Polysymplectic formulation for topologically massive Yang-Mills field theory

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Abstract

We analyze the De Donder-Weyl covariant field equations for the topologically massive Yang-Mills theory. These equations are obtained through the Poisson-Gerstenhaber bracket described within the polysymplectic framework. Even though the Lagrangian defining the system of our interest is singular, we show that by appropriately choosing the polymomenta one may obtain an equivalent regular Lagrangian, thus avoiding the standard analysis of constraints. Further, our simple treatment allows us to only consider the privileged \((n-1)\)-forms in order to obtain the correct field equations, in opposition to certain examples found in the literature.

1 Introduction

Non-Abelian gauge models like Yang-Mills and Chern-Simons field theories are very well suited to describe a huge diversity of phenomena in physics.

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On the one hand, the Yang-Mills field theory has been completely successful describing the standard model of particle physics [1], while the non-Abelian Chern-Simons field theory, on the other hand, is commonly associated to a wide variety of areas ranging from condensed matter and solid state physics to general relativity [2, 3, 4, 5, 6, 7, 8]. The topologically massive Yang-Mills field theory is obtained by adding a Chern-Simons invariant to the Yang-Mills theory [9, 10, 11], thus resulting in a mechanism to generate gauge field mass. As it is well known, topologically massive Yang-Mills theory provides a link between field theories and knot theory, and also has been introduced to model specific issues in several areas of physics. For example, it has been related to the quantum Hall effect [12] and superconductivity [13], while in the gravitational context is related to the topological massive gravity [14, 15, 16].

At the classical level, equations of motion for the topologically massive Yang-Mills theory has been studied from different perspectives, including the standard Dirac-Hamiltonian [17] and the Hamilton-Jacobi [18] approaches. In this article we analyze this topologically massive theory from the modern viewpoint of the polysymplectic framework for field theories (see [19, 20, 21, 22, 23, 24, 25] for generalities and the geometric description, and [24, 26, 27, 28] for some specific examples). Even though some of the various versions of the multisymplectic formalism may differ in their geometric ingredients, the vast majority of state-of-the-art versions rely on its construction on a jet bundle structure which, being naturally covariant, offers an outstanding perspective for the analysis of physically motivated field theories. Also, one of the main features of the jet bundle structure is that a field theory may be seen as a finite dimensional extended model. Besides, the polysymplectic formalism is strongly based on the De Donder-Weyl equations which are covariant in the sense that there is no privileged direction in the definition of the polymomenta (as opposed to the standard symplectic Hamiltonian approach where the time variable is treated as a privileged coordinate in the momenta definition). The De Donder-Weyl equations may be obtained from a generalization of the standard Poisson bracket, known as the Poisson-Gerstenhaber bracket and defined for Hamiltonian forms under the co-exterior product. Indeed, the Hamiltonian \((n-1)\)-forms play a special role under this Poisson-Gerstenhaber bracket as they guarantee closure of the bracket under the co-exterior product defined on a \((n-1)\)-dimensional subspace and they also are necessary in order to get the correct equations.

In the literature, the term multisymplectic is commonly associated to generic structures on fiber bundles which do not distinguish vertical and horizontal subspaces. On the contrary, the term polysymplectic is used whenever an explicit decomposition into these subspaces is considered.
of motion as described in [24, 29, 30, 31, 32] (see also [33] for a discussion within the multisymplectic formalism for lattice field theories).

In order to proceed our analysis for the topologically massive Yang-Mills theory we start by testing first the polysymplectic formalism for the Yang-Mills and the Chern-Simons theories, and then we continue systematically. We thus find that, by judiciously decomposing the polymomenta into its symmetric and anti-symmetric parts, the correct equations of motion may be obtained by considering only the Hamiltonian \((n-1)\)-forms. Further, for the three cases we obtained a vanishing divergence of the symmetric parts of the polymomenta. This in turn allow us to introduce an appropriate transformation of the polymomenta inducing an equivalent De Donder-Weyl Hamiltonian from which the equations of motion for each field may be straightforwardly obtained. Our treatment may be confronted with the analysis of the Maxwell field in reference [24] where, adversely, the Hamiltonian \((n-1)\)-forms are not considered as the fundamental forms in order to obtain the correct equations of motion.

The rest of the article is organized as follows. In Section 2 we summarize the polysymplectic formalism based on the explicit decomposition of the fiber bundles into its vertical and horizontal parts and describe the De Donder-Weyl equations of motion in order to briefly introduce all the required background and to set the our notation. In Section 3 we apply the polysymplectic formalism to the Yang-Mills, Chern-Simons and topologically massive Yang-Mills field theories. We emphasize that the three non-Abelian field theories contain analogous structures which allow us to describe these models systematically. Finally, in Section 4 we include some concluding remarks.

2 Polysymplectic structure and the Poisson-Gerstenhaber bracket

In this Section we briefly introduce the De Donder-Weyl Hamiltonian from the perspective of the polysymplectic formalism and its relation to the Poisson-Gerstenhaber bracket. By considering simplicity as our guiding principle, we proceed as close as possible to references [24, 29, 30, 31]. We encourage the reader to check these references for further details and, also, reference [34] for a technical construction of the vertical-horizontal splitting in the context of the polysymplectic formalism for nontrivial fiber bundles.

To start, we will consider an arbitrary smooth \(n\)-dimensional spacetime manifold \(\mathcal{M}\) with local coordinates \(\{x^\mu\}, \mu = 1, \ldots n\), and volume form
\( \omega \). We also will consider the fibered manifold \((E, \pi, \mathcal{M})\), where \(E\) denotes the total space manifold with local coordinates \(\{\phi^a\}\) (local sections around \(p \in \mathcal{M}\)) which may be physically interpreted as the classical gauge fields associated to a given theory (\(a = 1, \ldots, m\) denoting the set of internal degrees of freedom), and the map \(\pi : E \to \mathcal{M}\) stands for the canonical projection. Finally, we will consider the first jet manifold of \(\pi, J^1E\), with local coordinates \((x^\mu, \phi^a, \phi^\mu_a)\), where \(\phi^\mu_a := \partial \phi^a / \partial x^\mu\) stand for the field derivative coordinates.

In what follows, we will identify the first jet manifold with the configuration space of the theory.

In order to describe the dynamics of a given theory we will consider the Lagrangian density \(L: J^1E \to \mathbb{R}\) as the smooth function of the configuration space such that \(L(j^1_p \phi) = L(\phi^a, \partial_\mu \phi^a, x^\mu)\), where \(j^1_p\) is the first prolongation of the jet bundle. Next, we introduce the polymomenta given by

\[
\pi^\mu_a := \frac{\partial L}{\partial (\partial_\mu \phi^a)}.
\]

The polymomentum phase space is endowed with a canonical \(n\)-form \(\Theta_{DW}\) known as the Poincaré-Cartan form, given by \([35, 36, 37]\)

\[
\Theta_{DW} = \pi^\mu_a d\phi^a \wedge \omega_\mu - H_{DW} \omega,
\]

where \(\omega_\mu := \partial_\mu \llcorner \omega\) is the basis for the \((n-1)\)-form subspace, and the De Donder-Weyl Hamiltonian function \(H_{DW}\) is obtained by means of the covariant Legendre transformation

\[
H_{DW}(\phi^a, \pi^\mu_a, x^\mu) := \pi^\mu_a \partial_\mu \phi^a - L.
\]

Calculating the exterior differential of the Poincaré-Cartan form results in the \((n+1)\)-form

\[
\Omega_{DW} := d\Theta_{DW} = d\pi^\mu_a \wedge d\phi^a \wedge \omega_\mu - dH_{DW} \wedge \omega.
\]

As described in references \([22, 23, 38, 39]\), the classical dynamics of the fields is essentially encoded in the vertical components of the multivector field annihilating the canonical \((n+1)\)-form \(\Omega_{DW}\). Thus, we will only consider the vertical part of the Poincaré-Cartan \(n\)-form

\[
\Theta_{DW}^V := \pi^\mu_a d\phi^a \wedge \omega_\mu.
\]

Now, let us define the vertical exterior differential as

\[
d^V \Phi = \frac{1}{p!} \partial_\mu \Phi^{M_1 \cdots M_p} dz^v \wedge dz^{M_1} \wedge \cdots \wedge dz^{M_p},
\]
where \( z^M := (z^v, x^\mu) = (\phi^a, \pi^\mu_a, x^\mu) \) stands for the first jet bundle local coordinates. In this way we are able to calculate the vertical exterior differential of \( \Theta^V_{\text{DW}} \)

\[
\Omega^V_{\text{DW}} = d^V \Theta^V_{\text{DW}} = -d\phi^a \wedge d\pi^\mu_a \wedge \omega_\mu.
\]

This \( \Omega^V_{\text{DW}} \) is simply known as the polysymplectic \((n + 1)\)-form.

Let \( \mathcal{X}^V \) be a vertical multivector field, that is, a \( p \)-multivector field such that it has one vertical and \((p - 1)\) horizontal indices, namely \( \mathcal{X}^V = \mathcal{X}^{\nu_1 \ldots \nu_{p-1}}(z^M) \partial_\nu_1 \wedge \cdots \wedge \partial_\nu_{p-1} \). We will call the vertical multivector field \( \mathcal{X}^V \) a Hamiltonian multivector if there exists an horizontal \((n - p)\)-form \( n^{-p} \) such that

\[
\mathcal{X}^V \lrcorner \Omega^V_{\text{DW}} = d^{n-p} F.
\]

From now on, we will assume that all the multivector fields are vertical. As discussed in [24, 29, 30, 31], the space of Hamiltonian forms results not closed with respect to the exterior product, but it is closed with respect to the co-exterior product defined by

\[
\mathcal{F} \bullet \mathcal{G} := \ast^{-1}(\ast \mathcal{F} \wedge \ast \mathcal{G}),
\]

where \( \ast \) stands for the standard Hodge operator. In addition, for a pair of Hamiltonian forms one may define the Poisson-Gerstenhaber bracket, \( \{ \cdot, \cdot \} \),

\[
\{ \mathcal{F}_1, \mathcal{F}_2 \} = (-1)^{n-p} \mathcal{X}_1 \lrcorner \mathcal{X}_2 \lrcorner \Omega^V_{\text{DW}}.
\]

One may note that the Hamiltonian forms provided with the Poisson-Gerstenhaber bracket, \( \{ \cdot, \cdot \} \), close as a graded Poisson-Gerstenhaber algebra under the co-exterior product defined in [9]. In addition, we also note from the bracket (10) that the Hamiltonian \((n - 1)\)-forms will play a primordial role within this polysymplectic formalism as for them the bracket clearly results closed. This relevance is associated to the fact that we may construct Noether currents by means of the \((n - 1)\)-forms in order to construct the observables for a given theory [32, 33]. We may use the Poisson-Gerstenhaber bracket to induce the standard relations among the canonically conjugate variables

\[
\{ \pi^\mu_a \omega_\mu, \phi^b \} = \delta^b_a, \quad \{ \pi^\mu_a \omega_\mu, \phi^b \omega_\nu \} = \delta^b_\nu \omega_\mu, \quad \{ \pi^\mu_a, \phi^b \omega_\nu \} = \delta^\mu_b \delta^a_\nu.
\]

From our particular point of view, however, a very important application of the Poisson-Gerstenhaber bracket is that it allows us to write the De Donder-Weyl field equations for an arbitrary Hamiltonian \((n - 1)\)-form \( F = F^\mu \omega_\mu \) by means of the relation

\[
d \bullet F = -\sigma (-1)^n \{ H_{\text{DW}}, F \} + d^h \bullet F,
\]
where \( \sigma = \pm 1 \). The operation \( d\bullet \) is known as the total co-exterior differential and for an arbitrary \( p \)-form \( F \) reads

\[
d^{p} F := \frac{1}{(n-p)!} \partial_{\nu} F^{\mu_{1}...\mu_{n-p}} \partial_{\mu} z^{\nu} dx^{\mu} \bullet \partial_{\mu_{1}...\mu_{n-p}} \omega + d^{h} \bullet F,
\]

and \( d^{h} \bullet \) denotes the horizontal co-exterior differential given by

\[
d^{h} \bullet F := \frac{1}{(n-p)!} \partial_{\mu} F^{\mu_{1}...\mu_{n-p}} dx^{\mu} \bullet \partial_{\mu_{1}...\mu_{n-p}} \omega.
\]

Finally, by considering the canonical brackets (11) and the total co-exterior differential (12) for the components of the canonical variables, \( \pi_{a}^{\mu} \) and \( \phi^{a} \), we obtain the De Donder-Weyl field equations

\[
\partial_{\mu} \pi_{a}^{\mu} = -\frac{\partial H_{DW}}{\partial \phi^{a}}, \quad \partial_{\mu} \phi^{a} = \frac{\partial H_{DW}}{\partial \pi_{a}^{\mu}}.
\]

It is straightforward to show that the relations (15) are equivalent to the Lagrangian field equations if \( L \) is hyperregular, that is, whenever the covariant Legendre transformation is a diffeomorphism \([19, 35, 40, 41]\), thus, hyperregular Lagrangians on the first jet manifold, \( J^{1}E \), induce well defined De Donder-Weyl Hamiltonian systems on the dual jet \( J^{1*}E \), and vice versa. Even though for the cases we are interested below this last condition does not follow, it is possible to extend this formalism to these kind of systems due to the inherent symmetries, as will be discussed.

3 Non-Abelian field theories

The main purposes of this section are to describe, within the polysymplectic framework, the field equations associated to the topologically massive Yang-Mills field theory, on the one side, and to analyze the emerging constraints for this model, on the other side. In order to achieve this, first we will develop separately the polysymplectic and Poisson-Gerstenhaber structures, as described in the previous section, for a pair of related models, namely the Yang-Mills field theory and the non-Abelian Chern-Simons topological field theory. After doing that, we will couple the Yang-Mills and the Chern-Simons theories to obtain the model of our interest. From now on, we will assume that all the fields of our interest are described in a three dimensional background space-time manifold \( \mathcal{M} \) endowed with a Minkowski metric \( \eta_{\mu\nu} = \text{diag}(1, -1, -1) \) and Latin indices which describe internal degrees of freedom are raised and lowered by the corresponding metric in the internal space.
3.1 Yang-Mills field theory

The Yang-Mills field theory has been previously analyzed within the context of the multisymplectic formulation in references [42], [43], where the main discussion was guided towards the quantum representation, and the constrained structure for this model was only considered heuristically. From our point of view, our intention is to elucidate the role of the constraint related to the symmetric part of polymomenta explicitly. To this end, let us begin with the well known 2 + 1 dimensional Yang-Mills model which is defined by means of the Lagrangian [44, 45]

\[ L_{YM} = -\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu}, \]  

where \( \mu = 1, 2, 3 \) denote space-time indices while lower-case Latin indices denote internal degrees of freedom. The components of the field strength are given by

\[ F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f_{abc} A_b^\mu A_c^\nu. \]  

Here \( A_\nu^a \) are the gauge or Yang-Mills fields, and generalize the vector potential in electrodynamics, \( g \) is a coupling constant and \( f_{abc} \) are the structure constants of the Lie algebra associated to the internal symmetry group. Now, by adapting definition (1) to our case, we obtain the polymomenta

\[ \pi_{\mu\nu}^a = \frac{\partial L_{YM}}{\partial (\partial_\mu A_\nu^a)} = -F_{\mu\nu}^a, \]  

which, due to the anti-symmetry of the field strength \( F_{\mu\nu}^a \), yields the conditions

\[ \pi_a^{(\mu\nu)} \approx 0. \]  

(Note that these conditions result analogous to the primary constraints within the Dirac formalism [46]. Henceforth, and by a slight abuse of language, we will simply refer to conditions (19) as primary constraints, and will endow the weak equality symbol with the same meaning as in Dirac approach. See [31] for further details in the Dirac treatment for constraints within the polysymplectic formalism). In the following, we will consider the symmetric and anti-symmetric parts of our polymomentum and field variables, so we can get the appropriate field equations. By means of the Legendre transformation (3) we obtain the De Donder-Weyl Hamiltonian for the Yang-Mills theory

\[ H_{DW}^{YM}(A, \pi, x) = \pi_{a^{[\mu\nu]}}^a \partial_{[\mu} A_{\nu]}^a - L_{YM} = -\frac{1}{4} \pi_{a^{[\mu\nu]}}^a \pi_{a^{[\mu\nu]}}^a - \frac{g}{2} f_{abc} A_\mu^a A_\nu^b A_\lambda^c A_{\lambda[\mu\nu]}^a. \]  

7
In analogy to the standard Dirac formalism for singular Lagrangians \[46\], we define the total De Donder-Weyl Hamiltonian, \( \tilde{H}^{\text{YM}}_{\text{DW}} \), as the De Donder-Weyl Hamiltonian subject to the primary constraints (19), that is,

\[
\tilde{H}^{\text{YM}}_{\text{DW}} = H^{\text{YM}}_{\text{DW}} + \lambda^a_{\mu \nu} \pi^a_{\mu \nu},
\]

(21)

where \( \lambda^a_{\mu \nu} \) are Lagrange multipliers enforcing the constraints (19). Of course, the Lagrange multipliers \( \lambda^a_{\mu \nu} \) are symmetric in Greek indices. We will use this total De Donder-Weyl Hamiltonian in order to analyze the correct equations of motion for the system. Indeed, by considering the \((n-1)\)-forms \( A^a_{\mu \nu} := A^a_{\mu} \omega^\nu \) and \( \pi^a_{\mu \nu} := \pi^a_{\mu \nu} \omega^\mu \) as the canonically conjugate variables (see (11)), the field equations (12) read

\[
d \bullet A^a_{\mu \nu} = - \{ [ \tilde{H}^{\text{YM}}_{\text{DW}}, A^a_{\mu \nu} ] \} = - \frac{1}{2} \pi^a_{[\mu \nu]} - \frac{g}{2} f^{b c} A^a_b A^\nu_c,
\]

\[
d \bullet \pi^a_{\mu} = - \{ [ \tilde{H}^{\text{YM}}_{\text{DW}}, \pi^a_{\mu} ] \} = - g f^c_{ab} A^b_a \pi^a_{[\nu \mu]},
\]

(22)

where the horizontal co-exterior differential terms in (12) automatically vanish as the background spacetime we are considering is Minkowski. In the subsequent models, this will also be the case. By considering the symmetric and anti-symmetric parts of these equations of motion we straightforwardly obtain the set of relations

\[
\partial^{[\mu} A^a_{\nu]} = - \frac{1}{2} \pi^a_{[\mu \nu]} - \frac{g}{2} f^{b c} A^a_b A^\nu_c,
\]

\[
\partial^{\mu} A^a_{\nu} = \lambda^a_{(\mu \nu)},
\]

\[
\partial_{\nu} \pi^a_{[\nu \mu]} - g f^c_{ab} A^b_a \pi^a_{\nu [\mu]} = 0,
\]

\[
\partial_{\nu} \pi^a_{(\nu \mu)} = 0.
\]

(23)

These relations allow us to fully characterize the dynamics of the system. Certainly, the first line of (23) only stands for the definition of the polymomenta (18). The second line fixes the Lagrangian multipliers in terms of the symmetric derivatives of the gauge field. By defining the covariant derivative for the gauge group as

\[
D^b_{\mu a} := \delta^b_a \partial_\mu - g f^b_{ac} A^c_\mu,
\]

(24)

where the \( \delta \) stands for the identity for the generators of the internal group, and by considering relation (18), \( \pi^a_{[\mu \nu]} = - F^a_{\mu \nu} \), we may identify the third
line above as the standard Yang-Mills field equations for the field strength $F^{\nu\mu}_a$, that is,

$$(D_\nu F^{\nu\mu})_a = 0.$$  

(25)

Finally, the last line may be interpreted, together with constraints (19), as the fact that the symmetric part of the polymomentum does not contain information about the dynamics of the model.

We observe that in the De Donder-Weyl formalism, the variation of the Hamiltonian only determines the divergence of the polymomenta (15), therefore, the equations of motion given in (23), remain invariant if the polymomenta transform as

$$\pi^{\mu\nu}_a \mapsto \pi'^{\mu\nu}_a := \pi^{\mu\nu}_a - \xi^{\mu\nu}_a,$$

(26)

where the term $\xi^{\mu\nu}_a$ must satisfies the divergenceless condition, $\partial_\mu \xi^{\mu\nu}_a = 0$. In order to see this, let us define a new Lagrangian, $L'_{YM}$, given by

$$L'_{YM} = L_{YM} - \xi^{\mu\nu}_a \partial_\mu A^a_\nu,$$

(27)

it is straightforward to see, that under the divergenceless condition of $\xi^{\mu\nu}_a$, the Lagrangian $L'_{YM}$ satisfies the same Euler-Lagrange equations as $L_{YM}$. Moreover, this transformation preserves the De Donder-Weyl Hamiltonian,

$$H^{YM}_{DW}(A', \pi', x) = \pi'^{\mu\nu}_a \partial_\mu A^a_\nu - L'_{YM}$$

$$= \pi^{\mu\nu}_a \partial_\mu A^a_\nu - \xi^{\mu\nu}_a \partial_\mu A^a_\nu - L_{YM} + \xi^{\mu\nu}_a \partial_\mu A^a_\nu = H^{YM}_{DW}(A, \pi, x).$$

(28)

This means that both polymomenta, $\pi^{\mu\nu}_a$ and $\pi'^{\mu\nu}_a$, result physically equivalent. Within this perspective, from the equations (23), we can observe that the symmetric part of the polymomenta satisfies the divergenceless condition, $\partial_\mu \pi^{(\mu\nu)}_a = 0$, this suggest that we can define a new set of polymomentum variables given by

$$\pi^{\mu\nu}_a \mapsto \pi'^{\mu\nu}_a := \pi^{\mu\nu}_a - \pi^{(\mu\nu)}_a,$$

(29)

such that the equations of motion remain invariant. These new variables correspond to the anti-symmetric part of the polymomenta, therefore, by removing the symmetric part, the weak condition given by the primary constraint (19), can be taken as a strong condition in the Dirac’s sense, thus avoiding the standard formalism for constrained systems. As a consequence of the ambiguity in the De Donder-Weyl equations for the polymomenta, we conclude that the dynamics of the system is fully encoded in its anti-symmetric part as it is expected.
3.2 Non-Abelian Chern-Simons model

In this subsection we will describe, from the polysymplectic point of view, the dynamics for the three dimensional Chern-Simons field theory. To this end, we start by considering the Lagrangian \[ L_{\text{CS}} = \epsilon^{\mu\nu\rho} \left( A_\mu^a \partial_\nu A_\rho^a + \frac{1}{3} f_{abc} A_\mu^a A_\nu^b A_\rho^c \right). \] (30)

As before, Greek and lower-case Latin indices stand for space-time and internal symmetry components, respectively. Also note that \( A_\mu^a \) represents the gauge fields, \( f_{abc} \) are fully anti-symmetric structure constants associated to the gauge algebra and, finally, \( \epsilon^{\mu\nu\rho} \) stands for the standard three dimensional Levi-Civita symbol. Proceeding as in the previous subsection, we start by considering the definition (1) in order to obtain the polymomenta

\[ \pi_{\nu}^{\mu} = \partial L_{\text{CS}} \] \[ = \epsilon^{\mu\nu\rho} A_\rho^a. \] (31)

From this last relation we note that, contrary to the Yang-Mills case, all the derivatives of the gauge fields are not invertible in terms of the polymomenta. Thus, we obtain a set of primary constraints that we separate for convenience into symmetric and anti-symmetric parts

\[ \pi_{\mu}^{(\nu)} \approx 0, \]
\[ \pi_{\nu}^{[\mu]} - \epsilon^{\mu\nu\rho} A_\rho^a \approx 0. \] (32)

Once again, by means of the covariant Legendre transformation (3) we compute the De Donder-Weyl Hamiltonian associated to the Chern-Simons model

\[ H_{\text{DW}}^{\text{CS}}(A, x) = \pi_{\nu}^{\mu} \partial_\mu A_\nu^a - L_{\text{CS}} \] \[ = -\frac{1}{3} \epsilon^{\mu\nu\rho} f_{abc} A_\mu^a A_\nu^b A_\rho^c, \] (33)

and, in analogy to the Yang-Mills case, we may introduce the total Hamiltonian

\[ \tilde{H}_{\text{DW}}^{\text{CS}} = H_{\text{DW}}^{\text{CS}} + \lambda_{\mu\nu}^a \pi_{\nu}^{(\mu)} + \eta_{\mu\nu}^a (\pi_{\nu}^{[\mu]} - \epsilon^{\mu\nu\rho} A_\rho^a), \] (34)

where the \( \lambda \)'s and the \( \eta \)'s are, respectively, symmetric and anti-symmetric Lagrange multipliers enforcing the constraints (32). As expected, we will use this total Hamiltonian in order to reproduce the correct equations of motion as defined in (12) by means of the Poisson-Gerstenhaber brackets.
To this end, we define the canonical \((n-1)\)-form variables \(A_{\mu
u}^a := A_{\mu}^a \omega^\nu\) and \(\pi_a^{\mu} := \pi^{\mu
u} \omega_\nu\). Thus, the field equations read

\[
\mathbf{d} \bullet A_{\mu
u}^a = - \left\{ \vec{H}_{DW}^{CS}, A_{\mu
u}^a \right\} = \lambda_{\mu
u}^a + \eta_{\mu
u}^a,
\]

\[
\mathbf{d} \bullet \pi_a^\mu = - \left\{ \vec{H}_{DW}^{CS}, \pi_a^\mu \right\} = \epsilon^{\mu \nu \rho} f_{abc} A_b^\nu A_c^\rho + \eta_{\mu \nu \rho} \epsilon^{\mu \nu \rho},
\]

(35)

Once again, one may consider the symmetric and anti-symmetric parts of these field equations, thus obtaining the relations

\[
\partial_{[\mu} A_{\nu]}^a = \eta_{\mu \nu}^a,
\]

\[
\partial_{(\mu} A_{\nu)}^a = \lambda_{\mu \nu}^a,
\]

\[
\partial_{\mu} \pi_{[\mu \nu]}^a = \epsilon^{\nu \mu \rho} f_{abc} A_b^\nu A_c^\rho - \eta_{\nu \mu \rho} \epsilon^{\nu \mu \rho},
\]

\[
\partial_{\nu} \pi_{(\nu \mu)}^a = 0.
\]

(36)

This set of relations may be interpreted as follows. The first two lines simply fix both Lagrange multipliers, \(\eta_{\mu \nu}^a\) and \(\lambda_{\mu \nu}^a\), in terms of the gradients of the gauge field. Likewise, in order to obtain the Chern-Simons field equations we only need to substitute, respectively, the second line of the constraints (32) on the left hand side of the third equation and also the first line of equations (36) in the same line to obtain in a direct manner

\[
\epsilon^{\alpha \mu \nu} F_{\alpha \mu \nu}^a = 0,
\]

(37)

where we defined the components of the Chern-Simons 3-form \(F_{\mu \nu}^a\) as in (17).

Of course, equations (37) may be recognized as the De Donder-Weyl field equations of the non-Abelian Chern-Simons field theory. Finally, in a similar fashion as for the Yang-Mills model, the first line of constraints (32), together with the last line of relations (36) tell us that the symmetric part of the polymomentum does not contain any dynamical information about the Chern-Simons model. As in the previous example, we can observe from the last equation in (36), that the polymomenta satisfy the divergenceless condition \(\partial_{\nu} \pi_a^{(\nu \mu)} = 0\). This means that the transformed polymomentum variables \(\pi_{\mu \nu}^a \mapsto \pi_{(\mu \nu)}^a := \pi_{\mu \nu}^a - \pi_{(\mu \nu)}^a\) leave the De Donder-Weyl equations invariant, therefore, the dynamics of the system is fully determined by its anti-symmetric part. Since the symmetric part of the polymomenta \(\pi_{(\mu \nu)}^a\) is related to the primary constraints, its divergenceless condition suggests that these constraints can be taken as strong constraints in the Dirac sense.
3.3 Topologically massive Yang-Mills theory

Based on the two previous examples, we will consider now the Lagrangian for the three dimensional topologically massive Yang-Mills field theory given by [18]

\[ L_{TMYM} = \frac{1}{4} F_{\alpha}^{\mu\nu} F_{\alpha}^{\mu\nu} + \frac{m}{4} \epsilon^{\mu\nu\rho} \left( F_{\alpha\mu\nu} A_{\rho}^{\alpha} - \frac{g}{3} f_{\alpha\beta\gamma} A_{\mu}^{\alpha} A_{\nu}^{\beta} A_{\rho}^{\gamma} \right), \]  

(38)

where \( m \) and \( g \) are free parameters commonly associated to the mass of the field and to the coupling constant, respectively, while \( A_{\mu}^{\alpha} \) stands for the gauge field. The components of the field strength \( F_{\mu\nu}^{\alpha} \) are analogously given by equation (17), and \( f_{\alpha\beta\gamma} \) are again the structure constants associated to the gauge symmetry of this model. In order to construct the De Donder-Weyl Hamiltonian first we will introduce the polymomenta by means of \[ \pi_{\mu\nu}^{\alpha} = \partial L_{TMYM} \partial (\partial_{\mu} A_{\nu}^{\alpha}) = -F_{\mu\nu}^{\alpha} + \frac{m}{2} \epsilon_{\mu\nu\rho} A_{\alpha}^{\rho}, \]  

(39)

which systematically may be decomposed into their symmetric and anti-symmetric parts as

\[ \pi_{\alpha}^{(\mu\nu)} \approx 0, \]

\[ \pi_{\alpha}^{[\mu\nu]} = -2 \partial_{[\mu} A^{\nu]}_{\alpha} - g f_{\alpha\beta\gamma} A_{\nu}^{\beta} A_{\rho}^{\gamma} \pi_{\alpha}^{[\mu\nu]} + \frac{m}{2} \epsilon_{\mu\nu\rho} A_{\alpha}^{\rho}. \]  

(40)

Note that only the symmetric part of the polymomenta \( \pi_{\alpha}^{\mu\nu} \) may be identified with a primary constraint as, from the last line above, we may clearly see that the derivatives of the gauge field \( A_{\mu}^{\alpha} \) are invertible in terms of the anti-symmetric components of the polymomenta. Next, we adapt Legendre transformation [3] to our system encountering the De Donder-Weyl Hamiltonian for the topologically massive Yang-Mills field

\[ H_{TMYM}^{DW}(A, \pi, x) = \pi_{\alpha}^{[\mu\nu]} \partial_{[\mu} A_{\nu]}^{\alpha} - L_{TMYM} \]

\[ = -\frac{1}{4} \pi_{[\mu\nu]}^{\alpha} A_{\alpha}^{[\mu\nu]} - \frac{g}{2} f_{\alpha\beta\gamma} A_{\nu}^{\alpha} A_{\rho}^{\beta} A_{\mu}^{\gamma} + \frac{m}{4} \epsilon_{\mu\nu\rho} A_{\alpha}^{\rho} \pi_{\alpha}^{[\mu\nu]} \]

\[ -\frac{m^{2}}{16} \epsilon_{\mu\nu\rho} \epsilon_{\mu\nu\rho} A_{\alpha}^{\mu} A_{\rho}^{\alpha} + \frac{mg}{12} \epsilon_{\mu\nu\rho} f_{\alpha\beta\gamma} A_{\mu}^{\gamma} A_{\nu}^{\beta} A_{\rho}^{\gamma}. \]  

(41)

Once again, by taking into account the constraints for the symmetric part of the polymomenta, we propose the total Hamiltonian

\[ \tilde{H}_{TMYM}^{DW} = H_{TMYM}^{DW} + \lambda^{\alpha}_{\mu\nu} \pi_{\alpha}^{(\mu\nu)}, \]  

(42)

where the \( \lambda^{\alpha}_{\mu\nu} \) stand for Lagrange multipliers enforcing the constraints, as it is common by now. The De Donder-Weyl equations are obtained from [12]
when applied to the canonical pair of \((n - 1)\)-forms \((A^\mu_a, \pi^{\mu}_a)\) defined as in the previous examples, thus yielding

\[
\begin{align*}
\textbf{d} \cdot A^\mu_a &= - \{ \textbf{H}^{\text{TMYM}}_{\text{DW}}, A^\mu_a \} \\
&= -\frac{1}{2} \pi_a^{[\mu|\nu]} - \frac{g}{2} \int_a^{bc} A_\mu^a A^\nu_c + \frac{m}{4} \epsilon^{\mu|\nu|\rho} A_\rho^a + \lambda^{\mu}_a, \\
\textbf{d} \cdot \pi^\mu_a &= - \{ \textbf{H}^{\text{TMYM}}_{\text{DW}}, \pi^\mu_a \} \\
&= \frac{g}{4} \int_a^{bc} \pi^{[\mu|\nu]} A_\rho^a - \frac{m}{4} \epsilon^{\mu|\nu|\rho} \pi^{\rho|\sigma}_a - \frac{m g}{4} \int_a^{bc} \epsilon^{\mu|\nu|\rho} A_\rho^b A_\sigma^c + 3 \frac{m^2}{4} A_a^\mu.
\end{align*}
\] (43)

Proceeding as in the previous subsections, for the sake of simplicity we decomposed the above equations into their symmetric and anti-symmetric parts, obtaining the identities

\[
\begin{align*}
\partial^\mu (A^\nu_a) &= -\frac{1}{2} \pi^{[\mu|\nu]} - \frac{g}{2} \int_a^{bc} A_\mu^a A^\nu_c + \frac{m}{4} \epsilon^{\mu|\nu|\rho} A_\rho^a, \\
\partial^\mu (A^\nu_a) &= \lambda^{\mu|\nu}_a, \\
\partial^\nu (\pi^{[\mu|\nu]}_a) &= g \int_a^{bc} \pi^{[\mu|\nu]} A_\rho^a - \frac{m}{4} \epsilon^{\mu|\nu|\rho} \pi^{\rho|\sigma}_a - \frac{m g}{4} \int_a^{bc} \epsilon^{\mu|\nu|\rho} A_\rho^b A_\sigma^c + 3 \frac{m^2}{4} A_a^\mu \\
\partial^\nu (\pi^{\nu|\mu}_a) &= 0. 
\end{align*}
\] (44)

Interpretation of the above identities follows in a similar manner as in the previous examples. The first line stands for the invertibility of the derivatives of the field in terms of the polymomenta and the gauge fields. The second line fixes the Lagrange multipliers \(\lambda^{\mu|\nu}_a\). Further, by considering the second line of (40) and the definition of the field strength \(F^{\mu\nu}_a\) adapted to this model in analogy to (17), we straightforwardly find the De Donder-Weyl field equations for the topologically massive Yang-Mills theory

\[
D_{\nu a}^b F_b^{\mu\nu} - \frac{m}{2} \epsilon^{\mu|\nu|\rho} F_\rho^{\mu\nu} = 0,
\] (45)

where \(D_{\nu a}^b\) is a covariant derivative defined in analogy to (24). Finally, as in the preceding two examples we expect that only the anti-symmetric part of the polymomentum is truly associated to the dynamics of the model. In order to see this, by using the last equation appearing in (44), we define a new set of polymomentum variables by \(\pi^{\mu\nu}_a \rightarrow \tilde{\pi}^{\mu\nu}_a := \pi^{\mu\nu}_a - \pi^{\nu\mu}_a\). The divergenceless condition of the polymomenta \(\partial^\nu \tilde{\pi}^{\nu|\mu}_a = 0\), means that its symmetric part can be removed from the De Donder-Weyl equations of motion without modifying
the dynamics of the system. In conclusion, due to the symmetric properties of this model we note that within our formulation we may avoid the standard Dirac procedure for constrained systems in order to obtain the correct De Donder-Weyl equations of motion which turn out to be totally equivalent to the Lagrangian field equations.

4 Conclusions

We analyzed several non-Abelian field theoretical models within the polysymplectic framework. In particular, we consider the topologically massive Yang-Mills field theory which is described by adding a Chern-Simons invariant to the Yang-Mills theory. In this way, we started by introducing the polysymplectic formalism for the Yang-Mills and Chern-Simons theories separately, and then we proceed systematically. Indeed, all of these theories present similar characteristics as all of them are described by a singular Lagrangian which, however, allowed us to find proper polymomenta in such a way that we were able to avoid the constraint analysis by introducing a completely equivalent De Donder-Weyl Hamiltonian. For any of the theories considered, their respective equivalent Hamiltonians may be inherited from the De Donder-Weyl formalism due to the presence of divergenceless of the symmetric part of the polymomenta. Further, for these new equivalent De Donder-Weyl Hamiltonians it was completely clear that only the anti-symmetric part of the polymomenta resulted relevant, as expected.

Besides, an important issue allowed by our separation of the canonical variables $(n-1)$-forms into their symmetric and anti-symmetric parts was to keep our description of the field equations in a closed way from the perspective of the Poisson-Gerstenhaber bracket, thus strengthen the primary role that the $(n-1)$ forms play within the polysymplectic formalism. This must be confronted with several other examples found in the literature (see for example, [24] and [28]) for which one may, in contrast, incorporate other than $(n-1)$-forms as canonical variables in order to get the correct equations of motion.

We hope that the introduced polysymplectic formalism for the topologically massive field theory may shed some light on the quantization for these kind of theories from the polysymplectic point of view. In particular, it will be relevant to test Kanatchikov’s Schrödinger-like quantum equation (see [30] for further details) as, in principle, one may wonder in which sense this quantum scheme may incorporate the cases for non-regular Lagrangian theories. Even though there are recent efforts in this direction (see for example [49] and [50] in the context of Einstein gravity), we think that the non-Abelian
field theories analyzed above may serve as a starting point for the analysis of the quantum version of non-regular Lagrangians from the polysymplectic perspective. This will be done elsewhere.

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