The purpose of this note is to determine the Picard group of the moduli space of vector bundles over an arbitrary algebraic surface. Since Donaldson's pioneer work on using moduli of vector bundles to define smooth invariants of an algebraic surface, there has been a surge of interest in understanding the geometry of this moduli space. Among other things, the study of line bundles on this moduli space plays a major role in this area. One important question remain open until now is to determine the Picard group of this moduli space. This is known in some special cases, for instance for projective plane [St], ruled surfaces [Qi2, Yo] and K3 surfaces [GH]. In this note, we will settle this question by providing a general construction of line bundles that will include virtually all line bundles on this moduli space, when the second Chern class of the sheaves parameterized by this moduli space is sufficiently large. The construction is again based on Knudsen and Mumford's recipe of determinant line bundle construction. The new input from this note is that instead of using complexes on surface $X$ we will use complexes on $X \times X$, which yields all previously known line bundles as well as new ones. After this construction, we will use our knowledge of the first two Betti numbers of this moduli space to argue that this construction contains virtually all line bundles on this moduli space. The proof relies heavily on the Grothendieck-Riemann-Roch theorem and the knowledge of the singularities of the moduli space.

The theorem we will prove is the following:

**Main theorem.** Let $(X, H)$ be any polarized algebraic surface and $I \in \text{Pic}(X)$. Suppose $H^2(X, \mathbb{Z})$ has no torsions, then there is an integer $N$ depending on $(X, I, H)$ such...
that for any \( d \geq N \) there is a homomorphism

\[
\Phi : \text{Pic}(X \times X)^\sigma \oplus \mathbb{Z} \longrightarrow \text{Pic}(\mathcal{M}_H(I, d)) \otimes \mathbb{Z}\left[\frac{1}{12}\right]
\]

that has finite kernel and cokernel.

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**Construction of line bundles on moduli spaces**

We begin with a general construction of line bundles on moduli space of stable sheaves based on Knudsen and Mumford’s determinant line bundle construction. One way of getting line bundles on the moduli space \( \mathbb{M} \) is by forming (perfect) complexes on \( \mathbb{M} \times X \) out of the universal sheaf \( \mathcal{F} \) and then take the determinant of its push-forward on \( \mathbb{M} \). This has been worked out by many people in various settings. It is interesting to observe that we can also construct complexes on \( \mathbb{M} \) that are push-forwards of complexes on \( \mathbb{M} \times X \times X \) constructed based on \( \pi_{12}^* \mathcal{F} \otimes \pi_{13}^* \mathcal{F} \), where \( \pi_{12}, \pi_{13} : \mathbb{M} \times X \times X \rightarrow \mathbb{M} \times X \) are projections. In this way, we recover the previously known line bundles as well as a group of new line bundles. We will work out the detail of this construction in this section.

We begin with some word on the convention that will be used throughout this paper. We let \( X \) be a fixed smooth algebraic surface over complex numbers \( \mathbb{C} \) and \( H \) a very ample line bundle on \( X \). For any choice of \( I \in \text{Pic}(X) \) and \( d \in \mathbb{Z}(\cong H^4(X; \mathbb{Z})) \), we form the (coarse) moduli scheme of \( H \)-stable rank 2 sheaves \( \mathcal{E} \) satisfying \( \det \mathcal{E} \cong I \) and \( c_2(\mathcal{E}) = d \), which will be denoted by \( \mathbb{M}_H(I, d) \). (We will abbreviated it to \( \mathbb{M}(I, d) \) when the choice of \( H \) is understood.) We will denote by \( \bar{\mathbb{M}}(I, d) \) the (coarse) moduli scheme of \( H \)-semistable sheaves satisfying the same restrain, modulo certain equivalence relation. Following Gieseker [Gi1], \( \mathbb{M}(I, d) \) is quasi-projective and \( \bar{\mathbb{M}}(I, d) \) is projective. Throughout this paper, we will use Roman letters to denote vector bundles and use Calligraphic letters to denote sheaves. We denote by \( A^i X \) (resp. \( A_i X \)) the Chow cohomology (resp. homology) group of \( X \). For any sheaf, we denote by \( \tilde{c}_i \) the Chern class taking value in the Chow cohomology group and by \( c_i \) the Chern class taking value in the (ordinary) cohomology group. We denote by \( \text{ch} \) the Chern character taking value in \( A^*_Q \), the Chow cohomology group with rational coefficient. When we assign a letter, say \( \pi \), to denote projection from a product space to its factor(s), we will use \( \pi_i \) to denote projection to the \( i \)-th copy and use \( \pi_{ij} \) to denote projection to the produce of \( i \)-th and \( j \)-th copies. Sometimes we will simply use, say \( \pi_X \), to denote projection to the factor \( X \).
Now we construct line bundles on $\mathcal{M}(I, d)$. For the moment, we assume $\mathcal{M}(I, d)$ admits a universal family, namely a sheaf $\mathcal{F}$ on $\mathcal{M}(I, d) \times X$ flat over $\mathcal{M}(I, d)$ whose restriction to fiber $\{w\} \times X$ is isomorphic to the sheaf represented by $w \in \mathcal{M}(I, d)$. Note that the choice of universal family is not unique. Any two such families differ by tensoring a pullback invertible sheaf on $\mathcal{M}(I, d)$. Let $K$ be the Grothendieck’s group of $X \times X$ and let $\tilde{K}$ be the kernel of $\xi : K \to \mathbb{Z}$,

\begin{align}
(1.1) \quad \xi(C) = \chi(C \otimes \pi_1^* \mathcal{E} \otimes \pi_2^* \mathcal{E})
\end{align}

for some $\mathcal{E} \in \mathcal{M}(I, d)$ and $\pi_i : X \times X \to X$ projection. (Here and later, we will use $\otimes$ to denote the tensor product of complexes.) We let $p_i$ be the projection from $\mathcal{M}(I, d) \times X \times X \to \mathcal{M}(I, d)$ to its factor(s) with self-explanatory index. Then we can define a homomorphism

\begin{align}
(1.2) \quad \tilde{K} \to \text{Pic}(\mathcal{M}(I, d))
\end{align}

as follows: Since the universal family $\mathcal{F}$ is flat over $\mathcal{M}(I, d)$, $\mathcal{F}$ is a perfect complex on $\mathcal{M}(I, d) \times X$. Then for any element in $\tilde{K}$ represented by a complex $C$ on $X \times X$, we can form the complex $p_{23}^* C \otimes p_{12}^* \mathcal{F} \otimes p_{23}^* \mathcal{F}$ and the push forward complex

\[ (p_1)_! (p_{23}^* C \otimes p_{12}^* \mathcal{F} \otimes p_{23}^* \mathcal{F}). \]

Following [KM], we can define the determinant line bundle

\begin{align}
(1.3) \quad \det \left( (p_1)_! (p_{23}^* C \otimes p_{12}^* \mathcal{F} \otimes p_{23}^* \mathcal{F}) \right).
\end{align}

The homomorphism (1.2) is the one sending $C \in \tilde{K}$ to the line bundle (1.3). Observe that if we replace $\mathcal{F}$ by a different universal family, say $\mathcal{F}' = \mathcal{F} \otimes \pi_1^* \mathcal{O}(L)$, $\pi_1 : \mathcal{M}(I, d) \times X \to \mathcal{M}(I, d)$, then

\[ \det \left( (p_1)_! (p_{23}^* C \otimes p_{12}^* \mathcal{F}' \otimes p_{23}^* \mathcal{F}') \right) = L^{\otimes \left( 2\xi(C) \right)} \otimes \det \left( (p_1)_! (p_{23}^* C \otimes p_{12}^* \mathcal{F} \otimes p_{23}^* \mathcal{F}) \right), \]

where $\xi(C)$ is as in (1.1). Hence (1.2) is independent of the choice of the universal family since $C \in \tilde{K}$ and then the homomorphism (1.3) is well-defined.

We will see shortly that the homomorphism (1.2) is far from injective. In the following we will use the Grothendieck-Riemann-Roch theorem (abbreviate GRR theorem) to give an explicit description of the image of (1.2). Since $p_1 : \mathcal{M}(I, d) \times X \times X \to \mathcal{M}(I, d)$ is smooth,

\begin{align}
(1.4) \quad \text{ch} \left( (p_1)_! (p_{23}^* C \otimes p_{12}^* \mathcal{F} \otimes p_{13}^* \mathcal{F}) \right) = p_{1!*} \left( \text{ch}(p_{23}^* C \otimes p_{12}^* \mathcal{F} \otimes p_{13}^* \mathcal{F}) \cdot p_{23}^* \text{td}(X \times X) \right).
\end{align}
by [Fu, Example 18.3.10]. By choosing \( C \in \tilde{K} \), the summand in \( A^0(\mathfrak{M}(I, d))_\mathbb{Q} \) of (1.4) is trivial and the summand in \( A^1(\mathfrak{M}(I, d))_\mathbb{Q} \) reads

\[
\tilde{c}_1 \left( (p_1)_*(p_{23}^*C \boxtimes p_{12}^*F \boxtimes p_{13}^*F) \right) = [p_1_*(\text{ch}(p_{23}^*C \boxtimes p_{12}^*F \boxtimes p_{13}^*F) \cdot p_{23}^*\text{td}(X \times X))]_{[1]},
\]

(1.5)

We will use \([\cdot]_{[i]}\) to denote its component in \( A^i \). A simple calculation shows that it is of the form

\[
[p_1_*(\left((p_{12}^*c_2(F) + p_{13}^*c_2(F)) \cdot \alpha_1 + (p_{12}^*c_3(F) + p_{13}^*c_3(F)) \cdot \alpha_2 \\
+ p_{12}^*c_2(F) \cdot p_{13}^*c_2(F) \cdot \alpha_3 + (p_{12}^*c_2(F) \cdot p_{13}^*c_3(F) + p_{13}^*c_2(F) \cdot p_{12}^*c_3(F)) \cdot \alpha_4)]]_{[1]},
\]

for some \( \alpha \in A^*(X \times X)_\mathbb{Q} \) based on type consideration and

\[
\text{ch}(F) \equiv 2 - \tilde{c}_2(F) + \frac{1}{2} \tilde{c}_3(F) \mod \tilde{c}_1(F).
\]

Hence (1.5), which is the Chern class of (1.3), is contained in the linear span of the images

\[
p_{M*}(\tilde{c}_2(F) \cdot p_X^*(\cdot)) : A^1(X) \rightarrow A^1(\mathfrak{M}(I, d))_\mathbb{Q},
\]

(1.6)

\[
p_{M*}(\tilde{c}_3(F) \cdot p_X^*(\cdot)) : A^0(X) \rightarrow A^1(\mathfrak{M}(I, d))_\mathbb{Q}
\]

(1.7)

\((p_M, p_X)\) are projections of \( \mathfrak{M}(I, d) \times X \) and

\[
p_{1*}(p_{12}^*\tilde{c}_2(F) \cdot p_{13}^*\tilde{c}_2(F) \cdot p_{23}^*(\cdot)) : A^1(X \times X) \rightarrow A^1(\mathfrak{M}(I, d))_\mathbb{Q}.
\]

(1.8)

It is easy to check that the image of (1.6) is contained in the image of (1.8) when \( 4d \geq I^2 \), which we assume in the sequel. Thus the Chern class of (1.3) is contained in the image of

\[
A^1(X \times X)_\mathbb{Q} \oplus \mathbb{Q} \rightarrow A^1(\mathfrak{M}(I, d))_\mathbb{Q}
\]

(1.9)

that is a direct sum of (1.8) and (1.7).

We remark that since \( \mathfrak{M}(I, d) \) is singular in general, the homomorphism given by the first Chern class

\[
\tilde{c}_1 : \text{Pic}(\mathfrak{M}(I, d)) \rightarrow A^1(\mathfrak{M}(I, d))_\mathbb{Q}
\]
may have infinite kernel. However, (1.9) suggests that very likely there should be a homomorphism

\[(1.10) \quad \Phi : \text{Pic}(X \times X)^\sigma \oplus \mathbb{Z} \longrightarrow \text{Pic}(\mathcal{M}(I,d))\]

compactible to (1.9) that will capture all (up to finite index) line bundles constructed in (1.3). Here \(\sigma : X \times X \rightarrow X \times X\) is the map exchanging factor and \(\text{Pic}(X \times X)^\sigma\) consists of line bundles invariant under \(\sigma\). We use \(\text{Pic}(X \times X)^\sigma\) instead of \(\text{Pic}(X \times X)\) in (1.10) since the map (1.2) so constructed is \(\sigma\) invariant.

We now construct this homomorphism explicitly. We first let

\[\psi : \text{Pic}(X \times X)^\sigma \oplus \mathbb{Z} \longrightarrow \tilde{K} \otimes \mathbb{Z} [\frac{1}{12}]\]

be the map sending \(L \in \text{Pic}(X \times X)^\sigma\) to

\[(\mathcal{O}_{X \times X} - \mathcal{O}(L)) + \frac{1}{2} (\mathcal{O}_{X \times X} - \mathcal{O}(L))^2 - \frac{1}{3} (\mathcal{O}_{X \times X} - \mathcal{O}(L))^3 + \alpha \mathbb{C}_{p_0} \in \tilde{K} \otimes \mathbb{Q}\]

and sending \(m \in \mathbb{Z}\) to

\[C_0 = m\mathcal{O}_{X \times \{x_0\}} + \alpha \mathbb{C}_{p_0} \in \tilde{K} \otimes \mathbb{Q},\]

where \(p_0 \in X \times X\) is a fixed point and \(\alpha \in \mathbb{Z}[\frac{1}{12}]\) is chosen so that \(\psi(L)\) and \(\psi(m)\) lie in \(\tilde{K} \otimes \mathbb{Z}[\frac{1}{12}]\). (\(\alpha \in \mathbb{Z}[\frac{1}{12}]\) follows from \(L\) being \(\sigma\)-invariant.) Note that this choice gives

\[\text{ch}(\psi(L)) = \tilde{c}_1(L) + \alpha [p_0]^\vee.\]

We then define \(\Phi(\alpha), \alpha \in \text{Pic}(X \times X)^\sigma \oplus \mathbb{Z},\) to be the \(\mathbb{Q}\)-line bundle

\[\det \left( (p_1)_* (p_{23}^* \psi(L) \boxtimes p_{12}^* F \boxtimes p_{13}^* F) \right) \in \text{Pic}(\mathcal{M}(I,d)) \otimes \mathbb{Z}[\frac{1}{12}],\]

This way we have defined

\[(1.11) \quad \Phi : \text{Pic}(X \times X)^\sigma \oplus \mathbb{Z} \longrightarrow \text{Pic}(\mathcal{M}(I,d)) \otimes \mathbb{Z}[\frac{1}{12}].\]

\(\Phi\) is a homomorphism, which can be proved similar to that on page 422 of [Li1] despite \(\psi\) is not necessarily a homomorphism. (One hint that this is true comes from the identity \(\text{ch}(\psi(L)) = \tilde{c}_1(L) + \alpha [p_0]^\vee\).) We shall omit the proof here since it is rather routine.
In the remainder of this section, we shall investigate how to define this homomorphism when there are no universal family on \( \mathcal{M}(I,d) \times X \) and under what condition does it extend to 

\[
\mathcal{F} : \text{Pic}(X \times X)^{\sigma} \oplus \mathbb{Z} \to \text{Pic}(\mathcal{M}(I,d)) \otimes \mathbb{Z}[\frac{1}{12}].
\]

We will accomplish this by taking \( \mathcal{M}(I,d) \) as a GIT quotient of a Quot-scheme and then apply the descent theory. Following [Gi1, Ma], there is a Quot-scheme \( \text{Quot} \) with a linear \( G = SL(N) \) action such that if we denote by \( \text{Quot}^{s} \) (resp. \( \text{Quot}^{ss} \)) the open subset of \( G \)-stable (resp. \( G \)-semistable) points, then 

\[
\text{Quot}^{s}/G = \mathcal{M}(I,d) \quad \text{and} \quad \text{Quot}^{ss}/G = \overline{\mathcal{M}}(I,d).
\]

Let \( \mathcal{E} \) be a universal quotient family on \( \text{Quot}^{ss} \times X \). Then we can define determinant line bundles on \( \text{Quot}^{ss} \) as we did for \( \mathcal{M}(I,d) \) by using the family \( \mathcal{E} \) and complexes in \( \tilde{\mathcal{K}} \). Let \( \tilde{p} \) be projections from \( \text{Quot}^{ss} \times X \times X \) to its factors with self-explanatory index. Then for any \( C \in \tilde{\mathcal{K}} \), we get a line bundle on \( \text{Quot}^{ss} \):

\[
L(C) = \det \left( (\tilde{p}_1)_{\ast}(\tilde{p}_{23}^{\ast}C \boxtimes \tilde{p}_{12}^{\ast}\mathcal{E} \boxtimes \tilde{p}_{13}^{\ast}\mathcal{E}) \right).
\]

\( L(C) \) admits a canonical \( G \)-linearization since the \( G \)-action on \( \text{Quot}^{ss} \) lifts to a \( G \)-action on \( \mathcal{E} \). By descent theory, \( L(C) \) descends to \( \mathcal{M}(I,d) \) if for any \( w \in \text{Quot}^{s} \) the stabilizer \( \text{stab}_w \subseteq G \) of \( w \) acts trivially on the fiber of \( L(C) \) over \( w \). For the same reason, it descends to \( \overline{\mathcal{M}}(I,d) \) if similar condition holds for all \( w \in \text{Quot}^{ss} \) having closed orbits \( G \cdot w \).

**Definition 1.1.** An ample line bundle \( H \) is said to be \((I,d)\)-generic if whenever \( J \in \text{Pic}(X) \) satisfies \((2c_1(J) - c_1(I)) \cdot c_1(H) = 0\), then \(-4c_1(J)^2 > 4d - c_1(I)^2\) unless \( c_1(J) \) is torsion.

Note that if \( H \) is \((I,d)\)-generic, then any strictly semistable sheaf \( \mathcal{F} \in \overline{\mathcal{M}}(I,d) \) is \( S \)-equivalent to \( \mathcal{F}_1 \oplus \mathcal{F}_2 \) with \( 2c_1(\mathcal{F}_1) \equiv c_1(I) \) modulo torsions.

**Proposition 1.2.** For any \( C \in \tilde{K} \), the line bundle \( L(C) \) descends to \( \mathcal{M}(I,d) \). Further if \( H \) is \((I,d)\)-generic, then it descends to \( \overline{\mathcal{M}}(I,d) \).

**Proof.** For stable point \( w \in \text{Quot}^{s} \), the stabilizer \( \text{stab}_w \) is \( \mathbb{Z}/N\mathbb{Z} \) acting on \( \mathcal{E} \otimes k(w) \) via multiplication by an \( N \)-th root of unity, say \( \epsilon \). Here \( \mathcal{E} \otimes k(w) \) is the restriction of \( \mathcal{E} \) to the fiber \( \{w\} \times X \). Then its induced action on \( L(C)_w \) (which is the restriction to the fiber over \( w \)) is a multiplication by \( \epsilon^m \), where \( m \) is given by (1.1) that is 0 since \( C \in \tilde{K} \). Therefore by descent theory \( L(C) \) descends to \( \mathcal{M}(I,d) \).
We now study descent problem on $\mathcal{M}(I,d)$. Let $w \in \text{Quot}^{ss}$ be a strictly semistable point with closed orbit $G \cdot w$. Then $\mathcal{E} \otimes k(w)$ is a direct sum of rank one sheaves, say $\mathcal{F}_1 \oplus \mathcal{F}_2$. If we further assume $H$ is $(I,d)$-generic, then $c_1(\mathcal{F}_1)$ and $c_1(\mathcal{F}_2)$ differ by a torsion element. In this case, it is straightforward to check, as did on page 426 of [Li1], that $\text{stab}_w$ acts trivially on $L(C)_w$. Therefore, $L(C)$ descends to $\mathcal{M}(I,d)$ as claimed. □

**Remark.** When $H$ is not $(I,d)$-generic, one can determine explicitly which line bundles do descend by using the descent argument, if one knows all strictly semistable sheaves.

Knowing that (1.12) always descends to $\mathcal{M}(I,d)$, we can define a homomorphism

$$\Phi : \text{Pic}(X \times X)^{\sigma} \oplus \mathbb{Z} \to \text{Pic}(\mathcal{M}(I,d)) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{12}]$$

that sends $\alpha$ in $\text{Pic}(X \times X)^{\sigma} \oplus \mathbb{Z}$ to the descent of the determinant line bundle associated to the complex $\psi(\alpha)$. Similarly, if $H$ is $(I,d)$-generic then the same construction yields a homomorphism

$$\overline{\Phi} : \text{Pic}(X \times X)^{\sigma} \oplus \mathbb{Z} \to \text{Pic}(\overline{\mathcal{M}}(I,d)) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{12}]$$

(1.13)

2. Injectivity of the homomorphism $\Phi$

In this section, we will show that for sufficiently large $d$, the homomorphism

$$\Phi : \text{Pic}(X \times X)^{\sigma} \oplus \mathbb{Z} \to \text{Pic}(\mathcal{M}(I,d)) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{12}]$$

has finite kernel. The method we will use is to construct a variety $W$ and a morphism

$$g : W \to \mathcal{M}(I,d)$$

(2.1)

and show that the composition

$$g^* \circ \Phi : \text{Pic}(X \times X)^{\sigma} \oplus \mathbb{Z} \to \text{Pic}(W) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{12}]$$

has finite kernel, using GRR theorem.

The variety $W$ we shall use is the one parameterizing non-locally free sheaves that are kernels of $\mathcal{E} \to \mathbb{C}_x \oplus \mathbb{C}_y$, where $\mathcal{E}$ is fixed and $x, y \in X$ are arbitrary closed points.
$W$ will be birational to a $\mathbb{P}^1 \times \mathbb{P}^1$ bundle over $X \times X$. Let $\mathcal{E}$ be a rank two locally free sheaf on $X$ and let $\mathbb{P}(\mathcal{E})$ be the associated projective bundle with projection

$$\pi : \mathbb{P}(\mathcal{E}) \to X$$

(we adopt the convention in [Ha, p162] that $\pi_*(\mathcal{O}(1)) = \mathcal{E}$) and let $W$ be the blowing-up of $\mathbb{P}(\mathcal{E}) \times \mathbb{P}(\mathcal{E})$ along the diagonal. We first construct a sheaf on $W \times X$ flat over $W$. Let $q_i$ be the projection

$$q_i : W \to \mathbb{P}(\mathcal{E}) \times \mathbb{P}(\mathcal{E}) \xrightarrow{pr_i} \mathbb{P}(\mathcal{E}), \quad i = 1, 2$$

and let

$$\iota_i : \text{id} \times (\pi \circ q_i) : W \to W \times X, \quad i = 1, 2.$$  

$\iota_i$ is a closed immersion. We let

$$\beta : \pi_X^* \mathcal{E} \to \iota_1^* q_1^* \mathcal{O}(1) \oplus \iota_2^* q_2^* \mathcal{O}(1)$$

be the homomorphism induced by

$$\pi_X^* \mathcal{E} \to \pi_X^* \mathcal{E} \otimes \mathcal{O}_{\iota_i(W)} = \iota_i^* q_i^* \pi^* \mathcal{E} \to \iota_i^* q_i^* \mathcal{O}(1),$$

where $\pi_X : W \times X \to X$ is the second projection. $\beta$ is surjective away from $S \times X$, where $S$ is the exceptional divisor of $W$. We let $G$ be the kernel of $\beta$ and let $\Omega$ be the cokernel of

$$G \to \pi_X^* \mathcal{E}.$$  

In this way, we obtain two exact sequences:

$$(2.3) \quad 0 \to G \to \pi_X^* \mathcal{E} \to \Omega \to 0$$

and

$$(2.4) \quad 0 \to \Omega \to \iota_1^* q_1^* \mathcal{O}(1) \oplus \iota_2^* q_2^* \mathcal{O}(1) \to \iota_1^* q_1^* \mathcal{O}(1)_{|S \times X} \to 0$$

since coker($\beta$) is isomorphic to the last term in (2.4).

We claim that $G$ is a family of torsion free sheaves flat over $W$. Because of the exact sequence (2.3), it suffices to show that $\Omega$ is flat over $W$ and

$$\text{Tor}(\Omega, \mathcal{O}_{(w) \times X}) = 0$$
for all closed \( w \in W \). But this is clear from the exact sequence (2.4) because \( \iota_i(W) \subseteq W \times X \) is smooth over \( W \), \( q_1^*\mathcal{O}(1) \) is an invertible sheaf on \( W \) and \( \iota_i(W) \cap (S \times X) \) is a divisor in \( \iota_i(W) \). This proves the claim.

Following the discussion in §1, we can form a homomorphism

\[
\Phi_W : \text{Pic}(X \times X)\sigma \oplus \mathbb{Z} \longrightarrow \text{Pic}(W)\otimes\mathbb{Z}[\frac{1}{12}]
\]

using the determinant line bundle construction outlined before (1.11) using the sheaf \( G \) over \( W \times X \) and complexes

\[
\psi : \text{Pic}(X \times X)\sigma \oplus \mathbb{Z} \longrightarrow \check{K} \otimes\mathbb{Z}[\frac{1}{12}],
\]

We will investigate the injectivity of this homomorphism by looking at their first Chern classes, using GRR. From the formula (1.5) and the one after, for \( L \in \text{Pic}(X \times X)\sigma \)

\[
\hat{c}_1(\Phi_W(L)) = \left[ p_{1*}\left( \text{ch}(p_{23}^*\psi(L) \boxtimes p_{12}^*G \boxtimes p_{13}^*G) \cdot p_{23}^*\text{td}(X \times X) \right) \right]_{[1]},
\]

where \( p_{1} \) are projections from \( W \times X \times X \) to its factor(s) with self-explanatory index. Since fibers of \( p_{1} \) are \( X \times X \), it suffices to determine

\[
\left[ \text{ch}(p_{23}^*\psi(L) \boxtimes p_{12}^*G \boxtimes p_{13}^*G) \cdot p_{23}^*\text{td}(X \times X) \right]_{[5]}
\]



\[
= \left[ p_{23}^*(\hat{c}_1(L) + \alpha[p_{0}]^\vee) \cdot p_{12}^*\text{ch}(G) \cdot p_{13}^*\text{ch}(G) \cdot p_{2}^*\text{td}(X) \cdot p_{3}^*\text{td}(X) \right]_{[5]}
\]

modulo the kernel of \( p_{1*} \). We first determine \( \text{ch}(G) \). By using (2.3) and (2.4) we get

\[
\text{ch}(G) = \text{ch}(\pi_X^*\mathcal{E}) - \text{ch}(\iota_1* q_1^*\mathcal{O}(1)) - \text{ch}(\iota_2* q_2^*\mathcal{O}(1)) + \text{ch}(\iota_1* q_1^*\mathcal{O}(1)_{S \times X}).
\]

(\( \pi \) is a projection from \( W \times X \) to its factor.) Applying GRR theorem to the proper morphism \( \iota_1 : W \to W \times X \), we obtain

\[
\text{ch}(\iota_1* q_1^*\mathcal{O}(1)) = \iota_1* \left( \text{ch}(q_1^*\mathcal{O}(1)) \cdot \text{td}(W) \right) \cdot \text{td}(W \times X)^{-1}
\]

\[
= \iota_1* q_1^* \hat{e}_1(\mathcal{O}(1)) \cdot \pi_W^* \text{td}(W) \cdot \text{td}(W \times X)^{-1}
\]

\[
= \iota_1* q_1^* \hat{e}_1(\mathcal{O}(1)) \cdot \pi_X^* \text{td}(X)^{-1}.
\]

Here the second identity holds because \( \iota_1(W) \) is a graph of \( \pi \circ q_1 : W \to X \). Similarly, we have

\[
\text{ch}(\iota_2* q_2^*\mathcal{O}(1)) = \iota_2* q_2^* \hat{e}_1(\mathcal{O}(1)) \cdot \pi_X^* \text{td}(X)^{-1}.
\]
In the following, we will choose a split $\mathcal{E}$ (i.e. $\mathcal{E}$ is a direct sum of invertible sheaves) to simplify our calculation. We assume $\det \mathcal{E} = I$. In this case, $\mathbb{P}(\mathcal{E})$ is isomorphic to $\mathbb{P}^1 \times X$, $\tilde{c}_1(\mathcal{O}(1))$ is dual to a section of $\mathbb{P}(\mathcal{E}) \to X$ and $\tilde{c}_1(\mathcal{O}(1))^2 = 0$.

We now reorganize terms in the expansion of (2.6) in terms of (2.7) for split $\mathcal{E}$. One term in the expansion of (2.6) is

$$\left[p_{23}^*(\tilde{c}_1(L) + \alpha[p_0])^\vee \cdot (p_{12} \circ \pi_X)^* \text{ch}(\mathcal{E}) \cdot (p_{13} \circ \pi_X)^* \text{ch}(\mathcal{E})\right]_{\overline{5}}$$

which is contained in the kernel of $p_{1*}$, following the vanishing of (1.1). (Recall $p_*$ is designated to projections from $W \times X \times X$ to its factor, $\pi_*$ is projection from $W \times X$ to its factor and $q_*$ is projection from $W$ to $\mathbb{P}(\mathcal{E})$.) The next terms are

$$\left[p_{23}^*(\tilde{c}_1(L) \cdot p_{12}^* (\iota_i \circ q_i^* e_{\tilde{c}_1(\mathcal{O}(1))}) \cdot \pi_X^* \text{td}(X)^{-1}) \cdot p_{13}^* (\iota_j \circ q_j^* e_{\tilde{c}_1(\mathcal{O}(1))}) \cdot \pi_X^* \text{td}(X)^{-1}) \cdot p_2^* \text{td}(X) \cdot p_3^* \text{td}(X)\right]_{\overline{5}},$$

where $(i, j)$ runs through pairs $(1, 1)$, $(1, 2)$, $(2, 1)$ and $(2, 2)$. After simplification, their images under $p_{1*}$ are

$$p_{1*}\left(p_{23}^* \iota_i(W) \cdot p_{12}^* [\iota_i(W)]^\vee \cdot p_{13}^* [\iota_j(W)]^\vee \right) = \begin{cases} q_{XX}^* \tilde{c}_1(L), & i \neq j; \\ q_{XX}^* \text{pr}_1^* (\tilde{c}_1(L|\Delta_1)), & i, j = 1; \\ q_{XX}^* \text{pr}_2^* (\tilde{c}_1(L|\Delta_2)), & i, j = 2, \end{cases}$$

since $L$ is invariant under permutation $\sigma$. Here $q_{XX} : W \to X \times X$ is the projection, $\Delta \subseteq X \times X$ is the diagonal, $\tilde{c}_1(L|_{\Delta})$ is in $A^1X$ by the isomorphism $\Delta = X$ and $\pi : X \times X \to X$ is projection. Note that $q_{XX}^* \tilde{c}_1(L)$ appears in the expression.

We need to take care of the remainder terms. We first look at another series of terms

$$(2.8) \quad - p_{23}^* \tilde{c}_1(L) \cdot (p_{11} \circ \pi_X)^* \text{ch}(\mathcal{E}) \cdot p_{1i}^* (\iota_1 \circ q_1^* e_{\tilde{c}_1(\mathcal{O}(1))}) \cdot \pi_X^* \text{td}(X)^{-1}$$

$$+ \iota_2 \circ q_2^* e_{\tilde{c}_1(\mathcal{O}(1))} \cdot \pi_X^* \text{td}(X)^{-1}) \cdot p_2^* \text{td}(X) \cdot p_3^* \text{td}(X)\right]_{\overline{5}},$$

where $(i, \bar{i})$ is either $(2, 3)$ or $(3, 2)$. For simplicity, we let $\alpha_2$ and $\alpha_3 \in A^*(X \times X)$ be such that

$$- p_{23}^* \alpha_i = p_{23}^* \tilde{c}_1(L) \cdot p_i^* \text{ch}(\mathcal{E}) \cdot p_i^* \text{td}(X).$$

For our purpose, it is sufficient to look at their components in $q_{XX}^* A^1(X \times X)$ under the decomposition

$$A^1(W) = q_{XX}^* A^1(X \times X) \oplus \mathbb{Z}[R_1]^\vee \oplus \mathbb{Z}[R_2]^\vee \oplus \mathbb{Z}[S]^\vee,$$
where $R_1, R_2 \subseteq W$ are birational to $P^1 \times \{pt\} \times X \times X$ and $\{pt\} \times P^1 \times X \times X$ respectively, $S \subseteq W$ is the exceptional divisor of $W \to P(\mathcal{E}) \times P(\mathcal{E})$. (Recall $W$ is a blowing up of $P^1 \times P^1 \times X \times X$ since $\mathcal{E}$ is split.) Since $\iota(W)$ is a graph of $W \to X$, the component in $q^*_X A^1(X \times X)$ of the image under $p_{1*}$ of (2.8) are

\[(2.9) \quad p_{1*}\left(p_{23}^* \alpha_2 \cdot p_{13}^*([\iota_1(W)]^\vee + [\iota_2(W)]^\vee)\right)_{[1]}\]

and

\[(2.10) \quad p_{1*}\left(p_{23}^* \alpha_3 \cdot p_{12}^*([\iota_1(W)]^\vee + [\iota_2(W)]^\vee)\right)_{[1]}\]

To simplify these, we proceed as follows: Consider the projection $f : W \to X \times X$ (from $W \to P(\mathcal{E}) \times P(\mathcal{E})$) and the projection $P = (f, \text{id}_X, \text{id}_X) : W \times X \times X \to X \times X \times X \times X$.

Then by definition of $\iota_1$ and $\iota_2$, we have

\[p_{12}^*([\iota_j(W)]^\vee) = P^* \circ \pi_{j3}^* ([|\Delta|]^\vee)\]

and

\[p_{13}^*([\iota_j(W)]^\vee) = P^* \circ \pi_{j4}^* ([|\Delta|]^\vee),\]

where $\pi_{ij} : X \times X \times X \times X \to X \times X$ is a projection. Then the component in $A^1(X \times X)$ of (2.9) is the pull back (under $q^*_X$) of

\[\pi_{12*} \circ P_* \left(\pi_{14}^* ([|\Delta|]^\vee + \pi_{24}^* ([|\Delta|]^\vee))\right)_{[1]},\]

where $\Sigma$ is a birational section of $W \to X \times X$. By projection formula, it is

\[\pi_{12*} \circ P_* \left(\pi_{14}^* ([|\Delta|]^\vee + \pi_{24}^* ([|\Delta|]^\vee))\right)_{[1]} = \left[\pi_{12}^* \left(\pi_{34}^* (\alpha_2) \cdot (\pi_{14}^* ([|\Delta|]^\vee) + \pi_{24}^* ([|\Delta|]^\vee))\right)\right]_{[1]},\]

Similarly, (2.10) is

\[\pi_{12}^* \left(\pi_{34}^* (\alpha_3) \cdot (\pi_{14}^* ([|\Delta|]^\vee) + \pi_{24}^* ([|\Delta|]^\vee))\right)_{[1]} = \left[\pi_{12}^* \left(\pi_{34}^* (\alpha_2) \cdot (\pi_{14}^* ([|\Delta|]^\vee) + \pi_{24}^* ([|\Delta|]^\vee))\right)\right]_{[1]},\]

Note that $\sigma^*(\alpha_2) = \sigma_3$, where $\sigma$ is the permutation. Hence both are

\[\pi_{12}^* (\alpha) + \pi_{24}^* (\alpha), \quad \alpha = \pi_{12}^* (\alpha_2).\]
The image under $p_{1*}$ of the last set of terms in (2.6) contains factor $\text{ch}(\iota_* q^*_X \mathcal{O}(1)_{S \times X})$ whose components in $q_X^* A^1(X \times X)$ are trivial. Put them together, we see that the component in $q_X^* A^1(X \times X)$ of $\tilde{c}_1(\Phi_W(L))$ is the pull-back of

\begin{equation}
2 \tilde{c}_1(L) + \text{pr}_1^* (\tilde{c}_1(L|\Delta)) + \text{pr}_2^* (\tilde{c}_1(L|\Delta)) + 2 \text{pr}_1^* (\alpha) + 2 \text{pr}_2^* (\alpha) \in A^1(X \times X).
\end{equation}

Now we look at the component in $q_X^* A^1(X \times X)$ of $\tilde{c}_1(\Phi_W(1))$. By replacing $\tilde{c}_1(L)$ in (2.6) with $[X \times \{x_0\} + \{x_0\} \times X] \setminus$ and going through the same line of analysis, we see that there is a $\beta \in A^1(X)$ depending only on $X$ and $I$ such that the component in $\tilde{c}_1(\Phi_W(1))$ is

\begin{equation}
q_X^* (\text{pr}_1^* \beta + \text{pr}_2^* \beta).
\end{equation}

Therefore, if $(L, n) \in \text{Pic}(X \times X)^\sigma \oplus \mathbb{Z}$ makes

$$\tilde{c}_1(\Phi((L, n))) = 0,$$

then $\tilde{c}_1(L)$ must be of the form $\text{pr}_1^* c + \text{pr}_2^* c$ for some $c \in A^1(X)$ satisfying the identity

\begin{equation}
F(c) + c_2(E)c + n\beta = 0,
\end{equation}

where $F : A^1 X \rightarrow A^1 X$ is a map depending on $X$ and $I$ but not $E$, from (2.11) and (2.12). (The exact form of (2.13) can be derived easily from (2.11) and (2.12) but we choose not to in part to emphasize the term $c_2(E)c$ that is crucial in our later discussion.)

Now we state and prove the main proposition of this section:

**Proposition 2.1.** For any choice of $(I, H)$, there is an $N$ such that whenever $d \geq N$, then the homomorphism

$$\Phi : \text{Pic}(X \times X)^\sigma \oplus \mathbb{Z} \longrightarrow \text{Pic}((\mathcal{M}(I, d)) \otimes \mathbb{Z}[\frac{1}{12}])$$

has finite kernel.

**Proof.** We let $\mathcal{E}_1$ be $\mathcal{O} \oplus \mathcal{O}(I)$ and let $\mathcal{E}_2$ be $\mathcal{O}(H) \oplus \mathcal{O}(I \otimes H^{-1})$. (When $H^2 = H \cdot I$ then we let $\mathcal{E}_2$ be $\mathcal{O}(H^{-1}) \oplus \mathcal{O}(I \otimes H)$.) Let $W_i$ be the variety constructed at the beginning of this section that is the blowing-up of $\mathbb{P}(\mathcal{E}_i) \times \mathbb{P}(\mathcal{E}_i)$ and let $\mathcal{G}_i$ be the sheaf on $W_i \times X$ constructed in (2.3). We claim that if

$$(L, n) \in \text{Pic}(X \times X)^\sigma \oplus \mathbb{Z}$$

then

$$\Phi((L, n)) = 0.$$
lies in the kernel of $\Phi_W$ in (2.5) for $i = 1, 2$, then $n = 0$ and $L$ is torsion. Indeed, by the previous argument, $\Phi_W((L, n)) = 0$ implies that $\tilde{c}_1(L) = \text{pr}_1^* c + \text{pr}_2^* c$ with $c \in A^1 X$ satisfying two identities that are (2.8) with $c_2(\mathcal{E})$ replaced by $c_2(\mathcal{E}_1) = 0$ and by $c_2(\mathcal{E}_2) \neq 0$. Clearly, this is possible only if $c$ is a torsion. Once we know that $c$ is torsion, then $\Phi_W((L, n)) = 0$ force $n = 0$, which can be checked directly. This establishes the claim.

Now we prove the proposition. Let $(L, n)$ be any element in the kernel of $\Phi$. It suffices to show that $\Phi_W((L, n)) = 0$ for both $i$. Let $\mathcal{E}$ be either $\mathcal{E}_1$ or $\mathcal{E}_2$. Then there is a constant $N \geq 0$ such that for any $d \geq N$ there is a subsheaf $\mathcal{F} \subseteq \mathcal{E}$ such that $\det \mathcal{F} = I$ and $c_2(\mathcal{F}) = d$ satisfying the following property: $\mathcal{F}$ admits a deformation, say $\mathcal{F}_t$, whose general members are locally free and $\mu$-stable. This can be proved as follows: First for large $d$, we can choose $\mathcal{F} \subseteq \mathcal{E}$ so that the traceless part $\text{Ext}^2(\mathcal{F}, \mathcal{F})^0 = 0$. Then we can deform $\mathcal{F}$ to locally free sheaves. When $d$ is large enough and the support of $\mathcal{E}/\mathcal{F}$ is in general position, then the argument on page 158 of [Gi2] shows that any locally free deformation of $\mathcal{F}$ are $\mu$-stable. This proves the existence of such deformation. Now let $0 \in T$ be a smooth curve and $\mathcal{F}_T$ a deformation of $\mathcal{F}_0 = \mathcal{F}$ just mentioned. Without loss of generality, we assume $\mathcal{F}_T$ is locally free over $T - 0$. Let $\Lambda \subseteq X$ be the finite set over which $\mathcal{F}_0$ is not locally free.

We now construct the corresponding family of varieties $W_T$ associated to $\mathcal{F}_T$. Let $\mathbb{P}(\mathcal{F}_T)$ be the projective bundle of $\mathcal{F}_T$ over $T \times X - \{0\} \times \Lambda$ and let $W_T$ be the blowing-up of the diagonal of $\mathbb{P}(\mathcal{F}_T) \times_T \mathbb{P}(\mathcal{F}_T)$. Similar to (2.3), we can define a sheaf $\mathcal{G}_T$ on $W_T \times X$ that is flat over $W_T$. Note that since $\mathcal{F}_t$ is $\mu$-stable for $t \neq 0$, the family $\mathcal{G}_{T^*}, T^* = T - 0$, that is the restriction of $\mathcal{G}_T$ to $W_T - W_0$ ($W_0$ is the fiber over $0 \in T$) is a family of $H$-stable sheaves with determinant $I$ and second Chern class $d$. Hence, it induces a morphism

$$\mu : W_T - W_0 \rightarrow \mathcal{M}(I, d).$$

Also, by applying the determinant line bundle construction to the family $\mathcal{G}_T$, we can form a homomorphism

$$\Phi_{W_T} : \text{Pic}(X \times X)^\sigma \oplus \mathbb{Z} \rightarrow \text{Pic}(W_T) \otimes \mathbb{Z}[\frac{1}{12}]$$

similar to (2.5). Since this construction commutes with base change,

$$\Phi_{W_T}((L, n))|_{W_T - W_0} \cong \mu^* \Phi((L, n)).$$

In particular, since $\Phi((L, n))$ is trivial, $\Phi_{W_T}((L, n))|_{W_T - W_0}$ is trivial and consequently $\Phi_{W_T}((L, n))|_{W_0}$ is trivial.
Assume $\mathcal{F}^\vee = \mathcal{E}$ is $\mathcal{E}_1$ we begin with. Then $W_0$ is canonically isomorphic to $W_1 - A$ for some closed $A \subseteq W_1$ of codimension two and further,

$$\Phi_{W_0}((L, n))_{|W_0} \cong \Phi_{W_1}((L, n))_{|W_1 - A}.$$ 

Hence for $(L, n) \in \ker(\Phi)$, $\Phi_{W_1}((L, n))$ is trivial as well, since $\text{codim}(A) = 2$ and $W_1$ is smooth. Similarly, if we choose $\mathcal{F}$ so that $\mathcal{F}^\vee = \mathcal{E}_2$, then we must have $\Phi_{W_2}((L, n)) = 0$. This shows that $\Phi_{W_i}((L, n))$ are trivial for $i = 1, 2$ and henceforth establishing the proposition. □

3. Surjectivity of $\Phi$

In this section, we shall show that when $H^2(X, \mathbb{Z})$ has no torsion, then for sufficiently large $d$ the so constructed homomorphism

$$\Phi : \text{Pic}(X \times X)^\sigma \oplus \mathbb{Z} \longrightarrow \text{Pic}(\mathcal{M}(I, d)) \otimes \mathbb{Z}[\frac{1}{12}]$$

has finite cokernel. The tactic we will use is to first show that under certain circumstances the restriction homomorphism

$$\text{Pic}(\mathcal{M}(I, d)) \longrightarrow \text{Pic}(\mathcal{M}(I, d))$$

is surjective, by studying extension problem. We will then use the knowledge of the first two Betti numbers of $\mathcal{M}(I, d)$ to determine the Picard group of $\mathcal{M}(I, d)$, up to finite index.

We first look at the problem of extending line bundle on $\mathcal{M}(I, d)$ to $\mathcal{M}(I, d)$. This in general is rather tricky due to the singularities of $\mathcal{M}(I, d)$. In out case, we can give an affirmative answer to this problem because of our knowledge of singularities of $\mathcal{M}_H(I, d)$. To this end, we will quickly review some relevant facts about the singularities of the moduli space $\mathcal{M}(I, d)$. Since the discussion of $\text{Pic}(\mathcal{M}(I, d))$ will simplify when the polarization $H$ is $(I, d)$-generic, we will work with a set of polarizations simultaneously. (Note that except when $\text{Pic}(X)/\text{Pic}^0(X)$ has rank one, no polarization is $(I, d)$ generic for all $d$.) In the following, we will use $\mathcal{M}_H(I, d)$ and $\mathcal{M}_H(I, d)$ to denote the moduli schemes of $H$-stable and $H$-semistable sheaves respectively. Also, we will speak freely of moduli scheme for $\mathcal{M}_H(I, d)$ and $\mathcal{M}_H(I, d)$ only depend on the ray $\mathcal{Q} \cdot c_1(H) \subseteq H^{1,1}(X, \mathbb{Q})$. We let $N_\mathcal{Q}^+$ be the $\mathcal{Q}$-cone in $H^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Q})$ spanned by Chern classes of $\mathcal{Q}$-ample line bundles. We say a neighborhood $\mathcal{C} \subseteq N_\mathcal{Q}^+$ is precompact if the closure of $\mathcal{C}$ in $H^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Q})$ is still contained in $N_\mathcal{Q}^+$. Note that for any $c \in N_\mathcal{Q}^+$, a Euclidean ball of sufficiently small radius centered at $c$ in
$N^+_Q$, after fixing an Euclidean metric on $H^{1,1}(X, \mathbb{R})$, is a precompact neighborhood of $c$. We will use $\mathcal{C} \subseteq N^+_Q$ to mean $\mathcal{C}$ is precompact. Also, by abuse of notation we will say a $\mathbb{Q}$-line bundle $H \in \mathcal{C}$ if $c_1(H) \in \mathcal{C}$.

We collect the relevant properties of $\overline{\mathcal{M}}(I, d)$ in the following lemmas:

**Lemma 3.1.** For any precompact $\mathcal{C} \subseteq N^+_Q$, there is an integer $N$ depending on $(X, I, \mathcal{C})$ such that for any $d \geq N$ and $\mathbb{Q}$-ample line bundle $H \in \mathcal{C}$ (i.e. $c_1(H) \in \mathcal{C}$),

1. $\overline{\mathcal{M}}_H(I, d)$ is normal, irreducible and of dimension predicted by Riemann-Roch theorem;
2. $\mathcal{M}_H(I, d)$ is a local complete intersection scheme and the codimension of the singular locus of $\mathcal{M}_H(I, d)$ is at least 5;
3. Let $\tilde{H} \in \mathcal{C}$ be another $\mathbb{Q}$-ample line bundle with $c_1(\tilde{H}) \in \mathcal{C}$, then there are closed subsets $U_H \subseteq \mathcal{M}_H(I, d)$ and $U_{\tilde{H}} \subseteq \mathcal{M}_{\tilde{H}}(I, d)$ having codimension at least 5 respectively such that $\mathcal{M}_H(I, d) - U_H$ is canonically isomorphic to $\mathcal{M}_{\tilde{H}}(I, d) - U_{\tilde{H}}$ as schemes, by identifying two points that represent isomorphic sheaves.

**Proof.** (1) and (2) are proved on page 10 of [Li3] and (3) is proved on page 458 of [Li1] by using [Qi2]. □

As was mentioned in §2, the moduli $\overline{\mathcal{M}}_H(I, d)$ is constructed as GIT quotient of a Grothendieck’s Quot-scheme, say $\text{Quot}_H(I, d)$, by a reductive group $G$. (We take $G = PGL(m)$ this time.) The property we need from this construction is stated in the following lemma.

**Lemma 3.2.** For any precompact $\mathcal{C} \subseteq N^+_Q$, there is an integer $N$ depending on $(X, I, \mathcal{C})$ such that for any $d \geq N$ and $H \in \mathcal{C}$, we can construct a Grothendieck’s Quot-scheme $\text{Quot}_H(I, d)$ and a reductive group $G$ acting linearly on $\text{Quot}_H(I, d)$ of which the following holds:

1. Let $\text{Quot}_H(I, d)^s$ and $\text{Quot}_H(I, d)^{ss}$ be the open subsets of $G$-stable and $G$-semistable (with respect to the given linearization) points in $\text{Quot}_H(I, d)$. Then

$$\text{Quot}_H(I, d)^s/G = \mathcal{M}_H(I, d), \quad \text{Quot}_H(I, d)^{ss}/G = \overline{\mathcal{M}}_H(I, d)$$

and the projection sends $w \in \text{Quot}_H(I, d)^{ss}$ to $F_w \in \mathcal{M}_H(I, d)$, where $F_w$ is the corresponding quotient sheaf of $w$;
2. $\text{Quot}_H(I, d)^{ss}$ is a local complete intersection scheme whose singular locus has codimension at least 5 in $\text{Quot}_H(I, d)^{ss}$;
3. Let $w \in \text{Quot}_H(I, d)^{ss}$ be any point with closed orbit $G \cdot w \subseteq \text{Quot}_H(I, d)^{ss}$, then $\text{stab}_w = \mathbb{C}^*$ if $F_w \cong J_1 \oplus J_2$ with $J_1 \neq J_2$, $\text{stab}_w = PGL(2, \mathbb{C})$ if $F_w \cong J \oplus J$ and $\text{stab}_w = \{1\}$ otherwise;
(4) Assume \( H \) is \((I, d)\)-generic, then
\[
\text{codim}(\text{Quot}_H(I, d)^{ss} - \text{Quot}_H(I, d)\text{, Quot}_H(I, d)^{ss}) \geq 8.
\]

**Proof.** (1) and (3) is proved by [Gi1] (see also [Ma]) and (2) is proved on page 5 of [Li2] and (4) follows from [Qi2]. □

The last is the information about the first two Betti numbers of the moduli space \( \mathcal{M}_H(I, d) \) proved recently in [Li3].

**Lemma 3.3.** For any precompact \( C \subseteq \mathbb{N}_+^\mathbb{Q} \), there is an integer \( N \) depending on \((I, d, C)\) such that for any \( d \geq N \) and \( H \in C \),
\[
H^1(\mathcal{M}_H(I, d); \mathbb{Q}) \cong H^1(X; \mathbb{Q})
\]
and
\[
H^2(\mathcal{M}_H(I, d); \mathbb{Q}) \cong H^2(X; \mathbb{Q}) \oplus \wedge^2 H^1(X; \mathbb{Q}) \oplus H^0(X; \mathbb{Q}).
\]

We also quote a lemma that concerns extending line bundles over local complete intersection singularities.

**Lemma 3.4.** Let \( W \) be any quasi-projective scheme having only local complete intersection singularity and \( \text{codim}(\text{Sing} W, W) \geq 4 \). Then for any closed subset \( \Lambda \subseteq W \) of codimension at least two, the canonical map induced by inclusion
\[
\text{Pic}(W) \longrightarrow \text{Pic}(W - \Lambda)
\]
is an isomorphism.

**Proof.** See [SGA 2] on page 132. □

Now we draw some easy consequence from these lemmas concerning the Picard group of \( \mathcal{M}_H(I, d) \). In the following, we fix an \( I \), a \( C \subseteq \mathbb{N}_+^\mathbb{Q} \) and the integer \( N \) specified in the previous lemmas.

**Lemma 3.5.** For any \( H_1, H_2 \in C \) and \( d \geq N \), \( \text{Pic}(\mathcal{M}_{H_1}(I, d)) \) is canonically isomorphic to \( \text{Pic}(\mathcal{M}_{H_2}(I, d)) \), compatible to the birational map in Lemma 3.1.

**Proof.** Let \( U_{H_1} \subseteq \mathcal{M}_{H_1}(I, d) \) and \( U_{H_2} \subseteq \mathcal{M}_{H_2}(I, d) \) be the closed subset given in Lemma 3.1. It suffices to show that the pull back homomorphism \( \text{Pic}(\mathcal{M}_{H_1}(I, d)) \rightarrow \text{Pic}(\mathcal{M}_{H_2}(I, d) - U_{H_2}) \) is an isomorphism, since \( \mathcal{M}_{H_1}(I, d) - U_{H_1} \) is canonically isomorphic to \( \mathcal{M}_{H_2}(I, d) - U_{H_2} \). But this follows from Lemma 3.4 and (2) of Lemma 3.1. This proves the lemma. □
Lemma 3.6. Suppose $H^2(X, \mathbb{Z})$ is torsion free. Then for $d \geq N$ and $(I, d)$-generic $H \in \mathcal{C}$, the pull back homomorphism

$$\mu : \text{Pic}(\mathcal{M}_H(I, d)) \rightarrow \text{Pic}(\mathcal{M}_H(I, d))$$

is an isomorphism.

Proof. It is easy to see that $\mu$ is injective. Indeed, assume $L$ is a line bundle on $\mathcal{M}_H(I, d)$ admitting a non-vanishing section over $\mathcal{M}_H(I, d)$. Since $\mathcal{M}_H(I, d)$ is normal and

$$\text{codim}(\mathcal{M}_H(I, d) - \mathcal{M}_H(I, d), \mathcal{M}_H(I, d)) \geq 2,$$

this section extends to a non-vanishing section over $\mathcal{M}_H(I, d)$. Hence $L$ is trivial itself.

Now we show that $\mu$ is surjective. Let $L$ be a line bundle on $\mathcal{M}_H(I, d)$. Then its pull back $\pi^*L$ on $\text{Quot}_H(I, d)^s$ admits a canonical $G$-linearization, where $\pi : \text{Quot}_H(I, d)^s \rightarrow \mathcal{M}_H(I, d)$. By Lemma 3.4 and Lemma 3.2, $\pi^*L$ extends to a line bundle $L$ on $\text{Quot}_H(I, d)^{ss}$ and the $G$-linearization of $\pi^*L$ extends as well. The key observation is that $L$ always descends to a line bundle on $\mathcal{M}_H(I, d)$, under our assumption on $H^2(X; \mathbb{Z})$. To see this, we need to check that for any $w \in \text{Quot}_H(I, d)^{ss}$ with closed orbit $G \cdot w$ and non-trivial stabilizer $\text{stab}_w$, $\text{stab}_w$ acts trivially on the fiber of $L$ over $w$. Indeed, since $L$ is a line bundle, this action is given by a character

$$\chi_w : \text{stab}_w \rightarrow \mathbb{C}^*.$$

By (3) of Lemma 3.2, $\text{stab}_w$ can only take two forms, $\mathbb{C}^*$ and $PGL(2)$, unless it is trivial. Since $PGL(2)$ has no non-trivial character, we are left to show that $\chi_w$ is trivial even when $\text{stab}_w = \mathbb{C}^*$. Following Drezet and Narasimhan [DN], we know that if $w_1, w_2$ are two points contained in the same connected component of

$$\{ w \in \text{Quot}_H(I, d)^{ss} | G \cdot w \text{ closed and } \text{stab}_w \neq \{1\} \},$$

then $\chi_{w_1} = 0$ if and only if $\chi_{w_2} = 0$. By (3) of Lemma 3.2, $G \cdot w$ closed and $\text{stab}_w \neq \{1\}$ implies $\mathcal{F}_w = \mathcal{F}_1 \oplus \mathcal{F}_2$. However, since $H$ is $(I, d)$-generic, $c_1(\mathcal{F}_1) - c_1(\mathcal{F}_2)$ is torsion, which is zero since $H^2(X, \mathbb{Z})$ has no torsions. Then we can deform $\mathcal{F}_1 \oplus \mathcal{F}_2$ within the subset of split quotient sheaves in $\text{Quot}_H(I, d)^{ss}$, then within (3.1), to $\mathcal{F} \oplus \mathcal{F}$. Therefore $w$ and the point $\tilde{w}$ associated to the quotient sheaf $\mathcal{F} \oplus \mathcal{F}$ lie in the same component of (3.1). Therefore $\chi_w = 0$ because $\text{stab}_{\tilde{w}} = PGL(2)$ has no non-trivial characters. This proves that $L$ descends to a line bundle on $\mathcal{M}_H(I, d)$, which is an extension of $L$. This proves that Pic$(\mathcal{M}_H(I, d)) \rightarrow \text{Pic}(\mathcal{M}_H(I, d))$ is surjective. □

As a consequence of this lemma, we see that for large $d$, the divisor $\Lambda \subseteq \mathcal{M}_H(I, d)$ consisting of non-locally free sheaves is Cartier when $H$ is $(I, d)$-generic and $H^2(X, \mathbb{Z})$ is torsion free. In the following, we shall show that the condition $H^2(X, \mathbb{Z})$ has no torsion is unnecessary in this case, which is crucial in applying local Lefschetz theorem to study the Betti numbers of $\mathcal{M}_H(I, d)$ carried out in [Li3].
Lemma 3.7. For any precompact neighborhood $\mathcal{C} \subseteq \mathbb{N}_Q^+$, there is a constant $N$ such that for any $d \geq N$ and $(I,d)$-generic $H \in \mathcal{C}$, the divisor $\Lambda \subseteq \mathcal{M}_H(I,d)$ consisting of non-locally free sheaves is Cartier.

Proof. Let $k$ be the number of torsion elements (including 0) in $H^2(X,Z)$ and let $\pi : \tilde{X} \to X$ be a $k$-fold (unramified) covering so that $H^2(\tilde{X},Z)$ is torsion free. Let $\tilde{I} = \pi^* I$ and $\tilde{C} \subseteq \mathbb{N}_Q^+(\tilde{X})$ be the image in $H^2(\tilde{X},\mathbb{Q})$ under pullback $H^2(X;\mathbb{Q}) \to H^2(\tilde{X};\mathbb{Q})$. $\tilde{C} \subseteq \mathbb{N}_Q^+(\tilde{X})$ is precompact. Therefore there is an $N$ such that for any $\tilde{d} \geq N$ and $H' \in \tilde{C}$, the moduli scheme $\mathcal{M}_{H'}(\tilde{I},kd)$ and the corresponding Quot-scheme $\text{Quot}_{H'}(\tilde{I},kd)$ satisfy the conclusion of Lemma 3.1 and 3.2. Let $\tilde{\Lambda} \subseteq \mathcal{M}_{H'}(\tilde{I},kd)$ be the closed subset of all non-locally free sheaves. Since $\pi : \tilde{X} \to X$ is a finite unramified covering, the morphism $\pi : \mathcal{M}_H(I,d) \to \mathcal{M}_{H'}(\tilde{I},kd)$ induced by sending $E \in \mathcal{M}_H(I,d)$ to $\pi^*(E)$ is an immersion, where $\tilde{H} = \pi^* H$. In particular, $\Lambda = (\pi)^{-1}(\tilde{\Lambda})$. We claim that $\tilde{\Lambda}$ is Cartier near $\pi(\Lambda)$. To establish this, following the proof of Lemma 3.6, we only need to show that for any split semistable sheaf $E = F_1 \oplus F_2$ in $\Lambda$, $\pi^*(F_1) \oplus \pi^*(F_2)$ can be deformed within the set of splitting semistable sheaves in $\mathcal{M}_{H'}(\tilde{I},kd)$ to $F \oplus F$. But this is always possible because $c_1(F_1) - c_1(F_2)$ is torsion implies $c_1(\pi^*(F_1)) = c_1(\pi^*(F_2))$. Therefore $\tilde{\Lambda}$ is Cartier near $\pi(\Lambda)$ and hence $\Lambda = \pi^{-1}(\tilde{\Lambda})$ is Cartier. This completes the proof of the lemma. $\square$

We now state and prove the main theorem of this note.

Theorem 3.8. Suppose $H^2(X,Z)$ has no torsions. Then for any precompact $\mathcal{C} \subseteq \mathbb{N}_Q^+$, there is an integer $N$ depending on $(X,I,C)$ such that for any $d \geq N$ and $(I,d)$-generic $H \in \mathcal{C}$, the homomorphism

$$\Phi : \text{Pic}(X \times X)^\sigma \oplus \mathbb{Z} \to \text{Pic}(\mathcal{M}_H(I,d)) \otimes \mathbb{Z}[1/12]$$

constructed in (1.13) has finite kernel and cokernel.

Corollary 3.9. Suppose $H^2(X,Z)$ has no torsions. Then for any precompact $\mathcal{C} \subseteq \mathbb{N}_Q^+$, there is an integer $N$ depending on $(X,I,C)$ such that for any $d \geq N$ and $H \in \mathcal{C}$, the homomorphism

$$\Phi : \text{Pic}(X \times X)^\sigma \oplus \mathbb{Z} \to \text{Pic}(\mathcal{M}_H(I,d)) \otimes \mathbb{Z}[1/12]$$

has finite kernel and cokernel.
Proof. By enlarging $C \subseteq N_Q^+$ if necessary, we can assume $C$ contains $\mathbb{Q}$-polarization $\tilde{H}$ that is $(I,d)$-generic for any $d \geq 0$. We then choose $N$ so that all previous results holds for this $N$. Now for any $d \geq N$, let $H \in C$ be $(I,d)$-generic. Then by Lemma 3.5, $\text{Pic}(\mathcal{M}_H(I,d))$ is isomorphic to $\text{Pic}(\mathcal{M}_{\tilde{H}}(I,d))$, which is isomorphic to $\text{Pic}(\mathcal{M}_{\tilde{H}}(I,d))$ by Lemma 3.6. On the other hand, the homomorphism $\Phi$ above certainly commutes

under these isomorphisms. Therefore, $\Phi$ has finite kernel and cokernel because $\Phi$ does. This completes the proof of the corollary. □

Before we prove the theorem, we need two more technical lemmas. The first concerns the Hodge decomposition of $H^2(\mathcal{M}_H(I,d);\mathbb{R})$. Since we know $H^2(\mathcal{M}_H(I,d);\mathbb{R})$ explicitly, we will determine its Hodge decomposition with the aid of

$$g: W \rightarrow \mathcal{M}_H(I,d)$$

that is constructed in the beginning of §2 based on a $\mu$-stable rank two locally free sheaf $E$ with $\text{det } E = I$ and $c_2(E) = d - 2$. Let

$$g^{\text{pic}}: \text{Pic}(\mathcal{M}_H(I,d)) \rightarrow \text{Pic}(W)$$

be the induced homomorphism.

**Lemma 3.9.** For any precompact $C \subseteq N_Q^+$, there is an integer $N$ depending on $(X,I,C)$ such that for any $d \geq N$, $H \in C$ and $g: W \rightarrow \mathcal{M}_H(I,d)$ as before, the image

$$c_1 \circ g^{\text{pic}}: \text{Pic}(\mathcal{M}_H(I,d)) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow H^2(W;\mathbb{Q})$$

is isomorphic to

$$H^{1,1}(X \times X;\mathbb{R})^\sigma \cap H^2(X \times X;\mathbb{Q}) \oplus \mathbb{Q}.$$

**Proof.** Let $\Lambda$ be the image of $c_1 \circ g^{\text{pic}}$. From the gauge theory and the proof of Lemma 3.3 in [Li3], the image of

$$g^*: H^2(\mathcal{M}_H(I,d);\mathbb{R}) \rightarrow H^2(W;\mathbb{R})$$

is spanned by the images

$$p_{W*}(c_2(G) \cup p_X^*(\cdot)): H^2(X;\mathbb{R}) \rightarrow H^2(W;\mathbb{R}),$$

$$p_{W*}(c_3(G) \cup p_X^*(\cdot)): H^0(X;\mathbb{R}) \rightarrow H^2(W;\mathbb{R})$$

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and the image
\[ \wedge^2 p_{W*} (c_2(G) \cup p_X^* (\cdot)) : \wedge^2 H^1(X; \mathbb{R}) \to H^1(W; \mathbb{R}). \]

where $G$ is the sheaf on $W \times X$ constructed in §2 and $p_W, p_X$ are projections of $W \times X$. Based on exact sequences (2.3) and (2.4), it is straightforward to check that the direct sum of these homomorphisms
\[ \mu : H^2(X; \mathbb{R}) \oplus \wedge^2 H^1(X; \mathbb{R}) \oplus H^0(X; \mathbb{R}) \to H^2(W; \mathbb{R}) \]
is injective. Since $c_1(G)$ is an integral class of pure Hodge type, $\mu$ preserves the integer lattice as well as the obvious Hodge structures on both sides. In particular,
\[ (3.2) \quad \Lambda \subseteq \text{Im} (\mu) \cap (H^{1,1}(W; \mathbb{R}) \cap H^2(W; \mathbb{Q})) \]
and the later is isomorphic to the direct sum of
\[ H^0(X; \mathbb{Q}) \]
\[ H^{1,1}(X; \mathbb{R}) \cap H^2(X; \mathbb{Q}) \]
and
\[ (H^{1,0}(X; \mathbb{R}) \wedge H^{0,1}(X; \mathbb{R})) \cap (\wedge^2 H^1(X; \mathbb{Q})). \]

On the other hand, by our construction of the homomorphism
\[ \Phi_\mathbb{Q} : \text{Pic}(X \times X) \otimes_{\mathbb{Z}} \mathbb{Q} \oplus \mathbb{Q} \to \text{Pic}(\mathfrak{M}(\mathcal{I}, d)) \otimes_{\mathbb{Z}} \mathbb{Q} \]
and the formulas of their Chern classes (see (1.6), (1.7) and (1.8)), we see that the homomorphism
\[ (3.3) \quad H^{1,1}(X \times X; \mathbb{R})^\sigma \cap H^2(X \times X; \mathbb{Q}) \oplus \mathbb{Q} \to H^2(W; \mathbb{Q}) \]
induced by
\[ g^{\text{pic}} \circ \Phi_\mathbb{Q} : \text{Pic}(X \times X)^\sigma \otimes_{\mathbb{Z}} \mathbb{Q} \oplus \mathbb{Q} \to \text{Pic}(W) \otimes_{\mathbb{Z}} \mathbb{Q} \]
is injective. Clearly, the image of (3.3) is contained in $\Lambda$. Therefore, if we can show that the left hand side of (3.3) is isomorphic to the right hand side of (3.2), then for dimensional reason the equality in (3.2) must hold and then the lemma follows immediately.

We now show that
\[ H^{1,1}(X \times X; \mathbb{R})^\sigma \cap H^2(X \times X; \mathbb{Q}) \cong H^{1,1}(X; \mathbb{R}) \cap H^2(X; \mathbb{Q}) \]
\[ \oplus (H^{1,0}(X; \mathbb{R}) \wedge H^{0,1}(X; \mathbb{R})) \cap (\wedge^2 H^1(X; \mathbb{Q})). \]
(3.4)
We will use the Kunneth decomposition
\[ H^2(X \times X; \mathbb{R}) = \bigoplus_{i+j=2} H^i(X; \mathbb{R}) \otimes H^j(X; \mathbb{R}). \]

It is straightforward to check that an element
\[ v = \sum_{i=1}^n (b_i (1 \otimes \alpha_i) + b'_i (\alpha_i \otimes 1)) + \sum_{i,j=1}^m a_{ij} (\beta_i \otimes \beta_j) \]
in \( H^2(X \times X; \mathbb{R}) \) is \( \sigma \) invariant if and only if
\[ b_i = b'_i \quad \text{and} \quad a_{ij} = -a_{ji}, \]
where \( \{\alpha_i\}^n \) and \( \{\beta_j\}^m \) are basis of \( H^2(X; \mathbb{R}) \) and \( H^1(X; \mathbb{R}) \) respectively. Hence
\[ H^2(X \times X; \mathbb{R})^\sigma \cong H^2(X; \mathbb{R}) \oplus \wedge^2 H^1(X; \mathbb{R}). \]

We can derive a similar formula for \( H^1(X \times X; \mathbb{R})^\sigma \) and \( H^1(X \times X; \mathbb{Q})^\sigma \). Combined, we get the desired identity (3.4). This completes the proof of Lemma 3.9. \( \square \)

The next lemma states that the restriction of \( c_1(L) \) to \( H^2(\mathcal{M}_H(I,d); \mathbb{Z}) \), where \( L \) is a line bundle on \( \mathcal{M}_H(I,d) \), determines \( c_1(L) \) completely. This is crucial to our study since we only know the structure of \( H^2(\mathcal{M}_H(I,d); \mathbb{R}) \) rather than \( H^2(\mathcal{M}_H(I,d); \mathbb{R}) \).

**Lemma 3.10.** Let the notation be as in Theorem 3.8. Then there is an \( N \) depending on \( (X,I,\mathcal{C}) \) such that for any \( d \geq N \) and \( (I,d) \)-generic \( H \in \mathcal{C} \) the following holds:

Let \( L \) be a line bundle on \( \mathcal{M}_H(I,d) \) such that the restriction of its Chern class to \( H^2(\mathcal{M}_H(I,d); \mathbb{Z}) \) is trivial, then \( L \) is derived from a representation \( \rho : \pi_1(\overline{\mathcal{M}}_H(I,d)) \to S^1 \).

**Proof.** Let \( Z \) be a desingularization of \( \overline{\mathcal{M}}_H(I,d) \) with projection \( \pi : Z \to \overline{\mathcal{M}}_H(I,d) \) and let \( D_1, \cdots, D_k \) be irreducible components of the exceptional divisor of \( \pi \), which we assume is normal crossing. Then the kernel of the composition
\[ H^2(Z; \mathbb{Z}) \overset{\pi^*}{\to} H^2(\overline{\mathcal{M}}_H(I,d); \mathbb{Z}) \overset{\text{rest}}{\to} H^2(\overline{\mathcal{M}}_H(I,d)_{\text{reg}}; \mathbb{Z}), \]
where \( \overline{\mathcal{M}}_H(I,d)_{\text{reg}} \) is the regular locus of \( \overline{\mathcal{M}}_H(I,d) \), is spanned by all \( c_1(\mathcal{O}(D_i)) \). In particular there are integers \( n_1, \cdots, n_k \) such that
\[ c_1(\pi^* L(\sum n_i D_i)) = 0 \in H^2(Z; \mathbb{Z}), \]
since the restriction to \( H^2(\mathcal{M}_H(I,d); \mathbb{Z}) \) of \( c_1(L) \) is trivial. Hence

\[
\tilde{L} = \pi^* L(\sum n_i D_i)
\]

is induced by a representation \( \tilde{\rho} : \pi_1(Z) \to S^1 \), or induced by a local system \( \tilde{\Sigma} \).

It remains to show that \( \tilde{\Sigma} \) descends to a local system \( \Sigma \) on \( \mathbb{M}_H(I,d) \), since then the line bundle induced by \( \Sigma \) is isomorphic to \( L \) on \( \mathbb{M}_H(I,d)_{\text{reg}} \) and then by 3.1, they are isomorphic on \( \mathbb{M}_H(I,d) \) as well. This will prove the lemma.

Instead of studying the descent of \( \Sigma \) via projection \( \pi : Z \to \mathbb{M}_H(I,d) \), which requires the knowledge of the singularity of \( \mathbb{M}_H(I,d) \), we will work with the quotient \( p : \text{Quot}_H(I,d)^{ss} \to \mathbb{M}_H(I,d) \) since virtually all local information of \( \mathbb{M}_H(I,d) \) is contained in this construction. First, since \( \pi : Z \to \mathbb{M}_H(I,d) \) is an isomorphism over \( \mathbb{M}_H(I,d)_{\text{reg}} \), we obtain a local system \( \Sigma' \) on \( \mathbb{M}_H(I,d)_{\text{reg}} \) and its pull-back \( p^*(\Sigma') \) on \( p^{-1}(\mathbb{M}_H(I,d)_{\text{reg}}) \subseteq \text{Quot}_H(I,d)^{ss} \). Clearly, \( p^*(\Sigma') \) is \( G \)-equivalent. Let

\[
R = \text{Quot}_H(I,d)^{ss} - p^{-1}(\mathbb{M}_H(I,d)_{\text{reg}}).
\]

By Lemma 3.1, \( R \) has codimension at least 5. Then for any closed \( x \in R \) and normal slice \( S \) of \( R \subseteq \text{Quot}_H(I,d)^{ss} \) at \( x \), \( \pi_1(S - x) = \{1\} \), because \( \text{Quot}_H(I,d)^{ss} \) is a local complete intersection scheme ((2) of Lemma 3.2) and \( \text{codim}(R) \geq 5 \), by the local Lefschetz theorem on page 155 of [GM]. Therefore \( p^*(\Sigma') \) extends to a local system, say \( \Sigma'' \), that is \( G \)-equivalent as well, since \( \text{Quot}_H(I,d)^{ss} \) is normal. To show that \( \Sigma'' \) descends to \( \Sigma \) on \( \mathbb{M}_H(I,d) \), it suffices to show that for any \( w \in \text{Quot}_H(I,d)^{ss} \) with closed orbit \( G \cdot w \), \( \text{stab}_w \) acts trivially on the fiber of \( \Sigma'' \) over \( w \). But this can be proved using the fact that \( H^2(X; \mathbb{Z}) \) has no torsion as we did in the proof of Lemma 3.6. This shows that \( \Sigma'' \) descends to a local system \( \Sigma \). This completes the proof of the Lemma. \( \square \)

**Proof of Theorem 3.8.** We let \( N \) be the integer that makes all previous results valid. Let \( d \geq N \) and let \( H \in \mathcal{C} \) be \( (I,d) \)-generic. By Lemma 3.8 for any line bundle \( L \) on \( \mathbb{M}_H(I,d) \), the restriction of its Chern class \( c_1(L) \) to \( H^2(\mathcal{M}_H(I,d); \mathbb{Q}) \) is contained in the image of

\[
c_1 \circ \Phi : \text{Pic}(X \times X)^\sigma \otimes \mathbb{Q} \oplus \mathbb{Q} \to H^2(\mathcal{M}_H(I,d); \mathbb{Q}).
\]

Hence we can find an element \( \alpha \in \text{Pic}(X \times X)^\sigma \oplus \mathbb{Z} \) and an integer \( k \) such that the restriction of \( c_1(L^\otimes k \otimes L_\alpha) \) to \( H^2(\mathcal{M}_H(I,d); \mathbb{Z}) \) is trivial, where \( L_\alpha \) is the line bundle \( F(\alpha) \) on \( \mathbb{M}_H(I,d) \). Here \( k \) can be chosen to depend only on the induced homomorphism

\[
H^2(X \times X; \mathbb{Z})^\sigma \oplus \mathbb{Z} \to H^2(\mathcal{M}_H(I,d); \mathbb{Z}[\frac{1}{12}]).
\]
By Lemma 3.10, $L^\otimes k \otimes L_\alpha$ is induced by a representation $\rho : \pi_1(\overline{\mathcal{M}}_H(I, d)) \to S^1$, or an element in $H^1(\overline{\mathcal{M}}_H(I, d); \mathbb{R})$. However, $\overline{\mathcal{M}}_H(I, d)$ is normal and $\overline{\mathcal{M}}_H(I, d) - \mathcal{M}_H(I, d)$ has codimension at least 2, hence

$$H^1(\overline{\mathcal{M}}_H(I, d); \mathbb{R}) \to H^1(\mathcal{M}_H(I, d); \mathbb{R})$$

is injective. On the other hand, we know

$$\dim H^1(\mathcal{M}_H(I, d); \mathbb{R}) = h^1(X)$$

and the restriction of $\Phi$

$$\Phi_0 : \text{Pic}^0(X \times X)^\sigma \to \text{Pic}^0(\overline{\mathcal{M}}_H(I, d))$$

has finite kernel since

$$g^{\text{pic}} \circ \Phi_0 : \text{Pic}(X \times X)^\sigma \to \text{Pic}(W)$$

has finite kernel, by direct check. This implies $\text{Im}(\Phi_0)$ has dimension $h^1(X)$. Therefore, $\Phi_0$ is surjective. This proves that

$$\Phi : \text{Pic}(X \times X)^\sigma \oplus \mathbb{Z} \to \text{Pic}(\overline{\mathcal{M}}_H(I, d)) \otimes \mathbb{Z}[\frac{1}{12}]$$

has finite cokernel. □

**Remark.** 1. The author inclined to believe the condition that $H^2(X; \mathbb{Z})$ is torsion free in this theorem is unnecessary, but is unable to remove it using the current technique.

2. It is interesting to find a bound $N$ in this theorem that can be determined effectively. After the effective bound of O’Grady [OG], we can find an effective bound that works for all results in this note that are independent of Lemma 3.3 whose proof relies on the result of C. Taubes that is not effective. It is interesting to get a purely algebraic proof of Lemma 3.3.

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