Choreographies in the discrete nonlinear Schrödinger equations

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Abstract. We study periodic solutions of the discrete nonlinear Schrödinger equation (DNLSE) that bifurcate from a symmetric polygonal relative equilibrium containing \( n \) sites. With specialized numerical continuation techniques and a varying physically relevant parameter we can locate interesting orbits, including infinitely many choreographies. Many of the orbits that correspond to choreographies are stable, as indicated by Floquet multipliers that are extracted as part of the numerical continuation scheme, and as verified a posteriori by simple numerical integration. We discuss the physical relevance and the implications of our results.

1 Introduction

In the last two decades there has been a growing interest in the study of choreographic solutions (“choreographies”) of the \( n \)-body and \( n \)-vortex problems. Choreographies are periodic solutions where the bodies or the vortices follows the same path. The first non-circular choreography was discovered numerically for the case of three bodies in [20], and its existence was rigorously proved in [8]. The term “choreography” was adopted for the \( n \)-body problem after the work of Simó [24]. Since then, variational methods [4,9], numerical minimization [7], numerical continuation [5], and computer-assisted proofs [14], have been used to determine choreographies of the \( n \)-body problem; see also the references in these papers. For vortices in the plane, choreographies have been constructed for 3 and 4 vortices in [1]. For \( n \) vortices in a general bounded domain, choreographic solutions have been found close to a stagnation point of a vortex [3], and close to the boundary of the domain [2].

In [5,6,13], choreographies for the \( n \)-body problem are found in dense sets of Lyapunov families that arise from the stationary \( n \)-polygon of bodies in a rotating frame. The existence of these choreographies depends only on the symmetries of the equations in rotating coordinates, i.e., these results can be extended to find choreographies of the \( n \)-vortex problem in the plane, in a disk, or on a surface of

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revolution. Such results can also be extended to the discrete nonlinear Schrödinger equation (DNLSE), which appears in the study of optical waveguide arrays and in Bose-Einstein condensates trapped in optical lattices [15].

While much research has been done on choreographies in the \( n \)-vortex and \( n \)-body problems, we are not aware of its extension to the study of the existence of choreographies in periodic lattices of \( n \) sites, as modeled by the DNLSE. The interest in \( n \)-body and \( n \)-vortex choreographies can perhaps be explained by the fact that variational methods are better suited for singular potentials. On the other hand, the continuation methods used in [5] are very well suited for locating choreographies for other symmetric potentials, such as in the DNLSE.

Evidence suggests that linearly stable choreographies in the \( n \)-body problem only exist for \( n = 3 \), and in the \( n \)-vortex problem for \( n = 3, \ldots, 7 \), as a consequence of the fact that the polygonal relative equilibrium of \( n \) bodies is unstable for \( n \geq 3 \) [5], and for vortices for \( n > 7 \) [10]. On the other hand, the DNLSE has dense sets of stable choreographies, as a consequence of the fact that the DNLSE has stable polygonal relative equilibria for all \( n \) [11]. We note that boundary value continuation methods can determine unstable choreographies as easily as stable ones; a property not shared by most other techniques. The aim of our paper is to investigate the existence of choreographies in the DNLSE using a boundary value continuation method. As illustrative examples we present a selection of choreographies, most of them stable, in a periodic lattice for the case of 9, 17, and 31 sites. These numbers have been chosen rather arbitrarily from the various numbers of sites that we have considered, which includes even numbers.

The lack of previous work on detecting choreographies in the DNLSE may be related to the difficulties encountered in the measurement of phases in physical problems modeled by the DNLSE. However, such difficulties appear to be surmountable in nonlinear optics. Within the context of nonlinear optics, several predictions of the DNLSE have been experimentally found in the last two decades [16]; in particular the formation of discrete solitons in waveguide arrays. The use of suitable optical techniques, known as laser heterodyne measurements, to detect such electric fields (field intensity and phase) [18,25], may open the door to experimental observation of stable choreographic solutions.

In Section 2, we consider Lyapunov families of periodic orbits, and their relation to the existence of choreographies in a periodic lattice of Schrödinger sites. In Section 3, we present methods to continue the Lyapunov families, and we exhibit a small selection of the many linearly stable choreographies that we have determined. Section 4 provides a brief summary of our results, as well as concluding remarks pertaining to the physical observation of stable choreographies.

### 2 Lyapunov families and choreographies

In a rotating frame with frequency \( \omega \), \( q_j(t) = e^{i\omega t}u_j(t) \), the equation that describes the dynamics of a lattice of \( n \) sites is given by the Hamiltonian system

\[
\dot{u} = i\partial_{\bar{u}}H_\omega(u), \quad \text{where} \quad H_\omega = \sum_{j=1}^{n} \left( \frac{1}{2} |u_j|^4 + \omega |u_j|^2 - |u_{j+1} - u_j|^2 \right). \quad (1)
\]

The sites \( u_j(t) \in \mathbb{C} \) satisfy periodicity conditions \( u_j(t) = u_{j+n}(t) \). The equation of motion has explicit polygonal equilibrium solutions given by

\[
a_j = ae^{i\frac{j}{n}(\alpha \zeta)}, \quad \zeta = \frac{2\pi}{n}, \quad \text{when} \quad \omega(a) = 4 \sin^2(\alpha \zeta/2) - a^2. \quad (2)
\]