On the research of the magnetic field between the plates of a plane capacitor

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Abstract. The purpose of the paper was to consider the impact of the displacement current in magnetic field in the vicinity of the plane capacitor plates. We obtained a magnetic field model that accounted for the effects of both displacement currents and conduction currents. Theoretical calculations of the surface on which the magnetic field changes the direction were performed.

1. Introduction
Working on the unification of magnetic and electrical theories, English physicist James Maxwell suggested that solenoidal magnetic field is produced not only by electric current, but also by a changing electric field. Maxwell named the additional summand reflecting this assumption in the Ampère's circuit law the displacement current [1].

Researching the displacement current is of the main interest in terms of studying the aspects of dielectric nano-objects’ “optical magnetism” [2], creating dielectric optical rectennae and intermediate emitters [3] and developing optical analogues for radiofrequency devices.

Research in this field is complicated by the impossibility of detecting the displacement current directly. Measurement of the magnetic field created simultaneously by both conduction and displacement currents is the main way of the experimental study of the displacement current [4].

The study focuses on the research into heterogeneity of the magnetic field that exists amid the coating of the plates of a plane AC-circuit connected capacitor, between which there is conflict of magnetic fields created by displacement currents and conduction currents. In the second section, we give the key theoretic equations, and justify the importance of accounting for the influence of magnetic field produced by the conduction currents when analysing the capacitors. In the third section, we centre around a two-dimensional capacitor model obtained, and describe the mathematics behind the heterogeneity of the induction of the magnetic field amid the coating of the plates of a plane symmetric model. An analytic function was derived for each target point for the space in-between the plates estimating the value and the direction of the magnetic induction vector.

2. On the magnetic field in the vicinity of the plane capacitor plates
Maxwell’s equation that depicts, in integral form, the circulation of the magnetic field strength, is used for mathematical description of the induction of the magnetic field B in the vicinity of a plane capacitor:
\[ \oint_{\mathcal{L}} \bar{H} \, d\mathcal{L} = \int_{\mathcal{S}} \left( j + \frac{\partial \bar{D}}{\partial t} \right) \, d\mathcal{S}, \]  

(2.1)

where \( \bar{H} \) is the magnetic field strength vector, \( \bar{B} = \mu_0 \mu \bar{H} \), \( \mu_0 \) is the permeability of free space, \( \mu \) is the magnetic permeability of the medium, \( j \) is the conduction current density, \( \bar{D} = \epsilon \epsilon_0 \bar{E} \) is the electric induction vector, \( \epsilon, \epsilon_0 \) are permittivity of a dielectric medium and permittivity of free space, respectively, \( \bar{E} \) is the electric field intensity vector. Generally, researchers [1, 5] indicate that the magnetic field that exists between the capacitor plates is created solely by the displacement currents (Fig. 2.1). The thorough analysis of the experimental data obtained in the studies [6, 7] and E. Purcell’s concepts on conduction currents magnetic fields between the plates [4] has led to the conclusion that the field that exists amid the coating of the plates of a plane capacitor is heterogenous: there are fields created by displacement currents as well as by conduction currents, and they are oppositely directed.

Thus, for further analysis a new model will be obtained that will allow us to account not only for magnetic fields generated by displacement currents, but also for magnetic fields produced by conduction currents in the plates of the capacitor and in the conductive wires. Fig. 2.1 shows only the magnetic fields created by the conductive currents, whereas Fig. 2.2 depicts the three components of the magnetic field that are necessary for further analysis: \( B_1 \) — magnetic field produced by conduction currents in conductive wires; \( B_2 \) — conduction currents in the plates of the capacitor, \( B_3 \) — displacement currents.

![Figure 2.1. The direction of the magnetic induction \( B_1 \) and \( B_2 \), created by the conduction currents.](image)

![Figure 2.2. Allocation of magnetic fields around the arms of a plane capacitor, inclusive of magnetic fields produced by displacement and conduction current.](image)

Note that in regard to conductive wires, the magnetic fields created by the conduction current and the displacement current are codirected, whereas the magnetic field of the plates’ conductivity current is oppositely directed (figure 2.2).

### 3. The analysis of the magnetic induction of in-between plates space superposition on a plane model

For the relevant analysis of the three-field superposition some calculations are necessary. Analytical mathematical three-dimensional modelling of the three fields (\( B_1, B_2, \) and \( B_3 \)) is rather complex, thus the problem will be solved from the two-dimensional point of view, making additional assumptions:

- only the radial electric current is observed on the plane disk-shaped coating;
- plates and conductive wires are homogenous and are made of the same substance;
- the wires and plates are infinitely thin;
- end effects can be ignored;
only the frequencies at which the electric field inside the capacitor is homogenous are analysed [8].

Analytically integrating the Biot-Savart-Laplace equation in the differential form is essential for every target point of the in-between plates space [9]:
\[
d\vec{B} = \frac{\mu_0 I}{4\pi \rho} [d\vec{l} \times \vec{p}],
\]
where \( \vec{B} \) is the magnetic induction vector, \( \mu \) is the magnetic permeability of the medium, \( \mu_0 \) is the permeability of free space, \( I \) is the current in the conductor, \( d\vec{l} \) is the element of the current conductor, \( \vec{p} \) — radius vector, created from the \( d\vec{l} \) to the observation point.

The plane image is divided into six parts: left and right finite length external wires outside the plates, upper and lower sections of the left plate, upper and lower sections of the right plate. The following system of notation was introduced:

- \( R \) — radius of the circular plates,
- \( L \) — finite length of the wire outside the capacitor,
- \( r \) — the two-dimensional parameter (corresponding to the transverse direction coordinate, \( 0 \leq r \leq R \)),
- \( d \) — the two-dimensional parameter (corresponding to the longitudinal coordinate, \( 0 \leq d \leq \Delta \)),
- \( \Delta \) — distance between capacitor plates.

Analytical integration of the upper-left section of the plate is as follows:
\[
dB = \frac{\mu_0 I}{4\pi d} \sin \alpha \cdot d\alpha. \tag{3.2}
\]

The integral form of the equation (3.2) can be presented as:
\[
B = \frac{\mu_0 I}{4\pi} \left( \cos \alpha_1 - \cos \alpha_2 \right).
\]

As for angles \( \alpha_1 \) and \( \alpha_2 \), their values are functions of \( r, d \). The value for them at each target point is determined geometrically:

\[
\begin{align*}
\alpha_1 &= \arctg \left( \frac{d}{r} \right), \\
\alpha_2 &= 90 + \arctg \left( \frac{R-r}{d} \right), \\
\cos \alpha_1 &= \frac{r}{\sqrt{r^2 + d^2}}, \\
\cos \alpha_2 &= -\frac{R-r}{\sqrt{(R-r)^2 + d^2}}.
\end{align*}
\]

Magnetic induction produced by the upper part of the left plate is defined according to the equation:
\[
B_{L,Upper} = \frac{\mu_0 I}{4\pi} \left( \frac{r}{\sqrt{r^2 + d^2}} + \frac{R-r}{\sqrt{(R-r)^2 + d^2}} \right) = f(I,r,d). \tag{3.3}
\]

Analytical integration for the lower part of the left plate at the target point A (figure 3.2) can be identified as follows. As consistent with (3.1) and in accordance with previously defined conception of the angles, it can be stated:
\[
B_{L,Lower} = \frac{\mu_0 I}{4\pi d} (-\cos \alpha_1 + \cos \alpha_3). \tag{3.4}
\]
According to basic vector rules, the magnetic field at the lower segment of the left plate is directed in the opposite direction at the target point A. Taking the direction of the magnetic field into account, the signs of the terms within the parentheses (3.4) must be changed.

The magnetic induction produced by the lower segment of the left plate:

\[ B_{L,\text{Lower}} = \frac{\mu_0 I}{4\pi} \left( \frac{r}{\sqrt{r^2 + d^2}} - \frac{r + R}{\sqrt{(R + r)^2 + d^2}} \right) = f(I, r, d). \tag{3.5} \]

Analytical integration for the magnetic induction of the finite length left external wire at the same target point can be presented as follows. As consistent with (3.1) and basing on universally recognized definitions of the magnetic induction, the equation can be stated as follows:

\[ B_{L,\text{Wire}} = \frac{\mu_0 I}{4\pi} \cos(\beta_1 - \beta_2). \tag{3.6} \]

The magnetic induction direction at the target point A does not coincide with the direction of the field created by the upper segment of the left plate as well. Taking the direction of the magnetic field into account, the signs of the terms within the parentheses (3.6) must be changed.

The magnetic induction produced by the external left wire:

\[ B_{L,\text{Wire}} = \frac{\mu_0 I}{4\pi} \left( \frac{L + d}{\sqrt{r^2 + (L + d)^2}} + \frac{d}{\sqrt{r^2 + d^2}} \right) = f(L, r, d). \tag{3.7} \]

Equations (3.3), (3.5), (3.7) define the magnetic fields superposition of the conductive current at the target point A in the in-between plates capacitor volume from the left part of the wire-plate system:

\[ B_{L,\text{Wire}} = B_{L,\text{Upper}} + B_{L,\text{Lower}} + B_{L,\text{Wire}}. \]

The value for the magnetic dependence between upper and lower segments of the plates as a function of \( r, d \) variables can be determined using vector rules for the right plate. Parameter \( 0 \leq d \leq \Delta \).

The value of the upper segment of the right plate magnetic induction:

\[ B_{R,\text{Upper}} = \frac{\mu_0 I}{4\pi} \frac{1}{\Delta - d} (\cos \delta_1 - \cos \delta_2). \]

Figure 3.2. For derivation of the value of magnetic induction produced by conduction currents in the lower part of the left plate.

Figure 3.3. For derivation of the value of magnetic induction produced by conduction currents in the finite length external left wire.
The value for the lower segment of the right plate magnetic induction:

\[ B_{R,Lower} = \frac{\mu_0 I}{4\pi} \frac{\rho - \rho_1}{\Delta - d} (\cos \gamma_1 - \cos \gamma_2) \]  

The value of the external part of the wire for the right part of the symmetric image magnetic induction (figure 3.5):

\[ B_{R,Wire} = \frac{\mu_0 I}{4\pi} \frac{r - \rho_2}{r} (\cos \psi_1 + \cos \psi_2) . \]

The geometric determination of the values of the angles allows us to obtain the estimated conduction current magnetic induction. The value of the magnetic induction of the upper segment of the right plate:

\[ B_{R,Upper} = \frac{\mu_0 I}{4\pi} \frac{R - r}{\Delta - d} \sqrt{(\Delta - d)^2 + (R - r)^2} + \frac{r}{\sqrt{(\Delta - d)^2 + (r)^2}} . \]  

(3.8)

The value of the magnetic induction of the lower segment of the right plate, with respect to the direction:

\[ B_{R,Lower} = \frac{\mu_0 I}{4\pi} \frac{r + R}{\Delta - d} \sqrt{(\Delta - d)^2 + (r + R)^2} + \frac{r}{\sqrt{(\Delta - d)^2 + (r)^2}} . \]  

(3.9)

The value of the magnetic induction for the external part of the wire from the right plate, with respect to the direction:

\[ B_{R,Wire} = \frac{\mu_0 I}{4\pi} \frac{\Delta - d}{r} \sqrt{(\Delta - d)^2 + (r)^2} - \frac{\Delta - d + L}{\sqrt{(\Delta - d + L)^2 + (r)^2}} . \]  

(3.10)

Magnetic induction superposition at the target point A of the right segment of the symmetric image:

\[ B_a = B_{R,Upper} + B_{R,Lower} + B_{R,Wire} \]

where \( B_{R,Upper} \), \( B_{R,Lower} \), \( B_{R,Wire} \) are defined by (3.8) – (3.10) equation. When analysing the results obtained above, it is necessary to distinguish between the shares the external wires and the capacitor plates contribute to the gradient of the magnetic induction. For the external wires and the plates (left and right), geometric functions are introduced. For the plates:

\[ B_i = B_{L,Upper} + B_{L,Lower} + B_{R,Upper} + B_{R,Lower} = \frac{\mu_0 I K_i(r, d)}{4\pi} , \]

where \( K_i(r, d) \) is a geometric function that determines the value for magnetic induction created by the plates for a target point, with the parameters \( r, d \). The geometric function is defined as follows:
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\[ K_i(r,d) = \frac{1}{d} \left[ \frac{2r}{\sqrt{r^2 + d^2}} - \frac{r + R}{\sqrt{(R + r)^2 + d^2}} + \frac{R - r}{\sqrt{(R - r)^2 + d^2}} \right] + \frac{1}{\Delta - d} \left[ \frac{2r}{\sqrt{(\Delta - d)^2 + (r + R)^2}} - \frac{R - r}{\sqrt{(\Delta - d)^2 + (\Delta - d + R - r)^2}} + \frac{r + R}{\sqrt{(\Delta - d)^2 + (r + R)^2}} \right]. \]  (3.11)

Likewise, the external wires magnetic induction is defined as follows:

\[ B_z = B_{z,\text{Wire}} + B_{z,\text{Wire}} = \frac{\mu_0 I}{4\pi} K_z(r,d), \]  (3.12)

where \( K_z(r,d) \) a geometric function that determines the value for magnetic induction created by the plates for a target point, with the parameters \( r \) and \( d \). The geometric function is defined as follows:

\[ K_z(r,d) = \frac{1}{r} \left[ \frac{\Delta - d}{\sqrt{(\Delta - d)^2 + (r)^2}} - \frac{\Delta - d + L}{\sqrt{(\Delta - d + L)^2 + (r)^2}} + \frac{L + d}{\sqrt{r^2 + (L + d)^2}} + \frac{d}{\sqrt{r^2 + d^2}} \right]. \]  (3.13)

Since the problem currently being resolved is symmetric, the plots of functions \( K_1 \) and \( K_2 \) must also be symmetric relative to the plane \( d = \Delta/2 \). Figure 3.6 shows the plots of the functions \( K_1(0.5R,d) \) and \( K_2(0.5R,d) \). Note that while deriving the analytical functions \( K_1(r,d) \) both the sign when integrating and the sign displaying the magnetic field direction were considered. It is quite evident that when the parameters take the values \( 0 < r < R \) and \( 0 < d < \Delta \), the functions take values \( K_2(r,d) \leq 0 \), \( K_1(r,d) \geq 0 \), while the function \( K_1(r,d) \) is by several orders of magnitude over the function \( K_2(r,d) \).

The magnetic field created by the displacement currents are estimated according to Maxwell equation (2.1):

\[ B_{\text{disp}} = \mu_0 \varepsilon_0 \frac{r}{2} \frac{\partial E}{\partial t} \approx 5.56 \cdot 10^{-18} \mu_0 \varepsilon_0 \frac{U_{\text{ac}}}{\Delta}, \]  (3.14)

where \( \mu_0, \varepsilon_0 \) are permeability and permittivity of the free space, \( \mu, \varepsilon \) are magnetic and dielectric permeability of the medium, respectively. The estimation of magnetic induction produced by the displacement currents for the amplitude values is performed according to the equation:

\[ B_{\text{disp}} \approx 5.56 \cdot 10^{-18} \mu_0 \varepsilon_0 \frac{U_{m}}{\Delta}, \]  (3.14)

where \( U_{m} \) is the voltage amplitude between the plates of the plane capacitor.

The magnetic induction superposition for the conductive currents target point is determined as follows:

\[ B_z = \frac{\mu_0 I}{4\pi} K_z(r,d), \]  (3.15)

where \( K(r,d) = K_1(r,d) + K_2(r,d) \) is a geometric function.

The value of the time variable current is included in the equation (3.15). In terms of amplitude values, it can be stated:

\[ B_{\Sigma m} = \frac{\mu_0 I}{4\pi} m K(r,d). \]  (3.16)
Thus, the analytical forms of all the three researched magnetic fields were developed. Applying the obtained equations, it is possible to derive the equation for the surface on which the absolute value of the vector of magnetic induction produced by conduction currents is equal to the absolute value of the vector of magnetic induction produced by displacement currents.

\[ |B_{\Sigma,m}| = |B_{m \, \text{disp}}|, \]

\[ \frac{\mu \mu_0 I_m K(r,d)}{4\pi} = \mu \mu_0 e \varepsilon_0 \omega \frac{r U_{mc}}{2 \Delta}, \]

\[ K(r,d) = \frac{e \varepsilon_0 2\pi \omega U_{mc}}{\Delta \Delta \omega \varepsilon_0} I_m. \]

Assume that the analysed circuit consists only of a generator and an ideal capacitor:

\[ \frac{U_{mc}}{I_m} = \frac{1}{\omega C}, \]

\[ \frac{e \varepsilon_0 2\pi \omega U_{mc}}{\Delta} \frac{I_m}{\Delta} = \frac{1}{\omega C} = \frac{2}{R^2}, \]

\[ \frac{K(r,d)}{r} = \frac{2}{R^2}. \]  

The surface described by the equation (3.18) will be named the prevalence surface.

The prevalence dependency surfaces at different values of the ratio $\Delta/R$ are shown on Figure 3.7.

\[ \text{Figure 3.7. The prevalence surfaces at different values of the ratio } \Delta/R: \]

(a) $\Delta/R = 1.5$; (b) $\Delta/R = 1$.

Knowing the direction of the magnetic field at every target point is essential when experimenting and detecting the magnetic field produced by displacement currents. For the analysis of the direction of the magnetic field it is necessary to plot the graph of the dependency between $B_{\text{disp}}$ and $B_c$ of the modulus of the magnetic induction vector on the radius at $d = \Delta/2$, considering the direction in relative terms. Thus, for the values $R = 0.25$ m, $\Delta = 1.5R$, $C = \frac{e \varepsilon_0 R^2}{\Delta}$, the graphs are presented at Figure 3.8.
According to the graph in Figure 3.8, at $d = \Delta / R = 1.5$, there is a shift in the direction of the magnetic field. At $0.178 \, m < r < 0.178 \, m$, the field is directed to one side, whereas at $r > 0.178 \, m$, to another side.

Generally, the capacitor can be non-ideal. It will result in the necessity of considering a more complex equation of the circuit, which will in turn lead to an increasing complexity of the equation (3.17).

4. Conclusion
A model for magnetic induction in the capacitor was obtained. Analytical functions allowing us to define the direction and the value of the magnetic field of conduction currents and displacement currents at any point in the space between the plates were determined when analysing the model.

The equation for the prevalence surface was determined. It was established that at values of radii close to values of radii of the plates in the vicinity of a plane capacitor, the magnetic field produced by the displacement currents plays a major role and starts to prevail over the magnetic field produced by the conduction currents. The results obtained might be used for the processing of the experimental results in the future.

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