The HRS tilting process and Grothendieck hearts of t-structures

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Abstract
In this paper we revisit the problem of determining when the heart of a t-structure is a Grothendieck category, with special attention to the case of the Happel-Reiten-Smalø (HSR) t-structure in the derived category of a Grothendieck category associated to a torsion pair in the latter. We revisit the HRS tilting process deriving from it a lot of information on the HRS t-structures which have a projective generator or an injective cogenerator, and obtain several bijections between classes of pairs (A, t) consisting of an abelian category and a torsion pair in it. We use these bijections to re-prove, by different methods, a recent result of Tilting Theory and the fact that if t = (T, F) is a torsion pair in a Grothendieck category G, then the heart of the associated HRS t-structure is itself a Grothendieck category if, and only if, t is of finite type. We survey this last problem and recent results after its solution.

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1 Introduction
The aim of this paper is twofold. On one side we want to give a summary of the main results related with the following question:

Question 1.1. When is the heart of a t-structure a Grothendieck category?

We shall mainly concentrate in the route leading to the answer to the question in the case when the ambient triangulated category is the (unbounded) derived category D(G) of a Grothendieck category G and the t-structure is the

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Happel-Reiten-Smalø (HRS) t-structure in $\mathcal{D}(\mathcal{G})$ associated to a torsion pair in $\mathcal{G}$ (see Example 2.4(2)). But we include a final short section, where we briefly summarize the main results for general triangulated categories with coproducts and t-structures. As a second goal, we want to revisit the HRS tilting process and show that it allows to prove in an easy way parts of recent results in the literature, and that, with the help of some recent notions about purity in arbitrary Grothendieck categories, one can re-prove the answer to Question 1.1 by methods completely different to those used to get the earlier answer.

All throughout the paper, unless otherwise stated, all categories will be additive. We will mainly use two types of those categories commonly studied in Homological Algebra, concretely abelian categories and triangulated categories (we refer to [S] and [N] for the respective definitions). The key concept for us is that of a t-structure in a triangulated category, introduced by Beilinson, Bernstein and Deligne [BBD] in their treatment of perverse sheaves. Roughly speaking a t-structure in the triangulated category $\mathcal{D}$ is a pair $\tau = (\mathcal{U}, \mathcal{W})$ of full subcategories satisfying some axioms (see Definition 3 for the details) which guarantee that the intersection $\mathcal{H}_\tau = \mathcal{U} \cap \mathcal{W}$ is an abelian category, commonly called the heart of the t-structure. This abelian category comes with a cohomological functor $H^0_\tau : \mathcal{D} \to \mathcal{H}_\tau$. In [BBD] the category of perverses sheaves on a variety $\mathcal{X}$ appeared as the heart of a t-structure in $\mathcal{D}^{\text{b}}(\mathcal{X})$, the bounded derived category of coherent sheaves on $\mathcal{X}$.

In several modern developments of Mathematics, as Motive Theory, the homological approach to Mirror Symmetry, Modular Representation of finite groups, Representation Theory of Algebras, among others, the role of t-structures is fundamental. For this reason it is important to know when the heart of a t-structure has nice properties as an abelian category. Vaguely speaking, one would ask: When is nice the heart of a given t-structure?. Trying to make sense of the adjective ‘nice’ here, one commonly uses the following “hierarchy” among abelian categories introduced by Grothendieck [G]. We say that an abelian category $\mathcal{A}$ is:

1. AB3 (resp. AB3*) when it has (arbitrary set-indexed) coproducts (resp. products);

2. AB4 (resp. AB4*) when it is AB3 (resp. AB3*) and the coproduct functor $\bigsqcup : [\Lambda, \mathcal{A}] \to \mathcal{A}$ (resp. product functor $\prod : [\Lambda, \mathcal{A}] \to \mathcal{A}$) is exact, for each set $\Lambda$;

3. AB5 (resp. AB5*) when it is AB3 (resp. AB3*) and the direct limit functor $\varinjlim : [\Lambda, \mathcal{A}] \to \mathcal{A}$ (resp. inverse limit functor $\varprojlim : [\Lambda^{\text{op}}, \mathcal{A}] \to \mathcal{A}$) is exact, for each directed set $\Lambda$.

4. a Grothendieck category, when it is AB5 and has a generator or, equivalently, a set of generators.

Grothendieck categories appear quite naturally in Algebra and Geometry and their behavior is, in many aspects, similar to that of module categories over
a ring (see [S, Chapter V]). For instance, such a category has enough injectives and every object in it has an injective envelope. Even more, by a famous theorem of Gabriel and Popescu (see [GP], and also [S, Theorem X.4.1]), such a category is always a Gabriel localisation of a module category, which roughly means that it is obtained from such a category by formally inverting some morphisms. This is the main reason why the study of when the heart of a t-structure is a Grothendieck category, i.e. Question 1.1 has deserved most of the attention, apart of the study of when it is a module category, that we barely touch in this paper. When one starts approaching the question, one quickly sees that it is hopeless unless some extra hypotheses are imposed on the ambient triangulated category \( D \) and/or on the t-structure \( \tau \) itself. For instance, it is unavoidable to require that \( D \) has coproducts or, at least, to guarantee that coproducts in \( D \) of objects in the heart of \( \tau \) always exist. On the other hand, the problem gets quite complicated if the coproduct in \( H_\tau \) and the coproduct in \( D \) of a given family of objects in \( H_\tau \) do not coincide. A way of ensuring that they coincide is to require that the t-structure be \textit{smashing}, i.e. that the co-aisle \( W \) of the t-structure is closed under coproducts in \( D \). Therefore, instead of the initial question, the following one has more hopes of being answered and has deserved a lot of attention in recent times (see Section 5):

**Question 1.2.** Let \( D \) be a triangulated category with coproducts and let \( \tau = (U, W) \) be a smashing t-structure in \( D \). When is the heart of \( \tau \) a Grothendieck category?

Although studied historically first, the question for the HRS t-structure is a particular case of this last question. Namely, if \( G \) is a Grothendieck category, then its derived category \( D(G) \) is the prototypical example of a triangulated category with coproducts (and also products). When a torsion pair \( t = (T, F) \) is given in \( G \), the associated HRS t-structure in \( D(G) \) is smashing. So restricted to this particular example, the last question is re-read as follows, and it is the main problem that we survey and re-visit in this paper:

**Question 1.3.** Let \( G \) be a Grothendieck category, let \( t = (T, F) \) be a torsion pair in \( G \) and let \( H_t \) be the heart of the associated HRS t-structure in \( D(G) \). When is \( H_t \) a Grothendieck category?

The organization of the paper goes as follows. In Section 2 we introduce the main concepts needed for the understanding of the paper, specially torsion pairs in abelian categories and t-structures in triangulated categories, with a look also at the HRS tilting process. It turns out that if one starts with a pair \( (A, t) \) consisting of an abelian category \( A \) and a torsion pair \( t \), then the new abelian category obtained by tilting \( A \) with respect to \( t \) need not have Hom sets. Corollary 2.8 gives the precise conditions to get Hom sets. In Section 3 we study when the heart of (the HRS t-structure associated to) a torsion pair in an abelian category has either a projective generator or an injective cogenerator. This leads naturally to quasi-(co)tilting torsion pairs (see Proposition 3.6) in abelian categories. Then it is proved that tilting (resp. cotilting) torsion pairs
in abelian categories are precisely the co-faithful (resp. faithful) torsion pairs for which the heart is AB3 (resp. AB3*) and has a projective generator (resp. injective cogenerator), see Theorems 3.7 and 3.9. As a particular case, one gets that torsion pairs given by classical 1-tilting sets (resp. objects) are precisely the co-faithful torsion pairs whose heart is the module category over a small pre-additive category (resp. over a ring). We end the section by giving a series of bijections induced by the HRS tilting process (see Proposition 3.13), some of which imply the case $n = 1$ a recent result of Positselski and Stovicek (see Corollary 3.15). In Section 4 we study Question 1.3. Subsections 4.1 and 4.2 are dedicated to show the milestones of the route that led to the solution of the problem in [PS2]. Subsection 4.3 briefly summarizes recent results by Bazzoni, Herzog, Prihoda, Saroch and Trlifaj about the same question in the particular case when the torsion pair is tilting. We end the section by re-proving, using a recent characterization of the AB5 condition by Positselski and Stovicek (see [Po-St]), the fact that if the heart of a torsion pair in a Grothendieck category is itself a Grothendieck category, then the torsion pair is of finite type. The final Section 5 shows the most recent results and the present state Question 1.2.

2 Preliminaries

In the rest of the paper, whenever $\mathcal{C}$ is an additive category and $\mathcal{X}$ is any class of objects, we shall denote by $\mathcal{X}^\perp$ (resp. $^\perp \mathcal{X}$) the full subcategory consisting of the objects $Y$ such that $\mathcal{C}(X, Y) = 0$ (resp. $\mathcal{C}(Y, X) = 0$), for all $X \in \mathcal{X}$. When $\mathcal{X} = \{X\}$ for simplicity we will write $X^\perp$ (resp. $^\perp X$) instead of $\mathcal{X}^\perp$ (resp. $^\perp \mathcal{X}$).

Unless explicitly said otherwise, in the rest of the paper the letter $\mathcal{A}$ will denote an abelian category.

2.1 Torsion pairs

Definition 1. A torsion pair in $\mathcal{A}$ is a pair $t = (\mathcal{T}, \mathcal{F})$ of full subcategories satisfying the following two conditions:

1) $\mathcal{A}(T, F) = 0$, for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$;

2) For each object $X$ of $\mathcal{A}$, there is an exact sequence

\[
0 \longrightarrow T_X \longrightarrow X \longrightarrow F_X \longrightarrow 0
\]

where $T_X \in \mathcal{T}$ and $F_X \in \mathcal{F}$.

A torsion class in $\mathcal{A}$ is one $\mathcal{T}$ that appears as first component of a torsion pair in $\mathcal{A}$. A torsionfree class $\mathcal{F}$ is defined dually. Note that in a torsion pair we have $\mathcal{F} = \mathcal{T}^\perp$ and $\mathcal{T} = ^\perp \mathcal{F}$. On the other hand, in the sequence above $T_X$ and $F_X$ depend functorially on $X$, so that the assignment $X \mapsto T_X$ (resp. $X \mapsto F_X$) underlies a functor $t : \mathcal{A} \rightarrow \mathcal{T}$ (resp. $(1 : t) : \mathcal{A} \rightarrow \mathcal{F}$), which is right (resp. left) adjoint of the inclusion functor $i_{\mathcal{T}} : \mathcal{T} \hookrightarrow \mathcal{A}$ (resp. $i_{\mathcal{F}} : \mathcal{F} \hookrightarrow \mathcal{A}$).
The composition \( \iota_T \circ t : \mathcal{A} \to \mathcal{A} \) (resp. \( \iota_F \circ (1 : t) : \mathcal{A} \to \mathcal{A} \)), which we will still denote by \( t \) (resp. \((1 : t)\)), is called the torsion radical (resp. torsion coradical) associated to \( t \).

In particular situations, torsion and torsionfree classes are identified by the satisfaction of some closure properties. Recall that an abelian category is called \textit{locally small} when the subobjects of any given object form a set.

**Proposition 2.1.** Let \( \mathcal{A} \) be an AB3 (resp. AB3*) abelian category which is locally small and let \( \mathcal{T} \subseteq \mathcal{A} \) (resp. \( \mathcal{F} \)) be a full subcategory. The following assertions are equivalent:

1. \( \mathcal{T} \) (resp. \( \mathcal{F} \)) is a torsion (resp. torsionfree) class;
2. \( \mathcal{T} \) (resp. \( \mathcal{F} \)) is closed under taking quotients (=epimorphic images), co-products (resp. subobjects, products) and extensions.

**Proof.** See [S, Proposition VI.2.1]. \( \square \)

Recall that, in any category \( \mathcal{C} \), a class of objects \( \mathcal{X} \) is called a \textit{generating} (resp. \textit{cogenerating}) class when, for each object \( C \in \text{Ob}(\mathcal{C}) \), there is an epimorphism \( X_C \to C \) (resp. monomorphism \( C \to X_C \)), for some object \( X_C \in \mathcal{X} \).

We recall some particular cases of torsion pairs:

**Definition 2.** Let \( t = (\mathcal{T}, \mathcal{F}) \) be a torsion pair in \( \mathcal{A} \). We will say that \( t \) is:

1. \textit{faithful} (resp. \textit{co-faithful}) when \( \mathcal{F} \) (resp. \( \mathcal{T} \)) is a generating (resp. cogenerating) class of \( \mathcal{A} \).
2. \textit{of finite type} when direct limits in \( \mathcal{A} \) of objects in \( \mathcal{F} \) exist and are in \( \mathcal{F} \).

**Remark 2.2.** In [HRS] faithful (resp. co-faithful) torsion pairs are called cotilting (resp. tilting). In this paper we separate from that terminology, reserving the term ‘cotilting’ (resp. ‘tilting’) for torsion pairs defined by 1-cotilting (resp. 1-tilting) objects (see Definition 8).

### 2.2 \( \tau \)-Structures

In the sequel, the letter \( \mathcal{D} \) will denote a triangulated category and \( ?[1] : \mathcal{D} \to \mathcal{D} \) its suspension functor. Moreover, we put \( ?[0] = 1_\mathcal{D} \) and \( ?[k] \) the \( k \)-th power of \( ?[1] \), for each integer \( k \). We will denote the triangles on \( \mathcal{D} \) by \( X \to Y \to Z \to X[1] \), or also, \( X \to Y \to Z \to X[1] \).

**Definition 3.** Let \( (\mathcal{D}, ?[1]) \) be a triangulated category. A \textit{\( \tau \)-structure} on \( \mathcal{D} \) is a couple of full subcategories closed under direct summands \( (\mathcal{U}, \mathcal{W}) \) such that:

1. \( \mathcal{U}[1] \subseteq \mathcal{U} \);
2. \( \mathcal{D}(U, W[-1]) = 0 \), for all \( U \in \mathcal{U} \) and \( W \in \mathcal{W} \).
3. For each $X \in \mathcal{D}$, there is a distinguished triangle:

$$U_X \rightarrow X \rightarrow V_X \rightarrow U_X[1]$$

with $U_X \in \mathcal{U}$ and $V_X \in \mathcal{W}[-1]$.

In such case, the subcategory $\mathcal{U}$ is called the \textit{aisle} of the t-structure, and $\mathcal{W}$ is called the \textit{coaisle}. Note that in such case, we have $\mathcal{W}[-1] = \mathcal{U}^\perp$ and $\mathcal{U} = \perp(\mathcal{W}[-1]) = \perp(\mathcal{U}^\perp)$. For this reason, we will write the t-structures using the following notation $(\mathcal{U}, \mathcal{U}^\perp[1])$. On the other hand, the objects $U_X$ and $V_X$ in the previous triangle are uniquely determined by $X$, up to isomorphism, so that the assignment $X \mapsto U_X$ (resp. $X \mapsto V_X$) underlies a functor $\tau_{\mathcal{U}}^\leq : \mathcal{D} \rightarrow \mathcal{U}$ (resp. $\tau_{\mathcal{U}}^\geq : \mathcal{D} \rightarrow \mathcal{U}^\perp$) which are right (resp. left) adjoint of the inclusion functor $\iota_{\mathcal{U}} : \mathcal{U} \hookrightarrow \mathcal{D}$ (resp. $\iota_{\mathcal{U}}^\perp : \mathcal{U}^\perp \hookrightarrow \mathcal{D}$). The composition $\iota_{\mathcal{U}} \circ \tau_{\mathcal{U}}^\leq : \mathcal{D} \rightarrow \mathcal{D}$ (resp. $\iota_{\mathcal{U}}^\perp \circ \tau_{\mathcal{U}}^\geq : \mathcal{D} \rightarrow \mathcal{D}$), which we will still denote by $\tau_{\mathcal{U}}^\leq$ (resp. $\tau_{\mathcal{U}}^\geq$) and it is called the \textit{left truncation} (resp. \textit{right truncation}) functor associated to the t-structure $(\mathcal{U}, \mathcal{U}^\perp[1])$. The full subcategory $\mathcal{H} = \mathcal{U} \cap \mathcal{U}^\perp[1]$ of $\mathcal{D}$ is called the \textit{heart} of the t-structure and it is an abelian category, where the short exact sequences are the triangles in $\mathcal{D}$ having their three terms in $\mathcal{H}$. In particular, we have $\text{Ext}^1_{\mathcal{H}}(M, N) = \mathcal{D}(M, N[1])$, for all objects $M$ and $N$ in $\mathcal{H}$. Moreover, the canonical morphism $\text{Ext}^2_{\mathcal{H}}(M, N) \rightarrow \mathcal{D}(M, N[2])$ is a monomorphism in $\text{Ab}$, for all objects $M, N \in \mathcal{H}$ (see [BBD, Remarque 3.1.17]).

The kernel and cokernel of a morphism $f : M \rightarrow N$ on the heart $\mathcal{H}$ are computed as follow: we complete $f$ to a triangle in $\mathcal{D}$ and consider the following diagram, where the row and column are triangles:

$$
\begin{array}{ccc}
\tau_{\mathcal{U}}^\leq(Z[-1])[1] & \longrightarrow & M \\
& \downarrow \tau_{\mathcal{U}}^\geq(Z[-1])[1] & \downarrow + \\
N & \longrightarrow & Z \\
& \downarrow + & \\
& \tau_{\mathcal{U}}^\leq(Z[-1])[1] & \longrightarrow I \\
& \downarrow + & \downarrow \iota_f \\
& \tau_{\mathcal{U}}^\geq(Z[-1])[1] & \longrightarrow N \\
& \downarrow + & \downarrow \perp \\
I & \longrightarrow & \tau_{\mathcal{U}}^\leq(Z[-1])[1] \\
& \downarrow + & \downarrow p_f \\
& I & \longrightarrow & M
\end{array}
$$

From the octahedral axiom, we obtain the following triangles in $\mathcal{D}$

$$\tau_{\mathcal{U}}^\leq(Z[-1]) \longrightarrow M \overset{p_f}{\longrightarrow} I \overset{+}{\longrightarrow} I \overset{\iota_f}{\longrightarrow} N \overset{\perp}{\longrightarrow} \tau_{\mathcal{U}}^\geq(Z[-1])[1] \overset{+}{\longrightarrow} ,$$

where all terms are in $\mathcal{H}$ (with $\iota_f \circ p_f = f$). Then we have that $\text{Ker}_{\mathcal{H}}(f) = \tau_{\mathcal{U}}^\leq(Z[-1])$ and $\text{Coker}_{\mathcal{H}}(f) = \tau_{\mathcal{U}}^\geq(Z[-1])[1]$.

Recall that if $\mathcal{D}$ and $\mathcal{A}$ are a triangulated and an abelian category, respectively, then an additive functor $H : \mathcal{D} \rightarrow \mathcal{A}$ is called \textit{cohomological
when any triangle \( X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{t} \) in \( \mathcal{D} \) induces an exact sequence \( H(X) \xrightarrow{H(u)} H(Y) \xrightarrow{H(v)} H(Z) \) in \( \mathcal{A} \). The following proposition, shown by Beilinson, Bernstein and Deligne in [BBD], associate to each t-structure in a triangulated category, an intrinsic homology theory.

**Proposition 2.3.** Let \( (\mathcal{D}, ?[1]) \) be a triangulated category. If \( \sigma = (\mathcal{U}, \mathcal{U}^\perp[1]) \) is a t-structure in \( \mathcal{D} \), then the assignments \( X \mapsto \tau^\sigma_{\mathcal{U}}(X[1]) \) and \( X \mapsto \tau^\sigma_{\mathcal{U}}(X)[-1][1] \) define two naturally isomorphic functors from \( \mathcal{D} \) to \( \mathcal{H} \), which are cohomological. In the sequel we will fix a (cohomological) functor \( H^n_\mathcal{U} : \mathcal{D} \to \mathcal{H} \) that is naturally isomorphic to those two functors.

**Examples 2.4.** The following examples of t-structures will be of great interest in this paper.

1. Let \( \mathcal{A} \) be an abelian category for which \( \mathcal{D}(\mathcal{A}) \) exists i.e. \( \mathcal{D}(\mathcal{A}) \) has Hom sets. For each \( m \in \mathbb{Z} \), we will denote by \( \mathcal{D}^{\leq m}(\mathcal{A}) \) (resp. \( \mathcal{D}^{\geq m}(\mathcal{A}) \)) the full subcategory of \( \mathcal{D}(\mathcal{A}) \) consisting of the cochain complexes \( X \) such that \( H^k(X) = 0 \), for all \( k > m \) (resp. \( k < m \)). Moreover, we put \( \mathcal{D}^{[a,b]}(\mathcal{A}) := \mathcal{D}^{\leq b}(\mathcal{A}) \cap \mathcal{D}^{\geq a}(\mathcal{A}) \) for any integers \( a \) and \( b \). Then, the pair \( (\mathcal{D}^{\leq m}(\mathcal{A}), \mathcal{D}^{\geq m}(\mathcal{A})) \) is a t-structure in \( \mathcal{D}(\mathcal{A}) \) whose heart is equivalent to \( \mathcal{A} \). The corresponding left and right truncation functors will be denoted by \( \tau^{\leq m} : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A}) \) and \( \tau^{>m} : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A}) \), respectively. For the case \( m = 0 \), the corresponding t-structure is known as the canonical t-structure in \( \mathcal{D}(\mathcal{A}) \).

2. (Happel-Reiten-Smalø) Let \( \mathcal{A} \) be an abelian category for which \( \mathcal{D}(\mathcal{A}) \) exists (i.e. \( \mathcal{D}(\mathcal{A}) \) has Hom sets) and let \( \mathbf{t} = (\mathcal{T}, \mathcal{F}) \) be a torsion pair in \( \mathcal{A} \). The classes \( \mathcal{U}_t := \{ X \in \mathcal{D}^{\leq 0}(\mathcal{A}) : H^0(X) \in \mathcal{T} \} \) and \( \mathcal{W}_t = \{ X \in \mathcal{D}^{\geq -1}(\mathcal{A}) : H^{-1}(X) \in \mathcal{F} \} \) give rise a t-structure in \( \mathcal{D}(\mathcal{A}) \), concretely, \( (\mathcal{U}_t, \mathcal{W}_t) = (\mathcal{U}_t, \mathcal{U}_t^+[1]) \). It is called the Happel-Reiten-Smalø t-structure associated to \( \mathbf{t} \). We will denote its heart by \( \mathcal{H}_t \). In next subsection we relax the hypothesis that \( \mathcal{D}(\mathcal{A}) \) has Hom sets, showing that the formation of \( \mathcal{H}_t \) from \( \mathcal{A} \) and \( \mathbf{t} \) is still possible sometimes.

3. Let \( \mathcal{D} \) be a triangulated category with coproducts. An object \( X \) in \( \mathcal{D} \) is called compact, when the functor \( \mathcal{D}(X, ?) : \mathcal{D} \to \text{Ab} \) preserves coproducts. If \( \mathcal{S} \) is a set of compact objects in \( \mathcal{D} \), then the pair \( (\mathcal{S}^{\perp_{\geq 0}}, \mathcal{S}^{\perp_{< 0}}) \) is a t-structure in \( \mathcal{D} \) (see [K], Theorem 12.1). A compactly generated t-structure in \( \mathcal{D} \) is any one obtained in this way from a set of compact objects. We put aisle \( \mathcal{S} := \mathcal{S}^{\perp_{= 0}} \), which is the smallest full subcategory of \( \mathcal{D} \) that contains \( \mathcal{S} \) and is closed under coproducts, extensions and non-negative shifts.

### 2.3 The Happel-Reiten-Smalø (HRS) tilting process

This process stems from the seminal work in [BBD] and was fully developed in [HRS]. In our treatment here we will work in a general framework, by allowing
ourselves the freedom of working for the moment with big triangulated categories, i.e. categories satisfying all axioms of triangulated categories, but where the groups of morphisms between two objects need not be sets. Note that this is also the case in [HRS] since the authors work with the bounded derived category $D^b(A)$ of an abelian category, which need not have Hom sets in general (see [CN]). This allows more flexibility for HRS tilting procedure and, increasing the universe if necessary, will pose no set-theoretical problems. This will involve the use of big abelian categories. In the rest of the paper we adopt the convention that the term category means a category with Hom sets. Whenever we allow Hom groups which are not sets we will use the term big category. So the expression ‘is a category’ will mean ‘is a category with Hom sets’.

The following result is the version for big triangulated categories of [Ma2, Proposition 3.1.1 and 3.1.4]. Recall that a t-structure $\tau = (\mathcal{U}, \mathcal{W})$ is left (resp. right) nondegenerate when $\bigcap_{n \in \mathbb{Z}} \mathcal{U}[n] = 0$ (resp. $\bigcap_{n \in \mathbb{Z}} \mathcal{W}[n] = 0$), and it is called nondegenerate when it is left and right nondegenerate. Mattiello’s proof is valid here and proves the result, except for the nondegeneracy of $\tau_t$.

**Proposition 2.5.** Let $D$ any big triangulated category, let $\tau = (\mathcal{U}, \mathcal{W})$ be a nondegenerate t-structure in it and denote by $A$ its heart, which is then a big abelian category, and denote by $H^0_\tau : D \rightarrow A$ be the associated cohomological functor. If $t := (T, F)$ is a torsion pair in $A$, then the pair $\tau_t := (\mathcal{U}_t, \mathcal{W}_t)$ given by the following classes is again a nondegenerate t-structure in $D$:

$$\mathcal{U}_t = \{ X \in D : H^k_\tau(X) = 0, \text{ for } k > 0, \text{ and } H^0_\tau(X) \in T \}$$

$$\mathcal{W}_t = \{ Y \in D : H^k_\tau(Y) = 0, \text{ for } k < -1, \text{ and } H^{-1}_\tau(Y) \in F \}.$$

Moreover, the pair $\bar{t} := (F[1], T)$ is a torsion pair in the heart $H_t$ of $\tau_t$.

**Proof.** (of the nondegeneracy of $\tau_t$) We clearly have that $\bigcap_{n \in \mathbb{Z}} \mathcal{U}_t$ consists of the objects $X$ such that $H^k_\tau(X) = 0$, for all $k \in \mathbb{Z}$. The nondegeneracy of $\tau$ then implies that $X = 0$ (see, e.g., [NSZ] Lemma 3.3, adapted to big triangulated categories). This gives the left nondegeneracy of $\tau_t$ and the right nondegeneracy follows dually.

**Definition 4.** The t-structure $\tau_t$ of last proposition is said to be the HRS-tilt of $\tau$ with respect to $t$. The HRS process in $D$ is a ‘map’ $\Phi_D$ defined on the class of pairs $(\tau, t)$, where $\tau$ is a nondegenerate t-structure in $D$ and $t$ is a torsion pair in the heart $H_\tau$ of $\tau$. It is defined by $\Phi_D(\tau, t) = (\tau_t, \bar{t})$.

The following is now a very natural question.

**Question 2.6.** In the situation of Proposition 2.5, assume that $A$ has Hom sets. When is it true that also $H_t$ has Hom sets?

The following is the answer:
Proposition 2.7. Let $\tau = (\mathcal{U}, W)$ be a nondegenerate $t$-structure in the big triangulated category $\mathcal{D}$ such that its heart $\mathcal{A}$ is a category. Let $\mathbf{t} = (T, F)$ be a torsion pair in $\mathcal{A}$. The following assertions are equivalent:

1. The heart $\mathcal{H}_\mathbf{t}$ of the tilted $t$-structure $\mathbf{t}$ is a category.

2. $\mathcal{D}(T, F[1]) \cong \text{Ext}^1_\mathcal{A}(T, F)$ is a set, for all $T \in \mathcal{T}$ and all $F \in \mathcal{F}$.

In such case, if one puts $\Phi^0_D(\tau, \mathbf{t}) := (\tau_n, \mathbf{t}_n)$, where $\Phi^0_D = \Phi_D \circ \cdots \circ \Phi_D$, then the heart of $\tau_n$ has Hom sets, for each $n > 0$.

Proof. (1) $\implies$ (2) is clear since, by [BBD], we have an isomorphism $\mathcal{H}_\mathbf{t}(T, F[1]) \cong \mathcal{D}(T, F[1])$.

(2) $\implies$ (1) Let $M, N$ be objects of $\mathcal{H}_\mathbf{t}$. We have a triangle $H^{-1}(N)[1] \to N \to H^0(N) \to \tau \in \mathcal{D}$, where, for simplicity, we put $H^k := H_\omega \circ (?[k])$ for each $k \in \mathbb{Z}$. An application of the cohomological functor $\mathcal{D}(M, ?)$ from $\mathcal{D}$ to the category $\text{AB}$ of big abelian groups gives an exact sequence

$$\mathcal{D}(M, H^{-1}(N)[1]) \to \mathcal{D}(M, N) \to \mathcal{D}(M, H^0(N)),$$

Therefore the proof is reduced to check that $\mathcal{D}(M, N)$ is a set when $N = F[1]$, for some $F \in \mathcal{F}$, or $N = T \in \mathcal{T}$, for some $T \in \mathcal{T}$.

Suppose that $N = T \in \mathcal{T}$. Taking the triangle (*) with $M$ instead of $N$ and applying to it the cohomological functor $\mathcal{D}(?, T)$, we obtain an exact sequence

$$\mathcal{D}(H^0(M), T) \to \mathcal{D}(M, T) \to \mathcal{D}(H^{-1}(M)[1], T) = 0.$$

But $\mathcal{D}(H^0(M), T) \cong \mathcal{A}(H^0(M), T)$ is a set by the hypothesis on $\mathcal{A}$. It then follows that $\mathcal{D}(M, T)$ is a set.

Suppose that $N = F[1] \in \mathcal{F}[1]$ and apply $\mathcal{D}(?, F[1])$ to the triangle of the previous paragraph. We get the exact sequence

$$0 = \mathcal{D}(H^{-1}(M)[2], F[1]) \to \mathcal{D}(H^0(M), F[1]) \to \mathcal{D}(M, F[1]) \to \mathcal{D}(H^{-1}(M)[1], F[1]).$$

But we have that $\mathcal{D}(H^{-1}(M)[1], F[1]) \cong \mathcal{D}(H^{-1}(M), F) \cong \mathcal{A}(H^{-1}(M), F)$, which is a set due to the hypothesis on $\mathcal{A}$. Then $\mathcal{D}(M, F[1])$ is a set since, by hypothesis, $\mathcal{D}(H^0(M), F[1])$ is a set.

Next we consider the tilted torsion pair $\mathbf{i} = (\mathcal{F}[1], \mathcal{T})$ in $\mathcal{H}_\mathbf{t}$. Then, by [BBD], we have that

$$\text{Ext}^1_{\mathcal{H}_\mathbf{t}}(F[1], T) \cong \mathcal{D}(F[1], T[1]) \cong \mathcal{D}(F, T) \cong \mathcal{A}(F, T),$$

which is a set due to the hypothesis on $\mathcal{A}$. Replacing now $\mathcal{A}$ by $\mathcal{H}_\mathbf{t}$ in the equivalence of assertions 1 and 2, we get that the tilted $t$-structure $\tau$ with respect to $\mathbf{t}$ has a heart $\mathcal{H}_\tau$ which is a category. The last statement of the proposition is then clear. \qed
As a particular case, when the big triangulated category is \( D = D(\mathcal{A}) \), where \( \mathcal{A} \) is an abelian category, one can consider the canonical t-structure \( \tau = (D^{\leq 0}(\mathcal{A}), D^{\geq 0}(\mathcal{A})) \) as initial one. Its heart is \( \mathcal{A} \) and the functor \( H^k : D(\mathcal{A}) \rightarrow \mathcal{A} \) is the classical \( k \)-th cohomology functor. As a direct consequence of last proposition, we get:

**Corollary 2.8.** Let \( \mathcal{A} \) be an abelian category and let \( t = (\mathcal{T}, \mathcal{F}) \) be a torsion pair in \( \mathcal{A} \). The following assertions are equivalent:

1. The heart \( \mathcal{H}_t \) is a category, i.e. with \( \text{Hom} \) sets.
2. \( \text{Ext}^1_{\mathcal{A}}(T, F) \) is a set (as opposite to a proper class), for all \( T \in \mathcal{T} \) and \( F \in \mathcal{F} \).  

The following is a consequence of last corollary for certain torsion pairs that will appear often in this paper (see Definition 6 below for the definition of the subcategories \( \text{Pres}(V) \) and \( \text{Copres}(Q) \)).

**Corollary 2.9.** In the situation of last corollary, suppose that \( \mathcal{T} = \text{Pres}(V) \) (resp. \( \mathcal{F} = \text{Copres}(Q) \)), for some object \( V \) (resp. \( Q \)) of \( \mathcal{A} \) such that all coproducts (resp. products) of copies of \( V \) (resp. \( Q \)) exist in \( \mathcal{A} \). The heart \( \mathcal{H}_t \) is a category if, and only if, \( \text{Ext}^1_{\mathcal{A}}(V, F) \) (resp. \( \text{Ext}^1_{\mathcal{A}}(T, Q) \)) is a set, for all \( F \in \mathcal{F} \) (resp. \( T \in \mathcal{T} \)).

**Proof.** The statement for \( Q \) is dual of the statement for \( V \), so we just prove the later one. Let’s take any \( T \in \mathcal{T} = \text{Pres}(V) \) and consider an exact sequence \( 0 \rightarrow T' \rightarrow V^{(I)} \rightarrow T \rightarrow 0 \), with \( T' \in \mathcal{T} \), for some set \( I \). If \( F \in \mathcal{F} \) is any object and we apply the contravariant functor \( \mathcal{A}(?, F) \), we obtain the exact sequence

\[
0 = \mathcal{A}(T', F) \rightarrow \text{Ext}^1_{\mathcal{A}}(T, F) \rightarrow \text{Ext}^1_{\mathcal{A}}(V^{(I)}, F),
\]

and an easy adaptation of an argument in the proof of Lemma 3.2 below proves that the canonical map \( \text{Ext}^1_{\mathcal{A}}(V^{(I)}, F) \rightarrow \text{Ext}^1_{\mathcal{A}}(V, F)^I \) is a monomorphism, for each \( X \in \text{Ob}(\mathcal{A}) \). Therefore if \( \text{Ext}^1_{\mathcal{A}}(V, F) \) is a set, then \( \text{Ext}^1_{\mathcal{A}}(T, F) \) is a set, for all \( T \in \mathcal{T} \) and Corollary 2.8 applies. \( \square \)

This allows us a re-interpretation of the HRS process, where the map \( \Phi \) acts instead on pairs \( (\mathcal{A}, t) \), where \( \mathcal{A} \) is an abelian category and \( t = (\mathcal{T}, \mathcal{F}) \) is torsion pair in \( \mathcal{A} \) satisfying the equivalent conditions of last corollary. Concretely:

**Definition 5.** A torsion pair \( t = (\mathcal{T}, \mathcal{F}) \) in an abelian category \( \mathcal{A} \) will be an **adequate torsion pair** when \( \text{Ext}^1_{\mathcal{A}}(T, F) \) is a set, as opposite to a proper class, for all \( T \in \mathcal{T} \) and \( F \in \mathcal{F} \). We will denote by \( (\mathbf{AB}, \text{tor}) \) the class of pairs \( (\mathcal{A}, t) \) consisting of an abelian category \( \mathcal{A} \) and an adequate torsion pair \( t \) in it. The **Happel-Reiten-Smalø (HSR) tilting process** is the map \( \Phi : (\mathbf{AB}, \text{tor}) \rightarrow (\mathbf{AB}, \text{tor}) \) given by \( \Phi[(\mathcal{A}, t)] = (\mathcal{H}_t, \mathcal{t}) \), where \( \mathcal{H}_t \) is the heart of the HRS tilt \( \tau_{t} \) of the canonical t-structure of \( D(\mathcal{A}) \) with respect to \( t \) (see Definition 4).
In particular if, under the hypotheses of last corollary, we assume that \( \text{Ext}^1_A(T, F) \) is a set, for all \( T \in \mathcal{T} \) and all \( F \in \mathcal{F} \), then, by HRS-tilting iteration, one gets the following diagram of abelian categories, all of them with Hom sets, and torsion pairs:

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\mathcal{H}_t} & \mathcal{H}_t \\
\tau & \sim & \tau \\
\end{array}
\]

\[
\tau = (\mathcal{T}, \mathcal{F}) \sim \tau = (\mathcal{F}[1], \mathcal{T}[0]) \sim \tau = (\mathcal{T}[1], \mathcal{F}[1])
\]

The following is [HRS, Proposition 3.2].

**Proposition 2.10.** Let \( \Phi : (\mathcal{AB}, \text{tor}) \to (\mathcal{AB}, \text{tor}) \) be the HRS tilting process map (see Definition 5). Let \( (\mathcal{A}, \mathcal{t}) \) be in \( (\mathcal{AB}, \text{tor}) \) and put \( (\mathcal{B}, \mathcal{t}) := \Phi[(\mathcal{A}, \mathcal{t})] \). The torsion pair \( \mathcal{t} \) is faithful (resp. co-faithful) if, and only if, \( \mathcal{t} \) is co-faithful (resp. faithful). In particular we have induced maps by restriction

\[
\begin{array}{ccc}
\mathcal{AB}, \text{tor}_{\text{faithful}} & \xrightarrow{\Phi} & \mathcal{AB}, \text{tor}_{\text{cofaithful}} \\
\end{array}
\]

denotes the class of adequate faithful (resp. co-faithful) torsion pairs.

Furthermore, we have a triangle functor \( G : \mathcal{D}^b(\mathcal{H}_t) \to \mathcal{D}^b(\mathcal{A}) \) whose restriction to \( \mathcal{H}_t \) is naturally isomorphic to the inclusion functor \( \mathcal{H}_t \to \mathcal{D}^b(\mathcal{A}) \). The functor \( G \) is called the realization functor. The following proposition shows that in some cases the heart \( \mathcal{H}_t \) is equivalent to \( \mathcal{A}[1] \).

**Proposition 2.11.** [HRS, Proposition 3.4] Let \( \mathcal{A} \) be an abelian category and let \( \mathcal{t} = (\mathcal{T}, \mathcal{F}) \) be a torsion pair in \( \mathcal{A} \) such that \( \mathcal{T} \) is a cogenerating class. The following assertions hold:

1. If \( \mathcal{H}_t \) has enough projectives, then \( \mathcal{H}_t \cong \mathcal{A}[1] \) via the realization functor;
2. If \( \mathcal{A} \) has enough injectives, then \( \mathcal{H}_t \cong \mathcal{A}[1] \) via the realization functor.

In particular, whenever \( \mathcal{A} \) is a category with enough projectives or with enough injectives \( \Phi^2[(\mathcal{A}, \mathcal{t})] \cong (\mathcal{A}, \mathcal{t}) \). That is, if \( \Phi^2[(\mathcal{A}, \mathcal{t})] = (\mathcal{A}', \mathcal{t}') \), then there is an equivalence of categories \( F : \mathcal{A} \xrightarrow{\sim} \mathcal{A}' \) which takes \( \mathcal{t} \) to \( \mathcal{t}' \).

## 3 Projective and injective objects in the heart. Quasi-(co)tilting torsion pairs

### 3.1 Quasi-(co)tilting objects and torsion pairs

We need to introduce a few subcategories associated to an object.

**Definition 6.** Let \( \mathcal{A} \) be an abelian category and let \( X \) and \( V \) be objects of \( \mathcal{A} \), where we assume that all (set-indexed) coproducts of copies of \( V \) exist in...
\(A\). We will say that \(X\) is \(V\)-generated (resp. \(V\)-presented) when there is an epimorphism of the form \(V(I) \to X\) (resp. an exact sequence \(V(J) \to V(I) \to X\)) for some set \(I\) (resp. sets \(I\) and \(J\)). We will denote by \(\text{Gen}(V)\) and \(\text{Pres}(V)\) the classes of \(V\)-generated and \(V\)-presented objects, respectively.

When \(Q \in \text{Ob}(A)\) is such that all products of copies of \(Q\) exist in \(A\), we get the dual notions of \(Q\)-cogenerated and \(Q\)-copresented object, and the corresponding subcategories \(\text{Cogen}(Q)\) and \(\text{Copres}(Q)\).

**Definition 7.** Let \(A\) be an abelian category and let \(V\) be an object such that all coproducts of copies of \(V\) exist in \(A\). We will say that an object \(X\) is \(V\)-subgenerated when it is isomorphic to a subobject of an object in \(\text{Gen}(V)\). The class of \(V\)-subgenerated objects will be denoted by \(\text{Gen}(V)\). On the other hand, the class of objects on \(A\) which are isomorphic to direct summands of (resp. finite) coproducts of copies of \(V\) will be denoted by \(\text{Add}(V)\) (resp. \(\text{add}(V)\)).

Dually, when \(Q\) is an object such that all products of copies of \(Q\) exist in \(A\), we define the concept of \(Q\)-subcogenerated object, with the corresponding class \(\text{Cogen}(Q)\), and the category \(\text{Prod}(Q)\) of objects isomorphic to direct summands of products of copies of \(Q\).

**Lemma 3.1.** Let \(A\) be an abelian category and let \(X\) be an object such that all coproducts (resp. products) of copies of \(X\) exist in \(A\). Then all coproducts (resp. products) of objects in \(\text{Pres}(X)\) (resp. \(\text{Copres}(X)\)) exist in \(A\).

**Proof.** The result for \(\text{Copres}(X)\) is dual of the one for \(\text{Pres}(X)\). We just do the latter one. Let \((T_\lambda)_{\lambda \in \Lambda}\) be a family in \(\text{Pres}(X)\), fix an exact sequence \(X(J_\lambda) \xrightarrow{f_\lambda} X(I_\lambda) \to T_\lambda \to 0\), with sets \(J_\lambda\) and \(I_\lambda\), for each \(\lambda \in \Lambda\). We then get an induced exact sequence \(\prod_{\lambda \in \Lambda} X(J_\lambda) \xrightarrow{\prod f_\lambda} \prod_{\lambda \in \Lambda} X(I_\lambda) \to \text{Coker}(\prod f_\lambda) \to 0\). This gives the following commutative diagram of functors \(A \to \text{Ab}\), with exact rows:

\[
\begin{array}{cccc}
0 & \to & A(\text{Coker}(\prod f_\lambda), ?) & \to & A(\prod_{\lambda \in \Lambda} X(I_\lambda), ?) & \to & A(\prod_{\lambda \in \Lambda} X(J_\lambda), ?) \\
& & \downarrow{\alpha} & & \downarrow{\iota} & & \\
0 & \to & \prod_{\lambda \in \Lambda} A(T_\lambda, ?) & \to & \prod_{\lambda \in \Lambda} A(X(I_\lambda), ?) & \to & \prod_{\lambda \in \Lambda} A(X(J_\lambda), ?)
\end{array}
\]

where the left vertical arrow \(\alpha\) exists by the universal property of kernels in \(\text{Ab}\). By definition of coproducts, the two right vertical arrows are isomorphisms, which in turn implies that \(\alpha\) an isomorphism. By Yoneda’s lemma, for each \(\mu \in \Lambda\), the composition \(A(\text{Coker}(\prod f_\lambda), ?) \xrightarrow{\alpha} \prod_{\lambda \in \Lambda} A(T_\lambda, ?) \xrightarrow{\pi_\mu} A(T_\mu, ?)\), where \(\pi_\mu\) is the projection, is of the form \(u_\mu^\lambda = A(u_\mu, ?)\), for an unique morphism \(u_\mu : T_\mu \to \text{Coker}(\prod f_\lambda)\). It immediately follows that \(\text{Coker}(\prod f_\lambda)\) together with the morphisms \((u_\lambda : T_\lambda \to \text{Coker}(\prod f_\lambda))\) is the coproduct of the \(T_\lambda\) in \(A\).

**Lemma 3.2.** Let \(A\) be an abelian category and let \(V\) (resp. \(Q\)) be an object in \(A\) such that all coproducts (resp. products) of copies of \(V\) (resp. \(Q\)) exist
in \( \mathcal{A} \). If \( \text{Gen}(V) \subseteq \text{Ker}(\text{Ext}_{\mathcal{A}}^1(V, ?)) \) (resp. \( \text{Cogen}(Q) \subseteq \text{Ker}(\text{Ext}_{\mathcal{A}}^1(? , Q)) \)), then the class \( \text{Gen}(V) \) (resp. \( \text{Cogen}(Q) \)) is a torsion (resp. torsionfree) class in \( \mathcal{A} \).

**Proof.** We will prove the assertion for \( \text{Gen}(V) \), the one for \( \text{Cogen}(Q) \) following by duality. We start by proving that \( \text{Gen}(V) \) is closed under extensions. Let \( 0 \to T' \to M \to T \to 0 \) be an exact sequence in \( \mathcal{A} \) with \( T, T' \in \text{Gen}(V) \). Now, let \( p : V^{(I)} \to T \) be an epimorphism, for some set \( I \), and consider the following commutative diagram, where the right square is a pullback:

\[
\begin{array}{ccc}
0 & \to & T' \\
\downarrow & & \downarrow \\
0 & \to & T' \\
\downarrow & & \downarrow \\
0 & \to & T \\
\end{array}
\]

We claim that the upper row of the diagram is a split exact sequence. Indeed, the fact that \( \text{Ext}_{\mathcal{A}}^1(V, T') = 0 \) implies that, for each \( j \in I \), the \( j \)-th inclusion \( \iota_j : V \to V^{(I)} \) factors through \( f : M' \to V^{(I)} \), thus yielding a morphism \( g_j : V \to M' \) such that \( f \circ g_j = \iota_j \). When \( j \) varies in \( I \), the universal property of coproducts yields a unique morphism \( g : V^{(I)} \to M' \) such that \( g \circ \iota_j = g_j \), for all \( j \in I \). We then get that \( f \circ g \circ \iota_j = \iota_j \), for all \( j \in I \), which implies that \( f \circ g = 1_{V^{(I)}} \) and settles our claim. Therefore we have that \( M' \cong T' \oplus V^{(I)} \), so that \( M' \in \text{Gen}(V) \) and, consequently, also \( M \in \text{Gen}(V) \).

It is clear that the classes \( T := \text{Gen}(V) \) and \( \mathcal{F} := T^\perp = V^\perp \) satisfy condition 1 of Definition 8. On the other hand, for any \( A \in \text{Ob}(\mathcal{A}) \), we may consider the canonical map \( \epsilon_A : V^{(\mathcal{A}(V, A))} \to A \). This is the unique morphism such that \( \epsilon_A \circ \iota_f = f \), where \( \iota_f : V \to V^{(\mathcal{A}(V, A))} \) is the \( f \)-th injection into the coproduct, for all \( f \in \mathcal{A}(V, A) \). Its image is usually called the trace of \( V \) in \( \mathcal{A} \) and is denoted by \( \text{tr}_V(A) \). We then get an exact sequence \( 0 \to \text{tr}_V(A) \to A \to A/\text{tr}_V(A) \to 0 \). We clearly have that \( \text{tr}_V(A) \in \text{Gen}(V) \). Moreover, we get an induced exact sequence of abelian groups \( 0 \to \mathcal{A}(V, \text{tr}_V(A)) \cong A(V, A) \to \mathcal{A}(V, A/\text{tr}_V(A)) \to \text{Ext}_{\mathcal{A}}^1(V, \text{tr}_V(A)) = 0 \). It then follows that \( A/\text{tr}_V(A) \in V^\perp = \mathcal{F} \), and so \( (T, \mathcal{F}) = (\text{Gen}(V), V^\perp) \) is a torsion pair in \( \mathcal{A} \). □

We are ready to introduce some types of objects which have special importance in the study of the heart of a t-structure. They are generalizations of corresponding notions in module categories.

**Definition 8.** Let \( \mathcal{A} \) be an abelian category. An object \( V \) (resp. \( Q \)) of \( \mathcal{A} \) will be called **quasi-tilting** (resp. **quasi-cotilting**) when all coproducts (resp. products) of copies of \( V \) (resp. \( Q \)) exist in \( \mathcal{A} \) and \( \text{Gen}(V) = \text{Gen}(V) \cap \text{Ker}(\text{Ext}_{\mathcal{A}}^1(V, ?)) \). (resp. \( \text{Cogen}(Q) = \text{Cogen}(Q) \cap \text{Ker}(\text{Ext}_{\mathcal{A}}^1(? , Q)) \)). The corresponding torsion pair \( \mathbf{t} = (\text{Gen}(V), V^\perp) \) (resp. \( \mathbf{t} = (\perp Q, \text{Cogen}(Q)) \)) (see Lemma 9.2) is called the **quasi-tilting** (resp. **quasi-cotilting**) torsion pair associated to \( V \) (resp. \( Q \)).

When, for such a \( V \) (resp. \( Q \)), one has \( \text{Gen}(V) = \text{Ker}(\text{Ext}_{\mathcal{A}}^1(V, ?)) \) (resp. \( \text{Cogen}(Q) = \text{Ker}(\text{Ext}_{\mathcal{A}}^1(? , Q)) \)) and this class is cogenerating (resp. generating) in \( \mathcal{A} \), we will say that \( V \) (resp \( Q \)) is a **1-tilting** (resp. **1-cotilting**) object. The
The corresponding torsion pair is called the tilting (resp. cotilting) torsion pair associated to \( V \) (resp. \( Q \)).

The proof of the following goes as in module categories (see [CDT, Proposition 2.1]).

**Corollary 3.3.** If \( \mathcal{A} \) is an abelian category and \( V \) (resp. \( Q \)) is a quasi-tilting (resp. quasi-cotilting) object of \( \mathcal{A} \), then \( \text{Gen}(V) = \text{Pres}(V) \) (resp. \( \text{Cogen}(Q) = \text{Copres}(Q) \)).

The natural question of when a quasi-tilting (resp. quasi-cotilting) torsion pair has a heart that is a category, i.e. has Hom sets, has a clear answer:

**Corollary 3.4.** Let \( V \) (resp. \( Q \)) be a quasi-tilting (resp. quasi-cotilting) object of the abelian category \( \mathcal{A} \), and let \( t = (T, F) \) the associated torsion pair in \( \mathcal{A} \).

The following assertions hold:

1. The heart \( \mathcal{H}_t \) is a category (i.e. has Hom sets) if, and only if, \( \text{Ext}^1_{\mathcal{A}}(V, F) \) (resp. \( \text{Ext}^1_{\mathcal{A}}(T, Q) \)) is a set, for all \( F \in \mathcal{F} \) (resp. \( T \in \mathcal{T} \)).

2. If \( V \) (resp. \( Q \)) is a 1-tilting (resp. 1-cotilting) object, then \( \text{Ext}^2_{\mathcal{A}}(V, ?) = 0 \) (resp. \( \text{Ext}^2_{\mathcal{A}}(?, Q) = 0 \)). One says that the projective (resp. injective) dimension of \( V \) (resp. \( Q \)) is less or equal than 1.

3. If \( V \) (resp. \( Q \)) is a 1-tilting (resp. 1-cotilting) object, then \( \mathcal{H}_t \) is a category, i.e. it has Hom set.

**Proof.** (1) It is a direct consequence of Corollaries 2.9 and 3.3.

(2) We just do the proof for \( V \), the one for \( Q \) being dual. Let \( 0 \to M \to X \xrightarrow{f} Y \to V \to 0 \) be an exact sequence in \( \mathcal{A} \), representing an element \( \epsilon \in \text{Ext}^2_{\mathcal{A}}(V, M) \). Since \( \mathcal{T} \) is a cogenerating class, we can fix a monomorphism \( \mu : X \to T \), with \( T \in \mathcal{T} \). By taking the pushout of \( \mu \) and \( f \) we immediately get an exact sequence \( 0 \to M \to T \xrightarrow{g} T' \to V \to 0 \), where \( T, T' \in \mathcal{T} \), which also represents \( \epsilon \). But then \( \epsilon = 0 \) since \( \text{Im}(g) \in \mathcal{T} = \text{Ker}(\text{Ext}^1_{\mathcal{A}}(V, ?)) \).

(3) Let \( F \in \mathcal{F} \) be any object. Using the cogenerating condition of \( \mathcal{T} \), we take an exact sequence \( 0 \to F \to T_0 \to T_1 \to 0 \), where \( T_0, T_1 \in \mathcal{T} \). We then get an exact sequence of (in principle big) abelian groups

\[
\mathcal{A}(V, T_1) \to \text{Ext}^1_{\mathcal{A}}(V, F) \to \text{Ext}^1_{\mathcal{A}}(V, T_0) = 0.
\]

It then follows that \( \text{Ext}^1_{\mathcal{A}}(V, F) \) is a set, which, by Corollary 2.8 implies that \( \mathcal{H}_t \) has Hom sets.

3.2 When does the heart of a co-faithful (resp. faithful) torsion pair have a projective generator (resp. injective cogenerator)?

To answer the question of the title of this subsection we need a few preliminary results.
Lemma 3.5. Let \( D \) be a big triangulated category and \( \tau = (U, W) \) be a t-structure in \( D \) whose heart \( A := U \cap W \) is a category, i.e. it has Hom sets. Let \( t = (T, F) \) be a torsion pair in \( A \) such that \( \text{Ext}^1_{\mathcal{A}}(T, F) \cong D(T, F[1]) \) is a set (as opposite to a proper class), for all \( T \in T \) and all \( F \in F \) (see Proposition 2.7). The following assertions hold, where \( \mathcal{H}_t \) denotes the heart of the tilted t-structure \( \tau_t \):

1. The functor \( (H_0^0)|_{\mathcal{H}_t} : \mathcal{H}_t \to \mathcal{A} \) is left adjoint of the functor \( A \to \mathcal{H}_t \) taking \( A \to t(A) \), where \( t : \mathcal{A} \to T \) is the torsion radical associated to \( t \). In particular \((H_0^0)|_{\mathcal{H}_t} : \mathcal{H}_t \to \mathcal{A} \) preserves all colimits that exist in \( \mathcal{H}_t \).

2. The functor \((H_1^{-1})|_{\mathcal{H}_t} : \mathcal{H}_t \to \mathcal{A} \) is right adjoint of the functor \( A \to \mathcal{H}_t \) taking \( A \to (1 : t)(A)[1] \). In particular \((H_1^{-1})|_{\mathcal{H}_t} : \mathcal{H}_t \to \mathcal{A} \) preserves all limits that exist in \( \mathcal{H}_t \).

Proof. We just prove assertion 1 since assertion 2 follows by duality. We know that \((\mathcal{F}[1], \mathcal{T})\) is a torsion pair in \( \mathcal{H}_t \). The associated torsion radical \( t \) and coradical \((1 : \bar{t})\) act on objects as \( M \to t(M) = H_1^{-1}(M)[1] \) and \( M \to (1 : \bar{t})(M) = H_0^0(M) \), respectively. We can then decompose \((H_0^0)|_{\mathcal{H}_t} : \mathcal{H}_t \to \mathcal{A} \) as the composition \( \mathcal{H}_t \xrightarrow{(1 : \bar{t})} \mathcal{T} \xrightarrow{t} \mathcal{A} \), where the right arrow is the inclusion functor. Each of the two functors in this composition has a right adjoint, which implies that \( H|_{\mathcal{H}_t} : \mathcal{H}_t \to \mathcal{A} \) has a right adjoint which is the composition \( \mathcal{A} \xrightarrow{t} \mathcal{T} \to \mathcal{H}_t \) \( \Box \).

The importance of quasi-(co)tilting objects in the study of hearts of HRS t-structures stems from the following fact:

Proposition 3.6. Let \( A \) be an abelian category and let \( t = (T, F) \) be a torsion pair in \( A \). If \( \mathcal{H}_t \) is an AB3 (resp. AB3*) abelian category with a projective generator (resp. injective cogenerator) \( P \) (resp. \( E \)), then \( H^0(P) \) (resp. \( H^{-1}(E) \)) is a quasi-tilting (resp. quasi-cotilting) object and \( t \) is the associated quasi-tilting (resp. quasi-cotilting) torsion pair.

Proof. The statement for the injective cogenerator is dual to the one for projective generator. We just do the last one. Let \( P \) be as above and let \( P^{(I)} \) denote the coproduct of \( I \) copies of it in \( \mathcal{H}_t \). We warn that it might not coincide with the corresponding coproduct in \( D(A) \), if this one exists. By Lemma 3.5 we have an isomorphism \( H^0(P^{(I)}) \cong H^0(P)^{(I)} = V^{(I)} \) in \( \mathcal{A} \), so that all coproducts of copies of \( V \) exist in \( \mathcal{A} \).

If \( T \in \mathcal{T} \) is any object, then, due to the fact that \( P \) is a projective generator of \( \mathcal{H}_t \), we have an exact sequence \( P^{(I)} \to P^{(J)} \to T[0] \to 0 \) in \( \mathcal{H}_t \). By last paragraph, we have an exact sequence \( H^0(P)^{(I)} \to H^0(P)^{(J)} \to H^0(T[0]) = T \to 0 \) in \( \mathcal{A} \). We then get that \( \mathcal{T} \subseteq \text{Pres}(V) \), the converse inclusion being obvious. So we have that \( T = \text{Gen}(V) = \text{Pres}(V) \).

Moreover, if we consider the short exact sequence \( 0 \to H^{-1}(P)[1] \to P \to V[0] \to 0 \) in \( \mathcal{H}_t \) and apply to it the long exact sequence of \( \text{Ext}^1_{\mathcal{H}_t}(?, T[0]) \), we
get an exact sequence

\[ 0 = \mathcal{H}_t(H^{-1}(P)[1], T[0]) \to \text{Ext}^1_{\mathcal{H}_t}(V[0], T[0]) \to \text{Ext}^1_{\mathcal{H}_t}(P, T[0]) = 0, \]

from which we get that \( \text{Ext}^1_{\mathcal{H}_t}(V, T[0]) \to \text{Ext}^1_{\mathcal{H}_t}(P, T[0]) = 0 \), and so \( T = \text{Gen}(V) \subseteq \text{Gen}(V) \cap \ker(\text{Ext}^1_{\mathcal{H}_t}(V, ?)) \).

For the reverse inclusion, given \( M \in \text{Gen}(V) \cap V^\perp \), there exist \( T_1, T_2 \in T \) and an exact sequence in \( \mathcal{A} \) as follows:

\[ 0 \to M \to T_1 \to T_2 \to 0. \]

Since \( \text{Pres}(V) = \text{Gen}(V) = T \), we can take an epimorphism \( q : V^{(\alpha)} \to T_2 \) whose kernel belongs to \( T \). Consider the following pullback diagram

\[ \begin{array}{ccc}
0 & \to & M \\
\downarrow & & \downarrow \\
Z & \to & V^{(\alpha)} \\
\downarrow & p.b. & \downarrow q \\
0 & \to & T_1 \\
0 & \to & T_2 \\
\end{array} \]

Notice that \( Z \) is an extension of \( T_1 \) and the kernel of \( q \), so that \( Z \in T \). Taking into account that \( M \in V^\perp = \ker(\text{Ext}^1_{\mathcal{A}}(V, ?)) = \ker(\text{Ext}^1_{\mathcal{A}}(V(I), ?)) \), for each set \( I \neq \emptyset \), we get that the first row in the above diagram splits, so that \( M \in T \).

A first lesson of last proposition is that, in order to identify torsion pairs whose associated heart is a Grothendieck category, one can restrict to the quasi-cotilting ones. The proposition also helps in the following answer to the title of the subsection:

**Theorem 3.7.** Let \( \mathcal{A} \) be an abelian category and let \( \mathfrak{t} = (T, F) \) be a torsion pair. The following assertions are equivalent:

1. \( \mathfrak{t} \) is a tilting torsion pair.

2. \( \mathfrak{t} \) is a co-faithful torsion pair whose heart \( \mathcal{H}_t \) is an AB3 abelian category with a projective generator.

3. \( \mathcal{H}_t \) is an AB3 abelian category with a projective generator and \( \mathfrak{t} = (F[1], T[0]) \) is a faithful torsion pair in \( \mathcal{H}_t \).

In this case, \( V \) is 1-tilting object such that \( T = \text{Gen}(V) \) if, and only if, \( V[0] \) is a projective generator of \( \mathcal{H}_t \). Moreover, an object \( P \) of \( \mathcal{H}_t \) is a projective generator of this latter category if, and only if, it is isomorphic to \( V[0] \) for some 1-tilting object \( V \) of \( \mathcal{A} \) such that \( T = \text{Gen}(V) \).

**Proof.** Note that in any of assertions (1)-(3) the class \( T \) is cogenerating in \( \mathcal{A} \). This is clear in assertions (1) and (2), and for assertion (3) it follows from Proposition 2.10.
(2) \iff (3) is a consequence of this last mentioned proposition (= Proposition 3.2).

(1) \implies (2) Let \( V \) be a 1-tilting object of \( \mathcal{A} \) such that \( \mathbf{t} = (\text{Gen}(V), V^\perp) \). We start by proving that \( V[0] \) is a projective object of \( \mathcal{H}_\mathbf{t} \), i.e. that \( \text{Ext}^1_{\mathcal{H}_\mathbf{t}}(V[0],M) = 0 \), for all \( M \in \mathcal{H}_\mathbf{t} \). But, taking into account the associated exact sequence \( 0 \to H^{-1}(M)[1] \to M \to H^0(M)[0] \to 0 \), the task reduces to the case when \( M \in \mathcal{T}[0] \cup \mathcal{F}[1] \). If \( M = \mathcal{T}[0] \), with \( T \in \mathcal{T} = \text{Ker} (\text{Ext}^1_{\mathcal{A}}(V,?)) \), then we have \( \text{Ext}^1_{\mathcal{H}_\mathbf{t}}(V[0],T[0]) \cong \text{Ext}^1_{\mathcal{A}}(V,T) \equal 0 \). On the other hand, if \( F \in \mathcal{F} \) we have \( \text{Ext}^1_{\mathcal{H}_\mathbf{t}}(V[0],F[1]) \cong \text{Ext}^2_{\mathcal{A}}(V,F) = 0 \) (see [BBD] Remarque 3.1.17 and Corollary 3.2). Note that what we have done with \( V \) can be done with \( V^{(I)} \), for any set \( I \neq \emptyset \). That is, the argument also proves that \( V^{(I)}[0] \) is a projective object of \( \mathcal{H}_\mathbf{t} \), for all sets \( I \neq \emptyset \), and hence for all sets \( I \).

Lemma 3.8 below says now that the stalk complex \( V^{(I)}[0] \) is the coproduct in \( \mathcal{H}_\mathbf{t} \) of \( I \) copies of \( V[0] \). Moreover \( \mathcal{T}[0] \) is a generating class in \( \mathcal{H}_\mathbf{t} \) since \( \mathbf{t} = (\mathcal{F}[1], \mathcal{T}[0]) \) is a faithful torsion pair due to Proposition 2.10. By the equality \( \mathcal{T} = \text{Pres}_{\mathcal{A}}(V) \), we then get that \( \mathcal{T}[0] \subseteq \text{Gen}_{\mathcal{H}_\mathbf{t}}(V[0]) \), from which one immediately gets that \( \mathcal{H}_\mathbf{t} = \text{Gen}_{\mathcal{H}_\mathbf{t}}(V[0]) = \text{Pres}_{\mathcal{H}_\mathbf{t}}(V[0]) \). Applying now Lemma 3.1 we conclude that arbitrary coproducts exist in \( \mathcal{H}_\mathbf{t} \), so that this is an AB3 abelian category, with \( V[0] \) as a projective generator.

(2) = (3) \implies (1) By Proposition 3.6 we know that \( \mathbf{t} \) is quasi-tilting torsion pair. Let \( V \) be a quasi-tilting object such that \( \mathcal{T} = \text{Gen}(V) \). Since \( \mathbf{t} \) is cofaithful, i.e. \( \mathcal{T} \) is a cogenerating class in \( \mathcal{A} \) we get that \( \text{Gen}(V) = \mathcal{A} \), which then implies that \( \text{Gen}(V) \cap \text{Ker}(\text{Ext}^1_{\mathcal{A}}(V,?)) = \text{Ker}(\text{Ext}^1_{\mathcal{A}}(V,?)) \), so that \( \text{Gen}(V) = \text{Ker}(\text{Ext}^1_{\mathcal{A}}(V,?)) \) and, hence, \( V \) is a 1-tilting object.

For the final statement, the proof of implication (1) \implies (2) shows that if \( V \) is a 1-tilting object of \( \mathcal{A} \) defining \( \mathbf{t} \), then \( V[0] \) is a projective generator of \( \mathcal{H}_\mathbf{t} \). It remains to prove that if \( P \) is projective generator of \( \mathcal{H}_\mathbf{t} \) then \( P \cong V[0] \) for such a 1-tilting object. The proof of Proposition 3.6 and that of implication (2) \implies (1) show that \( V := H^0(P) \) is a 1-tilting object of \( \mathcal{A} \) defining \( \mathbf{t} \). Then, by implication (1) \implies (2), we also know that \( V[0] \) is a projective generator of \( \mathcal{H}_\mathbf{t} \). It then follows that \( P \) is a direct summand of the coproduct in \( \mathcal{H}_\mathbf{t} \) of \( I \) copies of \( V[0] \), for some set \( I \). By Lemma 3.5 below, we then get that \( P \) is a direct summand of \( V^{(I)}[0] \), which implies that \( H^{-1}(P) = 0 \) and hence that \( P \cong V[0] \).

Lemma 3.8. Let \( \mathcal{A} \) be an abelian category, let \( V \) be a 1-tilting object, let \( \mathbf{t} = (\text{Gen}(V), V^\perp) \) the associated torsion pair in \( \mathcal{A} \) and let \( \mathcal{H}_\mathbf{t} \) be the heart of the associated HRS t-structure in \( \mathcal{D}(\mathcal{A}) \). For each set \( I \), the coproduct of \( I \) copies of \( V[0] \) exists in \( \mathcal{H}_\mathbf{t} \) and it is precisely the stalk complex \( V^{(I)}[0] \).

Proof. Let \( \iota_j : V \to V^{(I)} \) denote the \( j \)-th injection into the coproduct in \( \mathcal{A} \), for each \( j \in I \). For each \( N \in \mathcal{H}_\mathbf{t} \) we have an induced morphism \( \gamma_N : \mathcal{H}_\mathbf{t}(V^{(I)}[0], N) \to \mathcal{H}_\mathbf{t}(V[0], N)^I \), which is the unique morphism of abelian groups such that \( \pi_j \circ \gamma_N = (\iota_j[0])^*(N) = \mathcal{H}_\mathbf{t}(\iota_j[0], N) : \mathcal{H}_\mathbf{t}(V^{(I)}[0], N) \to \mathcal{H}_\mathbf{t}(V[0], N) \), for all \( j \in I \), where \( \pi_j : \mathcal{H}_\mathbf{t}(V[0], N)^I \to \mathcal{H}_\mathbf{t}(V[0], N) \) is the \( j \)-the projection.
Our task reduces to prove that $\gamma_N$ is an isomorphism, for all $N \in \mathcal{H}_t$. To do that we consider the exact sequence $0 \to H^{-1}(N)[1] \to N \to H^0(N)[0] \to 0$ in $\mathcal{H}_t$. Note that, by the first paragraph of the proof of last proposition, we know that $V^{(I)}[0]$ is projective in $\mathcal{H}_t$, for all sets $I$. This gives the following commutative diagram with exact rows:

$$
\begin{array}{cccc}
0 & \to & \mathcal{H}_t(V^{(I)}[0], H^{-1}(N)[1]) & \to & \mathcal{H}_t(V^{(I)}[0], N) & \to & \mathcal{H}_t(V^{(I)}[0], H^0(N)[0]) & \to & 0 \\
\gamma_{H^{-1}(N)[1]} & & \gamma_N & & \gamma_{H^0(N)[0]} & & \\
0 & \to & \mathcal{H}_t(V[0], H^{-1}(N)[1]) & \to & \mathcal{H}_t(V[0], N) & \to & \mathcal{H}_t(V[0], H^0(N)[0]) & \to & 0
\end{array}
$$

$\gamma_{H^0(N)[0]}$ is clearly an isomorphism since it can be identified with the canonical map $\mathcal{A}(V^{(I)}, H^0(N)) \to \mathcal{A}(V, H^0(N))$, which is an isomorphism by definition of the coproduct $V^{(I)}$ in $\mathcal{A}$. The task is further reduced to prove that $\gamma_{H^{-1}(N)[1]}$ is an isomorphism. But this latter map gets identified with the canonical morphism $\gamma_p' : \text{Ext}^1_{\mathcal{A}}(V^{(I)}, F) \to \text{Ext}^1_{\mathcal{A}}(V, F)$, where $F := H^{-1}(M)$. We just need to prove that $\gamma_p'$ is an isomorphism, for all $F \in \mathcal{F}$. For this we use the cogenerating condition of $\mathcal{T} = \text{Gen}(V)$ and, given $F \in \mathcal{F}$, we fix an exact sequence $0 \to F \to T \to T' \to 0$, with $T, T' \in \mathcal{T}$. Bearing in mind that $\text{Ext}^1_{\mathcal{A}}(V^{(I)}, F)|_{\mathcal{T}} = 0$, for all sets $J$, we get the following commutative diagram with exact rows, where the two left vertical arrows are the canonical isomorphisms induced by definition of the coproduct $V^{(I)}$ in $\mathcal{A}$:

$$
\begin{array}{cccc}
\mathcal{A}(V^{(I)}, T) & \to & \mathcal{A}(V^{(I)}, T') & \to & \text{Ext}^1_{\mathcal{A}}(V^{(I)}, F) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{A}(V, T) & \to & \mathcal{A}(V, T') & \to & \text{Ext}^1_{\mathcal{A}}(V, F) & \to & 0
\end{array}
$$

It follows that $\gamma_p'$ is also an isomorphism as desired. □

Due to its importance, it is worth stating explicitly the dual of Theorem 3.7.

**Theorem 3.9.** Let $\mathcal{A}$ be an abelian category and let $t = (\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$. The following assertions are equivalent:

1. $t$ is a cotilting torsion pair.
2. $t$ is a faithful torsion pair whose heart $\mathcal{H}_t$ is an $AB3^*$ abelian category with an injective cogenerator.
3. $\mathcal{H}_t$ is an $AB3^*$ abelian category with an injective cogenerator and $\tilde{t} = (\mathcal{F}[1], \mathcal{F}[0])$ is a co-faithful torsion pair in $\mathcal{H}_t$.

In this case $Q$ is a 1-cotilting object such that $\mathcal{F} = \text{Cogen}(Q)$ if, and only if, $Q[1]$ is an injective cogenerator of $\mathcal{H}_t$. Moreover, an object $E$ of $\mathcal{H}_t$ is an injective cogenerator of this category if, and only if, $E \cong Q[1]$ for some 1-cotilting object of $\mathcal{A}$ defining $t$. 
We have now the following sort of reverse consequence:

**Corollary 3.10.** Let \( \mathcal{A} \) be an abelian category and let \( \mathfrak{t} = (\mathcal{T}, \mathcal{F}) \) be a torsion pair in \( \mathcal{A} \). The following assertions hold:

1. \( \mathcal{A} \) is AB3 with a projective generator and \( \mathfrak{t} \) is a faithful torsion pair in \( \mathcal{A} \) if, and only if, \( \overline{\mathfrak{t}} = (\mathcal{F}[1], \mathcal{T}[0]) \) is a tilting torsion pair in \( \mathcal{H}_\mathfrak{t} \). In such case, \( P \) is a projective generator of \( \mathcal{A} \) if and only if \( P[1] \) is a 1-tilting object of \( \mathcal{H}_\mathfrak{t} \) such that \( \mathcal{F}[1] = \text{Gen}_{\mathcal{H}_\mathfrak{t}}(P[1]) \).

2. \( \mathcal{A} \) is AB3* with an injective cogenerator and \( \mathfrak{t} \) is a co-faithful torsion pair in \( \mathcal{A} \) if, and only if, \( \overline{\mathfrak{t}} = (\mathcal{F}[1], \mathcal{T}[0]) \) is a cotilting torsion pair in \( \mathcal{H}_\mathfrak{t} \). In such case, \( E \) is an injective cogenerator of \( \mathcal{A} \) if and only if \( E[0] \) is a 1-cotilting object of \( \mathcal{H}_\mathfrak{t} \) such that \( \mathcal{T}[0] = \text{Cogen}_{\mathcal{H}_\mathfrak{t}}(E[0]) \).

**Proof.** Obviously, each assertion is obtained from the other one by duality. We just prove assertion 2. By [HRS, Proposition 3.2] we know that \( \mathcal{T}[0] \) is a generating class in \( \mathcal{H}_\mathfrak{t} \), and, by [HRS, Proposition I.3.4] and using the terminology of that article, we have that \( \Phi[(\mathcal{H}_\mathfrak{t}, \mathfrak{t})] \) is equivalent to \( (\mathcal{A}, \mathfrak{t}) \), in fact it is equal to \( (\mathcal{A}[1], \mathfrak{t}[1]) \). Moreover, by [HRS, Theorem 3.3] we even know that \( \mathcal{D}^b(\mathcal{A}) \) and \( \mathcal{D}^b(\mathcal{H}_\mathfrak{t}) \) are equivalent triangulated categories. This allows us to apply Theorem 3.9, replacing \( \mathcal{A} \) by \( \mathcal{H}_\mathfrak{t} \) and \( \mathfrak{t} \) by \( \overline{\mathfrak{t}} \) in that theorem, to conclude that \( \overline{\mathfrak{t}} \) is a cotilting torsion pair in \( \mathcal{H}_\mathfrak{t} \). The last statement is also a consequence of Theorem 3.9. \( \square \)

### 3.3 Hearts that are module categories

In order to study those hearts which are module categories, we need the following concepts:

**Definition 9.** Let \( \mathcal{A} \) be an abelian category and \( \mathcal{T}_0 \) be a set of objects such that arbitrary coproducts of objects of \( \mathcal{T}_0 \) exist in \( \mathcal{A} \). We shall say that \( \mathcal{T}_0 \) is:

1. a 1-tilting set when \( \prod_{T \in \mathcal{T}_0} T \) is a 1-tilting object;
2. a self-small set when, for each \( T \in \mathcal{T}_0 \) and each family \( (T_\lambda)_{\lambda \in \Lambda} \) in \( \mathcal{T}_0 \), the canonical map \( \prod_{\lambda \in \Lambda} \mathcal{A}(T, T_\lambda) \to \mathcal{A}(T, \prod_{\lambda \in \Lambda} T_\lambda) \) is an isomorphism.
3. a classical 1-tilting set when it is 1-tilting and self-small.

When \( \mathcal{T}_0 = \{T\} \) we say that \( T \) is, respectively, a 1-tilting, a self-small and a classical 1-tilting object.

The following is the version that we will need of a theorem of Gabriel and Mitchell:

**Proposition 3.11.** Let \( \mathcal{A} \) be any category. The following assertions are equivalent:

1. \( \mathcal{A} \) is equivalent to \( \text{Mod} - \mathcal{B} \) (resp. \( \text{Mod} - R \)), for some small pre-additive category \( \mathcal{B} \) (resp. some ring \( R \));
2. \( \mathcal{A} \) is an AB3 abelian category that admits a self-small set of projective generators (resp. a self-small projective generator).

Proof. The equivalence for \( \text{Mod} - R \) is a particular case of the one for \( \text{Mod} - \mathcal{B} \), for \( \mathcal{B} \) a small pre-additive category, since a ring is the same as a pre-additive category with just one object. The classical version of Gabriel-Mitchell theorem states that assertion 1 holds if, and only if, \( \mathcal{A} \) is AB3 and has a set of small(=compact) projective generators (see, e.g., [Po, Corollary 3.6.4]). We just need to check that in any AB3 abelian category, if \( \mathcal{P}_0 \) is a self-small set of projective generators, then \( \mathcal{P}_0 \) consists of small objects. Indeed, let \( (\lambda \lambda)_{\lambda \in \Lambda} \) be any family of objects in \( \mathcal{A} \). For each \( \lambda \in \Lambda \), we then have an exact sequence

\[
\prod_{P \in \mathcal{P}_0} P(I_{P, \lambda}) \xrightarrow{J_{\lambda}} \prod_{P \in \mathcal{P}_0} P(I_{P, \lambda}) \xrightarrow{P_{\lambda}} A_\lambda \rightarrow 0 \quad \text{in} \quad \mathcal{A}.
\]

Due to right exactness of coproducts, we then get an exact sequence

\[
\prod_{\lambda \in \Lambda} \prod_{P \in \mathcal{P}_0} P(I_{P, \lambda}) \xrightarrow{\bigoplus_{\lambda \in \Lambda} J_{\lambda}} \prod_{\lambda \in \Lambda} \prod_{P \in \mathcal{P}_0} P(I_{P, \lambda}) \xrightarrow{\bigoplus_{\lambda \in \Lambda} P_{\lambda}} \prod_{\lambda \in \Lambda} A_\lambda \rightarrow 0.
\]

If now \( P' \in \mathcal{P}_0 \) is arbitrary and we apply \( \mathcal{A}(P', ?) \) to the last exact sequence, using the projectivity of \( P' \) and the self-smallness of \( \mathcal{P}_0 \) we readily get that the canonical map \( \prod_{\lambda \in \Lambda} \mathcal{A}(P', A_\lambda) \rightarrow \mathcal{A}(P', \prod_{\lambda \in \Lambda} A_\lambda) \) is an isomorphism, so that \( P' \) is small (=compact) in \( \mathcal{A} \).

\[\square\]

Corollary 3.12. Let \( \mathcal{A} \) be an abelian category and let \( t = (T, \mathcal{F}) \) be a torsion pair in \( \mathcal{A} \). The following assertions are equivalent:

1. There is a classical 1-tilting set \( T_0 \) (resp. a classical 1-tilting object \( V \)) such that \( T = \text{Gen}(T_0) \) (resp. \( T = \text{Gen}(V) \)).

2. \( t \) is a co-faithful torsion pair whose heart \( \mathcal{H}_t \) is equivalent to the module category over a small pre-additive category (resp. over a ring).

3. \( \mathcal{H}_t \) is equivalent to the module category over a small pre-additive category (resp. over a ring) and \( \bar{t} = (\mathcal{F}[1], T[0]) \) is a faithful torsion pair in \( \mathcal{H}_t \).

Proof. (2) \( \iff \) (3) is a consequence of Proposition 2.10.

(1) \( \implies \) (2) Since \( V := \prod_{T \in T_0} T \) is a 1-tilting object it follows from Theorem 3.7 that \( V[0] \) is a projective generator of \( \mathcal{H}_t \), which in turns implies that \( T_0[0] = \{T[0] : T \in T_0\} \) is a set of projective generators of \( \mathcal{H}_t \). An easy adaptation of the proof of Lemma 3.8 shows that if \( (\lambda \lambda)_{\lambda \in \Lambda} \) is a family in \( T_0 \), then the coproduct of the \( T_\lambda[0] \) in \( \mathcal{H}_t \) exists and is the stalk complex \( (\prod_{\lambda \in \Lambda} T_\lambda)[0] \). Given now \( T \in T_0 \) arbitrary, then we have a sequence of isomorphisms:

\[
\prod_{\lambda \in \Lambda} \mathcal{H}_t(T[0], T_\lambda[0]) \cong \prod_{\lambda \in \Lambda} A(T, T_\lambda) \xrightarrow{\text{canonical}} A(T, \prod_{\lambda \in \Lambda} T_\lambda) \cong \mathcal{H}_t(T[0], \prod_{\lambda \in \Lambda} T_\lambda[0]),
\]

whose composition is easily identified with the canonical morphism \( \prod_{\lambda \in \Lambda} \mathcal{H}_t(T[0], T_\lambda[0]) \rightarrow \mathcal{H}_t(T[0], \prod_{\lambda \in \Lambda} T_\lambda[0]) \). This morphism is then an isomorphism, which implies
that $T_0[0]$ is a self-small set of projective generators of $H_t$. By Proposition 3.11 we conclude that $H_t \cong \text{Mod} - A$, for some small pre-additive category.

(2) = (3) $\Rightarrow$ (1) Since, due to the co-faithful condition of $t$ and Proposition 2.10, $T[0]$ is a generating class in $H_t$ we get that any projective object of $H_t$ is in $T[0]$. Then any self-small set of projective generators of $H_t$ is of the form $T[0]$, for some set $T_0 \subset T$. By Theorem 3.7 and its proof, we get that $t$ is the tilting torsion pair defined by the 1-tilting object $\hat{T} := \coprod_{T \in T_0} T$. It just remains to check that $T_0$ is a self-small set. But this is a direct consequence of the self-smallness of $T_0[0]$ since we have an equivalence of categories $T \cong T[0]$ and coproducts in $T$ and $T[0]$ are calculated as in $A$ and $H_t$, respectively. □

3.4 Bijections induced by the HRS tilting process

The previous results and the HRS tilting process give rise to a nice series of bijections. We continue with the terminology of Definition 5 and Proposition 2.10.

Corollary 3.13. Let $\Phi : (AB, \text{tor}) \rightarrow (AB, \text{tor})$ be the map induced by the HRS tilting process (see Definition 5). By restriction, $\Phi$ defines bijections, which are inverse of themselves (i.e. $(\Phi \circ \Phi)|_C = 1_C$, for $C$ any subclass in the list):

1. Between $(AB, \text{tor}_{\text{tilt}})$ and $(AB^3_{\text{proj}}, \text{tor}_{\text{faithful}})$, where $\text{tor}_{\text{tilt}}$ and $\text{tor}_{\text{faithful}}$ denote the subclasses of $\text{tor}$ consisting of the tilting and the faithful torsion pairs, respectively, and $AB^3_{\text{proj}}$ denotes the class of $AB^3$ abelian categories with a projective generator;

2. Between $(AB, \text{tor}_{\text{cotilt}})$ and $(AB^3_{\text{inj}}, \text{tor}_{\text{cofaithful}})$, where $\text{tor}_{\text{cotilt}}$ and $\text{tor}_{\text{cofaithful}}$ denote the subclasses of $\text{tor}$ consisting of the cotilting and the co-faithful torsion pairs, respectively, and $AB^3_{\text{inj}}$ denotes the class of $AB^3*$ abelian categories with an injective cogenerator;

3. Between $(AB, \text{tor}_{\text{stilt - class}})$ and $(\text{Mod}_{\text{paddt}}, \text{tor}_{\text{faithful}})$, where $\text{tor}_{\text{stilt - class}}$ denotes the subclass of $\text{tor}_{\text{tilt}}$ consisting of those torsion pairs associated to a classical tilting set of objects and $\text{Mod}_{\text{paddt}}$ is the class of abelian categories which are equivalent to module categories over small pre-additive categories.

4. Between $(AB, \text{tor}_{\text{class - tilt}})$ and $(\text{Mod}_{\text{ring}}, \text{tor}_{\text{faithful}})$, where $\text{tor}_{\text{class - tilt}}$ denotes the subclass of $\text{tor}_{\text{tilt}}$ consisting of the torsion pairs associated to classical tilting objects and $\text{Mod}_{\text{ring}}$ denotes the class of categories equivalent to module categories over rings.

Proof. By Definition 5 and Proposition 2.10 we get induced maps $(AB, \text{tor}_{\text{faithful}}) \rightarrow (AB, \text{tor}_{\text{cofaithful}})$ and $(AB, \text{tor}_{\text{cofaithful}}) \rightarrow (AB, \text{tor}_{\text{faithful}})$. By Theorems 3.7 and 3.9 and Corollary 3.10 we get the bijections in 1 and 2. Finally, the bijections in 3 and 4 follow from Corollary 3.12. □
Now one can derive the bijection for \( n = 1 \) of \([Po-St2, \text{Corollary 3.12}]\), with the help of the following known result, for which we indicate the references for its proof.

**Proposition 3.14.** Let \( \mathcal{A} \) be an abelian category. The following assertions hold:

1. If \( \mathcal{A} \) is AB3 and has a projective generator, then \( \mathcal{A} \) is AB4* (and hence AB3*).

2. If \( \mathcal{A} \) is AB3* and has an injective cogenerator, then \( \mathcal{A} \) is AB4 (and hence AB3).

**Proof.** It is well-known that any AB3 category with an injective cogenerator is AB4 (see \([Po, \text{Corollary 3.2.9}]\)), and, by duality, an AB3* abelian category with a projective generator is also AB4*. Using the duality principle, it will be enough to prove assertion 2, for which we just need to prove that if \( \mathcal{A} \) is AB3* and has an injective cogenerator, then it is AB3. This amounts to prove that if \( I \) is any set, viewed as a small category, then the constant diagram functor \( \kappa : \mathcal{A} \to \mathcal{A}^I \) has a left adjoint. This follows from Freyd’s special adjoint theorem and its consequences (see \([Freyd, \text{Chapter 3, Exercises M, N}]\)). □

**Corollary 3.15** (Positselski-Stovicek). The HRS tilting process gives a one-to-one correspondence between:

1. The pairs \((\mathcal{A}, t)\) consisting of an AB3* abelian category \( \mathcal{A} \) with an injective cogenerator and a tilting torsion pair \( t \) in \( \mathcal{A} \);

2. The pairs \((\mathcal{B}, \bar{t})\) consisting on an AB3 abelian category \( \mathcal{B} \) with a projective generator and a cotilting torsion pair \( \bar{t} \) in \( \mathcal{B} \).

Moreover the categories of assertion 1 are also AB4 and those of assertion 2 are also AB4*.

**Proof.** We use the map \( \Phi \) of Corollary 3.13. We then have induced bijections \( \Phi : (\mathcal{A}, \text{tor}_\text{tilt}) \to (\mathcal{A}3\text{proj}, \text{tor}_\text{faithful}) \) and \( \Phi : (\mathcal{A}3\text{proj}, \text{tor}_\text{faithful}) \to (\mathcal{A}, \text{tor}_\text{cofaithful}). \) By restriction, we then get a bijection between the intersection of the domains and the intersection of the codomains. The intersection of the domains is precisely the class of pairs in 1 (note that the fact that \( T \) is cogenerating, equivalently that \( t \in \text{tor}_\text{cofaithful} \), is automatic). Similarly, the intersection of codomains is precisely the class of pairs in 2.

The last statement follows from Proposition 3.14. □

## 4 When is the heart of a torsion pair a Grothendieck category?

### 4.1 Initial results

The aim of this subsection is to show the milestones of the initial work on the problem, almost entirely due to a group of italian algebraists from the
Following generalization of Brenner-Butler’s theorem (see [BB]), due to Colpi of this paper, it goes as follows. Suppose that $A$ is an abelian category and $t$ is a classical tilting torsion pair in it. According to [HRS, Theorem 3.3] the realization functor gives an equivalent of triangulated categories $G : D^b(\mathcal{H}_t) \cong D^b(A)$. On the other hand, by Corollary 3.12 we know that $\mathcal{H}_t$ is a module category, actually via the equivalence of categories $\mathcal{H}_t(V[0],?) : \mathcal{H}_t \cong \text{Mod} - R$, where $R = \text{End}_{\mathcal{H}_t}(V[0]) \cong \text{End}_A(V)$. Then we also have an equivalence of triangulated categories $D^b(\text{Mod} - R) \cong D^b(\mathcal{H}_t)$, and taking the composition, we get an induced equivalence of triangulated categories $D^b(\text{Mod} - R) \cong D^b(A)$, taking $R$ to $V$. We can think of the inverse of this functor as a sort of tensor product by $V$. We can then think of the equivalence $\mathbb{L}T_V : \mathcal{D}(\text{Mod} - R) \cong \mathcal{D}(A)$ as a right derived functor of $R \mathcal{D}(A) \longrightarrow \mathcal{D}(\text{Mod} - R)$ of the canonical functor $H_V := \mathcal{A}(V,?) : A \longrightarrow \text{Mod} - R$. This functor turns out to have a left adjoint $T_V : \text{Mod} - R \longrightarrow A$ of which we can think as a sort of ‘tensor product’.

Due to the definition of the torsion pair $t$, one then has $\mathcal{R}H_V(T) = \mathcal{A}(V,T)[0]$ and $\mathcal{R}H_V(F) = \text{Ext}_A^1(V,F)[-1]$. This implies that the equivalence $\mathcal{R}H_V : \mathcal{D}(A) \longrightarrow \mathcal{D}(\text{Mod} - R)$ induces equivalences of categories

$$\mathcal{F}[1] \cong \mathcal{X} := \{ X \in \text{Mod} - R : X \cong \text{Ext}_A^1(V,F), \text{ with } F \in \mathcal{F} \}$$

$$\mathcal{T}[0] \cong \mathcal{Y} := \{ Y \in \text{Mod} - R : Y \cong \mathcal{A}(V,T), \text{ with } T \in \mathcal{T} \}.$$  

Note that we then get induced equivalences of categories $H_V = \mathcal{A}(V,?): \mathcal{T} \cong \mathcal{Y}$ and $H_V' = \text{Ext}_A^1(V,?): \mathcal{F} \cong \mathcal{X}$ whose quasi-inverses are necessarily $T_V : \mathcal{Y} \cong \mathcal{T}$ and $T_V' : \mathcal{X} \cong \mathcal{F}$. This essentially gives the proof of the following generalization of Brenner-Butler’s theorem (see [BB], due to Colpi and Fuller (see [CF] Theorem 3.2)):

**Theorem 4.1.** Let $\mathcal{A}$ be an abelian category, let $V$ be a classical 1-tilting object in $\mathcal{A}$ and let $R = \text{End}_A(V)$ the ring of endomorphisms of $V$. With the notation above, we have an equality of pairs $(\mathcal{X}, \mathcal{Y}) = (\text{Ker}(T_V), \text{Ker}(T_V'))$, and this is a faithful torsion pair $t'$ in $\text{Mod} - R$. Moreover, we have induced equivalences of categories $\mathcal{T} \cong \mathcal{Y}$ and $\mathcal{F} \cong \mathcal{X}$.
In addition, by the paragraphs above, the torsion pair \( t' \) is sent to \( \bar{t} = (\mathcal{F}[1], \mathcal{T}[0]) \) by the equivalence of categories \( \text{Mod} - R \sim H_t \). Then, using Proposition 2.11, one gets the following initial result:

**Proposition 4.2.** ([CGM, Corollary 2.4]) \( A \) is equivalent to \( H_{t'} \), where \( t' \) is as above.

On the other hand, we have a dual situation, that is, starting with \( (\text{Mod-} R, s) \) a pair in \( (\text{Mod-r} \text{ing}, \text{tor} \text{faithful}) \). It then follows that \( R[1] \) is a 1-tilting object of \( H_s \) (see Corollary 3.10) so that \( \Phi((\text{Mod-} R, s)) = (H_s, \mathcal{F}) \in (\text{AB}, \text{tortilt}) \). For this reason, the last result indicates that Question 1.3 for faithful torsion pairs in modules categories is equivalent to the question of when an abelian category \( A \) with a classical 1-tilting object is a Grothendieck category. This fact was exploited by Colpi, Gregorio and Mantese in [CGM] and, they obtained the first partial answer to the question 1.3.

**Theorem 4.3.** Let \( (A, t) \in (\text{AB}, \text{tortilt}) \) and we consider \( t' \) as above. Then, the following assertions are equivalent:

1. \( A \cong H_{t'} \) is a Grothendieck category;
2. for any direct system \( (X_\lambda) \) in \( H_{t'} \) the canonical morphism \( \lim_{\rightarrow \mathcal{H}_{t'}} H_V(X_\lambda) \to H_V(\lim_{\rightarrow \mathcal{H}_{t'}} X_\lambda) \) is a monomorphism;
3. the functor \( H_V \) preserve direct limits.

If \( t' \) is of finite type, then the previous conditions are equivalent to the condition that the functor \( T_V \circ H_V \) preserve direct limits.

On the other hand, in the same work [CGM] the authors gave necessary conditions for a faithful torsion pair in a module category to have a heart which is a Grothendieck category (see [CGM, Proposition 3.8]).

**Proposition 4.4.** Let \( R \) be a ring and let \( t = (T, F) \) be a faithful torsion pair in \( \text{Mod-} R \). If \( H_t \) is a Grothendieck category, then \( t = (T, F) \) is a cotilting torsion pair.

This initial work by the Italian algebraists ended with the following unpublished result of Colpi and Gregorio [CG] (see [Ma, Theorem 6.2]).

**Theorem 4.5.** ([Colpi and Gregorio Theorem]) Let \( R \) be a ring, let \( t = (T, F) \) be a faithful torsion pair in \( \text{Mod-} R \) and let \( H_t \) be the heart of the associated Happel-Reiten-Smalø l-structure in \( \mathcal{D}(\text{Mod-} R) \). Then \( H_t \) is a Grothendieck category if, and only if, \( t = (T, F) \) is a cotilting torsion pair.

### 4.2 The solution of the problem

The solution to the problem was given by the authors in [PS1] and [PS2]. We realized that the hard part of the problem was to deal with the AB5 condition on \( H_t \). This naturally led to a detailed study of direct limits in the heart. And, in order to understand those direct limits, it was a preliminary step to understand the behavior of the stalk complexes in the heart with respect to direct limits.
Proposition 4.6. [PS1, Lemma 4.1 and Proposition 4.2] Let $t = (T, F)$ be a torsion pair in the Grothendieck category $G$. The following assertions hold (see Lemma 3.5):

1. The functor $H^0 : H_t \to G$ is right exact and preserve coproducts;
2. The functor $H^{-1} : H_t \to G$ is left exact and preserve coproducts;
3. For every $(M_\lambda)$ direct system in $H_t$, the induced morphism $\lim_{\to} H^k(M_\lambda) \to H^k(\lim_{\to} M_\lambda)$ is an epimorphism, for $k = -1$, and an isomorphism, for $k \neq -1$.
4. The pair $t = (F[1], T[0])$ in $H_t$ is of finite type.
5. For each direct system $(F_\lambda)_\lambda$ in $F$, we have a canonical isomorphism $\lim_{\to} F_\lambda[1] \cong (1 : t)(\lim_{\to} F_\lambda)[1]$.

For instance, using the previous result and Proposition 2.11, we immediately get a necessary condition for a positive answer in the case of a co-faithful torsion pair.

Lemma 4.7. Let $G$ be a Grothendieck category and let $t = (T, F)$ be a co-faithful torsion pair. If $H_t$ is a Grothendieck category, then $t$ is of finite type.

Proof. Suppose that $H_t$ is a Grothendieck category. Since $G$ has enough injectives, from Proposition 2.11 we get that $H_T$ is equivalent to $G[1]$, via realization functor, where $\mathcal{T} = (T[1], T[0])$ is the corresponding torsion pair in $H_T$. Using assertion 4 of the previous proposition, we deduce that $\mathcal{T} = (T[1], F[1])$ is a torsion pair of finite type in $H_T \cong G[1]$. That is, we have that $\lim_{\to} F_\lambda[1] \in F[1]$, and this implies that $\lim_{\to} F_\lambda \in F$ due to the canonical equivalence $G \cong G[1]$, which restricts to one $\mathcal{F} \cong F[1]$.

Another point of the strategy of the authors was to use the canonical cohomology functors $H^k : D(G) \to G$ to approach the problem. In that way one gets sufficient conditions:

Proposition 4.8. [PS1, Proposition 3.4] Let $G$ be a Grothendieck category and let $\sigma = (U, U^\perp[1])$ be a t-structure on $D(G)$. We denote its heart by $H_\sigma$. If the classical cohomological functors $H^k : H_\sigma \to \text{Ab}$ preserve direct limits, for all integer $k$, then $H_\sigma$ is an AB5 abelian category.

The following is now a natural question that remains open.

Question 4.9. Given a Grothendieck category $G$ and a t-structure $\sigma = (U, U^\perp[1])$ in $D(G)$ such that its heart $H_\sigma$ is an AB5 abelian category. Do the cohomological functors $H^m : H_\sigma \to G$ preserve direct limits, for all $m \in \mathbb{Z}$?

The key point of Lemma 4.7 is the fact that $H_T$ is equivalent to $G[1]$. Recently, Chen, Han and Zhou have given necessary and sufficient conditions for such an equivalence to exist.
Theorem 4.10. [CHZ, Theorem A] Let $G$ be a Grothendieck category and let $t = (T, F)$ be a torsion pair in $G$. The following assertions are equivalent:

1. $H_t$ is equivalent to $G[1]$, via realization functor;

2. Each object $X$ in $G$ fits into an exact sequence

$$0 \rightarrow F^0 \rightarrow F^1 \rightarrow X \rightarrow T^0 \rightarrow T^1 \rightarrow 0$$

with $F^i \in F$ and $T^i \in T$, for $i = 0, 1$, and $\text{Ext}_G^3(T^1, F^0) = 0$.

The tilting and cotilting torsion pairs are covered in the previous theorem. Thus Lemma 4.7 and its proof are also valid for such torsion pairs.

The general answer to Question 1.3 was given by the authors.

Theorem 4.11. [PS2, Theorem 1.2] Let $G$ be a Grothendieck and let $t = (T, F)$ be a torsion pair in $G$. Then, $H_t$ is a Grothendieck category if, and only if, $t$ is of finite type.

The main and harder part was to prove that the finite type of $t$ is a necessary condition. That is done in [PS1, Theorem 4.8] and the preliminary results leading to it. For torsion pairs of finite type, the authors proved that $T = \text{Pres}(V)$, for some object $V$, and then, for a fixed generator $G$ of $G$, they showed in [PS1, Proposition 4.7] that the skeletally small subclass $N$ of $H_t$ consisting of those complexes $N$ such that $H^{-1}(N)$ is a subquotient of $G^m$ and $H^0(N) \cong V^n$, for some $m, n \in \mathbb{N}$, is a class of generators of $H_t$. Therefore, in order to answer Question 1.3 the only thing remaining was to prove that if $t$ is of finite type then $H_t$ is AB5. This was done in [PS2, Theorem 1.2].

4.3 A by-side problem: When is the heart of a tilting torsion pair a Grothendieck category?

Since tilting torsion pairs have hearts with a projective generator, it is good to know when that heart is a Grothendieck category, because such a heart would very close to a module category. The question of the title of this subsection has been recently answered [BHPST, Corollary 2.5] for the case when the ambient Grothendieck category $G$ is the module category over a ring. Recall that a pure exact sequence in $\text{Mod} - R$ is a short exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ that remains exact after applying the functor $\otimes_R X$, for all left $R$-modules $X$. A module $P \in \text{Mod} - R$ is pure-projective when the functor $\text{Hom}_R(P, ?) : \text{Mod}$-$R \rightarrow \text{Ab}$ preserves exactness of pure exact sequences.

Theorem 4.12. Let $R$ be a ring, let $V$ be a 1-tilting (right) $R$-module and let $t = (\text{Gen}(V), V^\perp)$ be the associated torsion pair in $\text{Mod} - R$. The following assertions are equivalent:

1. $V$ is pure-projective.

2. $t$ is of finite type (equivalently, the heart $H_t$ is a Grothendieck category).
It is well-known that a module is pure-projective if, and only if, it is a direct summand of a coproduct (=direct sum) of finitely presented modules. So when the heart of a tilting torsion pair $t = \langle \text{Gen}(V), V^\perp \rangle$ is a Grothendieck category, the projective generator of the heart $V[0]$ is determined by a set of ‘small objects’. Therefore the following question, first risen in [PS1, Question 5.5], is apropos. We will call two modules $M$ and $N$ $\text{Add}$-equivalent when $\text{Add}(M) = \text{Add}(N)$.

**Question 4.13.** Let $V$ be a $1$-tilting $R$-module whose associated torsion pair is of finite type (equivalently, such that the heart $\mathcal{H}_t$ is a Grothendieck category). Is $V$ $\text{Add}$-equivalent to a classical $1$-tilting module? Equivalently, is the heart $\mathcal{H}_t$ equivalent to the module category over a ring?

It turns out that the answer to this question is negative in general, with counterexamples already existing when $R$ is a noetherian ring (see [BHPST, Section 4]). However, the following is true:

**Theorem 4.14.** Let the ring $R$ be in one of the following classes:

1. That of commutative rings;
2. That of Krull-Schmidt rings, i.e. rings for which every finitely presented (right) module is a direct sum of modules with local endomorphism ring (e.g. right Artinian rings)
3. That of rings for which every pure-projective (right) module is a coproduct of finitely presented ones.

A $1$-tilting $R$-module is pure-projective if, and only if, it is $\text{Add}$-equivalent to a classical $1$-tilting $R$-modules. Said in equivalent words, the heart of a tilting torsion pair in $\text{Mod}_- R$ is a Grothendieck category if, and only if, it is equivalent to a module category over a ring.

**Proof.** See [BHPST, Corollary 2.8 and Theorem 3.7].

### 4.4 A new approach using purity

Using now a recent result of Positselski and Stovicek [Po-St] we can actually identify the cotilting torsion pairs in an abelian category for which the heart is actually an AB5 abelian category. We need the following definition.

**Definition 10.** Let $\mathcal{A}$ be any additive category. We shall say that an object $Y$ of $\mathcal{A}$ is $\text{pure-injective}$ if the following two conditions hold:

1. The product of $Y^I$ exists in $\mathcal{A}$, for all sets $I$;
2. For each nonempty set $I$, there is a map $\phi : Y^I \to Y$ such that $\phi \circ t_j = 1_Y$, for all $j \in I$. Here $t_j : Y \to Y^I$ is the unique morphism such that $\pi_i \circ t_j = \delta_{ij} 1_Y$, with $\delta_{ij}$ the Kronecker symbol and $\pi_i : Y^I \to Y$ the $i$-th projection.
We call the morphism \( \iota_j \) in \( j \)-th injection into the product.

**Corollary 4.15.** Let \( \mathcal{A} \) be an abelian category and \( t = (\mathcal{T}, \mathcal{F}) \) be a torsion pair in \( \mathcal{A} \). The following assertions are equivalent:

1. There is a pure-injective 1-cotilting object \( Q \) of \( \mathcal{A} \) such that \( F = \text{Cogen}(Q) \);
2. \( t \) is a faithful torsion pair in \( \mathcal{A} \) whose heart is an AB5 abelian category with an injective cogenerator;
3. The heart \( \mathcal{H}_t \) is an AB5 abelian category with an injective cogenerator and \( \bar{t} \) is a co-faithful torsion pair in \( \mathcal{H}_t \).

In particular, the HRS procedure gives a bijection ‘inverse of itself’

\[
(\text{AB, tor}_{\text{cotilt-pinj}}) \overset{\cong}{\rightarrow} (\text{AB5}_{\text{inj}}, \text{tor}_{\text{cofaithful}}),
\]

where:

(a) \( (\text{AB, tor}_{\text{cotilt-pinj}}) \) consists of the pairs \( (\mathcal{A}, t) \), where \( \mathcal{A} \) is an abelian category and \( t = (\mathcal{T}, \mathcal{F}) \) is a torsion pair, with \( F = \text{Cogen}(Q) \) for \( Q \) a 1-cotilting pure-injective object.

(b) \( (\text{AB5}_{\text{inj}}, \text{tor}_{\text{cofaithful}}) \) consists of the pairs \( (\mathcal{B}, \bar{t}) \), where \( \mathcal{B} \) is an AB5 abelian category with an injective cogenerator and \( \bar{t} = (\mathcal{X}, \mathcal{Y}) \) is a co-faithful torsion pair in \( \mathcal{B} \).

**Proof.** (2) \( \iff \) (3) It is a consequence of Proposition 2.10.

(1) \( \implies \) (2) By dualizing the proof of Theorem 3.7 and Lemma 3.8 we know that \( Q[1] \) is an injective cogenerator of \( \mathcal{H}_t \) and that the stalk complex \( Q'[1] \) is the product of \( I \) copies of \( Q[1] \) in \( \mathcal{H}_t \), for all sets \( I \). If \( \phi : Q' \rightarrow Q \) is a map such that \( \phi \circ \iota_j = 1_Y \), for all \( j \in I \), with the notation of Definition 10, we then get that \( \phi[1] : Q'[1] \rightarrow Q[1] \) satisfies that \( \phi[1] \circ \iota_j[1] = 1_{Y[1]} \), for all \( j \in I \). But \( \iota_j[1] : Y[1] \rightarrow Y'[1] \) is clearly in \( j \)-th injection into the product in \( \mathcal{H}_t \). Therefore \( Q[1] \) is pure-injective in \( \mathcal{H}_t \). Since, by Theorem 3.9, we know that \( \mathcal{H}_t \) is AB3*, we can apply [Po-St, dual of Theorem 3.3] to conclude that \( \mathcal{H}_t \) is AB5.

(2) \( \implies \) (1) By [Po-St] dual of Theorem 3.3 again, we know that \( \mathcal{H}_t \) admits an injective cogenerator \( E \) which is pure-injective. But since \( F[1] \) is a cogenerating class in \( \mathcal{H}_t \) we necessarily have that \( E = Q[1] \), for some \( Q \in \mathcal{F} \). Now the dual of the proof of (2) = (3) \( \implies \) (1) in Theorem 3.7 shows that \( Q \) is a 1-cotilting object of \( \mathcal{A} \) such that \( F = \text{Cogen}(Q) \) and \( \mathcal{F} \) is a generating class in \( \mathcal{A} \). The argument in the proof of (1) \( \implies \) (2) proves that \( Q \) is pure-injective in \( \mathcal{A} \) if and only if \( Q[1] = E \) is pure-injective in \( \mathcal{H}_t \), something that we know by hypothesis.

\[ \square \]

We are now in a position to re-prove the hard part of [PS2, Theorem 1.2], that is the proof of [PS1, Theorem 4.8], by using recent results in the literature:
Theorem 4.16. Let $\mathcal{G}$ be a Grothendieck category and let $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{G}$. If the heart $\mathcal{H}_\mathbf{t}$ of the associated Happel-Reiten-Smalø t-structure in $\mathcal{D}(\mathcal{G})$ is a Grothendieck category, then $\mathbf{t}$ is of finite type, i.e. $\mathcal{F}$ is closed under taking direct limits in $\mathcal{G}$.

Proof. By Proposition 3.16 we know that $\mathcal{F} = \text{Cogen}(Q) = \text{Copres}(Q)$, for some quasi-cotilting object $Q$. Consider now the subcategory $\mathcal{E} = \text{Cogen}(Q)$ of $\mathcal{G}$. This subcategory is clearly closed under taking subobjects, quotients and coproducts, so that it is an abelian exact subcategory where colimits are calculated as in $\mathcal{G}$. In particular $\mathcal{E}$ is an AB5 abelian category. Moreover, if $X$ is a generator of $\mathcal{G}$ one readily gets that $(1 : t)(X)$ is a generator of $\mathcal{E}$, so that this subcategory is actually a Grothendieck category.

Note also that the inclusion functor $\iota : \mathcal{E} \hookrightarrow \mathcal{G}$ has a right adjoint $\rho : \mathcal{G} \longrightarrow \mathcal{E}$. The action on objects is given by $\rho(M) = \text{tr}_\mathcal{F}(M)$, where $\text{tr}_\mathcal{F}(M)$ is the trace of $\mathcal{F}$ in $M$, i.e. the subobject sum of all subobjects of $M$ which are in $\mathcal{E}$. We leave to the reader the easy verification that $(\iota, \rho)$ is an adjoint pair. We can then derive these functors. Due to the exactness of $\iota$, the left derived of $\iota$, $\mathbb{L}\iota = \iota$ is’ $\iota$ itself, i.e. it just takes a complex $X^\bullet \in \mathcal{D}(\mathcal{F})$ to the same complex viewed as an object of $\mathcal{D}(\mathcal{G})$. The right derived $\mathbb{R}\rho : \mathcal{D}(\mathcal{G}) \longrightarrow \mathcal{D}(\mathcal{E})$ is defined in the usual way, namely, it is the composition $\mathcal{D}(\mathcal{G}) \xrightarrow{\iota} \mathcal{K}(\mathcal{G}) \xrightarrow{\rho} \mathcal{K}(\mathcal{F}) \xrightarrow{q} \mathcal{D}(\mathcal{E})$, where $i$ is the homotopically injective resolution functor, (abusing of notation) $\rho$ is the obvious functor induced at the level of homotopy categories, and $q$ is the canonical localization functor. Then, by classical properties of derived functors and derived categories, we get that $(\iota, \mathbb{R}\rho)$ is an adjoint pair of triangulated functors.

Consider now the restricted torsion pair $\mathbf{t}' = (\mathcal{T} \cap \mathcal{E}, \mathcal{F})$ in $\mathcal{E}$. By [PS1, Proposition 3.2], we know that its heart $\mathcal{H}_{\mathbf{t}'}$ is an AB3 abelian category. Moreover, the triangulated functor $\iota : \mathcal{D}(\mathcal{E}) \longrightarrow \mathcal{D}(\mathcal{G})$ clearly satisfies that $\iota(\mathcal{H}_{\mathbf{t}'}) \subseteq \mathcal{H}_{\mathbf{t}}$. We therefore get an induced functor $\iota : \mathcal{H}_{\mathbf{t}'} \longrightarrow \mathcal{H}_{\mathbf{t}}$, which is necessarily exact since short exact sequences in hearts are the triangles in the ambient triangulated category with their three vertices in that heart. We claim that the composition of functors $\rho' : \mathcal{H}_{\mathbf{t}'} \hookrightarrow \mathcal{D}(\mathcal{G}) \xrightarrow{\mathbb{R}\rho} \mathcal{D}(\mathcal{E}) \xrightarrow{H^0_{\mathbf{t}'}} \mathcal{H}_{\mathbf{t}'}$ is right adjoint of $\iota : \mathcal{H}_{\mathbf{t}'} \longrightarrow \mathcal{H}_{\mathbf{t}}$. Let $X \in \mathcal{H}_{\mathbf{t}'}$ and $M \in \mathcal{H}_{\mathbf{t}}$ be arbitrary objects. Note that $X$ can be identified by $\iota$ with a complex $M$ concentrated in degrees $\geq -1$. Then $\mathbb{R}\rho(M)$ is the complex $M \longrightarrow \rho(E^{-1}) \longrightarrow \rho(E^0) \longrightarrow \rho(E^1) \longrightarrow \cdots$. As a right adjoint, the functor $\rho : \mathcal{G} \longrightarrow \mathcal{E}$ is left exact, and this implies that $H^{-1}(\mathbb{R}\rho(M)) \cong \rho(H^{-1}(M)) \cong H^{-1}(M)$ since $H^{-1}(M) \in \mathcal{F}$. This implies that $\mathbb{R}\rho(M) \in \mathcal{W}_{\mathbf{t}'}$, where $\mathcal{W}_{\mathbf{t}'}$ is the coaisle of the HRS t-structure in $\mathcal{D}(\mathcal{E})$ associated to $\mathbf{t}'$. Remember that the restriction of $H^0_{\mathbf{t}'}$ to $\mathcal{W}_{\mathbf{t}'}$ is right adjoint of the inclusion functor $\mathcal{H}_{\mathbf{t}'} \hookrightarrow \mathcal{W}_{\mathbf{t}'}$ (see [PS1 Lemma 3.1(2)]). We then have a sequence of isomorphisms, natural on both variables:
\[\mathcal{H}_t'(X, \rho'(M)) = \mathcal{H}_t'(X, (H^0_t \circ \mathbb{R}\rho)(M)) \cong \mathcal{W}_t(X, \mathbb{R}\rho(M)) = D(\mathcal{F})(X, \mathbb{R}\rho(M)) \cong D(\mathcal{G})(\iota(X), M) \cong \mathcal{H}_t(\iota(X), M),\]

which implies that \((\iota, \rho')\) is an adjoint pair.

We then get that the exact functor \(\iota : \mathcal{H}_t' \to \mathcal{H}_t\) preserves direct limits. Moreover, it reflects zero objects since \(\iota(X) = 0\) means that \(X\) is acyclic, viewed as a complex of objects of \(\mathcal{G}\), which is the same as being acyclic when viewed as a complex of objects in \(\mathcal{F}\). It immediately follows from these considerations that direct limits are exact in \(\mathcal{H}_t'\) because they are so in \(\mathcal{H}_t\). That is \(\mathcal{H}_t'\) is an AB5 abelian category.

On the other hand, we claim that \(Q\) is a 1-cotilting object of \(\mathcal{F}\) and that \(t'\) is its associated torsion pair in \(\mathcal{F}\). Indeed, since we know that \(\mathcal{F} = \text{Cogen}(Q) = \mathcal{F} \cap \text{Ker}(\text{Ext}^1_{\mathcal{F}}(?, Q))\), it is enough to check that \(\mathcal{F} \cap \text{Ker}(\text{Ext}^1_{\mathcal{F}}(?, Q)) = \text{Ker}(\text{Ext}^1_{\mathcal{F}}(?, Q))\). The inclusion \(\subseteq\) is clear. For the converse, let \(M \in \text{Ker}(\text{Ext}^1_{\mathcal{F}}(?, Q))\) and fix two exact sequences

\[0 \to F' \xrightarrow{u} F \to M \to 0 \quad \text{and} \quad 0 \to F'' \to Q' \to Q'' \to 0\]

with \(F, F', F'' \in \mathcal{F}\) and \(I\) some set (where for the second exact sequence we used that \(\mathcal{F} = \text{Copres}(Q)\)). Taking the pushout of \(u\) and \(v\), we obtain the following commutative diagram with exact rows and columns:

\[
\begin{array}{cccccccc}
0 & & 0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & F' & \xrightarrow{u} & F & \rightarrow & M & \rightarrow & 0 \\
\downarrow & & \downarrow & \text{P.O.} & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & Q' & \rightarrow & X & \rightarrow & M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
F'' & \rightarrow & F'' & \rightarrow & F'' & \rightarrow & F'' & \rightarrow & 0 \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
\]

We then obtain that \(X \in \mathcal{F}\) (as it is an extension of \(F\) and \(F'' \in \mathcal{F}\)), so that \(Q', X\), and \(M\) all belong in \(\mathcal{F}\). By the choice of \(M\), the second row of the diagram splits, so that \(M \in \mathcal{F}\) since it is isomorphic to a direct summand of \(X\).

It now follows from Theorem 3.9, Corollary 4.15, and the proof of the latter that \(Q\) is pure-injective in \(\mathcal{F}\). But then \(\mathcal{F}\) is closed under taking direct limits in the Grothendieck category \(\mathcal{F}\) (see Con-Sil Theorem 3.9), which is equivalent to say that is is closed under taking direct limits in \(\mathcal{G}\). That is, \(t = (T, \mathcal{F})\) is a torsion pair of finite type, as desired.

\[\square\]
5 Beyond the HRS case: Some recent results

After Question 1.3 was solved, as said in the introduction, it is Question 1.2 the one that has deserved more attention. So far the work has mainly concentrated on the case when the t-structure is compactly generated, in which case one can even assume that the ambient triangulated category $\mathcal{D}$ is compactly generated. This is due to the fact that $\mathcal{L} := \text{Loc}_\mathcal{D}(\mathcal{U})$, the smallest triangulated subcategory of $\mathcal{D}$ containing $\mathcal{U}$ and closed under taking arbitrary coproducts, is compactly generated and the restricted t-structure $\tau' = (\mathcal{U}, \mathcal{U} \cap \mathcal{L})$ has the same heart as $\tau$.

Using different techniques, as functor categories ([AMV], [Bo, Initial versions of the paper]), stable $\infty$-categories [Lurie] and the theory of derivators [SSV], see also [PS3] and [Bazz] for particular cases, partial answers to the question in the compactly generated case were obtained. Joining all of them, it was in the ambient that, for all compactly generated t-structures appearing in nature, the heart is a Grothendieck category. The concluding result in this vein has been recently obtained independently in [Bo, version 7] and [SS]:

Theorem 5.1. ([Bo, v7] and [SS]) Let $\mathcal{D}$ be a triangulated category with coproducts and $\tau = (\mathcal{U}, \mathcal{W})$ be a compactly generated t-structure. The heart $\mathcal{H}_\tau = \mathcal{U} \cap \mathcal{W}$ is a Grothendieck category.

In the development via derivators of [SSV], the new concept of homotopically smashing t-structure (with respect to a strong stable derivator) was introduced. We refer to that reference for the definition and to [Groth] for all the terminology concerning derivators. All compactly generated t-structures that appear as the base of a strong stable derivator are homotopically smashing. These last t-structures are always smashing, but the converse is not true. For instance the HRS t-structure is always smashing, but it is homotopically smashing exactly when the torsion pair is of finite type (see [SSV, Proposition 6.1]). The following is a combination of [SSV, Theorems B and C], and we refer to that reference for all unexplained terminology appearing in the statement:

Theorem 5.2. Let $\mathbb{D} : \text{Cat}^{op} \to \text{CAT}$ be a strong stable derivator, with base $\mathcal{D} := \mathbb{D}(1)$, and let $\tau = (\mathcal{U}, \mathcal{W})$ be a t-structure in $\mathcal{D}$ that is homotopically smashing with respect to $\mathcal{D}$, then the heart $\mathcal{H}_\tau$ is an AB5 abelian category. When, in addition, $\mathbb{D}$ is the derivator associated to the homotopy category of a stable combinatorial model structure and $\tau$ is generated by a set, that heart is a Grothendieck category.

Soon after [SSV] appeared, Rosanna Laking [L] proved the following result:

Theorem 5.3. Let $\mathcal{D}$ be a compactly generated triangulated category that is the base of a strong stable derivator $\mathbb{D}$, and let $\tau = (\mathcal{U}, \mathcal{W})$ be a left nondegenerate t-structure in $\mathcal{D}$. The following assertions are equivalent

1. $\tau$ is homotopically smashing with respect to $\mathbb{D}$.
2. $\tau$ is smashing and the heart $\mathcal{H}_\tau$ of $\tau$ is a Grothendieck category.
The last two results suggest the following open question:

**Question 5.4.** Let $D$ be a well-generated triangulated category (e.g. a compactly generated one) that is the base of a strong stable derivator $\mathbb{D}$. Are the following two conditions equivalent for a $t$-structure $\tau = (U, W)$ in $D$?

1. $\tau$ is homotopically smashing with respect to $D$.
2. $\tau$ is smashing and the heart $H_\tau$ of $\tau$ is a Grothendieck category.

In order to get (a partial version of) this question in a derivator-free way, a hint comes from [L, Theorem 4.6] (see also [LV, Theorem 4.7]), where the author proves that ‘homotopically smashing’ and ‘definable’ are synonymous terms for the co-aisle of left nondegenerated $t$-structure, when the ambient triangulated category is the compactly generated base of a strong stable derivator (see [L] for the definition of definable subcategory of a compactly generated triangulated category). This suggests the following question:

**Question 5.5.** Let $D$ be a compactly generated triangulated category. Are the following two conditions equivalent for a $t$-structure $\tau = (U, W)$ in $D$?

1. $W$ is definable.
2. $\tau$ is smashing and the heart $H_\tau$ of $\tau$ is a Grothendieck category.

References

[AMV] L. Angeleri-Hügel, F. Marks, J. Vitória, *Torsion pairs in silting theory*, Pacific. J. Math. 2 (2017), 257-278.

[Bazz] S. Bazzoni, *The $t$-structure induced by an $n$-tilting module*. Trans. Amer. Math. Soc. 371 (2019), 6309-6340.

[BHPST] S. Bazzoni, I. Herzog, P. Prihoda, J. Saroch, J. Trlifaj, *Pure-projective tilting modules*. Available at https://arxiv.org/abs/1703.04745

[BBD] A. Beilinson, J. Bernstein, P. Deligne, “Faisceaux Pervers”. *Analysis and topology on singular spaces*, I, Luminy 1981. Astérisque 100, Soc. Math. France, Paris. (1982), 5-171.

[Bo] M. Bondarko, *On torsion pairs, (well-generated) weight structures, adjacent $t$-structures, and related (co)homological functors*. Preprint available at https://arxiv.org/abs/1611.00754

[BB] S. Brenner, M. Butler, *Generalizations of the Bernstein-Gelfand-Ponomarev reflections functors*, Representation Theory II, Ottawa 1979, Springer Lecture in Mathematics 832 (1980), 103-169.

[CN] C. Casacuberta, A. Neeman, *Brown representability theorem does not come for free*, Math. Research Lett. 16(1) (2009), 1-5.
[CHZ] X. Chen, Z. Han, Y. Zhou, Derived equivalences via HRS-tilting, preprint (2018), Preprint available at: https://arxiv.org/pdf/1804.05629.pdf

[CDT] R. Colpi, G. D’Este, A. Tonolo, Quasi-tilting modules and counter equivalences, Journal of Algebra, 191 (1997), 461-494.

[CF] R. Colpi, K.R. Fuller, Tilting objects in abelian categories and quasitilted rings, Trans. Amer. Math. Soc. 359 (2007), 741-765.

[CG] R. Colpi, E. Gregorio, The Heart of cotilting theory pair is a Grothendieck category, Preprint.

[CGM] R. Colpi, E. Gregorio, F. Mantese, On the Heart of a faithful torsion theory, J. Algebra 307 (2007), 841-863.

[Con-St] P. Coupek, J. Stovicek, Cotilting sheaves on noetherian schemes. Preprint available at https://arxiv.org/abs/1707.01677

[Freyd] P. Freyd, Abelian categories, Harper and Row (1964).

[HRS] D. Happel, I. Reiten, S.O. Smalø, Tilting in abelian categories and quasitilted algebras. Memoirs AMS 120 (1996).

[KN] B. Keller, P. Nicolás, Weight structures and simple dg modules for positive dg algebras. Int. Math. Res. Notes IRMN 5 (2013), 1028-1078.

[GP] P. Gabriel, N. Popescu, Caractérisation des catégories abéliennes avec générateurs et limites inductives exactes. C. R. Acad. Sci. Paris 258 (1964), 4188-4190.

[Groth] M. Groth, Derivators, pointed derivators and stable derivators, Algebr. and Geom. Topology 13(1) (2013), 313-374.

[G] A. Grothendieck, Sur quelques points d’Algèbre Homologique. Tohoku Math. J. 9(2) (1957), 119-221.

[L] R. Laking, Purity in compactly generated derivators and t-structures with Grothendieck hearts. Preprint available at https://arxiv.org/abs/1804.01326

[LV] R. Laking, J. Vitoria, Definability and approximations in triangulated categories. Preprint available at https://arxiv.org/abs/1811.00340

[Lurie] Higher Algebra (2017). Available at http://www.math.harvard.edu/~lurie/papers/HA.pdf

[Ma] F. Mattiello, On the heart associated to a faithful torsion pair, Available on www.algant.eu/documents/theses/mattiello.pdf, (2011).

[Ma2] F. Mattiello, New trends in Tilting Theory. PhD Thesis. University of Padova. 2015. Available at http://paduaresearch.cab.unipd.it/7499/
[N] A. Neeman, *Triangulated categories*, Annals of Mathematics studies. Princeton University Press, **148** (2001).

[NSZ] P. Nicolás, M. Saorín, A. Zvonareva, *Silting theory in triangulated categories with coproducts*, J. Pure and Appl. Algebra **223** (2019), 2273-2319.

[PS1] C. Parra, M. Saorín, *Direct limits in the heart of a t-structure: the case of a torsion pair*, Journal of Pure and Applied Algebra, Volume 219, **9** (2015) 4117-4143.

[PS2] C. Parra, M. Saorín, *Addendum to Direct limits in the heart of a t-structure: the case of a torsion pair*, [J. Pure App. Algebra, 219, **9** (2015), 4117-4143], Journal of Pure and Applied Algebra, Volume 220, **6** (2016), 2467-2469.

[PS3] C.E. Parra, M. Saorín, *Hearts of t-structures in the derived category of a commutative Noetherian ring*, Transactions of the American Mathematical Society, (2017), 7789-7827.

[Po] N. Popescu, *Abelian categories with applications to rings and modules*. Academic Press (1973).

[Po-St] L. Positselski, J. Stovicek, *Exactness of direct limits for abelian categories with an injective cogenerator*. J. Pure and Appl Alg, **223**(8) (2018). DOI: 10.1016/j.jpaa.2018.11.004

[Po-St2] L. Positselski, J. Stovicek, *The tilting-cotilting correspondence* Preprint available at https://arxiv.org/abs/1710.02230.

[SS] M. Saorín, J. Stovicek, *t-structures with Grothendieck hearts via functor categories*. In preparation.

[SSV] M. Saorín, J. Stovicek, S. Virili, *t-Structures on stable derivators and Grothendieck hearts*. Preprint available at https://arxiv.org/abs/1708.07540.

[S] B. Stenström, *Rings of quotients*, Grundlehren der math. Wissensch., **217**, Springer-Verlag, (1975).

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