Holonomy groups of pseudo-quaternionic-Kählerian manifolds of non-zero scalar curvature

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Abstract
The holonomy group $G$ of a pseudo-quaternionic-Kählerian manifold of signature $(4r, 4s)$ with non-zero scalar curvature is contained in $\text{Sp}(1) \cdot \text{Sp}(r, s)$ and it contains $\text{Sp}(1)$. It is proved that either $G$ is irreducible, or $s = r$ and $G$ preserves an isotropic subspace of dimension $4r$, in the last case, there are only two possibilities for the connected component of the identity of such $G$. This gives the classification of possible connected holonomy groups of pseudo-quaternionic-Kählerian manifolds of non-zero scalar curvature.

Keywords
Pseudo-quaternionic-Kählerian manifold · non-zero scalar curvature · holonomy group · holonomy algebra · curvature tensor · symmetric space

Mathematics Subject Classification (2000) 53C29 · 53C26

1 Introduction and the results

Quaternionic-Kählerian geometry is of increasing interest both in mathematics and mathematical physics, see e.g. [10, 11, 12]. A pseudo-quaternionic-Kählerian manifold is a pseudo-Riemannian manifold $(M, g)$ of signature $(4r, 4s)$, $r + s > 1$ together with a parallel quaternionic structure $Q \subset \mathfrak{so}(TM)$, i.e. three-dimensional linear Lie algebra $Q$ with a basis $I_1, I_2, I_3$ which satisfies the relations $I_1^2 = I_2^2 = I_3^2 = -\text{id}$, $I_3 = I_1I_2 = -I_2I_1$. The holonomy group $G$ of such manifold is contained in $\text{Sp}(1) \cdot \text{Sp}(r, s) = \text{Sp}(1) \times \text{Sp}(r, s)/\mathbb{Z}_2$. Conversely, any pseudo-Riemannian manifold with such holonomy group is pseudo-quaternionic-Kählerian. Denote by $G^0$ the restricted holonomy group of $(M, g)$, i.e. the connected component of identity of $G$.

The classification of connected holonomy groups of Riemannian manifolds is well known and it has a lot of applications both in geometry and physics, see e.g. [2, 3].

The corresponding problem for pseudo-Riemannian manifolds of arbitrary signature is
solved only in some partial cases, see the recent reviews [12]. The difficulty appears if the holonomy group preserves a degenerate subspace of the tangent space. Here we show that the holonomy group \( G \subset \text{Sp}(1) \cdot \text{Sp}(r,s) \) of a pseudo-quaternionic-Kählerian manifold with non-zero scalar curvature is irreducible if \( s \neq r \). If \( s = r \), then \( G \) may preserve a degenerate subspace of the tangent space, in this case there are only two possibilities for \( G \).

In [11] it is proved that the curvature tensor \( R \) of a pseudo-quaternionic-Kählerian manifold of signature \((4r, 4s)\) can be written as

\[
R(X, Y) = \nu R_0 + \mathcal{W},
\]

where \( \nu = \frac{\text{scal}}{4m(m+2)} \) \((m = 4r + 4s)\) is the reduced scalar curvature,

\[
R_0(X, Y) = \frac{1}{2} \sum_{\alpha=1}^{3} g(X, I_\alpha Y) I_\alpha + \frac{1}{4} \left( X \wedge Y + \sum_{\alpha=1}^{3} I_\alpha X \wedge I_\alpha Y \right),
\]

\( X, Y \in TM \), is the curvature tensor of the quaternionic projective space \( \mathbb{H}P^{r,s} \), and \( \mathcal{W} \) is an algebraic curvature tensor with zero Ricci tensor. It is proved that \( \text{scal} \neq 0 \) if and only if the holonomy group \( G \) contains \( \text{Sp}(1) \). In [11] it is proved also that any pseudo-quaternionic-Kählerian manifold with non-zero scalar curvature is locally indecomposable, i.e. it is not locally a product of two pseudo-Riemannian manifolds of positive dimension, or, equivalently, \( \mathcal{G}^0 \) does not preserve any proper non-degenerate vector subspace of the tangent space.

The tangent space to the manifold \((M, g)\) can be identified with the pseudo-Euclidean space \( \mathbb{R}^{4r,4s} \) endowed with the pseudo-Euclidean metric \( \eta \) and an \( \eta \)-orthogonal quaternionic structure \( I_1, I_2, I_3 \), or with the pseudo-quaternionic-Hermitian space \( \mathbb{H}^{r,s} \) endowed with a pseudo-quaternionic-Hermitian metric \( g \). Let \( s = r \) and \( W \subset \mathbb{H}^{r,r} \) be an isotropic subspace of quaternionic dimension \( r \). Let \( p_1, \ldots, p_r, q_1, \ldots, q_r \) be a basis of \( \mathbb{H}^{r,r} \) such that \( p_1, \ldots, p_r \) is a basis of \( W \) and the only non-zero values of the pseudo-quaternionic-Hermitian form \( g \) on \( \mathbb{H}^{r,r} \) are \( g(p_i, q_i) = g(q_i, p_i) = 1 \). The maximal subalgebra \( \mathfrak{sp}(r, r)_W \subset \mathfrak{sp}(r, r) \) preserving \( W \) can be identified with the following matrix Lie algebra:

\[
\mathfrak{sp}(r, r)_W = \left\{ \begin{pmatrix} C & B \\ 0 & -C^t \end{pmatrix} \Bigg| C, B \in \text{Mat}(r, \mathbb{H}) \right\},
\]

where \( \text{Mat}(r, \mathbb{H}) \) denotes the space of \( r \times r \) quaternionic matrices. Note the following.

Let \( \mathbb{H}^m \) be an \( m \)-dimensional quaternionic vector space and \( e_1, \ldots, e_m \) a basis of \( \mathbb{H}^m \). We identify an element \( X \in \mathbb{H}^m \) with the column \((X_t)\) of the left coordinates of \( X \) with respect to this basis, \( X = \sum_{t=1}^{m} X_t e_t \). Let \( f : \mathbb{H}^m \to \mathbb{H}^m \) be an \( \mathbb{H} \)-linear map. Define the matrix \( \text{Mat}_f \) of \( f \) by the relation \( f e_t = \sum_{\alpha=1}^{m} (\text{Mat}_f)_{\alpha t} e_\alpha \). Now if \( X \in \mathbb{H}^m \), then \( fX = (X^t \text{Mat}_f)^t \) and because of the non-commutativity of the quaternionic numbers this is not the same as \( \text{Mat}_f X \). Conversely, any \( m \times m \) quaternionic matrix defines an \( \mathbb{H} \)-linear map \( f : \mathbb{H}^m \to \mathbb{H}^m \). Denote by \( \mathfrak{sp}(r, r)_W \subset \mathfrak{sp}(r, r) \) the maximal connected Lie subgroup preserving \( W \), i.e. the connected Lie subgroup corresponding to the subalgebra \( \mathfrak{sp}(r, r)_W \subset \mathfrak{sp}(r, r) \). Define the Lie subalgebra

\[
\mathfrak{h}_0 = \mathfrak{sp}(1) \oplus \left\{ \begin{pmatrix} C & 0 \\ 0 & -C^t \end{pmatrix} \Bigg| C \in \text{Mat}(r, \mathbb{H}) \right\} \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(r, r)_W
\]

and denote by \( H_0 \) the corresponding connected Lie subgroup of \( \text{Sp}(1) \cdot \text{Sp}(r, r)_W \).

We prove the following two statements.
Theorem 1 Let \((M, g)\) be a pseudo-quaternionic-Kählerian manifold of non-zero scalar curvature and of signature \((4r, 4s)\). If its restricted holonomy group \(G^0\) is not irreducible, then \(s = r\), \(G^0\) preserves an isotropic quaternionic subspace \(W \subset H^{r,r}\) of quaternionic dimension \(r\) and either \(G^0 = H_0\), or \(G^0 = \text{Sp}(1) \cdot \text{Sp}(r, r) W\).

Proposition 1 Any pseudo-quaternionic-Kählerian manifold with the restricted holonomy group \(H_0\) is locally symmetric, i.e. its curvature tensor is parallel.

Note that if the manifold \((M, g)\) is not locally symmetric and its restricted holonomy group \(G^0\) is irreducible, then \(G^0 = G = \text{Sp}(1) \cdot \text{Sp}(r, s)\), e.g. [5].

Simply connected symmetric pseudo-quaternionic-Kählerian manifolds are classified in [8]. Each such space \((M, g)\) may be represented as \(M = F/K\), where \(F\) is the connected group generated by transvections and \(K\) is the stabilizer of a fixed point \(o \in M\). The holonomy group \((M, g)\) coincides with the isotropy representation of \(K\). These spaces are exhausted by

\[
\begin{align*}
G & \quad = \text{SU}(2) \cdot \text{U}(p, q), \\
G & \quad = \text{SU}(1,1) \times \text{GL}(r, \mathbb{R}), \\
G & \quad = \text{SO}(p+4, q), \\
G & \quad = \text{SO}^*(2l+4), \\
G & \quad = \text{Sp}(p+1, q), \\
G & \quad = \text{SU}(2) \cdot \text{Sp}(1) \cdot \text{Sp}(p, q), \\
G & \quad = \text{Sp}(2) \cdot \text{SU}(2) \cdot \text{Sp}(p, q).
\end{align*}
\]

Note that the subgroup \(H_0 \subset \text{Sp}(1) \cdot \text{Sp}(r, r) W\) is the holonomy group of the symmetric space \(\text{SL}(r+1, \mathbb{R})/\text{GL}(1, \mathbb{R}) \times \text{SL}(r, \mathbb{R})\). This shows that the holonomy groups of other symmetric spaces are irreducible.

We get the following corollaries.

Corollary 1 Let \((M, g)\) be a pseudo-quaternionic-Kählerian manifold of non-zero scalar curvature and of signature \((4r, 4s)\). Then either its restricted holonomy group coincides with \(\text{Sp}(1) \cdot \text{Sp}(r, s)\) or with \(\text{Sp}(1) \cdot \text{Sp}(r, r) W\), or \((M, g)\) is locally symmetric.

Corollary 2 Let \((M, g)\) be a complete pseudo-quaternionic-Kählerian manifold of non-zero scalar curvature and of signature \((4r, 4s)\). Then either its restricted holonomy group coincides with \(\text{Sp}(1) \cdot \text{Sp}(r, s)\) or with \(\text{Sp}(1) \cdot \text{Sp}(r, r) W\), or \((M, g)\) is the factor space of a symmetric space obtained in [8] by a freely acting discrete group \(\Gamma\).

Corollary 3 Let \((M, g)\) be a simply connected complete pseudo-quaternionic-Kählerian manifold of non-zero scalar curvature and of signature \((4r, 4s)\). If the holonomy group \(G\) of \((M, g)\) is irreducible, then either \(G = \text{Sp}(1) \cdot \text{Sp}(r, s)\), or \((M, g)\) is a symmetric space obtained in [8] different from \(\text{SL}(r+1, \mathbb{R})\). If \(G\) is not irreducible, then \(s = r\), and either \(G = \text{Sp}(1) \cdot \text{Sp}(r, r) W\), or \((M, g)\) is isometric to the symmetric space \(\text{SL}(r+1, \mathbb{R})/\text{GL}(1, \mathbb{R}) \times \text{SL}(r, \mathbb{R})\).

Theorem [8] gives only the list of possible connected holonomy groups. To complete the classification of all connected holonomy groups, one must show that \(\text{Sp}(1) \cdot \text{Sp}(r, r) W\) may appear as the holonomy group of a pseudo-quaternionic-Kählerian manifold.
2 Proof of Theorem 1

Let \((M, g)\) be a pseudo-quaternionic-Kählerian manifold of non-zero scalar curvature and of signature \((4, 4s)\). Let \(m = r + s\). Obviously, it is enough to prove the theorem in terms of the holonomy algebra \(g \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(r, s)\) of \((M, g)\). In [11] it is proved that \(g\) contains \(\mathfrak{sp}(1)\) and it does not preserve any proper non-degenerate subspace of \(\mathbb{R}^{4r, 4s}\).

For any subalgebra \(g \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(r, s)\) denote by \(\mathcal{R}(g)\) the space of algebraic curvature tensors of type \(g\), i.e. the space of linear maps from \(\wedge^2 \mathbb{R}^{4r,4s} \to g\) satisfying the first Bianchi identity

\[
R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0
\]

for all \(X, Y, Z \in \mathbb{R}^{4r, 4s}\). It is well-known that any \(R \in \mathcal{R}(g)\) satisfies

\[
\eta(R(X,Y)Z,U) = \eta(R(Z,U)X,Y)
\]

for all \(X, Y, Z, U \in \mathbb{R}^{4r, 4s}\). For example, if \((M, g)\) is a pseudo-quaternionic-Kählerian manifold, \(x \in M\), and \(g\) is the holonomy algebra of \((M, g)\) at the point \(x\), then identifying \(T_xM\) with \(\mathbb{R}^{4r, 4s}\), we get that the value \(R_x\) of the curvature tensor \(R\) of \((M, g)\) at the point \(x\) belongs to \(\mathcal{R}(g)\). From the Ambrose-Singer Theorem [3] it follows that if \(g \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(r, s)\) is the holonomy algebra of a pseudo-quaternionic-Kählerian manifold, then it is a Berger algebra, i.e. \(g\) is spanned by the images of the algebraic curvature tensors \(R \in \mathcal{R}(g)\). We will prove the theorem assuming that \(g \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(r, s)\) is a Berger algebra.

Suppose that \(g\) preserves a degenerate subspace \(W \subset \mathbb{R}^{4r, 4s}\). Then \(g\) preserves the isotropic subspace \(W \cap W^\perp\), i.e. we may assume that \(W\) is isotropic. Since \(g\) contains \(\mathfrak{sp}(1)\), \(W\) is a quaternionic subspace of \(\mathbb{H}^{r,s}\). Let \(\dim_G W = t\). Then \(W \subset W^\perp\) and \(\dim_G W^\perp = m - t > t\). Let \(E\) be a quaternionic subspace of \(W^\perp\) complementary to \(W\), then the restriction of \(g\) to \(E\) is non-degenerate, let \((r_0, s_0)\) be its signature. We have \(W \subset E^\perp\). Let \(W_1 \subset E^\perp\) be any isotropic subspace complementary to \(W\). Clearly, \(\dim_G W_1 = \dim_G W\). Let \(p_1 \ldots p_t, e_1, \ldots, e_{r_0+s_0}, q_1, \ldots, q_t \in W_1\), and the only non-zero values of the pseudo-quaternionic-Hermitian form \(g\) are \(g(p_i, q_i) = g(q_i, p_i) = 1\) (if \(1 \leq i \leq t\)), \(g(e_i, e_i) = -1\) (if \(1 \leq i \leq r_0\)), \(g(e_i, e_i) = 1\) (if \(r_0 + 1 \leq i \leq r_0 + s_0\)). Then the Lie algebra \(\mathfrak{sp}(r, s)\) can be identified with the following matrix Lie algebra:

\[
\mathfrak{sp}(r, s) = \left\{ \begin{array}{ccc}
C & -(E_{r_0,s_0}X)^t & B \\
Y & A & X \\
D & -(E_{r_0,s_0}Y)^t & -C^t \\
\end{array} \right| \begin{array}{c}
C \in \text{Mat}(r, \mathbb{H}), \quad B, D \in S(r, \mathbb{H}), \\
A \in \mathfrak{sp}(r_0, s_0), \quad X, Y \in \text{Mat}(r_0 + s_0, t, \mathbb{H}) \end{array} \right\},
\]

where \(\text{Mat}(r, \mathbb{H})\) denotes the space of \(r \times r\) quaternionic matrices,

\[
S(r, \mathbb{H}) = \{B \in \text{Mat}(r, \mathbb{H})\} \quad B^t = -B,
\]

\(\text{Mat}(r_0 + s_0, t, \mathbb{H})\) denotes the space of \(r_0 + s_0 \times t\) quaternionic matrices, \(E_{r_0,s_0} = \begin{pmatrix} -E_{r_0} & 0 \\ 0 & E_{s_0} \end{pmatrix}\), and \(E_1\) denotes the identity \(t \times t\) matrix. For the maximal subalgebra \(\mathfrak{sp}(r, s)_{W} \subset \mathfrak{sp}(r, s)\) preserving \(W\) we get

\[
\mathfrak{sp}(r, s)_{W} = \left\{ \begin{array}{ccc}
C & -(E_{r_0,s_0}X)^t & B \\
0 & A & X \\
0 & 0 & -C^t \\
\end{array} \right| \begin{array}{c}
C \in \text{Mat}(r, \mathbb{H}), \quad B \in S(r, \mathbb{H}), \\
A \in \mathfrak{sp}(r_0, s_0), \quad X \in \text{Mat}(r_0 + s_0, t, \mathbb{H}) \end{array} \right\}.
\]
We claim that if \( r_0 + s_0 \neq 0 \), then \( \mathcal{R}(\mathfrak{sp}(1) \oplus \mathfrak{sp}(r, s)_W) = \mathcal{R}(\mathfrak{sp}(r, s)_W) \). Let \( R \in \mathcal{R}(\mathfrak{sp}(1) \oplus \mathfrak{sp}(r, s)_W) \). From \( (1) \) it follows that \( R = \nu R_0 + W \), where \( \nu \in \mathbb{R} \), \( R_0 \) is given by \( (2) \) with \( X, Y \in \mathbb{R}^{4r, 4s} \), and \( W \in \mathcal{R}(\mathfrak{sp}(r, s)) \). Let \( p \in W \), \( X \in E \), and \( Y, Z \in \mathbb{R}^{4r, 4s} \). Using \( (1) \), we get

\[
\eta(R(p, X)Y, Z) = \eta(R(Y, Z)p, X) = 0,
\]

since \( R(Y, Z)p \in W \) and \( X \in E \subset W^\perp \). Consequently, \( R(p, X) = 0 \). Let \( X, Y \in E \), then using the Bianchi identity, we get

\[
R(X, Y)p = -R(Y, p)X - R(p, X)Y = 0.
\]

This shows that \( \nu \mathfrak{pr}_W \circ R_0(X, Y)|_W = -\mathfrak{pr}_W \circ \mathcal{W}(X, Y)|_W \). On the other hand, \( \mathfrak{pr}_W \circ \mathcal{W}(X, Y)|_W \in \mathfrak{gl}(W) = \mathfrak{gl}(r, \mathbb{H}) \), whilst \( \mathfrak{pr}_W \circ R_0(X, Y)|_W = \frac{1}{2} \sum_{\alpha=1}^3 \eta(X, I_\alpha Y)I_\alpha|_W \). Consequently, \( \nu \mathfrak{pr}_W \circ R_0(X, Y)|_W = -\mathfrak{pr}_W \circ \mathcal{W}(X, Y)|_W = 0 \). This means that \( \nu \sum_{\alpha=1}^3 \eta(X, I_\alpha Y)I_\alpha = 0 \). Taking \( X = e_1 \), \( Y = I_1 e_1 \), we get \( \nu = 0 \). Thus, \( R = \mathcal{W} \in \mathcal{R}(\mathfrak{sp}(r, s)_W) \). This shows that if \( r_0 + s_0 \neq 0 \), then any Berger subalgebra of \( \mathfrak{sp}(1) \oplus \mathfrak{sp}(r, s)_W \) is contained in \( \mathfrak{sp}(r, s)_W \).

Let \( r_0 = s_0 = 0 \). Then \( s = t = r \). From \( (1) \) it follows that

\[
\mathcal{R}(\mathfrak{sp}(1) \oplus \mathfrak{sp}(r, r)) = \mathcal{R}R_0 \oplus \mathcal{R}(\mathfrak{sp}(r, r)),
\]

where \( R_0 \) is given by \( (2) \) with \( X, Y \in \mathbb{R}^{4r, 4r} \). Consider the subalgebra

\[
\mathfrak{h}_0 = \mathfrak{sp}(1) \oplus \left\{ \begin{pmatrix} C & 0 \\ 0 & -C^t \end{pmatrix} \mid C \in \text{Mat}(r, \mathbb{H}) \right\} \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(r, r)_W.
\]

This algebra appears as the holonomy algebra of the pseudo-quaternionic-Kählerian symmetric space \( \text{SL}(r + 1, \mathbb{H})/\text{S}(\text{GL}(1, \mathbb{H}) \times \text{GL}(r, \mathbb{H})) \) \( \mathcal{T} \). This shows that \( \mathcal{R}(\mathfrak{h}_0) \) contains an element \( R_1 \) such that \( \mathfrak{h}_0 \) annihilates \( R_1 \in \mathcal{R}(\mathfrak{h}_0) \) and the image of \( R_1 \) spans \( \mathfrak{h}_0 \). Let \( \mathfrak{g} \subset \mathfrak{h}_0 \) and \( R \in \mathcal{R}(\mathfrak{g}) \). Let \( X, Y \in W \) and \( X_1 \in W_1 \). The Bianchi identity \( (3) \) implies \( R(X, Y)X_1 = 0 \) and

\[
R(X_1, Y)X = R(X_1, Y)X,
\]

i.e. \( R(X, Y) = 0 \), and for each fixed \( X_1 \in W_1 \), \( R(X_1, \cdot)|_W \) belongs to the first prolongation \( \mathfrak{g}(W)^{(1)} \) of the subalgebra \( \mathfrak{g} \subset \mathfrak{sp}(1) \oplus \mathfrak{gl}(r, \mathbb{H}) \subset \mathfrak{gl}(4r, \mathbb{R}) \). Similarly, if \( X_1, Y_1 \in W_1 \), then \( R(X_1, Y_1) = 0 \); if \( X \in W \), then \( R(X, \cdot)|_W \) belongs to \( (\mathfrak{g}(W)^{(1})) \). It holds \( \mathfrak{g}(\mathfrak{gl}(r, \mathbb{H}))^{(1)} = 0 \), hence \( \mathcal{R}(\mathfrak{h}_0 \cap \mathfrak{sp}(r, r)) = 0 \). From this and \( (5) \) it follows that \( \mathcal{R}(\mathfrak{h}_0) = \mathbb{R}R_1 \). This and \( (6) \) show that

\[
\mathcal{R}(\mathfrak{sp}(1) \oplus \mathfrak{sp}(r, r)_W) = \mathbb{R}R_1 \oplus \mathcal{R}(\mathfrak{sp}(r, r)_W).
\]

Let \( \mathfrak{g} \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(r, r)_W \) be a Berger subalgebra such that \( \mathfrak{g} \not\subset \mathfrak{sp}(r, r)_W \). Then there exists \( R \in \mathcal{R}(\mathfrak{g}) \) such that \( R = \nu R_1 + W \), \( \nu \neq 0 \) and \( W \in \mathcal{R}(\mathfrak{sp}(r, r)_W) \). Let \( p, X \in W \) and \( Y, Z \in \mathbb{R}^{4r, 4r} \). From \( (5) \) applied to \( W \), we obtain \( \mathcal{W}(p_X) = 0 \) for any \( X \in W_1 \). Hence, \( R(X, X_1)|_W = \nu R_1(X, X_1)|_W \). This shows that \( \mathfrak{g}(W)|_W = \mathfrak{h}_0 = \mathfrak{sp}(1)|_W \oplus \mathfrak{gl}(W) \). Suppose that \( \mathfrak{g} \neq \mathfrak{h}_0 \). Then for some \( B \in \text{S}(r, \mathbb{H}) \) the element \( \xi = \begin{pmatrix} 0 & B \\ 0 & -E_r \end{pmatrix} \) belongs to \( \mathfrak{g} \). If \( B \neq 0 \), then choosing the basis \( p_1, \ldots, p_r, q_1', \ldots, q_r', \) where \( q_i' = q_i - \frac{1}{2} \sum_{j=1}^r B_{ij} p_j \), we get that \( \xi = \begin{pmatrix} -E_r & 0 \\ 0 & E_r \end{pmatrix} \) belongs to \( \mathfrak{g} \).
Let $\xi_1 = \begin{pmatrix} C & B \\ 0 & -C^t \end{pmatrix} \in g$, where $C \in \text{Mat}(r, \mathbb{H})$ and $B \in S(r, \mathbb{H})$. Then, $[\xi, \xi_1] = \begin{pmatrix} 0 & 2B \\ 0 & 0 \end{pmatrix} \in g$. This shows that $g = h_0 \ltimes L$, where $L \subset \{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \mid B \in S(r, \mathbb{H}) \}$. The Lie brackets of elements from $h_0$ and $L$ are given by the representation of $\text{gl}(r, \mathbb{H})$ on $S(r, \mathbb{H})$. Since this representation is irreducible [5], $g = \text{sp}(1) \oplus \text{sp}(r, r)_W$. It is not hard to see that this algebra is a Berger algebra, hence it is a candidate to be a holonomy algebra.

3 Proof of Proposition 1

Let $(M, g)$ be pseudo-quaternionic-Kählerian manifold with the holonomy algebra $h_0$. Fix a point $x \in M$. Then $T_x M$ is identified with $\mathbb{R}^{4r, 4r}$. For the covariant derivative of the curvature tensor at the point $x$ we have $\nabla_x R_x \in \mathbb{R}(h_0)$ for any $X \in \mathbb{R}^{4r, 4r}$. Let $X, Y \in W$ and $X_1 \in W_1$. From the above, we get $\nabla_{X_1} R_x(X, Y) = 0$. This and the second Bianchi identity imply

$$\nabla_{X_1} R_x(X, Y) = \nabla_{Y_1} R_x(X_1, X),$$

i.e. $\nabla_{X_1} R_x(X_1, Y) \mid W \in W$ belongs to the second prolongation of the subalgebra $\text{sp}(1) \oplus \text{gl}(r, \mathbb{H}) \subset \text{gl}(4r, \mathbb{R})$, which is trivial, since the second prolongation of its complexification $\mathfrak{sl}(2, \mathbb{C}) \oplus \text{gl}(2r, \mathbb{C}) \subset \text{gl}(4r, \mathbb{C})$ is trivial [11]. We get that $\nabla_x R_x = 0$. By the same arguments, $\nabla_{X_1} R_x = 0$. Thus, $\nabla R_x = 0$ for any $x \in M$, i.e. $\nabla R = 0$ and $(M, g)$ is locally symmetric.

4 Proof of corollaries

Corollary 1 follows from the above results and from the fact that the only irreducible holonomy group of not locally symmetric pseudo-quaternionic-Kählerian manifolds is $\text{Sp}(1) \cdot \text{Sp}(r, s)$ [3]. Corollary 3 follows from Corollary 1 and from the results of [1]. To prove Corollary 2 consider the universal covering $(\tilde{M}, \tilde{g})$ of $(M, g)$. Then $(\tilde{M}, \tilde{g})$ is simply connected and complete, hence it is contained in the list of symmetric spaces from [1] given in Introduction. It is known that $M = \tilde{M}/\Gamma$ for some freely acting discrete group $\Gamma$ of isometries of $\tilde{M}$, e.g. [4].

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