Refined diamond norm bounds on the emergence of objectivity of observables

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Abstract
The theory of quantum Darwinism aims to explain how our objective classical reality arises from the quantum world, by analysing the distribution of information about a quantum system that is accessible to multiple observers, who probe the system by intercepting fragments of its environment. Previous work showed that, when the number of environmental fragments grows, the quantum channels modelling the information flow from system to observers become arbitrarily close—in terms of diamond norm distance—to ‘measure-and-prepare’ channels, ensuring objectivity of observables; the convergence is formalised by an upper bound on the diamond norm distance, which decreases with increasing number of fragments. Here, we derive tighter diamond norm bounds on the emergence of objectivity of observables for quantum systems of infinite dimension, providing an approach which can bridge between the finite- and the infinite-dimensional cases. Furthermore, we probe the tightness of our bounds by considering a specific model of a system-environment dynamics given by a pure loss channel. Finally, we generalise to infinite dimensions a result obtained by Brandão \textit{et al} (2015 \textit{Nat. Commun. 6} 7908), which provides an operational characterisation of quantum discord in terms of one-sided redistribution of correlations to many parties. Our results provide a unifying framework to benchmark quantitatively the rise of objectivity in the quantum-to-classical transition.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Quantum theory has proven to be extremely successful in describing the physical laws of microscopic objects. However, assuming the general validity of quantum theory, the apparent absence of quantum features (such as non-locality and superposition effects) in our everyday classical reality raises the issue of the quantum-to-classical transition: how do physical systems lose their ‘quantumness’ with increasing scales and become effectively classical?

The theory of decoherence [1–4], which developed significantly over the past decades, has pointed out the key role played in this transition by the interaction of the system with its environment: due to this interaction the two can become entangled, and the quantum correlations so established between the two parties cannot be observed at the level of the system alone. The entanglement with the environment thus defines the physical properties we can observe at the level of the system. In particular, only those states that are robust in spite of the interaction with the environment are observable in practice. The environmental monitoring therefore leads to the selection of preferred states (known as pointer states [5–7]) which represent the natural candidates for the classical states that are compatible with our everyday experience. However, decoherence alone does not explain how the striking contrast between classical and quantum states is overcome in the emergence of classicality. In fact, while classical states can be detected and agreed upon by initially ignorant observers without being perturbed, and thus exist objectively, quantum states are generally affected by the measurement process. It is therefore necessary to clarify how the information about pointer states becomes objective.

The theory of quantum Darwinism [8–13] provides a possible solution by promoting the environment from source of decoherence to carrier of information about the system. In fact, quantum Darwinism points out that a fundamental consequence of the system-environment interaction is the presence of information about the system encoded in the environment. By intercepting fragments of the environment, it is possible to acquire such information indirectly. In particular, quantum Darwinism explains how information about the pointer states proliferates in the environment, allowing multiple observers to detect these states without perturbing their existence.

The quantum Darwinism approach to the emergence of classicality has been explored theoretically in various specific models [14–26] and has also been the subject of recent experimental tests [27–29]. However, the range of applicability of such framework still represents an open issue in the quantum-to-classical transition problem. A recent result by Brandão et al [30] made a significant contribution to it, showing how some classical features emerge in a model-independent way from the quantum formalism alone. Such result relies on the splitting of the objectivity notion into the following statements:

• **Objectivity of observables**: multiple observers probing the same system can at most acquire classical information about one and the same measurement;

• **Objectivity of outcomes**: the observers will agree on the result obtained from the preferred measurement.

Brandão et al modelled the information flow from a (finite-dimensional) quantum system to the fragments of its environment via quantum channels, i.e., completely positive trace-preserving (cptp) maps. They showed that, when the number of fragments \( N \) becomes large
enough, most of these channels are well approximated by specific cptp maps, called ‘measure-and-prepare’. The form of such channels ensures the objectivity requirements. This is formalised by a bound on the distance (induced by the so-called diamond norm) between the system-environment channels and the measure-and-prepare ones. It is found that such distance goes to zero as $N \to \infty$, leading to convergence to objectivity of observables (in the following, we will refer to such a bound as objectivity bound). In [31], Knott et al overcame the finite-dimension restriction by showing that also infinite-dimensional systems, under appropriate energy constraints, exhibit objectivity of observables. Another interesting result in this context was recently obtained by Qi and Ranard [32]: they showed that, for finite-dimensional systems, the set of channels which do not converge to objectivity is of fixed size $O(1)$, instead of scaling with the number of environmental fragments $N$, as in [30, 31]. Their result, which incorporates insights from an earlier version of the present manuscript, provides to the best of our knowledge the tightest objectivity bound in the finite-dimensional scenario.

In this paper we extend the infinite-dimensional analysis of [31] and provide a unified approach to the emergence of the objectivity of observables in the interaction between a quantum system of arbitrary dimension and a large number of fragments of its environment.

Specifically, we first prove that the objectivity of observables holds true for a wide class of (energy-constrained) infinite-dimensional systems. For such class we obtain tighter bounds on the emergence of this classical feature, compared to those available in the literature. Moreover, our framework can act as a bridge between the finite- and infinite-dimensional scenarios. Our results rely on an infinite-dimensional version of the Choi–Jamiołkowski isomorphism, adapted to our set of energy-constrained states. This generalises what was done in [31] for a specific choice of the energy constraint. Moreover, our analysis exploits novel bounds relating the diamond-norm distance of two channels with the distance between their respective Choi–Jamiołkowski states—see Aubrun et al [33]. Such results are presented in section 2.

A relevant issue concerning the emergence of objectivity of observable, not tackled in references [30, 31], concerns the optimality of the rates at which the convergence to objectivity takes place. In fact, objectivity of observables is regarded as emergent whenever the upper bound on the distance between channels representing the system-environment information flow and the measure-and-prepare ones goes to zero asymptotically. But this does not give information on how well the objectivity bound approximates the considered diamond norm distance. To perform such optimality check, a possible strategy is to derive a lower bound for that diamond norm, which turns out to be an upper bound on the speed at which the emergence of objectivity of observables takes place. In section 3 we perform this analysis for the specific model of a system-environment dynamics given by a pure loss channel, and show that for such model the rate of convergence to objectivity of observables scales at least as the inverse of the number of environmental fragments.

The final point we address regards the extension to an infinite-dimensional scenario of the operational interpretation of quantum discord [34, 35] derived for finite-dimensional systems by Brandão et al [30]. In particular, it was proven that when information is distributed to many parties on one side of a bipartite system, the minimal average loss in correlations corresponds to the quantum discord. In section 4 we generalise this result to the infinite-dimensional case by exploiting the objectivity bounds proved in section 2.

In summary, the paper is organised as follows. Our improved objectivity bounds are presented in section 2, followed by the pure loss channel analysis in section 3, while the operational interpretation of quantum discord is found in section 4. Some technical details behind our proofs are deferred to the appendixes.
2. Improved bounds on the emergence of objectivity of observables

The scenario we consider consists of a system $A$, generally infinite-dimensional, and its environment $B$, which is described as a collection of $N$ (possibly infinite-dimensional) subsystems $B_1, \ldots, B_N$, namely the environment fragments. We assume that the system of interest $A$ is initially decorrelated from $B_1, \ldots, B_N$, and that the corresponding state has bounded mean energy (defined via an appropriate Hamiltonian—see below). The information flow from the system to the whole environment is modelled as a quantum channel, i.e., a ctp map $\Lambda : \mathcal{D}(A) \to \mathcal{D}(B_1 \otimes \cdots \otimes B_N)$, where $\mathcal{D}(Z)$ denotes the set of density matrices associated with a physical system $Z$. The transfer of quantum information from $A$ to the single environmental fragment $B_j$ is therefore described by the ‘subchannel’ $\Lambda_j = \text{Tr}_{B_i \setminus B_j} \circ \Lambda$. Objectivity of observables then arises whenever the maps $\Lambda_j$ become arbitrarily close to measure-and-prepare channels, which allow observers to acquire only classical information about one and the same measurement. These channels are defined as $\mathcal{E}_\beta(X) := \sum_j \text{Tr}(M_j X) \tau_{j\beta}$, where $\{M_j\}_i$ is a positive operator-valued measure (POVM)—crucially independent of the index $j$—and $\{\tau_{j\beta}\}_i$ is a set of states for subsystem $B_j$.

We shall quantify distinguishability in the space of channels via a distance called energy-constrained diamond norm $[36, 37]$. This is a modification of the standard diamond norm $[38–40]$, designed to implement a restriction on the average (initial) energy of the quantum system under examination. This is measured by a Hamiltonian, which we take to be an arbitrary self-adjoint operator $H$ with spectrum bounded from below. Without loss of generality, we assume its ground state energy to be positive, i.e.

$$\inf_{\lambda \in \text{sp}(H)} \lambda = E_0 > 0,$$ 

where $\text{sp}(H)$ is the spectrum of $H$.

**Definition 1.** Let $A'$ be a quantum system equipped with a Hamiltonian $H_{A'}$ that satisfies equation (1), and pick $E > E_0$. Then the energy-constrained diamond norm of an arbitrary Hermiticity-preserving linear map $\Lambda : \mathcal{D}(A') \to \mathcal{D}(B)$ is defined by

$$\|\Lambda\|_{H,E} := \sup_{\text{Tr}[\rho_{A'}] \leq E} \left\| (\text{id}_A \otimes \Lambda_{A'})[\rho_{A'}] \right\|_1,$$

where $A$ is an arbitrary ancillary system, and $\| \cdot \|_1$ is the one-norm. A recent result by Weis and Shirokov [41] ensures that the input state $\rho_{A'}$ in equation (2) can be taken to be pure.

In our analysis, we assume that the Hamiltonian admits a countable set of eigenvectors forming an orthonormal basis $\{|j\rangle, j = 0, 1, \ldots\}$ of the Hilbert space. The index $j$ is allowed to go to infinity, and the case of a finite-dimensional system will be treated by a suitable choice of the Hamiltonian eigenvalues (see below).

We want to stress that the assumption that a Hamiltonian $H$ has discrete spectrum and is bounded from below is physically well motivated; in fact, it is contained in the so-called Gibbs hypothesis [42]:

**Gibbs hypothesis.** A (possibly unbounded) self-adjoint operator $H$ is said to satisfy the Gibbs hypothesis if for every $\beta > 0$ the partition function $Z(\beta) := \text{Tr} e^{-\beta H}$ is finite. As a consequence, the state $\frac{1}{Z(\beta)} e^{-\beta H}$ has finite entropy. Moreover, for every eigenvalue $E$ of the Hamiltonian $H$, the (unique) maximiser $\rho$ of the entropy subjected to the constraint $\text{Tr} \rho H \leq E$ is the Gibbs state

$$\gamma(E) = \frac{1}{Z(\beta(E))} e^{-\beta(E) H}.$$
where \( \beta = \beta(E) \) is the solution to the equation \( \text{Tr} e^{-\beta H}(H - E) = 0 \).

By setting

\[
H = \sum_j f_j |j\rangle \langle j|,
\]

we also require that the (increasing) sequence of eigenvalues \( f_j \) diverges sufficiently rapidly, in formula

\[
\left| \sum_j \frac{1}{f_j} \log \frac{1}{f_j} \right| < \infty.
\]

In the following, we will refer to the entire spectrum \( \{f_j\} \) by using the short notation \( f = \{f_j\} \).

Equation (5) clearly implies that \( \sum_j 1/f_j < \infty \). Notably, this excludes the physically relevant case \( f_j = j \), corresponding to the canonical Hamiltonian on the Hilbert space of a harmonic oscillator. In spite of this drawback, our technical assumption allows us to explore a rich family of constraints that effectively extend and interpolate between previously known bounds. Moreover, a slight modification of our proof technique allows us to deal with the excluded case \( f_j = j \) as well; for details, see the end of this section.

Before proceeding with the presentation of our results, we want to clarify that the positive operator \( H \) and the scalar threshold \( E \) do not actually have any dynamical characterization. In fact, their role in our framework is simply to confine the bulk of a state’s probability mass to a finite-dimensional subspace in a smooth and well-defined way.

We now introduce some technical elements and definitions that will enter our main results on the emergence of objectivity of observables, stated in theorem 4. We start by considering a special class of entangled states featuring an \( f \)-dependent tail in the Hamiltonian eigenbasis:

\[
|\phi\rangle := c_f \sum_{j=0}^{\infty} \phi_j |j\rangle _{AA'},
\]

with \( \phi_j^2 := 1/f_j \) and \( c_f := \left( \sum_j 1/f_j \right)^{-1/2} \). In our derivation, an important role will be played by the local von Neumann entropy of \( |\phi\rangle \), given by

\[
\sigma := S \left( \text{Tr}_{A'}|\phi\rangle \langle \phi|_{AA'} \right) = -\sum_j \frac{\phi_j^2}{f_j} \log \left( \frac{\phi_j^2}{f_j} \right) < \infty,
\]

where the last inequality follows from equation (5).

A useful technical tool in our work is the \( d \)-dimensional truncation of our entangled state \( |\phi\rangle \), which can be obtained as \((\Pi_d \otimes \text{id})|\phi\rangle = (\text{id} \otimes \Pi_d)|\phi\rangle = c_f \sum_{j=0}^{d-1} \phi_j |j\rangle _{AA'} \), where  \( \Pi_d = \sum_{j=0}^{d-1} |j\rangle \langle j| \). The ‘approximation error’ associated with this truncation can be quantified as follows:

**Definition 2.** The *tail* of our entangled state \( |\phi\rangle \), dependent on the truncation dimension \( d \), is defined as

\[
e_d := ||(\text{id} - \Pi_d) \otimes \text{id})|\phi\rangle || = c_f \sqrt{\sum_{j=d}^{\infty} \frac{1}{f_j}}.
\]
The modified Choi–Jamiolkowski state of a ctpm map $\Lambda : \mathcal{D}(A) \to \mathcal{D}(B)$, for a given sequence of Hamiltonian eigenvalues $f = \{f_j\}$, is defined as

$$J_f(\Lambda) := \text{id}_A \otimes \Lambda_{\mathcal{N}}[\phi] \langle \phi |,$$

where $|\phi\rangle$ is given in equation (6).

Having introduced all the required ingredients, we can now state the following theorem.

**Theorem 4.** Let $A$ be a quantum system equipped with a Hamiltonian $H_A$ which satisfies the Gibbs hypothesis and which, when written as in equation (4), also satisfies equation (5). Consider an arbitrary ctpm map $\Lambda : \mathcal{D}(A) \to \mathcal{D}(B_1 \otimes \cdots \otimes B_N)$, and define the effective dynamics from $\mathcal{D}(A)$ to $\mathcal{D}(B_j)$ as $\Lambda_j := \text{Tr}_{B_1} \cdots \text{Tr}_{B_{j-1}} \otimes \Lambda \circ \text{Tr}_{B_{j+1}} \cdots \text{Tr}_{B_N}$. For an arbitrary number $0 < \delta < 1$, there exists a POVM $\{M_j\}$, and a set $S \subseteq \{1, \ldots, N\}$, with $|S| \geq (1 - \delta)N$, such that, for all $j \in S$ and for any integer truncation dimension $d \geq 0$, we have that

$$\|\Lambda_j - \mathcal{E}_j\|_{\text{H.E.}} \leq \frac{\zeta}{d},$$

where the measure-and-prepare channel $\mathcal{E}_j$ is given by

$$\mathcal{E}_j(X) := \sum_l \text{Tr}(M_l X) \tau_{jl},$$

for some family of states $\tau_{jl} \in \mathcal{D}(B_j)$, and

$$\zeta = \kappa d \left( \frac{E^2 \sigma}{N c_f^2} \right)^{1/3} + \frac{4E}{c_f^2} \epsilon_d,$$

where $c_f$ is the normalization factor introduced in equation (6); $\epsilon_d$ is given in definition 2; $\sigma$ is defined by equation (7) and $\kappa := 3(16 \ln(2))^{1/3}$ is a universal constant.

The complete proof is detailed in appendix A. In what follows we provide the key ideas behind it.

**Outline of the proof of theorem 4.** We start by proving that the one-norm of an operator $L$, given by the difference between two $f$-Choi states, can be bounded as follows:

$$\|L\|_1 \leq 4d^2 \max_{\mathcal{M}} \|\text{id} \otimes \mathcal{M}(L)\|_1 + 4\epsilon_d.$$  

(13)

Here, $\mathcal{M}$ is an arbitrary measurement, thought of as a quantum-to-classical channel, $d$ is the truncation dimension and $\epsilon_d$ is given in definition 2. We then show that the distance between two channels is bounded by that between their $f$-Choi states:

$$\|\Lambda_0 - \Lambda_1\|_{\text{H.E.}} \leq \frac{E}{c_f} \|J_f(\Lambda_0) - J_f(\Lambda_1)\|_1.$$  

(14)

The key ingredient of the proof is a result (lemma A12 in appendix A) which introduces a set of quantum-to-classical channels $\{M_j| j \in J\}$ acting on a subset $J$ of the environment fragments $B_1, \ldots, B_N$. Let $\zeta$ be the outcome of such set of measurements, then the state $\mathcal{E}_j \rho_A \otimes \rho_B_j$ can be proved to be the modified Choi–Jamiolkowski state of a measure-and-prepare channel $\mathcal{E}_j$ with POVM independent of $j \notin J$. The lemma bounds the quantity

$$\mathbb{E}_{j \in J} \max_{\mathcal{M}_j} \|\text{id} \otimes \mathcal{M}_j[\rho_{AB} - \mathbb{E}_j \rho_A \otimes \rho_{B_j}]\|_1.$$  

(15)
Figure 1. Case $f_j = j^2$. We plot the upper bound on $\|\Lambda_j - E_j\|_{\mathcal{H},E}$ for $E = 1$ and $\delta = 0.01$, obtained through numerical optimisation of equation (17) over the truncation dimension $d$.

through a function of the entropy for system $A$; in (15), the expectation value is with respect to the uniform distribution over $\{1, \ldots, N\} \setminus J$, and the maximum is taken over all quantum-to-classical channels.

Since $\rho_{AB} = J_f(\Lambda_j)$ and $E_z \rho_z \otimes \rho_{B_j} = J_f(E_j)$, by combining lemma A12 with the previous inequalities we find a bound for the quantity $E_j \|\Lambda_j - E_j\|_{\mathcal{H},E}$. We then easily obtain $E_j \|\Lambda_j - E_j\|_{\mathcal{H}, E} \leq \zeta$, where the index $j$ has uniform probability distribution over $\{1, \ldots, N\}$, and $\zeta$ is given by equation (12).

We conclude the proof by applying Markov’s inequality. In fact, the statement of the theorem is equivalent to the following one:

$$\mathbb{P}\left(\|\Lambda_j - E_j\|_{\mathcal{H}, E} \geq \frac{\zeta}{\delta}\right) \leq \delta. \quad (16)$$

The result of theorem 4 can be interpreted as follows. Fixing $0 < \delta < 1$ and $E$, and letting the number of environmental fragments $N$ tend to infinity, we have that the dynamical maps connecting the system to each of the fragments become indistinguishable from measure-and-prepare channels. This statement is true for at least a fraction $1 - \delta$ of the sub-environments. Moreover, the measure-and-prepare channels involved are all defined by the same POVM $\{M_l\}_l$. For $\delta \ll 1$ this means that almost all observers probing the system by intercepting fragments of the environment can at most acquire classical information about one and the same measurement $\{M_l\}_l$—i.e., objectivity of observables holds for such observers.

To illustrate the application of the results derived in this section to concrete physical models, we now consider some relevant examples.

Case $f_j = j^2$, with $j \geq 1$ (particle in a box). A quantum particle of mass $m$ confined in a box of length $L$ has Hamiltonian eigenvalues $f_j = \gamma j^2$, where $\gamma$ is a constant given by $\gamma = \frac{\hbar^2 \pi^2}{2 m L^2}$. Choosing units such that $\gamma = 1$ we have that $f_j = j^2$, and theorem 4 turns out to hold for

$$\zeta = \alpha \left(\frac{\sigma d^2 E^2}{N}\right)^{1/3} + \beta E \sqrt{\psi^{(d)}(1)}, \quad (17)$$

where $\psi^{(d)}(z)$ is the $d$-th derivative of the digamma function $\psi(z)$, $\sigma \approx 2.4$ and $\alpha, \beta$ are universal constants: $\alpha := (12 \pi^4)^{1/4}$, $\beta := \sqrt{\frac{24}{\pi}}$. In figure 1 we plot the objectivity bound $\frac{\zeta}{\delta}$ provided by a numerical optimisation of equation (17) over $d$, with $E = 1$ and $\delta = 0.01$. 

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Case of a D-dimensional system. In this example, we show that our methods can bridge finite and infinite dimensions. Specifically, let us consider the sequence of Hamiltonian eigenvalues

\[
f_j = \begin{cases} 
1 & \text{j} \leq D - 1, \\
\frac{e^{\omega j}}{1 - e^{-\omega}} & \text{j} \geq D,
\end{cases}
\]  

(18)

where \(D\) is a parameter that will turn out to be the actual Hilbert space dimension when \(\omega \to \infty\). We assume \(d \geq D\) for convenience. We obtain that

\[
\zeta = \left(\frac{432E^2(D + e^{-\omega D})^2d^3s}{N}\right)^{1/3} + 4E\sqrt{\frac{D + e^{-\omega D}}{e^d}}
\]  

(19)

where

\[
s := \ln(2)\sigma \\
= \ln(D + e^{-\omega D}) + \frac{\omega(1-D+De^{-\omega})}{(e^\omega - 1)(De^{-\omega} + 1)} - \ln(1-e^{-\omega}) \frac{\omega}{(1+De^{-\omega})}
\]

(see example A13 in appendix A for details). Taking the limit \(\omega \to \infty\) we have that

\[
\lim_{\omega \to \infty} \zeta = \left(\frac{432E^2D^2d^3 \ln D}{N}\right)^{1/3}.
\]  

(20)

In this scenario, \(\text{Tr}[\rho H] \leq E\) translates into the condition \(\text{Tr}[\rho] \leq 1\), plus the additional constraint that the support of \(\rho\) is contained in the \(D\)-dimensional subspace spanned by \(\{|0\}, |1\}, \ldots, |D-1\}\). Physically, the considered limit corresponds to raising all the Hamiltonian eigenvalues with \(j \geq D\) to unattainably high energies, so that only levels with \(j < D\) can be populated.

In the finite-dimensional scenario, the tightest objectivity bound to date has been recently obtained by Qi and Ranard [32]. The Qi–Ranard result can be compared to ours by making the substitution \(|R| = 1, |Q| = N\delta\) in equation (12) of reference [32]. For the present comparison, we have to consider two possible expressions for the parameter \(\Omega\) (which appears in equation (13) of reference [32]): \(\Omega = D^2\) and \(\Omega = 4D^{3/2}\), where \(D\) is the dimension of system \(A\); the corresponding expressions for the Qi–Ranard bound are the following:

\[
\Omega = D^2: \quad b_1 = \left(\frac{2D^6 \ln D}{N\delta}\right)^{1/2}
\]

(21)

\[
\Omega = 4D^{3/2}: \quad b_2 = 4\left(\frac{2D^5 \ln D}{N\delta}\right)^{1/2}
\]

(22)

Expressions \(b_1\) and \(b_2\) can be directly compared with our bound \(b := \zeta\), with \(\zeta\) given by equation (20), by taking \(d = D\) (which clearly gives us the tightest bound for the range \(d \geq D\)) and by choosing \(E = 1\). The comparison is meaningful only in the regime in which the bounds are non-trivial, namely smaller than 2. By applying this requirement to our bound \(b\) we obtain a threshold value for \(N\) which depends on \(D: N > \frac{24dD^3}{B}\ln D\). We find that our bound \(b\) is never stronger than \(b_2\) in its non-triviality regime. Note that \(b_2\) was derived by the authors of [32] on the basis of our suggestion to exploit the Aubrun et al result [33] in this context. Hence \(b_2\), which is the tightest known objectivity bound in finite dimensions, is a synthesis of independent insights from the analysis of Qi and Ranard and our work. The less tight bound \(b_1\) was
instead obtained by Qi and Ranard independently of the present work. There exists a regime in which our bound $b$ is non-trivial and stronger than $b_1$; however, this regime may be of little relevance in experimental contexts, as it requires a very large threshold for $N$, e.g. rising above Avogadro’s number for $\delta < 0.1$.

We now return to the case $f_i = j$, in which the condition $\sum \frac{1}{f_i} < \infty$ is not satisfied. In this case the $f$-Choi states cannot be defined, and we replace them with truncated (standard) ones. To derive the objectivity bound we go through the same conceptual steps followed by Knott et al in [31]. However, we bound the distance between truncated Choi–Jamiołkowski states more restrictively, by exploiting a result by Aubrun et al [33, corollary 9]. We are then able to derive an objectivity bound that, for $d > 16$, is tighter than the one obtained in [31]. In particular, we find that theorem 4 holds for $f_i = j$ with

$$\zeta = \lambda \left( \frac{d^5 \log(d)}{N} \right)^{1/3} + 4 \sqrt{\frac{E}{d}}$$

(23)

where $\lambda := 3(16 \ln(2))^{1/3}$. In figure 2 we compare the mean energy bound provided in [31] (blue dots, uppermost curve) with the refined one we obtain from equation (23) (red dots, lowermost curve). Both bounds are numerically optimised over $d$ by setting $E = 1$ and $\delta = 0.01$.

3. Testing optimality of the objectivity bound with an $N$-splitter

The emergence of objectivity of observables, as explored in the previous section as well as in [30, 31], is expressed by an upper bound on the distance between the effective dynamics $\Lambda_j$ and the measure-and-prepare channels $E_j$, which goes to zero as the number $N$ of environment fragments gets large. We now probe the optimality of such statement by looking at a lower bound for the distance between $\Lambda_j$ and $E_j$ in a specific example. This gives information on the speed at which the emergence of objectivity of observables takes place. We carry out this analysis for a system-environment interaction modelled by a pure loss channel. In detail, both our system $A$ and each of its sub-environments $B_1, \ldots, B_N$ will be single bosonic modes with associated annihilation operators $a_0$ and $a_1, \ldots, a_N$, respectively. The canonical commutation
relations read \([a_j, a_k^\dagger] = \delta_{jk}\). We consider the quantum channel
\[
\Lambda_{A\rightarrow B_1, \ldots B_N}(\cdot) := U \left( (\cdot) \otimes_{j=2}^N |0\rangle \langle 0|_{B_j} \right) U^\dagger,
\]
where \(U\) is the symplectic unitary which implements an \(N\)-splitter from \(\mathcal{D}(A \otimes B_2 \otimes \cdots \otimes B_N)\) to \(\mathcal{D}(B_1 \otimes \cdots \otimes B_N)\). In terms of bosonic operators (in the Heisenberg picture) this transformation takes the explicit form \(U\) is a unitary that depends on the initial environment state and the two-mode squeezed vacuum states. Upon straightforward calculation, one obtains equation (24) corresponds to a pure loss channel of parameter \(\frac{1}{N}\) [44]. Varying the environment state one obtains instead a general attenuator [45–50]. The reduced map \(\Lambda_j : \mathcal{D}(A) \rightarrow \mathcal{D}(B_j)\) is given by \(\Lambda_j = \text{Tr}_{B_1, \ldots, B_{j-1}, B_{j+1}, \ldots, B_N} \circ \Lambda\) and has the same form for all \(j\), as shown in appendix B.1. As in the previous section, we assume that system \(A\) has bounded mean energy. As is typically the case in optical systems, the relevant Hamiltonian is obtained by setting \(f_j = j\) (where \(j\) may be interpreted as the number of photons). We show that, for a maximum energy threshold \(E\) on system \(A\) satisfying \(E \geq \frac{2}{N}\), the channels \(\Lambda_j\) approach the measure-and-prepare ones no faster than \(\sim N^{-1}\). In particular, we can prove the following proposition.

**Proposition 5.** Consider the ctp map \(\Lambda : \mathcal{D}(A) \rightarrow \mathcal{D}(B_1 \otimes \cdots \otimes B_N)\) given by equation (24), and define \(\Lambda_j : = \text{Tr}_{B_1, \ldots, B_{j-1}, B_{j+1}, \ldots, B_N} \circ \Lambda\) as the effective dynamics from \(\mathcal{D}(A)\) to \(\mathcal{D}(B_j)\). Let \(E\) be the energy bound for system \(A\), which is assumed to satisfy \(E \geq \frac{2}{N}\). Then, for all POVMs \(\{M_j\}_j\) and states \(\{\tau_j\}_j \in \mathcal{D}(B_j)\), it holds that
\[
\min_{j=1,\ldots,N} \|\Lambda_j - \mathcal{E}_j\|_{\sigma;E} \geq \frac{1}{2N},
\]
where the measure-and-prepare channel \(\mathcal{E}_j\) is given by equation (12).

**Remark 6.** The assumption \(E \geq \frac{2}{N}\) in proposition 5 is not strictly necessary yet it significantly simplifies the calculation.

**Outline of the proof of proposition 5.** We look at the quantity
\[
\mu(\Lambda) := \inf_{M, \tau} \|\Lambda_j - \mathcal{E}_j \|_{\sigma;E},
\]
where the infimum is over the set of possible POVMs \(M = \{M_j\}_j\) and states \(\tau_j = \{\tau_j\}_j\) entering the definition of the measure-and-prepare channel \(\mathcal{E}_j\). We start by restricting the evaluation of the diamond norm to two-mode squeezed vacuum states: \(|\psi_n\rangle = \frac{1}{\cosh(\theta)} \sum_{m} \tanh(\theta)^{|n|} ^{|m|}\rangle\). Since the channels \(\mathcal{E}_j\) are entanglement-breaking, the infimum on \(M\) and \(\tau\) translates into an infimum on the set of separable states (with respect to the bipartition \(C : B_j\), where \(C\) is the ancillary system entering the definition of the diamond norm): \((\text{id} \otimes \mathcal{E}_j)[|\psi_\tau\rangle] = \omega \in \mathcal{E}\), with \(\psi_\tau := |\psi_\tau\rangle\langle\psi_\tau|\). We thus obtain that
\[
\mu(\Lambda) \geq \inf_{\omega \in \mathcal{E}} \sup_{|X|, \sinh(\theta) \leq E} \|\text{id} \otimes \Lambda_j [\psi_\tau] - \omega\|_1.
\]
A lower bound for the one-norm in equation (27) is estimated through the inequality \(\|X\|_1 \geq 2\|X\|_\infty\), which holds true for any operator \(X\) with \(\text{Tr} X = 0\). The operator norm on the rhs is bounded from below by looking at the matrix entries with respect to a second set of two-mode squeezed vacuum states. Upon straightforward calculations, one obtains equation (25). Details are provided in appendix B.2.

It is interesting to compare the lower bound in equation (25) with the upper bounds on the convergence rate we have found so far, with the goal of estimating the rate at which emergence...
of objectivity actually takes place. To estimate an upper bound for the distance $\|\Lambda_j - \mathcal{E}_j\|_{\mathcal{H}E}$ we optimise equation (23) over $d$ by using the inequality $\ln(d) \leq d$, and exploit the fact that, for the model we are considering, all the reduced maps $\Lambda_j$ have the same form. We thus obtain the following range:

$$
\frac{1}{2N} \leq \|\Lambda_j - \mathcal{E}_j\|_{\mathcal{H}E} \leq \mu \left(\frac{E_0}{N}\right)^{\frac{1}{\mu}},
$$

where $\mu < 10$ is a constant.

4. Quantum discord from local redistribution of quantum correlations in infinite dimension

Quantum discord [34, 35] is regarded as a measure of the purely quantum part of correlations between systems [51, 52]. Consider two systems $A$ and $B$, collectively described by a state $\rho$; the total amount of correlations between them is quantified by the mutual information $I(A : B) = S(A) + S(B) - S(AB)$, where $S$ denotes the von Neumann entropy: $S(A) = -\text{Tr}[\rho_A \log \rho_A]$. The quantum discord between $A$ and $B$ (from the perspective of subsystem $B$) is then defined by

$$
D(A|B)_\rho := I(A:B)_\rho - \max_{\Gamma \in \mathcal{QC}} I(A:B)_{\rho \otimes \Gamma},
$$

where $\mathcal{QC}$ refers to quantum-to-classical channels having the form $\Gamma(X) := \sum_k \text{Tr}[N_k X] |k\rangle \langle k|$, with POVM $\{N_k\}_k$. The quantum discord $D(A|B)_\rho$ thus represents the amount of correlations that is inevitably lost when $B$ is subject to a minimally disturbing local measurement, or, in other words, when $B$ encodes its part of information into a classical system; in this respect, $D(A|B)_\rho$ can be thought of as the purely quantum part of correlations between $A$ and $B$ in the state $\rho$. In [30] Brandão et al derived an interesting operational interpretation of quantum discord in terms of redistribution of quantum information to many parties. In particular they showed that

$$
\lim_{N \to \infty} \max_{\Lambda_N} \mathbb{E}_j I(A:B)_{\rho \otimes \Lambda_N} = \max_{\Gamma \in \mathcal{QC}} I(A:B)_{\rho \otimes \Gamma},
$$

where the maximisation is over all maps $\Lambda_N : \mathcal{D}(B) \to \mathcal{D}(B_1 \otimes \cdots \otimes B_N)$, and $\mathbb{E}_j I(A:B)_\rho$ is the average mutual information between $A$ and $B_j$ for the uniform probability distribution over $j$. Equation (30) shows that, when the share of correlations of $B$ is redistributed to infinitely many parties $\{B_j\}$, the maximum average mutual information accessible through each one of the parties $B_j$ corresponds to the purely classical part of correlations. This result is at the heart of the operational characterisation of quantum discord provided by Brandão et al [30]. In fact, from equation (30) it follows that

$$
D(A|B)_\rho = \lim_{N \to \infty} \mathbb{E}_j \left( I(A:B)_\rho - I(A:B)_\rho \right),
$$

i.e., $D(A|B)_\rho$ is characterised as the minimal average loss in mutual information when $B$ locally redistributes its share of correlations. Brandão et al derived equation (30) as a corollary of the theorem through which they proved emergence of objectivity of observables in finite dimensions [30, corollary 4]. In that context, $B$ can be interpreted as the environment of system $A$, which splits into fragments $\{B_j\}$.

We generalise the above result to an infinite-dimensional scenario, for systems subjected to generic energy constraints. In particular, as infinite-dimensional counterpart of [30, corollary 4], we prove the following corollary of our theorem 4:
Corollary 7. Let $A$ be a quantum system equipped with a Hamiltonian $H_A$, and $B$ a quantum system equipped with Hamiltonian $H_B$, both satisfying the Gibbs hypothesis. We also assume that $H_B$, when written as in equation (4), satisfies equation (5). Let $\Lambda_N : D(B) \to D(B_1 \otimes \cdots \otimes B_N)$ be a cptp map, and define $\Lambda_j := \text{Tr}_{B \setminus B_j} \circ \Lambda_N$ as the effective dynamics from $D(B)$ to $D(B_j)$. Then for every $\delta > 0$ there exists a set $S \subseteq \{1, \ldots, N\}$ with $|S| \geq (1 - \delta)N$ such that for all $j \in S$ and all states $\rho \in D(A \otimes B)$ with $\text{Tr}[\rho H_A] \leq E_A$, $\text{Tr}[\rho_H B] \leq E_B$,

$$I(A : B)(\rho \otimes \Lambda_j \rho) \leq \max_{\Gamma \in QC} I(A : B)_{\text{id} \otimes \Gamma(\rho)} + (2\epsilon' + 4\Delta) S(\gamma(E_A / \Delta))$$

$$+ (1 + \epsilon') h\left( \frac{\epsilon'}{1 + \epsilon'} \right) + 2h(\Delta),$$

where $\epsilon' = \frac{\epsilon}{1 + \epsilon}$, $\Delta = \frac{\epsilon}{1 + \epsilon}$, $\gamma(E)$ is the Gibbs state for system $A$ defined in equation (3), and the maximum on the rhs is over quantum-to-classical channels $\Gamma(X) := \sum_j \text{Tr}(N_j |j\rangle \langle j|)$, with $\{N_j\}_j$ a POVM and $\{|j\rangle\}_j$ a set of orthonormal states. As a consequence,

$$\lim_{N \to \infty} \max_{\Lambda_N} \mathbb{E}_{\rho \in QC} \mathbb{I}(A : B)_{\text{id} \otimes \Lambda\rho} = \max_{\Gamma \in QC} \mathbb{I}(A : B)_{\text{id} \otimes \Gamma(\rho)}$$

Outline of the proof of corollary 7. We follow the conceptual steps of the proof of [30, corollary 4], adapting them to our infinite-dimensional framework. In particular, our argument relies on a continuity bound for the conditional entropy of infinite-dimensional systems subjected to energy constraints [42, lemma 17]. We apply it to the states $\tau = (\text{id} \otimes \Lambda_j)(\rho)$ and $\sigma = (\text{id} \otimes E^\phi_j)(\rho)$, which are close in one-norm by virtue of theorem 4. Since the reduced entropies on the $A$ subsystems are the same for $\tau$ and $\sigma$, the continuity bound for the conditional entropy holds true for the mutual information as well. We then obtain equation (32). To prove equation (33), we exploit equation (32) to show that the lhs is no larger that the rhs; this concludes the proof, as the reverse (rhs no larger than lhs) is trivial. The complete proof is given in appendix C.

Remark 8. The result of corollary 7 also applies to a Hamiltonian $H_B$ that takes the form (4) with $f_j = j$, and therefore does not satisfy equation (5). In fact, the proof remains valid when the objectivity bound of theorem 4 is replaced with the one given by (23).

As mentioned before, equation (33) implies that quantum discord can be interpreted as the minimal average loss in mutual information when one of the two parties asymptotically redistributes its share of correlations. In the framework of quantum Darwinism this means that, when the number of environment fragments grows significantly, the correlations established between the (infinite-dimensional) system of interest $A$ and each of the observers (who in turn has access only to a fragment $B_j$ of the environment) can be at most classical.

5. Conclusions and outlook

In this paper we investigated the generic characteristics of the objectivity of observables arising in the quantum-to-classical transition within the premises of quantum Darwinism. Going beyond recent studies for finite- and infinite-dimensional systems [30, 31], we presented a unified approach to derive bounds on the emergence of such objectivity in quantum systems of arbitrary dimension, probed by multiple observers each accessing a fragment of the environment. In the particular case of a system-environment dynamics specified by a pure loss channel, we derived lower and upper bounds on the rate at which objectivity of observables emerges as a function of the number of environmental fragments. Furthermore, we proved that,
even when the system under observation is infinite-dimensional, it cannot share quantum correlations with asymptotically many observers, as the maximum correlation each observer can establish with the system is, on average, of purely classical nature. This observation, which extends to the infinite-dimensional scenario an operational interpretation for quantum discord put forward in [30], can also be seen as a quantitative manifestation of the quantum-to-classical transition, seen exclusively from the balance of correlations, without having to analyse the system-environment interaction.

The role of quantum discord in understanding the quantum-to-classical transition has also been recently investigated in reference [13]. In particular, the authors showed the equivalence between the so-called strong quantum Darwinism and spectrum broadcasting, another framework aiming for the modelling and interpretation of ‘objectivity’ [53]. Exploring deeper connections between these studies and our results, with the aim to achieve an even more fundamental (and quantitative) understanding of the emergence of objectivity and classicality, is certainly an endeavour worthy of further investigation. Another fascinating perspective could be to study the applicability of our methods—which are rooted in quantum information theory and related, e.g., to no-broadcasting and monogamy properties of genuinely quantum correlations—to cosmological scenarios [54], in order to cast new light on the black hole information paradox and related issues remaining unsolved at the quantum/classical/general-relativistic triple border.

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Appendix A. $f$-dependent objectivity bounds

The proof of theorem 4 involves a generalisation of the concept of Choi–Jamiołkowski isomorphism, which relies on a class of infinite-dimensional entangled states depending on the underlying system’s Hamiltonian. For clarity, we recall here the definition of modified Choi–Jamiołkowski state associated to such class of states.

**Definition 3 (Restatement).** The modified Choi–Jamiołkowski state of a ctpm map \( \Lambda : D(A') \rightarrow D(B) \), for a given sequence of Hamiltonian eigenvalues \( f = \{ f_j \} \), is defined as

\[
J_f(\Lambda) := \id_A \otimes \Lambda_{AA'} [\phi \langle \phi |],
\]

where the entangled state \( |\phi \rangle \) reads

\[
|\phi \rangle := c_f \sum_{j=0}^{\infty} \phi_j |j, f_j \rangle_{AA'},
\]

with \( \phi_j := 1/f_j \) and \( c_f := \left( \sum \frac{1}{f_j} \right)^{-\frac{1}{2}} \).
We start by proving lemma A9, which bounds the distance between two $f$-Choi states as a function of $d$ (the truncation dimension), and lemma A11, which relates the distance between two channels to that between their $f$-Choi states. Our preparations are completed by the rather technical lemma A12: there we show that a crucial inequality exploited in reference [30], whose original formulation explicitly relies on finite-dimensional techniques, may be suitably modified to fit our infinite-dimensional scenario. To achieve the latter result, we exploit the assumption that the $f$-Choi states have finite local entropy. Once all the above ingredients are in place, we present the proof of theorem 4. We conclude the section by presenting additional details on the calculations behind equation (19), obtained for the sequence $\{f_j\}$ which bridges the finite- and infinite-dimensional cases.

Lemma A9. Given $L = \tau - \sigma$, where $\tau = J_f(\Lambda_1)$ and $\sigma = J_f(\Lambda_2)$ are modified Choi–Jamiołkowski states for the cptp maps $\Lambda_1$ and $\Lambda_2$, we have that

$$\|L\|_1 \leq 4d^2 \max_M \|\text{id} \otimes M[L]\|_1 + 4\epsilon_d,$$

(A.2)

where $\epsilon_d$ is given in definition 2, $d$ is the corresponding truncation dimension, and the maximum on the rhs is over quantum-to-classical channels $M(Y) = \sum_i \text{Tr}(N_iY)|i\rangle\langle i|$, with POVM $\{N_i\}$ and orthonormal states $\{|i\rangle\}$.

Proof. By writing $L$ in the form $L = \sum_{i,j=0}^{\infty} |i\rangle\langle j| \otimes L_{ij}$ we have that

$$\|L\|_1 \leq \|\Pi_d \otimes \text{id}[L]\|_1 + \|((\text{id} - \Pi_d) \otimes \text{id})[L]\|_1,$$

$$= \|\Pi_d \otimes \text{id}[L]\|_1 + \left\| \sum_{\min(i,j) \geq d} |i\rangle\langle j| \otimes L_{ij} \right\|_1,$$

$$\leq 4d^2 \max_M \|\Pi_d \otimes M[L]\|_1 + \sum_{\min(i,j) \geq d} \|i\rangle\langle j| \otimes L_{ij}\|_1,$$

$$\leq 4d^2 \max_M \|\text{id} \otimes M[L]\|_1 + \sum_{\min(i,j) \geq d} \|i\rangle\langle j| \otimes L_{ij}\|_1. \tag{A.3}$$

Note that in 1 we used the triangle inequality. In 2, instead, we applied a result by Aubrun et al [33, corollary 9]: in fact, for an arbitrary bipartite operator $Z$, it holds that

$$\max_M \|U \otimes M[Z]\|_1 = \|Z\|_{\text{LOCC}} \geq \|Z\|_{\text{LO}},$$

where the maximization on the lhs is as usual over local measurements, while the quantities (a) $\|\cdot\|_{\text{LOCC}}$ and (b) $\|\cdot\|_{\text{LO}}$ are the distinguishability norms [55, 56] associated with the sets of (a) local operations assisted by classical communication from the second system to the first; and (b) local operations alone. By [33, equation (40)], it holds that $\|Z\|_{\text{LO}} \geq \frac{1}{4n^{\frac{3}{2}}}|Z|_1$, where $n$ denotes the smaller of the local dimensions. The inequality in 2 is just an application
of this, with $Z := (\Pi_d \otimes \text{id})[L]$ and hence $n \leq d$. Furthermore, in 3 we applied the pinching theorem [57, equation (4.52)], or, alternatively, the data processing inequality for the trace distance—note that $X \mapsto \Pi X \Pi$ is a ctp map for every projector $\Pi$. Finally, multiple applications of the triangle inequality yield

$$\left\| \sum_{\min(i,j) \geq d} |i\rangle \langle j| \otimes L_{ij} \right\|_1 = \|L - (\Pi_d \otimes \text{id})L(\Pi_d \otimes \text{id})\|_1$$

$$= \| (\tau - \sigma) - (\Pi_d \otimes \text{id})(\tau - \sigma)(\Pi_d \otimes \text{id}) \|_1$$

$$= \| \tau - \tau_d - (\tau - \sigma_d) \|_1$$

$$\leq \| \tau - \tau_d \|_1 + \| \sigma - \sigma_d \|_1,$$

where $\tau_d := (\Pi_d \otimes \text{id})\tau(\Pi_d \otimes \text{id})$ and $\sigma_d := (\Pi_d \otimes \text{id})\sigma(\Pi_d \otimes \text{id})$. The one-norms on the rhs can be bounded from above by exploiting the result of proposition S2 in the supplemental material of [31], suitably adapted to our modified Choi–Jamiołkowski state. In particular, by replacing the coefficients $\phi_j = e^{-\frac{i}{c} \omega}$ in [31, proposition S2] with our $\phi_j = f_j \frac{1}{\omega}$ we have that, for $\rho = J_f(\Lambda)$,

$$\| \rho - \rho_d \|_1 \leq 2\epsilon_d,$$

(A.4)

where $\rho_d := (\Pi_d \otimes \text{id})\rho(\Pi_d \otimes \text{id})$. We then obtain

$$\left\| \sum_{\min(i,j) \geq d} |i\rangle \langle j| \otimes L_{ij} \right\|_1 \leq 4\epsilon_d.$$

Before proceeding with the proof, we restate for clarity the definition of energy-constrained diamond norm (see definition 1 in the main text) for the specific case of a Hamiltonian which satisfies the Gibbs hypothesis and is written as in equation (4).

**Definition A10.** Let $\Lambda'$ be a quantum system equipped with a Hamiltonian $H_{\Lambda'}$ satisfying the Gibbs hypothesis and written as in equation (4), and pick $E > E_0 = f_0$. Then the energy-constrained diamond norm of an arbitrary Hermiticity-preserving linear map $\Lambda : \mathcal{D}(\Lambda') \rightarrow \mathcal{D}(B)$ is defined by

$$\|\Lambda\|_{\mathcal{D}(\Lambda') \Rightarrow E} := \sup_{\sum_j \phi_j \rho_j \otimes \psi_j \leq E} \| (\text{id}_A \otimes \Lambda')(\rho_{AA'}) \|_1,$$

(A.5)

where $A$ is an arbitrary ancillary system, and $\| \cdot \|_1$ is the one-norm. A recent result by Weis and Shirokov [41] ensures that the input state $\rho_{AA'}$ in equation (A.5) can be taken to be pure.

**Lemma A11** (Generalisation of lemma S6 in the supplemental material of [31] for a Hamiltonian given by equations (4) and (5)). For ctp maps $\Lambda_0$ and $\Lambda_1$ whose input system is equipped with a Hamiltonian $H$ which satisfies the Gibbs hypothesis, takes the form as in equation (4) and satisfies equation (5), we have that

$$\|\Lambda_0 - \Lambda_1\|_{\mathcal{D}(\Lambda') \Rightarrow E} \leq \frac{E}{c_f} \| J_f(\Lambda_0) - J_f(\Lambda_1) \|_1,$$

(A.6)

where the modified Choi–Jamiołkowski state $J_f(\Lambda)$ of $\Lambda$ is constructed as in definition 3.
\textbf{Proof.} Lemma A11 can be proved by adapting the argument in the proof of [31, lemma S6] to our choice of the input system’s Hamiltonian, i.e., by replacing the definition of modified Choi–Jamiołkowski states used there with the one given in definition 3.

\textbf{Lemma A12 (Adapted from equation (16) in the supplementary notes of [30]).} Let $\Lambda$ be a cptp map, and let the corresponding modified Choi–Jamiołkowski state given by definition 3 be denoted with $\rho_{AB_1\ldots B_N} := \id_A \otimes \Lambda_{\psi}(\langle \phi | \phi \rangle)$, where $|\phi \rangle$ is given in equation (6). Fix an integer $m \leq N$. Then there exists a set of indices $J := \{j_1, \ldots, j_{42}\}$, where $q \leq m$, and quantum-to-classical channels $M_{j_1}, \ldots, M_{j_{42}}$ such that

$$\| \rho_{AB_1\ldots B_N} - \mathbb{E}_{s,\delta} \rho_A^s \otimes \rho_B^s \|_1 \leq \sqrt{\frac{2 \ln(2)}{m}}$$

where: $\sigma$ is given in equation (7); the expectation value is with respect to the uniform distribution over $\{1, \ldots, N\} \setminus J$; the maximum runs over all quantum-to-classical channels; $z$ is a random variable that represents the outcome of the measurements $M_{j_1}, \ldots, M_{j_{42}}$ on $\rho_{AB_1\ldots B_N}$; and $\rho_A, \rho_B^s$ are the corresponding post-measurement states.

\textbf{Proof.} It suffices to adapt the derivation of equation (16) in the supplementary notes of [30] to our infinite-dimensional scenario: the Choi–Jamiołkowski state of $\Lambda$ is replaced with the $f$-Choi state of definition 3, and the entropy $\log d_A$ with $S(\rho_A)$. Since $\Lambda$ is trace preserving, $\rho_A = \Tr_A[|\phi \rangle \langle \phi | _{\Lambda A}]$, and $S(\rho_A) = \sigma$ by definition of $\sigma$.

\textbf{Theorem 4 (Restatement).} Let $A$ be a quantum system equipped with a Hamiltonian $H_A$ which satisfies the Gibbs hypothesis and which, when written as in equation (4), also satisfies equation (5). Consider an arbitrary cptp map $\Lambda : \mathcal{D}(A) \rightarrow \mathcal{D}(B_1 \otimes \cdots \otimes B_N)$, and define the effective dynamics from $\mathcal{D}(A)$ to $\mathcal{D}(B_j)$ as $\Lambda_j := \Tr_{B_1 \otimes \cdots \otimes B_{j-1}} \circ \Lambda$. For an arbitrary number $0 < \delta < 1$, there exists a POVM $\{M_j\}$ and a set $S \subseteq \{1, \ldots, N\}$, with $|S| \geq (1 - \delta)N$, such that, for all $j \in S$ and for any integer truncation dimension $d \geq 0$, we have that

$$\| \Lambda_j - \mathcal{E}_j \|_{\text{H.E.}} \leq \frac{\zeta}{\delta},$$

where the measure-and-prepare channel $\mathcal{E}_j$ is given by

$$\mathcal{E}_j(X) := \sum_i \Tr(M_i X) r_{ij},$$

for some family of states $r_{ij} \in \mathcal{D}(B_j)$, and

$$\zeta = \kappa d \left( \frac{E^2 \sigma}{N c_f^2} \right)^{1/3} + \frac{4E}{c_f} \epsilon_d,$$

where $c_f$ is the normalization factor introduced in equation (6); $\epsilon_d$ is given in definition 2; $\sigma$ is defined by equation (7) and $\kappa := 3(16 \ln(2))^{1/3}$ is a universal constant.

\textbf{Proof.} As we will see below, the states $\rho_{AB_1}$ and $\mathbb{E}_{s\delta} \rho_A^s \otimes \rho_B^s$ defined in lemma A12 are modified Choi–Jamiołkowski states. By applying lemma A9 to them we have that

$$\| \rho_{AB_1} - \mathbb{E}_{s\delta} \rho_A^s \otimes \rho_B^s \|_1 \leq 4d^2 \max_{\Lambda_j} \| \id_A \otimes \Lambda_j (\rho_{AB_1} - \mathbb{E}_{s\delta} \rho_A^s \otimes \rho_B^s) \|_1 + 4\epsilon_d.$$
By combining equation (A.8) with lemma A12 we then obtain

\[
E_{j \in J} \| \rho_{ABj} - E_{c} \rho_{A}^j \otimes \rho_{B}^j \|_1 \leq 4 d^{\frac{3}{2}} E_{j \in J} \max_{M_j} \| \mathrm{id}_A \otimes M_j \left[ \rho_{ABj} - E_{c} \rho_{A}^j \otimes \rho_{B}^j \right] \|_1 + 4 \epsilon_d \\
\leq 4 d^{\frac{3}{2}} \sqrt{\frac{2 \ln(2) \sigma}{m}} + 4 \epsilon_d.
\]  
(A.9)

We now show that \( E_{z} \rho_{A}^j \otimes \rho_{B}^j \) is the modified Choi–Jamiołkowski state of a quantum-to-classical channel, explicitly given by

\[
E_{j}(X) := c_{j}^{-2} E_{z} \mathrm{Tr} \left[ (\rho_{A}^{j})^{H} X H \right] \rho_{B}^{j},
\]  
(A.10)

In fact,

\[
\left( \mathrm{id}_A \otimes E_{j} \right) (|\phi \rangle \langle \phi|) = c_{j}^{-1} \sum_{jk} \frac{1}{j k^{\frac{3}{2}} k^{\frac{1}{2}}} |j \rangle \langle k| \otimes E_{j}(|j \rangle \langle k|) = E_{j} \sum_{jk} \langle j | \rho_{A}^{j} | k \rangle |j \rangle \langle k| \otimes \rho_{B}^{j} = E_{j} \rho_{A}^{j} \otimes \rho_{B}^{j}.
\]

Note that the measurement appearing in equation (A.10) is independent of \( j \notin J \). In fact, calling \( N_{B_{b_{1}} \ldots B_{b_{q-1}}} \) the POVM element corresponding to the outcome \( z \) of the measurement \( M_{j_{1}} \otimes \cdots \otimes M_{j_{q-1}} \), the POVM appearing in equation (A.10) can be expressed as \( \left\{ c_{j}^{-2} p(z)(H_{j}^{\alpha} \rho_{A}^{j})^{H} H_{j}^{\alpha} \right\}_{z} \), where \( p(z) = \mathrm{Tr} \left[ \rho_{AB_{1} \ldots B_{N}} N_{B_{b_{1}} \ldots B_{b_{q-1}}} \right] \). Now the claim follows because

\[
p(z)(\rho_{A}^{j})^{T} = \mathrm{Tr}_{B_{1} \ldots B_{N}} \left[ \rho_{AB_{1} \ldots B_{N}} N_{B_{b_{1}} \ldots B_{b_{q-1}}} \right]
\]  
(A.11)

is independent of \( j \notin J \).

Since \( \rho_{ABj} \) is, by definition, the modified Choi–Jamiołkowski state of \( \Lambda_{j} \), from lemma A11 it follows that

\[
\| \Lambda_{j} - E_{j} \|_{\diamond H.E} \leq \frac{E}{c_{j}} \| \rho_{ABj} - E_{c} \rho_{A}^j \otimes \rho_{B}^j \|_1.
\]  
(A.12)

This, combined with equation (A.9), gives

\[
E_{j \notin J} \| \Lambda_{j} - E_{j} \|_{\diamond H.E} \leq \frac{E}{c_{j}} E_{j \notin J} \| \rho_{ABj} - E_{c} \rho_{A}^j \otimes \rho_{B}^j \|_1 \\
\leq \frac{E}{c_{j}} \left( 4 d^{\frac{3}{2}} \sqrt{\frac{2 \ln(2) \sigma}{m}} + 4 \epsilon_d \right)
\]

\[
= \sqrt{\frac{32 \ln(2) E^{2} d^{3} \sigma}{m c_{j}^{2}}} + 4 \frac{E}{c_{j}} \epsilon_d.
\]
From the previous result we then find that

$$
\mathbb{E}_{j}[\Lambda_j - \mathcal{E}]_{\mathcal{H},E} \leq \mathbb{E}_{j}[\Lambda_j - \mathcal{E}_j]_{\mathcal{H},E} + \frac{m}{N} \mathbb{E}_{j}[\Lambda_j - \mathcal{E}_j]_{\mathcal{H},E} \\
\leq \sqrt{\frac{32}{N} \ln(2) \frac{E^2 d^3 \sigma}{m c^4} + \frac{4E}{c^2} \epsilon_d + \frac{2m}{N}}.
$$

The right-hand side, minimised with respect to $m$, gives the quantity

$$
\zeta = \kappa d \left( \frac{E^2 \sigma}{N c^2} \right)^{1/3} + \frac{4E}{c^2} \epsilon_d, \quad (A.13)
$$

where $\kappa = 3(16 \ln(2))^{1/3}$. To complete the proof, we apply Markov’s inequality: $P(X \geq a) \leq \frac{E(X)}{a}$, where $X$ is a non-negative random variable, $E(X)$ its expectation value, and $a > 0$. In our case, $X = \|\Lambda_j - \mathcal{E}\|_{\mathcal{H},E}$, with $j$ being uniformly distributed, and $a = \frac{\zeta}{\delta}$, which leads us to

$$
P\left(\|\Lambda_j - \mathcal{E}_j\|_{\mathcal{H},E} \geq \frac{\zeta}{\delta}\right) \leq \delta, \quad (A.14)
$$

completing the proof.

\[ \square \]

**Example A13** (Case study: bridging finite and infinite dimensions). We calculate the quantity given by equation (12) for the sequence of Hamiltonian eigenvalues

$$
f_j = \begin{cases} 
1 & j \leq D - 1, \\
\frac{e^{\omega j}}{1 - e^{\omega}} & j \geq D.
\end{cases} \quad (A.15)
$$

We have that

$$
\epsilon_f = \left( D + e^{-\omega D} \right)^{-1/2}, \quad (A.16)
$$

$$
\epsilon_d = \left( \frac{e^{-\omega d}}{D + e^{-\omega D}} \right)^{1/2} = e^{-\omega d/2} \epsilon_f, \quad (A.17)
$$

$$
s := \ln(2) \sigma = \ln(D + e^{-\omega D}) + \frac{\omega(1 - D + De^{-\omega})}{(e^{\omega} - 1)(De^{\omega D} + 1)} - \frac{\ln(1 - e^{-\omega})}{(1 + De^{\omega D})}, \quad (A.18)
$$

where we assumed $d \geq D$. We then obtain

$$
\zeta = \left( \frac{432 E^2 (D + e^{-\omega D})^2 d^3 s}{N} \right)^{1/2} + 4E \sqrt{\frac{D + e^{-\omega D}}{e^{\omega d}}}, \quad (A.19)
$$

which is valid for any $d \geq D$. 

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Appendix B. Properties of the pure loss channel

B.1. Symmetry of the reduced dynamics

We show that, when the dynamics from system $A$ to the environment fragments $B_1, \ldots, B_N$ is given by

$$\Lambda_{A\to B_1, \ldots, B_N}(\cdot) \equiv U \left( \cdot \otimes \bigotimes_{j=2}^N |0\rangle \langle 0|_{B_j} \right) U^\dagger, \quad (B.1)$$

with $U$ the symplectic unitary implementing an $N$-splitter, the reduced dynamics $\Lambda_j = \operatorname{Tr}_{B_1, \ldots, B_j} \circ \Lambda$ have the same form for all $j$. We start by introducing the Weyl displacement operator:

$$D(\vec{\alpha}) := \exp \left[ \sum_j \left( \alpha_j a_j^\dagger - \alpha_j^* a_j \right) \right], \quad (B.2)$$

where $\vec{\alpha}$ denotes a complex vector in $\mathbb{C}^N$, with $N$ the number of modes. A quantum state $\rho$ can be described in terms of the characteristic function

$$\chi_{\rho}(\vec{\alpha}) := \operatorname{Tr} \left[ \rho D(\vec{\alpha}) \right], \quad (B.3)$$

by means of which the state $\rho$ can be reconstructed as

$$\rho = \int \frac{d^{2N} \alpha}{\pi^N} \chi_{\rho}(\vec{\alpha}) D(-\vec{\alpha}). \quad (B.4)$$

We will describe the channel in equation (B.1) as the unitary operation on $\mathcal{D}(A \otimes B_2 \otimes \cdots \otimes B_N)$ given by

$$\rho_{\text{in}} \rightarrow \rho_{\text{out}} = U \rho_{\text{in}} U^\dagger, \quad (B.5)$$

with

$$\rho_{\text{in}} = \rho_A \otimes \bigotimes_{j=2}^N |0\rangle \langle 0|_{B_j}. \quad (B.6)$$

The characteristic function for the input state is given by

$$\chi_{\rho_{\text{in}}}(\vec{\alpha}) = \operatorname{Tr} \left[ \rho_{\text{in}} D(\vec{\alpha}) \right] = \chi_{\rho_A}(\alpha_1) \chi_{|0\rangle \langle 0|_{B_2}}(\alpha_2) \cdots \chi_{|0\rangle \langle 0|_{B_N}}(\alpha_N)$$

$$= \chi_{\rho_A}(\alpha_1) \exp \left[ -\frac{1}{2} \left( ||\vec{\alpha}||^2 - |\alpha_1|^2 \right) \right] \quad (B.7)$$

and for the output state we have

$$\chi_{\rho_{\text{out}}}(\vec{\alpha}) = \operatorname{Tr} \left[ \rho_{\text{out}} D(\vec{\alpha}) \right] = \operatorname{Tr} \left[ U \rho_{\text{in}} U^\dagger D(\vec{\alpha}) \right] = \operatorname{Tr} \left[ \rho_{\text{in}} U^\dagger D(\vec{\alpha}) U \right]. \quad (B.8)$$

Since $U^\dagger D(\vec{\alpha}) U = D(V^\dagger \vec{\alpha})$, we obtain that

$$\chi_{\rho_{\text{out}}}(\vec{\alpha}) = \operatorname{Tr} \left[ \rho_{\text{in}} D(V^\dagger \vec{\alpha}) \right] = \chi_{\rho_A} \left( \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_N}{\sqrt{N}} \right) \exp$$

$$\times \left[ -\frac{1}{2} \left( \sum_j |\alpha_j|^2 - \frac{1}{N} \sum_j |\alpha_j|^2 \right) \right]. \quad (B.9)$$
The characteristic function of the output state $\rho_{\text{out}} = \text{Tr}_{B_j} [\rho_{\text{in}}]$ is obtained by setting $\alpha_{i \neq j} = 0$:

$$
\chi_{\rho_{\text{out}}}(\alpha_j) = \chi_{\rho_{\text{in}}} \left( \frac{\alpha_j^2}{N} \right) \exp \left[ - \frac{1}{2} \left( \frac{N - 1}{N} \right) |\alpha_j|^2 \right].
$$

(B.10)

It has the same form for all $j$, and the same property is therefore true for the reduced channel $\Lambda_j = \text{Tr}_{B_j} \circ \Lambda$.

**B.2. Lower bound for the objectivity range of a pure loss channel**

The statement

$$
\exists \{M_j\} : \forall j \in S, \exists \{\tau_{j\ell}\} : \| \Lambda_j - \mathcal{E}_{M,j} \|_{\text{HE}} \leq \frac{1}{\delta} \zeta,
$$

(B.11)

is equivalent to the inequality

$$
\inf_{M} \sup_{1 \leq j \leq N} \inf_{\tau_{j\ell}} \| \Lambda_j - \mathcal{E}_{M,j} \|_{\text{HE}} \leq \frac{1}{\delta} \zeta.
$$

(B.12)

Note that we made explicit the dependence of the measure-and-prepare channels from POVM $M = \{M_j\}$ and set of states $\tau_{j\ell} = \{|\tau_{j\ell}\rangle\}$ through the notation $\mathcal{E}_{M,j}(X) = \sum_{\ell} \text{Tr}(M_j X) |\tau_{j\ell}\rangle \langle \tau_{j\ell}|$. To investigate the optimality of the objectivity bound in equation (B.11) we thus need to estimate a lower bound for the lhs of equation (B.12). We will perform this analysis for the channel in equation (B.1). Since the reduced dynamics $\Lambda_j$ have the same form for all $j$ we can get rid of the supremum on $j$ and look at a lower bound for the quantity

$$
\mu(\Lambda) := \inf_{M,j} \| \Lambda_j - \mathcal{E}_{M,j} \|_{\text{HE}}.
$$

(B.13)

By substituting the definition of diamond norm we obtain

$$
\mu(\Lambda) = \inf_{M,j} \| \Lambda_j - \mathcal{E}_{M,j} \|_{\text{HE}} = \inf_{M,j} \| \text{id}_C \otimes (\Lambda_j - \mathcal{E}_{M,j}) \lambda [\rho] \|_1,
$$

(B.14)

where $C$ is an arbitrary ancillary system (see definition 1 in the main text). To simplify the notation, in the following we suppress the explicit reference to the bipartition $C : A$. We can choose $\rho = \psi_i := |\psi_i\rangle \langle \psi_i|$ with $|\psi_i\rangle := \sum_{n=1}^N \text{tanh} (r)^n |nn\rangle$, which is a two-mode squeezed vacuum state, and (noting that $\text{Tr} [\rho H_A] = \text{sinh} (r^2)$) find the inequality

$$
\mu(\Lambda) \geq \inf_{M,j} \sup_{|\psi_i\rangle, \text{sinh}(r^2) \leq E} \| \text{id} \otimes (\Lambda_j - \mathcal{E}_{M,j}) |\psi_i\rangle \|_1.
$$

(B.15)

The channel $\mathcal{E}_{M,j}$ is entanglement breaking, so $\text{id} \otimes \mathcal{E}_{M,j} |\psi_i\rangle$ is a separable state: $\text{id} \otimes \mathcal{E}_{M,j} |\psi_i\rangle = \omega \in \text{SEP}$. Since $\|X\|_1 \geq 2 \|X\|_\infty$ if $\text{Tr}[X] = 0$, we have that

$$
\mu(\Lambda) \geq \inf_{\omega \in \text{SEP}, \text{sinh}(r^2) \leq E} \sup_{|\psi_i\rangle, \text{sinh}(r^2) \leq E} \| \text{id} \otimes \Lambda_j |\psi_i\rangle - \omega \|_\infty
$$

$$
= 2 \inf_{\omega \in \text{SEP}, \text{sinh}(r^2) \leq E} \sup_{|\psi_i\rangle, \text{sinh}(r^2) \leq E} \sup_{|\phi\rangle \neq |\psi_i\rangle} \| \langle \phi | \text{id} \otimes \Lambda_j |\psi_i\rangle |\phi\rangle - \langle \phi | \omega |\phi\rangle \|,
$$

(B.16)

where we substituted the definition of the infinity norm. We can choose, as $|\phi\rangle$, a two-mode squeezed vacuum state $|\phi_j\rangle = \sum_{n=1}^N \text{tanh} (s|^n|nn\rangle$, and get rid of the modulus to obtain
For a separable state $\omega$, $|\langle \omega | \phi \rangle| \leq \lambda_{\text{max}}$, where $\lambda$ is defined by the Schmidt decomposition: $|\phi\rangle = \sum_i \sqrt{\lambda_i} |e_i f_i\rangle$. Hence $|\langle \phi_r | \omega | \phi_r \rangle| \leq \lambda_{\text{max}}(\phi_r) = \frac{1}{\cosh(s)}$, and we have that

$$\mu(\Lambda) \geq 2 \inf_{\omega} \sup_{\rho} \left( \sup_{r:sinh(r)^2 \leq E} \left( \langle \phi_r | id \otimes \Lambda [\psi_r] | \phi_r \rangle - \frac{1}{\cosh(s)^2} \right) \right). \quad (B.17)$$

A calculation of the quantity $\langle \phi_r | id \otimes \Lambda [\psi_r] | \phi_r \rangle$ can be found in [58]. By exploiting that result we find

$$\mu(\Lambda) \geq 2 \sup_r \left( \sup_{s:|\sinh(s)|^2 \leq E} \left( \frac{N}{\sqrt{N} \cosh(r) \cosh(s) - \sinh(r) \sinh(s)} \right) - \frac{1}{\cosh(s)^2} \right). \quad (B.18)$$

For a given $s$, the supremum of the function

$$\frac{N}{\sqrt{N} \cosh(r) \cosh(s) - \sinh(r) \sinh(s)}\quad (B.20)$$

is reached for $r = \tilde{r}$ such that $\tilde{r} = \frac{\tanh(s)^2}{N - \tanh(s)}$. Since $\tilde{E} \leq \frac{1}{N - 1} \leq \frac{\lambda}{2}$ for $N \geq 2$ (and noting that $N \geq 2$ by definition of the channel $\Lambda$), we can choose $\tilde{E} \geq \frac{\lambda}{4}$ in order to have $\tilde{E} \leq \tilde{E}$ satisfied for all possible values of $N$. This is equivalent to evaluate an unconstrained supremum, for which we can use the calculation performed in [58] to obtain

$$\mu(\Lambda) \geq 2 \sup_s \left( \sup_{|\sinh(s)|^2 \leq E} \left[ \frac{N}{N \cosh(s) - \sinh(s)} - \frac{1}{\cosh(s)^2} \right] \right) = 2 \sup_s \frac{\tanh(s)^2}{N \cosh(s) - \sinh(s)} \quad (B.21)$$

Our analysis therefore led to the following result: when the dynamics from $A$ to $B_1, \ldots, B_N$ is given by equation (B.1) and the maximum energy of system $A$ satisfies $\tilde{E} \geq \frac{\lambda}{8}$, for all $f$ and for all POVM $\{M_f\}$ and sets $\{\gamma, j\}$ entering the definition of $\mathcal{E}$, it holds that

$$\| \Lambda_j - \mathcal{E}_{ij} \|_{\mathrm{tr,H}_E} \geq \frac{1}{2N}. \quad (B.22)$$

**Appendix C. Proof of corollary 7**

To prove corollary 7, it suffices to adapt to our infinite-dimensional setting the argument in the proof of [30, corollary 4]. The success of this programme depends crucially on a fundamental result by Winter [42, lemma 17], reported below as lemma C14, which expresses a continuity bound for the conditional entropy of infinite-dimensional systems subjected to energy constraints.

**Lemma C14** (42, lemma 17). For a Hamiltonian $H$ on $A$ satisfying the Gibbs hypothesis and any two states $\tau$ and $\sigma$ on the bipartite system $A \otimes B$ with $\text{Tr}(\tau H), \text{Tr}(\sigma H) \leq \tilde{E}$, $\frac{1}{2} \| \tau - \sigma \|_1 \leq \epsilon < \epsilon' \leq 1$ and $\Delta = \frac{\epsilon'}{\epsilon}$. \[\text{Tr}(\sigma H) \leq \tilde{E}, \quad \frac{1}{2} \| \tau - \sigma \|_1 \leq \epsilon < \epsilon' \leq 1, \quad \Delta = \frac{\epsilon'}{\epsilon}.\]
\[ |S(A|B)_{\tau} - S(A|B)_{\sigma}| \leq (2\epsilon' + 4\Delta)S(\gamma(E/\Delta)) + (1 + \epsilon') h \left( \frac{\epsilon'}{1 + \epsilon'} \right) + 2h(\Delta). \]  
(C.1)

**Proof of corollary 7.** Let 0 < \delta < 1 be a fixed number. Theorem 4 allows us to construct a POVM \( \{M_i\} \), a set \( S \subseteq \{1, \ldots, N\} \) of cardinality at least \( |S| \geq (1 - \delta)N \), and ensembles of states \( \{\tau_j\} \) such that the corresponding measure-and-prepare channels \( \mathcal{E}_j \) defined in equation (11) satisfy equations (10) and (12) for all \( j \in S \). Now, consider the states \( \tau = (\text{id} \otimes \Lambda_j)(\rho) \) and \( \sigma = (\text{id} \otimes \mathcal{E}_j)(\rho) \). By definition of \( f \)-diamond norm it follows that

\[ \frac{1}{2} \|\tau - \sigma\|_1 = \frac{1}{2} \| (\text{id} \otimes \Lambda_j)(\rho) - (\text{id} \otimes \mathcal{E}_j)(\rho) \|_1 \]
\[ \leq \frac{1}{2} \| \Lambda_j - \mathcal{E}_j \|_{\mathcal{H}_B, \mathcal{E}_B} \]
\[ \leq \epsilon < \epsilon' \leq 1, \]

where the inequalities in the last line follow from theorem 4, and we set \( \epsilon' := 2\epsilon := \frac{\zeta}{2} \), with \( \zeta \) given in equation (12).

Applying lemma C.14 to states \( \tau \) and \( \sigma \), we deduce that

\[ |S(A|B)_{\text{id} \otimes \Lambda_j(\rho)} - S(A|B)_{\text{id} \otimes \mathcal{E}_j(\rho)}| \leq (2\epsilon' + 4\Delta)S(\gamma(E_A/\Delta)) + (1 + \epsilon') h \left( \frac{\epsilon'}{1 + \epsilon'} \right) + 2h(\Delta) \]

(C.2)

where \( \Delta := \frac{1}{1 + \epsilon'} \). Since the reduced entropies on the \( A \) subsystems are the same for \( \tau \) and \( \sigma \), this translates to

\[ |I(A : B)_{\text{id} \otimes \Lambda_j(\rho)} - I(A : B)_{\text{id} \otimes \mathcal{E}_j(\rho)}| \leq (2\epsilon' + 4\Delta)S(\gamma(E_A/\Delta)) + (1 + \epsilon') h \left( \frac{\epsilon'}{1 + \epsilon'} \right) + 2h(\Delta), \]

and therefore

\[ I(A : B)_{\text{id} \otimes \Lambda_j(\rho)} \]
\[ \leq I(A : B)_{\text{id} \otimes \mathcal{E}_j(\rho)} + (2\epsilon' + 4\Delta)S(\gamma(E_A/\Delta)) + (1 + \epsilon') h \left( \frac{\epsilon'}{1 + \epsilon'} \right) + 2h(\Delta) \]
\[ \leq \max_{\Gamma \in \mathcal{QC}} I(A : B)_{\text{id} \otimes \Gamma(\rho)} + (2\epsilon' + 4\Delta)S(\gamma(E_A/\Delta)) + (1 + \epsilon') h \left( \frac{\epsilon'}{1 + \epsilon'} \right) + 2h(\Delta), \]

where the last inequality follows because any measure-and-prepare channel can be obtained by post-processing from a quantum-to-classical channel, and the mutual information obeys the data processing inequality.

We now move on the proof of equation (33). The fact that the right-hand side is no larger than the left-hand side is well known; to prove it, it suffices to choose as \( \Lambda \) the quantum-to-classical map that attains the accessible information \( I(A : B_J) := \max_{\Gamma \in \mathcal{QC}} I(A : B)_{\text{id} \otimes \Gamma(\rho)} \) makes \( N \) copies of the classical result, and stores it in \( N \) registers \( B_1, \ldots, B_N \).

As it turns out, we only have to prove that the lhs of equation (33) is no larger than the rhs. By using equation (32) we can write
$\mathbb{E}_J I(A : B_j)$

\[
\leq \frac{1}{N} \left[ (1 - \delta)N \left( I(A : B_a) + (2\epsilon' + 4\Delta)S(\gamma(E_A/\Delta)) + (1 + \epsilon')h \left( \frac{\epsilon'}{1 + \epsilon'} \right) + 2h(\Delta) \right) \\
+ \delta N 2S(A) \right] \\
= (1 - \delta) \left( I(A : B_a) + (2\epsilon' + 4\Delta)S(\gamma(E_A/\Delta)) + (1 + \epsilon')h \left( \frac{\epsilon'}{1 + \epsilon'} \right) + 2h(\Delta) \right) \\
+ \delta 2S(A),
\]

(C.3)

where we used the notation $I(A : B_j) : = I(A : B_j)_{\Lambda \in \Lambda_{B1-Bj}}$. We can choose $\delta = \sqrt{\zeta}$, then

\[
\epsilon' = 2\epsilon = \frac{\zeta}{\delta} \rightarrow 0,
\]

(C.4)

and therefore

\[
\Delta = \frac{1}{2} \frac{\epsilon'}{1 + \epsilon'} \rightarrow 0.
\]

(C.5)

Moreover, since $S(\gamma(E_A)) = o(E_A)$ [42], we have that $\Delta S(\gamma(E_A/\Delta)) \rightarrow 0$, as well as $\epsilon' S(\gamma(E_A/\Delta)) \rightarrow 0$ (since $\epsilon' = O(\Delta)$). As a consequence, for our choice of $\delta$,

\[
\mathbb{E}_J I(A : B_j)
\]

\[
\leq (1 - \delta) \left( I(A : B_a) + (2\epsilon' + 4\Delta)S(\gamma(E_A/\Delta)) + (1 + \epsilon')h \left( \frac{\epsilon'}{1 + \epsilon'} \right) + 2h(\Delta) \right) + \delta 2S(A)
\]

\[
\rightarrow_{N \rightarrow \infty} I(A : B_a),
\]

(C.6)

independently of the choice of $\Lambda = \Lambda_{B1-B2-BN}$. By considering the maximum of $\mathbb{E}_J I(A : B_j)$ over $\Lambda_{B1-B2-BN}$ and then the limit $N \rightarrow \infty$ we therefore obtain that

\[
\lim_{N \rightarrow \infty} \max_{\Lambda_{B1-B2-BN}} \mathbb{E}_J I(A : B_j) \leq I(A : B_a).
\]

(C.7)

\[\square\]

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References

[1] Joos E, Zeh H, Kiefer C, Giulini D, Kupsch J and Stamatescu I 2003 Decoherence and the Appearance of a Classical World in Quantum Theory (Berlin: Springer)

[2] Zurek W 2003 Rev. Mod. Phys. 75 715–75

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[3] Schlosshauer M 2005 Rev. Mod. Phys. 76 1267–305
[4] Schlosshauer M 2007 Decoherence and the Quantum-to-Classical Transition (The Frontiers Collection) (Berlin: Springer)
[5] Zurek W H 1981 Phys. Rev. D 24 1516–25
[6] Zurek W H 1982 Phys. Rev. D 26 1862–80
[7] Zurek W H 2003 Phys. Today 44 36
[8] Zurek W H 2009 Nat. Phys. 5 181
[9] Ollivier H, Poulin D and Zurek W H 2004 Phys. Rev. Lett. 93 220401
[10] Ollivier H, Poulin D and Zurek W H 2005 Phys. Rev. A 72 042113
[11] Blume-Kohout R and Zurek W 2006 Phys. Rev. A 73 062310
[12] Horodecki R, Korbicz J K and Horodecki P 2015 Phys. Rev. A 91 032122
[13] Le T P and Olaya-Castro A 2019 Phys. Rev. Lett. 122 010403
[14] Blume-Kohout R and Zurek W 2006 Phys. Rev. A 73 062310
[15] Horodecki R, Korbicz J K and Horodecki P 2015 Phys. Rev. A 91 032122
[16] Balaneskovic N and Zurek W H 2004 Phys. Rev. A 71 062340
[17] Balaneskovic N and Zurek W H 2005 Phys. Rev. A 72 042113
[18] Blume-Kohout R and Zurek W 2008 Phys. Rev. Lett. 101 240405
[19] Zwolak M, Quan H and Zurek W 2009 Phys. Rev. Lett. 103 110402
[20] Riedel C and Zurek W H 2010 Phys. Rev. Lett. 105 020404
[21] Riedel C, Zurek W and Zwolak M 2012 New J. Phys. 14 083010
[22] Galve F, Zambrini R and Maniscalco S 2015 Sci. Rep. 6 19607
[23] Le T P and Olaya-Castro A 2019 Phys. Rev. Lett. 122 010403
[24] Balaneskovic N and Mendler M 2016 Phys. Rev. A 93 220401
[25] Pleasance G and Garraway B M 2017 Phys. Rev. A 96 062105
[26] Le T and Olaya-Castro A 2018 Phys. Rev. A 98 032103
[27] Chen M C, Zhong H S, Li Y, Wu D, Wang X L, Li L, Liu N L, Lu C Y and Pan J W 2019 Sci. Bull. 64 580–5
[28] Ciampi M A, Pinna G, Mataloni P and Paternostro M 2018 Phys. Rev. A 98 020101
[29] Unden T K, Louzon D, Zwolak M, Zurek W H and Jelezko F 2019 Phys. Rev. Lett. 123 140402
[30] Brandao F, Piani M and Horodecki P 2015 Nat. Commun. 6 7908
[31] Qi X and Ranard D 2020 Emergent classicality in general multipartite states and channels (arXiv:2001.01507)
[32] Aubrun G, Lami L, Palazuelos C, Szarek S and Winter A 2020 Commun. Math. Phys. 375 679–724
[33] Ollivier H and Zurek W H 2001 Phys. Rev. Lett. 88 017901
[34] Winter A 2018 Probl. Inf. Transm. 54 20–33
[35] Winter A 2017 arXiv:1712.11097
[36] Aharonov D, Kitaev A and Nisan N 1998 Proc. of the 13th Annual ACM Symp. on Theory of Computing STOC ’98 (New York: ACM) pp 20–30
[37] Watrous J 2009 Theory Comput. 5 217–38
[38] Watrous J 2016 Commun. Math. Phys. 347 291–313
[39] Watrous J 2017 Lecture notes—theory of quantum information https://cs.uwaterloo.ca/watrous/LectureNotes.html
[40] Holevo A 2012 Quantum Systems, Channels, Information: A Mathematical Introduction (De Gruyter Studies in Mathematical Physics) (Berlin: de Gruyter & Co)
[41] Winter A 2015 J. Math. Phys. 56 022201
[42] De Palma G and Trevisan D 2018 Commun. Math. Phys. 360 639–62
[43] Sabapathy K and Winter A 2017 Phys. Rev. A 95 062309
[44] Lami L, Sabapathy K and Winter A 2018 New J. Phys. 20 113012
[45] Lim Y, Lee S, Kim J and Jeong K 2019 Phys. Rev. A 99 052326
[46] Becker S, Datta N, Lami L and Rouzé C 2019 arXiv:1912.06129
[47] Modti K, Brodutch A, Cable H, Paterek T and Vedral V 2012 Rev. Mod. Phys. 84 1655
[48] Adesso G, Bromley T and Cianciaruso M 2016 J. Phys. A 49 473001
[49] Horodecki R, Korbicz J K and Horodecki P 2015 Phys. Rev. A 91 032122
[54] Arrasmith A, Albrecht A and Zurek W 2019 Nat. Commun. 10 1024
[55] Matthews W, Wehner S and Winter A 2009 Commun. Math. Phys. 291 813–43
[56] Lami L, Palazuelos C and Winter A 2018 Commun. Math. Phys. 361 661–708
[57] Bhatia R 2013 Matrix Analysis (Graduate Texts in Mathematics) (New York: Springer)
[58] Lami L, Khatri S, Adesso G and Wilde M 2019 Phys. Rev. Lett. 123 050501