On the critical $p$-Kirchhoff equation*

Erisa Hasani and Kanishka Perera
Department of Mathematical Sciences
Florida Institute of Technology
Melbourne, FL 32901, USA
ehasani2016@my.fit.edu & kperera@fit.edu

Abstract

We study a nonlocal elliptic equation of $p$-Kirchhoff type involving the critical Sobolev exponent. First we give sufficient conditions for the (PS) condition to hold. Then we prove some existence and multiplicity results using tools from Morse theory, in particular, the notion of a cohomological local splitting and eigenvalues based on the Fadell-Rabinowitz cohomological index.

1 Introduction and statement of results

Nonlocal elliptic equations of $p$-Kirchhoff type involving critical Sobolev exponents have been recently studied in the literature (see, e.g., Hamydy et al. [7], Ourraoui [14], Zhou and Song [21], Li et al. [10], Li et al. [9], and the references therein). In this paper we study the existence and multiplicity of solutions to the critical $p$-Kirchhoff equation

\[
\begin{aligned}
- h\left(\int_\Omega |\nabla u|^p \, dx\right) \Delta_p u &= f(x, u) + |u|^{p^*-2} u \quad \text{in } \Omega \\
 u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

(1.1)

where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, $\Delta_p u = \text{div}(\nabla u |^{p-2} \nabla u)$ is the $p$-Laplace of $u$, $1 < p < N$, $h : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function, $p^* = Np/(N-p)$ is the critical Sobolev exponent, and $f$ is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying the subcritical growth condition

\[
|f(x, t)| \leq a_1 |t|^{q-1} + a_2 \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}
\]

(1.2)

for some constants $a_1, a_2 > 0$ and $1 < q < p^*$. A model case is

\[
h(t) = 1 + bt^{\gamma-1},
\]

(1.3)

*MSC2010: Primary 35J92, Secondary 35B33, 58E05

Key Words and Phrases: $p$-Kirchhoff equation, critical Sobolev exponent, existence, multiplicity, Morse theory, Fadell-Rabinowitz cohomological index, cohomological local splitting
where $b > 0$ and $\gamma > 1$.

Weak solutions of problem (1.1) coincide with critical points of the $C^1$-functional
\[
E(u) = \frac{1}{p} H\left(\int_\Omega |\nabla u|^p \, dx\right) - \int_\Omega F(x, u) \, dx - \frac{1}{p^*} \int_\Omega |u|^{p^*} \, dx, \quad u \in W^{1,p}_0(\Omega),
\]
where
\[
H(t) = \int_0^t h(s) \, ds, \quad F(x, t) = \int_0^t f(x, s) \, ds
\]
are the primitives of $h$ and $f$, respectively. As is typical with problems of critical growth, the main difficulty here is the lack of compactness. Recall that the functional $E$ satisfies the Palais-Smale compactness condition, or the (PS) condition for short, if every sequence $(u_j) \subset W^{1,p}_0(\Omega)$ such that $E(u_j)$ is bounded and $E'(u_j) \to 0$, called a (PS) sequence, has a convergent subsequence. First we give a sufficient condition for every bounded (PS) sequence to have a convergent subsequence. Let
\[
S_{N,p} = \inf_{u \in W^{1,p}_0(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^p \, dx}{\left(\int_\Omega |u|^{p^*} \, dx\right)^{p/p^*}}
\]
be the best Sobolev constant.

**Theorem 1.1.** If (1.2) holds and
\[
h(t) > S_{N,p}^{-p/p^*} t^{p^*/p-1} \quad \forall t > 0,
\]
then every bounded sequence $(u_j) \subset W^{1,p}_0(\Omega)$ such that $E'(u_j) \to 0$ has a convergent subsequence.

In the model case (1.3), the inequality (1.6) holds in each of the following cases:

(i) $\gamma = p^*/p$ and $b \geq S_{N,p}^{-p/p^*}$,

(ii) $\gamma > p^*/p$ and
\[
(p^*/p)^{p^*/p-1} > \frac{(\gamma - p^*/p)^{\gamma-p^*/p}}{(\gamma - 1)^{\gamma-1}} S_{N,p}^{-p^*/p}(\gamma-1).
\]

**Remark 1.2.** The case where $1 < \gamma < p^*/p$ can be handled using arguments similar to those used in [3], which only considered the semilinear case $p = 2$.

Next we give sufficient conditions for the existence of a solution to problem (1.1).

**Theorem 1.3.** If (1.2) and (1.6) hold, and
\[
H(t) \geq \frac{p}{p^*} S_{N,p}^{-p^*/p} t^{p^*/p} + a_3 t^r - a_4 \quad \forall t \geq 0
\]
for some constants $a_3, a_4 > 0$ and $r > q/p$, then problem (1.1) has a solution.
In particular, we have the following existence result for the model problem

\[
\begin{cases}
- \left[ 1 + b \left( \int_{\Omega} |\nabla u|^p \, dx \right)^{\gamma-1} \right] \Delta_p u = f(x, u) + |u|^{p^*-2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\] (1.8)

**Corollary 1.4.** If (1.2) holds, then problem (1.8) has a solution in each of the following cases:

(i) \( \gamma = p^*/p \), \( b = S_{N, p}^{-p^*/p} \), and \( q < p \),

(ii) \( \gamma = p^*/p \) and \( b > S_{N, p}^{-p^*/p} \),

(iii) \( \gamma > p^*/p \) and

\[
p^{-\gamma/p-1} > \frac{(\gamma - p^*/p)\gamma p^*/p (p^*/p - 1)p^*/p-1}{(\gamma - 1)^{\gamma-1}} S_{N, p}^{-p^*/p}(\gamma-1).
\]

Now we assume that

\[
\lim_{t \to 0} \frac{H(t)}{t} = 1 \tag{1.9}
\]

and

\[
\lim_{t \to 0} \frac{p F(x, t)}{|t|^p} = \lambda \text{ uniformly a.e. in } \Omega. \tag{1.10}
\]

Then problem (1.4) has the trivial solution \( u \equiv 0 \), and we seek nontrivial solutions. The location of \( \lambda \) with respect to the spectrum of the \( p \)-Laplacian will play an important role in our results. We recall that the spectrum \( \sigma(-\Delta_p) \) consists of those \( \lambda \in \mathbb{R} \) for which the eigenvalue problem

\[
\begin{cases}
- \Delta_p u = \lambda |u|^{p^*-2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\] (1.11)

has a nontrivial solution. The first eigenvalue

\[
\lambda_1 = \inf \sigma(-\Delta_p) \tag{1.12}
\]

is positive, simple, and has an associated eigenfunction \( \varphi_1 \) that is positive in \( \Omega \) (see Anane [2] and Lindqvist [11, 12]). Moreover, \( \lambda_1 \) is isolated in the spectrum, so the second eigenvalue

\[
\lambda_2 = \inf \sigma(-\Delta_p) \cap (\lambda_1, \infty) \tag{1.13}
\]

is well-defined (see Anane and Tsouli [1]). We have the following theorem.
Theorem 1.5. Assume (1.2), (1.6), (1.7), (1.9), and (1.10) with \( \lambda \notin \sigma(-\Delta_p) \).

(i) If \( \lambda > \lambda_1 \), then problem (1.1) has a nontrivial solution.

(ii) If \( \lambda > \lambda_2 \), then problem (1.1) has two nontrivial solutions.

In particular, we have the following corollary for the model problem (1.8).

Corollary 1.6. Assume (1.2) and (1.10) with \( \lambda \notin \sigma(-\Delta_p) \).

(i) If \( \lambda > \lambda_1 \), then problem (1.8) has a nontrivial solution in each of the three cases in Corollary 1.4.

(ii) If \( \lambda > \lambda_2 \), then problem (1.8) has two nontrivial solutions in each of the three cases in Corollary 1.4.

The proof of Theorem 1.5 uses tools from Morse theory, in particular, the notion of a cohomological local splitting (see Degiovanni et al. [5], Perera et al. [18], and Perera [16]) and eigenvalues based on the cohomological index (see Perera [17] and Perera and Szulkin [19]). We will recall these tools in the next section.

2 Preliminaries

Let \( E \) be a \( C^1 \)-functional defined on a Banach space \( W \). For \( a \in \mathbb{R} \), we denote by \( E^a \) the sublevel set \( \{ u \in W : E(u) \leq a \} \), and we denote by \( H \) the Alexander-Spanier cohomology with \( \mathbb{Z}_2 \) coefficients (see Spanier [20]). In Morse theory, the local behavior of \( E \) near a critical point \( u_0 \) is described by the sequence of critical groups

\[
C^q(E, u_0) = H^q(E^c, E^c \setminus \{ u_0 \}), \quad q \geq 0,
\]

where \( c = E(u_0) \) is the corresponding critical value (see Chang [4], Mawhin and Willem [13], and Perera et al. [18]).

Recall that \( E \) satisfies the Palais-Smale compactness condition at the level \( c \in \mathbb{R} \), or the \((PS)_c\) condition for short, if every sequence \( (u_j) \subset W \) such that \( E(u_j) \to c \) and \( E'(u_j) \to 0 \) has a convergent subsequence. The proof of Theorem 1.5 will make use of the following alternative proved in Perera [15] (see also Perera et al. [18, Proposition 3.28(ii)]).

Proposition 2.1. Assume that zero is a critical point of \( E \) with \( E(0) = 0 \) and \( C^k(E, 0) \neq 0 \) for some \( k \geq 1 \), and that there are regular values \( -\infty < a < 0 < b \leq +\infty \) of \( E \) with \( H^k(E^b, E^a) = 0 \) such that \( E \) has only a finite number of critical points in \( E^{-1}([a, b]) \) and \( E \) satisfies the \((PS)_c\) condition for all \( c \in [a, b] \cap \mathbb{R} \). Then \( E \) has a nontrivial critical point \( u_1 \) with either

\[
a < E(u_1) < 0 \quad \text{and} \quad C^{k-1}(E, u_1) \neq 0,
\]

or

\[
0 < E(u_1) < b \quad \text{and} \quad C^{k+1}(E, u_1) \neq 0.
\]
To obtain a nontrivial critical group at zero in the absence of a suitable direct sum decomposition, we will use a cohomological local splitting. For a symmetric set $A \subset W \setminus \{0\}$, let $\overline{A} = A/\mathbb{Z}_2$ be the quotient space of $A$ with each $u$ and $-u$ identified, let $f : \overline{A} \to \mathbb{R}P^\infty$ be the classifying map of $\overline{A}$, and let $f^* : H^*(\mathbb{R}P^\infty) \to H^*(\overline{A})$ be the induced homomorphism of the cohomology rings. The $\mathbb{Z}_2$-cohomological index of $A$ is defined by

$$i(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ \sup \{m \geq 1 : f^*(\omega^{m-1}) \neq 0\} & \text{if } A \neq \emptyset, \end{cases}$$

where $\omega \in H^1(\mathbb{R}P^\infty)$ is the generator of the polynomial ring $H^*(\mathbb{R}P^\infty) = \mathbb{Z}_2[\omega]$ (see Fadell and Rabinowitz [1]).

**Definition 2.2.** We say that $E$ has a cohomological local splitting near zero in dimension $0 \leq k < \infty$ if there are disjoint nonempty closed symmetric subsets $A_0$ and $B_0$ of the unit sphere $S = \{u \in W : \|u\| = 1\}$ with

$$i(A_0) = i(S \setminus B_0) = k \quad (2.1)$$

and $\rho > 0$ such that, setting $A = \{tu : u \in A_0, 0 \leq t \leq \rho\}$ and $B = \{tu : u \in B_0, 0 \leq t \leq \rho\}$, we have

$$\sup_A E \leq 0 \leq \inf_B E. \quad (2.2)$$

This definition was given, in an equivalent form, in Degiovanni et al. [5] and is a slight variant of Perera et al. [18, Definition 3.33], which in turn is a variant of the homological local linking of Perera [16]. The following proposition was proved in Degiovanni et al. [5] (see also Perera et al. [18, Proposition 3.34] and Perera [16]).

**Proposition 2.3.** If zero is an isolated critical point of $E$ and $E$ has a cohomological local splitting near zero in dimension $k$, then $C^k(E,0) \neq 0$.

To show that the functional $E$ in (1.4) has a cohomological local splitting near zero when $\lambda \notin \sigma(-\Delta_p)$, we will make use of a sequence of eigenvalues based on the cohomological index that was first introduced in Perera [17] (see also Perera and Szulkin [19]). Recall that eigenvalues of problem (1.11) coincide with critical values of the functional

$$\Psi(u) = \frac{1}{\int_\Omega |u|^p \, dx}, \quad u \in S = \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega |\nabla u|^p \, dx = 1 \right\}. \quad (2.3)$$

Denote by $\mathcal{F}$ the class of symmetric subsets of $S$, let

$$\mathcal{F}_k = \{ M \in \mathcal{F} : i(M) \geq k \},$$

and set

$$\lambda_k = \inf_{M \in \mathcal{F}_k} \sup_{u \in M} \Psi(u), \quad k \geq 1. \quad (2.3)$$
Then $\lambda_1$ and $\lambda_2$ agree with (1.12) and (1.13), respectively, and $(\lambda_k)$ is a nondecreasing and unbounded sequence of eigenvalues. Moreover, denoting by 

$$\Psi^a = \{u \in S : \Psi(u) \leq a\}, \quad \Psi_a = \{u \in S : \Psi(u) \geq a\}$$

the sublevel and superlevel sets of $\Psi$, respectively, we have

$$\lambda_k < \lambda_{k+1} \implies i(\Psi^{\lambda_k}) = i(S \setminus \Psi_{\lambda_{k+1}}) = k$$

(see Perera et al. [18, Theorem 4.6]). In the next section, we will make use of (2.4) to show that if $\lambda_k < \lambda < \lambda_{k+1}$, then $E$ has a cohomological local splitting near zero in dimension $k$.

3 Proofs

In this section we prove Theorem 1.1, Theorem 1.3, and Theorem 1.5.

Proof of Theorem 1.1. Since $(u_j)$ is bounded, for a renamed subsequence,

$$u_j \rightharpoonup u, \quad \|u_j - u\|^p \to t$$

for some $u \in W^{1,p}_0(\Omega)$ and $t \geq 0$. Since $E'(u_j) \to 0,$

$$h\left( \int_\Omega |\nabla u_j|^p \, dx \right) \int_\Omega |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla v \, dx - \int_\Omega f(x, u_j) v \, dx - \int_\Omega |u_j|^{p'} u_j v \, dx = o(\|v\|)$$

$$\forall v \in W^{1,p}_0(\Omega). \quad (3.2)$$

By the Brézis-Lieb lemma (see 3) and (3.1),

$$\int_\Omega |\nabla u_j|^p \, dx \to \int_\Omega |\nabla u|^p \, dx + t =: s.$$

For a further subsequence, $u_j \to u$ strongly in $L^2(\Omega)$ and a.e. in $\Omega$. So taking $v = u_j$ in (3.2) gives

$$h(s) \left( \int_\Omega |\nabla u|^p \, dx + t \right) - \int_\Omega u f(x, u) \, dx - \int_\Omega |u_j|^{p'} \, dx = o(1), \quad (3.3)$$

while taking $v = u$ and passing to the limit gives

$$h(s) \int_\Omega |\nabla u|^p \, dx - \int_\Omega u f(x, u) \, dx - \int_\Omega |u|^{p'} \, dx = 0. \quad (3.4)$$

Since

$$\int_\Omega |u_j|^{p'} \, dx - \int_\Omega |u|^{p'} \, dx = \int_\Omega |u_j - u|^{p'} \, dx + o(1)$$

by the Brézis-Lieb lemma, subtracting (3.4) from (3.3) and using (1.3) gives

$$th(s) = \int_\Omega |u_j - u|^{p'} \, dx + o(1) \leq S_{N, p}^{-p'/p} \left( \int_\Omega |\nabla (u_j - u)|^p \, dx \right)^{p'/p} + o(1).$$

Noting that $h(s) \geq h(t)$ since $s \geq t$ and $h$ is nondecreasing, and passing to the limit gives $th(t) \leq S_{N, p}^{-p'/p} t^{p'/p},$ which together with (1.6) implies that $t = 0$. So $u_j \to u.$

□
Proof of Theorem 1.3. The inequality (1.7) together with (1.2) and (1.5) gives
\[
E(u) \geq a_5 \left( \int_{\Omega} |\nabla u|^p \, dx \right)^r - a_6 \int_{\Omega} |u|^q \, dx - a_7
\]
for some constants \(a_5, a_6, a_7 > 0\). Since \(q < p^*\) and \(r > q/p\), this together with the Sobolev embedding implies that \(E\) is bounded from below and coercive. Coercivity implies that every (PS) sequence is bounded, so \(E\) satisfies the (PS) condition by Theorem 1.1. So \(E\) has a global minimizer.

Proof of Theorem 1.5. As in the proof of Theorem 1.3, \(E\) is bounded from below and has a global minimizer \(u_0\). We may assume without loss of generality that \(E\) has only a finite number of critical points and hence all critical points of \(E\) are isolated. Then the critical groups of \(E\) at \(u_0\) are given by
\[
C^q(E, u_0) = \begin{cases} 
\mathbb{Z}_2 & \text{if } q = 0 \\
0 & \text{otherwise.}
\end{cases} \tag{3.5}
\]

Next we show that \(E\) has a cohomological local splitting near zero when \(\lambda > \lambda_1\). Let \((\lambda_k)\) be the sequence of eigenvalues in (2.3). Since \(\lambda \notin \sigma(-\Delta_p)\), we have \(\lambda_k < \lambda < \lambda_{k+1}\) for some \(k \geq 1\). By (2.4),
\[
i(\Psi^{\lambda_k}) = i(S \setminus \Psi^{\lambda_{k+1}}) = k.
\]
Let
\[
A_0 = \Psi^{\lambda_k}, \quad B_0 = \Psi^{\lambda_{k+1}}.
\]
Then \(A_0\) and \(B_0\) are disjoint nonempty closed symmetric subsets of \(S\) satisfying (2.4). Fix \(\varepsilon > 0\) so small that \(\lambda - \varepsilon > \lambda_k\) and \(\lambda + \varepsilon < \lambda_{k+1}\). By (1.10) and (1.2), \(\exists a_\varepsilon > 0\) such that
\[
\left| F(x, t) - \frac{\lambda}{p} |t|^p \right| \leq \frac{\varepsilon}{p} |t|^p + a_\varepsilon |t|^{p^*} \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}. \tag{3.6}
\]
For \(u \in S\) and \(t \geq 0\),
\[
H\left( \int_{\Omega} |\nabla (tu)|^p \, dx \right) = H(t^p) = t^p + o(t^p) \text{ as } t \to 0
\]
by (1.9), and
\[
\left| \int_{\Omega} F(x, tu) \, dx - \frac{\lambda t^p}{p} \int_{\Omega} |u|^p \, dx \right| \leq \frac{\varepsilon t^p}{p} \int_{\Omega} |u|^p \, dx + o(t^p) \text{ as } t \to 0
\]
by (3.6) and the Sobolev embedding, so
\[
\frac{t^p}{p} \left( 1 - \frac{\lambda + \varepsilon}{\Psi(u)} \right) + o(t^p) \leq E(tu) \leq \frac{t^p}{p} \left( 1 - \frac{\lambda - \varepsilon}{\Psi(u)} \right) + o(t^p) \text{ as } t \to 0.
\]
In particular,
\[ E(tu) \leq -\frac{tp}{p} \left( \frac{\lambda - \varepsilon}{\lambda_k} - 1 \right) + o(tp) \] (3.7)
for \( u \in A_0 \), and
\[ E(tu) \geq \frac{tp}{p} \left( 1 - \frac{\lambda + \varepsilon}{\lambda_{k+1}} \right) + o(tp) \]
for \( u \in B_0 \), so (2.2) holds for sufficiently small \( \rho > 0 \). So \( E \) has a cohomological local splitting near zero in dimension \( k \).

(i) By (3.7), \( E(tu) < 0 \) for \( u \in A_0 \) and \( t > 0 \) sufficiently small, so \( E(u_0) = \inf E < 0 \) and hence \( u_0 \) is nontrivial.

(ii) Since \( E \) has a cohomological local splitting near zero in dimension \( k \), \( C^k(E,0) \neq 0 \) by Proposition 2.3. For \( a < \inf E \) and \( b = +\infty \),
\[ H^k(E^b, E^a) = H^k(W^{1,p}_0(\Omega)) = 0 \]
since \( k \geq 1 \), so \( E \) has a nontrivial critical point \( u_1 \) with either \( E(u_1) < 0 \) and \( C^{k-1}(E, u_1) \neq 0 \), or \( E(u_1) > 0 \) and \( C^{k+1}(E, u_1) \neq 0 \) by Proposition 2.1. Since \( \lambda > \lambda_2 \), \( k \geq 2 \) and hence \( C^{k-1}(E, u_0) = 0 = C^{k+1}(E, u_0) \) by (3.3), so \( u_1 \) is distinct from \( u_0 \).

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