Next-to-leading resummation of cosmological perturbations via the Lagrangian picture: 2-loop correction in real and redshift spaces

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Abstract. We present an improved prediction of the nonlinear perturbation theory (PT) via the Lagrangian picture, which was originally proposed by Matsubara (2008). Based on the relations between the power spectrum in standard PT and that in Lagrangian PT, we derive analytic expressions for the power spectrum in Lagrangian PT up to 2-loop order in both real and redshift spaces. Comparing the improved prediction of Lagrangian PT with $N$-body simulations in real space, we find that the 2-loop corrections can extend the valid range of wave numbers where we can predict the power spectrum within 1% accuracy by a factor of 1.0 ($z = 0.5$), 1.3 (1), 1.6 (2) and 1.8 (3) varied with 1-loop Lagrangian PT results. On the other hand, in all redshift ranges, the higher-order corrections are shown to be less significant on the two-point correlation functions around the baryon acoustic peak, because the 1-loop Lagrangian PT is already accurate enough to explain the nonlinearity on those scales in $N$-body simulations.

Keywords: galaxy clustering, baryon acoustic oscillation, power spectrum

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1 Introduction

In the forthcoming era of precision cosmology, accurate and precise predictions of cosmological theories play crucial roles not only to estimate the cosmological parameters but also to detect a tiny signal from the observational data. The large-scale structure of the universe is a powerful probe of cosmology, and recently the baryon acoustic oscillations (BAOs) imprinted on the large-scale structure [1–3] have attracted much attention to constrain the nature of dark energy.

There are many ongoing and planned redshift surveys aiming at clarifying the nature of the dark energy through a precision measurement of BAOs, including the Sloan Digital Sky Baryon Oscillation Spectroscopic Survey (BOSS) [8], Hobby-Eberly Telescope Dark Energy Experiment (HETDEX) [9], BigBOSS [10], Subaru Measurement of Images and Redshifts (SuMIRe) [12], Wide-Field Infrared Survey Telescope (WFIRST) [11], Euclid [13]. To measure the characteristic scales of BAO precisely enough for investigating the dark energy, accurate theoretical template for the power spectrum and correlation function on BAO scales is quite essential.

Although the nonlinear gravitational clustering of galaxies on BAO scales is rather moderate, the linear theory of density perturbations [14] is inadequate to quantitatively describe the BAOs [15, 16]. The standard perturbation theory (SPT) provides a systematic way to investigate the gravitational clustering of dark matter in nonlinear regime [17–23, 25, 26]. Even though the SPT describes the nonlinearity fairly well at sufficiently high redshifts [27], it is still insufficient to describe the clustering on BAO scales at observationally relevant redshifts, \( z = 0–3 \). To overcome such situation, various methods to improve the SPT are proposed, partially resuming infinite series of higher-order perturbations. The renormalized perturbation theory (RPT) [28–30] is a representative pioneering work, and since it appeared, various approaches have been subsequently proposed [31–38]. The quantitative predictions of those approaches are compared with N-body simulations, and actually shown to excellently
reproduce the $N$-body results quite well in a certain range of power spectrum and two-point correlation function. The accuracy and precision of the predictions actually depend on the treatment of each method, and a comprehensive test of various methods is examined in ref. [46].

Most of the above newly developed perturbation theories predict the nonlinear gravitational clustering of dark matter in real space. However, the observable quantity in actual redshift surveys is the clustering of galaxies in redshift space. There are two ingredients to be added to the perturbation theory in order to compare the theoretical predictions with observations: redshift-space distortions [47–49] and biasing [51, 52]. While the redshift distortions induce the clustering anisotropies, the galaxy biasing can produce a scale-dependent enhancement (or suppression) in the clustering amplitude. Hence, coupled with the effect of gravitational clustering, both the amplitude and shape of the power spectrum and two-point correlation function would be significantly changed, and an accurate modeling of BAOs that ensures a percent-level precision seems non-trivial.

The resummed PT via the Lagrangian picture provides a unique framework to include both the ingredients into the fundamental formulation, and does not have any phenomenological parameter of dynamics once the biasing model is fixed [37, 53, 54] (but see [50]). In contrast to the SPT formulated as Eulerian PT, the resummed PT by refs. [37, 53, 54] is based on the framework of the Lagrangian perturbation theory (LPT) [55–63], in which the displacements of fluid element are treated as dynamical variables. It is known that the first-order LPT reproduces the classic Zel’dovich approximation [64]. In both Eulerian and Lagrangian PTs, the small variables in the perturbative expansion correspond to the initial density fluctuations. Therefore, when the theoretical predictions of LPT are made in the Eulerian space with the full expansion of perturbations, LPT should give mathematically equivalent results to the SPT expansion [54]. In the Lagrangian resummation method, correlations of the displacement field at zero-lag are kept in the exponent, and other contributions to the power spectrum are expanded as usual in Eulerian space. This partial expansion corresponds to the resummation of a class of infinite higher-order perturbations. As a result, an exponential prefactor naturally arises in the expression of the nonlinear power spectrum [37]. The exponential prefactor plays an important role in describing the nonlinear smearing effects of BAO in both real and redshift spaces.

In previous works, the Lagrangian resummation method has been investigated only up to 1-loop order in perturbations. The purpose of this paper is to generalize the previous work and to carry out 2-loop calculations of the same resummation method. We derive analytic expressions of the 2-loop results in both real and redshift spaces. To see how well the 2-loop corrections improve the previous 1-loop results, we focus on the power spectrum of dark matter in real space, and compare the predictions with numerical simulations. Although we leave a quantitative analysis in redshift space for future work, the improvement of the predictions shown in this paper is an important step, and would lead to a strong motivation to actually evaluate the effects of redshift-space distortions and the biasing via the LPT up to 2-loop order.

This paper is organized as follows. In section 2, useful equations and concepts, which are mostly derived in previous work, are summarized. In section 3, our main results of the 2-loop calculations both in real and redshift spaces are presented. In section 4, the analytic expressions are compared with $N$-body simulations of dark matter. In section 5, our results and conclusions are summarized.
2 Lagrangian Perturbation Theory and Standard Perturbation Theory

In this section, we summarize some concepts and equations of Lagrangian resummations method, which we use in this paper. Readers find derivations of the following equations in refs. [37, 53].

According to the result of 1-loop resummation via LPT, the following identity is confirmed by straightforward calculations of 1-loop SPT and LPT:

\[ P_{\text{LPT}}(k) = \exp\left[ -\frac{k^2}{6\pi^2} \int dp \, P_L(p) \right] \times \left[ P_L(k) + P_{\text{SPT}}^{\text{1-loop}}(k) + \frac{k^2}{6\pi^2} P_L(k) \int dp \, P_L(p) \right], \quad (2.1) \]

where \( P_{\text{LPT}}(k) \) is the nonlinear power spectrum predicted by the 1-loop resummation via LPT, \( P_L(k) \) is the linear power spectrum, and \( P_{\text{SPT}}^{\text{1-loop}}(k) \) is the 1-loop contributions (without linear contribution) to the power spectrum predicted by the SPT. Note that the time dependence of the power spectrum \( P_{\text{LPT}} \) is implicitly encoded in the linearly extrapolated power spectrum \( P_L(k) \) evaluated at the redshift of our interest.

As we mentioned in section 1, the displacement field is the fundamental variable in LPT, and is treated as a small perturbative quantity. In computing the power spectrum, we need to evaluate the correlation of displacement fields at different positions in the exponent. The resummation method proposed by refs. [37, 53] gives a specific but efficient recipe to treat this. That is, the correlation of the displacement fields at a single position is kept in the exponent, while the correlation among components of separated displacement vectors, which is expected to be small compared to zero-lag correlation as long as we consider a large separation, is expanded from the exponent. The exponential prefactor in eq. (2.1) indeed represents the zero-lag correlation up to the 1-loop level. It is easily checked that expanding the exponential prefactor and truncating the series up to the 1-loop order exactly recovers the power spectrum of 1-loop SPT.

Note that in the language of RPT [28, 29], the resummation treatment by refs. [37, 53] corresponds to a partial vertex renormalization of the SPT expansion [37], and the exponential prefactor in eq. (2.1) contains the information beyond the 1-loop SPT. Further, the resultant functional form of the exponential prefactor quite resembles the renormalized propagator at high-\( k \) limit [29], and is shown to play a physically important role in describing the nonlinear smearing effect of BAOs. Nevertheless, because of a rather strong damping, the resultant predictions for the power spectrum ceases to work well at higher-\( k \) modes, where the non-linear enhancement of the power spectrum amplitude becomes significant. Thus, similar to RPT and other improved PT treatment, the accuracy of the Lagrangian resummation treatment will be limited to a specific range \( k \lesssim k_{\text{damp}} \), where \( k_{\text{damp}} \) is the characteristic scale of the exponential damping factor. But, this characteristic scale depends on the order of loop corrections included in the power spectrum expression, and including the next-to-leading order will definitely improve the predictions applicable to some higher-\( k \) modes.

The main purpose of this paper is to extend the 1-loop expression (2.1) to a general relation between the power spectrum in SPT and that in LPT for arbitrary loop-order, and is to explicitly derive the 2-loop expression. To do so, we first notice that the exponential prefactor corresponds to the square of a Fourier transform of the one-point probability distribution function of the displacement field, \( e^{-i k \cdot \Psi} \), where \( \Psi(q) = x(q) - q \) is the displacement field, \( x \) and \( q \) are respectively Eulerian and Lagrangian coordinates of a fluid
element [53, 54]. According to the cumulant expansion theorem [65], this factor reduces to

\[
\langle e^{-ik\Psi} \rangle = \exp \left[ -2 \sum_{n=1}^{\infty} \frac{k_{i1} \cdots k_{i2n}}{(2n)!} A_{i1 \cdots i2n}^{(2n)} \right],
\]

(2.2)

where

\[
A_{i1 \cdots i2n}^{(2n)} = \left[ \prod_{i=1}^{2n} \int \frac{d^3p_i}{(2\pi)^3} \right] \delta^3 \left( \sum_{i=1}^{2n} p_i \right) C_{i1 \cdots i2n} (p_1, \cdots, p_{2n}),
\]

(2.3)

and \( C_{i1 \cdots i2n} \) is a 2\( n \)-order cumulant of the displacement field in Fourier space \( \tilde{\Psi} \), defined by

\[
\langle \tilde{\Psi}_{i1}(p_1) \cdots \tilde{\Psi}_{i2n}(p_{2n}) \rangle_c = (-1)^{n-1}(2\pi)^3 \int \sum_{i=1}^{2n} p_i \right) C_{i1 \cdots i2n} (p_1, \cdots, p_{2n}).
\]

(2.4)

The square of eq. (2.2) is naturally factorized out in LPT for the nonlinear power spectrum. On the other hand, the LPT and SPT should give the same result when the perturbations are completely expanded in Eulerian space, since the fundamental equations of motion, and small perturbations of the expansion are the same in both perturbation schemes [54]. Thus, when we factorize out the square of eq. (2.2) from SPT series of the power spectrum, we have the following identity:

\[
P(k) = \exp \left[ -2 \sum_{n=1}^{\infty} \frac{k_{i1} \cdots k_{i2n}}{(2n)!} A_{i1 \cdots i2n}^{(2n)} \right] \times \left\{ P_L(k) + \sum_{m=1}^{\infty} P_{\text{SPT}}^m(k) \right\} \left[ 1 + \sum_{l=1}^{\infty} \frac{1}{l!} \left( 2 \sum_{n=1}^{\infty} \frac{k_{i1} \cdots k_{i2n}}{(2n)!} A_{i1 \cdots i2n}^{(2n)} \right)^l \right],
\]

(2.5)

where \( P_{\text{SPT}}^m(k) \) is an \( m \)-loop contribution to the power spectrum in SPT. The indices \( i, j, i_1, i_2 \) etc. correspond to spatial components and run over 1, 2, 3. Repeated appearance of \( i_1, i_2, \ldots \) implicitly means those indices should be summed over.

As we mentioned, in the Lagrangian resummation method, the exponential prefactor is kept unexpanded, and perturbative truncations are made in the exponent. The quantity in the last bracket is normally expanded and truncated as usual. In the \( N \)-loop approximation of the Lagrangian resummation, the series in the exponent is expanded up to \( O(P_L)^N \), and the remaining factor is expanded up to \( O(P_L)^{N+1} \). The case of \( N = 1 \) just reduces to eq. (2.1).

In this paper, we generalize eq. (2.1) to the case of 2-loop approximation, \( N = 2 \). From the consideration above, we do not need to calculate LPT from the beginning. Instead, we first calculate \( F_{\text{SPT}}^m(k) \) for \( m = 1, 2 \), and use eq. (2.5) truncated at the \( N = 2 \) order to obtain 2-loop predictions of the Lagrangian resummation method.

We need the polyspectra \( C_{i1 \cdots i2n} \) of only \( n = 1 \), which are given by

\[
C_{ij}(p, -p) \equiv C_{ij}(p) = C_{ij}^{(11)}(p) + C_{ij}^{(22)}(p) + C_{ij}^{(13)}(p) + C_{ij}^{(31)}(p),
\]

(2.6)
where

\[
C_{ij}^{(11)}(p) = L_i^{(1)}(p)L_j^{(2)}(p)L_L(p),
\]

\[
C_{ij}^{(22)}(p) = \frac{1}{2} \int \frac{d^3p'}{(2\pi)^3} L_i^{(2)}(p', p - p')L_j^{(2)}(p', p - p')P_L(p')P_L(|p - p'|),
\]

\[
C_{ij}^{(13)}(p) = C_{ji}^{(31)}(p) = \frac{1}{2}L_i^{(1)}(p)P_L(p) \int \frac{d^3p'}{(2\pi)^3} L_j^{(3)}(p, -p', p')P_L(p'),
\]

up to \(O(P_L)^2\). The perturbative kernels in LPT up to third order are given by [61, 62]

\[
L_i^{(1)}(p) = \frac{k_i}{k^3},
\]

\[
L_i^{(2)}(p_1, p_2) = \frac{3}{7}k_i^2 \left[ 1 - \left( \frac{p_1 \cdot p_2}{p_1p_2} \right)^2 \right],
\]

\[
L_i^{(3a)}(p_1, p_2, p_3) = \frac{5}{3}k_i^2 \left[ 1 - \left( \frac{p_1 \cdot p_2}{p_1p_2} \right)^2 \right] \left[ 1 - \left( \frac{p_1 + p_2 + p_3}{|p_1 + p_2 + p_3|} \right)^2 \right]
\]

\[
- \frac{1}{3}k_i \left[ 1 - 3 \left( \frac{p_1 \cdot p_2}{p_1p_2} \right)^2 + 2 \frac{(p_1 \cdot p_2)(p_2 \cdot p_3)(p_3 \cdot p_1)}{p_1^2p_2^2p_3^2} \right]
\]

\[
+ k \times T(p_1, p_2, p_3),
\]

where \(k = p_1 + \cdots + p_n\) for \(L_i^{(n)}\), and a vector \(T\) represents a transverse part whose expression is not needed in the following application. The third-order kernel \(L^{(3)}\) in eq. (2.9) is a symmetrized one in terms of their arguments:

\[
L_i^{(3)}(p_1, p_2, p_3) = \frac{1}{3} \left[ L_i^{(3a)}(p_1, p_2, p_3) + \text{perm.} \right].
\]

Next consider the power spectrum in redshift space. In the Lagrangian picture, the displacement field in redshift space, \(\Psi^s\), is distorted by the peculiar velocities, and mapped from that in real space as

\[
\Psi^s = \Psi + \frac{\hat{z} \cdot \ddot{\Psi}}{H} \hat{z},
\]

where \(H\) is the Hubble parameter, the unit vector \(\hat{z}\) indicates the line-of-sight direction and \(\dot{\Psi} = d\Psi/dt\). Using the fact that the time-dependence of the perturbative kernel in real space is approximately described by \(\Psi^{(n)} \propto D^n\) with \(D\) being the linear growth factor, the displacement field of each perturbation order in redshift space is given by

\[
\Psi^{s(n)} = \Psi^{(n)} + n f \left( \hat{z} \cdot \Psi^{(n)} \right) \hat{z},
\]

where \(f = d \ln D / d \ln a\) is the logarithmic derivative of the linear growth factor. Then, the mappings from real space to redshift space of the order-by-order cumulants are given by

\[
C_{ij}^{s(nm)} = R_{ik}^{(n)} R_{jl}^{(m)} C_{kl}^{(mm)},
\]
where

\[ R_{ij}^{(n)} = \delta_{ij} + nf \hat{z}_i \hat{z}_j. \]  

(2.17)

Thanks to the linear mapping (2.15), the perturbation calculation of the displacement field is rather straightforward in redshift space, provided the perturbative solutions in real-space. Nevertheless, the resultant form of the redshift-space power spectrum is rather complicated because of the coupling between velocity and density fields. The general expressions for redshift-space power spectrum is formally written as

\[
P_s(k) = \exp \left[ -2 \sum_{n=1}^{\infty} \frac{k_{i_1} \cdots k_{i_{2n}}}{(2n)!} A_{i_1 \cdots i_{2n}}^{s(2n)} \right] \times \left\{ (1 + f \mu^2)^2 P_L(k) + \sum_{m=1}^{\infty} P_{m\text{-loop}}^{s\text{SPT}}(k) \right\} \times \left[ 1 + \sum_{l=1}^{\infty} \frac{1}{l!} \left( 2 \sum_{n=1}^{\infty} \frac{k_{i_1} \cdots k_{i_{2n}}}{(2n)!} A_{i_1 \cdots i_{2n}}^{s(2n)} \right)^l \right],
\]  

(2.18)

where \( P_{m\text{-loop}}^{s\text{SPT}}(k) \) is an \( m \)-loop contribution to the redshift-space power spectrum in SPT. The quantity \( A_{i_1 \cdots i_{2n}}^{s(2n)} \) is defined by

\[
A_{i_1 \cdots i_{2n}}^{s(2n)} = \prod_{i=1}^{2n} \int d^3 p_i \frac{1}{(2\pi)^3} \delta_D \left( \sum_{i=1}^{2n} p_i \right) C_{i_1 \cdots i_{2n}}^{s}(p_1, \cdots, p_{2n}),
\]  

(2.19)

where \( C_{i_1 \cdots i_{2n}}^{s} \) is defined by a similar mapping as that of eq. (2.16).

Although the above result involves many additional terms compared to the real-space power spectrum, several important terms in characterizing redshift distortions is identified in eq. (2.18). One is the enhancement factor, \((1 + f \mu^2)^2\), known as the linear Kaiser factor [47]. The corrections to the exponential prefactor give an additional damping of the power spectrum amplitude, which roughly corresponds to the Finger-of-God effect driven by the random motion of fluid element. In section 3.2, we will explicitly see the corrections characterizing the redshift distortion up to the 2-loop order.

### 3 Analytic expressions of the 2-loop power spectrum in Lagrangian re-summation method

Based on the relation between the power spectrum in the SPT and that in the LPT, the 2-loop LPT power spectrum can be expressed as a combination of 1- and 2-loop SPT power spectra. In this section, we derive analytic expressions of the real- and redshift-space power spectra up to the 2-loop order in LPT.

#### 3.1 Real space

Let us first evaluate the power spectrum in real space. The only task is to calculate the terms in the exponent of eq. (2.5) up to the demanding order in \( P_L(k) \). Up to the second order in
$P_L(k)$, the exponent of eq. (2.5) is expanded as \(^1\)

\[
-2 \sum_{n=1}^{\infty} \frac{k_i \cdots k_{i_2n}}{(2n)!} A_{i_1 \cdots i_2n}^{(2n)} = -k_i k_j A_{ij}^{(2)} \frac{1}{12} k_i k_j k_l k_m A_{ijklm}^{(4)} + \cdots
\]

\[
\simeq -\frac{k^2}{6\pi^2} \left[ A^{(11)} + A^{(22)} + 2A^{(13)} \right]. \quad (3.1)
\]

The quantity $A^{(a\beta)}$ is calculated from the polyspectra $C_{ij}^{(a\beta)}$ as

\[
A^{(a\beta)} = \int_0^{\infty} p^2 dp \, C^{(a\beta)}(p), \quad (3.2)
\]

where we define $C_{ij}^{(a\beta)}(p) \equiv C^{a\beta}(p) \hat{p}_i \hat{p}_j$ and $\hat{p}_i = p_i/p$. The explicit expressions for $C^{a\beta}$ are presented in eqs. (A9)–(A11) of ref. [37]. In the above, we used the fact that $\int d^3p/(2\pi)^3 \hat{p}_i \hat{p}_j f(p) = \delta_{ij}/(6\pi^2) \int p^2 dp f(p)$ for an arbitrary function $f(p)$.

Based on the perturbative kernels in LPT defined by eqs. (2.10)-(2.13), the 1-loop correction term, $A^{(11)}$, is explicitly written as [37]

\[
A^{(11)} = \int_0^{\infty} dp \, P_L(p). \quad (3.3)
\]

The next-to-leading order contributions, $A^{(22)}$ and $A^{(13)}$, are also given by

\[
A^{(22)} = \int_0^{\infty} p^2 dp \, C^{(22)}(p)
\]

\[
= \int_0^{\infty} p^2 dp \frac{1}{2} \int \frac{d^3p_1}{(2\pi)^3} \delta_{ij} \left[ L_i^{(2)}(p_1, p - p_1) \right] P_L(p_1) P_L([p - p_1])
\]

\[
= \frac{9}{392\pi^2} \int_0^{\infty} p_1 p_2 dp_1 dp_2 P_L(p_1) P_L(p_2) K \left| (p_1/p_2 + p_2/p_1)/2 \right|, \quad (3.4)
\]

\[
A^{(13)} = \int_0^{\infty} p^2 dp \, C^{(13)}(p)
\]

\[
= \int_0^{\infty} p^2 dp \frac{1}{2} \delta_{ij} L_i^{(1)}(p) P_L(p) \int \frac{d^3p_1}{(2\pi)^3} L_j^{(3)}(p, -p_1, p_1) P_L(p_1)
\]

\[
= \frac{5}{84\pi^2} \int_0^{\infty} p_1 p_2 dp_1 dp_2 P_L(p_1) P_L(p_2) K \left| (p_1/p_2 + p_2/p_1)/2 \right|, \quad (3.5)
\]

where the function $K(y)$ is defined by

\[
K(y) = \frac{5}{3} y - y^3 - \frac{(y^2 - 1)^2}{2} \ln \left| \frac{y - 1}{y + 1} \right|. \quad (3.6)
\]

We note that the function $K(y)$ is always positive in a range $y > 0$. Also, the 2-loop corrections, $A^{(13)}$ and $A^{(22)}$, are shown to be positive.

\(^1A_{ijklm}^{(4)}\) does not have a $O(P_L)^2$ connected cumulant.
Collecting the 2-loop corrections given above, the power spectrum up to 2-loop order in LPT is expressed as

\[
P_{2-LPT}^{\text{2-loop}}(k) = \exp \left[ -\frac{k^2}{6\pi^2} \left( A^{(11)} + A^{(22)} + 2A^{(13)} \right) \right]
\]

\[
\times \left[ P_L(k) + P_{\text{SPT}}^{\text{1-loop}}(k) + P_{\text{SPT}}^{\text{2-loop}}(k) \right]
\]

\[
+ P_L(k) \left\{ \frac{k^2}{6\pi^2} \left( A^{(11)} + A^{(22)} + 2A^{(13)} \right) + \frac{k^4}{72\pi^4} \left( A^{(11)} \right)^2 \right\} + \frac{k^2}{6\pi^2} P_{\text{SPT}}^{\text{1-loop}}(k) A^{(11)} \right],
\]

where \( P_{\text{SPT}}^{\text{1-loop}}(k) \) and \( P_{\text{SPT}}^{\text{2-loop}}(k) \) are 1- and 2-loop contributions to the real space power spectrum in SPT respectively.

### 3.2 Redshift space

Next consider the power spectrum in observable redshift space. In similar manner to the real space, we start with eq. (2.18) by expanding the exponent up to the second order in \( P_L(k) \):

\[
-2\sum_{n=1}^{\infty} \frac{k_{1n} \cdots k_{n2}}{(2n)!} A_{11 \cdots 2n}^{(2n)} = -k_i k_j A_{ij}^{(2)} - \frac{1}{12} k_i k_j k_l k_m A_{ijlm}^{(4)} + \cdots
\]

\[
\simeq -\frac{k_i k_j}{6\pi^2} \left[ A_{ij}^{(11)} + A_{ij}^{(22)} + A_{ij}^{(13)} + A_{ij}^{(31)} \right].
\]

The quantity \( A^{(\alpha\beta)} \) represents a momentum integration of a polyspectrum in redshift space. Based on the relation between the polyspectra in real and redshift spaces, we obtain

\[
A_{ij}^{(\alpha\beta)} = 6\pi^2 \int \frac{d^3 p}{(2\pi)^3} C_{ij}^{(\alpha\beta)}(p)
\]

\[
= \left\{ \delta_{ij} + (\alpha + \beta) \delta \hat{z} \hat{z} + \alpha\beta f^2 \hat{z} \hat{z} \right\} A^{(\alpha\beta)}.
\]

Taking an inner product of \( A_{ij}^{(\alpha\beta)} \) with \( k_i k_j \), we have

\[
k_i k_j A_{ij}^{(\alpha\beta)} = k^2 D^{(\alpha\beta)}(f, \mu) A^{(\alpha\beta)}.
\]

The quantity \( \mu = \hat{z} \cdot \hat{k} / k \) is the directional cosine of the wave vector \( k \) with respect to the line of sight. The function \( D^{(\alpha\beta)}(\mu, f) \) which describes the redshift distortion is defined by

\[
D^{(\alpha\beta)}(\mu, f) = D^{(2\alpha)}(\mu, f) \equiv 1 + (\alpha + \beta) f^2 \mu^2.
\]

From these relations, we can get the non-linear power spectrum in redshift space up to 2-loop order. The result is

\[
P_{2-SLPT}^{\text{2-loop}}(k) = \exp \left[ -\frac{k^2}{6\pi^2} \left( D^{(11)} A^{(11)} + D^{(22)} A^{(22)} + 2D^{(13)} A^{(13)} \right) \right]
\]

\[
\times \left[ (1 + f^2 \mu^2)^2 P_L(k) + P_{\text{SPT}}^{\text{1-loop}}(k) + P_{\text{SPT}}^{\text{2-loop}}(k) \right]
\]

\[
+ (1 + f^2 \mu^2)^2 P_L(k) \left\{ \frac{k^2}{6\pi^2} \left( D^{(11)} A^{(11)} + D^{(22)} A^{(22)} + 2D^{(13)} A^{(13)} \right) \right\}
\]

\[
+ \frac{k^4}{72\pi^4} \left( D^{(11)} A^{(11)} \right)^2 \right] + \frac{k^2}{6\pi^2} P_{\text{SPT}}^{\text{1-loop}}(k) D^{(11)} A^{(11)} \right].
\]
where \( P_{sSPT}^{1\text{-}\text{loop}}(k) \) and \( P_{sSPT}^{2\text{-}\text{loop}}(k) \) are 1- and 2-loop contributions to the redshift-space power spectrum in SPT, respectively.

In eq. (3.12), the terms proportional to \( f \mu^2 \) represent the corrections characterizing the redshift distortions. These corrections appear not only in the spatial-correlation terms, which have been expanded from the exponent, but also in the exponential prefactor. Thus, the resultant redshift-space power spectrum naturally possesses an additional damping effect responsible for the Finger-of-God (FoG) effect. Note that a naive perturbative treatment based on the SPT usually fails to describe the FoG effect, which should be phenomenologically introduced by hand (but see [50]). By contrast, in LPT, the redshift distortion is described by a simple linear mapping from real space, (2.14) or (2.15), and with the resummation method, the power spectrum damping by the FoG effect is naturally incorporated into the redshift-space power spectrum in a non-perturbative manner.

While the resultant expression for redshift-space power spectrum seems to contain physically relevant effects of non-linear redshift distortions, an explicit evaluation of eq. (3.12) needs a bit technical aspect. To be specific, a difficult part is the term \( P_{sSPT}^{2\text{-}\text{loop}}(k) \). It involves six-dimensional integral, and is characterized as a function of \( k \) and directional cosine \( \mu \). No one has so far tried to evaluate it, and at present, the explicit calculation of \( P_{sSPT}^{2\text{-}\text{loop}}(k) \) seems infeasible unless we develop an efficient technique to compute \( P_{sSPT}^{2\text{-}\text{loop}}(k) \). Hence, we leave the discussion on the numerical evaluation of the redshift-space power spectrum for future work, and we hereafter focus on the quantitative prediction of real-space power spectrum.

### 3.3 Evaluations of the 1- and 2-loop power spectra in SPT

The expression of real-space power spectrum given in eq. (3.7) involves the 1- and 2-loop power spectra in SPT, and we need to directly evaluate them for a quantitative prediction of the power spectrum. Here, we briefly explain how to compute these two contributions. The explicit expressions for the 1- and 2-loop corrections in SPT are decomposed into several pieces as

\[
\begin{align*}
P_{\text{SPT}}^{1\text{-}\text{loop}}(k) &= P_{\text{SPT}}^{(22)}(k) + P_{\text{SPT}}^{(13)}(k), \\
P_{\text{SPT}}^{2\text{-}\text{loop}}(k) &= P_{\text{SPT}}^{(33)}(k) + P_{\text{SPT}}^{(24)}(k) + P_{\text{SPT}}^{(15)}(k),
\end{align*}
\]  

where the quantities \( P_{\text{SPT}}^{(mn)} \) imply the ensemble averages obtained from the \( m \)-th and \( n \)-th order perturbative solutions of SPT. They are expressed as a collection of multi-dimensional integrals involving the perturbative kernel of the higher-order solutions (e.g., [6, 24, 46]):

\[
\begin{align*}
P_{\text{SPT}}^{(22)}(k) &= 2 \int \frac{d^3q}{(2\pi)^3} \left\{ \mathcal{F}^{(2)}(q,k-q) \right\}^2 P_L(q) P_L(|k-q|), \\
P_{\text{SPT}}^{(13)}(k) &= 6 P_L(k) \int \frac{d^3q}{(2\pi)^3} \mathcal{F}^{(3)}(k,q,-q) P_L(q), \\
P_{\text{SPT}}^{(33)}(k) &= 9 P_L(k) \left\{ \int \frac{d^3p}{(2\pi)^3} \mathcal{F}^{(3)}(k,p,-p) P_L(p) \right\}^2 \\
&\quad + 6 \int \frac{d^3pd^3q}{(2\pi)^6} \left\{ \mathcal{F}^{(3)}(p,q,k-p-q) \right\}^2 P_L(p) P_L(q) P_L(|k-p-q|), \\
P_{\text{SPT}}^{(24)}(k) &= 24 \int \frac{d^3pd^3q}{(2\pi)^6} \mathcal{F}^{(2)}(p,k-p) \mathcal{F}^{(4)}(p,q,-q,k-p) P_L(p) P_L(q) P_L(|k-p|).
\end{align*}
\]
Table 1. Parameters of N-body simulations.

| Name           | $L_{\text{box}}$              | # of particles | $z_{\text{init}}$ | # of runs |
|----------------|-------------------------------|----------------|-------------------|-----------|
| Main           | $1,000 h^{-1} \text{Mpc}$     | 512$^3$        | 31                | 30        |
| High (L11-N11) | $2,048 h^{-1} \text{Mpc}$     | 2048$^3$       | 99                | 1         |
| (L12-N11)      | $4,096 h^{-1} \text{Mpc}$     | 2048$^3$       | 99                | 1         |

$$P_{\text{SPT}}^{(15)}(k) = 30P_L(k) \int \frac{d^3pd^3q}{(2\pi)^6} \mathcal{F}^{(5)}(p, q, k, -p, -q) P_L(p) P_L(q).$$  

(3.18)

Here, $\mathcal{F}^{(n)}(k_1, \ldots, k_n)$ is the symmetrized kernel of the $n$-th order solutions for the density field (see [6, 26] for derivation and explicit expressions).

Note that the expressions of the 1-loop power spectra $P_{\text{SPT}}^{1\text{-loop}}$ can be further reduced to the one-dimensional and two-dimensional integral for $P^{(13)}$ and $P^{(22)}$, respectively (e.g., [21–23]). For the explicit computation of power spectrum in next section, we use the method of Gaussian quadratures for numerical integration of 1-loop power spectra. On the other hand, for the 2-loop power spectra, except for the first term in $P^{(33)}$, the integrals are no longer simplified, and we directly evaluate the five-dimensional integration (due to the symmetry around the vector $k$, one of the azimuthal angles is trivially integrated). In this paper, following the approach by [6], we adopt the Monte-Carlo technique of quasi-random sampling using the library *Cuba*\(^2\). The symmetrized kernels for perturbative solutions higher than the third order are too complicated to express analytically, and we numerically generated the symmetrized kernels using the recursion relation of perturbative solutions (e.g., [6, 20, 26, 28]).

4 Comparison with N-body simulations in real space

In this section, in order to quantify the effect of the next-to-leading order corrections in LPT, we explicitly compute the 2-loop corrections, and compare the analytic predictions with N-body simulations. We adopt a cosmological model with the WMAP 5yr [4] parameters $\Omega_m = 0.279$, $\Omega_\Lambda = 0.721$, $\Omega_b = 0.046$, $h = 0.701$, $n_s = 0.96$ and $\sigma_8 = 0.817$, and the linear power spectrum $P_L(k)$ is calculated from the output of the CAMB code [5].

4.1 N-body simulations

To compare the LPT results with N-body simulations, we use the subset of the N-body data presented in [6]. The data were created by a public N-body code \texttt{GADGET2} [42] with cubic boxes of side length $1 h^{-1} \text{Gpc}$, 512$^3$ particles. The initial conditions were generated by the \texttt{2LPT} code [43] at $z_{\text{init}} = 31$, and the output data of the 30 independent realizations were stored at redshifts $z = 3, 2, 1, \text{and } 0.5$. In addition to the main simulations, we also use the results of high-resolution simulation data presented in ref. [44], which were similarly created by \texttt{GADGET2} with the initial condition generator, \texttt{2LPT}. We summarize our sets of N-body simulation in Table 1.

The power spectra of these simulations were computed by combining several output results with different box sizes, and for the purpose to probe BAOs, we specifically use the

\(^2\)http://www.feynarts.de/cuba/
results of $N$-body runs called L11-N11 and L12-N11 in ref. [44], whose box sizes are 2,048 and 4,096 $h^{-1}$ Mpc, respectively. The simulations were evolved from the redshift $z_{\text{init}} = 99$ and the output results were stored at $z = 3$ and 1. Although the realization of these simulations is only one, each simulation contains 2,048$^3$ particles, and we preferentially use the L11-N11 run to measure the small-scale power spectrum.

We measure the matter power spectrum adopting the same treatment as described in ref. [40]. That is, we calculate the power spectrum from the Fourier transform of the density field assigned on the 1,024$^3$ grids with the Cloud-in-Cells interpolation. To reduce the effect of finite-mode sampling advocated by ref. [39], the resultant power spectrum is multiplied by the ratio $P_{\text{lin}}(k)/\hat{P}_{\text{PT}}(k)$ at $k \lesssim 0.1h\text{Mpc}^{-1}$, where the quantity $\hat{P}_{\text{PT}}(k)$ is calculated from the SPT up to the third-order in density field, and $P_{\text{lin}}(k)$ is the input linear power spectrum extrapolated to a given output redshift. In computing $\hat{P}_{\text{PT}}(k)$, we use the Gaussian-sampled density field used to generate the initial condition of each $N$-body run (see refs. [39, 40] in detail). As for the estimation of two-point correlation function, we use the grid-based calculation using the Fast Fourier Transformation (FFT) [6]. Similar to the power spectrum analysis, we first compute the square of the density field on each grid of Fourier space. Then, applying the inverse Fourier transformation, we take the average over realization, and obtain the two-point correlation function. Note that the finite-mode sampling also affects the calculation of the two-point correlation function. We correct it by subtracting and adding the extrapolated linear density field as, $\hat{\xi}(r) - \hat{\xi}_{\text{lin}}(r) + \xi_{\text{lin}}(r)$, where $\hat{\xi}_{\text{lin}}$ is the correlation function estimated from the Gaussian density field, and $\xi_{\text{lin}}$ is the linear theory prediction of two-point correlation function.

4.2 Power spectrum

Figure 1 shows the ratio of power spectrum to a smooth reference spectrum, $P(k)/P_{\text{nw}}(k)$, where the function $P_{\text{nw}}(k)$ is the linear power spectrum calculated from the smooth transfer function neglecting the BAO feature in ref. [7]. From top to bottom, the results at redshifts $z = 3, 2, 1$ and 0.5 are shown. In each panel, the power spectra from linear theory (black dotted), 1-loop SPT (blue dashed), 2-loop SPT (blue solid), 1-loop LPT (red dashed) and 2-loop LPT (red solid) are plotted, and are compared with $N$-body simulations. Results of the main and high-resolution simulations are respectively shown as black and gray symbols with error bars. Note that owing to 30 independent realizations and the correction of the finite-mode sampling by ref. [39], the scatter of the $N$-body results is rather reduced, and the size of each error bar becomes hard to see visually. Overall, the predictions of the 2-loop LPT show better agreement with $N$-body simulations. The range of the agreement is wider than that of the 1-loop LPT at all redshifts, indicating that the 2-loop corrections in LPT can give an important contribution to the nonlinear enhancement of the power spectrum at high $k$. On the other hand, the 2-loop correction in SPT slightly reduces the amplitude of power spectrum, and the predictions of 2-loop SPT tend to reproduce the $N$-body results quite well compared to the 1-loop SPT. However, a closer look at the behaviors on small scales reveals that the predictions overestimate the $N$-body results at high redshift, and then turn to slightly underestimate at low redshift. Note that the power spectra obtained from the main and high-resolution simulations show somewhat different behaviors at $z = 3$. At $k \gtrsim 0.2h\text{Mpc}^{-1}$, the result of main simulations gets smaller power, and the discrepancy between simulation and the 2-loop predictions apparently manifests. Presumably, this would be explained by a transient phenomenon due to the lack of number of particles. Low resolution simulations with a small number of particles usually suffer from the discreteness effect, and the
Figure 1. Ratio of power spectrum to smoothed reference spectrum, $P(k)/P_{nw}(k)$, given at redshifts $z = 3$ (top left), 2 (top right), 1 (bottom left) and 0.5 (bottom right). The power spectra from linear theory (black dotted), 1-loop SPT (blue dashed), 2-loop SPT (blue solid), 1-loop LPT (red dashed) and 2-loop LPT (red solid) are compared with $N$-body simulations. Results of the main and high-resolution simulations are shown as black and gray symbols with error bars, respectively. The reference power spectrum $P_{nw}(k)$ is calculated from the no-wiggle formula of the linear transfer function in ref. [7].

$N$-body data on the scales below the particle mean separation are not reliable particularly at an early time. Later, as the gravitational clustering develops, the non-linear mode transfer is expected to eventually dominate the initial power, and the small-scale structure tend to catch up the actual nonlinear growth (e.g., [45]). Although this qualitative picture is specifically relevant for the small-scales clustering, the apparent lack of initial power in low-resolution simulations may be influential even on relatively large scales, $k \gtrsim 0.2 \, h\, \text{Mpc}^{-1}$, as shown in detail by ref. [44] (see figure 3 of their paper). It can affect the estimation of the validity range of PT results, and we need to take care of the systematics.

In figure 2, to investigate the agreement in more quantitative ways, we show the differences of the power spectrum between simulation and PT prediction normalized by the smooth reference spectrum, i.e., $(P_{N\text{-body}} - P_{PT})/P_{nw}(k)$. Here, the differences between the main simulation and predictions are mainly shown, but we also plot the case with high-resolution simulation for 2-loop LPT (main vs 2-loop LPT: red, high-resolution vs 2-loop LPT: light red). In each panel of figure 2, the green and red vertical arrows respectively represent the characteristic wave numbers.
Figure 2. Difference between the results of main $N$-body simulation and PT (2-loop SPT: blue, 1-loop LPT: green, 2-loop LPT: red) divided by the reference spectrum, $[P_{N-body}(k) - P_{PT}(k)]/P_{nw}(k)$. We also show the difference between the result of high-resolution simulation and 2-loop LPT with light red symbols. In each panel, green and red vertical arrows represent the wave number $k_{nl}^{1-loop}/2$ and $k_{nl}^{2-loop}/2$, which are respectively estimated from the exponential pre-factor in the 1- and 2-loop expressions for real-space power spectrum in LPT [see eqs. (4.1) and (4.2)].

$k_{nl}^{1-loop}/2$ and $k_{nl}^{2-loop}/2$ defined by

$$k_{nl}^{1-loop} = \left( \frac{A^{(11)}}{6\pi^2} \right)^{-1/2},$$  \hspace{1cm} (4.1)

$$k_{nl}^{2-loop} = \left[ \frac{A^{(11)} + A^{(22)} + 2A^{(13)}}{6\pi^2} \right]^{-1/2}. \hspace{1cm} (4.2)$$

These wave numbers indicate the characteristic damping scales of the power spectrum in LPT, which are natural extension of the definition of the nonlinear scale, $k_{nl}$, in ref. [37] [see eq. (38) of this paper].

Compared to the predictions of 1-loop LPT, the 2-loop corrections in LPT extend the range of agreement with 1% precision by a factor of 1.0 ($z = 0.5$), 1.3 (1), 1.6 (2) and 1.8 (3). The ongoing (e.g., HETDEX [9]) and proposed surveys (e.g., BigBOSS [10], WFIRST [11], and SuMIRe [12]) plan to precisely measure the BAOs at $z = 1–2$, and thus the improvement with the 2-loop correction at high-$z$ is practically important and encouraging for an accurate determination of acoustic scales. Figure 2 also implies that LPT has a better convergence.
property for the higher-order corrections, since the range of agreement becomes monotonically wider as we include the higher-order corrections. Apparently, at lower redshifts, the range of agreement of the 2-loop SPT becomes comparable to that of the 2-loop LPT. As seen in figure 1, however, the 2-loop corrections in SPT can give a negative contribution at large $k$, and it is not guaranteed that the inclusion of higher-order corrections in SPT monotonically improves the prediction. In this respect, the agreement in SPT shown at lower redshifts may be regarded as accidental one. As the non-linearity significantly develops, even the 2-loop LRT is insufficient to improve the validity range of prediction. Although we did not store the simulation data at $z = 0$, it is expected that the validity range of 2-loop LPT at $z = 0$ becomes rather comparable to that of 1-loop LPT, and the 1% precision would be achieved only at $k \lesssim 0.7 h^{-1}\text{Mpc}$ (see ref. [46]).

Finally, it is worth mentioning a relation between the validity range of LPT and the exponential damping scale. Ref. [37] pointed out that the damping scale is considered as a criteria for the validity range of the 1-loop LPT, and we can see that the $k_{1\text{-loop}}/2$ becomes a plausible indicator of the agreement with $N$-body simulations. On the other hand, figure 2 shows that such a criteria does not simply work for the 2-loop LPT. Though the damping scale of the 2-loop LPT is always smaller than that of the 1-loop LPT, due to the higher-order corrections in the exponential prefactor, the validity range of the 2-loop LPT becomes wider than the scale indicated by $k_{1\text{-loop}}^2/2$ or $k_{1\text{-loop}}/2$. In this sense, the damping scale $k_{1\text{-loop}}^2/2$ may be regarded as the scale where the prediction of 2-loop LPT starts to deviate from that of the 2-loop SPT.

### 4.3 Correlation function

The two-point correlation function $\xi(r)$ is given by Fourier transform of the power spectrum:

$$\xi(r) = \int_0^\infty \frac{k^2dk}{2\pi^2} \sin( kr) \frac{P(k)}{kr}.$$  \hfill (4.3)

Figure 3 shows the two-point correlation functions around the baryon acoustic peak at redshifts $z = 3$ (top left), 2 (top right), 1 (bottom left), and 0.5 (bottom right). In each panel, the predictions calculated from linear theory (black dotted with error bars), 1-loop LPT (green dashed) and 2-loop LPT (red solid) are plotted. Also, figure 4 shows the fractional difference between the $N$-body and Lagrangian PT up to 1-loop (green) and 2-loop (red) results respectively, i.e., $[\xi_{N\text{-body}}(r) - \xi_{\text{PT}}(r)]/\xi_{N\text{-body}}$. Note again that the error bars of the $N$-body results are rather small, and are visually hard to see in figure 3.

Apart from the large scales beyond the location of acoustic peak, where the reliability of $N$-body simulations becomes subtle due to the limited simulation boxsize [6], a good agreement between the $N$-body simulations and predictions is found for both 1-loop and 2-loop LPT. As decreasing redshifts, the acoustic peak structure tends to be erased and the peak position is shifted to a small scale. These are purely the result of non-linear mode coupling, and the LPT explains this smearing effect quite well [37]. On the other hand, the shift of the baryon acoustic peak position is at a sub-percent level. Even at redshift $z = 0$ [41], it is difficult to quantify this effect in our results of $N$-body simulations. An interesting point is here that the correlation function is hardly affected by the 2-loop correction compared to the power spectrum. This trend has been also reported in other PT treatment (e.g., closure theory [6] and RPT [30]). The difference between $N$-body simulations and both 1-loop and 2-loop results of LPT is small enough around the acoustic peak ($r \sim 105 h^{-1}\text{Mpc}$), and the accuracy of both LPT predictions reaches at several percents level, which is sufficient for
ongoing and future surveys [6]. A closer look at the baryon acoustic peak at lower redshifts, however, reveals that 2-loop LPT tends to oversmear the peak structure, and the agreement with N-body simulation looks somewhat degraded. Although we have not yet succeeded to identify the origin of this discrepancy, the result may suggests either the significance of higher-order corrections at lower redshift or the need for delicate numerical treatment in evaluating the multi-dimensional integration. By contrast, a noticeable effect of the 2-loop corrections can be seen on small scales ($r \lesssim 50\, h^{-1}\text{Mpc}$), where the nonlinear enhancement of the correlation function becomes prominent at lower redshifts. Even though the 1-loop LPT tends to deviate from N-body results, the 2-loop LPT reproduces the N-body trends quite well within 1% precision.

5 Summary

In this paper, we present an improved prediction of the nonlinear perturbation theory via the Lagrangian picture, originally proposed by ref. [37]. Based on the previous result [37], we derive a general relation between the power spectrum in SPT and that in LPT for arbitrary loop-order. Using this relation, we explicitly write down the analytic expression of the power spectrum in LPT up to the 2-loop order in both real and redshift spaces.
Comparing the results in LPT with $N$-body simulations in real space, precisely, we quantitative study the validity range of LPT at various redshifts. For power spectrum, the higher-order corrections enhance the power of high-$k$. Including the 2-loop corrections, the range of validity, where the LPT prediction agrees with $N$-body simulation within 1% precision, is improved by a factor of 1.0 ($z = 0.5$), 1.3 (1), 1.6 (2) and 1.8 (3), compared with the 1-loop LPT. On the other hand, the two-point correlation functions around the baryon acoustic peak are less sensitive to the higher-order corrections in all redshift range. This implies that the 1-loop LPT is sufficient to accurately describe the baryon acoustic peak in correlation functions.

Finally, we should note remaining issues and future prospects of our work. We postponed the evaluation of redshift-space power spectrum and the check of its accuracy. The expression of redshift-space power spectrum involves a difficult term to evaluate, and we need to develop an efficient algorithm to compute it. On the other hand, in practice, it is inevitable to properly take account of the galaxy biasing in the real BAO survey. The LPT provides a way to treat the nonlinear galaxy biasing in a self-consistent manner, and the previous study [53] has reported the advantage of LPT based on the 1-loop calculations. The extension to the 2-loop calculations would be presumably a straightforward task, and will be reported elsewhere.

Figure 4. Fractional differences between $N$-body and Lagrangian PT results, $[\xi_{N\text{-body}}(r) - \xi_{\text{PT}}(r)]/\xi_{N\text{-body}}(r)$; 1-loop: green, 2-loop: red.
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