Approximating Boolean Functions with Disjunctive Normal Form

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Abstract

The theorem states that: Every Boolean function can be $\epsilon$-approximated by a Disjunctive Normal Form (DNF) of size $O,(2^n/\log n)$. This paper will demonstrate this theorem in detail by showing how this theorem is generated and proving its correctness. We will also dive into some specific Boolean functions and explore how these Boolean functions can be approximated by a DNF whose size is within the universal bound $O,(2^n/\log n)$. The Boolean functions we interested in are:

- Parity Function: the parity function can be $\epsilon$-approximated by a DNF of width $(1-2\epsilon)n$ and size $2^{(1-2\epsilon)n}$. Furthermore, we will explore the lower bounds on the DNF’s size and width.
- Majority Function: for every constant $1/2 < \epsilon < 1$, there is a DNF of size $2^{O(\sqrt{n})}$ that can $\epsilon$-approximated the Majority Function on $n$ bits.
- Monotone Functions: every monotone function $f$ can be $\epsilon$-approximated by a DNF $g$ of size $2^{n-\Omega(n)}$ satisfying $g(x) \neq f(x)$ for all $x$.

1 Universal Bounds

Definition 1.1 Disjunctive Normal Form: a canonical normal form of a logical formula consisting of a disjunction (OR) of conjunctions (AND).

The Lupanov’s Theorem states that every Boolean function with $n$ variables can be computed by a DNF with size $2^{n-1}$ and width $n$. Then, there is a question about whether we can find a DNF with smaller size that can compute most of the input correctly. In another word, we want to use a DNF to approximate other Boolean functions.

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**Definition 1.2** \( \epsilon \)-close: The functions \( f, g : 0, 1^n \to 0, 1 \) are \( \epsilon \)-close if \(|x \in 0, 1^n : f(x) \neq g(x)| \leq \epsilon 2^n \)

**Definition 1.3** \( \epsilon \)-approximate: A DNF \( \epsilon \)-approximates to \( f : 0, 1^n \to 0, 1 \) if the function it computes is \( \epsilon \)-close to \( f \).

We are interested in the universal upper bounds of Disjunctive Normal Form for approximating any Boolean function. It is a strong argument since there are many different kinds of Boolean functions, and this theorem is applicable to all of them. There definitely are many special Boolean functions that have tighter upper bounds, which we will discuss in Section 2, 3 and 4. The following theorem gives a tight universal upper bound for all Boolean functions.

In order to compute the optimal upper bound, they constructed a two-stage process and ensure that both states happen with high probability by choosing the appropriate parameters, as shown in Theorem 1. In the first stage, the algorithm selects a random subset \( S \) of \( f^{-1}(0) \), and define a function \( g \) which equals 1 on every input in \( f^{-1}(1) \cup S \). The second stage selects a random subset of sub-cubes that are 1-monochromatic in \( g \). The union of sub-cubes corresponds to a DNF that computes a function \( h \). When \( S \) is small enough, function \( h \) is close to \( f \), and elements in \( f^{-1}(1) \) are covered by those sub-cubes.

**Theorem 1.1** Let \( \epsilon > 10/n \). Every Boolean function \( f : \{0, 1\}^n \to \{0, 1\} \) can be \( \epsilon \)-approximated by a DNF of size \( 4 \log(4/\epsilon) \times 2^n - d \) where:

\[
d = \log \log \frac{n}{\log(4/\epsilon)}(\log \log_2(\epsilon(n)))
\]

Which means, every \( f \) can be \( \epsilon \)-approximated by a DNF of size \( O(2^n/\log n) \)

Now, let’s start the proof of Theorem 1:

Assume \( \min Pr[f(x) = 0], Pr[f(x) = 1] \geq \epsilon \). Let \( g : \{0, 1\}^n \to \{0, 1\} \) be the random functions that \( g(x) = 1 \) if \( x \in f^{-1}(1) \). Then, for each \( x \in f^{-1}(0) \), set \( g(x) = 1 \) with probability \( \epsilon/2 \). Let \( G \) denote the induced distribution over all Boolean functions and apply the Chernoff bound we can get:

\[
Pr_G[Pr_{f^{-1}(0)}[f(x) \neq g(x)] \geq \epsilon] \leq e^{-\epsilon^2 
\times 2^n/3}
\]

**Definition 1.4** Special: Let \( C \) be a sub-cube, \( C \) is special if \( C \) has dimension exactly \( d \) and the \( d \) free coordinates of \( c \) are \( dk + 1, ..., dk + d \) for some \( k = 0, ..., \lfloor n/d \rfloor - 1 \).

By definition, Special sub-cubes would satisfy the following properties:

- there are \( \lfloor n/d \rfloor \times 2^n - d \) special sub-cubes
- each sub-cube is included with probability \( (\epsilon/2)^2 d \)

Let \( h : \{0, 1\}^n \to \{0, 1\} \) be the union of a random subset of the 1-monochromatic special sub-cubes in \( g \) and each special sub-cube in \( g \) is included in \( h \) with probability \( (\epsilon/2)^\lambda \), \( \lambda \) = number of \( x \) in sub-cube \( C \) such that \( f(x) = 1 \). Note that:
Let $x \in f^{-1}(1)$, the probability of $h(x) = 0$ equals to the probability that none of the special sub-cubes containing $x$ are included in $h$. Since any two special sub-cubes only intersect at $x$, so:

$$Pr_G[h(x) = 0] = (1 - (\epsilon/2)^{2^d})^{\lceil n/d \rceil} \leq \exp^{-(\epsilon/2)^{2^d} n/d} < \epsilon/4 \quad (2)$$

This shows:

$$E_G[Pr_{f^{-1}(1)}[f(x) \neq h(x)]] < \epsilon/4 \quad (3)$$

and therefore

$$Pr_G[Pr_{f^{-1}(1)}[f(x) \neq h(x)] \geq \epsilon] \geq 1/4 \quad (4)$$

Then, from the properties of special sub-cubes we can get the following:

$$E_G[DNF - size[h]] = (\epsilon/2)^{2^d} \cdot \lceil n/d \rceil \cdot 2^{n-d} \leq 2 \ln(4/\epsilon) \cdot 2^{n-d} \quad (5)$$

$$Pr_G[DNF - size[h] \geq 4 \ln(4/\epsilon) \cdot 2^{n-d}] \leq 1/2 \quad (6)$$

From Equation 1, 4 and 6, combine the result by union the bounds, we can conclude that

there exists a function $h$ such that $DNF - size[h] \leq 4 \ln(4/\epsilon) \cdot 2^{n-d}$, and $Pr[f(x) \neq h(x)] \leq \epsilon$.

Complete proving Theorem 1.

Furthermore, there is a more intuitive version of Theorem 1.

**Theorem 1.2** Every function $f$ can be 0.1-approximated by a DNF of size less or equal to $2^n / \log(n)$

To prove Theorem 2, first we can flip each 0-input to 1 independently with probability $\epsilon/2$. On this condition, error on 0-inputs is less or equal to $\epsilon$. Second, let $d = \log \log(n)$, partition $[n]$ into $n/d$ blocks of size $d$. Every $x$ is contained in $n/d$ special sub-cubes. So, we have

$$Pr[x \ is \ not \ covered] = (1 - \epsilon^{2^d})^{n/d} \leq \epsilon/4 \quad (7)$$

Third, note that each special sub-cube included with probability exactly $\epsilon^{2^d}$. Therefore,

$$Pr[\lambda] = \epsilon^{2^d} \cdot n/d \cdot 2^{n-d} \approx 2^n / \log(n) \quad (8)$$

Proved.
2 Approximating Parity Function with DNF

We are interested in whether the universal bound we showed above can apply to every Boolean functions. First, we choose Parity function and compute the upper bound of size of the DNF which approximates the Parity function.

**Definition 2.1** Parity: a parity function is a Boolean function whose value is 1 if and only if the input vector has an odd number of 1s. \( P A R_n \) refers to the parity function with \( n \) bits.

The parity function of two inputs is also known as the XOR function. The output of parity function is called the Parity bit.

First, from the Lupanov’s Theorem we can observe a fact that every function can be \( \epsilon \)-approximated by a DNF of size \((1 - \epsilon)2^{n - 1}\) and width \( n \).

Second, the theorem from Boppana-Hastad (1997) states that every DNF that can \( .01 - \) approximate Parity function has size at least \( 2^{n/16} \) and width at least \( n/16 \).

These bounds are definitely not tight enough and not within the universal bound we discussed in Section 1. So, we want to show that the universal bound can apply to the approximation of parity functions. This can be easily done by the following steps:

- Flip each 0 to X (a symbol to represent an unknown value) with a probability \( \epsilon \).
- Add all the sub-cubes of dimension \( \log \log n \) that cover only 1 and X.

Thus we know that there is a DNF of size \( O(2^n / \log n) \) that can \( \epsilon \)-approximate the parity function.

However, the universal bound is still not tight enough. Next, we want to show a tighter upper bound of approximating parity functions.

**Theorem 2.1** Parity functions can be \( \epsilon \)-approximated by a DNF of size \( 2^{(1-2\epsilon)n} \) and width \( (1-2\epsilon)n \).

Theorem 2.1 states the upper bound of a DNF that can approximate the Parity function \( P A R_n \). Based on Theorem 2.1, the size of the DNF is within the universal bound that stated in Theorem 1.1.

Let \( \epsilon = 1/4 \), we can construct a DNF Approximator for \( P A R_n \) and argue that the size of the DNF is \( 2^{n/2} \) and width is \( n/2 \). In order to do this, select an input \( x \), which \( x \) has \( n \) bits. Then partition \( x \) into two equal-length parts \( y \) and \( z \),

\[
P A R(x) = P A R(y) \oplus P A R(z)
\]  

Consider \( f(x) = P A R(y) \lor P A R(z) \), then we know that \( Pr[f(x) = P A R(x)] = 3/4 \):

- \( P A R(x) = 1 \rightarrow f(x) = 1 \)
PAR(x) = 0 → f(x) = 0 with probability 1/2

Since PAR(y) and PAR(z) have trivial DNFs of size $2^{n/2-1}$ and width $n/2$. We get the DNF with size $2^{n/2}$ and width $n/2$ that $\epsilon$-approximates $PAR_n$. Hence the construction is finished.

3 Approximating Monotone Boolean Functions

Definition 3.1 For two bitstrings $x, y \in \{0, 1\}^n$, we say that $x \preceq y$ if $x_i \leq y_i$ for all $i \in [n]$.

Monotone boolean functions are a large family of boolean functions with the requirement that $f(x) \leq f(y)$ for all $x \preceq y$. Given this requirement, a natural question is can we achieve a tighter bound than $O(2^n / \log n)$ for $\epsilon$-approximating monotone boolean functions? The answer is yes:

Theorem 3.1 Every monotone function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ can be $\epsilon$-approximated by a monotone function $g$ of DNF size $2^{n - \Omega(\sqrt{n})}$, satisfying $g(x) \leq f(x)$ for all $x \in \{0, 1\}^n$.

This is proven using two lemmas. They will need a few definitions first.

Definition 3.2 A $k$-regular DNF is a DNF where all of its terms have a width of $k$. A regular DNF is a DNF that is $k$-regular for some $k$.

Definition 3.3 A lower $\epsilon$-approximator for a function $f$ is an $\epsilon$-approximator $g$ such that $g(x) \leq f(x)$ for all $x$.

Definition 3.4 Let $f$ be a boolean function and $k \in [n]$. The density of $f$ at level $k$ is defined as $\mu_k(f) := \Pr[\|x\|=k][f(x) = 1]$. Note: if $f$ is monotone, then $\mu_k(f) \geq \mu_{k-1}(f)$.

Lemma 3.2 For any $\epsilon > 0$, every monotone function $f$ is $\epsilon$-close to the disjunction $g$ of monotone DNFs, $g(x) = g_1(x) \lor \cdots \lor g_t(x)$, where

1. $t \leq 2/\epsilon$
2. each $g_i$ is $k_i$-regular for some $k_i \in \left[\frac{n}{2} - \sqrt{n\ln(4/\epsilon)/2}, \frac{n}{2} + \sqrt{n\ln(4/\epsilon)/2}\right]$
3. the DNF size of $g_i$ is at least $(\epsilon/2)\binom{n}{k_i}$
4. $g(x) \leq f(x)$ for all $x \in \{0, 1\}^n$

Proof of Lemma 3.2
Set $l := \sqrt{\frac{n\ln(4/\epsilon)}{2}}$
For $k \in [n]$, define $f_k(x) := \lor\{ T_x :\| x \| = k \text{ and } T_x \text{ is a minterm of } f\}$. Where $T_x(y) = 1$ iff $y \succeq x$. 
By the Chernoff bound, \( \Pr_{x}[\|x\| \geq \frac{2}{3}] \leq \frac{2}{3} \).

So, \( f^*(x) := f_{\frac{2}{3} - 1}(x) \lor \cdots \lor f_{\frac{2}{3} + l}(x) \) is a lower \( \frac{2}{3} \)-approximator of \( f \).

By the triangle inequality, it suffices to show a \( g \) that is \( \frac{2}{3} \)-close to \( f^* \) which satisfies all four conditions.

We set the output of the following algorithm to \( g \):

\[
\text{for } k \in \{ \frac{2}{3} - l, \frac{2}{3} + l \} \text{ do } \quad \text{if } \Pr_{\|x\|=k}[T_x \text{ is a minterm of } f^*] < \frac{2}{3} \text{ then } \quad \text{Set } f^*(x) = 0 \text{ for all } T_x \text{ where } \|x\| = k \quad \text{end}
\]

Intuitively, this algorithm outputs a function \( g \) that is equal to \( f^* \) except with outputs set to 0 if less than a \( \frac{2}{3} \) fraction of the inputs at a given layer define a minterm \( T_x \) in \( f^* \).

The next steps will demonstrate that \( g \) fulfills all of the conditions.

**g is a \( \epsilon \)-approximator of \( f \):** At any given layer only a \( \frac{2}{3} \) fraction of the inputs are altered, meaning \( g \) is \( \frac{2}{3} \)-close to \( f^* \), making \( g \) a lower \( \epsilon \)-approximator of \( f \).

**Condition 1:** We will prove that \( g \) fulfills all four conditions.

**Condition 2:** Define each \( g_i(x) := \lor \{ T_x : \|x\| = k_i \text{ and } T_x \text{ is a min term of } g \} \).

Each \( g_i \) is \( k_i \)-regular since each \( T_x \) has width \( \|x\| \). Also, \( g_i \) only has weight when \( k_i \in [\frac{2}{3} - l, \frac{2}{3} + l] \).

**Condition 3:** Each \( g_i \) only has weight when a \( \frac{2}{3} \) fraction of the \( T_x \) are in \( g \) for \( \|x\| = k_i \). So the size of each \( g_i \) is \( \frac{2}{3} \binom{n}{k_i} \).

**Condition 4:** The only edit the algorithm makes is setting \( f^*(x) = 0 \), so \( g(x) \leq f^*(x) \) for all \( x \), meaning \( g(x) \leq f(x) \).

**Condition 1:** We will prove that \( \mu_{k_i}(g_1 \lor \cdots \lor g_t) \geq \frac{t\epsilon}{2} \) for all \( i \in [t] \). This implies condition 1 because

\[
\mu_{k_i}(g_1 \lor \cdots \lor g_t) \leq 1
\]

\[
\frac{t\epsilon}{2} \leq 1
\]

\[
t \leq \frac{2}{\epsilon}
\]

Assume without loss of generality that \( k_1 < \cdots < k_t \).

Suppose \( \mu_{k_i}(g_1 \lor \cdots \lor g_i) \geq \frac{t\epsilon}{2} \) for some \( i < t \).

Because the \( g \)'s are monotone functions, \( \mu_{k_i+1}(g_1 \lor \cdots \lor g_t) \geq \mu_{k_i}(g_1 \lor \cdots \lor g_i) \geq \frac{t\epsilon}{2} \).

To find \( \mu_{k_{i+1}}(g_1 \lor \cdots \lor g_{i+1}) \), note that the terms of \( g_{i+1} \) are disjoint from \( g_1 \lor \cdots \lor g_i \), because all of the \( g_{i+1} \) terms have width of \( k_{i+1} \). Thus:

\[
\mu_{k_{i+1}}(g_1 \lor \cdots \lor g_{i+1}) = \mu_{k_{i+1}}(g_1 \lor \cdots \lor g_i) + \mu_{k_{i+1}}(g_{i+1})
\]

\[
\geq \frac{t\epsilon}{2} + \frac{\epsilon}{2} = (i + 1)\frac{\epsilon}{2}
\]

6
Therefore, $\mu_{k+1}(g) \geq \frac{3}{4}$.
This concludes the proof of Lemma 3.2.

**Lemma 3.3** Let $f$ be a regular monotone function. For every $\epsilon > 0$ there exists a monotone DNF $g$ of size $2^{n - \Omega(\sqrt{n}/\log(n))}$ that is a lower $\epsilon$-approximator for $f$.

**Proof of Lemma 3.3**

We may assume that $\epsilon \geq \frac{C \log n}{\sqrt{n}}$ because otherwise, the size would be $2^{n - \Omega(1)}$ which is already true by Theorem 1.2.

Let $f$ be a regular monotone function for some $k \in [n]$.

Our approximator $g$ will be disjunctions of some terms $T_y$ where each $y \in f^{-1}(1)$ and $T_y(x) = 1$ for all $x \geq y$. This construction makes $g$ a lower approximator for $f$.

This proof will first divide the inputs by hamming weight and then reduce the problem to only consider a smaller subset of the inputs. Note that the hamming weight of a uniformly distributed input is the same as sampling from a binomial distribution.

By the Chernoff bound, $\Pr_x[\| x \| \geq \frac{n}{2} + t \sqrt{\frac{n \ln(3/\epsilon)}{2}}] \leq e^{-t^2/2}$. Setting $t = \sqrt{2 \ln(\frac{1}{\epsilon})}$, we get $\Pr_x[\| x \| \geq \frac{n}{2} + \sqrt{\frac{n \ln(3/\epsilon)}{2}}] \leq \frac{\epsilon}{3}$.

By the anti-concentration of the Binomial, for an interval $I \subseteq [0, n]$ of width at most $\epsilon \sqrt{n}$, we have $\Pr_x[\| x \| \in I] \leq 2\epsilon$. Using an interval of $[k, k + \frac{\epsilon \sqrt{n}}{6}]$, $\Pr_x[\| x \| \in [k, k + \frac{\epsilon \sqrt{n}}{6}]] \leq \frac{\epsilon}{3}$.

Notice that if $\| x \| < k$, then $f(x) = 0$ because $f$ is regular. Also, if our approximator outputs 0 whenever $\| x \| \geq \frac{n}{2} + \sqrt{\frac{n \ln(3/\epsilon)}{2}}$ or $\| x \| \in [k, k + \frac{\epsilon \sqrt{n}}{6}]$, it will be wrong with an extra probability of at most $\frac{2\epsilon}{3}$. So for the remaining interval, $A := \{ x \in \{0, 1\}^n : \| x \| \in [k + \frac{\epsilon \sqrt{n}}{6}, \frac{n}{2} + \sqrt{\frac{n \ln(3/\epsilon)}{2}}]\}$, $\Pr_x[\| x \| \in A]$.

For $l \in [n - k]$, we say $S_l$ is the set of 1-inputs with hamming weight $k + l$.

For each $l \geq \frac{\epsilon \sqrt{n}}{6}$, there exists a monotone DNF $g_l$ satisfying:

(i) minterms of $g_l$ are of the form $T_y$ for $y \in S_{l/2}$.

(ii) size of $g_l = O(2^{n - l/2}) \leq 2^{n - \Omega(\sqrt{n})}$
(iii) \( \Pr_{x \in S_l} [g_l(x) = 0] \leq \frac{\epsilon}{9} \)

Then by setting \( g \) to be the disjunction of all \( g_l \) where \( k + l \in [k + \frac{\epsilon \sqrt{n}}{6}, \frac{9}{2} + \sqrt{\frac{n \ln(3/\epsilon)}{2}}] \), the size of \( g \) will be at most \( n * 2^{n-\Omega(\epsilon \sqrt{n})} \leq 2^n - \Omega(\epsilon \sqrt{n} - \log n) \). And by (iii), \( \Pr_{x \in A} [g(x) \neq f(x)] \leq \frac{\epsilon}{9} \), which would complete the proof.

We generate each \( g_l \) by sampling from the following distribution \( D \). For each \( y \in S_{l/2} \), include \( T_y \) as a minterm of \( g_l \) with probability \( p := 2^{-l/2} \). Now we will show that this construction obeys the three conditions.

(i) is satisfied by the definition.

(ii) The size of the term \( g_l \) follows a binomial distribution, so \( E_{g_l \sim D} [g_l \text{ size}] = p * |S_{l/2}| < p2^n = 2^{n-l/2} \). By Markov’s inequality, set \( a = 3 * 2^{n-l/2} \) and

\[
\Pr\{X \leq a\} \geq 1 - \frac{E[X]}{a}
\]

\[
\Pr\{g_l \text{ size} \leq 3 * 2^{n-l/2}\} \geq \frac{2}{3}
\]

So with positive probability, the size is upper bounded by \( O(2^{n-l/2}) \).

(iii) Take any \( x \in S_l \), there must exist a \( z \in S_0 \) such that \( z \prec x \) because the \( z \) will correspond to a minterm in \( f \) which all have hamming weight \( k \). Also, there are \( \binom{l}{l/2} \) many \( y \in S_{l/2} \) where \( z \prec y \prec x \) because \( x \) is 1 on exactly \( l \) bits, and \( y \) is 1 on exactly \( l \) of its bits.

The probability of a \( g_l \) sampled from \( D \) outputting \( g_l(x) = 0 \) is at most the probability of picking none of the \( y \in S_{l/2} \) described above. So, \( P_{g_l \sim D} [g_l(x) = 0] \leq (1 - p)^{\Theta(2^{l/\sqrt{12}})} \)

\[
eq e^{-\Omega(2^{l/2}/\sqrt{12})}
\]

\[
< e^{-\Omega(2^{\sqrt{n}/12})/\sqrt{n}} < \frac{\epsilon}{9}
\]

Therefore, \( E_{g_l \sim D} [\Pr_{x \in S_l} [g_l(x) = 0]] \leq \frac{\epsilon}{9} \). And by Markov’s inequality, \( \Pr_{g_l \sim D} [\Pr_{x \in S_l} [g_l(x) = 0] \leq \frac{\epsilon}{9}] \geq 1 - \frac{\epsilon}{9} \). So in all, the probability that \( g_l \) is the correct size (ii) and is correct on enough inputs (iii) is the inverse of:

\[
\Pr\{\text{not satisfied} \lor (\text{iii) not satisfied}\} \leq \Pr\{\text{not satisfied}\} + \Pr\{\text{not satisfied}\} \leq \frac{1}{3} + \frac{1}{3} = \frac{2}{3}
\]

So the probability that both (ii) and (iii) are satisfied is \( \geq 1 - \frac{2}{3} = \frac{1}{3} \).

Thus, there is a positive probability that \( g_l \) satisfies all three conditions which means it exists.

This concludes the proof of Lemma 3.3.
3.1 Proof of Theorem 3.1

By Lemma 3.2, every monotone $f$ has a lower $\epsilon$-approximator $g(x) = g_1(x) \lor \ldots g_t(x)$ where $t \leq 2/\epsilon$ and each $g_i(x)$ is a regular monotone function.

By Lemma 3.3, each $g_i(x)$ has a lower $\epsilon/2t$-approximator $h_i$ of size $2^{n - \Omega((\epsilon \sqrt{n}/t) - \log(n))}$.

By using the union bound, we get:

$$\Pr_x[g(x) \neq h(x)] = \Pr_x[h_1(x) \neq g_1(x) \lor \cdots h_t(x) \neq g_t(x)] \leq \frac{t \epsilon}{2t} = \frac{\epsilon}{2}$$

By using the triangle inequality, we get:

$$\Pr_x[h(x) \neq f(x)] \leq \Pr_x[g(x) \neq f(x)] + \Pr_x[h(x) \neq g(x)] = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

And so $h$ is a lower $\epsilon$-approximator of $f$ with size $\leq t \cdot 2^{n - \Omega((\epsilon \sqrt{n}/t) - \log(n))} = 2^{n - \Omega(\sqrt{n})}$.

This concludes the proof of Theorem 3.1.

4 Approximating Majority Function with DNF

Majority Function $\text{Maj}_n$ is one of the most common Boolean functions that used in computer science. $\text{Maj}_n$ can also be approximated by DNF. We are interested in what is the upper bound of the size of the DNF that can approximate Majority Function, and whether this upper bound is inside the universal bound that we proved in Section 1.

First, what is the upper bound of the DNF that approximates Majority Function $\text{Maj}_n$?

**Theorem 4.1** The DNF of size $2^{O(\sqrt{n}/\epsilon)}$ can $\epsilon$-approximates Majority on $n$ bits $\text{Maj}_n$.

In order to prove this theorem, we give a construction of a DNF whose size is $2^{O(\sqrt{n}/\epsilon)}$ and show this DNF approximates $\text{Maj}_n$. The construction is inspired by the random DNF construction of Talagrand.

**Theorem 4.2** For all $\epsilon \geq 1/\sqrt{n}$, there is a DNF of width $w = 1/\epsilon \sqrt{n}$ and size $(ln2)2^w$ that $O(\epsilon)$ – approximates Majority with $n$ bits $\text{Maj}_n$.

To prove this theorem, let $D$ be a randomly chosen DNF with $(ln2)2^w$ terms, where each term is chosen by picking $w$ variables independently with replacement. Then, to prove Theorem 4.2, we can just show that

$$E_D[\Pr_x[D(x) \neq \text{Maj}(x)]] \leq O(\epsilon) \quad (10)$$

This is equivalent with showing

$$E_x[\Pr_D[D(x) \neq \text{Maj}(x)]] \leq O(\epsilon) \quad (11)$$
Let $t \in [-\sqrt{n}, \sqrt{n}]$, given a string $x \in 0, 1^n$, the fraction of 1's in $x$ is $1/2 + 1/2(t/\sqrt{n})$ because of the Central Limit Theorem. Since $\text{Maj}(x) = 1$ if and only if $t > 0$, by construction, $Pr_D[D(x) = 1]$ only depends on $t$. So we have:

$$Pr_D[D(x) = 1] = 1 - (1 - 2^{-2}(1 + t/\sqrt{n})^w)^{(\ln 2)^2} \quad (12)$$

Now, in order to prove Theorem 4.2, it is sufficient to show that

$$E_x[(1 - 2^{-w}(1 + t/\sqrt{n}))^{(\ln 2)^2} | t > 0] \leq O(\epsilon)$$

and

$$E_x[1 - (1 - 2^{-w}(1 + t/\sqrt{n}))^{(\ln 2)^2} | t > 0] \leq O(\epsilon)$$

Note the fact that $(1 - x)^y \leq \exp(-xy)$ and $(1 - x)^y \geq 1 - xy$, using $w = 1/\epsilon \sqrt{n}$, we can get

$$(1 - 2^{-w}(1 + t/\sqrt{n}))^{(\ln 2)^2} \leq (1/2)^{(1+t/\sqrt{n})^w} \leq (1/2)^{1+t/\epsilon}$$

and

$$1 - (1 - 2^{-w}(1 + t/\sqrt{n}))^{(\ln 2)^2} \leq (\ln 2) \exp(wt/\sqrt{n}) = (\ln 2) \exp(t/\epsilon)$$

Then there is only one thing remains to show, which is $E_x[\exp(-|t|/\epsilon)] \leq O(\epsilon)$. Note the fact that for each $i = 0, 1, 2, ..., \frac{\log(n)}{2\sqrt{n}} (\epsilon)$, we get

$$Pr[|t| \in [2^i \epsilon, 2^{i+1} \epsilon]] = Pr[|\text{Normal}(0, 1)| \in [2^i \epsilon, 2^{i+1} \epsilon]] + O(1/\sqrt{n})$$

where $O(1/\sqrt{n})$ is negligible since $\epsilon \geq 1/\sqrt{n}$. Using $Pr[|t| \in [0, \epsilon]] \leq O(\epsilon)$, we get

$$E_x[\exp(-|t|/\epsilon)] \leq O(\epsilon) + \sum_{n=1}^{\infty} \exp(-2^n)O(2^n \epsilon) \leq O(\epsilon) \quad (13)$$

Hence, we finished the proof.

After we proved the upper bound of DNF that approximates $\text{Maj}_n$, we are going to show that this upper bound is inside the universal upper bound $O(2^{\sqrt{n}/\log(n)})$. As we proved in this section, the DNF that approximates Majority has an upper bound of size $2^{O(\sqrt{n}/\epsilon)}$. Let $a = 2^{\sqrt{n}}$, $b = 2^n / \log(n)$, show that $a/b \leq 1$.

$$a/b = 2^{\sqrt{n}}/(2^n / \log(n)) = 2^{\sqrt{n}} \log(n)/2^n = \log(n)/2^{\sqrt{n}} \quad (14)$$

Clearly, when $n$ becomes larger, $\log(n)/2^{\sqrt{n}}$ becomes smaller and $\log(n)/2^{\sqrt{n}} \leq 1$. This indicates that $2^{O(\sqrt{n}/\epsilon)} \leq 2^n / \log(n)$. Hence the upper bound of DNF that approximates Majority is inside the universal bound of DNF approximation.
5 Conclusion

The Disjunctive Normal Form is a strong Boolean function that can be used to approximate other Boolean functions. The size of DNF that for approximation is always within a specific universal upper bound. By showing the DNF approximation of Parity function, Monotone function and Majority function in Section 2, 3 and 4. We can observe that the universal bound does apply to these well-known Boolean functions. In addition, the universal bound typically is not strict enough for some specific Boolean functions. For example, the Parity function and Majority both have a tighter upper bound. However, those upper bounds are specific to the Boolean function, which cannot be applied to Boolean functions in general.
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