A Generalized Geometric Measurement of Quantum Discord: Exact Treatment

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A generalization of the geometric measure of quantum discord is introduced in this article, based on Hellinger distance, which has virtue of computability and independence of local measurements. In addition our definition also does not suffer from the recently raised critiques about quantum discord. Importantly the exact result can be obtained for bipartite pure states with arbitrary levels, which is completely determined by the Schmidt decomposition of the states. For bipartite mixed states the exact result can also be found for X type case, of which the sub matrices are spanned by the Schmidt-type states. Furthermore this definition has a natural generalization into multipartite states. As for symmetric case, permutational or translational invariance, we shown that it can be evaluated exactly by supposing that the “nearest” completely classical state shares the same symmetry of the state. In addition we show that our definition can also be used to mark the appearance of quantum phase transition in many-body systems.

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I. INTRODUCTION

Quantum systems can exhibit non-classical correlation by a different way from quantum entanglement, which is known as quantum discord (QD). The quantum discord characterizes the minimal perturbation induced by single-party von Neumann measurement \[1\]. And thus there exists non-entangled state with non-zero QD. Importantly QD is now shown as a resource to speed up the quantum information processing. For instance the determined quantum computation with one qubit \[2\] and quantum metrology with noised states \[3\] have been demonstrated an advantage over classical computation even without entanglement. Moreover QD has also inspired great attention in other diverse contexts \[4\].

However QD is so hard to determine exactly because of the optimization in the definition that there exist very few exact results, even for the simplest two-qubit case \[5\]. (Recently an exact evaluation of QD is proposed in Ref \[6\]. However it is pointed out in \[7\] that this approach is not exactly correct.) It has been shown that the computation of quantum discord is NP-complete \[8\]; the running time of the computation of QD is increased exponentially with the dimension of the Hilbert space. Thus one has to find another way to measure QD. With respect of this fact, geometric measure of QD is introduced by Dakić and the coauthors, which is defined as the square form of the shortest distance between arbitrary state \(\rho\) and zero-discord state \(\chi\) in Hilbert space \[9\]. By this feature the optimization in the geometric definition of discord (GD) can be reduce greatly, and more a tight lower bound of GD can be determined for arbitrary states \[10\]. However the square form of GD is not monotonic under local operations, of which the value can increase by local operations \[9\] \[11\]. This deficit raises the question whether GD and furthermore QD can unambiguously manifest the non-classical correlation in quantum states \[11\] \[12\]. In order to overcome this problem, many generalization of GD have been proposed. For instance a rescaled GD is defined by rescaling the density operator with its norm in \[13\]. Furthermore the so-called Schatten \(p\)-norm is introduced to qualify the distance \[13\], instead 2-norm in Ref. \[9\]. In addition the Bures distance is also introduced \[15\]. However it is difficult genially by these generalization to find the analytical expression for GD since their involved evaluation and optimization of the eigenvalue and corresponding eigenvectors in the expressions. Recently an interesting generalization is proposed by introducing the Hellinger distance \[16\] \[18\]. This definition has a simple structure and can be evaluated readily. Moreover it is also monotonically non-increasing (contractivity) under local operations \[17\] \[18\].

In addition it is interesting how to generalize this definition into multipartite case. A direct way is to introduce the three-tangle as a generalization of QD for tripartite case \[19\] and its generalization for four-qubit case \[20\], as have done in studying tripartite entanglement. This approach has virtue of clear discrimination of bipartite and multipartite quantum correlation. However it is difficult to determine analytically and furthermore to generalize into many-body case. Another way is to find the minimum of the QD between arbitrary single party and the others \[21\], termed as global QD. However it do not consider the other possible bipartite correlation, e.g. \((n, N − n)\) division of the system with total \(N\) parties, and thus is not a comprehensive measurement of QD. Additionally the exact treatment of global QD is still difficult since one has to find optimal single party von Neumann measurements for all parties. Recently a geometric generalization of global QD is proposed by finding the shortest distance from the zero global QD state \[22\]. However the author adopt the 2-norm of distance, which suffers from the critique of non-contractivity under local operations \[11\].

With respect of these facts, we present an alternative approach to QD by a generalization of Hellinger distance \[16\] \[18\], of which satisfies the requirements of a good measure of quantum correlation \[17\] \[18\]. Furthermore in order to avoid the critique in Ref. \[12\], the shortest distance is defined instead as from the completely classical state \[23\], which hence becomes independent on the local measurements. By this generalization QD can be exactly evaluated for arbitrary
bipartite pure states and for some special mixed states, as shown in this article. Moreover this generalization can be readily applied to multipartite case. As for symmetric multipartite states the exact results can be founded. This article is divided into several sections. The definition is presented in Section II, and a general expression for this geometric QD is also presented. Then Section III presents the exact results for bipartite pure states and for a special type of mixed states. In Section IV we apply the definition for multipartite states and the exact result can be obtained for symmetric states. In addition we also show the ability of marking the quantum phase transition in many-body systems. Conclusion and discussion are presented in final section.

II. DEFINITION AND TECHNICAL PREPARATION

We first present the definition.

Definition 1  Given arbitrary state $\rho$ and completely classical state $\sigma$, we define geometric measure of QD

$$D^H = \frac{1}{2} \min_{\sigma} \| \sqrt{\rho} - \sqrt{\sigma} \|^2,$$  \hspace{1cm} (1)

where $\| \cdot \|$ is the Hilbert-Schmidt norm and the superscript means the Hellinger distance.

For qubit system, the completely classical state $\sigma$ is the probabilistic mixture of a special computational basis $\{ |+\rangle, |\rangle \} \otimes N$, the single-party states $|+\rangle = \cos \frac{\theta}{2} |1\rangle + \exp(-i\phi) \sin \frac{\theta}{2} |0\rangle$ and $|\rangle = \exp(i\phi) \sin \frac{\theta}{2} |1\rangle - \cos \frac{\theta}{2} |0\rangle$, in which $\theta \in [0, \pi)$ and $\phi \in [0, 2\pi)$. As for multi-level case, the similar definition can be constructed readily, which is not shown here.

Now we are ready to find a general expression of $D^H(\rho)$. First the definition Eq. (1) can be reduced to

$$D^H = 1 - \max_{\sigma} \text{Tr} \left[ \sqrt{\rho} \sqrt{\sigma} \right]. \hspace{1cm} (2)$$

Hence the evaluation of $D^H(\rho)$ is reduced to find the maximal overlap of $\sqrt{\rho}$ and $\sqrt{\sigma}$. Second $\sigma$ can be written generally as the probabilistic mixture of computational basis

$$\sigma = \sum_n p_n |\sigma_n\rangle \langle \sigma_n|.$$ \hspace{1cm} (3)

$\{ |\sigma_n\rangle \}$ denotes the general computational basis state, which, for instance of qu-bit case, is composed from the single-party basis states $|+\rangle$ and $|\rangle$. In follow we will present the general expressions for pure and mixed state cases respectively. For convenience we set $N$ as the total number of the computational basis states and $p_N = 1 - \sum_{n=1}^{N-1} p_n$.

-Pure state- For a pure state $|\psi\rangle$, one gets

$$D^H = 1 - \max_{\{p_n, |\sigma_n\rangle \}} \sum_n |\langle \psi | \sigma_n \rangle|^2 \sqrt{p_n}.$$ \hspace{1cm} (4)

The extremal value points happen when

$$\frac{\partial D^H}{\partial p_i} = 0 \Rightarrow \frac{|\langle \psi | \sigma_i \rangle|^2}{\sqrt{p_i}} = \frac{|\langle \psi | \sigma_N \rangle|^2}{\sqrt{p_N}}, \ i = 1, 2, \cdots, N - 1. \hspace{1cm} (5)$$

It is not difficult to find $\frac{\partial^2 D^H(\rho)}{\partial p_i \partial p_j} > 0$. Then $D^H(\rho)$ has the minimal value when

$$p_i = \frac{|\langle \psi | \sigma_i \rangle|^4}{\sum_n |\langle \psi | \sigma_n \rangle|^4}, \ i = 1, 2, \cdots, N. \hspace{1cm} (6)$$

Consequently $D^H$ reduces to

$$D^H = 1 - \max_{\{\sigma_n\}} \sqrt{\sum_n |\langle \psi | \sigma_n \rangle|^4}. \hspace{1cm} (7)$$

-Mixed state- As for the spectrum decomposition $\rho = \sum_k \lambda_k |\phi_k\rangle \langle \phi_k|$, Eq. (7) can be written as

$$D^H = 1 - \max_{\{p_n, |\sigma_n\rangle \}} \sum_n \sqrt{\lambda_k} \sqrt{p_n} |\langle \phi_k | \sigma_n \rangle|^2. \hspace{1cm} (8)$$

As for $p_i$, the extremal value points happen when

$$\frac{\partial D^H}{\partial p_i} = 0 \Rightarrow \sum_k \sqrt{\lambda_k} |\langle \phi_k | \sigma_i \rangle|^2 = \sum_k \sqrt{\lambda_k} |\langle \phi_k | \sigma_N \rangle|^2 \sqrt{p_N}, \ i = 1, 2, \cdots, N - 1. \hspace{1cm} (9)$$

Directly $\frac{\partial^2 D^H}{\partial p_i \partial p_j} > 0$. Then $D^H$ reduces to

$$D^H = 1 - \max_{\{\sigma_n\}} \left[ \frac{\sum_n \left( \sum_k \sqrt{\lambda_k} |\langle \phi_k | \sigma_n \rangle|^2 \right)^2}{\sum_n \left( \sum_k \sqrt{\lambda_k} |\langle \phi_k | \sigma_n \rangle|^2 \right)^2} \right], \ i = 1, 2, \cdots, N. \hspace{1cm} (10)$$

With these preparations, we are ready to evaluate $D^H(\rho)$ explicitly. It should be pointed out that these expressions above is suitable for arbitrary multi-level case. However for simplicity the following discussions would focus only on qubit case since the extensive interest on quantum information procession. The extension into multi-level case is direct.

III. BIPARTITE STATE: EXACT TREATMENT

-Pure state- From Eq. (7) the determination of $D^H(\rho)$ reduces to find the maximal overlap $|\langle \psi | \sigma_n \rangle|$. It is known
that there exists Schmidt decomposition form for any bipartite pure state, which mathematically corresponds to the minimal expansion of a pure state. The corresponding Schmidt states then construct the simplest subspace, in which the state is a vector. Hence in order to find the maximal overlap between $|\psi\rangle$ and $\sigma$, it is necessary to be a vector or the mixed combination of the vectors in this subspace. With respect of classicality of $\sigma_n$ and that the superposition of Schmidt states cannot be separable, the only reasonable choice of $\sigma$ is to be the mixed combination of Schmidt states.

As an example, we try to find $D^H$ for

$$
|\psi\rangle_{AB} = \frac{1}{\sqrt{2}} |1\rangle_A \left( \frac{1}{2} |1\rangle_B + \frac{\sqrt{3}}{2} |0\rangle_B \right) + \frac{1}{\sqrt{2}} |0\rangle_A \left( \frac{\sqrt{3}}{2} |1\rangle_B + \frac{1}{2} |0\rangle_B \right),
$$

of which the Schmidt form is

$$
|\psi\rangle_{AB} = \frac{\sqrt{2} + \sqrt{3}}{2} |1\rangle_A + |0\rangle_A |1\rangle_B + |0\rangle_B \sqrt{\frac{2}{3}}
$$

which clearly is the sum of the fourth power of the Schmidt coefficients.

Then one can obtain that the maximal overlap is 7/8, which happens when the "nearest" $\sigma$ has the form, obtained by setting $\theta_1 = \theta_2 = \pi/2$ and $\phi_1 = \phi_2 = 0$, then

$$
\sigma_{AB} = \frac{7 + \sqrt{3}}{14} |1\rangle_A \langle 1| \otimes |1\rangle_B \langle 1| B
$$

in which $|1\rangle_B = \frac{\sqrt{2}}{2} (|1\rangle + |0\rangle)$ and $|0\rangle_B = \frac{\sqrt{2}}{2} (|1\rangle - |0\rangle)$. The $p_i$ can be determined easily by Eq. (6). So $D^H = 1 - \frac{7}{8}$.

Then we obtain the first conclusion

**Conclusion 1** For arbitrary pure bipartite state, which has Schmidt decomposition

$$
|\psi\rangle_{AB} = \sum_n c_n |n\rangle_A \langle \bar{n}| B,
$$

the "nearest" completely classical state $\sigma$ can be written as

$$
\sigma = \frac{1}{c} \sum_n |c_n|^4 |n\rangle_A \langle n| \otimes |\bar{n}\rangle_B \langle \bar{n}|,
$$

in which $c = \sum_n |c_n|^2$. Then

$$
D^H = 1 - \sqrt{c}
$$

-Mixed state- For $\rho = \sum_k \lambda_k |\phi_k\rangle \langle \phi_k|$, in general these pure states $|\phi_k\rangle$ do not share the same Schmidt basis. Thus we cannot find a general method to determine $D^H$. However an exceptional case is that the density matrix shows "X" form

$$
\rho_X = \begin{pmatrix}
\rho_{11} & 0 & \cdots & 0 & 0 & \rho_{1n} \\
0 & \rho_{22} & \cdots & 0 & \rho_{2(n-1)} & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \rho_{(n-1)2} & \cdots & 0 & \rho_{(n-1)(n-1)} & 0 \\
\rho_{n1} & \cdots & \cdots & \cdots & \cdots & \rho_{nn}
\end{pmatrix}
$$

which actually is the direct sum of sub-matrices. It should be pointed out that the basis of Eq. (22) is not necessarily
the computational basis. The only restriction is that the basis states for any sub-matrix are orthonormalized as Schmidt form. Thus the eigenstates of sub-matrix are Schmidt-typed in their own form. More importantly since the basis states for different sub-matrices have no overlap, then the “nearest” \(\sigma\) is necessarily a mixed combination of the basis states in all sub-matrices.

As an example, we try to find \(D^H\) for Werner state

\[
\rho_W = \frac{1}{4} \left( \begin{array}{cccc} 1 - r & 0 & 0 & 0 \\ 0 & 1 + r & 2r & 0 \\ 0 & 2r & 1 + r & 0 \\ 0 & 0 & 0 & 1 - r \end{array} \right),
\]

in which \(r \in [0, 1]\) and \(|\psi^+\rangle = \frac{1}{\sqrt{2}} (|10\rangle + |01\rangle)\). It is obvious that \(\rho_W\) has a "X" form on the basis \(|\{11\}, |10\rangle, |01\rangle, |00\rangle\}

\[
\rho_W = \frac{1}{4} \left( \begin{array}{cccc} 1 - r & 0 & 0 & 0 \\ 0 & 1 + r & 2r & 0 \\ 0 & 2r & 1 + r & 0 \\ 0 & 0 & 0 & 1 - r \end{array} \right),
\]

that generally is direct sum of two sub-matrices

\[
\rho_1 = \frac{1}{4} \left( \begin{array}{cccc} 1 - r & 0 & 0 & 0 \\ 0 & 1 - r & 0 & 0 \\ 0 & 0 & 1 - r & 0 \\ 0 & 0 & 0 & 1 - r \end{array} \right); \rho_2 = \frac{1}{4} \left( \begin{array}{cccc} 1 + r & 2r & 0 & 0 \\ 2r & 1 + r & 0 & 0 \\ 0 & 0 & 1 + r & 0 \\ 0 & 0 & 0 & 1 - r \end{array} \right),
\]

defined on the basis \(|\{11\}, |00\rangle\} and \(|\{10\}, |01\rangle\} respectively. Then there are four eigenstates

\[
|1\rangle = |11\rangle; |2\rangle = |00\rangle \\
|3\rangle = \frac{1}{\sqrt{2}} (|10\rangle - |01\rangle) \\
|4\rangle = \frac{1}{\sqrt{2}} (|10\rangle + |01\rangle)
\]

which in their own forms are already Schmidt forms.

By Eq.(10), one obtains

\[
\max \text{Tr} [\sqrt{\rho_W} \sqrt{\sigma}] = 2 \max \sqrt{c_0^2 + c_1^2}
\]

in which \(c_0 = \left(3\sqrt{1 - r^2} + \sqrt{1 + 3r^2}\right)/8\), \(c_1 = \cos \Omega \left( \sqrt{1 - r^2} - \sqrt{1 + 3r^2} \right)/8\) and \(\cos \Omega = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \cos (\phi_1 - \phi_2)\). Then one has \(D^H = 1 - \frac{1}{2} \sqrt{3 - r + \sqrt{(1 - r^2)(1 + 3r^2)}}\) when \(\cos \Omega = \pm 1\), which can occur, for example if \(\theta_1 = \theta_2 = 0\) and \(\phi_1 = \phi_2\). The corresponding "nearest" \(\sigma\) has the form

\[
\sigma = \frac{1}{2} \left[ \frac{1 + r}{3 + r + \sqrt{(1 - r^2)(1 + 3r^2)}} |10\rangle \langle 10| + |01\rangle \langle 01| \right] + \frac{1}{3 + r + \sqrt{(1 - r^2)(1 + 3r^2)}} |11\rangle \langle 11| + |00\rangle \langle 00|,
\]

which just is a mixedness of the Schmidt basis of the two sub-matrices. It should be pointed out the the choice of values of \(\theta_{1(2)}\) and \(\phi_{1(2)}\) is not unique.

Although the simplicity of the example, we can obtain the second conclusion in this article

**Conclusion 2** For a X-type density operator Eq. (22) the "nearest" completely classical state \(\sigma\) is necessarily the mixed combination of the basis of the density matrix. The probability is determined by the eigenstates of \(\rho_X\), as shown in Eq.(11).
It is not difficult to find that the extremal points appear when
\[ \sin \delta_1 = 0. \]
Furthermore one notes that a, b, c and d are invariant under transformation \( \theta_i \leftrightarrow \pi - \theta_i \). Thus the extremal points occurs when \( \theta_i \) equals to 0, \( \pi \) or \( \pi/2 \) and an the meanwhile \( \delta_i \) is 0 or \( \pi \). There are many choices for \( \theta_i \)’s and \( \delta_i \)’s. Our calculation shows that except that \( \theta_i \) equals to \( \pi/2 \) for any \( i = 1, 2, 3 \), one can always obtain the extremal value for other possible choices of \( \theta_i \). Then one can find the choice that \( \theta_1 = \theta_2 = \theta_3 = 0 \) and \( \phi_1 = \phi_2 = \phi_3 = 0 \). Thus \( D_H = 1 - 1/\sqrt{2} \) and
\[
\sigma = \frac{1}{4} \left( |1\rangle \langle 1| \otimes |0\rangle \langle 0| + |0\rangle \langle 0| \otimes |1\rangle \langle 1| + |1\rangle \langle 1| \otimes |0\rangle \langle 0| + |0\rangle \langle 0| \otimes |1\rangle \langle 1| \right),
\]
which obviously is also permutationally invariant.

B. 4-qubit case

As for 4-qubit states, there exist another symmetry besides of permutational invariance, named as translation symmetry. The definition of translation of state is similar to that in solid systems. However the difference is that it is defined for single-party state in Hilbert space, instead of single particle in real configuration [23]. We will display by two exemplifications that the “nearest” \( \sigma \) for the state of translational invariance is necessary also translational invariant. And \( D_H \) can also be determined readily by setting the parameters have the same value respectively, as shown in 3-qubit case.

\[ \langle \text{GHZ}_1 \rangle_4 \text{ state-}, \]

which is defined as \( | \text{GHZ}_1 \rangle_4 = \frac{1}{\sqrt{2}} (|1010\rangle + |0101\rangle) \). It is obvious that the state is actually constructed by cyclic permutation of 1010, which is named as cyclic unit. It is not difficult to find
\[
\sum_n | \langle 4, 2 | \sigma_n \rangle |^4 = \frac{1}{16} \left( 1 + a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 \right),
\]
in which
\[
a = \prod_{i=1}^{4} \cos \theta_i, \quad b = \cos \theta_1 \cos \theta_2 c = \cos \theta_1 \cos \theta_3 d = \cos \theta_1 \cos \theta_4, \quad e = \cos \theta_2 \cos \theta_3 f = \cos \theta_2 \cos \theta_4 g = \cos \theta_3 \cos \theta_4.
\]

It is not difficult to find that the overlap has maximal value 1/2 and then \( D_H = 1 - 1/\sqrt{2} \) when cos \( \theta_i = \pm 1 \) (i = 1, 2, 3, 4). \( \phi_i \)’s can be any values and are set to be zero for simplicity. Consequently the “nearest” \( \sigma = \frac{1}{2} \left( |1010\rangle \langle 1010| + |0101\rangle \langle 0101| \right) \), which is also translational invariant with the same cyclic unit to that of \( | \text{GHZ}_1 \rangle_4 \).

\[ | \text{W}_2 \rangle_4 \text{ state-}, \]

which is defined as
\[
| \text{W}_2 \rangle_4 = \frac{1}{2} \left( |1100\rangle + |0110\rangle + |0011\rangle + |1001\rangle \right).
\]
The state is actually constructed by cyclic unit 1100. Moreover it is bi-seperable since \( | \text{W}_2 \rangle_4 = \frac{1}{\sqrt{7}} (|10\rangle + |01\rangle)_{13} \otimes \frac{1}{\sqrt{2}} (|10\rangle + |01\rangle)_{24} \). Thus the “nearest” \( \sigma \) can also be constructed by two parts, i.e. \( \sigma = \sigma_{13} \otimes \sigma_{24} \), in which \( \sigma_{13} \) and \( \sigma_{24} \) are the “nearest” completely classical states for \( \frac{1}{\sqrt{2}} (|10\rangle + |01\rangle)_{13} \) and \( \frac{1}{\sqrt{2}} (|10\rangle + |01\rangle)_{24} \) respectively. By Conclusion 1 one can obtain \( D_H = 1/2 \) and
\[
\sigma = \frac{1}{2} (|10\rangle \langle 10| + |01\rangle \langle 01|)_{13} \otimes \frac{1}{2} (|10\rangle \langle 10| + |01\rangle \langle 01|)_{24} = \frac{1}{4} \left( |1100\rangle \langle 1100| + |0110\rangle \langle 0110| + |0011\rangle \langle 0011| + \langle 1001| \langle 1001| \right),
\]
which is obviously translationally invariant.

C. A short discussion

By the previous exemplifications, we can obtain the third conclusion

\textbf{Conclusion 3} For multipartite state with permutational or translational invariance, the “nearest” \( \sigma \) necessarily has the same invariance, which can be determined by setting \( \theta_i \) in \( \sigma \) to be the same, so do for \( \phi_i \).

It should be pointed out that the form of \( \sigma \) is not necessary to comply with the superposition terms in the states. As an example, we try to find the \( \sigma \) for Dicke state \(|4, 2\rangle = \frac{1}{\sqrt{5}} \sum_{\text{perm}} |1100\rangle \), which is the equally weighted sums of all permutations of computational basis states with two qubits being \(|1\rangle \) and two qubits being \(|0\rangle \). By explicit calculation, one obtain
\[
\sum_n | \langle 4, 2 | \sigma_n \rangle |^4 = \frac{217 - 96 \cos(2\theta) + 108 \cos(4\theta) + 27 \cos(8\theta)}{1536},
\]
which obviously has maximal value when \( \theta = \pi/2 \). Then \( D_H \approx 0.46 \) and
\[
\sigma = 0.482 \left( |1_x\rangle \langle 1_x| \otimes |0_x\rangle \langle 0_x| \right) \\
+ 0.006 \left( \sum_{\text{perm}} |1_x\rangle \langle 1_x| \otimes |0_x\rangle \langle 0_x| \right),
\]
in which \(|1_x\rangle = \frac{1}{\sqrt{2}} (|1\rangle + |0\rangle) \) and \(|0_x\rangle = \frac{1}{\sqrt{2}} (|1\rangle - |0\rangle) \). The second term is the equally weighted sums of all permutations of the density operators with two qubits being \(|1_x\rangle \) and the other two being \(|0_x\rangle \). It is obvious that \( \sigma \) displays more complex form than that of \(|4, 2\rangle \).

As for the mixed case, the similar conclusion can be obtained since its eigenstates are also symmetric. A simple example is the discussion for Werner state \( \rho_W \) in Sec. III.

V. \( D_H \) AND QUANTUM PHASE TRANSITION IN MANY-BODY SYSTEM

In this section, we show that \( D_H \) can also mark the quantum phase transition in many-body systems. For clarity and
simplicity, this discussion focuses on two popular models, Lipkin-Meshkov-Glick (LMG) [25] and Dicke models [28], of which the ground states can be determined analytically and additionally both are permutation invariant.

A. Lipkin-Meshkov-Glick model

The LMG model describes a set of spin-half particles coupled to all others with an interaction independent of the position and the nature of the elements. The Hamiltonian can be written as

\[ H = -\frac{\lambda}{N} (S_x^2 + \gamma S_y^2) - h_z S_z, \]

in which \( S_{\alpha} = \sum_{i=1}^{N} \sigma_{\alpha}^i/2 (\alpha = x, y, z) \) and the \( \sigma_{\alpha} \) denotes the Pauli operator, and \( N \) is the total particle number in this system. The prefactor \( 1/N \) is essential to ensure the convergence of the free energy per spin in the thermodynamic limit. It is known that there is a second-order transition at \( h = h_z/|\lambda| = 1 \) for the ferromagnetic case (\( \lambda > 0 \)) and a first-order one at \( h = 0 \) for the antiferromagnetic case (\( \lambda < 0 \)) [26][27]. The following discussion is divided into two parts by \( \gamma = 1 \) or not.

\( \gamma = 1 \) In this case the model can be solved exactly, of which the eigenstate is \( |N/2, n\rangle \), in which \( n \) denotes the quantum number of angular moment \( S_z \), and the corresponding eigenenergy is \( E_n = -\frac{\lambda}{N} (\frac{N}{2} + 1) + \frac{\lambda}{N} n^2 - h_z n (h = 1) \). For \( \lambda > 0 \), the minimal value of \( E_n \) appears when \( n = \left[ \frac{h_z N}{\lambda} \right] \). Then the ground state is \( |N/2, N/2\rangle \). For \( \lambda \geq 1 \), the ground state is \( |N/2, \left[ \frac{h_z N}{\lambda} \right] \rangle \), which actually is a Dicke state

\[ |N, m\rangle = \frac{1}{\sqrt{N!C_m}} \sum_{\text{perm}} \left( \frac{1}{m} \right) \left( \frac{N-m}{2} \right), \]

in which \( m = \left[ \frac{N}{2} \right] + \left[ \frac{h_z N}{\lambda} \right] \). Thus one has

\[ D^h = \left\{ \begin{array}{ll}
0, & \frac{h_z}{\lambda} > 1;
0, & 0 < \frac{h_z}{\lambda} < 1;
\end{array} \right. \]

which is plotted for \( 0 < h_z/\lambda < 1 \) in Fig.1.

As for \( \lambda < 0 \), the minimal value of \( E_n \) appears when \( n = - \left[ \frac{h_z N}{\lambda} \right] \). Then the ground state is \( |N/2, -N/2\rangle \) for \( \lambda \geq 1 \) and \( |N/2, -N/2\rangle \) for \( \lambda < 0 \) in angular moment picture, which both have vanishing GD.

\( \gamma \in [0, 1) \) The ground state is in this case [29]

\[ |g\rangle = \frac{1}{\sqrt{c}} \sum_{n=0}^{[N/2]} (-1)^n \sqrt{\frac{(2n-1)!!}{2n!!}} \tanh^n x |2n\rangle \]

\[ c^2 = \sum_{n=0}^{[N/2]} (-1)^n \frac{(2n-1)!!}{2n!!} \tanh^{2n} x, \]

in which

\[ \tanh 2x = \left\{ \begin{array}{ll}
\frac{1 - \gamma}{2h_z - \gamma}, & h_z > 1
\frac{\gamma}{2h_z - \gamma}, & 0 \leq h_z < 1
\end{array} \right. \]

Figure 1: (Color online) \( D^H \) of the ground state for \( \gamma = 1 \) and \( \lambda > 0 \) in LMG model.

\[ \tanh 2x = \frac{1 - \gamma}{1 + \gamma + 2|h_z|}. \]

\[ \Gamma(\lambda_0) = -\frac{1 - 5\lambda_0^2}{4} + h_x \frac{\lambda_0(2 - \lambda_0^2)}{8(1 - \lambda_0^2)^{3/2}} \]

\[ \Delta(\lambda_0) = h_z - \frac{1 - 7\lambda_0^2}{2} + h_x \frac{\lambda_0(4 - 3\lambda_0^2)}{4(1 - \lambda_0^2)^{3/2}}. \]

and \( \lambda_0 \) is determined by the equation

\[ \lambda_0 h_x - \frac{h_x(1 - 2\lambda_0^2)}{2\sqrt{1 - \lambda_0^2}} - \lambda_0(1 - 2\lambda_0^2) = 0. \]

There are two critical points, \( h_x = 0 \) for \( h_z = 1 \), which corresponds to a second order quantum phase transition and \( h_x = 0 \) for \( 0 < h_z < 1 \), a first order one. As shown in Fig.3 \( D^H \) can unambiguously manifest the appearance of critical points.

B. Dicke model

In this subsection we discuss another well-known model, Dicke model [28]. Dicke model is related to many fundamental issues in quantum optics, quantum mechanics and condensed matter physics, such as the coherent spontaneous radiation [30], the dissipation of quantum system [31], quantum chaos [32] and atomic self-organization in a cavity [33].
The multipartite entanglement in Dicke model has also been discussed [34]. The Hamiltonian for single-model Dicke model reads

\[ H = \omega a^\dagger a + \omega_0 \frac{\lambda}{N} \sum_{i=1}^{N} (\sigma_i^+ + \sigma_i^-) (a^\dagger + a) \]

\[ = \omega_0 J_z + \omega a^\dagger a + \frac{\lambda}{\sqrt{N}} (a^\dagger + a) (J_+ + J_-), \]  

(48)

where \( J_z = \frac{1}{2} \sum_{i=1}^{N} \sigma_i^z \) and \( J_\pm = \frac{1}{2} \sum_{i=1}^{N} \sigma_i^\pm \) are the collective angular momentum operators. There are two distinct phases for ground state, normal phase and superradiant phase, separated by critical point \( \lambda_c = \sqrt{\omega \omega_0} / 2 \).

By the method in Ref. [35], the reduce density operator of atom system for the ground state can be obtained analytically. As shown in Fig. 3 \( D^H \) clearly marks the appearance of quantum phase transition.

VI. CONCLUSION AND DISCUSSION

In conclusion, a generalization of the geometric measure of quantum discord is introduced in this article. This definition has virtue that it can be generalized into multipartite case directly. Moreover since the independence of local measurements and the square root form in Eq. (1), it does not suffered from the critiques, raised recently in Refs. [9, 11, 12]. One important conclusion in this article is that our generalized geometric discord can be exactly evaluated for pure bipartite state of arbitrary levels, as shown in Conclusion 1, which is com-
mutation or translation of single-party states. And then the nearest \( \sigma \) shows the same invariance. Finally we show that our new definition can be used to mark the quantum phase transitions in many-body systems.

It seems a natural conjecture from Conclusion [1] that one could define the generalized Schmidt decomposition for multipartite state, which would be the coherent superposition of the terms in the "nearest" \( \sigma \). However as shown for Dicke state [4, 2] in Sec.IVC, \( \sigma \) in this case has more complicate form than that of the measured state. This point is also a manifestation of the fact of matter that the understanding of Schmidt decomposition of bipartite pure state cannot directly be generalized into multipartite case. Thus it is an interesting open question how to define the generalized Schmidt decomposition of multipartite states from a geometric point, which is expected to deepen the understanding of multipartite correlation.

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