Lax pair for the Adler (lattice Krichever-Novikov) System

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Abstract

In the paper [V. Adler, IMRN 1 (1998) 1–4] a lattice version of the Krichever-Novikov equation was constructed. We present in this note its Lax pair and discuss its elliptic form.
1 Introduction

Integrable nonlinear evolution equations that can be solved in terms of linear problems with spectral parameter on an elliptic curve have been known since a number of years, the most well-known example being the Landau-Lifschitz (LL) equation, cf. [1], i.e.

\[ S_t = S \times S_{xx} + S \times JS . \tag{1.1} \]

Here \( S \) is the normalised spin-vector \( S \in \mathbb{R}^3 \), and \( J = \text{diag}(J_1, J_2, J_3) \) is the diagonal matrix of anisotropy parameters.

Another well-known example is the Krichever-Novikov (KN) equation which reads

\[ \xi_t = \frac{1}{4} \xi_{xxx} + \frac{3(1 - \xi_x^2)}{8 \xi_x} - \frac{3}{2} \varphi(2\xi) \xi_x^3 , \tag{1.2} \]

with \( \varphi \) being the Weierstrass \( \wp \)-function. Eq. (1.2) arose in [2] from the general problem of describing finite-gap solutions of the Kadomtsev-Petviashvili (KP) equation associated with commuting differential operators in the KP hierarchy. Eq. (1.2) applies to the particular case that these operators describe a subalgebra generated by second order differential operators, i.e. the common eigenspaces are of dimension 2 (i.e. the “rank 2” situation) and obey an algebraic relation which determines an elliptic curve, cf. also [3].

In its equivalent rational form the equation reads

\[ u_t = \frac{1}{4} u_{xxx} + \frac{3 r(u) - u_x^2}{8 u_x} , \tag{1.3} \]

for \( u \equiv \varphi(\xi) \) and where \( r(u) = 4u^3 - g_2u - g_3 \) is the polynomial of the standard Weierstrass curve

\[ \Gamma : \quad U^2 = r(u) = 4u^3 - g_2u - g_3 \tag{1.4} \]

(in principle by performing homographic transformations of the type

\[ u \rightarrow \frac{\alpha u + \beta}{\gamma u + \delta} \]

\( r(u) \) can be replaced by a general quartic polynomial). Degenerate cases (when the curve (1.4) reduces to a rational curve) can be mapped to the Schwarzian KdV (SKdV) equation

\[ \frac{z_t}{z_x} = \frac{1}{4} \{ z , x \} , \quad \{ z , x \} \equiv \frac{z_{xxx}}{z_x} - \frac{3 z_x^2}{2 z_x^4} , \tag{1.5} \]
the brackets denoting the Schwarzian derivative. The latter equation has a beautiful
discrete counterpart on the 2-dimensional lattice, namely
\[
\frac{(z_{n,m} - z_{n+1,m})(z_{n,m+1} - z_{n+1,m+1})}{(z_{n,m} - z_{n,m+1})(z_{n+1,m} - z_{n+1,m+1})} = \frac{q^2}{p^2} \quad (1.6)
\]
which (together with its Lax pair) was first given in \[4\], cf. also \([5]\). The r.h.s. of eq. (1.6) taking the form of the canonical cross-ratio of four points in the complex plane, this
discrete equation turned out to play the defining role in a theory of discrete conformal
maps, cf. \([6]\). It also played an earlier role in the theory of Padé Approximants, cf. e.g. \([7]\).

As far as integrable discrete systems associated with elliptic curves is concerned, a
number of results exists on nonlinear chains, i.e. differential-difference equations (D∆E’s),
mostly due to the group from Ufa, cf. e.g. \([8, 9]\). One particular case of such a D∆E is
the following example due to Yamilov, cf. \([8]\),
\[
\frac{du_n}{dt} = (4u_n^3 - g_2u_n - g_3) \left( \frac{1}{u_{n+1} - u_n} + \frac{1}{u_n - u_{n-1}} \right), \quad (1.7)
\]
which constitutes a differential-difference analogue of the KN equation. The elliptic form
of this equation is given by:
\[
\partial_t \xi_n = \zeta(\xi_n + \xi_{n+1}) + \zeta(\xi_n - \xi_{n+1}) - \zeta(\xi_n + \xi_{n-1}) - \zeta(\xi_n - \xi_{n-1}) \quad (1.8)
\]
with \(u_n = \wp(\xi_n)\) and where \(\zeta\) denotes the Weierstrass \(\wp\)-function. In this context an ex-
tension of the theory developed in \([2]\) to commuting difference equations that was recently
proposed in \([10]\) should be mentioned as well.

On the level of integrable partial difference equations (P∆E’s), i.e. equations with
both spatial as well as temporal independent variable discrete, the situation is much less
developed. To my knowledge the first example of a fully discrete system (discrete in
space as well as time) was given in \([11]\), where a lattice version of the LL equations was
constructed from an Ansatz for a Lax pair. This resulting lattice system can be retrieved
from the basic equations:
\[
\begin{align*}
T_0 \hat{S} - \hat{S} \times JT &= \tilde{T}_0 S - J\tilde{T} \times S \quad (1.9a) \\
S_0 \hat{T} - \hat{T} \times JS &= \tilde{S}_0 S - J\tilde{S} \times T \quad (1.9b) \\
\hat{S}_0 T_0 - \tilde{T}_0 S_0 &= \alpha \left( \hat{S} \cdot J^{-1}T - \tilde{T} \cdot J^{-1}S \right) \quad (1.9c)
\end{align*}
\]
with \( \alpha = J_1 J_2 J_3 \) and where \( S_0, T_0 \) are scalar fields and the three-vectors \( S, T \) in \( \mathbb{R}^3 \) are normalised to have unit length. How this system is related to other discretisations of the LL equations, cf. e.g. \cite{12}, is yet unknown.

In the present letter we investigate a more recent and simpler example of a P\( \Delta \)E associated with elliptic curves that was found by V. Adler in \cite{13} on the basis of Bäcklund transformations (BT’s) for the KN equation. The main point of this letter is to present the Lax pair for this lattice system and to exhibit its connection to other lattice systems, notably its elliptic form.

### 2 The lattice KN System

A lattice version of the KN equation \((1.3)\) was found recently by V. Adler in \cite{13} as the permutability condition for the BT’s of the KN equation. For the purpose of fixing the notations, we we briefly recall here its construction.

The form of the auto-BT mapping solutions of eq. \((1.2)\) to solutions of itself was proposed to be of the following form:

\[
u_x \tilde{u}_x = \frac{1}{A} H(u, \tilde{u}, a) \tag{2.1}\]

with

\[
H(x, y, z) \equiv \left( xy + xz + yz + \frac{g_2}{4} \right)^2 - (x + y + z)(4xyz - g_3) \tag{2.2}
\]

and where \((a, A) = (\wp(\alpha), \wp'(\alpha))\) is a point on the elliptic curve \( \Gamma \).

**Remark:** As indicated in \cite{13} this result can be generalised to include non-auto-Bäcklund transformations between solutions of different versions of the KN equations associated with different elliptic curves. In fact, if \( h(u, v) \) is any biquadratic polynomial, then the equation \( h = 0 \) defines a birational correspondence between the curves \( w^2 = r(u) \) and \( z^2 = R(v) \) obtained from:

\[
r(u) = h_u^2 - 2hh_uu \quad \text{and} \quad R(v) = h_v^2 - 2hh_vv .
\]
The permutability condition for the BT, i.e. the condition that the BT given by

\[ u_x \hat{u}_x = \frac{1}{B} H(u, \hat{u}, b) \]  

(2.3)

(where again \((b, B) = (\varphi(\beta), \varphi'(\beta))\) is a point on the curve \(\Gamma\)) \textit{commutes} with the original BT leads to the algebraic equation of the form:

\[ k_0 u \hat{u} \hat{u} \hat{u} - k_1 (u \hat{u} \hat{u} + u \hat{u} \hat{u} + \hat{u} u \hat{u}) + k_2 (\hat{u} u + u \hat{u}) \]

\[ -k_3 (u \hat{u} + \hat{u} \hat{u}) - k_4 (u \hat{u} + \hat{u} \hat{u}) + k_5 (u + \hat{u} + \hat{u} + \hat{u}) + k_6 = 0 \]  

(2.4)

In fact, eliminating the derivatives from (2.1) and (2.3) one obtains the expression:

\[ B^2 H(u, \hat{u}, a) H(\hat{u}, \hat{u}, a) - A^2 H(u, \hat{u}, b) H(\hat{u}, \hat{u}, b) = 0 \]

the l.h.s. of which factorises into two branches, one being the l.h.s. of (2.4) and the other being a similar expression with \(B \to -B\). Thus one obtains the parametrisation of the coefficients:

\[ k_0 = A + B \quad , \quad k_1 = aB + bA \quad , \quad k_2 = a^2 B + b^2 A \]

\[ k_3 = \frac{AB(A + B)}{2(b - a)} - b^2 A + B \left(2a^2 - \frac{g_2}{4}\right) \]

\[ k_4 = \frac{AB(A + B)}{2(a - b)} - a^2 B + A \left(2b^2 - \frac{g_2}{4}\right) \]

\[ k_5 = \frac{g_3}{2} k_0 + \frac{g_2}{4} k_1 \quad , \quad k_6 = \frac{g_3}{16} k_0 + g_3 k_1 \]

(2.5)

where both \((a, A)\) and \((b, B)\) are points of \(\Gamma\), i.e. we have the relations \(A^2 = r(a), B^2 = r(b)\).

In eq. (2.4) we have adopted our preferred short-hand notation for lattice systems with \(u = u_{n,m}, \; \hat{u} = u_{n+1,m}, \; \hat{u} = u_{n,m+1}\) and \(\hat{u} = u_{n+1,m+1}\) denoting the values of the dependent variable \(u\) around an elementary plaquette, cf. Figure 1. The \(\alpha, \beta\) in Figure 1 are lattice parameters representing the grid size. Alternatively, we can think of the lattice parameters in the case of (2.4) to take values as points on the elliptic curve (1.4), i.e. they are parametrised by \(\alpha\) and \(\beta\) as

\[ (a, A) = (\varphi(\alpha), \varphi'(\alpha)) \quad , \quad (b, B) = (\varphi(\beta), \varphi'(\beta)) \]  

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Figure 1: Configuration of lattice points in the lattice equation of KdV type.

We mention that Adler’s lattice system (2.4) can be written in the following remarkable form:

\[
A \left[ (u - b)(\tilde{u} - b) - (a - b)(c - b) \right] - \left[ (\tilde{u} - b)(\tilde{\tilde{u}} - b) - (a - b)(c - b) \right] \\
+ B \left[ (u - a)(\tilde{u} - a) - (b - a)(c - a) \right] - \left[ (\tilde{u} - a)(\tilde{\tilde{u}} - a) - (b - a)(c - a) \right] = \\
= ABC(a - b) 
\]

(2.6)
in which \( c = \wp(\beta - \alpha) \), \( C = \wp'(\beta - \alpha) \). To obtain (2.6) we note the following relations among the coefficients \( k_i = k_i(\alpha, \beta) \) in (2.5)

\[
k_3 + k_2 = B(a - b)(a - c) \quad , \quad k_4 + k_2 = A(b - a)(b - c) \quad , 
\]
as well as

\[
k_5 = a(k_2 + k_3) + b(k_2 + k_4) - Ab^3 - Ba^3 \quad , \\
k_6 = A(b^2 - (a - b)(c - b))^2 + B(a^2 - (b - a)(c - a))^2 \\
+ AB \left[ A(b - c) + B(a - c) \right] , 
\]

eliminating the moduli \( g_2, g_3 \). Use is made of the addition formulae for the Weierstrass \( \wp \)-function in the form:

\[
A(c - b) + B(c - a) = C(a - b) . 
\]

We explain in the next section in what sense we consider an equation of the type (2.4) to be integrable, and present a derivation of its Lax pair.
3 Integrability of the Adler system

We will here recollect a few general points regarding lattice systems of the type presented in the previous section. The points made here are not new and were presented and illustrated for different lattice systems at numerous occasions, cf. e.g. [14, 15, 16].

Eq. (2.4) is an example in “general position” of a large class of partial difference equations (PΔE’s) of the following canonical form:

\[ f(u, \tilde{u}, \hat{u}, \tilde{\hat{u}}; \alpha, \beta) = 0, \]

where we adopt the notation of the previous section to indicate the vertices around an elementary plaquette on a rectangular lattice:

\[ u := u_{n,m}, \quad \tilde{u} = u_{n+1,m} \]

\[ \hat{u} := u_{n,m+1}, \quad \tilde{\hat{u}} = u_{n+1,m+1} \]

cf. Figure 1. In eq. (3.1) \( \alpha, \beta \) denote (as in the previous section) lattice parameters, i.e. parameters that are singled out from other (fixed) parameters by being associated with the lattice shifts

\[ u \xrightarrow{\alpha} \tilde{u} \]

\[ u \xrightarrow{\beta} \hat{u} \]

In fact, we will assume in what follows that the lattice equations (3.1) are covariant with respect to the simultaneous interchangement of both lattice variables as well as lattice parameters. As is well-known integrable lattice equations of the type above can be reinterpreted as the permutability condition of Bäcklund transformations, identifying the lattice parameters as the corresponding Bäcklund parameters. Furthermore, the dependence on these parameters can be exploited to obtain continuum limits of the lattice equations to semi-continuous and fully continuous equations that are compatible with the original fully discrete equation, cf. [17, 18].

Examples of (3.1) have been studied since many years, cf. [4] for a review. Two important questions that arise in considering integrable cases of such equations are the following:
1. When does a P∆E of the form (3.1) make sense from the point of view of initial value problems (IVP’s)? This is the question on the well-posedness of the IVP on the lattice.

2. What is the characteristic property of such equations associated with its integrability? We have in mind here the analogous property to the existence of conservation laws and higher symmetries.

Both questions have been answered (for canonical examples) in recent years. The first question was answered in a series of papers a decade ago, cf. e.g. [19, 20]. Clearly, if the equation \( f = 0 \) can be solved uniquely at each vertex of the plaquette (i.e. if it is linear in each of the dependent variables associated with its vertices), we can define initial value problems on configurations like:

![Figure 2: Configuration of initial data on the lattice for lattice equations of KdV type.](image)

The second question, regarding the characteristic of integrability of the lattice equations, was discussed in [14] (as well as in various public lectures presented by the author over the past few years). There, the main characteristic of the integrability is expressed by the fact that in the integrable cases the equation (3.1) represents actually a parameter-family of consistent equations on a multi-dimensional lattice. It is at this point that the lattice parameters \( \alpha, \beta \) will play a crucial role: instead of fixing them once and forever and solving the equation for specific values of the parameters, we leave them free and associate new lattice directions with each new value of the corresponding lattice parameter. The fact that this can actually be done in a consistent way, imposing a multitude of equations of the same form (but with different parameters and in different lattice directions) on one
and the same dependent variable $u$ is the true characteristic of the integrability of the lattice equation: it is a purely combinatorial property of the lattice system that forms the precise analogue of the existence of higher symmetries of continuous evolution equations of KdV type.

**Proposition:** The lattice equation (2.4), with the parametrisation of coefficients given by (2.5), represents a *compatible parameter-family of partial difference equations*, i.e. the equation can be embedded in a consistent way into a multidimensional lattice, on each twodimensional sublattice of shifts of which an equation of the form (2.4) holds for one and the same independent variable $u$.

Let us explain how the consistency of this embedding manifests itself, cf. [14]. As stated above, with each value of the lattice parameters $\alpha, \beta$ we can associate a discrete variable corresponding to a direction in a multidimensional lattice, s.t. the solution $u$ can be considered as a *function* on this multidimensional lattice, i.e. a single-valued object

$$u = u_{n,m,h,...} = u(n,m,h,...; \alpha, \beta, \gamma,...)$$

with the notation for the lattice shifts

$$\tilde{u} := u_{n+1,m,h,...}, \quad \hat{u} := u_{n,m+1,h,...}, \quad \overline{u} := u_{n,m,h+1,...}$$

It suffices to investigate how the embedding the lattice equation works in a three-dimensional lattice. Thus we impose the system of three equations:

$$f(u, \tilde{u}, \hat{u}, \overline{u}; \alpha, \beta) = 0 \quad (3.2a)$$
$$f(u, \tilde{u}, \overline{u}, \alpha, \gamma) = 0 \quad (3.2b)$$
$$f(u, \hat{u}, \overline{u}, \beta, \gamma) = 0 \quad (3.2c)$$

and investigate the iteration scheme around an elementary cube for given initial values at $u$, $\tilde{u}$, $\hat{u}$ and $\overline{u}$ (see Figure 3). Iterating along the faces of the cube by using the three equations (3.2a), (3.2b) and (3.2c) we arrive at a possible point of conflict for the value of $\hat{\overline{u}}$ which, in principle, can be calculated in three different ways. If, generically, these three different ways give the same answer, i.e. if the evaluation of the iteration scheme at
that corner of the cube is single-valued, the system is consistent. It is in this case that we consider the original lattice equation to be integrable.

\[ u \]

\[ \hat{u} \]

\[ \tilde{u} \]

\[ \tilde{\tilde{u}} \]

\[ \tilde{\hat{u}} \]

\[ \tilde{\tilde{\hat{u}}} \]

**Figure 3:** Consistency of the lattice system (2.4) on the three-dimensional lattice.

For all previously considered lattice equations, e.g. the case (1.6), the consistency in the sense explained above can be easily verified by hand, and calculating the variable at the outer point of the elementary cube in terms of the initial values leads to an expression that is invariant under the interchange of lattice directions, cf. [21] for details, (in the paper [14] we gave the lattice modified KdV equation as an example, where the scheme works precisely in the same way). We mention that in a recent submission, [22], the consistency around the cube was reiterated in a slightly different context. The relevant formulae pertaining to the case of the lattice Schwarzian KdV of [4], eq. (1.6), were given in the thesis [21], and have also been recovered in that paper.

Coming back to the main issue of the present paper, the integrability of the Adler system (2.4), in the sense of the consistency on the multidimensional lattice, is straightforward but technically much more involved than the previously considered cases of KdV type. The difficulty stems from the algebraic relations between the various lattice parameters on the elliptic curve, and consequently the use of algebraic manipulation tools becomes inevitable in this case. The verification that the Adler system (2.4) obeys the
consistency around the cube has been verified explicitly with the help of J. Hietarinta using REDUCE. Thus, in the sense of this notion of the existence of commuting discrete flows in the multi-dimensional lattice, the Adler system is integrable.

We will now derive the Lax pair for the Adler system from the above considerations. This is achieved by noting that, as an immediate consequence of the statement that the integrable lattice equations are consistently embedded in a multidimensional lattice, one can state that in a sense (viewed as a parameter-family of equations rather than a single equation on a 2D lattice) the lattice equation forms its own Lax pair. This point of view was presented in a number of recent seminars on the subject, cf. [15, 16], and was implemented explicitly for various cases (including e.g. the case of the lattice Boussinesq equation), cf. also the thesis [21] where the derivation for the case of the lattice SKdV equation (1.6) was presented. Let us outline the steps needed in order to implement this idea; they are as follows:

1. Fix a “virtual” direction on the 3D lattice, e.g. the direction associated with a shift denoted by $u \to \overline{u}$ and a lattice parameter $\kappa$, with corresponding equations given by (3.2b) and (3.2c).

2. Regard now the shifted object in the virtual direction $\overline{u}$ to be a new quantity, while $\tilde{u}$ and $\hat{u}$ represent the “physical” shifts on the original variable, and solve for $\overline{u}$ and $\overline{v}$ (which can be done due to the multilinearity of the lattice equation). This yields expressions which are fractional linear in $u$.

3. Resolve the resulting “discrete Riccati” equations by linearisation, i.e. set $\overline{u} = f/g$ and separate into linear eqs. for $f$ and $g$, leading to a $2 \times 2$ matrix system; this will produce the Lax matrices up to a common factor $D$ which needs to be specified by a determinantal condition.

In this way we obtain in a straightforward way the Lax pair for the given lattice equation in the form

\[
\begin{align*}
\tilde{\varphi} &= L\varphi \\
\hat{\varphi} &= M\varphi
\end{align*}
\]

(3.3)
with \( \varphi \equiv (f, g)^T \), and where the auxiliary lattice parameter \( \kappa \) plays now the role of the spectral parameter. Of course, the above scheme only applies to the case where the lattice equation can be consistently embedded into a multidimensional lattice.

Thus, having assessed the integrability in the above sense for the Adler system (2.4), it is an easy exercise to implement the steps 1-3 for this lattice system. This leads to a Lax representation of the form (3.3) where one part of the Lax pair is given by

\[
\tilde{\varphi} = L_\kappa(\tilde{u}, u; \alpha) \varphi \tag{3.4a}
\]

with Lax matrix:

\[
L_\kappa = \frac{1}{D} \begin{pmatrix}
    k_1 u \tilde{u} - k_2 \tilde{u} + k_4 u - k_5 & k_3 u \tilde{u} - k_5 (u + \tilde{u}) - k_6 \\
    k_0 u \tilde{u} - k_1 (u + \tilde{u}) - k_3 & -k_1 u \tilde{u} + k_2 u - k_4 \tilde{u} + k_5
\end{pmatrix}. \tag{3.4b}
\]

The coefficients in (3.4b) can be inferred immediately from the form of the lattice equations itself, setting \( k_i = k_i(\alpha, \kappa) \), i.e. replacing \( \beta \) by the spectral parameter \( \kappa \) in the formulae (2.5) for the \( k_i \). The only subtlety is the choice of the prefactor \( D = D(u, \tilde{u}, \alpha, \kappa) \) which follows from the determinant of the matrix in (3.4b), which is calculated to be:

\[
(a - k)^2 K K' H(u, \tilde{u}, a) \quad \text{with} \quad k = \varphi(\kappa) \; , \; K = \varphi'(\kappa) \; , \; K' = \varphi'(\kappa - \alpha).
\]

Choosing \( D \) to be equal to the square root of this expression to ensure that \( \det(L_\kappa) = 1 \). Taking the other part of the Lax pair in the same form apart from the obvious replacements: \( \tilde{\sim} \to \tilde{\sim}, \alpha \to \beta \), we have the compatibility relations

\[
L_\kappa(\tilde{u}, \tilde{u}; \alpha) L_\kappa(\tilde{u}, u; \beta) = L_\kappa(\tilde{u}, \tilde{u}; \beta) L_\kappa(\tilde{u}, u; \alpha)
\]

which are the Lax equations for Adler system (2.4).

One further exercise is to obtain the Lax pair for the BT (2.1) itself. Obviously, we can think of this equation as a differential-difference equation representing a continuous commuting flow to the fully discrete lattice system (2.4). The corresponding Lax matrix is obtained in precisely the same way as the discrete Lax matrices: solve the Riccati equation associated with the virtual shift \( \tilde{\sim} \), i.e. eq. (2.1) with \( a \) replaced by \( k \), \( A \) by \( K \), and \( \tilde{u} \) by \( \tilde{u} \), and then making the same identifications as before to obtain a \( 2 \times 2 \) differential matrix.
There is now an additive freedom built into the system, which is fixed by imposing that the Lax matrix be traceless. Thus, we obtain an additional linear equation given by

\[ \varphi_x = U_\kappa \varphi \]  

(3.5a)

with

\[ U_\kappa = \frac{1}{K u_x} \begin{pmatrix} \frac{1}{2} g_3 - (u + k)(uk - \frac{1}{4} g_2) & g_3(u + k) + (uk + \frac{1}{4} g_2)^2 \\ -(u - k)^2 & -\frac{1}{2} g_3 + (u + k)(uk - \frac{1}{4} g_2) \end{pmatrix} \]  

(3.5b)

which supplements the discrete Lax pair (3.4). The linear equation (3.5) is the spatial part of the Lax pair for the continuous KN equation (1.3), which can be recovered from the original Lax pair given in the paper [2].

Thus, we have obtained the Lax pair the lattice KN equation (2.4) of Adler directly from the equation itself. After having obtained these results, which were presented during the workshop on Discrete Systems and Integrability at the Isaac Newton Institute (September 2001), cf. [16], V. Adler communicated to the author that similar (yet unpublished) results had been obtained by Yu. Suris and himself, [23].

4 Discussion

In this paper we presented a novel Lax pair for the lattice Krichener-Novikov equation of Adler. The derivation is based on the simple observation that in a precise sense the lattice equation is its own Lax pair. This is based on the nontrivial fact that the lattice equation under consideration represent a consistent parameter-family of equations which are embedded in a lattice of arbitrary dimensionality. The “consistency around a cube”, cf. Figure 3, which demonstrates this property was first presented in a paper [14] and the derivations of Lax pair in cases such as the lattice BSQ and SKdV equations was implemented in the thesis [21]. In the recent submission [22] some of these results were given a geometrical context. equation that were already presented in [21].

Although the above observation is very simple, it is also quite deep in that it constitutes really the precise anologue of the existence of hierarchies and of infinite sequences of conservation laws that are the hallmark of soliton systems.
Having obtained the Lax pair for the Adler system one can now embark on a more systematic study of this new elliptic system, including its finite-dimensional and similarity reductions leading to finite-dimensional mappings associated with spectral problems on the torus and (possibly) to Painlevé-type of equations possessing isomonodromic deformation problems on the torus. The first part of such a study is already underway, cf. [24].

Let us finish by making some remarks on the elliptic version of some of the equations mentioned. It is easily noted that the BT (2.1) from which the Adler system originated can be cast into elliptic form by going over to dependent variables $u = \wp(\xi)$ as in the case of the continuous KN equation (i.e. the correspondence between the forms (1.2) and (1.3)). We, thus, obtain for $\xi$ the equation:

$$\frac{-\xi_x^2}{\xi_x} = \frac{\sigma(\xi + \tilde{\xi} + \alpha)\sigma(\xi + \tilde{\xi} - \alpha)\sigma(\xi - \tilde{\xi} + \alpha)\sigma(\xi - \tilde{\xi} - \alpha)}{\sigma(2\alpha)\sigma(2\xi)\sigma(2\tilde{\xi})}. \tag{4.1}$$

This elliptic form can be found from the original equation for the BT by employing a number of relations, notably the identity:

$$H(u, v, a) = (u - v)^2 \left[ \frac{1}{4} \left( \frac{U - V}{u - v} \right)^2 - (u + v + a) \right] \left[ \frac{1}{4} \left( \frac{U + V}{u - v} \right)^2 - (u + v + a) \right] \tag{4.2}$$

in which $U^2 \equiv r(u)$, $V^2 \equiv r(v)$, as well as the addition formula for the Weierstrass $\wp$-function, i.e.

$$\wp(\xi) + \wp(\eta) + \wp(\xi + \eta) = \frac{1}{4} \left( \frac{\wp'(\xi) - \wp'(\eta)}{\wp(\xi) - \wp(\eta)} \right)^2,$$

and the relations

$$\wp(\xi) - \wp(\eta) = \frac{\sigma(\eta + \xi)\sigma(\eta - \xi)}{\sigma^2(\eta)\sigma^2(\xi)}, \quad \wp'(\xi) = -\frac{\sigma(2\xi)}{\sigma^4(\xi)},$$

in which $\sigma$ is the Weierstrass $\sigma$-function.

Eliminating the derivatives from the BT (4.1) together with a second BT with $\tilde{\xi}$ replaced with $\hat{\xi}$ and $\alpha$ by $\beta$ we obtain

$$\frac{\sigma(\xi + \tilde{\xi} + \alpha)\sigma(\xi + \tilde{\xi} - \alpha)\sigma(\xi - \tilde{\xi} + \alpha)\sigma(\xi - \tilde{\xi} - \alpha)}{\sigma(\xi + \hat{\xi} + \beta)\sigma(\xi + \hat{\xi} - \beta)\sigma(\xi - \hat{\xi} + \beta)\sigma(\xi - \hat{\xi} - \beta)} \times \frac{\sigma(\hat{\xi} + \tilde{\xi} + \alpha)\sigma(\hat{\xi} + \tilde{\xi} - \alpha)\sigma(\hat{\xi} - \tilde{\xi} + \alpha)\sigma(\hat{\xi} - \tilde{\xi} - \alpha)}{\sigma(\hat{\xi} + \hat{\xi} + \beta)\sigma(\hat{\xi} + \hat{\xi} - \beta)\sigma(\hat{\xi} - \hat{\xi} + \beta)\sigma(\hat{\xi} - \hat{\xi} - \beta)} = \frac{\sigma^2(2\alpha)}{\sigma^2(2\beta)}. \tag{4.3}$$
We remark that eq. (4.3) cannot yet be regarded as the elliptic form of the Adler system (2.4), as the latter is obtained from a factorised form of the permutability condition of BT’s. In fact, an elliptic version can be obtained in a straightforward way from the formula (2.6), but we haven’t managed yet to cast it into an elegant form, so we omit it here. Alternatively, a so-called “three-leg” form of the Adler system was announced in the recent submission [22], but it is yet unclear how this is related to the formula (4.3). The purpose of these ways of rewriting the Adler system is to obtain a better insight into the underlying structure of the equation, including the origin of the Lax pair. A more systematic understanding of the relationship between the various members in the KN family of equations seems to be necessary and we intend to address these problems more at length in a future publication, [25].

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