TREES, LINEAR ORDERS AND GÂTEAUX SMOOTH NORMS

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Abstract. We introduce a linearly ordered set \( Z \) and use it to prove a necessity condition for the existence of a Gâteaux smooth norm on \( C_0(\Upsilon) \), where \( \Upsilon \) is a tree. This criterion is directly analogous to the corresponding equivalent condition for Fréchet smooth norms. In addition, we prove that if \( C_0(\Upsilon) \) admits a Gâteaux smooth lattice norm then it also admits a lattice norm with strictly convex dual norm.

1. Introduction and Preliminaries

Among the most well-established geometrical properties of norms are smoothness and strict convexity. A norm \( \| \cdot \| \) on a Banach space \( X \) is called Gâteaux smooth, or just Gâteaux, if, given any \( x \in X \setminus \{0\} \), there exists a functional in \( X^* \), denoted by \( \|x\|' \), such that
\[
\lim_{\lambda \to 0} \frac{\|x + \lambda h\| - \|x\|}{\lambda} = \|x\|'(h)
\]
for all \( h \in X \). In addition, if the limit above is uniform for \( h \) in the unit sphere \( S_X \), then \( \| \cdot \| \) is called Fréchet smooth, or simply Fréchet.

Turning now to properties of strict convexity, we say that \( \| \cdot \| \) is strictly convex if, given \( x, y \in X \) satisfying \( \|x\| = \frac{1}{2}\|x + y\| = \|y\| \), we have \( x = y \). Of the many stronger cousins of strictly convex norms, we mention one. The norm \( \| \cdot \| \) is locally uniformly rotund, or LUR, if, given a point \( x \in S_X \) and a sequence \( (x_n) \subseteq S_X \) satisfying \( \|x + x_n\| \to 2 \), we have \( \|x - x_n\| \to 0 \).

Renorming theory is a branch of functional analysis that seeks to determine the extent to which a given Banach space can be endowed with equivalent norms sporting certain geometrical properties, such as the ones above. In this paper, a norm on a given Banach space is always assumed to be equivalent to the canonical norm. We refer the reader to [1] for a comprehensive account of this field up to 1993, together with the more recent surveys [2] and [12].

In recent years, trees have assumed an important role in the field, both as a source of counterexamples to existing questions and as a vehicle for exploring new avenues of research; see, for example [3], [4] and [5]. We say that a partially ordered set \( (\Upsilon, \preceq) \) is a tree if, given arbitrary \( t \in \Upsilon \), the set of predecessors \( \{s \in \Upsilon \mid s \preceq t\} \), denoted by the interval \( (0, t) \), is well-ordered. The set of immediate successors of \( t \in \Upsilon \) is denoted by \( t^+ \). In this way, trees are a natural generalisation of ordinal numbers. As well as \( (0, t) \), we define the interval \( (s, t] = (0, t) \setminus \{0, s\} \) for \( s \preceq t \), the wedge \( [t, \infty) = \{u \in \Upsilon \mid t \preceq u\} \) and finally \( (t, \infty) = [t, \infty) \setminus \{t\} \). We remark that the symbols 0 and \( \infty \) are, in this context, convenient notational devices and not themselves elements of \( \Upsilon \).

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The scattered locally compact interval topology on $\Upsilon$ is the coarsest topology for which all intervals $(0, t]$ are both open and closed. This topology agrees with the standard interval topology of any ordinal $\Omega$, if we consider $\Omega$ as a tree. To ensure that this topology is also Hausdorff, we restrict our attention to trees $\Upsilon$ with the property that every non-empty, linearly ordered set in $\Upsilon$ has at most one minimal upper bound. With this topology in mind, we consider the Banach space $C_0(\Upsilon)$ of continuous real-valued functions vanishing at infinity, and the dual space of measures. We remark that as $\Upsilon$ is scattered, the weak topology and the topology of pointwise convergence agree on norm-bounded subsets of $C_0(\Upsilon)$.

Trees and linearly ordered sets enjoy close ties. For a comprehensive review of these relationships, we refer the reader to [11]. Given partial orders $P$ and $Q$, we say that the map $\rho : P \to Q$ is called increasing (respectively strictly increasing) if $\rho(s) \preceq \rho(t)$ (respectively $\rho(s) \prec \rho(t)$) whenever $s \prec t$. Decreasing and strictly decreasing functions are defined analogously. If there exists a strictly increasing map from $P$ to a linear order $Q$, we say that $P$ is $Q$-embeddable, or $P \preceq Q$. Evidently, in this context, $\preceq$ is a transitive relation on the class of partial orders.

In much of what follows, $P$ will be a tree and $Q$ a linear order. It is well known that $\Upsilon \preceq Q$ if and only if $\Upsilon$ is special, which means that $\Upsilon$ can be written as a countable union of antichains (cf. [11, Theorem 9.1]). Special trees tend to have very good properties; for example, the following result can be found in [9].

**Theorem 1.** Given a tree $\Upsilon$, the space $C_0(\Upsilon)$ admits a norm with LUR dual norm if and only if $\Upsilon$ is special.

We introduce a couple of combinatorial ideas used extensively in [6].

**Definition 2.** Given an increasing function $\rho : \Upsilon \to \mathbb{R}$, we say that $t \in \Upsilon$ is a bad point for $\rho$ if there exists a sequence of distinct points $(u_n) \subseteq t^+$, such that $\rho(u_n) \to \rho(t)$.

Bad points are so named because their presence often indicates that the given $C_0(\Upsilon)$ space has negative renorming properties. An analogue of the next simple result appears at the beginning of Section 3.

**Proposition 1** ([Haydon]). The tree $\Upsilon$ is special if and only if $\Upsilon \preceq \mathbb{R}$ and there exists an increasing map $\rho : \Upsilon \to \mathbb{R}$ that has no bad points.

We move on to the second combinatorial property taken from [6].

**Definition 3.** A subset $E$ of a tree is said to be ever-branching if each element of $E$ has a pair of strict successors in $E$ that are incomparable in the tree order.

It is easy to see that within every ever-branching subset can be found a dyadic tree of height $\omega$; that is, a tree with a single minimal element, no limit elements, and with the property that each element has exactly two immediate successors.

Many types of norm on $C_0(\Upsilon)$ can be characterised in terms of increasing real-valued functions on $\Upsilon$, with further combinatorial properties that can be expressed in terms of bad points and ever-branching subsets. Of particular interest to us is the following result.

**Theorem 4** ([Haydon [6]]). Given a tree $\Upsilon$, the space $C_0(\Upsilon)$ admits a Fréchet norm if and only if there exists an increasing function $\rho : \Upsilon \to \mathbb{R}$ that has no bad points and is not constant on any ever-branching subset.
In order to exhibit a tree that does not satisfy the statement of Theorem 4, we introduce a fundamental construction, due to Kurepa. Given a linear order \( \Sigma \), we define the Hausdorff tree
\[
\sigma \Sigma = \{ A \subseteq \Sigma \mid A \text{ is well-ordered} \}.
\]

We remark that some authors demand the additional requirement that elements of \( \sigma \Sigma \) are bounded above. One of the reasons why Kurepa’s construction is so important in the theory of trees is summed up by the following theorem.

**Theorem 5** ((Kurepa [7])). If \( \Sigma \) is a linear order then \( \sigma \Sigma \not\approx \Sigma \).

From Theorem 5, \( \sigma \mathbb{Q} \) is not special. On the other hand, if we take an enumeration \( (q_n) \) of the rationals and consider the map \( A \mapsto \sum_{a \in A} 2^{-n} \), we see that \( \sigma \mathbb{Q} \not\approx \mathbb{R} \).

It follows that, by Proposition 1, every increasing, real-valued function defined on \( \sigma \mathbb{Q} \) has a bad point.

**Corollary 1** ((Haydon)). The space \( C_0(\sigma \mathbb{Q}) \) admits no Fréchet norm.

While many types of norm are accounted for in [6], equivalent conditions for the existence of norms on \( C_0(\Upsilon) \) with strictly convex dual, or Gâteaux norms, cannot be adequately expressed in terms of increasing real-valued functions. In all that follows, \( \omega_1 \) denotes the first uncountable ordinal. The following linearly ordered set is introduced in [9].

**Definition 6.** Let \( Y \) be the set of all strictly increasing, continuous, transfinite sequences \( x = (x_\xi)_{\xi \leq \beta} \) of real numbers, where \( 0 \leq \beta < \omega_1 \). Order \( Y \) by declaring that \( x < y \) if and only if either \( y \) strictly extends \( x \), or if there is some ordinal \( \alpha \) such that \( x_\xi = y_\xi \) for \( \xi < \alpha \) and \( y_\alpha < x_\alpha \).

Observe that \( Y \) is not ordered in the usual lexicographic way. Compared to the real line, \( Y \) is large.

**Proposition 2** ((Smith [9])). If \( \beta < \omega_1 \) then \( Y^\beta \not\approx Y \), where \( Y^\beta \) is ordered lexicographically.

As \( \mathbb{R} \not\approx Y \), we see that \( \mathbb{R}^\beta \not\approx Y \) for all \( \beta < \omega_1 \). On the other hand, it can be shown that \( Y \) contains no well-ordered or conversely well-ordered subsets. The next theorem is the main result of [9].

**Theorem 7** ((Smith [9])). Given a tree \( \Upsilon \), the Banach space \( C_0(\Upsilon) \) admits a norm with strictly convex dual norm if and only if \( \Upsilon \not\approx Y \).

Theorem 7 is a direct analogue of Theorem 1. In [9], it is shown that the spaces \( C_0(\sigma(\mathbb{R}^\beta)) \), where \( \mathbb{R}^\beta \) is ordered lexicographically, admit norms with strictly convex duals provided \( \beta < \omega_1 \). On the other hand, by Theorem 5, \( C_0(\sigma(Y)) \) does not admit such a norm.

The order \( Y \) can also be used to give an improved sufficient condition for the existence of Gâteaux norms in the context of trees.

**Theorem 8** ((Smith [8])). If there exists an increasing function \( \rho : \Upsilon \rightarrow Y \) that is not constant on any ever-branching subset then \( C_0(\Upsilon) \) admits a Gâteaux norm.

We end our review of the existing literature by presenting what was hitherto the best known necessary condition for Gâteaux norms in this context. Given a tree \( \Upsilon \), the forcing topology on \( \Upsilon \) takes as its basis the set of all wedges \([t, \infty)\),
A subset $B \subseteq \Upsilon$ is called Baire if it is a Baire space with respect to the induced forcing topology; that is, any countable intersection of relatively dense, open subsets of $B$ is again dense. When referring to the Baire property, we will only consider subsets that are perfect with respect to the forcing topology; in other words those without isolated points or, equivalently, maximal elements. Arguably the simplest example of such an object is the ordinal $\omega_1$, though more interesting ones that have no uncountable linearly ordered subsets can be found in [11, Lemma 9.12] (cf. [5]).

Theorems 4 and 8 applied to a constant function on $\omega_1$ demonstrate that, by itself, the Baire property cannot destroy Gâteaux renormability. Instead, we have the following result.

**Theorem 9** (Haydon [5]). If $C_0(\Upsilon)$ admits a Gâteaux norm then $\Upsilon$ contains no ever-branching Baire subsets.

We turn now to the results of this paper. In order to properly express our necessary condition for Gâteaux renormability, we must introduce a second linearly ordered set.

**Definition 10.** Let $Z$ be the set of all increasing, continuous sequences $x = (x_\xi)_{\xi \leq \beta}$ of real numbers, where $0 \leq \beta < \omega_1$, and such that $x$ is strictly increasing on $[0, \beta)$. The order of $Z$ follows that of $\Upsilon$; $x < y$ if and only if either $y$ strictly extends $x$, or if there is some ordinal $\alpha$ such that $x_\xi = y_\xi$ for $\xi < \alpha$ and $y_\alpha < x_\alpha$.

The elements of $Z$ that are not in $\Upsilon$ are exactly those of the form $x = (x_\xi)_{\xi \leq \beta+1}$, where $(x_\xi)_{\xi \leq \beta} \in \Upsilon$ and $x_\beta = x_{\beta+1}$. This order is a partial Dedekind completion of $\Upsilon$. We also need a natural definition of bad points with respect to $Z$.

**Definition 11.** Given an increasing function $\rho : \Upsilon \rightarrow Z$, we say that $t \in \Upsilon$ is $Z$-bad for $\rho$ if there exists a sequence of distinct points $(u_n) \subseteq t^+$ such that $\rho(u_n) \to \rho(t)$ in the order topology of $Z$.

Using $Z$-bad points, we obtain a direct analogy to the necessity part of Theorem 4; the following is the main result of this paper.

**Theorem 12.** If the space $C_0(\Upsilon)$ admits a Gâteaux norm, then there exists an increasing function $\rho : \Upsilon \rightarrow Z$ that has no $Z$-bad points and is not constant on any ever-branching subset.

In some sense, $\Upsilon$ is to $\mathbb{Q}$ what $Z$ is to $\mathbb{R}$, and these relationships correspond well to those of Theorems [7, 11, 12] and [4] respectively.

The following corollary of Theorem 12 generalises a result from [3], which states that $C_0([0, \omega_1))$ does not admit any Gâteaux lattice norm.

**Corollary 2.** If $C_0(\Upsilon)$ admits a Gâteaux lattice norm then $\Upsilon \not\subseteq \mathbb{Q}$ and, consequently, $C_0(\Upsilon)$ admits a lattice norm with strictly convex dual.

We end Section 2 by proving the next proposition, which shows that Theorem 9 is a corollary of Theorem 12.

**Proposition 3.** If $\rho : \Upsilon \rightarrow Z$ is an increasing function that is not constant on any ever-branching subset, then $\Upsilon$ does not admit any ever-branching Baire subsets.

The final section, devoted to examples, begins with a proof that Theorem 9 is strictly implied by Theorem 12.
Proposition 4. The tree $\sigma Y$ is $Z$-embeddable, but every increasing function $\rho : Y \rightarrow Z$ has a $Z$-bad point. In particular, $C_0(\sigma Y)$ does not admit a Gâteaux norm.

Proposition 4 is analogous to Corollary 1. Section 3 ends with Example 15, which shows there is a gap between the conditions of Theorems 8 and 12. This, together with the analogies presented above and the author’s bias, prompts the following problem.

Problem 1. If there exists an increasing function $\rho : Y \rightarrow Z$ that has no $Z$-bad points and is not constant on any ever-branching subset, does $C_0(Y)$ admit a Gâteaux norm?

Recently, the author gave a purely topological formulation of Theorem 7. Given a tree $Y$, the space $C_0(Y)$ admits a norm with strictly convex dual norm if and only if $Y$ is a so-called Gruenhage space, with respect to its interval topology [10].

Problem 2. Is there an internal characterisation of trees $Y$, with the property that $C_0(Y)$ admits a Gâteaux norm?

Problem 2 may be restated in terms of Fréchet norms, Kadec norms and others. This section closes with further problem, motivated by Corollary 2.

Problem 3. If $L$ is locally compact and $C_0(L)$ admits a Gâteaux lattice norm, does $C_0(L)$ admit a norm with strictly convex dual? Is this statement also true with respect to a general Banach lattice?

2. Necessity conditions for Gâteaux renormability

To help familiarise the reader with $Z$ and $Z$-bad points, we begin by briefly describing some forms of sequential convergence in $Z$. First observe that if $x \in Y$, $y \in Z$ and $y > x$ is sufficiently close to $x$ in the order topology of $Z$, then $y$ must be a strict extension of $x$. On the other hand, if $x \in Z \setminus Y$ then $x$ has no strict extensions in $Z$. The proof of the next lemma is a simple exercise in elementary analysis and is omitted.

Lemma 1. Let $x \in Z$ and suppose $(z^n) \subseteq Z$ is a sequence satisfying $x < z^n$. We have the following rules for the convergence of $(z^n)$ to $x$:

1. if $x = (x_\xi)_{\xi \leq \beta} \in Y$ then $z^n \rightarrow x$ if and only if $z^n$ strictly extends $x$ for large enough $n$, and $z^n_{\beta+1} \rightarrow \infty$.

If $x = (x_\xi)_{\xi \leq \beta+1} \in Z \setminus Y$ then since $x$ has no strict extensions, there exists $\alpha_n \leq \beta$ such that $z^n_\xi = x_\xi$ for $\xi < \alpha_n$ and $z^n_{\alpha_n} < x_{\alpha_n}$. In this case, we have:

2. if $\beta = 0$ or $\beta = \alpha + 1$ for some $\alpha$, then $z^n \rightarrow x$ if and only if $\alpha_n = \beta$ for large enough $n$, and $z^n_{\beta} \rightarrow x_\beta$;

3. if $\beta$ is a limit ordinal, then $z^n \rightarrow x$ if and only if $\alpha_n \rightarrow \beta$.

We present a simple application of Lemma 1. If $\pi : Y \rightarrow Y$ is a strictly increasing map then it could have $Z$-bad points. However, if we fix an order isomorphism $\theta : \mathbb{R} \rightarrow (0,1)$ and define, for $x = (x_\xi)_{\xi \leq \beta} \in Y$, $\Theta(x_\xi) = \theta(x_\xi)$ whenever $\xi \leq \beta$, then by Lemma 1 part (1), the strictly increasing $Y$-valued map $\Theta \circ \pi$ has no $Z$-bad points. Thus, some $Z$-bad points are easily removed by making simple adjustments. More details of how $Z$ operates can be found in Section 3.

Now, for the rest of this section, we fix a norm $\| \cdot \|$ on $C_0(Y)$. We continue by introducing a concept that features in both [3] and [6]. Given $t \in Y$, let $C_t$ be the set of all $f \in C_0(Y)$ such that $f$ vanishes outside $(0,t]$ and increasing on $(0,t]$. 
**Definition 13.** If \( f \in C_t \) and \( \delta \geq 0 \), the increasing function \( \mu(f, \delta, \cdot) \) is defined on the wedge \([t, \infty)\) by

\[
\mu(f, \delta, \cdot) = \inf \{ \| f + (f(t) + \delta)1_{(t,u]} + \varphi \| \mid \varphi \in \mathcal{C}_0(\Upsilon) \text{ and } \text{supp} \varphi \subseteq (u, \infty) \}
\]

where \( 1_A \) denotes the indicator function of the set \( A \) and \( \text{supp} \varphi \) is the support of \( \varphi \). We also define the abbreviation \( \mu(f, \cdot) \) by \( \mu(f, u) = \mu(f,0,u) \) and the associated function \( \mu \), given by \( \mu(t) = \inf \{ \| 1_{[0,t]} + \varphi \| \mid \varphi \in \mathcal{C}_0(\Upsilon) \text{ and } \text{supp} \varphi \subseteq (t, \infty) \} \).

Attainment of the infimum in the definition of these so-called \( \mu \)-functions has important consequences for the renormability of \( \mathcal{C}_0(\Upsilon) \), and bad points and ever-branching subsets come into play. The first consequence of the following lemma is trivial, and the second and third are immediate generalisations of \([6, \text{Lemma 3.1}]\) and \([6, \text{Proposition 3.4}]\) respectively.

**Lemma 2** ([Haydon [6]]). Suppose \( t \in \Upsilon \), \( f \in C_t \) and \( \delta \geq 0 \). Then:

1. if \( \| \cdot \| \) is a lattice norm then \( \| f + (f(t) + \delta)1_{(t,u]} \| = \mu(f, \delta, u) \) for all \( u \succ t \);
2. if \( u \succ t \) is a bad point for \( \mu(f, \delta, \cdot) \) then \( \| f + (f(t) + \delta)1_{(t,u]} \| = \mu(f, \delta, u) \);
3. if \( \mu(f, \delta, \cdot) \) is constant on some ever-branching subset \( E \subseteq (u, \infty) \), where \( u \succ t \), then there exists \( \varphi \in \mathcal{C}_0(\Upsilon) \) with

\[
\text{supp} \varphi \subseteq \{ v \in (u, \infty) \mid v \ll w \text{ for some } w \in E \}
\]

and \( \mu(f, \delta, u) = \| f + (f(t) + \delta)(1_{(t,u]} + \varphi) \| \).

We continue with an idea from [9].

**Definition 14.** A subset \( V \subseteq \Upsilon \) is called a plateau if \( V \) has a least element \( 0_V \) and \( V = \bigcup_{t \in V} [0_V, t] \). A partition \( \mathcal{P} \) of \( \Upsilon \) consisting solely of plateaux is called a plateau partition.

Observe that if \( V \) is a plateau then \( V \setminus \{0_V\} \) is open. It follows that if we have a plateau partition \( \mathcal{P} \) and define the set of least elements \( H = \{ 0_V \mid V \in \mathcal{P} \} \), then \( H \) is closed in \( \Upsilon \). Of course, \( H \) may be regarded as a tree in its own right, with its own interval topology. Plateaux are stable under taking arbitrary intersections.

**Proposition 5** ([Smith [9], Proposition 10]). Let \( \Upsilon \) be a tree and \( \mathcal{F} \) a family of plateaux of \( \Upsilon \) with non-empty intersection \( W \). Then \( W \) is a plateau and \( 0_W = \sup_{V \in \mathcal{F}} 0_V \).

The connection between increasing functions and plateaux is given by the next proposition.

**Proposition 6** ([Smith [9], Proposition 9]). Let \( \rho : \Upsilon \to \Sigma \) be an increasing function into a linear order \( \Sigma \). Then the equivalence relation \( \sim \), given by \( s \sim t \) if and only if there exists \( r \ll s, t \) such that \( \rho(s) = \rho(r) = \rho(t) \), defines the plateau partition of \( \Upsilon \), with respect to \( \rho \). Moreover, the restriction of \( \rho \) to the set of least elements \( H = \{ 0_V \mid V \in \mathcal{P} \} \) is strictly increasing.

Proposition 6 applies equally well to decreasing functions. As the \( \mu \)-functions from Definition 13 are increasing on their respective domains, they may be analysed using plateaux. Elements of the following technical lemma appear implicitly in the proof of [6, Theorem 8.1].
**Lemma 3.** Let $||\cdot||$ be Gâteaux smooth and suppose that $\varepsilon||\cdot||_\infty \leq ||\cdot|| \leq ||\cdot||_\infty$ for some $\varepsilon \in (0, 1)$. Moreover, suppose $V$ is a plateau, $f \in C_0$, and $\mu(f, \cdot)$ is constant on $V$. We define a function $\lambda$ on $V \setminus \{0_V\}$ by setting

$$
\lambda(t) = \sup\{\delta \geq 0 | \mu(f, \delta, t) \leq \mu(f, 0_V) + \frac{1}{2}\varepsilon\delta\}.
$$

We check that $\lambda$ is well-defined and satisfies the following properties:

1. $\lambda$ is decreasing on $V \setminus \{0_V\}$;
2. if $\lambda$ takes constant value $\nu$ on the plateau $W \subseteq V \setminus \{0_V\}$ then $\mu(f, \nu, \cdot)$ takes constant value $\mu(f, 0_V) + \frac{1}{2}\varepsilon\nu$ on $W$;
3. if $\mathcal{P}$ is the plateau partition of $V \setminus \{0_V\}$ with respect to $\lambda$, supplied by Proposition $\Box$ $W \in \mathcal{P}$, and $f_W \in C_{0W}$ is defined by

$$
f_W = f + (f(0_V) + \lambda(0_W))1_{\{0_V, 0_W\}}
$$

then $\mu(f_W, \cdot)$ takes constant value $\mu(f, 0_V) + \frac{1}{2}\varepsilon\lambda(0_W)$ on $W$;
4. if the infimum in the definition of $\mu(f, t)$ is attained then $\lambda(t) > 0$.

**Proof.** Fix $t \in V \setminus \{0_V\}$ and, for $\delta \geq 0$, define $F(\delta) = \mu(f, \delta, t) - \mu(f, 0_V) - \frac{1}{2}\varepsilon\delta$. Observe that $F$ is continuous and $F(0) = 0$. Moreover, if $\sup \varphi$ is a subset of $(t, \infty)$, we estimate that $||f + f(t) + \varphi|| \geq \varepsilon\delta - ||f + f(t)1_{\{0_V, 0_W\}}||$, whence $F(\delta)$ tends to $\infty$ as $\delta$ does. As a result, $\lambda(t)$ is well-defined.

Now we can check the properties of $\lambda$. We see that $\mu(f, \lambda(t), t) = \mu(f, 0_V) + \frac{1}{2}\varepsilon\lambda(t)$ for any $t \in V \setminus \{0_V\}$. Therefore, if $t \leq u$ then, as $\mu(f, \lambda(u), \cdot)$ is increasing, we have

$$
\mu(f, \lambda(u), t) \leq \mu(f, \lambda(u), u) = \mu(f, 0_V) + \frac{1}{2}\varepsilon\lambda(u)
$$

which shows that $\lambda(t) \geq \lambda(u)$, giving us property (1).

The second property follows immediately and the third follows from the second. To prove property (4), we let $g = f + f(t)1_{\{0_V, t\}} + \varphi$ with $\sup \varphi \subseteq (t, \infty)$, such that $||g|| = \mu(f, t) = \mu(f, 0_V)$. Observe that as the infimum $\mu(f, 0_V)$ is attained, we have

$$
||g'||(1_{\{0_V, t\}}) = \lim_{\delta \to 0^+} \frac{||g + \delta 1_{\{0_V, t\}}|| - ||g||}{\delta} \geq 0
$$

and similarly for $-1_{\{0_V, t\}}$, whence $||g'||(1_{\{0_V, t\}}) = 0$. Now it is evident that there exists $\delta > 0$ satisfying

$$
\mu(f, \delta, t) \leq ||g + \delta 1_{\{0_V, t\}}|| \leq ||g|| + \frac{1}{2}\varepsilon\delta = \mu(f, 0_V) + \frac{1}{2}\varepsilon\delta
$$

which means that $\lambda(t) \geq \delta > 0$. □

While noting property (4) above, we stress that sometimes $\lambda$ does vanish, and it is necessary to analyse what happens in this case.

**Lemma 4.** Suppose $V$, $f$, $\mu(f, \cdot)$, $\lambda$ and the partition $\mathcal{P}$ are as in Lemma $\Box$. If $\lambda(t) = 0$ for some $t \in W \cap \mathcal{P}$, then:

1. $W = [0_W, \infty) \cap V$;
2. $W$ is finitely-branching, in other words, $u^+ \cap W$ is finite whenever $u \in W$;
3. $W$ contains no ever-branching subsets.

**Proof.** The first property follows because $\lambda \geq 0$ and is decreasing. To prove property (2), we suppose that $u \in V$ is such that $u^+ \cap V$ is infinite. Then $u$ is a bad point for $\mu(f, \cdot)$ as $\mu(f, v) = \mu(f, u)$ for infinitely many $v \in u^+$. Consequently, the infimum in the definition of $\mu(f, u)$ is attained by part (2) of Lemma $\Box$ and it follows from Lemma $\Box$ part (4) that $\lambda(u) > 0$. As a result, $u \notin W$. For property (3), it is
Let $P_1$ since each $f$ together with Proposition 6, furnishes us with the plateau partition of $\lambda(u) > 0$. Indeed, given such $u$ and $E$, by part (3) of Lemma 2 the infimum in the definition of $\mu(f,u)$ is attained. Therefore, by part (4) of Lemma 3 $\lambda(u) > 0$. 

The proof of Theorem 12 is similar to that of Theorem 7, in that it employs monotone real-valued functions to recursively define a refining sequence of plateau partitions of the given tree. This sequence is used to define a $Z$-valued function or, in the case of Theorem 7 or Corollary 2, a $Y$-valued function. We will see that we must make use of the elements in $Z \setminus Y$ precisely when our $\lambda$-functions from Lemma 3 vanish.

of Theorem 12 Let $||f||$ be Gâteaux smooth and suppose that $\varepsilon(||f||) \leq ||f|| \leq ||f||$ for some $\varepsilon \in (0,1)$. We assemble, for each $\beta < \omega_1$, a plateau partition $\mathcal{P}_\beta$, and for each $V \in \mathcal{P}_\beta$, a function $f_{(\beta,V)} \in C_{0V}$ such that:

1. $\mu(f_{(\beta,V)}, \cdot)$ takes constant value $\mu(f_{(\beta,V)}, 0_V)$ on $V$;
2. $\mu(f_{(\beta,V)}, 0_V) = 1 - \frac{1}{2} \varepsilon(||f_{(\beta,V)}||) - 1$.

Following this, we define a function $\pi : Y \to Z$ and prove that it possesses a number of properties. Our final function $\rho$ will be a modification of $\pi$.

We begin by constructing $\mathcal{P}_0$. Recall the increasing function $\mu$ from Definition 13. Let $\mathcal{P}_0$ be its plateau partition, courtesy of Proposition 6 and define $f_{(0,V)} = 1_{(0,0_V)}$ for $V \in \mathcal{P}_0$. It follows that $\mu(f_{(0,V)}, \cdot)$ takes constant value $\mu(f_{(0,V)}, 0_V) = \mu(0_V)$ on $V$, and that

$$
\mu(f_{(0,V)}, 0_V) - 1 \leq ||f_{(0,V)}|| - 1 \leq 0 = \frac{1}{2}\varepsilon(||f_{(0,V)}||) - 1.
$$

Now suppose $\mathcal{P}_\beta$ and the associated $f_{(\beta,V)}$ have been built. Let $V \in \mathcal{P}_\beta$. If $V = \{0_V\}$ then set $\mathcal{P}_V = \{V\}$ and $f_{(\beta+1,V)} = f_{(\beta,V)}$. Otherwise, Lemma 3 together with Proposition 6 furnishes us with the plateau partition of $V \setminus \{0_V\}$ associated with the $\lambda$-function. We augment this with the single element $\{0_V\}$ to give a plateau partition $\mathcal{P}_\beta$ of $V$. Set $\mathcal{P}_{\beta+1} = \bigcup\{\mathcal{P}_V \mid V \in \mathcal{P}_\beta\}$. If $W \in \mathcal{P}_\beta$ then either $W = \{0_V\}$ or $W \subseteq V \setminus \{0_V\}$. In the former case let $f_{(\beta+1,W)} = f_{(\beta,V)}$; it is easy to see that $f_{(\beta+1,W)}$ satisfies conditions (1) and (2) above. In the latter case, let $f_{(\beta+1,W)} = f_W$, where $f_W$ is as in Lemma 3 part (3). We observe condition (1) is satisfied, again by Lemma 3 part (3). To see that condition (2) holds, note that

$$
\mu(f_{(\beta+1,W)}, 0_V) - \mu(f_{(\beta,V)}, 0_V) = \frac{1}{2}\varepsilon(0_V) = \frac{1}{2}\varepsilon(||f_{(\beta+1,W)}||_\infty - ||f_{(\beta,V)}||_\infty)
$$

and apply the inductive hypothesis.

We move on to the limit case. Suppose that $\beta < \omega_1$ is a limit ordinal and that all has been constructed for $\alpha < \beta$. Given $t \in Y$, we let $V_{\alpha} \in \mathcal{P}_\alpha$ be such that $t \in V_{\alpha}$. Set $\mathcal{P}_\beta = \{\bigcap_{\alpha < \beta} V_{\alpha}^t \mid t \in \mathcal{P}_\beta\}$. Fix some $V \in \mathcal{P}_\beta$. Let $t = 0_V$, $V_{\alpha} = V_{\alpha}^t$, $t_{\alpha} = 0_{V_{\alpha}}$ and $f_{\alpha} = f_{(\alpha,V_{\alpha})}$. Then $t = \sup_{\alpha < t_{\alpha}} t_{\alpha}$ by Proposition 6. What we would like to do is define $f_{(\beta,V)} = f \in C_0(Y)$ to be the unique function supported on $(0,t)$, such that its restriction to $(0,t_{\alpha})$ is $f_{\alpha}$. This can indeed be done, provided that $(||f_{\alpha}||_\infty)_{\alpha < \beta}$ is bounded. Observe that if $g \in C_t$ satisfies condition (2) above then

$$
\varepsilon(||g||_\infty - 1 \leq \mu(g,u) - 1 \leq \frac{1}{2}\varepsilon(||g||_\infty - 1)
$$

giving $||g||_\infty \leq \frac{3}{2} - 1$. Therefore $(||f_{\alpha}||_\infty)_{\alpha < \beta}$ is bounded as required. Moreover, since each $f_{\alpha} \in C_{t_{\alpha}}$, we have $f \in C_1$. Now set $g_{\alpha} = f_{\alpha} + f_{\alpha}(t_{\alpha})1_{(t_{\alpha},t)}$. Of course, as $f_{\alpha}$ is increasing on $(0,t_{\alpha})$ and vanishes elsewhere, we have $||g_{\alpha}||_\infty = ||f_{\alpha}||_\infty$. 


Moreover, as Μ(fα+1,t_{α+1}) takes constant value Μ(fα,t_{α}) on V_{α} by inductive hypothesis, and Μ(gα,u) = Μ(fα,u) whenever u \in V \subseteq V_{α}, it follows that Μ(gα,·) takes constant value Μ(fα,t_{α}) on V. The reader can verify that, as \((gα)_{α<β}\) converges in norm to \(f\), \((Μ(gα,·))_{α<β}\) converges uniformly to \(Μ(f,·)\) (cf. \[6\] Lemma 3.6). As a result, \(f\) satisfies conditions (1) and (2) above. This ends the recursion.

Now we define π. Given \(t \in \mathcal{Y}\), let \(V^\beta_t\) be as above. In addition, we let \(λ^\beta_t\) be the λ-function associated with \(V^\beta_{t}\) and \(f(β,V^\beta_{t})\), provided \(V^\beta_{t}\) is not a singleton. Set \(π(t)_0 = −μ(t)\). If \(β > 0\), let \(π(t)_β = μ(f(β,V^\beta_{t}),t)\) as long as \(0 < V^\beta_{t} < t\) for all \(α < β\) and \(λ^\beta_t(α) > 0\) whenever \(α + 1 < β\). Otherwise, we leave \(π(t)_β\) undefined.

We verify that \(π(0)\) is an element of \(Z\). Observe that if \(π(t)_β\) is defined, then so is \(π(t)_{α}\) whenever \(α < β\). If \(0 < α < β\) then \(π(t)_0 < 0 < π(t)_{α}\) and moreover

\[
π(t)_{α+1} = μ(f(α,V^\alpha_{t}),t) = μ(f(α,V^{\alpha+1}_{t}),t) + \frac{1}{2}ελ^\alpha_t(0V^\alpha_{t+1}) = π(t)_{α} + \frac{1}{2}ελ^\alpha_t(t)
\]

whence \(π(t)_{α+1} \geq π(t)_{α}\). In addition, if \(α + 1 < β\) then \(π(t)_{α+1} > π(t)_{α}\) by our definition of π. Now, if \(β\) is a limit ordinal and \(π(t)_{α}\) is defined for all \(α < β\), so is \(π(t)_{β}\). Moreover, by applying the uniform convergence of the μ-functions at limit stages of the partition construction, we see that \(π(t)_β = μ(f(β,V^\beta_{t}),t) = \lim_{α<β} μ(f(α,V^\alpha_{t}),t) = \lim_{α<β} π(t)_{α}\). This is enough to prove that \(π(t) \in Z\).

We observe our first property of π, namely that it is increasing. Let \(s,t \in \mathcal{Y}\) with \(s < t\). We set γ to be the least ordinal such that \(π(s)_{γ}\) and \(π(t)_{γ}\) are not both defined and equal. If \(γ = 0\) then, as μ is increasing, it follows that \(π(s)_0 > π(t)_0\), whence \(π(s) = π(t)\). If \(γ > 0\) then, by continuity, \(γ = β + 1\) for some \(β\). By transfinite induction, \(V^α_{s} = V^α_{t}\) for all \(α ≤ β\). Indeed, \(μ(s) = −π(s)_0 = −π(t)_0 = μ(t)\), so \(V^α_{s} = V^α_{t}\). If \(V^α_{s} = U = V^α_{t}\) and \(α < β\), set \(λ^α_s = λ = λ^α_t\). Remembering property (2) of Lemma \[4\] we have

\[
(1) \quad \frac{1}{2}ελ(s) = π(s)_{α+1} − π(s)_{α} = π(t)_{α+1} − π(t)_{α} = \frac{1}{2}ελ(t)
\]

whence \(λ(s) = λ(t)\) and \(V^α_{s+1} = V^α_{t+1}\). Limit stages of the induction follow by taking intersections.

Now let \(V^α_{t} = V = V^β_{t}\), \(λ^α_t = λ = λ^β_t\) and observe that \(0 < t < t\). There are two cases to consider: either \(π(t)_{β+1}\) is defined or it is not. First of all, we suppose that \(π(t)_{β+1}\) is defined and prove that \(π(s) = π(t)\) in this case. Indeed, if \(π(s)_{β+1}\) is not defined then we are done, as \(π(t)\) strictly extends \(π(s)\). On the other hand, if \(π(s)_{β+1}\) is defined then since \(π(s)_{β+1} ≠ π(t)_{β+1}\) and \(λ\) is decreasing, it must be that \(π(s)_{β+1} > π(t)_{β+1}\). Therefore \(π(s) = π(t)\).

The other option is that \(π(t)_{β+1}\) is undefined. In this case, since \(0 < t < t\), it must be that \(λ^α_t(α) = 0\) for some \(α + 1 < β + 1\), by the definition of π. As \(π(t)_{β}\) is defined then, again by the definition of π, it follows that \(α + 1 = β\). Let \(V^α_{s} = U = V^α_{t}\) and \(λ^α_s = λ = λ^α_t\). Then by Eqn. \[4\] above, we have \(λ′(s) = λ′(t) = 0\), meaning \(π(s)_{β+1}\) is not defined either. Consequently, \(π(s) = π(t)\).

We have established that π is an increasing function. Now we show that it is not constant on any ever-branching subset and, given \(t \in \mathcal{Y}\), there are only finitely many \(u \in t^+\) such that \(π(u) = π(t)\). To prove this claim, consider \(t \in \mathcal{Y}\) and the plateau \(W = \{u \in [t,∞) | π(u) = π(t)\}\). If \(W\) is the singleton \(\{t\}\) then there is nothing to prove, so we suppose that there exists some \(u \in W\) with \(t < u\). Let
both \( \pi(t) \) and \( \pi(u) \) be defined on \([0, \beta]\) and fix \( V = V' \). In just the same way as above, we have that \( V'_\alpha = V'\beta \) whenever \( \alpha \leq \beta \) and, in particular, \( V'_\beta = V \).

Observe that, as a consequence, \( W \subseteq V \). Moreover, just as above, as \( \pi(u)_{\beta+1} \) is undefined and \( 0_{V'_\beta} < t < u \), we have \( \beta = \alpha + 1 \) for some \( \alpha \). It follows that if we set \( V'_\alpha = U = V'\alpha \) and \( \lambda'_\alpha = \lambda' = \lambda'_\alpha \), then \( \lambda'(t) = \lambda'(u) = 0 \).

Now we can appeal to parts (2) and (3) of Lemma \( \square \) applied to \( U \), \( f_{(\alpha, \alpha)} \), \( \mu(f_{(\alpha, \alpha)}) \cdot \) and \( \lambda' \) to conclude that \( V \) is finitely-branching and contains no ever-branching subsets. As \( W \subseteq V \), we are done.

We finish our appraisal of \( \pi \) by showing that it does not admit certain types of \( Z \)-bad points. First of all, if \( \pi(t) \in Y \) then \( t \) cannot be \( Z \)-bad for \( \pi \). Indeed, by Lemma \( \square \) part (1) and the fact that the elements of \( \text{ran} \pi \) are uniformly bounded sequences, the only way that \( t \) can be \( Z \)-bad for \( \pi \) is if there are infinitely many \( u \in t^+ \) such that \( \pi(u) = \pi(t) \). Now suppose that \( \pi(t) = (\pi(t)_{\xi})_{\xi \leq \beta+1} \in Z \setminus Y \), where \( \beta \) is a limit ordinal. We prove that \( t \) is not \( Z \)-bad for \( \pi \).

We know already that \( \pi(u) = \pi(t) \) for only finitely many \( u \in t^+ \) so, for a contradiction, we must suppose that there is a sequence of distinct points \( (u_n) \subseteq t^+ \) such that \( \pi(t) < \pi(u_n) \) and \( (u_n) \rightarrow t \). We have that \( \pi(t)_{\beta} = (\pi(t)_{\beta})_{\beta+1} \). Let \( V = V'_j \), where \( V_j \) is the unique element \( V \in \mathcal{P}_\beta \) containing \( t \), and let \( f = f_{(\beta, V)} \). Observe that if \( \lambda = \text{the function from Lemma \( \square \)} \) associated with \( f \) and \( V \) then, necessarily, \( \lambda(0) = 0 \). Indeed, by the definition of \( \pi \), we have \( \frac{1}{2} \lambda(t) = (\pi(t)_{\beta})_{\beta+1} - (\pi(t)_{\beta})_{\beta} \). By Lemma \( \square \) part (3), there exist ordinals \( \alpha_n < \beta \) such that \( \alpha_n \rightarrow \beta \), \( \pi(u_n)_{\xi} = \pi(t)_{\xi} \) whenever \( \xi < \alpha_n \) and \( \pi(u_n)_{\alpha_n} < \pi(t)_{\alpha_n} \).

By continuity and transfinite induction, \( \alpha_n = \xi_n + 1 \) for some ordinals \( \xi_n \) and \( V'_n = V'_{n+1} \). Set \( V_n = V'_{\xi_n} \) and \( f_n = f_{(\xi_n, V_n)} \). As \( \alpha_n \rightarrow \beta \), it follows that \( V = \bigcap_n V_n \) and the functions \( f_n + f_n(0_{V_n})1_{(0_{V_n}, t]} \) converge in norm to \( f + f(0_V)1_{(0_v, t]} \). Moreover \( \mu(f_n, u_n) = \pi(u_n)_{\xi_n} = \pi(t)_{\xi_n} \rightarrow \pi(t)_{\beta} = \mu(f, t) \). Now choose \( \varphi_n \in \mathcal{C}_0(Y) \) to satisfy \( \text{supp} \varphi_n \subseteq (u_n, \infty) \) and \( \|f_n + f_n(0_{V_n})1_{(0_{V_n}, u_n)} + \varphi_n\| \leq \mu(f_n, u_n) + 2^{-n} = \mu(f_n, t) + 2^{-n} \). As the \( u_n \) are distinct, it follows that \( (f_n + f_n(0_{V_n})1_{(0_{V_n}, u_n)} + \varphi_n) \) converges to \( f + f(0_V)1_{(0_v, t]} \) in the pointwise topology of \( \mathcal{C}_0(Y) \). As \( Y \) is scattered and this sequence is norm-bounded, it converges in the weak topology too. Therefore \( \|f + f(0_V)1_{(0_v, t]}\| = \mu(f, t) \). However, by part (4) of Lemma \( \square \) the attainment of the infimum forces \( \lambda(t) > 0 \), which is not the case. It follows that \( t \) cannot be a \( Z \)-bad point for \( \pi \).

One case remains untreated. If \( \pi(t) = (\pi(t)_{\xi})_{\xi \leq \beta+1} \in Z \setminus Y \) and \( \beta \) is not a limit ordinal, it is possible that \( t \) is \( Z \)-bad for \( \pi \). Fortunately, by making an adjustment to \( \pi \) akin to that given after Lemma \( \square \) we can remove \( Z \)-bad points of this kind.

Given \( x = (x_{\xi})_{\xi \leq \beta} \in Z \), define

\[
\Phi(x)_{\xi} = \begin{cases} 
2x_0 & \text{if } \xi = 0 \\
x_{\xi} + x_{\xi-1} + 1 & \text{if } \xi \text{ is a successor ordinal} \\
2x_{\xi} + 1 & \text{otherwise}
\end{cases}
\]

for \( \xi \leq \beta \). It is easy to establish that \( \Phi \) takes values in \( Z \) and is strictly increasing. Set \( \rho = \Phi \circ \pi \). As \( \Phi \) is strictly increasing, \( \rho \) is increasing and, if we consider Proposition \( \square \) partitions \( Y \) in exactly the same way as \( \pi \). In particular, \( \rho \) is not constant on any ever-branching subset of \( Y \). Again, as \( \Phi \) is strictly increasing, if \( t \) is \( Z \)-bad for \( \rho \) then it is also \( Z \)-bad for \( \pi \). Therefore, to prove that \( \rho \) has no \( Z \)-bad points, we suppose that \( \pi(t) = (\pi(t)_{\xi})_{\xi \leq \beta+1} \in Z \setminus Y \) and \( \beta \) is not a limit ordinal. We have that \( \pi(t)_{\beta} = \pi(t)_{\beta+1} \) so, by the construction of \( \pi \), there exists an ordinal \( \alpha \) such that \( \beta = \alpha + 1 \). Therefore, \( \pi(t)_{\alpha} < \pi(t)_{\beta} \) and thus \( \rho(t)_{\beta} < \rho(t)_{\beta+1} \),
giving \( \rho(t) \in Y \). Again by appealing to Lemma 1 part (1), if \( t \) is \( Z \)-bad for \( \rho \) then \( \rho(u) = \rho(t) \) for infinitely many \( u \in t^+ \). However, that would force \( \pi(u) = \pi(t) \) for infinitely many \( u \in t^+ \), and we have already established that this is impossible. □

of Corollary 4. If \( || \cdot || \) is a lattice norm then, by part (1) of Lemma 2 the infima in the definition of the \( \mu \)-functions are always attained. It follows that the \( \lambda \)-functions of Lemma 3 never vanish. Now, we prove that in this case, the map \( \pi \) defined in the proof of Theorem 12 is \( Y \)-valued and strictly increasing. Indeed, if we return to the point where we prove that \( \pi(t) \in Z \), we see that, as the \( \lambda \)-functions never vanish, \( \pi(t)_\alpha < \pi(t)_{\alpha+1} \) whenever \( \alpha + 1 \leq \beta \). Consequently \( \pi(t) \in Y \). To show that \( \pi \) is strictly increasing, we let \( s \prec t \) and return to the point in the proof where \( \pi \) is shown to be increasing, specifically, where \( \gamma \) is defined. If \( \gamma = 0 \) then we are done. Otherwise, \( \gamma = \beta + 1 \) for some \( \beta \). Since the \( \lambda \)-functions never vanish, it is impossible that \( \pi(t)_{\beta+1} \) is undefined, therefore \( \pi(s) < \pi(t) \). This proves that \( \Upsilon \not\preceq Y \). The second statement of Corollary 2 holds because the strictly convex dual norm constructed in Theorem 11 is a lattice norm. □

We finish the section with a proof of Proposition 3. It will help to introduce a useful game-theoretic characterisation of Baire trees 5. Players \( A \) and \( B \) take turns to nominate elements of a tree \( \Upsilon \), beginning with \( t_0 \) played by \( B \). In general, \( A \) follows \( t_{2n} \) with \( t_{2n+1} \geq t_{2n} \), and \( B \) responds with \( t_{2n+2} \geq t_{2n+1} \). The game is won by \( B \) if the sequence \((t_n)\) has no upper bound in \( \Upsilon \). The tree \( \Upsilon \) is Baire if and only if \( B \) has no winning strategy in this so-called \( \Upsilon \)-game. Using this game, it is possible to prove the following result.

**Proposition 7** ([Haydon 5 Proposition 1.4]). If \( \Upsilon \) is Baire and \( \rho : \Upsilon \rightarrow \mathbb{R} \) is increasing, then there exists \( t \in \Upsilon \) such that \( \rho \) is constant on the wedge \( [t, \infty) \).

One trivial consequence of Proposition 3 is that if the increasing map \( \rho : \Upsilon \rightarrow \mathbb{R} \) is not constant on any ever-branching subset then \( \Upsilon \) contains no ever-branching Baire subsets. Indeed, if \( E \subseteq \Upsilon \) were ever-branching and Baire then, by Proposition 7 we could find \( t \in E \) such that \( \rho \) is constant on \( [t, \infty) \cap E \), which is an ever-branching subset of \( \Upsilon \). We observe that the same holds if we replace \( \mathbb{R} \) with any linear order \( \Sigma \) satisfying the statement of Proposition 3. Therefore, to establish Proposition 3 it is enough to prove the following result.

**Proposition 8.** If \( \Upsilon \) is Baire and \( \rho : \Upsilon \rightarrow Z \) is increasing, then there exists \( t \in \Upsilon \) such that \( \rho \) is constant on \( [t, \infty) \).

**Proof.** The following order will be used in this and a subsequent proof. Define \( Z_0 = \{ x = (x_\alpha)_{\alpha \leq \beta} \in Z \mid x \subseteq [0, 1], x_0 = 0 \text{ and } \beta \text{ is a limit whenever } x_\beta = 1 \} \).

By considering the map \( \Theta \), introduced after Lemma 1 we observe that \( Z \not\preceq Z_0 \) and, accordingly, we can assume that our increasing function \( \rho \) takes values in \( Z_0 \).

We show that \( \rho \) is constant on some wedge of \( \Upsilon \) by playing the \( \Upsilon \)-game with a particular strategy for \( B \). Given \( u \in \Upsilon \) and an ordinal \( \alpha \), we call \((\alpha,u)\) a fixed pair if \( \rho(v)_\xi \) is defined and equal to \( \rho(u)_\xi \) whenever \( v \in [u, \infty) \) and \( \xi \leq \alpha \). If \( (\alpha,u) \) is fixed, \( v \in [u, \infty) \) and \( \xi \leq \alpha \), then \( (\xi,v) \) is also fixed. Let \( B \) play arbitrary \( t_0 \) as the first move and put \( \alpha_0 = 0 \). Note that \((0,t_0)\) is fixed. Now suppose that \( n \geq 1 \) and that moves \( t_0 \preceq t_1 \preceq \ldots \preceq t_{2n-1} \) have been played alternately by \( B \) and \( A \). We choose the next move \( t_{2n} \) played by \( B \), together with \( \alpha_n \), in the following manner.
Let
\[ r_n = \sup\{\rho(u)_{\alpha} \mid u \geq t_{2n-1} \text{ and } (\alpha, u) \text{ is a fixed pair}\} .\]

Let \( B \) choose fixed \((\alpha_n, t_{2n})\) such that \( t_{2n} \geq t_{2n-1} \) and \( \rho(t_{2n})_{\alpha_n} > r_{n-2} - n \). This strategy does not guarantee a win for \( B \), so there exist moves \((t_{2n+1})\) of \( A \) such that \((t_n)\) has an upper bound \( u \in Y \). If \( \alpha = \sup \alpha_n \), we see that \((\alpha, u)\) is fixed. This follows by continuity and the fact that \((\alpha_n, u)\) is fixed for all \( n \).

If \( \rho_{\alpha+1} \) is not defined for any \( v \) then \( \rho \) takes constant value \( \rho(u) \) on \([u, \infty)\), and we are done. Suppose instead that \( \rho(v)_{\alpha+1} \) exists for some \( v \geq u \). Because \((\alpha, v)\) is fixed and \( \rho \) is increasing, the real-valued map \( \rho(\cdot)_{\alpha+1} \) must be decreasing on \([v, \infty)\). As the forcing-open set \([v, \infty)\) is Baire, by Proposition 7 there exists \( w \geq v \) such that \( \rho(\cdot)_{\alpha+1} \) is constant on \([w, \infty)\), and it follows that \((\alpha+1, w)\) is a fixed pair. We note that the inequalities
\[ r_n - 2^{-n} < \rho(t_{2n})_{\alpha_n} = \rho(w)_{\alpha_n} \leq \rho(w)_{\alpha} \leq \rho(w)_{\alpha+1} \leq r_n \]
hold for all \( n \), and conclude that \( \rho(w)_{\alpha+1} = \rho(w)_{\alpha} \). Consequently, by the definition of elements of \( Z \), \( \rho \) takes constant value \( \rho(w) \) on \([w, \infty)\).

\[ \square \]

3. EXAMPLES

In this section, we prove Proposition 4 and present Example 15. Before giving the proof of Proposition 4, we make an observation about embeddability and \( Z \)-bad points that is analogous to Proposition 1.

Given a tree \( \Upsilon \), let \( \Upsilon \not\leq Z \) and suppose that there is an increasing function \( \rho : \Upsilon \to Z \) with no \( Z \)-bad points. We claim that if this is the case then \( \Upsilon \not\leq Y \).

In order to prove this claim, we introduce the following algebraic operation on \( Z \). Recall the order isomorphism \( \theta : R \to (0, 1) \), fixed after Lemma 1. For \( x = (x_\xi)_{\xi \leq \alpha} \) and \( y = (y_\xi)_{\xi \leq \beta} \) of \( Z \), define \( x \cdot y \) for \( \xi \leq max\{\alpha, \beta\} \) by
\[
(x \cdot y)_\xi = \begin{cases} 
\theta^{-1}(\theta(x_\xi)\theta(y_\xi)) & \text{if } \xi \leq min\{\alpha, \beta\} \\
x_\xi & \text{if } \alpha < \xi \leq \beta \\
y_\xi & \text{if } \beta < \xi \leq \alpha 
\end{cases}
\]

where \( \theta(x_\xi)\theta(y_\xi) \) is an ordinary real product. We leave the reader with the simple task of verifying that \( \cdot \) is a semigroup operation on \( Z \) that respects the order; in other words, if \( x \leq y \) and \( u \leq v \) then \( x \cdot u \leq y \cdot v \) and, moreover, the third inequality is strict if either of the first two are. Now, let the increasing function \( \nu : \Upsilon \to Z \) have no \( Z \)-bad points and suppose \( \pi : \Upsilon \to Z \) is strictly increasing. As \( \cdot \) respects order, it follows that the pointwise product \( \pi \cdot \tau \) is strictly increasing and has no \( Z \)-bad points. By Lemma 1, any element of \( Z \) can be approached from above by a strictly decreasing sequence. Therefore, as \( t \in \Upsilon \) is not a \( Z \)-bad point for \( \pi \), there exists \( \pi^*(t) \in Z \) such that \( \pi(t) < \pi^*(t) \leq \pi(u) \) whenever \( u \in t^+ \). Finally, since \( Y \) is dense in \( Z \), we can pick \( \rho(t) \in Y \) between \( \pi(t) \) and \( \pi^*(t) \); the resulting function \( \rho \) is strictly increasing.

**Proof of Proposition 4**. In the light of Theorem 5 and our observation above, all we need to do is prove that \( \sigma Y \not\leq Z \). Recall the order \( Z_0 \) from the proof of Proposition 8. As \( Z \not\leq Z_0 \), elements of \( \sigma Y \) can and are considered as subsets of \( Z_0 \). Our proof that \( \sigma Y \not\leq Z \) rests on the claim that \( Z_0 \) is Dedekind complete; that is, each subset of \( A \) of \( Z_0 \) has a least upper bound, denoted by \( sup A \).

For now, we assume that this claim holds and define a strictly increasing map \( \rho : \sigma Y \to Z \). Given \( A \in \sigma Y \), treated as a subset of \( Z_0 \), let \( \rho(A) = sup A \) if \( sup A \in \)}
it follows that $\beta > \alpha$, $x$ is a limit, let $\alpha$, $x$ where $(\beta, x)$ is some fixed pair. By the nature of fixed pairs, this is well-defined. If $(\beta, x)$ is fixed, subject to the condition that there is no fixed pair $(\beta, x)$. As $A$ is non-empty and $(0, x)$ is fixed whenever $x \in A$, it follows that $\beta > 0$. We define a sequence $z = (z_\alpha)_{\alpha \leq \beta}$. If $\alpha < \beta$, let $z_\alpha = x_\alpha$, where $(\alpha, x)$ is some fixed pair. By the nature of fixed pairs, this is well-defined. If $\beta$ is a limit, let $z_\beta = \sup_{\alpha < \beta} z_\alpha$. Instead, if $\beta = \alpha + 1$ for some $\alpha$ then, as $A$ has no greatest element, there exists a fixed pair $(\alpha, x)$, such that $x_\beta$ is defined. Let $z_\beta$ be the infimum of all such $x_\beta$. It is easy to verify that $z \in Z_0$; it can be that $z_\beta = 1$, but only if $\beta$ is a limit ordinal. We omit the pedestrian task of proving that $z$ is the least upper bound of $A$. 

Our last task is to show that there is a tree $\Psi$ satisfying the condition of Theorem 9 but not that of Theorem 8. Before doing so, we must make some remarks. Recall the plateau partitions of Proposition 6 and note the following slightly reworded version of a result from [8].

**Proposition 9** ((Smith [8], Corollary 3)). *Suppose that $\Upsilon$ is a tree, $\Sigma$ a linear order, and $\rho : \Upsilon \rightarrow \Sigma$ an increasing function that is not constant on any ever-branching subset of $\Upsilon$. Then there exists an increasing function $\pi : \Upsilon \rightarrow \Sigma \times \omega$, such that the plateau partition $\mathcal{P}$ of $\Upsilon$ with respect to $\pi$ consists solely of linearly ordered subsets.*

Let $\Upsilon$, $\Sigma$, $\pi$ and $\mathcal{P}$ be as in Proposition 9 and, moreover, let us suppose that $\Upsilon$ admits no uncountable linearly ordered subsets. In this case, each $V \in \mathcal{P}$ identifies with a finite or countable ordinal and, therefore, there exists a strictly increasing function $\pi_V : V \rightarrow \mathbb{Q}$. It is apparent that the function $\tau : \Upsilon \rightarrow \Sigma \times \omega \times \mathbb{Q}$, defined by $\tau(t) = (\pi(t), \pi_V(t))$, where $V_t$ is the unique element of $\mathcal{P}$ containing $t$, is strictly increasing. As $\omega \times \mathbb{Q} \cong \mathbb{Q}$, it follows that $\Upsilon \cong \Sigma \times \mathbb{Q}$.

**Example 15.** Observe that $Y$ has cardinality continuum $c$. If $A \in \sigma Y$ then $A^+$ identifies with the set $u(A)$ of all upper bounds of $A$ and, thus, has cardinality $c$ if $u(A)$ is non-empty. Fix a well-order $\sqsubseteq$ of $Y$, and let $\Psi = \sigma Y \times c$. We order $\Psi$ by declaring that $(A, \alpha) \sqsubseteq (B, \beta)$ if and only if either $A = B$ and $\alpha \leq \beta$, or if $A < B$ and $\alpha$ is no greater than the order type of $\{x \in u(A) \mid x \sqsubset \min(B \setminus A, \leq)\}$, with respect to $\sqsubseteq$.

With respect to this order, each element of $\Psi$ has between one and two immediate successors. Indeed, if $(A, \alpha) \in \Psi$ then $(A, \alpha + 1)$ is always an immediate successor. If $u(A)$ is non-empty then $(A \cup \{y\}, 0)$ is also such a successor, where $y \in u(A)$ and
\(\{x \in u(A) \mid x \sqsubseteq y\}\) has order type \(\alpha\). The set \(\sigma Y \times \{0\}\) is a natural copy of \(\sigma Y\) inside \(\Psi\) that is closed with respect to the interval topology.

Now, by Proposition 4, there exists a strictly increasing map \(\pi : \sigma Y \to Z\). Define \(\rho : \Psi \to Z\) by \(\rho(A, \alpha) = \pi(A)\). By Proposition 6, the plateau partition of \(\Psi\) with respect to \(\rho\) consists exactly of the sets \(\{(A, \alpha) \mid \alpha < \epsilon\}\), where \(A \in \sigma Y\). Therefore, \(\rho\) is not constant on any ever-branching subset. Because the number of immediate successors of any element of \(\Psi\) is at most two, \(\rho\) has no \(Z\)-bad points either. Therefore \(\Psi\) satisfies the condition of Proposition 12.

On the other hand, there exists no increasing \(Y\)-valued function on \(\Psi\) that is not constant on any ever-branching subset. Indeed, if there were such a function, by considering its restriction to \(\sigma Y \times \{0\}\), there would be a map \(\tau : \sigma Y \to Y\), also not constant on any ever-branching subset. However, by following a similar argument to that given after Proposition 7, being \(Z\)-embeddable, \(\sigma Y\) has no perfect Baire subsets. In particular, \(\sigma Y\) does not contain a copy of \(\omega_1\). Therefore, by Proposition 2 and the remarks following Proposition 8, we would have \(\sigma Y \preceq Y \times \mathbb{Q} \preceq Y\) which, by Theorem 5 is impossible.

We recall Problem 1 and conjecture that \(C_0(\Psi)\) admits a Gâteaux norm. The Gâteaux norms presented in [8] are built by combining norms obtained from existing techniques, namely the Fréchet norms of Talagrand and Haydon, and norms with strictly convex duals. In the author’s opinion, if Problem 1 is to be resolved positively, we require a method of constructing Gâteaux norms on \(C(K)\) spaces that unifies these techniques on a more fundamental level.

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