DYSON-SCHWINGER EQUATIONS AND THEIR APPLICATION TO NONPERTURBATIVE FIELD THEORY

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Abstract

Two examples of recent progress in applications of the Dyson–Schwinger equation (DSE) formalism are presented:

(1) Strong coupling quantum electrodynamics in 4 dimensions (QED$_4$) is an often studied model, which is of interest both in its own right and as an abelian model of quantum chromodynamics (QCD). We present results from a study of subtractive renormalization of the fermion propagator Dyson-Schwinger equation (DSE) in massive strong-coupling quenched QED$_4$. Results are compared for three different fermion-photon proper vertex Ans{"a}tze: bare $\gamma^\mu$, minimal Ball-Chiu, and Curtis-Pennington. The procedure is straightforward to implement and numerically stable. This is the first study in which this technique is used and it should prove useful in future DSE studies, whenever renormalization is required in numerical work.

(2) The Bethe-Salpeter equation (BSE) with a class of non-ladder scattering kernels is solved in Minkowski space in terms of the perturbation theory integral representation (PTIR). We consider a bound state of two spinless particles with the formal expression of the full scattering kernel in a $\phi^2\sigma$ scalar model. Making use of the PTIR we isolate the possible kinematical singularities of the bound state amplitude in Minkowski space. The resulting BS amplitude is written as a parametric integral of its weight function in this approach. We derive an integral equation for the weight function with a real kernel. We compare numerical solutions of the non-ladder scattering kernel with that of the massive scalar exchange kernel in the ladder approximation.

1 Quenched Massive QED$_4$

Strong coupling QED in three space and one time dimension has been studied within the Dyson-Schwinger Equation (DSE) formalism for some time [1, 2, 3], in order to see whether there may be a phase transition to a nontrivial “local” theory at high momenta [2, 4], as a model for dynamical chiral symmetry breaking (DCSB) in walking technicolor theories [5, 6], and also as an abelianized model for nonperturbative phenomena in QCD [7, 8]. For a recent
review of Dyson-Schwinger equations and their application see Ref. [9]. The usual approach is to write the DSE for the fermion propagator or self-energy, possibly including equations for the photon vacuum polarization [10, 11] or the fermion-photon proper vertex [1]. An appropriate Ansatz is made for the undefined Green’s functions that contain the infinite continuation of the tower of DSE’s. The resulting nonlinear integral equations are converted to Euclidean metric in the usual way [12] and solved numerically by iteration from an initial guess. DCSB occurs when the fermion propagator develops a nonzero scalar self-energy in the absence of an explicit chiral symmetry breaking (ECSB) fermion mass. We refer to coupling constants strong enough to induce DCSB as supercritical and those weaker are called subcritical. We write the fermion propagator as

\[ S(p) = \frac{Z(p^2)}{p - M(p^2)} = \frac{1}{A(p^2)} (p - B(p^2)) \]

with \( Z(p^2) \) the finite momentum-dependent fermion renormalization, and \( B(p^2) \) the scalar self-energy. In the massless theory (i.e., in the absence of ECSB) by definition DCSB occurs when \( B(p^2) \neq 0 \). Note that \( A(p^2) \equiv 1/Z(p^2) \) and \( M(p^2) \equiv B(p^2)/A(p^2) \).

Many studies, even until quite recently, have used the bare vertex as an Ansatz for the one-particle irreducible (1-PI) vertex \( \Gamma^\nu(k, p) \) [2, 4, 6, 7, 10, 11, 13], despite the fact that this violates the Ward-Takahashi Identity (WTI) [14]. It is also common, especially in studies motivated by walking technicolor theories [5], to find vertex Ansätze which claim to solve the WTI, but which still possess kinematic singularities in the limit of zero photon momentum \( q^2 = (k - p)^2 \to 0 \) [15, 16]. With any of these Ansätze the resulting fermion propagator is not gauge-covariant, i.e., physical quantities such as the critical coupling for dynamical symmetry breaking, or the mass itself, are gauge-dependent [15]. A general form for \( \Gamma^\nu(k, p) \) which does satisfy the Ward Identity was given by Ball and Chiu in 1980 [17]; it consists of a minimal longitudinally constrained term which satisfies the WTI, and a set of tensors spanning the subspace transverse to the photon momentum \( q \).

Although the WTI is necessary for gauge-invariance, it is not a sufficient condition; further, with many of these vertex Ansätze the fermion propagator DSE develops overlapping logarithms so that it is not multiplicatively renormalizable. There has been much recent research on the use of the transverse parts of the vertex to ensure both gauge-covariant and multiplicatively renormalizable solutions [15]–[23], some of which will be discussed below.

What is common to all of the studies so far is that the fermion propagator is not really subtractively renormalized. Most of these studies have assumed an initially massless theory and have renormalized at the ultraviolet cutoff of the integrations, taking \( Z_1 = Z_2 = 1 \). Where a nonzero bare mass has been used [4, 10], it has simply been added to the scalar term in the propagator. Although there have been formal discussions of the renormalization [4, 21], the important step of subtractive renormalization has not been performed.

1.1 DSE, and Vertex Ansätze

The DSE for the renormalized fermion propagator, in a general covariant gauge, is

\[ S^{-1}(p^2) = Z_2(\mu, \Lambda)[p - m^0(\Lambda)] - iZ_1(\mu, \Lambda)e^2 \int_\Lambda \frac{d^4k}{(2\pi)^4} \gamma^\mu S(k)\Gamma^\nu(k, p)D_{\mu\nu}(q) ; \]

here \( q = k - p \) is the photon momentum, \( \mu \) is the renormalization point, and \( \Lambda \) is a regularizing parameter (taken here to be an ultraviolet momentum cutoff). We write \( m_0(\Lambda) \) for the

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regularization-parameter dependent bare mass. In the massless theory (i.e., in the absence of an ECSB) the bare mass is zero, \( m_0(\Lambda) = 0 \). The physical charge is \( e \) (as opposed to the bare charge \( e_0 \)), and the general form for the photon propagator is

\[
D_{\mu\nu}(q) = \left\{ -g_{\mu\nu} + \frac{q_{\mu}q_{\nu}}{q^2} \right\} \left( \frac{1}{1 - \Pi(q^2)} - \xi \frac{q_{\mu}q_{\nu}}{q^2} \right),
\]

with \( \xi \) the covariant gauge parameter. Since we will work in the quenched approximation and the Landau gauge we have \( e^2 \equiv e_0^2 = 4\pi\alpha_0 \) and

\[
D_{\mu\nu}(q) \rightarrow D_{0\mu\nu}(q) = \left( -g_{\mu\nu} + \frac{q_{\mu}q_{\nu}}{q^2} \right) \frac{1}{q^2},
\]

for the photon propagator.

### 1.1.1 Vertex Ansatz

The requirement of gauge invariance in QED leads to the Ward-Takahashi Identities (WTI); the WTI for the fermion-photon vertex is

\[
q_\mu \Gamma^\mu(k, p) = S^{-1}(k) - S^{-1}(p),
\]

where \( q = k - p \). This is a generalization of the original differential Ward identity, which expresses the effect of inserting a zero-momentum photon vertex into the fermion propagator,

\[
\frac{\partial S^{-1}(p)}{\partial p_\nu} = \Gamma^\nu(p, p).
\]

In particular, it guarantees the equality of the propagator and vertex renormalization constants, \( Z_2 \equiv Z_1 \). The Ward-Takahashi Identity is easily shown to be satisfied order-by-order in perturbation theory and can also be derived nonperturbatively.

As discussed in [9], this can be thought of as just one of a set of six general requirements on the vertex: the vertex must satisfy the WTI; it should contain no kinematic singularities; it should transform under charge conjugation (\( C \)), parity inversion (\( P \)), and time reversal (\( T \)) in the same way as the bare vertex, e.g.,

\[
C^\dagger \Gamma^\mu(k, p)C = -\Gamma^T_\mu
\]

(where the superscript \( T \) indicates the transpose); it should reduce to the bare vertex in the weak-coupling limit; it should ensure multiplicative renormalizability of the DSE in Eq. (2); the transverse part of the vertex should be specified to ensure gauge-covariance of the DSE.

Ball and Chiu [17] have given a description of the most general fermion-photon vertex that satisfies the WTI; it consists of a longitudinally-constrained (i.e., “Ball-Chiu”) part \( \Gamma_{BC}^\mu \), which is a minimal solution of the WTI, and a basis set of eight transverse vectors \( T^\mu_i(k, p) \), which span the hyperplane specified by \( q_\mu T^\mu_i(k, p) = 0, q \equiv k - p \). The minimal longitudinally constrained part of the vertex is given by

\[
\Gamma_{BC}^\mu(k, p) = \frac{1}{2} \left\{ \left( A(k^2) + A(p^2) \right) \gamma^\mu + \frac{(k + p)^\mu}{k^2 - p^2} \left\{ \left[ A(k^2) - A(p^2) \right] \frac{k + p}{2} - \left[ B(k^2) - B(p^2) \right] \right\} \right\}.
\]
A general vertex is then written as
\[
\Gamma^\mu(k, p) = \Gamma^\mu_{BC}(k, p) + \sum_{i=1}^{8} \tau_i(k^2, p^2, q^2) T^\mu_i(k, p),
\]
where the \(\tau_i\) are functions which must be chosen to give the correct \(C\), \(P\), and \(T\) invariance properties.

The work of Curtis and Pennington \[21\] was mentioned above in connection with the specification of the transverse vertex terms in order to produce gauge-invariant and multiplicatively renormalizable solutions to the DSE. In the framework of massless QED, they eliminate four of the transverse vectors since they are Dirac-even and must generate a scalar term. By requiring that the vertex \(\Gamma^\mu(k, p)\) reduce to the leading log result for \(k \gg p\) they are led to eliminate all the transverse basis vectors except \(T^\mu_6\), with a dynamic coefficient chosen to make the DSE multiplicatively renormalizable. This coefficient has the form
\[
\tau_6(k^2, p^2, q^2) = \frac{1}{2} \left[ A(k^2) - A(p^2) \right] / d(k, p),
\]
where \(d(k, p)\) is a symmetric, singularity free function of \(k\) and \(p\), with the limiting behavior \(\lim_{k^2 \gg p^2} d(k, p) = k^2\). [Here, \(A(p^2) \equiv 1/Z(p^2)\) is their \(1/F(p^2)\).] For purely massless QED, they find a suitable form, \(d(k, p) = (k^2 - p^2)^2 / (k^2 + p^2)\). This is generalized to the case with a dynamical mass \(M(p^2)\), to give
\[
d(k, p) = \frac{(k^2 - p^2)^2 + [M^2(k^2) + M^2(p^2)]^2}{k^2 + p^2}.
\]
They establish that multiplicative renormalizability is retained up to next-to-leading-log order in the DCSB case. Subsequent papers establish the form of the solutions for the renormalization and the mass \[21\], and demonstrate the gauge-covariance of the solutions \[21\]. (However, Dong, Munczek and Roberts \[22\] have recently contested this). Bashir and Pennington \[23\] have recently described a vertex Ansatz which makes the fermion self-energy exactly gauge-covariant, in the sense that the critical point for the chiral phase transition is independent of gauge.

In this work we will primarily compare the Curtis-Pennington Ansatz with results using the bare vertex. Some solutions are also obtained with the minimal Ball-Chiu vertex which, like the Curtis-Pennington vertex, satisfies the WTI, but which does not lead to gauge-invariant solutions.

For each vertex Ansatz, the equations are separated into a Dirac-odd part describing the finite propagator renormalization \(A(p^2)\), and a Dirac-even part for the scalar self-energy, by taking \(\frac{1}{4} \text{Tr}\) of the DSE multiplied by \(\bar{p}/p^2\) and 1, respectively. These equations are rotated to Euclidean metric, giving equations for the spacelike momenta only. The volume integrals \(\int d^4k\) are separated into angle integrals and an integral \(\int dk^2\); the angle integrals are easy to perform analytically, yielding the two equations which will be solved numerically.

### 1.2 The Subtractive Renormalization

The subtractive renormalization of the fermion propagator DSE proceeds similarly to the one-loop renormalization of the propagator in QED. (This is discussed in \[9\] and standard texts.) One first determines a finite, regularized self-energy, which depends on both a regularization
parameter and the renormalization point; then one performs a subtraction at the renormalization point, in order to define the renormalization parameters $Z_1$, $Z_2$, $Z_3$ which give the full (renormalized) theory in terms of the regularized calculation.

A review of the literature of DSEs in QED shows, however, that this step is never actually performed. Curtis and Pennington [21] for example, define their renormalization point at the UV cutoff. Miransky [4] gives a formal discussion of the variation of the mass renormalization $Z_{m}^{\mu, \Lambda}$, but does not implement it numerically.

Many studies take $Z_1 = Z_2 = 1$ [18, 20, 21]; this is a reasonable approximation in cases where the coupling $\alpha_0$ is small (i.e., $\alpha_0 \lesssim 1.15$), but if $\alpha_0$ is chosen large enough, the value of the dynamical mass at the renormalization point may be significantly large compared with its maximum in the infrared. For instance, in [21], figures for the fermion mass are given with $\alpha_0 = 0.97, 1.00, 1.15$ and 2.00 in various gauges. For $\alpha_0 = 2.00$, the fermion mass at the cutoff is down by only an order of magnitude from its limiting value in the infrared.

Repeating the arguments from [9], one defines a regularized self-energy $\Sigma'(\mu, \Lambda; p)$, leading to the DSE for the renormalized fermion propagator,

$$\tilde{S}^{-1}(p) = Z_2(\mu, \Lambda)\left[p - m_0(\Lambda)\right] - \Sigma'(\mu, \Lambda; p) = p - m(\mu) - \tilde{S}(\mu; p),$$

(12)

where the (regularized) self-energy is

$$\Sigma'(\mu, \Lambda; p) = iZ_1(\mu, \Lambda)e^2 \int_{\Lambda}^{\mu} \frac{d^4k}{(2\pi)^4} \gamma^\lambda \tilde{S}(\mu; k) \tilde{\Gamma}^\nu(\mu; k, p) \tilde{D}_{\lambda\nu}(\mu; (p - k)).$$

(13)

[To avoid confusion we will follow Ref. [4] and in this section only we will denote regularized quantities with a prime and renormalized ones with a tilde, e.g. $\Sigma'(\mu, \Lambda; p)$ is the regularized self-energy depending on both the renormalization point $\mu$ and regularization parameter $\Lambda$ and $\tilde{\Sigma}(\mu; p)$ is the renormalized self-energy.] As suggested by the notation (i.e., the omission of the $\Lambda$-dependence) renormalized quantities must become independent of the regularization-parameter as the regularization is removed (i.e., as $\Lambda \to \infty$). The self-energies are decomposed into Dirac and scalar parts,

$$\Sigma'(\mu, \Lambda; p) = \Sigma'_d(\mu, \Lambda; p^2) \ p + \Sigma'_s(\mu, \Lambda; p^2)$$

(and similarly for the renormalized quantity, $\tilde{\Sigma}(\mu; p)$). By imposing the renormalization boundary condition,

$$\left.\tilde{S}^{-1}(p)\right|_{p^2 = \mu^2} = p - m(\mu),$$

(14)

one gets the relations

$$\tilde{\Sigma}_{d,s}(\mu; p^2) = \Sigma'_d(\mu, \Lambda; p^2) - \Sigma'_s(\mu, \Lambda; \mu^2)$$

(15)

for the self-energy,

$$Z_2(\mu, \Lambda) = 1 + \Sigma'_d(\mu, \Lambda; \mu^2)$$

(16)

for the renormalization, and

$$m_0(\Lambda) = \frac{m(\mu) - \Sigma'_s(\mu, \Lambda; \mu^2)}{Z_2(\mu, \Lambda)}$$

(17)

for the bare mass. In order to reproduce the case with no ECSB mass, for a given cutoff $\Lambda$, one chooses $m(\mu) = \Sigma'_s(\mu, \Lambda; \mu^2)$ so that the bare mass $m_0(\Lambda)$ is zero. The mass renormalization constant is given by

$$Z_m(\mu, \Lambda) = m_0(\Lambda)/m(\mu),$$

(18)
i.e., as the ratio of the bare to renormalized mass.

The vertex renormalization, \( Z_1(\mu, \Lambda) \) is identical to \( Z_2(\mu, \Lambda) \) as long as the vertex Ansatz satisfies the Ward Identity; this is how it is recovered for multiplication into \( \Sigma'(\mu, \Lambda; p) \) in Eq. (13). It will be noticed that this is inappropriate for the bare-vertex Ansatz since it fails to satisfy the WTI; nonetheless, since for the bare vertex case there is no way to determine \( Z_1(\mu, \Lambda) \) independently we will use \( Z_1 = Z_2 \) for the sake of comparison. In the Landau gauge for the bare vertex these will then both be 1, since in this case \( \Sigma'(\mu, \Lambda; p^2) = 0 \) for all \( p^2 \) as is well known [9].

### 1.3 Results

Solutions were obtained for the DSE with the Curtis-Pennington and bare vertices, for couplings \( \alpha_0 \) from 0.1 to 1.75; solutions were also obtained for the minimal Ball-Chiu vertex, with couplings \( \alpha_0 \) from 0.1 to 0.6 (for larger couplings the DSE with this vertex was susceptible to numerical noise). In Landau gauge, the critical coupling for the DSE with bare vertex is \( \alpha_{\text{bare}}^c = \pi/3 \); the critical coupling for the Curtis-Pennington vertex is \( \alpha_{\text{CP}}^c = 0.933667 \) [24], and that for the Ball-Chiu vertex is expected to be close to these two values. A full discussion of the numerical results with detailed figures can be found elsewhere [25]. A brief summary of these is given here in the conclusions.

### 2 Minkowski-space Bethe-Salpeter Equation

Considerable interest has been recently attached to the covariant description of bound states in conjunction with model calculations of high-energy processes, such as deep inelastic scattering. A fully covariant description of composite bound states is essential. The bound state nature of hadrons is described by the appropriate vertex function, or equivalently the Bethe-Salpeter (BS) amplitude. In a relativistic field theory two-body bound states are described by the Bethe-Salpeter (BS) amplitude [26]. The BS amplitude obeys an integral equation whose kernel has singularities due to the Minkowski metric. The resultant solutions are mathematically no longer functions but “distributions”. This singular structure makes it difficult to handle the BS equation numerically. In order to handle such a singular integral equation the analytic continuation of the relative-energy variable, which is called “Wick rotation”, is widely used [27]. The ladder BS amplitude is solved as a function of Euclidean relative momentum in the standard approach. If one uses a “dressed” propagator for constituent particles or more complicated kernels in the BS equation, the validity of the Wick rotation becomes highly nontrivial, e.g., almost all the dressed propagator studied previously in the Dyson-Schwinger equation approach contains pathological complex “ghost” poles[4]. It is therefore preferable to formulate and solve the BS equation directly in Minkowski space.

We present a method to solve the BS equation without Wick rotation by making use of the perturbation theory integral representation (PTIR) for the BS amplitude [28]. This integral representation has been studied for a scalar-scalar bound state in the ladder approximation [29]. We extend this method to a wide class of non-ladder kernels. We rewrite the BS equation as the integral equation of the weight function for the BS amplitude and discuss the singularity structure of the kernel function for the weight function.
2.1 Scalar-Scalar BS Equation

Let us consider a bound state of two spinless particles $\phi$ having a mass $m$. They interact each other through the exchange of another spinless particle $\sigma$ with a mass $\mu$. Let the interaction between $\phi$ and $\sigma$ be the Yukawa coupling: $L_{\text{int}} = -g\phi^2\sigma$. The Bethe-Salpeter amplitude $\Phi(p, P)$ for the bound state having the total momentum $P$ and the relative one $2p$ obeys the following equation:

$$[D(P/2 + p)D(P/2 - p)]^{-1}\Phi(p, P) = \int \frac{d^4q}{(2\pi)^4} I(p, q; P)\Phi(q, P)$$

(19)

where $D(q)$ is the propagator of $\phi$-particle and we approximate it with the tree one: $D_0(q) = 1/(m^2 - q^2 - i\epsilon)$. The scattering kernel $I(p, q; P)$ describes the process: $\phi_1\phi_2 \rightarrow \phi_3\phi_4$ and the momentum $2p$ and $2q$ are the relative momentum of initial and final states. We consider the following formal expression for the full scattering kernel:

$$I(p, q; P) = \int_0^\infty d\gamma \int_\Delta d\vec{\xi} \rho_{ch}(\gamma, \vec{\xi}; g) \frac{\gamma - \sum_{i=1}^4 \xi_i q_i^2 + \xi_5 s + \xi_6 t - i\epsilon}{(4\pi)^2}$$

(20)

where $q_i$ is the momentum carried by $\phi_i$ and $s,t$ and $u$ are Mandelstam variables. The symbol $\Delta$ denotes the integral region of 6 dimensionless Feynman parameters $\xi_i$ such that $\Delta \equiv \{\xi_i | \xi_i \geq 0, \sum \xi_i = 1 (i = 1, \ldots, 6)\}$. The “mass” parameter $\gamma$ represents a spectrum of the scattering kernel. The function $\rho_{ch}(\gamma, \vec{\xi}; g)$ gives the weight of the spectrum in a different channel; $ch = \{st\}, \{tu\}, \{us\}$. This expression has been derived by Nakanishi and is called the perturbation theory integral representation (PTIR)\[28\]. It should be mentioned that any perturbative Feynman diagram for the scattering kernel can be written in this form, so that the weight function $\rho_i$ is, in principle, calculable as a power series of coupling constant $g$.

2.2 PTIR for BS amplitude

Let us consider the $s$-wave bound state for simplicity\[1\]. We assume that the BS amplitude $\Phi(p, P)$ has an integral representation of the form:

$$\Phi(p, P) = -i \int_{-\infty}^{\infty} d\alpha \int_{-1}^{1} dz \frac{\varphi_n(\alpha, z)}{[m^2 + \alpha - (p^2 + zp \cdot P + P^2/4) - i\epsilon]^{n+2}}$$

(21)

where the non-negative integer parameter $n$ is a dummy parameter, since a partial integration with respect to $\alpha$ changes the power. We can utilize this artificial degree of freedom for a numerical study. Substituting the above expression into Eq.(19) together with Eq.(20), we obtain the following integral equation for $\varphi_n(\alpha, z)$ as follows:

$$\varphi_n(\bar{\alpha}, \bar{z}) = \int_{-\infty}^{\infty} d\alpha \int_{-1}^{1} dz \sum_{ch} \int_0^\infty d\gamma \int_\Delta d\vec{\xi} \rho_{ch}(\gamma, \vec{\xi}; g) \frac{\varphi_n(\alpha, z)}{(4\pi)^2} K_n(\alpha, z; \alpha, z) \varphi_n(\alpha, z),$$

(22)

\[ Extension to higher partial wave solutions is straightforward.
where we have suppressed the dependence on the kernel parameters. Note that the Eq.(22) is frame-independent. The real kernel function $K_n(\tilde{\alpha}, \tilde{z}; \alpha, z)$ with a fixed kernel parameter set $(\gamma, \xi)$ has the following structure:

$$
K_n(\tilde{\alpha}, \tilde{z}; \alpha, z) = \frac{\partial}{\partial \tilde{\alpha}} (\tilde{\alpha}^n \theta(\tilde{\alpha})) h_n(\alpha, z) - \theta (\alpha - \omega_1(\tilde{\alpha}, \tilde{z}, z))(\alpha - \omega_2(\tilde{\alpha}, \tilde{z}, z))
$$

$$
\times \left\{ \frac{g_n(\tilde{\alpha}, \tilde{z}; \alpha, z)}{\sqrt{(\alpha - \omega_1(\tilde{\alpha}, \tilde{z}, z))(\alpha - \omega_2(\tilde{\alpha}, \tilde{z}, z))}} + k_n(\tilde{\alpha}, \tilde{z}; \alpha, z) \right\}
$$

(23)

where $\omega_1(\tilde{\alpha}, \tilde{z}, z)$, $g_n(\tilde{\alpha}, \tilde{z}; \alpha, z)$ and $h_n(\alpha, z)$ are regular functions. The function $k_n(\tilde{\alpha}, \tilde{z}; \alpha, z)$ is also regular in the simple one-σ-exchange ladder kernel, but in general it contains a singularity such as $\text{Pf} \cdot 1/(\alpha - \tau(\tilde{\alpha}, \tilde{z}, z))^n$ where the symbol $\text{Pf} \cdot$ stands for Hadamard’s finite part and $\tau(\tilde{\alpha}, \tilde{z}, z)$ is a regular function. Thus the first term of Eq.(23) contains a δ-function singularity at $\tilde{\alpha} = 0$, only if $n = 0$. This singularity, independent of $\tilde{z}, \alpha$ and $z$, corresponds to the pole singularity in $\Phi(p, P)$ which comes from the free propagation of two φ’s. On the other hand, the second term contains a discontinuity due to the step function depending on $\tilde{\alpha}, \tilde{z}, \alpha$ and $z$. In addition to the singularity in $k_n(\tilde{\alpha}, \tilde{z}; \alpha, z)$ as mentioned above, the term has a square root line singularity at the boundary of its support. Since the Hadamard’s finite part $\text{Pf} \cdot 1/x$ coincides Cauchy’s principal value, the kernel function $K_{n=1}(\tilde{\alpha}, \tilde{z}; \alpha, z)$ is integrable, so that the integral equation (22) with a constant kernel parameter set is numerically tractable by setting the dummy parameter $n = 1$, provided that the weight function $\varphi_1(\alpha, z)$ is differentiable at the singular points of $K_1(\tilde{\alpha}, \tilde{z}; \alpha, z)$. As a check on our method we have reproduced the known results for the one-σ-exchange ladder kernel by solving Eq.(24) numerically. The numerical solution has been extended also to a class of non-ladder (“generalized ladder”) kernels. The case for the most general scattering kernel, whose weight function can contain a derivative of δ-function, is currently under investigation.

3 Conclusions

We have described preliminary results in a study of four-dimensional quenched QED, with subtractive renormalization performed numerically, on-the-fly during the calculation. We believe that this is the first calculation of its kind, and the technique described here will be applicable elsewhere (e.g., in both QED and QCD), whenever numerical renormalization is needed.

The Curtis-Pennington vertex has been the primary focus of this study, since it has the desirable properties of making the solutions approximately gauge-invariant and also multiplicatively renormalizable up to next-to-leading log order. Solutions have been obtained for comparison purposes, using the minimal Ball-Chiu vertex and using the bare vertex Ansätze (with $Z_1 = Z_2$). For the Ball-Chiu vertex, couplings in the range from $\alpha_0 = 0.1$ to 0.6 were used, while couplings up to $\alpha_0 = 1.75$ were used with both the bare and the Curtis-Pennington vertices. Various renormalization points and renormalized masses were studied.

The subtractive renormalization procedure is straightforward to implement. The solutions are stable and the renormalized quantities become independent of regularization as the regularization is removed, which is as expected. For example, the mass function $M(p^2)$ and momentum-dependent renormalization $Z(p^2)$ are unchanged to within the numerical accuracy of the computation as the integration cutoff is increased by many orders of magnitude. For the range of couplings $\alpha_0$ considered, the values of the renormalization constant
$Z_1(\mu, \Lambda) = Z_2(\mu, \Lambda)$ are never very far from 1 and vary relatively weakly with the choice of renormalized mass and cutoff as expected in Landau gauge. For subcritical couplings, using the Curtis-Pennington vertex, we also find that the mass renormalization $Z_m(\mu, \Lambda)$ scales approximately as $Z_m(\mu, \Lambda) \propto (\mu^2/\Lambda^2)^{\frac{1}{2}-\gamma_{CP}(\alpha_0)}$, where e.g. $\gamma_{CP}(0.5) = 0.358$.

An unexpected feature of the equations is that for any set $\{ \alpha_0, \mu^2, m(\mu) \}$, as the cutoff is increased there is a region where the dynamical mass is negative. The significance of this for QED is not completely understood. One possibility is that it may signal the failure of multiplicative renormalizability for the model DSE, in which case further refinements of the vertex Ansatz may be called for. It should be emphasized that this negative dip in the scalar self-energy is very small unless the coupling $\alpha_0$ approaches 2. For instance, for $\alpha_0 = 1.00$, the negative peak was $\sim 2.4 \times 10^{-5}$ the size of the renormalized mass.

Extensions of this work to include other gauges and vertices of the Bashir-Pennington type are underway [23].

For the Bethe-Salpeter equation we have derived a real integral equation of the weight function for the scalar-scalar BS amplitude with a formal expression of the full scattering kernel. We found that our integral equation is numerically tractable for a class of non-ladder scattering kernels. We have verified that our numerical solutions agree with those previously obtained from a Euclidean treatment of the pure ladder limit. This represents a powerful new approach to obtaining solutions of the BSE and a more detailed discussion will appear soon [30].

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