Holographic equivalence between the first law of entanglement entropy and the linearized gravitational equations

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We use the intertwining properties of integral transformations to provide a compact proof of the holographic equivalence between the first law of entanglement entropy and the linearized gravitational equations, in the context of the AdS/CFT correspondence. We build upon the framework developed by Faulkner et al. [1] using the the Wald formalism, and exploit the symmetries of the vacuum modular Hamiltonian of ball-shaped boundary regions.

I. INTRODUCTION AND SUMMARY

The AdS/CFT correspondence is the conjecture that under certain conditions, a theory of gravity in a \((d+1)\)-dimensional asymptotically Anti de Sitter (AdS) space-time (the bulk) is dual to a “hologram”, a strongly coupled large-N conformal field theory (CFT) living on the \(d\)-dimensional asymptotic boundary [2].

The AdS/CFT correspondence provides a framework for describing gravity in the bulk in terms of the CFT and may be the way to describe -- and learn about -- quantum gravity. As a step in that direction, much effort has been put into “deriving” the classical gravitational laws from CFT properties and vice versa [1,2,3].

It is hypothesized that spacetime itself can be seen as a geometrization of the entanglement structure of the CFT [2]. For CFTs dual to Einstein gravity, the entanglement entropy of a boundary subregion \(B\) is dual to the area of the bulk extremal surface \(\tilde{B}\) that ends on \(-\) and is homological to \(- B\) (to leading order in \(1/N\)) [7,11,16].

\[
S(B)_{\text{HT}} = \frac{\text{Area}(\tilde{B})}{4G_N},
\]

where \(G_N\) is Newton’s constant. We will refer to the extremal surface \(\tilde{B}\) as the Ryu-Takayanagi, or RT-surface, also outside the realm of Einstein gravity. For a generalized theory of gravity (where the Lagrangian is a contraction of Riemann tensors), the entanglement entropy is thought to be dual the Wald-functional evaluated at the extremal surface, up to terms involving the extrinsic curvature [11,13].

The holographic entanglement entropy [1] thus furnishes a direct relation between properties of the CFT and the geometry of the bulk. This points to a relation between the “dynamics” of boundary entanglement and the dynamics of the bulk geometry: gravity. In [1] it was shown that the first law of entanglement entropy,

\[
\delta(H_{\text{mod}}) = \delta S,
\]

where \(H_{\text{mod}}\) is the modular Hamiltonian, implies that the linearized gravitational equations must be satisfied in the bulk. The key ingredient in the derivation of [1] was that for ball-shaped boundary subregions \(B\), there exists a \((d-1)\)-form \(\chi\) such that

\[
\delta H_{\text{mod}}(B) = \delta S(B) - \int_\Sigma d\chi \cdot d\chi = \star(-2\delta E_{ab}\xi^b) \tag{3}
\]

where the \(\delta E_{ab}\) are the linearized gravitational equations of motion without matter coupling and \(\Sigma\) is a Cauchy surface that ends on the RT-surface \(\tilde{B}\) (see figure [2]). The first law [2] now implies

\[
\int_{\Sigma(\tilde{B})} d\chi = \int_{\Sigma(\tilde{B})} \star(-2\delta E_{ab}\xi^b) = 0 \tag{4}
\]

Subsequently it is argued that the linearized gravitational equations must vanish locally by taking derivatives with respect to the ball radii.

In this article, we present a framework that combines the approach of [1] with methods in integral geometry [14]. The key ingredient is to make optimal use of the symmetry properties of the vacuum modular Hamiltonian of a boundary ball, which satisfies two Casimir eigenvalue equations:

\[
(L_{SO(d,2)}^2 + 2d)H_{\text{mod}} = 0, \quad (L_{SO(d,1)}^2 + d)H_{\text{mod}} = 0.
\]

The second equation holds for all constant-time slices that can be associated to the spherical entangling surface \(\partial B\). Both these Casimir eigenvalue equation operators “annihilate” the left-hand side of equation [3] and thus provide a relation between \(\delta S\) and the integral involving the linearized equations of motion. We will refer to these Casimir eigenvalue equation operators as Casimir equations in what follows.

We show that the Casimir equations project the integral of \(d\chi\) onto an integral of \(\delta E_{ab}\) over the surface \(\tilde{B}\). In particular, the \(SO(d,1)\) Casimir equation yields

\[
0 = \left(L_{SO(d,1)}^2 + d\right)\int_\Sigma d\chi = 4\pi R\delta E_{\text{el}}(\tilde{B}). \tag{5}
\]

where \(R\) is the Radon Transform. The Radon transform is invertible on a constant-time slice [15,16], and the fact that equation [5] holds for all boundary balls on all constant-time slices then implies that the linearized gravitational equations must be satisfied.

This simple framework, summarized in figure [11], that does not require a specific choice of coordinates or gauge, will be the subject of this article.
II. WALD FORMALISM

Relation (3) can be derived via the Wald formalism \[1, 19, 20\], which we will briefly review.

Consider a theory with Lagrangian \((d + 1)\)-form \(L\), a functional of the fields \{\(\phi\)\}, which variation is given by

\[
\delta L = E(\delta \phi) + d\Theta(\delta \phi),
\]

where \(E(\delta \phi)\) is the \((d+1)\)-form containing the equations of motion and \(\Theta(\delta \phi)\) is called the symplectic potential, which appears as a boundary term in the action.

For a diffeomorphism invariant theory, the change of the Lagrangian under a diffeomorphism generated by a vector field \(\xi\) is a total derivative

\[
\delta L = \mathcal{L}_\xi L = d\xi \cdot L + \xi \cdot dL = d(\xi \cdot L),
\]

where the dot \(\cdot\) stands for interior multiplication. The second equality is a manifestation of Cartan’s formula.

Equating (6) and (7) allows for the construction of a current \(d\)-form

\[
J_\xi = \Theta(\mathcal{L}_\xi \phi) - \xi \cdot L, \quad \text{with} \quad dJ_\xi = -E(\mathcal{L}_\xi \phi).
\]

The current \(d\)-form \(J_\xi\) is conserved \emph{on-shell}, for every diffeomorphism generating vector field \(\xi\). As a consequence, there exists a \((d-1)\)-form \(Q_\xi\), called the “Noether charge \((d-1)\)-form”, such that, \emph{on-shell}

\[
J_\xi = dQ_\xi.
\]

We also define the \emph{symplectic current} \(d\)-form \(\Omega\)

\[
\Omega(\delta_1 \phi, \delta_2 \phi) = \delta_1 \Theta(\delta_2 \phi) - \delta_2 \Theta(\delta_1 \phi).
\]

Using the equations above, it can be checked that

\[
\delta J_\xi = \Omega(\mathcal{L}_\xi \phi, \delta \phi) + d(\xi \cdot \Theta(\delta \phi)).
\]

One can also define a Noether charge \emph{off-shell}

\[
J_\xi = dQ_\xi + \xi^a C_a, \quad \text{with} \quad d(\xi^a C_a) = -E(\mathcal{L}_\xi \phi)
\]

where \(\xi^a C_a\) is a \(d\)-form that contains the equations of motion for all the fields, except scalar fields \[1, 20\].

This equation will form the basis for the remainder of this article.

III. THE HOLOGRAPHY OF BOUNDARY BALLS

The RT-surface \(\tilde{B}\) for a ball-shaped boundary region \(B\) is highly symmetric: it is the bifurcation surface of a Killing vector field \(\xi(B)\) \[1\]. This observation sets the stage for a natural application of the Wald formalism, from which equation (3) can be derived.

The entanglement entropy of ball-shaped boundary subregions is also interesting for the following reasons:

- the reduced density matrix \(\rho_B\) is thermal with respect to the Hamiltonian \(H_{\text{mod}} = -\log \rho_B\), which is the charge associated with the modular flow generating Killing vector field \(\xi(B)\) \[1, 21\];
- \(H_{\text{mod}}(B)\) can be written as the integral of a smeared, local operator, the stress tensor \(T\) \[21\]:

\[
H_{\text{mod}}(B) = \int_B *j, \quad j_a = T_{ab} \xi^b
\]

- the Killing vector field \(\xi(B)\) can be uniquely continued into the bulk causal wedge, the AdS-Rindler wedge (see figure 4). We normalize \(\xi(B)\) to have surface gravity \(2\pi\).

The boundary stress tensor’s bulk dual is a functional of the bulk metric (perturbation) \[22, 24\]. An important observation in \[1\] is that the holographic dual of the modular Hamiltonian is given by the contribution from the asymptotic boundary to equation (14):

\[
H_{\text{mod}}(B) = \int_\infty (\delta Q_\xi - \xi \cdot \Theta).
\]
For surfaces with vanishing extrinsic curvature, the holographic entanglement entropy is given by [11, 12]:

$$\delta S(B) = \int_B (\delta Q_\xi - \xi \cdot \Theta) = \int_B \delta Q_\xi$$

(17)

where we used that $\xi|_{\tilde{B}} = 0$. These geometrical identifications set the stage for the translation between entanglement dynamics [2] and gravitational dynamics.

### IV. THE CASIMIR EQUATION

The quadratic Casimir of the conformal group $SO(d,2)$, $L^2_{SO(d,2)}$, has eigen operators that are labeled by their dimension $\Delta$ and spin $l$. The eigenvalues are given by [25]

$$C^\Delta_{SO(d,2)} = -\Delta(\Delta - d) - l(l + d - 2).$$

(18)

The modular Hamiltonian $H_{\text{mod}}$ is a smeared integral of the boundary stress tensor [15], which has dimension $\Delta = d$ and spin $l = 2$, so $H_{\text{mod}}$ satisfies

$$\left(L^2_{SO(d,2)} + 2d\right) H_{\text{mod}} = 0.$$ 

(19)

The modular Hamiltonian also satisfies a second type of Casimir equation for every constant-time slice that contains $\partial B$. Constant-time slices on the boundary are spacelike surfaces whose non-degenerate, timelike normal vector is Killing. They naturally extend to the bulk, and by symmetry, the RT-surface must lie on the bulk extension of the boundary constant-time slice. Thus, from now on, the term “constant-time slice” will also refer to its extension into the bulk, which has hyperbolic geometry. The stabilizer of a constant-time slice is the subgroup $\text{Isom}(\mathbb{H}^d) = SO(d,1) \subset SO(d,2)$.

On a constant-time slice with normal vector $t$, the modular Hamiltonian [15] can be written as

$$H_{\text{mod}} = \int_B d^{d-1}x \, T_{tt}[\xi(B)].$$

(20)

Note that $t^a = \frac{x^a}{\sqrt{\text{det}(g)}}$, except on the bifurcation surface $\tilde{B}$. The stress tensor component $T_{tt}$ transforms as a scalar under isometries that preserve the constant-time slice. The modular Hamiltonian thus also satisfies an $SO(d,1)$ Casimir equation on each constant-time slice $\Sigma(B)$:

$$\left(L^2_{SO(d,1)} + d\right) H_{\text{mod}}(B) = 0,$$ 

(21)

where the eigenvalue is given by equation (18) with $d$ replaced by $d - 1$ and $l = 0$. $T_{tt}$ carries dimension $\Delta$ under $SO(d,1)$ transformations, as it inherits its scaling behavior from the full $SO(d,2)$ group.

### V. INTERTWINEMENT

The modular Hamiltonian $H_{\text{mod}}(B)$ can be seen as a function on the space of boundary balls, which we will call kinematic space [14, 26]. The space of boundary balls on a single constant-time slice has a de Sitter (dS) geometry and the $SO(d,1)$ Casimir is represented as the d’Alambertian $\Box_{\text{dS}}$ and the Casimir equation (21) can be written as a de Sitter wave equation: $\Box_{\text{dS}} + d) H_{\text{mod}} = 0$ [14, 27].

We will exploit the intertwining properties of integral transformations, which relate the d’Alembertian $\Box_{\text{dS}}$ on kinematic space to the d’Alembertian $\Box_{\text{dS}}$ on the constant-time slice, in order to prove equation (5).

The Radon transform $R$ and a second useful integral transform $R_T$ of a function $f$ are given by:

$$R f(\tilde{B}) \equiv \int_B f, \quad R_T f(\tilde{B}) \equiv \int_{\Sigma_t(\tilde{B})} f[\xi(B)].$$

(22)

where $\Sigma_t \subset \mathbb{H}^d$ is taken to be on a constant-time slice. It is well-known that the Radon transform satisfies an intertwining relation [14, 16, 28, 29]

$$L^2_{SO(d,1)} \cdot R f = RL^2_{SO(d,1)} \cdot f.$$

(23)

where $L^2_{SO(d,1)}$ is the quadratic $\mathfrak{so}(d,1)$ Casimir, and the $\cdot$ denotes its action on the object on its right hand side. For (homogeneous) coset spaces $G/H$, the quadratic Casimir of a (semi-simple) Lie group $G$ is represented on functions by the Laplacian, up to an overall scaling [30]. Both hyperbolic space $\mathbb{H}^d$ as well as its kinematic space $d\text{S}_d$ are coset spaces of $G \equiv SO(d,1)$ with $H=SO(d)$ and $H=SO(d-1,1)$ respectively. In coordinates we have (see appendix C for a review of intertwining)

$$\Box_{\text{dS}} R f = -R \Box_{\text{dS}} f.$$

(24)
The second type of transform \( \text{see appendix } \boxed{} \) has a similar intertwining property:
\[
\square_{\text{dS}} R_\xi f = -R_\xi \square_{\text{dS}} f.
\] (25)
It follows from a double partial integration and the Killing property of \( \xi \):
\[
(L_{\text{dS}(d,1)}^2 + d) R_\xi f = -2\pi Rf.
\] (26)
Note that intertwenment rules can be used to translate the dynamics of local fields on AdS spacetime to the dynamics of fields on kinematic space. The AdS/CFT correspondence provides a natural set of non-local, diffeomorphism invariant bulk probes that have a definite dual in the CFT. CFT Wilson loops, OPE-blocks and the two-point functions of heavy operators can be associated to extremal bulk quantities \[14, \ 31, \ 32\], in addition to aforementioned entanglement entropies of boundary subregions. These non-local probes can (perturbatively) be seen as integral transformations of bulk fields, which also have intertwining properties; the derivation of intertwenment merely relies on diffeomorphism invariance (see appendix \[\boxed{}\]).

VI. GRAVITY

At last we will combine the geometrical analysis \[\boxed{}\] and the intertwenment rules \[\boxed{}\] to derive the linearized gravitational equations.

At leading order in \(1/N\), we can ignore matter fields. In that case, we have \(\delta \mathcal{H}(\xi, \delta g) = 0\), since per definition \(\mathcal{L}_\xi g = 0\). As a consequence, the left-hand side of equation \[\boxed{}\] vanishes, so we have
\[
\int \delta S_{\text{bulk}}(B) = \delta S(B)
\]
\[
\int_\infty (\delta Q_{\xi} - \xi \cdot \Theta(\delta \phi)) = \int_B \delta Q_{\xi} - \xi \cdot \Theta(\delta \phi) - \int_\Sigma d\chi,
\] (27)
where (see appendix \[\boxed{}\] for a review) \[\boxed{}\]:
\[
d\chi \equiv \xi^a \delta C^a = \star(-2\delta E^a_{\text{bulk}} \xi^a).
\] (28)
The linearized gravitational equations satisfy \(\nabla^a \delta E_{ab} = 0\) by virtue of the Noether identity \[\boxed{}\]. Conservation of \(\delta E^a_{\text{bulk}} \xi^a\) implies that the integral of \(d\chi\) does not depend on the choice of the Cauchy surface \(\Sigma(B)\), which we take to be on a constant-time slice:
\[
\int_{\Sigma(B)} d\chi = \int_{\Sigma_{t}} d\chi = -2 \int_{\Sigma_{t}} \delta E_{tt} |\xi|.
\] (29)
In \[\boxed{}\], the first law \[\boxed{}\] and equation \[\boxed{}\] are used directly to argue that the integral of the \(d\)-form \(d\chi\) must vanish. Subsequently, an appropriate combination of kinematic space derivatives is taken to argue that the equations of motion must vanish locally: \(\delta E_{ab} = 0\).

Here, we take a different approach: the key ingredient is equation \[\boxed{}\], whose left-hand side is annihilated by the Casimir equation \[\boxed{}\], such that
\[
(L_{\text{dS}(d,1)}^2 + d) \delta S(B) = (L_{\text{dS}(d,1)}^2 + d) \int_{\Sigma_t} d\chi.
\] (30)
From equation \[\boxed{}\] we see directly that the Casimir equation “projects” the integral over the Cauchy slice \(\Sigma_t(B)\) (right hand side) onto an integral that only has support on the RT-surface \(B\) only (left-hand side). Concretely, we recognize that equation \[\boxed{}\] is of the type \[\boxed{}\]. Applying intertwenment rule \[\boxed{}\] directly gives
\[
0 = (L_{\text{dS}(d,1)}^2 + d) \int_{\Sigma_t} d\chi = 4\pi R \delta E_{tt}(B).
\] (31)
The Radon transform is known to be invertible on hyperbolic space \[\boxed{}\]. Equation \[\boxed{}\] holds for every boundary ball on every constant time-slice, so in every point and for every timelike vector \(t^a\) we have
\[
\delta E_{ab} t^a t^b = 0 \Rightarrow \frac{d}{dt^a} \frac{d}{dt^b} (\delta E_{ab} t^a t^b) = 0.
\] (32)
We conclude that the symmetric part of \(\delta E_{ab}\) must vanish, which is equivalent to the condition that the linearized equations of motion must be satisfied.

In \[\boxed{}\], the Wald formalism and equation \[\boxed{}\] are not used. Instead, it is shown that for theories with \(S = (4G_N)^{-1} A\) the first law \[\boxed{}\] leads to the linearized Einstein equations by directly applying intertwenment rules to \(\delta S\), which is the longitudinal Radon transform \[\boxed{}\] of the metric perturbation \(\delta g_{ab}\). Intertwinement rules are developed for the traceless and trace parts of the metric perturbation \(\delta g_{ab}\), showing that
\[
\int \delta S(B) = 4\pi R \delta E_{tt}(B),
\] (33)
which, in our approach directly follows for generalized theories of gravity from equations \[\boxed{}\] and \[\boxed{}\].

VII. BULK MATTER

From the perspective of entanglement entropy, the quantum corrections of the holographic entanglement entropy \[\boxed{}\] are, to first subleading order in \(1/N\), given by the FLM-formula \[\boxed{}\]:
\[
S(B) = S_{\text{RT}}(B) + S_{\text{bulk}}(\Sigma(B)) + S_{\text{Wald-like}}(B)
\] (34)
where \(S_{\text{bulk}}\) is the bulk entanglement entropy for the AdS-Rindler wedge (see figure \[\boxed{}\]). The terms at subleading order in \(G_N\) conspire to \[\boxed{}\],
\[
\delta S_{\text{bulk}}(\Sigma(B)) + \delta S_{\text{Wald-like}}(B) = \int_{\Sigma(B)} \star j \cdot j_a = \delta(T_{ab}) \xi^a
\]
where \( T_{ab} \) is the Hilbert stress tensor, which appears on the right hand side of the gravitational equations. Note that \( T_{ab} \) contains a contribution from the graviton. This term modifies equation (17) at first subleading order in \( 1/N \) and is of the form (22), on a constant-time slice. This means that intertwinement rule (26) can be used, such that equation (31) becomes

\[
2\pi R \left( 2\delta E_{tt} - \delta \langle T_{tt} \rangle \right) (\tilde{B}) = 0, \tag{35}
\]

which implies, by virtue of the invertibility of the Radon transform on \( \mathbb{H}^d \) and equation (32), that the linearized equations of motion must also be satisfied in the presence of matter.

Conversely, if we assume the linearized gravitational equations with matter coupling, then the invertibility of the Radon transform fixes the \( 1/N \) correction to be of the form (35), which was first shown for Einstein gravity in [33].

\section*{VIII. OUTLOOK}

In the above, we used the constant-time slice Casimir equation (21). We could have used the conformal Casimir equation (19) to “annihilate” \( H_{\text{mod}} \).

\section*{Appendix A: \( \xi^a C_a \) and the Noether Identity}

This section is based on appendix B of [1]. Under a diffeomorphism generated by a vector field \( \xi \), the variation of the Lagrangian n-form is given by

\[
\mathcal{L}_\xi L = E(\mathcal{L}_\xi \phi) + d\Theta(\mathcal{L}_\xi \phi). \tag{A1}
\]

The equation of motion n-form \( E(\mathcal{L}_\xi \phi) \) contains all the fields. For an \((r, s)\)-tensor field \( \phi \), the contribution to \( E(\mathcal{L}_\xi \phi) \) is given by

\[
E(\mathcal{L}_\xi \phi) = \star E_{a_1 \cdots a_r}^b \mathcal{L}_\xi \phi_{a_1 \cdots a_r}^{b_1 \cdots b_s} \tag{A2}
\]

where \( E_{a_1 \cdots a_r}^b \) is the “equation of motion tensor”. For example, for the metric field \( g \), and Einstein gravity we have

\[
E_g^{ab} = \frac{1}{16\pi G_N} \left( G_{ab} + g_{ab} \Lambda \right). \tag{A3}
\]

Expanding the Lie-derivate in equation (A2) gives

In the appendix we show that

\[
0 = \left( L_{\mathcal{L}_\xi} \mathcal{L}_\xi \phi + 2d \right) \left( \int_\Sigma d\chi + \int_\Sigma \star (\delta \langle T_{ab} \rangle \xi^b) \right) \tag{36}
\]

where \( R^\perp \) is the perpendicular Radon transform, the integral of the projection of a tensor to the two-dimensional normal plane of \( \tilde{B} \). Unfortunately, the inversion and injectivity properties of \( R^\perp \) are still unknown.

Finally, an interesting observation is that for other homogeneous spaces the same framework can be applied, if the terms in equation (14) can be identified as \( \delta H \) (16) and \( \delta S \) (17), to establish the equivalence of the linearized gravitational equations and the first law (2). One can also apply the Casimir equation in conjunction with intertwinement rules to the terms up to second order in the perturbation, among which the canonical energy \( \mathcal{E} \) (36, 37), in order to obtain gravitational equations at second order in the perturbation. This will be the subject of future work.

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Equation (A5) holds for any vector field $\xi$, so we must have

$$\sum_{\phi} \left( (E^b)^{a_1 \cdots a_r}_{a_1 \cdots a_r} \nabla_c \phi_{b_1 \cdots b_s} \right) + \nabla_c \left( \sum_{i=1}^{r} (E^b)^{a_1 \cdots a_r}_{a_1 \cdots a_r} \phi_{b_1 \cdots b_i} \delta^d \right) = 0 \quad \text{(A6)}$$

This is the "Noether identity".

Now consider a theory in which only the metric appears. The Noether Identity (A6) becomes

$$\nabla_d \left( E^a_{\cdot d} g^{a_1 \cdot d} + E^a_{\cdot a_2} g^{a_1 \cdot d} \right) = 0 \quad \text{(A7)}$$

If we assume that the unperturbed equations of motion are satisfied, and we expand in the perturbation it follows that $\nabla^b \delta F_{ab} = 0$. In other words, the first order perturbation of the equations of motion is a conserved symmetric two-tensor.

Using equations (A2), (A5) and the Noether identity (A6) we conclude

$$E(L_\xi \phi) = \star \nabla_c \left( \sum_{i=1}^{r} (E^b)^{a_1 \cdots a_r}_{a_1 \cdots a_r} \phi_{b_1 \cdots b_i} \delta^d + \sum_{i=1}^{s} (E^b)^{a_1 \cdots a_r}_{a_1 \cdots a_r} \phi_{b_1 \cdots b_i} \delta^d \right)$$

$$= \star \star d \star F \quad \text{with:} \quad F^c = \sum_{i=1}^{r} (E^b)^{a_1 \cdots a_r}_{a_1 \cdots a_r} \phi_{b_1 \cdots b_i} \delta^d + \sum_{i=1}^{s} (E^b)^{a_1 \cdots a_r}_{a_1 \cdots a_r} \phi_{b_1 \cdots b_i} \delta^d$$

$$= d \star F \equiv -d \langle \xi^a C_a \rangle \quad \text{(A8)}$$

The $C$ are defined, to be consistent with (1), as:

$$\xi^a C_a = (-) \star F = (-) \star \left( \sum_{i=1}^{s} (E^b)^{a_1 \cdots a_r}_{a_1 \cdots a_r} \phi_{b_1 \cdots b_i} \delta^d - \sum_{i=1}^{r} (E^b)^{a_1 \cdots a_r}_{a_1 \cdots a_r} \phi_{b_1 \cdots b_i} \delta^d \right)$$

For a theory with only the metric field, we have

$$E(L_\xi g) = d \star F = -d \langle \xi^a C^a \rangle \quad \text{with:} \quad F_b = 2\xi^a E^a_{ab} \quad \text{(A9)}$$

Clearly, for the unperturbed metric, both sides vanish (on-shell). At the linear level, we have

$$F_b = 2\xi^a \delta E^a_{ab} \quad \chi = \star (-2\xi^a \delta E_{ab}) \quad \text{(A10)}$$

**Appendix B: Killing Vector $\xi(B)$**

For completeness, we give the expression of the Killing vector $\xi$ in Poincaré coordinates. When the caustics of the boundary ball are parametrized by $x_{1,2} = (t_0 \pm R, \vec{x}_0)$, where $R$ corresponds to the radius of the boundary ball and $(t_0, \vec{x}_0)$ to the center, then

$$\xi(B(x_1, x_2)) = 2\pi \frac{R^2 - (t - t_0)^2 - (\vec{x} - \vec{x}_0)^2 - z^2}{2R} \partial_t - 2\pi \frac{(t - t_0)((\vec{x} - \vec{x}_0)\partial_{\vec{x}} + z\partial_z)}{R} \quad \text{(B1)}$$
The Killing vector \( \xi \) also can be expressed in terms of vectors on embedding space \( \mathbb{R}^{d,2} \). Let \( N_{1,2} \) be the embedding space null vectors “pointing towards” the points \( x_{1,2} \) on the asymptotic boundary of the AdS-hyperbola defined by \( X^2 = -1 \). The Killing vector \( \xi \) is now given by:

\[
\xi^A = \frac{(N_2 \cdot X)N_1^A - (N_1 \cdot X)N_2^A}{N_1 \cdot N_2}. \tag{B2}
\]

A more general expression in Poincaré coordinates, in terms of the boundary points \( x_{1,2} \) is given by

\[
\xi^\mu = \frac{(z^2 + (x - x_1)^2) - (z^2 + (x - x_2)^2)}{(x_1 - x_2)^2}, \quad \xi^\nu = \frac{(z^2 + (x - x_1)^2)(x^\mu - x_1^\mu) - (z^2 + (x - x_2)^2)(x^\nu - x_2^\nu)}{(x_1 - x_2)^2}. \tag{B3}
\]

### Appendix C: Intertwinement

#### 1. Review of Intertwinement

Under a diffeomorphism \( x \mapsto x'(x) \) (that leaves the constant-time slice invariant) we have

\[
Rf(\tilde{B}) \rightarrow Rf'(\tilde{B}') = Rf(\tilde{B}), \tag{C1}
\]

or in terms of the group element \( g \in SO(d,1) \)

\[
Rg \cdot f(g \cdot \tilde{B}) = Rf(\tilde{B}), \tag{C2}
\]

where \( \cdot \) denotes the action of the group element on the object on its right hand side (function, surface,...). Equivalently to equation (C2), we can write

\[
Rg \cdot f(\tilde{B}) = Rf(g^{-1} \cdot \tilde{B}). \tag{C3}
\]

Since the isometry group \( \text{Iso}(\mathbb{H}^d) = SO(d,1) \) is a Lie group, we can also write equation (C8) in infinitesimal form in terms of the generators of \( \mathfrak{so}(d,1) \):

\[
RL_{AB} \cdot f(\tilde{B}) = -Rf(L_{AB} \cdot \tilde{B}), \tag{C4}
\]

where now the \( \cdot \) denotes the action of the Lie-algebra element \( L_{AB} \) on \( C_0(\mathbb{H}^d) \) and \( C(K_{t=0}) \) respectively. Exploiting relation (C4) twice, we find

\[
RL^2 \cdot f(\tilde{B}) = Rf(L^2 \cdot \tilde{B}) = L^2 \cdot Rf(\tilde{B}), \tag{C5}
\]

where \( L^2 \) is the quadratic Casimir operator. For homogeneous coset spaces \( G/H \), the quadratic Casimir of the (semisimple) Lie group \( G \) is represented by the d’Alembertian [30]. Both \( \mathbb{H}^d \) as well as the kinematic space \( \text{Isom}(\mathbb{H}^d) = dS_d \) are of the form \( G/H \), with \( G = SO(d,1) \) and \( H = SO(d) \) and \( H = SO(d-1,1) \) respectively. The relative scaling is fixed by considering the Cartan Killing form on \( \mathfrak{so}(d,1) \) explicitly, or simply by checking the intertwinement property in a coordinate basis. One can check that equation (C5) becomes

\[
\Box_{dS_d} Rf(\tilde{B}) = -R \Box_{dS_d} f(\tilde{B}). \tag{C6}
\]

Nowhere did we use specific properties of \( R \), so for

\[
R_\xi f(\tilde{B}) = -\int_{\Sigma(\tilde{B})} f[\xi(\tilde{B})] \text{ we have similarly: } \Box_{dS_d} R_\xi f(\tilde{B}) = -R_\xi \Box_{dS_d} f(\tilde{B}). \tag{C7}
\]

#### 2. Intertwinement for \( R_\xi \)

In this subsection we show that for a conserved symmetric two-tensor \( W \)

\[
\left( L_{SO(d,1)} + d \right) \int_{\Sigma(\tilde{B})} * (W_{ab} \xi^b) = -\int_{\tilde{B}} W_{ab} \frac{\xi^a \xi^b}{|\xi|^2}. \tag{C8}
\]
where $\tilde{B}$ is the bifurcation surface for the Killing vector $\xi$, and $\Sigma$ is any Cauchy surface that ends on $\tilde{B}$ and $B$. This is the more general form of intertwinement property (26) for (22). For completeness, we will also derive relation (26) here, not using any specific coordinate set.

First, we note that the integral does not depend on the choice of $\Sigma(\tilde{B})$, by virtue of the conservation of $W_{ab}\xi^b$:

$$\nabla^a(W_{ab}\xi^b) = (\nabla^aW_{ab})\xi^b + W_{ab}\nabla^a\xi^b = 0,$$

by virtue of the conservation of $W$ and the Killing equation; we choose $\Sigma$ to be a surface orthogonal to $\xi$:

$$\int_{\Sigma(\tilde{B})} \ast(W_{ab}\xi^b) = \int_{\Sigma(\tilde{B})} \left( W_{ab} \frac{\xi^b}{|\xi|} \right) |\xi| \equiv \int_{\Sigma(\tilde{B})} f|\xi| \equiv R_\xi f(\tilde{B})$$

From the intertwinement property (25) we have

$$L_{SO(d,1)}^2 \int_{\Sigma(\tilde{B})} f|\xi| = - \int_{\tilde{B}} (D_a D^a f)|\xi|,$$

where $D$ is the induced covariant derivative on the $\mathbb{H}^d$. We can further simplify this result (25) by using

$$(D_a D^a f)|\xi| = D_a(|\xi| D^a f) - D_a(f D^a|\xi|) + f D_a D^a |\xi|,$$

One can check that

$$D_a D^a |\xi| = \left( g^{ab} + \frac{\xi^a \xi^b}{|\xi|^2} \right) \nabla_a \nabla_b \sqrt{|\xi|^2} = \left( g^{ab} + \frac{\xi^a \xi^b}{|\xi|^2} \right) \nabla_a \left( \frac{\xi^c \nabla_b \xi^c}{\sqrt{|\xi|^2}} \right)$$

$$= \left( g^{ab} + \frac{\xi^a \xi^b}{|\xi|^2} \right) \left( \frac{\nabla^c \xi^d (\nabla_b \xi^c) \nabla_d \xi^b}{\sqrt{|\xi|^2}} \right) - \left( g^{ab} + \frac{\xi^a \xi^b}{|\xi|^2} \right) \left( \frac{\xi^c \nabla_b \xi^c (\nabla_a \xi_d)}{\sqrt{\xi^2}} \right)$$

$$+ \left( g^{ab} + \frac{\xi^a \xi^b}{|\xi|^2} \right) \frac{\xi^c \nabla_a \nabla_b \xi_c}{\sqrt{|\xi|^2}}$$

Now for terms in (C13) we use that a surface orthogonal to $\xi$ has vanishing extrinsic curvature

$$K_{ab} = \left( g^{ac} + \frac{\xi^a \xi^c}{|\xi|^2} \right) \nabla_c \xi_b = 0$$

and for the last term (C14) we use that for Killing vectors on AdS-spacetime

$$\nabla_a \nabla_b \xi_c = R_{cbad} \xi^d \overset{AdS}{=} -g_{ac} \xi_b + \xi_c g_{ab},$$

such that

$$\square_{\Sigma}|\xi| = D_a D^a |\xi| = d|\xi|.$$ (C17)

It follows that from equation (C12) and (C17)

$$\int_{\Sigma} (C_{\Sigma}[f]|\xi|) = d \int_{\Sigma} f|\xi| - \int_{\tilde{B}} f N^a D_a |\xi| + \text{boundary terms at } \infty$$

We assume the other boundary terms to vanish, for sufficiently rapidly falling off $f$ at $\infty$ and using $\xi|\tilde{B} = 0$. In summary, we have used partial integration (C12) and equation (C17) to get

$$(L_{SO(d,1)}^2 + d) \int_{\Sigma(\tilde{B})} f|\xi| = \int_{\tilde{B}} f N^a D_a |\xi|.$$ (C19)

On the surface $\tilde{B}$, we have $\nabla_a \xi_b = \kappa n_{ab}$ where $n_{ab}$ is the anti-symmetric binormal and the surface gravity $\kappa = 2\pi$. Since we integrated on a surface orthogonal to $\xi$, the normal vector $N$ is orthogonal to $\xi$ as well:

$$N^a D_a |\xi| = N^a \frac{\xi^c}{|\xi|} \nabla_a \xi_c = 2\pi N^a \frac{\xi^c}{|\xi|} n_{ac} = -\pi n^{ab} n_{ab} = -2\pi$$

So finally we have,

$$\left( L_{SO(d,1)}^2 + d \right) \int_{\Sigma(\tilde{B})} f|\xi| = \int_{\tilde{B}} f N^a D_a |\xi| = -2\pi \int_{\tilde{B}} f,$$

or

$$\left( L_{SO(d,1)}^2 + d \right) \int_{\Sigma(\tilde{B})} \ast(W_{ab}\xi^b) = -2\pi \int_{\tilde{B}} W_{ab} \frac{\xi^b}{|\xi|^2}.$$ (C21)
Appendix D: The SO(d,2) Casimir Equation

Here we prove equation (36). First we note that equation (C5) follows from diffeomorphism invariance only; specific details of the transformation $R$, the field $f$ and the isomorphism group were not used. So similar to equation (C5), we also have

$$RL^2_{SO(d,2)} \cdot f(\tilde{B}) = L^2_{SO(d,2)} \cdot R f(\tilde{B}),$$  \hspace{1cm} (D1)

for any diffeomorphism invariant transformation $R$ of the field $f$, which can also be a tensor field.

Now we consider a particular transform $\tilde{R}_\xi$ that maps a conserved symmetric two-tensors $W_{ab}$ on AdS to a function on kinematic space:

$$\tilde{R}_\xi(\tilde{B}) \equiv \int_{\Sigma(\tilde{B})} \star(W_{ab}\xi^b(\tilde{B}))$$  \hspace{1cm} (D2)

where $\Sigma(\tilde{B})$ is a Cauchy surface that ends on $\tilde{B}$. Note that (D2) does not depend on the specific choice of Cauchy surface by virtue of the conservation of $W_{ab}\xi^b(\tilde{B})$. Both the integral of $d\chi$ (see 4) as well as the FLM-formula (34) are of this form. Below, we will derive the intertwining properties of the transform (D2).

On general tensors, the conformal Casimir $L^2_{SO(d,2)}$ is represented as \cite{30,38}:

$$-\Box_{AdS} - l(l + d - 1),$$  \hspace{1cm} (D3)

where $l = 0$ for functions and $l = 2$ for the traceless symmetric part of a two-tensor. Thus, decomposing $W_{ab}$ in its trace and traceless components

$$W^\text{trace}_{ab} = \frac{W}{d + 1} g_{ab}, \quad W = g^{ab} W_{ab}, \quad W^\text{traceless}_{ab} = W_{ab} - W^\text{trace}_{ab}$$  \hspace{1cm} (D4)

we have

$$(L^2_{SO(d,2)} + 2d)\tilde{R}_\xi W = \tilde{R}_\xi \left(- (\Box_{AdS} W) + 2dW - 2(d+1)W^\text{traceless}\right)$$  \hspace{1cm} (D5)

It follows after some algebra that

$$\tilde{R}_\xi \left(- (\Box_{AdS} W) + 2dW - 2(d+1)W^\text{traceless}\right) = \int_{\Sigma(\tilde{B})} \star \left(\Delta(W_{ab}\xi^b) + 2\nabla^b \xi^b W_{ab}\right),$$  \hspace{1cm} (D6)

where $\Delta \equiv \delta d + d\delta$ is the Hodge Laplacian (and $\delta$ is the co-differential).

Two important identities used for the derivation of equation (D6) are given by:

$$(\Delta \omega)_a = -(\Box \omega)_a + R^b_a \omega_b, \text{ for a one form } \omega$$  \hspace{1cm} (D7)

where $\Box = \nabla^a \nabla_a$ and

$$\nabla^b \nabla_a W_{ab} = [\nabla_b, \nabla_a] W_{ad} g^{bd} = -(d + 1)W^\text{traceless}_{ab}.$$  \hspace{1cm} (D8)

The second contribution to the RHS of equation (D6) vanishes, because under a diffeomorphism generated by $\xi$,

$$W_{ab} \rightarrow \tilde{W}_{ab} = W_{ab} + (\xi \nabla)_{ab} + \ldots \hspace{1cm} (D9)$$

$$\nabla^b W_{ab} \rightarrow \tilde{\nabla}^b \tilde{W}_{ab} = \nabla^b \tilde{W}_{ab} = \nabla^b (W_{ab} + (\xi \nabla)_{ab} + \ldots) \hspace{1cm} (D10)$$

where we use that $\xi$ is Killing, which implies that $\nabla = \tilde{\nabla}$. Conservation of $W$ requires $\nabla^b W_{ab} = \tilde{\nabla}^b \tilde{W}_{ab} = 0$, so it follows from equation (D10) that $\nabla^b \xi^b W_{ab} = 0$. In summary, we have

$$\left(L^2_{SO(d,2)} + 2d\right)\tilde{R}_\xi W = \int_{\Sigma(\tilde{B})} \star (\Delta(W_{ab}\xi^b)).$$  \hspace{1cm} (D11)

After some manipulation, using the definition of the Hodge Laplacian and Stokes theorem, it follows that
\[ \left( L_{\text{SO}(d,2)}^2 + 2d \right) \tilde{R}_\xi W = -(-1)^{d+1} \int_{\Sigma(B)} \ast \ast d \ast d(W_{ab} \xi^b) \quad \text{Note: } \delta(W_{ab} \xi^b) = (-1)^d \nabla^a(W_{ab} \xi^b) = 0 \]

\[ = - \int_{\tilde{B}} \ast d(W_{ab} \xi^b) \quad \text{Note: } \ast \ast \omega_p = -(-1)^{p(n-p)} \omega_p \]

\[ = - \int_{\tilde{B}} \ast (W_b^\mu \nabla_\mu \xi_c - W_a^\mu \nabla_\mu \xi_b) \quad \text{Note: } \xi|_{\tilde{B}} = 0 \]

\[ = -2 \int_{\tilde{B}} n^{ab} \tilde{W}_b^c \nabla_a \xi_c, \]

where \( n \) is the antisymmetric binormal. On \( \tilde{B}, \nabla_a \xi_b = \kappa n_{ab} = 2 \pi n_{ab} \), so it follows that

\[ \left( L_{\text{SO}(d,2)}^2 + 2d \right) \tilde{R}_\xi W = 2 \pi \int_{\tilde{B}} s^{ab} W_{ab} \equiv 2 \pi R^\perp W, \tag{D13} \]

where \( s^{ab} = (g^{ab} - h^{ab}) \) is the symmetric binormal and \( R^\perp \) is the transverse Radon transform \([32]\). Both \( \delta E_{ab} \) and \( \delta T_{ab} \) are conserved symmetric two-tensors, so

\[ 0 \text{ Eqn. (D19)} = \left( L_{\text{SO}(d,2)}^2 + 2d \right) H_{\text{mod}}(B) \]

\[ 1^\text{st law} = \left( L_{\text{SO}(d,2)}^2 + 2d \right) \delta S(B) \]

\[ \text{Eqn. (D14)} = \left( L_{\text{SO}(d,2)}^2 + 2d \right) \tilde{R}_\xi (-2 \delta E + \delta T) \]

\[ = -2 \pi R^\perp (2 \delta E - \delta T) \]

This finalizes the proof of equation (D19). This was first shown for Einstein gravity via a different method in \((33)\). Our result holds for a generalized theory of gravity.

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