A Novel Spectral Method for Burgers Equation on The Real Line

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Received 20 November 2019; Accepted (in revised version) 7 February 2020.

Abstract. A spectral method for the Burgers equation on the whole real line based on generalised Hermite functions is proposed. The generalised stability and convergence of the method are proved. Numerical results confirm the theoretical findings and demonstrate the efficiency of the algorithm.

AMS subject classifications: 41A30, 76M22, 65M70, 33C45, 65M12

Key words: Burger equation on the real line, spectral method, nonlinear problem, generalised Hermite function.

1. Introduction

The main object of our interest is the equation, which originated in the work of Bateman\textsuperscript{[6]} in 1915 and was lately named after Burgers, who employed it to study turbulent fluids\textsuperscript{[8]}. Nowadays this equation is widely used in various areas of applied mathematics, including gas dynamics, waves in shallow water and turbulence in fluid dynamics\textsuperscript{[2,7,21,32–34,36,37,39,40]}. A vast literature is devoted to analytic solutions of the Burgers equation on bounded and unbounded domains with different initial and boundary conditions — cf. Refs.\textsuperscript{[1,5,9,15,22,35,41,44]}. Numerical methods such as finite difference methods\textsuperscript{[4,10,26,29] and finite element approaches\textsuperscript{[3,11,12,14,23,27,45] have been also extensively studied. Another popular group of numerical methods for the Burgers equation — viz. spectral methods, attracted a considerable attention as well. In particular, Maday and Quarteroni\textsuperscript{[30,31]} considered Legendre and Chebyshev spectral and pseudospectral methods, Ma and Guo\textsuperscript{[28]} studied Chebyshev spectral method for Burgers-like equation, Wu et al.\textsuperscript{[42]} presented a Chebyshev-Legendre collocation method for generalised Burgers equation and

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Khater et al. [25] discussed Chebyshev spectral collocation methods for Burgers-type equations. Now the challenge is to develop efficient algorithms for the Burgers equation on unbounded domains and to study their stability and convergence. One approach is related to employing finite difference/finite element methods for equation with artificial boundary conditions [13] and another is domain decomposition methods [19]. However, these approaches may bring additional errors and complicate theoretical analysis and actual computations. Thus the use of certain orthogonal systems on unbounded domains is more suitable. Guo and Xu [20] employed Hermite polynomials and Guo et al. [18] developed spectral and pseudospectral methods based on Hermite functions for problems on the real line.

In this work, we engage generalised Hermite functions to approximate the Burgers equation on the real line. The benefit of this approach is three folds. The first is the direct approximation of the solutions on the whole line, so that there is no need to employ domain decomposition or variable transformations. Second, this simplifies theoretical analysis and produces sparse systems. Finally, numerical solutions have the spectral accuracy in space.

The remainder of the paper is organised as follows. In Section 2, we recall approximation results for generalised Hermite functions. A spectral scheme for the Burgers equation is introduced and analysed in Section 3. Numerical results presented in Section 4, demonstrate the efficiency of this algorithm, whilst Section 5 contains some concluding remarks.

2. Preliminaries

Let \( \Lambda = \{\xi | -\infty < \xi < \infty \} \) and \( \chi(\xi) \) be a weight function. For an integer \( \mu \geq 0 \), we denote by \((u, v)_{\mu, \chi, \Lambda}, |u|_{\mu, \chi, \Lambda} \) and \( \|u\|_{\mu, \chi, \Lambda} \) the inner product, semi-norm and norm of the space \( H^\mu_{\chi}(\Lambda) \), respectively. In particular, \( H^0_{\chi}(\Lambda) = L^2_{\chi}(\Lambda) \) has inner product \((u, v)_{\chi, \Lambda}\) and the norm \( \|u\|_{\chi, \Lambda} \). For simplicity, we omit subscript \( \chi \) whenever \( \chi(\xi) \equiv 1 \). The \( l \)-th order Hermite polynomial is given by

\[
h_l(\xi) = (-1)^l e^{\xi^2} \frac{1}{\sqrt{2^l l!}} \frac{d^l}{d\xi^l} (e^{\xi^2}).
\]

Such polynomials are mutually orthogonal with respect to the weight \( \chi(\xi) = e^{-\xi^2} \). Following [43], we now consider the generalised Hermite functions

\[
h^\sigma_l(\xi) = \frac{1}{\sqrt{2^l l!}} h_l(\sigma \xi) e^{-(1/2)\sigma^2 \xi^2}, \quad l \geq 0,
\]

where \( \sigma > 0 \) is a constant. According to [38, Eq. (7.65)] and the definition (2.1), we have

\[
h^\sigma_l(\xi) = \sqrt{2^l l!} \sum_{k=0}^{[l/2]} \frac{(-1)^k}{2^{2k} k!(l-2k)!} ((\sigma \xi)^l - 2k e^{-(\sigma^2 \xi^2)/2}).
\]

Obviously,

\[
\lim_{|\xi| \to \infty} h^\sigma_l(\xi) = 0, \quad l \geq 0.
\]
The set of \( h_1^\sigma(\xi) \) forms a complete \( L^2(\Lambda) \)-orthogonal system — viz.

\[
\int_{\Lambda} h_1^\sigma(\xi) \overline{h_1^\sigma(\xi)} d\xi = \frac{\sqrt{\pi}}{\sigma} \delta_{1k},
\]

(2.3)

where \( \delta_{1k} \) is the Kronecker delta. The generalised Hermite function \( h_1^\sigma(\xi) \) is an \( l \)-th eigenfunction of the Sturm-Liouville problem

\[
e^{(1/2)\sigma^2 \xi^2} \partial_\xi \left( e^{-\sigma^2 \xi^2} \partial_\xi \left( e^{(1/2)\sigma^2 \xi^2} v(\xi) \right) \right) + \lambda_1 v(\xi) = 0,
\]

for the eigenvalue \( \lambda_1 = \lambda_{l,\sigma} := 2\sigma^2 l \). Moreover, it satisfies the recurrence relation

\[
\partial_\xi h_1^\sigma(\xi) = -\sigma \sqrt{\frac{l + 1}{2}} h_{l+1}^\sigma(\xi) + \sigma \sqrt{\frac{l}{2}} h_{l-1}^\sigma(\xi) \\
= -\frac{1}{2} \sqrt{\lambda_{l+1}} h_{l+1}^\sigma(\xi) + \frac{1}{2} \sqrt{\lambda_l} h_{l-1}^\sigma(\xi).
\]

(2.4)

Any function \( v(\xi) \in L^2(\Lambda) \) can be represented in the form

\[
v(\xi) = \sum_{l=0}^{\infty} \hat{v}_l h_1^\sigma(\xi),
\]

where

\[
\hat{v}_l = \frac{\sigma}{\sqrt{\pi}} \int_{\Lambda} v(\xi) h_1^\sigma(\xi) d\xi.
\]

For any non-negative integer \( N \), let \( \mathbb{P}_N(\Lambda) \) stand for the set of all algebraic polynomials of degree at most \( N \). Setting

\[
\mathcal{Q}_{N,\sigma}(\Lambda) := \operatorname{span} \{ h_0^\sigma(\xi), h_1^\sigma(\xi), \ldots, h_N^\sigma(\xi) \} = \left\{ \psi(\xi) e^{-(1/2)\sigma^2 \xi^2} | \psi(\xi) \in \mathbb{P}_N(\Lambda) \right\},
\]

we note that for any \( \phi \in \mathcal{Q}_{N,\sigma}(\Lambda) \), the inequality

\[
\| \partial_\xi \phi \|_\Lambda \leq c \sigma^N N^{\mu/2} \| \phi \|_\Lambda
\]

holds — cf. [43, Lemma 2.1].

To describe the approximation error, we consider the normed space

\[
H_{A,\sigma}(\Lambda) := \left\{ v \| v \|_{H_{A,\sigma}(\Lambda)} < \infty \right\}, \quad \mu \geq 0,
\]

where

\[
\| v \|_{H_{A,\sigma}(\Lambda)} = \left( \sum_{k=0}^{\mu} \left\| \left( \sigma^4 \xi^2 + \sigma^2 \right)^{(\mu-k)/2} \partial_\xi^k v \right\|_\Lambda^2 \right)^{1/2}.
\]

The orthogonal projection \( \Pi_{N,\sigma,\Lambda} : L^2(\Lambda) \to \mathcal{Q}_{N,\sigma}(\Lambda) \) is defined by

\[
\left( \Pi_{N,\sigma,\Lambda} v - v, \phi \right) = 0, \quad \forall \phi \in \mathcal{Q}_{N,\sigma}(\Lambda).
\]
Lemma 2.1 (cf. Xiang & Wang [43, Theorem 2.1]). If \( v \in H_{\lambda,\sigma}^{\mu}(\Lambda) \) and \( v \in [0,\mu] \), then
\[
\| v - \Pi_{N,\sigma,\lambda} v \|_{\nu,\Lambda} \leq c \left( \sigma^2 N \right)^{(\nu - \mu)/2} \| v \|_{H_{\lambda,\sigma}^{\mu}(\Lambda)}.
\]

We also recall the estimate (2.8) in [24] for the norm \( \Pi_{N,\sigma,\lambda} v \), that is
\[
\| \Pi_{N,\sigma,\lambda} v \|_{L^\infty(\Lambda)} \leq \| v \|_{L^\infty(\Lambda)} + c \left( \sigma^2 N \right)^{(1-\mu)/2} \| v \|_{H_{\lambda,\sigma}^{\mu}(\Lambda)}.
\] (2.5)

According to [24, Lemma 2.3], for any \( \phi \in \mathcal{Q}_{N,\sigma}(\Lambda) \) and \( 1 \leq p \leq q \leq \infty \) the following inverse inequality holds:
\[
\| \phi e^{(\sigma^2/2 - \sigma^2/q)\xi^2} \|_{L^q(\Lambda)} \leq c \left( \sigma N^{5/6} \right)^{1/p-1/q} \| \phi e^{(\sigma^2/2 - \sigma^2/p)\xi^2} \|_{L^p(\Lambda)}.
\] (2.6)

In what follows we also need the following lemma.

Lemma 2.2 (Guo et al. [18, cf. Lemma 4.1]). Assume that:

(i) \( d \) and \( d_k \) are non-negative constants.

(ii) \( \mathcal{E}(\eta) \) is a non-negative function of \( \eta \).

(iii) For \( \rho \geq 0 \) and for all \( 0 \leq \eta \leq \eta_1 \), the inequality
\[
\mathcal{E}(\eta) \leq \rho + d \int_0^\eta \left( \mathcal{E}(\zeta) + \sum_{k=2}^n N^{d_k} \mathcal{E}^k(\zeta) \right) d\zeta
\]
holds.

(iv) There is \( \eta_1 > 0 \) such that
\[
\rho e^{d_0 \eta_1} \leq \min_{2 \leq k \leq n} N^{-d_0/(k-1)}.
\]

Then for all \( \eta \in [0,\eta_1] \), the estimate
\[
\mathcal{E}(\eta) \leq \rho e^{d_0 \eta}
\]
holds.

3. A Spectral Scheme and Error Analysis

Here we introduce a spectral scheme for the underlying problem and prove its stability and convergence. Let \( \lambda > 0 \) be the kinetic viscosity, \( T \) a fixed positive number, \( g(\xi, \tau) \) and \( V_0(\xi) \) the source term and initial state, respectively. Consider the Burgers equation
\[
\partial_\tau V(\xi, \tau) + \frac{1}{2} \partial_\xi (V^2(\xi, \tau)) - \lambda \partial_\xi^2 V(\xi, \tau) = g(\xi, \tau), \quad \xi \in \Lambda, \quad 0 < \tau \leq T,
\]
\[
\lim_{\xi \to \pm \infty} V(\xi, \tau) = \lim_{\xi \to \pm \infty} \partial_\xi V(\xi, \tau) = 0, \quad 0 < \tau \leq T,
\]
\[
V(\xi, 0) = V_0(\xi), \quad \xi \in \Lambda,
\] (3.1)
where $V_0(\xi)$ and $\partial_\xi V_0(\xi)$ tend to zero as $\xi \to \pm \infty$. Let

$$a(u, v) = \int_A \partial_\xi u \partial_\xi v d\xi.$$ 

A weak formulation of (3.1) is to find

$$V \in L^2(0, T; H^1(\Lambda)) \cap L^\infty(0, T; L^2(\Lambda))$$

such that

$$(\partial_\xi V(\xi, \tau), w(\xi)) + \frac{1}{2} \left( \partial_\xi (V^2(\xi, \tau)), w(\xi) \right) + \lambda a(V(\xi, \tau), w(\xi)) = (g(\xi, \tau), w(\xi))$$

for all $w(\xi) \in H^1(\Lambda)$, $0 < \tau \leq T$, $V(\xi, 0) = V_0(\xi)$, $\xi \in \Lambda$. \hspace{1cm} (3.2)

The spectral scheme for (3.2) consists in finding $v_N \in \mathcal{B}_{N,\sigma}(\Lambda)$ such that

$$(\partial_\tau v_N(\xi, \tau), \phi(\xi)) + \frac{1}{2} \left( \partial_\xi \left( v_N^2(\xi, \tau) \right), \phi(\xi) \right) + \lambda a(v_N(\xi, \tau), \phi(\xi)) = (g(\xi, \tau), \phi(\xi))$$

for all $\phi(\xi) \in \mathcal{B}_{N,\sigma}(\Lambda)$, $0 < \tau \leq T$, $v_N(\xi, 0) = v_{N,0} = \Pi_{N,\sigma} V_0(\xi)$, $\xi \in \Lambda$. \hspace{1cm} (3.3)

Following the proof of Lemma 4.1 in [17], we can show the boundedness of solutions of the problems (3.2), (3.3).

Now, we can deal with the stability and convergence of the scheme (3.3). Since the problem (3.3) is nonlinear, we consider the generalised stability defined in [16]. Let $\tilde{v}_{N,0}$ and $\tilde{g}$ be the errors of $v_N(\xi, 0)$ and $g$, respectively. They induce the error $\tilde{v}_N$ of the numerical solution $v_N$. It follows from (3.3) that

$$(\partial_\tau \tilde{v}_N(\tau), \phi) + \lambda a(\tilde{v}_N(\tau), \phi) = (\tilde{g}(\xi, \tau), \phi) + \sum_{j=1}^3 G_j(\tau, \phi)$$

for all $\phi \in \mathcal{B}_{N,\sigma}(\Lambda)$, $0 < \tau \leq T$, $\tilde{v}_{N,0} = \tilde{v}_N(\xi, 0)$, $\xi \in \Lambda$. \hspace{1cm} (3.4)

where

$$G_1(\tau, \phi) = -\left( \partial_\xi (v_N(\xi, \tau) \tilde{v}_N(\tau)), \phi \right),$$

$$G_2(\tau, \phi) = -\left( v_N(\tau) \partial_\xi \tilde{v}_N(\tau), \phi \right),$$

$$G_3(\tau, \phi) = -\left( \tilde{v}_N(\tau) \partial_\xi \tilde{v}_N(\tau), \phi \right).$$

Choosing $\phi = 2 \tilde{v}_N(\tau)$ in (3.4) yields

$$\partial_\tau \| \tilde{v}_N(\tau) \|^2_{L^2(\Lambda)} + 2\lambda \| \tilde{v}_N(\tau) \|_{L,\Lambda}^2 = 2(\tilde{g}(\tau), \tilde{v}_N(\tau)) + 2 \sum_{j=1}^3 G_j(\tau, \tilde{v}_N(\tau)).$$

(3.5)
Substituting (3.6)-(3.9) into (3.5), we arrive at the estimate

\[
|2(\tilde{g}(\tau), \tilde{v}_N(\tau))| \leq 2\|\tilde{g}(\tau)\|_{L^2(A)}\|\tilde{v}_N(\tau)\|_{L^2(A)}
\]

\[
\leq \|\tilde{g}(\tau)\|_{L^2(A)}^2 + \|\tilde{v}_N(\tau)\|_{L^2(A)}^2.
\]

(3.6)

In addition, the Hölder inequality and (2.6) with \( p = 2, q = 4 \) lead to the estimates

\[
|2G_1(\tau, 2\tilde{v}_N(\tau))| \leq 2\|v_N(\tau)\|_{1,\Lambda}\|\tilde{v}_N(\tau)\|_{L^2(\Lambda)}^2
\]

\[
\leq c \left( \sigma N^{5/6} \right)^{1/2} \|\tilde{v}_N(\tau)\|_{L^2(\Lambda)}^2 \|v_N(\tau)\|_{1,\Lambda}
\]

\[
\leq \|v_N(\tau)\|_{1,\Lambda}^2 + c \left( \sigma N^{5/6} \right) \|\tilde{v}_N(\tau)\|_{L^2(\Lambda)}^4.
\]

(3.7)

\[
|2G_2(\tau, 2\tilde{v}_N(\tau))| \leq 2\|v_N(\tau)\|_{L^\infty(\Lambda)}\|\tilde{v}_N(\tau)\|_{L^2(\Lambda)}\|\tilde{v}_N(\tau)\|_{1,\Lambda}
\]

\[
\leq \frac{\lambda}{2}\|\tilde{v}_N(\tau)\|_{1,\Lambda}^2 + \frac{c}{\lambda} \|v_N(\tau)\|_{L^\infty(\Lambda)}^2 \|\tilde{v}_N(\tau)\|_{L^2(\Lambda)}^2.
\]

(3.8)

Substituting (3.6)-(3.9) into (3.5), we arrive at the estimate

\[
\partial_\tau \|\tilde{v}_N(\tau)\|_{L^2(\Lambda)}^2 + \lambda \|\tilde{v}_N(\tau)\|_{1,\Lambda}^2
\]

\[
\leq \|\tilde{g}(\tau)\|_{L^2(\Lambda)}^2 + \|v_N(\tau)\|_{1,\Lambda}^2 + \left( 1 + \frac{c}{\lambda} \|v_N(\tau)\|_{L^\infty(\Lambda)}^2 \right) \|\tilde{v}_N(\tau)\|_{L^2(\Lambda)}^2
\]

\[
+ \frac{c}{\lambda} \left( \sigma N^{5/6} \right) \|\tilde{v}_N(\tau)\|_{L^2(\Lambda)}^4.
\]

(3.10)

Let

\[
\mathcal{E}(w, \lambda, \tau) = \|w\|_{L^2(\Lambda)}^2 + \lambda \int_0^\tau |w(\zeta)|_{1,\Lambda}^2 d\zeta.
\]

(3.11)

Integrating the inequality (3.10) with respect to \( \tau \) in \([0, \tau]\) gives

\[
\mathcal{E}(\tilde{v}_N, \lambda, \tau) \leq \rho(\tau) + c^* \int_0^\tau \left( \mathcal{E}(\tilde{v}_N, \lambda, \zeta) + N^{5/6} \mathcal{E}(\tilde{v}_N, \lambda, \zeta) \right) d\zeta,
\]

where

\[
\rho(\tau) = \|\tilde{v}_{N,0}\|_{L^2(\Lambda)}^2 + \int_0^\tau \left( \tilde{g}(\zeta) + |v_N(\zeta)|_{1,\Lambda}^2 \right) d\zeta,
\]

\[
c^* = c \left( 1 + \frac{1}{\lambda} + \frac{c}{\lambda} \|v_N(\tau)\|_{L^\infty(\Lambda)}^2 \right).
\]

Then, by Lemma (2.2) we have the following stability result.
Theorem 3.1. If $v_N(\tau)$ is the solution of (3.3) and $\tau_1$ is a positive number such that
\[ \rho(\tilde{v}_{N,0}, \tilde{g}, \tau) \leq \frac{e^{-5/6}\tau_1}{N^{5/6}}, \]
then for any $\tau \in (0, \tau_1]$ the inequality
\[ \delta'(\tilde{v}_N, \lambda, 0) \leq \rho(\tilde{v}_{N,0}, \tilde{g}, \tau)e^{(5/6)\tau} \]
holds.

Next, we consider the convergence of the scheme (3.3). Let $V$ and $V_N = \Pi_{N,\sigma} V$ be the solution of (3.2) and its $L^2$-orthogonal projection, respectively. It follows from (3.2) that
\[
(\partial_\tau V_N(\tau), \phi) + \frac{1}{2} \left( \partial_\xi (V_N^2(\tau), \phi) + \lambda a(V_N(\tau), \phi) \right) \\
= (g(\tau), \phi) + \sum_{j=1}^{3} Q_j(\tau, \phi) \quad \text{for all} \quad \phi \in \mathcal{Q}_{N,\sigma}(\Lambda), \quad 0 < \tau \leq T, (3.12)
\]
where
\[
Q_1(\tau, \phi) = (\partial_\tau (V_N(\tau) - V(\tau)), \phi), \\
Q_2(\tau, \phi) = \frac{1}{2} \left( \partial_\xi \left( V_N^2(\tau) - V^2(\tau) \right), \phi \right), \\
Q_3(\tau, \phi) = \lambda a(V_N(\tau) - V(\tau), \phi).
\]
Subtracting (3.12) from (3.3) and setting $\tilde{V}_N = v_N - V_N$ gives
\[
\left( \partial_\tau \tilde{V}_N(\tau), \phi \right) + \frac{1}{2} \left( \partial_\xi \left( \tilde{v}_N^2(\tau) - V_N^2(\tau) \right), \phi \right) + \lambda a \left( \tilde{V}_N(\tau), \phi \right) \\
= \sum_{j=1}^{3} Q_j(\tau, \phi) \quad \text{for all} \quad \phi \in \mathcal{Q}_{N,\sigma}(\Lambda), \quad 0 < \tau \leq T, (3.13)
\]
Choosing $\phi = 2\tilde{V}_N$ in (3.13) yields
\[
\| \partial_\tau \tilde{V}_N(\tau) \|_{L^2(\Lambda)}^2 + \left( \partial_\xi \left( \tilde{v}_N^2 - V_N^2 \right), \tilde{V}_N \right) + 2\lambda |\tilde{V}_N|_{1,\Lambda}^2 = 2 \sum_{j=1}^{3} Q_j(\tau, \tilde{V}_N). (3.14)
\]
Recalling relation (2.2) and using integration by parts, the Hölder inequality, the inequalities (2.6) with $q = 4, p = 2$ and (2.5) with $\mu = 1$ give
\[
\left| \left( \partial_\xi \left( \tilde{v}_N^2 - V_N^2 \right), \tilde{V}_N \right) \right| = \left| \left( \tilde{v}_N^2 - V_N^2, \partial_\xi \tilde{V}_N \right) \right| = \left| \left( \tilde{V}_N^2 + 2V_N \tilde{V}_N, \partial_\xi \tilde{V}_N \right) \right|
\]
\[ \text{Spectral Method for Burgers Equation} \]

Now we want to estimate the terms \( Q_j(\tau, \tilde{N}) \), \( 1 \leq j \leq 3 \). For this, we employ the Hölder inequality, Lemma 2.1 with \( \nu = 0 \) and \( \nu = 1 \), the inequalities (2.6) with \( q = 4, p = 2 \) and (2.5) with \( \mu = 1 \), and integration by parts thus obtaining

\[
\begin{align*}
|2Q_1(\tau, 2\tilde{N})| &\leq 2|\mathcal{T}_1(\tilde{V} - V_N)|_{L^2(\Omega)} |\tilde{V}_N|_{H^{1/2}(\Omega)} \\
&\leq c \left( (\sigma^2 N)^{-\mu/2} \right) |\mathcal{T}_2(\tilde{V} - V_N)|_{H^{1/2}(\Omega)} |\tilde{V}_N|_{L^2(\Omega)} \\
&\leq c \left( (\sigma^2 N)^{-\mu} \right) |\mathcal{T}_3(\tilde{V} - V_N)|_{H^{1/2}(\Omega)} |\tilde{V}_N|_{L^2(\Omega)},
\end{align*}
\]

(3.16)

\[
\begin{align*}
|2Q_3(\tau, 2\tilde{N})| &\leq |V - V_N|_{L^1(\Omega)} |\tilde{V}_N|_{H^{1/2}(\Omega)} \\
&\leq \left( (\sigma^2 N)^{(1-\mu)/2} \right) |V|_{H^{1/2}(\Omega)} |\tilde{V}_N|_{H^{1/2}(\Omega)},
\end{align*}
\]

(3.17)

\[
\begin{align*}
|2Q_2(\tau, 2\tilde{N})| &= |V - V_N, \partial_t \tilde{V}_N| \\
&\leq |\tilde{V}_N|_{H^{1/2}(\Omega)} |\mathcal{T}_4(\tilde{V} - V_N)|_{L^2(\Omega)} \\
&= |\tilde{V}_N|_{H^{1/2}(\Omega)} |\mathcal{T}_5(\tilde{V} - V_N)|_{L^2(\Omega)} + 2|\tilde{V}_N|_{H^{1/2}(\Omega)} |\mathcal{T}_6(\tilde{V} - V_N)|_{L^2(\Omega)} \\
&\leq \left( (\sigma^2 N)^{5/6} \right) |V - V_N|_{L^2(\Omega)} |\tilde{V}_N|_{H^{1/2}(\Omega)} \\
&\quad + 2c \left( (\sigma^2 N)^{-\mu/2} \right) |V|_{H^{1/2}(\Omega)} |\tilde{V}_N|_{H^{1/2}(\Omega)} \\
&\leq \left( (\sigma^2 N)^{5/12 - \mu/2} \right) |V|_{H^{1/2}(\Omega)} |\tilde{V}_N|_{H^{1/2}(\Omega)} \\
&\quad + 2c \left( (\sigma^2 N)^{-\mu/2} \right) |V|_{H^{1/2}(\Omega)} |\tilde{V}_N|_{H^{1/2}(\Omega)} \\
&\leq \left( \frac{\lambda}{2} |\tilde{V}_N|_{H^{1/2}(\Omega)} + \frac{c}{\lambda} \right) \left( (\sigma^2 N)^{5/6 - \mu} \right) |V|_{H^{1/2}(\Omega)} \\
&\quad + \left( \frac{\lambda}{2} |\tilde{V}_N|_{H^{1/2}(\Omega)} + \frac{c}{\lambda} \right) \left( (\sigma^2 N)^{-\mu} \right) A_{h,\sigma}^2(V),
\end{align*}
\]

(3.18)

where

\[
A_{h,\sigma}^2(V) = \left| \left( V - V_N \right)^{1/2} \right| _{L^2(\Omega)} \left| \left( V - V_N \right)^{1/2} \right| _{L^2(\Omega)} \left| \left( V - V_N \right)^{1/2} \right| _{L^2(\Omega)} \\
\quad + \left| \left( V - V_N \right)^{1/2} \right| _{L^2(\Omega)} \left| \left( V - V_N \right)^{1/2} \right| _{L^2(\Omega)} \left| \left( V - V_N \right)^{1/2} \right| _{L^2(\Omega)}.
\]
Substituting (3.15)-(3.18) into (3.14), we obtain
\[
\frac{d}{d\tau} \| \tilde{V}_N \|^2_{L^2(\Lambda)} + \lambda \| \tilde{V}_N \|^2_{L^2(\Lambda)} \leq c \left(1 + \frac{1}{\lambda} \right) \left( (\sigma^2 N)^{1-\mu} + \sigma^4 N^{5/6-\mu} \right) B_{\mu,\sigma}^2(V) \\
+ c \left(1 + \frac{1}{\lambda} \right) \left( \| V \|^2_{L^2(\Lambda)} + \| V \|_{H^\mu_{A,\sigma}(\Lambda)}^2 \right) \| \tilde{V}_N \|^2_{L^2(\Lambda)} \\
+ \frac{c}{\lambda} \left( \sigma^2 N^{5/6} \right) \| \tilde{V}_N \|^4_{L^2(\Lambda)}
\]
where
\[
B_{\mu,\sigma}^2(V) = \| \frac{d}{d\tau} V \|^2_{L^2(0,T;H^\mu_{A,\sigma}(\Lambda))} + \| V \|^2_{L^2(0,T;H^\mu_{A,\sigma}(\Lambda))} + A^2_{\mu,\sigma}(V).
\]
Integrating the above inequality over interval \((0, \tau)\) in \(\tau\) and using the Eq. (3.11) leads to the estimate
\[
\mathcal{E}(\tilde{V}_N, \lambda, \tau) \leq c \tau \left(1 + \frac{1}{\lambda} \right) \left( (\sigma^2 N)^{1-\mu} + \sigma^4 N^{5/6-\mu} \right) B_{\mu,\sigma}^2(V) \\
+ c \left(1 + \frac{1}{\lambda} \right) \left( \frac{\sigma}{\lambda} \right) \left( \| V \|^2_{L^2(0,T;L^2(\Lambda))} + \| V \|_{L^2(0,T;H^\mu_{A,\sigma}(\Lambda))}^2 \right) \| \tilde{V}_N \|^2_{L^2(\Lambda)} \\
\times \int_0^\tau \mathcal{E}(\tilde{V}_N, \lambda, \zeta) + N^{5/6} \mathcal{E}^2(\tilde{V}_N, \lambda, \zeta) d\zeta.
\]
Recalling Lemma 2.2, we arrive at the following theorem.

**Theorem 3.2.** If
\[
\mu > \frac{11}{6}, \quad V \in L^\infty(0,T;H^1(\Lambda)) \bigcap L^\infty(0,T;H^\mu_{A,\sigma}(\Lambda)) \bigcap H^1(0,T;H^\mu_{A,\sigma}(\Lambda)),
\]
then for all \(0 \leq \tau \leq T\) we have
\[
\| V(\tau) - v_N(\tau) \|^2_{L^2(\Lambda)} + \int_0^\tau \| V(\zeta) - v_N(\zeta) \|^2_{L^2(\Lambda)} d\zeta \leq c^* \left( (\sigma^2 N)^{1-\mu} + \sigma^4 N^{5/6-\mu} \right),
\]
where \(c^*\) is a positive constant, which only depends on \(\lambda, T\) and the norm of \(V\) in the spaces mentioned.

### 4. Numerical Results

In this section, we show how to implement the method and present numerical results demonstrating its efficiency. Let \(\xi_{N,A,l}\) and \(\delta_{N,A,l}\) be the nodes and weights of the standard Hermite-Gauss interpolation. Definition (2.1) leads to the nodes and weights of the generalised Hermite function interpolation — viz.

\[
\xi_{N,A,l}^\sigma = \frac{\xi_{N,A,l}}{\sigma}, \quad \delta_{N,A,l}^\sigma = \delta_{N,A,l} e^{\frac{\xi_{N,A,l}^2}{\sigma}}.
\]
The numerical solution \(v_N(\xi, \tau)\) is sought in the form

\[
v_N(\xi, \tau) = \sum_{l=0}^{N} \delta_l(\tau) h_l^\sigma(\xi).
\]

Choosing \(\phi = h_k^\sigma(\xi)\) in (3.3) gives

\[
\sum_{l=0}^{N} (h_l^\sigma(\xi), h_k^\sigma(\xi)) \partial_\xi \delta_l(\tau) + \lambda \sum_{l=0}^{N} (\partial_\xi h_l^\sigma(\xi), \partial_\xi h_k^\sigma(\xi)) \delta_l(\tau) = \frac{1}{2} (v_N^2(\xi, \tau), \partial_\xi h_k^\sigma(\xi)) + (g(\xi, \tau), h_k^\sigma(\xi)), \quad 0 \leq k \leq N.
\]

(4.2)

Consider the matrices \(M = (m_{k,l}), S = (s_{k,l})\) and the vectors \(Y(\tau) = (\hat{v}_0(\tau), \ldots, \hat{v}_N(\tau))^T\), \(N(\tau) = (n_0(\tau), \ldots, n_N(\tau))^T\) and \(F(\tau) = (g_0(\tau), \ldots, g_N(\tau))^T\) with the entries

\[
m_{k,l} = \int_{\Lambda} h_k^\sigma(\xi) h_l^\sigma(\xi) d\xi = \begin{cases} \frac{\sqrt{\pi}}{\sigma}, & k = l, \\ 0, & \text{otherwise}, \end{cases} \quad 0 \leq k, l \leq N,
\]

\[
s_{k,l} = \int_{\Lambda} \partial_\xi h_k^\sigma(\xi) \partial_\xi h_l^\sigma(\xi) d\xi = \begin{cases} -\frac{\sqrt{\pi}}{2} \sqrt{\pi l(l-1)}, & k = l-2, \\ \frac{\sqrt{\pi}}{2} \sqrt{\pi l(l+1)(l+2)}, & k = l+2, \\ 0, & \text{otherwise}, \end{cases} \quad 0 \leq k, l \leq N,
\]

\[
n_k(\tau) = \frac{1}{2} \int_{\Lambda} v_N^2(\xi, \tau) \partial_\xi h_k^\sigma(\xi) d\xi \approx \frac{1}{2} \sum_{i=0}^{N} v_N^2 \left( \xi^\sigma_{N,\Lambda,i}, \tau \right) \partial_\xi h_k^\sigma \left( \xi^\sigma_{N,\Lambda,i}, \tau \right) \delta^\sigma_{N,\Lambda,i}, \quad 0 \leq k \leq N,
\]

\[
g_k(\tau) = \int_{\Lambda} g(\xi, \tau) h_k^\sigma(\xi) d\xi \approx \sum_{i=0}^{N} g \left( \xi^\sigma_{N,\Lambda,i}, \tau \right) h_k^\sigma \left( \xi^\sigma_{N,\Lambda,i}, \tau \right) \delta^\sigma_{N,\Lambda,i}, \quad 0 \leq k, l \leq N.
\]

It follows from (2.3), (2.4) and (4.1) that the system (4.2) can be written as

\[
M \partial_\tau Y(\tau) + \lambda SY(\tau) = N(\tau) + F(\tau), \quad 0 < \tau \leq T.
\]

Remark 4.1. Since \(\{h_k^\sigma(\xi)\}\) is an orthogonal set, the mass matrix \(M\) is diagonal. Besides, the orthogonality of \(\{h_k^\sigma(\xi)\}\) and the Eq. (2.4) yield that the stiffness matrix \(S\) is a pentadiagonal with the minor diagonals zero. The entries of the vectors \(N(\tau)\) and \(F(\tau)\) are computed by numerical quadrature.

In actual implementation, we use the explicit fourth order Runge-Kutta method in time \(\tau\) with the step \(\Delta \tau\). The numerical errors are evaluated in discrete \(L^2\)-norm, — i.e.

\[
E_{N,\Delta \tau}(\tau) := \|V(\tau) - v_N(\tau)\|_{N,\Lambda} = \left( \sum_{i=0}^{N} |V(\xi^\sigma_{N,\Lambda,i}, \tau) - v_N(\xi^\sigma_{N,\Lambda,i}, \tau)|^2 \delta^\sigma_{N,\Lambda,i} \right)^{1/2}.
\]
Example 4.1. We consider the test function
\[ V(\xi, \tau) = \text{sech}^2(\alpha \xi - \beta \tau - \gamma) \]
with \( \alpha = 1, \beta = 1, \gamma = 1 \).

Fig. 1 displays the errors \( \log_{10} E_{N, \Delta \tau}(\tau) \) versus modes \( \sqrt{N} \) with \( \lambda = 1, \tau = 1, \Delta \tau = 0.005, \Delta \tau = 0.001 \) and \( \Delta \tau = 0.0001 \). The errors rapidly decay as \( N \) grows and \( \Delta \tau \) decreases. For a fixed \( \Delta \tau \) the numerical error is dominated by the approximation error in space, so that they rapidly decay as \( N \) increases. On the other hand, for \( N > 100 \) the total numerical error is dominated by the approximation error in time, and decays as \( \Delta \tau \) decreases, consistent with theoretical analysis in Section 3 and showing the spectral accuracy of the scheme (3.3) in space.

Fig. 2 shows the errors \( \log_{10} E_{N, \Delta \tau}(\tau) \) versus modes \( \sqrt{N} \) with \( \lambda = 0.001, \tau = 1, \Delta \tau = 0.005, \Delta \tau = 0.001 \) and \( \Delta \tau = 0.0001 \). Obviously, the scheme (3.3) works well for small \( \lambda \).

Example 4.2. We now consider the test function
\[ V(\xi, \tau) = \sin(\omega \xi + \tau)e^{-\xi^2} \]  
(4.3)

In Fig. 3, we draw the errors \( \log_{10} E_{N, \Delta \tau}(\tau) \) against modes \( \sqrt{N} \) with \( \omega = 2, \lambda = 1, \tau = 1, \Delta \tau = 0.005, \Delta \tau = 0.001 \) and \( \Delta \tau = 0.0001 \). The curves show that exponential convergence rate is achieved, and the efficiency of our new algorithm. Fig. 4 displays the errors \( \log_{10} E_{N, \Delta \tau}(\tau) \) against modes \( \sqrt{N} \) with \( \omega = 2, \lambda = 0.001, \tau = 1, \Delta \tau = 0.005, \Delta \tau = 0.001 \) and \( \Delta \tau = 0.0001 \). We observe that even for small \( \lambda \) the algorithm is still efficient.

Example 4.3. We consider the test function
\[ V(\xi, \eta) = \frac{\sin(\omega \xi + \eta)}{(\xi + b)^a} \]
Remark 4.2. Figs. 1-5 displayed the errors \( \log_{10} E_{N, \Delta \tau}(\tau) \) against modes \( \sqrt{N} \) indicate that the convergent rates behave as

\[
\| V - v_N \|_{N, \Lambda} \sim e^{-c \sqrt{N}},
\]

where \( c \) is a positive constant. This is consistent with the theoretical analysis above.

In order to show the efficiency of the new algorithm, we present numerical results obtained by a mapped Chebyshev spectral method. Let \( \xi \in \Lambda = (−\infty, \infty) \), \( y \in I = (−1, 1) \), \( s > 0 \) be a scaling factor and \( \omega(y) = (1 - y^2)^{-1/2} \) the Chebyshev weight function. The
functions mapping $\Lambda$ into $I$ and vice versa have the form — cf. [38, Eqs. (7.160)]:

$$\xi = f(y; s) = \frac{s}{2} \ln \frac{1 + y}{1 - y}, \quad y = h(\xi; s) = \tanh(\xi/s).$$

We denote $W(y) = V(\xi) = V(f(y; s))$. Obviously

$$\frac{\partial^2 \psi}{\partial y^2}(\xi) = \frac{1}{f'(y; s)^2} \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial y} \right) = \frac{1}{s^2} (1 - y^2) \partial_y \left( (1 - y^2) \partial_y W(y) \right),$$

Choosing $s = 1$, we write the Eqs. (3.1) as

$$\begin{align*}
\partial_\xi W(y, \tau) &+ \frac{1}{2} (1 - y^2) \partial_y \left( W^2(y, \tau) \right) \\
= &- \lambda (1 - y^2) \partial_y \left( (1 - y^2) \partial_y W(y, \tau) \right) \\
G(y, \tau), & \quad y \in I, \quad 0 < \tau \leq T, \\
W(\pm 1, \tau) & = \lim_{\xi \to \pm \infty} V(h(\xi; 1), \tau) = \partial_\xi W(\pm 1, \tau) \\
= & \lim_{\xi \to \pm \infty} \partial_\xi V(h(\xi; 1), \tau) = 0, \quad 0 < \tau \leq T, \\
W(y, 0) & = W_0(y), \quad y \in I,
\end{align*}$$

(4.4)

where $G(y, \tau) = g(f(y; 1), \tau)$. Let

$$\mathcal{F}(I) = \{w| \text{there exists finite trace of } w(y) \text{ at } y = \pm 1\},$$

$$L^2_{0, \omega}(I) = \mathcal{F}(I) \cap \{w| w \in L^2_{\omega}(I) \text{ and } w(\pm 1) = 0\},$$

$$H^1_{0, \omega}(I) = \mathcal{F}(I) \cap \{w| w \in H^1_{\omega}(I) \text{ and } w(\pm 1) = \partial_w w(\pm 1) = 0\}.$$

The Chebyshev polynomial of degree $n$ denoted by $T_n(y)$. The functions $T_n(y)$ are mutually orthogonal with respect the weight function $\omega(y)$, i.e.

$$\int_I T_n(y)T_m(y)\omega(y)dy = \frac{c_n\pi}{2} \delta_{nm},$$

where $c_0 = 2$, $c_n = 1$ for $n \geq 1$ and $\delta_{nm}$ is the Kronecker delta. Introducing the functions

$$\psi_n(y) = T_n(y) - \frac{2(n + 2)}{n + 3}T_{n+2}(y) + \frac{n + 1}{n + 3}T_{n+4}(y), \quad n = 0, 1, \ldots,$$

we note that $\psi_n(\pm 1) = \partial_\xi \psi_n(\pm 1) = 0$, $n = 0, 1, \ldots$, cf. [38].

We now consider the spaces

$$\tilde{\mathcal{F}}^0_N(I) = \text{span} \{\psi_0(y), \psi_1(y), \ldots, \psi_{N-4}(y)\},$$

$$\mathcal{F}^0_N(I) = \tilde{\mathcal{F}}^0_N(I) \cap H^1_{0, \omega}(I),$$
and the orthogonal projections \( \Pi_{N,\omega}^{1,0} : \mathcal{H}_{0,\omega}(I) \rightarrow \gamma_{N}^{0}(I) \) defined by

\[
\left( \Pi_{N,\omega}^{1,0} w - w, v \right)_{\omega} + \left( \partial_y \left( \Pi_{N,\omega}^{1,0} w - w \right), \partial_y v \right)_{\omega} = 0 \quad \text{for all} \quad v \in \gamma_{N}^{0}(I).
\]

The weak form of the problem (4.4) consists in finding

\[
W \in L^2 \left( 0, T; \mathcal{H}_{0,\omega}(I) \right) \bigcap L^\infty \left( 0, T; \mathcal{L}^2_{0,\omega}(I) \right)
\]

such that

\[
(\partial_t W, v)_{\omega} + \mu \left( (1 - y^2) \partial_y W, \omega^{-1} \partial_y \left( (1 - y^2) v \omega \right) \right)_{\omega} = (F, v)_{\omega} - \frac{1}{2} \left( (1 - y^2) \partial_y (W^2), v \right)_{\omega} \quad \text{for all} \quad v \in \mathcal{H}_{0,\omega}^1(I),
\]

\[
W(y, 0) = W_0(y) \quad \text{on} \quad I.
\]

The corresponding spectral scheme is to find \( w_N \in \gamma_{N}^{0}(I) \) for all \( 0 < \tau \leq T \) such that

\[
(\partial_t w_N, \psi_m)_{\omega} + \mu \left( (1 - y^2) \partial_y w_N, \omega^{-1} \partial_y \left( (1 - y^2) \psi_m \omega \right) \right)_{\omega} = (F, \psi_m)_{\omega} - \frac{1}{2} \left( (1 - y^2) \partial_y (w_N^2), \psi_m \right)_{\omega} \quad \text{for all} \quad \psi_m \in \gamma_{N}^{0}(I),
\]

\[
w_N(y, 0) = \Pi_{N,\omega}^{1,0} W_0(y), \quad \text{on} \quad I.
\]

In actual computations, the numerical solution is sought in the form

\[
w_N(y, \tau) = \sum_{i=0}^{N-1} \hat{w}_i(\tau) \psi_i(y),
\]

and we use the explicit fourth order Rugge-Kutta method in time \( \tau \) with the step \( \Delta \tau \).

We consider the test function

\[
W(y, \tau) = \sin \left( \frac{\omega}{2} \ln \frac{1 + y}{1 - y} + \tau \right) e^{-\left(1/4)\ln(1+y)/(1-y)\right)^2},
\]

which corresponds the test function (4.3).

Table 1 shows the \( L^2 \)-errors \( E_{N,\Delta \tau}(\tau) \) of the Hermite spectral method (EHSM) with the test function (4.3) and the \( L^2 \)-errors \( E_{N,\Delta \tau}(\tau) \) of the mapped Chebyshev spectral method (EMCSM) with the test function (4.5). It is clear that the Hermite spectral method outperforms the mapped Chebyshev spectral method.

| \( N \) | 12 | 28 | 44 | 60 | 76 | 92 | 108 | 124 |
|--------|----|----|----|----|----|----|-----|-----|
| EHSM   | 8.22e-2 | 5.09e-3 | 4.17e-4 | 2.86e-05 | 1.70e-6 | 9.10e-8 | 4.46e-9 | 2.04e-10 |
| EMCSM  | 5.92e-1 | 1.01e-2 | 5.15e-4 | 8.24e-05 | 6.70e-6 | 3.17e-6 | 1.14e-6 | 3.75e-7 |
5. Concluding Discussion

We propose a spectral method based on generalised Hermite functions. It is applied to the Burgers equation on the whole real line and allows us to avoid using the domain decomposition technique and variable transformations. In addition, it simplifies the theoretical analysis and actual computations. We prove the generalised stability, the convergence of the method and show that for smooth solutions it has spectral accuracy in space and outperforms the mapped Chebyshev spectral method. Numerical results demonstrate the efficiency of the algorithm.

Acknowledgments

The author would like to thank the anonymous referees for their valuable suggestions and comments.

This work is supported in part by NSFC grants (Nos.11771299, 11371123, 11571151).

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