MULTIPLECTIES OF NOETHERIAN DEFORMATIONS

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Abstract. The Noetherian class is a wide class of functions defined in terms of polynomial partial differential equations. It includes functions appearing naturally in various branches of mathematics (exponential, elliptic, modular, etc.). A conjecture by Khovanskii states that the local geometry of sets defined using Noetherian equations admits effective estimates analogous to the effective global bounds of algebraic geometry.

We make a major step in the development of the theory of Noetherian functions by providing an effective upper bound for the local number of isolated solutions of a Noetherian system of equations depending on a parameter $\epsilon$, which remains valid even when the system degenerates at $\epsilon = 0$. An estimate of this sort has played the key role in the development of the theory of Pfaffian functions, and is expected to lead to similar results in the Noetherian setting. We illustrate this by deducing from our main result an effective form of the Łojasiewicz inequality for Noetherian functions.

1. Introduction

One of the cornerstones of algebraic geometry is the Bezout theorem: a system of polynomial equations in a complex projective space always admits a specified number of solutions depending on their degrees. This statement has profound implications for the algebraic category: essentially every geometric and topological property of an algebraic variety can be estimated in terms of the degrees of the equations defining it.

Moving beyond the algebraic category, Khovanskii has defined the class of real Pfaffian functions. In his theory of Fewnomials [16], Khovanskii has shown that the number of solutions of a system of real Pfaffian equations admits an effective upper bound in terms of the degrees of the equations. This fundamental result has been the basis of many subsequent works, showing that the geometry of real Pfaffian sets is tame and admits effective estimates in terms of degrees [25, 23, 14].

The real Pfaffian class consists of functions satisfying a system of differential equations with a certain triangularity condition. It is surprisingly general, and includes many important transcendental functions — most notably the real exponential. On the other hand, not all differential systems appearing naturally in mathematics are real Pfaffian. We mention a few key examples:

- Exponential maps of (complex) commutative algebraic groups, for instance complex exponentials and elliptic functions. Solutions of equations involving such functions have been studied in transcendental number theory, [17, 15, 20].
- Abelian integrals and iterated Abelian integrals. Solutions of equations involving such functions have been studied in relations to perturbations of Hamiltonian systems and their limit cycles [2, 5].
- Functions of modular type, for instance the modular invariant $J$ and Ramanujan’s functions $P, Q, R$. Solutions of equations involving such functions have been studied in transcendental number theory, [21, 11].
- Hamiltonian flow maps in completely integrable systems.
At the time of the development of the theory of Fewnomials, Khovanskii also considered
the more general notion of Noetherian functions, excluding the triangularity condition from
the definition of the Pfaffian functions. One may loosely say that a collection of functions is
Noetherian if each of their derivatives can be expressed algebraically in terms of the functions
themselves (see §1.1 for two equivalent precise definitions). All functions listed above, while
not real-Pfaffian, do form Noetherian systems.

Noetherian functions do not satisfy global estimates similar to those obtained in the theory
of Fewnomials. However, Khovanskii has conjectured that a local analog of these estimates
continues to hold in the class of Noetherian functions, which would form a type of local analog
of the Pfaffian class. As in the Pfaffian case, the key step is to establish an upper bound for
the number of solutions of a system of equations involving Noetherian functions in terms of
degrees — this time in a suitably defined local sense.

In this paper we make a main step in the development of the theory of Noetherian functions
by establishing an upper bound for the local number of solutions of a system of Noetherian
equations depending on a Noetherian parameter \( \varepsilon \). The novelty of our result is that the
estimate is valid even if the system degenerates and has non-isolated solutions when \( \varepsilon = 0 \).
A result of this type has been established by Gabrielov in the (complex) Pfaffian case, and
subsequently used to derive estimates for many topological and geometric properties in the
Pfaffian category (see [14] for a survey). As an illustrative example, we use our estimate to
establish an effective Łojasiewicz inequality in the Noetherian category following Gabrielov.
We expect many other results of [14] to follow similarly.

1.1. Noetherian functions. We begin by defining our principal object of investigation,
namely the rings of Noetherian functions.

**Definition 1** (Noetherian functions, algebraic [24]). A ring \( S \) of analytic functions in a
domain \( U \subset \mathbb{C}^n \) is called a ring of Noetherian functions if it is generated over the polynomial
ring \( \mathbb{C}_n \) by functions \( \phi_1, \ldots, \phi_m \in S \) and closed under differentiation with respect to each
variable \( x_i \).

Any element of \( \psi \in S \) is called a Noetherian function. We will say that it has degree \( d \)
with respect to \( \phi_1, \ldots, \phi_m \) if \( d \) is the minimal degree of a polynomial \( P \in \mathbb{C}_{n+m} \) such that
\( \psi = P(x_1, \ldots, x_n, \phi_1, \ldots, \phi_m) \). When there is no risk of confusion we will simply call this the
degree of \( \psi \).

Alternatively, equip \( \mathbb{C}^{n+m} \) with the affine coordinates \( (x_1, \ldots, x_n, f_1, \ldots, f_m) \) and denote
by \( \mathbb{C}_{n+m} \) the ring of polynomials in these variables. Consider a distribution \( \Xi \subset T\mathbb{C}^{n+m} \)
defined by

\[
\Xi = \langle V_1, \ldots, V_n \rangle, \quad V_i = \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} g_{ij} \frac{\partial}{\partial f_j}, \quad \text{for } i = 1, \ldots, n.
\]

where \( g_{ij} \in \mathbb{C}_{n+m} \). We call \( \delta = \max_j \deg g_{ij} \) the degree of the chain. For any point where \( \Xi \)
is integrable, we will denote by \( \Lambda_p \) the germ of an integral manifold through \( p \).

**Definition 2** (Noetherian functions, geometric [12]). A tuple \( \phi_1, \ldots, \phi_m \) of analytic functions
on a domain \( U \subset \mathbb{C}^n \) is called a Noetherian chain if its graph forms an integral manifold of
the distribution \( \Xi \) (for some choice of the coefficients \( g_{ij} \in \mathbb{C}_{n+m} \)). In other words, if the
following system of differential equations is satisfied

\[
\frac{\partial \phi_j}{\partial x_i} = g_{ij}(x_1, \ldots, x_n, \phi_1, \ldots, \phi_m).
\]
Any function \( \psi = P(x_1, \ldots, x_n, \phi_1, \ldots, \phi_m) \) where \( P \in \mathbb{C}^{n+m} \) is called a Noetherian function. We will say that it has degree \( d \) with respect to \( \phi_1, \ldots, \phi_m \) if \( d \) is the minimal degree of such a polynomial \( P \in \mathbb{C}^{n+m} \). When there is no risk of confusion we will simply call this the degree of \( \psi \).

It is straightforward to check that the generators of a ring of Noetherian functions forms a Noetherian chain and vice versa.

**Remark 3.** Note that by the above, if \( \Lambda \) is an integral manifold of (1) then the projection onto the \( x \)-variables gives a system of coordinates on \( \Lambda \), and with respect to these coordinates the Noetherian functions are simply the restrictions of polynomials \( P \in \mathbb{C}^{n+m} \) to \( \Lambda \). This viewpoint is particularly convenient for our purposes and will be used frequently.

We define a Noetherian set to be the common zero locus of any collection of Noetherian functions with a common domain of definition \( U \).

As solutions of non-linear differential equations, Noetherian functions do not admit good global behavior — in fact, their domains of definition may include movable singularities and natural boundaries. It is natural therefore to begin the study of these functions with their local properties: what can be said about germs of Noetherian functions and sets in terms of the system (1) that defines them?

In order to be more concrete, consider any system (1) with the parameters \( m, n, \delta \) and an \( n \)-tuple \( \psi_1, \ldots, \psi_n \) of Noetherian functions of degrees bounded by \( d \). Let \( N(m, n, \delta, d; 0) \) denote the maximal possible multiplicity of a common zero of the equations \( \psi_1 = \cdots = \psi_n = 0 \), assuming that the zero is isolated. More generally, consider any family \( \psi_1^\varepsilon, \ldots, \psi_n^\varepsilon \) of Noetherian functions of degrees bounded by \( d \) (for each fixed \( \varepsilon \)) and depending analytically on \( \varepsilon \). Let \( N(m, n, \delta, d) \) denote the maximal possible number of isolated zeros (counted with multiplicities) of the equations \( \psi_1^\varepsilon = \cdots = \psi_n^\varepsilon = 0 \) which converge to a given point as \( \varepsilon \to 0 \). Crucially, here it is not assumed that the given point is an isolated solution of the limiting system at \( \varepsilon = 0 \).

It is a general principle that estimates for \( N(m, n, \delta, d; 0) \) imply estimates for the local topological and analytic structure of functions and sets, by a combination of topological arguments (e.g. Morse theory) and analytic-geometric arguments (e.g. the study of polar curves). The more restrictive \( N(m, n, \delta, d; 0) \) does not control the topological and analytic-geometric structure to the same extent, but it is a convenient first approximation which is often easier to study and still provides useful information.

Instances of the problem of estimating \( N(m, n, \delta, d; 0) \) and \( N(m, n, \delta, d) \) have been studied by authors in various areas of mathematics. Below we present an outline of some of the main contributions.

1.2. **Historical sketch.** The case \( n = 1 \) is special and has attracted considerable attention. In this case the problem reduces to the study of polynomial functions on the trajectory of a (non-singular) polynomial vector field at a point. Moreover, by an argument due to Khovanskii [10], in this case the quantities \( N(m, n, \delta, d; 0) \) and \( N(m, n, \delta, d) \) coincide. Therefore the problem is to estimate the multiplicity of the restriction of a polynomial to the trajectory of a non-singular vector field at a point.

Brownawell and Masser [6, 7] and Nesterenko [19] have studied this problem with motivations from transcendental number theory. In particular, Nesterenko has obtained an asymptotic estimate for \( N(m, 1, \delta, d) \) with respect to \( d \) which is sharp up to a multiplicative constant. These authors have also considered the case where one takes into account the multiplicity at several (rather than just one) fixed points.
Risler [22] has considered the problem in the context of control theory and specifically non-holonomic systems. Gabrielov [10] refined this work and found a surprising reformulation in terms of the Milnor fibers of certain deformations, leading to an estimate for $N(m, 1, \delta, d)$ which is simply-exponential $m$ and polynomial in $d$ and $\delta$.

Novikov and Yakovenko [21] have considered the problem with motivations in the study of abelian integrals, and have produced a bound which is valid not only for local multiplicity but for the number of zeros (counted with multiplicities) in a small ball of a specified radius. Yomdin [27] has obtained a similar result motivated by the study of cyclicities in dynamical systems. For both of these works, the principal difficulty is to obtain an estimate in an interval whose radius does not degenerate along deformations where the polynomial becomes identically vanishing on the trajectory.

The case $n > 1$ is considerably more involved, because germs of functions and sets in several variables can degenerate in much more complicated ways. Khovanskii [16] has studied systems (1) satisfying an additional triangularity condition and defined over the reals, leading to the influential theory of Pfaffian functions. Khovanskii has shown that the global geometry of Pfaffian sets (e.g. number of connected components, sum of Betti numbers) can be estimated in terms of the degrees of the coefficients of the system (1).

Gabrielov [8] has established a local complex analog of Khovanskii's estimates for the number of solutions of a system of Pfaffian equations, namely an upper bound for $N(m, n, \delta, d)$ (for the Pfaffian case). Gabrielov used this result to produce an effective form of the Lojasiewicz inequality for Pfaffian functions, and later in a series of joint works with Vorobjov (for example see [14, 13, 9]) established many results on the complexity of various geometric constructions within the Pfaffian class (e.g. stratification, cellular decomposition, closure and frontier).

In the early 1980s Khovanskii conjectured that the quantity $N(m, n, \delta, d)$ admits an effective upper bound (see [12] for the history and another equivalent form of this conjecture). The conjecture in this generality has remained unsolved.

Building upon the one-dimensional approach of Gabrielov [10], Gabrielov and Khovanskii [12] have established an estimate for $N(m, n, \delta, d; 0)$ using a topological deformation technique. In our previous work [3] we have established a weaker estimate for this quantity using algebraic techniques, and extended these local estimates to an estimate on the number of zeros (counted with multiplicities) in a sufficiently small ball, provided that the ball is not too close to a non-isolated solution of the system. However, neither of these approaches give an estimate for the more delicate quantity $N(m, n, \delta, d)$.

In [4] we began the investigation of non-isolated solutions of systems of Noetherian equations for $n = 2$. We have obtained an explicit upper bound for an appropriately defined “non-isolated intersection multiplicity”. However, this bound is not sufficient if one is interested in estimating the number of solutions born from a deformation of a non-isolated solution.

1.3. Statement of our result. Consider a system (1) and the corresponding ring of Noetherian functions $S$ defined in a domain $U \subset \mathbb{C}^n$, and let $p \in U$.

Let $\rho \in S$ and let $X \subset U$ be a germ of a Noetherian set

$$X = \{x \in U : \psi_1(x) = \cdots = \psi_{n-1}(x) = 0\}$$

at the point $p$, where $\psi_i \in S$ for $i = 1, \ldots, k$.

**Definition 4.** The deformation multiplicity, or deflicity of $X$ with respect to $\rho$ is the number of isolated points in $\rho^{-1}(y) \cap X$ (counted with multiplicities) which converge to $p$ as $y \to \rho(p)$. 


We let $\mathcal{M}(m,n,\delta,d)$ denote the maximum possible deformation multiplicity for any system $(\mathbb{1})$ with parameters $m, n, \delta$ and any Noetherian set defined as above with $\deg \psi_i \leq d$ and $\deg \rho \leq d$.

**Remark 5.** This notion is closely related to the quantity $\mathcal{N}(m,n,\delta,d)$ defined in section $(\mathbb{1})$. Indeed, let a family $\psi_1^\varepsilon, \ldots, \psi_n^\varepsilon$ be given where $\psi_i^\varepsilon \in S$ and $\deg \psi_i^\varepsilon \leq d$. Let us assume further that the dependence on $\varepsilon$ is Noetherian, i.e., that these functions may be viewed as Noetherian functions in the variables $x_1, \ldots, x_n, \varepsilon$ and have degrees bounded by $d$ in this sense as well. Then the number of isolated zeros of $\psi_1^\varepsilon = \cdots = \psi_n^\varepsilon = 0$ which converge to $p$ as $\varepsilon \to 0$ coincides with the deflicity of the set

$$X = \{(x,\varepsilon) \in U : \psi_1(x,\varepsilon) = \cdots = \psi_n(x,\varepsilon) = 0\}$$

with respect to $\rho \equiv \varepsilon$.

We can now state our main result.

**Theorem 1.** The quantity $\mathcal{M}(m,n,\delta,d)$ admits an effective upper bound,

$$\mathcal{M}(m,n,\delta,d) \leq (\max\{d,\delta\}(m+n))^{16(m+n)^{20n+3}} = (\max\{d,\delta\}(m+n))^{O(n)}$$

Recall that according to Khovanskii’s conjecture, the quantity $\mathcal{N}(m,n,\delta,d)$ admits an effective upper bound. While our result applies only under the additional condition that the dependence of the deformation on $\varepsilon$ is Noetherian, in most applications one needs to deal only with explicitly presented deformations which do depend in a Noetherian manner on $\varepsilon$. One can therefore expect that Theorem 1 will suffice for the investigation of many local topological and analytic-geometric properties of Noetherian sets and functions.

**Remark 6.** One can also consider systems $(\mathbb{1})$ involving rational (rather than polynomial) coefficients, as well as rational $P,R$. As long as one considers points $p$ away from the polar locus of these functions, Theorem 2 remains valid and can be proved in the same way. On the other hand, when $p$ belongs to the singular locus of the system, the situation is considerably more involved.

1.4. **Application: Lojasiewicz inequality.** As an example of applications of Theorem 1 we prove an effective version of Lojasiewicz inequalities for Noetherian functions.

Let $\Xi_\mathbb{R} \subset T\mathbb{R}^{n+m}$ be a real distribution spanned by real vector fields $V_i$ as in $(\mathbb{1})$, with $g_{ij} \in \mathbb{R}_{n+m}$ of degree $\leq \delta$, where $\mathbb{R}_{n+m} = \mathbb{R}[x_1, \ldots, x_n, f_1, \ldots, f_m]$, and let $\Lambda_p \subset \mathbb{R}^{n+m}$ be a germ of its integral manifold through $p$. The restrictions of real polynomials of degree $d$ to $\Lambda_p$ are called real Noetherian functions of degree $d$.

**Theorem 2.** Let $f,g$ be real Noetherian functions of $n$ variables of degree $d$ defined by the same Noetherian chain, $f(p) = 0$, and assume that $\{f = 0\} \subset \{g = 0\}$ near $p$. Then there exists a constant $0 < k \leq (\max\{d,\delta\}(m+n))^{O(n)}$ such that $|f| > |g|^k$ near $p$.

The proof below is essentially the proof from [8], with Theorem 1 replacing [8] Theorem 2.1.

Denote the complexifications of $f,g$ by the same letters. They are evidently Noetherian functions, defined by the (complexification of the) same Noetherian chain. Denote by $\Delta \subset \mathbb{C}^2$ the polar curve of $f$ with respect to $g$, i.e. the set of critical values of the mapping $(f,g) : \Lambda_p \to \mathbb{C}^2$.

**Lemma 7.** Let $f = \sum c_i g^{\lambda_i}$ be the Puiseux expansion of an irreducible component $\Delta' \neq \{f = 0\}$ of $\Delta$. Let $s$ be the least common denominator of $\lambda_i$, and let $r = s \lambda_1$. Then

$$r, s \leq (\max\{d,\delta\}(m+n))^{16(m+n)^{20n+4}}$$
Proof. Let \( \Sigma' \subset \Lambda_p \) be an irreducible component of the critical set of the mapping \((f,g)\) whose image is \( \Delta' \). Then \( r \) is at most the number of points in \( \{f(x) = \epsilon, x \in \Sigma'\} \) converging to \( p \) as \( \epsilon \to 0 \), and \( s \) is at most the number of points in \( \{g(x) = \epsilon, x \in \Sigma'\} \) converging to \( p \) as \( \epsilon \to 0 \). We can assume that these points are isolated: after cutting by a linear space \( L' \) of suitable dimension, the polar curve of the restriction of \( f \) to \( L' \) with respect to the restriction of \( g \) to \( L' \) will have \( \Delta' \) as its irreducible component and the problem is reduced to a similar one in smaller dimension. Therefore it is enough to bound from above the deflicity of \( X = \{dg \wedge df = 0\} \) with respect to \( f \) or \( g \). But \( X \) is a Noetherian set of degree \( 2(d+\delta-1) \), so Theorem 1 implies the claim of the Lemma.

Proof of Theorem 2. Let \( \Sigma \) be the union of points of minima of restrictions of \(|f|\) to the intersection of level curves of \(|g|\) with a small ball around \( p \). We can assume that the closure of \( \Sigma \) contains \( p \), otherwise the problem is reduced to a similar one of lower dimension. Therefore the image of \( \Sigma \) under \((f,g)\) belongs to a polar curve of \( f \) relative to \( g \), and not to \( \{f = 0\}\) by the condition. Therefore Theorem 2 follows from the previous Lemma.

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2. Background

2.1. Noetherian functions: integrability and multiplicity estimates. A system of the form \([I]\) does not necessarily admit an integral manifold through every point of \( \mathbb{C}^{n+m} \), since we do not assume that the vector fields \( V_i \) commute globally. The integrability locus \( \mathcal{IL} \subset \mathbb{C}^{n+m} \) of \([I]\) is defined to be the union of all the integral manifolds of the system. In \([12]\) it was shown that \( \mathcal{IL} \) is algebraic for any system \([I]\), and moreover that the following theorem holds.

Theorem 3. Let the system \([I]\) have parameters \( m,n,\delta \). Then \( \mathcal{IL} \) can be defined as the zero locus of a set of polynomials of degrees not exceeding

\[
d_{\mathcal{IL}} = \frac{(m+1)(\delta-1)}{2} \left[ 2\delta(n+m+2) - 2m - 2(2m+2) + \delta(n+2) - 1 \right]
\]

We now state the main result of \([12]\), namely an upper bound for the multiplicity of an isolated common zero of a tuple of Noetherian functions.

Theorem 4. Let the system \([I]\) have parameters \( m,n,\delta \) and let \( \psi_1,\ldots,\psi_n \) be Noetherian functions of degrees at most \( d \) with respect to this system. Then the multiplicity of any isolated solution of the equations \( \psi_1 = \cdots = \psi_n = 0 \) does not exceed the maximum of the following two numbers:

\[
\frac{1}{2} Q \left( (m+1)(\delta-1) \left[ 2\delta(n+m+2) - 2m - 2(2m+2) + 2\delta(n+2) - 2 \right] \right)^{2(m+n)}
\]

\[
\frac{1}{2} Q \left( 2(Q+n)^{n} (d + Q(\delta-1)) \right)^{2(m+n)}, \quad \text{where} \quad Q = c n \left( \frac{\epsilon(n+m)}{\sqrt{n}} \right)^{ln n+1} \left( \frac{2}{3} \right)^{n}
\]

2.2. Multiplicity operators. We shall make extensive use of the notion of multiplicity operators introduced in \([3]\). For the convenience of the reader we recall in this section the basic definitions and properties that will be used.

Let \( F = (F_1,\ldots,F_n) : B^n(0,1) \to \mathbb{C}^n \) be a holomorphic mapping extendable to a neighborhood of the unit ball \( B^n(0,1) \). We will denote by \( \|\cdot\| \) the maximum norm on the unit ball.

For every \( k \in \mathbb{N} \) there is a finite family of polynomial differential operators \( \{M_B^k\} \) called the basic multiplicity operators. Any operator \( M^{(k)} \) in the convex hull of this set is called
a *multiplicity operator* of order $k$. Every multiplicity operator of order $k$ is a polynomial differential operator of order $k$ and degree $\dim J_{n,k} - k$, where $J_{n,k}$ denotes the space of $k$-jets of functions in $n$ variables.

We will use the convention that when applying a multiplicity operator $M^{(k)}$ to a polynomial $P \in \mathbb{C}_{n+m}$, the derivatives of the variables $f_1, \ldots, f_m$ with respect to $x_1, \ldots, x_n$ shall be given by the Noetherian system (1). In this way $M^{(k)}(P) \in \mathbb{C}_{n+m}$ and moreover, if $\deg P = d$ then $\deg M^{(k)}(P) \leq d_M(n,\delta,d,k)$, where

$$d_M(n,\delta,d,k) = \binom{n+k}{k}(d+k\delta) - k$$

At points where (1) is integrable, this agrees of course with applying $M^{(k)}$ to $P$ viewed as a Noetherian function of $x_1, \ldots, x_n$ by restriction of $P$ to the integral manifold of (1) through the point. At points where (1) is not integrable $M^{(k)}(P)$ has no intrinsic meaning and will in general depend on the order in which we choose to evaluate repeated derivatives. We fix this order arbitrarily. In fact we will only consider $M^{(k)}(P)$ on points where the system is integrable, so this arbitrary choice will make no difference in our arguments.

We will sometimes need to evaluate multiplicity operators with respect to a linear subspace of the $x$-coordinates. If $\mathcal{S}$ is such a linear subspace and $M^{(k)}$ a multiplicity operator of dimension $\dim \mathcal{S}$, we will denote the corresponding multiplicity operator by $M^{(k)}_{\mathcal{S}}$ (viewed again as an operator acting on $\mathbb{C}_{n+m}$). Also, to simplify the notation we denote by $M^{(k)}_p$ (resp. $M^{(k)}_0$) the differential functional obtained by applying $M^{(k)}$ (resp. $M^{(k)}_0$), followed by evaluation at the point $p$.

The following proposition is the basic property of the multiplicity operators.

**Proposition 8.** We have $\text{mult} F_p > k$ if and only if $M^{(k)}_p(F) = 0$ for all multiplicity operators of order $k$.

We now give a list of fundamental theorems concerning multiplicity operators that shall play a role in this paper.

**Theorem 5.** Assume that $\|F\| \leq 1$ and that $F$ has $k+1$ zeros (counted with multiplicities) in the polydisc $D^*_n$. Then for every $k$-th multiplicity operator $M^{(k)}$,

$$C_{n,k}^2 |M^{(k)}_0(F)| < r$$

where $C_{n,k}^2$ is a universal constant.

**Theorem 6.** Let $\|F\| \leq 1$ and let $s := |M^{(k)}_0(F)| \neq 0$ be a $k$-th multiplicity operator. There exist positive universal constants $A_{n,k}, B_{n,k}$ with the following property:

For every $r < s$ there exists $A_{n,k}r < \tilde{r} < r$ such that

$$\|F(z)\| \geq B_{n,k}s^k \text{ for every } \|z\| = \tilde{r}$$

**Corollary 9.** Assume that $\|F\|, \|G\| \leq 1$ and let $s := |M^{(k)}_0(F)| \neq 0$ be a $k$-th multiplicity operator and $0 < \varepsilon < 1$. There exist universal constants $A''_{n,k}, B''_{n,k}$ with the following property:

If $\|G(0)\| < \varepsilon$ then for every $0 < r < B''_{n,k}s$ there exists $A''_{n,k}r < \tilde{r} < r$ such that

$$\|F(z)\| > \|G^{k+1}(z)\| \text{ for every } \|z\| = \tilde{r},$$

and in particular

$$\# \{z : F(z) = 0, \|z\| < \tilde{r} \} = \# \{z : F(z) + G^{k+1}(z) = 0, \|z\| < \tilde{r} \}.$$  

**Theorem 7.** Let $M^{(k)}$ be a multiplicity operator of order $k$ and $\gamma \subset (\mathbb{C}^n,0)$ a germ of an analytic curve. Then

$$\text{ord}_\gamma M^{(k)}(F) \geq \min \{\text{ord}_\gamma f_i : i = 1, \ldots, n \} - k$$
3. Structure of the proof

Consider a system \([1]\) with parameters \(n, m, \delta\) and the corresponding ring of Noetherian functions \(S\). By Remark \([3]\) we view \(S\) as the restriction of the ring \(C_{n+m}\) to an integral manifold \(\Lambda_p\) of \([4]\).

Let \(\rho \in S\) with \(\deg \rho \leq d\) and let \(X \subset \Lambda_p\) be a germ of a Noetherian set

\[
X = \{ x \in \Lambda : \psi_1(x) = \cdots = \psi_{n-1}(x) = 0 \}
\]

at the point \(p\), where \(\psi_i \in S\) and \(\deg \psi_i \leq d\) for \(i = 1, \ldots, k\).

By definition, \(\rho = R|\Lambda_p\) and \(\psi_i = P_i|\Lambda_p\) where \(R, P_1, \ldots, P_{n-1}\) are elements of \(C_{n+m}\) of degrees bounded by \(d\). To simplify the notation we will let \(P\) denote the tuple \(P_1, \ldots, P_{n-1}\). When the point \(p\) and the integral manifold \(\Lambda_p\) are clear from the context, we will refer to the deflicity of \(X\) with respect to \(\rho\) at \(p\) simply as the deflicity of \(P, R\).

3.1. Analytic expression for the deflicity. Since Noetherian functions are analytic, the germ \(X\) admits a decomposition into irreducible analytic components. Let \(\{\gamma_i\}\) denote the components of \(X\) which are curves and such that \(\rho|_{\gamma_i} \neq \text{const}\) (where each curve is counted with an associated multiplicity \(m_i\)). We will call these curves the good curves of \(X\) with respect to \(\rho\). All other components (whether they are curves on which \(\rho\) is constant or higher-dimensional components) will be called bad components.

To motivate this definition, note that since \(\rho\) is an analytic function, it cannot admit isolated zeros on components of \(X\) that have dimension greater than one, and therefore the bad components do not contribute any isolated zeros in the definition of the deflicity of \(X\) with respect to \(\rho\). On the other hand, the number of solutions of \(\rho = y\) on a good curve \(\gamma_i\) converging to \(p\) as \(y \to \rho(p)\) is equal by definition to \(\text{mult}_{\gamma_i}(\rho - \rho(p))\). Since each good curve is transversal to \(\rho^{-1}(\varepsilon)\) for all sufficiently small \(\varepsilon\), each solution of \(\rho = \varepsilon\) on \(\gamma_i\) should be counted with multiplicity \(m_i\). To conclude, we have following proposition.

**Proposition 10.** The deflicity of \(X\) with respect to \(\rho\) at the point \(p\) is given by

\[
\sum_i m_i \text{mult}_{\gamma_i}(\rho - \rho(p))
\]

where the sum is taken over the good components \(\gamma_i\).

3.2. The set of non-isolated intersections. Recall that \(P = (P_1, \ldots, P_{n-1})\) and \(R\) denote polynomials of degree bounded by \(d\).

**Definition 11.** The set of non-isolated intersection of \(P\) with respect to \(R\), denoted by \(\mathcal{NI}(P; R)\), is the set of all points \(q \in C^{n+m}\) such that \(q\) belongs to a bad components of the set \(\{P|_{\Lambda_q} = 0\}\) with respect to \(R|_{\Lambda_q}\).

The set of bad component on each particular integral manifold \(\Lambda_q\) is analytic, but generally not algebraic. However, the following proposition shows that all of these sets together do form an algebraic set.

**Proposition 12.** The set \(\mathcal{NI}(P; R)\) is algebraic. Moreover, it can de defined by equations of degrees not exceeding

\[
d_{\mathcal{NI}} = \max[d_{\mathcal{NI}}, d_M(n, \delta, d, N(m, n, \delta, d; 0))]
\]

**Proof.** A point \(q \in C^{n+m}\) belongs to \(\mathcal{NI}(P; R)\) if \([1]\) is integrable at the point, and the system of equations

\[
P|_{\Lambda_q} = (R - R(q))|_{\Lambda_q} = 0
\]
admits a non-isolated solution through.q. We can write down equations of degree \( d_\Delta \) to guarantee integrability by Theorem 3.

Since the maximal possible multiplicity of an isolated solution is bounded by \( k = N(m, n, \delta, d; 0) \), a solution is non-isolated if and only if it has multiplicity greater than this number. By Proposition 3 this is equivalent to the vanishing of \( M^{(k)}(P; R - R(q)) \) for every multiplicity operator \( M^{(k)} \) of order \( k \). It remains to note that these expressions evaluate to polynomials (as functions of \( q \)) of the required degrees.

The real analytic function \( \text{dist}(. , N\mathfrak{J}(P; R)) \) will play a key role in our arguments. We will refer to it as the critical distance.

3.3. The inductive step. We will prove our main result, Theorem 1 by induction on the dimension of the set \( N\mathfrak{J}(P; R) \). To facilitate this induction we let \( M(m, n, \delta, d; e) \) denote the maximum possible deformation multiplicity at \( p \) for any system \( P \) with parameters \( m, n, \delta \) and any pair \( X, \rho \) defined as in the beginning of section 3 with \( \deg P \leq d \) and \( \deg R \leq d \), and such that the dimension of \( N\mathfrak{J}(P; R) \) near \( p \) is bounded by \( e \).

The case \( e = 0 \) corresponds to systems where the intersection at \( p \) is in fact isolated. In this case the deformation multiplicity is given by the usual multiplicity of \( p \) as a solution of the equations \( P = R - R(p) = 0 \). Thus \( M(m, n, \delta, d; 0) \) is bounded by Theorem 4.

Theorem 1 now follows by an easy induction from the following.

**Theorem 8.** The quantity \( M(m, n, \delta, d; e) \) satisfies
\[
(17) \quad M(m, n, \delta, d; e) \leq M(m, n, \delta, d', e - 1)
\]
where
\[
(18) \quad d' = (Ad_E + B)(k + 1) + n + m,
\]
\( A, B \) are defined in Proposition 15, \( d_E \) in Proposition 23 and \( k = N(m, n, \delta, d; 0) \) as in Proposition 14.

We now present a schematic proof of this statement, taking for granted Propositions 14, 15, 16 and 23. The reader is advised to review this proof before reading the details of the aforementioned propositions in order to gain perspective on the context in which they are used. The propositions are presented in the three subsections of Section 4 and in Section 5 respectively, and each can be read independently of the others.

**Proof.** Consider a system \( P \) with parameters \( (n, m, \delta) \), an integral manifold \( \Lambda_p \), and a pair \( P, R \) of degree \( d \) such that \( \dim N\mathfrak{J}(P; R) = e \). Let \( \{\gamma_i\} \) denote the good curves of \( P, R \) on \( \Lambda_p \) and let \( \{\mathfrak{N}\mathfrak{C}_i\} \) denote the irreducible components of \( N\mathfrak{J}(P; R) \). We assume that \( R|_{\Lambda_p} \neq \text{const} \) (otherwise there is nothing to prove). We will establish the claim by constructing a new pair \( P', R \) of degree \( d' \) such that

(i). The deflicity of \( P', R \) is no smaller than that of \( P, R \) on \( \Lambda_p \).

(ii). We have \( \dim N\mathfrak{J}(P'; R) < \dim N\mathfrak{J}(P; R) = e \).

Proposition 14 establishes a condition under which a perturbation of \( P \) does not decrease the deflicity. Roughly, the order of the perturbations along each good curve \( \gamma_i \) must be smaller than a certain prescribed order \( \nu_i \). Under this condition, we show by a Rouche-type argument that the perturbation cannot decrease the number roots converging to \( p \).

Proposition 15 relates the orders \( \nu_i \) to the critical distance. Namely, we give two explicit constants \( A, B \) such that for any good curve \( \gamma_i \) the order of the critical distance function on \( \gamma_i \) is at least \( \frac{\nu_i - B}{A} \). Thus, to satisfy the requirements of Proposition 14 it essentially suffices to find a function \( E \) minorizing the critical distance.
The construction of $E$ is carried out in Proposition 23 where we construct a polynomial $E \in \mathbb{C}^{n+m}$ of an explicitly bounded degree $d_E$ which minorizes the critical distance along each good curve and does not vanish identically on any of the $\mathcal{NC}_i$. We choose a generic affine-linear functional $\ell \in (\mathbb{C}^{n+m})^*$ which vanishes at $p$, and does not vanish identically on any of the $\mathcal{NC}_i$ and define
\begin{equation}
E' = E^A \cdot \ell^B
\end{equation}
where $A, B$ are the coefficients given in Proposition 15. Then $E'$ satisfies the asymptotic conditions of Proposition 14.

Our second goal is condition (ii). We achieve this condition in two steps. Namely, it would clearly suffice to construct a perturbation $P'$ satisfying the following two conditions:

(A). $\mathcal{N}(P'; R) \subseteq \mathcal{N}(P; R)$.

(B). None of the components $\mathcal{NC}_i$ are contained in $\mathcal{N}(P'; R)$.

We establish condition (A) by a Sard-type argument, essentially using the fact that the occurrence of a non-isolated intersection is a condition of infinite codimension. Specifically, if we choose $Q_1, \ldots, Q_{n-1}$ to be sufficiently generic polynomials of degree $n + m$ and define
\begin{equation}
P_j' = P_j + Q_j(E')^{k+1}
\end{equation}
then according to Proposition 16 condition (A) is satisfied. Moreover, $P'$ clearly satisfies the asymptotic conditions of Proposition 14 and hence condition (i).

It remains to establish condition (B). Consider a component $\mathcal{NC}_i$. Recall that $E'$ does not vanish at generic points of $\mathcal{NC}_i$ by construction. And $Q_1$ being chosen to be sufficiently generic, certainly does not either. Since $P_1$ vanishes at every point of $\mathcal{NC}_i$ by definition, we see that $P_1'$ does not vanish at generic points of $\mathcal{NC}_i$, and hence condition (B) is satisfied. This concludes the proof.

**Proof of Theorem 7** We claim that in notations of Theorem 8
\begin{equation}
d' \leq d^{32(m+n)^3}(m + n)^{40(m+n)^5} \quad \text{if} \quad d > \delta.
\end{equation}
Indeed, rough estimate for $\mathcal{N}(m, n, \delta, d; 0)$ in Theorem 4 gives
\begin{equation}
\mathcal{N}(m, n, \delta, d; 0) < (\delta + d)^{8(m+n)^2}(m + n)^{8(m+n)^3}.
\end{equation}

Let denote the right hand side by $C$. Therefore, $B, (k+1) \leq C$ by Proposition 15. Then $d_M(n, \delta, d, B) < C^{n+1}(d + \delta)$ by (7), and $d_{\mathcal{L}} < \delta^{2(m+n)}(m + n)^{6(m+n)}$, by (4). Therefore, according to Proposition 12 $d_{\mathcal{L}} < C^{m+n}$ and $A < C^{(m+n)^2}$.

From (12) we have $d_H < C^{m+n}$. Therefore, from (53), $d_E < C^{2(m+n)^2}$. Therefore
\begin{equation}
d' \leq \left(C^{3(m+n)^2} + C\right)C + m + n < C^{4(m+n)^2} \leq (\max\{d, \delta\})^{32(m+n)^4}(m + n)^{40(m+n)^5}.
\end{equation}

Now, applying (21) at most $n$ times, we get
\begin{equation}
\mathcal{M}(m, n, \delta, d; c) \leq \mathcal{M}(m, n, \delta, \tilde{d}; 0), \quad \tilde{d} \leq (\max\{d, \delta\})^{8n}(m + n)^{20n},
\end{equation}
so, by (22) we get the required bound.

**4. Good perturbations**

Let $P = (P_1, \ldots, P_{n-1})$ and $R$ be polynomials of degree bounded by $d$, and let $\Lambda_p$ be the germ of an integral manifold of (1) at the point $p$. Let $\{\gamma_i\}$ denote the set of good curves of $P, R$ through the point $p$ with associated multiplicities $m_i$. Let $\ell \in (\mathbb{C}^n)^*$ be a generic linear functional on the $x$-plane which is non-vanishing on vectors tangent to each $\gamma_i$ at $p$, and let $T = \ker \ell.$
At generic point \( q \) of each \( \gamma_i \), the multiplicity \( m_i \) is equal to the multiplicity of the isolated intersection \( P = 0 \) through the \( n - 1 \)-dimensional \( \mathcal{T} \)-plane. In particular, it is bounded by \( k = N(m, n, \delta; d; 0) \).

**Remark 13.** In fact, since the multiplicity is taken through the \( n - 1 \)-dimensional plane \( \mathcal{T} \), and the restriction of \( P \) to \( \mathcal{T} \) may be though of as Noetherian function in the ambient space \( \mathbb{C}^{n+m-1} \) cut out by \( \ell = 0 \), we could have chosen \( k = N(m, n - 1, \delta; d; 0) \). However, since this does not affect our overall estimates and complicates the notation we use the weaker bound above.

Let \( M^{(k)}_j \) be a generic multiplicity operator through the \( \mathcal{T} \) plane of order \( k \). By the above, \( M = M^{(k)}_j(P) \) does not vanish identically on any \( \gamma_i \).

### 4.1. Deflicity preserving perturbations

Let \( E \) and \( Q_1, \ldots, Q_{n-1} \) be holomorphic functions on \( \Lambda_p \).

**Proposition 14.** Suppose that for every \( \gamma_i \)
\[
\text{ord}_{\gamma_i} E > \max[\text{ord}_{\gamma_i} M, \text{ord}_{\gamma_i}(R - R(p))] \tag{23}
\]
and let
\[
P'_j = P_j + Q_j E^{k+1} \quad j = 1, \ldots, n - 1. \tag{24}
\]
Then the deflicity of \( P', R \) is no smaller than the deflicity of \( P, R \).

**Proof.** We assume for simplicity of the notation that \( \ell(p) = R(p) = 0 \). Recall that the deflicity of \( P, R \) is equal to
\[
\sum_{\gamma_i} \text{mult}_{\gamma_i} R = \sum_{\gamma_{ij}} \text{ord}_{\gamma_{ij}} R \tag{25}
\]
where \( \gamma_{ij} \) denote the set of pro-branches of the complex curve \( \gamma_i \) with respect to some generic linear functional. Here, if a curve \( \gamma_i \) appears with multiplicity \( m_i \) then we consider each of its pro-branches as appearing \( m_i \) times in the set \( \gamma_{ij} \).

Similarly, if we let \( \gamma'_{ij} \) denote the good curves corresponding to \( P', R \) and \( \gamma_{ij} \) their pro-branches (taking multiplicities into account), then the deflicity of \( P', R \) is equal to
\[
\sum_{\gamma'_{ij}} \text{mult}_{\gamma'_{ij}} R = \sum_{\gamma_{ij}} \text{ord}_{\gamma_{ij}} R \tag{26}
\]
We will show that if \( E \) satisfies the growth condition \( (23) \) then there is an injective map \( \iota \) from the set \( \{\gamma_{ij}\} \) to the set \( \{\gamma'_{ij}\} \) such that
\[
\text{ord}_{\gamma_{ij}} R = \text{ord}_{(\gamma_{ij})} R. \tag{27}
\]
Therefore the right-hand side of \( (26) \) is no smaller than the right-hand side of \( (25) \), and the conclusion of the proposition follows.

Since \( \ell \) is transversal to the curves \( \gamma_i \) at \( p \), each pro-branch \( \gamma_{ij} \) intersects \( \{\ell = s\} \) at exactly one point \( \gamma_{ij}(s) \), and \( s \) is a natural parameter on \( \gamma_{ij} \).

Introduce an equivalence relation \( \sim \) on \( \{\gamma_{ij}\} \) by letting \( \gamma_{ij} \sim \gamma_{ij'} \) if and only if
\[
\text{ord}_s \text{dist}(\gamma_{ij}(s), \gamma_{ij'}(s)) > \max[\text{ord}_{\gamma_{ij}} M, \text{ord}_{\gamma_{ij}} R] \tag{28}
\]
In other words, two pro-branches are equivalent if and only if the distance between them is smaller by an order of magnitude than \( M \) and \( R \) (evaluated at one of them). It is easy to check that this is indeed an equivalence relation. We call the equivalence classes \( C_{\alpha} \) of \( \sim \) clusters.
Choose a representative $\gamma_\alpha$ for each cluster $C_\alpha$. By (28) we can choose an order $\nu_\alpha \in \mathbb{Q}$ such that
\[
\min_{\nu_\alpha} \{\text{ord}_\nu (\gamma_\alpha(s), \gamma_{ij}(s))\} > \nu_\alpha > \max\{\text{ord}_\nu M_\alpha, \text{ord}_\nu R_\alpha\},
\]
and by (23) we can also require that
\[
\text{ord}_\nu E > \nu_\alpha.
\]

Consider the ball $B_\alpha(s) \subset \{\ell = s\}$ with center at $\gamma_\alpha(s)$ and radius $s^{\nu_\alpha}$. By (29) and the definition of $\sim$ it follows that for sufficiently small, $B_\alpha(s)$ meets the pro-branches $\gamma_{ij}$ in the cluster $C_\alpha$ and only them.

Given (29) and (30), Corollary 9 applies with $r = s^{\nu_\alpha}/A_{\alpha}^{\nu}$ for sufficiently small. It follows that the number of zeros of $P^m = 0$ in the ball $B_\alpha(s) \subset \{\ell = s\}$ with center at $\gamma_\alpha(s)$ and radius $s^{\nu_\alpha}/A_{\alpha}^{\nu}$ is at least the size of the cluster $C_\alpha$.

By (29) the balls $B'_\alpha(s)$ are disjoint for different $\alpha$ and sufficiently small $s$. Consider now the pro-branches $\gamma_{ij}$. For sufficiently small $s$, at least $\#C_\alpha$ of them must lie in $B'_\alpha(s)$. Let $\iota$ be an arbitrary injection from the pro-branches $\gamma_{ij}$ belonging to $C_\alpha$ to the pro-branches $\gamma_{ij}'$ lying in $B'_\alpha(s)$ for sufficiently small $s$.

The construction will be finished if we prove that $\iota$ satisfies (27). But this is clear, since by (29) the radius of $B'_\alpha(s)$ is an order of magnitude smaller than $R$ on $\gamma_\alpha$ and therefore any curve $\gamma$ lying in $B'_\alpha(s)$ for sufficiently small $s$ must satisfy
\[
\text{ord}_\gamma R = \text{ord}_\gamma R.
\]

4.2. Minorizing the growth conditions by the critical distance. Proposition 14 allows us to construct a perturbation of $P,R$ which does not decrease the deflicity, given a function $E$ which is small on the good curves $\gamma_i$ compared to $R$ and $M$.

In this subsection we show that $R$ and $M$ are minorized by the critical distance, and hence in order to apply Proposition 14 it suffices to minorize the critical distance. More precisely,

**Proposition 15.** For every good curve $\gamma_i$ we have
\[
\max\{\text{ord}_{\gamma_i} M, \text{ord}_{\gamma_i} (R - R(p))\} \leq A \text{ord}_{\gamma_i} \text{dist}(\cdot, \mathcal{N}(P, R - R(p))) + B
\]
where
\[
A := \max\{d_{N_\delta}, d_M(n, \delta, d, B)\}^{n+m},
B := \mathcal{N}(m, n, \delta, d; 0)
\]

**Proof.** We start with the estimate for the function $R - R(p)$. Let $k = N(m, n, \delta, d; 0)$ and let $I_R$ generated by the polynomials $M^{(k)}(P, R - R(p))$ (where $M^{(k)}$ ranges over all multiplicity operators of order $k$) and the polynomials defining the integrability locus $\mathcal{J}_L$ provided by Theorem 3. We denote the set of all of these generators by $\{G_i\}$.

Arguing in the same manner as in the proof of Proposition 14, we see that the zero locus of $I_R$ is contained in $\mathcal{N}(P; R)$ (in fact, it consists of those points in $\mathcal{N}(P; R)$ where $R = R(p)$).

By the effective Lojasiewicz inequality [15], there exists a constant $C > 0$ such that
\[
\max_i |G_i(\cdot)| \geq C \text{dist}(\cdot, Z(I_R))^A \geq C \text{dist}(\cdot, \mathcal{N}(P; R))^A.
\]

Let $\gamma_i$ be a good curve. Then $\gamma_i \subset \mathcal{J}_L$ and the polynomials defining $\mathcal{J}_L$ vanish identically on it. Therefore along $\gamma_i$ the maximum must be attained for one of the generators given by the multiplicity operator $M^{(k)}$, and hence
\[
\text{ord}_{\gamma_i} M^{(k)}(P, R - R(p)) \leq A \text{ord}_{\gamma_i} \text{dist}(\cdot, \mathcal{N}(P; R)).
\]
Lemma 17. Suppose that at least one of the zero locus of \( M \) is analytic. Since \( \dim U = 1 \) for any manifold \( \Lambda \) and any point \( q \in \{ P|\Lambda = 0 \} \), if \( q \notin \mathcal{N}(P; R) \) then the zero locus of \( P|\Lambda \) near \( q \) must be a curve, and at least one of \( \mathcal{T}_1, \ldots, \mathcal{T}_n \) must not be parallel to this curve. Then there exists a multiplicity operator of order \( k \) in the \( \mathcal{T}_i \) direction which is not vanishing at \( q \).

Let \( I_M \) denote the ideal generated by the polynomials defining the integrability locus and all multiplicity operators of order \( k \) through any of spaces \( \mathcal{T}_1, \ldots, \mathcal{T}_n \) of \( P \). By the above, the zero locus of \( I_M \) is contained in \( \mathcal{N}(P; R) \). Moreover, the integrability conditions vanish identically on \( \gamma_i \), and all other generators of \( I_M \) have orders no smaller than the order of \( M^{(k)}(P) \) (since our original choice of \( \mathcal{T} \) and \( M^{(k)} \) was generic).

We can now complete the proof in a manner analogous to the argument we used for \( I_R \).

\[ \square \]

4.3. Sard-type claim for generic perturbations. We are interested in applying Proposition 14 in order to produce a perturbation of \( P, R \) which does not decrease deflicity, and which reduces the set of non-isolated intersections \( \mathcal{N}(P; R) \). The first step is to show that our perturbation does not create new non-isolated intersections. We will show that for a sufficiently generic choice of the coefficients \( Q_j \) this will be the case.

Let \( E \) be a Noetherian function. Let \( R = n \) and denote by \( \mathcal{P}_\beta \) the space of polynomials of degree bounded by \( \beta \) in \( n \) variables. Finally let \( (Q_1, \ldots, Q_n) \in \mathcal{P}^n_\beta \), and set \( P_i' = P_i + Q_iE \) for \( i = 1, \ldots, n \).

Proposition 16. Assume \( R|\Lambda_q \neq \text{const} \). Then for a generic tuple \( Q = (Q_1, \ldots, Q_n) \in \mathcal{P}^n_\beta \), there exists a neighborhood \( U_p \subset \mathbb{C}^{n+m} \) of \( p \) such that

\[ \mathcal{N}(P^i; R) \cap U_p \subseteq \mathcal{N}(P; R) \cap U_p \]

where \( P_i' \) is defined as above.

Proof. Fix a neighborhood \( U_p \) of \( p \) such that \( R \) is not constant on any integral manifold of \( \mathcal{I} \) in \( U_p \). Then for every point in \( q \in U_p \cap \mathcal{I} \) either Lemma 17 or Lemma 18 below is applicable, depending on whether \( E(q) = 0 \), with parameters \( \beta = n + m \) and \( l = n \). In both cases it follows that there exists a set \( B(q) \subset \mathcal{P}^n_{n+m} \) of codimension \( n + m + 1 \), such that if \( Q \notin B(q) \) then \( q \notin \mathcal{N}(P; R) \) implies \( q \notin \mathcal{N}(P^i; R) \). Moreover, as noted in the proof of Proposition 17 the graph of the relation \( Q \in B(q) \), i.e. the set

\[ G = \{(Q, q) \in \mathcal{P}^n_{n+m} \times U_p : Q \in B(q)\} \]

is analytic. Since \( \dim U_p = n + m \) it follows that the projection of \( G \) to \( \mathcal{P}^n_{n+m} \) has codimension at least 1. Any \( Q \) outside this projection will satisfy (35). \( \square \)

It remains to state and prove the following two lemmas.

Lemma 17. Suppose that \( q \in \mathcal{I}, E(q) \neq 0 \) and suppose further that \( R|\Lambda_q \neq R(q) \). Let \( 0 \leq l \leq n \).

For \( (Q_1, \ldots, Q_l) \) outside a set of \( B_l \) of codimension \( \beta + 1 \) in \( \mathcal{P}^l_\beta \), the set

\[ Z_l := \{ P_1' = \cdots = P_l' = 0, R = R(q) \} \cap \Lambda_q \]

is empty or has pure codimension \( l + 1 \) (in a neighborhood of \( q \)).

**Proof.** Let \( B_l \) be the set of \( Q_1 \) violating the condition. It is defined by the conditions \( M^{(k)}_q(F^1_1,\ldots,F^1_n) = 0 \) for all multiplicity operators \( M^{(k)} \) of every order \( k \), where \( (F^i_l)_{i=1}^n \) are \( n \)-tuples consisting of \( P^i_1|_{\Lambda_q},\ldots,P^i_l|_{\Lambda_q} \), \( R|_{\Lambda_q} - R(q) \) and \( n-l-1 \) generic linear functions vanishing at \( q \). These expressions are polynomial in the coefficients of \( Q_1,\ldots,Q_l \), so the set \( B_l \) is algebraic and its dimension is well defined. We note that the conditions are analytic with respect to \( q \) as well.

We prove the claim by induction. The case \( l = 0 \) corresponds precisely to our assumption \( R|_{\Lambda_q} \not\equiv R(q) \).

Suppose that the claim is proved for \( l-1 \). Consider the projection \( \pi : \mathcal{P}^l_{\beta} \to \mathcal{P}^{l-1}_{\beta} \) forgetting the last coordinate. The set \( \pi^{-1}(B_{l-1}) \) has codimension at least \( \beta + 1 \) by induction. The claim will follow if we show that the fiber of each point outside \( B_{l-1} \) also has codimension at least \( \beta + 1 \).

Let \( (Q_1,\ldots,Q_{l-1}) \not\in B_{l-1} \). Then \( Z_{l-1} \) is either empty or has pure codimension \( l \). If it is empty, there is nothing to prove. Otherwise \( (Q_1,\ldots,Q_l) \) will belong to the fiber if and only if \( P^l_l \) vanishes identically on some irreducible component of \( Z_{l-1} \). Since there are finitely many such components, it will suffice to check that identical vanishing on each of them has codimension at least \( \beta + 1 \). Let \( Z_{l-1}' \) be one such component.

Since \( E(q) \neq 0 \), we have \( P^l_l|_{Z_{l-1}'} \equiv 0 \) if and only if \( Q_l \equiv -P_l/E \) identically on \( Z_{l-1}' \). This is an affine-linear condition on \( Q_l \) of codimension at least \( \dim \mathcal{P}^l_{\beta}|_{Z_{l-1}'} \). It remains only to note that this dimension is at least \( \beta + 1 \): for instance if \( x_j \) is a coordinate not identically vanishing on \( Z_{l-1}' \) then clearly \( 1, x_1,\ldots, x^\beta_j \) are linearly independent as functions defined on \( Z_{l-1}' \).

**Lemma 18.** Suppose that \( q \in \mathfrak{J}, q \not\in \mathcal{N}(P;R) \) and \( E(q) = 0 \). Let \( 0 \leq l \leq n \).

For \( (Q_1,\ldots,Q_l) \) outside a set of \( B_l \) of codimension \( \beta + 1 \) in \( \mathcal{P}^l_{\beta} \), the set

\[ Z_l := \{ P^1_1 = \cdots = P^l_l = P_{l+1} = \cdots = P_n = 0, R = R(q) \} \cap \Lambda_q \]

is zero-dimensional in a neighborhood of \( q \) (that is, contains only, possibly, \( q \)).

**Proof.** Let \( B_l \) be the set of \( Q_l \) violating the condition. We can check that \( B_l \) is algebraic as in the proof of Lemma 17.

We prove the claim by induction. The case \( l = 0 \) corresponds precisely to our assumption that \( q \not\in \mathcal{N}(P;R) \).

Suppose that the claim is proved for \( l-1 \). Consider the projection \( \pi : \mathcal{P}^l_{\beta} \to \mathcal{P}^{l-1}_{\beta} \) forgetting the last coordinate. The set \( \pi^{-1}(B_{l-1}) \) has codimension at least \( \beta + 1 \) by induction. The claim will follow if we show that the fiber of each point outside \( B_{l-1} \) also has codimension at least \( \beta + 1 \).

Let \( (Q_1,\ldots,Q_{l-1}) \not\in B_{l-1} \). Then the set \( Z_{l-1} \) is zero dimensional (in a neighborhood of \( q \)). If it is in fact empty, then one of the equations defining it is non-vanishing at \( q \). In this case, since \( E(q) = 0 \) by assumption, \( Z_{l-1} \) is empty as well.

We therefore must consider the case that \( Z_{l-1} = \{ q \} \) (in a neighborhood of \( q \)). In this case, the equations

\[ \{ P^1_1 = \cdots = P^l_{l-1} = P_{l+1} = \cdots = P_n = 0, R = R(q) \} \cap \Lambda_q \]

define a curve \( \gamma \subset \Lambda_q \). Thus \( (Q_1,\ldots,Q_l) \) will belong to \( B_l \) if and only if \( P^l_l \) vanishes identically on an irreducible component of this curve. Since there are finitely many such components, it will suffice to check that identical vanishing on each of them has codimension at least \( \beta + 1 \). Let \( \gamma' \) be one such component.
We have $P'|_{\gamma'} \equiv 0$ if and only if $EQ_t \equiv -P_t$ identically on $\gamma'$. This is an affine-linear condition on $Q_t$ of codimension at least $\dim P|_{\gamma'}$, if $E \not\equiv 0$ on $\gamma'$ then this is clear, and otherwise the condition is never satisfied because $S_{t-1}$ is a regular sequence and hence $P_t$ does not vanish identically on $\gamma$. The conclusion now follows as in the proof of Lemma 17.

5. Minimizing the critical distance

Let $P = (P_1, \ldots, P_{n-1})$ and $R$ be polynomials of degree bounded by $d$, and let $\Lambda_p$ be the germ of an integral manifold of (1) at the point $p$. Let $\{\gamma_i\}$ denote the set of good curves of $P, R$ through the point $p$ with associated multiplicities $m_i$.

Our goal in this section is to construct a Noetherian function $E$ of bounded degree such that $E$ minimizes the critical distance on good curves, and does not vanish identically on any of the top-dimensional components of $\mathcal{N}(P, R)$. The main step in the construction is the following Lemma.

We introduce some notations to facilitate our proof. If $\mathcal{T}$ is a linear subspace of the $x$-coordinates, then for every point $q \in \mathcal{L}$ we will denote by $\mathcal{T}_q \subset \Lambda_q$ the integral submanifold of the sub-distribution of (1) corresponding to $\mathcal{T}$. Similarly $B_{\mathcal{T}}(q, r) \subset \mathcal{T}_q$ will denote the ball of radius $r$ around $q$ in $\mathcal{T}_q$.

The $x$-plane provides natural coordinates on the integral manifolds of (1) in a neighborhood $U$ of $p$. In particular for any two integral manifolds $\Lambda_q, \Lambda_{q2} \subset U$ we have a map $\tau_{q1}^{q2} : \Lambda_q \rightarrow \Lambda_{q2}$ mapping each point in $\Lambda_q$ to the point with the same $x$-coordinates in $\Lambda_{q2}$. If we choose $U$ small enough, then by the analytic dependence of flows on initial conditions, for every $q \in U$ we have

$$\text{dist}(q, \tau_{q1}^{q2}q) \leq 2\text{dist}(q_1, q_2).$$

**Lemma 19.** Let $\mathcal{N} \subset \mathcal{N}(P, R)$ be an irreducible component of $\mathcal{N}(P, R)$. There exists a polynomial $H \in \mathbb{C}[n+m]$ such that

1. For every good curve $\gamma$ we have
   $$\text{ord}_\gamma H \geq \text{ord}_\gamma \text{dist}(\cdot, \mathcal{N}_e)$$
2. $H$ does not vanish identically on $\mathcal{N}_e$.
3. The degree of $H$ is bounded by
   $$d_H := (k + 1)d_M(n, \delta, d, K)$$
   where $k = N(m, n, \delta, d; 0)$

**Proof.** For any $q \in \mathcal{N}_e$ the set $\mathcal{N}(P, R) \cap \Lambda_q$ consists of those components of $\{P|_{\Lambda_q} = 0\}$ which have dimension greater than 1, or where $R$ is constant. In particular, $\mathcal{N}_e \cap \Lambda_q$ is a union of such components. Assume now that $q \in \mathcal{N}_e$ is generic. Then locally near $q$ we will have $\mathcal{N}_e \cap \Lambda_q = \{P|_{\Lambda_q} = 0\}$. Thus if $\text{codim}_{\Lambda_q}(\mathcal{N}_e \cap \Lambda_q) = l$, then letting $\Phi = (P_1, \ldots, P_l)$ (up to a reordering of $P_j$), we have $\mathcal{N}_e \cap \Lambda_q = \{\Phi|_{\Lambda_q} = 0\}$.

Let now $\mathcal{T}$ be a generic $l$-dimensional subspace of the $x$ coordinates. Denote by $k$ the multiplicity of the isolated zero $\{\Phi|_{\mathcal{T}_q} = 0\}$. In particular,

$$k \leq N(m, n, \delta, d; 0).$$

By Proposition 6 there exists a multiplicity operator $M^{(k)}$ of order $k$ such that $M^{(k)}(\Phi) \neq 0$. Let $M = M^{(k)}(\Phi)$.

**Claim 20.** At any point $q \in \mathcal{N}_e$ such that $M(q) \neq 0$, we have

1. $\text{mult}_q \Phi|_{\mathcal{T}_q} = k$.
2. $\mathcal{N}_e \cap \Lambda_q = \{\Phi = 0\} \cap \Lambda_q$ in a neighborhood of $q$. 

Proof of the claim. Since $M(q) \neq 0$, the set $\{\Phi = 0\} \cap \Lambda_q$ has codimension $l$ (near $q$). Since $l$ was chosen to be the generic (hence maximal) codimension of $NC \cap \Lambda_q$ intersected with any integral manifold, the codimension of $NC \cap \Lambda_q$ is at most $l$. The point $q$ must be a zero of multiplicity at least $k$, since $k$ was chosen as the generic (and hence minimal) multiplicity of a zero of $\Phi|_{\mathcal{T}_q'}$ at a point of $NC$. The multiplicity cannot exceed $k$ because $M(q) \neq 0$.

Clearly $NC \cap \Lambda_q \subset \{\Phi = 0\} \cap \Lambda_q$, so in fact their dimensions coincide and $NC \cap \Lambda_q$ must be a union of irreducible components of $\{\Phi = 0\} \cap \Lambda_q$. Suppose toward contradiction that this set has another component $C$ through $q$.

Since $M(q) \neq 0$, the set $\{\Phi = 0\} \cap \Lambda_q$ has an isolated intersection with $\mathcal{T}_q$ at $q$. Consider generic $q' \in \Lambda_q$ arbitrarily close to $q$. Then $\mathcal{T}_{q'}$ will meet $C$ at some point $q_1$ close to $q$, and the set $NC \cap \Lambda_q$ at some other point $q_2$ close to $q$. Both points $q_1, q_2$ correspond to zeros of $\Phi|_{\mathcal{T}_{q'}}$. But the point $q_2$ must again be a zero of multiplicity at least $k$ (by the same argument we used for $q$).

The points $q_1, q_2$ tend to $q$ as $q'$ tends to $q$. Thus $\Phi|_{\mathcal{T}_{q'}}$ has at least $k + 1$ zeros (counted with multiplicities) in a ball whose radius tends to zero as $q'$ tends to $q$. By Theorem 6 $M(q) = \lim_{q' \to q} M(q') = 0$, contradicting our assumption.

Let $\gamma$ be any good curve, and denote by $\gamma(t)$ a pro-branch (where $\gamma(0) = p$). Since $M$ is analytic, we can fix $t_0 > 0$ so small that for any $0 < t_1 < t_0$ we have

$$|M(\gamma(t_0))| > \frac{1}{2} |M(\gamma(t_1))|$$

(if $M(p) \neq 0$ this is obvious, and otherwise $|M(\gamma(t))|$ is eventually monotonic). We claim that

$$\text{ord}_r \text{dist}(\cdot, NC) \leq (k + 1) \text{ord}_r M$$

Assuming the contrary, will produce a sequence of points in $NC$ converging to $\gamma(t_0)$, which is impossible since $NC$ is closed and $\gamma$ is a good curve.

Suppose toward contradiction that for sufficiently small $t$ there exist points $y(t) \in NC$ such that

$$\text{ord}_r(t) > (k + 1) \text{ord}_r M(\gamma(t)) \quad \rho(t) := \text{dist}(\gamma(t), y(t)).$$

Then one can certainly choose a function $r(t)$ such that

$$\text{ord}_r r(t) > \text{ord}_r M(\gamma(t)), \quad \text{ord}_r(t) > k \text{ord}_r(t) + \text{ord}(M(\gamma(t))).$$

Moreover, by analyticity of $M$ we have $\text{ord}_r M(\gamma(t)) = \text{ord}_r M(y(t))$. In particular,

$$\text{ord}_r r(t) > \text{ord}_r M(y(t)), \quad \text{ord}_r(t) > k \text{ord}_r(t) + \text{ord}(M(y(t)))$$

By Theorem 6 there exists a radius $A_{n,k} r(t) < \tilde{r}(t) < r(t)$ such that that minimum of $\|\Phi\|$ over $\partial B_{\tilde{r}}(y(t), \tilde{r}(t))$ is greater than $B_{n,k} M(y(t)) \tilde{r}^k(t)$ which, by (48), is asymptotically larger than $\rho(t)$. In particular, by Rouché’s principle any perturbation of $\Phi|_{\mathcal{T}_s}(y(t), \tilde{r}(t))$ of order $\rho(t)$ does not change the number of zeros.

Consider the family of balls $B_s$ connecting $B_0$ and $B_1$,

$$B_0 = B_{\mathcal{T}_s}(y(t), \tilde{r}(t)), \quad B_1 = B_{\mathcal{T}_s}(\tau^p_{\tilde{r}(t)}(\gamma(t)), \tilde{r}(t))$$

by linear motion of the center. By (30) we know that $\text{dist}(y(t), \tau^p_{\tilde{r}(t)}(\gamma(t)))$ is of the same order as $\rho(t)$, and therefore the restriction $\Phi|_{B_s}$ of $\Phi$ to $B_s$ may be viewed as a perturbation of order $\rho(t)$ of $\Phi_0$. By the previous paragraph, it follows that that number of zeros of $\Phi_s$ remains constant.

\footnote{In fact $y(t)$ be be chosen to be an analytic curve, though our argument does not require this.}
In $B_0$ we have the point $y(t) \in \mathcal{N}c \cap \Lambda_{y(t)}$, and \( \text{ord } M(y(t)) < \text{ord } r(t) \) by (48). By Claim 20, \( y(t) \) is a zero of $\Phi|_{B_0}$ of multiplicity $k$, and by Theorem 5 there are no other zeros in $B_0$. By the previous paragraph, the number of zeros of $\Phi_s$ remains $k$ throughout the family.

Again by Claim 20, $\mathcal{N}c \cap \Lambda_{y(t)} = \{ \Phi = 0 \} \cap \Lambda_{y(t)}$ locally near $y(t)$. It follows that the zero \( y(t) \) cannot bifurcate for small values of $s$: it must remain an element of $\mathcal{N}c \cap \Lambda_{y(t)}$, and $k$ is the minimal possible multiplicity for such a root (see the proof of Claim 20).

We can now apply Lemma 21 below to $\Phi_s$, and conclude that $\Phi_s$ has a zero $y_s(t)$ of multiplicity $k$. By analyticity, since $y_s(t) \in \mathcal{N}c \cap \Lambda_{y(t)}$ for small $s$, the same must hold for every $s \in [0, 1]$. In particular, $y_1(t)$ is a zero of $\Phi_1$ and $y_1(t) \in \mathcal{N}c \cap \Lambda_{y(t)}$.

Denote
\[
\gamma'_t(s) := \tau^P_{y(t)}y(s) \quad \text{for } t < s < t_0
\]
and let
\[
B'_s = B_{\gamma'_t(s)}(\bar{\gamma}(t)), \quad \Phi'_s = \Phi|_{B'_s} \quad \text{for } t < s < t_0.
\]
Note that $B'_t = B_1$. We would now like to apply Lemma 21 to the family $\Phi'_s$. The conditions are verified in a similar manner. Briefly, (40) implies that dist($\gamma'_t(s)$, $\gamma(s)$) is of the same order as $\rho(t)$ and combining with (44), we see that \( \text{ord } M(\gamma'_t(s)) \leq \text{ord } M(y(t)) \) (uniformly in $s$). It then follows as before that for every $s$ in the family, $\Phi'_s$ has no more than $k$ zeros, and as a small perturbation of $\Phi|_{\mathcal{T},s}$ it has at least one zero (corresponding to the zero $\gamma(s)$ of $\Phi|_{\mathcal{T},s}$). Finally, the non-bifurcation condition follows in exactly the same manner as before.

We thus apply Lemma 21 and conclude in the same way as before that $\Phi'_1$ has a zero $y'_1(t)$, and moreover that $y'_1(t) \in \mathcal{N}c \cap B'_1$. As $t$ tends to zero the center of $B'_1$ tends to $\gamma(1)$ and its radius tends to zero, so it follows that $y'_1(t)$ tends to $\gamma(1)$. As $y'_1(t) \in \mathcal{N}c$ (for every sufficiently small $t$) we obtained the desired contradiction. Therefore (45) is proved, and taking $H = M^{k+1}$ gives the desired conclusion. \( \square \)

The proof of (19) will be completed once we prove the following Lemma.

**Lemma 21.** Let $U \subset \mathbb{C}^l$ be an open domain, and $\Phi_s : \tilde{U} \to \mathbb{C}^l$ be an analytic family of holomorphic mappings $s \in [0, 1]$. Assume that

1. the closure $\tilde{U}$ contains at most $p$ zeros of $\Phi$, counting multiplicities, for all $s$,
2. $\Phi_s$ has at least one zero in $U$ for all $s$,
3. $\Phi_0$ has a zero $y_0$ of multiplicity $p$ in $U$,

\[
\gamma'_t(s) := \tau^P_{y(t)}y(s) \quad \text{for } t < s < t_0
\]
(4) the zero $y_0$ lies on a germ of a curve $y_*$ of zeros of multiplicity $p$ of $\Phi_*$ (i.e. $y_0$ doesn’t bifurcate into several zeros for small values of $s$).

Then $y_*$ can be analytically extended to a curve of zeros of multiplicity $p$ of $\Phi$ lying in $U$ for all $s \in [0, 1]$.

Proof of the Lemma. Indeed, the curve $y_*$ can be analytically extended in $s$ as long as it doesn’t leave $U$, as the non-bifurcating condition is analytic. Now, due to the first condition $U$ contains at most $p$ zeros of $\Phi_*$, so, as $s$ grows, no zero of $\Phi_*$ can cross into $U$ unless $y_*$ left $U$. But at that moment $U$ would contain more than $p$ zeros. Therefore no new zeros can cross into $U$, so $y_* \in U$ for all $s \in [0, 1]$. □

We require one more standard fact, whose proof we include for the convenience of the reader.

Fact 22. Let $V \subset \mathbb{C}^N$ be an affine variety of degree $D$. Then $V$ is set-theoretically cut out by polynomials of degree $D$.

Proof. Let $x \in \mathbb{C}^N \setminus V$. We will find a polynomial of degree bounded by $D$ that does not vanish at $x$. If $V$ is a hypersurface then it is the zero locus of a polynomial of degree $D$ and the claim is obvious. Otherwise choose a generic projection $\pi : \mathbb{C}^N \to \mathbb{C}^{\dim V + 1}$, such that $\pi(x) \notin \pi(V)$. Then $\pi(V)$ is a hypersurface of degree $D$, and the previous argument produces a polynomial of degree $D$ which vanishes on $\pi^{-1} \pi(V)$. Since $x$ is not contained in this set, the proof is completed. □

Finally we can present the construction of the function $E$.

Proposition 23. There exists a polynomial $E \in \mathbb{C}_{n+m}$ such that

1. For every good curve $\gamma$ we have

$$\text{ord}_\gamma, E \geq \text{ord}_\gamma, \text{dist}(\cdot, \mathcal{N}(P; R))$$

2. $H$ does not vanish identically on any irreducible component of $\mathcal{N}(P; R)$.

3. The degree of $E$ is bounded by

$$d_E := d_\mathcal{N}^{2(n+m)} + d_H$$

Proof. Let $\mathcal{N}(P; R) = \bigcup_{i=1}^{s} \mathcal{N}_i$ be the irreducible decomposition of $\mathcal{N}(P; R)$. By Proposition 12 the set $\mathcal{N}(P; R)$ can be defined by polynomial equations of degree $d_\mathcal{N}_j$. Therefore, $s \leq d_\mathcal{N}_j$ and any irreducible component $\mathcal{N}_i$ of this set has degree bounded by $d_\mathcal{N}_j$. Choose a polynomial $Q_i$ of this degree which vanishes on $\mathcal{N}_i$ and not on any $\mathcal{N}_j$ for $j \neq i$. Also construct for each $\mathcal{N}_i$ the polynomial $H_i$ provided by Lemma 19.

Let

$$E = \sum_{i=1}^s H_i \prod_{j \neq i} Q_j.$$ 

Let $\gamma$ be a good curve, and suppose that $\text{dist}(\cdot, \mathcal{N}(P; R))|_\gamma$ attains its minimum on the component $\mathcal{N}_i$. Then

$$\text{ord}_\gamma, H_i, \text{ord}_\gamma, Q_i \geq \text{ord}_\gamma, \text{dist}(\cdot, \mathcal{N}(P; R))$$

and since each summand in (54) is a product containing either $H_i$ or $Q_i$,

$$\text{ord}_\gamma, E \geq \text{ord}_\gamma, \text{dist}(\cdot, \mathcal{N}(P; R)).$$

Moreover, for each component $\mathcal{N}_i$, all summands other than the $i$-th vanish identically on $\mathcal{N}_i$, whereas the $i$-th summand does not. Therefore $E$ does not vanish identically on any component $\mathcal{N}_i$, and the proposition is proved. □
References

[1] Katia Barré-Sirieix, Guy Diaz, François Gramain, and Georges Philibert. Une preuve de la conjecture de Mahler-Manin. Invent. Math., 124(1-3):1–9, 1996.

[2] Sergeybegini and Dmitry Novikov. On the number of zeros of Melnikov functions. Ann. Fac. Sci. Toulouse Math. (6), 20(3):465–491, 2011.

[3] Gal Binyamini and Dmitry Novikov. Multiplicity operators. submitted to Isr. J. Math.

[4] Gal Binyamini and Dmitry Novikov. Intersection multiplicities of Noetherian functions. Adv. Math., 231(6):3079–3093, 2012.

[5] Gal Binyamini, Dmitry Novikov, and Sergei Yakovenko. On the number of zeros of Abelian integrals. Invent. Math., 181(2):227–289, 2010.

[6] W. D. Brownawell and D. W. Masser. Multiplicity estimates for analytic functions. I. J. Reine Angew. Math., 314:200–216, 1980.

[7] W. D. Brownawell and D. W. Masser. Multiplicity estimates for analytic functions. II. Duke Math. J., 47(2):273–295, 1980.

[8] A. G. Khovanski˘ı. Fewnomials, volume 88 of Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, 1991. Translated from the Russian by Smilka Zdravkovska.

[9] A. G. Khovanski˘ı. Fewnomials, volume 88 of Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, 1991. Translated from the Russian by Smilka Zdravkovska.