Open mushrooms: stickiness revisited

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Abstract

We investigate mushroom billiards, a class of dynamical systems with sharply divided phase space. For typical values of the control parameter of the system $\rho$, an infinite number of marginally unstable periodic orbits (MUPOs) exist making the system sticky in the sense that unstable orbits approach regular regions in phase space and thus exhibit quasi-regular behaviour for long periods of time. The problem of finding these MUPOs is expressed as the well-known problem of finding optimal rational approximations of a real number, subject to some system-specific constraints. By introducing a generalized mushroom and using properties of continued fractions, we describe for the first time a zero measure set of control parameter values $\rho \in (0,1)$ for which all MUPOs are destroyed and therefore the system is less sticky, leading to a different power law exponent for the Poincaré recurrence time distribution statistics. The open mushroom (billiard with a hole) is then considered in order to quantify the stickiness exhibited due to MUPOs and exact leading order expressions for the algebraic decay of the survival probability function $P(t) \sim C/t$ are calculated for mushrooms with triangular and rectangular stems. Numerical simulations are also performed which confirm our predictions for both sticky and less sticky mushrooms.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Billiards [1–3] are systems in which a point particle alternates between motion in a straight line and specular reflections from the walls of its container. Because they demonstrate a broad variety of behaviours (regular, chaotic and mixed phase space dynamics) they have been readily used as models in theoretical and experimental physics [4–7]. They are widely applicable mainly because their dynamics corresponds to the classical (short-wavelength)
Figure 1. (a) Simple mushroom, (b) elliptic mushroom with triangular stem, (c) ‘Honey mushroom’ with four integrable islands and two ergodic (chaotic) components.

limit of the wave equations for light, sound or quantum particles in a homogeneous cavity. There also exist many other mathematically and physically motivated modifications of the usual classical billiard dynamics to include for example soft collisions [8], few [9] and many particle systems [10], curved trajectories due to curved spaces [11] and magnetic fields [12], non-specular reflections [13, 14], time-dependent boundaries [15], and other wave phenomena such as diffraction effects [16] and the Goos–Hänchen effect [17].

The mushroom billiard is constructed by a convex semi-elliptical (including semi-circular) ‘hat’ attached to a ‘stem’ such that their intersection is smaller than the diameter of the hat. Examples are shown in figure 1. It is special in that under certain conditions [18, 19], it forms a class of dynamical systems with sharply divided phase space which are easy to visualize and analyse. For example, the phase space of the mushroom shown in figure 1(a) is composed of a single completely regular (integrable) invariant component and a single connected chaotic and ergodic component (see figure 2(b)), in contrast with other generic mixed systems such as the standard map [20], where KAM hierarchical islands form a dense family in the neighbourhood of each other. Interestingly, mushrooms can also be designed to have an arbitrary number of integrable and chaotic ergodic components (see figure 1(c)) [18]. Therefore, mushroom billiards are paradigmatic models for studying the phase space dynamics near the boundary of integrable islands but can also be used to study the so-called LAB effect [21], where even for systems with interacting particles the stationary distribution can be nonuniform [9]. One must note however that small perturbations (imperfections) to the mushroom’s boundary can cause the emergence of KAM islands or even complete chaos [22] (see also [19] for rigorous arguments).

Because of their unusually simple divided classical phase space, mushroom billiards are of increasing interest to the quantum chaos community. As a result, this has facilitated the numerical verification [23] of Percival’s conjecture which states that in the semiclassical limit, eigenmodes localize to one or another invariant region of phase space (regular or chaotic), with occurrence in proportion to the respective phase space volumes [24]; recently, this has been applied to generalize the boundary term in Weyl’s law [25]. Similarly, the mechanism of dynamical tunnelling between classically isolated phase space regions has also been investigated in the context of mushrooms [26] and has been observed in microwave mushrooms [27].

Although the classical phase space of mushroom billiards is sharply divided, generic ‘simple’ mushrooms (figure 1(a)) exhibit long power-law tails of order $\sim t^{-2}$ in the Poincaré recurrence time statistics [28]. These tails have been attributed to the presence of one-parameter families of marginally unstable periodic orbits (MUPOs, see the definition in
section 2), ‘embedded’ in the ergodic component of the phase space [28]. The flow close to these orbits is strongly reminiscent of that close to KAM islands [29] and therefore causes the system to display the phenomenon of ‘stickiness’, where chaotic orbits stick close to regions of stability for long periods of time causing the emergence of power-law tails (more details in the next section). In fact, the stickiness of chaotic trajectories was shown (using continued fraction representations) to occur through an infinite number of MUPOs concentrating near the border with the regular island [30] (also see [35] for further discussions). Although the quantum analogue of stickiness is not well understood, MUPOs seem to play an important role both in the density of states of microwave billiards [30] and the directionality of dielectric micro-cavities [31, 32].

It is worth mentioning that non-sticky mushrooms have been previously constructed using elliptical hats and non-rectangular stems [19, 33] but have not been studied experimentally yet. Their non-stickiness arises from the fact that each focus of the semi-elliptical hat provides a sharp boundary between rotational and librational orbits and may be used as the end point of the entrance to the foot. However, in such a case, some care is needed with the stem’s length and its base width to ensure sufficient defocusing. In addition to this, the size of the opening of the stem must also ensure a bounded number of maximum possible collisions in the hat [33]. Further details concerning the defocusing mechanism and hyperbolicity in billiards are given in [34].

In this paper we focus on classes of mushrooms with circular hats, in which stickiness to leading order is due to the presence of MUPOs. In the first part of the paper, we follow the more detailed reference [35] as well as [30] and express the problem of finding MUPOs as the well-known problem of finding optimal rational approximations of a number (section 2.1). This interesting connection made with number theory allows us to introduce and characterize a zero measure set of control parameter values, using continued fractions, for which all MUPOs are completely removed (section 2.2). This set, not previously discussed in the literature, corresponds to mushrooms with a less sticky hat, the implications of which are yet to be studied classically or quantum mechanically and are likely to be relevant to applications mentioned above. We obtain upper bounds for MUPO-free and finitely sticky irrational mushrooms and also give an explicit example of a MUPO-free mushroom billiard (section 2.3).

In the second part of this paper we attempt to quantify the stickiness exhibited due to MUPOs in the mushroom by placing holes through which particles may escape. This approach dates back to the early 1980s when mathematicians suggested investigating ‘open’ dynamical systems (systems with holes or leakages) as a means of generating transient chaos [36], retrieving information from distributions and deducing facts about the equivalent closed systems (see [37, 38] and references therein). As also discussed in [39], placing a small hole on a billiard’s boundary allows one to ‘peep’ into the system’s dynamics. The smaller the hole, the smaller the observational effect (whether quantum or classical) on the dynamics. We therefore take this approach and ‘open’ the mushroom billiard and look at the survival probability function $P(t)$, given a uniform initial distribution of particles projected onto the billiard boundary. By considering linear perturbations of MUPOs, we obtain exact leading order expressions for the asymptotic algebraic decay of $P(t) \sim C/t$. This is done for two separate cases, firstly for MUPOs in the semicircular hat of the mushroom (section 3.1) and then for bouncing ball-type MUPOs in the case of a rectangular stem (section 3.2). The explicit form of these expressions depends on the geometrical parameters of the billiard and in turn allows us not only to predict but also to calibrate the survival probability function. Furthermore, in connection with the results of section 2.2 we reach to the conclusion that a MUPO-free mushroom will have $C = 0$, therefore displaying a power law decay of order $\sim t^{-2}$ or faster. Finally, the results are confirmed numerically (section 3.3) and then discussed
 briefly, including higher-dimensional mushrooms in connection with the results of section 2 and other implications of this work (section 4).

2. Stickiness in closed mushrooms

2.1. Mushrooms with MUPOs

The term ‘stickiness’ is used when chaotic orbits stick close to regions of stability for long periods of time [40]. This is due to points or regions of phase space having vanishingly small local [41] Lyapunov exponents. Such points are typically found in the vicinity of KAM elliptic islands or in the very close neighbourhood of MUPOs: parabolic periodic orbits whose linear stability (monodromy matrix) eigenvalues are equal to 1. Hence, any small perturbation of a MUPO will initially grow linearly (not exponentially) with time. If these perturbed orbits are in the ergodic component of the system’s phase space, they will exhibit long periods of quasi-regular behaviour, intermittently interrupted by relatively short chaotic bursts and are thus also associated with the phenomenon of intermittency (see [42] and references therein). Because of their linear stability, such periodic orbits are only marginally unstable. In billiards, MUPOs come in families of non-isolated periodic orbits along the billiard’s boundary, occupying zero volume in phase space. Perturbations in their angle cause the orbits to leave the vicinity of the periodic orbit and explore the rest of the ergodic component uniformly. A well-known and well-studied family of MUPOs are the bouncing ball orbits present in the chaotic Bunimovich stadium billiard [43] forming a continuous family of period 2 orbits trapped forever between the stadium’s parallel walls.

Even though MUPOs do not affect the overall ergodicity of the system, they govern long time statistical properties of the system, such as the Poincaré recurrence times distribution $Q(t) \sim t^{-2}$ [28] which is intimately related [44] with the long time survival probability $P(t) \sim t^{-1}$ [29] given a carefully positioned ‘hole’, as well as the rate of mixing (the rate of the decay of correlations) $C(t) \sim t^{-1}$ [45]. Furthermore, the exponents of these power
laws appear to be a universal fingerprint of nonuniform hyperbolicity and stickiness, at least for one- and two-dimensional Hamiltonian systems with sharply divided phase space [29]. The MUPOs in the mushroom’s hat and in the annular billiard were extensively studied by Altmann in his PhD thesis [35] and more briefly in [28–30] and occur in many billiards with circular arcs.

As discussed in the introduction, a generic, simple mushroom billiard’s phase space consists of a single integrable and a single ergodic component with an infinite number of MUPOs populating close to the boundary of the ergodic component. These MUPOs are best understood when introduced geometrically. The dashed red semicircle of radius $r$ in figure 2(a) corresponds to the border between the ergodic and regular component of the mushroom’s phase space (see figure 2(b)). Any orbit intersecting this semicircle will sooner or later fall into the mushroom’s stem and is therefore unstable and lies in the ergodic component of the phase space. Any non-periodic orbit not intersecting the dashed semicircle does not ‘see’ the stem and thus exhibits regular motion remaining forever in the mushroom’s hat. MUPOs, as shown in figure 2(a), are periodic orbits which do intersect the dashed semicircle and therefore are unstable though always remain in the mushroom’s hat. A compact way of describing them was given in [28]:

$$\alpha_{s,j} = \cos \frac{j\pi}{s} \leq r \frac{R}{\cos \frac{\pi}{2s}} = \beta_{s,j}, \quad (1)$$

where

$$s \geq 3, \quad 1 \leq j \leq \begin{cases} \frac{s}{2} - 1, & \text{if } s \text{ is even,} \\ \frac{s-1}{2}, & \text{if } s \text{ is odd,} \end{cases}, \quad \lambda = \begin{cases} 1, & \text{if } s \text{ is even,} \\ 2, & \text{if } s \text{ is odd.} \end{cases} \quad (2)$$

In equation (1), $r$ and $R$ are as defined in figure 2. The coprime integers $s$ and $j$ describe periodic orbits of the circle billiard with angles of incidence $\theta_{s,j} = \frac{\pi}{2} - \frac{j\pi}{s}$. More specifically, $s$ is the period and $j$ the rotation number of the orbit. $R\alpha_{s,j}$ is the shortest distance from the periodic orbit $(s,j)$ to the origin. $R\beta_{s,j}$ is half the longest straight line passing through the origin which intersects the unfolded (along the hat’s base) periodic orbit $(s,j)$ at equal distances on either side. Hence, (1) guarantees that $(s,j)$ is a MUPO and can be oriented in such a way as not to enter the stem while still intersecting the dashed semicircle. Let $S_{\rho}$ denote the set of periodic orbits which are marginally unstable for a given $\rho = r\frac{R}{\cos \frac{\pi}{2s}}$.

A small perturbation $\eta$ with respect to the incidence angle $\theta_{s,j}$ of a MUPO will cause the orbit to precess in the opposite direction, following the corresponding orbit in a semicircle billiard and will eventually ‘fall’ into mushroom’s stem causing it to feel the strong chaotic effect of the defocusing mechanism. However, since the precessing angular velocity is proportional to the perturbation strength $\eta$ which may be arbitrarily small, the orbit will behave in a quasi-periodic fashion and entry into the stem may take an unbounded amount of time. Hence the term ‘stickiness’, meaning that orbits in the immediate vicinity of MUPOs stick close to the regular component of phase space for long periods of time. Note however that although these periodic orbits are dynamically marginally unstable, they are not structurally robust against parameter perturbations of $\rho$.

The intervals $(\alpha_{s,j}, \beta_{s,j})$ are shrinking quadratically with increasing $s$. We see this by rearranging (1) into

$$\frac{j}{s} \geq \frac{1}{\pi} \arccos \rho > \frac{1}{\pi} \arccos \left(\cos \frac{\lambda s}{2} \frac{2\pi}{s} \frac{1}{\cos \frac{\pi}{2s}}\right), \quad (3)$$

5
expanding for large $s$

$$\frac{j}{s} \geq \vartheta^* > \frac{j}{s} - \left( \frac{\pi \cot \frac{j\pi}{s}}{2} \right) \frac{1}{\lambda^2 s^2} + \mathcal{O} \left( \frac{1}{s^4} \right),$$

where we have set $\vartheta^* = \frac{1}{s} \arccos \rho$ and rearranging once more to get

$$0 \leq \frac{j}{s} - \vartheta^* < \left( \frac{\pi \cot \frac{j\pi}{s}}{2\lambda^2} \right) \frac{1}{s^2},$$

where we have neglected the positive terms of order $\sim s^{-4}$, thus possibly losing some of the MUPOs; we give explicit bounds on this term in the next section. In this way the problem of finding the elements of $\mathcal{S}_\rho$ is expressed as the well-known number theoretic problem of finding rational approximations $\frac{j}{s}$ of $\vartheta^* \in (0, \frac{1}{2})$ [30]. However, in this case we have a couple of complications: the approximations are one-sided and the tolerance depends both on the numerical value of $\frac{j}{s}$ and the parity of $s$ through $\lambda$ in (2).

The interesting connection made here allows one to apply well-known results from number theory to the present dynamical system and infer useful dynamical properties about it. Altmann et al [30] showed using continued fractions representations that for almost all (a set of measure one) $\rho \in (0, 1)$ there exist infinitely many MUPOs (for more details see [35]). Hence, orbits in a generic mushroom exhibit stickiness causing the Poincaré recurrence times distribution to decay as $\sim t^{-2}$. The density of MUPOs can be graphically seen in figure 3 where the intervals $(\alpha_{s,j}, \beta_{s,j})$ for $s \leq 100$ are plotted and seem to cover more and more of the unit interval in an overlapping fashion (for a similar representation see [28]).

In the following section, we remove the parity-dependent $\lambda$ in the context of a more general mushroom model. This in turn allows us to use properties of continued fractions more effectively to derive a sufficient condition so that (1) has no solutions and hence destroy all MUPOs in the hat of the mushroom.
2.2. Mushrooms without MUPOs

2.2.1. Generalized mushroom. We have seen that MUPOs in the mushroom billiard are directly related to number theory through (5). In this section we propose a generalization of the mushroom billiard studied in section 2.1 within the class of billiards proposed in [46] (see also [19]), which will allow us to efficiently use properties of continued fractions without having to worry about the parity of MUPOs. The main result here will be to prove the existence of a zero measure set of $\rho = \frac{r}{R}$ values for which the mushroom’s hat is MUPO-free. Furthermore, we shall obtain a sufficient condition which explicitly describes a subset of this set.

Consider the ‘elementary cell’ obtained by slicing the mushroom along its vertical axis of symmetry. Then the period of the corresponding $(s, j)$ orbit is $s\lambda/2$. Similarly, since we are currently only interested in collisions with the curved segment of the billiard, we introduce the parameter $\alpha \in (0, 1)$ which allows the mushroom to have circular hats of variable size. This billiard, shown in figure 4, was shown to have a sharply divided phase space in [19] as long as $L > 0$. The boundary between the two components is given by the dashed arc of radius $r \in (0, R)$. Note that because the stem is triangular, there are no bouncing ball orbits present.

Periodic orbits in the hat of the proposed mushroom will now have incidence angles with the curved boundary given by

$$|\theta_q,p| = \pi/2 - \alpha \pi \frac{q}{p}$$

for some coprime $p$ and $q$, and equation (1) becomes

$$\cos \alpha \pi \frac{q}{p} \leq \frac{r}{R} < \frac{\cos \frac{\alpha \pi q}{p}}{\cos \frac{\alpha \pi q}{q}}.$$  

(6)

Note that there is no longer a parity-dependent $\lambda$. Similarly (5) becomes

$$0 \leq \frac{p}{q} - \frac{\vartheta^*}{\alpha} < \frac{\alpha \pi \cot \frac{\alpha \pi q}{q}}{2q^2} + \frac{R_2(q)}{q^2},$$

(7)

where $\vartheta^* = \frac{1}{2} \arccos \rho$, and $R_2(q)$ is the remainder term obtained from the Taylor expansion for large $q$. In the following we bound the argument of the cotangent by $\pi \vartheta^*$ and bound $R_2$, so that for $q \geq Q$ we have

$$0 \leq \frac{p}{q} - \frac{\vartheta^*}{\alpha} < \frac{\alpha \pi \cot \frac{\alpha \pi q}{q}}{2q^2} + \frac{R_2(q)}{q^2} < \left[ \frac{1}{q^2} \frac{\alpha \pi \rho}{2\sqrt{1 - \rho^2}} \right] + \hat{R}_2(q, Q) \frac{1}{q^2},$$

(8)
where $R_2(q)$ is bounded by

$$
\hat{R}_2(q, Q) = \frac{\alpha^2 \pi^2}{\cos^2\left(\frac{\alpha \pi}{Q}\right) q^2} \left[ \left( \tan^2\left(\frac{\alpha \pi}{Q}\right) + \frac{4}{3} \right) \sqrt{1 - \rho^2} + \frac{\rho^3}{2(1 - \rho^2)^2} \right] \\
+ \left( 1 + \frac{\alpha^2 \pi^2}{\cos\left(\frac{\alpha \pi}{Q}\right)^2 Q^2} \right) \frac{\rho^3}{2 \cos\left(\frac{\alpha \pi}{Q}\right)^2 (1 - \rho^2 \left(1 + \frac{\alpha^2 \pi^2}{\cos\left(\frac{\alpha \pi}{Q}\right)}\right)^2 \right)^{1/2}},
$$

(9)

as obtained in appendix A. Here, $Q$ is a fixed number up to which (8) must be checked numerically. It must be greater than $\max(\alpha \pi, \frac{\alpha \pi}{\cos\left(\frac{\alpha \pi}{Q}\right)} \sqrt{1 - \rho^2})$, following from appendix A.

2.2.2. MUPO-free condition. We now turn to some number theory and introduce some basic concepts. It is well known that the best rational approximations of a real number $\xi$ are obtained through its continued fraction representation [47]

$$
\xi = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}} = [a_0; a_1, a_2, \ldots],
$$

(10)

where the quantities $a_0, a_1, a_2, \ldots$ are called ‘partial quotients’ and are usually taken to be positive integers. Irrational numbers have an infinite continued fraction representation while rationals have finite. The $n$th truncation of a continued fraction representation gives the $n$th ‘convergent’ $\frac{A_n}{B_n}$ of $\xi$. Hence irrational numbers have an infinite number of convergents while rationals finite. Convergents are ‘best approximations’ to $\xi$, meaning that there is no other fraction with denominator smaller than $B_n$ which approximates $\xi$ better.

For the mushroom, we would like to find values of $\vartheta^* = [a_0; a_1, a_2, \ldots]$ for which

$$
0 \leq \frac{p}{q} - \frac{\vartheta^*}{\alpha} < \frac{K(Q, Q)}{q^2}
$$

(11)

has no solutions since this would also imply no solutions to (6). Solutions to (11), if any, are only given by the convergents of $\frac{\vartheta^*}{\alpha}$ if $0 < K(Q, Q) \leq \frac{1}{2}$ [48]. Obviously, if $\frac{\vartheta^*}{\alpha}$ is rational then there is only a finite number of solutions to (11) and the corresponding mushroom is ‘finitely’ sticky. However, if $\frac{\vartheta^*}{\alpha}$ is irrational the answer is not so simple. We focus on the convergents $\frac{A_n}{B_n}$ of $\frac{\vartheta^*}{\alpha}$ and express it in terms of them such that

$$
\frac{\vartheta^*}{\alpha} = \frac{\zeta_{n+1} A_n + A_{n-1}}{\zeta_{n+1} B_n + B_{n-1}},
$$

(12)

where $\zeta_n = [a_n; a_{n+1}, a_{n+2}, \ldots]$ is the $n$th ‘complete quotient’ of $\frac{\vartheta^*}{\alpha}$. Hence,

$$
\frac{A_n}{B_n} - \frac{\vartheta^*}{\alpha} = \frac{A_n B_{n-1} - A_{n-1} B_n}{(\zeta_{n+1} + \frac{B_{n+1}}{B_n}) B_n^2} = \frac{(-1)^{n-1}}{(\zeta_{n+1} + \frac{B_{n+1}}{B_n}) B_n^2}.
$$

(13)

It is easy to see that if $n$ is even, then $\frac{A_n}{B_n} - \frac{\vartheta^*}{\alpha} < 0$. Therefore, equation (11) will not have any solutions if

$$
K(Q, Q) < \frac{1}{\zeta_{n+1} + \frac{B_{n+1}}{B_n}}
$$

(14)

for all odd $n$. Since $\zeta_{n+1} < \zeta_{n+1} + 1$ and $\frac{B_{n+1}}{B_n} < 1$, it follows that

$$
K(Q, Q) < \frac{1}{q^2 + 2},
$$

(15)
where \( \varphi = \max(d_{2\alpha}) \) is a sufficient condition for (11) and therefore (6) not to have any solutions. The condition is never satisfied if \( d_{2\alpha} \) is unbounded.

The set of numbers with bounded even partial quotients as derived above has zero measure \([48]\) and has Hausdorff dimension one as \( \varphi \) is unbounded as \( \rho \rightarrow 0 \) \([49]\). As shown in \([30]\), a generic mushroom will be ‘infinitely sticky’ in the sense that it has infinitely many MUPOs (for more details see \([35]\)). However, we have shown here that there are infinitely many solutions. The condition is never satisfied if \( \varphi \) is 1, for the original mushroom with \( \alpha = \frac{1}{2} \), if \( \rho < \left( \frac{1}{(\rho_{\min})^2} + 1 \right)^{-\frac{1}{2}} \approx 0.390683 \), (15) gives a sufficient condition for (11) not to have any solutions and therefore describes a mushroom with no MUPOs in its hat. Nevertheless, a MUPO-free mushroom is still expected to exhibit stickiness through orbits which are just inside the ergodic component of the phase space and therefore just intersecting the dashed semicircle of radius \( r \) of figure 2(a). What this means is that points with zero local Lyapunov exponents are more sparsely distributed as they are no longer supported by periodic orbits.

How this lack of MUPOs affects the power-law decays of different statistical observables is a natural question which we attempt to answer in the context of an ‘open’ mushroom in section 3.

2.3. Mushrooms with large stems

2.3.1. MUPO-free example. For larger values, \( \frac{1}{2} < K(Q, Q) \leq 1 \), solutions to (11), if any, are given by the convergents \( \frac{B_n}{A_n} \) and also by the so-called intermediate convergents of the form \( \frac{cA_{n+1} + A_n}{cB_{n+1} + B_n} \) \([48]\), where \( c \) is an integer such that \( 1 < c < a_{n+2} \). The increased ‘easiness’ in finding good approximations and therefore solutions to (11) is immediately and graphically apparent from the increased frequency of overlaps and density for larger values of \( \rho \) in figure 3. There are however values of \( \vartheta^* \) with \( K(Q, Q) > \frac{1}{2} \) satisfying (14) such that the corresponding mushrooms will have no MUPOs. An example of such a mushroom is \( \rho = \cos\left( \frac{\sqrt{2}}{23}\pi \right) \approx 0.64013 \) which has \( K(q, 95) < 0.6549 \) and \( 2\vartheta^* = [0; 1, 1, 3, \{1, 4\}] \) (where we have numerically checked the absence of MUPOs up to \( q = 95 \)). Here, the odd convergents of \( 2\vartheta^* \) satisfy \( 0 < \frac{1}{\zeta_n} - \frac{\nu}{\nu'} < \frac{1}{\zeta_n} \) where \( \zeta_n = (\zeta_{n+1} + \frac{B_n}{A_n})^{-1} \) for odd \( n \geq 3 \), where \( \zeta_{n+1} = [1; 4, [1, 4]] = \frac{1}{2}(1 + \sqrt{2}) \). It is an easy exercise to show that for all odd \( n \geq 3 \)

\[
B_n = \frac{1}{2}(\alpha + \lambda \sqrt{2} - \alpha \lambda + \sqrt{2}),
B_{n+1} = \frac{1}{2}(\beta + \lambda \sqrt{2} + \beta \lambda + \sqrt{2}),
\]
(16)

where \( \lambda = 3 \pm 2\sqrt{2}, \alpha = 12 \pm 7\sqrt{2} \) and \( \beta = \pm 26 + 19\sqrt{2} \) are all positive numbers. Hence, \( \frac{b_{n+1}}{A_n} = \frac{1}{2}(\beta + \lambda \sqrt{2} + \beta \lambda + \sqrt{2}) \) is strictly decreasing with \( n \) and therefore \( K_n \) is bounded by

\[
K(q, 95) < K_5 < K_n < \frac{1}{\sqrt{2}} \]
(17)

for all odd \( n \geq 5 \), where \( K_5 \approx 0.706 \). Similarly for the intermediate convergents of \( 2\vartheta^* \) we have that

\[
\frac{cA_{n+1} + A_n}{cB_{n+1} + B_n} - 2\vartheta^* = \frac{cA_{n+1} + A_n}{cB_{n+1} + B_n} - \frac{\zeta_{n+2}A_{n+1} + A_n}{\zeta_{n+2}B_{n+1} + B_n},
\]
(18)

which for odd \( n \geq 5 \) simplifies to

\[
\frac{2 + 2\sqrt{2} - c}{(cB_{n+1} + B_n)(2 + 2\sqrt{2} + B_n)} = \frac{K_n(c)}{(cB_{n+1} + B_n)^2},
\]
(19)

9
since \( \zeta_{n+2} = [4; 1, [4, 1]] = 2 + \frac{1}{2+\sqrt{2}} \). Hence, using (16) and a similar argument as above \( K_{n}(c) = \frac{m(4m^2 + 2m)}{4\sqrt{2}} - \frac{(2+\sqrt{2})(c+2\sqrt{2})}{8(5-\sqrt{2})} \). \( \left( \frac{1}{\sqrt{2}} \right)^2 \) is bounded by
\[
K(q, 95) < \bar{K}_5(1) \leq K_{n}(c) < \frac{4 + 4c - c^2}{4\sqrt{2}} \tag{20}
\]
for \( c = 1, 2, 3 \) and odd \( n \geq 5 \) where \( \bar{K}_5(1) \approx 1.237 \). Therefore, \( \rho = \cos \left( \frac{\pi}{1233} \right) \) describes a mushroom with no MUPOs in its chaotic region.

2.3.2. Supremum of MUPO-free values. From the example above we can now use similar arguments to establish that MUPO-free values of \( \rho \) exist up to \( \frac{1}{\sqrt{2}} \). In other words, \( \operatorname{sup} \{ \rho \in (0, 1) : S_{\rho} = \emptyset \} = \frac{1}{\sqrt{2}} \). To see this, let \( \hat{K}(Q, Q) \) denote the value of \( K(Q, Q) \) at \( \rho = \frac{1}{\sqrt{2}} \). Then from equations (8) and (9) \( K(Q, Q) < \hat{K}(Q, Q) = \frac{1}{2} + \frac{\pi}{1233} + O(Q^{-3}) \) for \( 0 < \rho < \frac{1}{\sqrt{2}} \). Now consider for \( n \in \mathbb{Z}^+ \) large
\[
2 \hat{\theta}^* = [0; 1, [1, m]] = \frac{m + 2 + \sqrt{m^2 + 4m}}{4m - 4} = \frac{1}{2} + \frac{1}{4m} + O \left( \frac{1}{m^2} \right), \tag{21}
\]
so that \( \rho = \cos \pi \hat{\theta}^* = \frac{1}{\sqrt{2}} - \frac{\pi}{2} \sqrt{mn} + O(m^{-2}) \). We first look at the odd convergents of \( 2 \hat{\theta}^* \) as in (13)
\[
0 < \frac{A_n}{B_n} - 2 \hat{\theta}^* = \frac{K_n}{B_n^2}, \tag{22}
\]
where \( K_n = (\zeta_{n+1} + \frac{B_{n-1}}{B_n})^{-1} \) and \( \zeta_{n+1} = [1; m, [1, m]] = \frac{1}{2} + \sqrt{1 + \frac{1}{m}} \). Via a similar manipulation as in (16) we obtain that for odd \( n \geq 3 \)
\[
\frac{B_{n-1}}{B_n} = \frac{(2 - \lambda_{-}) + (\lambda_{+} - 2)(\frac{1}{\lambda_{+}})^{\frac{1}{n+1}}}{(1 + 2m - \lambda_{-}) - (1 + 2x - \lambda_{+})(\frac{1}{\lambda_{+}})^{\frac{1}{n+1}}}, \tag{23}
\]
where \( \lambda_{\pm} = \frac{1}{2}(2 + m \pm \sqrt{4m^2 + m^2}) \). Thus, \( K_n \geq K_1 = \frac{2m \sqrt{m+\sqrt{4m^2 + m^2}}}{2m+ \sqrt{4m^2 + m^2}} = 1 - \frac{2}{m} + O(m^{-2}) \) converges exponentially to \( \frac{1}{2} \left( 1 - \frac{2}{m} + O(m^{-2}) \right) \) with \( n \) and therefore \( K_n > \hat{K}(q, B_n) \approx \frac{q}{B_n} \) for large enough \( m \) and \( n \). Similarly, when looking at the intermediate convergents of \( 2 \hat{\theta}^* \) as in (18) and (19) such that
\[
\hat{K}_{n}(c) = \frac{(\zeta_{n+2} - c)(c + \frac{B_{n-1}}{B_n})}{(\zeta_{n+2} + \frac{B_{n-1}}{B_n})}, \tag{24}
\]
where \( c = 1, 2, \ldots (m - 1), \) \( \zeta_{n+2} = [m; 1, [m, 1]] = \frac{m}{2} + \sqrt{\frac{m^2}{4} + m} \) and \( \frac{B_{n-1}}{B_n} \) can be obtained from (23), we find that \( \hat{K}_{1}(1) \leq \hat{K}_{n}(c) \) and \( \hat{K}_{n+1}(c) < \hat{K}_{n}(c) \). Hence, since \( \hat{K}_{n}(1) \) converges exponentially to \( 1 - \frac{2}{m} + O(m^{-3}) \) with \( n \), \( \hat{K}_{n}(1) > \hat{K}(q, cB_{n-1} + B_n) \approx \frac{q}{B_n} \) for large enough \( m \) and \( n \), thus verifying our claim above the supremum of MUPO-free mushrooms.

2.3.3. Supremum of finitely sticky irrational values. If \( \frac{a^*}{m} \in \mathbb{Q} \), then the corresponding mushroom has a finite number of MUPOs and is thus ‘finitely’ sticky. This is because rational numbers have a finite continued fraction representation and there is no other way of approximating a rational \( \frac{a^*}{m} \) by rational \( \frac{p}{q} \) that is faster than \( q^{-2} \). Therefore, there
are however infinitely many $\frac{\theta}{\pi} \not\in \mathbb{Q}$ which are also finitely sticky [35]. Furthermore, the set of finitely sticky mushrooms is of measure 0 and dimension 1, just like the MUPO-free set. Such mushrooms may be constructed by simply adding periodic tails of small even partial quotients to the continued fraction expansion of $\frac{\theta}{\pi}$. Therefore, we find that $\sup \{ \rho : \# S_{\rho} < \infty, \alpha = 1/2, \frac{1}{2} \arccos \rho \not\in \mathbb{Q} \} = \frac{2}{\sqrt{16+\pi^2}} \approx 0.7864$. To see this take the leading order term of $K(Q, Q)$ as $Q \to \infty$ and equate it to 1, so that $\frac{a_{\pi,\rho}}{2\sqrt{1-\rho^2}} \equiv 1$. Now since $\rho = \cos \pi \theta^* = \frac{1}{2} \arccos \frac{\pi}{\sqrt{16+\pi^2}} = [0; 2; 2, 1, 3, 1, 2, 1, \{1, m\}] = 2\tilde{\theta}^*$. It follows that the corresponding $\tilde{K}(Q, Q) < 1$ in the limit $Q \to \infty$. Now since we may augment the tail of the continued fraction expansion of $2\tilde{\theta}^*$ as done above by the transformations $a_v \to a_v + 1$ and $\zeta_{v+2} \to [1; m, [1, m]]$ for any even $v$, then $K_n \equiv B_n^2 (\frac{\pi}{\sqrt{16+\pi^2}} - 2\tilde{\theta}^*)$ will converge exponentially to some function $f(m) = 1 - \frac{k}{m} + R_1(m)$ with $n$ for some constant $k$ and $R_1(m) = \ell(\xi)_{m}$ for some $0 < \xi < m$. Therefore, as $v \to \infty$, $(\tilde{\theta}^* - \theta^*) \to 0^+$ and $\tilde{K}(Q, Q) \to 1$. However, we may always choose $m$ and $n$ big enough such that $K_n > \tilde{K}(B_n, B_n)$. A similar statement can be made for the intermediate convergents of $2\tilde{\theta}^*$. Therefore, for values of $\rho > \frac{4}{\sqrt{16+\pi^2}}$, $K(Q, Q) > 1$ and therefore all convergents of $\frac{\theta}{\pi} \not\in \mathbb{Q}$ are solutions of (11) [47], hence describing mushrooms with infinitely many MUPOs.

In the next section, we investigate the stickiness due to MUPOs in the hat and stem of the mushroom in the context of escape through a small hole placed on the stem of the mushroom.

3. Escape from the mushroom

In the previous sections we have investigated the dynamics of the mushroom billiard and specifically focused on the chaotic region of phase space, close to the regular island. We have seen how MUPOs come into existence, how they affect the dynamics of orbits in their specifically focused on the chaotic region of phase space, close to the regular island. We have seen how MUPOs come into existence, how they affect the dynamics of orbits in their immediate vicinity and also how they can be removed. In this section we shall address the problem of escape through a small hole placed on the stem of the mushroom. We shall derive exact expressions, to leading order, for the survival probability for two specific cases of the mushroom.

The uniform (Liouville) distribution projected onto the billiard boundary has the form $(2|\partial Q|)^{-1} \, dz \, d \sin \theta$, where $|\partial Q|$ is the perimeter of the billiard while $z \in (0, |\partial Q|)$ is the length parametrization round the billiard boundary and $\theta \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right)$ is the angle of incidence with it. This is the most natural choice for an initial distribution of particles. Given such a distribution, the probability $P(t)$ that a particle survives (i.e. does not escape through $k$ small holes) in a fully chaotic billiard up to time $t$ decays exponentially $\sim e^{-\gamma t}$ due to the ‘random’, chaotic behaviour of the particles, with $\gamma \approx \frac{\ell_{\pi(\frac{\pi}{\sqrt{16+\pi^2}})}}{\ell(\frac{\pi}{\sqrt{16+\pi^2}})}$ [39] to leading order, where $\langle t \rangle$ is the mean free path, $\ell$ is the length of each hole and $|\partial Q|$ the area of the billiard. $|\partial Q|$ is easy to calculate and hence for the remainder of the paper we leave it in general form to allow for different shaped stems.

For the mushroom billiard, assuming that the holes are placed well in the ergodic component of its phase space, $P(t)$ is expected to be composed of a constant term, corresponding to the initial conditions (ICs) trapped forever in the mushroom’s hat and a time-dependent term corresponding to ergodic ICs. Typical chaotic orbits will decay exponentially as before, while sticky orbits will decay with a power law of order $\sim t^{-1}$ [29]. All these behaviours coexist and are formulated below:

$$P(t) \approx A + B \left( e^{-\gamma t} + \frac{C}{t} \right),$$

(25)
Figure 5. Left: mushroom with a triangular stem. Right: image reconstruction trick at the base of the mushroom hat. Orbits entering the lower semicircle through the thick blue horizontal line of length $2r$ are assumed to escape through the hole $\epsilon$ soon thereafter.

where we have neglected terms of order $t^{-2}$. In equation (25), $A$ is the measure of the integrable island given by

$$A = 4(2|\partial Q|)^{-1} \left[ R \sqrt{1 - \rho^2} - \rho R \arccos \rho + \frac{\pi}{2} R (1 - \rho) \right],$$

and $B$ is its complement ($B = 1 - A$). The exponential escape rate is given by

$$\bar{\gamma} \approx \sum_{i=1}^{k} \epsilon_i \langle \bar{\tau} \rangle_{B|\partial Q},$$

while the mean free path in the ergodic component is

$$\langle \bar{\tau} \rangle = \frac{c_v}{c_\mu} = \frac{\pi (|Q_s| + R^2 \arcsin \rho + \rho R^2 \sqrt{1 - \rho^2})}{B|\partial Q|},$$

where $|Q_s|$ is the area of the mushroom’s stem while $c_v$ and $c_\mu$ are the invariant probability measures of the ergodic component for the billiard flow and map respectively.

Algebraic decays, of the form $\frac{C}{t}$ in (25), originate from the stickiness exhibited, and in particular due to the presence of MUPOs as discussed in the previous sections. It is a geometrical description of the constant $C$ that we seek here. In the case of the stadium billiard, for example, near-bouncing ball orbits were studied and such a constant was successfully calculated in [50]. In the following two subsections we attempt to do the same, first for the MUPOs living in the mushroom’s hat and then for near-bouncing ball orbits present in mushrooms with rectangular stems.

3.1. Sticky hat

We consider a mushroom with a central triangular stem and circular hat as shown in figure 5, hence removing any bouncing ball orbits between parallel walls. The asymptotic algebraic decay $\frac{C}{t}$ should be equal to the measure (relative volume occupied in phase space) of the set of quasi-periodic ICs which do not enter the stem until a time $t$. Such an assumption is justified by the apparent ‘reluctance’ [51] displayed by orbits to (re-)enter a ‘sticky’ mode. This reluctance to (re-)enter as well as to leave sticky modes is demanded by ergodicity, which requires trajectories to fill the phase space uniformly. When the mushroom is opened, the exponential decay of (25) prevents most of these orbits from (re-)entering the sticky modes surrounding MUPOs.
Figure 6. Phase space plots of initial conditions that do not escape from the hat for at least \( N = 200 \) collisions with the boundary at \( \rho = 0.815 \). Plots (b), (c) and (d) are magnifications of (a), showing in more detail the MUPOs (4,1), (5,1), (66,13) and their surrounding sticky orbits. MUPOs accumulate close to the boundary of the integrable island at \( \theta = \arcsin \rho \) (see also figure 2(b)). The red lines in (b) and (c) are the analytic prediction given by (30)–(33) and can be used to integrate the enclosed areas.

We use the image reconstruction trick [18] and neglect collisions with the base of the mushroom’s hat. Hence, the dynamics in the hat remains unchanged while a horizontal slit of length \( 2\rho \) centred at the origin corresponds to the stem’s opening. We parametrize the now circular boundary by the angle \( \phi \), where \( \phi \in (0, 2\pi) \) increases anticlockwise as shown in the right panel of figure 5. Now, each IC \((\phi, \theta_{s,j})\) is a MUPO if the collision coordinate \( \phi \) satisfies

\[
\phi \in \bigcup_{k=0}^{\lambda_s-1} (\phi_1(\theta_{s,j}, k), \phi_2(\theta_{s,j}, k)),
\]

with

\[
\phi_1(\theta_{s,j}, k) = \theta_{s,j} + \frac{\pi}{\lambda_s} + \arccos(\rho^{-1} \sin \theta_{s,j}) + (k - 1) \frac{2\pi}{\lambda_s}, \quad (30)
\]

\[
\phi_2(\theta_{s,j}, k) = \theta_{s,j} + \frac{\pi}{\lambda_s} - \arccos(\rho^{-1} \sin \theta_{s,j}) + k \frac{2\pi}{\lambda_s}, \quad (31)
\]

where \( \rho = \frac{\rho}{R} \) and the angles \( \phi_i \) are taken modulo \( 2\pi \). Each MUPO then defines a dashed, horizontal line in the \( \phi - \theta \) plane (the phase space), and each dashed line has length \( \phi_2 - \phi_1 = \frac{2\pi}{\lambda_s} - 2\arccos(\rho^{-1} \sin \theta_{s,j}) \). Note that \( \phi_1 \) and \( \phi_2 \) are not defined if \( \sin \theta > \rho \).

To help visualize how the long surviving ICs near the above-described MUPOs populate the phase space, we turn to some computer simulations. Initial conditions near the integrable island’s boundary are chosen randomly so that \( \phi \in (0, 2\pi) \) and \( \theta \in (0, \arcsin \frac{\rho}{2}) \). The ones that survive for at least \( N \) collisions with the boundary are shown in the top-left panel of figure 6 for parameters \( N = 200 \) and \( \rho = 0.815 \). We note that for the selected value of \( \rho \), the...
most dominant MUPO is the square with \((s, j) = (4, 1)\) (see also figure 6(b)). In figure 6(c) one can identify the pentagon orbit \((s, j) = (5, 1)\) which like all odd \(s\)-orbits has twice its period \((\lambda s = 10)\) of surviving intervals along the horizontal line \(\theta_{s,j}\). Further magnification into the phase space reveals the \((s, j) = (6, 1)\) orbit (see also figure 6(d)) and then an accumulation of higher order orbits closer to the island’s boundary at \(\arcsin \rho\). The next MUPO is \((s, j) = (920, 181)\).

We introduce a small perturbation \(\eta \ll 1\) in the angle \(\theta_{s,j}\) of each MUPO and expand (30)–(31) to leading order:

\[
\phi_1(\theta_{s,j} + \eta, k) = \phi_1(\theta_{s,j}, k) + \left(1 - \frac{\cos \theta_{s,j}}{\sqrt{\rho^2 - \sin^2 \theta_{s,j}}}\right) \eta + O(\eta^2),
\]

\[
\phi_2(\theta_{s,j} + \eta, k) = \phi_2(\theta_{s,j}, k) + \left(1 + \frac{\cos \theta_{s,j}}{\sqrt{\rho^2 - \sin^2 \theta_{s,j}}}\right) \eta + O(\eta^2).
\]

We also impose a time constraint such that the perturbed MUPO will survive up to time \(t\) by requiring that

\[
\phi \geq \phi_1(\theta_{s,j} + \eta, k) + 2\eta N, \tag{34}
\]

\[
\phi \leq \phi_2(\theta_{s,j} + \eta, k) + 2\eta N, \tag{35}
\]

where \(N = \lceil \frac{t}{2R \cos(\theta_{s,j} + \eta)} \rceil\) is the number of collisions in time \(t\). Expanding (34) and (35) to leading order together with (32) and (33) defines in total four lines which form a quadrilateral in phase space with area \(\Delta_{s,j}\) which can be integrated with respect to the invariant measure \((2|\partial Q|B)^{-1} \, d\phi\, ds\) in \(\theta\) to give

\[
\Delta_{s,j} = \frac{8R \cos^2 \theta_{s,j} (\pi - s \lambda \arccos(\rho \sin \theta_{s,j}))^2}{2s^2 \lambda^2 |\partial Q|Bt} + O\left(\frac{1}{t^2}\right), \tag{36}
\]

to leading order in \(t\). There are \(2\lambda s\) such quadrilaterals due to \(\theta\)-symmetry; however, only half of the total area for each MUPO lies in \(\phi \in (0, \pi)\), which corresponds to the actual mushroom’s hat. As for the ICs on the straight segments of the hat, since the billiard map is measure preserving, only \(2\lambda j\) quadrilaterals are mapped onto them. Hence overall we obtain

\[
\frac{C}{t} = \sum_{(s,j) \in S_\rho} \lambda(s + 2j)(\Delta_{s,j} - \delta_{s,j}), \tag{37}
\]

where \(S_\rho\) was defined in section 2.1 and

\[
\delta_{s,j} = \begin{cases} \Delta_{s,j}/2, & \text{if } \cos \frac{\Delta_{s,j}}{t} = \rho, \\ 0, & \text{otherwise} \end{cases}
\]

accounts for the possibility that a MUPO is situated exactly on the border of the chaotic region and therefore can only be perturbed from one side. The sum in (37) converges since the elements of \(S_\rho\), if any, are distributed with a bounded density with respect to \(\ln s\). Also note that \(C\) does not depend on the size or position of the hole on the stem.

If there are no MUPOs in the mushroom’s hat \((S_\rho = \emptyset)\) and stem, then \(C = 0\). In such a case, as discussed at the end of section 2.2, a MUPO-free mushroom would still exhibit stickiness, realized by some faster power law exponent. The stickiness is due to orbits which only slightly intersect the dashed semicircle of radius \(r\) and therefore satisfy \(\rho - \varepsilon \leq \sin \theta < \rho\), where \(0 < \varepsilon \ll 1\). The measure of this set is obviously proportional to \(\varepsilon\), of which \(\sigma \propto \arccos \frac{\varepsilon}{\rho} = \sqrt{2\varepsilon/\rho} + O(\varepsilon^{3/2})\) will enter the mushroom’s stem at each
forward iteration of the billiard map. Since the motion of these ICs is quasi-periodic, their typical lifetime before entering the mushroom’s stem is $\sim \sigma^{-1}$ and therefore $\sim \epsilon^{-1/2}$. For this reason we expect the power law exponent of the stickiness in a MUPO-free mushroom to be equal to 2 and that equation (25) should then read $P(t) = A + B(e^{-\gamma t} + \frac{2}{7})$. Such an algebraic tail is related to other important statistical observables which quantify stickiness such as Poincaré recurrence times (see the appendix of [29]) and decay of correlations (see [35], p 24). Numerical simulations of the survival probability function for different $\rho$ values are performed and discussed in section 3.3.

3.2. Sticky stem

In the previous section, we derived an expression to leading order for the asymptotic behaviour of $P(t)$ (see equations (25)–(29) and (36)–(38)) for a mushroom with a triangular stem. Here, we investigate the stickiness introduced by the bouncing ball orbits present in mushrooms with rectangular stems of length $L$ and a hole of size $\epsilon$ on one of the two parallel segments as shown in figure 7. A method for calculating the contribution of these orbits to $P(t)$ was devised and explained in detail in [50]. Here, we follow this method and obtain an exact expression to leading order for the survival probability of the mushroom billiard. In doing so we discover an interesting discontinuous dependence of $P(t)$ on $\rho = dR$ and also show that in the limit $\rho \to 1$ the expression for $P(t)$ reduces to the one obtained in [50] for the stadium billiard.

We first split the billiard’s boundary $\partial Q$ into four, non-overlapping, connected segments: $\partial Q_h^t$, $\partial Q_w^t$, $\partial Q_h^h$, and $\partial Q_h^h$, referring to the stem’s base, the stem’s parallel walls, the hat’s base and the hat’s curved segment, respectively. We parametrize the right parallel wall of $\partial Q_w^t$ by $s \in (0, L)$ such that the interval $(h^-, h^+)$ defines the hole of size $\epsilon$ as shown in figure 7. We note that ICs $(x, \theta)$ with $0 < x < h^-$ and $0 < \theta < \arctan \frac{\epsilon}{4\rho}$ cannot jump over the hole and therefore do not interact with the mushroom’s hat. Such orbits behave in a completely regular manner and therefore can be integrated directly to give

$$
\frac{2}{2|\partial Q|B} \left( \int_0^{\arcsin(h^-/t)} \int_{\arcsin(h^-/t)}^{\arcsin(h^-)} \cos \theta \, ds \, d\theta + \int_0^{\arcsin(2h^-/t)} \int_{\arcsin(2h^-/t)}^{\arcsin(2h^-)} \cos \theta \, ds \, d\theta \right)
= \frac{(2h^-)^2 + (h^-)^2}{2|\partial Q|Br},
$$

(39)
where we have neglected terms of order $\sim t^{-2}$ and multiplied by 2 due to the horizontal symmetry of the billiard. Similarly, ICs with $h^* < x < L$ and $0 > \theta > -\arctan \frac{1}{2\pi R}$ give

$$
\frac{2}{2|\partial Q|^2} \int_0^{\arcsin(L-h^*)/t} \int_{r \sin \theta}^{L-h^*} \cos \theta \, ds \, d\theta = \frac{(L-h^*)^2}{2|\partial Q|^2 |\partial B|^2}.
$$

(40)

ICs from $\partial Q^h_0$ have contributions of order $\sim t^{-2}$ to $P(t)$ and therefore are ignored.

As expected, the survival probability at long times is proportional to the square of the available length on either side of the hole. For the remainder of this section we consider ICs $(x_i, \theta_i)$ such that $h^* < x_i < L$ and $0 < \theta_i < 1$, and investigate how they contribute to $P(t)$. We let $n$ denote the number of collisions a particle experiences from straight to straight segment before entering the hat of the mushroom, and define $d_1 = L - (x_i + 2\pi n \tan \theta_i) > 0$ as the distance from the edge of the straight to the point of the last straight wall collision. We can see that $n = \left\lfloor \frac{L-x_i}{2\pi \tan \theta_i} \right\rfloor$, where $\lfloor \cdot \rfloor$ are the floor and ceiling functions, respectively. Note that $0 < d_1 < 2\pi \tan \theta_i$. Once a particle enters the hat of the mushroom it is advantageous to switch to coordinates suitable for the circle billiard map given by $(\phi, \psi) \rightarrow (\phi + \pi - 2\psi, \psi)$ such that $\phi$ is the angular collision coordinate and increases from zero in an anticlockwise fashion as shown in figure 7, while $\psi \in (-\pi, \pi)$ is the angle of reflection. Note that $\phi$ is different from what was used in section 3.1. Also $\psi$ is used instead of $\theta$ here to distinguish between collisions on the curved segment of the billiard boundary $(\partial Q^h_0)$ and collisions elsewhere. Once in the hat, we neglect collisions with the vertical base $\partial Q^c_h$, by using the image reconstruction trick as before. We find that the particle entering the hat will first collide with $\partial Q^c_h$ at

$$
\phi = -\frac{d_1}{R} + (1 + \rho) \theta_i > 0,
$$

(41)

and its angle will be

$$
\psi = -\frac{d_1}{R} + \rho \theta_i.
$$

(42)

Let $\theta_f$ be the final angle obtained when the orbit re-enters the stem of the mushroom after experiencing a reflection process (a series of $k \in \mathbb{Z}^+$ collisions with $\partial Q^c_h$) in the hat. We thus find that

$$
\theta_f = \frac{2k d_1}{R} = \left(2k \rho + 1\right) \theta_i.
$$

(43)

By carefully investigating the reflection process we find that $k$ can be either equal to 1, $\left\lceil \frac{\theta_i}{\pi} \right\rceil$ or $\left\lceil \frac{\theta_i}{\pi} \right\rceil + 1$, depending on the ICs $(x_i, \theta_i)$, which agree with the so-called magic numbers from [46]. This can be seen if one looks at the least number of iterations of the circle billiard map before the orbit described by (41) and (42) intersects the horizontal slit hole:

$$
k = \inf \left\{ j \in \mathbb{Z}^+ : \frac{\psi}{(2j - 1) \psi + \phi} < \rho \right\}.
$$

(44)

In equation (45), we have substituted the possible values of $k$ into (43) and also calculated the values of $\theta_i$ for which each collision scenario corresponds to

$$
\begin{align*}
\theta_f &= \begin{cases} 
\frac{2d_1}{R} - (2\rho + 1) \theta_i < 0, & \frac{(2\rho + 1)d_1}{2\rho(\rho + 1)R} < \theta_i, \quad k = 1 \text{ collision} \\
\frac{2d_1}{R} - (2\rho + 1) \theta_i > 0, & \frac{d_1}{2\rho R} < \theta_i < \frac{2(\rho - 1)d_1}{2\rho^2 R}, \quad k = \zeta \text{ collisions} \\
\frac{2(\zeta + 1)d_1}{R} - (2(\zeta + 1) \rho + 1) \theta_i > 0, & \frac{d_1}{2(\rho + 1)R} < \theta_i < \frac{(2\rho + 1)d_1}{2\rho(\rho + 1)R}, \quad k = (\zeta + 1) \text{ collisions}
\end{cases}
\end{align*}
$$

(45)
where we have set $\zeta = \left\lceil \frac{R}{\rho} \right\rceil$. Note that $d_i$ is a function of both $x_i$ and $\theta_i$. The first inequality on $\theta_i$ ($k = 1$ collision) seems to suggest that $\theta_i$ is unbounded; however, this is not the case. This can be seen in an example situation plotted in figure 8 where we have made the substitution $\omega = \frac{d_i}{R} \in (0, \rho)$. Note that if $\rho^{-1}$ is an integer, then the $\zeta$-collision process in (45) is no longer attainable and we only have two possible collision scenarios. It is interesting to note that if $\rho = 1$, equation (45) reduces to equations (10)–(11) of [50] which refer to the stadium billiard’s reflection process with the curved segment.

We now formulate the time of escape for ICs $(x_i, \theta_i)$:

$$t(x_i, \theta_i, k) \approx \frac{L - x_i}{\theta_i} + \frac{L - h^*}{|\theta_f|} + 2R(\rho + k + 1),$$

where we have taken small angle approximations and substitute the values of $\theta_f$ and $k$ for each collision scenario to get three equations for the time to escape. Each one of these equations describes conic sections since they are quadratic in both $x_i$ and $\theta_i$ variables. Rearranging to make $\theta_i$ the subject, we obtain three hyperbolae in the $x_i-\theta_i$ plane, describing the ICs that escape exactly at large times $t$. It is important to know the domain of each hyperbola. This can be obtained by substituting for the $d_i$ variables into the inequalities of (45), and then rearranging for $\theta_i$. These inequalities are given below for the corresponding collision scenarios:

$$k = \zeta \text{ collisions, } \quad \frac{L - x_i}{2\rho(1+n)R} < \theta_i < \frac{(2\xi \rho - 1)(L - x_i)}{2\rho \xi R(1 + 2n) - 2\rho n R^2},$$

$$k = (\zeta + 1) \text{ collisions, } \quad \frac{(2\xi \rho - 1)(L - x_i)}{2\rho \xi R(1 + 2n) - 2\rho n R} < \theta_i < \frac{(2\rho + 1)(L - x_i)}{2\rho R(\rho + 2n + 1 + n)},$$

$$k = 1 \text{ collision, } \quad \frac{(2\rho + 1)(L - x_i)}{2\rho R(\rho + 2n + 1 + n)} < \theta_i.$$  \hspace{1cm} (47)

For $n = 0, 1, \ldots$ and for $t$ large, we plot the three hyperbolae from (46) subject to (47) and the three straight lines from (45) onto the $x_i - \theta_i$ plane (see figure 9). These define an area in phase space which corresponds to the ICs that survive at least until time $t$ for fixed $n$. The various colours indicate the type of reflection process $k$ that the ICs experience in consistence with the ones in figure 8. Note that as the number of collisions $n$ with the straight
Figure 9. Area enclosed by equations (46) subject to (47) and equations (45) in the \( x_i - \theta_i \) phase space for \( n = 0 \) and 1, using \( \rho = 0.6 \), \( L = 1 \) and \( t = 50 \). The colours used are in consistence with the ones in figure 8. The dotted, dashed and solid black straight lines come from the inequalities in equation (45). The corners \( A - G \) are defined in appendix B and are each highlighted by a black dot for \( n = 0 \). The dashed vertical line at \( x_i = h^+ \) shows how the hole truncates the area of interest. The area defined for all \( n \) corresponds to the ICs that survive at least until time \( t = 50 \).

As the segments increases, the area of interest tilts and stretches in a non-overlapping fashion. To obtain the contribution to \( P(t) \) of these long surviving ICs, we must integrate each non-overlapping area and sum them all up. Note that the invariant measure will be assumed to be \( d\mu = (2|Q| \mu)^{-1} \ d\theta_i \ dx_i \) here since \( \theta_i \) is small and thus \( d\sin \theta_i \approx d\theta_i \).

The corners of each enclosed area \( A - G \), as shown in figure 9 for each value of \( n \) are given in appendix B. There are various issues which one needs to consider in order to obtain correct asymptotic expressions for the areas. Firstly, one needs to approximate all the hyperbolae by straight lines. This is done by joining the corners \( A - G \) and thus forming an irregular polygon. For example, the hyperbola between \( A \) and \( F \), which comes from \( t(x_i, \theta_i, 1) \), is approximated by a straight line joining \( A \) and \( F \) and similarly, for the hyperbola joining \( B \) and \( C \), which comes from \( t(x_i, \theta_i, \zeta) \), and for the hyperbola joining \( D \) and \( E \) which comes from \( t(x_i, \theta_i, \zeta + 1) \). The remaining edges are already straight lines and thus need no approximating. As shown in [50], the error in these approximations is \( O(t^{-2}) \) and hence meets our required asymptotic accuracy.

Another issue to be dealt with is the position of the hole which restricts the irregular polygons in \( x_i \in (h^+, L) \). This forces a deformation by truncating each polygon from the left each time one of its corners surpasses the hole’s position as seen for example in figure 9. This is due to the tilting effect caused as \( n \) is increased. Following [50] again, we expect seven different sums since there are six corners \( (A - F) \), each of which will intersect the hole at \( h^+ \) at different values of \( n \). We thus solve for \( n \) and find that the leftmost corner \( A_{x_i} = h^+ \) when \( n = n_A = \left\lfloor \frac{-2(1+2\rho)-2-20\rho-12\rho^2-4\rho^3}{4\rho+12\rho^2-8\rho^3} \right\rfloor \). Similar expressions have been obtained for all other corners \( (B - F) \) and are given in appendix C. Interestingly, we find that the order in which the corners \( A - F \) coincide with the hole’s position depends on the system’s control parameter \( \rho = \frac{r}{R} \). Their order alternates between \( n_A < n_B < n_D < n_E < n_C < n_F \) for \( \rho \in (0, 1) \), which is shown in figure 10. This is due to the discontinuity introduced by the ceiling function in \( \zeta \) for \( n_B = n_E \); hence, the lower and
Figure 10. \( n_A - n_F \) are defined in the text above and given in appendix D. The figure shows how they vary discontinuously as a function of \( \rho = \frac{r}{R} \in (0, 1) \) for \( t = 10^4 \).

upper bounds of the seven sums will depend on the above order, and so will their arguments. Altogether we write

\[
\sum_{n=0}^{n_A} \hat{P}_1 + \sum_{n=n_A+1}^{n_B} \hat{P}_2 + \sum_{n=n_B+1}^{n_C} \hat{P}_3 + \sum_{n=n_C+1}^{n_D} \hat{P}_4 + \sum_{n=n_D+1}^{n_E} \hat{P}_5 + \sum_{n=n_E+1}^{n_F} \hat{P}_6 + \sum_{n=n_F+1}^{\infty} \hat{P}_7, \tag{48}
\]

\[
\sum_{n=0}^{n_A} \tilde{P}_1 + \sum_{n=n_A+1}^{n_B} \tilde{P}_2 + \sum_{n=n_B+1}^{n_C} \tilde{P}_3 + \sum_{n=n_C+1}^{n_D} \tilde{P}_4 + \sum_{n=n_D+1}^{n_E} \tilde{P}_5 + \sum_{n=n_E+1}^{n_F} \tilde{P}_6 + \sum_{n=n_F+1}^{\infty} \tilde{P}_7, \tag{49}
\]

where \( \hat{\cdot} \) and \( \tilde{\cdot} \) are used to distinguish between the two orderings described above. \( \hat{P}_i \) and \( \tilde{P}_i \), \( i = 1, \ldots, 7 \), are the respective areas of the polygons which we are summing over. Note that \( \hat{P}_1 = \tilde{P}_1 \) and \( \hat{P}_7 = \tilde{P}_7 \). The process of finding all the \( \hat{P}_i \) and \( \tilde{P}_i \) is long but fairly elementary.

We now obtain leading order expressions for each sum in \( t \). The way to do this is similar as in [50], where a more detailed explanation of the method can be found. First we substitute \( t = \frac{1}{u} \), and then \( n = \frac{v}{u} \) into the \( \hat{P}_i \) and the \( \tilde{P}_i \), such that \( u \) is small and \( v = O(1) \). We Taylor expand \( \hat{P}_i \) and \( \tilde{P}_i \) into series up to order \( u^2 \) and then reverse the substitution by setting \( v = nu \), thus effectively incorporating the large \( n \) into the leading order term of each series expansion. Now each sum can be simplified into expressions involving polygamma functions of order 0 and 1.

The polygamma function of order \( i \) is defined as the \((i + 1)\)th derivative of the logarithm of the gamma function:

\[
\psi^{(i)}(z) = \frac{d^{(i+1)}}{dz^{(i+1)}} \ln \Gamma(z). \tag{50}
\]

The arguments of the polygamma functions are of the form \( z = \frac{a}{bu} + c \), where \( a, b \) and \( c \) are real constants, and can thus be expanded as a Taylor series to leading order as follows:

\[
\psi^{(0)} \left( \frac{a}{bu} + c \right) = \ln \left( \left| \frac{a}{bu} \right| \right) + O(u), \tag{51}
\]

\[
\psi^{(i \geq 1)} \left( \frac{a}{bu} + c \right) = (-1)^{i-1}(i-1)! \left( \frac{bu}{a} \right)^i + O(u^{i+1}). \tag{52}
\]
With these approximations at hand, we obtain expressions for the sums in (48) and (49). We only present here the first of the approximated sums and include the rest in appendix D:

$$\sum_{n=0}^{\infty} \hat{P}_1 = \sum_{n=0}^{\infty} \tilde{P}_1 = \frac{(h^* - L)^2}{2(2\rho + 1)t},$$

(53)

where we have neglected terms of order $\sim t^{-2}$. Altogether (48) and (49) take the form

$$\left(\frac{L - h^*}{2(1 + \zeta)}\right) = \frac{1}{4t} \left[ \frac{\varepsilon_1 \rho + \varepsilon_2 \rho^2 + \varepsilon_3 \rho^3 + \varepsilon_4 \rho^4}{2(2\rho + 1)(2\zeta \rho - 1)^2} \right] + \ln \left(\frac{2\rho + 1}{(2\zeta \rho - 1)^2}\right),$$

(54)

where the coefficients $\varepsilon_i$ ($i = 1, \ldots, 4$) and $j_j$ ($j = 1, 2$) are given in appendix E for both orderings $\hat{\cdot}$ and $\tilde{\cdot}$. It remains to multiply (50) by 2 due to the horizontal symmetry of the mushroom and normalize by $2|\partial B|$ to obtain a probability. The sum of expressions (39), (40) and (54) depending on the value of $\zeta$ therefore gives the asymptotic contribution of the long surviving near-bouncing ball orbits $C_t$ to the mushroom’s survival probability $P(t)$.

Interestingly yet reassuringly, in the limit of $\rho \to 1$, $\zeta = 2$ and the complicated expression for (54) reduces to

$$\frac{1}{t} \left( \frac{7}{2}(L - h^*)^2 - 2(1 + \zeta)(L - h^*)^2 \rho + 3\zeta(L - h^*)^2 \rho^2 \right) + O\left(\frac{1}{t^2}\right),$$

(55)

and

$$\frac{1}{t} \left( -\frac{5}{2}(L - h^*)^2 + 4(1 + \zeta)(L - h^*)^2 \rho - 4(2\zeta - 3)(L - h^*)^2 \rho^2 \right) + O\left(\frac{1}{t^2}\right),$$

(56)

for the two orderings $\hat{\cdot}$ and $\tilde{\cdot}$ respectively, indicating that the discontinuous dependence on $\zeta$ persists; hence, this limit is in some sense ill-defined.

### 3.3. Numerical simulations

Having obtained exact leading order analytic expressions for all the parameters appearing in (25) we now numerically test their validity by plotting the conditional probability $P_e(t)$ that a particle survives up to time $t$ given that the particle is chosen uniformly from the ergodic component of the billiard flow (see figure 11):

$$P_e(t) = (P(t) - A)/B = e^{-\bar{\gamma}t} + \frac{C}{t} + O\left(\frac{1}{t^2}\right).$$

(57)

The plots are purposely chosen (from many more) to portray and verify the results obtained in this paper. Three different mushrooms are simulated: one with a finite number of MUPOs present only in the hat (top), one with bouncing ball orbits in the stem and a MUPO-free hat (middle), and one with no MUPOs at all (bottom). The empty black circles in the plots correspond to the numerical data while the blue curves give the analytic predictions of (57). Different hole sizes and total perimeters give different exponential escape rates $\bar{\gamma}$ and in turn crossover times to a power law decay. Note the huge contribution to $C$ from the bouncing ball orbits. Although each simulation consists of $\sim 10^8$ chaotic ICs, we were unable to detect any power law decay in the MUPO-free mushroom (bottom) and hence any clues of stickiness.
Figure 11. Numerical simulations of $P_e(t)$ defined in (57) are plotted on a logarithmic scale using $10^8$ chaotic ICs as a function of $t$. The parameters $(r, R, L, \epsilon)$ used for the triangular stem (top) are $(\cos 0.3484\pi, 1, 1, 0.048)$ such that $S_\rho = \{(20, 7), (66, 23), (376, 131)\}$, while for the rectangular stem (middle) $(\cos (\frac{\sqrt{2}+\pi}{\sqrt{2}}), 1, 1, 0.02)$, with $h^* = 0.3$ such that the mushroom’s hat has no MUPOs. The MUPO-free mushroom (bottom) has parameters $(\cos (\frac{5+\sqrt{2}}{\sqrt{2}}\pi), 1, 1, 0.02)$ and appears not to have a power-law tail. The blue curves are the analytic predictions while the numerical data correspond to the empty circles. The insets are plots of $tP_e(t)$ showing the agreement with the analytic expressions for the constant $C$. 
4. Conclusions and discussion

In this paper, we have attempted to quantify the stickiness due to MUPOs observed in the mushroom billiard by placing a hole in its ergodic component and looking at the survival probability function $P(t)$ at long times (see equation (25)). Our analytic predictions are in good agreement with the numerical simulations performed and therefore confirm that $P(t) \propto \frac{C}{t}$ for long enough times. Also their good agreement with the constants $C$ derived in sections 3.1 and 3.2 for MUPOs present in the hat and in the stem, respectively, implies that these MUPOs are indeed the primary causes of the power-law decay $\mathcal{O}(t^{-1})$. This observation in turn applies to the Poincaré recurrence times distribution $Q(t)$ and the rate of mixing of the ergodic component [45].

The explicit expressions obtained here for $C$ allow one not only to predict but also to calibrate the asymptotic behaviour of $P(t)$. Also we have shown that these distributions as well as the overall existence of MUPOs in the hat are sensitive to the system’s control parameter $\rho = \frac{r}{R}$, whilst only the near-bouncing ball orbits’ contribution to $P(t)$ depends on the hole’s position and size. The reason for this is that the hole intersects the sticky region generated in phase space by the period-2 bouncing ball orbits. This creates a fictitious, time-dependent ‘island of stability’ in the mushroom’s ergodic component. Although orbits in it are unstable, they only experience up to one nonlinear collision process before escaping, thus allowing us to approximate their occupancy in phase space with polygonal ‘spikes’ which we could then integrate over. In the case of the MUPOs in the mushroom’s hat, we could easily bound the long surviving orbits by assuming that they will escape exponentially fast once in the stem.

It is expected that the methods used here can be further generalized and applied to other mushrooms with elliptical hats for instance, or even to other billiards such as the annular or drive-belt stadium billiards where polygonal-type MUPOs act as scaffolding for sticky orbits to cling onto.

A major result of this paper is the introduction of a zero measure set which describes MUPO-free mushrooms (see section 2.2) which to the best of our knowledge possess the simplest mixed phase space in two dimensions. The interesting connection between mushrooms and number theory (see equation (5)) first appearing in [30, 35] cannot be directly exploited due to a sensitive parity dependence of the periodic orbits of the mushroom. We have overcome this complication by considering a generalized mushroom with a variable-sized hat and triangular stem (see figure 4). This allowed us to efficiently use properties of continued fractions and characterize a subset of the infinitely many MUPO-free mushrooms. We thus obtained upper bounds for MUPO-free and finitely sticky irrational mushrooms and also gave an explicit example of a MUPO-free mushroom billiard (see section 2.3). Furthermore, unlike the non-sticky elliptical mushrooms mentioned in the introduction, we expect that the MUPO-free mushroom exhibits a reduced amount of stickiness (larger scaling exponent since $C = 0$). This is attributed to the difficulty in ‘finding’ the foot of the mushroom by chaotic orbits which are just inside the dashed circular arc of radius $r$. However, despite our extensive numerical simulations performed we have been unable to detect any power law decay of $P(t)$ thus far.

The results of section 2, with the exception of the generalized mushroom, should also hold in the case of a three-dimensional mushroom billiard with a hemispherical hat of radius $R$, a cylindrical stem of radius $r$ and height $h > 0$ and a cuboidal pedestal of base length $l \geq 2r$ to break angular momentum conservation (see figure 12). Orbits inside a three-dimensional spherical billiard always lie in the same two-dimensional plane containing the centre of the corresponding sphere. Hence, due of the axial symmetry of the hat and stem opening, the conditions for the existence of MUPOs are exactly the same in higher dimensions.
as in equations (1) and (2). The remaining (zero-measure) MUPOs in such a system are of the bouncing ball type and are found both in stem and pedestal. We conjecture that the corresponding mushroom has a sharply divided phase space with a MUPO-free hat\(^1\).

Finally, one would expect to see that the classical dynamical features of the less sticky mushroom introduced in section 2 appear in the analogous quantum mushroom in accordance with Bohr’s correspondence principle. Some possible directions for investigating such effects are the localization of wavefunctions (scars) \([23]\), dynamical quantum tunnelling rates from regular to chaotic regions of phase space \([26]\) as well as experimentally in the emission directionality of mushroom microlasers \([31]\). We also hope that the exact results obtained for the classical survival probability function \(P(t)\) in section 3 will be of benefit to future semiclassical treatments of open systems in quantum chaos \([53]\).

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Appendix A. A bound on \(R_2\)

To obtain a bound on \(R_2\) in equation (8) we shall use the remainder term from Taylor’s theorem several times. Taylor’s theorem states that if \(f\) is a function which is \(n\) times differentiable on the closed interval \([a, x]\) and \(n + 1\) times differentiable on the open interval \((a, x)\), then

\[
f(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + R_n(x),
\]

where \(R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - a)^{n+1}\) for some \(a < \xi < x\).\(^1\)

\(^1\) L A Bunimovich and G Del Magno have also considered this model, but have not yet proved ergodicity of the chaotic component \([52]\).
Let \( c = \cos \left( \frac{\alpha \pi}{q} \right) \leq \cos \pi \theta^* \), and \( \varepsilon = c \left( \frac{1}{\cos \left( \frac{\alpha \pi}{q} \right)^2} - 1 \right) \) such that equation (6) corresponds to \( c \leq \rho < c + \varepsilon \). We have to compute a bound for \( \arccos(c + \varepsilon) = \arccos \left( \frac{\cos(\alphaqz)}{\cos(\alphaqz)} \right) \). The Taylor expansion of \( \arccos(c + \varepsilon) \) at \( \varepsilon = 0 \) is

\[
\arccos(c + \varepsilon) = \arccos(c) - \frac{\varepsilon}{\sqrt{1 - c^2}} + A_1, \tag{A.1}
\]

with \( A_1 = -\frac{c + x}{2(1-(c^2)\varepsilon^2)} \varepsilon < 0 \) and \( 0 \leq x \leq \varepsilon \). Now, for \( q \to \infty \), we have

\[
\frac{\varepsilon}{c} = \left( \frac{1}{\cos \left( \frac{\alpha \pi}{q} \right)^2} - 1 \right) = \frac{\alpha \pi^2}{2q^2} + B_2 = B_0, \tag{A.2}
\]

with \( q^3 B_2 = \frac{\sin(\alpha \pi z)\cos^2(\alpha \pi z)}{\cos(\alpha \pi z)^4} + \frac{5\sin(\alpha \pi z)\cos^2(\alpha \pi z)}{6\cos(\alpha \pi z)^5} \) and \( q B_0 = \frac{\sin(\alpha \pi w)\cos(\alpha \pi z)^4}{\cos(\alpha \pi w)^5} \) for some \( 0 \leq z, w \leq \frac{1}{q} \). Since for \( q \geq Q \) we have \( 1 < \frac{1}{\cos(\alpha \pi z)} \leq \frac{1}{\cos \left( \frac{\alpha \pi}{q} \right)} \) and \( 0 < \sin(\alpha \pi z) \leq \sin \left( \frac{\alpha \pi}{q} \right) \). Hence,

\[
0 < q^3 B_2 \leq \frac{\sin(\alpha \pi z)^2}{\cos(\alpha \pi z)^4} + 5 \frac{\sin(\alpha \pi z)\cos^2(\alpha \pi z)}{6\cos(\alpha \pi z)^5},
\]

and similarly \( 0 < q B_0 \leq \frac{\sin(\alpha \pi z)^2}{\cos(\alpha \pi z)^4} \). This, together with (A.1) and (A.2), gives \( \arccos(c + \varepsilon) = \arccos(c) - \frac{\varepsilon}{\sqrt{1 - c^2}} + C_1 + A_1 \), where \( C_1 = -\frac{c B_0}{\sqrt{1-c^2}} < 0 \) is bounded by

\[
|C_1| \leq \left( \frac{\sin(\alpha \pi z)^2}{\cos(\alpha \pi z)^4} + 5 \frac{\sin(\alpha \pi z)\cos^2(\alpha \pi z)}{6\cos(\alpha \pi z)^5} \right) \frac{\alpha \pi^4}{q^2} \frac{c}{\sqrt{1 - c^2}}.
\]

and \( A_1 < 0 \) by

\[
|A_1| \leq \frac{c (1 + B_0)}{2(1 - c^2)(1 + B_0)^2} c^2 B_0 \leq \left( 1 + \frac{\alpha \pi^2}{\cos(\alpha \pi z)^2} Q^2 \right) \frac{\alpha \pi^4}{2 \cos(\alpha \pi z)^4} \frac{c^3}{(1 - c^2)^2} \frac{1}{q^4}
\]

\[
\leq \left( 1 + \frac{\alpha \pi^2}{\cos(\alpha \pi z)^2} Q^2 \right) \frac{\alpha \pi^4}{2 \cos(\alpha \pi z)^4} \frac{1}{(1 - c^2)^2} \frac{1}{q^4}
\]

for \( q > \frac{\alpha \pi}{\cos(\alpha \pi z)} \sqrt{1 - (\cos \pi \theta^*)^2} \). In the same way,

\[
\frac{\alpha \pi^2}{2 \sqrt{1 - c^2}} \frac{1}{q^2} \varepsilon \leq \frac{\alpha \pi^2}{2 \sqrt{1 - c^2}} \frac{1}{q^2} + C_2, \tag{A.3}
\]

with \( C_2 < 0 \) and bounded by

\[
|C_2| \leq \frac{\alpha \pi^2}{2 \sqrt{1 - c^2}} \frac{1}{q^2} \leq \frac{\alpha \pi^2}{2 \sqrt{1 - (\cos \pi \theta^*)^2} q^2} \frac{1}{q^2} \leq \frac{\alpha \pi^4}{2 \sqrt{1 - (\cos \pi \theta^*)^2} q^2} \frac{1}{q^4}.
\]

Finally, we must bound \( \frac{1}{\sqrt{1 - (\cos \pi \theta^*)^2 + v}} = \frac{1}{\sqrt{1 - (\cos \pi \theta^*)^2}} + D_0 \)
where $D_0 = \frac{v}{2(1 - (\cos \pi \theta^* + y)^2)}$ with $0 \leq y \leq v$ and so it is bounded by

$$|D_0| \leq \frac{v}{2k^2} \leq \frac{2\pi \cos \pi \theta^*}{(1 - (\cos \pi \theta^*)^2)^2} \leq \frac{\alpha^2 \pi^2 (\cos \pi \theta^*)^2}{\cos(\frac{\alpha \pi}{Q})^2 (1 - (\cos \pi \theta^*)^2)^2} \leq \frac{1}{q^2}.$$

Therefore, for (A.3) we have

$$\frac{\alpha^2 \pi^2 \epsilon}{2\sqrt{1 - \epsilon^2}} = \frac{\alpha^2 \pi^2 \cos \pi \theta^*}{2\sqrt{1 - (\cos \pi \theta^*)^2}} \leq \frac{1}{q^2} + C_2 + C_3,$$

where

$$|C_3| \leq \frac{\alpha^4 \pi^4 (\cos \pi \theta^*)^3}{2 \cos(\frac{\alpha \pi}{Q})^2 (1 - (\cos \pi \theta^*)^2)^2} \frac{1}{q^2}.$$

Putting everything together for $q \geq \max(Q, \frac{\alpha \pi}{\sqrt{\cos \pi \theta^*(1 - \cos \pi \theta^*)}})$, we have

$$\arccos(c + \epsilon) = \arccos(c) = \frac{\alpha^2 \pi^2}{2\sqrt{1 - \epsilon^2}} - \frac{1}{q^2} + C_1 + A_1 + C_2 + C_3,$$

where the remainders have magnitudes bounded by

$$|C_1| \leq \frac{\sin(\frac{\alpha \pi}{Q})^2}{\cos(\frac{\alpha \pi}{Q})^3} \frac{5}{6 \cos(\frac{\alpha \pi}{Q})^2} \frac{\alpha^4 \pi^4 \cot \pi \theta^*}{q^4},$$

$$|A_1| \leq \left(1 + \frac{\alpha^2 \pi^2}{\cos(\frac{\alpha \pi}{Q})^3} Q^2\right) \frac{\alpha^4 \pi^4 (\cos \pi \theta^*)^3}{2 \cos(\frac{\alpha \pi}{Q})^2 (1 - (\cos \pi \theta^*)^2(1 + \frac{\alpha^2 \pi^2}{\cos(\frac{\alpha \pi}{Q})^2})^2) q^4},$$

$$|C_2| \leq \frac{\alpha^4 \pi^4 \cot \pi \theta^*}{2 \cos(\frac{\alpha \pi}{Q})^3} q^4,$$

$$|C_3| \leq \frac{\alpha^4 \pi^4 (\cot \pi \theta^*)^3}{2 \cos(\frac{\alpha \pi}{Q})^3} q^4,$$

and are all negative.

**Appendix B. Corners of the polygonal area**

The corners of the polygons (for fixed $n$) as shown in figure 9 are found by solving for the intersections of the various curves and lines obtained from equations (45) and (46):

$$A_{x_i} = \frac{-(h^* \rho + L)((1 + 2\rho)(1 + n + \rho + 2n\rho) + L(1 + \rho)(1 + 2(2 + n)\rho + 24n\rho^2)) + R(-2L\rho R - LR)t}{2 + 2(4 + n)\rho + 6 + 4n}\rho^2 + R(-2\rho R - R)t},$$

(B.1)

$$A_{\theta} = \frac{(h^* - L)(1 + 2\rho)^2}{2R(1 + \rho(4 + n + 3\rho + 2n\rho)) - t(1 + 2\rho^2)},$$

(B.2)

$$B_{x_i} = \frac{-(2h^* R\rho((1 + 2\rho)(1 + n + \rho + 2n\rho) + L(-2(1 + \xi)R + t + 4(1 + 3\xi + 21(1 + \xi)n)\rho^3)) + 2\rho(t - \xi(1 + 2\rho) + R(-3 + 2\xi^2 + 2L(n + \xi(4 + 2\xi + n)\rho)))}{(2\xi - 1)(-t(1 + 2\rho) + 2R(1 + \xi + 2\xi\rho + \rho(4 + n + 3\rho + 2n\rho)))},$$

(B.3)
\[ B_{\xi} = \frac{(h^+ - L)(2\rho + 1)^2}{(2\xi \rho - 1)(-t(1 + 2\rho) + 2R(1 + \zeta + 2\xi \rho + \rho(4 + n + 3\rho + 2n\rho)))}, \]  
\[ C_{\xi} = L - \frac{2(h^+ - L)R\rho(2\xi \rho - 1)(\xi \rho + n(2\xi \rho - 1))}{(1 + 2\rho)(-2\xi^2 \rho + 2R(-1 - \zeta + (-1 + 2\xi(1 + \zeta) - n)\rho + \xi(\zeta + 3(2n)\rho^3))}, \]
\[ D_{\xi} = \frac{(h^+ - L)(1 - 2\xi \rho)^2}{(1 + 2\rho)(t - 2\xi \rho + 2R(-1 - \zeta + (-1 + 2\xi(1 + \zeta) - n)\rho + \xi(3 + 2n)\rho^2))}, \]
\[ E_{\xi} = \frac{(h^+ - L)(1 - 2\xi \rho)^2}{(1 + 2\rho)(t - 2\xi \rho + 2R(-1 - \zeta + (-1 + 2\xi(1 + \zeta) - n)\rho + \xi(3 + 2n)\rho^2))}, \]
\[ F_{\xi} = \frac{(h^+ - L)(1 - 2\xi \rho)^2}{(1 + 2\rho)(t - 2\xi \rho + 2R(-1 - \zeta + (-1 + 2\xi(1 + \zeta) - n)\rho + \xi(3 + 2n)\rho^2))}, \]
\[ G_{\xi} = L, \]
\[ G_{\theta} = 0, \]
where \( \zeta = \lceil \frac{\rho}{R} \rceil \) and \( \rho = \frac{\zeta}{R} \).

**Appendix C. Values of n when corners hit the hole**

The upper and lower limits of the sums in expressions (48) and (49) are the solutions for \( n \) when the \( x_i \) coordinate of the corners \( A - F \) exceeds \( h^+ \):
\[ n_B = \left[ \frac{(2 + \xi - \frac{\rho}{R}) + (6 - 4\xi^2 + \frac{\xi}{R}(2\xi - 2))\rho + (-16\xi - 8\xi^2 + 4\xi^3 \frac{\rho}{R})\rho^2 + (-4 - 12\xi)\rho^3}{4(1 + \xi^2)(2\rho + 1)\rho^2} \right], \]
\[ n_C = \left[ \frac{(2 + \xi - \frac{\rho}{R}) + (6 - 4\xi^2 + \frac{\xi}{R}(2\xi - 2))\rho + (4 - 12\xi - 8\xi^2 + 4\xi^3 \frac{\rho}{R})\rho^2 + (-12 - 4\xi^2)\rho^3}{4(1 + \xi^2)(2\rho - 1)\rho^2} \right], \]
\[ n_D = \left[ \frac{2\xi - \frac{\rho}{R} + (2 - 4\xi^2 + \frac{\xi}{R}(2\xi - 2))\rho - 4\xi\rho^2 - 4\xi^2 \rho^3}{4\xi(2\rho - 1)\rho^2} \right], \]
where \( \zeta = \left\lfloor \frac{E}{F} \right\rfloor \) and \( \rho = \frac{F}{2} \).

Appendix D. Leading order approximations of sums in equations (48) and (49)

\[
\sum_{n=a_t+1}^{n_e} \hat{P}_2 = \frac{(h^* - L)^2}{4\rho t} \left[ \frac{(4 + 2z)\rho + (-8z - 2z^2)\rho^2 + 4z^2\rho^3}{(2z\rho - 1)(2\rho + 1)} + \ln(2z\rho - 1) \right],
\]

\[
\sum_{n=a_t+1}^{n_e} \hat{P}_3 = \frac{(h^* - L)^2}{4\rho t} \left[ \frac{2\rho(-1 + 2z(\rho - 1)(2\rho + \zeta(-1 + \rho + 2z\rho - 2\rho^2))}{(1 + 2\rho)(1 - 2z\rho^2)} \right.

\[ \left. + (2 + \zeta) \ln \left( \frac{2\rho + 1}{(2z\rho - 1)^2} \right) \right],
\]

\[
\sum_{n=a_t+1}^{n_e} \hat{P}_4 = \frac{(h^* - L)^2}{4\rho t} \left[ \frac{-2\rho(1 + z - z^2 + 2z^3\rho)(1 + \rho(1 + 2z(-1 + \rho)))}{(1 + 2\rho)(1 - 2z\rho^2)} \right.

\[ \left. + (1 + 3z + z^2) \ln(2\rho - 1) \right],
\]

\[
\sum_{n=a_t+1}^{n_e} \hat{P}_5 = \frac{(h^* - L)^2}{4\rho t} \left[ \frac{2\rho(-1 + 2z(-1 + \rho))(2\rho + \zeta(-1 + \rho + 2z\rho - 2\rho^2))}{(1 + 2\rho)(1 - 2z\rho^2)} \right.

\[ \left. + (2 + \zeta) \ln \left( \frac{2\rho + 1}{(2z\rho - 1)^2} \right) \right],
\]

\[
\sum_{n=a_t+1}^{n_e} \hat{P}_6 = \frac{(h^* - L)^2}{4\rho t} \left[ \frac{2\rho(\zeta\rho - 1)(\zeta(2\rho - 1) - 2)}{(1 + 2\rho)(2\zeta\rho - 1)} + \ln(2\zeta\rho - 1) \right],
\]

\[
\sum_{n=a_t+1}^{n_e} \hat{P}_7 = \frac{(h^* - L)^2}{(2\rho + 1)t},
\]

\[
\sum_{n=a_t+1}^{n_e} \hat{P}_8 = \frac{(h^* - L)^2}{4\rho t} \left[ \frac{(2 - 2z)\rho + (2 - 4z + 2z^2)\rho^2}{(2z\rho - 1)(2\rho + 1)} + \ln \left( \frac{2\rho + 1}{2z\rho - 1} \right) \right],
\]

\[
\sum_{n=a_t+1}^{n_e} \hat{P}_9 = \frac{(h^* - L)^2}{4\zeta t\rho} \left[ \frac{2\rho(1 + \rho + 2z^2\rho - (1 + \rho))(1 - 2z(\zeta\rho - 1))}{(1 + 2\rho)(1 - 2z\rho^2)} \right.

\[ \left. + (1 + \zeta) \ln \left( \frac{(2\zeta\rho - 1)^2}{(2\rho + 1)} \right) \right].
\]
\[ \sum_{n=n_A+1}^{n_F} \tilde{P}_4 = \frac{(h^2 - L)^2}{4\xi (1 + \xi)} \left[ \frac{4(1 + \zeta)\rho (1 - (\zeta - 1)\rho)(1 + \zeta - \zeta^2 + 2\zeta^3\rho)}{(1 + 2\rho)(1 - 2\zeta\rho)^2} \right. \\
\left. + (1 + 3\zeta + \zeta^2) \ln \left( \frac{(2\zeta\rho - 1)^2}{(2\rho + 1)^2} \right) \right], \quad (D.9) \]

\[ \sum_{n=n_C+1}^{n_F} \tilde{P}_3 = \frac{(h^2 - L)^2}{4\xi \rho} \left[ \frac{2\rho (1 + \rho + 2\zeta^2\rho - \zeta (1 + \rho))(1 - 2\zeta(\zeta\rho - 1))}{(1 + 2\rho)(1 - 2\zeta\rho)^2} \right. \\
\left. + (1 + \zeta) \ln \left( \frac{(2\zeta\rho - 1)^2}{2\rho + 1} \right) \right], \quad (D.10) \]

\[ \sum_{n=n_A+1}^{n_F} \tilde{P}_6 = \frac{(h^2 - L)^2}{4\rho t} \left[ \frac{(2 - 2\xi)\rho + (2 - 4\xi + 2\zeta^2)\rho^2}{(1 + 2\rho)(2\zeta\rho - 1)} \right] + \ln \left( \frac{2\rho + 1}{2\zeta\rho - 1} \right). \quad (D.11) \]

\[ \sum_{n=n_A+1}^{\infty} \tilde{P}_7 = \frac{(h^2 - L)^2 (\rho + 1)}{(2\rho + 1)t}, \quad (D.12) \]

where \(\zeta = \left\lfloor \frac{\theta}{\tau} \right\rfloor\) and \(\rho = \frac{\tau}{T} \).

Appendix E. Coefficients of equation (54)

The coefficients of equation (54) for the two different orderings \(\tilde{\zeta}\) and \(\bar{\zeta}\) of \(n_A - n_F\) are
\[ \tilde{\varepsilon}_1 = -2 - 4\zeta + 4\zeta^3 \quad \tilde{\varepsilon}_1 = 8 + 18\zeta + 2\zeta^2 - 8\zeta^3 \quad (E.1) \]
\[ \tilde{\varepsilon}_2 = -2(1 + \zeta + \zeta^2 + 2\zeta^3 + 6\zeta^4) \quad \tilde{\varepsilon}_2 = 8 + 12\zeta - 20\zeta^2 + 24\zeta^4 \quad (E.2) \]
\[ \tilde{\varepsilon}_3 = -12\zeta^2 - 8\zeta^3 - 4\zeta^4 + 8\zeta^5 \quad \tilde{\varepsilon}_3 = -16\zeta^2 + 8\zeta^3 + 8\zeta^4 - 16\zeta^5 \quad (E.3) \]
\[ \tilde{\varepsilon}_4 = 16\zeta^3 + 16\zeta^4 + 8\zeta^5 \quad \tilde{\varepsilon}_4 = 16(\zeta^3 + \zeta^4) \quad (E.4) \]
\[ \tilde{j}_1 = 1 + 7\zeta + 3\zeta^2 \quad \tilde{j}_1 = -4 - 8\zeta - 2\zeta^2 \quad (E.5) \]
\[ \tilde{j}_2 = -6\zeta - 2\zeta^2 \quad \tilde{j}_2 = 6 + 12\zeta + 4\zeta^2 \quad (E.6) \]

where \(\zeta = \left\lfloor \frac{\theta}{\tau} \right\rfloor\).

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