Radial drift in warped protoplanetary disks

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ABSTRACT

The meter-size barrier in protoplanetary disks is a major challenge in planet formation, for which many solutions were suggested. One of the leading solutions is dust traps, that halt or slow the inward migration of dust particles. The source and profile of these traps are still not completely known. Warped disks are ubiquitous among accretion disks in general and protoplanetary disks in particular, and the warping could lead naturally to the formation of dust traps. Dust traps in warped disks could rise not only from pressure gradients, but also due to different precession rates between gas and dust. Here we derive analytically the radial drift in warped disks, and demonstrate derivation for some specific conditions. The radial drift in warped protoplanetary disks is qualitatively different, and depending on the structure of the disk, dust traps could form due to the warping. Similar processes could lead to the formation of traps also in other accretion disks such as AGN disks.

1. INTRODUCTION

Protoplanetary disks do not form and evolve in isolation. Most stars are formed in groups or stellar clusters (Lada & Lada 2003), in which many stars are born and evolve. Planets form in disks of gas and dust surrounding these stars, such that the environment of planet formation is far from being quiescent, and the interactions with the environment could play a crucial role in shaping the conditions and properties of planet formation. Protoplanetary disks could be affected by flybys, binarity of the host star and magnetic interactions with the protostar (e.g. Foucart & Lai 2011; Lubow & Martin 2018; Nealon et al. 2018; Kraus et al. 2020). All these interactions leave signatures on the disk, such as disk warping and even breaking and extreme cases. And indeed, there is governing observational evidence for distorted protoplanetary disks (e.g. Benisty et al. 2015, 2018; Bi et al. 2020), that indicates the importance of modeling planet formation in them. However, the majority of planet formation theories consider flat, isolated disks.

The growth of small dust grains to full-sized planets is assumed to be hierarchical, i.e. from dust grains to pebbles, planetesimals and so on, and includes several orders of magnitude. The initial stages, up to cm size, as well as the final stages, from km-size to planetesimals, are well explained by the current planet formation theories – by sticking (Wurm & Blum 1998) and pebble accretion (Ormel & Klahr 2010; Perets & Murray-Clay 2011; Lambrechts & Johansen 2012), correspondingly. However, meter-size objects have to overcome several barriers in order to enhance their growth, and although many solutions were suggested to this problem, it is still an open question and a subject of ongoing research (see review Morbidelli & Raymond 2016). One of the major barriers in planet formation is the radial-drift barrier, which describes the rapid migration of objects toward the star, due to the pressure gradient in the disk, in shorter timescales than the expected growth timescales (Weidenschilling 1977). Further barriers include collisional fragmentation, erosion (e.g. Güttler et al. 2010) and aeolian-erosion (e.g. Demirci et al. 2020; Rozner et al. 2020; Grishin et al. 2020). There are several suggested solutions to the meter-size barriers, we will list some of the major ones.

Streaming-instability is a promising suggested solution to overcome the meter-size barriers (Youdin & Goodman 2005), in which the local concentration of solids in the disk is catalyzed to a point where gravitational collapse could take place and give rise to the formation of larger objects. Another suggested solution is the seeding of already-formed objects, including km-size objects. In this approach, low-probability events of planet formation are sufficient to explain the total number of exoplanets, due to an external transport of planet-seeds, such that together with streaming-instability for example, the whole population of planets could be explained (Grishin et al. 2019). Another suggested solution is dust traps. The first dust traps to be
suggested were pressure traps, in which local pressure maxima in the disk induce zero pressure gradient, which suppresses the usual radial drift and leads to a dust pile-up (Nakagawa et al. 1986; Whipple 1972). Several mechanisms were suggested as sources for these traps, including vortices (e.g. Barge & Sommeria 1995), planet gap edges (e.g. Pinilla et al. 2012) and self-induced (Gonzalez et al. 2017). Recently, it was suggested that a dust trap could be formed not only by pressure gradient, but also due to precession difference Aly & Lodato (2020); Longarini et al. (2021). In distorted disks, the gas and dust precess differently, such that dust rings are formed, and there are locations in which the relative velocity between gas and dust becomes zero and the radial drift is suppressed.

In this paper, we extend and generalize the analytical study of dust traps in warped disks and the radial drift there in general. We derive the modified equations of motion for dust and gas particles in warped disks, and present the steady-state solutions in a closed analytic form under some conditions. By that, we generalize the standard radial drift expression used in flat disks, and give a complementary perspective to the hydrodynamical works on warped disks mentioned earlier. Our derivation is general and should apply to any warped disk, regardless of the warping source. We manifest the derivation for typical parameters of warped circum-binary disks and discuss implications for these disks and other.

The usual radial drift equations are derived usually based on the assumptions of a flat, axisymmetric disk (e.g. Nakagawa et al. 1986), here we relieve some of the initial assumptions to describe analytically the radial drift in warped disks and discuss the implications of the modified radial drift. Due to the asymmetry of warped disks, dust traps could form not only due to the existence of pressure gradients but also due to the structure of the disk, which will lead to another kind of dust traps, distinguished from the pressure traps by its nature.

In section 2 we describe the structure of a warped disk. In section 3 we describe the coupled evolution of gas and dust and the unique behaviour in warped disks. In section we present the modifications for the standard radial drift in the case of circumbinary warped disks. In section 5 we discuss the radial drift in several additional examples of warped disks and discuss further implications. In section 6 we discuss the caveats of our model. In section 7 we summarize the paper and conclude.

2. THE STRUCTURE OF A WARPED DISK

Consider a disk that extends from $r_{\text{in}}$ to $r_{\text{out}}$. The disk density profile and thickness are given by

\[ \Sigma_g(r) \propto r^{-p}, \quad \frac{H}{r} \propto r^{(2p-1)/4} \]  

where we use $p = 3/2$ following Armitage (2011). The orientation of the disk is specified by the normalized angular momentum vector $\hat{l}(r)$,

\[ \hat{l}(r) = (\sin \beta \cos \gamma, \sin \beta \sin \gamma, \cos \beta) \]  

where $\beta(r)$ is the warp angle and $\gamma(r)$ is the twist angle. At $r = r_{\text{in}}, r_{\text{out}}$, the angular momentum directions are $\hat{l}_{\text{in}}, \hat{l}_{\text{out}}$ correspondingly. A disk is defined as warped if $\beta(r)$ varies with the radius.

There are two regimes of warsps propagation: the diffusive regime ($\alpha > h/r$) and the wave-like regime ($\alpha < h/r$) in which the waves propagate as bending waves and pressure forces drive the evolution. $\alpha$ is the Shakura Sunayev parameter (Shakura & Sunyaev 1973) and $h/r$ is the aspect ratio of the gas. Warped protoplanetary disks are described by the wave-like regime (e.g. Papaloizou & Lin (1995)): Lubow & Ogilvie (2000) and references therein), while AGN disks for example are in the diffusive regime (e.g. Papaloizou et al. 1998; Lodato & Price 2010). The steepness of the warp could be described by the warp amplitude dimensionless radial change in the angular momentum, $\psi := r|\partial \hat{l}/\partial r|$. When $\psi/(h/r) \gg 1$, the bending waves become non-linear.

3. GAS & DUST EVOLUTION

Objects in protoplanetary disks experience gas drag, which depends on the size of the object, its velocity relative to the gas and the gas properties. The overall gas drag in the different regimes could be written as (e.g. Perets & Murray-Clay 2011)

\[ \mathbf{F}_D = -\frac{1}{2}C_D(Re)\pi R^2 \rho_g v_{\text{rel}}^2 \mathbf{\hat{v}}_{\text{rel}} \]  

such that $R$ is the size of the object, $\rho_g$ is the gas density, $v_{\text{rel}}$ is the relative velocity between the object and the gas and $C_D(Re)$ is a function that depends on the Reynolds number $Re = 2Rv_{\text{rel}}/(0.5\rho v_{\text{th}}\lambda)$. $v_{\text{th}} = \sqrt{8/\pi c_s}$ is the thermal velocity, $c_s$ is the sound speed and $\lambda$ is the mean free path of the gas. For small objects, the force scales linearly with the relative velocity, i.e. $\mathbf{F}_D \propto -v_{\text{rel}}$, while for large objects the force scales quadratically, i.e. $\mathbf{F}_D \propto -v_{\text{rel}}^2 \mathbf{\hat{v}}_{\text{rel}}$.

The dynamics of dust is affected by the interaction with the gas. While small dust grains are well-coupled to the gas, larger grains acquire relative velocity. The coupling to the gas is encapsulated by the Stokes number, defined by
where we consider rotation of the velocity in a flat disk by angle \( \gamma \). The stopping time of the gas such that the drag force on the dust is given by \( \eta = \frac{3\bar{v}_h\lambda}{2\rho_d R^2} \)

\[
St = \Omega_K t_{\text{stop}}, \quad t_{\text{stop}} = \frac{|mv_{\text{rel}}|}{|F_D|} \tag{4}
\]

where \( \Omega_K \) is the Keplerian orbital frequency, \( t_{\text{stop}} \) is the stopping time, \( v_{\text{rel}} \) is the relative velocity and \( F_D \) is the drag force. The dependence of \( t_{\text{stop}} \) on the size of the grain, \( R \), varies with the corresponding regime, i.e. Epstein \((R \lesssim \lambda_{\text{mfp}})\), Stokes \((R \gtrsim \lambda_{\text{mfp}}, \ Re < 1 \) where \( Re \) is the Reynolds number) or ram-pressure \((Re \gtrsim 800) \) where \( \lambda_{\text{mfp}} \) is the mean free path of the gas. For the Epstein and Stokes regimes (e.g. Perets & Murray-Clay 2011),

\[
t_{\text{stop}} = \begin{cases} 
\frac{\rho_g R}{\rho_p v_{\text{th}}}, & \text{Epstein,} \\
\frac{4\rho_g}{9\rho_p} \frac{R^2}{\lambda_{\text{mfp}} v_{\text{th}}}, & \text{Stokes}
\end{cases} \tag{5}
\]

where \( \rho_g \) is the gas density, \( \rho_p \) is the object’s density and \( v_{\text{th}} = \sqrt{8k_B T/(\pi \mu m_H)} \) is the thermal velocity, \( T \) is the temperature of the gas and \( \mu m_H \) is the mean molecular weight.

In this paper, we focus on the regime \( R \gtrsim \lambda \), in which the drag force on the dust is given by \( \eta = \frac{3\bar{v}_h\lambda}{2\rho_d R^2} \).

1986 and references therein),

\[
\mathbf{F}_D = -A\rho_d (\mathbf{v}_d - \mathbf{v}_g), \tag{6}
\]

where similar equations could be written for the gas, replacing \( \rho_d \) with \( \rho_g \). \( R \) is the size of the object. Unless stated otherwise, the fiducial parameters we consider are \( \rho_g = 3 \times 10^{-9} (a/a_0)^{-16/7} \) g cm\(^{-3}\), \( \lambda = 1 (a/a_0)^{16/7} \) AU, \( \bar{v}_{\text{th}} = \sqrt{8k_B T/\mu m_H \pi} \), \( T = 120 (a/a_0)^{-3/7} \) K, \( \mu m_H = 3.9 \times 10^{-24} \) g, \( \rho_g = 1 \) g cm\(^{-3}\) (Nakagawa et al. 1986; Perets & Murray-Clay 2011 and references therein), where \( a_0 \) is a characteristic length-scale, and we focus on meter-size objects, i.e. \( R = 1 \) m. The dust-to-gas ratio is taken to be 1\% (Chiang & Youdin 2010).

The steady-state radial drift of the dust, when effects of backreaction are neglected, is given by (e.g. Nakagawa et al. 1986; Birnstiel et al. 2016)

\[
\frac{dr}{dt} = v_r = -\frac{\rho_g}{\rho_g + \rho_d} \frac{2D\Omega_K}{\eta_v K}, \tag{8}
\]

\[
\eta = -\frac{1}{2} \frac{\partial P_g}{\partial r} v_K \Omega_K \tag{9}
\]

where \( v_K \) is the Keplerian velocity, \( \Omega_K \) is the Keplerian angular velocity, \( \mathbf{D} = A(\rho_g + \rho_d) \), \( P_g \) is the gas pressure and \( \eta \) is the gas pressure support parameter.

The gas and dust velocity fields could be thought as a rotation of the velocity fields in a flat Cartesian disk (see also Longarini et al. (2021)).

\[
\mathbf{u}_d = \left( \begin{array}{c}
u_{d,r}' (\cos \phi \cos \beta \cos \gamma_d - \sin \gamma_d \sin \phi) - u_{d,\phi}' (\sin \phi \cos \beta \cos \gamma_d + \cos \phi \sin \gamma_d) \\
u_{d,r}' (\cos \phi \sin \gamma_d \cos \beta + \cos \gamma_d \sin \phi) + u_{d,\phi}' (\cos \phi \cos \gamma_d - \sin \phi \sin \gamma_d \cos \beta) \\
-u_{d,r}' \sin \beta \cos \phi + u_{d,\phi}' \sin \phi \sin \beta
\end{array} \right), \tag{10}
\]

\[
\mathbf{u}_g = \left( \begin{array}{c}
u_{g,r}' (\cos \phi \cos \beta \cos \gamma_g - \sin \gamma_g \sin \phi) - u_{g,\phi}' (\sin \phi \cos \beta \cos \gamma_g + \cos \phi \sin \gamma_g) \\
u_{g,r}' (\cos \phi \sin \gamma_g \cos \beta + \cos \gamma_g \sin \phi) + u_{g,\phi}' (\cos \phi \cos \gamma_g - \sin \phi \sin \gamma_g \cos \beta) \\
-u_{g,r}' \sin \beta \cos \phi + u_{g,\phi}' \sin \phi \sin \beta
\end{array} \right) \tag{11}
\]

where we consider \( \beta = \beta_g = \beta_d \), \( \Omega_g \) is the rigid precession of the gas such that \( \gamma_g = \Omega_g t \) and \( \Omega_{\text{ext}} \) is the dust precession, \( \gamma_d = \Omega_{\text{ext}} t \) and \( \phi \) is the polar angle along the disk. The velocity field in a warped disk is then a rotation of the velocity in a flat disk by angle \( \beta \) around the y-axis and \( \gamma_i \) around the z-axis. The velocity components in the flat coordinate system are given by \( u_{i,j}' \), where subindex of \( \text{d} \) relates to dust components, \( g \) to gas components and \( r \) and \( \phi \) to the spatial component in a cylindrical coordinate system. For simplicity, we
could form not only due to pressure maxima (disk. density is given by also due to precession (C)
its profile differ significantly from the usual radial drift of parameters, the radial drift won’t vanish completely, and \( \gamma \):= \( \frac{G\phi_r}{D[1 + (G_r + G_\phi)D - 0.5G_\phi\Omega_K] + \Omega_K^2} \) (16)
where \( A \) is the gas drag coefficient and these equations could be solved substituting the decomposition introduced in eq. 10, see Appendix A for further details of the derivation. The radial drift in a flat disk (eq. 8) is then modified, and a full analytical solution could be derived. In the co-precession radius, in which the dust ring is formed, the modification of the radial drift could be given by

\[
v_{dr} = -\frac{\rho_g}{\rho_d + \rho_g} \frac{2D\Omega_K}{D^2 + \Omega_K^2} \eta v_K \xi(\gamma, \beta),
\]

(15)

\( G_\phi, G_r, C_r, \) and \( C_\phi \) are given explicitly in Appendix A and \( \gamma := \gamma_d = \gamma_g \), such that for \( G_\phi = 1, G_r = 0, C_r = 1 \) and \( C_\phi = 0 \) we retrieve the usual radial drift for a flat disk.

As can be seen from eq. 15, in warped disks a dust trap could form not only due to pressure maxima (\( \eta = 0 \)) but also due to precession (\( \xi = 0 \)). Even if for some choices of parameters, the radial drift won’t vanish completely, its profile differ significantly from the usual radial drift considered in flat disks.

The precession frequencies are determined by the structure of the disk and the external torques applied on the disk. Given an external torque with a corresponding external precession frequency \( \Omega_{ext} \), the external torque density is given by \( T = \Omega_p \times L \), where \( L = \Sigma r^2\Omega_l \) is the angular momentum of the disk per unit area. \( \Sigma \) is the surface density, \( \Omega \) is the angular frequency and \( I(r) \) is a unit vector in the local direction of the angular momentum. The global precession frequency of the disk \( \Omega_p \) (rigid precession), is defined by \( T_{tot} = \Omega_p L_{tot} \), where for convenience we set \( \hat{z} := \Omega_{ext} \), which enables us to move to a non-vector equation. We will follow briefly the derivation in Lodato & Facchini (2013). For small warps, the overall torque and angular momentum could be calculated by

\[
T_{tot} \approx \int_{r_{in}}^{r_{out}} \Omega_{ext}(r)L(|r)|2\pi r dr,
\]

(17)

\[
L_{tot} \approx \int_{r_{in}}^{r_{out}} L(|r)|2\pi r dr
\]

(18)
such that \( \Omega_p \) is given by

\[
\Omega_p = \frac{\int_{r_{in}}^{r_{out}} \Omega_{ext}(r)L(|r)|2\pi r dr}{\int_{r_{in}}^{r_{out}} L(|r)|2\pi r dr}
\]

(19)

Assuming \( \Omega_{ext}(r) \propto r^{-s} \) and \( L(|r)| \propto \Sigma \Omega \sqrt{r^2 l^2 + l_y^2} \propto r^{1/2-p} \), the expression could be simplified for \( 0 < p < 5/2 \)

\[
\Omega_p = \Omega_{ext}(r_{in}) \frac{\int_{r_{in}}^{r_{out}/r_{in}} x^{3/2-p-s} dx}{\int_{1}^{r_{out}/r_{in}} x^{3/2-p-s} dx} = \frac{\Omega_{ext}(r_{in})}{(r_{out}/r_{in})^{5/2-p-s}} \frac{5/2 - p}{s + p - 5/2}
\]

(20)

4. CIRCUMBINARY/CIRCUMTRIPLE DISKS

Radial drift in circumbinary disks is even faster than in single-star disks (Zagaria et al. 2021). In general, there were set severe constraints on the possibility of planet formation in circumbinary disks (e.g. Thébault et al. 2006 and references therein). The lifetime of a circumbinary disk is shorter due to the massive depletion of solids, which leaves a narrower available parameter space for planets’ growth. However, there are many observations of planets around binary systems (e.g. Marzari & Thebault 2019). These together give greater importance to understanding dust traps in binary disks, either pressure traps (Nakagawa et al. 1986; Whipple 1972) or traffic jams (Aly & Lodato 2020; Aly et al. 2021; Longarini et al. 2021).

Consider a circumbinary containing companions with masses \( m_1 \) and \( m_2 \) and a separation of \( a_{bin} \). The binary imposes a torque per unit area on the disk element at a given radius \( r \), averaged over a binary orbital period and the disk azimuthal direction, is specified by (Foucart & Lai 2013)
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\[ \mathbf{T}_{\text{bin}}(r) = -\frac{3Gm_1m_2\Sigma_0a_{\text{bin}}^2}{4(m_1 + m_2)r^3}(\hat{\mathbf{l}}_b \cdot \hat{\mathbf{l}})(\hat{\mathbf{l}}_b \times \hat{\mathbf{l}}) \]  

(21)

where \( \hat{\mathbf{l}}_b \) is the direction of the angular momentum of the binary. Then,

\[ \Omega_{\text{ext}}(r) = \frac{3Gm_1m_2a_{\text{bin}}^2}{4(m_1 + m_2)r^4}\Omega_{\text{bin}}(r) \propto r^{-7/2}, \]

(22)

\[ \Omega_{\text{bin}} = \frac{G(m_1 + m_2)}{r^3} \]

(23)

and following eq. 20,

\[ \Omega_{\text{p}} = \frac{2}{5}\Omega_{\text{ext}}(r_{\text{in}}) \frac{1 - (r_{\text{out}}/r_{\text{in}})^{-5/2}}{(r_{\text{out}}/r_{\text{in}}) - 1} \]  

(24)

As discussed in Aly & Lodato (2020); Aly et al. (2021); Longarini et al. (2021), dust traps are expected to form in circumbinary disks and here we extended their model analytically.

Consider a circumbinary disk with \( m_1 = m_2 = 1 M_\odot \), and following Longarini et al. (2021), \( a_{\text{bin}} = 10 \text{ AU} \), \( \beta = \pi/6 \), \( R_{\text{in}} = 15 \text{ AU} \) and \( R_{\text{out}} = 150 \text{ AU} \).

**Figure 1.** The differential precession frequency of the dusty disk \( \Omega_{\text{ext}} \) and the rigid precession frequency of the gaseous disk \( \Omega_{\text{p}} \) for \( m_1 = m_2 = 1 M_\odot \), \( a_{\text{bin}} = 10 \text{ AU} \), \( R_{\text{in}} = 15 \text{ AU} \) and \( R_{\text{out}} = 150 \text{ AU} \).

In the following, we will discuss radial drift in several examples of warped disks, distorted disks in general and implications.

### 5.1. Distant stellar companion

Consider a central star with mass \( M_1 \), surrounded by a circumstellar disk. This system is in orbit with a distant binary companion of mass \( M_2 \), with a separation \( a_{\text{bin}} \). The binary companion exerts a perturbing torque on the disk. Averaging over the orbital period of the disk annulus and the binary, the torque per unit mass is given by (Zanazzi & Lai 2018)

**Figure 2.** The correction to the standard radial drift in a warped circumbinary disk with \( m_1 = m_2 = 1 M_\odot \), \( a_{\text{bin}} = 10 \text{ AU} \), \( R_{\text{in}} = 15 \text{ AU} \) and \( R_{\text{out}} = 150 \text{ AU} \), for different tilt angles \( \beta \), calculated in the co-precession radius.

As can be seen, while for some tilt angles the radial drift for objects smaller than meter-size is larger than the standard radial drift derived in Weidenschilling (1977), for some other angles the correction due to warping could lead to a significantly suppressed radial drift and even radial drift in the opposite direction. Our results could explain also the enhanced radial drift found in the other numerical simulations of circumbinary disks (e.g. Zagaria et al. 2021). As can be seen, the radial drift is essentially different in warped disks, whether it is suppressed or enhanced, and hence the initial stages of planet formation in these disks.

Recently, a misaligned circumtriple protoplanetary disk was found (Bi et al. 2020; Smallwood et al. 2021). The triple structure is more complicated and gives rise potentially to warps and breaks in the disk. Hence, enhanced formation of dust traps is expected also in circumtriple disks.
\[ T_{\text{ext}} = -r^2 \Omega \omega_{db}(\hat{b} \cdot \hat{l})(\hat{l} \times \hat{l}), \]  
\[ \Omega(r) \approx \sqrt{\frac{GM_\star}{r^3}}, \quad \omega_{db}(r) = \frac{3GM_\star}{4a_{\text{bin}}^3\Omega} \]  
(25)  
(26)

For this case, \( \Omega_{\text{ext}} = \omega_{db} \) and \( s = 3/2 \) in the notation of the previous subsection. Hence, the rigid precession is given by

\[ \Omega_p = 2\Omega_{\text{ext}}(r_{\text{in}}) \frac{1 - \sqrt{r_{\text{in}}/r_{\text{out}}}}{(r_{\text{out}}/r_{\text{in}}) - 1} \]  
(27)

The dust ring will then form where \( \Omega_{\text{ext}} = \Omega_p \),

\[ r_{\text{dr}} = 2^{2/3}r_{\text{in}} \left( \frac{1 - \sqrt{r_{\text{in}}/r_{\text{out}}}}{(r_{\text{out}}/r_{\text{in}}) - 1} \right)^{2/3} \]  
(28)

Here, for some choices of parameters, the dust ring radius could be smaller than the inner radius, indicating that there would be no formation of a dust ring. However, the radial drift profile will still change according to the equations of motion of the gas and dust we introduced.

5.2. Eccentric disks

Eccentric disks are another class of distorted disks, and as such, the radial drift in them might be modified as well. The eccentricity of the disk could be perturbed again by an external torque or planet embedded in the disk. In principle, the radial drift on these disks could be derived similarly to the derivation we introduced in Appendix A, but with different coefficients for the equation.

5.3. Flybys

While binary interactions could maintain long-living significant warps, flybys could lead only to more moderate effects (Nealon et al. 2020). The effect of flybys on planet formation is expected to be more limited, since the warp-damping timescale in this case could be shorter than the typical growth timescale of a planet. Even in this case, pebbles might be able to grow to a large enough size to reduce the effect of the radial drift. Moreover, several flybys could have a cumulative effect on the planet formation in the disk.

5.4. Symbiotic relations with other processes in the disk

Perturbations in the disk such as warps also affect other growth/destruction processes in planet formation apart from the formation of pressure bumps. Once the dust-to-gas ratio in the vicinity of the trap is large enough, a runaway process will start and the growth will be enhanced even more, due to the formation of a dust trap (Gonzalez et al. 2017). As we discussed in subsection 3, the relative velocity between the dust and the gas changes in the vicinity of the warp, as a result of the change in the gas surface density. Consequently, aeolian-erosion (e.g. Rozner et al. 2020) would be suppressed, although it might still function if the disk is turbulent. Moreover, streaming-instability (Youdin & Goodman 2005) could be affected by the different structures of the disk and the new redistribution of solids in the vicinity of the warp and in further areas that are affected as well. Breaking the disk into several small disks might limit the size abundance and by that increase the efficiency of streaming-instability (Martin & Lubow 2022). Misalignments could lead also to a change in the relative velocities between objects in the disk, such that destructive collisions and collisional growth might be affected as well. Furthermore, the size distribution and mass segregation of objects in the disk could be modified due to the warping.

6. CAVEATS

Here we will briefly discuss potential caveats of our model:

- The formation of dust traps could lead to a backreaction in the dust evolution/scale-height, which in turn will strengthen the trapping of dust particles and the efficiency of the planet formation and one should take into account the combined evolution of the backreaction and the formation of dust ring due to the perturbation (see discussion on these topics in Gonzalez et al. 2017; Longarini et al. 2021).

- The warping could modify also the pressure gradient, which will add another correction to the radial drift. This is not taken into account in our current derivation.

- We neglected the vertical motion of the objects, but in general the warping will change the behavior in this direction as well.

7. SUMMARY

In this paper, we discussed the radial drift in warped disks. We derived analytically the expression for dust radial drift in these disks and demonstrated for some examples of warped disks. Radial drift in warped disks is essentially different from the one in flat disk and might
have important consequences on the nature of planet formation in these disks. Not only that warped disks could give rise to dust traps in which the radial drift halts, but they also modify the radial drift outside the traps.

While most of the studies of dust traps focus on pressure maxima, in distorted disks in general and warped disks especially, there could be another type of dust traps, which rises from the different precession rates of the gas and the dust. These traps could interact also with other growth and destruction mechanisms of planets, and should be taken into account as an integral part of the planet formation theory in distorted disks. The characteristics of the perturbed profile are derived from the origin of the perturbation, which could be either transient (e.g. for flybys) or long-lasting (e.g. circumbinary disks).

The traditional planet formation research focused on coplanar single star disks, but there is a wealth of distorted disks, which should be included in planet formation theories, especially due to the piling-up evidence of these disks. Not only the radial drift changes, but also further steps in the evolution of planets and planetesimals, and the physical picture is far from being complete.

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APPENDIX

A. RADIAL DRIFT IN A WARPED DISK – FULL DERIVATION

Here we derive the radial drift for a warped disk, generalizing the derivation for a flat disk in Nakagawa et al. (1986). The equations of motion for the dust and gas particles are given by (e.g. Nakagawa et al. 1986)

\[
\frac{du_d}{dt} = -A \rho_d (u_d - u_g) - \frac{GM_d}{r_d^3} r_d,
\]

(A1)

\[
\frac{du_g}{dt} = -A \rho_g (u_d - u_g) - \frac{GM_g}{r_g^3} r_g - \nabla P_g
\]

(A2)

where \( A \) is the gas drag coefficient and from now parameters related to the dust will be noted with subindex \( d \) and parameters related to the gas with subindex \( g \). To obtain the gas and dust velocities in a warped disk, we start from these velocities in a flat disk, and then rotate it by an angle \( \beta = \beta_g = \beta_d \) around the \( y \)-axis and \( \gamma \) around the \( z \) axis, where \( \gamma_g = \Omega_g t \) for the gas and \( \gamma_d = \Omega_{\text{ext}} t \) for the dust. To generalize the derivation specified in Nakagawa et al. (1986) for a warped disk, we will introduce the tilt and twist angle \( \beta \) and \( \gamma_i \) correspondingly and rotate the general flat velocity fields, when \( u'_i \) refer to the velocity components in the flat system.

\[
\mathbf{u}_d = \begin{pmatrix}
  u'_{d,r} (\cos \phi \cos \beta \cos \gamma_d - \sin \gamma_d \sin \phi) - u'_{d,\phi} (\sin \phi \cos \beta \cos \gamma_d + \cos \phi \sin \gamma_d) \\
  u'_{d,r} (\cos \phi \sin \gamma_d \cos \beta + \cos \gamma_d \sin \phi) + u'_{d,\phi} (\cos \phi \cos \gamma_d - \sin \phi \sin \gamma_d \cos \beta) \\
  -u'_{d,r} \sin \beta \cos \phi + u'_{d,\phi} \sin \phi \sin \beta
\end{pmatrix},
\]

(A3)

\[
\mathbf{u}_g = \begin{pmatrix}
  u'_{g,r} (\cos \phi \cos \gamma_g - \sin \gamma_g \sin \phi) - u'_{g,\phi} (\sin \phi \cos \gamma_g + \cos \phi \sin \gamma_g) \\
  u'_{g,r} (\cos \phi \sin \gamma_g \cos \beta + \cos \gamma_g \sin \phi) + u'_{g,\phi} (\cos \phi \cos \gamma_g - \sin \phi \sin \gamma_g \cos \beta) \\
  -u'_{g,r} \sin \beta \cos \phi + u'_{g,\phi} \sin \phi \sin \beta
\end{pmatrix},
\]

(A4)

The equations of motion in the flat system of coordinates are then given by, when we neglect the vertical dynamics, i.e. the \( z \) components of the equations and define \( \mathbf{v}_d = \mathbf{u}_d - \mathbf{v}_{\text{Kep}}, \mathbf{v}_g = \mathbf{u}_g - \mathbf{v}_{\text{Kep}} \) are given by

\[
\frac{d_r \mathbf{r}}{dt} \cdot \mathbf{\dot{r}}' = \begin{bmatrix}
  -A \rho_g (\mathbf{v}_d - \mathbf{v}_g) - \frac{GM_d}{r_d^3} r_d - \frac{\partial \mathbf{v}_{\text{Kep}}}{\partial t} \\
  -A \rho_g (\mathbf{v}_g - \mathbf{v}_d) - \frac{GM_g}{r_g^3} r_g - \frac{\partial \mathbf{v}_{\text{Kep}}}{\partial t}
\end{bmatrix} \cdot \mathbf{\dot{r}}',
\]

(A6)

\[
\frac{d_\phi \mathbf{\phi}}{dt} \cdot \mathbf{\dot{\phi}}' = \begin{bmatrix}
  -A \rho_g (\mathbf{v}_d - \mathbf{v}_g) - \frac{GM_d}{r_d^3} r_d - \frac{\partial \mathbf{v}_{\text{Kep}}}{\partial t} \\
  -A \rho_g (\mathbf{v}_g - \mathbf{v}_d) - \frac{GM_g}{r_g^3} r_g - \frac{\partial \mathbf{v}_{\text{Kep}}}{\partial t}
\end{bmatrix} \cdot \mathbf{\dot{\phi}}',
\]

(A7)

\[
\frac{d_r \mathbf{\phi}}{dt} \cdot \mathbf{\dot{r}}' = \begin{bmatrix}
  -A \rho_g (\mathbf{v}_d - \mathbf{v}_g) - \frac{GM_d}{r_d^3} r_d - \frac{\partial \mathbf{v}_{\text{Kep}}}{\partial t} - \nabla P_g \\
  -A \rho_g (\mathbf{v}_g - \mathbf{v}_d) - \frac{GM_g}{r_g^3} r_g - \frac{\partial \mathbf{v}_{\text{Kep}}}{\partial t}
\end{bmatrix} \cdot \mathbf{\dot{\phi}}',
\]

(A8)

\[
\frac{d_\phi \mathbf{\phi}}{dt} \cdot \mathbf{\dot{\phi}}' = \begin{bmatrix}
  -A \rho_g (\mathbf{v}_d - \mathbf{v}_g) - \frac{GM_d}{r_d^3} r_d - \frac{\partial \mathbf{v}_{\text{Kep}}}{\partial t} - \nabla P_g \\
  -A \rho_g (\mathbf{v}_g - \mathbf{v}_d) - \frac{GM_g}{r_g^3} r_g - \frac{\partial \mathbf{v}_{\text{Kep}}}{\partial t}
\end{bmatrix} \cdot \mathbf{\dot{\phi}}'.
\]

(A9)

Using the velocity decomposition in terms of the velocity components in the flat coordinate system specified in eq. A3,
\[
\begin{align*}
\frac{d}{dt} \dot{r} & : 
B_r(\gamma_d) \frac{\partial v_{d,r}^r}{\partial t} + B_\phi(\gamma_d) \frac{\partial v_{d,\phi}^r}{\partial t} 
\approx -\lambda \rho_g \left[ C_r(\gamma_d) v_{d,r}^r + C_\phi(\gamma_d) v_{d,\phi}^r - C_r(\gamma_d) v_{d,r}^r - C_\phi(\gamma_d) v_{d,\phi}^r \right] + Q(\gamma_d) 2 \Omega_K v_{d,\phi}^r - S(\gamma_d) \frac{1}{2} \Omega_K v_{d,r}^r, \
\frac{d}{dt} \dot{\phi} & : 
F_r(\gamma_d) \frac{\partial v_{d,r}^\phi}{\partial t} + F_\phi(\gamma_d) \frac{\partial v_{d,\phi}^\phi}{\partial t} 
\approx -\lambda \rho_g \left[ G_r(\gamma_d) v_{d,r}^\phi + G_\phi(\gamma_d) v_{d,\phi}^\phi - G_r(\gamma_d) v_{d,r}^\phi - G_\phi(\gamma_d) v_{d,\phi}^\phi \right] + U(\gamma_d) 2 \Omega_K v_{d,\phi}^\phi - V(\gamma_d) \frac{1}{2} \Omega_K v_{d,r}^\phi, \
\frac{d}{dt} \dot{t} & : 
B_r(\gamma_d) \frac{\partial v_{d,r}^t}{\partial t} + B_\phi(\gamma_d) \frac{\partial v_{d,\phi}^t}{\partial t} 
\approx -\lambda \rho_g \left[ C_r(\gamma_d) v_{d,r}^t + C_\phi(\gamma_d) v_{d,\phi}^t - C_r(\gamma_d) v_{d,r}^t - C_\phi(\gamma_d) v_{d,\phi}^t \right] + Q(\gamma_d) 2 \Omega_K v_{d,\phi}^t - S(\gamma_d) \frac{1}{2} \Omega_K v_{d,r}^t - \frac{1}{\rho_g} \frac{\partial P_g}{\partial t}, \
\frac{d}{dt} \dot{v}_t & : 
F_r(\gamma_d) \frac{\partial v_{d,r}^{v_t}}{\partial t} + F_\phi(\gamma_d) \frac{\partial v_{d,\phi}^{v_t}}{\partial t} 
\approx -\lambda \rho_g \left[ G_r(\gamma_d) v_{d,r}^{v_t} + G_\phi(\gamma_d) v_{d,\phi}^{v_t} - G_r(\gamma_d) v_{d,r}^{v_t} - G_\phi(\gamma_d) v_{d,\phi}^{v_t} \right] + U(\gamma_d) 2 \Omega_K v_{d,\phi}^{v_t} - V(\gamma_d) \frac{1}{2} \Omega_K v_{d,r}^{v_t}.
\end{align*}
\]

When the coefficients are calculated by projecting the velocity vectors on the ‘flat’ coordinate system and extracting the coefficients: \( \dot{r} \cdot \hat{u}_i := C_r(\gamma_i) u_{i,r}^r + C_\phi(\gamma_i) u_{i,\phi}^r \) and \( \dot{\phi} \cdot \hat{u}_i = G_r(\gamma_i) u_{i,r}^\phi + G_\phi(\gamma_i) u_{i,\phi}^\phi \) which yields

\[
\begin{align*}
C_r(\gamma) &= \cos \beta \cos \phi \cos (\phi - \gamma) + \sin \phi \sin (\phi - \gamma), \\ C_\phi(\gamma) &= \cos \gamma_i \sin^2 \beta \sin (2\phi) - \sin \gamma_i \left( \cos \gamma_i \cos \beta \sin \phi \right), \\ G_r(\gamma) &= \cos (\phi - \gamma) \sin \phi - \cos \beta \cos \phi \sin (\phi - \gamma), \\ G_\phi(\gamma) &= \cos (\phi - \gamma) \cos \phi + \cos \beta \sin \phi \sin (\phi - \gamma).
\end{align*}
\]

Then, a full generalized analytical solution for the dust radial drift at a steady-state (i.e. the partial derivatives equal to zero) could be derived (although complicated). Hence, we will examine some simplifying assumptions: \( \gamma_g = \gamma_d \) i.e \( C_r := C_r(\gamma_d) = C_r(\gamma_g), \ C_\phi := C_\phi(\gamma_d) = C_\phi(\gamma_g) \) and \( G_r := G_r(\gamma_d) = G_r(\gamma_g), \ G_\phi := G_\phi(\gamma_d) = G_\phi(\gamma_g), \ Q(\gamma_d) = Q(\gamma_g) = 1, \ S(\gamma_g) = S(\gamma_d) = 0, \ U(\gamma_d) = U(\gamma_g) = 0, \ V(\gamma_d) = V(\gamma_g) = 0 \) — which takes place (but not only) at the co-precession radius, and neglecting the corrections to the derivatives on the left side, the radial drift in the flat system is
Then, the correction factor to the radial drift and the orbit-averaged coefficients are

\[
\frac{v_{d,r}}{0.5\Omega_K \rho_g C_r(-2G_{\phi} \rho_d - 2G_{\phi} \rho_d - 2G_r \rho_d - 2G_r \rho_d)(AC_r(G_{\rho_d} \rho_d + G_{\rho_d} \rho_d + G_r \rho_d) - 0.5C_{\phi} \Omega_K) + A^2C_r \Omega_K^2 \rho_g^2 (-G_{\phi} - G_r)} =
\]

\[
= -2\eta \Omega_K \frac{v_K}{0.5A^3 \Omega_K \rho_g^2 C_r(-2G_{\phi} \rho_d - 2G_{\phi} \rho_d - 2G_r \rho_d - 2G_r \rho_d)(AC_r(G_{\rho_d} \rho_d + G_{\rho_d} \rho_d + G_r \rho_d) - 0.5C_{\phi} \Omega_K) + A^2C_r \Omega_K^2 \rho_g^2 (-G_{\phi} - G_r)} =
\]

\[
= -2\eta \Omega_K \frac{v_K}{-A^3C_r \Omega_K \rho_g^2 C_r(G_{\phi} + G_r)^2} =
\]

\[
= -2\eta \Omega_K \frac{v_K}{D(G_{\phi} + G_r) |C_r(G_{\phi} + G_r)D - 0.5C_{\phi} \Omega_K| - A^2C_r \Omega_K \rho_g^2 (G_{\phi} + G_r)} =
\]

where \( D = A(\rho_g + \rho_d) \) and the last reduction is not trivial and holds only when \( G_r + G_{\phi}, C_r \neq 0 \).

**B. LIMITING CASES**

Here we will examine our results in some limiting cases.

**B.1. The standard radial drift limit \((\beta, \gamma_i \to 0)\)**

For this case, in which the tilt angle and the precession are negligible,

\[
C_r, G_{\phi} \to 1, C_{\phi}, G_r \to 0
\]

The equations then become

\[
d, \dot{r}' : 0 \approx -A \rho_g (v_{d,r}' - v_{g,r}') + 2\Omega_K v_{d,\phi}'
\]

\[
d, \dot{\phi}' : 0 \approx -A \rho_g (v_{d,\phi}' - v_{g,\phi}') - \frac{1}{2} \Omega_K v_{d,r}'
\]

\[
g_r, \dot{r}' : 0 \approx -A \rho_g (v_{g,r}' - v_{d,r}') + 2\Omega_K v_{d,\phi}' - \frac{1}{2} \Omega_K v_{g,r}'
\]

\[
g_r, \dot{\phi}' : 0 \approx -A \rho_g (v_{g,\phi}' - v_{d,\phi}') - \frac{1}{2} \Omega_K v_{g,\phi}'
\]

which are the standard equations for radial drift in a flat disk (e.g. Nakagawa et al. 1986), i.e. \( \xi \to 1 \).

**B.2. Small warping, negligible precession \((0 < \beta \ll 1, \gamma_i \to 0)\)**

In the limit of small warping and negligible precession, the coefficients are given by

\[
C_r \approx 1 - \frac{\beta^2}{2} \cos^2 \phi,
\]

\[
C_{\phi} \approx \frac{\beta^2}{2} \sin(2\phi),
\]

\[
G_r \approx \frac{\beta^2}{2} \cos \phi \sin \phi,
\]

\[
G_{\phi} \approx 1 - \frac{\beta^2}{2} \sin^2 \phi
\]

and the orbit-averaged coefficients are

\[
\bar{C}_r = \bar{G}_{\phi} \approx 1 - \frac{\beta^2}{4},
\]

\[
\bar{C}_{\phi} = \bar{G}_r \approx 0
\]

Then, the correction factor to the radial drift \( \xi \) is given by

\[
\xi \approx \frac{1}{1 - \frac{\beta^2}{4} \frac{D^2 + \Omega_K^2}{D^2 + \left( \frac{\Omega_K}{1 - \frac{\beta^2}{4}} \right)^2}}
\]