Abstract

We show how to compute real space renormalization group flows in lattice field theory by a self-consistent method which is designed to preserve the basic stability properties of a Boltzmann factor. Particular attention is paid to controlling the errors which come from truncating the action to a manageable form. In each step, the integration over the fluctuation field (high frequency components of the field) is performed by a saddle point method. The saddle point depends on the block-spin. Higher powers of derivatives of the field are neglected in the actions, but no polynomial approximation in the field is made. The flow preserves a simple parameterization of the action. In the first part the method is described and numerical results are presented. In the second part we discuss an improvement of the method where the saddle point approximation is preceded by self-consistent normal ordering, i.e. solution of a gap equation. In the third part we describe a general procedure to obtain higher order corrections with the help of Schwinger Dyson equations.

In this paper we treat scalar field theories as an example. The basic limitations of the method are also discussed. They come from a possible breakdown of stability which may occur when a composite block-spin or block variables for domain walls would be needed.

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1 Renormalization group flow on the lattice

Suppose we start from the Euclidean action $S_0[\phi^0]$ of a field theory which lives on some lattice $\Lambda^0$ or on the continuum. It depends on a field $\phi^0$ on $\Lambda^0$. We wish to compute a sequence of effective actions $S_i[\phi^i]$ which live on lattices $\Lambda^i$ of increasing lattice spacing $a_i$, $i=0,1,2,\ldots$.

In principle the sequence of actions is defined once we fix a block-spin definition. The block-spin definition determines a field $\Phi = \phi^{i+1}$ on the coarser lattice $\Lambda^{i+1}$ as some kind of average

$$\Phi = C_i \phi^i$$

of the field $\phi = \phi^i$ on the lattice $\Lambda^i$. The actions are defined recursively by

$$Z^i[\Phi] = e^{-S^{i+1}[\Phi]} = \int D\phi \delta(C^i \phi - \Phi) e^{-S^i[\phi]}.$$  \hspace{1cm} (1)

For given $\Phi$, $Z^i[\Phi]$ is the partition function of an auxiliary theory in which only those variables are integrated out which are to be interpreted later as high frequency modes of the field.

We will also need certain expectation values of this auxiliary theory. If $A = A[\phi]$ is an observable, we set

$$\langle A \rangle^i = Z^i[\Phi]^{-1} \int D\phi A[\phi] \delta(C^i \phi - \Phi) e^{-S^i[\phi]}.$$  \hspace{1cm} (2)

These expectation values depend on $\Phi$, but we neglect to indicate this dependence explicitly. In the calculations, two auxiliary $\Phi$-dependent quantities will play an important role, the background field $\Psi^i[\Phi]$ and the fluctuation propagator $\Gamma^i[\Psi^i]$. The fluctuation propagator is considered to depend on the block-spin $\Phi$ through the background field $\Psi^i = \Psi^i[\Phi]$. Both auxiliary quantities are defined as expectation values of the auxiliary theory.

$$\Psi^i(z) = \Psi^i[\Phi](z) = \langle \phi(z) \rangle^i,$$  \hspace{1cm} (3)

$$\Gamma^i[\Psi^i](z,w) = \langle \phi(z) \phi(w) \rangle^i - \langle \phi(z) \rangle^i \langle \phi(w) \rangle^i.$$  \hspace{1cm} (4)

These definitions differ from those of Gawedzki and Kupiainen, cp. [8,15]. In their work, expectation values of a free theory were used. The present approach is more in the spirit of Balaban’s work on gauge theories, where expansions of the full action around its minimum were used for reasons of gauge invariance.

The validity of the method requires that the auxiliary theory has an infrared cutoff of order $a_i^{-1}_{i+1}$. Therefore the fluctuation propagator must decay exponentially with decay length no larger than about $a_{i+1}$. When this condition is violated, it signals a bad choice of the block-spin definition.

We are not able to calculate the functional integrals (1,2) exactly. Therefore approximations are necessary. The calculation is done in a self-consistent way. Both the background field and the fluctuation propagator are used in the
calculation. Conditions will be imposed on them which ensure that equations (3) and (4) are fulfilled within the accuracy of the approximation. We discuss in subsection 2.4 how a truncation of the effective action to a manageable form can be justified. No expansion in powers of fields is involved.

It follows from the definition (4) that the fluctuation propagator satisfies the constraints

$$C^i \Gamma^i[\Psi^i] = 0 = \Gamma^i[\Psi^i]C^i.$$  \hspace{1cm} (5)

The simplest choice of block-spin for a theory of scalar fields is as follows. One identifies the sites $x$ of the coarser lattice $\Lambda^{i+1}$ with disjoint hypercubes in the lattice $\Lambda^i$, and one chooses the block-spin as the average of the original field on these hypercubes

$$\Phi(x) = \int_z C^i(x, z) \phi(z) = \text{av} \phi(z).$$

One can try to improve on the locality properties of the effective actions by modifying the block-spin procedure [12].

We may imagine starting from a lattice of finite volume. After a finite number of steps we arrive at a lattice $\Lambda^N$ which consists of only a single site. The field $\Phi$ on this lattice is some average of an average ... of an average of the original field $\phi^0$. Let us interpret it as magnetization. The action $S^N[\Phi]$ will give us the constraint effective potential – i.e. the free energy – as a function of the magnetization.

We will make the following assumptions on the choice of block-spin in this paper. We assume that $C^i$ is a linear map, and that $C^iC^{i\ast}$ is invertible. (For gauge theories, linearity will have to be given up.) We assume also that $C^{i\ast}$ interpolates constant fields to constant fields, and we impose the following normalization condition

$$\int_{x \in \Lambda^{i+1}} C^i \phi(x) = \int_{z \in \Lambda^i} \phi(z)$$

for constant fields. For the above mentioned choice of block-spin, the equation is an identity for arbitrary fields $\phi$.

Let $\mathcal{H}^i$ be the Hilbert space of square summable functions on $\Lambda^i$, and let $\mathcal{H}_C^i$ be the subspace of functions $\zeta^i$ with vanishing block average $C^i \zeta^i = 0$. Invertibility of $C^iC^{i\ast}$ ensures the existence of the orthogonal decomposition

$$\mathcal{H}^i = \tilde{C}^{i\ast} \mathcal{H}^{i+1} \oplus \mathcal{H}_C^i,$$

where $\tilde{C}^i = (C^iC^{i\ast})^{-1}C^i$. This is true because every $\phi \in \mathcal{H}^i$ can be decomposed as

$$\phi = \tilde{C}^{i\ast} \Phi + \zeta^i$$

with $\Phi = C^i \phi \in \mathcal{H}^{i+1}$ and $\zeta^i = \phi - \tilde{C}^{i\ast} \Phi$. The first summand is obviously in $\tilde{C}^{i\ast} \mathcal{H}^{i+1}$ and the second is in $\mathcal{H}_C^i$ because it vanishes when we apply $C^i$: $C^i \zeta^i = C^i \phi - C^i \tilde{C}^{i\ast} \Phi = \Phi - \Phi = 0$. Also the orthogonal decomposition is true because the scalar product vanishes: $(\zeta^i, \tilde{C}^{i\ast} \Phi) = ((C^iC^{i\ast})^{-1}C^i \zeta^i, \Phi) = 0$.

Notation: We will use letters $z, w...$ for sites in $\Lambda^i$ and $x, y...$ for sites in $\Lambda^{i+1}$. 

3
2 The saddle point approximation

2.1 Parameterization of the action

We wish to compute the action $S_{i+1}[\phi]$ which depends on a field $\Phi$ on the lattice $\Lambda^{i+1}$ from the action $S_i[\phi]$ on the finer lattice $\Lambda^i$. We will make some approximations to perform the calculation. We present a parameterization of the actions which will be preserved by the approximate renormalization group flow.

The action $S_{i+1}[\phi]$ will depend on the field $\Phi$ through a functional $\Psi_i[\Phi]$. The field $\Psi_i$ lives on the lattice $\Lambda^i$. It is called the background field.

The recursion formula will have the form

$$S_{i+1}[\Phi] = S_i[\Psi_i] - \frac{1}{2} \text{tr} \ln \Gamma_i[\Psi_i],$$

$$\Psi_i = \Psi_i[\Phi].$$

Approximations will be made such that

1. The parameterization of the action preserves its form

2. The fluctuation propagator, the effective action and its first two field derivatives, and the background field $\Psi_i[\Phi]$ and its derivative $\Psi_i'[\Phi]$ with respect to the field $\Phi$ will enter into the recursion relations, but they need only be calculated for constant fields. This computational problem is fit for a PC.

We will use a dual notation for derivatives with respect to fields

$$S_i''[\phi](z) \equiv S_{i,z}''[\phi] = \frac{\delta}{\delta \phi(z)} S_i'[\phi],$$

and similarly for second derivatives $S_i'_{z,w}$.

The background field $\Psi_i$ will be determined as a functional of $\Phi$ by the saddle point condition which involves the previous action

$$S_i[\Psi_i] = \min \text{ subject to } \mathcal{C}_i \Psi_i = \Phi.$$

The fluctuation field propagator $\Gamma_i$ lives on lattice $\Lambda^i$. It is a self-adjoint integral operator with kernel

$$\Gamma_i[\phi](z, w), \quad z, w \in \Lambda^i$$

and is defined as pseudoinverse of $S_i''[\phi]$ for all $\phi$ in the sense that

$$\Gamma_i[\phi] S_i''[\phi] \Gamma_i[\phi] = \Gamma_i[\phi].$$

In the case where the field $\phi$ is equal to the background field $\Psi_i[\Phi]$, the fluctuation field propagator is positive definite on $\mathcal{H}_C^i$. It satisfies the constraints (9), viz.

$$\mathcal{C}_i \Gamma_i[\Psi_i] \mathcal{C}_i^* = \Gamma_i[\Psi_i] \mathcal{C}_i^*.$$
Therefore the space $\mathcal{H}^i$ is mapped into $\tilde{\mathcal{H}}^i$; by $\Gamma^i$, and $\Gamma^i$ vanishes on $\tilde{\mathcal{C}}^i$. The trace $\text{tr}$ which appears in the action (6) is to be understood as a trace over $\tilde{\mathcal{H}}^i$.

We will explain later on how the background field and the fluctuation propagator can be evaluated by a self-consistent approximation which involves neglect of higher order terms in gradients of fields.

It is the crux of any real space renormalization group method to find truncated forms of the effective actions which can be parameterized in a manageable form.

Often this is done in an ad hoc fashion which throws away some pieces which are irrelevant in a perturbative sense—higher powers of fields, for instance. This is not really justified because existing irrelevant terms get suppressed in the next RG-step. But new irrelevant terms are created by the marginal and relevant ones at the same time. As a consequence, there is a kind of equilibrium, so that (along the renormalized trajectory) the irrelevant terms are determined by the relevant and marginal ones. They are not necessarily very small and they influence the flow of the marginal and relevant coupling constants. Throwing them away after each renormalization group step introduces therefore a systematic error which accumulates.

In principle there is a better way. One could determine the irrelevant pieces as a function of the marginal and relevant ones by solving a fixed point equation. But until now this is practical only in simplified (hierarchical) models [21].

The motivation for our method of truncation will be described below. We will argue that our approximation becomes more and more accurate the larger the scaling factor (ratio of lattice spacings) $s = a_{i+1}/a_i$. We will present numerical evidence which confirms this.

On the other hand, the saddle point approximation becomes exact in the limit $s \to 1$ (when the nature of the cutoff permits such a limit). It becomes less accurate with increasing scaling factor because the phase space for high frequency modes increases.

The truncation consists of a local approximation to the second derivative $S'' = -W$ of the action. For more precise notation, set

\begin{align*}
-S^i_{z,w}[\phi] &= W^i[\phi](z, w) \\
(W^i[\phi,f]) (z) &= \int_w W[\phi](z, w)f(w)
\end{align*}

We will approximate $W^i$ by a function $\overline{W}^i(z, w|\xi)$ which depends only on the value $\xi$ of the field at one site of the lattice $\Lambda^i$. If $f \in \mathcal{H}^i$ is a function on $\Lambda^i$ then we approximate

\begin{equation}
(W^i[\phi,f]) (z) = \frac{1}{2} \int_w \left[ \overline{W}^i(z, w|\phi(z)) + \overline{W}^i(z, w|\phi(w)) \right] f(w).
\end{equation}

$\overline{W}^i$ can be determined from the knowledge of $W^i$ for constant fields $\phi$. Symmetrization is performed to maintain hermiticity of $W$. 

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To understand the meaning of the approximation, suppose that $S^i$ is the action of a $\phi^4$-theory. Then

$$(S''^i f)(z) = \left[-\Delta + m^2 + \frac{1}{2} \lambda \phi(z)^2\right]f(z). \quad (13)$$

We see that our approximation is exact in this case because the field dependent part of $S''^i$ only involves the field $\phi$ at a single point $z$.

In particular, the approximation is exact for free field theories. Since the saddle point approximation is also exact for free fields, our method is exact for free field theory.

A general action can be decomposed into a potential and a term whose first derivative with respect to the field vanishes for constant fields. We call it a generalized kinetic term.

$$S^i[\phi] = \text{generalized kinetic term} + V^i[\phi]. \quad (14)$$

A field independent contribution to $V^i$ is of no interest. Therefore it suffices to know the derivative $V^i'$. It is uniquely determined if we know the derivative $S^i'$ of the action for constant fields.

Our method consists in deriving recursion relations for $V^i$ and $W^i$ i.e. for the first and second derivatives $S^i''$ and $W^i = -S^i''$ of the action evaluated at constant fields.

Moreover we wish to obtain the effective actions as a function of the block-spin and not only as a function of the background field. Therefore we will need information on the block-spin dependence of the background field. It turns out that the use of the recursion formulae requires knowledge of the derivative of the background field with respect to the block-spin. There is a formula for this:

$$\Psi^i, x[\Phi](z) \equiv \frac{\delta}{\delta \Phi(x)} \Psi^i[\Phi] = (1 - \Gamma^i S^i'')\tilde{C}^i(z, x). \quad (15)$$

For a block-spin $\overline{\Phi}$ which lives on a lattice made of a single site, translation invariance implies that there exists a saddle point $\Psi^i = \Psi^i[\overline{\Phi}]$ of the action which is also constant. Because of the constraint $C^i \Psi^i = \Phi$ and our normalization conventions it follows that

$$\Psi^i[\overline{\Phi}] = \tilde{C}^i \overline{\Phi}. \quad (16)$$

So we know the background field in this case, assuming the saddle point is the minimum. If it is not, this signals a breakdown of stability. We will come back to such possibilities. They are familiar from the Maxwell construction in thermodynamics.

The argument can be extended to show that to any block-spin $\overline{\Phi}$ which is constant on $\Lambda^{i+1}$ there exists a saddle point $\Psi^i[\overline{\Phi}]$ which is constant, assuming that translational invariance is not spontaneously broken. If the invariance under translations by block lattice vectors is not spontaneously broken in the auxiliary theory with expectation values $\langle \overline{\Phi} \rangle$, then the appropriate saddle point
\[ \Psi \] will be invariant under block lattice translations. Therefore it is equal to
the constraint minimum of the action on a single block with periodic boundary
conditions. Now we can appeal to translation invariance of the action under
translations on the fine lattice \( \Lambda^i \) again to conclude that there exists a constant
saddle point unless translational symmetry is spontaneously broken. It may
happen that it is spontaneously broken or the saddle point is not a minimum.
Again, this signals a breakdown of stability.

In view of its interpretation as a propagator for the high frequency modes
of the field which are integrated out in a renormalization group step, the kernel
\( \Gamma^i[\phi](z, w) \) of the fluctuation field propagator must decay exponentially with
distance \( |z - w| \) with decay length at most one block lattice spacing \( a_{i+1} \) (see
[1]). This is a condition which limits the range of applicability of the present
method. It could be monitored during the computation of the renormalization
group flow. When it is violated, this will typically entail a violation of the
locality properties of the next action \( S^{i+1} \) as well. Locality requires that \( S^i_{zw} \)
decays exponentially with the distance between \( z \) and \( w \) with decay length no
larger than one lattice spacing \( a_i \).

When this is violated it is typically a sign that the choice of the block-spin
definition is not appropriate. This can happen for several reasons, see section
8.

2.2 The detailed procedure of saddle point approximation

Let \( S^i \) be a generic action. We now want to evaluate the effective action \( S^{i+1} \)

\[ e^{-S^{i+1}[\Phi]} = \int D\phi \delta(C^i\phi - \Phi)e^{-S^i[\phi]} \]  \hspace{1cm} (17)

in a saddle point approximation. For this purpose we split the field
\( \phi(z) = \Psi^i[\Phi](z) + \zeta^i(z) \) \hspace{1cm} (z \in \Lambda^i).

into the background field \( \Psi^i[\Phi] \) and a fluctuation field \( \zeta^i \). The background field
has also the meaning of a mean field, because in the Gaussian approximation
that we will use for the action \( S^i \) eq.(3) holds

The mean field \( \Psi^i \) is determined as a functional of the block-spin field \( \Phi \) by
a saddle point condition which involves the previous action

\[ S^i[\Psi^i] = \text{min} \text{ subject to } C^i\Psi^i = \Phi \hspace{1cm} . \]

The result is a nonlinear equation for \( \Psi^i[\Phi] \)

\[ S^i[\Psi^i] = C^i\lambda^i \] \hspace{1cm} (18)

\[ C^i\Psi^i = \Phi \] \hspace{1cm} (19)

\( \lambda^i[\Phi](x) \) are Lagrange multipliers, \( x \in \Lambda^{i+1} \).
Expanding the action around the mean field to second order in $\zeta^i$ yields

$$S^i[\Psi_i + \zeta^i] = S^i[\Psi_i] + \int_z S^i_{i,z}[\Psi_i] \zeta^i(z) + \frac{1}{2} \int_{z,w} S^i_{i,zw}[\Psi_i] \zeta^i(z) \zeta^i(w).$$

(20)

The linear term in the expansion vanishes

$$(S^i, \zeta^i) = (C^i \lambda^i, \zeta^i) = (\lambda^i, C^i \zeta^i) = 0,$$

since due to (19) $C^i \zeta^i = 0$. Therefore we get

$$e^{-S^i[\Psi_i]} \approx \int_{H^i} D\zeta^i e^{-\frac{1}{2} \zeta^i S^i[\Psi_i] \zeta^i} e^{-\frac{1}{2} \operatorname{tr} \ln \Gamma^i[\Psi_i]}.$$

(21)

Note that $S^i''$ is a linear operator on $H^i$. In appendix A.1 we show that $\Gamma^i$ can be written as

$$\Gamma^i = S^i'' - \frac{1}{2} \operatorname{tr} \ln \Gamma^i[\Psi_i].$$

(23)

Because of $C^i \Gamma^i = 0 = \Gamma^i C^i$ the fluctuation propagator vanishes on $C^i H^{i+1}$ and is a linear operator $H^i \rightarrow H^i_C$. Therefore we can use the quantity $(\Gamma^i)^{-1} = S^{i''} H_C^{i+1}$ in (22). The extension of $\Gamma^i$ to all of $H^i$ is a pseudoinverse of $S^{i''}$ as described by [8].

The remaining fluctuation integral is then Gaussian with covariance $\Gamma^i[\Psi_i]$ and may easily be performed. The result is

$$S^{i+1}[\Phi] = S^i[\Psi_i] - \frac{1}{2} \operatorname{tr} \ln \Gamma^i[\Psi_i].$$

(24)

The trace $\operatorname{tr}$ is to be understood as a trace over $H^i_C$.

If one uses in (21)

$$\delta(\phi) = \lim_{\kappa \to \infty} e^{-\frac{\kappa}{2} \int \phi(z)^2},$$

one obtains the following alternative expression for the fluctuation propagator.

$$\Gamma^i[\Psi^i[\Phi]] = \lim_{\kappa \to \infty} \left( S^{i''}[\Psi^i[\Phi]] + \kappa C^i C^i \right)^{-1}.$$

(25)

### 2.3 Recursion relations

By iterating the saddle point approximation we obtain $S^{i+2}$ from $S^{i+1}$:

$$S^{i+2}[\varphi] = S^{i+1}[\Psi^{i+1}[\varphi]] - \frac{1}{2} \operatorname{tr} \ln \Gamma^{i+1}[\Psi^{i+1}[\varphi]].$$

Now we need $\Gamma^{i+1}$. It is defined in terms of $S^{i+1''}$ by an equation similar to (23). Therefore we try to get a recursion relation for

$$W^{i}[\phi] = -S^{i''}[\phi].$$
In appendix A.5 the following recursion relation will be deduced by differ-
ing eq.(6). (We neglect to write functional $\Psi^i[\Phi]$-dependencies on the right
hand side.)

\begin{equation}
W^{i+1}[\Phi](x, y) = \\
\int_{z, w \in \Lambda^i} W^i(z, w) \Psi^i, y(z) \Psi^i, x(w) \\
+ \frac{1}{2} \int_{z, w, z_1 \in \Lambda^i} \Gamma^i(z_1, z_2) W^i, z_1(z_2, z_3) \Gamma^i(z_3, z_4) W^i, w(z_4, z_1) \Psi^i, x(z) \Psi^i, y(w) \\
+ \frac{1}{2} \int_{z, w, z_1, z_2 \in \Lambda^i} \Gamma^i(z_1, z_2) W^i, z_1(z_2, z_1) \Gamma^i(z, z_3) W^i, w(z_3, z_4) \Psi^i, y(w) \Psi^i, x(z_4).
\end{equation}

This functional relation is an exact consequence of eq.(6), no localiz ation
approximation has been made yet.

We see that we need an equation for the derivative $\Psi^i$ of the background
field. In appendix A.2 it will be shown that

\begin{equation}
\Psi^i = (1 - \Gamma^i S^i) \tilde{C}^i. 
\end{equation}

The recursion relation is shown in graphical form in figure 2. The graphical
notation is described in figure 1.

We see that the recursion relation for $S^i$ contains a 1-particle reducible
term. This is an artifact of the simple saddle point approximation which will
disappear in the improved version of section 4.

As described in section 2.1 the action $S^i$ can be split into a potential term
$V^i[\phi]$ and a generalized kinetic term whose first field derivative vanishes for
constant fields; the potential term is determined by the derivative $S^i$ evaluated
at constant field.

We give the recursion relation for the first derivative $S^i$ in a general form. Using (see appendix A.3)

\begin{equation}
\Gamma^i, z = \Gamma^i W^i, z \Gamma^i.
\end{equation}

one gets by differentiation of (24)

\begin{equation}
S^{i+1}, x[\Phi] = \int_{z} \Psi^i, x[\Phi](z) \left\{ S^i, z[\Psi^i[\Phi]] - \frac{1}{2} \text{tr} \Gamma^i[\Psi^i[\Phi]] W^i, z[\Psi^i[\Phi]] \right\}. 
\end{equation}

The graphical illustration is shown in figure 3.
\[
- \frac{\delta^n}{\delta \phi(z_1) \cdots \delta \phi(z_n)} S^i[\phi] = \begin{array}{c}
\text{Diagram 1}
\end{array}
\]

\[
- \frac{\delta^n}{\delta \Phi(x_1) \cdots \delta \Phi(x_n)} S^{i+1}[\Phi] = \begin{array}{c}
\text{Diagram 2}
\end{array}
\]

\[
\frac{\delta}{\delta \Phi(x)} \Psi^i[\Phi](z) = \begin{array}{c}
\text{Diagram 3}
\end{array}
\]

Figure 1: Definition of graphical notation for the derivatives of the action and of the background field.

Figure 2: Recursion relation for \(W^i = -S^{i''}\) in saddle point approximation.

Figure 3: Recursion relation for \(-S^{i+1'}\) in saddle point approximation.
For constant block-spin field the derivative of the generalized kinetic term vanishes. Therefore the recursion relation for the potential term has the same form as (29):

$$V_{i+1}^{i+1}[\Phi] = \int_z \Psi_{i+1}^i[z][\Phi](z) \left\{ V^i_{i+1}^{i+1}[\Phi] - \frac{1}{2} \text{tr} \Gamma^i[\Psi^i][W^i_{i+1}^{i+1}[\Psi^i[\Phi]]] \right\}. \quad (30)$$

### 2.4 Justification of the localization approximation; Application to constant fields

Let us summarize what we have achieved so far. Given the functionals $V^i$ and $W^i$ we can compute the fluctuation propagator $\Gamma^i$ via equation (23) and the derivative of the background field $\Psi^i$ via equation (27). This enables us to calculate $W^{i+1}$ and $V^{i+1}$ by means of recursion relations (26) and (30). At least this can be done in principle. Because of the functional nature of these equations the calculation is too complicated for numerical purposes. We will arrive at a significant simplification by arguing that we may focus attention on constant fields.

Fields $\Phi$ with very large derivatives have very small probability because of the kinetic term in the action. We may therefore assume that $\Phi$ does not have such violent behavior.

$\Psi^i[\Phi](z)$ is an interpolation of the field $\Phi$ from the coarser lattice $\Lambda_i^{i+1}$ to the lattice $\Lambda^i$. The interpolation is determined by a minimality condition on the action. Because of the kinetic term in the action, we expect that the result of the interpolation is a smooth function on $\Lambda^i$. This means that we may consider $\Psi^i[\Phi]$ as nearly constant over distances $a_i$, assuming that the scaling factor $s = a_{i+1}/a_i$ is big enough.

By locality $S^i_{z,w}[\Psi^i[\Phi]]$ is very nearly zero if $z$ is not within about one lattice spacing of $w$, and it depends only on the field within a neighbourhood of about one lattice spacing $a_i$ of $z$ and $w$. Therefore we may approximate $W^i[\Psi^i]$ by a hermitian operator $\overline{W}^i$ which involves only the value $\xi = \Psi^i(z)$ of the field at one point (see also (12)),

$$W^i[\Psi^i](z, w) \approx \frac{1}{2} \left( \overline{W}^i(z, w|\Psi^i(z)) + \overline{W}^i(z, w|\Psi^i(w)) \right), \quad (31)$$

This justifies the localization approximation for $W^{i+1}$. $\overline{W}^i$ has locality properties similar to a Laplacian.

To determine $W^{i+1}[\Phi]$ we need $\Gamma^i[\phi]$ for $\phi = \Psi^i[\Phi]$. $\Gamma^i(x, y)$ depends on the field $\phi$ on a domain of diameter about $a_{i+1}$. Unless $\Phi$ is constant, the field $\phi$ need not be nearly constant over such a big domain. But $\Gamma^i[\phi]$ is determined by $S^{ii}[\phi] = -W^i$ and we can use the localization approximation for this latter quantity in order to determine $\Gamma^i$ for general field $\Phi$ if we need it.

Looking now at the recursion relation for $W^{i+1}(x, y)$ we see that three different quantities are involved:

- The fluctuation propagator has an exponential decay of one block lattice spacing $a_{i+1}$ (see [1] and figure [4]).
- The operator $W_i(z, w)$ and its derivatives are very local by assumption and have therefore a range of one lattice spacing $a_i$.

- The derivative of the background field $\Psi_i'$ is also determined by these quantities and has exponential decay with decay length $a_i+1$.

Combining these arguments we see that $W_i^{i+1}[\Phi]$ has similar locality properties on the scale of the new lattice spacing $a_{i+1}$ as $W_i^i$ had on the scale $a_i$.

Therefore we will be entitled to make similar approximations on the next level as before.

Now we can perform functional derivatives of $W^i$ for almost constant fields:

\[
W^i_{,z}[\phi](z_1, z_2) = \frac{1}{2} \left( \overline{W}^{i'}(z_1, z_2|\phi(z_1))\delta(z - z_1) \\
+ \overline{W}^{i'}(z_1, z_2|\phi(z_2))\delta(z - z_2) \right) \\
+ \overline{W}^{i''}(z_1, z_2|\phi(z_1))\delta(z - z_1)\delta(w - z_2) \\
+ \overline{W}^{i''}(z_1, z_2|\phi(z_2))\delta(z - z_2)\delta(w - z_2),
\]

where $'$ denotes the derivative of $\overline{W}(z_1, z_2|\xi)$ with respect to the single variable $\xi$. These formulas are valid when smeared with test functions in variables $z$ and $w$ which are nearly constant over distances $a_i$. 

\[
W^i_{,w}[\phi](z_1, z_2) = \frac{1}{2} \left( \overline{W}^{i''}(z_1, z_2|\phi(z_1))\delta(z - z_1) \\
+ \overline{W}^{i''}(z_1, z_2|\phi(z_2))\delta(z - z_2)\delta(w - z_2) \\
+ \overline{W}^{i''}(z_1, z_2|\phi(z_2))\delta(z - z_2)\delta(w - z_2),
\]

Figure 4: The decay of the fluctuation propagator.
Setting the block-spin field constant the fluctuation propagator becomes a function of the field value. By virtue of (27) the derivative of the background field can be written as a simple derivative:

\[ \Psi^{i,x}(z, x) \equiv \Psi^{i}(z, x) \Phi = \int_{z_1} \delta(z, z_1) + \int_{w} \Gamma^{i}(z, w) \bar{W}^{i}(w, z_1) \bar{C}^{i}(z_1, x). \]  

(34)

For constant block-spin fields, inserting (32) and (33) into the recursion relation (26) for \( W \) gives therefore a recursion relation for \( \bar{W}^{i} \):

\[
\bar{W}^{i+1}(x, y | \Phi) = \int_{z, w \in \Lambda^{i}} \bar{W}^{i}(z, w | \Phi) \Psi^{i}(z, y | \Phi) \Psi^{i}(w, x | \Phi) \\
+ \frac{1}{8} \int_{z_1 \in \Lambda^{i}} \Gamma^{i}(z_1, z_2 | \Phi) \bar{W}^{i}(z_2, z_3 | \Phi) \Gamma^{i}(z_3, z_4 | \Phi) \bar{W}^{i}(z_4, z_1 | \Phi) \\
\times \left\{ \begin{array}{l}
\Psi^{i}(z_2, x | \Phi) \Psi^{i}(z_4, y | \Phi) + \Psi^{i}(z_2, x | \Phi) \Psi^{i}(z_1, y | \Phi) \\
+ \Psi^{i}(z_3, x | \Phi) \Psi^{i}(z_4, y | \Phi) + \Psi^{i}(z_3, x | \Phi) \Psi^{i}(z_1, y | \Phi) \end{array} \right\} \\
+ \frac{1}{4} \int_{z_1, z_2 \in \Lambda^{i}} \Gamma^{i}(z_1, z_2 | \Phi) \bar{W}^{i}(z_2, z_1 | \Phi) \\
\times \left\{ \begin{array}{l}
\Psi^{i}(z_1, x | \Phi) \Psi^{i}(z_1, y | \Phi) + \Psi^{i}(z_2, x | \Phi) \Psi^{i}(z_2, y | \Phi) \\
+ \Psi^{i}(z_3, x | \Phi) \Psi^{i}(z_3, y | \Phi) + \Psi^{i}(z_3, x | \Phi) \Psi^{i}(z_3, y | \Phi) \end{array} \right\} \\
+ \frac{1}{8} \int_{z_1, z_2 \in \Lambda^{i}} \Gamma^{i}(z_1, z_2 | \Phi) \bar{W}^{i}(z_2, z_1 | \Phi) \bar{W}^{i}(z_3, z_4 | \Phi) \Psi^{i}(z_4, z_3 | \Phi) \\
\times \left\{ \begin{array}{l}
\Gamma^{i}(z_2, z_3 | \Phi) \Psi^{i}(z_3, y | \Phi) + \Gamma^{i}(z_2, z_3 | \Phi) \Psi^{i}(z_4, y | \Phi) \\
+ \Gamma^{i}(z_1, z_3 | \Phi) \Psi^{i}(z_3, y | \Phi) + \Gamma^{i}(z_1, z_3 | \Phi) \Psi^{i}(z_4, y | \Phi) \end{array} \right\}.
\]

(35)

For constant fields we have translation symmetry. Therefore the derivative of the potential must be independent of the coordinate:

\[ V^{i,z}(\bar{\Phi}) = \bar{V}^{i}(\bar{\Phi}) = \text{const.} \]  

(36)

To obtain the recursion relation of the effective potential we insert the local approximation for \( \bar{W}^{i} \):

\[
\bar{V}^{i+1}(\bar{\Phi}) = \int_{z} \bar{V}^{i,x}(z | \bar{\Phi}) \bar{V}^{i}(\bar{\Phi}) \\
- \frac{1}{4} \int_{z_1, z_2} \Gamma^{i}(z_1, z_2 | \bar{\Phi}) \bar{W}^{i}(z_2, z_1 | \bar{\Phi}) \left( \bar{V}^{i,x}(z_1 | \bar{\Phi}) + \bar{V}^{i,x}(z_2 | \bar{\Phi}) \right). 
\]  

(37)

We simplify this. By assumption of section [1]

\[ \int_{x \in \Lambda^{i+1}} C^{i}(x, z) = 1. \]
Differentiating the constraint (19)

\[ \int_z C^i(x', z) \Psi^{i,x}(z) = \delta(x' - x), \]  
and integrating over \( x' \) we see that

\[ \int_{z \in \Lambda} \Psi^{i,x}[\Phi](z) = 1. \]

This leads to the simpler recursion relation

\[ V^{i+1}[\Phi] = V^i[\Phi] - \frac{1}{4} \int_{z_1, z_2} \Gamma^i(z_1, z_2|\Phi) \Psi^{i,x}(z_1|\Phi) + \Psi^{i,x}(z_2|\Phi) \]  

(39)

One can use this to determine the constraint effective potential, i.e. the potential on the last lattice \( \Lambda^N \) which consists of one point only.

3 Numerical results

3.1 Getting started

We want to calculate the constraint effective potential on the lattice numerically. We consider an example. For starting action we take the standard \( \phi^4 \)-theory in two dimensions,

\[ S^0(\phi^0) = -\int_{z,w \in \Lambda^0} \frac{1}{2} \phi^0(z) \Delta(z, w) \phi^0(w) + \int_{z \in \Lambda^0} \left( \frac{1}{2} m_0^2 \phi^0(z)^2 + \frac{\lambda_0}{4!} \phi^0(z)^4 \right). \]  

(40)

To calculate \( W^{i}[\Phi] \) and \( V^{i}[\Phi] \) we need \( W^{0|\phi^0}, W^{0|\phi^0}, W^{0|\phi^0}, W^{0|\phi^0} \) and \( \Gamma^0[\phi^0] \) for constant fields \( \phi^0 = \bar{\phi} \). We write \( W^{0} \) (see also (13)) in a symmetric form:

\[ W^{0}(z_1, z_2|\bar{\phi}) = \Delta(z_1, z_2) - \left( m_0^2 + \frac{\lambda_0}{2} \bar{\phi}(z_1)^2 \right) \delta(z_1 - z_2) \]  

(41)

\[ W^{0}(z_1, z_2|\bar{\phi}) = \Delta(z_1, z_2) - \left( m_0^2 + \frac{\lambda_0}{4} (\phi^0(z_1)^2 + \phi^0(z_2)^2) \right) \delta(z_1 - z_2) \]  

(42)

Therefore we have for \( W^{0} \)

\[ W^{0}(z_1, z_2|\bar{\phi}) = \Delta(z_1, z_2) - \left( m_0^2 + \frac{\lambda_0}{2} \bar{\phi}^2 \right) \delta(z_1 - z_2) \]  

(43)

which is not an approximation because equation (31) is *exact* in this case. Now it follows for the constant field \( \bar{\phi} \)

\[ W^{0}(z_1, z_2|\bar{\phi}) = \Delta(z_1, z_2) - \left( m_0^2 + \frac{\lambda_0}{2} \bar{\phi}^2 \right) \delta(z_1 - z_2) \]  

(44)

\[ W^{0}(z_1, z_2|\bar{\phi}) = -\lambda \bar{\phi} \delta(z_1 - z_2) \]  

(45)

\[ W^{0}(z_1, z_2|\bar{\phi}) = -\lambda \delta(z_1 - z_2) \]  

(46)
Similarly, the potential
\[ V^0(\phi) = V^0(\phi^0) \bigg|_{\phi^0} = \frac{\lambda}{3!} \phi^3 \]  
\tag{47}

### 3.2 Relations in momentum space

We have fixed the starting point. Next we want to calculate the recursion relations for \( \overrightarrow{W}^i \) and \( \overrightarrow{V}^i \). To reduce the amount of work we switch to momentum space. Because we insert constant fields we have translation symmetry. Therefore the Fourier transformation looks quite simple. For the notation see appendix C.

\[ \overrightarrow{W}^i(z_1, z_2|\phi) = \int_p \overrightarrow{W}^i(p|\phi)e^{i(p(z_1-z_2))} \]  \tag{48}

\[ \Psi^{i,x}(z|\phi) = \int_p \Psi^i(p|\phi)e^{i(p(z-x))} \]  \tag{49}

\[ \Gamma^i(z_1, z_2|\phi) = \int_{l,q,l'} \Gamma^i(l, q, l'|\phi)e^{i((q+l)z_1-(q+l')z_2)} \]  \tag{50}

and similar for the derivatives, which are now simple derivatives with respect to \( \phi \). \( \Gamma^i \) has only translation symmetry on the block lattice. Therefore it is not diagonal in momentum space (see appendix C).

Transforming the recursion relations into momentum space leads to

\[ \overrightarrow{W}^{i+1}(q|\phi) = \int_l \Psi^{i*}(q + l|\phi)\overrightarrow{W}^i(q + l|\phi)\Psi^{i}(q + l|\phi) \]

\[ + \frac{1}{8} \int_{q',l_1,\ldots,l_4} \Gamma^i(l_1, q', l_2|\phi)\Gamma^i(l_3, q + l_2, l_4|\phi)\Psi^{i*}(q - l_2 + l_3|\phi)\Psi^{i*}(q - l_1 + l_4|\phi) \]

\[ \left\{ \overrightarrow{W}^{i*}(q + q' + l_3|\phi) + \overrightarrow{W}^{i*}(q + l_2|\phi) \right\} \left\{ \overrightarrow{W}^{i*}(q + l_1|\phi) + \overrightarrow{W}^{i*}(q + q' + l_4|\phi) \right\} \]

\[ + \frac{1}{4} \int_{q',l_1,l_2,l_3} \Gamma^i(l_1, q', l_2|\phi)\Psi^{i*}(q - l_2 + l_3|\phi)\Psi^{i*}(q - l_1 + l_3|\phi) \]

\[ \left\{ \overrightarrow{W}^{\mu*}(q' + l_1|\phi) + \overrightarrow{W}^{\mu*}(q' + l_2|\phi) \right\} \]

\[ + \frac{1}{8} \int_{q',l_1,\ldots,l_4} \Gamma^i(l_1, q', l_2|\phi)\Gamma^i(-l_1 + l_2, 0, l_4|\phi)\Psi^{i*}(q + l_3|\phi)\Psi^{i*}(q + l_3 + l_4|\phi) \]

\[ \left\{ \overrightarrow{W}^{\mu*}(q' + l_1|\phi) + \overrightarrow{W}^{\mu*}(q' + l_2|\phi) \right\} \left\{ \overrightarrow{W}^{\mu*}(q + l_3|\phi) + \overrightarrow{W}^{\mu*}(l_4|\phi) \right\} \]  \tag{51}

and

\[ \overrightarrow{V}^{i+1}(\phi) = \overrightarrow{V}^{i*}(\phi) - \frac{1}{4} \int_{q,l,l'} \Gamma^i(l, q, l'|\phi)\Psi^{i*}(-l + l'|\phi) \]

\[ \left\{ \overrightarrow{W}^{\nu*}(q + l|\phi) + \overrightarrow{W}^{\nu*}(q + l'|\phi) \right\}. \]  \tag{52}
We need also the formula (27) in momentum space:

\[ \Psi^\prime_i(q + l|\phi) = C^{i*}(q + l) + \int_{l'} \Gamma_i(l, q, l'|\phi) \Psi^\prime(q + l'|\phi) C^{i*}(q + l') \]  

(53)

3.3 Algorithm

We are now able to calculate all quantities needed. The following algorithm gives an overview of the procedure:

0. \( i = 0 \): Given the starting action fix the values of \( W_0(p|\phi_j) \) and \( V_{0'}(\phi_j) \) for a number of constant field values \( \phi_1, \ldots, \phi_n \) and for all momenta \( p \).

1. Calculate \( W_i(p|\phi_j) \) and \( W_{i'}(p|\phi_j) \) by numerical differentiation.

2. Determine \( \Gamma_i(l, q, l'|\phi_j) \) with the help of the explicit formula (23).

3. Calculate \( \Psi^\prime_i(q + l|\phi_j) \) by (53).

4. Insert all into recursion relations for \( W_{i+1}(q|\phi_j) \) and \( V_{i'+1}(\phi_j) \).

5. Increment \( i \) and goto 1 until \( i = N \).

We want to see numerically how this procedure works to produce the correct – derivative of the – effective potential. Therefore we wrote three programs to compare various methods. A Monte-Carlo program gives us reference values. A combined heatbath and metropolis algorithm is used. The heatbath is used with the kinetic term of the action to produce a Gaussian distributed random value for the fluctuation field value. The following metropolis decides on the basis of the whole action if this value is accepted.

As an alternative we calculated the effective potential perturbatively, using the Gawedzki-Kupianen formalism [8]. The action \( S_i[\phi] \) is split into a kinetic term \( \frac{1}{2} \phi(v')^{-1} \phi \) and a potential \( V_i(\phi) \). \( v' \) is the massless free propagator. Again the notion of fluctuation field \( \zeta^i \) with \( C^i \zeta^i = 0 \) is used and we have a background field \( \Psi^i \):

\[ \phi = \Psi^i + \zeta^i \]  

(54)

The background field is an interpolation of the block-spin field:

\[ \Psi^i = A^i \Phi. \]

The interpolation operator \( A^i \) is defined as

\[ A^i = v^i C^{i*}(v^{i+1})^{-1} \]  

(55)

here \( v^{i+1} \) is the free propagator for the block-spin field:

\[ v^{i+1} = C^i v^i C^{i*} \]  

(56)
\[ V^{i+1} = + + + \]

\[ + \times + \times + \Delta + \times + + \]

\[ + \times + \times + \Delta + \times + + \]

\[ \ldots \]

Figure 5: Perturbative expansion of the effective potential. External lines represent \( A_i^i \Phi \) and internal lines \( \Gamma^i \). 2-point vertices have the weight \( m_i^2 \), 4-point vertices have the weight \( \lambda_i \) and 6-point vertices have the weight \( \gamma_i \).

Inserting the split of the field \( \phi \) into (17) we get for the fluctuation integral

\[ e^{-S^{i+1}[\Phi]} = e^{-\frac{1}{2} \Phi(v^{i+1})^{-1} \Phi - V^{i+1}[\Phi]} = \int_{\mathcal{H}_c^i} D\zeta^i e^{-\frac{1}{2} \zeta^i \Gamma^i \zeta^i - \zeta^i v^{i}[\zeta^i + \zeta^i]} \]

where the fluctuation propagator is defined as

\[ \Gamma^i = v^i - A^i_v v^{i+1} A^{i+} \].

The fluctuation integral can be rewritten as

\[ e^{-V^{i+1}[\phi]} = e^{\frac{1}{2} \sum_{\Psi^i} \bar{\Psi}^i \frac{\Gamma^i}{\Psi^i} e^{-V^i[\Psi^i]} \}

This can be expanded. All graphs in one loop order and up to three fluctuation propagators are taken into account. This leads to the graphs in figure 5.

The scale factor (block length) is the same as for the saddle point approximation and we used a polynomial parameterization of the potentials:

\[ V^i[\phi] = \int_{z \in \Lambda^i} \left( \frac{1}{2} m_i^2 \phi(z)^2 + \frac{\lambda_i}{4!} \phi(z)^4 + \frac{\gamma_i}{6!} \phi(z)^6 \right) \]

\[^1\text{Inserting (54) and (56) one sees the same formula for } \Gamma^i \text{ as in (23). For free fields } v^i \text{ and } S^{i \sigma -1} \text{ are identical.}\]
Figure 6: The derivative of the constraint effective potential is plotted against the block-spin $\Phi$. The bottom line is the original potential. The next upper line is the saddle point approximation and the top line the perturbative calculation. The open squares represent the Monte Carlo results. This was calculated on a $16 \times 16$ lattice with blocking factor 2, $m_0^2 = -0.8$ and $\lambda_0 = 0.4$. Note that the minimum of the potential is where $V' = 0$ (see figure 20.)

3.4 Result

As a result we get the following picture:

- For small field values the perturbative calculation is comparable in accuracy to our method.
- For larger field values the perturbative calculation goes rapidly wrong, while the saddle point approximation remains accurate and has the correct asymptotics (see figure 6).
- For the values of coupling constants which we consider, larger blocks are better than small blocks (see figure 7). This confirms our expectation from the discussion of the localization approximation.

In figure 8 the flow of the potential is shown.
Figure 7: The derivative of the constraint effective potential is plotted against the block-spin $\Phi$. The bottom line is the original potential. The next upper line is the saddle point approximation with blocking factor 2. The top line is the saddle point approximation with blocking factor 4. It is better than with blocking factor 2. The open squares represent the Monte Carlo results. This was calculated on a $16 \times 16$ for $m_0^2 = -0.4$ and $\lambda_0 = 0.4$. 
Figure 8: The flow of the derivative of the constraint effective potential is plotted against the block-spin $\Phi$ in the vicinity of the minimum of the potential. The lines represent $\nabla^i V'$ for $i = 0, 1, 2, 3, 4$ from the bottom and the open squares the Monte Carlo results. This was calculated on a $16 \times 16$ lattice with blocking factor 2, $m_0^2 = -0.4$ and $\lambda_0 = 0.4$. The extension of the lattice is getting smaller after every blocking step by factor 2. On the other side the lattice spacing grows by factor 2, so that the lattice volume stays constant. No rescaling to the unit lattice is performed.
4 Improved saddle point method with normal ordering

4.1 Advanced method with normal ordering

The localization approximation becomes more and more accurate the larger the scaling factor (ratio of lattice spacings) \( s = a_{i+1}/a_i \). On the other hand, the saddle point approximation becomes exact in the limit \( s \to 1 \) (when the nature of the cutoff permits such a limit). It becomes less accurate with increasing scaling factor because the phase space for high frequency modes increases. This is one reason why we choose to consider a fairly sophisticated improvement of the saddle point method.

The self consistent improvement of the saddle point integration consists of two steps. In the first step, the old action \( S^i[\phi] = S^i[\Psi^i + \zeta^i] \) is self-consistently normal ordered in the fluctuation field \( \zeta^i \); at the same time the background field \( \Psi^i = \Psi^i[\Phi] \) is determined. Symbolically

\[
S^i[\phi] =: T^i[\Psi^i, \zeta^i] : \tag{59}
\]

The precise definition and properties of the normal ordered form \( T^i \) of the action \( S^i \) will be described presently. In the second step, the fluctuation field integration is performed, neglecting normal products of higher order than second in the fluctuation field. The integral is again Gaussian, and the result reads

\[
S^{i+1}[\Phi] = T^i[\Psi^i, 0] - \frac{1}{2} \text{tr} \ln \Gamma^i[\Psi^i], \tag{60}
\]

The trace is again over the space \( \mathcal{H}_C^i \) of functions \( \zeta^i \) on \( \Lambda^i \) with vanishing block average, \( C_i \zeta^i = 0 \).

The method of self-consistent normal ordering is very old - see \[22\]; sometimes it is called the Feynman Bogoliubov method. Therefore we will refer to our method here as Feynman Bogoliubov approximation. The definition and properties of the normal ordered action \( T^i \) are as follows.

Let \( d\mu_{\Gamma^i} \) be the normalized Gaussian measure (free field measure) with covariance (propagator) \( \Gamma^i \). Given the fluctuation propagator \( \Gamma^i \), the normal ordered amplitude \( T^i \) is defined as

\[
T^i[\Psi^i, \zeta^i] = \int d\mu_{\Gamma^i[\Psi^i]}(\xi) S[\Psi^i + \zeta^i + \xi]
\]

\[
= \exp \left( \frac{1}{2} \xi^2 \Gamma^i[\Psi^i] \frac{\delta}{\delta \xi} \right) S[\Psi^i + \zeta^i + \xi] \bigg|_{\xi = 0} \tag{62}
\]

The second formula is valid when \( S^i \) is differentiable. The first formula is more general, it can be used to define a normal product expansion also for discontinuous actions \( S^i \) (cp. the section \[5\]).

We use the following notation for functionals with two arguments:

\[
\frac{\delta}{\delta \Psi^i(z)} A[\Psi^i, \zeta^i] = A_{;z}[\Psi^i, \zeta^i], \quad \frac{\delta}{\delta \zeta^i(z)} A[\Psi^i, \zeta^i] = A_{;z}[\Psi^i, \zeta^i].
\]
The notation is a little sloppy. Because \( C^i \zeta^i = 0 \), the second expression only makes sense after smearing with a test function \( f \) which obeys \( C^i f = 0 \), viz.

\[
\int_z f(z) \frac{\delta}{\delta \zeta^i(z)} A[\Psi^i, \zeta^i] = \frac{d}{d\tau} A[\Psi^i, \zeta^i + \tau f] \bigg| \tau = 0.
\]

The background field \( \Psi^i = \Psi^i[\Phi] \) will be determined by the condition that the expansion of the action \( T^i \) in powers of the fluctuation field has no linear term:

\[
T^i[\Psi^i, \zeta^i] = T^i[\Psi^i, 0] + \frac{1}{2} \zeta^i T^i_{;;}[\Psi^i, 0] \zeta^i + \ldots \quad \text{for } C^i \zeta^i = 0. \quad (63)
\]

That is the saddle point condition

\[
\int_z \zeta^i(z) T^i_{;;}[\Psi^i] = 0 \quad \text{if } C^i \zeta^i = 0 \quad (64)
\]

which is equivalent to

\[
T^i[\Psi^i] = C^i \lambda^i, \quad (65)
\]

\[
C^i \Psi^i = \Phi. \quad (66)
\]

\( \lambda^i[\Phi](x) \) are Lagrange multipliers, \( x \in \Lambda^{i+1} \).

Now we specify the fluctuation propagator \( \Gamma^i \) as pseudo inverse of \( T^i_{;;} \) so that

\[
\Gamma^i[\Psi^i] T^i_{;;}[\Psi^i] \Gamma^i[\Psi^i] = \Gamma^i[\Psi^i]. \quad (67)
\]

\[
C^i \Gamma^i[\Psi^i] = C^i \Gamma^i[\Psi^i] C^i = 0. \quad (68)
\]

As in the saddle point approximation the fluctuation propagator satisfies the constraints (5), is positive definite on \( \mathcal{H}^i_{\zeta^i} \), and equal to the inverse of \( T^i_{;;} \) on \( \mathcal{H}^i_{\zeta^i} \). But \( T^i_{;;} \) depends itself on the propagator \( \Gamma^i \). Therefore (67) leads to a gap equation for \( \Gamma^i \).

One sees that the background field \( \Psi^i \) is the minimum of the normal ordered form \( T^i \) of the action \( S^i \) on the surface with prescribed block-spin \( \Phi = C^i \phi \). The Hessian is a quadratic form on the tangent space \( \mathcal{H}^i_{\zeta^i} \) to the surface.

The determination of the fluctuation field propagator \( \Gamma^i \) involves solving a gap equation. The gap equation is obtained by differentiating eq. (62) twice at \( \zeta^i = 0 \).

\[
T^i_{;;} = e^{\left( \frac{1}{2} \xi \xi^i \Gamma^i[\Psi^i] \right)} S^{ii}[\Psi^i + \xi] \bigg| \xi = 0 \quad (69)
\]

\[
\Gamma^i[\Psi^i] = T^i_{;;}[\Psi^i, 0]^{-1} \text{ on } \mathcal{H}^i_{\zeta^i}. \quad (70)
\]

The gap equation is shown in graphical form in figure 11. Figure 9 explains the graphical notation. Further derivatives can be applied and we obtain a generalization of eq. (63) as shown in figure 10.
\[ - \frac{\delta^n}{\delta \xi^i(z_1) \cdots \delta \xi^i(z_n)} T^i[\psi^i, \zeta^i] \bigg|_{\xi^i = 0} = \bigg|_{\xi^i = 0} \]

Figure 9: Graphical notation for normal ordered amplitudes

\[ \bigg|_{\xi^i = 0} = \frac{1}{2} \bigg|_{\xi^i = 0} + \frac{1}{8} \bigg|_{\xi^i = 0} + \cdots \]

Figure 10: Definition of the normal ordered amplitudes. Further legs generated by \( \psi \)-derivatives will be marked by primes.

\[ \bigg|_{\xi^i = 0} = \bigg|_{\xi^i = 0} \]

Figure 11: Gap equation for the fluctuation propagator \( \Gamma^i \) in Feynman-Bogoliubov approximation
This completes the discussion of the improved saddle point approximation. Eq. (60) is still a formidable functional recursion formula. We need to make it practical by our second approximation of localization. The localization approximation has to be made for the second derivative $T^{i;}$ and not for $S''$. In addition to the formula for the second derivative $S''$ there is a formula which expresses the first derivative $S'$ of the ordinary action in terms of the first derivative $T^{i;}$ of the previous normal ordered action; this furnishes the potential up to an additive constant.

We have a similar formula for the derivative of the background field:

$$\Psi^i_{,x}(z) = (1 - \Gamma_B^{i} T^{i;}) \tilde{C}^i(z, x).$$  \hspace{1cm} (71)

where $\Gamma_B^{i}$ is defined as inverse of $T^{i;}$ on $H_i$. In contrast to $\Gamma^{i}$ it is defined as inverse of the mixed derivative $T^{i;}$, which involves derivatives with respect to the fluctuation field and the background field.

**Remark:** It can be shown with some effort that the two propagators are actually equal, i.e. $\Gamma_B^{i}[\Psi^i] = \Gamma^{i}[\Psi]$ when $\Psi^i$ is the background field. See appendix B.

### 4.2 Gaussian integration

We wish to evaluate the integral (64) for $S^{i+1}$, using the quadratic approximation (59,63) for $S^i$, viz.

$$S^i[\Psi^i + \zeta^i] = T^i[\Psi^i, 0] + \frac{1}{2} : \zeta^i T^i[\Psi^i, 0] \zeta^i :$$

We undo the normal ordering again. Dropping arguments $\xi = 0$ we obtain

$$S^i[\Psi^i + \zeta^i] = T^i[\Psi^i] + \frac{1}{2} \zeta^i T^i[\Psi^i] \zeta^i - \frac{1}{2} \text{tr} T^i[\Psi^i] \Gamma^{i}[\Psi^i].$$  \hspace{1cm} (72)

The trace is over $H_i$. By arguments similar to appendix A.1 the fluctuation propagator has the explicit form

$$\Gamma^{i} = T^{i;}^{-1} - T^{i;}^{-1} C^{i*}(C^{i}T^{i;}-1C^{i*})^{-1}C^{i}T^{i;}-1.$$  \hspace{1cm} (73)

As in the saddle point approximation there is an alternative formula for the fluctuation propagator

$$\Gamma^{i}[\Psi^i[\Phi]] = \lim_{k \to \infty} (T^{i;}[\Psi^i[\Phi]] + \kappa \tilde{C}^{i*} \tilde{C}^i)^{-1}. $$  \hspace{1cm} (74)

We argue that the last term in (72) is a constant independent of $\Psi^i$. Multiplying $T^i$ from the right of (73) gives

$$\Gamma^{i} T^{i;} = 1 - T^{i;}^{-1} C^{i*}(C^{i}T^{i;}-1C^{i*})^{-1}C^{i}. $$  \hspace{1cm} (75)
We take the trace of (75)
\[\text{tr}(\Gamma T^i_{\cdot i}) = \text{tr} 1 - \text{tr} \left( T^i_{\cdot i}^{-1} C^i_* (C^i_* T^i_{\cdot i}^{-1} C^i) \right)\]
\[= \text{tr} \delta_z(0) - \text{tr} \delta_x(0)\]
\[= \sum_z 1 - \sum_x 1 = \text{const.} \quad \text{q.e.d.}\]

Now we can again evaluate the functional integral (1) in Gaussian approximation with the result
\[S_{i+1}[\Phi] = T^i[\Psi^i[\Phi],0] - \frac{1}{2} \text{tr} \ln \Gamma^i[\Psi^i]. \tag{76}\]

4.3 Relations between derivatives of the normal ordered action

We derive a relation between the \(\Psi^i\)- and \(\xi\)-derivative by using the following well known change of covariance lemma for Gaussian measures \([9]\). If the covariance \(\Gamma_s\) depends parametrically on a variable \(s\), then
\[\frac{d}{ds} \int d\mu_{\Gamma_s}(\phi) A[\phi] = \frac{1}{2} \int d\mu_{\Gamma_s}(\phi) \frac{\delta}{\delta \phi} \Gamma_s \frac{\delta}{\delta \phi} A[\phi], \tag{77}\]
where \(\dot{\Gamma}_s = d\Gamma_s/ds\).

We denote \(\Psi^i\)-derivatives by commas \(\cdot\) and \(\xi\)-derivatives by semicolons \(\cdot\); as before. Applying the change of covariance lemma to the definition (61) of the normal ordered amplitude we obtain the relation
\[T^i_{\cdot z}[\Psi^i] = T^i_{\cdot z}[\Psi^i] + \frac{1}{2} \text{tr} \Gamma^i_{\cdot z}[\Psi^i] T^i_{\cdot w}[\Psi^i]. \tag{78}\]

After another \(\xi\)-derivative we get
\[T^i_{\cdot z;w}[\Psi^i] = T^i_{\cdot w;}[\Psi^i] + \frac{1}{2} \text{tr} \Gamma^i_{\cdot z}[\Psi^i] T^i_{\cdot w;}[\Psi^i]. \tag{79}\]

4.4 Recursion relation for the first and second derivatives of the action

We differentiate the recursion relation (76) to obtain a formula for the first derivative of the action \(S^i+1\). The derivative of the \(\text{tr} \ln\) is computed in the same way as before. The result reads
\[S^i+1, x = \int_z \Psi^i, x(z) \left( T^i_{\cdot z}[\Psi^i] - \frac{1}{2} \text{tr} \Gamma^i_{\cdot z}[\Psi^i] T^i_{\cdot z;}[\Psi^i] \right). \tag{80}\]

We re-express the \(\Psi^i\)-derivative in the first term using eq. (78). A cancellation occurs and we obtain the final form of the recursion relation for the first derivative of the action.
\[S^i+1, x = \int_z \Psi^i, x(z) T^i_{\cdot z;}[\Psi^i]. \tag{81}\]
Next we turn to the second derivative. We differentiate eq. (81) once more. Since $T^{i,z}$ is the $\xi$-derivative of the normal ordered action we can use eq. (78) again to evaluate the derivative of $T^{i,z}$ with the help of

$$\Gamma^{i,z}[\Psi^i] = -\Gamma^i[\Psi^i]T^{i,z}[\Psi^i]\Gamma^i[\Psi^i]. \quad (82)$$

Differentiating the factor $\Psi^i_{,x}(z)$ produces a term

$$\int_z \Psi^i_{,xy}(z) T^i_{;z}. \quad (83)$$

Differentiating the constraint (66) twice shows that $\Psi^i_{,xy}(z) \in H^i_\Omega$ because

$$\int_z C^i(x', z) \Psi^i_{,x}(z) = \delta(x' - x), \quad (84)$$

implies

$$\int_z C^i(x', z) \Psi^i_{,x}(z) = 0. \quad (85)$$

The saddle point condition (64) implies therefore that expression (83) vanishes. As a result we obtain the recursion formula for the second derivative of the action

$$S^{i+1}_{,xy} = \int_{z,w} \Psi^i_{,x}(z) T^i_{;z}, w \Psi^i_{,y}(w)\quad (86)$$

$$= \int_{z,w} \Psi^i_{,z}(z) \left( T^i_{;zw} - \frac{1}{2} \text{tr}(\Gamma^i T^i_{;w}) \Gamma^i T^i_{;w} \right) \Psi^i_{,y}(w) \quad (87)$$

We neglected to indicate arguments $\Psi^i$ of propagators and normal ordered action. We see that there is no 1-particle reducible contribution.

A subtle feature of this formula is the appearance of two different vertices in the last term; one involves a $\Psi^i$-derivative and the other a $\xi$-derivative to create the external leg. This feature is familiar from Schwinger Dyson equations, cp. section 5 below. Neglecting the difference would only neglect a two loop correction, though.

The recursion relations in Feynman Bogoliubov approximation are shown in figures 12 and 13. They involve propagators which are to be determined as solutions of the gap equation (69), shown graphically in figure 11.

### 4.5 Derivative of the background field

The recursion relations involve the derivative $\Psi^i_{,x}$ of the background field. Differentiating the saddle point condition (64) we obtain

$$\int_{z,w} \zeta^i(w) \Psi^i_{,x}(z) T^i_{;w}, z[\Psi^i] = 0 \quad (88)$$
Figure 12: Recursion relation for $S^{i+1'}$ in the Feynman Bogoliubov approximation

Figure 13: Recursion relations for the $S^{i+1''}$ in the Feynman Bogoliubov approximation. The second formula is obtained from the first with the help of eq. (73).
We claim that this equation and the constraint (84) are solved by
\[ \Psi^{i,x}(z) = \tilde{\mathcal{C}}^{i*}(z, x) - \int_{w,u} \Gamma_{B^{i}}(z, w)T^{i; w,u}\tilde{\mathcal{C}}^{i*}(u, x), \] (89)
where \( \tilde{\mathcal{C}}^{i*} = \mathcal{C}^{i*}(\mathcal{C}\mathcal{C}^{i*})^{-1} \) as before and \( \Gamma_{B^{i}} \) is defined as the inverse of \( T^{i} \), on \( \mathcal{H}_{\mathcal{C}}^{i} \). The proof proceeds as in the saddle point section.

The remark at the end of the last subsection leads to a simplification of (89) because \( \Gamma^{i} \) can be substituted for \( \Gamma_{B^{i}} \). See appendix B.

4.6 Discussion of the localization approximation

There are new aspects of the localization approximation which need to be discussed. The consideration of this section assumed that the starting amplitude \( S^{i} \) is accurately known. Because of this, it is appropriate to regard the recursion relations as recursion relations for the normal ordered amplitudes \( T^{i} \), and to make the localization approximation for these amplitudes, not for \( S^{i} \). The localization approximation for \( T^{i} \) is justified by appealing to the same arguments as before: In computing \( S^{i+1} \) we only need \( T^{i} \); for smooth fields.

Thus we may regard one step in the recursion relations as composed of three parts

1. Localization approximation on \( T^{i} \) for smooth fields.
2. Computation of \( S^{i+1} \) from \( T^{i} \) for smooth fields.
3. Computation of \( T^{i+1} \) from \( S^{i+1} \).

When proceeding in this way, the result for \( S^{i+1} \) must be substituted into the (gap) equations which determine the normal ordering. In general there result graphs which contain propagators from two successive length scales \( i \) and \( i + 1 \). This complication does not arise when only 1-loop graphs are retained.

There is one further technical complication. In contrast with \( S^{i''} \), \( T^{i''} \) is not necessarily exactly invariant under arbitrary translations by lattice vectors in \( \Lambda^{i} \) but only under block lattice translations. Therefore it is not easy to calculate its pseudoinverse (the fluctuation field propagator) exactly. One can proceed as follows, however.

Because \( T^{i} \) and \( \Gamma^{i} \) are determined by a gap equation one can start with some guess for \( T^{i} \), e.g. with the pseudoinverse of \( S^{i''} \) which was used in the saddle point approximation. Then one iterates the gap equation.

The gap equations are here considered for reasons of accuracy and not because there are infrared problems. Quite on the contrary, the propagators must have exponential decay. Therefore the iteration can be expected to converge very fast.
5 Schwinger Dyson Equations

In principle one can improve on the approximations for the fluctuation field integral if the action $S^i$ is known. In any case, more accurate methods are applicable in the first step, since the starting action $S^0$ is known by assumption.

If $S^i$ is only known approximately, there can be problems because the correction terms might depend on details of the action which are neglected when a localization approximation is made. This is the crucial point to discuss. It will turn out that inclusion of two loop corrections requires that also the localization approximation is improved to the next order. One needs to consider third derivatives of the normal ordered action evaluated at nearly constant fields.

Here we discuss the evaluation of the fluctuation field integral by solution of Schwinger Dyson equations.

Let $\Gamma$ be a positive semidefinite operator on $H$. Then the normalized Gaussian measure $d\mu_\Gamma$ on $H$ is defined. Its characteristic function is

$$\int d\mu_\Gamma(\phi) \exp iq\phi = \exp \left(-\frac{1}{2}q\Gamma q\right).\quad (90)$$

If $A[\phi]$ is a function of $\phi$ which is integrable with respect to the Gaussian measure $d\mu_\Gamma$, it can be written in normal ordered form, as we know.

$$A[\phi] = :B[\phi]: \quad (91)$$

$$B[\phi] = \int d\mu_\Gamma(\xi)A[\xi + \phi] \quad (92)$$

The normal ordering prescription depends on $\Gamma$.

If $\Gamma$ is strictly positive and bounded away from zero, then $B[\phi]$ is an entire analytic function of $\phi$ and can therefore be expanded in a power series. This yields the expansion of $A$ in normal products of $\phi$. It exists whether or not $A$ is continuous or differentiable. This is important when one wishes to deal with singular cases such as a discrete Gaussian model, where the action depends on the integer part of a real lattice field.

Here we are interested in cases where $\Gamma$ has zero eigenvalues. The 1-dimensional Dirac measure $\delta(x)dx$ is the prototype of such a measure; it is not a pathological case. In this degenerate case $\Gamma$ projects on a subspace $H_C$ of $H$ and $B[\phi + \xi]$ is an entire analytic function of $\xi \in H_C$, assuming the restriction $\Gamma_C$ of $\Gamma$ to $H_C$ is bounded away from zero.

The normal ordering commutes with shifts of the field and with derivatives with respect to $\phi$.

We need the following general integration by parts formula

$$\int d\mu_\Gamma(\phi)A[\phi]e^{-V[\phi]} = B[\Gamma \frac{\delta}{\delta \xi}] \int d\mu_\Gamma(\phi)e^{-V[\phi+\xi]} \bigg|_{\xi = 0}.\quad (93)$$

This generalizes the well known special case $[3]$

$$\int d\mu_\Gamma(\phi)\phi(z)e^{-V[\phi]} = \int_w \Gamma(z, w)\frac{\delta}{\delta \xi(w)} \int d\mu_\Gamma(\phi)e^{-V[\phi+\xi]} \bigg|_{\xi = 0}. \quad (94)$$
and can be readily proven with the help of eq. (90), as follows.

Proof of the general integration by parts formula: We use the Taylor formula in compact form

\[ e^{a \delta \phi} F(\phi) = F(\phi + a). \]  

(95)

Since an arbitrary functional \( B[\phi] \) admits a functional Fourier transform, it suffices to prove the formula (93) for the special case

\[ B[\phi] = e^{i q \phi}. \]

We compute

\[
\int d\mu (\phi) e^{i q \Gamma \delta \phi} A(\phi) \\
= N_\Gamma \int D\phi A[\phi] e^{-i q \Gamma \delta \phi} e^{-\frac{i}{2} \phi \Gamma^{-1} \phi} \\
= N_\Gamma \int D\phi A[\phi] e^{-\frac{i}{2} (\phi - i q \Gamma)^{-1} (\phi - i q \Gamma)} \\
= \int d\mu (\phi) A[\phi] e^{i q \phi + \frac{i}{2} \phi \Gamma q} \\
= \int d\mu (\phi) : e^{i q \phi} : \\
\]

Q.E.D.

The general integration by parts formula has many uses. Suppose we define expectation values of an interacting theory

\[
\langle A \rangle = Z^{-1} \int d\mu (\phi) A[\phi] e^{-V[\phi]}, \quad (97)
\]

\[
Z = \int d\mu (\phi) e^{-V[\phi]}, \quad (98)
\]

and the generating function \( G[\xi] \) of amputated Green functions of the interacting theory

\[
e^{G[\xi]} = \int d\mu (\phi) e^{-V[\phi+\xi]}.
\]

(99)

We preferred to introduce an unnormalized generating function, so that \( G[0] = \ln Z \). The normalized generating function is

\[
K[\xi] = G[\xi] - G[0].
\]

We may then use the integration by parts formula to write

\[
\langle A \rangle = e^{-G[0]} \left( \delta \frac{\delta}{\delta \xi} e^{G[\xi]} \right) \bigg|_{\xi = 0}
\]

(100)

The Schwinger Dyson equation for the generating function \( G \) is another application of the integration by parts formula. It is obtained by differentiating the
definition of $e^G$ with respect to $\xi$ and converting the resulting downstairs factor $\frac{\delta}{\delta \xi} V$ into a differential operator. Suppose

$$V[\phi] =: W[\phi] :$$

Then

$$- \frac{\delta}{\delta \xi(z)} G[\xi] = e^{-G[\xi]} W_z [\xi + \Gamma \frac{\delta}{\delta \xi} e^{G[\xi+\zeta]}] \bigg|_{\xi = 0} = 0 \tag{101}$$

If $\Gamma$ has zero modes, this is only valid for the directional derivative in $H_C$ directions, i.e. when multiplied from the left with $\Gamma$. Additive constants in $G$ drop out.

Let

$$V^i[\Psi, \zeta] = S^i[\Psi + \zeta] - \frac{1}{2} \zeta \Gamma^i_C[\Psi]^{-1} \zeta, \tag{102}$$

$$e^{G^i[\Psi, \zeta]} = \int d\mu(z) e^{-V^i[\Psi, \zeta + \xi]} \text{ for } \xi \in H^i_C, \tag{103}$$

$$\Gamma = \Gamma^i[\Psi^i[\Phi]]. \tag{104}$$

We chose to work with an unnormalized generating function for amputated Greens functions again; the normalized generating function is

$$K^i[\Psi, \xi] = G^i[\Psi, \xi] - G^i[\Psi, 0]$$

$\Gamma^i_C[\Psi]^{-1}$ is the inverse of the restriction of $\Gamma^i[\Psi]$ to its range $H^i_C$. Dropping an additive constant, the exact effective action is

$$S^{i+1}[\Phi] = -G^i[\Psi, 0] - \frac{1}{2} tr \ln \Gamma^i[\Psi], \tag{105}$$

$$\Psi = \Psi^i[\Phi]. \tag{106}$$

The second term compensates for the transition from the unnormalized Gaussian measure in definition (1) to a normalized one. It involves a trace over $H^i_C$, cp. section 2.

The Schwinger Dyson equation (101) carries over literally; $V, W, G$ all depend parametrically on $\Psi$. The single power of $\zeta$ which arises from differentiating the second term in eq. (102) converts into a differential operator. As a result one obtains

$$S^i[\Psi + \xi] = : T^i[\Psi, \xi] : \tag{107}$$

$$\xi(z) = e^{-G^i[\Psi, \xi]} \int_w \Gamma^i(z, w) T^i_{zw}[\Psi, \xi + \Gamma^i \frac{\delta}{\delta \xi} e^{G^i[\Psi, \xi + \zeta]}] \bigg|_{\xi = 0} = 0 \tag{108}$$

$$T^i_{zw} = \frac{\delta}{\delta \xi(w)} T^i. \tag{109}$$
The above Schwinger Dyson equations are true for any choice of the background field $\Psi$ and for any choice of the propagator $\Gamma^i$. A great simplification results if they are chosen judiciously. It is customary to call the amputated Green functions vertices. $n$-point vertices are obtained as $n$-th $\xi$-derivatives of $K^i$ at $\xi = 0$. They depend parametrically on the background field. In this terminology the first and second derivative $K^i_{\xi_1}$ and $K^i_{\xi_2}$ of $K^i$ with respect to $\xi$ are the one- and two point vertices. Only the directional derivatives in directions in $\mathcal{H}_\xi^i$ are defined. We require

$$\zeta_1 K^i_{\xi},[\Psi,\xi=0] = 0, \quad (110)$$

$$\zeta_1 K^i_{\xi},[\Psi,\xi=0][\zeta_2] = 0, \quad (111)$$

at $\Psi = \Psi^i[\Phi]$ and for arbitrary $\zeta_1, \zeta_2 \in \mathcal{H}_\xi^i$. These conditions are shown graphically in figure 14. In the figures, the vertices obtained by differentiating $K^i$ are cross hatched. The Schwinger Dyson equations for the $n$-point functions are obtained by differentiating eq. (108) with respect to $\xi$ at $\xi = 0$. The resulting equations for one, two and three point functions are shown in graphical form in figure 15.

In principle the vertices can be computed from the Schwinger Dyson equations by iteration. As a consequence of conditions (110,111) the equations for the one- and two point functions substitute for the earlier equations for the background field $\Psi$ and fluctuation propagator $\Gamma^i$. In zeroth approximation, the vertices with three and more legs are equal to the corresponding normal ordered amplitudes ("bare vertices"). The iteration proceeds by inserting the previous approximation for the vertices on the right hand side of the equations for the three-point functions, and similarly in higher ones.

Because we have an infrared cutoff $a_i^{-1}$, we can expect that the iteration converges fast.

The effective action can be computed from eq. (105). We determine its $\Phi$-derivative by differentiating the definition (103) of $G^i[\Psi,0]$. Using the change
Figure 15: Schwinger Dyson equations for the one-, two- and three-point vertices. The first two equations serve to determine the background field and the fluctuation field propagator.
Figure 16: Schwinger Dyson Gap equation in 1-loop approximation

of covariance lemma (107) again, we obtain

\[ \frac{\delta}{\delta \Psi(z)} G^i[\Psi,0] = e^{-G[\Psi,0]} \int d\mu_T(\zeta) \left( \frac{1}{2} \delta \Gamma \cdot \delta \zeta - V^i,\zeta[\Psi,\zeta] \right) e^{-V[\Psi,\zeta]} \] (112)

where \( \Gamma = \Gamma^i[\Psi] \). We restrict attention to \( \Phi \)-derivatives. From the definition of \( V^i \) we get

\[ \frac{\delta}{\delta \Phi(x)} V^i[\Psi^i[\Phi],\zeta] = \int \Psi_{,x}(z) \left( S^i,\zeta + \frac{1}{2} \zeta \Gamma^{-1}_C \Gamma \Gamma^{-1}_C \zeta \right) \] (113)

This is inserted into eq. (112). We convert again the downstairs factors of \( \zeta \) into differential operators, using the integration by parts formula. One term cancels against the second term in eq. (105) and we obtain the final result

\[ \frac{\delta}{\delta \Phi(x)} Q^i[\Psi,\zeta] = T^i,\zeta[\Psi,\zeta + \Gamma \delta \delta \zeta] e^{K^i[\Psi,\zeta]} \] (114)

The result is shown in graphical notation in figure 17. The derivative of the effective action can be computed when the \( K^i \)-vertices have been determined from the Schwinger Dyson equation.

We consider next the equation for the background field. Comparing the above definition of \( Q^i \) with the Schwinger Dyson equation (eq. (108) at \( \xi = 0 \)) for the 1-point function we see that the equation for the background field takes the form

\[ \zeta Q^i[\Psi] = 0 \quad \text{for all} \quad \zeta \in \mathcal{H}_C^i \] (116)

The Schwinger Dyson equation for the 2-point function implies

\[ \Gamma^i Q^i \Gamma^i = \Gamma^i. \] (117)

One deduces (in the by now familiar way) the formula for the derivative of the background field,

\[ \Psi^i_{,x}(z) = \tilde{C}^i(x, z) - \int_{u,w} \Gamma_B^i(z, w) Q^i_{;u}(w) \tilde{C}^i_{;u}(u, x). \] (118)

34
\[ N + 1 + \frac{1}{6} N + \cdots \]

Figure 17: Recursion relation for \( S'' \) in the Schwinger Dyson approach.

\[ \Gamma_{B^i} \] is defined as inverse of the mixed second derivative \( Q^i ;_z ;_z \). Also we have

\[ \Gamma^i , z = -\Gamma^i Q^i , z \Gamma^i . \quad (119) \]

One may finally differentiate (114) once more to get an equation for the second derivative of the effective action. The term involving the second derivative of the background field vanishes again because the constraints imply that \( \Psi^i , xy \in \mathcal{H}_C \) (see (85)). As a result

\[
\frac{\delta}{\delta \Phi(x) \delta \Phi(y)} S^{i+1} = \\
\int_{zw} \Psi_{x,z}(z) \left( T^i , z , w [\Psi, \Gamma \delta \delta \zeta] + \int_u T^i , z , u [\Psi, \Gamma \delta \delta \zeta] \left( \Gamma , w \frac{\delta}{\delta \zeta} \right) (u) \\
\right. \\
+ \left. T^i , z [\Psi, \Gamma \delta \delta \zeta] \right) e^{K^i [\Psi, \zeta]} \bigg| \zeta = 0 \Psi , y (w) \\
\]

One can insert (78) and eq. (119) for the first term in order to eliminate a reference to the first \( \xi - \)derivative of \( T^i \).

The equations for the first and second derivative of the effective action are shown in Figures 17 and 18. Again, there are no 1-particle reducible contributions.

From these equations one can see what are the leading corrections to the approximations which were considered in the previous section. To make the comparison we take the zeroth loop order of the Schwinger Dyson equations. After inserting b) into c) in figure 15 in zeroth order we get the simplified vertices shown in figure 19. Using these in the recursion relation for the derivatives of the effective action and keeping only 1-loop graphs leads to the familiar result of the previous section (figures 12 and 13).

5.1 Two loop corrections

When we make appropriate localization approximations, the Schwinger Dyson equations should become integro-differential equations for matrix functions of a single variable. We see, however, that the equations involve normal ordered amplitudes with more than two high frequency legs.
Figure 18: Recursion relation for $S''$ in the Schwinger Dyson approach. The prime denotes a derivation with respect to the background field.

Figure 19: Schwinger Dyson equations of the first three vertices in zeroth order
The Schwinger Dyson equations for the vertices and the formulae for the derivatives of the effective action require a separate discussion.

We see that the two loop terms in the recursion relation for the second derivative of the effective action involves normal ordered amplitudes with three high-frequency (“hard”) legs. They are not known if we only know the second derivative of the normal ordered action evaluated at nearly constant field.

Therefore we need consider third derivatives of normal ordered actions evaluated at nearly constant fields. This introduces quantities $\mathcal{W}(z, w, u|\eta)$. They have exponential decay with the tree distance of $z, w, u$ with decay length not bigger than one lattice spacing.

We will also need recursion relations for these 3-point quantities. They can be obtained from eq. (120) by differentiating once more. We will not write the result explicitly, the graphs involved are similar to those in the Schwinger Dyson equation (figure 13) for the 3-point vertices, except that the external legs are soft ones. It appears that this equation involves a normal ordered amplitude with four hard legs (in the last graph), and so we seem to be in trouble again. But this is actually not really so. Because the external legs are soft, there are actually only two independent hard relative momenta in this graph as in the remaining graphs. The arguments of two hard legs which join to different soft external legs may therefore be identified, and the knowledge of up to third derivatives of the normal ordered action suffices to determine this vertex accurately enough.

The strategy for the Schwinger Dyson equations is different. There are vertices with still more hard legs. We regard the determination of the fluctuation propagator and of the (cross hatched) vertices as one task, whose input is the effective action $S^\prime$. We propose to make localization approximations for the normal ordered actions $T^\prime$, but not for $S^\prime$. Therefore $S^\prime$ may be regarded as accurately known for arbitrary fields from the previous application of the recursion formula. Therefore one may take derivatives of any order to create hard legs. One needs to insert the resulting formulae into the Schwinger Dyson equations. As a result, one has to evaluate graphs involving propagators at two different length scales. This complication arose already in the Feynman Bogoliubov approximation in section 4.

The practical evaluation of all the two loop graphs requires a serious effort in high performance computing, unless one is willing to approximate the rather complicated exact formulae for the bare propagators and vertices. We are not prepared to elaborate on this. In this paper we only discuss the actual evaluation of 1-loop graphs.

6 Relation to other approaches

6.1 Flow equation for one particle irreducible average actions

In this paper we are chiefly interested in blocking to discrete “flows” from lattice to lattice because we wish to investigate the accuracy of variants of the method
by comparison with Monte Carlo simulations. But the method itself is not limited to this situation. We could consider actions $S^K$ which depend on a continuously variable cutoff $K$ in place of the discrete length scales $a_i$, and we may leave $\kappa$ finite, or take the limit $\kappa \to \infty$ at the end. The flow will depend on a substitute for the projector

$$R_K = \kappa C^i C^i.$$  \hspace{1cm} (121)

Differentiating the recursion relation in saddle point approximation, eq. (6) with respect to $t = \ln K$, we get from the formula (25) for the fluctuation propagator

$$\frac{\partial}{\partial t} S^K = \frac{1}{2} \text{tr} \left( S''^K + R_K \right)^{-1} \frac{\partial}{\partial t} R_K.$$  \hspace{1cm} (122)

This is Wetterich’s flow equation for average actions [24]. The initial condition is also the same. We conclude that Wetterich’s effective actions can be interpreted as Wilson effective actions, considered as functions of the background field.

This is a remarkable result because Wetterich’s average actions are not by definition equal to a Wilson effective action. They are one-particle irreducible while a computation of a Wilsonian effective action in perturbation theory contains one particle reducible pieces. A one particle reducible piece occurs also in Polchinski’s version of a flow equation [20].

However, there are no one-particle reducible pieces in the recursion relations when the Wilson effective action is considered as a function of the background field. For given values of the block-spin, the background field depends not only on the block-spin but also on the action. When the background field is determined accurately enough (e.g. in Feynman Bogoliubov approximation) no one particle reducible graph appears.

6.2 Perfect actions

Let us also clarify the relation of our approach with the work of Hasenfratz and Niedermayer [12] on perfect actions. They also use a background field $\Psi$. Let $\sigma : \Lambda^i \to \Lambda^{i+1}$ be the map which scales the coordinates of every lattice point by a factor $s = a_{i+1}/a_i$. This induces a map of fields $\sigma^* : \mathcal{H}^{i+1} \to \mathcal{H}^i$ defined by $\sigma^* \Phi(z) = s^i \Phi(\sigma z)$, where $l$ is determined by the dimension of the field. When seeking a perfect action in tree approximation, the $\text{tr} \ln \Gamma^i$ in the recursion relation is neglected, and the fixed point condition which defines a perfect action in tree approximation reads

$$S^i(\Psi^i[\Phi]) = S^i(\sigma^* \Phi).$$  \hspace{1cm} (123)

6.3 Legendre transforms of higher order

It was pointed out to us by Yu. Pismak [19] that our method invites the application of the formalism of Legendre transform of second order [13], and of higher order when higher order corrections are considered. This yields expansions in skeleton graphs similar to those familiar from renormalization theory
with two-point functions and vertices which are determined as solutions of the Schwinger Dyson equations.

7 Theories with fermions

In principle the considerations of this paper apply also to theories with fermions, if one knows how to block them. The $\delta$-function in the defining recursion relation (17) for the actions has only a symbolic meaning in this case, but the split of the integration over Fermi fields into integration over high and low frequencies is nevertheless possible in the same way as for Bose fields. But because of the different formula for Grassmannian Gaussian integrals, the second term in the recursion relation (11) changes sign. In the formulae for normal ordered actions and for derivatives of actions, this is taken into account by the familiar rule

\[ a \text{ factor } (-1) \text{ for every closed loop} \]

It is known how to block from fermions with the right number of flavors in the continuum to Kogut Susskind lattice fermions, and on from there. How to do this in a way which preserves locality was first discovered by G. Mai and is reviewed in [17]; Bietenholz and Wiese found a similar scheme in their studies of perfect actions [4].

In fermionic theories the improvement of the saddle point method by self-consistent normal ordering can be important, see the next section.

8 How locality can fail

Things can go badly wrong when locality of the effective action fails due to a bad choice of block-spin. It is therefore essential to monitor whether the method can be expected to give reliable results by monitoring the range

\[
\text{average } \frac{\int_{w} |(z-w)W(z,w|\Psi)|}{\int_{w} |W(z,w|\Psi)|} \quad (124)
\]

of the interaction for each value of the constant field which enters into the computation of the desired quantity.

Failures of locality can occur for several reasons which require different remedies.

One sees from the recursion relation for the derivatives of the action that good locality properties can only be expected if the fluctuation propagator $\Gamma^i(z,w|\Psi)$ decays with distance $|z-w|$ with decay length no more than one block lattice spacing.

For a $\phi^4$-action

\[
\Gamma = \lim_{\kappa \to \infty} \left( -\Delta + m^2 + \frac{1}{2} \lambda \phi(z)^2 + \kappa \mathcal{C}^i \mathcal{C}^i \right)^{-1}. \quad (125)
\]
Following Balaban \[\text{\cite{1}}\], one can estimate the decay properties by the lowest eigenvalue of the operator in brackets with $\Delta$ replaced by the Laplacian $\Delta^N$ with Neumann boundary conditions on the block boundary; basically this eigenvalue should be strictly positive and of order at least one in units where $a_{i+1} = 1$. We are interested chiefly in (nearly) constant $\phi$; in this case one can also find the decay by Fourier analysis cp. ref. \[\text{\cite{8}}\]. The $\kappa C^*C^*$-term effectively eliminates the zero mode of $\Delta^N$ (constants) and the next eigenvalue of $\Delta^N$ has the desired magnitude. Therefore things go wrong when $m^2 + \frac{1}{2} \lambda \phi(z)^2$ becomes too strongly negative. This can happen when $\phi \approx 0$ and $m^2$ is too strongly negative - i.e. near the top of a Mexican hat which is too high, cp. figure \[\text{\cite{20}}\].

When $m^2$ is negative enough, something still more terrible happens. Numerical work revealed that the minimal action for constant block-spin zero is reached for background fields which have periodic domain walls of alternating slope. In other words, the auxiliary theory of section \[\text{\cite{8}}\] shows spontaneous
breaking of translation symmetry by block lattice vectors.

At intermediate RG-steps one could try to remedy this by choosing block-spins of fixed length - i.e. integrating out the modulus of the field - to obtain nonlinear $\sigma$-model type effective actions.

But anyway we cannot calculate the dependence of the constraint effective potential on the magnetization in the unstable region (figure 20) because the assumption of no spontaneous breaking of translational symmetry is physically wrong there.

Nonlocalities can appear in another more interesting way which is relevant for studies of a dynamical Higgs mechanism, in particular superconductivity. The remedy in this case is the introduction of a composite Bose field as a block-spin. In superconductivity it represents Cooper pairs. This mechanism was studied in some detail by Grabowski [11].

Nonlocalities of 3-point functions will also induce nonlocalities of $S^{i''}$. The formal solution of the Schwinger Dyson equation for the 3-point function involves (among others) the famous chain-of-bubble diagrams. The fluctuation propagators in these diagrams have exponential decay. Nevertheless the sum of these diagrams can fail to have the desired locality properties due to a pole in momentum space below the UV-cutoff $a_{+1}^{-1}$ of the new action. This occurs in fermionic theories when the coupling gets strong enough. In the 2-dimensional Gross Neveu model and in models of superconductivity, one is always driven into this domain by the renormalization group flow. When this happens, one is forced to introduce a composite block-spin [11].

9 Motivation for the improvement of the saddle point method

In a $\phi^4$-theory the simple saddle point method works quite well. But this is due to the simple form of the kinetic term in this theory.

Consider instead an action for an $n$-component field $\phi$ in two dimensions of the form

$$S[\phi] = \int \left( \frac{2n}{f} (1 + \phi^2)^{-2} (\nabla \phi)^2 + n\delta(0) \ln(1 + \phi^2) \right) \tag{126}$$

where $f$ is a coupling constant. In the continuum, $\delta(0)$ is quadratically divergent, on the lattice $\delta(0) = a^{-2}$.

This action comes from the $O(n + 1)$-symmetric nonlinear $\sigma$-model in stereographic coordinates; the term proportional $\delta(0)$ comes from the measure on the sphere. But let us forget where the action came from.

Suppose we wish to block from the continuum to the lattice. For simplicity, consider first what happens if we apply a simple saddle point treatment to the whole theory. Expanding the action up to terms quadratic in the field, we get

$$S[\phi] \approx \int \left( \frac{2n}{f} (\nabla \phi)^2 + n\delta(0)\phi^2 \right) + ... \tag{127}$$
Evidently we met with disaster. There is a quadratically divergent coefficient in a quadratic action. The same problem appears if we do the high frequency integral to compute an effective lattice action from the continuum action.

If we normal order first before expanding in the field, the situation is different. Let us imagine a finite volume and normal order with respect to the massless free propagator $v_{CB}$ without its zero mode. Normal ordering the leading term $f(\nabla \phi)^2 \phi^2$ produces a quadratic term $-f \Delta v_{CB}(0) \phi^2 = f \delta(0) \phi^2$ which cancels against the quadratically divergent term from the measure.

The disease of the simple saddle point method comes from its lack of covariance under field reparameterization. This can lead to catastrophic violations of Ward identities in theories with symmetries. Since we shall want to apply our method to such theories in the future, it was essential to go beyond a simple saddle point approximation.

Another advantage of the normal ordered version of our method (section 4) consists in the fact that it enlarges the class of theories which can be dealt with to theories in which the field assumes a discrete set of values, such as the discrete Gaussian model, Ising and Potts models etc. This is true because any action can be expanded in normal ordered products of the fields.

10 Models whose fields assume discrete values

Here we wish to explain how normal ordering can help to deal with models whose fields assume a discrete set of values. Only the main idea will be presented; detailed numerical studies remain to be done.

Consider as an example the discrete Gaussian Model in two or three dimensions. It lives on a lattice $\Lambda^0$. The field assumes integer multiples of $2\pi$ as its values. The starting action is

$$S^0(n) = \frac{1}{2\beta} \int_z [\nabla_\mu n(z)]^2, \quad n(z) \in 2\pi \mathbb{Z}$$

In two dimensions, this model is related by a duality transformation to the plane rotator model (with Villain action $[23]$) which has a Kosterlitz Thouless phase transition. In three dimensions it is the dual transform of a U(1)-lattice gauge theory. The renormalization group flow of this model is well understood and lead to a rigorous proof that the three dimensional U(1)-lattice gauge theory shows linear confinement for arbitrary values of the coupling constant $[10]$. The overall factor in the action is written as $1/\beta$ because the duality transformation interchanges high and low temperatures.

The basic idea is to regard the discrete field as a function of a continuous field $\phi$. One regards the action as a function of this continuous field. One normal orders it in a self-consistent way. This furnishes at the same time a split into a free action which is quadratic in $\phi$ and an interaction. In favorable cases, a good approximation to the self-consistent split can be guessed a priori.

---

2It is known that Ward identities guarantee a correct cancellation of divergences in the nonlinear $\sigma$-model $[6]$. 42
The normal ordered action is an entire function of the field and can be expanded in powers of the field components which one wishes to integrate out.

Let \( N(\xi) \) be the periodic function of \( \xi \in \mathbb{R} \) with period \( 2\pi \beta^{-1/2} \) which is defined by

\[
N(\xi) = \xi \text{ for } -\pi \beta^{-1/2} < \xi < \pi \beta^{-1/2}.
\]

We note that \( N(\xi) \) is small for all values of \( \xi \) if \( \beta \) is large. Now we may write

\[
n(\xi) = \beta^{1/2} [\phi(\xi) - N(\phi(\xi))]
\]

We may substitute an integration over \( \phi(\xi) \) for the sum over \( n(\xi) \). The action takes the form

\[
S^0(\phi) = \frac{1}{2} \int [\nabla_\mu \phi(z)]^2 + V_1^0(\phi) + V_2^0(\phi),
\]

\[
V_1^0(\phi) = \int N(\phi(z)) \Delta \phi(z),
\]

\[
V_2^0(\phi) = \frac{1}{2} \int [\nabla_\mu N(\phi(z))]^2.
\]

Given the propagator \( \Gamma \) with respect to which we want to normal order, the normal ordered form of the action is determined by Gaussian integration (61).

In the case at hand, the result can be computed by Fourier transformation. (There are also more direct methods which exploit the fact that the Lagrangian depends only on the field at two lattice points [16]).

The periodic function \( N(\varphi) \) of \( \varphi \in \mathbb{R} \) admits a Fourier expansion

\[
N(\varphi) = \sum_{m \in \mathbb{Z} \setminus \{0\}} N_m e^{im\beta^{1/2} \varphi},
\]

\[
N_m = \frac{i}{m} \beta^{-1/2} (-1)^m.
\]

We wish to normal order in the fluctuation field \( \zeta \) which is to be integrated out in one RG-step. We decompose as usual \( \phi = \Psi + \zeta, \Psi = \Psi^0[\Phi] \). Now we can write the three terms in the action in normal ordered form. The quadratic term is trivial and remains quadratic. We use partial integration and the formula (90) for the characteristic function of a Gaussian measure to compute the normal ordered form of \( V_1^0 \).

\[
\int d\mu_\Gamma(\xi) V_1^0(\phi + \xi) =
\]

\[
\int d\mu(\xi) \int_\mathbb{R} \left( \Delta \phi(z) N((\phi + \xi)(z)) + \int_\mathbb{R} \Delta \Gamma(w, z) \frac{\delta}{\delta \xi(z)} N((\phi + \xi)(z)) \right)
= \sum_{m > 0} 2\pi N_m \int e^{-\beta m^2 \Gamma(z, z)/2}
\]

\[
\left( \Delta \phi(z) \sin(m\beta^{1/2} \phi(z)) + \beta^{1/2} m \int \Delta \Gamma(w, z) \cos(m\beta^{1/2} \phi(z)) \right)
\]

\[
\phi = \Psi + \zeta.
\]
The second term $V_2^0$ can be treated in a similar fashion. First the lattice derivatives must be written as differences of two terms. After that, the Fourier expansions are inserted and the Gaussian integration can be performed as before. We will not give the somewhat complicated result in full, but only an approximation to the resulting effective action which is valid when $\beta$ is large. Although it is not quantitatively correct when $\beta$ is not large, it will suffice to discuss qualitative features for orientation. The $V_2^0$-term is small compared to the $V_1^0$-term in this limit.

The crucial aspect of the model is the breaking of the symmetry under field translations

$$\phi(z) \mapsto \phi(z) + 2\pi\beta^{-1/2}a.$$  

The free action is symmetric under translations with $a \in \mathbb{R}$, but the interaction breaks the symmetry down to translations with $a \in \mathbb{Z}$.

When $\beta$ is large, the expressions $\exp\left(-\beta m^2 \Gamma(z,z)/2\right)$ are exponentially small for $m \neq 0$, and the leading symmetry breaking terms are those with $m = \pm 1$. We neglect to write the others, and also the small symmetry breaking corrections to the generalized kinetic term.

The effective action $S^1$ after one step is given by eq. (60). It depends on the normal ordered form of $S^0$ at $\zeta = 0$, on the fluctuation propagator and on the background field as we know. It takes the following form in $d$ dimensions

$$S^1(\Phi) = \int_z \left(\frac{1}{2} (\nabla \Phi(z))^2 + \lambda(z) \cos(\beta^{1/2} \Phi(z)) - \frac{1}{2} \ln \Gamma(z,z) + ...\right)$$  

$$\lambda(z) = 2e^{-\beta \Gamma(z,z)/2} \int_w \Delta \Gamma(w,z).$$  

For a translation invariant propagator $\Gamma$, $\lambda(z)$ would be constant. The propagator depends on the dimension $d$.

Below we will briefly discuss some qualitative implications on the basis of this approximate expression, neglecting a possible $\Phi$-dependence of $\Gamma$. Quantitative calculations could start from the exact normal ordered expression for $S^0$. The exact 1-loop effective action after one step is still given by eq. (60), which can be differentiated. A localization approximation is made after the first RG-step, and the calculation proceeds in the usual way from there on.

The fluctuation propagator depends on an infrared cutoff $M = a_1^{-1}$. For orientation, we may imagine that the cutoff is lowered by as much as we please in one step; we discuss what happens if $M \to 0$.

In $d = 3$ dimensions, the propagator $\Gamma$ has a limit when $M \to 0$. Therefore, also $\lambda$ has a limit. However, to judge the RG-flow, we need to rescale to the unit lattice. Therefore we need to consider the dimensionless quantity $a_1^d \lambda$. It goes to infinity. Therefore the cosine shaped potential wells become infinitely

---

The standard free $\Phi$-independent fluctuation propagator of Kupiainen and Gawedzki gives a good a priori guess for the self-consistent split in this particular model.
high in the infrared limit, and there will always be spontaneous breaking of the \( Z \)-symmetry with a nonvanishing surface tension. There is also a curvature at the minimum which produces a mass. In the dual picture, the surface tension becomes the string tension of the \( U(1) \)-lattice gauge theory.

In \( d = 2 \) dimensions, \( \Gamma \) diverges logarithmically when \( M \to 0 \). The asymptotic behavior of \( a_\beta^2 \lambda \) as \( M \to 0 \) depends then on \( \beta \). There will be a critical value \( \beta_c \). Its precise value depends on details which were neglected in (138). For \( \beta < \beta_c \), \( a_\beta^2 \lambda \) tends to infinity, and for \( \beta > \beta_c \) it tends to zero. This corresponds to the two phases of the plane rotator which are separated by the Kosterlitz-Thouless phase transition. The \( Z \)-symmetry is either spontaneously broken or enhanced to \( R \). This phenomenon of symmetry enhancement was discovered long ago by Fröhlich and Spencer [7].

It might also be of interest to consider the dual transform of the 4-dimensional \( U(1) \)-lattice gauge theory which is a \( Z \)-gauge theory. In principle it could be treated with our method. Discussions on how to deal with gauge theories are found in [2, 3, 14].

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## A Formulas for saddle point approximation

### A.1 Form of \( \Gamma \)

We are looking for the propagator which corresponds to the quadratic term

\[
\frac{1}{2} \zeta^i S^{i\nu} \Psi^i | \Phi | \zeta^i
\]

in (21). The fluctuation fields \( \zeta^i \) obey the constraint \( C^i \zeta^i = 0 \). Let

\[
\Gamma^i = S^{i\nu - 1} - S^{i\nu - 1} C^i (C^i S^{i\nu - 1} C^i)^{-1} C^i S^{i\nu - 1}.
\]  

(139)

It obeys the constraint \( C^i \Gamma^i = 0 \). Therefore \( \Gamma^i : \mathcal{H}^i \to \mathcal{H}_C^i \). We now want to show that

\[
(\Gamma^i)^{-1} = S^{i\nu}
\]

This means that

\[
\Gamma^i S^{i\nu} \bigg| \mathcal{H}_C^i = 1 \mathcal{H}_C^i = S^{i\nu} \Gamma^i \bigg| \mathcal{H}_C^i.
\]

The first equation is equivalent to

\[
\Gamma^i S^{i\nu} \zeta^i = \zeta^i \quad \forall \ z^i \in \mathcal{H}_C^i.
\]

Applying \( S^{i\nu} \zeta^i \) to (139) from the right confirms this.

The second equation is true because it is the adjoint of the first.

---

\( S^{i\nu} \) has in general no zero modes. If there are some zero modes (e.g., for \( \phi = 0 \)) then they can be eliminated with the help of projection operators before building the inverse \( S^{i\nu - 1} \). This is well known (see [8]).
### A.2 Proof of the relation for the background field

Differentiating the defining equation (18) for the background field by the chain rule, we obtain

\[ S_i''[\Psi] \Psi_i' = C_i^* \lambda_i'' \quad (140) \]

for some \( \lambda_i'' \in \mathcal{H}^{i+1} \). Multiplying (140) with \( S_i'' \) gives

\[ C_i \Psi_i'' = C_i S_i'' C_i^* \lambda_i'' \]

By virtue of \( C_i \Psi_i'' = 1 \) it follows

\[ 1 = (C_i S_i'' C_i^*) \lambda_i'' \]

Therefore

\[ (C_i S_i'' C_i^*)^{-1} = \lambda_i'' \quad (142) \]

We have now one possible form of \( \Psi_i'' \):

\[ \Psi_i'' = S_i''^{-1} C_i^* (C_i S_i'' C_i^*)^{-1} \quad (143) \]

Multiplying \( \Gamma_i \) in (139) with \( S_i'' \) from right and using (142) gives

\[ \Gamma_i S_i'' = 1 - S_i''^{-1} C_i^* (C_i S_i'' C_i^*)^{-1} C_i \]

Inserting (141) leads to

\[ \Gamma_i S_i'' = 1 - \Psi_i'' C_i. \quad (144) \]

By multiplying \( C_i^* \) from the right we get the result

\[ \Psi_i'' = (1 - \Gamma_i S_i'') C_i^*. \quad (145) \]

### A.3 Derivative of \( \Gamma_i \)

Multiplying (144) with \( \Gamma_i \) from the right and using \( C_i \Gamma = 0 \) gives

\[ \Gamma_i S_i'' \Gamma_i = \Gamma_i. \]

Differentiation leads to

\[ \Gamma_i'' = \Gamma_i S_i'' \Gamma_i + \Gamma_i S_i'' \Gamma_i + \Gamma_i S_i'' \Gamma_i. \]

\( S_i'' \Gamma_i \) and \( \Gamma_i S_i'' \) can be replaced now by (144). Using \( C_i \Gamma_i'' = 0 \) leads to the result

\[ \Gamma_i'' = -\Gamma_i S_i'' \Gamma_i \]

or expressed with \( W^i \)

\[ \Gamma_i'',z = \Gamma_i W^i,z \Gamma_i. \quad (146) \]
A.4 Determination of the background field

An equation for $Ψ^i, x[Φ](z)$ was already derived in \[A.2\]. It involves quantities which are obtained together with the propagator. Expressed in terms of $W^i$ it looks

$$Ψ^i = (1 + Γ^i W^i) \tilde{C}^i.$$ \[1\]

To determine the recursion relation of $W^i$ we need also the second derivative. Using the formula for the derivative of the fluctuation propagator (146) we find

$$Ψ^i, x[Φ](z) = \int_{w, z_1, z_2 \in \Lambda^i} Γ^i(z, z_1) W^i, w(z_1, z_2) Ψ^i, y[Φ](w) Ψ^i, x[Φ](z_2).$$ \[147\]

A.5 Recursion relation for $W^i$

To get $W^{i+1}$ out of (24) we need the second derivative of $S^{i+1}$.

Consider first

$$\frac{δ}{δΦ(x)} \text{tr ln } Γ^i[Ψ^i[Φ]] = \text{tr } Γ^i[Ψ^i[Φ]] - \frac{δ}{δΦ(x)} W^i[Ψ^i[Φ]]$$

$$= \int z \text{ tr } \left( Γ^i[Ψ^i[Φ]] W^{i+1}[Ψ^i[Φ]] \right) Ψ^i, x[Φ](z)$$

where we have used (146) and $W^i, x[Ψ^i]$ or $Ψ^i, x[Φ]$ denote the functional derivatives

$$\frac{δ}{δΨ^i(z)} W^i[Ψ^i], \quad \frac{δ}{δΦ(x)} Ψ^i[Φ].$$

We differentiate once more to obtain the second derivative and omit the func-
Now we consider $S_i[\Psi^i[\Phi]]$. By the chain rule

$$\frac{\delta^2}{\delta \Phi(x) \delta \Phi(y)} \text{tr} \ln \Gamma^i[\Psi^i[\Phi]] = \Psi^i, x[\Phi]^* S_i''[\Psi^i[\Phi]] \Psi^i, y[\Phi] + S_i'[\Psi^i[\Phi]] \Psi^i, xy[\Phi].$$

(149)

By the saddle point condition, $S_i'[\Psi^i[\Phi]] = C^i \lambda^i$. Therefore the last term looks in coordinates

$$\int_{z \in \Lambda^i} \frac{\delta^2}{\delta \Phi(x) \delta \Phi(y)} \Psi^i[\Phi](z) \int_{x_1 \in \Lambda^{i+1}} C^i(z, x_1) \lambda^i(x_1).$$

Inserting (147) shows that this term vanishes because of $\Gamma^i C^i = 0$. Therefore

$$W^{i+1}[\Phi](x, y) = \Psi^i, x[\Phi]^* W^i[\Psi^i[\Phi]] \Psi^i, y[\Phi] + \frac{1}{2} \frac{\delta^2}{\delta \Phi(x) \delta \Phi(y)} \text{tr} \ln \Gamma^i[\Psi^i[\Phi]].$$

Using (148) and (147) leads to the result (26).

### B Simplification of mixed derivatives

In the case of Feynman Bogoliubov approximation we defined the propagator $\Gamma^i$ for arbitrary fields $\phi$ as inverse of $S''$ on $H^i_{\phi}$ and derived some formulas in a general manner. Afterwards we used special fields namely the background field...
which depends on the block-spin. Now we regard it only as a function of the background field. It is convenient to extend this definition to arbitrary fields by specifying that it is independent of the high frequency components of the field, 

\[ \Gamma^i[\Psi^i + \zeta^i] = \Gamma^i[\Psi^i] \]  

(150)

for \( \zeta^i \in \mathcal{H}_C^i \).

It is important that the formula for the derivative of the propagator

\[ \Gamma^i, z[\Psi^i] = \delta \frac{\delta}{\delta \Psi^i(z)} \Gamma^i[\Psi^i] = -\Gamma^i T^i, z \Gamma^i \]  

(151)

\[ T^i, z = \frac{\delta}{\delta \Psi^i(z)} T^i \]  

(152)

is then only valid in the restricted sense that it can be used for evaluating \( \Phi \)-derivatives of the propagator.

With this specification of the propagator, the definition (62) of the normal ordered amplitude implies that

\[ T[\Psi^i + \zeta^i, \xi - \zeta^i] = T[\Psi^i, \xi] \]  

(153)

for \( \zeta^i \in \mathcal{H}_C^i \). In other words, \( T[\Psi^i, \xi] \) depends actually only on \( \Psi^i + \xi \) so long as \( \xi \in \mathcal{H}_C^i \).

We find it convenient, however, to extend the definition (61) of \( T[\Psi^i, \xi] \) to arbitrary \( \xi \). Eq. (153) remains valid, of course, but \( T[\Psi^i, \xi] \) is no longer only a function of the sum of its arguments.

When evaluating \( \Phi \) derivatives, eq. (151) can be used to express the derivative \( \Gamma^i, z \) of the propagator. On the other hand, \( \Psi^i \)-derivatives in \( \mathcal{H}_C^i \) directions are the same as \( \xi \)-derivatives; the last term does not contribute in this case because of eq. (150).

Also the equation (89) for the derivative of the background field simplifies, i.e. \( \Gamma_B^i = \Gamma^i \):

\[ \Gamma^i, z(\Psi^i, x) = \tilde{\Gamma}^i z(\xi, x) - \int_{w,u} \Gamma^i(z, w) T^i, w, u \tilde{\Gamma}^i(u, x), \]  

(154)

It fulfills also (88) and the constraint (84). One needs the identity

\[ \int_{z,w} \Gamma^i(\xi, *) \frac{\delta}{\delta \Psi^i(z)} T^i, w \Gamma^i(w, *) = \Gamma^i(\xi, *) \]  

(155)

\( \Psi^i \)- and \( \xi \)-derivatives of \( T^i \) agree in \( \mathcal{H}_C^i \)-directions, and the range of \( \Gamma^i \) is \( \mathcal{H}_C^i \). Therefore we may convert the \( \Psi^i \)-derivative into a \( \xi \)-derivative. The assertion follows now from the gap equation (67).

### C Momentum space representation

Let \( \Lambda^j \) be a \( D \)-dimenional position space lattice with lattice spacing \( a_i \) and length \( L_i = N_i a_i \). The corresponding dual lattice \( \tilde{\Lambda}^j \) has a lattice spacing \( \tilde{a}_i = 2\pi/L_i \).
and length \( L_i = 2\pi/a_i \). The block factor between two successive lattices shall be denoted by \( s = a_{i+1}/a_i \). It is convenient to introduce an additional momentum space lattice \( \Lambda_{i,i+1} \) with spacing \( \tilde{a}_{i,i+1} = \tilde{L}_i \) and length \( \tilde{L}_{i,i+1} = sL_i \). Then any \( p \in \tilde{\Lambda}^i \) can be decomposed into \( p = q + l \) with \( q \in \tilde{\Lambda}^{i+1} \) and \( l \in \tilde{\Lambda}^{i,i+1} \).

The definitions

\[
\int_p := \frac{\tilde{a}_i^D}{(2\pi)^D} \sum_{p \in \tilde{\Lambda}^i}, \quad \int_q := \frac{\tilde{a}_{i+1}^D}{(2\pi)^D} \sum_{q \in \tilde{\Lambda}^{i+1}}, \quad \int_l := \sum_{l \in \tilde{\Lambda}^{i,i+1}}
\]

imply the split

\[
\int_p = \int_q \int_l.
\]

We now want to derive the momentum space representation \( (\ref{eq:momentum_space}) \) of the fluctuation propagator. First note that any linear operator \( L : \mathcal{H}^i \to \mathcal{H}^i \) with \( L(z,z') = L(z-z') \) can be written as

\[
L(z-z') = \int_p L(p)e^{ip(z-z')}.
\]

Examples are \( \overline{W}^i(z,z'|\phi) \) in the saddle point approximation and the laplacian \( \Delta^i(z,z') \). In the latter case one gets

\[
\Delta^i(p) = -\hat{p}^2, \quad \hat{p}_\mu = \frac{2}{a_i} \sin \left( \frac{p_\mu a_i}{2} \right).
\]

The representation \( (\ref{eq:momentum_space}) \) also holds for the averaging operator \( C_i(x,z) \) since \( x \in \mathcal{H}^{i+1} \subset \mathcal{H}^i \) with

\[
C_i(p) = \prod_{\mu=1}^D \left( \frac{\sin(p_\mu a_i/2)}{s \sin(p_\mu a_i/2)} e^{ip_\mu(s-1)a_i/2} \right)
\]

The fluctuation propagator \( \Gamma^i \) on the other hand is only invariant under translations \( \Gamma^i(z,z') = \Gamma^i(z-x,z'-x) \) with \( x \in \Lambda^{i+1} \) and therefore has the more complicated Fourier representation

\[
\Gamma^i(z,z') = \int_{l,q,l'} \Gamma^i(l,q,l')e^{i(q+l)z-i(q+l')z'}
\]

with

\[
\Gamma^i(l,q,l') = v^i(q+l)\delta_{l,l'} - u^i(q+l)C^{i*}(q+l)u^i(q+l')v^i(q+l')
\]

\[
u^i(q) = \int_l C^i(q+l)v^i(q+l)C^{i*}(q+l)
\]

\[
v^i(p) = \overline{W}^i(\phi)^{-1}(p)
\]
References

[1] T. Balaban, Regularity and Decay of Lattice Green's Functions, Commun. Math. Phys. **89** (1983), p. 571

[2] T. Balaban, Commun. Math. Phys. **109** (1983) 571

[3] T. Balaban, A. Jaffe, Constructive Gauge Theory, in: Erice School Math. Phys (1985) 207

[4] W. Bietenholz and U. Wiese, Nucl.Phys.Proc.Suppl. 34, 516-521 (1994)

[5] J. D. Bjorken and S. D. Drell, vol 2, Relativistic Quantum Fields, Mc Graw-Hill Book Inc., New York, 1965

[6] E. Brézin, J. Zinn-Justin and J. C. Le Guillou, Phys. Rev **B14**, 4976 (1976)

[7] J. Fröhlich, T. Spencer, Phase diagrams and critical properties of (classical) Coulomb systems. Erice lectures 1980
J. Stat.Phys **24** (1981) 617

[8] K. Gawedzki and A. Kupiainen, A Rigorous Block Spin Approach to Massless Lattice Theories, Commun. Math. Phys. **77**, 31-64 (1980)

[9] J. Glimm and A. Jaffe, Quantum Physics: A Functional Integral Point of View, Springer Verlag, New York 1987, second edition

[10] M. Göpfert, G. Mack, Proof of confinement of static quarks in 3-dimensional $U(1)$-lattice gauge theory for all values of the coupling constant, Commun. Math. Phys. **82** (1982) 545

[11] M. Grabowski, Low Energy Effective Actions with Composite Fields, PhD thesis Hamburg, DESY 94-146

[12] P. Hasenfratz, F. Niedermayer et al. , Nucl. Phys. B414 (1994), 785-814, Nucl. Phys. B454 (1995), 615-637, Nucl. Phys. B454 (1995), 638-644, Nucl. Phys. B454 (1995), 587-614

[13] Richard W. Haymaker, Variational Methods for Composite Operators, Riv.Nuovo Cim.14, no.8: 1-89, 1991

[14] U. Kerres, G. Mack, G. Palma, Nucl. Phys. (Proc. Suppl.) B42 (1995), 584-586, and: Perfect 3-dimensional lattice actions for 4-dimensional quantum field theories at finite temperatures, DESY 94-226, submitted to Nucl. Phys. B.

[15] G. Mack, Multigrid methods in Quantum Field Theory, in G. ’t Hooft et al (eds), Nonperturbative Quantum Field Theory (Cargese Proceedings 1987) Plenum Press New York 1988
[16] G. Mack, A. Pordt, Convergent weak coupling expansions for lattice field theory that look like perturbation series, Rev. Math. Phys 1, 47-87, (1989), esp. Appendix A

[17] G. Mai, Ein Blockspin für $2^d/2$ Fermionen, Diplomarbeit Universität Hamburg, 1989.
G. Mack, T. Kalkreuter, G. Palma, M. Speh, Effective Field Theories, in H. Gausterer, C.B. Lang (eds.) Computational Methods in Field Theory, Lecture Notes in Physics 409, Springer Heidelberg 1992

[18] G. Mack. I.Todorov, Conformal invariant Green functions without ultraviolet divergences, Phys. Rev. D8 (1971) 1764

[19] Yu. Pismak, private communication

[20] J. Polchinski, Renormalization and Effective Lagrangians, Nucl. Phys. B231 (1984) 269-295

[21] A. Pordt and C. Wieczerkowski, Nonassociative Algebras and Nonperturbative Field Theory for Hierarchical Models, 74 pages, report MS-TPI-94-04

[22] W. Rühl, Mean Fields and Self Consistent Normal Ordering of Lattice Spin and Gauge Field Theories, Z. Phys. C (1986), 32, p. 265–278

[23] J. Villain, J. Phys (Paris) 36 (1975) 631

[24] C. Wetterich, Exact Evolution Equation for the Effective Potential, Physics Letters B301 (1993) 80-94