Violation of general Bell inequalities by a pure bipartite quantum state

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Abstract

In the present article, based on the formalism introduced in [Loubenets, J. Math. Phys. 53, 022201 (2012)], we derive for a pure bipartite quantum state a new upper bound on its maximal violation of general Bell inequalities. This new bound indicates that, for an infinite dimensional pure bipartite state with a finite sum of its Schmidt coefficients, violation of any general Bell inequality is bounded from above by the value independent on a number of settings and a type of outcomes, continuous or discrete, specific to this Bell inequality. As an example, we apply our new general results to specifying upper bounds on the maximal violation of general Bell inequalities by infinite dimensional bipartite states having the Bell states like forms comprised of two binary coherent states $|\alpha\rangle, |\alpha\rangle$, with $\alpha > 0$. We show that, for each of these bipartite coherent states, the maximal violation of general Bell inequalities cannot exceed the value 3 and analyse numerically the dependence of the derived analytical upper bounds on a parameter $\alpha > 0$.

1 Introduction

Ever since the seminal paper of Bell\textsuperscript{1} quantum violation of Bell inequalities was analysed, analytically and numerically, in many papers and is now used in many quantum information processing tasks. The maximal quantum violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality\textsuperscript{2} is well known\textsuperscript{3,4,5,6,7} and constitutes $\sqrt{2}$, for any bipartite quantum state, possibly infinite dimensional. It was also recently proved\textsuperscript{8,9} that the maximal quantum violation of the original Bell inequality\textsuperscript{11} is equal to $\frac{3}{2}$.

\textsuperscript{1}For the original Bell inequality see also\textsuperscript{10}.

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More generally, quantum violation of any (unconditional) correlation bipartite Bell inequality cannot exceed the real Grothendieck’s constant \( K^R_G \in [1.676, 1.783] \) but this is not already the case for quantum violation of general bipartite Bell inequalities, in particular, bipartite Bell inequalities on joint probabilities, and last years this problem was intensively discussed in the literature within different mathematical approaches.

For an arbitrary bipartite state \( \rho \) on a Hilbert space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \), with dimensions \( d_n := \dim \mathcal{H}_n \), the upper bound on the maximal violation of general bipartite Bell inequalities for \( S_n \) measurements with any type of outcomes at each \( n = 1, 2 \) site follows from the general upper bounds derived in [14, 20, 21, 22] for an \( N \)-partite quantum state and reads:

\[
2 \min\{d_1, d_2, S_1, S_2\} - 1
\]

– in case of generalized quantum measurements and

\[
\min\left\{ \sqrt{d}, 3 \right\}, \quad \text{for } S = 2, \quad \text{min}\left\{ \sqrt{d^S}, 2 \min\{d, S\} - 1 \right\}, \quad \text{for } S \geq 3,
\]

– in the case \( d_1 = d_2 = d, S_1 = S_2 = S \) and projective quantum measurements at both sites.

From bound (1) it follows that, for an arbitrary two-qudit state, pure or mixed, violation of any general Bell inequality cannot exceed the value

\[
2 \min\{d_1, d_2\} - 1,
\]

while, for an arbitrary infinite dimensional bipartite quantum state, pure or mixed, violation of any general Bell inequality with \( S_1, S_2 \geq 1 \) settings at the corresponding sites cannot exceed

\[
2 \min\{S_1, S_2\} - 1.
\]

For the two-qudit state \( \frac{1}{\sqrt{d}} \sum_{j=1}^{d} |j\rangle \otimes |j\rangle \), it was also found in [15] (see there Theorem 0.3) via the operator space formalism that the maximal quantum violation of all general Bell inequalities cannot exceed \( C \frac{d}{\sqrt{\ln d}} \), where \( C \) is an unknown constant independent on a dimension \( d \).

In the present article, based on the local quasi hidden variable (LqHV) formalism introduced and developed in [14, 20, 21, 22, 23, 24], we derive a new upper bound on the maximal violation of general Bell inequalities by an arbitrary pure bipartite quantum state, possibly infinite dimensional.

This new upper bound, expressed in terms of the Schmidt coefficients for a pure bipartite state, is consistent with the general upper bound valid for an arbitrary bipartite state, pure or mixed, and indicates that, for an infinite dimensional pure bipartite state

\[ \text{This follows from the definition of the Grothendieck’s constant } K^R_G \text{ and Theorem 2.1 in [5].} \]

\[ \text{For bound (1), see Eq. (64) in [14].} \]

\[ \text{For bound (2), see Eq. (22) in [20] and Eq. (31) in [22].} \]
with a finite sum of its Schmidt coefficients, violation of any general Bell inequality is bounded from above by the value independent on a number of settings and a type of outcomes, continuous or discrete, specific to this Bell inequality.

As an example, we apply our new result to finding an upper bound on the maximal violation of general Bell inequalities by infinite dimensional entangled bipartite states having the Bell states like forms comprised of two binary coherent states $|\alpha\rangle$, $|-\alpha\rangle$, $\alpha > 0$. These entangled bipartite coherent states have been intensively discussed in the literature $[26, 27, 28, 29, 30, 31, 32, 33]$ in view of their experimental implementations. We show that, for each of these bipartite coherent states, the maximal violation of all general Bell inequalities cannot exceed the value 3 and analyse numerically the dependence of the derived upper bound on a parameter $\alpha > 0$.

The article is organized as follows.

In Section 2, we recall (in relation to a bipartite case) the notion of a general Bell inequality $[11]$ and describe the state parameters $[14]$ characterizing (Theorem 1) the maximal violation of general Bell inequalities by an arbitrary quantum state.

In Section 3, we specify the notion of a source operator for a bipartite state (introduced for a general $N$-partite state in $[14, 34]$) and present (Theorem 2) the analytical upper bound on the maximal violation of general Bell inequalities by an arbitrary bipartite state, which is expressed in terms of source operators of this state and follows from the more tight upper bound, presented by Eq. (53) in $[14]$.

In Section 4, for the maximal violation of general Bell inequalities by a pure bipartite state, we derive a new upper bound (Theorem 3) expressed in terms of the Schmidt coefficients of this pure state, and discuss the main consequences (Corollaries 1,2) following from this new result.

In Section 5, we apply our new results to specifying upper bounds (Proposition 1) on the maximal violations of general Bell inequalities by infinite dimensional pure bipartite states, having the Bell states like forms comprised of two binary coherent states $|\alpha\rangle$, $|-\alpha\rangle$, $\alpha > 0$. We prove that, for each of these bipartite coherent states, the maximal violation of general Bell inequalities cannot exceed the value 3 and analyse numerically the dependence of the derived upper bounds on a parameter $\alpha > 0$.

In Section 6, we summarize the main results of the present article.

## 2 Preliminaries: quantum violation of Bell inequalities

Considered$^5$ a general bipartite correlation scenario where each of two participants performs $S_n \geq 1$, $n = 1, 2$, different measurements, indexed by numbers $s_n = 1, ..., S_n$ and with outcomes $\lambda_n \in \Lambda_n$. We refer to this correlation scenario as $S_1 \times S_2$-setting and denote by $P_{(s_1, s_2)}(\cdot)$ the probability distribution of outcomes $(\lambda_1, \lambda_2) \in \Lambda := \Lambda_1 \times \Lambda_2$ under the joint measurement specified by a tuple $(s_1, s_2)$ of settings where each $n$-th participant performs a measurement $s_n$ at the $n$-th site. The complete probabilistic description of

$^5$On the probabilistic description of a general correlation scenario, see $[35]$. 

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such an $S_1 \times S_2$-setting correlation scenario is given by the family
\begin{equation}
\mathcal{P}_{S,\Lambda} := \{ P_{(s_1, s_2)} \mid s_n = 1, \ldots, S_n, \quad n = 1, 2 \}, \quad (5)
\end{equation}
of joint probability distributions.

A correlation scenario admits a local hidden variable (LHV) model if each of its joint probability distributions $P_{(s_1, s_2)} \in \mathcal{P}_{S,\Lambda}$ admits the representation
\begin{equation}
P_{(s_1, s_2)}(d\lambda_1 \times d\lambda_2) = \int_{\Omega} P_{1,s_1}(d\lambda_1 | \omega) \cdot P_{2,s_2}(d\lambda_2 | \omega) \nu(d\omega) \quad (6)
\end{equation}
in terms of a unique probability distribution $\nu(d\omega)$ of some variables $\omega \in \Omega$ and conditional probability distributions $P_{n,s_n}(\cdot | \omega)$, referred to as “local” in the sense that each $P_{n,s_n}(\cdot | \omega)$ at an $n$-th site depends only on a measurement $s_n = 1, \ldots, S_n$ at an $n$-th site.

Under an $S_1 \times S_2$-setting correlation scenario described by a family of joint probability distributions (5), consider a linear combination
\begin{equation}
\mathcal{B}_{\Phi_{S,\Lambda}}(\mathcal{P}_{S,\Lambda}) := \sum_{s_1, s_2} \langle \phi_{(s_1, s_2)}(\lambda_1, \lambda_2) \rangle_{P_{(s_1, s_2)}} \quad (7)
\end{equation}
of the mathematical expectations
\begin{equation}
\langle \phi_{(s_1, s_2)}(\lambda_1, \lambda_2) \rangle_{P_{(s_1, s_2)}} := \int_{\Lambda} \phi_{(s_1, s_2)}(\lambda_1, \lambda_2) P_{(s_1, s_2)}(d\lambda_1 \times d\lambda_2) \quad (8)
\end{equation}
of an arbitrary form, specified by a collection
\begin{equation}
\Phi_{S,\Lambda} = \{ \phi_{(s_1, s_2)} : \Lambda \rightarrow \mathbb{R} \mid s_n = 1, \ldots, S_n; \quad n = 1, 2 \} \quad (9)
\end{equation}
of bounded real-valued functions $\phi_{(s_1, s_2)}$ on $\Lambda = \Lambda_1 \times \Lambda_2$. Depending on a choice of a bounded function $\phi_{(s_1, s_2)}$ and types of outcome sets $\Lambda_n$, $n = 1, 2$, expression (8) can constitute either the probability of some observed event or if $\Lambda_n \subset \mathbb{R}$, $n = 1, 2$, the mathematical expectation (mean) of the product of observed outcomes (called in quantum information as a correlation function) or have a more complicated form.

If an $S_1 \times S_2$-setting correlation scenario (5) admits the LHV modelling in the sense of representation (6), then every linear combination (7) of its mathematical expectations (8) satisfies the “tight” LHV constraints
\begin{equation}
\begin{align*}
\mathcal{B}_{\Phi_{S,\Lambda}}^{\inf}(\mathcal{P}_{S,\Lambda})_{lhv} &\leq \mathcal{B}_{\Phi_{S,\Lambda}}(\mathcal{P}_{S,\Lambda})_{lhv} \leq \mathcal{B}_{\Phi_{S,\Lambda}}^{\sup}(\mathcal{P}_{S,\Lambda})_{lhv}, \\
\mathcal{B}_{\Phi_{S,\Lambda}}^{\inf}(\mathcal{P}_{S,\Lambda})_{lhv} &\leq \mathcal{B}_{\Phi_{S,\Lambda}}^{lhv} := \max \left\{ \left| \mathcal{B}_{\Phi_{S,\Lambda}}^{\inf} \right|, \left| \mathcal{B}_{\Phi_{S,\Lambda}}^{\sup} \right| \right\}.
\end{align*}
\end{equation}

For the definition of this notion under a general correlation scenario, see Definition 4 [35].
where constants
\[
B_{\Phi S, \Lambda}^{\sup} := \sup_{P_S, \Lambda \in \Theta_{S, \Lambda}^{lhv}} B_{\Phi S, \Lambda}(P_{S, \Lambda})
\]
\[
= \sup_{\lambda^{(s_n)} \in \Lambda_n, \forall s_n, n=1,2} \sum_{s_1, s_2} \phi(s_1, s_2)(\lambda_1^{(s_1)}, \lambda_2^{(s_2)}),
\]
\[
B_{\Phi S, \Lambda}^{\inf} := \inf_{P_S, \Lambda \in \Theta_{S, \Lambda}^{lhv}} B_{\Phi S, \Lambda}(P_{S, \Lambda})
\]
\[
= \inf_{\lambda^{(s_n)} \in \Lambda_n, \forall s_n, n=1,2} \sum_{s_1, s_2} \phi(s_1, s_2)(\lambda_1^{(s_1)}, \lambda_2^{(s_2)}).
\]

Here, \(\Theta_{S, \Lambda}^{lhv}\) denotes the set of all families (5) of joint probability distributions describing

\(S_1 \times S_2\)-setting correlation scenarios with outcomes in \(\Lambda = \Lambda_1 \times \Lambda_2\) admitting the LHV modelling.

Depending on a form of functional (7), which is specified by a family \(\Phi_{S, \Lambda}\) of bounded functions (10), some of the LHV constraints in (10) can hold for a wider (than LHV) class of correlation scenarios, some may be simply trivial, i.e. fulfilled under all correlation scenarios.

**Definition 1** \([11, 14]\) Each of the tight linear LHV constraints in (10) that can be violated under a non-LHV correlation scenario is referred to as a general Bell inequality.

Bell inequalities on correlation functions (like the CHSH inequality) and Bell inequalities on joint probabilities constitute particular classes of general Bell inequalities. The general form of \(N\)-partite Bell inequalities on correlation functions (called correlation Bell inequalities) and the general form of \(N\)-partite Bell inequalities on joint probabilities are introduced in \([11]\).

If, under a bipartite correlation scenario, all joint measurements \((s_1, s_2)\) are performed on a quantum state \(\rho\) on a Hilbert space \(\mathcal{H}_1 \otimes \mathcal{H}_2\), then each joint probability distribution \(P_{s_1, s_2}\) in (5) takes the form
\[
P_{s_1, s_2}(d\lambda_1 \times d\lambda_2) = \text{tr}[\rho\{M_1^{(s_1)}(d\lambda_1) \otimes M_2^{(s_2)}(d\lambda_2)\}],
\]
where \(M_n^{(s_n)}(\cdot)\), \(M_n^{(s_n)}(\Lambda_n) = I_{\mathcal{H}_n}\), is a normalized positive operator-valued (POV) measure, describing \(s_n\)-th quantum measurement at \(n\)-th site. For this correlation scenario, we denote the family (5) of joint probability distributions by
\[
P_{s_1, s_2}(\rho, m_{S, \Lambda}) := \left\{ \text{tr}[\rho\{M_1^{(s_1)}(d\lambda_1) \otimes M_2^{(s_2)}(d\lambda_2)\}], s_n = 1, ..., S_n, n \in 1, 2 \right\},
\]
where
\[
m_{S, \Lambda} := \left\{ M_n^{(s_n)} \mid s_n = 1, ..., S_n, n \in 1, 2 \right\}
\]
is the collection of all local POV measures at two sites, describing this quantum correlation scenario.

The following statement follows if \(N = 2\) from the general statements for an \(N\)-partite case in \([11]\) (see there Eq. (48) and Lemma 3).
Theorem 1 [14] For an $S_1 \times S_2$-setting quantum correlation scenario [13] performed on a state $\rho$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$, possibly infinite dimensional, every linear combination [7] of its mathematical expectations [3] satisfies the “tight” constraints:

$$B^{\text{inf}}_{\Phi, S, A} - \frac{\gamma_{S_1 \times S_2}^{(\rho, \Lambda)}}{2} (B^{\text{sup}}_{\Phi, S, A} - B^{\text{inf}}_{\Phi, S, A})$$

$$\leq B^{\text{sup}}_{\Phi, S, A} (P^{\rho, \text{mS}, A}_{S, A})$$

$$\leq B^{\text{sup}}_{\Phi, S, A} + \frac{\gamma_{S_1 \times S_2}^{(\rho, \Lambda)}}{2} (B^{\text{sup}}_{\Phi, S, A} - B^{\text{inf}}_{\Phi, S, A}),$$

where

$$1 \leq \gamma_{S_1 \times S_2}^{(\rho, \Lambda)} := \sup_{m, A, \Phi, S, A \text{-setting}} \frac{|B^{\Phi, S, A}_{\Phi, S, A} (P^{\rho, \text{mS}, A}_{S, A})|}{B^{\Phi, S, A}_{\Phi, S, A}}$$

is the maximal violation by a state $\rho$ of all $S_1 \times S_2$-setting general Bell inequalities with outcomes $(\lambda_1, \lambda_2) \in \Lambda$.

3 General upper bounds

Let $T_{S_1 \times S_2}^{(\rho)}$ be a self-adjoint trace class dilation of a state $\rho$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ to the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2$. By its definition

$$\text{tr} \left[ T_{S_1 \times S_2}^{(\rho)} \left( \mathbb{1}_{\mathcal{H}_1 \otimes S_1} \otimes X_1 \otimes \mathbb{1}_{\mathcal{H}_2 \otimes (S_1 - 1 \otimes S_2)} \otimes \mathbb{1}_{\mathcal{H}_2 \otimes (S_2 - 1 \otimes S_2)} \right) \right]$$

$$= \text{tr} [\rho (X_1 \otimes X_2)], \quad k_n = 0, \ldots, (S_n - 1), \quad n = 1, 2,$$

for all bounded operators $X_n$ on $\mathcal{H}_n$, $n = 1, 2$. Clearly, $T_{S_1 \times S_2}^{(\rho)} = \rho$, $\text{tr}[T_{S_1 \times S_2}^{(\rho)}] = 1$ and $\|T_{S_1 \times S_2}^{(\rho)}\|_1 \geq 1$, where $\|\cdot\|_1$ means the trace norm. In [14, 34], we call a self-adjoint trace class operator $T_{S_1 \times S_2}^{(\rho)}$ as an $S_1 \times S_2$-setting source operator for a state $\rho$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$. As proved in [13], for every state $\rho$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ and arbitrary integers $S_1, S_2 \geq 1$, a source operator $T_{S_1 \times S_2}^{(\rho)}$ exists.

For a separable quantum state, there always exists a positive source operator. However, for an arbitrary bipartite quantum state, a source operator $T_{S_1 \times S_2}^{(\rho)}$ does not need to be either positive or, more generally, tensor positive. The latter general notion introduced in [13] means that

$$\text{tr} \left[ T_{S_1 \times S_2}^{(\rho)} \left( A_1 \otimes \cdots \otimes A_{S_1} \otimes B_1 \otimes \cdots \otimes B_{S_2} \right) \right] \geq 0$$

for all positive bounded operators $A_k, B_m$ on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

Theorem 3 in [13] implies.

\footnote{See Proposition 1 in [13] for a general N-partite case.}
Theorem 2 For an arbitrary state $\rho$ on a complex Hilbert space $H_1 \otimes H_2$, possibly infinite dimensional, and any integers $S_n \geq 1$, $n = 1, 2$, the maximal violation

$$\Upsilon_{S_1 \times S_2}^{(\rho)} := \sup_{\Lambda} \Upsilon_{S_1 \times S_2}^{(\rho, \Lambda)}$$

by a state $\rho$ of all $S_1 \times S_2$-setting general Bell inequalities for any type of outcomes, discrete or continuous, at each of two sites satisfies the relation

$$1 \leq \Upsilon_{S_1 \times S_2}^{(\rho)} \leq \min \left\{ \inf_{T_{S_1 \times S_2}} \| T_{S_1 \times S_2}^{(\rho)} \|_1, \inf_{T_{S_1 \times S_2}} \| T_{S_1 \times S_2}^{(\rho)} \|_1 \right\} \leq \inf_{T_{S_1 \times S_2}} \| T_{S_1 \times S_2}^{(\rho)} \|_1,$$

where $T_{S_1 \times S_2}$, $T_{S_1 \times S_2}$, and $T_{S_1 \times S_2}$ are source operators of a state $\rho$ on Hilbert spaces $H_1 \otimes S_1 \otimes H_2$, $H_1 \otimes S_2 \otimes H_2$, and $H_1 \otimes S_1 \otimes H_2$, respectively.

If for a state $\rho$, the maximal violation parameter $\Upsilon_{S_1 \times S_2}^{(\rho)}$ is bounded from above by a value independent on numbers $S_1, S_2 \geq 1$ of settings at each of two sites, then the parameter

$$1 \leq \Upsilon_{\rho} := \sup_{S_1, S_2} \Upsilon_{S_1 \times S_2}^{(\rho)}$$

constitutes the maximal violation by a state $\rho$ of all possible general Bell inequalities.

The general upper bounds on parameter $\Upsilon_{S_1 \times S_2}^{(\rho)}$ valid for an arbitrary state $\rho$, pure or mixed, are presented by relations (1) in Introduction. In the following Section, based on Theorem 2, we derive for the maximal quantum violation of all general $S_1 \times S_2$-setting Bell inequalities, a new upper bound which is consistent with bound (1) but is true only for pure bipartite states.

4 Upper bounds for a pure bipartite state

Recall that, for any pure bipartite state $|\psi\rangle\langle \psi|$ on $H_1 \otimes H_2$, the non-zero eigenvalues $0 < \lambda_k(\psi) \leq 1$ of its reduced states on $H_1$ and $H_2$ coincide and have the same multiplicity while vector $|\psi\rangle \in H_1 \otimes H_2$ admits the Schmidt decomposition

$$|\psi\rangle = \sum_{1 \leq k \leq r_{sch}} \sqrt{\lambda_k(\psi)} |e_k^{(1)}\rangle \otimes |e_k^{(2)}\rangle, \quad \sum_{1 \leq k \leq r_{sch}} \lambda_k(\psi) = 1,$$

where each eigenvalue of the reduced states is taken in this sum so many times what is its multiplicity and $|e_k^{(n)}\rangle \in H_n$, $n = 1, 2$, are the normalized eigenvectors of the reduced states on $H_n$ of a pure state $|\psi\rangle\langle \psi|$ on $H_1 \otimes H_2$.

Parameters $\sqrt{\lambda_k(\psi)}$ and $1 \leq r_{sch} \leq \min\{d_1, d_2\}$ are called the Schmidt coefficients and the Schmidt rank of $|\psi\rangle$, respectively. For a separable pure bipartite state, the Schmidt rank equals to 1.
From Eqs. (27)–(30) in [21] it follows and it is also easy to check that the self-adjoint trace class operators
\[ T_{1 \times S_2}^{(\psi)} := \sum_{k,k_1} \sqrt{\lambda_k(\psi)\lambda_{k_1}(\psi)} |\psi_{k_1}) \rangle \langle e_{k_1}^{(1)} | \otimes |W_{k_1k}^{(2,S_2)}|, \]  
\[ T_{S_1 \times 1}^{(\psi)} := \sum_{k,k_1} \sqrt{\lambda_k(\psi)\lambda_{k_1}(\psi)} W_{k_1k}^{(1,S_1)} \otimes |e_{k_1}^{(2)} \rangle \langle e_{k_1}^{(1)} |, \]
on \mathcal{H}_1 \otimes \mathcal{H}_2^{\otimes S_2} \text{ and } \mathcal{H}_1^\otimes S_1 \otimes \mathcal{H}_2, \text{ respectively, where}
\[ W_{kk,S_n}^{(n)} := (|e_k^{(n)} \rangle \langle e_k^{(n)}|) \otimes \mathcal{S}_n, \]
\[ W_{k\neq k_1,S_n}^{(n)} := \left( \frac{|e_k^{(n)} + e_{k_1}^{(n)} \rangle \langle e_k^{(n)} + e_{k_1}^{(n)}|}{2S_n + 1} \right)^{\otimes \mathcal{S}_n} - \left( \frac{|e_k^{(n)} - e_{k_1}^{(n)} \rangle \langle e_k^{(n)} - e_{k_1}^{(n)}|}{2S_n + 1} \right)^{\otimes \mathcal{S}_n} \]
\[ + i \frac{(|e_k^{(n)} + ie_{k_1}^{(n)} \rangle \langle e_k^{(n)} + ie_{k_1}^{(n)}|)^{\otimes \mathcal{S}_n}}{2S_n + 1} - i \frac{(|e_k^{(n)} - ie_{k_1}^{(n)} \rangle \langle e_k^{(n)} - ie_{k_1}^{(n)}|)^{\otimes \mathcal{S}_n}}{2S_n + 1}, \]
\[ n = 1, 2, \]
constitute, correspondingly, the 1 × S₂-setting and S₁ × 1-setting source operators of a pure bipartite state |\psi\rangle \langle \psi|.

Specified for a bipartite case, Eq. (31) in [21] implies that for either of source operators in [23], the trace norms
\[ \left\| T_{1 \times S_2}^{(\psi)} \right\|_1 \left\| T_{S_1 \times 1}^{(\psi)} \right\|_1 \leq 2 \sum_k \sqrt{\lambda_k(\psi)}^2 - 1, \]  
where the right hand sides do not depend on numbers S₁, S₂ of measurement settings at each of two sites. Note that
\[ 2 \left( \sum_k \sqrt{\lambda_k(\psi)}^2 \right) - 1 \leq 2 r_{sch}^{(\psi)} - 1. \]  

In view of relations (20), (25) and (26), bound (4), we derive the following new result.

**Theorem 3** For an arbitrary pure bipartite state |\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2, the maximal violation \( \mathcal{T}_{S_1 \times S_2}^{(\psi)} \) of S₁ × S₂-setting general Bell inequalities for any number/type of outcomes at each site admits the upper bound
\[ \mathcal{T}_{S_1 \times S_2}^{(\psi)} \leq 2 \min \left\{ \left( \sum_k \sqrt{\lambda_k(\psi)}^2 \right)^2, S_1, S_2 \right\} - 1 \]  
\[ \leq 2 \min \left\{ r_{sch}^{(\psi)}, S_1, S_2 \right\} - 1, \]
where \( \lambda_k(\psi) \) and \( r_{sch}^{(\psi)} \) are, correspondingly, the Schmidt coefficients and the Schmidt rank of a pure bipartite state |\psi⟩.
In view of the relation
\[ 2 \min \left\{ r_{sch}^{(\psi)}, S_1, S_2 \right\} - 1 \leq 2 \min \{ d_1, d_2, S_1, S_2 \} - 1, \]  
(29)
the upper bounds (27), (28) are consistent with the general upper bound (1) valid for all states, pure or mixed.

Theorem 3 and relation (21) imply

**Corollary 1** For a pure bipartite state \(|\psi\rangle\langle\psi|\), possibly infinite dimensional, with a finite sum of Schmidt coefficients, the maximal violation \(\Upsilon_{|\psi\rangle\langle\psi|}\) of all general Bell inequalities admits the bound
\[ \Upsilon_{|\psi\rangle\langle\psi|} \leq 2 \left( \sum_{k} \sqrt{\lambda_k(\psi)} \right)^2 - 1. \]  
(30)

Let some vectors \(|\psi_1\rangle, |\psi_2\rangle \in H\) be normalized and linear independent. For each of the following pure entangled states having the Bell states like forms
\[
|\Psi_{00}\rangle = \frac{|\psi_1\rangle \otimes |\psi_1\rangle + |\psi_2\rangle \otimes |\psi_2\rangle}{\sqrt{2 \left( 1 + |\langle \psi_1 | \psi_2 \rangle|^2 \right)}},
\]
\[
|\Psi_{01}\rangle = \frac{|\psi_1\rangle \otimes |\psi_2\rangle + |\psi_2\rangle \otimes |\psi_1\rangle}{\sqrt{2 \left( 1 + |\langle \psi_1 | \psi_2 \rangle|^2 \right)}},
\]
\[
|\Psi_{10}\rangle = \frac{|\psi_1\rangle \otimes |\psi_1\rangle - |\psi_2\rangle \otimes |\psi_2\rangle}{\sqrt{2 \left( 1 + |\langle \psi_1 | \psi_2 \rangle|^2 \right)}},
\]
\[
|\Psi_{11}\rangle = \frac{|\psi_1\rangle \otimes |\psi_2\rangle - |\psi_2\rangle \otimes |\psi_1\rangle}{\sqrt{2 \left( 1 + |\langle \psi_1 | \psi_2 \rangle|^2 \right)}},
\]
the Schmidt rank \(r_{sch}^{(\Psi_{jk})} = 2, j, k = 0, 1\). This and Eq. (28) imply.

**Corollary 2** For each of pure entangled states (31), possibly infinite dimensional, the maximal violation (21) of general Bell inequalities satisfies the relation
\[ \Upsilon_{|\Psi_{jk}\rangle\langle\Psi_{jk}|} \leq 3, \quad j, k = 0, 1. \]  
(32)

This bound is, in particular, true for each of the Bell states \(|\beta_{jk}\rangle\) on \(\mathbb{C}^2 \otimes \mathbb{C}^2\).
5 Example: bipartite coherent states

In this Section, we proceed to apply the new results of Theorem 3 and Corollary 1 to finding upper bounds on the maximal violation of general Bell inequalities by infinite dimensional entangled bipartite coherent states of the form (31):

\[
|\Phi_1(\alpha)\rangle = \frac{|\alpha\rangle \otimes |\alpha\rangle + |-\alpha\rangle \otimes |-\alpha\rangle}{\sqrt{2(1 + e^{-4\alpha^2})}},
\]

\[
|\Phi_2(\alpha)\rangle = \frac{|\alpha\rangle \otimes |-\alpha\rangle + |-\alpha\rangle \otimes |\alpha\rangle}{\sqrt{2(1 + e^{-4\alpha^2})}},
\]

\[
|\Phi_3(\alpha)\rangle = \frac{|\alpha\rangle \otimes |\alpha\rangle - |-\alpha\rangle \otimes |-\alpha\rangle}{\sqrt{2(1 - e^{-4\alpha^2})}},
\]

\[
|\Phi_4(\alpha)\rangle = \frac{|\alpha\rangle \otimes |-\alpha\rangle - |-\alpha\rangle \otimes |\alpha\rangle}{\sqrt{2(1 - e^{-4\alpha^2})}},
\]

where

\[
|\pm\alpha\rangle = e^{-\frac{\alpha^2}{2}} \sum_{m=0}^{\infty} \frac{(\pm\alpha)^m}{\sqrt{m!}} |m\rangle
\]

are the normalized binary coherent states with parameter \(\alpha > 0\) and \{|m\rangle, \ m = 0, 1, \ldots\}\ are the Fock vectors.

For \(\alpha \to 0\), each of bipartite coherent states (33) tends to the product state \(|0\rangle \otimes |0\rangle\) whereas states (34) tend to the corresponding Bell states:

\[
|\Phi_3(\alpha)\rangle = \frac{\alpha}{\sqrt{\sinh(2\alpha^2)}} \{|1\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + o(\alpha)\} \to_{\alpha \to 0} |\beta_{01}\rangle := \frac{1}{\sqrt{2}}(|1\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle),
\]

\[
|\Phi_4(\alpha)\rangle = \frac{\alpha}{\sqrt{\sinh(2\alpha^2)}} \{|1\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle + o(\alpha)\} \to_{\alpha \to 0} -|\beta_{11}\rangle := \frac{1}{\sqrt{2}}(|1\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle).
\]

Here, \(o(\alpha)\) is a high-order term of \(\alpha\).

For states (33), the nonzero eigenvalues of their reduced states are nondegenerate and read (see in Appendix)

\[
\lambda_{\pm}(\Phi_1(\alpha)) = \lambda_{\pm}(\Phi_2(\alpha)) = \left(1 \pm e^{-2\alpha^2}\right)^2 \frac{2}{2(1 + e^{-4\alpha^2})}
\]

for all \(\alpha > 0\). For each of states (34), the nonzero eigenvalue of the reduced states is equal to \(\frac{1}{2}\) for all \(\alpha > 0\) and has multiplicity 2. The Schmidt ranks of states (33) and (34) are equal to 2. This and Corollary 1 imply.
Proposition 1  For each of infinite dimensional bipartite coherent states $|\Phi_j(\alpha)\rangle$, $j = 1, \ldots, 4$, the maximal violation (38) of general Bell inequalities

$$
\Upsilon_{|\Phi_j\rangle\langle\Phi_j|} \leq 3 - e^{-4\alpha^2}, \quad j = 1, 2,
$$

(38)

$$
\Upsilon_{|\Phi_j\rangle\langle\Phi_j|} \leq 3, \quad j = 3, 4,
$$

(39)

where

$$
\frac{3 - e^{-4\alpha^2}}{1 + e^{-4\alpha^2}} \leq 3
$$

(40)

for all $\alpha > 0$.

The numerical calculation of the analytical upper bound in (38) is presented on Fig. 1.

6 Conclusion

In the present article, we find a new upper bound (Theorem 3) on the maximal violation of general Bell inequalities by a pure bipartite state. This new upper bound, expressed in terms of the Schmidt coefficients of a pure bipartite state, is consistent with the upper bound (1) valid for all bipartite states, pure or mixed, and implies (Corollary 1) that the maximal violation of all general Bell inequalities by a pure bipartite state, possibly
infinite dimensional, with a finite sum $\sum_k \sqrt{\lambda_k(\psi)}$ of Schmidt coefficients cannot exceed the value $2 \left( \sum_k \sqrt{\lambda_k(\psi)} \right)^2 - 1$.

As an example, we analyse analytically (Proposition 1) and numerically (see Fig.1) the upper bounds on the maximal violation of general Bell inequalities by infinite dimensional bipartite states (33), (34), having the Bell states like forms comprised of two binary coherent states $|\alpha\rangle, |\alpha\rangle$ where $\alpha > 0$.

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**Appendix**

The vectors

$$|u_1\rangle := |\alpha\rangle, \quad |u_2\rangle := \frac{|-\alpha\rangle - \langle u_1| - \alpha\rangle |u_1\rangle}{\sqrt{1 - |\langle u_1| - \alpha\rangle|^2}} \quad (A1)$$

where vector $|u_2\rangle$ is due to the Gram-Schmidt orthonormalisation process between vectors $|\alpha\rangle$ and $|-\alpha\rangle$, constitute the orthonormal basis of the linear span of vectors $|\alpha\rangle$ and $|-\alpha\rangle$. For $\alpha > 0$,

$$|u_2\rangle = \frac{|-\alpha\rangle - e^{-2\alpha^2}|\alpha\rangle}{\sqrt{1 - e^{-4\alpha^2}}} \quad (A2)$$
and bipartite coherent states (33) and (34) admit the following decompositions:

\[
|\Phi_1(\alpha)\rangle = (1 + e^{-4\alpha^2})|u_1\rangle \otimes |u_1\rangle + e^{-2\alpha^2} \sqrt{1 - e^{-4\alpha^2}} |u_1\rangle \otimes |u_2\rangle \sqrt{2(1 + e^{-4\alpha^2})}, \tag{A3}
\]

\[
|\Phi_2(\alpha)\rangle = 2e^{-2\alpha^2}|u_1\rangle \otimes |u_1\rangle + \sqrt{1 - e^{-4\alpha^2}} (|u_1\rangle \otimes |u_2\rangle + |u_2\rangle \otimes |u_1\rangle),
\]

\[
|\Phi_3(\alpha)\rangle = (1 - e^{-4\alpha^2})|u_1\rangle \otimes |u_1\rangle - e^{-2\alpha^2} \sqrt{1 - e^{-4\alpha^2}} |u_1\rangle \otimes |u_2\rangle \sqrt{2(1 - e^{-4\alpha^2})},
\]

\[
|\Phi_4(\alpha)\rangle = \frac{1}{\sqrt{2}}(|u_1\rangle \otimes |u_2\rangle - |u_2\rangle \otimes |u_1\rangle).
\]

The nonzero eigenvalues of the reduced states of $|\Phi_j(\alpha)\rangle\langle\Phi_j(\alpha)|$, $j = 1, 2$ can be easily calculated and are nongenerate and are given by

\[
\lambda_{\pm}(\Phi_j(\alpha)) = \frac{(1 \pm e^{-2\alpha^2})^2}{2(1 + e^{-4\alpha^2})}. \tag{A4}
\]

The nonzero eigenvalue of the reduced states of $|\Phi_j(\alpha)\rangle\langle\Phi_j(\alpha)|$, $j = 3, 4$, equals to $\frac{1}{2}$ and has multiplicity 2.

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