SEMICROSSED PRODUCTS OF C*-ALGEBRAS AND THEIR C*-ENVELOPES

EVGENIOS T.A. KAKARIADIS

Abstract. Let $C$ be a C*-algebra and $\alpha : C \to C$ a unital *-endomorphism. There is a natural way to construct operator algebras which are called semicrossed products, using a convolution induced by the action of $\alpha$ on $C$. We show that the C*-envelope of a semicrossed product is (a full corner of) a crossed product. As a consequence, we get that, when $\alpha$ is *-injective, the semicrossed products are completely isometrically isomorphic and share the same C*-envelope, the crossed product $C_\infty \rtimes_{\alpha,\infty} \mathbb{Z}$.

We show that minimality of the dynamical system $(C, \alpha)$ is equivalent to non-existence of non-trivial Fourier invariant ideals in the C*-envelope. We get sharper results for commutative dynamical systems.

1. Introduction and preliminaries

The purpose of this paper is to give a clear picture of how non-selfadjoint operator algebras arise by a dynamical system consisting of a C*-algebra and a *-homomorphism, i.e. a semicrossed product. Our examination focusses on finding the C*-envelope of the semicrossed products subject to a covariance relation, for which we give a full answer in every possible case. We see the C*-envelope as the appropriate candidate for a C*-algebra that inherits some of the properties of the dynamical system has and we show how this is justified for commutative systems. For completeness we have included in section 4 an application of the joint work [17] with Elias Katsoulis. We prove some of the results of [17] needed here, in an ad-hoc manner, avoiding using the language of C*-correspondences, hence preserving self-containment of the paper.

Given a dynamical system $(C, \alpha)$ there are various ways of considering universal C*-algebras over collections of pairs $(\pi, V)$, so that $(H, \pi)$ is a representation of $C$, $V$ an operator in $B(H)$ and “a covariance relation” holds. Some of the forms the covariance relation may have are

(1) $\pi(\alpha(c)) = V^* \pi(c) V$, \hspace{1cm} (“implements”)
(2) $V \pi(\alpha(c)) = \pi(c) V$, \hspace{1cm} (“intertwines”)
(3) $V \pi(\alpha(c)) V^* = \pi(c)$, \hspace{1cm} (“undoes”)

2000 Mathematics Subject Classification. 47L55, 47L40, 46L05, 37B20.
Key words and phrases. semicrossed product, crossed product, C*-envelope.

The author was supported by the “Irakleitos II” program (co-financed by the European Social Fund, the European Union and Greece).
and some possible choices for $V$ is to be a contraction, an isometry, a co-isometry or a unitary\footnote{The presentation we give here follows the list presented by M. Lamoureux in his talk in GPOTS (1999), and I thank A. Katavolos for bringing this to my attention.}. For example, when $\alpha$ is a *-automorphism and $V$ is considered a unitary, then all three relations are equivalent and the universal $\mathrm{C}^*$-algebra is nothing else but the usual crossed product $\mathcal{C} \rtimes_\alpha \mathbb{N}$. Also, when $\mathcal{C}$ is unital, $\alpha$ is injective and we consider relation (1) for co-isometries $V$, then we get the crossed product by an endomorphism of Stacey [27] (we assume that $\alpha$ is non-unital in that case, otherwise relation (1) would force $V$ to be unitary). Moreover, it seems that some relations are absurd; for example relation (1) cannot hold for arbitrary contractions, since this would imply that there is a completely isometric homomorphism $\iota$ such as, commutativity of $\mathcal{C} \rightarrow \mathcal{C}$, and $\iota(\mathcal{C})$ generates $\mathcal{C}$ as a $\mathrm{C}^*$-algebra, i.e. $\mathcal{C} = \iota(\mathcal{C})$. If $\mathcal{C}$ is a $\mathrm{C}^*$-cover for $\mathcal{A}$, then $\mathcal{J}_{\mathcal{A}}$ will denote the Šilov ideal of $\mathcal{A}$ in $\mathcal{C}$. Thus, $\mathcal{C}_{\mathrm{env}}(\mathcal{A}) = \mathcal{C}/\mathcal{J}_{\mathcal{A}}$ and the restriction of the natural projection $q: \mathcal{C} \rightarrow \mathcal{C}/\mathcal{J}_{\mathcal{A}}$ on $\mathcal{A}$ is a completely isometric representation of $\mathcal{A}$. (Any ideal $\mathcal{J} \subseteq \mathcal{C}$, with the property that the restriction of the natural projection $\mathcal{C} \rightarrow \mathcal{C}/\mathcal{J}$ on $\mathcal{A}$ is a complete isometry, is called a boundary ideal and $\mathcal{J}_{\mathcal{A}}$ is the largest such ideal.) An equivalent way to define the $\mathrm{C}^*$-envelope
of an operator algebra is through the following universal property: for any C*-cover \((\mathcal{C}, \iota)\) of \(\mathcal{A}\) there is a *-epimorphism \(\Phi : \mathcal{C} \to \mathcal{C}_{\text{env}}(\mathcal{A})\), such that \(\Phi(\iota(a)) = a\) for any \(a \in \mathcal{A}\).

Apart from the interest in semicrossed products as non-selfadjoint algebras (see [11] and remark 6.8), we use them to propose a candidate for a C*-algebra generated by a dynamical system. As known, when \(\alpha : \mathcal{C} \to \mathcal{C}\) is a *-isomorphism, the crossed product is the appropriate C*-algebra as it captures some of the properties of the dynamical system \((\mathcal{C}, \alpha)\). In the case where \(\alpha : \mathcal{C} \to \mathcal{C}\) is just a *-homomorphism there are more than one C*-algebras that are generated by the dynamical system, as the choices for the covariance relation and \(V\) are numerous.

For this reason we examine these choices and define various non-selfadjoint algebras by mimicking the way the crossed product is constructed. As we show, when \(\alpha : \mathcal{C} \to \mathcal{C}\) is injective they all are completely isometrically isomorphic, and there is a unique “smallest” C*-algebra generated by them, i.e. their C*-envelope (see remark 5.2). Then we show that the C*-envelope captures some natural properties of the dynamical system. Moreover, for the commutative case we prove that minimality of the C*-envelope is equivalent to the compact Hausdorff space being infinite and the dynamical system being minimal. As a consequence, one gets that all the ideals in any C*-cover of the semicrossed product is boundary with respect to the semicrossed product.

The structure of this paper is as follows. In section 2 we give the main definitions for the operator algebras that we call the semicrossed products of \((\mathcal{C}, \alpha)\), by using left (resp. right) covariant pairs. Also, we develop an argument of duality that enables us to examine just the left or the right case. Thus we start with four operator algebras \(\mathfrak{A}(\mathcal{C}, \alpha, \text{contr})\), \(\mathfrak{A}(\mathcal{C}, \alpha, \text{is})\), \(\mathfrak{A}(\mathcal{C}, \alpha, \text{co-is})\) and \(\mathfrak{A}(\mathcal{C}, \alpha, \text{un})\).

In section 3 we show that \(\mathfrak{A}(\mathcal{C}, \alpha, \text{co-is})\) and \(\mathfrak{A}(\mathcal{C}, \alpha, \text{un})\) are completely isometrically isomorphic and that their C*-envelope is a crossed product. We mention that the semicrossed product \(\mathfrak{A}(\mathcal{C}, \alpha, \text{is})\) is Peters’ semicrossed product introduced in [22]. In that paper the author was asking about the case of the right covariance relation; this is exactly the semicrossed product \(\mathfrak{A}(\mathcal{C}, \alpha, \text{is})\) that is explored here. Theorem 3.6 generalizes [23, Theorem 4].

Section 4 is an application of joint work with Elias Katsoulis (see [17, Example 4.3]). There we have shown that the C*-envelope of \(\mathfrak{A}(\mathcal{C}, \alpha, \text{contr})\) and \(\mathfrak{A}(\mathcal{C}, \alpha, \text{is})\) is a full corner of a crossed product, using the language of C*-correspondences. Here, we do the same using ad-hoc versions of the arguments of [17].

In section 5 we present some remarks that arise naturally from the work of the previous two sections. For example we show that the semicrossed products we construct with respect to left covariant pairs are completely isometrically isomorphic (in a natural way) if and only if the *-endomorphism \(\alpha\) is injective.

In section 6 we show that the C*-envelope has no Fourier-invariant ideals (see definition 6.4) if and only if the dynamical system is minimal (see definition 6.1). In this case, all four semicrossed products are completely isometrically isomorphic and their C*-envelope is the crossed product \(\mathcal{C}_\infty \rtimes_{\alpha_\infty} \mathbb{Z}\). Moreover, when \(\mathcal{C}\) is a commutative C*-algebra \(C(X)\) over a compact Hausdorff space \(X\), the C*-envelope is simple if, and only if, the dynamical system is minimal and \(X\) is infinite.

Finally we mention that all the dynamical systems \((\mathcal{C}, \alpha)\) are considered unital, meaning that \(\mathcal{C}\) has a unit \(e\) and \(\alpha(e) = e\). Therefore, the operator algebras examined here are always...
unital. Analogous result may be obtained (with similar methods) for non-unital cases, when \( \alpha \) is non-degenerate. Remark 6.2 shows why we cannot go beyond non-unital cases in the context of section 6.

In what follows \( \mathbb{Z}_+ = \{0, 1, 2, \ldots \} \) and we use the symbol \( s \) for the unilateral shift on \( \ell^2(\mathbb{Z}_+) \), given by \( s(e_n) = e_{n+1} \). Before we begin, we present some basic facts and constructions that are associated to a pair \( (C, \alpha) \).

First of all, for such a pair we define the ideal \( R_\alpha = \bigcup_n \ker \alpha^n \); it is called the radical of \( (C, \alpha) \) and an element \( c \in C \) is in \( R_\alpha \) if, and only if, \( \lim_n \alpha^n(c) = 0 \). Thus \( \alpha(R_\alpha) \subseteq R_\alpha \) and \( R_\alpha \) is the radical of \( (C, \alpha) \); hence we can define the injective \( \ast \)-homomorphism \( \bar{\alpha} : C/R_\alpha \rightarrow C/\alpha(c) + R_\alpha \).

Note that \( R_\alpha = (0) \) if, and only if, \( \alpha \) is injective.

**Remark 1.1.** There are cases where the ideal \( R_\alpha \) may be “too large”, in the sense that \( C/R_\alpha \) may be \( \mathbb{C} \). For example, let \( X = \mathbb{R}^+ \cup \{+\infty\} \) be the one-point compactification of \( \mathbb{R}^+ \) and let \( C = C(X) \). Then \( C \) is the unitalization of \( C_0(\mathbb{R}^+) \). Define the map \( \phi : X \rightarrow X \), by \( \phi(x) = x + 1 \) and \( \phi(\infty) = \infty \) and consider the \( \ast \)-homomorphism \( \alpha : C \rightarrow C \) given by \( \alpha(f) = f \circ \phi \).

For any pair \((C, \alpha)\) we can define the direct limit \( C^\ast \)-algebra \( C_\infty \) of the direct sequence

\[
C \xrightarrow{\alpha} C \xrightarrow{\alpha} C \xrightarrow{\alpha} \ldots
\]

So \( C_\infty = \lim(C_n, \alpha_n) \), where \( C_n = C \) and \( \alpha_n = \alpha \), for every \( n \). If

\[
\iota_n : C_n \rightarrow C_\infty : c \mapsto [0, \ldots, c, \alpha(c), \ldots]
\]

are the induced \( \ast \)-homomorphisms, where \([0, \ldots, c, \alpha(c), \ldots] \) is the equivalence class in \( C_\infty \) containing \((0, \ldots, c, \alpha(c), \ldots)\), then \( \ker \iota_n = R_\alpha \). Indeed, \([0, \ldots, c, \alpha(c), \ldots] = 0 \) if, and only if, \( \lim_n ||\alpha^n(c) - 0|| = 0 \), by definition.

Let \( \alpha_\infty : C_\infty \rightarrow C_\infty \) be the \( \ast \)-homomorphism of \( C_\infty \), given by

\[
\alpha_\infty[0, \ldots, c, \alpha(c), \ldots] = [0, \ldots, \alpha(c), \alpha^2(c), \ldots].
\]

It is easy to see that \( \alpha_\infty \) is always a \( \ast \)-automorphism.

**Remarks 1.2.** Since \( \iota_n(C_n) \cong C/R_\alpha \), the direct limit may be the trivial \( C^\ast \)-algebra \( \mathbb{C} \) (for example let \((C, \alpha)\) be as in remark 1.1). On the other hand, when \( \alpha : C \rightarrow C \) is injective, then \( C_\infty \) contains a copy of \( C \) and \( \alpha_\infty|_C = \alpha \). In this case the pair \((C_\infty, \alpha_\infty)\) is an extension of the pair \((C, \alpha)\).

Recall the construction of the enveloping operator algebra of a unital Banach algebra (see 2.4.6 and 2.4.7). In a few words, let \( B \) be a unital Banach algebra and let \( \mathcal{F} \) be a collection of (possibly degenerate) contractive representations \((H_\pi, \pi)\) of \( B \), where \( H_\pi \) is a Hilbert space. For any integer \( \nu \geq 1 \) and any matrix \([F_{ij}]\in \mathcal{M}_\nu(B)\), we define

\[
\omega_\nu([F_{ij}]) = \sup \left\{ \|\pi(F_{ij})\|_{\mathcal{M}_\nu(B(H_\pi))} : (H_\pi, \pi) \in \mathcal{F} \right\}.
\]

It is easy to see that each \( \omega_\nu \) is a seminorm. If \( \mathcal{N} = \ker \omega_1 \), then we can define the induced \( \nu \)-norms on the quotient \( B/\mathcal{N} \); we let \( \|F + \mathcal{N}\|_\infty := \omega_1(F) \). The enveloping operator algebra \( \mathcal{O}(B, \mathcal{F}) \) of \( B \) with respect to the collection \( \mathcal{F} \), is the completion of the quotient \( B/\mathcal{N} \) with
respect to the norm $\| \cdot \|_\infty$. It is the operator algebra with the following (universal) property: there is a completely contractive and unital homomorphism $\iota : B \to \mathcal{O}(B, \mathcal{F})$, whose range is dense, such that for any contractive representation $(H_\pi, \pi) \in \mathcal{F}$ there exists a (necessarily unique) completely contractive homomorphism $\tilde{\pi} : \mathcal{O}(B, \mathcal{F}) \to \mathcal{B}(H_\pi)$ such that $\tilde{\pi} \circ \iota = \pi$.

In order to construct the operator algebras that we will call semicrossed products of $(\mathcal{C}, \alpha)$ (see definition 2.1), we first define the following $\ell^1$-Banach algebras. Note that this is the analogue of the procedure that one follows to build up the crossed product of a C*-algebra by a *-automorphism. First we equip the linear space $c_{00}(\mathbb{Z}_+) \otimes \mathcal{C}$ (the algebraic tensor product of linear spaces) with the left multiplication

$$(\delta_n \otimes c) *_l \delta_m \otimes y = \delta_{n+m} \otimes (a^m(c) \cdot y),$$

and we denote by $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l$ the Banach algebra that is obtained by completing with respect to the $1$-norm

$$\sum_{n=0}^k \delta_n \otimes c_n \| = \sum_{n=0}^k \| c_n \| c.$$ 

In an analogous way, we equip the linear space $\mathcal{C} \otimes c_{00}(\mathbb{Z}_+)$ with the right multiplication

$$(c \otimes \delta_n) *_r (y \otimes \delta_m) = (c \cdot a^n(y)) \otimes \delta_{n+m},$$

and we denote by $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_r$, the Banach algebra obtained.

Note that $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l$ and $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_r$ are isometrically isomorphic as Banach spaces but not as Banach algebras. Also, if $e$ is the unit of $\mathcal{C}$ then it is easy to check that $\delta_0 \otimes e$ is the unit for both algebras.

As we will see in definition 2.1, the semicrossed products of a pair $(\mathcal{C}, \alpha)$ are the enveloping operator algebras of the $\ell^1$-Banach algebras $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l$ and $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_r$, with respect to various collections of contractive representations.

Finally, we will use the following notation concerning crossed products (see, for example, [28]). Let $\alpha : \mathcal{C} \to \mathcal{C}$ be a *-isomorphism of the C*-algebra $\mathcal{C}$ and fix a faithful representation $(H_0, \pi)$ of $\mathcal{C}$. For the Hilbert space $H = H_0 \otimes \ell^2(\mathbb{Z})$, we define $\hat{\pi} : \mathcal{C} \to \mathcal{B}(H)$, so that $\hat{\pi}(c) = \text{diag}\{\pi(\alpha^n(c)) : n \in \mathbb{Z}\}$ and $U = 1_{H_0} \otimes u$, where $u$ is the bilateral shift on $\ell^2(\mathbb{Z})$, ie.

$$U = \begin{bmatrix} \cdots & 0 & \cdots \\ 0 & 1_{H_0} & 0 \\ \cdots \\ \end{bmatrix} \text{ and } \hat{\pi}(c) = \begin{bmatrix} \cdots & \pi(\alpha^{-1}(c)) & \pi(c) & \pi(\alpha(c)) & \cdots \\ \end{bmatrix}.$$ 

The representation $(U \times \hat{\pi})$ of the crossed product $\mathcal{C} \rtimes_\alpha \mathbb{Z}$ that integrates the pair $(\hat{\pi}, U)$ is called the left regular representation. Analogously, the representation $(\hat{\pi} \times U^*)$ that integrates the pair $(\hat{\pi}, U^*)$ is called the right regular representation. It is known that the C*-algebras that are generated by the ranges of $(U \times \hat{\pi})$ and $(\hat{\pi} \times U^*)$ are both *-isomorphic to the crossed product $\mathcal{C} \rtimes_\alpha \mathbb{Z}$. To sketch the proof, there is a *-isomorphism $\Psi : \text{range}(U \times \hat{\pi}) \to \text{range}(\hat{\pi} \times U^*)$, such that $\Psi(U^n\hat{\pi}(c)) = \hat{\pi}(\alpha^{-n}(c))U^{-n}$. Also, a gauge-invariance uniqueness theorem gives that $\text{range}(U \times \hat{\pi}) \simeq \mathcal{C} \rtimes_\alpha \mathbb{Z}$.
2. Definitions

Before we give the main definitions we have to take a closer look at the representations of the \( \ell^1 \)-Banach algebras. Let us consider first the case of \( \ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l \). Let \( \rho : \ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l \to \mathcal{B}(H) \) be a \( | \cdot |_1 \)-contractive representation (denoted simply by \( | \cdot |_1 \)-representation, from now on); then the restriction of \( (H, \rho) \) to the \( C^* \)-algebra \( \mathcal{C} \) defines a contractive homomorphism (thus a *-representation) of \( \mathcal{C} \). Also, let \( V = \rho(\delta_1 \otimes e) \); then \( V \) is a contraction in \( \mathcal{B}(H) \) and the definition of the left multiplication gives

\[
\pi(c)V = \rho(\delta_0 \otimes c)\rho(\delta_1 \otimes e) = \rho(\delta_1 \otimes (\alpha(c) \cdot e)) = V\pi(\alpha(c)),
\]

for all \( c \in \mathcal{C} \). Conversely, let \( (H, \pi) \) be a *-representation of \( \mathcal{C} \) and \( V \) be a contraction in \( H \) such that the following equality holds

\[
(1) \quad \pi(c)V = V\pi(\alpha(c)), \quad c \in \mathcal{C}.
\]

We define the map

\[
(V \times \pi) : \ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l \to \mathcal{B}(H) : \sum_{n=0}^{\infty} \delta_n \otimes c_n \mapsto \sum_{n=0}^{\infty} V^n \pi(c_n).
\]

It is easy to check that \( (V \times \pi) \) is a \( | \cdot |_1 \)-representation of \( \ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l \).

Hence, \( (H, \rho) \) is a \( | \cdot |_1 \)-representation of \( \ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l \) if, and only if, \( \rho = (V \times \pi) \) for a pair \( (\pi, V) \), where \( (H, \pi) \) is a *-representation of \( \mathcal{C} \), \( V \) is a contraction in \( H \) and equality \( (1) \) holds. Such pairs \( (\pi, V) \) are called left covariant contractive, isometric, co-isometric or unitary if \( V \in \mathcal{B}(H) \) is a contraction, an isometry, a co-isometry or a unitary, respectively. We refer to equality \( (1) \) as the left covariance relation.

Analogously, there exists a bijection between the representations of \( \ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_r \) and the right covariant pairs \( (\pi, V) \), i.e. pairs satisfying the right covariance relation \( V\pi(c) = \pi(\alpha(c))V, \ c \in \mathcal{C} \). In this case we write

\[
(\pi \times V)(c \otimes \delta_n) := \pi(c)V^n.
\]

**Definition 2.1.** Let \( \mathcal{C} \) be a unital \( C^* \)-algebra and \( \alpha : \mathcal{C} \to \mathcal{C} \) a unital *-homomorphism. We define the following enveloping operator algebras of \( \ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l \),

\( \mathfrak{A}(\mathcal{C}, \alpha, \text{contr})_l \): with respect to the collection of representations

\[
\{ (V \times \pi) : (\pi, V) \text{ is a left covariant contractive pair} \},
\]

\( \mathfrak{A}(\mathcal{C}, \alpha, \text{isom})_l \): with respect to the collection of representations

\[
\{ (V \times \pi) : (\pi, V) \text{ is a left covariant isometric pair} \},
\]

\( \mathfrak{A}(\mathcal{C}, \alpha, \text{co-isom})_l \): with respect to the collection of representations

\[
\{ (V \times \pi) : (\pi, V) \text{ is a left covariant co-isometric pair} \},
\]

\( \mathfrak{A}(\mathcal{C}, \alpha, \text{un})_l \): with respect to the collection of representations

\[
\{ (V \times \pi) : (\pi, V) \text{ is a left covariant unitary pair} \}.
\]

where for every representation \( (H, \pi) \) we assume that \( \dim H \leq \dim H_u \), \( H_u \) being the Hilbert space of the universal representation of \( \mathcal{C} \). This assumption is just to ensure that the collections considered are sets.
For the right-covariant case, we define the enveloping operator algebras $\mathfrak{A}(\mathcal{C}, \alpha, \text{contr})_r$, $\mathfrak{A}(\mathcal{C}, \alpha, \text{isom})_r$, $\mathfrak{A}(\mathcal{C}, \alpha, \text{co-is})_r$, and $\mathfrak{A}(\mathcal{C}, \alpha, \text{un})_r$ of $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_r$, analogously.

\textbf{Remark 2.2.} There is a bijection between the left covariant pairs and the right covariant pairs. More precisely

1. $(\pi, V)$ is a left covariant \textbf{contractive} pair if, and only if, $(\pi, V^*)$ is a right covariant \textbf{contractive} pair,
2. $(\pi, V)$ is a left covariant \textbf{isometric} pair if, and only if, $(\pi, V^*)$ is a right covariant \textbf{co-isometric} pair,
3. $(\pi, V)$ is a left covariant \textbf{co-isometric} pair if, and only if, $(\pi, V^*)$ is a right covariant \textbf{isometric} pair,
4. $(\pi, V)$ is a left covariant \textbf{unitary} pair if, and only if, $(\pi, V^*)$ is a right covariant \textbf{unitary} pair.

Indeed, by taking adjoints in the relation (11) we get that $V^*\pi(c^*) = \pi(\alpha(c))V^*$, for any $c \in \mathcal{C}$. Thus $V^*\pi(c) = \pi(\alpha(c))V^*$, for any $c \in \mathcal{C}$, since $\mathcal{C}$ is selfadjoint. But, the map $c \mapsto \pi(c^*)$ is not a *-homomorphism and we cannot pass from the representations of $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l$ to the representations of $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_r$, simply by taking adjoints. Nevertheless, the following trick, establishes a duality that simplifies our proofs.

For convenience, we use the symbol $\mathcal{F}_{t,l}$, $t = 1, 2, 3, 4$, for the collection of the left covariant \textit{contractive}, \textit{isometric}, \textit{co-isometric} and \textit{unitary} pairs, respectively. Also, we use the symbol $\mathcal{F}_{t,r}$, $t = 1, 2, 3, 4$, for the right covariant \textit{contractive}, \textit{co-isometric}, \textit{isometric} and \textit{unitary} pairs, respectively. We define the antilinear bijection

$$
\# : c_{00}(\mathbb{Z}_+) \odot \mathcal{C} \to \mathcal{C} \odot c_{00}(\mathbb{Z}_+),
$$

so that $(\delta_n \odot c)^\# = c^* \odot \delta_n$, for every $c \in \mathcal{C}$. Abusing notation we write $(F^\#)^\# = F$, for every $F \in c_{00}(\mathbb{Z}_+) \odot \mathcal{C}$. This bijection is an isometry and extends to an isometry of $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l$ onto $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_r$, for which we use the same symbol. Moreover, for every representation $(H, \rho)$ of $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_r$, we define

$$
\rho^\# : \ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l \to \mathcal{B}(H) : \rho^\#(F) = \rho(F^\#)^*.
$$

It is routine to see that $(H, \rho^\#)$ is a $| \cdot |_1$-representation of $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l$. Also, it is obvious that $\rho \in \mathcal{F}_{t,r}$ if, and only if, $\rho^\# \in \mathcal{F}_{t,l}$.

\textbf{Lemma 2.3.} Let $\rho \in \mathcal{F}_{t,r}$, $t = 1, 2, 3, 4$ and $[F_{ij}] \in \mathcal{M}_\nu(\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_r)$, $\nu \geq 1$. Then $\rho^\# \in \mathcal{F}_{t,l}$ and

$$
\|[\rho(F_{ij})]_\mathcal{B}(H^\nu) = \|[\rho^\#(F^\#_{ij})]_\mathcal{B}(H^\nu).
$$

\textit{Proof.} We prove equality of the norms. Recall that the transpose map $A \mapsto A^t$ is isometric; hence for every $[F_{ij}] \in \mathcal{M}_\nu(\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_r)$, we have

$$
\|[\rho^\#(F^\#_{ij})]_\mathcal{B}(H^\nu) = \|[\rho((F^\#_{ij})^\#)^*]_\mathcal{B}(H^\nu) = \|[\rho(F_{ij})^*]_\mathcal{B}(H^\nu) = \|[\rho(F_{ij})]_\mathcal{B}(H^\nu) = \|[\rho(F_{ij})^t]_\mathcal{B}(H^\nu) = \|[\rho(F_{ij})]_\mathcal{B}(H^\nu),
$$

and the proof is complete. \hfill \Box

7
Remark 2.4. One has to be careful in the connection between the left case and the right case. Following [23] Remark 4 we get that even if $\alpha$ is a $\ast$-automorphism and $C$ is a commutative $C^*$-algebra then $\mathcal{A}(C, \alpha, \text{isom})_r$ is not always isometrically isomorphic to $\mathcal{A}(C, \alpha, \text{isom})_l$, otherwise $\alpha$ would be always conjugate to its inverse. For a counterexample see [14].

3. Semicrossed products over left co-isometric and left unitary covariant pairs

Let us start by examining the semicrossed products $\mathcal{A}(C, \alpha, \text{is})_l$ and $\mathcal{A}(C, \alpha, \text{un})_l$. We show that they are completely isometrically isomorphic and that their $C^*$-envelope is a crossed product. To do so, it is easier first to consider the enveloping operator algebras $\mathcal{A}(C, \alpha, \text{is})_r$ and $\mathcal{A}(C, \alpha, \text{un})_r$, and then use lemma [23] to pass to the left case.

We recall that $\mathcal{A}(C, \alpha, \text{is})_r$ is the enveloping operator algebra of $\ell^1(Z_+, C, \alpha)_r$ with respect to the representations $(\pi \times V)$, where $(\pi, V)$ is a right covariant isometric pair. For every $\nu \geq 1$ and $[F_{ij}] \in \mathcal{M}_\nu(\ell^1(Z_+, C, \alpha)_r)$, we define the seminorms

$$\omega_\nu([F_{ij}]) = \sup \left\{ \|[\pi \times V](F_{ij})]\|_{B(H^\nu)} : (\pi, V) \text{ r.cov.isom. pair} \right\}. $$

Let $\mathcal{N} = \{ F \in \ell^1(Z_+, C, \alpha)_r : \omega_1(F) = 0 \}$. Then the seminorm $\omega_1$ induces a norm on the quotient $\ell^1(Z_+, C, \alpha)_r / \mathcal{N}$, given by $\|F + \mathcal{N}\|_\infty := \omega_1(F)$.

An analogous procedure is followed for the definition of the semicrossed product $\mathcal{A}(C, \alpha, \text{un})_r$.

**Proposition 3.1.** The semicrossed products $\mathcal{A}(C, \alpha, \text{is})_r$ and $\mathcal{A}(C, \alpha, \text{un})_r$ are completely isometrically isomorphic.

**Proof.** Since every right covariant unitary pair is a right covariant isometric pair, it suffices to prove that every right covariant isometric pair dilates to a right covariant unitary pair. But this is established in the proof of [27] Proposition 2.3]. □

**Remark 3.2.** The seminorms $\omega_\nu$ are not norms in general. For example, assume that the $\ast$-homomorphism $\alpha$ has non-trivial kernel and let $c \in \ker \alpha$. Then $V \pi(c) = \pi(\alpha(c))V = 0$ for every right covariant isometric pair $(\pi, V)$. Since $V$ is an isometry we get that $\pi(c) = 0$. Hence $\omega_1(c \otimes \delta_0) = 0$. Note that the same holds for every $c \in R_\alpha$.

The next proposition shows the connection between the radical $R_\alpha$ and the kernel $\mathcal{N}$. In its proof we prove also the existence of a non-trivial right covariant unitary pair. As a consequence $\mathcal{A}(C, \alpha, \text{is})_r$ and $\mathcal{A}(C, \alpha, \text{un})_r$ are not zero.

**Proposition 3.3.** Let $\mathcal{N} = \{ F \in \ell^1(Z_+, C, \alpha)_r : \omega_1(F) = 0 \}$. Then $\mathcal{N} = \ell^1(Z_+, R_\alpha, \alpha)_r$.

**Proof.** First of all, note that $\ell^1(Z_+, R_\alpha, \alpha)_r$ is a $|\cdot|_1$-closed ideal of $\ell^1(Z_+, C, \alpha)_r$ that is contained in $\mathcal{N}$. Now, consider the crossed product $C_\infty \rtimes_{\alpha_\infty} Z$ (see the definitions in the introduction). Then the right regular representation $(\pi \times U)$ of the crossed product induces a right covariant unitary pair for $(C, \alpha)$. Indeed, it suffices to prove that $\pi$ induces a representation of $C$. Let $q : C \rightarrow C/R_\alpha$ be the canonical $\ast$-epimorphism and recall that $C/R_\alpha$ embeds in $C_\infty$. Thus $(\pi \circ q, U)$ is a right covariant unitary pair of $(C, \alpha)$. 

8
Let $F \in \mathcal{N}$ with $F = | \cdot |_1 - \lim_N \sum_{n=0}^{N} c_n \otimes \delta_n$. Then
\[
\lim_N \sum_{n=0}^{N} \pi(q(c_n))U^n = \lim_N \sum_{n=0}^{N} (((\pi \circ q) \times U) (c_n \otimes \delta_n) = ((\pi \circ q) \times U) (F).
\]
But $F \in \mathcal{N}$, hence $((\pi \circ q) \times U) (F) = 0$. For $\xi, \eta \in H$ we have
\[
\langle \pi \circ q(c_n)(\xi), \eta \rangle = \lim_N \sum_{k=0}^{N} \langle \pi(q(c_k))U^k(\xi \otimes e_n), \eta \otimes e_0 \rangle
\]
\[
= \langle ((\pi \circ q) \times U)(F)(\xi \otimes e_n), \eta \otimes e_0 \rangle = 0,
\]
hence $\pi(q(c_n)) = 0$, so $q(c_n) = 0$. Thus $c_n \in \mathcal{R}_\alpha$, for every $n \geq 0$. \hfill $\square$

For the next proposition, recall that we can define the injective $^*$-homomorphism $\hat{\alpha} : \mathcal{C}/\mathcal{R}_\alpha \to \mathcal{C}/\mathcal{R}_\alpha$, with $\hat{\alpha}(c + \mathcal{R}_\alpha) = \alpha(c) + \mathcal{R}_\alpha$.

**Proposition 3.4.** The semicrossed product $\mathfrak{A}(\mathcal{C}, \alpha, is)_r$ is completely isometrically isomorphic to the semicrossed product $\mathfrak{A}(\mathcal{C}/\mathcal{R}_\alpha, \hat{\alpha}, is)_r$.

**Proof.** It suffices to show that the map
\[
Q : \ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_r / \mathcal{N} \to \ell^1(\mathcal{C}/\mathcal{R}_\alpha, \hat{\alpha}, \mathbb{Z}_+)_r
\]
\[
c \otimes \delta_n + \mathcal{N} \mapsto (c + \mathcal{R}_\alpha) \otimes \delta_n,
\]
is completely isometric. To this end, let $F = \sum_{n=0}^{k} c_n \otimes \delta_n$ and $G = Q(F + \mathcal{N}) = \sum_{n=0}^{k} (c_n + \mathcal{R}_\alpha) \otimes \delta_n$. If $(\pi, V)$ is a right covariant isometric pair for $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_r$ acting on $H$, then $\pi(\mathcal{R}_\alpha)H = 0$ by remark 3.2. Thus $\pi$ induces a representation $\sigma : \mathcal{C}/\mathcal{R}_\alpha \to \mathcal{B}(H)$ with $\sigma(c + \mathcal{R}_\alpha) = \pi(c)$. Hence, $(\sigma, V)$ is a right covariant isometric pair of $\ell^1(\mathcal{C}/\mathcal{R}_\alpha, \hat{\alpha}, \mathbb{Z}_+)_r$, and $(\pi \times V)(F) = (\sigma \times V)(G)$. Thus,
\[
\|((\pi \times V)(F))\| = \|((\sigma \times V)(G))\| \leq \|G\|_\infty.
\]
Hence $\omega_1(F) \leq \|G\|_\infty$ and so $\|F + \mathcal{N}\|_\infty = \omega_1(F) \leq \|G\|_\infty$.

On the other hand, let $(\rho, V)$ be a right covariant isometric pair of $\ell^1(\mathcal{C}/\mathcal{R}_\alpha, \hat{\alpha}, \mathbb{Z}_+)_r$; then the map
\[
\pi := \rho \circ q : \mathcal{C} \to \mathcal{B}(H) : c \mapsto \rho(c + \mathcal{R}_\alpha)
\]
is a representation of $\mathcal{C}$. It is easy to see that $(\pi, V)$ is a right covariant isometric pair of $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_r$ and that $(\pi \times V)(F) = (\rho \times V)(G)$. Hence,
\[
\|((\rho \times V)(G))\| = \|((\pi \times V)(F))\| \leq \omega_1(F) = \|F + \mathcal{N}\|_\infty.
\]
Thus $\|G\|_\infty \leq \|F + \mathcal{N}\|_\infty$. The same arguments work for $[F_{ij}] \in \mathcal{M}_\nu(\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_r)$, $\nu \geq 1$, and the proof is complete. \hfill $\square$

Hence, we can assume that $\alpha : \mathcal{C} \to \mathcal{C}$ is injective. Then $\omega_1 = \|\cdot\|_\infty$ and $\mathcal{C}$ embeds in $\mathcal{C}_\infty$.

**Proposition 3.5.** Let $\alpha : \mathcal{C} \to \mathcal{C}$ be an injective $^*$-homomorphism. Then, the $C^*$-envelope of $\mathfrak{A}(\mathcal{C}, \alpha, un)_r$ is the crossed product $\mathcal{C}_\infty \rtimes_{\alpha_\infty} \mathbb{Z}$. 


Proof. First we will show that $C_\infty \rtimes_{\alpha_\infty} \mathbb{Z}$ is a $C^*$-cover of $\mathfrak{A}(C, \alpha, \text{un})_r$. It is clear that $\ell^1(\mathbb{Z}_+, C, \alpha)_r \subseteq \ell^1(\mathbb{Z}, C_\infty, \alpha_\infty)_r$, and that by restricting any right covariant unitary pair of $(C_\infty, \alpha_\infty)$ we get a right covariant unitary pair of $\ell^1(\mathbb{Z}_+, C, \alpha)_r$. Thus $\|F\|_{C_\infty \rtimes_{\alpha_\infty} \mathbb{Z}} \leq \|F\|_{\infty}$, for every $F$ in $\ell^1(\mathbb{Z}_+, C, \alpha)_r$.

On the other hand, let $(\rho, U)$ be a right covariant unitary pair of $\ell^1(\mathbb{Z}_+, C, \alpha)_r$. We can extend $(H, \rho)$ to a representation of $C_\infty$ in the following way: let $x \in C_\infty$ be such that $\alpha_n^\infty(x) \in C$ for some $n$ and let $\rho'(x) = U^n \rho(\alpha_n^\infty(x)) (U^*)^n$. It is easy to see that $\rho'(x)$ is independent of the choice of $n$, hence $\rho'$ extends to a representation of $C_\infty$. Moreover, $(\rho', U)$ is a right covariant unitary pair of $\ell^1(\mathbb{Z}, C_\infty, \alpha_\infty)_r$. Indeed, let $x \in C_\infty$ such that $\alpha_n^\infty(x) \in C$; then $\alpha(\alpha_n^\infty(x)) = \alpha_{n+1}^\infty(x)$, since $\alpha_\infty$ extends $\alpha$. Thus,

$$U \rho'(x) = U \cdot U^n \rho(\alpha_n^\infty(x)) (U^*)^n = U^n \cdot U \rho(\alpha_n^\infty(x)) (U^*)^n \cdot (U^*)^n = U^n \rho(\alpha_n^\infty(x)) (U^*)^n \cdot U = \rho'(\alpha_n^\infty(x)) U .$$

Hence, for every $F$ in $\ell^1(\mathbb{Z}_+, C, \alpha)_r$, and every right covariant unitary pair $(\rho, U)$ of $\ell^1(\mathbb{Z}_+, C, \alpha)_r$, we have that

$$\| (\rho \times U)(F) \| = \| (\rho' \times U)(F) \| \leq \| F\|_{C_\infty \rtimes_{\alpha_\infty} \mathbb{Z}} ;$$

so $\| F\|_{\infty} \leq \| F\|_{C_\infty \rtimes_{\alpha_\infty} \mathbb{Z}}$. Note that the same arguments work for $[F_{ij}] \in \mathcal{M}_\nu(\ell^1(\mathbb{Z}_+, C, \alpha)_r)$, $\nu \geq 1$. Hence, if $(\hat{\pi} \times U)$ is the right regular representation of the crossed product, the map

$$\mathfrak{A}(C, \alpha, \text{un})_r \to C_\infty \rtimes_{\alpha_\infty} \mathbb{Z} : \sum_{n=0}^k c \otimes \delta_n \mapsto \sum_{n=0}^k \hat{\pi}(c) U^n$$

is a completely isometric homomorphism.

In order to conclude that the crossed product is a $C^*$-cover, it suffices to prove that every element of the form $\hat{\pi}[0, \ldots, c, \alpha(c), \ldots]$ is in the $C^*$-algebra $C^*(\hat{\pi}, U)$ generated by the range of $(\hat{\pi} \times U)$. Indeed, we know that $\hat{\pi}(\alpha_n^\infty(x)) = U \hat{\pi}(x) U^*$ for every $x \in C_\infty$, so $\hat{\pi}(x) = U \hat{\pi}(\alpha_1^\infty(x)) U^*$, for every $x \in C_\infty$. Thus

$$\hat{\pi}[0, c, \alpha(c), \ldots] = U \hat{\pi}(\alpha_1^\infty[0, c, \ldots]) U^* = U \hat{\pi}[c, \alpha(c), \ldots] U^* \in C^*(\hat{\pi}, U).$$

Induction shows that every element of the form $\hat{\pi}[0, \ldots, c, \alpha(c), \ldots]$ is in $C^*(\hat{\pi}, U)$.

To end the proof, let $C^*_e$ be the $C^*$-envelope of $\mathfrak{A}(C, \alpha, \text{un})_r$ and $\Phi$ be the $^*$-epimorphism $\Phi : C_\infty \rtimes_{\alpha_\infty} \mathbb{Z} \to C^*_e$, such that $\Phi(\hat{\pi}[c, \alpha(c), \ldots]) = c \otimes \delta_0$ for every $c \in C$. Assume that the Šilov ideal $J = \text{ker } \Phi$ is non-trivial. Then it is invariant by the gauge action for the right regular representation. Hence, it has non-trivial intersection with the fixed point algebra of the gauge action, which is exactly $C_\infty$. So, there is an $n$ such that $J \cap \alpha_n^\infty(C) \neq (0)$. Let $0 \neq c \in C$ such that $\hat{\pi}[0, \ldots, c, \ldots] \in J$. Then $\hat{\pi}[c, \alpha(c), \ldots] = (U^*)^n \hat{\pi}[0, \ldots, c, \ldots] U^n \in J$, so $J \cap C \neq (0)$. But then

$$0 = \| \hat{\pi}[c, \alpha(c), \ldots] + J \| = \| \Phi(\hat{\pi}[c, \alpha(c), \ldots]) \| = \| c \otimes \delta_0 \|_{\infty} = \| c \|_{C},$$

which is a contradiction, since $C$ is contained isometrically in the semicrossed product. Thus $J = (0)$. \hfill $\square$

Using propositions 3.1, 3.4 and 3.5 we get the following theorem for the general case.
Theorem 3.6. Let $\alpha : \mathcal{C} \to \mathcal{C}$ be a *-homomorphism. Then the C*-envelope of the semicrossed products $\mathfrak{A}(\mathcal{C}, \alpha, is)_r$ and $\mathfrak{A}(\mathcal{C}, \alpha, un)_r$ is the crossed product $(\mathcal{C}/\mathcal{R}_\alpha)_\infty \rtimes \hat{\alpha}\infty \mathbb{Z}$.

Proof. By propositions 3.4 and 3.1 we have that

$$\mathfrak{A}(\mathcal{C}, \alpha, is)_r \simeq \mathfrak{A}(\mathcal{C}/\mathcal{R}_\alpha, \hat{\alpha}, is)_r \simeq \mathfrak{A}(\mathcal{C}/\mathcal{R}_\alpha, \hat{\alpha}, un)_r,$$

thus they have the same C*-envelope. By proposition 3.3 this is the crossed product $(\mathcal{C}/\mathcal{R}_\alpha)_\infty \rtimes \hat{\alpha}\infty \mathbb{Z}$.

Now, for the left case, let $\mathcal{N}_{3,l}$ be the kernel w.r.t the collection $\mathcal{F}_{3,l}$. Then $\mathcal{N}_{3,l} = \mathcal{N}$, by lemma 2.3 and because $\mathcal{R}_\alpha$ is self-adjoint. The same lemma gives also that

$$\| [F_{ij} + \mathcal{N}] \|_\nu = \| [F_{ij}^\# + \mathcal{N}] \|_\nu,$$

for every $[F_{ij}] \in \mathcal{M}_\nu(\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l)$. Hence, we get the next proposition.

Proposition 3.7. The semicrossed product $\mathfrak{A}(\mathcal{C}, \alpha, co-is)_l$ is completely isometrically isomorphic to the semicrossed product $\mathfrak{A}(\mathcal{C}/\mathcal{R}_\alpha, \hat{\alpha}, co-is)_l$.

Theorem 3.8. Let $\alpha : \mathcal{C} \to \mathcal{C}$ be a *-homomorphism. Then the C*-envelope of the semicrossed products $\mathfrak{A}(\mathcal{C}, \alpha, co-is)_l$ and $\mathfrak{A}(\mathcal{C}, \alpha, un)_l$ is the crossed product $(\mathcal{C}/\mathcal{R}_\alpha)_\infty \rtimes \hat{\alpha}\infty \mathbb{Z}$.

Proof. Without loss of generality, we can assume that $\alpha : \mathcal{C} \to \mathcal{C}$ is injective. We will show that there is a completely isometric homomorphism of $\mathfrak{A}(\mathcal{C}, \alpha, co-is)_l$ into $\mathcal{C}_\infty \rtimes \hat{\alpha}\infty \mathbb{Z}$. Let $[F_{ij}] \in \mathcal{M}_\nu(\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l)$ and let $(\pi, V)$ be a left covariant co-isometric pair. Then, $(\pi, V^*)$ is a right covariant isometric pair, hence it dilates to a right covariant unitary pair $(\Pi, U)$ of $(\mathcal{C}_\infty, \hat{\alpha}\infty)$. Thus, $(\Pi \times U)$ is a dilation of the representation $(\pi \times V^*) = (V \times \pi)^\#$. So, the pair $(\Pi, U^*)$ is a left covariant unitary pair of $(\mathcal{C}_\infty, \hat{\alpha}\infty)$, hence $(\Pi \times U)^\#$ is a representation of the crossed product $\mathcal{C}_\infty \rtimes \hat{\alpha}\infty \mathbb{Z}$. Thus, we have that

$$\| [(V \times \pi)(F_{ij})] \| = \| [(V \times \pi)^\#(F_{ij}^\#)] \| \leq \| ((\Pi \times U)(F_{ij}^\#)) \| = \| [F_{ij}^\#] \|_{\mathcal{M}_\nu(\mathcal{C}_\infty \rtimes \hat{\alpha}\infty \mathbb{Z})}.$$

Thus $\| [F_{ij}] \|_\nu \leq \| [F_{ij}] \|_{\mathcal{M}_\nu(\mathcal{C}_\infty \rtimes \hat{\alpha}\infty \mathbb{Z})}$. Also, it is easy to show that $\| [F_{ij}] \|_{\mathcal{M}_\nu(\mathcal{C}_\infty \rtimes \hat{\alpha}\infty \mathbb{Z})} \leq \| [F_{ij}] \|_\nu$. Hence, the identity map $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l \to \mathcal{C}_\infty \rtimes \hat{\alpha}\infty \mathbb{Z}$ is a completely isometric homomorphism. Since we have assumed that $\alpha$ is injective we get that $\mathcal{N}_{3,l} = (0)$. Thus the identity map extends to a completely isometric homomorphism of $\mathfrak{A}(\mathcal{C}, \alpha, co-is)_l$. The rest of the proof goes in a similar way with the one of proposition 3.3. □

Remark 3.9. From a first look, it seems that the theory we used here could be related to the crossed products by an endomorphism $\mathcal{T}(\mathcal{C}, \alpha, \mathcal{L})$ examined in [12], but there is a significant difference. In [12] Definition 3.1 Exel considers a universal C*-algebra for which the representation theory consists of pairs $(\pi, S)$ such that $S\pi(c) = \pi(\alpha(c))S$, but also $S^*\pi(c)S = \pi(\mathcal{L}(c))$, for a chosen transfer operator $\mathcal{L}$. On the other hand the pairs we examined do define a transfer operator on every Hilbert space (just by defining $\mathcal{L}(\pi(c)) = S^*\pi(c)S$), but $\mathcal{L}$ is not the same for every such pair. Nevertheless, our theory is applicable to $\mathcal{U}(\mathcal{C}, \alpha)$ defined in [12] Definition 4.4]. Let the operator algebra generated by the analytic polynomials in $\mathcal{U}(\mathcal{C}, \alpha)$. Then, every representation consists of pairs $(\pi, S)$ where $S$ is an isometry, subject to the relation $\pi(\alpha(c)) = S\pi(c)S^*$. Hence, for any such a pair we get that $\pi(\alpha(c))S = S\pi(c)$ and we can apply theorem 3.6.
4. Semicrossed products over left contractive and left isometric covariant pairs

We recall that $\mathfrak{A}(\mathcal{C}, \alpha, \text{contr})$ is the enveloping operator algebra of $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l$ with respect to the family of the representations $(V \times \pi)$, where $(\pi, V)$ ranges over left covariant contractive pairs. For every $\nu \geq 1$ and $[F_{ij}] \in M_{\nu}(\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_r)$, we define the seminorms

$$
\| [F_{ij}] \|_{\nu} = \sup \{ \| (\pi \times V)(F_{ij}) \|_{B(H^U)} : (\pi, V) \text{ r.cov.contr. pair} \}.
$$

**Proposition 4.1.** The semicrossed products $\mathfrak{A}(\mathcal{C}, \alpha, \text{contr})_l$ and $\mathfrak{A}(\mathcal{C}, \alpha, \text{is})_l$ are completely isometrically isomorphic.

**Proof.** Since every left covariant isometric pair is a left covariant contractive pair, it suffices to prove that every left covariant contractive pair dilates to a left covariant isometric pair. But this is established in [20].

The following is an example of a left covariant isometric pair. It follows that the semicrossed products $\mathfrak{A}(\mathcal{C}, \alpha, \text{contr})_l$ and $\mathfrak{A}(\mathcal{C}, \alpha, \text{is})_l$ are not zero; moreover, that the $\nu$-seminorms are in fact norms. In theorem 4.3 we show that the following construction gives a completely isometric representation of $\mathfrak{A}(\mathcal{C}, \alpha, \text{contr})_l$ and $\mathfrak{A}(\mathcal{C}, \alpha, \text{is})_l$.

**Example 4.2.** Let $(H_0, \pi)$ be a representation of $\mathcal{C}$. We define $\tilde{\pi}(c) = \text{diag}\{ \pi(\alpha^n(c)) : n \in \mathbb{Z}_+ \}$ acting on the Hilbert space $H_0 \otimes \ell^2(\mathbb{Z}_+)$. Also, let $S = I_{H_0} \otimes s$, where $s$ is the unilateral shift. Thus,

$$
S = \begin{bmatrix}
0 & 0 & \cdots \\
1_{H_0} & 0 & \cdots \\
1_{H_0} & 0 & \cdots \\
\cdots & \cdots & \cdots 
\end{bmatrix}, \quad \tilde{\pi}(c) = \begin{bmatrix}
\pi(c) \\
\pi(\alpha(c)) \\
\pi(\alpha^2(c)) \\
\cdots 
\end{bmatrix},
$$

for $c \in \mathcal{C}$. One can easily check that $(\tilde{\pi}, S)$ is a left covariant isometric pair.

If $(H_0, \pi)$ is faithful, then the induced representation $(S \times \tilde{\pi})$ is faithful on $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l$.

Indeed, let $F = | \cdot |_1 - \lim_N \sum_{n=0}^N \delta_n \otimes c_n$, such that $(S \times \tilde{\pi})(F) = 0$. Then for every $\xi, \eta \in H_0$ we have that

$$
\langle \pi(c_n) \xi, \eta \rangle = \lim_N \sum_{k=0}^N \langle S^k \tilde{\pi}(c_k)(\xi \otimes e_0), \eta \otimes e_n \rangle = \langle (S \times \tilde{\pi})(F)(\xi \otimes e_0), \eta \otimes e_n \rangle = 0.
$$

Since $(H_0, \pi)$ is faithful, we get that $c_n = 0$, for every $n$, thus $F = 0$.

For every left covariant isometric pair $(\pi, V)$ we denote by $C^*(\pi, V)$ the $C^*$-algebra generated by the range of the representation $(V \times \pi)$. Due to the left covariance relation, $C^*(\pi, V)$ is the closure of the polynomials

$$
\sum_{n,m} V^n \pi(c_{n,m})(V^*)^m, \quad c_{n,m} \in \mathcal{C}, \quad n, m \in \mathbb{Z}_+.
$$

Let $H_u = \bigoplus_i H_i$, $\pi_u = \bigoplus_i \pi_i$ and $V_u = \bigoplus_i V_i$, where the summand ranges over the left covariant isometric pairs $(\pi_i, V_i)$ that act on Hilbert spaces $H_i$. Then the semicrossed product $\mathfrak{A}(\mathcal{C}, \alpha, \text{is})_l$ is the closure of the polynomials $\sum_{n=0}^k V_u^n \pi_u(c_n), \quad c_n \in \mathcal{C}, \quad k \in \mathbb{Z}_+.$
The $C^*$-algebra $C^*(\pi_u, V_u)$ has the following universal property: for every left covariant isometric pair $(\pi, V)$ there is a $*$-epimorphism $\Phi : C^*(\pi_u, V_u) \to C^*(\pi, V)$, such that $\Phi \circ \pi_u = \pi$ and $\Phi \circ V_u = V$. (The $*$-epimorphism $\Phi$ is induced by restricting $\pi_u$ and $V_u$ to $H \subseteq H_u$.)

For any $z \in \mathbb{T}$ we define a $*$-automorphism $\beta_z$ of $C^*(\pi_u, V_u)$, such that $\beta_z(\pi_u(c)) = \pi_u(c)$, $c \in \mathcal{C}$ and $\beta_z(V_u^n) = z^n V_u^n$, $n \in \mathbb{Z}_+$. An $\epsilon/3$-argument, along with the fact that $C^*(\pi_u, V_u)$ is the closed linear span of the monomials $V_u^n \pi(\alpha)c(V_u)^n$, shows that the family $\{\beta_z\}_{z \in \mathbb{T}}$ is point-norm continuous. Thus, we can define the conditional expectation $\mathcal{E} : C^*(\pi_u, V_u) \to C^*(\pi_u, V_u)$, by $\mathcal{E}(F) := \int_{\mathbb{T}} \beta_z(F)dz$, $F \in C^*(\pi_u, V_u)$, where $dz$ is Haar measure on the unit circle $\mathbb{T}$. It is easy to see that the fixed point algebra $C^*(\pi_u, V_u)^\beta$, i.e. the range of $\mathcal{E}$, is the closed linear span of $\sum_{n=0}^k V_u^n \pi_u(c_n)(V_u)^n$, $c_n \in \mathcal{C}$. Hence, the fixed point algebra is (the inductive limit) $\bigcup_k B_k$ of the $C^*$-subalgebras $B_k = \overline{\text{span}}\{\sum_{n=0}^k V_u^n \pi_u(c_n)(V_u)^n : c_n \in \mathcal{C}\}$. Also, it is a routine to check that $\mathcal{E}$ is a norm-continuous, faithful projection onto the fixed point algebra.

Now, let us fix a faithful representation $(H_0, \pi)$ of $\mathcal{C}$, and let $(\bar{\pi}, S)$ be as in example 1.2.

For every $z \in \mathbb{T}$ we define the unitary operator $u_z : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$, by $u_z(e_n) = z^n e_n$. Let $U_z = 1_{H_0} \otimes u_z$; the map $\gamma_z = ad_{U_z}$ satisfies $\gamma_z(\bar{\pi}(c)) = \bar{\pi}(c)$, $c \in \mathcal{C}$ and $\gamma_z(S^n) = z^n S^n$, $n \in \mathbb{Z}_+$, hence defines a $*$-automorphism of $C^*(\bar{\pi}, S)$. Again, an $\epsilon/3$-argument shows that $\{\gamma_z\}_{z \in \mathbb{T}}$ is point-norm continuous family. Hence, we can define the conditional expectation $E(F) := \int_{\mathbb{T}} \gamma_z(F)dz$, $F \in C^*(\bar{\pi}, S)$. The map $E$ is a norm-continuous, faithful projection onto the fixed algebra $C^*(\bar{\pi}, S)^\gamma = \overline{\text{span}}\{S^n \pi(c)(S^*)^n : c \in \mathcal{C}\}$.

For the canonical $*$-epimorphism $\Phi : C^*(\pi_u, V_u) \to C^*(\bar{\pi}, S)$ we have that $\Phi \circ \beta_z = \gamma_z \circ \Phi$, hence $\Phi \circ \mathcal{E} = E \circ \Phi$. Thus the restriction of $\Phi$ on $C^*(\pi_u, V_u)^\beta$ is a $*$-homomorphism onto $C^*(\bar{\pi}, S)^\gamma$. The following is a revision of ([15, Theorem 1.4]).

**Theorem 4.3.** The $*$-epimorphism $\Phi : C^*(\pi_u, V_u) \to C^*(\bar{\pi}, S)$ is injective, hence a $*$-isomorphism. Consequently, the semicrossed products $\mathfrak{A}(\mathcal{C}, \alpha, \text{contr})_l$ and $\mathfrak{A}(\mathcal{C}, \alpha, \text{is})_l$ are completely isometrically isomorphic to the closure (in $C^*(\bar{\pi}, S)$) of the polynomials $\sum_{n=0}^k S^n \bar{\pi}(c_n)$, $c_n \in \mathcal{C}$, for any faithful representation $(H_0, \pi)$ of $\mathcal{C}$.

**Proof.** First we will show that the restriction of $\Phi$ to $C^*(\pi_u, V_u)^\beta$ is injective. Assume that $\ker(\Phi|_{C^*(\pi_u, V_u)^\beta})$ is not trivial. Since $C^*(\pi_u, V_u)^\beta$ is an inductive limit there is a $k$ such that $B_k \cap \ker(\Phi|_{C^*(\pi_u, V_u)^\beta}) \neq \{0\}$, thus there is a non-zero analytic polynomial $F = \sum_{n=0}^k V_u^n \pi_u(c_n)(V_u)^n$ such that $\Phi(F) = \sum_{n=0}^k S^n \pi(c_n)(S^*)^n = 0$. Note that $S^n \pi(c)(S^*)^n = \text{diag}(0, \ldots, 0, \pi(c), \pi(\alpha(c)), \ldots)$, hence the $(m, m)$-element $(\Phi(F))_{m,m}$ of $\Phi(F)$ equals to

$$
(\Phi(F))_{m,m} = \begin{cases} 
\pi \left( \min \{m,k\} \sum_{j=0}^{\min \{m,k\}} \alpha^{m-j}(c_{m-j}) \right), & \text{for } m < k, \\
\pi \left( \alpha^m(c_0) + \alpha^{m-1}(c_1) + \cdots + c_m \right), & \text{for } m \geq k,
\end{cases}
$$

Since $\Phi(F) = 0$, we have that $(\Phi(F))_{0,0} = 0$, hence $\pi(c_0) = 0$. Since $(H_0, \pi)$ is injective we get that $c_0 = 0$. Thus $(\Phi(F))_{1,1} = \pi(c_1)$ and arguing as previously we get that $c_1 = 0$. Continuing
in the same way we get that \( c_m = 0 \) for all \( m = 0, 1, \ldots, k \), hence \( F = \sum_{n=0}^k V_u^n \hat\pi_u(c_n)(V_u^*)^n = 0 \), which is a contradiction. Thus the restriction of \( \Phi \) to \( C^*(\pi_u, V_u)^\beta \) is injective.

Now, let \( F \in \ker \Phi \), then \( F^*F \in \ker \Phi \). Hence, \( \Phi \circ \mathcal{E}(F^*F) = E \circ \Phi(F^*F) = 0 \). But, \( \mathcal{E}(F^*F) \in C^*(\pi_u, V_u)^\beta \), and the restriction of \( \Phi \) to \( C^*(\pi_u, V_u)^\beta \) is injective; thus \( \mathcal{E}(F^*F) = 0 \). So, \( F = 0 \), since \( \mathcal{E} \) is faithful.

By using lemma \([23]\) as in proposition \([3,8]\) we get the next theorem.

**Theorem 4.4.** The semicrossed products \( \mathfrak{A}(C, \alpha, \text{contr}) \) and \( \mathfrak{A}(C, \alpha, \text{co-is}) \) are completely isometrically isomorphic to the closed linear span of the polynomials \( \sum_{n=0}^k \hat\pi(c_n)(S^*)^n, c_n \in C \), for any faithful representation \((H, \pi)\) of \( C \) and \((\hat\pi, S)\) as in the example \([7,2]\).

We now proceed to the determination of the \( C^*\)-envelope of the semicrossed products \( \mathfrak{A}(C, \alpha, \text{contr}) \) and \( \mathfrak{A}(C, \alpha, \text{is}) \). As mentioned in the introduction, in \([23]\) Peters computes the \( C^*\)-envelope of \( \mathfrak{A}(C, \alpha, \text{is}) \), when \( C \) is commutative and \( \alpha \) is injective. In \([16]\) we prove a similar theorem without the assumption of commutativity, but still assuming that \( \alpha \) is a *-automorphism. In \([17]\) with Elias Katsoulis we give the result for the general case (see theorem \([4,8]\) below), by extending the method of “adding tails” introduced in \([21]\). In particular \([17]\) Proposition 3.12] shows the necessity of that extension.

Let \( M \equiv \mathcal{M}(\ker \alpha) \) be the multiplier algebra of \( \ker \alpha \), and \( \theta : C \to M \) be the unique unital *-homomorphism extending the natural embedding \( \ker \alpha \hookrightarrow M \). Also, consider the \( C^*\)-algebra \( T = c_0(\theta(C)) \); we use the letters \( \hat{x}, \hat{y}, \) e.t.c. for the elements \((x_n), (y_n) \in T \) and the symbol \( \hat{0} \) for the zero sequence \((0) \in T \). For the \( C^*\)-algebra \( B = C \oplus T \) we define the map \( \beta : B \to B \) by

\[
\beta(c, \hat{x}) \equiv \beta(c, (x_n)) = (\alpha(c), \theta(c), x_1, x_2, \ldots) \equiv (\alpha(c), \theta(c), \hat{x}),
\]

for every \( c \in C \) and \( \hat{x} \equiv (x_n) \in c_0(\theta(C)) \). Note that \( B \) contains \( C \), but \( \beta \) does not extend \( \alpha \). Also, \( \beta \) is an injective *-homomorphism. Indeed, let \((c, \hat{x}) \in \ker \beta \); then \( x_n = 0, \theta(c) = 0 \) and \( \alpha(c) = 0 \). Thus, \( c \in \ker \alpha \), so \( c = \theta(c) = 0 \). Hence \((c, \hat{0}) = 0 \). Finally, if \( e \) is the unit of \( C \), we have

\[
\beta^m(e, \hat{0})(c, \hat{0}) = (e, 1_M, \ldots, 1_M, 0, \ldots)(c, \hat{0}) = (c, \hat{0}),
\]

for every \( m \in \mathbb{Z}_+ \).

For the *-automorphism \( \beta_\infty : B_\infty \to B_\infty \) (which is an extension of \( \beta \)), consider the crossed product

\[
B_\infty \rtimes_{\beta_\infty} \mathbb{Z} = \operatorname{span}\{U^n \hat{\pi}(y) : y \in B_\infty, n \in \mathbb{Z}\},
\]

where \((\hat{\pi}, U)\) is the pair that induces the left regular representation of \((B_\infty, \beta_\infty)\). Since \( B_\infty = \bigcup_n \beta_\infty^n(B) \), and due to the left covariance relation, we get

\[
B_\infty \rtimes_{\beta_\infty} \mathbb{Z} = \operatorname{span}\{U^n \hat{\pi}(b)(U^*)^m : n, m \in \mathbb{Z}_+, b \in B\}.
\]

Let \( \mathfrak{A} \) be the \( C^*\)-subalgebra of the crossed product generated by \( U \hat{\pi}(e, \hat{0}) \) and \( \hat{\pi}(c, \hat{0}), c \in C \). Then \( \mathfrak{A} \) is the closed linear span of the monomials \( U^n \hat{\pi}(c, \hat{0})(U^*)^m, n, m \in \mathbb{Z}_+, c \in C \). Also, note that

\[
U \hat{\pi}(e, \hat{0}) = \hat{\pi}(e, \hat{0})U \hat{\pi}(e, \hat{0}).
\]
Lemma 4.5. [17] Lemma 3.4] Every element $U^n\hat{\pi}(b)(U^*)^n$, $b \in B$, can be written as $A + \hat{\pi}(0, \vec{y})$, where $A \in A$.

Proof. For $n = 0$, we have $\hat{\pi}(b) = \hat{\pi}(c, \vec{x}) = \hat{\pi}(c, \vec{0}) + \hat{\pi}(0, \vec{x})$. For $n = 1$,

$$U\hat{\pi}(b)U^* = U\hat{\pi}(c, \vec{x})U^* = U\hat{\pi}(c, \vec{0})U^* + U\hat{\pi}(0, \vec{x})U^*.$$

Let $c' \in C$ such that $\theta(c') = x_1$; then

$$\beta_\infty(c', x_2, x_3, \ldots) = \beta(c', x_2, x_3, \ldots)$$

$$= (\alpha(c'), \theta(c'), x_2, \ldots) = (\alpha(c'), \vec{0}) + (0, \vec{x}).$$

Thus $(c', x_2, x_3, \ldots) = \beta_\infty^{-1}(\alpha(c'), \vec{0}) + \beta_\infty^{-1}(0, \vec{x})$, hence

$$U\hat{\pi}(0, \vec{x})U^* = \hat{\pi}(\beta_\infty^{-1}(0, \vec{x})) = \hat{\pi}(c', x_2, \ldots) - \hat{\pi}(\beta_\infty^{-1}(\alpha(c'), \vec{0}))$$

$$= \hat{\pi}(c', \vec{0}) - U\hat{\pi}(\alpha(c'), \vec{0})U^* + \hat{\pi}(0, x_2, \ldots).$$

Thus, $U\hat{\pi}(b)U^* = A + \hat{\pi}(0, x_2, \ldots)$. The proof is completed using induction on $n$. \hfill \Box

Proposition 4.6. [17] Lemma 3.7] The semicrossed product $\mathfrak{A}(C, \alpha, is_i)$ is completely isometrically isomorphic to the closure of the polynomials $\sum_{n=0}^k U^n\hat{\pi}(c_n, \vec{0})$. Hence, $\mathfrak{A}$ is a $C^*$-cover of $\mathfrak{A}(C, \alpha, is_i)$.

Proof. Let $(H_0, \pi)$ be a faithful representation of $B_\infty$ and let $(\hat{\pi}, U)$ be the unitary covariant pair in $H = H_0 \otimes \ell^2(\mathbb{Z})$, that gives the left regular representation of the crossed product. For simplicity, let $\phi$ be the representation of $\hat{C}$ given by

$$\phi(c) := \hat{\pi}(c, \vec{0}) = \left[ \begin{array}{c} \pi(\beta_\infty^{-1}(c, \vec{0})) \\ \pi(c, \vec{0}) \\ \pi(\beta_\infty(c, \vec{0})) \\ \pi(\beta_\infty^{-1}(c, \vec{0})) \end{array} \right].$$

Then $(\phi, U\phi(c))$ is a left covariant contractive pair for $(\hat{C}, \alpha)$. Indeed,

$$\phi(c) \cdot U\phi(e) = \hat{\pi}(c, \vec{0})U \cdot \hat{\pi}((e, \vec{0})) = U\hat{\pi}(\beta_\infty(c, \vec{0})) \cdot \hat{\pi}((e, \vec{0}))$$

$$= U\hat{\pi}(\beta_\infty(c, \vec{0}))(e, \vec{0}) = U\hat{\pi}(\alpha(c), \vec{0}) = U\phi(e) \cdot \phi(\alpha(c)).$$

Hence, for every $\nu \geq 1$ and $[F_{ij}] \in M_\nu(\ell^1(\mathbb{Z}_+, \hat{C}, \alpha))$, we have that

$$\|[(U\phi(e) \times \phi)(F_{ij})]\| \leq \|[F_{ij}]\|_\infty. \tag{4}$$

Using the equation (3), we note that

$$(U\phi(e) \times \phi)(\delta_n \otimes c) = (U\phi(e))^n\phi(c) = U^n\phi(c),$$

for every $n \in \mathbb{Z}_+$ and $c \in \mathcal{C}$. Thus, we have to prove that, for every $\nu \geq 1$, the inequality (4) is an equality.
Let $K = [e_n : n \geq 0] \subseteq \ell^2(\mathbb{Z})$. It is easy to see that $H_0 \otimes K$ is a reducing subspace of $(H, \phi)$. Then the restriction of $\phi$ on $H_0 \otimes K$ is

$$\phi(c)|_{H_0 \otimes K} = \begin{bmatrix} \pi(c, \bar{\theta}) \\ \pi(\beta_\infty(c, \bar{\theta})) \\ \pi(\beta_\infty^2(c, \bar{\theta})) \\ \vdots \end{bmatrix}.$$ 

This representation is faithful, since we have assumed that $\pi$ is faithful. Also, the pair $(P_{H_0 \otimes K} U \phi(e)|_{H_0 \otimes K}, \phi|_{H_0 \otimes K})$ satisfies the left covariance relation for $(\mathcal{C}, \alpha)$; indeed, let $c \in \mathcal{C}$, then

$$\phi(c)|_{H_0 \otimes K} \cdot \left(P_{H_0 \otimes K} U \phi(e)|_{H_0 \otimes K}\right) = P_{H_0 \otimes K} \left(\phi(c) U \phi(e)\right)|_{H_0 \otimes K},$$

$$= P_{H_0 \otimes K} \left(U \phi(e) \cdot \phi(\alpha(c))\right)|_{H_0 \otimes K},$$

$$= \left(P_{H_0 \otimes K} U \phi(e)|_{H_0 \otimes K}\right) \cdot \phi(\alpha(c))|_{H_0 \otimes K}.$$ 

We note that

$$P_{H_0 \otimes K} U|_{H_0 \otimes K} = (1_{H_0} \otimes P_K)(1_{H_0} \otimes u)|_{H_0 \otimes K} = 1_{H_0} \otimes s = S,$$ 

hence, using induction on the relation (3), we get that

$$(P_{H_0 \otimes K} U \phi(e)|_{H_0 \otimes K})^n = \left(P_{H_0 \otimes K} U P_{H_0 \otimes K}\right)^n \phi(e)|_{H_0 \otimes K} = S^n \phi(e)|_{H_0 \otimes K}.$$

Let $\mathcal{C}$ be the $C^*$-algebra $C^*(\phi|_{H_0 \otimes K}, P_{H_0 \otimes K} U \phi(e)|_{H_0 \otimes K})$ and $\Phi$ be the *-epimorphism from $C^*(\pi_u, V_u)$ onto $\mathcal{C}$. We will show that $\Phi$ is injective. For $z \in \mathbb{T}$, let $\gamma_z = ad_{U_z}$, where $U_z(\xi \otimes e_k) = z^k \xi \otimes e_k$, $k \in \mathbb{Z}_+$. Then $\gamma_z$ is a *-automorphism of $\mathcal{C}$; indeed,

$$\gamma_z(\Phi(c)|_{H_0 \otimes K}) = \Phi(c)|_{H_0 \otimes K},$$

and

$$\gamma_z(P_{H_0 \otimes K} (U \phi(e)|_{H_0 \otimes K}) = \gamma_z(S \phi(e)|_{H_0 \otimes K}) = \gamma_z(S) \gamma_z(\phi(e)|_{H_0 \otimes K}),$$

$$= z S \phi(e)|_{H_0 \otimes K} = z \left(P_{H_0 \otimes K} U \phi(e)|_{H_0 \otimes K}\right).$$

As in the proof of theorem 4.3 it suffices to show that the restriction of $\Phi$ to the fixed point algebra is injective. Since

$$(P_{H_0 \otimes K} (U \phi(e)|_{H_0 \otimes K})^n \phi(c)|_{H_0 \otimes K} \left(P_{H_0 \otimes K} (U \phi(e)|_{H_0 \otimes K})^*\right)^n = S^n \phi(c)|_{H_0 \otimes K} \phi(e)(S^*)^n = S^n \phi(c)|_{H_0 \otimes K},$$

we get that the fixed point algebra of $\mathcal{C}$ is the closed linear span of these monomials. Recall that $(H_0, \pi)$ is a faithful representation of $\mathcal{C}$; hence we can follow *mutatis mutandis* the arguments of the proof of theorem 4.3 and conclude that the restriction of $\Phi$ to the fixed point algebra $\mathcal{C}^*$ is injective. Hence $\Phi$ is a *-isomorphism. Thus, for every $F \in \ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_t$ we get

$$\|F\|_\infty = \left\|\left(P_{H_0 \otimes K} U \phi(e)|_{H_0 \otimes K}\right) \times \phi|_{H_0 \otimes K}\right\|.$$
But,

\[
\left( (P_{H_0 \otimes K} U \phi(e) |_{H_0 \otimes K}) \times \phi |_{H_0 \otimes K} \right) = P_{H_0 \otimes K} \left( U \phi(e) \times \phi \right) |_{H_0 \otimes K}
\]

and eventually, for any \( F \in \ell^1(\mathbb{Z}_+ , \mathcal{C}, \alpha)_l \), we have

\[
\|F\|_{\infty} = \left\| \left( (P_{H_0 \otimes K} U \phi(e) |_{H_0 \otimes K}) \times \phi |_{H_0 \otimes K} \right) (F) \right\| = \left\| P_{H_0 \otimes K} \left( U \phi(e) \times \phi \right) |_{H_0 \otimes K} (F) \right\| \leq \left\| (U \phi(e) \times \phi) (F) \right\| \leq \|F\|_{\infty}.
\]

Hence, \( \|F\|_{\infty} = \|(U \phi(e) \times \phi)(F)\| \), \( F \in \ell^1(\mathbb{Z}_+ , \mathcal{C}, \alpha)_l \). The same argument can be used for any matrix \([F_{ij}]\), and the proof is complete. \( \square \)

We will show that \( \mathfrak{A} \) is a full corner of \( B_\infty \rtimes_{\beta_\infty} \mathbb{Z} \). Recall that for a projection \( p \) in the multiplier algebra of a C*-algebra \( \mathcal{C} \), the C*-subalgebra \( p \mathcal{C} p \) is called a corner of \( \mathcal{C} \). A corner is called full if the linear span of \( \mathcal{C} p \mathcal{C} \) is dense in \( \mathcal{C} \). Equivalently, if \( p \mathcal{C} p \) is not contained in any proper ideal of \( \mathcal{C} \).

Let \( p = \hat{\pi}(e, \tilde{0}) \). Then it is easy to see that \( p \) is a projection in (the multiplier algebra of) \( B_\infty \rtimes_{\beta_\infty} \mathbb{Z} \), and that

1. \( p^\pi(c, \tilde{0}) = \hat{\pi}(c, \tilde{0}) = \hat{\pi}(c, \tilde{0})p \), for every \( c \in \mathcal{C} \),
2. \( pU^n \hat{\pi}(b) = U^n \hat{\pi}(c, x_1, \ldots, x_n, \tilde{0}) \), for every \( b = (c, \tilde{x}) \in \mathcal{B} \),
3. \( U^n \hat{\pi}(b)p = U^n \hat{\pi}(c, \tilde{0}) \), for every \( b = (c, \tilde{x}) \in \mathcal{B} \),
4. \( pAp = A \), for every \( A \in \mathfrak{A} \); hence \( p\mathfrak{A}p = \mathfrak{A} \).

**Proposition 4.7.** [17] Theorem 3.10] The corner \( p(B_\infty \rtimes_{\beta_\infty} \mathbb{Z})p \) is full and equals to \( \mathfrak{A} \).

**Proof.** Let \( U^n \hat{\pi}(b)(U^*)^m \) be a monomial in the crossed product. If \( n \geq m \), by lemma 4.5 we have that

\[
p(U^n \hat{\pi}(b)(U^*)^m)p = p(U^{n-m}U^m \hat{\pi}(b)(U^*)^m)p = p(U^{n-m}(A + \hat{\pi}(0, \tilde{y}))p = pU^{n-m}Ap = U^{n-m}A,
\]

for some \( A \in \mathfrak{A} \). In the same way we get that \( p(U^n \hat{\pi}(b)(U^*)^m)p = A(U^*)^{m-n} \), for some \( A \in \mathfrak{A} \), when \( n < m \). Hence, \( \mathfrak{p}(B_\infty \rtimes_{\beta_\infty} \mathbb{Z})p \subseteq \mathfrak{A} \). On the other hand, \( \mathfrak{A} \subseteq B_\infty \rtimes_{\beta_\infty} \mathbb{Z} \), hence \( \mathfrak{A} = p\mathfrak{A}p \subseteq p(B_\infty \rtimes_{\beta_\infty} \mathbb{Z})p \). Thus \( \mathfrak{A} \) is the corner \( p(B_\infty \rtimes_{\beta_\infty} \mathbb{Z})p \).

To prove that it is also full, let \( \mathcal{I} \) be an ideal in the crossed product, such that \( \mathfrak{A} \subseteq \mathcal{I} \). We will prove that \( \mathcal{I} \) is not non-trivial. To this end, it suffices to prove that \( \hat{\pi}(\mathcal{B}_\infty) \subseteq \mathcal{I} \). Since \( \mathcal{B}_\infty = \bigcup \beta_\infty^{-1}(\mathcal{B}) \) and due to the left covariance relation, it suffices to show that \( \hat{\pi}(\mathcal{B}) \subseteq \mathcal{I} \). First, \( \hat{\pi}(\mathcal{C}) \in \mathfrak{A} \subseteq \mathcal{I} \). In order to prove that \( \hat{\pi}(c_0(\theta(\mathcal{C}))) \subseteq \mathcal{I} \), it suffices to show that
\( \hat{\pi}(0, 0, \ldots, 0, 1_M, 0, \ldots) \in \mathcal{I} \), for every \( n \in \mathbb{Z}_+ \). Note that
\[
\hat{\pi}(0, 0, \ldots, 0, 1_M, 0, \ldots) = \hat{\pi}(\beta^n(0, 1_M, \varnothing)) = \hat{\pi}\left((\beta^n(0, 1_M, \varnothing)\right) = (U^n)\hat{\pi}(0, 1_M, \varnothing)U^n.
\]
Hence, it suffices to show that \( \hat{\pi}(0, 1_M, \varnothing) \in \mathcal{I} \). Indeed,
\[
\hat{\pi}(0, 1_M, \varnothing) = \hat{\pi}(\beta(e, \varnothing) - (e, \varnothing)) = \hat{\pi}(\beta(e, \varnothing)) - \hat{\pi}(e, \varnothing)
\]
\[
= \hat{\pi}(\beta_\infty(e, \varnothing)) - \hat{\pi}(e, \varnothing) = U^n\hat{\pi}(e, \varnothing)U - \hat{\pi}(e, \varnothing) \in \mathcal{I},
\]
since \( \hat{\pi}(e, \varnothing) \in \mathcal{I} \).

Now, let \( E : B_\infty \rtimes_{\beta_\infty} \mathbb{Z} \to (B_\infty \rtimes_{\beta_\infty} \mathbb{Z})^\beta \) be the conditional expectation of the crossed product, and let \( E' \) be the restriction of \( E \) to \( \mathcal{A} = p(B_\infty \rtimes_{\beta_\infty} \mathbb{Z})p \). It is easy to see that range \( E' = p(\text{range } E)p \). But, range \( E = B_\infty = \bigcup_n \hat{\pi}(\beta_\infty^n(\mathcal{B})) \). Thus,
\[
\text{range } E' = p(\bigcup_n \hat{\pi}(\beta_\infty^n(\mathcal{B})))p = \bigcup_n p(\hat{\pi}(\beta_\infty^n(\mathcal{B})))p.
\]
Let \( \mathcal{J} \) be an ideal in range \( E' \). Recall that \( \beta_\infty^n(\mathcal{B}) \) is an increasing sequence, so \( p(\hat{\pi}(\beta_\infty^n(\mathcal{B})))p \) is also increasing. Thus, if \( \mathcal{J} \) is not trivial, then it must intersect some \( p(\hat{\pi}(\beta_\infty^n(\mathcal{B})))p \). If \( b = (c, \varnothing) \in \mathcal{B} \), such that \( p(\hat{\pi}(\beta_\infty^n(b)))p \in \mathcal{J} \), then
\[
\mathcal{J} \ni (\hat{\pi}(e, \varnothing)U^n - p(\hat{\pi}(\beta_\infty^n(b))p \cdot (U^n\hat{\pi}(e, \varnothing)))
\]
\[
= \hat{\pi}(e, \varnothing)\pi(\beta_\infty^n(\mathcal{B}))U^n\hat{\pi}(e, \varnothing) = \hat{\pi}(e, \varnothing)\hat{\pi}(b)\hat{\pi}(e, \varnothing) = \hat{\pi}(e, \varnothing).
\]
Hence, \( \mathcal{J} \) must intersect \( \{ \hat{\pi}(e, \varnothing) : c \in \mathcal{C} \} \).

**Theorem 4.8.** [17 Theorem 4.6] The \( C^* \)-envelope of the semicrossed products \( \mathcal{A}(C, \alpha, \text{contr})_l \) and \( \mathcal{A}(C, \alpha, \text{is})_l \) is the full corner \( p(B_\infty \rtimes_{\beta_\infty} \mathbb{Z})p \) of the crossed product \( B_\infty \rtimes_{\beta_\infty} \mathbb{Z} \), where \( p = \hat{\pi}(e, \varnothing) \).

**Proof.** We have shown that \( \mathcal{A} = p(B_\infty \rtimes_{\beta_\infty} \mathbb{Z})p \) is a \( C^* \)-cover of the semicrossed products. Let \( \mathcal{J} \neq \varnothing \) be the Silov ideal. Since, \( \mathcal{J} \) is \( \beta_\infty \)-invariant, there is an \( 0 \neq x \in \mathcal{J} \cap \text{range } E' \). By the previous remarks, there is an \( 0 \neq y \in \mathcal{C} \), with \( \hat{\pi}(y, \varnothing) \in \mathcal{J} \). Hence, the ideal \( \langle \hat{\pi}(y, \varnothing) \rangle \) is a boundary ideal for the semicrossed products. Thus,
\[
\|y\| = \|\hat{\pi}(y, \varnothing)\| = \|\hat{\pi}(y, \varnothing) + \langle \hat{\pi}(y, \varnothing) \| = 0,
\]
which is a contradiction. \( \square \)

5. An overview

In this section we gather some useful remarks concerning the semicrossed products we have defined. We present them just for the semicrossed products that satisfy the left covariance relation. Of course one can get the analogues for the right case.

By propositions [3,7] and [11] we have that \( \mathcal{A}(C, \alpha, \text{contr})_l \simeq \mathcal{A}(C, \alpha, \text{is})_l \) and \( \mathcal{A}(C, \alpha, \text{co-is})_l \simeq \mathcal{A}(C, \alpha, \text{un})_l \), and by the universal property of \( \mathcal{A}(C, \alpha, \text{contr})_l \), the identity map \( \ell^1(\mathbb{Z}_+, C, \alpha)_l \to \ell^1(\mathbb{Z}_+, C, \alpha)_l \) extends to a u.c.c. homomorphism of \( \mathcal{A}(C, \alpha, \text{contr})_l \) onto \( \mathcal{A}(C, \alpha, \text{co-is})_l \) (since
every covariant co-isometric pair is a covariant contractive pair). But there are cases where the semicrossed products are not completely isometrically isomorphic. Indeed, consider the dynamical system \((\mathcal{C}, \alpha)\) of example 1.1. Then the semicrossed products \(\mathfrak{A}(\mathcal{C}, \alpha, \text{co-is})_1\) and \(\mathfrak{A}(\mathcal{C}, \alpha, \text{un})_1\) are completely isometrically isomorphic to the disc algebra \(A(\mathbb{D})\). On the other hand, by theorem 4.3, the semicrossed products \(\mathfrak{A}(\mathcal{C}, \alpha, \text{contr})_1\) and \(\mathfrak{A}(\mathcal{C}, \alpha, \text{is})_1\) contain a copy of \(\mathcal{C}\). Since \(A(\mathbb{D})\) does not contain a copy of \(C(\mathbb{R}_+ \cup \{\infty\})\) (the only \(C^*\)-algebra that lives in \(A(\mathbb{D})\) is \(\mathbb{C}\), we have that \(\mathfrak{A}(\mathcal{C}, \alpha, \text{is})_1\) and \(\mathfrak{A}(\mathcal{C}, \alpha, \text{co-is})_1\) cannot be completely isometrically isomorphic. The following proposition shows that this happens because \(\alpha\) is not injective.

**Proposition 5.1.** There is a u.c.is.is. \(\Phi : \mathfrak{A}(\mathcal{C}, \alpha, \text{contr})_1 \to \mathfrak{A}(\mathcal{C}, \alpha, \text{co-is})_1\), such that \(\Phi\) fixes \(\mathcal{C}\) pointwise if, and only if, \(\alpha\) is injective. The same holds for the right case.

**Proof.** Let \(\Phi : \mathfrak{A}(\mathcal{C}, \alpha, \text{contr})_1 \to \mathfrak{A}(\mathcal{C}, \alpha, \text{co-is})_1\) be such that \(\Phi(\delta_0 \otimes c) = \delta_0 \otimes c\). Then for \(c \in \mathcal{R}_\alpha\) we get

\[
0 = \|\delta_0 \otimes c\|_{\mathfrak{A}(\mathcal{C}, \alpha, \text{co-is})_1} = \|\Phi(\delta_0 \otimes c)\|_{\mathfrak{A}(\mathcal{C}, \alpha, \text{co-is})_1} = \|\delta_0 \otimes c\|_{\mathfrak{A}(\mathcal{C}, \alpha, \text{contr})_1} = \|c\|_{\mathcal{C}}.
\]

Hence, \(\mathcal{R}_\alpha = (0)\), so \(\alpha\) is injective.

For the converse, assume that \(\alpha : \mathcal{C} \to \mathcal{C}\) is injective. Then the ideal \(\mathcal{R}_\alpha\) is trivial and the \(C^*\)-algebra \(\mathcal{B}\) of section 4 is exactly \(\mathcal{C}\). Hence, theorems 3.8 and 4.3 give that the mapping \(\delta_n \otimes c \mapsto \delta_n \otimes c \in \mathcal{C}_\infty \rtimes_{\alpha_\infty} \mathbb{Z}\) extends to a completely isometric homomorphism from \(\mathfrak{A}(\mathcal{C}, \alpha, \text{contr})_1\) onto \(\mathfrak{A}(\mathcal{C}, \alpha, \text{co-is})_1\).

**Remark 5.2.** By the previous proposition the mapping \(\delta_n \otimes c \mapsto \delta_n \otimes c\) defines a completely isometric isomorphism between all pairs of semicrossed products if, and only if, \(\alpha\) is injective. In this case, they all share the same \(C^*\)-envelope, the crossed product \(\mathcal{C}_\infty \rtimes_{\alpha_\infty} \mathbb{Z}\).

Finally, we note that when \(\alpha : \mathcal{C} \to \mathcal{C}\) is a \(*\)-automorphism, then \(\mathcal{C}_\infty = \mathcal{C}\) and \(\alpha_\infty = \alpha\), thus the \(C^*\)-envelope of the semicrossed products is the crossed product \(\mathcal{C} \rtimes_\alpha \mathbb{Z}\) (see [15, Theorem 1.5]).

**Remark 5.3.** Theorem 4.6 and theorem 4.8 (which is a case of [17, Theorem 4.6]) generalize [23, Theorem 4]. In [23] Peters studies the semicrossed products \(\mathfrak{A}(\mathcal{C}, \alpha, \text{is})_1\) and \(\mathfrak{A}(\mathcal{C}, \alpha, \text{is})_r\), where \(\mathcal{C}\) is a commutative \(C^*\)-algebra \(C(X)\) and \(\alpha(f) = f \circ \phi\), with \(\phi : X \to X\) surjective, i.e. \(\alpha\) is injective. In this case, one can construct an extension \((\tilde{X}, \tilde{\phi})\) of \((X, \phi)\), where \(\tilde{\phi}\) is an homeomorphism. In a few words, let \(\prod_{n \in \mathbb{Z}_+} X\) with its usual topology and define the subset

\[
\tilde{X} = \{(x_n) \in \prod_{n \in \mathbb{Z}_+} X : \phi(x_{n+1}) = x_n\},
\]

which is a compact Hausdorff space. Consider the continuous map

\[
\tilde{\phi} : \tilde{X} \to \tilde{X} : (x_n) \mapsto \tilde{\phi}((x_n)) = (\phi(x_1), (x_n)).
\]

We can see that \(\tilde{\phi}\) is a homeomorphism of \(\tilde{X}\) and that the natural projection \(p : \tilde{X} \to X\) given by \(p((x_n)) = x_1\) is an open, surjective continuous map. Moreover, it intertwines \(\phi\) and \(\tilde{\phi}\), i.e. \(\phi \circ p = p \circ \tilde{\phi}\). Then by [23, Theorem 4] the \(C^*\)-envelope of \(\mathfrak{A}(\mathcal{C}, \alpha, \text{is})_1\) and \(\mathfrak{A}(\mathcal{C}, \alpha, \text{is})_r\) is \(C(\tilde{X}) \rtimes_{\tilde{\phi}} \mathbb{Z}\). By [23, Corollary 4] the dynamical system \((\tilde{X}, \tilde{\phi})\) is conjugate to the one induced by \((C(X)_\infty, \alpha_\infty\)). Hence, the \(C^*\)-envelope \(C(\tilde{X}) \rtimes_{\tilde{\phi}} \mathbb{Z}\) is \(*\)-isomorphic to the crossed product \((C(X))_\infty \rtimes_{\alpha_\infty} \mathbb{Z}\).
6. Minimality

In this section, we indicate why we believe that the C*-envelope of a semicrossed product is an appropriate candidate for the C*-algebra generated by a dynamical system. The reason is that properties of \((\mathcal{C}, \alpha)\) pass naturally to the dynamical systems \((\mathcal{B}, \beta)\) and \((\mathcal{B}_\infty, \beta_\infty)\) and vice versa. To give an example, we prove a result that connects a notion of minimality to a notion of simplicity. There are a number of similar results in the literature. For example, Davidson and Roydor have proved that, for commutative multivariable dynamical systems, minimality is equivalent to the simplicity of the C*-envelope of the tensor product (see [10, Definition 5.1, Proposition 5.4]). Also, there are well known criteria that give equivalence of simplicity of a crossed product to properties of a dynamical system by a *-automorphism (for example see [2, 18, 25] and/or [26]). Fortunately (!), none of these results fits in our case. For example [10, Proposition 5.4] is proved for dynamical systems where at least two *-endomorphisms participate. Moreover, in our context we deal with dynamical systems \((\mathcal{C}, \alpha)\) where \(\alpha\) is just a *-homomorphism (or *-injective, at most). On the other hand in most of the cases the C*-envelope of a semicrossed product is a crossed product of a larger dynamical system and our intention is to show how results for crossed products can be used for semicrossed products with a little effort.

Let us fix the notion of minimality that will be used throughout this section. Recall that \((\mathcal{C}, \alpha)\) is unital.

**Definition 6.1.** A dynamical system \((\mathcal{C}, \alpha)\) is called minimal if there are no non-trivial ideals \(\mathcal{I}\) of \(\mathcal{C}\) that are \(\alpha\)-invariant. Moreover, when \(\alpha\) is a *-automorphism, it is called bi-minimal if there are no non-trivial \(\alpha\)-bi-invariant ideals \(\mathcal{I}\) of \(\mathcal{C}\), i.e. that satisfy \(\alpha(\mathcal{I}) = \mathcal{I}\).

**Remark 6.2.** When \(\alpha\) is a *-automorphism, then \((\mathcal{C}, \alpha)\) is minimal if and only if it is bi-minimal. Indeed, let \(\mathcal{I}\) such that \(\alpha(\mathcal{I}) \subseteq \mathcal{I}\), and define \(\mathcal{I} = \bigcup_n \alpha^{-n}(\mathcal{I})\). Then \(\mathcal{I}\) is a \(\alpha\)-bi-invariant ideal, hence if \((\mathcal{C}, \alpha)\) is bi-minimal, then \(e_C \in \mathcal{I}\). Thus, for \(\epsilon > 0\), there is \(n_0\) and \(x \in \mathcal{C}\) such that \(\|e_C - \alpha^{-n_0}(x)\| < \epsilon\). Since \(\alpha\) is isometric and unital we get
\[
\|e_C - x\| = \|\alpha^{-n_0}(e_C - x)\| = \|e_C - \alpha^{-n_0}(x)\| < \epsilon.
\]
Since \(\epsilon > 0\) was arbitrary and \(x \in \mathcal{C}\) we get that \(e_C \in \mathcal{C}\), hence \(\mathcal{J} = \mathcal{C}\). Thus \((\mathcal{C}, \alpha)\) is minimal. The converse is trivial.

This is not true for non-unital cases. For example, let \(\mathcal{C} = C_0(\mathbb{R})\) and \(\alpha(f) = f \circ \phi\) where \(\phi(t) = t + 1\) for \(t \in \mathbb{R}\). Then \((C_0(\mathbb{R}), \alpha)\) is bi-minimal but any \(C_0([r, +\infty))\), for \(r \in \mathbb{R}\), is an \(\alpha\)-invariant ideal.

We use the constructions of section 4. Note that, when \(\alpha\) is not injective then \(\mathcal{B}\) is not minimal, since the tail \(T\) is a \(\beta\)-invariant ideal of \(\mathcal{B}\), with \((\mathcal{B}, \beta)\) as defined in section 4.

**Proposition 6.3.** The dynamical system \((\mathcal{C}, \alpha)\) is minimal if, and only if, \((\mathcal{B}, \beta)\) is minimal if, and only, \((\mathcal{B}_\infty, \beta_\infty)\) is minimal if, and only, \((\mathcal{B}_\infty, \beta_\infty)\) is bi-minimal. In this case we have that \(\alpha\) is injective and \((\mathcal{C}, \alpha) = (\mathcal{B}, \beta)\).

**Proof.** If \((\mathcal{C}, \alpha)\) is minimal, then \(\alpha\) is injective, otherwise \(\ker \alpha\) would be an \(\alpha\)-invariant ideal. Hence, \(T = 0\) which gives that \((\mathcal{B}, \beta) = (\mathcal{C}, \alpha)\), thus \((\mathcal{B}, \beta)\) is minimal.

Conversely, assume that \((\mathcal{B}, \beta)\) is minimal. Then \(T = 0\), since \(T\) is \(\beta\)-invariant and cannot be equal to \(\mathcal{B}\). Thus \((\mathcal{B}, \beta) = (\mathcal{C}, \alpha)\), hence \((\mathcal{C}, \alpha)\) is minimal. So \(\alpha\) is injective, otherwise \(\mathcal{M}(\ker \alpha) \neq (0)\) which leads to the contradiction \(T \neq (0)\).
Assume that \((\mathcal{C}, \alpha)\) is minimal (hence \(\alpha\) is injective), and let \(\mathcal{J} \neq (0)\) be a \(\alpha_{\infty}\)-invariant ideal in \(\mathcal{C}_{\infty}\). Then \(\mathcal{J}\) has non-trivial intersection with \(\mathcal{C}\). Indeed, assume that \(\mathcal{J} \cap \mathcal{C} = (0)\) and let \(0 \neq c \in \mathcal{J} \cap \mathcal{C}_{n_0}\) (if there is not such an \(n_0\) then \(\mathcal{J} = (0)\)). Then
\[
\alpha_{\infty}^{n_0}(c) \in \alpha_{\infty}^{n_0}(\mathcal{J}) \cap \alpha_{\infty}^{n_0}(\mathcal{C}_{n_0}) \subseteq \mathcal{J} \cap \mathcal{C} = (0).
\]
Hence, \(c \in \ker \alpha_{\infty}^{n_0}\), which leads to \(\ker \alpha_{\infty} \neq (0)\), a contradiction. Now, let \(I = \mathcal{J} \cap \mathcal{C} \neq (0)\) which is an ideal in \(\mathcal{C}\); then
\[
\alpha(I) = \alpha_{\infty}(I) \subseteq \alpha_{\infty}(\mathcal{J}) \cap \alpha_{\infty}(\mathcal{C}) \subseteq \mathcal{J} \cap \mathcal{C} = I,
\]
hence, \(I\) is a non-zero \(\alpha\)-invariant ideal of \(\mathcal{C}\). Thus \(I = \mathcal{C}\), hence \(e_{\mathcal{C}_{\infty}} = e_c \in I \subseteq \mathcal{J}\). So \(\mathcal{J} = \mathcal{C}_{\infty}\).

To end the proof, assume that \((\mathcal{B}_{\infty}, \beta_{\infty})\) is minimal, and let \(J \neq (0)\) be a \(\beta\)-invariant ideal in \(\mathcal{B}\). Let \(\mathcal{J}_{\infty}\) be the \(\mathcal{C}^{*}\)-subalgebra of \(\mathcal{B}_{\infty}\) defined by the direct system
\[
J \xrightarrow{\beta} J \xrightarrow{\beta} J \xrightarrow{\beta} \ldots.
\]
This is well defined because \(\beta\) restricts to an injective \(*\)-endomorphism of \(J\). It is easy to see that \(\mathcal{J}_{\infty}\) is a non-zero ideal in \(\mathcal{B}_{\infty}\) and, moreover, that is \(\beta_{\infty}\)-invariant. Hence \(\mathcal{J}_{\infty} = \mathcal{B}_{\infty}\). Then \(J = \mathcal{J}_{\infty} \cap \mathcal{B} = \mathcal{B}\). \(\square\)

Before we proceed to the first main theorem of this section, let us briefly discuss the Fourier transform of a crossed product. Let \((\mathcal{C}, \alpha)\) be a dynamical system such that \(\alpha : \mathcal{C} \to \mathcal{C}\) is a \(*\)-automorphism. If \(E : \mathcal{C} \rtimes_{\alpha, \mathbb{Z}} \mathbb{Z} \to \mathcal{C} \rtimes_{\alpha, \mathbb{Z}} \mathbb{Z}^\beta \equiv \mathcal{C}\) is the conditional expectation of the crossed product, we define the Fourier co-efficients
\[
E_n : \mathcal{C} \rtimes_{\alpha, \mathbb{Z}} \mathbb{Z} \to \mathcal{C} \rtimes_{\alpha, \mathbb{Z}} \mathbb{Z}^\beta \equiv \mathcal{C} : F \mapsto E_n(F) := E(U^{-n}F), n \in \mathbb{Z}.
\]
A Féjer-type Lemma shows that the Cesàro means of the Fourier monomials \(U^nE_n(F)\) converge to \(F\) in norm.

If \(\mathcal{I}\) is a non-zero ideal of \(\mathcal{C} \rtimes_{\alpha, \mathbb{Z}} \mathbb{Z}\), then \(E_n(\mathcal{I})\) is a non-zero \(\alpha\)-bi-invariant ideal in \(\mathcal{C}\). For example, for \(c \in \mathcal{C}\) and \(F \in \mathcal{I}\) we get
\[
c \cdot E_n(F) = c \cdot E(U^{-n}F) = \int_{\mathbb{T}} c \beta_z(U^{-n}F)dz = \int_{\mathbb{T}} \beta_z(cU^{-n}F)dz
\]
\[
= \int_{\mathbb{T}} \beta_z(U^{-n}\alpha^{-n}(c)F)dz = E_n(\alpha^{-n}(c)F) \in E_n(\mathcal{I}).
\]
In a similar way obtain \(E_n(F)c = E_n(Fc) \in E_n(\mathcal{I})\). Also,
\[
\alpha(E_n(F)) = UF_E_n(F)U^* = UE(U^{-n}F)U^* = \int_{\mathbb{T}} U\beta_z(U^{-n}F)U^*dz
\]
\[
= \int_{\mathbb{T}} \beta_z(U^{-n}FU^*)dz = \int_{\mathbb{T}} \beta_z(U^{-n+1}FU^*)dz \in E_n(\mathcal{I}),
\]
and similarly \(\alpha^{-1}(E_n(F)) = U^*E_n(F)U \in E_n(\mathcal{I})\).

Finally, if \(E_n(\mathcal{I}) = (0)\) then \(E(U^{-n}F^*F) = E_n(F^*F) = 0\), for every \(0 \neq F \in \mathcal{I}\). But \(E\) is faithful, hence \(U^{-n}F^*F = 0\), thus \(F^*F = 0\) since \(U\) is unitary. So \(F = 0\), which leads to the contradiction that \(\mathcal{I} = (0)\).

**Definition 6.4.** Let \((\mathcal{C}, \alpha)\) be a dynamical system such that \(\alpha : \mathcal{C} \to \mathcal{C}\) is a \(*\)-automorphism. An ideal \(\mathcal{I}\) of \(\mathcal{C} \rtimes_{\alpha} \mathbb{Z}\) is called **Fourier-invariant** if \(E_n(\mathcal{I}) \subseteq \mathcal{I}\) for all \(n \in \mathbb{Z}\).
Note that this is equivalent to saying that the Fourier monomials \( U^n E_n(F) \) are in \( \mathcal{I} \), for every \( F \in \mathcal{I} \) and \( n \in \mathbb{Z} \).

**Theorem 6.5.** The following are equivalent:

1. \( (\mathcal{C}, \alpha) \) is minimal,
2. \( (\mathcal{B}, \beta) \) is minimal,
3. \( (\mathcal{B}_\infty, \beta_\infty) \) is bi-minimal,
4. the crossed product \( \mathcal{B}_\infty \rtimes_{\beta_\infty} \mathbb{Z} \) has no non-trivial Fourier-invariant ideals.

If any of the previous conditions holds, then \( \alpha \) is injective and \( \mathcal{C} = \mathcal{B} \). Moreover, the semicrossed products we have defined with respect to the collections \( \mathcal{F}_{i,t} \), for \( t = 1, 2, 3, 4 \), are completely isometrically isomorphic to one another and share the same \( C^* \)-envelope \( \mathcal{C}_\infty \rtimes_{\alpha_\infty} \mathbb{Z} \).

**Proof.** It suffices to prove that \((3) \Leftrightarrow (4)\). This is a standard result for crossed products. To give a short proof, assume that the crossed product \( \mathcal{B}_\infty \rtimes_{\beta_\infty} \mathbb{Z} \) has no non-trivial Fourier-invariant ideals and let \( \mathcal{J} \) be a non-trivial \( \alpha_\infty \)-bi-invariant ideal of \( \mathcal{C}_\infty \equiv \mathcal{B}_\infty \). Then, the crossed product \( \mathcal{J} \rtimes_{\alpha_\infty} \mathbb{Z} \) is a non-trivial Fourier-invariant ideal in the crossed product. Indeed, it is a Fourier invariant ideal by definition and it cannot be the zero ideal (otherwise \( \mathcal{J} = (0) \)). Hence, \( \mathcal{J} \rtimes_{\alpha_\infty} \mathbb{Z} = \mathcal{C}_\infty \rtimes_{\alpha_\infty} \mathbb{Z} \). By using gauge action we get that the corresponding fixed point algebras must be equal, thus \( \mathcal{J} = \mathcal{C}_\infty \).

For the converse, assume that \( (\mathcal{C}_\infty, \alpha_\infty) \) is bi-minimal and let \( \mathcal{I} \) be a non-trivial Fourier-invariant ideal in the crossed product \( \mathcal{C}_\infty \rtimes_{\alpha_\infty} \mathbb{Z} \). Then every \( E_n(\mathcal{I}) \) is a non-zero \( \alpha_\infty \)-bi-invariant ideal in \( \mathcal{C}_\infty \). Hence \( E_n(\mathcal{I}) = \mathcal{C}_\infty \) for every \( n \in \mathbb{Z} \). By Césaro summability we obtain \( \mathcal{I} = \mathcal{C}_\infty \rtimes_{\alpha_\infty} \mathbb{Z} \). \( \square \)

**Remark 6.6.** By what we have proved it is trivial to see that simplicity of the crossed product implies minimality of the dynamical system. But there is no hope of proving the converse for the general case. As Davidson and Roydor point out in [10] Remark 5.12 if we consider the minimal dynamical system \( (\mathcal{C}, \alpha) \) where \( \mathcal{C} = C(\{x\}) \) and \( \alpha = \text{id} \), then \( \mathcal{B}_\infty \rtimes_{\beta_\infty} \mathbb{Z} \simeq \mathcal{C} \rtimes_{\text{id}} \mathbb{Z} \simeq C(\mathbb{T}) \) which is not simple. To go even further, assume that we are given the dynamical system \( (\mathcal{C}, \text{id}) \), where \( \mathcal{C} \) is simple (thus the dynamical system is bi-minimal). Then \( \mathcal{C} \rtimes_{\text{id}} \mathbb{Z} \) is isomorphic to \( \mathcal{C} \otimes C(\mathbb{T}) \). Hence, given a non-trivial ideal \( \mathcal{J} \subset C(\mathbb{T}) \) we have that \( \mathcal{C} \otimes \mathcal{J} \) is a non-trivial ideal in the crossed product.

On the other hand, there are well known dynamical systems with \(*\)-automorphisms, such that the crossed product they produce is a simple \( C^* \)-algebra. An example is \( A_\theta \) with \( \theta \in \mathbb{R} \setminus \mathbb{Q} \) (see [7] Theorem VIII.3.9).

Let us drop to the case of commutative dynamical systems. So, let \( X \) be a compact Hausdorff space and \( \phi : X \to X \) a continuous map. For simplicity we write \( (X, \phi) \) instead of \( (\mathcal{C}, \alpha) \), where \( \alpha(f) := f \circ \phi \).

Assume that \( \phi \) is not surjective, i.e. \( \alpha \) is not injective, then we can add the tail we produced in section [4] We use notation as in [10]. Define \( \mathcal{U} = X \setminus \phi(X) \), \( T = \{(u, k) : u \in \mathcal{U}, k < 0\} \) and \( X_T = X \sqcup T \). Then the continuous mapping \( \phi^T : X_T \to X_T \) with \( \phi^T|_X = \phi \) and

\[
\phi^T(u, k) = (u, k + 1), \text{ for } k < -1, \quad \text{and } \phi^T(u, -1) = u
\]

is surjective. Moreover, \( (X_T, \phi^T) \) is the dynamical system \( (\mathcal{B}, \beta) \) we construct in section [4] for \( (C(X), \alpha) \).
When \( \phi \) is surjective, we can use the projective limit of Peters described in remark [\ref{remark:peters-projective}](#remark:peters-projective) and get the dynamical system \( (\tilde{X}, \tilde{\phi}) \).

**Theorem 6.7.** Let \((X, \phi)\) be a dynamical system, where \( X \) is a compact Hausdorff space. Minimality implies surjectivity and the following are equivalent:

1. \((X, \phi)\) is minimal and \( X \) is infinite,
2. \((\tilde{X}, \tilde{\phi})\) is minimal and \( \tilde{X} \) is infinite,
3. the crossed product \( C(\tilde{X}) \rtimes \tilde{\phi} \mathbb{Z} \) is simple.

If any of the previous conditions hold, the semicrossed products we have defined with respect to the collections \( \mathcal{F}_{t,l} \), for \( t = 1, 2, 3, 4 \), are completely isometrically isomorphic to one another and share the same C*-envelope \( C(\tilde{X}) \rtimes \tilde{\phi} \mathbb{Z} \).

**Proof.** Note that \( X \) is infinite if and only if \( \tilde{X} \) is infinite. So, theorem 6.5 gives the equivalence \([1) \Leftrightarrow (2)\). Also \([2) \Rightarrow (3)\] is [\cite{[1]}](#footnote)](#footnote) Theorem VIII.3.9. To finish the proof, assume that \( C(\tilde{X}) \rtimes \tilde{\phi} \mathbb{Z} \) is simple. Then by [\cite{[2]}](#footnote) Corollary \((\tilde{X}, \tilde{\phi})\) is minimal and topologically free, i.e. for every \( 0 \neq k \in \mathbb{Z} \) the open set \( \{ x \in \tilde{X} : \tilde{\phi}^k(x) \neq x \} \) is dense in \( \tilde{X} \) (see [\cite{[2]}](#footnote) Definition 1] and the remarks following it). If \( \tilde{X} \) were finite, then by the pigeonhole principle there is at least one periodic point \( x_0 \in \tilde{X} \), say with period \( m \). Then \( x_0 \notin \{ x \in \tilde{X} : \tilde{\phi}^m(x) \neq x \} \), and since \( \{ x \in \tilde{X} : \tilde{\phi}^m(x) \neq x \} \) is dense in \( \tilde{X} \), thus nonempty, we can assume that \( \{ x \in \tilde{X} : \tilde{\phi}^m(x) \neq x \} = \{ x_1, \ldots , x_t \} \). Since \( X \) is Hausdorff there is an open neighborhood \( U \) of \( x_0 \) such that none of \( x_1, \ldots , x_t \) is in \( U \). Hence, \( \{ x \in \tilde{X} : \tilde{\phi}^m(x) \neq x \} \) is not dense in \( \tilde{X} \), which is a contradiction. Thus \( \tilde{X} \) is infinite and the proof is complete. \( \square \)

**Remark 6.8.** Simplicity of the C*-envelope induces semi-simplicity of the semicrossed product (see [\cite{[23]}](#footnote) Proposition 3]), but the converse is false. For example, assume \( X = \mathbb{T} \) and \( \phi : \mathbb{T} \to \mathbb{T} \) be rotation by a rational angle \( \theta \). It is obvious that \( \phi \) is surjective, thus the semicrossed products are c.is.is. Every point in \( \mathbb{T} \) is recurrent, hence by [\cite{[11]}](#footnote) Theorem 10] the semicrossed products are semisimple. But \( \mathcal{A}_\theta \) is not simple.

Theorem 6.7 provides the identification of all ideals of any C*-cover of the semicrossed products. Recall that if \( \mathcal{C} \) is a C*-cover of an operator algebra \( \mathcal{A} \), then an ideal \( \mathcal{I} \triangleleft \mathcal{C} \) is called boundary if \( \| [a_{ij}] \| = \| [a_{ij}] + \mathcal{I} \| \) for any \( a_{ij} \in \mathcal{A} \).

**Corollary 6.9.** Let \((X, \phi)\) be a dynamical system, where \( X \) is a compact Hausdorff space. If any of \( (1) - (3) \) of theorem 6.7 holds, then the semicrossed products are u.c.is.is, and if \( \mathcal{I} \) is an ideal in a C*-cover \( \mathcal{C} \) of the semicrossed product, then it is boundary.

**Proof.** Let \( \mathcal{I} \triangleleft \mathcal{C} \) and the *-epimorphism \( \Phi : \mathcal{C} \to C(\tilde{X}) \rtimes \tilde{\phi} \mathbb{Z} \). Since the C*-envelope is simple, then \( \Phi(\mathcal{I}) = (0) \), hence \( \mathcal{I} \subseteq \ker \Phi \); thus it is a boundary ideal for the semicrossed product, since \( \ker \Phi \) is the Šilov ideal, i.e. the biggest boundary ideal. \( \square \)

**Question.** Note that for the proof of theorem 6.7 we have used [\cite{[2]}](#footnote) Corollary], which holds for commutative dynamical systems. For non-commutative and automorphic dynamical systems there exist criterias that lead to simplicity of the crossed product. Hence, the natural question that is raised here is if there is an analogue of theorem 6.7 for non-commutative dynamical systems \( (\mathcal{C}, \alpha) \) at least when the spectrum \( \tilde{\mathcal{C}} \) is Hausdorff. The first step to that direction would be to answer the following question.
Q. Let \((\mathcal{C}, \alpha)\) be a unital dynamical system such that the spectrum \(\widehat{\mathcal{C}}\) is a Hausdorff space. Then \((\mathcal{C}, \alpha)\) is minimal and topologically free if, and only if, \((\mathcal{C}_\infty, \alpha_\infty)\) is minimal and topologically free.

An arbitrary dynamical system \((\mathcal{C}, \alpha)\) is called topological free if for any \(n_1, \ldots, n_k \in \mathbb{Z}_+ \setminus \{0\}\), \(\cap_{i=1}^k \{x \in \widehat{\mathcal{C}} : x \circ \alpha^{n_i} \neq x\}\) is dense in \(\widehat{\mathcal{C}}\). When \(\widehat{\mathcal{C}}\) is Hausdorff this is equivalent to saying that \(\{x \in \widehat{\mathcal{C}} : x \circ \alpha^n = x\}\) has empty interior for any \(n \geq 1\) (by the remarks following [2 Proposition 1]). In the case where \(\alpha\) is a \(*\)-automorphism, this is equivalent to the usual topological freeness [2 Definition 1], since \(\{x \in \widehat{\mathcal{C}} : x \circ \alpha^n \neq x\}\) = \(\{x \in \widehat{\mathcal{C}} : x \circ \alpha^{-n} \neq x\}\) for any \(n \geq 1\).

Acknowledgement. I wish to give my sincere thanks to E. Katsoulis for his helpful remarks and advices. I also wish to thank A. Katavolos for his kind help and advice during the preparation of this paper. Finally, I wish to thank Judy Vs. for the support and inspiration.

References

[1] M. Alaimia and J. Peters, 'Semicrossed products generated by two commuting automorphisms' J. Math. Anal. Appl. 285 (2003), 128–140.
[2] R.J. Archbold, J.S. Spielberg, 'Topologically free actions and ideals in discrete C*-dynamical systems', Proc. Edinb. Math. Soc. (1) 37 (1993), 119–124.
[3] W. Arveson, 'Operator algebras and measure preserving automorphisms', Acta Math. 118, (1967), 95–109.
[4] W. Arveson and K. Josephson, 'Operator algebras and measure preserving automorphisms II', J. Functional Analysis 4, (1969), 100–134.
[5] D. P. Blecher and C. Le Merdy, 'Operator algebras and their modules—an operator space approach', volume 30 of London Mathematical Society Monographs, New Series, The Clarendon Press Oxford University Press, Oxford.
[6] L. DeAlba and J. Peters, 'Classification of semicrossed products of finite-dimensional C*-algebras, Proc. Amer. Math. Soc. 95 (1985), 557–564.
[7] K. R. Davidson, 'C*-algebras by Example', volume 6 of Fields Institute Monographs, American Mathematical Society & Providence, 1996.
[8] K. R. Davidson, E. G. Katsoulis, 'Isomorphisms between topological conjugacy algebras', J. reine angew. Math. 621 (2008), 29-51.
[9] K. R. Davidson, E. G. Katsoulis, 'Nonself-adjoint crossed products and dynamical systems', Contemporary Mathematics, to appear.
[10] K. R. Davidson, J. Roydor, 'C*-envelopes of tensor algebras for multivariable dynamics', Proc. Edinb. Math. Soc. (2) 53 (2010), 333-351.
[11] A. Donsig, A. Katavolos, A. Manousos, 'The Jacobson radical for Analytic Crossed Products', J. Funct. Anal. 187 (2001), 129–145.
[12] R. Exel, 'A new look at the crossed-product of a C*-algebra by an endomorphism', Ergodic Theory Dynam. Systems 23(6) (2003), 1733–1750.
[13] D. Hadwin and T. Hoover, 'Operator algebras and the conjugacy of transformations', J. Funct. Anal. 77 (1988), 112–122.
[14] H. Hoare, W. Parry, 'Affine transformations with quasi-discrete spectrum. I', J. London Math. Soc. 41 (1966), 88–96.
[15] E. T.A. Kakariadis, 'Semicrossed Products and Reflexivity', Journal of Operator Theory, to appear.
[16] E. T.A. Kakariadis, E. G. Katsoulis, 'Semicrossed Products of Operator Algebras and their C*-envelopes', J. Funct. Anal., 262(7) (2012), p.p. 3108–3124.
[17] E. T.A. Kakariadis, E. G. Katsoulis, 'Contributions to the theory of C*-Correspondences with applications to multivariable dynamics', Trans. Amer. Math. Soc., to appear.
[18] S. Kawamura, J. Tomiyama, 'Properties of topological dynamical systems and corresponding C*-algebras', Tokyo J. Math. 13 (1990), 251–257.
[19] M. McAsey and P. Muhly, 'Representations of nonselfadjoint crossed products', Proc. London Math. Soc. 47(3) (1983), 128–144.
[20] P. S. Muhly, B. Solel, 'Extensions and dilations for C*-dynamical systems', Operator theory, operator algebras and applications, Contenp. Math. 414 (2006), p.p. 375–381.
[21] P. S. Muhly, M. Tomforde, 'Adding tails to C*-correspondences', Doc. Math., 9 (2004), p.p. 79–106.
[22] J. R. Peters, 'Semicrossed products of C*-algebras', J. Funct. Anal., 59(3) (1984), p.p. 498–534.
[23] J. R. Peters, 'The C*-envelope of a semicrossed product and Nest Representations', 2008, arXiv.org:0810.5364.
[24] S.C. Power, 'Classification of analytic crossed product algebras', Bull. London Math. Soc. 105 (1989), 368–372.
[25] J. Renault, 'The ideal structure of groupoid crossed product C*-algebras', J. Operator Theory 25 (1991), 3–36.
[26] A. Sierakowski, 'Discrete crossed product C*-algebras', Ph.D. thesis, University of Copenhagen (2009).
[27] P. J. Stacey, 'Crossed products of C*-algebras by *-endomorphisms', J. Austral. Math. Soc. Ser. A, 54 (1993), p.p. 204-212.
[28] D. P. Williams, 'Crossed products of C*-algebras', volume 134 of Mathematical Surveys and Monographs, American Mathematical Society.

PURE MATH. DEPT., U. WATERLOO, WATERLOO, ON N2L–3G1, CANADA
E-mail address: ekakaria@uwaterloo.ca