Chiral Perturbation Theory: Introduction and Recent Results in the One-Nucleon Sector

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Abstract

We provide an introduction to the basic concepts of chiral perturbation theory and discuss some recent developments in the manifestly Lorentz-invariant formulation of the one-nucleon sector.

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1 Introduction

Effective field theory (EFT) is a powerful tool in the description of the strong interactions at low energies. The central idea is due to Weinberg [Weinberg, 1979]:

"... if one writes down the most general possible Lagrangian, including all terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S–matrix consistent with analyticity, perturbative unitarity, cluster decomposition and the assumed symmetry principles."

In general, an EFT is an approximation to a (more) fundamental theory, designed to be valid in a certain kinematical domain. Instead of solving the underlying theory, the processes under investigation are described in terms of a suitable set of effective degrees of freedom, dominating the phenomena in the particular energy region. In the context of the strong interactions, the underlying theory is
quantum chromodynamics (QCD)—a gauge theory with color SU(3) as the gauge group. Under normal conditions, the fundamental degrees of freedom of QCD, namely, quarks and gluons, do not show up as free particles. One assumes that any asymptotically observed hadron must be in a color-singlet state, i.e., a physically observable state is invariant under SU(3) color transformations. The strong increase of the running coupling for large distances possibly provides a mechanism for the color confinement.

For the low-energy properties of the strong interactions and the setting up of a corresponding EFT description, another phenomenon is of vital importance. The masses of the up and down quarks and, to a lesser extent, also of the strange quark are sufficiently small that the dynamics of QCD in the chiral limit, i.e., for massless quarks, is believed to resemble that of the "real" world. Although a rigorous mathematical proof is not yet available, there are good reasons to assume that a dynamical spontaneous symmetry breaking emerges from the chiral limit. Examples of indications for this to happen are the comparatively small masses of the pseudoscalar octet, the absence of a parity doubling in the low-energy spectrum of hadrons, and a non-vanishing scalar singlet quark condensate.

According to the Goldstone theorem, a breakdown of the chiral SU(3)_L × SU(3)_R symmetry at the Lagrangian level to the SU(3)_V symmetry in the ground state implies the existence of eight massless pseudoscalar Goldstone bosons. The finite masses of the pseudoscalar octet of the real world are attributed to the explicit chiral symmetry breaking due to the quark masses in the QCD Lagrangian. Due to the vanishing of the Goldstone boson masses in the chiral limit in combination with their vanishing interactions in the zero-energy limit, a derivative and quark-mass expansion is the natural scenario for an EFT. The corresponding method is called (mesonic) chiral perturbation theory (ChPT) [Gasser and Leutwyler, 1984], with the Goldstone bosons as the relevant effective degrees of freedom (see Table 1).

Using these effective degrees of freedom, physical quantities are calculated in terms of an expansion in \( q/\Lambda_\chi \), where \( q \) stands for momenta or masses of the pseudoscalar octet that are smaller than the energy/mass scale \( \Lambda_\chi = \mathcal{O}(1 \, \text{GeV}) \) associated with spontaneous symmetry breaking. Since an EFT is based on the most general Lagrangian, which includes all terms that are compatible with the symmetries of the underlying theory, the corresponding Lagrangian contains an infinite number of terms, where each term is accompanied by a low-energy coupling constant (LEC). The method that allows one to decide which terms contribute in a calculation up to a certain accuracy is called Weinberg’s power counting. In the mesonic sector, the combination of dimensional regularization with the modified minimal subtraction scheme of ChPT leads to a straightforward correspondence between the loop expansion and the chiral expansion in terms of momenta and quark masses at a fixed ratio. In actual calculations only a finite number of terms in the expansion in \( q/\Lambda_\chi \) has to be considered and thus one has predictive power. What distinguishes the EFT approach from purely phenomenological approaches is the possibility of a systematic improvement. Mesonic ChPT has been tremendously successful and may be considered as a full-grown and mature area of low-energy particle physics.

The situation gets more complicated once other hadronic degrees of freedom beyond the Goldstone bosons are considered. Together with such hadrons, another scale of the order of the chiral symmetry breaking scale \( \Lambda_\chi \) enters the problem and the methods of the pure Goldstone-boson sector cannot be

| Theoretical framework | Fundamental theory | Effective field theory |
|-----------------------|--------------------|------------------------|
| Degrees of freedom    | Quarks and gluons  | Goldstone bosons (+ other hadrons) |
| Parameters            | \( g_3 \) + quark masses | Low-energy coupling constants + quark masses |

Table 1: Comparison of QCD and ChPT.
transferred one to one. For example, in the extension to the one-nucleon sector the correspondence between the loop expansion and the chiral expansion, at first sight, seems to be lost: higher-loop diagrams can contribute to terms as low as $O(q^2)$ \cite{Gasser:1988}. For a long time this was interpreted as the absence of a systematic power counting in the relativistic formulation of ChPT. However, over the last decade new developments in devising a suitable renormalization scheme have led to a simple and consistent power counting for the renormalized diagrams of a manifestly Lorentz-invariant approach.

The purpose of this article is to first provide a pedagogical introduction to the basic concepts of ChPT and to then present the more recent developments of a manifestly Lorentz-invariant approach to the one-nucleon sector. It is definitely not intended to give a survey of the vast literature on ChPT and its various extensions in terms of chiral effective field theories. For further information the interested reader is referred to review articles and lecture notes addressing different topics with various priorities \cite{Bijnens:1993}, \cite{Georgi:1993}, \cite{Ecker:1995}, \cite{Pich:1995}, \cite{Bernard:1995}, \cite{Hammert:1998}, \cite{Burgess:2000}, \cite{Scherer:2003}, \cite{Scherer:2005}, \cite{Epelbaum:2006}, \cite{Bijnens:2007}, \cite{Bernard:2007}, \cite{Bernard:2008}.

The article is organized as follows. Section 2 contains a discussion of the chiral symmetry of QCD, spontaneous symmetry breaking, and the Goldstone theorem. In Sec. 3 the basic concepts of mesonic ChPT are developed. Section 4 is devoted to baryonic ChPT. The power-counting problem is illustrated and solutions in terms of suitable renormalization conditions are presented. Section 5 contains a few selected applications of the manifestly Lorentz-invariant approach to nucleon properties.

2 Chiral symmetry and spontaneous symmetry breaking

The essential ingredients to setting up chiral perturbation theory as the effective field theory of the strong interactions are the chiral SU(3)$_L \times$ SU(3)$_R$ symmetry of QCD for massless u, d, and s quarks and the emergence of a spontaneous breakdown to the vectorial subgroup SU(3)$_V$.

2.1 Quantum chromodynamics and chiral symmetry

QCD is the gauge theory of the strong interactions \cite{Gross:1973}, \cite{Weinberg:1973}, \cite{Fritzsch:1973} with color SU(3) as the underlying gauge group. Historically, the color degree of freedom was introduced into the quark model to account for the Pauli principle in the description of baryons as three-quark states. The matter fields of QCD are the so-called quarks which are spin-1/2 fermions, with six different flavors (u, d, s, c, b, t) in addition to their three possible colors (see Table 2). Since quarks have not been observed as asymptotically free states, the meaning of quark masses and their numerical values are tightly connected with the method by which they are extracted from hadronic properties (see Ref. \cite{Manohar:2008} for a thorough discussion).

2.1.1 The QCD Lagrangian

The QCD Lagrangian can be obtained from the Lagrangian for free quarks by applying the gauge principle with respect to the group SU(3) of all unitary, unimodular, 3 × 3 matrices. Denoting the quark field components by $q_{f,A,\alpha}$, where $f = 1, \ldots, 6$ refers to the flavor index, $A = 1, 2, 3$ to the color index, and $\alpha = 1, \ldots, 4$ to the Dirac spinor index, respectively, the “free” quark Lagrangian without interaction may be regarded as the sum of $6 \times 3 = 18$ free fermion Lagrangians:

$$\mathcal{L}_{\text{free quarks}} = \sum_{f=1}^{6} \sum_{A=1}^{3} \sum_{\alpha,\alpha'=1}^{4} \left( \bar{q}_{f,A,\alpha}(\gamma_\mu^\alpha \partial_\mu - m_f \delta_{\alpha\alpha'}) q_{f,A,\alpha'} \right). \quad (1)$$
Table 2: Quark flavors and their charges and masses. See [Manohar and Sachrajda, 2008] for details.

| Flavor | u   | d   | s   |
|--------|-----|-----|-----|
| Charge [e] | 2/3 | -1/3 | -1/3 |
| Mass [MeV] | 1.5 - 3.3 | 3.5 - 6.0 | 70 - 130 |

| Flavor | c | b | t |
|--------|---|---|---|
| Charge [e] | 2/3 | -1/3 | 2/3 |
| Mass [GeV] | 1.27^{+0.07}_{-0.11} | 4.20^{+0.17}_{-0.07} | 171.2 ± 2.1 |

Suppressing the Dirac spinor index and introducing for each quark flavor $f$ a color triplet
\[ q_f = \begin{pmatrix} q_{f,1} \\ q_{f,2} \\ q_{f,3} \end{pmatrix}, \]  
the gauge principle is applied with respect to the group SU(3), i.e., all $q_f$ are subject to the same local SU(3) transformation:
\[ q_f \mapsto q_f' = \exp \left( -i \sum_{a=1}^{8} \Theta_a \frac{\lambda^c_a}{2} \right) q_f = U q_f, \]  
where the eight $\lambda^c_a$ denote Gell-Mann matrices acting in color space and the $\Theta_a$ are smooth, real functions in Minkowski space. Whenever convenient, we will make use of the summation convention implying a summation over repeated indices. Introducing eight gauge potentials $A_{a\mu}$, transforming as
\[ A_{a\mu} = U A_{a\mu} U^\dagger + i g_3 \partial_{\mu} U U^\dagger, \]  
the covariant derivative of the quark field, by construction, transforms as the quark field:
\[ D_{\mu} q_f \equiv (\partial_\mu + ig_3 A_\mu) q_f \mapsto (D_{\mu} q_f)' = D_{\mu}' q_f' = U D_{\mu} q_f. \]  
In Eq. (5), $g_3$ denotes the strong coupling constant. In order to treat the gauge potentials as dynamical degrees of freedom, one defines a generalization of the field strength tensor to the non-Abelian case as
\[ G_{a\mu} = \partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} - g_3 f_{abc} A_{b\mu} A_{c\nu}, \]  
where, suppressing the superscript $c$ in the Gell-Mann matrices, the standard totally antisymmetric SU(3) structure constants are given by (see Table 3)
\[ f_{abc} = \frac{1}{4 \delta} \text{Tr}([\lambda_a, \lambda_b, \lambda_c]). \]  
Given Eq. (4), the field strength tensor transforms under SU(3) as
\[ G_{a\mu} \equiv G_{a\mu} \frac{\lambda^c_a}{2} \mapsto U G_{a\mu} U^\dagger. \]  
The QCD Lagrangian obtained by applying the gauge principle to the free Lagrangian of Eq. (1), finally, reads
\[ \mathcal{L}_{\text{QCD}} = \sum_{f = u,d,s,c,b,t} \bar{q}_f (i \slashed{D} - m_f) q_f - \frac{1}{4} G_{a\mu} G_a^{\mu\nu}. \]
where we will neglect the three heavy quarks have relatively small masses in comparison to a typical hadronic scale of the order of 1 GeV. On the other

contraction of the covariant derivative with the so-called $\theta$ term of Eq. \(10\) implies an explicit $P$ and $CP$ violation of the strong interactions which, for example, would give rise to an electric dipole moment of the neutron. The present empirical information indicates that the $\theta$ term is small and, in the following, we will omit Eq. \(10\) from our discussion.

2.1.2 Chiral limit

The terminology chiral limit refers to massless quarks, resulting in an important additional global symmetry of the QCD Lagrangian which will be discussed in the following. We introduce the chirality matrix $\gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5^\dagger$, $\{\gamma^\mu, \gamma_5\} = 0$, $\gamma_5^2 = 1$, and define the projection operators

$$P_L = \frac{1}{2}(1 - \gamma_5) = P_L^\dagger, \quad P_R = \frac{1}{2}(1 + \gamma_5) = P_R^\dagger.$$  \hspace{1cm} (11)

These operators satisfy the completeness relation $P_L + P_R = 1$, are idempotent, $P_L^2 = P_L$, $P_R^2 = P_R$, and respect the orthogonality relations $P_LP_R = P_RP_L = 0$. When applied to the solutions of the free massless Dirac equation, the operators $P_R$ and $P_L$ project to the positive and negative helicity eigenstates, hence the subscripts $R$ and $L$ for right-handed and left-handed, respectively.

Omitting color and flavor indices, we introduce left- and right-handed quark fields as

$$q_L = P_Lq \quad \text{and} \quad q_R = P_Rq.$$  \hspace{1cm} (12)

A quadratic form containing any of the 16 independent $4 \times 4$ matrices $\{1, \gamma^\mu, \gamma_5, \gamma^\mu\gamma_5, \sigma^{\mu\nu}\}$ can be decomposed as

$$\bar{q}\Gamma_iq = \begin{cases} \bar{q}_L\Gamma_1q_L + \bar{q}_R\Gamma_1q_R & \text{for} \quad \Gamma_1 \in \{\gamma^\mu, \gamma^\mu\gamma_5\} \\ \bar{q}_R\Gamma_2q_L + \bar{q}_L\Gamma_2q_R & \text{for} \quad \Gamma_2 \in \{1, \gamma_5, \sigma^{\mu\nu}\} \end{cases}.$$  \hspace{1cm} (13)

where $\bar{q}_R = \bar{q}P_L$ and $\bar{q}_L = \bar{q}P_R$.

The validity of Eq. \(13\) is general and does not refer to “massless” quark fields.

From a phenomenological point of view the $u$ and $d$ quarks and to a lesser extent also the $s$ quark have relatively small masses in comparison to a typical hadronic scale of the order of 1 GeV. On the other hand, we will neglect the three heavy quarks $c$, $b$, and $t$, because we will restrict ourselves to energies well below the production threshold of particles containing a heavy (anti-) quark. In the following, we will approximate the full QCD Lagrangian by its light-flavor version, and will consider the chiral limit for the three light quarks $u$, $d$, and $s$. To that end, we apply Eq. \(13\) to the term containing the contraction of the covariant derivative with $\gamma^\mu$. This quadratic quark form decouples into the sum of two terms which connect only left-handed with left-handed and right-handed with right-handed quark fields. The QCD Lagrangian in the chiral limit can then be written as

$$\mathcal{L}_{\text{QCD}}^0 = \sum_{l=u,d,s} (\bar{q}_{R,l}i\not{D}\not{q}_{R,l} + \bar{q}_{L,l}i\not{D}\not{q}_{L,l}) - \frac{1}{4}G_{a\mu\nu}G^{a\mu\nu}.$$  \hspace{1cm} (14)
Note that because of Eq. (13) the quark-mass term generates a coupling between left- and right-handed quark fields.

2.1.3 Global symmetry currents of the light quark sector

Due to the flavor independence of the covariant derivative, $L_{QCD}^0$ is invariant under the infinitesimal global transformations of the left- and right-handed quark fields,

$$
q_L \equiv \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \mapsto \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} + \frac{i}{2} \sum_{a=1}^{3} \epsilon_a \lambda_a q_L,
$$

$$
q_R \equiv \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} \mapsto \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} + \frac{i}{2} \sum_{a=1}^{3} \epsilon_a \lambda_a q_R.
$$

(15)

Note that the Gell-Mann matrices act in flavor space. $L_{QCD}^0$ is said to have a classical global $U(3)_L \times U(3)_R$ symmetry. Applying Noether’s theorem [Noether, 1918], [Hill, 1951], [Gell-Mann and Lévy, 1960], from such an invariance one would expect a total of $2 \times (8 + 1) = 18$ conserved currents:

$$
L^\mu_a = \bar{q}_L \gamma^\mu \lambda_a^q q_L, \quad L^\mu = \bar{q}_L \gamma^\mu q_L, \quad R^\mu_a = \bar{q}_R \gamma^\mu \lambda_a^q q_R, \quad R^\mu = \bar{q}_R \gamma^\mu q_R.
$$

(16)

Making use of

$$
P_L \gamma^\mu P_R = P_R \gamma^\mu P_L = \left\{ \begin{array}{c} \gamma^\mu \\
\gamma^\mu \gamma_5 \end{array} \right\},
$$

we introduce the linear combinations

$$
V^\mu_a = R^\mu_a + L^\mu_a = \bar{q}_R \gamma^\mu \lambda_a^q q_L,
$$

$$
A^\mu_a = R^\mu_a - L^\mu_a = \bar{q}_R \gamma^\mu \gamma_5 \lambda_a^q q_L,
$$

(17)

(18)

which under a parity transformation of the quark fields, $q(t, \vec{x}) \mapsto \gamma_0 q(t, -\vec{x})$, transform as vector and axial-vector current densities, respectively,

$$
P : V^\mu_a(t, \vec{x}) \mapsto V^\mu_a(t, -\vec{x}),
$$

$$
P : A^\mu_a(t, \vec{x}) \mapsto -A^\mu_a(t, -\vec{x}).
$$

(19)

(20)

The conserved singlet vector current results from a transformation of all left-handed and right-handed quark fields by the same phase,

$$
V^\mu = R^\mu + L^\mu = \bar{q}_R \gamma^\mu q_L.
$$

(21)

The singlet axial-vector current originates from a transformation of all left-handed quark fields with one phase and all right-handed with the opposite phase,

$$
A^\mu = R^\mu - L^\mu = \bar{q}_R \gamma^\mu \gamma_5 q_L.
$$

(22)

Quantum fluctuations destroy the singlet axial-vector current conservation and there will be extra terms, referred to as anomalies [Bell and Jackiw, 1969], [Adler, 1969], [Adler and Bardeen, 1969], resulting in

$$
\partial_\mu A^\mu = \frac{3g_3^2}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} G_{\mu\nu}^a G_{\rho\sigma}^a, \quad \epsilon_{0123} = 1.
$$

The factor of three originates from the number of flavors. In the large $N_c$ (number of colors) limit of Ref. [t Hooft, 1974] the singlet axial-vector current is conserved, because the strong coupling constant behaves as $g_3^2 \sim N_c^{-1}$. 

7
2.1.4 Chiral algebra

The invariance of \( L^0_{\text{QCD}} \) under global \( SU(3)_L \times SU(3)_R \times U(1)_V \) transformations implies that also the QCD Hamilton operator in the chiral limit, \( H^0_{\text{QCD}} \), exhibits a global \( SU(3)_L \times SU(3)_R \times U(1)_V \) symmetry. As usual, the charge operators are defined as the space integrals of the charge densities,

\[
Q_{aL}(t) = \int d^3x \, q^\dagger_L(t, \vec{x}) \frac{\lambda_a}{2} q_L(t, \vec{x}), \\
Q_{aR}(t) = \int d^3x \, q^\dagger_R(t, \vec{x}) \frac{\lambda_a}{2} q_R(t, \vec{x}), \\
Q_V(t) = \int d^3x \, \left[ q^\dagger_L(t, \vec{x}) q_L(t, \vec{x}) + q^\dagger_R(t, \vec{x}) q_R(t, \vec{x}) \right].
\]

For conserved symmetry currents, these operators are time independent, i.e., they commute with the Hamiltonian,

\[
[Q_{aL}, H^0_{\text{QCD}}] = [Q_{aR}, H^0_{\text{QCD}}] = [Q_V, H^0_{\text{QCD}}] = 0.
\]

The commutation relations among the charge operators reflect the underlying Lie algebra of \( SU(3)_L \times SU(3)_R \times U(1)_V \),

\[
[Q_{aL}, Q_{bL}] = i f_{abc} Q_{cL}, \\
[Q_{aR}, Q_{bR}] = i f_{abc} Q_{cR}, \\
[Q_{aL}, Q_{bR}] = 0, \\
[Q_{aL}, Q_V] = [Q_{aR}, Q_V] = 0.
\]

Equations (27) - (30) are verified by expressing the commutators in terms of equal-time anti-commutation relations of the quark fields.

It should be stressed that, even without being able to explicitly solve the equation of motion of the quark fields entering the charge operators of Eqs. (27) - (30), we know from the equal-time commutation relations and the symmetry of the Lagrangian that these charge operators are the generators of infinitesimal transformations of the Hilbert space associated with \( H^0_{\text{QCD}} \). Furthermore, their commutation relations with a given operator specify the transformation behavior of the operator in question under the group \( SU(3)_L \times SU(3)_R \times U(1)_V \).

2.1.5 Quark masses and chiral symmetry breaking

So far, we have discussed an idealized world with massless light quarks. The finite \( u \)-, \( d \)-, and \( s \)-quark masses explicitly break the chiral symmetry and generate divergences of the symmetry currents. As a consequence, the charge operators are, in general, no longer time independent. However, as first pointed out by Gell-Mann\[Gell-Mann, 1962\], the equal-time commutation relations still play an important role even if the symmetry is explicitly broken.

Defining the quark-mass matrix as

\[
\mathcal{M} = \text{diag}(m_u, m_d, m_s),
\]

the quark-mass term in the QCD Lagrangian leads to a mixing of left- and right-handed fields [see Eq. (13)],

\[
\mathcal{L}_M = -\bar{q} \mathcal{M} q = -\left( \bar{q}_R \mathcal{M} q_L + \bar{q}_L \mathcal{M} q_R \right).
\]

Inserting the transformations of Eqs. (15) into the quark-mass term of Eq. (31) results in the variation \( \delta \mathcal{L}_M \), from which one obtains for the divergences

\[
\partial_\mu L^\mu_a = \frac{\partial \delta \mathcal{L}_M}{\partial \epsilon_a^L} = -i \left( \bar{q}_L \frac{\lambda_a}{2} \mathcal{M} q_R - \bar{q}_R \mathcal{M} \frac{\lambda_a}{2} q_L \right), \\
\partial_\mu L^a = \frac{\partial \delta \mathcal{L}_M}{\partial \epsilon^L} = -i \left( \bar{q}_L \mathcal{M} q_R - \bar{q}_R \mathcal{M} q_L \right).
\]
The analogous expressions for $\partial_\mu R^a_\mu$ and $\partial_\mu R^a_\mu$ are obtained from Eqs. (32) through the substitution $R \leftrightarrow L$. The divergences are proportional to the mass parameters which is the origin of the expression current-quark mass. In terms of the vector and axial-vector currents the divergences read

$$
\begin{align*}
\partial_\mu V^\mu_a &= i\bar{q}\gamma_\mu \frac{\lambda_a}{2} q, \\
\partial_\mu A^\mu_a &= i\bar{q}\{\gamma_\mu, \frac{\lambda_a}{2}\} \gamma_5 q, \\
\partial_\mu V^\mu &= 0, \\
\partial_\mu A^\mu &= 2i\bar{q}\gamma_5 q + \frac{3g_3^2}{32\pi^2} \epsilon_{\mu\rho\sigma\tau} G^\mu_\rho G^\tau_\sigma, \quad \epsilon_{0123} = 1.
\end{align*}
$$

We are now in the position to summarize the various (approximate) symmetries of the strong interactions in combination with the corresponding currents and their divergences.

- In the limit of massless quarks, the sixteen currents $L^\mu_a$ and $R^\mu_a$ or, alternatively, $V^\mu_a$ and $A^\mu_a$ are conserved. The same is true for the singlet vector current $V^\mu$, whereas the singlet axial-vector current $A^\mu$ has an anomaly.

- For any value of quark masses, the individual flavor currents $\bar{u}\gamma_\mu u$, $\bar{d}\gamma_\mu d$, and $\bar{s}\gamma_\mu s$ are always conserved in the strong interactions reflecting the flavor independence of the strong coupling and the diagonality of the quark-mass matrix. Of course, the singlet vector current $V^\mu$, being the sum of the three flavor currents, is always conserved.

- In addition to the anomaly, the singlet axial-vector current has an explicit divergence due to the quark masses.

- For equal quark masses, $m_u = m_d = m_s$, the eight vector currents $V^\mu_a$ are conserved, because $[\lambda_a, 1] = 0$. Such a scenario is the origin of the SU(3) symmetry originally proposed by Gell-Mann and Ne’eman [Gell-Mann and Ne’eman, 1964]. The eight axial-vector currents $A^\mu_a$ are not conserved. The divergences of the octet axial-vector currents of Eq. (33) are proportional to pseudoscalar quadratic forms. This can be interpreted as the microscopic origin of the PCAC relation (partially conserved axial-vector current) [Gell-Mann, 1964], [Adler and Dashen, 1968] which states that the divergences of the axial-vector currents are proportional to renormalized field operators representing the lowest-lying pseudoscalar octet.

- Taking $m_u = m_d \neq m_s$ reduces SU(3) flavor symmetry to SU(2) isospin symmetry.

- Taking $m_u \neq m_d$ leads to isospin symmetry breaking.

- Various symmetry-breaking patterns are discussed in great detail in Ref. [Pagels, 1975].

### 2.1.6 Green functions, chiral Ward identities, and generating functional

For conserved currents, the spatial integrals of the charge densities are time independent, i.e., in a quantized theory the corresponding charge operators commute with the Hamilton operator. These operators are generators of infinitesimal transformations on the Hilbert space of the theory. The mass eigenstates should organize themselves in degenerate multiplets with dimensionalities corresponding to irreducible representations of the Lie group in question. For the moment, we assume that the dynamical system described by the Hamiltonian does not lead to a spontaneous symmetry breakdown. We will come back to this point later. Which irreducible representations ultimately appear, and what the actual energy eigenvalues are, is determined by the dynamics of the Hamiltonian. For example, SU(2) isospin
symmetry of the strong interactions reflects itself in degenerate SU(2) multiplets such as the nucleon doublet, the pion triplet, and so on. Ultimately, the actual masses of the nucleon and the pion should follow from QCD.

It is also well-known that symmetries imply relations between S-matrix elements. For example, applying the Wigner-Eckart theorem to pion-nucleon scattering, assuming the strong-interaction Hamiltonian to be an isoscalar, it is sufficient to consider two isospin amplitudes describing transitions between states of total isospin $I = 1/2$ or $I = 3/2$. All the dynamical information is contained in these isospin amplitudes and the results for physical processes can be expressed in terms of these amplitudes together with geometrical coefficients, namely, the Clebsch-Gordan coefficients.

In quantum field theory, the objects of interest are the Green functions which are vacuum expectation values of time-ordered products. Later on, we will also refer to matrix elements of time-ordered products between states other than the vacuum as Green functions. The physical scattering amplitudes are obtained from the Green functions using the reduction formalism \cite{Lehmann}. Symmetries provide strong constraints not only for scattering amplitudes, i.e. their transformation behavior, but, more generally speaking, also for Green functions and, in particular, among Green functions. Even if a symmetry is broken, i.e., the infinitesimal generators are time dependent, conditions related to the symmetry-breaking terms can still be obtained using equal-time commutation relations.

The symmetry currents relevant to the global $SU(3)_L \times SU(3)_R \times U(1)_V$ of QCD are given in Eqs. (17), (18), and (21). Moreover, since we also want to discuss explicit symmetry breaking, we introduce the scalar and pseudoscalar densities

$$S_a = \bar{q} \lambda_a q, \quad P_a = i \bar{q} \gamma_5 \lambda_a q, \quad a = 0, \ldots, 8,$$  

(34)

where $\lambda_0 = \sqrt{2/3} \mathbb{1}$. For example, linear combinations of $S_a$ and $P_a$ are needed to describe the divergences of the currents in Eqs. (33). Whenever it is more convenient, we will also use

$$S(x) = \bar{q}(x)q(x), \quad P(x) = i \bar{q}(x)\gamma_5 q(x),$$  

(35)

instead of $S_0$ and $P_0$.

For example, the following Green functions of the “vacuum” sector

$$\langle 0|T[A_\mu^a(x)P_b(y)]|0\rangle,$$

$$\langle 0|T[P_a(x)J_\mu^a(y)P_c(z)]|0\rangle,$$

$$\langle 0|T[P_a(w)P_b(x)P_c(y)P_d(z)]|0\rangle$$

are related to pion decay, the pion electromagnetic form factor ($J_\mu$ is the electromagnetic current), and pion-pion scattering, respectively. One may also consider similar time-ordered products evaluated between a single nucleon in the initial and final states in addition to the vacuum Green functions. This allows one to discuss properties of the nucleon as well as dynamical processes involving a single nucleon, such as

$$\langle N|J_\mu(x)|N\rangle \leftrightarrow \text{nucleon electromagnetic form factors},$$

$$\langle N|A_\mu^a(x)|N\rangle \leftrightarrow \text{axial form factor + induced pseudoscalar form factor},$$

$$\langle N|T[J_\mu(x)J_\nu(y)]|N\rangle \leftrightarrow \text{Compton scattering},$$

$$\langle N|T[J_\mu(x)P_a(y)]|N\rangle \leftrightarrow \text{pion electroproduction}.$$
as a simple example the two-point Green function involving an axial-vector current and a pseudoscalar density,
\[ G_{AP,ab}^\mu(x, y) = \langle 0| T[A_\mu^a(x)P_b(y)]|0\rangle = \Theta(x_0 - y_0)\langle 0|A_\mu^a(x)P_b(y)|0\rangle + \Theta(y_0 - x_0)\langle 0|P_b(y)A_\mu^a(x)|0\rangle, \] (36)
and evaluate the divergence
\[
\partial_\mu G_{AP,ab}^\mu(x, y) = \partial_\mu [\Theta(x_0 - y_0)\langle 0|A_\mu^a(x)P_b(y)|0\rangle + \Theta(y_0 - x_0)\langle 0|P_b(y)A_\mu^a(x)|0\rangle]
\]
\[= \delta(x_0 - y_0)\langle 0|A_\mu^a(x)P_b(y)|0\rangle - \delta(x_0 - y_0)\langle 0|P_b(y)A_\mu^a(x)|0\rangle
\]
\[+ \Theta(x_0 - y_0)\langle 0|\partial_\mu A_\mu^a(x)P_b(y)|0\rangle + \Theta(y_0 - x_0)\langle 0|P_b(y)\partial_\mu A_\mu^a(x)|0\rangle
\]
\[= \delta(x_0 - y_0)\langle 0|[A_\mu^a(x), P_b(y)]|0\rangle + \langle 0|T[\partial_\mu A_\mu^a(x)P_b(y)]|0\rangle,
\]
where we made use of \( \partial_\mu^a \Theta(x_0 - y_0) = \delta(x_0 - y_0)g_{0\mu} = -\partial_\mu^a \Theta(y_0 - x_0) \). This simple example already shows the main features of (chiral) Ward identities. From the differentiation of the theta functions one obtains equal-time commutators between a charge density and the remaining quadratic forms. The results of such commutators are a reflection of the underlying symmetry. As a second term, one obtains the divergence of the current operator in question. If the symmetry is perfect, such terms vanish identically. If the symmetry is only approximate, an additional term involving the symmetry breaking appears. For a soft breaking such a divergence can be treated as a perturbation.

The time ordering of \( n + 1 \) points \( x, x_1, \ldots, x_n \) gives rise to \((n+1)!\) distinct orderings, each involving products of \( n \) theta functions. Via induction, the generalization of the above simple example to an \((n+1)\)-point Green function is symbolically of the form
\[
\partial_\mu \langle 0| T\{J^\mu(x)A_1(x_1) \cdots A_n(x_n)\}|0\rangle = \\
\langle 0| T \{[\partial_\mu^a J^\mu(x)]A_1(x_1) \cdots A_n(x_n)\}|0\rangle \\
+ \delta(x_0 - x_1)\langle 0| T\{[J_0(x), A_1(x_1)]A_2(x_2) \cdots A_n(x_n)\}|0\rangle \\
+ \delta(x_0 - x_2)\langle 0| T\{A_1(x_1)[J_0(x), A_2(x_2)] \cdots A_n(x_n)\}|0\rangle \\
+ \cdots + \delta(x_0 - x_n)\langle 0| T\{A_1(x_1) \cdots [J_0(x), A_n(x_n)]\}|0\rangle,
\] (37)
where \( J^\mu \) stands generically for any of the Noether currents.

The discussion so far assumes that one explicitly works out the particular chiral Ward identity one is interested in. However, there is an elegant way of obtaining all chiral Ward identities from a single expression. To that end we introduce into the Lagrangian of QCD the couplings of the nine vector currents, eight axial-vector currents, nine scalar quark densities, and nine pseudoscalar quark densities to external c-number fields [Gasser and Leutwyler, 1985]:
\[
\mathcal{L} = \mathcal{L}_{QCD}^0 + \mathcal{L}_{ext},
\] (38)
where
\[
\mathcal{L}_{ext} = \sum_{a=1}^{8} v_\mu^a \bar{q} \gamma_\mu \frac{\lambda_a}{2} q + v_\mu^a \frac{i}{3} \bar{q} \gamma_\mu \gamma_5 \frac{\lambda_a}{2} q + \sum_{a=1}^{8} a_\mu^a \bar{q} \gamma_\mu \gamma_5 \frac{\lambda_a}{2} q - \sum_{a=0}^{8} s_a \bar{q} \lambda_a q + \sum_{a=0}^{8} p_a i \bar{q} \gamma_5 \lambda_a q
\]
\[= \bar{q} \gamma_\mu (v_\mu^a + \frac{1}{3} v_\mu^a + \gamma_5 a_\mu) q - \bar{q}(s - i \gamma_5 p) q.
\] (39)
The 35 real functions \( v_\mu^a(x), v_\mu^a(x), a_\mu^a(x), s_a(x), \) and \( p_a(x) \), will collectively be denoted by \([v, a, s, p]\). A precursor of this method was already used by Bell and Jackiw [Bell and Jackiw, 1969] in their discussion of the anomalous divergences in the \( \pi^0 \to \gamma \gamma \) decay. The Green functions of the vacuum sector may be combined in the generating functional
\[
\exp(iZ[v, a, s, p]) = \langle 0| T \exp \left[ i \int d^4x \mathcal{L}_{ext}(x) \right] |0\rangle.
\] (40)
Note that both the quark field operators \( q \) in \( L_{\text{ext}} \) and the ground state \(|0\rangle \) refer to the chiral limit, indicated by the subscript 0 in Eq. (40). A particular Green function is then obtained through a functional derivative with respect to the external fields. As an example, suppose we are interested in the scalar \( u \)-quark condensate in the chiral limit, \( \langle 0|\bar{u}u|0\rangle_0 \). We express \( \bar{u}u \) as

\[
\bar{u}u = \frac{1}{2} \sqrt{2} \bar{q} \lambda_0 q + \frac{1}{2} \bar{q} \lambda_3 q + \frac{1}{2} \sqrt{3} \bar{q} \lambda_8 q
\]

and obtain

\[
\langle 0|\bar{u}(x)u(x)|0\rangle_0 = \frac{i}{2} \left[ \frac{2}{\sqrt{3}} \delta_{s_0(x)} + \frac{\delta}{\delta s_3(x) + \frac{1}{\sqrt{3}} \delta s_8(x)} \right] \exp(iZ[v, a, s, p]) \bigg|_{v=a=s=p=0}.
\]

From the generating functional, we can even obtain Green functions of the “real world,” where the quark fields and the ground state are those with finite quark masses. For example, the two-point function of two axial-vector currents of the “real world,” i.e., for \( s = \text{diag}(m_u, m_d, m_s) \), and the “true vacuum” \(|0\rangle \), is given by

\[
\langle 0|T[A^\mu_\alpha(x)A^\nu_\beta(0)]|0\rangle = (-i)^2 \frac{\delta^2}{\delta a^\mu_\alpha(x) \delta a^\nu_\beta(0)} \exp(iZ[v, a, s, p]) \bigg|_{v=a=p=0, s=\text{diag}(m_u, m_d, m_s)}.
\]

Note that the left-hand side involves the quark fields and the ground state of the “real world,” whereas the right-hand side is the generating functional defined in terms of the quark fields and the ground state of the chiral limit. The actual value of the generating functional for a given configuration of external fields \( v, a, s, \) and \( p \) reflects the dynamics generated by the QCD Lagrangian.

The (infinite) set of all chiral Ward identities resides in an invariance of the generating functional under a local transformation of the external fields [Gasser and Leutwyler, 1984], [Leutwyler, 1994]. The use of local transformations allows one to also consider divergences of Green functions. We require \( L \) of Eq. (38) to be a Hermitian Lorentz scalar, to be even under \( P, C, \) and \( T \), and to be invariant under local chiral transformations. In fact, it is sufficient to consider \( P \) and \( C \), only, because \( T \) is then automatically incorporated owing to the CPT theorem.

Under parity, the quark fields transform as

\[
q_f(t, \vec{x}) \overset{P}{\rightarrow} \gamma_0 q_f(t, -\vec{x}),
\]

and the requirement of parity conservation,

\[
L(t, \vec{x}) \overset{P}{\rightarrow} L(t, -\vec{x}),
\]

leads, using the results of Table IV to the following constraints for the external fields,

\[
v^\mu \overset{P}{\rightarrow} v_\mu, \quad v^\mu_\alpha \overset{P}{\rightarrow} v^\nu_\alpha, \quad a^\mu \overset{P}{\rightarrow} -a_\mu, \quad s \overset{P}{\rightarrow} s, \quad p \overset{P}{\rightarrow} -p.
\]

In Eq. (44) it is understood that the arguments change from \((t, \vec{x})\) to \((t, -\vec{x})\).

Similarly, under charge conjugation the quark fields transform as

\[
q_{f,\alpha} \overset{C}{\rightarrow} C_{\alpha\beta} q_{f,\beta}, \quad \bar{q}_{f,\alpha} \overset{C}{\rightarrow} -q_{f,\beta} C_{\beta\alpha}^{-1},
\]

where the subscripts \( \alpha \) and \( \beta \) are Dirac spinor indices,

\[
C = i\gamma^2 \gamma^0 = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]
Table 4: Transformation properties of the Dirac matrices $\Gamma$ under parity.

| $\Gamma$ | 1 | $\gamma^\mu$ | $\sigma^{\mu\nu}$ | $\gamma_5$ | $\gamma^\mu\gamma_5$ |
|---------|---|--------------|-------------------|--------|------------------|
| $\gamma_0\Gamma\gamma_0$ | 1 | $\gamma_\mu$ | $\sigma_{\mu\nu}$ | $-\gamma_5$ | $-\gamma_\mu\gamma_5$ |

Table 5: Transformation properties of the Dirac matrices $\Gamma$ under charge conjugation.

is the usual charge conjugation matrix, and $f$ refers to flavor. Taking Fermi statistics into account, one obtains

$$\bar{q}\Gamma F q = -\bar{q} CT^T C F^T q,$$

where $F$ denotes a matrix in flavor space. In combination with Table 5 it is straightforward to show that invariance of $L_{\text{ext}}$ under charge conjugation requires the transformation properties

$$v_\mu \overset{C}{\rightarrow} -v_\mu^T, \quad v_\mu^{(s)} \overset{C}{\rightarrow} -v_\mu^{(s)T}, \quad a_\mu \overset{C}{\rightarrow} a_\mu^T, \quad s, p \overset{C}{\rightarrow} s^T, p^T,$$

where the transposition refers to the flavor space.

Finally, we need to discuss the requirements to be met by the external fields under local SU(3)$_L \times$ SU(3)$_R \times$ U(1)$_V$ transformations. In a first step, we write Eq. (39) in terms of the left- and right-handed quark fields. Using the projection operators of Eq. (11) the Lagrangian of Eq. (39) reads

$$L = L^0_{\text{QCD}} + \bar{q}_L \gamma^\mu \left( l_\mu + \frac{1}{3} v_\mu^{(s)} \right) q_L + \bar{q}_R \gamma^\mu \left( r_\mu + \frac{1}{3} v_\mu^{(s)} \right) q_R - \bar{q}_R (s + ip) q_L - \bar{q}_L (s - ip) q_R.$$  (47)

Equation (47) remains invariant under local transformations

$$q_R \mapsto \exp \left( -i \frac{\Theta(x)}{3} \right) V_R(x) q_R,$$

$$q_L \mapsto \exp \left( -i \frac{\Theta(x)}{3} \right) V_L(x) q_L,$$  (48)

where $V_R(x)$ and $V_L(x)$ are independent space-time-dependent SU(3) matrices, provided the external fields are subject to the transformations

$$r_\mu \mapsto V_R r_\mu V_R^\dagger - i \partial_\mu V_R V_R^\dagger,$$

$$l_\mu \mapsto V_L l_\mu V_L^\dagger - i \partial_\mu V_L V_L^\dagger,$$

$$v_\mu^{(s)} \mapsto v_\mu^{(s)} - \partial_\mu \Theta,$$

$$s + ip \mapsto V_R (s + ip) V_R^\dagger,$$

$$s - ip \mapsto V_L (s - ip) V_L^\dagger.$$  (49)

The derivative terms in Eq. (49) serve the same purpose as in the construction of gauge theories, i.e., they cancel analogous terms originating from the kinetic part of the quark Lagrangian. Note that the external currents are coupled with an “opposite” sign in comparison with our convention for gauge theories.
There is another, yet, more practical aspect of the local invariance, namely: such a procedure allows one to also discuss a coupling to external gauge fields in the transition to the effective theory to be discussed later. For example, a coupling of the electromagnetic field to point-like fundamental particles results from gauging a U(1) symmetry. Here, the corresponding U(1) group is to be understood as a subgroup of a local SU(3)$_L \times$ SU(3)$_R$. Another example deals with the interaction of the light quarks with the charged and neutral gauge bosons of the weak interactions.

Let us consider both examples explicitly. The coupling of quarks to an external electromagnetic field $\mathbf{A}_\mu$ is given by

$$r_\mu = l_\mu = -eQ \mathbf{A}_\mu,$$

where $Q = \text{diag}(2/3, -1/3, -1/3)$ is the quark charge matrix and $e > 0$ the elementary charge:

$$\mathcal{L}_{\text{ext}} = -eA_\mu(\bar{q}LQ\gamma^\mu q_L + \bar{q}_RQ\gamma^\mu q_R) = -eA_\mu\bar{q}Q\gamma^\mu q = -eA_\mu\left(\frac{2}{3}\bar{u}\gamma^\mu u - \frac{1}{3}\bar{d}\gamma^\mu d - \frac{1}{3}\bar{s}\gamma^\mu s\right).$$

On the other hand, if one considers only the two-flavor version of QCD one has to insert for the external fields

$$r_\mu = l_\mu = -e\tau_3^\mu A_\mu, \quad \nu_\mu = -\frac{e}{2}A_\mu.$$

In the description of semi-leptonic interactions such as $\pi^- \to \mu^-\bar{\nu}_\mu$, $\pi^- \to \pi^0e^-\bar{\nu}_e$, or neutron decay $n \to p e^-\bar{\nu}_e$, one needs the interaction of quarks with the massive charged weak bosons $\mathcal{W}_\mu^\pm = (\mathcal{W}_{1\mu} \mp i\mathcal{W}_{2\mu})/\sqrt{2}$,

$$r_\mu = 0, \quad l_\mu = -\frac{g}{\sqrt{2}}(\mathcal{W}_\mu^+T_+ + \text{H.c.}),$$

where H.c. refers to the Hermitian conjugate and

$$T_+ = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  

Here, $V_{ij}$ denote the elements of the Cabibbo-Kobayashi-Maskawa quark-mixing matrix describing the transformation between the mass eigenstates of QCD and the weak eigenstates [Amsler et al., 2008].

$$|V_{ud}| = 0.97418 \pm 0.00027, \quad |V_{us}| = 0.2255 \pm 0.0019.$$

At lowest order in perturbation theory, the Fermi constant is related to the gauge coupling $g$ and the $W$ mass as

$$G_F = \sqrt{2} \frac{g^2}{8M_W^2} = 1.16637(1) \times 10^{-5} \text{GeV}^{-2}.$$

Making use of

$$\bar{q}_L\gamma^\mu\mathcal{W}_\mu^+T_+q_L = \frac{1}{2}\mathcal{W}_\mu^+[V_{ud}\bar{u}\gamma^\mu(1 - \gamma_5)d + V_{us}\bar{u}\gamma^\mu(1 - \gamma_5)s],$$

we see that inserting Eq. (52) into Eq. (47) leads to the standard charged-current weak interaction in the light-quark sector,

$$\mathcal{L}_{\text{ext}} = -\frac{g}{2\sqrt{2}} \left(\mathcal{W}_\mu^+[V_{ud}\bar{u}\gamma^\mu(1 - \gamma_5)d + V_{us}\bar{u}\gamma^\mu(1 - \gamma_5)s] + \text{H.c.} \right).$$

The situation is slightly different for the neutral weak interaction. Here, the three-flavor version requires a coupling to the singlet axial-vector current which, because of the anomaly of Eq. (33), we
have dropped from our discussion. On the other hand, in the two-flavor version the axial-vector current part is traceless and we have

$$ r_\mu = e \tan(\theta_W) \frac{\tau_3}{2} \mathcal{Z}_\mu, $$

$$ l_\mu = -\frac{g}{\cos(\theta_W)} \frac{\tau_3}{2} \mathcal{Z}_\mu + e \tan(\theta_W) \frac{\tau_3}{2} \mathcal{Z}_\mu, $$

$$ v^{(s)}_\mu = \frac{e \tan(\theta_W)}{2} \mathcal{Z}_\mu, $$

(54)

where $\theta_W$ is the weak angle. With these external fields, we obtain the standard weak neutral-current interaction

$$ \mathcal{L}_{\text{ext}} = -\frac{g}{2 \cos(\theta_W)} \mathcal{Z}_\mu \left( \bar{u} \gamma^\mu \left\{ \frac{1}{2} - \frac{4}{3} \sin^2(\theta_W) \right\} 1 - \frac{1}{2} \gamma_5 \right) u + \bar{d} \gamma^\mu \left\{ \left[ -\frac{1}{2} + \frac{2}{3} \sin^2(\theta_W) \right] 1 + \frac{1}{2} \gamma_5 \right\} d, $$

where we made use of $e = g \sin(\theta_W)$.

### 2.1.7 PCAC in the presence of an external electromagnetic field

Finally, the technique of coupling the QCD Lagrangian to external fields also allows us to determine the current divergences for rigid external fields, i.e., fields which are not simultaneously transformed. For the sake of simplicity we restrict ourselves to the two-flavor sector. (The generalization to the three-flavor case is straightforward.)

Consider a global chiral transformation only and assume that the external fields are not simultaneously transformed. In this case the divergences of the currents read [Fuchs and Scherer, 2003]

$$ \partial_\mu V^\mu_i = i \bar{q} \gamma^\mu \left[ \frac{\tau_i}{2}, v_\mu \right] q + i \bar{q} \gamma^\mu \gamma_5 \left[ \frac{\tau_i}{2}, a_\mu \right] q - i \bar{q} \left[ \frac{\tau_i}{2}, s \right] q - \bar{q} \gamma_5 \left[ \frac{\tau_i}{2}, p \right] q, $$

(55)

$$ \partial_\mu A^\mu_i = i \bar{q} \gamma^\mu \gamma_5 \left[ \frac{\tau_i}{2}, v_\mu \right] q + i \bar{q} \gamma^\mu \gamma_5 \left[ \frac{\tau_i}{2}, a_\mu \right] q + i \bar{q} \gamma_5 \left[ \frac{\tau_i}{2}, s \right] q + \bar{q} \left[ \frac{\tau_i}{2}, p \right] q. $$

(56)

As an example, let us consider the QCD Lagrangian for a finite light quark mass $\hat{m} = m_u = m_d$ in combination with a coupling to an external electromagnetic field $A_\mu$ [see Eq. (51), $a_\mu = 0 = p$]. The expressions for the divergence of the vector and axial-vector currents, respectively, are given by [Fuchs and Scherer, 2003]

$$ \partial_\mu V^\mu_i = -\epsilon_{3ij} e A_\mu \bar{q} \gamma^\mu \frac{\tau_j}{2} q = -\epsilon_{3ij} e A_\mu V^\mu_j, $$

(57)

$$ \partial_\mu A^\mu_i = -e A_\mu \epsilon_{3ij} \bar{q} \gamma^\mu \gamma_5 \frac{\tau_j}{2} q + 2 \hat{m} \bar{q} \gamma_5 \frac{\tau_j}{2} q = -e A_\mu \epsilon_{3ij} A^\mu_j + \hat{m} P_i, $$

(58)

with the isovector pseudoscalar density $P_i = i \bar{q} \gamma_5 \tau_i q$. In fact, Eq. (58) is incomplete, because the third component of the axial-vector current, $A^\mu_3$, has an anomaly which is related to the decay $\pi^0 \rightarrow \gamma \gamma$. The full equation reads

$$ \partial_\mu A^\mu_i = \hat{m} P_i - e A_\mu \epsilon_{3ij} A^\mu_j + \delta_{i3} \frac{e^2}{32 \pi^2} \epsilon_{\mu \nu \rho \sigma} F^{\mu \nu} F^{\rho \sigma}, \quad \epsilon_{0123} = 1, $$

(59)

where $F^{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor.

We emphasize the formal similarity of Eq. (58) to the (pre-QCD) PCAC (Partially Conserved Axial-Vector Current) relation obtained by Adler [Adler, 1965] through the inclusion of the electromagnetic interactions with minimal electromagnetic coupling. Since in QCD the quarks are taken as truly elementary, their interaction with an (external) electromagnetic field is of such a minimal type. In Adler’s version, the right-hand side of Eq. (59) contains a renormalized field operator creating and destroying pions instead of $\hat{m} P_i$. From a modern point of view, the combination $\hat{m} P_i / (M_F^2 F_u)$ serves as an interpolating pion field. Furthermore, the anomaly term is not yet present in Ref. [Adler, 1965].

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2.2 Spontaneous symmetry breaking and Goldstone theorem

2.2.1 Linear sigma model

Spontaneous symmetry breaking occurs if the ground state has a lower symmetry than the Hamiltonian. For example, in the linear sigma model [Schwinger, 1957], [Gell-Mann and Lévy, 1960], the Lagrangian is constructed in terms of the O(4) multiplet \((\sigma, \pi_1, \pi_2, \pi_3)\),

\[
\mathcal{L} = \frac{1}{2} \left( \partial_\mu \sigma \partial^\mu \sigma + \partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} \right) - \frac{m^2}{2} (\sigma^2 + \vec{\pi}^2) - \lambda \left(\sigma^2 + \vec{\pi}^2\right)^2,
\]

(60)

where \(\lambda > 0\). Under parity we assume \(\sigma(t, \vec{x}) \mapsto \sigma(t, -\vec{x})\) and \(\pi_i(t, \vec{x}) \mapsto -\pi_i(t, -\vec{x})\). The Lagrangian is invariant under the infinitesimal transformations

\[
\begin{pmatrix}
\sigma \\
\pi_1 \\
\pi_2 \\
\pi_3
\end{pmatrix}
\mapsto
\begin{pmatrix}
\sigma' \\
\pi'_1 \\
\pi'_2 \\
\pi'_3
\end{pmatrix}
= \begin{pmatrix} 1 & -i \sum_{a=1}^{6} \epsilon_a T_a \end{pmatrix}
\begin{pmatrix}
\sigma \\
\pi_1 \\
\pi_2 \\
\pi_3
\end{pmatrix},
\]

where the six \(4 \times 4\) matrices are given by

\[
T_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -i \\
0 & 0 & i & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & i \\
0 & 0 & 0 & 0 \\
0 & -i & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & -i & 0 \\
0 & i & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
T_4 = \begin{pmatrix} 0 & -i & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}, \quad T_5 = \begin{pmatrix} 0 & 0 & -i & 0 \\
0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}, \quad T_6 = \begin{pmatrix} 0 & 0 & 0 & -i \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 \end{pmatrix}.
\]

The linear combinations \(R_i = (T_i + T_{i+3})/2\) and \(L_i = (T_i - T_{i+3})/2, i = 1, 2, 3\), satisfy the commutation relations corresponding to an SU(2) \(\times\) SU(2) Lie group. The multiplet \((\sigma, \pi_1, \pi_2, \pi_3)\) transforms according to the \((\frac{1}{2}, \frac{1}{2})\) representation. By choosing \(m^2 < 0\), the symmetry is realized in the Nambu-Goldstone mode [Nambu, 1960], [Goldstone, 1961]. Let us assume that the ground state is characterized by the vacuum expectation values

\[
\langle \sigma \rangle = v \equiv -\sqrt{-\frac{m^2}{\lambda}}, \quad \langle \pi_i \rangle = 0.
\]

(61)

Introducing \(\sigma = v + \sigma'\), the Lagrangian reads

\[
\mathcal{L} = \frac{1}{2} \left( \partial_\mu \sigma' \partial^\mu \sigma' + \partial_\mu \vec{\pi}' \cdot \partial^\mu \vec{\pi}' \right) + \frac{1}{2} (-2m^2) \sigma'^2 + \lambda v \sigma' \left(\pi_1^2 + \pi_2^2 + \pi_3^2 + \sigma'^2\right) + \frac{\lambda}{4} \left(\pi_1^2 + \pi_2^2 + \pi_3^2 + \sigma'^2\right)^2 - \frac{\lambda}{4} v^4.
\]

(62)

The ground-state configuration is no longer invariant under the full group O(4). While the generators \(T_1, T_2,\) and \(T_3\) annihilate the ground state of Eq. (61), the generators \(T_4, T_5,\) and \(T_6\) do not. The model-independent feature of the above example is given by the fact that for each of the three generators \(T_4, T_5,\) and \(T_6\) which do not annihilate the ground state one obtains a massless Goldstone boson. This is why Eq. (62) contains no mass terms for the pions. In fact, the number of Goldstone bosons is determined by the structure of the symmetry groups [Goldstone et al., 1962]. Let \(G\) denote the symmetry group of the Lagrangian with \(n_G\) generators and the subgroup \(H\) the symmetry group of the ground state with \(n_H\) generators. For each generator which does not annihilate the vacuum one obtains a massless Goldstone boson, i.e., the total number of Goldstone bosons equals \(n_G - n_H\).

The Lagrangians used in motivating the phenomenon of a spontaneous symmetry breakdown are typically constructed in such a fashion that the degeneracy of the ground states is built into the potential
at the classical level (the prototype being the “Mexican hat” potential). As in the above case, it is then argued that an elementary Hermitian field of a multiplet transforming non-trivially under the symmetry group \( G \) acquires a vacuum expectation value signaling a spontaneous symmetry breakdown. However, there also exist theories such as QCD where one cannot infer from inspection of the Lagrangian whether the theory exhibits spontaneous symmetry breaking. Rather, the criterion for spontaneous symmetry breaking is a non-vanishing vacuum expectation value of some Hermitian operator, not an elementary field, which emerges through the dynamics of the underlying theory. In particular, we will see that the quantities developing a vacuum expectation value may also be local Hermitian operators composed of more fundamental degrees of freedom of the theory.

While the model of Eq. (60) is constructed to illustrate the concept of a spontaneous symmetry breaking, it is not fully understood theoretically why QCD should exhibit this phenomenon. We will first motivate why experimental input, the hadron spectrum of the “real” world, indicates that spontaneous symmetry breaking happens in QCD. Secondly, we will show that a non-vanishing singlet scalar quark condensate is a sufficient condition for a spontaneous symmetry breaking.

### 2.2.2 The hadron spectrum

We saw in Section 2.1.3 that the QCD Lagrangian possesses an SU(3)\(_L\) × SU(3)\(_R\) × U(1)\(_V\) symmetry in the chiral limit in which the light quark masses vanish. From symmetry considerations involving the Hamiltonian \( H^0_{\text{QCD}} \) only, one would naively expect that hadrons organize themselves into approximately degenerate multiplets fitting the dimensionalities of irreducible representations of the group SU(3)\(_L\) × SU(3)\(_R\) × U(1)\(_V\). The U(1)\(_V\) symmetry results in baryon number conservation and leads to a classification of hadrons into mesons (\( B = 0 \)) and baryons (\( B = 1 \)). The linear combinations \( Q_{aV} = Q_{aR} + Q_{aL} \) and \( Q_{aA} = Q_{aR} - Q_{aL} \) of the left- and right-handed charge operators commute with \( H^0_{\text{QCD}} \), have opposite parity, and thus for states of positive parity one would expect the existence of degenerate states of negative parity (parity doubling) which can be seen as follows.

Let \( |\alpha, +\rangle \) denote an eigenstate of \( H^0_{\text{QCD}} \) and parity with eigenvalues \( E_\alpha \) and +1, respectively,

\[
H^0_{\text{QCD}} |\alpha, +\rangle = E_\alpha |\alpha, +\rangle,
\]

\[
P |\alpha, +\rangle = |\alpha, +\rangle,
\]

such as, e.g., a member of the ground-state baryon octet (in the chiral limit). Defining \( |\phi_{aa}\rangle = Q_{aA} |\alpha, +\rangle \), because of \( [H^0_{\text{QCD}}, Q_{aA}] = 0 \), we have

\[
H^0_{\text{QCD}} |\phi_{aa}\rangle = H^0_{\text{QCD}} Q_{aA} |\alpha, +\rangle = Q_{aA} H^0_{\text{QCD}} |\alpha, +\rangle = E_\alpha Q_{aA} |\alpha, +\rangle = E_\alpha |\phi_{aa}\rangle,
\]

\[
P |\phi_{aa}\rangle = P Q_{aA} P^{-1} |\alpha, +\rangle = -Q_{aA} |\alpha, +\rangle = -|\phi_{aa}\rangle.
\]

The state \( |\phi_{aa}\rangle \) can be expanded in terms of the members of a multiplet with negative parity,

\[
|\phi_{aa}\rangle = Q_{aA} |\beta, -\rangle = \langle \beta, - | Q_{aA} |\alpha, +\rangle = t_{a\beta\alpha} |\beta, -\rangle.
\]

However, the low-energy spectrum of baryons does not contain a degenerate baryon octet of negative parity. Naturally the question arises whether the above chain of arguments is incomplete. Indeed, we have tacitly assumed that the ground state of QCD is annihilated by the generators \( Q_{aA} \). Let \( b^\dag_{\alpha+} \) denote an operator creating quanta with the quantum numbers of the state \( |\alpha, +\rangle \). Similarly, let \( b^\dagger_{\alpha-} \) create degenerate quanta of opposite parity. Expanding

\[
[Q_{aA}, b^\dagger_{\alpha+}] = b^\dagger_{\beta-} t_{a\beta\alpha},
\]

the usual chain of arguments then works as

\[
Q_{aA} |\alpha, +\rangle = Q_{aA} b^\dagger_{\alpha+} |0\rangle = \left( [Q_{aA}, b^\dagger_{\alpha+}] + b^\dagger_{\alpha+} Q_{aA} \right) |0\rangle = t_{a\beta\alpha} b^\dagger_{\beta-} |0\rangle,
\]

(63)
However, if the ground state is not annihilated by $Q_{aA}$, the reasoning of Eq. (63) does no longer apply. In that case the ground state is not invariant under the full symmetry group of the Lagrangian resulting in a spontaneous symmetry breaking. In other words, the non-existence of degenerate multiplets of opposite parity points to the fact that $SU(3)_V$ instead of $SU(3)_L \times SU(3)_R$ is approximately realized as a symmetry of the hadrons. Furthermore, the octet of the pseudoscalar mesons is special in the sense that the masses of its members are small in comparison with the corresponding $1^-$ vector mesons. They are the candidates for the Goldstone bosons of a spontaneous symmetry breaking.

According to the Coleman theorem [Coleman, 1966], the symmetry of the ground state determines the symmetry of the spectrum, i.e.

$$Q_{aV}|0\rangle = Q_{V}|0\rangle = 0$$

implies $SU(3)_V$ multiplets which can be classified according to their baryon number. In the reverse conclusion, the symmetry of the ground state can be inferred from the symmetry of the spectrum. Figures 1 and 2 show the octets of the lowest-lying pseudoscalar-meson states and the lowest-lying baryon states of spin-parity $\frac{1}{2}^+$, respectively.

The axial charges satisfy the commutation relations

$$[Q_{aA}, Q_{bA}] = i f_{abc} Q_{cV},$$

$$[Q_{aV}, Q_{bA}] = i f_{abc} Q_{cA}. $$

Since the parity doubling is not observed for the low-lying states, one assumes that the $Q_{aA}$ do not annihilate the ground state,

$$Q_{aA}|0\rangle \neq 0,$$

i.e., the ground state of QCD is not invariant under "axial" transformations. In the present case, $G = SU(3)_L \times SU(3)_R$ with $n_G = 16$ and $H = SU(3)_V$ with $n_H = 8$ and we expect eight Goldstone bosons. According to the Goldstone theorem [Goldstone, 1961], [Goldstone et al., 1962], to each axial generator $Q_{aA}$, which does not annihilate the ground state, corresponds a massless Goldstone boson field $\phi_a(x)$ with spin 0, whose symmetry properties are tightly connected to the generator in question. The Goldstone bosons have the same transformation behavior under parity,

$$\phi_a(t, \vec{x}) \stackrel{P}{\rightarrow} -\phi_a(t, -\vec{x}),$$
i.e., they are pseudoscalars, and transform under the subgroup $H = SU(3)_V$, which leaves the vacuum invariant, as an octet [see Eq. (66)]:

$$[Q_aV, \phi_b(x)] = i f_{abc} \phi_c(x).$$

(69)

### 2.2.3 The scalar singlet quark condensate

In the following, we will show that a non-vanishing scalar singlet quark condensate in the chiral limit is a sufficient (but not a necessary) condition for a spontaneous symmetry breaking in QCD. In this section all physical quantities such as the ground state, the quark operators etc. are considered in the chiral limit.

Let us first recall the definition of the nine scalar and pseudoscalar quark densities:

$$S_a(y) = \bar{q}(y)\lambda_a q(y), \quad a = 0, \ldots, 8,$$

(70)

$$P_a(y) = i \bar{q}(y)\gamma_5\lambda_a q(y), \quad a = 0, \ldots, 8,$$

(71)

where $\lambda_0 = \sqrt{2/3} \mathbb{1}$. We need the equal-time commutation relation of two quark operators of the form $A_i(x) = q^\dagger(x)\hat{A}_i q(x)$, where $\hat{A}_i$ symbolically denotes Dirac and flavor matrices and a summation over color indices is implied:

$$[A_1(t, \vec{x}), A_2(t, \vec{y})] = \delta^3(\vec{x} - \vec{y})q^\dagger(x)[\hat{A}_1, \hat{A}_2]q(x).$$

(72)

With the definition

$$Q_{aV}(t) = \int d^3x q^\dagger(t, \vec{x})\frac{\lambda_a}{2} q(t, \vec{x}),$$

and using

$$[\frac{\lambda_a}{2}, \gamma_0\lambda_b] = 0, \quad [\frac{\lambda_a}{2}, \gamma_0\lambda_b] = \gamma_0 i f_{abc} \lambda_c,$$

we see, after integration of Eq. (72) over $\vec{x}$, that the scalar quark densities of Eq. (70) transform under $SU(3)_V$ as a singlet and as an octet, respectively,

$$[Q_{aV}(t), S_0(y)] = 0, \quad a = 1, \ldots, 8,$$

(73)
\[ [Q_aV(t), S_b(y)] = i \sum_{c=1}^{8} f_{abc} S_c(y), \quad a, b = 1, \ldots, 8, \quad (74) \]

with analogous results for the pseudoscalar quark densities. Using the relation

\[ \sum_{a,b=1}^{8} f_{abc} f_{abd} = 3 \delta_{cd} \quad (75) \]

for the structure constants of SU(3), we re-express the octet components of the scalar quark densities as

\[ S_a(y) = -\frac{i}{3} \sum_{b,c=1}^{8} f_{abc} [Q_bV(t), S_c(y)]. \quad (76) \]

In the chiral limit the ground state is necessarily invariant under SU(3) \[ V \], i.e., \( Q_aV |0\rangle = 0 \), and we obtain from Eq. (76)

\[ \langle 0|S_a(y)|0\rangle = \langle 0|S_a(0)|0\rangle \equiv \langle S_a \rangle = 0, \quad a = 1, \ldots, 8, \quad (77) \]

where we made use of translational invariance of the ground state. As an intermediate result we see that the octet components of the scalar quark condensate must vanish in the chiral limit. From Eq. (77), we obtain for \( a = 3 \)

\[ \langle \bar{u}u\rangle - \langle \bar{d}d\rangle = 0, \]

e.g. \( \langle \bar{u}u\rangle = \langle \bar{d}d\rangle \) and for \( a = 8 \)

\[ \langle \bar{u}u\rangle + \langle \bar{d}d\rangle - 2 \langle \bar{s}s\rangle = 0, \]

e.g. \( \langle \bar{u}u\rangle = \langle \bar{d}d\rangle = \langle \bar{s}s\rangle \).

Because of Eq. (73) a similar argument cannot be used for the singlet condensate, and if we assume a non-vanishing singlet scalar quark condensate in the chiral limit, we find using \( \langle \bar{u}u\rangle = \langle \bar{d}d\rangle = \langle \bar{s}s\rangle \):

\[ 0 \neq \langle \bar{q}q \rangle = 3 \langle \bar{u}u + \bar{d}d + \bar{s}s \rangle = 3 \langle \bar{u}u \rangle = 3 \langle \bar{d}d \rangle = 3 \langle \bar{s}s \rangle. \quad (78) \]

Finally, we make use of (no summation implied!)

\[ (i)^2[\gamma_5 \frac{\lambda_a}{2}, \gamma_0 \gamma_5 \lambda_a] = \lambda_a^2 \gamma_0 \]

in combination with

\[ \lambda_1^2 = \lambda_2^2 = \lambda_3^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4^2 = \lambda_5^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \lambda_6^2 = \lambda_7^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \lambda_8^2 = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \]

to obtain

\[ i[Q_aA(t), P_a(y)] = \begin{cases} \bar{u}u + \bar{d}d, & a = 1, 2, 3 \\ \bar{u}u + \bar{s}s, & a = 4, 5 \\ \bar{d}d + \bar{s}s, & a = 6, 7 \\ \frac{1}{3} (\bar{u}u + \bar{d}d + 4\bar{s}s), & a = 8 \end{cases} \quad (79) \]

where we have suppressed the \( y \) dependence on the right-hand side. We evaluate Eq. (79) for a ground state which is invariant under SU(3) \[ V \], assuming a non-vanishing singlet scalar quark condensate,

\[ \langle 0|i[Q_aA(t), P_a(y)]|0\rangle = \frac{2}{3} \langle \bar{q}q \rangle, \quad a = 1, \ldots, 8, \quad (80) \]
where, because of translational invariance, the right-hand side is independent of \( y \). Inserting a complete set of states into the commutator of Eq. (80) yields that both the pseudoscalar density \( P_a(y) \) as well as the axial charge operators \( Q_{aA} \) must have a non-vanishing matrix element between the vacuum and massless one-particle states \( |\phi_b\rangle \). In particular, because of Lorentz covariance, the matrix element of the axial-vector current operator between the vacuum and these massless states, appropriately normalized, can be written as

\[
\langle 0 | A^\mu_a(0) | \phi_b(p) \rangle = ip^\mu F_0 \delta_{ab}, \tag{81}
\]

where \( F_0 \approx 93 \text{ MeV} \) denotes the “decay” constant of the Goldstone bosons in the chiral limit. From Eq. (81) we see that a non-zero value of \( F_0 \) is a necessary and sufficient criterion for spontaneous chiral symmetry breaking. On the other hand, because of Eq. (80) a non-vanishing scalar quark condensate \( \langle \bar{q}q \rangle \) is a sufficient (but not a necessary) condition for a spontaneous symmetry breakdown in QCD.

### 3 Mesonic chiral perturbation theory

Our goal is the construction of the most general theory describing the dynamics of the Goldstone bosons associated with the spontaneous symmetry breakdown in QCD. In the chiral limit, we want the effective Lagrangian to be invariant under \( G = SU(3)_L \times SU(3)_R \times U(1)_V \). It should contain exactly eight pseudoscalar degrees of freedom transforming as an octet under the subgroup \( H = SU(3)_V \). Moreover, taking account of spontaneous symmetry breaking, the ground state should only be invariant under \( SU(3)_V \times U(1)_V \).

#### 3.1 Transformation properties of the Goldstone bosons

The purpose of this section is to discuss the transformation properties of the field variables describing the Goldstone bosons [Weinberg, 1968], [Coleman et al., 1969], [Callan et al., 1969]. We will need the concept of a *nonlinear realization* of a group in addition to a *representation* of a group which one usually encounters in Physics. We will first discuss a few general group-theoretical properties before specializing to QCD.

##### 3.1.1 General considerations

Let us consider a physical system described by a Lagrangian which is invariant under a compact Lie group \( G \). We assume the ground state of the system to be invariant under only a subgroup \( H \) of \( G \), giving rise to \( n = n_G - n_H \) Goldstone bosons. Each of these Goldstone bosons will be described by an independent field \( \phi_i \) which is a smooth real function on Minkowski space \( M^4 \). These fields are collected in an \( n \)-component vector \( \Phi = (\phi_1, \cdots, \phi_n) \), defining the vector space \( M_1 \). Our aim is to find a mapping \( \varphi \) which uniquely associates with each pair \((g, \Phi) \in G \times M_1\) an element \( \varphi(g, \Phi) \in M_1 \) with the following properties:

\[
\varphi(e, \Phi) = \Phi \ \forall \ \Phi \in M_1, \ e \text{ identity of } G, \tag{82}
\]

\[
\varphi(g_1, \varphi(g_2, \Phi)) = \varphi(g_1g_2, \Phi) \ \forall \ g_1, g_2 \in G, \ \forall \Phi \in M_1. \tag{83}
\]

Such a mapping defines an *operation* of the group \( G \) on \( M_1 \). The construction proceeds as follows [Leutwyler, 1992]. Let \( \Phi = 0 \) denote the “origin” of \( M_1 \) which, in a theory containing Goldstone bosons only, loosely speaking corresponds to the ground state configuration. Since the ground state is supposed to be invariant under the subgroup \( H \) we require the mapping \( \varphi \) to be such that all elements \( h \in H \) map the origin onto itself. In this context the subgroup \( H \) is also known as the little group of \( \Phi = 0 \).

We will establish a connection between the Goldstone boson fields and the set of all left cosets \( \{gH|g \in G\} \) which is also referred to as the quotient \( G/H \). For a subgroup \( H \) of \( G \) the set \( gH = \)

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Goldstone boson fields. Of course, the Goldstone boson fields are not constant vectors in $g$. However, this implies the conclusion is that there exists an isomorphic mapping between the quotient $G/H$ and and the Goldstone boson fields. Of course, the Goldstone boson fields are not constant vectors in $\mathbb{R}^n$ but functions on Minkowski space. This is accomplished by allowing the cosets $gH$ to also depend on $x$.

Now let us discuss the transformation behavior of the Goldstone boson fields under an arbitrary $g \in G$ in terms of the isomorphism established above. To each $\Phi$ corresponds a coset $\tilde{g}H$ with appropriate $\tilde{g}$. Let $f = \tilde{g}h \in \tilde{g}H$ denote a representative of this coset such that

$$\Phi = \varphi(f, 0) = \varphi(\tilde{g}h, 0).$$

Now apply the mapping $\varphi(g)$ to $\Phi$:

$$\varphi(g, \Phi) = \varphi(g, \varphi(\tilde{g}h, 0)) = \varphi(\tilde{g}gh, 0) = \varphi(f', 0) = \Phi', \quad f' \in g(\tilde{g}H).$$

In order to obtain the transformed $\Phi'$ from a given $\Phi$ we simply need to multiply the left coset $\tilde{g}H$ representing $\Phi$ by $g$ in order to obtain the new left coset representing $\Phi'$:

$$\Phi \xrightarrow{g} \Phi'$$

$$\downarrow \quad \uparrow$$

$$\tilde{g}H \xrightarrow{g} \tilde{g}gH$$

This procedure uniquely determines the transformation behavior of the Goldstone bosons up to an appropriate choice of variables parameterizing the elements of the quotient $G/H$.

### 3.1.2 Application to QCD

The symmetry groups relevant to the application in QCD are

$$G = SU(N) \times SU(N) = \{(L, R)|L \in SU(N), R \in SU(N)\} \text{ and } H = \{(V, V)|V \in SU(N)\} \cong SU(N).$$

Let $\tilde{g} = (\tilde{L}, \tilde{R}) \in G$. We characterize the left coset $\tilde{g}H$ through the $SU(N)$ matrix $U = \tilde{R}\tilde{L}^\dagger$ [Balachandran et al., 1991] such that $\tilde{g}H = (1, \tilde{R}\tilde{L}^\dagger)H$. This corresponds to the convention of choosing as the representative of the element which has the unit matrix in its first argument. The transformation behavior of $U$ under $g = (L, R) \in G$ is obtained by multiplication in the left coset:

$$g\tilde{g}H = (L, R\tilde{R}\tilde{L}^\dagger)H = (1, R\tilde{R}\tilde{L}^\dagger L^\dagger)(L, L)H = (1, R(\tilde{R}\tilde{L}^\dagger)L^\dagger)H,$$

i.e.

$$U = \tilde{R}\tilde{L}^\dagger \mapsto U' = R(\tilde{R}\tilde{L}^\dagger)L^\dagger = RUL^\dagger. \quad (84)$$
As mentioned above, we need to introduce an $x$ dependence to account for the fact that we are dealing with fields:

$$U(x) \mapsto RU(x)L^\dagger.$$  

(85)

For the physically relevant cases the corresponding unitary matrices may be parameterized as

$$U(x) = \exp \left( i \frac{\phi(x)}{F_0} \right),$$  

(86)

where, for $N = 2$,

$$\phi = \sum_{i=1}^{3} \phi_i \tau_i \equiv \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}$$  

(87)

and, for $N = 3$,

$$\phi = \sum_{a=1}^{8} \phi_a \lambda_a \equiv \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}} \eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}} \eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}K^0 & -\frac{2}{\sqrt{3}} \eta \end{pmatrix}.$$  

(88)

The origin $\phi(x) = 0$, i.e. $U_0 = 1$, denotes the ground state of the system. Under transformations of the subgroup $H = \{(V,V)|V \in SU(N)\}$ corresponding to rotating both left- and right-handed quark fields in QCD by the same $V$, the ground state remains invariant,

$$U_0 \mapsto VU_0V^\dagger = VV^\dagger = 1 = U_0.$$  

On the other hand, under “axial transformations,” i.e. rotating the left-handed quarks by $A$ and the right-handed quarks by $A^\dagger$, the ground state does not remain invariant,

$$U_0 \mapsto A^\dagger U_0A^\dagger = A^\dagger A^\dagger \neq U_0,$$

which is consistent with the assumed spontaneous symmetry breakdown.

The traceless and Hermitian matrices of Eqs. (87) and (88) contain the Goldstone boson fields. We want to discuss their transformation behavior under the subgroup $H = \{(V,V)|V \in SU(N)\}$. Expanding

$$U = 1 + i \frac{\phi}{F_0} - \frac{\phi^2}{2F_0^2} + \cdots,$$

we immediately see that the transformation behavior of Eq. (85) restricted to the subgroup $H$,

$$1 + i \frac{\phi}{F_0} - \frac{\phi^2}{2F_0^2} + \cdots \mapsto V(1 + i \frac{\phi}{F_0} - \frac{\phi^2}{2F_0^2} + \cdots)V^\dagger = 1 + i \frac{V\phi V^\dagger}{F_0} - \frac{V\phi V^\dagger V\phi V^\dagger}{2F_0^2} + \cdots,$$

implies

$$\phi \mapsto V\phi V^\dagger.$$  

(89)

However, this corresponds exactly to the fact that the Goldstone bosons transform according to the adjoint representation under $SU(3)_V$, i.e. they transform as an octet.

For group elements of $G$ of the form $(A,A^\dagger)$ one may proceed in a completely analogous fashion. However, one finds that the fields $\phi_a$ do not have a simple transformation behavior under these group elements.


### 3.2 Effective Lagrangian and power-counting scheme

The application of effective field theory (EFT) to strong interaction processes has become one of the most important theoretical tools in the low-energy regime. The basic idea consists of writing down the most general possible Lagrangian, including all terms consistent with assumed symmetry principles, and then calculating matrix elements with this Lagrangian within some perturbative scheme [Weinberg, 1979]. A successful application of this program thus requires two main ingredients:

1. A knowledge of the most general effective Lagrangian;

2. an expansion scheme for observables in terms of a consistent power-counting method.

#### 3.2.1 The lowest-order effective Lagrangian

In terms of the SU(3) matrix $U(x)$ of Eqs. (86) and (88) the most general, chirally invariant, effective Lagrangian with the minimal number of derivatives reads

$$\mathcal{L}_{\text{eff}} = \frac{F_0^2}{4} \text{Tr} \left( \partial_\mu U \partial^\mu U^\dagger \right),$$

where $F_0 \approx 93$ MeV is a free parameter which later on will be related to the pion decay $\pi^+ \to \mu^+\nu_\mu$. Because of the trace property $\text{Tr}(AB) = \text{Tr}(BA)$, the Lagrangian is invariant under the global $\text{SU}(3)_L \times \text{SU}(3)_R$ transformation $U \mapsto RUL^\dagger$ of Eq. (85). The global $U(1)_V$ invariance is trivially satisfied, because the Goldstone bosons have baryon number zero, thus transforming as $\phi \mapsto \phi$ under $U(1)_V$ which also implies $U \mapsto U$.

The substitution $\phi_a(t, \vec{x}) \mapsto -\phi_a(t, \vec{x})$ or, equivalently, $U(t, \vec{x}) \mapsto U^\dagger(t, \vec{x})$ provides a simple method of testing, whether an expression is of so-called even or odd intrinsic parity, i.e., even or odd in the number of Goldstone boson fields. For example, the Lagrangian of Eq. (90) is even. Since the Goldstone bosons of QCD are pseudoscalars, the true parity transformation is given by $\phi_a(t, \vec{x}) \mapsto -\phi_a(t, -\vec{x})$ or, equivalently, $U(t, \vec{x}) \mapsto U^\dagger(t, -\vec{x})$.

The purpose of the multiplicative constant $F_0^2/4$ in Eq. (90) is to generate the standard form of the kinetic term $\frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a$, which can be seen by expanding the exponential $U = 1 + i\phi/F_0 + \cdots$, $\partial_\mu U = i\partial_\mu \phi/F_0 + \cdots$, resulting in

$$\mathcal{L}_{\text{eff}} = \frac{F_0^2}{4} \text{Tr} \left[ i\partial_\mu \phi \left( \frac{i\partial^\mu \phi}{F_0} - \frac{\partial^\mu \phi}{F_0} \right) \right] + \cdots = \frac{1}{4} \partial_\mu \phi_a \partial^\mu \phi_a \text{Tr}(\lambda_a \lambda_b) + \cdots = \frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a + \mathcal{L}_{\text{int}},$$

where we made use of $\text{Tr}(\lambda_a \lambda_b) = 2\delta_{ab}$. Since there are no other terms containing only two fields, the eight fields $\phi_a$ indeed describe eight independent massless particles.

There are no other independent, chirally invariant terms containing only two derivatives. A term of the type $\text{Tr}[(\partial_\mu \partial^\mu U)U^\dagger]$ may be re-expressed as

$$\text{Tr}[(\partial_\mu \partial^\mu U)U^\dagger] = \partial_\mu [\text{Tr}(\partial^\mu UU^\dagger)] - \text{Tr}(\partial^\mu U \partial_\mu U^\dagger),$$

i.e., up to a total derivative it is proportional to the Lagrangian of Eq. (90). However, in the present context, total derivatives do not have a dynamical significance, i.e. they leave the equations of motion unchanged and can thus be dropped. The product of two invariant traces is excluded at lowest order, because $\text{Tr}(\partial_\mu UU^\dagger) = 0$.

Let us turn to the vector and axial-vector currents associated with the global $\text{SU}(3)_L \times \text{SU}(3)_R$ symmetry of the effective Lagrangian of Eq. (89). To that end, we parameterize

$$L = 1 - i\epsilon_a^L \frac{\lambda_a}{2}, \quad (91)$$

$$R = 1 - i\epsilon_a^R \frac{\lambda_a}{2}, \quad (92)$$
In order to construct \( J_{aL}^\mu \), set \( \epsilon_a^R = 0 \) and choose \( \epsilon_a^L = \epsilon_a^L(x) \). Using
\[
U^\dagger U = 1 \quad \Rightarrow \quad \partial^\mu (U^\dagger U) = 0 \quad \Rightarrow \quad \partial^\mu U^\dagger U = -U^\dagger \partial^\mu U,
\]
the variation of the Lagrangian can be brought into the form
\[
\delta \mathcal{L}_{\text{eff}} = \frac{F_0^2}{4} i \partial_\mu \epsilon_a^L \text{Tr} \left( \lambda_a \partial^\mu U^\dagger U \right).
\]
Applying the method of Ref. [Gell-Mann and Lévy, 1960], we obtain for the left currents
\[
J_{aL}^\mu = \frac{\partial \delta \mathcal{L}_{\text{eff}}}{\partial \partial_\mu \epsilon_a^L} = i \frac{F_0^2}{4} \text{Tr} \left( \lambda_a \partial^\mu U^\dagger U \right),
\]
and, completely analogously, choosing \( \epsilon_a^L = 0 \) and \( \epsilon_a^R = \epsilon_a^R(x) \), for the right currents
\[
J_{aR}^\mu = \frac{\partial \delta \mathcal{L}_{\text{eff}}}{\partial \partial_\mu \epsilon_a^R} = -i \frac{F_0^2}{4} \text{Tr} \left( \lambda_a U \partial^\mu U \right).
\]
Combining Eqs. (94) and (95) the vector and axial-vector currents read
\[
J_{aV}^\mu = J_{aR}^\mu + J_{aL}^\mu = -i \frac{F_0^2}{4} \text{Tr} \left( \lambda_a \{ U, \partial^\mu U \} \right),
\]
\[
J_{aA}^\mu = J_{aR}^\mu - J_{aL}^\mu = -i \frac{F_0^2}{4} \text{Tr} \left( \lambda_a \{ U, \partial^\mu U \} \right).
\]
Furthermore, because of the symmetry of \( \mathcal{L}_{\text{eff}} \) under SU(3)\(_L\) \( \times \) SU(3)\(_R\), both vector and axial-vector currents are conserved. Using the substitution \( U \leftrightarrow U^\dagger \) and Eq. (93), the vector current densities \( J_{aV}^\mu \) of Eq. (96) contain only terms with an even number of Goldstone bosons and the axial-vector current densities \( J_{aA}^\mu \) of Eq. (97) only terms with an odd number of Goldstone bosons. To find the leading term let us expand Eq. (97) in the fields,
\[
J_{aA}^\mu = -i \frac{F_0^2}{4} \text{Tr} \left( \lambda_a \left\{ 1 + \cdots, -i \frac{\lambda_b \partial^\mu \phi_b}{F_0} + \cdots \right\} \right) = -F_0 \partial^\mu \phi_a + \cdots.
\]
We conclude that the axial-vector current has a non-vanishing matrix element when evaluated between the vacuum and a one-Goldstone boson state:
\[
\langle 0 | J_{aA}^\mu(x) | \phi_b(p) \rangle = \langle 0 | -F_0 \partial^\mu \phi_a(x) | \phi_b(p) \rangle = -F_0 \partial^\mu \exp(-ip \cdot x) \delta_{ab} = ip^\mu F_0 \exp(-ip \cdot x) \delta_{ab}.
\]
Equation (98) is the manifestation of Eq. (81) at lowest order in the effective field theory.

### 3.2.2 Symmetry breaking through the quark masses

Up to now, we have assumed a perfect SU(3)\(_L\) \( \times \) SU(3)\(_R\) symmetry. As has been discussed in Section 2.1.5, the quark-mass term of QCD results in an explicit symmetry breaking.

\[
\mathcal{L}_\mathcal{M} = -\bar{q}_R \mathcal{M} q_L - \bar{q}_L \mathcal{M}^\dagger q_R, \quad \mathcal{M} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}.
\]

In order to incorporate the consequences of Eq. (99) into the effective-Lagrangian framework, one makes use of the following argument [Georgi, 1984]: Although \( \mathcal{M} \) is in reality just a constant matrix and does not transform along with the quark fields, \( \mathcal{L}_\mathcal{M} \) of Eq. (99) would be invariant if \( \mathcal{M} \) transformed as
\[
\mathcal{M} \rightarrow R \mathcal{M} L^\dagger.
\]
One then constructs the most general Lagrangian $\mathcal{L}(U, \mathcal{M})$ which is invariant under Eqs. (85) and (100) and expands this function in powers of $\mathcal{M}$. At lowest order in $\mathcal{M}$ one obtains

$$
\mathcal{L}_{\text{s.b.}} = \frac{F_0^2 B_0}{2} \text{Tr}(\mathcal{M} U^\dagger + U \mathcal{M}^\dagger),
$$

(101)

where the subscript s.b. refers to symmetry breaking. In order to interpret the new parameter $B_0$, let us consider the Hamiltonian density corresponding to the sum of the Lagrangians of Eq. (90) and (101):

$$
\mathcal{H}_{\text{eff}} = \frac{F_0^2}{4} \text{Tr}(\dot{U} \dot{U}^\dagger) + \frac{F_0^2}{4} \text{Tr}(\nabla U \cdot \nabla U^\dagger) - \mathcal{L}_{\text{s.b.}}.
$$

Since the first two terms are always larger or equal to zero, $\mathcal{H}_{\text{eff}}$ is minimized by constant and uniform fields. Using the ansatz

$$
\phi = \phi_0 + \frac{1}{F_0^2} \phi_2 + \frac{1}{F_0^4} \phi_4 + \cdots
$$

for the minimizing field values and organizing the individual terms in powers of $1/F_0^2$, one finds $\phi = 0$ as the classical solution even in the presence of quark-mass terms. Now consider the energy density of the ground state ($U_{\text{min}} = U_0 = \mathbb{1}$),

$$
\langle \mathcal{H}_{\text{eff}} \rangle_{\text{min}} = -F_0^2 B_0 (m_u + m_d + m_s),
$$

(102)

and compare its derivative with respect to (any of) the light quark masses $m_q$ with the corresponding quantity in QCD,

$$
\frac{\partial \langle 0 | \mathcal{H}_{\text{QCD}} | 0 \rangle}{\partial m_q} \bigg|_{m_u = m_d = m_s = 0} = \frac{1}{3} \langle 0 | \bar{q} q | 0 \rangle_0 = \frac{1}{3} \langle \bar{q} q \rangle_0,
$$

where $\langle \bar{q} q \rangle_0$ is the chiral quark condensate of Eq. (78). Within the framework of the lowest-order effective Lagrangian, the constant $B_0$ is thus related to the chiral quark condensate as

$$
3 F_0^2 B_0 = -\langle \bar{q} q \rangle_0.
$$

(103)

Let us add a few remarks.

1. A term $\text{Tr}(\mathcal{M})$ by itself is not invariant.

2. The combination $\text{Tr}(\mathcal{M} U^\dagger - U \mathcal{M}^\dagger)$ has the wrong behavior under parity $\phi(t, \vec{x}) \rightarrow -\phi(t, -\vec{x})$.

3. Because $\mathcal{M} = \mathcal{M}^\dagger$, $\mathcal{L}_{\text{s.b.}}$ contains only terms even in $\phi$.

In order to determine the masses of the Goldstone bosons, we identify the terms of second order in the fields in $\mathcal{L}_{\text{s.b.}},$

$$
\mathcal{L}_{\text{s.b.}} = -\frac{B_0}{2} \text{Tr}(\phi^2 \mathcal{M}) + \cdots
$$

(104)

For the sake of simplicity we consider the isospin-symmetric limit $m_u = m_d = \hat{m}$ so that the $\pi^0\eta$ term vanishes and there is no $\pi^0$-$\eta$ mixing. The masses of the Goldstone bosons, to lowest order in the quark masses, are then given by

$$
M_\pi^2 = 2B_0 \hat{m},
$$

(105)

$$
M_\eta^2 = B_0 (\hat{m} + m_s),
$$

(106)

$$
M_K^2 = \frac{2}{3} B_0 (\hat{m} + 2m_s).
$$

(107)
Table 6: Comparison between the symmetry-breaking patterns of a Heisenberg ferromagnet and QCD.

These results, in combination with Eq. (103), \(B_0 = -\langle \bar{q}q \rangle_0 / (3F_0^2)\), correspond to relations obtained in Ref. [Gell-Mann et al., 1968] and are referred to as the Gell-Mann, Oakes, and Renner relations. Without additional input regarding the numerical value of \(B_0\), Eqs. (105) - (107) do not allow for an extraction of the absolute values of the quark masses \(\hat{m}\) and \(m_s\), because re-scaling \(B_0 \rightarrow \lambda B_0\) in combination with \(m_q \rightarrow m_q / \lambda\) leaves the relations invariant. For the ratio of the quark masses one obtains, using the empirical values \(M_\pi = 135\) MeV, \(M_K = 496\) MeV, and \(M_\eta = 547\) MeV,

\[
\frac{M_K^2}{M_\pi^2} = \frac{\hat{m} + m_s}{2\hat{m}} \Rightarrow \frac{m_s}{\hat{m}} = 25.9,
\]

\[
\frac{M_\eta^2}{M_\pi^2} = \frac{2m_s + \hat{m}}{3\hat{m}} \Rightarrow \frac{m_s}{\hat{m}} = 24.3.
\]

Let us conclude this section with a remark on \(\langle \bar{q}q \rangle_0\). A non-vanishing quark condensate in the chiral limit is a sufficient but not a necessary condition for a spontaneous chiral symmetry breaking. The effective Lagrangian term of Eq. (101) not only results in a shift of the vacuum energy but also in finite Goldstone boson masses and both effects are proportional to the parameter \(B_0\). We recall that it was a symmetry argument which excluded a term \(\text{Tr}(\mathcal{M})\) which, at leading order in \(\mathcal{M}\), would decouple the vacuum energy shift from the Goldstone boson masses. The scenario underlying \(\mathcal{L}_{\text{s.b.}}\) of Eq. (101) is similar to that of a Heisenberg ferromagnet which exhibits a spontaneous magnetization \(\langle \vec{M} \rangle\), breaking the \(O(3)\) symmetry of the Heisenberg Hamiltonian down to \(O(2)\). In the present case the analogue of the order parameter \(\langle \vec{M} \rangle\) is the quark condensate \(\langle \bar{q}q \rangle_0\). In the case of the ferromagnet, the interaction with an external magnetic field \(\vec{H}\) is given by \(-\langle \vec{M} \rangle \cdot \vec{H}\), which corresponds to Eq. (102), with the quark masses playing the role of the external field \(\vec{H}\) (see Table 6). However, in principle, it is also possible that \(B_0\) vanishes or is rather small. In such a case the quadratic masses of the Goldstone bosons might be dominated by terms which are nonlinear in the quark masses, i.e., by higher-order terms in the expansion of \(\mathcal{L}(U, \mathcal{M})\). Such a scenario is the origin of the so-called generalized chiral perturbation theory [Knecht et al., 1995]. The analogue would be an antiferromagnet which shows a spontaneous symmetry breaking but with \(\langle \vec{M} \rangle = 0\). The analysis of the s-wave \(\pi\pi\)-scattering lengths [Colangelo et al., 2000], [Colangelo et al., 2001a] supports the conjecture that the quark condensate is indeed the leading order parameter of the spontaneously broken chiral symmetry (see also Sec. 3.2.4).

### 3.2.3 Construction of invariants

So far, we have discussed the lowest-order effective Lagrangian for a global \(SU(3)_L \times SU(3)_R\) symmetry. In Sec. 2.1.6 we stated that the Ward identities of QCD are obtained from a locally invariant generating functional involving a coupling to external fields. Therefore, following Refs. [Gasser and Leutwyler, 1984]
Gasser and Leutwyler, 1985, we will promote the global symmetry of the effective Lagrangian to a local one,
\[ L \rightarrow V_L(x), \quad R \rightarrow V_R(x), \]
and introduce a coupling to the same external fields \( v, a, s, \) and \( p \) as in QCD [see Eq. (47)]. The transformation behavior of the special unitary matrix \( U \) of Eqs. (86) and (88) under \( G = SU(3)_L \times SU(3)_R \), parity \( P \), and charge conjugation \( C \) is given by
\[ U \xrightarrow{G} V_R U V_L^\dagger, \quad U(\vec{x}, t) \xrightarrow{P} U^\dagger(-\vec{x}, t), \quad U \xrightarrow{C} U^T. \]
Given an object \( A \) transforming as \( V_R A V_L^\dagger \), such as \( U \) or \( \chi \), the covariant derivative of \( A, D_\mu A \), is defined as
\[ D_\mu A \equiv \partial_\mu A - i r_\mu A + i A l_\mu \rightarrow D_\mu' A' = V_R(D_\mu A)V_L^\dagger. \tag{109} \]
The defining property is that the covariant derivative should transform as the object it acts on. In particular, the covariant derivative of \( U \) is given by
\[ D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu. \tag{110} \]
For the external fields we introduce corresponding field strength tensors in matrix form as
\[ f_{\mu\nu}^R \equiv \partial_\mu r_\nu - \partial_\nu r_\mu - i[r_\mu, r_\nu] \xrightarrow{G} V_R f_{\mu\nu}^R V_R^\dagger, \tag{111} \]
\[ f_{\mu\nu}^L \equiv \partial_\mu l_\nu - \partial_\nu l_\mu - i[l_\mu, l_\nu] \xrightarrow{G} V_L f_{\mu\nu}^L V_L^\dagger. \tag{112} \]
They are traceless, because \( \text{Tr}(l_\mu) = \text{Tr}(r_\mu) = 0 \) and the trace of any commutator vanishes. Finally, we introduce the linear combination \( \chi = 2B_0(s + ip) \), where, e.g., pure QCD is given by \( \chi = 2B_0\text{diag}(m_u, m_d, m_s) \).

The effective Lagrangian is constructed in terms of \( U, U^\dagger, \chi, \chi^\dagger, f_{\mu\nu}^R, f_{\mu\nu}^L \) and covariant derivatives of these objects. Suppose we have matrices \( A, B, C, \cdots \), all of which transform as
\[ A \xrightarrow{G} V_R A V_L^\dagger, \quad B \xrightarrow{G} V_R B V_L^\dagger, \quad \cdots. \]
Invariants may be formed by “multiplying” in the following way:
\[ \text{Tr}(AB^\dagger) \xrightarrow{G} \text{Tr}(V_R A V_L^\dagger V_L^\dagger V_R^\dagger) = \text{Tr}(V_R^\dagger V_R A B^\dagger) = \text{Tr}(AB^\dagger), \]
where the generalization to a longer string of terms is obvious and the product of invariant traces is also invariant:
\[ \text{Tr}(AB^\dagger CD^\dagger), \quad \text{Tr}(AB^\dagger)\text{Tr}(CD^\dagger), \quad \cdots. \tag{113} \]
In the chiral counting scheme the elements count as
\[ U = \mathcal{O}(q^0), \quad D_\mu U = \mathcal{O}(q), \quad r_\mu, l_\mu = \mathcal{O}(q), \quad f_{\mu\nu}^{L/R} = \mathcal{O}(q^2), \quad \chi = \mathcal{O}(q^2). \]
Any additional covariant derivative counts as \( \mathcal{O}(q) \). The list of objects \( A \) up to and including order \( q^2 \) which transform as \( A' = V_R A V_L^\dagger \) reads
\[ U, D_\mu U, D_\mu D_\nu U, \chi, U f_{\mu\nu}, f_{\mu\nu}^R U. \]
The construction of chirally invariant expressions up to and including order \( q^2 \) proceeds as follows. At \( \mathcal{O}(q^0) \) the only invariant term is a constant, \( \text{Tr}(U U^\dagger) = \text{Tr}(1) = \text{const} \). Because of \( \text{Tr}(D_\mu U U^\dagger) = 0 \), terms of the type \( \text{Tr}[\mathcal{O}(q)] \times \text{Tr}(\cdots) \) are excluded. At \( \mathcal{O}(q^2) \) we have
\[ \text{Tr}(D_\mu D_\nu U U^\dagger) = -\text{Tr}[D_\nu U(D_\mu U)^\dagger], \quad \text{Tr}[D_\mu U(D_\nu U)^\dagger], \quad \text{Tr}[U(D_\mu D_\nu U)^\dagger] = -\text{Tr}[D_\nu U(D_\mu U)^\dagger], \]
\[ \text{Tr}(\chi U^\dagger), \quad \text{Tr}(U \chi^\dagger), \quad \text{Tr}[(U f_{\mu\nu}) U^\dagger] = \text{Tr}(f_{\mu\nu}^L) = 0, \quad \text{Tr}(f_{\mu\nu}^R) = 0. \]
Because of Lorentz invariance, indices have to be contracted and the remaining three candidates are

$$\text{Tr} \left[ D_{\mu} U (D^\mu U)^\dagger \right], \quad \text{Tr} \left( \chi U^\dagger \pm U \chi^\dagger \right).$$

Finally, due to parity conservation,

$$\mathcal{L}(\vec{x}, t) \overset{P}{\rightarrow} \mathcal{L}(-\vec{x}, t).$$

Tr($\chi U^\dagger - U \chi^\dagger$) has to be excluded because of the wrong parity. At $\mathcal{O}(q^2)$, charge conjugation does not generate any additional constraint.

The locally invariant lowest-order Lagrangian $\mathcal{L}_2$ is given by

$$\mathcal{L}_2 = \frac{F_0^2}{4} \text{Tr} \left[ D_{\mu} U (D^\mu U)^\dagger \right] + \frac{F_0^2}{4} \text{Tr} \left( \chi U^\dagger + U \chi^\dagger \right). \quad (114)$$

At $\mathcal{O}(q^2)$ it contains two parameters: the SU(3) chiral limit of the Goldstone boson decay constant $F_0 \approx 93$ MeV, and, hidden in the definition of $\chi$, $B_0 = -\langle 0|\bar{q}q|0\rangle/(3F_0^2)$.

The lowest-order equation of motion corresponding to Eq. (114) is obtained by considering small variations of the SU(3) matrix,

$$U'(x) = U(x) + \delta U(x) = \left( 1 + i \sum_{a=1}^{8} \Delta_a(x) \lambda_a \right) U(x), \quad (115)$$

where the $\Delta_a(x)$ are real functions. The matrix $U'$ satisfies both conditions $U'U'^\dagger = 1$ and $\text{det}(U') = 1$ up to and including terms linear in $\Delta_a$. Applying the principle of stationary action, the variation of the action reads

$$\delta S = i \frac{F_0^2}{4} \int_{t_1}^{t_2} dt \int d^3x \sum_{a=1}^{8} \Delta_a(x) \text{Tr} \left\{ \lambda_a [D_{\mu} D^\mu U U^\dagger - U (D_{\mu} D^\mu U)^\dagger - \chi U^\dagger + U \chi^\dagger] \right\},$$

where we made use of partial integration, the standard boundary conditions $\Delta_a(t_1, \vec{x}) = \Delta_a(t_2, \vec{x}) = 0$, the divergence theorem, and the definition of the covariant derivative of Eq. (109). Since the test functions $\Delta_a(x)$ may be chosen arbitrarily, we obtain eight Euler-Lagrange equations

$$\text{Tr} \left\{ \lambda_a [D^2 U U^\dagger - U (D^2 U)^\dagger - \chi U^\dagger + U \chi^\dagger] \right\} = 0, \quad a = 1, \ldots, 8, \quad (116)$$

which may be combined into a compact matrix form

$$\mathcal{O}_{\text{EOM}}^{(2)}(U) \equiv D^2 U U^\dagger - U (D^2 U)^\dagger - \chi U^\dagger + U \chi^\dagger + \frac{1}{3} \text{Tr}(\chi U^\dagger - U \chi^\dagger) = 0. \quad (117)$$

The trace term in Eq. (117) appears, because Eq. (116) contains eight and not nine independent equations.

### 3.2.4 Two simple applications: Pion decay and $\pi\pi$ scattering

The Lagrangian $\mathcal{L}_2$ of Eq. (114) has predictive power, once the low-energy coupling constant $F_0$ is identified. This LEC may be obtained from the weak decay of the pion, $\pi^+ \rightarrow \mu^+ \nu_\mu$. For that purpose we insert the corresponding external fields of Eq. (52), describing the interaction of quarks with the massive charged weak bosons, into $\mathcal{L}_2$. The coupling of a single $W$ boson to a single Goldstone boson originates from the covariant derivatives in $\mathcal{L}_2$,

$$\frac{F_0^2}{4} \text{Tr}[D_{\mu} U (D^\mu U)^\dagger] = i \frac{F_0^2}{2} \text{Tr}(l_{\mu} \partial^\mu U U^\dagger) + \cdots = \frac{F_0}{2} \text{Tr}(l_{\mu} \partial^\mu \phi) + \cdots,$$
and is given by
\[ \mathcal{L}_{W\phi} = -\frac{g}{\sqrt{2}} \frac{F_0}{2} \text{Tr}[(W_+ T_+ + W_- T_-) \partial^\mu \phi] = -\frac{g}{2} \frac{F_0}{2} [W_+ (V_{ud} \partial^\mu \pi^- + V_{us} \partial^\mu K^-) + W_- (V_{ud} \partial^\mu \pi^+ + V_{us} \partial^\mu K^+)]. \]  

(118)

The invariant amplitude of the weak pion decay is of the structure “leptonic vertex \times W propagator \times hadronic vertex,”

\[ \mathcal{M} = i \left[ -\frac{g}{2\sqrt{2}} \bar{u}_\mu \gamma^\rho (1 - \gamma_5) v_{\mu^+} \right] \frac{i g_{\rho\sigma}}{M_W^2} \left[ -\frac{g}{2} F_0 V_{ud}(-ip^\sigma) \right] = -G_F V_{ud} F_0 \bar{u}_\mu \gamma^\rho (1 - \gamma_5) v_{\mu^+}, \]  

(119)

where \( G_F \) is the Fermi constant of Eq. (53) and \( p \) denotes the four-momentum of the pion. In the gauge-boson propagator, momenta \( p \) have been neglected in comparison to the gauge-boson mass \( M_W \).

The corresponding decay rate is

\[ \frac{1}{\tau} = \frac{G^2 V^2}{4\pi} \frac{F_0^2 M_W m_\mu^2}{M^2} \left( 1 - \frac{m_\mu^2}{M^2} \right)^2. \]

The constant \( F_0 \) is referred to as the pion-decay constant in the chiral limit. It measures the strength of the matrix element of the axial-vector current operator between a one-Goldstone-boson state and the vacuum [see Eq. (81)]. Since the interaction of the \( W \) boson with the quarks is of the \( V - A \) type and the vector current operator does not contribute to the matrix element between a single pion and the vacuum, pion decay is completely determined by the axial-vector current. The degeneracy of a single \( W \) vacuum \[ \text{see Eq. (81)} \]. Since the interaction of the \( \phi \) with the \( \pi \) is completely determined by the axial-vector current, the degeneracy of a single \( \phi \) vacuum \[ \text{see Eq. (81)} \].

Now that the LEC \( F_0 \) has been identified, we will show how the lowest-order Lagrangian predicts the prototype of a Goldstone-boson reaction, namely, \( \pi \pi \) scattering. We consider \( \mathcal{L}_2 \) in the \( SU(2)_L \times SU(2)_R \) sector with \( r_\mu = l_\mu = 0 \),

\[ \mathcal{L}_2 = \frac{F^2}{4} \text{Tr} \left( \partial^\mu U \partial^\mu U^\dagger \right) + \frac{F^2}{4} \text{Tr} \left( \chi U^\dagger + U \chi^\dagger \right), \]

where

\[ \chi = 2B \begin{pmatrix} \hat{m} & 0 \\ 0 & \hat{m} \end{pmatrix}, \quad U = \exp \left( i \frac{\phi}{F} \right), \quad \phi = \sum_{i=1}^3 \phi_i \tau_i = \begin{pmatrix} \pi^0 & \sqrt{2} \pi^+ \\ -\sqrt{2} \pi^- & -\pi^0 \end{pmatrix}. \]

In the \( SU(2)_L \times SU(2)_R \) sector it is common to express quantities in the chiral limit without index 0, e.g., \( F \) and \( B \). By this one means the \( SU(2)_L \times SU(2)_R \) chiral limit, i.e., \( m_u = m_d = 0 \) but \( m_s \) at its physical value. In the \( SU(3)_L \times SU(3)_R \) sector the quantities \( F_0 \) and \( B_0 \) denote the chiral limit for all three quarks: \( m_u = m_d = m_s = 0 \). Using the substitution \( U \leftrightarrow U^\dagger \), we see that \( \mathcal{L}_2 \) contains even powers of \( \phi \) only:

\[ \mathcal{L}_2 = \mathcal{L}^{2\phi}_2 + \mathcal{L}^{4\phi}_2 + \cdots. \]

Since \( \mathcal{L}_2 \) does not produce a vertex with three Goldstone bosons, there are no \( s \)-, \( u \)-, and \( t \)-channel pole diagrams, i.e., at \( D = 2 \), \( \pi \pi \) scattering is entirely generated by a four-Goldstone-boson-interaction term. Expanding

\[ U = 1 + i \frac{\phi}{F} - \frac{1}{2} \frac{\phi^2}{F^2} - \frac{i}{6} \frac{\phi^3}{F^3} + \frac{1}{24} \frac{\phi^4}{F^4} + \cdots, \]

the interaction term \( \mathcal{L}^{4\phi}_2 \) is identified as

\[ \mathcal{L}^{4\phi}_2 = \frac{1}{48F^2} \left[ \text{Tr}(\phi \partial_\mu \phi)(\phi \partial^\mu \phi) + 2B \text{Tr}(\mathcal{M}\phi^4) \right]. \]
Figure 3: Lowest-order Feynman diagram for $\pi\pi$ scattering. The vertex is derived from $\mathcal{L}_2$, denoted by 2 in the interaction blob.

We note that substituting $F \to F_0$, $B \to B_0$ and the relevant expressions for $\phi$ and the quark-mass matrix $\mathcal{M}$ the corresponding formula for SU(3)$_L \times$ SU(3)$_R$ looks identical. Inserting $\phi = \phi_i \tau_i$ and working out the traces yields

$$\mathcal{L}_2^{4\phi} = \frac{1}{6F^2}(\phi_i \partial^\mu \phi_i \partial_\mu \phi_j - \phi_i \phi_i \partial_\mu \phi_j \partial^\mu \phi_j) + \frac{M^2}{24F^2}\phi_i \phi_i \phi_j \phi_j,$$

where $M^2 = 2B\hat{m}$. The Feynman rule derived from $\mathcal{L}_2^{4\phi}$ for Cartesian isospin indices $a, b, c,$ and $d$ reads (see Fig. 3)

$$\mathcal{M} = i \left[ \delta_{ab} \delta_{cd} \frac{s - M^2}{F^2} + \delta_{ac} \delta_{bd} \frac{t - M^2}{F^2} + \delta_{ad} \delta_{bc} \frac{u - M^2}{F^2} \right]$$

$$- \frac{i}{3F^2} (\delta_{ab} \delta_{cd} + \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) (\Lambda_a + \Lambda_b + \Lambda_c + \Lambda_d),$$

where $\Lambda_k = \frac{p_k^2 - M^2}{4\hat{m}}$ and $s, t,$ and $u$ are the usual Mandelstam variables,

$$s = (p_a + p_b)^2, \quad t = (p_a - p_c)^2, \quad u = (p_a - p_d)^2.$$ 

In general, the $T$-matrix element for the scattering process $\pi_a(p_a) + \pi_b(p_b) \to \pi_c(p_c) + \pi_d(p_d)$ can be parameterized as

$$T^{abcd}(p_a, p_b, p_c, p_d) = \delta^{abc} A(s, t, u) + \delta^{ac} \delta^{bd} A(t, s, u) + \delta^{ad} \delta^{bc} A(u, t, s),$$

where the function $A$ satisfies $A(s, t, u) = A(s, u, t)$ [Weinberg, 1966]. Since the last line of the Feynman rule of Eq. (120) disappears, if the external lines satisfy on-mass-shell conditions, at $\mathcal{O}(q^2)$ the prediction for the function $A$ is given by

$$A(s, t, u) = \frac{s - M^2}{F^2}.$$ 

In Eq. (122) we substituted $F_\pi$ for $F$ and $M_\pi$ for $M$, because the difference is of $\mathcal{O}(q^4)$ in $T$. Equation (122) illustrates an important general property of Goldstone-boson interactions. If we consider the (theoretical) limit $M_\pi^2, s, t, u \to 0$, the $T$ matrix vanishes, $T \to 0$. In other words, the strength of Goldstone-boson interactions vanishes in the zero-energy and mass limit.

Usually, $\pi\pi$ scattering is discussed in terms of its isospin decomposition. Since the pions form an isospin triplet, the two isovectors of both the initial and final states may be coupled to $I = 0, 1, 2$. For $m_u = m_d = \hat{m}$ the strong interactions are invariant under isospin transformations, implying that scattering matrix elements can be decomposed as

$$\langle I', I'_3 | T | I, I_3 \rangle = T^I \delta_{I'I_3}.$$ 

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For the case of $\pi\pi$ scattering the three isospin amplitudes are given in terms of the invariant amplitude $A$ of Eq. (121) by Gasser and Leutwyler, 1984

\[
T^{I=0} = 3A(s, t, u) + A(t, u, s) + A(u, s, t), \\
T^{I=1} = A(t, u, s) - A(u, s, t), \\
T^{I=2} = A(t, u, s) + A(u, s, t).
\] (124)

For example, the physical $\pi^+\pi^+$ scattering process is described by $T^{I=2}$. Other physical processes are obtained using the appropriate Clebsch-Gordan coefficients.

Evaluating the $T$ matrices at threshold, one obtains the s-wave $\pi\pi$-scattering lengths

\[
T^{I=0}|_{\text{thr}} = 32\pi a_0^0, \quad T^{I=2}|_{\text{thr}} = 32\pi a_0^2,
\] (125)

where the subscript $0$ refers to s wave and the superscript to the isospin. ($T^{I=1}|_{\text{thr}}$ vanishes because of Bose symmetry). The convention in ChPT differs by a factor $(-\pi)$ from the usual definition of a scattering length in the effective range expansion. The current-algebra prediction of Ref. Weinberg, 1966 is identical with the lowest-order result obtained from Eq. (122),

\[
a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} = 0.159, \quad a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2} = -0.0454,
\] (126)

where we made use of the numerical values $F_\pi = 92.4$ MeV and $M_\pi = M_{\pi^+} = 139.57$ MeV. In order to obtain the results of Eq. (126), use has been made of $s_{\text{thr}} = 4M_\pi^2$ and $t_{\text{thr}} = u_{\text{thr}} = 0$. Equations (126) represent an absolute prediction of chiral symmetry. Once $F_\pi$ is known (from pion decay), the scattering lengths are predicted. The s-wave $\pi\pi$-scattering lengths have been calculated at next-to-leading (NL) order Gasser and Leutwyler, 1984 and at next-to-next-to-leading order Bijnens et al., 1996, Bijnens et al., 1997. By matching the chiral representation of the scattering amplitude with a dispersive representation Roy 1971, Ananthanarayan et al., 2001, the predictions for the s-wave $\pi\pi$-scattering lengths are Colangelo et al., 2000, Colangelo et al., 2001b

\[
a_0^0 = 0.220 \pm 0.005, \quad a_0^2 = -0.0444 \pm 0.0010.
\] (127)

The empirical results for the s-wave $\pi\pi$-scattering lengths have been obtained from various sources. In the $K_{e4}$ decay $K^+ \rightarrow \pi^+\pi^- e^+\nu_e$, the connection with low-energy $\pi\pi$ scattering stems from a partial-wave analysis of the form factors relevant for the $K_{e4}$ decay in terms of $\pi\pi$ angular momentum eigenstates. In the low-energy regime the phases of these form factors are related by (a generalization of) Watson’s theorem Watson, 1954 to the corresponding phases of $I = 0$ s-wave and $I = 1$ p-wave elastic scattering Colangelo et al., 2001a. Using effective field theory techniques, isospin-breaking effects generated by real and virtual photons, and by the mass difference of the up and down quarks were discussed in Ref. Colangelo et al., 2009. Performing a combined analysis of the Geneva-Saclay data Rosselet et al., 1977, the BNL-E865 data Pislak et al., 2001, Pislak et al., 2003, and the NA48/2 data Batley et al., 2008 results in Colangelo et al., 2009

\[
a_0^0 = 0.217 \pm 0.008_{\text{exp}} \pm 0.006_{\text{th}}
\] (128)

which is in excellent agreement with the prediction of Eq. (127). The $\pi^+p \rightarrow \pi^+\pi^0 n$ reactions require an extrapolation to the pion pole to extract the $\pi\pi$ amplitude and are thus regarded to contain more model dependence, $a_0^0 = 0.204 \pm 0.014_{\text{stat}} \pm 0.008_{\text{syst}}$ Kermani et al., 1998. The DIRAC Collaboration Adeva et al., 2005 makes use of a lifetime measurement of pionium to extract $|a_0^0 - a_0^2| = 0.264_{-0.020}^{+0.033}$. Finally, in the $K^\pm \rightarrow \pi^\pm\pi^0\pi^0$ decay, isospin-symmetry breaking leads to a cusp structure $\sim a_0 - a_2$ in the $\pi^0\pi^0$ invariant mass distribution near $s_{\pi^0\pi^0} \approx 4M_{\pi^+}^2$ Cabibbo, 2004, Cabibbo and Isidori, 2005.
Based on the model of [Cabibbo and Isidori, 2005], the NA48/2 Collaboration extract $a_0^0 - a_2^0 = 0.268 \pm 0.010 \text{ (stat)} \pm 0.004 \text{ (syst)} \pm 0.013 \text{ (ext)}$. A more sophisticated analysis of the cusps in $K \to 3\pi$ within an effective field theory framework can be found in Refs. [Colangelo et al., 2006], [Bissegger et al., 2008], and [Bissegger et al., 2009].

In particular, when analyzing the data of Ref. [Pislak et al., 2001 in combination with the Roy equations, an upper limit $|\bar{l}_3| \leq 16$ was obtained in Ref. [Colangelo et al., 2001a] for the scale-independent low-energy coupling constant which is related to $l_3$ of the SU(2)$_L \times$ SU(2)$_R$ Lagrangian of Gasser and Leutwyler [Gasser and Leutwyler, 1984]. The great interest generated by this result is to be understood in the context of the pion mass at $O(q^4)$

$$M^2_{\pi} = M^2 - \frac{\bar{l}_3}{32\pi^2 F^2} M^4 + O(M^6),$$

(129)

where $M^2 = 2\hat{m}B$. Recall that the constant $B$ is related to the scalar quark condensate in the chiral limit and that a non-vanishing quark condensate is a sufficient criterion for spontaneous chiral symmetry breakdown in QCD. If the expansion of $M^2_{\pi}$ in powers of the quark masses is dominated by the linear term in Eq. (129), the result is often referred to as the Gell-Mann-Oakes-Renner relation [Gell-Mann et al., 1968]. If the terms of order $\hat{m}^2$ were comparable or even larger than the linear terms, a different power counting or bookkeeping in ChPT would be required [Knecht et al., 1995]. The estimate $|\bar{l}_3| \leq 16$ implies that the Gell-Mann-Oakes-Renner relation is indeed a decent starting point, because the contribution of the second term of Eq. (129) to the pion mass is approximately given by

$$-\frac{\bar{l}_3 M^2_{\pi}}{64\pi^2 F^2_{\pi}} M_{\pi} = -0.054 M_{\pi}$$

for $\bar{l}_3 = 16$,

i.e., more than 94% of the pion mass must stem from the quark condensate [Colangelo et al., 2001a].

### 3.2.5 Primer to dimensional regularization

If we want to use the Lagrangian of Eq. (114) beyond the tree level, we will encounter ultraviolet divergences from loop integrals. For the regularization of the loop diagrams we will make use of dimensional regularization ["t Hooft and Veltman, 1972], [Leibbrandt, 1975], ["t Hooft and Veltman, 1979], because it preserves algebraic relations between the Green functions (Ward identities). We will illustrate the method by considering the following simple example,

$$I = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - M^2 + i0^+},$$

(130)

which shows up in the generic diagram of Fig. 4. Introducing $a \equiv \sqrt{\bar{k}^2 + M^2} > 0$, we define $f(k_0) = \{[k_0 + (a - i0^+)] [k_0 - (a - i0^+)]\}^{-1}$. In order to determine $\int_{-\infty}^{\infty} dk_0 f(k_0)$ as part of the calculation of $I$, we consider $f$ in the complex $k_0$ plane and make use of Cauchy’s theorem $\oint_C dz f(z) = 0$ for functions
which are differentiable in every point inside the closed contour \( C \). Choosing the path as shown in Fig. 5 and taking account of the fact that the quarter circles at infinity do not contribute, we obtain the so-called Wick rotation

\[
\int_{-\infty}^{\infty} dt f(t) = -i \int_{-\infty}^{\infty} dt f(it) = i \int_{-\infty}^{\infty} dt f(it). \tag{131}
\]

As an intermediate result, the integral of Eq. (130) reads

\[
I = \int \frac{d^4l}{(2\pi)^4} \frac{1}{l^2 + M^2},
\]

where \( l^2 = l_1^2 + l_2^2 + l_3^2 + l_4^2 \) denotes a Euclidian scalar product in four dimensions. Performing the angular integration in four dimensions and introducing a cutoff \( \Lambda \) for the radial integration, the integral \( I \) diverges quadratically for large values of \( l \) (ultraviolet divergence):

\[
I(\Lambda) = \frac{1}{8\pi^2} \int_0^\Lambda l^2 \frac{l^3}{l^2 + M^2} = \frac{\Lambda^2}{4\pi^2} \ln \left( \frac{M^2}{\Lambda^2 + M^2} \right) = \frac{M^2}{(4\pi)^2} \left[ \frac{1}{x} + \ln(x) - \ln(1 + x) \right], \tag{132}
\]

where \( x = M^2/\Lambda^2 \to 0 \) as \( \Lambda \to \infty \). In dimensional regularization, we generalize the integral from 4 to \( n \) dimensions and introduce polar coordinates

\[
\begin{align*}
l_1 & = l \cos(\theta_1), \quad l_2 = l \sin(\theta_1) \cos(\theta_2), \quad l_3 = l \sin(\theta_1) \sin(\theta_2) \cos(\theta_3), \\
& \vdots \\
l_{n-1} & = l \sin(\theta_1) \sin(\theta_2) \cdots \cos(\theta_{n-1}), \quad l_n = l \sin(\theta_1) \sin(\theta_2) \cdots \sin(\theta_{n-1}),
\end{align*}
\]

where \( 0 \leq l, \theta_i \in [0, \pi] \ (i = 1, \cdots, n - 2), \) and \( \theta_{n-1} \in [0, 2\pi] \). A general integral is then symbolically of the form

\[
\int d^n l \cdots = \int_0^\infty dl l^{n-1} \int_0^{2\pi} d\theta_{n-1} \int_0^\pi d\theta_{n-2} \sin(\theta_{n-2}) \cdots \int_0^\pi d\theta_1 \sin^{n-2}(\theta_1) \cdots.
\]

If the integrand does not depend on the angles, the angular integration can explicitly be carried out:

\[
\int d\Omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}.
\]
3.2.6 Power-counting scheme

We define the integral for \( n \) dimensions (\( n \) integer) as

\[
I_n(M^2, \mu^2) = \mu^{4-n} \int \frac{d^n k}{(2\pi)^n} \frac{i}{k^2 - M^2 + i0^+},
\]

where the scale \( \mu \) (t'Hooft parameter, renormalization scale) has been introduced so that the integral has the same dimension for arbitrary \( n \). The integral formally reads

\[
I_n(M^2, \mu^2) = \mu^{4-n} \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \frac{1}{(2\pi)^n} \int_0^\infty \frac{dt}{t^2 + M^2} l^{n-1} \Gamma\left(\frac{n}{2}\right) \Gamma\left(1 - \frac{n}{2}\right)
\]

\[
= \mu^{4-n} \frac{\pi^{\frac{n}{2}}}{(4\pi)^{\frac{n}{2}}} (M^2)^{\frac{n}{2}-1} \Gamma\left(1 - \frac{n}{2}\right)
\]

(133)

Since \( \Gamma(z) \) is an analytic function in the complex plane except for poles of first order in \( 0, -1, -2, \cdots \), and \( a^\ast = \exp[\ln(a)z], a \in \mathbb{R}^+ \) is an analytic function in \( \mathbb{C} \), the right-hand side of Eq. (133) can be thought of as a function of a complex variable \( n \) which is analytic in \( \mathbb{C} \) except for poles of first order for \( n = 2, 4, 6, \cdots \). The analytic continuation for complex \( n \) reads

\[
I(M^2, \mu^2, n) = \frac{M^2}{(4\pi)^2} \left(\frac{4\pi \mu^2}{M^2}\right)^{2 - \frac{n}{2}} \Gamma\left(1 - \frac{n}{2}\right) = \frac{M^2}{16\pi^2} \left[R + \ln\left(\frac{M^2}{\mu^2}\right)\right] + O(n - 4),
\]

(134)

where

\[
R = \frac{2}{n - 4} - [\ln(4\pi) + \Gamma'(1) + 1].
\]

(135)

The comparison between Eqs. (134) and (132) illustrates the following general observations: in dimensional regularization power-law divergences are analytically continued to zero and logarithmic ultraviolet divergences of one-loop integrals show up as single poles in \( \epsilon = 4 - n \).

3.2.6 Power-counting scheme

The Lagrangian \( \mathcal{L}_{\text{eff}} \) of mesonic chiral perturbation theory is organized as a string of terms with an increasing number of derivatives and quark-mass terms,

\[
\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \cdots,
\]

(136)

where the subscripts refer to the order in the momentum and quark-mass expansion. The index 2, for example, denotes either two derivatives or one quark-mass term. In terms of Feynman rules, derivatives generate four-momenta. A quark-mass term counts as two derivatives because of Eqs. (105) - (107) \( (M^2 \sim m_q) \) in combination with the on-shell condition \( p^2 = M^2 \). We will generically count a small four-momentum—or the corresponding derivative—and a Goldstone-boson mass as of \( O(q) \). The chiral orders in Eq. (136) are all even \( [O(q^{2k}), k \geq 1] \), because Lorentz indices of derivatives always have to be contracted and quark-mass terms count as \( O(q^2) \).

Besides the knowledge of the most general Lagrangian, we need a method which allows one to assess the importance of different renormalized diagrams contributing to a given process. For that purpose we analyze a given diagram under a simultaneous re-scaling of all external momenta, \( p_i \mapsto tp_i \), and the light-quark masses, \( m_q \mapsto t^2 m_q \) (corresponds to \( M^2 \mapsto t^2 M^2 \)). The chiral dimension \( D \) of a given diagram is defined as

\[
\mathcal{M}(tp_i, t^2 m_q, t\mu) = t^D \mathcal{M}(p_i, m_q, \mu) = O(q^D).
\]

(137)
\[ D = 4 \cdot 2 - 2 \cdot 3 + 2 \cdot 2 = 6. \]
\[ D = 4 \cdot 2 - 2 \cdot 3 + 2 \cdot 1 + 4 \cdot 1 = 8. \]
\[ D = 4 \cdot 4 - 2 \cdot 5 + 2 \cdot 2 = 10. \]

Figure 6: Application of the power-counting formula of Eq. (138) in \( n = 4 \) dimensions.

For small enough momenta (and masses) contributions with increasing \( D \) become less important. The chiral dimension is given by

\[
D = nN_L - 2N_I + \sum_{k=1}^{\infty} 2kN_{2k}\]

\[
= 2 + (n - 2)N_L + \sum_{k=1}^{\infty} 2(k - 1)N_{2k}\]

\[
\geq 2 \quad \text{in 4 dimensions},
\]

where \( n \) is the number of space-time dimensions, \( N_L \) the number of independent loops, \( N_I \) the number of internal Goldstone boson lines, and \( N_{2k} \) the number of vertices from \( \mathcal{L}_{2k} \). Equation (139) establishes a relation between the momentum and loop expansion, because at each chiral order, the maximum number of loops is bounded from above. In other words, we have a perturbative scheme in terms of external momenta and masses which are small compared to some scale [here: \( \Lambda_\chi = 4\pi F_0 = \mathcal{O} (1 \text{ GeV}) \)]. Examples of the application of the power-counting formula are shown in Fig. 6.

In order to prove the power-counting formula we start from the Feynman rules for evaluating the S-matrix element and investigate the behavior of the individual building blocks. Internal lines are described by a propagator in \( n \) dimensions which under re-scaling behaves as

\[
\int \frac{d^n k}{(2\pi)^n} \frac{i}{k^2 - M^2 + i0^+} \rightarrow \int \frac{d^n k}{(2\pi)^n} \frac{i}{t^2(k^2/t^2 - M^2 + i0^+)} k = tk', t^{n-2} \int \frac{d^n k'}{(2\pi)^n} \frac{i}{k'^2 - M^2 + i0^+}.
\]

Vertices with \( 2k \) derivatives or \( k \) quark-mass terms re-scale as

\[
\delta^n(q)q^{2k} \rightarrow t^{2k-n}\delta^n(q)q^{2k},
\]

since \( p \rightarrow tp \) if \( q \) is an external momentum, and \( k = tk' \) if \( q \) is an internal momentum (see above). These are the rules to calculate \( S \sim \delta^n(P)\mathcal{M} \). We need to add \( n \) to compensate for the overall momentum-conserving delta function. Applying these rules, the scaling behavior of the contribution to \( \mathcal{M} \) of a given diagram reads

\[
D = n + (n - 2)N_I + \sum_{k=1}^{\infty} N_{2k}(2k - n).
\]

The relation between the number of independent loops, the number of internal lines, and the total number of vertices \( N_V = \sum_{k=1}^{\infty} N_{2k} \) is given by \( N_L = N_I - (N_V - 1) \). The product of \( N_V \) momentum-conserving \( \delta \) functions contains overall momentum conservation. Therefore, one has \( N_V - 1 \) rather than \( N_V \) restrictions on the internal momenta. Applying

\[
-n \sum_{k=1}^{\infty} N_{2k} = -nN_V = n(N_L - N_I - 1)
\]
3.3 Next-to-leading order

Already in 1967 it was shown by Weinberg [Weinberg, 1967] that an effective Lagrangian is a convenient tool for reproducing the results of current algebra in terms of tree-level calculations. In the purely mesonic sector, $\mathcal{L}_2$ of Eq. (114) represents the corresponding Lagrangian. It was noted by Li and Pagels [Li and Pagels, 1971] that a perturbation theory around a symmetry which is realized in the Nambu-Goldstone mode, in general, leads to observables which are non-analytic functions of the symmetry-breaking parameters, here the quark masses. In 1979 Weinberg initiated the application of an effective-field-theory program beyond the tree level allowing for a systematic calculation of corrections to the chiral limit [Weinberg, 1979]. When calculating one-loop graphs, using vertices from $\mathcal{L}_2$, one generates ultraviolet divergences which in the framework of dimensional regularization appear as poles at space-time dimension $n = 4$. The loop diagrams are renormalized by absorbing the infinite parts into the redefinition of the fields and the parameters of the most general Lagrangian. Since $\mathcal{L}_2$ is not renormalizable in the traditional sense, the infinities cannot be absorbed by a renormalization of the coefficients $F_0$ and $B_0$. However, to quote from Ref. [Weinberg, 1995]: "... the cancelation of ultraviolet divergences does not really depend on renormalizability; as long as we include every one of the infinite number of interactions allowed by symmetries, the so-called non-renormalizable theories are actually just as renormalizable as renormalizable theories." According to Weinberg’s power counting of Eq. (139), one-loop graphs with vertices from $\mathcal{L}_2$ are of $\mathcal{O}(q^4)$. The conclusion is that one needs to
construct the most general Lagrangian $\mathcal{L}_4$ and to adjust (renormalize) its parameters to cancel one-loop infinities.

Beyond the quantum corrections to processes already described by $\mathcal{L}_2$, at next-to-leading order we will encounter another important feature, namely, the effective Wess-Zumino-Witten (WZW) action. The WZW action provides an effective description of the constraints due to the anomalous Ward identities. In general, anomalies arise if the symmetries of the Lagrangian at the classical level are not supported by the quantized theory after renormalization.

### 3.3.1 The $O(q^4)$ Lagrangian of Gasser and Leutwyler

The most general $SU(3)_L \times SU(3)_R$-invariant Lagrangian at $O(q^4)$ is given by [Gasser and Leutwyler, 1985]

\[
\mathcal{L}_4 = L_1 \left\{ \text{Tr} [D_\mu U (D^\mu U)^\dagger] \right\}^2 + L_2 \text{Tr} \left[ D_\mu U (D_\nu U)^\dagger \right] \text{Tr} \left[ D^\mu U (D^\nu U)^\dagger \right] \\
+ L_3 \text{Tr} \left[ D_\mu U (D^\mu U)^\dagger D_\nu U (D^\nu U)^\dagger \right] + L_4 \text{Tr} \left[ D_\mu U (D^\mu U)^\dagger \right] \text{Tr} \left( \chi^\dagger U + U \chi \right) \\
+ L_5 \text{Tr} \left[ D_\mu U (D^\mu U)^\dagger (\chi^\dagger U + U \chi) \right] + L_6 \left[ \text{Tr} \left( \chi^\dagger U + U \chi \right) \right]^2 \\
+ L_7 \left[ \text{Tr} \left( \chi^\dagger U - U \chi \right) \right]^2 + L_8 \text{Tr} \left( U \chi^\dagger U \chi^\dagger + \chi^\dagger U \chi \right) \\
- i L_9 \text{Tr} \left[ f^{R \mu \nu}_{\mu \nu} U (D^\mu U)^\dagger + f^{L \mu \nu}_{\mu \nu} (D^\mu U)^\dagger D^\nu U \right] + L_{10} \text{Tr} \left( U f^{L \mu \nu}_{\mu \nu} U^\dagger f^{\mu \nu}_{\mu \nu} \right) \\
+ H_1 \text{Tr} \left( f^{R \mu \nu}_{\mu \nu} f^{\mu \nu}_{\mu \nu} + f^{L \mu \nu}_{\mu \nu} f^{\mu \nu}_{\mu \nu} \right) + H_2 \text{Tr} \left( \chi \chi \right). \\
\]

(140)

The numerical values of the low-energy coupling constants $L_i$ are not determined by chiral symmetry. In analogy to $F_0$ and $B_0$ of $\mathcal{L}_2$ they are parameters containing information on the underlying dynamics and should, in principle, be calculable in terms of the (remaining) parameters of QCD, namely, the heavy-quark masses and the QCD scale $\Lambda_{QCD}$. In practice, they parameterize our inability to solve the dynamics of QCD in the non-perturbative regime. So far they have either been fixed using empirical input or theoretically using QCD-inspired models, meson-resonance saturation [Ecker et al., 1989a] [Pich, 2008], and lattice QCD (see Ref. [Necco, (2009)] for a recent overview).

By construction Eq. (140) represents the most general Lagrangian at $O(q^4)$, and it is thus possible to absorb the one-loop divergences by an appropriate renormalization of the coefficients $L_i$ and $H_i$:

\[
L_i = L_i^r + \frac{\Gamma_i}{32\pi^2} R \quad (i = 1, \cdots, 10), \quad H_i = H_i^r + \frac{\Delta_i}{32\pi^2} R \quad (i = 1, 2),
\]

(141)

where $R$ has already been defined in Eq. (135):

\[
R = \frac{2}{n-4} - \left[ \ln(4\pi) + \Gamma'(1) + 1 \right],
\]

with $n$ denoting the number of space-time dimensions and $\gamma_E = -\Gamma'(1)$ being Euler’s constant. The constants $\Gamma_i$ and $\Delta_i$ are given in Table 7. Except for $L_3$ and $L_7$, the low-energy coupling constants $L_i$ and the “contact terms” —i.e., pure external field terms—$H_1$ and $H_2$ are required in the renormalization of the one-loop graphs. Since $H_1$ and $H_2$ contain only external fields, they are of no physical relevance. The idea of renormalization consists of adjusting the parameters of the counter terms of the most general effective Lagrangian so that they cancel the divergences of (multi-) loop diagrams. In doing so, one still has the freedom of choosing a suitable renormalization condition. For example, in the minimal subtraction scheme (MS) one would fix the parameters of the counter term Lagrangian such that they would precisely absorb the contributions proportional to $2/(n-4)$ in $R$, while the modified minimal subtraction scheme of ChPT (MS) would, in addition, cancel the term in the square brackets.
| Coefficient | Empirical Value | $\Gamma_i$ |
|-------------|-----------------|-----------|
| $L^r_1$     | $0.4 \pm 0.3$   | $\frac{3}{16}$ |
| $L^r_2$     | $1.35 \pm 0.3$  | $\frac{3}{16}$ |
| $L^r_3$     | $-3.5 \pm 1.1$  | $0$       |
| $L^r_4$     | $-0.3 \pm 0.5$  | $\frac{1}{144}$ |
| $L^r_5$     | $1.4 \pm 0.5$   | $\frac{1}{144}$ |
| $L^r_6$     | $-0.2 \pm 0.3$  | $\frac{5}{48}$ |
| $L^r_7$     | $-0.4 \pm 0.2$  | $0$       |
| $L^r_8$     | $0.9 \pm 0.3$   | $\frac{7}{5}$ |
| $L^r_9$     | $6.9 \pm 0.7$   | $\frac{1}{5}$ |
| $L^r_{10}$  | $-5.5 \pm 0.7$  | $\frac{1}{5}$ |

Table 7: Renormalized low-energy coupling constants $L^r_i$ in units of $10^{-3}$ at the scale $\mu = M_\rho$, see [Bijnens et al., 1995]. $\Delta_1 = -1/8$, $\Delta_2 = 5/24$.

The renormalized coefficients $L^r_i$ depend on the scale $\mu$ introduced by dimensional regularization [see Eq. (134)] and their values at two different scales $\mu_1$ and $\mu_2$ are related by

$$L^r_i(\mu_2) = L^r_i(\mu_1) + \frac{\Gamma_i}{16\pi^2} \ln \left( \frac{\mu_1}{\mu_2} \right).$$

(142)

We will see that the scale dependence of the coefficients and the finite part of the loop-diagrams compensate each other in such a way that physical observables are scale independent.

A discussion of the two-flavor Lagrangian at $\mathcal{O}(q^4)$ [Gasser and Leutwyler, 1984] can be found in Appendix D of Ref. [Scherer, 2003]. For the construction of the $\mathcal{O}(q^6)$ Lagrangian of even intrinsic parity, see Refs. [Scherer and Fearing, 1995], [Fearing and Scherer, 1996] and [Bijnens et al., 1999]. For a status report on mesonic chiral perturbation theory beyond the one-loop level, we refer the reader to Ref. [Bijnens, 2007].

### 3.3.2 The effective Wess-Zumino-Witten action

The Lagrangians discussed so far have a larger symmetry than QCD [Witten, 1983]. For example, if we consider the case of “pure” QCD, i.e., no external fields except for the quark-mass term $\chi = 2B_0 M$, $\mathcal{L}_2$ and $\mathcal{L}_4$ contain interaction terms with an even number of Goldstone bosons only (even intrinsic parity). In other words, they cannot describe, e.g., $K^+K^- \to \pi^+\pi^-\pi^0$. Analogously, $\mathcal{L}_2$ and $\mathcal{L}_4$ including a coupling to electromagnetic fields cannot describe $\pi^0 \to \gamma\gamma$.

In order to overcome this shortcoming, Witten suggested to add the simplest term possible which breaks the symmetry of having only an even number of Goldstone bosons at the Lagrangian level. For the case of massless Goldstone bosons without any external fields the modified equation of motion reads

$$\partial_\mu \left( \frac{F^2_0}{2} U \partial^\mu U^\dagger \right) + \lambda e^{\mu\nu\rho\sigma} U \partial_\mu U^\dagger U \partial_\nu U^\dagger U \partial_\rho U^\dagger U \partial_\sigma U^\dagger = 0,$$

(143)

where $\lambda$ is a (purely imaginary) constant. For the purpose of writing down an action corresponding to Eq. (143), we extend the range of definition of the fields to a hypothetical fifth dimension,

$$U(y) = \exp \left( i\alpha \frac{\phi(x)}{F_0} \right), \quad y^i = (x^\mu, \alpha), \quad i = 0, \cdots, 4, \quad 0 \leq \alpha \leq 1,$$

(144)
where Minkowski space is defined as the surface of the five-dimensional space for $\alpha = 1$. The action in the absence of external fields (denoted by a superscript 0) is given by [Witten, 1983]

$$S^0_{\text{ano}} = n S^0_{\text{WZW}}, \quad S^0_{\text{WZW}} = -\frac{i}{240\pi^2} \int_0^1 d\alpha \int d^4x e^{ijklm} \text{Tr} \left( U^i_l \cdots U^m_l \right),$$  \hspace{1cm} (145)

where

$$\epsilon_{01234} = -\epsilon^{01234} = 1, \quad U^L_i = U^\dagger_i \frac{\partial U}{\partial y^i}, \quad \lambda = \frac{in}{48\pi^2}.$$  

A rather unusual and surprising feature of Eq. (145) is that the action functional corresponding to the new term cannot be written as the four-dimensional integral of a Lagrangian expressed in terms of $U$ and its derivatives. Wess and Zumino derived consistency or integrability relations which are satisfied by the anomalous Ward identities and then explicitly constructed a functional involving the pseudoscalar octet which satisfies the anomalous Ward identities [Wess and Zumino, 1971]. In particular, Wess and Zumino emphasized that their interaction Lagrangians cannot be obtained as part of a chiral invariant Lagrangian. Using topological arguments Witten showed that the constant $n$ appearing in Eq. (145) must be an integer. However, it was pointed out in Ref. [Bär and Wiese, 2001] that the traditional argument relating $n$ with the number of colors $N_c$ is incomplete. Before discussing this argument, let us investigate the consequences of $S^0_{\text{WZW}}$.

Expanding the SU(3) matrix $U(y)$ in terms of the Goldstone boson fields, $U(y) = 1 + i\alpha \phi(x)/F_0 + O(\phi^2)$, one obtains an infinite series of terms, each involving an odd number of Goldstone bosons, i.e., the WZW action $S^0_{\text{WZW}}$ is of odd intrinsic parity. For each individual term the $\alpha$ integration can be performed explicitly resulting in an ordinary action in terms of a four-dimensional integral of a local Lagrangian. For example, the term with the smallest number of Goldstone bosons reads

$$S^0_{\text{WZW}} = \frac{1}{240\pi^2 F_0^5} \int d^4x \epsilon^{\mu\nu\rho\sigma} \text{Tr} (\phi \partial_\mu \phi \partial_\nu \phi \partial_\rho \phi \partial_\sigma \phi),$$  \hspace{1cm} (146)

which describes, e.g., $K^+ K^- \rightarrow \pi^+ \pi^- \pi^0$. In particular, the WZW action without external fields involves at least five Goldstone bosons [Wess and Zumino, 1971].

The connection to the number of colors $N_c$ is established by introducing a coupling to electromagnetism [Wess and Zumino, 1971], [Witten, 1983]. In the presence of external fields there will be an additional term in the anomalous action,

$$S_{\text{ano}} = S^0_{\text{ano}} + S^\text{ext}_{\text{ano}} = n (S^0_{\text{WZW}} + S^\text{ext}_{\text{WZW}}),$$  \hspace{1cm} (147)

given by (see, e.g., Ref. Bijnens, 1993)

$$S^\text{ext}_{\text{WZW}} = -\frac{i}{48\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} \text{Tr} (Z_{\mu\nu\rho\sigma})$$  \hspace{1cm} (148)

with

$$Z_{\mu\nu\rho\sigma} = \frac{1}{2} \left[ U_\mu U^\dagger_\nu U_\rho L_\sigma + U_\mu U^\dagger_\nu L_\rho U_\sigma - U^\dagger_\mu L_\nu U_\rho U_\sigma 
+ iU_\mu \partial_\nu L_\rho U_\sigma - iU^\dagger_\mu \partial_\nu U_\rho U_\sigma + i\partial_\nu U_\mu U^\dagger_\rho U_\sigma 
- iU^\dagger_\mu L_\nu U_\rho U_\sigma + iU_\mu U^\dagger_\nu L_\rho U_\sigma 
- iU_\mu U^\dagger_\nu U_\rho L_\sigma + iU^\dagger_\mu L_\nu U_\rho L_\sigma 
- \frac{1}{2} U_\mu U^\dagger_\nu U_\rho U_\sigma \right]$$  

$$\hspace{2cm} - \frac{1}{2} U^\dagger_\mu U_\nu U_\rho U_\sigma + \frac{1}{2} U_\mu U^\dagger_\nu U_\rho U_\sigma - \frac{1}{2} U^\dagger_\mu U_\nu U_\rho U_\sigma$$  

$$\hspace{2cm} + iU_\mu U^\dagger_\nu U_\rho \partial_\sigma L_\sigma + iU^\dagger_\mu U_\nu U_\rho \partial_\sigma L_\sigma 
+ \frac{1}{2} U_\mu U^\dagger_\nu U_\rho \partial_\sigma L_\sigma + \frac{1}{2} U^\dagger_\mu U_\nu U_\rho \partial_\sigma L_\sigma 
- \frac{1}{2} U_\mu U^\dagger_\nu U_\rho L_\sigma + \frac{1}{2} U^\dagger_\mu U_\nu U_\rho L_\sigma$$  

$$\hspace{2cm} - \frac{1}{2} U_\mu U^\dagger_\nu U_\rho L_\sigma + \frac{1}{2} U^\dagger_\mu U_\nu U_\rho L_\sigma - \frac{1}{2} U_\mu U^\dagger_\nu L_\rho U_\sigma$$  

$$\hspace{2cm} + \frac{1}{2} U^\dagger_\mu U_\nu L_\rho U_\sigma - \frac{1}{2} U_\mu U^\dagger_\nu L_\rho U_\sigma - \frac{1}{2} U^\dagger_\mu U_\nu L_\rho U_\sigma$$  

$$\hspace{2cm} + iU_\mu U^\dagger_\nu L_\rho \partial_\sigma L_\sigma + iU^\dagger_\mu U_\nu L_\rho \partial_\sigma L_\sigma 
- U_\mu U^\dagger_\nu L_\rho \partial_\sigma L_\sigma + U^\dagger_\mu U_\nu L_\rho \partial_\sigma L_\sigma$$  

$$\hspace{2cm} - iU_\mu U^\dagger_\nu L_\rho L_\sigma + iU^\dagger_\mu U_\nu L_\rho L_\sigma,$$  \hspace{1cm} (149)
where we defined the abbreviations $U^\mu_\alpha = U^\dagger \partial_\mu U$ and $U^\mu_R = U \partial_\mu U^\dagger$.

As a special case, let us consider a coupling to external electromagnetic fields by inserting

$$r_\mu = l_\mu = -e Q A_\mu,$$

where $Q$ is the quark-charge matrix. The terms involving three and four electromagnetic four-potentials vanish upon contraction with the totally antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma}$, because their contributions to $Z_{\mu\nu\rho\sigma}$ are symmetric in at least two indices, and we obtain

$$n \mathcal{L}_{\text{WZW}}^\text{ext} = -en A_\mu J^\mu + i \frac{ne^2}{48\pi^2} \epsilon^{\mu\nu\rho\sigma} \partial_\nu A_\rho A_\sigma \text{Tr}[2Q^2(U_{\partial_\mu} U^\dagger - U^\dagger_{\partial_\mu} U) - QU^\dagger Q_{\partial_\mu} U + QUQ_{\partial_\mu} U^\dagger].$$  \hspace{1cm} (150)

We note that the current

$$J^\mu = \frac{\epsilon^{\mu\nu\rho\sigma}}{48\pi^2} \text{Tr}(Q_{\partial_\nu} U U^\dagger_{\partial_\rho} U U^\dagger_{\partial_\sigma} U U^\dagger_{\partial_\sigma} U) + QU^\dagger Q_{\partial_\mu} U + QUQ^\dagger_{\partial_\mu} U^\dagger), \quad \epsilon_{0123} = 1,$$  \hspace{1cm} (151)

by itself is not gauge invariant and the additional terms of Eq. (150) are required to obtain a gauge-invariant action. The standard procedure of determining $n$ is to investigate the interaction Lagrangian which is relevant to the decay $\pi^0 \rightarrow \gamma\gamma$ by expanding $U = 1 + i \text{diag}(\pi^0, -\pi^0, 0)/F_0 + \cdots$. However, as pointed out by Bär and Wiese, when considering the electromagnetic interaction for an arbitrary number of colors one should replace the ordinary quark charge matrix by

$$Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \rightarrow \begin{pmatrix} \frac{1}{2N_c} + \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2N_c} - \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2N_c} - \frac{1}{2} \end{pmatrix}.$$

The corresponding effective Lagrangian for $\pi^0 \rightarrow \gamma\gamma$ decay,

$$\mathcal{L}_{\pi^0\gamma\gamma} = -\frac{n}{N_c} \frac{e^2}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \mathcal{F}_{\mu\nu} \mathcal{F}_{\rho\sigma} \frac{\pi^0}{F_0},$$

results in the decay rate

$$\Gamma_{\pi^0 \rightarrow \gamma\gamma} = \frac{\alpha^2 M_{\pi^0}^2}{64\pi^3 F_0^2} \frac{n^2}{N_c^2} = 7.6 \text{ eV} \times \left(\frac{n}{N_c}\right)^2$$

in good agreement with the experimental value $(7.7 \pm 0.6)$ eV for $n = N_c$. However, the result is no indication for $N_c = 3$ \cite{BarWiese2001}. The conclusion from their analysis is that one should rather consider three-flavor processes such as $\eta \rightarrow \pi^+\pi^-\gamma$ or $K\gamma \rightarrow K\pi$ to test the expected $N_c$ dependence in a low-energy reaction. For example, the Lagrangian relevant to the decay $\eta \rightarrow \pi^+\pi^-\gamma$ is given by

$$\mathcal{L}_{\eta\pi^+\pi^-\gamma} = \frac{ie n}{12\sqrt{3}\pi^2 F_0^3} (Q_u - Q_d) \epsilon^{\mu\nu\rho\sigma} A_\mu \partial_\nu \eta \partial_\rho \pi^+ \partial_\sigma \pi^-,$$

where the quark-charge difference $Q_u - Q_d = 1$ is independent of $N_c$. However, by investigating the corresponding $\eta$ and $\eta'$ decays up to next-to-leading order in the framework of the combined $1/N_c$ and chiral expansions, Borasoy and Lipartia have concluded that the number of colors cannot be determined from these decays due to the importance of sub-leading terms which are needed to account for the experimental decay widths and photon spectra \cite{BorasoyLipartia2005}.

For a discussion of the $\mathcal{O}(q^0)$ Lagrangian of odd intrinsic parity see Refs. \cite{Ebertshaeuer2002} and \cite{Bijnens2002}. 

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3.3.3 Masses of the Goldstone bosons

A discussion of the masses at $O(q^4)$ is one of the simplest applications of chiral perturbation theory beyond the tree level. For that purpose let us consider $\mathcal{L}_2 + \mathcal{L}_4$ for QCD with finite quark masses but in the absence of external fields. We restrict ourselves to the limit of isospin symmetry, i.e., $m_u = m_d = \hat{m}$. In order to determine the masses we calculate the self energies $\Sigma(p^2)$ of the Goldstone bosons.

Let

\begin{equation}
\Delta_{\phi_F}(p) = \frac{1}{p^2 - M_{\phi,2}^2 + i0^+}, \quad \phi = \pi, K, \eta,
\end{equation}

(152)

denote the Feynman propagator containing the lowest-order masses,

\[ M_{\pi,2}^2 = 2B_0\hat{m}, \quad M_{K,2}^2 = B_0(\hat{m} + m_s), \quad M_{\eta,2}^2 = \frac{2}{3}B_0(\hat{m} + 2m_s). \]

(The subscript 2 refers to chiral order 2.) The proper self-energy insertions, $-i\Sigma_\phi(p^2)$, consist of one-particle-irreducible diagrams only, i.e., diagrams which do not fall apart into two separate pieces when cutting an arbitrary internal line. At chiral order $D = 4$, the contributions to $-i\Sigma_{\phi,4}(p^2)$ are those shown in Fig. 8. In general, the full (unrenormalized) propagator may be summed using a geometric series (see Fig. 9):

\begin{equation}
\Delta_\phi(p) = \frac{i}{p^2 - M_{\phi,2}^2 + i0^+} + \frac{i}{p^2 - M_{\phi,2}^2 + i0^+}[ -i\Sigma_\phi(p^2) ] \frac{i}{p^2 - M_{\phi,2}^2 + i0^+} + \cdots
\end{equation}

(153)

The physical mass, including the interaction, is defined as the pole of Eq. (153),

\begin{equation}
M_\phi^2 - M_{\phi,2}^2 - \Sigma_\phi(M_\phi^2) \equiv 0,
\end{equation}

(154)

where the precision of the determination of $M_\phi^2$ depends on the precision of the calculation of $\Sigma_\phi$.

For our particular application with exactly two external meson lines, the relevant interaction Lagrangians can be written as

\[ \mathcal{L}_{\text{int}} = \mathcal{L}_2^{4\phi} + \mathcal{L}_4^{2\phi}, \]

(155)

where

\[ \mathcal{L}_2^{4\phi} = \frac{1}{24F_0^2} \left\{ \text{Tr}[\phi, \partial_\mu \phi] \phi \phi^\mu \phi + B_0 \text{Tr}(\mathcal{M} \phi^4) \right\}, \]

(156)
Figure 10: Contribution of the pion loops to the $\pi^0$ self energy.

$$L_4^{2\phi} = \frac{1}{2} (a_\pi \pi^0 \pi^0 + b_\pi \partial_\mu \pi^0 \partial^\mu \pi^0) - a_\pi \pi^+ \pi^- - b_\pi \partial_\mu \pi^+ \partial^\mu \pi^-$$

$$- a_K K^+ K^- - b_K \partial_\mu K^+ \partial^\mu K^- - a_K K^0 K^0 - b_K \partial_\mu K^0 \partial^\mu \bar{K}^0$$

$$- \frac{1}{2} (a_\eta \eta^2 + b_\eta \partial_\mu \eta \partial^\mu \eta).$$ \hspace{1cm} (157)

The constants $a_\phi$ and $b_\phi$ are given by

$$a_\pi = \frac{64 B_0^2}{F_0^2} \left[ (2\hat{m} + m_s)\hat{m} L_6 + \hat{m}^2 L_8 \right],$$

$$b_\pi = -\frac{16 B_0}{F_0^2} \left[ (2\hat{m} + m_s) L_4 + \hat{m} L_5 \right],$$

$$a_K = \frac{32 B_0^2}{F_0^2} \left[ (2\hat{m} + m_s)(\hat{m} + m_s) L_6 + \frac{1}{2} (\hat{m} + m_s)^2 L_8 \right],$$

$$b_K = -\frac{16 B_0}{F_0^2} \left[ (2\hat{m} + m_s) L_4 + \frac{1}{2} (\hat{m} + m_s) L_5 \right]$$

$$a_\eta = \frac{64 B_0^2}{3 F_0^2} \left[ (2\hat{m} + m_s)(\hat{m} + 2 m_s) L_6 + 2(\hat{m} - m_s)^2 L_7 + (\hat{m}^2 + 2 m_s^2) L_8 \right],$$

$$b_\eta = -\frac{16 B_0}{F_0^2} \left[ (2\hat{m} + m_s) L_4 + \frac{1}{3} (\hat{m} + 2 m_s) L_5 \right].$$ \hspace{1cm} (158)

At $O(q^4)$ the self energies are of the form

$$\Sigma_{\phi,4}(p^2) = A_\phi + B_\phi p^2,$$ \hspace{1cm} (159)

where the constants $A_\phi$ and $B_\phi$ receive a tree-level contribution from $L_4$ and a one-loop contribution with a vertex from $L_2$ (see Fig. 8). For the tree-level contribution of $L_4$ this is easily seen, because the Lagrangians of Eq. (157) contain either exactly two derivatives of the fields or no derivatives at all. For example, the contact contribution for the $\eta$ reads

$$-i \Sigma_{\eta,4}^{\text{tree}}(p^2) = -i (a_\eta + b_\eta p^2).$$

For the one-loop contribution the argument is as follows. The Lagrangian $L_4^{2\phi}$ contains either two derivatives or no derivatives at all which, symbolically, can be written as $\phi \phi \partial^2 \phi$ and $\phi^4$, respectively. The first term results in $M^2$ or $p^2$, depending on whether the $\phi$ or the $\partial \phi$ are contracted with the external fields. The “mixed” situation vanishes upon integration. The second term, $\phi^4$, does not generate a momentum dependence.

As a specific example, we evaluate the pion-loop contribution to the $\pi^0$ self energy (see Fig. 10) by applying the Feynman rule of Eq. (120) for $a = c = 3, p_a = p_c = p, b = d = j,$ and $p_b = p_d = k$:
\[
\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{3F_0} \left[ -4p^2 - 4k^2 + 5M_{\pi,2}^2 \right] \frac{i}{k^2 - M_{\pi,2}^2 + i0^+},
\]
where 1/2 is a symmetry factor. Since the integral diverges, we consider its extension to \(n\) dimensions. In addition to the loop-integral of Eq. (134), we need

\[
\mu^{4-n}i \int \frac{d^n k}{(2\pi)^n} \frac{k^2}{k^2 - M^2 + i0^+} = \mu^{4-n}i \int \frac{d^n k}{(2\pi)^n} \frac{k^2 - M^2 + M^2}{k^2 - M^2 + i0^+},
\]
where we have added 0 = \(M^2 + M^2\) in the numerator. We make use of

\[
\mu^{4-n}i \int \frac{d^n k}{(2\pi)^n} = 0
\]
in dimensional regularization which is “shown” as follows. Consider the (more general) integral

\[
\int d^n k (k^2)^p,
\]
substitute \(k = \lambda k' (\lambda > 0)\), and relabel \(k' = k\)

\[
= \lambda^{n+2p} \int d^n k (k^2)^p.
\]
Since \(\lambda > 0\) is arbitrary and, for fixed \(p\), the result is to hold for arbitrary \(n\), Eq. (160) is set to zero in dimensional regularization. We emphasize that the vanishing of Eq. (160) has the character of a prescription. The integral does not depend on any scale and its analytic continuation is ill defined in the sense that there is no dimension \(n\) where it is meaningful. It is ultraviolet divergent for \(n + 2p \geq 0\) and infrared divergent for \(n + 2p \leq 0\).

We then obtain

\[
\mu^{4-n}i \int \frac{d^n k}{(2\pi)^n} \frac{k^2}{k^2 - M^2 + i0^+} = M^2 I(M^2, \mu^2, n),
\]
with \(I(M^2, \mu^2, n)\) of Eq. (134). The pion-loop contribution to the \(\pi^0\) self energy is thus

\[
\frac{i}{6F_0^2} (-4p^2 + M_{\pi,2}^2) I(M_{\pi,2}^2, \mu^2, n),
\]
which is indeed of the type discussed in Eq. (159) and diverges as \(n \to 4\).

After analyzing all loop contributions and combining them with the tree-level contributions of Eqs. (158), the constants \(A_\phi\) and \(B_\phi\) of Eq. (159) are given by

\[
A_\pi = \frac{M_{\pi,2}^2}{F_0^2} \left\{ \frac{-1}{6} I(M_{\pi,2}^2) - \frac{1}{6} I(M_{\eta,2}^2) - \frac{1}{3} I(M_{K,2}^2) + 32[(2\hat{m} + m_s)B_0 L_6 + \hat{m} B_0 L_5] \right\},
\]
loop contribution

\[
B_\pi = \frac{2 I(M_{\pi,2}^2)}{3} + \frac{1 I(M_{K,2}^2)}{3} - \frac{16 B_0}{F_0^2} [(2\hat{m} + m_s)L_4 + \hat{m}L_5],
\]

\[
A_K = \frac{M_{K,2}^2}{F_0^2} \left\{ \frac{1}{12} I(M_{\eta,2}^2) - \frac{1}{4} I(M_{K,2}^2) + \frac{1}{2} I(M_{K,2}^2) + 32[(2\hat{m} + m_s)B_0 L_6 + \frac{1}{2} (\hat{m} + m_s) B_0 L_8] \right\},
\]

\[
B_K = \frac{1 I(M_{\eta,2}^2)}{4} + \frac{1 I(M_{\pi,2}^2)}{4} + \frac{1 I(M_{K,2}^2)}{2} - \frac{16 B_0}{F_0^2} [(2\hat{m} + m_s)L_4 + \frac{1}{2} (\hat{m} + m_s)L_5],
\]

\[
A_\eta = \frac{M_{\eta,2}^2}{F_0^2} \left[ \frac{-2}{3} I(M_{\eta,2}^2) + \frac{M_{\eta,2}^2}{F_0^2} \frac{1}{6} I(M_{\eta,2}^2) - \frac{1}{2} I(M_{\pi,2}^2) + \frac{1}{3} I(M_{K,2}^2) \right] + \frac{M_{\eta,2}^2}{F_0^2} [16 M_{\eta,2}^2 L_8 + 32(2\hat{m} + m_s)B_0 L_6] + \frac{128 B_0^2 (\hat{m} - m_s)^2}{9 F_0^2} (3L_7 + L_8),
\]

\[
B_\eta = \frac{I(M_{K,2}^2)}{F_0^2} - \frac{16}{F_0^2} (2\hat{m} + m_s) B_0 L_4 - 8 \frac{M_{\eta,2}^2}{F_0^2} L_5,
\]

(161)
where, for simplicity, we have suppressed the dependence on the scale \( \mu \) and the number of dimensions \( n \) in the integrals \( I(M^2, \mu^2, n) \) [see Eq. (134)]. Both the integrals \( I \) and the bare coefficients \( L_i \) (with the exception of \( L_7 \)) have \( 1/(n-4) \) poles and finite pieces. In particular, the coefficients \( A_\phi \) and \( B_\phi \) are not finite as \( n \to 4 \) showing that they do not correspond to observables.

The masses at \( \mathcal{O}(q^4) \) are determined by solving Eq. (154) with the predictions of Eq. (159) for the self energies,

\[
M^2_\phi = M^2_{\phi,2} + A_\phi + B_\phi M^2_\phi,
\]

from which we obtain

\[
M^2_\phi = M^2_{\phi,2} \frac{A_\phi}{1 - B_\phi} = M^2_{\phi,2} (1 + B_\phi) + A_\phi + \mathcal{O}(q^6),
\]

because \( A_\phi = \mathcal{O}(q^4) \) and \( \{B_\phi, M^2_{\phi,2}\} = \mathcal{O}(q^2) \). Expressing the bare coefficients \( L_i \) in Eq. (161) in terms of the renormalized coefficients by using Eq. (141), the results for the masses of the Goldstone bosons at \( \mathcal{O}(q^4) \) read [Gasser and Leutwyler, 1985]

\[
\begin{align*}
M^2_{\pi,4} &= M^2_{\pi,2} \left[ 1 + \frac{M^2_{\pi,2}}{32\pi^2 F_0^2} \ln \left( \frac{M^2_{\pi,2}}{\mu^2} \right) - \frac{M^2_{\eta,2}}{96\pi^2 F_0^2} \ln \left( \frac{M^2_{\eta,2}}{\mu^2} \right) \right] \\
&\quad + \frac{16}{F_0^2} \left[ (2\hat{m} + m_s) B_0 (2L^r_6 - L^r_4) + \hat{m} B_0 (2L^r_8 - L^r_5) \right],
\end{align*}
\]

\[
\begin{align*}
M^2_{K,4} &= M^2_{K,2} \left[ 1 + \frac{M^2_{\eta,2}}{48\pi^2 F_0^2} \ln \left( \frac{M^2_{\eta,2}}{\mu^2} \right) \right] \\
&\quad + \frac{16}{F_0^2} \left[ (2\hat{m} + m_s) B_0 (2L^r_6 - L^r_4) + \frac{1}{2} (\hat{m} + m_s) B_0 (2L^r_8 - L^r_5) \right],
\end{align*}
\]

\[
\begin{align*}
M^2_{\eta,4} &= M^2_{\eta,2} \left[ 1 + \frac{M^2_{\eta,2}}{16\pi^2 F_0^2} \ln \left( \frac{M^2_{\eta,2}}{\mu^2} \right) - \frac{M^2_{\eta,2}}{24\pi^2 F_0^2} \ln \left( \frac{M^2_{\eta,2}}{\mu^2} \right) \right] \\
&\quad + \frac{16}{F_0^2} (2\hat{m} + m_s) B_0 (2L^r_6 - L^r_4) + \frac{M^2_{\eta,2}}{F_0^2} (2L^r_8 - L^r_5) \\
&\quad + \frac{M^2_{\pi,2}}{96\pi^2 F_0^2} \ln \left( \frac{M^2_{\eta,2}}{\mu^2} \right) - \frac{M^2_{\pi,2}}{32\pi^2 F_0^2} \ln \left( \frac{M^2_{\pi,2}}{\mu^2} \right) + \frac{M^2_{K,2}}{48\pi^2 F_0^2} \ln \left( \frac{M^2_{K,2}}{\mu^2} \right) \right] \\
&\quad + \frac{128}{9} \frac{B_0^2 (\hat{m} - m_s)^2}{F_0^2} (3L^r_7 + L^r_5). 
\end{align*}
\]

First of all, we note that the expressions for the masses are finite. The infinite parts of the coefficients \( L_i \) of the Lagrangian of Gasser and Leutwyler exactly cancel the divergent terms resulting from the integrals. This is the reason why the bare coefficients \( L_i \) must be infinite. Furthermore, at \( \mathcal{O}(q^4) \) the masses of the Goldstone bosons vanish, if the quark masses are sent to zero. This is, of course, what we had expected from QCD in the chiral limit but it is comforting to see that the self interaction in \( L_2 \) (in the absence of quark masses) does not generate Goldstone boson masses at higher order. At \( \mathcal{O}(q^4) \), the squared Goldstone boson masses contain terms which are analytic in the quark masses, namely, of the form \( m_q^2 \) multiplied by the renormalized low-energy coupling constants \( L_i^r \). However, there are also non-analytic terms of the type \( m_q^2 \ln(m_q) \)—so-called chiral logarithms—which do not involve new parameters. Such a behavior is an illustration of the mechanism found by Li and Pagels [Li and Pagels, 1971], who noticed that a perturbation theory around a symmetry which is realized in the Nambu-Goldstone mode results in both analytic as well as non-analytic expressions in the perturbation. Finally, the scale dependence of the renormalized coefficients \( L_i^r \) of Eq. (141) is by construction such that it cancels the scale dependence of the chiral logarithms. Thus, physical observables do not depend on the scale \( \mu \). It
is straightforward to verify this statement by differentiating Eqs. (162) - (164) with respect to $\mu$ and by making use of

$$\frac{dL_i^\gamma(\mu)}{d\mu} = -\frac{\Gamma_i}{16\pi^2\mu'},$$

where the $\Gamma_i$ are given in Table 7.

### 3.3.4 Electromagnetic polarizabilities of the pion

Another strong constraint provided by chiral symmetry is the connection between the electromagnetic polarizabilities of the charged pion and the radiative pion beta decay. In the framework of classical electrodynamics, the electric and magnetic polarizabilities $\alpha$ and $\beta$ describe the response of a system to a static, uniform, external electric and magnetic field in terms of induced electric and magnetic dipole moments. In principle, empirical information on the pion polarizabilities can be obtained from the differential cross section of low-energy Compton scattering on a charged pion,

$$\frac{d\sigma}{d\Omega_{\text{lab}}} = \left(\frac{\omega'}{\omega}\right)^2 \frac{e^2}{4\pi M_\pi} \left\{ \frac{e^2}{4\pi M_\pi} \frac{1}{2} \left[ (\alpha + \beta)_{\pi^+}(1 + z)^2 + (\alpha - \beta)_{\pi^+}(1 - z)^2 \right] \right\} + \cdots,$$

where $z = \hat{q} \cdot \hat{q}'$ and $\omega'/\omega = [1 + \omega(1 - z)/M_\pi]$. The forward and backward differential cross sections are sensitive to $(\alpha + \beta)_{\pi^+}$ and $(\alpha - \beta)_{\pi^+}$, respectively.

Within the framework of the partially conserved axial-vector (PCAC) hypothesis and current algebra the electromagnetic polarizabilities of the charged pion are related to the radiative charged-pion beta decay $\pi^+ \rightarrow e^+ e^0 \gamma$ [Terent’ev, 1973]. The result obtained using ChPT at leading non-trivial order ($\mathcal{O}(q^4)$) [Bijnens and Cornet, 1988] is equivalent to the original PCAC result,

$$\alpha_{\pi^+} = \beta_{\pi^+} = \frac{2e^2}{4\pi} \frac{1}{(4\pi F_\pi^2)^2 M_\pi} \frac{\bar{l}_\Delta}{6},$$

where $\bar{l}_\Delta \equiv (\bar{l}_6 - \bar{l}_5)$ is a linear combination of scale-independent parameters of the two-flavor $\mathcal{O}(q^4)$ Lagrangian [Gasser and Leutwyler, 1984]. At $\mathcal{O}(q^4)$ this difference is related to the ratio $\gamma = F_A/F_V$ of the pion axial-vector form factor $F_A$ and the vector form factor $F_V$ of radiative pion beta decay [Gasser and Leutwyler, 1984], $\gamma = \bar{l}_\Delta/6$. Once this ratio is known, chiral symmetry makes an absolute prediction for the polarizabilities. This situation is similar to the s-wave $\pi\pi$-scattering lengths of Eq. (126) which are predicted once $F_\pi$ is known. Using the most recent determination $\gamma = 0.443 \pm 0.015$ by the PIBETA Collaboration [Friez et al., 2004] (assuming $F_V = 0.0259$ obtained from the conserved vector current hypothesis) results in the $\mathcal{O}(q^4)$ prediction $\alpha_{\pi^+} = (2.64 \pm 0.09) \times 10^{-4} \text{fm}^3$, where the estimate of the error is only the one due to the error of $\gamma$ and does not include effects from higher orders in the quark-mass expansion.

Corrections to the leading-order PCAC result have been calculated at $\mathcal{O}(q^6)$ and turn out to be rather small [Bürgi, 1996], [Gasser et al., 2006]. Using updated values for the LECs, the predictions of [Gasser et al., 2006] are

$$\begin{align*}
(\alpha + \beta)_{\pi^+} &= 0.16 \times 10^{-4} \text{fm}^3, \\
(\alpha - \beta)_{\pi^+} &= (5.7 \pm 1.0) \times 10^{-4} \text{fm}^3.
\end{align*}$$

The corresponding corrections to the $\mathcal{O}(q^4)$ result indicate a similar rate of convergence as for the $\pi\pi$-scattering lengths [Gasser and Leutwyler, 1984], [Bijnens et al., 1996]. The error for $(\alpha + \beta)_{\pi^+}$ is of the order $0.1 \times 10^{-4} \text{fm}^3$, mostly from the dependence on the scale at which the $\mathcal{O}(q^6)$ low-energy coupling constants are estimated by resonance saturation.

As there is no stable pion target, empirical information about the pion polarizabilities is not easy to obtain. For that purpose, one has to consider reactions which contain the Compton scattering amplitude
Figure 11: The reaction $\gamma p \rightarrow \gamma \pi^+ n$ contains Compton scattering on a pion as a sub diagram in the $t$ channel, where $t = (p_n - p_p)^2$.

Figure 12: Differential cross section averaged over 537 MeV $< E_\gamma < 817$ MeV and $1.5 M^2_\pi < s_1 < 5M^2_\pi$. Solid line: model 1; dashed line: model 2; dotted line: fit to experimental data.

The potential of studying the influence of the pion polarizabilities on radiative pion photoproduction from the proton was extensively studied in [Drechsel and Fil'kov, 1994]. In terms of Feynman diagrams, the reaction $\gamma p \rightarrow \gamma \pi^+ n$ contains real Compton scattering on a charged pion as a pion pole diagram (see Fig. 11). In the recent experiment on $\gamma p \rightarrow \gamma \pi^+ n$ at the Mainz Microtron MAMI [Ahrens et al., 2005], the cross section was obtained in the kinematic region 537 MeV $< E_\gamma < 817$ MeV, $140^\circ \leq \theta^\text{cm}_{\gamma\gamma'} \leq 180^\circ$. Figure 12 shows the experimental data, averaged over the full photon beam energy interval and over the squared pion-photon center-of-mass energy $s_1$ from $1.5 M^2_\pi$ to $5 M^2_\pi$ as a function of the squared pion momentum transfer $t$ in units of $M^2_\pi$. For such small values of $s_1$, the differential cross section is expected to be insensitive to the pion polarizabilities. Also shown are two model calculations: model 1 (solid curve) is a simple Born approximation using the pseudoscalar pion-nucleon interaction including the anomalous magnetic moments of the nucleon; model 2 (dashed curve) consists of pole terms without the anomalous magnetic moments but including contributions from the resonances $\Delta(1232)$, $P_{11}(1440)$, $D_{13}(1520)$ and $S_{11}(1535)$. The dotted curve is a fit to the experimental data.

The kinematic region where the polarizability contribution is biggest is given by $5M^2_\pi < s_1 < 15M^2_\pi$ and $-12M^2_\pi < t < -2M^2_\pi$. Figure 13 shows the cross section as a function of the beam energy integrated over $s_1$ and $t$ in this second region. The dashed and solid lines (dashed-dotted and dotted lines) refer
Figure 13: The cross section of the process $\gamma p \rightarrow \gamma \pi^+ n$ integrated over $s_1$ and $t$ in the region where the contribution of the pion polarizability is biggest and the difference between the predictions of the theoretical models under consideration does not exceed 3%. The dashed and dashed-dotted lines are predictions of model 1 and the solid and dotted lines of model 2 for $(\alpha - \beta)_{\pi^+} = 0$ and $(\alpha - \beta)_{\pi^+} = 14 \times 10^{-4}$ fm$^3$, respectively.

to models 1 and 2, respectively, each with $(\alpha - \beta)_{\pi^+} = 0$ ($(\alpha - \beta)_{\pi^+} = 14 \times 10^{-4}$ fm$^3$). By comparing the experimental data of the 12 points with the predictions of the models, the corresponding values of $(\alpha - \beta)_{\pi^+}$ for each data point have been determined in combination with the corresponding statistical and systematic errors. The result extracted from the combined analysis of the 12 data points reads

$$ (\alpha - \beta)_{\pi^+} = (11.6 \pm 1.5_{\text{stat}} \pm 3.0_{\text{syst}} \pm 0.5_{\text{mod}}) \times 10^{-4} \text{fm}^3 $$

(167)

and has to be compared with the ChPT result of $(5.7 \pm 1.0) \times 10^{-4}$ fm$^3$, which deviates by 2 standard deviations from the experimental result.

Clearly, the model-dependent input to the result of Eq. (167) deserves further study. In particular, the model error was estimated by comparing the analysis with two specific models. In Ref. [Kao et al., 2007] radiative pion photoproduction was studied in the framework of heavy-baryon chiral perturbation theory at the one-loop level. Unfortunately, the kinematical conditions of the MAMI experiment were not explicitly considered. It was argued that the extraction of pion polarizabilities is, in principle, possible and that the main uncertainty in the extraction arises from the effect of two structures of the $O(q^4)$ Lagrangian.

The Primakoff method was used at Serpukhov with the result [Antipov et al., 1983]

$$ (\alpha - \beta)_{\pi^+} = (13.6 \pm 2.8_{\text{stat}} \pm 2.4_{\text{syst}}) \times 10^{-4} \text{fm}^3 $$

(168)

in agreement with the value from MAMI. Recently, also the COMPASS Collaboration at CERN has investigated this reaction, and the data analysis is underway [Guskov, 2008]. Unfortunately, the third method based on the reactions $e^+ e^- \rightarrow \gamma \gamma \rightarrow \pi^+ \pi^-$, has led to even more contradictory results (see Ref. [Gasser et al., 2006]).

Also on the theoretical side there has been a long-standing problem. The application of dispersion sum rules as performed in [Fil’kov and Kashevarow, 1999], [Fil’kov and Kashevarow, 2006] yields $(\alpha - \beta)_{\pi^+} = (13.0^{+2.8}_{-1.9}) \times 10^{-4}$ fm$^3$ which provides an even more pronounced discrepancy with the predictions of chiral perturbation theory than the MAMI result [Gasser et al., 2006]. These dispersion relations are based on specific forms for the absorptive part of the Compton amplitudes. In Ref. [Pasquini et al., 2008], the analytic properties of these forms have been examined and the strong enhancement of intermediate-meson contributions was shown to be connected with spurious singularities.
It was shown that the results of dispersion theory and effective field theory are not in conflict, once the basic requirements of dispersion relations are taken into account.

Clearly, the electromagnetic polarizabilities of the charged pion remain one of the challenging topics of hadronic physics in the low-energy domain. Chiral symmetry provides a strong constraint in terms of radiative pion beta decay and mesonic chiral perturbation theory makes a firm prediction beyond the current algebra result at the two-loop level. Both the experimental determination as well as the theoretical extraction from experiment require further efforts.

4 Baryonic chiral perturbation theory

4.1 Lagrangian

So far we have considered the purely mesonic sector involving the interaction of Goldstone bosons with each other and with the external fields. Now we want to describe matrix elements with a single baryon in the initial and final states.

4.1.1 Transformation properties of the fields

Our aim is the most general description of the interaction of baryons with Goldstone bosons and external fields at low energies. For that purpose we not only need to specify the transformation behavior of the Goldstone bosons and external fields but also of the remaining dynamical fields entering the Lagrangian. Our discussion follows Refs. [Georgi, 1984], [Gasser et al., 1988]. Consider the nucleon doublet and the octet of $\frac{1}{2}^+$ baryons (see Fig. 2),

\[
\Psi = \begin{pmatrix} p \\ n \end{pmatrix},
\]

\[
B = \sum_{a=1}^{8} B_a \lambda_a = \begin{pmatrix} \frac{1}{\sqrt{2}} \Sigma^0 + \frac{1}{\sqrt{6}} \Lambda \\ \Sigma^- \\ \Xi^- \\ p \\ \Sigma^+ \\ \Xi^0 \\ n \\ -\frac{1}{\sqrt{2}} \Sigma^0 + \frac{1}{\sqrt{6}} \Lambda \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \Sigma^0 + \frac{1}{\sqrt{6}} \Lambda \\ \Sigma^- \\ \Xi^- \\ p \\ \Sigma^+ \\ \Xi^0 \\ n \\ -\frac{1}{\sqrt{2}} \Sigma^0 + \frac{1}{\sqrt{6}} \Lambda \end{pmatrix}.
\]

Each entry of $\Psi$ and $B$, respectively, is a complex, four-component Dirac field. In contradistinction to the case of the Goldstone-boson matrix $\phi$ of Eq. (88), we have $B \neq B^\dagger$. The representation of the isospin group SU(2)$_V$ and the flavor group SU(3)$_V$ on $\{\Psi\}$ and $\{B\}$, respectively, is given by

\[
\Psi \mapsto V \Psi, \quad V \in \text{SU}(2)_V, \tag{171}
\]

\[
B \mapsto VBV^\dagger, \quad V \in \text{SU}(3)_V, \tag{172}
\]

i.e., $\Psi$ transforms under the fundamental representation of SU(2) and $B$ transforms under the adjoint representation of SU(3). Starting from Eqs. (171) and (172) we will discuss realizations of SU(2)$_L \times$ SU(2)$_R$ and SU(3)$_L \times$ SU(3)$_R$ on $\{\Psi\}$ and $\{B\}$, respectively.

Let us begin with $G = \text{SU}(2)_L \times \text{SU}(2)_R$. Recall that the transformation of Eq. (85),

\[
U \mapsto RUL^\dagger,
\]

defines a nonlinear realization of $G$ on $\{U\}$. Introducing $u^2 = U$, we define the SU(2)-valued function $K(L, R, U)$ by

\[
u \mapsto u' = \sqrt{R}UL^\dagger \equiv RuK^{-1}(L, R, U), \quad \text{i.e.,} \quad K(L, R, U) = u'^{-1}Ru = \sqrt{R}UL^\dagger^{-1}R\sqrt{U}. \tag{173}
\]
The transformation
\[ \varphi(g) : \begin{pmatrix} U \\ \Psi \end{pmatrix} \mapsto \begin{pmatrix} U' \\ \Psi' \end{pmatrix} = \begin{pmatrix} RUL^\dagger \\ K(L, R, U)\Psi \end{pmatrix} \] (174)
defines an operation of the group \( G \) on the set \( \{ (U, \Psi) \} \). This is true, because (a) the identity of \( G \) leaves any pair \( (U, \Psi) \) invariant and (b) the transformation satisfies the homomorphism property
\[ \varphi(g_1)\varphi(g_2) \begin{pmatrix} U \\ \Psi \end{pmatrix} = \varphi(g_1) \begin{pmatrix} R_2UL_2^\dagger \\ K(L_2, R_2, U)\Psi \end{pmatrix} = \begin{pmatrix} R_1R_2UL_2^\dagger \\ K(L_1, R_1, R_2UL_2^\dagger)K(L_2, R_2, U)\Psi \end{pmatrix} \]
where we made use of
\[ K(L_1, R_1, R_2UL_2^\dagger)K(L_2, R_2, U) = K((L_1L_2), (R_1R_2), U). \]

Note that for a general group element \( g = (L, R) \in G \) the transformation behavior of \( \Psi \) depends on \( U \). The exception to this rule is the case of an isospin transformation \( R = L = V \), where, because of \( U' = u'^2 = V_uV^\dagger V_uV^\dagger = V_uV^\dagger V_uV^\dagger \), one has \( u' = V_uV^\dagger \). Comparing with Eq. (173), we obtain \( K^{-1}(V, V, U) = V^\dagger \) or \( K(V, V, U) = V \). This is consistent with our starting point that \( \Psi \) transforms linearly as an isospin doublet under the isospin subgroup \( H = SU(2)_V \) of \( G = SU(2)_L \times SU(2)_R \). Recall that the symmetry of the vacuum determines the multiplet structure of the spectrum [Coleman, 1966].

For \( G = SU(3)_L \times SU(3)_R \) one uses
\[ \varphi(g) : \begin{pmatrix} U \\ B \end{pmatrix} \mapsto \begin{pmatrix} U' \\ B' \end{pmatrix} = \begin{pmatrix} RUL^\dagger \\ K(L, R, U)BK^\dagger(L, R, U) \end{pmatrix} \] (175)
where \( K \) is defined completely analogously to Eq. (173) after inserting the corresponding SU(3) matrices.

The generalization to other multiplets is straightforward. One first specifies the transformation behavior under the subgroup \( H \) in terms of \( V \) and \( V^\dagger \). In order to find the transformation behavior under \( G \) one simply replaces \( V \rightarrow K \) and \( V^\dagger \rightarrow K^\dagger \).

### 4.1.2 Baryonic effective Lagrangian at lowest order

Given the dynamical fields of Eqs. (174) and (175) and their transformation properties, we will now discuss the most general effective baryonic Lagrangian at lowest order. We will start with the effective \( \pi N \) Lagrangian \( \mathcal{L}_{\pi N}^{(1)} \) which we demand to have a local \( SU(2)_L \times SU(2)_R \times U(1)_V \) symmetry. The transformation behavior of the external fields is given in Eq. (49), whereas \( U \) and the nucleon doublet transform as
\[ \begin{pmatrix} U(x) \\ \Psi(x) \end{pmatrix} \mapsto \begin{pmatrix} \exp[-i\Theta(x)]K[V_L(x), V_R(x), U(x)]\Psi(x) \\ V_R(x)U(x)V_L^\dagger(x) \end{pmatrix} \] (176)
The local character of the transformation implies that we need to introduce a covariant derivative \( D_\mu \Psi \) with the usual property that it transforms in the same way as \( \Psi \),
\[ D_\mu \Psi = (\partial_\mu + \Gamma_\mu - iv^{(\mu)}_\nu)\Psi, \] (177)
where the so-called connection is given by
\[ \Gamma_\mu = \frac{1}{2} [u^\dagger(\partial_\mu - iv^{(\mu)}_\nu)u + u(\partial_\mu - iv^{(\mu)}_\nu)u^\dagger]. \] (178)
Since $K$ not only depends on $V_L$ and $V_R$ but also on $U$, the covariant derivative contains besides the external fields also $u$ and $u^\dagger$ and their derivatives. At $\mathcal{O}(q)$ there exists another Hermitian building block, the so-called vielbein,

$$u_\mu \equiv i \left[ u^\dagger (\partial_\mu - i r_\mu) u - u (\partial_\mu - i l_\mu) u^\dagger \right],$$

which under parity transforms as an axial vector, $u_\mu \mapsto -u^\mu$, and under $SU(2)_L \times SU(2)_R \times U(1)_V$, transforms as $u_\mu \mapsto K u_\mu K^\dagger$.

The structure of the most general effective $\pi N$ Lagrangian describing processes with a single nucleon in the initial and final states is of the type $\hat{\Psi} \hat{O} \Psi$, where $\hat{O}$ is an operator acting in Dirac and flavor space, transforming under $SU(2)_L \times SU(2)_R \times U(1)_V$ as $K \hat{O} K^\dagger$. The Lagrangian must be a Hermitian Lorentz scalar which is even under the discrete symmetries $C$, $P$, and $T$. The most general such Lagrangian with the smallest number of derivatives is given by [Gasser et al., 1988]

$$\mathcal{L}^{(1)}_{\pi N} = \bar{\Psi} \left( i \not\partial - m + \frac{g_A}{2} \gamma^\mu \gamma_5 u_\mu \right) \Psi.$$  

It contains two parameters not determined by chiral symmetry: the chiral limit $m$ of the nucleon mass $m_N$ and the chiral limit $g_A$ of the axial-vector coupling constant $g_A$. The physical value of $g_A$ is determined from neutron beta decay and is given by $g_A = 1.2695 \pm 0.0029$. The overall normalization of the Lagrangian is chosen such that in the case of no external fields and no pion fields it reduces to that of a free nucleon of mass $m$.

Similarly as in the mesonic case, the Lagrangian $\mathcal{L}^{(1)}_{\pi N}$ has predictive power once the two parameters have been identified. For example, a tree-level calculation of pion-nucleon scattering produces the famous Weinberg-Tomozawa relation for the $s$-wave $\pi N$-scattering lengths [Weinberg, 1966], [Tomozawa, 1966].

$$a^I = -\frac{M_\pi}{8\pi(1+\mu)F_\pi^2} \left[ I(I+1) - \frac{3}{4} - 2 \right],$$

where $I = 1/2$ or $I = 3/2$ refers to the total isospin of the $\pi N$ system. As in $\pi\pi$ scattering, the $s$-wave $\pi N$-scattering lengths vanish in the chiral limit, i.e., Goldstone bosons interact “weakly” with other hadrons in the zero-energy and mass limit.

Since the nucleon mass $m_N$ does not vanish in the chiral limit, the zeroth component $\partial^0$ of the partial derivative acting on the nucleon field does not produce a “small” quantity. This results in new features of the chiral power counting in the baryonic sector. The counting of the external fields as well as of covariant derivatives acting on the mesonic fields remains the same as in mesonic chiral perturbation theory. On the other hand, the counting of bilinears $\bar{\Psi} \Gamma \Psi$ is probably easiest understood by investigating the matrix elements of positive-energy plane-wave solutions to the free Dirac equation in the Dirac representation:

$$\psi^{(+)}(t, \vec{x}) = \exp(-i \vec{p} \cdot \vec{x}) \sqrt{E + m_N} \left( \frac{\chi}{E + m_N} \right),$$

where $\chi$ denotes a two-component Pauli spinor and $p^\mu = (E, \vec{p})$ with $E = \sqrt{\vec{p}^2 + m_N^2}$. In the low-energy limit, i.e., for non-relativistic kinematics, the lower (small) component is suppressed as $|\vec{p}|/m_N$ in comparison with the upper (large) component. For the analysis of the bi-linears it is convenient to divide the 16 Dirac matrices into even and odd ones, $\mathcal{E} = \{ 1, \gamma_0, \gamma_5 \gamma_i, \sigma_{ij} \}$ and $\mathcal{O} = \{ \gamma_5, \gamma_5 \gamma_0, \gamma_i, \sigma_{0i} \}$ [Foldy and Wouthuysen, 1950], respectively, where odd matrices couple large and small components but not large with large, whereas even matrices do the opposite. Finally, $i\partial^\mu$ acting on the nucleon solution produces $p^\mu$ which we write symbolically as $p^\mu = (m_N, \vec{0}) + (E - m_N, \vec{p})$, where we count the second
term as $\mathcal{O}(q)$, i.e., as a small quantity. We are now in the position to summarize the chiral counting scheme for the (new) elements of baryon chiral perturbation theory [Krause, 1990]:

$$\Psi, \bar{\Psi} = \mathcal{O}(q^0), D_\mu \Psi = \mathcal{O}(q^0), (iD - m)\Psi = \mathcal{O}(q),$$

where the order given is the minimal one. For example, $\gamma_\mu$ has both an $\mathcal{O}(q^0)$ piece, $\gamma_0$, as well as an $\mathcal{O}(q)$ piece, $\gamma_i$. Note that because of the additional spin degree of freedom the baryonic effective Lagrangian contains both odd and even chiral orders. A rigorous non-relativistic reduction may be achieved in the framework of the Foldy-Wouthuysen method [Foldy and Wouthuysen, 1950], [Fearing et al., 1994] or the heavy-baryon approach [Jenkins and Manohar, 1991], [Bernard et al., 1992a].

The construction of the SU(3)$_L \times$ SU(3)$_R$ Lagrangian proceeds similarly except for the fact that the baryon fields are contained in the $3 \times 3$ matrix of Eq. (170) transforming as $K B K$. As in the mesonic sector, the building blocks are written as products transforming as $K \cdots K$ with a trace taken at the end. The lowest-order Lagrangian reads [Georgi, 1984], [Krause, 1990]

$$L^{(1)}_{MB} = \text{Tr} \left[ \bar{B} \left( iD - M_0 \right) B \right] - \frac{D}{2} \text{Tr} \left( \bar{B} \gamma^\mu \gamma_5 \{u_\mu, B\} \right) - \frac{F}{2} \text{Tr} \left( \bar{B} \gamma^\mu \gamma_5 [u_\mu, B] \right),$$

where $M_0$ denotes the mass of the baryon octet in the chiral limit. The covariant derivative of $B$ is defined as

$$D_\mu B = \partial_\mu B + [\Gamma_\mu, B],$$

with $\Gamma_\mu$ of Eq. (178) [for SU(3)$_L \times$ SU(3)$_R$]. The constants $D$ and $F$ may be determined by fitting the semi-leptonic decays $B \to B' + e^- + \bar{\nu}_e$ at tree level [Borasoy, 1999]:

$$D = 0.80, \quad F = 0.50.$$  

4.2 Renormalization and power counting

In the following discussion we will restrict ourselves to the two-flavor case. The effective Lagrangian relevant to the one-nucleon sector consists of the sum of the purely mesonic and $\pi N$ Lagrangians, respectively,

$$L_{\text{eff}} = L_\pi + L_{\pi N} = L^{(2)}_\pi + L^{(4)}_\pi + \cdots + L^{(1)}_{\pi N} + L^{(2)}_{\pi N} + \cdots,$$

which are organized in a derivative and quark-mass expansion. Tree-level calculations involving the sum $L^{(2)}_\pi + L^{(1)}_{\pi N}$ reproduce the current algebra results. The higher-order Lagrangians of the $\pi N$ sector can be found in [Gasser et al., 1988], [Ecker and Mojzis, 1996], [Fettes et al., 2000]. When studying higher orders in perturbation theory in terms of loop corrections one encounters ultraviolet divergences. As a preliminary step, the loop integrals are regularized, typically by means of dimensional regularization. In the process of renormalization the counter terms are adjusted such that they absorb all the ultraviolet divergences occurring in the calculation of loop diagrams. This will be possible, because we include in the Lagrangian all of the infinite number of interactions allowed by symmetries [Weinberg, 1995]. At the end the regularization is removed by taking the limit $n \to 4$. Moreover, when renormalizing, we still have the freedom of choosing a renormalization condition. As we will see, the power counting is intimately connected with choosing a suitable renormalization condition.

4.2.1 The generation of counter terms

Before discussing the power-counting problem and its solution, let us briefly recall the principles of the renormalization procedure which will then allow us to set up a consistent power counting. At the beginning, the Lagrangian is written down in terms of bare, i.e., unrenormalized parameters and
fields. In order to illustrate the procedure let us discuss \( \mathcal{L}^{(1)}_{\pi N} \) of Eq. (180) and consider the free part in combination with the \( \pi N \) interaction term with the smallest number of pion fields,

\[
\mathcal{L}^{(1)}_{\pi N} = \bar{\Psi}_B \left( i \gamma^\mu \partial_\mu - m_B - \frac{1}{2} \frac{g_{A B}}{F_B} \gamma^\mu \gamma_5 \partial_\mu \phi_i B \tau_i \right) \Psi_B + \cdots ,
\]

(188)

where the subscript \( B \) denotes bare quantities. The renormalization is performed by expressing all the bare parameters and bare fields of the effective Lagrangian in terms of renormalized quantities (see, e.g., Refs. [Collins, 1984], [Weinberg, 1995] for details). Introducing the renormalized fields through

\[
\Psi = \frac{\Psi_B}{\sqrt{Z_\Psi}}, \quad \phi_i = \frac{\phi_i B}{\sqrt{Z_\phi}},
\]

(189)

we express the field redefinition constants \( \sqrt{Z_\Psi} \) and \( \sqrt{Z_\phi} \) and the bare quantities in terms of renormalized parameters:

\[
\begin{align*}
Z_\Psi &= 1 + \delta Z_\Psi (m, g_A, g_i, \nu), \\
Z_\phi &= 1 + \delta Z_\phi (m, g_A, g_i, \nu), \\
m_B &= m(\nu) + \delta m (m, g_A, g_i, \nu), \\
g_{A B} &= g_A(\nu) + \delta g_A (m, g_A, g_i, \nu),
\end{align*}
\]

(190)

where \( g_i, i = 1, \cdots, \infty \), collectively denote all the renormalized parameters which correspond to bare parameters \( g_i B \) of the full effective Lagrangian of Eq. (187). The parameter \( \nu \) indicates the dependence on the choice of the renormalization condition. We emphasize that the usual choice \( m(\nu) = m \), where \( m \) is the nucleon pole mass in the chiral limit, is only one among an infinite number of possibilities. Substituting Eqs. (189) and (190) into Eq. (188), we obtain

\[
\mathcal{L}^{(1)}_{\pi N} = \mathcal{L}_{\text{basic}} + \mathcal{L}_{\text{ct}} + \cdots
\]

(191)

with the so-called basic and counter-term Lagrangians, respectively,

\[
\begin{align*}
\mathcal{L}_{\text{basic}} &= \bar{\Psi} \left( i \gamma^\mu \partial_\mu - m - \frac{1}{2} \frac{g_A}{F} \gamma^\mu \gamma_5 \partial_\mu \phi_i \tau_i \right) \Psi, \\
\mathcal{L}_{\text{ct}} &= \delta Z_\Psi \bar{\Psi} i \gamma^\mu \partial_\mu \Psi - \delta \{ m \} \bar{\Psi} \Psi - \frac{1}{2} \delta \left\{ \frac{g_A}{F} \right\} \bar{\Psi} \gamma^\mu \gamma_5 \partial_\mu \phi_i \tau_i \Psi,
\end{align*}
\]

(192, 193)

where we introduced the abbreviations

\[
\begin{align*}
\delta \{ m \} &\equiv \delta Z_\Psi m + Z_\Psi \delta m, \\
\delta \left\{ \frac{g_A}{F} \right\} &\equiv \delta Z_\Psi \frac{g_A}{F} \sqrt{Z_\Psi} + Z_\Psi \left( \frac{g_{A B}}{F_B} - \frac{g_A}{F} \right) \sqrt{Z_\phi} + \frac{g_A}{F}(\sqrt{Z_\phi} - 1).
\end{align*}
\]

In Eq. (192), \( m, g_A \), and \( F \) denote the chiral limit of the physical nucleon mass, the axial-vector coupling constant, and the pion-decay constant, respectively. Expanding the counter-term Lagrangian of Eq. (193) in powers of the renormalized coupling constants generates an infinite series. By adjusting the expansion coefficients suitably, the individual terms are responsible for the subtractions of loop diagrams. For example, the divergences occurring in dimensionally regularized one-loop calculations involving vertices of \( \mathcal{L}_2 \) are absorbed in the renormalization of the bare coefficients \( L_i \) [see Eq. (141)].
Figure 14: One-loop contributions to the nucleon self energy. The number 1 in the interaction blobs refers to $L^{(1)}_{\pi N}$.

### 4.2.2 Power counting for renormalized diagrams

In the following, whenever we speak of renormalized diagrams, we refer to diagrams which have been calculated with a basic Lagrangian and to which the contribution of the counter-term Lagrangian has been added. Counter-term contributions are typically denoted by a cross. One also says that the diagram has been subtracted, i.e., the unwanted contribution has been removed with the understanding that this can be achieved by a suitable choice for the coefficient of the counter-term Lagrangian. In this context we will adjust the finite pieces of the renormalized couplings such that renormalized diagrams satisfy the following power counting: a loop integration in $n$ dimensions counts as $q^n$, pion and fermion propagators count as $q^{-2}$ and $q^{-1}$, respectively, vertices derived from $L^{(2k)}_{\pi}$ and $L^{(k)}_{\pi N}$ count as $q^{2k}$ and $q^k$, respectively. Here, $q$ collectively stands for a small quantity such as the pion mass, small external four-momenta of the pion, and small external three-momenta of the nucleon. The power counting does not uniquely fix the renormalization scheme, i.e., there are different renormalization schemes leading to the above specified power counting.

### 4.2.3 The power-counting problem

In the mesonic sector, the combination of dimensional regularization and the modified minimal subtraction scheme $\tilde{\text{MS}}$ [see Eq. (141)] leads to a straightforward correspondence between the chiral and loop expansions. By studying the one-loop contributions of Fig. 14 to the nucleon self energy, we will see that this correspondence, at first sight, seems to be lost in the baryonic sector.

In the following we will calculate the mass $m_N$ of the nucleon up to and including $O(q^3)$. As in the case of the Goldstone bosons, the physical mass is defined through the pole of the full propagator, but here at $p = m_N$. In terms of the nucleon self energy $\Sigma(p)$ we will solve the equation

$$m_N - m - \Sigma(m_N) = 0$$

(194)

where $m$ denotes the nucleon mass in the chiral limit.

According to the power counting specified above, we need to calculate the two types of one-loop contributions shown in Fig. 14 together with the corresponding counter-term contribution and a tree-level contribution. After renormalization, we would like to have the orders $D = n \cdot 1 - 2 \cdot 1 - 1 \cdot 1 + 2 \cdot 1 = n - 1$ for the first loop diagram and $n \cdot 1 - 2 \cdot 1 + 1 \cdot 1 = n - 1$ for the second loop diagram.

The basic interaction Lagrangian obtained from expanding $L^{(1)}_{\pi N}$ up to and including two pion fields reads

$$L^{(1)}_{\text{int}} = -\frac{1}{2} \frac{g_A}{F} \bar{\Psi} \gamma^\mu \gamma_5 \partial_\mu \phi_i \tau_i \Psi - \frac{1}{4F^2} \bar{\Psi} \gamma^\mu \vec{\phi} \times \partial_\mu \vec{\phi} \cdot \vec{\tau} \Psi.$$

The corresponding Feynman rules are given by
From the next-to-leading-order $\pi N$ Lagrangian $L^{(2)}_{\pi N}$ we only need one term, namely,

$$L^{(2)}_{\pi N} = c_1 \text{Tr}(\chi U^\dagger + U\chi^\dagger)\bar{\Psi}\Psi + \cdots,$$

resulting in the constant tree-level contribution

$$\Sigma_{\text{tree}}^2 = -4c_1 M^2.$$  \hfill (198)

Moreover, there are no tree-level contributions from the Lagrangian $L^{(3)}_{\pi N}$. The second diagram of Fig. 14 is zero, because the contraction $\epsilon_{abc} = 0$ in the Feynman rule of Eq. (196) vanishes. In dimensional regularization, the first diagram of Fig. 14 generates the contribution

$$-i\Sigma_{\text{loop}}(p) = -i\frac{3g_A^2}{4F^2} \mu^{4-n} \int \frac{d^n k}{(2\pi)^n} \frac{k(\phi - m - k)k}{[(p - k)^2 - m^2 + i0^+](k^2 - M^2 + i0^+)}.$$

Using $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$, the numerator of the integrand is written as

$$-(\phi + m)k^2 + (p^2 - m^2)k - [(p - k)^2 - m^2] k,$$

yielding the intermediate result

$$\Sigma_{\text{loop}}(\phi) = \frac{3g_A^2}{4F^2} \mu^{4-n} i \left\{ -\left[ (\phi + m)\mu^{4-n} i \int \frac{d^n k}{(2\pi)^n} \frac{1}{(p - k)^2 - m^2 + i0^+} \right] 
\right.$$

\left. - \left[ (\phi + m)M^2 \mu^{4-n} i \int \frac{d^n k}{(2\pi)^n} \frac{1}{[(p - k)^2 - m^2 + i0^+](k^2 - M^2 + i0^+)} \right] 
\right.

\left. + (p^2 - m^2) \mu^{4-n} i \int \frac{d^n k}{(2\pi)^n} \frac{k}{[(p - k)^2 - m^2 + i0^+](k^2 - M^2 + i0^+)} \right\}.$$  \hfill (200)

The last term in Eq. (200) vanishes since the integrand is odd in $k$. We use the following convention for scalar loop integrals,

$$I_{N\cdots}(p_1, \cdots, q_1, \cdots) = \mu^{4-n} i \int \frac{d^n k}{(2\pi)^n} \frac{1}{[(k + p_1)^2 - m^2 + i0^+] \cdots [(k + q_1)^2 - M^2 + i0^+] \cdots}.$$  \hfill (201)

The vector integral in the third line of Eq. (200) is determined using the ansatz

$$\mu^{4-n} i \int \frac{d^n k}{(2\pi)^n} \frac{k_\mu}{[(p - k)^2 - m^2 + i0^+](k^2 - M^2 + i0^+)} = p_\mu C.$$  \hfill (202)
Multiplying Eq. (202) by $p^\mu$, one obtains for $C$,

$$
C = \frac{1}{2p^2} \left[ I_N - I_\pi + (p^2 - m^2 + M^2)I_{N\pi}(-p, 0) \right].
$$

(203)

In terms of the above convention for the scalar loop integrals the loop contribution to the nucleon self energy reads

$$
\Sigma^{\text{loop}}(\phi) = -\frac{3g_A^2}{4F^2}\left\{ (\phi + m)I_N + (\phi + m)M^2I_{N\pi}(-p, 0)
- (p^2 - m^2)\frac{\phi}{2p^2}\left[ I_N - I_\pi + (p^2 - m^2 + M^2)I_{N\pi}(-p, 0) \right]\right\}.
$$

(204)

The explicit expressions for the integrals are given by

$$
I_\pi = \frac{M^2}{16\pi^2}\left[ R + \ln\left( \frac{M^2}{\mu^2} \right) \right], \quad I_N = \frac{m^2}{16\pi^2}\left[ R + \ln\left( \frac{m^2}{\mu^2} \right) \right],
$$

$$
I_{N\pi}(p, 0) = \frac{1}{16\pi^2}\left[ R + \ln\left( \frac{m^2}{\mu^2} \right) - 1 + \frac{p^2 - m^2 - M^2}{p^2} \ln\left( \frac{M}{m} \right) + \frac{2mM}{p^2}F(\Omega) \right],
$$

(205)

where $R$ is given in Eq. (135), $\Omega$ is defined as

$$
\Omega = \frac{p^2 - m^2 - M^2}{2mM},
$$

and

$$
F(\Omega) = \begin{cases} 
\sqrt{\Omega^2 - 1} \ln(-\Omega - \sqrt{\Omega^2 - 1}), & \Omega \leq -1, \\
\sqrt{1 - \Omega^2} \arccos(-\Omega), & -1 \leq \Omega \leq 1, \\
\sqrt{\Omega^2 - 1} \ln(\Omega + \sqrt{\Omega^2 - 1}) - i\pi \sqrt{\Omega^2 - 1}, & 1 \leq \Omega.
\end{cases}
$$

Because of the terms proportional to $R$, the result for the self energy contains divergences as $n \to 4$, so it has to be renormalized. The counter-term Lagrangian must produce structures which precisely cancel the divergences, because otherwise the result for the nucleon mass will not be finite. For convenience, we choose the renormalization parameter $\mu = m$.

In the modified minimal subtraction scheme $\overline{\text{MS}}$ all the contributions proportional to $R$ are canceled by corresponding contributions generated by the counter-term Lagrangian of Eq. (193), but also by counter-term Lagrangians resulting from higher-order terms of Eq. (187). Operationally this means that we simply drop all terms proportional to $R$ and indicate the renormalized coupling constants by a subscript $r$. Again, this is possible, because we include in the Lagrangian all of the infinite number of interactions allowed by symmetries [Weinberg, 1995]. The renormalized diagram is depicted in Fig. 15 where the cross generically denotes counter-term contributions. The $\overline{\text{MS}}$-renormalized self-energy contribution then reads

$$
\Sigma^{\text{loop}}_r(\phi) = \frac{3g_A^2}{4F^2}\left\{ (\phi + m)M^2I_{N\pi}^r(-p, 0) - (p^2 - m^2)\frac{\phi}{2p^2}\left[ (p^2 - m^2 + M^2)I_{N\pi}^r(-p, 0) - I_\pi^r \right]\right\},
$$

(206)
where the superscript \( r \) on the integrals means that the terms proportional to \( R \) have been dropped. Writing \( \phi + m = 2m + (\phi - m) \) and comparing the first term of Eq. (201) with Eq. (206), we note that among other terms, the \( \overline{\text{MS}} \) renormalization involves (even in the chiral limit) an infinite renormalization yielding the relation between the bare and the renormalized mass \( \text{[Gasser et al., 1988]} \)

\[
m_B = m + \frac{3g_A^2}{32\pi^2F^2}m^3R + \cdots.
\]

Using \( I_{N\pi}^r(-p,0) = -\frac{1}{16\pi^2} + \cdots \), we see that the \( \overline{\text{MS}} \)-renormalized self energy produces a contribution of \( \mathcal{O}(q^2) \) which is in conflict with the power counting assigned above. For a long time this was interpreted as the absence of a systematic power counting in the relativistic formulation of ChPT.

We can now solve Eq. (194) for the nucleon mass,

\[
m_N = m + \Sigma_{\text{tree}}(m_N) + \Sigma_{\text{loop}}^r(m_N) = m - 4c_1rM^2 + \Sigma_{\text{loop}}^r(m_N).
\]

We have for the difference \( m_N - m = \mathcal{O}(q^2) \). Since our calculation is only valid up to \( \mathcal{O}(q^3) \), it is sufficient to determine \( \Sigma_{\text{loop}}^r(m_N) \) to that order. In fact, using \( \arccos(-\Omega) = \frac{\pi}{2} + \cdots \), the expansion of \( I_{N\pi}^r \) is given by

\[
I_{N\pi}^r = \frac{1}{16\pi^2} (-1 + \frac{\pi M}{m} + \cdots),
\]

from which we obtain for the nucleon mass in the \( \overline{\text{MS}} \) scheme \( \text{[Gasser et al., 1988]} \),

\[
m_N = m - 4c_1rM^2 + \frac{3g_A^2M^2}{32\pi^2F^2}m - \frac{3g_A^2M^3}{32\pi^2F^2}.
\]

The solution to the power-counting problem is the observation that the term violating the power counting, namely, the third on the right-hand side of Eq. (209), is analytic in the quark mass and can thus be absorbed in counter terms. In addition to the \( \overline{\text{MS}} \) scheme we have to perform an additional finite renormalization. For that purpose we rewrite

\[
c_1r = c_1 + \delta c_1, \quad \delta c_1 = \frac{3mg_A^2}{128\pi^2F^2} + \cdots
\]

in Eq. (209) which then gives the final result for the nucleon mass at \( \mathcal{O}(q^3) \):

\[
m_N = m - 4c_1M^2 - \frac{3g_A^2M^3}{32\pi^2F^2}.
\]

To summarize, we have shown that the validity of a power-counting scheme is intimately connected with a suitable renormalization condition. In the case of the nucleon mass, the \( \overline{\text{MS}} \) scheme alone does not suffice to bring about a consistent power counting. We will shortly outline two methods, the infrared renormalization \( \text{[Becher and Leutwyler, 1999]} \) and the extended on-mass-shell renormalization \( \text{[Fuchs et al., 2003a]} \), which both produce a systematic power counting in a manifestly Lorentz-invariant framework.

### 4.3 Solutions to the power-counting problem

#### 4.3.1 Heavy-baryon approach

The first solution to the power-counting problem was provided by the heavy-baryon formulation of ChPT \( \text{[Jenkins and Manohar, 1991]} \), \( \text{[Bernard et al., 1992a]} \). The basic idea consists in dividing an external
nucleon four-momentum into a large piece close to on-shell kinematics and a soft residual contribution: 
\[ p = mv + k, \quad v^2 = 1, \quad v^0 \geq 1 \] 
[often \( v^\mu = (1, 0, 0, 0) \)]. The relativistic nucleon field is expressed in terms of velocity-dependent fields,
\[
\Psi(x) = e^{-imv \cdot x}(\mathcal{N}_v + \mathcal{H}_v),
\]
with
\[
\mathcal{N}_v = e^{+imv \cdot x} \frac{1}{2} (1 + \gamma^v) \Psi, \quad \mathcal{H}_v = e^{+imv \cdot x} \frac{1}{2} (1 - \gamma^v) \Psi.
\]
Using the equation of motion for \( \mathcal{H}_v \), one can eliminate \( \mathcal{H}_v \) and obtain a Lagrangian for \( \mathcal{N}_v \) which, to lowest order, reads [Bernard et al., 1992a]
\[
\hat{\mathcal{L}}^{(1)}_{\pi N} = \bar{N}_v (iv \cdot D + g_A S_v \cdot u) \mathcal{N}_v + O(1/m), \quad S^\mu_v = \frac{i}{2} \gamma_5 \sigma^{\mu v} v_\nu.
\]
The result of the heavy-baryon reduction is a \( 1/m \) expansion of the Lagrangian similar to a Foldy-Wouthuysen expansion [Foldy and Wouthuysen, 1950]. In higher orders in the chiral expansion, the expressions due to \( 1/m \) corrections of the Lagrangian become increasingly complicated [Fettes et al., 2000].

Moreover—and what is more important—the approach sometimes generates problems regarding analyticity which can be illustrated by considering the example of pion-nucleon scattering [Becher, 2002]. The invariant amplitudes describing the scattering amplitude develop poles for \( \mathcal{S} = m_N^2 \) and \( u = m_N^2 \). For example, the singularity due to the nucleon pole in the \( s \) channel (see Fig. 16) is understood in terms of the relativistic propagator
\[
\frac{1}{(p + q)^2 - m_N^2} = \frac{1}{2p \cdot q + M_\pi^2},
\]
which, of course, has a pole at \( 2p \cdot q = -M_\pi^2 \) or, equivalently, \( s = m_N^2 \). Analogously, a second pole results from the \( u \) channel at \( u = m_N^2 \). Although both poles are not in the physical region of pion-nucleon scattering, analyticity of the invariant amplitudes requires these poles to be present in the amplitudes.

Let us compare the situation with a heavy-baryon type of expansion, where, for simplicity, we choose as the four-velocity \( p^\mu = m_N v^\mu \),
\[
\frac{1}{2p \cdot q + M_\pi^2} = \frac{1}{2m_N v \cdot q + \frac{M_\pi^2}{2m_N}} = \frac{1}{2m_N v \cdot q} \left( 1 - \frac{M_\pi^2}{2m_N v \cdot q} + \cdots \right).
\]
Clearly, to any finite order the heavy-baryon expansion produces poles at \( v \cdot q = 0 \) instead of a simple pole at \( v \cdot q = -M_\pi^2/(2m_N) \) and will thus not generate the (nucleon) pole structures of the invariant amplitudes unless an infinite number of diagrams is summed. For a comprehensive overview of calculations performed in the heavy-baryon framework the reader is referred to Ref. [Bernard et al., 1995].
4.3.2 Master integral

We have seen that the modified minimal subtraction scheme $\overline{\text{MS}}$ does not produce the desired power counting. We will discuss the power-counting problem in terms of the dimensionally regularized one-loop integral

$$H(p^2, m^2, M^2; n) \equiv -i \int \frac{d^n k}{(2\pi)^n} \frac{1}{[(k - p)^2 - m^2 + i0^+][k^2 - M^2 + i0^+]}.$$

We are interested in nucleon four-momenta close to the mass-shell condition, $p^2 \approx m^2$, counting $p^2 - m^2$ as $O(q)$ and $M^2$ as $O(q^2)$. In order to conform with Ref. [Becher and Leutwyler, 1999], we have omitted the factor $\mu^{4-n}$ and have reversed the overall sign in comparison with our previous definition of $I_{N\pi}$. Let us turn to the discussion of $H(p^2, m^2, M^2; n)$. To that end, we make use of the Feynman parametrization

$$\frac{1}{ab} = \int_0^1 dz \frac{1}{(az + b(1-z))^2} \quad (215)$$

with $a = (k - p)^2 - m^2 + i0^+$ and $b = k^2 - M^2 + i0^+$, interchange the order of integrations, and perform the shift $k \to k + z p$, to obtain

$$H(p^2, m^2, M^2; n) = -i \int_0^1 dz \int \frac{d^n k}{(2\pi)^n} \frac{1}{[k^2 - A(z) + i0^+]^2},$$

where

$$A(z) = z^2 p^2 - z(p^2 - m^2 + M^2) + M^2.$$

Making use of

$$\int \frac{d^n k}{(2\pi)^n} \frac{(k^2)^p}{(q^2 - A)^q} = \frac{i(-)^{p-q} \Gamma(p + \frac{q}{2}) \Gamma(q - p - \frac{q}{2}) \Gamma(\frac{q}{2}) \Gamma(q)}{\Gamma(\frac{n}{2})} A^{p+\frac{q}{2} - q}$$

with $p = 0$ and $q = 2$, we find

$$H(p^2, m^2, M^2; n) = \frac{1}{(4\pi)^{\frac{n}{2}}} \Gamma\left(2 - \frac{n}{2}\right) \int_0^1 dz [A(z) - i0^+]^{\frac{n}{2} - 2}. \quad (216)$$

The relevant properties can nicely be displayed at the threshold $p_{th}^2 = (m + M)^2$, where $A(z) = [z(m + M) - M]^2$ is particularly simple. The small imaginary part can be dropped in this case, because $A(z)$ is never negative. Splitting the integration interval into $[0, z_0]$ and $[z_0, 1]$ with $z_0 = M/(m + M)$, we have, for $n > 3$,

$$\int_0^1 dz[A(z)]^{\frac{n}{2} - 2} = \int_0^{z_0} dz [M - z(m + M)]^{n-4} + \int_{z_0}^1 dz [z(m + M) - M]^{n-4}$$

$$= \frac{1}{(n-3)(m+M)} (M^{n-3} + m^{n-3}),$$

yielding, through analytic continuation, for arbitrary $n$

$$H((m + M)^2, m^2, M^2; n) = \frac{\Gamma\left(2 - \frac{n}{2}\right)}{(4\pi)^{\frac{n}{2}}(n-3)} \left( \frac{M^{n-3}}{m + M} + \frac{m^{n-3}}{m + M} \right). \quad (217)$$

The first term, proportional to $M^{n-3}$, is defined as the so-called infrared singular part $I$. Since $M \to 0$ implies $p_{th}^2 \to m^2$ this term is singular for $n \leq 3$. The second term, proportional to $m^{n-3}$, is defined as the infrared regular part $R$. Note that for non-integer $n$ the infrared singular part contains non-integer powers of $M$, while an expansion of the regular part always contains non-negative integer powers of $M$ only.
4.3.3 Infrared regularization

Let us now turn to a formal definition of the infrared singular and regular parts for arbitrary $p^2$ [Becher and Leutwyler, 1999] which makes use of the Feynman parametrization of Eq. (216). Introducing the dimensionless variables

$$\alpha = \frac{M}{m} = \mathcal{O}(q), \quad \Omega = \frac{p^2 - m^2 - M^2}{2mM} = \mathcal{O}(q^0),$$

we rewrite $A(z)$ as

$$A(z) = m^2[z^2 - 2\alpha \Omega z(1 - z) + \alpha^2(1 - z)^2] \equiv m^2 C(z),$$

so that $H$ is now given by

$$H(p^2, m^2, M^2; n) = \kappa(m; n) \int_0^1 dz[C(z) - i0^+]^{\frac{2}{n} - 2},$$

where

$$\kappa(m; n) = \frac{\Gamma\left(\frac{2 - n}{2}\right)}{(4\pi)^\frac{n}{2}} m^{n-4}.$$  \hspace{1cm} (220)

The infrared singularity originates from small values of $z$, where the function $C(z)$ goes to zero as $M \to 0$. In order to isolate the divergent part one scales the integration variable $z \equiv \alpha x$ so that the upper limit $z = 1$ in Eq. (219) corresponds to $x = 1/\alpha \to \infty$ as $M \to 0$. An integral $I$ having the same infrared singularity as $H$ is then defined which is identical to $H$ except that the upper limit is replaced by $\infty$:

$$I \equiv \kappa(m; n) \int_0^\infty dz[C(z) - i0^+]^{\frac{2}{n} - 2} = \kappa(m; n) \alpha^{n-3} \int_0^\infty dx[D(x) - i0^+]^{\frac{2}{n} - 2},$$

where

$$D(x) = 1 - 2\Omega x + x^2 + 2\alpha x(\Omega x - 1) + \alpha^2 x^2.$$  \hspace{1cm} (221)

(The pion mass $M$ is not sent to zero.) Accordingly, the regular part of $H$ is defined as

$$R \equiv -\kappa(m; n) \int_1^\infty dz[C(z) - i0^+]^{\frac{2}{n} - 2},$$

so that

$$H = I + R.$$ \hspace{1cm} (223)

Let us verify that the definitions of Eqs. (221) and (222) indeed reproduce the behavior of Eq. (217). To that end we make use of $\Omega_{\text{thr}} = 1$, yielding

$$I_{\text{thr}} = \kappa(m; n) \alpha^{n-3} \int_0^\infty dx \left\{ [(1 + \alpha)x - 1]^2 - i0^+ \right\}^{\frac{2}{n} - 2},$$

which converges for $n < 3$. In order to continue the integral to $n > 3$, write Becher and Leutwyler, 1999

$$\left\{ [(1 + \alpha)x - 1]^2 - i0^+ \right\}^{\frac{2}{n} - 2} = \frac{(1 + \alpha)x - 1}{(1 + \alpha)(n - 4)} \frac{d}{dx} \left\{ [(1 + \alpha)x - 1]^2 - i0^+ \right\}^{\frac{2}{n} - 2},$$

and make use of a partial integration

$$\int_0^\infty dx \left\{ [(1 + \alpha)x - 1]^2 - i0^+ \right\}^{\frac{2}{n} - 2} =$$

$$\left[ \frac{(1 + \alpha)x - 1}{(1 + \alpha)(n - 4)} \left\{ [(1 + \alpha)x - 1]^2 - i0^+ \right\}^{\frac{2}{n} - 2} \right]_0^\infty - \frac{1}{n - 4} \int_0^\infty dx \left\{ [(1 + \alpha)x - 1]^2 - i0^+ \right\}^{\frac{2}{n} - 2}.$$  \hspace{1cm} (224)
For $n < 3$, the first expression vanishes at the upper limit and, at the lower limit, yields $1/[(1+\alpha)(n-4)]$. Bringing the second expression to the left-hand side, we may then continue the integral analytically as

$$
\int_0^\infty dx \left\{ [(1+\alpha)x - 1]^2 - i0^+ \right\}^{\frac{n}{2}-2} = \frac{1}{(n-3)(1+\alpha)},
$$

so that we obtain for $I_{\text{thr}}$

$$
I_{\text{thr}} = k(m; n)\alpha^{n-3} \frac{1}{(n-3)(1+\alpha)} = \frac{\Gamma \left( 2 - \frac{n}{2} \right)}{(4\pi)^{\frac{n}{2}}(n-3) m + M},
$$

which agrees with the infrared singular part $I$ of Eq. (217).

The threshold value of the regular part of Eq. (222) is obtained by analytic continuation from $n < 3$ to $n > 3$: 

$$
R_{\text{thr}} = -\frac{\Gamma \left( 2 - \frac{n}{2} \right)}{(4\pi)^{\frac{n}{2}}} \int_1^\infty dz \left[ z(m + M) - M_\pi \right]^{n-4}
$$

$$
= -\frac{\Gamma \left( 2 - \frac{n}{2} \right)}{(4\pi)^{\frac{n}{2}}(n-3)(m + M)} (\infty^{n-3} - m^{n-3})
$$

$$
n \leq 3 \quad \frac{\Gamma \left( 2 - \frac{n}{2} \right)}{(4\pi)^{\frac{n}{2}}(n-3) m + M},
$$

which is indeed the regular part $R$ of Eq. (217).

What distinguishes $I$ from $R$ is that, for non-integer values of $n$, the chiral expansion of $I$ gives rise to non-integer powers of $O(q)$, whereas the regular part $R$ may be expanded in an ordinary Taylor series. For the threshold integral, this can nicely be seen by expanding $I_{\text{thr}}$ and $R_{\text{thr}}$ in the pion mass counting as $O(q)$. On the other hand, it is the regular part which does not satisfy the counting rules. The basic idea of the infrared renormalization consists of replacing the general integral $H$ of Eq. (216) by its infrared singular part $I$, defined in Eq. (221), and dropping the regular part $R$, defined in Eq. (222). In the low-energy region $H$ and $I$ have the same analytic properties whereas the contribution of $R$, which is of the type of an infinite series in the momenta, can be included by adjusting the coefficients of the most general effective Lagrangian. This is the infrared renormalization condition.

As discussed in detail in Ref. [Becher and Leutwyler, 1999], the method can be generalized to an arbitrary one-loop graph (see also Ref. [Semke and Lutz, 2006]). It is first argued that tensor integrals involving an expression of the type $k^{a_1} \cdots k^{a_2}$ in the numerator may always be reduced to scalar loop integrals of the form

$$
-i \int \frac{d^n k}{(2\pi)^n} \frac{1}{a_1 \cdots a_m \ b_1 \cdots b_n},
$$

where $a_i = (q_i + k)^2 - M^2 + i0^+$ and $b_i = (p_i - k)^2 - m^2 + i0^+$ are inverse meson and nucleon propagators, respectively. Here, the $q_i$ refer to four-momenta of $O(q)$ and the $p_i$ are four-momenta which are not far off the nucleon mass shell, i.e., $p_i^2 = m^2 + O(q)$. Using the Feynman parametrization, all pion propagators and all nucleon propagators are separately combined, and the result is written in such a way that it is obtained by applying $(m-1)$ and $(n-1)$ partial derivatives with respect to $M^2$ and $m^2$, respectively, to a master formula. A simple illustration is given by

$$
\frac{1}{a_1 a_2} = \int_0^1 dz \frac{1}{a_1 z + a_2 (1-z)} = \frac{\partial}{\partial M^2} \int_0^1 dz \frac{1}{a_1 z + a_2 (1-z)} ,
$$

where $a_i = (q_i + k)^2 - M^2 + i0^+$. Of course, the expressions become more complicated for larger numbers of propagators. The relevant property of the above procedure is that the result of combining the meson
propagators is of the type \( 1/A \) with \( A = (k + q)^2 - M^2 + i0^+ \), where \( q \) is a linear combination of the \( m \) momenta \( q_i \), with an analogous expression \( 1/B \) for the nucleon propagators. Finally, the expression

\[
-i \int \frac{d^n k}{(2\pi)^n AB} \frac{1}{AB}
\]

may then be treated in complete analogy to \( H \) of Eq. (214), i.e., the denominators are combined as in Eq. (215), and the infrared singular and regular pieces are identified by writing \( f_i^1 dz \cdots = f_i^0 dz \cdots - f_i^2 dz \cdots \).

4.3.4 Extended on-mass-shell scheme

In the following, we will concentrate on yet another solution which has been motivated in Ref. [Gegelia and Japaridze, 1999] and has been worked out in detail in Ref. [Fuchs et al., 2003a]. The central idea of the extended on-mass-shell (EOMS) scheme consists of performing additional subtractions beyond the \( \tilde{\text{MS}} \) scheme such that renormalized diagrams satisfy the power counting. Terms violating the power counting are analytic in small quantities and can thus be absorbed in a renormalization of counter terms.

In order to illustrate the approach, let us consider a simplified version of the integral \( H \), namely its value in the chiral limit,

\[
H(p^2, m^2, 0; n) = -i \int \frac{d^n k}{(2\pi)^n} \frac{1}{[(k - p)^2 - m^2 + i0^+][(k^2 + i0^+)]},
\]

where

\[
\Delta = \frac{p^2 - m^2}{m^2} = O(q)
\]

is a small quantity. Applying the power-counting rules of Sec. 4.2.2, we want the renormalized integral to be of order \( D = n - 1 - 2 = n - 3 \). Introducing \( C(z, \Delta) = z^2 - \Delta z(1 - z) - i0^+ \), we obtain

\[
H(p^2, m^2, 0; n) = \kappa(m; n) \int_0^1 dz [C(z, \Delta)]^{\frac{1}{2} - 2},
\]

where \( \kappa(m; n) \) is given in Eq. (220). For the purpose of evaluating the integral of Eq. (228) we write

\[
\int_0^1 dz [C(z, \Delta)]^{\frac{1}{2} - 2} = (-\Delta)^{\frac{n}{2} - 2} \int_0^1 dz z^{\frac{1}{2} - 2} \left( 1 - \frac{1 + \Delta}{\Delta} z \right)^{\frac{1}{2} - 2}
\]

and apply Eqs. 15.3.1 and 15.3.4 of Ref. [Abramowitz and Stegun, 1972] to obtain

\[
H(p^2, m^2, 0; n) = \kappa(m; n) \frac{\Gamma \left( \frac{n}{2} - 1 \right)}{\Gamma \left( \frac{n}{2} \right)} F \left( 1, 2 - \frac{n}{2}; \frac{n}{2}; \frac{p^2}{m^2} \right),
\]

where \( F(a, b; c; z) \) is the hypergeometric function [Abramowitz and Stegun, 1972]. In order to discuss the power-counting properties of \( H \) (in the chiral limit) in terms of \( \Delta \), we make use of Eq. 15.3.6 of Ref. [Abramowitz and Stegun, 1972] to re-write Eq. (229) as

\[
H(p^2, m^2, 0; n) = \frac{m^{n-4}}{(4\pi)^2} \left[ \frac{\Gamma \left( \frac{2 - n}{2} \right)}{\Gamma \left( n - 3 \right)} F \left( 1, 2 - \frac{n}{2}; 4 - n; -\Delta \right) + (-\Delta)^{n-3} \Gamma \left( \frac{n}{2} - 1 \right) \Gamma(3 - n) F \left( \frac{n}{2} - 1; n - 2; n - 2; -\Delta \right) \right].
\]
The subtracted term of Eq. (233) is local in the external momentum \( p \), i.e., it is a polynomial in \( p^2 \) and can thus be obtained by a finite number of counter terms in the most general effective Lagrangian.

We have seen in Eq. (230) that the one-loop integral is of the type

\[
H \sim F(n, \Delta) + \Delta^{n-3} G(n, \Delta),
\]

where \( F \) and \( G \) are hypergeometric functions and are analytic in \( \Delta \) for any \( n \). The observation central for the setting up of a systematic method is the fact that the part proportional to \( F \) can be obtained by first expanding the integrand in small quantities and then performing the integration for each term \cite{Gegelia et al., 1994} (see Sec. 4.3.5 for an illustration of the general method). We now apply a conventional renormalization prescription which allows us to identify those terms which we subtract from a given integral without explicitly calculating the integral beforehand. In essence we work with a modified integrand which is obtained from the original integrand by subtracting a suitable number of counter terms. The meaning of suitable in the present context will be explained in a moment. To that end we consider the series

\[
\sum_{l=0}^{\infty} \frac{(p^2 - m^2)^l}{l!} \left[ \left( \frac{1}{2p^2 \partial p_\mu} \frac{\partial}{\partial p_\mu} \right)^l \frac{1}{(k^2 - 2k \cdot p + (p^2 - m^2) + i 0^+)(k^2 + i 0^+)} \right]_{p^2 = m^2} = \frac{1}{(k^2 - 2k \cdot p + i 0^+)(k^2 + i 0^+)} \bigg|_{p^2 = m^2}
\]

\[
+ (p^2 - m^2) \left[ \frac{1}{2m^2 (k^2 - 2k \cdot p + i 0^+)^2} \right]_{p^2 = m^2} \quad + \cdots,
\]

where \([ \cdots ]_{p^2 = m^2}\) means that we consider the coefficients of \((p^2 - m^2)^l\) only for four-momenta \( p^\mu \) satisfying the on-mass-shell condition. Although the coefficients still depend on the direction of \( p^\mu \), after integration of this series with respect to the loop momentum \( k \) and evaluation of the resulting coefficients for
\( p^2 = m^2 \), the integrated series is a function of \( p^2 \) only. In fact, as was shown in Ref. \[Gegelia et al., 1994\], the integrated series exactly reproduces the first term of Eq. \((230)\). At this point we stress that

\[
-i \int \frac{d^n k}{(2\pi)^n (k^2 - 2k \cdot p + i0^+)(k^2 + i0^+)} \bigg|_{p^2=m^2}
\]

and

\[
\left[ -i \int \frac{d^n k}{(2\pi)^n (k^2 - 2k \cdot p + p^2 - m^2 + i0^+)(k^2 + i0^+)} \right] \bigg|_{p^2=m^2}
\]

are not the same for \( n \leq 3 \).

The formal definition of the EOMS renormalization scheme is then as follows: we subtract from the integrand of \( H(p^2, m^2, 0; n) \) those terms of the series of Eq. \((235)\) which violate the power counting. These terms are always analytic in the small parameter and do not contain infrared singularities. In the above example we only need to subtract the first term. All the higher-order terms contain infrared singularities. For example, the last term of the second coefficient would generate a behavior \( k^3/k^4 \) of the integrand for \( n = 4 \). The integral of the first term of Eq. \((235)\) is given by Eq. \((233)\), and we end up with Eq. \((234)\) for the renormalized integral:

\[
H_R = H - H_{\text{subtr}} = O(q^{n-3}).
\]

Since the subtraction point is \( p^2 = m^2 \), the renormalization condition is denoted “extended on-mass-shell” (EOMS) scheme in analogy with the on-mass-shell renormalization scheme in renormalizable theories. In the general case including the pion mass, one would consider the series

\[
\frac{1}{(k^2 - 2k \cdot p + i0^+)(k^2 + i0^+)} \bigg|_{p^2=m^2} + (p^2 - m^2) \left[ \frac{1}{2m^2 (k^2 - 2k \cdot p + i0^+)^2} + \cdots \right]_{p^2=m^2} + M^2 \frac{1}{(k^2 - 2k \cdot p + i0^+)(k^2 + i0^+)^2} \bigg|_{p^2=m^2} + \cdots
\]

instead of Eq. \((235)\). However, it would still be only the contribution resulting from the first term that were to be subtracted.

Within the EOMS framework it is straightforward to obtain a consistent power counting in manifestly Lorentz-invariant baryon ChPT including, e.g., vector mesons \[Fuchs et al., 2003b\] or the \( \Delta(1232) \) resonance \[Hacker et al., 2005\] as explicit degrees of freedom. Moreover, the infrared regularization of Becher and Leutwyler can be reformulated in a form analogous to the EOMS renormalization scheme and can thus be applied straightforwardly to multi-loop diagrams with an arbitrary number of particles with arbitrary masses \[Schindler et al., 2004a\] (see also Refs. \[Lehmann and Prezeau, 2002\], \[Bruns and Mei\'nner, 2005\], \[Bruns and Mei\'nner, 2008\]). The application of both infrared and extended on-mass-shell renormalization schemes to multi-loop diagrams was explicitly demonstrated by means of a two-loop self-energy diagram \[Schindler et al., 2004b\].

### 4.3.5 Dimensional counting analysis

In this section we provide an illustration of the dimensional counting analysis \[Gegelia et al., 1994\] in terms of a specific example. To that end let us consider the one-loop integral of Eq. \((214)\),

\[
H(p^2, m^2, M^2; n) = -i \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 - 2p \cdot k + p^2 - m^2 + i0^+} \frac{1}{k^2 - M^2 + i0^+}.
\]
One would like to know how the integral behaves for small values of $M$ and/or $p^2 - m^2$ as a function of $n$. If we consider, for fixed $p^2 \neq m^2$, the limit $M \to 0$, the integral $H$ can be represented as

$$
H(p^2, m^2, M^2; n) = \sum_i M^{\beta_i} F_i(p^2, m^2, M^2; n),
$$

(237)

where the functions $F_i$ are analytic in $M^2$ and are obtained as follows. First, one re-writes the integration variable as $k = M^{\alpha_i} \tilde{k}$, where $\alpha_i$ is an arbitrary non-negative real number. Next, one isolates the overall factor of $M^{\beta_i}$ so that the remaining integrand can be expanded in positive powers of $M^2$ and interchanges the integration and summation. The resulting series represents the expansion of $F_i(p^2, m^2, M^2; n)$ in powers of $M^2$. The sum of all possible re-scalings with subsequent expansions with non-trivial coefficients then reproduces the expansion of the result of the original integral.

To be specific, let us apply this program to $H$:

$$
H(p^2, m^2, M^2; n) = -i \int \frac{d^n k}{(2\pi)^n} \frac{1}{\tilde{k}^2 M^{2\alpha_i} - 2p \cdot \tilde{k} M^{\alpha_i} + p^2 - m^2 + i0^+} \frac{1}{\tilde{k}^2 M^{2\alpha_i} - M^2 + i0^+}. 
$$

(238)

From Eq. (238) we see that the first fraction does not contribute to the overall factor $M^{\beta_i}$ for any $\alpha_i$. It will be expanded in (positive) powers of $(\tilde{k}^2 M^{2\alpha_i} - 2p \cdot \tilde{k} M^{\alpha_i})$ except for $\alpha_i = 0$. For $0 < \alpha_i < 1$, we re-write the second fraction as

$$
\frac{1}{M^{2\alpha_i}} \frac{1}{\tilde{k}^2 - M^2 - 2\alpha_i + i0^+} = \frac{1}{M^{2\alpha_i}} \frac{1}{\tilde{k}^2 + i0^+} \left(1 + \frac{M^{2-2\alpha_i}}{\tilde{k}^2 + i0^+} + \cdots\right).
$$

(239)

On the other hand, if $1 < \alpha_i$ we re-write the second fraction as

$$
\frac{1}{M^2} \frac{1}{\tilde{k}^2 - M^2 - 2\alpha_i - 2 + i0^+} = -\frac{1}{M^2} \left(1 + \tilde{k}^2 M^{2\alpha_i - 2} + \cdots\right).
$$

(240)

In both cases one obtains integrals of the type $\int d^n \tilde{k} \tilde{k}^{\mu_1} \cdots \tilde{k}^{\mu_m}$ as the coefficients of the expansion. However, such integrals vanish in dimensional regularization. Therefore, the only non-trivial terms in the sum of Eq. (237) correspond to either $\alpha_i = 0$ or $\alpha_i = 1$. Thus we obtain

$$
H(p^2, m^2, M^2; n) = H^{(0)}(p^2, m^2, M^2; n) + H^{(1)}(p^2, m^2, M^2; n),
$$

(241)

where

$$
H^{(0)}(p^2, m^2, M^2; n) = -i \sum_{j=0}^{\infty} \left(M^2\right)^j \int \frac{d^n k}{(2\pi)^n} \frac{1}{\tilde{k}^2 - 2p \cdot k + p^2 - m^2 + i0^+} \frac{1}{(k^2 + i0^+)^{j+1}},
$$

(242)

and

$$
H^{(1)}(p^2, m^2, M^2; n) = -i \frac{M^{n-2}}{p^2 - m^2 + i0^+} \sum_{j=0}^{\infty} \frac{(-1)^j M^j}{(p^2 - m^2 + i0^+)^j} \int \frac{d^n k}{(2\pi)^n} \frac{(\tilde{k}^2 M - 2p \cdot \tilde{k})^j}{\tilde{k}^2 - 1 + i0^+}.
$$

(243)

A comparison with the direct calculation of $H$ shows that the dimensional counting method indeed leads to the correct expressions [Gegelia et al., 1994]. While the loop integrals of Eq. (243) have a simple analytic structure in $p^2 - m^2$, the same technique can be repeated for the loop integrals of Eq. (242) when $p^2 - m^2 \to 0$, now using the change of variable $k = (p^2 - m^2)^{\gamma_i} \tilde{k}$ with arbitrary non-negative real numbers $\gamma_i$.

5 Applications

In the following we will illustrate a few selected applications of the manifestly Lorentz-invariant framework to the one-nucleon sector.
Figure 17: Contributions to the nucleon self energy at $O(q^4)$. The number $n$ in the interaction blobs refers to $\mathcal{L}_{\pi N}^{(n)}$. The Lagrangian $\mathcal{L}_{\pi N}^{(2)}$ does not produce a contribution to the $\pi NN$ vertex.

5.1 Nucleon mass and sigma term at $O(q^4)$

A full one-loop calculation of the nucleon mass also includes $O(q^4)$ terms (see Fig. 17). The quark-mass expansion up to and including $O(q^4)$ is given by

$$m_N = m + k_1 M^2 + k_2 M^3 + k_3 M^4 \ln \left( \frac{M}{m} \right) + k_4 M^4 + O(M^5),$$

where the coefficients $k_i$ in the EOMS scheme read [Fuchs et al., 2003a]

$$k_1 = -4c_1, \quad k_2 = -\frac{3g_A^2}{32\pi F^2}, \quad k_3 = -\frac{3}{32\pi^2 F^2 m} (g_A^2 - 8c_1 m + c_2 m + 4c_3 m),$$

$$k_4 = \frac{3g_A^2}{32\pi^2 F^2 m} (1 + 4c_1 m) + \frac{3}{128\pi^2 F^2} c_2 - \hat{e}_1. \quad (245)$$

Here, $\hat{e}_1 = 16c_{38} + 2c_{115} + 2c_{116}$ is a linear combination of $O(q^4)$ coefficients [Fettes et al., 2000]. A comparison with the results using the infrared regularization [Becher and Leutwyler, 1999] shows that the lowest-order correction ($k_1$ term) and those terms which are non-analytic in the quark mass $\hat{m}$ ($k_2$ and $k_3$ terms) coincide. On the other hand, the analytic $k_4$ term ($\sim M^4$) is different. This is not surprising; although both renormalization schemes satisfy the power counting specified in Sec. 4.2.2, the use of different renormalization conditions is compensated by different values of the renormalized parameters.

For an estimate of the various contributions of Eq. (244) to the nucleon mass, we make use of the parameter set

$$c_1 = -0.9 m_N^{-1}, \quad c_2 = 2.5 m_N^{-1}, \quad c_3 = -4.2 m_N^{-1}, \quad c_4 = 2.3 m_N^{-1},$$

which was obtained in Ref. [Becher and Leutwyler, 2001] from a (tree-level) fit to the $\pi N$ scattering threshold parameters. Using the numerical values

$$g_A = 1.267, \quad F_\pi = 92.4 \text{ MeV}, \quad m_N = m_p = 938.3 \text{ MeV}, \quad M_\pi = M_{\pi^+} = 139.6 \text{ MeV},$$

one obtains for the mass of nucleon in the chiral limit (at fixed $m_s \neq 0$):

$$m = m_N - \Delta m = [938.3 - 74.8 + 15.3 + 4.7 + 1.6 - 2.3 \pm 4] \text{ MeV} = (883 \pm 4) \text{ MeV} \quad (248)$$

with $\Delta m = (55.5 \pm 4) \text{ MeV}$. Here, we have made use of an estimate for $\hat{e}_1 M^4 = (2.3 \pm 4) \text{ MeV}$ obtained from the $\sigma$ term. (Note that errors due to higher-order corrections are not taken into account.) In terms of the $SU(2)_L \times SU(2)_R$-chiral-symmetry-breaking mass term of the QCD Hamiltonian,

$$\mathcal{H}_{ab} = \hat{m}(\bar{u}u + \bar{d}d), \quad (249)$$

the pion-nucleon $\sigma$ term is defined as the proton matrix element

$$\sigma = \frac{1}{2m_p} \langle p(p, s) | \mathcal{H}_{ab}(0) | p(p, s) \rangle \quad (250)$$
at zero momentum transfer. The $\sigma$ term provides a sensitive measure of explicit chiral symmetry breaking in QCD, because it is a correction to a null result in the chiral limit rather than a small correction to a non-trivial result [Pagels, 1975]. The quark-mass expansion of the $\sigma$ term reads

$$\sigma = \sigma_1 M^2 + \sigma_2 M^3 + \sigma_3 M^4 \ln \left( \frac{M}{m} \right) + \sigma_4 M^4 + \mathcal{O}(M^5),$$

(251)

with

$$\begin{align*}
\sigma_1 &= -4c_1, \\
\sigma_2 &= -\frac{9g_A^2}{64\pi F^2}, \\
\sigma_3 &= -\frac{3}{16\pi^2 F^2 m} \left( g_A^2 - 8c_1 m + c_2 m + 4c_3 m \right), \\
\sigma_4 &= \frac{3}{8\pi^2 F^2 m} \left[ \frac{3g_A^2}{8} + c_1 m(1 + 2g_A^2) - \frac{c_3 m}{2} \right] - 2\hat{e}_1.
\end{align*}$$

(252)

We obtain [with $\hat{e}_1 = 0$ in Eq. (252)]

$$\sigma = (74.8 - 22.9 - 9.4 - 2.0) \text{ MeV} = 40.5 \text{ MeV}.$$

(253)

The result of Eq. (253) has to be compared with, e.g., the dispersive analysis $\sigma = (45 \pm 8) \text{ MeV}$ of Ref. [Gasser et al., 1991] which would imply, neglecting higher-order terms, $-2\hat{e}_1 M^4 \approx (4.5 \pm 8) \text{ MeV}$. As has been discussed, e.g., in Ref. [Becher and Leutwyler, 1999], a fully consistent description would also require to determine the low-energy coupling constant $c_1$ from a complete $\mathcal{O}(q^4)$ calculation of, say, $\pi N$ scattering. The results of Eqs. (215) and (222) satisfy the constraints as implied by the application of the Hellmann-Feynman theorem to the nucleon mass [Gasser et al., 1988],

$$\sigma = M^2 \frac{\partial m_N}{\partial M^2}.$$

(254)

5.2 Chiral expansion of the nucleon mass to $\mathcal{O}(q^6)$

So far, essentially all of the manifestly Lorentz-invariant calculations have been restricted to the one-loop level. One of the exceptions is the chiral expansion of the nucleon mass which, in the framework of the reformulated infrared regularization, has been calculated up to and including $\mathcal{O}(q^6)$ [Schindler et al., 2007b, Schindler et al., 2008]:

$$m_N = m + k_1 M^2 + k_2 M^3 + k_3 M^4 \ln \frac{M}{\mu} + k_4 M^4 + k_5 M^5 \ln \frac{M}{\mu} + k_6 M^5 + k_7 M^6 \ln^2 \frac{M}{\mu} + k_8 M^6 \ln \frac{M}{\mu} + k_9 M^6.$$

(255)

We refrain from displaying the lengthy expressions for the coefficients $k_i$ but rather want to discuss a few general implications [Schindler et al., 2008]. Chiral expansions like Eq. (255) currently play an important role in the extrapolation of lattice QCD results to physical quark masses. Unfortunately, the numerical contributions from higher-order terms cannot be calculated so far since, starting with $k_4$, most expressions in Eq. (255) contain unknown low-energy coupling constants (LECs) from the Lagrangians of $\mathcal{O}(q^4)$ and higher. The coefficient $k_5$ is free of higher-order LECs and is given in terms of the axial-vector coupling constant $g_A$ and the pion-decay constant $F$:

$$k_5 = \frac{3g_A^2}{1024\pi^3 F^4} \left( 16g_A^2 - 3 \right).$$

While the values for both $g_A$ and $F$ should be taken in the chiral limit, we evaluate $k_5$ using the physical values $g_A = 1.2695(29)$ and $F = 92.42(26)$ MeV. Setting $\mu = m_N$, $m_N = (m_p + m_n)/2 = 938.92$ MeV, and $M = M_{\pi^+} = 139.57$ MeV we obtain $k_5 M^5 \ln(M/m_N) = -4.8$ MeV. This amounts to approximately 31% of the leading non-analytic contribution at one-loop order, $k_2 M^3$. Figure 18...
Figure 18: Pion mass dependence of the term $k_5 M^5 \ln(M/m_N)$ (solid line) for $M < 400 \text{MeV}$. For comparison also the term $k_2 M^3$ (dashed line) is shown.

shows the pion mass dependence of the term $k_5 M^5 \ln(M/m_N)$ (solid line) in comparison with the term $k_2 M^3$ (dashed line) for pion masses below 400 MeV which is considered a region where chiral extrapolations are valid (see, e.g., Refs. [Meißner, 2006], [Djukanovic et al., 2006]). We see that already at $M \approx 360 \text{MeV}$ the term $k_5 M^5 \ln(M/m_N)$ becomes as large as the leading non-analytic term at one-loop order, $k_2 M^3$, indicating the importance of the fifth-order terms at unphysical pion masses. Our results for the renormalization-scheme-independent terms agree with the heavy-baryon ChPT results of Ref. [McGovern and Birse, 1999].

5.3 Form factors of the nucleon

5.3.1 Scalar form factor

The pion-nucleon $\sigma$ term corresponds to the kinematical point $t = 0$ of the scalar form factor which is defined as

$$\langle p(p', s') | \mathcal{H}_{ab}(0) | p(p, s) \rangle = \bar{u}(p', s') u(p, s) \sigma(t), \quad t = (p' - p)^2.$$ 

The numerical results for the real and imaginary parts of the scalar form factor at $O(q^4)$ are shown in Fig. 19 for the extended on-mass-shell scheme (solid lines) and the infrared regularization scheme (dashed lines). While the imaginary parts are identical in both schemes, the differences in the real parts are practically indistinguishable. Note that for both calculations $\sigma(0)$ and $\Delta_\sigma \equiv \sigma(2M_\pi^2) - \sigma(0)$ have been adjusted to the dispersion results of Ref. [Gasser et al., 1991], $\Delta_\sigma = (15.2 \pm 0.4) \text{MeV}$.

Figure 20 contains an enlargement near $t \approx 4M_\pi^2$ for the results at $O(q^3)$ which clearly displays how the heavy-baryon calculation fails to produce the correct analytic behavior not only at the tree level but also in higher-order loop diagrams. Both real and imaginary parts diverge as $t \rightarrow 4M_\pi^2$.

5.3.2 Electromagnetic form factors

Imposing the relevant symmetries such as translational invariance, Lorentz covariance, the discrete symmetries, and current conservation, the nucleon matrix element of the electromagnetic current operator $\mathcal{J}^\mu(x)$,

$$\mathcal{J}^\mu(x) = \frac{2}{3} \bar{u}(x) \gamma^\mu u(x) - \frac{1}{3} \bar{d}(x) \gamma^\mu d(x),$$


can be parameterized in terms of two form factors,

\[
\langle N(p', s')|J^\mu(0)|N(p, s)\rangle = \bar{u}(p', s') \left[ F_1^N(Q^2)\gamma^\mu + i\frac{\sigma^{\mu\nu}q_\nu}{2m_p} F_2^N(Q^2) \right] u(p, s), \quad N = p, n,
\]

where \(q = p' - p\), \(Q^2 = -q^2\), and \(m_p\) is the proton mass. At \(Q^2 = 0\), the so-called Dirac and Pauli form factors \(F_1\) and \(F_2\) reduce to the charge and anomalous magnetic moment in units of the elementary charge \(e\) and the nuclear magneton \(e/(2m_p)\), respectively,

\[
F_1^p(0) = 1, \quad F_1^n(0) = 0, \quad F_2^p(0) = 1.793, \quad F_2^n(0) = -1.913.
\]

The Sachs form factors \(G_E\) and \(G_M\) are linear combinations of \(F_1\) and \(F_2\),

\[
G_E^N(Q^2) = F_1^N(Q^2) - \frac{Q^2}{4m_p^2} F_2^N(Q^2), \quad G_M^N(Q^2) = F_1^N(Q^2) + F_2^N(Q^2), \quad N = p, n,
\]

and, in the non-relativistic limit, their Fourier transforms are commonly interpreted as the distribution of charge and magnetization inside the nucleon. For a covariant interpretation in terms of the transverse charge density see Refs. [Miller, 2007], [Carlson and Vanderhaeghen, 2008]. The description of the electromagnetic form factors of the nucleon presents a stringent test for any theory or model of the strong interactions (see, e.g., Ref. [Perdrisat et al., 2007] for a recent review).

In the framework of chiral perturbation theory, the electromagnetic form factors were calculated in the early relativistic approach [Gasser et al., 1988], the heavy-baryon approach [Bernard et al., 1992a, Fearing et al., 1997], the small-scale expansion [Bernard et al., 1998], the infrared regularization [Kubis and Meißner, 2001], and the EOMS scheme [Fuchs et al., 2004]. All these calculations have in common that they fail to describe the proton and nucleon form factors for momentum transfers beyond...
Figure 21: The Sachs form factors of the nucleon in manifestly Lorentz-invariant chiral perturbation theory at $O(q^4)$ without vector mesons. Full lines: results in the extended on-mass-shell scheme; dashed lines: results in infrared regularization. The experimental data are taken from Ref. Friedrich and Walcher, 2003.

$Q^2 \sim 0.1 \text{GeV}^2$. Moreover, up to and including $O(q^4)$, the most general effective Lagrangian provides sufficiently many independent parameters such that the empirical values of the anomalous magnetic moments and the charge and magnetic radii are fitted rather than predicted. Figure 21 shows the Sachs form factors in the momentum transfer region $0 \text{GeV}^2 \leq Q^2 \leq 0.4 \text{GeV}^2$ in the EOMS scheme and the reformulated infrared regularization [Schindler et al., 2005].

In Ref. Kubis and Meißner, 2001 it was shown that the inclusion of vector mesons can result in the re-summation of important higher-order contributions. In standard ChPT, such vector meson contributions manifest themselves in terms of the values of the low-energy coupling constants. Symbolically, the contributions to certain LECs originate from the expansion of the vector-meson propagator,

$$\frac{1}{q^2 - M_V^2} = -\frac{1}{M_V^2} \left[ 1 + \frac{q^2}{M_V^2} + \left( \frac{q^2}{M_V^2} \right)^2 + O(q^6) \right]$$

combined with the relevant vector-meson vertices. However, diagrams with internal vector-meson lines inside loops were not considered, because a generalization of ChPT which fully includes the effects of vector mesons as intermediate states in loops was not yet available [Kubis and Meißner, 2001]. On the other hand, the EOMS renormalization scheme of Ref. Fuchs et al., 2003a and the reformulated version of infrared regularization of Ref. Schindler et al., 2004a both allow to include virtual vector mesons systematically in the region of the applicability of baryon chiral perturbation theory [Fuchs et al., 2003b (see also Ref. Bruns and Meißner, 2008)]. The standard power counting determines which diagrams (including diagrams with vector mesons appearing in loops) should be taken into account to a given order in the chiral expansion.

In Ref. Schindler et al., 2005 the electromagnetic form factors were calculated with the $\rho$, $\omega$, and $\phi$ mesons as explicit degrees of freedom. In the vector-field representation of Ref. Ecker et al., 1989b the $\rho$ meson is represented by $\rho^\mu = \rho^\mu_i \tau_i$ and the $\omega$ and $\phi$ mesons by $\omega^\mu$ and $\phi^\mu$, respectively. The coupling of the vector mesons to pions and external fields is at least of $O(q^3)$,

$$L^{(3)}_{\pi V} = -f_\rho \text{Tr}(\rho^{\mu\nu} f_{\mu\nu}^+) - f_\omega \omega^{\mu\nu} f_{\mu\nu}^{(s)} - f_\phi \phi^{\mu\nu} f_{\mu\nu}^{(s)} + \cdots$$

(257)

where the field strength tensors are given by

$$f_{\mu\nu}^{(s)} = \partial_\mu v_\nu^{(s)} - \partial_\nu v_\mu^{(s)} , \quad f_{\mu\nu}^+ = u^+ f_{\mu\nu} R u + u f_{\mu\nu} L u^+ ,$$

70
Figure 22: Feynman diagrams including vector mesons that contribute to the electromagnetic form factors of the nucleon up to and including $O(q^4)$. External leg corrections are not shown. Solid, wiggly, and double lines refer to nucleons, photons, and vector mesons, respectively. The numbers in the interaction blobs denote the order of the Lagrangian from which they are obtained. The direct coupling of the photon to the nucleon is obtained from $L_{\pi N}^{(1)}$ and $L_{\pi N}^{(2)}$.

with $f_{\mu\nu}^R$ and $f_{\mu\nu}^L$ defined in Eqs. (111) and (112), respectively. For the case of a coupling to an external electromagnetic potential $A_\mu$, the external fields are given by Eq. (51). Furthermore, in terms of the connection $\Gamma^\mu$ of Eq. (178), we define

$$\rho^{\mu\nu} = \nabla^\mu \rho^\nu - \nabla^\nu \rho^\mu,$$
$$\nabla^\mu \rho^\nu = \partial^\mu \rho^\nu + [\Gamma^\mu, \rho^\nu],$$

and, finally,

$$\omega^{\mu\nu} = \partial^\mu \omega^\nu - \partial^\nu \omega^\mu,$$
$$\phi^{\mu\nu} = \partial^\mu \phi^\nu - \partial^\nu \phi^\mu.$$

The lowest-order Lagrangian for the coupling to the nucleon is given by

$$L_{VN}^{(0)} = \frac{1}{2} \sum_{V=\rho,\omega,\phi} g_V \bar{\Psi} \gamma^\mu V^\mu \Psi, \quad (258)$$

and the $O(q)$ Lagrangian reads

$$L_{VN}^{(1)} = \frac{1}{4} \sum_{V=\rho,\omega,\phi} G_V \bar{\Psi} \sigma^{\mu\nu} V_{\mu\nu} \Psi, \quad (259)$$

The additional power-counting rules state that vertices from $L_{eV}^{(3)}$ count as $O(q^3)$ and vertices from $L_{VN}^{(4)}$ as $O(q^4)$, respectively, while the vector-meson propagators count as $O(q^0)$. The additional diagrams involving vector mesons that contribute in the calculation of the form factors up to and including $O(q^4)$ using the Lagrangians of Eqs. (257), (258), and (259) are shown in Fig. 22. The parameters of the vector-meson Lagrangian of Eq. (257) for the coupling to external fields have been taken from Ref. [Ecker et al., 1989b], and those of Eqs. (258) and (259) for the coupling of vector mesons to the nucleon from the dispersion relations of Refs. [Mergell et al., 1996], [Hammer and Meißner, 2004].
As expected on phenomenological grounds, the quantitative description of the data has improved considerably for $Q^2 \geq 0.1$ GeV$^2$ (see Fig. 23). The small difference between the two renormalization schemes is due to the way how the regular higher-order terms of loop integrals are treated. Numerically, the results are similar to those of Ref. [Kubis and Meißner, 2001]. Due to the renormalization condition, the contribution of the vector-meson loop diagrams either vanishes (IR) or turns out to be small (EOMS). Thus, in hindsight our approach puts the traditional phenomenological vector-meson-dominance model on a more solid theoretical basis. In the sense of a strict chiral expansion in terms of small external momenta $q$ and quark masses $m_q$ at a fixed ratio $m_q/q^2$ [Gasser and Leutwyler, 1984], up to and including $O(q^4)$ the results with and without explicit vector mesons are completely equivalent. The additional vector-meson contributions up to this order are compensated by a readjustment of the low-energy constants pertaining to the theory including vector mesons as dynamical degrees of freedom. On the other hand, the inclusion of vector-meson degrees of freedom in the present framework results in a reordering of terms which, in an ordinary chiral expansion, would show up at higher orders beyond $O(q^4)$. It is these terms which change the form factor results favorably for larger values of $Q^2$. It should be noted, however, that this re-organization proceeds according to well-defined rules so that a controlled, order-by-order, calculation of corrections is made possible. In contrast to the calculation without vector mesons, the Sachs form factors $G_E^p$, $G_M^p$, and $G_M^n$ now show sufficient curvature to generate a more accurate phenomenology for values of $Q^2$, where the ordinary chiral expansion to the same order is no longer reliable.

### 5.3.3 Axial and induced pseudoscalar form factors

Assuming isospin symmetry, the most general parametrization of the isovector axial-vector current evaluated between one-nucleon states is given by

$$\langle N(p', s')|A_\mu^B(0)|N(p, s)\rangle = \bar{u}(p', s') \left[ \gamma_\mu \gamma_5 G_A(Q^2) + \frac{q_\mu}{2m_N} \gamma_5 G_P(Q^2) \right] \frac{\tau_i}{2} u(p, s),$$

(260)

where $q = p' - p$, $Q^2 = -q^2$, and $m_N$ denotes the nucleon mass. $G_A(Q^2)$ is called the axial form factor and $G_P(Q^2)$ is the induced pseudoscalar form factor. The value of the axial form factor at zero momentum transfer is defined as the axial-vector coupling constant, $g_A = G_A(Q^2 = 0) = 1.2695(29)$ [Amsler et al., 2008], and is quite precisely determined from neutron beta decay. The $Q^2$ dependence

Figure 23: The Sachs form factors of the nucleon in manifestly Lorentz-invariant chiral perturbation theory at $O(q^4)$ including vector mesons as explicit degrees of freedom. Full lines: results in the extended on-mass-shell scheme; dashed lines: results in infrared regularization. The experimental data are taken from Ref. [Friedrich and Walcher, 2003].
of the axial form factor can be obtained either through neutrino scattering or pion electroproduction (see [Bernard et al., 2002] and references therein). The second method makes use of the so-called Adler-Gilman relation [Adler and Gilman, 1966] which provides a chiral Ward identity establishing a connection between charged pion electroproduction at threshold and the isovector axial-vector current evaluated between single-nucleon states (see, e.g., Ref. [Scherer and Koch, 1991] for more details). The induced pseudoscalar form factor $G_P(Q^2)$ has been investigated in ordinary and radiative muon capture as well as pion electroproduction (see Ref. [Gorringe and Fearing, 2004] for a review).

For the analysis of experimental data, $G_A(Q^2)$ is conventionally parameterized using a dipole form as

$$G_A(Q^2) = \frac{g_A}{(1 + \frac{Q^2}{M_A^2})^2},$$

(261)

where the axial mass $M_A$ is related to the axial root-mean-square radius by $\langle r^2_A \rangle^{\frac{1}{2}} = 2\sqrt{3}/M_A$. The global average for the axial mass extracted from neutrino scattering experiments given in Refs. [Liesenfeld et al., 1999], [Bernard et al., 2002] is

$$M_A = (1.026 \pm 0.021) \text{ GeV}.$$

(262)

The extraction of the axial mean-square radius from charged pion electroproduction at threshold is motivated by the current algebra results and the PCAC hypothesis. At threshold (the spatial components of) the center-of-mass transition current for pion electroproduction can be written in terms of two s-wave amplitudes $E_{0+}$ and $L_{0+}$,

$$e\vec{M}_{|_{\text{thr}}} = \frac{4\pi W}{m_N} \left[ i\vec{\sigma}_\perp E_{0+}(k^2) + i\vec{\sigma}_\parallel L_{0+}(k^2) \right],$$

where $W$ is the total center-of-mass energy, $k^2$ is the four momentum transfer squared of the virtual photon, and $\vec{\sigma}_\perp = \vec{\sigma} \cdot \hat{k} \hat{k}$ and $\vec{\sigma}_\parallel = \vec{\sigma} - \vec{\sigma}_\perp$. The reaction $p(e,e'\pi^+)n$ has been measured at MAMI at an invariant mass of $W = 1125$ MeV (corresponding to a pion center-of-mass momentum of $|q^*| = 112$ MeV) and photon four-momentum transfers of $-k^2 = 0.117, 0.195$ and $0.273 \text{ GeV}^2$ [Liesenfeld et al., 1999]. Using an effective-Lagrangian model an axial mass of

$$\bar{M}_A = (1.077 \pm 0.039) \text{ GeV}$$

was extracted, where the bar is used to distinguish the result from the neutrino scattering value. In the meantime, the experiment has been repeated including an additional value of $-k^2 = 0.058 \text{ GeV}^2$ and is currently being analyzed. The global average from several pion electroproduction experiments is given by [Bernard et al., 2002]

$$\bar{M}_A = (1.069 \pm 0.016) \text{ GeV}.$$

(263)

It can be seen that the values of Eqs. (262) for the neutrino scattering experiments are smaller than Eq. (263) for the pion electroproduction experiments. The discrepancy was explained in heavy-baryon chiral perturbation theory [Bernard et al., 1992b]. It was shown that at $\mathcal{O}(q^3)$ pion loop contributions modify the $k^2$ dependence of the electric dipole amplitude from which $\bar{M}_A$ is extracted. These contributions result in a change of

$$\Delta M_A = 0.056 \text{ GeV},$$

(264)

brining the neutrino scattering and pion electroproduction results for the axial mass into agreement. In a recent analysis [Bodek et al., 2008] updated expressions for the vector form factors have been taken into account together with the hadronic correction of Eq. (264) to produce an average from both neutrino and electroproduction experiments,

$$M_A = (1.014 \pm 0.014) \text{ GeV}.$$

(265)
Earlier calculations of the axial form factor were performed in the framework of heavy-baryon ChPT \cite{Bernard:1992we,Fearing:1997fa} and the small-scale expansion \cite{Bernard:1998cg}. In Ref. \cite{Schindler:2007em} the form factors $G_A$ and $G_P$ have been calculated in manifestly Lorentz-invariant baryon ChPT up to and including $\mathcal{O}(q^4)$. The axial form factor can be written as

$$G_A(Q^2) = g_A - \frac{1}{6} g_A \langle r_A^2 \rangle Q^2 + \frac{g_A^4}{4F^2} L(Q^2),$$

where $\langle r_A^2 \rangle$ is the axial mean-square radius and $L$ contains loop contributions and satisfies $L(0) = L'(0) = 0$. The result for $G_A$ in the momentum-transfer region $0 \text{ GeV}^2 \leq Q^2 \leq 0.4 \text{ GeV}^2$ is shown in Fig. 24. The parameters have been determined such as to reproduce the axial mean-square radius corresponding to the dipole parametrization with $M_A = 1.026$ GeV (dashed line). The dotted and dashed-dotted lines refer to dipole parameterizations with $M_A = 0.95$ GeV and $M_A = 1.20$ GeV, respectively. The loop contributions from $L(Q^2)$ are small and the result does not produce enough curvature to describe the data for momentum transfers $Q^2 \geq 0.1 \text{ GeV}^2$. The situation is similar to the electromagnetic case of Fig. 21, where ChPT at $\mathcal{O}(q^4)$ also fails to describe the form factors beyond $Q^2 \geq 0.1 \text{ GeV}^2$.

In addition to the standard treatment including the nucleon and pions, the axial-vector meson $a_1(1260)$ has also been considered as an explicit degree of freedom \cite{Schindler:2007em}. In the vector-field formulation of \cite{Ecker:1989yg} the $a_1(1260)$ meson is represented by $A^\mu = A^\mu_T$. The advantage of this formulation is that the coupling of the axial-vector mesons to pions and external sources is at least of $\mathcal{O}(q^3)$. The calculation of the contributions to the isovector axial-vector form factors only requires the term

$$L^{(3)}_{\pi A} = \frac{f_A}{4} \text{Tr}(A_{\mu\nu} f_{\mu\nu}^-),$$

where the field strength tensor is defined as

$$f_{\mu\nu}^- = u^+ f_{\mu\nu}^R u - u f_{\mu\nu}^L u^+,$$
Figure 25: Diagram containing the axial-vector meson (double line) contributing to the form factors at $O(q^4)$.

and

$$A_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu, \quad \nabla_\mu A_\nu = \partial_\mu A_\nu + [\Gamma_\mu, A_\nu],$$

with the connection of Eq. (178). The coupling of the axial-vector meson to the nucleon starts at $O(q^0)$. The corresponding Lagrangian reads

$$L_{AN}^{(0)} = \frac{g_{A1}}{2} \bar{\Psi} \gamma^\mu \gamma_5 A_\mu \Psi.$$  (268)

A calculation up to and including $O(q^4)$ would in principle also require the Lagrangian of $O(q^3)$. However, there is no term at this order that is allowed by the symmetries. The additional power-counting rules are as in Section 5.3.2 for the vector mesons. We count the axial-vector meson propagator as $O(q^0)$, vertices from $L_{\pi A}^{(3)}$ as $O(q^3)$, and vertices from $L_{AN}^{(0)}$ as $O(q^0)$, respectively. The contributions of the axial-vector meson to the form factors $G_A$ and $G_P$ at $O(q^4)$ originate from the diagram in Fig. 25. The inclusion of the axial-vector meson effectively results in one additional low-energy coupling constant which has been determined by a fit to the data for $G_A(Q^2)$. The inclusion of the axial-vector meson results in an improved description of the experimental data for $G_A$ (see Fig. 26), while the contribution to $G_P$ is small.

5.3.4 Pion-nucleon form factor

The pion-nucleon form factor $G_{\pi N}(Q^2)$ may be defined in terms of the pseudoscalar quark density $P_i = iq\gamma_5\tau_i q$ and the average light-quark mass $\bar{m}$ as [Gasser et al., 1988]

$$\bar{m} \langle N(p', s')|P_i(0)|N(p, s)\rangle = \frac{M_\pi^2 F_\pi}{M_\pi^2 + Q^2} G_{\pi N}(Q^2) i\bar{u}(p', s')\gamma_5\tau_i u(p, s),$$  (269)

where $q = p' - p$, $Q^2 = -q^2$, and $\Phi_i(x) \equiv \frac{m_{P_i}(x)}{M_\pi F_\pi}$ is the corresponding interpolating pion field. The pion-nucleon coupling constant is given by $g_{\pi N} = G_{\pi N}(-M_\pi^2)$. Using the (QCD-) partially conserved axial-vector current (PCAC) relation, $\partial_\mu A_\mu^i = \bar{m} P_i$, the pion-nucleon form factor is completely given in terms of the axial and the induced pseudoscalar form factors,

$$2m_N G_A(Q^2) - \frac{Q^2}{2m_N} G_P(Q^2) = 2 \frac{M_\pi^2 F_\pi}{M_\pi^2 + Q^2} G_{\pi N}(Q^2).$$

This is an exact relation which holds true for any value of $Q^2$. The result at $O(q^4)$ is given by [Schindler et al., 2007a]

$$G_{\pi N}(Q^2) = \frac{m_N g_A}{F_\pi} - g_{\pi N} \Delta \frac{Q^2}{M_\pi^2} + \cdots$$

where $\Delta = 1 - \frac{m_N g_A}{F_\pi g_{\pi N}}$ denotes the Goldberger-Treiman discrepancy. The chiral expansion of the pion-nucleon coupling constant can be found in Ref. [Schindler et al., 2007a].
Figure 26: Left panel: Axial form factor $G_A$ in manifestly Lorentz-invariant ChPT at $\mathcal{O}(q^4)$ including the axial-vector meson $a_1(1260)$ explicitly. Full line: result in infrared renormalization, dashed line: dipole parametrization. The experimental data are taken from Ref. [Bernard et al., 2002]. Right panel: The induced pseudoscalar form factor $G_P$ in manifestly Lorentz-invariant ChPT at $\mathcal{O}(q^4)$ including the axial-vector meson $a_1(1260)$ explicitly. Full line: result with axial-vector meson; dashed line: result without axial-vector meson. One can clearly see the dominant pion pole contribution at $Q^2 \approx -M_{\pi}^2$.

6 Conclusion

Effective field theory has become a very important tool for investigating the dynamics of the strong interactions. In particular, mesonic chiral perturbation theory is a full-grown and mature area of low-energy particle physics which has successfully been applied at the two-loop level. Whether the predictions for the electromagnetic polarizabilities of the charged pion are really in conflict with empirical data remains to be seen. In the baryonic sector new renormalization conditions have reconciled the manifestly Lorentz-invariant approach with the standard power counting. Phenomenological extensions allowing for the rigorous inclusion of (axial-)vector-meson degrees of freedom (and also of the $\Delta(1232)$ resonance) have opened the door to an extended kinematic region. Unfortunately, the question of convergence in the three-flavor sector remains a controversial issue [Lehnhart et al., 2005], even though the manifestly Lorentz-invariant approach might yield better phenomenological results [Geng et al., 2008]. Finally, beyond the one-nucleon sector the covariant framework has been used in the discussion of relativistic corrections to the nucleon-nucleon potential (see, e.g., Refs. [Higa and Robilotta, 2003], [Robilotta, 2007]) or may be applied to the nuclear many-body problem (see, e.g., Refs. [Furnstahl, 2004], [Serot, 2004]).

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