QUANTIZATION DIMENSIONS OF COMPACTLY SUPPORTED
PROBABILITY MEASURES VIA RÉNYI DIMENSIONS

MARC KESSEBÖHMER, ALJOSCHA NIEMANN, AND SANGUO ZHU

Abstract. We provide a complete picture of the upper quantization dimension in terms
of the Rényi dimension by proving that the upper quantization dimension
$D_r(\nu)$ of order $r > 0$ for an arbitrary compactly supported Borel probability measure $\nu$ is given by its
Rényi dimension at the point $q_r$ where the $L^q$-spectrum of $\nu$ and the line through the
origin with slope $r$ intersect. In particular, this proves the continuity of $r \mapsto D_r(\nu)$ as
conjectured by Lindsay (2001). This viewpoint also sheds new light on the connection of
the quantization problem with other concepts from fractal geometry in that we obtain a one-
to-one correspondence of the upper quantization dimension and the
$L^q$-spectrum restricted to $(0, 1)$. We give sufficient conditions in terms of the
$L^q$-spectrum for the existence of the
quantization dimension. In this way we show as a byproduct that the quantization dimension
exists for every Gibbs measure with respect to a $C^1$-self-conformal iterated function system
on $\mathbb{R}^d$ without any assumption on the separation conditions as well as for inhomogeneous
self-similar measures under the inhomogeneous open sets condition. Some known general
bounds on the quantization dimension in terms of other fractal dimensions can readily be
derived from our new approach, some can be improved.

1. Introduction and statement of main results

The quantization problem for probability measures originates from information theory,
e. g. image compression and data processing, and was subsequently studied in great detail
by many mathematicians. Recently, this theory has again attracted increasing attention
in applications such as optimal transport problems [JP22], numerical integration [Pag15;
ENLP22] and mathematical finance [BPW10; PW12; Hof+14; BFP16; FPS19]. The core
problem is to study the asymptotics of the errors in approximating a given random variable
with a quantized version of the random variable (i. e. taking only finitely many values),
in the sense of $r$-means. The quantization dimension then reflects the exponential rate of
this convergence and it has been studied by various authors, for example [Gra02; LM02;
Del+04], [Roy13; Zhu15a; Zhu15b; KZ15; ZZS16; KZ16; ZZS17; KZ17; Zhu18; Zhu20;
ZZ21]. To be more specific, let $X$ be a bounded $\mathbb{R}^d$-valued random variable, $d \in \mathbb{N}$, on a
probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with distribution $\nu := \mathbb{P} \circ X^{-1}$. For a given $n \in \mathbb{N}$, let $\mathcal{F}_n$ denote
the set of all Borel measurable functions $f : \mathbb{R}^d \to \mathbb{R}^d$ with card $f(\mathbb{R}^d) \leq n$; we call an
element of $\mathcal{F}_n$ an $n$-quantizer. Our aim is to approximate $X$ with a quantized version of $X$, i. e. $X$
will be approximated by $f \circ X$ with $f \in \mathcal{F}_n$ where we quantify this approximation with respect to the $r$-quasinorm. More precisely, we are interested in the $n$-th quantization



FB 3 – Mathematik und Informatik, University of Bremen, Bibliothekstr. 1, 28359 Bremen, Germany
School of Mathematics and Physics, Jiangsu University of Technology, Changzhou 213001, China
E-mail addresses: mhk@uni-bremen.de, niemann1@uni-bremen.de, sgzhu@jsut.edu.cn.
2000 Mathematics Subject Classification. 28A80; 42B35; 45D05.
Key words and phrases. quantization dimension, Rényi dimension, $L^q$-spectrum, Minkowski dimension,
partition function, coarse multifractal formalism.

1
error of \( \nu \) of order \( r > 0 \) given by

\[
\varepsilon_{n,r}(\nu) := \inf_{f \in \mathcal{F}_x} \left( \int_{\Omega} |x - f(x)|^r \, dx \right)^{1/r} = \inf_{f \in \mathcal{F}_x} \left( \int |x - f(x)|^r \, d\nu(x) \right)^{1/r}.
\]

As pointed out for instance in [LM02], ‘the problem of determining the quantization dimension function for a general probability is open.’ With the present paper we want to close this gap completely for the upper quantization dimension and, under additional regularity conditions, also for the lower quantization dimension. In fact, we identify the upper quantization dimension of \( \nu \) of order \( r > 0 \) as the Rényi dimension of \( \nu \) evaluated at the point \( q_r \) where the \( L^r \)-spectrum of \( \nu \) and the line through the origin with slope \( r \) intersect. Building on a result of [PS00; Fen07], we prove the existence of the quantization dimension of Gibbs measures with respect to conformal iterated function systems without any separation conditions and, by a result of [Lis14], we prove the same statement for inhomogeneous self-similar measures under the iOSC (see below for definitions).

The starting point of our investigations was the observation that for particular measures the quantization dimension \( D_1(\nu) \) of order \( r = 1 \) is connected to the upper spectral dimension \( \bar{\nu} \) of the Krein–Feller operator associated to \( \nu \) for \( d = 1 \) as determined in [KN22c] via the identity \( D_1(\nu) = \bar{\nu}/(1 - \bar{\nu}) \). Indeed, for \( r \neq 1 \), we also expect similar connections to higher dimensional polyharmonic operators as considered in [KN22a].

In the following we assume that \( \nu \) is a compactly supported Borel probability measure and let \( D_\nu \) denote the partition of \( \mathbb{R}^d \) by half-open cubes of the form \( \bigcup_{i=1}^d (k_i 2^{-n}, (k_i + 1) 2^{-n}] \) with \( k \in \mathbb{Z}^d \). We set \( \mathcal{D} := \bigcup_{n \in \mathbb{N}} D_\nu \), which defines a semiring of sets. For every \( n \in \mathbb{N} \), we write \( \mathcal{A}_n := \{ \alpha \subset \mathbb{R}^d : 1 \leq \text{card}(\alpha) \leq n \} \). Due to [GL00b, Lemma 3.1] an equivalent formulation of the \( n \)-th quantization error of \( \nu \) of order \( r \) is given by

\[
\varepsilon_{n,r}(\nu) = \begin{cases} 
\inf_{\alpha \in \mathcal{A}_n} \left( \int d(x,\alpha)^r d\nu(x) \right)^{1/r}, & r > 0, \\
\inf_{\alpha \in \mathcal{A}_n} \exp \int \log d(x,\alpha) d\nu(x), & r = 0,
\end{cases}
\]

with \( d(x,\alpha) := \min_{y \in \alpha} \|x-y\| \). By [GL00b, Lemma 6.1], we have \( \varepsilon_{n,r}(\nu) \to 0 \). In fact, it is well known that \( \varepsilon_{n,r}(\nu) = O(n^{-1/d}) \) and, if \( \nu \) is singular with respect to Lebesgue, then \( \varepsilon_{n,r}(\nu) = o\left(n^{-1/d}\right) \) (see [GL00b, Theorem 6.2]). It is a natural question to find the ‘right exponent’ for the convergence order. This will be the main concern of this paper. For this we define the upper and lower quantization dimension for \( \nu \) of order \( r \) by

\[
\overline{D}_r(\nu) := \limsup_{n \to \infty} \frac{\log n}{- \log \varepsilon_{n,r}(\nu)}, \quad \underline{D}_r(\nu) := \liminf_{n \to \infty} \frac{\log n}{- \log \varepsilon_{n,r}(\nu)}.
\]

If \( \overline{D}_r(\nu) = \underline{D}_r(\nu) \), we call the common value the quantization dimension for \( \nu \) of order \( r \) and denote it by \( D_r(\nu) \). Note that without loss of the generality, we can (and for ease of exposition, we will) assume that the support of \( \nu \) is contained in \( \Omega := (0, 1]^d \). To see this fix \( a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}^d \) and let \( \Phi_{a,b}(x) := ax + b, x \in \mathbb{R}^d \) such that \( \Phi_{a,b}(\text{supp}(\nu)) \subset (0, 1]^d \). Then,

\[
\varepsilon_{n,r}(\nu \circ \Phi_{a,b}^{-1}) = \inf_{f \in \mathcal{F}_x} \left( \int |\Phi_{a,b}(x) - f(\Phi_{a,b}(x))|^r \, d\nu \right)^{1/r} = \inf_{f \in \mathcal{F}_x} \left( \int |ax + b - f(\Phi_{a,b}(x))|^r \, d\nu \right)^{1/r} = |a| \inf_{f \in \mathcal{F}_x} \left( \int |x - \Phi_{1/a,-b/a}(f(\Phi_{a,b}(x)))|^r \, d\nu \right)^{1/r} = |a| \varepsilon_{n,r}(\nu),
\]

where we have used the fact that \( f \mapsto \Phi_{1/a,-b/a} \circ f \circ \Phi_{a,b} \) defines a surjection on \( \mathcal{F}_x \).

The following classical result in quantization theory goes back to Zador [Zad82] and was
generalized by Bucklew and Wise in [BW82]; we refer to [GL00b, Theorem 6.2] for a rigorous proof:

Let \( \nu \) be a Borel probability measure with bounded support and non-vanishing absolutely continuous part with density \( h \). Then, for \( r > 0 \),

\[
\lim_{n \to \infty} n^{-r/d} e_{n,r}(\nu)^q = C(r,d) \left( \int h^{1/r}(x) \, dx \right)^{q/r},
\]

where \( C(r,d) \) is a constant independent of \( \nu \).

Interestingly, there is an analog result on the eigenvalue counting function for polyharmonic operators (see for instance [BS70]). While engineers are mainly dealing with absolutely continuous distributions, the quantization problem is significant for all Borel probability measures with bounded support.

In this note we follow some ideas that are developed in [KN22c; KN22d; KN22b; KN22a] to tackle spectral asymptotics for Kreǐn–Feller operators. One of the central objects is the \( L^q \)-spectrum \( \beta_q \) of \( \nu \) given, for \( q \in \mathbb{R}_{\geq 0} \), by

\[
\beta_q(q) := \limsup_{n \to \infty} \beta_{q,n}(q) \quad \text{with} \quad \beta_{q,n}(q) := \frac{\log \left( \sum_{C \in D_q} \nu(C)^q \right)}{\log(2^n)}.
\]

Define

\[
q_r := \inf \{ q > 0 : \beta_q(q) < rq \}.
\]

Further, the (upper) generalized Rényi dimension of \( \nu \) [BR56] is given, for \( q \in \mathbb{R}_{\geq 0} \), by

\[
\overline{R}_q(q) := \begin{cases} 
\beta_q(q)/(1-q), & \text{for } q \neq 1, \\
\limsup_{n \to \infty} \left( \sum_{C \in D_q} \nu(C) \log \nu(C) \right) / \log(2^{-n}), & \text{for } q = 1.
\end{cases}
\]

It turns out that, for \( q \neq 1 \) (and similar for \( q = 1 \)), the Rényi dimension can be expressed also in terms of the Hentschel–Procaccia generalized dimension [HP83]

\[
\overline{R}_q(q) = \frac{1}{1-q} \limsup_{r \to 0} \frac{\log \int \nu(B_r(x))^{q-1} \, dx}{-\log r}.
\]

It is easy to construct purely atomic measures such that \( q_r = 0 \) for all \( r > 0 \) and \( \beta_q(0) > 0 \), see [KN22c]. In this case, the upper quantization dimension is also 0. Further, we will need some ideas from entropy theory: Let us define the set of \( \nu \)-partitions \( \Pi \) to be the set of finite collections \( P \) of dyadic cubes such that there exists a partition \( \overline{P} \) of \( Q \) by dyadic cubes from \( D \) with \( P = \{ Q \in \overline{P} : \nu(Q) > 0 \} \). We define

\[
\mathcal{M}_{\nu,r}(x) := \inf \left\{ \text{card}(P) : P \in \Pi_r, \max_{Q \in P} \nu(Q) \Lambda (Q)^{r/d} < 1/x \right\},
\]

and

\[
\overline{h}_{\nu,r} := \limsup_{x \to \infty} \frac{\log \mathcal{M}_{\nu,r}(x)}{\log x}, \quad h_{\nu,r} := \liminf_{x \to \infty} \frac{\log \mathcal{M}_{\nu,r}(x)}{\log x},
\]

will be called the upper, resp. lower, \((\nu,r)\)-partition entropy. We write \( \overline{h}_r := \overline{h}_{\nu,r} \) and \( h_r := h_{\nu,r} \).

For all \( n \in \mathbb{N} \) and \( \alpha > 0 \), we define

\[
N_{\nu,\alpha,r}(n) := \text{card} N_{\nu,\alpha,r}(n), \quad N_{\nu,\alpha,r}(n) := \left\{ Q \in D_n : \nu(Q) \Lambda (Q)^{r/d} \geq 2^{-\alpha n} \right\},
\]

and set

\[
\overline{F}_{\nu,r}(a) := \limsup_n \frac{\log^+ (N_{\nu,\alpha,r}(n))}{\log 2^n} \quad \text{and} \quad \underline{F}_{\nu,r}(a) := \liminf_n \frac{\log^+ (N_{\nu,\alpha,r}(n))}{\log 2^n}.
\]
Following [KN22d], we refer to the quantities
\[ F_r := F_{\nu,r} := \sup_{\alpha > 0} \frac{F_{\nu,r}(\alpha)}{\alpha} \quad \text{and} \quad F_r := F_{\nu,r} := \sup_{\alpha > 0} \frac{F_{\nu,r}(\alpha)}{\alpha} \]
as the \((\nu, r)\)-upper, resp. lower, optimized coarse multifractal dimension. We will see in Section 3 that we always have
\[ F_r = q_r = \bar{h}_r \geq h_r \geq F_r. \quad (1.3) \]

Now, we are in the position to state our main results. All proofs of this section will be postponed to Section 4.3.

**Theorem 1.1.** Let \( \nu \) be compactly supported probability measure on \( \mathbb{R}^d \). If \( \sup_{x \in (0, 1)} \beta_{\nu}(x) > 0 \), then for every \( r > 0 \),
\[ \frac{rF_r}{1 - F_r} \leq D_r(\nu) \leq \frac{rh_r}{1 - h_r} \leq \overline{D}_r(\nu) = \overline{\beta}_r(q_r). \]

Otherwise, if \( \beta_{\nu}(x) = 0 \) for all \( x > 0 \), then \( D_r(\nu) = 0 \) for all \( r \geq 0 \).

**Remark 1.2.** We would like to emphasize here that the proof of the upper bound of \( D_r(\nu) \) is based on an adaptive approximation algorithm in the sense of [DeV87; HKY00]. This can be used to implement a straightforward procedure for finding a quantizer \( f \in F_n, n \in \mathbb{N} \) that provides an upper bound, which is optimal with respect to the upper exponential rate.

**Remark 1.3.**

1. The following useful identities are a consequence of (1.3) and apply independently of the condition on \( \beta_{\nu} \):
\[ \overline{D}_r(\nu) = \frac{r\bar{h}_r}{1 - \bar{h}_r} = \frac{rq_r}{1 - q_r} = \frac{rF_r}{1 - F_r}. \]

   We will make use of this observation in the proof of Theorem 1.1.

2. At least for special cases, it has been observed in [KZ15, p. 6] that the upper quantization dimension is often determined by a critical value; we are now in the position to determine this critical value for arbitrary compactly supported probability measures as
\[ \overline{D}_r(\nu) = \inf \left\{ q > 0 : \sum_{Q \in D} (\nu(Q)\Lambda(Q))^q \frac{\mu(Q)}{\mu(\mathbb{R}^d)}^{q/(q+r)} < \infty \right\}. \]

   This follows from the definition of \( q_r \) and the identities given in (1) (see also [KN22d, Lemma 3.3]).

3. For \( r = 0 \) a general result of [Zhu12] is applicable yielding
\[ \dim_H(\nu) \leq D_0(\nu) \leq \overline{D}_0(\nu) \leq \dim^*_p(\nu) \leq -\partial^+ \beta(1), \]
where \(-\partial^+ \beta(1)\) denotes the left-sided derivative of \( \beta \) in 1,
\[ \dim^*(\nu) := \inf \left\{ \dim(A) : \mu(A) = \mu(\mathbb{R}^d) \right\}, \]
\[ \dim(\nu) := \inf \left\{ \dim(A) : \mu(A) > 0 \right\} \]
with \( \dim \) denoting either the Hausdorff dimension \( \dim_H \) or the packing dimension \( \dim_P \).
Remark 1.4. By Theorem 1.1 we infer the following one-to-one correspondence between \( \overline{D}_r (v) \), \( r > 0 \) and \( \beta_r (q) \), \( q \in (0, 1) \). For this note that \( q_r = \overline{D}_r (v) / (r + \overline{D}_r (v)) \). Hence, if \( \overline{D}_r (v) > 0 \), then

\[
\beta_r \left( \frac{\overline{D}_r (v)}{\overline{D}_r (v) + r} \right) = \frac{r \overline{D}_r (v)}{\overline{D}_r (v) + r}.
\]

If we now set \( q_0 := \sup \left\{ \overline{D}_r (v) / (r + \overline{D}_r (v)) : r > 0 \right\} \), then for all \( q \in (q_0, 1) \) we have \( \beta_r (q) = 0 \) and for \( q \in (0, q_0) \), we have \( \beta_r (q) > 0 \) and

\[
\overline{D}_{\beta_r \left( q \right) / q} (v) = \frac{\beta_r (q)}{1 - q}.
\]

At the end of this section in Example 1.15, we exploit the fact that the upper quantization dimension is known for inhomogeneous self-similar measures supported on self-similar sets (see \[Zhu16\]) and derive from this its \( L^0 \)-spectrum on \((0, 1)\).

Further, using \( \overline{D}_r (v) = r q_r / (1 - q_r) \), we can affirm a conjecture of Lindsay stated in his PhD thesis \[Lin01\]:

**Corollary 1.5 (Continuity).** The map \( r \mapsto \overline{D}_r (v) \) is continuous on \((0, \infty)\).

As another easy consequence, we reproduce a previous result of [KZ07] on the stability of quantization dimension.

**Corollary 1.6 (Finite stability).** For any two compactly supported probability measures \( v_1 \) and \( v_2 \), \( p \in (0, 1) \) and each \( r > 0 \), we have

\[
\overline{D}_r \left( p v_1 + (1 - p) v_2 \right) = \max \left\{ \overline{D}_r (v_1), \overline{D}_r (v_2) \right\}.
\]

Our approach gives also rise to a slight improvement for the upper bound of the lower quantization dimension. For this we introduce the following quantity

\[
\dim_{\infty} (v) := \liminf_{n \to \infty} \frac{\log \max_{Q \in E_n} v (Q)}{-\log 2^n}
\]

to sharpen (1.4) from [Pö01].

**Proposition 1.7.** For \( r > 0 \) such that \( \dim_{\infty} (v) / (r + \dim_{\infty} (v)) < 1 \), we have

\[
\overline{D}_r (v) \leq \frac{r \dim_{\infty} (v)}{r + \dim_{\infty} (v) - \dim_{\infty} (v)}.
\]

**Remark 1.8.** If the quantization dimension exists for some \( r > 0 \), then it is given by purely measure-geometric data encoded by the \((v, r)\)-partition entropy, namely we have \( h_r = h_r = h_r \) and this value determines the quantization dimension \( D_r (v) = r h_r / (1 - h_r) \).

As a further consequence of our formalism, we derive a rigidity statement that forces the quantization dimension to be constant in \( r > 0 \).
Corollary 1.9. Assume $\tilde{\dim}_M(\nu) > 0$. Then the following statements are equivalent:

1. $\tilde{D}_r(\nu) = \overline{\dim}_R(\nu)$ for some $r > 0$
2. $\tilde{R}_r(q) = \overline{\dim}_M(\nu)$ for some $q \in (0, 1)$.
3. $\tilde{\alpha}_r(1) \leq -\overline{\dim}_M(\nu)$.
4. The $L^q$-spectrum is linear on $[0, 1]$, i.e., $\beta_r(q) = \overline{\dim}_M(\nu)(1 - q)$. $q \in [0, 1]$.
5. $\tilde{D}_r(\nu) = \overline{\dim}_M(\nu)$ for all $r > 0$.

In particular, we find the necessary condition $\overline{\dim}_B(\mu) = \overline{\dim}_M(\nu) + \overline{\dim}_H(\mu) = \overline{\dim}_M(\nu)$, which would imply (3) of the above statements.

As a second main result, we find often easily verifiable necessary conditions that guarantee upper and lower quantization dimension to match.

Definition 1.10. We define two notions of regularity.

1. We call $\nu$ multifractal-regular in $r$ ($r$-MF-regular), if $F_{\nu,r} = F_{\nu}$. 
2. We call the measure $\nu$ $L^q$-regular for $r$ if

   a. $\beta_r(q) = \liminf r \beta_{r/q} = \overline{\dim}_M(\nu)$, for all $q \in (q_r - \epsilon, q_r)$, for some $\epsilon > 0$,
   b. $\beta_r(q_r) = \liminf r \beta_{r/q}$ and $\beta_r$ is differentiable in $q_r$.

The following theorem shows that the spectral partition function is a valuable auxiliary concept to determine the quantization dimension for a given measure $\nu$.

Theorem 1.11. The following regularity implications hold for $r > 0$:

$\nu$ is $L^q$-regular for $r \implies \nu$ is r-MF-regular $\implies \tilde{D}_r(\nu) = \tilde{D}_r(\nu) = \overline{\dim}_M(\nu)$.

Special cases for the following corollary have a long history where the results concerning the quantization dimension were obtained under only the open set condition, strong open set condition or even the strong separation condition [GL97; GL00a; GL02; Zhu11; ZZS17; KZ17]. Unlike here, these papers investigated also the quantization coefficients, which provide more accurate information about the approximation rate. Recall e.g. Graf and Luschgy’s results [GL00a; GL04] for self-similar measures: Let $(f_i)_{i=1}^N$ be a family of contractive similitudes on $\mathbb{R}^d$ with contraction ratios $(r_i)^N_{i=1} \in (0, 1)^N$. According to [Hut81], for a given probability vector $(p_i)_{i=1}^N$, there exists a unique compactly supported Borel probability measure on $\mathbb{R}^d$ such that $\nu = \sum_{i=1}^N p_i \cdot v \circ f_i^{-1}$. This measure is called the self-similar measure. We say that $(f_i)_{i=1}^N$ satisfies the open set condition (OSC) if there is a non-empty open set $U$ such that $f_i(U) \cap f_j(U) = \emptyset$ for all $i \neq j$, and $f_i(U) \subset U$ for all $i = 1, \ldots, N$. If additionally $\sup \nu \cap U \neq \emptyset$ for the open set $U$ in the OSC, we say that the strong open set condition (SOSC) holds. For $r \in (0, \infty)$, let $k$, be the positive real number given by

$$\sum_{i=1}^N (p_i r_i)^{1/(k_r + r)} = 1,$$  \hfill (1.5)

then, assuming the OSC, the quantization dimension is given by $D_r(\nu) = k_r$. This result has been generalized by Lindsay ([Lin01]) to conformal IFS under the strong open set condition (see also [LM02]) using the notion of the associated pressure function. We are able to generalize this result further as follows:

Let $U \subset \mathbb{R}^d$ be an open set. We say a $C^1$-map $S : U \to \mathbb{R}^d$ is conformal if for every $x \in U$ the matrix $S'(x)$, giving the total derivative of $S$ in $x$, satisfies $|S'(x) \cdot y| = |S'(x)||y|$ for all $y \in \mathbb{R}^d$ and $\left|S'(x) \cdot z\right| := \sup_{z \in \mathbb{R}^d} |S'(x) \cdot z| > 0$. A family of conformal mappings $(S_i : K \to K)_{i=1,\ldots,N}$, for some compact set $K$ and $N \geq 2$ is a $C^1$-conformal iterated function system ($C^1$-cIFS) if for each $i \in I$, the contraction $S_i$ extends to an injective conformal map.
Figure 1.1. The $L^q$-spectrum $\beta_r$ for the self-similar measure $\nu$ supported on the Sierpiński tetraeder in $\mathbb{R}^3$ with all four contraction ratios equal $1/2$ and with probability vector $(0.66, 0.2, 0.08, 0.06)$. $\beta_r (0) = \dim_H (\nu) = 2$.

Let us now consider an inhomogeneous self-similar measure $\nu$ given as the unique solution to

$$\nu = p_0 \mu + \sum_{i = 1}^{N} p_i \nu \circ f_i^{-1}. \quad (1.6)$$

In here, $\mu$ is a compactly supported Borel probability measure, called the condensation measure, $(f_i)_{i=1}^{N}$ is a family of contractive similitudes on $\mathbb{R}^d$ with contraction ratios $(r_i)_{i=1}^{N}$.
Without loss of generality we assume \( \max_{i} q \) with \( L(2a) \) of the \( \nu \) is defined in terms of the \( L^\beta \)-spectrum of \( \nu \), which is given, for \( q \in (0, 1) \), by

\[
\beta_r(q) = \max \left\{ \beta_r(q), \gamma_r(q) \right\}.
\]

In this example the \( L^\beta \)-spectrum is not necessarily differentiable in \( q_r \). Nevertheless, part \( (2a) \) of the \( L^\beta \)-regularity for \( r \) is still valid for \( \beta_r \), under the assumptions of Corollary 1.14. Note that the assumption on \( \mu \) guarantees that the quantization dimension of \( \mu \) of order \( r > 0 \) exists.

**Example 1.15.** In our final example we consider an inhomogeneous self-similar measure \( \nu \) as above, where instead of the iOSC we assume that \( \mu \) itself is a self-similar measure with respect to the same IFS \( (f_i)_{i=1}^{N} \) and probability vector \( \left(t_i\right)_{i=1}^{N} \). We will demonstrate how our result can be used to find the \( L^\beta \)-spectrum on \((0,1)\) knowing the quantization dimension. For fixed \( r > 0 \) let \( \varepsilon_{1,r} \) and \( \varepsilon_{2,r} \) be implicitly given by

\[
\sum_{i=1}^{N} (t_i s_i^r)^{\varepsilon_{1,r}} = 1 \quad \text{and} \quad \sum_{i=1}^{N} (p_i s_i^r)^{\varepsilon_{2,r}} = 1.
\]

Then, by [Zhu16, Theorem 1.2], we have \( D_r(\nu) = \max \{\varepsilon_{1,r}, \varepsilon_{2,r}\} > 0 \). Now, for \( r = \beta_r(q)/q \) with \( q \in (0, 1) \), we have by Theorem 1.1

\[
q = q_r = \frac{D_r(\nu)}{(\varepsilon_{1,r} + r)}.
\]

Without loss of generality we assume \( \max \{\varepsilon_{1,r}, \varepsilon_{2,r}\} = \varepsilon_{1,r} \). Then, \( q = \varepsilon_{1,r} / (r + \varepsilon_{1,r}) \) and

\[
1 = \sum_{i=1}^{N} (t_i s_i^r)^{\varepsilon_{1,r}} = \sum_{i=1}^{N} \left( t_i s_i^r \right)^{\varepsilon_{1,r}} = \sum_{i=1}^{N} p_i q_{\beta_r(q)}^{\varepsilon_i}.
\]

Since \( x \mapsto x/(x + r) \) is increasing on \( \mathbb{R}_{\geq 0} \), it follows that \( \beta_r(q) = \max \{\varrho_{1,q}, \varrho_{2,q}\} \) where \( \varrho_{1,q}, \varrho_{2,q} \) denote the unique solutions of

\[
\sum_{i=1}^{N} t_i^{\varrho_{1,q}} = 1 \quad \text{and} \quad \sum_{i=1}^{N} p_i q_{\varrho_{2,q}}^{\varepsilon_i} = 1.
\]

This shows that also for this special case of inhomogeneous self-similar measure the \( L^\beta \)-spectrum is given by the same formula as under iOSC provided in Corollary 1.14.
2. Partition functions, coarse multifractal formalism, and $L^q$-spectra

In the sequel, we will refer to some results of [KN22c] and for this reason we recall the newly introduced notion of partition functions with respect to a non-negative, monotone set function $\mathcal{J}$ defined on the dyadic cubes $\mathcal{D}$. The full generality of this approach is not needed, and for the purposes of this paper we will restrict ourselves throughout to the particular choice

$$\mathcal{J} := \mathcal{J}_{\nu,r} : \mathcal{Q} \mapsto \nu(Q)(\Lambda(Q))^{r/d}, \quad \text{for } r \geq 0.$$  

The partition function with respect to $\mathcal{J}$ is then given, for $q \geq 0$, by

$$\tau_{\mathcal{J}}(q) := \limsup_{n \to \infty} \tau_{\mathcal{J},n}(q), \quad \text{with } \tau_{\mathcal{J},n}(q) := \frac{\log \left( \sum_{C \in \mathcal{D}_n} \mathcal{J}(C)^q \right)}{\log(2^n)},$$

where we use the convention $0^0 = 0$, that is for $q = 0$ we neglect the summands with $\mathcal{J}(Q) = 0$ in the definition of $\tau_{\mathcal{J}}$. We define the critical value

$$q_{\mathcal{J}} := \inf \{ q \geq 0 : \tau_{\mathcal{J}}(q) < 0 \}$$

which is connected to the $L^q$-spectrum and our notation from Section 1 via

$$\tau_{\mathcal{J},r}(q) = \beta_r(q) - rq \quad \text{and} \quad q_r = q_{\mathcal{J},r}.$$  

In order to set up the optimize coarse multifractal dimension with respect to $\mathcal{J}$ in this context, we introduce the following terminology to simplify the notation and also to make the correspondence with the relevant results in [KN22d] explicit. For $n \in \mathbb{N}$ and $\alpha > 0$, we set

$$N_{\alpha, \mathcal{J},r}(n) := \{ C \in \mathcal{D}_n : \mathcal{J}_{\nu,r}(C) \geq 2^{-\alpha n} \} = N_{\nu,r}(n)$$

and to express the dependence on $\mathcal{J}_{\nu,r}$ for all relevant quantities we also use the notation $N_{\alpha, \mathcal{J},r}(n) := N_{\nu,r}(n)$, $\mathcal{F}_{\mathcal{J},r}(\alpha) := \mathcal{F}_{\nu,r}(\alpha)$, $\mathcal{F}_{\mathcal{J},r}(\alpha) := \mathcal{F}_{\nu,r}(\alpha)$, and write $\mathcal{F}_{\mathcal{J},r} := \mathcal{F}_{\nu,r}$ and $\mathcal{F}_{\mathcal{J},r} := \mathcal{F}_{\nu,r}$ to denote the optimize coarse multifractal dimension with respect to $\mathcal{J}_{\nu,r}$.

Similarly, for $\alpha > 1/\mathcal{J}_{\nu,r}(Q)$, we write $M_{\mathcal{J},r}(x) := M_{\nu,r}(x)$ as given in (1.2) and for the corresponding partition entropies $h_{\mathcal{J},r} := h_{\nu,r}$ and $h_{\mathcal{J},r} := h_{\nu,r}$, respectively.

We end this section with some important facts about the $L^q$-spectrum of $\nu$:

- $\beta_r(1) = 0$.
- $\beta_r(0) = \dim_M(\nu)$.
- $\dim_0(\nu) \leq d$.
- For $q \geq 0$, we have $-qd \leq \beta_r(q)$.
- If $\nu$ is absolutely continuous, then $\beta_r(q) = d(1 - q)$, for all $q \in [0,1]$.

3. Optimal partitions and partition entropy

3.1. Upper bounds. We make use of an observation from [KN22d] which is valid for set functions $\mathcal{J} : \mathcal{D} \to \mathbb{R}_{\geq 0}$ on the dyadic cubes $\mathcal{D}$, which are monotone, uniformly vanishing and locally non-vanishing with $\mathcal{J}(Q) > 0$ and such that $\liminf_{n} \tau_{\mathcal{J},n}(q) \in \mathbb{R}$ for some $q > 0$. Here, uniformly vanishing means $\lim_{n \to \infty} \sup_{Q \in \cup_{D \in \mathcal{D}_n}} \mathcal{J}(Q) = 0$ and locally non-vanishing means $\mathcal{J}(Q) > 0$ implies that there exists $Q' \in \{ R \in \mathcal{D} : R \subseteq Q \}$ with $\mathcal{J}(Q') > 0$. It is important to note that all these conditions on the set function are fulfilled for our particular choice $\mathcal{J} = \mathcal{J}_{\nu,r}$.

Now we are in the position to state the upper bounds in terms of optimal partitions denoted by $P_r$.
Proposition 3.1 ([KN22d, Prop. 4.1]). For \(0 < t < \mathfrak{A}(Q)\), we have that
\[
P_t := \left\{ Q \in D : \mathfrak{A}(Q) < t \& \exists Q' \in D^Y_{[\log_2 \Lambda(Q)]^{d-1}} : Q' \supset Q \& \mathfrak{A}(Q') \geq t \right\}
\]
is a finite partition of dyadic cubes of \(Q\), and we have
\[
F_{3} \leq h_{\mathfrak{A}} \leq \limsup_{t \downarrow 0} \frac{\log(\text{card}(P_t))}{-\log(t)} \leq q_{3},
\]
and
\[
F_{3} \leq h_{\mathfrak{A}} \leq \liminf_{t \downarrow 0} \frac{\log(\text{card}(P_t))}{-\log(t)}.
\]

3.2. The dual problem. In this section we recall the close connection to the shrinking rate of the following quantity
\[
\gamma_{3,n} := \inf_{P \in \Pi, \text{card}(P) \leq n} \max_{Q \in P} \mathfrak{A}(Q)
\]
as elaborated in [KN22d]. As before, let \(3 := 3_{\nu,r}, r > 0\). Then [KN22d, Cor. 4.10] gives in this situation the following.

Lemma 3.2. For \(r > 0\), we have
\[
\gamma_{3,r,n} = O\left(n^{-(1+r/d)}\right) \quad \text{and} \quad M_{3,r}(x) = O\left(x^{d/(d+r)}\right).
\]
If additionally \(\nu\) is singular, then
\[
\gamma_{3,r,n} = o\left(n^{-(1+r/d)}\right) \quad \text{and} \quad M_{3,r}(x) = o\left(x^{d/(d+r)}\right).
\]

We obtain the following estimate for the upper exponent of divergence of \(\gamma_{3,n}\) given by
\[
\alpha_{3} := \limsup_{n \rightarrow \infty} \frac{\log(\gamma_{3,n})}{\log(n)} \quad \text{and} \quad \alpha_{3} := \liminf_{n \rightarrow \infty} \frac{\log(\gamma_{3,n})}{\log(n)}.
\]

Proposition 3.3 ([KN22d, Prop. 4.11]). For \(r > 0\), we have
\[
-\frac{1}{h_{\mathfrak{A}}} = \overline{\alpha}_{\mathfrak{A}} \leq \frac{1}{q_{3}} \leq \frac{\dim_M(\nu) + r}{\dim_M(\nu)} \leq -(1 + r/d) \quad \text{and} \quad -\frac{1}{h_{\mathfrak{A}}} = \underline{\alpha}_{\mathfrak{A}}.
\]

3.3. Lower bounds. We have seen so far that
\[
F_{r} = F_{3,r} \leq q_{3,r} = q_{r}.
\]

For the lower bounds we use ideas from the coarse multifractal analysis as developed in [KN22c; KN22d] for the particular choice \(3 = 3_{\nu,r}\). Since the conditions stated [KN22d, Prop. 5.7 and 5.8] are fulfilled for \(3_{\nu,r}\), we obtain the following two equalities.

Proposition 3.4. For \(r > 0\), we always have \(F_{r} = q_{r}\) and if \(\nu\) is \(L^{\beta}\)-regular for \(r\), then \(F_{r} = q_{r}\).

4. Application to the Quantization Dimension

In this section we give a proof for the upper and lower bound of the quantization dimension of order \(r > 0\) as stated in Theorem 1.1.
4.1. Upper bounds. We start by proving the upper bound for the upper quantization dimension.

**Proposition 4.1.** For all \( n \in \mathbb{N} \), we have
\[
e_{n,r}(v) \leq \sqrt{d} n \gamma_{3r,n}.
\]
In particular,
\[
D_r(v) \leq \frac{r \overline{h}_{3r}}{1 - \overline{h}_{3r}} = \frac{rq_r}{1 - q_r} = \overline{R}_r(q_r) \leq \overline{\dim}_M(v),
\]
and
\[
D_r(v) \leq \frac{r \overline{h}_{3r}}{1 - \overline{h}_{3r}}.
\]

**Proof.** We only consider the case \( q_r > 0 \). The case \( q_r = 0 \) follows analogously. Let \( P \in \Pi_v \) with \( \text{card}(P) \leq n \). Write \( P = \{Q_1, \ldots, Q_{\text{card}(P)}\} \) and let \( m_i \) denote the middle point of the dyadic cube \( Q_r, i \leq \text{card}(P) \) and set \( a_n := (m_1, \ldots, m_{\text{card}(P)}) \). Then we have
\[
e_{n,r}(v) \leq \left( \int \sum_{i} d(x, a_n)^r \, dv(x) \right)^{1/r} = \left( \sum_{i} \int_{Q_i} d(x, a_n)^r \, dv(x) \right)^{1/r}
\]
\[
\leq \sqrt{d} \left( \sum_{i} v(Q_i) \Lambda(Q_i)^{r/d} \right)^{1/r} \leq \sqrt{dn^{1/r}} \left( \max_{Q \in P} v(Q) \Lambda(Q)^{r/d} \right)^{1/r}.
\]
Now, taking the infimum over all \( P \in \Pi_v \) with \( \text{card}(P) \leq n \) delivers
\[
e_{n,r}^*(v) \leq \sqrt{dn} \gamma_{3r,n}.
\]
Note that by Proposition 3.3, for every \( \varepsilon > 0 \), we have for \( n \) large
\[
n \gamma_{3r,n} \leq n^{1 - 1/q_r + \varepsilon},
\]
and there exists a subsequence \( (n_k) \) such that
\[
n_k \gamma_{3r,n_k} \leq n_k^{1 - 1/2q_r + \varepsilon},
\]
implying
\[
\limsup_{n \to \infty} \frac{- \log(n_k)}{\log(e_{n,r}(v))} \leq \frac{r \overline{h}_{3r}}{1 - \overline{h}_{3r}} = \frac{rq_r}{1 - q_r} \quad \text{and} \quad \liminf_{n \to \infty} \frac{- \log(n_k)}{\log(e_{n,r}(v))} \leq \frac{r \overline{h}_{3r}}{1 - \overline{h}_{3r}}.
\]
Moreover, Proposition 3.3 implies \(-1/q_r \leq - (\overline{\dim}_M(v) + r) / \overline{\dim}_M(v) \) which proves the last inequality.

**Corollary 4.2.** If \( v \) is singular, then \( \lim_{n \to \infty} n^{1/d} e_{n,r}(v) = 0 \).

**Proof.** Since \( v \) is singular, by Lemma 3.2 and Proposition 4.1, we have
\[
e_{n,r}(v) \leq \left( n \gamma_{3r,n} \right)^{1/r} = o \left( n^{-1/d} \right).
\]

Using the simple fact \( \overline{D}_r(v) \leq \overline{D}_r(v) \), \( \overline{D}_r(v) \leq \overline{D}_r(v) \) (see also [GL04, Lemma 3.5]) and Proposition 4.1, we obtain the following bounds for the geometric mean error.

**Corollary 4.3.** We have \( \overline{D}_0(v) \leq -\overline{\beta}(1) \) and \( \overline{D}_0(v) \leq \liminf_{r \uparrow 0} r \overline{h}_{3r} / \left( 1 - \overline{h}_{3r} \right) \).
4.2. Lower bounds. Recall, for \( s > 0 \) we let \( \langle Q \rangle_s \) denote the cube centered and parallel with respect to \( Q \) such that \( \Lambda(Q) = s^d \Lambda(\langle Q \rangle_s) \), \( s > 0 \). The following lemma was implicitly used in [KN22c].

**Lemma 4.4.** For fixed \( \alpha > 0 \) there exists a sequence \((E_{\alpha,n})\) with \( E_{\alpha,n} \subset N_{\alpha,3}(n) \), \( c_{\alpha,n} := \text{card}(E_{\alpha,n}) \geq \left\lfloor \frac{N_{\alpha,3}(n)}{5^d} \right\rfloor \) and for all cubes \( Q, Q' \in E_{\alpha,n} \) with \( Q \neq Q' \) we have \( \langle Q \rangle_3 \cap \langle Q' \rangle_3 = \emptyset \).

**Proof.** If \( N_{\alpha,3}(n) = \emptyset \), nothing needs to be shown. Hence, we assume \( N_{\alpha,3}(n) \neq \emptyset \) and construct inductively a subset \( E_n \) of \( N_{\alpha,3}(n) \) of cardinality \( c_n := \text{card}(E_n) \geq \left\lfloor \frac{N_{\alpha,3}(n)}{5^d} \right\rfloor \) such that for all cubes \( Q, Q' \in E_n \) with \( Q \neq Q' \) we have \( \langle Q \rangle_3 \cap \langle Q' \rangle_3 = \emptyset \). At the beginning of the induction we set \( D^{(0)} := N_{\alpha,3}(n) \). Assume we have constructed \( D^{(0)} \supseteq D^{(1)} \supseteq \ldots \supseteq D^{(j-1)} \) such that \( \langle Q \rangle_3 \cap \langle Q \rangle_3 = \emptyset \), for some \( Q, Q_j \in D^{(j-1)} \) with \( Q \neq Q_j \). In this case we set

\[
D^{(j)} := \{ C \in D^{(j-1)} : \langle C \rangle \cap \langle Q_j \rangle_3 = \emptyset \} \cup \{ Q_j \}.
\]

By this construction, we have \( \text{card}(D^{(j)}) < \text{card}(D^{(j-1)}) \), since \( C \cap \langle Q_j \rangle_3 = \emptyset \). Otherwise, if \( \langle Q \rangle_3 \cap \langle Q \rangle_3 = \emptyset \) for all \( Q, Q' \in D^{(j-1)} \) with \( Q \neq Q' \), then the finite induction terminates and we set \( E_{\alpha,n} = D^{(j-1)} \). In each inductive step, we remove at most \( 5^d - 1 \) elements of \( D^{(j-1)} \), while one element, namely \( Q_j \), is kept. This implies \( \text{card}(E_{\alpha,n}) \geq \left\lfloor \frac{N_{\alpha,3}(n)}{5^d} \right\rfloor \). \( \square \)

**Proposition 4.5.** For \( r > 0 \), we have

\[
\overline{D}_r(n) \geq \frac{rF_{3\nu,r}}{1 - d_r} \quad \text{and} \quad D_r(n) \geq \frac{rF_{3\nu,r}}{1 - F_{3\nu,r}}.
\]

**Proof.** Fix \( \alpha > 0 \) such that \( F_{3\nu,r}(\alpha) > 0 \). Further, let \((n_k)_k\) be such that

\[
F_{3\nu,r}(\alpha) = \lim_{k \to \infty} \frac{\log^+ (N_{\alpha,3\nu}(n_k))}{\log 2n_k}
\]

and let \( c_{\alpha,n_k} := \text{card}(E_{\alpha,n_k}) \) be given as in Lemma 4.4 for \( \beta = 3\nu,r \). Notice that by our assumption \( F_{3\nu,r}(\alpha) > 0 \) and it follows that \( \lim_k c_{\alpha,n_k} = \infty \). Let \( A \) be of cardinality at most \( c_{\alpha,n_k}/2 \) and

\[
E'_{\alpha,n_k} := \left\{ Q \in E_{\alpha,n_k} : \inf_{a \in A} d(a,Q) \geq 2^{-n_k} \right\}.
\]

Since, for all \( Q_1, Q_2 \in E_{\alpha,n_k} \) we have \( \langle Q_1 \rangle_3 \cap \langle Q_2 \rangle_3 = \emptyset \) it follows that if \( d(a,Q) < 2^{-n_k} \) for some \( a \in A \) and \( Q \in E_{\alpha,n_k} \), then \( d(a,Q') \geq 2^{-n_k} \) for all \( Q' \in E_{\alpha,n_k} \setminus \{Q\} \) and therefore,

\[
\text{card} \left\{ Q \in E_{\alpha,n_k} : \inf_{a \in A} d(a,Q) < 2^{-n_k} \right\} \leq \text{card}(A).
\]

Hence, \( \text{card}(E'_{\alpha,n_k}) \geq c_{\alpha,n_k}/2 \) and

\[
\int d(x,A)^r \, dv(x) \geq \sum_{Q \in E_{\alpha,n_k}} \int_Q d(x,A)^r \, dv(x) \geq \sum_{Q \in E_{\alpha,n_k}} \nu(Q) \Lambda(Q)^{r/d} \geq c_{\alpha,n_k} 2^{-\alpha n_k - 1}.
\]
Consequently, \( c_{\lfloor cn/k \rfloor,r}(v) \geq c_{\alpha,n_k} 2^{-an_k} \) and we obtain for the first claim
\[
\limsup_{k} \frac{-\log c_{\lfloor cn/k \rfloor,r}(v)}{\log \lfloor cn/k \rfloor} \geq \limsup_{k} \frac{r \log (c_{\lfloor cn/k \rfloor,r}/4)}{-\log c_{\lfloor cn/k \rfloor,r}(v)} \geq \limsup_{k} \frac{r \left( \log \left( N_{\alpha,3r}(n_k)/5^d - 1 \right) - \log 4 \right)}{-\log \left( N_{\alpha,3r}(n_k)/5^d - 1 \right) + (\alpha n_k + 1) \log 2}
\]
\[
= \frac{r F_{3r}(\alpha)/\alpha}{1 - F_{3r}(\alpha)/\alpha}.
\]
This gives
\[
\limsup_{n} \frac{-\log n}{\log c_{n,r}(v)} \geq \sup_{\alpha > 0} \limsup_{k} \frac{-\log c_{\lfloor cn/k \rfloor,r}(v)}{\log \lfloor cn/k \rfloor} \geq \sup_{\alpha > 0} \frac{r F_{3r}(\alpha)/\alpha}{1 - F_{3r}(\alpha)/\alpha} = \frac{r F_{3r}(\alpha)}{1 - F_{3r}(\alpha)}.
\]
For the lower limit assume \( F_{3r}(\alpha) > 0 \) and note that for every \( \varepsilon \in (0, F_{3r}(\alpha)) \) and all \( n \) large
\[
c_{\alpha,n} := \left[ 5^{-d} N_{\alpha,3r}(n) \right] \geq 2^{\varepsilon \log(\varepsilon)}.
\]
Now, for \( k \in \mathbb{N} \), we define
\[
n_k := \left\lfloor \frac{\log (2k)}{(1 - F_{3r}(\alpha) - \varepsilon) \log(2)} \right\rfloor.
\]
Clearly, this gives \( c_{\alpha,n_k} \geq 2k \). Then for any subset \( A \) with \( \text{card}(A) \leq k \leq c_{\alpha,n_k}/2 \) we have as above \( \text{card} \left( E_{\alpha,n_k} \right) \geq c_{\alpha,n_k}/2 \geq k \). Then
\[
\int d(x,A)^r \, dv(x) \geq \sum_{Q \in E_{\alpha,n_k}} \int d(x,A)^r \, dv(x) \geq \sum_{Q \in E_{\alpha,n_k}} v(Q) \Lambda(Q)^r / d \geq c_{\alpha,n_k} 2^{-an_k} \geq k^{-an_k}.
\]
Taking the infimum over \( A \) with \( \text{card}(A) \leq k \) we obtain \( c_{\alpha,r}^e(v) \geq k^{-an_k} \). This gives
\[
\frac{-\log k}{-\log c_{n,r}(v)} \geq \frac{r \log k}{-\log k + \alpha n_k \log 2} \geq \frac{r \log k}{-\log k + \alpha \log(2k)/\left( F_{3r}(\alpha) - \varepsilon \right)}.
\]
Taking the lower limit over \( k \) and letting \( \varepsilon \) tend to zero, yields
\[
D_e(v) \geq \frac{r}{1 + \alpha/F_{3r}(\alpha)} = \frac{r F_{3r}(\alpha)/\alpha}{1 - F_{3r}(\alpha)/\alpha}.
\]
Finally, taking the supremum for \( \alpha > 0 \) gives
\[
D_e(v) \geq \frac{r F_{3r}(\alpha)}{1 - F_{3r}(\alpha)}.
\]
\[\square\]

4.3. Proofs of main results.

Proof of Theorem 1.1. The main theorem follows by combining Proposition 4.1 and Proposition 4.5. \[\square\]
Proof of Corollary 1.5. This follows from the fact that \( r \mapsto q_r \) is continuous. Indeed, if \( q_r = 0 \) for some \( r > 0 \), then \( 0 \leq \beta_r(q) < r q \) for all \( q \in (0, 1) \). Consequently, \( \lim_{q \uparrow 0} \beta_r(q) = 0 \) and combined with the convexity of \( \beta_r \) and \( \beta_r(1) = 0 \), we infer \( \beta_r(q) = 0 \) for all \( q > 0 \). Therefore, \( q_r = 0 \) for all \( r > 0 \). The case \( q_r > 0 \) follows from the fact that \( \beta_r \) is continuous and decreasing on \( (0, 1) \) with \( \beta_r(1) = 0 \). \( \square \)

Proof of Corollary 1.19. This follows from the convexity of \( \beta_r \) combined with Theorem 1.1.

Proof of Theorem 1.10. By Proposition 3.4, we have \( F_{\beta_r} = q_r \). Hence, we can infer from Theorem 1.1

\[
\frac{rq_r}{1-q_r} = \frac{rF_{\beta_r}}{1-F_{\beta_r}} \leq D_r(\nu) \leq \frac{rq_r}{1-q_r}.
\]

\( \square \)

Proof of Corollary 1.16. This follows from the convexity of \( \beta_r \) combined with Theorem 1.1.

Proof of Corollary 1.12. The corollary follows by combining Theorem 1.1 with Theorem 1.11 and the regularity result on the \( L^q \)-spectrum obtained in [Fen07].

Proof of Proposition 1.17. We only consider the case \( \dim_\nu > 0 \). The case \( \dim_\nu = 0 \) follows along the same lines. Let \( 0 < s < \dim_\nu(\nu) \). Then, for \( n \) large, we have

\[
\max_{Q \in \mathcal{D}_n} \mathbb{E}(Q) \Lambda(Q)^{s/d} \leq 2^{-(s+r)n} < 2^{-(s+r)n+1}.
\]

This implies \( M_{\beta_r}(2^{-(s+r)n+1}) \leq \frac{q_r}{2^{s+r}(0)}. \) Therefore, we obtain

\[
\frac{h_{\beta_r}}{\log (M_{\beta_r}(2^{(s+r)n-1}))} \leq \liminf_{n \to \infty} \frac{\log (M_{\beta_r}(2^{(s+r)n-1}))}{\log (2^{(s+r)n-1})} = \dim_\nu(\nu).
\]

Now, \( s \uparrow \dim_\nu(\nu) \) proves the claim. \( \square \)

Proof of Corollary 1.14. This follows immediately from [Lis14, Corollary 3.10], where the existence of \( \beta_r \) as a limit and \( \beta_r = \max \{ \beta_{r'}, \nu \} \) is shown, combined with Theorem 1.1 and Theorem 1.11.

\[\text{References}\]

[BFP16] O. Bardou, N. Frikha, and G. Pagès. CVaR hedging using quantization-based stochastic approximation algorithm. *Math. Finance* 26.1 (2016), 184–229. doi: 10.1111/mafi.12049.

[BPW10] A. L. Bronstein, G. Pagès, and B. Wilbertz. How to speed up the quantization tree algorithm with an application to swing options. *Quant. Finance* 10.9 (2010), 995–1007. doi: 10.1080/14697680903508487.

[BR56] J. Balatoni and A. Rényi. Remarks on entropy. Hungarian. *Publ. Math. Inst. Hung. Acad. Sci.* 1 (1956), 9–40.

[BS70] M. v. Birman and M. Z. Solomjak. The principal term of the spectral asymptotics for “non-smooth” elliptic problems. *Funkcional. Anal. i Priložen.* 4.4 (1970), 1–13.
[BW82] J. A. Bucklew and G. L. Wise. Multidimensional asymptotic quantization theory with $r$th power distortion measures. *IEEE Trans. Inform. Theory* 28.2 (1982), 239–247. doi: 10.1109/TIT.1982.1056486.

[Del+04] S. Delattre, S. Graf, H. Luschgy, and G. Pagès. Quantization of probability distributions under norm-based distortion measures. *Statist. Decisions* 22.4 (2004), 261–282. doi: 10.1524/stand.22.4.261.64314.

[DeV87] R. A. DeVore. “A note on adaptive approximation”. *Proceedings of China-U.S. Joint Conference on Approximation Theory* (Hangzhou, 1985). Vol. 3. 4. 1987, pp. 74–78.

[ENLP22] R. El Nmeir, H. Luschgy, and G. Pagès. New approach to greedy vector quantization. *Bernoulli* 28.1 (2022), 424–452. doi: 10.3150/21-bej1350.

[Fen07] D.-J. Feng. Gibbs properties of self-conformal measures and the multifractal formalism. *Ergodic Theory Dynam. Systems* 27.3 (2007), 787–812. doi: 10.1017/S0143385706000952.

[FPS19] L. Fiorin, G. Pagès, and A. Sagna. Product Markovian quantization of a diffusion process with applications to finance. *Methodol. Comput. Appl. Probab.* 21.4 (2019), 1087–1118. doi: 10.1007/s11009-018-9652-1.

[GL00a] S. Graf and H. Luschgy. Asymptotics of the quantization errors for self-similar probabilities. *Real Anal. Exchange* 26.2 (2000/01), 795–810.

[GL00b] S. Graf and H. Luschgy. *Foundations of quantization for probability distributions*. Vol. 1730. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2000, pp. x+230. doi: 10.1007/BFb0103945.

[GL02] S. Graf and H. Luschgy. The quantization dimension of self-similar probabilities. *Math. Nachr.* 241 (2002), 103–109. doi: 10.1002/1522-2616(200207)241:1<103::AID-MANA103>3.0.CO;2-J.

[GL04] S. Graf and H. Luschgy. Quantization for probability measures with respect to the geometric mean error. *Math. Proc. Cambridge Philos. Soc.* 136.3 (2004), 687–717. doi: 10.1017/S0305004103007229.

[GL97] S. Graf and H. Luschgy. The quantization of the Cantor distribution. *Math. Nachr.* 183 (1997), 113–133. doi: 10.1002/mana.19971830108.

[Gra02] S. Graf. The quantization of probability distributions: Some results and open problems. 26th Summer Symposium Conference, suppl. Report on the Summer Symposium in Real Analysis XXVI. 2002, pp. 87–102.

[HKY00] Y. Hu, K. A. Kopotun, and X. M. Yu. Modified adaptive algorithms. *SIAM J. Numer. Anal.* 38.3 (2000), 1013–1033. doi: 10.1137/S0036142999353569.

[Hof+14] M. Hoffmann, M. Labadie, C.-A. Lehalle, G. Pagès, H. Pham, and M. Rosenbaum. Optimization and statistical methods for high frequency finance. *Congrès SMAI 2013*. Vol. 45. ESAIM Proc. Surveys. EDP Sci., Les Ulis, 2014, pp. 219–228. doi: 10.1051/proc/201445022.

[HP83] H. G. E. Hentschel and I. Procaccia. The infinite number of generalized dimensions of fractals and strange attractors. *Phys. D* 8.3 (1983), 435–444. doi: 10.1016/0167-2789(83)90235-X.

[Hut81] J. E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.* 30.5 (1981), 713–747. doi: 10.1512/iiumj.1981.30.30055.

[JP22] B. Jourdain and G. Pagès. Quantization and martingale couplings. *ALEA Lat. Am. J. Probab. Math. Stat.* 19.1 (2022), 1–22. doi: 10.30757/alea.v19-01.
[KN22a] M. Kesseböhmer and A. Niemann. Approximation order of Kolmogorov diameters via $L^q$-spectra and applications to polyharmonic operators. *J. Funct. Anal.* 283.7 (2022), Paper No. 109598, 22. doi: 10.1016/j.jfa.2022.109598.

[KN22b] M. Kesseböhmer and A. Niemann. Spectral asymptotics of Kreĭn-Feller operators for weak Gibbs measures on self-conformal fractals with overlaps. *Adv. Math.* 403 (2022), Paper No. 108384. doi: 10.1016/j.aim.2022.108384.

[KN22c] M. Kesseböhmer and A. Niemann. Spectral dimensions of Kreĭn–Feller operators and $L^q$-spectra. *Adv. Math.* 399 (2022), Paper No. 108253. doi: 10.1016/j.aim.2022.108253.

[KN22d] M. Kesseböhmer and A. Niemann. Spectral dimensions of Kreĭn–Feller operators in higher dimensions. *arXiv* (2022). doi: 10.48550/arXiv.2202.05247.

[KZ07] M. Kesseböhmer and S. Zhu. Stability of quantization dimension and quantization for homogeneous Cantor measures. *Math. Nachr.* 280.8 (2007), 866–881. doi: 10.1002/mana.200410519.

[KZ15] M. Kesseböhmer and S. Zhu. Some recent developments in quantization of fractal measures. *Fractal geometry and stochastics V*. Vol. 70. Progr. Probab. Birkhäuser/Springer, Cham, 2015, pp. 105–120. doi: 10.1007/978-3-319-18660-3_7.

[KZ16] M. Kesseböhmer and S. Zhu. On the quantization for self-affine measures on Bedford-McMullen carpets. *Math. Z.* 283.1-2 (2016), 39–58. doi: 10.1007/s00209-015-1588-3.

[KZ17] M. Kesseböhmer and S. Zhu. The upper and lower quantization coefficient for Markov-type measures. *Math. Nachr.* 290.5-6 (2017), 827–839. doi: 10.1002/mana.201500328.

[Lin01] L. J. Lindsay. “Quantization dimension for probability distributions”. PhD thesis. University of North Texas, 2001, p. 77.

[Lis14] P. Liszka. The $L^q$ spectra and Rényi dimension of generalized inhomogeneous self-similar measures. *Cent. Eur. J. Math.* 12.9 (2014), 1305–1319. doi: 10.2478/s11533-014-0414-1.

[LM02] L. J. Lindsay and R. D. Mauldin. Quantization dimension for conformal iterated function systems. *Nonlinearity* 15.1 (2002), 189–199. doi: 10.1088/0951-7715/15/1/309.

[LN98] K.-S. Lau and S.-M. Ngai. $L^q$-spectrum of the Bernoulli convolution associated with the golden ratio. *Studia Math.* 131.3 (1998), 225–251.

[NX19] S.-M. Ngai and Y. Xie. $L^q$-spectrum of self-similar measures with overlaps in the absence of second-order identities. *J. Aust. Math. Soc.* 106.1 (2019), 56–103. doi: 10.1017/S1446788718000034.

[OS07] L. Olsen and N. Snigireva. $L^q$ spectra and Rényi dimensions of in-homogeneous self-similar measures. *Nonlinearity* 20.1 (2007), 151–175. doi: 10.1088/0951-7715/20/1/010.

[OS08] L. Olsen and N. Snigireva. Multifractal spectra of in-homogenous self-similar measures. *Indiana Univ. Math. J.* 57.4 (2008), 1789–1843. doi: 10.1512/iiumj.2008.57.3622.

[Pag15] G. Pagès. “Introduction to optimal vector quantization and its applications for numerics. CEMRACS 2013-modelling and simulation of complex systems: Stochastic and deterministic approaches”. ESAIM, 2015.

[Pöt01] K. Pötzelberger. The quantization dimension of distributions. *Math. Proc. Cambridge Philos. Soc.* 131.3 (2001), 507–519. doi: 10.1017/S0305004101005357.
[PS00] Y. Peres and B. Solomyak. Existence of $L^q$ dimensions and entropy dimension for self-conformal measures. *Indiana Univ. Math. J.* 49.4 (2000), 1603–1621. doi: 10.1512/iumj.2000.49.1851.

[PW12] G. Pagès and B. Wilbertz. Optimal Delaunay and Voronoi quantization schemes for pricing American style options. *Numerical methods in finance.* Vol. 12. Springer Proc. Math. Springer, Heidelberg, 2012, pp. 171–213. doi: 10.1007/978-3-642-25746-9_6.

[Roy13] M. K. Roychowdhury. Quantization dimension estimate of inhomogeneous self-similar measures. *Bull. Pol. Acad. Sci. Math.* 61.1 (2013), 35–45. doi: 10.4064/ba61-1-5.

[Zad82] P. L. Zador. Asymptotic quantization error of continuous signals and the quantization dimension. *IEEE Trans. Inform. Theory* 28.2 (1982), 139–149. doi: 10.1109/TIT.1982.1056490.

[Zhu11] S. Zhu. Asymptotic uniformity of the quantization error of self-similar measures. *Math. Z.* 267.3–4 (2011), 915–929. doi: 10.1007/s00209-009-0653-1.

[Zhu12] S. Zhu. A note on the quantization for probability measures with respect to the geometric mean error. *Monatsh. Math.* 167.2 (2012), 291–305. doi: 10.1007/s00605-011-0307-3.

[Zhu15a] S. Zhu. Convergence order of the geometric mean errors for Markov-type measures. *Chaos Solitons Fractals* 71 (2015), 14–21. doi: 10.1016/j.chaos.2014.11.015.

[Zhu15b] S. Zhu. The quantization for in-homogeneous self-similar measures with inhomogeneous open set condition. *Internat. J. Math.* 26.5 (2015), 1550030, 23. doi: 10.1142/S0129167X15500305.

[Zhu16] S. Zhu. Asymptotics of the quantization errors for in-homogeneous self-similar measures supported on self-similar sets. *Sci. China Math.* 59.2 (2016), 337–350. doi: 10.1007/s11425-015-5045-x.

[Zhu18] S. Zhu. Asymptotic order of the quantization errors for a class of self-affine measures. *Proc. Amer. Math. Soc.* 146.2 (2018), 637–651. doi: 10.1090/proc/13756.

[Zhu20] S. Zhu. Asymptotic uniformity of the quantization for the Ahlfors-David probability measures. *Sci. China Math.* 63.6 (2020), 1039–1056. doi: 10.1007/s11425-017-9436-4.

[ZZ21] S. Zhu and Y. Zhou. On the optimal Voronoi partitions for Ahlfors-David measures with respect to the geometric mean error. *J. Math. Anal. Appl.* 498.2 (2021), Paper No. 124897, 20. doi: 10.1016/j.jmaa.2020.124897.

[ZZZS16] S. Zhu, Y. Zhou, and Y. Sheng. Asymptotics of the geometric mean error in the quantization for inhomogeneous self-similar measures. *J. Math. Anal. Appl.* 434.2 (2016), 1394–1418. doi: 10.1016/j.jmaa.2015.09.081.

[ZZZS17] S. Zhu, Y. Zhou, and Y. Sheng. Exact convergence order of the $L_r$-quantization error for Markov-type measures. *Chaos Solitons Fractals* 98 (2017), 152–157. doi: 10.1016/j.chaos.2017.03.020.