Kähler–Einstein metrics with edge singularities

Thalia D. Jeffres, Rafe Mazzeo, and Yanir A. Rubinstein

with an appendix by Chi Li and Yanir A. Rubinstein

Abstract

This article considers the existence and regularity of Kähler–Einstein metrics on a compact Kähler manifold $M$ with edge singularities with cone angle $2\pi \beta$ along a smooth divisor $D$. We prove existence of such metrics with negative, zero and some positive cases for all cone angles $2\pi \beta \leq 2\pi$. The results in the positive case parallel those in the smooth case. We also establish that solutions of this problem are polyhomogeneous, i.e., have a complete asymptotic expansion with smooth coefficients along $D$ for all $2\pi \beta < 2\pi$. This work rests on a recent advance by Donaldson [19]; certain of the existence results overlap those in other recent articles [6, 14, 15].

1 Introduction

Let $D \subset M$ be a smooth divisor in a compact Kähler manifold. A Kähler edge metric on $M$ with angle $2\pi \beta$ along $D$ is a Kähler metric on $M \setminus D$ that is asymptotically equivalent at $D$ to the model edge metric

$$g_\beta := |z_1|^{2\beta-2} |dz_1|^2 + \sum_{j=2}^n |dz_j|^2,$$

here $z_1, z_2, \ldots, z_n$ are holomorphic coordinates such that $D = \{z_1 = 0\}$ locally. We always assume that $0 < \beta < 1$.

Of particular interest is the existence and geometry of metrics of this type which are also Einstein. The existence of Kähler–Einstein (KE) edge metrics was first conjectured by Tian in the mid '90's [48]. In fact, Tian conjectured the existence of KE metrics with ‘crossing’ edge singularities when $D$ has simple normal crossings. One motivation was his observation that these metrics could be used to prove various inequalities in algebraic geometry; in particular, the Miyaoka–Yau inequality could be proved by deforming the cone angle of Kähler–Einstein edge metrics with negative curvature to $2\pi$. Furthermore, these metrics can be used to bound the degree of immersed curves in general type varieties. He also anticipated that the complete Tian–Yau KE metric on the complement of a divisor should be the limit of the Kähler–Einstein edge metrics as the angle $2\pi \beta$ tends to 0. Recently, Donaldson [19] proposed using these metrics in a similar way to construct smooth Kähler–Einstein metrics on Fano manifolds by deforming the cone angle of Kähler–Einstein metrics of positive curvature, and more generally to relate this approach to the much-studied obstructions to existence of smooth Kähler–Einstein metrics.

One of the main results in this article is a proof of Tian’s conjecture on the existence of Kähler–Einstein edge metrics when $D$ is smooth. In a sequel to this article we shall prove the general case [39]; this involves certain additional complications, but can be achieved by an elaboration of the methods introduced here.
In the lowest dimensional setting, $M$ is a Riemann surface and the problem is to find constant curvature metrics with prescribed conic singularities at a finite collection of points. This was accomplished in general by McOwen and Troyanov [40, 53]; as part of this, Troyanov found some interesting restrictions on the cone angles necessary for the existence of spherical cone metrics. Later, Luo and Tian [34] established the uniqueness of these metrics. For the problem in higher dimensions, we focus only on the case where $D$ is smooth, unless explicitly stated. The case of Kähler–Einstein edge metrics with negative curvature was studied by the first named author [25, 26], where it was already realized that some of the a priori estimates of Aubin and Yau [1, 57] should carry over to this setting. An announcement for the existence in that negative case with $\beta \in (0, \frac{1}{2}]$ was made over ten years ago by the first and second named authors [38]. There were several analytic issues described in that announcement which seemed to complicate the argument substantially and details never appeared.

Quite recently there has been a renewed interest in these problems stemming from an important advance by Donaldson [19], alluded to just above, whose insightful observations make it possible to establish good linear estimates. He proves a deformation theorem, showing that the set of attainable cone angles for KE edge metrics is open. The key to his work is the identification of a function space on which the linearized Monge–Ampère equation is solvable in the class of bounded functions.

We realized, following the appearance of [19], that only a slight change of perspective suggested by his advance makes it possible to apply the theory of elliptic edge operators from [36] in a rather direct manner so as to circumvent the difficulties surrounding the openness part of the argument proposed in [38]. Indeed, we show that estimates equivalent to those of Donaldson (but on slightly different function spaces) follow directly from some of the basic results in that theory, and we explain this at some length. This alternate approach to the linear theory allows us to go somewhat further, and we use it to show that solutions are polyhomogeneous, i.e., have complete asymptotic expansions. This was announced in [38] and speculated on in [19], and the existence of this higher regularity should be very helpful in the further study of these metrics.

In fact, this higher regularity plays a fundamental role in the nonlinear a priori estimates. In order to obtain our existence results it is necessary to work with minimal assumptions on the reference geometry, and polyhomogeneity is one of the key ingredients which allows us to do so. As a result, we are able to establish the existence theorem for all cone angles less than $2\pi$, which we carry out in the negative, zero and in certain positive curvature cases.

More precisely, what we achieve here is the following. We prove existence of Kähler–Einstein edge metrics with cone angle $2\pi\beta$ that have negative, zero and positive curvature, as appropriate, for all cone angles $2\pi\beta \leq 2\pi$, when $D$ is smooth. Existence in the positive case is proved under the condition that the twisted Mabuchi K-energy is proper, in parallel to Tian’s result in the smooth case [49]. Next, we prove that solutions of a general class of complex Monge–Ampère equations are polyhomogeneous, i.e., have complete asymptotic expansions with smooth coefficients. We provide a sharper identification of the function space defined by Donaldson for his deformation result. As we have briefly noted above, there are two somewhat different scales of Hölder spaces which can be used for this type of problem. One, used in [25, 19], we call the wedge Hölder spaces; the other, from [36], are the edge Hölder spaces. The latter behave more naturally with respect to a certain dilation structure which is inherent in this problem. This makes the linear theory, and certain parts of the nonlinear theory, more transparent. However, we stress that we provide two independent proofs of the a priori estimates, one in each class of spaces. The proof of higher regularity (polyhomogeneity) is directly related to special properties of the edge spaces, but holds for solutions in either
class of spaces, and because of this, the two approaches to the a priori estimates may be used interchangeably. Finally, in proving the a priori estimates we have made an effort to extend various classical arguments and bounds to this singular setting with minimal assumptions on the background geometry. In particular, we employ a particular continuity method closely related to the Ricci iteration [42], that together with the Chern–Lu inequality allows us to obtain the a priori estimate on the Laplacian assuming only that the reference edge metric has bisectional curvature bounded above. We then explain how the Evans–Krylov theory together with our asymptotic expansion imply a priori Hölder bounds on the second derivatives for all cone angles with no further curvature assumptions. Reducing the dependence of the estimates for the existence of a Kähler–Einstein metric to only an upper bound on the bisectional curvature of the reference metric does not seem to have been observed previously even in the smooth setting, where traditionally a lower bound on the bisectional curvature is required, or at least an upper bound on the bisectional curvature together with a lower bound on some curvature. Finally, in the case of positive curvature, we show how to control the Sobolev constant and infimum of the Green function, which are both needed for the uniform estimate. In an Appendix it is shown that the bisectional curvature of the reference metric is bounded from above on $M \setminus D$ whenever $\beta \in (0, 1]$. Before stating our results, let us mention some other recent articles concerning existence. Quickly capitalizing on the linear estimates in [19], and originally part of the collaboration leading to the current article, Brendle [14] adapted the classical estimates of Aubin and Yau to obtain existence of Ricci-flat KE edge metrics with $\beta \in (0, \frac{1}{2}]$. In a different direction, and allowing $D$ to have simple normal crossings, Berman [6] showed how to bypass the openness problem and produce KE metrics whose volume form is asymptotic to that of an edge metric by using a variational approach. Finally, a quite different approach to existence and also allowing divisors with simple normal crossings, based on approximation by smooth metrics, has appeared in a recent article by Campana, Guenancia and Păun [15] in the case of nonpositive curvature with $\beta \in (0, \frac{1}{2}]$. However, neither of these methods give good information about the regularity of the solution metric at the divisor.

We now state our main results more precisely. All the relevant notation is defined in Sections 2 and 3.

**Theorem 1.** (Asymptotic expansion of solutions) Let $\omega$ be a polyhomogeneous Kähler edge metric with angle $2\pi \beta \in (0, 2\pi]$. Suppose that $u \in D_{s,\gamma}^0 \cap \text{PSH}(M, \omega)$, $s = w$ or $e$, is a solution of the complex Monge–Ampère equation

$$\omega^n_u = \omega^n F(z, u), \quad \text{on } M \setminus D,$$

where $\omega_u = \omega + \sqrt{-1} \partial \bar{\partial} u$ and $F$ is polyhomogeneous in its arguments and is such that if $u \in A^0_{\text{phg}}$ then $F(z, u) \in A^0_{\text{phg}}$. Then $u$ is polyhomogeneous, i.e., $u \in A^0_{\text{phg}}(X)$.

**Theorem 2.** (Kähler–Einstein edge metrics) Let $(M, \omega_0)$ be a compact Kähler manifold with $D \subset M$ a smooth divisor, and suppose that $\mu[\omega_0] + (1 - \beta)[D] = c_1(M)$, where $\beta \in (0, 1]$ and $\mu \in \mathbb{R}$. If $\mu > 0$, suppose in addition that the twisted K-energy $E^\beta_0$ is proper. Then there exists a unique (when $\mu \neq 0$) Kähler–Einstein edge metric $\omega_{\text{KE}}$ with Ricci curvature $\mu$ and with angle $2\pi \beta$ along $D$. This metric is polyhomogeneous, namely, $\varphi_{\text{KE}}$ admits a complete asymptotic expansion with smooth coefficients as $r \to 0$ of the form

$$\varphi_{\text{KE}}(r, \theta, Z) \sim \sum_{j,k \geq 0} \sum_{\ell=0}^{N_{j,k}} a_{j,k,\ell}(\theta, Z) r^{j+k/\beta}(\log r)^\ell,$$
where $r = |z_1|^{\beta}/\beta$ and $\theta = \arg z_1$, and with each $a_{jkl} \in C^\infty$. There are no terms of the form $r^\gamma (\log r)^{\ell}$ with $\ell > 0$ if $\gamma \leq 2$. In particular, $\varphi_{KE}$ has infinite conormal regularity and a precise Hölder regularity as measured relative to the reference edge metric $\omega$, which is encoded by $\varphi_{KE} \in A^0 \cap D^{0,\gamma}_w$.

To put these conclusions about regularity into words, we prove infinite ‘conormal’ regularity $(A^0)$, which asserts that the solution is tangentially smooth and also infinitely differentiable with respect to the vector field $r \partial_r$, then Hölder continuity of some second derivatives with respect to the model metric $(D^{0,\gamma}_w)$, and finally the existence of an asymptotic expansion in powers of the distance to the edge $(A^0_{plug})$. This expansion also leads to the precise asymptotics of the curvature tensor and its covariant derivatives. For example, when $\beta \leq \frac{1}{2}$ we have $\varphi_{KE} \in C^{2,\frac{1}{2}-2}_w$, and the curvature tensor of $\omega_{\varphi_{KE}}$ is in $C^{0,\frac{1}{2}-2}_w$.

Acknowledgements

The authors are grateful to Gang Tian for his advice and encouragement throughout the course of this project. They also thank Simon Donaldson for his interest in this work. The NSF supported this research through grants DMS-0805529 (R.M.) and DMS-0802923 (Y.A.R.).

2 Preliminaries

We set the stage for the rest of the article with a collection of facts and results needed later. First consider the flat model situation, where $M = \mathbb{C}^n$ with linear coordinates $(z_1, \ldots, z_n)$, and $D$ the linear subspace $\{z_1 = 0\}$. For brevity we often write $Z = (z_2, \ldots, z_n)$. The model Kähler form and Kähler metric are given by

$$\omega_\beta = \frac{1}{2} \sqrt{-1} \left( |z_1|^{2\beta-2} dz^1 \wedge \overline{dz^1} + \sum_{j=2}^{n} dz^j \wedge \overline{dz^j} \right), \quad \text{(2)}$$

$$g_\beta = |z_1|^{2\beta-2} |dz^1|^2 + \sum_{j=2}^{n} |dz^j|^2. \quad \text{(3)}$$

This is the product of a flat one complex dimensional conic metric with cone angle $2\pi \beta$ with $\mathbb{C}^{n-1}$. We always assume that $0 < \beta \leq 1$; the expressions above make sense for any real $\beta$, but their geometries are quite different for $\beta$ outside of this range.

Now suppose that $M$ is a compact Kähler manifold and $D$ a smooth divisor. Fix $\beta \in (0,1]$ and $\mu \in \mathbb{R}$, and assume that there is a Kähler class $\Omega = \Omega_{\mu,\beta}$ such that

$$\mu \Omega + (1 - \beta) c_1(L_D) = c_1(M). \quad \text{(4)}$$

Here, $L_D$ is the line bundle associated to $D$. Thus, $c_1(M) - (1 - \beta) c_1(L_D)$ is a positive or negative class if $\mu > 0$ or $\mu < 0$. If $\mu = 0$, $\Omega$ is an arbitrary Kähler class.

Let $g$ be any Kähler metric which is smooth (or of some fixed finite regularity) on $M \setminus D$. We shall say that $g$ is a Kähler edge metric with angle $2\pi \beta$ if, in any local holomorphic coordinate system near $D$ where $D = \{z_1 = 0\}$, and $z_1 = \rho e^{\sqrt{-1} \theta}$,

$$g_{\overline{i}i} = F \rho^{2\beta-2}, \quad g_{ij} = g_{\overline{i}i} = O(\rho^{\beta-1+\eta'}), \quad \text{and all other } g_{ij} = O(1), \quad \text{(5)}$$

for some $\eta' > 0$, where $F$ is a bounded function which is at least continuous at $D$ (and which will have some specified regularity). If this is the case, we say that $g$ is asymptotically equivalent
to $g_\beta$, and that its associated K"ahler form $\omega$ (which by abuse of terminology we also call a metric sometimes) is asymptotically equivalent to $\omega_\beta$. There are slightly weaker hypotheses under which it is reasonable to say that $g$ has angle $2\pi\beta$ at $D$, but the definition we have given here is sufficient for our purposes.

**Definition 2.1.** With all notation as above, a K"ahler current $\omega$, with associated singular K"ahler metric $g$, is called a K"ahler--Einstein edge current, respectively metric, with angle $2\pi\beta$ and curvature $\mu$ if $\omega$ and $g$ are asymptotically equivalent to $\omega_\beta$ and $g_\beta$, and if

$$\text{Ric} \omega - (1 - \beta)[D] = \mu \omega,$$

where $[D]$ is the current associated to integration along $D$.

In this section we present some preliminary facts about the geometry and analysis of the class of K"ahler edge metrics. We first review some different coordinate charts near the edge $D$ used extensively below. Many calculations in this article are most easily done in a singular real coordinate chart, although when the complex structure is particularly relevant to a calculation, we use certain adapted complex coordinate charts. While all of this is quite elementary, there are some identifications that can be confusing, so it is helpful to make all of this very explicit.

We calculate the curvature tensor for any one such metric $g$, assuming it is sufficiently regular. We then introduce the relevant class of K"ahler edge potentials and describe the continuity method that will be used for the existence theory. As we recall, this particular continuity method is closely related to the Ricci iteration, that, naturally, we also treat simultaneously in this article. We conclude the section with a fairly lengthy description of the various function spaces that will be used later. Rather than a purely technical matter, this discussion gets to the heart of some of the more important analytic and geometric issues that must be faced here. There are two rather different choices of Hölder spaces; one is naturally associated to this class of K"ahler edge metrics and was employed, albeit in a slightly different guise, by Donaldson [19], while the other, from [36], is well adapted to this edge geometry because of its naturality under dilations and has been used in many other analytic and geometric problems where edges appear.

### 2.1 Coordinate systems

As above, fix local complex coordinates $(z_1, \ldots, z_n) = (z_1, Z)$ with $D = \{z_1 = 0\}$ locally. There are two other coordinate systems which are quite useful for certain purposes. The first is a singular holomorphic coordinate chart, where we replace $z_1$ by $\zeta = z_1^\beta / \beta$. Of course, $\zeta$ is multi-valued, but we can work locally in the logarithmic Riemann surface which uniformizes this variable. Thus if $z_1 = \rho e^{\sqrt{-1}\theta}$, then $\zeta = \rho e^{\sqrt{-1}\beta \theta}$, where $r = \rho^\beta / \beta$ and $\bar{\theta} = \beta \theta$. The second is the real cylindrical coordinate system $(r, \theta, y)$ around $D$, where $r = |\zeta|$ as above, $\theta$ is the argument of $z_1$, and $(y_1, \ldots, y_{2n-2}) = (\text{Re} Z, \text{Im} Z)$. Note that $re^{\sqrt{-1}\beta \theta} = z_1^\beta / \beta$. We use either $(z_1, Z)$ or $(\zeta, Z)$ in situations where the formalism of complex analysis is useful, and $(r, \theta, y)$ elsewhere. For later purposes, note that

$$d\zeta = z_1^{\beta - 1} dz_1 \iff dz_1 = (\beta \zeta)^{\beta - 1} d\zeta, \quad \frac{\partial \zeta}{\partial z_1} = z_1^{\beta - 1} \iff \frac{\partial z_1}{\partial \zeta} = (\beta \zeta)^{\beta - 1}.$$  

(7)

One big advantage of either of these other coordinate systems is that they make the model metric $g_\beta$ appear less singular. Indeed,

$$g_\beta = |d\zeta|^2 + |dZ|^2 = dr^2 + \beta^2 r^2 d\theta^2 + |dy|^2.$$  

(8)
In either case, one may regard the coordinate change as encoding the singularity of the metric via a singular coordinate system. This is only possible for edges of real codimension two, and there are many places, both in [19] and here, where we take advantage of this special situation. For edges of higher codimension, one cannot conceal the singular geometry so easily, see [36]. The expression for \( g_\beta \) in cylindrical coordinates makes clear that for any \( \beta, \beta' \), we have \( C_1 g_\beta \leq g_{\beta'} \leq C_2 g_\beta \); the corresponding inequality in the original \( z \) coordinates must be stated slightly differently, as \( C_1 g_\beta \leq \Phi^* g_{\beta'} \leq C_2 g_\beta \), where \( \Phi(z_1, \ldots, z_n) = (z_1^{\beta'/\beta}, z_2, \ldots, z_n) \).

We now compute the complex derivatives in these coordinates. We have

\[
\partial_{z^1} = \frac{1}{2} e^{-\sqrt{-1} \theta} (\partial_{\rho} - \sqrt{-1} \rho \partial_{\theta}) = \frac{1}{2} e^{-\sqrt{-1} \theta} (\beta r)^{1-\frac{1}{\beta}} (\partial_{r} - \sqrt{-1} \frac{1}{\beta r} \partial_{\theta}),
\]

and then

\[
\partial^2_{z^1 \bar{z}^1} = (\beta r)^{2-\frac{2}{\beta}} \left( \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{\beta^2 r^2} \partial_\theta^2 \right).
\]

The other mixed complex partials \( \partial^2_{z^i \bar{z}^j}, \partial^2_{z^i z^j} \) are compositions of the operators in [9] and their conjugates and certain combinations of the \( \partial_{\mu} \). From this we obtain that

\[
\Delta_{g_\beta} u = \sum_{i,j=1}^n (g_\beta)^{ij} u_{ij} = \left( \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{\beta^2 r^2} \partial_\theta^2 + \Delta_y \right) u,
\]

since \( (g_\beta)^{11} = r^{2-2\beta} = (\beta r)^{\frac{2}{\beta}-2} \) and \( (g_\beta)^{ij}, (g_\beta)^{\bar{i}\bar{j}} = 0 \) and all other \( (g_\beta)^{ij} = \delta^{ij} \).

As already described, we shall work with the class of Kähler metrics \( g \) that satisfy condition [5], and which we call asymptotically equivalent to \( g_\beta \). If \( g \) is of this type, then

\[
g^{11} = F^{-1} r^{2-2\beta}, \quad g^{1j}, g^{\bar{i}\bar{j}} = O(\rho^{n+1-\beta}), \quad \text{and all other} \ g^{ij} = O(1)
\]

for some \( \eta' > 0 \), hence

\[
\Delta_g = F^{-1} \left( \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{\beta^2 r^2} \partial_\theta^2 + \sum_{r,s=1}^{2n-2} c_{rs}(r, \theta, y) \partial_{gr,y_s}^2 + E \right),
\]

where

\[
E := r^{n-2} \sum_{i+j+\mu \leq 2} a_{ij\mu}(r, \theta, y) (r \partial_r)^i (\partial_\theta)^j (r \partial_\theta)^\mu.
\]

Here \( \eta > 0 \) is determined from \( \eta' \) and \( \beta \), and all coefficients have some specified regularity down to \( r = 0 \). In particular, the coefficient matrix \( (c_{rs}) \) is positive definite, with \( c_{rs}(0, \theta, y) \) independent of \( \theta \), and the coefficients \( a_{ij\mu} \) are bounded as \( r \to 0 \). Thus there are no cross-terms to leading order, and the 11 part of this operator is ‘standard’ once we multiply the entire operator by \( F \).

One way that this asymptotic structure will be used is as follows. Fundamental to this work is the role of the family of dilations \( S_\lambda : (r, \theta, y) \mapsto (\lambda r, \theta, \lambda y) \) centered at some point \( p \in D \) corresponding to \( y = 0 \). If we push forward this operator by \( S_\lambda \), which has the effect of expanding a very small neighbourhood of \( p \), then the principal part scales approximately like \( \lambda^2 \) while \( E \) scales like \( \lambda^{2-\eta} \). Hence, after a linear change of the \( y \) coordinates,

\[
A \lambda^{-2}(S_\lambda)_* \Delta_g \longrightarrow \Delta_{g_\beta} \quad \text{as} \quad \lambda \to \infty
\]
where \( A = F(p) \). In particular, \( E \) scales away completely.

One important comment is that if the derivatives \( u_{ij} \) are all bounded, and if \( g \) satisfies these asymptotic conditions, then so does \( \tilde{g} \), where \( \tilde{g}_{ij} = g_{ij} + u_{ij} \).

A key point in the treatment below, exploited by Donaldson [19], is that for any Kähler metric \( g \), \( \Delta_g \) only involves combinations of the following second order operators:

\[
\begin{align*}
P_{11} &= (\partial^2_i + \frac{1}{r} \partial_r + \frac{1}{\beta r^2} \partial^2_{\theta}), \\
P_{1\bar{\ell}} &= (\partial_r - \frac{\sqrt{-1}}{\beta r} \partial_{\theta}) \partial_{\bar{\ell}}, \\
P_{\ell1} &= (\partial_r + \frac{\sqrt{-1}}{\beta r} \partial_{\theta}) \partial_{\bar{\ell}}, \quad \ell = 2, \ldots, n, \text{ and} \\
P_{\ell\bar{k}} &= \partial^2_{z_{\bar{k}z_{\bar{k}}}}, \quad \ell, \bar{k} = 2, \ldots, n.
\end{align*}
\]

The precise combinations of the derivatives with respect to \( z_2, \ldots, z_n \) into their \((1, 0)\) and \((0, 1)\) parts is mostly irrelevant. For later reference, we therefore set

\[
\begin{align*}
\mathcal{Q} &= \{ Q_{i} \}_{i \in \mathcal{I}} = \{ \partial_r, r^{-1} \partial_{\theta}, \partial_{y_{k}}, \partial_{y_{k}y_{k}}, \partial_{\bar{r}} \partial_{y_{k}}, \partial_{r} \partial_{\theta}, \partial_{\bar{r}} \partial_{\theta}, \partial_{\bar{r}} \partial_{y_{k}} \}, \\
\mathcal{Q}^* &= \{ Q^*_{i} \}_{i \in \mathcal{I}^*} = \mathcal{Q} \cup \{ \partial^2_{\theta}, (1/r) \partial_r, (1/r^2) \partial^2_{\theta} \}.
\end{align*}
\]

As a final note, let us record the form of the complex Monge–Ampère operator in these coordinates, for any Kähler metric which satisfies the decay assumptions above. We have

\[
(\omega + \sqrt{-1} \partial \bar{\partial} u)^n / \omega^n = \frac{\det(g_{ij} + \sqrt{-1} u_{ij})}{\det g_{ij}} = \det(\delta_{ij} + \sqrt{-1} u_{ij}^i),
\]

where \( u_{ij}^i = u_{ik} g^{jk} \). Using the calculations above, we have

\[
\begin{align*}
u_1^1 &= F^{-1} P_{11} u + O(r^n) u_{11}, \\
u_1^j &= e^{-\sqrt{-1} \theta (\beta r)^{1/\beta - 1}} g^{jk} P_{1k} u + O(r^{n+1/\beta - 1}) P_{11} u, \\
u_i^1 &= F^{-1} e^{-\sqrt{-1} \theta (\beta r)^{1/\beta - 1}} P_{11} u + O(r^{n+1/\beta - 1}) u_{ij}, \\
u_i^j &= g^{jk} P_{1k} u + O(r^n) u_{1j}.
\end{align*}
\]

Hence if we multiply every column but the first in \((\delta_{ij} + \sqrt{-1} u_{ij}^i)\) by \( e^{\sqrt{-1} \theta (\beta r)^{1/\beta - 1}} \) and every row but the first by \( e^{-\sqrt{-1} \theta (\beta r)^{1/\beta - 1}} \), then the determinant remains the same, and we have shown that

\[
\begin{align*}
\det(g_{ij} + \sqrt{-1} u_{ij}) / \det g_{ij} &= \det \left( 1 + F^{-1} P_{11} u \quad F^{-1} P_{12} u \quad \ldots \quad F^{-1} P_{1n} u \right) + R, \\
&\quad \vdots \\
&\quad g^{nk} P_{1k} u \quad \ldots \quad \ldots \quad + g^{nf} P_{1n} u.
\end{align*}
\]

where \( R = r^n R_0(u_{pq}) \), with \( R_0 \) polynomial in its entries.

### 2.2 Kähler edge potentials

Fix a smooth Kähler form \( \omega_0 \) with \( [\omega_0] \in \Omega = \Omega_{\mu, \beta} \). Consider the space of all Kähler potentials relative to \( \omega_0 \), asymptotically equivalent to the model metric,

\[
\mathcal{H}_{\omega_0} := \{ \varphi \in C^\infty(M \setminus D) \cap C^0(M) : \omega_{\varphi} := \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi > 0 \text{ on } M \quad \text{and } \omega_{\varphi} \text{ asymptotically equivalent to } \omega_{\beta} \}. \quad (18)
\]
Lemma 2.2. Let $\beta \in (0,1]$. Then $\mathcal{H}_{\omega_0}$ is non-empty.

Proof. Let $h$ be a smooth Hermitian metric on $L_D$. We claim that for $c > 0$ sufficiently small,

$$\phi_0 := c|s|^{2\beta}_h = c(|s|^{2\beta}_h)^{2\beta} \in \mathcal{H}_{\omega_0}. \quad (19)$$

To prove this, it suffices to consider $p \in M \setminus D$ near $D$. Use a local holomorphic frame $e$ for $L_D$ and local holomorphic coordinates $\{z_i\}_{i=1}^n$ valid in a neighborhood of $p$, such that $s = z_1 e$, so that locally $D$ is cut out by $z_1$. Let $a := |e|^2_h$, and set $H := a^\beta$, so $|s|^{2\beta}_h = H|z_1|^{2\beta}$. Note that $H$ is smooth and positive. Then

$$\sqrt{-1}\partial\bar{\partial}|s|^{2\beta}_h = \beta^2 H|z_1|^{2\beta-2}\sqrt{-1}d\zeta \wedge d\bar{\zeta} + 2\beta \text{Re}(|z_1|^{2\beta}z_1^{-1}\sqrt{-1}d\zeta \wedge \partial H) + |z_1|^{2\beta}\sqrt{-1}\partial\bar{\partial}H. \quad (20)$$

For $c > 0$ small, the form $\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi_0$ is positive definite and satisfies the conditions of (21), hence is asymptotically equivalent to $g_\beta$. \hfill \Box

It is useful to record the form of $\omega_{\phi_0}$ in the $(\zeta, Z)$ coordinates as well. First note that if $\psi_0$ is a Kähler potential for $\omega_0$, then using (4),

$$\sqrt{-1}\partial\bar{\partial}\psi_0 = (\psi_0)_{z_1z_1}|\beta\zeta^{\frac{1}{\beta}-2}\sqrt{-1}d\zeta \wedge d\bar{\zeta} + 2\text{Re}((\psi_0)_{z_1z_1}\beta\zeta^{\frac{1}{\beta}-1}\sqrt{-1}d\zeta \wedge dz) + (\psi_0)_{z_1z_1}\zeta \sqrt{-1}dz \wedge \bar{dz}. \quad (21)$$

Next, $|s|^{2\beta}_h = \beta^2 H|\zeta|^2$, hence

$$\sqrt{-1}\partial\bar{\partial}(c|s|^{2\beta}_h) = c\beta^2(\sqrt{-1}Hd\zeta \wedge d\bar{\zeta} + 2\text{Re}(\zeta \sqrt{-1}d\zeta \wedge \partial H) + |\zeta|^2\sqrt{-1}\partial\bar{\partial}H). \quad (22)$$

From these two expressions, it is clear once again that $\phi_0 \in \mathcal{H}_{\omega_0}$ when $c$ is sufficiently small. Putting these expressions together shows that $\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi_0$ is locally equal to

$$\left(\beta\zeta^{\frac{1}{\beta}-2}(\psi_0)_{z_1z_1} + c\beta^2 H + c|\beta\zeta^{\frac{1}{\beta}} H_{z_1z_1} + 2c\beta^{\frac{1}{\beta}+1}\text{Re}(\zeta^{\frac{1}{\beta}}H)\right)\sqrt{-1}d\zeta \wedge d\bar{\zeta} + 2\text{Re}\left((\beta\zeta^{\frac{1}{\beta}-1}(\psi_0)_{z_1z_1} + c\beta^2 \zeta H_{z_1z_1} + c\beta^{\frac{1}{\beta}+1}\zeta \zeta H_{z_1z_1}\right)\sqrt{-1}d\zeta \wedge dz \wedge \bar{dz} + \sqrt{-1}\partial\bar{\partial}(\zeta H) \quad (23)$$

The reason for writing the derivatives of $\psi_0$ and $H$ with respect to $z_1$ rather than $\zeta$ is because we know that both of these functions are smooth in the original $z$ coordinates, and hence so are its derivatives with respect to $z$.

We now use this expression to deduce some properties of the curvature tensor of $g$. This turns out to be simple in this singular holomorphic coordinate system. The coefficients of the $(0,4)$ curvature tensor are given by

$$R_{ijk\bar{l}} = -g_{ij,k\bar{l}} + g_{jk}^{s\bar{k}} g_{i\bar{l},k} g_{s\bar{j},\bar{l}}, \quad (24)$$

where the indices after a comma indicate differentiation with respect to a variable. In the following, contrary to previous notation, we temporarily use the subscripts 1 and $\bar{1}$ to denote components of the metric or derivatives with respect to $\zeta$ and $\zeta$, not $z_1$ or $\bar{z}_1$. \hfill \Box
Lemma 2.3. The curvature tensor of $\omega = \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi_0$ is uniformly bounded on $M \setminus D$ provided $\beta \in (0, \frac{1}{2}]$.

Proof. Since in the $(\zeta, Z)$ coordinates, $cI < |g_{ij}| < CI$, it suffices to show that $|R_{ijkl}| < C$. From (23),

$$g_{11,1} = O(|\zeta|^{\frac{5}{2} - 3}), \quad g_{ij,1} = O(|\zeta|^{\frac{1}{2} - 2}), \quad g_{ij,1} = O(|\zeta|^{\frac{1}{2} - 1}),$$

$$g_{11,k} = O(1), \quad g_{ij,k} = O(|\zeta| + |\zeta|^{\frac{1}{2} - 1}), \quad g_{ij,k} = O(1).$$

Similarly, $|g_{ij,kl}| \leq C (1 + |\zeta|^{\frac{1}{2} - 2} + |\zeta|^{\frac{5}{2} - 4})$.

As indicated by Donaldson, there seem to be genuine cohomological obstructions to finding reference edge metrics with bounded curvature when $\beta > \frac{1}{2}$. Nevertheless, in Proposition A.1 it is shown that the bisectional curvature of $\omega$ is bounded from above on $M \setminus D$ provided $\beta \leq 1$.

2.3 The twisted Ricci potential

From now on (except when otherwise stated) we denote

$$\omega := \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi_0 \in \mathcal{H}_{\omega_0},$$

(25)

with $\phi_0$ given by (19). In the remainder of this article, we refer to $\omega$ as the reference metric. Define $f_\omega$ by

$$\sqrt{-1} \partial \bar{\partial} f_\omega = \text{Ric} \omega - (1 - \beta)[D] - \mu \omega,$$

(26)

where $[D]$ denotes the current of integration along $D$, and with the normalization

$$\frac{1}{V} \int_M e^{f_\omega} \omega^n = 1, \quad \text{where} \quad V := \int_M \omega^n.$$

(27)

We call this the twisted Ricci potential; this terminology refers to the fact that the adjoint bundle $K_M + (1 - \beta) L_D$ takes the place of the canonical bundle $K_M$. Alternatively, one can also think of $\text{Ric} \omega - (1 - \beta)[D]$ as a kind of Bakry-Émery Ricci tensor. By (20),

$$\omega^n/(n! \sqrt{-1} dz \wedge d\bar{z}) = \det \left[ \frac{\partial^2 (\psi_0 + \phi_0)}{\partial z^i \partial \bar{z}^j} \right] = \sum_{k=0}^{n} f_{0k} |z_1|^{2k\beta} + \sum_{k=1}^{n} (f_{1k} + f_{2k} z_1 + f_{3k} \bar{z}_1)|z_1|^{2k\beta - 2},$$

(28)

where all $f_{jk}$ are smooth functions of $(z_1, \ldots, z_n)$, and $dz := dz_1 \wedge \cdots \wedge dz_n$.

2.4 The continuity method for the twisted Kähler–Einstein equation

The existence of Kähler–Einstein metrics asymptotically equivalent to $g_\beta$ is governed by the Monge–Ampère equation

$$\omega_\phi^n = e^{f_\omega - \mu \phi} \omega^n.$$  

(29)

We seek a solution $\phi \in \mathcal{H}_{\omega_0}$, and shall do so using the classical continuity method. We consider a continuity path in the space of metrics $\mathcal{H}_\omega$ (with some specified regularity) obtained
essentially by concatenating (and extending) two previously studied paths, one by Aubin [2] in the positive case and the other by Tian–Yau [51] in the negative case. The path is given by

$$\omega^n_\varphi = \omega^n e^{f_\omega - s\varphi}, \quad s \in (-\infty, \mu], \quad (30)$$

where $$\varphi(-\infty) = 0$$, and $$\omega_\varphi(-\infty) = \omega$$. By a result of Wu [55], for $$s \to -\infty$$, there exists a solution $$\varphi(s)$$ of the form $$s^{-1}f_\omega + o(1/s)$$. A key feature of this continuity path is that

$$\text{Ric} \omega_\varphi = s\omega_\varphi + (\mu - s)\omega + (1 - \beta)[D], \quad (31)$$

which implies the very useful property that for all solutions $$\varphi(s)$$ along this path, the Ricci curvature is bounded below on $$M \setminus D$$, i.e., $$\text{Ric} \omega_\varphi > s\omega_\varphi$$. As we explain in §6.3 another important property is that the Mabuchi K-energy is monotone along this path.

Much of the remainder of this article is directed toward analyzing this family of Monge–Ampère equations: §3 describes the linear analysis needed to understand the openness part of the continuity argument as well as the regularity theory; §4 uses this linear analysis to prove that solutions are automatically ‘smooth’ at $$D$$, by which we mean that they are polyhomogeneous (see below); the a priori estimates needed to obtain the closedness of the continuity argument are derived in the remaining sections of the article, and the proof is concluded in §9.

We will pursue a somewhat parallel development of this proof using two different scales of Hölder spaces since one goal of this article is to illustrate the relative merits of each of these classes of function spaces, with future applications in mind. Certain aspects of the proof work much more easily in one rather than the other, but we give a complete proof of the existence in either framework. The proof of higher regularity, which shows that these two approaches are ultimately equivalent, is very closely tied, however, to only one of these scales of spaces.

**Remark 2.4.** The continuity path (30) has several useful properties, some already noted above, which are necessary for the proof of Theorem 2 when $$\beta > 1/2$$. However, we also consider the two-parameter family of equations

$$\omega^n_\varphi = e^{tf_\omega + ct - s\varphi} \omega^n, \quad \int_M e^{tf_\omega + ct} \omega^n = V, \quad c_t \in \mathbb{R}, \quad (s,t) \in A, \quad (32)$$

where $$A := (-\infty, 0] \times [0, 1] \cup (0, \mu] \times \{1\}$$. This incorporates the continuity path $$s = \mu, 0 \leq t \leq 1$$, which is the common one in the literature. The analysis required to study this two-parameter family requires little extra effort, and has been included since it may be useful elsewhere.

It provides an opportunity to use the Chern–Lu inequality in its full generality (see §7). In addition, we have already noted that one cannot obtain openness for (30) at $$s = -\infty$$ directly, but must produce a solution for $$s$$ (very) negative by some other method. Wu [55] accomplishes this by a perturbation argument; this augmented continuity argument gives yet another means to do this, but works only when $$\beta \leq 1/2$$. We refer to §9 for more details.

We emphasize that our proof of Theorem 2 in full generality requires the path (30) (i.e., fixing $$t = 1$$).

### 2.5 The twisted Ricci iteration

The idea of using the particular continuity method (30) to prove the existence of Kähler–Einstein metrics for all $$\mu$$ (independently of sign) was suggested in [42, p. 1533]. As explained there and recalled below, this path arises from discretizing the Ricci flow via the Ricci iteration. Naturally then, after treating this continuity path we will be in a position to prove smooth edge convergence of the (twisted) Ricci iteration to the Kähler–Einstein edge metric.
One Kähler–Ricci flow in our setting is
\[ \frac{\partial \omega(t)}{\partial t} = -\text{Ric} \omega(t) + (1 - \beta)[D] + \mu \omega(t), \quad \omega(0) = \omega \in \mathcal{H}_\omega. \]

Let \( \tau \in (0, \infty) \). The (time \( \tau \)) Ricci iteration, introduced in [42], is the sequence \( \{ \omega_{k\tau} \}_{k \in \mathbb{N}} \subset \mathcal{H}_\omega \), satisfying the equations
\[ \omega_{k\tau} = \omega_{(k-1)\tau} + \tau \mu \omega_{k\tau} - \tau \text{Ric} \omega_{k\tau} + \tau (1 - \beta)[D], \quad \omega_{0\tau} = \omega, \]
for each \( k \in \mathbb{N} \) for which a solution exists in \( \mathcal{H}_\omega \). Equivalently, let \( \omega_{k\tau} = \omega_{\psi_{k\tau}} \), with \( \psi_{k\tau} = \sum_{l=1}^{k} \varphi_{l\tau} \). Then,
\[ \omega^n_{\psi_{k\tau}} = \omega^n e^{f_{\omega} - \mu \psi_{k\tau} + \frac{1}{\tau} \varphi_{k\tau}}. \]

Since the first step is simply \( \omega^n_{\psi_{\tau}} = \omega^n e^{f_{\omega} + (1 - \mu) \varphi_{\tau}} \), the Ricci iteration exists (uniquely) once a solution exists (uniquely) for (30) for \( s = \mu - \frac{1}{\tau} \). Thus, much like for the Ricci flow, a key point is to prove uniformity of the a priori estimates as \( k \) tends to infinity. The convergence to the Kähler–Einstein metric then follows essentially by the monotonicity of the twisted K-energy if the Kähler–Einstein metric is unique.

As noted above, our choice of the particular continuity path (30) allows us to treat the continuity method and the Ricci iteration in a unified manner. When \( \mu \leq 0 \), our estimates for (30), the arguments of [42], and the higher regularity developed in [41] imply the uniqueness, existence and edge smooth convergence of the iteration, for all \( \tau \). When \( \mu > 0 \) the uniqueness of the (twisted) Ricci iteration was proven recently by Berndtsson [9], and it follows from his result that whenever the twisted K-energy \( E_0^\beta \) is proper then also the Kähler–Einstein edge metric must be unique. Given this, our analysis in this article and in [42] then immediately implies edge smooth convergence of the iteration for large enough times steps, more specifically, provided \( \tau > 1/\mu \) and \( E_0^\beta \) is proper, or else provided \( \tau > 1/\alpha_{\Omega,\omega} \), and \( \alpha_{\Omega,\omega} > \mu \), where \( \alpha_{\Omega,\omega} \) is Tian’s invariant defined in §6.3 (note that by Lemma 6.9 this assumption implies \( E_0^\beta \) is proper). As pointed out to us by Berman, given the results of [42], the remaining cases follow immediately in the same manner by using one additional very useful pluripotential estimate contained in [8, Lemma 6.4] and stated explicitly in [6], and recalled in Lemma 6.8 below. As already observed in [7] this estimate gives in an elegant manner a uniform estimate on the oscillation of solutions along the iteration, and is used in [7] to prove convergence of the twisted Ricci iteration and flow in very general singular settings, smoothly away from the singular set, and global \( C^0 \) convergence on the level of potentials. Our result below, in the case \( \mu > 0 \), is complementary to theirs since it shows how to use their uniform estimate and our analysis to obtain smooth convergence near the edge. We thank Berman for his encouragement to include this result here, prior to the appearance of [7].

To summarize, we have the following statement.

**Theorem 2.5.** Under the assumptions of Theorem 2, the Ricci iteration (33) exists uniquely and converges in \( D_0^{0,\gamma} \cap A^0 \) to a unique Kähler–Einstein edge metric in \( \mathcal{H}_\omega \).

These function spaces encode the strongest possible convergence for this problem, and are defined directly below.

### 2.6 Function spaces

To conclude this section of preliminary material, we review the various function spaces used below. These are the ‘wedge and edge’ Hölder spaces, as well as the spaces of conormal and
polyhomogeneous functions necessary for our treatment of the the higher regularity theory. The wedge Hölder spaces are the ones used in [26, 19], and are naturally associated to the incomplete edge geometry. The edge spaces, introduced in [36], are also naturally associated to this geometry and have some particularly favorable properties stemming from their invariance under dilations. Using this, certain parts of the proofs below become quite simple. The wedge Hölder spaces, on the other hand, are closer to standard Hölder spaces, and indeed reduce to them when $\beta = 1$. They impose stronger regularity conditions. Since we use both types of spaces here, we describe many of the proofs below with respect in both settings in hopes of giving the reader a better sense of their relative advantages.

Before giving any of the formal definitions below, let us recall that a Hölder space is naturally associated to a distance function $d$ via the Hölder seminorm

$$[u]_{d,a,\gamma} := \sup_{p \neq p', d(p, p') \leq 1} \frac{|u(p) - u(p')|}{d(p, p')^\gamma}.$$  

We only need to take the supremum over points with distance at most 1 apart, since if $d(p, p') > 1$, then this quotient is bounded by $2\sup |u|$. The two different spaces below differ simply through the different choices of distance function $d$.

### 2.6.1 Wedge Hölder spaces

First consider the distance function $d_1$ associated to the model metric $g_\beta$; note that it is clearly equivalent to replace the actual $g_\beta$ distance function with any other function on $M \times M$ which is uniformly equivalent, and it is simplest to use the one defined in the coordinates $(r, \theta, y)$ by

$$d_1((r, \theta, y), (r', \theta', y')) = \sqrt{|r - r'|^2 + (r + r')^2|\theta - \theta'|^2 + |y - y'|^2}.$$  

Note that the angle parameter $\beta$ does not appear explicitly in this formula, but if we were to have included it, there would be a factor of $\beta^2$ before $(r + r')^2|\theta - \theta'|^2$. This changes $d_1$ at most by a factor, so we may as well omit it altogether.

Now define the wedge Hölder space $C_w^{0,\gamma} \equiv C_w^{0,\gamma}(M)$ to consist of all functions $u$ on $M \setminus D$ for which

$$\|u\|_{w;0,\gamma} := \sup |u| + [u]_{d_1,a,\gamma} < \infty.$$  

The spaces with higher regularity are defined using differentiations with respect to unit length vector fields with respect to $g_\beta$; these vector fields are spanned by $\partial_r$, $r^{-1}\partial_\theta$ and $\partial_y$.

Thus

$$C_w^{k,\gamma}(M) = \{ u : \partial_r^i (r^{-1}\partial_\theta)^j \partial_y^\mu u \in C_w^{k,\gamma}(M) \quad \forall \; i + j + |\mu| \leq k \}.$$  

There is one potentially confusing point about these spaces, especially if we compare this definition with the equivalent one given in [19]. As is evident from the definition above, the space $C_w^{0,\gamma}$ above does not depend on the cone angle parameter $\beta$ (at least so long as $\beta$ stays bounded away from 0 and $\infty$). However, suppose we consider the (apparently) fixed function $f = |z_1|^a = R^a$ for some $a > 0$ in terms of the original holomorphic coordinates. In terms of the cylindrical coordinates $(r, \theta, y)$, we have $f = cr^{a/\beta}$ for some constant $c$, and hence $f \in C_w^{0,\gamma}$ if and only if $a/\beta \geq \gamma$, i.e., $a \geq \beta \gamma$. Inequalities of this type appear in [19]. This seems inconsistent with the claim that the Hölder space is independent of $\beta$; the discrepancy between these statements is explained by observing that the singular coordinate change does depend on $\beta$, and while the function $R^a$ is independent of $\beta$, its composition with this coordinate change is not. Equivalently, if we pull back the function space $C_w^{0,\gamma}$ via this coordinate change, then we
get a varying family of function spaces on \( M \). We prefer, however, to think of \( M \setminus D \) as a fixed but singular geometric object, with smooth structure determined by the coordinates \((r, \theta, y)\), and with a single scale of naturally associated Hölder spaces.

### 2.6.2 Edge Hölder spaces

Now consider the distance function \( d_2 \) associated to the complete metric

\[
\hat{g}_\beta := r^{-2}g_\beta = \frac{dr^2 + |dy|^2}{r^2} + \beta^2 d\theta^2.
\]

As before, we can replace the distance \( d_2 \) by another metric uniformly equivalent to it, which we take as

\[
d_2((r, \theta, y), (r', \theta', y')) = (r + r')^{-1} \sqrt{|r - r'|^2 + (r + r')^2|\theta - \theta'|^2 + |y - y'|^2},
\]

where we only consider \( r, r' \leq C \). As before, we do not need to include any factors of \( \beta \).

The Hölder norm \( ||u||_{e;0,\gamma} \) is now defined using the seminorm associated to \( d_2 \). The higher Hölder norms are defined using unit-length vector fields with respect to \( \hat{g}_\beta \), which are spanned by \( \{r\partial_r, \partial_\theta, \partial_y\} \). The corresponding spaces of functions for which these norms are finite are denoted \( C^{k;\gamma}_e \equiv C^{k;\gamma}_e(M) \). We use the subscript \( e \) here to follow long-established notation [36].

The key property of this distance function is that it is invariant with respect to the scaling

\[
(r, \theta, y) \mapsto (\lambda r, \theta, \lambda y)
\]

for any \( \lambda > 0 \). The vector fields \( r\partial_r, \partial_\theta \) and \( r\partial_y \) are also invariant with respect to these dilations. This means that if \( u_{\lambda,0}(r, \theta, y) = u(\lambda^{-1}r, \theta, \lambda^{-1}y + y_0) \), then \( ||u_{\lambda,0}||_{e;\gamma} = ||u||_{e;\gamma} \).

(We assume, of course, that both \((r, \theta, y)\) and \((\lambda^{-1}r, \theta, \lambda^{-1}y + y_0)\) lie in the domain of \( u \).) One way to interpret this is as follows. Consider the annulus

\[
B_{\lambda,0} := \{ (r, \theta, y) : 0 < \lambda < r < 2\lambda, |y - y_0| < \lambda \},
\]

for \( \lambda \) small. The image of this annulus under translation by \( y_0 \) and dilation by \( \lambda^{-1} \) is the standard annulus \( B_{1,0} \). Hence if \( u \) is supported in \( B_{\lambda,0} \), then \( u_{\lambda,0} \) is defined in \( B_{1,0} \) and

\[
||u||_{e;\gamma} = ||u_{\lambda,0}||_{e;\gamma}.
\]

For any \( \nu \in \mathbb{R} \), we also define weighted edge Hölder spaces

\[
r^\nu C^{k;\gamma}_e(X) = \{ u = r^\nu v : v \in C^{k;\gamma}_e(X) \}.
\]

Although \( C^{0;\gamma}_e(X) \subset L^\infty(X) \), elements of \( C^{0;\gamma}_e(X) \) need not be continuous at \( r = 0 \); an easy example is the function \( \sin \log r \), which lies in \( C^{1;\gamma}_e \) for all \( k \). On the other hand, elements of \( r^\nu C^{0;\gamma}_e \) are continuous and vanish at \( D \) if \( \nu > 0 \).

### 2.6.3 Comparison between the wedge and edge Hölder spaces

We now comment on the differences between these spaces. Since \( r, r' \leq C \), we have \( d_1 \leq C^{-1}d_2 \), and hence

\[
||u||_{e;\gamma} \leq C^\gamma ||u||_{w;\gamma},
\]

or equivalently,

\[
C^{k;\gamma}_w \subset C^{k;\gamma}_e. \tag{34}
\]
In fact, elements in the wedge Hölder space enjoy significantly more regularity than those in the edge Hölder space. We have already commented that while elements of $C_e^{0,\gamma}$ are bounded, they need not be continuous up to $D$, whereas elements of $C_w^{0,\gamma}$ certainly are. Moreover, if $u \in C_w^{0,\gamma}$, then $u(0,\theta,y)$ is independent of $\theta$ and lies in $C^0,\gamma(D)$, while by contrast, if $u \in C_e^{0,\gamma}$, then the ‘tangential’ difference quotient $|u(r,\theta,y) - u(r,\theta,y')|/|y-y'|^\gamma$ can blow up like $r^{-\gamma}$.

Given these considerations, the utility of the edge Hölder spaces is not obvious. However, there are also some big advantages to using them. First of all note that if $\mu \in (0,1)$, the function $r^\mu$ lies in $C_w^{0,\gamma}$ only if $\mu \leq \gamma$, while $r^\mu \in C_e^{0,\gamma}$ for all $\gamma \in (0,1)$. Next, passing to the higher Hölder norms, the function $r^\mu$ never lies in $C_w^{k,\gamma}$ if $k \geq 1$, but since $(r\partial_r)^i r^\mu = \mu^i r^\mu$, we see that $r^\mu \in C_e^{k,\gamma}$ for all $k \geq 0$. In other words, the edge spaces more naturally accommodate noninteger exponents. This is a big advantage when dealing with the singular elliptic equations under consideration here because solutions of such equations typically involve noninteger powers of $r$, and it is quite reasonable to think of these solutions as being infinitely differentiable. Thus, while it is possible to use the wedge Hölder spaces for the existence and deformation theory of KE edge metrics, one is then forced to work with a restricted range of Hölder exponents, and is then prevented from easily studying the higher differentiability properties of solutions. We shall see below how these issues are handled easily with the edge Hölder spaces.

In the remainder of this article, whenever our discussion applies to both of these spaces, we refer to the ‘generic’ singular Hölder space $C_e^{s,\gamma}$, where

$$s \text{ equals either } w \text{ or } e.$$  

This $s$ should not be confused with the parameter $s$ along the continuity path $[30]$.

### 2.6.4 Hybrid Hölder spaces

Before proceeding, we also recall some intermediate spaces based on the edge geometry but which provide better control in the tangential ($\partial_y$) directions. As remarked above, if $|\mu| \leq \ell$, then

$$u \in C_e^{\ell,\gamma} \Rightarrow \partial_y^i u \in r^{-|\mu|}C_e^{\ell-|\mu|,\gamma}. $$

To remedy this, we introduce the following hybrid spaces:

For any $\nu \in \mathbb{R}$, $0 < \gamma < 1$ and $0 \leq \ell \leq \ell'$, define

$$r^\nu C_e^{k,\gamma,k'}(X) = \{ u = r^\nu v : (r\partial_r)^i \partial_y^j (r\partial_y)^{\mu} \partial_y^{\mu'} v \in C_e^{0,\gamma} \text{ for } i+j+|\mu|+|\mu'| \leq k, |\mu'| \leq k' \}. $$

In other words, we still only allow $k + \gamma$ derivatives, but up to $k'$ of them may be taken with respect to $\partial_y$ while all the rest must be taken with respect to $r\partial_r$, $\partial_\theta$ and $r\partial_y$.

### 2.6.5 Conormal and polyhomogeneous functions

The final set of spaces we define are the spaces of conormal and polyhomogeneous functions.

**Definition 2.6.** For any $\nu \in \mathbb{R}$, define

$$A^\nu(X) = \bigcap_{\ell \geq \nu' \geq 0} r^{\nu'} C_e^{\ell,\gamma,\ell'}(X).$$

This is the space of conormal functions (of weight $\nu$). Thus elements of $A^\nu$ are bounded by $r^{\nu'}$, as are all of their derivatives with respect to the vector fields $r\partial_r$, $\partial_\theta$ and $\partial_y$. 

14
Next, we say that \( u \in A^\nu(X) \) is polyhomogeneous, and write \( u \in A^\nu_{phg}(X) \), if it has an expansion of the form

\[
u \sim \sum_{j=0}^{\infty} \sum_{p=0}^{N_j} a_{jp}(\theta,y) r^{\sigma_j} (\log r)^p
\]

where the coefficients \( a_{jp} \) are all \( C^\infty \), and \( \{ \sigma_j \} \) is a discrete sequence of complex numbers such that \( \Re \sigma_j \to \infty \), with \( \Re \sigma_j \geq \nu \) for all \( j \) and \( N_j = 0 \) if \( \Re \sigma_j = \nu \). This expansion can be differentiated arbitrarily many times with the corresponding differentiated remainder. We say that \( u \) has a nonnegative index set if \( u \in A^0_{phg} \). Note finally that if \( u \in A^0_{phg} \), then \( u \) is bounded.

These spaces are the correct analogues of the spaces of infinitely differentiable functions in this context. Unlike in the smooth setting, however, we make a distinction between functions which are infinitely differential (conormal) and those which have “Taylor series” expansions (i.e. are polyhomogeneous). These function spaces accommodate behavior typical for solutions of degenerate elliptic edge problems, e.g., functions like \( r^{\sigma} (\log r)^p a(\theta,y) \) where \( a \) is smooth, \( p \) is a nonnegative integer and \( \sigma \in \mathbb{C} \).

### 3 Linear analysis

We now present the key facts about the linear elliptic theory needed to handle the existence, deformation and regularity theory for canonical edge metrics. We discuss this from two points of view, on the one hand reviewing the estimates obtained by Donaldson in the wedge H"older spaces, and also proving the analogous estimates in the edge H"older spaces as consequences of basic properties of edge pseudodifferential operators, as developed in [36].

Fix a K"ahler edge metric \( g \) on \( M \) with cone angle \( 2\pi \beta \) along the smooth divisor \( D \); we initially suppose that the metric \( g \) is polyhomogeneous along \( D \), though this will be relaxed later. As in the introduction, we consider the Laplacian \( \Delta_g \), or slightly more generally, any operator of the form \( L = \Delta_g + V \) where \( V \) is a polyhomogeneous function with nonnegative index set (so that \( V \) is bounded), or even just a bounded function with some given Hölder regularity.

Our goal here is to explain how the known polyhomogeneous structure of the Schwartz kernel of the inverse for the Friedrichs extension of \( L \) leads to the estimates we need later. This inverse is a pseudodifferential edge operator, and the theory in [36] yields rather detailed information about its Schwartz kernel. We now recall certain aspects of this theory and explain how it applies to the problem at hand. In a parallel discussion we also recall Donaldson’s estimates and explain the relationship between these two approaches.

#### 3.1 Edge structures and edge operators

As we have already indicated in the definitions of the Hölder spaces above, there is some advantage to considering the pair \((M,D)\) as a genuinely singular (stratified) object; the general notion of an edge structure is a good way to formulate this.

The general setting is as follows. Let \( X \) be a compact manifold with boundary, such that \( \partial X \) is the total space of a fibration, with fibre \( F \) and base \( Y \). For us, \( X \) is the real blowup of \( M \) around \( D \). This is a manifold with boundary, where \( \partial X \) is the unit normal sphere bundle \( \text{SND} \) of \( D \) in \( M \), and hence is the total space of an \( S^1 \) bundle over \( D \). More precisely, the real blowup \( X := [M;D] \) is the disjoint union \((M \setminus D) \sqcup \text{SND}\), endowed with the unique
smallest topological and differential structure so that the lifts of smooth functions on $M$ and polar coordinates around $D$ are smooth. This comes equipped with a smooth blowdown map $B : X \to M$.

There is a slight subtlety in the description of this smooth structure which we already encountered above, which concerns the fact that there is an incompatibility between the smooth structure provided by the holomorphic coordinates $(z_1, \ldots, z_n)$, where $D = \{z_1 = 0\}$ locally, in terms of which the metrics we consider have the simplest descriptions as Kähler metrics, and the smooth structure provided by the cylindrical coordinate system $(r, \theta, y)$ we defined earlier. Indeed, since $r = |z_1|^{\beta}/\beta$, we see that functions smooth with respect to $z$ are not necessarily smooth with respect to $(r, \theta, y)$ and vice versa, although there is an equivalence between these two smooth structures (provided by this coordinate transformation). This is closely analogous to a similar change of smooth structures that explains two descriptions of the asymptotic behaviour of the Bergman and Kähler–Einstein metrics near the boundary of a strictly pseudoconvex domain [20].

The best way to come to terms with this inconsistency is to realize that what is important is not the smooth structure on $X$ but rather the ‘polyhomogeneous structure’, i.e., the ring of polyhomogeneous functions, since this is invariant under this coordinate change. In any case, continuing on, $\partial X$ is defined by $\{\rho = 0\}$, and the $S^1$ fibres are defined by $\{y = \text{const.}\}$. Functions on $X$ are polyhomogeneous if they are polyhomogeneous either with respect to $(r, \theta, y)$ or else with respect to $\rho = |z_1|$, $\theta$, $y$.

To continue with our description of edge structures, define the space of smooth edge vector fields on $X$. This is the space $\mathcal{V}_e(X)$ consisting of all smooth vector fields on $X$ which are unconstrained in the interior, but which lie tangent to the fibres at the boundary. For $X = [M, D]$, in cylindrical coordinates, $\mathcal{V}_e(X)$ is generated by $r \partial_r$, $\partial_\theta$ and $r \partial_y$. The space of differential edge operators $\text{Diff}^e_e(X)$ is then the set of operators which can be written as locally finite sums of products of elements of $\mathcal{V}_e(X)$. Thus again for $X = [M, D]$, for any $m \geq 0$, the typical element of $\text{Diff}^m_e(X)$ has the form

$$A = \sum_{j+k+|\mu| \leq m} a_{jk\mu}(r, \theta, y)(r \partial_r)^j \partial_\theta^k (r \partial_y)^\mu. \quad (35)$$

We henceforth restrict attention solely to the case $X = [M; D]$ and $m = 2$, though the main linear results below hold in greater generality.

If $g$ is an incomplete edge metric on $M$ with cone angle $\beta$, then $\Delta_g + V$ can be written out as in (11), with a principal part and an error term $E$. The operator $A = r^2 L$ is an edge operator in the sense defined just above.

A differential edge operator $A$ is called elliptic if it is an ‘elliptic combination’ of elements of $\mathcal{V}_e(X)$, e.g., a sum of squares of a generating set of sections plus lower order terms. This is the case here and we refer to [36] for a formulation of ellipticity in this setting, and in particular a coordinate-invariant notion of an edge symbol.

### 3.2 Normal and indicial operators

If $A$ is an elliptic edge operator, its mapping properties are governed not only by its ellipticity, but by two model operators defined at any point of $D$. In the general theory these are called the indicial and normal operators, $I(A)$ and $N(A)$, respectively, and there is an invariant description for these too. Here we simply record that for $A = r^2 L$,

$$N(A) = (s \partial_s)^2 + \beta^{-2} \partial_\theta^2 + s^2 \Delta_w, \quad \text{and} \quad I(A) = (s \partial_s)^2 + \beta^{-2} \partial_\theta^2,$$

16
where \((s, w)\) are global affine coordinates on a half-space \(\mathbb{R}^+_s \times \mathbb{R}^{2n-2}_w\) and \(\theta \in S^1_{2\pi}\). Formally, \(N(A)\) is obtained by dropping the error term \(E\), freezing coordinates at a given point \(y_0 \in D\), and replacing the local coordinates \((r, y)\) by the global affine coordinates \((s, w)\); more invariantly, \(N(A)\) acts on functions on the inward-pointing normal bundle of the fiber of \(\partial X\) over \(y_0\). Similarly, \(I(A)\) is defined by dropping the terms in \(N(A)\) which have the property that they map any function \(s^a v(\theta, w)\) (with \(v\) smooth) to a function which vanishes faster than \(r^a\). The only term in the operator \(N(A)\) above which has this effect is \(r^2 \Delta_w\). In general, both \(N(A)\) and \(I(A)\) could depend on \(y_0\) (for example, if the cone angle varies along \(D\)), but here they do not. Note that

\[
N(A) = s^2 L_\beta \quad \text{where} \quad L_\beta = \partial_s^2 + \frac{1}{s} \partial_s + \frac{1}{\beta^2 s^2} \partial_\theta^2 + \Delta_w
\]

is the Laplacian of the flat model metric \(g_\beta\).

A number \(a \in \mathbb{C}\) is called an indicial root of \(A\) (and also of \(L\)) if there is some nontrivial \(\psi(\theta)\) such that \(I(A)s^a \psi(\theta) = 0\); for \(A = r^2 L\),

\[
I(A)s^a \psi(\theta) = (\beta^{-2} \partial_\theta^2 + a^2)\psi = 0
\]

\[
\iff \left\{ a \in \{j/\beta : j \in \mathbb{Z}\}, \quad \psi_j(\theta) = a_j \cos j \theta + b_j \sin j \theta, \ j \geq 1, \ \psi_0(\theta) = 1. \right. \]

Hence, 0 is a ‘double’ indicial root, which means that both \(I(A)(s^0)\) and \(I(A)(s^0 \log s) = 0\), which is special to the fact that \(D\) has codimension 2. It is important that \(\theta\) lies on a compact manifold (namely, \(S^1\)), since these indicial roots are just the square roots of the eigenvalues of \(\beta^{-2} \partial_\theta^2\). Note also that for any \(a\) and \(\psi(\theta, y) \in C^\infty\), we always have \(A(r^a \psi(\theta, y)) = O(r^a)\), and \(a\) is an indicial root if and only if \(A(r^a \psi_j) = O(r^{a+1})\) and \(\psi = \psi_j\) (multiplied by an arbitrary function of \(y\)).

### 3.3 Mapping properties and the Friedrichs domain

We next describe the basic mapping properties of \(L\) on weighted Hölder spaces; they follow from [36, Corollary 6.4] applied to the operator \(A = r^2 L\).

**Proposition 3.1.** The mapping

\[
L : r^\nu C^\ell_{c, \gamma} \rightarrow r^{\nu - 2} C^\ell_{c, \gamma}
\]

has closed range if and only if \(\nu \notin \frac{1}{2j+1}, j \in \mathbb{Z}\).

The weights corresponding to indicial roots are excluded because, as it is not difficult to see, at these values this mapping does not have closed range. The emphasis in [36] is on the Fredholm and semi-Fredholm theory (in fact, [3.1] is never Fredholm in our case: when \(\nu < 0\), the nullspace is infinite dimensional, while if \(\nu > 0\) is nonindicial, the cokernel is closed but infinite dimensional). This lack of a ‘Fredholm range’ is again special to an edge of real codimension two.

It is more useful, however, to consider \(L\) as an unbounded operator acting on a single space, rather than as an operator between two differently weighted spaces. The main point is that (assuming \(V\) is real), \(L\) is a symmetric operator on the core domain \(C^\infty_0(M \setminus D)\). It is well-known that if the codimension of \(D\) is at least 4, then there is a unique closed extension of this symmetric operator, and it is self-adjoint. In our case, however, there are many self-adjoint
extensions, and even more closed extensions. These are in bijective correspondence with the
Lagrangian (with respect to a certain natural symplectic structure) and closed subspaces of the
quotient $\mathcal{D}_{\text{max}}(L)/\mathcal{D}_{\text{min}}(L)$, respectively, where the maximal and minimal domains for $L$
are defined by
\[
\mathcal{D}_{\text{max}}(L) = \{ u \in L^2(M) : Lu \in L^2(M) \} \\
\mathcal{D}_{\text{min}}(L) = \{ u \in L^2(M) : \exists u_j \in C_0^\infty(M \setminus D) \text{ s.t. } u_j \to u, \ Lu_j \to f \text{ in } L^2 \}
\]
In the definition of $\mathcal{D}_{\text{max}}$, $Lu$ is defined distributionally, but only on the smooth part of $M$. The
closed operators $(L, \mathcal{D}_{\text{max}})$ and $(L, \mathcal{D}_{\text{min}})$ are Hilbert-space adjoints. The canonical self-adjoint
Friedrichs extension is defined in the customary way using the coercive quadratic form
\[
\langle u, v \rangle = \int_M (\nabla u \cdot \nabla v - Vu) \ dV_g.
\]
In our setting we can identify the Friedrichs domain $\mathcal{D}_{\text{Fr}}(L)$, i.e., the domain of this Friedrichs
extension, quite explicitly. One can show, see [36, §7], that any $u \in \mathcal{D}_{\text{Fr}}(L)$ has a ‘weak’ partial
expansion $u \sim u_0(y) + \tilde{u}$, where $\tilde{u} = O(r^\mu)$ for some $\mu > 0$; it is weak in the sense that it is an
asymptotic expansion in the usual sense (in particular, with decaying remainder) only if both
sides are paired with a test function $\chi(y)$ (depending only on $y$). In other words, $u$ lies in this
Friedrichs domain if and only if
\[
(r, \theta) \mapsto \langle u(r, \theta, \cdot), \chi(y) \rangle = \langle u_0(y), \chi(y) \rangle + O(r^\mu).
\]
In general, if $u$ does not lie in the Friedrichs domain, then this expansion would have an extra
term on the right of the form $\langle u_{01}(y), \chi(y) \rangle \log r$, so the Friedrichs condition is simply that this
coefficient of $\log r$ vanish for all $\chi$, i.e., that the distribution $u_{01}$ vanish. A principal source of
the difficulties reported in [38] revolved around the issue of working with such weak expansions.
Henceforth we only work with the Friedrichs extension of $L$, and denote it simply by $L$. It is
straightforward to deduce using Hardy-type estimates that the domain $\mathcal{D}_{\text{Fr}}(L)$ is compactly
contained in $L^2$, which means that $L$ has discrete spectrum. The nullspace is finite dimensional
and consists of bounded polyhomogeneous (provided both $g$ and $V$ are polyhomogeneous)
functions $G$ the generalized inverse of $L$, so
\[
LG = GL = \text{Id} - \Pi,
\]
where $\Pi$ is the finite rank orthogonal projector onto the nullspace.
In contrast to Proposition [3.1] we now focus on the mapping
\[
L : L^2(M, dV_g) \supset \mathcal{D}_{\text{Fr}}(L) \rightarrow L^2(M, dV_g),
\]
which is invertible at least on a subspace of finite codimension. We commenced with the $L^2$
theory, which is well-known, but now shift to the analogous but less well known analogue in
Hölder spaces. Thus, proceeding by analogy with these $L^2$ definitions, set
\[
\mathcal{D}_{s}^{0,\gamma}(L) := \{ u \in C_s^{2,\gamma} : Lu \in C_s^{0,\gamma} \}, \quad s = w \text{ or } e.
\]
Note that if $u$ is only assumed to be bounded and to satisfy $Lu \in C_s^{0,\gamma}$, then for $s = e$, it
is a standard consequence of regularity in the edge theory that $u \in C_e^{2,\gamma}$. However, for an
arbitrary element of $C_e^{2,\gamma}$, it is definitely not true that $Lu$ is bounded, since typically it lies only
in $r^{-2}C_e^{2,\gamma}$. The analogous statement for $s = w$ is also not necessarily true (although it is true
that any such \( u \) lies in \( C^{2,\gamma} \) away from \( r = 0 \). Indeed, as we explain below, \( D^{\alpha,\gamma}_s \) contains the function \( v = r^{1/\beta} e^{i\theta} \), and if \( \beta > 1/2 \), then \( \partial^2_r v \) is not bounded either. Because functions of this type arise naturally here, we henceforth assume that

\[
\text{if } s = w, \text{ then } \gamma \in [0, 1) \cap [0, \frac{1}{\beta} - 1]
\]

Finally, note that any such \( u \) lies in \( C^{2,\gamma} \) locally away from \( r = 0 \).

The main issue in understanding the (essentially invertible) mappings

\[
L : D^{\alpha,\gamma}_s(L) \rightarrow C^{\alpha,\gamma}_s
\]

is to obtain more explicit characterizations of these singular ‘Hölder-Friedrichs’ domains. The Green function \( G \) provides a first step in this direction.

**Proposition 3.2.** Let \( K \) be the nullspace of \( L \) in \( L^2(M, dV_g) \) (which coincides with the nullspace in \( C^{2,\gamma}_s \)). Then

\[
D^{\alpha,\gamma}_s(L) = G(C^{\alpha,\gamma}_s) \oplus K = \{ u = Gf : f \in C^{\alpha,\gamma}_s \} \oplus K,
\]

**Proof.** Since \( C^{\alpha,\gamma}_s \subset L^2(M, dV_g) \), the space on the right is well-defined. If \( u \) is in the space on the right, then clearly \( Lu = f \in C^{\alpha,\gamma}_s \). Conversely, if \( u \in C^{2,\gamma}_s \), \( f \in C^{\alpha,\gamma}_s \) and \( Lu = f \) distributionally, then \( u \) is in the \( L^2 \) Friedrichs domain, and hence \( u = Gf + v \) for some \( v \in K \).

3.4 Finer properties of functions in the Hölder-Friedrichs domains

This last proposition sets the stage for the more detailed study of the regularity of functions in these domains. In this subsection we first briefly recall Donaldson’s estimates, which provide a characterization of \( D^{\alpha,\gamma}_w \), and then state the corresponding results for \( D^{\alpha,\gamma}_v(L) \), with the proofs deferred to the next subsection. We also describe some auxiliary regularity properties of elements in the edge domain which will be useful later.

We prove two main things here. First, \( D^{\alpha,\gamma}_w \) is strictly smaller than \( C^{2,\gamma}_s \), while \( D^{\alpha,\gamma}_w \) is strictly larger than \( C^{2,\gamma}_s \); we characterize these domains in terms of which derivatives lie in \( C^{\alpha,\gamma}_s \). Second, we show that this domain is independent of the operator \( L \) in the sense that it remains the same if we replace the polyhomogeneous Kähler edge metric \( \omega \) by any metric \( \omega_u \), where the Kähler potential \( u \) itself only lies in \( D^{\alpha,\gamma}_w \), and \( V \in C^{\alpha,\gamma}_s \).

The first of these issues is the analogue of a classical result that the Friedrichs domain of the Laplacian on a closed smooth manifold \( M \) is just \( H^2(M) \); in other words, if \( u \) is in the Friedrichs domain, then every first and second derivative lies in \( L^2 \), and conversely. This follows from the classical elliptic estimates in Sobolev spaces, of course, but is also equivalent to the classical fact that the Riesz potentials \( \partial^2_r \circ \Delta^{-1} \) are bounded operators on \( L^2 \). For operators with smooth coefficients, the analogous fact is an immediate consequence of the \( L^2 \) boundedness of pseudodifferential operators of order 0, since the analogous ‘variable curvature Riesz potentials’ lie in this class. For metrics and operators with rougher coefficients, we could use that Riesz potentials are singular integral operators (i.e., Calderon-Zygmund operators), and invoke the standard boundedness properties of that class of operators on either \( L^2 \) or Hölder spaces. This outline inverts the history, since the theory of singular integral and pseudodifferential operators, and in particular their boundedness properties, was developed precisely to answer and generalize questions of this type.

If \( u \) is an arbitrary function in \( D^{\alpha,\gamma}_s(L) \), then it is not true that every second derivative which appears in the operator \( L \) applied to \( u \) lands in \( C^{\alpha,\gamma}_s \). Donaldson’s simple yet crucial
observation [19] is that this is not necessary! As described in §2, the Monge–Ampère operator only involves the particular combinations of second derivatives of the form $g^{ij} u_{ij}$ (no summation), or equivalently, using the notation introduced there, the expressions $P_{ij} u$. Fortuitously, these operators characterize the Hölder–Friedrichs domain.

**Proposition 3.3.** Let $\mathcal{Q}$ be the set of operators [15]. Then

$$\mathcal{D}^{0,\gamma}_s(L) = \{ u \in C^{0,\gamma}_s : Q_i u \in C^{0,\gamma}_s, \forall Q_i \in \mathcal{Q} \}.$$  

(When $s = w$ we are assuming (36).) Equivalently, each of the maps

$$Q_i \circ G : C^{0,\gamma}_s \to C^{0,\gamma}_s, \quad Q_i \in \mathcal{Q},$$  

is bounded. If $\beta \leq 1/2$, then we have the stronger statements that

$$\mathcal{D}^{0,\gamma}_s(L) = \{ u \in C^{0,\gamma}_s : Q^*_i u \in C^{0,\gamma}_s, \forall Q^*_i \in \mathcal{Q}^* \},$$  

and each

$$Q^*_i \circ G : C^{0,\gamma}_s \to C^{0,\gamma}_s, \quad Q^*_i \in \mathcal{Q}^*,$$

is bounded.

**Remark 3.4.** To state the characterization in this Proposition even more plainly, then if $u \in \mathcal{D}^{0,\gamma}_s(L)$ then

$$||g^{ij} u_{ij}||_{s,0,\gamma} \leq C(||Lu||_{s,0,\gamma} + ||u||_{C^0}),$$

(no summation) which corresponds precisely to the result in [19] when $s = w$.

There is another useful way to phrase this. Define the norms

$$||u||_{\mathcal{D}^{0,\gamma}_s} = ||u||_{s,0,\gamma} + \sum_{Q_i \in \mathcal{Q}} ||Q_i u||_{s,0,\gamma} \quad \text{and} \quad ||u||_{\mathcal{D}^{0,\gamma}_s}^* = ||u||_{s,0,\gamma} + \sum_{Q^*_i \in \mathcal{Q}^*} ||Q^*_i u||_{s,0,\gamma}. \quad \text{(39)}$$

Later on we also use the seminorms $|| \cdot ||_{\mathcal{D}^{0,\gamma}_s}$ and $|| \cdot ||_{\mathcal{D}^{0,\gamma}_s}^*$, which are defined by omitting the initial $|| \cdot ||_{s,0,\gamma}$. This proposition implies that $|| \cdot ||_{\mathcal{D}^{0,\gamma}_s}$ and $|| \cdot ||_{\mathcal{D}^{0,\gamma}_s}^*$ induce Banach space structures on $\mathcal{D}^{0,\gamma}_s$ when $\beta > 1/2$ and $\beta \leq 1/2$, respectively. The space $\mathcal{D}^{0,\gamma}_u$ is essentially the one introduced by Donaldson [19] and denoted $\mathcal{C}^{2,\gamma,\beta}$ in that article. Note, however, the characterization of $\mathcal{D}^{0,\gamma}_u$ when $\beta \leq 1/2$, which includes the individual derivatives $\partial_r^2$, $r^{-1} \partial_r$, and $r^{-2} \partial_\theta^2$; this is sharper than what seems to be possible to obtain using the methods in [19]. We also see below that applying these derivatives to elements of $\mathcal{D}^{0,\gamma}_s$ definitely does not give bounded functions when $\beta > 1/2$.

Noting that the complex operators $P_{ij}$ are sums of the operators $Q_i$, proving one direction of this proposition is trivial: if $u$ and every $Q_i u$ lie in $C^{0,\gamma}_s$, then obviously $Lu \in C^{0,\gamma}_s$ since $Lu$ is just a sum of these terms with coefficients in $C^{0,\gamma}_s$ (or better). The other direction is equivalent to the boundedness properties of the compositions $Q_i \circ G$, or $Q^*_i \circ G$ if $\beta \leq 1/2$. For $s = w$, this is accomplished by Donaldson for the model problem by direct methods; he then indicates that the passage from these estimates for the model problem to those for the global ‘curved’ problem can be handled by patching together the local estimates. For $s = e$, we use the structure of these operators as pseudodifferential edge operators of order 0 and then invoke the basic boundedness results for such operators. We shall explain all of this more carefully in the next subsection.

Assuming this result, we deduce the following:
Corollary 3.5. Let $\phi \in D_s^{0,\gamma}(L) \cap \text{PSH}(M, \omega)$. If $\tilde{L} := \Delta_{\omega_\phi} + \tilde{V}$, where $\tilde{V} \in \mathcal{C}_s^{0,\gamma}$, then
\[ D_s^{0,\gamma}(\tilde{L}) = D_s^{0,\gamma}(L). \]

Proof. If $u \in D_s^{0,\gamma}(L)$, then by Proposition 3.3 $u$ and every $Q_i u$ lie in $\mathcal{C}_s^{0,\gamma}$, therefore $\tilde{L}u \in \mathcal{C}_s^{0,\gamma}$ since this is a linear combination of $u$ and $P_{ij} u$ with coefficients in $\mathcal{C}_s^{0,\gamma}$, and the $P_{ij} u$ are similar combinations of the $Q_i u$.

The converse is the assertion that if $\tilde{L}u = f \in \mathcal{C}_s^{0,\gamma}$, then every $Q_i u \in \mathcal{C}_s^{0,\gamma}$, which in turn is equivalent to the a priori estimate
\[ \|Q_i u\|_{s,0,\gamma} \leq C\|f\|_{s,0,\gamma}. \]

Suppose first that $\|\phi\|_{s,2,\gamma}$ is small. Then write $g^{ij}_\phi = (1 + e^{ij})g^{ij}$, where $\|e^{ij}\|_{s,0,\gamma}$ is small (note (36) is needed here). Decompose $\Delta_{\omega_\phi} = \Delta_\omega + E$ accordingly. Rewriting $Lu = f$ as $\Delta_{\omega_\phi} u = f - \tilde{V} u \in \mathcal{C}_s^{0,\gamma}$, we may as well suppose that $\tilde{V} = 0$. Now, with $G$ the Green function for $\Delta_{\omega}$,
\[ \tilde{L} \circ G = (\Delta_{\omega} + E) \circ G = \text{Id} + E \circ G - \Pi, \]
where $\Pi$ is the rank one projection onto constants. But $E \circ G$ is a bounded operator on $\mathcal{C}_s^{0,\gamma}$ with small norm, hence $\tilde{G} = G \circ (\text{Id} + E \circ G)^{-1}$ is an inverse to $\Delta_{\omega_\phi}$ up to a rank one error. Furthermore, the image of $\tilde{G}$ is the same as the image of $G$, hence equals $D_s^{0,\gamma}(L)$.

If $\phi$ has large norm, then we consider the path of metrics $\omega_{\sigma \theta}$, $0 \leq \sigma \leq 1$. The argument above shows that the domain remains locally constant, so by compactness of the interval, the domain remains constant throughout. This proves the other direction. \qed

We also prove one further result about the pointwise properties of this domain for $s = e$, since as we have already commented, even the continuity of these functions is not immediately obvious from the basic definition.

Corollary 3.6. If $u \in D_e^{0,\gamma}$, then $u$, $\partial_\theta u$ and every $\partial_y u$ is continuous on up to $r = 0$, and has a well-defined restriction on $D$ which is independent of $\theta$.

Proof. By Proposition 3.3 $\partial_r u$, $\partial_\sigma \partial_r u$, $\partial_\sigma \partial_\sigma u$, $\partial_r \partial_\sigma u$, $r^{-1} \partial_\theta u \in \mathcal{C}_e^{0,\gamma}$; in particular, these functions are all bounded. Now integrate from $r = r_0$ to $r = 0$ to obtain that these functions are continuous to $r = 0$. If this ‘boundary value’ were to depend on $\theta$ nontrivially, then $r^{-1} \partial_\theta u$ would be unbounded, which we know is not the case. \qed

It is obvious that $D_e^{0,\gamma} \subset \mathcal{C}_e^{2,\gamma}$. Also, by (31) and Proposition 3.2
\[ D_w^{0,\gamma} \subset D_e^{0,\gamma}. \tag{40} \]

3.5 Pseudodifferential edge operators and their boundedness

We describe the proof of Proposition 3.3 in this subsection. The main point is to understand the structure of the Green function $G$, or more specifically, the precise pointwise structure of its Schwartz kernel $G(z, z')$. This is then used to obtain the necessary bounds on the integrals
\[ Q_i u(z) = \int_X Q_i G(z, z') f(z') \, dV_q(z'), \quad f \in \mathcal{C}_s^{0,\gamma}. \tag{41} \]
The main point of the discussion below is that the operators $Q_i \circ G$ are pseudodifferential edge operators of ‘nonnegative type’ (which we define precisely below), and it is a general fact that if $A$ is any such operator, then $A$ is bounded on $\mathcal{C}_w^{0,\gamma}$. Donaldson [19] proved the boundedness of the special operators $P_{ij} \circ G$ on $\mathcal{C}_w^{0,\gamma}$. On the other hand, if $Q$ is an arbitrary second derivative, that $Q \circ G$ does not always have nonnegative index sets, as we explain below, and the boundedness will not be true then.

Viewed more broadly, the polyhomogeneous structure of $G$ allows one to deduce the corresponding polyhomogeneous structure of the ‘Riesz potentials’ $Q_i \circ G$; the boundedness of these operators on various natural function spaces is then a basic feature of the edge calculus. Donaldson derived the polyhomogeneous structure of the Green function for the flat model problem $G_{ij}$ by explicit calculation (some form of which is similar to the calculations in [36]), and then proved the Hölder estimates on the wedge spaces by hand. As noted earlier, the edge calculus is simply a systematization of the perturbation arguments which allow one to pass from this flat model to the actual curved problem.

The edge calculus $\Psi_\epsilon^*(X)$ is a space of pseudodifferential operators on $X$, elements of which have degeneracies at $\partial X$ similar to the ones exhibited by differential edge operators as in [55]. We use $X$ systematically now rather than $M$ since it is important that we work on a manifold with boundary. This space of operators is large enough to contain not only all differential edge operators $A$, but also parametrices and generalized inverses for $A$, provided $A$ is elliptic in this category, and also for incomplete elliptic edge operators like $L = r^{-2}A$. The term ‘calculus’ is used to indicate that $\Psi_\epsilon^*(X)$ is almost closed under composition, with the caveat that not every pair of elements may be composed due to growth properties of Schwartz kernels in the incoming and outgoing variables which prevent the corresponding integrals from converging.

An element $B \in \Psi_\epsilon^*(X)$ is characterized by the regularity properties of its Schwartz kernel $B(z, z')$ as a distribution on $X \times X$; the superscript * is a placeholder for a set of indices which indicate the singularity structure of this distribution in various regimes. By definition, any such $B(z, z')$ is the pushforward of a distribution $K_B$ defined on a space $X^2_\epsilon$, called the edge double space, which is a resolution of $X^2$ obtained by performing a (real) blow-up of the fiber diagonal of $\partial X^2$. This distribution $K_B$ has a standard pseudodifferential singularity along the lifted diagonal (by which we mean a polyhomogeneous expansion in powers of the distance to this submanifold), as well as polyhomogeneous expansions at all boundary hypersurfaces of $X^2_\epsilon$. and product-type expansions at the higher codimension corners. We have defined polyhomogeneity on manifolds with boundary earlier, and will extend this to manifolds with corners below. The detailed notation $B \in \Psi_\epsilon^{m,k,\text{Ext},\text{Ext}}(X)$ records the pseudodifferential order $m$ along the diagonal and and the exponent sets in the expansions at the various boundary faces. We explain this in more detail now.

We first construct the blowup $X^2_\epsilon$. The product $X^2 = X \times X$ is a manifold with corners up to codimension two. The corner $(\partial X)^2$ has a distinguished submanifold, denoted $\text{fdiag}_{\partial X}$, which is the fiber diagonal. We blow this up normally, to obtain a space $[X^2, \text{fdiag}_{\partial X}] := X^2_\epsilon$. Using the local coordinates $(r, \theta, y)$ above on the first factor of $X$ and an identical copy $(r', \theta', y')$ on the second copy, the corner is the submanifold $\{r = r' = 0\}$, and $\text{fdiag}_{\partial X} = \{r = r' = 0, y = y'\}$. The blowup may be thought of as introducing polar coordinates around this submanifold:

$$R = |(r, r', y - y')|, \quad \omega = R^{-1}(r, r', y - y'),$$

$$R \geq 0, \quad \omega \in S_+^{2n-1} = \{\omega = (\omega_1, \omega_2, \omega): \omega_1, \omega_2 \geq 0\},$$

supplemented by $y', \theta, \theta'$ to make a full coordinate system. Thus $X^2_\epsilon$ has a new boundary hypersurface, defined as $\{R = 0\}$, called the ‘front face’ $ff$, and the lifts of the two original
boundary hypersurfaces, given as \( \{ \omega_1 = 0 \} \) and \( \{ \omega_2 = 0 \} \), called the right and left faces, \( rf \) and \( lf \), respectively. We write defining functions for these faces as \( \rho_{lf}, \rho_{rt} \) and \( \rho_{lf} \). The diagonal of \( X^2 \) lifts to the submanifold \( diag_\epsilon = \{ \omega = (1/\sqrt{2}, 1/\sqrt{2}, 0), \theta = \theta', R \geq 0 \} \).

Here are some motivations for this construction. First, Schwartz kernels of pseudodifferential operators are singular along the diagonal in \( X^2 \), but the fact that this diagonal intersects the corner nontransversely makes these singularities hard to describe near this intersection. By contrast, the lifted diagonal \( diag_\epsilon \) intersects the boundary of \( X^2 \) only in the interior of \( ff \), and this intersection is transversal. Another feature is that \( X^2_\epsilon \) captures the homogeneity under dilations inherent in this problem. The flat model operator

\[
L_\beta = \partial_r^2 + r^{-1} \partial_r + (\beta r)^{-2} \partial_\theta^2 + \Delta_y,
\]

on the product space \([0, \infty)_r \times S^1_\theta \times \mathbb{R}^2_{y^n-2}\) is homogeneous of order \(-2\) with respect to the dilations \((r, \theta, y) \mapsto (\lambda r, \theta, \lambda y)\), and is also translation invariant in \(y\). It follows that the Schwartz kernel \( G_\beta(z, z') \) of the inverse for the Friedrichs extension of \( L_\beta \) commutes with translations in \(y\), i.e., depends only on the difference \( y - y'\) rather than \(y\) and \(y'\) individually, and is homogeneous of order \(-2n + 2\) in the sense that

\[
G_\beta(\lambda r, \lambda r', \lambda(y-y'), \theta, \theta') = \lambda^{-2n+2} G_\beta(r, r', y-y', \theta, \theta').
\]

In the polar coordinate system above, this simply says that

\[
G_\beta(r, r', y-y', \theta, \theta') = G_\beta(\omega, \theta, \theta') R^{-2n+2}.
\]

This implies that \( G_\beta \) lifts to the double space \((\mathbb{R}^+ \times S^1 \times \mathbb{R}^{2n-2})_C^2\) as a product of \( R^{-n+2}\) and an ‘angular part’ \( G_\beta \). A further analysis shows that \( G_\beta \) has a singularity at \( \omega = (1/\sqrt{2}, 1/\sqrt{2}, 0) \) and polyhomogeneous expansions along the side faces \( \{ \omega_1 = 0 \} \) and \( \{ \omega_2 = 0 \} \).

We now recall the general definition of polyhomogeneity. We do this only on the model orthant \( \mathcal{O} = (\mathbb{R}^+)^k \times \mathbb{R}^\ell \), with linear coordinates \((x_1, \ldots, x_k, y_1, \ldots, y_\ell)\), but it is easy to see that this definition is coordinate-invariant, so this discussion translates immediately to arbitrary manifolds with corners. First, let

\[
V_0(\mathcal{O}) = \text{span}_{C^\infty}(x_1 \partial x_1, \ldots, x_k \partial x_k, \partial y_1, \ldots, \partial y_\ell)
\]

be the space of all smooth vector fields tangent to all boundaries of this space. We may as well assume that all distributions are supported in a ball \( \{|x|^2 + |y|^2 \leq 1\} \). If \( \nu = (\nu_1, \ldots, \nu_k) \in \mathbb{R}^k \), then \( u \) is conormal of order \( \nu \), \( u \in \mathcal{A}^\nu(\mathcal{O}) \), if

\[
V_1 \ldots V_j u \in x^\nu L^\infty(\mathcal{O}) \ \forall \ j \geq 0 \quad \text{and for all} \ V_i \in V_0(\mathcal{O}).
\]

Next, \( u \) is polyhomogeneous if near the origin in \( \mathcal{O} \), \( u \) has an expansion of the form

\[
u \sim \sum u_{\gamma,p}(y)(\log x)^p,
\]

where \( \{ \gamma \} \) is a sequence of elements in \( \mathbb{C}^k \) such that any region \( \text{Re}(\zeta_j) < C_j \), \( j = 1, \ldots, k \), contains only finitely many elements of this set; here \( (\log x)^p = (\log x_1)^{p_1} \ldots (\log x_k)^{p_k} \), with each \( p \in \mathbb{N}_0^k \), and such that \( |p| \leq N(\gamma) \). The coefficients \( a_{\gamma,p}(y) \) are smooth. We associate to such an expansion an index family \( E = \{ E^\ell \}, \ell = 1, \ldots, k \), consisting of all pairs of multi-indices \( \{(\gamma, p)\} \) of exponents which occur in this expansion, and denote by \( \mathcal{A}^E_{\text{pfg}} \) the space of all such distributions. As in the codimension one case, we say that \( u \in \mathcal{A}^0_{\text{pfg}} \) if \( u \) is
polyhomogeneous and if each index set $E^{(i)}$ is greater than or equal to 0 in the sense described earlier. We also write the simple index set $\{(\gamma + \ell, 0) : \ell \in \mathbb{N}_0\}$ simply as $(\gamma)$; thus $A_{\text{phg}}^{(\gamma)} = x^\gamma C^\infty$, i.e., $u = x^\gamma v$ where $v$ is $C^\infty$ up to that face. (Even more specifically, a function which is smooth in the traditional sense up to the boundary and corners has index set $(0)$.)

We now define the space of pseudodifferential edge operators on $X$.

**Definition 3.7.** We say that $B \in \Psi_e^{m,r,E_{rf},E_{\text{lf}}}(X)$ if the Schwartz kernel of $B$ is the pushforward from $X^2_\rho$ to $X^2_\rho$ of a distribution $K_B$ on $X^2_\rho$ which has the following properties. $K_B$ decomposes as a sum $K_B^{(1)} + K_B^{(2)}$ where $K_B^{(1)} = \rho^{-2n+r} K'_B$ is supported in a neighbourhood of diag$_e$ which does not intersect the side faces, and which has a classical pseudodifferential singularity of order $m$ along this lifted diagonal which is smoothly extendible across ff. (This simply says that the strength of the pseudodifferential singularity is uniform up to ff, after removing the factor $\rho^{-2n+r}$.) On the other hand, $K_B^{(2)}$ is polyhomogeneous on $X^2_\rho$ with index sets $(\nu - 2n)$ at ff, $E_{rf}$ at rf and $E_{\text{lf}}$ at lf.

The point of this decomposition is simply to isolate the part of $K_B$ which contains the diagonal singularity and to emphasize the uniformity of this singularity up to ff. The shift of the order at the front face by $-2n$ is a normalization so that the Schwartz kernel of the identity operator, which is $r^{-1} \delta(r - r') \delta(\theta - \theta') \delta(y - y')$ relative to the measure $dV_g \equiv rdrd\theta dy$, has order 0. Indeed, using that $\delta(r - r')$ is homogeneous of degree $-1$ and $\delta(y - y')$ is homogeneous of degree $2 - 2n$, this Schwartz kernel is homogeneous of order $-2n$.

Finally we may state the basic structure theorem for the Green function of $L$.

**Proposition 3.8.** Let $g$ be a polyhomogeneous edge metric with angle $\beta$ along $D$ and $L = -\Delta_g + V$ where $V$ is polyhomogeneous and bounded on $X$, and suppose that $G$ is the generalized inverse to the Friedrichs extension of $L$. Then $G \in \Psi^{-2,2,E,E}(X)$, where the index set $E$ is determined by the indicial roots of $L_\beta$ and by the index sets of $g$ and $V$. In particular, if $g$ and $V$ are smooth (i.e., both have index set $(0)$ at $\partial X$), then

$$E \subset \{(j/\beta + k, \ell) : j, k, \ell \in \mathbb{N}_0 \text{ and } \ell = 0 \text{ for } j + k \leq 1\} \setminus \{(1, 0, 0)\}. \quad (44)$$

Moreover, even if (as happens for our eventual solutions), $g$ and $V$ are polyhomogeneous with index set contained in the index set $(44)$, then the index set $E$ for $G$ is also contained in the index set $(44)$.

**Remark 3.9.** The fact that $G$ has the same index set $E$ at the left and right faces is natural since $G$ is symmetric. The index set $E$ may be slightly more complicated when $\beta \in \mathbb{Q}$ since in that case $j/\beta$ can equal a positive integer for certain $j$, and this creates extra logarithmic factors in the expansion (i.e., elements of $E$ of the form $(k, 1)$), but these all occur outside the range $\text{Re} \zeta \leq 1$, so do not enter into the considerations below. These log terms are absent if $g$ is an orbifold metric.

Finally, we emphasize that despite the somewhat elaborate language needed to state this result, this structure theorem for $G$ is essentially the same as, or rather a slight refinement of, the one stated by Donaldson [19].

The proof of this result can be extracted from [36]: it is simply the elliptic parametrix construction in the edge calculus, altered slightly so as to accomodate operators such as the Laplacian of an incomplete, instead of a complete (as in [36]), edge metric. As with any parametrix construction, one must obtain detailed information about the solution operator for the model problem $\Delta_{g\beta}$, or in other words about the model Green function $G_\beta$. This is the
technical core, and the rest of the argument uses pseudodifferential calculus to show how to write the actual Green function for $L$ as a perturbation of $G_{\beta}$. The specific information we need to obtain, then, is that the Schwartz kernel of $G_{\beta}$ has the same polyhomogeneous structure as in the statement of Proposition 3.3. This may be approached in several ways. One, as pursued in [36], is to take the Fourier transform in $y$, thus reducing $\Delta_{\beta}$ to the family of operators

$$\hat{\Delta}_{\beta} = \partial_{r}^{2} + r^{-1}\partial_{r} + (\beta r)^{-2}\partial_{\theta}^{2} - |\eta|^{2}$$

on $\mathbb{R}^{+} \times S^{1}$ where $\eta$ is the variable dual to $y$. This is analyzed directly either by separating variables or (what turns out to be more advantageous) multiplying by $r^{2}$, setting $\rho = \log r$ and $s = r|\eta|$ to convert this to

$$\partial_{s}^{2} + \beta^{-2}\partial_{\theta}^{2} - 1.$$ 

Taking one further Fourier transform in the $s$ variable, this reduces finally to the resolvent of the Laplacian on the circle, $\beta^{-2}\partial_{\theta}^{2} - (1 + \sigma^{2})$ which is known explicitly. Chasing back through these transformations yields a tractable expression for $G_{\beta}$. We refer to [36] for details. An alternate approach employed in [19] is to write $G_{\beta}$ as an integral over $0 < t < \infty$ of the heat kernel $\exp(t\Delta_{\beta})$. This heat kernel is the product of the heat kernel on the model two dimensional cone with cone angle $2\pi\beta$ and the Euclidean heat kernel on $\mathbb{R}^{2n-2}$. The former of these is known classically, albeit as an infinite sum involving Bessel functions, see [19] and [41], while the latter is the standard Gaussian. Either method requires about the same amount of work.

One seemingly minor point in the statement of Proposition 3.8 which is important below is the fact that the index set $E$ does not contain the element $(1,0)$, so in other words, the terms $r^{1}$ and $(r')^{1}$ do not appear in the expansion of $G$ at the left and right faces. This can be explained as follows. As a distribution, $G(r,\theta,y,r',\theta',y')$ satisfies

$$LG = r^{-1}\delta(r-r')\delta(\theta-\theta')\delta(y-y').$$

Calculating locally in the interior of $rf$, away from the front face, we see that $LG = 0$ there. Since we know at this point that $G$ is polyhomogeneous, we can calculate formally, letting $L$ act on the series expansion. It is then easy to see that $L$ cannot annihilate the term $a(\theta,y)r$ because $r^{-1}\partial_{r}(r) = r^{-1}$ (and $r^{-1}\partial_{r}$ appears in $L$). This proves the claim.

We now state the basic boundedness theorem needed to prove Proposition 3.3.

**Proposition 3.10.** Let $B \in \Psi_{\infty,0}^{m,r,E,E'}(X)$. If $m \leq 0$, $r \geq 0$ and both $E$ and $E'$ are nonnegative (i.e. contain no terms $(\gamma,p)$ with $\text{Re}\gamma < 0$ or $p > 0$ if $\text{Re}\gamma = 0$), then

$$B : C^{k,\gamma}_{e}(X) \rightarrow C^{k,\gamma}_{e}(X)$$

is bounded for any $\ell \in \mathbb{N}_{0}$.

**Remark 3.11.** For brevity, we shall often refer to the conditions here, that $m \leq 0$, $r \geq 0$ and that the index sets at the left and right faces are nonnegative by simply saying that $B$ has nonnegative type, or even just that $B$ is nonnegative.

This is proved as follows. First decompose $B$ into a sum of two operators, $B_{1} + B_{2}$, where the Schwartz kernel of $B_{1}$ is supported near the lifted diagonal $\text{diag}_{e}$ in $X_{e}^{2}$ and that of $B_{2}$ is supported away from this lifted diagonal. Hence $B_{1}$ carries the full pseudodifferential singularity, while the Schwartz kernel of $B_{2}$ is polyhomogeneous on $X_{e}^{2}$. If $f \in C^{k,\gamma}_{e}$, then the fact that $B_{1}f \in C^{k,\gamma}_{e}$ can be proved by reducing to the standard boundedness of (ordinary,
nondegenerate) pseudodifferential operators on Hölder spaces using scaling arguments. This uses that \( m \leq 0 \) and \( r \geq 0 \). On the other hand, \( B_2f \in \mathcal{C}_e^{s,\gamma} \) for all \( s \in \mathbb{N}_0 \), hence is \( \mathcal{C}^\infty \) in the interior. This is proved by a simple direct analysis of the integral defining \( B_2f \). This uses that the index sets at the side faces are nonnegative.

**Proof of Proposition 3.3.** For the proof of the assertion when each \( Q_i \circ G \) acts on \( \mathcal{C}_w^{0,\gamma} \), we refer to [19].

As for the case when these operators act on \( \mathcal{C}_e^{0,\gamma} \), the proof follows directly from Proposition 3.10 once we show that each \( Q_i \circ G \) satisfies the hypotheses there, specifically that it has nonpositive pseudodifferential order (this is clear since \( Q_i \) has order less than or equal to 2 while \( G \) has order \(-2\)) and that its index sets are all nonnegative. We claim that this is true for every \( Q_i \in \mathcal{Q} \), and if \( \beta \leq 1/2 \), then it is true for every \( Q_i^* \in \mathcal{Q}^* \).

The key point is that each of the vector fields \( r\partial_r, \partial_\theta \) and \( r\partial_{y_j} \) on \( X \) lift smoothly via the blowdown map \( B \) to \( X_e^2 \): indeed, each of these lifts is a vector field on this blown up space which is tangent to all boundary faces. Thus \( \rho_{\ell f}B^*\partial_{y_k} \) is tangent to all faces, as is \( \rho_{\ell f}B^*r^{-1}\partial_\theta \), while \( \rho_{\ell f}B^*\partial_r \) differentiates transversely to the left face, but is tangent to all other faces. From this we can deduce the claim. Indeed, one simply needs to check that each operator \( Q_i \in \mathcal{Q} \) lifts to an operator on \( X_e^2 \) of the form \( \rho_{\ell f}^{-2}Q_i \) where \( Q_i \) acts tangentially along \( \ell f \) and which differentiates transversely to \( \ell f \) in such a way that it annihilates the leading terms of the expansion there so that the resulting index set remains nonnegative. For example, examining the action along \( \ell f \), \( \partial_r \) kills the leading term \( a_0r^0 \), and the derivative of the next term \( a_1r^{1/\beta} \) is still bounded; for the action of \( r^{-1}\partial_\theta \), we note that the leading coefficient \( a_0 \) of \( r^0 \) is independent of \( \theta \), so this term is also good; a similar case by case check ensures that the lifts of all the \( Q_i \) have the appropriate properties.

When \( \beta \leq 1/2 \), we have a bit more leeway since the expansion along \( \ell f \) still has leading term \( a_0r^0 \), but the next term is \( a_2r^2 \). Therefore, we can also act on this with \( \partial_r^2, r^{-1}\partial_r \) and \( r^{-2}\partial_\theta^2 \) and still produce an operator with nonnegative index sets.

We remark that Donaldson only checks the boundedness of the operators \( P_{ij} \circ G \) for \( ij \neq 11 \), and then uses that \( P_{11} \) is a combination of \( \Delta_{x_i} \) and the other \( P_{ij} \). Our approach, however, shows that the operators \( Q_i \circ G \) are bounded, and when \( \beta \leq 1/2 \), produces the simpler result that every \( Q_i^* \circ G \) is bounded on \( \mathcal{C}_e^{0,\gamma} \); in other words, in this case, all wedge second derivatives of any \( u \in \mathcal{D}_e^{0,\gamma} \) lie in \( \mathcal{C}_e^{0,\gamma} \).

### 3.6 A comparison of methods

The previous subsections consist of little more than a lot of terminology and the statements of basic results about edge operators, but once these are given, the actual proof of Proposition 3.3 is immediate. Donaldson’s article [19] seems to give a more elementary proof of this same conclusion in the case of \( \mathcal{C}_w^{0,\gamma} \), and it is worth saying a bit more about the similarities and differences between the approaches, as well as the advantages of each.

We have already commented on the two different, but eventually equivalent, methods for constructing the kernel \( G_\beta \). There is perhaps little to recommend one method over the other.

The other two steps of the argument in [19] are in some sense inverted relative to the development here. Our parametrix construction is simply a systematic way to pass from the polyhomogeneous structure of the model inverse \( G_\beta \) to the corresponding structure for the actual inverse \( G \). This requires a certain amount of technical overhead, but the benefit is that one obtains the polyhomogeneous structure of \( G \) itself, which can be quite useful. The Hölder boundedness for \( G \) and \( G_\beta \) are then deduced from the same general result about boundedness.
of edge pseudodifferential operators. As we explained earlier, this in turn can be reduced by scaling to the boundedness of ordinary pseudodifferential operators on ordinary Hölder spaces. This takes advantage of the approximate homogeneity structure of these operators and of the edge function spaces. Donaldson’s argument, by contrast, first establishes the Hölder estimates for the model operator $G_\beta$ using scaling arguments. He then briefly notes that these estimates can be patched together to obtain the Hölder estimates for the differentiated kernels $P_{ij}G$. Thus the core difference between the approaches is that the patching (or transition from the model to the actual inverse) is done either at the level of the parametrix in the first approach, but only at the level of a priori estimates in the second.

4 Higher regularity for solutions of the Monge–Ampère equation

We now use the machinery of the last section to prove one of our main results, that under reasonable initial hypotheses, solutions of the complex Monge-Ampère equation are polyhomogeneous (Theorem 1). This type of proof is quite standard now, and we list as precursors the proof of polyhomogeneity for complete Bergman and Kähler–Einstein metrics on strictly pseudoconvex domains by Lee and Melrose [30], and the existence of or obstructions to polyhomogeneity for solutions of the singular Yamabe problem [37]. The proof here is close to the one in this last reference. This regularity result was part of the announcement [38], and the proof here is the one envisioned there. We now work exclusively in the edge Hölder spaces since they turn out to be the more natural tools for this problem.

We turn to the proof of Theorem 1. There are three main steps. The first is to show that $u \in C^{k,\gamma}$ for every $k \in \mathbb{N}$; the second is to improve this to full conormality, i.e., to show that $u \in A^0$; finally, we improve this conormality to the existence of a polyhomogeneous expansion. The first step is equivalent to standard higher elliptic regularity for Monge-Ampère equations; this uses the dilation invariance properties of the edge Hölder spaces in a crucial way. The second step then breaks this dilation invariance by showing that we may also differentiate arbitrarily many times along $D$. This uses the hybrid spaces introduced in §2.4. The final step uses an iteration to show that $u$ has a longer and longer partial polyhomogeneous expansion. These last two steps require us to state a few further results about the boundedness of pseudodifferential edge operators which are needed in the course of the proof.

Let us begin, then, by quoting a consequence of the Evans–Krylov–Safonov theory concerning solutions of Monge–Ampère equations [20, 21, 28].

**Theorem 4.1.** Let $\omega$ be a smooth Kähler metric in a ball $B \subset \mathbb{C}^n$ and $F \in C^\infty(B \times \mathbb{R})$. Suppose that $u \in C^2(B)$ is a solution of $\omega^n_u = F(z,u)$ on $B$. Then for any $k \geq 2$, there is a constant $C$ depending on $k$, $\omega$, $\sup_B |u|$ and $\sup_B |\Delta \omega u|$ such that if $B'$ is a ball with the same center as $B$ but with half the radius, then

$$||u||_{C^{k,\gamma}(B')} \leq C.$$ 

The constant $C$ depends uniformly on the $C^{k+3}(B)$ norm of the coefficients of $\omega$.

In fact, the Evans–Krylov theorem gives the $C^{2,\gamma}(B)$ estimate. The higher regularity is obtained by a straightforward bootstrap, since differentiating the equation with respect to any coordinate vector field $W$ gives a linear equation for $Wu$ with coefficients depending on at most the second derivatives of $u$, to which we can apply ordinary Schauder estimates. We shall review a proof of an extension of this theorem to the singular setting in §11 below.
To use this in our setting, we first observe that the Monge–Ampère equation is invariant under the scaling \((r, \theta, y) \rightarrow (\lambda r, \theta, \lambda y)\), or in the original complex coordinates, \((z_1, \ldots, z_n) \rightarrow (\lambda^{1/\beta} z_1, \lambda z_2, \ldots, \lambda z_n)\). Denote this scaling map by \(S_\lambda\). We see from (5) that the rescalings of \(g\) converge up to a factor. Indeed, 
\[
\lambda^{-2} S_\lambda^* g \longrightarrow g_\beta.
\]
Now let \(B\) be the ball of radius \(r_0/2\) centered at some point \((r_0, y_0)\) in the coordinates \((r, y)\), where \(r_0\) is small, and let \(B'\) be the product of the ball of half this radius. We consider the sets \(B \times S^1\) and \(B' \times S^1\). The pointwise norm of the curvature tensor of \(g\) in \(B \times S^1\) is bounded by \(Cr_0^{-2}\). Choose coordinates so that \(y_0 = 0\), and consider the family of metrics 
\[
gr_0 = r_0^{-2} S_\lambda g.
\]
These are defined in \(\tilde{B} \times S^1\) where \(\tilde{B}\) is a ball of radius 1/2 centered at \((1, 0)\). Let \(\tilde{B}'\) be the ball of radius 1/4 centered at this same point. Finally, we consider the function \(u_{r_0}(r, \theta, y) = u(r_0 r, \theta, r_0 y)\), also defined in \(\tilde{B} \times S^1\). Then \(u_{r_0}\) satisfies the Monge–Ampère equation with respect to the metrics \(g_{r_0}\) in this standard ball, and the Evans-Krylov estimate then gives that 
\[
\|u_{r_0}\|_{k, \gamma; \tilde{B}' \times S^1} \leq C,
\]
where \(C\) depends in particular on \(g_{r_0}\), \(\sup |u_{r_0}|\), and \(\sup |\Delta g_{r_0} u_{r_0}|\). The assumption that \(u \in C^2\) implies that these quantities are uniformly bounded as \(r_0 \to 0\), and hence \(u_{r_0}\) is uniformly bounded in any \(C^{k, \gamma}\) norm in \(\tilde{B}' \times S^1\). Finally, recall that the edge Hölder norms are invariant under this rescaling. We have now proved that \(u \in C^{k, \gamma}_e\) for any \(k \geq 2\).

We have proved in particular that \((r \partial_r) (\partial_y u)\) is bounded for any \(j, \ell \geq 0\), but a priori, we only know that \((\partial_y)^j \partial_y \partial_y u\) is bounded, and hence \((\partial_y)^j \partial_y u\), blow up like some power of \(r\). We now address this and show that these tangential derivatives are bounded too. Write the Monge–Ampère equation as 
\[
\log \det(g_{ij} + \sqrt{-1} u_{ij}) = \log \det(g_{ij}) + \log F(z, u).
\]
Applying \(\partial_y\) to both sides and using the well-known formula for the derivative of a logarithmic determinant, we find that 
\[
(\Delta_{\tilde{g}} + V) \partial_y u = f,
\]
where \(\tilde{g}_{ij} = g_{ij} + \sqrt{-1} u_{ij}\), \(V = F_u(z, u)/F(z, u)\) and \(f = \partial_y \log \det(g_{ij}) + F_z(z, u)/F(z, u)\).

Since \(u \in D^{0, \gamma}_e\), we see that both \(V\) and \(f\) lie in \(C^{0, \gamma}_e\). Recalling also that \(\partial_y u\) is bounded, we can now apply Corollary \((43)\) which proves that \(\partial_y u \in D^{0, \gamma}_e\).

To carry out this next iteration, let us define the higher hybrid domain spaces 
\[
D^{k, \gamma; k'}_e(L) = \{u : \Delta_g u \in C^{k, \gamma; k'}_e\}
\]
for any \(0 \leq k' \leq k\). We have thus proved in the last paragraph that \(u \in D^{k, \gamma; 1}_e\) for any \(k \geq 0\). In order to proceed, we must establish the analogue of Corollary \((33)\) namely that if \(L\) has coefficients in \(C^{k, \gamma; k'}_e\) and \(Lu = f \in C^{k, \gamma; k'}_e\), then \(u \in D^{k, \gamma; k'}_e\); in addition, we need that 
\[
D^{k, \gamma; k'}_e \subset C^{k+2, \gamma; k'}_e.
\]

Assuming these two claims, then we see exactly as above that \(\partial_y u \in D^{k, \gamma; k'}_e\), and hence \(u \in C^{k+3, \gamma; k'+1}_e\), for all \(k\). Continuing, we obtain that \(u \in D^{k, \gamma; k'}_e\) for all \(k' \leq k\), and hence \(u \in A^0(X)\).
Let us prove these claims. Suppose we know that \( G : C^k_{e^0} \to C^k_{e^1} \) for some \( k' \). Fix \( f \in C^k_{e^0} \) and set \( u = Gf \). Then \( \partial_y u = G(\partial_y f) + [G, \partial_y]f \). Proposition 3.30 in \cite{36} asserts that the commutator \([G, \partial_y] \) lies in \( \Psi^{-2,2,E,E'}(X) \); in other words, not only is there the usual cancellation at the diagonal so that this is still an operator of order \(-2\), but there is also a cancellation at the front face, so this still vanishes to order 2 there. By induction, we obtain both that \([G, \partial_y]f \in C^{k+2,2,E,E'}_{e^1} \) and in addition \( G(\partial_y f) \in C^{k+1,1,E,E'}_{e^1} \). Therefore, \( \partial_y u \in C^{k+1,1,E,E'}_{e^1} \), or equivalently, \( u \in C^{k+2,2,E,E'}_{e^1} \). Now we may follow the steps in the proofs of Proposition 3.3 and Corollary 3.5 almost verbatim. The inclusion in (45) is obtained from the boundedness of the operators \( Q \circ G \) on every hybrid space \( C^{k,\gamma}_{e^1} \), which is proved by the same induction as above.

We come to the final step, that \( u \) is polyhomogeneous. This requires two more boundedness properties of edge pseudodifferential operators, namely that this class of operators preserves the spaces of conormal and of polyhomogeneous functions. In particular, if \( B \) is any pseudodifferential edge operator, with nonnegative order at the front face and nonnegative index sets at the side faces, then

\[
B : A^0(X) \to A^0(X) \quad \text{and} \quad B : A^0_{\text{phg}}(X) \to A^0_{\text{phg}}(X). \tag{46}
\]

We do not need to worry about the order of \( m \) along the diagonal since we are applying \( B \) to functions which are infinitely differentiable in an appropriate sense anyway. In fact, there is a refinement of these results which is what actually gives the improvement in the argument that follows.

**Lemma 4.2.** Let \( B \in \Psi^{m,2,E,E'}_{e^1}(X) \), where \( E \) and \( E' \) are nonnegative. Then

\[
B : A^0(X) \to A^0_{\text{phg}}(X) + A^2(X),
\]

and more generally, if \( \nu \geq 0 \),

\[
B : A^0_{\text{phg}}(X) + A^\nu(X) \to A^0_{\text{phg}}(X) + A^{\nu+2}(X).
\]

**Proof.** The second assertion is an easy consequence of the first. To prove this first assertion, if \( B \) has index set with all exponents greater than or equal to 2 at the left \((r \to 0)\) face, then since \( B \) vanishes to order 2 at the front face, we can write \( B = r^2 \tilde{B} \) where \( \tilde{B} \) is nonnegative. Hence in that case, \( B : A^0 \to A^2 \).

Now suppose that the exponents in the expansion of \( B \) at the left face of \( X^2 \) which lie in the range \([0,2)\) are \( \gamma_1, \ldots, \gamma_N \), and assume that there are no log terms in these expansions for simplicity. Then

\[
B^{(N)} := (r \partial_r - \gamma_1)(r \partial_r - \gamma_2) \cdots (r \partial_r - \gamma_N)B \in \Psi^{m+N,2,E(2),E'(2)}_{e^1}(X),
\]

where \( E(2) \) is some new index set derived from \( E \) which has all elements greater than or equal to 2. Thus we can apply the previous observation to see that \( B^{(N)} : A^0 \to A^2 \), or said slightly differently, if \( f \) is bounded and conormal, so \( f \in A^0 \), then \( B^{(N)} f = u^{(N)} \) is of the form \( r^v v \) where \( v \) is bounded and conormal. Now we can integrate the ODE \((r \partial_r - \gamma_1) \cdots (r \partial_r - \gamma_N)\) to see that \( u = B f \) has a partial polyhomogeneous expansion with all terms of the form \( r^\gamma_j \), \( j = 1, \ldots, N \), since each of these terms are killed by \( r \partial_r - \gamma_j \). \( \square \)
We wish to apply this lemma when $G$ is the Green function for $\Delta_g + V$, where $g$ and $V$ are polyhomogeneous. It is straightforward to extend this result slightly to show that it remains valid for some fixed $\nu$ provided both $g$ and $V$ only lie in $A^{0}_{\text{phg}} + A^\nu$. We leave details of this extension to the reader.

Finally, let us apply this to the equation $L \partial_y u = f$. We know initially that $f \in A^0$, hence at the first step, $\partial_y u \in A^0_{\text{phg}} + A^2$. But this now gives that $f$ and the coefficients of $L$ lie in $A^0_{\text{phg}} + A^2$, hence $\partial_y u \in A^0_{\text{phg}} + A^4$. Continuing on in this manner gives a complete expansion for $\partial_y u$, and from this we deduce also that $u$ is polyhomogeneous. This concludes the proof of Theorem [1].

Let us remark what is really going on in this proof. Once we have established that $u$ is conormal, i.e., that it is infinitely differentiable with respect to $r \partial_r$, $\partial_\theta$ and $\partial_y$, then we can treat the Monge–Ampère equation satisfied by $u$ as an ODE in the $r$ direction; all dependence in the other directions can be treated parametrically, and in particular, $y$ and $\theta$ directions are harmless.

Donaldson [19] already indicated this sort of iteration method to obtain full differentiability with respect to $y$, using his estimates for $G$ on $C^0_w$. However, his method may not easily allow one to deduce the full conormality simply because it is based on differentiating with respect to $\partial_r$ rather than $r \partial_r$. This may seem like a minor thing, but since we must deal with functions with involve both integer powers of $r$ and $r^{1/\beta}$, then inevitably all expressions involving both of these types of monomials is only finitely differentiable in the wedge spaces. By contrast, a basic point when using edge spaces is that any function $r^\gamma$ should be regarded as ‘smooth’, for any $\gamma \in \mathbb{C}$.

**Determination of leading terms**

For various applications below, in particular the refinement of the calculation of curvature, we must determine the first few terms of the expansion of a solution of the Monge–Ampère equation.

**Proposition 4.3.** Let $\varphi$ be a solution of the Monge–Ampère equation \((\Delta_g + V)\varphi = 0\). Suppose that $\varphi \in D^0_{\text{phg}}$, and hence by that theorem, $\varphi \in A^0_{\text{phg}}$, then the asymptotic expansion of $\varphi$ takes the form

$$\varphi(r, \theta, y) \sim \sum_{j,k,\ell \geq 0} a_{jk\ell}(\theta, y) r^{j+k+\frac{\beta}{2}} (\log r)^\ell$$

(48)

as $r \searrow 0$. Certain coefficients are always absent; for example, $a_{00\ell} = 0$ for $\ell > 0$ and $a_{10\ell} \equiv 0$ for all $\ell$. If $a_{jk\ell} = 0$ for some $j, k$ for all $\ell > 0$, then we write this coefficient simply as $a_{jk}$. When $0 < \beta < 1/2$,

$$\varphi(r, \theta, y) \sim a_{00}(y) + a_{20}(y) r^2 + (a_{01}(y) \sin \theta + b_{01}(y) \cos \theta) r^{1+\frac{\beta}{2}} + a_{40}(y) r^4 + O(r^{4+\epsilon})$$

(49)

for some $\epsilon = \epsilon(\beta) > 0$; when $\beta = 1/2$, the asymptotic sum on the right includes an extra term $(a_{02}(y) \sin 2\theta + b_{02}(y) \cos 2\theta) r^4$; finally, if $1/2 < \beta < 1$, then

$$\varphi(r, \theta, y) = a_{00}(y) + (a_{01}(y) \sin \theta + b_{01}(y) \cos \theta) r^{1+\frac{\beta}{2}} + a_{20}(y) r^2 + O(r^{2+\epsilon})$$

(50)

for some $\epsilon = \epsilon(\beta) > 0$.

We begin with a lemma.
Lemma 4.4. The twisted Ricci potential $f_\omega$ has an expansion

$$f_\omega = \sum_{k=-1}^{n-1} c_{0k}(\theta, y)r^{2k+\frac{\Delta}{2}} + \sum_{k=0}^{n-1} (c_{1k}(\theta, y) + c_{2k}r \cos \theta + c_{3k}r \sin \theta)r^{2k}$$

where each $c_{jk}$ is a smooth function of its arguments.

Proof. It follows from (28) that

$$\frac{(\omega_0 + \sqrt{-1} \overline{\partial} \overline{\partial} \phi_0)^n}{|s|^{23-2} \omega_0^n} = \sum_{k=-1}^{n-1} \tilde{f}_{00}r^{2k+\frac{\Delta}{2}} + \sum_{k=0}^{n-1} (\tilde{f}_{1k} + \tilde{f}_{2k}r \cos \theta + \tilde{f}_{3k}r \sin \theta)r^{2k}.$$  (51)

where each $\tilde{f}_{jk}$ is a smooth function of the arguments $r \cos \theta, r \sin \theta$ and $y$. In addition, we have already noted that $\phi_0 = r^2 \Phi_0$ where $\Phi_0$ is also smooth as a function of $r \cos \theta, r \sin \theta$ and $y$. The result now follows directly from the equation

$$e^{-f_\omega} = \frac{(\omega_0 + \sqrt{-1} \overline{\partial} \overline{\partial} \phi_0)^n}{|s|^{23-2} \omega_0^n} e^{\mu \phi_0 - F_{\omega_0}},$$

where $F_{\omega_0}$ is defined by $\sqrt{-1} \overline{\partial} F_{\omega_0} = \text{Ric} \omega_0 - \omega_0 + (1 - \beta)\sqrt{-1} \overline{\partial} \log a$, (here we use the notation of Lemma 2.2) and the equation itself fixes a normalization for $F_{\omega_0}$, and again $F_{\omega_0}$ is smooth in these same arguments.

Proof of Proposition 4.3. The idea is quite simple. Since we now know that $\varphi$ has an asymptotic expansion, we simply substitute a ‘general’ expansion into the equation

$$\omega^n_\varphi/\omega^n = F(z, \varphi)$$  (52)

and determine the unknown exponents and coefficients. Since our main case of interest is when $F(z, \varphi) = e^{f_\omega - \mu \varphi}$, we shall explain the argument for this special function, but it should be clear that the same type of argument works in general.

Using the precise form of the expansion for $f_\omega$ determined above, the index set for $\varphi$ must be contained in

$$\Gamma := \{(j + k/\beta, \ell) : j, k, \ell \in \mathbb{N}_0\},$$

or in other words, the only terms which appear are of the form $a_{jk\ell}(\theta, y)r^{j+\frac{k}{\beta}}(\log r)\ell$. This is done inductively. Supposing that we know that this is true for all $j, k$ such that $j + k/\beta \leq A$, then we only need consider the action of $P_{11}$ on the next term in the series $a_{\gamma\ell}r^\gamma(\log r)\ell$. This must either be annihilated by $P_{11}$, i.e., $\gamma$ is an integer multiple of $1/\beta$, or else it must match a previous term in the expansion, i.e. $\gamma - 2 = j' + k'/\beta$. In either case, the form of the expansion propagates one step further.

Since the solution $\varphi$ is bounded, there are no terms $a_{000}(\log r)^\ell$ with $\ell > 0$, so using the convention in the statement of the theorem, the leading term is simply $a_{000}0$. Note further that $a_{00}$ depends only on $y$ but not on $\theta$. This can be seen by substituting in the equation. If $a_{00}$ were to depend nontrivially on $\theta$, then the term $P_{11}\varphi$ would contain $r^{-2}\partial^2_\theta a_{00}$, and this is not cancelled by any other term in the equation. Hence $a_{00} = a_{00}(y)$.

Similar reasoning can be applied to the next few terms in the expansion. We use discreteness of the set of exponents to progressively isolate the most singular terms after we substitute the putative expansion for $\varphi$ into the equation. Since $a_{00}$ is independent of $\theta$, $P_{11}a_{00}$, $P_{11}a_{00}$ and $P_{11}a_{00}$ are bounded. Hence if the next term in the expansion is $a_{\gamma\ell}r^\gamma(\log r)\ell$ with $\gamma \leq 2$, then
apply $P_{11}$ to it produces as its most singular term $r^{\gamma-2}(\log r)^{\ell} (\gamma^2 + \partial_\theta^2) a_{\gamma\ell}$. This shows immediately that either $\gamma$ must be an indicial root, i.e., $\gamma = 1/\beta$ if $\beta > 1/2$ with $a_{\gamma\ell}$ a linear combination of $\cos \theta$ and $\sin \theta$, or else $\gamma = 2$. Note that this also shows that $a_{10\ell} \equiv 0$ for all $\ell \geq 0$.

Assuming $\gamma < 2$ and $\ell > 0$, then using the leading order cancellation, then next most singular term in $P_{11} a_{01\ell} r^{\frac{1}{2}} (\log r)^{\ell} \equiv \gamma r^{\gamma-2}(\log r)^{\ell-1} a_{01\ell}$ with no other term to cancel it. This is impossible, so we have ruled out all terms with $\ell > 0$. If $\gamma = 2$ and $\ell > 0$, there is no longer a leading order cancellation, but we are left with the singular term $a_{20\ell}(r \log r)^{\ell}$, so $a_{20\ell} = 0$ when $\ell > 0$.

Now consider what happens to the term $a_{20} r^2$. It interacts with the leading order terms $a_{00}$ in $\varphi$ and $c_{00}$ in $f_\omega$ only. Neither of these depend on $\theta$, so we find that $a_{20}$ is a function of $y$ alone.

We can continue this same reasoning further. Applying $P_{11}$ to the next term in the expansion $a_{\gamma\ell} r^\gamma (\log r)^{\ell}$ beyond $a_{20} r^2$ produces a leading order term which is a nonzero multiple of $a_{\gamma\ell} r^{\gamma-2}(\log r)^{\ell}$ if $\ell > 0$. Even though this term is bounded now, there are no other log terms at the level $r^{\gamma-2}$ in $[52]$. On the other hand, if $\ell = 0$, then we end up with a term $r^{\gamma-2}(\gamma^2 + \partial_\theta^2) a_{\gamma0}$, and there are no terms in $[52]$ to cancel it either. Hence $\gamma$ must be one of the two indicial roots $k/\beta$, $k = 1$ or 2, and the coefficient must be a linear combination of $\cos k\theta$ and $\sin k\theta$.

We comment further on the cases $\beta = 1/2$ or $\beta = 1/4$. In the former, one might suspect that one would need a term $r^2 \log r a_{021}$ because applying $P_{11}$ to this should match the $r^0$ term coming from the leading coefficients of $\varphi$ and $f_\omega$. However, those coefficients do not depend on $\theta$, whereas $a_{021}$ would be a combination of $\cos 2\theta$ and $\sin 2\theta$, as above, so there is no interaction. This is also true for $\beta = 1/4$. \qed

**Remark 4.5.** It is worth noting that there are more terms in the expansion of the solution metric than in the expansion of the reference metric. Because of this, the computations in the appendix do not apply and one cannot conclude any boundedness of the bisectional curvatures when $\beta > 1/2$.

## 5 Maximum principle and the uniform estimate

We now recall the formulation of the maximum principle in this edge setting. The main issue is to find barrier functions which allow one to reduce to the classical maximum principle on $M \setminus D$. These barrier functions were introduced in $[20]$.

**Lemma 5.1.** Let $f$ be continuous on $M$ and satisfy $|f(r, \theta, y) - a(y)| \leq Cr^\gamma$ for some $a \in C^0(D)$ and $0 < \gamma < 1$. Then for $\epsilon$ small enough,

(i) For any $C \neq 0$, $f + C|s|_h^\epsilon$ achieves its maximum and minimum in $M \setminus D$;

(ii) For some $c > 0$, $c|s|_h^\epsilon \in \text{PSH}(M, \omega)$.

**Proof.** (i) The function $|s|_h^\epsilon$ is comparable to $r^{\epsilon/\beta}$, so for $C > 0$, $r \mapsto f(r, \theta, y) + C|s|_h^\epsilon$ strictly increases, hence cannot reach its maximum at $r = 0$.

(ii) If $h$ is a smooth Hermitian metric on $L_D$ with global holomorphic section $s$ so that $D = s^{-1}(0)$, then for any $b > 0$, we have $\sqrt{-1} \bar{\partial} \partial b \geq b \sqrt{-1} \bar{\partial} \partial \log b$. Setting $b := |s|_h^\epsilon$ gives

$$\sqrt{-1} \bar{\partial} \partial b \geq \sqrt{-1} \epsilon |s|_h^\epsilon \partial \bar{\partial} \log |s|_h = -\frac{1}{2} \epsilon |s|_h^2 R(h) > -C \omega,$$

where $C$ depends only on the choice of $\omega, h, s, \epsilon$, and $C^{-1} b \in \text{PSH}(M, \omega)$. \qed
The assumption on $f$ above holds in particular for $f \in \mathcal{C}_{0,\gamma}^0$, and more generally for $f$ and $\Delta_\omega f$ when $f \in \mathcal{D}_{0,\gamma}^0$. This lemma is used as follows. Replacing $|s|^\gamma_h$ by $c|s|^\gamma_h$ and letting $c$ tend to 0, we obtain estimates which are the same as those one would expect from the maximum principle on $M\setminus D$. See the proofs of Lemmas 5.1, 7.2, and 7.2 below for more on this. The uniqueness and a priori $C^0$ estimate when $\mu \leq 0$ are now immediate consequences.

**Lemma 5.2.** Solutions to the Monge–Ampère equation (32) with $s \leq 0$ are unique in either $\mathcal{D}_{0,\gamma}^0 \cap \text{PSH}(M,\omega)$ or $\mathcal{D}_{0,\gamma}^0 \cap \text{PSH}(M,\omega)$, and satisfy

$$||\varphi(s,t)||_{C^0(M)} \leq C = C(||f_\omega||_{C^0(M)}, M, \omega).$$

**Proof.** Uniqueness when $s < 0$ is proved in [26]; that argument carries over directly to this Monge–Ampère equation and either of the types of function spaces we are using here, because of Lemma 5.1. Finally, when $s = 0$ the result of Blocki [12] gives uniqueness in $L^\infty(M)$ up to a constant, but then that constant is uniquely determined since $\sup \varphi(0, t) = \lim_{s \to 0^-} \sup \varphi(s, t)$.

The same argument also shows that $||\varphi(s, t)||_{C^0(M)} \leq -2s^{-1}||f_\omega||_{C^0(M)}$, for each $s < 0$. One can then obtain a uniform estimate for all $s \leq 0$ as follows. First, by the above, we may assume that $s > S$, for some $S < 0$. With respect to the fixed smooth Kähler form $\omega_0$, (32) can be rewritten as

$$\omega^n_n = \omega_0^n F|s|^{2\beta - 2} e^{tf_\omega + ct - s\varphi},$$

where $F \in C^0(M)$. By the previous estimate, $||e^{tf_\omega + ct - s\varphi}||_{C^0(M)} \leq C$ uniformly in $s$. It follows that $||F|s|^{2\beta - 2} e^{tf_\omega + ct - s\varphi}||_{L^p(M, \omega^n_0)} \leq C_p$, for all $p \in (1, 1/(1 - \beta))$, with $C_p$ independent of $s$. Assuming this, by Kołodziej’s estimate [27] $\text{osc} \varphi(s, t) \leq C$, with $C > 0$ independent of $s, t$, and since by (32) $\varphi(s, t)$ changes sign then also $|\varphi(s, t)| \leq C$. \hfill \Box

### 6 The uniform estimate in the positive case

In contrast to the nonpositive curvature cases, when $\mu > 0$, there are well-known obstructions to the existence of an a priori $C^0$ estimate along the continuity path. In this section we review the standard theory due to Tian and others [49, 50] along with the necessary modifications to adapt it to our setting. For an alternative variational approach that can be applied to more general classes of plurisubharmonic functions we refer to [6].

#### 6.1 Poincaré and Sobolev inequalities

In this subsection we show that along the continuity path (30) one has uniform Poincaré and Sobolev inequalities.

We first prove that a uniform Poincaré inequality holds as soon as $s > \epsilon > 0$. The following argument is the analogue of [50] Lemma 6.12 in this edge setting, and also generalizes [34], Lemma 3] to higher dimensions. The second part is the same assertion as [19] Proposition 8]. The proof here takes advantage of the fine regularity results for solutions available to us.

**Lemma 6.1.** Denote by $\Delta_\omega \varphi(s)$ the Friedrichs extension of the Laplacian associated to $\omega_\varphi(s)$.

1. For any $s \in (0, \mu)$, $\lambda_1(-\Delta_\omega \varphi(s)) > s$.
2. For $s = \mu$, $\lambda_1(-\Delta_\omega \varphi(\mu)) \geq \mu$. If $(\Delta_\omega \varphi(\mu) + \mu)\psi = 0$ then $\nabla^1_{g_\varphi(\mu)} \psi$ is a holomorphic vector field tangent to $D$.  

33
Proof. (i) Let \( \psi \) be an eigenfunction of \( \Delta_{\omega(s)} \) with eigenvalue \(-\lambda_1\). Since \( \varphi(s) \) is polyhomogeneous, then the eigenfunctions of \( \Delta_{\omega(s)} \) are also polyhomogeneous. This is a special case of the main regularity theorem for linear elliptic differential edge operators from [36]. The proof uses the same pseudodifferential machinery described in §3 (although for this particular result it is possible to give a more elementary proof). The key fact we need, that is a consequence of the use of the Friedrichs extension, is that \( \psi \sim a_0 r^0 + a_1 r^{\frac{1}{2}} + a_2 r^2 + O(r^{2+\eta}) \) for some \( \eta > 0 \), and in particular there is no log \( r \) in this expansion.

A standard Weitzenböck formula states that

\[
\frac{1}{2} \Delta_g |\nabla_g f|_g^2 = \text{Ric}_g (\nabla_g f, \nabla_g f) + |\nabla^2 f|_g^2 + \nabla (\Delta_g f).
\]

Since \( \Delta_g = 2 \Delta_\omega \) and \( |\nabla^2 f|_g^2 = 2|\nabla^{1,0} \nabla^{1,0} f|^2 + 2(\Delta_\omega f)^2 \), this becomes

\[
\Delta_\omega |\nabla^{1,0} \psi|_g^2 = 2 \text{Ric}_g (\nabla^{1,0} \psi, \nabla^{0,1} \psi) + 2|\nabla^{1,0} \nabla^{1,0} \psi|^2 + 2\lambda_1^2 \psi^2 - 4\lambda_1 |\nabla^{1,0} \psi|_\omega^2.
\]

We now claim that

\[
\int_M \Delta_\omega^s |\nabla^{1,0} \psi|_{\omega^s,\omega^s}^2 = 0.
\]

This follows directly from the expansion of \( \psi \), since the worst term in the expansion of \( \nabla^{1,0} \psi \) is \( r^{\frac{1}{2} - 1} \). Hence if we integrate over \( r \geq \epsilon \) then the boundary term is of order \( c r^{\frac{1}{2} - 2} \) (taking into account the measure \( r d\theta dy \) on this boundary), and this tends to 0 with \( \epsilon \). This proves the claim. Thus integrating (54) and using that \( \text{Ric}_\omega(s) > s \omega(s) \) when \( s < \mu \) we see that \( \lambda_1 > s \).

(ii) When \( s = \mu \) this same argument yields \( \lambda_1 \geq \mu \). Moreover, equality holds precisely when \( \nabla^{1,0} \nabla^{1,0} \psi = 0 \) on \( M \setminus D \), i.e., \( \nabla^{1,0} \psi \) is a holomorphic vector field on \( M \setminus D \). Using the asymptotic expansion, \( \nabla^{1,0} \psi \) is continuous up to \( D \), and hence extends holomorphically to \( M \). Now, the coefficient of its \( \frac{\partial}{\partial \zeta} \) component equals \( g^{1j} \psi_j \). By [12] \( g^{1j} = O(r^{n'}) \), i.e., vanishes on \( D \) for \( j \neq 1 \), (and \( \psi \) is infinitely differentiable in the \( j \neq 1 \) directions), while \( g^{11} \) is uniformly positive however by the asymptotic expansion \( \psi_1 = O(r^{\frac{1}{2} - 1}) \). In conclusion \( \nabla^{1,0} \psi \) has no \( \frac{\partial}{\partial \zeta} \) component, hence is tangent to \( D \).

Let us now turn to an estimation of the Sobolev constant. First observe that the Sobolev inequality holds for the model edge metric \( g_\beta \), i.e.,

\[
||f||_{L^{\frac{2n}{n-2}}(M,g_\beta)} \leq C_\beta ||f||_{W^{1,2}(M,g_\beta)},
\]

and hence also for any metric uniformly equivalent to it. To check this, it suffices to show that this inequality holds in a neighbourhood of any point, and we only need to consider neighbourhoods around points of \( D \). But in these we can use the \((\zeta, Z)\) coordinate system to reduce to the standard Euclidean case. An alternate proof can be given which uses that the heat kernel on \((C^n, g_\beta)\) blows up no faster than \( t^{-n} \).

The point here, however, is that we must show that the Sobolev constant is uniform in \( s \) when \( s > \epsilon \). This can be done in a few different ways. The first is the most classical. It is well-known that a bound on the constant in the isoperimetric inequality implies a bound of the Sobolev constant, so we concentrate on the isoperimetric inequality. Croke [18] proved that if \((M,g)\) is smooth (and compact) and \( \text{Ric}(g) \geq c > 0 \), then the isoperimetric constant has a bound depending only on the dimension, the volume, and the constant \( c \). To prove Croke’s result in this setting we need to establish some facts about the geodesic flow. Since \((M,g)\) is incomplete, we need to be cautious of the geodesics which hit the edge. Using that \( g \) is
polyhomogeneous, and that the cone angle $2\pi \beta$ is less than $2\pi$, we can prove that if $p$ and $q$ are any two points in $M \setminus D$, then there is a minimizing geodesic which connects $p$ to $q$, and furthermore, every such geodesic does not intersect $D$. This follows from a simple comparison argument: if a geodesic $\gamma$ approaches $D$, then approximating by the model metric $g_\beta$, one can find strictly shorter comparison curves which are bounded away from $D$. This can also be used to show that the set of initial directions of geodesics emanating from a given point $p \in M \setminus D$ which hit $D$ has measure zero in the unit sphere bundle $S_pM \subset T_pM$. With these two observations, one may check that Croke’s bound on the isoperimetric constant is valid here too, and that this constant is controlled by the dimension and lower bound on the Ricci curvature only (since the volume is fixed in the Kähler class).

Using this behavior of geodesics we also obtain a uniform diameter bound. Indeed, if $p, q \in M \setminus D$, and if $\gamma$ is a minimizing geodesic connecting them, then since $\gamma$ does not hit $D$ we can apply the argument leading to Myer’s theorem to obtain that if the length of $\gamma$ is too large, then it must contain a conjugate point in its interior, which contradicts its minimality.

Another route to bounding the Sobolev constant uniformly uses a rather general result by Hajlasz–Koskela [24] (see also Saloff-Coste [43, Theorem 2.1]). This states that if $M$ is a metric measure space which has a volume doubling constant (i.e., there exists a constant $C_d > 0$ so that $\text{Vol}(B_p(2r)) \leq C_d \text{Vol}(B_p(r))$ for all $p \in M$ and $r > 0$), then if $M$ satisfies a weak local Poincaré inequality, then there is also a global Sobolev inequality, with perhaps a suboptimal exponent, and with constant depending only on the volume, diameter bound, Poincaré constant and doubling constant $C_d$. We have control of all the other quantities, so we must show only that $C_d$ is also controlled. This is done as follows. If $p \in M \setminus D$ and $R < \frac{1}{2}\text{dist}_{\omega_\phi(s)}(p, D)$, then this follows from standard volume comparison properties, which depend on the lower Ricci bound. If the radius is larger, or if $p \in D$, then we may approximate by the model metric $g_\beta$, where the computation is straightforward.

In either case, we have proved:

**Lemma 6.2.** Let $s > \epsilon$. There exists uniform constants $C > 0$ and $k > 2$ depending only on $(M, \omega, n)$ and $\epsilon$ so that for any $f \in W^{1,2}(M, \omega_\phi(s))$,

$$||f||_{L^{\frac{2k}{k-2}}(B(x,r), \omega_\phi(s))} \leq C||f||_{W^{1,2}(M, \omega_\phi(s))}.$$

**Remark 6.3.** We conclude this discussion by noting one other approach to the estimation of the Sobolev constants in the case when $D$ satisfies some positivity assumption (e.g., when $D$ is ample). Then general results of Demailly imply that $[D]$ can be approximated by cohomologous smooth $(1,1)$ forms with small negative part. Using this, and solving a regularized version of our Monge–Ampère equation, $\omega_\phi(s)$ can be approximated smoothly on compact subsets of $M \setminus D$ by Kähler forms $\omega_{\phi,s,j}$ whose Ricci curvature is positive and uniformly bounded away from zero. It is then not difficult to verify that the uniform bound on the Sobolev constant of $\omega_{\phi,s,j}$ implies that of $\omega_{\phi(s)}$.

### 6.2 Energy functionals

Unlike in the previous cases, there are well-known obstructions to obtaining a $C^0$ estimate in the positive case. The existence of such an estimate is then described in terms of the behavior of certain energy functionals. For more background we refer to [2, 4, 45, 50].
The energy functionals $I, J$, introduced by Aubin [2], are defined by

$$
I(\omega, \omega_\phi) = \frac{1}{V} \int_M \sqrt{-1} \partial \bar{\partial} \phi \wedge \omega^{n-1} \wedge \omega_\phi = \frac{1}{V} \int_M \phi (\omega^n - \omega^n_\phi),
$$

$$
J(\omega, \omega_\phi) = \frac{V^{-1}}{n+1} \int_M \sqrt{-1} \partial \bar{\partial} \phi \wedge \omega^{n-1} \wedge (n-l)\omega^{n-l-1} \wedge \omega_\phi^l.
$$

This definition certainly makes sense for pairs of smooth Kähler forms, and by Corollary 3.6 and the continuity of the mixed Monge–Ampère operators on $\text{PSH}(M, \omega_0) \cap C^0(M)$ [5, Proposition 2.3], these functionals can be uniquely extended to pairs $(\omega_0, \omega_\phi)$, with $\omega_0$ smooth and $\omega_\phi \in \mathcal{H}_{\omega_0}$, and hence also to $\mathcal{H}_\omega \times \mathcal{H}_\omega$, where now by $\omega$ we mean the reference metric given by (25). These functionals are nonnegative and equivalent,

$$
\frac{1}{n} J \leq I - J \leq \frac{n}{n+1} I \leq nJ.
$$

(56)

One use of these functionals is in deriving a conditional $C^0$ estimate.

**Lemma 6.4.** Let $s \in (0, \mu)$. Any $C^0(M) \cap \text{PSH}(M, \omega)$ solution $\phi(s)$ to (30) is unique. Moreover, if $\phi(s) \in D^{0,\gamma}_s$ then $\|\phi(s)\|_{C^0(M)} \leq C(1 + I(\omega, \omega_\phi(s)))$, for all $s \in (\epsilon, \mu)$.

**Proof.** The uniqueness is due to Berndtsson [9].

We now prove the estimate. Using the uniform estimates on the Poincaré and Sobolev constants, the arguments proceed much as in the smooth case [50, Lemma 6.19].

First, let $G_\omega$ be the Green function of $-\Delta_\omega$, i.e., $-\Delta_\omega G_\omega = -G_\omega \Delta_\omega = 1\text{d} - \Pi$, where $\Pi$ is the orthogonal projector onto the constants. (Note that this is contrary to our previous sign convention for $G$, but conforms with the usual convention for this estimate.) Necessarily, $\int_M G_\omega(\cdot, \bar{z})\omega^n(\bar{z}) = 0$. We claim that $A_\omega := -\inf_{M \times M} G_\omega < \infty$. Assuming this for the moment, we can write

$$
\varphi_s(z) = V^{-1} \int_M \varphi_s \omega^n - \int_M G_\omega(x, y) \Delta_\omega \varphi_s(y) \omega^n(y).
$$

Hence, since $-n < \Delta_\omega \varphi_s$,

$$
\sup \varphi(s) \leq \frac{1}{V} \int_M \varphi(s) \omega^n + nVA_\omega,
$$

(57)

To prove this claim about the Green function, recall that

$$
G(z, \bar{z}) = \int_0^\infty (H(t, z, \bar{z}) - \Pi(z, \bar{z})) \text{d}t,
$$

where $H$ is the heat kernel associated to this (Friedrichs) Laplacian, and $\Pi(z, \bar{z})$ is the Schwartz kernel of this rank one projector. This integral converges absolutely for any $z \neq \bar{z}$. We rewrite this as

$$
G(z, \bar{z}) = \int_0^1 H(t, z, \bar{z}) \text{d}t - \Pi(z, \bar{z}) + \int_1^\infty (H(t, z, \bar{z}) - \Pi(z, \bar{z})) \text{d}t.
$$

(58)

It follows easily from standard estimates that the integral from 1 to $\infty$ converges to a bounded function. On the other hand, by the maximum principle, $H > 0$, so the first term on the right is nonpositive. Finally, $\Pi(z, \bar{z}) = V^{-1}$ is just a constant, so $G$ is bounded below.

36
To conclude the proof, it suffices to prove \(-\inf \varphi(s) \leq -\frac{C}{\sqrt{t}} \int M \varphi(s)\omega^n_{\varphi(s)}\) (indeed, \(\varphi(s)\) changes sign by the normalization \([26]\) of \(f_\omega\), so \(||\varphi(s)||_{C^0(M)} \leq \text{osc } \varphi(s)\)). This can be shown in one of two ways. The first is by noting that Bando–Mabuchi’s Green’s function lower bound \([16]\) extends to our present setting, and thus \(A_{\omega(s)} < C\) uniformly in \(s\) and \(-\inf \varphi(s) \leq -\frac{1}{\sqrt{t}} \int M \varphi(s)\omega^n_{\varphi(s)} + nVC\). Indeed, the proof of their bound relies on an estimate of Cheng–Li \([16]\) of the heat kernel \(H_{\omega(s)}(t, z, \bar{z})\) \(-\frac{C}{\sqrt{t}} \int M \varphi(s)\omega^n_{\varphi(s)}\). The following is an extension of a formula \(\omega = 2\varphi\) from Lemma 6.1, when \(f_\omega = f - \mu\varphi - \frac{\log \omega^n_{\varphi}}{\omega_{\varphi}} + c\varphi\) with \(c\varphi\) a constant. The formula then follows since both sides vanish when \(\omega_{\varphi} = \omega\).

\[\text{Lemma 6.5.} \quad \text{One has,} \quad E^0_\omega(\omega, \omega) = \frac{1}{V} \int_M \log \frac{\omega^n_{\varphi}}{\omega^n_{\varphi}} - \mu(I - J)(\omega, \omega) + \frac{1}{V} \int_M f_\omega(\omega^n - \omega^n_{\varphi}). \tag{60}\]

Proof. For any smooth \((t)\) path \(\{\omega_{\varphi}\} \subset H_{\omega_{\varphi}}\) connecting \(\omega\) and \(\omega_{\varphi}\) \([50\), p. 70\],

\[\int M \Delta_{\varphi}\dot{\varphi}\left(\log \frac{\omega^n_{\varphi}}{\omega^n} + 1 + \mu\varphi - f_\omega\right)\omega^n_{\varphi}, \tag{61}\]

and this coincides with \(dE^0_\omega(\dot{\varphi})\) since \(f_{\omega_{\varphi}} = f_\omega - \mu\varphi - \frac{\log \omega^n_{\varphi}}{\omega_{\varphi}} + c\varphi\) with \(c\varphi\) a constant. The formula then follows since both sides vanish when \(\omega_{\varphi} = \omega\).

As noted in the Introduction, a key property of the continuity path \([31]\) is the monotonicity of \(E^\beta_0\). Monotonicity of similar twisted K-energy functionals was noted, e.g., in \([42]\), and the following is the analogue of \([42]\) Lemma 9.3.

\[\text{Lemma 6.6.} \quad E^\beta_0\text{ is monotonically decreasing along the continuity path }[31]. \]

Proof. By \([31]\), \(\sqrt{-1}\partial\bar{\partial}f_{\omega_{\varphi}} = -(\mu - s)\sqrt{-1}\partial\bar{\partial}\varphi\), and from \([31]\) we have \((\Delta_{\varphi} + s)\dot{\varphi} = -\varphi\). It follows that

\[\frac{d}{ds} E^0_\omega(\omega, \omega_{\varphi(s)}) = -\frac{\mu - s}{V} \int_M \dot{\varphi}\Delta_{\varphi}(\Delta_{\varphi} + s)\dot{\varphi}\omega^n_{\varphi}, \tag{62}\]

and this is nonpositive by the positivity of \(\Delta_{\varphi} + s\Delta_{\varphi}\), which is immediate for \(s \leq 0\), and follows from Lemma \([42]\) when \(s \in (0, \mu)\).
We say, following Tian [49], that $E_0^\beta$ is proper if $\lim_{j \to \infty} (I - J)(\omega, \omega_j) = \infty$ implies $\lim_{j \to \infty} E_0^\beta(\omega, \omega_j) = \infty$. From Lemmas 6.4 and 6.6 we have:

**Corollary 6.7.** Let $\varphi(s) \in D^{0,1}_s \cap \text{PSH}(M, \omega)$. If $E_0^\beta$ is proper then $||\varphi(s)||_{c^0(M)} \leq C$, independently of $s \in (\epsilon, \mu)$.

We also note that as observed by Berman [6], an alternative proof of Corollary 6.7 follows by combining Kołodziej’s estimate [27] and the following result contained in [8] Lemma 6.4] and [6] (note that $\varphi(s)$ change sign).

**Lemma 6.8.** [8, 6] Suppose $J(\omega, \omega, \varphi) \leq C$. Then for each $t > 0$ there exists $C' = C'(C, M, \omega, t)$ such that $\int_M e^{-t(\varphi - \sup \varphi)} \omega_n \leq C'$.

We next recall the definition of Tian’s invariants [45, 46]

$$\alpha_{\Omega, \chi} := \sup \left\{ a : \sup_{\varphi \in PSH(\Omega, \omega_0)} \int_M e^{-a(\varphi - \sup \varphi)} \chi_n < \infty \right\}, \quad \alpha(M) := \alpha_{c_1(M), \omega_0},$$

$$\beta_{\Omega, \omega} := \sup \left\{ b : \text{Ric} \chi \geq b \chi, \chi \in \mathcal{H}_\omega \right\}, \quad \beta(M) := \sup \left\{ b : \text{Ric} \lambda \geq b \lambda, \lambda \in \mathcal{H}_c \right\},$$

where the measure $\chi_n$ is assumed to have density in $L^p(M, \omega_0^p)$, for some $p > 1$, and where, for emphasis, $\mathcal{H}_\omega$ is given by (15) and, when $M$ is Fano, $\mathcal{H}_c$ denotes the space of smooth Kähler forms representing $c_1(M)$ (and finally, as always $\Omega = [\omega_0] = [\omega]$, $\omega_0$ a smooth Kähler form, and $\omega = \omega(\beta)$ the reference Kähler edge current). These invariants are always positive as shown by Tian when $\chi$ is smooth, and hence by the Hölder inequality also in general. For some relations between $\alpha_{\Omega, \omega}$ and $\alpha_{c_1, \omega_0}$ we refer to [6] where such invariants for singular measures were studied in depth.

**Lemma 6.9.** Suppose that $\alpha_{\Omega, \omega} - \frac{\mu}{n+1} > \epsilon$. Then $E_0^\beta \geq \epsilon I - C$, for some $C \geq 0$.

**Proof.** Again we follow the classical argument [48]. p. 164]. [50], p. 95]. Note that for any $a \in (0, \alpha_\omega)$ there exists a constant $C_a$ such that $\frac{1}{a} \int_M \log \frac{\varphi_0^a}{\varphi^a} \omega_n^a \geq aI(\omega, \omega_0) - C_a$. Indeed, by [57], and Jensen’s inequality,

$$e^{C_a} \geq \frac{1}{a} \int_M e^{-a(\varphi - \frac{1}{a} \int_M \varphi \omega_n^a)} \omega_n^a$$

$$= \frac{1}{a} \int_M e^{-\log \frac{\varphi_0^a}{\varphi^a} - a(\varphi - \frac{1}{a} \int_M \varphi \omega_n^a)} \omega_n^a \geq e^{-\frac{1}{a} \int_M (\log \frac{\varphi_0^a}{\varphi^a} + a(\varphi - \frac{1}{a} \int_M \varphi \omega_n^a)) \omega_n^a}.$$ 

By [58] and (60) it then follows that $E_0^\beta \geq (a - \frac{\mu}{n+1}) I - C$. \hfill \square

**Corollary 6.10.** For all $s \in (-\infty, \frac{n+1}{n} \alpha_{\Omega, \omega}) \cap (-\infty, \mu]$, we have $||\varphi(s)||_{c^0(M)} \leq C$, with $C$ independent of $s$.

**Proof.** When $\alpha_{\Omega, \omega} > \frac{\mu}{n+1}$ the result follows from Lemma 6.9 and Corollary 6.7 (note, as explained in [49], that there is no difficulty in treating the interval $s \in (0, \epsilon)$ for some $\epsilon > 0$). In general the classical derivation [50] carries over. \hfill \square

This conditional $C^0$ estimate implies of course, given the other ingredients of the proof of Theorem 2, that $\beta_{\Omega, \omega} \geq \min \{ \mu, \frac{n+1}{n} \alpha_{\Omega, \omega} \}$, just as in the smooth setting. We remark that Donaldson [19] conjectured that when $D \subset M$ is a smooth anticanonical divisor of a Fano manifold, then $\beta(M) = \sup \{ \beta : (29) \text{ admits a solution with } \mu = \beta \}$. Our results have direct bearings on this problem that we will discuss elsewhere.
7 The Laplacian estimate

Let \( f : M \to N \) be a holomorphic map between two complex manifolds. The Chern–Lu inequality was originally used by Lu to bound \( |\partial f|^2 \) when the target manifold has negative bisectional curvature [33] under some technical assumptions. This inequality was later used by Yau [56] together with his maximum principle to greatly generalize the result to the case where \((M, \omega)\) is a complete Kähler manifold with a lower bound \( C_1 \) on the Ricci curvature, and \((N, \eta)\) is a Hermitian manifold whose bisectional curvature is bounded above by a negative constant \(-C_3\). These results lead to Yau’s Schwarz lemma, which says that the map \( f \) decreases distances in a manner depending only on \( C_3 > 0 \) and \( C_1 \).

In a related direction, the use of the Chern–Lu inequality to prove a Laplacian estimate for complex Monge–Ampère equations seems to go back in print at least to Bando–Kobayashi [3], who considered the case \( \text{Ric} \omega \geq -C_2 \eta \) and \( C_3 \) arbitrary (not necessarily positive) in the context of constructing a Ricci flat metric on the complement of a divisor. Next, the case \( \text{Ric} \omega \geq -C_1 \omega \) (and again \( C_3 \) arbitrary) was used in proving the a priori Laplacian estimate for the Ricci iteration [42].

The point of Proposition 7.1 below is to state the Chern–Lu inequality in a unified manner that applies to a wide range of Monge–Ampère equations that appear naturally in Kähler geometry. It makes the Laplacian estimate in these settings slightly simpler, and the explicit dependence on the geometry more transparent. In addition, the Chern–Lu inequality applies in some situations where the standard derivation [1, 57, 44] of the Aubin–Yau Laplacian estimate may fail (as in the case of the Ricci iteration) or give an estimate with different dependence on the geometry (which will be crucial in our setting). While Proposition 7.1 below is certainly folklore among experts, it seems that it is less well-known than it deserves to be. In particular, we are not aware of a treatment of the Aubin–Yau or Calabi–Yau Theorems that uses it.

7.1 The Chern–Lu inequality

Let \((M, \omega), (N, \eta)\) be compact Kähler manifolds and let \( f : M \to N \) be a holomorphic map with \( \partial f \neq 0 \). The Chern–Lu inequality [17, 33] is

\[
\Delta_\omega \log |\partial f|^2 \geq \frac{\text{Ric} \omega \otimes \eta(\partial f, \bar{\partial} f)}{|\partial f|^2} - \frac{\omega \otimes R^N(\partial f, \bar{\partial} f, \partial f, \bar{\partial} f)}{|\partial f|^2}, \quad \text{on } M.
\]

Since the original statement [32, (7.13)], [33, (4.13)] contains a misprint, we include a direct and slightly simplified derivation (since we restrict to the Kähler setting) for completeness.

Write \( \partial f : T^{1,0}M \to T^{1,0}N \). Then \( \partial f \) is a section of \( T^{1,0}M \otimes T^{1,0}N \) given in local holomorphic coordinates by \( \partial f = \frac{\partial f}{\partial z^i} dz^j \otimes \frac{\partial}{\partial w^i} \). With respect to the metric induced by \( \omega \) and \( \eta \) on the product bundle above,

\[
u := |\partial f|^2 = g^{ij} h_{jk} \frac{\partial f}{\partial z^i} \frac{\partial f}{\partial z^j}.\]

Compute in normal coordinates at a point

\[
\Delta_\omega u = \sum_p u_{p\bar{p}} = -\sum_{i,j,p} g^{ij} h_{jk} f_{i,j}^k + \sum_{i,p} h_{jk,d,m} f_{i,p}^d f_{i,j}^m f_{i,j}^k + \sum_{i,j,p} f_{i,j,p}^j f_{i,j}^k = \text{Ric} \omega \otimes \eta(\partial f, \bar{\partial} f) - \omega \otimes R^N(\partial f, \bar{\partial} f, \partial f, \bar{\partial} f) + \sum_{i,j,p} f_{i,j,p}^j f_{i,j}^k.
\]
By the Cauchy–Schwarz inequality, \( u^2 \sum_{i,j,p} f^j_{,ip} f^j_{,ip} \geq \sum_k u_k u_k \), and since \( \Delta_\omega \log u = \Delta_\omega u / u - \sum_k u_k u_k / u^2 \), the desired inequality follows.

One particularly useful form of the Chern–Lu inequality is when \( f \) is the identity map.

\[ \text{Proposition 7.1.} \quad \text{In the above, let } f = \text{id} : (M, \omega) \to (M, \eta) \text{ be the identity map, and assume that } \text{Ric}_\omega \geq -C_1 \omega - C_2 \eta \text{ and that } \text{Bisecc}_\eta \leq C_3, \text{ for some } C_1, C_2, C_3 \in \mathbb{R}. \text{ Then,} \]

\[ \Delta_\omega \log |\partial f|^2 \geq -C_1 - (C_2 + 2C_3)|\partial f|^2. \]  \( (64) \)

In particular, if \( \omega = \eta + \sqrt{-1} \partial \bar{\partial} \varphi \) then

\[ \Delta_\omega \left( \log \text{tr}_\omega \eta - (C_2 + 2C_3 + 1) \varphi \right) \geq -C_1 - (C_2 + 2C_3 + 1)n + \text{tr}_\omega \eta. \]  \( (65) \)

Hence, \( \omega \geq C \eta \) for some \( C > 0 \) depending only on \( C_1, C_2, C_3, n \) and \( ||\varphi||_{C^0(M)} \).

**Proof.** By \((63)\), \( u = \text{tr}_\omega \eta \). The assumption on \( \text{Ric}_\omega \) implies that

\[
\text{Ric}_\omega \otimes \eta(\partial f, \bar{\partial} f) = g^{ij}g^{kl}R_{ij}h_{kl} \geq -C_1 g^{ij}g^{kl}g_{ij}h_{kl} - C_2 g^{ij}g^{kl}h_{ij}h_{kl}
= -C_1 \text{tr}_\omega \eta - C_2 (\eta, \eta)_\omega \geq -C_1 \text{tr}_\omega \eta - C_2 (\text{tr}_\omega \eta)^2,
\]

where the last inequality follows from the identity [10, Lemma 2.77]

\[
(\eta, \eta)_\omega = (\text{tr}_\omega \eta)^2 - n(n - 1) \eta \wedge \eta \wedge \omega^{n-2}. \]

Similarly, we also have

\[
-\omega \otimes R^N (\partial f, \bar{\partial} f, \partial f, \bar{\partial} f) = g^{ij}g^{kl}R_{ij}^{Nkl} \geq -C_3 g^{ij}g^{kl}(h_{ij}h_{kl} + h_{il}h_{kj}) \geq -2C_3 (\text{tr}_\omega \eta)^2.
\]

Thus, \((64)\) follows from \((62)\). Since \( \text{tr}_\omega \eta = n - \Delta_\omega \varphi \), equation \((65)\) follows from \((64)\). \( \square \)

### 7.2 The Laplacian estimate in the singular Hölder spaces

We now apply the Chern–Lu inequality to obtain an a priori Laplacian estimate for the continuity method \((32)\). For solutions of \((30)\) it gives a bound depending, in addition to the uniform norm, on an upper bound on the sectional curvature of the reference metric; in contrast the well-known Aubin–Yau bound depends on a lower bound for the sectional curvature \([1, 57, 44]\).

\[ \text{Lemma 7.2.} \quad \text{Suppose that there exists a reference metric } \omega \in \mathcal{H}_{\omega_0} \text{ with } \text{Ric}_\omega \geq -C_2 \omega \text{ and } \text{Bisecc}_\omega \leq C_3, \text{ on } M \setminus D, \text{ where } C_2 \in \mathbb{R} \cup \{ \infty \}, C_3 \in \mathbb{R}. \text{ Let } s > S. \text{ Solutions to } (32) \text{ in } \mathcal{D}^{1,\gamma}_{s,t} \cap \text{PSH}(M,\omega), \text{ satisfy} \]

\[ ||\Delta_\omega \varphi(s,t)||_{C^0(M)} \leq C(||f_\omega||_{C^0(M)}, S, (1-t)C_2, C_3), \]  \( (66) \)

where \((1-t)C_2\) is understood to be \(0\) when \( t = 1 \). Moreover, \( \frac{1}{C} \omega \leq \omega_{\varphi(s)} \leq C\omega \).

**Proof.** Along the continuity path \((32)\),

\[
\text{Ric}_\omega \varphi = (1-t)\text{Ric}_\omega + s\omega_\varphi + (\mu t - s)\omega + (1-\beta)[D] \geq S\omega_{\varphi_{s,t}} - (1-t)C_2 \omega,
\]
Hence, the assumptions of Proposition 7.1 are satisfied, and the desired estimates follow directly from (65) if the maximum of \( \log \text{tr} \omega - A \varphi \) takes place in \( M \setminus D \).

Next, suppose the maximum is attained on \( D \). We claim that \( \log \text{tr} \omega - A \varphi \in C^{0,\tilde{\gamma}} \), for any \( \tilde{\gamma} \leq \min\{\frac{1}{\beta} - 1, \gamma\} \). Indeed, in the local coordinates \( z_1, \ldots, z_n \), \( g_{i\overline{j}} = \frac{1}{\det g} A_{i\overline{j}} \), where \( A \) is the cofactor matrix of \( [g] \). Since \( A_{i\overline{j}} \) is a polynomial in the components \( g_{k\overline{k}} \), it too lies in \( C^{0,\tilde{\gamma}} \). In addition, \( 1/\det g = e^{-f - \epsilon_t + s\varphi} / \det g = |z|^2 - 2\beta F \) for some \( F \in C^{0,\tilde{\gamma}} \), hence this lies in \( C^{0,\tilde{\gamma}} \) for \( \tilde{\gamma} \leq \frac{1}{\beta} - 1 \). Hence \( \log \text{tr} \omega = g_{i\overline{j}} g_{j\overline{i}} \in C^{0,\tilde{\gamma}} \), proving the claim.

Now by Lemma 5.1 applied to \( f := \log \text{tr} \omega - A \varphi \), we have that \( f + |s|_h \) achieves its maximum away from \( D \) for \( \epsilon < \beta \gamma \). By (65) and Lemma 5.1 (ii) we have

\[
\Delta \varphi(f + N^{-1}|s|_h) \geq -C_1 - (2C_3 + 1)n + (1 - C/N)\text{tr} \omega.
\]

The maximum principle thus implies \( \text{tr} \omega \leq C = C(C_1, C_3, ||\varphi||_{C^0(M), \omega}) \), and so \( \omega \geq C \omega \). Going back to (30), we have \( \omega \leq C \omega^0 \) (with \( C \) depending on \( ||f_\omega||_{C^0(M)} \) and \( ||\varphi(s, t)||_{C^0(M)} \)), and so also \( \omega \leq \omega \).

\[
\square
\]

8 H"older estimates for second derivatives

In the interior of \( M \setminus D \) one may apply the Evans–Krylov regularity theorem for Monge–Ampère equations (Theorem 4.1) to conclude the existence of an a priori interior \( C^{2,\gamma} \) estimate for a solution \( \varphi \) on any ball \( B' \) provided one knows a \( C^0 \) estimate for \( \varphi \) and for \( \Delta \omega \varphi \). This relies crucially on uniform ellipticity of the Laplacian, however, and hence does not apparently hold near \( D \).

In this section we explain how to adapt Theorem 4.1 to this setting. In fact, we do this in two quite different ways. We first show that this estimate carries over almost directly in the setting of edge H"older spaces, using scaling arguments. This is quite simple, but of course hides the details of the Evans–Krylov theorem itself. Another approach, which works in the setting of wedge H"older spaces, is to adapt the proof of Theorem 4.1 to work in these spaces. Since this is of independent interest, we present it too. Note, however, that because we have proved the theorem about higher regularity of solutions, it does not matter at all whether we work in the edge or wedge H"older spaces. This is because if we use either of these a priori estimates to pass to a limit and find a solution in \( D^{0,\gamma'} \) for some \( \gamma' < \gamma \), then we know by Theorem 1 that this solution is polyhomogeneous, regardless of whether \( s = e \) or \( w \), due to the inclusion \( D^{0,\gamma} \subset D^{0,\gamma'} \) (see (40)).

8.1 \( C^2_{e,\gamma} \) estimates

**Theorem 8.1.** Fix any \( 0 < \beta < 1 \). Let \( \varphi = \varphi(s, t) \in C^{2,\gamma}_e \cap \text{PSH}(M, \omega) \) be a solution to (92); by Theorem 4.1 is polyhomogeneous. Suppose that \( ||\varphi||_{C^0(M)} \leq C_0, ||\varphi||_{e,2,0} \leq C_2 \). Then,

\[
||\varphi||_{e,2,\gamma} \leq C' = C_0(C_0, C_2).
\]

**Proof.** We have already stated the Evans–Krylov bound in Theorem 4.1 and in our use of that theorem there we described a rescaling procedure for solutions of this Monge–Ampère equation near an edge. Pulling back by the dilation map \( S_\lambda \), we obtain a family of metrics

\[
\lambda^{-2}S_\lambda^2 g_{\varphi(s)};
\]

\[
\square
\]
restricting \( g_{\varphi(s)} \) to a set \( B \times S^1 \), where \( B \) is a ball centered at \((r_0, y_0)\) with radius \( \frac{1}{2}r_0 \), then this family of pullbacks is a family of metrics on \( B_1 \times S^1 \) where \( B_1 \) is a ball centered at \((1, 0)\) with radius \( \frac{1}{2} \). For each \( s \), this family converges to \( g_\beta \) as \( \lambda \to 0 \). However, the \( C^0 \) bound on the Laplacian of \( \varphi(s) \) shows that this convergence is actually uniform and depends only on the constants \( C_0, C_2 \).

Now apply the Evans–Krylov estimate in the half-ball \( B'_1 \times S^1 \), and scale back to half of the original ball. The edge Hölder norm is invariant under this rescaling, and we conclude that

\[
||\varphi(s)||_{C^{2,\gamma}} = \sup_{B \times S^1} ||\varphi(s)||_{2,\gamma; B \times S^1} \leq C'_2,
\]

as claimed.

8.2 \( D^{0,\gamma}_w \) estimates

To conclude our treatment of estimates for this problem, we now show how to obtain a priori \( D^{0,\gamma}_w \) estimates for a sequence of solutions. In fact, we can do this simply by working in singular holomorphic coordinates and following the standard proof. There is one delicate point in carrying this out, which is handled by appealing to the form of the asymptotic expansion for solutions (Proposition 4.3).

We follow the lecture notes of Siu [44] and Blocki [13, §5], as well as the treatment of the real Monge–Ampère equation by Gilbarg and Trudinger [22], with the modification by Wang–Jiang [54], Blocki [11], that formulates the problem in divergence form, and allows us to work with any cone angle in \((0, 2\pi]\).

**Theorem 8.2.** Let \( \varphi = \varphi(s, t) \in D^{0,\gamma}_w \cap \text{PSH}(M, \omega) \) be a solution to (32). Then,

\[
[\varphi]_{D^{0,\gamma}_w} \leq C,
\]

where \( \gamma > 0 \) and \( C \) depend only on \( M, \omega, \beta, ||\Delta_\omega \varphi||_{C^0}, ||\varphi||_{C^0} \).

**Proof.** Away from \( D \) the estimates are derived by the standard proof. Thus consider

\[
\log \det(\psi_{ij} + \varphi_{ij}) = tf_\omega - s\varphi + \log \det \psi_{ij} =: \log h,
\]

where \( \psi \) is a local Kähler potential for \( \omega \) on some neighborhood intersecting \( D \) non-trivially. We work with \( u := \psi + \varphi \). By Lemma 2.2 \( \psi \in C^{2,\gamma}_w \).

Let \( \eta \in \mathbb{C}^n \) be a unit vector, and consider \( u \) as a function of \((\zeta, Z)\). Then

\[
(\log \det \nabla^{1,1} u)_{\eta\bar{\eta}} = -u^{ij}u^{kl}_\eta u^{\eta\bar{l}}u_{\eta\bar{\eta}ij} + u^{ij}_{\eta\bar{\eta}ij}.
\]

Thus, letting \( w := u_{\eta\bar{\eta}} \), we have

\[
u^{ij}w_{ij} \geq (\log h)_{\eta\bar{\eta}} = \frac{h_{\eta\bar{\eta}}}{h} - \frac{|h_{\eta\bar{\eta}}|^2}{h^2};
\]

which can be rewritten in divergence form,

\[
(hu^{ij}w_i)_j \geq \eta^j(\eta^k h_k)_l - g, \quad g := \frac{|h_{\eta\bar{\eta}}|^2}{h}.
\]

We state estimates for \( h \), of which we only use the Lipschitz bound.
Lemma 8.3. For all $\beta \leq 1$, \[ ||h||_{D^0_w} \leq C, \] while \[ ||h||_{D^{0,0}_w} \leq C \] for $\beta \leq 1/2$. 

Proof. By (23), 
\[
det \psi_{ij} = F_1 + F_2|\zeta|^2 + F_3|\zeta|^{q-2} + F_4|\zeta|^{1+q/2} + F_5|\zeta|^{q/2} + o(|\zeta|^{q/2}). \] (72)

Thus $\log \det \nabla^1 \psi$ is in $C^{0,1}$, and belongs to $C^{2,0}_w$ if $\beta \leq 1/2$. Since $f_\omega, \varphi(s,t)$ are uniformly in $D^{2,0}_w$ by (2.3) and Lemma 7.2 then $||(\log h)_{\eta\eta}|| \leq C$, and as $\eta$ was arbitrary, $||\log h||_{D^{2,0}_w} \leq C$, whenever $\beta \leq 1/2$. 

Theorem 8.4. [22, Theorem 8.18] Let $\Omega \subset \mathbb{R}^m$, and assume $B_{4\rho} = B_{4\rho}(y) \subset \Omega$. Let $L = D_i(a^{ij}D_j + b^i) + c^iD_i + d$ be strictly elliptic $\lambda I < [a^{ij}]$, with $a^{ij}, b^i, c^i, d \in L^\infty(\Omega)$, satisfying 
\[
\sum_{i,j} |a^{ij}|^2 < \Lambda^2, \quad \lambda^{-2} \sum_{i,j} (|b^i|^2 + |c^i|^2) + \lambda^{-1}|d| \leq \nu^2.
\]

Then if $U \in W^{1,2}(\Omega)$ is nonnegative and satisfies $LU \leq g + D_i f^i$, with $f^i \in L^q, g \in L^{q/2}$ with $q > m$, then for any $p \in [1, m/m-2)$, 
\[
\rho^{-m/p}||U||_{L^p(B_{2\rho})} < C(\inf_{B_{2\rho}} U + \rho^{1-m/q}||f||_{L^q(B_{2\rho})} + \rho^{2-2n/q}||g||_{L^{q/2}(B_{2\rho})})
\]
\[
< C(\inf_{B_{2\rho}} U + \rho||f||_{L^\infty(B_{2\rho})} + \rho^2||g||_{L^\infty(B_{2\rho})}),
\]
with $C = C(n, \Lambda/\lambda, \nu, q, p)$. 

Lemma 8.5. One has 
\[
\sup_{B_{2\rho}} w - \frac{1}{|B_\rho|} \int_{B_\rho} w \leq C \left( \sup_{B_{2\rho}} w - \sup_{B_\rho} w + \rho(\rho + 1) \right),
\] (73)
with $C = C(\beta, M, \omega)$. 

Note that here integration is with respect to the volume form $\omega^n$, which in singular coordinates is equivalent to the Euclidean volume form for $|d\zeta|^2 + |dZ|^2$. 

Proof. By (70) $v := \sup_{B_{2\rho}} w - w$ satisfies 
\[
(hu^{ij}v_i)_j \leq g - \eta^j(\eta^k h_k)_l.
\] (74)

By Lemma 7.2 we have uniform positive bounds on $[u_{ij}]$ and $[u^{ij}]$ as well as on $[a^{ij}] := h[a^{ij}]$ and its inverse, in these coordinates. This, together with Lemma 8.3 and Theorem 8.4 gives the desired inequality provided that $v \in W^{1,2}$. But by the form of the asymptotic expansion of solutions (50) we have $v_j = u_{\eta\eta j} \in L^2$ for any $\eta$ and $j$: indeed $((a_{01}(Z) \sin \theta + b_{01}(Z) \cos \theta) r^{\frac{1}{m}})_{11} = 0$, hence $u_{11} = o_{20}(Z) + O(r^2)$ for some $\epsilon = \epsilon(\beta) > 0$ when $r$ is sufficiently small, and so $u_{11} = O(r^{-1}) \in L^2$, while $u_{131} = O(r^{\frac{m}{m-2}}) \in L^2$, and other third derivatives of $u$ are even more regular. 

43
The rest of the proof of Theorem 8.2 is now identical to the standard proof. Let \( \{V_j\}_{j=1}^n \) be smooth vector fields on \( M \setminus D \) that span \( T^{1,0}M \) over \( M \setminus D \) and that on a local chart near \( D \) are given by \( V_1 := \frac{1 - \beta}{\partial \bar{z}}, V_k := \frac{\partial}{\partial z_k}, \ k = 2, \ldots, n \), and denote

\[
M(\rho) := \sup_{|\zeta|, |Z| \in (0, \rho)} \sum_{j=1}^n V_j \zeta_j u, \quad m(\rho) := \inf_{|\zeta|, |Z| \in (0, \rho)} \sum_{j=1}^n V_j \zeta_j u
\]

Our goal is to show that \( \nu(\rho) := M(\rho) - m(\rho) \) is Hölder continuous with respect to \( g_\beta \), i.e., \( \nu(\rho) \leq C \rho^\gamma \), for some \( \gamma > 0 \), or equivalently that \( \nu(\rho) \leq (1 - \epsilon)\nu(2\rho) + \sigma(\rho) \), for some \( \epsilon \in (0, 1) \) and some non-decreasing function \( \sigma \) [22, Lemma 8.23]. Let

\[
M_\eta(\rho) := \sup_{|\zeta|, |Z| \in (0, \rho)} u_{\eta \bar{\eta}}, \quad m_\eta(\rho) := \inf_{|\zeta|, |Z| \in (0, \rho)} u_{\eta \bar{\eta}}, \quad \nu_\eta(\rho) := M_\eta(\rho) - m_\eta(\rho).
\]

Equation (73) implies

\[
\sup_{B_\rho} \frac{1}{|B_\rho|} \int_{B_\rho} w \leq C (\nu_\eta(2\rho) - \nu_\eta(\rho) + \rho(\rho + 1)),
\]

and so it remains to obtain a uniform inequality for \( w - \inf_{B_\rho} w \).

Note that \( DF|_{A}(A - B) \leq F(A) - F(B) \), by concavity of \( F(A) := \log \det A \) on the space of positive Hermitian matrices. Since \( DF|_{\nabla^1,1_u} = (\nabla^{1,1}u)^{-1} \), we have

\[
u^{ij}(y)(u_{ij}(y) - u_{ij}(x)) \leq \log \det u_{ij}(y) - \log \det u_{ij}(x) \leq [h]_{w,0,1}|y - x|.
\]

We now decompose \((u^{ij})\) as a sum of rank one matrices. This will result in the previous equation being the sum of pure second derivatives for which we can apply our estimate from the previous step. By uniform ellipticity this decomposition can be done uniformly in \( y \) [14, p. 103]. Namely, we can fix a set \( \{\gamma_k\}_{k=1}^N \) of unit vectors in \( \mathbb{C}^n \) (which we can assume contains \( \gamma_1 = \eta \) as well as a unitary frame of which \( \eta \) is an element) and write

\[
(u^{ij}(y)) = \sum_{k=1}^N \beta_k(y) \gamma_k^* \gamma_k,
\]

with \( \beta_k(y) \) uniformly positive depending only on \( n, \lambda \) and \( A \). Thus (74) and (77) give

\[
w(y) - w(x) \leq C|y - x| - \sum_{k=2}^N \beta_k(y)(u_{\gamma_k \bar{\gamma}_k}(y) - u_{\gamma_k \bar{\gamma}_k}(x))
\]

\[
\leq C|y - x| + \sum_{k=2}^N \beta_k(y)(\sup_{B_\rho} u_{\gamma_k \bar{\gamma}_k} - u_{\gamma_k \bar{\gamma}_k}(y)).
\]

Now let \( w(x) = \inf_{B_\rho} w \), and average over \( B_\rho \) to get, using (75),

\[
\frac{1}{|B_\rho|} \int_{B_\rho} w - \inf_{B_\rho} w \leq C \left( \sum_{k=2}^N \nu_{\eta \bar{\eta}}(2\rho) - \nu_{\eta \bar{\eta}}(\rho) + \rho(\rho + 1) \right).
\]

Combining this with (75), and summing over \( k = 1, \ldots, N \) we thus obtain an estimate on \( \eta(\rho) \) of the desired form. Hence \( \Delta_{w} \varphi(s) \in C^0_{w}\gamma \) for some \( \gamma > 0 \). In fact our proof actually showed that \( \varphi_{\eta \bar{\eta}} \in C^0_{w}\gamma \) for any \( \eta \). Hence, by polarization we deduce that also \( \varphi_{ij} \in C^0_{w}\gamma \), for any \( i, j \).

Hence, \( [\varphi(s)]_{D^0_{0,\gamma}} \leq C = C(|M, \omega, \beta|, ||\Delta_{w} \varphi(s)||_{C^0(M)}, ||\varphi(s)||_{C^0(M)}) \). This concludes the proof of Theorem 8.2.
9 Existence of Kähler–Einstein edge metrics

We now conclude the proof of Theorem 2 on the existence of Kähler–Einstein edge metrics, as well as of the convergence of the twisted Ricci iteration (Theorem 2.5). We then describe the additional regularity properties as stated in Theorem 2.

Starting the continuity path. We cannot apply the implicit function theorem directly when \( s = -\infty \) since if we reparametrize (30) by setting \( \sigma = 1/s \), then the linearization degenerates at \( \sigma = 0 \). We deal with this by showing that there exists a solution for some large negative value of \( s \). This can be done by a perturbation argument due to Wu, which has the advantage that it requires very little about the initial reference metric and applies for all \( \beta < 1 \); when \( \beta \leq 1/2 \), we can also do this using the modified continuity family (see Remark 2.4).

Wu proves that a solution to (30) with \( s = S_0 \ll -1 \) can be produced using a standard Newton iteration method [55, Proposition 7.3]. His proof is written in a different context, for a certain class of complete Kähler metrics, but the argument is a standard one and it is straightforward to check that it can be carried over directly to our setting by substituting the Schauder estimates stated in Remark 3.4 and recalled in (79) below. We emphasize yet again (this was also noted in [55]) that this argument does use any curvature assumptions.

Second, as described in §2.4, one can embed the original continuity path (30) into a two parameter family (32) where solutions exist trivially for the finite parameter values \((s,0)\). We can then carry out the rest of the existence proof from this if \( \beta \leq 1/2 \). This restriction arises because the closedness part of the argument does not hold for the full two-parameter continuity family when \( \beta > 1/2 \).

Openness. Define \( M_{s,t} : D_s^{0,\gamma} \to C_s^{0,\gamma} \) by

\[
M_{s,t}(\varphi) := \log \frac{\omega^n}{\omega^n} - tf + s\varphi, \quad (s,t) \in A = (-\infty,0] \times [0,1] \cup [0,\mu] \times \{1\}.
\]

Note that \( M_{s,0}(0) = 0 \). If \( \varphi(s,t) \in D_s^{0,\gamma} \cap \text{PSH}(M,\omega) \) is a solution of (32), we claim that its linearization

\[
DM_{s,t}|_{\varphi(s,t)} = \Delta_{\varphi(s,t)} + s : D_s^{0,\gamma} \to C_s^{0,\gamma}, \quad (s,t) \in A,
\]

is an isomorphism when \( s \neq 0 \). If \( s = 0 \), this map is an isomorphism if we restrict on each side to the codimension one subspace of functions with integral equal to 0. Furthermore, we also claim that \((s,t,\varphi) \mapsto M_{s,t}(\varphi)\) is a \( C^1 \) mapping into \( C_s^{0,\gamma} \). Given these claims, the Implicit Function Theorem then guarantees the existence of a solution \( \varphi(s,\tilde{t}) \in D_s^{0,\gamma} \) for all \((s,\tilde{t}) \in A\) sufficiently close to \((s,t)\).

Proposition 3.2 asserts that (78) is Fredholm of index 0 for any \((s,t) \in A\), and by Proposition 3.3

\[
\|u\|_{D_s^{0,\gamma}} \leq C(\|DM_{s,t}u\|_{C_s^{0,\gamma}} + \|u\|_{C^0}),
\]

its nullspace \( K \) is clearly trivial when \( s < 0 \), and also by Lemma 6.1 for \((s,1)\) with \( s \in (0,\mu)\); finally, when \( s = 0 \) it consists of constants. Thus \( DM_{s,t} \) is an isomorphism when \( s \neq 0 \), and is an isomorphism on the \( L^2 \) orthogonal complement to the constants when \( s = 0 \). This proves the first claim.

The second claim follows even more easily from the explicit formula for \( DM_{s,t} \) in (78) once we observe that by Corollary 3.5, the domains of these linearizations at different \( \varphi \) are all the same.

Note finally that using (79) nearby solutions remain in \( \text{PSH}(M,\omega) \).
Closedness. Fix some $S < 0$ and denote $A_S := \{(s,t) \in A : s \in (S,0]\}$. Let $\{(s_j,t_j)\}$ be a sequence in $\text{int}A_S$ converging to $(s,t) \in \overline{A_S}$, and let $\varphi(s_j,t_j) \in \mathcal{D}^{0,\gamma}_{\omega} \cap \text{PSH}(M,\omega)$ be solutions to (32). Under the assumptions of Theorem 2, the results of [5, 7] and [8] imply that $\|\varphi(s_j,t_j)\|_{\mathcal{D}^{0,\gamma}_{\omega}} \leq C$, where $C$ depends on $S$, a lower bound on the Ricci curvature of $\omega$ times $(1 - \min_j t_j)$, and an upper bound on its bisectional curvature, both over $M \setminus D$; alternatively, using the Aubin–Yau Laplacian estimate [1, 57, 14] gives a bound depending on $S$ and a lower bound on the bisectional curvature of $\omega$ over $M \setminus D$. Thus, when $\beta \in (0, \frac{2}{3}] \cup \{1\}$, Lemma 2.3 implies that either type of bounds give a uniform estimate $\|\varphi(s,t)\|_{\mathcal{D}^{0,\gamma}_{\omega}} \leq C$, for all $(s,t) \in A_S$. In general, restrict to the path (30) (i.e., let $t_j = 1$ for all $j$) and then $\|\varphi(s,1)\|_{\mathcal{D}^{0,\gamma}_{\omega}} \leq C$, for all $s \in (S,0]$ by using Proposition A.1. For any $\gamma' \in (0, \gamma)$, we can then extract a subsequence which converges to some function $\varphi(s,t) \in \mathcal{D}^{0,\gamma'}_s$, and clearly $M_{s,t}(\varphi(s,t)) = 0$. Letting $S \to -\infty$, we obtain a solution for all $(s,t) \in A_\infty$ in the case $\beta \in (0, \frac{2}{3}] \cup \{1\}$, and for all $(s,1) \in (-\infty,0] \times \{1\}$ in the general case. Now by openness in $A$ about the solution at $(0,1)$ (cf. [2, 4]), there exist solutions also for $[0, \epsilon) \times \{1\} \subset A$. Then by Corollary 6.7 and the previous arguments we obtain solutions for all $(s,t) \in A$ when $\beta \in (0, \frac{2}{3}] \cup \{1\}$, and for all $(s,1) \in (-\infty, \mu] \times \{1\}$ in the general case. By Theorem 1, these solutions are polyhomogeneous. Finally, $\varphi(s,t) \in \text{PSH}(M,\omega)$.

Regularity. Using the steps above, we obtain a solution $\varphi := \varphi(\mu,1) \in \mathcal{A}^0_{\text{phy}} \cap \text{PSH}(M,\omega)$ to (30). Denote by $g_\mu$ the associated Kähler–Einstein edge metric. Using Proposition 4.3 and [53], $g_\mu$ is asymptotically equivalent to the reference metric $g$, and moreover, by the explicit form of the expansion and the fact that $\mathcal{P}_{i\bar{j}}$ annihilates the $r^0$ and $r^{\frac{1}{2}}$ terms, we obtain that $\varphi \in \mathcal{A}^0 \cap \mathcal{D}^{0,\frac{1}{2} - 1}_w$. This completes the proof of Theorem 2.

Note that when $\beta \leq 1/2$ also $\varphi \in \mathcal{C}^{0,\frac{1}{2} - 2}_w$. When $\beta \leq 1/2$, $g_\mu$ has uniformly bounded curvature on $M \setminus D$. Indeed, if $\varphi = \varphi(s)$ is any solution of (30) in $\text{PSH}(M,\omega)$, then writing $(g_{\mu})_{ij} := \psi_{ij} + \varphi_{ij}$ where $\omega = -\sqrt{-1} \partial \bar{\partial} \psi$ locally, then the Laplacian estimate shows that $g_{\mu}^{ij}(\psi_{ij} + \varphi_{ij})$ (no summation) is bounded. By Proposition 4.3, $|\varphi_{ij,kl}| = O(1 + r^{\frac{3}{2} - 2})$, and $|\psi_{ij,k}| = O(1 + r^{\frac{3}{2} - 1})$, so by Lemma 2.3, $R_{ijkl} \in \mathcal{C}^{0,\frac{1}{2} - 2}_w$.

Convergence of the Ricci iteration. We use the notation of [2, 5]. As noted there, $\mu - \frac{1}{\tau}$ plays the role of $s$. Consider first the case $\mu \leq 0$. By the earlier analysis of (30), for any $\tau > 0$ the iteration exists uniquely and $\{\psi_{kr}\}_{k \in \mathbb{N}} \subset \mathcal{C}^{0,\gamma}_\gamma$. By Lemma 5.1 the inductive maximum principle argument of [42] yields $|\psi_{kr}| \leq C$. Along the iteration, just as for the path (30), the Ricci curvature is bounded from below by $\mu - \frac{1}{\tau}$, hence Proposition 7.1 and Lemma 5.1 show that $|\Delta_{\omega,k} \psi_{kr}| \leq C$ (we consider the maps $id : (M, \omega_{kr}) \to (M, \omega)$). Going back to the equation (33) and using the $C^0$ estimate then shows that $|\Delta_\omega \psi_{kr}| \leq C$, hence by Theorem 8.1, $|\psi_{kr}| \in \mathcal{C}^{2,\gamma}_\gamma$. Thus a subsequence converges in $\mathcal{C}^{2,\gamma}_\gamma$ to an element $\psi_\infty$ of $\mathcal{D}^{0,\gamma}_\gamma$. Since each step in the iteration follows a continuity path of the form (30) with $\omega$ replaced by $\omega_{kr}$, Lemma 5.6 implies that $E_0^\beta(\omega_{kr}) < 0$ (unless $\omega$ was already Kähler–Einstein). Since $E_0^\beta$ is an exact energy functional, i.e., satisfies a cocyle condition [35], then $E_0^\beta(\omega, \omega_{kr}) = \sum_{j=1}^{k} E_0^\beta(\omega_{j-1} \tau, \omega_{j \tau}) < 0$. Therefore, $\psi_\infty$ is a fixed point of $E_0^\beta$, hence a Kähler–Einstein edge metric. By Lemma 5.2 such Kähler–Einstein metrics are unique; we conclude that the original iteration converges to $\psi_\infty$ both in $\mathcal{A}_0$ and in $\mathcal{D}^{0,\gamma}_\gamma$.

Next, consider the case $\mu > 0$, and take $\mu = 1$ for simplicity. By the properness assumption, Corollary 6.7 implies the iteration exists (uniquely by Lemma 6.4) for each $\tau \in (0, \infty)$ and then the monotonicity of $E_0^\beta$ implies that $J(\omega, \omega_{kr}) \leq C$. To obtain the uniform estimate on $\text{osc} \psi_{kr}$
we will employ the argument of [7] as explained to us by Berman. By Lemma 6.8 have \( \int_M e^{-p(\psi_{k\tau} - \sup \psi_{k\tau})} \omega^n \leq C \), where \( p/3 = \max\{1 - \frac{1}{2}, \frac{1}{2}\} \). Now rewrite (33) as

\[
\omega^n_{\psi_{k\tau}} = \omega^n e^{-\frac{1}{2}p(\psi_{k\tau} - \frac{1}{2}p\psi_{k(\tau-1)r})}.
\]

(80)

Using Kolodziej’s estimate and the Hölder inequality this yields the uniform estimate \( \text{osc} \psi_{k\tau} \leq C \). Unlike for solutions of (30), the functions \( \psi_{k\tau} \) need not be changing signs. Therefore we let \( \tilde{\psi}_{k\tau} := \psi_{k\tau} - \frac{1}{2} \int_M \psi_{k\tau} \omega^n \). As in the previous paragraph we obtain a uniform estimate \( \text{tr} \omega\psi_{k\tau} \omega \leq C \). However, to conclude that \( \text{tr} \omega \omega_{k\tau} \omega \leq C \) from (80) we must show that \( \left| (1 - \frac{1}{2})\psi_{k\tau} - \frac{1}{2} \psi_{(k-1)r} \right| \leq C \). This is shown in [52, p. 599]. Thus, as before, we conclude that \( \{\tilde{\psi}_{k\tau}\} \) subconverges to the potential of a Kähler–Einstein edge metric. Whenever it is unique, the iteration itself necessarily converges. Berndtsson’s generalized Bando–Mabuchi Theorem [4, 9] shows uniqueness of Kähler–Einstein edge metrics up to an automorphism (which must be tangent to \( D \) by [6] or Lemma 6.1), but then a well-known argument (cf., e.g., [6]) shows that such automorphisms cannot exist when \( E^β_0 \) is proper. This concludes the proof of Theorem 2.5.

A Appendix

Chi Li and Yanir A. Rubinstein

Proposition A.1. Let \( \beta \in (0, 1] \), and let \( \omega = \omega_0 + \sqrt{-1} \partial \bar{\partial} |s|^{2\beta} \) be given by (25). The bisectional curvature of \( \omega \) is bounded from above on \( M \setminus D \).

We denote throughout by \( \hat{g}, g \) the Kähler metrics associated to \( \omega_0, \omega \), respectively. As in [52], to simplify the calculation and estimates we need a lemma to choose an appropriate local holomorphic frame and coordinate system, whose elementary proof we include for the reader’s convenience. We thank Gang Tian for pointing out to us the calculations in [52] which were helpful in writing this Appendix.

Lemma A.2. [52, p. 599] There exists \( \epsilon_0 > 0 \) such that if \( 0 < \text{dist}_{\hat{g}}(p, D) \leq \epsilon_0 \), then we can choose a local holomorphic frame \( e \) of \( L_D \) and local holomorphic coordinates \( \{z_i\}_{i=1}^n \) valid in a neighborhood of \( p \), such that (i) \( s = z_1 e \), and \( a := |e|^2_h \) satisfies \( a(p) = 1, da(p) = 0, \) \( \frac{\partial^2 a}{\partial c_i \partial z_j} a(p) = 0 \), and (ii) \( \hat{g}_{j,k}(p) = \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \omega_0(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}) \big|_{p} = 0 \), whenever \( j \neq 1 \).

Proof. (i) Fix any point \( q \in D \), and choose a local holomorphic frame \( e' \) and holomorphic coordinates \( \{w_i\}_{i=1}^n \) in \( B_q(q, \epsilon(q)) \) for \( \epsilon(q) \ll 1 \). Let \( s = f' e' \) with \( f' \) a holomorphic function and \( |e'|_F^2 = c \). Let \( e = F e' \) for some nonvanishing holomorphic function \( F \) to be specified later. Then \( a = |Fe'|_F^2 = |F|^2 c \). Now fix any point \( p \in B_q(q, \epsilon(q)) \setminus \{q\} \). In order for \( a \) to satisfy the vanishing properties with respect to the variables \( \{w_i\}_{i=1}^n \) at a point \( p \), we can just choose \( F \) such that \( F(p) = c(p)^{-1/2} \), and

\[
\frac{\partial}{\partial w_i} F(p) = -c^{-1} F \frac{\partial}{\partial w_i} c(p) = -c^{-3/2} \frac{\partial}{\partial w_i} c(p) \\
\frac{\partial}{\partial w_i} \frac{\partial}{\partial w_j} F(p) = -c^{-1} \left( F \frac{\partial}{\partial w_i} \frac{\partial}{\partial w_j} c + \frac{\partial}{\partial w_i} c \frac{\partial}{\partial w_j} F + \frac{\partial}{\partial w_i} c \frac{\partial}{\partial w_j} F \right)(p) \\
= -c^{-3/2} \frac{\partial}{\partial w_i} \frac{\partial}{\partial w_j} c(p) + 2c^{-5/2} \frac{\partial}{\partial w_i} c \frac{\partial}{\partial w_j} c(p).
\]

Since \( c = |e'|_h^2 \) is never zero, when \( \epsilon(q) \) is small, which implies \( |w - w(p)| \) is small, we can assume \( F \neq 0 \) in \( B_q(q, \epsilon(q)) \). Now \( s = fe = f' e' \) with \( f = f' F^{-1} \) a holomorphic function. Since \( D = \{s = 0\} \) is a smooth divisor, we can assume \( \frac{\partial}{\partial w_i} f(q) \neq 0 \), and choosing \( \epsilon(q) \) sufficiently
small, we can assume that $\partial_{w^i} f \neq 0$ in $B_{\delta}(q, \epsilon(q))$. Thus by the inverse function theorem, $z_1 = f(w_1, \ldots, w_n), z_2 = w_2, \ldots, z_n = w_n$ are holomorphic coordinates in $B_{\delta}(q, \epsilon(q)/2)$ and now $s = f(w)e = z_1e$. By the chain rule, it then follows that $a$ satisfies $a(p) = 1, \partial_a a(p) = \partial_s \partial_z a(p) = 0$.

Now cover $D$ by $\cup_{q \in D} B_{\delta}(q, \epsilon(q)/2)$. By compactness of $D$ the conclusion follows.

(ii) Denote by $\{w^i\}_{i=1}^n$ the coordinates obtained in (i). Following [23] p. 108, let $\tilde{z} := w^k - w^k(p) + \frac{1}{2} b^k_{st} (w^s - w^s(p))(w^t - w^t(p))$, with $b^k_{st} = b^k_{st}$, define a new coordinate system. Then, $\omega_0(\partial_{w^k}, \partial_{w^k}) = \omega_0(\partial_{\tilde{z}^k}, \partial_{\tilde{z}^k}) + \hat{\gamma}_{ijk} b^j_{kp} w^p + \hat{\gamma}_{ijk} b^j_{kp} w^p + O(\sum_{i=1}^n |w^i - w^i(p)|^2)$, and

$$d_{ijk} := \frac{\partial}{\partial w^k} \omega_0(\partial_{w^i}, \partial_{w^i})|_p = \frac{\partial}{\partial \tilde{z}^k} \omega_0(\partial_{\tilde{z}^i}, \partial_{\tilde{z}^i})|_p + \hat{\gamma}_{ijk} b^j_{ik}.$$  

Let $\hat{g}^{rs}_{st} := \hat{g}_{rs}$, for each $r, s > 1$, and denote the inverse of the $(n-1)\times(n-1)$ matrix $[\hat{g}^{rs}_{st}]$ by $[\hat{g}^{rs}]$. Let $b^k_{ik} = 0$. Then, for each $j > 1$, the equations can be rewritten as $d_{ijk} - \sum_{t=1}^n \hat{g}^{ij}_{st}(p)b^k_{ik} = e_{ijk}$. Hence, $\sum_{j=1}^n \hat{g}^{sij} e_{ijk} = \sum_{j=1}^n \hat{g}^{sij} d_{ijk} - b^s_{ik} > 1$. For each $s > 1$, define $b^s_{ik}$ so that the right hand side vanishes. Multiplying the equations by $[\hat{g}^{rs}_{st}]$, we obtain $e_{ijk} = 0$ for each $t > 1$. Finally, set $z^1 := z^1 + w^1(p), i = 1, \ldots, n$. Since $b^1_{ik} = 0$, we have $z^1 = w^1$, and therefore these coordinates satisfy both properties (i) and (ii) of the statement, as desired.

Let $H := a^\beta$, then $|s|^2\beta_\beta = |z_1|^2\beta_\beta = |Hz|^2\beta_\beta$. Note that both $a$ and $H$ are locally defined smooth positive functions. Let $\omega = \sqrt{\frac{-1}{2}} g_{ij} dz^i \wedge d\bar{z}^j, \omega_0 = \sqrt{\frac{-1}{2}} \hat{g}_{ij} dz^i \wedge d\bar{z}^j$, and write $z \equiv z_1$ and $\rho := |z|$. Using the symmetry for subindices, we can calculate in a straightforward manner:

$$g_{ij} = \hat{g}_{ij} + H_{ij}|z|^{2\beta} + \beta H_i \delta_{1j}|z|^{2\beta-2}z + \beta^2 H |z|^{2\beta-2} \delta_{i1} \delta_{1j},$$

$$g_{ij,k} = \hat{g}_{ij,k} + H_{ijk}|z|^{2\beta} + \beta H_{ik} \delta_{1j}|z|^{2\beta-2}z + \beta(H_{kj} \delta_{1i} + H_{ij} \delta_{1k})|z|^{2\beta-2} \delta_{i1}$$

$$+ \beta^2 (H_{i1} \delta_{1j} \delta_{1k} + H_{k1} \delta_{1i} \delta_{1j} + H_{j1} \delta_{1i} \delta_{1k})|z|^{2\beta-2} \delta_{i1} \delta_{1j} \delta_{1k},$$

$$g_{ij,k\bar{k}} = \hat{g}_{ij,k\bar{k}} + H_{ijk\bar{k}}|z|^{2\beta} + \beta \left[H_{ik\bar{k}} \delta_{1j} + H_{j\bar{k}k} \delta_{1i} \right]|z|^{2\beta-2}z + \left[H_{ji\bar{k}} \delta_{1k} + H_{i\bar{k}j} \delta_{1k} \right]|z|^{2\beta-2}$$

$$+ \beta^2 (H_{j1\bar{k}} \delta_{1i} \delta_{1j} + H_{i1\bar{k}} \delta_{1i} \delta_{1k} + H_{k1\bar{k}} \delta_{1j} \delta_{1k})|z|^{2\beta-2} \delta_{i1} \delta_{1j} \delta_{1k},$$

$$+ \beta^2 (\beta - 1) \left[H_{i1\bar{k}} \delta_{1j} \delta_{1k} \right]|z|^{2\beta-4}z + \left[H_{j1\bar{k}} \delta_{1l} \delta_{1j} \delta_{1k} \right]|z|^{2\beta-4}z.$$  

Let $p \in M \setminus D$ satisfy $\text{dist}_{\tilde{g}}(p, D) \leq \epsilon_0$. The lemma implies in particular $H(p) = 1, H_i(p) = H_{ij}(p) = 0$, and the expressions above simplify to:

$$g_{ij}(p) = \hat{g}_{ij} + H_{ij}|z|^{2\beta} + \beta^2 |z|^{2\beta-2} \delta_{i1} \delta_{1j},$$

$$g_{ij,k}(p) = \hat{g}_{ij,k} + H_{ijk}|z|^{2\beta} + \beta (\delta_{1i} H_{kj} + \delta_{1k} H_{ij})|z|^{2\beta-2}z + \beta^2 (\beta - 1) \delta_{1i} \delta_{1j} \delta_{1k}|z|^{2\beta-4}z,$$

$$g_{ij,k\bar{k}}(p) = \hat{g}_{ij,k\bar{k}} + H_{ijk\bar{k}}|z|^{2\beta} + \beta (\delta_{1i} H_{k\bar{k}} + \delta_{1k} H_{ij})|z|^{2\beta-2}z + \beta^2 (\beta - 1) \delta_{1i} \delta_{1j} \delta_{1k} H_{ij} |z|^{2\beta-2}$$

$$+ \beta^2 (\beta - 1)^2 \delta_{1i} \delta_{1j} \delta_{1k} \delta_{1\bar{l}} |z|^{2\beta-4}.$$
It follows that
\[ g^r_s(p) = O(1), \quad g^1_s(p) = O(\rho^{2-2\beta}), \tag{81} \]
and,
\[ g^{11}(p) = \beta^{-2} \rho^{2-2\beta} \frac{1}{1 + b(p)\rho^{2-2\beta}} + O(\rho^2), \tag{82} \]
where \( O(\rho^2) < C_3\rho^2 \) and \( b(p) := \beta^{-2} \det[H^{ij}] / \det[H^{rs}_{r,s>1}]_p \) with \( 0 < C_1 < b(p) < C_2 \), and \( C_1, C_2, C_3 \) independent of \( p \in \mathcal{M} \setminus \mathcal{D} \).

Take two unit vectors \( \eta = \eta^i \frac{\partial}{\partial z^i}, \nu = \nu^i \frac{\partial}{\partial z^i} \in T^1_0 \mathcal{M} \), so that \( g(\eta, \eta)|_p = g(\nu, \nu)|_p = 1 \). Then from the expression of \( g^{ij} \) we have
\[ \eta^i, \nu^i = O(\rho^{1-\beta}) \quad \eta^r, \nu^r = O(1), \quad \text{for } r > 1. \tag{83} \]

Set
\[ \text{Bisec}_\omega(\eta, \nu) = R(\eta, \bar{\eta}, \nu, \bar{\nu}) = R_{ijkl}\bar{\eta}^i\nu^j\bar{\nu}^k\bar{\eta}^l = \sum_{i,j,k,l} \Lambda_{ijkl} + \Pi_{ijkl}, \]
with \( \Lambda_{ijkl} := -g^{ijkl}\eta^i\nu^j\bar{\nu}^k\bar{\eta}^l \), and \( \Pi_{ijkl} := g^{kl}g^{ij}g^{kl}\eta^i\nu^j\bar{\nu}^k\bar{\eta}^l \) (no summations). By (81)–(83) we have \( |\Lambda_{ijkl}| \leq C \) except for \( \Lambda_{1111} = -\beta^2(1-\beta)^2|z|^{2\beta-4}||\eta^1||^2||\nu^1||^2 \), hence
\[ \sum_{i,j,k,l} \Lambda_{ijkl}(p) = O(1) + \Lambda_{1111}(p) = O(1) - \beta^2(1-\beta)^2|z|^{2\beta-4}||\eta^1||^2||\nu^1||^2. \tag{84} \]

The Proposition follows immediately by combining (84) and the following estimate.

**Lemma A.3.** There exists a uniform constant \( C > 0 \) such that for every \( p \in \mathcal{M} \setminus \mathcal{D} \),
\[ \sum_{i,j,k,l} \Pi_{ijkl}(p) \leq C + \beta^2(1-\beta)^2|z|^{2\beta-4}||\eta^1||^2||\nu^1||^2. \]

**Proof.** Define a bilinear Hermitian form of two tensors \( a = [a_{ijk}], \quad b = [b_{pq}] \in (\mathbb{C}^n)^3 \) satisfying \( a_{ijk} = a_{kji} \) and \( b_{pqr} = b_{rqp} \) by setting
\[ \langle [a_{ijk}], [b_{pq}] \rangle := \sum_{i,j,k,p,q,r} g^{ij}(\eta^i a_{ijk} \nu^k)(\bar{\eta}^p b_{pq} \bar{\nu}^r). \]
It is easy to see that this is a nonnegative bilinear form. We denote by \( \| \cdot \| \) the associated norm. Then \( \sum_{i,j,k,l} \Pi_{ijkl} = \|[g_{ij,k}]\|^2 \). Write,
\[ g_{ijk} = A_{ijk} + B_{ijk} + D_{ijk} + E_{ijk}, \]
with \( A_{ijk} := g_{ijk}, \quad B_{ijk} := H_{ijk}|z|^{2\beta}, \quad D_{ijk} := \beta(\delta_{i1}H_{kj} + \delta_{k1}H_{ij})|z|^{2\beta-2}z, \) and \( E_{ijk} := \beta^2(\beta - 1)\delta_{i1}\delta_{j1}\delta_{k1}|z|^{2\beta-4}z \). Denote \( A = [A_{ijk}] \) and similarly \( B, D, E \). Using (81),
\[ \langle D, E \rangle \leq C \sum_j g^{1j}|\eta^1|^2|\nu^j|^2|z|^{2\beta-4} \leq C \rho^{1-\beta}, \]
and similarly we conclude that \( \|[g_{ij,k}]\|^2 \leq C + \|A + E\|^2 \). Now, since \( \sqrt{\epsilon} A - \sqrt{\epsilon} E \geq 0 \), we obtain \( \|A + E\|^2 \leq (1 + \frac{1}{\epsilon})\|A\|^2 + (1 + \epsilon)\|E\|^2 \). Note now that by (82)
\[ \|E\|^2 = g^{11}|E_{111}|^2|\eta^1|^2|\nu^1|^2 \leq C + \frac{\beta^2(1-\beta)^2}{1 + b(p)\rho^{2-2\beta}}|\eta^1|^2|\nu^1|^2. \]

49
Thus, letting $\epsilon = \epsilon(p) = b(p)\rho^{2-2\beta}$, we will have proved the lemma provided we can bound $(1 + \rho^{2\beta-2})\|A\|^2$. Now, by (82) and Lemma A.2 (ii),

$$\rho^{2\beta-2}\|A\|^2 = \sum_{i,k,p,r} \rho^{2\beta-2}\hat{g}_{i1,k}\hat{g}_{1j,l}\hat{g}^{11}\eta^{i}\eta^{j}\nu^{k}\nu^{l} \leq C.$$ 

This concludes the proof of Lemma A.3.

References

[1] T. Aubin, Équations du type Monge–Ampère sur les variétés kählériennes compactes, Bull. Sci. Math. 102 (1978), 63–95.

[2] T. Aubin, Réduction du cas positif de l’équation de Monge–Ampère sur les variétés kählériennes compactes à la démonstration d’une inégalité, J. Funct. Anal. 57 (1984), 143–153.

[3] S. Bando, R. Kobayashi, Ricci–flat Kähler metrics on affine algebraic manifolds, Lecture notes in Math. 1339 (1988), 20–31.

[4] S. Bando, T. Mabuchi, Uniqueness of Kähler–Einstein Metrics Modulo Connected Group Actions, in: Algebraic Geometry, Sendai, 1985, Advanced Studies in Pure Mathematics 10, 1987, pp. 11–40.

[5] E. Bedford, B.A. Taylor, The Dirichlet problem for a complex Monge–Ampère equation, Inv. Math. 37 (1976), 1–44.

[6] R.J. Berman, A thermodynamical formalism for Monge–Ampère equations, Moser–Trudinger inequalities and Kähler–Einstein metrics, preprint, arXiv:1011.3976.

[7] R. Berman, S. Boucksom, P. Eyssidieux, V. Guedj, A. Zeriahi, Mean field equations and the Kähler–Ricci flow, preprint in preparation.

[8] R. Berman, S. Boucksom, V. Guedj, A. Zeriahi, A variational approach to complex Monge-Ampère equations, preprint, arXiv:0907.4490v1.

[9] B. Berndtsson, A Brunn–Minkowski type inequality for Fano manifolds and the Bando–Mabuchi uniqueness theorem, preprint, arXiv:1103.0923v3.

[10] A.L. Besse, Einstein manifolds, Springer, 1987.

[11] Z. Błocki, Interior regularity of the complex Monge–Ampère equation in convex domains, Duke Math. J. 105 (2000), 167–181.

[12] Z. Błocki, Uniqueness and stability for the complex Monge–Ampère equation on compact Kähler manifolds, Indiana Univ. Math. J. 52 (2003), 1697–1701.

[13] Z. Błocki, The complex Monge–Ampère equation on compact Kähler manifolds, Lecture notes, February 2007.

[14] S. Brendle, Ricci flat Kähler metrics with edge singularities, preprint, arXiv:1103.5454v2.

[15] F. Campana, H. Guenancia, M. Păun, Metrics with cone singularities along normal crossing divisors and holomorphic tensor fields, preprint, arXiv:1104.4879v2.

[16] S.-Y. Cheng, P. Li, Heat kernel estimates and lower bound of eigenvalues, Comment. Math. Helv. 56 (1981), 327–338.
[17] S.-S. Chern, On holomorphic mappings of Hermitian manifolds of the same dimension, Proc. Symp. Pure Math. 11, American Mathematical Society, 1968, pp. 157–170.

[18] C.B. Croke, Some isoperimetric inequalities and eigenvalue estimates, Ann. scient. Éc. Norm. Sup. 13 (1980), 419–435.

[19] S.K. Donaldson, Kähler metrics with cone singularities along a divisor, preprint, arXiv:1102.1196v2.

[20] C.L. Epstein, R. Melrose, G. Mendoza, Resolvent of the Laplacian on strictly pseudoconvex domains, Acta Math. 167 (1991), 1–106.

[21] L.C. Evans, Classical solutions of fully nonlinear, convex, second-order elliptic equations, Comm. Pure Appl. Math. 35 (1982), 333–363.

[22] D. Gilbarg, N.S. Trudinger, Elliptic partial differential equations of second order, Springer, 2001.

[23] P. Griffiths, J. Harris, Principles of algebraic geometry, Wiley Interscience, 1978.

[24] P. Hajłasz, P. Koskela, Sobolev meets Poincaré, C. R. Acad. Sci. Paris Sér. I 320 (1995), 1211–1215.

[25] T.D. Jeffres, Kähler–Einstein cone metrics, Ph.D. Thesis, Stony Brook University, 1996.

[26] T.D. Jeffres, Uniqueness of Kähler–Einstein cone metrics, Publ. Math. 44 (2000), 437–448.

[27] S. Kołodziej, The complex Monge–Ampère equation, Acta Math. 180 (1998), 69–117.

[28] N.V. Krylov, Boundedly inhomogeneous elliptic and parabolic equations (Russian), Izv. Akad. Nauk SSSR Ser. Mat. 46 (1982), 487–523.

[29] N.V. Krylov, M.V. Safonov, An estimate for the probability of a diffusion process hitting a set of positive measure (Russian), Dokl. Akad. Nauk SSSR 245 (1979), 18–20.

[30] J. Lee, R. Melrose, Boundary behavior of the complex Monge–Ampère equation, Acta Math. 148 (1982), 159–192.

[31] C. Li, Remarks on logarithmic K-stability, preprint, arXiv:1104.0428v1.

[32] Y.-C. Lu, On holomorphic mappings of complex manifolds, Ph.D. Thesis, University of California, Berkeley, 1967.

[33] Y.-C. Lu, Holomorphic mappings of complex manifolds, J. Diff. Geom. 2 (1968), 299–312.

[34] F. Luo, G. Tian, Liouville equation and spherical convex polytopes, Proc. Amer. Math. Soc. 116 (1992), 1119–1129.

[35] T. Mabuchi, K-energy maps integrating Futaki invariants, Tôhoku Math. J. 38 (1986), 575–593.

[36] R. Mazzeo, Elliptic theory of differential edge operators, I, Comm. PDE 16 (1991), 1616–1664.

[37] R. Mazzeo, Regularity for the singular Yamabe problem, Indiana Univ. Math. J. 40 (1991), 1277–1299.

[38] R. Mazzeo, Kähler–Einstein metrics singular along a smooth divisor, Journées Équations aux dérivées partielles (1999), 1–10.

[39] R. Mazzeo, Y.A. Rubinstein, Kähler–Einstein metrics with crossing edge singularities, preprint in preparation.
[40] R.C. McOwen, Point singularities and conformal metrics on Riemann surfaces. Proc. Amer. Math. Soc. 103 (1988), 222–224.

[41] E. Mooers, Heat kernel asymptotics on manifolds with conic singularities, J. Anal. Math. 78 (1999), 1–36.

[42] Y.A. Rubinstein, Some discretizations of geometric evolution equations and the Ricci iteration on the space of Kähler metrics, Adv. Math. 218 (2008), 1526–1565.

[43] L. Saloff-Coste, A note on Poincaré, Sobolev, and Harnack inequalities, Internat. Math. Res. Notices (1992), 27–38.

[44] Y.-T. Siu, Lectures on Hermitian–Einstein metrics for stable bundles and Kähler–Einstein metrics, Birkhäuser, 1987.

[45] G. Tian, On Kähler–Einstein metrics on certain Kähler manifolds with $C^1(M) > 0$, Inv. Math. 89 (1987), 225–246.

[46] G. Tian, On stability of the tangent bundles of Fano varieties, Internat. J. Math. 3 (1992), 401–413.

[47] G. Tian, The K-energy on hypersurfaces and stability, Comm. Anal. Geom. 2 (1994), 239–265.

[48] G. Tian, Kähler–Einstein metrics on algebraic manifolds, in: Transcendental methods in algebraic geometry (Cetraro 1994), Lecture Notes in Math. 1646, pp. 143–185.

[49] G. Tian, Kähler-Einstein metrics with positive scalar curvature, Inv. Math. 130 (1997), 1–37.

[50] G. Tian, Canonical Metrics in Kähler Geometry, Birkhäuser, 2000.

[51] G. Tian, S.-T. Yau, Existence of Kähler–Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry, in: Mathematical Aspects of String Theory (S.-T. Yau, Ed.), World Scientific, 1987, pp. 574–628.

[52] G. Tian, S.-T. Yau, Complete Kähler manifolds with zero Ricci curvature, I, J. Amer. Math. Soc. 3 (1990), 579–609.

[53] M. Troyanov, Prescribing curvature on compact surfaces with conic singularities, Trans. Amer. Math. Soc. 324 (1991), 793–821.

[54] R. Wang, J. Jiang, Another approach to the Dirichlet problem for equations of Monge–Ampère type, Northeastern Math. J. 1 (1985), 27–40.

[55] D. Wu, Kähler–Einstein metrics of negative Ricci curvature on general quasi-projective manifolds, Comm. Anal. Geom. 16 (2008), 395–435.

[56] S.-T. Yau, A general Schwarz lemma for Kähler manifolds, Amer. J. Math. 100 (1978), 197–203.

[57] S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the Complex Monge–Ampère equation, I, Comm. Pure Appl. Math. 31 (1978), 339–411.

Wichita State University
jeffres@math.wichita.edu

Stanford University
mazzeo@math.stanford.edu, yanir@member.ams.org