An asymptotic approximation to the free-surface elevation around a cylinder in long waves

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Abstract

Strong nonlinear effects are known to contribute to the wave run-up caused when a progressive wave impinges on a vertical surface piercing cylinder. The magnitude of the wave run-up is largely dependent on the coupling of the cylinder slenderness, $ka$, and wave steepness, $kA$, parameters. This present work proposes an analytical solution to the free-surface elevation around a circular cylinder in plane progressive waves. It is assumed throughout that the horizontal extent of the cylinder is much smaller than the incident wavelength and of the same order of magnitude as the incident wave amplitude. A perturbation expansion of the velocity potential and free-surface boundary condition is invoked and solved to third-order in terms of $ka$ and $kA$. The validity of this approach is investigated through a comparison with canonical second-order diffraction theory and existing experimental results. We find that for small $ka$, the long wavelength theory is valid up to $kA \approx 0.16 - 0.2$ on the up wave side of the cylinder. However, this domain is significantly reduced to $kA < 0.06$ when an arbitrary position around the cylinder is considered. An important feature of this work is an improved account of the first-harmonic of the free-surface elevation over linear diffraction theory.

1 Introduction

When a monochromatic progressive wave of fundamental frequency $\omega$ interacts with a body penetrating the free-surface, energy is transferred to the harmonics of the carrier wave and the vertical displacement of the surrounding free-surface is enhanced. The elevation of the free-surface in contact with the body is known as the wave run-up, denoted $R$, and is measured from the mean water level.

We consider a vertical circular cylinder in deep water waves whereby the incident wave is scattered in all directions along the horizontal plane of the free-surface. The extent of this scattering is described by the scattering parameter $ka$ – where $k$ and $a$ denote the wave number and cylinder radius respectively. Provided the wave steepness is small and the cylinder non-absorbing, the transfer of wave energy implies that the wave run-up is somewhat less than $2A$. If on the other hand, the incident wave steepness is sufficiently large, but bounded by the wave breaking limit, a vertical jet of water at the body surface will form. This is an extreme form of wave run-up.

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Linear scattering of water waves by a fixed vertical cylinder was first considered by Havelock [1] and extended to second-order by Hunt and Baddour [2]. In both theories, the nonlinear free-surface boundary condition

\[ \phi_{tt} + g \phi_z = -2\nabla \phi \cdot \nabla \phi_t - \frac{1}{2} \nabla \phi \cdot \nabla (\nabla \phi)^2, \quad \text{on } z = \zeta, \quad (1) \]

where \( \phi \) denotes the velocity potential and \( \zeta \) the vertical free-surface displacement — is treated by a Stokes expansion in terms of \( kA \) and are consistent to \( \mathcal{O}(A) \) and \( \mathcal{O}(A^2) \) respectively. The problem of wave scattering by a cylinder has attracted attention through two alternative treatments of the nonlinear free-surface boundary condition to \( \mathcal{O}(A^3) \) in the work of Faltinsen, Newman, and Vinje [3] and Malenica and Molin [4]. Of the two schemes, both assume incident waves of small steepness \( kA \), however, the infinite water depth theory of Faltinsen et al. [3] (hereafter referred to as ‘FNV’) is restricted to \( ka \ll 1 \), while that of Malenica and Molin [4] is based on classical Stokes perturbation expansion and is valid for arbitrary wavelength and water depth. These two works were stimulated by observations of the excitation of resonant modes in offshore structures exhibiting high natural-frequencies by higher-harmonic nonlinear wave loads. Due to the restriction of \( ka \ll 1 \), the work of FNV is strictly a long wavelength (LWL) theory.

In realistic ocean environments comprising long wavelengths, the incident wave amplitude can be of comparable magnitude to the horizontal extent of a partially immersed cylinder. Consequently, the free-surface Keulegan-Carpenter number, \( K_c = \pi A/a \), is of \( \mathcal{O}(1) \). This clearly violates the Stokes perturbation expansion of the free-surface boundary condition which implicitly restricts \( K_c \ll 1 \). With this in mind, FNV reconsiders the governing relative length scales to the problem in a consistent fashion such that \( kA = \mathcal{O}(\varepsilon) \), \( ka = \mathcal{O}(\varepsilon) \) and \( A/a = \mathcal{O}(1) \) — where \( \varepsilon \) is a perturbation quantity of order \( \ll 1 \). Whilst the restriction placed on \( ka \) essentially implies that the incident wavelength is asymptotically large when compared to the cylinder radius, it also implies that the incident wave amplitude is of the same order of magnitude as the cylinder radius.

In computing wave forces, this LWL approach has been generalised by Newman [5] for the case of unidirectional irregular waves, whereby the wavelength of each spectral component is assumed large when compared to the cylinder radius. A further generalisation by Faltinsen [6], employing a Lewis conformal mapping, accounts for the third-harmonic LWL force associated with square cylinders comprising rounded edges.

While for the wave run-up, Newman and Lee [7] compared an FNV second-order free-surface elevation prediction with second-order diffraction computations. This demonstrated that the two methods are in reasonable agreement for long wavelengths. Moreover, experimental evidence provided by Morris-Thomas and Thiagarajan [8] demonstrates that intermediate to long wavelengths are critical for the wave run-up, particularly when the amplitudes of these waves are of comparable magnitude to the cylinder radius. In this regime, third-harmonic components of \( \mathcal{O}(A^3) \) become important, accounting for up to 10% of the overall wave run-up. Furthermore, nonlinear effects operating at the first-harmonic have been clearly observed for increasing values of \( kA \) [8]. Classical linear and second-order diffraction theories do not account for these nonlin-
earities. The goal of this present work is to illustrate that these nonlinearities appear and can be partially accounted for in the long wavelength regime.

In this present work the LWL theory of FNV is extended to account for the free-surface elevation correct to $O(\varepsilon^3)$ for monochromatic progressive waves of amplitude $A$ incident upon a fixed vertical surface piercing cylinder of radius $a$ in deep water. Following a somewhat different path than that considered in FNV, we treat the free-surface boundary condition via two perturbation schemes corresponding to an expansion in terms of $ka$ and $kA$. The analytical results are compared to measured wave run-up values and second-order diffraction calculations.

2 Problem Definition

A fluid domain of infinite horizontal extent is considered. The fluid is assumed ideal and of infinite depth but bounded by Equation (1) at the free-surface. A circular cylinder of radius $a$ is partially immersed in the fluid. The coordinate system adopted for the problem is defined in Figure 1 where $(x, y) = (r \cos \theta, r \sin \theta)$. The boundary value problem for the velocity potential $\phi(r, \theta, z, t)$ is defined:

\begin{align*}
\nabla^2 \phi &= 0, & \text{for } r \in [a, \infty) \cup [0, 2\pi] \cup z \in [\zeta, -\infty), \\
\phi_r &= 0, & \text{at } r = a, \\
|\nabla \phi| &\to 0, & \text{as } r \to \infty \text{ and } z \to -\infty,
\end{align*}

along with the free-surface boundary condition (1) which must be satisfied at the instantaneous free-surface position $z = \zeta(r, \theta, t)$.

To treat Equation (1), it is convenient to express $\phi$ in terms of a power series with coefficients \{\varepsilon^n; n = 1, 2, 3, \ldots\} such that $\varepsilon \ll 1$. In canonical water wave diffraction theory one usually selects $\varepsilon = kA$ — the wave steepness. All horizontal dimensions are then scaled according to $ka$, the diffraction parameter whereby $a = O(1)$. These two assumptions implicitly imply that $A/a \ll 1$.

Observations in large waves relative to the horizontal dimensions of a surface piercing body suggest that $A/a \ll 1$ will not always be the case. Therefore, we impose the alternative restriction that $ka = O(\varepsilon)$ and, by consequence, $A/a$ is now of $O(1)$ \[3\]. In large waves, this is clearly the case because typical wavelengths are always much larger than both $a$ and $A$. These two assumptions, $kA \ll 1$ and $ka \ll 1$, are essential and form the basis for what is to follow.

To treat the boundary value problem, we formally select the following perturbation parameters:

\begin{equation}
\varepsilon_1 = kA, \quad \text{and} \quad \varepsilon_2 = ka,
\end{equation}

which are both of $O(\varepsilon)$ — here, and elsewhere $\varepsilon^{m+n} \equiv \varepsilon_1^m \varepsilon_2^n$. These represent a parameter and a coordinate expansion respectively. Similar to canonical diffraction theory we now define a power series expansion for $\phi$. However, given that we must also take into account $\varepsilon_2$, the expansion
assumes a slightly different form and is written

$$\phi = \varepsilon_1 \phi_1 + \varepsilon_1 \varepsilon_2 \phi_2 + \varepsilon_1 \varepsilon_2 \phi_3 + \varepsilon_2 \varepsilon_2 \phi_2 + \varepsilon_3 \varepsilon_3 \phi_3 + \mathcal{O}(\varepsilon^4),$$

(6)

where \( \phi_n \) are linear potentials in terms of the wave steepness and \( \psi_2 \) and \( \psi_3 \) are of second- and third-order respectively. Similarly for the free-surface elevation, we write the following expansion

$$\zeta = \zeta_1 + \zeta_2 + \zeta_3 + \mathcal{O}(\varepsilon^4),$$

(7)

where \( \{ \zeta_n; n = 1, 2, 3 \} \) are defined.

The expression for \( \zeta_3 \) is particularly interesting because it contains a term that is proportional to \( \varepsilon_4^1/\varepsilon_2 \). In canonical water wave diffraction theory, this term would presumably appear at fourth-order in the wave steepness. However, in the dual expansion considered here, it is strictly of third-order. In the following sections we solve for \( \phi_n \), \( \psi_2 \) and \( \psi_3 \) and write a more explicit expression for \( \zeta \) the free-surface elevation.

3 Linear Potentials

We first consider \( \{ \phi_n; n = 1, 2, 3 \} \). According to the theory of Stokes, a plane progressive wave of fundamental circular frequency \( \omega \) propagating from \( x = -\infty \) can be expressed by the well known potential

$$\phi(x, y, z, t) = \Re \left\{ \frac{gA}{\omega} e^{kz} e^{i(\omega t - kx)} \right\} + \mathcal{O}(A^4),$$

(9)

where the dispersion relation correct to third-order in wave steepness is defined \( k_0 = k(1 + k^2 A^2) \) where \( k_0 = \omega^2/g \) [9, Art. 250]. When a wave described by Equation (9) interacts with a vertical cylinder, centred at the origin of our coordinate system (see Figure 1), the classical linear diffraction potential that describes the fluid motion is written [1, Eq. 17]:

$$\phi^{(1)} = \Re \left\{ \frac{gA}{\omega} e^{(kz + i\omega t)} \sum_{m=0}^{\infty} \epsilon_m i^{-m} C_m(kr) \cos m\theta \right\},$$

(10)

where \( \epsilon_m \) denotes the Neumann symbol defined by \( \epsilon_0 = 1 \) and \( \epsilon_m = 2 \) for \( m > 0 \) and

$$C_m(kr) = J_m(kr) - \frac{\mathcal{J}_m'(ka)}{H_m^{(2)/}(ka)} H_m^{(2)}(kr),$$

(11)

where \( J_m \) and \( H_m \) denote Bessel and Hankel functions of order \( m \) respectively and the primes denotes differentiation with respect to the argument. Equation (10) satisfies the boundary
conditions (2)–(4) and the homogeneous form of Equation (1) prescribed on the plane \( z = 0 \). When both \( ka \) and \( kr \) are small, Equation (10) can be expanded into the following approximate form [10, Eq. 72]:

\[
\phi^{(1)} = \Re \left\{ \frac{g A}{\omega} e^{(kz+i\omega t)} \left( C_0(kr) - 2i C_1(kr) \cos \theta - 2 C_2(kr) \cos 2\theta \right) \right\},
\]

where the first three coefficients from Equation (11) are

\[
C_0(kr) = 1 + \frac{1}{2} (ka)^2 \left( \ln \frac{1}{2} kr + \gamma + \pi i/2 \right) - \frac{1}{4} (kr)^2 + \ldots,
\]

\[
C_1(kr) = \frac{k}{2} \left( r + \frac{a^2}{r} \right) + \ldots,
\]

and

\[
C_2(kr) = \frac{k}{8} \left( r^2 + \frac{a^4}{r^2} \right) + \ldots,
\]

where \( \gamma \approx 0.57722 \) denotes the Euler-Mascheroni constant. Upon considering our perturbation parameters, we can see that Equation (12) is correct to order \( \varepsilon_1 \varepsilon_2^2 \) and contains contributions of order \( \varepsilon_1 \) and \( \varepsilon_1 \varepsilon_2 \). Therefore, we can then equate like terms from Equation (12) with those of the series expansion (6) and recover the following expressions

\[
O(\varepsilon_1) : \quad \phi_1 = \frac{\omega}{k^2} e^{kz} \cos \omega t,
\]

\[
O(\varepsilon_1 \varepsilon_2) : \quad \phi_2 = \frac{\omega}{k^2} e^{kz} \sin \omega t \left( \frac{r}{a} + \frac{a^2}{r} \right) \cos \theta,
\]

\[
O(\varepsilon_1 \varepsilon_2^2) : \quad \phi_3 = \frac{\omega}{k^2} e^{kz+i\omega t} \left[ \frac{1}{2} \left( \ln \frac{1}{2} kr + \gamma + \frac{\pi i}{2} \right) - \frac{1}{4} \left( \frac{r^2}{a^2} - \cos 2\theta \left( \frac{r^2}{a^2} + \frac{a^2}{r^2} \right) \right) \right].
\]

The expansion (12), and expressions \( \{ \phi_n; n = 1, 2, 3 \} \), are valid for \( kr = O(\varepsilon_2) \) and, in terms of the wave steepness, purely linear. Moreover, one may also notice that but virtue of \( \phi_1 \), the linear vertical free-surface displacement on the plane \( z = 0 \) is simply

\[
\zeta_1 = \varepsilon_1 \eta_{1,0} = -\varepsilon_1 \frac{1}{g} \partial_t \phi_1 = A \sin \omega t.
\]

Equations (16a–16c) provide the first three terms of the potential \( \phi \). Consequently, what remains is for us to determine the nonlinear potentials \( \psi_2 \) and \( \psi_3 \) to complete the problem to \( O(\varepsilon^3) \).

## 4 Nonlinear Potentials

In solving for \( \psi_2 \) and \( \psi_3 \), we must first transfer the free-surface boundary condition (1) to the moving plane \( \zeta_1 = A \sin \omega t \). This is the essential feature of the approach adopted by Faltinsen et al. [3] and we adopt it here. The reasoning behind it is that the radial gradients on the right hand side of Equation (1) are amplified by our choice of coordinate expansion \( \varepsilon_2 = ka \). Consequently, the leading order contribution from simply substituting \( \{ \phi_n; n = 1, 2, 3 \} \) into the RHS of Equation (1) about the plane \( z = 0 \) is of \( O(\varepsilon^2) \). This is not what is required. To
overcome this apparent disparate length scale, we can affect the transfer about the plane \( z = \zeta_1 \) thereby recovering, on the RHS of Equation (1), the leading order contribution of \( \mathcal{O}(\varepsilon^3) \) as required. More specifically, this leading order contribution is in fact of \( \mathcal{O}(\varepsilon^2_1 \varepsilon_2) \) which implies that this expansion is valid only in the region of \( r = \mathcal{O}(a) \). With this in mind, we formally adopt the normalised coordinates \( (\hat{r}, \theta, \hat{z}) \) for the nonlinear potentials:

\[
\hat{r}(r) = \frac{r}{a}, \quad \text{and} \quad \hat{z}(z, t) = -\frac{z + \zeta_1}{a}.
\]  

(18)

With the aid of Expansion (6), the transfer of Equation (1) to the plane \( \zeta_1 = A \sin \omega t \) is affected by employing a Taylor series expansion about the moving plane \( \hat{z} = 0 \). Equating like terms to \( \mathcal{O}(\varepsilon^3) \), provides the following two boundary conditions

\[
\mathcal{O}(\varepsilon^2_1 \varepsilon_2): \quad \frac{g}{k} \partial_z \psi_2 = 2 \left( \partial_r \phi_2 \partial_r \phi_2 + \frac{1}{r^2} \partial_\theta \phi_2 \partial_\theta \phi_2 + \partial_z \phi_1 \partial_z \phi_1 \right),
\]

(19)

\[
\mathcal{O}(\varepsilon^3_1): \quad \frac{g}{k^3} \partial_z \psi_3 = \frac{1}{2} \left( \partial_r \phi_2 \partial_r + \frac{1}{r^2} \partial_\theta \phi_2 \partial_\theta \right) \left( (\partial_r \phi_2)^2 + (\partial_\theta \phi_2)^2 \right),
\]

(20)

on \( \hat{z} = 0 \). Physically, Equations (19) and (20) represent a vertical velocity distribution imposed on an harmonically oscillating free-surface. By substituting \( \phi_1 \) and \( \phi_2 \) into Equations (19) and (20) more explicit expressions result, whence

\[
\partial_z \psi_2 = \frac{\omega}{k^2} \sin 2\omega t \left( \frac{1}{r^4} - \frac{2}{r^2} \cos 2\theta \right),
\]

(21)

and

\[
\partial_z \psi_3 = 2 \frac{\omega}{k^2} \sin^3 \omega t \left( \cos \theta \left( \frac{1}{r^4} - \frac{2}{r^6} \right) + \frac{1}{r^3} \cos 3\theta \right).
\]

(22)

Apart from the change of notation, the linear combination of Expressions (21) and (22), the form of \( \varepsilon^2_1 \varepsilon_2 \psi_2 + \varepsilon^3_1 \psi_3 \), agrees with that of Faltinsen et al. [3, Eq. 3.10].

4.1 The Boundary Value Problem

The form of the free-surface boundary conditions for \( \psi_2 \) and \( \psi_3 \) suggest implicit azimuthal symmetry and oscillatory time dependence. Consequently, this prompts solutions for \( \psi_2 \) and \( \psi_3 \) in the following forms:

\[
\psi_2 = \frac{\omega}{k^2} \sin 2\omega t \left( \alpha_0(\hat{r}, \hat{z}) + \alpha_2(\hat{r}, \hat{z}) \cos 2\theta \right),
\]

(23)

and

\[
\psi_3 = \frac{\omega}{k^2} \sin^3 \omega t \left( \alpha_1(\hat{r}, \hat{z}) \cos \theta + \alpha_3(\hat{r}, \hat{z}) \cos 3\theta \right).
\]

(24)

where the unknown set of coefficients \( \{\alpha_m(\hat{r}, \hat{z}): m = 0, 1, 2, 3\} \) are required and must be chosen to satisfy Equations (21) and (22). Formally, four two-dimensional boundary value problems for
\( \alpha_m(\hat{r}, \hat{z}) \) are defined as follows:

\[
\nabla^2 \alpha_m(\hat{r}, \hat{z}) = 0, \quad \hat{r} \in [1, \infty) \cup \hat{z} \in [0, \infty), \tag{25}\]

\[
\partial_r \alpha_m = 0, \quad \text{on } \hat{r} = 1, \tag{26}\]

\[
|\nabla \alpha_m| \to 0, \quad \text{as } \hat{r} \to \infty, \tag{27}\]

\[
\partial_z \alpha_m = f_m(\hat{r}), \quad \text{on } \hat{z} = 0, \tag{28}\]

where \( f_m(\hat{r}) \) is defined

\[
\{1/\hat{r}^4, -4/\hat{r}^5 + 2/\hat{r}^7, -2/\hat{r}^2, 2/\hat{r}^3\}. \tag{29}\]

For these class of problems, a generalised Fourier-Bessel integral transform is appropriate [11, page 161]:

\[
f(r) = \int_0^\infty \varphi_\lambda(\hat{r}) \varphi_\lambda(\lambda \rho) \frac{d\lambda}{|H_\nu(\lambda a)|^2} \int_a^\infty f(\rho) \varphi_\lambda(\rho) \rho d\rho, \quad a < r < \infty, \tag{30}\]

where \( H_\nu \) denotes a Hankel function of the first kind and \( \varphi_\lambda(r) \) involves the linear combination

\[
\varphi_\lambda(r) = Y'_\nu(\lambda a) J_\nu(\lambda r) - J'_\nu(\lambda a) Y_\nu(\lambda r) \tag{31}\]

such that \( \varphi_\lambda(a) = 2\pi/a \) and \( \varphi_\lambda'(a) = 0 \) thereby satisfying the requirements of the Neumann condition at the cylinder surface. Provided \( \sqrt{\hat{r}} f(\hat{r}) \) and \( \sqrt{\hat{r}} f'(\hat{r}) \to 0 \) as \( \hat{r} \to \infty \), the integral transform (30) is valid and the radiation condition, for \( \hat{r} \to \infty \), is then implicitly satisfied.

In solving the boundary value problem, the integral transform is applied by multiplying the field equation (25) by \( \hat{r} \varphi_\lambda(\hat{r}) \) and integrating from 1 to \( \infty \). Taking into account the behaviour of the various functions as \( \hat{r} \to \infty \), the following ordinary differential equation results

\[
\partial_{\hat{r}} \hat{\alpha}_m(\lambda, \hat{z}) - \lambda^2 \hat{\alpha}_m = \frac{2}{\pi} \partial_r \alpha_m(1, \hat{z}), \tag{32}\]

with the following transformed free-surface condition (28):

\[
\partial_{\hat{z}} \hat{\alpha}_m = \int_1^\infty f_m(\hat{r}) \hat{r} \varphi_\lambda(\hat{r}) d\hat{r} \tag{33}\]

on \( \hat{z} = 0 \) for each integer \( m \). The particular solution of Equation (32) that satisfies Equation (33), and by implication the boundary value problem (25) – (27), is then

\[
\hat{\alpha}_m(\lambda, \hat{z}) = -\frac{e^{-\lambda \hat{z}}}{\lambda} \int_1^\infty f_m(\hat{r}) \varphi_\lambda(\hat{r}) \hat{r} d\hat{r}. \tag{34}\]

To employ the inversion formula (30), we must first obtain more explicit expressions for the integrals shown in Equation (34). First, we consider those involving \( \hat{\alpha}_2 \) and \( \hat{\alpha}_3 \). These can be reduced quite easily by making the substitution \( \xi = \lambda \hat{r} \), and applying the appropriate recurrence relations for cylinder functions [12, Eq. 9.1.27 and Eq. 9.1.30], arriving at the following results:

\[
\hat{\alpha}_2(\lambda, \hat{z}) = \frac{e^{-\lambda \hat{z}}}{\lambda} (Y_1(\lambda)J_3(\lambda) - J_1(\lambda)Y_3(\lambda)), \tag{35}\]
The nature of the integrands for \( \hat{\alpha}_0 \) and \( \hat{\alpha}_1 \) are somewhat more troublesome and do not permit a reduction by cylinder recurrence relations. Consequently, we employ the following integral relation [13, Eq. 5]

\[
\int x^\mu \mathcal{E}_\nu(x) \, dx = x((\mu + \nu - 1)\mathcal{E}_\nu(x)S_{\mu-1,\nu-1} - \mathcal{E}_{\nu-1}(x)S_{\mu,\nu}),
\]

(37)

where \( S_{\mu,\nu}(x) \) denotes the Lommel function of the second-kind of argument \( z \) and \( \mathcal{E}_\nu \) is any cylinder function of order \( \nu \). In solving for \( \hat{\alpha}_0 \), we utilise the substitution \( \xi = \lambda \hat{r} \), as before, and apply the integral relation provided by Equation (37). After making use of the Wronskian, and noting that the series \( S_{\mu,\nu} \sim x^{\mu-1} \) for \( \xi \to \infty \), thus ensuring that there is no contribution from the upper limit of integration, we have

\[
\hat{\alpha}_0(\lambda, \hat{z}) = -\frac{8}{\pi} \lambda^2 e^{-\lambda \hat{z}} S_{-4,1}(\lambda).
\]

(38)

An expression for the function \( \hat{\alpha}_1 \) can be developed in a similar manner but with added complexity due to the form of \( f_1(\hat{r}) \). With some manipulation, the following expression is emitted

\[
\hat{\alpha}_1 = \lambda^4 e^{-\lambda \hat{z}} \left( \frac{32}{\pi \lambda} S_{-5,0} - \frac{24}{\pi \lambda} S_{-7,0} + (2S_{-4,1} - \lambda^2 S_{-6,1})(J_2 Y_0 - Y_2 J_0) \right),
\]

(39)

where the argument \( \lambda \) of the Bessel and Lommel functions is implied but omitted for brevity.

Finally, expressions for \( \{\alpha_m(\hat{r}, \hat{z}); m = 0, 1, 2, 3\} \) are determined by inversion in accordance with Equation (30), whence

\[
\alpha_m(\hat{r}, \hat{z}) = \int_0^\infty \hat{\alpha}_m(\lambda, \hat{z}) \varphi_\lambda(\hat{r}) \frac{\lambda d\lambda}{|H_m^\nu(\lambda)|^2}.
\]

(40)

For \( \alpha_0 \) and \( \alpha_1 \) we represent the series \( S_{\mu,\nu}(\lambda) \) using an appropriate hypergeometric form [14, Eq. 8.4.27.3]:

\[
S_{\mu,\nu}(2\sqrt{x}) = \frac{2^{\mu-1}}{c(\mu, \nu)} G_{1,3}^{1,1} \left( \begin{array}{c}
\mu + 1/2, \\ \mu + 1/2, \nu/2, -\nu/2
\end{array} \mid (\mu + \nu)/2, (\mu + 1)/2, \nu/2, -\nu/2 \right),
\]

(41)

with

\[
c(\mu, \nu) = \Gamma((1 - \mu - \nu)/2) \Gamma((1 - \mu + \nu)/2),
\]

(42)

where \( G \) and \( \Gamma \) denote Meijer \( G \)- and Gamma functions respectively. Although such a generalised representation of \( S_{\mu,\nu} \) does pose some numerical difficulties, this is our only recourse due to the fact that the addition of \( \mu + \nu \) amounts to odd negative integer values for each appearance of a Lommel function in \( \alpha_0 \) and \( \alpha_1 \). Consequently, more convenient expressions for \( S_{\mu,\nu} \) [13, § 3] are not possible.

The integrands (40) for \( m = 0, 1, 2 \) and 3 are sufficiently smooth and rapidly approach zero as \( \lambda \to \infty \) suggesting that its principal contribution arises in the vicinity of the lower limit of
integration. The numerical integration was conducted in Sage [15] which provides a wrapper for the GNU Scientific Library’s [16] adaptive Gaussian-Kronrod quadrature algorithm QAG\(^1\). The Meijer G-function (41) was tackled with the mpmath [17] Python library for arbitrary-precision floating-point arithmetic.

Results of the numerical integration for \(\{\alpha_m; m = 0, 1, 2, 3\}\) on \(\hat{z} = 0\) with \(1 \leq \hat{r} < 2\) are illustrated in Figure 2. Although each coefficient monotonically decreases for increasing \(\hat{r}\), this attenuation is rather slow and suggests a significant nonlinear forcing in the neighbourhood of the cylinder.

5 The Free-surface Elevation

We now turn our attention to the free-surface elevation (7) and determine the coefficients \(\eta_{m,n}\) of the expansion. In its exact form, the free surface elevation follows from the Bernoulli equation

\[
\zeta(r, \theta, t) = -\frac{1}{g}\left(\phi_t + \frac{1}{2} \nabla \phi \cdot \nabla \phi\right),
\]

where \(z - \zeta = 0\) defines the instantaneous free-surface position. As is customary, we transfer Equation (43) to the plane \(z = 0\) by employing a Taylor series expansion. In terms of our normalised coordinate system for the nonlinear potentials, this plane corresponds to \(\zeta_1 = \hat{z}a\). Incorporating the expansion for the velocity potential (6) along with the perturbation series for the free-surface elevation (7) in the Bernoulli equation (43) emits the following coefficients for Expressions (8a)–(8c) \(O(\varepsilon)\):

\[
\eta_{1,0} = -\frac{1}{g} \partial_t \phi_1,
\]
\[
\eta_{1,1} = -\frac{1}{g} \partial_t \phi_2,
\]
\[
\eta_{2,0} = -\frac{1}{g}\left(\frac{k^2}{2} \phi_1^2 + (\partial_t \phi_2)^2 + \frac{1}{\hat{r}^2} (\partial_\theta \phi_2)^2\right) - \frac{1}{g} (\partial_t \phi_1)^2,
\]
\[
\eta_{1,2} = -\frac{1}{g} \partial_t \phi_3,
\]
\[
\eta_{2,1} = -\frac{1}{g}\left(\partial_t \psi_2 - \frac{2k}{g} \partial_t \phi_1 \partial_t \phi_2 + k^2 \left(\phi_1 \phi_2 + \partial_t \phi_2 \partial_t \phi_3 + \frac{1}{\hat{r}^2} \partial_\theta \phi_2 \partial_\theta \phi_3\right)\right),
\]

\(^1\text{Numerical integration algorithms in the GNU Scientific Library are based on QUADPACK, see http://nines.cs.kuleuven.be/software/QUADPACK/}.\)
\[ \eta_{3,0} = - \frac{1}{g} \left( \partial_t \psi_3 + \partial_z \psi_2 \omega \cos \omega t + k^2 \left( \partial_t \phi_2 \partial_t \psi_2 + \frac{1}{f^2} \partial_t \phi_2 \partial_t \psi_2 - \phi_1 \partial_z \psi_2 \right) + \right. \]
\[ + \frac{3}{2} \frac{k^2}{g^2} \partial_t \phi_1^2 - \frac{1}{2} \frac{k^3}{g} \partial_t \phi_1 \phi_1^2 - \frac{3}{2} \frac{k^3}{g} \left( \partial_t \phi_1 (\partial_t \phi_2)^2 + \frac{1}{f^2} \partial_t \phi_1 \partial_t \phi_2 \right) \right), \]
\[ \eta_{4,-1} = - \frac{1}{g} \left( \partial_z \psi_3 \omega \cos \omega t + k^2 \left( \partial_t \phi_2 \partial_t \psi_3 + \frac{1}{f^2} \partial_t \phi_2 \partial_t \psi_3 - \phi_1 \partial_z \psi_3 \right) \right). \] 

More explicit expressions for the free-surface elevation follow by substituting \( \phi_1, \phi_2, \phi_3, \psi_1 \) and \( \psi_2 \) (16a)–(16c), (23) and (24) into the above equations for \( \eta_{m,n} \).

For the free-surface elevation at the surface of the cylinder we set \( \hat{r} = 1 \), and to second-order we require \( \eta_{1,1} \) and \( \eta_{2,0} \). The linear combination of these in accordance with Equation (8b) gives

\[ \zeta_2 = \frac{1}{2} kA^2 (\cos 2\theta - \frac{1}{2}) - 2kAa \cos \theta \cos \omega t + \frac{1}{2} kA^2 (\cos 2\theta - \frac{1}{2}) \cos 2\omega t \] 

which essentially agrees with Faltinsen et al. [3, Eq. 3.13 with \( R = 1 \) in their notation]. Similarly, for the third-order free-surface elevation \( \zeta_3 \) at \( \hat{r} = 1 \) we can make use of Expansion (8c) with Equations (47)–(50). After organising \( \zeta_3 \) into its harmonic components, the following compact expression results

\[ \zeta_3 = \sum_{m=1}^{3} a_m(\theta) \sin m\omega t + \sum_{m=0}^{4} b_m(\theta) \cos m\omega t, \] 

where the odd and even coefficients of the series are defined

\[ a_1 = \frac{1}{2} k^2 Aa^2 \left( \ln \left( \frac{kA}{2} \right) + \gamma - \frac{1}{2} - \cos 2\theta \right) + \frac{1}{4} k^2 A^3 (9 \cos 2\theta - 5), \]
\[ a_2 = -\frac{1}{2} k^2 A^2a \left( 5 \cos \theta + \cos 3\theta \right), \]
\[ a_3 = \frac{1}{4} k^2 A^3 (1 - 3 \cos 2\theta), \]

and

\[ b_0 = -\frac{3k^2 A^4}{8a} \left( \alpha_1 + (3\alpha_3 - \alpha_1) \cos 2\theta - 3\alpha_3 \cos 4\theta \right), \]
\[ b_1 = \frac{\pi k^2}{4} Aa^2 - k^2 A^3 \left( \left( \frac{3}{4} \alpha_1 + \alpha_2 \right) \cos \theta + \left( \frac{3}{4} \alpha_3 - \alpha_2 \right) \cos 3\theta \right), \]
\[ b_2 = -2k^2 A^2a(\alpha_0 + \alpha_2 \cos 2\theta) + \frac{k^2 A^4}{2a} \left( \alpha_1 + (3\alpha_3 - \alpha_1) \cos 2\theta - 3\alpha_3 \cos 4\theta \right), \]
\[ b_3 = k^2 A^3 \left( \left( \alpha_2 + \frac{3}{4} \alpha_1 \right) \cos \theta - \left( \alpha_2 - \frac{3}{4} \alpha_3 \right) \cos 3\theta \right), \]
\[ b_4 = -\frac{k^2 A^4}{8a} \left( \alpha_1 + (3\alpha_3 - \alpha_1) \cos 2\theta - 3\alpha_3 \cos 4\theta \right), \]

In the above expressions, it is understood that \( \alpha_m \) is defined on the plane \( \hat{z} = 1 \) and the dispersion relation \( \omega^2/g = k(1 + k^2 A^2) \) holds.

It is interesting to note that while \( \eta_{3,0} \) contributes to both the first- and third-harmonics, \( \eta_{4,-1} \) contributes to the zeroth-, second- and fourth-harmonics. These contributions are consistent
with a conventional Stokes expansion in terms of $\varepsilon_1$ only. In terms of $\varepsilon_2$, $\varepsilon_2^3 \eta_{3,0}$ is the leading order contribution of a Stokes expansion to $O(A^3)$. Moreover, although $\varepsilon_1^4 / \varepsilon_2 \eta_{4,-1}$ is of fourth-order in wave amplitude, it is strictly of $O(\varepsilon^3)$ in the dual expansion considered here.

6 Discussion

It is important to reiterate that a conventional Stokes expansion, whereby $A \ll L$, when applied to the free-surface boundary condition, implicitly restricts $A \ll a$. Ostensibly, this is by virtue of the fact that $a = O(L)$. In contrast, the theory presented here explicitly assumes $A/a = O(1)$. This is a salient point and constitutes the underlying difference between the two schemes.

Consequently, certain contributions present in both $\psi_2$ and $\psi_3$ only arise due to the intrinsic assumption $A/a = O(1)$. For example, additional first- and second-harmonic contributions, not present in either $\zeta_1$ or $\zeta_2$, result from the forcing of both Equations (16a) and (16b) at the free-surface. Some of these additional contributions would in fact be accounted for in canonical diffraction theory to $O(A^2)$ and $O(A^3)$ while those of $O(A^4/a)$ would not. In other words, contributions of $O(A^4/a)$ are strictly inconsistent with a Stokes expansion to third-order wave steepness.

To illustrate the wave run-up at $\theta = \pi$, produced by the present theory, we present a time trace, Figure 3, for $ka = 0.208$ and $kA = 0.4$. Although this wave is in fact approaching the wave breaking limit in deep water, $kA_{\text{max}} = 0.14\pi$, it does serve to illustrate third-order contributions arising from $\zeta_3$. In comparison to $\zeta_1$, second- and third-order components are approximately $0.37\zeta_1$ and $0.08\zeta_1$ respectively. It is worth noting that $\zeta_3$ emits one time independent term that does not contribute to the wave run-up at $\theta = \pi$ due to an equal and opposite forcing at the free-surface. Consequently, only $\zeta_2$ contributes to the zeroth-harmonic at $\theta = \pi$ and is purely of $O(A^2)$ in origin.

More general results for the free-surface elevation at $(r, \theta) = (a, \pi)$ are presented in Figure 5. We observe that the wave run-up is magnified with both the scattering parameter and the wave steepness. However, this magnification with wave steepness is considerably more dramatic and gradually increases with increasing $ka$. In conventional diffraction theory [1], the wave run-up at $\theta = \pi$ should approach 2 as $ka \to \infty$. In the present theory, however, this is not the case because it is only valid for asymptotically small $ka$.

To investigate the effectiveness of the present theory in predicting the harmonic components of the wave run-up we first consider the fundamental frequency. Figure 6 compares the first-harmonic prediction,

$$\zeta^{(1)}(1, \pi, t) = \varepsilon_1 \eta_{1,0} + \varepsilon_1 \varepsilon_2 \eta_{1,1} + \chi_{1,1} \sin \omega t + \chi_{2,1} \cos \omega t,$$

with linear diffraction theory [1] and measured first-harmonics of the wave run-up on a circular cylinder [8] — the first-harmonic wave run-up is denoted $R(1)$. We observe that although the present formulation under predicts the measured first-harmonic for large $kA$ it appears to be an improvement over the linear diffraction formulation. This observation is particularly evident.
for values of $ka = 0.208$ and $0.417$. On the other hand, the LWL theory is clearly invalid for $ka = 0.698$. This is ostensibly because the assumption of $ka \ll 1$ has been violated and has therefore compromised the comparison.

We now investigate the validity of the present theory by comparing the wave run-up to measured values [8]. To accommodate this, we present Figure 7 which shows the normalised wave run-up $R/A$, at the position $\theta = \pi$, versus the infinite depth wave steepness $k_0A$. Also included are, correct to second-order in wave steepness, diffraction computations from the boundary integral equation method program WAMIT and linear diffraction theory [1]. Our first observation is that, for $ka = 0.698$, the present theory distinctly over-predicts the wave run-up. However, this is not surprising as the condition $ka \ll 1$ is not sufficiently satisfied and local diffraction effects associated solely with $ka$ are clearly important here.

In contrast, the present theory performs well for both $ka = 0.417$ and $0.208$ in the domain of $k_0A < 0.15 - 0.2$ (see Figure 7). Interestingly, both second-order diffraction computations and the present third-order theory agree for $ka = 0.417$ and, in addition, both compare favourably to the measured results. However, for $ka = 0.208$, the present theory agrees more favourably with measured values than second-order diffraction computations. In particular, this observation distinctly holds true for $k_0A$ less than approximately $0.15$. Moreover, this suggests than third-order wave steepness nonlinearities are significant in long waves and in fact more important than nonlinearities associated with local wave diffraction associated with $ka$.

We now turn our attention to the free-surface elevation around the boundary of the cylinder. Figure 8 illustrates two plots of $R(\theta)/A$ versus $\theta/\pi$ for the moderate wave steepness $kA = 0.016$. While the first plot concerns the relatively small scattering $ka = 0.208$, the second concerns a more moderate value $ka = 0.417$. Of the two, the first is indicative of the range of validity of the present theory. Whereby the incident wavelength is much larger than the cylinder’s diameter.

In all cases we observe a local minimum of $R(\theta)/A$ in the region of $0 < \theta/\pi < 0.5$ (Figure 8). Physically, this results from an increase in fluid momentum as the flow negotiates the cylinder, hence to conserve energy, the free-surface must decrease in elevation accordingly. It appears as though this local minimum is a function of both local diffraction $ka$ and wave steepness $kA$. Presumably, this minimum shifts closer to $\theta/\pi = 0.5$ in the limit of $ka \to 0$ because the cylinder becomes increasingly transparent to the onset flow. This observation is consistent with the present theory.

Of the two plots, Figure 8a demonstrates that the present theory captures the trend exhibited by the measured data when local diffraction effects, associated with $ka$, are small. Moreover, the theory also agrees with second-order diffraction computations. Also shown, is the improvement by the present theory over linear diffraction theory (Figure 8). For more moderate scattering, $ka = 0.417$ for instance, while the present theory correctly predicts the amplification of the wave run-up at $\theta = \pi$, it slightly under-predicts the the amplification at $\theta/\pi = 0.5$ – by approximately $9\%$ (see Figure 8b). This under-prediction is likely caused by dominant $\cos \theta$ symmetry terms associated with the second-harmonic contributions in the expression for $\zeta_3$. Despite this, the present theory does capture the overall trend of the measured data.
7 Conclusions

We have described a long wavelength theory, correct to $O(\varepsilon^3)$, for the free-surface elevation around a vertical cylinder in plane progressive waves. It provides an efficient means to evaluate the wave run-up provided that the intrinsic assumptions of $A/a = O(1)$, $ka \ll 1$ and $kA \ll 1$ are observed. Indeed, after utilising measured results, we have demonstrated that the present theory performs well provided these assumptions are satisfied.

Salient features of the solution to the free-surface elevation are nonlinear effects of $O(A^3)$ – operating at the first- and third-harmonics, and contributions of $O(A^4/a)$ – operating at the zeroth-, second-, and fourth-harmonics. For instance, these nonlinear effects provide an improved account of the first-harmonic wave run-up over linear diffraction theory when local diffraction effects associated with $ka$ are small. Moreover, when these local diffraction effects are small the present theory agrees more favourably with the overall wave run-up at $\theta = \pi$ than second-order diffraction theory computations.

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References

[1] T. H. Havelock. The pressure of water waves upon a fixed obstacle. Proc. Roy. Soc. Lond. A, 175:409–421, 1940. doi: 10.1098/rspa.1940.0066.

[2] J. N. Hunt and R. E. Baddour. The diffraction of a nonlinear progressive wave by a vertical cylinder. Quarterly Journal of Mechanics and Applied Mathematics, 34(1):69–87, 1981. doi: 10.1093/qjmam/34.1.69.

[3] O. M. Faltinsen, J. N. Newman, and T. Vinje. Nonlinear wave loads on a slender vertical cylinder. J. Fluid Mech., 289:179–198, 1995. doi: 10.1017/S0022112095001297.

[4] Š. Malenica and B. Molin. Third-harmonic wave diffraction by a vertical cylinder. J. Fluid Mech., 302:203–229, 1995. doi: 10.1017/S0022112095004071.

[5] J. N. Newman. Nonlinear scattering of long waves by a vertical cylinder. Waves and Non-linear Processes in Hydrodynamics, pages 91–102, 1996. Edited by J. Grue et al.

[6] O. M. Faltinsen. Ringing loads on a slender vertical cylinder of general cross-section. J. Eng. Math., 35:199–217, 1999. doi: 10.1023/A:1004362827262.
[7] J. N. Newman and C.-H. Lee. Runup on a vertical cylinder in long waves. In R. Eatock Taylor, editor, *10th Intl. Workshop on Water Waves and Floating Bodies*, pages 198–191, Oxford, UK, April 2-5 1995.

[8] M. T. Morris-Thomas and K. P. Thiagarajan. The run-up on a cylinder in progressive surface gravity waves: harmonic components. *Appl. Ocean Res.*, 26:98–113, 2004. doi: 10.1016/j.apor.2004.11.002.

[9] H. Lamb. *Hydrodynamics*. Dover Publications, New York, 6th edition, 1932.

[10] M. J. Lighthill. Waves and hydrodynamic loading. *BOSS’79*, 1:1–40, 1979.

[11] N. N. Lebedev, I. P. Skalskaya, and Y. S. Uflyand. *Problems of Mathematical Physics*. Prentice-Hall, Englewood Cliffs, N.J., USA, 1965. Russian Translation by R.A. Silverman.

[12] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover Publications, New York, 1965. URL http://www.math.sfu.ca/~cbm/aands/.

[13] J. Steinig. The sign of Lommel’s function. *Transactions of the American Mathematical Society*, 163:123–129, 1972. doi: 10.1090/S0002-9947-1972-0284625-X.

[14] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev. *Integrals and Series Volume 3: More Special Functions*. Gordon and Breach Science Publishers, New York, 1990.

[15] W. A. Stein et al. *Sage Mathematics Software (Version 4.6.1)*. The Sage Development Team, 2011. URL http://www.sagemath.org.

[16] M. Galassi, J. Davies, J. Theiler, B. Gough, G. Jungman, M. Booth, and F. Rossi. *GNU Scientific Library Reference Manual*. Free Software Foundation, Inc., 51 Franklin Street, Fifth Floor, Boston, MA 02111, USA, edition 1.11, for gsl version 1.11 edition, 5 February 2008. URL http://www.gnu.org/software/gsl/manual/.

[17] F. Johansson et al. *mpmath: a Python library for arbitrary-precision floating-point arithmetic (version 0.14)*, February 2010. URL http://code.google.com/p/mpmath/.
\[ \zeta_1 = A \sin \omega t \]

Figure 1: Schematic of the coordinate system adopted for long wavelength theory.

Figure 2: The coefficients \( \alpha_m(\hat{r}, 0) \) as a function of the scaled radial distance for: \( m = 0 \) (——); \( m = 1 \) (-----); \( m = 2 \) (···); and \( m = 3 \) (····).
Figure 3: Time series representation of the wave run-up on a vertical cylinder at $\theta = \pi$ for $ka = 0.208$ and $kA = 0.4$. The top figure shows the contributions of each order $\zeta_1$, $\zeta_2$ and $\zeta_3$. The bottom figure shows the systematic addition of each of these components.
Figure 4: Surface and corresponding contour plots of the maximum free-surface elevation \( \zeta / A \) on the upwave side of the cylinder surface from LWL theory. Here, \( \zeta / A \) is evaluated at the position \((r, \theta) = (a, \pi)\) and plotted against the scattering parameter \(ka\) and wave steepness \(kA = ka(A/a)\).

Figure 5: Contour plots of the harmonic components of the free-surface elevation computed at the position \((r, \theta) = (a, \pi)\) plotted against the scattering parameter \(ka\) and wave steepness \(kA = ka(A/a)\): (a) the first-harmonic; (b) the second-harmonic; and (c) the third-harmonic.
Figure 6: The normalised modulus of the first-harmonic component of the wave run-up \(|R(\omega_1)|/A\), evaluated at the incident wave side of the cylinder \((r = a, \theta = \pi)\), versus the wave steepness \(kA\). The two axes correspond to scattering parameters of (a) \(ka = 0.208\) and (b) \(ka = 0.417\). The data sets are: (—–) present third-order LWL computations; (– – –) linear diffraction theory; and (*) measured values [8].

Figure 7: The maximum wave run-up \(R/A\) evaluated at \((r = a, \theta = \pi)\) plotted against the wave steepness \(kA\). The present long wavelength theory is denoted 1LWL and 3LWL for the first- and third-order computations; LDT and 2DT correspond to linear and second-order diffraction results respectively. The data \((\mu \pm \sigma)\) corresponds to measured values of the run-up in plane progressive waves [8]; \(\mu\) and \(\sigma\) denote the mean and standard deviation of the measured \(R\) respectively.
Figure 8: The maximum wave run-up around a truncated vertical cylinder \((d/a = 2.53)\) in plane progressive waves of steepness \(kA = 0.016\) for scattering parameters: \(ka = 0.208\) (a); and \(ka = 0.417\) (b). The variation in the maximum free-surface elevation \(R/A\) with \(\theta/\pi\) is shown for the present third-order long wavelength (3LWL) theory, linear (LDT) and second-order (2DT) diffraction theories, and measured data [8].