The Hopf Bifurcation Theorem in Hilbert Spaces for Abstract Semilinear Equations

Tadashi Kawanago

Received: 12 March 2021 / Revised: 12 March 2021 / Accepted: 14 October 2021 / Published online: 9 November 2021
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2021

Abstract
We prove a Hopf bifurcation theorem in Hilbert spaces for abstract semilinear equations, which improves a classical result by Crandall and Rabinowitz in the case where basic spaces are Hilbert spaces. Actually, our theorem does not need any compactness conditions, which leads to wider applications. In particular, our theorem can be applied to semilinear equations in unbounded domains of $\mathbb{R}^n$.

Keywords Hopf bifurcation theorem · Abstract semilinear equations · Hilbert spaces · Compactness condition

1 Introduction
Concerning the Hopf bifurcation theorems in infinite dimensions, a lot of versions have been proved until now (see e.g. [1,4,5,11,13] and the references therein). Among them [4, Theorem 1.11] by Crandall and Rabinowitz is one of most important results. It is a theorem for abstract semilinear equations and has been well applied so far to various studies because of its generality (see e.g. [5,14]). It needs, however, some compactness condition, and, consequently, can not be applied to partial differential equations in unbounded domains of $\mathbb{R}^n$.

On the other hand, Hopf bifurcation in partial differential equations in unbounded domain of $\mathbb{R}^n$ has been studied more recently and Hopf bifurcation theorems applicable to such studies were proven (see e.g. [2,10,12]). As far as the author knows, although each of them can be applied to a specific type of equations, it does not seem to have generality applicable to various studies.

In this paper we prove a general Hopf bifurcation theorem for abstract nonlinear equations which does not need any compactness conditions and is applicable to semilinear differential equations in unbounded domains of $\mathbb{R}^n$. Actually, our theorem can be considered to be an improvement of [4, Theorem 1.11] in the case where basic spaces are Hilbert spaces.

Tadashi Kawanago

tadashi@cc.saga-u.ac.jp

1 Faculty of Education, Saga University, Saga 840-8502, Japan
We consider the next abstract semilinear equation in Hilbert spaces:
\[ u_t = Au + h(\lambda, u), \] (1.1)
where the linear operator \( A \) and the map \( h \) are described in Sect. 2 below.

The assumptions of our main theorem (Theorem 2.2 below) are weaker than those of \([4, \text{Theorem 1.11}]\). Actually, our result has the following features:

- We do not assume that \( A \) generates a \( C_0 \)-semigroup.
- We do not assume that \( A \) has compact resolvents.

These features contribute to wider applications (see Sect. 5 below). More concretely, the former has merit since Hopf bifurcation can occur even for the case where \( A \) is not densely defined (see \([11]\) and the references therein), for example. The latter feature makes it possible to apply our Theorem 2.2 to semilinear equations in unbounded domains of \( \mathbb{R}^n \). Actually, we treat the Cauchy problem for a system of semilinear heat equations as a concrete example in Sect. 5 below.

The plan of our paper is the following. In Sect. 2 we describe our main results and discuss the features of our results. In Sect. 3 we describe some preliminary results to prove our main results. We prove our main result in Sect. 4. In Sect. 5 we present some concrete examples.

## 2 The Hopf Bifurcation Theorems in Infinite Dimensions

In this section we present a new bifurcation theorem (Theorem 2.2 below), which is an infinite dimensional version of the classical Hopf bifurcation theorem.

Let \( V \) be a real Banach space and \( V_c = V + iV \) be its complexification. Let \( A \) be a closed linear operator on \( V \) with a bounded inverse \( A^{-1} \). We denote its domain by \( \mathcal{D}(A) \), range by \( \mathcal{R}(A) \), null space by \( \mathcal{N}(A) \) and the extension of \( A \) on \( V_c \) by \( A_c \). If \( W \) is another Banach space, \( \mathcal{L}(V, W) \) denotes the set of bounded linear operators from \( V \) to \( W \). We simply write \( \mathcal{L}(V) := \mathcal{L}(V, V) \).

We consider the Eq. (1.1). First we describe a known result \([4, \text{Theorem 1.11}]\). We assume the following conditions (H1)–(H4).

(H1) The operator \( A \) is the generator of a \( C_0 \)-semigroup on \( V \).
(H2) \( \exp(tA_c) \) is a holomorphic semigroup on \( V_c \).
(H3) The resolvent \( (z - A_c)^{-1} \) is compact for any \( z \in \rho(A_c) \).

It follows from (H1) and (H2) that if \( r > \text{Re}z \) for all \( z \in \sigma(A_c) \), then the fractional powers \( (r - A)\alpha \) are defined for \( \alpha \geq 0 \). We can define the Banach space \( V_\alpha \subset V \) with norms \( \| \cdot \|_\alpha \) by \( V_\alpha := \mathcal{D}((r - A)^\alpha) \) with \( \| v \|_\alpha := \| (r - A)^\alpha v \|_V \) for \( v \in V_\alpha \).

(H4) There exist an \( \alpha \in [0, 1) \) and an open neighborhood \( \Omega \) of \((0, 0)\) in \( \mathbb{R} \times V_\alpha \) such that \( h \in C^2(\Omega, V) \). Moreover, \( h_t(0, 0) = 0 \) and \( h(\lambda, 0) = 0 \) if \((\lambda, 0) \in \Omega \).

We also assume the following (B1)–(B3):

(B1) \( \pm i \) are the simple eigenvalues of \( A_c \), i.e.
\[
\begin{align*}
\dim \mathcal{N}(i - A_c) &= 1 = \text{codim} \mathcal{R}(i - A_c), \\
\psi \in \mathcal{N}(i - A_c) - \{0\} &\implies \psi \notin \mathcal{R}(i - A_c).
\end{align*}
\]
So, by the implicit function theorem, $A_c + h_u(\lambda, 0)$ has an eigenvalue $\mu(\lambda) \in \mathbb{C}$ and eigenfunction $\psi(\lambda) \in \mathcal{D}(A_c)$ corresponding to $\mu(\lambda)$ for any $\lambda$ in a small neighborhood of 0 such that $\mu(0) = i$ and that $\mu(\lambda)$ and $\psi(\lambda)$ are functions of class $C^2$.

(B2) (Transversality condition of eigenvalues) $\Re \mu'(0) \neq 0$.

(B3) $ik \in \rho(A_c)$ for $k \in \mathbb{Z} - \{-1, 1\}$.

**Theorem 2.1** ([4, Theorem 1.11]) We assume (H1)--(H4) and (B1)--(B3). Then, $(\lambda, u) = (0, 0)$ is a Hopf bifurcation point of (1.1).

Here, we omit the description in [4, Theorem 1.11] on the uniqueness of the branch of bifurcating periodic solutions.

Next, we state our new results. We consider the case in which $V$ is a real Hilbert space and $0 \in \rho(A_c)$. We define the real Hilbert space $U := \mathcal{D}(A) \subset V$ with the norm $\|u\|_U := \|Au\|_V$ for $u \in U$.

We set the real Hilbert spaces $X$ and $Y$ by

$$X := H^1_{per}(0, 2\pi), V) \cap L^2((0, 2\pi), U) \quad \text{and} \quad Y := L^2((0, 2\pi), V). \quad (2.1)$$

Here, $H^1_{per}(0, 2\pi), V) := \{u \in H^1((0, 2\pi), V); u(0) = u(2\pi)\}$.

We assume (B1)--(B3) and the following (K1), (K2-1)--(K2-4):

(K1) There exists $M \in (0, \infty)$ such that

$$\|(in - A_c)^{-1}\|_{V_c \to V_c} \leq \frac{M}{n} \quad \text{for} \quad n = 2, 3, 4, \ldots.$$  

(K2-1) There is an open interval $K$ in $\mathbb{R}$ such that $0 \in K$ and $h$ is a map from $K \times U$ to $V$.

For any $(\lambda, u) \in K \times X$, we set $[h(\lambda, u)](t) := h(\lambda, u(t))$ for a.e. $t \in (0, 2\pi)$.

(K2-2) $h(\lambda, u) \in Y$ for any $(\lambda, u) \in K \times X$.

We define the map $\Phi$: $(\lambda, u) \in K \times X \mapsto h(\lambda, u) \in Y$.

(K2-3) $\Phi \in C^2(K \times X, Y)$.

**Remark 2.1** We can regard $U$ (resp. $V$) as the closed subspace of $X$ (resp. $Y$) which consists of constant functions in $X$ (resp. $Y$). Then, we verify that (K2-3) implies $h \in C^2(K \times U, V)$ with

$$[\Phi_u(\lambda, u)v](t) = h_u(\lambda, u(t))v(t) \quad \text{in} \quad V, \quad (2.2)$$

and so on for $\lambda \in K$, $u, v, w \in X$ and a.e. $t \in (0, 2\pi)$.

(K2-4) $h_u(0, 0) = 0$ and $h(\lambda, 0) = 0$ if $\lambda \in K$.

In what follows we simply denote (K2-1)--(K2-4) by (K2). Now, we shortly state our result:

**Proposition 2.1** Let $V$ be a real Hilbert space and $A$ be a closed linear operator on $V$. We assume (K1), (K2) and (B1)--(B3). Then, $(\lambda, u) = (0, 0)$ is a Hopf bifurcation point of (1.1).

**Remark 2.2** The conditions (H1) and (H2) imply (K1). Though we do not assume (H1) in Proposition 2.1, we note that (K1) and (H1) imply (H2).
Proposition 2.1 is a short version of our main result Theorem 2.2 below, which shows that the branch of bifurcating periodic solutions are unique in a neighborhood of $(\lambda, u) = (0, 0)$.

Next, we make preparation to state our main result. Let $m \in \mathbb{Z}$, $n \in \mathbb{N}$ and $u \in V_c$. We write $e_m(t) := e^{imt}$, $c_n(t) := \cos nt$ and $s_n(t) := \sin nt$ for $t \in \mathbb{R}$. We denote $(u \otimes e_m)(t) := ue_m(t) = ue^{imt}$ ($t \in \mathbb{R}$). Similarly, $(u \otimes c_n)(t) := u \cos nt$ and $(u \otimes s_n)(t) := u \sin nt$ ($t \in \mathbb{R}$). We set $X_1 := \{u \otimes c_1 + v \otimes s_1 ; u, v \in U\}$ as a subspace of $X$. We define the translation operator $\tau_{\theta}$ by $(\tau_{\theta}u)(t) := u(t - \theta)$ for any $\theta \in \mathbb{R}$. For simplicity, we set $f(\lambda, u) = Au + h(\lambda, u)$. If $u(t)$ is a $2\pi$-periodic solution of the next equation:

$$u_t = (\sigma + 1)f(\lambda, u) \quad (2.4)$$

then $u(t/(\sigma + 1))$ is a $2\pi(\sigma + 1)$-periodic solution of (1.1).

Our main theorem is the following:

**Theorem 2.2** We assume all conditions in Proposition 2.1. Then, there exist $a, \varepsilon > 0, u_* \in X_1 - \{0\}$ and functions $\zeta = (\lambda, \sigma) \in C^1([0, a), \mathbb{R}^2), \eta \in C^1([0, a), X)$ with the following properties:

(a) $(\lambda, \sigma, u) = (\zeta(\alpha), \alpha u_* + \alpha \eta(\alpha))$ is a solution of (2.4) for any $\alpha \in [0, a)$,

(b) $\zeta(0) = \zeta'(0) = (0, 0)$ and $\eta(0) = 0$,

(c) If $(\lambda, v)$ is a solution of (1.1) of period $2\pi(\sigma + 1)$, $|\lambda| < \varepsilon, |\sigma| < \varepsilon, \tilde{v} \in X$ and $\|\tilde{v}\|_X < \varepsilon$, where $\tilde{v}(t) := v((\sigma + 1)t)$ for $t \in \mathbb{R}$, then there exist $\alpha \in (0, a)$ and $\theta \in [0, 2\pi)$ such that $(\lambda, \sigma) = (\zeta(\alpha) + v((\sigma + 1)t) = \alpha u_* (t + \theta) + \alpha \eta(\alpha)(t + \theta)$ for any $t \in \mathbb{R}$.

**Remark 2.3** (i) In our Theorem 2.2 we do not need the compact resolvent condition (H3). This contributes to wider applications. See concrete examples in Sect. 5.

(ii) We can naturally extend the domain of $\zeta$ and $\eta$ in Theorem 2.2 for $\alpha \in (-a, 0)$ as $C^1$ maps satisfying (a). See the proof of Theorem 2.2 for details. ☐

At the end of this section we explain the feature of our proof of Theorem 2.2, which is described in Sect. 4. The proof of our main result Theorem 2.2 is based on Theorem 3.1 below, which is an abstract bifurcation theorem. The technique of our proof differs from the standard ones used in the proofs of the related known results. In general the derivation of infinite-dimensional bifurcation theorem is so far based mainly on two techniques. One is to analyze directly the infinite-dimensional space. Crandall and Rabinowitz [4] adapted this way by using the semigroup theory. Another is to reduce the problem to a finite-dimensional problem by using the Lyapunov–Schmidt method, the center manifold theorem and so on (see e.g. [9,11]). Accordingly, we reduce our problem to the analysis on two infinite-dimensional spaces by using our Theorem 3.1 mentioned above. Actually we can express $X$ (the real space of $2\pi$-periodic $V$-valued functions defined in (2.1)) as the direct sum of a low-frequency subspace and a high-frequency subspace. We reduce our problem to the analysis on each subspace. Analysis on the low-frequency subspace is as follows: The complex space $V_c$ and the real space of $V$-valued simple harmonic oscillation are isomorphic as real linear spaces (see Proposition 3.1 below). The seemingly difficult points of analysis for the low-frequency subspace can be reduced to the linear algebraic properties of the isomorphism. This analysis is so simple that it always works well without the choice of functional spaces. Indeed it works well even if $V$ is a general Banach space. On the other hand, the analysis on the high-frequency subspace is based on the Fourier analysis. Whether it works well or not seems to depend much on the choice of functional spaces. We need here our technical assumption that $V$ is a Hilbert space.
3 Preliminary Results

To begin with, we describe [8, Theorem 3] for the case \( m = 2 \). The proof of our main theorem (Theorem 2.2) is based on this result.

Let \( \mathcal{X} \) and \( \mathcal{Y} \) be real Banach spaces and \( O \) be an open neighborhood of 0 in \( \mathcal{X} \). Let \( J \) be an open neighborhood of \((0, 0)\) in \( \mathbb{R}^2 \). Let \( g \in C^2(J \times O, \mathcal{Y}) \) be a map such that

\[
g(\Lambda, 0) = 0 \quad \text{for any} \quad \Lambda = (\Lambda_1, \Lambda_2) \in J.
\]

We define \( H : J \times \mathcal{X} \rightarrow \mathbb{R}^2 \times \mathcal{Y} \) by

\[
H \left( \frac{\Lambda}{u} \right) := \left( \frac{l u - e_1}{g(u, 0)u} \right).
\]

Here, \( l = (l^1, l^2) \in \mathcal{L}(\mathcal{X}, \mathbb{R}^2) \) and \( e_1 := (1, 0) \). We define \( G : J \times O \rightarrow \mathbb{R} \times \mathcal{Y} \) by

\[
G \left( \frac{\Lambda}{u} \right) := \left( \frac{l^2 u}{g(\Lambda, u)} \right).
\]

We set \( Z := \{ u \in \mathcal{X} ; l^1 u = 0 \} \).

**Remark 3.1** The system such as \( H(\Lambda, u) = 0 \) and \( G(\Lambda, u) = 0 \) above is called an extended system in general in the field of numerical analysis (see e.g. [6,8]).

**Theorem 3.1** ([8, Theorem 3] for the case \( m = 2 \)) In addition to the assumptions above we assume that there exists \( u_* \in O \) such that

\[
\text{the extended system } H(\Lambda, u) = 0 \text{ has an isolated solution } (\Lambda, u) = (0, u_*).
\]

Then there exist an open neighborhood \( W \) of \((0,0)\) in \( \mathbb{R}^2 \times \mathcal{X} \), \( a \in (0, \infty) \) and functions \( \zeta \in C^1((-a, a), \mathbb{R}^2) \), \( \eta \in C^1((-a, a), Z) \) such that \( \zeta(0) = 0, \eta(0) = 0 \) and

\[
G^{-1}(0) \cap W = \{(\Lambda, 0) ; (\Lambda, 0) \in W\} \cup \{ (\zeta(\alpha), a u_* + a \eta(\alpha)) ; |\alpha| < a \}.
\]

**Remark 3.2** To show that the Hopf bifurcation actually occurs for given concrete examples, Theorem 3.1 is often more practical than Proposition 2.1. See e.g. [8].

In what follows in this section, we use the same notation in Sect. 2. We set \( Y_1 := \{ u \otimes c_1 + v \otimes s_1 ; u, v \in V \} \) as a closed subspace of \( Y \). We define \( L_1 : V_c \rightarrow Y_1 \) by \( L_1 \psi := \text{Re}(\psi \otimes e_1) \) for any \( \psi \in V_c \) and \( T_1 : X_1 \rightarrow Y_1 \) by \( T_1 w := \dot{w} - Aw \) for any \( w \in X_1 \). Namely,

\[
L_1(a + ib) = a \otimes c_1 - b \otimes s_1 \quad \text{for any} \quad a, b \in V, \tag{3.4}
\]

\[
T_1(a \otimes c_1 + b \otimes s_1) = (b - a a) \otimes c_1 - (a + a b) \otimes s_1 \quad \text{for any} \quad a, b \in U. \tag{3.5}
\]

In view of (3.4) the following result clearly holds:

**Proposition 3.1** (i) The operator \( L_1 \) is isomorphic as a real linear operator from \( V_c \) to \( Y_1 \).

Here, we regard \( V_c \) as a real linear space.

(ii) The operator \( L_1|_{U_c} : U_c \rightarrow X_1 \) is isomorphic as a real linear operator from \( U_c \) to \( X_1 \).

Here, we regard \( U_c \) as a real linear space.

**Proposition 3.2** (i) \( L_1 \mathcal{N}(i - A_c) = \mathcal{N}(T_1) \).

(ii) \( L_1 \mathcal{R}(i - A_c) = \mathcal{R}(T_1) \).
We easily verify from Proposition 3.1 (ii) that if \( w \in X_1 \) then there exists a unique \( \psi \in U_c \) such that \( w = L_1 \psi \) and \( T_1 w = L_1 (i - A_c) \psi \).

(i) Let \( w \in L_1 \mathcal{N}(i - A_c) \). Then, \( w \in X_1 \). So, there exists a unique \( \psi \in U_c \) such that \( w = L_1 \psi \), which leads to \( T_1 w = L_1 (i - A_c) \psi \). Since \( L_1 \) is one to one, \( \psi \in \mathcal{N}(i - A_c) \).

Therefore, \( T_1 w = L_1 0 = 0 \). So, \( w \in \mathcal{N}(T_1) \) and \( L_1 \mathcal{N}(i - A_c) \subset \mathcal{N}(T_1) \).

Conversely, let \( w \in \mathcal{N}(T_1) \). Then, there exists a unique \( \psi \in U_c \) such that \( w = L_1 \psi \). It follows that \( L_1 (i - A_c) \psi = T_1 w = 0 \). Since \( L_1 \) is one to one, \( (i - A_c) \psi = 0 \). Therefore, \( \psi \in \mathcal{N}(i - A_c) \) and \( w \in L_1 \mathcal{N}(i - A_c) \). We conclude that \( \mathcal{N}(T_1) \subset L_1 \mathcal{N}(i - A_c) \).

(ii) Simple argument by linear algebra leads to the desired conclusion, as in the proof of (i). So, we leave the proof to the reader. \( \square \)

**Proposition 3.3** Let \( \psi \in U_c \) and \( w = L_1 \psi \).

(i) \( L_1 (i \psi) = \dot{w} \),

(ii) If \( \psi \in \mathcal{N}(i - A_c) \), then \( L_1 (i \psi) = Aw \).

**Proof** (i) \( L_1 (i \psi) = \text{Re} \left[ \frac{d}{dt} (\psi \otimes e_1) \right] = \frac{d}{dt} \text{Re} (\psi \otimes e_1) = \dot{w} \).

(ii) We immediately obtain the desired conclusion from (i) and Proposition 3.2 (i). \( \square \)

**4 Proof of Theorem 2.2**

Let \( X \) and \( Y \) be real Hilbert spaces defined by (2.1). We denote the \( n \)-th Fourier coefficient of \( \varphi \in Y_c \) by

\[
\hat{\varphi}(n) := \frac{1}{2\pi} \int_0^{2\pi} e^{-int} \varphi(t) dt \quad (n \in \mathbb{Z}).
\]

We set

\[
X_0 := U \quad \text{and} \quad X_\infty := \{ \varphi \in X : \hat{\varphi}(n) = 0 \text{ for } n = -1, 0, 1 \}
\]

as closed subspaces of \( X \),

\[
Y_0 := V \quad \text{and} \quad Y_\infty := \{ \varphi \in Y : \hat{\varphi}(n) = 0 \text{ for } n = -1, 0, 1 \}
\]

as closed subspaces of \( Y \). Let \( X_1 \) (resp. \( Y_1 \)) be a closed subspace of \( X \) (resp. \( Y \)) defined in Sect. 2 (resp. Sect. 3).

**Proof of Theorem 2.2** We apply the notation in Sects. 2 and 3. We denote \( \Lambda = (\lambda, \sigma) \in K \times \mathbb{R} \). We define \( g \in C^2(K \times \mathbb{R} \times X, Y) \) by \( g(\Lambda, u) = u_t - (\sigma + 1) f(\lambda, u) \).

We set \( u_* := L_1 \psi_* = \text{Re}(\psi_* \otimes e_1) \in X_1 \). Then, \( Lu_* = (1, 0) = e_1 \). Let \( H : K \times \mathbb{R} \times X \rightarrow \mathbb{R}^2 \times Y \) be the operator defined by (3.1). Then, by (K2-4) and Proposition 3.2
We note that $T_u^{4.2}$, to $\mu(\lambda)$ $f$

If $\alpha < (\lambda, v)$ let (4.1)

reduced to the case: $q$

consider the case: $l$

Then, $\lambda$

where $f_{\lambda u}^0 := f_{\lambda u}(0, 0)$. We verify that $S := DH^*|_{\mathbb{R}^2 \otimes X_0 + X_1} : \mathbb{R}^2 \otimes X_0 + X_1 \to \mathbb{R}^2 \otimes Y_0 + Y_1$ and $T := DH^*|_{X_0} : X_0 \to Y_0$ are well-defined by Remark 2.1 and that $DH^* = S \oplus T$. We note that $Tu = u_i - Au$ for any $u \in X_0$. In view of the below Lemmas 4.1 and 4.2, $DH^*$ is bijective. So, by Theorem 3.1 $(\lambda, u) = (0, 0)$ is a Hopf bifurcation point and there exist an open neighborhood $W$ of $(0, 0)$ in $\mathbb{R}^2 \times X$, $a \in \mathbb{R}^2$, and functions $\xi \in C^1((-a, a), \mathbb{R}^2)$, $\eta \in C^1((-a, a), Z)$ such that $\xi(0) = 0$, $\eta(0) = 0$ and (3.3) holds. Here, $Z := \{u \in X ; l_i u = 0\}$. So, (a) holds.

Next, we show the following (4.5) in preparation to prove (b) and (c).

\[
\xi(-\alpha) = \xi(\alpha) \quad \text{and} \quad \eta(-\alpha) = -\tau_\pi(\eta(\alpha)) \quad \text{for any} \quad \alpha \in [0, a).
\]

(4.5)

We set $U(\alpha) := \alpha u_* + \alpha \eta(\alpha) \in X$ for any $\alpha \in (-a, a)$. We define $V(\alpha) \in X$ by $V(\alpha) := \tau_\pi(U(\alpha))$. Let $\gamma \in (0, a)$ be a constant such that $(\xi(\alpha), V(\alpha)) : \alpha \in [0, \gamma] \subset W$. Then, $(\xi(\alpha), V(\alpha)) \in G^{-1}(0) \cap W$ for any $\alpha \in [0, \gamma)$. So, by Theorem 3.1 for any $\alpha \in [0, \gamma)$ there exists $\beta \in (-a, a)$ such that $(\xi(\alpha), V(\alpha)) = (\xi(\beta), U(\beta))$. On the other hand, $l_i V(\alpha) = -\alpha$ and $l_i U(\beta) = \beta$. Therefore, $(\xi(-\alpha), U(-\alpha)) = (\xi(\alpha), V(\alpha))$ for any $\alpha \in [0, \gamma)$. Actually, we easily verify from commonly used argument by contradiction that

\[
a = \sup \{q \in (0, a) ; (\xi(-\alpha), U(-\alpha)) = (\xi(\alpha), V(\alpha)) \quad \text{for any} \quad \alpha \in [0, q]\}
\]

which implies (4.5).

By (4.5), $\xi'(0) = 0$. So, (b) holds. Finally, we show (c). Let $\epsilon$ be a positive constant such that if $(\lambda, \sigma, v) \in \mathbb{R}^2 \times X$ satisfies $|\lambda| < \epsilon$, $|\sigma| < \epsilon$ and $\|v\|_X < \epsilon$ then $(\lambda, \sigma, v) \in W$. Now, let $(\lambda, v)$ be a solution of (1.1) of period $2\pi(\sigma + 1)$, $|\lambda| < \epsilon$, $|\sigma| < \epsilon$, $\tilde{v} \in X$ and $\|\tilde{v}\|_X < \epsilon$, where $\tilde{v}(t) := v((\sigma + 1)t)$ for $t \in \mathbb{R}$. For simplicity, we set $(p, q) := l \tilde{v} = (l_1^{\circ} \tilde{v}, l_2^{\circ} \tilde{v})$. First we consider the case: $q = 0$. Then $(\lambda, \sigma, \tilde{v}) \in W$ is a solution of $G(\Lambda, u) := (l_i^2 u, g(\Lambda, u)) = 0$. By Theorem 3.1 there exists $\alpha \in (-a, a)$ such that $(\lambda, \sigma, \tilde{v}) = (\xi(\alpha), \tilde{v} = \alpha u_* + \alpha \eta(\alpha)$. If $\alpha < 0$ then $\tilde{v} = \tau_\pi((-\alpha)u_* + (-\alpha)\eta(-\alpha))$ in view of (4.5) and $\tau_\pi u_* = -u_*$. Next, we consider the case: $q \neq 0$. There exists $\theta \in (0, 2\pi)$ such that $e^{i\theta} = (p - iq)/\sqrt{p^2 + q^2}$. Then, $l_1^{\circ} \tau_\theta \tilde{v} = 0$ and $(\lambda, \sigma, \tau_\theta \tilde{v}) \in W$ is a solution of $G(\Lambda, u) = 0$. So, the present case is reduced to the case: $q = 0$. Therefore, (c) holds.

In the above proof, we use the following two lemmas:

Lemma 4.1 The operator $S$ is bijective.

Lemma 4.2 The operator $T$ is bijective.

Proof of Lemma 4.1 By (B1), Remark 2.1 and the implicit function theorem (see e.g. [3, Theorem A1]) $f_u(\lambda, 0)$ has an eigenvalue $\mu(\lambda) \in \mathbb{C}$ and an eigenfunction $\psi(\lambda) \in U_c$ corresponding to $\mu(\lambda)$ for any $\lambda$ in a small open interval $K_1$ such that $0 \in K_1 \subset K$, $\mu(0) = i$, $\psi(0) = \psi_*$, $\mu(\cdot) \in C^2(K_1, \mathbb{C})$ and $\psi(\cdot) \in C^2(K_1, U_c)$. Differentiating $f_u(\lambda, 0)\psi(\lambda) = \mu(\lambda)\psi(\lambda)$ with respect to $\lambda$, we have

\[
\mu'(0)\psi_* + i\psi'(0) = f_{\lambda u}^0 \psi_* + A_c \psi'(0).
\]

(4.6)
We set $p := \text{Re} \mu'(0) \neq 0$ by (B2), $q = \text{Im} \mu'(0)$ and $u_2 := L_1 \psi'(0) \in X_1$. It follows from (4.6) and Proposition 3.3 that
\[ f^0_{x,a} u_* = p u_* + q A u_* + T_1 u_2. \] (4.7)

Let $u_0 \in X_0$, $u_1 \in X_1$ and $u = u_0 + u_1$. In view of (4.4) and (4.7), we have
\[ S \begin{pmatrix} \lambda, \sigma \\ u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 - \lambda u_* \\ -A u_0 \end{pmatrix} + \lambda p u_* - (\sigma + \lambda q) A u_*. \] (4.8)

By (B1), we have $\mathcal{R}(i - A_c) \oplus \text{span}\{u_*\} = V_c$. It follows from Proposition 3.1 (i), Proposition 3.2 (ii) and Proposition 3.3 that
\[ \mathcal{R}(T_1) \oplus \text{span}\{u_*, A u_*\} = Y_1. \] (4.9)

First, we show that $S$ is one to one. Let $S(\lambda, \sigma, u) = 0$. It follows from (B2), (4.8), (4.9) and $0 \in \rho(A)$ that $u_0 = 0$, $\lambda = \sigma = 0$,
\[ l u_1 = 0 \quad \text{and} \quad T_1 u_1 = 0. \] (4.10)

Let $\psi_1 := L_1^{-1} u_1 \in U_c$. Then by (4.10) and Proposition 3.2 (i),
\[ \psi_1 \in N(i - A_c) \quad \text{and} \quad (d, \psi_1)_{U_c} = 0. \] (4.11)

It follows from (4.11), (B1) and $(d, \psi_*)_{U_c} = 1$ that $\psi_1 = 0$, which implies $u_1 = 0$. So, $S$ is one to one.

Next, we show that $S$ is onto. Let $(a, b, y_0, y_1) \in \mathbb{R}^2 \oplus Y_0 \oplus Y_1$. In view of $0 \in \rho(A)$, there exists $x_0 \in X_0$ such that $-A x_0 = y_0$. By (4.9) there exist $w \in \mathcal{R}(T_1)$ and $(\gamma, \delta) \in \mathbb{R}^2$ such that
\[ w + \gamma u_* + \delta A u_* = y_1. \] (4.12)

We set $\lambda_0 := -\gamma / p$ and $\sigma_0 := -\delta + \gamma q / p$. There exists $v_1 \in X_1$ such that $T_1 (v_1 - \lambda_0 u_2) = w$. Let $(\alpha, \beta) := lu_1 \in \mathbb{R}^2$ and $x_1 := v_1 + (a - \alpha) u_* + (\beta - b) A u_*$. By Proposition 3.2 (i) and Proposition 3.3 (ii), we have $A u_* = L_1 (i \psi_* \in N(T_1))$. So, $l A u_* = (0, -1)$. It follows from $l u_* = e_1$, Proposition 3.2 (i), (4.8) and (4.12) that $S(\lambda_0, \sigma_0, x_0, x_1) = (a, b, y_0, y_1)$. Therefore, $S$ is onto.

**Proof of Lemma 4.2** Let $\mathcal{X} := (X_\infty)_c$ and $\mathcal{Y} := (Y_\infty)_c$ (i.e. $\mathcal{X}$ and $\mathcal{Y}$ be the complexification of $X_\infty$ and $Y_\infty$, respectively.) It suffices to show that $T_{c*} : \mathcal{X} \to \mathcal{Y}$ is bijective. Let $z \in \mathcal{Y}$. Then, $z = \sum_{|n| \geq 2} p_n \otimes e_n$ in $\mathcal{Y}$, where $p_n := \hat{z}(n)$. It suffices to show that the following Eq. (4.13) has a unique solution in $\mathcal{X}$:
\[ u_t - A_c u = z \] (4.13)

If a solution of (4.13) exists, we obtain formally from the Fourier analysis that $u = \sum_{|n| \geq 2} q_n \otimes e_n$, where $q_n := (i n - A_c)^{-1} p_n$. The proof is complete if we show $u \in \mathcal{X}$, i.e.
\[ \sum_{|n| \geq 2} \left( |n|^2 \| q_n \|_{V_c}^2 + \| q_n \|_{U_c}^2 \right) < \infty. \] (4.14)

It follows from (K1) that
\[ |n| \| q_n \|_{V_c} \leq |n| \|(i n - A_c)^{-1} \|_{V_c \rightarrow V_c} \| p_n \|_{V_c} \leq M \| p_n \|_{V_c}, \] (4.15)
\[ \| q_n \|_{U_c} = \| A_c (i n - A_c)^{-1} p_n \|_{V_c} = \| i n (i n - A_c)^{-1} p_n - p_n \|_{V_c} \leq (M + 1) \| p_n \|_{V_c}. \] (4.16)
By (4.15), (4.16) and Parseval’s identity we have (4.14). □

5 Examples

In this section we freely use the notation used in Sect. 4.

Example 1 We consider the following Cauchy problem:

\[\begin{align*}
\frac{d}{dx} u &= -u_{xx} - v - \rho u + u(\lambda \kappa^2 - u^2 - v^2) \quad \text{for} \quad (x,t) \in \mathbb{R} \times [0,\infty), \\
\frac{d}{dx} v &= v_{xx} + u - \rho v + v(\lambda \kappa^2 - u^2 - v^2) \quad \text{for} \quad (x,t) \in \mathbb{R} \times [0,\infty),
\end{align*}\]

Here, \(\rho\) and \(\kappa\) are functions on \(\mathbb{R}\) defined by \(\rho(x) := \frac{1}{4} \frac{2 \tanh^2(x/2) - 1}{\kappa(x)}\) and \(\kappa(x) := \text{sech}(x/2)\).

For the Eq. (5.1) the branch of periodic solutions \((u, v) = (u_\lambda, v_\lambda) (\lambda > 0)\) bifurcates at \(\lambda = 0\) from the branch of trivial solutions. Here, \(u_\lambda(x,t) := \sqrt{\lambda} \kappa(x) \cos t\) and \(v_\lambda(x,t) := \sqrt{\lambda} \kappa(x) \sin t\).

We can not apply [4, Theorem 1.11] to (5.1) since the linear operator in (5.1) does not have compact resolvents. We show in what follows that we can apply our Proposition 2.1 to (5.1) to verify the above Hopf bifurcation.

Let \(V := L^2(\mathbb{R}) \times L^2(\mathbb{R})\). We define \(A : V \to V\) and \(A_0 : V \to V\) by the following:

\[A\left(\begin{pmatrix} \phi \\ \psi \end{pmatrix}\right) := \begin{pmatrix} \psi_{xx} - \psi - \rho \phi \\ \psi_{xx} + \phi - \rho \psi \end{pmatrix} \quad \text{for} \quad (\phi, \psi) \in D(A) := H^2(\mathbb{R}) \times H^2(\mathbb{R}), \]

\[A_0\left(\begin{pmatrix} \phi \\ \psi \end{pmatrix}\right) := \begin{pmatrix} \psi_{xx} - \psi \\ \psi_{xx} + \phi \end{pmatrix} \quad \text{for} \quad (\phi, \psi) \in D(A_0) := H^2(\mathbb{R}) \times H^2(\mathbb{R}), \]

\[A_\infty\left(\begin{pmatrix} \phi \\ \psi \end{pmatrix}\right) := \begin{pmatrix} \phi_{xx} - \phi - 4/4 \\ \psi_{xx} + \phi - \psi - 4/4 \end{pmatrix} \quad \text{for} \quad (\phi, \psi) \in D(A_\infty) := H^2(\mathbb{R}) \times H^2(\mathbb{R}), \]

We set \(U := D(A) = H^2(\mathbb{R}) \times H^2(\mathbb{R})\). By the Sobolev embedding theorem, \(H^1(\mathbb{R})\) is embedded in \(L^6(\mathbb{R})\). So, we can well define the map \(h : \mathbb{R} \times U \to V\) by

\[h(\lambda, \phi) := \begin{pmatrix} \phi_{xx} - \phi - 4/4 \\ \psi_{xx} + \phi - \psi - 4/4 \end{pmatrix} \quad \text{for} \quad \lambda \in \mathbb{R} \quad \text{and} \quad \phi(\psi) \in U. \quad (5.2)\]

Thus, (K2 - 1) holds.

We explain the correspondence between the present example and the description in Sect. 4. We define \(u_\ast \in X_1\) by \(u_\ast(x,t) := (\kappa(x) \cos t, \kappa(x) \sin t)\) and \(\psi_\ast \in U_\ast\) by

\[\psi_\ast := L^{-1}_0(u_\ast) = (\kappa, -i\kappa). \quad (5.3)\]

We verify that \(u_\ast \in N(T_1)\) and \(\psi_\ast \in N(A_c - i)\) (see Proposition 3.2 (i)). We set \(\lambda(\alpha) = \alpha^2\), \(\sigma(\alpha) \equiv 0\) and \(\eta(\alpha) \equiv (0,0)\) for \(\alpha \in \mathbb{R}\). Then, we verify that \((\lambda, \sigma, u) = (\lambda(\alpha), \sigma(\alpha), \alpha u_\ast + \alpha \eta(\alpha))\) is a solution of (2.4).

In order to verify the applicability of our Proposition 2.1 to (5.1) we describe the outline of the derivation of (K1), (K2), (B1)–(B3) in what follows.

Derivation of (K2)

We showed that (K2 - 1) holds. Let \(V\) be a Hilbert space. We set \(X := H^1((0,2\pi), L^2(\mathbb{R})) \cap L^2((0,2\pi), H^1(\mathbb{R}))\) and \(Y := L^2((0,2\pi), L^2(\mathbb{R}))\). We use the next embedding inequality
(5.4) from $H^1((0, 2\pi), \mathcal{V})$ into $C([0, 2\pi], \mathcal{V})$ and the Gagliardo-Nirenberg interpolation inequality (5.5):

\[
\|u\|_{C([0,2\pi],\mathcal{V})} \leq C_1 \|u\|_{H^1((0,2\pi),\mathcal{V})} \quad \text{for any } u \in H^1((0,2\pi),\mathcal{V}),
\]

\[
\|\phi\|_{L^6(\mathbb{R})} \leq \|\phi\|_{L^{2/3}(\mathbb{R})}^{2/3} \quad \text{for any } \phi \in H^1(\mathbb{R}).
\]

Here, $C_1 > 0$ is a certain constant independent of $u$. It follows from (5.5), (5.4) and Hölder’s inequality that if $f, g, h \in \mathcal{X}$ then $fgh \in \mathcal{Y}$ with the estimate

\[
\|fgh\|_{\mathcal{Y}} \leq C_2 \|f\|_{\mathcal{X}} \|g\|_{\mathcal{X}} \|h\|_{\mathcal{X}} \quad \text{for any } f, g, h \in \mathcal{X}.
\]

Here, $C_2 := C_1^{2/3} > 0$. In view of (5.6), we have (K2 - 2). In the present case the map $\Phi: \mathbb{R} \times \mathcal{X} \to \mathcal{Y}$ is defined by

\[
\Phi(\lambda, u) := \left( u(\lambda k^2 - u^2 - v^2) \right) \quad \text{for } \lambda \in \mathbb{R} \text{ and } u = (u, v) \in \mathcal{X}.
\]

We omit the derivation of (K2 - 3) since it is not difficult by using (5.6). Finally, (K2 - 4) clearly holds.

Next, we make preparation to derive (B3) and (K1):

**Lemma 5.1** The following holds:

(i) $k = 0$ or $|k| \geq 2 \implies ik \in \rho(A_{0c})$ and $\|(A_{0c} - ik)^{-1}\|_{\mathcal{V}_c \to \mathcal{V}_c} \leq 1$.

(ii) $|k| \geq 4 \implies \|(A_{0c} - ik)^{-1}\|_{\mathcal{V}_c \to \mathcal{V}_c} \leq \frac{\sqrt{2}}{|k|}$.

**Proof** Since $A_0$ is a real operator, it suffices to prove (i) and (ii) only for $k \geq 0$.

(i) Let $k = 0$ or $k \geq 2$. We consider the following equation:

\[
(A_{0c} - ik)(\phi, \psi) = (\gamma, \omega).
\]

For any given $(\gamma, \omega) \in \mathcal{V}_c$ this equation has a unique solution $(\phi, \psi) \in U_c$ such that

\[
\hat{\phi}(\xi) = \frac{-i(\xi^2 + ik)\hat{\gamma}(\xi) + \hat{\omega}(\xi)}{1 + (\xi^2 + ik)^2} \quad \text{and} \quad \hat{\psi}(\xi) = \frac{-i(\xi^2 + ik)\hat{\gamma}(\xi) + \hat{\omega}(\xi)}{1 + (\xi^2 + ik)^2}.
\]

By Schwarz inequality

\[
|\hat{\phi}(\xi)|^2 + |\hat{\psi}(\xi)|^2 \leq \|J(k, \xi)\| \{ |\hat{\gamma}(\xi)|^2 + |\hat{\omega}(\xi)|^2 \}, \quad \text{where} \quad J(k, \xi) := \frac{\xi^4 + (k + 1)^2}{\xi^8 + 2(k^2 + 1)\xi^4 + (1 - k^2)^2}.
\]

We verify that $J(k, \xi) \leq 1$ for any $\xi \in \mathbb{R}$. So, we have the desired result.

(ii) Let $k \geq 4$. Then, we verify that $J(k, \xi) \leq 2/k^2$ for any $\xi \in \mathbb{R}$. By this and (5.10), we have the desired result. □

**Derivation of (B3) and (K1)**

We define $B \in \mathcal{L}(\mathcal{V})$ by $B(\phi, \psi) = -(\rho \phi, \rho \psi)$ for $(\phi, \psi) \in \mathcal{V}$. Then, we have $B_c = (A_c - ik) - (A_{0c} - ik) \in \mathcal{D}(A_{0c}) (= \mathcal{D}(A_c))$. By Lemma 5.1 (i) we have $ik \in \rho(A_{0c})$ and $\|B_c\|\|(A_{0c} - ik)^{-1}\| \leq 1/4$ for any $k \in \mathbb{Z} - \{\pm 1\}$. So, (B3) holds in view of stability property of bounded inverse operator (see e.g. [7, Corollary 2.4.1]).
Next, by Lemma 5.1 (ii) we have \((A_0 - ik)^{-1}\|_{V_c} \leq 1/2\sqrt{2} < 1/2\) for \(k \geq 4\). It follows from [7, Corollary 2.4.1] that
\[
\| (A_c - ik)^{-1} \| \leq \frac{\| (A_0 - ik)^{-1} \|}{1 - \| B_c \| (A_0 - ik)^{-1}} \leq \frac{8\sqrt{2}}{7k} \quad \text{for} \quad k \geq 4.
\]

In view of this estimate and (B3), we obtain (K1).

We make preparation to derive (B1).

**Lemma 5.2**

(i) \(A_c - i\) is a Fredholm operator of index 0.

(ii) \(N(A_c - i) = \{ \psi_\ast \}\).

**Proof**

(i) Step 1. By using the Fourier analysis as in the proof of Lemma 5.1 (i), we verify that for any \((\gamma, \omega) \in V_c\) the equation \((A_{\infty c} - i)(\phi, \psi) = (\gamma, \omega)\) has a unique solution \((\phi, \psi) \in U_c\). So, \(A_{\infty c} - i\) is bijective.

Step 2. By Step 1, \(A_{\infty c} - i\) is a Fredholm operator of index 0. We denote by \(\hat{A}, \hat{A}_\infty \in \mathcal{L}(U, V)\) by \(\hat{A}\phi = A\phi\) for \(\phi \in U\) and by \(\hat{A}_\infty \phi = A_\infty \phi\) for \(\phi \in U\). In order to complete the proof it suffices to show that \(\hat{A}_c - i\) is a Fredholm operator of index 0. We denote by \(\chi_R\) the identity function of \((-R, R)\) for \(R > 0\). Let \(g(x) := (1 - \tanh^2(x/2))/2\). We define \(H_R \in \mathcal{L}(U_c, V_c)\) by \(H_R := (\hat{A}_\infty - i) - (1 - \chi_R)g\) for \(R > 0\). Then, \(\hat{A}_c - i = H_R - \chi_{RG}\).

In view of stability property of bijective operator (e.g. [7, Corollary 2.4.1]) \(H_R\) is bijective for a sufficiently large constant \(R > 0\). Since the multiplication operator \(\chi_{RG} : U_c \rightarrow V_c\) is compact, \(\hat{A}_c - i\) is a Fredholm operator of index 0 by the stability property of Fredholm operator.

(ii) We consider the following equation:
\[
(A_c - i)(\phi, \psi) = (0, 0).
\]

Let \(\mathcal{H} := \{ (\phi, \psi) \in \{C^2(\mathbb{R})\}^2 \}; \quad (5.11)\) holds\} be a complex linear space. We note that \(\mathcal{N}(A_c - i) \subset \mathcal{H}\) and \(\mathcal{N}(A_c - i) = \mathcal{H} \cap U_c\). Let \(\mathcal{H}_1\) and \(\mathcal{H}_2\) be subspaces of \(\mathcal{H}\) defined by
\[
\mathcal{H}_1 := \{ (\phi, -i\phi) \in \mathcal{H}; \quad \phi'' - \rho \phi = 0 \quad \text{on} \quad \mathbb{R} \},
\]
\[
\mathcal{H}_2 := \{ (\phi, i\phi) \in \mathcal{H}; \quad \phi'' - (2i + \rho) \phi = 0 \quad \text{on} \quad \mathbb{R} \}.
\]

Then we verify \(\text{dim} \mathcal{H} = 4\) and \(\text{dim} \mathcal{H}_1 = \text{dim} \mathcal{H}_2 = 2\) from the foundational theorem on existence and uniqueness of solutions for ODEs and standard techniques on ODEs. Actually we have \(\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2\) since four vectors \((1, 0, \pm i, 0)\) and \((0, 1, 0, \pm i)\) are linearly independent and two solutions satisfying \((\phi(0), \phi'(0), \psi(0), \psi'(0)) = (1, 0, -i, 0), (0, 1, 0, -i)\) (resp. \((1, 0, i, 0), (0, 1, 0, i)\)) belong to \(\mathcal{H}_1\) (resp. \(\mathcal{H}_2\)). We note that \(\phi = (\phi, -i\phi) \in \mathcal{H}_1\), \(\psi = (\psi, i\psi) \in \mathcal{H}_2\) and \(\phi + \psi = (\phi + \psi, i(\psi - \phi)) \in U_c\) implies \(\phi, \psi \in H^2(\mathbb{R})\) and \(\phi, \psi \in U_c\). Combining this and \(\psi_\ast \in \mathcal{H}_1 \cap U_c\), the proof is complete if we show that \(\mathcal{H}_1 \not\subseteq U_c\) and \(\mathcal{H}_2 \cap U_c = \{0\}\). First, we show \(\mathcal{H}_1 \not\subseteq U_c\). To this end, it suffices to show that there exists \(z = (z, -iz) \in \mathcal{H}_1\) such that
\[
z(x) \geq w(x) := e^{x/2} + 1 \quad \text{for any} \quad x \geq 6.
\]

We verify that \(w''(x) - \rho z = 0\) for \(x \geq 6\). Let \(x \geq 6\). Let \(w(x) := -(1/16)(e^x + e^{-x} - 8e^{x/2} - 6)\) sech \(x/2 < 0\) for any \(x \geq 6\). Then, \(w(x) \geq 6\). Then, \(z := -w\) satisfies \(\phi'' - \rho \phi \geq 0\) on \([6, \infty)\). Let \(a \in (6, \infty)\). By \(\phi'' > 0\) and the maximum principle, \(\phi(y)\) achieves the maximum value at \(y = a\) on the interval \([6, a]\). So, \(\phi(y)\) is actually monotone increasing for \(y \geq 6\) and (5.12) holds. Therefore, \(\mathcal{H}_1 \not\subseteq U_c\).
Next, we show $\mathcal{H}_2 \cap U_c = \{0\}$. We define the operators $H, H_{\infty}: L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ by

$$H\phi := \phi'' - (2i + \rho)\phi, \quad H_{\infty}\phi := \phi'' - (2i + 1/4)\phi$$

for $\phi \in \mathcal{D}(H) = \mathcal{D}(H_{\infty}) := H^2(\mathbb{R})$. We verify from the standard Fourier analysis that for any given $\psi \in L^2(\mathbb{R})$ the equation $H_{\infty}\phi = \psi$ has a unique solution $\phi \in H^2(\mathbb{R})$ such that $\hat{\phi}(\xi) = -\hat{\psi}(\xi)/(2i + 1/4 + \xi^2)$ for any $\xi \in \mathbb{R}$. So, $H_{\infty}$ is bijective, $\|H_{\infty}^{-1}\| \leq 1/\sqrt{(1/4)^2 + 4}$ and $\|B\| \leq 1/2$. Here, $B \in \mathcal{L}(L^2(\mathbb{R}))$ is the multiplication operator satisfying $B = H - H_{\infty}$ on $\mathcal{D}(H)$. In view of $\|H_{\infty}^{-1}\|\|B\| \leq 1$ and and the stability of inverse operators (e.g. [7, Corollary 2.4.1]) $H$ is also bijective. Therefore, $\mathcal{H}_2 \cap U_c = \{0\}$. \hfill \Box

**Derivation of (B1)**

It suffices to show that $i$ is the simple eigenvalue of $A_c$. In view of Lemma 5.2, the proof is complete if we show $\psi_\ast \notin \mathcal{R}(A_c - i)$. We proceed by contradiction. Suppose $\psi_\ast \in \mathcal{R}(A_c - i)$. We verify that $\psi_\ast \in \mathcal{N}(A_c^2 + i)$, which implies $(\psi_\ast, \psi_\ast)_{V_c} = 0$. This contradicts $\psi_\ast \in V_c - \{0\}$. \hfill \Box

**Derivation of (B2)**

As in the proof of Lemma 4.1, it follows from (B1), Remark 2.1 and the implicit function theorem that $A_c + \lambda k^2$ has an eigenvalue $\mu(\lambda) \in \mathbb{C}$ and eigenfunction $\hat{\psi}(\lambda) \in \mathcal{D}(A_c)$ corresponding to $\mu(\lambda)$ for any $\lambda$ in a small neighborhood of 0 such that $\mu(\cdot)$ and $\psi(\cdot)$ are functions of class $C^2$ with $\mu(0) = i$ and $\psi(0) = \psi_\ast$. It follows from (4.6) that $\mu'(0)\psi_\ast = \int_{-\infty}^{\infty} \hat{\psi}(\lambda)/2\pi d\lambda = i \psi_\ast$. Combining this and $\psi_\ast \in \mathcal{N}((A_c - i)^2)$,

$$\mu'(0)\|\psi_\ast\|^2_{V_c} = (\psi_\ast, \int_{-\infty}^{\infty} \hat{\psi}(\lambda)/2\pi d\lambda \psi_\ast) = 2\int_{\mathbb{R}} \operatorname{sech}^4(x/2) dx > 0.$$ 

So, $\mu'(0) > 0$. \hfill \Box

**Example 2** We consider the following system of Fitzhugh-Nagumo type:

$$\left\{
\begin{array}{ll}
\dot{u} &= v - u \quad \text{for } (x, t) \in I \times [0, \infty),
\dot{v} &= v_{xx} - 2u + 2v + u(\lambda \sin^2 x - 2u^2 + 2uv - v^2) \quad \text{for } (x, t) \in I \times [0, \infty),
\end{array}
\right.$$

(5.13)

Here, we set $I := (0, \pi)$. The branch of periodic solutions $(u, v) = (u_\lambda, v_\lambda) (\lambda > 0)$ bifurcates at $\lambda = 0$ from the branch of trivial solutions. Here, $u_\lambda(x, t) := \sqrt{\lambda} \cos t \sin x$ and $v_\lambda(x, t) := \sqrt{\lambda}(\cos t - \sin t) \sin x$. Let $V := H^1_0(I) \times L^2(I)$. We define $A : V \rightarrow V$ by

$$Au := \begin{pmatrix}
\begin{bmatrix}
v - u \\
v_{xx} - 2u + 2v
\end{bmatrix}
\end{pmatrix}
$$

for any $u := \begin{pmatrix}
u \\
u_v
\end{pmatrix} \in U = \mathcal{D}(A) := H^1_0(I) \times (H^1_0 \cap H^2)(I)$. We define $A : V \rightarrow V$ by

For any $z \in \rho(A)$, the resolvent $(z - A)^{-1}$ is not compact. Actually, we have

$$(z - A)^{-1}\begin{pmatrix}
\frac{n+1}{n} \sin nx, & -2n \sin nx
\end{pmatrix} = \left(\frac{\sin nx}{n}, 0\right) \quad \text{for any } n \in \mathbb{N}.$$ 

So, $(z - A)^{-1}$ maps a bounded sequence in $V$ to a non-precompact sequence. Therefore, we can not apply [4, Theorem 1.11] to our problem (5.13).

We show that we can apply our Proposition 2.1 to verify the above Hopf bifurcation. To this end, we describe the outline of the derivation of (K1), (K2), (B1)–(B3) in what follows. In the same way as Example 1, we can well define the map $h : \mathbb{R} \times U \rightarrow V$ by

$$h(\lambda, u) := (0, u(\lambda \sin^2 x - 2u^2 + 2uv - v^2)) \quad \text{for any } \lambda \in \mathbb{R} \text{ and } u := (u, v) \in U.$$ 

\[ Springer \]
Thus, (K2-1) holds.

Let \( X \) and \( Y \) be real Hilbert spaces defined by (2.1). We define \( u_\ast \in X_1 \) by \( u_\ast(x, t) := (\cos t \sin x, (\cos t - \sin t) \sin x) \) and \( \psi_\ast \in U_\ast \) by \( \psi_\ast(x, t) := L^{-1}_1(u_\ast) = (\sin x, (1+i) \sin x) \). Then, we verify that \( u_\ast \in \mathcal{N}(T_1) \) and \( \psi_\ast \in \mathcal{N}(A_\ast - i) \) (see Proposition 3.2 (i)) and that \((\lambda, \sigma, u) = (\lambda(\alpha), \sigma(\alpha), au_\ast + \alpha \eta(\alpha))\) is a solution of (2.4). Here, we set \( \lambda(\alpha) = \alpha^2 \), \( \sigma(\alpha) \equiv 0 \) and \( \eta(\alpha) \equiv 0 \) for \( \alpha \in \mathbb{R} \).

Derivation of (K2)

We showed that (K2 - 1) holds. We set \( \mathcal{X} := H^1((0, 2\pi), L^2(I)) \cap L^2((0, 2\pi), H^1_0(I)) \) and \( \mathcal{Y} := L^2((0, 2\pi), L^2(I)) \). It follows from (5.5) that
\[
\|\phi\|_{L^6(I)} \leq \|\phi\|^{2/3}_{L^2(I)}\|\phi''\|^{1/3}_{L^2(I)} \quad \text{for any } \phi \in H^1_0(I).
\] (5.14)

In view of (5.4), (5.14) and Hölder’s inequality, if \( f, g, h \in \mathcal{X} \) then \( fgh \in \mathcal{Y} \) with the estimate (5.6). By using (5.6), we verify that (K2 - 2) and (K2 - 3) hold. Clearly, (K2 - 4) holds.

Derivation of (B1)

Let \( (u, v) \in U_\ast \) satisfy \((i - A_\ast)(u, v) = (0, 0)\). Then, eliminating \( v \) from this equation, we have \( u_{xx} + u = 0 \). It follows that \( \mathcal{N}(i - A_\ast) = \text{span} \{\psi_\ast\} \). Now, it suffices to show the following:

For any \( p \in V \) there exist \( u \in U_\ast \) and a unique \( \alpha \in \mathbb{C} \) such that
\[
(i - A_\ast)u + \alpha \psi_\ast = p. \tag{5.15}
\]

Let \( u := (u, v) = (\sum u_n \sin nx, \sum v_n \sin nx) \) and \( p := (p, q) = (\sum p_n \sin nx, \sum q_n \sin nx) \) be Fourier expansion of \( u \) and \( p \). Here, \( u_n, v_n, p_n \) and \( q_n \) are complex numbers. Then, by elementary Fourier analysis, the solutions of the equation in (5.15) are given by the following:
\[
u = c \sin x + \sum_{n=2}^{\infty} \frac{2p_n}{(i+1)(n^2-1)} \sin nx, \quad \text{and} \quad \alpha = \frac{(1+i)p_1 - iq_1}{2}, \tag{5.16}
\]
where \( c \) is any complex number. So, (5.15) holds.

Derivation of (B2)

By the integration by parts, we verify \( \psi_\# := ((1 + i) \sin x, \sin x) \in \mathcal{N}((i - A_\ast)^*). \) In the same way as Example 1, \( \mu'(0)(\psi_\#, \psi_\ast)_V = (\psi_\#, f_{\lambda, \mu} \psi_\ast)_V \) It follows that \( \mu'(0) = (1/\pi) \int_0^\pi \sin^2 x \, dx > 0 \).

Derivation of (B3) and (K1)

It is similar to the derivation of (B1). Let \( k \in \mathbb{N} \cup \{0\} \) and \( a = (a, b) \in V \). We consider the equations \( (A_\ast - ki)u = a \), where \( u = (u, v) \in U \). We set \( u := (u, v) = (\sum u_n \sin nx, \sum v_n \sin nx) \) and \( b - 2a/(1 + ki) = \sum d_n \sin nx \). Then, it follows from elementary Fourier analysis that
\[
u = \frac{v - a}{1 + ki} \quad \text{and} \quad \left\{-n^2 + \left(2 - ki - \frac{2}{1 + ki}\right)\right\} v_n = d_n. \tag{5.17}
\]

First, we consider the case: \( k = 0 \). In this case, \( u \in U \) is uniquely given by the first equality of (5.17) and \( v = -\sum (d_n/n^2) \sin nx \). So, 0 \( \in \rho(A_\ast) \).

Next, we consider the case: \( k \geq 2 \). By (5.17), \( u \in U \) is uniquely given by the first equality of (5.17) and
\[
v_n = -\frac{d_n}{\left\{n^2 - 2 + 2/(1 + k^2)\right\} + ik\left\{1 - 2/(1 + k^2)\right\}}. \tag{5.18}
\]
It follows from (5.18) that \( v \in H^1_0 \cap H^2(I) \), which leads to \( u \in U \). So, \( k_i \in \rho(A_c) \) for \( k \geq 2 \). Therefore, (B3) holds.

Finally, let \( k \geq 2 \) and we denote by \( \| \cdot \| \) the norm of \( L^2(I) \) and by \( \| \cdot \|_1 \) the norm of \( H^1_0(I) \). Here, \( \| h \|_1 := \| h_x \| \) for \( h \in H^1_0(I) \). We easily verify \( n|v_n| \leq |d_n| \) for \( n \in \mathbb{N} \). So, \( \| v_x \| \leq 2\|a\|/\sqrt{1+k^2} + \|b\| \). By this and (5.17), we have

\[
\|u_x\| \leq \frac{1}{\sqrt{1+k^2}} \left( \frac{2\|a\|}{\sqrt{5}} + \|b\| + \|a_x\| \right). \tag{5.19}
\]

On the other hand, by (5.18)

\[
|v_n| \leq \frac{|d_n|}{k\{1 - 2/(1 + k^2)\}} \leq \frac{5}{3k} |d_n| \quad \text{for} \quad n \in \mathbb{N}.
\]

It follows that

\[
\|v\| \leq \frac{5}{3k} \left( \frac{2\|a\|}{\sqrt{1+k^2}} + \|b\| \right). \tag{5.20}
\]

Therefore, (K1) holds in view of (5.19), (5.20) and Poincaré inequality \( \|a\| \leq \|a_x\| \). \( \square \)

**Acknowledgements** The author is grateful to Professor Masato Iida and Professor Yoshitaka Yamamoto for useful discussion.

**References**

1. Amann, H.: Hopf Bifurcation in Quasilinear Reaction–Diffusion Systems. In Delay Differential Equations and Dynamical Systems. Lecture Notes in Mathematics, vol. 1475, pp. 53–63. Springer, Berlin (1991)
2. Brand, T., Kunze, M., Schneider, G., Seelbach, T.: Hopf bifurcation and exchange of stability in diffusive media. Arch. Rat. Mech. Anal. 171, 263–296 (2004)
3. Crandall, M.G., Rabinowitz, P.H.: Bifurcation from simple eigenvalues. J. Funct. Anal. 8, 321–340 (1971)
4. Crandall, M.G., Rabinowitz, P.H.: The Hopf bifurcation theorem in infinite dimensions. Arch. Rat. Mech. Anal. 67, 53–72 (1977)
5. Gomez, D., Mei, L., Wei, J.: Stable and unstable periodic spiky solutions for the Gray–Scott system and the Schnakenberg system. J. Dyn. Differ. Equ. 32, 441–481 (2020)
6. Kawanago, T.: A symmetry-breaking bifurcation theorem and some related theorems applicable to maps having unbounded derivatives. Japan J. Indust. Appl. Math. 21, 57–74 (2004). ([Corrigendum to this paper: Japan J. Indust. Appl. Math. 22, 147 (2005)](http://doi.org/10.1007/s10215-005-1397-6))
7. Kawanago, T.: Computer assisted proof to symmetry-breaking bifurcation phenomena in nonlinear vibration. Japan J. Ind. Appl. Math. 21, 75–108 (2004)
8. Kawanago, T.: Codimension-\( m \) bifurcation theorems applicable to the numerical verification methods. Adv. Numer. Anal. 2013, Article ID 420897 (2013)
9. Kielhöfer, H.: Bifurcation Theory. An Introduction with Applications to Partial Differential Equations. Second Edition. Applied Mathematical Sciences, vol. 156. Springer, New York (2012)
10. Li, H., Zhao, X., Yan, W.: Bifurcation of time-periodic solutions for the incompressible flow of nematic liquid crystals in three dimension. Adv. Nonlinear Anal. 9, 1315–1332 (2020)
11. Liu, Z., Magal, P., Ruan, S.: Hopf bifurcation for non-densely defined Cauchy problems. Z. Angew. Math. Phys. 62, 191–222 (2011)
12. Melcher, A., Schneider, G.: A Hopf-bifurcation theorem for the vorticity formulation of the Navier–Stokes equations in \( \mathbb{R}^3 \). Commun. Partial Differ. Equ. 33(4–6), 772–783 (2008)
13. Nishida, T., Teramoto, Y., Yoshihara, H.: Hopf bifurcation in viscous incompressible flow down an inclined plane. J. Math. Fluid Mech. 7, 29–71 (2005)
14. Wang, Q., Yang, J., Zhang, L.: Time-periodic and stable patterns of a two-competing-species Keller–Segel chemotaxis model: effect of cellular growth. Discrete Contin. Dyn. Syst. Ser. B 22, 3547–3574 (2017)

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.