REDUCIBILITY OF QUANTUM HARMONIC OSCILLATOR ON $\mathbb{R}^d$
PERTURBED BY A QUASI-PERIODIC POTENTIAL WITH
LOGARITHMOMIC DECAY

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Abstract. We prove the reducibility of quantum harmonic oscillators in $\mathbb{R}^d$ perturbed by a quasi-periodic in time potential $V(x,\omega t)$ with logarithmic decay. By a new estimate built for solving the homological equation we improve the reducibility result by Grébert-Paturel(Annales de la Faculté des sciences de Toulouse : Mathématiques. 28, 2019).

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1. Introduction and Main Results

1.1. Statement of the results. In this paper we consider the linear equation

$$i\partial_t u = (-\Delta + |x|^2) u + \epsilon V(x, \omega t) u, \quad u = u(t, x), \quad t \in \mathbb{R}, x \in \mathbb{R}^d,$$

where $\epsilon \geq 0$ is a small parameter and frequency vector $\omega$ of the forced oscillations is regarded as a parameter in $\mathcal{D}_0 := [0, 2\pi)^n$. We assume that the potential $V : \mathbb{R}^d \times \mathbb{T}^n \ni (x, \varphi) \mapsto V(x, \varphi) \in \mathbb{R}$ is continuous in all its variables and analytic in $\varphi$, where $\mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$ denotes the $n$-dimensional torus. For $\sigma > 0$, the function $V(x, \varphi)$ analytically in $\varphi$ extends to the strip $\mathbb{T}_\sigma^0 = \{(a + bi) \in \mathbb{C}^n / 2\pi \mathbb{Z}^n : |b| < \sigma\}$ and for all $(x, \varphi) \in \mathbb{R}^d \times \mathbb{T}_\sigma^0$ verifies

$$|V(x, \varphi)| \leq C(1 + \ln(1 + |x|^2))^{-2\sigma},$$

where $\sigma > 0$ and $C > 0$.

Before state the main results, we need some notations. For readers’ convenience we will follow the notations in [28]. Let $H_0 = -\Delta + |x|^2 = -\Delta + x_1^2 + x_2^2 + \cdots + x_d^2$ be the $d$-dimensional quantum harmonic oscillator. Its spectrum is the sum of $d$ copies of the odd integers set, i.e. the spectrum of $H_0$ equals $\hat{E} := \{d, d + 2, d + 4, \cdots\}$. Denote by $E_j$ for $j \in \hat{E}$ the associated eigenspace whose dimension equals

$$\# \{(i_1, i_2, \cdots, i_d) \in (2\mathbb{N} + 1)^d : i_1 + i_2 + \cdots + i_d = j\} := d_j \leq j^{d-1},$$

where $\mathbb{N} = \{0, 1, 2, \cdots\}$. Denote by $\{\Phi_{j,l}, l = 1, 2, \cdots, d_j\}$ the basis of $E_j$ obtained by $d$-tensor product of Hermite functions: $\Phi_{j,l} = \varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_d}$ for some choice of $i_1 + i_2 + \cdots + i_d = j$. Then setting

$$\mathcal{E} := \{(j, l) \in \hat{E} \times \mathbb{N} : l = 1, 2, \cdots, d_j\} \quad \text{and} \quad w_{j,l} = j \quad \text{for} \quad (j, l) \in \mathcal{E},$$

$(\Phi_a)_{a \in \mathcal{E}}$ forms the Hermite basis of $L^2(\mathbb{R}^d)$, verifying $H_0 \Phi_a = w_a \Phi_a, \quad a \in \mathcal{E}$. We define on $\mathcal{E}$ an equivalent relation: $a \sim b \iff w_a = w_b$ and denote by $[a]$ the equivalence class

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corresponding with \( a \in \mathcal{E} \). We notice for later use that \( \#[a] \leq w_a^{d-1} \) and that abbreviate the eigenspace \( E_{w_a} \) as \( E[a] \).

For \( s \geq 0 \) denote by \( \mathcal{H}^s \) the form domain of \( H_0^s \) and the domain of \( H_0^{s/2} \) endowed by the graph norm. For negative \( s \), the space \( \mathcal{H}^s \) is the dual of \( \mathcal{H}^{-s} \). In particular, for \( s \geq 0 \) an integer we have

\[
\mathcal{H}^s = \{ f \in L^2(\mathbb{R}^d) : x^\alpha \partial^\beta f \in L^2(\mathbb{R}^d), \forall \alpha, \beta \in \mathbb{N}^d, |\alpha| + |\beta| \leq s \}.
\]

For Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), we will denote by \( \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \) the space of bounded linear operators from \( \mathcal{H}_1 \) to \( \mathcal{H}_2 \) and write \( \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1) \) as \( \mathcal{B}(\mathcal{H}) \) for simplicity.

To a function \( u \in \mathcal{H}^s \) we associate the sequence \( \xi \) of its Hermite coefficients by the formula \( u(x) = \sum_{a \in \mathcal{E}} \xi_a \Phi_a(x) \) and define \( \ell^s_2 := \{(\xi_a)_{a \in \mathcal{E}} : \sum_{a \in \mathcal{E}} w_a^s |\xi_a|^2 < \infty \} \). Next we will identify \( \mathcal{H}^s \) with \( \ell^s_2 \) by endowing both spaces with the norm

\[
||u||_{\mathcal{H}^s} = ||\xi||_s = \left( \sum_{a \in \mathcal{E}} w_a^s |\xi_a|^2 \right)^{\frac{1}{2}}.
\]

Our main theorem is the following.

**Theorem 1.1.** Assume that \( V \) satisfies (1.2) with \( \nu \geq \frac{n+d+1}{2} \). There exists \( \epsilon_* > 0 \) such that for all \( 0 \leq \epsilon < \epsilon_* \) there exists \( D_\epsilon \subset D_0 := [0, 2\pi]^n \) such that for all \( \omega \in D_\epsilon \), the linear Schrödinger equation (1.1) reduces to a linear equation with constant coefficients in \( \mathcal{H}^1 \).

More precisely, for any \( \omega \in D_\epsilon \), there exists a linear isomorphism \( \Psi(\varphi) = \Psi_{\omega, \epsilon}(\varphi) \in \mathcal{B}(\mathcal{H}^p), \) for \( p \in [0, 1], \) unitary on \( L^2(\mathbb{R}^d) \), which analytically depends on \( \varphi \in \mathbb{T}^{n/2}_s \) and a bounded Hermitian operator \( \mathcal{W} = \mathcal{W}_{\omega, \epsilon} \in \mathcal{B}(\mathcal{H}^s) \), for \( s \geq 0 \), such that \( t \mapsto u(t, \cdot) \in \mathcal{H}^1 \) satisfies

\[
id\partial_t u = (-\Delta + |x|^2)u + \epsilon V(x, \omega t)u
\]

if and only if \( t \mapsto v(t, \cdot) = \Psi(\omega t)^{-1} u(t, \cdot) \) satisfies the autonomous equation

\[
id\partial_t v = (-\Delta + |x|^2)v + \mathcal{W}(v).
\]

Furthermore, there exists \( C > 0 \) such that

\[
\text{Meas}(D_0 \setminus D_\epsilon) \leq C \epsilon^\frac{1}{2}, \\
||\mathcal{W}||_{\mathcal{B}(\mathcal{H}^s)} \leq C \epsilon, \\
||\Psi(\varphi) - \text{Id}||_{\mathcal{B}(\mathcal{H}^p)} \leq C \epsilon^\frac{1}{2}, \quad \forall \varphi \in \mathbb{T}^{n/2}_s, \quad \forall p \in [0, 1].
\]

**Remark 1.2.** The infinite matrix \( (W_a^b)_{a, b \in \mathcal{E}} \) of the operator \( \mathcal{W} \) written in the Hermite basis \( (W_a^b = \int_{\mathbb{R}^d} \Phi_a W(\Phi_b) dx) \) is block diagonal and satisfies for some \( C > 0 \):

\[
||W_{[a]}|| \leq \frac{C \epsilon}{(1 + \ln w_a)^{\frac{1}{2}}}, \quad \forall a \in \mathcal{E}.
\]

As a consequence, we obtain the following corollaries concerning the Sobolev norm estimations on the solution of (1.1) and the spectra of the corresponding Floquet operator defined by

\[
K = -i \sum_{j=1}^n \omega_j \frac{\partial}{\partial \varphi_j} - \Delta + |x|^2 + \epsilon V(x, \varphi).
\]
Corollary 1.3. Assume that $V$ satisfies (1.2) with $\iota \geq \frac{n+d+1}{2}$. There exists $\epsilon_*>0$ such that for all $0 \leq \epsilon < \epsilon_*$ and $\omega \in \mathcal{D}_\epsilon$, there exists a unique solution $u \in C(\mathbb{R}, \mathcal{H}^1)$ such that $u(0) = u_0$. Moreover, $u$ is almost-periodic in time and satisfies

$$\|u_0\|_{\mathcal{H}^1} \leq \|u(t)\|_{\mathcal{H}^1} \leq (1+C\epsilon)\|u_0\|_{\mathcal{H}^1},$$

for some $C > 0$.

Corollary 1.4. Assume that $V$ satisfies (1.2) with $\iota \geq \frac{n+d+1}{2}$. There exists $\epsilon_*>0$ such that for all $0 \leq \epsilon < \epsilon_*$ and $\omega \in \mathcal{D}_\epsilon$, the spectrum of the Floquet operator $K$ is a pure point.

1.2. An introduction of the proof and related results. In this paper we will follow the strategy in [28] or [19]. As [28], if we endow the phase space $\mathcal{H}^s \times \mathcal{H}^s$ with the symplectic structure $\text{id} u \wedge d\bar{u}$, the equation (1.3) can be rewritten as the Hamiltonian system associated with the Hamiltonian

$$H(u, \bar{u}) = h(u, \bar{u}) + \epsilon q(\omega t, u, \bar{u}),$$

where $h(u, \bar{u}) = \int_{\mathbb{R}^d} |\nabla u|^2 + |x|^2 \bar{u}u dx$ and $q(\omega t, u, \bar{u}) = \int_{\mathbb{R}^d} V(x, \omega t)u\bar{u} dx$. Expanding $u$ and $\bar{u}$ on the Hermite basis of real valued functions $u = \sum_{a \in \mathcal{E}} \xi_a \Phi_a$, $\bar{u} = \sum_{a \in \mathcal{E}} \eta_a \Phi_a$, the Hamiltonian reads as $h = \sum_{a \in \mathcal{E}} w_a \xi_a \eta_a$, $q = \langle \xi, Q(\omega t)\eta \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the natural scalar product (no complex conjugation) and $Q$ is the infinite matrix whose entries are

$$Q^a_b(\omega t) = \int_{\mathbb{R}^d} V(x, \omega t)\Phi_a(x)\Phi_b(x) dx,$$

and $(\xi, \eta) \in Y_s$, in which $Y_s = \{ \zeta = (\zeta_a \in \mathbb{C}^2; a \in \mathcal{E}) : \|\zeta\|_s < \infty \}$. Therefore, Theorem 1.1 is equivalent to the reducibility problem for the non-autonomous Hamiltonian system associated with the Hamiltonian

$$\sum_{a \in \mathcal{E}} w_a \xi_a \eta_a + \epsilon \langle \xi, Q(\omega t)\eta \rangle.$$

As [28] we will construct a canonical change of variables and conjugate the Hamiltonian system with the Hamiltonian (1.8) to the Hamiltonian system associated with an autonomous Hamiltonian $\sum_{a \in \mathcal{E}} w_a \xi_a \eta_a + \langle \xi, W\eta \rangle$, where $W$ is block diagonal and will be clear in the following sections.

Here we would like to compare our approaches with those of Grébert and Paturel [28]. By and large, because of the so-called small-divisor problem, in both [28] and the present paper the KAM technique is used to eliminate the dependence on $\varphi$ of the perturbation $\langle \xi, Q(\varphi)\eta \rangle$. There is, however, an apparent difference. In [28], the perturbation matrix $Q(\varphi)$ belongs to $\mathcal{M}_{s,\beta}$. In other words, for any $a, b \in \mathcal{E}$,

$$w^\beta_a w^\beta_b \|Q^b_a\| \left(\sqrt{\min \{w_a, w_b\}} + |w_a - w_b|\right)^\frac{\beta}{2} \leq C.$$

The estimate comes from Cor. 3.2 from [30], the assumption of the perturbation and a delicate analysis. In this paper, we have a weaker assumption on $V(x, \varphi)$. Similarly, we also need an estimate on the perturbation matrix element $Q^b_a$ for every $a, b \in \mathcal{E}$. The following is the corresponding estimate:

Lemma 1.5. Assume that $V$ satisfies (1.2) with $\iota \geq 0$, then we have

$$\left|\int_{\mathbb{R}^d} V(x, \varphi)\Psi_a(x)\Psi_b(x) dx\right| \leq \frac{C}{(1 + \ln w_a)(1 + \ln w_b)^{\iota}},$$

(1.10)
for all $\Psi_a \in E_{[a]}$, $\Psi_b \in E_{[b]}$ and $\|\Psi_a\| = \|\Psi_b\| = 1$, where the constant $C \equiv C(d, \epsilon) > 0$ and $\| \cdot \|$ denotes the $L^2(\mathbb{R}^d)$ norm.

**Remark 1.6.** The above estimate was firstly proved by Z. Wang and one of the authors in [43] when $d = 1$.

**Remark 1.7.** From the above lemma, we obtain
\[
(1 + \ln w_a)^\beta (1 + \ln w_b)^\beta \| Q_{[a]}^{[b]} \| \leq C.
\]
Thus we need to introduce a new space $M_\beta$ which can be compared with the corresponding one $\overline{M}_{s, \beta}$ in [28]. See section 2 for the definition of $M_\beta$.

In [28] Grébert and Paturel used Lemma 4.3 called the key lemma to solve the homological equation, in which they assume $|\mu_a - \lambda_a| \leq \frac{C_0}{w_a^\delta}$ for all $a \in \mathcal{E}$. We remark this lemma was first built up in Proposition 2.2.4 [17] and also applied in [23] and [27]. But in our case we only have a weaker assumption $|\mu_a - \lambda_a| \leq \frac{C_0}{(1 + \ln w_a)^\beta}$ for all $a \in \mathcal{E}$ which comes from $Q \in M_\beta$. On the first try one can follow the proof of Lemma 4.3 in [28], the estimate should be
\[
\|B(k)_{[a]}^{[b]}\| \leq \frac{C_d K^{d-1} \exp\{C \delta, \kappa^{-\frac{1}{2}}\}}{\kappa(1 + |w_a - w_b|)} \| A_{[a]}^{[b]} \|.
\]
If we choose $\kappa \sim e^\alpha$, the term $\exp\{C \delta, \kappa^{-\frac{1}{2}}\}$ is too big for KAM iteration whatever $\alpha > 0$ is chosen. If one chooses $\kappa \sim \frac{C_0}{(1 + \ln w_a)\delta}$, it follows $\exp\{C \delta, \kappa^{-\frac{1}{2}}\} \sim e^{-100}$ which seems to be enough for KAM. But from a further investigation we find that one will face big troubles relative with measure estimates. Now it is clear that we need to develop some new estimate for $\|B(k)_{[a]}^{[b]}\|$. In the following we denote by $M$ the set of infinite matrices $A : \mathcal{E} \times \mathcal{E} \rightarrow \mathbb{C}$ which satisfies $\sup_{a,b \in \mathcal{E}} \| A_{[a]}^{[b]} \| < \infty$, where $A_{[a]}^{[b]}$ denotes the restriction of $A$ on the block $[a] \times [b]$ and $\| \cdot \|$ denotes the operator norm. Here is the new estimate.

**Lemma 1.8.** Let $A \in M$ and $B(k)$ defined for $k \in \mathbb{Z}^n$ with $|k| \leq K$ by
\[
B(k)_j^l = \frac{A_j^l}{k \cdot \omega - \mu_j + \mu_l}, \quad j \in [a], l \in [b],
\]
where $\omega \in [0, 2\pi)^n$ and $(\mu_a)_{a \in \mathcal{E}}$ is a sequence of positive real numbers satisfying
\[
|\mu_a - \lambda_a| \leq \frac{C_0}{(1 + \ln w_a)^\delta}, \quad \text{for all } a \in \mathcal{E}
\]
(1.13)
such that for all $a, b \in \mathcal{E}$ and all $|k| \leq K$,
\[
|k \cdot \omega - \lambda_a + \lambda_b| \geq \gamma(1 + |w_a - w_b|),
\]
(1.14)
\[
|k \cdot \omega - \mu_j + \mu_l| \geq \kappa(1 + |w_a - w_b|), \quad j \in [a], l \in [b]
\]
(1.15)
where $\gamma, \kappa, \delta, C_0 > 0$. Then $B(k) \in M$ and there exists a positive constant $C_{\delta, d}$ such that
\[
\|B(k)_{[a]}^{[b]}\| \leq \exp\{C_{\delta, d}^{-1/6}\} \frac{\exp\{C \delta, \kappa^{-1/6}\}}{\kappa(1 + |w_a - w_b|)} \| A_{[a]}^{[b]} \|,
\]
where $\| \cdot \|$ denotes the operator norm and $C_{\delta, d} = \frac{d - 1}{2} (2C_0)^{\frac{1}{2}}$. 
Remark 1.9. The term \( \text{exp}\{C_\delta \delta^{-1/\delta}\} \) is clearly better than the one \( \text{exp}\{C_\delta \delta^{-1/\delta}\} \) in (1.11). We explain it in a heuristic way. For example, if choose \( \gamma \sim K^{-\alpha} \), it follows that \( \text{exp}\{C_\delta \delta^{-1/\delta}\} \sim \text{exp}\{K^{\hat{s}}\} \) and \( \text{exp}\{K^{\hat{s}}\} \leq C\varepsilon^{1/2} \) when \( \alpha < \delta \). The KAM iteration can be easily set up as [28].

In the end let us review the previous works on the reducibility of some important PDEs and the behaviors of solutions in Sobolev spaces. We begin with the quantum harmonic oscillators (for short “QHO”). See [15, 29, 43, 44] for the reducibility results on 1-d QHO with bounded perturbations. Bambusi [3, 4] firstly built the reducibility results for 1-d QHO with unbounded perturbations. His proof was based on the pseudodifferential calculus, in which he generalized the ideas from [1, 40](see also [11, 25, 38]). It seems that the pseudodifferential calculus method hasn’t been applied for the unbounded perturbation terms such as \( \langle x \rangle^\mu \cos(x - \omega t)(\mu > 0) \) (Remark 2.7 in [3]), which have been solved recently by Luo and one of the authors ([31]) when \( 0 \leq \mu < \frac{1}{2} \). For a general 1-d Schrödinger equation we recall the classical results [7] in which the potential grows at infinity like \( \langle x \rangle^{2\ell} \) with \( \ell > 1 \) and the perturbation is bounded by \( \langle x \rangle^\beta \) with \( \beta < \ell - 1 \). The limit case was solved by Liu-Yuan ([35]). Recently the upper bound for \( \beta \) was improved further by Bambusi([3, 4]). When the perturbation is limited to a class of terms such as \( \langle x \rangle^\mu \cos(x - \omega t) \) we([33]) can improve the original upper boundedness from \( \ell - 1 \) to at least \( \ell - \frac{5}{4} \).

Reducibility for PDEs in high dimension was initiated by Eliasson-Kuksin [19]. We can refer to [28] and [32] for higher-dimensional QHO with bounded potential. The reducibility result for n-d QHO was first built in [9] by Bambusi-Grêbert-Maspero-Robert. Towards other PDEs Montalto [39] obtained the first reducibility result for linear wave equations with unbounded perturbations on \( \mathbb{T}^d \), which can be applied to the linearized Kirchhoff equation in higher dimension. Bambusi, Langella and Montalto [5] obtained the reducibility results for transportation equations with unbounded perturbations([22]). See also [23, 24] for a linear Schrödinger equation on zoll manifold with unbounded potential. We remark that by implementing the above techniques the KAM-type results of quasi - linear PDEs such as incompressible Euler flows in 3D [2] and forced Kirchhoff equation on \( \mathbb{T}^d \) [14] have been built recently.

The reducibility results usually imply the boundedness of Sobolev norms. For the growth rate of the solutions with time in \( H^s \)–norm, see [9] and [26] for a \( t^s \)– polynomial growth for 1-d QHO with time periodic perturbations. Delort [16] constructed a \( t^{s/2} \)– polynomial growth for 1-d QHO with certain time periodic order zero perturbation(see [36] for a short proof). Combining with the ideas in [9] and [18], Zhao, Zhou and one of the authors [34] obtained the precise dynamics for one class of 1-d QHO with quasi - periodic in time quadratic perturbations, in which they also presented an exponential growth in time for 1-d QHO. Recently, for 2-d QHO with perturbation which is decaying in \( t \), Faou-Raphaël [21] constructed a solution whose \( H^s \)–norm presents logarithmic growth with \( t \). For 2-d QHO with perturbation being the projection onto Bargmann-Fock space, Thomann [42] constructed explicitly a traveling wave whose Sobolev norm presents polynomial growth with \( t \), based on the study in [41] for linear Lowest Landau equations(LLL) with a time-dependent potential. There are also many literatures, e.g. [6, 8, 10, 12, 13, 20, 37, 45], which are closely relative to the upper growth bound of the solution in Sobolev space.

The rest paper will be organized as follows. In Section 2, a new reducibility theorem is presented. In Section 3, we check all the hypothesis of the reducibility theorem are
satisfied which follows the main theorem. In Section 4, after present the proof of key lemma we prove the reducibility theorem. Finally, the appendix contains some technical Lemmas.

2. Reducibility Theorem

In this section we state an abstract reducibility theorem for quadratic quasiperiodic in time Hamiltonians of a general form \( \sum_{a \in \mathcal{E}} \lambda_a \xi_a \dot{a} + \epsilon(\xi, Q(\omega t)\eta) \).

2.1. Setting. Following [28], we will introduce some spaces and relative algebraic properties for later use.

**Linear space.** Let \( s \in \mathbb{R} \), we consider the complex weighted \( \ell^2 \) space
\[
\ell^2_s = \{ \xi = (\xi_a \in \mathbb{C}, a \in \mathcal{E}) : \|\xi\|_s < \infty \}, \text{ where } \|\xi\|_s^2 = \sum_{a \in \mathcal{E}} w_a^s|\xi_a|^2.
\]
Then we define
\[
Y_s = \ell^2_s \times \ell^2_s = \{ \zeta = (\zeta_a = (\xi_a, \eta_a) \in \mathbb{C}^2, a \in \mathcal{E}) : \|\zeta\|_s < \infty \},
\]
where
\[
\|\zeta\|_s^2 = \sum_{a \in \mathcal{E}} w_a^s|\xi_a|^2 = \sum_{a \in \mathcal{E}} w_a^s(|\xi_a|^2 + |\eta_a|^2).
\]
We equip the space \( Y_s, s \geq 0 \), with the symplectic structure \( \text{id}\xi \wedge d\eta \). Let \( f \) be a \( C^1 \) smooth function, defined on a domain \( \mathcal{O} \subset Y_s \), then we have the associated Hamiltonian system
\[
\begin{align*}
    \dot{\xi} &= -i \nabla_\eta f(\xi, \eta), \\
    \dot{\eta} &= i \nabla_\xi f(\xi, \eta),
\end{align*}
\]
where \( \nabla f = (\nabla_\xi f, \nabla_\eta f)^T \) is the gradient with respect to the scalar product in \( Y_0 \). Naturally, define the Poisson bracket for any \( C^1 \) smooth functions \( f \) and \( g \), defined on a domain \( \mathcal{O} \subset Y_s \)
\[
\{f, g\} = -i \left( \sum_{a \in \mathcal{E}} \frac{\partial f}{\partial \xi_a} \frac{\partial g}{\partial \eta_a} - \frac{\partial f}{\partial \eta_a} \frac{\partial g}{\partial \xi_a} \right).
\]

**Infinite matrices.** Let \( \beta \geq 0 \), we denote by \( \mathcal{M}_\beta \) the set of infinite matrices \( A : \mathcal{E} \times \mathcal{E} \mapsto \mathbb{C} \) that satisfy
\[
|A|_\beta := \sup_{a, b \in \mathcal{E}} (1 + \ln w_a)^\beta (1 + \ln w_b)^\beta \|A_{[a]}^{[b]}\| < \infty,
\]
where \( A_{[a]}^{[b]} \) denotes the restriction of \( A \) on the block \([a] \times [b]\) and \( \| \cdot \| \) denotes the operator norm. Further we denote \( \mathcal{M} = \mathcal{M}_0 \). We will also need the more regular space \( \mathcal{M}_\beta^+ \subset \mathcal{M}_\beta \): an infinite matrix \( A \in \mathcal{M} \) belongs to \( \mathcal{M}_\beta^+ \) if
\[
|A|_{\beta^+} := \sup_{a, b \in \mathcal{E}} (1 + |w_a - w_b|)(1 + \ln w_a)^\beta (1 + \ln w_b)^\beta \|A_{[a]}^{[b]}\| < \infty.
\]
The following structural lemma is proved in Appendix 5.1:

**Lemma 2.1.** Let \( \beta > \frac{1}{2} \), there exists an absolute constant \( C \equiv C(\beta) > 0 \) such that
(i). Let \( A \in \mathcal{M}_\beta \) and \( B \in \mathcal{M}_\beta^+ \). Then \( AB \) and \( BA \) belong to \( \mathcal{M}_\beta \) and
\[
|AB|_\beta, |BA|_\beta \leq C|A|_\beta|B|_{\beta^+}.
\]
(ii). Let \( A, B \in \mathcal{M}_\beta^+ \). Then \( AB \) belongs to \( \mathcal{M}_\beta^+ \) and \( |AB|_{\beta^+} \leq C|A|_{\beta^+}|B|_{\beta^+} \).
(iii). Let \( A \in \mathcal{M}_\beta^+ \). Then \( e^A - \text{Id} \) belongs to \( \mathcal{M}_\beta^+ \) and 
\[ |e^A - \text{Id}|_{\beta^+} \leq e^{|A|_{\beta^+}} |A|_{\beta^+}. \]
(iv). Let \( A \in \mathcal{M}_\beta \). Then for any \( s \geq 1 \), \( A \in \mathcal{B}(l^2_s, l^2_s) \) and 
\[ \|A\|_{\mathcal{B}(l^2_s, l^2_s)} \leq C |A|_\beta. \]
(v). Let \( A \in \mathcal{M}_\beta^+ \). Then for any \( s \in [-1, 1] \), \( A \in \mathcal{B}(l^2_s) \) and 
\[ \|A\|_{\mathcal{B}(l^2_s)} \leq C |A|_{\beta^+}. \]

**Normal form.**

**Definition 2.2.** A matrix \( Q : \mathcal{E} \times \mathcal{E} \mapsto \mathbb{C} \) is in normal form, denoted by \( Q \in \mathcal{NF} \), if

(i). \( Q \) is block diagonal, i.e. \( Q^b_a = 0 \) for all \( w_a \neq w_b \).

(ii). \( Q \) is Hermitian, i.e. \( Q^b_a = \overline{Q^b_a} \).

Notice that a block diagonal matrix with bounded blocks in operator norm defines a bounded operator on \( l^2_s \), \( \forall s \in \mathbb{R} \), which implies \( \mathcal{M}_\beta \cap \mathcal{NF} \subset \mathcal{B}(l^2_s) \). To a matrix \( Q = (Q^b_a)_{a,b \in \mathcal{E}} \in \mathcal{B}(l^2_s) \) for \( s \geq 0 \), define in a unique way a quadratic form on \( Y_s \equiv (\zeta_a)_{a \in \mathcal{E}} = (\zeta_a, \eta_a)_{a \in \mathcal{E}} \) by the formula 
\[ g(\xi, \eta) = \langle \xi, \eta \rangle = \sum_{a,b \in \mathcal{E}} Q^b_a \zeta_a \eta_b. \]
A straightforward computation leads to
\[ \{q_1, q_2\}(\xi, \eta) = i \langle \xi, [Q_1, Q_2] \eta \rangle, \quad (2.1) \]
where \([Q_1, Q_2]\) is the commutator of two matrices \( Q_1 \) and \( Q_2 \).

In the following, by abuse of language, we will call both of \( q \) and \( Q \) the Hamiltonian. **Parameter.** In all the paper \( \omega \) will play the role of a parameter belonging to \( \mathcal{D}_0 = [0, 2\pi]^n \). All the constructed maps will be dependent on \( \omega \) with \( C^1 \) regularity. When a map is only defined on a Cantor subset of \( \mathcal{D}_0 \) the regularity has to be understood in the Whitney sense.

**A class of quadratic Hamiltonian.** Let \( \beta > \frac{1}{2}, \sigma > 0 \) and \( \mathcal{D} \subset \mathcal{D}_0 \) and denote by \( \mathcal{M}_\beta(\mathcal{D}, \sigma) \) the set of \( C^1 \) mappings \( \mathcal{D} \times \mathcal{T}_\sigma^o \ni (\omega, \varphi) \mapsto Q(\omega, \varphi) \in \mathcal{M}_\beta \), which is real analytic in \( \varphi \in \mathcal{T}_\sigma^o \). Endow the space with the norm
\[ [Q]^{D, \sigma}_\beta := \sup_{\omega \in \mathcal{D}, j=0,1} |\partial^j_{\omega} Q(\omega, \varphi)|_\beta. \]
In view of Lemma 2.1 (iv), to a matrix \( Q \in \mathcal{M}_\beta(\mathcal{D}, \sigma) \) define the quadratic form in \( Y_1 \)
\[ g(\xi, \eta; \omega, \varphi) = \langle \xi, Q(\omega, \varphi) \eta \rangle \]
and a straightforward computation leads to
\[ |g(\xi, \eta; \omega, \varphi)| \leq C |[Q]^{D, \sigma}_\beta||(\xi, \eta)||_2^2, \quad \text{for} \ (\xi, \eta) \in Y_1, \ (\omega, \varphi) \in \mathcal{D} \times \mathcal{T}_\sigma^o. \]
Denote by \( \mathcal{M}_\beta^+(\mathcal{D}, \sigma) \) the subspace of \( \mathcal{M}_\beta(\mathcal{D}, \sigma) \) formed by Hamiltonians \( S \), verifying \( S(\omega, \varphi) \in \mathcal{M}_\beta^+ \) and endow it with the norm \([S]^{D, \sigma}_\beta := \sup_{\omega \in \mathcal{D}, j=0,1} |\partial^j_{\omega} S(\omega, \varphi)|_\beta^+ \). The space of Hamiltonians \( N \in \mathcal{M}_\beta(\mathcal{D}, \sigma) \) that are independent on \( \varphi \) will be denoted by \( \mathcal{M}_\beta(\mathcal{D}) \) and equipped with the norm \([N]^{D, \sigma}_\beta := \sup_{\omega \in \mathcal{D}, j=0,1} |\partial^j_{\omega} N(\omega)|_\beta \).

**Hamiltonian flow.** To any \( R \in \mathcal{M}_\beta^+ \) with \( \beta > \frac{1}{2} \), define in a unique way the symplectic linear change of variables on \( Y_s : (\xi, \eta) \mapsto (e^{iR^T \xi}, e^{iR^T \eta}) \). It is well-defined and invertible in \( \mathcal{B}(l^2_s) \) for all \( s \in [-1, 1] \) by the assertion (v) of Lemma 2.1. Concretely, the change of variables can be regarded as the time-one flow generated by the quadratic Hamiltonian \( \chi(\xi, \eta) = \langle \xi, R \eta \rangle \) and it preserves the symmetry \( \eta = \xi \) for any initial datum considered in the paper if and only if \( R \) is a Hermitian matrix, i.e.
\[ R^T = \overline{R}. \quad (2.2) \]
When \( R \) also depends smoothly on \( \varphi, \mathcal{T}_\sigma^o \ni \varphi \mapsto R(\varphi) \in \mathcal{M}_\beta^+ \) we associate to \( R \) the symplectic linear change of variables on the extended phase space
\[ \phi^1_\chi(y, \varphi, \xi, \eta) = (\tilde{y}, \varphi, e^{-iR^T \xi}, e^{iR^T \eta}), \quad (2.3) \]
where the Hamiltonian $\chi(y, \varphi, \xi, \eta) = \langle \xi, R(\varphi)\eta \rangle$ and $\check{y} = y - \int_0^1(e^{-itR^*}\xi, \nabla_{\varphi}R e^{it\eta})dt$. 

2.2. **Hypothesis on the spectrum.** Now we present our hypothesis on $\lambda_a, a \in \mathcal{E}$:

**Hypothesis H1** (asymptotics). Assume that there exists an absolute constant $c_0 > 0$ such that

$$\lambda_a \geq c_0 \text{ and } |\lambda_a - \lambda_b| \geq c_0|w_a - w_b|, \text{ for all } a, b \in \mathcal{E}. \quad (2.4)$$

**Hypothesis H2** (second Melnikov condition in measure). There exist absolute positive constants $\tau_1, \tau_2$ and $C$ such that the following holds: for each $\gamma > 0$ and $K \geq 1$ there exists a closed subset $\mathcal{D}' \subset \mathcal{D}_0$ satisfying

$$\text{Meas}(\mathcal{D}_0 \setminus \mathcal{D}') \leq C K^n \gamma^{-\tau_2}$$

such that for all $\omega \in \mathcal{D}'$, all $k \in \mathbb{Z}^n$ with $0 < |k| \leq K$ and all $a, b \in \mathcal{E}$ we have

$$|k \cdot \omega - \lambda_a + \lambda_b| \geq \gamma(1 + |w_a - w_b|).$$

2.3. **The reducibility theorem.** Consider the Hamiltonian

$$H_\omega(t, \xi, \eta) = \sum_{a \in \mathcal{E}} \lambda_a \xi_a \eta_a + \epsilon \langle \xi, Q(\omega t)\eta \rangle \quad (2.5)$$

and the associated non-autonomous Hamiltonian system on $Y_\omega$:

$$\begin{align*}
\dot{\xi} &= -iN_0 \xi - icQ^T(\omega t)\xi, \\
\dot{\eta} &= iN_0 \eta + icQ(\omega t)\eta,
\end{align*} \quad (2.6)$$

where $N_0 = \text{diag}(\lambda_a : a \in \mathcal{E})$.

**Theorem 2.3.** Let $\alpha = \frac{\alpha_1 + \alpha_2}{2} + 1$ with $\alpha_1 = \max\{\tau_1, n + d\}$ and $\alpha_2 = \max\{\tau_2, 1\}$. Fix $\beta \geq \frac{\alpha}{2}$ and $\sigma > 0$. Assume that $(\lambda_a)_{a \in \mathcal{E}}$ satisfies Hypothesis H1, H2 and that $Q \in \mathcal{M}_\beta(\mathcal{D}_0, \sigma)$, then there exists $\epsilon_* > 0$ such that for all $0 < \epsilon < \epsilon_*$, there exists $C > 0$ and

(i). a Cantor set $\mathcal{D}_\epsilon \subset \mathcal{D}_0$ of asymptotically full measure:

$$\text{Meas}(\mathcal{D}_0 \setminus \mathcal{D}_\epsilon) \leq C e^{\alpha_2/16}; \quad (2.7)$$

(ii). a $C^1$ family (in $\omega \in \mathcal{D}_\epsilon$) of real analytic (in $\varphi \in \mathbb{T}_{\sigma/2}^n$) linear, unitary and symplectic coordinate transformations on $Y_0$:

$$\Phi_\omega(\varphi) : (\xi, \eta) \mapsto (\mathcal{M}_\omega(\varphi)\xi, \mathcal{M}_\omega(\varphi)\eta), \quad (\omega, \varphi) \in \mathcal{D}_\epsilon \times \mathbb{T}_{\sigma/2}^n;$$

(iii). a $C^1$ family of quadratic autonomous Hamiltonians in normal form $\mathcal{H}_\omega = \langle \xi, N_\omega \eta \rangle$, where $N_\omega \in \mathcal{N}F$ is close to $N_0$:

$$\|N_\omega - N_0\|_\beta \leq C \epsilon, \quad \omega \in \mathcal{D}_\epsilon, \quad (2.8)$$

such that $t \mapsto (\xi(t), \eta(t)) \in Y_1$ is a solution of autonomous Hamiltonian system associated with $\mathcal{H}_\omega$:

$$\begin{align*}
\dot{\xi} &= -iN_\omega^T \xi, \\
\dot{\eta} &= iN_\omega \eta
\end{align*} \quad (2.9)$$

if and only if $t \mapsto \Phi_\omega(\omega t)(\xi(t), \eta(t)) \in Y_1$ is a solution of the original Hamiltonian system (2.6). Furthermore, $\Phi_\omega(\varphi)$ is a bounded operator from $Y_\omega$ into itself for all $p \in [0, 1]$ and close to identity:

$$\|\mathcal{M}_\omega(\varphi) - \text{Id}\|_{g(\xi)} \leq C e^{\frac{\alpha}{16}}.$$
3. Applications to the quantum harmonic oscillator on $\mathbb{R}^d$

In this section we will prove Theorem 2.3 which concludes Theorem 1.1.

3.1. Verification of the hypothesis. We first verify the Hypothesis of Theorem 2.3 for the quantum harmonic oscillator equation (1.3).

**Lemma 3.1** (see [27]). When $\lambda_a = w_a$, $a \in \mathcal{E}$. Hypothesis H1 and H2 hold true with $c_0 = 1$ and $\mathcal{D}_0 = [0, 2\pi]^n$ and $\tau_1 = n + 1, \tau_2 = 1$.

Proof of Lemma 1.5: Let $\Psi_a \in E_{[a]}, \Psi_b \in E_{[b]}$ and $\|\Psi_a\| = \|\Psi_b\| = 1$, then

$$\left| \int_{\mathbb{R}^d} V(x) \Psi_a(x) \Psi_b(x) dx \right| \leq \|V^{\frac{1}{2}}(x, \varphi)\| \|\Psi_a\||\|\Psi_b\|.$$  (3.1)

By the symmetry we only need to estimate $\|V^{\frac{1}{2}}(x, \varphi)\|$. Let $\mathcal{D}_a = \{x \in \mathbb{R}^d : |x| \leq w_a^{1/4d}\}$ and denote by $\mathcal{D}_c$ its complement, then one has

$$\|V^{\frac{1}{2}}(x, \varphi)\| \leq \|V^{\frac{1}{2}}(x, \varphi)\|_{L^2(\mathcal{D}_a)} + \|V^{\frac{1}{2}}(x, \varphi)\|_{L^2(\mathcal{D}_c)}.$$  (3.2)

Clearly, since $|V(x, \varphi)| \leq C$ then one has

$$\|V^{\frac{1}{2}}(x, \varphi)\|_{L^2(\mathcal{D}_a)} \leq \|\Psi_a\|_{L^p(\mathcal{D}_a)} \|V^{\frac{1}{2}}(x, \varphi)\|_{L^q(\mathcal{D}_a)} \leq C \|\Psi_a\|_{L^p(\mathcal{D}_a)} \|V^{\frac{1}{2}}(x, \varphi)\|_{L^q(\mathcal{D}_a)}.$$  (3.3)

By (1.2) one has $|V(x, \varphi)| \leq C \left(\frac{1 + \ln(1 + |x|^2)}{(1 + \ln(1 + |x|^2))^2} \leq \frac{C_{d,e}}{(1 + \ln w_a)^{d+2}}\right)$ for $x \in \mathcal{D}_a$. Hence, we obtain

$$\|V^{\frac{1}{2}}(x, \varphi)\|_{L^2(\mathcal{D}_c)} \leq \frac{C_{d,e}}{(1 + \ln w_a)^{d+2}} \|\Psi_a\|_{L^2(\mathbb{R}^d)}.\)$$

Combining (3.2), (3.3) with the last estimation we have

$$\|V^{\frac{1}{2}}(x, \varphi)\| \leq \frac{C_{d,e}}{(1 + \ln w_a)^{d+2}}.$$  (3.4)

Collecting (3.1) and (3.4) concludes the result (1.10).

**Remark 3.2.** Using Lemma 1.5 and a similar method as [27] we have $Q_0^b(\varphi) = \int_{\mathbb{R}^d} V(x, \varphi) \Phi_a(x) \Phi_b(x) dx$, belongs to $\mathcal{M}_t(D_0, \sigma)$.

**Remark 3.3.** For quantum harmonic oscillator equation (1.3), we have $\alpha = n + d + 1$ since $\alpha = \frac{n}{a_2} + 1$ with $\alpha_1 = \max\{\tau_1, n + d\}$ and $\alpha_2 = \max\{\tau_2, 1\}$, where $\tau_1 = n + 1$ and $\tau_2 = 1$.  \[\square\]
3.2. Proof of Theorem 1.1. The Schrödinger equation (1.3) is a Hamiltonian system on \( \mathcal{H}^s \times \mathcal{H}^s \) \((s \geq 1)\) associated with the Hamiltonian function (1.6). Written on the orthonormal basis \((\Phi_a)_{a \in \mathcal{E}}\), it is equivalent to the Hamiltonian system on \( Y_s \) associated with (1.8) which reads as (2.3) with \( \lambda_a = w_a \) and \( Q \) given by (1.7). By Lemma 3.1 and Remark 3.2, if \( V \) satisfies (1.2) with \( \nu \geq \frac{a + b}{s - 1} \), we can apply Theorem 2.3 to the Hamiltonian (1.8) which concludes Theorem 1.1. More precisely, in the new coordinates under a unitary transformation given by Theorem 2.3, \((\xi, \eta) = (M_\omega(\omega t)\xi', M_\omega(\omega t)\eta')\), the original system

\[
\begin{align*}
\begin{cases}
\dot{\xi}_a &= -iw_a\xi_a - i\varepsilon(Q^T(\omega t)\xi)_a, \\
\dot{\eta}_a &= iw_a\eta_a + i\varepsilon(Q(\omega t)\eta)_a,
\end{cases}
\quad a \in \mathcal{E}
\end{align*}
\]

conjugates to an autonomous system as follows:

\[
\begin{align*}
\begin{cases}
\dot{\xi}'_a &= -i(N^T_\omega \xi')_a, \\
\dot{\eta}'_a &= i(N_\omega \eta')_a,
\end{cases}
\quad a \in \mathcal{E},
\end{align*}
\]

where \( N_\omega \in \mathcal{N} \mathcal{F} \) and \( M_\omega(\omega t)M_\omega^T(\omega t) = \text{Id} \). Furthermore, corresponding to the initial datum \( u_0(x) = \sum_{a \in \mathcal{E}} \xi(0)_a \Phi_a(x) \in \mathcal{H}^s \) the solution \( u(t, x) \) of (1.3) reads

\[
u(t, x) = \sum_{a \in \mathcal{E}} \xi(t)_a \Phi_a(x) \quad \text{with} \quad \xi(t) = M_\omega(\omega t)e^{-itN_\omega^T}M_\omega^T(0)\xi(0).
\]

Concretely, define the transformation \( \Psi(\varphi) \in \mathcal{B} (\mathcal{H}^s) \) by

\[
\Psi(\varphi) \left( \sum_{a \in \mathcal{E}} \xi'_a \Phi_a(x) \right) = \sum_{a \in \mathcal{E}} \left( M_\omega(\varphi) \xi'_a \right) \Phi_a(x).
\]

Then \( \nu(t, x) \) satisfies (1.4) if and only if \( u(t, x) = \Psi(\omega t)\nu(t, x) \) satisfies the original equation (1.3), where \( \mathcal{W} \) is defined by

\[
\mathcal{W} \left( \sum_{a \in \mathcal{E}} \xi'_a \Phi_a(x) \right) = \sum_{a \in \mathcal{E}} (W \xi'_a) \Phi_a(x) \quad \text{with} \quad W = N_\omega - N_0.
\]

By construction collecting (2.7)-(2.9) leads to (1.5) in Theorem 1.1. □

For the proofs of Corollary 1.3 and 1.4 refer to [27].

4. PROOF OF REDUCIBILITY THEOREM

In this section we will prove the reducibility theorem presented in Sec. 2 by the KAM methods. As we mentioned before, Lemma 1.8 is very important for solving homological equations, whose proof we present below.

4.1. Proof of Lemma 1.8.

\textit{Proof.} Denote by \( D_{[a]} \) the diagonal (square) matrix with entries \( \mu_j \) for \( j \in [a] \). The equation (1.12) reads

\[
k \cdot \omega B_{[a]}^{[b]} - D_{[a]}B_{[a]}^{[b]} + B_{[a]}^{[b]}D_{[b]} = A_{[a]}^{[b]}.
\]

From (1.14) for all \( a, b \in \mathcal{E} \) and all \( |k| \leq K \), \( k \cdot \omega - \lambda_a + \lambda_b \neq 0 \). We distinguish two cases. \textbf{Case 1:} suppose that \( k \cdot \omega - \lambda_a + \lambda_b > 0 \). Clearly, in this case by (1.14) we have
\[ k \cdot \omega - \lambda_a + \lambda_b \geq \gamma (1 + |w_a - w_b|) \geq \gamma. \]

By (1.13) for \( j \in [a] \) and \( l \in [b] \), if \( \min \{ w_a, w_b \} > \exp \left\{ \left( \frac{2C_0}{\gamma} \right)^{1/\delta} \right\} \), we obtain

\[
\begin{align*}
&k \cdot \omega - \mu_j + \mu_l = k \cdot \omega - \lambda_a + \lambda_b - \mu_j + \lambda_a - \lambda_b + \mu_l \\
&\geq \gamma - \frac{C_0}{(1 + \ln w_a)^{\delta}} - \frac{C_0}{(1 + \ln w_b)^{\delta}} \\
&\geq \gamma - \frac{2C_0}{(1 + \ln \min \{ w_a, w_b \})^{\delta}} > 0.
\end{align*}
\]

It follows that

\[
\begin{align*}
k \cdot \omega + \min_{l \in [b]} \{ \mu_l \} > \max_{j \in [a]} \{ \mu_j \} > 0. \tag{4.2}
\end{align*}
\]

Subcase 1: \( \min \{ w_a, w_b \} > \exp \left\{ \left( \frac{2C_0}{\gamma} \right)^{1/\delta} \right\} \). (4.2) proves that \( k \cdot \omega I_{[b]} + D_{[b]} \) is an invertible operator and \( \| (k \cdot \omega I_{[b]} + D_{[b]})^{-1} \| = \frac{1}{k \cdot \omega + \min_{l \in [b]} \{ \mu_l \}}. \) Thus (4.1) is equivalent to

\[
B_{[a]}^{[b]} - D_{[a]}^{[b]}B_{[a]}^{[b]}(k \cdot \omega I_{[b]} + D_{[b]})^{-1} = A_{[a]}^{[b]}(k \cdot \omega I_{[b]} + D_{[b]})^{-1}.
\]

Denote by \( \mathcal{L}_{[a] \times [b]} \) the operator acting on matrices of size \([a] \times [b]\) such that

\[
\mathcal{L}_{[a] \times [b]}(B_{[a]}^{[b]}) = D_{[a]}^{[b]}B_{[a]}^{[b]}(k \cdot \omega I_{[b]} + D_{[b]})^{-1}.
\]

Then we have \( \| \mathcal{L}_{[a] \times [b]}(B_{[a]}^{[b]}) \| \leq \frac{\max_{j \in [a]} \{ \mu_j \}}{k \cdot \omega + \min_{l \in [b]} \{ \mu_l \}} \| B_{[a]}^{[b]} \| \) and \( \| \mathcal{L}_{[a] \times [b]} \| < 1 \) by (4.2).

The operator \( \text{Id} - \mathcal{L}_{[a] \times [b]} \) is invertible and thus

\[
\begin{align*}
\| B(k)_{[a]}^{[b]} \| &= \| (\text{Id} - \mathcal{L}_{[a] \times [b]})^{-1} A_{[a]}^{[b]}(k \cdot \omega I_{[b]} + D_{[b]})^{-1} \| \\
&\leq \frac{1}{1 - \frac{\max_{j \in [a]} \{ \mu_j \}}{k \cdot \omega + \min_{l \in [b]} \{ \mu_l \}}} \cdot \frac{\| A_{[a]}^{[b]} \|}{\| B_{[a]}^{[b]} \|} \\
&= \frac{\| A_{[a]}^{[b]} \|}{\| B_{[a]}^{[b]} \|} \frac{\| A_{[a]}^{[b]} \|}{k \cdot \omega + \min_{l \in [b]} \{ \mu_l \} - \max_{j \in [a]} \{ \mu_j \}} \tag{4.3} \\
&\leq \frac{\| A_{[a]}^{[b]} \|}{\kappa(1 + |w_a - w_b|)} \quad \text{by (1.15) and (4.2)}.
\end{align*}
\]

Subcase 2: \( \min \{ w_a, w_b \} \leq \exp \left\{ \left( \frac{2C_0}{\gamma} \right)^{1/\delta} \right\} \). In this situation we have \( |B(k)_{j}^{[b]}| \leq \frac{|A_{j}^{[b]}|}{\kappa(1 + |w_a - w_b|)}. \)

Then for any \( \xi \in \ell_2^b \)

\[
\| B(k)_{[a]}^{[b]} \xi_{[b]} \|^2 = \sum_{j \in [a]} \left( \sum_{l \in [b]} B(k)_{j}^{[b]} \xi_{l} \right)^2 \leq \frac{1}{\kappa^2(1 + |w_a - w_b|)^2} \sum_{j \in [a]} \left( \sum_{l \in [b]} |A_{j}^{[b]} \xi_{l}| \right)^2.
\]
On the other hand,

\[
\sum_{j \in [a]} \left( \sum_{l \in [b]} |A^l_j||\xi_l| \right)^2 \leq \sum_{j \in [a]} \left( \sum_{l \in [b]} |A^l_j|^2 \right) \left( \sum_{l \in [b]} |\xi_l|^2 \right) \leq \|\xi_{[b]}\|^2 \sum_{j \in [a]} \sum_{l \in [b]} |A^l_j|^2
\]

\[
\leq \|\xi_{[b]}\|^2 \sum_{j \in [a]} \|A^{[b]}_{[a]}\|^2 \leq w^d_{a} \|A^{[b]}_{[a]}\|^2 \|\xi_{[b]}\|^2.
\]

Similarly,

\[
\sum_{j \in [a]} \left( \sum_{l \in [b]} |A^l_j||\xi_l| \right)^2 \leq \|\xi_{[b]}\|^2 \sum_{j \in [a]} \sum_{l \in [b]} |A^l_j|^2 \leq \|\xi_{[b]}\|^2 \sum_{l \in [b]} \sum_{j \in [a]} |A^l_j|^2
\]

\[
\leq \|\xi_{[b]}\|^2 \sum_{l \in [b]} \|A^{[b]}_{[a]}\|^2 \leq w^d_{b} \|A^{[b]}_{[a]}\|^2 \|\xi_{[b]}\|^2.
\]

It follows that

\[
\sum_{j \in [a]} \left( \sum_{l \in [b]} |A^l_j||\xi_l| \right)^2 \leq \left( \min\{w_a, w_b\}\right)^d \|A^{[b]}_{[a]}\|^2 \|\xi_{[b]}\|^2.
\]

Therefore,

\[
\|B(k)^{[b]}_{[a]}\| \leq \frac{\exp\{C_{\delta,d}^{-1/\delta}\}}{\kappa(1 + |w_a - w_b|)} \|A^{[b]}_{[a]}\|,
\]

where \(C_{\delta,d} = (d-1)2C_0)^{1/\delta}\).

**Case 2:** suppose that \(k \cdot \omega - \lambda_a + \lambda_b < 0\). Similarly, by (1.14) we have \(\lambda_a - \lambda_b - k \cdot \omega \geq \gamma(1 + |w_a - w_b|) \geq \gamma\). Then (4.1) is equivalent to

\[-(D_{[a]} - k \cdot \omega I_{[a]})^\prime B_{[a]}^{[b]} + B_{[a]}^{[b]} D_{[b]} = A_{[a]}^{[b]}\].

The following proof is similar as case 1 in which we use \(L'_{[a] \times [b]}\) instead of \(L_{[a] \times [b]}\), where

\[L'_{[a] \times [b]}(B_{[a]}^{[b]}) = (D_{[a]} - k \cdot \omega I_{[a]})^{-1} B_{[a]}^{[b]} D_{[b]}\).

\[\square\]

**Remark 4.1.** In (4.3), we use a cancellation.

4.2. **Homological equation.** In this section consider a homological equation of the form

\[-\omega \cdot \nabla \varphi S + i[N, S] + Q = \text{remainder}\]

where \(N \in \mathcal{NF}\) close to \(N_0\) and \(Q \in \mathcal{M}_\beta\). We will construct a solution \(S \in \mathcal{M}_\beta^+\) in the following proposition.

**Proposition 4.2.** Denote \(D \subset D_0\). Let \(D \ni \omega \mapsto N(\omega) \in \mathcal{NF}\) be a \(C^1\) mapping that verifies

\[|N - N_0|_{\beta}^3 \leq 2\epsilon_0\]  \hspace{1cm} (4.4)

and \(Q \in \mathcal{M}_\beta(D, \sigma)\). Assume that \(K \geq 1\) and \(0 < \kappa \leq \gamma \leq \frac{\epsilon_0}{8d}\), verifying

\[
\exp \left\{ 8d \left( \frac{\epsilon_0}{\gamma} \right)^{\frac{1}{7}} \right\} \kappa \leq \gamma.
\]  \hspace{1cm} (4.5)
Then there exists a subset $\mathcal{D}' = \mathcal{D}'(\gamma, \kappa, K)$, satisfying
\[
\text{Meas}(\mathcal{D} \setminus \mathcal{D}') \leq CK^{\alpha_1 + \alpha_2}\quad (4.6)
\]
and $C^1$ mappings $\tilde{N} : \mathcal{D}' \mapsto \mathcal{M}_\beta \cap \mathcal{N} \mathcal{F}$, $R : \mathcal{D}' \times \mathbb{T}_n^\alpha \mapsto \mathcal{M}_\beta$ and $S : \mathcal{D}' \times \mathbb{T}_n^\alpha \mapsto \mathcal{M}_\beta^+$, Hermitian and analytic in $\varphi$, such that
\[
- \omega \cdot \nabla_\varphi S + i[N, S] = \tilde{N} - Q + R
\]
and for any $0 < \sigma' < \sigma$
\[
[\tilde{N}]_{\beta}^{D', \sigma'} \leq [Q]_{\beta}^{D, \sigma},
\]
\[
[R]_{\beta}^{D', \sigma'} \leq \frac{Ce^{-\frac{\delta}{2}(\sigma - \sigma')}}{\sigma - \sigma'} [Q]_{\beta}^{D, \sigma},
\]
\[
[S]_{\beta}^{D', \sigma'} \leq \frac{CK\gamma}{\kappa^3(\sigma - \sigma')^n} [Q]_{\beta}^{D, \sigma},
\]
where the constant $C > 0$ depends on $n, d, \beta$ and $\alpha_1 = \max\{\tau_1, n + d\}$, $\alpha_2 = \max\{\tau_2, 1\}$.

**Proof.** Written in Fourier variables (w.r.t. $\varphi$), the homological equation (4.7) reads
\[
-ik \cdot \omega \tilde{S}(k) + i[N, \tilde{S}(k)] = \delta_{k,0} \tilde{N} - \hat{Q}(k) + \hat{R}(k),
\]
where $\delta_{k,j}$ denotes the Kronecker symbol.

Decompose the equation on each product block $[a] \times [b]$
\[
L \tilde{S}_{[a]}^{[b]}(k) = i\delta_{k,0} N_{[a]}^{[b]} - i \hat{Q}_{[a]}^{[b]}(k) + i \hat{R}_{[a]}^{[b]}(k),
\]
where $L := L(k, [a], [b], \omega)$ is the linear operator, acting on the space of complex $[a] \times [b]$-matrices defined by
\[
LM = (k \cdot \omega - N_{[a]}(\omega)) M + MN_{[b]}(\omega) \quad \text{with} \quad N_{[a]} = N_{[a]}^{[a]}.
\]
First solve this equation when $|k| + |w_a - w_b| = 0$ (i.e. $k = 0$, $w_a = w_b$) by defining
\[
\tilde{S}^{[a]}_{[a]}(0) = 0, \quad \hat{R}^{[a]}_{[a]}(0) = 0 \quad \text{and} \quad \tilde{N}^{[a]}_{[a]} = \hat{Q}^{[a]}_{[a]}(0).
\]
Then setting $\tilde{N}_{[a]}^{[b]} = 0$ for $w_a \neq w_b$ we obtain $\tilde{N} \in \mathcal{M}_\beta \cap \mathcal{N} \mathcal{F}$ satisfies $|\tilde{N}|_{\beta} \leq |\hat{Q}(0)|_{\beta}$. The estimates of the derivatives (w.r.t. $\omega$) are obtained by differentiating the expressions of $\tilde{N}$. Taking all the estimates leads to (4.8).

Now turn to the other cases when $|k| + |w_a - w_b| > 0$. Diagonalize the (Hermitian) matrix $N_{[a]}$ in an orthonormal basis: $\overline{P}_{[a]}^T N_{[a]} P_{[a]} = D_{[a]}$ and denote $\tilde{S}^{[b]}_{[a]} = \overline{P}_{[a]}^T \tilde{S}^{[b]}_{[a]} P_{[b]}$, $\hat{Q}^{[b]}_{[a]} = \overline{P}_{[a]}^T \hat{Q}^{[b]}_{[a]} P_{[b]}$ and $\hat{R}^{[b]}_{[a]} = \overline{P}_{[a]}^T \hat{R}^{[b]}_{[a]} P_{[b]}$. Here we note for later use that $\|\hat{M}^{[b]}_{[a]}\| = \|\hat{M}^{[b]}_{[a]}\|$ for $M = S, Q, R$. In this new variables the homological equation (4.11) reads
\[
(k \cdot \omega - D_{[a]}) \tilde{S}^{[b]}_{[a]}(k) + \tilde{S}^{[b]}_{[a]}(k) D_{[b]} = -i \hat{Q}^{[b]}_{[a]}(k) + i \hat{R}^{[b]}_{[a]}(k).
\]
We solve it term by term: let $a, b \in \mathcal{E}$ and set
\[
\hat{R}^{[b]}_{[a]}(k) = 0, \quad \text{for} \quad |k| \leq K,
\]
\[
\hat{R}^{[b]}_{jl}(k) = \hat{Q}^{[b]}_{jl}(k), \quad j \in [a], l \in [b], \quad \text{for} \quad |k| > K
\]
(4.13)
and
\[
\hat{S}_{[a]}^{[b]}(k) = 0, \quad \text{for } |k| > K \text{ or } |k| + |w_a - w_b| = 0,
\]
\[
\left(\hat{S}_{[a]}^{[b]}(k)\right)_{jl} = \frac{-i \left(\hat{Q}_{[a]}^{[b]}(k)\right)_{jl}}{k \cdot \omega - \alpha_j + \beta_l},
\]
in the other cases,

where \(\alpha_j(\omega)\) and \(\beta_l(\omega)\) denote eigenvalues of \(N_{[a]}(\omega)\) and \(N_{[b]}(\omega)\), respectively. Before the estimations of such matrices, first remark the following assertion:

\[
\left(\hat{Q}_{[a]}^{[b]}(k)\right)_{jl} = \left(\hat{Q}_{[b]}^{[a]}(-k)\right)_{lj} \Rightarrow \left(\hat{S}_{[a]}^{[b]}(k)\right)_{jl} = \left(\hat{S}_{[b]}^{[a]}(-k)\right)_{lj}.
\]

Hence, if \(Q\) verifies condition (2.2), so it is with \(S\) which implies the flow generated by \(S\) preserves the symmetry \(\eta = \xi\).

Canonically, (4.13) leads to

\[
|R(\varphi)|_\beta = |R'(\varphi)|_\beta \leq C e^{-\frac{\bar{S}(\sigma - \sigma')}{(\sigma - \sigma')}^{\alpha_0}} \sup_{|\Re \varphi| < \sigma'} |Q(\varphi)|_\beta, \text{ for } |\Re \varphi| < \sigma'.
\] (4.15)

Facing the small divisors

\[
k \cdot \omega - \alpha_j(\omega) + \beta_l(\omega), \quad j \in [a], l \in [b] \text{ and } [a], [b] \in \hat{\cE},
\]
we distinguish two cases depending on whether \(k = 0\) or not.

**The case** \(k = 0\). In this case we know that \(w_a \neq w_b\) which implies \(|w_a - w_b| \geq 2\). Using (2.4) and (4.4) we get that, if \(\kappa \leq \gamma \leq \frac{\epsilon_0}{4}\) and \(\epsilon_0 \leq \frac{\epsilon_0}{4}\),

\[
| - \lambda_a + \lambda_b| \geq c_0 |w_a - w_b| \geq \frac{c_0}{2} (1 + |w_a - w_b|) \geq 2\gamma (1 + |w_a - w_b|)
\]

and

\[
| - \alpha_j(\omega) + \beta_l(\omega)| \geq | - \lambda_a + \lambda_b| - 4\epsilon_0 \geq \frac{c_0}{2} |w_a - w_b| \geq \kappa (1 + |w_a - w_b|).
\]

Collecting the last two estimates and condition (4.4) allows us to utilize Lemma 1.8 to conclude that

\[
|\hat{S}(0)|_{\beta+} \leq \kappa^{-1} \exp \left\{ d \left(\frac{\epsilon_0}{\gamma}\right)^{\frac{2}{\beta}} \right\} |\hat{Q}(0)|_{\beta}.
\] (4.16)

**The case** \(k \neq 0\). Concretely, in this case we only solve the main terms of Fourier series truncated at order \(K\). Utilizing Hypothesis H2, for any \(\gamma > 0\), there exists a subset \(D_1 = D(2\gamma, K)\), satisfying \(\text{Meas}(D_0 \setminus D_1) \leq C K^{-1} \gamma^2\), such that for all \(\omega \in D_1\) and \(0 < |k| \leq K\), \(|k \cdot \omega - \lambda_a + \lambda_b| \geq 2\gamma (1 + |w_a - w_b|)\). By (4.4) this implies

\[
|k \cdot \omega - \alpha_j(\omega) + \beta_l(\omega)| \geq |k \cdot \omega - \lambda_a + \lambda_b| - |\alpha_j(\omega) - \lambda_a| - |\beta_l(\omega) - \lambda_b| \geq 2\gamma (1 + |w_a - w_b|) - \frac{2\epsilon_0}{(1 + \ln w_a)^{2\beta}} - \frac{2\epsilon_0}{(1 + \ln w_b)^{2\beta}} \geq \gamma (1 + |w_a - w_b|), \quad \text{if } w_b \geq w_a \geq \exp \left\{ \left(\frac{4\epsilon_0}{\gamma}\right)^{\frac{1}{\beta}} \right\}.
\]
Now let \( w_a \leq \exp \left\{ \left( \frac{4\epsilon_0}{\gamma} \right) \right\} \). In fact if \( |w_a - w_b| \geq CK \), we can prove that
\[
|k \cdot \omega - \alpha_j(\omega) + \beta_l(\omega)| \geq \kappa(1 + |w_a - w_b|).
\]

We face the case \( |w_a - w_b| \leq CK \) which follows \( w_b \leq CK \exp \left\{ \left( \frac{4\epsilon_0}{\gamma} \right) \right\} \). Since
\[
|\partial_\omega (k \cdot \omega) (\frac{k}{|k|})| = |k| \geq 1, \text{ condition (4.4) implies } |\partial_\omega (k \cdot \omega - \alpha_j(\omega) + \beta_l(\omega)) (\frac{k}{|k|})| \geq \frac{|k|}{2}.
\]
The last estimate allows us to use Lemma 5.2 to conclude that
\[
|k \cdot \omega - \alpha_j(\omega) + \beta_l(\omega)| \geq \kappa(1 + |w_a - w_b|), \quad \forall j \in [a], l \in [b]
\]
extcept a set \( F_{[a],[b],k} \) whose measure is smaller than \( Cw_a^{-1}w_b^{d-1}k(1+|w_a-w_b|) \). Denoting \( F \) be the union of \( F_{[a],[b],k} \) for \([a],[b] \in \mathcal{E} \) and \( 0 < |k| \leq K \) such that \( w_a \leq \exp \left\{ \left( \frac{4\epsilon_0}{\gamma} \right) \right\} \)
and \( w_b \leq CK \exp \left\{ \left( \frac{4\epsilon_0}{\gamma} \right) \right\} \) with \( |w_a - w_b| \leq CK \), condition (4.5) leads to
\[
\text{Meas}(F) \leq Cw_a^{-d}w_b^{d}K^n \kappa \leq CK^{n+d} \exp \left\{ 8d \left( \frac{\epsilon_0}{\gamma} \right) \right\} \kappa \leq CK^{n+d}\gamma.
\]

Let \( \mathcal{D}' = D_1 \cup D_2 \) with \( D_2 = D \setminus F \) and \( \alpha_1 = \max\{\tau_1, n + d\} \) and \( \alpha_2 = \max\{\tau_2, 1\} \), then
\[
\text{Meas}(\mathcal{D} \setminus \mathcal{D}') \leq \text{Meas}(\mathcal{D}_0 \setminus \mathcal{D}_1) + \text{Meas}(F) \leq CK^{n} \gamma^{\tau_2} + CK^{n+d}\gamma \leq CK^{\alpha_1}\gamma^{\alpha_2}.
\]

Further, by construction, for all \( \omega \in \mathcal{D}', 0 < |k| \leq K, a,b \in \mathcal{E} \) and \( j \in [a], l \in [b] \) we have
\[
|k \cdot \omega - \lambda_a + \lambda_b| \geq 2\gamma(1 + |w_a - w_b|) \quad \text{and} \quad |k \cdot \omega - \alpha_j(\omega) + \beta_l(\omega)| \geq \kappa(1 + |w_a - w_b|).
\]

Hence, in view of (4.14), utilizing Lemma 1.8 concludes that \( \hat{S}(k) \in M_{\hat{1}}^\gamma \) satisfies
\[
|\hat{S}(\omega)|_{\beta^+} = |\hat{S}(\omega)|_{\beta^+} \leq \kappa^{-1} \exp \left\{ d \left( \frac{\epsilon_0}{\gamma} \right) \right\} |\hat{Q}(\omega)|_{\beta}, \quad 0 < |k| \leq K.
\]

Combining the last estimate with (4.16) we obtain a solution \( S \) satisfying for all \( |3\varphi| < \sigma' \)
\[
|S(\varphi)|_{\beta^+} \leq \frac{C}{\kappa(\sigma - \sigma')^n} \exp \left\{ d \left( \frac{\epsilon_0}{\gamma} \right) \right\} \sup_{|3\varphi| < \sigma} |Q(\varphi)|_{\beta}.
\]

To obtain the estimates for the derivative (w.r.t. \( \omega \)) we differentiate (4.11):
\[
L(\partial_\omega \hat{S}_{[a]}(k,\omega)) = -(\partial_\omega L) \hat{S}_{[a]}(k,\omega) - i\partial_\omega \hat{Q}_{[a]}(k,\omega) + i\partial_\omega \hat{R}_{[a]}(k,\omega)
\]
which is an equation of the same type as (4.11) for \( \partial_\omega \hat{S}_{[a]}(k,\omega) \) and \( \partial_\omega \hat{R}_{[a]}(k,\omega) \) where
\(-i\hat{Q}_{[a]}(k,\omega) \) is replaced by \( B_{[a]}^{[b]}(k,\omega) \) and \( \partial_\omega \hat{R}_{[a]}(k,\omega) \) where
\[
\hat{S}_{[a]}(k,\omega) = \chi_{|k| \leq K} L^{-1}(k, [a], [b], \omega) B_{[a]}^{[b]}(k,\omega), \quad \hat{R}_{[a]}(k,\omega) = i\chi_{|k| > K} B_{[a]}^{[b]}(k,\omega).
\]
Collecting condition (4.4) and the definition (4.12) leads to $|\langle \partial_\omega L \hat{S}(k, \omega) \rangle_\beta | \leq CK |\hat{S}(k, \omega)\rangle_\beta$ which implies

$$|B(k, \omega)|_\beta \leq \frac{CK}{\kappa} \exp \left\{ \frac{d (\frac{\epsilon_0}{\gamma})}{\frac{1}{\kappa}} \right\} \left( |\hat{Q}(k)|_\beta + |\partial_\omega \hat{Q}(k)|_\beta \right).$$

Following the same strategy as in the resolution of (4.11) we get for $N$

$$N_{16} \quad ZHIQIANG WANG \quad 4.3.$$

To (4.9) and (4.10).

By construction, if $Q \in \mathcal{D}_0$ and the quadratic perturbation $q_0(\varphi, \xi, \eta) = \langle \xi, Q_0(\varphi) \eta \rangle$ with $Q_0 = eQ \in \mathcal{M}_\beta(\mathcal{D}_0, \sigma_0)$ and $\sigma_0 = \sigma$. Building iteratively the change of variables $\phi_\beta$, we obtain the normal form $h_m = \omega \cdot y + \langle \xi, N_m(\omega) \eta \rangle$ and the perturbation $q_m = \langle \xi, Q_m(\omega, \varphi) \eta \rangle$ with $Q_m \in \mathcal{M}_\beta(\mathcal{D}_m, \sigma_m)$ as follows: assume that the construction has been built up to step $m \geq 0$ then

(i). we utilize Proposition 4.2 to construct $S_{m+1}(\omega, \varphi)$ solution of the homological equation verifying for $(\omega, \varphi) \in \mathcal{D}_{m+1} \times \mathbb{T}^n_{\sigma_{m+1}}$

$$- \omega \cdot \nabla_\varphi S_{m+1} + i[N_m, S_{m+1}] = \tilde{N}_m - Q_m + R_m$$

where $\tilde{N}_m(\omega)$, $R_m(\omega, \varphi)$ defined for $(\omega, \varphi) \in \mathcal{D}_{m+1} \times \mathbb{T}^n_{\sigma_{m+1}}$ by

$$\tilde{N}_m(\omega) = \left( \delta_{|\omega|} \mid j \rangle Q_m(0) \langle j \mid \right),$$

$$R_m(\omega, \varphi) = \sum_{|k| > K_{m+1}} \hat{Q}_m(\omega, k) e^{ik\varphi};$$

(ii). we define $N_{m+1}, Q_{m+1}$ for $(\omega, \varphi) \in \mathcal{D}_{m+1} \times \mathbb{T}^n_{\sigma_{m+1}}$ by

$$N_{m+1} = N_m + \tilde{N}_m,$$

$$Q_{m+1} = R_m + \int_0^1 e^{-itS_{m+1}}[(1-t)(\tilde{N}_m + R_m) + tQ_m, S_{m+1}] e^{itS_{m+1}} dt.$$  

By construction, if $Q_m$ and $N_m$ are Hermitian, so it is with all of $\tilde{N}_m$, $R_m$ and $S_{m+1}$, by resolution of the homological equation, and all $N_{m+1}$ and $Q_{m+1}$. Let

$$h_{m+1}(y, \varphi, \xi, \eta; \omega) = \omega \cdot y + \langle \xi, N_{m+1}(\omega) \eta \rangle,$$

$$x_{m+1}(y, \varphi, \xi, \eta; \omega) = \langle \xi, S_{m+1}(\omega, \varphi) \eta \rangle,$$

$$q_{m+1}(y, \varphi, \xi, \eta; \omega) = \langle \xi, Q_{m+1}(\omega, \varphi) \eta \rangle.$$
Recall that $\phi^t_x$ denotes the time $t$ flow generated by $S$ (see (2.3)), then

$$f \circ \phi^t_{x_{m+1}} = f + \int_0^1 \{f, \chi_{m+1}\} \circ \phi^t_{x_{m+1}} dt$$

or

$$f \circ \phi^t_{x_{m+1}} = f + \{f, \chi_{m+1}\} + \int_0^1 (1 - t)\{f, \chi_{m+1}\}, \chi_{m+1}\} \circ \phi^t_{x_{m+1}} dt$$

Therefore collecting (2.1) and (2.3) leads to for $\omega \in D_{m+1}$

$$(h_m + q_m) \circ \phi^t_{x_{m+1}} = h_m \circ \phi^t_{x_{m+1}} + q_m \circ \phi^t_{x_{m+1}}$$

$$= h_m + \{h_m, \chi_{m+1}\} + q_m + \int_0^1 \{(1 - t)\{h_m, \chi_{m+1}\} + q_m, \chi_{m+1}\} \circ \phi^t_{x_{m+1}} dt$$

$$= h_m + (\xi, (\tilde{N}_m + R_m)\eta) + \int_0^1 \{(\xi, ((1 - t)(\tilde{N}_m + R_m) + tQ_m)\eta), \chi_{m+1}\} \circ \phi^t_{x_{m+1}} dt$$

$$:= h_{m+1} + q_{m+1}$$

4.4. Iterative lemma. Following the general iterative procedures (4.19)-(4.24) we have

$$(h_0 + q_0) \circ \phi^t_{x_1} \circ \phi^t_{x_2} \cdots \circ \phi^t_{x_m} = h_m + q_m$$

where $h_m = \omega \cdot y + (\xi, N_m\eta)$ with $N_m \in \mathcal{N} \mathcal{F}$ and $q_m = (\xi, Q_m\eta)$ with $Q_m \in \mathcal{M}_\beta(D_m, \sigma_m)$. At the step $m$ the Fourier series are truncated at order $K_m$ and the small divisors are controlled by $\gamma_m$ and $\kappa_m$. Specifically, we choose all of the parameters for $m \geq 0$ in term of $\epsilon_m$ which will control $[Q_m]_{\beta, \sigma_m}^{D_m}$ as follows: first define $\sigma_0 = \sigma$ and $\epsilon_0$ verifying $[Q_0]_{\beta, \sigma_0}^{D_0} \leq \epsilon_0$ and denote $\alpha = \frac{m}{\alpha_2} + 1$, then for $m \geq 1$ let

$$\epsilon_m = \epsilon_m^{5/4}, \kappa_m = \epsilon_m^{1/4}, \gamma_m = \epsilon_0^{1/6}(\ln \epsilon_m^{-1})^{-\alpha}$$

$$\sigma_m - \sigma_0 = C_s \sigma_0 m^{-2}, \ K_m = 2(\sigma_m - \sigma_0)^{-1} \ln \epsilon_m^{-1},$$

where $(C_s)^{-1} = 2 \sum_{m \geq 1} m^{-2}$ and $\alpha_1 = \max\{\tau_1, n + d\}, \alpha_2 = \max\{\tau_2, 1\}.$

Remark 4.3. From (4.25), the assumption (4.5) in the KAM iteration is equivalent to

$$8d \epsilon_0^{\frac{3}{22}} (\ln \epsilon_m^{-1})^{\frac{11}{22}} + \frac{1}{6} \ln \epsilon_0^{-1} + \alpha \ln (\ln \epsilon_m^{-1}) \leq \frac{1}{4} \ln \epsilon_m^{-1}, \ \forall \ m \geq 1.$$  

Clearly, if $\beta \geq \frac{3}{2},$ (4.26) or (4.5) always holds true for $\epsilon_0 \ll 1.$

Lemma 4.4. Let $\alpha = \frac{m}{\alpha_2} + 1$ with $\alpha_1 = \max\{\tau_1, n + d\}, \alpha_2 = \max\{\tau_2, 1\}$ and $\beta \geq \frac{3}{2}.\ \alpha_1 = \max\{\tau_1, n + d\}, \alpha_2 = \max\{\tau_2, 1\}$ and $\beta \geq \frac{3}{2}.\ \alpha_1 = \max\{\tau_1, n + d\}, \alpha_2 = \max\{\tau_2, 1\}$ and $\beta \geq \frac{3}{2}.$ There exists $\epsilon_0$ depending on $\sigma, d, \kappa, \tau_1, \tau_2$ and $h_0$ such that, for $0 \leq \epsilon_0 < \epsilon_0$ and $\epsilon_m = \epsilon_0^{(5/4)m}$, $m \geq 0$, the followings hold for all $m \geq 1$: there exists $D_m \subset D_{m-1}, S_m \in \mathcal{M}_\beta(D_m, \sigma_m), h_m = \omega \cdot y + (\xi, N_m\eta)$ with $N_m \in \mathcal{N} \mathcal{F}$ and $Q_m \in \mathcal{M}_\beta(D_m, \sigma_m)$ such that

(i). for $p \in [-1, 1]$ the transformation

$$\phi_m(\cdot, \omega, \varphi) := \phi^1_{x_m} : Y_p \mapsto Y_p, \ \forall (\omega, \varphi) \in D_m \times T_{\sigma_m}^\sigma$$

is linear (unitary in $Y_0$) isomorphism conjugating the Hamiltonian at step $m - 1$ to the Hamiltonian at step $m$, i.e.

$$(h_{m-1} + q_{m-1}) \circ \phi_m = h_m + q_m,$$
(ii). the following estimates hold:

\[
\text{Meas}(D_{m-1} \setminus D_m) \leq \epsilon_0^{\alpha/2/(\ln \epsilon_0^{-1})^{-\alpha/4}} \quad (4.28)
\]

\[
\frac{N_{m-1}}{\beta}^D \leq \epsilon_{m-1},
\]

\[
\frac{Q_m}{\beta}^{D_m, \sigma_m} \leq \epsilon_m, \quad (4.29)
\]

\[
\frac{S_m}{\beta}^{D_m, \sigma_m} \leq \epsilon_0^{1/6}(\ln \epsilon_0^{-1})^{-\alpha/2} \epsilon_{m-1}^{1/4} \quad (4.30)
\]

and for all \( p \in [-1, 1] \) the transformation satisfies

\[
\|\phi_m(\cdot, \omega, \varphi) - \text{Id}\|_{L_p(\mathcal{Y})} \leq \epsilon_0^{1/6}(\ln \epsilon_0^{-1})^{-\alpha/2} \epsilon_{m-1}^{1/4} \quad \forall (\omega, \varphi) \in D_m \times \mathbb{T}_{\sigma_m}^n. \quad (4.31)
\]

**Proof.** At step 1, the initial \( h_0 = \omega \cdot y + \langle \xi, N_0 \eta \rangle \) and thus condition (4.4) is trivially satisfied. Remark 4.3 allows us to apply Proposition 4.2 for constructing \( S_1, \tilde{N}_0, R_0 \) and \( D_1, \sigma_1 \) such that for \( (\omega, \varphi) \in D_1 \times \mathbb{T}_{\sigma_1}^n, -\omega \cdot \nabla \varphi S_1 + i[N_0, S_1] = \tilde{N}_0 - Q_0 + R_0. \) Then utilizing (4.6), we obtain for \( \epsilon_0 \ll 1 \)

\[
\text{Meas}(D_0 \setminus D_1) \leq CK_1^{\alpha_1} \leq C\epsilon_0^{\alpha_2/6}(\ln \epsilon_0^{-1})^{-\alpha_2} \leq \epsilon_0^{\alpha_2/6}(\ln \epsilon_0^{-1})^{-\alpha_2/4}.
\]

Due to (4.10) we have for \( \epsilon_0 \ll 1 \)

\[
[S_1]^{D_1, \sigma_1} \leq \frac{CK_1^{\alpha_1}}{\kappa_1^{\alpha_1}} [Q_0]^{D_0, \sigma_0} \leq C\epsilon_0^{1/6}(\ln \epsilon_0^{-1})^{-\alpha_2} \epsilon_0^{1/4} \leq \epsilon_0^{1/6}(\ln \epsilon_0^{-1})^{-\alpha_2} \epsilon_0^{1/4}.
\]

Thus, in view of (2.3), (4.27) assertion (iii),(v) of Lemma 2.1 we get for all \( p \in [-1, 1] \)

\[
\|\phi_1(\cdot, \omega, \varphi) - \text{Id}\|_{L_p(\mathcal{Y})} \leq C\epsilon_0^{\alpha_2/6}(\ln \epsilon_0^{-1})^{-\alpha_2} \epsilon_0^{1/4}, \quad \text{if } \epsilon_0 \ll 1.
\]

Collecting (4.8) and (4.9) leads to \( \tilde{N}_0 \leq \epsilon_0 \) and for \( \epsilon_0 \ll 1 \)

\[
[R_0]^{D_1, \sigma_1} \leq \frac{C\epsilon_0^{-\alpha_2}(\sigma_0 - \sigma_1)}{\kappa_1^{\alpha_1}((\sigma_0 - \sigma_1)\alpha_1)} [Q_0]^{D_0, \sigma_0} \leq C\epsilon_0^2 \leq \frac{1}{2} \epsilon_0^5 = \epsilon_0.
\]

Besides, (4.23) infers that for \( \epsilon_0 \ll 1 \)

\[
[S_1]^{D_1, \sigma_1} \leq [R_0]^{D_1, \sigma_1} + C[Q_0]^{D_0, \sigma_0} [S_1]^{D_1, \sigma_1} \leq \frac{1}{2} + C\epsilon_0^{5/4+1/6} \leq \epsilon_1.
\]

Now assume that we have verified Lemma 4.4 up to step \( m \), then we consider the step \( m+1 \). Since \( \bar{h}_m = \omega \cdot y + \langle \xi, N_m \eta \rangle \) and \( [N_m - N_0]_{\beta} \leq \sum_{l=0}^{m-1} [\tilde{N}_l]^{D_{l+1}} \leq \sum_{l=0}^{m-1} \epsilon_l \leq 2\epsilon_0, \) if \( \epsilon_0 \ll 1 \). Then condition (4.4) verifies and Remark 4.3 allows us to apply Proposition 4.2 for constructing \( S_{m+1}, \tilde{N}_m, R_m \) and \( D_{m+1}, \sigma_{m+1} \) such that for \( (\omega, \varphi) \in D_{m+1} \times \mathbb{T}_{\sigma_{m+1}}^n \)

\[-\omega \cdot \nabla \varphi S_{m+1} + i[N_m, S_{m+1}] = \tilde{N}_m - Q_m + R_m.
\]

Similarly, utilizing (4.6) we obtain for \( \epsilon_0 \ll 1 \)

\[
\text{Meas}(D_m \setminus D_{m+1}) \leq CK_{m+1}^{\alpha_1} \leq C\epsilon_0^{\alpha_2/6}(m+1)^{2\alpha_1}(\ln \epsilon_0^{-1})^{-\alpha_2} \leq \epsilon_0^{\alpha_2/6}(\ln \epsilon_0^{-1})^{-\alpha_2}.
\]

Due to (4.10) we have for \( \epsilon_0 \ll 1 \)

\[
[S_{m+1}]^{D_{m+1}, \sigma_{m+1}} \leq \frac{CK_{m+1}^{\alpha_1+m+1}}{\kappa_1^{\alpha_1+m+1}(\sigma_m - \sigma_{m+1})^{m+1}} [Q_m]^{D_m, \sigma_m} \leq \epsilon_0^{1/6}(\ln \epsilon_0^{-1})^{-\alpha_2/2} \epsilon_0^{1/4}.
\]

Thus, in view of (2.3),(4.27) and assertion (iii),(v) of Lemma 2.1 we get for \( p \in [-1, 1] \)

\[
\|\phi_{m+1}(\cdot, \omega, \varphi) - \text{Id}\|_{L_p(\mathcal{Y})} \leq C[S_{m+1}]^{D_{m+1}, \sigma_{m+1}} \leq \epsilon_0^{1/6}(\ln \epsilon_0^{-1})^{-\alpha_2/2} \epsilon_0^{1/4}, \quad \text{if } \epsilon_0 \ll 1.
\]
Collecting (4.8) and (4.9) leads to $|\tilde{N}_m|^{D_{m+1}} \leq \epsilon_m$ and

$$[R_m]^{D_{m+1} \sigma_{m+1}}_\beta \leq \frac{C_2^{-1} \epsilon_{m+1}}{(m - 1)^n} |Q_m|^{D_m \sigma_m} \leq \frac{\epsilon_{m+1}}{2}, \text{ if } \epsilon_0 \ll 1.$$  

In addition, (4.23) implies that for $\epsilon_0 \ll 1$

$$[Q_{m+1}]^{D_{m+1} \sigma_{m+1}}_\beta \leq \frac{[R_m]^{D_{m+1} \sigma_{m+1}}_\beta + C[Q_m]^{D_m \sigma_m}[S_m]^{D_{m+1} \sigma_{m+1}}_\beta}{\epsilon_{m+1}} \leq \frac{\epsilon_{m+1}}{2} + C\epsilon_0 \frac{1}{(\ln \epsilon_0)^{1 - \frac{1}{16}}} \epsilon_0^{5/4} \leq \epsilon_{m+1}. \square$$

4.5. Transition to the limit and proof of reducibility theorem. Let $D_\epsilon = \cap_{m \geq 0} D_m$. In view of (4.28), this is a Borel set satisfying for $\epsilon_0 \ll 1$

$$\text{Meas}(D_0 \setminus D_\epsilon) \leq \sum_{m=0}^{\infty} \epsilon_0^{3/2} (\ln \epsilon_0)^{1 - \frac{3}{2}} \epsilon_0^{1/2} \sum_{m=0}^{\infty} \left(\frac{1}{5} \epsilon_0^{-\frac{1}{2}}\right)^m \leq \epsilon_0^{3/2}.$$  

This leads to the assertion (i) of Theorem 2.3.

In the following, let $p \in [0, 1], (\omega, \varphi) \in \mathcal{D}_\epsilon \times \mathbb{T}^n_{\sigma/2}$ and $\epsilon_0 \ll 1$. Collecting (4.30) and (4.29) we conclude the direct lemmas as follows:

**Lemma 4.5.** $\{Q_m(\omega, \varphi)\}_{m \geq 1}$ is a Cauchy sequence in $\mathcal{M}_\beta$ and $Q_m(\omega, \varphi) \to 0$ when $m \to \infty$. Furthermore, (4.30) infers the uniformly convergence on $(\omega, \varphi)$.

**Lemma 4.6.** $\{N_m(\omega) - N_0\}_{m \geq 1}$ is a Cauchy sequence in $\mathcal{M}_\beta$. Letting $W(\omega) \in \mathcal{M}_\beta$ be the limit mapping we have $N_m(\omega) - N_0 \to W(\omega)$ when $m \to \infty$. Moreover, (4.29) implies the uniformly convergence on $\varphi$, which leads to the $C^1$ regularity.

To estimate the change of variables, we need the following two lemmas.

**Lemma 4.7.** Let $\Phi_m = \phi_1 \circ \phi_2 \circ \cdots \circ \phi_m$ for $m \geq 1$, then we have

$$\|\Phi_m(\cdot, \omega, \varphi) - \text{Id}\|_{\mathcal{B}(Y_p)} \leq \epsilon_0^{1/6} (\ln \epsilon_0)^{-\frac{m_1}{16}} \sum_{l=0}^{m-1} \epsilon_0^{1/4}. \quad (4.32)$$

**Proof.** First (4.31) implies the above estimate (4.32) holds true for $m = 1$. Now assume that we have verified (4.32) up to $m > 1$, then we consider the case for $m = 1$. From the definition,

$$\Phi_m - \text{Id} = \Phi_m \circ \phi_m - \text{Id} = \Phi_m \circ (\phi_m - \text{Id}) + \Phi_m - \text{Id}.$$  

Therefore, collecting the assumption and (4.31) leads to

$$\|\Phi_{m+1}(\cdot, \omega, \varphi) - \text{Id}\|_{\mathcal{B}(Y_p)} \leq \|\Phi_m(\cdot, \omega, \varphi) - \text{Id}\|_{\mathcal{B}(Y_p)} + C\|\phi_{m+1}(\cdot, \omega, \varphi) - \text{Id}\|_{\mathcal{B}(Y_p)}$$

$$\leq \epsilon_0^{1/6} (\ln \epsilon_0)^{-\frac{m_1}{16}} \sum_{l=0}^{m} \epsilon_0^{1/4} + C\epsilon_0^{1/6} (\ln \epsilon_0)^{-\frac{m_1}{16}} \epsilon_0^{1/4} \leq \epsilon_0^{1/6} (\ln \epsilon_0)^{-\frac{m_1}{16}} \sum_{l=0}^{m} \epsilon_0^{1/4}.$$  

By induction, we complete the proof. \square

**Lemma 4.8.** $\{\Phi_m(\cdot, \omega, \varphi)\}_{m \geq 1}$ is a Cauchy sequence in $\mathcal{B}(Y_p)$. Letting $\Phi_\infty(\cdot, \omega, \varphi) \in \mathcal{B}(Y_p)$ be the limit mapping we have $\Phi_m(\cdot, \omega, \varphi) \to \Phi_\infty(\cdot, \omega, \varphi)$ when $m \to \infty$. Furthermore, (4.32) implies the uniformly convergence on $(\omega, \varphi)$, which leads to $\Phi_\infty(\cdot, \omega, \varphi)$ is analytic in $\varphi$ and $C^1$ in $\omega$. Moreover, (4.32) implies

$$\|\Phi_\infty(\cdot, \omega, \varphi) - \text{Id}\|_{\mathcal{B}(Y_p)} \leq \epsilon_0^{5/12}. \quad (4.33)$$
Proof. From the definition we have
\[ \Phi_{m+1} - \Phi_m = \Phi_m \circ \phi_{m+1} - \Phi_m = \Phi_m \circ (\phi_{m+1} - \text{Id}). \]
Collecting (4.31) and (4.32) leads to
\[ \|\Phi_{m+1}(\cdot, \omega, \varphi) - \Phi_m(\cdot, \omega, \varphi)\|_{B(Y_p)} \leq C\|\phi_{m+1}(\cdot, \omega, \varphi) - \text{Id}\|_{B(Y_p)} \leq \epsilon_1^{1/4}. \]
Hence, given \( m_2 \geq m_1 \geq 1 \) we have
\[ \|\Phi_{m_2}(\cdot, \omega, \varphi) - \Phi_{m_1}(\cdot, \omega, \varphi)\|_{B(Y_p)} \leq \sum_{l=m_1}^{m_2-1} \epsilon_1^{1/4} \leq 2\epsilon_{m_1} \to 0 \text{ when } m_1 \to \infty. \]
The last estimate implies the Cauchy sequence, which concludes the results. \( \square \)

Define \( B(\ell^p_p) \ni M_m(\omega, \varphi) = e^{iS_1(\omega, \varphi)} \circ e^{iS_2(\omega, \varphi)} \circ \cdots \circ e^{iS_m(\omega, \varphi)} \) for \( m \geq 1 \). In view of (2.3) we get that
\[ \phi_m(\xi, \eta, \omega, \varphi) = \left( e^{-iS_m(\omega, \varphi)} \xi, e^{iS_m(\omega, \varphi)} \eta \right) \]
and
\[ \Phi_m(\xi, \eta, \omega, \varphi) = \left( M_m(\omega, \varphi) \xi, M_m(\omega, \varphi) \eta \right). \quad (4.34) \]
From a straightforward computation, given \( m_2 \geq m_1 \geq 1 \)
\[ \|M_{m_2}(\omega, \varphi) - M_{m_1}(\omega, \varphi)\|_{B(\ell^p_p)} \leq \|\Phi_{m_2}(\cdot, \omega, \varphi) - \Phi_{m_1}(\cdot, \omega, \varphi)\|_{B(Y_p)}. \]
This implies that \( \{M_m(\omega, \varphi)\}_{m \geq 1} \) is a Cauchy sequence in \( B(\ell^p_p) \). Letting \( M_\infty(\omega, \varphi) \) be the limit mapping, the uniformly convergence leads to that \( (\omega, \varphi) \mapsto M_\infty(\omega, \varphi) \) is analytic in \( \varphi \) and \( C^1 \) in \( \omega \). Moreover, due to (4.34)
\[ \Phi_\infty(\xi, \eta, \omega, \varphi) = \left( M_\infty(\omega, \varphi) \xi, M_\infty(\omega, \varphi) \eta \right) \]
and taking into account (4.33) leads to
\[ \|M_\infty(\omega, \varphi) - \text{Id}\|_{B(\ell^p_p)} = \|M_\infty(\omega, \varphi) - \text{Id}\|_{B(\ell^p_p)} \leq \|\Phi_\infty(\cdot, \omega, \varphi) - \text{Id}\|_{B(Y_p)} \leq \epsilon_{0}^{5/12}. \quad (4.36) \]
By construction the map \( \Phi_m(\cdot, \omega, \omega) \) conjugates the original Hamiltonian system associated with \( H_m(t, \xi, \eta) = \langle \xi, N_0 \eta \rangle + \epsilon \langle \xi, Q(\omega t) \eta \rangle \) into the Hamiltonian system associated with \( H_m(t, \xi, \eta) = \langle \xi, N_m(\omega) \eta \rangle + \langle \xi, Q_m(\omega, \omega t) \eta \rangle \). Collecting Lemma 4.5 and 4.6 one concludes \( Q_m(\omega, \omega t) \to 0 \) and \( N_m(\omega) \to N_\omega \) when \( m \to \infty \), where the operator \( N_\omega \equiv N_\infty(\omega) = N_0 + W(\omega) \in \mathcal{N}\mathcal{F} \) is \( C^1 \) in \( \omega \) with
\[ |W|_\beta = |N_\omega - N_0|_\beta \leq 2\epsilon_0. \quad (4.37) \]
Let \( \Phi_\omega(\varphi) := \Phi_\infty(\cdot, \omega, \varphi) \) and \( M_\omega(\varphi) = M_\infty(\omega, \varphi) \), then (4.35) reads \( \Phi_\omega(\varphi)(\xi, \eta) = \left( M_\omega(\varphi) \xi, M_\omega(\varphi) \eta \right) \). Furthermore, denoting the limiting Hamiltonian \( H_\omega = \langle \xi, N_\omega \eta \rangle = \langle \xi, N_0 \eta \rangle + \langle \xi, W \eta \rangle \), the symplectic coordinate transformation \( \Phi_\omega(\varphi) \) conjugates the original Hamiltonian system associated with \( H_\omega \) into the autonomous Hamiltonian system associated with \( H_\omega \). Collecting (4.36) and (4.37) leads to (2.8) and (2.9) in Theorem 2.3. \( \square \)
5. Appendix

5.1. Proof of Lemma 2.1. Recall that $\beta > \frac{1}{2}$.

(i). The proof results from Lemma 5.1 with $\beta > \frac{1}{2}$ and

$$\sum_{c \in \mathcal{E}} \frac{1}{(1 + \ln w_c)^{2\beta}(1 + |w_b - w_c|)} \leq C. \quad (5.1)$$

(ii). Similarly, collecting (5.1) and

(iii). Use assertion (ii) of Lemma 2.1.

(iv). Let $A \in \mathcal{M}_\beta$ and $s \geq 1$. Then for any $\xi \in \ell_2^s$, we have

$$\|A\xi\|^2_s = \sum_{a \in \mathcal{E}} w_a^{-s} \sum_{b \in \mathcal{E}} A_a^{[b]} \xi_{[b]} \leq \sum_{a \in \mathcal{E}} w_a^{-s} \left( \sum_{b \in \mathcal{E}} \|A_{[a]}^{[b]} \| \| \xi_{[b]} \| \right)^2$$

$$\leq |A|_\beta^2 \sum_{a \in \mathcal{E}} w_a^{-s} \left( \sum_{b \in \mathcal{E}} \frac{w_b^{s/2} \| \xi_{[b]} \|}{(1 + \ln w_a)^{\beta}(1 + \ln w_b)^{\beta} w_b^{s/2}} \right)^2$$

$$\leq |A|_\beta^2 \sum_{a \in \mathcal{E}} \frac{1}{(1 + \ln w_a)^{2\beta} w_a^s} \left( \sum_{b \in \mathcal{E}} \frac{1}{(1 + \ln w_b)^{2\beta} w_b^s} \right) \left( \sum_{b \in \mathcal{E}} w_b^s \| \xi_{[b]} \|^2 \right)$$

$$\leq |A|_\beta^2 \sum_{a \in \mathcal{E}} \frac{1}{(1 + \ln w_a)^{2\beta} w_a^s} \left( \sum_{b \in \mathcal{E}} \frac{1}{(1 + \ln w_b)^{2\beta} w_b^s} \right) \| \xi \|^2_s$$

$$\leq C^2 |A|_\beta^2 \| \xi \|^2_s.$$

(v). Case 1: $s \in [0, 1]$. In this case we first prove

$$(I) := \sum_{b \in \mathcal{E}} \frac{(w_a/w_b)^s}{(1 + \ln w_b)^{2\beta}(1 + |w_a - w_b|)} \leq C, \quad \forall s \in [0, 1] \text{ and } \beta > \frac{1}{2}. \quad (5.2)$$

We split the series above into two parts as follows:

$$(I) = \left( \sum_{w_b > \frac{w_a}{2}} + \sum_{\frac{w_a}{2} \leq w_b} \right) \frac{(w_a/w_b)^s}{(1 + \ln w_b)^{2\beta}(1 + |w_a - w_b|)} := (I_1) + (I_2).$$

For the former, $(w_a/w_b)^s \leq 2^s \leq 2$. Thus

$$(I_1) \leq \sum_{b \in \mathcal{E}} \frac{2}{(1 + \ln w_b)^{2\beta}(1 + |w_a - w_b|)} \leq C. \quad (5.3)$$

Then turn to the latter. Since $w_b \leq \frac{w_a}{2}$, then $1 + |w_a - w_b| \geq w_a - w_b \geq \frac{w_a}{2} \geq w_b$.

Thus $1 + |w_a - w_b| = (1 + |w_a - w_b|)(1 + |w_a - w_b|) \geq (w_a/2)^s w_b^1 - s \geq \frac{1}{2} w_a^s w_b^1 - s$.

Therefore, one obtains $(I_2) \leq \frac{2}{1 + \ln w_b)^{2\beta} w_b \leq C$. Collecting the last estimate and (5.3) leads to the results (5.2).

Now, we prepare to prove the assertion of (v) when $s \in [0, 1]$. Since $A \in \mathcal{M}_\beta^+$,
then for any $\xi \in l^2_s$,
\[
\|A\xi\|^2 \leq \sum_{a \in \mathcal{E}} w^a \left( \sum_{b \in \mathcal{E}} (\|A^b\| \cdot \|\xi^{[b]}\|)^2 \right)
\[
\leq \sum_{a \in \mathcal{E}} \frac{|A|^2_{+}}{(1 + \ln w_a)^{2\beta}} \left( \sum_{b \in \mathcal{E}} \frac{(w_a/w_b)^s}{(1 + \ln w_b)^{2\beta}(1 + w_a/w_b)^{1/2}} (\|\xi^{[b]}\|^2) \right)^2
\[
\leq C|A|^2_{+} \sum_{b \in \mathcal{E}} w^b \|\xi^{[b]}\|^2 \sum_{a \in \mathcal{E}} \frac{1}{(1 + \ln w_a)^{2\beta}(1 + w_a/w_b)} \quad \text{by (5.2)}
\[
\leq C^2|A|^2_{+} \|\xi\|^2.
\]

Next turn to the other case: $s \in [-1,0)$. Repeating similar procedures as the first case and noting that
\[
\sum_{a \in \mathcal{E}} \frac{(w_a/w_b)^s}{(1 + \ln w_a)^{2\beta}(1 + w_a/w_b)} \leq C, \quad \forall s \in [-1,0) \text{ and } \beta > \frac{1}{2},
\]
we complete the proof. \qed

5.2. Some auxiliary lemmas.

**Lemma 5.1** (see Lemma A1 in [43]). For $j \geq 1$ and $\delta > 1$, there exists a positive constant $C \equiv C(\delta)$ independent of $j$ such that $\sum_{l \geq 1} \frac{1}{(1 + \ln n)^{2\delta(1 + |l - j|)}} \leq C$.

The following lemma is classical.

**Lemma 5.2.** Let $f : [0,1] \to \mathbb{R}$ be a $C^1$ map satisfying $|f'(x)| \geq \delta$ for all $x \in [0,1]$ and let $\kappa > 0$ then $\text{Meas}\left( \{ x \in [0,1] : |f(x)| \leq \kappa \} \right) \leq \frac{2\kappa}{\delta}$.

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**References**

[1] Baldi, P., Berti, M., Montalto, R.: KAM for quasi-linear and fully nonlinear forced perturbations of Airy equation. Math. Ann. **359**, 471-536 (2014)
[2] Baldi, P., Montalto, R.: Quasi-periodic incompressible Euler flows in 3D. Advances in Mathematics. **384**, 107730 (2021)
[3] Bambusi, D.: Reducibility of 1-d Schrödinger equation with time quasiperiodic unbounded perturbations. II. Commum. Math. Phys. **353**, 353-378 (2017)
[4] Bambusi, D.: Reducibility of 1-d Schrödinger equation with time quasiperiodic unbounded perturbations. I. Trans. Amer. Math. Soc. **370**, 1823-1865 (2018)
[5] Bambusi, D., Langella, D., Montalto, R.: Reducibility of non-resonant transport equation on with unbounded perturbations. Ann. Henri Poincaré, **20**, 1893-1929 (2019).
[6] Bambusi, D., Langella, D., Montalto, R.: Growth of Sobolev norms for unbounded perturbations of the Laplacian on flat tori. arXiv:2012.02654. 
[7] Bambusi, D., Graffi, S.: Time quasi-periodic unbounded perturbations of Schrödinger operators and KAM methods. Commun. Math. Phys. 219, 465-480 (2001)
[8] Bambusi, D., Grébert, B., Maspero, A., Robert, D.: Growth of Sobolev norms for abstract linear Schrödinger equations. J. Eur. Math. Soc. 23, 557-583 (2021)
[9] Bambusi, D., Grébert, B., Maspero, A., Robert, D.: Reducibility of the quantum harmonic oscillator in d-dimensions with polynomial time-dependent perturbation. Anal. & PDE 11, 775-799 (2018)
[10] Berti, M., Maspero, A.: Long time dynamics of Schrödinger and wave equations on flat tori. J. Diff. Eqs., 267(2), 1167-1200 (2019).
[11] Berti, M., Montalto, R.: Quasi-periodic standing wave solutions for gravity-capillary water waves. Memoirs of the American Mathematical Society, Volume 263, Number 1273, 2020.
[12] Bourgain, J.: Growth of Sobolev norms in linear Schrödinger equations with smooth time dependent potentials. J. Anal. Math., 77, 315-348 (1999).
[13] Bourgain, J.: Growth of Sobolev norms in linear Schrödinger equations with quasi-periodic potential. Commun. Math. Phys., 204(1), 207-247 (1999).
[14] Corsi, L., Montalto, R.: Quasi-periodic solutions for the forced Kirchhoff equation on T^d. Nonlinearity, 31, 5075-5109 (2018)
[15] Combescure, M.: The quantum stability problem for time-periodic perturbations of the harmonic oscillator. Ann. Inst. H. Poincaré Phys. Théor. 47(1), 63-83 (1987); Erratum: Ann. Inst. H. Poincaré Phys. Théor. 47(4), 451-454 (1987)
[16] Delort, J.-M.: Growth of Sobolev norms for solutions of time dependent Schrödinger operators with harmonic oscillator potential. Commun. PDE 39, 1-33 (2014)
[17] Delort, J.-M., Szeftel, J.: Long-time existence for small data nonlinear Klein-Gordon equations on tori and spheres. Int. Math. Res. Not. 37, 1897-1966 (2004)
[18] Eliasson, L.H.: Floquet solutions for the 1-dimensional quasi-periodic Schrödinger equation. Commun. Math. Phys. 146, 447-482 (1992)
[19] Eliasson, H.L., Kuksin, S. B.: On reducibility of Schrödinger equations with quasiperiodic in time potentials. Commun. Math. Phys. 286, 125-135 (2009)
[20] Fang, D., Zhang, Q.: On growth of Sobolev norms in linear Schrödinger equations with time dependent Gevrey potentials. J. Dynam. Differential Equations., 24(2), 151-180 (2012).
[21] Faou, E., Raphael, P.: On weakly turbulent solutions to the perturbed linear harmonic oscillator. arXiv: 2006.08206 (2020)
[22] Feola, R., Giuliani, F., Montalto, R., Procesi, M.: Reducibility of first order linear operators on tori via Moser’s theorem. J. Funct. Anal., 276(3), 932-970 (2019).
[23] Feola, R., Grébert, B.: Reducibility of Schrödinger equation on the sphere. Int. Math. Res. Not. 0, 1-39 (2020)
[24] Feola, R., Grébert, B., Nguyen, T.: Reducibility of Schrödinger equation on a Zoll manifold with unbounded potential. J. Math. Phys. 61 (7), 071501 (2020)
[25] Feola, R., Procesi, M.: Quasi-periodic solutions for fully nonlinear forced reversible Schrödinger equations. J. Differ. Equ. 259, 3389-3447 (2015)
[26] Graffi, S., Yamada, K.: Absolute continuity of the Floquet spectrum for a nonlinearly forced harmonic oscillator. Commun. Math. Phys. 215 (2), 245-250 (2000)
[27] Grébert, B., Paturel, E.: KAM for the Klein Gordon equation on S^d. Boll. Unione Mat. Ital. 9, 237-288 (2016)
[28] Grébert, B., Paturel, E.: On reducibility of quantum harmonic oscillator on R^d with quasiperiodic in time potential. Annales de la Faculté des sciences de Toulouse : Mathématiques. 28, 977-1014 (2019)
[29] Grébert, B., Thomann, L.: KAM for the quantum harmonic oscillator. Commun. Math. Phys. 307, 383-427 (2011)
[30] Koch, H., Tataru, D.: L^p eigenfunction bounds for the Hermite operator. Duke Math. J. 128, 369-392 (2005)
[31] Liang, Z., Luo, J.: Reducibility of 1-d quantum harmonic oscillator equation with unbounded oscillation perturbations. J. Differ. Equ. 270, 343-389 (2021)
[32] Liang, Z., Wang, Z.: Reducibility of quantum harmonic oscillator on R^d with differential and quasi-periodic in time potential. J. Differ. Equ. 267, 3355-3395 (2019)
[33] Liang, Z., Wang, Z.Q.: Reducibility of 1-d Schrödinger equation with unbounded oscillation perturbations. arXiv: 2003.13022v3 (2020)
[34] Liang, Z., Zhao, Z., Zhou, Q.: 1-d quasi-periodic quantum harmonic oscillator with quadratic time-dependent perturbations: Reducibility and growth of Sobolev norms. J. Math. Pures Appl. 146, 158-182 (2021)
[35] Liu, J., Yuan, X.: Spectrum for quantum duffing oscillator and small-divisor equation with large-variable coefficient. Commun. Pure Appl. Math. 63, 1145-1172 (2010)
[36] Maspero, A.: Lower bounds on the growth of Sobolev norms in some linear time dependent Schrödinger equations. Math. Res. Lett. 26, 1197-1215 (2019)
[37] Maspero, A., Robert, D.: On time dependent Schrödinger equations: Global well-posedness and growth of Sobolev norms. J. Func. Anal., 273(2), 721-781 (2017).
[38] Montalto, R.: KAM for quasi-linear and fully nonlinear perturbations of Airy and KdV equations. PhD Thesis, SISSA - ISAS, 2014.
[39] Montalto, R.: A reducibility result for a class of linear wave equations on $\mathbb{T}^d$. Int. Math. Res. Notices, 2019(6), 1788-1862 (2019).
[40] Plotnikov, P.I., Toland, J.F.: Nash-Moser theory for standing water waves. Arch. Rational Mech. Anal. 159, 1-83 (2001)
[41] Schwinte, V., Thomann, L.: Growth of Sobolev norms for coupled Lowest Landau Level equations. Pure Appl. Anal. 3, 189-222 (2021)
[42] Thomann, L.: Growth of Sobolev norms for linear Schrödinger operators. To appear in Pure Appl. Anal. arXiv: 2006.02674 (2020)
[43] Wang, Z., Liang, Z.: Reducibility of 1D quantum harmonic oscillator perturbed by a quasiperiodic potential with logarithmic decay. Nonlinearity. 30, 1405-1448 (2017)
[44] Wang, W.-M.: Pure point spectrum of the Floquet Hamiltonian for the quantum harmonic oscillator under time quasi-periodic perturbations. Commun. Math. Phys. 277, 459-496 (2008)
[45] Wang, W.-M.: Logarithmic bounds on Sobolev norms for time dependent linear Schrödinger equations. Comm. Partial Differential Equations, 33(12), 2164-2179 (2008).