DISCONTINUOUS PHENOMENA IN BIOREACTOR SYSTEM

HANY A. HOSHAM ∗
Department of Mathematics, Faculty of Science, Taibah University
Yanbu 41911, Saudi Arabia
Department of Mathematics, Faculty of Science, Al-Azhar University
Assiut 71524, Egypt

EMAN D. ABOU ELELA
Department of Mathematics, Faculty of Science, Taibah University
Yanbu 41911, Saudi Arabia

(Communicated by Miguel Sanjuan)

ABSTRACT. This paper critically examines discontinuous bifurcation and stability issues in model of methane gas production from organic waste via decaying process in two cases, namely sliding and non-sliding flow. The presence of certain types of discontinuities in Monod curve lead to discontinuous system and therefore the criteria for the existence and stability of equilibrium points are established. The analysis highlights the presence of several types of border collision bifurcations depending upon the effect of the dilution factor, biomass concentration and solid-liquid-gas separator efficiency, like nonsmooth fold, persistence and grazing-sliding scenarios. In addition, numerical simulations are carried out to illustrate and validate the results.

1. Introduction. The discontinuous systems (DS) are usually described by a collection of indexed differential or difference equations such that the state space or phase space splits into several domains \( D_i \) separated by surfaces \( M_j \), i.e.,

\[
\dot{\xi} = f_i(\eta, \alpha), \quad \xi \in D_i \subset \mathbb{R}^n, \alpha \in \mathbb{R}^n, \mathbb{R}^n = (\cup D_i) \cup (\cup M_j)
\]

where \( D_i, \ i = 1, 2, ..., N \) are finitely generated open domains of an \( n \)-dimensional phase space, \( M_j \) are \((n-1)\)-dimensional manifolds separating the domains \( D_j \). This subject of DS has been developed as an important topic in scientific and engineering research. Because it was shown that DS occur as models in many applications, including mechanical models (Coulomb friction, intermittent operation, hysteresis), electrical circuits (relays or transistors, switched power converter), thermal systems (phase changes), and control designs (adaptive sliding mode controller). In all of these cases the assumptions behind most of the results in bifurcation theory for smooth systems are violated and many new phenomena are observed, see [3, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 20, 21] and literatures therein.

The modeling and control of a nonlinear smooth or regular bioreactors systems are known to undergo a variety of bifurcation phenomena ranging from multiplicity

---

2010 Mathematics Subject Classification. Primary: 34A36, 34C23, 34K18, 34K20, 92B05; Secondary: 34K21, 74H60.

Key words and phrases. Discontinuous system, discontinuous bifurcation, Grazing-sliding bifurcation, UASB reactor.

∗ Corresponding author: Hany A. hosham.
and stability of steady states to sustained oscillations. Bioreactors are used in processing wastewaters that typically contain high concentrations of biodegradable organic material. Anaerobic methods are applied to reduce water contamination problems. Depuration through anaerobic treatments converts organic matter in wastewater into methane (\( CH_4 \)-biogas) and carbonic gas (\( CO_2 \)). Methane can be used as an energetic component because it offers good calorific power, and \( CO_2 \) can be recirculated to the bioreactor to improve the percentages of biogas yield, thus decreasing organic loads.

Several mathematical models have been proposed for the various bioreactor types that are used for methane production via decaying. The most notable models are based upon the Monod specifics growth rate expression. In this context, the researchers paid a lot of attention to study the dynamic behavior, the steady state, the stability and the control of the wastewater models [1, 2, 4, 5, 17, 18, 19].

This paper consequently documents the existence of different bifurcation scenarios in a modified mathematical model for gas production from organic waste. The present model has a lack of global differentiability that is responsible for the appearance of bifurcations that are forbidden in smooth systems called discontinuity-induced bifurcation (DIB). The DIB occurs as a result of the collision of a pseudo and admissible equilibrium branch on the discontinuity manifold. Further, it can be divided into two specific generic scenarios, respectively characterized by the persistence and the nonsmooth fold of the solution trajectories. Specifically, at the boundary equilibria in DS, persistence bifurcation is the case in which the branch of admissible equilibrium hits the switching surface \( M \) and turns into branch of a virtual (or pseudo) equilibrium, while the collision and disappearance of the branches admissible equilibrium with a coexisting of branch of pseudo-equilibrium occurs in the nonsmooth fold case. As well, we discuss the existence of the abrupt transitions between the trajectory and the boundary of sliding motion when the trajectory reaches a boundary tangentially which is referred to grazing-sliding bifurcation. Furthermore, we will continue to investigate the effect of design parameters (such as the dilution factor, biomass and SLG separator deficiency) variations in two main cases, namely, sliding and non-sliding flow on the nonlinear dynamic behavior and derive the analytical conditions for persistence and nonsmooth fold, grazing-sliding bifurcation scenarios.

The rest of the paper is organized as follows: In Section 2, we first propose the discontinuous system, and give some basic definitions and preliminaries regarding dynamical behavior. Then, we present a brief description of bioreactor system and its design into an entire bio-process. Also, we set up the mathematical model as a discontinuous system and identify biological parameters regarding bioreactor performance. In Sections 3 and 4, respectively, we investigate the discontinuous behaviour of the proposed model, including the existence of all the possible equilibria, their stability and related discontinuous bifurcation in two main modes, namely non-sliding and sliding mode. In section 5, we discuss the normal form of generalized Poincaré map at a grazing-sliding bifurcation point. Finally in Section 6, we draw biological conclusion on the results of this work.

2. Model of Methane gas production from organic waste. To start, let us consider the DS having a single discontinuity surface such that system (1) can be explicitly written as:

\[
\dot{\eta} = \begin{cases} 
    f_+(\eta, \alpha), & h(\eta, \alpha) > 0, \\
    f_-(\eta, \alpha), & h(\eta, \alpha) < 0,
\end{cases}
\]  

(2)
where \( f_{\pm} \in C^1(U, \mathbb{R}^n) \) and \( h(\eta, \alpha) \) is defined the discontinuity surface as \( \mathcal{M} := \{ \eta \in U \mid h(\eta, \alpha) = 0 \} \), \( \alpha \in \mathbb{R} \). Further, the two regions correspond to system (2) are given by \( \mathcal{D}_1 := \{ \eta \in U \mid h(\eta, \alpha) > 0 \} \), \( \mathcal{D}_2 := \{ \eta \in U \mid h(\eta, \alpha) < 0 \} \).

Let \( \rho(\eta, \alpha) = (n^T(\eta, \alpha)f_+ (\eta, \alpha))(n^T(\eta, \alpha)f_- (\eta, \alpha)) \), where the normal vector \( n(\eta, \alpha) \) perpendicular to the manifold \( \mathcal{M} \) is given by \( n(\eta, \alpha) = \frac{\nabla \eta(\eta, \alpha)}{\| \nabla \eta(\eta, \alpha) \|} \), \( ||n(\eta, \alpha)|| = 1 \). Then, we distinguish the following domains on \( \mathcal{M} \):

(i): The crossing region \( \mathcal{M}_c := \{ \eta \in \mathcal{M} | \rho(\eta, \alpha) > 0 \} \) implies that all trajectories of (2) approaching the discontinuity surface \( \mathcal{M} \) cross it immediately. Therefore, for such starting point, there is a unique absolutely continuous flow.

(ii): The sliding region \( \mathcal{M}_s := \{ \eta \in \mathcal{M} | \rho(\eta, \alpha) \leq 0, n^T(\eta, \alpha)f_+ (\eta, \alpha) < 0 \} \) implies that a flow of (2) reaching \( \mathcal{M}_s \) has to stay in \( \mathcal{M}_s \) until it reaches the boundary of \( \mathcal{M}_s \).

(iii): The escaping region \( \tilde{\mathcal{M}}_s := \{ \eta \in \mathcal{M} | \rho(\eta, \alpha) \leq 0, n^T(\eta, \alpha)f_+ (\eta, \alpha) > 0 \} \).

These conditions meaning that any trajectories starting in or reaching \( \mathcal{M}_s \) are directed away from the surface; hence, the flow cannot be continued uniquely.

On the discontinuous manifold, the flow in \( \mathcal{M}_s \) itself is governed by Filippov’s extension [8]:

\[
\dot{\eta} = F_{s}(\eta, \alpha) = \lambda(\eta, \alpha)f_+ (\eta, \alpha) + (1 - \lambda(\eta, \alpha))f_- (\eta, \alpha), \lambda \in [0, 1],
\tag{3}
\]

with such that \( n^T(\eta, \alpha)F_{s}(\eta, \alpha) = 0 \).

The existence and uniqueness of solutions of (2) and (3) are guaranteed by the theory of differential inclusions [8]. Then, we assume that when \( \alpha = 0 \), the origin is an equilibrium. Since the origin lies on the switching manifold, it is an equilibrium of both sub-systems.

Methane gas is a kind of excellent eco-friendly energy where the gas can be used for many purposes instead of petroleum, coal and other traditional energy types. It is used for heating and lighting, as fuel is used in kitchens for food preparation. Methane, the chief component of natural gas, is produced in nature by the bacterial decay of vegetation and animal wastes in the absence of atmospheric oxygen.

Several models have been designed based on specific growth rate of biomass, including those of Monod, Andrews, Tessier, and Contois kinetics see [5, 17, 18, 19] and references therein). In general, the Monod model is used in ideal laboratory conditions in bioreactor engineering. Further, the basic reason of mathematical model to be discontinuous system is the presence of certain types of discontinuities in Monod curve. Therefore, bioreactors are typical examples of processes that can exhibit nonlinear phenomena in discontinuous system.

In this process, we note that our model (Figure 1) has the following remarks:

(i): The wastewater enters the reactor through its lower section and exits through the upper section.

(ii): The reactor has no filling to support biological growth.

(iii): The Solid-Liquid Gas (SLG) separator is fundamental in order to maintain settled sludge, a clarified effluent (gas-free), and properly separated gases.

(iv): Anaerobic digestion: This is performed by bacteria when oxygen is not present. Fermentation sub-products are a mixture of gases (mainly \( CO_2 \) and \( CH_4 \), biogas) and also some biomass.
Control of an anaerobic digester through a nonlinear controller taking advantage of the knowledge of the bifurcation scenarios. Thus bifurcation and stability issues are important for a good design and control.

The reactions in the UASB reactor are considered in nonlinear form and illustrative way by modeling the system such that the bacterial growth (Monod kinetics curve) is approximated by discontinuous straight line.

We assume that the state space is split in two subregions separated discontinuity line implicitly defined by a smooth function \( h : \mathbb{R}^2 \rightarrow \mathbb{R}, \) so that \( \mathcal{M} := \{ \chi \in \mathbb{R}^2 \mid h(\chi, \gamma) = 0 \} \) such that \( h(\chi, \gamma) = y - \gamma, \) and \( \chi = (x, y)^T. \) Therefore, the process of producing Methane from organic waste can be described as follows:

\[
\dot{\chi} = \begin{cases} 
  f_-(\chi, \alpha), & h(\chi, \gamma) < 0, \\
  f_+(\chi, \alpha), & h(\chi, \gamma) > 0,
\end{cases}
\]

where

\[
f_-(\chi, \alpha) = \begin{pmatrix} \alpha_1 (m - \alpha_6 x) + \alpha_2 x y \\
\alpha_1 (\xi_{in} - y) - r \alpha_2 x y \end{pmatrix},
\]

\[
f_+(\chi, \alpha) = \begin{pmatrix} \alpha_1 (m - \alpha_6 x) + (\mu_1 + \alpha_3 (y - \alpha_4)) x \\
\alpha_1 (\xi_{in} - y) - r (\mu_1 + \alpha_3 (y - \alpha_4)) x \end{pmatrix}.
\]

In this system, the \( x \) is the biomass concentration in the reactor, \( y \) is the substrate concentration in the reactor, \( \alpha_1 \) is the dilution factor and represents the influent volumetric flow per unit of reactor volume, \( r \) is the substrate yield coefficient, \( \xi_{in} \) is the substrate concentration in the input flow, \( m \) is the biomass concentration in the input flow, \( \alpha_6 \) is the SLG separator deficiency, \( \alpha_2 = \frac{\mu_1}{\alpha_4}, \) \( \alpha_3 = \frac{\mu_2 - \mu_1}{\alpha_5 - \alpha_4}, \) where \( \mu_1 \) and \( \mu_2 \) are the rate of microbial growth, \( \alpha_i, i = 4, 5 \) are control parameters. We should note that, all parameters in the model are strictly non-negative.
The discontinuity line of (4) is \( \mathcal{M} := \{ \chi \in \mathbb{R}^2 \mid y = \gamma \} \). Then, the two regions correspond to system (4) are given by:

\[
D_1 := \{ \chi \in \mathbb{R}^2 \mid y > \gamma \}, \quad D_2 := \{ \chi \in \mathbb{R}^2 \mid y < \gamma \}.
\]

Let \( \xi_1 = x, \xi_2 = y - \gamma \) and we rewrite the system (4) as:

\[
\dot{\xi} = \begin{cases} 
  f_+(\xi,\alpha) = F(\xi,\alpha), & \xi_2 > 0 \\
  f_-(\xi,\alpha) = F(\xi,\alpha) + G(\xi,\alpha), & \xi_2 < 0,
\end{cases}
\]

where, \( G \) is typically non-zero when \( \xi \in \mathcal{M} \) (i.e., \( \xi_2 = 0 \)) and

\[
F(\xi,\alpha) = \begin{pmatrix}
  \alpha_1(m - \alpha_6) + (\mu_1 + \alpha_3(\xi_2 + \gamma - \alpha_4))\xi_1 \\
  \alpha_1(\xi_1 - \xi_2 - \gamma) - r(\mu_1 + \alpha_3(\xi_2 + \gamma - \alpha_4))\xi_1
\end{pmatrix}.
\]

(6)

\[
G(\xi,\alpha) = \begin{pmatrix}
  (\alpha_2(\xi_2 + \gamma) - \mu_1 - \alpha_3(\xi_2 + \gamma - \alpha_4))\xi_1 \\
  -r(\alpha_2(\xi_2 + \gamma) - \mu_1 - \alpha_3(\xi_2 + \gamma - \alpha_4))\xi_1
\end{pmatrix}.
\]

(7)

The sliding and crossing sets correspond to system (5) are given by:

\[
\mathcal{M}_s := \{ \xi \in \mathcal{M} \mid (\alpha_1(\xi_1 - \gamma) - r\alpha_2\gamma\xi_1)(\alpha_1(\xi_1 - \gamma) - r\mu_1 + \alpha_3(\gamma - \alpha_4))\xi_1 < 0 \},
\]

\[
\mathcal{M}_c := \{ \xi \in \mathcal{M} \mid (\alpha_1(\xi_1 - \gamma) - r\alpha_2\gamma\xi_1)(\alpha_1(\xi_1 - \gamma) - r\mu_1 + \alpha_3(\gamma - \alpha_4))\xi_1 > 0 \}.
\]

In the following two situations, we investigate the possible dynamic that can occur due to existence of equilibrium points.

3. **Non-sliding flow.** When \( \gamma = \alpha_4 \), then we find \( \mathcal{M}_s := \{ \phi \} \), and the system (4) became a continuous piecewise-smooth flow (CPWS) as:

\[
\dot{\xi} = \begin{cases} 
  f_+(\xi,\alpha) = F(\xi,\alpha), & \xi_2 \geq 0 \\
  f_-(\xi,\alpha) = F(\xi,\alpha) + \xi_2G(\xi,\alpha), & \xi_2 < 0,
\end{cases}
\]

(8)

We note that if \( \xi \in \mathcal{M} \), then \( f_+ = f_- \), where

\[
F(\xi,\alpha) = \begin{pmatrix}
  \alpha_1(m - \alpha_6) + (\mu_1 + \alpha_3\xi_2)\xi_1 \\
  \alpha_1(\xi_1 - \xi_2 - \gamma) - r(\mu_1 + \alpha_3\xi_2)\xi_1
\end{pmatrix}.
\]

and

\[
G(\xi,\alpha) = \begin{pmatrix}
  (\alpha_2 - \alpha_3)\xi_1 \\
  (\alpha_3 - \alpha_2)r\xi_1
\end{pmatrix}.
\]

The roots of the nonlinear system \( F(\xi,\alpha) = 0 \) given as roots of quadratic equation involving only one unknown \( \xi_1 \):

\[
r\alpha_3\alpha_6\xi_1^2 + (\alpha_1\alpha_6 + \alpha_3\alpha_4 - \alpha_3\xi_1 - \mu_1 - r\alpha_3)\xi_1 - \alpha_1m = 0,
\]

(9)

where

\[
\tilde{\xi}_1 = \frac{(\alpha_1\alpha_6 - \mu_1)\xi_1 - \alpha_1m}{\alpha_3\xi_1}.
\]

(10)

In similar way, there are two families of equilibrium points of nonlinear system \( F(\xi,\alpha) + \xi_2G(\xi,\alpha) = 0 \), given as:

\[
r\alpha_2\alpha_6\xi_1^2 + (\alpha_1\alpha_6 - \alpha_2\xi_1 - r\alpha_2)\xi_1 - \alpha_1m = 0,
\]

(11)

where

\[
\tilde{\xi}_2 = \frac{(\alpha_1\alpha_6 - \alpha_2\gamma)\xi_1 - \alpha_1m}{\alpha_2\xi_1}.
\]

(12)
It is useful to classify equilibrium points based on their stability. Equilibria $\xi$ can be classified by looking at the signs of the eigenvalues of the linearization of the both system (4) around $\bar{\xi}$.

**Lemma 3.1.** Assume that $\bar{\xi}$ are equilibrium points that are given by (9), (10). Then the $\ominus$-system has stable equilibrium points $\bar{\xi}$ if:

$$
\alpha_1(1 + \alpha_6) - \alpha_2(\bar{\xi}_2 + \gamma - r\bar{\xi}_1) > 0,
$$

$$
\alpha_1(\alpha_1\alpha_6 - \alpha_2(\bar{\xi}_2 + \gamma) + r\alpha_2\alpha_6\bar{\xi}_1) > 0.
$$

otherwise $\bar{\xi}$ are unstable.

**Proof.** The jacobian matrix of the $\ominus$-system evaluated at equilibrium point $\bar{\xi}$ given as

$$
J = \begin{pmatrix}
-\alpha_1\alpha_6 + \alpha_2(\bar{\xi}_2 + \gamma) & \alpha_2\bar{\xi}_1 \\
-r\alpha_2(\bar{\xi}_2 + \gamma) & -\alpha_1 - r\alpha_2\bar{\xi}_1
\end{pmatrix}.
$$

Then, the characteristic equation of the $J$ is given by:

$$
H(\lambda) = \lambda^2 + [\alpha_1(1 + \alpha_6) - \alpha_2(\bar{\xi}_2 + \gamma - r\bar{\xi}_1)]\lambda + \alpha_1[\alpha_1\alpha_6 - \alpha_2(\bar{\xi}_2 + \gamma) + r\alpha_2\alpha_6\bar{\xi}_1].
$$

Hence, the eigenvalues are negative if $[\alpha_1(1 + \alpha_6) - \alpha_2(\bar{\xi}_2 + \gamma - r\bar{\xi}_1)] > 0$ and $\alpha_1[\alpha_1\alpha_6 - \alpha_2(\bar{\xi}_2 + \gamma) + r\alpha_2\alpha_6\bar{\xi}_1] > 0$ (Routh-Hurwitz stability criterion).

**Lemma 3.2.** Assume that $\bar{\xi}$ are equilibrium points that are given by (11), (12). Then the $\oplus$-system has stable equilibrium points $\bar{\xi}$ if:

$$
\alpha_1(1 + \alpha_6) + \alpha_3(r\bar{\xi}_1 - \bar{\xi}_2) - \mu_1 > 0,
$$

$$
\alpha_1(\alpha_1\alpha_6 + \alpha_3(r\alpha_6\bar{\xi}_1 - \bar{\xi}_2) - \mu_1) > 0,
$$

otherwise $\bar{\xi}$ are unstable.

**Proof.** The proof is similar to Lemma 3.1.
Moreover, we apply the bifurcation analysis and numerical simulation of model (8) relies on the dynamics of the discontinuous line. Hence, under the variation of parameters, we deal with two categories of bifurcation, namely bifurcation that rely on the collapse or change of stability of equilibrium points and bifurcation related to the change of the admissible/virtual character of equilibrium points.

**Definition 3.3.** Assume that $\xi \in \mathcal{M}$, then the system (4) has a fixed point which is called as:

- Admissible fixed point if $f_-(\xi, \alpha) = 0$, $\xi_2 < 0$ or $f_+(\xi, \alpha) = 0$, $\xi_2 > 0$, respectively.
- A virtual fixed point if $f_-(\xi, \alpha) = 0$, $\xi_2 > 0$ or $f_+(\xi, \alpha) = 0$, $\xi_2 < 0$, respectively.
- A boundary fixed point if $\xi_2 = 0$ and $f_-(\xi, \alpha) = 0$ or $f_+(\xi, \alpha) = 0$, respectively.

Hence, we give the following corollary concerning the equilibrium points of (8).

**Corollary 1.** The equilibrium $\xi$ is an admissible equilibrium of $\oplus$-system if it exists for $\frac{(\alpha_1 \alpha_6 - \mu_1) \xi_1 - \alpha_1 m}{\alpha_2 \xi_1} > 0$ ($\bar{\xi}_1$ is given by (9)), otherwise it is a virtual equilibrium. Similarly $\xi$ is an admissible equilibrium of $\ominus$-system if it exists for $\frac{(\alpha_1 \alpha_6 - \alpha_2) \bar{\xi}_1 - \alpha_1 m}{\alpha_2 \xi_1} < 0$ (where $\bar{\xi}_1$ is given by (11)), otherwise it is a virtual equilibrium.

Our model depends on many parameters, we fixed the following parameters according to experiments data [20]. $\alpha_1 = 2.5$, $\xi_{in} = 860$, $r = 3.35$, $m = 240$, $\mu_1 = 0.32$, $\mu_2 = 0.5544$, $\alpha_4 = 1800$, $\alpha_5 = 4000$. Hence, we expect to observe birth of an admissible and a virtual equilibrium. To show this let us calculate $\xi$ as a function in the SLG separator deficiency $\alpha_6$. In Figure 2(a) shows a birth of an admissible and a virtual equilibrium of $\ominus$-system. For $\oplus$-system, we note that one equilibrium point is admissible for all values of $\alpha_6$ where it is everywhere positive, but the other point has a transition from a virtual to admissible state Figure 2(b).

On the other hand, the boundary equilibrium point (BEP) is given as $\bar{\xi}^B = (\frac{\alpha_1 m}{\alpha_1 \alpha_6 - \mu_1}, 0)$, which can be considered as the critical point between admissible and virtual equilibria. Therefore, in the current situation the BEP lies on $\mathcal{M}$ and there is a change of the nature of equilibrium point form being admissible to virtual and vice versa upon crossing of $\mathcal{M}$.

**Theorem 3.4.** Assume that $\bar{\xi}^B$ is a boundary equilibrium point and $\alpha = \alpha_6$ is a bifurcation parameter. Then the CPWS (8) has only a persistence bifurcation.

**Proof.** The persistence bifurcation in non-sliding flow take place at $\bar{\xi}^B$ if the following conditions are satisfied:

$$det(\tilde{A}) \neq 0, \tilde{N} - \tilde{C} \tilde{A}^{-1} \tilde{M} \neq 0, \delta = 1 + \tilde{C} \tilde{A}^{-1} \tilde{B} > 0,$$

where $\tilde{A} = F_\xi, \tilde{N} = h_\alpha, \tilde{C} = h_\xi, \tilde{M} = F_\alpha, \tilde{B} = G$ are all evaluated at $\bar{\xi}^B$.

Further, we assume that the bifurcation parameter is SLG (i.e., $\alpha = \alpha_6$), and we calculate the bifurcation of equilibrium $\xi$ according to the above conditions. We get:

$$det(\tilde{A}) = \frac{\alpha_1 (\alpha_1^2 \alpha_5^2 + \mu_1^2 + r m \alpha_1 \alpha_3 \alpha_6)}{(\alpha_1 \alpha_5 + \mu_1)} \neq 0,$$

and

$$\tilde{N} - \tilde{C} \tilde{A}^{-1} \tilde{M} = -\frac{r m \alpha_1 \mu_1}{(\alpha_1 \alpha_6 - \mu_1)^2 + r m \alpha_1 \alpha_3 \alpha_6} \neq 0.$$
so that the collision is transversal. Further, the bifurcation function is given as:

$\delta = 1 - \frac{rma_1a_6(a_3 - a_2)}{(a_1a_6 - \mu_1)^2 + rma_1a_3a_6}.$

For $\delta > 0$ we get $\frac{rma_1a_6(a_3 - a_2)}{(a_1a_6 - \mu_1)^2 + rma_1a_3a_6} < 1$ that is satisfied for all values of $a_6$ where $(a_1a_6 - \mu_1)^2 + rma_1a_3a_6 > 0$. Hence, the CPWS (8) has only a persistence bifurcation at the collision.

In Figure 3, we show that the bimodal (8) exhibits persistence bifurcation at $a_6 = 0.01824$. Further, the increasing the biomass $m$ leading to increasing the value of $a_6$ for which the persistence bifurcation occurred, and this agree with the fact that the higher biomass amount at the bioreactor input (it also means that increasing the bacterial concentration at the bioreactor input) leading to the SLG sedimentier is less efficient.

Now, we investigate the effects of dilution factor ($a_1$) on bioreactor design. If
Given as \( \bar{\alpha} \), this pseudo-equilibrium point is called admissible if \( \bar{\alpha} = 0 \). Further, we use \( m = 0 \) to find \( \alpha_1^{\text{max}} \) at the persistence bifurcation point, hence \( \alpha_1^{\text{max}} = 2.639 \). In this case \( \alpha_1 \) is too high, therefore the bacterial cells cannot max grow fast enough to reach steady-state and washout will occur.

We note that the case of no dilution (\( \alpha_1 = 0 \)) or the case of maximum dilution factor (\( \alpha_1 = \alpha_1^{\text{max}} \)) has a negative influence on the genetically mixed populations strains of anaerobic bacteria which could determine the selection of cultures or reduce their rate of specific growth, and therefore they might have a significant effect on both the degree of degradation of the organic load of the waste and on the delay imposed in the biogas production.

In the next section, we discuss the dynamical behaviour of the proposed model (5), when the sliding motion takes place on \( \mathcal{M}_s \).

4. **Sliding flow.** In this case \( \gamma \neq \alpha_4 \), the model (5) is discontinuous at \( \xi_2 = 0 \) where the vector fields are given by (6), (7). Therefore, the equilibrium points of \( F(\xi, \alpha) = 0 \) are given as:

\[
\xi_2 = \frac{\alpha_1 \alpha_6 - \mu_1 + \alpha_3(\alpha_4 - \gamma)}{\alpha_3 \xi_1}, \quad (14)
\]

where \( \xi_1 \) is given as a roots of (9). Further, the equilibrium points of \( F(\xi, \alpha) + G(\xi, \alpha) = 0 \) are the same points that are given by (11), (12).

At the discontinuity \( \xi \in \mathcal{M}_s \), the flow in \( \mathcal{M}^s \) itself is governed by the following extension as:

\[
\dot{\xi} = F_s(\xi, \alpha) = F(\xi, \alpha) + \lambda G(\xi, \alpha), \quad \lambda \in [0, 1], \quad (15)
\]

where \( \lambda = -\frac{\xi_j F}{\xi_j G}(\xi, \alpha) \) and \( F, G \) are given by (6), (7), respectively. Therefore, we have an explicit form of the one-dimension sliding vector field as:

\[
\dot{\xi} = F_s(\xi, \alpha) = \frac{\alpha_1}{r}(r m + \xi_{\text{in}} - \gamma - r \alpha_6 \xi_1). \quad (16)
\]

It should be point out that the sliding flow is linear and stable. Further, the pseudo-equilibrium point is given as:

\[
\xi_1 = \frac{r m + \xi_{\text{in}} - \gamma}{r \alpha_6}.
\]

This pseudo-equilibrium point is called admissible if \( 0 \leq \lambda \leq 1 \), otherwise it is called a virtual. Further, we deduce from (15) that either \( \lambda = 0 \) or \( \lambda = 1 \), that is, the \( \xi \) is also the the boundary of the sliding region \( \mathcal{M}^s \). Hence, at \( \lambda = 0 \) the boundary point is given as \( \xi_1^B = \left( \frac{\alpha_1 m}{\alpha_4} - \mu_1 + \alpha_3(\alpha_4 - \gamma), 0 \right) \) and at \( \lambda = 1 \) the boundary point is given as \( \xi_1^B = \left( \frac{\alpha_1 m}{\alpha_4}, 0 \right) \).

Now, we investigate the occurrence of the possible border collision bifurcations in sliding flow (5).

**Theorem 4.1.** Assume that \( \xi_1^B \) is a boundary equilibrium point and \( \alpha = \alpha_6 \) is a bifurcation parameter of system (5). Then

- If \( \alpha_2 \gamma + \alpha_3 \alpha_4 < \mu_1 + \alpha_3 \gamma \), the sliding flow has a persistence bifurcation.
- If \( \alpha_2 \gamma + \alpha_3 \alpha_4 > \mu_1 + \alpha_3 \gamma \) the sliding flow has a non-smooth fold bifurcation.

**Proof.** These types of bifurcations are observed if the following conditions are satisfied

\[
det(\bar{\mathbf{A}}) \neq 0, \quad \bar{\mathbf{N}} - \bar{C} \bar{\mathbf{A}}^{-1} \bar{\mathbf{M}} \neq 0, \quad \delta = \bar{C} \bar{\mathbf{A}}^{-1} \bar{\mathbf{B}} \neq 0, \quad (17)
\]
while the elements are as before, but $F$ and $G$ are defined by (6),(7) should be taken into account. Let $\alpha = \alpha_6$ is a bifurcation parameter of system (5), to show the effect of the SLG separator deficiency parameter according to the above conditions, we find that:

$$\det(\bar{A}) = \frac{\alpha_1(\mu_1^2 + \alpha_3^2(\gamma^2 + \alpha_4^2)) + \alpha_7^2\alpha_6(\alpha_1\alpha_6 + r\alpha_3)}{\alpha_1\alpha_6 + \mu_1 + \alpha_3(\gamma + \alpha_4)} \neq 0,$$

Then $\bar{A}$ is invertible. As well, we get

$$\bar{N} - \bar{C}\bar{A}^{-1}\bar{M} = -\frac{r\alpha_1\alpha_3(\mu_1 + \alpha_3(\gamma - \alpha_4))}{(\alpha_1\alpha_6 - \mu_1 + \alpha_3(\gamma - \alpha_4))^2 + r\alpha_1\alpha_3\alpha_6} \neq 0.$$

Thus, the collision is transversal, and regarding to the non-degeneracy condition, we obtain the bifurcation function as:

$$\delta = \frac{r\alpha_1\alpha_3\alpha_6(\alpha_2\gamma + \alpha_3\alpha_4 - (\mu_1 + \alpha_3\gamma))}{(\alpha_1\alpha_6 - \mu_1 + \alpha_3(\gamma - \alpha_4))^2 + r\alpha_1\alpha_3\alpha_6} \neq 0.$$

Then depends on the sign of the right hand side we have a different kind of a border collision bifurcation. Thus, for $\delta > 0$, i.e., $\alpha_2\gamma + \alpha_3\alpha_4 > \mu_1 + \alpha_3\gamma$, the sliding flow has a nonsmooth fold bifurcation, whilst for $\delta < 0$, i.e., $\alpha_2\gamma + \alpha_3\alpha_4 < \mu_1 + \alpha_3\gamma$, there is persistence bifurcation or transition between one admissible equilibrium and one pseudo-equilibrium.

The nonsmooth fold bifurcation refers to the conversion collision of two branches of admissible upon intersection of $\mathcal{M}$ to two branches of virtual equilibria. Assume that we fix the parameters values as before such that $\gamma = \alpha_4 + 100$, and take $\alpha_6$ as our bifurcation parameter. Therefore, at a certain parameter value $\alpha_6 = 0.04802$ with $\lambda = 0$, the system (5) undergoes a nonsmooth fold bifurcation, see Figure 5. In Figure 6 shows nonsmooth fold bifurcation occurs at $\lambda = 1$ such that $\alpha_6 = 0.04905$. As well the system (5) undergoes a persistence bifurcation if $\gamma = \alpha_4 - 100$ at a certain parameter value $\alpha_6 = 0.0255$ with $\lambda = 0$, see Figure 7. In Figure 8 shows persistence bifurcation occurs at $\alpha_6 = 0.02496$ such that $\lambda = 1$.

We will now move to bifurcation that involve trajectory of a sub-system under the effect of parameter variation crosses the discontinuity manifold $\mathcal{M}$ transversally at the boundary of $\mathcal{M}_s$. 

![Figure 5. Existence of nonsmooth bifurcation of sliding flow (5) at $\lambda = 0, \alpha_6 = 0.04802$](image-url)
Figure 6. Existence of nonsmooth bifurcation of sliding flow (5) at $\lambda = 1, \alpha_6 = 0.04905$

Figure 7. Existence of persistence bifurcation of sliding flow (5) at $\lambda = 0, \alpha_6 = 0.0255$

Figure 8. Existence of persistence bifurcation of sliding flow (5) at $\lambda = 1, \alpha_6 = 0.02496$
Theorem 4.2. Consider the sliding system (5) subject to the constraints $\bar{\xi} \in M_s$, and $\alpha_6$ is a bifurcation parameter. Then the sliding system undergoing grazing-sliding bifurcation if

- $\alpha_1 \alpha_6 (\xi_{in} - \gamma) > (\mu_1 + \alpha_3 (\gamma - \alpha_4)) (rm + \xi_{in} - \gamma)$ with $\lambda = 0$.
- $(\alpha_1 \alpha_6 - \alpha_1 \alpha_2 \gamma) (\xi_{in} - \gamma) < rm \alpha_2 \gamma$, with $\lambda = 1$.

Proof. The grazing-sliding bifurcation is defined as tangential intersection between trajectory and the boundary of $M_s$.

Without loss of generality, we assume that the boundary bifurcation point lies on $M_s$ (i.e., $F(\bar{\xi}, \alpha) = F_s(\bar{\xi}, \alpha)$, that is $\lambda = 0$ in (15)). In this situation, we should have

$$\bar{\xi} \in M, \ n^T(\xi, \alpha) F(\xi, \alpha)|_{\bar{\xi}} = 0, \ n^T(\xi, \alpha) \nabla F(\xi, \alpha). F(\xi, \alpha)|_{\bar{\xi}} > 0 \quad (18)$$

These conditions ensure that the bifurcation point is located on the boundary of $M_s$ and that it is a tangent point.

The grazing bifurcation point $\bar{\xi}$ is determined as $\bar{\xi} = (\frac{\alpha_1 (\xi_{in} - \gamma)}{r (\mu_1 + \alpha_3 (\gamma - \alpha_4))}, 0)$. At this point, the sliding system (5) satisfies all above conditions such that first item in Theorem 4.2 holds.

In the case when it is $F(\bar{\xi}, \alpha) + G(\bar{\xi}, \alpha) = F_s(\bar{\xi}, \alpha)$, (that is $\lambda = 1$ in (15)) that grazes at $\bar{\xi} \in M_s$ we then would expect an analogous conditions (18) are obtained with opposite sign. Therefore, the grazing-sliding bifurcation point $\bar{\xi}$ is determined as $\bar{\xi} = (\frac{\alpha_1 (\xi_{in} - \gamma)}{rm \alpha_2 \gamma}, 0)$ and therefore the second item in Theorem 4.2 holds.

Figure 9 shows three trajectories of system (5) with a focus on the effect of the dilation factor $\alpha_6$. At $\alpha_6 = \alpha_6^{graz} = 0.03$ the flow hits the boundary of $M_s$, and if $\alpha_6 < \alpha_6^{graz}$, then the trajectory shown has no intersection points with $M$. Further, if $\alpha_6 > \alpha_6^{graz}$, then the trajectory shown has a segment of sliding motion.

The dynamical behavior of such a system undergoes a grazing-sliding bifurcation will be discussed in the next section.

5. State dependent discontinuity and dynamics. In order to describe the dynamics around the grazing-sliding bifurcation observed in model of methane gas production (5), we use the generalized Poincaré discontinuous map. Assume that $\psi(\xi, t)$ is a general solution corresponding to equations (5) and (15), respectively as:
Without loss of generality, we assume that $\psi_1(\xi, t)$ reaches $\mathcal{M}$. Then we distinguish cases whether or not $\xi$ element of $\mathcal{M}_s$ or $\mathcal{M}_c$:  

(a): Suppose that the case $\alpha < \alpha^{graz}$. Then there is a continuous time trajectory generated by the Poincaré map which is defined as: $P(\xi) := \varphi(\xi, t)$, such that $n^T(\xi)F(\xi) > 0$ for all $\xi \in \mathbb{R}^2$.

(b): Whenever $\alpha > \alpha^{graz}$, then the trajectory shown has a segment of sliding motion. Therefore, the discontinuous behavior for composed motion is defined by $P(\xi) = \psi_1 \circ \psi_s \circ \psi_1(\xi, t)$ and the linearization of $P$ is obtained as:

$$\frac{\partial P}{\partial \xi} = Y_1(t_2, \xi^2).J^s(t_1, \xi^1).Y_1(t_1, \xi^1),$$

where $(\xi^i, t^i), i = 1, 2$ are boundary points, $Y_1$ is the fundamental matrix solution for the linearized equation $Y = \frac{2F}{n} Y$ with $Y(0) = I$, and $J^s$ is a jump matrix or correction of the trajectory, see [11, 15] where

$$J^s = I + \frac{(F_s(\xi^1) - F(\xi^1))n^T}{n^TF(\xi^1)}.$$  

Note that $n^T F(\xi^1) \neq 0$ and the sliding flow leaves $\mathcal{M}_s$ tangentially. Further, in a situation where there is transversal intersection, the term $F_s(\xi^1) - F(\xi^1)$ in $J^s$ is replaced by $G(\xi^1)$.

(c): If $\alpha = \alpha^{graz}$, the trajectory hits a discontinuity line and the jump matrix becomes singular due to $n^TF(\xi) = 0$. Therefore, the continuous state jumps instantaneously to a new value specified by the rest map which is called zero discontinuous map (ZDM). The ZDM is the mapping that captures the effect of grazing contact such that the total elapsed time at $\xi$ is zero (i.e., the trajectory has zero length of sliding segment). Further, a smooth projection $S_p$ of the ZDM onto a local Poincaré section leads to define Poincaré discontinuity map (PDM), such that $PDM = S_p(ZDM)$. For details of deriving analytical these maps and related properties, see [6].

In our model (5), according to the results of Theorem 4.2, the point $\bar{\xi}$ is a regular grazing point of (5) and $G(\bar{\xi}) \neq 0$. Then following [6], we can summarize the important results as follows:

The PDM is given by:

$$\xi \mapsto PDM = \begin{cases} 
\xi, & \text{if } h_{min} \geq 0, \\
\xi + h_{min} \left[ \frac{F+G}{h_{\xi}G} - \frac{(h_{\xi}F_{\xi} + (F+G))}{(h_{\xi}F_{\xi} + F)} \right] \xi = \bar{\xi}, & \text{if } h_{min} < 0. 
\end{cases}$$

Further, the normal form map is given by:

$$P_N(\xi, \alpha) = \begin{cases} 
\varphi_1(\xi, T), & \text{when } h(\varphi(\xi, T)) \geq 0, \\
PDM(\varphi_1(\xi, T)), & \text{when } h(\varphi(\xi, T)) < 0, 
\end{cases}$$

where $h_{min}(\xi) = h_\xi \xi + O(|\xi|^2)$.

The authors in [6] have established a bifurcation function to unfold the grazing-sliding bifurcation by extended analysis of the above mapping, that is:

$$\varphi(\alpha) = 1 + C^T N(\tilde{\lambda} I - \tilde{N})^{-1} \tilde{E},$$

where $C = (\psi_1, \psi_s, \psi_1)'$ is the grazing-sliding bifurcation by extended analysis of the above mapping, that is:
where \( C^T = h_x(\xi), \hat{N} = ((\hat{\lambda} I - \frac{\xi V}{\xi^2})\hat{J})|_{\xi, \alpha_{graz}}, \hat{J} = \varphi_1(\xi, T, \alpha_{graz}), V = (h_\xi F)(\xi, \alpha_{graz}), \varrho_0(\alpha) = VF, \) and when \( \hat{\lambda} = 1 \) refer to the trajectory has a periodic orbit such that \( P_N(\xi, \alpha_{graz}) = \hat{\lambda}\xi. \) Moreover, depending on the sign of \( \varrho, \) we distinguish between the main categories of grazing-sliding bifurcation scenarios as:

- If \( \varrho > 0, \) then the non-sliding orbit exists for \( \varrho_0 > 0 \) and the sliding orbit exists for \( \varrho_0 < 0. \)
- If \( \varrho < 0, \) then the both orbits exist for \( \varrho_0 > 0. \)
- A persistence or non-smooth transition occurs for the primary trajectory undergoing the grazing-sliding bifurcation if \( \varrho > 0 \) or \( \varrho < 0, \) respectively.

Note that, it is possible to continue with general values of parameters, but resulting formulas are rather awkward. Therefore we apply the above results of a situation that is governed by Theorem 4.2. We find that: \( \varrho(\alpha_{graz}) = 0.78 > 0, \) and \( \varrho_0 = 0.086, \) at \( \alpha_{graz} = 0.03. \) Then the system undergoing grazing-sliding bifurcation. When we move the bifurcation parameter to \( \alpha_6 = 0.031 > \alpha_{graz}, \) we find \( \varrho_0 = -6.4. \) Then the trajectory has a segment of sliding motion. These results agree well with Theorem 4.2.

The following is a brief summary of the main results.

6. Conclusion. The existence of discontinuous bifurcation in a nonlinear model of the anaerobic digestion of organic waste has been investigated. Due to the presence of discontinuities in Monod curve the mathematical model have been described as a DS. The existence and stability of equilibrium points of DS were used to investigate how the bioreactor performance depends on process parameters. As we found that the dilution factor(\( \alpha_1 \)) and SLG separator deficiency (\( \alpha_6 \)) played a significant role in the efficiency of waste reduction and gas production. When \( \alpha_6 \) was taken as a bifurcation parameter, the CPWS underwent only persistence bifurcation that guaranteed by Theorem 3.4. It was shown at the persistence bifurcation point, the increasing of the biomass leading to increasing of SLG parameter. This leads to decreased methane productivity and lower process efficiency. These results agree with the fact that, when the biomass is much higher, the bacterial has a higher concentration at the bioreactor input which leads to negative effect on anaerobic digestion process efficiency. Further, we have shown, the dependence of the concentration of biomass on the dilution factor, the concentration of biomass decreases with an increase in the dilution factor until the reactor undergoes washout. It has been observed that the case of no or maximum dilution factor, there was a negative effects on the growth rate of anaerobic bacteria and hence the population dynamics in the bioreactor.

Furthermore, we found that the bioreactor model that represented as DS guided the discontinuity line to capture a more complex dynamics, such as three different transition of the trajectory behavior caused by a small change of SLG parameter. This situation associated with another important feature of the bifurcation scenario observed in the system is that there exists a grazing-sliding bifurcation. We should mention that the significant change in the behavior at the grazing point due to high sensitivity of SLG separator efficiency. Therefore, this analysis of bifurcation allows us to identify the minimum control of SLG parameter gain that guarantees the stability of the bioreactor design. Our numerical computations have illustrated the characteristic features of the model and related discontinuous bifurcation.
Acknowledgments. The authors would like to thank the anonymous reviewers and editors for their helpful and insightful comments that greatly contributed to improving the final version of the paper.

REFERENCES

[1] A. H. Ajbar, M. ALAhmad and E. Ali, On the dynamics of biodegradation of wastewater in aerated continuous bioreactors, *Mathl. Comput. Model.*, 54 (2011), 1930–1942.
[2] R. T. Alqahtani, *Modelling of Biological Wastewater Treatment*, Ph.D. Thesis. University of Wollongong, Australia, 2013.
[3] J. Awrejcewicz and C. Lamarque, *Bifurcation and Chaos in Nonsmooth Mechanical Systems*, World Scientific Publishing Co., Inc., River Edge, NJ, 2003.
[4] A. Bornhöft, R. Hanke-Rauschenbach and K. Sundmacher, Steady-state analysis of the anaerobic digestion model no. 1 (ADM1), *Nonlinear Dyn.*, 73 (2013), 535–549.
[5] B. Benyahia, T. Sari, B. Cherki and J. Harmand, Bifurcation and stability analysis of a two step model for monitoring anaerobic digestion processes, *Journal of Process Control*, 22 (2012), 1008–1019.
[6] M. di Bernardo, C. Budd, A. R. Champneys and P. Kowalczyk, *Piecewise-smooth Dynamical Systems: Theory and Applications*, Springer-Verlag, London, 2008.
[7] M. Fečkan and M. Pospíšil, *Poincaré-Andronov-Melnikov Analysis for Non-Smooth Systems*, Academic Press is an imprint of Elsevier, London, 2016.
[8] A. F. Filippov, Differential equations with discontinuous right-hand side, *American Mathematical Society Translations*, 2 (1964), 199–231.
[9] Y. Gao, X. Meng and Q. Lu, Border collision bifurcations in 3D piecewise smooth chaotic circuit, *Appl. Math. Mech.-Engl. Ed.*, 37 (2016), 1239–1250.
[10] H. A. Hosham, *Cone-like Invariant Manifolds for Nonsmooth Systems*, Ph.D. Thesis. Universität zu Köln, Germany, 2011.
[11] H. A. Hosham, Bifurcation of periodic orbits in discontinuous systems, *Nonlinear Dyn.*, 87 (2017), 135–148.
[12] T. Küpper and H. A. Hosham, Reduction to invariant cones for nonsmooth systems, *Math. Comput. Simul.*, 81 (2011), 980–995.
[13] T. Küpper, H. A. Hosham and K. Dudtschenko, The dynamics of bells as impacting system, *J. Mech. Eng. Sci.*, 225 (2011), 2436–2443.
[14] T. Küpper, H. A. Hosham and D. Weiss, Bifurcation for nonsmooth dynamical systems via reduction methods, in: *Recent Trends in Dynamical Systems, Proceedings in Mathematics and Statistics*, Springer-Verlag, 35 (2013), 79–105.
[15] R. I. Leine and H. Nijmeijer, *Dynamics and Bifurcations of Non-Smooth Mechanical Systems*, Springer-Verlag, Berlin, Germany, 2004.
[16] Y. Li, L. Yuan and Z. Du, Bifurcation of nonhyperbolic limit cycles in piecewise smooth planar systems with finitely many zones, *Int. J. Bifurcation and Chaos, Appl. Sci. Engrg.*, 27 (2017), 1750162, 14 pp.
[17] L. A. Melo-Varela, S. Casanova-Trujillo and G. Olivar-Tost, Dynamics of a bioreactor with a bacteria piecewise-linear growth model in a methane-producing process, *Math. Prob. in Engin.*, 2013 (2013), Art. ID 685452, 8 pp.
[18] R. Muñoz, *Design and Implementation of a COD Control System of a Prototype UASB Reactor for Treating Leachates*, M.S. thesis, National University of Colombia, 2006.
[19] S. Shen, G. C. Premier, A. Guwy and R. Dinsdale, Bifurcation and stability analysis of an anaerobic digestion model, *Nonlinear Dyn.*, 48 (2007), 391–408.
[20] D. Weiss, T. Küpper and H. A. Hosham, Invariant manifolds for nonsmooth systems, *Physica D: Nonlinear Phenomena*, 241 (2012), 1895–1902.
[21] D. Weiss, T. Küpper and H. A. Hosham, Invariant manifolds for nonsmooth systems with sliding mode, *Math. Comput. Simul.*, 110 (2015), 15–32.

Received November 2017; revised April 2018.

E-mail address: hanyalbadrey@azhar.edu.eg or hanyalbadrey@yahoo.com
E-mail address: ybakit@taibahu.edu.sa