Singularities and the distribution of density in the Burgers/adhesion model

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We are interested in the tail behavior of the pdf of mass density within the one and \( d \)-dimensional Burgers/adhesion model used, e.g., to model the formation of large-scale structures in the Universe after baryon-photon decoupling. We show that large densities are localized near “kurtoparabolic” singularities residing on space-time manifolds of codimension two \((d \leq 2)\) or higher \((d \geq 3)\). For smooth initial conditions, such singularities are obtained from the convex hull of the Lagrangian potential (the initial velocity potential minus a parabolic term). The singularities contribute universal power-law tails to the density pdf when the initial conditions are random. In one dimension the singularities are preshocks (nascent shocks), whereas in two and three dimensions they persist in time and correspond to boundaries of shocks; in all cases the corresponding density pdf has the exponent \(-7/2\), originally proposed by E. Khanin, Mazel and Sinai (1997 Phys. Rev. Lett. 78, 1904) for the pdf of velocity gradients in one-dimensional forced Burgers turbulence. We also briefly consider models permitting particle crossings and thus multi-stream solutions, such as the Zel’dovich approximation and the (Jeans)–Vlasov–Poisson equation with single-stream initial data: they have singularities of codimension one, yielding power-law tails with exponent \(-3\).

su quell’ immenso baratro di stelle
sopra quei gruppi, sopra quegli ammassi,
quel seminio, quel baleno di stelle

Giovanni Pascoli, from La Vertigine

I. INTRODUCTION

In 1970 Zel’dovich introduced a simple model for explaining features of the nonlinear formation of large-scale structures in the Universe \([1]\). Just after the baryon-photon decoupling in the early Universe, there may have been a rarefied medium formed by collisionless dustlike particles without pressure, interacting only via Newtonian gravity \([3]\). The appropriate mathematical description, the equation for a self-gravitating gas in an expanding three-dimensional universe, has so far been studied mostly by numerical simulations. The Zel’dovich approximation, to which we shall return in Section \([V]\), is far simpler and involves basically particles moving in straight lines, just as rays in geometrical optics (see also Refs. \([1][3]\)). As a consequence caustics are formed, the simplest of which are pancakes, near which the mass density is very large. Zel’dovich, Arnold and their collaborators were mostly interested in the nature of the singularities resulting from the model and classified them using catastrophe theory and Lagrangian singularity theory \([4][6][8]\). Kofman et al. \([3]\) studied the probabilistic aspects and determined the probability density function (pdf) \( p(\rho) \) of the density of matter. For Gaussian initial fluctuations (at decoupling) they obtained a \( \rho^{-3} \) law for the tail at large densities. They also gave a simple heuristic argument relating the \( \rho^{-3} \) law to the divergence with exponent \(-1/2\) near the pancakes.

One difficulty with the Zel’dovich approximation is that the pancake structures, which are formed after the first particle crossing occurs, rapidly smear out (see Section \([IV]\), whereas in reality massive pancake-like structures are found to be quite long-lived. The gravitational dynamics of pancakes is indeed incorrectly captured within the Zel’dovich approximation (see, e.g., Ref. \([1]\)). This has led to the introduction of the adhesion model of Gurbatov and Saichev \([11][12]\) in which particle, upon crossing, stick together (adhere). The adhesion model is just the multi-dimensional Burgers equation, taken in the inviscid limit \( \nu \to 0 \)

\[\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \nu \nabla^2 \mathbf{v} \] (1)

\[\mathbf{v} = -\nabla \psi, \] (2)

\[\partial_t \rho^{(E)} + \nabla \cdot (\rho^{(E)} \mathbf{v}) = 0. \] (3)

Here, \( \mathbf{v} = (u, v, w) \) is the velocity, \( \psi \) the (velocity) potential and \( \rho^{(E)} \) the Eulerian (mass) density (the initial Lagrangian density, which is quasi-uniform in the cosmological problem, is denoted \( \rho_0 \)). As is well known, the Burgers equation in the limit \( \nu \to 0 \) produces shocks along surfaces (in three dimensions) on which the density is infinite and across which the velocity is discontinuous.

The question we intend to address is the behavior at large \( \rho's \) of the pdf of the density when using the one- and multi-dimensional Burgers/adhesion model with random and smooth initial conditions, having, e.g., Gaussian statistics with an exponentially decreasing spectrum. (The case of non-smooth (e.g. Brownian) initial conditions has been considered in Refs. \([13][14]\).) In one dimensional decaying Burgers turbulence, the density and the Eulerian velocity gradient have a simple relation (cf. \([13]\)). Hence, the density pdf is deducible from the pdf...
of the velocity gradient. It was shown in Ref. [16] that the latter has a tail at large negative values which is a power law with exponent $-7/2$. Actually, this law has been first proposed in Ref. [17] for randomly forced Burgers turbulence. For the forced case, the functional form of the pdf has been the subject of considerable controversy and the question is not yet completely resolved (see Refs. [17–28]). In particular E and Vanden Eijnden [22,23] developed a probabilistic formalism that copes with the delicate problems arising in the limit of vanishing viscosity when shocks are present, proved that $\alpha < -3$ and made a good case for $\alpha = -7/2$.

The paper is organized as follows. In Section II we present general background material about the multi-dimensional Burgers equation and show why, paradoxically, shocks are not generally responsible for large (but finite) densities, which generically arise from other types of singularities. In Sections III and IV we present the one-dimensional and the multi-dimensional case, respectively. In the final Section V we compare the predictions of various models used in the cosmological literature: the Burgers/adhesion model, the Zel’dovich model and the (Jeans)–Vlasov–Poisson model.

II. THE SOLUTION OF THE BURGERS EQUATION AND ITS GEOMETRICAL CONSTRUCTION

We shall here be exclusively interested in the solution to the $d$-dimensional Burgers equation (4)-(6), in the limit $\nu \to 0$, with given initial data $v_0(x) = v(x,0) = -\nabla \psi_0(x)$ and $\rho_0 = \rho^{(E)}(x,0)$.

We begin with the velocity. The Hopf [29] and Cole [30] transformation allows an explicit integral representation of the solution for $\nu > 0$. By steepest descent, a well known “maximum representation” is obtained in the limit $\nu \to 0$ [31,22]

$$\psi(x,t) = \max_q \left[ \psi_0(q) - \frac{(x - q)^2}{2t} \right].$$

It is easily seen that the point $q$ at which the maximum is achieved is the Lagrangian point associated to the (Eulerian) point $x$ at time $t$. Indeed, by setting the gradient of the r.h.s. of (4) to zero and using (4), we obtain

$$x = q + tv_0(q).$$

In other words, $x$ is the position at time $t$ of the fluid particle starting at $q$ and retaining its initial velocity $v_0(q)$.

The problem is that [4] is valid only for regular Lagrangian points, that is points which have not been captured by a shock by time $t$. There is another construction of the solution which brings out the geometrical nature of the problem with shocks. Let us define the Lagrangian potential

$$\varphi(q,t) = -\frac{|q|^2}{2} + t\psi_0(q).$$

It follows from (4) that

$$t\psi(x,t) + \frac{|x|^2}{2} = \max_q [\varphi(q,t) + x \cdot q].$$

The r.h.s. of (7) is seen to be the Legendre transform of the Lagrangian potential [33].

An important property of the Legendre transformation is that the r.h.s. of (7) is unaffected if we replace the Lagrangian potential $\varphi(q,t)$ by its convex hull $\varphi_c(q,t)$. The Eulerian singularities of the solution are determined by the structure of the convex hull.

Since we shall make extensive use of convex hulls in this paper, let us define the matter precisely. Let $g(q)$ be a real function of $q$ defined over a convex domain $D$ of $\mathbb{R}^d$ (for example, the whole space). We say that $g(q)$ is convex if, for any $q \in D$, $q' \in D$ and $0 \leq \theta \leq 1$, we have

$$g(\theta q' + (1 - \theta)q) \geq \theta g(q') + (1 - \theta)g(q).$$

Let $\varphi(q)$ be an arbitrary function. We define its convex hull $\varphi_c(q)$ as

$$\varphi_c(q) \equiv \min g(q),$$

the minimum being taken over all functions $g(\cdot) \geq \varphi(\cdot)$ which are convex.

In other words, the graph of $\varphi_c(q)$ is obtained by tightly pulling a string (in one dimension) or an elastic sheet (in two dimensions) around the graph of $\varphi(q)$ (see Figs. 1 and 2).

![FIG. 1. Lagrangian potential and its convex hull in one dimension. The graph of the convex hull contains regular parts of the Lagrangian potential and segments touching the original graph at two points, lying over shock intervals.](image-url)
Indeed, the Hessian (determinant of the Hessian matrix by (6) is necessarily convex when we assume in this paper), the Lagrangian potential given potential \( \psi \) gayevski of Ref. [12].

In one dimension the graph of the convex hull contains (i) parts of the original graph at two points. The former correspond to regular (Lagrangian) points and the latter to points having fallen into a shock. In two dimensions the convex hull consists generically of four kind of objects: (i) parts of the original graph, (ii) pieces of ruled surfaces, (iii) “kurtoparabolic points”, to which we shall come back, and (iv) triangles (see Fig. 2). The associated Eulerian objects are, respectively, (i) regular points, (ii) shock lines, (iii) end points of shocks and (iv) shock nodes. For the three-dimensional case, see Ref. [34] and references therein.

A more complete description of singularities is obtained by considering the metamorphoses of singularities as time elapses. A complete classification in two and three dimensions may be found in the appendix (supplement 2) by V.I. Arnold, Yu.M. Baryshnikov and I.A. Bogayevski of Ref. [12].

Let us just observe at this stage that, when the initial potential \( \psi_0(q) \) is a sufficiently smooth function of \( q \) (as we assume in this paper), the Lagrangian potential given by (3) is necessarily convex when \( t \) is sufficiently small. Indeed, the Hessian (determinant of the Hessian matrix of second space derivatives)

\[
H(q, t) \equiv \det \left( \frac{\partial^2 \varphi(q, t)}{\partial q_i \partial q_j} \right) = t^d \det \left( -\frac{1}{t} \delta_{ij} + \frac{\partial^2 \psi_0(q)}{\partial q_i \partial q_j} \right)
\]

remains very close to its initial value \((-1)^d\) for short times. As long as the Hessian does not change sign, the convexity is the same as for the initial paraboloid \( \varphi(q, 0) = -|q|^2/2 \).

We also introduce two Lagrangian maps from \( q \) to \( x \). The naive Lagrangian map \( L_t \) is just given by \( \mathcal{L}_t \). The (proper) Lagrangian map is

\[
\mathcal{L}_t : q \mapsto -\nabla_q \varphi_c(q, t) = x(q, t).
\]

At regular points, where \( \varphi(q, t) \) and \( \varphi_c(q, t) \) coincide, so do the two Lagrangian maps. If, however, \( q \) is not a regular point, the Lagrangian map transforms it into the appropriate Eulerian shock location, whose determination requires the knowledge of the convex hull, that is, a global geometrical construction. The map \( q \mapsto x = \mathcal{L}_t q \) is invertible only at Eulerian points which are not on shocks (otherwise there is more than one Lagrangian point which is mapped into \( x \)). The Jacobian of the Lagrangian map at regular points is defined as

\[
J(q, t) \equiv \det \left( \frac{\partial x_i}{\partial q_j} \right).
\]

It follows from (11) and (12) that it is just \((-1)^d\) times the Hessian \( H(q, t) \) of the Lagrangian potential.

We turn now to the determination of the (Eulerian) density \( \rho(E)(x, t) \). Mass conservation implies that, if \( x \) is not on a shock,

\[
\rho(E)(x, t) = \frac{\rho_0}{J(q, t)}, \quad q = \mathcal{L}_t^{-1} x.
\]

If \( x \) is on a shock, the density is of course infinite. This does not, however, imply that large but finite densities are generally obtained near shocks. Indeed, for \( \rho(E)(x, t) \) to be large, the Jacobian \( J(\mathcal{L}_t^{-1} x, t) \) and thus the Hessian \( H(\mathcal{L}_t^{-1} x, t) \) must be small. At any given time \( t \), this happens only near the \((d - 1)\)-dimensional manifold of vanishing Hessian. Arbitrarily close to such a “parabolic” point there are generically hyperbolic points where the surface defined by \( \varphi(q) \) crosses its tangent (hyper)plane and which, therefore, do not belong to its convex hull. As we shall see in Sections III and IV there can be “kurtoparabolic points” with vanishing Hessian which are at the boundary of regular regions. (In Greek, κυρτόσστρα means “convex”; hence the proposed name. In Ref. [34] they are called \( A_3 \)-points.) In one dimension kurtoparabolic points correspond to preshocks \( \mathcal{L}_t \) and exist only at discrete times; in two and three dimensions they generally persist for a finite time and are associated, in the Eulerian space, to boundaries of shocks (termination points in two dimensions and edges in three dimensions). Large densities are obtained exclusively in the neighborhood of kurtoparabolic points.

### III. THE ONE-DIMENSIONAL CASE
A. Preshocks

In one dimension, following fluid dynamical tradition, we denote the Lagrangian coordinate by $a$ and the velocity by $u$. We denote by $u_0(a) = -d\psi_0(a)/da$ the initial velocity, which is assumed to be random and sufficiently smooth. It is convenient, but not essential, to assume periodic boundary conditions, homogeneity and a vanishing mean velocity. We denote by $\rho_0$ the initial background density, taken deterministic and uniform. At regular points (outside of shocks) the Eulerian velocity and density are given implicitly by

$$u(x,t) = u_0(a), \quad \rho^{(E)}(x,t) = \frac{\rho_0}{\partial_a u}, \quad (14)$$

$$x = a + tu_0(a). \quad (15)$$

Since $\partial_x u = (\partial_x u_0) (\partial_a x)^{-1}$, it follows that the Eulerian density can be written in terms of the Eulerian velocity gradient $\partial_x u$ 12

$$\rho^{(E)}(x,t) = \rho_0 \left(1 - t \partial_x u(x,t) \right). \quad (16)$$

In Ref. 14 it was shown that the pdf of the Eulerian velocity gradient has a $-7/2$ power law at large negative values. Hence, the pdf $p(\rho)$ of $\rho^{(E)}$ has also a $-7/2$ power law, but at large positive values. The proof given in Ref. 14 was rather detailed. Here, we give a simplified derivation, adapted to the case of the density. Furthermore, we shall work mostly with the potential and normal forms near singularities, to prepare the ground for the multi-dimensional case.

From (14) and (15) we have, at regular points,

$$\rho^{(E)}(L_t a, t) = \frac{\rho_0}{1 - td^2 \psi_0(a)/da^2}. \quad (17)$$

Large values of $\rho^{(E)}$ are thus obtained in the neighborhood of Lagrangian points with vanishing Jacobian, where $d^2 \psi_0(a)/da^2 = 1/t$. Once mature shocks have formed, the Lagrangian points with vanishing Jacobian are inside shock intervals and thus not regular. The only kurtoparabolic points (points with vanishing Jacobian at the boundary of regular regions) are obtained at preshocks, that is when a new shock is just born at some time $t_*$. Preshocks, play a central role in the $-7/2$ law of E, Khanin, Mazel and Sinai for the forced case 17. Such points, denoted by $a_*$, are local negative minima of the initial velocity gradient, characterized by the following relations

$$\frac{d^2 \psi_0}{da^2}(a_*) = \frac{1}{t_*}, \quad \frac{d^3 \psi_0}{da^3}(a_*) = 0, \quad \frac{d^4 \psi_0}{da^4}(a_*) < 0. \quad (18)$$

There is however an additional global regularity condition that the preshock point $a_*$ has not been captured before $t_*$ by a mature shock. This may be written in terms of the convex hull $\varphi_c$ of the Lagrangian potential $\varphi$, as

$$\varphi(a_*, t_*) = -a_*/2 + t \psi_0(a_*) = \varphi_c(a_*, t_*) \quad (19)$$

As shown in 14, this global condition affects only constants and not the scaling properties of $\rho(\rho)$ at large $\rho$'s.

We can now Taylor expand the Lagrangian potential and the Lagrangian map near the space-time location $(a_*, t_*)$. By adding a suitable constant to the initial potential, by performing a suitable translation and also a Galilean transformation canceling the initial velocity at $a_*$, we may assume that $a_* = \psi_0(a_*) = d\psi_0(a_*)/da = 0$. We then obtain, to the relevant leading order, the following “preshock normal forms”

$$\varphi(a, t) \simeq \frac{\tau a^2}{2} + \zeta a^4, \quad (20)$$

$$x(a, t) \simeq -\tau a - 4\zeta a^3, \quad (21)$$

$$J(a, t) \simeq -\tau - 12\zeta a^2, \quad (22)$$

where

$$\tau = \frac{t - t_*}{t_*}, \quad \zeta = \frac{t_* d^4 \psi_0}{24 da^4}(0) < 0. \quad (23)$$

The Lagrangian potential, together with its convex hull, are shown in Fig. 3.

Note that at $t = t_*$ there is a degenerate maximum with quartic behavior and that, immediately after $t_*$, for $\tau > 0$, convexity is lost and a shock interval is born. Given the symmetry, resulting from our choice of coordinates, the convex hull contains a horizontal segment extending between the two maxima $a_{\pm} = \pm(-\tau/(4\zeta))^{1/2}$. Note that for $\tau > 0$ the Jacobian vanishes at two locations $\pm(-\tau/(12\zeta))^{1/2}$ which are within the shock interval and are therefore irrelevant as far as the Burgers/adhesion model is concerned (although they become relevant when particle crossing is permitted; see Section A).

From (22) we see that the density $\rho_0/J$ has a $a^{-2}$ singularity in Lagrangian coordinates at $t = t_* (\tau = 0)$. Since,
by (24), the relation between \(a\) and \(x\) is cubic at \(\tau = 0\), the density \(\rho^{(E)}(x, t_*) \propto |x|^{-2/3}\) which is unbounded. For any \(t \neq t_*\) the density remains bounded, except at the shock location. For \(\tau < 0\), this follows immediately from (22), which implies \(\rho^{(E)} \leq \rho_0/|\tau|\). For \(\tau > 0\), the exclusion of the shock interval requires \(|a| > a_+\). Hence, \(\rho^{(E)} \leq \rho_0/(2\tau)\). It is clear that large densities are obtained only in the immediate neighborhood of the preshock. More precisely, it follows from (21) and (22) that \(\rho^{(E)} > \rho\) requires simultaneously

\[
|\tau| < \frac{\rho_0}{\rho} \quad \text{and} \quad |x| < (-12\zeta)^{-1/2} \left(\frac{\rho_0}{\rho}\right)^{3/2},
\]

which become very small intervals around the spatio-temporal location of preshocks when \(\rho\) is large.

**B. The -7/2 law in one dimension**

So far, we have looked at the question of large densities from a deterministic point of view. We turn to the probability to have a large density at a given Eulerian point \(x\) and a given time \(t\). We shall calculate the cumulative distribution

\[
P^>(\rho; x, t) \equiv \text{Prob} \left\{ \rho^{(E)}(x, t) > \rho \right\},
\]

from which we obtain the pdf \(p(\rho; x, t) = -\partial_\rho P^>(\rho; x, t)\). In the random case each preshock has a random Eulerian location \(x_*\), occurs at a random time \(t_*\) and has a random \(\zeta < 0\) coefficient (there is also a random velocity \(u_0\) of the preshock but this is easily seen to be irrelevant for our purposes). Only those realizations such that \(x_*\) and \(t_*\) are sufficiently close to \(x\) and \(t\) will contribute large densities. Denoting by \(p_3(x_*, t_*, \zeta)\) the joint pdf of the three arguments, which is understood to vanish unless \(\zeta < 0\), we have

\[
P^>(\rho; x, t) = \int_{\rho^{(E)}(x,t) > \rho} p_3(x_*, t_*, \zeta) \, dx_* \, dt_* \, d\zeta.
\]

(If homogeneity is assumed \(p_3\) does not depend on \(x_*\); the case of homogeneity extending over the whole space can be obtained by letting the spatial period \(L \to \infty\); we must then also replace \(p_3\) by \(n(t_*, \zeta)/L\) where \(n\) is a number of preshocks per unit length rather than a probability; similarly, \(P^>\) is then a probability per unit length.) Because of the very sharp localization near preshocks implied by (24), for large \(\rho\)'s, we may replace \(p_3(x_*, t_*, \zeta)\) by \(p_3(x, t, \zeta)\). Using then, in a suitable frame, the normal forms (21)–(22) we can rewrite (24) as an integral over local Lagrangian variables \(a\) and \(\tau\) and obtain

\[
P^>(\rho; x, t) \simeq \int_D t ( -\tau - 12\zeta a^2 ) p_3(x, t, \zeta) \, da \, d\tau \, d\zeta.
\]

Here, the domain \(D\) is the set of \((a, \tau, \zeta)\) such that

\[
\frac{\tau}{-4\zeta} < a^2 < \frac{1}{12\zeta} \left(\frac{\rho_0}{\rho} + \tau\right)
\]

The right part of (25) expresses that the density exceeds the value \(\rho\), while the left part (which is trivial when \(\tau < 0\) since \(\zeta < 0\)) excludes the shock interval \([a_-, a_+]\). In (27) the factor \(-\tau - 12\zeta a^2\) is a Jacobian stemming from the change to Lagrangian space variables and the factor \(t\) stems from the change of temporal variables. The integration over \(a\) and \(\tau\) can be carried out explicitly, yielding

\[
P^>(\rho; x, t) \simeq C(x, t) \left(\frac{\rho_0}{\rho}\right)^{5/2},
\]

\[
C(x, t) \equiv At \, \int_{-\infty}^{0} |\zeta|^{-1/2} p_3(x, t, \zeta) \, d\zeta,
\]

where \(A\) is a positive numerical constant. Thus, for any \(x\) and \(t\), the cumulative probability of the density follows a \(\rho^{-5/2}\) law. Hence, \(p(\rho; x, t) \propto \rho^{-7/2}\), as \(\rho \to \infty\), which establishes the \(-7/2\) law for the pdf. Note that, contrary to the derivation in Ref. [10], we did not use homogeneity. With this additional assumption, \(p_3\) and thus \(C(x, t)\) become independent of \(x\).

Taking into account higher-order singularities does not influence this result. Consier, e.g., quintic-root preshocks arising from degenerate inflection points in the initial velocity, at which the second, third and fourth space derivatives all vanish. In one dimension such singularities are not generic in the deterministic case but could nevertheless contribute in the random case, as happens in Berry’s “battle of catastrophes” [30]. The exact vanishing of two more derivatives has probability zero but there is a finite probability that the third and fourth velocity derivatives have values small enough to give the preshock an approximately quintic-root structure. We found that such events contribute only subdominant corrections to the \(-7/2\) law.

**IV. TWO DIMENSIONS AND BEYOND**

A. Preshocks in two dimensions

In two dimensions we use the notation \(a = (a, b)\) and \(x = (x, y)\) for Lagrangian and Eulerian coordinates. It follows from (10) that the first singularity happens at the time \(t_*\) which is the inverse of the largest positive eigenvalue of the Hessian matrix of the initial potential \(\psi_0(a, b)\). No generality is lost by making the following assumptions: (i) the maximum is achieved at the origin, (ii) the potential and its gradient vanish at the origin, (iii) the maximum eigenvalue is equal to one \((t_* = 1)\) and (iv) the eigendirections of the matrix of second derivatives are the \(a\)-axis for the eigenvalue 1 and the \(b\)-axis for the other eigenvalue \(1 - \mu\) with \(\mu > 0\). Using this, we can Taylor expand the initial potential to the relevant (fourth) order:
\[ \psi_0(a,b) \simeq \frac{a^2}{2} + (1 - \mu)\frac{b^2}{2} + \alpha a^3 + \beta a^2 b + \gamma ab^2 + \delta b^3 + \zeta a^4 + \eta a^3 b + \theta a^2 b^2 + \kappa ab^3 + \rho b^4. \] (31)

Expressing that the matrix of second derivatives has its largest eigenvalue at \( a = b = 0 \), we find that \( \alpha = \beta = 0 \) and that the quadratic form

\[ Q(a,b) \equiv 12\mu\zeta a^2 + 6\mu\eta ab + (2\mu\theta + 4\gamma^2)b^2 \] (32)

must be definite negative, thereby putting certain restrictions on \( \mu, \zeta, \theta \) and \( \gamma \) which will henceforth be assumed.

We can now write the corresponding normal form of the Lagrangian potential \( t\psi_0(a,b) - (a^2 + b^2)/2 \), at time \( t = 1 + \tau \) for small \( \tau \), which includes all the relevant terms

\[ \varphi(a,b,t) \simeq \tau \frac{a^2}{2} - \frac{b^2}{2} + \gamma ab^2 + \zeta a^4 + \eta a^3 b + \theta a^2 b^2. \] (33)

For \( \tau < 0 \) the surface defined by the Lagrangian potential has a single maximum at \( a = b = 0 \). At \( \tau = 0 \) this maximum still exists but is quartically degenerated in the \( a \)-direction. For \( \tau > 0 \), the origin turns into a saddle and two new maxima appear (for values of \( a = O(\tau^{1/2}) \) and of \( b = O(\tau^{3/2}) \)). The general aspect of the surface is shown in Fig. 4. It is clearly not convex; hence, a new shock is born.

Consider first the situation at \( t = t_* \). The Hessian vanishes and, hence, the density becomes infinite at the origin. This is the singularity called \( A_3 \) by Arnold [6,8,33], for which the mean density in a small disk of radius \( r \) around the origin (in the the Eulerian space) is easily shown to diverge as \( \bar{\rho}(r) \propto r^{-2/3} \).

Shortly after \( t_* \), if we (incorrectly) use \( \varphi(a,b,t) \) rather than its convex hull, we find that the density becomes infinite in Lagrangian coordinates along a zero-Hessian curve of approximately elliptical shape. On the surface defined by the Lagrangian potential \( \varphi(a,b,t) \), the corresponding points are parabolic (they are shown as a dotted line on Fig. 4). When approaching such a line of parabolic points, one has an \( A_2 \) singularity in the sense of Arnold, for which \( \bar{\rho}(r) \propto r^{-1/2} \). Actually, nearly all the points on this line are “hidden under the convex hull”. Constructing the convex hull of \( \varphi(a,b,t) \) is not a local operation and thus, in general, not elementary. Let us just illustrate what can happen in the simpler case where \( \gamma = \eta = 0 \), which has an additional symmetry. We then have

\[ \varphi(a,b,t) \simeq \tau \frac{a^2}{2} - \frac{b^2}{2} + \zeta a^4 + \theta a^2 b^2, \] (35)

which is even in both \( a \) and \( b \). The conditions of negative definiteness of the quadratic form \( Q \) are then \( \zeta < 0 \) and \( \theta < 0 \). It follows from the symmetries and the fact that lines of constant \( a \) are parabolas that the convex hull contains a piece of ruled surface made of segments parallel to the \( a \)-axis. These segments are connecting the two maxima of sections at constant \( b \), which exist for any \( b^2 < -\tau/(2\theta) \). The horizontal projections of these end points define the separatrix between regular Lagrangian points and points absorbed into the newly created shock. It is the ellipse

\[ \tau + 4\zeta a^2 + 2\theta b^2 = 0, \] (36)

obtained by requiring \( \partial_t \varphi(a,b,t) = 0 \). The associated points on the surface are shown as a continuous line on Fig. 4. The corresponding Eulerian structure is easily seen to be an embryonic shock line, parallel to the \( y \)-axis, with a length \( O(\tau^{1/2}) \) and a velocity jump also \( O(\tau^{1/2}) \), except near its end points, where it vanishes.

It is now easily checked that the separatrix ellipse and the zero-Hessian ellipse are tangent at the points \( a = 0, b = \pm (-\tau/(2\theta))^{1/2}, \) denoted \( A \) and \( A' \) on Fig. 4. These points of vanishing Hessian, which belong to the edge of the regular region, are the only kurtoparabolic points in the sense of Section 4. Arbitrarily large densities are obtained in their neighborhood. Contrary to the one-dimensional case, the condition \( \rho(E) > \rho \), for large \( \rho \), does not put an upper bound, similar to (24), on the time \( \tau \) elapsed since \( t_* \). Actually, in two (and more) dimensions, kurtoparabolic points persist generically for at least a finite time, irrespective of the presence of the ad-
ditional symmetry assumed in the simple example given above.

B. Kurtoparabolic points in two dimensions

We recall our definition of a kurtoparabolic point $A$ as a point (i) where the Hessian of the Lagrangian potential $\varphi(a, b, t)$ vanishes and (ii) which belongs to the boundary of the regular part of the convex hull $\varphi_\epsilon(a, b, t)$. This requires two local constraints: that $A$ be parabolic and that the surface defined by $\varphi(a, b, t)$ should not cross the tangent plane at $A$. It also requires a global constraint, namely that $A$ should not be situated below a piece of the convex hull, which would correspond to $A$ having been absorbed by a mature shock before the current time $t$. The latter condition is automatically satisfied at the birth of the first singularity and will then persist for at least a finite time. The former can be expressed purely in terms of the local properties of the Lagrangian potential.

For this, let us find the normal form associated to a kurtoparabolic point at an arbitrary time $t$ (not necessarily close to $t_\star$). In what follows the time is purely a parameter which will not be written. The vanishing of the Hessian requires the vanishing of at least one eigenvalue of the Hessian matrix of second derivatives. We assume, again without loss of generality, that $A$ is the origin, that the potential and its gradient vanish at $A$, that the vanishing eigenvalue of the Hessian matrix corresponds to the $a$-axis and that the other eigenvalue is $-\mu < 0$. We now write the local Taylor expansion of the Lagrangian potential.

Clearly, we need to include terms up to fourth order in $a$, but we shall see that the relevant order in $b$ is two (because $b = O(a^2)$). Hence, we can write

$$\varphi(a, b) \simeq -\mu \frac{b^2}{2} + \alpha a^3 + \beta a^2b + \zeta a^4. \quad (37)$$

We now require that, at least locally (i.e. for small $a$ and $b$), the surface defined by $\varphi(a, b)$ should be below its tangent plane at the origin. This amounts just to $\varphi(a, b) \leq 0$ and is equivalent to

$$\alpha = 0, \quad \zeta < 0, \quad \beta^2 < -2\mu \zeta. \quad (38)$$

(Note that $\beta \neq 0$ since its vanishing would correspond to having a preshock.) The only “sharp” condition is the vanishing of the coefficient $\alpha$ of $a^3$. Indeed, for $\alpha \neq 0$ the surface crosses its tangent plane at the origin. Hence, there are two sharp conditions for the existence of a kurtoparabolic point: the vanishing of the Hessian and of the coefficient of $a^3$. Thus, we expect that kurtoparabolic points are found on codimension two manifolds in spacetime, that is, time-dependent discrete locations which persist for at least a finite time.

Actually, kurtoparabolic points are generally born with the first singularity which is itself such a point, albeit one with a higher degree of degeneracy (the coefficient $\beta$ also vanishes). Generically, kurtoparabolic points disappear at large times when only a network of shocks subsists. This is seen, for example, in the study of flame-front cracks in Ref. [37].

The typical local aspect of a kurtoparabolic point is shown in Fig. 5.

![FIG. 5. The Lagrangian potential in two dimensions, in the neighborhood of a kurtoparabolic point. The continuous line is the separatrix between the regular part and the ruled surface of the convex hull. The dotted line corresponds to the vanishing of the Jacobian of the Lagrangian map.](image)

The $a \mapsto -a$ symmetry, which is here generic, and the convexity of sections by planes of constant $a$ allows a straightforward construction of the convex hull of $\varphi(a, b)$. The convex hull contains, as in Section IV A, a ruled surface made of segments parallel to the $a$-axis. The separatrix is now a parabola of equation

$$b = -\frac{\zeta}{\beta} a^2, \quad (39)$$

shown as a continuous line in Fig. 3. The line of vanishing Hessian

$$H(a, b) = -2\mu \beta b - 4(3\mu \zeta + \beta^2) a^2 = 0 \quad (40)$$

is also a parabola (shown as a dotted line) which touches the former at the kurtoparabolic point. The corresponding Eulerian structure is a shock line with an end point of the kind shown in Fig. 2. High densities $\rho^{(E)} = \rho_0/H(a, b)$ are obtained in the neighborhood of the kurtoparabolic point. Specifically, the set of regular Lagrangian point such that $\rho^{(E)} > \rho$ is defined by the following inequalities

$$-\frac{\beta b}{2\zeta} < a^2 < -\frac{\rho_0/\rho + 2\mu \beta b}{4(3\mu \zeta + \beta^2)}. \quad (41)$$

Using the Lagrangian map $x = -\partial_a \varphi$, $y = -\partial_b \varphi$ it may be shown that, in the Eulerian space, the mean density in a disk of small radius $r$ around the end of the shock line $\bar{\rho}(r) \propto r^{-2/3}$.

7
It follows from (41) that, when \( \rho \) is large \( a \) and \( b \) are restricted to being very close to the kurtoparabolic point.

C. The -7/2 law in two dimensions

We now assume random and smooth initial conditions and proceed along the same general lines as in Section IIIA in particular, spatial periodicity is assumed (here and in the next section), although this is not essential. The cumulative probability to have \( \rho^{(E)} > \rho \) is now expressed in terms of the joint pdf of all the relevant parameters at a kurtoparabolic point (its position \( \mathbf{x}_s \), and the three parameters \( \mu, \beta \) and \( \zeta \)) in the normal form:

\[
P > (\rho; \mathbf{x}, t) = \int_{\rho^{(E)}(\mathbf{x}, t) > \rho} p(a, b, \mu, \beta, \zeta; t) \, d\mathbf{x}, \, d\mu \, d\beta \, d\zeta.
\]

(Additional dependences of the probability density on the velocity at the kurtoparabolic point and on the orientation of the axis corresponding to the vanishing eigenvalue are omitted for simplicity, since they are irrelevant for the calculation of the density.) An important difference with the one-dimensional case, due to the persistence of kurtoparabolic points, is the lack of a \( t \nu \) argument in \( p_a \). The probability to have a kurtoparabolic point at \( \mathbf{x}_s \) may still depend on \( \mathbf{x}_s \) (unless the initial condition is homogeneous) and will certainly depend on the time \( t \). With Gaussian initial conditions it may be shown that \( p_a \) is non-vanishing for any \( t \rangle 0 \).

Next, we use the very sharp localization of high-densities near the Eulerian points which are associated to the kurtoparabolic points and change from Eulerian to (local normal form) Lagrangian coordinates, rewriting (42) as

\[
P > (\rho; \mathbf{x}, t) \approx \int_{D} H(a, b, t) \, p_4(\mathbf{x}_s, \mu, \beta, \zeta; t) \, da \, db \, d\mu \, d\beta \, d\zeta,
\]

where the domain \( D \) is the set of \( (a, b, \mu, \beta, \zeta) \) such that (11) holds and \( H(a, b, t) \), given by (13), is the Jacobian of the Lagrangian map. In (13) the integration over \( a \) and \( b \) can be carried out explicitly. This gives \( P > (\rho; \mathbf{x}, t) \propto (\rho_0/\rho)^{5/2} \) and, hence,

\[
p(\rho; \mathbf{x}, t) \propto \left( \frac{\rho}{\rho_0} \right)^{-7/2}, \quad \rho \to \infty,
\]

which is the two-dimensional -7/2 law. (The constant in front of the power law, which involves integrals over \( \mu, \beta \) and \( \zeta \), is not written.)

It is of interest to note that we have obtained exactly the same scaling as in one dimension. The reason for this is rather interesting. In one dimension, the dominant singularities are preshocks which are isolated events in space-time, while in two dimensions they are kurtoparabolic points which are persistent in time. Nevertheless, if we compare the integrals (27) and (43) and the conditions on the integration domains (28) and (41), we find that, in two dimensions the spatial \( b \)-variable plays exactly the same role as the temporal \( \tau \)-variable in one dimension. We can see this also by examining Fig. 3, cuts at constant \( b \) will have the same shape as the curves shown in Fig. 3 for the one-dimensional case, changing from a single maximum for \( b \rangle 0 \), to a quartically degenerate maximum for \( b = 0 \) and to two symmetrical maxima for \( b \langle 0 \).

We must stress that in two (and more) dimensions, the dominant contribution to the tail of the density does not come from preshocks as in one dimension but comes from the entire life span of kurtoparabolic points which are just born at preshocks. Actually, the contribution of a small time interval straddling a preshock gives a \( \rho^{-4} \) intermediate asymptotic law, going over to a \( \rho^{-7/2} \) law at very large \( \rho \).

D. Higher dimensions

In dimensions \( d \rangle 2 \), kurtoparabolic points are now space-time manifolds of codimension two. The corresponding normal form (in suitable coordinates) can be written, using \( a \) for the coordinate in the direction of the vanishing eigenvalue of the Hessian matrix and \( \mathbf{b} = b_i \) \( (i = 1, \ldots, d - 1) \) for the \( d - 1 \) remaining coordinates. The analog of (37) with a vanishing \( a\alpha^3 \) term is now:

\[
\varphi(a, \mathbf{b}) \simeq \zeta a^4 + \sum_{i=1}^{d-1} \left( -\mu_i b_i^2 + \beta_i a^2 b_i \right), \tag{45}
\]

where

\[
\zeta < 0, \quad \mu_i > 0, \quad \sum_{i=1}^{d-1} \beta_i b_i < -2\zeta \tag{46}
\]

are the analogs of (68). An easy calculation shows that the Jacobian \( J = (-1)^d H(a, \mathbf{b}) \) of the Lagrangian map is given by

\[
J(a, \mathbf{b}) = -\left( \prod_{i=1}^{d-1} \mu_i \right) \left[ 4 \left( 3\zeta + \sum_{i=1}^{d-1} \beta_i b_i \right) a^2 + 2 \sum_{i=1}^{d-1} \beta_i b_i \right]. \tag{47}
\]

Because of the \( a \to -a \) symmetry and the convexity in all the other variables, the separatrix of the convex hull is again obtained by setting \( \partial_a \varphi(a, \mathbf{b}) = 0 \), thereby obtaining

\[
2\zeta a^2 + \sum_{i=1}^{d-1} \beta_i b_i = 0. \tag{48}
\]
By proceeding as in two dimensions, we can write the analog of \((43)\) in which \(b, \mu\) and \(\beta\) must now be reinterpreted as \((d - 1)\)-dimensional vectors. As to the integration domain \(\mathcal{D}\), it is now defined by \(J(a, b) < \rho_0/\rho\) and the negativity of the l.h.s. of \((43)\), which expresses the belonging to the regular domain.

We now observe that \(\sum_{i=1}^{d-1} \beta_i b_i\) plays here the same role as \(\beta b\) in the two-dimensional case. There is no \(b\)-dependence other than this. Hence, assuming that not all the \(\beta_i\)’s vanish (otherwise we have a preshock) we can change variables in the \(b\)-space, taking one axis in the direction of \(\sum_{i=1}^{d-1} \beta_i b_i\). The integration in all the \(b\)-directions orthogonal to this direction is trivial and gives order unity contributions. The remaining integral is just the same as in two dimensions. Hence, in any dimension \(d > 2\), the contribution stemming from such kurtoparabolic manifolds of codimension two is again a \(\rho^{-7/2}\) tail in the pdf of the density.

For \(d = 3\) another type of singularity with vanishing Hessian has been identified, denoted by \(A_1 A_3\) in Ref. [33]. It corresponds to a kurtoparabolic point at which the tangent (hyper)plane has another point of tangency with the graph of the Lagrangian potential. Such points do not contribute to the leading order of the pdf for large densities. For \(d > 3\), higher-order singularities such as the \(A_3\)-points of Ref. [34] can appear generically and we do not know how they affect the \(-7/2\) law.

V. CONCLUDING REMARKS

In this paper we determined the pdf of the mass density for the Burgers equation in the inviscid limit for smooth initial conditions which are random but not necessarily homogeneous. We showed that in one, two and three dimensions this pdf has a power-law tail with exponent \(-7/2\). In one dimension this tail originates from preshocks, that is nascent shocks which take place at discrete times, whereas in two and three dimensions the tail comes from time-persistent boundaries of shocks (associated in Lagrangian space to kurtoparabolic singularities). In the neighborhood of such points, arbitrary large but finite densities are present. Note that, in any case, the \(-7/2\) law is due to a phenomenon of shock germination, either in space-time or just in space.

The density pdf will of course be modified if, instead of the limit of vanishing viscosity, we assume a finite but small viscosity. A similar question has already been considered by Gotô and Kraichnan [23] for the pdf of large (negative) velocity gradients \(\xi\) in one-dimensional forced Burgers turbulence; they found that, at very large values of \(\xi\) there is a power-law range with exponent \(-1\), different from the exponent prevailing at those values of \(\xi\) where the inviscid limit is achieved. Similarly, we expect that, due to the large shear inside shocks, the \(-7/2\) tail in the density pdf would become just an intermediate asymptotic range, beyond which another law should prevail.

As mentioned in the Introduction, the Burgers equation with vanishing viscosity is used by cosmologists, under the name of adhesion model, to approach the problem of the formation of large scale structures. In principle, the appropriate mathematical framework should involve partial differential equations for density and velocity, coupled to the Poisson equation for the gravitational potential. The initial velocity (in an expanding universe at decoupling) is then determined uniquely in terms of initial density fluctuations (see, e.g., Section 2.2.2 of Ref. [15]). When dealing with dustlike collisionless matter, a hydrodynamical description cannot be justified on the usual grounds that local thermodynamic equilibrium is quickly achieved. However, as long as particles do not cross, a quasi-hydrodynamical description, without any viscous diffusion term, is suitable. After crossing, the multi-stream situation may be described pseudo-microscopically in terms of a distribution function \(f(x, v, t)\) in the position-velocity phase space satisfying the \(d\)-dimensional (Jeans)–Vlasov–Poisson [35]

\[
\partial_t f(x, v, t) + (v \cdot \nabla_x - \nabla_x \cdot \nabla_v) f = 0, \quad (49)
\]

supplemented by the Poisson equation, relating the gravitational potential to the density \(\rho(x, t) = \int f(x, v, t) dv\). (We omitted the expansion factor for simplicity.) In \((49)\), the position and velocity variables are in principle independent, but the relevant solutions for the pseudo-microscopic description are of the “single-speed” type. By this we understand that the distribution \(f\) has its support on a \(d\)-dimensional submanifold of the \(2d\)-dimensional phase space such that, initially, there is a single velocity \(u_0(x)\) associated to a given position \(x\). Such single-speed solutions may, after particle crossing, possess more than one velocity for a given \(x\). The distribution \(f\) does however remain well-defined (see Fig. 1), which corresponds to the one-dimensional case. For smooth initial data, the support of \(f\) remains smooth even after particle crossing (see Ref. [36] for the one-dimensional case). In one dimension, it is easy to show that, near a point \(x_*\) where the tangent to the graph of
the support is parallel to the v-axis (such as points \( x_\perp \) on Fig. 5), the density has a singularity \( \propto |x - x_\perp|^{-1/2} \). When \( x_* \) is random with a probability density at \( x_* = x \), we infer by an argument similar to that of Section 11B that the pdf \( p(\rho; x, t) \propto \rho^{-3} \) for \( \rho \to \infty \). This argument carries over to higher dimensions; the \( x_* \)-points are then on the \((d - 1)\)-dimensional manifold where the Jacobian of the Lagrangian map vanishes.

The same \( \rho^{-3} \) law holds within the \( d\)-dimensional Zel’dovich approximation [2]. The latter is indeed equivalent to a modified prescription for determining the gravitational potential; it is obtained from the Poisson equation only initially and its gradient (the gravitational force) is then taken constant along particle trajectories. Clearly, the modified prescription has the only effect that it changes the precise position of the support (after the first crossing) but not the nature of the ensuing density singularities. As noticed in Ref. [40] the power law first crossing) but not the nature of the ensuing density singularities if one allows multi-valued solutions; this is precisely the case in collisionless physical situations.

The Burgers/adhesion model has been found useful for describing large-scale features of collisionless dynamics, such as the positions and slow thickening of pancakes. It is nevertheless intrinsically a hydrodynamical model and, as such, better suited to handle low-pressure collision-dominated matter. As far as their singularities are concerned, the Zel’dovich approximation and the Burgers/adhesion model are in different universality classes and have different tail behavior for the density pdf. We finally mention that with the three-dimensional adhesion model, various other quantities may be calculated analytically, which are more directly related to what can be determined by cosmological observations and/or by N-body simulations, such as the density correlation function (at small distances) and the pdf of the mass in a ball of small radius.

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