Notes on character stacks for non-orientable surfaces

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Abstract

We give a counterexample to a formula suggested by the work of Letellier and Rodriguez-Villegas [28] for the mixed Poincaré series of character stacks for non-orientable surfaces. The counterexample is obtained by an explicit description of these character stacks for (real) elliptic curves.
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1 Introduction

Let $K$ be an algebraically closed field, $r,k \geq 1$ be non-negative integers and $C = (O_1, \ldots, O_k)$ a $k$-tuple of semisimple orbits of $G = \text{GL}_n(K)$. Consider a couple $(\Sigma, \sigma)$ where $\Sigma$ is a Riemann surface of genus $r-1$ and $\sigma : \Sigma \to \Sigma$ an antiholomorphic involution without fixed points. The character stack $M^r_C$ associated to such a couple $(\Sigma, \sigma)$ and $C$ is the stacky quotient

$$M^r_C := \left\{ \{D_1, \ldots, D_r \in G, Z_1 \in O_1, \ldots, Z_k \in O_k \mid D_1 \theta(D_1) \cdots D_r \theta(D_r) Z_1 \cdots Z_k = 1\} / G \right\}$$

(1.0.1)

where $\theta : G \to G$ is the Cartan involution $\theta(A) = (A)^{-1}$. For a more detailed definition and the relation between $M^r_C$ and representations of $\pi_1(\Sigma)$ see Section §2.2.

The stacks $M^r_C$ are deeply related to branes inside the moduli space of Higgs bundles: the computation of cohomology and geometry of branes is a key part in understanding mirror symmetry for the Hitchin system. References about the subject can be found for example in [4],[5],[8],[7].

Recently Letellier and Rodriguez-Villegas (see [28, Theorem 4.6]) computed the E-series $E(M^r_C, q)$ of these stacks over $\mathbb{C}$ when $C$ is generic (for a definition of generic $k$-tuples of orbits see Definition 2.3.1). The E-series is a specialization of the whole (compactly supported) mixed-Poincaré series $H_c(M^r_C, q, t)$ obtained by plugging $t = -1$. For a definition of the mixed-Poincaré series see §2.3.1. E-series give important information such as the number of irreducible components or non emptiness.

The computation of [28, Theorem 4.6] is obtained via reduction over $\mathbb{F}_q$ and point counting. The authors consider the $\mathbb{F}_q$- stack $M^r_C, q$ and compute in an explicit way a rational function $Q(t)$ such that

$$Q(q^n) = \# M^r_C, q^n(\mathbb{F}_q^n)$$

(1.0.2)

In this case there is an equality $E(M^r_{C, C}, q) = Q(q)$, as shown for example in [28, Theorem 2.8]. Surprisingly, the functions $Q(t)$ appearing in this context are very similar to the ones computing E-series of character stacks for Riemann surfaces.

Consider $g \geq 0$, $k \geq 1$ and $C = (O_1, \ldots, O_k)$ as above. The associated character stack $M_C$ for a Riemann surface $\Sigma$ of genus $g$ is the stacky quotient

$$M_C := \left\{ \{A_1, B_1, \ldots, A_g, B_g \in G, X_1 \in O_1, \ldots, X_k \in O_k \mid [A_1, B_1] \cdots [A_g, B_g] X_1 \cdots X_k = 1\} / G \right\}$$

This can alternatively be described as the quotient stack

$$\left\{ \{\rho : \pi_1(\Sigma - \{x_1, \ldots, x_k\}) \to G \mid \rho(y_i) \in O_i\} / G \right\}$$

where $S = \{x_1, \ldots, x_k\}$ is a set of $k$ points of $\Sigma$ and each $y_i$ is a small loop around $x_i$. E-series for these stacks and generic orbits were computed in [21, Theorem 1.2.3]. As observed in [28, Remark 1.5], for $r = 2h$ we have an equality $E(M^r_C, q) = E(M_C, q)$ where $M_C$ is associated to a Riemann surface of genus $h$. Even for $r$ odd, the formulas for the E-series of $M^r_C$ are very similar to those of $E(M_C, q)$ (see §2.3.3 for more details).

There is a longstanding conjecture about the whole mixed Poincaré series of character stacks
for Riemann surfaces [21 Conjecture 1.2.1]: in [28 Theorem 4.8] the authors verified that a completely analogous formula holds for \(M'_e\) for \(r = 1\) and \(k = 1\) (see §2.3.3 for more details). It would then have been natural to expect the conjecture to hold for all \(r\). The main result of this paper (see §3.1.3) is an explicit description of some of these spaces and their cohomology in the case \(r = 2\) giving a counterexample to the expected formula. The main theorem is:

**Theorem 1.0.1.** Put \(r = 2, k = 1\) and consider the orbit \(\{e^{\frac{ad}{\pi i}}\}\) where \((n, d) = 1\) and \(d\) is even. Then \(M'_e\) is a \(B(\mu_2)\)-gerbe over \(\mathbb{C}^*\). In particular, its mixed Poincaré series is

\[
H_c(M'_e, q, t) = qt^2 + t
\]

To prove Theorem 1.0.1 we need some results of independent interest concerning the geometry of the spaces \(M'_e\) for \(k = 1\) and the orbit \(\{e^{\frac{ad}{\pi i}}\}\) (see §3.1.1).

To summarize these results, let \(M^e_{n,d}\) be the GIT quotient associated to \(M'_e\) and \(M_{n,d}\) be the GIT quotient of the character stack associated to the Riemann surface \(\tilde{\Sigma}\) (of genus \(r - 1\)) for \(k = 1\) and the orbit \(\{e^{\frac{ad}{\pi i}}\}\). There is an involution on \(M_{n,d}\), which we denote again by \(\sigma\), which sends a representation \(\tilde{\rho} \in M_{n,d}\) to \(\sigma(\tilde{\rho}) = \theta(\tilde{\rho})(\sigma_*)\) (for more details and a definition of \(\sigma_*\) see §2.1, §2.2). In §3.1.1 we show that:

**Theorem 1.0.2.** If \(r\) is odd, the fixed point locus \(M^\sigma_{n,d}\) is isomorphic to \(M^e_{n,d}\). If \(r\) is even, there is an open-closed decomposition \(M^\sigma_{n,d} = M^\sigma_{n,d}^+ \sqcup M^\sigma_{n,d}^-\) such that \(M^\sigma_{n,d}^+ \cong M^\sigma_{n,d}^- \cong M^e_{n,d}\).

The Theorem 1.0.2 (and the others in §3.1.1) are probably known to the experts but we could not locate a reference in the literature. We review them here for the sake of completeness. In the paragraph §3.1.3 we describe the variety \(M^\sigma_{n,d}\) for \(r = 2\) and find an isomorphism \(M^\sigma_{n,d}^+ \cong \mathbb{C}^*\) which allows to prove Theorem 1.0.1.

Finally, we still lack of another conjectural formula for the mixed Poincaré series of these character stacks. One of the main difficulties when trying to formulate such a conjecture lies in the impossibility of extending the standard approach used to compute the Poincaré polynomials of character varieties for Riemann surfaces (as done for example by Mellit in [30] or by Garcia-Prada in [17]) to the context of antiholomorphic involutions. Basically, for the character stacks \(M'_e\) the corresponding moduli space of (real) Higgs bundles is no more an algebraic variety and this makes computations very different in nature from the setting of [30] or [17] (see §4 for more details).

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## 2 Preliminaries and notations

### 2.1 Fundamental groups of punctured non-orientable surfaces

Let \(\tilde{\Sigma}\) be a Riemann surface of genus \(g\) and \(\sigma : \tilde{\Sigma} \to \tilde{\Sigma}\) be an antiholomorphic involution \(\sigma\) without fixed points. The quotient \(\Sigma := \tilde{\Sigma}/\sigma\) is endowed with the structure of a non-orientable
surface: topologically $\Sigma$ is the connected sum of $r := g + 1$ real projective planes. We denote by $p$ the quotient map $p : \tilde{\Sigma} \to \Sigma$.

Let $S = \{z_1, \ldots, z_k\}$ be a set of $k$ points on $\Sigma$. We fix also a basepoint $x_0 \in \Sigma - S$ and a point $\tilde{x}_0$ in the fiber $p^{-1}(x_0)$. We denote the fundamental groups of $\Sigma - S, \tilde{\Sigma} - p^{-1}(S)$ with basepoints $x_0, \tilde{x}_0$ by $\pi_1(\Sigma - S)$ and $\pi_1(\tilde{\Sigma} - p^{-1}(S))$ respectively. The map $p$ induces an immersion $p_* : \pi_1(\tilde{\Sigma} - p^{-1}(S)) \to \pi_1(\Sigma - S)$. We fix also a path $\lambda_{\sigma} : \tilde{x}_0 \to \sigma(\tilde{x}_0)$ inside $\tilde{\Sigma} - p^{-1}(S)$; its projection determines a closed path $p(\lambda_{\sigma}) = \gamma_{\sigma} \in \pi_1(\Sigma - S)$. Finally, we denote by $\sigma_{\ast}$ the morphism on $\pi_1(\Sigma - p^{-1}(S))$ given by

$$\sigma_{\ast}(c) = \lambda^{-1}_{\sigma}(c)\lambda_{\sigma}.$$ 

We recall that there are explicit presentations of the above fundamental groups:

$$\Pi := \pi_1(\Sigma - S) = \langle d_1^2 \cdots d_k^2 z_1 \cdots z_k = 1 \rangle$$

and

$$\Pi := \pi_1(\tilde{\Sigma} - p^{-1}(S)) = \langle [a_1, b_1] \cdots [a_g, b_g] x_1 \cdots x_{2k} = 1 \rangle.$$ 

**Example 2.1.1.** Let us look at the case of $r = 2$ and $k = 1$. We consider the elliptic curve $\tilde{E}$ associated to the lattice $\langle 1, i \rangle \subseteq \mathbb{C}$ i.e $\tilde{E} \cong \mathbb{C}/\langle 1, i \rangle$ and let $\pi$ be the projection $\pi : \mathbb{C} \to E$. Let $\sigma : \tilde{E} \to \tilde{E}$ be the involution without fixed points defined by $\sigma(z) = \bar{z} + \frac{1}{2}$, and $E = \tilde{E}/\sigma$ be the associated quotient.

We fix a point $z_1 \in E$ and we let its preimage in $\tilde{E}$ be $p^{-1}(z_1) = \{y_1, y_2\}$. Put $x_0 = p(0)$ and $\bar{x}_0 = 0$ as base points and $\lambda_{\sigma} = \pi(\gamma(t))$ where $\gamma(t) = \frac{1}{2}t$. Denoting by $a, b$ the paths $a(t) = \pi(it)$ and $b(t) = \pi(t)$, the fundamental group $\pi_1(\tilde{E} - \{y_1, y_2\})$ admits the presentation

$$\langle b^{-1}a^{-1}ba = x_2x_1 \rangle.$$ 

where $x_1, x_2$ are two loops around $y_1, y_2$. It is not difficult to compute that

$$\sigma_{\ast}(a) = x_1a^{-1}$$

and

$$\sigma_{\ast}(b) = b$$

Moreover, the following equalities hold: $\lambda_{\sigma}^{-1}\sigma(x_1)\lambda_{\sigma} = ax_1^{-1}x_2^{-1}x_1a^{-1}$ and $\lambda_{\sigma}^{-1}\sigma(x_2)\lambda_{\sigma} = bax_1^{-1}a^{-1}b^{-1}$.

### 2.2 Character stacks for non-orientable surfaces

We fix an algebraically field $K$ (which for us will be either $\mathbb{C}$ or $\mathbb{F}_q$). We denote by $G$ the general linear group $GL_n(K)$ and by $\theta$ the Cartan involution $g \to (g)^{-1}$. The corresponding semidirect product will be denoted by $G^+ = G \rtimes (\theta)$. Let $C = (O_1, \ldots, O_k)$ be a $k$-tuple of
Let $	ext{Remark 4.2}$.

Remark 4.2. It is necessary to precisely describe monodromies around the punctures, as explained in [28, Remark 4.2]. A representation $\rho$ is therefore natural to ask conversely which representations $\sigma$ admit an alternative description in terms of the so-called real $\sigma$-invariant representations (which can be found in [10, Section 2][31, Section 3, 3.2] and [28, Remark 4.2]). A representation $\rho \in \text{Hom}_C^e(\Pi, G^+)$ gives by restriction a representation $\tilde{\rho} : \tilde{\Pi} \to G$ such that the following diagram commutes

$$
\begin{array}{ccc}
1 & \longrightarrow & \tilde{\Pi} \\
\downarrow \tilde{\rho} & & \downarrow \chi \longrightarrow (\theta) \\
1 & \longrightarrow & G
\end{array}
\quad (2.2.2)
$$

where $p : G \to (\theta)$ is the natural projection and $\chi : G \to G^+$ the natural inclusion. Given the explicit presentation of $\pi_1(\Sigma - S)$ we can rewrite $\text{Hom}_C^e(\Pi, G^+)$ as

$$\text{Hom}_C^e(\Pi, G^+) = \{D_1, \ldots, D_r \in G \text{ and } Z_1 \in O_1, \ldots, Z_k \in O_k \mid D_1 \theta(D_1) \cdots D_r \theta(D_r) Z_1 \cdots Z_k = 1\}. $$

The variety $\text{Hom}_C^e(\Pi, G^+)$ is endowed with a $G$-action defined by:

$$g \cdot D_i = g D_i g^{-1}, \quad g \cdot Z_i = g Z_i g^{-1}. \quad (2.2.1)$$

The character stacks we will consider are the quotient stacks

$$M_C^e = [\text{Hom}_C^e(\Pi, G^+)]/G.$$

As $\text{Hom}_C^e(\Pi, G^+)$ is affine, we can also consider the GIT quotient $M_C^e := \text{Hom}_C^e(\Pi, G^+)//G$ and the universal map $q : M_C^e \to M_C^e$.

Remark 2.2.1. The stacks $M_C^e$ admit an alternative description in terms of the so-called real $\sigma$-invariant representations (which can be found in [10, Section 2][31, Section 3, 3.2] and [28, Remark 4.2]). A representation $\rho \in \text{Hom}_C^e(\Pi, G^+)$ gives by restriction a representation $\tilde{\rho} : \tilde{\Pi} \to G$ such that the following diagram commutes

$$
\begin{array}{ccc}
1 & \longrightarrow & \tilde{\Pi} \\
\downarrow \tilde{\rho} & & \downarrow \chi \longrightarrow (\theta) \\
1 & \longrightarrow & G
\end{array}
\quad (2.2.2)
$$

It is therefore natural to ask conversely which representations $\tilde{\rho}$ of $\tilde{\Pi}$ can be lifted to a morphism $\rho : \Pi \to G^+$ which makes the diagram (2.2.2) commute. To answer to the question, it is necessary to precisely describe monodromies around the punctures, as explained in [28, Remark 4.2].

Let $\tilde{S} = \{y_{1,1}, \ldots, y_{1,k}, y_{2,1}, \ldots y_{k,2}\}$ where $\sigma(y_{i,1}) = y_{i,2}$ for $i = 1, \ldots, k$. We can rewrite the standard presentation $2.1.2$ of $\Pi$ as

$$\langle [a_1, b_1] \cdots [a_g, b_g] x_{1,1} x_{1,2} \cdots x_{k,1} x_{1,2} \cdots x_{k,2} = 1 \rangle,$$

where each $x_{i,j}$ is a path around $y_{i,j}$. Let $\tilde{C}$ be the $2k$-tuple $\tilde{C} = (O_1, \ldots, O_k, O_1, \ldots O_k)$ and $\text{Hom}_C^e(\tilde{\Pi}, G)$ be the affine variety

$$\text{Hom}_C^e(\tilde{\Pi}, G) := \{\tilde{\rho} : \tilde{\Pi} \to G \mid \tilde{\rho}(x_{i,1}) \in O_i \text{ and } \tilde{\rho}(x_{i,2}) \in O_i\}.$$

For a representation $\tilde{\rho} \in \text{Hom}_C^e(\tilde{\Pi}, G)$ we say that $\tilde{\rho}$ is $\sigma$-invariant if $\tilde{\rho} \cong \theta \tilde{\rho}(\sigma_a)$. This is
equivalent to ask for an element $h_\sigma \in G$ which verifies
\[ h_\sigma \tilde{\rho} h_\sigma^{-1} = \theta(\sigma). \] (2.2.3)

Given such a couple $(\tilde{\rho}, h_\sigma)$, say that the representation $\tilde{\rho}$ is real if we have
\[ \tilde{\rho}(\sigma(\lambda)\lambda_\sigma) = h_\sigma^{-1}\theta(h_\sigma^{-1}). \] (2.2.4)

If the conditions of Equations (2.2.3), (2.2.4) are satisfied, the couple $(\tilde{\rho}, h_\sigma)$ can be extended to a map $\rho \in \text{Hom}_C(\tilde{\Pi}, G^+)$ such that the diagram (2.2.2) commutes. Let $\tilde{\mathcal{U}}_C$ be the variety
\[ \tilde{\mathcal{U}}_C = \{(\tilde{\rho}, h_\sigma) \in \text{Hom}_C(\tilde{\Pi}, G) \times G \text{ which verify Equations (2.2.3, 2.2.4)} \}. \] (2.2.5)

The variety $\tilde{\mathcal{U}}_C$ is endowed with a $G$-action defined by
\[ g \cdot (\tilde{\rho}, h_\sigma) = (g\tilde{\rho}g^{-1}, \theta(g)hg^{-1}). \] (2.2.6)

The arguments above imply that there is an isomorphism of quotient stacks $\mathcal{M}_C^\circ \cong [\tilde{\mathcal{U}}_C/G]$.

**Remark 2.2.2.** If $\tilde{\rho}$ is an irreducible representation and $h$ is such that there is an equality $h\tilde{\rho}h^{-1} = \theta(\tilde{\rho})$, then either $h^{-1}\theta(h^{-1}) = \tilde{\rho}(\sigma(\lambda)\lambda_\sigma)$ or $h^{-1}\theta(h^{-1}) = -\tilde{\rho}(\sigma(\lambda)\lambda_\sigma)$ and only one of the two is true. (see Shu’s thesis [11, III.5.1.3]). As said above, in the $+$ case we say that $\tilde{\rho}$ is real and in the $-$ case we say that $\tilde{\rho}$ is quaternionic.

**Remark 2.2.3.** It is natural to consider the stack $\mathcal{M}_C^\circ := [\text{Hom}_C(\tilde{\Pi}, G)/G]$ and the associated GIT quotient $M_\rho$. The stacks $M_\rho$ and $\mathcal{M}_C^\circ$ admit an involution, which we denote again by $\sigma$, induced by the map $\sigma(\tilde{\rho}) := \theta(\tilde{\rho}(\sigma))$.

By Remark 2.2.1, we can define a morphism $f : M_\rho \to M_\rho^\sigma$ which maps a couple $(\tilde{\rho}, h)$ as in Equation (2.2.5) to the representation $\tilde{\rho}$. In a slightly more involved way, it would be possible to lift the map $f$ to a morphism of quotient stacks $F : \mathcal{M}_C^\circ \to \mathcal{M}_C^\circ$. These morphisms are in general not even surjective.

### 2.3 Cohomology computation

In this paragraph we will briefly review the results obtained by Letellier and Rodríguez-Villegas in [28] about the character stacks $\mathcal{M}_C^\circ$. Let us first recall the definition of the E-series and the mixed Poincaré series of an algebraic stack and the combinatorics needed for the formulas for the E-series $E(M_\rho^\circ, q)$.

#### 2.3.1 Mixed Poincaré series

Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $k$. For $K = \mathbb{C}$, we will consider the compactly-supported cohomology groups $H^*_c(\mathcal{X}) := H^*_c(\mathcal{X}, \mathbb{C})$ with
coefficients in \( \mathbb{C} \). For \( K = \mathbb{F}_q \), we will denote by \( H_c^*(X) \) the compactly supported étale cohomology with coefficients in \( \overline{\mathbb{Q}}_\ell \).

When \( K = \mathbb{C} \), each vector space \( H_c^k(X) \) is endowed with the weight filtration \( W^k \) (see [12, Chapter 8] for a definition and [28, Section 2.2] for the analogous one for stacks \( X \) over \( \mathbb{F}_q \)).

We define the mixed-Poincaré series \( H_c(X, q, t) \) as

\[
H_c(X, q, t) := \sum_{k,m} \dim(W^k_m/W^{k-1}_m) q^m t^k.
\]

(2.3.1)

The specialization \( H_c(X, 1, t) \) of \( H_c(X, q, t) \) at \( q = 1 \) is equal to the Poincaré series of the stack \( X \). When \( \sum (-1)^k \dim(W^k_m/W^{k-1}_m) \) is finite for each \( m \), we define the E-series:

\[
E(X, q) := H_c(X; q, -1) = \sum_{m,k} \dim(W^k_m/W^{k-1}_m) (-1)^k q^m t^k.
\]

(2.3.2)

For a quotient stack \( X = [X/G] \) where \( G \) is a connected linear algebraic group and \( X \) an affine variety, the E-series \( E(X, q) \) is well defined and \( E(X, q) = E(X, q)E(BG, q) \) where \( BG \) is the classifying stack of \( G \); for a proof see [28, Theorem 2.5].

2.3.2 Combinatorics

We fix integers \( m, k \geq 0 \) and we denote by \( \mathcal{P} \) the set of partitions. Let \( x_1 = \{x_{1,1}, x_{1,2}, \ldots\} \), \ldots, \( x_k = \{x_{k,1}, \ldots\} \) be \( k \) sets of infinitely many variables and let us denote by \( \Lambda = \Lambda(x_1, \ldots, x_k) \) the ring of functions separately symmetric in each set of variables. On \( \Lambda \) there is a natural bilinear form obtained by extending by linearity

\[
\langle f_1(x_1) \cdots f_k(x_k), g_1(x_1) \cdots g_k(x_k) \rangle = \prod_{i=1}^k \langle f_i, g_i \rangle
\]

where \( \langle, \rangle \) is the bilinear form making the Schur functions \( s_\mu \) an orthonormal basis. For a multipartition \( \mu = (\mu_1, \ldots, \mu_k) \in \mathcal{P}^k \) we denote by \( h_\mu = h_{\mu_1}(x_1) \cdots h_{\mu_k}(x_k) \) the associated complete symmetric function and similarly \( m_\mu = m_{\mu_1}(x_1) \cdots m_{\mu_k}(x_k) \).

We consider the hook functions

\[
H_{m,\lambda}(z, w) = \prod_{s \in \lambda} \frac{(z^{2a(s)+1} - w^{2l(s)+1})m}{(z^{2a(s)+2} - w^{2l(s)})((z^{2a(s)} - w^{2l(s)+2})}
\]

(2.3.3)

and the associated series

\[
\Omega_m(z, w) = \sum_{\lambda \in \mathcal{P}} H_{m,\lambda}(z, w) \prod_{i=1}^k H_\lambda(x_i, q, t)
\]

(2.3.4)

where \( H_\lambda(x_i, q, t) \) are the (modified) Macdonald symmetric polynomials (for a definition see
We define the functions $H_{\mu,m}(z,w)$ by the following formula:

$$H_{\mu,m}(z,w) := (z^2 - 1)(1 - w^2) \langle \log(\Omega_m(z,w)), h_{\mu} \rangle$$  \hspace{1cm} (2.3.5)$$

where $\log$ is the plethystic logarithm (for a definition see for example [21, Section 2.3.3]).

### 2.3.3 E-polynomials and conjectures

Let us now explain one of the main results of [28] about the stacks $\mathcal{M}_C$. Let $C = (O_1, \ldots, O_k)$ be a $k$-tuple of semisimple orbits of $G$.

**Definition 2.3.1.** The $k$-tuple $C$ is said to be generic if the following property holds. Given a subspace $V$ of $k^n$ which is stabilized by some $X_i \in O_i$, for each $i = 1, \ldots, k$, such that

$$\prod_{i=1}^k \det(X_i|_V) = 1$$

then either $V = \{0\}$ or $V = k^n$.

In [28, Theorem 4.6] the authors showed that for a generic $C$, the following equality holds:

$$E(\mathcal{M}_C^e, q) = \frac{q^{d_{\mu}}}{q - 1} H_{\mu,2} \left( -\sqrt{q}, \frac{1}{\sqrt{q}} \right)$$  \hspace{1cm} (2.3.6)$$

where $\mu = (\mu_1, \ldots, \mu_k)$ is the multipartition given by the multiplicities of the eigenvalues of $O_1, \ldots, O_k$ respectively and

$$d_{\mu} = n^2(r - 2 + k) + 2 - \sum_{i,j} (\mu_j^i)^2.$$  

This result is surprisingly similar to the analogous one obtained in [21, Theorem 1.2.3] about character stacks for Riemann surfaces. Fix a Riemann surface $\Sigma$ of genus $g$ and a set of $k$-points $S = \{y_1, \ldots, y_k\} \subseteq \Sigma$. The associated character stack $\mathcal{M}_C$ is the quotient stack $\mathcal{M}_C := [\{A_1, B_1, \ldots, A_g, B_g \in G, X_1 \in O_1, \ldots X_k \in O_k | [A_1, B_1] \cdots [A_g, B_g]X_1 \cdots X_k = 1]/G]$.

In [21] Theorem 1.2.3] the authors showed that the following equality holds:

$$E(\mathcal{M}_C, q) = \frac{q^{d_{\mu}}}{q - 1} H_{\mu,2g} \left( 1 \sqrt{q}, \sqrt{q} \right)$$  \hspace{1cm} (2.3.7)$$

where $d_{\mu} = n^2(2g - 2 + k) + 2 - \sum_{i,j} (\mu_j^i)^2$. Notice that for $r = 2h$ the E-polynomial of $\mathcal{M}_C^e$ agrees thus with the one of $\mathcal{M}_C$ for a Riemann surface $\Sigma$ of genus $h$.

In the same paper [21, Conjecture 1.2.1], the authors were also able to give a conjectural formula for the whole mixed Poincaré series of the stacks $\mathcal{M}_C$, naturally deforming Equation (2.3.7). The conjectural identity for the mixed Poincaré series of $\mathcal{M}_C$ is

$$H_e(\mathcal{M}_C, q,t) = \frac{(qt^2)^{d_{\mu}}}{q t^2 - 1} H_{\mu,2g} \left( -\frac{1}{\sqrt{q}}, t\sqrt{q} \right).$$  \hspace{1cm} (2.3.8)$$
The conjectural identity (2.3.8) is generally believed to be true. The Poincaré series of the stacks $\mathcal{M}_C$ were computed by Mellit in [30, Theorem 7.12] and his result agrees with the specialization of Formula (2.3.8) at $q = 1$. In [28, Theorem 4.6] it is proved that a formula analogous to Formula (2.3.8) holds in the non-orientable setting for $r = k = 1$, i.e. that the following equality holds

$$
H_c(\mathcal{M}_C^\epsilon, q, t) = \frac{(qt^2)^{\frac{d}{2}}}{qt^2 - 1} \mathbb{H}_{\mu, 1} \left( t \sqrt{q}, -\frac{1}{\sqrt{q}} \right).
$$

(2.3.9)

It would therefore have been natural to expect that such a formula holds for any $r, k$, i.e that

$$
H_c(\mathcal{M}_C^\epsilon, q, t) = \frac{(qt^2)^{\frac{d}{2}}}{qt^2 - 1} \mathbb{H}_{\mu, r} \left( t \sqrt{q}, -\frac{1}{\sqrt{q}} \right).
$$

(2.3.10)

for a generic $C$. The main result of this paper is a counterexample to Formula (2.3.10), obtained by an explicit description of these spaces in the case $r = 2$.

3 Main results

3.1 Generic case for $k = 1$

3.1.1 Geometric description

We fix $k = 1, r \geq 1$ and $d, n$ such that $d$ is even and $(d, n) = 1$. In this section we assume that $K = \mathbb{C}$. Let $C$ be the generic orbit $\{ e^{\frac{2\pi i}{d}} I_n \}$. We fix a Riemann surface $\tilde{\Sigma}$ of genus $g := r - 1$ with an antiholomorphic involution $\sigma$ and we denote the associated character stacks in this case by $\mathcal{M}^\epsilon_{n, d} := \mathcal{M}^\epsilon_C$ and $\mathcal{M}_{n, d} := \mathcal{M}_C$ respectively and similarly the associated GIT quotients by $M^\epsilon_{n, d}$ and $M_{n, d}$ respectively.

As $C$ is a central orbit, the character stack $\mathcal{M}_{n, d}$ is the twisted character stack

$$
\left\{ \{ A_1, B_1, \ldots, A_g, B_g \in \text{GL}_n \mid [A_1, B_1] \cdots [A_g, B_g] = e^{\frac{2\pi i d}{n}} \} / \text{GL}_n \right\}.
$$

considered for example in [21]. As $d$ and $n$ are coprime, the representations $\tilde{\rho} \in \mathcal{M}_{n, d}$ are irreducible (as shown in [24, Lemma 2.2.6]): in this case, given a $\sigma$-invariant couple $\rho = (\tilde{\rho}, h_\sigma)$ with $\tilde{\rho} \in \text{Hom}_G(\tilde{\Pi}, G)$ we have $\text{Stab}_G(\rho) = \pm 1$ (see Shu’s thesis [11, III.5.1.3]). The stack $\mathcal{M}^\epsilon_{n, d}$ is thus a $B(\mu_2)$ gerbe over the affine variety $M^\epsilon_{n, d}$.

Remark 3.1.1. The canonical morphism $q : \mathcal{M}^\epsilon_{n, d} \to M^\epsilon_{n, d}$, being a $B(\mu_2)$ gerbe, is proper. The proper base change for Artin stacks implies that for every $x \in M^\epsilon_{n, d}$ and for every $i \in \mathbb{Z}$ we have

$$
(R^i q_* \mathbb{C}) |_x = H^i(B(\mu_2)).
$$

As the higher cohomology of $B(\mu_2)$ vanishes, $R^i q_* \mathbb{C} = 0$ if $i \neq 0$ and $q_* \mathbb{C} = \mathbb{C}$. The Leray spectral sequence for cohomology with compact support implies that $H^p_c(\mathcal{M}^\epsilon_{n, d}) \cong H^p_c(M^\epsilon_{n, d})$. The cohomology of the quotient stack is isomorphic to that of the GIT quotient: in particular, the (compactly-supported) cohomology of $\mathcal{M}^\epsilon_{n, d}$ is 0 in negative degrees.
The main result of this paragraph is the following proposition:

**Proposition 3.1.2.**

(i) If \( r \) is odd there are no quaternionic representations inside \( M_{n,d}^\sigma \). If \( r \) is even, \( M_{n,d}^\sigma \) admits a decomposition into open-closed subvarieties

\[
M_{n,d}^\sigma = M_{n,d}^{\sigma,+} \sqcup M_{n,d}^{\sigma,-}
\]

where \( M_{n,d}^{\sigma,+}, M_{n,d}^{\sigma,-} \) are given by real/quaternionic representations respectively and there is an isomorphism \( M_{n,d}^{\sigma,+} \cong M_{n,d}^{\sigma,-} \).

(ii) The map \( f : M_{n,d}^\epsilon \to M_{n,d}^{\sigma,+} \) introduced in Remark 2.2.3 is an isomorphism.

Before proving Proposition 3.1.2, we notice that the quaternionic and the real representations form disjoint subsets by Remark 2.2.2. To see that there are no quaternionic representations for \( r \) odd, we will use the equivalence between quaternionic representations and quaternionic Higgs bundles. As this correspondence is crucial for the study of the varieties \( M_{n,d}^\epsilon \), let us briefly review it here. For more details, see for example [8], [32], [1], [5], [7].

### 3.1.2 (Pseudo)-real Higgs bundles

A Higgs bundle over \( \tilde{\Sigma} \) is a couple \((\mathcal{E}, \Phi)\) with \( \Phi : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{\Sigma} \). The moduli space of (stable) Higgs bundle over \( \tilde{\Sigma} \) of rank \( n \) and degree \( d \) is denoted by \( M_{Dol,n,d}^\sigma \) (for a definition of stability see for example [4] Section 4.1). It is a fundamental result (see for example [34]) that there is an homeomorphism (called non abelian Hodge correspondence)

\[
M_{Dol,n,d}^\sigma \cong M_{n,d}.
\]  

(3.1.1)

We consider the involution on \( M_{Dol,n,d} \), which we denote again by \( \sigma \), given by

\[
\sigma((\mathcal{E}, \Phi)) = (\sigma^*(\mathcal{E}), -\sigma^*(\Phi))
\]

and we say that a Higgs field \( (\mathcal{E}, \Phi) \) is \( \sigma \)-invariant if there exists an isomorphism \( \alpha : (\mathcal{E}, \Phi) \to \sigma((\mathcal{E}, \Phi)) \). Pseudo-real Higgs bundles are couples \((\mathcal{E}, \Phi, \alpha)\) such that

\[
\sigma^*(\mathcal{E}) \alpha = I_{\mathcal{E}}.
\]

In a similar way, quaternionic Higgs bundles are defined by asking for the equality

\[
\sigma^*(\mathcal{E}) \alpha = -I_{\mathcal{E}}.
\]

In [8] Proposition 5.6] [7] Theorem 4.8] it is shown that the homomorphism (3.1.1) restricts to an homeomorphism \( M_{n,d}^\sigma \cong M_{Dol,n,d}^\sigma \). In loc. cit it is shown moreover that this bijection sends real/quaternionic representations into real/quaternionic Higgs bundles respectively. We will denote the subsets of \( M_{Dol,n,d} \) given by real/quaternionic Higgs bundles by \( M_{Dol,n,d}^{\sigma,+} / M_{Dol,n,d}^{\sigma,-} \).
respectively. In general, $M^\sigma_{Dal,n,d}$ is not an algebraic variety but just a real analytic submanifold.

For odd $r$, if a quaternionic couple $(\bar{\rho}, h)$ existed (i.e. $M^\sigma_{n,d}$ is not empty) there would exist a stable quaternionic Higgs field $(\mathcal{E}, \Phi)$ on $\Sigma$. Its determinant $\text{det}(\mathcal{E})$ would be a quaternionic line bundle of degree $d$ over $\Sigma$: the quaternionic condition is preserved under taking the determinant as $n$ is odd. The existence of a quaternionic line bundle for odd $r$ is ruled out by the topological criterion of [31, Theorem 2.4].

To prove Proposition 3.1.2 we will need the following preliminary Lemma.

**Lemma 3.1.3.** Put $\text{Hom}_{n,d}(\tilde{\Pi}, G) := \text{Hom}_q(\tilde{\Pi}, G)$ and let us consider the varieties $Y, Z$ defined by

$$Y := \text{Hom}_{n,d}(\tilde{\Pi}, G) \times_{M_{n,d}} \text{Hom}_{n,d}(\tilde{\Pi}, G) = \{ (\bar{\rho}_1, \bar{\rho}_2) | \bar{\rho}_1 \cong \bar{\rho}_2 \}$$

and

$$Z := \{ (\bar{\rho}_1, \bar{\rho}_2, h) | (\bar{\rho}_1, \bar{\rho}_2) \in Y, h \in \text{GL}(n) | h\bar{\rho}_1 h^{-1} = \bar{\rho}_2 \}.$$  

The projection map $\psi : Z \to Y$ is a principal $\mathbb{G}_m$-bundle for the étale topology.

**Proof.** The variety $Z$ is endowed with the $\mathbb{G}_m$ action $t \cdot (\rho_1, \rho_2, h) = (\rho_1, \rho_2, th)$. This action is free and transitive as all the representations in $\text{Hom}_{n,d}(\tilde{\Pi}, G)$ are irreducible. Moreover $\psi(t \cdot z) = \psi(z)$ for all $z \in Z$. We are thus reduced to show that $\psi$ is locally trivial for the étale topology.

As the map $\tilde{q} : \text{Hom}_{n,d}(\tilde{\Pi}, G) \to M_{n,d}$ is a principal $\mathbb{P} \text{GL}_n$-bundle for the étale topology, there exists an étale open covering $\{ U_i \}_{i \in I}$ of $M_{n,d}$ such that $\tilde{q}^{-1}(U_i) \cong U_i \times \mathbb{P} \text{GL}_n$ for each $i \in I$. Put $Y_{U_i} := Y \times_{M_{n,d}} U_i$ and similarly $Z_{U_i} := Z \times_{M_{n,d}} U_i$: it is enough to show that the pullback map $\psi : Z_{U_i} \to Y_{U_i}$ is locally trivial in the étale topology for each $i \in I$. Fix then $i \in I$ and put $U_i = U$. Notice that the variety $Y_U$ admits the following isomorphism:

$$Y_U = \tilde{q}^{-1}(U) \times_U \tilde{q}^{-1}(U) \cong (U \times \mathbb{P} \text{GL}_n) \times_U (U \times \mathbb{P} \text{GL}_n) \cong U \times \mathbb{P} \text{GL}_n \times \mathbb{P} \text{GL}_n.$$  

In a similar way, the variety $Z_U$ is isomorphic to

$$Z_U = \psi^{-1}(Y_U) = \{ (u, g, h, s) \in U \times \mathbb{P} \text{GL}_n \times \mathbb{P} \text{GL}_n \times \mathbb{P} \text{GL}_n | gh^{-1} = [s] \}$$

in such a way that $\psi$ corresponds to the morphism $\psi(u, g, h, s) = (u, g, h)$. We can view $Y_U$ as a subset of $U \times \mathbb{P} \text{GL}_n \times \mathbb{P} \text{GL}_n \times \mathbb{P} \text{GL}_n$ as

$$Y_U = \{ (u, g, h, s) \in U \times \mathbb{P} \text{GL}_n \times \mathbb{P} \text{GL}_n \times \mathbb{P} \text{GL}_n | gh^{-1} = s \}.$$  

Via these identifications, the map $\psi$ corresponds to the restriction of the morphism $U \times \mathbb{P} \text{GL}_n \times \mathbb{P} \text{GL}_n \times \mathbb{P} \text{GL}_n \to U \times \mathbb{P} \text{GL}_n \times \mathbb{P} \text{GL}_n \times \mathbb{P} \text{GL}_n$ given by the identity on the first three factors and the quotient map $\text{GL}_n \to \mathbb{P} \text{GL}_n$ on the last one. This is a principal $\mathbb{G}_m$-bundle because $\text{GL}_n \to \mathbb{P} \text{GL}_n$ is so.

\qed
We now prove Proposition 3.1.2; we keep the notations of Lemma 3.1.3.

Proof of Proposition 3.1.2.

Put $\text{Hom}_{n,d}(\tilde{\Pi}, G)^\sigma = \tilde{q}^{-1}(M_{n,d}^\sigma)$ and $\text{Hom}_{n,d}(\tilde{\Pi}, G)^{\sigma,+} = \tilde{q}^{-1}(M_{n,d}^{\sigma,+})$ and similarly for quaternionic representations $\text{Hom}_{n,d}(\tilde{\Pi}, G)^{\sigma,-} = \tilde{q}^{-1}(M_{n,d}^{\sigma,-})$. The variety $\text{Hom}_{n,d}(\tilde{\Pi}, G)^\sigma$ is isomorphic to the closed subvariety $Y^\sigma$ of $Y$ given by:

$$Y^\sigma = \{(\tilde{\rho}_1, \tilde{\rho}_2) \in Y \mid \tilde{\rho}_2 = \theta \tilde{\rho}_1 \sigma_s\}$$

via the map $p_1|_{Y^\sigma} : Y^\sigma \to \text{Hom}_{n,d}(\tilde{\Pi}, G)^\sigma$, where $p_1$ is the projection onto the first factor of $Y$. Put $Y^{\sigma,+} = p_1^{-1}(\text{Hom}_{n,d}(\tilde{\Pi}, G)^{\sigma,+})$ and similarly $Y^{\sigma,-}$. From Remark 2.2.2 there is a well-defined morphism $p_3 : \psi^{-1}(Y^\sigma) \to \{I_n, -I_n\}$

$$(\tilde{\rho}_1, \tilde{\rho}_2, h) \to \theta(h)h\tilde{\rho}(\sigma(\lambda)\lambda_s).$$

Notice that $Y^{\sigma,+} = \psi(p_3^{-1}(I_n))$ and $Y^{\sigma,-} = \psi(p_3^{-1}(-I_n))$: as $\psi$ is open, we deduce that $\text{Hom}_{n,d}(\tilde{\Pi}, G)^{\sigma,+}$, $\text{Hom}_{n,d}(\tilde{\Pi}, G)^{\sigma,-}$ are disjoint and open and so closed too inside $\text{Hom}_{n,d}(\tilde{\Pi}, G)^\sigma$. The same is true then for $M_{n,d}^{\sigma,+}, M_{n,d}^{\sigma,-}$. The projection $(\tilde{\rho}_1, \tilde{\rho}_2, h) \to (\tilde{\rho}_1, h)$ induces an isomorphism $\psi^{-1}(Y^{\sigma,+}) = \text{Hom}_{n,d}(\tilde{\Pi}, G^{\sigma})^\epsilon$. By Proposition 3.1.3 the morphism

$$\text{Hom}_{n,d}(\tilde{\Pi}, G^\epsilon)^\epsilon \to \text{Hom}_{n,d}(\tilde{\Pi}, G)^{\sigma,+}$$

is thus a principal $G_m$-bundle. The $G$ action on $\text{Hom}_{n,d}(\tilde{\Pi}, G^\epsilon)^\epsilon$ defined by the Formula (2.2.6) induces an action of its center $Z_G = G_m$ which differs from the one coming from the principal $G_m$-bundle structure by a square factor. The morphism $\text{Hom}_{n,d}(\tilde{\Pi}, G^\epsilon) \to \text{Hom}_{n,d}(\tilde{\Pi}, G)^{\sigma,+}$ induces thus a $G$-equivariant isomorphism

$$\text{Hom}_{n,d}(\tilde{\Pi}, G^\epsilon)^\epsilon/(G_m/(\pm I_n)) \cong \text{Hom}_{n,d}(\tilde{\Pi}, G)^{\sigma,+}$$

(3.1.2)

We deduce the following chain of isomorphisms:

$$M_{n,d}^\epsilon = \text{Hom}_{n,d}(\tilde{\Pi}, G^\epsilon)^\epsilon/(G/(\pm I_n)) \cong \text{Hom}_{n,d}(\tilde{\Pi}, G^\epsilon)^\epsilon/((G/G_m)/(G_m/(\pm I_n))) \cong M_{n,d}^{\sigma,+}.$$

To end the proof of Proposition 3.1.2 it actually remains to show that $M_{n,d}^{\sigma,+}, M_{n,d}^{\sigma,-}$ are isomorphic if $r$ is even. For $r$ even there exists a quaternionic representation $\tau \in M_{4,0}^{\sigma,+}$ of rank 1 over $\tilde{\Sigma}$ (see [32] Theorem 2.4)). Taking the tensor product by $\tau$ gives then an isomorphism $- \otimes \tau : M_{n,d}^{\sigma,+} \to M_{n,d}^{\sigma,-}$: the same proof was carried out for real and quaternionic vector bundles in [32] Theorem 1.1].

3.1.3 Character stacks for (real) elliptic curves

We focus now on the case $r = 2$: we consider the elliptic curve $\tilde{E}$ and the antiholomorphic involution $\sigma$ introduced in Example 2.1.1. We keep the notations introduced in the Example 2.1.1.
In [24, Lemma 2.2.6] it is shown that for \((n, d) = 1\) there is an isomorphism
\[
M_{n,d} = \mathbb{C}^* \times \mathbb{C}^*.
\] (3.1.3)

To see this, notice that a representation \(\tilde{\rho} \in M_{n,d}\) corresponds to a couple of matrices \(A, B\) such that
\[
B^{-1}A^{-1}BA = e^{\frac{2\pi id}{n}} I_n.
\]
where \(\tilde{\rho}(a) = A\) and \(\tilde{\rho}(b) = B\). Let \(z, w \in \mathbb{C}^*\) such that \(A^n = zI_n\) and \(B^n = wI_n\)(see [24, Theorem 2.2.17]). The isomorphism (3.1.3) is obtained by mapping \(\tilde{\rho}\) to the couple \((z, w)\).

Via this identification, the involution \(\sigma\) is given by:
\[
\sigma(z, w) = (z, w^{-1})
\]
and so \(M_{n,d}^\sigma = \mathbb{C}^* \sqcup \mathbb{C}^*\). From Equation (2.1.4) we deduce indeed that
\[
\theta(\tilde{\rho}(\sigma^*(b))) = \theta(\tilde{\rho}(b)) = \theta(B)
\]
and so \((\theta(\tilde{\rho}(\sigma^*(b))))^n = \theta(B)^n = w^{-1}I_n\). By Equation (2.1.5) the following equality holds:
\[
\theta(\tilde{\rho}(\sigma^*(a))) = \theta(\tilde{\rho}(x^1a^{-1})) = \theta(\tilde{\rho}(x_1))\theta(A^{-1}) = e^{-\frac{n}{2\pi}} A^t
\]
and so \((\theta(\tilde{\rho}(\sigma^*(a))))^n = (A^t)^n = zI_n\). By Proposition 3.1.2 we deduce the following result:

**Proposition 3.1.4.** For \(r = 2\), the character variety \(M_{n,d}^\epsilon\) is isomorphic to \(\mathbb{C}^*\) and the character stack \(\mathcal{M}_{n,d}^\epsilon\) is a \(B(\mu_2)\)-gerbe over \(\mathbb{C}^*\).

By Remark 3.1.1 for \(r = 2\) the following identity holds:
\[
H_c(\mathcal{M}_{n,d}^\epsilon, q, t) = qt^2 + t.
\] (3.1.4)

As suggested in the introduction, this does not agree with the expected formula (2.3.10). If the Formula (2.3.10) were true, the following identity would hold
\[
H_c(\mathcal{M}_{n,d}^\epsilon, q, t) = \frac{qt^2}{qt^2 - 1} \mathbb{H}_{n,2}(t\sqrt{q}, -\frac{1}{\sqrt{q}})
\]
where \(\mathbb{H}_{n,2}(z, w)\) are the functions defined by Equation (2.3.5) for \(\mu = ((n))\). The functions \(\mathbb{H}_{n,2}(z, w)\) have been explicitly computed by Carlsson in [9, Theorem 1.0.2]: his result agrees with the conjectural formula (2.3.8) for the mixed Poincaré series of character varieties \(\mathcal{M}_{n,d}\) for elliptic curves, i.e. \((qt^2)\mathbb{H}_{n,2}(t\sqrt{q}, -\frac{1}{\sqrt{q}}) = (qt^2 + t)^2\). This implies that
\[
\frac{(qt^2 + t)^2}{qt^2 - 1} = \frac{qt^2}{qt^2 - 1} \mathbb{H}_{n,2}(t\sqrt{q}, -\frac{1}{\sqrt{q}}) \neq (qt^2 + t)
\] (3.1.5)
giving a counterexample to the conjectural formula (2.3.10).
4 Final remarks

We still lack of another conjecture for the mixed Poincaré series of the stacks $\mathcal{M}_{n,d}^r$ for $r > 2$ and more generally of the stacks $\mathcal{M}_C^r$ for a generic $C$ and of an explanation for the surprising similarity between $E$-series of character stacks $\mathcal{M}_C^r$ and $\mathcal{M}_C$. A first step could be the computation of the Poincaré series $P_r(\mathcal{M}_{n,d}^r, t)$. The computation of the cohomology of the character stacks $\mathcal{M}_{n,d}, \mathcal{M}_C$ has been achieved in a series of papers in the last years (see [17], [33], [30]) using non abelian Hodge correspondence.

In the paragraph §3.1.2 we introduced the Dolbeault moduli space of Higgs bundles $M_{Dol,n,d}$ and recalled that there is a canonical homeomorphism $M_{Dol,n,d} \simeq M_{n,d}$, called non abelian Hodge correspondence. More generally, for any $C$ there is a corresponding Dolbeault moduli space of Higgs bundles $M_{Dol}$ and a corresponding version of the non abelian Hodge correspondence homeomorphism. The main results about the cohomology $\mathcal{M}_{n,d}$ and $\mathcal{M}_C$ have been obtained via the computation of the cohomology of the corresponding Dolbeaut moduli space $M_{Dol}$.

In general, the rich geometric structure of the Dolbeaut moduli spaces $M_{Dol}$ allows to compute their cohomology in a different number of ways. The $\mathbb{C}^*$-action defined by $t \cdot (E, \Phi) = (E, t\Phi)$ allows, for example, to use Morse theory to compute cohomology at least in small rank $n$. Historically, this was the first strategy used to compute cohomology of the varieties $M_{Dol,n,d}$: see for example [25], [17].

Schiffmann obtained a general formula for the Poincaré series of the character stacks $\mathcal{M}_{n,d}$ (see [33] Corollary 1.6,1.7) by counting $\mathbb{F}_q$-rational points of the varieties $M_{Dol,n,d}$. Mellit later greatly generalized Schiffmann’s results and by counting the $\mathbb{F}_q$ rational points of the varieties $M_{Dol}$, he obtained a closed formula for the Poincaré series of $\mathcal{M}_C$ for any generic $C$ (see [30] Theorem 7.10). His results agree with the specialization of the conjectural formula (2.3.8) at $q = 1$.

Notice however that for the varieties $\mathcal{M}_{n,d}^r$, the corresponding moduli space of (real) Higgs bundles $M_{Dol,n,d}^{\mathbb{R}}$ is not an algebraic variety: this prevents us from employing both of these techniques. On one hand, as $M_{Dol,n,d}^{\mathbb{R}}$ is not an algebraic variety, it is impossible to use reduction over $\mathbb{F}_q$ as done by Mellit in [30].

On the other hand, the $\mathbb{C}^*$-action on $M_{Dol,n,d}$ induces just an $\mathbb{R}^*$-action on $M_{Dol,n,d}^{\mathbb{R}}$: the absence of a real version of the Bialynicki-Birula decomposition makes hard to extend the tools used in the papers [25] or [17] to the case of $\mathcal{M}_{n,d}^r$. Recently, there have been attempts to compute the cohomology with coefficients in $\mathbb{Z}/(2)$ of the varieties $M_{n,d}^r$: for $\mathbb{Z}/(2)$, some of the Morse theory results can be extended to real algebraic varieties too (see for example [2], [1]). It is not clear how to extend the results of these papers to chomology with coefficients in $\mathbb{C}$.

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