Dirac-like Monopoles in Three Dimensions and Their Possible Influences on the Dynamics of Particles

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Abstract

Dirac-like monopoles are studied in three-dimensional Abelian Maxwell and Maxwell-Chern-Simons models. Their scalar nature is highlighted and discussed through a dimensional reduction of four-dimensional electrodynamics with electric and magnetic sources. Some general properties and similarities of them whenever considered in Minkowski or Euclidian space are mentioned. However, by virtue of the structure of the space-time in which they are studied a number of differences among them take place. Furthermore, we pay attention to some consequences of these objects whenever acting upon usual particles. Among other subjects, special attention is given to the study of a Lorentz-violating non-minimal coupling between neutral fermions and the field generated by a monopole alone. In addition, an analogue of the Aharonov-Casher effect is discussed in this framework.

1 Introduction and Motivation

The idea that magnetic monopoles, as stable particles carrying magnetic charges, ought to exist has proved to be remarkably durable. In (3+1) dimensions, a persuasive argument
was first put forward by Dirac in 1931 [1], who invoked such objects in order to provide a theoretical explanation why electric charges appear only as multiples of the elementary one.

Furthermore, ‘t Hooft [2] and Polyakov [3] discovered that the existence of magnetic monopoles follows from quite general ideas about the unification of the fundamental interactions. Nowadays, it is well-known that such objects emerge from general “grand unified” theories of particle physics whose gauge group is suitably broken-down to the $U(1)$-factor. Indeed, Dirac had proved the consistency of structureless magnetic monopoles with quantum electrodynamics. On the other hand, some properties of the ‘t Hooft-Polyakov monopole, such as its size and mass, are determined by the distance scale of the spontaneous symmetry breakdown of a grand unified theory. The magnetic charge, $g$, of the monopole is typically the “Dirac charge”, $g_D = 1/2e$, which is distributed over a core with a radius of order $M_X^{-1}$ (the unification distance scale) while its mass is comparable to the magnetostatic potential energy of the core. An excellent review on these subjects may be found in Ref. [4].

In turn, the study of three-dimensional field theories has attracted a great deal of efforts since nearly two decades [5, 6]. Even though such studies were initially motivated by the theoretical connection between such models and their four-dimensional analogues at high temperature, planar physics enjoys nowadays the status of an interesting and self-contained topic in itself. This position was achieved, in part, thanks to some peculiar features that take place in this space-time, such as the coexistence of massive vector gauge bosons and gauge invariance, and the possibility of having objects displaying charge and statistical fractionization [7, 8]. On the other hand, the interest in planar physical models was also remarkably motivated by Condensed Matter phenomena that display planar dynamics. Among these ones, we may quote the Quantum Hall Effect [9] and the High-Tc Superconductivity [8, 10].

Of particular interest is also the study of topological objects in this framework. For example, topologically magnetic vortex-like solutions naturally appear attached to electric charges whenever we are dealing with a Chern-Simons-like electrodynamics (the so-called Maxwell-Chern-Simons (MCS) model). In addition, it is well-known that this composite entity (electric charge + magnetic vortex) may present anyonic statistics thanks to the magnetic flux induced by the vortex [8, 11].

Another sort of topological entities shows up whenever breaking Bianchi identity. These are generally characterized by a potential, $A_\mu$, which carries a singular structure. As it is well-known, such a kind of potentials first appeared in Dirac’s paper on magnetic
monopole [1]. Actually, while in (3+1) dimensions the simplest solution appears like as a point-like magnetic monopole, we shall see that in the (2+1)-dimensional case, the breaking of the Bianchi identity leads us to a wider class of solutions, not restricted to magnetic ones (this is the reason why we call them Dirac-like objects).

Indeed, some works have dealt with such issues in both Euclidean [12, 13] and Minkowskian [14] three-dimensional spaces. Here, it is worthy mentioning that the mass parameter was shown to be quantized in the Abelian version of the Maxwell-Chern-Simons model whenever Dirac-like monopoles interact with usual charges [12] (similarly to the result already known for theories whose gauge groups presented non-trivial third homotopy group [6]). In addition, classical and quantum consequences of the monopole potential acting upon a charged particle were recently analyzed [14].

In this article we wish to go further into this subject and investigate some issues concerning the nature of such objects in three dimensions, as well as some of their influences on the dynamics of particles. Then, in Section II we introduce a dimensional reduction of (3+1)D electrodynamics with magnetic sources to (2+1) dimensions. Such a presentation is interesting for highlighting the scalar nature of these sources in the planar case. Indeed, such a scheme yields two Abelian “electrodynamics” which do not have any explicit interplay between them. In addition, we point out the differences between these models, particularly in their magnetic sectors.

Section III is devoted to the subject of the Dirac-like monopoles itself. There, we present a brief review of such objects introduced in Minkowskian and Euclidean spaces. Attention is given to the differences between them. We also present an analysis of the solutions admitted by the differential equation that shows up whenever Bianchi identity is broken in (2 + 1) dimensions.

In Section IV, we deal with the interaction between a Dirac-like monopole and a usual particle. More precisely, our attention is focused on a Lorentz-violating non-minimal term, which couples monopole field-strength to neutral matter. Although violating Lorentz, it is shown to be invariant under CPT-symmetry. In addition, the equations of motion are similar to those we have for the case of a charged particle minimally interacting with the vector potential produced by a magnetic vortex. Indeed, by virtue of this similarity, such an interaction leads us to a Aharonov-Casher-like effect on the usual particle, produced by the tangential electric field of the monopole.

Finally, our paper is closed by pointing out our Conclusions and Prospects for future investigation.
2 The origin of the scalar nature of planar Dirac-like objects

Here, we intend to give an alternative view of the scalar nature of the Dirac-like monopoles in (2+1) dimensions. The proper study of the breaking of the Bianchi identity in planar Abelian Maxwell and Maxwell-Chern-Simons frameworks will be the goal of the next section, where we shall pay attention, among others, to the tangential (azimuthal) behavior of the electric-like field generated by a point-like “magnetic source” [13, 14].

In order to trace back the scalar nature of (2+1)D magnetic current to its four-dimensional ancestor, we propose to carry out a plain dimensional reduction of the (3+1)D Maxwell theory with electric ($j^\mu$) and magnetic ($k^\mu$) sources, equations (1-2) below, to the planar case. Hereafter, we shall work in Minkowski space-time, but no difficulty arises in carrying out a similar plain in the Euclidian case.

We start off from:

\begin{align}
\partial_\mu F^{\mu\nu} &= j^\nu , \quad (1) \\
\partial_\mu \tilde{F}^{\mu\nu} &= k^\nu , \quad (2)
\end{align}

with $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\kappa\lambda} F_{\kappa\lambda}$.

First of all, we reduct the potential and currents like the ‘splitting’ below:

\begin{align}
A^\mu &\longrightarrow (A^\mu; \ A^3 \equiv S) , \\
j^\mu &\longrightarrow (j^\mu; \ j^3 \equiv \lambda) , \\
k^\mu &\longrightarrow (k^\mu; \ k^3 \equiv \chi).
\end{align}

Then, we realize that the (3+1) dimensional quantities are reducted to (2+1)D ones. For instance, $A^\mu$ yields to a (2+1)D-vector, $A^\mu = (A^0, A^1, A^2)$, and to an extra scalar potential, $A^3 \equiv S$. Notice, in addition, that from the point of view of a (2+1) dimensional frame the fields $A^\mu$ and $S$ are, at principle, completely independents (the same is valid for the currents). Similarly, $j^\mu$ and $k^\mu$ are the (2+1)D electric and magnetic currents, while $j^3 \equiv \lambda$ and $k^3 \equiv \chi$ represent the survivors of the 3rd components of the electric and

\footnote{Our conventions read: $\mu, \nu, \text{etc} = 0, 1, 2, 3$, $\text{diag} \eta_{\mu\nu} = (+, -, -, -)$, and $\epsilon^{0123} = -\epsilon_{0123} = +1$. In addition, $\mu\nu, \text{etc} = 0, 1, 2$, $\text{diag} g_{\mu\nu} = (+, -, -)$, and $\epsilon^{012} = \epsilon_{012} = +1$; while the planar spatial indices are labeled like as: $i, j = 1, 2$ and $\epsilon^{ij} = \epsilon_{ij} = +1$.}
magnetic genuine 4-currents, respectively.

In addition, adopting the reduction ansatz that the quantities do not depend on the 3rd-spatial coordinate, say, \( \partial_3(f) = 0 \), where \( f \) represents any potential or current, the field-strengths take the following forms after the dimensional reduction:

\[
\begin{align*}
F_{\mu \nu} &\rightarrow (F_{\mu \nu}; F_{\mu 3} \equiv G^\mu), \\
\tilde{F}_{\mu \nu} &\rightarrow (\tilde{F}_{\mu \nu} \equiv \tilde{G}^{\mu \nu}; \tilde{F}_{\mu 3} = \tilde{F}^\mu),
\end{align*}
\]

(6) (7)

where the new field-strengths are defined as: \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), \( F_{\mu 3} = G^\mu \), \( \tilde{F}_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \kappa} F_{\kappa \nu} \), \( G^\mu = \partial_\mu S \) and \( \tilde{G}^{\mu \nu} = \epsilon^{\mu \nu \kappa} G_\kappa \).

Notice also that the usual planar electric and magnetic fields are contained in the former field, say, \( \tilde{F}^\mu = (-B; -\epsilon^{ij} E^j) \). In turn, the fields \( G^\mu = \partial_\mu S \) and \( \tilde{G}^{\mu \nu} = \epsilon^{\mu \nu \kappa} G_\kappa \) answer for the appearance of another “electrodynamic” model, like below.

Now, taking into account relations (4)-(7), expressions (1-2) lead us to the two following sets of equations:

\[
\begin{align*}
\partial_\mu F_{\mu \nu} &= j^\nu \quad \text{and} \quad \partial_\mu \tilde{F}^\mu = \chi, \\
\partial_\mu G^\mu &= \lambda \quad \text{and} \quad \partial_\mu \tilde{G}^{\mu \nu} = k^\nu,
\end{align*}
\]

(8) (9)

from what we may still write down:

\[
\begin{align*}
\epsilon^{ij} \partial_i B &= \partial_i E^j + j^j, \\
\nabla \cdot \vec{E} &= j^0 = \rho, \\
\partial_t B + \epsilon^{ij} \partial_i E^j &= \chi, \\
\epsilon^{ij} \partial_i b &= \epsilon^{ij} \partial_i e^j + k^j, \\
\epsilon^{ij} \partial_i e^j &= k^0 = \rho_m, \\
\partial_t b - \nabla \cdot \vec{e} &= \lambda,
\end{align*}
\]

(8) (9)

where the fields above are defined like below:

\[
E^i = - \partial^i A^0 - \partial_t A^i \quad \text{and} \quad B = \epsilon^{ij} \partial^i A^j,
\]

\[
e^i = - \partial^i S \quad \text{and} \quad b = \partial_t S.
\]

Therefore, we realize that after dimensional reduction is implemented we get two independent electrodynamic-like models in (2+1) dimensions, each of them with its proper
electric and magnetic sources. Indeed, the appearance of two non-coupled Abelian factor is nothing but a natural consequence of the reduction scheme. For instance, the latter one is equivalent to select the zero-mode sector of a more general dimensional reduction proposal, namely, the Kaluza-Klein ansatz that relies on the compactness of the 3rd-spatial coordinate [13]. Thus, the natural $SO(2)$-symmetry associated to such a component is kept in (2+1)D, since the scalar field, $S$, is clearly invariant under rotations in the plane. We should also notice that the number of on-shell degrees of freedom is conserved in the reduction scheme. The two physical components of $A^\mu$ lies, after dimensional reduction, in $A^\mu$ and in $S$, each of them carrying a unique degree of freedom.

Furthermore, it is important to stress here that the breaking-down of the Bianchi identity in (2+1)D and what we interpret as its associated magnetic source in the planar world is the (2+1)D-manifestation of the 3rd component of the genuine magnetic 4-current[14]. This is how we understand the argument by Henneaux and Teitelboim [12], that this charge rather behaves like an instanton in the planar case. In addition, it is worthy noticing that $\chi$-charge is a pseudo-scalar, say, it changes its signal under parity: $\chi \rightarrow \chi^P = -\chi$, what is consistent with the equations of motion and with the fact that it appears, after dimensional reduction, as a reminiscent of the magnetic 4-current, which is a pseudo-vector. Similar behavior also occurs to all other currents and fields above. It would also be interesting to understand now, if possible to accomplish such a program, how the (3+1)-dimensional Dirac quantization condition may induce an analogue on the $\chi$-charge.

So, our claim is that, once we start with a 4D Maxwell theory enriched by the presence of magnetic monopoles and, if some physical system is considered such that non-planar effects are negligible in comparison with planar effects, such a system may reveal particles that interact via two quantum numbers and one of them may induce an electric field with azimuthal configuration (see Section III, for details).

3 Analyzing the breaking of Bianchi identity

Dirac-like objects come about through breaking Bianchi identity, as we have already mentioned. In (3+1) dimensions, when we consider Maxwell electrodynamics with magnetic

\[\text{For this, notice that we are considering, as usually is done, the set of equations (8) as being the (2+1)-dimensional counterpart of the standard electrodynamics in 4 dimensions. The other Abelian sector, (9), that comes from the scalar potential, $S$, is then merely considered as being the partner of planar electromagnetism after the reduction procedure, even though the set (1) is the one that keeps the 'genuine' (2+1)D reminiscent of the magnetic 4-current.}\]
sources, we have the following equations $\partial_\mu F^{\mu \nu} = j^\nu$ and $\partial_\mu \tilde{F}^{\mu \nu} = k^\nu$. There, the magnetic Gauss law, $\nabla \cdot \vec{B} = \chi^0$, whenever taken for a point-like source, $\chi^0 = g\delta^3(\vec{x})$, leads us to the concept of a genuine magnetic monopole since $\tilde{B} = g\vec{x}/4\pi |\vec{x}|^3$, in analogy to the electric field produced by an isolated point-like electric charge. Clearly, such a similarity takes place because of the duality between electric and magnetic sectors, say, $\vec{E}$ and $\vec{B}$ are rank-1 tensors (notice that this happens only in 4 dimensions!).

On the other hand, when considered in $(2 + 1)$ dimensions, the broken version of Bianchi identity yields to

$$\partial^\mu \tilde{F}_\mu = \partial_t B + \epsilon^{ij} \partial^i E^j = \chi.$$  

Here, there is no Gauss law for the magnetic field, which implies, in turn, that magnetic monopoles like as those we encountered in 4 dimensions, are no longer present. Thus, although rising up like as genuine magnetic sources in $(3 + 1)$-dimensional electromagnetism, the present objects are expected to exhibit several differences whenever compared to the first ones.

Furthermore, in dealing with the massless case the breaking of the Bianchi identity causes no effect on the equations of motion, i.e., electric current is automatically conserved,

$$\partial_\nu \partial_\mu F^{\mu \nu} = \partial_\nu j^\nu = 0.$$  

Nevertheless, when the Chern-Simons term, $\mathcal{L}_{CS} = mA_\mu \tilde{F}^{\mu}$, is taken into account, things change deeply. Now, the equations of motion acquire an extra (topological) current term,

$$\partial_\mu F^{\mu \nu} = j^\nu + m\tilde{F}^\nu,$$  

which yields to $\nabla \cdot \vec{E} = \rho + mB$ and $\epsilon^{ij} \partial^i B = \partial_t E^j + j^j + m\epsilon^{ij} E^i$.

Now, contrary to the massless case, if Dirac-like objects are introduced, $\partial^\mu \tilde{F}_\mu = \chi$, then current is no longer conserved, say:

$$\partial_\nu \partial_\mu F^{\mu \nu} = m\chi,$$  

and gauge symmetry is lost. Thus, in order to restore such a symmetry we should suppose that the appearance of Dirac-like entities naturally induces an extra electric current,

$$j^\nu_M = -m\tilde{F}^\nu,$$  

so that equation (12) is modified to

$$\partial_\mu F^{\mu \nu} = J^\nu + m\tilde{F}^\nu$$  

(15)
and it is now, identically conserved

$$\partial_\nu \partial_\mu F^{\mu\nu} = \partial_\nu J^\nu + m \partial_\nu \tilde{F}^\nu = 0,$$  \hspace{1cm} (16)

where $J^\nu = j^\nu + j^\nu_M$ is the total \(\text{usual} + \text{topologically induced}\) electric current (for further details, see Refs. [12, 13]).

On the other hand, in the 3-dimensional Euclidean space, we have that:

$$\partial_\mu \tilde{F}^\mu = \partial_\tau \tilde{F}^0 + \partial_i \tilde{F}^i = \chi,$$  \hspace{1cm} (17)

Now, writing $\tilde{F}^\mu = -\partial^\mu \phi$, and taking $\chi$ as being a point, $\chi = g \delta^3(\vec{x})$, we get $\partial^2 \phi = -g \delta^3(\vec{x})$, whose solution reads

$$\phi(\vec{x}) = -g/4\pi |\vec{x}|,$$  \hspace{1cm} (18)

where $|\vec{x}| = \sqrt{\tau^2 + x_i^2}$.

The fields, in turn, are given by $\tilde{F}^\mu = -\partial^\mu \phi = gx^\mu/4\pi |\vec{x}|^3$, or still (let us recall that $\tilde{F}^\mu = (-B; -\epsilon^{ij} E^j)$):

$$B = -\frac{g}{4\pi} \frac{x^0}{|\vec{x}|^3} \quad \text{and} \quad E_i = -\frac{g}{4\pi} \frac{\epsilon_{ij} x_j}{|\vec{x}|^3},$$  \hspace{1cm} (19)

what clearly shows us that genuine magnetic monopoles, whose only effect is the production of a magnetic field, as we realized in 4 dimensions, no longer take place here.

It is particularly noticeable the tangential character of the electric field above (like as its analogue in Minkowski case, below), in contrast to what we expect from usual electric or even magnetic poles. When working in the MCS framework, the induced electric current, equation (14), is readily found to be

$$\rho_M = -\frac{mg}{4\pi} \frac{\tau}{|\vec{x}|^3} \quad \text{and} \quad j^i_M = -\frac{mg}{4\pi} \frac{x^i}{|\vec{x}|^3},$$  \hspace{1cm} (20)

which presents radial-like dependence. Further details about such a subject, including investigation on non-Abelian versions of such entities may be found in Ref. [13].

Now, let us return to Minkowski space-time and let us analyze the structure and solutions of equation (10) in details. Rewriting this equation in components we get (hereafter, we shall use a subscript $g$ in order to distinguish these fields from the usual electric and magnetic ones and from their Euclidean counterparts),

$$\partial_i B_g + \epsilon^{ij} \partial^0 E^j_g = \chi,$$  \hspace{1cm} (21)
whose point-like solutions may be obtained by considering special situations. First, considering the static limit of the fields in equation above, we obtain

\[ \epsilon^{ij} \partial_i E^j_g = g \delta^2(x) , \]  

(22)

which, when written in terms of the potential \( \vec{E}_g = -\nabla \Phi_g \) gets the following form:

\[ [\partial_x, \partial_y] \Phi_g = -g \delta^2(\vec{x}) , \]  

(23)

whose solution reads (with \( r = |\vec{r}| = \sqrt{x^2 + y^2} \) and \( \theta = \arctan(y/x) \), as usual)

\[ \Phi_g(\vec{x}) = -\frac{g}{2\pi} \arctan\left( \frac{y}{x} \right) = -\frac{g}{2\pi} \theta . \]  

(24)

Notice the remarkable feature of such a potential: it has angular rather than radial dependence. Notice also its singular structure: the angle-function is not well-defined at the origin, like the string-like presented by the vector potential associated to a genuine magnetic monopole in (3+1) dimensions. Besides, it is a multivalued function and the corresponding electric field (see below) is not a conservative one, fact already indicated by equation (22). Indeed, its associated electric field reads:

\[ \vec{E}_g = \frac{g}{2\pi} \frac{x \hat{j} - y \hat{i}}{x^2 + y^2} = \frac{g}{2\pi} \frac{\hat{e}_\theta}{r} , \]  

(25)

like as in Ref.[14], which has an azimuthal rather than a radial-like vector behavior. In addition, by demanding null radiation at this static limit, \( \int_V \partial_i \epsilon_{ij} E_j B dV = 0 \), it readily follows that in this case \( B_g \) must vanish. Therefore this static solution appears much as due to a peculiar topological electric charge, rather than to a magnetic monopole.

Furthermore, if we compare it with the vector potential associated to a magnetic vortex (\( \Phi_B \) being its magnetic flux),

\[ \vec{A}_v(\vec{x}) = \frac{\Phi_B}{2\pi r} \hat{e}_\theta \]  

(26)

we may identify a kind of “duality” between them. Actually, the magnetic vortex may be obtained from a Dirac-like monopole, equation (25), by interchanging the vectors \( \vec{A}_v \) and \( \vec{E}_g \), together with \( g \) and \( \Phi_B \) (let us recall that in the case of a usual electric charge a similar identification requires the interchanging between \( \vec{A}_v \) and the dual of \( \vec{E} \)).

Before carrying on the analysis of other possible solutions, let us pay attention to the (topological) electric current induced by the appearance of this monopole in the MCS framework. From equations (14) and (25) it follows that

\[ \rho_M = 0 \quad \text{and} \quad j^i_M = \frac{mg}{2\pi} \frac{x^i}{|\vec{x}|^2} . \]  

(27)
which is radial, and implies that equation (16) is satisfied.

The second situation is the radial-like electric field. Now, searching for solutions of equation (21) that present $\varepsilon^{ij} \partial^i E^j = 0$, we are left with

$$\partial_t B_g = \chi.$$  \hspace{1cm} (28)

Here, let us take the simplest time-dependent configuration for $\chi$-charge, $\chi = g \delta(t) \delta^2(\vec{x})$, which is similar to the one we have taken in Euclidean space. Such a case is readily solved by taking:

$$B_g(\vec{x}, t) = g \delta^2(\vec{x}) \Theta(t),$$  \hspace{1cm} (29)

what is clearly the magnetic field due to a vortex-like object with flux equal to $2 \pi g$ (created at $t = 0$).

We may also think about a configuration which reverses the direction of the magnetic flux, say

$$B_g(\vec{x}, t) = \frac{g}{2} \delta^2(\vec{x}) \Theta(t) - \Theta(-t),$$  \hspace{1cm} (30)

which clearly represents a magnetic vortex with flux $-g/2$ that changes its signal at $t = 0$, or still, the destruction of a $-g/2$-flux vortex at $t = 0$ with the simultaneous creation of another one with flux $g/2$.

For such an object the equation (29), its topologically induced electric current, takes the form

$$\rho_M = mg \delta^2(\vec{x}) \Theta(t) \quad \text{and} \quad j_M^i = 0,$$  \hspace{1cm} (31)

which represents a point-like electric charge of strength $-mg$ created at $t = 0$. In addition, since $B$ and $\rho_M$ above are located at $\vec{x} = 0$, we conclude that in MCS case, the appearance of a composite vortex-electric charge may be alternatively provided through the introduction of a vortex-like solution, like as (29), whenever breaking the Bianchi identity.

A more general solution associated to equation (21) is obtained by taking a “mixture” of previous ones. Let be $\chi = g \delta^2(\vec{x}) \delta(t)$ and let us combine previous solutions, like below:

$$B_g(\vec{x}, t) = \frac{g}{2} \delta^2(\vec{x}) \Theta(t) \quad \text{and} \quad \vec{E}_g = \frac{g}{4\pi} \frac{\delta}{|\vec{x}|} \delta(t).$$  \hspace{1cm} (32)

As it is clear, such expressions take together the solutions associated to the vortex-like, created at $t = 0$, and to the Dirac-like monopole, only at $t = 0$. The electric field above induces

$$\vec{j}_M = \frac{mg}{4\pi} \frac{\vec{x}}{|\vec{x}|^2} \delta(t),$$  \hspace{1cm} (33)
which takes electric charges away from the origin at \( t = 0 \), while

\[
\rho_M = \frac{mg}{2} \delta^2(\vec{x}) \Theta(t),
\]

(34)
corresponds to the induced charge at \( \vec{x} = 0 \), provided by \( \vec{j}_M \).

Let us compare solution above with that we have in Euclidean space, equation (19). Monopole-like solution in Euclidean space, equation (19) represents an object that produces a tangential electric field and a ‘radial-like’ magnetic field, both of them proportional to \( 1/|\vec{x}|^2 \). On the other hand, if we consider one of its analogue in Minkowskian space-time, solution (32), we realize that in this case the monopole-like solution gives us a magnetic field confined to a point in space, a vortex, and a tangentially directed electric field which is proportional to \( 1/|\vec{x}| \) and, in addition, takes place only at \( t = 0 \). Therefore, we conclude that the dimension and structure (topology, etc.) of the space-time is decisive for the solutions of the fields associated to Dirac-like objects.

Before close this section, let us pay attention to the issue concerning the introduction of such entities in electrodynamical-like models, namely, three-dimensional Abelian gauge theories. First of all, notice that in the Bianchi identity breaking scenario, no space is reserved to the appearance of a mass term, say, we could not provide a mass gap for the radiation associated to the monopole-like field (for the time being, we are supposing different radiation for dynamical and geometrical sectors of the equations of motion).

Now, in the case of the pure Maxwell (massless) model, the breaking of Bianchi identity causes no additional trouble in the dynamical sector, for instance, electric current remains conserved. Therefore, in this case, nothing prevent us from taking into account that we have indeed a unique (massless) radiation which mediates the interaction among usual electric charges (usual electric and magnetic fields), among Dirac-like objects \( (\vec{E}_g \text{ and } B_g) \), and also among the first and the latter ones. It is worthy noticing that such an identification of apparently distinct sorts of interaction as being manifestation of only one kind of radiation is possible here because of all the required potentials and fields are gapless.

However, if we try to apply a similar identification in the MCS framework we meet serious troubles. Here, usual radiation is naturally massive. For example, the electric field between two static electric charges is proportional to \( K_0(m|\vec{x}|) \) (with \( K_0 \) being the modified Bessel function of 2\text{nd} kind at 0\text{th} order), and so it is a short-range interaction. In deep contrast, the tangential electric field due to a monopole-like solution carries no hint about mass, see equations (19) and (27). Actually, as far as we have tried, no way was found in order to identify both types of interaction as produced by the same radia-
tion. This would require an action which has already enclosed usual and monopole-like potentials as its basic ingredients, and so, answer whether is required one or two kinds of radiation.

4 Neutral particles non-minimally coupled to monopole field and the Aharonov-Casher effect

In this section we shall consider a non-minimal coupling of a spinor field with the electric field generated by a ‘static monopole’, equation (25). First, however, we shortly review some basic aspects of the usual non-minimal case, mainly those concerning the Aharonov-Casher effect. Indeed, it is a peculiarity of (2+1) dimensions that even spinless particles may carry anomalous magnetic momentum, whenever interacting with an electromagnetic field. This lies in the fact that the momentum may be naturally supplemented by the dual field-strength, say:

\[ \partial_\mu \rightarrow \partial_\mu + i h \tilde{F}_\mu, \tag{35} \]

where \( h \) measures the planar anomalous magnetic momentum of the matter (see, for example, Refs. [16, 17], for further details).

Now, let us take the electromagnetic field, \( \tilde{F}_\mu = (-B; \epsilon_{ij} E_j) \), produced by a usual point-like electric charge, say, \( B = 0 \) and \( \vec{E} = q\vec{x}/2\pi|\vec{x}|^2 \). Then, if we consider the interaction of such a field with a given particle (mass \( m \)), we find that the energy-operator of the latter reads

\[ H = \frac{1}{2m}(\partial_i + i h \tilde{F}_i)^2 = \frac{1}{2m}(\partial_i + i h \epsilon_{ij} E_j)^2. \tag{36} \]

In addition, if the “free” wave-functions associated to the particle satisfy

\[ \left(i\partial_0 + \frac{1}{2m}\nabla^2\right) \psi^{(0)} = 0, \tag{37} \]

then, the WKB approximation yields the new functions:

\[ \psi(\vec{x}, t) = \psi^{(0)}(\vec{x}, t) \exp \left[-i \int dx_\mu h \tilde{F}_\mu \right]. \tag{38} \]

Thus, we realize that, the addition of the field \( \tilde{F}_\mu \) to the usual momentum is equivalent to introduce (at WKB-level) a non-integrable phase to the former wave-functions. Clearly, a similar plain also holds in the case of the minimal coupling, \( \partial_\mu \rightarrow \partial_\mu + ieA_\mu \) (see Ref. [17]), which is responsible for the appearance of the so-celebrated Aharonov-Bohm (AB)
effect\cite{18} and, in (2+1) dimensions magnetic flux-carrying particles leads to fractional statistics\cite{8}.

The interesting point to be noticed here is that, if we consider the particle performs a spatial loop, say $\theta$, around the charge $q$, then

$$\theta = h \oint dl_i \tilde{F}^i = h \int dl_i \epsilon^{ij} E^j = \frac{h}{2\pi} \oint dl_i \epsilon^{ij} \frac{x^j}{|\underline{x}|^2}.$$ \hspace{1cm} (39)

Now, since $dl_i = \epsilon_{ij} dx_j$ ($d\underline{x}$ is radial) and $\nabla \cdot \underline{\underline{x}}/|\underline{x}|^2 = 2\pi\delta^2(\underline{x})$ we finally obtain:

$$\theta = \frac{hq}{2\pi} \int_S dS \nabla \cdot \frac{\underline{x}}{|\underline{x}|^2} = hq.$$

Therefore, we have that $\psi(\underline{x},t) = \psi^{(0)}(\underline{x},t) e^{i\theta}$, where $\theta$ is the Aharonov-Casher (AC) phase provided by the electric field $\tilde{E} = q\underline{x}/2\pi|\underline{x}|^2$ (for further details see Refs.\cite{17,19,21} and related references therein).

In our present case, the counterpart of the electric field above reads like equation (25), $\tilde{E}_g = g (\hat{x} - \hat{y})/2\pi|\underline{x}|^2$. Then, $\tilde{F}^i_g = -\epsilon^{ij} E^j_g = g\underline{x}/2\pi|\underline{x}|^2$ is already radial. In this case, a similar loop as in the previous case, $\theta'$, vanishes:

$$\theta' = h \oint dl_i \tilde{F}^i = \frac{hq}{2\pi} \oint dl_i \frac{x^i}{|\underline{x}|^2} = 0.$$ \hspace{1cm} (41)

Then, our monopole does not induce an AC-phase on a given particle if they interact in the usual non-minimal way. In addition, we should notice that the contrast between the cases above comes from the fact that, in the first one, say, $\tilde{E} = q\underline{x}/2\pi|\underline{x}|^2$, the dual operation induced whenever taking $\tilde{F}^i = -\epsilon^{ij} E^j$ is exactly compensated by an extra one associated to $dl_i = \epsilon_{ij} dx_j$.

In view of such an aspect, we shall consider here the (Lorentz-odd) non-minimal term like below (coupled, for concreteness, to spinors):

$$\mathcal{L}' = \overline{\psi} (i\partial_\mu \gamma^\mu - M + ia\gamma^0 \gamma^\mu \tilde{F}_\mu) \psi,$$ \hspace{1cm} (42)

whose equation of motion reads:

$$(i\partial_\mu \gamma^\mu - M + ia\gamma^0 \gamma^\mu \tilde{F}_\mu) \psi = 0$$ \hspace{1cm} (43)

Before studying some properties of such a term, like as its connection with AC effect, we shall give attention to its behavior under special properties, say, gauge invariance, Charge Conjugation (C), Parity (P) and Time Reversal (T). For this, let us take $\gamma^0 = \sigma^z$,
$\gamma^1 = i\sigma^x$ and $\gamma^2 = i\sigma^y$ as the representation of the Dirac matrices in $(2+1)$-dimensions.

First, analyzing the behavior of $\mathcal{L}'$ under gauge transformations, we may clearly realize its gauge-invariance, since

\[
\begin{align*}
\delta \bar{\psi} &= \epsilon \bar{\psi} , \\
\delta \psi &= -\epsilon \psi , \\
\delta E_x &= \epsilon \gamma^1 \eta , \\
\delta E_y &= 2\epsilon \gamma^2 \eta , \\
\delta B &= \epsilon \eta ,
\end{align*}
\]

(44)

where $\epsilon$ is a global gauge parameter and $\eta$ is a local auxiliary field which helps in the gauge invariance.

On the other hand, using the identity $\gamma^\mu \gamma^\nu = \eta^\mu\nu - i\epsilon^\mu\nu\kappa \gamma_\kappa$, we may write:

\[
\begin{align*}
ia \bar{\psi} \gamma^0 \gamma^\mu \bar{F}_\mu \psi &= -ia B \bar{\psi} \psi + a \bar{\psi} \gamma^0 \bar{\psi} , \\
\end{align*}
\]

(45)

Now, let us see how the terms above behave under $C$, $P$ and $T$ operations. Let us strat off from:

\[
\begin{align*}
ia B \bar{\psi} \psi &\xrightarrow{C} -ia B \bar{\psi} \psi , \\
\end{align*}
\]

(46)

\[
\begin{align*}
ia B \bar{\psi} \psi &\xrightarrow{P} -ia B \bar{\psi} \psi , \\
\end{align*}
\]

(47)

\[
\begin{align*}
ia B \bar{\psi} \psi &\xrightarrow{T} +ia B \bar{\psi} \psi , \\
\end{align*}
\]

(48)

then, although breaking $C$ and $P$, such a term keeps $T$-invariance and so $CPT$-symmetry is preserved. In addition, let us notice that this term provides an extra (imaginary) mass for the fermions when $B \neq 0$ (while its usual counterpart, $f B \bar{\psi} \gamma^0 \psi$, couples to the electric charge$^3$).

On the other hand, the spatial components behave like follows:

\[
\begin{align*}
\bar{a} \psi \gamma^i \bar{E}_i \psi &\xrightarrow{C} +\bar{a} \psi \gamma^i \bar{E}_i \psi , \\
\bar{a} \psi \gamma^i \bar{E}_i \psi &\xrightarrow{P} +a \bar{\psi} \gamma^i \bar{E}_i \psi , \\
\bar{a} \psi \gamma^i \bar{E}_i \psi &\xrightarrow{T} +a \bar{\psi} \gamma^i \bar{E}_i \psi ,
\end{align*}
\]

(49)

(50)

(51)

which state us that the term above, coupling the current to the electric field, preserves all of these symmetries above, and CPT is obviously kept. Here, it should be noted that its usual counterpart, $f B \bar{\psi} \gamma^i \psi$, which couples the current density to the dual electric

\[3\text{Then, in view of its imaginary nature, we should take it away from equation (42) in order to maintain the real character of this Lagrangian. This is done in what follows (see eq. (52), and related discussion).} \]
field is $P$ and $T$-odd (while respects $CPT$, since it is $C$-even). Then, when Lorentz and CPT-symmetries are taken into account, we recognize profound differences between the present and the usual non-minimal couplings. Thus, our proposal may be viewed as a low-energy alternative to the standard term, particularly in those cases in which neither $P$ nor $T$ operation is broken.

Hereafter we shall focus our attention to the field produced by the monopole, equation (25), and its consequences concerning AC phase as well. Thus, we shall work with the (Lorentz-violating) Lagrangian below:

$$
\mathcal{L} = \bar{\psi} (i \partial_t \gamma^0 - i \partial_i \gamma^i - M + i a \gamma^0 \gamma^i \tilde{F}_i) \psi ,
$$

which leads to the following eqs. of motion:

$$
i \partial_t \psi (x) = [\gamma_0 \vec{\gamma} \cdot ( -i \vec{\nabla} + a \vec{E}_g ) + M \gamma_0 ] \psi (x).
$$

In addition, it is easy to show that Lagrangian (52) is invariant under gauge transformations, say:

$$
\begin{aligned}
d\bar{\psi} &= \epsilon \bar{\psi} , \\
d\psi &= - \epsilon \psi , \\
d \delta E_x &= \epsilon \gamma^2 \phi , \\
d \delta E_y &= \epsilon \gamma^1 \phi 
\end{aligned}
$$

where $\phi$ is a local auxiliary field analog to the $\eta$ field described above.

Now, taking the field generated by a point-like monopole, $E_i = (E_g)_i = g \epsilon_{ij} x_j / 2\pi |\vec{x}|^2$, and working in the WKB approximation, we have that:

$$
\psi (\vec{x}, t) = \psi^{(0)} (\vec{x}, t) \exp \left[ -i a \int dl_i E^i_g \right],
$$

with $\psi^{(0)}$ satisfying equation (53) when $\vec{E}_g$ is vanishing. Now, supposing that the fermion perform a loop, $\alpha_g$, around the monopole,

$$
\alpha_g = a \oint dl_i E^i_g = \frac{ag}{2\pi} \int \epsilon_{ij} dx_j \epsilon^{ik} E^k_g = ag,
$$

which clearly represents the AC phase on the fermion wave-function produced by the monopole field, $\vec{E}_g$. We should stress, once more, that such a phase comes from a $C$, $P$ and $T$-invariant non-minimal coupling.

In this way we have carried out the duality symmetry found at the previous section between the electric field produced by a static monopole and the vector potential of a
magnetic vortex to the level of quantum mechanics.

Now, in order to study the behavior of the wave-functions it is more convenient to work with the second-order differential equation in polar coordinates \(x = r \cos \varphi\) and \(y = r \sin \varphi\), like follows:

\[
\left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \left( \frac{\partial}{\partial \varphi} + i\alpha \right)^2 + \right.
\left. - \alpha s \sigma_z \frac{1}{r} \delta(r) + k^2 \right] \psi(r, \phi) = 0 ,
\]

(57)

where \(\alpha = \frac{ae}{2\pi}\), \(k^2 = E^2 - M^2\), and we are using \(s = +1\) for “spin up” and \(s = -1\) for “spin up” (the actual spin of the spinors is \(s/2\)).

This equation is well known from the Aharonov-Bohm (AB) effect for relativistic particles. In fact, this is a sort of Aharonov-Casher (AC) effect since we have a neutral particle in the presence of an electric field, as we have discussed above. Furthermore, the presence of spin leads to the \(\delta(r)\) interaction, which mimics the interaction of the spin of the particle with a magnetic vortex (Zeeman effect), and may be interpreted as a contact interaction of the spin with the monopole itself. Although this residual interaction term vanishes outside the location of the monopole, the influence of the monopole on the dynamics of the particles is still felt by the induction of a non-trivial phase, equations (55-56) and, consequently, on the phase shift of the scattered wave-function.

In the works of Refs.\[20, 21, 22\] such a problem is treated in the context of the AB effect. The authors adopt different approaches to regularize the delta function potential, which in our case is equivalent to suppose that the radius \(R\) of the monopole is finite and is taken to zero at the end of the calculations.

To quote the main results, we consider the upper component of \(\psi(r, \varphi)\) and expand it as

\[
\psi_1 = \sum_{m=-\infty}^{m=+\infty} f_m(r) e^{im\varphi} ,
\]

(58)

where \(f_m(r)\) obeys the following equation

\[
\left[ \frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \right) - \frac{(m + \alpha)^2}{r^2} + \right.
\left. - \frac{\alpha}{R} \delta(r - R) + k^2 \right] f_m(r) = 0 ,
\]

(59)
with the following boundary conditions

$$f_m(R - \varepsilon) = f_m(R + \varepsilon),$$

$$R \frac{d}{dr} f_m|_{R-\varepsilon} = \alpha f_m(R),$$

which incorporate the effect of the delta function.

By writing the $f_m(r)$ in terms of Bessel functions

$$f_m(r) = \begin{cases} A_m J_{|m+\alpha|}(k r) + B_m J_{-|m+\alpha|}(k r) & r > R \\ C_m J_m(k r) & r < R \end{cases}$$

and using equations (60) to determine the coefficients of the Bessel functions this renormalization method allows for the irregular function $J_{-|m+\alpha|}$ to contribute if the following relations

$$|m| + |m + \alpha| = -\alpha s$$

and

$$|m| + \alpha s + 1 > 0,$$

are simultaneously satisfied $[20, 21]$.

The same kind of problem was analyzed in $[25]$ by using the self-adjoint method and found to be equivalent to the renormalization method if some relations between the self-adjoint parameter and the renormalized coupling constant are satisfied $[26]$.

5 Conclusions and Prospects

In the present paper, attention was given to Dirac-like monopoles in three-dimensional Abelian Maxwell and Maxwell-Chern-Simons models. Initially, we gave an alternative view on the scalar nature of such objects in planar world. This was done by carrying out the dimensional reduction of four-dimensional Maxwell theory, enriched by magnetic sources, to three dimensions. There, we realized the appearance of two independent Abelian factors, one related to the usual $A^\mu$-potential ($\vec{E}$ and $B$ fields), while the another is implemented by a scalar potential, $S$. In addition, we have also verified that the broken Bianchi identity of the $A^\mu$-sector (usual planar electromagnetism), $\partial_\mu \tilde{F}_\mu = \chi$, presents a pseudo-scalar that is the survivor of the 3rd component of the genuine magnetic 4-current.
Furthermore, in analyzing the structure of the solutions of $\partial_\mu \tilde{F}_\mu = \chi$, we have realized it admits a wider class of solutions than so far considered in the literature. Indeed, in Minkowski space-time, we have seen that not only the azimuthal-like electric field shows up, but also, magnetic vortex-like solutions may appear as well.

In addition, when neutral matter interacts non-minimally with Dirac-like monopoles in a particular way, say, via $i\gamma^0 \gamma^j \tilde{F}_j$, then an analogous to the Aharonov-Casher effect is exhibited by such particles. We have also found some subtleties which have to be taken into account carefully because of the effective delta function potential whose origin rests on the “contact” interaction between the particle and the monopole. This is still under study as well as the consequences of the allowed solutions on the angular momentum of the particle and perhaps on the quantization of the parameter $\alpha$.

Before pointing out our Prospects, we would like to take once more the issue concerning the Bianchi identity in (2+1) dimensions. First of all, let us suppose it holds. Now, let us consider a physical system in which magnetic vortex are created. For instance, when the external magnetic field is suitably increased in high-Tc superconductors samples. More precisely, let us imagine one vortex is created at $t = t_1$ and at spatial origin. Then, the superconductor is supplemented by $B_1(\vec{x}, t) = b_1 \delta^2(\vec{x}) \Theta(t - t_1)$. On the other hand, since $\partial_\mu \tilde{F}^\mu = \partial_t B - \nabla \wedge \vec{E} = 0$ then $B_1$ above must induce a tangential-like electric field, $\vec{E}_1 = b_1 \frac{\vec{\xi}}{r} \delta(t - t_1)$, in order to prevent the breaking of Bianchi identity. Hence, we conclude that the azimuthal-like electric field may appear even in standard planar electromagnetism, say, without Dirac-like objects. Actually, $\vec{E}_1$ above survived only at $t = t_1$ because we have supposed that the creation of the vortex is also instantaneous. Now, if a finite time is needed for creating such a vortex, then we expect that this electric field will also take place during all this time.

As perspectives for future investigation, we may quote, among others, the issue concerning the effect of the dimensional reduction on the so-called Dirac quantization condition in (3+1) dimensions and which would be its counterpart in the planar world (as far as we have understood, the Henneaux-Teitelboim condition [12] does not answer for such a point, since it seems to be valid only when the topological mass is non-vanishing).

The relevance of the scalar field, $S$, appearing in Section 2 is also under investigation in the context of the so-called statistical field. Actually, by taking an Abelian Lagrangian which contains the usual Maxwell and the $\theta$-term as well, $\theta \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu}$, in (3+1) dimensions, we have seen that, after a suitable dimensional reduction scheme, we naturally generate, in (2+1) dimensions, a model which encloses the kinetic terms for $A_\mu$ and $S$ as well as another one that links both of these fields by means of a Chern-Simons-like term. Indeed,
by identifying $a_\mu = \partial_\mu S$, we clearly realize that such a subsequent model is actually that for the (non-dynamical) statistical field, $a_\mu$, in which this field enters in order to restore Parity symmetry. Further results will appear elsewhere [27].

The discussion raised up in the preceding paragraphs may also lead us to interesting results, in particular, for providing a link between fundamental aspects of planar Abelian electrodynamics and Condensed Matter phenomena, namely the up-to-date topic of high-Tc superconductivity. Still in this line, the study of the interaction between usual particles and Dirac-like objects may be useful in connection to low-energy problems.

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