POSETS WITH COVER GRAPH OF PATHWIDTH TWO HAVE BOUNDED DIMENSION

CSABA BIRÓ, MITCHEL T. KELLER, AND STEPHEN J. YOUNG

Abstract. Joret, Micek, Milans, Trotter, Walczak, and Wang recently asked if there exists a constant \( d \) such that if \( P \) is a poset with cover graph of \( P \) of pathwidth at most 2, then \( \dim(P) \leq d \). We answer this question in the affirmative by showing that \( d = 17 \) is sufficient. We also show that if \( P \) is a poset containing the standard example \( S_5 \) as a subposet, then the cover graph of \( P \) has treewidth at least 3.

1. Introduction

Although the dimension of a poset and the treewidth of a graph have been prominent subjects of mathematical study for many years, it is only recently that the impact of the treewidth of graphs on poset dimension has received any real attention. This new interest in connections between these topics has led to recasting an old result in terms of treewidth. It is natural to phrase the following result from 1977 in terms of treewidth, which had been defined (using a different name) by Halin in [5] a year earlier. However, the importance of treewidth (and the use of that name) only became widely known through the work of Robertson and Seymour [9] nearly a decade later.

Theorem 1.1 (Trotter and Moore [15]). If \( P \) is a poset such that the cover graph of \( P \) is a tree, then \( \dim(P) \leq 3 \). Equivalently, if \( P \) is a poset having connected cover graph of treewidth at most 1, then \( \dim(P) \leq 3 \).

Recently there have been a number of papers on the dimension of planar posets [3, 4, 12]. This work naturally led to the question of bounding a poset’s dimension in terms of the treewidth of its cover graph. Over 30 years ago, Kelly showed in [7] that there are planar posets having arbitrarily large dimension by constructing a planar poset containing \( S_d \), the standard example of dimension \( d \), as a subposet. These examples use large height to stretch out \( S_d \) to allow a planar embedding. Joret et al. [6] point out that the pathwidth of Kelly’s examples is 3 for \( d \geq 5 \). Thus, any bound on dimension solely in terms of pathwidth or treewidth is impossible. However, they were able to show that it suffices to add a bound on the height in order to bound the dimension. In particular, they proved the following:

Theorem 1.2 (Joret et al. [6]). For every pair of positive integers \( (t, h) \), there exists a least positive integer \( d = d(t, h) \) so that if \( P \) is a poset of height at most \( h \) and the treewidth of the cover graph of \( P \) is at most \( t \), then \( \dim(P) \leq d \).
Motivated by the observation about the pathwidth of Kelly’s examples, Joret et al. concluded their paper by asking if there is a constant $d$ such that if $P$ is a poset whose cover graph has pathwidth at most 2, then $\dim(P) \leq d$. They also asked this question with treewidth replacing pathwidth. (An affirmative answer to the latter question would imply an affirmative answer to the former.) In this paper, we show that the answer for pathwidth 2 is in fact “yes” with the following result:

**Theorem 1.3.** Let $P$ be a poset. If the cover graph of $P$ has pathwidth at most 2, then $\dim(P) \leq 17$.

In fact, the precise version of this result (Theorem 4.6) is intermediate between answering the pathwidth question and answering the treewidth question, as we only need to exclude six of the 110 forbidden minors that characterize the graphs of pathwidth at most 2. (Treewidth at most 2 is characterized simply by forbidding $K_4$ as a minor.)

We show in Theorem 5.2 that any poset containing the standard example $S_5$ has treewidth at least 3. This provides a small piece of evidence in favor of the idea that if the treewidth of a poset is at most 2, then the poset’s dimension is bounded.

Before proceeding to our proofs, we provide some definitions for completeness. We then establish some essential properties of the 2-connected blocks of a graph of pathwidth at most 2. We then prove the more general version of Theorem 1.3 and conclude with the rather technical proof that posets containing $S_5$ have cover graphs of treewidth at least 3.

### 2. Definitions and Pathwidth 2 Obstructions

Let $P$ be a poset. If $x < y$ in $P$ and there is no $z \in P$ such that $x < z < y$ in $P$, we say that $x$ is covered by $y$ (or $y$ covers $x$) and write $x \lessdot y$. The cover graph of $P$ is the graph $G$ with the elements of $P$ as its vertices and $x$ is adjacent to $y$ in $G$ if and only if $x \lessdot y$ or $y \lessdot x$. (If we view the order diagram of $P$ as a graph, that graph is $P$’s cover graph.) The dimension of $P$ is the least $t$ such that there exist $t$ linear extensions—collectively known as a realizer—$L_1, \ldots, L_t$ of $P$ with the property that $x <_P y$ if and only if $x <_{L_i} y$ for $i = 1, \ldots, t$. An incomparable pair $(x, y)$ of $P$ is said to be reversed by a linear extension $L$ if $y <_L x$.

To show that a set $R$ of linear extensions of a poset $P$ is a realizer, it suffices to show that each incomparable pair is reversed by some linear extension in $R$. For further background on the combinatorics of partially ordered sets, refer to Trotter’s monograph [13].

Let $G = (V,E)$ be a graph. A pair $(T, \mathcal{V})$, where $T$ is a tree and $\mathcal{V} = (V_t)_{t \in T}$ with $V_t \subseteq V$ for all $t \in T$, is a tree-decomposition of $G$ if

1. $V(G)$ is the union of all the $V_t$;
2. for every $e \in E$, there exists a vertex $t$ of $T$ such that $e \subseteq V_t$; and
3. if $t_1, t_2, t_3$ are vertices of $T$ and $t_2$ lies on the unique path from $t_1$ to $t_3$ in $T$, then $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$.

The sets $V_t$ are often referred to as the bags of the tree-decomposition. The width of $(T, \mathcal{V})$ is $\max_t |V_t| - 1$. The treewidth of $G$, which we denote by $\text{tw}(G)$, is the minimum width of a tree-decomposition of $G$. A path-decomposition of a graph is a tree-decomposition in which the tree $T$ is a path. The pathwidth of $G$, denoted by $\text{pw}(G)$, is the minimum width of a path-decomposition of $G$. 


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Following Diestel [2], we make the following definition of a special type of path because these types of paths will be frequently referenced in our argument. If $G$ is a graph and $H$ is a subgraph of $G$, we say that a path $P$ is an $H$-path if $P$ is nontrivial and intersects $H$ precisely at its two end vertices. The length of a path is the number of edges it contains. We will also freely use terminology regarding the block structure of graphs. Readers unfamiliar with this terminology should consult Diestel’s text [2], in particular Chapter 3.

The set of graphs of pathwidth at most $k$ is a minor closed family. Therefore, by the Graph Minor Theorem [10], this set of graphs can be characterized by forbidding a finite set of graphs as minors. For $k = 2$, Kinnersley and Langston found the entire set of 110 obstructions in [8]. The proof of this paper’s main result relies on only six graphs from their list, but having the whole list at hand was critical to the development of our proof. Besides the obvious obstruction $K_4$, the other five we must exclude are depicted in Figure 1. It is elementary to verify that these graphs have pathwidth 3. We will refer to these graphs in the proof by the names shown and use $\mathcal{F}$ to denote \{$K_4$, $T_1$, $T_2$, $T_3$, $T_4$, $T_5$\}. If a graph $G$ does not contain an element of $\mathcal{F}$ as a minor, we will say that $G$ is $\mathcal{F}$-minor free.

3. Properties of the 2-connected blocks

We begin without restricting our attention to only cover graphs. In this section, we consider a graph $G$ such that $\text{pw}(G) \leq 2$ and prove strong properties of the block structure. This structure is essential in the proof of our main theorem. To establish this structural result, we first make the following definition.

**Definition 3.1.** A parallel nearly outerplanar graph is a graph that consists of a longest cycle $C$ with vertices labelled (in order) as $x_1, x_2, \ldots, x_k, y_1, y_1, \ldots, y_l$ along with some chords and chords subdivided exactly once. The chords and subdivided chords have attachment points $x_{i_1}, y_{j_1}, \ldots, x_{i_m}, y_{j_m}$ such that $i_1 \leq \cdots \leq i_m$ and $j_1 \leq \cdots \leq j_m$.

An example of a parallel nearly outerplanar graph is shown in Figure 2. We think of the vertices along the bottom of the cycle as being the $x_i$ and those along the top as being the $y_j$. Vertices to the left of the leftmost chord and to the right of the rightmost chord could be either $x_i$'s or $y_j$'s.
Lemma 3.2. A graph $G$ is a parallel nearly outerplanar graph if and only if $G$ is 2-connected and $\text{pw}(G) \leq 2$.

Proof. It is easy to see that every parallel nearly outerplanar graph is 2-connected and has pathwidth at most 2. A path-decomposition of width 2 can be obtained by starting with the bag containing $x_1$ and $y_1$ and working through the $x_i$ and $y_j$ by increasing subscript. After all edges incident with $x_i$ have had their other attachment point included in a bag with $x_i$, the bag $\{x_i, x_{i+1}, y_j\}$, where $y_j$ is the “current” vertex from the other side of the cycle, covers the edge $x_ix_{i+1}$. We can then remove $x_i$ from the bag and continue. A symmetric process is used to move from $y_j$ to $y_{j+1}$ after covering all edges incident with $y_j$. The internal vertex of a subdivided chord appears in a bag with precisely its two attachment points.

For the converse, let $C$ be a longest cycle in $G$. A $C$-path will be called an ear. We first note that $C$ cannot have crossing ears. More precisely, if $P$ and $Q$ are ears, $V(C) \cap V(P) = \{v_1, v_2\}$, and $V(C) \cap V(Q) = \{q_1, q_2\}$, then the order of these intersection vertices on $C$ must be $p_1, p_{3-i}, q_j, q_{3-j}$ for some $i, j = 1, 2$. If this were not the case, then $G$ would have a $K_4$-minor, forcing $\text{pw}(G) \geq 3$.

Next we show that no ear may have more than one internal vertex. Indeed, if $P$ is an ear with at least two internal vertices and $V(C) \cap P = \{v_1, v_2\}$, then both paths between $v_1$ and $v_2$ on $C$ must contain at least two internal vertices, for otherwise $C$ is not the longest cycle. If this occurs, then $G$ has a $T_2$-minor.

We now show that the internal vertex, if one exists, of any ear is of degree 2. Let $v$ be the internal vertex of the ear $xvy$, and suppose that the degree of $v$ is at least 3. Let $H$ be the subgraph induced by the vertices of $C$ and the vertex $v$. If $v$ has degree at least 3 in $H$, then $H$ contains a $K_4$-minor. Otherwise, there is a $v' \in V(G)$ such that $v'v \in E(G)$, but $v' \notin V(H)$. Let $H'$ be the subgraph of $G$ formed from $H$ by adding the vertex $v'$ and edge $vv'$. Since $G$ is 2-connected and $H'$ is not, there is an $H'$-path $P$ (possibly just a single edge) with one endpoint being $v'$. The other endpoint may only be $x$ or $y$, since otherwise we have a $K_4$-minor. Without loss of generality, the other endpoint is $x$, which implies that $xPv'y$ is an ear with at least two internal vertices, a contradiction.

We have now shown that $G$ contains a (longest) cycle and some non-crossing ears with at most one inner vertex which must have degree two. The only thing that remains to be shown is that the vertices of the cycle may be labeled as in the definition, effectively placing an ordered structure on the ears. If this were not true, there would be three ears with attachment points $a_1, b_1, a_2, b_2$, and $a_3, b_3$ that appear around the longest cycle of $G$ ordered as $a_1, b_1, a_2, b_2, a_3, b_3$ around $C$, with
the possibility that \( b_i = a_{i+1} \) for any \( i \) (cyclically). In this case, \( G \) contains the forbidden minor \( T_1 \), which gives our final contradiction. \( \square \)

We observe that our proof of the “if” direction of Lemma 3.2 only requires that \( G \) be 2-connected and not contain \( K_4, T_1, \) or \( T_2 \) as a minor. Furthermore, the cycle bounding the infinite face may be chosen to be any longest cycle of the graph, a fact which we will use in the proof of Lemma 3.3.

We note that after proving Lemma 3.2, we discovered that Barát et al. [1] had previously proved this fact while working to simplify the characterization of graphs of pathwidth 2. They used the name *track* for what we call a parallel nearly outerplanar graph. We use the latter name because it is more evocative of the aspects of the structure that are important in our proof and include the proof of Lemma 3.2 for completeness.

By Lemma 3.2 each 2-connected block of a graph of pathwidth two is a parallel nearly outerplanar graph. Our next lemma establishes that the vertices where these blocks join together lie on the parallel nearly outerplanar graphs’ longest cycles.

**Lemma 3.3.** Let \( G \) be a connected graph that does not contain an element of \( F \) as a minor. Let \( B \) be a 2-connected block of \( G \). There exists a longest cycle \( C \) of \( B \) such that if there is a vertex \( v \) of \( B \) adjacent to a vertex \( v' \) not in \( B \), then \( v \in V(C) \).

**Proof.** Let \( C \) be a longest cycle of \( B \) that minimizes the number of internal vertices of ears adjacent to vertices outside \( B \). Let \( v \) be an internal vertex of an ear \( xvy \), i.e., \( v \notin V(C) \), and suppose \( v \) is adjacent a vertex \( v' \) not in \( B \). Deletion of \( x \) and \( y \) from the cycle \( C \) leaves two paths, which we will call \( C_1 \) and \( C_2 \). If both \( C_1 \) and \( C_2 \) contain at least two vertices, then \( G \) has a \( T_3 \)-minor, because we have assumed that \( v \) has a neighbor \( v' \) not in \( B \). Thus, suppose \( C_1 \) contains a single vertex \( u \). If the degree of \( u \) in \( G \) is two, then the cycle formed from \( C \) by replacing \( u \) by \( v \) is also a longest cycle of \( B \), and has fewer internal vertices of ears adjacent to vertices outside \( B \). If the degree of \( u \) in \( B \) were 3, then there would be an ear \( uzx \) or \( uzy \). In either case, \( C \) would not be a longest cycle, as the edge \( uz \) (or \( uy \)) could be replaced by the path \( uzx \) (or \( uzy \)). Therefore, we may assume that \( u \) is adjacent to a vertex \( u' \) not in \( B \). Furthermore, \( u' \neq v' \), and there is no path from \( u' \) to \( v' \) in \( G \) that does not go through \( B \). If \( C_2 \) contains at least two vertices, then \( G \) contains a \( T_4 \)-minor. If \( C_2 \) is a single vertex \( w \), then it must have degree 2 in \( G \) to avoid having \( T_5 \) as a minor. But then \{\( x, v, y, u \}\} is a longest cycle of \( B \) with fewer internal vertices of ears adjacent to vertices outside \( B \) than \( C \). \( \square \)

4. **Posets with Cover Graphs of Pathwidth 2**

**Definition 4.1.** Let \( P \) be a poset. A *subdivision* of the cover relation \( x < : y \) in \( P \) is the addition of new points \( z_1, z_2, \ldots, z_l \) such that \( x < z_1 < \cdots < z_l < y \) and the new points \( z_i \) are incomparable with all points of \( P \) that are not greater than \( y \) or less than \( x \). We say that \( Q \) is a *subdivision* of \( P \) if \( Q \) can be constructed from \( P \) by subdividing some of its cover relations.

In light of what we know from the previous section about the structure of graphs of pathwidth at most 2, it is tempting to consider the effect of subdivision on dimension. Since such an approach would allow us to deal with some of the subdivided chords preventing the cover graph from being outerplanar, we might be inclined to
hope that if \( Q \) is a subdivision of \( P \), then \( \dim(Q) \leq c \dim(P) \) for some absolute constant \( c \). (Perhaps even \( c = 2 \).) However, this is not the case. In fact, Spinrad showed in [11] that this construction can increase dimension by an arbitrarily large factor. Fortunately, as we show in Lemma 4.2 there is a subdivision-like operation on the graphs of relevance to our result that has a small effect on dimension.

Our proof requires that we first introduce some additional terminology. Let \( G \) be a parallel nearly outerplanar graph that is the cover graph of a poset \( P \), and let \( C \) be a longest cycle provided by Lemma 3.3. An ear with no inner vertex is simply called a chord. We call an ear \( xyz \) undirected if \( x < z < y \) or \( y < z < x \) in \( P \). Otherwise we call the ear a beak. An upbeak is an ear with \( x < z > y \) in \( P \), and a downbeak is an ear with \( x > z < y \) in \( P \). (In either case, \( x \parallel y \).) We call the internal point of a beak a beak peak. Our first step will be to address undirected ears. We will then turn our attention to the issue of beaks.

**Lemma 4.2.** Let \( P \) be a poset with cover graph \( G \). Suppose that \( G \) is \( \mathcal{F} \)-minor free and fix an embedding of \( G \) in the plane. If \( Z \) is the collection of points that are not on the unbounded face of \( G \) and are neither minimal nor maximal in \( P \), then \( \dim(P) \leq 2 \dim(P - Z) + 1 \).

**Proof.** First notice that our definition of \( Z \) means that every element of \( Z \) is the internal vertex of a undirected ear \( x < z < y \) in \( P \). If the relation \( x < y \) in \( P - Z \) is a cover, then \( z \) is a subdividing point of the cover relation \( x < y \) in \( P \). Note, however, that \( P \) is not necessarily a subdivision of \( P - Z \), as some of the undirected ears may not correspond to cover relations in \( P - Z \). Nevertheless, we will refer to \( Z \) as the set of subdividing points of \( P \) and an element of \( Z \) will be called a subdividing point of \( P \) even if the relation involved is not a cover of \( P \). When \( xzy \) is an undirected ear of \( P \) with \( x < z < y \) in \( P \), we will refer to \( x \) as the lower element of \( z \). Similarly, \( y \) will be called the upper element of \( z \).

Let \( \{L_1, \ldots, L_d\} \) be a realizer of \( P - Z \) with \( d = \dim(P - Z) \). For each \( L_i \), we will construct two linear extensions \( L_i' \) and \( L_i'' \) of \( P \) by inserting the subdividing elements appropriately, and we will show that most incomparable pairs will be reversed in one of these linear extensions. We will create one extra linear extension to reverse the rest of the incomparable pairs.

To construct \( L_i' \), we place each subdividing point of \( P \) immediately above its lower element in \( L_i \). We form \( L_i'' \) by placing the each subdividing point immediately below its upper element in \( L_i \). There may be some ambiguity in this definition if subdividing points share upper or lower elements. To deal with such situations, let \( z_1, \ldots, z_k \) be subdividing points of \( P \) that share the lower element \( x \). For \( i = 1, \ldots, k \), let the upper element of \( z_i \) be \( y_i \). We may assume that these upper elements are distinct, since the removal of one point of a pair of points with duplicated holdings does not impact dimension (other than in the irrelevant case of a two-element antichain). Let \( \sigma \) be a permutation of \( \{1, \ldots, k\} \) such that \( y_{\sigma(1)} < \cdots < y_{\sigma(k)} \) in \( L_i \). In \( L_i' \) we insert the subdividing points so that \( x < z_{\sigma(k)} < \cdots < z_{\sigma(1)} \). For \( L_i'' \), our concern is with subdividing points \( z_1, \ldots, z_k \) sharing the upper element \( y \). Let \( x_i \) be the lower element of \( z_i \), and let \( \sigma \) be a permutation of \( \{1, \ldots, k\} \) such that \( x_{\sigma(1)} < \cdots < x_{\sigma(k)} \) in \( L_i \). To form \( L_i'' \), we insert the subdividing elements so that \( z_{\sigma(k)} < \cdots < z_{\sigma(1)} < y \) in \( L_i'' \).

Consider an incomparable pair \( (a, b) \). If \( a, b \in P - Z \), then obviously there is a linear extension \( L_i' \) (and an \( L_i'' \)) with \( a > b \). Suppose \( a \in P - Z \) and \( b \in Z \) and let \( x \) be the lower element of \( b \). Then \( a \notin x \) in \( P - Z \) implies that there in an \( L_i \) in
which \( a > x \), and hence \( a > b \) in \( L'_i \). Similarly, if \( a \in Z \) and \( b \in P - Z \), there exists an \( L''_i \) with \( a > b \).

If \( a, b \in Z \) have the same lower element, then their order in \( L'_i \) will be opposite of their order in \( L''_i \). Hence, one of \( L'_i \) and \( L''_i \) has \( a > b \). A similar argument works when \( a \) and \( b \) have the same upper element.

Next we assume that \( a, b \in Z \) have distinct upper and lower elements. Specifically, let \( x_a, y_a, x_b, y_b \) be the lower elements and upper elements of \( a \) and \( b \), respectively. If \( x_a \neq x_b \), then \( x_a > x_b \) in some \( L_i \), and hence \( a > b \) in \( L'_i \). Similarly, if \( y_a \neq y_b \), then \( a > b \) in some \( L''_i \).

At this stage, we have shown that the incomparable pair \((a, b)\) will be reversed, unless all of the following conditions are satisfied:

1. \( a, b \in Z \);
2. \( a \) and \( b \) have distinct upper and lower elements \( x_a, y_a, x_b, y_b \), respectively;
3. \( x_a < x_b \) and \( y_a < y_b \).

We say such a pair \((a, b)\) is in a bad diamond. We will prove that there exists a single linear extension that reverses all such pairs.

We do this by viewing the poset \( P \) as an acyclic directed graph \( D \), with directed edges corresponding to covers and pointing from smaller elements to larger elements. For each incomparable pair \((a, b)\) in a bad diamond, we introduce a new directed edge \( ba \). We call these new edges, and the directed graph formed from \( D \) by adding these new edges is denoted by \( D' \).

We claim that \( D' \) does not contain a directed cycle. Suppose for a contradiction that it does, and let \( C \) be a directed cycle in \( D' \) that contains as few new edges as possible. Notice that \( C \) must contain at least one new edge since \( D \) is acyclic. Let \( P_1 \) be a maximal path in \( C \) that consists entirely of new edges. Suppose that \( P_1 \)’s initial point is \( b \) and its terminal point is \( a \). Since \( G \) is \( F \)-minor free, \( C \) must lie entirely within a 2-connected block of \( G \), i.e., a parallel nearly outerplanar graph. Hence, \( C \) must contain either a directed path from \( y_a \) to \( y_b \) or a directed path from \( x_a \) to \( x_b \); without loss of generality, we assume the latter. Consequently, \( C \) must contain a directed path \( P_2 \) from \( a \) to \( x_a \). Adding the edge \( x_a a \) to \( P_2 \) yields a directed cycle in \( D' \) containing fewer new edges than \( C \), a contradiction.

Since \( D' \) is acyclic, there is a total order \( L_0 \) on its vertices that respects the orientation of its edges. By construction, \( L_0 \) is then a linear extension of \( P \) that reverses all incomparable pairs that are in bad diamonds. Therefore, we may conclude that \( \{L_0, L'_1, \ldots, L'_d, L''_1, \ldots, L''_d\} \) is a realizer of \( P \) and \( \dim(P) \leq 2\dim(P-Z) + 1 \).

To address the case of beaks in the cover graph, we will form two extensions of the poset and show that their intersection is \( P - Z \). We will then apply Lemma 4.2 to \( P \) and use what we know about the extensions of \( P - Z \) to bound its dimension. Note that in the remainder of this section, we often view the poset as a directed graph and refer to a chain of covers as a directed path.

**Lemma 4.3.** Let \( P \) be a poset with cover graph \( G \). If \( G \) is \( F \)-minor free, then \( P \) has extensions \( \Upsilon \) and \( \Delta \) with cover graphs \( G_\Upsilon \) and \( G_\Delta \) that are outerplanar except with some chords replaced by directed paths of length 2.

**Proof.** To construct \( \Upsilon \) and \( \Delta \), we consider the 2-connected blocks of the cover graph of \( P \) one at a time. In each block, we consider the beaks \( xzy \) and introduce a relation between \( x \) and \( y \). It is clear that if we are able to do this, beaks in \( G \) will
become edges in $G_\Upsilon$ and $G_\Delta$ and a pendant vertex (corresponding to the beak peak) will be added to one of the beak attachment points. Thus, the only obstruction to $G_\Upsilon$ and $G_\Delta$ being outerplanar will come from unidirected ears, corresponding to replacing chords of an outerplanar graph by directed paths of length 2.

We introduce relations between beak attachment points for all beaks in such a way that we maintain consistency of these new relations. Since two blocks intersect in at most a single point on their longest cycles, introducing a new comparability within one block cannot force two incomparable beak attachment points in another block to become comparable by transitivity. Therefore, we may define the extensions on the blocks independently.

Consider a 2-connected block $B$. Since $B$ is parallel nearly outerplanar, a fixed plane embedding provides (up to duality) a natural left-to-right ordering on its beaks as suggested in Figure 2. Fix one of these orders and number the $k$ beaks of $B$ accordingly from 1 to $k$. Denote the attachment points for beak $i$ by $x_i$ and $y_i$, with the $x_i$ all lying on the same side of the outer cycle of $B$ and the $y_i$ lying on the other.

We now show that there exists an extension of the subposet induced by the vertices of $B$ in which $x_i < y_i$ for all $i = 1, \ldots, k$. Let $D$ be the digraph defined by the subposet induced by the vertices of $B$. More specifically, $V(D) = V(B)$ and there is a directed edge $uv$ in $D$ if and only if $(u, v)$ is a cover in $P$. To prove such an extension exists, it suffices to show that if we construct $D'$ by adding the directed edges $x_i y_i$ to $D$, then $D'$ contains no directed cycle. By a slight abuse of terminology, we will call the added directed edge $x_i y_i$ a beak.

Suppose for a contradiction that $D'$ contains a directed cycle $C'$. Notice that $C'$ must contain at least one beak, because $D$ is an acyclic graph. In fact, $C'$ has to contain at least two beaks, for if the only beak it contains were $x_i y_i$, then $y_i < x_i$ in $P$, which would contradict the fact that $x_i y_i$ is a beak. Therefore, $C'$ contains the beaks $x_i y_i$ and $x_j y_j$. As a consequence, $C'$ must contain a directed path between $y_i$ and $x_j$. This path forces $x_i$ and $y_j$ to belong to different (topological) components, contradicting the existence of $C'$ as a directed cycle.

By a symmetric argument, there exists an extension of the subposet induced by the vertices of $B$ in which $y_i < x_i$ for all $i = 1, \ldots, k$.

Now we can define the extensions $\Upsilon$ and $\Delta$ of $P$. In a given embedding with left-right orientations of the 2-connected blocks, construct $\Upsilon$ by adding, for each block, the relations $x_i < y_i$ for all $i$. Similarly, construct $\Delta$ by adding the relations $y_i < x_i$ for all $i$ in each block. □

The final major step in our argument is to prove that $P = \Upsilon \cap \Delta$, as then we may use realizers of $\Upsilon$ and $\Delta$ to construct a realizer of $P$, thereby bounding the dimension.

**Lemma 4.4.** Let $P$ be a poset with $F$-minor-free cover graph. If $\Upsilon$ and $\Delta$ are extensions of $P$ as defined in the proof of Lemma 4.3, then $P = \Upsilon \cap \Delta$.

**Proof.** It is sufficient to show that if $w \not\leq w'$ in $P$, then one of the extensions preserves this (non)relation. We begin by considering the situation where $w$ and $w'$ are in the same 2-connected block of the cover graph. We first address the case where $w$ and $w'$ are both on the outer cycle of a 2-connected block and then reduce the remaining cases to this one. We conclude by addressing what happens when $w$ and $w'$ are in different blocks.
Case I Suppose \( w \) and \( w' \) are both on the outer cycle \( C \) of a 2-connected block \( B \) and that \( w < w' \) in both \( \Upsilon \) and \( \Delta \). There are directed paths (chains) from \( w \) to \( w' \) in both \( \Upsilon \) and \( \Delta \). We consider the shortest of these paths in the sense of containing the fewest beaks. Let \( x_iy_i \) be the last beak on the path in \( \Upsilon \), and \( y_jx_j \) be the last beak on the path in \( \Delta \). First note that \( y_i \neq y_j \), since otherwise \( y_i < w' \) in \( P \), implying there is a shorter path in \( \Delta \) that skips \( y_jx_j \). For a similar reason, \( x_i \neq x_j \).

Without loss of generality, assume that \( i < j \). If \( w' \) is right of \( y_jx_j \) (allowing \( w' = x_j \)), then there is a path in \( P \) from \( y_i \) to \( w' \). By minimality, this path cannot pass through \( y_j \), because then \( y_jx_j \) could be skipped. Hence, the path separates \( x_i \) from \( y_j \). Notice that \( w \) is not on the path from \( y_i \) to \( w' \), as this would imply \( w < w' \) in \( P \). Therefore, \( w \) would have to be in both (topological) components, which is a contradiction. A similar contradiction can be derived if \( w' \) is left of \( x_iy_i \) or \( w' = y_i \). In that case the path from \( x_i \) to \( w' \) in \( P \) would separate \( x_i \) from \( y_j \).

This leaves only the possibility that \( w' \) is between the two beaks. If \( w' \) is on the \( x_iy_i \) arc of the outer cycle, then the path from \( y_i \) to \( w' \) separates \( x_i \) from \( y_j \), and if \( w' \) is on the path from \( y_i \) to \( y_j \), then the \( x_jw' \) path performs the separation. Therefore, we may conclude that \( w \not< w' \) in \( \Upsilon \) or \( \Delta \).

Case II Still assuming \( w \) and \( w' \) are in the same 2-connected block, we now suppose that exactly one of them is on the longest cycle \( C \). Specifically, we will consider the case when \( w \) is on \( C \) and \( w' \) is not, and the ear containing \( w' \) is right of \( w \). This is just for convenience of discussion; the other three possibilities have identical proofs.

Suppose there is a directed path from \( w \) to \( w' \) in both \( \Upsilon \) and \( \Delta \); consider one of these that goes through the minimum number of (newly-directed) beaks. First note that \( w' \) cannot be a peak of a downbeak, since that would make \( w' \) minimal in \( P \) and thus in \( \Upsilon \) and \( \Delta \). If \( w' \) is a subdividing point of a unidirected ear, then let \( u < w' \) be its attachment point. We have \( w \not< u \) in \( P \), so by Case I, we maintain this in one of \( \Upsilon \) or \( \Delta \). That extension preserves \( w \not< w' \).

The remaining possibility in this case is that \( w' \) is the peak of an upbeak. By the minimality of the path from \( w \) to \( w' \), the path uses no beaks right of the beak containing \( w' \). For the purpose of the argument, we may ignore all ears, chords, and points of \( C \) strictly right from the beak of \( w' \). By so doing, \( w' \) becomes a point on the outer cycle, and by Case I, one of \( \Upsilon \) or \( \Delta \) will preserve \( w \not< w' \).

Case III To conclude the scenario where both \( w \) and \( w' \) are in the same 2-connected block, it remains only to address the case when neither of them is on \( C \). Without loss of generality assume that \( w \) is left of \( w' \). Considering a path from \( w \) to \( w' \) through the fewest number of beaks, we may assume that this path does not touch any part of the block left of \( w \) and right of \( w' \). (If either \( w \) or \( w' \) is part of a unidirected ear, using these portions would imply the existence of a directed cycle, and for beak peaks the path can be shortened by going via the other attachment point.) By ignoring the parts of the block left of \( w \) and right of \( w' \), we place \( w \) and \( w' \) on an outer cycle, and thus Case I guarantees one of \( \Upsilon \) and \( \Delta \) preserves \( w \not< w' \).

Case IV It remains only to consider the case where \( w \) and \( w' \) are in different 2-connected blocks. If \( w \) and \( w' \) lie in different components of the cover graph, both \( \Upsilon \) and \( \Delta \) preserve \( w \not< w' \). Hence, we may assume there exists a path in the cover graph from \( w \) to \( w' \). (Since \( w \not< w' \) in \( P \), this path is not a directed path.) Let the 2-connected blocks that the path passes through be called \( B_1, B_2, \ldots, B_t \), possibly
including the block of \( w \) and the block of \( w' \) (if either is 2-connected). Note that we allow \( l = 0 \) if the path does not pass through any 2-connected blocks. Let \( a_i \) and \( b_i \) be the (uniquely-determined) entry and exit vertices of the path into and out of \( B_i \); if \( w \in B_1 \), then let \( a_1 = w \), and if \( w' \in B_l \), then let \( b_l = w' \).

If \( a_i \leq b_i \) in \( P \) for all \( i = 1, 2, \ldots, l \), then since the path from \( w \) to \( w' \) in the cover graph of \( P \) is not directed, both \( \Upsilon \) and \( \Delta \) preserve \( w \neq w' \). On the other hand, if there exists an \( i_0 \) such that \( a_{i_0} \not< b_{i_0} \), then this (non)relation is preserved in one of \( \Upsilon \) or \( \Delta \). Since any directed path from \( w \) to \( w' \) must pass through the points \( a_i, b_i \), we have shown \( w \not< w' \) in one of \( \Upsilon \) or \( \Delta \). □

As we combine the three preceding lemmas to prove our main theorem, we will reduce to a poset with outerplanar cover graph. The following result guarantees that such posets have small dimension.

**Theorem 4.5** (Felsner, Trotter, and Wiechert [4]). If a poset \( P \) has an outerplanar cover graph, then \( \dim(P) \leq 4 \).

We are finally ready to state the full version of our main theorem.

**Theorem 4.6.** Let \( P \) be a poset with cover graph \( G \). If \( G \) is \( \mathcal{F} \)-minor free, then \( \dim(P) \leq 17 \).

**Proof.** Begin by fixing an embedding of \( G \) in the plane and, as in Lemma 4.2, let \( Z \) be the collection of points that are not on the unbounded face of \( G \) and are neither minimal nor maximal in \( P \). By Lemma 4.2, we know that \( \dim(P) \leq 2 \dim(P-Z) + 1 \). We now claim that \( \dim(P-Z) \leq 8 \), which will prove the theorem.

Applying Lemmas 4.3 and 4.4 to \( P-Z \), we find that \( P-Z \) has two extensions \( \Upsilon \) and \( \Delta \) for which \( P-Z = \Upsilon \cap \Delta \). Furthermore, since \( P-Z \) does not contain any unidirected ears, the process of constructing \( \Upsilon \) and \( \Delta \) cannot introduce unidirected ears, and the comparabilities added to form \( \Upsilon \) and \( \Delta \) turn beak peaks into vertices of degree 1 in the cover graphs, we have that \( \Upsilon \) and \( \Delta \) have outerplanar cover graphs. Therefore, by Theorem 4.5, there are realizers \( R_\Upsilon \) and \( R_\Delta \) of \( \Upsilon \) and \( \Delta \), respectively, with \( |R_\Upsilon|, |R_\Delta| \leq 4 \). Since \( P-Z = \Upsilon \cap \Delta \), we know that \( R_\Upsilon \cup R_\Delta \) is a realizer of \( P-Z \). Therefore, \( \dim(P-Z) \leq 8 \) and \( \dim(P) \leq 17 \). □

To obtain Theorem 1.3, we now note that if \( P \) is a poset with cover graph \( G \) of pathwidth at most 2, then \( G \) is \( \mathcal{F} \)-minor free, so Theorem 4.6 implies \( \dim(P) \leq 17 \). It is natural to wonder whether the bound of Theorem 4.6 is best possible. We have no reason to believe the result is optimal is and suspect it may be possible to reduce the bound to 4 with more work. That would be best possible, as Felsner, Trotter, and Wiechert give a 4-dimensional poset having cover graph with pathwidth 2 in [4].

We also note that Trotter [14] has subsequently made an observation regarding the relationship between dimension and the block structure of the cover graph, making it possible to drop \( T_3, T_4 \), and \( T_5 \) from the list of forbidden minors. However, that approach leads to a considerably weaker bound on the dimension than the one we offer here.

5. **Standard Examples and Treewidth**

A second question posed in [6] remains open.
Question 5.1. Is there a constant $d$ such that if $P$ is a poset with cover graph $G$ and $\text{tw}(G) \leq 2$, then $\dim(P) \leq d$?

The following theorem provides some weak evidence for an affirmative answer to this question, since the theorem implies that if the answer to Question 5.1 is “no”, a counterexample cannot be constructed using large standard examples.

**Theorem 5.2.** If $P$ is a poset that contains the standard example $S_5$ as a subposet, then the cover graph of $P$ has treewidth at least 3.

**Proof.** Since $\text{tw}(K_4) = 3$, it will suffice to show that the cover graph of $P$ has a $K_4$-minor. (In fact, more is true, in that $K_4$ is the only forbidden minor required to characterize graphs of treewidth 2.) To find a $K_4$-minor in the cover graph of $P$, we build up a series of internally vertex-disjoint paths which we join to create a $K_4$-minor. We denote such a path between any two elements $x$ and $y$ by $P(x,y)$.

In fact, we restrict our attention to paths where $x$ and $y$ are comparable and such that the path represents a maximal chain between $x$ and $y$ in $P$.

Let $\{a_1, \ldots, a_5\}$ and $\{b_1, \ldots, b_3\}$ be elements of the subposet of $P$ isomorphic to $S_5$ with the standard ordering, that is, $a_i < b_j$ if and only if $i \neq j$. As $a_1 < b_2, b_3$ there exist two maximal chains $P(a_1, b_2)$ and $P(a_1, b_3)$. Since these paths need not be internally disjoint, let $c_1$ be the maximal element occurring in both $P(a_1, b_2)$ and $P(a_1, b_3)$ and fix three internally disjoint paths $P(a_1, c_1)$, $P(c_1, b_2)$, and $P(c_1, b_3)$. Restricting our focus to the $S_3$ determined by $\{a_1, a_2, a_3, b_1, b_2, b_3\}$, define $c_2$ and $c_3$ analogously. Notice that $\{c_1, c_2, c_3\}$ is an antichain in $P$ since $a_i$ is incomparable to $b_i$ for all $i$. In a dual manner we notice that $P(c_2, b_1)$ and $P(c_3, b_1)$ may not be internally disjoint. Define $d_1$ as the minimal element in both chains, and define $d_2$ and $d_3$ analogously. Thus the poset $P$ contains four (not necessarily disjoint) antichains $\{a_1, a_2, a_3\}$, $\{b_1, b_2, b_3\}$, $\{c_1, c_2, c_3\}$, and $\{d_1, d_2, d_3\}$ together with internally disjoint paths $P(a_i, c_i)$ and $P(d_i, b_i)$ for $i \in \{1, 2, 3\}$ and paths $P(c_i, d_j)$ for $i, j \in \{1, 2, 3\}$ with $i \neq j$.

See Figure 3. We call the subposet on these elements $S$.

![Figure 3](image)

**Figure 3.** The subposet $S$ with vertices internal to chains/paths not shown.

After noting that $P(c_i, d_j)$ is disjoint from $P(c_{i'}, d_{j'})$ when $(i, j) \neq (i', j')$, it is easy to see that $P(c_1, d_2), P(d_2, c_3), P(c_3, d_1), P(d_1, c_2), P(c_2, d_3), P(d_3, c_1)$ is a cycle $C$ in the cover graph of $P$. Thus, if any element of the poset is connected to this cycle by three vertex-disjoint paths, then the cover graph contains a $K_4$-minor, as desired. Noting that $a_4 < b_1, b_2, b_3$ we now consider the relationship between $a_4$ and $S$. Suppose first that $a_4$ is not less than any element of $\{c_1, c_2, c_3\}$. To preserve
the incomparabilities \( a_i \parallel b_i \), every element of \( C - \{c_1, c_2, c_3\} \) is less than precisely one element of \( \{b_1, b_2, b_3\} \). Hence, there exist three paths \( P_1, P_2, P_3 \) in the cover graph from \( d_4 \) to \( C \). (Note that these paths may use the paths \( P(b_i, d_i) \) if \( a_4 \) is not less than some of the \( d_i \).) On each path \( P_i \) there is a last vertex \( x_i \) that is shared between two of these paths. (It may be that \( x_i = a_4 \).) Now contract the paths between \( x_i \) and \( a_4 \) for each \( i \) into one vertex \( x \); we now have 3 vertex disjoint paths from \( x \) to \( C \), creating a \( K_4 \)-minor.

Therefore, we may assume that \( a_4 \) is less than one element of \( \{c_1, c_2, c_3\} \), say \( c_1 \). By a similar argument, we may assume \( b_4 \) is greater than an element of \( \{d_1, d_2, d_3\} \). Furthermore, since \( b_4 \) is incomparable to \( a_4 \) while \( d_2 \) and \( d_3 \) are comparable to \( c_1 \), our assumption that \( a_4 < c_1 \) forces \( d_1 \) to be the element of \( \{d_1, d_2, d_3\} \) that is less than \( b_4 \). Note that the incomparability between \( a_4 \) and \( b_4 \) implies that \( a_4 \) is incomparable to \( c_2 \) and \( c_3 \) and \( b_4 \) is incomparable to \( d_2 \) and \( d_3 \). Thus, there is a vertex \( \beta_4 \) on \( P(d_1, b_1) \) such that \( \beta_4 < b_4 \) and a vertex \( \alpha'_4 \) on \( P(a_1, c_1) \) such that \( a_4 < \alpha'_4 \). Since \( a_4 < b_1 \) and \( a_4 \) is incomparable to \( b_4 \), there is some element \( \alpha_4 \) on \( P(d_1, b_1) \) with \( \alpha_4 > \beta_4 \) and \( a_4 < \alpha_4 \). Similarly, there is an element \( \beta'_4 \) on \( P(a_1, c_1) \) with \( \beta'_4 < \alpha'_4 \) and \( \beta'_4 < b_4 \). See Figure 4 for an illustration of the relationship between these points. In a similar manner, we can find a \( j \in \{1, 2, 3\} \) and elements \( \beta_5 \) on \( P(d_j, b_j) \) and \( \beta'_5 \) on \( P(a_j, c_j) \) such that \( \beta_5, \beta'_5 < b_5 \). There are also elements \( \alpha_5, \alpha'_5 > a_5 \) such that \( \alpha_5 > \beta_5 \) on \( P(d_j, b_j) \) and \( \alpha'_5 > \beta'_5 \) on \( P(a_j, c_j) \). If there are multiple choices for \( \beta_j \), \( \beta'_j \), \( \alpha_j \), and \( \alpha'_j \) that satisfy all these requirements, we choose \( \beta_j \) and \( \beta'_j \) to be maximal and \( \alpha_j \) and \( \alpha'_j \) to be minimal among the possible choices.

![Figure 4. Expanding S by adding \( a_4, b_4, \alpha_4, \beta_4, \alpha'_4, \beta'_4 \).](image)

We consider the cases where \( j \neq 1 \) and \( j = 1 \) separately. For the former, suppose without loss of generality that \( j = 3 \), as depicted in Figure 5. In this case, if the following six paths are internally disjoint, they form a \( K_4 \)-minor with corners\(^1\) \( c_1, d_1, c_3, \) and \( d_3 \):

- \( P(d_3, c_2) P(c_2, d_3) \),
- \( P(d_3, \beta_5) P(\beta_5, b_5) P(b_5, \beta'_5) P(\beta'_5, c_3) \),
- \( P(c_3, d_2) P(d_2, c_1) \),
- \( P(c_1, \alpha'_4) P(\alpha'_4, a_4) P(a_4, \alpha_4) P(\alpha_4, d_1) \),
- \( P(d_1, c_3) \), and
- \( P(d_3, c_1) \).

\(^1\)We use “corners” here to mean that the given vertices (or sets of vertices) are in distinct branch sets of the \( K_4 \)-minor without fully specifying the branch sets.
The internal disjointness of each pair of the paths above is clear with the possible exception of the second path and the fourth path. However, if these paths fail to be disjoint, their intersection point has 3 paths to distinct vertices of \( C \), creating a \( K_4 \)-minor.

The most delicate part of our argument remains in the case where \( j = 1 \). We consider now the paths that enter \( P(a_4, b_1) \). Specifically, we examine the relationships between \( P(a_4, a_4), P(a_5, a_5), P(b_4, b_4), \) and \( P(b_5, b_5) \). The paths entering \( P(a_1, c_1) \) featuring the \( \alpha_4' \) and \( \beta_4' \) will interact identically by duality. It is clear that \( P(a_4, a_4) \) and \( P(b_4, b_4) \) do not intersect, as otherwise \( a_4 < b_4 \). Suppose then that \( P(a_5, a_5) \) and \( P(b_5, b_5) \) intersect at some point \( x \), while \( P(a_4, a_4) \) and \( P(b_4, b_4) \) do not intersect. Furthermore, if the paths \( P(a_4, a_4) \) and \( P(b_5, b_5) \) intersect more than once, we will assume that \( x \) is the minimal such intersection (in terms of the poset).

Now consider rerouting of the path \( P(d_1, b_1) \) through \( x \). The new path will be the concatenation of \( P(d_1, \beta_4), P(\beta_5, x), P(x, \alpha_4), \) and \( P(\alpha_4, b_1) \). We then choose the new vertices \( \tilde{\alpha}_4, \tilde{\alpha}_5, \tilde{\beta}_4, \tilde{\beta}_5 \) appropriately, recalling that they are chosen to be maximal or minimal amongst possible options. The new paths \( P(a_4, \tilde{\alpha}_4) \) and \( P(b_5, \tilde{\beta}_5) \) are internally disjoint by construction, and it is straightforward to construct paths \( P(\tilde{\alpha}_4, \alpha_4) \) and \( P(\tilde{\beta}_5, \beta_4) \) that are disjoint unless \( \beta_5 < \alpha_5 \leq \beta_4 < \alpha_4 \). In this case, consider the cycle formed by \( P(\beta_5, x), P(x, \alpha_4), \) and \( P(\alpha_4, \beta_5) \). (See Figure 6.) Observe that there are three disjoint paths—namely, \( P(x, \alpha_4), P(\beta_5, d_1), \) and \( P(\alpha_5, a_5) \)—emanating from the cycle. Since \( a_4, a_5, \) and \( d_1 \) all connect to the
path \( P(a_1, c_1) \), these three vertices are all in the same connected component after deleting the cycle. Therefore, we obtain a \( K_4 \)-minor by contracting the path \( P(a_1, c_1) \) to a single vertex.

Now consider the case where, in addition, \( P(a_5, \alpha_5) \) and \( P(b_4, \beta_4) \) intersect at some point \( y \), again choosing \( y \) as the minimal intersection point. Since \( \beta_5 < x < \alpha_4, \beta_4 < y < \alpha_5, \beta_4 < \alpha_4, \) and \( \beta_5 < \alpha_5 \), we have that \( \{ \beta_4, \beta_5 \} < \{ \alpha_4, \alpha_5 \} \). Since \( a_i \) is incomparable to \( b_i \) in the poset, we must have that \( x \) and \( y \) are incomparable as well. This implies that any intersection between \( P(x, \beta_5) \) and \( P(y, \beta_4) \) occurs at a point less than both \( x \) and \( y \) on these paths. Similarly, any intersection between \( P(x, \alpha_4) \) and \( P(y, \alpha_5) \) must be greater than both \( x \) and \( y \). Let \( K_{4-e} \) denote the graph that results from deleting any edge from \( K_4 \). It is then easy to see that there is a \((K_4-e)\)-minor with corners \( x, y, \{ \beta_4, \beta_5 \}, \) and \( \{ \alpha_4, \alpha_5 \} \), possibly adding intersection points between \( P(x, \beta_5) \) and \( P(y, \beta_4) \) to \( \{ \beta_4, \beta_5 \} \) and intersection points between \( P(x, \alpha_4) \) and \( P(y, \alpha_5) \) to \( \{ \alpha_4, \alpha_5 \} \). The missing connection to complete a \( K_4 \)-minor is between \( x \) and \( y \). However, notice that \( x \) and \( y \) are connected by a path through \( a_4, a_5, \) and \( P(a_1, c_1) \), giving the needed path to complete the minor.

We are now able to make a fairly strong assumption about the pairwise intersections of \( P(a_4, \alpha_4) \), \( P(a_5, \alpha_5) \), \( P(b_4, \beta_4) \), and \( P(b_5, \beta_5) \). Of the six possible crossings, the only two that can occur are \( P(a_4, \alpha_4) \) with \( P(a_5, \alpha_5) \) and \( P(b_4, \beta_4) \) with \( P(b_5, \beta_5) \). Furthermore, these intersections imply that \( \alpha_4 = \alpha_5 \) or \( \beta_4 = \beta_5 \), respectively.

Having established these intersection limitations (and the corresponding ones for the \( \alpha_i' \) and \( \beta_i' \)), we consider the graph formed by contracting each of \( P(a_i, a_i') \), \( P(a_i, \alpha_i') \), \( P(b_i, \beta_i) \), and \( P(b_i, \beta_i') \) for \( i = 4, 5 \) to a single edge. In fact, we go further and contract (arbitrarily) all the edges we can while ensuring that the \( \alpha_i, \beta_i, \alpha_i', \beta_i' \), \( a_i \), and \( b_i \) are not identified for \( i = 4, 5 \). Since it is possible for some of these vertices to have been equal at the outset, we are then left with a graph with at most 12 vertices. (We refer to the vertices as having labels to allow that, for example, \( \alpha_4 \) and \( \alpha_5 \) may refer to the same vertex.) The resulting graph is built up from a path in which the vertices with labels \( V = \{ \alpha_4, \alpha_5, \beta_4, \beta_5 \} \) appear consecutively, as do the vertices with labels \( V' = \{ \alpha_4', \alpha_5', \beta_4', \beta_5' \} \). In addition to this primary path, the graph resulting from the contraction contains a collection of 4 paths of length 2 (via the \( a_i \) and \( b_i \) connecting vertices in \( V \) and \( V' \).

We now show that in all but one case to be described later, this graph has a \( K_4 \)-minor. Let \( M' \) be the labels of the maximum vertices (with respect to the poset) of those with labels in \( V' \) and similarly define \( M \) as the labels of the minimal vertices of those with labels in \( V \). Since \( M \) and \( M' \) each correspond to a single vertex, they cannot contain two labels with the same subscript and must respect the ordering on elements with the same subscript. Thus, \( M \in \{ \{ \beta_4 \}, \{ \beta_5 \}, \{ \beta_4, \beta_5 \} \} \) and \( M' \in \{ \{ \alpha_4' \}, \{ \alpha_5' \}, \{ \alpha_4', \alpha_5' \} \} \). We now construct the \( K_4 \)-minor using \( M, V-M, M', \) and \( V'-M' \) as the corners. Figure 7 makes clear that (with appropriate contractions), \( V-M, M, M', V'-M' \) is a path of length 3.

To construct the \( K_4 \)-minor, it suffices to show that there are connections between (1) \( M \) and \( V'-M' \), (2) \( M' \) and \( V-M \), and (3) \( V-M \) and \( V'-M' \). Since \( \alpha_i \) and \( \alpha_i' \) are connected via \( a_i \) for \( i = 4, 5 \), and \( \beta_i \) and \( \beta_i' \) are connected via \( b_i \) for \( i = 4, 5 \), the first two cases are immediately resolved because of the possible contents of \( M \) and \( M' \). Furthermore, for the third pair, the only situation where we do not
immediately see a connection between $V - M$ and $V' - M'$ is when $M = \{\beta_4, \beta_5\}$ and $M' = \{\alpha'_4, \alpha'_5\}$.

In this case, the poset must contain the paths depicted in Figure 8. However, in this case there is a path between $a_4$ and $b_5$ since $a_4 < b_5$ in the poset. This path is not depicted in Figure 8. Furthermore, this path is disjoint from $C$ because $a_i$ is incomparable to $b_i$ for $i = 4, 5$. With these observations, it is straightforward to verify there is a $K_4$-minor with corners $a_4, b_5, \alpha'_4,$ and $\beta_5$. □

We conclude this section by observing that for any $x \in S_5$, there is a poset containing $S_5 - x$ and having a cover graph of treewidth 2. We show an example in Figure 9 with the poset on the left and a redrawing of the cover graph on the right. Notice that $S_5 - x$ is the subposet formed by the elements other than $u$ and $v$. (Since the graph is clearly $K_4$-minor-free, it has treewidth at most 2.) This implies that Theorem 5.2 is best possible.

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Figure 9. A poset containing $S_5 - x$ (left) with cover graph $G$ (right) redrawn to help show $G$ does not contain a $K_4$-minor, and therefore $\text{tw}(G) = 2$.

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Department of Mathematics, University of Louisville, Louisville, KY 40292
E-mail address: csaba.biro@louisville.edu

Department of Mathematics, Washington and Lee University, Lexington, VA 24450
E-mail address: kellermt@wlu.edu

Department of Mathematics, University of Louisville, Louisville, KY 40292
E-mail address: stephen.young@louisville.edu