Thermal Hawking Broadening and Statistical Entropy of Black Hole Wave Packet

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(Dated: June 20, 2013)

The quantum mechanical structure of Schwarzschild black hole is probed, in the mini super spacetime, by means of a non-singular minimal uncertainty Hartle-Hawking wave packet. The Compton width of the microstate probability distribution is translated into a thermal Hawking broadening of the mass spectrum. The statistical entropy is analytically calculated using the Fowler prescription. While the exact Bekenstein-Hawking entropy is recovered at the semi classical limit, the accompanying logarithmic tail gives rise to a Planck size minimal entropy black wave packet.

The Bekenstein-Hawking black hole area entropy is

\[ S_{BH} = \frac{k_B c^3}{4G\hbar} A \]  

constitutes a triple point in the phase of physical theories, touching gravity, even beyond general relativity, quantum mechanics, and statistical mechanics. However, despite its central role in physics, and several illuminating derivations, its statistical mechanical roots have not been fully revealed. There are only a few exceptions where one can actually account, at some limit, for the number of micro states. As far as the prototype Schwarzschild black hole is concerned, we still do not know where the micro states are hiding and even what we are about to enumerate. A black hole is classically characterized by its event horizon, but once \( h \) is switched on, even an innocent looking question like 'where is this horizon located' still lacks a meaningful answer in the quantum or even semi classical sense. In this paper, we examine the possibility that the Schwarzschild black hole states are apparently hidden simply because they degenerate into one single general relativistic state at the \( h \rightarrow 0 \) limit. We carry out our analysis within the framework of the mini super spacetime (a variant of the mini superspace approach, see [10]). For example, the gauge fixing option, still at our disposal, has to be exercised with caution, in particular at the mini super spacetime where the general relativistic action is integrated out over time and solid angle into the mini action \( \int L(T, T', S, S', R) d\tau \). The trail that takes us from here all the way to the reduced Hamiltonian eq.(7) has already been presented in the literature. However, for the sake of coherence, the crucial steps must be briefly outlined. A word of caution is in order: Throughout this paper we treat \( \int L(q, q', r) d\tau \) in full mathematical analogy with \( \int L(q, \dot{q}, t) dt \). Technically, the \( t \)-evolution is traded for the \( r \)-evolution, both classically as well as quantum mechanically, with the notions of Lagrangian and Hamiltonian being adapted accordingly. A similar \( t \leftrightarrow r \) technique was adopted by York and Schmekel.

Unlike the forbidden gauge prefixing of the 'lapse' function \( R(r) \), which kills the Hamiltonian constraint and introduces an unphysical degree of freedom, it is apparently harmless to prefix (say) \( S(r) \) at least at the mini super spacetime level (for an alternative mini superspace approach, see [10]). For example, the gauge choice \( S(r) = r \) defines the tenable radial marker whose geometrical interpretation is \( T, R \)-independent. One can verify that up to a total derivative, and up to an overall factor which can always be absorbed by \( T \), the emerging \( r \)-dependent mini Lagrangian takes the form

\[ L(T, R, R', r) = (r R' + R - 1) \sqrt{\frac{T}{R}} \]  

A simple check verifies that this Lagrangian yields classically the Schwarzschild solution and nothing else. Having in mind the Hamiltonian formalism, however, the trouble is that the conjugate momenta \( p_R = \frac{\partial L}{\partial R'} \) and \( p_T = \frac{\partial L}{\partial T'} \) fail to determine the velocities \( R' \) and \( T' \), giving instead rise to two primary constraints

\[ \phi_1 = p_R - r \sqrt{\frac{T}{R}} \approx 0 \quad , \quad \phi_2 = p_T \approx 0 \]  

The fact that their Poisson brackets do not vanish, that is \( \{ \phi_1, \phi_2 \} = -\frac{r^2}{2\sqrt{TR}} \neq 0 \), makes them second class and invites the Dirac procedure for dealing with constraint systems. The point is that the naive Hamiltonian \( H_{naive} = p_R R' + p_T T' - L \) is not uniquely determined,
and one may add to it any linear combination of the φ’s, which are zero, and go over to \( \mathcal{H}^* = \mathcal{H}_{naive} + \sum u_i \phi_i \). Consistency then requires the constraints be constants of motion, and as such, they must weakly obey \( \frac{d \phi_i}{d r} = 0 \). Once the \( u_i \) coefficients are calculated [8], the so-called total Hamiltonian finally makes its appearance

\[
\mathcal{H}_{total} = (1 - R) \left( \frac{T}{R} + \frac{1}{r} \left( p_R - r \sqrt{\frac{T}{R}} \right) + \frac{T p_T}{r R} \right).
\]  

(5)

Among the associated non-vanishing Dirac brackets, to be replaced by commutation relations in the quantum theory, we find the conventional \( \{ R, p_R \}_D = 1 \), as well as the unconventional \( \{ R, T \}_D = 2 \sqrt{TR}/r \). Explicitly imposing now the \( \phi_{1,2} \) constraints (thereby importing them to the quantum level), and substituting

\[
T = \frac{p_R R p_R}{r^2},
\]

(6)

we are finally led to the reduced on-shell Hamiltonian

\[
\mathcal{H}(R, p, r) = \frac{1}{r} (1 - R) p
\]

(7)

where \( p \) stands for \( p_R \) for the sake of clarity. As a preliminary check, one may solve \( \frac{\partial \mathcal{H}}{\partial p} = R', \) \( \frac{\partial \mathcal{H}}{\partial \mathcal{H}} = -p' \) to confirm the Schwarzschild solution \( R = 1 - 2m/r, \) \( p = \omega r \) (ω can be absorbed by rescaling the time coordinate \( t \)).

The quantization procedure is next. The r-dependent Schrödinger equation associated with the symmetrized reduced Hamiltonian is given by

\[
- \frac{i\hbar}{2} \left( (1 - R) \frac{\partial}{\partial R} + \frac{\partial}{\partial R} (1 - R) \right) \psi = \frac{i\hbar r}{\partial r} \psi.
\]

(8)

Reflecting the fact that the Hamiltonian is linear in the momentum, our Schrödinger equation is apparently \( R \)-independent. It is therefore essential to keep in mind the commutation relation \( \{ R, p \} = \hbar \). The corresponding ‘energy’ eigenstates, studied by Berry and Keating [12] in search for a system where Riemann’s ζ-function zeroes can be physically realized, are not square integrable. This is however not necessarily a problem here because we are after the ‘most classical’ \( \Delta R \Delta p = \frac{1}{2} \hbar \) wave packet solution

\[
\psi(R, r) = \sqrt{\frac{r}{a}} \left( \frac{2}{\pi} \right)^{1/4} e^{-\frac{r^2}{a^2}} \left( 1 - R - \frac{b}{r} \right)^2
\]

(9)

which is an integral over the Berry-Keating states

\[
\psi(R, r) = \int_{-\infty}^{\infty} \frac{f(u) r^{-\frac{1}{2}u} du}{(1 - R)^{\frac{1}{2}+\frac{1}{2}u}},
\]

(10)

for some weight function \( f(u) \). Multiplying \( \psi \) by a phase factor \( e^{i\omega r (1 - R + b/r)} \), in charge of \( \langle p \rangle = \hbar \omega r \), is optional, but would not affect our main conclusions. The wave packet eq. (9), the first one in a tower of orthonormal \( \Delta R \Delta p = (2n + 1)\hbar/2 \) wave packets, is non singular, neither at the \( r \to 0 \) limit nor at the boundaries where \( \psi(\pm \infty, r) \to 0 \).

The general relativity limit is approached when the width \( \sigma(r) = a/2r \) of the wave packet tends to zero. \( \psi^\dagger \psi \) becomes in this limit a narrow Dirac delta function peaked at the classical Schwarzschild solution. In turn, recalling that \( \langle T \rangle \sim \langle R \rangle = 1 - b/r \), we easily identify

\[
b = \frac{2Gm}{c^2}.
\]

(11)

The width must then vanish at the \( \hbar \to 0 \) limit. Following Bekenstein’s insight, one further expects the width to be purely quantum mechanical. It has been argued that adding one bit of information to a heavy black hole increases its \( G \)-dependent Schwarzschild radius by an amount set by its \( G \)-independent Compton length scale (strikingly not by Planck scale). This paves the way for

\[
\eta = \frac{2\hbar}{mc},
\]

(12)

where \( \eta \) is a dimensionless constant to be determined below. The Planck scale enters the game via \( ab = 4n_P^2 \). For example, associated with the Schwarzschild mass operator \( m = \frac{\gamma c^2}{2G} (1 - R) \) are the averages

\[
\langle M \rangle = m^2, \quad \langle M^2 \rangle = m^2 + \frac{\eta^2 \hbar^2 c^2}{4G^2 m^2},
\]

(13)

setting up a fundamental lower bound \( \langle M^2 \rangle_{min} = \eta \eta_P^2 \) which will soon be translated into a minimal entropy.

A close inspection reveals that the black wave packet probability density \( \psi^\dagger \psi \) can be directly translated into the statistical mechanics normalized energy distribution

\[
\rho(E, m) = \frac{\sqrt{2Gm}}{\sqrt{\pi \eta \hbar c^3}} e^{-\frac{2G^2 m^2}{\eta^2 \hbar^2 c^6} (E - mc^2)^2},
\]

(14)

where \( E = Mc^2 \). While a positive \( m \) is a matter of choice, like in the Schwarzschild solution, the mass distribution must cover now the full range \(-\infty < M < \infty \), negative masses included. However, only for \( m > 0 \), the negative masses in the Gaussian tail come with non-negative probabilities. This way, the most probable mass is also the average mass, and the \( \langle M \rangle \to 0 \) limit is accessible.

The time is ripe now for the question where is the horizon actually located? Counter intuitively, as far as our wave packet is concerned, there is nothing special going on in the neighborhood of \( r = 2Gm/c^2 \). So, quantum mechanically, the answer may well be that there is no horizon whatsoever; the horizon is just a purely classical gravitational concept. Semi classically, however, one may interpret eq. (13) as the quantum mechanical profile of the horizon, with a probability density \( \rho(M, m) \) to
find the horizon at radius $2GM/c^2$. Following this line, it makes sense to define an information extract function $I(r, m)$. Classically, no (all) information can be extracted from the black hole interior (exerior), so $I(r, m)$ must approach the Heaviside step function $\theta(r - 2GM/c^2)$ at the general relativistic limit. Upon switching on $\hbar$, $I(r, m)$ is extended to $I(r, m) = \int_{-\infty}^{\infty} \rho(M, m) \theta(r - 2GM/c^2) dM$. More explicitly

$$I(r, m) = \frac{1}{2} \left( 1 + \text{erf} \left( \frac{mc}{\sqrt{2\eta h}}(r - \frac{2GM}{c^2}) \right) \right), \quad (15)$$

indicating that partial information can be extracted from $r < 2GM/c^2$ regions, while some information emanating from $r > 2GM/c^2$ regions gets blocked. This may suggest that black hole radiation cannot be purely thermal.

The energy distribution variance $\Delta E = \sqrt{\langle E^2 \rangle - \langle E \rangle^2}$ determines the Hawking temperature, to be specific

$$\Delta E = \frac{\eta \hbar c^3}{2GM} = k_B T_H \equiv \frac{1}{\beta} , \quad (16)$$

thereby fixing the parameter $\eta = \frac{1}{4\pi}$, and a posteriori justifying eq.\((12)\). Eq.\((16)\) defines the thermal Hawking broadening of the black wave packet. Such a $\Delta E$ is consistent with Hawking’s Euclidean path integral formalism, where the central role is played by the imaginary time periodicity $c\Delta \tau = \frac{\hbar c}{\Delta E} = \frac{8\pi Gm}{c^2}$. Notice that the entire spectrum eq.\((14)\) degenerates into a Dirac $\delta(E - mc^2)$ function at the GR limit when $h \rightarrow 0$, thereby hiding all quantum mechanical degrees of freedom which govern black hole thermodynamics.

Recalling that the average mass $m$ is $T_H$-dependent, one faces a continuous black hole energy spectrum whose levels are in fact temperature dependent. This puts us in a less familiar statistical mechanics territory and calls for extra caution. According to Fowler prescription [7], the naive partition function should be modified in this case

$$\sum_n \rho_n e^{-\beta E_n} \rightarrow \sum_n \rho_n e^{-\beta F_n} , \quad (17)$$

with the Boltzmann factor being traded for the Gibbs-Helmholtz factor. The Helmholtz free energy function $F$ obeys the Gibbs-Helmholtz equation

$$F + \beta \frac{\partial F}{\partial \beta} = E(\beta) \Rightarrow F(\beta) = \frac{1}{\beta} \int_{\beta_0}^{\beta} E(b) db . \quad (18)$$

$\beta_0$ is a constant of integration. Whereas a constant energy $E$ returns $\beta F = \beta E + \text{const}$ as expected, a temperature dependent energy (say) $E \sim \beta$ remarkably results in $\beta F \sim \frac{1}{2} \beta^2 + \text{const}$. Expressed now in a statistical mechanical language, the semi-classical face of this coin is familiar from the conventional (large $m$) approach. To be specific, the first law $dE = T_H dS$, interpreted as $dm \sim m^{-1} dS$, implies $S \sim \frac{1}{2} m^2$ rather than prematurely $S \sim m^2$. In fact, had not we invoked the Fowler prescription eq.\((17)\), we would have wrongly faced, at the large-$m$ regime $S = 2S_{BH} + \text{const}$, which is twice the amount of the Bekenstein-Hawking area entropy.

We proceed now to calculate the exact quantum mechanical Schwarzschild black hole entropy. First, we divide the normalized distribution $\rho(E, m)$ into $N$ equal probability and temperature independent sections, each of which representing a wide energy level, such that

$$\int_{E_n}^{E_{n+1}} \rho(E, m) dE = \frac{1}{N} . \quad (19)$$

This equation is formally solved by invoking the inverse error function $\text{erf}^{-1} x$, that is

$$E_n(\beta) = \frac{\hbar c^5 \beta}{8\pi G} \left[ \sqrt{2} \beta \text{erf}^{-1} (1 - \frac{2n}{N}) \right] , \quad (20)$$

for $n = 0, 1, ..., N$. The condensation of the black hole states at low Hawking temperatures is now manifest. A straightforward solution of the differential Gibbs-Helmholtz eq.\((18)\), with $E_n(\beta)$ eq.\((20)\) serving as the source term, reveals the Helmholtz free energy associated with the $n$-th level

$$\beta F_n = \frac{\hbar c^5 (\beta^2 - \beta_0^2)}{16\pi G} - \sqrt{2} \log \beta \frac{\beta}{\beta_0} \text{erf}^{-1} (1 - \frac{2n}{N}) . \quad (21)$$

The next step is to calculate the partition function

$$Z = e^{\frac{\hbar c^5 (\beta^2 - \beta_0^2)}{16\pi G}} \sum_{n=0}^{N} \frac{1}{N} e^{\sqrt{2} \log \beta \frac{\beta}{\beta_0} \text{erf}^{-1} (1 - \frac{2n}{N})} . \quad (22)$$

We let $N \rightarrow \infty$, and define a continuous integration variable $x = \frac{n}{N}$. The above sum is subsequently replaced by $\int_{0}^{1} e^{\sqrt{2} \log \beta \frac{\beta}{\beta_0} \text{erf}^{-1} (1 - 2x)} dx$, leading upon integration to

$$Z = e^{-\frac{\hbar c^5 (\beta^2 - \beta_0^2)}{16\pi G}} + \frac{1}{2} (\log \frac{\beta}{\beta_0})^2 . \quad (23)$$
The entropy \( S = k_B(1 - \beta \frac{\partial}{\partial \beta}) \log Z \) associated with this partition function is given explicitly by

\[
\frac{S(\beta)}{k_B} = \frac{\hbar c^2 (\beta^2 + \beta_0^2)}{16\pi G} + \frac{1}{2} (\log \beta_0 - \log \beta)
\]

(24)

where the leading term is identified to be the exact (factor \( \frac{1}{4} \) included) Bekenstein-Hawking area entropy eq. (13).

Note that eq. (24) is an exact quantum mechanical formula, and not just a perturbative expansion. Whereas, for large-\( m \), it only supplements the leading Bekenstein-Hawking limit by a novel (log)\(^2\) term (various log-terms have been discussed in the literature [13]), it opens a new window into small-\( m \) black hole thermodynamics.

\[
\beta_0 > \beta_c \quad \text{implies} \quad S_{\text{min}} > 0. \quad \text{While the} \quad \beta > \beta_{\text{min}} \quad \text{branch}
\]

exhibits a familiar black hole feature, namely a negative specific heat \( C = -\beta \frac{\partial S}{\partial \beta} < 0 \), it gets smoothly connected with a novel branch, associated with the \( \beta < \beta_{\text{min}} \) regime, for which the specific heat is counter intuitively, at least in the black hole sense, positive. The emerging, what seems to be a realization of the so-called UV/IR connection [15], takes us into an intriguing yet unfamiliar territory. If \( \beta_0 < \beta_c \), on the other hand, one encounters a region characterized by a negative entropy. We find this situation unacceptable recalling the fact that the entropy, being a logarithm measure of the total number of configurations, is non-negative definite. In other words, there is a gap disconnecting now the two branches mentioned earlier. Based on the above arguments, \( \beta_0 = \beta_c \) becomes our choice of preference if the ground state is non-degenerate.

To summarize, the Schwarzschild black hole states are apparently hidden simply because they degenerate into one single general relativistic state at the \( h \rightarrow 0 \) limit. Once \( h \) is switched on, the Compton width of the Hartle-Hawking wave packet gets revealed, with Schwarzschild geometry becoming just the most probable (as well as the average) solution. Converting the probability density into a statistical mechanics energy distribution, the variance of the latter means thermal Hawking broadening of the wave packet, thereby paving the way for calculating the statistical entropy (adopting Fowler’s prescription on technical grounds). While the exact Bekenstein-Hawking entropy is consistently recovered at the semi classical limit, its logarithmic tail gives rise to a minimal entropy Planck size black wave packet. The inclusion of a cosmological constant \( \Lambda \) is straightforward, including the BTZ and AdS cases, but several other questions are still open and deserve further clarification. For example, the role played by the negative masses in the spectrum, the physical meaning of the \( 0 \leq m < M_{\text{Pl}} \) sub Planckian branch (unphysical? quantum foam? elementary particles?), and the possibility of elevating the minimal entropy configuration to the level of the fundamental black hole building block. It remains to be seen if our work is somehow relevant for artificial black holes [16] as well.

Special thanks to BGU president Prof. Rivka Carmi for her kind support. Valuable conversations with Prof. Doron Cohen, Dr. Ilya Gurwich and Dr. Shimon Rubin are appreciated.

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