Holonomic Quantum Computing Based on the Stark Effect

B. A. Bernevig and Shou-Cheng Zhang

Department of Physics, Stanford University, Stanford, CA 94305

We propose a spin manipulation technique based entirely on electric fields applied to acceptor states in $p$-type semiconductors with spin-orbit coupling. While interesting in its own right, the technique can also be used to implement fault-resilient holonomic quantum computing. We explicitly compute adiabatic transformation matrix (holonomy) of the degenerate states and comment on the feasibility of the scheme as an experimental technique.

PACS numbers: 03.67.Lx, 71.70.Ej, 71.55.Eq

The physical realization of quantum computing rests on the ability to reversibly manipulate two level systems called qubits. While the promise of high computational power is certainly a tantalizing one, the intrinsic challenges associated with decoherence, adiabatic evolution, control and noise errors in quantum gate operations are still to be mastered.

One ingenious way to overcome quantum noise errors is the use of Non-abelian Holonomic (Geometric) Quantum Computation schemes [1]. In these procedures, through the slow tuning of some external parameters such as applied magnetic or electric fields, the qubit evolves adiabatically (with constant energy) around a path that changes its eigenstate from an initial to a final state. Generically, this quantum evolution is free of dynamical factors and is geometric in nature, depending only on the path in parameter space. Geometric holonomy could constitutes a fault-tolerant way to perform quantum computation [2]. Although some experimental systems that would exhibit such behavior have been proposed [3, 4], holonomic quantum computing overall still lacks the variety of concrete application proposals that conventional quantum computation enjoys.

In this paper we propose using electric fields to manipulate the spin of acceptor states in semiconductors with spin-orbit coupling. While interesting in its own right, the technique could be used to achieve spin manipulation in the absence of any external field. The holonomic qubits discussed here are based on the same principle as the recently discovered dissipationless spin current in hole doped semiconductors [5]. The individual qubits can therefore be coupled to each other by a quantum bus architecture based on the dissipationless spin current, offering exciting new possibilities towards the realization of an all solid-state holonomic quantum computer.

In the thesis of holonomic quantum computing, quantum information is encoded in an $n$-fold degenerate Hilbert space of a hamiltonian $H_A$ dependent on some external 'control' parameters (fields) $\lambda$. Upon a cyclical change of these parameters around a loop $C$ during time $T$ such that $\lambda_{in} = \lambda_{out}$, the system will evolve between the initial state (a vector) $|\psi\rangle_{in}$ into $|\psi\rangle_{out} = e^{i\epsilon_0T}\Gamma(C)|\psi\rangle_{in}$, where $\epsilon_0$ is the initial eigenvalue $H_{\lambda_{in}}|\psi\rangle_{in} = \epsilon_0|\psi\rangle_{in}$. The first factor is just the dynamical phase, and will be omitted, while the second factor is the non-abelian Wilczek-Zee curvature connection (matrix):

$$\Gamma(C) = \mathbf{P} e^{A^\mu d\lambda^\mu}, \quad A^\mu_{ij} = \langle \psi^i(\lambda)|\frac{\partial}{\partial \lambda^\mu}|\psi^j(\lambda)\rangle,$$

(1)

where $i, j = 1, \ldots, n$ and where $\mathbf{P}$ represents the path ordering due to the fact that the gauge connection $A^\mu$ is now a matrix acting on the degenerate space of Hamiltonian eigenstates ($\mu$ denotes the different control parameters). The degenerate Hilbert space of the Hamiltonian encodes the quantum information where the eigenstates are the codewords while the non-trivial holonomies associated with it represent the unitary transformations or 'computations' over the code. Zanardi and Rasetti [11] showed that this prescription is sufficient to implement quantum computation on single qubit holonomic gates.

In subsequent papers [1] [2] [3] [11] [12] several schemes for realizing holonomic computation have been proposed. The schemes involve geometric manipulation of trapped ions [12], charge pumping within Josephson junction net-
works \[4\], and Josephson charge qubits \[3\]. Controlled manipulation of \(U(1)\) holonomies (Berry phases) using nuclear magnetic resonance on a system of weakly coupled \(H^1\) and \(C^{13}\) nuclei has been experimentally achieved with great accuracy by Jones et al \[13\].

It would be of great advantage to have a conventional solid-state system where holonomic computation can be implemented by using only electric fields. In this paper we are concerned only with single qubit holonomy, leaving multi-qubit ones for a later publication \[14\]. We look at p-type cubic symmetry semiconductors such as Ge, GaAs and Si. The strong spin-orbit coupling present in these systems breaks up the valence bands into two doubly degenerate bands of spin 3/2 with helicity \(\pm 1/2\) and \(\pm 3/2\). The double degeneracy is nothing else than Kramers degeneracy and is guaranteed by \(T\)-invariance. \(T\)-invariance is maintained even when acceptor impurities (B, Al, Ga, In) are introduced in the semiconductors. The holes that bind to these impurities will maintain a certain symmetry subgroup of the original cubic symmetry of the valence bands they came from. Let us now consider the effect of an applied external electric field \(E\) on the acceptor-bound hole state. For large electric fields, the field distortion near the impurity ion can be safely neglected and the acceptor-hole state has the cubic symmetry of the crystal \(T_d \times I\), giving rise to a quadratic Stark effect \[7\]:

\[
H_{E^2} = \frac{p_0^2}{\varepsilon_i}(\alpha E^2 I + \beta E_x S_x^2 + E_y S_y^2 + E_z^2 S_z^2 - \frac{5}{4} E^2 I) + \\
\frac{2}{\sqrt{3}} \delta(E_y E_z \{S_y, S_z\} + E_x E_z \{S_z, S_x\} + E_x E_y \{S_x, S_y\}) ,
\]

where \(\varepsilon_i\) is the ionization energy, \(p_0 = e\tau\) the dipole moment (\(\tau\) being the mean radius of the ground state), and \(\{S\}\) are the spin-3/2 matrices, describing the valence band states, which essentially have \(P_{3/2}\) character. We have also defined \(\{A, B\} = (AB + BA)/2\). Readers familiar with semiconductor theory will recognize in the form of \(H_{E^2}\) the Luttinger Hamiltonian structure, with the substitution \(k \rightarrow E\). This is no coincidence since the symmetry group of both Hamiltonians is the same. For small applied electric field, we must take into consideration the local field of the ions, thereby reducing the symmetry from \(T_d \times I\) to \(T_d\) and giving rise to a linear Stark effect \[7\]:

\[
H_E = \frac{2p\chi}{\sqrt{3}}(E_x \{S_y, S_z\} + E_y \{S_z, S_x\} + E_z \{S_x, S_y\}) ,
\]

where \(p = e a_B\) with \(a_B\) the Bohr radius. The constants \(\alpha, \beta, \delta, \chi, \tau\) are given in Table 1, although the estimates for \(\chi\) in the literature vary considerably (\(\chi = 0.26\) according to Kopf and Lassman \[15\] so the value in Table 1 should be taken as a lower limit). We want to mention that the donor and acceptor Hamiltonians and physics are essentially different, with the donors undergoing only a quadratic Stark shift as opposed to the acceptor combination of the above linear and quadratic shifts.

Although for some field \(E\) the acceptor Hamiltonian will be a weighted sum of linear and quadratic Stark effects, we prefer, without any loss of generality, to work in either of the two regimes and not in the intermediate one. Each of the Hamiltonians above has two doubly degenerate Hilbert spaces, roughly corresponding to values of the \(z\)-component of the spin \(S_z\) being either \(\pm 1/2\) or \(\pm 3/2\) (this would be exactly true if the Hamiltonians were isotropic).

The 'control' parameters are the components of the electric field \(E\). We must now show we can achieve 'quantum computations over the code'. These are represented by \(SU(2)\) holonomies over each degenerate Hilbert space (equivalently, we must show that we can move within an energy subspace by adiabatically changing \(E\)).

We now prove that such holonomies do indeed exist in our system and give an explicit generic procedure to calculate them. While we could just brute-force diagonalize the Hamiltonians above and treat each of them separately, we prefer to use a more elegant approach that reveals more of the Hilbert space structure. This was developed by Demler and Zhang \[16\] in the context of the SO(5) theory of high Tc superconductivity, and extended by Murakami, Nagaosa and Zhang \[17\] to the case of hole band in semi-conductors. Readers not interested in the derivation can jump to the next page where we give the expression for the \(SU(2)\) holonomies. Out of the spin-3/2 \(J_x, J_y, J_z\) we can define the new 4 × 4 matrices:

\[
\Gamma^1 = \frac{2}{\sqrt{3}}(S_y, S_z), \quad \Gamma^2 = \frac{2}{\sqrt{3}}(S_z, S_x), \quad \Gamma^3 = \frac{2}{\sqrt{3}}(S_y, S_x)
\]

\[
\Gamma^4 = \frac{1}{\sqrt{3}}(S_x^2 - S_y^2), \quad \Gamma^5 = S_z^2 - \frac{5}{4} I_{4 \times 4} ,
\]

which satisfy the SO(5) Clifford algebra \(\Gamma^a \Gamma^b + \Gamma^b \Gamma^a = 2\delta_{ab} I_{4 \times 4}\). In explicit form, these matrices are:

\[
\Gamma^i = \begin{pmatrix}
0 & i\sigma^i \\
-i\sigma^i & 0
\end{pmatrix}; \Gamma^4 = \begin{pmatrix}
0 & I \\
I & 0
\end{pmatrix}; \Gamma^5 = \begin{pmatrix}
I & 0 \\
0 & -I
\end{pmatrix} ,
\]

\[
(5)
\]

where \(\sigma^i, i = 1, 2, 3\) are the usual Pauli matrices and \(I\) is the identity matrix \((2 \times 2\) in this case). We observe that the Hamiltonians \(H_{E^2}\) and \(H_{E^2}\) can now be cast into

| \(\alpha\) | \(\beta\) | \(\delta\) | \(\chi\) | \(\tau(M)\) |
|---|---|---|---|---|
| Ge | 1 | -0.3 | -0.36 | 0.7 \times 10^{-3} | 91 |
| Si | 1 | -0.2 | -0.42 | 1 \times 10^{-2} | 34.4 |

Table I: Values of the coefficients in the linear and quadratic Stark Hamiltonians \[8\].
a new clean form
\[ H_E = d^a_E I + d^a_E \Gamma^a; \quad H_{E2} = d^a_{E2} I + d^a_{E2} \Gamma^a; \quad a = 1, \ldots, 5 \]
where \( I \) is the 4 \& 4 identity matrix and where the following identities hold:
\[ d^0_E = d^0_{E2} = d^0_{E3} = 0, \quad d^1_E = p \chi E_x, \quad d^2_E = p \chi E_y, \quad d^3_E = p \chi E_z \]
\[ d^0_{E2} = -\frac{p^2}{\varepsilon_i} \alpha E^2, \quad d^1_{E2} = -\frac{p^2}{\varepsilon_i} \delta E_x E_y, \quad d^2_{E2} = -\frac{p^2}{\varepsilon_i} \delta E_z E_x \]
\[ d^3_{E2} = -\frac{p^2}{\varepsilon_i} \delta E_x E_y, \quad d^4_{E2} = -\frac{p^2}{\varepsilon_i} \frac{\sqrt{3}}{2} \beta (E^2_x - E^2_y) \]
\[ d^5_{E2} = -\frac{p^2}{\varepsilon_i} \frac{1}{2} \beta (2E^2_z - E^2_x - E^2_y) \]
Since the two Hamiltonians for small and large field have been brought to the same symbolic form, we can manipulate them together and only substitute for the values of \( d^a \) at the end of the calculation. The deep physical reason as to why the two apparently different Hamiltonians are actually very similar is the unbroken \( T \)-invariance of the system that leads to Kramers degeneracy. The total gauge field strength:
\[ \Gamma(C) = \mathbf{P} \exp\left( \oint A^a dE^a \right) \mathbf{P}^\dagger \exp\left( -\oint \frac{1}{2} \frac{d}{dE} \frac{d^a}{dE} \Gamma_{ab} dE_b \right) \]
Let us, without any loss of generality momentarily focus on the \( \epsilon_+ \) subspace. By choosing specific rotations (specific contours \( C \)) of the field \( E \), we can change an initial state \( |\psi\rangle_{in} = (1, 0, 0, 0) \) into the degenerate state within the same energy level, i.e., \( |\psi\rangle_{out} = \Gamma(C)|\psi\rangle_{in} = (0, 1, 0, 0) \). In fact, in the general case, starting from an arbitrary \( |\psi\rangle_{in} \) we can reach, through carefully choosing the contour \( C \), any other eigenstate within the degenerate subspace by electric field manipulation. In a physical intuitive picture, the spin within the \( \epsilon_+ \) subspace will follow the electric field as it tries to stay within the energy subspace. We have hence achieved spin manipulation with electric fields and showed that holonomic computation is possible in semiconductors with spin-orbit coupling.
In general, due to the non-abelian nature of \( A_t \), the path ordered integral has to be done numerically, over infinitesimal segments in parameter space and taking into account that different components of \( A \) do not commute with each other. While this is more of a nuisance than an intellectual challenge, it is comforting to know that for certain curves the expression can be simplified and path ordering can be easily implemented while still maintaining the full capability to transform the eigenstates into one another. We give such examples for both the linear and the quadratic Stark effect below.
For the linear Stark effect, again working in the \( \epsilon_+ \) energy subspace the expression for the holonomy \( \Gamma_E(C) \) becomes particularly simple:
\[ \Gamma_E(C) = \mathbf{P} \exp\left( -\frac{1}{2} \oint \frac{1}{E^2} \epsilon_{ijk} \sigma_k E_j dE_i \right) \] where the \( \sigma_k \) are the 3 pauli matrices. In polar coordinates \( \vec{E} = (E \sin \theta \cos \phi, E \sin \theta \sin \phi, E \cos \theta) \) for contours \( C \) which keep constant the absolute value of the electric field, we find that spherical triangles between the points \( A (\theta = 0, \phi = \phi_1), B (\theta = \pi/2, \phi = \phi_2), \) and \( C (\theta = \pi/2, \phi = \phi_2) \) are particularly easy to path order. Since we are changing only one angle at a time achieving this technologically should be easier than trying to implement variations in both angles (although, as Zee points out, there is a bit of confusion on how to go ‘around the corners’). For the case of the quadratic Stark effect things are more complicated. While finding a nice form for the holonomy factors in the general case is almost impossible due to the anisotropy in the Hamiltonian \( H_{E2} \), we can look at the idealized spherical symmetric situation for which \( \beta = \delta/\sqrt{3} \). This does not introduce large errors, as the anisotropy in these materials, although sig-
significant, is still small enough so that the spherical approximation works well. In this case we find, in units of $\frac{\hbar^2}{8m^3}$, $d^\mu_\sigma \Gamma^\alpha = \beta(\vec{E} \cdot \vec{S})^2 - 5/4\beta E^2 I_4 \times 4$. The holonomy structure resides exclusively in the first term. In fact, with the electric field replaced by a magnetic field, this is exactly the Hamiltonian studied by Zee in explaining a pioneering experiment by Tycko. The gauge field in polar coordinates is $A_\phi = \cos \theta \sigma_3/2 - \sin \theta \sigma_1$ and $A_\theta = \sigma_2$. For spherical triangles starting at $\theta = 0$ going to some value $\theta$ on an arc of fixed $\phi$, (which we can choose to be zero for convenience) then going at fixed $\theta$ on an arc to some non-zero $\phi$ and then back to the north pole along constant longitude, the holonomy reads $\Gamma_{E^2}(C) = W_1^{-1} VW$. where:

$$W_1^{-1} = \exp(-i \theta (\cos \phi \sigma_2 - \sin \phi \sigma_1)), \quad W = \exp(i \sigma_2 \theta)$$

$$V = \exp(-i \frac{\phi}{2} \sigma_3) \exp\left(\frac{\phi}{2} (\cos \theta \sigma_3 - 2 \sin \theta \sigma_1)\right)$$

We now turn to the problem of the feasibility of the scheme proposed for spin-manipulation by the Stark effect. We need that the coherence time of spins of bound holes be larger than the time in which we can adiabatically rotate the electric field. New experiments showed that the coherence time is larger than $1 \text{ms}$, $10^{-3}$, justifying the use of acceptor-bound-hole wavefunctions as qubits. It is indeed difficult to perform experiments which probe non-abelian phase factors. The original work of Tycko and subsequently the more complete experiment by Zwanziger, Koenig and Pines on nuclear magnetic quadrupole resonance proved the existence of the Wilczeck and Zee non-abelian transport of degenerate states. Instead of rotating the applied fields and keeping the sample fixed, these experiments kept the applied field fixed and rotated the sample, which is an equivalent procedure. The rotation frequencies were of the order of a few kHz (2020 Hz in Zwanziger et al). Imagining an electric-field version of this experiment, the rotation period of the field is already less than the bound hole spin coherence time, but further improvement may be necessary for a realistic measurement. We also need to guarantee, during the field rotation, that the adiabatic approximation is accurately maintained and that the acceptor impurity is not ionized. The ionization energies for acceptor states are of the order 10 – 60 meV (see Table 2). The splitting between the two levels $e_+$ and $e_-$ can be computed from our expressions for their energies (using the constants in Table 1) and are of the order 10 meV for Ge in a field of $10^9$ V/m. Hence the ionization and splitting energies are roughly the same size and much larger than the applied electric field frequency of rotation. The frequencies required for varying the electric field are hence low enough as to cause neither ionization of the impurity-hole system nor a breakdown of adiabaticity. The variation of dynamical phases over the sample volume, which usually leads to extensive dephasing can be overcome by an electric field variant of the double-sweep spin-echo techniques which refocus inhomogeneities in the dynamical phase but double the effect of the geometric phase.

In conclusion, this paper presents a novel way to manipulate the spins of acceptor impurity-bound hole states in p-type semiconductors with strong spin-orbit coupling using electric fields. Depending on its magnitude, the electric field couples both linearly and quadratically to the spin of the acceptor state through the Stark effect, but although apparently opposite, the two effects still maintain the T-invariance of the underlying semiconductor. The spin manipulation is completely geometric and realizes, in a practical solid-state system, the theoretical proposal for holonomic quantum computing. We have obtained an explicit and general form for the holonomy matrix which transforms adiabatically transports degenerate eigenstates. While our analysis is specific to spin 3/2 it is trivially generalized for any spin, provided the Stark effect is present. We have also briefly analyzed the experimental feasibility of the scheme. In a future work, we explore the idea of different qubits communicating via a bus architecture based on the dissipation-less spin current.

| TABLE II: Ionization energies in meV for different acceptor impurities (B, Al, Ga) in Si and Ge |
|--------|--------|--------|
|        | B      | Al     | Ga     |
| Ge     | 10.4   | 10.2   | 10.8   |
| Si     | 45.0   | 57.0   | 65.0   |

In closing, we want to stress that the wide variety of options available for spin manipulation in semiconductors with spin-orbit coupling makes them particularly suitable for the realization of quantum computers. Aside from electric field manipulation, the spin of the acceptor states (as well as that of the free holes) is heavily influenced by interior and exterior applied strain on the material, so that the same analysis presented above can be done for strain-induced holonomies and spin-manipulation.

In the closing stages of this work, we noticed the independent, recent work by Yuri Serebrennikov which presents similar ideas to the ones exposed here.

The authors wish to thank H.D. Chen, R.B. Laughlin and D.I. Santiago for many stimulating discussions and input. The author acknowledges support from the Stanford Graduate Fellowship Program. This work is supported by the NSF under grant numbers DMR-0342832 and the US Department of Energy, Office of Basic Energy Sciences under contract DE-AC03-76SF00515.
[1] P. Zanardi and M. Rasetti, Phys. Lett. A, 264, 94 (1999).
[2] Y. A. Kitaev, quant-phys/9707021.
[3] J. Preskill, quant-phys/9712048
[4] L. Faoro, J. Siewert and R. Fazio, Phys. Rev. Lett. 90, 028301 (2003).
[5] M.S. Choi, J. Phys: Condens. Matter 15, 7823 (2003).
[6] G. L. Bir, E. I. Butikov and G. E. Pikus, J. Phys. Che, Solids 24, 1467 (1963).
[7] G. L. Bir, E. I. Butikov and G. E. Pikus, J. Phys. Che, Solids 24, 1475 (1963).
[8] J. Pachos, P. Zanardi and M. Rasetti, Phys. Rev. A, 61, 010305 (1999).
[9] S. Murakami, N. Nagaosa and S.C. Zhang, Science, 301, 1348 (2003)
[10] F. Wilczek and A. Zee, Phys. Rev. Lett. 52, 2111 (1984).
[11] J. Pachos, quant-ph/0204117
[12] L.-M. Duan, J. I. Cirac, P. Zoller, Science, 292, 1695 (2001).
[13] J. A. Jones, et al., Nature, 403, 869 (2000).
[14] B.A. Bernevig et. al., in preparation.
[15] A. Kopf and K. Lassmann, Phys. Rev. Lett. 69, 1580 (1992)
[16] E. Demler and S.C. Zhang, Ann. Phys. 271, 83 (1999)
[17] S. Murakami, N. Nagaosa, S.C. Zhang, cond-mat/0310005
[18] A. Zee, Phys. Rev. A, 38, 1 (1988).
[19] G.D.J. Smit et. al., cond-mat/0310492
[20] B. Golding and M. I. Dykman, cond-mat/0309147
[21] R. Tycko, Phys. Rev. Lett. 58, 2281 (1987).
[22] J. W. Zwanziger, M. Koenig and A. Pines, Phys. Rev. A 42, 3107 (1990).
[23] Y. A. Serebrennikov, cond-mat/0402319