Inverse analysis of the residual stress in laser-assisted milling

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Abstract
In laser-assisted milling, higher temperature in shear zone softens the material potentially resulting in a shift of mean residual stress, which significantly affects the damage tolerance and fatigue performance of product. In order to guide the selection of laser and cutting parameters based on the preferred mean residual stress, inverse analysis is conducted by predicting residual stress based on guessed process parameters, which is defined as the forward problem, and applying iterative gradient search to find process parameters for next iteration, which is defined as the inverse problem. An analytical inverse analysis is therefore proposed for the mean residual stress in laser-assisted milling. The forward problem is solved by analytical prediction of mean residual stress after laser-assisted milling. The residual stress profile is predicted through the calculation of thermal stress, by treating laser beam as heat source, and plastic stress by first assuming pure elastic stress in loading process, then obtaining true stress with kinematic hardening followed by the stress relaxation. The variance-based recursive method is applied to solve inverse problem by updating process parameters to match the measured mean residual stress. Three cutting parameters including depth of cut, feed per tooth, and cutting speed, and two laser parameters including laser-tool distance and laser power, are updated with respected to the minimization of resulting residual stress and measurement in each iteration. Experimental measurements are referred on the laser-assisted milling of Ti–6Al–4V and Si₃N₄. The percentage difference between experiments and predictions is less than 5% for both materials, and the selection is completed within 50 loops.

Keywords Residual stress · Inverse analysis · Laser-assisted milling · Ti–6Al–4V · Si₃N₄ · Iterative gradient search

Nomenclature

| Symbol | Description |
|--------|-------------|
| ρ      | Density     |
| ξ      | Absorption ratio |
| α      | Thermal diffusivity |
| θ      | Rotation angle |
| εₚ     | Peak strain |
| μ      | Friction coefficient |
| φ      | Shear angle |
| cₚ     | Specific heat |
| Cs     | Side cutting edge angle |
| d      | Grain size |
| dₐ     | Axial depth of milling |

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The residual stress can largely affect the machined workpiece in terms of fatigue resistance. With the use of laser, higher temperature in shear zone softens the material potentially resulting in a shift of mean residual stress after laser-assisted milling [1], which significantly changes the damage tolerance and fatigue performance of product. Therefore, an inverse analysis is conducted on the mean residual stress after laser-assisted milling, in order to guide the selection of laser and cutting parameters based on the preferred mean residual stress. The forward problem, which is defined as the prediction of residual stress based on guessed process parameters needs to be solved first. The methodology of solving forward problem in both conventional and laser-assisted milling has been studied through experiments [2–5] and numerical simulations [6, 7], but these methods have low efficiency when applied in inverse problem, which is defined as the prediction of process parameters for next iteration through iterative gradient search. Analytical models for residual stress in the conventional milling process have been validated for different materials [8–12]. However, when the effect of additional heat source by laser is considered, the microstructure evolution can be triggered and affect the residual stress. The overall forward problem methodology of the residual stress prediction considering laser effect is summarized in Fig. 1. The heat source is calculated according to the size of laser spot and the laser power, and temperature field after laser preheating is calculated based on the conduction within workpiece [13, 14]. The geometry of milling tool is simplified as in orthogonal cutting at each instance [15], in order to predict the flow stress dependent on microstructure evolution, followed by cutting forces and machining temperature predictions. The residual stress is then predicted through mechanical loading based on forces, thermal loading based on temperature, and relaxation [16].

Iterative gradient search method guesses the process parameters based on the difference between predicted target performance and experimental measurement. This procedure has been widely applied in inverse analysis of hydraulic parameters [17], material properties [18, 19], torque [20], and constitutive equation constants [21–23], due to relatively simplified forward problem. The predictions based on initial guesses are close to measurements, and the forward problem is solved by empirical model or numerical simulation within 10 iterations. However, the prediction of mean residual stress in laser-assisted milling is a complex procedure, which takes up to several days if solved by numerical simulation such as finite element analysis [24]. In addition, the resultant mean residual stress is very sensitive to cutting and laser parameters, and the predicted value could be far away from the measurement even though the initial guesses are close, which takes more iterations to locate the desired parameters. Therefore, a gain coefficient is included in the proposed model. The coefficient is able to speed up the progress if the initial gap is huge and avoid convergence if the stopping criteria has not been reached, which enhances the computational efficiency and accuracy. Inverse analysis has been applied to satisfy residual stress [25, 26] but not mean residual stress requirements in laser-assisted milling. The inverse analysis method conducted in current study solves the forward problem through an analytical prediction model of mean residual stress in laser-assisted milling and includes a new iterative gradient search algorithm, which has been previously used by the authors on cutting force [27], tool life [28], and surface roughness [29], to solve the inverse problem. Three cutting parameters including depth of cut, feed per tooth, and cutting speed, and two laser parameters, including laser-tool distance and laser power, are selected as process parameters. The analysis is conducted on measured mean residual stress after laser-assisted milling of...
2 Forward problem methodology

2.1 Laser preheating temperature field

It is assumed that the material removal process during milling only occurs in plane at each instance. Therefore, a plane strain condition is assumed within the cutting (X) and depth (Z) directions. The temperature field is calculated within a plane containing the cutting path and perpendicular to top surface of workpiece (y = 0). The laser is treated as a heat source at top surface in addition to convection, and isothermal boundary conditions are assumed. The temperature rise rate at surface [13] is

\[
\Delta T_{\text{laser}}(x, y, 0) = \frac{q(x, y) - h(T_{\text{laser}}(x, y, 0) - T_0)}{\rho c_p}
\]  

(1)

where \( T_{\text{laser}} \) is the preheating temperature, \( \rho \) is density, \( c_p \) is specific heat, \( h \) is heat transfer coefficient, \( T_0 \) is the environment temperature, and \( q(x) \) is the heat generation rate of laser described by the Gaussian equation [13] as

\[
q(x, y) = \frac{\xi K_{\text{intensity}} P}{\pi r^2} \exp\left(-\frac{2(x^2 + y^2)}{r^2}\right)
\]

(2)

where \( P \) is the laser power, and \( r \) is the radius of laser spot. \( \xi \) is the absorption ratio dependent on many factors including absorptivity of the material, laser wavelength, surface texture, and environment heat convection. \( K_{\text{intensity}} \) is the intensity distribution coefficient. \( \xi \) and \( K_{\text{intensity}} \) are assumed to be 0.8 and 2.5, respectively [33]. The heat generation rate follows Gaussian distribution depending on position, laser power, and laser spot size. Temperature rise inside the workpiece due to heat conduction is described by

\[
\Delta T_{\text{laser}} = \alpha \nabla^2 T_{\text{laser}} - V_f \frac{\partial T_{\text{laser}}}{\partial x}
\]

(3)

where \( \alpha \) is thermal diffusivity and \( V_f \) is the feed rate of laser spot. The first term on the right describes the two-dimensional heat conduction, while the second term considers the effect of moving laser beam in cutting direction [34]. At boundaries, isothermal condition is assumed in cutting direction, and temperature rise due to heat conduction in Z direction is

\[
\Delta T_{\text{laser}} = \alpha \frac{\partial^2 T_{\text{laser}}}{\partial z^2}
\]

(4)

The initial temperature of workpiece as well as the environment temperature is 25 °C. An example of predicted overall temperature field is shown in Fig. 2.
2.2 Microstructure evolution and grain growth

Under the preheating temperature, the grain growth is considered in flow stress model [35], and milling tool and process parameters are recalculated assuming orthogonal cutting. The average cutting depth $t_c$ is

$$t_c = \frac{1}{2n_t} \frac{V_f}{RPM}$$

where $RPM$ is the rotation speed and $n_t$ is the number of flute. The instantaneous equivalent depth of cut is

$$t_c(\theta) = \sqrt{2} \times \frac{t_c}{2} \times \sin(\theta)$$

Fig. 3 Transformation from milling to equivalent orthogonal cutting. a Milling configuration with axial depth of milling $d_a$ and tool edge radius $r_{corner}$ b Equivalent orthogonal cutting with equivalent side cutting edge angle $C_s^*$ and equivalent cutting width $w^*$, the tangential cutting force is perpendicular to cutting edge instead of machined surface. c Transformation of cutting depth
where $\theta = 2\pi \times \text{RPM} \times t$ is the rotation angle, and $t$ is the cutting time. As shown in Fig. 3a, the corner edge radius $r_{\text{corner}}$ is equivalent to a side cutting edge angle of $C_s^*$ in Fig. 3b. The equivalent cutting width $w^*$ is dependent on axial depth of milling $d_a$ and the tool geometry shown in Fig. 3c as

$$w^* = \frac{d_a}{\cos(C_s^*)} \quad (7)$$

The equivalent cutting depth $t_c^*$ is also related to $C_s^*$ and $t_c$ by

$$t_c^* = t_c(\theta) \times \cos(C_s^*) \quad (8)$$

The equivalent cutting speed is

$$V(\theta) = \sqrt{V_x^2 + V_r^2 + 2V_xV_r\cos\theta} \quad (9)$$

where $V_x = 2\pi R_t \times \text{RPM}$ is the cutting speed, and $R_t$ is the tool radius. Other tool parameters are recalculated as well [34], and cutting force $F_c$, tangential force $F_t$, and plowing forces $P_{\text{thrust}}$ and $P_{\text{cut}}$ [35] are predicted based on flow stress.

The grain growth also decides flow stress under high temperature [24, 36]. The average grain size $d$ is given by

$$d = d_{\text{drex}}X_{\text{drex}} + d_0(1-X_{\text{drex}}) \quad (10)$$

where $d_0$ is the original average grain size, $d_{\text{drex}}$ is the dynamically recrystallized grain size, and $X_{\text{drex}}$ is the recrystallized volume fraction. Dynamic recrystallization is triggered if $\varepsilon$ is larger than $a_{2\varepsilon_p}\varepsilon_p$ is the peak strain as

$$\varepsilon_p = a_1a_0^{h_1}\varepsilon_0^{m_1} \exp(Q_{acm_1}/R_gT) + c_1 \quad (11)$$

where $R_g$ is the gas constant, and $Q_{ac}$ is the activation energy. $X_{\text{drex}}$ is described by the Avrami equation as

$$X_{\text{drex}} = 1 - \exp\left[-\beta_d\left(\frac{\varepsilon - \varepsilon_{0.5}}{\varepsilon_{0.5}}\right)^{k_d}\right] \quad (12)$$

where $\varepsilon_{0.5}$ is the strain when $X_{\text{drex}}=0.5$ as

$$\varepsilon_{0.5} = a_5d_0^{h_5}\varepsilon_0^{m_5} \exp(Q_{acm_5}/R_gT) + c_5 \quad (13)$$

$d_{\text{drex}}$ is given by

$$d_{\text{drex}} = a_8a_0^{h_8}\varepsilon_0^{m_8} \exp(Q_{acm_8}/R_gT) + c_8 \quad (14)$$

All the constant coefficients for Ti-6Al-4V could be determined by experiments and regression analysis and are listed in Table 1 [37]. As described in Fig. 1, mechanical stresses are calculated based on cutting force prediction model [34] dependent on laser preheating temperature and recrystallization effect, while the thermal stresses are calculated from machining temperature prediction model [38] under laser preheating.

### 2.3 Elastic stresses under mechanical and thermal loads

For loading process, the elastic stresses are first predicted under mechanical and thermal loads. As shown in Fig. 4, two sources of mechanical loads include the shear stress on shear plane and the friction between tool nose and machined surface. Hertzian rolling contact is assumed within $-a < x < a$, and normal stress $p(s)$ and tangential stress $q(s)$ are calculated at $x = s$ shown in Fig. 5. The mechanical stresses are

$$\sigma_x = \frac{2z}{\pi} \int_{-a}^{a} p(s)(x-s)^2 \text{ds} - \frac{z}{\pi} \int_{-a}^{a} q(s)(x-s)^3 \text{ds} \quad (15)$$

Normal stress $p(s)$ at tool-workpiece interface is $p_{\text{tool - edge}}$ by

$$p_{\text{tool - edge}} = \frac{2P_{\text{thrust}}}{\pi(\text{wa})} \quad (16)$$

where $w$ is the cutting width, $P_{\text{thrust}}$ is the normal plowing force, and $a$ is half of the contact width between tool and chip. Tangential stress $q(s)$ is $q_{\text{tool - edge}}$ as

$$q_{\text{tool - edge}} = \mu \left(\frac{P_{\text{cut}}}{wCA}\right) \quad (17)$$

where $CA$ is the contact width, $P_{\text{cut}}$ is the cutting plowing force, and $\mu$ is the friction coefficient.

At the shear plane, normal stress $p(s)$ becomes $p_x$ as

$$p_x = \frac{F_c \sin\phi + F_t \cos\phi}{L_{\text{AB}w}} \quad (18)$$

where $F_c$ and $F_t$ are machining forces in cutting and thrust directions, $L_{\text{AB}}$ is shear length, and $\phi$ is shear angle. Tangential stress $q(s)$ becomes $q_x$ as

$$q_x = k_{\text{AB}} \quad (19)$$
Table 1 Ti–6Al–4V material constants

| Peak strain | $a_1 d_{0l}$ | $m_1$ | $Q_{0m} m_1$ | $c_1$ |
|-------------|--------------|-------|--------------|-------|
|             | 0.0064×10$^{-2}$ | 0.0801 | 30.579 J/mol | 0     |

| Recrystallization kinematics | $\beta_l$ | $k_l$ | $a_{l0}$ | $a_2$ |
|-----------------------------|----------|-------|----------|-------|
|                             | 0.9339   | 0.5994 |          |       |

| Required strain | $a_s$ | $k_s$ | $Q_{0m} m_s$ | $m_s$ | $n_s$ | $c_s$ |
|-----------------|-------|-------|--------------|-------|-------|-------|
| $0.022 \times 10^{-3}$ | 0     | 26.430 J/mol | 0.11146 | 0     | 0     |

| Grain size | $a_4$ | $h_4$ | $Q_{0m} m_8$ | $m_8$ | $n_8$ | $c_8$ |
|------------|-------|-------|--------------|-------|-------|-------|
| $150 \times 10^{10}$ | 0     | $-6540$ J/mol | $0.03$ | 0     | 0     |

where $k_{AB}$ is the predicted flow stress. The total mechanical stress is

$$[\sigma_{\text{total}}] = [\sigma_{\text{shear-X-Z}}] + [\sigma_{\text{tooleedge-X-Z}}]$$

(20)

where $[\sigma_{\text{tooleedge-X-Z}}]$ is from substitution of Eqs. (16) and (17) into (15), and $[\sigma_{\text{shear-X-Z}}]$ is

$$[\sigma_{\text{shear-X-Z}}] = [Q] \begin{bmatrix} \sigma_{X'} \\ \tau_{X'Z'} \\ \sigma_{Z'} \end{bmatrix} [Q^T]$$

(21)

where $[Q] = \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix}$

(22)

Besides mechanical loads, sources for thermal loads include thermal stresses in two directions, surface tension, and hydrostatic pressure, where $\alpha$ is the thermal expansion coefficient by $\alpha = \frac{K \rho}{C_p}$ in which $K$, $\rho$, and $C_p$ are thermal conductivity, density, and specific heat. Young’s modulus of $E = 110$ GPa and Poisson’s ratio $\nu = 0.36$ are used. The total thermal stress is then the superposition of the three stress sources as

$$\sigma_{xx}^{\text{thermal}}(x, z) = -\frac{\alpha E}{1-2\nu} \int_{-\infty}^{\infty} \left( G_{zh} \frac{\partial T}{\partial x} (x, z) + G_{ch} \frac{\partial T}{\partial x} (x, z) \right) dx dz
+ \frac{2 \pi}{\nu} \frac{p(t)(t-x)^2}{(t-x)^2 + z^2} \left( T - \frac{\alpha E T(x, z)}{1-2\nu} \right) dt$$

$$\sigma_{yy}^{\text{thermal}}(x, z) = -\frac{\alpha E}{1-2\nu} \int_{-\infty}^{\infty} \left( G_{zh} \frac{\partial T}{\partial y} (x, z) + G_{ch} \frac{\partial T}{\partial y} (x, z) \right) dx dz
+ \frac{2 \pi}{\nu} \frac{p(t)(t-x)^2}{(t-x)^2 + z^2} \left( T - \frac{\alpha E T(x, z)}{1-2\nu} \right) dt$$

$$\sigma_{zz}^{\text{thermal}}(x, z) = -\frac{\alpha E}{1-2\nu} \int_{-\infty}^{\infty} \left( G_{zh} \frac{\partial T}{\partial z} (x, z) + G_{ch} \frac{\partial T}{\partial z} (x, z) \right) dx dz
+ \frac{2 \pi}{\nu} \frac{p(t)(t-x)^2}{(t-x)^2 + z^2} \left( T - \frac{\alpha E T(x, z)}{1-2\nu} \right) dt$$

(23)

where,

$$p(t) = \frac{\alpha E T(x, z = 0)}{1-2\nu}$$

(24)

$G_{zh}, G_{ch}, G_{zz}, G_{zdh}, G_{zh},$ and $G_{zch}$ are the Green’s functions, and $T$ is the temperature field [38]. The overall elastic stresses are calculated as

$$[\sigma_{\text{total}}] = [\sigma_{\text{mech}}] + [\sigma_{\text{thermal}}]$$

(25)

where,

$$[\sigma_{\text{thermal}}] = \begin{bmatrix} \sigma_{xx}^{\text{thermal}} \\ \sigma_{yy}^{\text{thermal}} \\ \sigma_{zz}^{\text{thermal}} \end{bmatrix}$$

(26)

The use of laser could affect both mechanical and thermal stresses. On one hand, the higher temperature under laser softens the material, which lowers the flow stress based on constitutive equation. On the other hand, the temperature field is largely decided by the laser preheating temperature field, which decides the thermal stresses as in Eq. (23).

2.4 Kinematic hardening and relaxation process

In real scenario, plastic deformation always happens beyond elastic stresses, which is indicated by whether von Mises yield criterion is satisfied. According to the definition, the effective strain and strain rate are

$$\varepsilon_{\text{eff}}^P = \frac{\sqrt{2}}{3} \left( \varepsilon_{xx}^P - \varepsilon_{yy}^P \right)^2 + \left( \varepsilon_{yy}^P - \varepsilon_{zz}^P \right)^2 + \left( \varepsilon_{zz}^P - \varepsilon_{xx}^P \right)^2 + 6\left( \varepsilon_{zz}^P \right)^2$$

$$\dot{\varepsilon}_{\text{eff}}^P = \frac{\sqrt{2}}{3} \left( \dot{\varepsilon}_{xx}^P \right)^2 + \left( \dot{\varepsilon}_{yy}^P \right)^2 + \left( \dot{\varepsilon}_{zz}^P \right)^2 + 2\left( \dot{\varepsilon}_{zz}^P \right)^2$$

(27)

(28)

In kinematic hardening, the yield surface is being translated while remaining the same size, instead of being expanded as in isotropic hardening. Kinematic hardening
is able to better describe the plastic behavior of the material in milling [39]. The kinematic hardening criterion is

$$F = \frac{3}{2} \left(S_{ij} - \alpha_{ij}\right)\left(S_{ij} - \alpha_{ij}\right) - R^2 = 0 \tag{29}$$

where $S_{ij} = \sigma_{ij} - (\sigma_{kk}/3)\delta_{ij}$ is the deviatoric stress, $R$ is the shear yield strength, and $\alpha_{ij} = \langle S_k n_k \rangle n_{ij}$ is the back stress, where $\langle \rangle$ is MacCauley symbol defined as $\langle x \rangle = 0.5 (x + |x|)$. $n_{ij} = \frac{S_{ij} - \alpha_{ij}}{\sqrt{2\lambda}}$, where $k$ is the average flow stress. The real strain rate in cutting direction is calculated by a mixed function of

$$\dot{\varepsilon}_{xx} = \frac{1}{E} \left[ \sigma_{xx} - \nu \left( \sigma_{xx} + \sigma_{zz}^* \right) \right] + \alpha \Delta T + \frac{1}{h_p} \left( \sigma_{xx} n_{xx} + \sigma_{yy} n_{yy} + \sigma_{zz}^* n_{zz} + 2 \tau_{xz}^* n_{xz} \right) n_{xx}$$

$$= \Psi \left[ \frac{1}{E} \left( \sigma_{xx} - \nu \left( \sigma_{xx} + \sigma_{zz}^* \right) \right) + \alpha \Delta T + \frac{1}{h_p} \left( \sigma_{xx} n_{xx} + \sigma_{yy} n_{yy} + \sigma_{zz}^* n_{zz} + 2 \tau_{xz}^* n_{xz} \right) n_{xx} \right]$$

where $\kappa=0.15$ [16], $h_p$ is plastic modulus, and $G_s$ is shear modulus. After the laser-assisted milling, the stress release is considered as fully elastic. When $F \leq 0$ and $dS_{ij}n_{ij} \geq 0$, the relaxation of stresses is governed by the Hook’s law as
\begin{equation}
\begin{aligned}
\Delta \sigma_{xx} &= \frac{E \Delta \varepsilon_{xx} + (1 + \nu)(\Delta \sigma_{zz} \nu - E \Delta T)}{(1-\nu^2)} \\
\Delta \sigma_{yy} &= \frac{\nu E \Delta \varepsilon_{xx} + (1 + \nu)(\Delta \sigma_{zz} \nu - E \Delta T)}{(1-\nu^2)} \\
\end{aligned}
\end{equation}

When \( F > 0 \), the released stress is

\begin{equation}
\Delta \sigma_{xx} = E \Delta \varepsilon_{xx} + (1 + \nu)(\Delta \sigma_{zz} \nu - E \Delta T)
\end{equation}

\begin{equation}
\Delta \sigma_{yy} = \nu E \Delta \varepsilon_{xx} + (1 + \nu)(\Delta \sigma_{zz} \nu - E \Delta T)
\end{equation}

The machining-induced temperature rise, elastic stress, and strain are released first, then the laser preheating temperature rise is released to get the residual stress. The mean residual stresses are then calculated by averaging the residual stress profiles for comparison of measurements on Ti–6Al–4V. The surface residual stresses are compared with measurements on Si3N4.

### 3 Inverse problem methodology

In inverse analysis, iterative gradient search method is used to find desired target performance and corresponding process parameters [9]. After an initial guess \( X_0 \), depth of cut \( d_a \), feed per tooth \( f_z \), cutting speed \( V_r \), laser-tool distance \( L \), and laser power \( P \) are updated in each loop as

\begin{equation}
X_n = (d_a^n, f_z^n, V_r^n, L^n, P^n)^T
\end{equation}

The process parameters in next loop are dependent on the gap between predicted and measured residual stress as

\begin{equation}
X_{n+1} = X_n + K_n (\sigma_{R}^{\text{exp}} - \sigma_{R}^{\text{R}}) G
\end{equation}

where \( \sigma_{R}^{\text{exp}} \) is the mean residual stresses in machining and feed directions from experiments and \( \sigma_{R}^{\text{R}} \) is the mean residual stress predicted under \( X_n \), \( G \) is called gain coefficient, and \( K_n \) is called Kalman gain matrix [40–42] being updated in each loop as

\begin{equation}
K_n = P_n \left( \frac{\Delta \sigma_{R}^{n-1}}{\Delta X_{n-1}} \right)^T R^{-1}
\end{equation}

The error covariance matrix \( R \) is

\begin{equation}
R = \begin{pmatrix}
\sigma_{R,\text{machining}}^2 & 0 \\
0 & \sigma_{R,\text{feed}}^2
\end{pmatrix}
\end{equation}

The derivative matrix \( \left( \frac{\Delta \sigma_{R}^{n-1}}{\Delta X_{n-1}} \right)^T \) is

\begin{equation}
\left( \frac{\Delta \sigma_{R}^{n-1}}{\Delta X_{n-1}} \right) = \begin{pmatrix}
\frac{n}{d_a} - \frac{n}{d_a} & \frac{n}{f_z} - \frac{n}{f_z} & \frac{n}{V_r} - \frac{n}{V_r} & \frac{n}{L} - \frac{n}{L} & \frac{n}{P} - \frac{n}{P}
\end{pmatrix}
\end{equation}

where \( \Delta X_0 = X_0 \) and \( \Delta \sigma_{R}^{n-1} = \sigma_{R}^{n-1} \). The simulation covariance matrix \( P_n \) has an initial value of

\begin{equation}
P_0 = \begin{pmatrix}
(\Delta d_a)^2 & 0 & 0 & 0 & 0 \\
0 & (\Delta f_z)^2 & 0 & 0 & 0 \\
0 & 0 & (\Delta V_r)^2 & 0 & 0 \\
0 & 0 & 0 & (\Delta L)^2 & 0 \\
0 & 0 & 0 & 0 & (\Delta P)^2
\end{pmatrix}
\end{equation}

where \( \Delta d_a, \Delta f_z, \Delta V_r, \Delta L, \) and \( \Delta P \) are the expected variance ranges of each process parameter. \( P_n \) is updated in each loop as
The gain coefficient $G$ in Eq. (37) is

$$G = \frac{\sum (\exp - \bar{\sigma}_R)_{n-1}}{\sum (\bar{\sigma}_R - \bar{\sigma}_R)_{n-1}} \frac{1}{C_18/C_19}$$

The denominator of $G$ avoids local convergence if the stopping criteria has not been reached. The numerator of $G$ speeds up the gradient search process when the difference between measurement and guess is large, which enhances computational efficiency. Therefore, the proposed new iterative gradient search method is able to avoid convergence, more adaptive, and more robust. Both forward and inverse problems are solved analytically in one algorithm with details of procedure summarized in Fig. 6.

4 Experimental validation and estimation of residual stress via inverse problem

4.1 Ti–6Al–4V

Experimental measurements are first referred on laser-assisted milling of Ti–6Al–4V grade 5 and grade 23 (ELI) [30, 31] for validation of inverse analysis. The spindle rotation speed is 1253 RPM. The size of laser spot is 2.5 mm × 3.6 mm, the laser power $P$ is 185 W, and the laser-tool distance $L$ is 3.5 mm. The tool has a rake angle of 15°, diameter of 19.05 mm, and nose...
radius of 0.8 mm. The cutting speed is 75 m/min, the feed per tooth is 0.1 mm/tooth, the depth of cut is 1 mm, and the width of cut is 3 mm.

The constitutive model of Ti–6Al–4V for flow stress prediction is

\[ \sigma = \left( A_{hp} + K_{hp} \varepsilon^{0.5} + B \varepsilon \right) \left( 1 + C n \frac{T - T_0}{T_m - T_0} \right)^m \]

(44)

where \( T_m \) is the melting temperature. All the material constants are listed in Table 2 [37], and \( d_0 \) is decided to be 10 \( \mu \)m.

For Ti–6Al–4 V grade 5, the measured mean residual stress as well as prediction from inverse analysis in laser-assisted milling are shown in Fig. 7. Measurements are collected every 50 \( \mu \)m up to 200 \( \mu \)m depth with even distribution for every 50 \( \mu \)m and averaged for mean residual stress. The initial guesses for inverse analysis are cutting depth of 1 mm, feed per tooth of 0.22 mm/s, cutting speed of 2.75 m/s, laser-tool distance of 7.7 mm, and laser power of 1034 W. In machining direction, the mean residual stress from experiments is 191 MPa in compression, and the prediction through inverse analysis is \(-132.80\) MPa with a percentage difference of 1.99%. The process parameters in final iteration are cutting depth of 0.79 mm, feed per tooth of 0.11 mm/s, cutting speed of 1.62 m/s, laser-tool distance of 6.03 mm, and laser power of 816.6 W. As observed in Fig. 7, the proposed algorithm is able to jump out of local extreme several times throughout the inverse analysis until the stopping criteria are reached.

The measured mean residual stress as well as prediction from inverse analysis in laser-assisted milling of Ti–6Al–4 V ELI are shown in Fig. 8. The initial guesses for inverse analysis are cutting depth of 1 mm, feed per tooth of 0.1 mm/s, cutting speed of 1.25 m/s, laser-tool distance of 3.5 mm, and laser power of 470 W. In machining direction, the mean residual stress from experiments is 175 MPa in compression, and the prediction through inverse analysis is \(-171.5\) MPa. A close match is found after 27 iterations with a percentage difference of 2% as listed in Table 3. In feed direction, the mean residual stress from experiments is 143.5 MPa in compression, and the prediction through inverse analysis is \(-146.68\) MPa with a percentage difference of 2.21%. The process parameters in final iteration are cutting depth of 0.51 mm, feed per tooth of 0.06 mm/s, cutting speed of 0.74 m/s, laser-tool distance of 1.78 mm, and laser power of 237.37 W. Again, the proposed inverse analysis method is able to reach both high computational efficiency and accuracy.

### Table 2
Constitutive model parameters for Ti–6Al–4V and Si3N4 [43]

| Material  | \( A_{hp} \) (MPa) | \( K_{hp} \) (MPa) | \( B \) (MPa) | \( C \) | \( m \) | \( n \) | \( T_m \) (°C) | \( \dot{\varepsilon}_0 \) (s\(^{-1}\)) |
|-----------|------------------|------------------|--------------|--------|-----|-----|------------|------------------|
| Ti–6Al–4V Grade 5 | 803.22 | 401.61 | 653.1 | 0.015 | 0.6 | 0.45 | 1668 | 1 |
| Ti–6Al–4V ELI | 803.22 | 401.61 | 653.1 | 0.025 | 0.8 | 0.45 | 1630 | 1 |
| Si3N4 | 25 | 800 | 8.31447 | 0.3 | 0.061 | 0.4 | 1 |

Fig. 7 Estimation of mean residual stress via inverse analysis on Ti–6Al–4V grade 5

![Fig. 7 Estimation of mean residual stress via inverse analysis on Ti–6Al–4V grade 5](image-url)
Since the inverse analysis is an iterative process, large error could be observed at some iterations. The proposed algorithm updates the process parameters based on the current prediction and target performance. However, the overall influence of five parameters on residual stress is complicated, and it is possible that the change of parameters will result in a larger deviation in next loop. As long as the final difference is within the criteria, the model is proven to be able to match the desired performance. In addition, the accuracy of the residual stress under determined parameters is not within discussion of the current study, as it is part of the validation of forward model. The forward predictive model including validation has been proposed [43].

When comparing the process parameters in final loop to initial guesses and experimental values, it is observed that although the process parameters are relatively close to initial guesses, they may be very different than experimental process parameters since the mean residual stress solutions may not be unique. In addition, the model-predicted residual stresses under experimental process parameters have been calculated by solving forward problem only [43]. The percentage errors for Ti–6Al–4V grade 5 and ELI are higher than 10% in both directions, which also indicates that the proposed inverse analysis method is highly accurate as the errors are mainly from the forward model.

### 4.2 Si₃N₄

Additional experimental measurements of Si₃N₄ [32] on surface residual stress are also referred. The cutting depth is 0.2 mm, feed rate is 6 mm/min, cutting speed is 1 m/s, laser-tool distance is 3 mm, and laser power is 300, 410, or 470 W. The constitutive model of Si₃N₄ for flow stress prediction is

$$
\sigma = \sigma_0 \left\{ 1 + \left( \frac{\dot{\varepsilon}}{\dot{\varepsilon}_0} \right)^n \right\} \left( \frac{\dot{\varepsilon}}{\dot{\varepsilon}_0} \exp \left( \frac{Q_{act}}{R_S T} \right) \right)^m
$$

where $\sigma_0$ is the yield stress measured at $\dot{\varepsilon} = 1 \times 10^{-8}$ s⁻¹. All the material constants are listed in Table 2. The comparison between measurements and predictions from inverse analysis is shown in Table 4. The surface residual stress in machining and feed directions is recorded under three levels of laser power. The analysis is done within 50 iterations, and the percentage difference is less than 5% for all cases. The average difference is 1.84% in machining direction and 2.53% in feed direction, which further demonstrates the high accuracy of proposed method.

**Table 3** Comparison of mean residual stress between experimental measurements and predictions through inverse analysis

| Ti–6Al–4V Grade 5 | Prediction from inverse analysis (MPa) | Percentage difference (%) |
|-------------------|---------------------------------------|---------------------------|
| Machining direction | −191 | −182.72 | 4.33 |
| Feed direction    | −130.2 | −132.80 | 1.99 |
| Ti–6Al–4V ELI | Prediction from inverse analysis (MPa) | Percentage difference (%) |
| Machining direction | −175 | −171.50 | 2.00 |
| Feed direction    | −143.5 | −146.68 | 2.21 |

**Fig. 8** Estimation of mean residual stress via inverse analysis on Ti–6Al–4V ELI
5 Conclusions

An inverse analysis is conducted on the mean residual stress in laser-assisted milling which solves the forward problem of predicting residual stress based on guessed process parameters and the inverse problem of finding process parameters for next iteration by applying iterative gradient search. The forward problem is solved by predicting residual stress considering material recrystallization under laser effect. The laser beam is treated as a heat source on top. The milling tool and process parameters are recalculated in orthogonal cutting. The recrystallization and grain growth are described by calibrated models showing the dependency of strain, strain rate, and temperature on recrystallization. For the loading process, the elastic stresses are first predicted, and the real stresses are calculated considering kinematic hardening. The mean residual stress is then predicted after the stress relaxation. The variance-based recursive method is applied to solve inverse problem and update process parameters to match the measurements. Three cutting parameters including depth of cut, feed per tooth, and cutting speed, and two laser parameters including laser-tool distance and laser power are updated in each iteration. The proposed iterative gradient search method introduces the gain coefficient that updates the parameters according to the difference between measurement and prediction, as well as the differences of predicted mean residual stress over loops, which makes the model able to avoid convergence, more adaptive, and more robust. The proposed model is validated through experimental measurements on the laser-assisted milling of Ti–6Al–4V and Si₃N₄. The percentage difference between experiments and predictions is less than 5%, and the process is completed within 50 loops. Therefore, the proposed inverse analysis model is also highly accurate and computationally efficient. The selected process parameters may be very different than experimental process parameters due to the multiple solutions issue. In addition, when comparing the model-predicted residual stresses under experimental process parameters to measurements, it is again concluded that the proposed inverse analysis method is highly accurate as the errors are mainly from the forward model.

The proposed inverse analysis is the first approach of laser-assisted milling to satisfy mean residual stress requirement, which provides a reliable reference for the selection of process parameters when desirable mean residual stress is needed.

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