Matroid Prophet Inequalities

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Abstract

Consider a gambler who observes a sequence of independent, non-negative random numbers and is allowed to stop the sequence at any time, claiming a reward equal to the most recent observation. The famous prophet inequality of Krengel, Sucheston, and Garling asserts that a gambler who knows the distribution of each random variable can achieve at least half as much reward, in expectation, as a “prophet” who knows the sampled values of each random variable and can choose the largest one. We generalize this result to the setting in which the gambler and the prophet are allowed to make more than one selection, subject to a matroid constraint. We show that the gambler can still achieve at least half as much reward as the prophet; this result is the best possible, since it is known that the ratio cannot be improved even in the original prophet inequality, which corresponds to the special case of rank-one matroids. Generalizing the result still further, we show that under an intersection of \( p \) matroid constraints, the prophet’s reward exceeds the gambler’s by a factor of at most \( O(p) \), and this factor is also tight.

Beyond their interest as theorems about pure online algorithms or optimal stopping rules, these results also have applications to mechanism design. Our results imply improved bounds on the ability of sequential posted-price mechanisms to approximate Bayesian optimal mechanisms in both single-parameter and multi-parameter settings. In particular, our results imply the first efficiently computable constant-factor approximations to the Bayesian optimal revenue in certain multi-parameter settings.

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1 Introduction

In 1978, Krengel, Sucheston and Garling [17] proved a surprising and fundamental result about the relative power of online and offline algorithms in Bayesian settings. They showed that if $X_1, X_2, \ldots, X_n$ is a sequence of independent, non-negative, real-valued random variables and $E[\max_i X_i] < \infty$, then there exists a stopping rule $\tau$ such that

$$2 \cdot E[X_\tau] \geq E[\max_i X_i].$$

(1)

In other words, if we consider a game in which a player observes the sequence $X_1, X_2, \ldots, X_n$ and is allowed to terminate the game at any time, collecting the most recently observed reward, then a prophet who can foretell the entire sequence and stop at its maximum value can gain at most twice as much payoff as a player who must choose the stopping time based only on the current and past observations. The inequality (1) became the first of many “prophet inequalities” in optimal stopping theory. Expressed in computer science terms, these inequalities compare the performance of online algorithms versus the offline optimum for problems that involve selecting one or more elements from a random sequence, in a Bayesian setting where the algorithm knows the distribution from which the sequence will be sampled whereas the offline optimum knows the values of the samples themselves and chooses among them optimally. Not surprisingly, these inequalities have important applications in the design and analysis of algorithms, especially in algorithmic mechanism design, a connection that we discuss further below.

In this paper, we prove a prophet inequality for matroids, generalizing the original inequality (1) which corresponds to the special case of rank-one matroids. More specifically, we analyze the following online selection problem. One is given a matroid whose elements have random weights sampled independently from (not necessarily identical) probability distributions on $\mathbb{R}_+$. An online algorithm, initialized with knowledge of the matroid structure and of the distribution of each element’s weight, must select an independent subset of the matroid by observing the sampled value of each element (in a fixed, prespecified order) and making an immediate decision whether or not to select it before observing the next element. The algorithm’s payoff is defined to be the sum of the weights of the selected elements. We prove in this paper that for every matroid, there is an online algorithm whose expected payoff is least half of the expected weight of the maximum-weight basis. It is well known that the factor 2 in Krengel, Sucheston, and Garling’s inequality (1) cannot be improved (see Section 5 for a lower bound example) and therefore our result for matroids is the best possible, even in the rank-one case.

Our algorithm is quite simple. At its heart lies a new algorithm for achieving the optimal factor 2 in rank-one matroids: compute a threshold value $T = E[\max_i X_i]/2$ and accept the first element whose weight exceeds this threshold. This is very similar to the algorithm of Samuel-Cahn [19], which uses a threshold $T$ such that $\Pr(\max_i X_i > T) = \frac{1}{2}$ but is otherwise the same, and which also achieves the optimal factor 2. It is hard to surpass the elegance of Samuel-Cahn’s proof, and indeed our proof, though short and simple, is not as elegant. On the other hand, our algorithm for rank-one matroids has a crucial advantage over Samuel-Cahn’s: it generalizes to arbitrary matroids without weakening its approximation factor. The generalization is as follows. The algorithm pretends that the online selection process is Phase 1 of a two-phase game; after each $X_i$ has been revealed in Phase 1 and the algorithm has accepted some set $A_1$, Phase 2 begins. In Phase 2, a new weight will be sampled for every matroid element, independently of the Phase 1 weights, and the algorithm will play the role of the prophet on the Phase 2 weights, choosing the max-weight subset $A_2$ such that $A_1 \cup A_2$ is independent. However, the payoff for choosing an element in Phase 2 is only half of its weight. When observing element $i$ and deciding whether to select it, our algorithm can be interpreted as making the choice that would maximize its expected payoff if Phase 1 were to end immediately after making this decision and Phase 2 were to begin. Of course, Phase 2 is purely fictional: it never actually takes place, but it plays a key role in both the design and the analysis of the algorithm. Note that this algorithm, specialized to rank-one matroids, is precisely the one proposed at the start of this paragraph: the expected value of

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More precisely, it was the second prophet inequality. The same inequality with a factor of 4, instead of 2, was discovered a year earlier by Krengel and Sucheston [10].
proceeding to Phase 2 without selecting anything would be $T = \mathbb{E} [\max_i X_i]/2$, hence our algorithm picks an element if and only if its weight exceeds $T$.

We next extend our algorithm to the case in which the feasibility constraint is given by a matroid intersection rather than a single matroid. For intersections of $p$ matroids, we present an online algorithm whose expected payoff is at least $\frac{1}{4p - 2}$ times the expected maximum weight of a feasible set. The algorithm is a natural extension of the one described earlier. It again imagines a fictional Phase 2 in which new independent random weights are sampled for all elements and revealed simultaneously, and the payoff for selecting an element in Phase 2 equals half of its weight. This time, we let $M_2$ denote the max-weight feasible set of Phase 2 elements, designate one of the $p$ matroids uniformly at random, and allow the algorithm to choose any $A_2 \subseteq M_2$ such that $A_1 \cup A_2$ is independent in the designated matroid. Observe that this is in fact a generalization of our algorithm for a single matroid, as enforcing $A_2 \subseteq M_2$ is a vacuous constraint for a single matroid. In Section 5 we show that our result for matroid intersections is almost tight: we present a lower bound demonstrating that the ratio $4p - 2$ cannot be improved by more than a constant factor.

As mentioned earlier, Bayesian optimal mechanism design problems provide a compelling application of prophet inequalities in computer science and economics. In Bayesian optimal mechanism design, one has a collection of $n$ agents with independent private types sampled from known distributions, and the goal is to design a mechanism for allocating resources and charging prices to the agents, given their reported types, so as to maximize the seller’s expected revenue in equilibrium. Chawla et al. [5] pioneered the study of approximation guarantees for sequential posted pricings (SPMs), a very simple class of mechanisms in which the seller makes a sequence of take-it-or-leave-it offers to the agents, with each offer specifying an item and a price that the agent must pay in order to win the item. Despite their simplicity, sequential posted pricings were shown in [5] to approximate the optimal revenue in many different settings. Prophet inequalities constitute a key technique underlying these results; instead of directly analyzing the revenue of the SPM, one analyzes the so-called virtual values of the winning bids, proving via prophet inequalities that the combined expected virtual value accumulated by the SPM approximates the offline optimum. Translating this virtual-value approximation guarantee into a revenue guarantee is an application of standard Bayesian mechanism design techniques introduced by Roger Myerson [18]. In the course of developing these results, Chawla et al. prove a type of prophet inequality for matroids that is of considerable interest in its own right: they show that if the algorithm is allowed to specify the order in which the matroid elements are observed, then it can guarantee an expected payoff at least half as large as the prophet’s. Our result can be seen as a strengthening of theirs, achieving the same approximation bound without allowing the algorithm to reorder the elements. Unlike our setting, in which the factor 2 is known to be tight, the best known lower bound for algorithms that may reorder the elements is $\sqrt{\pi/2} \approx 1.25$.

Extending the aforementioned results from single-parameter to multi-parameter domains, Chawla et al. define in [5] a general class of multi-parameter mechanism design problems called Bayesian multi-parameter unit-demand (BMUMD). SPMs in this setting are not truthful but can be modified to yield mechanisms that approximate the Bayesian optimal revenue with respect to a weaker solution concept: implementation in undominated strategies. A narrower class of mechanisms called oblivious posted pricings (OPMs) yields truthful mechanisms, but typically with weaker approximation guarantees; for example, it is not known whether OPMs can yield constant-factor approximations to the Bayesian optimal revenue in matroid settings, except for special cases such as graphic matroids. Without resolving this question, our results lead to an equally strong positive result for BMUMD: truthful mechanisms that 2-approximate the Bayesian optimal revenue in matroid settings and $(4p - 2)$-approximate it in settings defined by an intersection of $p$ matroid constraints.

1.1 Related work

The genesis of prophet inequalities in the work of Krengel, Sucheston, and Garling [16, 17] was discussed earlier. It would be impossible in this amount of space to do justice to the extensive literature on prophet inequalities. Of particular relevance to our work are the so-called multiple-choice prophet inequalities in
which either the gambler, the prophet, or both are given the power to choose more than one element. While several papers have been written on this topic, e.g. [12] [13] [14], the near-optimal solution of the most natural case, in which both the gambler and the prophet have $k > 1$ choices, was not completed until the work of Alaei [1], who gave a factor-$(1 - 1/\sqrt{k + 3})^{-1}$ prophet inequality for $k$-choice optimal stopping; a nearly-matching lower bound of $1 + \Omega(k^{-1/2})$ was already known from prior work.

Research on the relationship between algorithmic mechanism design and prophet inequalities was initiated by Hajiaghayi, Kleinberg, and Sandholm [11], who observed that algorithms used in the derivation of prophet inequalities, owing to their monotonicity properties, could be interpreted as truthful online auction mechanisms and that the prophet inequality in turn could be interpreted as the mechanism’s approximation guarantee. Chawla et al. [5] discovered a much subtler relation between the two subjects: questions about the approximability of offline Bayesian optimal mechanisms by sequential posted-price mechanisms could be translated into questions about prophet inequalities, via the use of virtual valuation functions. A fuller discussion of their contributions appears earlier in this section. Recent work by Alaei [1] deepens still further the connections between these two research areas, obtaining a near-optimal $k$-choice prophet inequality and applying it to a much more general Bayesian combinatorial auction framework than that studied in [5].

While not directly related to our work, the matroid secretary problem [3] also concerns relations between optimal stopping and matroids, this time under the assumption of a randomly ordered input, rather than independent random numbers in a fixed order. In fact, the “hard examples” for many natural examples in the matroid-secretary setting also translate into hard examples for the prophet inequality setting. In light of this relation, it is intriguing that our work solves the matroid prophet inequality problem whereas the matroid secretary problem remains unsolved, despite intriguing progress on special cases [7] [15], general matroids [4], and relaxed versions of the problem [21].

Finally, the Bayesian online selection problem that we consider here can be formulated as an exponential-sized Markov decision process, whose state reflects the entire set of decisions made prior to a specified point during the algorithm’s execution. Thus, our paper can be interpreted as a contribution to the growing CS literature on approximate solutions of exponential-sized Markov decision processes, e.g. [6] [9] [10]. Most of these papers use LP-based techniques. Combinatorial algorithms based on simple thresholding rules, such as ours, are comparatively rare although there are some other examples in the literature on such problems, for example [8].

2 Preliminaries

Bayesian online selection problems. An instance of the Bayesian online selection problem (BOSP) is specified by a ground set $\mathcal{U}$, a downward-closed set system $\mathcal{I} \subseteq 2^\mathcal{U}$, and for each $x \in \mathcal{U}$ a probability distribution $F_x$ supported on the set $\mathbb{R}_+$ of non-negative real numbers. These data determine a probability distribution over functions $w : \mathcal{U} \rightarrow \mathbb{R}_+$, in which the random variables $\{w(x) \mid x \in \mathcal{U}\}$ are independent and $w(x)$ has distribution $F_x$. We refer to $w(x)$ as the weight of $x$, and we extend $w$ to an additive set function defined on $2^\mathcal{U}$ by $w(A) = \sum_{x \in A} w(x)$. Elements of $\mathcal{I}$ are called feasible sets. For a given assignment of weights, $w$, we let $\text{MAX}(w)$ denote the maximum-weight feasible set and $\text{OPT}(w)$ denotes its weight; we will abbreviate these to $\text{MAX}$ and $\text{OPT}$ when the weights $w$ are clear from context.

An input sequence is a sequence $\sigma$ of ordered pairs $(x_i, w_i) i = 1, \ldots, n$, each belonging to $\mathcal{U} \times \mathbb{R}_+$, such that every element of $\mathcal{U}$ occurs exactly once in the sequence $x_1, \ldots, x_n$. A deterministic online selection algorithm is a function $A$ mapping every input sequence $\sigma$ to a set $A(\sigma) \in \mathcal{I}$ such that for any two input sequences $\sigma, \sigma'$ that match on the first $i$ pairs $(x_1, w_1), \ldots, (x_i, w_i)$, the sets $A_i(\sigma) = A(\sigma) \cap \{1, \ldots, i\}$ and $A_i(\sigma') = A(\sigma') \cap \{1, \ldots, i\}$ are identical. A randomized online selection algorithm is a probability distribution over deterministic ones. The algorithm’s choices define decision variables $b_i(\sigma)$ which are indicator functions of the events $x_i \in A(\sigma)$. An algorithm is monotone if increasing the value of $w_i$ (while leaving the rest of $\sigma$ unchanged) cannot decrease the value of $\mathbb{E}[b_i(\sigma)]$, where the expectation is over the algorithm’s internal randomness but not the randomness of $\sigma$ (if any). A monotone deterministic online selection algorithm
can be completely described by a sequence of thresholds \( T_1(\sigma), \ldots, T_n(\sigma) \), where \( T_i(\sigma) \in \mathbb{R}_+ \cup \{\infty\} \) is the infimum of the set of weights \( w \) such that \( i \in A(\sigma') \) when \( \sigma' \) is obtained from \( \sigma \) by modifying \( w_i \) to \( w \). Conversely, for any sequence of threshold functions \( T_1, \ldots, T_n \) such that \( T_i(\sigma) \) depends only on the first \( i-1 \) elements of \( \sigma \) and \( T_i(\sigma) = \infty \) whenever \( A_{i-1}(\sigma) \cup \{i\} \notin I \), there is a corresponding monotone deterministic online selection algorithm that selects \( x_i \) whenever \( w_i \geq T_i(\sigma) \).

Notice that an algorithm as defined above is agnostic to the order in which the matroid elements will be presented, i.e. it has a well-defined behavior no matter what order the elements appear in the input sequence. One could also consider order-aware algorithms that know the entire sequence \( x_1, \ldots, x_n \) in advance (but not the weights \( w_1, \ldots, w_n \)). In the matroid setting, our factor-2 prophet inequality for order-agnostic algorithms reveals that order-aware algorithms have no advantage over order-agnostic ones in the worst case; it is an interesting open question whether the same lack of advantage holds more generally.

One can similarly distinguish between adversaries with respect to their power to choose the ordering of the sequence. The original BOSP treated in previous work \cite{16, 17} considers a fixed-order adversary. That is, the adversary chooses an ordering (or distribution over orderings) for revealing the elements of \( U \) without knowing any of the weights \( w(x) \). Our main result is an algorithm that achieves \( \frac{1}{2} \text{OPT} \) (or \( \frac{1}{4} \text{OPT} \)) against a fixed-order adversary. This result combined with the techniques of \cite{5} immediately yields OPMs for single-parameter mechanism design. To extend our results to BMUMD, we must consider a stronger type of adversary. There are many ways that an adversary could adapt to the sampled weights and/or the algorithm’s decisions, some more powerful than others. The type of adaptivity that is relevant to our paper will be called an online weight-adaptive adversary. An online weight-adaptive adversary chooses the next element of \( U \) to reveal one at a time. After choosing \( x_1, \ldots, x_{i-1} \) and learning \( w(x_1), \ldots, w(x_{i-1}) \), the online weight-adaptive adversary chooses the next \( x_i \) to reveal without knowing the weight \( w(x_i) \) (or any weights besides \( w(x_1), \ldots, w(x_{i-1}) \)). Fortunately, the same exact proof shows that our algorithm, without any modification, also achieves \( \frac{1}{2} \text{OPT} \) (or \( \frac{1}{4} \text{OPT} \)) against an online weight-adaptive adversary. The connection between BMUMD and online weight-adaptive adversaries is not trivial, and is explained in Section 6.

Matroids. A matroid \( M \) consists of a ground set \( U \) and a nonempty downward-closed set system \( \mathcal{I} \subseteq 2^U \) satisfying the matroid exchange axiom: for all pairs of sets \( I, J \in \mathcal{I} \) such that \(|I| < |J|\), there exists an element \( x \in J \) such that \( I \cup \{x\} \in \mathcal{I} \). Elements of \( \mathcal{I} \) are called independent sets when \( \mathcal{I} = (U, \mathcal{I}) \) is a matroid. A maximal independent set is called a basis. If \( A \) is a subset of \( U \), its rank, denoted by rank(\( A \)), is the maximum cardinality of an independent subset of \( A \). Its closure or span, denoted by cl(\( A \)), is the set of all \( x \in U \) such that rank(\( A \cup \{x\} \)) = rank(\( A \)). It is well known that the following greedy algorithm selects a maximum-weight basis of a matroid: number the elements of \( U \) as \( x_1, \ldots, x_n \) in decreasing order of weight, and select the set of all \( x_i \) such that \( x_i \notin \text{cl}\{x_1, \ldots, x_{i-1}\} \).

3 Algorithms for Matroids

In this section we prove our main theorem, asserting the existence of algorithms whose expected reward is at least \( \frac{1}{2} \text{OPT} \) when playing against any online weight-adaptive adversary. Here is some intuition as to the considerations guiding the design of our algorithm. Imagine a prophet that is forced to start by accepting the set \( A \), and let the remainder of \( A \) (denoted \( R(A) \), defined formally in the following section) denote the subset that the restricted prophet adds to \( A \). Let the cost of \( A \) (denoted \( C(A) \), defined formally in the following section) denote the subset that the unrestricted prophet selected in place of \( A \). Then the restricted prophet makes \( w(A) + E[w(R(A))] \) in expectation, while the unrestricted prophet makes \( E[w(A)] + E[w(R(A))] \). So if \( A \) satisfies \( w(A) \geq \frac{1}{2} E[w(C(A))] \) for a small constant \( a \), it is not so bad to get stuck holding set \( A \). However, just because \( A \) is not a bad set to start with does not mean we shouldn’t accept anything that comes later. After all, the empty set is not a bad set to start with. If we can choose \( A \) in a way such that for any \( V \) we reject with \( A \cup V \notin \mathcal{I} \), \( w(V) \leq \frac{1}{4} E[w(R(A))] \), then \( A \) is not a bad set to finish with. Simply put, we want to choose thresholds that are large enough to guarantee that \( w(A) \) compares well to
\[ \mathbb{E}[w(C(A))] \], but small enough to guarantee that everything we reject is not too heavy. Indeed, the first step in our analysis is to define this property formally and show that an algorithm with this property obtains a \( \frac{1}{\alpha} \)-approximation.

### 3.1 Detour: The rank-one case

To introduce the ideas underlying our algorithm and its analysis, we start with a very simple analysis of the case of rank-one matroids. This is the special case of the problem in which the algorithm is only allowed to make one selection, i.e. the same setting as the original prophet inequality (1). Thus, the algorithm given in this section can be regarded as providing a new and simple proof of that inequality.

Let the random weights of the elements by denoted by \( X_1, \ldots, X_n \), and let \( T = \mathbb{E}[\max_i X_i]/2 \). We will show that an algorithm that stops at the first time \( \tau \) such that \( X_\tau \geq T \) makes at least \( T \) in expectation. Let \( p = \Pr[\max_i X_i \geq T] \). Then we get the following inequality, for any \( x > T \):

\[
\Pr[X_\tau > x] \geq (1-p) \sum_{i=1}^n \Pr[X_i > x]
\]

This is true because with probability \( 1-p \) the algorithm accepts nothing, so with probability at least \( (1-p) \) it has accepted nothing by the time it processes \( X_i \). So the probability that the algorithm accepts \( X_i \) and that \( X_i > x \) is at least \( (1-p) \Pr[X_i > x] \). It is also clear, by the union bound, that

\[
\sum_{i=1}^n \Pr[X_i > x] \geq \Pr[\max_i X_i > x]
\]

and therefore, for all \( x > T \),

\[
\Pr[X_\tau > x] \geq (1-p) \Pr[\max_i X_i > x].
\]

Now, observe that \( \mathbb{E}[\max_i X_i] = \int_0^T \Pr[\max_i X_i > x] \, dx + \int_T^\infty \Pr[\max_i X_i > x] \, dx = 2T \). As the first term is clearly at most \( T \), the second term must be at least \( T \). So finally, we write:

\[
\mathbb{E}[X_\tau] = \int_0^T \Pr[X_\tau > x] \, dx + \int_T^\infty \Pr[X_\tau > x] \, dx \\
\geq pT + (1-p) \int_T^\infty \Pr[\max_i X_i > x] \, dx \\
\geq pT + (1-p)T = T = \frac{1}{2} \mathbb{E}[\max_i X_i]
\]

which completes the proof of (1).

### 3.2 A property guaranteeing \( \alpha \)-approximation

To design and analyze algorithms for general matroids, we begin by defining a property of a deterministic monotone algorithm that we refer to as \( \alpha \)-balanced thresholds. In this section we prove that the expected reward of any such algorithm is at least \( \frac{1}{\alpha} \text{OPT} \). In the following section we construct an algorithm with 2-balanced thresholds, completing the proof of the main theorem.

To define \( \alpha \)-balanced thresholds, we must first define some notation. Let \( w, w' : \mathcal{U} \to \mathbb{R}_+ \) denote two assignments of weights to \( \mathcal{U} \), both sampled independently from the given distribution. We consider running the algorithm on an input sequence \( \sigma = (x_1, w(x_1)), \ldots, (x_n, w(x_n)) \) and comparing the value of its selected set, \( A = A(\sigma) \), with that of the basis \( B \) that maximizes \( w'(B) \). The matroid exchange axiom ensures that there is at least one way to partition \( B \) into disjoint subsets \( C, R \) such that \( A \cup R \) is also a basis of \( \mathcal{M} \). (Consider adding elements of \( B \) one-by-one to \( A \), preserving membership in \( \mathcal{I} \), until the two sets have equal cardinality, and let \( R \) be the set of elements added to \( A \).) Among all such partitions, let \( C(A), R(A) \) denote the one that maximizes \( w'(R) \).
**Definition 1.** For a parameter $\alpha > 0$, a deterministic monotone algorithm has $\alpha$-balanced thresholds if it has the following property. For every input sequence $\sigma$, if $A = A(\sigma)$ and $V$ is a set disjoint from $A$ such that $A \cup V \in \mathcal{I}$, then

$$\sum_{x_i \in A} T_i(\sigma) \geq \left( \frac{1}{\alpha} \right) \cdot \mathbb{E}[w'(C(A))]$$

(2)

$$\sum_{x_i \in V} T_i(\sigma) \leq \left( 1 - \frac{1}{\alpha} \right) \cdot \mathbb{E}[w'(R(A))],$$

(3)

where the expectation is over the random choice of $w'$.

**Proposition 1.** If a monotone algorithm has $\alpha$-balanced thresholds, then it satisfies the following approximation guarantee against online weight-adaptive adversaries:

$$\mathbb{E}[w(A)] \geq \frac{1}{\alpha} \text{OPT}.$$  

(4)

**Proof.** We have

$$\text{OPT} = \mathbb{E}[w'(C(A)) + w'(R(A))]$$

(5)

because $C(A) \cup R(A)$ is a maximum-weight basis with respect to $w'$, and $w'$ has the same distribution as $w$. For any real number $z$, we will use the notation $(z)^+$ to denote $\max\{z, 0\}$. The proof will consist of deriving the following three inequalities, in which $w_i$ stands for $w(x_i)$.

$$\mathbb{E} \left[ \sum_{x_i \in A} T_i \right] \geq \frac{1}{\alpha} \mathbb{E}[w'(C(A))]$$

(6)

$$\mathbb{E} \left[ \sum_{x_i \in A} (w_i - T_i)^+ \right] \geq \mathbb{E} \left[ \sum_{x_i \in R(A)} (w'(x_i) - T_i)^+ \right]$$

(7)

$$\mathbb{E} \left[ \sum_{x_i \in R(A)} (w'(x_i) - T_i)^+ \right] \geq \frac{1}{\alpha} \mathbb{E}[w'(R(A))].$$

(8)

Summing (6)-(8) and using the fact that $T_i + (w_i - T_i)^+ = w_i$ for all $x_i \in A$, we obtain

$$\mathbb{E}[w(A)] \geq \frac{1}{\alpha} \mathbb{E}[w'(C(A))] + \frac{1}{\alpha} \mathbb{E}[w'(R(A))].$$

Inequality (4) is a restatement of the definition of $\alpha$-balanced thresholds. Inequality (7) is deduced from the following observations. First, the algorithm selects every $i$ such that $w_i > T_i$, so $\sum_{x_i \in A} (w_i - T_i)^+ = \sum_{i=1}^n (w_i - T_i)^+$. Second, the online property of the algorithm and the fact that weight-adaptive adversaries do not learn $w_i$ before choosing to reveal $x_i$ imply that $T_i$ depends only on $(x_1, w_1), \ldots, (x_{i-1}, w_{i-1})$ and that the random variables $w(x_i), w'(x_i), T_i$ are independent. As $w_i = w(x_i)$ and $w'(x_i)$ are identically distributed, it follows that

$$\mathbb{E} \left[ \sum_{i=1}^n (w_i - T_i)^+ \right] = \mathbb{E} \left[ \sum_{i=1}^n (w'(x_i) - T_i)^+ \right] \geq \mathbb{E} \left[ \sum_{x_i \in R(A)} (w'(x_i) - T_i)^+ \right],$$

and (7) is established. Finally, we apply Property (3) of $\alpha$-balanced thresholds, using the set $V = R(A)$, to
deduce that
\[
\mathbb{E}\left[ \sum_{x_i \in R(A)} w'(x_i) \right] \leq \mathbb{E}\left[ \sum_{x_i \in R(A)} T_i \right] + \mathbb{E}\left[ \sum_{x_i \in R(A)} (w'(x_i) - T_i)^+ \right]
\]
\[
\leq \left(1 - \frac{1}{\alpha}\right) \mathbb{E}\left[ \sum_{x_i \in R(A)} w'(x_i) \right] + \mathbb{E}\left[ \sum_{x_i \in R(A)} (w'(x_i) - T_i)^+ \right]
\]
\[
\frac{1}{\alpha} \mathbb{E}\left[ \sum_{x_i \in R(A)} w'(x_i) \right] \leq \mathbb{E}\left[ \sum_{x_i \in R(A)} (w'(x_i) - T_i)^+ \right]
\]
Consequently (8) holds, which concludes the proof. \qed

3.3 Achieving 2-balanced thresholds

This section presents an algorithm with 2-balanced thresholds. The algorithm is quite simple. In step \(i\), having already selected the (possibly empty) set \(A_{i-1}\), we set threshold \(T_i = \infty\) if \(A_{i-1} \cup \{x_i\} \notin \mathcal{I}\), and otherwise
\[
T_i = \frac{1}{2} \mathbb{E}[w'(R(A_{i-1})) - w'(R(A_{i-1} \cup \{x_i\}))]
\]
(9)
\[
= \frac{1}{2} \mathbb{E}[w'(C(A_{i-1} \cup \{x_i\})) - w'(C(A_{i-1}))]
\]
(10)
The algorithm selects element \(x_i\) if and only if \(w_i \geq T_i\). The fact that both (9) and (10) define the same value of \(T_i\) is easy to verify. Let \(B\) denote the maximum weight basis of \(\mathcal{M}\) with weights \(w'\).
\[
w'(C(A_{i-1} \cup \{x_i\})) = w'(R(A_{i-1} \cup \{x_i\})) + w'(C(A_{i-1} \cup \{x_i\}))
\]
\[
w'(R(A_{i-1} \cup \{x_i\})) - w'(R(A_{i-1} \cup \{x_i\})) = w'(C(A_{i-1} \cup \{x_i\})) - w'(C(A_{i-1}))
\]
Property (2) in the definition of \(\alpha\)-balanced thresholds follows from a telescoping sum.
\[
\sum_{x_i \in A} T_i = \frac{1}{2} \sum_{x_i \in A} \mathbb{E}[w'(C(A_{i-1} \cup \{x_i\})) - w'(C(A_{i-1}))]
\]
\[
= \frac{1}{2} \sum_{x_i \in A} \mathbb{E}[w'(C(A_{i})) - w'(C(A_{i-1}))]
\]
\[
= \frac{1}{2} \mathbb{E}[w'(C(A)) - w'(C(A_0))] = \frac{1}{2} \mathbb{E}[w'(C(A))].
\]
The remainder of this section is devoted to proving Property (2) in the definition of \(\alpha\)-balanced thresholds. In the present context, with \(\alpha = 2\) and thresholds \(T_i\) defined by (9), the property simply asserts that for every pair of disjoint sets \(A, V\) such that \(A \cup V \in \mathcal{I}\),
\[
\mathbb{E}\left[ \sum_{x_i \in V} w'(R(A_{i-1})) - w'(R(A_{i-1} \cup \{x_i\})) \right] = 2 \sum_{x_i \in V} T_i(\sigma) \leq \mathbb{E}[w'(R(A))]
\]
We will show, in fact, that this inequality holds for every non-negative weight assignment \(w'\) and not merely in expectation. The proof appears in Proposition 2 below. To establish it, we will need some basic properties of matroids.

Definition 2 (20, Section 39.3). If \(\mathcal{M}\) is a matroid and \(S\) is a subset of its ground set, the deletion \(\mathcal{M} - S\) and the contraction \(\mathcal{M}/S\) are two matroids with ground set \(\mathcal{M} - S\). A set \(T\) is independent in \(\mathcal{M} - S\) if \(T\) is independent in \(\mathcal{M}\), whereas \(T\) is independent in \(\mathcal{M}/S\) if \(T \cup S_0\) is independent in \(\mathcal{M}\), where \(S_0\) is any maximal independent subset of \(S\).
Lemma 1. Suppose \( M = (U, \mathcal{I}) \) is a matroid and \( V, R \in \mathcal{I} \) are two independent sets of equal cardinality.

1. There is a bijection \( \phi : V \to R \) such that for every \( v \in V \), \((R - \{\phi(v)\}) \cup \{v\} \) is an independent set.
2. For a weight function \( w' : U \to \mathbb{R} \), suppose that \( R \) has the maximum weight of all \(|R|\)-element independent subsets of \( V \cup R \). Then the bijection \( \phi \) in part 1 also satisfies \( w'(\phi(v)) \geq w'(v) \).

Proof. Part 1 is Corollary 39.12a in [2]. To prove part 2 simply observe that the weight of \((R - \{\phi(v)\})\cup\{v\}\) cannot be greater than the weight of \( R \), by our assumptions on \( R \) and \( \phi \).

The next two lemmas establish basic properties of the function \( S \mapsto R(S) \).

Lemma 2. For any independent set \( A \), the set \( R(A) \) is equal to the maximum weight basis of \( M/A \).

Proof. Let \( B \) be the maximum-weight basis of \( M \). Among all bases of \( M/A \) that are contained in \( B \), the set \( R(A) \) is, by definition, the one of maximum weight. Therefore, if it is not the maximum-weight basis of \( M/A \), the only reason can be that there is another basis of \( M/A \), not contained in \( B \), having strictly greater weight. But we know that the maximum-weight basis of \( M/A \) is selected by the greedy algorithm, which iterates through the list \( y_1, \ldots, y_k \) of elements of \( U - A \) sorted in order of decreasing weight, and picks each element \( y_i \) that is not contained in \( cl(A \cup \{y_1, \ldots, y_{i-1}\}) \). In particular, every \( y_i \) chosen by the greedy algorithm on \( M/A \) satisfies \( y_i \notin cl(\{y_1, \ldots, y_{i-1}\}) \) and therefore belongs to \( B \). Thus the maximum-weight basis of \( M/A \) is contained in \( B \) and must equal \( R(A) \).

Lemma 3. For any independent set \( J \), the function \( f(S) = w'(R(S)) \) is a submodular set function on subsets of \( J \).

Proof. For notational convenience, in this proof we will denote the union of two sets by ‘+’ rather than ‘∪’. Also, we will not distinguish between an element \( x \) and the singleton set \( \{x\} \).

To prove submodularity it suffices to consider an independent set \( S + x + y \) and to prove that \( f(S) - f(S + x) \leq f(S + y) - f(S + x + y) \). Replacing \( M \) by \( M/S \), we can reduce to the case that \( S = \emptyset \) and prove that \( f(\emptyset) - f(x) \leq f(y) - f(x + y) \) whenever \( \{x, y\} \) is a two-element independent set.

What is the interpretation of \( f(\emptyset) - f(x) \)? Recall that \( f(\emptyset) = w'(R(\emptyset)) \) is the weight of the maximum-weight basis \( B \) of \( M \). Similarly, \( f(x) \) is the weight of the maximum-weight basis \( B_x \) of \( M/\{x\} \). Let \( b_1, b_2, \ldots, b_r \) denote the elements of \( B \) in decreasing order of weight. Consider running two executions of the greedy algorithm to select \( B \) and \( B_x \) in parallel. The only step in which the algorithms make differing decisions is the first step \( i \) in which \( \{b_1, \ldots, b_i\} \cup \{x\} \) contains a circuit. In this step, \( b_i \) is included in \( B \) but excluded from \( B_x \). Similarly, when we run two executions of the greedy algorithm to select \( B_y \) and \( B_{xy} \) — the maximum-weight bases of \( M/\{y\} \) and \( M/\{x, y\} \), respectively — the only step in which differing decisions are made is the earliest step \( j \) in which \( \{b_1, \ldots, b_j\} \cup \{x, y\} \) contains a circuit. But \( j \) certainly cannot be later than \( i \), since \( \{b_1, \ldots, b_j\} \cup \{x, y\} \) is a superset of \( \{b_1, \ldots, b_i\} \cup \{x\} \) and hence contains a circuit. We may conclude that

\[
f(\emptyset) - f(x) = b_i \leq b_j = f(y) - f(x + y),
\]

and hence \( f \) is submodular as claimed.

Proposition 2. For any disjoint sets \( A, V \) such that \( A \cup V \in \mathcal{I} \),

\[
\sum_{x_i \in V} w'(R(A_i - 1)) - w'(R(A_{i-1} \cup \{x_i\})) \leq w'(R(A)).
\]

Proof. The function \( f(S) = w'(R(S)) \) is submodular on subsets \( S \subseteq A \cup V \), by Lemma 3. Hence

\[
\sum_{x_i \in V} w'(R(A_{i-1})) - w'(R(A_{i-1} \cup \{x_i\})) \leq \sum_{x \in V} w'(R(A)) - w'(R(A \cup \{x\})).
\] (11)
Apply Lemma 1 to the independent sets $V, R(A)$ in $\mathcal{M}/A$ to obtain a bijection $\phi$ such that $w'(\phi(x)) \geq w'(x)$ and $A \cup (R(A) - \phi(x)) \cup \{x\} \in \mathcal{I}$ for all $x \in V$. By definition of $R(\cdot)$, we know that $A \cup \{x\} \cup R(A \cup \{x\})$ is the maximum weight independent subset of $A \cup \{x\} \cup B$ that contains $A \cup \{x\}$. One such set is $A \cup (R(A) - \phi(x)) \cup \{x\}$, so

$$
w'(A) + w'(R(A)) - w'(\phi(x)) + w'(x) \leq w'(A) + w'(R(A \cup \{x\})) + w'(x)
$$

$$
w'(R(A)) - w'(R(A \cup \{x\})) \leq w'(\phi(x))
$$

$$
\sum_{x \in V} w'(R(A)) - w'(R(A \cup \{x\})) \leq \sum_{x \in V} w'(\phi(x)) = w'(R).
$$

The proposition follows by combining (11) and (12).

\[ \square \]

4 Matroid intersections

Our algorithm and proof for matroid intersections is quite similar. We need to modify some definitions and extend some proofs, but the spirit is the same.

4.1 A generalization of $\alpha$-balanced thresholds

We first have to extend our notation a bit. Denote the independent sets for the $p$ matroids as $\mathcal{I}_1, \ldots, \mathcal{I}_p$. Denote the “truly independent” sets as $\mathcal{I} = \cap_j \mathcal{I}_j$. Still let $w, w' : \mathcal{U} \rightarrow \mathbb{R}_+$ denote two assignments of weights to $\mathcal{U}$, both sampled independently from the given distribution. We consider running the algorithm on an input sequence $\sigma = (x_1, w(x_1)), \ldots, (x_n, w(x_n))$ and comparing the value of its selected set, $A = A(\sigma)$, with that of the $B \in \mathcal{I}$ that maximizes $w'(B)$. For all $j$, the matroid exchange axiom ensures that there is at least one way to partition $B$ into disjoint subsets $C_j, R_j$ such that $A \cup R_j \in \mathcal{I}_j$, and $B \subseteq \text{cl}_j(A \cup R_j)$. Among all such partitions, let $C_j(A), R_j(A)$ denote the one that maximizes $w'(R_j)$ (greedily add elements from $B$ to $R_j$ unless it creates a dependency in $\mathcal{I}_j$). We denote by $R(A) = \cap_j R_j(A)$ and $C(A) = \cup_j C_j(A)$.

**Definition 3.** For a parameter $\alpha > 0$, a deterministic monotone algorithm has $\alpha$-balanced thresholds if it has the following property. For every input sequence $\sigma$, if $A = A(\sigma)$ and $V$ is a set disjoint from $A$ such that $A \cup V \in \mathcal{I}$, then

$$
\sum_{x_i \in A} T_i(\sigma) \geq \left( \frac{1}{\alpha} \right) \cdot \mathbb{E} \left[ \sum_j w'(C_j(A)) \right]
$$

$$
\sum_{x_i \in V} T_i(\sigma) \leq \left( \frac{1}{\alpha} \right) \cdot \mathbb{E} \left[ \sum_j w'(R_j(A)) \right],
$$

where the expectation is over the random choice of $w'$.

**Proposition 3.** If a monotone algorithm has $\alpha$-balanced thresholds for $\alpha \geq 2$, then it satisfies the following approximation guarantee against weight-adaptive adversaries when $\mathcal{I}$ is the intersection of $p$ matroids:

$$
\mathbb{E}[w(A)] \geq \frac{\alpha - p}{\alpha(\alpha - 1)} \text{OPT}.
$$

The proof closely parallels the proof of Proposition 1 and is given in the appendix.
4.2 Obtaining $\alpha$-balanced thresholds

This section presents an algorithm obtaining $\alpha$-balanced thresholds for any $\alpha > 1$. One can take a derivative to see that the optimal choice of $\alpha$ for the intersection of $p$ matroids is $\alpha_p = p + \sqrt{p(p-1)}$. For simplicity, we will instead just use $\alpha = 2p$, as this is nearly optimal and always at least 2. When $\alpha = 2p$, the approximation guarantee from Proposition 3 is $\frac{1}{2p-2}$.

We now define our thresholds. Let

$$T(A, i, j) = \frac{1}{\alpha} \mathbb{E}[w'(R_j(A)) - w'(R_j(A \cup \{x_i\}))]$$

$$= \frac{1}{\alpha} \mathbb{E}[w'(C_j(A \cup \{x_i\})) - w'(C_j(A))]$$

$$T(A, i) = \sum_j T(A, i, j).$$

In step $i$, having already selected the (possibly empty) set $A_{i-1}$, we set threshold $T_i = \infty$ if $A_{i-1} \cup \{i\} \notin \mathcal{I}$, and $T_i = T(A_{i-1}, i)$ otherwise. In other words, each $T(A, i, j)$ is basically the same as the threshold used for the single matroid algorithm if $\mathcal{I}_j$ was the only matroid constraint. It is not exactly the same, because $R(A)$ when $\mathcal{I}_j$ is the only matroid is not the same as $R_j(A)$ in the presence of other matroid constraints. $T(A, i)$ just sums $T(A, i, j)$ over all matroids.

The proof of Equation (13) follows exactly the proof of Equation (2).

The proof of Equation (14) follows from Proposition 2 although perhaps not obviously. As $A \cup V \in \mathcal{I}$, we clearly have $A \cup V \in \mathcal{I}_j$ for all $j$. So the hypotheses of Proposition 2 are satisfied for all $j$. Summing the bound we get in Proposition 2 over all $j$ gives us Equation (14).

5 Lower Bounds

Here we provide two examples. The first is the well-known example of [16] showing that the factor of 2 is tight for matroids. We present their construction here for completeness. The second shows that the ratio $O(p)$ is tight for the intersection of $p$ matroids.

We start with the well-known example of [16]. Consider the 1-uniform matroid over 2 elements. We have $w(1) = 1$ with probability 1, $w(2) = n$ with probability $1/n$ and 0 otherwise. Then the prophet obtains $2 - 1/n$ in expectation, but the gambler obtains at most 1, as his optimal strategy is just to take the first element always.

The example for the intersection of $p$ matroids has appeared in other forms in [3, 5]. Let $q$ be a prime between $p/2$ and $p$. Then let $\mathcal{U} = \{(i, j) : 0 \leq i \leq q^2 - 1, 0 \leq j \leq q - 1\}$. Then let $\mathcal{I}$ contain all sets of the form $\{(i, j_1), \ldots, (i, j_x)\}$. Now let $w(i, j) = 1$ with probability $1/q$, and $w(i, j) = 0$ otherwise, for all $i, j$. Reveal the elements in any order. No matter what strategy the gambler uses to pick the first element, his optimal strategy from that point on is to just accept every remaining element with the same first coordinate. However the gambler winds up with his first element, he makes at most $1 - 1/q$ in expectation from the remaining elements he is allowed to pick (as there are at most $q - 1$ remaining elements, and each has $\mathbb{E}[w(i, j)] = 1/q$). Therefore, the expected payoff to the gambler is less than 2. However, with probability at least $1 - 1/e$, there exists an $i$ such that $w(i, j) = 1$ for all $j$ (as the probability that this occurs for a fixed $i$ is $1/q^2$ and there are $q^2$ different $i$’s). So the expected payoff to the prophet is $\Theta(q)$.

Finally, we just have to show that $\mathcal{I}$ can be written as the intersection of $q$ matroids. Let $\mathcal{I}_x$ be the partition matroid that partitions $\mathcal{U}$ into $\cup_j S_j = \cup_j \cup_i \{(i, xi + j \pmod{q})\}$, and requires that only one element of each $S_j$ be chosen. Then clearly, $\mathcal{I} \subseteq \cap_{x \in \mathbb{Z}_q} \mathcal{I}_x$ as any two elements with the same first coordinate lie in different partitions in each of the $\mathcal{I}_x$. In addition, $\cap_{x \in \mathbb{Z}_q} \mathcal{I}_x \subseteq \mathcal{I}$. Consider any $(i, j)$ and $(i', j')$ with $i \neq i'$. Then when $(j - j') \pmod{q} = x(i - i') \pmod{q}$, $(i, j)$ and $(i', j')$ are in the same partition of $\mathcal{I}_x$. As
q is prime, this equation always has a solution. Therefore, we have shown that $I = \cap_{x \in \mathbb{Z}_q} I_x$, and $I$ can be written as the intersection of $q \leq p$ matroids. As the prophet obtains $\Theta(p)$ in expectation, and the gambler obtains less than 2 in expectation, no algorithm can achieve an approximation factor better than $O(p)$.

6 Interpretation as OPMs

Here, we describe how to use our algorithm to design OPMs for unit-demand multi-parameter bidders under matroid and matroid intersection feasibility constraints. We begin by recalling the definition of Bayesian multi-parameter unit-demand mechanism design (BMUMD) from [5]. In any such mechanism design problem, there is a set of services, $U$, partitioned into disjoint subsets $J_1, \ldots, J_n$, one for each bidder. The mechanism must allocate a set of services, subject to downward-closed feasibility constraints given by a collection $I$ of feasible subsets. We assume that the feasibility constraints guarantee that no bidder receives more than a single service, i.e. that the intersection of any feasible set with one of the sets $J_i$ contains no more than one element. (If this property is not already implied by the given feasibility constraints, it can be ensured by intersecting the given constraints with one additional partition matroid constraint.)

As in the work of Chawla et al. [5], we assume that each bidder $i$'s values for the services in set $J_i$ are independent random variables, and we analyze BMUMD mechanisms for any such distribution by exploring a closely-related single-parameter domain that we denote by $I^\text{copies}$. In $I^\text{copies}$ there are $|U|$ bidders, each of whom wants just a single service $x$ and has a value $v_x$ for receiving that service. The feasibility constraints are the same in both domains — the mechanism may select any set of services that belongs to $I$ — and the joint distribution of the values $v_x$ ($x \in I$) is the same as well; the only difference between the two domains is that an individual bidder $i$ in the BMUMD problem becomes a set of competing bidders (corresponding to the elements of $J_i$) in the domain $I^\text{copies}$. As might be expected, the increase in competition between bidders results in an increase in revenue for the optimal mechanism; indeed, the following lemma from [5] will be a key step in our analysis.

Lemma 4. Let $A$ be any individually rational and truthful deterministic mechanism for instance $I$ of BMUMD. Then the expected revenue of $A$ is no more than the expected revenue of the optimal mechanism for $I^\text{copies}$.

A second technique that we will borrow from [5] (and, ultimately, from Myerson’s original paper on optimal mechanism design [18]), is the technique of analyzing the expected revenue of mechanisms indirectly via their virtual surplus. We begin by reviewing the definitions of virtual valuations and virtual surplus. Assume that $v_x$, the value of bidder $i$ for item $x \in J_i$, has cumulative distribution function $F_x$ whose density $f_x$ is well-defined and positive on the interval on which $v_x$ is supported. Then the virtual valuation function $\phi_x$ is defined by

$$\phi_x(v) = v - \frac{1 - F_x(v)}{f_x(v)},$$

and the virtual surplus of an allocation $A \in I$ is defined to be the sum $\sum_{x \in A} \phi_x(v_x)$. Myerson [18] proved the following:

Lemma 5. In single-parameter domains whose bidders have independent valuations with monotone increasing virtual valuation functions, the expected revenue of any mechanism in Bayes-Nash equilibrium is equal to its expected virtual surplus.

The distribution of $v_x$ is said to be regular when the virtual valuation function $\phi_x$ is monotonically increasing. We will assume throughout the rest of this section that bidders’ values have regular distributions, in order to apply Lemma 5. To deal with non-regular distributions, it is necessary to use a technique known as ironing, also due to Myerson [18], which in our context translates into randomized pricing via a recipe described in Lemma 2 of [5].
Algorithm 1: Mechanism \( \mathcal{M} \) for unit-demand multi-dimensional bidders

1: Initialize \( A = \emptyset \).
2: for \( i = 1, 2, \ldots, n \) do
3: for all \( x \in J_i \) do
4: Set price \( p_x = \begin{cases} 
\phi_x^{-1}(T(A,x)) & \text{if } A \cup \{x\} \in \mathcal{I} \\
\infty & \text{otherwise.}
\end{cases} 
\)
5: end for
6: Post price vector \( (p_x)_{x \in J_i} \).
7: Bidder \( i \) chooses an element \( x \in J_i \) (or nothing) at these posted prices.
8: if \( x \) is chosen then
9: Allocate \( x \) to bidder \( i \) and charge price \( p_x \).
10: \( A \leftarrow A \cup \{x\} \)
11: else
12: Allocate nothing to bidder \( i \) and charge price 0.
13: end if
14: end for

Our plan is now to design truthful mechanisms \( \mathcal{M} \) and \( \mathcal{M}^{\text{copies}} \) for the BMUMD domain \( \mathcal{I} \) and the associated single-parameter domain \( \mathcal{I}^{\text{copies}} \), respectively, and to relate them to the optimal mechanisms for those domains via the following chain of inequalities.

\[
R(\mathcal{M}) \geq R(\mathcal{M}^{\text{copies}}) = \Phi(\mathcal{M}^{\text{copies}}) \geq \frac{1}{\alpha} \Phi(\text{OPT}^{\text{copies}}) = \frac{1}{\alpha} R(\text{OPT}^{\text{copies}}) \geq \frac{1}{\alpha} R(\text{OPT}). \tag{16}
\]

Here, \( R(\cdot) \) and \( \Phi(\cdot) \) denote the expected revenue and expected virtual surplus of a mechanism, respectively, and \( \alpha \) denotes the approximation guarantee of a prophet inequality algorithm embedded in our mechanism. Thus, \( \alpha = 2 \) when \( \mathcal{I} \) is a matroid, and more generally \( \alpha = 4p - 2 \) when \( \mathcal{I} \) is given by an intersection of \( p \) matroid constraints.

Most of the steps in line (16) are already justified by the lemmas from prior work discussed above. The relation \( R = \Phi \) for mechanisms \( \mathcal{M}^{\text{copies}} \) and \( \text{OPT}^{\text{copies}} \) is a consequence of Lemma 5, while the relation \( R(\text{OPT}^{\text{copies}}) \geq R(\text{OPT}) \) is Lemma 4. We will naturally derive the relation \( \Phi(\mathcal{M}^{\text{copies}}) \geq \frac{1}{\alpha} \Phi(\text{OPT}^{\text{copies}}) \) as a consequence of the prophet inequality. To do so, it suffices to define mechanism \( \mathcal{M}^{\text{copies}} \) such that its allocation decisions result from running the prophet inequality algorithm on an input sequence consisting of the virtual valuations \( \phi_x(v_x) \), presented in an order determined by an online weight-adaptive adversary. The crux of our proof will consist of designing said adversary to ensure that the relation \( R(\mathcal{M}) \geq R(\mathcal{M}^{\text{copies}}) \) also holds.

Given these preliminaries, we now describe the mechanisms \( \mathcal{M} \) and \( \mathcal{M}^{\text{copies}} \). Central to both mechanisms is a pricing scheme using thresholds \( T(A,x) \), defined as the threshold \( T_s \) that our online algorithm would use at step \( s \) when \( x_s = x \) and the algorithm has accepted the set \( A \) so far. (Contrary to previous sections of the paper in which steps of the online algorithm’s execution were denoted by \( i \), here we reserve the variable \( i \) to refer to bidders in the mechanism, using \( s \) instead to denote a step of the online algorithm. Note that the thresholds assigned by our algorithm depend only on \( A \) and \( x \), not on \( s \), hence the notation \( T(A,x) \) is justified.) Mechanism \( \mathcal{M} \), described by the pseudocode in Algorithm 1, simply makes posted-price offers to bidders \( 1, 2, \ldots, n \) in that order, defining the posted price for each item by applying its inverse-virtual-valuation function to the threshold that the prophet inequality algorithm sets for that item.

To define mechanism \( \mathcal{M}^{\text{copies}} \), we first define an online weight-adaptive adversary and then run the prophet inequality algorithm on the input sequence presented by this adversary, using its thresholds to define posted prices exactly as in mechanism \( \mathcal{M} \) above. The adversary is designed to minimize the mechanism’s revenue, subject to the constraint that the elements are presented in an order that runs through all of the
elements of $J_1$, then the elements of $J_2$, and so on. In fact, it is easy to compute this worst-case ordering by backward induction, which yields a dynamic program presented in pseudocode as Algorithm 2. The dynamic programming table consists of entries $V(A, i)$ denoting the expected revenue that $M^{\text{copies}}$ will gain from selling elements of the set $J_{i+1} \cup \cdots \cup J_n$, given that it has already allocated the elements of $A$. Computing and storing these values requires exponential time and space, but we are not concerned with making $M^{\text{copies}}$ into a computationally efficient mechanism because its role in this paper is merely to provide an intermediate step in the analysis of mechanism $M$.

The formula for $V(A, i)$ is guided by the following considerations. Since $M^{\text{copies}}$ will post prices $p_x = \phi_x^{-1}(T(A, x))$ for all $x \in J_{i+1}$ given that it has already allocated $A$, it will not allocate any element of $J_{i+1}$ if $v_x < p_x$ for all $x \in J_{i+1}$, and otherwise it will allocate some element $x \in J_{i+1}$. In the former case, its expected revenue from the remaining elements will be $V(A, i+1)$. In the latter case, it extracts revenue $p_x$ from bidder $i+1$ and expected revenue $V(A \cup \{x\}, i+1)$ from the remaining bidders. Thus, an adversary who wishes to minimize the revenue obtained by the mechanism will order the elements $x \in J_{i+1}$ in increasing order of $p_x + V(A \cup \{x\}, i+1)$. Denoting the elements of $J_{i+1}$ in this order by $x_1, x_2, \ldots, x_k$, we obtain the formula

$$V(A, i) = \left( \prod_{j=1}^{k} F_{x_j}(p_{x_j}) \right) \cdot V(A, i+1) + \sum_{\ell=1}^{k} \left( \prod_{j=1}^{\ell-1} F_{x_j}(p_{x_j}) \right) \cdot (1-F_{x_{\ell}}(p_{x_{\ell}})) \cdot (p_{x_{\ell}} + V(A \cup \{x_{\ell}\}, i+1)). \quad (17)$$

The first term on the right side accounts for the possibility that bidder $i+1$ buys nothing, while the sum accounts for the possibility that bidder $i+1$ buys $x_{\ell}$, for each $\ell = 1, \ldots, k$.

Mechanism $M^{\text{copies}}$ has already been described above, and is specified by pseudocode in Algorithm 3. We note that $M^{\text{copies}}$ does not satisfy the definition of an OPM in [3], since the price $p_x$ for $x \in J_i$ may depend on the bids $b_y$ for $y \in J_1 \cup \cdots \cup J_{i-1}$. However, it retains a key property of OPMs that make them
suitable for analyzing multi-parameter mechanisms: the prices of elements of $J_i$ are predetermined before any of the bids in $J_i$ are revealed.

**Theorem 1.** Mechanism $M$ for BMUMD settings with independent regular valuations obtains a 2-approximation to the revenue of the optimal deterministic mechanism for matroid feasibility constraints, and a $(4p - 2)$-approximation to the revenue of the optimal deterministic mechanism for feasibility constraints that are the intersection of $p$ matroids.

**Proof.** Both $M$ and $M_{\text{copies}}$ are posted-price (hence, truthful) mechanisms that always output a feasible allocation. To prove that the allocation is always feasible, one can argue by contradiction: if not, there must be a step in which the set $A$ becomes infeasible through adding an element $x$. However, in both $M$ and $M_{\text{copies}}$, we can see that the price $p_x$ is infinite in that case, while bid $b_x$ is greater than or equal to $p_x$, a contradiction.

The proof of the approximate revenue guarantee follows the outline given by equation (16) above. As explained earlier, the only two steps in that equation that do not follow from prior work are the relations

\[
R(M) \geq R(M_{\text{copies}}) \quad \text{(18)}
\]

\[
\Phi(M_{\text{copies}}) \geq \frac{1}{\alpha} \Phi(\text{OPT}_{\text{copies}}). \quad \text{(19)}
\]

To justify the second line, observe that the “adversary” (Algorithm 2) that computes the ordering of the bids is an online weight-adaptive adversary. This is because the adversary does not need to observe the values $v_x (x \in J_i)$ in order to sort the elements of $J_i$ in order of increasing $p_x + V(A \cup \{x\}, i - 1)$. Thus, the prophet inequality algorithm running on the input sequence specified by the adversary achieves an expected virtual surplus that is at least $\frac{1}{\alpha} \Phi(\text{OPT}_{\text{copies}})$. Furthermore, the set of elements selected by $M_{\text{copies}}$ is exactly the same as the set of elements selected by the prophet inequality algorithm — the criterion $b_x \geq p_x$ is equivalent to the criterion $w(x) \geq T(A, x)$ because $w(x) = \phi_x(b_x)$, $T(A, x) = \phi_x(p_x)$, and $\phi$ is monotone increasing. This completes the proof of (19).

To prove (18) we use an argument that, in effect, justifies our claim that Algorithm 2 is a worst-case adversary for mechanism $M_{\text{copies}}$. Specifically, for each $i = 0, \ldots, n$ and each feasible set $A \subseteq J_1 \cup \cdots \cup J_i$, let $R(M, A, i)$ and $R(M_{\text{copies}}, A, i)$ denote the expected revenue that $M$ (respectively, $M_{\text{copies}}$) obtains from

---

**Algorithm 3: Mechanism $M_{\text{copies}}$ for single-parameter domain $T_{\text{copies}}$.**

1: Obtain bids $b_x$ for all bidders $x \in U$.
2: Run Algorithm 2 using $v_x = b_x$ for all $x$, to obtain an ordering of $U$.
3: Set $w(x) = \phi_x(b_x)$ for all $x \in U$.
4: Present the pairs $(x, w(x))$ to the prophet inequality algorithm, in the order computed above.
5: Obtain thresholds $T(A, x)$ from the prophet inequality algorithm.
6: Set price $p_x = \phi_x^{-1}(T(A, x))$ for all $x \in U$.
7: // Determine allocation and payments
selling items in \( J_{i+1} \cup \cdots \cup J_n \) conditional on having allocated set \( A \) while processing the bids in \( J_1 \cup \cdots \cup J_i \). (In evaluating the expected revenue of the two mechanisms, we assume that the bidders are presented to \( \mathcal{M} \) in the order \( i = 1, \ldots, n \), and that they are presented to \( \mathcal{M}^{\text{copies}} \) in the order determined by the adversary, Algorithm \([2]\).) We will prove, by downward induction on \( i \) in the order \( i \leq n \), that

\[
\forall i, A \quad R(M, A, i) \geq R(M^{\text{copies}}, A, i) = V(A, i)
\]

and then \([18]\) follows by specializing to \( i = 0, A = \emptyset \). When \( i = n \), we have \( R(M, A, i) = R(M^{\text{copies}}, A, i) = V(A, i) = 0 \) so the base case of the induction is trivial. The relation \( R(M^{\text{copies}}, A, i) = V(A, i) \) for \( i < n \) is justified by the discussion preceding equation \([17]\). To prove \( R(M, A, i) \geq R(M^{\text{copies}}, A, i) \), suppose that both mechanisms have allocated set \( A \) while processing the bids in \( J_1 \cup \cdots \cup J_i \). Conditional on the set of \( x \in J_{i+1} \) such that \( v_x \geq p_x \) being equal to any specified set \( K \), we will prove that \( M \) obtains at least as much expected revenue as \( M^{\text{copies}} \) from selling the elements of \( J_{i+1} \cup \cdots \cup J_n \). If \( K \) is empty, then the two mechanisms will obtain expected revenue \( R(M, A, i+1) \) and \( R(M^{\text{copies}}, A, i+1) \), respectively, from elements of \( J_{i+1} \cup \cdots \cup J_n \), and the claim follows from the induction hypothesis. Otherwise, \( M^{\text{copies}} \) obtains expected revenue \( \min\{p_y + V(A \cup \{x\}, i+1) \mid x \in K \} \) while \( M \) obtains expected revenue \( p_y + R(M, A \cup \{y\}, i+1) \) where \( y \in K \) is the element of \( K \) chosen by bidder \( i+1 \) when presented with the menu of posted prices for the elements of \( J_{i+1} \). The induction hypothesis implies

\[
p_y + R(M, A \cup \{y\}, i+1) \geq p_y + V(A \cup \{y\}, i+1) \geq \min\{p_x + V(A \cup \{x\}, i+1) \mid x \in K \},
\]

and this completes the proof.

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A Proof of Proposition 3

Proof. We have

\[
\text{OPT} = \mathbb{E}[w'(C_j(A)) + w'(R_j(A))] \quad \forall j
\]

(20)

\[
\text{OPT} = \mathbb{E}[w'(C(A)) + w'(R(A))]
\]

(21)

because \(C_j(A) \cup R_j(A)\) is a maximum-weight independent set with respect to \(w'\) for all \(j\), as is \(C(A) \cup R(A)\), and \(w'\) has the same distribution as \(w\). The proof will again consist of deriving the following three inequalities.

\[
\mathbb{E}\left[ \sum_{x_i \in A} T_i \right] \geq \frac{1}{\alpha} \mathbb{E}\left[ \sum_j w'(C_j(A)) \right]
\]

(22)

\[
\mathbb{E}\left[ \sum_{x_i \in A} (w_i - T_i)^+ \right] \geq \mathbb{E}\left[ \sum_{x_i \in R(A)} (w'(x_i) - T_i)^+ \right]
\]

(23)

\[
\mathbb{E}\left[ \sum_{x_i \in R(A)} (w'(x_i) - T_i)^+ \right] \geq \mathbb{E}[w'(R(A))] - \frac{1}{\alpha} \mathbb{E}\left[ \sum_j w'(R_j(A)) \right].
\]

(24)

Summing (22) + (23) + \(\frac{1}{\alpha-1}(24)\) and using the fact that \(T_i + (w_i - T_i)^+ = w_i\) for all \(x_i \in A\), we obtain

\[
\mathbb{E}\left[ w(A) \right] \geq \left(\frac{1}{\alpha - 1} - \frac{1}{\alpha(\alpha - 1)}\right) \mathbb{E}\left[ \sum_j w'(C_j(A)) \right] + \frac{\alpha - 2}{\alpha - 1} \mathbb{E}\left[ \sum_{x_i \in R(A)} (w'(x_i) - T_i)^+ \right]
\]

\[
+ \frac{1}{\alpha - 1} \mathbb{E}[w'(R(A))] - \frac{1}{\alpha(\alpha - 1)} \mathbb{E}\left[ \sum_j w'(R_j(A)) \right].
\]

Subsituting in Equations (20) and (21) (and observing that \(\frac{\alpha - 2}{\alpha - 1} \geq 0\) whenever \(\alpha \geq 2\)), we get:

\[
\mathbb{E}\left[ w(A) \right] \geq \frac{1}{\alpha - 1} \text{OPT} - \frac{p}{\alpha(\alpha - 1)} \text{OPT} = \frac{\alpha - p}{\alpha(\alpha - 1)} \text{OPT}
\]

It remains to show that Equations (22) - (24) hold for any \(\alpha\)-balanced thresholds. Equation (22) is again a restatement of the definition of \(\alpha\)-balanced thresholds. Inequality (23) is deduced from the same observations as Equation (7). Finally, as in Proposition 1, we apply Property (14) of \(\alpha\)-balanced thresholds, using the set \(V = R(A)\), to deduce that

\[
\mathbb{E}\left[ \sum_{x_i \in R(A)} w'(x_i) \right] \leq \mathbb{E}\left[ \sum_{x_i \in R(A)} T_i \right] + \mathbb{E}\left[ \sum_{x_i \in R(A)} (w'(x_i) - T_i)^+ \right]
\]

\[
\leq \frac{1}{\alpha} \mathbb{E}\left[ \sum_j w'(R_j(A)) \right] + \mathbb{E}\left[ \sum_{x_i \in R(A)} (w'(x_i) - T_i)^+ \right]
\]

\[
\mathbb{E}[w'(R(A))] - \frac{1}{\alpha} \mathbb{E}\left[ \sum_j w'(R_j(A)) \right] \leq \mathbb{E}\left[ \sum_{x_i \in R(A)} (w'(x_i) - T_i)^+ \right]
\]

Consequently (24) holds, which concludes the proof. \(\square\)