RADIAL MULTIPLIERS AND RESTRICTION TO SURFACES OF THE FOURIER TRANSFORM IN MIXED-NORM SPACES

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Abstract. In this article we revisit some classical conjectures in harmonic analysis in the setting of mixed norm spaces $L^p_{\text{rad}}L^2_{\text{ang}}(\mathbb{R}^n)$. We produce sharp bounds for the restriction of the Fourier transform to compact hypersurfaces of revolution in the mixed norm setting and study an extension of the disc multiplier. We also present some results for the discrete restriction conjecture and state an intriguing open problem.

1. Introduction

The well-known restriction conjecture, first proposed by E. M. Stein, asserts that the restriction of the Fourier transform of a given integrable function $f$ to the unit sphere, $\hat{f}|_{S^{n-1}}$, yields a bounded operator from $L^p(\mathbb{R}^n)$, $n \geq 2$, to $L^q(S^{n-1})$ so long as

$$1 \leq p < \frac{2n}{n+1}, \quad \frac{1}{q} \geq \frac{n+1}{n-1} \left(1 - \frac{1}{p}\right).$$

This conjecture has been fully proved only in dimension $n = 2$ by C. Fefferman [7] (see also [4] for an alternative geometrical proof). In higher dimensions, the best known result is the particular case $q = 2$ and $1 \leq p \leq \frac{2(n+1)}{n+3}$, which proof was obtained independently by P. Tomas and E. M. Stein [13].

The periodic analogue, i.e. for Fourier series, was observed by A. Zygmund [16], but also in two dimensions. It asserts that for any trigonometric polynomial

$$P(x) = \sum_{|\nu|=R} a_{\nu} e^{2\pi i \nu \cdot x}, \ \nu \in \mathbb{Z}^2,$$

the following inequality holds:

$$\|P\|_{L^4(Q)} \lesssim \|P\|_{L^2(Q)},$$

uniformly on $R > 0$ and where $Q$ is any unit square in the plane.

The alternative proof given in [4] allows us to connect both the periodic and the nonperiodic restriction theorems, explaining the reason for the apparently different numerologies of the corresponding $(p, q)$ exponent ranges. It also raises an interesting question about the location of lattice points in small arcs of circles [3].

The first result in this paper goes further in that direction: Given $\{\xi_j\}$ a finite set of points in the circle $\{|\xi| = R\}$ of the plane, let us consider

$$M := \sup_j \# \left\{ \xi_k, \ |\xi_k - \xi_j| \leq R^\frac{\epsilon}{4} \right\}.$$ 

We have:

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Theorem 1. The following inequality holds

\[
\sup_{\mu(Q)=1} \left[ \int_Q \left| \sum a_k e^{2\pi i \xi_k \cdot x} \right|^4 d\mu(x) \right]^{\frac{1}{4}} \lesssim M^{\frac{1}{2}} \left( \sum |a_k|^2 \right)^{\frac{1}{2}},
\]

where the supremum is taken over all unit squares of \( \mathbb{R}^2 \) and \( \mu \) corresponds to the Lebesgue measure.

The corresponding result in higher dimensions (\( n \geq 3 \)) is an interesting open problem:

Conjecture 2. Let \( \{\xi_j\} \subset S^{n-1}_R \) and \( M := \sup_j \# \left\{ \xi_k, \| \xi_k - \xi_j \| \leq R^\frac{2}{p} \right\} \), is it true that

\[
\sup_{\mu(Q)=1} \left[ \int_Q \left| \sum a_k e^{2\pi i \xi_k \cdot x} \right|^{\frac{2n}{n-1}} d\mu(x) \right]^{\frac{n-1}{2n}} \lesssim M^{\frac{1}{2}} \left( \sum |a_k|^2 \right)^{\frac{1}{2}}.
\]

Although there are many interesting publications by several authors throwing some light on the restriction conjecture, its proof remains open in dimension \( n \geq 3 \). One of the more remarkable improvements was B. Barcelo’s thesis \([12]\). He proved that Fefferman’s result also holds for the cone in \( \mathbb{R}^3 \). Another interesting result was given by L. Vega in his Ph.D. thesis \([14]\), where he obtained the best result in the Stein-Tomas restriction inequality when the space \( L^p(\mathbb{R}^n) \) is replaced by \( L^p_{rad} L^2_{ang}(\mathbb{R}^n) \).

Here we shall consider the restriction of the Fourier transform to other surfaces of revolution in these mixed norm spaces. Several special cases have already been treated \([8, 9]\) but we present a more general and unified proof for “all” compact surfaces of revolution:

\[ \Gamma = \{(g(z), \theta, z) \in \mathbb{R}^{n+1}, \theta \in S^{n-1}, a \leq z \leq b, 0 \leq g \in C^1(a, b)\}. \]

That is, in \( \mathbb{R}^{n+1}, n \geq 2 \), we consider cylindrical coordinates \((r, \theta, z)\) where the first components \((r, \theta)\) correspond to the standard polar coordinates in \( \mathbb{R}^n \): \( 0 < r < \infty, \theta \in S^{n-1}, \) and \( z \in \mathbb{R} \) denotes the zenithal coordinate. In this coordinate system, the \( L^p_{rad} L^2_{ang}(\mathbb{R}^{n+1}) \) norm is given by

\[
\left( \int_0^\infty r^{n-1} \left( \int_{-\infty}^{\infty} \sum_{S^{n-1}} |f(r, \theta, z)|^2 d\theta dz \right)^{\frac{2}{n}} dr \right)^{\frac{1}{2}}.
\]

We can state our result.

Theorem 3. Let \( \Gamma \) be a compact surface of revolution, then the restriction of the Fourier transform to \( \Gamma \) is a bounded operator from \( L^p_{rad} L^2_{ang}(\mathbb{R}^{n+1}) \) to \( L^2(\Gamma) \), i.e. there exists a finite constant \( C_p \) such that

\[
\left( \int_{-\infty}^{\infty} \int_{S^{n-1}} g(z)^{n-1} \sqrt{1 + g'(z)^2} \left| \hat{f}(g(z), \theta, z) \right|^2 d\theta dz \right)^{\frac{1}{2}} \lesssim C_p \left\| f \right\|_{L^p_{rad} L^2_{ang}(\mathbb{R}^{n+1})},
\]

so long as \( 1 \leq p < \frac{2n}{n+1} \).

A central point in this area is C. Fefferman’s observation that the disc multiplier in \( \mathbb{R}^n \) for \( n \geq 2 \), given by the formula

\[
\hat{T}_0 f(\xi) = \chi_{B(0,1)}(\xi) \hat{f}(\xi),
\]

where \( \chi_{B(0,1)}(\xi) = 1 \) if \( \xi \in B(0,1) \) and \( \chi_{B(0,1)}(\xi) = 0 \) otherwise, and \( \hat{T}_0 f(\xi) \) is the Fourier transform of \( f \) restricted to the unit ball. This observation leads to a method for proving restriction inequalities for other surfaces of revolution.
is bounded on $L^p(\mathbb{R}^n)$ only in the trivial case $p = 2$. However, it was later proved (see ref [6] and [10]) that $T_0$ is bounded on the mixed norm spaces $L^p_{rad}L^2_{ang}(\mathbb{R}^n)$ if and only if $\frac{2n}{n+1} < p < \frac{2n}{n-1}$. Here we extend that result to a more general class of radial multipliers.

**Theorem 4.** Let $T_m$ be a Fourier multiplier defined by

$$ (T_m f)^\wedge(\xi) := m(|\xi|) \hat{f}(\xi), $$

(1.4)

for all rapidly decreasing smooth functions $f$, where $m$ satisfies the following hypothesis:

1. $\text{Supp}(m) \subset [a, b] \subset \mathbb{R}^+$, and $m$ is differentiable in the interior $(a, b)$.
2. $\int_a^b |m'(x)| \, dx < \infty$.

$T_m$ is then bounded in $L^p_{rad}L^2_{ang}(\mathbb{R}^n)$ so long as $\frac{2n}{n+1} < p < \frac{2n}{n-1}$.

Finally, let us observe that Theorem 4 admits different extensions taking into account Littlewood-Paley theory. Some vector valued and weighted inequalities are satisfied by $T_0$ and the so called universal Kakeya maximal function acting on radial functions.

2. Restriction in the discrete setting

**Proof of Theorem 4.** First let us observe that, by an easy argument, we can assume $M = 1$ without loss of generality. Next we take a smooth cut-off $\varphi$ so that

$$ \begin{align*}
\varphi &\equiv 1 \text{ on } B \left(0, \frac{1}{2}\right), \\
\varphi &\equiv 0 \text{ when } ||x|| \geq 1, \\
\varphi &\in C_0^\infty(\mathbb{R}^2).
\end{align*} $$

We can then write

$$ \begin{align*}
f(\xi) &= \sum_k a_k \varphi(\xi + \xi_k) e^{2\pi i \xi \cdot q} \\
&= \sum_k a_k \varphi_k(\xi) e^{2\pi i \xi \cdot q},
\end{align*} $$

where $q$ is a point in $\mathbb{R}^2$. We have

$$ \hat{f}(x) = \sum_k a_k \hat{\varphi}(x - q) e^{2\pi i \xi \cdot (x - q)}. $$

Note that the $L^4$ norm of $\hat{f}$ majorizes the left hand side of (1.1),

$$ \int |\hat{f}(x)|^4 \, dx \geq \int_{x - q \in Q_0} \left| \sum_k a_k e^{2\pi i \xi \cdot (x - q)} \hat{\varphi}(x - q) \right|^4 \, dx $$

$$ \gtrsim \int_Q \left| \sum_k a_k e^{2\pi i \xi \cdot x} \right|^4 \, dx, $$

where $Q_0 = \left[-\frac{1}{2}, \frac{1}{2}\right]^2$ and $Q = q + Q_0$. 

On the other hand, we have
\[
\int |f(x)|^4 \, dx = \int |f * f(\xi)|^2 \, d\xi
\]
\[
= \int \left| \sum_{k,j} a_k a_j \varphi_k * \varphi_j(\xi) e^{i\xi \cdot q} \right|^2 \, d\xi.
\]
Furthermore, because the supports of \(\varphi_k\) and \(\varphi_j\) have a finite overlapping, uniformly on the radius \(R\).
\[
\int |f(x)|^4 \, dx \lesssim \left( \sum |a_k|^2 \right)^2,
\]
q.e.d. \(\square\)

Using similar arguments we can obtain the following analogous result: In \(\mathbb{R}^2\) let us consider the parabola \(\gamma(t) = (t, t^2)\) and a set of real numbers \(\{\xi_j\}\) so that \(|t_{j+1} - t_j| \geq 1\), then
\[
\sup_{\mu(Q) = 1} \left\| \sum_j a_j e^{2\pi i \gamma(t_j) \cdot x} \right\|_{L^4(Q)} \lesssim \left( \sum |a_j|^2 \right)^{\frac{1}{4}}.
\]
An interesting open question is to decide if the \(L^4\) norm could be replaced by an \(L^p\) norm \((p > 4)\) in the inequality above. It is known that \(p = 6\) fails, but for \(4 < p < 6\) it is, as far as we know, an interesting open problem \([2]\).

3. The restriction conjecture in mixed norm spaces

Recall that in \(\mathbb{R}^{n+1}\) we establish cylindrical coordinates \((r, \theta, z)\), where \((r, \theta)\) corresponds to the usual spherical coordinates in \(\mathbb{R}^n\) and \(z \in \mathbb{R}\) denotes the zenithal component. We will also use the notation \((\rho, \phi, \zeta)\) to refer to the same coordinate system.

The \(L^p_{\text{rad}} L^2_{\text{zen}} L^2_{\text{ang}} (\mathbb{R}^{n+1})\) norm is therefore given by
\[
\|f\|_{L^p_{\text{rad}} L^2_{\text{zen}} L^2_{\text{ang}} (\mathbb{R}^{n+1})} = \left( \int_0^{\infty} r^{n-1} \left( \int_{\mathbb{S}^{n-1}} |f(r, \theta, z)|^2 \, d\theta \, dz \right)^{\frac{p}{2}} \, dr \right)^{\frac{1}{p}}. \tag{3.1}
\]

Let \(g\) be a continuous positive function supported on a compact interval \(I\) of the real line that is almost everywhere differentiable, and consider the surface of revolution in \(\mathbb{R}^{n+1}\) given by
\[
\Gamma := \{(g(z), \theta, z) \in \mathbb{R}^{n+1}, \theta \in \mathbb{S}^{n-1}, -\infty < z < \infty\}. \tag{3.2}
\]
We are interested in the restriction to \(\Gamma\) of the Fourier transform of functions in the Schwartz class \(\mathcal{S}(\mathbb{R}^{n+1})\). The restriction inequality
\[
\|\hat{f}\|_{L^2(\Gamma)} \leq C_p \|f\|_{L^p_{\text{rad}} L^2_{\text{zen}} (\mathbb{R}^{n+1})}
\]
for \(1 \leq p < \frac{2n}{n+1}\) is, by duality, equivalent to the extension estimate:
\[
\left\| \hat{f} \right\|_{L^q_{\text{rad}} L^2_{\text{zen}} (\mathbb{R}^{n+1})} \leq C_q \|f\|_{L^2(\Gamma)}
\]
for \(q > \frac{2n}{n-1}\).
To compute $\hat{fd}_\Gamma$, let us recall

$$d_\Gamma = g(z)^{n-1} \sqrt{1 + (g'(z))^2} dz d\theta$$

so that

$$\hat{fd}_\Gamma (\rho, \phi, \zeta) = \int_{-\infty}^{\infty} \int_{S^{n-1}} G_1(z) f(z, \theta) e^{-iz\zeta} e^{-i(\rho g(z))\theta} d\theta dz.$$ (3.3)

Next we use the spherical harmonic expansion

$$f(z, \theta) = \sum_{k,j} a_{k,j}(z) Y_{j}^k(\theta),$$

where for each $k$, $\{Y_{j}^k\}_{j=1,...,d(k)}$ is an orthonormal basis of the spherical harmonics degree $k$. We then obtain:

$$\hat{fd}_\Gamma (\rho, \phi, \zeta) = \sum_{k,j} 2\pi i^k Y_{j}^k(\phi) \rho^{-\frac{n-2}{2}} \int_{-\infty}^{\infty} g(z)^{\frac{n}{2}} \left( 1 + (g'(z))^2 \right)^{\frac{1}{2}} \cdot a_{k,j}(z) J_{k+\frac{n-2}{2}}(\rho g(z)) e^{-iz\zeta} dz,$$

where $J_{\nu}$ denotes Bessel’s function of order $\nu$ (see ref. [15]). Denoting by $G_2(z) := g(z)^{\frac{n}{2}} \left( 1 + (g'(z))^2 \right)^{\frac{1}{2}}$, the Fourier transform $\hat{fd}_\Gamma$ becomes

$$\sum_{k,j} 2\pi i^k Y_{j}^k(\phi) \rho^{-\frac{n-2}{2}} \int_{-\infty}^{\infty} G_2(z) a_{k,j}(z) J_{k+\frac{n-2}{2}}(\rho g(z)) e^{-iz\zeta} dz.$$ (3.4)

Taking into account the orthogonality of the elements of the basis $\{Y_{j}^k\}$ together with Plancherel’s Theorem in the $z$-variable, we obtain that the mixed norm $\|\hat{fd}_\Gamma\|^q_{L^q_{\rho,2,2}}$ is up to a constant equal to

$$\int_0^{\infty} \rho^{-\frac{n-2}{2} + n - 1} \left( \sum_{k,j} g(\zeta)^n \left| 1 + (g'(\zeta))^2 \right| a_{k,j}(\zeta)^2 |J_{\nu_k}(\rho g(\zeta))|^2 d\zeta \right)^{\frac{q}{2}} dp,$$

where $\nu_k = k + \frac{n-2}{2}$. On the other hand we have

$$\int_{\Gamma} |f|^2 = \int_{-\infty}^{\infty} \int_{S^{n-1}} \sum_{j,k} a_{k,j}(z) Y_{j}^k(\theta)^2 \cdot g(z)^{n-1} \sqrt{1 + g'(z)^2} d\theta dz$$

$$\sum_{j,k} \int_{-\infty}^{\infty} |a_{k,j}(z)|^2 g(z)^{n-1} \sqrt{1 + g'(z)^2} dz.$$ (3.6)

Therefore our theorem will be a consequence of the following fact:
Lemma 5. Given any sequence of positive indices \( \{\nu_j\} \) with \( \nu_j \geq \frac{n-2}{2} \) for all \( j \) and Schwartz functions \( a_j \), the following inequality holds:

\[
\int_0^\infty \rho^{-q + \frac{n-2}{2} + n-1} \left( \sum_j \int_{-\infty}^\infty |g(z)|^n \left| 1 + (g'(z))^2 \right| |a_j(z)|^2 |J_{\nu_j}(\rho g(z))|^2 \, dz \right)^{\frac{q}{2}} \, d\rho \\
\lesssim \left( \sum_j \int_{-\infty}^\infty |g(z)|^{n-1} \left( 1 + (g'(z))^2 \right)^{\frac{q}{2}} |a_j(z)|^2 \, dz \right)^{\frac{q}{2}}, \tag{3.7}
\]

for \( q > \frac{2n}{n-1} \).

Remark 6. Taking into account the hypothesis about \( g \) we will look for estimates depending upon \( A = \sup_{x \in I} |g(x)| \) and \( B = \sup_{x \in I} |g'(x)| \), where \( I \) is the compact support of \( g \). It is also easy to see that we can reduce ourselves to consider the sums over the family of indices \( \{\nu_j\}_{j=1}^\infty \) such that \( \nu_j \geq \frac{n-2}{2} \). Therefore it is enough to show

\[
\int_0^\infty \rho^{-q + \frac{n-2}{2} + n-1} \left( \sum_j \int_{-\infty}^\infty |b_j(z)|^2 |J_{\nu_j}(\rho g(z))|^2 \, dz \right)^{\frac{q}{2}} \, d\rho \\
\lesssim \left( \sum_j \int_{-\infty}^\infty |b_j(z)|^2 \, dz \right)^{\frac{q}{2}}, \tag{3.8}
\]

for a family of smooth functions \( \{b_j\}_j \) and indexes \( \nu_j \geq \frac{n-2}{2} \).

In order to show (3.8) we will need a sharp control of the decay of Bessel functions; namely the following estimates:

Lemma 7. The following estimates hold for \( \nu \geq 1 \):

1. \( J_\nu(r) \leq \frac{1}{r^\nu} \), when \( r \geq 2\nu \).
2. \( J_\nu(r) \leq \frac{1}{r^{\nu}} \), when \( r \leq \frac{1}{2\nu} \).
3. \( J_\nu(\nu + \rho^{1/3}) \leq \frac{1}{\rho^{1/3} + \nu^{1/3}} \), when \( 0 \leq \rho \leq \frac{1}{2}\nu^{2/3} \).
4. \( J_\nu(\nu - \rho^{1/3}) \leq \frac{1}{\rho^{1/3} + \nu^{1/3}} \), when \( 1 \leq \rho \leq \frac{1}{2}\nu^{2/3} \).
5. \( J_\nu(r) \leq r^{\nu} \), as \( r \to 0 \).

These asymptotics follow by the stationary phase method as it is shown in [15], [11] and [13].

Proof of Lemma 7. To prove (3.8) we shall first decompose the \( \rho \)-integration in dyadic parts: \( [0, \infty) = [0, 1) \cup \bigcup_{n=0}^{\infty} [2^n, 2^{n+1}) \).

\[
\int_0^1 \rho^{-q + \frac{n-2}{2} + n-1} \left( \sum_j \int_{-\infty}^\infty |b_j(z)|^2 |J_{\nu_j}(\rho g(z))|^2 \, dz \right)^{\frac{q}{2}} \, d\rho \\
+ \sum_M \int_M^{2M} \rho^{-q + \frac{n-2}{2} + n-1} \left( \sum_j \int_{-\infty}^\infty |b_j(z)|^2 |J_{\nu_j}(\rho g(z))|^2 \, dz \right)^{\frac{q}{2}} \, d\rho, \tag{3.9}
\]
where $M = 2^m$, $m = 0, 1, \ldots$

For the lower integrand, we have the following splitting:

\[
\int_0^1 \rho^{-\frac{n}{n-2}+n-1} \ldots \frac{q}{2} d\rho = \int_0^1 \rho^{-\frac{n}{n-2}+n-1} \ldots \frac{q}{2} d\rho + \int_\frac{1}{2}^1 \rho^{-\frac{n}{n-2}+n-1} \ldots \frac{q}{2} d\rho
\]

\[
= I + II.
\]

In order to bound $I$ we invoke Minkowski’s inequality and property 5. of Lemma 7.

\[
I \lesssim \left[ \int_{-\infty}^{\infty} \sum_j \left( \int_0^1 \rho^{-\frac{n}{n-2}+\frac{q}{2} (n-1)} |b_j(z)|^2 |J_{\nu_j}(\rho z)|^2 \right)^{\frac{q}{2}} d\rho \right]^{\frac{q}{2}}.
\]

\[
\leq \left[ \int_{-\infty}^{\infty} \sum_j |b_j(z)|^2 A^{2\nu_j} \left( \int_0^1 \rho^{-\frac{n}{n-2}+(n-1)+q\nu_j} d\rho \right) ^{\frac{q}{2}} dz \right]^{\frac{q}{2}},
\]

where $A = \|g\|_{\infty}$. Since the sum is taken over all $\nu_j \geq \frac{n-2}{2}$, the inner integrand is well defined and we can bound

\[
I \lesssim A^{\frac{n-1}{n}} \left[ \sum_j \int_{-\infty}^{\infty} |b_j(z)|^2 dz \right]^{\frac{q}{2}}.
\]

(3.10)

The second part is similarly bounded

\[
II \lesssim \left( 1 + A^{\frac{n-1}{n}} \right) \left[ \sum_j \int_{-\infty}^{\infty} |b_j(z)|^2 dz \right]^{\frac{q}{2}}.
\]

(3.11)

Then Lemma 5 will be a consequence of the following claim:

**Claim 8.** For all $q > 4$, the following inequality holds true

\[
\int_M^{2M} \rho \left( \sum_j \int_{-\infty}^{\infty} |b_j(z)|^2 |J_{\nu_j}(\rho g(z))|^2 dz \right)^{\frac{q}{2}} d\rho
\]

\[
\lesssim M^{\frac{4-q}{2}} \left( \int_{-\infty}^{\infty} \sum_j |b_j(z)|^2 dz \right)^{\frac{q}{2}}.
\]

(3.12)

Indeed, if $q > 4$ we need only to note that

\[
\int_M^{2M} \rho^{-\frac{n}{n-2}+n-1} \left( \sum_j \int_{-\infty}^{\infty} |b_j(z)|^2 |J_{\nu_j}(\rho g(z))|^2 dz \right)^{\frac{q}{2}} d\rho
\]

\[
\lesssim M^{(n-2)(-\frac{q}{2}+1)} \int_M^{2M} \rho \left( \sum_j \int_{-\infty}^{\infty} |b_j(z)|^2 |J_{\nu_j}(\rho g(z))|^2 dz \right)^{\frac{q}{2}} d\rho,
\]
invoke our claim and sum over all dyadic intervals in (3.9):

\[
\sum_m \int_{2^m}^{2^{m+1}} \rho^{-q \frac{n-2}{n-1} + n - 1} \left( \sum_j \int_{-\infty}^{\infty} |b_j(z)|^2 |J_{\nu_j}(\rho g(z))|^2 \, dz \right) \, \frac{d\rho}{\rho}
\]

\[
\lesssim \sum_m 2^{m(2 - q + q + \frac{q}{2})} \left( \int_{-\infty}^{\infty} \sum_j |b_j(z)|^2 \, dz \right) \left( \int_{-\infty}^{\infty} \sum_j |b_j(z)|^2 \, dz \right)^{\frac{q}{2}}.
\] (3.13)

It is then a simple matter to check that the exponent is negative for 
\( q > \frac{2n}{n-1} \).

If the exponent \( q \) is however smaller, \( \frac{2n}{n-1} < q \leq 4 \), we need to use an extra trick. Note that equation (3.12) implies

\[
\int_{2^m}^{2^{m+1}} \left( \sum_j \int_{-\infty}^{\infty} |b_j(z)|^2 |J_{\nu_j}(\rho g(z))|^2 \, dz \right) \, \frac{d\rho}{\rho} \lesssim M^{1 - \frac{q}{2n} - \frac{q}{n}} \left( \int_{-\infty}^{\infty} \sum_j |b_j(z)|^2 \, dz \right)^{\frac{q}{2n}},
\]

for all \( q_1 > 4 \). Then using Hölder’s inequality and the previous inequality,

\[
\int_{M}^{2M} \left( \sum_j \int_{-\infty}^{\infty} |b_j(z)|^2 |J_{\nu_j}(\rho g(z))|^2 \, dz \right) \, \frac{d\rho}{\rho} \lesssim M^{1 - \frac{q}{2n} - \frac{q}{n}} \left( \int_{M}^{2M} \left( \sum_j \int_{-\infty}^{\infty} |b_j(z)|^2 |J_{\nu_j}(\rho g(z))|^2 \, dz \right) \, \frac{d\rho}{\rho} \right)^{\frac{q}{2n}}.
\]

Therefore, summing over all intervals, we obtain

\[
\sum_m \int_{2^m}^{2^{m+1}} \rho^{-q \frac{n-2}{n-1} + n - 1} \left( \sum_j \int_{-\infty}^{\infty} |b_j(z)|^2 |J_{\nu_j}(\rho g(z))|^2 \, dz \right) \, \frac{d\rho}{\rho}
\]

\[
\lesssim \sum_m 2^{m\left(-q \frac{n-2}{n-1} + n - 1 + 1 - \frac{q}{2}\right)} \left( \int_{-\infty}^{\infty} \sum_j |b_j(z)|^2 \, dz \right)^{\frac{q}{2}},
\]

where the exponent \( -q \frac{n-1}{2} + n \) is negative for all \( q > \frac{2n}{n-1} \).
To prove Claim 8 let us split each dyadic integrand in (3.9) in three parts corresponding to the different ranges of control of Bessel functions.

\[
\int_M^2 \rho \left( \int_{-\infty}^{\infty} \sum_{\nu_j \in I^c} |b_j(z)|^2 |J_{\nu_j}(\rho g(z))|^2 \right)^{1/2} d\rho \\
+ \int_M^2 \rho \left( \int_{-\infty}^{\infty} \sum_{\nu_j \in I^c} |b_j(z)|^2 |J_{\nu_j}(\rho g(z))|^2 \right)^{1/2} d\rho \\
+ \int_M^2 \rho \left( \int_{-\infty}^{\infty} \sum_{\nu_j \in I^c} |b_j(z)|^2 |J_{\nu_j}(\rho g(z))|^2 \right)^{1/2} d\rho \\
= \sum_M (I_M^0 + I_M^< + I_M^>) ,
\]

where \( I_M^0 = [0, M g(z)/2] \), \( I_M^< = [M g(z)/2, 4 M g(z)] \), and \( I_M^> = [4 M g(z), \infty) \).

Recall that if \( 2k < r, |J_k(r)| \leq r^{-1/2} \) in \( I_M^0 \) we have \( 2 \nu_j < M g(z) < \rho g(z) \), hence

\[
I_M^0 \leq A^{-1/2} \int_M^2 \rho^{1-2} \left( \int_{-\infty}^{\infty} \sum_{\nu_j \in I^c} |b_j(z)|^2 \right)^{1/2} d\rho \\
\leq A^{-1/2} M^{4-\frac{q}{q-2}} \left( \int_{-\infty}^{\infty} \sum_{\nu_j} |b_j(z)|^2 \right)^{1/2} . \quad (3.14)
\]

Similarly, \( I_M^< \) is also easily bounded as if \( k > 2r, |J_k(r)| \leq k^{-1} \), and in \( I_M^> \), \( k > 4 M g(z) > 2 \rho g(z) \). Furthermore, since \( \rho g(z) > 1, (\rho g(z))^{-2} < (\rho g(z))^{-1} \), and, in \( I_M^> \), we have \( |J_k(\rho g(z))|^2 \leq (\rho g(z))^{-1} \). This shows that again

\[
I_M^< \leq A^{-1/2} M^{4-\frac{q}{2}} \left( \int_{-\infty}^{\infty} \sum_{\nu_j} |b_j(z)|^2 \right)^{1/2} . \quad (3.15)
\]

Finally, we need to work a little bit harder than in the previous cases to obtain a suitable estimate for \( I_M^< \). First of all note that Minkowski’s inequality yields

\[
I_M^< \leq \left[ \int_{-\infty}^{\infty} \left( \int_M^1 \rho \left( \sum_{\nu_j \in I^c} |b_j(z)|^2 |J_{\nu_j}(\rho g(z))|^2 \right)^{1/2} d\rho \right)^{1/2} dz \right]^{1/2} . \quad (3.16)
\]

In \( I_M^< \) we want to use estimate (3) of Lemma 7 we thus need to split the inner integral so that \( \rho g(z) \sim \nu_j + \alpha \nu_j \) in the according range of \( \alpha \). Consider the family of sets

\[
G_{\alpha} = \left[ \frac{M}{2} + \alpha M^{\frac{1}{4}} g(z)^{-\frac{4}{3}}, \frac{M}{2} + (\alpha + 1) M^{\frac{1}{4}} g(z)^{-\frac{4}{3}} \right] ,
\]
for \( \alpha = 0, 1, 2, \ldots \), so that \( \bigcup G_\alpha \supseteq [M, 2M] \) and in each interval \( \rho g(z) \sim \nu_j + \alpha \nu_j \), and split in the following way

\[
I_M^c \lesssim \left[ \int_{-\infty}^{\infty} \left\{ \sum_\alpha \int_{G_\alpha} \rho \left( \sum_{\nu_j \in I_c} |b_j(z)|^2 |\mathcal{J}_{\nu_j} (\rho g(z))|^2 \right)^{\frac{q}{2}} \right\}^{\frac{q}{2}} \right]^{\frac{q}{2}},
\]

Let us also define

\[
A_\beta = \sum_{\nu_j \in G_\beta} |b_j(z)|^2.
\]

We can then invoke Lemma 7 and rearrange the sums to bound \( I_M^c \) by

\[
\left[ \int_{-\infty}^{\infty} \left\{ \sum_\alpha \int_{G_\alpha} \rho \left( \sum_{\beta \leq \alpha} A_\beta \frac{1}{(|\alpha - \beta| + 1)^{1/2} M^{\frac{q}{q+1}} g(z)^{-\frac{q}{q+1}}} \right)^{\frac{q}{2}} \right\}^{\frac{q}{2}} \rho d\rho \right]^{\frac{q}{2}} + \left[ \int_{-\infty}^{\infty} \left\{ \sum_\alpha \int_{G_\alpha} \rho \left( \sum_{\beta \geq \alpha} A_\beta \frac{1}{(|\alpha - \beta| + 1)^2 M^{\frac{q}{q+1}} g(z)^{-\frac{q}{q+1}}} \right)^{\frac{q}{2}} \right\}^{\frac{q}{2}} \rho d\rho \right]^{\frac{q}{2}}.
\]

Note that the second sum is easier to control than the first. We shall, therefore, focus on the first term, \( I_M^{c,1} \). Since the intervals \( G_\alpha \) have length \( M^{\frac{q}{q+1}} g(z)^{-\frac{q}{q+1}} \),

\[
I_M^{c,1} \lesssim M^{\frac{4-q}{q+1} A^2 (\frac{q}{q-1})^{-\frac{q}{q+1}}} \left[ \int_{-\infty}^{\infty} \left\{ \sum_\alpha \left( \sum_{\beta \geq \alpha} A_\beta \frac{1}{(|\alpha - \beta| + 1)^{2}} \right)^{\frac{q}{2}} \right\}^{\frac{q}{2}} \right]^{\frac{q}{2}}.
\]

Furthermore, using Young’s inequality, since \( q > 4 \), taking \( 2/q = 1/s - 1/2 \) we obtain

\[
\sum_\alpha \left( \sum_{\beta \geq \alpha} A_\beta \frac{1}{(|\alpha - \beta| + 1)^{2}} \right)^{\frac{q}{2}} \lesssim \left( \sum_\gamma A_\gamma \right)^{\frac{q}{2}} \lesssim \left( \sum_\gamma A_\gamma \right)^{\frac{q}{2}}.
\]

We have thus showed that the central integrand \( I_M^c \) can also be bounded in the desired way;

\[
I_M^{c,1} \lesssim A^{2 (\frac{q}{q+1})} M^{\frac{4-q}{q+1}} \left[ \int_{-\infty}^{\infty} \sum_{k \in I_M^C} |a_k|^2 \right]^{\frac{q}{2}} \text{. (3.17)}
\]

q.e.d. \( \Box \)
4. Generalized Disc Multiplier

In the late 80’s it was proved independently in \[6, 10\] that the disc multiplier operator is bounded in the mixed norm spaces $L_p^{\text{rad}}L_q^{\text{ang}}(\mathbb{R}^n)$ for all $\frac{2n}{n+1} < p < \frac{2n}{n+1}$.

Let us here explore further the theory of radial Fourier multipliers following the ideas presented in the aforementioned articles.

Let $m$ be a radial function and consider the Fourier multiplier $$(T_m f)(\xi) = m(|\xi|) \hat{f}(\xi).$$

Once again, recall the expansion of a given function $f$ in terms of its spherical harmonics,

$$f(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{d(k)} f_{k,j} (|x|) Y_k^j \left( \frac{x}{|x|} \right).$$

Then, the classical formula relating the Fourier transform and the spherical harmonics expansion, \[11\], yields

$$\hat{f}(\xi) = \sum_{k=0}^{\infty} \sum_{j=1}^{d(k)} \left( \frac{\xi}{|\xi|} \right) \left( k+\frac{n-2}{2} \right) \int_0^\infty f_{k,j}(t) J_{k+\frac{n-2}{2}} (2\pi |\xi| t) t^{\frac{k+\frac{n-2}{2}}{2}} dt.$$ 

The expression of $T_m$ in terms of its spherical harmonics expansion is then

$$T_m f(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{d(k)} 2\pi i^k \int_{\mathbb{R}^n} e^{2\pi i x \xi} m(|\xi|) Y_k^j \left( \frac{\xi}{|\xi|} \right) |\xi|^{-(k+\frac{n-2}{2})}$$

$$\int_0^\infty f_{k,j}(t) J_{k+\frac{n-2}{2}} (2\pi |\xi| t) t^{k+\frac{n-2}{2}} dt d\xi.$$ 

Exchanging the order of integration, the previous expression becomes

$$\sum_{k=0}^{\infty} \sum_{j=1}^{d(k)} 2\pi i^k \int_0^\infty f_{k,j}(t) t^{k+\frac{n-2}{2}} \hat{g}_t(x) dx,$$

where

$$g_t(\xi) = m(|\xi|) J_{k+\frac{n-2}{2}} (2\pi |\xi| t) |\xi|^{-(k+\frac{n-2}{2})} Y_k^j \left( \frac{\xi}{|\xi|} \right).$$

Therefore, computing once more the Fourier transform of a radial function,

$$T_m f(\rho \theta) = \sum_{k=0}^{\infty} \sum_{j=1}^{d(k)} 4\pi^2 (-1)^k Y_k^j (\theta) T_m^{k,j} f(r),$$

with

$$T_m^{k,j} f(r) = \int_0^\infty f_{k,j}(t) t^{\frac{n-2k-1}{2}} r^{\frac{n-2k-1}{2}} K_{k+\frac{n-2}{2}} (t, r) dt,$$

where

$$K_{\nu}(t, r) = \sqrt{rt} \int_a^b m(s) J_{\nu} (2\pi ts) J_{\nu} (2\pi rs) s ds.$$ 

In order to simplify the notation, note that

$$T_m f(\rho \theta) \approx \sum_{k=0}^{\infty} \sum_{j=1}^{d(k)} Y_k^j (\theta) T_m^{k,j} f(r) \quad (4.1)$$
with $T_m^{k,j}$ defined as before, but

$$K_\nu (t, r) = \sqrt{rt} \int_a^b m(s) J_\nu (ts) J_\nu (rs) \, ds.$$ 

Let us take a closer look at the kernel of the operator $K_\alpha$,

$$K_\alpha (t, r) = \sqrt{rt} \int_a^b m(s) J_\alpha (ts) J_\alpha (rs) \, ds. \quad (4.2)$$

It is suitable to decode these kernels in terms of an auxiliary function $U_r (s) = \sqrt{rs} J_\alpha (rs)$. The use of Bessel’s equation yields

$$\frac{\partial}{\partial s} \{ U_r (s) U'_r (s) - U_t (s) U'_t (s) \} = (t^2 - r^2) \sqrt{tr} J_\alpha (rs) J_\alpha (ts) s.$$

Therefore, after an integration by parts in $(4.2)$, we obtain

$$K_\alpha (t, r) = \left[ m(s) - \frac{1}{t^2 - r^2} \{ U_r (s) U'_r (s) - U_t (s) U'_t (s) \} \right]_a^b \right. $$

$$- \int_a^b m'(s) \frac{1}{t^2 - r^2} \{ U_r (s) U'_r (s) - U_t (s) U'_t (s) \} \, ds.$$ 

Hence, we express the modified disc multiplier in the following way

$$T_m f (r \theta) = \sum_{k,j} Y_k^j (\theta) \int_0^\infty f_{k,j} (t) t^{\frac{n-2k-1}{2}} r^{\frac{n-2k-3}{2}} k(r, t, s) \, dt, \quad (4.3)$$

where $k_\alpha (t, r, s)$ denotes the kernel $\frac{1}{t^2 - r^2} \{ U_r (s) U'_r (s) - U_t (s) U'_t (s) \}$. A simple expansion of $k_\alpha$ reveals the underlying singularities of the operator $K_\alpha$:

$$k_\alpha (t, r, s) = \left( \frac{\sqrt{t} J_\alpha (ts) J_\alpha (rs)}{2 (t+r)} + s \frac{\sqrt{t} J_\alpha (ts) J_\alpha (rs)}{2 (t+r)} \right). \quad (4.4)$$

A thorough study of the kernel $k_\alpha (r, t, 1)$ was carried out in [5] using the decay properties of Bessel functions (Lemma (7)) in order to show that the disc multiplier is bounded in the mixed norm spaces $L^p_{\text{rad}} L^2_{\text{ang}} (\mathbb{R}^n)$ in the optimal range $\frac{2n}{n+1} < p < \frac{2n}{n-1}$.

Although nothing really new has been done, we have brought to light a more general family of operators underlying the disc multiplier, that is the family of operators $T^s$ defined as

$$T^s f (r \theta) = \sum_{k,j} Y_k^j (\theta) \int_0^\infty f_{k,j} (t) t^{\frac{n-2k-1}{2}} r^{\frac{n-2k-3}{2}} k(r, t, s) \, dt. \quad (4.5)$$

Indeed, any bound on operators $T^s$ that is uniform in $s$ implies a bound on $T_m$ for a suitable $m$. 


**Proposition 9.** Let \( f \) be a rapidly decreasing function then, for every \( \frac{2n}{n+1} < p < \frac{2n}{n-1} \)

\[ \|T^s f\|_{p,2} \leq C_{p,n} \|f\|_{p,2}, \quad (4.6) \]

where the constant \( C_{p,n} \) is uniform in \( s \).

**Proof.** In order to simplify the expressions we will just write one of the four core kernels of \( k_\alpha \) apparent in \( [4.3] \), that is

\[ T^s f (r_\theta) \sim \sum_{k,j} Y_{kl}^j (\theta) \int_0^\infty f_{k,j} (t) t^{n+2k-1} r^{n+2k-1} s \sqrt{t} J'_\alpha (ts) J_\alpha (rs) \sqrt{r} \, dt, \quad (4.7) \]

for any fixed \( s \in \mathbb{R} \). The orthonormality in \( L^2 (\mathbb{S}^{n-1}) \) of spherical harmonics can now be used in our advantage to compute the \( L^p_{\text{rad}} L^2_{\text{ang}} \) norm of \( T^s \). Indeed, \( \|T^s f\|_{p,2} \) is up to the notation reduction equal to

\[ \left( \int_0^\infty r^{n-1} \left\{ \sum_{k,j} \left| \int_0^\infty f_{k,j} (t) t^{n+2k-1} r^{n+2k-1} s \sqrt{t} J'_\alpha (ts) J_\alpha (rs) \sqrt{r} \, dt \right|^2 \right\} ^{\frac{p}{2}} \right) ^{\frac{1}{p}} dt. \]

Two simple changes of variables, \( t' = st \) and \( r' = sr \), yield

\[ s^{-\frac{n-1}{p}} \left( \int_0^\infty r^{n-1} \left\{ \sum_{k,j} \left| \int_0^\infty f_{k,j} \left( t' \frac{s}{r} \right) t'^{n+2k-1} r'^{n+2k-1} s \sqrt{t'} J'_\alpha (t's) J_\alpha (rs) \sqrt{r'} \, dt' \right|^2 \right\} ^{\frac{p}{2}} \right) ^{\frac{1}{p}} dr'. \]

Note that this expression corresponds to that of the disc multiplier \( T_0 \) analyzed by in \( [4] \). We can therefore bound it by

\[ C_{p,n} s^{-\frac{n-1}{p}} \left( \int_0^\infty r^{n-1} \left\{ \sum_{k,j} \left| \int_0^\infty f_{k,j} \left( t' \frac{s}{r} \right) \right|^2 \right\} ^{\frac{p}{2}} \right) ^{\frac{1}{p}} dr', \]

for every \( \frac{2n}{n+1} < p < \frac{2n}{n-1} \). One last change of variables produces the estimate

\[ \|T^s f\|_{p,2} \leq C \|f\|_{p,2}, \]

where \( C \) is uniform on \( s \). \( \square \)

It is now a simple matter to produce a bound for the operator \( T_m \).

\[ \|T_m f\|_{p,2} \leq |m (b)| \|T^b f\|_{p,2} + |m (a)| \|T^b f\|_{p,2} + \int_a^b |m' (s)| \|T^s f\|_{p,2} \, ds, \quad (4.8) \]

and Theorem \( [4] \) follows from the uniformity in the bound \( (4.6) \). That is

\[ \|T_m f\|_{p,2} \leq C \left( \sup_{s \in [a,b]} |m (s)| + \int_a^b |m' (s)| \, ds \right) \|f\|_{p,2}. \]
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