + 2}\epsilon} \sqrt{\frac{2}{\mu} \mathbb{E} [D_{\phi}(\alpha x^*, y_t)]} \\
\leq 1 + 5d \log_{\frac{4\alpha R + 2\sqrt{2R^2d} \left( \frac{\alpha^2 \beta^2 d^2}{2} + \frac{\gamma^2 L^2 R^2 d^3}{2}\right) + 3\eta \alpha LRT}{\epsilon}} \\
= 1 + 5d \log_{\frac{4\alpha R + 2\sqrt{2R^2d} \left( \frac{\alpha^2 \beta^2 d^2}{2} + \frac{\gamma^2 L^2 R^2 d^3}{2}\right) + 3\eta \alpha LRT}{\epsilon}}.
\end{align*}
\]

Therefore the expected total number of calls to the oracle \(\tilde{O}_K^n\) in \(T\) rounds is at most

\[
(1 + \gamma T) \left( 1 + 5d \log_{\frac{4\alpha R + 2\sqrt{2R^2d} \left( \frac{\alpha^2 \beta^2 d^2}{2} + \frac{\gamma^2 L^2 R^2 d^3}{2}\right) + 3\eta \alpha LRT}{\epsilon}} \right).
\]

The second part of the theorem can be directly verified using the specific choices of \(\eta, \epsilon\) and \(\gamma\) and noting \(\log(\text{poly}(\beta d)) = O(\log T)\) since \(T > \beta^2 d^3\).

\section{Conclusion and Open Problems}

We have described two different algorithmic approaches to reducing regret minimization to offline approximation algorithms and maintaining optimal regret and poly-logarithmic oracle complexity per iteration, resolving previously stated open questions.

An intriguing open problem remaining is to find an efficient algorithm in the bandit setting that guarantees both \(\tilde{O}(\sqrt{T})\) regret and \(\text{poly}(\log T)\) oracle complexity per iteration (at least on average).

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Appendix

A Proof of Lemma 2.5

Proof of Lemma 2.5. Let $H = \{x \in \mathbb{R}^d : w^\top x \geq b\}$ for a unit vector $w \in \mathbb{R}^d$ and $b \in \mathbb{R}$, and assume without loss of generality that $x^* = 0$. Consider the one-dimensional random variable $Y := w^\top X - b$, where $X \sim p$. Denote by $q : \mathbb{R} \rightarrow \mathbb{R}$ the density function of $y$. Then we have

$$
\int_H p(x) dx = \int_0^\infty q(y) dy.
$$

Let $y^* := \mathbb{E}[Y] = w^\top x^* - b = -b$. By our assumption, we know $|y^*| \leq \frac{1}{2e}$. Moreover, since log-concavity is preserved under linear transformations (Prékopa, 1973), we know that $y$ also follows a log-concave distribution, and it is easy to see that it is also isotropic. Using Lemma 5.4 in (Lovász and Vempala, 2007), we know $\int_{y^*}^\infty q(y) dy \geq \frac{1}{e}$. In addition, from Lemma 5.5 in (Lovász and Vempala, 2007) we know $q(y) \leq 1$ (for all $y \in \mathbb{R}$). Therefore, we have

$$
\frac{1}{e} - \int_0^{y^*} q(y) dy \leq \int_{y^*}^{\infty} q(y) dy - \int_0^{\infty} q(y) dy \leq |y^*| \sup_{y \in \mathbb{R}} q(y) \leq \frac{1}{2e},
$$

which implies $\int_0^{\infty} q(y) dy \geq \frac{1}{2e}$, completing the proof.

B Proof of Lemma 4.9

Proof of Lemma 4.9. Since $W$ is a convex cone and $\Pi_W(x) \in W$, we have $w + \Pi_W(x) \in W$ ($\forall w \in W$). By Lemma 2.1, we have $(\Pi_W(x) - x)^\top (y - \Pi_W(x)) \geq 0$ ($\forall y \in W$). Letting $y = w + \Pi_W(x) \in W$ ($\forall w \in W$), we get $(\Pi_W(x) - x)^\top w \geq 0$, which means $\Pi_W(x) - x \in W^\circ$. 

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C Proof of Theorem 5.1

We first show the following lemma using a similar argument in Lemma 4.8.

Lemma C.1. For $x \in B(0, R)$ and $\epsilon \in (0, 2R]$, if $SOD(x, \epsilon, W)$ returns a decomposition $(V, p)$, then for all unit vector $w \in W$, we have $\min_{i \in [k]} w^T(v_i - x) \leq 3\epsilon$.

Proof. Suppose that there exists a unit vector $h \in W$ such that $\min_{i \in [k]} h^T(v_i - x) > 3\epsilon$. Note that $\|v_i - x\| \leq \|v_i\| + \|x\| \leq 2R$. Denoting $r = \frac{3\epsilon}{4R}$, we have

$$\forall h' \in \frac{h}{2} + (W \cap B(0, r)) : \min_{i \in [k]} h'^T(v_i - x) > \epsilon.$$ 

Since $r \leq \frac{1}{2}$ for $\epsilon \leq 2R$, we have $\frac{h}{2} + (W \cap B(0, r)) \subseteq \frac{h}{2} + (W \cap B(0, 1/2)) \subseteq W \cap B(0, 1) = W_1$. Because the algorithm returns a decomposition, we have that after the last iteration,

$$\forall w \in W_1 \setminus W_{k+1} : \exists i \in [k], \text{ s.t. } w^T(v_i - x) \leq \epsilon.$$ 

Therefore, we must have $\frac{h}{2} + (W \cap B(0, r)) \subseteq W_{k+1}$. We also have $\text{Vol}(W_{i+1}) \leq (1 - 1/2\epsilon)\text{Vol}(W_i)$ from Lemma 2.6 since $W_{i+1}$ is the intersection of $W_i$ with a half-space that does not contain $W_i$’s centroid. Then we have

$$\text{Vol}(W_1) = \text{Vol}(W \cap B(0, 1)) = r^{-d}\text{Vol}(W \cap B(0, r)) \leq r^{-d}\text{Vol}(W_{k+1})$$

$$\leq r^{-d}(1 - 1/2\epsilon)^k\text{Vol}(W_1) < \text{Vol}(W_1),$$

where the last step is due to $k \geq 5d\log\frac{1}{r} = 5d\log\frac{4R}{\epsilon}$. Therefore we have a contradiction. 

Proof of Theorem 5.1. If $SOD(x, \epsilon, W)$ returns a decomposition $(V, p)$, by Lemmas C.1 and 4.10, we know that there exists $c \in W^0$ such that $\|\sum_{i=1}^k p_i v_i + c - x\| \leq 3\epsilon$.

If $SOD(x, \epsilon, W)$ returns a separating hyperplane $(w, b)$ at iteration $i \in [k]$, we know $w_i^T x \leq w_i^T v_i - \epsilon$. Since $w = \frac{w_i}{\|w_i\|}$, $b = w^T v_i$ and $\|w_i\| \leq 1$, we have $w_i^T x \leq w_i^T v_i - \|w_i\| \leq b - \epsilon$. By the guarantee of $O_{K,K^*}$, we have $b - \epsilon = w_i^T v_i - \epsilon \leq \min_{x \in K^*} w_i^T x^* - \epsilon$.

The number of calls of $O_{K,K^*}$ is clearly upper bounded by $k = \lceil 5d\log\frac{4R}{\epsilon} \rceil$ since there are at most $k$ iterations and each iteration only calls $O_{K,K^*}$ once. 

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