Quantum Fourier Transform Over Galois Rings

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Abstract

Galois rings are regarded as “building blocks” of a finite commutative ring with identity. There have been many papers on classical error correction codes over Galois rings published. As an important warm-up before exploring quantum algorithms and quantum error correction codes over Galois rings, we study the quantum Fourier transform (QFT) over Galois rings and prove it can be efficiently preformed on a quantum computer. The properties of the QFT over Galois rings lead to the quantum algorithm for hidden linear structures over Galois rings.

Key Words: Galois Rings, Quantum Fourier Transform, Quantum Algorithms, Quantum Error Correction Codes

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1 Introduction

Quantum Fourier transform (QFT) is a main tool in constructing some quantum algorithms, for example, Shor’s algorithm [1]. Readers are invited to refer to the Chapter 5 in the textbook by Nielsen and Chuang [2] for the definition of the QFT and its applications. The QFT over finite fields was introduced in two papers [3, 4]: the one by De Beaudrap, Cleve and Watrous and the other by Van Dam and Hallgren. The properties of the QFT over finite fields directly give rise to quantum algorithms for hidden linear/non-linear structures over finite fields [3, 5].

A ring contains more algebraic structures than a field: every field is a ring but not every ring is a field. Galois rings [6, 7, 8] are regarded as “building blocks” of a ring. Quantum information and computation [2] over Galois rings is to be a very meaningful research topic. The QFT over Galois rings possibly leads to interesting quantum algorithms, for examples, quantum algorithms for hidden linear/non-linear structures over Galois rings (or even a ring). Quantum error correction codes [9] over Galois rings can be explored because there are many classical error correction codes over Galois rings, see [7, 8, 10] for relevant references. Hence our research on the QFT over Galois rings is an important warm-up to study quantum algorithms and quantum error correction codes over Galois rings.

The remainder of this paper is organized as follows. Section 2 collects basic facts on Galois rings used in the following sections. Section 3 defines the QFT over Galois rings and analyzes its main properties. The second proof for the lemma 3.4 comprehensively exploits various properties of Galois rings. Section 4 proves that the QFT over Galois rings can be efficiently implemented on a quantum computer. The properties of the discriminant matrix over Galois rings are discussed from different points of view. Last section remarks the QFT over a ring.

2 Preliminary on Galois rings

A ring $A$ is a set equipped with addition and multiplication. It is an abelian group with the unit 0 under addition, denoted by $(A, +)$. It is a semigroup with the unit 1 under multiplication, denoted by $(A, \cdot)$, in which an invertible element is called a unit but some elements may have no inverses. A nonzero element $a \in A$ is called a zero divisor if there is another nonzero element $b \in A$ satisfying $ab = 0$.

Our notations on Galois rings are taken from the book [8] by Wan. $R \equiv \mathbb{Z}_{p^s}$ denotes the residue class ring of integers $\mathbb{Z}$ modulo $p^s$ for a prime number $p$ and an integer $s \geq 1$, i.e.,

$$R \equiv \mathbb{Z}_{p^s} = \{0, 1, 2, \cdots, p^s - 1\}. \quad (2.1)$$
$R' \cong GR(p^s, p^{sm})$ denotes a Galois ring of characteristic $p^s$ and cardinality $p^{sm}$, where $m$ is some integer $m \geq 1$. The ring $R$ is a subring of $R'$, and the ring $R'$ is an extension of the ring $R$.

For $m = 1$, the Galois ring $R'$ corresponds to a residue class ring $R'|_{m=1} = \mathbb{Z}_{p^s}$, and for $s = 1$, it corresponds to a finite field $R'|_{s=1} = \mathbb{F}_{p^m}$.

An arbitrary element $\alpha$ of the Galois ring $R'$ can be expressed in two ways. In the additive formalism, $\alpha$ is uniquely expressed as

$$\alpha = \sum_{i=0}^{m-1} a_i \xi^i \quad \text{with} \quad a_i \in R,$$

where $\xi$ is a root of a monic basic primitive polynomial,

$$h(X) = h_0 + h_1X + \cdots + h_{m-1}X^{m-1} + X^m \in R[X]$$

of degree $m$ over $R$.

In the $p$-adic formalism, $\alpha$ is uniquely expressed as

$$\alpha = \sum_{i=0}^{s-1} t_i p^i \quad \text{with} \quad t_i \in T_{p^m} = \{0, 1, \ldots, \xi^{p^m-2}\},$$

where the set $T_{p^m}$ is referred to as the Teichmüller set.

In the $p$-adic formalism (2.3), $\alpha$ is a unit if and only if $t_0 \neq 0$, and it is a zero divisor or 0 if and only if $t_0 = 0$.

**Lemma 2.5.** Given an arbitrary zero divisor $\alpha$ of the Galois ring $R'$, it can be expressed as $\alpha = p^j \alpha'$, $1 \leq j \leq s - 1$ where $\alpha'$ is a unit of $R'$.

**Proof.** In the $p$-adic formalism, the zero divisor $\alpha \in R'$ has a unique form

$$\alpha = t_j p^j + \cdots + t_{s-1} p^{s-1} = p^j \alpha', \quad t_j \neq 0, \quad 1 \leq j \leq s - 1$$

where $\alpha' = t_j + \cdots + t_{s-1} p^{s-j-1}$ is a unit of $R'$.

The Frobenius automorphism of the Galois ring $R'$ over $R$ is a map $\phi$ uniquely defined by $\phi(\xi) = \xi^p$ and $\phi(r) = r$ for all $r \in R$. Define a composition of the Frobenius automorphism $\phi$ as

$$\phi^{i+1} = \phi \circ \phi, \quad \phi^0 = 1, \quad i = 0, 1, 2, \ldots, m - 1.$$

Observe that $\phi^m = 1$. The Galois group $\text{Gal}(R'/R)$ of the ring extension $R'/R$ is the cyclic group $\langle \phi \rangle$ of order $m$ generated by $\phi$. The trace $\text{Tr} \equiv \text{Tr}_{R'/R}$ of this ring extension is defined by

$$\text{Tr} : R' \to R, \quad \text{Tr}(\alpha) = \sum_{\phi \in \text{Gal}(R'/R)} \varphi(\alpha).$$
It satisfies the following properties:

\[
\text{Tr}(\alpha + \beta) = \text{Tr}(\alpha) + \text{Tr}(\beta), \quad \text{Tr}(\phi(\alpha)) = \text{Tr}(\alpha),
\]
\[
\text{Tr}(\alpha) = \alpha \text{Tr}(a), \quad \text{Tr}(a) = ma \quad \text{for all } a \in R.
\]

Hence the trace mapping \(\text{Tr}\) is a surjective homomorphism from the additive group \((R', +)\) to the additive group \((R, +)\).

3 The QFT over the Galois ring \(R'\)

We introduce the QFT over the Galois ring \(R'\), study its main properties, and present two types of proofs for the lemma 3.4.

3.1 Notations

The Galois ring \(R'\) is a module over its subring \(R\), namely \(R \times R' \to R'\), see the book [11] by Shoup for our notations on module and matrix over a ring.

In the additive formalism \((2.2)\) of the Galois ring \(R'\), the set \(\{\xi_i\}_{i=0}^{m-1}\) forms a basis of this module on \(R\), and its element \(x \in R'\) has a simpler notation \(x = \bar{x}^T \cdot \bar{\xi}\) with the row vector \(\bar{x}^T \in R^{1 \times m}\) and the column vector \(\bar{\xi} \in R^{m \times 1}\),

\[
\bar{x}^T = (x_0, \ldots, x_{m-1}), \quad \bar{\xi}^T = (\xi^0, \ldots, \xi^{m-1})
\]

where \(T\) denotes the matrix transpose.

The set \(\{|x_i\}_{x_i \in R}\) of all Dirac kets is an orthonormal basis of the Hilbert space \(C^{p^s}\), and hence the set of all \(m\)-tuple tensor products of Dirac kets \(|x_i\rangle\), \(i = 1, \ldots, m - 1\), denoted by

\[
|x\rangle \equiv |x_0\rangle \otimes |x_1\rangle \otimes \cdots \otimes |x_{m-1}\rangle, \quad x \in R'
\]
gives rise to an orthonormal basis of the Hilbert space \((C^{p^s})^\otimes m\). The set \(\{|x\rangle\}\) satisfies

\[
\sum_{x \in R'} |x\rangle \langle x| = I_{d_{ps}^m}
\]

where \(I_{d_{ps}^m}\) denotes the \(p^s\)-dimensional identity.

Introduce a character \(\chi_\alpha\) for the finite abelian group \((R', +)\) by

\[
\chi_\alpha(u) = (\omega_{ps})^{\text{Tr}(\alpha u)}, \quad \omega_{ps} = e^{i\frac{2\pi}{ps}}, \quad u \in R'
\]

with the multiplication law given by \(\chi_\alpha \circ \chi_\beta \equiv \chi_{\alpha + \beta}, \alpha, \beta \in R'\). It satisfies

\[
\chi_\alpha(u + v) = \chi_\alpha(u)\chi_\alpha(v), \quad u, v \in R',
\]

and so it is a group homomorphism from the additive group \((R', +)\) to the multiplicative semigroup \((R', \cdot)\).

Let \(\mathcal{F}_{R'}\) denote the QFT over the Galois ring \(R'\). We use the additive character \(\chi_\alpha\) \((3.2)\) to define \(\mathcal{F}_{R'}\) as

\[
\mathcal{F}_{R'} \equiv \frac{1}{\sqrt{ps^m}} \sum_{\alpha, u \in R'} \chi_\alpha(u)\langle \alpha|u\rangle.
\]

At \(m = 1\), \(\mathcal{F}_{R'}\) is the QFT \(\mathcal{F}_R\) over \(R = \mathbb{Z}_{p^s}\) [2]. At \(s = 1\), \(\mathcal{F}_{R'}\) is the QFT over the finite field \(\mathbb{F}_{ps}\) [3, 4].

4
3.2 Properties of the QFT over $R'$

We describe the properties of the $F_{R'}$ (3.3) in three corollaries of the lemma (3.4).

**Lemma 3.4.** Let $R'$ denote the Galois ring $GR(p^s, p^{sm})$ and $\chi_\alpha$ denote the additive character, $\alpha \in R'$. Then the character $\chi_\alpha(u)$ has the property

$$\sum_{u \in R'} \chi_\alpha(u) = p^{sm} \delta_{\alpha, 0} = \begin{cases} p^{sm}, & \alpha = 0 \\ 0, & \alpha \neq 0 \end{cases}.$$  \hspace{1cm} (3.5)

**Proof.** It is obvious for $\alpha = 0$. For $\alpha \neq 0$, there exists an element $v \in R'$ satisfying $\chi_\alpha(v) \neq 1$, and we have

$$\sum_{u \in R'} \chi_\alpha(u) = \sum_{u \in R'} \chi_\alpha(u + v) = \chi_\alpha(v) \sum_{u \in R'} \chi_\alpha(u) \implies \sum_{u \in R'} \chi_\alpha(u) = 0$$

since $\chi_\alpha$ (3.2) is a nontrivial character of the additive group $(R', +)$. \hfill \Box

**Corollary 3.6.** The set of all $p^{sm}$-dimensional normalized vectors,

$$\tilde{\chi}_\alpha = \frac{1}{\sqrt{p^{sm}}} (\chi_\alpha(u))_{u \in R'}, \quad \alpha \in R'$$

is an orthonormal basis of the Hilbert space $\mathbb{C}^{p^{sm}}$, and we have

$$\tilde{\chi}_\alpha(\tilde{\chi}_\beta) = \frac{1}{p^{sm}} \sum_{u \in R'} \chi_\alpha(u) \chi_\beta^*(u) = \delta_{\alpha, \beta}, \quad \alpha, \beta \in R'$$

where $\dagger$ denotes the Hermitian conjugation.

Hence the $p^{sm} \times p^{sm}$ matrix $\frac{1}{\sqrt{p^{sm}}} (\chi_\alpha(u))_{\alpha, u \in R'}$ is a unitary matrix, and the QFT over $R'$, $F_{R'}$ (3.3) is a unitary transformation, namely,

$$F_{R'} F_{R'}^\dagger = F_{R'}^\dagger F_{R'} = I_{p^{sm}}$$

in the Hilbert space $(\mathbb{C}^{p^s})^{\otimes m}$.

**Corollary 3.7.** The shift operator $S_\alpha$ on the Galois ring $R'$, defined by

$$S_\alpha \equiv \sum_{u \in R'} |u + \alpha\rangle \langle u|, \quad \alpha \in R',$$

is diagonalized by the QFT over $R'$, $F_{R'}$ (3.3), namely,

$$F_{R'} S_\alpha F_{R'}^\dagger = \sum_{u \in R'} \chi_\alpha(u) |u\rangle \langle u|. $$
Proof. After some algebra, we have
\[
\mathcal{F}_{R'}^* S_\alpha \mathcal{F}_{R'}^+ = \frac{1}{p^{sm}} \sum_{u, v, t \in R'} \chi_\alpha(u) \chi_{u-t}(v) |u\rangle\langle t|
\]
and then prove the corollary with the lemma 3.4. □

Corollary 3.9. Let \( A_r \) and \( B_r \) denote the control additive gates: \( A_r|x\rangle|y\rangle \equiv |x\rangle|y + rx\rangle \) and \( B_r|x\rangle|y\rangle \equiv |x + ry\rangle|y\rangle \), \( x, y, r \in R' \). Then they have the control/target inversion property given by
\[
(\mathcal{F}_{R'}^+ \otimes \mathcal{F}_{R'}^*) A_r(\mathcal{F}_{R'} \otimes \mathcal{F}_{R'}^+ ) = B_r
\]
Proof. We use the same methodology [3] of proving the control/target inversion property for control additive gates over finite fields. The proof is an application of the corollary 3.7. □

De Beaudrap and coauthors [3] introduced control additive gates over finite fields and realized that the control/target inversion property derives the quantum algorithm for hidden linear structures over finite fields. Therefore the corollary 3.9 leads to the same quantum algorithm for hidden linear structures over Galois rings, see [3] for this algorithm.

3.3 The second proof for the lemma 3.4

The proof for the lemma 3.4 is based on the fact that \( \chi_\alpha \) (3.2) is the character of the additive group \((R', +)\). On the other hand, the additive character \( \chi_\alpha \) (3.2) contains information on the multiplicative semigroup \((R', \cdot)\), and therefore the lemma 3.4 can be proved only with properties of the Galois ring \( R' \).

Given an element \( \alpha \in R' \), it is either 0 or 1 or a non-identity unit or a zero divisor. Hence we prove the lemma 3.4 in the following four steps.

Denote \( \chi(\alpha u) \equiv \chi_\alpha(u) \), \( u \in R' \).

1). \( \alpha = 0 \). We have \( \chi(0) = 0 \) then \( \sum_{u \in R'} 1 = p^{sm} \) to prove the lemma.

2). \( \alpha = 1 \). The trace mapping \( Tr \) over the Galois ring \( R' \) relative to \( R \) is a surjective additive group homomorphism from \((R', +)\) to \((R, +)\). Denote the kernel of this homomorphism by
\[
ker(Tr) = \{ v \in R' | Tr(v) = 0 \}
\]
and then the quotient group \( R'/ker(Tr) \) is isomorphic to \( R = \mathbb{Z}_{p^s} \). The isomorphism gives rise to a partition of the Galois ring \( R' \) as a disjoint union of the kernel \( ker(Tr) \) and cosets \( (z_i + Ker(Tr)) \) with \( Tr(z_i) = i \), \( i = 1, \cdots, p^s - 1 \). This partition derives the cardinality of \( ker(Tr) \) or \( (z_i + ker(Tr)) \) as \( p^{(m-1)s} \). Hence we have
\[
\sum_{u \in R'} \chi(u) = p^{(m-1)s} \sum_{i=0}^{p^s-1} (\omega_{p^s})^i = 0.
\]

(3.10)
3). As \( \alpha \) is a unit, the mapping \( u \mapsto v = \alpha u \) is bijective due to the existence of \( \alpha^{-1} \), and we have
\[
\sum_{u \in R'} \chi(\alpha u) = \sum_{v \in R'} \chi(v) = 0
\]
which uses the statement in the step 2).

4). As \( \alpha \) is a zero divisor, with the lemma 2.5, it has the form of \( \alpha = p^j \alpha' \), \( 1 \leq j \leq s - 1 \), where \( \alpha' \) is a unit of the Galois ring \( R' \). We have
\[
\sum_{u \in R'} \chi(\alpha u) = \sum_{u \in R'} (\omega_{p^s})^{Tr(\alpha' u)} = \sum_{v \in R'} (\omega_{p^s})^{Tr(v)} = 0 \tag{3.11}
\]
which exploits the steps 2) and 3).

A nice additive character for the additive group of a ring has properties of its multiplicative semigroup so that the related QFT over this ring has an efficient implementation on a quantum computer. The second proof for the lemma 3.4 and Section 4 suggest the character \( \chi_\alpha \) (3.2) as an example for the nice additive character.

4 An efficient implementation of \( \mathcal{F}_{R'} \)

We study the factorization of \( \mathcal{F}_{R'} \) (4.3) in terms of \( \mathcal{F}_R = \mathcal{F}_{R'}|_{m=1} \) and then prove that it can be efficiently performed on a quantum computer. We collect basic facts on the discriminant matrix \( D \) (4.1) of the Galois ring \( R' \).

4.1 Factorization of \( \mathcal{F}_{R'} \)

An \( m \times m \) matrix associated with the basis \( \{\xi^i\}_{i=0}^{m-1} \) of the module \( R' \) on \( R \),
\[
D = (D_{ij})_{0 \leq i,j \leq m-1}, \quad D_{ij} = \text{Tr}(\xi^{i+j}),
\]
is called the discriminant matrix over the Galois ring \( R' \), and it is the Hankel matrix satisfying \( D_{ij} = D_{i+1,j-1} \). We express the trace of the product of two elements \( x, y \in R' \) as
\[
\text{Tr}(x \cdot y) = \bar{x}^T D \bar{y} = \bar{x}'^T \bar{y}', \quad x'_i = (D \bar{x})_i = \text{Tr}(x \xi^i). \tag{4.2}
\]
Namely, we decomposes the trace of \( x \cdot y \) as a linear summation of the products of two elements \( x'_i, y_i \in R \).

Let \( \mathcal{F}_R \) denote the QFT over the residue class ring of integers \( R (\mathbb{Z}_{p^s}) \),
\[
\mathcal{F}_R \equiv \frac{1}{\sqrt{p^s}} \sum_{x, y \in R} (\omega_{p^s})^{x \cdot y |y_i \rangle \langle x_i|}, \quad 1 \leq i \leq m - 1, \tag{4.3}
\]
and then we describe \( \mathcal{F}_{R'} \) (4.3) as the composition of an \( m \)-fold tensor product of \( \mathcal{F}_R \) and a shift operator \( \mathcal{U}_D \),
\[
\mathcal{F}_{R'} = (\mathcal{F}_R)^{\otimes m} \circ \mathcal{U}_D, \quad \mathcal{U}_D \equiv \sum_{x \in R'} |x'\rangle \langle x|, \tag{4.4}
\]
where \( x' = (D\vec{x})^T \cdot \vec{\xi} \). Obviously, the properties of \( U_D \) are determined by the discriminant matrix \( D \).

### 4.2 The discriminant matrix \( D \) is invertible

The set \( \{|x\rangle \}_{x \in R'} \) is an orthonormal basis of the Hilbert space \((\mathbb{C}^p)^\otimes m\). As the discriminant matrix \( D \) is invertible, the map \( \vec{x} \mapsto \vec{x}' = D\vec{x} \) is bijective, and the set \( \{|x'\rangle \}_{x' \in R'} \) also forms an orthonormal basis of \((\mathbb{C}^p)^\otimes m\). Hence the shift operator \( U_D \) is a unitary transformation. Furthermore, we can derive \( U_D^\dagger = U_D^{-1} \).

**Lemma 4.5.** The discriminant matrix \( D \) (4.1) is invertible.

**Proof.** The discriminant matrix \( D \) is invertible if and only if its rows form a basis of \( R^{1 \times m} \), or equivalently, the following equations

\[
\sum_{i=0}^{m-1} b_i \cdot \text{row}_i(D) = 0 \iff \sum_{i=0}^{m-1} b_i D_{ij} = 0, \quad j = 0, \cdots, m - 1
\]  

(4.6)

admit only \( b_i = 0 \) as a solution.

Assume a nonzero solution \( \vec{b}^T = (b_0, \cdots, b_{m-1}) \) of the equation \( \vec{b}^T \cdot D = 0 \) and a corresponding nonzero element \( \beta = \vec{b}^T : \vec{\xi} \in R' \). With another arbitrary element \( \alpha = \vec{a}^T : \vec{\xi} \in R' \), we calculate

\[
\text{Tr}(\beta \cdot \alpha) = \vec{b}^T \cdot D \cdot \vec{a} = 0
\]

(4.7)

where \( \vec{a}, \vec{b} \in R^{m \times 1} \). As \( \beta \) is a unit of \( R' \), i.e., its inverse \( \beta^{-1} \) exists, replacing \( \alpha \) with \( \beta^{-1} \alpha \) gives rise to \( \text{Tr}(\alpha) = 0 \) for \( \alpha \in R' \). As \( \beta \) is a zero divisor of \( R' \), with the lemma 2.5, it has a form of \( \beta = p^k \beta' \), \( 1 \leq k \leq s - 1 \) where \( \beta' \) is a unit with the inverse \((\beta')^{-1} \), and then we have

\[
p^k \text{Tr}(\beta' \alpha) = 0 \Rightarrow p^k \text{Tr}(\alpha) = 0
\]

(4.8)

suggesting that \( \text{Tr}(\alpha) \) is either zero or a zero divisor of \( R \).

Hence, if \( \beta \neq 0 \), then \( \text{Tr}(\alpha) \) for \( \alpha \in R' \) is a zero divisor or zero. This contradicts with the fact that the trace map \( \text{Tr} : R' \rightarrow R \) is surjective. Therefore, \( \beta = 0 \), namely the equation \( \vec{b}^T \cdot D = 0 \) only has a zero solution \( \vec{b} = 0 \), which is equivalent to the existence of \( D^{-1} \), the inverse of the \( D \) matrix. \( \square \)

This proof suggests: if \( \{\xi^i\}_{i=0}^{m-1} \) is a basis of \( R' \) then \( D^{-1} \) exists. On the other hand, it is easy to prove: if \( D^{-1} \) exists then \( \{\xi^i\}_{i=0}^{m-1} \) forms a basis of \( R' \). The matrix \( D \) is hence called the discriminant matrix associated with the basis \( \{\xi^i\}_{i=0}^{m-1} \) of the Galois ring \( R' \).
4.3 Remarks on the discriminant matrix \( D \)

The lemma 4.5 the existence of \( D^{-1} \) over the Galois ring \( R' \), can be proved in the other way. If \( D^{-1} \) exists, then the map \( D : \bar{z} \mapsto D\bar{z} \) is bijective. This means the kernel of this map \( D \) is trivial, namely \( D\bar{z} = 0 \) if and only if \( \bar{z} = 0 \). Assume a nonzero \( y \) satisfying \( D\bar{y} = 0 \). As \( y \) is a unit of \( R' \), the set \( \{ y\xi^i \}_{i=0}^{m-1} \) is a new basis of the Galois ring \( R' \). We expand \( \alpha \in R' \) with the new basis, \( \alpha = \sum_{i=0}^{m-1} a_i(y\xi^i) \), and then apply the trace map to get \( Tr(\alpha) = 0 \) for \( \alpha \in R' \)

Due to the bijective map, \( D \) entries of the discriminant matrix \( F \)

\[ D \text{ is invertible.} \]

\[ \text{Denote} \quad Tr(\xi^i \bar{h}') = 0, \quad \text{Tr}(\bar{h}') = 0. \]

Assume a nonzero \( \bar{y} \) such that \( Tr(\bar{y} D) = 0 \). As \( y \) is a unit of \( R' \), with the help of the formula (4.9), we denote \( y = p^k y' \), \( 1 \leq k \leq s-1 \) with \( y' \) a unit. We have \( p^k Tr(y'\xi^i) = 0 \) due to \( Tr(y^i) = 0 \).

Expand \( \alpha \in R' \) with the new basis \( \{ y'\xi^i \}_{i=0}^{m-1} \), namely \( \alpha = \sum_{i=0}^{m-1} a_i(y'\xi^i) \), and we have \( p^k Tr(\alpha) = 0 \) which suggests \( Tr(\alpha) \) either a zero divisor or zero. Since the trace map \( Tr \) is surjective, the kernel of this map \( D \) has to be trivial, and hence \( D \) is invertible.

Here, we make a sketch on how to compute the discriminant matrix \( D \) over the Galois ring \( R' \). Given a basic primitive polynomial \( \xi^m = \bar{h}^T \cdot \bar{\xi} \) from (2.3) with roots \( \xi, \xi^p, \ldots, \xi^{p^{m-1}} \). Compute \( \xi^k \) in a recursive procedure,

\( \xi^k = (\bar{h}^{(k-m)})^T \cdot \bar{\xi}, \quad \bar{h}^{(0)} = \bar{h}, \quad m \leq k \leq p^m - 2 \) (4.9)

where \( \bar{h}^{(k-m)} \) is calculated via

\[
\bar{h}^{(k-m)} = V^{k-m+1} = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{pmatrix}, \quad V = \begin{pmatrix}
0 & 1 & \cdots & 0 & h_0 \\
1 & 0 & \cdots & 0 & h_1 \\
0 & 1 & \cdots & 0 & h_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & h_{m-1}
\end{pmatrix}.
\] (4.10)

2). Compute \( Tr(\xi^i), 1 \leq i \leq m - 1 \) in terms of \( \xi^i, 1 \leq j \leq p^m - 2 \), with the definition of the trace (2.7). 3). Compute \( Tr(\xi^j), m \leq i \leq 2m - 2 \) in terms of \( Tr(\xi^i), 1 \leq j \leq m - 1 \), with the help of the formula (4.9). 4). We obtain all entries of the discriminant matrix \( D \).

4.4 Complexity analysis of implementing \( F_{R'} \)

Denote \( n = \log p^s \). Assume that the discriminator matrix \( D \) is known via relevant classical computation.

The factorization formalism (4.4) of the QFT \( F_{R'} \) describes an efficient quantum circuit for the implementation of \( F_{R'} \). It is known that the QFT \( F_R \) can be efficiently approximated \( [12] \). The invertible discriminant matrix \( D \) gives rise to the bijective map, \( D : R^\otimes m \rightarrow R^\otimes m \). This map can be efficiently performed as a permutation on a classical computer, and hence the corresponding unitary transformation,

\[ U_D : (\mathbb{C}^{p^s})^\otimes m \rightarrow (\mathbb{C}^{p^s})^\otimes m \]

can be efficiently performed on a quantum computer \( [13] \).
The bijective map from $\vec{x}$ to $D\vec{x}$ ensures that the vector $\vec{x}$ can be computed in a polynomial time with the known $D$ and $D\vec{x}$. The vector $D\vec{x}$ can be computed in time $O(m^2)$. Hales and Hallgren [12] proved that there exists a quantum algorithm to approximate the QFT $F_R$ over $R = \mathbb{Z}_{p^s}$ within accuracy $\epsilon$ which runs in time $O(n \log \frac{n}{\epsilon} + \log^2 \frac{1}{\epsilon})$. Hence $F_{R'}$ can be performed in a polynomial time $O(m^2) + mO(n \log \frac{n}{\epsilon} + \log^2 \frac{1}{\epsilon})$ within accuracy $\epsilon$.

Let $C(p^s, \epsilon)$ denote the minimum size of a quantum circuit approximating the QFT $F_R$ over $R$ within accuracy $\epsilon$, and then performing $F_{R'}^\otimes m$ needs a quantum circuit with the size $mC(p^s, \epsilon)$. The matrix operation $D\vec{x}$ can be performed in a circuit with size $O(m^2n^2)$, namely, each arithmetic operation needs a circuit with size $n^2$. Hence $F_{R'}$ is performed on a quantum circuit with the size $O(m^2n^2) + mC(p^s, \epsilon)$.

Therefore, the QFT $F_{R'}$ (4.3) over the Galois ring $R'$ can be performed within accuracy $\epsilon$ in a polynomial time

$$O(m^2) + mO\left(n \log \frac{n}{\epsilon} + \log^2 \frac{1}{\epsilon}\right)$$

and by a quantum circuit of the size $O(m^2n^2) + mC(p^s, \epsilon)$.

5 Comments on the QFT over a ring

With the help of the QFT over Galois rings, the QFT over a finite commutative ring with identity can be defined in principle.

A finite commutative ring with identity is expressed as a direct sum of local rings, and a local commutative ring can be characterized as a homomorphic image of a polynomial ring over a Galois ring, see [6, 7] for related theorems and proofs. The simplest example is the fundamental theorem of arithmetics: given a unique prime factorization of the integer $m$ by

$$m = p_1^{n_1}p_2^{n_2} \cdots p_k^{n_k}, \quad n_i \in \mathbb{N}, p_i \text{ prime}, \quad 1 \leq i \leq k,$$

there is a ring isomorphism

$$\mathbb{Z}_m \cong \mathbb{Z}_{p_1^{n_1}} \oplus \mathbb{Z}_{p_2^{n_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{n_k}}$$

which defines the QFT over $\mathbb{Z}_m$ in terms of the QFTs over $\mathbb{Z}_{p^s}$.

De Beaudrap and coauthors [3] proved that if the QFT over a ring has the property of the control/target inversion then the QFT over the matrix ring has the same property. A matrix ring over a finite commutative ring is often a noncommutative ring, and hence the QFT over a noncommutative ring can be discussed via the QFT over a finite commutative ring with identity.
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