On characteristic classes of $Q$-manifolds

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Abstract

We define the notion of characteristic classes for supermanifolds endowed with a homological vector field $Q$. These take values in the cohomology of the Lie derivative operator $L_Q$ acting on arbitrary tensor fields. We formulate a classification theorem for intrinsic characteristic classes and give their explicit description.

1. Let $M$ be a smooth supermanifold and $\mathcal{T}(M) = \bigoplus_{n,m \in \mathbb{N}} \mathcal{T}^{(n,m)}(M)$ be its tensor algebra; here $\mathcal{T}^{(n,m)}(M)$ is the space of $n$-times contravariant and $m$-times covariant tensor fields on $M$. The elements of $\mathcal{T}^{(1,1)}(M)$ are naturally identified with the endomorphisms of the $C^\infty(M)$-module $\mathcal{T}^{(1,0)}(M)$. The composition of two endomorphisms endows $\mathcal{T}^{(1,1)}(M)$ with the structure of an associative algebra over $C^\infty(M)$. We will denote this algebra $\mathcal{A}$. There is a natural trace on $\mathcal{A}$, which is a $C^\infty(M)$-linear map $\text{Str} : \mathcal{A} \to C^\infty(M)$ vanishing on supercommutators.

2. An odd vector field $Q \in \mathcal{T}^{(1,0)}(M)$ is said to be homological, if

$$Q^2 = \frac{1}{2} [Q, Q] = 0.$$  

(1)

By definition [7], the pair $(M, Q)$ is called a $Q$-manifold.

The simplest example of a $Q$-manifold is an odd tangent bundle $\Pi T N$ (i.e., the tangent bundle of $N$ with reversed parity of fibers). In this case, the supercommutative algebra of functions $C^\infty(\Pi T N)$ is naturally isomorphic to the exterior algebra of differential forms on $N$, with the de Rham differential being the (canonical) homological vector field on $\Pi T N$. A great number of interesting examples of $Q$-manifolds is provided by Lie algebroids [8] and various gauge systems [1 4 7]. For a recent discussion of homological vector fields in the category of graded supermanifolds we refer the reader to [5].

Given a homological vector field $Q$, one can view $\mathcal{T}(M)$ as a differential group with the coboundary operator

$$\delta A = L_Q A, \quad \forall A \in \mathcal{T}(M).$$  

(2)

Here $L_Q$ is the Lie derivative w.r.t. $Q$. The property $\delta^2 = 0$ follows from the identity $L_Q^2 = L_{Q^2} = 0$. Denote by $H_Q(M)$ the group of $\delta$-cohomology. Since $\delta$ differentiates

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the tensor product, the group \( H_Q(M) = \bigoplus_{n,m \in \mathbb{N}} H_Q^{(n,m)}(M) \) inherits the structure of a bigraded associative algebra over \( \mathbb{R} \).

3. Let \( \nabla \) be a symmetric connection on \( M \) with the curvature tensor \( R_{XY} \in \mathcal{A} \),

\[
R_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad \forall X, Y \in \mathcal{T}^{(1,0)}(M).
\]

Later on we will need the following relations characterizing the geometry of \( Q \)-manifolds with symmetric connection:

\[
\nabla_Q Q = 0, \quad \nabla_Q \Lambda = \frac{1}{2} R_{QQ} + \Lambda^2, \quad \nabla_X R_{QQ} = 2(R_{[X,Q]Q} - \nabla_Q R_{QX}).
\]

Here \( X \) is an arbitrary vector field and \( \Lambda \in \mathcal{A} \) is an odd endomorphism defined by the rule \( \Lambda(X) = \nabla_X Q \). With the tensor \( \Lambda \), we have a simple relation between the Lie and covariant derivatives of an endomorphism \( A \in \mathcal{A} \) w.r.t. the homological vector field:

\[
\nabla_Q A = L_Q A + [\Lambda, A].
\]

A \( Q \)-manifold is said to be flat if it admits a flat connection.

4. Under universal cocycles of a \( Q \)-manifold \( M \) we understand \( \delta \)-cocycles \( C_{\nabla}[Q] \in \mathcal{T}(M) \) that are given, at each coordinate chart, by polynomials in the components of the homological field \( Q \), Christoffel symbols of \( \nabla \), and their partial derivatives up to some finite order. The adjective “universal” emphasizes the fact that the closeness condition \( \delta(C_{\nabla}[Q]) = 0 \) is assumed to be satisfied by virtue of equation (1) without using a particular structure of \( Q \), \( \nabla \), and \( M \).

For example, the tensor powers of homological vector field \( Q^\otimes n \in \mathcal{T}^{(n,0)}(M) \) exhaust all universal cocycles that are independent of connection. A less trivial example of universal cocycles is the function

\[
P_n = \text{Str}((R_{QQ})^{2n}) \in C^\infty(M), \quad n \in \mathbb{N},
\]

which is nothing but the \( n \)th Pontrjagin’s character of the cotangent bundle \( TM \) evaluated on the homological vector field \( Q \). (For the definition and discussion of the characteristic classes of supervector bundles see [9].)

5. The characteristic classes of \( Q \)-manifolds are, by definition, the elements of \( H_Q(M) \) that are represented by universal cocycles.

**Theorem 1.** The cohomology classes of universal cocycles do not depend on the choice of symmetric connection, and hence they are invariants of a \( Q \)-manifold itself.

**Proof.** Let \( C_{\nabla_0}[Q] \) and \( C_{\nabla_1}[Q] \) be two universal cocycles that differ only by the choice of connection. Consider the direct product of \( M \) and the linear superspace \( \mathbb{R}^{1,1} \) with one even coordinate \( t \) and one odd coordinate \( \theta \). Equip the supermanifold \( \tilde{M} = M \times \mathbb{R}^{1,1} \) with the homological vector field \( \tilde{Q} = Q + \theta \partial_t \) and connection \( \tilde{\nabla} = \nabla_t \oplus \nabla' \), where \( \nabla_t = t \nabla_1 + (1 - t) \nabla_0 \) is a one-parameter family of connections on \( M \) and \( \nabla' \) is a flat connection on \( \mathbb{R}^{1,1} \). By universality, the tensor \( C_{\tilde{\nabla}}[\tilde{Q}] \in \mathcal{T}(\tilde{M}) \) is closed w.r.t. \( \tilde{\delta} = L_{\tilde{Q}} \). Since \( \theta^2 = 0 \), one can see that \( C_{\tilde{\nabla}}[\tilde{Q}] = C_{\nabla_t}[Q] + \theta \Psi_t \), where \( \Psi_t \) is some expression
depending on $Q$, $\nabla_0$, $\nabla_1$, and $t$. The closeness condition $\tilde{\delta}(C_\nabla[Q]) = 0$ is then equivalent to the following relations:

$$\delta(C_\nabla_t[Q]) = 0, \quad \partial_t C_\nabla_t[Q] = \delta \Psi_t.$$  \hfill (7)

Integrating the second relation over $t$ from 0 to 1, we get

$$C_\nabla_1[Q] - C_\nabla_0[Q] = \delta \int_0^1 dt \Psi_t.$$  \hfill (8)

Thus, $C_\nabla_0[Q]$ is cohomologous to $C_\nabla_1[Q]$. □

The first nontrivial series of charclasses of $Q$-manifolds was proposed in [4], the so-called principal series. The universal cocycles of this series involve the first covariant derivatives of the homological vector field and have the following form:

$$C^\infty(M) \ni A_n = \text{Str}(\Lambda^{n+1}) + \text{(curvature dependent terms)}, \quad \forall n \in \mathbb{N}. \hfill (9)$$

It was also shown that the class $A_0$ has a direct relationship to one-loop anomalies in the BV quantization method of gauge theories. In a particular case of homological vector fields corresponding to Lie algebroids [8], formula (9) reproduces the characteristic classes of Lie algebroids introduced by Fernandes [2].

6. In this note, we present two infinite series of universal cocycles, which essentially involve the second covariant derivatives of the homological vector field. Together with the universal cocycles (9) these new cocycles generate and exhaust, in essence, all interesting characteristic classes of $Q$-manifolds. In order to write them down in an explicit form we identify $T^{(1,n+1)}(M)$ with $T^{(0,n)}(M) \otimes A$ and treat the elements of the latter space as $n$-forms on $TM$ with values in $A$.

**Lemma.** For any vector field $X$, set $\Omega_X \equiv \nabla_X \Lambda - R_{XQ} \in A$. The tensor $\Omega \in T^{(0,1)}(M) \otimes A$ is a universal cocycle, i.e., $\delta \Omega = 0$.

**Proof.** It follows from Rels. (4) that

$$\nabla_Q \Omega_X = \Omega_{[Q,X]} + [\Lambda, \Omega_X], \quad \forall X \in T^{(1,0)}(M). \hfill (10)$$

Using successively the definition [2], Rel. (5), and the last identity, we find

$$(\delta \Omega)_X = (L_Q \Omega)_X = L_Q(\Omega_X) - \Omega_{[Q,X]} = \nabla_Q \Omega_X - [\Lambda, \Omega_X] - \Omega_{[Q,X]} = 0.$$  \hfill (11)$$

□.

Since the differential $\delta$ is compatible with contraction of tensor indices, we have immediately

**Corollary 1.** For any $n \in \mathbb{N}$, define $B_n \in T^{(0,n)}(M) \otimes A$ as

$$B_n(X_1, \ldots, X_n) = \Omega_{X_1} \Omega_{X_2} \cdots \Omega_{X_n}, \quad \text{for} \quad n > 0,$$  \hfill (12)

and $B_0 = 1 \in A$. Then $\delta B_n = 0.$
Corollary 2. The \( n \)-forms

\[
C_n(X_1, ..., X_n) = \text{Str} (\Omega_{X_1} \Omega_{X_2} \cdots \Omega_{X_n})
\]

are \( \delta \)-closed and invariant, up to sign, under the cyclic permutation of their arguments:

\[
C_n(X_n, X_1, ..., X_{n-1}) = (-1)^{\varepsilon_1 \varepsilon_2} C_n(X_1, ..., X_n).
\]

Here \( \varepsilon_1 = \varepsilon(X_n) + 1, \varepsilon_2 = \sum_{k=1}^{n-1} (\varepsilon(X_k) + 1) \), and \( \varepsilon(X_k) \) denotes the parity of the vector field \( X_k \).

We will refer to the cohomology classes of universal cocycles (9), (12), and (13) as the characteristic classes of \( A, B, \) and \( C \) series, respectively.

7. We say that a characteristic class \([ C_V[Q] ] \in H_Q(M)\) is intrinsic, if it does not vanish identically upon setting the curvature of \( \nabla \) to zero. In other words, the intrinsic charclasses survive on flat \( Q \)-manifolds. For example, all the characteristic classes from the \( A, B, \) and \( C \) series are intrinsic, while the \( \delta \)-cohomology classes of (6) are not. Clearly, the intrinsic charclasses constitute a subalgebra \( H^\text{int}_Q(M) \) in \( H_Q(M) \). As the next theorem shows, this subalgebra admits a fairly simple description.

**Theorem 2.** The algebra \( H^\text{int}_Q(M) \) is freely generated by the characteristic classes of \( A, B, \) and \( C \) series, together with the cohomology class \([ Q ] \in H^{(1,0)}_Q(M) \) of the homological vector field itself.

**Remark.** It might be well to point out that certain of the intrinsic charclasses may vanish for a particular \( Q \)-manifold, e.g. by dimensional reasons. The theorem above states just the absence of universal nontrivial relations between the generators of \( H^\text{int}_Q(M) \).

Let \( V \) denote the typical fiber of the tangent bundle \( TM \). The proof of the theorem is based on construction of a classifying \( Q \)-map from a given flat \( Q \)-manifold to the infinite-dimensional, linear \( Q \)-manifold associated to the Lie superalgebra \( L_0(V) \) of formal vector fields on \( V \) vanishing at the origin. This reduces the problem to computation of stable cohomologies of \( L_0(V) \) with tensor coefficients. In the special case that \( V \) is an ordinary (even) linear space, the last problem was completely solved in [3] and the method of that paper applies to the super case as well. The last but not least step involves extension of the “flat” universal cocycles to arbitrary (not necessarily flat) \( Q \)-manifolds.

The details of the proof will be given elsewhere, along with various applications and interpretations of the intrinsic characteristic classes.

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