Abelian Group Codes for Source Coding and Channel Coding

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Abstract

In this paper, we study the asymptotic performance of Abelian group codes for the lossy source coding problem for arbitrary discrete (finite alphabet) memoryless sources as well as the channel coding problem for arbitrary discrete (finite alphabet) memoryless channels. For the source coding problem, we derive an achievable rate-distortion function that is characterized in a single-letter information-theoretic form using the ensemble of Abelian group codes. When the underlying group is a field, it simplifies to the symmetric rate-distortion function. Similarly, for the channel coding problem, we find an achievable rate characterized in a single-letter information-theoretic form using group codes. This simplifies to the symmetric capacity of the channel when the underlying group is a field. We compute the rate-distortion function and the achievable rate for several examples of sources and channels. Due to the non-symmetric nature of the sources and channels considered, our analysis uses a synergy of information theoretic and group-theoretic tools.

I. INTRODUCTION

Approaching information theoretic performance limits of communication using structured codes has been of great interest for the last several decades [1], [6], [12], [13]. The earlier attempts to design computationally efficient encoding and decoding algorithms for point-to-point communication (both channel coding and source coding) resulted in injection of finite field structures to the coding schemes [10]. In the channel coding problem [22], the channel input alphabets are replaced with algebraic fields

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and encoders are replaced with matrices. Similarly in source coding problem [16], the reconstruction
alphabets are replaced with a finite fields and decoders are replaced with matrices. Later these coding
approaches were extended to weaker algebraic structures such as rings and groups [2], [3], [9], [17],
[18]. The motivation for this are two fold: a) finite fields exist only for alphabets with size equal to a
prime power, and b) for communication under certain constraints, codes with weaker algebraic structures
have better properties. For example, when communicating over an additive white Gaussian noise channel
with 8-PSK constellation, codes over $\mathbb{Z}_8$, the cyclic group of size 8, are more desirable over binary linear
codes because the structure of the code is matched to the structure of the signal set [3], and hence the
former have superior error correcting properties. As another example, construction of polar codes over
alphabets of size $p^r$, for $r > 1$ and $p$ prime, is simpler with a module structure rather than a vector space
structure [26], [29], [31]. Subsequently, as interest in network information theory grew, these codes were
used to approach the information-theoretic performance limits of certain special cases of multi-terminal
communication problems [15], [28], [34], [35]. These limits were obtained earlier using the random
coding ensembles in the information theory literature.

In 1979, Korner and Marton, in a significant departure from tradition, showed that for a binary
distributed source coding problem, the asymptotic average performance of binary linear code ensembles
can be superior to that of the standard random coding ensembles. Although, structured codes were
being used in communication mainly for computational complexity reasons, the duo showed that, in
contrast, even when computational complexity is a non-issue, the use of structured codes leads to
superior asymptotic performance limits in multi-terminal communication problems. In the recent past,
such gains were shown for a wide class of problems [4], [21], [23], [27], [33]. In our prior work,
we developed an inner bound to the optimal rate-distortion region for the distributed source coding
problem [21], [30] in which Abelian group codes were used as building blocks in the coding schemes.
Similar coding approaches were applied for the interference channel and the broadcast channel in [24],
[25]. The motivation for studying Abelian group codes beyond the non-existence of finite fields over
arbitrary alphabets is the following. The algebraic structure of the code imposes certain restrictions on
the performance. For certain problems, linear codes were shown to be not optimal [21], and Abelian group
codes exhibit a superior performance. For example, consider a distributed source coding problem with
two statistically correlated but individually uniform quaternary sources $X$ and $Y$ that are related via the
relation $X = Y + Z$, where $+$ denotes addition modulo-4 and $Z$ is a hidden quaternary random variable

\footnote{Note that this is an incomplete list. There is a vast body of work on group codes. See [10] for a more complete bibliography.}
that has a non-uniform distribution and is independent of \( Y \). The joint decoder wishes to reconstruct \( Z \) losslessly. In this problem, codes over \( \mathbb{Z}_4 \) perform better than linear codes over the Galois field of size 4. In summary, the main reason for using algebraic structured codes in this context is performance rather than complexity of encoding and decoding. Hence information-theoretic characterizations of asymptotic performance of Abelian group code ensembles for various communication problems and under various decoding constraints became important.

Such performance limits have been characterized in certain special cases. It is well-known that binary linear codes achieve the capacity of binary symmetric channels [14]. More generally, it has also been shown that \( q \)-ary linear codes can achieve the capacity of symmetric channels [12] and linear codes can be used to compress a source losslessly down to its entropy [20]. Goblick [1] showed that binary linear codes achieve the rate-distortion function of binary uniform sources with Hamming distortion criterion. Group codes were first studied by Slepian [32] for the Gaussian channel. In [5], the capacity of group codes for certain classes of channels has been computed. Further results on the capacity of group codes were established in [6], [7]. The capacity of group codes over a class of channels exhibiting symmetries with respect to the action of a finite Abelian group has been investigated in [9].

In this work, we focus on two problems. In the first, we consider the lossy source coding problem for arbitrary discrete memoryless sources with single-letter distortion measures and the reconstruction alphabet being equipped with the structure of a finite Abelian group \( G \). We derive an upper bound on the rate-distortion function achievable using group codes which are subgroups of \( G^n \), where \( n \) denotes the block length of encoding which is arbitrarily large. The average performance of the ensemble is shown to be the symmetric rate-distortion function of the source when the underlying group is a field i.e. the Shannon rate-distortion function with the additional constraint that the reconstruction variable is uniformly distributed. For the general case, it turns out that several additional terms appear corresponding to subgroups of the underlying group in the form of a maximization and this can result in a larger rate compared to the symmetric rate for a given distortion level.

In the second part, we consider the channel coding problem for arbitrary discrete memoryless channels. We assume that the channel input alphabet is equipped with the structure of a finite Abelian group \( G \). We derive a lower bound on the capacity of such channels achievable using group codes which are subgroups of \( G^n \). We show that the achievable rate is equal to the symmetric capacity of the channel when the underlying group is a field; i.e. it is equal to the Shannon mutual information between the channel input and the channel output when the channel input is uniformly distributed. Similar to the source coding, we show that in the general case, several additional terms appear corresponding to subgroups
of the underlying group in the form of a minimization and the achievable rate can be smaller than the symmetric capacity of the channel.

It can be noted that the bounds on the performance limits as mentioned above apply to any arbitrary discrete memoryless case. Moreover, we use joint typicality encoding and decoding [11] for both problems at hand. This will make the analysis more tractable. In this approach we use a synergy of information-theoretic and group-theoretic tools. The traditional approaches have looked at encoding and decoding of structured codes based on either minimum distance or maximum likelihood. We introduce two information quantities that capture the performance limits achievable using Abelian group codes that are analogous to the mutual information which captures the Shannon performance limits when no algebraic structure is enforced on the codes. They are source coding group mutual information and channel coding group mutual information. The converse bounds for both problems will be addressed in a future correspondence.

The paper is organized as follows: In section II, some definitions and basic facts are stated which are used in the paper. In Section III, we introduce the ensemble of Abelian group codes used in the paper. In section IV, we state the main results of the paper for both the source coding problem as well as the channel coding problem. We also simplify the expressions for the case where the underlying group is a $\mathbb{Z}_p^r$ ring. In Section V, we prove the results for the source coding problem and similarly, in Section VI, we prove the results for the channel coding problem. In Section VII, we show that for the source coding problem, when the underlying group is a field, the rate-distortion function achievable using Abelian group codes is equal to the symmetric rate-distortion function of the source and for the channel coding problem, the rate achievable using Abelian group codes is equal to the symmetric capacity of the channel. We also provide several examples dealing with non-field groups in this section. We conclude in Section VIII.

II. PRELIMINARIES

1) Source Model: The source is modeled as a discrete-time memoryless random process $X$ with each sample taking values from a finite set $\mathcal{X}$ called alphabet according to the distribution $p_X$. The reconstruction alphabet is denoted by $\mathcal{U}$ and the quality of reconstruction is measured by a single-letter distortion functions $d : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^+$. We denote this source by $(\mathcal{X}, \mathcal{U}, p_X, d)$.

2) Channel Model: We consider discrete memoryless channels used without feedback. We associate two finite sets $\mathcal{X}$ and $\mathcal{Y}$ with the channel as the channel input and output alphabets. The input-output relation of the channel is characterized by a conditional probability law $W_{Y|X}(y|x)$ for $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. The channel is specified by $(\mathcal{X}, \mathcal{Y}, W_{Y|X})$. 
3) Groups: All groups referred to in this paper are *Abelian groups*. Given a group \((G, +)\), a subset \(H\) of \(G\) is called a *subgroup* of \(G\) if it is closed under the group operation. In this case, \((H, +)\) is a group in its own right. This is denoted by \(H \leq G\). A *coset* \(C\) of a subgroup \(H\) is a shift of \(H\) by an arbitrary element \(a \in G\) (i.e. \(C = a + H\) for some \(a \in G\)). For a subgroup \(H\) of \(G\), the number of cosets of \(H\) in \(G\) is called the *index* of \(H\) in \(G\) and is denoted by \(|G : H|\). The index of \(H\) in \(G\) is equal to \(|G|/|H|\) where \(|G|\) and \(|H|\) are the cardinality or size of \(G\) and \(H\) respectively. For a prime \(p\) dividing the cardinality of \(G\), the *Sylow*-\(p\) subgroup of \(G\) is the largest subgroup of \(G\) whose cardinality is a power of \(p\). Group isomorphism is denoted by \(\cong\).

4) Group Codes: Given a group \(G\), a group code \(C\) over \(G\) with block length \(n\) is any subgroup of \(G^n\). A shifted group code over \(G\), \(C + B\) is a translation of a group code \(C\) by a fixed vector \(B \in G^n\). Group codes generalize the notion of linear codes over fields to sources with reconstruction alphabets (and channels with input alphabets) having composite sizes.

5) Achievability for Source Coding and the Rate-Distortion Function: For a group \(G\), a group transmission system with parameters \((n, \Theta, \Delta, \tau)\) for compressing a given source \((X, U = G, P_X, d)\) consists of a codebook, an encoding mapping and a decoding mapping. The codebook \(C\) is a shifted subgroup of \(G^n\) whose size is equal to \(\Theta\) and the mappings are defined as

\[
\begin{align*}
\text{Enc} : X^n &\rightarrow \{1, 2, \ldots, \Theta\}, \\
\text{Dec} : \{1, 2, \ldots, \Theta\} &\rightarrow C
\end{align*}
\]

such that

\[
P[d(X^n, \text{Dec(Enc}(X^n))) > \Delta] \leq \tau
\]

where \(X^n\) is the random vector of length \(n\) generated by the source. In this transmission system, \(n\) denotes the block length, \(\log \Theta\) denotes the number of “channel uses”, \(\Delta\) denotes the distortion level and \(\tau\) is a bound on the probability of exceeding the distortion level \(\Delta\).

Given a source \((X, U = G, P_X, d)\), a pair of non-negative real numbers \((R, D)\) is said to be achievable using group codes if for every \(\epsilon > 0\) and for all sufficiently large numbers \(n\), there exists a group transmission system with parameters \((n, \Theta, \Delta, \tau)\) for compressing the source such that

\[
\frac{1}{n} \log \Theta \leq R + \epsilon, \quad \Delta \leq D + \epsilon, \quad \tau \leq \epsilon
\]
The optimal group rate-distortion function $R^*(D)$ of the source is given by the infimum of the rates $R$ such that $(R,D)$ is achievable using group codes.

6) Achievability for Channel Coding: For a group $G$, a group transmission system with parameters $(n, \Theta, \tau)$ for reliable communication over a given channel $(\mathcal{X} = G, \mathcal{Y}, W_{Y|X})$ consists of a codebook, an encoding mapping and a decoding mapping. The codebook $C$ is a shifted subgroup of $G^n$ whose size is equal to $\Theta$ and the mappings are defined as

$$\text{Enc} : \{1, 2, \ldots, \Theta\} \rightarrow C$$

$$\text{Dec} : Y^n \rightarrow \{1, 2, \ldots, \Theta\}$$

such that

$$\sum_{m=1}^{\Theta} \frac{1}{\Theta} \sum_{x \in \mathcal{X}^n} \sum_{y \in \mathcal{Y}^n} 1_{\{x = \text{Enc}(m)\}} 1_{\{m \neq \text{Dec}(y)\}} W^n(y|x) \leq \tau$$

Given a channel $(\mathcal{X} = G, \mathcal{Y}, W_{Y|X})$, the rate $R$ is said to be achievable using group codes if for all $\epsilon > 0$ and for all sufficiently large $n$, there exists a group transmission system for reliable communication with parameters $(n, \Theta, \tau)$ such that

$$\frac{1}{n} \log \Theta \geq R - \epsilon, \quad \tau \leq \epsilon$$

The group capacity of the channel $C$ is defined as the supremum of the set of all achievable rates using group codes.

7) Typicality: Consider two random variables $X$ and $Y$ with joint probability mass function $p_{X,Y}(x,y)$ over $\mathcal{X} \times \mathcal{Y}$. Let $n$ be an integer and $\epsilon$ be a positive real number. The sequence pair $(x^n, y^n)$ belonging to $\mathcal{X}^n \times \mathcal{Y}^n$ is said to be jointly $\epsilon$-typical with respect to $p_{X,Y}(x,y)$ if

$$\forall a \in \mathcal{X}, \forall b \in \mathcal{Y} : \left| \frac{1}{n} N(a, b|x^n, y^n) - p_{X,Y}(a, b) \right| \leq \frac{\epsilon}{|\mathcal{X}||\mathcal{Y}|}$$

and none of the pairs $(a, b)$ with $p_{X,Y}(a, b) = 0$ occurs in $(x^n, y^n)$. Here, $N(a, b|x^n, y^n)$ counts the number of occurrences of the pair $(a, b)$ in the sequence pair $(x^n, y^n)$. We denote the set of all jointly $\epsilon$-typical sequence pairs in $\mathcal{X}^n \times \mathcal{Y}^n$ by $A^n_\epsilon(X,Y)$.

Given a sequence $x^n \in A^n_\epsilon(X)$, the set of conditionally $\epsilon$-typical sequences $A^n_\epsilon(Y|x^n)$ is defined as

$$A^n_\epsilon(Y|x^n) = \{ y^n \in \mathcal{Y}^n | (x^n, y^n) \in A^n_\epsilon(X,Y) \}$$
8) **Notation:** In our notation, $O(\epsilon)$ is any function of $\epsilon$ such that $\lim_{\epsilon \to 0} O(\epsilon) = 0$, $\mathbb{P}$ is the set of all primes, $\mathbb{Z}^+$ is the set of positive integers and $\mathbb{R}^+$ is the set of non-negative reals. Since we deal with summations over several groups in this paper, when not clear from the context, we indicate the underlying group in each summation; e.g. summation over the group $G$ is denoted by $\sum_{(G)}$. Direct sum of groups is denoted by $\bigoplus$ and direct product of sets is denoted by $\bigotimes$.

### III. ABELIAN GROUP CODE ENSEMBLE

In this section, we use a standard characterization of Abelian groups and introduce the ensemble of Abelian group codes used in the paper.

#### A. Abelian Groups

For an Abelian group $G$, let $\mathcal{P}(G)$ denote the set of all distinct primes which divide $|G|$ and for a prime $p \in \mathcal{P}(G)$ let $S_p(G)$ be the corresponding Sylow subgroup of $G$. It is known [19, Theorem 3.3.1] that any Abelian group $G$ can be decomposed as a direct sum of its Sylow subgroups in the following manner

$$G = \bigoplus_{p \in \mathcal{P}(G)} S_p(G)$$  \hspace{1cm} (1)

Furthermore, each Sylow subgroup $S_p(G)$ can be decomposed into $\mathbb{Z}_{p^r}$ groups as follows:

$$S_p(G) \cong \bigoplus_{r \in \mathcal{R}_p(G)} \mathbb{Z}_{p^r}^{M_{p,r}}$$  \hspace{1cm} (2)

where $\mathcal{R}_p(G) \subseteq \mathbb{Z}^+$ and for $r \in \mathcal{R}_p(G)$, $M_{p,r}$ is a positive integer. Note that $\mathbb{Z}_{p^r}^{M_{p,r}}$ is defined as the direct sum of the ring $\mathbb{Z}_{p^r}$ with itself for $M_{p,r}$ times. Combining Equations (1) and (2), we can represent any Abelian group as follows:

$$G \cong \bigoplus_{p \in \mathcal{P}(G)} \bigoplus_{r \in \mathcal{R}_p(G)} \mathbb{Z}_{p^r}^{M_{p,r}} = \bigoplus_{p \in \mathcal{P}(G)} \bigoplus_{r \in \mathcal{R}_p(G)} \bigoplus_{m=1}^{M_{p,r}} \mathbb{Z}_{p^r}^{(m)}$$  \hspace{1cm} (3)

where $\mathbb{Z}_{p^r}^{(m)}$ is called the $m^{th}$ $\mathbb{Z}_{p^r}$ ring of $G$ or the $(p, r, m)^{th}$ ring of $G$. Equivalently, this can be written as follows

$$G \cong \bigoplus_{(p, r, m) \in \mathcal{G}(G)} \mathbb{Z}_{p^r}^{(m)}$$

where $\mathcal{G}(G) \subseteq \mathbb{P} \times \mathbb{Z}^+ \times \mathbb{Z}^+$ is defined as:

$$\mathcal{G}(G) = \{(p, r, m) \in \mathbb{P} \times \mathbb{Z}^+ \times \mathbb{Z}^+ | p \in \mathcal{P}(G), r \in \mathcal{R}_p(G), m \in \{1, 2, \ldots, M_{p,r}\}\}$$
This means any element \( a \) of the Abelian group can be regarded as a vector whose components are indexed by \((p, r, m) \in G(G)\) and whose \((p, r, m)\)th component \( a_{p,r,m} \) takes values from the ring \( \mathbb{Z}_{p^r} \). With a slight abuse of notation, we represent an element \( a \) of \( G \) as
\[
a = \bigoplus_{(p, r, m) \in G(G)} a_{p,r,m}
\]
Furthermore, for two elements \( a, b \in G \), we have
\[
a + b = \bigoplus_{(p, r, m) \in G(G)} a_{p,r,m} +_{p^r} b_{p,r,m}
\]
where \(+\) denotes the group operation and \(+_{p^r}\) denotes addition mod-\( p^r \). More generally, let \( a, b, \ldots, z \) be any number of elements of \( G \). Then we have
\[
a + b + \cdots + z = \bigoplus_{(p, r, m) \in G(G)} (a_{p,r,m} +_{p^r} b_{p,r,m} +_{p^r} \cdots +_{p^r} z_{p,r,m})
\]
This can equivalently be written as
\[
[a + b + \cdots + z]_{p,r,m} = a_{p,r,m} +_{p^r} b_{p,r,m} +_{p^r} \cdots +_{p^r} z_{p,r,m}
\]
where \([\cdot]_{p,r,m}\) denotes the \((p, r, m)\)th component of it’s argument.

Let \( \mathbb{I}_{G,p,r,m} \in G \) be a generator for the group which is isomorphic to the \((p, r, m)\)th ring of \( G \). Then we have
\[
a = \sum_{(p, r, m) \in G(G)} a_{p,r,m} \mathbb{I}_{G,p,r,m}
\]
where the summations are done with respect to the group operation and the multiplication \( a_{p,r,m} \mathbb{I}_{G,p,r,m} \) is by definition the summation (with respect to the group operation) of \( \mathbb{I}_{G,p,r,m} \) to itself for \( a_{p,r,m} \) times. In other words, \( a_{p,r,m} \mathbb{I}_{G,p,r,m} \) is the short hand notation for
\[
a_{p,r,m} \mathbb{I}_{G,p,r,m} = \sum_{i \in \{1, \ldots, a_{p,r,m}\}} \mathbb{I}_{G,p,r,m}
\]
where the summation is the group operation.

**Example:** Let \( G = \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_9^2 \). Then we have \( \mathcal{P}(G) = \{2, 3\} \), \( S_2(G) = \mathbb{Z}_4 \) and \( S_3(G) = \mathbb{Z}_3 \oplus \mathbb{Z}_9^2 \), \( \mathcal{R}_2(G) = \{2\} \), \( \mathcal{R}_3(G) = \{1, 2\} \), \( M_{2,2} = 1 \), \( M_{3,1} = 1 \), \( M_{3,2} = 2 \) and
\[
G(G) = \{(2, 2, 1), (3, 1, 1), (3, 2, 1), (3, 2, 2)\}
\]
Each element \( a \) of \( G \) can be represented by a quadruple \( (a_{2,1}, a_{3,1,1}, a_{3,2,1}, a_{3,2,2}) \) where \( a_{2,1} \in \mathbb{Z}_4 \), \( a_{3,1,1} \in \mathbb{Z}_3 \) and \( a_{3,2,1}, a_{3,2,2} \in \mathbb{Z}_9 \). Finally, we have \( I_{G;2,1} = (1, 0, 0, 0) \), \( I_{G;3,1,1} = (0, 1, 0, 0) \), \( I_{G;3,2,1} = (0, 0, 1, 0) \), \( I_{G;3,2,2} = (0, 0, 0, 1) \) so that Equation (5) holds.

In the following section, we introduce the ensemble of Abelian group codes which we use in the paper.

B. The Image Ensemble

Recall that for a positive integer \( n \), an Abelian group code of length \( n \) over the group \( G \) is a subgroup of \( G^n \). Our ensemble of codes consists of all Abelian group codes over \( G \); i.e. we consider all subgroups of \( G^n \). We use the following fact to characterize all subgroups of \( G^n \):

Lemma III.1. For an Abelian group \( \tilde{G} \), let \( \phi : J \to \tilde{G} \) be a homomorphism from some Abelian group \( J \) to \( \tilde{G} \). Then \( \phi(J) \leq \tilde{G} \); i.e. the image of the homomorphism is a subgroup of \( \tilde{G} \). Moreover, for any subgroup \( \tilde{H} \) of \( \tilde{G} \) there exists a corresponding Abelian group \( J \) and a homomorphism \( \phi : J \to \tilde{G} \) such that \( \tilde{H} = \phi(J) \).

**Proof:** The first part of the lemma is proved in [8, Theorem 12-1]. For the second part, Let \( J \) be isomorphic to \( \tilde{H} \) and let \( \phi \) be the identity mapping (more rigorously, let \( \phi \) be the isomorphism between \( J \) and \( \tilde{H} \)).

In order to use the above lemma to construct the ensemble of subgroups of \( G^n \), we need to identify all groups \( J \) from which there exist non-trivial homomorphisms to \( G^n \). Then the above lemma implies that for each such \( J \) and for each homomorphism \( \phi : J \to G^n \), the image of the homomorphism is a group code over \( G \) of length \( n \) and for each group code \( C \leq G^n \), there exists a group \( J \) and a homomorphism such that \( C \) is the image of the homomorphism. This ensemble corresponds to the ensemble of linear codes characterized by their generator matrix when the underlying group is a field of prime size. Note that as in the case of standard ensembles of linear codes, the correspondence between this ensemble and the set of Abelian group codes over \( G \) of length \( n \) may not be one-to-one.

Let \( \tilde{G} \) and \( J \) be two Abelian groups with decompositions:

\[
\tilde{G} = \bigoplus_{(p,r,m) \in \mathcal{G}(\tilde{G})} \mathbb{Z}_{p^r}^{(m)}
\]

\[
J = \bigoplus_{(q,s,l) \in \mathcal{G}(J)} \mathbb{Z}_{q^s}^{(l)}
\]
and let $\phi$ be a homomorphism from $J$ to $\tilde{G}$. For $(q, s, l) \in G(J)$ and $(p, r, m) \in G(\tilde{G})$, let

$$g_{(q, s, l) \to (p, r, m)} = [\phi(\mathbb{I}_{J:q,s,l})]_{p,r,m}$$

where $\mathbb{I}_{J:q,s,l} \in J$ is the standard generator for the $(q, s, l)$th ring of $J$ and $[\phi(\mathbb{I}_{J:q,s,l})]_{p,r,m}$ is the $(p, r, m)$th component of $\phi(\mathbb{I}_{J:q,s,l}) \in \tilde{G}$. For $(q, s, l) \in G(J)$ and $(p, r, m) \in G(\tilde{G})$, let $b = \phi(a)$ and write $b = \bigoplus_{(p, r, m) \in G(\tilde{G})} b_{p,r,m}$. Note that as in Equation (5), we can write:

$$a = \bigoplus_{(q, s, l) \in G(J)} a_{q, s, l} \mathbb{I}_{J:q,s,l}$$

where the summations are the group summations. We have

$$b_{p,r,m} = [\phi(a)]_{p,r,m}$$

Note that $(a)$ follows since $\phi$ is a homomorphism; $(b)$ follows from Equation (4); and $(c)$ follows by using $a_{q, s, l} [\phi(\mathbb{I}_{J:q,s,l})]_{p,r,m}$ as the short hand notation for the summation of $[\phi(\mathbb{I}_{J:q,s,l})]_{p,r,m}$ to itself for $a_{q, s, l}$ times.

Note that $g_{(q, s, l) \to (p, r, m)}$ represents the effect of the $(q, s, l)$th component of $a$ on the $(p, r, m)$th component of $b$ dictated by the homomorphism. This means that the homomorphism $\phi$ can be represented...
by

$$\phi(a) = \bigoplus_{(p,r,m) \in G} \sum_{(q,s,l) \in \tilde{G}(J)} a_{q,s,l} g_{(q,s,l) \to (p,r,m)} \quad (6)$$

where $a_{q,s,l} g_{(q,s,l) \to (p,r,m)}$ is the short-hand notation for the mod-$p^r$ addition of $g_{(q,s,l) \to (p,r,m)}$ to itself for $a_{q,s,l}$ times. We have the following lemma on $g_{(q,s,l) \to (p,r,m)}$:

**Lemma III.2.** For a homomorphism described by (6), we have

$$g_{(q,s,l) \to (p,r,m)} = 0 \quad \text{If } p \neq q$$

$$g_{(q,s,l) \to (p,r,m)} \in p^{r-s} \mathbb{Z}_{p^r} \quad \text{If } p = q, r \geq s$$

Moreover, any mapping described by (6) and satisfying these conditions is a homomorphism.

**Proof:** The proof is provided in Appendix IX-A.

This lemma implies that in order to construct a subgroup of $\tilde{G}$, we only need to consider homomorphisms from an Abelian group $J$ to $\tilde{G}$ such that

$$\mathcal{P}(J) \subseteq \mathcal{P}(\tilde{G})$$

since if for some $(q, s, l) \in \mathcal{G}(J)$, $q \notin \mathcal{P}(\tilde{G})$ then $\phi(a)$ would not depend on $a_{q,s,l}$. For $p \in \mathcal{P}(\tilde{G})$, define

$$r_p = \max \mathcal{R}_p(G) \quad (7)$$

We show that we can restrict ourselves to $J$’s such that for all $(q, s, l) \in \mathcal{G}(J)$, $s \leq r_q$. Let $(p, r, m) \in \mathcal{G}(\tilde{G})$ be such that $p = q$. Since $g_{(q,s,l) \to (p,r,m)} \in \mathbb{Z}_{p^r}$ and $r \leq r_q$, we have

$$\left( a_{q,s,l} g_{(q,s,l) \to (p,r,m)} \right) \mod p^r = \left( (a_{q,s,l}) \mod p^r \right) g_{(q,s,l) \to (p,r,m)} \mod p^r$$

This implies that for all $a \in J$ and all $(q, s, l) \in \mathcal{G}(J)$, in the expression for the $(p, r, m)$th component of $\phi(a)$ with $p = q$, $a_{q,s,l}$ appears as $(a_{q,s,l}) \mod q^r$. Therefore, it suffices for $a_{q,s,l}$ to take values from $\mathbb{Z}_{q^r}$ and this happens if $s \leq r_q$.

To construct Abelian group codes of length $n$ over $G$, let $\tilde{G} = G^n$. we have

$$G^n \cong \bigoplus_{p \in \mathcal{P}(G)} \bigoplus_{r \in \mathcal{R}_p} \mathbb{Z}_{p^r}^{n \mathcal{M}_{p,r}} = \bigoplus_{p \in \mathcal{P}(G)} \bigoplus_{r \in \mathcal{R}_p} \mathbb{Z}_{p^r}^{(m)} \bigoplus_{(p,r,m) \in \mathcal{G}(G^n)} \mathbb{Z}_{p^r}^{(m)} \quad (8)$$
Define $J$ as

$$J = \bigoplus_{q \in \mathcal{P}(G)} \bigoplus_{s=1}^{r_q} \mathbb{Z}_{q,s}^{k_{q,s}} = \bigoplus_{q \in \mathcal{P}(G)} \bigoplus_{s=1}^{r_q} \mathbb{Z}_{q,s}^{(l)} = \bigoplus_{(q,s,l) \in G(J)} \mathbb{Z}_{q,s}^{(l)}$$  \hspace{1cm} (9)$$

for some positive integers $k_{q,s}$.

**Example:** Let $G = \mathbb{Z}_8 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_5$. Then we have

$$J = \mathbb{Z}_2^{k_{2,1}} \oplus \mathbb{Z}_2^{k_{2,2}} \oplus \mathbb{Z}_8^{k_{2,3}} \oplus \mathbb{Z}_3^{k_{3,1}} \oplus \mathbb{Z}_9^{k_{3,2}} \oplus \mathbb{Z}_5^{k_{5,3}}$$

Define

$$k = \sum_{q \in \mathcal{P}(G)} \sum_{s=1}^{r_q} k_{q,s}$$

and $w_{q,s} = \frac{k_{q,s}}{k}$ for $q \in \mathcal{P}(G)$ and $s = 1, \cdots, r_q$ so that we can write

$$J = \bigoplus_{q \in \mathcal{P}(G)} \bigoplus_{s=1}^{r_q} k_{q,s} \mathbb{Z}_{q,s}^{(l)}$$  \hspace{1cm} (10)$$

for some constants $w_{q,s}$ adding up to one.

The ensemble of Abelian group encoders consists of all mappings $\phi : J \to G^n$ of the form

$$\phi(a) = \bigoplus_{(p,r,m) \in G^n(\mathbb{Z}_p^r)} \sum_{(q,s,l) \in G(J)} a_{q,s,l} g_{(q,s,l)\rightarrow(p,r,m)}$$  \hspace{1cm} (11)$$

for $a \in J$ where $g_{(q,s,l)\rightarrow(p,r,m)} = 0$ if $p \neq q$. $g_{(q,s,l)\rightarrow(p,r,m)}$ is a uniform random variable over $\mathbb{Z}_{p^r}$ if $p = q, r \leq s$, and $g_{(q,s,l)\rightarrow(p,r,m)}$ is a uniform random variable over $p^{r-s} \mathbb{Z}_{p^r}$ if $p = q, r \geq s$. The corresponding shifted group code is defined by

$$C = \{\phi(a) + B | a \in J\}$$  \hspace{1cm} (12)$$

where $B$ is a uniform random variable over $G^n$. The rate of this code is given by

$$R = \frac{1}{n} \log |J| = \frac{k}{n} \sum_{q \in \mathcal{P}(G)} \sum_{s=1}^{r_q} s w_{q,s} \log q$$  \hspace{1cm} (13)$$

**Remark III.3.** An alternate approach to constructing Abelian group codes is to consider kernels of homomorphisms (the kernel ensemble). To construct the ensemble of Abelian group codes in this manner, let $\phi$ be a homomorphism from $J$ into $G^n$ such that for $a \in G^n$,

$$\phi(a) = \bigoplus_{(q,s,l) \in G(J)} \sum_{(p,r,m) \in G(G^n)} a_{p,r,m} g_{(p,r,m)\rightarrow(q,s,l)}$$
where \( g_{(p,r,m)\rightarrow(q,s,l)} = 0 \) if \( q \neq p \), \( g_{(p,r,m)\rightarrow(q,s,l)} \) is a uniform random variable over \( \mathbb{Z}_{q^r} \) if \( q = p, s \leq r \), and \( g_{(p,r,m)\rightarrow(q,s,l)} \) is a uniform random variable over \( p^{s-r}\mathbb{Z}_{q^r} \) if \( q = p, s \geq r \). The code is given by
\[
C = \{ a \in G^n | \phi(a) = c \}
\]
where \( c \) is a uniform random variable over \( J \).

In this paper, we use the image ensemble for both the channel and the source coding problem; however, similar results can be derived using the kernel ensemble as well.

IV. MAIN RESULTS

In this section, we provide an upper bound on the optimal rate-distortion function for a given source and a lower bound on the capacity of a given channel using group codes when the underlying group is an arbitrary Abelian group represented by Equation (3). We start by defining seven objects and then state two theorems using these objects, and finally provide an interpretation of the results and these objects with two examples.

A. Definitions

For \( q \in \mathcal{P}(G) \), let \( \mathcal{S}_q(G) = \{1, 2, \cdots, r_q\} \) where \( r_q \) is defined as
\[
r_q = \max \mathcal{R}_q(G) \tag{14}
\]
Define
\[
\mathcal{S}(G) = \{(q, s)|q \in \mathcal{P}(G), s \in \mathcal{S}_q(G)\} \tag{15}
\]
\[
\mathcal{Q}(G) = \{(p, r)|p \in \mathcal{P}(G), r \in \mathcal{R}_p(G)\} \tag{16}
\]
We denote vectors \( \hat{\theta} \) and \( w \) whose components are indexed by \( (q, s) \in \mathcal{S}(G) \) by \( (\hat{\theta}_{q,s})_{(q,s)\in\mathcal{S}(G)} \) and \( (w_{q,s})_{(q,s)\in\mathcal{S}(G)} \) respectively and a vector \( \theta \) whose components are indexed by \( (p, r) \in \mathcal{Q}(G) \) by \( (\theta_{p,r})_{(p,r)\in\mathcal{Q}(G)} \).

For \( \hat{\theta} = (\hat{\theta}_{q,s})_{(q,s)\in\mathcal{S}(G)} \), define
\[
\theta(\hat{\theta}) = \left( \min_{\substack{(q,s)\in\mathcal{S}(G) \\ q=p \\ w_{q,s}\neq0}} |r - s|^+ + \hat{\theta}_{q,s} \right)_{(p,r)\in\mathcal{Q}(G)}
\]
and let
\[
\Theta(w) = \left\{ \theta(\hat{\theta}) \mid (\hat{\theta}_{q,s})_{(q,s)\in\mathcal{S}(G)} : 0 \leq \hat{\theta}_{q,s} \leq s \right\} \tag{17}
\]
This set corresponds to a collection of subgroups of $G$ which appear in the rate-distortion function. In other words, depending on the weights $w$, certain subgroups of the group become important in the rate-distortion function. This will be clarified in the proof of the theorem. For $\theta \in \Theta(w)$, define

$$\omega_\theta = \frac{\sum_{(q,s) \in S} \max_{(p,r) \in \mathcal{Q}(G)} (\theta_{p,r} - |r - s|)^+ \cdot w_{q,s} \log q}{\sum_{(q,s) \in S} w_{q,s} \log q}$$

(18)

and let $H_\theta$ be a subgroup of $G$ defined as

$$H_\theta = \bigoplus_{(p,r,m) \in G(G)} p^{\theta_{p,r}} Z_{\mu}^{(m)}$$

(19)

Let $X$ and $U$ be jointly distributed random variables such that $U$ is uniform over $G$ and let $[U]_\theta = U + H_\theta$ be a random variable taking values from the cosets of $H_\theta$ in $G$. We define the source coding group mutual information between $U$ and $X$ as

$$I_{s.c.}^G (U; X) = \min_{w_{q,s}, (q,s) \in \mathcal{S}(G)} \max_{\theta \in \Theta(w)} \frac{1}{\omega_\theta} I([U]_\theta; X)$$

(20)

where $\mathbf{0}$ is a vector whose components are indexed by $(p,r) \in \mathcal{Q}(G)$ and whose $(p,r)^{th}$ component is equal to 0.

Let $X$ and $Y$ be jointly distributed random variables such that $X$ is uniform over $G$ and let $[X]_\theta = X + H_\theta$ be a random variable taking values from the cosets of $H_\theta$ in $G$. We define the channel coding group mutual information between $X$ and $Y$ as

$$I_{c.c.}^G (X; Y) = \max_{w_{q,s}, (q,s) \in \mathcal{S}(G)} \min_{\theta \in \Theta(w)} \frac{1}{1 - \omega_\theta} I(X; Y|[X]_\theta)$$

(21)

where $\mathbf{r}$ is a vector whose components are indexed by $(p,r) \in \mathcal{Q}(G)$ and whose $(p,r)^{th}$ component is equal to $r$.

### B. Main Results

The following theorem is the first main result of this paper.

**Theorem IV.1.** For a source $(\mathcal{X}, \mathcal{U} = G, p_X, d)$ and a given distortion level $D$, let $p_{XU}$ be a joint distribution over $\mathcal{X} \times \mathcal{U}$ such that its first marginal is equal to the source distribution $p_X$, its second marginal $p_U$ is uniform over $\mathcal{U} = G$ and such that $\mathbb{E}\{d(X,U)\} \leq D$. Then the rate-distortion pair $(R, D)$ is achievable where $R = I_{s.c.}^G (U; X)$.

**Proof:** The proof is provided in Section V-B.
When the underlying group is a $\mathbb{Z}_p^r$ ring, this result can be simplified. We state this result in the form of a corollary:

**Corollary IV.2.** Let $X, U$ be jointly distributed random variables such that $U$ is uniform over $\mathcal{U} = G = \mathbb{Z}_p^r$ for some prime $p$ and positive integer $r$. For $\theta = 1, 2, \cdots, r$, let $H_\theta$ be a subgroup of $\mathbb{Z}_p^r$ defined by $H_\theta = p^\theta \mathbb{Z}_p^r$ and let $[U]_\theta = U + H_\theta$. Then,

$$I_{s.c.}^G(U;X) = \max_{\theta=1}^{r} \frac{r}{\theta} I([U]_\theta;X)$$

*Proof:* The proof is provided in Section V-C.

The following theorem is the second main result of this paper.

**Theorem IV.3.** For a channel $(\mathcal{X} = G, \mathcal{Y}, W_{Y|X})$, the rate $R = I_{c.c.}^G(X;Y)$ is achievable using group codes over $G$.

*Proof:* The proof is provided in Section VI-B.

When the underlying group is a $\mathbb{Z}_p^r$ ring, this result can be simplified. We state this result in the form of a corollary:

**Corollary IV.4.** Let $X, Y$ be jointly distributed random variables such that $X$ is uniform over $\mathcal{X} = G = \mathbb{Z}_p^r$ for some prime $p$ and a positive integer $r$. For $\theta = 0, 1, \cdots, r-1$, let $H_\theta$ be a subgroup of $\mathbb{Z}_p^r$ defined by $H_\theta = p^\theta \mathbb{Z}_p^r$ and let $[X]_\theta = X + H_\theta$. Then,

$$I_{c.c.}^G(X;Y) = \max_{\theta=0}^{r-1} \frac{r-1}{r-\theta} I(X;Y|[X]_\theta)$$

*Proof:* The proof is provided in Section VI-C.

When dealing with group codes for the purpose of channel coding, an important case is when the channel exhibits some sort of symmetry. The capacity of group codes for channels with some notion of symmetry is found in [9]. The next corollary states that the result of this paper simplifies to the result of [9] when the channel is symmetric in the sense defined in [9].

**Corollary IV.5.** When the channel $(\mathcal{X} = G, \mathcal{Y}, W_{Y|X})$ is $G$-symmetric in the sense defined in [9], i.e. if

1) $G$ acts simply transitively on $\mathcal{X}$ (trivially holds for this case)

2) $G$ acts isometrically on $\mathcal{Y}$

3) For all $x, g \in G, y \in \mathcal{Y}$, $W(y|x) = W(g \cdot y|g + x)$

then $I_{c.c.}^G(X;Y)$ is equal to the rate provided in [9, Equation (33)].
Proof: The proof is provided in Section VI-D.

C. Interpretation of the Result

In this section, we try to give some intuition about the result and the quantities defined above using several examples. At a high level, \( w_{q,s} \) denotes the normalized weight given to the \( \mathbb{Z}_{q^s} \) component of the input group \( J \) in constructing the homomorphism from \( J \) to \( G^n \), and \( \theta \) indexes a subgroup \( H_\theta \) of \( G \) that comes from a collection \( \Theta(w) \) governed by the choice of \( w_{q,s} \). \( \frac{1}{\omega(\theta)} \mathbb{I}(\theta_X;\theta) \) in source coding and \( \frac{1}{\omega(\theta)} \mathbb{I}(\theta_Y|\theta_X) \) in channel coding denote the rate constraints imposed by the subgroup \( H_\theta \).

Due to the algebraic structure of the code, in the ensemble two random codewords corresponding to two distinct indexes are statistically dependent, unless \( G \) is a finite field. For the source coding problem, when the code is chosen randomly, consider the event that all components of their difference belong to a proper subgroup \( H_\theta \) of \( G \). Then if one of them is a poor representation of a given source sequence, so is the other with a probability that is higher than the case when no algebraic structure on the code is enforced. This means that the code size has to be larger so that with high probability one can find a good representation of the source. For the channel coding problem, when a random codeword corresponding to a given message index is transmitted over the channel, consider the event that all components of the difference between the codeword transmitted and a random codeword corresponding to another message index belong to a proper subgroup \( H_\theta \) of \( G \). Then the probability that the latter is decoded instead of the former is higher than the case when no algebraic structure on the code is enforced.

Example: We start with the simple example where \( G = \mathbb{Z}_8 \). In this case, we have \( \mathcal{P}(G) = \{2\} \), \( r_2 = 3 \), \( S_2 = \{1, 2, 3\} \), \( S = \{(2, 1), (2, 2), (2, 3)\} \), and \( Q(G) = \{(2, 3)\} \). For vectors \( w, \tilde{\theta} \) and \( \theta \) defined as above, we have \( w = (w_{2,1}, w_{2,2}, w_{2,3}) \), \( \tilde{\theta} = (\hat{\theta}_{2,1}, \hat{\theta}_{2,2}, \hat{\theta}_{2,3}) \) and \( \theta = \theta_{2,3} \). Recall that the ensemble of Abelian group codes used in the random coding argument consists of the set of all homomorphisms from some \( J = \mathbb{Z}_2^{k_{w,1}} \oplus \mathbb{Z}_4^{k_{w,2}} \oplus \mathbb{Z}_8^{k_{w,3}} \) and hence the vector of weights \( w \) determines the input group of the homomorphism. Depending on the values of the weights, the structure of the input group can be different; for example, if \( w_{2,1} = 0, w_{2,2} \neq 0 \) and \( w_{2,3} \neq 0 \), the input group will only have \( \mathbb{Z}_4 \) and \( \mathbb{Z}_8 \) components. Any vector \( \tilde{\theta} = (\hat{\theta}_{2,1}, \hat{\theta}_{2,2}, \hat{\theta}_{2,3}) \) with \( 0 \leq \hat{\theta}_{2,1} \leq 1 \), \( 0 \leq \hat{\theta}_{2,2} \leq 2 \) and \( 0 \leq \hat{\theta}_{2,3} \leq 3 \) corresponds to a subgroup \( K_{\tilde{\theta}} \) of the input group \( J \) given by

\[
K_{\tilde{\theta}} = 2\hat{\theta}_{2,1} \mathbb{Z}_2^{k_{w,1}} \oplus 2\hat{\theta}_{2,2} \mathbb{Z}_4^{k_{w,2}} \oplus 2\hat{\theta}_{2,3} \mathbb{Z}_8^{k_{w,3}}
\]

Similarly, any \( \theta = \theta_{2,3} \) with \( 0 \leq \theta_{2,3} \leq 3 \) corresponds to a subgroup \( H_\theta \) of the group space \( G^n \) given by

\[
H_\theta = 2\theta_{2,3} \mathbb{Z}_8^n
\]
Let us assume $w_{2,1} = 0$, $w_{2,2} \neq 0$ and $w_{2,3} \neq 0$ so that $K_{\theta} = 2^{\hat{\theta}_{2,2}}Z_4^{kw_{2,2}} \oplus 2^{\hat{\theta}_{2,3}}Z_8^{kw_{2,3}}$. It turns out that if

$$\theta = \theta(\hat{\theta}) = \min \left(1 + \hat{\theta}_{2,2}, \hat{\theta}_{2,3}\right)$$

then for any random homomorphism $\phi$ from $J$ into $G^n$, and for any $a = (\alpha, \beta) \in J$ with $\alpha \in 2^{\hat{\theta}_{2,2}}Z_4^{kw_{2,2}} \setminus 2^{\hat{\theta}_{2,2} + 1}Z_4^{kw_{2,2}}$ and $\beta \in 2^{\hat{\theta}_{2,3}}Z_8^{kw_{2,3}} \setminus 2^{\hat{\theta}_{2,3} + 1}Z_8^{kw_{2,3}}$, $\phi(a)$ is uniformly distributed over $H^n_{\theta}$. The set $\Theta(w)$ consists of all vectors $\theta$ for which there exists at least one such $a$. Note that this set corresponds to a collection of subgroups of $G^n$. The quantity $1 - \omega_{\theta}$ is a measure of the number of elements $a$ of $J$ for which $\phi(a)$ is uniform over $H_{\theta}$. It turns out that for this example, $\Theta(w) = \{0, 1, 2, 3\}$ and $\omega_0 = 0$, $\omega_1 = \frac{w_{2,3}}{2w_{2,2} + 3w_{2,3}}$, $\omega_2 = \frac{w_{2,2} + 2w_{2,3}}{2w_{2,2} + 3w_{2,3}}$ and $\omega_3 = 1$.

**Example:** Next, we consider the case where $G = \mathbb{Z}_4 \oplus \mathbb{Z}_3$. In this case, we have $P(G) = \{2, 3\}$, $r_2 = 2$, $r_3 = 1$, $S_2 = \{1, 2\}$, $S_3 = \{1\}$, $\mathcal{S} = \{(2, 1), (2, 2), (3, 1)\}$, and $Q(G) = \{(2, 2), (3, 1)\}$. For vectors $w$, $\hat{\theta}$ and $\theta$ defined as before, we have $w = (w_{2,1}, w_{2,2}, w_{2,3})$, $\hat{\theta} = (\hat{\theta}_{2,1}, \hat{\theta}_{2,2}, \hat{\theta}_{3,1})$ and $\theta = (\theta_{2,2}, \theta_{3,1})$. The ensemble of Abelian group codes consists of the set of all homomorphisms from some $J = \mathbb{Z}_2^{kw_{2,1}} \oplus \mathbb{Z}_4^{kw_{2,2}} \oplus \mathbb{Z}_3^{kw_{3,1}}$. Any vector $\hat{\theta} = (\hat{\theta}_{2,1}, \hat{\theta}_{2,2}, \hat{\theta}_{3,1})$ with $0 \leq \hat{\theta}_{2,1} \leq 1$, $0 \leq \hat{\theta}_{2,2} \leq 2$ and $0 \leq \hat{\theta}_{3,1} \leq 1$ corresponds to a subgroup $K_{\hat{\theta}}$ of the input group $J$ given by

$$K_{\hat{\theta}} = 2^{\hat{\theta}_{2,1}}Z_2^{kw_{2,1}} \oplus 2^{\hat{\theta}_{2,2}}Z_4^{kw_{2,2}} \oplus 3^{\hat{\theta}_{3,1}}Z_3^{kw_{3,1}}$$

Similarly, any $\theta = (\theta_{2,2}, \theta_{3,1})$ with $0 \leq \theta_{2,2} \leq 2$ and $0 \leq \theta_{3,1} \leq 1$ corresponds to a subgroup $H_{\theta}$ of the group space $G^n$ given by

$$H_{\theta} = 2^{\theta_{2,2}}Z_4^n \oplus 3^{\theta_{3,1}}Z_3^n$$

Let us assume $w_{2,1}, w_{2,2}, w_{2,3}$ are all non-zero. It turns out that if

$$\theta_{2,2} = \min \left(1 + \hat{\theta}_{2,1}, \hat{\theta}_{2,2}\right)$$

$$\theta_{3,1} = \hat{\theta}_{3,1}$$

then for any random homomorphism $\phi$ from $J$ into $G^n$, and for any $a = (\alpha, \beta, \gamma) \in J$ with $\alpha \in 2^{\hat{\theta}_{2,1}}Z_2^{kw_{2,1}} \setminus 2^{\hat{\theta}_{2,1} + 1}Z_2^{kw_{2,1}}$, $\beta \in 2^{\hat{\theta}_{2,2}}Z_4^{kw_{2,2}} \setminus 2^{\hat{\theta}_{2,2} + 1}Z_4^{kw_{2,2}}$ and $\gamma \in 3^{\hat{\theta}_{3,1}}Z_3^{kw_{3,1}} \setminus 3^{\hat{\theta}_{3,1} + 1}Z_3^{kw_{3,1}}$, $\phi(a)$ is uniformly distributed over $H^n_{\theta}$. Moreover, for this example we have

$$\Theta(w) = \{(0, 0), (1, 0), (2, 0), (0, 1), (1, 1), (2, 1)\}$$
V. PROOF OF SOURCE CODING

A. Encoding and Decoding

Following the analysis of Section III-B, we construct the ensemble of group codes of length $n$ over $G$ as the image of all homomorphisms $\phi$ from some Abelian group $J$ into $G^n$ where $J$ and $G^n$ are as in Equations (10) and (8) respectively. The random homomorphism $\phi$ is described in Equation (11).

To find an achievable rate for a distortion level $D$, we use a random coding argument in which the random encoder is characterized by the random homomorphism $\phi$, a random vector $B$ uniformly distributed over $G^n$ and a joint distribution $p_{XU}$ over $X \times U$ such that its first marginal is equal to the source distribution $p_X$, its second marginal $p_U$ is uniform over $U = G$ and such that $E\{d(X,U)\} \leq D$. The code is defined as in (12) and its rate is given by (13).

Given the source output sequence $x \in \mathcal{X}^n$, the random encoder looks for a codeword $u \in C$ such that $u$ is jointly typical with $x$ with respect to $p_{XU}$. If it finds at least one such $u$, it encodes $x$ to $u$ (if it finds more than one such $u$ it picks one of them at random). Otherwise, it declares error. The decoder outputs $u$ as the source reconstruction.

B. Error Analysis

Let $x = (x_1, \ldots, x_n)$ and $u = (u_1, \ldots, u_n)$ be the source output and the encoder/decoder output respectively. Note that if the encoder declares no error then since $x$ and $u$ are jointly typical, $(d(x_i, u_i))_{i=1, \ldots, n}$ is typical with respect to the distribution of $d(X,U)$. Therefore for large $n$, $\frac{1}{n} \sum_{i=1}^n d(x_i, u_i) \approx E\{d(X,U)\} \leq D$. It remains to show that the rate can be as small as $I_{s.c.}^G(X;U)$ while keeping the probability of encoding error small.

Given the source output $x \in \mathcal{X}^n$, define

$$\alpha(x) = \sum_{u \in A^n(U|x)} 1_{\{u \in C\}} = \sum_{u \in A^n(U|x)} \sum_{a \in J} 1_{\{\phi(a)+B=u\}}$$

An encoding error occurs if and only if $\alpha(x) = 0$. We use the following Chebyshev's inequality to show that under certain conditions the probability of error can be made arbitrarily small:

$$P(\alpha(x) = 0) \leq \frac{\text{var}\{\alpha(x)\}}{E\{\alpha(x)\}^2}$$

We need the following lemmas to proceed:
Lemma V.1. For \( a, \tilde{a} \in J \), \( u, \tilde{u} \in G^n \) and for \( (q, s, l) \in \mathcal{G}(J) \), let \( \hat{\theta}_{q,s,l} \in \{0, 1, \cdots, s\} \) be such that \( \tilde{a}_{q,s,l} - a_{q,s,l} \in q^{\hat{\theta}_{q,s,l}} \mathbb{Z}_{q^s} \setminus q^{\hat{\theta}_{q,s,l}+1} \mathbb{Z}_{q^s} \). For \( (p, r) \in \mathcal{Q}(G) \) define
\[
\theta_{p,r}(a, \tilde{a}) = \min_{(q, s, l) \in \mathcal{G}(J)} |r - s|^+ + \hat{\theta}_{q,s,l}
\]
and let \( \theta_{p,r} = \theta_{p,r}(a, \tilde{a}) \). Define the subgroup \( H_{\theta} \) of \( G \) as
\[
H_{\theta} = \bigoplus_{(p, r, m) \in \mathcal{Q}(G)} p^{\theta_{p,r}(m)} Z_{p^m}^{(m)}
\]
Then,
\[
P(\phi(a) + B = u, \phi(\tilde{a}) + B = \tilde{u}) = \begin{cases} \frac{1}{|G|^n} & \text{if } \tilde{u} - u \in H_{\theta}^0 \\ 0 & \text{otherwise} \end{cases}
\]

**Proof:** The proof is provided in Appendix IX-B.

Lemma V.2. For \( a \in J \) and \( \theta = (\theta_{p,r})_{(p, r) \in \mathcal{Q}(G)} \), let
\[
T_{\theta}(a) = \{ \tilde{a} \in J | \forall (p, r) \in \mathcal{Q}(G), \theta_{p,r}(a, \tilde{a}) = \theta_{p,r} \}
\]
where \( \theta_{p,r}(a, \tilde{a}) \) is defined as in the previous lemma. Then we have
\[
|T_{\theta}(a)| \leq \prod_{(q, s, l) \in \mathcal{G}(J)} q^{\sum_{(p, r) \in \mathcal{Q}(G)} (\theta_{p,r} - |r - s|^+)^+}
\]

**Proof:** The proof is provided in Appendix IX-C.

Lemma V.3. For \( a \in J \) and \( u \in G^n \), we have
\[
P(\phi(a) + B = u) = \frac{1}{|G|^n}
\]

**Proof:** Immediate from Lemma V.1.

Lemma V.4. For fixed \( w = (w_{q,s})_{(q, s) \in \mathcal{S}(G)} \) and for any \( a \in J = \bigoplus_{(q, s) \in \mathcal{S}(G)} \bigoplus_{l=1}^{k_{w_{q,s}}} Z_{q^l}^{(l)} \),
\[
\{ \theta = (\theta_{p,r})_{(p, r) \in \mathcal{Q}(G)} | |T_{\theta}(a)| \neq 0 \} = \Theta(w)
\]
where \( \Theta(w) \) is defined in Equation (17).

Proof: Provided in the Appendix IX-D.

We have

\[
\mathbb{E}\{\alpha(x)\} = \sum_{u \in A^n(U|x)} \sum_{a \in J} P(\phi(a) + B = u) \\
= \frac{|A^n(U|x)| \cdot |J|}{|G|^n}
\]

and

\[
\mathbb{E}\{\alpha(x)^2\} = \mathbb{E}\left\{ \sum_{u, \tilde{a} \in A^n(U|x)} \sum_{a \in J} 1\{\phi(a) + B = u, \phi(\tilde{a}) + B = \tilde{u}\} \right\} \\
= \sum_{u, \tilde{a} \in A^n(U|x)} \sum_{a, \tilde{a} \in J} P(\{\phi(a) + B = u, \phi(\tilde{a}) + B = \tilde{u}\}) \\
= \sum_{\theta \in \Theta(u)} \sum_{a \in J} \sum_{u \in A^n(U|x)} \sum_{\tilde{a} \in T_\theta(a)} \sum_{\tilde{u} \in H^n_\theta} \frac{1}{|G|^n} \cdot \frac{1}{|H_\theta|^n}
\]

Note that the term corresponding to \( \theta = 0 \) is upper bounded by \( \mathbb{E}\{\alpha(x)\}^2 \). Using Lemma IX.2, we have

\[
|A^n(U|x) \cap (u + H^n_\theta)| \leq 2^{n[H(U|[U]_\theta,X) + O(\epsilon)]}
\]

Therefore,

\[
\text{var}\{\alpha\} = \mathbb{E}\{\alpha(x)^2\} - \mathbb{E}\{\alpha(x)\}^2 \\
\leq \sum_{\theta \neq 0} |J| \cdot |A^n(U|x)| \prod_{(q,s) \in S(G)} q \left( s - \max_{p=q} (\theta_{p,r} - |r - s|^+) \right)^{k w_{q,s}^{-}} \frac{2^{n[H(U|[U]_\theta,X) + O(\epsilon)]}}{|G|^n \cdot |H_\theta|^n}
\]

Therefore,

\[
P(\alpha(x) = 0) \leq \frac{\text{var}\{\alpha(x)\}}{\mathbb{E}\{\alpha(x)\}^2} \\
\leq \sum_{\theta \neq 0} \prod_{(q,s) \in S(G)} q \left( s - \max_{p=q} (\theta_{p,r} - |r - s|^+) \right)^{k w_{q,s}^{-}} \frac{2^{-n[H(U|X) - H(U|[U]_\theta,X) - O(\epsilon)]}|G|^n}{|J| \cdot |H_\theta|^n}
\]

Note that \( H(U|X) - H(U|[U]_\theta,X) = H([U]_\theta|X) \) and

\[
|J| = \prod_{(q,s) \in S(G)} q^{k w_{q,s}^{-}}
\]
Therefore,

\[ P (\alpha(x) = 0) \leq \sum_{\theta \in \Theta(w)} \exp \left\{-n \left[ H([U]_\theta|X) - \log |G : H_\theta| + \frac{k}{n} \sum_{(q,s) \in S(G)} w_{q,s} \log q \max_{p=q} \left( \theta_{p,r} - |r-s|^+ \right) + O(\epsilon) \right] \right\} \]

In order for the probability of error to go to zero as \( n \) increases, we require the exponent of all the terms to be negative; or equivalently, for \( \theta \in \Theta(w) \) and \( \theta \neq 0 \),

\[ R \sum_{(q,s) \in S(G)} \max_{p=q} \left( \theta_{p,r} - |r-s|^+ \right) w_{q,s} \log q > \log |G : H_\theta| - H([U]_\theta|X) \]

Therefore, the achievability condition is

\[ R > \frac{1}{\omega_\theta} (\log |G : H_\theta| - H([U]_\theta|X)) \]

with the convention \( \frac{1}{0} = \infty \) and where

\[ \omega_\theta = \frac{\sum_{(q,s) \in S(G)} \max_{p=q} \left( \theta_{p,r} - |r-s|^+ \right) w_{q,s} \log q}{\sum_{(q,s) \in S(G)} w_{q,s} \log q} \]

Therefore, the achievable rate is equal to

\[ R = \min_{w_{q,s}} \frac{1}{\omega_\theta} \max_{\theta \in \Theta(w)} \frac{1}{\omega_\theta} I([U]_\theta; X) \]

C. Simplification of the Rate for the \( \mathbb{Z}_{p^r} \) Case

In this section, we provide a proof of Corollary IV.2 by showing that when \( G = \mathbb{Z}_{p^r} \) for some prime \( p \) and positive integer \( r \), then \( I_{s.c.}^G(U; X) = R_1 \) where

\[ R_1 = \max_{\theta=1} \frac{1}{\theta} I([U]_\theta; X) \] (25)

When \( G = \mathbb{Z}_{p^r} \) for some prime \( p \) and positive integer \( r \), we have \( S(G) = S_p(G) = \{1, 2, \cdots, r\} \). For fixed weights \( w_s, s \in S(G) \) adding up to one, define \( \bar{r} = \max\{s \in S(G)|w_s \neq 0\} \). We have

\[ J = \bigoplus_{s=1}^{\bar{r}} \bigoplus_{l=1}^{k w_s} \mathbb{Z}_{p^l} \]

For \( a \in J \) and for \( \theta = 1, \cdots, r \), let \( T_\theta(a) \) be defined as in Lemma V.1; i.e.

\[ T_\theta(a) = \{ \tilde{a} \in J | \min_{s=1}^{\bar{r}} \min_{l=1}^{k w_s} r - s + \hat{\theta}_{s,l}(a, \tilde{a}) = \theta \} \]
where for \( a, \tilde{a} \in J \) and for \( s = 1, \ldots, \tilde{r}, l = 1 \ldots, k w_s, \; \tilde{\theta}_{s,l}(a, \tilde{a}) = \min \{ 0 \leq \tilde{\theta}_{s,l} \leq s | \tilde{a}_{s,l} - a_{s,l} \in p^{\tilde{\theta}_{s,l}} Z_{p^s} \} \). Note that for \( \theta = r \), we have \( T_{\theta}(a) = \{ a \} \) and for \( 0 \leq \theta < r \),

\[
T_{\theta}(a) = \{ \tilde{a} \in J | \forall s = 1, \ldots, \tilde{r}, l = 1, \ldots, k w_s : \tilde{a}_{s,l} - a_{s,l} \in p^{[\theta + s - r]} Z_{p^s} \} 
\]

Note that \( p^{[\theta + s - r]} Z_{p^s} = \min(p^{\theta - \theta}, p^{s}) \) and \( p^{[\theta + 1 + s - r]} Z_{p^s} = \min(p^{\theta - \theta}, p^{s}) \). Therefore,

\[
| T_{\theta}(a) | = \left( \prod_{s=1}^{\tilde{r}} \prod_{l=1}^{k w_s} \min(p^{\theta - \theta}, p^{s}) \right) - \left( \prod_{s=1}^{\tilde{r}} \prod_{l=1}^{k w_s} \min(p^{\theta - \theta}, p^{s}) \right)
\]

\[
= \left( \prod_{s=1}^{\tilde{r}} \prod_{l=1}^{k w_s} p^{\theta - \theta} \right) - \left( \prod_{s=1}^{\tilde{r}} \prod_{l=1}^{k w_s} p^{\theta - \theta} \right) \cdot \left( \prod_{s=1}^{\tilde{r}} \prod_{l=1}^{k w_s} p^{\theta - \theta} \right)
\]

This means for \( \theta < r - \tilde{r}, | T_{\theta} | = 0 \) and for \( r - \tilde{r} \leq \theta \leq r \) and \( | T_{\theta} | \neq 0 \). Therefore \( \Theta(w) = \{ r - \tilde{r}, \ldots, r \} \).

The achievable rate is given by

\[
R = \min_{w_1, \ldots, w_r} \max_{\theta \in \Theta(w)} \frac{1}{1 - \omega_{\theta}} I([U]; X)
\]

(26)

where for \( \theta \in \Theta(w) \),

\[
1 - \omega_{\theta} = 1 - \frac{\sum_{s=r-\theta}^{\tilde{r}} \theta + s - r \sum_{s=1}^{s} w_s}{\sum_{s=1}^{s} w_s}
\]

(27)

Note that for \( \theta \geq r - \tilde{r} \), we have

\[
\frac{r - \theta}{r} \sum_{s=1}^{\tilde{r}} s w_s = \frac{r - \theta}{\tilde{r}} \sum_{s=1}^{\tilde{r}} s w_s + \sum_{s=r-\theta+1}^{\tilde{r}} \frac{s}{\tilde{r}} (r - \theta) w_s
\]

\[
\leq \sum_{s=1}^{\tilde{r}} s w_s + \sum_{s=r-\theta+1}^{\tilde{r}} (r - \theta) w_s
\]

\[
= (1 - \tilde{\omega}_{\theta}) \sum_{s=1}^{\tilde{r}} s w_s
\]

Therefore, it follows that \( 1 - \omega_{\theta} \geq \frac{r - \theta}{r} \) or equivalently, \( \omega_{\theta} \leq \frac{\theta + \tilde{r} - r}{r} \). Let \( \tilde{w}_1 = \cdots = \tilde{w}_{\tilde{r}-1} = \tilde{w}_{\tilde{r}+1} = \tilde{w}_r = 0 \) and let \( \tilde{w}_{\tilde{r}} = 1 \). Define \( \tilde{\omega}_{\theta} \) using Equation (27) replacing \( w \)‘s with \( \tilde{w} \)’s to get \( \tilde{\omega}_{\theta} = \frac{\theta + \tilde{r} - r}{r} \). It follows that we always have \( \omega_{\theta} \leq \tilde{\omega}_{\theta} \) and therefore, it is always optimal to put all the weight on \( \tilde{w} \) if we are confined to have \( w_{\tilde{r}+1} = \cdots = w_r = 0 \). It follows that the achievable rate is equivalent to

\[
R = \min_{\tilde{r} \in \{1, \ldots, r\}} \max_{\theta \in \{r - \tilde{r}, \ldots, r\}} \frac{\tilde{r}}{\theta + \tilde{r} - r} I([U]; X)
\]

(28)
For $\hat{r} < r$, since by convention $\frac{1}{0} = \infty$, the corresponding term is infinity. It implies that $\hat{r} = r$ achieves the optimal rate. Hence,

$$I_{s.c} = \max_{\theta \in \{1, \cdots, r\}} \frac{1}{\theta} I(U_\theta; X)$$ (29)

VI. PROOF OF CHANNEL CODING

A. Encoding and Decoding

Following the analysis of Section III-B, we construct the ensemble of group codes of length $n$ over $G$ as the image of all homomorphisms $\phi$ from some Abelian group $J$ into $G^n$ where $J$ and $G^n$ are as in Equations (10) and (8) respectively. The random homomorphism $\phi$ is described in Equation (11).

To find an achievable rate, we use a random coding argument in which the random encoder is characterized by the random homomorphism $\phi$ and a random vector $B$ uniformly distributed over $G^n$. Given a message $u \in J$, the encoder maps it to $x = \phi(u) + B$ and $x$ is then fed to the channel. At the receiver, after receiving the channel output $y \in Y^n$, the decoder looks for a unique $\tilde{u} \in J$ such that $\phi(\tilde{u}) + B$ is jointly typical with $y$ with respect to the distribution $p_X W_{Y|X}$ where $p_X$ is uniform over $G$. If the decoder does not find such $\tilde{u}$ or if such $\tilde{u}$ is not unique, it declares error.

B. Error Analysis

Let $u, x$ and $y$ be the message, the channel input and the channel output respectively. The error event can be characterized by the union of two events: $E(u) = E_1(u) \cup E_2(u)$ where $E_1(u)$ is the event that $\phi(u) + B$ is not jointly typical with $y$ and $E_2(u)$ is the event that there exists a $\tilde{u} \neq u$ such that $\phi(\tilde{u}) + B$ is jointly typical with $y$. We can provide an upper bound on the probability of the error event as $P(E(u)) \leq P(E_1(u)) + P(E_2(u) \cap (E_1(u))^c)$. Using the standard approach, one can show that $P(E_1(u)) \to 0$ as $n \to \infty$. The probability of the error event $E_2(u) \cap (E_1(u))^c$ averaged over all messages can be written as

$$P_{avg}(E_2(u) \cap (E_1(u))^c) = \sum_{u \in J} \frac{1}{|J|} \sum_{x \in G^n} \mathbb{1}_{\{\phi(u) + B = x\}} \sum_{y \in A^n_Y(y|x)} W_{Y|X}^n(y|x) \mathbb{1}_{\{\exists \tilde{u} \in J: \tilde{u} \neq u, \phi(\tilde{u}) + B \in A^n_Y(x|y)\}}$$

The expected value of this probability over the ensemble is given by $E\{P_{avg}(E_2(u) \cap (E_1(u))^c)\} = P_{err}$ where

$$P_{err} = \sum_{u \in J} \frac{1}{|J|} \sum_{x \in G^n} \sum_{y \in A^n_Y(y|x)} W_{Y|X}^n(y|x) P(\phi(u) + B = x, \exists \tilde{u} \in J: \tilde{u} \neq u, \phi(\tilde{u}) + B \in A^n_Y(X|y))$$
Using the union bound, we have
\[
P_{\text{err}} \leq \sum_{u \in J} \frac{1}{|J|} \sum_{x \in \mathcal{G}} \sum_{y \in \mathcal{A}_t} \sum_{\theta \in \Theta} \sum_{u \in T_0(u)} \sum_{\tilde{x} \in \mathcal{A}_t(Y|x)} W_{y|x}^n(y|x) P(\phi(u) + B = x, \phi(\tilde{u}) + B = \tilde{x})
\]
Define \( \Theta(w) \) as in Equation (17) and for \( \theta \in \Theta(w) \) and \( u \in J \), define \( T_0(u) \) as in Lemma V.2. It follows that
\[
P_{\text{err}} \leq \sum_{\theta \in \Theta(w)} \sum_{u \in J} \frac{1}{|J|} \sum_{x \in \mathcal{G}} \sum_{y \in \mathcal{A}_t(Y|x)} P(\theta) \sum_{\tilde{x} \in \mathcal{A}_t(X|y)} \sum_{u \in T_0(u)} W_{y|x}^n(y|x) P(\phi(u) + B = x, \phi(\tilde{u}) + B = \tilde{x})
\]
Using Lemmas V.1, IX.2 and V.2, we have
\[
P_{\text{err}} \leq \sum_{\theta \in \Theta(w)} \sum_{u \in J} \frac{1}{|J|} \sum_{x \in \mathcal{G}} \sum_{y \in \mathcal{A}_t(Y|x)} P(\theta) \sum_{\tilde{x} \in \mathcal{A}_t(X|y)} \sum_{u \in T_0(u)} W_{y|x}^n(y|x) \frac{1}{|\mathcal{G}|^n} \frac{1}{|H_\theta|^n} 2^n H(X|Y|X_\theta) + O(\epsilon) \]
\[
\leq \sum_{\theta \in \Theta(w)} \sum_{u \in J} \frac{1}{|J|} \prod_{(q,s) \in \mathcal{S}(G)} q \left( s - \max_{(p,r) \in \mathcal{Q}(G)} (\theta_{p,r} - |r - s|^+) \right) w_{q,s} 2^n H(X|Y|X_\theta) + O(\epsilon) \]
Equivalently, this can be written as
\[
P_{\text{err}} \leq \sum_{\theta \in \Theta(w)} \sum_{u \in J} \frac{1}{|J|} \prod_{(q,s) \in \mathcal{S}(G)} q \left( s - \max_{(p,r) \in \mathcal{Q}(G)} (\theta_{p,r} - |r - s|^+) \right) w_{q,s} \log q - H(X|Y|X_\theta) + \log |H_\theta| - O(\epsilon)
\]
Recall that \( R = \frac{k}{n} \sum_{(q,s) \in \mathcal{S}(G)} s w_{q,s} \log q \). In order for the probability of error to go to zero, we require the exponent of all the terms to be negative; or equivalently, for \( \theta \in \Theta(w) \) and \( \theta \neq r \),
\[
\sum_{(q,s) \in \mathcal{S}(G)} \left( s - \max_{(p,r) \in \mathcal{Q}(G)} (\theta_{p,r} - |r - s|^+) \right) w_{q,s} \log q < \log |H_\theta| - H(X|Y|X_\theta)
\]
Therefore, the achievability conditions are
\[
R \leq \frac{1}{1 - \omega_\theta} I(X; Y|X_\theta)
\]
for all $\theta \in \Theta(w)$ such that $\theta \neq \mathbf{r}$ where $\omega_\theta$ is defined in (18). This means that the following rate is achievable

$$R = \min_{\theta \in \Theta(w), \theta \neq \mathbf{r}} \frac{1}{1 - \omega_\theta} I(X; Y | [X]_\theta)$$

If we maximize over the choice of $w$, we can conclude that the rate $R = I^G_{\text{c.c.}}(X; Y)$ is achievable.

### C. Simplification of the Rate for the $\mathbb{Z}_{p^r}$ Case

In this section, we provide a proof of Corollary IV.4 by showing that if $G = \mathbb{Z}_{p^r}$ for some prime $p$ and a positive integer $r$, then $I^G_{\text{c.c.}}(X; Y) = R_1$ where

$$R_1 = \min_{\theta = 0}^{r-1} \frac{r}{r - \theta} I(X; Y | [X]_\theta)$$

(30)

First, we show that the achievable rate is equivalent to

$$R_2 = \max_{\tilde{r} = 1} \min_{\theta = r - \tilde{r}}^{r-1} I(X; Y | [X]_\theta)$$

(31)

When $G = \mathbb{Z}_{p^r}$ for some prime $p$ and positive integer $r$, we have $S(G) = S_p(G) = \{1, 2, \cdots, r\}$. For fixed weights $w_s, s \in S(G)$ adding up to one, define $\tilde{r} = \max\{s \in S(G) | w_s \neq 0\}$. Similarly to the source coding case, we can show that for $\theta \in \Theta(w) = \{r - \tilde{r}, \cdots, r\}$, we have $1 - w_\theta \geq \frac{\omega_\theta}{\tilde{r}}$ and it is always optimal to put all the weight on $w_{\tilde{r}}$ if we are confined to have $w_{\tilde{r}+1} = \cdots = w_r = 0$. It follows that the achievable rate provided in Equation (31) is equal to $I^G_{\text{c.c.}}(X; Y)$. Next, we show that the rate in Equation (31) is equal to the rate in Equation (30). We need the following lemma:

**Lemma VI.1.** Let $\theta$ and $\tilde{\theta}$ be such that $0 \leq \tilde{\theta} \leq \theta \leq r$. Then

$$I(X; Y | [X]_\theta) \leq I(X; Y | [X]_{\tilde{\theta}})$$

**Proof:** Note that $[X]_\theta$ and $[X]_{\tilde{\theta}}$ are both functions of $X$ and therefore

$$I(X; Y | [X]_\theta) = I(X; Y) - I([X]_\theta; Y)$$

$$I(X; Y | [X]_{\tilde{\theta}}) = I(X; Y) - I([X]_{\tilde{\theta}}; Y)$$

Furthermore, note that since $\tilde{\theta} \leq \theta$ the Markov chain $[X]_{\tilde{\theta}} \leftrightarrow [X]_\theta \leftrightarrow Y$ holds and therefore, $I([X]_\theta; Y) \geq I([X]_{\tilde{\theta}}; Y)$. This proves the lemma.
Let $\theta^*$ be the minimizer in Equation (30). For $r - \theta^* \leq \tilde{r} < r$ we have:

$$\min_{\theta = r - \tilde{r}} \frac{\tilde{r}}{r - \theta} I(X; Y|X[\theta]) \leq \left[ \frac{\tilde{r}}{r - \theta^*} I(X; Y|X[\theta^*]) \right]_{\theta = \theta^*}$$

$$= \frac{r}{r - \theta^*} I(X; Y|X[\theta^*])$$

For $\tilde{r} < r - \theta^*$ we have:

$$\frac{r - 1}{\min_{\theta = r - \tilde{r}} \frac{\tilde{r}}{r - \theta}} I(X; Y|X[\theta]) \leq \left[ \frac{\tilde{r}}{r - \theta} I(X; Y|X[\theta]) \right]_{\theta = r - \tilde{r}}$$

$$= I(X; Y|[X]_{r - \tilde{r}})$$

$$\leq I(X; Y|[X]_{\theta^*})$$

$$\leq R_1$$

Therefore, it follows that the rate $R_1$ is equivalent to the rate $R_2$ and hence $I_{c.c.}^G(X; Y) = R_1$.

**D. $G$-Symmetric Channels**

In this section, we provide a proof of corollary IV.5. Note that since we take $X = G$, we can take the action of $G$ on $X$ to be the group operation. We need to show that for all subgroups $H$ of $G$, $I(X; Y|[X]) = C_H$ where $X = X + H$ and $C_H$ is the mutual information between the channel input and the channel output when the input is uniformly distributed over $H$; in other words, $C_H = I(X; Y|[X] = H)$. This in turn follows by showing that for all $g \in G$

$$I(X; Y|[X] = g + H) = I(X; Y|[X] = H)$$

This can be shown as follows:

$$I(X; Y|[X] = g + H) = \sum_{x \in g + H} \sum_{y \in Y} \frac{1}{|H|} W(y|x) \log \frac{W(y|x)}{P(y)}$$

$$= \sum_{x \in H} \sum_{y \in Y} \frac{1}{|H|} W(y|x + g) \log \frac{W(y|x + g)}{P(y)}$$

$$\overset{(a)}{=} \sum_{x \in H} \sum_{y \in Y} \frac{1}{|H|} W(g \cdot y | x + g) \log \frac{W(g \cdot y | x + g)}{P(y)}$$

$$\overset{(b)}{=} \sum_{x \in H} \sum_{y \in Y} \frac{1}{|H|} W(y | x) \log \frac{W(y | x)}{P(y)}$$

$$= I(X; Y|[X] = H)$$
where (a) follows since the action of $g$ on $\mathcal{Y}$ is a bijection of $\mathcal{Y}$ and (b) follows from the symmetric property of the channel. Using this result, it can be shown that the rate provided in [9, Equation (33)] is equal to $I_{c.c.}^G(X;Y)$. The difference in the appearance of the two expressions is due to the fact that in [9, Equation (33)] the minimization is carried out over the subgroups of the input group whereas in the expression for $I_{c.c.}^G(X;Y)$ the minimization is carried out over the resulting subgroups of the output group.

VII. EXAMPLES

In this section, we provide a few examples for both the source coding problem as well as the channel coding problem. We show that when the underlying group is a field, the source coding group mutual information and the channel coding group mutual information are both equal to the Shannon mutual information. We also provide several non-field examples for both problems.

A. Examples for Source Coding

In this section, we find the rate-distortion region for a few examples. First, we consider the case where the underlying group is a field i.e. when $G = \mathbb{Z}_p^m$ for some prime $p$ and positive integer $m$. In this case, we have $\mathcal{P}(G) = \{p\}$, $\mathcal{R}_p(G) = \{1\}$, $M_{p,1} = m$ and $\mathcal{S} = \mathcal{S}_p(G) = \{1\}$. Since the set $\mathcal{S}$ is a singleton, the only choice for the weights is $w = w_{p,1} = 1$ for which

$$\Theta(w) = \{0,1\}$$

and for $\theta = 1$, we have $w_\theta = 0$ and $[U]_\theta = U$. Hence

$$I_{s.c.}^G = I(U;X)$$

This means when the underlying group is a field, the rate is equal to the regular mutual information between $U$ and $X$ when $U$ is a uniform random variable.

Next, we consider the case where the reconstruction alphabet is $\mathbb{Z}_4$. In this case, we have $p = 2$ and $r = 2$. Therefore,

$$R = \max_{\theta=1} 2 \frac{1}{\theta} I([U]_\theta;X)$$

$$= \max(2I([U]_1;X), I(U;X))$$

where $U$ is uniform over $\mathbb{Z}_4$, $X$ is the source output and $[U]_1 = U + 2^1 \mathbb{Z}_4 = X + \{0,2\}$ and the joint distribution is such that $\mathbb{E}\{d(U,X)\} \leq D$. Therefore,

$$2I([U]_1;X) = I(U + \{0,2\};X) + I(U + \{1,3\};X)$$
Hence,

$$R = \max (I(U; X), I(U + \{0, 2\}; X) + I(U + \{1, 3\}; X))$$

Next, we consider the case where the reconstruction alphabet is $Z_8$. For this source, we have $p = 2$ and $r = 3$. Following a similar argument as above we have:

$$R = \max \left( I(U; X), \frac{3}{2} I([U]_2; X), 3I([U]_1; X) \right)$$

where $U$ is uniform over $Z_8$, $X$ is the source output, $[U]_1 = U + \{0, 2, 4, 6\}$ and $[U]_2 = U + \{0, 4\}$. Similarly, for channels with input $Z_9$, we have $p = 3$, $r = 2$ and

$$R = \max (I(U; X), 2I([U]_1; X))$$

where $U$ is uniform over $Z_9$, $X$ is the source output and $[U]_1 = U + \{0, 3, 6\}$.

Finally, we consider $G = Z_2 \times Z_4$. In this case, $\mathcal{P}(G) = \{2\}$, $\mathcal{R}_2(G) = \{1, 2\}$, $\mathcal{S}(G) = \mathcal{S}_2(G) = \{1, 2\}$, $\mathbf{0} = (0, 0)$ and $w = (w_1, w_2)$ such that $w_1 + w_2 = 1$. We have three cases for $\Theta(w)$:

1. If $w_2 = 0$ (and $w_1 = 1$), we have $\Theta(w) = \{(0, 1), (1, 2)\}$. For $\theta = (0, 1)$ we have $\omega_\theta = 1$. Since by convention $\frac{1}{0} = \infty$, this implies that this case cannot be optimal.

2. If $w_1 = 0$ (and $w_2 = 1$), we have $\Theta(w) = \{(0, 0), (1, 1), (1, 2)\}$. For $\theta = (1, 1)$ we have $\omega_\theta = \frac{1}{2}$ and for $\theta = (1, 2)$ we have $\omega_\theta = 0$ therefore,

$$R_2 = \max \left( 2I([U]_{\theta=(1,1)}; X), I([U]_{\theta=(1,2)}; X) \right)$$

$$= \max \left( 2I([U]_{\theta=(1,1)}; X), I(X; Y) \right)$$

since $[U]_{\theta=(1,2)} = U$.

Finally, 3. If $0 < w_1 < 1$ (and $w_2 = 1 - w_1$), we have $\Theta(w) = \{(0, 0), (0, 1), (1, 1), (1, 2)\}$. For $\theta = (0, 1)$ we have $\omega_\theta = \frac{w_1 + w_2}{w_1 + 2w_2} = \frac{1}{1+w_2}$, for $\theta = (1, 1)$ we have $\omega_\theta = \frac{w_1}{1+w_2}$, and for $\theta = (1, 2)$ we have $\omega_\theta = 0$; therefore,

$$R_3 = \min_{w_1, w_2} \max \left( (1 + w_2)I([U]_{\theta=(0,1)}; X), \frac{1 + w_2}{w_2} I([U]_{\theta=(1,1)}; X), I([U]_{\theta=(1,2)}; X) \right)$$

$$= \min_{w_1, w_2} \max \left( (1 + w_2)I([U]_{\theta=(0,1)}; X), \frac{1 + w_2}{w_2} I([U]_{\theta=(1,1)}; X), I(U; X) \right)$$

The maximum of $R_3$ is achieved when

$$(1 + w_2)I([U]_{\theta=(0,1)}; X) = \frac{1 + w_2}{w_2} I([U]_{\theta=(1,1)}; X)$$
or equivalently

\[ w_2 = \frac{I([U]_{\theta=(1,1)}; X)}{I([U]_{\theta=(0,1)}; X)} \]

Therefore,

\[ R_3 = \max \{ I([U]_{\theta=(1,1)}; X) + I([U]_{\theta=(0,1)}; X), I(X; Y) \} \]

Note that similarly to the proof of Lemma VI.1 and by noting that \([U]_{\theta=(0,1)} \leftrightarrow [U]_{\theta=(1,1)} \leftrightarrow U \leftrightarrow X\) forms a Markov chain, we can show that

\[ I([U]_{\theta=(0,1)}; X) \leq I([U]_{\theta=(1,1)}; X) \leq I(X; Y) \]

This implies that \(R_1 \geq R_3\) and \(R_2 \geq R_3\). Therefore,

\[ R = R_3 = \max \{ I([U]_{\theta=(1,1)}; X) + I([U]_{\theta=(0,1)}; X), I(X; Y) \} \]

B. Examples for Channel Coding

In this section, we find the achievable rate for a few examples: First, we consider the case where the underlying group is a field i.e. when \(G = \mathbb{Z}_p^m\) for some prime \(p\) and positive integer \(m\). As in the source coding case, the only choice for the weights is \(w = w_{p,1} = 1\) for which \(\Theta(w) = \{0,1\}\). For \(\theta = 0\), we have \(w_\theta = 1\) and \([U]_\theta\) is a trivial random variable. Hence

\[ I_{s.c.}^G = I(U; X) \]

This means when the underlying group is a field, the rate is equal to the regular mutual information between \(U\) and \(X\) when \(U\) is a uniform random variable.

Next, we consider the case where the channel input alphabet is \(\mathbb{Z}_4\). In this case, we have \(p = 2\) and \(r = 2\). Therefore,

\[ R = \min_{\theta=0} \frac{1}{2 - \theta} I(X; Y | [X]_\theta) \]

\[ = \min(I(X; Y), 2I(X; Y | [X]_1)) \]

where the channel input \(X\) is uniform over \(\mathbb{Z}_4\), \(Y\) is the channel output and \([X]_1 = X + 2^1\mathbb{Z}_4 = X + \{0,2\}\). Therefore,

\[ 2I(X; Y | [X]_1) = I(X; Y | X \in \{0,2\}) + I(X; Y | X \in \{1,3\}) \]

Hence,

\[ R = \min(I(X; Y), I(X; Y | X \in \{0,2\}) + I(X; Y | X \in \{1,3\})) \]
Next, we consider a channel of input alphabet $\mathcal{Z}_8$. For this channel we have $p = 2$ and $r = 3$. Following a similar argument as above we have:

$$R = \min \left( I(X; Y), \frac{3}{2} I(X; Y || X)_1, 3I(X; Y || X)_2 \right)$$

where the channel input $X$ is uniform over $\mathcal{Z}_8$, $Y$ is the channel output, $[X]_1 = X + \{0, 2, 4, 6\}$ and $[X]_2 = X + \{0, 4\}$.

Similarly, for channels with input $\mathcal{Z}_9$, we have $p = 3$, $r = 2$ and

$$R = \min \left( I(X; Y), 2I(X; Y || X)_1 \right)$$

where the channel input $X$ is uniform over $\mathcal{Z}_9$, $Y$ is the channel output and $[X]_1 = X + \{0, 3, 6\}$.

Finally, we consider $G = \mathbb{Z}_2 \times \mathbb{Z}_4$. In this case, $\mathcal{P}(G) = \{2\}$, $\mathcal{R}_2(G) = \{1, 2\}$, $\mathcal{S}(G) = \mathcal{S}_2(G) = \{1, 2\}$, $r = (1, 2)$ and $w = (w_1, w_2)$ such that $w_1 + w_2 = 1$. We have three cases for $\Theta(w)$:

1. If $w_2 = 0$ (and $w_1 = 1$), we have $\Theta(w) = \{(0, 1), (1, 2)\}$. For $\theta = (0, 1)$ we have $\omega_{\theta} = 1$; therefore,

$$R_1 = \frac{1}{\omega_{\theta}} I(X; Y || X)_{\theta} = I(X; Y || X)_{\theta = (0, 1)}$$

2. If $w_1 = 0$ (and $w_2 = 1$), we have $\Theta(w) = \{(0, 0), (1, 1), (1, 2)\}$. For $\theta = (1, 1)$ we have $\omega_{\theta} = \frac{1}{2}$ and for $\theta = (0, 0)$ we have $\omega_{\theta} = 1$; therefore,

$$R_2 = \min \left( 2I(X; Y || X)_{\theta = (1, 1)}, I(X; Y || X)_{\theta = (0, 0)} \right)$$

$$= \min \left( 2I(X; Y || X)_{\theta = (1, 1)}, I(X; Y) \right)$$

since $[X]_{\theta = (0, 0)}$ is a trivial random variable.

Finally, (3) If $0 < w_1 < 1$ (and $w_2 = 1 - w_1$), we have $\Theta(w) = \{(0, 0), (0, 1), (1, 1), (1, 2)\}$. For $\theta = (0, 0)$ we have $\omega_{\theta} = 1$, for $\theta = (0, 1)$ we have $\omega_{\theta} = \frac{w_1 + w_2}{w_1 + 2w_2} = \frac{1}{1 + w_2}$ and for $\theta = (1, 1)$ we have $\omega_{\theta} = \frac{w_2}{1 + w_2}$ therefore,

$$R_3 = \max_{w_1, w_2} \min \left( \frac{1 + w_2}{w_2} I(X; Y || X)_{\theta = (1, 1)}, (1 + w_2) I(X; Y || X)_{\theta = (0, 1)}, I(X; Y || X)_{\theta = (0, 0)} \right)$$

$$= \max_{w_1, w_2} \min \left( \frac{1 + w_2}{w_2} I(X; Y || X)_{\theta = (1, 1)}, (1 + w_2) I(X; Y || X)_{\theta = (0, 1)}, I(X; Y) \right)$$

The maximum of $R_3$ is achieved when

$$\frac{1 + w_2}{w_2} I(X; Y || X)_{\theta = (1, 1)} = (1 + w_2) I(X; Y || X)_{\theta = (0, 1)}$$
or equivalently

\[ w_2 = \frac{I(X; Y| [X]_{\theta=(1,1)})}{I(X; Y| [X]_{\theta=(0,1)})} \]

Therefore,

\[ R_3 = \min \left( I(X; Y| [X]_{\theta=(1,1)}) + I(X; Y| [X]_{\theta=(0,1)}), I(X; Y) \right) \]

Note that similarly to the proof of Lemma VI.1 and by noting that \([X]_{\theta=(0,1)} \leftrightarrow [X]_{\theta=(1,1)} \leftrightarrow X \leftrightarrow Y\) forms a Markov chain, we can show that

\[ I(X; Y| [X]_{\theta=(1,1)}) \leq I(X; Y| [X]_{\theta=(0,1)}) \leq I(X; Y) \]

This implies that \(R_1 \leq R_3\) and \(R_2 \leq R_3\). Therefore,

\[ R = R_3 = \min \left( I(X; Y| [X]_{\theta=(1,1)}) + I(X; Y| [X]_{\theta=(0,1)}), I(X; Y) \right) \]

VIII. Conclusion

We derived the achievable rate-distortion function using Abelian group codes for arbitrary discrete memoryless sources. We showed that when the underlying group is a field, these group codes are linear codes, and this function is equivalent to the symmetric rate-distortion function i.e. the Shannon rate-distortion function with the additional constraint that the reconstruction random variable is uniformly distributed. We showed that when the underlying group is not a field, due to the algebraic structure of the code, certain subgroups of the group appear in the rate-distortion function and cause a larger rate for a given distortion level. We derived a similar result for the channel coding problem; i.e. an achievable rate using Abelian group codes for arbitrary discrete memoryless channels. We showed that in the case of linear codes, it simplifies to the symmetric capacity of the channel i.e. the Shannon capacity with the additional constraint that the channel input distribution is uniformly distributed. For the case where the underlying group is not a field, as in the source coding case, we observe that several subgroups of the group appear in the achievable rate and this causes the rate to be smaller than the symmetric capacity of the channel in general.
IX. APPENDIX

A. Proof of Lemma III.2

We first prove that for a homomorphism \( \phi \), \( g_{(q,s,l)\rightarrow(p,r,m)} \) satisfies the above conditions. First assume \( p \neq q \). Note that the only nonzero component of \( \mathbb{I}_{J:q,s,l} \) takes values from \( \mathbb{Z}_{q^s} \) and therefore

\[
q^s \mathbb{I}_{J:q,s,l} = \sum_{i=1,\ldots,q^s} \mathbb{I}_{J:q,s,l} = 0
\]

Note that since \( \phi \) is a homomorphism, we have \( \phi(q^s \mathbb{I}_{J:q,s,l}) = 0 \). On the other hand,

\[
\phi(q^s \mathbb{I}_{J:q,s,l}) = \phi\left( \sum_{i=1,\ldots,q^s} \mathbb{I}_{J:q,s,l} \right)
\]

\[
= \sum_{i=1,\ldots,q^s} \phi(\mathbb{I}_{J:q,s,l})
\]

\[
= \bigoplus_{(p,r,m)\in\mathcal{G}(\tilde{G})} \left[ \sum_{i=1,\ldots,q^s} \phi(\mathbb{I}_{J:q,s,l}) \right]_{p,r,m}
\]

\[
= \bigoplus_{(p,r,m)\in\mathcal{G}(\tilde{G})} q^s \left[ \phi(\mathbb{I}_{J:q,s,l}) \right]_{p,r,m}
\]

\[
= \bigoplus_{(p,r,m)\in\mathcal{G}(\tilde{G})} q^s g_{(q,s,l)\rightarrow(p,r,m)}
\]

Therefore, we have \( q^s g_{(q,s,l)\rightarrow(p,r,m)} = 0 \pmod{p^r} \) or equivalently \( q^s g_{(q,s,l)\rightarrow(p,r,m)} = C p^r \) for some integer \( C \). Since \( p \neq q \), this implies \( p^r | g_{(q,s,l)\rightarrow(p,r,m)} \) and since \( g_{(q,s,l)\rightarrow(p,r,m)} \) takes values from \( \mathbb{Z}_{p^r} \), we have \( g_{(q,s,l)\rightarrow(p,r,m)} = 0 \).

Next, assume \( p = q \) and \( r \geq s \). Note that same as above, we have \( \phi(q^s \mathbb{I}_{J:q,s,l}) = 0 \) and

\[
\phi(q^s \mathbb{I}_{J:q,s,l}) = \bigoplus_{(p,r,m)\in\mathcal{G}(\tilde{G})} q^s g_{(q,s,l)\rightarrow(p,r,m)}
\]

and therefore, \( q^s g_{(q,s,l)\rightarrow(p,r,m)} = 0 \pmod{p^r} \). Since \( g_{(q,s,l)\rightarrow(p,r,m)} \) takes values from \( \mathbb{Z}_{p^r} \) and \( p = q \), this implies \( p^{r-s} | g_{(q,s,l)\rightarrow(p,r,m)} \) or equivalently \( g_{(q,s,l)\rightarrow(p,r,m)} \in p^{r-s} \mathbb{Z}_{p^r} \).
Next we show that any mapping described by (6) satisfying the conditions of the lemma is a homomorphism. For two elements \( a, b \in J \) and for \((p, r, m) \in \mathcal{G}(\tilde{G})\) we have

\[
[\phi(a + b)]_{p, r, m} = \left[ \phi \left( \bigoplus_{(q, s, l) \in \mathcal{G}(J)} (a_{q, s, l} + q^r b_{q, s, l}) \right) \right]_{p, r, m}
\]

\[
= \left[ \phi \left( \sum_{(q, s, l) \in \mathcal{G}(J)} (a_{q, s, l} + q^r b_{q, s, l}) \mathbb{I}_{J:q,s,l} \right) \right]_{p, r, m}
\]

\[
= \left[ \phi \left( \sum_{(q, s, l) \in \mathcal{G}(J)} \sum_{i=1}^{(\tilde{G})} (a_{q, s, l} + q^r b_{q, s, l}) \mathbb{I}_{J:q,s,l} \right) \right]_{p, r, m}
\]

\[
= \sum_{(q, s, l) \in \mathcal{G}(J)} \sum_{i=1}^{(\tilde{G})} \phi(\mathbb{I}_{J:q,s,l}) g(q, s, l) \rightarrow (p, r, m)
\]

\[
\text{On the other hand, we have}
\]

\[
[\phi(a) + \phi(b)]_{p, r, m} = \left[ \phi(a) \right]_{p, r, m} + p^r \left[ \phi(b) \right]_{p, r, m}
\]

\[
= \left( \sum_{(q, s, l) \in \mathcal{G}(J)} a_{q, s, l} g(q, s, l) \rightarrow (p, r, m) \right) + p^r \left( \sum_{(q, s, l) \in \mathcal{G}(J)} b_{q, s, l} g(q, s, l) \rightarrow (p, r, m) \right)
\]

\[
= \left( \sum_{(q, s, l) \in \mathcal{G}(J)} \sum_{i=1}^{(\tilde{G})} g(q, s, l) \rightarrow (p, r, m) \right) + p^r \left( \sum_{(q, s, l) \in \mathcal{G}(J)} \sum_{i=1}^{(\tilde{G})} g(q, s, l) \rightarrow (p, r, m) \right)
\]

\[
= \sum_{(q, s, l) \in \mathcal{G}(J)} \sum_{i=1}^{(\tilde{G})} g(q, s, l) \rightarrow (p, r, m)
\]

\[
\text{where the addition in } a_{q, s, l} + b_{q, s, l} \text{ is the integer addition.}
\]

In order to show that \( \phi \) is a homomorphism, it suffices to show that under the conditions of the lemma, Equations (32) and (33) are equivalent. We show that for a fixed \((q, s, l) \in \mathcal{G}(J)\), if the conditions of the
lemma are satisfied, then
\[
\sum_{i=1}^{G} g(q, s, l) \to (p, r, m) = \sum_{i=1}^{G} g(q, s, l) \to (p, r, m)
\] (34)

Note that if \( p \neq q \), then both summations are zero. Note that we have
\[
\sum_{i=1}^{G} g(q, s, l) \to (p, r, m) = \sum_{i=1}^{G} g(q, s, l) \to (p, r, m) \pmod{p^{r}}
\]
and
\[
\sum_{i=1}^{G} g(q, s, l) \to (p, r, m) = \sum_{i=1}^{G} g(q, s, l) \to (p, r, m) \pmod{p^{r}}
\]
If \( p = q \) and \( r \leq s \), then we have \((a_{q, s, l} + b_{q, s, l} \mod p^{r}) = (a_{q, s, l} + b_{q, s, l} \mod p^{r})\) and hence it follows that Equation (34) is satisfied. If \( p = q \) and \( r \geq s \), since \( g(q, s, l) \to (p, r, m) \in p^{r-s}G_{p^{r}} \) we have
\[
\sum_{i=1}^{G} g(q, s, l) \to (p, r, m) = \sum_{i=1}^{G} g(q, s, l) \to (p, r, m) \pmod{p^{r}}
\]
and hence it follows that Equation (34) is satisfied.

B. Proof of Lemma V.1

Note that since \( g(q, s, l) \to (p, r, m) \)'s and \( B \) are uniformly distributed, in order to find the desired joint probability, we need to count the number of choices for \( g(q, s, l) \to (p, r, m) \)'s and \( B \) such that for \((p, r, m) \in G(G^{n})\),
\[
\left( \sum_{(q, s, l) \in G(J)} a_{q, s, l} g(q, s, l) \to (p, r, m) \right) p^{r} B_{p, r, m} = u_{p, r, m}
\]
\[
\left( \sum_{(q, s, l) \in G(J)} \tilde{a}_{q, s, l} g(q, s, l) \to (p, r, m) \right) p^{r} B_{p, r, m} = \tilde{u}_{p, r, m}
\]
and divide this number by the total number of choices which is equal to
\[
|G|^{n} \cdot \prod_{(p, r, m) \in G(G^{n})} \prod_{(q, s, l) \in G(J)} p^{\min(r, s)} = |G|^{n} \cdot \prod_{(p, r, m) \in G(G)} \prod_{(q, s, l) \in G(J)} p^{\min(r, s)}
\]
where the term $p_{\min(r,s)}$ appears since the number of choices for $g_{(q,s,l)\to(p,r,m)}$ is $p^r$ if $p = q, r \leq s$ and is equal to $p^s$ if $p = q, r \geq s$. Since $B$ can take values arbitrarily from $G^m$, the number of choices for the above set of conditions is equal to the number of choices for $g_{(q,s,l)\to(p,r,m)}$’s such that,

$$
\left(\sum_{(q,s,l)\in G(J)} (\tilde{a}_{q,s,l} - a_{q,s,l}) g_{(q,s,l)\to(p,r,m)} \right) = \tilde{u}_{p,r,m} - u_{p,r,m}
$$

Note that for all $(q, s, l) \in G(J)$, $(\tilde{a}_{q,s,l} - a_{q,s,l}) g_{(q,s,l)\to(p,r,m)} \in p^{\theta_{p,r}} Z_{p^r}$. Therefore we require $\tilde{u}_{p,r,m} - u_{p,r,m} \in p^{\theta_{p,r}} Z_{p^r}$ and therefore we require $\tilde{u} - u \in H^0_b$ or otherwise the probability would be zero.

For fixed $p \in P(G)$ and $r \in R_p(G)$, let $(q, s^*, l^*) \in G(J)$ be such that $q^* = p$ and

$$
\theta_{p,r} = |r - s^*|^+ + \hat{\theta}_{q^*, s^*, l^*}
$$

For fixed $(p, r, m) \in G(G^m)$, and for $(q, s, l) \neq (q^*, s^*, l^*)$, choose $g_{(q,s,l)\to(p,r,m)}$ arbitrarily from its domain. The number of choices for this is equal to

$$
\left[ \prod_{(p,r,m)\in G(G^m)} \prod_{(q,s,l)\in G(J)} \prod_{q=p} (q, s, l) \neq (q^*, s^*, l^*) p_{\min(r,s)} \right] ^ n
$$

For each $(p, r, m) \in G(G^m)$, we need to have

$$
(\tilde{a}_{q^*, s^*, l^*} - a_{q^*, s^*, l^*}) g_{(q^*, s^*, l^*)\to(p,r,m)} = \tilde{u}_{p,r,m} - u_{p,r,m} - \sum_{(q,s,l)\in G(J)} (\tilde{a}_{q,s,l} - a_{q,s,l}) g_{(q,s,l)\to(p,r,m)}
$$

Note that the right hand side is included in $p^{\theta_{q^*, s^*, l^*}} Z_{p^r}$ and $(\tilde{a}_{q^*, s^*, l^*} - a_{q^*, s^*, l^*})$ is included in $p^{\theta_{q^*, s^*, l^*}} Z_{p^r}$. We need to count the number of solutions for $g_{(q^*, s^*, l^*)\to(p,r,m)}$ in $p^{\theta_{q^*, s^*, l^*}} Z_{p^r}$. Using Lemma IX.1, we can show that the number of solutions is equal to $p^{\theta_{q^*, s^*, l^*}}$. The total number of solutions for $\phi$ is equal to

$$
\left[ \prod_{(p,r,m)\in G(G^m)} \prod_{(q,s,l)\in G(J)} \prod_{q=p} (q, s, l) \neq (q^*, s^*, l^*) p_{\min(r,s)} \right] ^ n
$$
Hence we have

\[
P(\phi(a) + B = u, \phi(\tilde{a}) + B = \tilde{u}) = \left[ \prod_{(p,r,m) \in \mathcal{G}} \left( \prod_{q \neq p} \prod_{(q,s,l) \in \mathcal{G}(J) \setminus \mathcal{G}(J)} p_{\min(r,s)}^\hat{\theta}_{q^*,s^*,l^*} \right) \right]^n
\]

Note that for \((q, s, l) = (q^*, s^*, l^*)\) we have

\[
\min(r, s) = \min(r, s^*) = r - |r - s^*|^+ = r - (\theta_{p,r} - \hat{\theta}_{q^*,s^*,l^*})
\]

Therefore, the above probability is equal to

\[
\left[ \prod_{(p,r,m) \in \mathcal{G}} \prod_{(q,s,l) \in \mathcal{G}(J) \setminus \mathcal{G}(J)} \frac{p_{\min(r,s)}^\hat{\theta}_{q^*,s^*,l^*}}{p^r - \theta_{p,r}} \right]^n = \left[ \prod_{(p,r,m) \in \mathcal{G}} \prod_{(q,s,l) \in \mathcal{G}(J) \setminus \mathcal{G}(J)} \frac{1}{p^r - \theta_{p,r}} \right]^n
\]

Since the dither \(B\) is uniform, we conclude that

\[
P\left( \phi(u) + B = x, \phi(\tilde{u}) + B = \tilde{x} \right) = \frac{1}{|\mathcal{G}|^n} \frac{1}{|H_\theta|^n}
\]

C. Proof of Lemma V.2

Let \(\tilde{a} \in T_\theta(a)\) be such that for \((q, s, l) \in \mathcal{G}(J),\)

\[
\tilde{a}_{q,s,l} - a_{q,s,l} \in q_{\tilde{\theta}_{q^*,s^*,l^*}}^\ast Z_{q^*} \setminus q_{\tilde{\theta}_{q^*,s^*,l^*} + 1} Z_{q^*}
\]

for some \(0 \leq \tilde{\theta}_{q,s,l} \leq s\). Since for all \(\tilde{a} \in T_\theta(a)\) and all \((p, r) \in \mathcal{Q}(G)\)

\[
\min_{(q,s,l) \in \mathcal{G}(J)} |r - s|^+ + \tilde{\theta}_{q,s,l}
\]

we require \(|r - s|^+ + \tilde{\theta}_{q,s,l} \geq \theta_{p,r}\) or equivalently \(\tilde{\theta}_{q,s,l} \geq \max_{(p,r) \in \mathcal{Q}(G)} |\theta_{p,r} - |r - s|^+|^+\) for all \((q, s, l) \in \mathcal{G}(J)\). This means for \((q, s, l) \in \mathcal{G}(J),\) \(\tilde{a}_{q,s,l}\) can only take values from

\[
a_{q,s,l} + q_{\max_{(p,r) \in \mathcal{Q}(G)} |\theta_{p,r} - |r - s|^+|^+} Z_{q^*}
\]
The cardinality of this set is equal to
\[ q^{s-\max(p, r)} \text{ for } (q, s, l) \in G(J) \]

Therefore,
\[ |T_\theta(a)| \leq \prod_{(q, s, l) \in G(J)} q^{s-(\theta_{p, r}-|r-s|^+)\}

D. Proof of Lemma IX-D

For \( \theta = (\theta_{p, r})_{(p, r) \in \mathbb{Q}(G)} \), if \( |T_\theta(a)| \neq 0 \), let \( \tilde{a} \in T_\theta(a) \) such that for \( (q, s, l) \in G(J) \),
\[ \tilde{a}_{q, s, l} - a_{q, s, l} \in q^{\hat{\theta}_{q, s, l}} \mathbb{Z}_{p^s} - q^{\hat{\theta}_{q, s, l}+1} \mathbb{Z}_{p^s} \]

for some \( 0 \leq \hat{\theta}_{q, s, l} \leq s \). For all \( (p, r) \in \mathbb{Q}(G) \), \( \tilde{a} \in T_\theta(a) \) implies
\[ \theta_{p, r} = \min_{(q, s, l) \in G(J)} |r - s|^+ + \hat{\theta}_{q, s, l} \]

Equivalently since
\[ \theta_{p, r} = \min_{q=p} \min_{(q, s, l) \in S} |r - s|^+ + \hat{\theta}_{q, s, l} \]
\[ = \min_{(q, s) \in S} |r - s|^+ + \min_{l=1, \ldots, kw_{q, s}} \hat{\theta}_{q, s, l} \]

This implies \( \theta \in \Theta(w) \). The converse part of the proof is similar and is omitted.

E. Useful Lemmas

**Lemma IX.1.** Let \( p \) be a prime and \( s, r \) a positive integer such that \( s \leq r \). For \( a \in \mathbb{Z}_{p^s} \) and \( b \in \mathbb{Z}_{p^r} \), let \( 0 \leq \hat{\theta} \leq s \) and \( \hat{\theta} \leq \theta \leq r \) be such that
\[ a \in p^\hat{\theta} \mathbb{Z}_{p^s} \backslash p^{\hat{\theta}+1} \mathbb{Z}_{p^s} \]
\[ b \in p^\theta \mathbb{Z}_{p^r} \]

Write \( a = p^{\hat{\theta}} \alpha \) for some invertible element \( \alpha \in \mathbb{Z}_{p^s} \) and \( b = p^\theta \beta \) for some \( \beta \in \beta \in \{0, 1, \ldots, p^{r-\theta} - 1\} \).

Then, the set of solutions to the equation \( ax \pmod{p^r} = b \) is
\[ \left\{ p^{\theta-\theta} \alpha^{-1} \beta + i \alpha^{-1} p^{r-\hat{\theta}} | i = 0, 1, \ldots, p^{\theta-1} \right\} \]

**Proof:** Note that the representation of \( b \) as \( b = p^{\theta} \beta \) is not unique and for any \( \tilde{\beta} \) of the form \( \tilde{\beta} = \beta + ip^{r-\theta} \) for \( i = 0, 1, \ldots, p^{\theta} - 1 \), \( b \) can be written as \( p^{\theta} \tilde{\beta} \). Also, the representation of \( a \) as \( a = p^{\hat{\theta}} \alpha \)
is not unique and for any \( \tilde{\alpha} = \alpha + ip^{r-\theta} \) for \( i = 0, 1, \ldots, p^{\hat{\theta}} - 1 \), we have \( a = p^{\hat{\theta}} \tilde{\alpha} \). The set of solutions to \( ax = b \) is identical to the set of solutions to \( p^{\hat{\theta}} x = p^\theta \alpha^{-1} \beta \). The set of solutions to the latter is

\[
\left\{ p^{\theta-\theta} \alpha^{-1} \beta + i \alpha^{-1} p^{r-\theta} | i = 0, 1, \ldots, p^{\hat{\theta}} - 1 \right\}
\]

It remains to show that this set of solutions is independent of the choice of \( \alpha \) and \( \beta \). First, we show that the set of solutions is independent of the choice of \( \beta \). For \( \tilde{\beta} = \beta + jp^{r-\theta} \) for some \( j \in \{0, 1, \ldots, p^{\theta_s} - 1\} \), we have

\[
\begin{align*}
\left\{ p^{\theta-\theta} \alpha^{-1} \tilde{\beta} + i \alpha^{-1} p^{r-\theta} | i = 0, 1, \ldots, p^{\hat{\theta}} - 1 \right\} &= \left\{ p^{\theta-\theta} \alpha^{-1} (\beta + jp^{r-\theta}) + i \alpha^{-1} p^{r-\theta} | i = 0, 1, \ldots, p^{\hat{\theta}} - 1 \right\} \\
&= \left\{ p^{\theta-\theta} \alpha^{-1} \beta + (i + j) \alpha^{-1} p^{r-\theta} | i = 0, 1, \ldots, p^{\hat{\theta}} - 1 \right\} \\
&\equiv (a) \left\{ p^{\theta-\theta} \alpha^{-1} \beta + i \alpha^{-1} p^{r-\theta} | i = 0, 1, \ldots, p^{\hat{\theta}} - 1 \right\}
\end{align*}
\]

where (a) follows since the set \( p^{r-\theta} \{0, 1, \ldots, p^{\hat{\theta}} - 1\} \) is a subgroup of \( \mathbb{Z}_{p^r} \) and \( jp^{r-\theta} \) lies in this set.

Next, we show that the set of solutions is independent of the choice of \( \alpha \). For \( \tilde{\alpha} = \alpha + jp^{r-\theta} \) for some \( j \in \{0, 1, \ldots, p^{\hat{\theta}} - 1\} \), we have

\[
\tilde{\alpha} \left( \alpha^{-1} - \alpha^{-1} jp^{r-\theta} \tilde{\alpha}^{-1} \right) = 1
\]

Therefore, it follows that the unique inverse of \( \tilde{\alpha} \) satisfies \( \alpha^{-1} - \tilde{\alpha}^{-1} \in \alpha^{-1} p^{r-\theta} \mathbb{Z}_{p^r} \). Assume \( \tilde{\alpha}^{-1} = \alpha^{-1} + k \alpha^{-1} p^{r-\theta} \). We have,

\[
\begin{align*}
\left\{ p^{\theta-\theta} \alpha^{-1} \beta + i \alpha^{-1} p^{r-\theta} | i = 0, 1, \ldots, p^{\hat{\theta}} - 1 \right\} &= \left\{ p^{\theta-\theta} \left( \alpha^{-1} + k \alpha^{-1} p^{r-\theta} \right) \beta + i \left( \alpha^{-1} + k \alpha^{-1} p^{r-\theta} \right) p^{r-\theta} | i = 0, 1, \ldots, p^{\hat{\theta}} - 1 \right\} \\
&= \left\{ p^{\theta-\theta} \alpha^{-1} \beta + \left( i + ik p^{r-\theta} + k \beta p^{r-\theta} \right) \alpha^{-1} p^{r-\theta} | i = 0, 1, \ldots, p^{\hat{\theta}} - 1 \right\} \\
&\equiv (a) \left\{ p^{\theta-\theta} \alpha^{-1} \beta + i \alpha^{-1} p^{r-\theta} | i = 0, 1, \ldots, p^{\hat{\theta}} - 1 \right\}
\end{align*}
\]

where same as above, (a) follows since the set \( p^{r-\theta} \{0, 1, \ldots, p^{\hat{\theta}} - 1\} \) is a subgroup of \( \mathbb{Z}_{p^r} \) and \( (ik p^{r-\theta} + k \beta p^{r-\theta})p^{r-\theta} \) lies in this set.

Lemma IX.2. Let \( X \) be a random variable taking values from the group \( G \) and for a subgroup \( H \) of \( G \), define \( [X] = X + H \). For \( y \in A^n_{\epsilon}(Y) \) and \( x \in A^n_{\epsilon}(X|y) \), let \( z = [x] = x + H^n \). Then we have

\[
(x + H^n) \cap A^n_{\epsilon}(X|y) = A^n_{\epsilon}(X|zy)
\]
and

\[(1 - \epsilon) 2^{n[H(X|Y) - O(\epsilon)]} \leq |(x + H^n) \cap A^n_v(X|y)| \leq 2^{n[H(X|Y) + O(\epsilon)]}
\]

**Proof:** First, we show that \((x + H^n) \cap A^n_v(X|y)\) is contained in \(A^n_v(X|zy)\). Since \(z\) is a function of \(x\), we have \((x, z, y) \in A^n_v(X, [X], Y)\). For \(x' \in (x + H^n) \cap A^n_v(X|y)\), we have \([x'] = x' + H^n = x + H^n = z\) and \((x', z, y) = (x', [x'], y) \in A^n_v(X, [X], Y)\). Therefore, \(x' \in A^n_v(X|zy)\) and hence,

\[(x + H^n) \cap A^n_v(X|y) \subseteq A^n_v(X|zy)
\]

Conversely, for \(x' \in A^n_v(X|zy)\), since \((x, z) \in A^n_v(X, [X])\) where \([X]\) is a function of \(X\), we have \([x'] = z\). This implies \(x' \in z + H^n = x + H^n\). Clearly, we also have \(x' \in A^n_v(X|y)\). The claim on the size of the set follows since \((z, y) \in A^n_v([X]Y)\).

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