Janus field theories from multiple M2 branes

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Abstract

Based on the recent proposal of $\mathcal{N} = 8$ superconformal gauge theories of the multiple M2 branes, we derive (2+1)-dimensional supersymmetric Janus field theories with a space-time dependent coupling constant. From the original Bagger-Lambert model, we get a supersymmetric field theory with the same action as that of the $N$ D2 branes, but the coupling depends on the space-time as a function of the light-cone coordinate, $g(t+x)$. Half of the supersymmetries are preserved in this case. We further investigate the M2 brane action deformed by mass and Myers-like terms. In this case, the final YM action is deformed by mass and Myers terms and the coupling behaves as $\exp(\mu x)$ where $\mu$ is a constant mass parameter. Weak coupling gauge theory is continuously changed to strong coupling in the large $x$ region.
1 Introduction

There has been a remarkable progress recently in constructing maximally supersymmetric (2+1) field theories with SO(8) R-symmetry by Bagger and Lambert [1] and Gustavsson [2]. This model has 8 scalar fields and it is conjectured to be an effective field theory of multiple M2-branes in d=11. An essential ingredient is the generalization of the Lie algebraic structure in ordinary gauge theories to the Lie 3-algebras [3]. As expected [4] the Lagrangian contains a Chern-Simons term and a sextic potential for scalars.

The 3-algebraic structure is naturally expected for the M2 brane because the Schild form of the bosonic membrane action is written in terms of the Nambu-Poisson bracket

\[ S \sim \int d^3 \sigma \{ X^I, X^J, X^K \}^2, \]

(1.1)

where the Nambu-Poisson bracket [5] is given by \( \{ X^I, X^J, X^K \} = \epsilon_{ijk} \partial_i X^I \partial_j X^J \partial_k X^K \). Then its quantum version must be written as

\[ S \sim \text{Tr} [X^I, X^J, X^K]^2 \]

(1.2)

where the 3-algebra for the generators \( T^a \) is given by \([T^a, T^b, T^c] = f^{abc} T^d\). The structure constant must obey the fundamental identity so that the action by Bagger-Lambert is invariant under supersymmetry and gauge transformations.

Despite many efforts [6, 7, 8], the quantization of the Nambu bracket is very hard and the only known example (satisfying the so called fundamental identity) is the algebra \( \mathcal{A}_4 \) [7] with 4 generators. This is because the requirement that the 3-algebra has a positive definite metric is very strong. It was conjectured [8] and proved [10] that the only nontrivial positive definite 3-algebra is \( \mathcal{A}_4 \). In order to circumvent this difficulty, it was recently shown [11, 12, 13] that if we relax the condition of the positivity of the metric we can construct 3-algebras containing the ordinary Lie algebra as a sub-algebra. This is a remarkable progress. The algebra contains 2 extra generators \( T^{-1}, T^0 \) in addition to the generators of Lie algebra \( T^i \). (Here we use the convention of [13].) The 3-algebra for them is given by

\[ [T^{-1}, T^a, T^b] = 0, \]

(1.3)

\[ [T^0, T^i, T^j] = f^{ij}_{\phantom{ij}k} T^k, \]

(1.4)

\[ [T^i, T^j, T^k] = f^{ijk} T^{-1}, \]

(1.5)

where \( a, b = \{-1, 0, i\} \). \( T^i \) are generators of the ordinary Lie algebra with the structure constant \( f^{ij}_{\phantom{ij}k} \). We can show that this satisfies the fundamental identity. The metric \( h^{ab} = \text{Tr}(T^a, T^b) \) is given by

\[ \text{Tr}(T^{-1}, T^{-1}) = \text{Tr}(T^{-1}, T^i) = 0, \quad \text{Tr}(T^{-1}, T^0) = -1, \]

\[ \text{Tr}(T^0, T^i) = 0, \quad \text{Tr}(T^0, T^0) = 0, \quad \text{Tr}(T^i, T^j) = h^{ij}. \]

(1.6)
Since the model contains negative metric, we may worry that the model based on the above 3-algebra will contain ghost modes and they violate the unitarity of the theory. This ghost modes are associated with the special components of algebra generators $T^{-1}$ and $T^0$. Remarkably the authors showed that the modes associated with the $T^{-1}$ generator become Lagrange multipliers and the integration gives a constraint $\partial^2 X_0^I = 0$ for the other problematic modes associated with $T^0$. Then the would-be ghost modes can be decoupled from the rest and the theory becomes unitary.

The constraint $\partial^2 X_0^I = 0$ is solved as $X_0^I = v\delta^I_{10}$ where $v$ is a constant. For a nonvanishing $v$, this breaks the $SO(8)$ R-symmetry to $SO(7)$. After integrating non-dynamical modes of the gauge field, the gauge theory action of $N$ D2 branes is derived. The original model does not contain any tunable parameter, but the value of $v$ gives the coupling constant for the D2 brane effective action.

In this paper we revisit the constraint equation. The constraint equation $\partial^2 X_0^I = 0$ is a massless wave equation and a general function of the light cone coordinate, $X_0^I = f(t + x) \delta^I_{10}$, solves the constraint. The integration of the non-dynamical gauge field can be similarly performed and the resulting theory becomes a (2+1)-dimensional Janus gauge theory. This breaks half of the original 16 supersymmetries.

In the Janus field theory, the coupling constant has the dependence on coordinates. Originally it was considered to be a dual of supergravity solutions with a space-time dependent dilaton field [15], and it has two different “faces” at the boundary. If there are two boundaries and there are different coupling constant for each boundary, we should include interface terms which makes gauge couplings non-constant. Supersymmetric field theories with the interface terms are constructed in [16, 17, 18].

We will further investigate the mass deformation of the BLG models. This model was studied by [19, 20] as a model of the matrix theory of type IIB plane waves. The deformed model has desirable maximal supersymmetries as well as other bosonic symmetries. In this case, the constraint equation is modified to $(\partial^2 - \mu^2)X_0^I = 0$ and the solution of this constraint is given by $X_0^I = \exp(\mu x)\delta^I_{10}$ where $x$ is a space direction. This preserves half of the original supersymmetries. The nondynamical gauge modes can be integrated out again and the theory becomes a supersymmetric Janus field theory with a Myers-term added. The gauge coupling constant changes from weak to strong as we move along the coordinate $x$ from $-\infty$ to $+\infty$.

The organization of the paper is as follows. In section 2, we first review the Bagger-Lambert model based on the realization of 3-algebra with a negative component of the metric. We also comment that the constraint equation has more generic solutions with the coupling constant depending on the space-time as a function of the light-cone coordinate. In section 3, we extend the model including a mass and Myers-like term and investigate the model similarly.

There are many other interesting developments of Multiple M2-brane [21].

*The idea of getting the D2-brane effective action by giving the vev was originally given in [14].
2 Bagger-Lambert theory

2.1 Brief review of BL model

We first briefly review the Bagger-Lambert action and its symmetry properties. It is a (2+1)-dimensional nonabelian gauge theory with $N = 8$ supersymmetries. It contains 8 real scalar fields $X^I = \sum_a X^I_a T^a$, $I = 3, ..., 10$, gauge fields $A^\mu = \sum_{ab} A^\mu_{ab} T^a \otimes T^b$, $\mu = 0, 1, 2$ with two internal indices and 11-dimensional Majorana spinor fields $\Psi = \sum_a \Psi_a T^a$ with a chirality condition $\Gamma_{012} \Psi = \Psi$. The action proposed by Bagger and Lambert is given by

$$L = -\frac{1}{2} \text{Tr}(D^\mu X^I, D_\mu X^I) + \frac{i}{2} \text{Tr}(\bar{\Psi}, \Sigma^\mu D_\mu \Psi) + \frac{i}{4} \text{Tr}(\bar{\Psi}, \Gamma_{IJ}[X^I, X^J, \Psi]) - V(X) + L_{CS}. \quad (2.7)$$

where $D_\mu$ is the covariant derivative defined by:

$$(D^\mu X^I)_a = \partial_\mu X^I_a - f^{cdh}_{ab} A_{\mu cd}(x) X^I_b. \quad (2.8)$$

$V(X)$ is the sextic potential term

$$V(X) = \frac{1}{12} \text{Tr}([X^I, X^J, X^K], [X^I, X^J, X^K]), \quad (2.9)$$

and Chern-Simons term for the gauge potential is given by

$$L_{CS} = \frac{1}{2} \epsilon^{\mu \nu \lambda} (f^{abcd} A_{\mu ab} \partial_\nu A_{\lambda cd} + \frac{2}{3} f^{cdh}_{ab} f^{efg}_{fh} A_{\mu ab} A_{\mu cd} A_{\mu ef}). \quad (2.10)$$

This action is invariant under the SUSY transformation

$$\delta X^I_a = i \epsilon^I \Psi_a, \quad \delta \Psi_a = D_\mu X^I_a \Gamma^\mu \epsilon - \frac{1}{6} X^I_a X^J_b X^K_c f^{abcd} \Gamma_{IJ} \epsilon, \quad (2.11)$$

and the gauge transformation

$$\delta X^I = \Lambda_{ab} [T^a, T^b, X^I], \quad \delta \Psi = \Lambda_{ab} [T^a, T^b, \Psi], \quad \delta \tilde{A}^{b}_{\mu a} = \Lambda_{cd} f^{cdh}_{ab}, \quad (2.12)$$

provided that the triple product $[A, B, C]$ has the fundamental identity and $\text{Tr}$ satisfies the property discussed in the next subsection. The most peculiar property of the model is that the gauge transformation and the associated gauge fields have two internal indices. This must come from the volume preserving diffeomorphism of the membrane action [22, 23] but the concrete realization of the gauge symmetry from the supermembrane action is not yet clear.
2.2 A specific realization of 3-algebra

This theory is based on an antisymmetric 3-algebraic structure \( G \) with generators \( T^a \)
\[
[T^a, T^b, T^c] = f^{abc}d T^d. \tag{2.13}
\]

Here we take the specific realization of the 3-algebra containing the ordinary Lie algebra as a sub-algebra. The most fundamental identity of the algebra is the generalized Jacobi identity. It is called the fundamental identity and given by
\[
[T^a, T^b, [T^c, T^d, T^e]] + [T^c, [T^a, T^b, T^d], T^e] + [T^c, T^d, [T^a, T^b, T^e]] = 0. \tag{2.14}
\]

If this identity holds, we can show that the gauge transformations generated by \( T^a \otimes T^b \) form a Lie algebra. Namely, if we write \( \tilde{T}^{ab}X = [T^a, T^b, X] \), a commutator closes among the generators \( \tilde{T}^{ab} \);
\[
[\tilde{T}^{ab}, \tilde{T}^{cd}]X = [T^a, T^b, [T^c, T^d, X]] - [T^c, T^d, [T^a, T^b, X]]
= [[T^a, T^b, T^c], T^d, X] + [T^c, [T^a, T^b, T^d], X]
= (f^{abc}e f^{ced} + f^{abd}e f^{ce}d)X. \tag{2.15}
\]

A specific choice of the 3-algebra satisfying the fundamental identity is given by \[11, 12, 13\]. It contains an ordinary set of Lie algebra generators as well as two extra generators \( T^{-1} \) and \( T^0 \). The algebra is given by
\[
[T^{-1}, T^a, T^b] = 0, \tag{2.16}
\]
\[
[T^0, T^i, T^j] = f^{ij}k T^k, \tag{2.17}
\]
\[
[T^i, T^j, T^k] = f^{ijk}T^{-1}, \tag{2.18}
\]
where \( a, b = \{-1, 0, i\} \). \( T^i \) is a generator of the Lie algebra and \( f^{ijk} \) is its structure constants. Here \( T^{-1} \) is the central generator meaning that its 3-algebra with any other generators vanish. \( T^0 \) is also special since it is not generated by the 3-algebra and does not appear in the right hand side of the algebra. One can easily check that this 3-algebra satisfies the fundamental identity. In order to construct a gauge invariant field theory Lagrangian, we need the trace operation with the identity
\[
\text{Tr}([T^a, T^b, T^c], T^d) + \text{Tr}(T^c, [T^a, T^b, T^d]) = 0. \tag{2.19}
\]

After a suitable redefinition of generators, such a trace can be given by
\[
\text{Tr}(T^{-1}, T^{-1}) = \text{Tr}(T^{-1}, T^i) = 0, \quad \text{Tr}(T^{-1}, T^0) = -1,
\]
\[
\text{Tr}(T^0, T^i) = 0, \quad \text{Tr}(T^0, T^0) = 0, \text{Tr}(T^i, T^j) = h^{ij}. \tag{2.20}
\]
If we define \( f^{abcd} \) as \( f^{abcd} = f^{abc}e h^{ed} \), \( f^{abcd} \) is totally antisymmetry.

The construction of the 3-algebra contains the ordinary Lie algebra as a sub-algebra. The generators of the gauge transformation can be classified into 3 classes.
\[ I = \{ T^{-1} \otimes T^a, a = 0, i \} \]
\[ A = \{ T^0 \otimes T^i \} \]
\[ B = \{ T^i \otimes T^j \} \]

Then it is easy to show that
\[ [I, I] = [I, A] = [I, B] = 0, \quad [A, A] = A, \quad [A, B] = B \]
and hence the generators of \( A \) form a sub-algebra, which can be identified as the Lie algebra of \( N \) D2-branes.

### 2.3 BL model to D2 branes

In the specific realization of the 3-algebra, we can decompose the modes of the fields as

\[ X^I = X'^I T^0 + X'^I T^{i-1} + X'^I T^i, \]
\[ \Psi = \Psi^0 T^0 + \Psi^i T^i, \]
\[ A_\mu = T^{-1} \otimes A_\mu(-1) - A_\mu(1) \otimes T^{-1} + A_{\mu 0 j} T^0 \otimes T^j - A_{\mu j 0} T^j \otimes T^0 + A_{\mu ij} T^i \otimes T^j. \]

It will be convenient to define the following fields as in \[13\]

\[ \hat{X}^I = X'^I T^i, \quad \hat{\Psi} = \Psi^i T^i \]
\[ \hat{A} = 2 A_{\mu 0 j} T^j, \quad B_\mu = f^{ij k} A_{\mu ij} T^k. \]

The gauge field \( A_\mu(-1) \) is decoupled from the action and we drop it in the following discussions. The gauge field \( \hat{A}_\mu \) is associated with the gauge transformation of the sub-algebra \( A \). Another gauge field \( B_\mu \) will play a role of the \( B \)-field of the BF theory and can be integrated out. With these expression the Bagger-Lambert action (2.7) can be rewritten as

\[ L = \text{Tr} \left( -\frac{1}{2} (\hat{D} \mu \hat{X}^I - B_\mu X'^I_0)^2 + \frac{i}{2} \hat{\Psi} \Gamma^\mu \hat{D}_\mu \hat{\Psi} + i \Psi^0 \Gamma^\mu B_\mu \bar{\Psi} + \frac{1}{4} (X'^K_0)^2 ([\hat{X}^I, \hat{X}^J])^2 - \frac{1}{2} (X'^I_0 [\hat{X}^I, \hat{X}^J])^2 - \frac{i}{4} \bar{\Psi}^0 \hat{X}^I [\hat{X}^J, \Gamma_{IJ} \hat{\Psi}] + \frac{i}{2} \bar{\Psi} X'^I_0 [\hat{X}^J, \Gamma_{IJ} \hat{\Psi}] + \frac{1}{2} \epsilon^{\mu \nu \lambda} \hat{F}_{\mu \nu} B_\lambda \right) 
+ L_{gh}, \]

where the ghost term is

\[ L_{gh} = -\partial_\mu X'^I_0 B_\mu \hat{X}^I + (\partial_\mu X'^I_0) (\partial^\mu X'^I_{-1}) - i \bar{\Psi} \Gamma^\mu \partial_\mu \Psi_0. \]

The covariant derivative and the field strength

\[ \hat{D}_\mu \equiv \partial_\mu \hat{X}^I - [\hat{A}_\mu, \hat{X}^I], \quad \hat{D}_\mu \Psi \equiv \partial_\mu \hat{\Psi} - [\hat{A}_\mu, \hat{\Psi}], \quad \hat{F}_{\mu \nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - [\hat{A}_\mu, \hat{A}_\nu] \]
are the ordinary covariant derivative and field strength for the sub-algebra \( \mathcal{A} \). As emphasized in [11, 12, 13], a coupling constant can be always absorbed by the field redefinition and there is no tunable parameters in this model.

The supersymmetry transformations for each mode are given by

\[
\begin{align*}
\delta X^I_0 &= i\bar{\epsilon} \Gamma^I \Psi_0, \\
\delta X^I_{-1} &= i\bar{\epsilon} \Gamma^I \Psi_{-1}, \\
\delta \hat{X}^I &= i\bar{\epsilon} \Gamma^I \hat{\Psi}, \\
\delta \Psi_0 &= \partial_\mu X^I_0 \Gamma^\mu \Gamma^I \epsilon, \\
\delta \Psi_{-1} &= \{\partial_\mu X^I_{-1} - \text{Tr}(B_\mu, \hat{X}^I]\} \Gamma^\mu \Gamma^I \epsilon - \frac{1}{6} \text{Tr}(\hat{X}^I_0 \{X^J_0, \hat{X}^K_0\}) \Gamma^{IJK} \epsilon, \\
\delta \hat{\Psi} &= \hat{D}_\mu \hat{X}^I_0 \Gamma^\mu \Gamma^I \epsilon - B_\mu X^I_0 \Gamma^\mu \Gamma^I \epsilon - \frac{1}{2} X^I_0 \{\hat{X}^J_0, \hat{X}^K_0\} \Gamma^{IJK} \epsilon, \\
\delta \hat{A}_\mu &= i\bar{\epsilon} \Gamma_\mu \Gamma^I \hat{\Psi} - \hat{X}^I \Psi_0, \\
\delta B_\mu &= i\bar{\epsilon} \Gamma_\mu \Gamma^I [\hat{X}^I, \hat{\Psi}] .
\end{align*}
\]

Here note that \( X^I_{-1} \) and \( \Psi_{-1} \) appear only linearly in the Lagrangian and thus they are Lagrange multipliers. By integrating out these fields, we have the following constraints for the other problematic fields associated with \( T^0 \);

\[
\partial^2 X^I_0 = 0, \quad \Gamma^\mu \partial_\mu \Psi_0 = 0.
\]

This should be understood as a physical state condition \( \partial^2 X^I_0 |_{\text{phys}} = 0 \). In the path integral formulation, these constraints appear as a delta function \( \delta(\partial^2 X^I_0) \) and those fields are constrained to satisfy the massless wave equations. In order to fully quantize the theory, we need to sum all the solutions satisfying the constraints, but we here take a special solution of the constraint and see what kind of field theory can be obtained in the following analysis.

The simplest solution is given by

\[
X^I_0 = v \delta^I_{10}, \quad \Psi_0 = 0,
\]

where \( v \) is some constant. This solution was considered in [11, 12, 13] and preserves all the 16 supersymmetries, the gauge symmetry generated by the subalgebra \( \mathcal{A} \), and \( SO(7) \) R-symmetry rotating \( X^A, A = 3, ..., 9 \). Another interesting solution is given by

\[
X^I_0 = v(x^0 + x^1) \delta^I_{10}, \quad \Psi_0 = 0
\]

where \( v(x^0 + x^1) \) is a general function on the light cone coordinate. As we see the supersymmetry transformation for \( \Psi_0 \),

\[
\delta \Psi_0 = \partial_\mu X^I_0 \Gamma^\mu \Gamma^I \epsilon,
\]

the solution \( X^I_0 = v(x^0 + x^1) \delta^I_{10} \) preserves half of the supersymmetries.
In both cases, if we fix the fields $X_0^I$ and $\Psi_0$ as above, we can integrate over the gauge field $B_\mu$ and obtain the effective action for $N$ D2 branes

$$\mathcal{L} = -\frac{1}{2}(\hat{D}_\mu \hat{X}^A)^2 + \frac{1}{4}v^2(\hat{X}^A, \hat{X}^B)^2 + \frac{i}{2}\bar{\Psi} \Gamma^\mu \hat{D}_\mu \hat{\Psi} - \frac{1}{4}v^2 \hat{F}_{\mu\nu}^2 + \frac{i}{2}v\bar{\Psi}[\hat{X}^A, \Gamma_{10,A}\hat{\Psi}], \quad (2.40)$$

where $A, B = 3, \cdots , 9$. The coupling constant $v$ is given by the vev of $X_0^{10}$ and either a constant or an arbitrary function on the light-cone $v(x^0 + x^1)$. This may be identified as the compactification radius of 11-th direction in M-theory; $v = 2\pi g_s l_s$. The supersymmetric YM theories with a space-time dependent coupling constant are known as Janus field theories \cite{15}.

A salient feature is that the 10-th spacial fields $X^{10}$ completely disappear from the Lagrangian by integrating out the redundant gauge field $B_\mu$. It is interesting that Janus field theories are naturally obtained from the Bagger-Lambert field theories.

The $v \to 0$ limit cannot be taken after integrating the redundant gauge field $B_\mu$. In the case of vanishing $v$, the Lagrangian is simply given by

$$\mathcal{L} = -\frac{1}{2}(\hat{D}_\mu \hat{X}^I)^2 + \frac{i}{2}\bar{\Psi} \Gamma^\mu \hat{D}_\mu \hat{\Psi} \quad (2.41)$$

with a constraint $\hat{F}_{\mu\nu} = 0$. The action is of course invariant under the full $SO(8)$ R-symmetry.

### 3 Mass deformation and Janus solutions

#### 3.1 Mass deformation of BL

The BL model in the previous section gives a well-know effective action for $N$ D2 branes with either a constant or a traveling coupling.

In this section we start from a mass deformed Bagger-Lambert action given by \cite{19,20} and show that supersymmetric Janus field theories with a Myers-term are obtained.

One parameter deformation of the Bagger-Lambert action preserving the full supersymmetries is given by adding the following mass and flux terms to the original Lagrangian. The mass term is given by

$$\mathcal{L}_{mass} = -\frac{1}{2}\mu^2 \text{Tr}(X^I, X^I) + \frac{\mu}{2} \text{Tr}(\bar{\Psi} \Gamma_{3456, \Psi}), \quad (3.42)$$

and a flux term is

$$\mathcal{L}_{flux} = -\frac{1}{6}\mu\epsilon_{EFGH} \text{Tr}([X^E, X^F, X^G], X^H) - \frac{1}{6}\mu\epsilon_{E'F'G'H'} \text{Tr}([X^{E'}, X^{F'}, X^{G'}, X^{H'}]). \quad (3.43)$$

Here $E, F, G, H = 3, 4, 5, 6$ and $E', F', G', H' = 7, 8, 9, 10$. This action is invariant under the original gauge transformation and the deformed SUSY transformation

$$\delta X^I = i\epsilon \Gamma^I \Psi,$$

$$\delta \Psi = (D_\mu X^I) \Gamma^\mu \epsilon - \frac{1}{6}[X^I, X^J, X^K] \Gamma_{IJK} \epsilon - \mu \Gamma_{3456} \Gamma^I X^I \epsilon,$$

$$\delta \hat{A}_{\mu}^b = i\epsilon \Gamma_\mu \Gamma_l X^l_\epsilon \Gamma_{d}^b a.$$

(3.44)
This deformed theory breaks the original $SO(8)$ $R$-symmetry down to $SO(4) \times SO(4)$. By setting $\mu \to 0$ both the action and SUSY transformation reduce to the original Bagger-Lambert action. In addition there is another supersymmetry transformation:

$$\delta X^I_a = 0, \quad \delta \tilde{A}^b_\mu a = 0,$$
$$\delta \Psi = \exp(-\frac{\mu}{3} \Gamma_{3456} \Gamma_\mu x^\mu) T^{-1} \eta,$$  \hspace{1cm} (3.45)

where $x^\mu$ is the coordinates of the world volume.

### 3.2 Deformed BL to Janus

This model can be similarly investigated by expanding the fields into modes with internal indices $a = (-1, 0, i)$. The mode expansions of the mass and the flux terms become

$$\mathcal{L}_{\text{mass}} = \mu^2 X^I_{-1} X^I_0 - \frac{\mu^2}{2} \text{Tr}(\dot{X}^I, \dot{X}^I) - i\mu \tilde{\Psi}_0 \Gamma_{3456} \Psi_0 + \frac{i}{2} \mu \text{Tr}(\dot{\tilde{\Psi}} \Gamma_{3456}, \dot{\Psi}),$$
and

$$\mathcal{L}_{\text{flux}} = -\frac{2}{3} \mu \epsilon_{EFGH} X^E_0 \text{Tr}(\dot{X}^F, [\dot{X}^G, \dot{X}^H]) - \frac{2}{3} \mu \epsilon_{E'F'G'H'} X^E_0 \text{Tr}(\dot{X}^{E'}, [\dot{X}^{G'}, \dot{X}^{H'}]).$$

(3.46) (3.47)

Now $X^I_{-1}$ and $\Psi_{-1}$ again appear linearly in the action, and they are Lagrange multipliers. Because of the mass terms, the constraint equations are modified to

$$(\partial^2 - \mu^2) X^I_0 = 0, \quad (\Gamma^\mu \partial_\mu + \mu \Gamma_{3456}) \Psi_0 = 0.$$  \hspace{1cm} (3.48)

Namely the fields with the $T^0$ component are constrained to obey the massive wave equations. Since $X^I$ are real fields, instead of the plane waves $\exp(ik_\mu x^\mu)$ with a time-like vector $k_\mu$, we take the following solution to the constraint equation;

$$X^I_0 = f e^{p_\mu x^\mu} \delta^I_{10} = v(x) \delta^I_{10}, \quad \Psi_0 = 0,$$  \hspace{1cm} (3.49)

where $f$ is an arbitrary constant and $p_\mu$ is a spacelike vector satisfying $p^2 = \mu^2$. Without loss of generality, we can take $p_\mu = (0, \mu, 0)$. This configuration preserves half of the 16 supersymmetries, since $\Psi_0$ transforms as:

$$\delta \Psi_0 = v(x) \mu (\Gamma^1 - \Gamma_{3456}) \Gamma^{10} \epsilon.$$  \hspace{1cm} (3.50)

Hence around the above configuration, we will get Janus gauge field theories with 8 supersymmetries. (We can take $X^I_0$ as a sum of two functions, then more supersymmetries are broken.)

Inserting this configuration to the action, one can again integrate the redundant gauge field $B_\mu$. Terms involving $B_\mu$ are given by:

$$-\frac{1}{2} (\dot{D}_\mu X^{10} - B_\mu X^{10}_0)^2 + \frac{1}{2} \epsilon^{\mu\nu\lambda} \dot{F}_{\mu\nu} B_\lambda - p^\mu v B_\mu \dot{X}^{10}. $$

(3.51)
Integrating $B_\mu$ gives
\[
\frac{1}{2v} \epsilon^{\mu\nu\lambda} \hat{F}_{\mu
u} \hat{p}_\lambda \hat{X}^{10} + \frac{1}{8v^2} (\epsilon^{\mu\nu\lambda} \hat{F}_{\mu
u} - 2v \hat{X}^{10} \hat{p}_\lambda)^2 \\
= -\frac{1}{4v^2} \hat{F}_{\mu
u}^2 + \frac{\mu^2}{2} \text{Tr}(\hat{X}^{10}, \hat{X}^{10}).
\]

(3.52)

Interestingly the second term is cancelled by the mass term of $\hat{X}^{10}$ and all the terms involving $\hat{X}^{10}$ have disappeared. To summarize, the resultant effective Lagrangian is given by:
\[
\mathcal{L} = -\frac{1}{2} \text{Tr}(\hat{D}_\mu \hat{X}^A)^2 - \frac{\mu^2}{2} \text{Tr}(\hat{X}^A, \hat{X}^A) + \frac{1}{4} v^2 \hat{X}^A \hat{X}^B] + \hat{F}_{\mu
u}^2 + \frac{\mu^2}{2} \text{Tr}(\hat{X}^A, \hat{X}^A) + \frac{1}{4} v^2 \hat{X}^A \hat{X}^B] + \hat{F}_{\mu
u}^2.
\]

(3.53)

This is a Janus field theory whose coupling constant is given by $v = f \exp(\mu x^1)$. The Lagrangian is invariant under the following 8 supersymmetries
\[
\delta \hat{X}^A = i \bar{\epsilon} \Gamma^A \hat{\Psi}, \\
\delta \hat{\Psi} = \hat{D}_\mu \hat{X}^A \Gamma^\mu \Gamma^\lambda \epsilon - \frac{1}{2} v \epsilon^{\mu\nu\lambda} \hat{F}_{\mu\nu} \hat{X}^A \hat{X}^B] \Gamma^\lambda \Gamma^1 \epsilon - \mu \Gamma_{3456} \Gamma^A \hat{X}^A \epsilon, \\
\delta \hat{A}_\mu = i \bar{\epsilon} \Gamma^A \Gamma^\mu \hat{\Psi}, \\
\delta \hat{A}_\mu = 0, \\
\delta \hat{B}_\mu = i \bar{\epsilon} \Gamma^I \hat{X}^I] \hat{\Psi}.
\]

(3.54)

Finally if $v$ vanishes, i.e. for $X^I_0 = 0$ and $\Psi_0 = 0$, the Lagrangian becomes
\[
\mathcal{L} = -\frac{1}{2} (\hat{D}_\mu \hat{X}^I)^2 + \frac{1}{2} \bar{\epsilon} \Gamma^\mu \hat{D}_\mu \hat{\Psi} + \frac{1}{2} \epsilon^{\mu\nu\lambda} \hat{F}_{\mu\nu} \hat{X}^B] + \frac{1}{4} \hat{F}_{\mu\nu}^2 - \mu \Gamma_{3456} \Gamma^I \hat{X}^I \epsilon.
\]

(3.55)

(3.56)

with a constraint $\hat{F}_{\mu\nu} = 0$. The supersymmetry transformation is given by
\[
\delta \hat{X}^I = i \bar{\epsilon} \Gamma^I \hat{\Psi}, \\
\delta \hat{\Psi} = \hat{D}_\mu \hat{X}^I \Gamma^\mu \Gamma^\lambda \epsilon - \mu \Gamma_{3456} \Gamma^I \hat{X}^I \epsilon, \\
\delta \hat{A}_\mu = 0, \\
\delta \hat{B}_\mu = i \bar{\epsilon} \Gamma^I \hat{X}^I \hat{\Psi}\]

(3.57)

(3.58)

(3.59)

(3.60)

and the Lagrangian has the $SO(4) \times SO(4)$ R-symmetry.

4 Conclusions and discussions

In this paper, we have derived Janus field theories from the Bagger-Lambert field theory with the specific realization of 3-algebra given by \[11,12,13\]. By integrating redundant fields, we obtained supersymmetric field theories whose coupling constant varies with the space-time coordinate. A similar analysis was also done for the mass-deformed Bagger-Lambert model. In this case,
we obtained a mass-deformed supersymmetric Yang-Mills theory with an exponentially growing coupling constant along one of the spatial direction.

The analysis in this paper became possible by the remarkable discovery of the realization of the 3-algebra. The roles played by the fields associated with the internal indices $T^{-1}, T^0$ and $T^i$ are completely different, and this is the origin of the success that the D2 brane effective theory can be reproduced from the very strangely looking model of Bagger-Lambert.

One of the most important directions will be to construct a matrix model of M-theory with $SO(10,1)$ symmetry. In the case of matrix models for superstrings, a superstring world sheet action is related to the D-brane gauge theories through matrix models $[24,25]$. Similarly we may expect that the supermembrane world volume action must be related to the Bagger-Lambert gauge theories of multiple M2-branes through a new class of matrix models. A natural guess $[26]$ is

$$S = \text{Tr} \left( -\frac{1}{6} [X^I, X^J, X^K]^2 + \frac{1}{2} \bar{\Psi} \Gamma_{I,J} [X^I, X^J, \Psi] \right),$$

(4.61)

where $I$ runs from 0 to 10, but the action is not invariant under supersymmetry transformations. This action is closely related to both of the supermembrane action and the Bagger-Lambert action, but unfortunately it seems different from both of them. The difficulty in the supermembrane action is that we cannot fix the $\kappa$-symmetry without breaking $SO(10,1)$ rotation. The difficulty to construct a gauge theory is how to exactly identify the gauge fields of the Bagger-Lambert model and its supersymmetry transformation in terms of the matrix model. The recently discovered 3-algebraic structure suggests that the embedding of the space-time in the internal space is more complicated than the case of the matrix models (i.e. large N reduction). We want to come back to this problem in near future.

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