Rethinking Renormalization

John R. Klauder*
Department of Physics and
Department of Mathematics
University of Florida
Gainesville, FL 32611-8440

Abstract

As applied to quantum theories, the program of renormalization is successful for ‘renormalizable models’ but fails for ‘nonrenormalizable models’. After some conceptual discussion and analysis, an enhanced program of renormalization is proposed that is designed to bring the ‘nonrenormalizable models’ under control as well. The new principles are developed by studying several, carefully chosen, soluble examples, and include a recognition of a ‘hard-core’ behavior of the interaction and, in special cases, an extremely elementary procedure to remove the source of all divergences. Our discussion provides the background for a recent proposal for a nontrivial quantization of nonrenormalizable scalar quantum field models, which is briefly summarized as well.

Dedication: It is a pleasure to dedicate this article to the memory of Prof. Alladi Ramakrishnan who, besides his own important contributions to science, played a crucial role in the development of modern scientific research and education in his native India. Besides a number of recent informative discussions during his yearly visits to the University of Florida, the present author had the pleasure much earlier of hosting Prof. Alladi during his visit and lecture at Bell Telephone Laboratories.

*klauder@phys.ufl.edu
Introduction

Renormalization has been a very successful paradigm for dealing with an important class of quantum theories. Its basic principles are easily stated: The parameters of a classical theory are different from those of a quantum theory because of additional self interaction that arises in a quantum theory. In practical terms, the interacting system is commonly treated as a perturbation of a free system, and the power series in the nonlinear coupling often displays divergent terms that need to be canceled and counterterms of a suitable nature are introduced to do just this. If a finite number of distinct counterterms can be found so that every term in the power series expansion is rendered finite, then the theory is called renormalizable, and many such theories have had highly successful applications and in several cases have led to astonishingly accurate predictions when compared to experimental measurements. This aspect of the program of renormalization is considered to be a resounding success and deservedly so. It is natural of course that a successful program such as renormalization has also been proposed to study a wider class of theories than its proponents originally intended, and this is indeed the case. A certain family of field theories fall into the class of being “nonrenormalizable”, an attribute that asserts that the procedures usually ascribed to the program of renormalization are unsuccessful in dealing with certain model problems. If such examples were confined to esoteric models with no potential application to the real world, it would be permissible to ignore those models that are classified as nonrenormalizable. But that is not the case. The most famous example corresponds to the Einstein gravitational field for which the general consensus is that quantum gravity is perturbatively nonrenormalizable. Since the standard procedures of renormalization have failed for such an important case, there have been proposed elaborate alternative theories that entail additional fields or degrees of freedom that are designed to produce a theory that is term-by-term finite within a perturbation analysis. Superstring theory is one such program, and $N = 8$ supergravity is another. In so doing, these alternative theories have introduced additional fields, which, thanks to the differing properties of fermions and bosons can lead to cancellations among the old, divergent contributions of the original theory and well chosen, new, divergent contributions from the carefully selected additional fields. This general approach is sufficiently broad that it would seem to cover all possible situations regarding how interactions and auxiliary counterterms can appear and interact with each other.
However, there is one important class of models that is in practice not covered by the preceding characterization. Admittedly, it is not obvious where one should look if such an overlooked class of examples is to be found. A clue to the overlooked class emerges if we recall that the traditional procedures of regularization and renormalization entail the implicit assumption that if the perturbative interaction is reduced in strength, say by the usual device of reducing the value of the associated coupling constant, then, in the limit that the coupling constant vanishes and the effect of the interaction is formally eliminated, the resulting theory in the limit of a vanishing coupling constant is identical to the free theory with which one started. Stated otherwise, and perhaps more directly, this is the implicit assumption that the set of interacting theories defined as the set that is produced for all nonzero (typically positive) values of the coupling constant is such that as the coupling constant goes to zero, the limit of that set of interacting theories is the free theory itself, i.e., the interacting theories are \textit{continuously connected to the free theory}. This highly natural, implicit assumption covers a lot of the important cases but it certainly does not cover all possibilities some of which may have some ultimate physical relevance. It is an important feature of this paper that we focus on these outlier model theories, which are typically nonrenormalizable models.

\textbf{Overview of the Present Paper}

The features ascribed to the renormalization program are not limited to quantum field theory but also arise in quantum mechanical analogues. As such, one can gain real insight into the distinction among super renormalizable, strictly renormalizable, and nonrenormalizable models. A common feature of the latter theories is the occurrence of a hard-core potential. From a (Euclidean) functional integral viewpoint, the nonlinear interaction acts partially as a hard core projecting out certain paths that would otherwise appear in the free theory. This fact – which we believe is a defining characteristic for a large class of nonrenormalizable interactions – means that an interacting theory is \textit{not continuously connected} to the free theory as the coupling constant is reduced to zero. This property of the quantum theory is also seen in the classical theory itself by the fact that, generally speaking, the set of solutions of the interacting classical theory does not reduce to the set of solutions that characterizes the free solutions. This aspect will be illustrated for particle systems as well as field systems.
The full dynamics of a classical system involves the action functional and its stationary variation to derive the equations of motion. In a (Euclidean) functional integral formalism, the classical action again plays an important role in the quantum dynamics. Regularization is essential in order to give a functional integral meaning, and it is customary to use a lattice approximation for the time for particle mechanics or for spacetime for field models. The lattice action induces a lattice Hamiltonian operator and in turn a lattice ground state for that Hamiltonian. It is natural that a model can be characterized by either the action, the Hamiltonian, or the ground state. It is important to remark that we focus heavily on the ground state in our analysis. When we take up the discussion of field problems, we will present an argument that shows an important role that the ground state plays.

However, before dealing with fields, we wish to illustrate how the issue of renormalization arises in elementary one dimensional examples.

**One Dimensional Example**

Consider a classical system for a single, phase space, degree of freedom \((p, q)\) with a classical Hamiltonian given by

\[
H_\lambda(p, q) = \frac{1}{2}(p^2 + q^2) + \lambda|q|^{-\alpha}.
\]

For any \(\alpha > 0\), it follows, just from energy considerations, that the motion of the particle can never be such as to reach the origin \(q = 0\) let alone pass through the value \(q = 0\). This situation holds for all values of the coupling constant \(\lambda > 0\), and as a consequence, as \(\lambda \to 0\), the set of classical solutions of the interacting theory do not correspond to the set of classical solutions of the free theory, namely, that of the free harmonic oscillator given by \(q(t) = A\cos(t - a)\). Specifically, for any choice of the amplitude \(A\) and the phase \(a\) there will be for every solution of the free theory a time \(t\) for which the solution vanishes and even crosses the line \(q = 0\). In contrast, the solutions of the interacting theory for which \(\lambda > 0\), all pass by continuity to solutions not of the free theory but to those which are rectified in the sense that they are of the form \(q(t) = \pm|A\cos(t - a)|\) and are all strictly different over time from the usual free theories. We give the name pseudofree to the name of the theory, different from the free theory, to which the interacting theory is continuously connected as the nonlinear coupling constant goes to zero. Clearly, if one reintroduces the interaction starting from the pseudofree
theory, the form of the new solutions is indeed continuously connected to that of the pseudofree theory.

The easiest way to characterize the pseudofree quantum theory is by its Hamiltonian which is the same as that of the free harmonic oscillator augmented by Dirichlet boundary conditions at \( x = 0 \). If one were contemplating a perturbation series representation of the interacting solution, that power series should not be about the free theory (to which the interacting solutions are not continuously connected!) but rather about the pseudofree theory.

Regarding the quantization of such a model, there are some surprises that can arise. For example, when \( 0 < \alpha < 1 \), it follows that the interacting quantum solution is in fact continuously connected to the free quantum theory unlike the situation for the classical case. For \( \alpha > 2 \), on the other hand, there is no modification of the theory that can be made to prevent the theory from passing to a pseudofree theory as the parameter \( \lambda \to 0 \). In other words, for \( \alpha > 2 \), the interacting quantum theory passes to a pseudofree theory with a set of eigenfunctions and eigenvalues that are generally different from those that characterize the free theory. What happens in the interval \( 1 \leq \alpha \leq 2 \) is quite interesting and to some extent open to different conclusions. With an eye toward maintaining a continuous connection of the interacting theories to the free theory, it is possible to choose a regularized form for the interaction, namely, a set of potentials of the form \( V_\epsilon(q, \lambda) \) that have the property that as \( \epsilon \to 0 \), the regularized potentials

\[
V_\epsilon(q, \lambda) \to \lambda |q|^{-\alpha}, \quad q \neq 0.
\]

These regularized forms of the potential are rather strictly constrained and they involve polynomial contributions in the coupling constant \( \lambda \). It is not difficult to determine the general form of the regularized potential simply on the basis of dimensional arguments. In particular, the dimensions of the Hamiltonian are those of the first term \( p^2 \), and taking Planck’s constant \( \hbar = 1 \) for the present time, the dimensions are that of \( L^{-2} \) where \( L \) denotes the dimension of length. With the regularization parameter \( \epsilon > 0 \) entering initially in the interaction as

\[
\lambda |q|^{-\alpha} \to \lambda (|q| + \epsilon)^{-\alpha},
\]

it follows that the dimension of \( \epsilon \), like \( q \), is \( L \). In order that the interaction terms have the right dimensions, i.e., \( L^{-2} \), it follows that the dimension of
\[ \lambda \text{ is that of } L^{\alpha - 2}. \] For regularization terms we restrict ourselves to terms of the form
\[ k_j \lambda^j e^{-p_j} \delta(q), \]
where \( \delta(q) \) is a Dirac delta function. With \( k_j \) chosen as an unknown dimensionless factor, and since \( \delta(q) \) has dimensions \( L^{-1} \), it follows that the power \( p_j = 1 - (2 - \alpha)j \) in order to ensure that the regularization terms above each have the desired dimension of the Hamiltonian, namely \( L^{-2} \). Hence the regularized form of the potential is given by
\[ V_\epsilon(q, \lambda) = \lambda(|q| + \epsilon)^{-\alpha} - \sum_{j=1}^{J} k_j \lambda^j \epsilon^{(2-\alpha)j-1} \delta(q). \]

The factor \( J \) denotes the upper limit of the sum which occurs whenever \( (2 - \alpha)^{-1} \) is nonintegral and \( (2 - \alpha)J < 1 < (2 - \alpha)(J + 1) \) for then all further regularization terms vanish as \( \epsilon \to 0 \). In this case further analysis shows that the factors \( k_j \) are given by \( k_1 = 2/((\alpha - 1) \) and then
\[ k_j = \frac{1}{[1 - j(2 - \alpha)]} \sum_{q=1}^{j-1} k_{j-q} k_q; \]
if instead, \( (2 - \alpha)^{-1} = J \) is an integer, then the last factor \( k_J \) involves a natural logarithm; see [1]. For \( \alpha = 2 \), \( J = \infty \), and all \( p_j = 1 \). For all \( \alpha \leq 2 \) such a series provides a regularized potential for which the interacting theory is continuously connected to the free theory as \( \lambda \to 0 \). It is noteworthy that when \( \alpha < 2 \) a finite series of counterterms, each with a diminishing divergence (i.e., \( p_{j+1} < p_j \)), provides the proper regularized potential, a property similar to that encountered when dealing with super renormalizable quantum field theories. When \( \alpha = 2 \) an infinite series of counterterms, all of equal divergence (i.e., \( p_{j+1} = p_j \)), leads to a suitable regularized potential, a property similar to that of so-called strictly renormalizable quantum field theories. For \( \alpha > 2 \), on the other hand, there is no regularized potential that leads to an interacting theory that is continuously connected to the free theory. Of course, the proposed regularization terms based simply on dimensionality do not know this fact, and it may be said that they do their best to signal their inability to provide a solution to the problem by the fact that when \( \alpha > 2 \), the term-by-term divergence actually increases (i.e., \( p_{j+1} > p_j \)), and moreover, \( p_j \to \infty \) as \( j \to \infty \), a property which is reminiscent of the behavior of nonrenormalizable quantum field theories.
A brief summary

We have discussed this simple quantum mechanical model in some detail in order to show what kind of singular behavior is possible even in quantum mechanics. In particular, we observe that for $\alpha < 1$, there is no anomalous behavior in the quantum theory although there is anomalous classical behavior. For $1 \leq \alpha \leq 2$, it can be arranged that there is no anomalous quantum behavior although there always will be anomalous classical behavior. The price to pay for this good quantum behavior is the introduction of regularized quantum terms that entail a power series in the coupling constant $\lambda$. For $\alpha > 2$, on the other hand, there is no escaping the anomalous quantum behavior no matter how one tries to regularize the quantum theory.

Field theory analog – a brief detour

We claim there is an analog with the above story for quantum mechanics that plays out in quantum field theory as well. For sufficiently weak perturbations, the interaction can be renormalized so that the resultant interacting theory is continuously connected to the free theory as the coupling constant is reduced to zero; this is the situation that applies to super renormalizable and possibly to strictly renormalizable theories. For sufficiently strong perturbations, the interaction cannot be renormalized so that the interacting theory is continuously connected to the free theory. Instead, for such strong perturbations, the interacting theories are connected to an appropriate pseudofree theory. Later, we will bolster the argument that this is the situation which should apply to nonrenormalizable theories. To make this leap of faith from a singular family of classical problems and their associated quantum problems to a wide class of quantum field theories, it will be helpful to develop a primary principle that captures the essence of the singular nature of the interaction that leads to either a continuous connection with the original free theory or instead leads to a continuous connection with a pseudofree theory.

Path Integral Formulation

The principle we adopt to describe the appearance of pseudofree theories is that of a hard-core interaction. The concept behind this principle is most simply appreciated in a functional integral representation of the associated quantum system. This analysis works for either a real time or an imaginary
time functional integral, and for its better mathematical structure, we shall choose the latter form. For the quantum mechanical problem that we have so far been discussing, the associated imaginary time (Euclidean) functional integral is given by

\[ \mathcal{N} \int e^{-\int \left\{ \frac{1}{2} (\dot{x}^2 + x^2) + \lambda V(x) \right\} dt} \mathcal{D}x. \]

Although the Brownian-like paths \( x(t) \) that enter this functional integral have a nowhere defined (i.e., divergent) derivative – a feature that is surely unlike the classical theory – it is noteworthy that the distinction between the behavior for \( \alpha < 2 \) and \( \alpha > 2 \) can nevertheless be won by simple classical arguments. For classical paths consider the following simple inequality

\[ |x(t_2) - x(t_1)| = \left| \int_{t_1}^{t_2} \dot{x}(t) \, dt \right| \leq |t_2 - t_1|^{1/2} \left[ \int_{t_1}^{t_2} \dot{x}^2(t) \, dt \right]^{1/2}. \]

Assuming a finite value for the kinetic energy, it follows, for some \( K < \infty \), that

\[ |x(t_2) - xt_1|^{-\alpha} \geq K |t_2 - t_1|^{-\alpha/2}. \]

Setting \( x(t_2) = 0 \), the location of the singularity, we see that

\[ \int |x(t)|^{-\alpha} \, dt \geq K \int |t|^{-\alpha/2} \, dt. \]

This inequality implies that for \( \alpha > 2 \) the integral over the interaction term diverges, while for \( \alpha < 2 \) that is not necessarily the case. When the integral over the interaction diverges, the contribution of that path is projected out (by the factor \( e^{-\infty} \)) for any positive value of the coupling constant. And as the coupling constant is reduced to zero, the contribution of that path is never restored leading to the exclusion of that path in the definition of the pseudofree theory. For the quantum mechanical problem previously discussed, this means that whenever \( \alpha > 2 \), the contribution of all paths that reach or cross the axis \( x = 0 \) are projected out of the functional integral; that is the meaning of the statement that the interaction acts in part like a hard core. Our simple argument involving the inequality derived from classical paths does not have anything to say about what happens for \( \alpha < 2 \), but that does not diminish its importance for the region \( \alpha > 2 \).

Before proceeding, let us restate some important issues that arose in our analysis of the one dimensional quantum problem as discussed above. The
model we studied had a clearly defined free theory (with \( \lambda \equiv 0 \)) which is just the usual harmonic oscillator. The free propagator (in imaginary time for convenience) is readily given by the sum

\[
\langle x'', T|x', 0 \rangle = \sum_{n=0}^{\infty} h_n(x'') e^{-\frac{(n+1/2)T}{2}} h_n(x'),
\]

where the set of functions \( \{h_n(x)\}_{n=0}^{\infty} \) are the Hermite functions defined by the generating function

\[
\exp(-s^2 + 2sx - \frac{1}{2}x^2) = \pi^{1/4} \sum_{n=0}^{\infty} (n!)^{-1/2} (s\sqrt{2})^n h_n(x).
\]

In the present case the pseudofree theory (denoted by a prime \( \prime \)) has a propagator defined by the expression

\[
\langle x'', T|x', 0 \rangle' = \theta(x''x') \sum_{n=0}^{\infty} h_n(x'') e^{-\frac{(n+1/2)T}{2}} [h_n(x') - h_n(-x')],
\]

where the function \( \theta(u) = 1 \) if \( u > 0 \) and \( \theta(u) = 0 \) if \( u < 0 \). It is the latter expression that incorporates the hard core, projecting out all those paths in the free harmonic oscillator propagator that reach or cross the value \( x = 0 \). Note well: It is the pseudofree theory to which the interacting theories are continuously connected as the coupling constant is reduced to zero. It is the pseudofree theory around which a meaningful perturbation theory for the singular perturbation can be constructed. From the point of view of a Euclidean functional integral, if one attempted to expand a partially hard core interaction about the free theory, this would lead to a series composed of ever more divergent expressions. Regularization of that series would serve to render those terms finite but it would also falsely imply that the interacting theory was continuously connected to the free theory because the regularized power series would reduce to the free theory when the coupling constant is reduced to zero. This property of the regularized perturbation series is entirely erroneous and misleading.

Moreover, the seed of the discontinuous nature of the perturbation about the free theory is already evident in the classical theory itself. This situation holds because the classical solutions of the interacting theory already do not reduce to the solutions of the classical free theory as \( \lambda \to 0 \). Instead they pass
to the classical solutions of the pseudofree theory as noted above. This result has the important consequence that an indelible imprint of the fact that one could be dealing with a discontinuous perturbation (of the free theory) can be determined from an analysis of the classical interacting theory itself! The nature of such an analysis is not too difficult; it rests on the determination that the set of solutions of the interacting theory for arbitrarily small coupling constant is not equivalent to the set of solutions of the free theory itself.

The criterion that a classical pseudofree theory be different from the classical free theory is necessary for a quantum pseudofree theory to be different from a quantum free theory. However, the one dimensional example with $0 < \alpha < 1$ demonstrates that such a criterion is not sufficient to ensure that the quantum theory also involves a pseudofree theory different from the free theory.

**Shifting the singularity from $x = 0$ to $x = c$**

Suppose, instead of the singularity being at $x = 0$, we moved it to the point $x = c$, where without loss of generality we can assume that $c > 0$. This means that our basic potential is $\lambda|x - c|^{-\alpha}$. We now briefly summarize the main changes that occur. First, the classical story. In this case, the free solution given by $q(t) = A\cos(t - a)$ may remain unchanged if the overall classical energy is sufficiently small, which occurs when $|A| \leq c$. When $|A| > c$, two solutions are possible, one of the form $q(t) = \max[A\cos(t - a), c]$ with the phase $a$ adjusted so that the classical path continues to obey the equation of motion. The second path is given by $q(t) = \min[A\cos(t - a), c]$ with the phase again adjusted so that the classical path solves the equation of motion. The quantum theory for this case is such that the pseudofree theory is defined by the harmonic oscillator Hamiltonian augmented by Dirichlet boundary conditions at $x = c$. As a consequence, the eigenfunctions and eigenvalues of the free harmonic oscillator are almost never relevant in the construction of the pseudofree Hamiltonian. The same conclusions would be drawn from an analysis of the Euclidean functional integral formulation of the quantum theory. For $\alpha \leq 2$, a regularized potential qualitatively similar to that discussed before, should be suitable to define an interaction that is continuously connected to the free theory. For $\alpha > 2$, however, no regularized form of the potential leads to interacting theories that are continuously connected to the free theory as the coupling constant passes to zero. Any perturbation analysis of the interacting theory when $\alpha > 2$ must
take place about the pseudofree theory. It is noteworthy in this example that as \( c \to \infty \) the classical solutions all tend to those of the free theory. It is also true that as \( c \to \infty \), the pseudofree quantum theory passes to the free quantum theory.

**A remark on higher dimensional examples**

Although these facts have been illustrated for a comparatively simple one-dimensional classical/quantum model, it is not difficult to imagine analogous situations in higher dimensional mechanical systems that lead to a corresponding behavior. For example, a two-dimensional configuration space may have a singular potential of the form \( \lambda(x^2 + y^2)^{-\alpha} \). However, this example does not lead to a discontinuous perturbation since, although there are Brownian motion paths that pass through the singular point \( x = y = 0 \) and which therefore need to be discarded, the set of such paths is only of measure zero. To achieve a discontinuous perturbation, one would need a singularity of co-dimension one such as offered by the potential \( \lambda|\sqrt{x^2 + y^2} - 1|^\alpha \), for example. There is a rich set of examples of this sort, but we shall not dwell on them for we are after still bigger game, namely, those that arise for an infinite number of variables!

**Classical and Quantum Field Theory**

Until now, we have seen simple models for which the interacting theory is not continuously connected to the free theory as the coupling constant is reduced to zero. In the classical regime, such a situation can be seen by comparing the set of solutions allowed by the free classical theory with the set of solutions allowed by the pseudofree classical theory. In those cases where the set of solutions of the pseudofree classical theory is a proper subset of the set of solutions of the free classical theory, we have a genuine situation where the interacting theory has left an indelible imprint on the classical theory as the coupling constant is reduced to zero. When it comes to an analysis of the associated quantum theories, however, the classical results offer only a partial guide. In certain cases, the interacting quantum theory is continuously connected to the free theory, and thus there is no distinct pseudofree quantum theory, even though the classical pseudofree and free theories differ from one another; for example, this is the case for the one dimensional model
when \(0 < \alpha < 1\). In such a case, it is natural that a quantum perturbation series about the free theory would be the proper choice. However, there is still another option, and this is the one to which we wish to draw attention, namely when the pseudofree quantum theory is distinct from the free quantum theory. It is for such situations that the interacting quantum theory is not continuously connected to the free quantum theory as the coupling constant is reduced toward zero. It is in such cases that a perturbation series of the interaction taken about the free theory would be wrong while a perturbation series about the pseudofree theory would be the proper choice; for example, this is the case for the one dimensional models when \(\alpha > 2\).

**Focus on the Ground State**

We aim to carry these concepts from one dimensional systems to field theoretic systems. Functional integral formulations entail regularization such as that offered by a lattice.

Consider the spacetime lattice formulation of a general problem phrased as a scalar field theory. Let \(\phi_k\) denote the field value at the lattice point \(k = (k_0, k_1, k_2, \ldots, k_s)\), where \(k_j \in \{0, \pm 1, \pm 2, \ldots\} \equiv \mathbb{Z}\), \(k_0\) refers to the (future) temporal direction, and the remaining \(k_j, 1 \leq j \leq s\), denote the \(s\) spatial directions; for a quantum mechanical problem, \(s = 0\). Assume that spacetime is replaced by a periodic, hypercubic lattice with \(L\) points on an edge and \(L^s \equiv N'\) lattice points in a spatial slice.

In this section we first wish to argue that moments of expressions of interest in the full spacetime distribution can be bounded by suitable averages of related quantities in the ground state distribution. In particular, let the full spacetime average on a lattice be given by

\[
\langle [\Sigma_{k_0} F(\phi, a)]^p \rangle \equiv M \int [\Sigma_{k_0} F(\phi, a)]^p e^{-I(\phi, a, \hbar)} \Pi_k d\phi_k,
\]

where \(I\) is the lattice action, \(\Sigma_{k_0}\) denotes a summation over the temporal direction \(k_0\) only, and \(F(\phi, a)\) is an expression that depends only on fields \(\phi_k\) at a fixed value of \(k_0\). For example, one may consider \(F(\phi, a) = \Sigma_k \phi_k^4 a^8\) or \(F(\phi, a) = \Sigma_{k,l} \Omega_{k,l} \phi_k \phi_l a^{2s}\), for some \(c\)-number kernel \(\Omega_{k,l}\), etc., where the primed sum implies summation over a spatial slice at fixed \(k_0\). It follows that

\[
\langle [\Sigma_{k_0} F(\phi, a)]^p \rangle = \Sigma_{k_0 \ldots k_0} a^p \langle F(\phi_1, a) \cdots F(\phi_p, a) \rangle,
\]

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where each $\phi_j$ refers to the fields at Euclidean time "$k_0 = j". A straightforward inequality shows that

$$|\langle F(\phi_1, a) \cdots F(\phi_p, a) \rangle| \leq |\langle F(\phi_1, a)^p \rangle \cdots \langle F(\phi_p, a)^p \rangle|^{1/p}.$$ 

Finally, for sufficiently large $N'(ba^*)$, we note that

$$\langle F(\phi, a)^p \rangle = \int F(\phi, a)^p \Psi(\phi)^2 \Pi'_k d\phi_k,$$

namely, an average in the ground state distribution. The argument behind the last equation is as follows. Quite generally,

$$\langle F(\phi, a)^p \rangle = M \sum_l \int \langle \phi | l \rangle e^{-E_l T} \langle l | \phi \rangle F(\phi, a)^p \Pi'_k d\phi_k,$$

where we have used the resolution of unity $1 = \int |\phi \rangle \langle \phi | \Pi'_k d\phi_k$ for states for which $\hat{\phi}(x) |\phi \rangle = \phi(x) |\phi \rangle$, as well as the eigenvectors $|l\rangle$ and eigenvalues $E_l$ for which $\mathcal{H} |l\rangle = E_l |l\rangle$. For asymptotically large $T$, it follows that only the (unique) ground state contributes, and the former expression becomes

$$\langle F(\phi, a)^p \rangle = \int F(\phi, a)^p |\langle \phi | 0 \rangle|^2 \Pi'_k d\phi_k,$$

now with $M = 1$, which is just the expression given above.

In summary, for a finite, hypercubic lattice with periodic boundary conditions, we have derived an important result: If the sharp time average of $[F(\phi, a)]^p$ is finite, then it follows that the spacetime average of $[\sum_{k_0} F(\phi, a)]^p$ is also finite.

**Ultralocal Scalar Quantum Fields**

As we have done before, we want to illustrate the existence of a pseudofree quantum field theory distinct from any free quantum field theory by means of a straightforward and soluble example. The example we have in mind is the so-called ultralocal scalar quantum field theory. This model has been rigorously solved previously, and its most complete story can be found in Chap. 10 of [1]. We start with a brief summary of this model based on that rigorous, nonperturbative analysis. Later we show how a simple and natural argument arrives at a completely satisfactory solution as well. The advantage
of having this simple, alternative argument is that it can be generalized to realistic, relativistically covariant model quantum field theories.

The classical Hamiltonian for a scalar ultralocal field theory with a quartic nonlinear interaction is given by

\[ H = \int \left\{ \frac{1}{2} \pi(t, x)^2 + m_0^2 \phi(t, x)^2 + g_0 \phi(t, x)^4 \right\} d^s x. \]

Here, \( s \) is the number of spatial dimensions which is one less than the number \( n \) of spacetime dimensions, \( s = n - 1 \). Note well the absence of spatial derivatives in this expression. Clearly this is not a relativistic model; rather it is a mathematical model that will teach us a great deal when it is successfully quantized.

Initially, we note that there are many functions \( \phi(t, x) \) such that

\[ \int [\dot{\phi}(t, x)^2 + m_0^2 \phi(t, x)^2] dt d^s x < \infty, \quad \int \phi(t, x)^4 dt d^s x = \infty, \]

a fact which implies that there is a classical pseudofree theory distinct from the classical free theory. This is an important preliminary remark as we try to determine the status of the quantum theory.

However, let us first make a few remarks about the classical properties of such models.

**Classical features**

The classical equations of motion for this model are given by

\[ \ddot{\phi}(t, x) + m_0^2 \phi(t, x) + 4g_0 \phi(t, x)^3 = 0. \]

Indeed, the variable \( x \) is strictly a spectator variable in this equation, and we can relegate it to a subsidiary role simply by rewriting the equation of motion as

\[ \ddot{\phi}_x(t) + m_0^2 \phi_x(t) + 4g_0 \phi_x(t)^3 = 0, \]

which shows the equation of motion is simply that of an independent anharmonic oscillator at each point of space. Its solution is given by \( \phi(t, x) \equiv \phi_x(t) \), where the latter function is based on the initial data, e.g., \( \phi(0, x) \equiv \phi_x(0) \) and \( \dot{\phi}(0, x) \equiv \dot{\phi}_x(0) \), two functions of \( x \) which may be taken to be continuous in \( x \), but need not be so.

Indeed, thanks to the independence of the solution for distinct \( x \) values, one may readily discretize this model by replacing the spatial continuum by
a hypercubic spatial lattice with a lattice spacing $a$ and $L$ sites on each edge, which leads to a spatial volume given by $V' \equiv (La)^s \equiv N' a^s$. To begin, we may replace the classical Hamiltonian by a lattice regularized version given by

$$H_{\text{reg}} = \sum_k \left\{ \frac{1}{2} [\pi_k(t)^2 + m_0^2 \phi_k(t)^2] + g_0 \phi_k(t)^4 \right\} a^s,$$

where $k \in \mathbb{Z}^s$; this expression is nothing but a Riemann sum approximation to the integral given above, and it will converge to the former with $x = \lim ka$, as the lattice spacing $a$ converges to zero. This regularized Hamiltonian gives rise to the regularized equations of motion

$$\ddot{\phi}_k(t) + m_0^2 \dot{\phi}_k(t) + 4 g_0 \phi_k(t)^3 = 0,$$

and even this set of discrete equations of motion converge to the continuum form of the equation of motion as $a \to 0$ and $ka \to x$.

**Free ultralocal field theory**

An important limiting case arises when $g_0 = 0$ which is the free theory given by the free Hamiltonian

$$H_0 = \frac{1}{2} \int [\pi(t, x)^2 + m_0^2 \phi(t, x)^2] \, dx .$$

The associated free equations of motion are given by

$$\ddot{\phi}(t, x) + m_0^2 \phi(t, x) = 0 ,$$

with a solution given in terms of the initial data $\phi(0, x) \equiv \dot{\phi}_x(0)$ and $\dot{\phi}(0, x) \equiv \phi_x(0)$, by the relation

$$\phi(t, x) = \phi_x(0) \cos(m_0 t) + m_0^{-1} \dot{\phi}_x(0) \sin(m_0 t) ,$$

along with $\pi(t, x) = \dot{\phi}(t, x)$, or specifically by

$$\pi(t, x) = -m_0 \phi_x(0) \sin(m_0 t) + \dot{\phi}_x(0) \cos(m_0 t) .$$

The lattice regulated free Hamiltonian and the associated free solution is also easily given by

$$H_0 = \frac{1}{2} \sum_k [\pi_k(t)^2 + m_0^2 \phi_k(t)^2] a^s ,$$

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The free model is therefore nothing but an infinite number of identical harmonic oscillators all with the same angular frequency $m_0$. Clearly, as $a \to 0$ and $ka \to x$, the regularized solutions $\phi_k(t)$ and $\pi_k(t)$ converge to the continuum solutions $\phi(t,x)$ and $\pi(t,x)$.

Quantum Theory – First Look

We start the discussion of the quantum theory with the free theory. We promote the classical field at time $t = 0$ (and then suppress the time argument) to an operator field $\phi(x) \to \hat{\phi}(x)$ as well as promote the classical momentum $\pi(x) \to \hat{\pi}(x)$, subject to the canonical commutation relation (in units where $\hbar = 1$)

$$[\hat{\phi}(x), \hat{\pi}(y)] = i\delta(x - y).$$

The free quantum Hamiltonian $\mathcal{H}_0$ is then written as

$$\mathcal{H}_0 = \frac{1}{2} \int [ : \hat{\pi}(x)^2 + m_0^2 \hat{\phi}(x)^2 : ] \, dx,$$

where, as usual, the notation $: (\cdot) :$ denotes normal ordering (all creation operators to the left of all annihilation operators). We denote by $|0_0\rangle$ the nondegenerate ground state of $\mathcal{H}_0$ for which $\mathcal{H}_0 |0_0\rangle = 0$ holds, thanks to the normal ordering which removes the (infinite) zero-point energy.

An important relation that characterizes the ground state eigenstate is the expectation functional

$$E_0(f) \equiv \langle 0_0 | e^{i\int \hat{\phi}(x) f(x) \, dx} | 0_0 \rangle = e^{-\left(1/4m_0\right) \int f(x)^2 \, dx}.$$

Indeed, the structure of this functional as the exponential of a local integral of $f(x)$ is dictated by the fact that the temporal development of the operators at any point $x$ is ultralocal, i.e., the temporal development at $x$ is completely independent of the time development at a different spatial point $x'$. This behavior carries over to the case of the interacting ultralocal model as well, and one expects that whatever the full Hamiltonian operator $\mathcal{H}$ is,
and whatever the associated ground state $|0\rangle$ is, for which $\mathcal{H}|0\rangle = 0$ holds, the ground state expectation functional has the form

$$E(f) = \langle 0| e^{i\int \hat{\phi}(x) f(x) \, dx} |0\rangle = e^{-\int L[f(x)] \, dx},$$

for some suitable choice of the function $L[u]$.

A canonical representation for the function $L[u]$ is readily determined. We focus on those cases that are even functions $L[-u] = L[u]$, which are then real and satisfy $L[0] = 0$ and otherwise $L[u] \geq 0$. Let $f(x) \equiv p \chi_\Delta(x)$, where $\chi_\Delta(x) \equiv 1$ if $x \in \Delta$ and zero otherwise; moreover, as a modest abuse of notation, we also set $\int \chi_\Delta(x) \, dx = \Delta$ as well. Thus

$$\langle 0| e^{i\int \hat{\phi}(x) f(x) \, dx} |0\rangle = e^{-\Delta L[p]} \equiv \int \cos(p \lambda) \, d\mu_\Delta(\lambda),$$

where we have made use of the symmetry of $L[u]$, and the fact that for each $\Delta > 0$ we are dealing with a characteristic function (Fourier transform of a probability measure $\mu_\Delta$). Thus,

$$L[p] = \lim_{\Delta \to 0} \Delta^{-1} \int [1 - \cos(p \lambda)] \, d\mu_\Delta(\lambda).$$

Based on this expression, and assuming convergence, it is clear that the most general function $L[u]$ is given by the relation

$$L[u] = a u^2 + \int_{\lambda \neq 0} [1 - \cos(u \lambda)] \, d\sigma(\lambda),$$

where $a \geq 0$ and $\sigma(\lambda)$ is a nonnegative measure such that

$$\int_{\lambda \neq 0} [\lambda^2/(1 + \lambda^2)] \, d\sigma(\lambda) < \infty.$$

The free model solution obtained above is one for which $a = 1/(4m_0)$ and $\sigma = 0$. Let us assume hereafter that $a = 0$ and $\sigma \neq 0$. Observe that it is possible that

$$\int_{\lambda \neq 0} d\sigma(\lambda) = \infty,$$

and in fact this will be the case for the solutions of interest to us because we insist that the spectrum of the field operator $\hat{\phi}(x)$ is absolutely continuous, and thus for any $\Delta > 0$, it is necessary that

$$\lim_{p \to \infty} e^{-\Delta L[p]} = 0.$$
For simplicity in what follows, we assume that the measure $\sigma(\lambda)$ is absolutely continuous, and we respect that assumption by setting

$$d\sigma(\lambda) = c(\lambda)^2 d\lambda,$$

where $c(\lambda)$ is known as the “model function”. It has been found that the choice of the model function completely characterizes the ultralocal model under consideration, and, importantly, apart from the free model, all nonlinear ultralocal models are described by the situation where $a = 0$ and the model function $c(\lambda) > 0$ [1].

**Model function**

To ensure that the model function $c(\lambda)$ has a suitable singularity at $\lambda = 0$, we focus our attention on model functions of the form

$$c(\lambda) = (b\Delta)^{1/2} \frac{e^{-y(\lambda)/2}}{|\lambda|^{\gamma}},$$

where $y(0) = 0$, $\gamma = 1/2$, and $b$ is a positive constant with dimensions $L^{-s}$.

[Remark: Other $\gamma$ values in the range $1/2 < \gamma < 3/2$, which are discussed in [1], can be obtained by suitable, invertible, changes of variables from the case where $\gamma = 1/2$.] As a consequence, it follows that

$$E(p) \equiv \langle 0 | e^{ipQ} | 0 \rangle$$

$$= e^{-(b\Delta) \int [1 - \cos(p \lambda)] \frac{e^{-y(\lambda)}}{|\lambda|^{1-2\Delta}}} d\lambda$$

$$\simeq (b\Delta) \int \cos(p \lambda) \frac{e^{-y(\lambda)}}{|\lambda|^{1-2\Delta}} d\lambda,$$

where $Q \equiv \int \dot{\phi}(x) \chi_{\Delta}(x) d^s x$ and the last relation holds when $0 < b\Delta \ll 1$. Observe that the prefactor $b\Delta$ in the last expression is an approximate normalization factor (and an asymptotically correct one!) for the ground state distribution.

This latter form of the expectation function for a single degree of freedom readily extends to an infinite set of such fields, with $p = \{p_k\}$ now, such that

$$E_\Delta(p) = \prod_k \left[ (b\Delta) \int \cos(p_k \phi_k) \frac{e^{-y(\phi_k)}}{|\phi_k|^{1-2\Delta}} d\phi_k \right].$$
Let us consider $\Sigma_k p_k \chi_\Delta(x - ka)$, where here we have in mind that $\chi_\Delta(x)$ denotes a small hypercubic cell around the origin of area $\Delta = a^s$. As $\Delta = a^s \to 0$ and $\Sigma_k p_k \chi_\Delta(x - ka) \to f(x)$, it follows that

$$
\lim_{\Delta \to 0} E_\Delta(p) = E(f) = \langle 0 | e^{i \int \hat{\phi}(x) f(x) \, dx} | 0 \rangle 
= \exp \{ - \int dx \int [1 - \cos(f(x) \lambda)] \, e^{-y(\lambda)} \, d\lambda/|\lambda| \}.
$$

This last relation allows us to identify the regularized ground state of a general ultralocal theory as given (with $\hbar$ temporarily restored) by the expression

$$
\Psi(\phi) \equiv \prod_k (b\Delta)^{1/2} \frac{e^{-y(\phi_k, a, \hbar)/2\hbar}}{|\phi_k|^{1/2 - b\Delta}} \equiv \prod_k \Psi_k(\phi_k) .
$$

Given that this expression represents the ground state, it then follows that the regularized Hamiltonian is given by

$$
\mathcal{H}_\Delta = \sum_k \left[ -\frac{1}{2} \hbar^2 \frac{\partial^2}{\partial \phi_k^2} a^{-s} + \frac{1}{2} \hbar^2 \frac{1}{\Psi_k(\phi_k)} \frac{\partial^2 \Psi_k(\phi_k)}{\partial \phi_k^2} a^{-s} \right]
\equiv -\frac{1}{2} \sum_k \hbar^2 \frac{\partial^2}{\partial \phi_k^2} a^{-s} + \mathcal{V}(\phi) ,
$$

where, for the choice of $\Psi(\phi)$ given above,

$$
\mathcal{V}(\phi) \equiv \sum_k \left[ \frac{1}{8} y''(\phi_k, a, \hbar) - \frac{1}{4} \hbar y'''(\phi_k, a, \hbar) + \frac{1}{2} \hbar \gamma_r y'(\phi_k, a, \hbar) \phi_k^{-1} + \frac{1}{2} \hbar^2 \gamma_r (\gamma_r + 1) \phi_k^{-2} \right] ;
$$

\begin{align*}
\gamma_r & \equiv \frac{1}{2} - b\Delta = \frac{1}{2} - ba^s .
\end{align*}

Consider the pseudofree ultralocal case for which

$$
y(\phi_k, a, \hbar) = m_0 \phi_k^2 a^s .
$$

For this choice, it follows that

$$
\mathcal{V}_{pf}(\phi) \equiv \frac{1}{2} \sum_k \left[ m_0^2 \phi_k^2 a^s - \hbar m_0 (1 - 2\gamma_r) + \hbar^2 \gamma_r (\gamma_r + 1) \phi_k^{-2} a^{-s} \right] .
$$
Given the Hamiltonian for this case we can immediately determine the lattice action for this pseudofree ultralocal model. In particular, it follows that
\[ I_{pf} = \sum_k \left\{ \frac{1}{2} (\phi_{k\#} - \phi_k)^2 a^{n-2} + m_0^2 \phi_k^2 a^n + \hbar^2 (1 - ba^s)(\frac{3}{2} - ba^s) a^{-2s} \phi_k^{-2} a^n \right\} \] .

In this expression the factor \( k\# \) signifies the next lattice point advanced by one unit in the time direction, i.e., if \( k = (k_0, k_1, \ldots, k_s) \) then \( k\# = (k_0 + 1, k_1, \ldots, k_s) \). Note well that any constant term (zero point energy) in the Hamiltonian cancels out with a similar term in the normalization factor in the functional integral and need not be included in the lattice action. Observe that the classical limit for which \( \hbar \to 0 \) accompanied by the continuum limit leads to the classical (Euclidean) action for the free ultralocal model.

**Interacting ultralocal models**

Drawing on the foregoing analysis of the pseudofree ultralocal model, we may give a brief discussion of interacting ultralocal models. The quartic interaction in the lattice action leads to a lattice Hamiltonian of the form
\[ \mathcal{H} = -\frac{1}{2} \hbar^2 \sum_k \frac{\partial^2}{\partial \phi_k^2} + V(\phi) , \]
where
\[ V(\phi) = \sum_k \left[ \frac{1}{2} m_0^2 \phi_k^2 a^s + \lambda_0 \phi_k^4 a^s + \frac{1}{2} \hbar^2 \gamma_r (1 + \gamma_r) \phi_k^{-2} a^{-s} \right] - E . \]
The constant \( E \) is chosen so that the ground state \( \Psi(\phi) \) fulfills \( \mathcal{H} \Psi(\phi) = 0 \). Unfortunately, the form of the expression \( y(\phi, a, \hbar) \) that is part of the ground state function is unknown, but it surely has the property that as \( \lambda_0 \to 0 \), then \( y(\phi, a, \hbar) \to m_0 \phi^2 a^s \) appropriate to the pseudofree model. Stated otherwise, the quartic interacting theory is continuously connected to the pseudofree model as advertised.

Although we can not analytically describe the ground state for the quartic ultralocal model, we can, as another example, choose a nonquadratic form for \( y(\phi, a, \hbar) \) and see to what interacting model it belongs. For example, let us consider
\[ y(\phi, a, \hbar) = m_0 \phi^2 a^s + g_0 \phi^4 a^s , \]
which leads to the potential
\[ V(\phi) = \sum_k \left[ \frac{1}{2} m_0^2 \phi_k^2 + 4m_0 g_0 \phi_k^4 + 4g_0^2 \phi_k^8 - \frac{1}{2} \hbar [m_0(1 - 2\gamma_r) + 2g_0(2\gamma_r - 3)\phi_k^2] + \hbar^2 \gamma_r (1 + \gamma_r) \phi_k^{-2} \right] a^s . \]
Evidently this choice describes a model with a mixed quadratic, quartic, and sixth order potential. The first three terms – those without ℏ as a coefficient – survive in the classical limit as ℏ → 0. Again, as the nonlinear coupling \( g_0 \rightarrow 0 \), it follows that this interacting model is continuously connected to the pseudofree model.

**Another Route to Quantize Ultralocal Models**

Let us now derive the pseudofree ultralocal model by an alternative argument. First, we recognize the free model and its ground state on a regularizing lattice as given by

\[
\Psi_0(\phi) = \sqrt{K} e^{-\frac{1}{2} m_0 \Sigma'_k \phi_k^2 a^s},
\]

which gives rise to the ground state expectation functional

\[
E_0(f) = \lim_{\Delta \rightarrow 0} K \int e^{i \Sigma'_k p_k \phi_k a^s - m_0 \Sigma'_k \phi_k^2 a^s} \Pi'_k d\phi_k
= e^{- (1/4m_0) \int f(x)^2 dx}.
\]

Perturbations in the mass for example would involve expressions of the form

\[
I_p(m_0) \equiv K \int [\Sigma'_k \phi_k^2 a^s]^p e^{-m_0 \Sigma'_k \phi_k^2 a^s} \Pi'_k d\phi_k,
\]

for which the result is clearly divergent in the continuum limit where the number \( N' \) of spatial lattice points diverges. It is instructive to see just where that \( N' \) factor originates, and to do so we pass to hyper-spherical coordinates defined by the expressions

\[
\phi_k \equiv \kappa \eta_k , \quad \kappa \geq 0 , \quad -1 \leq \eta_k \leq 1 ,
\]

\[
\kappa^2 \equiv \Sigma'_k \phi_k^2 , \quad 1 = \Sigma'_k \eta_k^2.
\]

In terms of these variables, it follows that

\[
I_p(m_0) = 2K \int [\kappa^2 a^s]^p e^{-m_0 \kappa^2 a^s} \kappa^{N'-1} d\kappa \delta(1 - \Sigma'_k \eta_k^2) \Pi'_k d\eta_k.
\]

For large \( N' \), this integral may be estimated by steepest descent methods as

\[
I_p(m_0) = O((N'/m_0)^p) I_0(m_0).
\]
Moreover, in a perturbation calculation of $I_1(m_0)$ about $I_1(1)$ (say) it follows that
\[
I_1(m_0) = I_1(1) - \delta m_0 I_2(1) + \frac{1}{2} \delta m_0^2 I_3(1) - \cdots ,
\]
where $\delta m_0 \equiv m_0 - 1$. Clearly this series is divergent as $N' \to \infty$, i.e., in the continuum limit. Note well that $N'$ makes an explicit appearance in this series only in the factor $\kappa^{N' - 1}$ that arises from the measure $\Pi'_k d\phi_k$ put into hyper-spherical coordinates.

To eliminate those divergences we need to eliminate that appearance of the factor $N'$. The only way to eliminate that factor is to change the ground state from that of the free system to that of the pseudofree system that takes account of the hard core. To attack the hard core directly is difficult and has so far not been a productive direction to follow. But, and here is the main point of this discussion: To eliminate the factor $N'$ that arises from the field measure it suffices to ensure that the ground state distribution for the pseudofree theory is such that
\[
\Psi_{pf}^2(\phi) \propto \kappa^{-(N' - R)} e^{-m_0 \Sigma_k \phi_k^2 a^s}
\]
for some finite parameter $R$.

For the ultralocal model, we shall more explicitly choose a ground state for the pseudofree model of the form
\[
\Psi_{pf}(\phi) = K' \Pi'_k |\phi_k|^{-(1-R/N')/2} e^{-\frac{1}{2}m_0 \phi_k^2 a^s},
\]
which leads to the desired form and respects the ultralocal symmetry of the model. How do we choose $R$? We require that this expression have an acceptable continuum limit, which we study by examining the characteristic function for the ground state distribution, i.e.,
\[
E_{pf}(f) = \lim_{a \to 0} \Pi'_k K \int e^{ip_k \phi_k a^s - m_0 \phi_k^2 a^s |\phi_k|^{-(1-R/N')} \Pi'_k d\phi_k}
\]
\[
= \lim_{a \to 0} \Pi'_k \{1 - K \int [1 - e^{ip_k \phi_k a^s}] e^{-m_0 \phi_k^2 a^s |\phi_k|^{-(1-R/N')} \Pi'_k d\phi_k} \}.
\]
The only way to achieve a meaningful continuum limit is, first, (effectively) choose $m_0 = (ba^s)m$, where $b$ is an arbitrary positive parameter with dimensions of $L^{-s}$, which, after a change of variables ($\phi_k \to a^{-s} \lambda_k$), yields to leading order,
\[
E_{pf}(f) = \lim_{a \to 0} \Pi'_k \{1 - K \int [1 - e^{ip_k \lambda_k}] e^{-bm \lambda_k^2} |\lambda_k|^{-(1-R/N')} \Pi_k d\lambda_k} \}.
\]
and, second, choose $K = c(ba^s)$ [which fixes $R$ to be $R = 2c(ba^s)N'$], and thus

$$E_{pf}(f) = e^{-cb\int d^nx\int\{1 - \cos[f(x)\lambda]\} e^{-bm\lambda^2} d\lambda/|\lambda|}.$$

Normally, the dimensionless factor $c$ has been chosen as $c = 1$ or $c = \frac{1}{2}$, but any positive value is acceptable.

It is of fundamental importance to observe that we have derived a correct version of the pseudofree ultralocal model by the simple act of choosing the pseudofree ground state distribution to cancel the unwanted factor $N'$, the very factor that causes the divergences in the first place, and then to ensure as meaningful a continuum limit as possible. This simple act ensures that all the moments of interest are now finite and no infinities arise whatsoever. Since this action has the effect of cancelling all divergences, it acts in all necessary ways as would the presumed hard core. In particular, the so-defined, divergence-free interacting theory does not pass continuously to the free theory but instead it passes to an alternative theory, namely, the pseudofree theory. That kind of limiting behavior is the biggest clue to the fact that the interaction acts as a (partial) hard core. Does the simple act of removing the offending factor $N'$ accurately correspond to including the effects of the hard core? In fact, it really doesn’t matter if the elimination of the factor $N'$ is an accurate realization of the hard core; the putative “hard core” has already rendered an important service by refocussing our attention beyond those counter terms that are suggested by perturbation theory. Additionally, the study of the soluble ultralocal models has helped us clarify the question of whether removing the factor $N'$ corresponds to accounting for the hard core. Specifically, the solution obtained from a rigorous viewpoint is identical to the one obtained by the supremely simple prescription of choosing a suitable pseudofree model that eliminates the offending factor $N'$. In this sense, the removal of the cause of the divergences, i.e., the factor $N'$, has rendered the theory finite in all respects, and since the result completely agrees with the rigorously obtained result, we are certainly entitled to assert that the removal of the factor $N'$ has accounted for the presence of the hard core in the case of ultralocal models.

We shall see that this breathtakingly elementary procedure, coupled with a judicious choice of further details of the pseudofree model, will provide a divergence-free formulation of additional examples of nonrenormalizable models, formulations that would be difficult to arrive at by any other means. It is reasonable that the procedure to eliminate the source of divergences
caused by the measure should apply to other models which, in some sense, are “close” to ultralocal models. It is also reasonable to expect that traditional nonrenormalizable models are good candidates on which to try a similar approach to deal with otherwise uncontrollable divergences.

Relativistic Models

The classical (Euclidean) action for covariant, quartic self interacting scalar fields is given by

\[ I = \int \left\{ \frac{1}{2} \left[ (\nabla \phi(x))^2 + m_0^2 \phi(x)^2 \right] + \lambda_0 \phi(x)^4 \right\} d^n x , \]

for an \( n \)-dimensional spacetime. To discuss the classical side of the pseudofree situation, we recall a classical Sobolev-type inequality (see, e.g., [1]) given by

\[ \left\{ \int \phi(x)^4 d^n x \right\}^{1/2} \leq c \int \left[ (\nabla \phi(x))^2 + m_0^2 \phi(x)^2 \right] d^n x , \]

which for \( n \leq 4 \) holds with \( c = 4/3 \) and for \( n \geq 5 \), requires that \( c = \infty \). This result implies that for \( n \geq 5 \), there are fields \( \phi(x) \) for which the free part of the classical action is finite but for which the quartic interaction diverges. These are just the conditions under which a classical pseudofree theory different from the classical free theory exists. Thus it is possible when \( n \geq 5 \) that the quantum theory also has a pseudofree theory different from its free theory.

We recall that a lattice regularized form of the Euclidean functional integral with only two free parameters \( m_0 \) and \( \lambda_0 \) has been shown to pass to a (generalized) free theory in the continuum limit [2]; thus a richer variety of renormalization counterterms is required to avoid triviality. Since, for \( n \geq 5 \) the quantum theories are perturbatively nonrenormalizable leading to a perturbation series composed of infinitely many distinct counterterms, such an approach does not resolve the problem. Our goal is to show that an unconventional counterterm suggested by what is needed to remove the source of the divergences can lead to a satisfactory resolution of all problems with the relativistic models. To that end we now turn our attention to a very different sort of lattice regularized functional integral formulation for self interacting relativistic scalar fields.

In particular, relativistic interacting scalar models admit an analogous treatment to that of the ultralocal models, and in our present discussion we
follow reference [3]. In begin with, let us introduce a lattice action defined by the expression

\[
I(\phi, a, h) \equiv \frac{1}{2} \sum_k (\phi_{k^*} - \phi_k)^2 a^{n-2} + \frac{1}{2} m_0^2 \sum_k \phi_k^2 a^n
+ \lambda_0 \sum_k \phi_k^4 a^n + \frac{1}{2} h^2 \sum_k F_k(\phi) a^n,
\]

where there is an implicit summation over all \(n\) nearest neighbors in the positive sense symbolized by the notation \(k^*\), and where the nonclassical counterterm is

\[
F_k(\phi) \equiv \frac{1}{4} \left( \frac{N' - 1}{N'} \right)^2 a^{-2s} \sum'_{r,l} \frac{J_{r,k} J_{t,l} \phi_k^2}{[\Sigma'_l J_{r,l} \phi_l^2] \Sigma_m J_{t,m} \phi_m^2]}
- \frac{1}{2} \left( \frac{N' - 1}{N'} \right)^2 a^{-2s} \sum'_{l} \frac{J_{l,k} \phi_k^2}{[\Sigma'_m J_{t,m} \phi_m^2]}
+ \left( \frac{N' - 1}{N'} \right)^2 a^{-2s} \sum'_{l} \frac{J_{l,k} \phi_k^2}{[\Sigma'_m J_{t,m} \phi_m^2]^2}.
\]

Here,

\[
J_{k,l} \equiv \frac{1}{2s + 1} \delta_{k,l \in \{k \cup k_{nn}\}},
\]

where \(\delta_{k,l}\) is a Kronecker delta. This latter notation means that an equal weight of \(1/(2s + 1)\) is given to the \(2s + 1\) points in the set composed of \(k\) and its \(2s\) nearest neighbors in the spatial sense only; \(J_{k,l} = 0\) for all other points in that spatial slice. [Specifically, we define \(J_{k,l} = 1/(2s + 1)\) for the points \(l = k = (k_0, k_1, k_2, \ldots, k_s), l = (k_0, k_1 \pm 1, k_2, \ldots, k_s), l = (k_0, k_1, k_2 \pm 1, \ldots, k_s), \ldots, l = (k_0, k_1, k_2, \ldots, k_s \pm 1).\) This definition implies that \(\Sigma_l J_{k,l} = 1.\)

For the ultralocal model, the analog of the constants \(J_{k,l}\) is the Kronecker delta, i.e., \(\delta_{k,l}\). In that case it was important to respect the physics of the ultralocal model with no interaction between fields at distinct (lattice) points. For the relativistic models, on the other hand, there is indeed communication between spatially neighboring points and we can use that fact to provide a lattice-symmetric, regularized form of the denominator factor. Moreover, the lack of integrability at \(\phi_k = 0\), for each \(k\), which was critical for the ultralocal models to ensure that the ground state becomes a generalized Poisson distribution in the continuum limit, is exactly what is not wanted in the case of the relativistic models. This latter fact is ensured by the factors \(J_{k,l}\) as chosen.
We first focus on our choice of the pseudofree model in the relativistic case, which is chosen somewhat differently than in the ultralocal case. Specifically, we define the generating function for the lattice regularized, covariant pseudofree model by
\[ S_{pf}(h) = M_{pf} \int \exp \left[ Z^{-1/2} \sum_k h_k \phi_k a^n / \hbar - \frac{1}{2} \sum_k (\phi_k^* - \phi_k)^2 a^{n-2} / \hbar \right. \]
\[ \left. - \frac{1}{2} \hbar \sum_k F_k(\phi) a^n \right] \Pi_k d\phi_k \]
here, \( Z \) denotes the so-called field strength renormalization constant to be discussed below. Associated with this choice of the pseudofree generating function is the lattice Hamiltonian for the pseudofree model, which (with the zero point energy subtracted) reads
\[ H_{pf} = -\frac{1}{2} \hbar^2 a^{-s} \sum_k \frac{\partial^2}{\partial \phi_k^2} + \frac{1}{2} \sum_k (\phi_k^* - \phi_k)^2 a^{s-2} + \frac{1}{2} \hbar^2 \sum_k F_k(\phi) a^s - E_0 . \]
Lastly, we introduce the expression for the pseudofree ground state
\[ \Psi_{pf}(\phi) = \sqrt{K} \frac{e^{-\sum_k \phi_k A_{k-l} \phi_l a^{2s}/2 \hbar - W(\phi a^{(s-1)/2}/\hbar^{1/2})/2}}{\Pi_k [\sum_l J_{k,l} \phi_l^2]^{(N'-1)/4N'}} \]
which, in effect, was chosen first, and then the lattice Hamiltonian and the lattice action were derived from it. We discuss the (unknown) function \( W \) below; however, we observe here that the other factors in \( \Psi_{pf}(\phi) \) properly account for both the large field and small field behavior of the ground state.

In the next section we discuss the continuum limit, and in doing so we are again guided by the discussion in [3].

**Continuum Limit**

Before focusing on the limit \( a \to 0 \) and \( L \to \infty \), we note several important facts about ground-state averages of the direction field variables \( \{ \eta_k \} \). First, we assume that such averages have two important symmetries: (i) averages of an odd number of \( \eta_k \) variables vanish, i.e.,
\[ \langle \eta_{k_1} \cdots \eta_{k_{2p+1}} \rangle = 0 , \]
and (ii) such averages are invariant under any spacetime translation, i.e.,
\[ \langle \eta_{k_1} \cdots \eta_{k_{2p}} \rangle = \langle \eta_{k_1+t} \cdots \eta_{k_{2p}+t} \rangle \]
for any \( l \in \mathbb{Z}^n \) due to a similar translational invariance of the lattice Hamiltonian. Second, we note that for any ground-state distribution, it is necessary that \( \langle \eta_k^2 \rangle = 1/N' \) for the simple reason that \( \Sigma_k \eta_k^2 = 1 \). Hence, \( |\langle \eta_k \eta_l \rangle| \leq 1/N' \) as follows from the Schwarz inequality. Since \( \langle [\Sigma_k \eta_k^2]^2 \rangle = 1 \), it follows that \( \langle \eta_k^2 \eta_l^2 \rangle = O(1/N'^2) \). Similar arguments show that for any ground-state distribution

\[
\langle \eta_k \cdots \eta_{k_2p} \rangle = O(1/N'^p),
\]

which will be useful almost immediately.

### Field strength renormalization

For \( \{ h_k \} \) a suitable spatial test sequence, we insist that expressions such as

\[
\int Z^{-p} [\Sigma_k' h_k \phi_k a^s]^{2p} \Psi_{pf}(\phi)^2 \Pi_k' d\phi_k
\]

are finite in the continuum limit. Due to the intermediate field relevance of the factor \( W \) in the pseudofree ground state, an approximate evaluation of the integral will be adequate for our purposes. Thus, we are led to consider

\[
K \int Z^{-p} [\Sigma_k' h_k \phi_k a^s]^{2p} \frac{e^{-\Sigma_{k,l} \phi_k A_{k-l} a^{2s}/\hbar - W}}{\Pi_k'[\Sigma_l' J_{k,l} \phi_l^2]^{(N'-1)/2N'}} \Pi_k' d\phi_k
\]

\[
\simeq 2K_0 \int Z^{-p} \kappa^{2p} [\Sigma_k' h_k \eta_k a^s]^{2p} \frac{e^{-\kappa^2 \Sigma_{k,l} \eta_k A_{k-l} a^{2s}/\hbar}}{\Pi_k'[\Sigma_l' J_{k,l} \eta_l^2]^{(N'-1)/2N'}} d\kappa \delta(1 - \Sigma_k' \eta_k^2) \Pi_k' d\eta_k,
\]

where \( K_0 \) is the normalization factor when \( W \) is dropped. Our goal is to use this integral to determine a value for the field strength renormalization constant \( Z \). To estimate this integral we first replace two factors with \( \eta \) variables by their appropriate averages. In particular, the quadratic expression in the exponent is estimated by

\[
\kappa^2 \Sigma_{k,l} \eta_k A_{k-l} \eta_l a^{2s} \simeq \kappa^2 \Sigma_{k,l} N'^{-1} A_{k-l} a^{2s} \propto \kappa^2 N' a^{2s} a^{-(s+1)},
\]

and the expression in the integrand is estimated by

\[
[\Sigma_k' h_k \eta_k a^s]^{2p} \simeq N'^{-p} [\Sigma_k' h_k a^s]^{2p}.
\]

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The integral over $\kappa$ is then estimated by first rescaling the variable $\kappa^2 \to \kappa^2/(N'a^{s-1}/\hbar)$, which then leads to an overall integral estimate proportional to

$$Z^{-p} [N'a^{s-1}]^{-p} N'^{-p} [\Sigma_k' h_k a^s]^{2p};$$

at this point, all factors of $a$ are now outside the integral. For this result to be meaningful in the continuum limit, we are led to choose $Z = N'^{-2} a^{-(s-1)}$. However, $Z$ must be dimensionless, so we introduce a fixed positive quantity $q$ with dimensions of an inverse length, which allows us to set

$$Z = N'^{-2} (qa)^{-(s-1)}.$$

**Mass and coupling constant renormalization**

A power series expansion of the mass and coupling constant terms lead to the expressions $\langle [m_0^2 \Sigma_k \phi_k^2 a^n]_p \rangle$ and $\langle [\lambda_0 \Sigma_k \phi_k^4 a^n]_p \rangle$ for $p \geq 1$, which we treat together as part of the larger family governed by $\langle [g_{0,r} \Sigma_k \phi_k^{2r} a^n]_p \rangle$ for integral $r \geq 1$. Thus we consider

$$K \int [g_{0,r} \Sigma_k' \phi_k^{2r} a^n]_p e^{-\sum_{k,l} \phi_k A_k \phi_l a^{2s}/\hbar - W' \Pi_k' \sum_{j,k,l} \phi_j^{2r} a^n (N' - 1)/2N'} d\phi_k \simeq 2K_0 \int [g_{0,r} \Sigma_k' \eta_k^{2r} a^n]_p \times e^{-\kappa^2 \sum_{k,l} \eta_k A_{k-l} \eta_l a^{2s}/\hbar - \sum_{k,l} \phi_j^{2r} a^n (N' - 1)/2N'} d\kappa \delta(1 - \sum_{k,l} \eta_k^{2r}) \Pi_k' d\eta_k / \Pi_k' \sum_{j,k,l} \phi_j^{2r} a^n (N' - 1)/2N'.
$$

The quadratic exponent is again estimated as

$$\kappa^2 \sum_{k,l} \eta_k A_{k-l} \eta_l a^{2s} \propto \kappa^2 N' a^{2s} a^{-(s+1)},$$

while the integrand factor

$$[\Sigma_k' \eta_k^{2r}]_p \simeq N'^p N'^{-rp}.$$

The same transformation of variables used above precedes the integral over $\kappa$, and the result is an integral, no longer depending on $a$, that is proportional to

$$g_{0,r}^p N'^{-(r-1)p} a^{sp}/N'^{rp} a^{(s-1)rp}.$$
To have an acceptable continuum limit, it suffices that
\[ g_{0,r} = N'(2r-1)(qa)^{(s-1)r-s}g_r, \]
where \( g_r \) may be called the physical coupling factor. Moreover, it is noteworthy that
\[ Z^r g_{0,r} = [N'(qa)^s]^{-1}g_r, \]
for all values of \( r \), which for a finite spatial volume \( V' = N' a^s \) leads to a finite nonzero result for \( Z^r g_{0,r} \). It should not be a surprise that there are no divergences for all such interactions because the source of all divergences has been neutralized!

We may specialize the general result established above to the two cases of interest to us. Namely, when \( r = 1 \) this last relation implies that \( m_0^2 = N'(qa)^{-1}m^2 \), while when \( r = 2 \), it follows that \( \lambda_0 = N'^3(qa)^{s-2}\lambda \). In these cases it also follows that \( Z m_0^2 = [N'(qa)^s]^{-1}m^2 \) and \( Z^2 \lambda_0 = [N'(qa)^s]^{-1}\lambda \), which for a finite spatial volume \( V' = N' a^s \) leads to finite nonzero results for \( Z m_0^2 \) and \( Z^2 \lambda_0 \), respectively.

**Conclusion**

For covariant scalar nonrenormalizable quantum field models, we have shown that the choice of a nonconventional counterterm, but one that is still nonclassical, leads to a formulation for which a perturbation analysis of both the mass term and the nonlinear interaction term, expanded about the appropriate pseudofree model, are term-by-term finite.

Coupled with the discussion for the ultralocal models, it is evident that the present analysis would suggest a related formulation for so-called Dicatrophic Quantum Field Theories introduced by the author in [4]. These models are distinguished by the fact that they can be viewed as fully relativistic models modified so that some (but not all) of the spatial derivatives are dropped; thus these models lie, in a certain sense, between the relativistic and ultralocal models.

It is also hoped that some of these ideas may have relevance in one or more formulations of quantum gravity, such as, for example, in the program of Affine Quantum Gravity introduced by the author; see [5].

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