Hitchin systems on singular curves II.
Gluing subschemes.

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Abstract

In this paper we continue our studies of Hitchin systems on singular curves (started in hep-th/0303069). We consider a rather general class of curves which can be obtained from the projective line by gluing two subschemes together (i.e. their affine part is: Spec \{ f ∈ \mathbb{C}[z] : f(A(ε)) = f((B(ε)); ε^N = 0 \}, where A(ε), B(ε) are arbitrary polynomials). The most simple examples are the generalized cusp curves which are projectivizations of Spec \{ f ∈ \mathbb{C}[z] : f'(0) = f''(0) = ... f^{N-1}(0) = 0 \}. We describe the geometry of such curves; in particular we calculate their genus (for some curves the calculation appears to be related with the iteration of polynomials A(ε), B(ε) defining the subschemes). We obtain the explicit description of moduli space of vector bundles, the dualizing sheaf, Higgs field and other ingredients of the Hitchin integrable systems; these results may deserve the independent interest. We prove the integrability of Hitchin systems on such curves. To do this we develop r-matrix formalism for the functions on the truncated loop group GL_n(\mathbb{C}[z]), z^N = 0. We also show how to obtain the Hitchin integrable systems on such curves as hamiltonian reduction from the more simple system on some finite-dimensional space.

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1 Introduction

In this paper we continue our studies of Hitchin systems on singular curves. Rational singular curves provide large class of explicit, but nontrivial examples of the description of the Hitchin system and all their ingredients: the moduli space of vector bundles, the dualizing sheaf, Higgs field etc. Such explicit examples are quite important, because Hitchin system, despite its importance, is far from being fully investigated. In such examples one can hope to work out methods for complete understanding of the subject.
In the previous paper [1] we have considered the Hitchin system on rational singular curves which can be obtained from the projective line by gluing several points together or by taking cusp singularities. The main idea of the present paper that the more general case of curves obtained by gluing subschemes can be treated along the same lines as it was done in the previous paper. So we largely extend the class of examples where Hitchin system can be explicitly described.

We hope that one can read this paper independently from [1], though sometimes it may be useful for the reader to look in it. We also assume the reader to be familiar with Hitchin’s original paper [2].

However we recall here some principal steps of the construction. Original Hitchin system lives on the cotangent bundle to the space $T^*\mathcal{M}$ of stable holomorphic bundles on nonsingular algebraic curves $\Sigma$. A point of the phase space corresponds to the pair $(E, \Phi)$ where $E$ is a holomorphic bundle and $\Phi$ is a cotangent vector to the moduli space. By standard arguments from deformation theory the tangent vector to the moduli space at the point $E$ can be identified with an element of $H^1(\Sigma, \text{End}(E))$. The Serre’s pairing provide a geometric description of cotangent vectors, indeed, a cotangent vector $\Phi$ can be identified with an element of $H^0(\Sigma, \text{End}(E) \otimes K)$ where $K$ is the canonical class of $\Sigma$.

One proceeds by constructing the following dynamical system which turns out to be integrable: one takes the canonical symplectic structure on the cotangent bundle and the family of functions $I_{k,l}$ which are the coefficients obtained by expanding $\text{Tr} \Phi^k(z)$ on the basic of $H^0(K^k)$. In this paper we continue to generalize this construction to the case of singular algebraic curves. In order to emphasize the analogy between the case of gluing points and subschemes, let us at first recall some of the results from [1], after that we describe the results of the present paper.

• Consider the curve $\Sigma^{proj}$ which results from gluing 2 distinct points $P_i$, $i = 1, 2$ on $\mathbb{C}P^1$ to one point (i.e. the curve which is obtained by adding the smooth point $\infty$ to the curve $\Sigma^{aff} = \text{Spec}\{f \in \mathbb{C}[z] : f(P_1) = f(P_2)\}$).

• A rank $r$ vector bundle on such a curve corresponds to a rank $r$ module $M_\Lambda$ over the affine part given by the subset of vector-valued functions $s(z)$ on $\mathbb{C}$ i.e. $s(z) \in \mathbb{C}[z]^r$ which satisfy the conditions: $s(P_1) = \Lambda s(P_2)$. The moduli space of vector bundles on $\Sigma^{proj}$ is the factor by $GL_r$ of the set of invertible matrices $\Lambda$ where $GL_r$ acts by conjugation.

• The space of global sections of the dualizing sheaf on $\Sigma^{proj}$ is one-dimensional and basic section can be described as meromorphic differential on $\mathbb{C}$ given by $\frac{dz}{z-P_1} - \frac{dz}{z-P_2}$.

• The endomorphisms of the module $M_\Lambda$ are matrix valued polynomials $\Phi(z)$ such that $\Phi(P_1) = \Lambda \Phi(P_2) \Lambda^{-1}$. The action of $\Phi(z)$ on $s(z)$ is: $s(z) \mapsto \Phi(z)s(z)$. The space $H^1(\text{End}(M_\Lambda))$ can be described as the space $gl[z]$ of matrix valued polynomials factorized by the subspaces: $\text{End}_{out} = \{\chi(z) \in gl[z] | \chi(z) = \text{const}\}$ and $\text{End}_{in} = \{\chi(z) \in gl[z] | \chi(P_1) = \Lambda \chi(P_2) \Lambda^{-1}\}$. The elements of $H^1(\text{End}(M_\Lambda))$ are the tangent vectors to the moduli space of vector bundles at the point $M_\Lambda$. The element $\chi(z)$
gives the following deformation of $\Lambda$:
\[
\delta_{\chi(z)} \Lambda = \chi(P_1) \Lambda - \Lambda \chi(P_2)
\] (1)

- The global sections of $H^0(End(M_\Lambda) \otimes \mathcal{K})$ ("Higgs fields") are described as
\[
\Phi(z) = \Lambda \Phi \Lambda^{-1} \frac{dz}{z - P_1} - \Phi \frac{dz}{z - P_2},
\] (2)
where $\Lambda \Phi \Lambda^{-1} - \Phi = 0$.

- The symplectic form on the cotangent bundle to the moduli space can be described as the reduction of the form on the space $\Lambda, \Phi$ given by
\[
Trd(\Lambda^{-1} \Phi) \wedge d\Lambda.
\] (3)

- The hamiltonians $Tr(\Phi(z)^k)$ Poisson commute on the non-reduced phase space with each other for any $k, z$, hence, due to their invariance, gives commuting family of hamiltonians on the reduced phase space.

Remarks: For these particular case of gluing two points the same Lax operator $\Phi(z)$ has been proposed by N. Nekrasov [3], though his methods are different from ours, and the explicit description of bundles, dualizing sheaf, endomorphisms etc are absent in his approach.

In the present paper instead of a points $P_i \in \mathbb{C}$ we consider a subschemes $A, B \in \mathbb{C}$. Algebraically the point $P$ is described as homomorphism $\mathbb{C}[z] \to \mathbb{C}$ given by $f \mapsto f(P)$. Analogously one can describe subschemes as homomorphisms of rings with units $\mathbb{C}[z] \to \mathbb{C}[\epsilon]$, where $\epsilon^N = 0$. Such homomorphisms are uniquely defined by the image of $z$ in $\mathbb{C}[\epsilon]$, which can be arbitrary polynomial $A(\epsilon) \in \mathbb{C}[\epsilon]$.

Notation. In this paper we will denote by subscheme $A(\epsilon)$, where $A(\epsilon)$ is an arbitrary polynomial, the subscheme in $\mathbb{C}$ or $\mathbb{C}P^1$ which are defined by the homomorphisms $\phi : \mathbb{C}[z] \to \mathbb{C}[\epsilon]$, where $\epsilon^N = 0$.

Notation. The number $N$ always denote exponent such that $\epsilon^N = 0$.

Let us summarize the main results of the present paper (in the main text we consider more general examples, but here we cite most illustrative ones).

- Consider the curve $\Sigma^{proj}$ which results from gluing 2 arbitrary subschemes $A(\epsilon), B(\epsilon)$ on $\mathbb{C}P^1$ to one point (i.e. the curve which is obtained by adding the smooth point $\infty$ to the curve $\Sigma^{aff} = Spec \{ f \in \mathbb{C}[z] : f(A(\epsilon)) = f(B(\epsilon)) \}$, where $\epsilon^N = 0$ ). In our paper for some $A(\epsilon), B(\epsilon)$ we calculate the genus (i.e. $dim H^1(\mathcal{O})$) of such curves (see section 2.1 proposition 2), which nontrivially depends on $A(\epsilon), B(\epsilon)$. The basic examples to keep in mind are the following.

  - Nilpotent: $A(\epsilon) = \epsilon, B(\epsilon) = 0$, the genus equals $N - 1$.
  - Root of unity: $A(\epsilon) = \epsilon, B(\epsilon) = \alpha \epsilon$, where $\alpha^k = 1$, the genus equals $N - 1 - [\frac{(N - 1)}{k}]$.

More generically one can consider: $A(\epsilon) = \epsilon$ and $B(\epsilon)$ such that $B(B(B...(B(\epsilon)))) = \epsilon \mod \epsilon^{N-1}$ then the genus will be the same.

k times
- Different geometric points: $A(\epsilon) = a_0 + a_1 \epsilon + ... + a_{N-1} \epsilon^{N-1}$, $B(\epsilon) = b_0 + b_1 \epsilon + ... + b_{N-1} \epsilon^{N-1}$, such that $a_0 \neq b_0$, the genus equals $N$.

- Rank $r$ vector bundles on such a curve correspond to some of rank $r$ modules $M_\Lambda$ over the affine part given by the subset of vector valued functions $s(z)$ on $\mathbb{C}$ i.e. $s(z) \in \mathbb{C}[z]^r$ which satisfy the conditions: $s(A(\epsilon)) = \Lambda(\epsilon)s(B(\epsilon))$, where $\Lambda(\epsilon) = \sum_{i=0}^{N-1} \Lambda_i \epsilon^i$ is matrix valued polynomial. The conditions of projectivity of the module $M_\Lambda$ (and hence the condition for corresponding sheaf over $\mathbb{C}P^1$ to be vector bundle) are the following (see section 3.1 proposition 4).

- Nilpotent: $A(\epsilon) = \epsilon$, $B(\epsilon) = 0$, the condition is: $\Lambda_0 = Id$.
- Root of unity: $A(\epsilon) = \epsilon$, $B(\epsilon) = \alpha \epsilon$, where $\alpha^k = 1$, the condition is: $\Lambda(\epsilon)\Lambda(\alpha \epsilon)\ldots\Lambda(\alpha^{k-1} \epsilon) = Id$.
- Different geometric points: $A(\epsilon) = a_0 + a_1 \epsilon + ... + a_{N-1} \epsilon^{N-1}$, $B(\epsilon) = b_0 + b_1 \epsilon + ... + b_{N-1} \epsilon^{N-1}$, in this case the only condition: $\Lambda_0$ must be invertible.

The moduli space of vector bundles on $\Sigma^{proj}$ is the factor by $GL_r$ of the set of $\Lambda(\epsilon)$ which satisfies the conditions above, where $GL_r$ acts by conjugation (see section 3.2 theorem 1).

- The global sections of the dualizing sheaf on $\Sigma^{proj}$ can be described as meromorphic differentials on $\mathbb{C}$ given by (see section 4.2 proposition 7):

$$\text{Res}_z \left( \frac{\phi(\epsilon)dz}{z - A(\epsilon)} - \frac{\phi(\epsilon)dz}{z - B(\epsilon)} \right),$$

where $\phi(\epsilon) = \sum_{i=0,...,N-1} \phi_i \epsilon^i$ is arbitrary. This expression should be understood expanding the denominators in geometric progression series:

$$\frac{1}{z - A(\epsilon)} = \frac{1}{z - a_0 - a_1 \epsilon - a_2 \epsilon^2 - \ldots} = \frac{1}{(z - a_0)\left(1 + \frac{a_1 \epsilon + a_2 \epsilon^2 + \ldots}{z - a_0} \right)^2 + \ldots},$$

and $\text{Res}_z$ means taking the coefficient at $\frac{1}{1}$. Our claim is that for all $\phi(\epsilon)$ the expression above gives global holomorphic differential on singular curve $\Sigma^{proj}$ and all the differentials can be obtained in such a way. In general the map from $\phi(\epsilon)$ to holomorphic differentials has a kernel.

- The endomorphisms of the module $M_\Lambda$ are matrix valued polynomials $\Phi(z)$ such that $\Phi(A(\epsilon)) = \Lambda(\epsilon)\Phi(B(\epsilon))\Lambda(\epsilon)^{-1}$. The action of $\Phi(z)$ on $s(z)$ is: $s(z) \mapsto \Phi(z)s(z)$. The space $H^1(\text{End}(M_\Lambda))$ can be described as the space $gl[z]$ of matrix valued polynomials factorized by the subspaces: $\text{End}_{out} = \{ \chi(z) \in gl[z] | \chi(z) = \text{const} \}$ and $\text{End}_{in} = \{ \chi(z) \in gl[z] | \chi(A(\epsilon)) = \Lambda(\epsilon)\chi(B(\epsilon))\Lambda(\epsilon)^{-1} \}$. The elements of $H^1(\text{End}(M_\Lambda))$ are the tangent vectors to the moduli space of vector bundles at the point $M_\Lambda$. The element $\chi(z)$ gives the following deformation of $\Lambda$:

$$\delta_{\chi(z)} \Lambda(\epsilon) = \chi(A(\epsilon))\Lambda(\epsilon) - \Lambda(\epsilon)\chi(B(\epsilon)).$$

\[ (5) \]
The global sections of $H^0(\text{End}(M_\Lambda) \otimes \mathcal{K})$ ("Higgs fields") are described as (see section 5.2 proposition 9):

$$\Phi(z) = \text{Res}_\epsilon \left( \frac{\Phi(\epsilon)}{z - A(\epsilon)} dz - \frac{\Lambda(\epsilon)^{-1}\Phi(\epsilon)\Lambda(\epsilon)}{z - B(\epsilon)} dz \right),$$

where $\text{Res}_\epsilon \Lambda(\epsilon)\Phi(\epsilon)\Lambda(\epsilon)^{-1} - \Phi(\epsilon) = 0$; and $\Phi(\epsilon) = \sum_i \Phi_i \epsilon_i + 1$ is a matrix valued polynomial. This expression should be understood expanding the denominators in geometric progression series, as it was explained above. And we also claim that all global sections from $H^0(\text{End}(M_\Lambda) \otimes \mathcal{K})$ can be obtained in such a way and for all $\Phi(\epsilon)$ the expression above gives global section from $H^0(\text{End}(M_\Lambda) \otimes \mathcal{K})$.

The symplectic form on the cotangent bundle to the moduli space can be described as the restriction and reduction of the form on the space $\Lambda(\epsilon), \Phi(\epsilon)$ given by (see section 6.1 theorem 2):

$$\text{Res}_\epsilon \text{Trd}(\Lambda(\epsilon)^{-1}\Phi(\epsilon)) \wedge d\Lambda(\epsilon).$$

So saying shortly:

**Result:** (see theorem 8) The Hitchin system on the curve $\Sigma^{\text{proj}}$ can be described as the system with a phase space which is the hamiltonian reduction of the space of $\Lambda(\epsilon), \Phi(\epsilon)$ (more precisely we should speak about its subspace defined by the conditions mentioned in the second item). Symplectic form is given by the formula 7. The reduction is taken by the group $GL(r)$, which acts by conjugation. The Lax operator is given by formula 6 (Hence hamiltonians are coefficients of the expansion of $\text{Tr}(\Phi(z)^k)$ at the basic of holomorphic $k$-differentials $H^0(\mathcal{K}^k)$). Let us emphasize that $\forall z, w, k, l$ it is true that $\text{Tr}(\Phi(z)^k)$ and $\text{Tr}(\Phi(w)^l)$ Poisson commute on the nonreduced phase space.

In order to prove integrability of Hitchin system in case of our singular curves one should prove that hamiltonians Poisson commute. This is done in section 7 by use of $r$-matrix technique, the propositions may be interesting by themselves so let us formulate them also.

The bracket between $\Phi(\epsilon)$ is of $r$-matrix type (see lemma 11):

$$\{\Phi(\epsilon) \otimes \Phi(\eta)\} = \left[ [\Phi(\epsilon) \otimes 1, \delta^\eta R] \right] = \left[ [\Phi(\epsilon) \otimes 1 + 1 \otimes \Phi(\eta), \delta^\eta R] \right].$$

Where $\{A \otimes B\}$ is standard St.-Petersburg’s notation for brackets between matrices meaning that the result is a matrix in the tensor product of spaces with matrix elements which are Poisson brackets between the matrix elements of matrices $A, B$. The matrix $R$ is just permutation matrix: $R(u \otimes v) = v \otimes u$. The function $\delta^\eta$ is equal to $\sum_{k=0}^{N-1} \frac{\partial}{\partial \eta^k}$ and $[,]_\eta$ is the truncation of a polynomial i.e. taking it’s part consisting of the monomials $\eta^k$, where $k < N$. The formula is easy to prove and it will be standard if there will be no truncations and all sums will be infinite i.e. $N = \infty$. 

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What is more surprising for us is that the following brackets are also of $r$-matrix form (see lemma 13):

$$\{\Phi(z, \epsilon) \otimes \Phi(w, \eta)\} = \left|\left[\Phi(z, \epsilon) \otimes 1, R\right] \delta^z_w \right|_{\{all \, variables\}}.$$ 

Where $\Phi(z, \epsilon) = \left|\Phi(\epsilon) \right|_{z-A(\epsilon)} - \frac{A^{-1}(\epsilon)\Phi(\epsilon)\Lambda(\epsilon)}{z-B(\epsilon)} \right|_{\{z\}}.$

And finally the desired bracket between the Lax operators for Hitchin system is also of $r$-matrix type (see lemma 14):

$$\{\Phi(z) \otimes \Phi(w)\} = \left|\left[\Phi(z) \otimes 1, R\right] \delta^z_w\right|_{\{z,w\}} = \left|\left[\Phi(z) \otimes 1 + 1 \otimes \Phi(w), R\right] \delta^z_w\right|_{w}.$$  

(9)

After that the commutativity of $Tr(\Phi(z)^k)$ is more or less standard game with $r$-matrices.

The functions $Tr(\Phi(z)^k)$ are invariant with respect to the action of $GL(n)$ by conjugation on the space of pairs $\Lambda(\epsilon), \Phi(\epsilon)$, so they can be pushed down to the reduced space. The main property of hamiltonian reduction is that it preserves the brackets of invariant functions. So we obtain the Poisson commuting family on the reduced space, which is as explained before the phase space of Hitchin system.

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2 Class of curves

2.1 Description

In this section we consider the curves defined as $\text{Spec}\{f \in \mathbb{C}[z], f(A(\epsilon)) = f(B(\epsilon))\}$, where $\epsilon^N = 0$, $A(\epsilon) = a_0 + a_1\epsilon + a_2\epsilon^2 + ... + a_{n-1}\epsilon^{N-1}$ and $B(\epsilon) = b_0 + b_1\epsilon + b_2\epsilon^2 + ... + b_{n-1}\epsilon^{N-1}$ are some fixed polynomial. We start with the following obvious lemma:

**Lemma 1** The condition $f(A(\epsilon)) = f(B(\epsilon))$ defines a subring in $\mathbb{C}[z]$.

These curves are singular curves with the only singular point. If $a_0 = b_0$ then the curve has the only one branch (like cusp $y^2 = x^3$), if $a_0 \neq b_0$ then the curve has two branches (like node $y^2 = x^2(x + 1)$). More precisely we consider projective curves $\Sigma^{proj}$ which are obtained from the curve above adding one smooth point $\infty$. (It can be obviously formalized). Concretely this implies the geometrical objects like differentials, endomorphisms etc. to do not have poles at $\infty$.

**Example 1** Consider $\text{Spec}\{f \in \mathbb{C}[z], f(1) = f(0)\}$. This is node (or double point) curve.
Example 2 Consider $\text{Spec}\{f \in \mathbb{C}[z], f(\epsilon) = f(0)\}$, where $\epsilon^2 = 0$. One has: $f(\epsilon) = f(0) + f'(0)\epsilon$, hence condition $f(\epsilon) = f(0)$ gives $f'(0) = 0$. So this curve is a cusp curve $y^2 = x^3$. So one can informally say that a cusp curve is result of gluing of two infinitely close to each other points $z = 0, z = \epsilon$.

Example 3 Consider $\text{Spec}\{f \in \mathbb{C}[z], f(\epsilon) = f(0)\}$, where $\epsilon^N = 0$. Analogously we obtain a curve defined by the conditions: $f'(0) = f''(0) = \ldots = f^{(N-1)}(0) = 0$.

Example 4 Consider $\text{Spec}\{f \in \mathbb{C}[z], f(\epsilon) = f(\alpha \epsilon)\}$, where $\epsilon^N = 0$, $\alpha$ is primitive root of unity of order $k > 1$, i.e. $\alpha^k = 1$. We obtain a curve defined by the conditions: $f^{(i)}(0) = 0$, for $l \neq k, 2k, 3k, \ldots$ and $l < N$.

Example 5 Consider $\text{Spec}\{f \in \mathbb{C}[z], f(\epsilon) = f(-\epsilon + b_2\epsilon^2 + b_3\epsilon^3)\}$, where $\epsilon^4 = 0$. We obtain a curve defined by the conditions: $f'(0) = 0$, $f''(0) = -3b_2f''(0)$. One can see that it does not depend on $b_3$.

Proposition 1 The condition $f(A(\epsilon)) = f(B(\epsilon))$, where $A(\epsilon) = a_0 + a_1\epsilon + \ldots$ and $a_1 \neq 0$, is equivalent to the condition $f(a_0 + \epsilon) = f(D(\epsilon))$ for some $D(\epsilon)$.

Actually, for $a_1 \neq 0$ one can obviously find such polynomial $C(\epsilon)$ that $A(C(\epsilon)) = a_0 + \epsilon$, (if $a_0 = 0$, then $C(\epsilon)$ is truncation of inverse formal power series for $A(\epsilon)$) so one sees that $D(\epsilon) = B(C(\epsilon))$.

Example 6 The condition $f(\epsilon + \epsilon^2) = f(-\epsilon + \epsilon^2)$ is equivalent to condition $f(\epsilon) = f(-\epsilon + 2\epsilon^2 - 4\epsilon^3)$, where $\epsilon^4 = 0$. Here we just look for $C(\epsilon)$ such that $A(C(\epsilon)) = \epsilon$. This means that $C(\epsilon) + C(\epsilon) = \epsilon$ hence $C(\epsilon) = \epsilon - 2\epsilon^2 + 4\epsilon^3$.

So one sees that in general position we can restrict ourselves to the curves, where $A(\epsilon) = a_0 + \epsilon$, by change of variables $z = \tilde{z} + a_0$ we can consider only the case $A(\epsilon) = \epsilon$.

Remark 1 The curves which are given by $\text{Spec}\{f \in \mathbb{C}[z], f(A(\epsilon)) = f(B(\epsilon))\}$ and $\text{Spec}\{f \in \mathbb{C}[z], f(A((\epsilon + c)) = f(B((\epsilon + c)))\}$, where $c \in \mathbb{C}$ in general are not isomorphic. For example $f(\epsilon) = f(0)$, $\epsilon^2 = 0$ is cusp curve, but $f(\epsilon + 1) = f(0)$ - is curve with two branches.

Let us calculate the genus of our singular curves.

Proposition 2 Here we consider three possible cases concerning the polynomials $A(\epsilon)$ and $B(\epsilon)$.

- **Nilpotent**: If $A(\epsilon) = \epsilon$ and $B(B(B(\ldots(B(\epsilon))))) = 0$ mod $\epsilon^{N-1}$, for some $k$ then $\dim H^1(\mathcal{O}) = N - 1$ (i.e. genus equals $N - 1$) for the curve $\Sigma$ defined above. This curve is equivalent to the curve defined by $B(\epsilon) = 0$ (i.e. the conditions defining the curve are: $f(\epsilon) = f(0)$ mod $\epsilon^{N-1}$).

- **Root of Unity**: If $A(\epsilon) = \epsilon$ and $B(B(B(\ldots(B(\epsilon))))) = \epsilon$ mod $\epsilon^{N-1}$ then $\dim H^1(\mathcal{O}) = N - 1 - [(N - 1)/k]$ (i.e. genus equals $N - 1 - [(N - 1)/k]$) for the curve $\Sigma$ defined
above. The curve depends nontrivially on the coefficients $b_1, ..., b_{N-1}$, (it does not depend on $b_N$ if $b_1 \neq 1$).

If for all $k \geq 1$ $B(B(B(...(B(ε)))) = \sum c_t ε^t$ and all the coefficients $c_t$ are not equal to zero

then the curve $Σ$ is the curve defined by the subring $f(0) = f''(0) = ... = f^{(N−1)}(0) = 0$.

So all such $B(ε)$ gives the same curve as $B(ε) = 0$.

• Different Points: In the case $a_0 ≠ b_0$, where $A(ε) = a_0 + a_1 ε + ...$, $B(ε) = b_0 + b_1 ε + ...$, $ε^N = 0$ the curve has two branches and $\dim H^1(\mathcal{O}) = N$ (i.e. genus equals $N$).

Let us does not analyze more complicated case: $B(B(B...(B(ε)))) = ε \mod ε^L$ for some $L < N − 1$.

The items 1 and 3 in proposition are quite obvious, the proof of the item 2 will be given elsewhere. Let us give only some motivation and example for item 2. From the condition $f(ε) = f(B(ε))$ follows that $f(ε) = f(B(ε)) = f(B(B(ε)))) = f(B(B(B(ε)))) = ...$ so it is natural to expect that if iteration process $B(B(...(B(ε))))$ does not stops, then one obtains infinitely many different points where the values of polynomial $f(ε)$ coincide so $f(ε) = Const \mod ε^N$. Let us give nontrivial example to illustrate our proposition.

Example 7 Consider the condition $f(ε) = f(−ε + b_2 ε^2 + b_3 ε^3 + b_4 ε^4 + b_5 ε^5 + b_6 ε^6)$, where $ε^7 = 0$. Rewriting it explicitly one obtains: $f′(0) = 0$ from coefficient at $ε^4$, no conditions from coefficient at $ε^2$; $f^3(0) = −3b_2 f^2(0)$ from $ε^3$; $b_2^2 = −b_3$ from $ε^4$ (we mean that if this condition is not satisfied then the curve will be $f^k(0) = 0$, $k < 7$ - this case is the same as $B(ε) = 0$ and we are not interested in it now); $f^5(0) = 5(−2b_2 f^4(0) + 12 f^2(0)(2b_2^2 − b_4))$ from $ε^5$; $2b_2^4 − 3b_2 b_4 − b_5 = 0$ from $ε^6$. The main miracle is that the same conditions for $b_i$ arises from the condition $B(B(ε)) = ε$, where $B(ε) = (−ε + b_2 ε^2 + b_3 ε^3 + b_4 ε^4 + b_5 ε^5 + b_6 ε^6)$ (i.e. $B(B(ε)) = ε − 2(b_2^2 + b_3) ε^3 + b_2(b_2^2 + b_3) ε^5 + 2(2b_2^4 − 3b_2 b_4 − b_5) ε^5 − 3b_2(2b_2^4 − 3b_2 b_4 − b_5) ε^6$ (in this calculation we substituted $(b_2^2 + b_3) = 0$, when we calculated terms at $ε^5, ε^6$)).

Remark 2 Also we see such a strange observation on the iteration of polynomials $B(ε) = −ε + ...$: assume that $B(B(ε)) = ε \mod ε^{2k+1}$ then expression for coefficients at $ε^{2k+2}$ and at $ε^{2k+3}$ are proportional to each other. Analogous situation should be with the iteration of polynomials of the type: $B(ε) = αε + ...$, where $α$ is root of unity.

Remark 3 If $B(B(B...(B(ε)))) = ε \mod ε^{N−1}$ then we can obviously choose such $b_{N−1}$

that $B(B(B...(B(ε)))) = ε \mod ε^N$. The curve $Σ$ does not depend on $b_{N−1}$ so we do not loose generality when we consider such $B(ε)$ that $B(B(B...(B(ε)))) = ε \mod ε^N$. 

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2.2 Geometric (Schematic) Interpretation

The curves defined as \( \text{Spec} \{ f \in \mathbb{C}[z], f(A) = f(B) \} \), where \( A, B \in \mathbb{C} \) geometrically can be described as gluing two points \( A, B \) to each other. So one should think about our curves \( \text{Spec} \{ f \in \mathbb{C}[z], f(A(\epsilon)) = f(B(\epsilon)) \} \), where \( \epsilon^N = 0 \), as gluing together not two geometric points \( A, B \in \mathbb{C} \) but two subschemes \( \text{Spec} \{ \mathbb{C}[\epsilon]/\epsilon^N \} \). This is leading analogy in exploring the properties of these singular curves i.e. description of dualizing sheaf, vector bundles and their endomorphisms. Subschemes are obviously given by \( \text{Spec} \{ \mathbb{C}[\epsilon]/\epsilon^N \} \rightarrow \text{Spec} \{ \mathbb{C}[z] \} : z \mapsto A(\epsilon) \) and another subscheme \( z \mapsto B(\epsilon) \).

For usual geometric point we can speak about "value of the function at a point" the same can be done for the schematic point i.e. "the value of function \( f \in \mathbb{C}[z] \) at a schematic point \( z \mapsto A(\epsilon) \)" is just \( f(A(\epsilon)) \) the "value" is not a number, but it is element of the structure ring of a point i.e. "value" belongs to \( \{ \mathbb{C}[\epsilon]/\epsilon^N \} \). So equality \( f(A(\epsilon)) = f(B(\epsilon)) \) means that the "values" of function \( f \) coincide in two schematic points. So this means that we have glued this two points together.

2.3 Further examples

One can glue many points:

\[ \text{Spec} \{ f \in \mathbb{C}[z], f(A_i(\epsilon)) = f(A_j(\epsilon)), f(B_i(\epsilon)) = f(B_j(\epsilon)), f(C_i(\epsilon)) = f(C_j(\epsilon)), \ldots \} \]

Most of our constructions can be generalized straightforwardly to this situation, we will not discuss it for simplicity.

2.4 How general is this class of singular curves?

Presumably not all singularities can be described as glueing subschemes. For example the condition \( f(A(\epsilon)) = f(B(\epsilon)) \) always implies \( f'(0) = 0 \). Possibly all the curves \( \text{Spec} \{ f \in \mathbb{C}[z], f(\epsilon) = f(B(\epsilon)) \} \), where \( B(B(...B(\epsilon)...)) = \epsilon \mod \epsilon^N \), are analytically equivalent to the curves \( \text{Spec} \{ f \in \mathbb{C}[z], f(\epsilon) = f(\alpha \epsilon) \} \), where \( \alpha^k = 1 \). There is another known example which does not fit into our description:

**Example 8** Let us consider the singular curve such that its affine part without \( \infty \) is described by the subring of \( \mathbb{C}[z] \) generated by \( 1, z^3, z^5 \). It corresponds to the Sylvester diagram 1, 0, 0, 1, 0, 1, 1, 0.

3 Moduli of bundles

3.1 Projective Modules

In “geometric-to-algebraic” dictionary vector bundles correspond to projective modules, it’s well known that projective modules are the same as locally (in Zariski topology) free modules. Recall that starting with vector bundles we consider the sheaf of its sections and thus obtain a sheaf of modules, which will be locally free (and so projective), vice
versa we consider sheaf of locally free modules and so we can find the glueing maps and so define the vector bundle. We assume that reader is more or less familiar with this geometric-to-algebraic correspondence. Algebraic language here is more preferable.

**Definition 1** Let us recall that the fiber at a point \( P \) of a module \( M \) over a ring \( R \) is defined as \( M^{\text{loc}}/I^{\text{loc}}M^{\text{loc}} \), where \( I \) is the maximal ideal of the point \( P \) and “\( \text{loc} \)” means “localization at point \( P \)”.

Recall that for locally free (projective) modules over algebras over \( \mathbb{C} \) the fibers at each points are of the same dimension as \( \mathbb{C} \) vector spaces. We will use this fact as a test for a module to be non projective.

**Proposition 3** The subset of vector-valued functions \( s(z) \) on \( \mathbb{C} \) i.e. \( s(z) \in \mathbb{C}[z]^{\mathbb{R}} \) which satisfy condition \( s(A(\epsilon)) = \Lambda(\epsilon)s(B(\epsilon)) \) for some fixed matrix-valued polynomial \( \Lambda(\epsilon) = \sum_{i=0,...,N-1} \Lambda_i\epsilon^i \) is a module over the algebra of functions \( \{ f \in \mathbb{C}[z] | f(A(\epsilon)) = f(B(\epsilon)) \} \).

We will denote this module \( M_{\Lambda} \). This module is not always projective (see next proposition). Obviously \( M_{\Lambda} \) is rank \( r \) for general \( \Lambda \).

**Example 9** Consider the double point curve: \( \Sigma = \text{Spec}\{ f \in \mathbb{C}[z] : f(1) = f(0) \} \). Then rank 1 modules (line bundles and rank one torsion free sheaf) are parameterized by the \( \lambda \in \mathbb{C} \). They are given by the condition \( \{ s(z) \in \mathbb{C}[z] : s(1) = \lambda s(0) \} \). Obviously \( M_\lambda \) are torsion free modules. For \( \lambda = 0 \) one can obviously see that it is not projective module, because the fiber at the point \( z = 0 \) jumps and becomes two-dimensional, this is impossible for locally free modules. It’s a nice exercise to calculate the divisor of the line bundle \( M_\lambda \). For \( \lambda \neq 0 \) one can obviously see that this module is locally free and hence projective. This example illustrates also that the moduli space of line bundles (the so-called generalized Jacobian) on singular curve is non compact (it is \( \mathbb{C}^* \) in this case and this isomorphism is also an isomorphism of groups, where as usually one considers the tensor product as a group operation on line bundles). The moduli space can be compactified by torsion free modules. In this case one should add one module corresponding to \( \lambda = 0 \) (it is isomorphic to the module \( \lambda = \infty \), i.e. the module \( \{ s \in \mathbb{C}[z] : 0 = s(0) \} \)). It can be shown that if one constructs properly the algebraic structure on the set of torsion free sheaves of rank 1 it coincides with the curve \( \Sigma^{\text{proj}} \) itself as a manifold. This can be done constructing the Poincaré line bundle on the product of curve with itself. This fact reflects the elliptic nature of such kind of curves.

**Example 10** Consider the cusp curve: \( \Sigma = \text{Spec}\{ f \in \mathbb{C}[z] : f(\epsilon) = f(0) \} \), \( \epsilon^2 = 0 \). The modules can be described by \( \{ s(z) \in \mathbb{C}[z] : s(\epsilon) = (\lambda_0 + \lambda_1\epsilon)s(0) \} \). Obviously if \( \lambda_0 \neq 1 \) then the module \( M_\lambda \) is the zero module. So we consider \( \lambda(\epsilon) = 1 + \lambda_1\epsilon \) and the modules \( M_\lambda \) can be described explicitly as \( \{ s \in \mathbb{C}[z] : s'(0) = \lambda_1 s(0) \} \). It coincides with the traditional description \( \{ \Pi, \text{example 6} \} \). In this example for all \( \lambda_1 \in \mathbb{C} \) these modules are projective. So \( \mathbb{C} \) is module space of line bundles. It can be compactified adding one point \( \lambda_1 = \infty \) (i.e. the module \( \{ s \in \mathbb{C}[z] : 0 = s(0) \} \), which is the same as the maximal ideal of the singular point \( z = 0 \), and the same as the direct image of \( \mathcal{O}^{\text{norma}} - \text{the structure sheaf of the normalized curve, and the same as just } \mathbb{C}[z] \text{ considered as a module over our algebra}. \) Properly introduced algebraic structure will show that this moduli space is \( \Sigma^{\text{proj}} \) itself, not \( CP^1 \) as one might think from the naive point of view.
Example 11  Consider the curve: $\Sigma = Spec\{f \in \mathbb{C}[z] : f(\epsilon) = f(0)\}, \epsilon^4 = 0$. Analogously one can consider modules: $\{s(z) \in \mathbb{C}[z] : s(\epsilon) = (1 + \lambda_1 \epsilon + \lambda_2 \epsilon^2 + ... + \lambda_{N-1} \epsilon^{N-1})s(0)\}$. It can be rewritten as $\{s(z) \in \mathbb{C}[z] : s'(0) = \lambda_i s(0)\}$. All of these modules will be projective. Corresponding line bundles over the $\Sigma^{proj}$ exhaust all the line bundles of degree zero.

Question  It seems that there is only one torsion free module which is not projective - i.e. maximal ideal of point $z = 0$ (the same as the direct image of $\mathcal{O}^{norm}$ and the same as just $\mathbb{C}[z]$ considered as a module over our algebra). So the compactification would be some singular one point compactification of $\mathbb{C}^{N-1}$, it would be nice to understand the local ring of this singular point.

Proposition 4  As in proposition 2 we treat three cases:

- **Nilpotent case**  $M_A$ is projective module over the algebra $\{f \in \mathbb{C}[z] : f(\epsilon) = f(B(\epsilon))\}$, where $B(B(\epsilon)) = 0; \epsilon^N = 0$ iff $\Lambda(0) = \Lambda_0 = Id$.

- **Root of unity case**  $M_A$ is projective module over the algebra $\{f \in \mathbb{C}[z] : f(\epsilon) = f(B(\epsilon))\}$, where $B(B(\epsilon)) = \epsilon; \epsilon^N = 0$

  iff $\Lambda(\epsilon)\Lambda(B(\epsilon))\Lambda(B(B(\epsilon)))...\Lambda(B(B(\epsilon))) = Id$, where $Id$ is identity matrix $k-1$ times (it does not depend on $\epsilon$).

- **Different Points**  $M_A$ is projective module over the algebra $\{f \in \mathbb{C}[z] : f(A(\epsilon)) = f(B(\epsilon))\}$, where $a_0 \neq b_0$ iff $\Lambda_0$ is invertible matrix.

The proof will be given elsewhere. The items 1 and 3 are obvious. Let us illustrate the item 2.

Example 12  Consider the curve: $\Sigma = Spec\{f \in \mathbb{C}[z] : f(\epsilon) = f(-\epsilon)\}, \epsilon^4 = 0$. So here $B(B(\epsilon)) = \epsilon$. Explicitly it can be described as $\Sigma = Spec\{1, z^2, z^4, z^5, ...\}$. One can consider modules: $\{s(z) \in \mathbb{C}[z] : s(\epsilon) = (1 + \lambda_1 \epsilon + \lambda_2 \epsilon^2 + \lambda_3 \epsilon^3)s(-\epsilon)\}$. From proposition 2 we know that $dimH^1(\mathcal{O}) = 2$, it’s well-known that $dimH^1(\mathcal{O})$ is tangent space to moduli space of line bundles, so moduli space of line bundles on our curve should be two-dimensional, on the other hand we see three parameter $\lambda_1, \lambda_2, \lambda_3$ which describes the rank 1 modules (which are obviously torsion free), so in order to solve the contradiction we should show that not all of modules are projective. Let us rewrite condition $s(\epsilon) = (1 + \lambda_1 \epsilon + \lambda_2 \epsilon^2 + \lambda_3 \epsilon^3)s(-\epsilon)$ explicitly. The coefficient at $\epsilon^1$ will give: $s'(0) = \lambda_1 s(0)/2$; the coefficient at $\epsilon^2$ will give: $\lambda_2^2/2s(0) = \lambda_2 s(0)$, we will show that if $\lambda_2 \neq \lambda_2^2/2$ then the module is not projective. The coefficient at $\epsilon^3$ gives: $s^{(3)}(0) = 3(\lambda_3 s(0) + \lambda_1 s^{(2)}(0)/2 - \lambda_3^3/4s(0))$ (we used $\lambda_2^2/2s(0) = \lambda_2 s(0)$ in the last term). So we see that if the condition $\lambda_2^2/2 = \lambda_2$ is not true then the module can be described as $s(0) = 0, s'(0) = 0, s^{(3)}(0) = 3/2\lambda_1 s^{(2)}(0)$. Hence it’s fiber at point zero is two-dimensional, (in general point it is 1-dimensional) so this module cannot be locally free (=projective).
We obtained the condition \( \lambda_2 = \frac{\lambda_1^2}{2} \). It is rather miraculous but simple to demonstrate that this condition is equivalent to the one stated above: \( \Lambda(\epsilon)\Lambda(B(\epsilon)) = Id \). On the other hand there is a following motivation: we request \( s(\epsilon) = \Lambda(\epsilon)s(B(\epsilon)) \), so

\[
\begin{align*}
s(\epsilon) &= \Lambda(\epsilon)s(B(\epsilon)) = \Lambda(\epsilon)\Lambda(B(\epsilon))s(B(B(\epsilon))) = \\
&= \Lambda(\epsilon)\Lambda(B(\epsilon))\Lambda(B(\epsilon))...\Lambda(B(B...B(\epsilon))))s((B(B...B(\epsilon)))) \quad \text{\(k\)-times} \\
&= \Lambda(\epsilon)\Lambda(B(\epsilon))\Lambda(B(\epsilon))...\Lambda(B(B...B(\epsilon))))s(\epsilon) \quad \text{\(k\)-times}.
\end{align*}
\]

The condition \( B(B...B(\epsilon)) = \epsilon \) makes it natural to expect that

\[
\Lambda(\epsilon)\Lambda(B(\epsilon))\Lambda(B(\epsilon))...\Lambda(B(B...B(\epsilon)))) = Id, \quad \text{\(k\)-times}
\]

otherwise as we have seen in the example the fiber at zero jumps and the module is not projective, this is true in general.

### 3.2 Vector Bundles

The modules \( M_\Lambda \) are equivalent, if there exists invertible map of modules \( K(z) : M_\Lambda \mapsto \tilde{M}_\Lambda \).

Recall that we denoted by \( \Sigma^{proj} \) the projective curve which we obtain from the affine curve \( Spec\{ f \in \mathbb{C}[z] : f(\epsilon) = f(B(\epsilon)) \} \) by adding one smooth point at infinity. The module \( M_\Lambda \) gives a vector bundle over \( \Sigma^{proj} \) in an obvious way: we define the sheaf which is trivial rank \( r \) module over the chart containing infinity and not containing singular point and which is module \( M_\Lambda \) (or more precisely its localization) over the chart which contains the singular point. The degree of such bundles equals zero. The vector bundles are equivalent if there exists the invertible map of modules \( K(z) : M_\Lambda \mapsto \tilde{M}_\Lambda \) over each chart. So we see that \( K(z) \) is matrix polynomial which should be regular both at infinity and zero so \( K(z) \) does not depend on \( z \) so it is constant. In this way we obtain:

**Proposition 5** The vector bundles \( M_\Lambda \) over \( \Sigma^{proj} \) are isomorphic if there exists a constant matrix \( K \) such that \( \Lambda(\epsilon) = K\tilde{\Lambda}(\epsilon)K^{-1} \).

As a corollary we obtain the following:

**Theorem 1** The open subset in the moduli space of vector bundles of degree zero and rank \( r \) over the curve \( \Sigma^{proj} \) is the space of matrix polynomials \( \Lambda(\epsilon) = \sum_{i=0}^{N-1} \Lambda_i \epsilon^i \), which satisfy the conditions below, factorized by the conjugation by constant matrices. The conditions for \( \Lambda(\epsilon) \) are the following:

- **Nilpotent case:** for the curves \( Spec\{ f \in \mathbb{C}[z] : f(\epsilon) = f(B(\epsilon)) \} \), where \( B(B(B...B(\epsilon))) = 0 \mod \epsilon^{N-1} \) we request \( \Lambda_0 = Id \).
• **Root of Unity case:** for the curves $\text{Spec}\{ f \in \mathbb{C}[z] : f(\epsilon) = f(B(\epsilon)) \}$, where $B(B(B\ldots(B(\epsilon)))) = \epsilon$, we request $\Lambda(\epsilon)\Lambda(B(\epsilon))\Lambda(B(B(\epsilon)))\ldots\Lambda(B(B(B\ldots(B(\epsilon)))) = \text{Id}$ and $\Lambda_0 = \text{Id}.$

• **Different points:** For the curves $\text{Spec}\{ f \in \mathbb{C}[z] : f(A(\epsilon)) = f(B(\epsilon)) \}$, where $a_0 \neq b_0$ we request $\Lambda_0$ to be invertible.

**Remark 4** Let us omit the questions of stability of bundles and what exactly we mean by ”factor”. Let us also note that all the bundles $M_\Lambda$ satisfy the property that $\pi^*M_\Lambda$ will be trivial bundle on $\mathbb{CP}^1$, where $\pi : \mathbb{CP}^1 \to \Sigma^{\text{proj}}$ is normalization map. Obviously there are lots of bundles $F$ of degree zero on $\Sigma^{\text{proj}}$ such that $\pi^*F$ are not trivial bundles but some bundles of the type $\oplus_{k=1,\ldots,k}O(t_k)$, such that $\sum t_k = 0$. So by no means we do not obtain all bundles on $\Sigma^{\text{proj}}$ as bundles $M_\Lambda$ for some $\Lambda$. But nevertheless general stable and possibly semistable bundle satisfy the property that $\pi^*F$ will be trivial bundle on $\mathbb{CP}^1$, and it is easy to see from our previous description of projective modules that general stable bundles can be obtained as bundles $M_\Lambda$ for some $\Lambda$.

3.3 **Geometric interpretation**

Geometrically the curve $\text{Spec}\{ f \in \mathbb{C}[z] : f(A(\epsilon)) = f(B(\epsilon)) \}$ is obtained by glueing two points $A$ and $B$ together. We described the modules over this curve as $s(z) : s(A(\epsilon)) = \Lambda(\epsilon)s(B(\epsilon))$. Geometrically this can be interpreted as glueing the fibers at point $A$ and $B$ of the trivial bundle by the map $\Lambda : \text{fiber at } A \mapsto \text{fiber at } B$. Indeed, fiber of the module $M$ at point $A$ is $M_A = M \otimes_{O(\Sigma)} O(A)$ in our case $M_B = M_A = \mathbb{C}[\epsilon]/\epsilon^n$ so we need to describe the map $\Lambda : \mathbb{C}[\epsilon]/\epsilon^n \mapsto \mathbb{C}[\epsilon]/\epsilon^n$ which is the map of modules over algebra $\mathbb{C}[\epsilon]/\epsilon^n$ it is given by it’s value at 1 which we denote by $\Lambda(\epsilon)$, so the map $\Lambda : \mathbb{C}[\epsilon]/\epsilon^n \mapsto \mathbb{C}[\epsilon]/\epsilon^n$ is given by the multiplication on $\Lambda(\epsilon)$. The condition for $s(z) : s(A(\epsilon)) = \Lambda(\epsilon)s(B(\epsilon))$ geometrically means that we consider such $s(z)$ that its value in point $A$ equals to $\Lambda$ multiplied its value at point $B$ for the map $\Lambda : M_B \mapsto M_A$ described above.

3.4 **On the notation $\frac{1}{\epsilon}$ for $\epsilon^n = 0$**

It’s obvious that if $\epsilon^n = 0$ then one cannot introduce the inverse element $\frac{1}{\epsilon}$ in a sense of the associative multiplication, (for example: let $\epsilon^2 = 0$ then $\frac{1}{\epsilon^2} = 0$). Nevertheless as we will see from the lemma below there is no problem in using formal expressions with negative degrees in $\epsilon$ to define the paring. It will be very convenient for us to use such notation in next sections where we will describe differentials on singular curves.

**Notation** Let us settle that $\text{Res}_sA = A_{-1}$, where $A = \sum_i A_i\epsilon^i$.

**Lemma 2** Let us consider $f(\epsilon) = \sum_{i \geq -N} f_i\epsilon^i$, $g(\epsilon) = \sum_{i \geq 0} g_i\epsilon^i$, $h(\epsilon) = \sum_{i \geq 0} h_i\epsilon^i$ then

$$\text{Res}_s((gh)f) = \text{Res}_s(g(hf))$$

(10)
The proof is obvious.

In the text below we will use terms containing $\frac{1}{\varepsilon}$ only in expressions like in the lemma above, so we do not need to care about the way of putting the brackets.

Let us make some remarks. What should be considered as a module of meromorphic differentials on the scheme $\text{Spec} \{ \mathbb{C}[\varepsilon] : \varepsilon^N = 0 \}$? It seems to be natural to denote by such differentials the expressions $f(\frac{1}{\varepsilon})d\varepsilon = \sum_{i=-N, \ldots, -1} f_i \varepsilon^i d\varepsilon$. The structure of module is given by the usual multiplication combined with the dropping out all the terms $\varepsilon^k$ where $k$ is out of range $k = -N, \ldots, -1$. As a module it is just trivial module. The use of such notations for the elements of the trivial module is convenient, because the pairing between the differentials $\omega$ and functions $f$ can be written in the same way as usually: $\text{Res}_{\varepsilon} f \omega$.

**Lemma 3** Consider $f(z, \varepsilon) = \sum_{i,j} f_{i,j} \varepsilon^i z^j$, it is obviously true that:

$$\text{Res}_{\varepsilon} \text{Res}_{z} f = \text{Res}_{z} \text{Res}_{\varepsilon} f = f_{-1,-1}$$

**Lemma 4** Consider $f(z) = \sum_{i \geq 0} f_i z^i$ and $A(\varepsilon) = \sum_i a_i \varepsilon^i$, where $\varepsilon^N = 0$, it is obviously true that:

$$\text{Res}_{z} \frac{f(z)}{z - A(\varepsilon)} = f(A(\varepsilon))$$

### 4 Dualizing Sheaf

#### 4.1 Construction

Recall that the dualizing sheaf $\mathcal{K}$ on manifold $X^n$ is such sheaf that there is isomorphism $t : H^n(X, \mathcal{K}) = \mathbb{C}$ and for any other coherent sheaf $F$ there is natural pairing $\text{Hom}(F, \mathcal{K}) \times H^n(X, F) \to H^n(X, \mathcal{K})$, which is combined with isomorphism $t$ gives isomorphism $\text{Hom}(F, \mathcal{K}) \to H^n(X, F)^*$, where $V^*$ is space dual to $V$. So for the flat sheaves $F$ we have isomorphism $H^0(K \otimes F^*) \to H^n(F)^*$.

On a nonsingular curve the dualizing sheaf is the sheaf of holomorphic 1-forms, but for a singular curve, one should specify what are $1 -$ forms and the naive definition i.e. the Kaehler differentials ([4] ch. 2 sect. 8) is not the right object. It’s known that (see [5] [6]):

**Proposition 6** The dualizing sheaf $\mathcal{K}$ on the singular curve $\Sigma$ can be described as: sheaf of meromorphic 1-forms $\alpha$ on normalization, such that $\forall f \forall P \in \Sigma$ it is true that:

$$\sum_{P_i \in \Sigma^{\text{norm}}, \phi(P_i) = P} \text{Res}_{P_i} f \alpha = 0,$$

where $f$ is the pullback of a function $f$ on the singular curve to its normalization, $P_i$ are such points on the normalization that they maps to the same point $P$ on the singular curve under the normalization map $\phi : \Sigma^{\text{norm}} \mapsto \Sigma$.

We will call the elements of $H^0(\mathcal{K})$ as holomorphic differentials on curve $\Sigma$. 

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Remark 5  The description of holomorphic differentials given above is very natural for the following reason: consider the nonsingular curve, then obviously $\text{Res}_pfw = 0$ for any point $p$, holomorphic function $f$ and fixed holomorphic differential $w$, in singular situation we see that there are some meromorphic differentials on normalization which satisfy analogous property that $\text{Res}_pfw = 0$ for any $f$ which comes from singular curve, so it is natural to guess that they should be treated as holomorphic. The more rigorous argument consists in demonstrating that this realization of the canonical sheaf really provide the duality

$$H^k(F \otimes K) \times H^{n-k}(F^*) \rightarrow \mathbb{C}.$$  

For some examples and demonstration see [1].

Remark 6  For all our curves $\text{Spec}\{ f \in \mathbb{C}[z] : f(A(\epsilon)) = f(B(\epsilon)) \}$ the normalization is the complex plane $\mathbb{C}$. So holomorphic differentials on such singular curves are meromorphic differentials on $\mathbb{C}$ which satisfy the condition [13].

Example 13  Consider the node (or double point) curve $\text{Spec}\{ f \in \mathbb{C}[z], f(1) = f(0) \}$. The holomorphic differentials on such curve are meromorphic differentials on $\mathbb{C}$ (which is a normalization of the node) of the kind $w(z) = c(\frac{1}{z-1} - \frac{1}{z})dz + f(z)dz$, where $c$ is a constant, $f(z)$ is holomorphic function on $\mathbb{C}$. One easily sees that the condition [13] which in this case says that the residues at points $z = 1$ and $z = 0$ are of the different sign, is satisfied. So global holomorphic differentials on the corresponding projective curve will be of the form $c(\frac{1}{z-1} - \frac{1}{z})dz$.

Example 14  Consider the cusp curve $\text{Spec}\{ f \in \mathbb{C}[z], f(\epsilon) = f(0) \}$, where $\epsilon^2 = 0$. Analogously to the previous example $w(z) = \frac{\partial f}{\partial \epsilon} + f(z)dz$. So global holomorphic differentials on the corresponding projective curve will be of the form $\frac{\partial f}{\partial \epsilon}$.

Remark 7  In two preceding examples the global holomorphic differentials can be obtained by degenerating the elliptic nonsingular curve and the holomorphic differentials on it (see [1] for comments).

Example 15  Consider $\text{Spec}\{ f \in \mathbb{C}[z], f(\epsilon) = f(0) \}$, where $\epsilon^N = 0$. Analogously to the previous example $w(z) = \sum_{i=1}^{N-1} \frac{\partial f}{\partial \epsilon} + f(z)dz$. Hence global holomorphic differentials on the corresponding projective curve will be of the form $\sum_{i=1}^{N-1} \frac{\partial f}{\partial \epsilon}$.

Example 16  Consider $\text{Spec}\{ f \in \mathbb{C}[z], f(\epsilon) = f(\alpha \epsilon) \}$, where $\epsilon^N = 0$, $\alpha$ is primitive root of unity of order $k > 1$, i.e. $\alpha^k = 1$. Holomorphic differentials are given by $w(z) = \sum_{i=1}^{N-1} \frac{\partial f}{\partial \epsilon} + f(z)dz$, where $\forall l c_{lk+1} = 0$.

Example 17  Analogously for the curve $\text{Spec}\mathbb{C}[1, z^{k_1}, z^{k_2}, ..., z^{k_n}]$ we see that holomorphic differentials will be: all $\frac{\partial f}{\partial \epsilon}$, such that $p \neq k_i$.

Example 18  Consider $\text{Spec}\{ f \in \mathbb{C}[z], f(\epsilon) = f(-\epsilon + b_2 \epsilon^2 + b_3 \epsilon^3) \}$, where $\epsilon^4 = 0$. Holomorphic differentials are given by $w(z) = \frac{\partial f}{\partial \epsilon} + c_2(\frac{1}{z^2} + \frac{b_2}{z_3})dz + f(z)dz$.

Example 19  Consider $\text{Spec}\{ f \in \mathbb{C}[z], f(\alpha \epsilon) = f(1-\beta \epsilon) \}$, where $\epsilon^2 = 0$. Holomorphic differentials are given by $w(z) = c_1(\frac{1}{z} - \frac{1}{z-1})dz + c_2(\frac{\alpha}{z^2} + \frac{\beta}{(z-1)^2})dz + f(z)dz$.  

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4.2 Global sections

In the proposition above the holomorphic differentials (i.e. the sections of dualizing sheaf) were given not explicitly, and to find them one should solve a system of linear equations. The key proposition (see below) of this section shows that for our curves \( \Sigma = \text{Spec}\{ f \in \mathbb{C}[z], f(A(\epsilon)) = f(B(\epsilon)) \} \) the differentials can be given explicitly, moreover we will see later that the symplectic form on the cotangent bundle to the moduli space in such coordinates will be written explicitly.

**Proposition 7** For all \( \omega_i \in \mathbb{C}, i = 0, ..., N - 1 \) and their generating function \( \omega(\epsilon) = \sum_{i=0}^{N-1} \omega_i \frac{1}{z^i} \), the expression

\[
\tilde{\omega}(z) = \text{Res}_z \frac{\omega(\epsilon)dz}{z - A(\epsilon)} - \frac{\omega(\epsilon)dz}{z - B(\epsilon)},
\]

(14)
gives a holomorphic differential (i.e. an elements of \( H^0(\mathcal{K}) \)) on the curve \( \Sigma^{\text{proj}} \), defined on the affine chart without infinity as \( \text{Spec}\{ f \in \mathbb{C}[z], f(A(\epsilon)) = f(B(\epsilon)) \} \). Moreover any holomorphic differential can be obtained in this way.

The correspondence \( \omega_i \mapsto \tilde{\omega}(z) \) is not in general injective (see examples below).

Expression \( \frac{1}{z - A(\epsilon)} \) should be understood expanding the geometric progression in the region \( z - a_0 \gg \epsilon \).

It is easy to see that for any \( f \in \mathbb{C}[z] \), which is a function on our curve, i.e. satisfies the conditions \( f(A(\epsilon)) = f(B(\epsilon)) \), it is true that: \( \text{Res}_z f \text{Res}_z (\frac{\omega(\epsilon)dz}{z - A(\epsilon)} - \frac{\omega(\epsilon)dz}{z - B(\epsilon)}) = 0 \), so \( \tilde{\omega}(z) \) is really a holomorphic differential on our singular curve. It can be shown that our construction gives all elements in \( H^0(\mathcal{K}) \).

Thus the coefficients \( \omega_i \) parameterize the space \( H^0(\mathcal{K}) \). In general they can be dependent, but it’s easy to choose independent parameters. We will see that in terms of \( \omega_i \) the symplectic form on the cotangent bundle to the moduli space of vector bundles on \( \Sigma \) can be written explicitly and quite simply.

**Example 20** Consider the node (or double point) curve \( \text{Spec}\{ f \in \mathbb{C}[z], f(1) = f(0) \} \). We take \( \omega(\epsilon) = \omega_0 \frac{1}{\epsilon} \). The formula [14] gives: \( \tilde{\omega}(z) = \text{Res}_z \omega \frac{\frac{dz}{\epsilon}}{z - 1} - \frac{\frac{dz}{\epsilon}}{z} = \omega_0 (\frac{dz}{\epsilon} - \frac{dz}{z}) \). It coincides with the holomorphic differential from example [13].

**Example 21** Consider the curve \( \text{Spec}\{ f \in \mathbb{C}[z], f(\epsilon) = f(0) \} \), where \( \epsilon^N = 0 \). We take \( \omega(\epsilon) = \omega_0 \frac{1}{\epsilon} + \omega_1 \frac{1}{\epsilon^2} + ... + \omega_{N-1} \frac{1}{\epsilon^{N-1}} \). The formula [14] gives:

\[
\tilde{\omega}(z) = \text{Res}_z \omega(\epsilon) \left( \frac{dz}{z} - \frac{dz}{\epsilon} \right) = \text{Res}_z \omega(\epsilon) \left( \frac{dz}{z(1 - \frac{\epsilon}{z})} - \frac{dz}{z} \right)
\]

\[
= \text{Res}_z \omega(\epsilon) \left( \frac{dz}{z} \frac{1}{1 - \frac{\epsilon}{z}} + \frac{\epsilon}{z^2} + ... + \frac{\epsilon^{N-1}}{z^{N-1}} \right) - \frac{dz}{z} \right)
\]

\[
= \text{Res}_z \omega(\epsilon) \frac{1}{\epsilon} + \omega_1 \frac{1}{\epsilon^2} + ... + \omega_{N-1} \frac{1}{\epsilon^{N-1}} \left( \sum_{i=1}^{N} \frac{\epsilon^i dz}{z^{i+1}} \right) = \sum_{i=1,...,N-1} \omega_i \frac{dz}{z^{i+1}}.
\]

It coincides with the holomorphic differentials from example [15].
Example 22  Consider $\text{Spec}\{f \in \mathbb{C}[z], f(\epsilon) = f(\alpha \epsilon)\}$, where $\epsilon^N = 0$, $\alpha$ is a primitive root of unity of order $k > 1$, i.e. $\alpha^k = 1$. We take $\omega(\epsilon) = \omega_0 + \omega_1 + \ldots + \omega_{N-1} \frac{1}{\epsilon}$. The formula 14 gives:

$$\tilde{\omega}(z) = \text{Res}_\epsilon \omega(\epsilon) \left( \frac{dz}{z - \epsilon} - \frac{dz}{z - \alpha \epsilon} \right) = \sum_{i=1, \ldots, N-1; i \neq k, 2k, 3k} \omega_i \frac{dz}{z^{i+1}}.$$

It coincides with the holomorphic differentials from example 16. And we see that the expression for holomorphic differentials does not depend on $\omega_i$, where $i \neq k, 2k, 3k$. So the map $\omega(\epsilon) \mapsto \tilde{\omega}(z)$ is not injective.

Example 23  Consider $\text{Spec}\{f \in \mathbb{C}[z], f(\epsilon) = f(-\epsilon + b_2 \epsilon^2 + b_3 \epsilon^3)\}$, where $\epsilon^4 = 0$. We take $\omega(\epsilon) = \omega_0 + \omega_1 \frac{1}{\epsilon} + \omega_2 \frac{1}{\epsilon^2} + \omega_3 \frac{1}{\epsilon^3}$.

The formula 14 gives:

$$\tilde{\omega}(z) = \text{Res}_\epsilon \omega(\epsilon) \left( \frac{dz}{z - \epsilon} - \frac{dz}{z - \epsilon^2} \right) = \left( \omega_1 - \omega_2 b_2 - \omega_3 b_3 \right) \frac{dz}{z^2} + \omega_3 \left( \frac{2}{z^4} + \frac{2b_2}{z^3} \right) dz.$$

It coincides with the holomorphic differentials from example 18. And we see that the expression for holomorphic differentials depends only on $\omega_3$ and the linear combination $(\omega_1 - \omega_2 b_2 - \omega_3 b_3)$.

### 4.3 Geometric interpretation ?

In previous section we saw the geometric sense of our constructions: curves were defined by glueing subschemes and modules were defined by glueing fibers of trivial modules. What is the geometric sense of our description of differentials (proposition 7) ? At the moment we do not know.

Let us speculate a little about the proposition 7 which we consider as rather nice. When one glues two ordinary points $\Sigma = \text{Spec}\{f \in \mathbb{C}[z], f(A) = f(B)\}$ we see that holomorphic differentials are given by $\left( \frac{1}{z-A} - \frac{1}{z-B} \right) dz$. The main property of this differential is that its resides at two points $A, B$ coincides up to sign. It would be tempting to do something similar in the case, when we glue two subschemes $A(\epsilon), B(\epsilon)$, i.e. it would be nice to introduce the notion of reside in schematic point and to have the following proposition: holomorphic differentials are those, whose resides in the schematic points coincide up to sign. But as far as we were able to learn there is no such notion of residue and our naive attempts to use for example Poincare formula, shows that it is impossible to have such proposition. But we suspect that the formula introduced ad hoc

$$\text{Res}_\epsilon \frac{\omega(\epsilon) dz}{z - A(\epsilon)} - \frac{\omega(\epsilon) dz}{z - B(\epsilon)}$$

should have some geometric sense and should be an example of some general principle.
5 Endomorphisms of $M_\Lambda$

5.1 Description

Proposition 8 Endomorphisms of the module $M_\Lambda$ can be described as matrix polynomials
$\Phi(z): s(z) \mapsto \Phi(z)s(z)$, which satisfy the condition

$$\Phi(A(\epsilon)) = \Lambda(\epsilon)\Phi(B(\epsilon))\Lambda(\epsilon)^{-1}. \quad (15)$$

Proof The condition above is obviously the condition for $\Phi(z)s(z)$ to satisfy the condition
$\Phi(A(\epsilon))s(A(\epsilon)) = \Lambda(\epsilon)\Phi(B(\epsilon))s(B(\epsilon))$, so $\Phi(z)s(z)$ is again an element of $M_\Lambda$ and
$\Phi(z): s(z) \mapsto \Phi(z)s(z)$ is an endomorphism of $M_\Lambda$.

Example 24 In abelian situation (i.e. rank 1 modules over any manifold) the condition above is empty and any element $\Phi(z)$ defines an endomorphism i.e. sheaf of endomorphisms of any rank 1 coherent sheaf is just $\mathcal{O}$.

Example 25 Consider the node (or double point) curve $Spec\{f \in \mathbb{C}[z], f(1) = f(0)\}$. The endomorphisms of the module $M_\Lambda$ (which is defined as $\{s \in \mathbb{C}[z]^r, s(1) = \Lambda s(0)\}$, for some matrix $\Lambda$, (see section 4.3 example 9) are given by such matrix-valued polynomials
$\Phi(z) = \Phi_0 + \Phi_1 z + \Phi_2 z^2 + ...$ which satisfy $\Phi(1) = \Lambda\Phi(0)\Lambda^{-1}$. Hence $\Phi(z) = \Phi_0 + (\Lambda\Phi_0\Lambda^{-1} - \Phi_0)z + z(z-1)\tilde{\Phi}(z)$, where $\tilde{\Phi}(z)$ is arbitrary. When one considers the projectivization of our curve and bundle corresponding to $M_\Lambda$ on it, one should only consider the constant endomorphism $\Phi(z) = \Phi_0$ in order to be regular at infinity. The condition $\Phi(1) = \Lambda\Phi(0)\Lambda^{-1}$ is satisfied if the matrices $\Phi_0$ commute with $\Lambda$. As a corollary we see that for general $\Lambda$ there are only $r$-dimensional space of such matrices. For the trivial bundle any matrix $\Phi_0$ gives its endomorphism.

Example 26 Consider the node (or double point) curve $Spec\{f \in \mathbb{C}[z], f(A) = f(B)\}$. The endomorphisms of the module $M_\Lambda$ are given by such matrix-valued polynomials
$\Phi(z) = \Phi_0 + \Phi_1 z + \Phi_2 z^2 + ...$ that $\Phi(A) = \Lambda\Phi(B)\Lambda^{-1}$. Hence $\Phi(z) = \Phi_0 + \Phi_1 z + (z-A)(z-B)\tilde{\Phi}(z)$, where $\Phi_0, \Phi_1$ should satisfy $\Phi_0 + A\Phi_1 = \Lambda(\Phi_0 + B\Phi_1)\Lambda^{-1}$ and $\tilde{\Phi}(z)$ is arbitrary. It’s impossible to express in a simple way $\Phi_0$ via $\Phi_1$, or vice versa. But it’s possible to express both of them throw the one arbitrary matrix $\Theta$. The most simple way to see it is the following: let us write $\Phi(z) = \Theta_1(z-A) - \Theta_2(z-B)$ so the condition $\Phi(A) = \Lambda\Phi(B)\Lambda^{-1}$ gives $\Theta_2 = \Lambda\Theta_1\Lambda^{-1}$. $\Theta_1$ can be taken arbitrary and one finds $\Phi(z) = \Theta_1(z-A) - \Lambda\Theta_1\Lambda^{-1}(z-B)$. Hence global endomorphisms are given by $\Phi(z) = \Theta_1(B-A)$, with such $\Theta_1$, which commute with $\Lambda$.

Question We saw in example above that in the case of glueing of ordinary points $f(A) = f(B)$ we have an explicit parameterization for endomorphisms of $M_\Lambda$ i.e. all solutions of equation $\Phi(A) = \Lambda\Phi(B)\Lambda^{-1}$ are given by $\Phi(z) = \Theta_1(z-A) - \Lambda\Theta_1\Lambda^{-1}(z-B) + (z-A)(z-B)\Phi(z)$ with $\Theta_1, \tilde{\Phi}(z)$ being arbitrary. The question: is it possible to give parameterization of endomorphisms in more general case of the curves given by glueing subschemes: $f(\epsilon) = f(\epsilon)$, i.e. to solve the equation $\Phi(A(\epsilon)) = \Lambda(\epsilon)\Phi(B(\epsilon))\Lambda^{-1}(\epsilon)$?

In the next section we give the explicit parameterization of the sections of $End(M_\Lambda) \otimes \mathcal{K}$. But strangely at the moment we do not know how to parameterize $End(M_\Lambda)$ itself.
Consider the cusp curve $\text{Spec}\{f \in \mathbb{C}[z], f(\epsilon) = f(0)\}$, where $\epsilon^2 = 0$. So the endomorphisms of the module $M_\Lambda$ (which is defined as $\{s \in \mathbb{C}[z]^r, s(\epsilon) = (\text{Id} + \Lambda_1\epsilon)s(0)\}$, for some matrix $\Lambda_1$, see section 3.1, example 10) are given by such matrix-valued polynomials $\Phi(z) = \Phi_0 + \Phi_1z + \Phi_2z^2 + \ldots$ that $\Phi(\epsilon) = \Lambda(\epsilon)\Phi(0)\Lambda^{-1}(\epsilon)$. This is equivalent to the condition $\Phi_1 = [\Lambda_1, \Phi_0]$. When one considers the projectivization of our curve and the bundle corresponding to $M_\Lambda$ on it, one should consider only constant endomorphisms $\Phi(z) = \Phi_0$. Thus one have $\Phi_1 = 0$ and $[\Lambda_1, \Phi_0] = 0$. Endomorphisms of the bundle $M_\Lambda$ are given by constant matrix polynomials $\Phi(z) = \Phi_0$, where matrices $\Phi_0$ commute with $\Lambda_1$. As a corollary we see that for general $\Lambda_1$ there is only $r$-dimensional space of such matrices (general bundles are semistable, but not stable). For the trivial bundle any matrix gives an endomorphism.

Consider $\text{Spec}\{f \in \mathbb{C}[z], f(\epsilon) = f(0)\}$, where $\epsilon^3 = 0$. The endomorphisms of the module $M_\Lambda$ (which is defined as $\{s \in \mathbb{C}[z]^r, s(\epsilon) = (\text{Id} + \Lambda_1\epsilon + \Lambda_2\epsilon^2)s(0)\}$, for some matrices $\Lambda_1, \Lambda_2$, see section 3.1, example 11) are given by such matrix-valued polynomials $\Phi(z) = \Phi_0 + \Phi_1z + \Phi_2z^2 + \ldots$, that $\Phi(\epsilon) = \Lambda(\epsilon)\Phi(0)\Lambda^{-1}(\epsilon)$. This is equivalent to the conditions: $\Phi_1 = [\Lambda_1, \Phi_0], \Phi_2 = [\Lambda_2, \Phi_0] + [\Phi_0, \Lambda_1]$. On the projectivization of our curve one should consider only constant endomorphism $\Phi(z) = \Phi_0$ of the bundle corresponding to $M_\Lambda$. One have $\Phi_1 = 0, \Phi_2 = 0$, hence $[\Lambda_1, \Phi_0] = 0, [\Lambda_2, \Phi_0] + [\Phi_0, \Lambda_1] = 0$. We have two conditions for one matrix $\Phi_0$. For general $\Lambda_i$ only scalar matrixes can satisfy both conditions, thus for general bundle $H^0(\text{End}(M_\Lambda)) = \mathbb{C}$ (general bundles are stable).

Remark 8 In our simple examples we see that for Calabi-Yau manifolds (the cusp curve in our case) general bundles are semistable, but not stable; for general type manifolds (canonical sheaf is ample) in our case, which is of genus 2) the general bundles are stable.

5.2 $\text{End}(M_\Lambda) \otimes \mathcal{K}$

Proposition 9 Sections of $\text{End}(M_\Lambda) \otimes \mathcal{K}$ over the chart without infinity can be described as matrix polynomials:

$$\Phi(z) = (\text{Res}_\epsilon \frac{\Phi(\epsilon)dz}{z - A(\epsilon)} - \frac{\Lambda^{-1}(\epsilon)\Phi(\epsilon)\Lambda(\epsilon)dz}{z - B(\epsilon)}) + \text{holomorphic in } z \text{ terms,}$$

(16)

where $\Phi(\epsilon) = \sum_{i=0}^{N-1} \Phi_i \frac{1}{\epsilon^i}$ is the formal generating function for matrices $\Phi_i$. So we say that taking an arbitrary $\Phi_i$ and considering $\Phi(z)$ given by the formula above, one obtains that $\Phi(z)$ is an element of $\text{End}(M_\Lambda) \otimes \mathcal{K}$. The global sections (regular everywhere including infinity) can be obtained requesting additional condition for $\Phi(\epsilon)$:

$$\text{Res}_\epsilon \Phi(\epsilon) - \Lambda^{-1}(\epsilon)\Phi(\epsilon)\Lambda(\epsilon) = 0.$$  

(17)

The proof will be given elsewhere. The part of the proposition about the global sections follows trivially from the first part, because the condition (7) is just the condition for the coefficient at $\frac{1}{z}$ in the expression (16) to be zero, which ensures that the expression (16) will
be holomorphic at infinity. It is important and we will show it latter that the condition coincides with the moment map equals zero condition for the hamiltonian reduction. The sections of \( \text{End}(M_\lambda) \otimes \mathcal{K} \) by definition are expressions of the type \( \sum_i \Phi_i(z) \otimes \omega_i \), where \( \Phi_i \in \text{End}(M_\lambda); \omega_i \in \mathcal{K} \); we claim that the expression can be represented in this form. The authors are unable to provide something like a simple formula for such decomposition, but it seems that it does exist. Let us give examples illustrating the proposition, and why it is not so trivial.

Example 29 Consider \( \text{Spec}\{ f \in \mathbb{C}[z], f(\epsilon) = f(0) \} \), where \( \epsilon^3 = 0 \). The endomorphisms of the module \( M_\lambda \) are given (see example 28) by \( \Theta(z) = \Theta + [\lambda_1, \Theta]z + ([\lambda_2, \Theta] + [\Theta, \lambda_1]z^2 + z^3 \Theta(z) \), where \( \Theta, \Theta(z) \) are arbitrary. The sections of the canonical module are given by \( \frac{\omega dz}{z^3} + \frac{\omega dz}{z^2} + c(z)dz \), where \( c(z) \) is holomorphic. Hence the sections of \( \text{End}(M_\lambda) \otimes \mathcal{K} \) can be described as

\[
\Phi(z) = (\Theta_1 + [\lambda_1, \Theta_1]z + ([\lambda_2, \Theta_1] + [\Theta_1, \lambda_1]z^2) \frac{dz}{z^2} + (\Theta_2 + [\lambda_1, \Theta_2]z + ([\lambda_2, \Theta_2] + [\Theta_2, \lambda_1]z^2) \frac{dz}{z^3} + \text{holomorphic in } z \text{ terms}
\]

\[
= \frac{\Theta_2 dz}{z^3} + \frac{(\Theta_1 + [\lambda_1, \Theta_2])dz}{z^2} + \frac{([\Theta_2, \lambda_1]z + [\lambda_2, \Theta_2] + [\lambda_1, \Theta_1])dz}{z} + \text{holomorphic in } z \text{ terms},
\]

for some matrices \( \Theta_1, \Theta_2 \). On the other hand let us consider the receipt given in proposition above:

\[
\Phi(z) = (\text{Res}_z \frac{\Phi(\epsilon)dz}{z - A(\epsilon)} - \frac{\Lambda^{-1}(\epsilon)\Phi(\epsilon)\Lambda(\epsilon)dz}{z - B(\epsilon)})
\]

\[
= \frac{\Theta_2 dz}{z^3} + \frac{(\Theta_1, \Phi_1)dz}{z^2} + \frac{([\lambda_1, \Phi_1] + [\lambda_2, \Phi_2] + [\lambda_1, \Phi_2, \lambda_1])dz}{z}.
\]

To prove our proposition in this example we should rewrite the expression above in terms of \( \Theta_i \). We see that there are three equations for the only two parameters \( \Phi_1, \Phi_2 \), but one can show that nevertheless the third equation is dependent of the first two. It means that if we take \( \Theta_2 = \Phi_2, \Theta_1 = \Phi_1 - [\lambda_1, \Phi_2] \) then the coefficients at \( \frac{dz}{z^2} \) are as they should be and the coefficient at \( \frac{dz}{z^3} \) is automatically the necessary one. So let us say again what we have get: the expression

\[
\Phi(z) = (\text{Res}_z \frac{\Phi(\epsilon)dz}{z - A(\epsilon)} - \frac{\Lambda^{-1}(\epsilon)\Phi(\epsilon)\Lambda(\epsilon)dz}{z - B(\epsilon)})
\]

can be represented in the form

\[
\Theta_1(z) \frac{dz}{z^2} + \Theta_2(z) \frac{dz}{z^3} + \text{holomorphic in } z \text{ terms},
\]

where

\[
\Theta_i(z) = (\Theta_i + [\lambda_1, \Theta_i]z + ([\lambda_2, \Theta_i] + [\Theta_i, \lambda_1]z^2)
\]

are elements of \( \text{End}(M_\lambda) \). To do this one puts \( \Theta_2 = \Phi_2, \Theta_1 = \Phi_1 - [\lambda_1, \Phi_2] \). Thus \( \Phi(z) \) is a section of \( \text{End}(M_\lambda) \otimes \mathcal{K} \).
Example 30  Consider the cusp curve $Spec\{ f \in \mathbb{C}[z], f(\epsilon) = f(0) \}$, where $\epsilon^2 = 0$. The endomorphisms of the module $M_\Lambda$ are given (see example 27) by $\Theta(z) = \Theta + [\Lambda_1, \Theta]z + z^2 \tilde{\Theta}(z)$, where $\Theta, \tilde{\Theta}(z)$ are arbitrary. The sections of the canonical module are given by $\frac{dz}{z^2} + c(z)dz$, where $c(z)$ is holomorphic. So the sections of $\text{End}(M_\Lambda) \otimes \mathcal{K}$ can be described as

$$\Phi(z) = (\Theta + [\Lambda, \Theta z]) \frac{dz}{z^2} + \text{holomorphic in } z \text{ terms.}$$

The global sections $H^0(\text{End}(M_\Lambda) \otimes \mathcal{K})$ are such $\Phi(z) = (\Theta + [\Lambda, \Theta z]) \frac{dz}{z^2}$, which are regular at infinity, hence $[\Theta, \Lambda] = 0$ and the global sections are: $\Phi(z) = (B - A) \frac{\Theta dz}{z^2}$, where $\Lambda \Theta = \Theta \Lambda$. On the other hand let us consider $\Phi(\epsilon) = \Phi_{\epsilon}$ and consider the receipt given in proposition above:

$$\Phi(z) = \left( \text{Res}_\epsilon \frac{\Phi(\epsilon)dz}{z - A(\epsilon)} - \frac{\Lambda^{-1}(\epsilon)\Phi(\epsilon)\Lambda(\epsilon)dz}{z - B(\epsilon)} \right) = \Phi_1 \frac{dz}{z^2} + \left[ \Lambda_1, \Phi_1 \right] \frac{dz}{z}.$$

We see that in this case the two expressions coincide.

Example 31  Consider the node curve $Spec\{ f \in \mathbb{C}[z], f(A) = f(B) \}$. The endomorphisms of the module $M_\Lambda$ are given (see example 26) by

$$\Theta(z) = \Theta(z - A) - \Lambda \Theta \Lambda^{-1}(z - B) + (z - A)(z - B) \tilde{\Theta}(z),$$

where $\Theta, \tilde{\Theta}(z)$ are arbitrary. The sections of the dualizing module are given by

$$\frac{cdz}{z - A} + \frac{cdz}{z - B} + \text{holomorphic in } z.$$

Hence the sections of $\text{End}(M_\Lambda) \otimes \mathcal{K}$ can be described as

$$\Phi(z) = (B - A)(\Lambda \Theta \Lambda^{-1} \frac{dz}{z - A} - \frac{\Theta dz}{z - B}) + \text{holomorphic in } z.$$

So the global sections $H^0(\text{End}(M_\Lambda) \otimes \mathcal{K})$ are such

$$\Phi(z) = (B - A)(\frac{\Lambda \Theta \Lambda^{-1} dz}{z - A} - \frac{\Theta dz}{z - B}),$$

which are regular at infinity, hence $\Lambda \Theta \Lambda^{-1} - \Theta = 0$ and the global sections are:

$$\Phi(z) = (B - A)(\frac{\Theta dz}{z - A} - \frac{\Theta dz}{z - B}),$$

where $\Lambda \Theta = \Theta \Lambda$. On the other hand let us consider $\Phi(\epsilon) = \Phi_{\epsilon}$ and consider the receipt given in proposition above:

$$\Phi(z) = \left( \text{Res}_\epsilon \frac{\Phi(\epsilon)dz}{z - A} - \frac{\Lambda^{-1}(\epsilon)\Phi(\epsilon)\Lambda(\epsilon)dz}{z - B} \right) = \frac{\Phi dz}{z - A} + \frac{\Lambda^{-1}(\epsilon)\Phi \Lambda dz}{z - B}.$$
5.3 $H^1(\text{End}(M_\Lambda))$

By definition $M_\Lambda$ is a submodule of $\mathbb{C}[z]^r$. So we have an exact sequence of modules $M_\Lambda \hookrightarrow \mathbb{C}[z]^r \rightarrow \mathbb{C}^p$ over an algebra $\{f \in \mathbb{C}[z], f(A(\epsilon)) = f(B(\epsilon))\}$, where $\epsilon^N = 0$. $\mathbb{C}[z]^r \rightarrow \mathbb{C}^p$ is a resolution for $M_\Lambda$. The complex of endomorphisms of a resolution gives a resolution of endomorphisms of $M_\Lambda$. This is one way to argue the proposition below, another is to use Čech description of $H^1(\text{End}(M_\Lambda))$. We will prove the proposition by the second method (see below).

**Proposition 10** The space of matrix polynomials $\chi(z) = \sum_{i=0}^{N-1} \chi_i z^i$ considered as endomorphisms of $\mathbb{C}[z]^r$, maps surjectively to $H^1(\text{End}(M_\Lambda))$. The kernel of this map consists of the sum of the two linear subspaces in the space of matrix polynomials $\chi(z)$: the first space is the space of constant polynomials $\chi(z) = \chi_0$ and the second space consists of such matrix polynomials which satisfy the condition: $\chi(A(\epsilon)) = \Lambda(\epsilon) \chi(B(\epsilon)) \Lambda(\epsilon)^{-1}$. (Let us mention that the intersection of these two subspaces is precisely $H^0(\text{End}(M_\Lambda))$ and the second subspace consists of such $\chi(z)$ which gives endomorphisms of the module $M_\Lambda$ over the affine chart without infinity.)

**Proof** The proposition is quite obvious from the point of view of Čech description of $H^1(\text{End}(M_\Lambda))$. Let us cover our singular curve by two charts $U_P = \Sigma \setminus \infty$, $U_\infty = \Sigma \setminus P$, where $P$ is the singular point. Then

$$H^1(\text{End}(M_\Lambda)) = \text{End}(M_\Lambda)(U_P \cap U_\infty)/\text{End}(M_\Lambda)(U_P) \oplus \text{End}(M_\Lambda)(U_\infty).$$

As we know from proposition 9 $\text{End}(M_\Lambda)(U_P)$ consists in polynomials $\chi(z)$, which satisfy $\chi(A(\epsilon)) = \Lambda(\epsilon) \chi(B(\epsilon)) \Lambda(\epsilon)^{-1}$. And this proves the proposition.

**Remark 9** For the node curve $f(A) = f(B)$ ($A, B \in \mathbb{C}$) the summation in the formula $\chi(z) = \sum_{i=0}^{N-1} \chi_i z^i$ must be understood in the interval $i = 0, 1$ (i.e. one should consider $\chi(z) = \chi_0 + \chi_1 z$), the same convention about summation in the case of the node curve should be accepted everywhere below (see example 33 for details.)

**Example 32** in the abelian case (i.e. when $M_\Lambda$ is a rank 1 module) for any $\Lambda$ it is known that $\text{End}(M_\Lambda)$ is just $\mathcal{O}$. So in the abelian case the proposition claims that $H^1(\mathcal{O})$ is the factor space of the space of all polynomials $\sum_{i=0}^{N-1} \chi_i z^i$ by the space of polynomials $\chi(z)$ which satisfy the condition $\chi(A(\epsilon)) = \chi(B(\epsilon))$. This is obviously true. For example this can be seen from the exact sequence: $\mathcal{O} \rightarrow \mathcal{O}^{\text{norm}} \rightarrow \mathbb{C}_P$, which gives that: $H^1(\mathcal{O}) = H^0(\mathbb{C}_P)$.

It’s well-known that the vector space $H^1(\text{End}(M_\Lambda))$ is the tangent space to deformations of $M_\Lambda$ as an algebraic vector bundle, on the other hand we know that all vector bundles are given by $\Lambda(\epsilon)$, so taking some $\delta \chi(z)$, where $\delta^2 = 0$, $\chi(z) = \chi_i z^i$ we must find the corresponding deformation of $M_\Lambda$. $\delta \chi(z)$ should deform $M_\Lambda$ to some $M_{\Lambda(\epsilon)}$, where $\tilde{\Lambda}(\epsilon) = \Lambda(\epsilon) + \delta \Delta_{\Lambda(\epsilon)}$. Our aim is to determine $\Delta_{\Lambda(\epsilon)}$.

**Proposition 11** Matrix polynomial $\chi(z) = \sum_{i=0}^{N-1} \chi_i z^i$, which is considered as an element of $H^1(\text{End}(M_\Lambda))$ due to the proposition above, gives the following deformation of $\Lambda(\epsilon)$:

$$\delta_{\chi(z)} \Lambda(\epsilon) = \chi(A(\epsilon)) \Lambda(\epsilon) - \Lambda(\epsilon) \chi(B(\epsilon)).$$

(18)
Remark 10  One knows that $H^2(Cohherent sheaves) = 0$ for the case of curves and so by general theory the map from $H^1(End(M_\Lambda))$ to the tangent space of deformations of the bundle $M_\Lambda$ is a bijection. The formula above can be taken as a definition for the map from the space of matrix polynomials $\chi(z) = \sum_i \chi_i z^i$ to the space $H^1(End(M_\Lambda))$. It means (it could be taken as a definition) that we associate with the matrix polynomial $\chi(z) = \sum_i \chi_i z^i$ such an element of $H^1(End(M_\Lambda))$ which deforms the bundle $M_\Lambda$ by the formula $\Lambda \mapsto \Lambda(e) + \chi(A(e))\Lambda(e) - \Lambda(e)\chi(B(e))$. What must be proved after such a definition is that one must describe the Serre’s pairing between $H^0(End(M_\Lambda) \otimes K)$ and the $H^1(End(M_\Lambda))$. We describe Serre’s pairing in proposition 12. The essential point consists in the interplay: if $\chi(z)$ acts on $\Lambda$ as above then the Serre’s pairing is described as in proposition 12.

Corollary 1  The matrix polynomial $\chi(z) = \chi_i z^i$, which satisfy the condition $\chi(A(e)) = \Lambda(e)\chi(B(e))\Lambda(e)^{-1}$ does not change $\Lambda$. This fact is in full agreement with the proposition 10 which says that such polynomial gives zero element in $H^1(End(M_\Lambda))$.

Corollary 2  The matrix polynomial $\chi(z) = \chi_0$, conjugates $\Lambda$ by the constant matrix, so it gives the same vector bundle. This fact is in full agreement with the proposition 10 which says that such polynomial gives zero element in $H^1(End(M_\Lambda))$.

The proposition may be argued as follows: consider an element $1 + \delta \chi(z)$, where $\delta^2 = 0$, it is an infinitesimal automorphism of the module $\mathbb{C}[z]^r$ corresponding to the endomorphism $\chi(z)$. Having such an automorphism it is clear how to deform the module $M_\Lambda$: new module is the set of elements $(1 + \delta \chi(z))s(z)$, where $s(z)$ is an element of $M_\Lambda$. The elements of the type $\tilde{s}(z) = (1 + \delta \chi(z))s(z)$ obviously satisfy the condition:

$$\tilde{s}(A(e)) = (1 + \delta \chi(A(e)))\Lambda(e)(1 + \delta \chi(B(e)))^{-1}\tilde{s}(B(e)),$$

hence

$$\tilde{s}(A(e)) = (\Lambda(e) + \delta(\chi(A(e))\Lambda(e) - \Lambda(e)\delta \chi(B(e))))\tilde{s}(B(e))$$

and we see that the new module is the module $M_{\Lambda(e)+\delta(\chi(A(e))\Lambda(e) - \Lambda(e)\delta \chi(B(e)))}$. Q.E.D.

There is another formal argument: the module $M_\Lambda$ is embedded in the module $\mathbb{C}[z]^r$, this module cannot be deformed, so deformations of $M_\Lambda$ are governed only by the deformation of the embedding $M_\Lambda \rightarrow \mathbb{C}[z]^r$. The elements of $H^1(End(M_\Lambda))$ are identified with some elements of $End(\mathbb{C}[z]^r)$ because of the fact that $\mathbb{C}[z]^r \rightarrow \mathbb{C}^p$ is a resolution of $M_\Lambda$. The elements of $End(\mathbb{C}[z]^r)$ obviously acts on the set of embeddings $M_\Lambda \rightarrow \mathbb{C}[z]^r$.

One can argue the proposition above more formally using the Čech description of cohomologies and sheaves. We consider the covering of our projective curve consisting of two charts: first is everything except the singular point, the second chart is not really an honest open set but a limit of the open sets - infinitesimal neighborhood of singular point i.e. $Spec\{ f \in \mathbb{C}(z), f$ is regular at $a_0$ and $b_0$ and $f(A(e)) = f(B(e)) \}$. (We need to consider such infinitesimal neighborhood because only in such neighborhood of singular point all modules become trivial, because it is a spectra of a local ring.) Hence any module on a singular curve can be given by glueing the two trivial modules by the glueing
function on the intersection of the two charts. Intersection of these two charts is the "general point" i.e. $\text{Spec} \{ \mathbb{C}(z) \}$. The first task is to describe the module $M_A$ by the gluing function. After that it’s obvious how to calculate what deformation corresponds to elements of $H^1(\text{End}(M_A))$. We represent an element of $H^1(\text{End}(M_A))$ as an element of $\chi \in \text{End}(M_A)$ on the intersection of two charts and one should simply multiply the gluing function by the element $1 + \delta \chi$. We obtain the new gluing function and the new bundle. It can be again represented in the form $M_A$. This gives the same results as above.

Let us give examples illustrating proposition 10 and proposition 11.

**Example 33** Consider the node curve with the affine part $\text{Spec} \{ f(A) = f(B) \}$, where $A, B \in \mathbb{C}$. Consider the matrix polynomial $\chi(z) = \chi_0 + \chi_1 z + (z - A)(z - B)\check{\chi}(z)$. According to proposition 10 it acts on $\Lambda$ by the formula $\delta \chi \Lambda = \chi(A)\Lambda - \Lambda\chi(B)$. The part $(z - A)(z - B)\check{\chi}(z)$ is zero in $A, B$ and it does not act on $\Lambda$. We can only consider $\chi(z) = \chi_0 + \chi_1 z$. According to proposition 10 $H^1(\text{End}(M_A))$ is the factor of the space $\chi_0 + \chi_1 z$ by the sum of the spaces $\chi(z) = \chi_0$ and $\chi(z) = \Theta(z - A) - \Lambda\Theta\Lambda^{-1}(z - B)$, where $\chi_0, \Theta$ are arbitrary matrices. It would be nice to have explicit parameterization of the orthogonal (with respect to Killing form) compliment to the sum of these two subspaces and to generalize it to the case of schematic points. The intersection of these two subspaces is the subspace of constant matrices $\chi(z) = \chi_0$, such that $\chi_0$ commutes with $\Lambda$ - this intersection is $H^0(\text{End}(M_A))$. We see that $\dim H^1(\text{End}(M_A)) = \dim H^0(\text{End}(M_A))$. This precisely coincides with the calculation from the Riemann-Roch theorem:

$$\dim H^0(\text{End}(M_A)) - \dim H^1(\text{End}(M_A)) = \deg(\text{End}(M_A)) - n^2(1 - \dim H^1(\mathcal{O})) = 0.$$

**Example 34** Consider the cusp curve $\Sigma^{\text{proj}}$. Recall that the affine part of $\Sigma^{\text{proj}}$ is given by $\text{Spec} \{ f(z) \in \mathbb{C}[z] : f(P + \epsilon) = f(P) \}$, the bundle $M_A$ corresponds to the module which is $\{ s(z) \in \mathbb{C}[z]^r : s(P + \epsilon) = (1 + \Lambda s(P) \}$ over the affine part. For the cusp curve proposition 10 explicitly means that: the space of matrix polynomials $\chi(z) = \chi_0 + \chi_1(z - P)$ maps surjectively to $H^1(\text{End}(M_A))$. The kernel of this map consists of the sum of the two linear subspaces in the space of matrix polynomials $\chi(z)$: the space of constant polynomials $\chi(z) = \chi_0$ and the space of such matrix polynomials which satisfy the condition: $\chi_1 = [\Lambda, \chi_0]$, for $i = 2, ..., N$. For the cusp curve proposition 10 explicitly means that: matrix polynomial $\chi(z) = \chi_0 + \chi_1(z - P)$ gives the following deformation of $\Lambda$:

$$\delta \chi(z)\Lambda = \Lambda + \chi_1 + [\chi_0, \Lambda].$$

Let us give an example (which seems a little amusing for us) illustrating lemma 10 and proposition 11.

**Example 35** Consider the curve with the affine part $\text{Spec} \{ f(\epsilon) = f(B(\epsilon)) \}$, where $B(B(\epsilon)) = \epsilon$, $e^N = 0$. According to theorem 1 the vector bundles are given by $k$ times such $\Lambda(\epsilon)$ that

$$\Lambda(\epsilon)\Lambda(B(\epsilon))\Lambda(B(B(\epsilon)))...\Lambda(B(B(\epsilon))) = \text{Id}$$

and $\Lambda_0 = \text{Id}$. (19)
Consider an arbitrary matrix valued polynomial \( \chi(z) = \sum_{i=0}^{N-1} \chi_i z^i \). Consider
\[
\tilde{\Lambda}(\epsilon) = \Lambda(\epsilon) + \delta(\chi(\epsilon)\Lambda(\epsilon) - \Lambda(\epsilon)\chi(B(\epsilon))),
\]
where \( \delta^2 = 0 \). Then \( \tilde{\Lambda} \) again satisfies the condition [19].

This can be seen simply rewriting
\[
\tilde{\Lambda}(\epsilon) = \exp(\delta \chi(\epsilon))\Lambda(\epsilon)\exp(-\delta \chi(B(\epsilon))).
\]

We can restate the example above as saying that the tangent space to the space of \( \Lambda(\epsilon) \) at the point \( \Lambda_0(\epsilon) \) which is defined by the equation [19] is the quotient of the space of all \( \chi(z) \) by the space \( \chi(z) = \Lambda_0(\epsilon)\chi(B(\epsilon))\Lambda_0(\epsilon)^{-1} \).

Let us describe the Serre’s duality.

**Proposition 12** The Serre’s pairing between \( H^0(End(M_\Lambda) \otimes K) \) and \( H^1(End(M_\Lambda)) \) can be written in terms of matrix polynomials \( \tilde{\Phi}(z) \) and \( \chi(z) \) very simply:
\[
Tr Re z \chi(z) \tilde{\Phi}(z).
\]

**Corollary 3** Consider a matrix polynomial \( \chi(z) = \sum_i \chi_i z^i \), which satisfy the condition \( \chi(A(\epsilon)) = \Lambda(\epsilon)\chi(B(\epsilon))\Lambda(\epsilon)^{-1} \). The Serre’s pairing (given by the formula [20]) between such \( \chi(z) \) and arbitrary \( \Phi(z) \in H^0(End(M_\Lambda) \otimes K) \) is identically zero. This fact is in a full agreement with proposition [17] which says that such polynomial gives the zero element in \( H^1(End(M_\Lambda)) \).

**Corollary 4** Consider a matrix polynomial \( \chi(z) = \chi_0 \), The Serre’s pairing (given by the formula [20]) between such \( \chi(z) \) and arbitrary \( \Phi(z) \in H^0(End(M_\Lambda) \otimes K) \) is identically zero. This fact is in a full agreement with proposition [17] which says that such polynomial gives the zero element in \( H^1(End(M_\Lambda)) \).

The lemma is quite obvious. To prove corollaries we use proposition [9] in order to represent \( \tilde{\Phi}(z) \) as a matrix polynomial:
\[
Res_\epsilon \frac{\Phi(\epsilon)dz}{z-A(\epsilon)} - \frac{\Lambda^{-1}(\epsilon)\Phi(\epsilon)\Lambda(\epsilon)dz}{z-B(\epsilon)}
\]
for some \( \Phi(\epsilon) \). And the corollaries follow obviously from the properties that \( Tr AB = Tr BA \) and \( Res_\epsilon Res_\zeta = Res_\zeta Res_\epsilon \).

**Remark 11** To prove the second corollary we should also use the condition
\[
Res_\epsilon(\Phi(\epsilon) - \Lambda^{-1}(\epsilon)\Phi(\epsilon)\Lambda(\epsilon)) = 0.
\]
The first corollary does not use this condition for \( \Phi(\epsilon) \) and is true for all \( \Phi(z) \) represented in the form
\[
Res_\epsilon \frac{\Phi(\epsilon)dz}{z-A(\epsilon)} - \frac{\Lambda^{-1}(\epsilon)\Phi(\epsilon)\Lambda(\epsilon)dz}{z-B(\epsilon)}
\]
with arbitrary \( \Phi(\epsilon) \).
6 Symplectic form

6.1 Description

In the expert’s language in this section we prove the following:

Claim: the canonical one-form on the cotangent bundle to the moduli space of vector bundles on the curve \( \Sigma_{A(\epsilon),B(\epsilon)} \) in terms of \( \Lambda(\epsilon), \Phi(\epsilon) \) coincides with the form:

\[
TrRes_{\epsilon}\Lambda(\epsilon)^{-1}\Phi(\epsilon)d\Lambda(\epsilon).
\] (21)

Let us formulate the claim above more exactly. In proposition 9 we showed that the elements of \( H^0(End_{M_\Lambda} \otimes K) \) can be described as

\[
\tilde{\Phi}(z) = \frac{\Phi(\epsilon)dz}{z-A(\epsilon)} - \frac{\Lambda^{-1}(\epsilon)\Phi(\epsilon)\Lambda(\epsilon)dz}{z-B(\epsilon)}
\]

where \( \Phi(\epsilon) \) is such that it satisfies the condition:

\[
Res_{\epsilon}\Phi(\epsilon) - \Lambda^{-1}(\epsilon)\Phi(\epsilon)\Lambda(\epsilon) = 0.
\] (22)

The space \( H^0(End_{M_\Lambda} \otimes K) \) is cotangent to the moduli space of vector bundles at the point \( M_\Lambda \). By proposition 9 we have the map

\[
p : \{\text{set of pairs}:(\Lambda(\epsilon), \Phi(\epsilon)) \text{which satisfy the condition}\} \rightarrow T^*Bun.
\] (23)

More exactly we also request \( \Lambda(\epsilon) \) to satisfy the condition as in theorem 1. The map \( p \) is not one-to-one but it is surjective. The pairs conjugated by the action of \( GL(n) \) gives the same point in \( T^*Bun \). We will show latter that the condition (22) is the same as to say that ”the moment map equals zero” for the natural action of \( GL(n) \) and the canonical symplectic form on the space \( (\Lambda(\epsilon), \Phi(\epsilon)) \). (This space is cotangent to the space \( (\Lambda(\epsilon)) \)).

Consider some pair \( \Lambda(\epsilon), \Phi(\epsilon) \) which satisfies the condition. Consider some tangent vector \( \delta\Lambda(\epsilon), \delta\Phi(\epsilon) \), such that it is tangent to the surface defined by the equation (22).

**Theorem 2** The canonical one-form on the cotangent bundle to the moduli space of vector bundles applied to the vector \( p(\delta\Lambda(\epsilon), \delta\Phi(\epsilon)) \) at a point \( p(\Lambda(\epsilon), \Phi(\epsilon)) \) is equal to

\[
TrRes_{\epsilon}\Lambda(\epsilon)^{-1}\Phi(\epsilon)d\Lambda(\epsilon).
\] (24)

**Proof** Let us take the tangent vector \( \delta\chi(z) \) considered in propositions 10, 11. Consider the tangent vector \( p(\delta\chi(z), \delta\Phi) \) with arbitrary \( \delta\Phi \) at arbitrary point \( p(\Lambda(\epsilon), \Phi(\epsilon)) \). By definition of the canonical 1-form on the cotangent bundle to the manifold its value on
the vector \( p(\delta(x), \delta \Phi) \) at the point \( p(\Lambda(\epsilon), \Phi(\epsilon)) \) equals to

\[
\langle p(\Phi(\epsilon) | p(\delta(x)) \rangle
\]

by propositions 9, 112 equals to:

\[
\text{TrRes}_e (\Phi(\epsilon)dz - \Lambda^{-1}(\epsilon)\Phi(\epsilon)\Lambda(\epsilon)dz) \chi(z) =
\]

\[
= \text{TrRes}_e \Phi(\epsilon) \chi(A(\epsilon)) - \Lambda^{-1}(\epsilon)\Phi(\epsilon)\Lambda(\epsilon) \chi(B(\epsilon)) =
\]

\[
= \text{TrRes}_e \Lambda^{-1}(\epsilon)\Phi(\epsilon) \left( \chi(A(\epsilon))\Lambda(\epsilon) - \Lambda(\epsilon) \chi(B(\epsilon)) \right) =
\]

by proposition 11 it equals to the desired result:

\[
= \text{TrRes}_e \Lambda^{-1}(\epsilon)\Phi(\epsilon) \delta(\epsilon). \quad (25)
\]

By proposition 11 all the tangent vectors can be represented as some \( \delta(x) \) and the theorem is proved □

### 6.2 Hamiltonian reduction, moment map and holomorphity condition

**Lemma 5** The action of the group \( GL(n) \) by conjugation on pairs \( \Lambda(\epsilon), \Phi(\epsilon) \) is hamiltonian with respect to the symplectic form \( \text{TrRes}_e d(\Lambda(\epsilon)^{-1}\Phi(\epsilon)) \wedge d\Lambda(\epsilon) \) and the moment map is given by the formula:

\[
\text{Res}_e \left( \Phi(\epsilon) - \Lambda(\epsilon)^{-1}\Phi(\epsilon)\Lambda(\epsilon) \right)
\]

The proof is standard and is left to the reader.

**Corollary 5** we see that the equation "moment equals zero" on \( \Phi(\epsilon) \) coincides with the holomorphity condition for \( \Phi(z) \in H^0(\text{End}(M_{\Lambda} \otimes K)) \) see equation (22).

As a corollary we obtain the following theorem:

**Theorem 3** The phase space of Hitchin system which is the total space of the cotangent bundle to the moduli space of vector bundles on a singular curve can be obtained as a hamiltonian reduction of the space of pairs \( \Lambda(\epsilon), \Phi(\epsilon) \) with the symplectic form \( \text{TrRes}_e d(\Lambda(\epsilon)^{-1}\Phi(\epsilon)) \wedge d\Lambda(\epsilon) \) by the action the of \( GL(n) \) by conjugation. (More precisely one should speak about submanifold in the space \( \Lambda(\epsilon), \Phi(\epsilon) \) as in theorem 1). The projection acts as follows: \( \Lambda(\epsilon) \) maps to vector the bundle \( M_{\Lambda} \) as described in theorem 11. \( \Phi(\epsilon) \) maps to the cotangent covector as it is described in proposition 12 (recall that \( H^0(\text{End}(M_{\Lambda} \otimes K)) = T_{M_{\Lambda}}^* \text{Bun} \)). Let us emphasize that this map respects the symplectic structures on both spaces which means that the canonical symplectic structure on the cotangent bundle to the moduli space of vector bundles is precisely the one obtained from

\[
\text{TrRes}_e d(\Lambda(\epsilon)^{-1}\Phi(\epsilon)) \wedge d\Lambda(\epsilon)
\]

on the space of pairs \( \Lambda(\epsilon), \Phi(\epsilon) \).

The Lax operator is defined as

\[
\Phi(z) = \text{Res}_e \left( \frac{\Phi(\epsilon)dz}{z - A(\epsilon)} - \frac{\Lambda^{-1}(\epsilon)\Phi(\epsilon)\Lambda(\epsilon)dz}{z - B(\epsilon)} \right)
\]

The generating functions for the Hitchin’s hamiltonians are defined as \( \text{Tr}\Phi(z)^k \).
7 Integrability

The aim of this section is to prove the commutativity of Hitchin’s Hamiltonians for the case of our singular curves. (Hitchin’s proof is not applicable in this case). In the previous section we described the Hitchin system as the Hamiltonian reduction. The aim of this section is to prove that $Tr \Phi(z)^k$ and $Tr \Phi(w)^l$ Poisson commute for all $z, w, k, l$. This functions are invariant with respect to the $GL(n)$ action, so they can be pushed down to the reduced space, which is by definition the phase space of Hitchin system and this functions are by definition Hitchin’s Hamiltonians. The main property of Hamiltonian reduction that it preserves the Poisson bracket between invariant functions. So it’s enough to prove that $Tr \Phi(z)^k$ and $Tr \Phi(w)^l$ Poisson commute on the nonreduced space, which is just the space of pairs $\Lambda(\epsilon), \Phi(\epsilon)$. This can be done by the $r$-matrix technique. It’s very easy and well-known among experts that the Poisson bracket between $\Phi(\epsilon)$ is of $r$-matrix form, but it is quite amusing for us that it’s also true for the

$$\Phi(z) = \text{Res}_\epsilon \left( \frac{\Phi(\epsilon)dz}{z - A(\epsilon)} - \frac{\Lambda^{-1}(\epsilon)\Phi(\epsilon)\Lambda(\epsilon)dz}{z - B(\epsilon)} \right)$$

(see lemma 14). It would be very interesting to understand the relation between our $r$-matrix approach in the case of singular curves and $r$-matrix approach proposed in [7], [8].

7.1 Truncated $\delta$-functions and their properties

The formula for the Poisson bracket between canonically conjugated variables $P(\eta)$ and $\Lambda(\epsilon)$ (see the next section) includes expression

$$\delta^\eta_\epsilon = \sum_{k=0,...,N-1} \frac{(\eta)^k}{(\epsilon)^{k+1}}.$$ 

Informally speaking one should think about it as about the delta-function. We will prove some elementary properties of it which we need in order to simplify expressions for Poisson brackets between $\Phi(\eta)$.

Lemma 6 For arbitrary function $f(\epsilon)$ and the ”delta”-function given by

$$\delta^\eta_\epsilon = \sum_{k=M,...,N-1} \frac{(\eta)^k}{(\epsilon)^{k+1}} \ (27)$$

the following equality is true:

$$|f(\epsilon)\delta^\eta_\epsilon|_\epsilon = |f(\eta)\delta^\eta_\epsilon|_\eta \ (28)$$

where $[...]_\epsilon$ is defined as a linear operation, which acts as follows on the basis of monomials:

$$\left| \frac{1}{\eta^ke^l} \right|_\eta = \begin{cases} \frac{1}{\eta^{k+1}e^{l+1}}, & \text{if } N \geq k \geq M+1 \text{ and } N \geq l \geq M+1 \\ 0, & \text{otherwise} \end{cases} \ (29)$$
Proof It’s easy to see that for \( f(\epsilon) = \epsilon^l, \ l \in \mathbb{Z} \) both sides of the equality can be rewritten as:
\[
\sum_{\max(M;l-N \leq k \leq \min(N-1;l-(M+1))} \frac{1}{\eta^{k+1}\epsilon^{l-k}}.
\]

Analogously we define \( |...|_\eta \) for the function of one variable as a linear operation defined on the monomials as follows
\[
\left| \frac{1}{\eta^k} \right|_\eta = \begin{cases} \frac{1}{\eta^k}, & \text{if } N \geq k \geq M + 1 \\ 0, & \text{otherwise} \end{cases} \quad (30)
\]

Obviously \( |f(\epsilon)g(\eta)|_\eta = |f(\epsilon)|_\eta|g(\eta)|_\eta \) where \( |...|_\eta \) for the function of two variables is defined by (29). We introduce another linear operation \( |...|_\eta \) for functions in \( \eta \) as follows:
\[
|\eta^k|_\eta = \begin{cases} \eta^k, & \text{if } -N \leq k < N \\ 0, & \text{otherwise} \end{cases} \quad (31)
\]

We will use this cutting only for polar expressions and in this case it is equivalent to the multiplication in \( \mathbb{C}[\frac{1}{\eta}] \frac{1}{\eta}\mathbb{N}^+ \).

Corollary 6 For the case \( M = 0 \), \( f(\epsilon) = \sum_{-N \leq i < N-1} f_ie^i, \ g(\epsilon) = \sum_{0 \leq i < N} g_ie^i \) the following is true:
\[
|f(\epsilon)g(\eta)\delta^\epsilon_\eta|_\eta = \left| \frac{f(\epsilon)\delta^\epsilon_\eta}{g(\eta)} \right|_\eta = \left| \frac{f(\epsilon)\delta^\epsilon_\eta}{g(\eta)} \right|_\eta = \left| \frac{f(\epsilon)\delta^\epsilon_\eta}{g(\eta)} \right|_\eta \quad (32)
\]

Remark 12 In the case \( M = -N \) obviously \( \delta^\eta_\eta = \delta^\epsilon_\eta \). In the case \( M = -\infty, N = +\infty \) we do need any cutting in the formula (28). And we obtain the well-known formula: \( f(\epsilon)\delta = f(\eta)\delta \).

Example 36 Let \( f(\epsilon) = \epsilon^{-4} \), and \( \delta^\eta_\eta = \sum_{k=0,1,...,N-1} (\frac{\eta^k}{\epsilon})^{k+1} \), where \( N \geq 1 \), then both sides of the equality (28) gives:
\[
\frac{1}{\epsilon^4\eta^4} + \frac{1}{\epsilon^4\eta^4}.
\]

Lemma 7 For \( f(z) = \sum_{i \in \mathbb{Z}} f_iz^i \) one obviously has:
\[
\text{Res}_zf(z)\delta^w_z = \sum_{i=M,...,N-1} f_iz^i \quad (33)
\]

Lemma 8 For \( M = 0 \) and \( f(\epsilon) = \sum_{-N \leq i < 0} f_ie^i \) one has:
\[
\left| (f(\epsilon) - f(\eta))\delta^\epsilon_\eta - |(f(\epsilon) - f(\eta))\delta^\epsilon_\eta|_\eta \right| = 0.
\]

Proof The expression in question is a positive in \( \epsilon \) part of \( (f(\epsilon) - f(\eta))\delta^\epsilon_\eta \). Let us test it for basic monomials:
\[
\left| (\epsilon^{-n} - \eta^{-n})\delta^\epsilon_\eta - |(\epsilon^{-n} - \eta^{-n})\delta^\epsilon_\eta|_\eta \right| = \sum_{k=n}^{N-1} \frac{\epsilon^{k-n}}{\eta^{k+1}} - \sum_{k=0}^{N-1-n} \frac{\epsilon^k}{\eta^{k+1+n}}
\]

which is zero by a simple change of indexes. □

Here we present the final version of the previous lemma which is also demonstrated by straightforward calculation.
Lemma 9  For $M = 0$ and $f(\epsilon) = \sum_{-N \leq i < N-1} f_i \epsilon^i$ one has:

$$|(f(\epsilon)) - f(\eta))\delta^\epsilon_\eta| - |(f(\epsilon)) - f(\eta))\delta^\epsilon_\eta|,_{\epsilon, \eta} = 0.$$ 

7.2 Non-reduced Poisson bracket and $r$-matrix

In what follows we use the notion of the reduced $\delta$-function with $M = 0$ and $N$ is defined by the construction of the curve, so that schematic points are morphisms to $\mathbb{C}[\epsilon]/\epsilon^N$. So by $\delta^\eta_\epsilon$ we mean

$$\delta^\eta_\epsilon = \sum_{k=0}^{N-1} \frac{\eta^k}{\epsilon^{k+1}}.$$ 

Now we calculate Poisson bracket for the ingredients of our construction. Let $\mathcal{L}$ be the space of $\Lambda$ subject to the relation "defining the module". It is the subset of invertible elements in $Mat_{K \times K}[\epsilon]/\epsilon^N$. The cotangent bundle $\mathcal{T}^*\mathcal{L}$ has a canonical parameterization with conjugated variables $P(\epsilon)$ and $\Lambda(\epsilon)$ such that the symplectic form is defined as

$$\omega = Res_\epsilon Tr(dP(\epsilon) \wedge d\Lambda(\epsilon)), \quad (34)$$

where

$$\Lambda(\epsilon) = \sum_{k=0}^{N-1} \Lambda_k \epsilon^k \quad P(\epsilon) = \sum_{k=0}^{N-1} P_k \epsilon^{-k-1}. \quad$$

Matrix coefficients $\Lambda^i_j$ and $P^i_j$ are canonically conjugated. We could write the Poisson bracket in the matrix form as follows:

$$\{P(\epsilon) \otimes \Lambda(\eta)\} = \delta^\eta_\epsilon R, \quad (35)$$

where $R$ is the trivial solution of the Yang-Baxter equation, which is the transposition matrix in the tensor product, so it acts as follows

$$R(v_1 \otimes v_2) = v_2 \otimes v_1$$

and in terms of matrix elements it could be expressed by

$$R = \sum_{i,j} e_{ij} \otimes e_{ji}.$$ 

It is worth to mention that

$$\{P(\epsilon) \otimes P(\eta)\} = \{\Lambda(\epsilon) \otimes \Lambda(\eta)\} = 0. \quad (36)$$

Now we introduce variables $\Phi(\epsilon)$ by the formula

$$\Phi(\epsilon) = |\Lambda(\epsilon)P(\epsilon)|,,-,$$

(obviously it is true that $P(\epsilon) = |\Lambda^{-1}(\epsilon)\Phi(\epsilon)|,,-$)
Lemma 10

\[
\{\Phi(\epsilon) \otimes \Phi(\eta)\} = \left|\Phi(\epsilon) \otimes 1, \delta_\eta^e R|_{-} = \left|\Phi(\epsilon) \otimes 1 + 1 \otimes \Phi(\eta), \delta_\eta^e R|_{\eta} \right. \quad \text{(37)}
\]

Proof

\[
\{\Phi(\epsilon) \otimes \Phi(\eta)\} = \{|\Lambda(\epsilon)P(\epsilon)|_{-} \otimes |\Lambda(\eta)P(\eta)|_{-}\} = \{|\Lambda(\epsilon)P(\epsilon) \otimes \Lambda(\eta)P(\eta)|_{-}\} = \]

\[
|\left(\Lambda(\epsilon) \otimes 1\right)\delta_\eta R(1 \otimes P(\eta)) - (1 \otimes \Lambda(\eta))\delta^e R(P(\epsilon) \otimes 1)|_{-} =
\]

\[
\left|\left(\Lambda(\epsilon)P(\epsilon) \otimes 1\right)\delta_\eta R|_{-} - |(1 \otimes \Lambda(\eta)P(\eta))\delta^e R|_{-} =
\right.
\]

\[
\text{}/ \text{ here we used corollary } \Box \text{ and the following property of the transposition operator }
\]

\[
R(1 \otimes A) = R(1 \otimes A)RR = (A \otimes 1)R/
\]

\[
\left|\left|\left(\Lambda(\epsilon)P(\epsilon)|_{-} \otimes 1\right)\delta_\eta R|_{-} - \left|1 \otimes |\Lambda(\eta)P(\eta)|_{-}\right.\right.\delta^e R|_{-} =
\]

\[
\left|\left(\Phi(\epsilon) \otimes 1\right)\delta_\eta^e R|_{-} - \left|1 \otimes \Phi(\eta)\right.\right.\delta^e R|_{-} =
\]

\[
\left|\left[\Phi(\epsilon) \otimes 1, \delta_\eta^e R|_{-} = \left|\left[1 \otimes \Phi(\eta), \delta^e R|_{-} =
\right.\right.\right.
\]

\[
\left|\left[\Phi(\eta) \otimes 1, \delta^e R|_{-} = \left|\left[1 \otimes \Phi(\eta), \delta^e R|_{-} =
\right.\right.\right.
\]

For proving the second equality we use the same strategy as in the infinite \(N\) case \[\Box\] but we have no more complex analysis intuition in virtue of the formality of all the expressions. By lemma \[\Box\] we have

\[
\left[\Phi(\epsilon) \otimes 1, \delta_\eta^e R\right] - \left\{\Phi(\epsilon) \otimes \Phi(\eta)\right\} = \left[\Phi(\eta) \otimes 1, \delta^e R|_{-} - \left[\Phi(\eta) \otimes 1, \delta^e R|_{-}.
\right.\right.\]

The \(|...|_{-}\) in the r.h.s. is zero and using

\[
\left[\Phi(\eta) \otimes 1, \delta^e R\right] = -\left[1 \otimes \Phi(\eta), \delta^e R\right]
\]

one obtains the result. \[\blacksquare\]

Now we introduce right-invariant vector fields:

\[
\Psi(\epsilon) = |P(\epsilon)\Lambda(\epsilon)|_{-} ; \quad P(\epsilon) = |\Psi(\epsilon)\Lambda^{-1}(\epsilon)|_{-} \quad \text{(38)}
\]

By the same method we obtain the following

Lemma 11

\[
\left\{\Psi(\epsilon) \otimes \Psi(\eta)\right\} = -\left[\Psi(\epsilon) \otimes 1, \delta_\eta^e R\right|_{-} = -\left[\Psi(\epsilon) \otimes 1 + 1 \otimes \Psi(\eta), \delta_\eta^e R\right|_{\eta}. \quad \text{(39)}
\]

Lemma 12

\[
\left\{\Phi(\epsilon) \otimes \Psi(\eta)\right\} = 0. \quad \text{(40)}
\]
Remark 13 In general this is true due to the fact that right- and left-invariant vector fields commute, but in our special case we prefer to give an explicit demonstration.

Proof

\[
\{ \Phi(\epsilon) \otimes \Psi(\eta) \} = |\{ \Lambda(\epsilon) P(\epsilon) \otimes P(\eta) \Lambda(\eta) \}|_-
= |\Lambda(\epsilon) \otimes P(\eta) \{ P(\epsilon) \otimes \Lambda(\eta) \} + \{ \Lambda(\epsilon) \otimes P(\eta) \} P(\epsilon) \otimes \Lambda(\eta) |_-
= |\Lambda(\epsilon) \otimes P(\eta) \delta_\epsilon^R \delta_\eta^R - \Lambda(\eta) \otimes P(\epsilon) \delta_\eta^R \delta_\epsilon^R |_-
\]

which is zero due to the corollary \(Q \quad \Box\)

Now we combine \(\Phi(\epsilon)\) and \(\Psi(\epsilon)\) into the expression

\[
\Phi(z, \epsilon) = \left| \frac{\Phi(\epsilon)}{z - A(\epsilon)} - \frac{\Psi(\epsilon)}{z - B(\epsilon)} \right|_{-\{\epsilon\}},
\]

formal in \(\epsilon\) and analytic in \(z\), where the decomposition order is always implied as

\[
\frac{1}{z - A(\epsilon)} = \sum_{k=0}^{\infty} \frac{A^k(\epsilon)}{z^{k+1}} \bigg|_\epsilon.
\]

We need also for the analytic half \(\delta\)-function

\[
\delta^z_w = \sum_{k=0}^{\infty} \frac{z^k}{w^{k+1}}.
\]

Lemma 13

\[
\{ \Phi(z, \epsilon) \otimes \Phi(w, \eta) \} = \left| [\Phi(z, \epsilon) \otimes 1, R] \delta^z_\epsilon \delta^z_w \right|_{\text{all variables}} \quad (41)
\]

Proof Using (37), (39), (40) one obtains

\[
\{ \Phi(z, \epsilon) \otimes \Phi(w, \eta) \} = \left| \frac{\{ \Phi(\epsilon) \otimes \Phi(\eta) \}}{(z - A(\epsilon))(w - A(\eta))} + \frac{\{ \Psi(\epsilon) \otimes \Psi(\eta) \}}{(z - B(\epsilon))(w - B(\eta))} \right|_{-\{\epsilon, \eta\}}
\]

\[
= \left| \frac{[\Phi(\epsilon) \otimes 1, \delta^z_\epsilon R]}{(z - A(\epsilon))(w - A(\eta))} - \left[ \frac{\Psi(\epsilon) \otimes 1, \delta^z_\eta R}{(z - B(\epsilon))(w - B(\eta))} \right] \right|_{-\{\epsilon, \eta\}}
\]

/using the fact that \(\frac{1}{(z - A(\epsilon))(w - A(\eta))}\) has only positive powers of \(\epsilon\) we continue/

\[
= \left| \frac{[\Phi(\epsilon) \otimes 1, \delta^z_\epsilon R]}{(z - A(\epsilon))(w - A(\eta))} - \frac{[\Psi(\epsilon) \otimes 1, \delta^z_\eta R]}{(z - B(\epsilon))(w - B(\eta))} \right|_{-\{\epsilon, \eta\}}
\]

/and using one more time corollary \(R\) we find/

\[
= \left| \frac{[\Phi(\epsilon) \otimes 1, \delta^z_\epsilon R]}{(z - A(\epsilon))(w - A(\epsilon))} - \frac{[\Psi(\epsilon) \otimes 1, \delta^z_\eta R]}{(z - B(\epsilon))(w - B(\epsilon))} \right|_{-\{\epsilon, \eta\}}
\]

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Here we need for some properties of formal expression \( \frac{1}{z - A(\epsilon)(w - A(\epsilon))} \)

\[
\frac{1}{z - A(\epsilon)} \frac{1}{w - A(\epsilon)} = \left[ \sum_{k=0}^{\infty} \frac{A^k(\epsilon)}{z^{k+1}} \right] \left[ \sum_{l=0}^{\infty} \frac{A^l(\epsilon)}{w^{l+1}} \right] = \left[ \sum_{m=0}^{\infty} A^m(\epsilon) \sum_{k=0}^{m} \frac{1}{z^{k+1}w^{m-k+1}} \right] \epsilon
\]

\[
= \left[ \sum_{m=0}^{\infty} A^m(\epsilon) \sum_{k=0}^{m} \frac{w^k}{z^{k+1}w^{m-k+1}} \right] \epsilon = \left[ \frac{1}{w - A(\epsilon)} \delta_w \right] \epsilon_{\{z, w\}} = \left[ \frac{1}{z - A(\epsilon)} \delta_z \right] \epsilon_{\{z, w\}}.
\]

Using this we proceed by

\[
\{ \Phi(z, \epsilon) \otimes \Phi(w, \eta) \} = \left[ \left( \frac{\Phi(\epsilon)}{z - A(\epsilon)} + \frac{\Psi(\epsilon)}{z - B(\epsilon)} \right) \otimes 1, R \right] \delta_{\eta \omega} = 0,
\]

and this finishes the proof because \( \delta^z_\eta \) has only positive powers in \( \epsilon \)

\( \Box \)

The geometrical object, i.e. the cotangent vector to the space of holomorphic bundles expresses as

\( \Phi(z) = \text{Res}_\epsilon \Phi(z, \epsilon) \).

We call this by the same letter implying that the \( z, w \) parameterize cotangent vectors while \( \epsilon, \eta \) are respective for the residue at schematic point. For this expression we obtain the following Poisson brackets:

**Lemma 14**

\[
\{ \Phi(z) \otimes \Phi(w) \} = \left[ \left( \frac{\Phi(z)}{z - A(\epsilon) + 1} \right) \otimes 1, R \right] \delta_{\omega \gamma} = 0.
\]

**Proof** The first equality follows from:

\[
\{ \Phi(z) \otimes \Phi(w) \} = \text{Res}_{\epsilon, \eta} \{ \Phi(z, \epsilon) \otimes \Phi(w, \eta) \} = \left[ \text{Res}_{\epsilon, \eta} \Phi(z, \epsilon) \otimes 1, R \right] \delta_{\omega \gamma}.
\]

where we have used [14]. The rest is proved as in lemma [10] \( \Box \)

### 7.3 Commutativity of Hitchin hamiltonians

**Definition 2** The quantities \( H_k(z) = \text{Tr} \Phi^k(z) \) are called the Hamiltonians of our system.

The following theorem is the precursor of the integrability.

**Theorem 4** The quantities \( H_k(z) \) Poisson commute

\[
\{ H_k(z), H_m(w) \} = 0.
\]

**Proof** We use the linearity of Poisson brackets, the \( |...|_w \) operation and the Leibnitz rule:

\[
\{ |\text{Tr}\Phi^k(z)|_z, |\text{Tr}\Phi^m(w)|_w \} = \{ |\text{Tr}\Phi^k(z), \text{Tr}\Phi^m(w)|_w, z \} = 0.
\]
We show now that interior $|...|_w$ is tautological. It is always true for function $f(w), g(w)$ with only negative powers in $w$ that

$$|f(w)g(w)|_w = |f(w)g(w)|_w.$$  

Applying this one obtains

$$\text{(43)} = (kl) * \text{Tr}_{1,2} \left[ \Phi^{i-1} \otimes \Phi^{j-1}(w) | \Phi^{i-1}(z) \otimes 1 + 1 \otimes \Phi^{j-1}(w), \delta^w_z R \right]_{w, z}$$

which is zero. ■

**Corollary 7** The Hitchin's Hamiltonians on the cotangent bundle to the moduli space of vector bundles on curves Poisson commute for the case of our singular curves also. And so they form the integrable system.

This follows from the fact that hamiltonian reduction preserves the Poisson bracket of invariant functions. Functions $H_k(z)$ Poisson commute on the space of pairs $\Lambda(\epsilon), \Phi(\epsilon)$ as we just proved, they are invariant with respect to the action of $GL(n)$. The cotangent bundle to the moduli space of vector bundles on our singular curves can be obtained as hamiltonian reduction from the space of pairs $\Lambda(\epsilon), \Phi(\epsilon)$ (see theorem 3). And finitely Hitchin hamiltonians commute on the reduced space. The calculation that the number of independent hamiltonians is the half of the dimension of the phase space is the same as in the case of nonsingular curves.

### 8 Discussion and relations with other works

A sort of similar constructions appears in the context of Beauville system [10, 11] which is an integrable system on the space of rational matrices. We have to mention that the specific choice of orbits in Beauville approach corresponds to Hitchin system on singular curves (see [12], examples 1,2). There will be interesting to generalize the way to choose orbits for more complicated singularities considered in the present paper. Another important analysis was effectuated in [13] where it was considered the limit procedure to obtain higher order poles in the Lax operator as a fusion of simple poles and it was obtained an elliptic analog of the Lax operator with double pole.

We should also remark that in [14], [15] there were obtained some infinite-dimensional hamiltonian quotient constructions do describe the finite-dimensional spaces of connections with poles. It would be interesting to work out analogous constructions for our phase spaces this may give immediate proof of integrability of some of our systems.
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