Asymptotic analysis of the Askey-scheme II: from Charlier to Hermite

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Abstract

We analyze the Hermite polynomials $H_n(\xi)$ and their zeros asymptotically as $n \to \infty$, using the limit relation between the Charlier and Hermite polynomials. Our formulas involve some special functions and they yield very accurate approximations.

Keywords: Hermite polynomials, Askey-scheme, asymptotic analysis, orthogonal polynomials, hypergeometric polynomials, special functions.

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1 Introduction

The Hermite polynomials $H_n(x)$ are defined by

$$H_n(x) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{(2x)^{n-2k}}{k!(n-2k)!}$$

for $n = 0, 1, \ldots$. They satisfy the orthogonality condition

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = \sqrt{\pi} 2^n n! \delta_{mn}$$

and the reflection formula

$$H_n(-x) = (-1)^n H_n(x).$$

The Hermite polynomials are special cases of the parabolic cylinder function $U(a, z)$,

$$H_n(x) = 2^n \exp \left( \frac{x^2}{2} \right) U \left( -n - \frac{1}{2}, \sqrt{2}x \right),$$

which was analyzed by Nico Temme in [5], [19] and [22].

The Hermite polynomials have been extensively studied since the pioneer article of C. Hermite [6] in 1864 (they were previously considered by Fourier and Chebyshev). They have many applications in the physical sciences and are particularly important in the quantum mechanical treatment of the harmonic oscillator [13] (see also [1], [7] and [24] for some extensions). We refer the interested reader to [3] and [15] for further properties and references.

There are several families of orthogonal polynomials which have asymptotic approximations in terms of $H_n(x)$. Some cases studied by Nico Temme include the Gegenbauer [17], Laguerre [12], [16], Tricomi-Carlitz and Jacobi polynomials [10]. He also considered the asymptotic representations of other families of polynomials such as the generalized Bernoulli, Euler, Bessel and Buchholz polynomials in [11].

A rich source of asymptotic relations between the $H_n(x)$ and other polynomials [4], [20], [21] is provided by the Askey-scheme of hypergeometric orthogonal polynomials [8].
where the arrows indicate limit relations between the polynomials.

In particular, the limit relation between the Charlier polynomials $C_n^{(a)}(x)$ and the Hermite polynomials is given by

$$
\lim_{a \to \infty} (-1)^n (2a)^{\frac{3}{2}} C_n^{(a)} \left( a + x \sqrt{2a} \right) = H_n(x).
$$

(3)

In this article we investigate the asymptotic behavior of $H_n(x)$ as $n \to \infty$, using (3) and the asymptotic results on the Charlier polynomials derived in [2]. We believe that our method provides a useful approach to asymptotic analysis and could be used for other families of polynomials of the Askey scheme.

2 Previous results

We define the Charlier polynomials by

$$
C_n^{(a)}(x) = {}_2F_0 \left( \begin{array}{c}
-n, -x \\
- \frac{1}{a}
\end{array} \right), \quad n = 0, 1, \ldots
$$

(4)

with $a > 0$. The following results were derived in [2].

**Theorem 1** As $n \to \infty$, $C_n^{(a)}(x)$ admits the following asymptotic approximations, with

$$
\Omega^\pm = (\sqrt{n} \pm \sqrt{a})^2.
$$

(5)
1. \( n = O(1) \).

\[
C_n^{(a)} \simeq \left(1 - \frac{x}{a}\right)^n
\]  \hspace{1cm} (6)

2. \( x < \Omega^-, \ 0 < n < a \).

\[
C_n^{(a)} \sim F_3(x) = \exp[\Psi_3(x)] \, L_3(x),
\]  \hspace{1cm} (7)

where

\[
\Psi_3(x) = x \ln \left(\frac{a + x - n + \Delta}{2a}\right) + n \ln \left(\frac{a - x + n + \Delta}{2a}\right) + \frac{1}{2} (a - x - n + \Delta),
\]  \hspace{1cm} (8)

and

\[
L_3(x) = \sqrt{\frac{a - x - n + \Delta}{2\Delta}},
\]  \hspace{1cm} (9)

with

\[
\Delta(x) = \sqrt{a^2 - 2a(x + n) + (x - n)^2}.
\]  \hspace{1cm} (10)

3. \( \Omega^+ < x \).

\[
C_n^{(a)} \sim F_4(x) = (-1)^n \exp[\Psi_4(x)] \, L_4(x),
\]  \hspace{1cm} (11)

where

\[
\Psi_4(x) = x \ln \left(\frac{a + x - n - \Delta}{2a}\right) + n \ln \left(\frac{x - a - n + \Delta}{2a}\right) + \frac{1}{2} (a - x - n + \Delta),
\]  \hspace{1cm} (12)

and

\[
L_4(x) = \sqrt{\frac{x - a + n + \Delta}{2\Delta}}.
\]  \hspace{1cm} (13)

4. \( x \approx \Omega^-, \ 0 < n < a \).

\[
C_n^{(a)} \sim \sqrt{2\pi} \left(\frac{n}{a}\right)^{\frac{1}{2}} \left(\frac{\sqrt{a} - \sqrt{n}}{\sqrt{\sqrt{a} - \sqrt{n}}}\right)^{\frac{3}{2}} \, \text{Ai} \left[ \left(\frac{n}{a}\right)^{\frac{1}{2}} \frac{(\Omega^- - x)}{(\sqrt{a} - \sqrt{n})^{\frac{3}{2}}} \right]
\]  \hspace{1cm} \times \exp \left[\frac{1}{2} \ln \left(\frac{n}{a}\right) + x \ln \left(1 - \sqrt{\frac{n}{a}}\right) + \sqrt{an} - \sqrt{n}\right],
\]  \hspace{1cm} (14)

where \( \text{Ai} (\cdot) \) is the Airy function.

5. \( \Omega^- < x < \Omega^+ \).

\[
C_n^{(a)} \sim F_{10}(x) = F_3(x) + F_4(x).
\]  \hspace{1cm} (15)
6. $x \approx \Omega^+.$

\[
C_n^{(a)} \sim \sqrt{2\pi} \left( \frac{n}{a} \right)^{\frac{1}{6}} \left( \sqrt{a} + \sqrt{n} \right)^{\frac{1}{2}} \mathrm{Ai} \left[ \left( \frac{n}{a} \right)^{\frac{1}{6}} \left( \frac{x - \Omega^+}{\sqrt{a} + \sqrt{n}} \right)^{\frac{1}{4}} \right] 
\times (-1)^n \exp \left[ \frac{1}{2} n \ln \left( \frac{n}{a} \right) + x \ln \left( 1 + \sqrt{\frac{n}{a}} \right) - \sqrt{an} - \sqrt{n} \right].
\]

(16)

3 Limit analysis

From (5) we have, as $a \to \infty$,

\[
\frac{\Omega^\pm - a}{\sqrt{2a}} \to \pm \sqrt{2n}.
\]

(17)

Hence, the six regions of Theorem 1 transform into the following regions:

1. Region I: $n = O(1)$.

Setting

\[
x = a + \xi \sqrt{2a}
\]

in (6) and using (3), we get

\[
H_n(\xi) \simeq (2\xi)^n.
\]

(19)

The formula above is exact for $n = 0, 1$ and is a good approximation when $\xi \gg n$.

2. Region II: $\xi < -\sqrt{2n}$.

From (10) and (18) we have

\[
\Delta \sim \sqrt{2}\sigma \sqrt{a} - \frac{\xi n}{\sigma}, \quad a \to \infty,
\]

(20)

with

\[
\sigma = \sqrt{\xi^2 - 2n}.
\]

(21)

Using (20) in (8) and (9), we get

\[
\frac{n}{2} \ln(2a) + \Psi_3(x) \sim \Phi_1(\xi) \equiv n \ln (\sigma - \xi) + \frac{\xi^2 + \sigma \xi - n}{2},
\]

(22)
\[ L_3(x) \sim U_1(\xi) = \sqrt{\frac{1}{2} \left( 1 - \frac{\xi}{\sigma} \right)}, \quad (23) \]

as \( a \to \infty \). Thus, from (3) we have
\[ H_n(\xi) \sim \Lambda_1(\xi) \equiv (-1)^n \exp [\Phi_1(\xi)] U_1(\xi). \quad (24) \]

3. Region III: \( \xi > \sqrt{2n} \).

Using (20) in (12) and (13), we obtain
\[ \frac{n}{2} \ln(2a) + \Psi_4(x) \sim \Phi_2(\xi) \equiv n \ln(\sigma + \xi) + \frac{\xi^2 - \sigma \xi - n}{2}, \quad (25) \]

\[ L_4(x) \sim U_2(\xi) = \sqrt{\frac{1}{2} \left( 1 + \frac{\xi}{\sigma} \right)}, \quad (26) \]

as \( a \to \infty \). Hence,
\[ H_n(\xi) \sim \Lambda_2(\xi) \equiv \exp [\Phi_2(\xi)] U_2(\xi). \quad (27) \]

Note that
\[ \Lambda_1(-\xi) = (-1)^n \Lambda_2(\xi), \]

as one would expect from (2).

4. Region IV: \( \xi \approx -\sqrt{2n} \).

Using (18) in (14) we have, as \( a \to \infty \),
\[ \frac{n}{2} \ln(2a) + \frac{1}{2} n \ln \left( \frac{n}{a} \right) + x \ln \left( 1 - \sqrt{\frac{n}{a}} \right) + n \sqrt{an} - \sqrt{n} \sim \Phi_3(\xi), \]

where
\[ \Phi_3(\xi) = \frac{n}{2} \ln(2n) - \frac{3}{2} n - \xi \sqrt{2n} = \frac{n}{2} \ln(2n) - \frac{3}{2} n - \xi \sqrt{2n}. \quad (28) \]

Also,
\[ \sqrt{2\pi} \left( \frac{n}{a} \right)^{\frac{1}{2}} \left( \sqrt{a} - \sqrt{n} \right)^{\frac{1}{2}} \sim n^{\frac{1}{2}} \]

and
\[ \left( \frac{n}{a} \right)^{\frac{1}{2}} \frac{(\Omega^+ - x)}{(\sqrt{a} - \sqrt{n})^{\frac{1}{2}}} \sim -n^{\frac{1}{2}} \sqrt{2} \left( \xi + \sqrt{2n} \right). \]
Therefore,
\[
H_n(\xi) \sim \Lambda_3(\xi) \equiv (-1)^n \sqrt{2\pi n^{\frac{1}{6}}} \exp[\Phi_3(\xi)] \text{Ai}
\left[-n^{\frac{1}{6}} \sqrt{2}(\xi + \sqrt{2n})\right].
\] (29)

5. Region V: \( \xi \approx \sqrt{2n} \).

Using (18) in (16) we have, as \( a \to \infty \),
\[
\frac{n}{2} \ln(2a) + \frac{1}{2} n \ln \left(\frac{n}{a}\right) + \ln \left(1 + \frac{n}{a}\right) - \sqrt{an} - \sqrt{n} \sim \Phi_4(\xi),
\]
with
\[
\Phi_4(\xi) = \frac{n}{2} \ln (2n) - \frac{3}{2} n - \xi \sqrt{2n}.
\] (30)

Also,
\[
\sqrt{2\pi} \left(\frac{n}{a}\right)^{\frac{1}{6}} \left(\sqrt{a} + \sqrt{n}\right)^{\frac{1}{2}} \sim n^{\frac{1}{6}}
\]
and
\[
\left(\frac{n}{a}\right)^{\frac{1}{6}} \frac{(x - \Omega^+)}{(\sqrt{a} + \sqrt{n})^{\frac{1}{2}}} \sim n^{\frac{1}{6}} \sqrt{2} \left(\xi - \sqrt{2n}\right).
\]

Therefore,
\[
H_n(\xi) \sim \Lambda_4(\xi) \equiv \sqrt{2\pi n^{\frac{1}{6}}} \exp[\Phi_4(\xi)] \text{Ai}
\left[n^{\frac{1}{6}} \sqrt{2}\left(\xi - \sqrt{2n}\right)\right].
\] (31)

Once again, we have
\[
\Lambda_3(-\xi) = (-1)^n \Lambda_4(\xi)
\]

6. Region VI: \( -\sqrt{2n} \ll \xi \ll \sqrt{2n} \).

From (15), we immediately obtain
\[
H_n(\xi) \sim \Lambda_5(\xi) \equiv \Lambda_1(\xi) + \Lambda_2(\xi).
\]

Since \(-1 < \frac{\xi}{\sqrt{2n}} < 1\), we set
\[
\xi = \sqrt{2n} \sin(\theta), \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.
\] (32)

From (21) we have
\[
\sigma = \sqrt{2n} \cos(\theta) \text{i}.
\]
Thus,

\[ n\pi i + \Phi_1 (\xi) = \frac{n}{2} [\ln (2n) - \cos (2\theta)] + n \left[ \frac{1}{2} \sin (2\theta) + \theta - \frac{\pi}{2} \right] i, \]

\[ \Phi_2 (\xi) = \frac{n}{2} [\ln (2n) - \cos (2\theta)] - n \left[ \frac{1}{2} \sin (2\theta) + \theta - \frac{\pi}{2} \right] i, \]

and

\[ U_1 (\xi) = \frac{\exp (\frac{\xi i}{2})}{\sqrt{2 \cos (\theta)}}, \quad U_2 (\xi) = \frac{\exp (-\frac{\xi i}{2})}{\sqrt{2 \cos (\theta)}}. \]

Hence,

\[ \Lambda_5 \left[ \sqrt{2n} \sin (\theta) \right] = \sqrt{\frac{2}{\cos (\theta)}} \exp [\Phi_5 (\theta)] \cos (\Theta), \quad (33) \]

with

\[ \Phi_5 (\theta) = \frac{n}{2} [\ln (2n) - \cos (2\theta)] \quad (34) \]

and

\[ \Theta = n \left[ \frac{1}{2} \sin (2\theta) + \theta - \frac{\pi}{2} \right] + \frac{\theta}{2}. \quad (35) \]

Using (32), we can write (34) and (35) in terms of \( \xi \)

\[ \Phi_5 (\theta) = \frac{n}{2} [\ln (2n) - 1] + \frac{\xi^2}{2}, \]

\[ \Theta = \frac{\xi}{2} \sqrt{2n - \xi^2} + \left( n + \frac{1}{2} \right) \arcsin \left( \frac{\xi}{\sqrt{2n}} \right) - n \frac{\pi}{2}. \]

Also,

\[ \sqrt{\frac{2}{\cos (\theta)} = \sqrt{2} \left( 1 - \frac{\xi^2}{2n} \right)^{-\frac{1}{2}}. \]

Therefore,

\[ \Lambda_5 (\xi) = \sqrt{2} \left( 1 - \frac{\xi^2}{2n} \right)^{-\frac{1}{4}} \exp \left\{ \frac{n}{2} [\ln (2n) - 1] + \frac{\xi^2}{2} \right\} \]

\[ \times \cos \left[ \frac{\xi}{2} \sqrt{2n - \xi^2} + \left( n + \frac{1}{2} \right) \arcsin \left( \frac{\xi}{\sqrt{2n}} \right) - n \frac{\pi}{2} \right]. \quad (36) \]
Considering the leading term of (36) as $n \to \infty$, we obtain

$$\Lambda_5(\xi) \sim \sqrt{2} \exp \left\{ \frac{n}{2} \left[ \ln (2n) - 1 \right] + \frac{\xi^2}{2} \right\} \cos \left( n\frac{\pi}{2} - \xi \sqrt{2n} \right)$$

in agreement with formula (4.14.9) in [9].

4 Zeros

Let us denote by $\zeta_1^n > \zeta_2^n > \cdots > \zeta_n^n$ the zeros of $H_n(\xi)$, enumerated in decreasing order. It then follows from (33) that $\zeta_j^n = \sqrt{2}n \sin (\tau_j^n)$, where $\tau_j^n$ satisfies

$$n \left[ \frac{1}{2} \sin (2\tau_j^n) + \tau_j^n - \frac{\pi}{2} \right] + \frac{\tau_j^n}{2} = \frac{\pi}{2} - j\pi, \quad 1 \leq j \leq n.$$

We can rewrite the equation

$$n \left[ \frac{1}{2} \sin (2t) + t - \frac{\pi}{2} \right] + \frac{t}{2} = A$$

as Kepler’s equation

$$E - \varepsilon \sin(E) = M, \quad (37)$$

with

$$E = 2t, \quad M = 2\frac{2A + n\pi}{2n + 1}, \quad \varepsilon = -\frac{2n}{2n + 1}. \quad (38)$$

It is well known [23] that the solution of (37) can be expressed as a Kapteyn series

$$E = M + 2 \sum_{k=1}^{\infty} \frac{1}{k} J_k (k\varepsilon) \sin (kM), \quad (39)$$

where $J_k (\cdot)$ is a Bessel function of the first kind.

Thus, using (38) in (39) with $A = \frac{\pi}{2} - j\pi$, we obtain

$$\tau_j^n = \pi \frac{1 + n - 2j}{2n + 1} + \sum_{k=1}^{\infty} \frac{1}{k} J_k \left( -\frac{2n}{2n + 1} k \right) \sin \left( 2\pi \frac{1 + n - 2j}{2n + 1} k \right), \quad (40)$$
for $1 \leq j \leq n$. Using the reflection formula \[14\] $J_k(-x) = (-1)^k J_k(x)$, we can write (40) as

$$
\tau_j^n = \frac{\pi}{2} - \frac{\pi}{2} (4j - 1) N^{-1} - \sum_{k=1}^{\infty} \frac{1}{k} J_k \left[ (1 - N^{-1}) k \right] \sin \left( \frac{4j - 1}{N} k\pi \right),
$$

(41)

where $N = 2n + 1$.

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