Abstract

In contrast with the scalar-weighted networks, where bipartite consensus can be achieved if and only if the underlying signed network is structurally balanced, the structural balance property is no longer a graph-theoretic equivalence to the bipartite consensus in the case of signed matrix-weighted networks. To re-establish the relationship between the network structure and the bipartite consensus solution, the non-trivial balancing set is introduced which is a set of edges whose sign negation can transform a structurally imbalanced network into a structurally balanced one and the weight matrices associated with edges in this set have a non-trivial intersection of null spaces. We show that necessary and/or sufficient conditions for bipartite consensus on matrix-weighted networks can be characterized by the uniqueness of the non-trivial balancing set, while the contribution of the associated non-trivial intersection of null spaces to the steady-state of the matrix-weighted network is examined. Moreover, for matrix-weighted networks with a positive-negative spanning tree, necessary and sufficient condition for bipartite consensus using the non-trivial balancing set is obtained. Simulation examples are provided to demonstrate the theoretical results.

Keywords: Matrix-weighted networks, bipartite consensus, balancing set, structural balance, null space

1. Introduction

The consensus problem of multi-agent networks has been extensively studied in the last two decades (Jadbabaie et al. [8], Olfati-Saber and Murray [11], Ren and Atkins [15], Mesbahi and Egerstedt [10]). The analysis of multi-agent networks, from a graph-theoretic perspective, emerges from the well-established algebraic graph theory Godsil et al. [6]. In Olfati-Saber and Murray [11], Ren et al. [16], Jadbabaie et al. [8], it was shown that a systematical unity is guaranteed under the consensus protocol whenever the communication graph with positive scalar-valued weights is (strongly) connected. An alteration was further made by Altafini [1] on the protocol which allows the scalar-valued weights to be either positive or negative while guaranteeing asymptotic stability of the network.

Recently, the consensus protocol has been examined in a broader context which in turn calls for the possibility of matrices as edge weights. A common practice is to adopt real symmetric matrices that are either positive (semi-)definite or negative (semi-)definite as edge weights. In fact, the involvement of matrix-valued weights arises naturally when characterizing the inter-dimensional communication amongst multi-dimensional agents, the scenarios being, for instance, graph effective resistance and its applications in distributed control and estimation Tuna [20], Barooah and Hespanha [3], opinion dynamics on multiple interdependent topics Friedkin et al. [5], Ye et al. [22], bearing-based formation control Zhao and Zelazo [23], coupled oscillators dynamics Tuna [21], and consensus and synchronization problems Tuna [19], Trinh et al. [18], Pan et al. [12]. The weight matrices inflict drastic change on the graph Laplacian thus urging many of the old topics to be re-investigated like the controllability of matrix-weighted networks Pan et al. [13] and its $H_2$ performance Foight et al. [4], de Badyn and Mesbahi [2].

In retrospect of the consensus algorithm on matrix-weighted networks, Trinh et al. [18] examined multi-agent networks that involve positive (semi-)definite matrices as edge weights. In this setting they have, among other things, proposed the positive spanning tree as a sufficient graph condition for the network consensus. Antagonistic interaction represented by negative (semi-)definite matrices was soon extended to both undirected and directed networks Pan et al. [12, 14]. It was shown that for the matrix-weighted network with a positive-negative spanning tree, it being structurally balanced is equivalent to admitting a bipartite consensus solution. Nevertheless, a missing correspondence between the network structure and its steady-state was pointed out in Pan et al. [12]. It was stated that structural balance is not sufficient in admitting the bipartite consensus in the presence of positive/negative
semi-definite weight matrices; while Su et al. [17] affirmed that the structural balance property is not a necessary condition either. Up to now, most of the research is done on sufficient graph-theoretic conditions for the bipartite consensus by ruling out the ramification of semi-definite weight matrices on the network. To the best of our knowledge, the bipartite consensus of general matrix-weighted networks is still deficient in any consistent graph-theoretic interpretation.

In this paper, we propose the non-trivial balancing set (NBS) as a tentative step to re-establish the relationship between the network structure and the bipartite consensus of matrix-weighted networks. The NBS defines a set of edges with non-trivial intersecting null spaces and, by their negation, restore the potential structural balance of the network. With the non-trivial balancing set, we would first study the matrix-weighted network in general, with or without structural balance or positive-negative spanning trees. The uniqueness of the non-trivial balancing set turns out to be a necessary yet insufficient condition for the bipartite consensus in this case. Inflicting a stronger precondition, the uniqueness of the NBS becomes both necessary and sufficient to achieve bipartite consensus for networks with positive-negative spanning trees. We extend from this well-defined case and discuss the sufficient condition to have the agents converge bipartitely in a more general setting.

The remainder of this paper is arranged as follows. Basic notions and definitions of graph theory and matrix theory are introduced in §2. In §3, we formulate the dynamical protocol and provide a simulation example to motivate our work, before we formally introduce the definition of the non-trivial balancing set in §4. §5 incorporates the main results in terms of the uniqueness of the non-trivial balancing set and its contribution to the bipartite consensus. §6 presents simulation results on the constructed graph that support the derived theories. Some concluding remarks are given in §7.

2. Preliminaries

2.1. Notation

Let $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Z}_+$ be the set of real numbers, natural numbers and positive integers, respectively. For $n \in \mathbb{Z}_+$, denote $\mathbb{Z} = \{1, 2, \cdots, n\}$. We note specifically that for sets, the notation $|\cdot|$ is used for cardinality. The symmetric matrix $Q \in \mathbb{R}^{n \times n}$ is positive (negative) definite if $z^T Q z > 0$ ($z^T Q z < 0$) for all $z \in \mathbb{R}^n$ and $z \neq 0$, in which case it is denoted by $Q > 0$ ($Q < 0$). When it is positive (negative) semi-definite, denoted by $Q \succeq 0$ ($Q \preceq 0$), if $z^T Q z \geq 0$ ($z^T Q z \leq 0$) for all $z \in \mathbb{R}^n$ and $z \neq 0$. We adopt an extra matrix-valued sign function $\text{sgn}(\cdot) : \mathbb{R}^{n \times n} \mapsto \{0, -1, 1\}$ to express this positive (negative) (semi-)definiteness of a symmetric matrix $Q$, it is defined such that $\text{sgn}(Q) = 1$ if $Q \succeq 0$ and $Q \neq 0$ or $Q > 0$, $\text{sgn}(Q) = -1$ if $Q \preceq 0$ and $Q \neq 0$ or $Q < 0$, and $\text{sgn}(Q) = 0$ if $Q = 0$. We shall employ $|\cdot|$ for such symmetric matrices to denote the operation $\text{sgn}(Q) \cdot Q$, namely, $|Q| = Q$ if $Q > 0$ or $Q \succeq 0$, $|Q| = -Q$ if $Q < 0$ or $Q \preceq 0$, and $|Q| = Q = -Q$ when $Q = 0$. Denote the null space of a matrix $Q \in \mathbb{R}^{n \times n}$ as $\text{null}(Q) = \{z \in \mathbb{R}^n | Qz = 0\}$. The notation $B = \text{blk}(\cdot)$ is used for the block matrix $B$ that is partitioned into the blocks in $\cdot$, and there is further $\text{blkdiag}(\cdot)$ to denote when all the non-zero blocks in $\cdot$ are on the diagonal of $B$; while $\text{blk}_{ij}(B)$ refers to the intersection of the $i$th row block and the $j$th column block of $B$.

2.2. Graph Theory

A multi-agent network can be characterized by a graph $G$ with node set $\mathcal{V} = \{V\}$ and edge set $\mathcal{E} \subseteq V \times V$, for which $e_{ij} = (i, j) \in \mathcal{E}$ if there is a connection between node $i$ and $j$ for $i, j \in V$. Define the matrix-weighted graph (network) $G$ as a triplet $G = (\mathcal{V}, \mathcal{E}, A)$, where $A$ is the set of all weight matrices. A subgraph of $G$ is a graph $G = (\mathcal{V}, \mathcal{E}, A)$ such that $\mathcal{V} \subseteq \mathcal{V}$, $\mathcal{E} \subseteq \mathcal{E}, A \subseteq A$. Let $W(e_{ij})$ denote the weight matrix assigned to edge $e_{ij}$ such that $W(e_{ij}) = A_{ij} \in A \subseteq \mathbb{R}^{d \times d}$. We shall refer to a matrix-weighted network $G = (\mathcal{V}, \mathcal{E}, A)$ with $n$ nodes and $d \times d$ weight matrices as $(n, d)$-matrix-weighted network. Reversely, $W^{-1}(A_{ij}) = e_{ij}$ maps from the weight matrix to the corresponding edge. In this paper, we use symmetric matrices for all edges in $G$, which are $A_{ij} \in \mathbb{R}^{d \times d}$ such that $|A_{ij}| \geq 0$ or $|A_{ij}| > 0$ if $(i, j) \in \mathcal{E}$ and $A_{ij} = 0$ otherwise for all $i, j \in \mathcal{V}$. Thereby the adjacency matrix for a matrix-weighted graph $A = [A_{ij}] \in \mathbb{R}^{d \times d \times d}$ is a block matrix such that the block on the $i$-th row and the $j$-th column is $A_{ij}$. We say an edge $(i, j) \in \mathcal{E}$ is positive (negative) definite or positive (negative) semi-definite if the corresponding weight matrix $A_{ij}$ is positive (negative) definite or positive (negative) semi-definite. Since the graphs considered are simple and undirected, we assume that $A_{ij} = A_{ji}$ for all $i \neq j \in \mathcal{V}$ and $A_{ii} = 0$ for all $i \in \mathcal{V}$. Let $\mathcal{N}_i = \{j \in \mathcal{V} | (i, j) \in \mathcal{E}\}$ be the neighbor set of an agent $i \in \mathcal{V}$. We use $C = \text{blkdiag}(C_1, C_2, \cdots, C_n) \in \mathbb{R}^{d \times d}$ to represent the matrix-weighted degree matrix of a graph where $C_i = \sum_{j \in \mathcal{N}_i} [A_{ij}] \in \mathbb{R}^{d \times d}$. The matrix-valued Laplacian matrix of a matrix-weighted graph is defined as $L(G) = C - A$, which is real and symmetric. The gauge transformation for $G$ is performed by the diagonal block matrix $D = \text{blkdiag} \{\sigma_1, \sigma_2, \cdots, \sigma_n\}$ where $\sigma_i = I_d$ or $\sigma_i = -I_d$. A gauge transformed Laplacian is a matrix $\hat{L}$ such that $\hat{L} = DLD$.

A path $P$ in a matrix-weighted graph $G = (\mathcal{V}, \mathcal{E}, A)$ is defined as a sequence of edges in the form of $(i_1, i_2), (i_2, i_3), \ldots, (i_{p-1}, i_p)$ where nodes $i_1, i_2, \ldots, i_p \in \mathcal{V}$ are all distinct and it is said that $i_1$ is reachable from $i_p$. A matrix-weighted graph $G$ is connected if any two distinct nodes in $G$ are reachable from each other. All graphs mentioned in this paper, unless stated otherwise, are assumed to be connected. The sign of a path $\text{sgn}(P)$ is defined as $\text{sgn}(A_{i_1i_2}) \cdots \text{sgn}(A_{i_{p-1}i_p})$, while the null
space of the path null(\(P\)) refers to \(\bigcup_{k=1}^{\mid P\mid} \text{null}(A_{i_k i_{k+1}})\). A path is said to be positive/negative definite if there is no semi-definite weight matrix on the path, i.e., null(\(P\)) = span\{0\}; otherwise, if null(\(P\)) \neq span\{0\}, the path is positive/negative semi-definite. A positive-negative tree in a matrix-weighted graph is a tree such that every edge in this tree is either positive definite or negative definite. A positive-negative spanning tree of a matrix-weighted graph \(G\) is a positive-negative tree containing all nodes in \(G\). A cycle \(C\) of \(G\) is a path that starts and ends with the same node, i.e., \(C\) = \{(1, 2), (2, 3), ..., (p−1, 1)\}. Note that a spanning-tree does not contain any circle. The sign of the cycle \(\text{sgn}(C)\) is defined similarly as that of the path, we say the cycle is negative if it contains an odd number of negative (semi-)definite weight matrices \(\text{sgn}(C) < 0\), and it is positive if the negative connections are of even number \(\text{sgn}(C) > 0\).

It is well-known that the structural balance of signed networks is a paramount graph-theoretic condition for achieving (bipartite) consensus. For matrix-weighted networks, there is an analogous definition as follows.

**Definition 1.** Pan et al. [12] A matrix-weighted network \(G = (V, E, A)\) is structurally balanced if there exists a bipartition of nodes \(V = V_1 \cup V_2\), \(V_1 \cap V_2 = \emptyset\), such that the matrix-valued weight between any two nodes within each subset is positive (semi-)definite, but negative (semi-)definite for edges connecting nodes of different subsets. A matrix-weighted network is structurally imbalanced if it is not structurally balanced.

By indexing the edges into \(E = \{e_1, ..., e_{\mid E\mid}\}\) along with their weight matrices \(A = \{A_1, ..., A_{\mid E\mid}\}\), we have the following definition of signed incidence matrix for matrix-weighted networks.

**Definition 2.** A signed incidence matrix \(H = \text{blk}[\{I_{d}, -I_{d}, 0_{d \times d}\}]\) of a matrix-weighted network \(G = (V, E, A)\) is an \(\mid E\mid d \times nd\) block matrix for which, the \(k\)-th \(d \times dn\) row block \(H^k, k \in [\mid E\mid]\), corresponds to the edge \(e_k\) with weight matrix \(A_{ij}\) between agent \(i\) and \(j\). The \(i\)-th and \(j\)-th blocks of \(H^k\) are \(I_{d}\) and \(-I_{d}\) respectively if \(A_{ij} \geq 0\), while let them be \(I_{d}\) and \(I_{d}\) if \(A_{ij} \sim 0\); any other block would be \(0_{d \times d}\).

**Lemma 1.** Let \(H\) be the signed incidence matrix of a matrix-weighted network \(G = (V, E, A)\). Then the matrix-valued Laplacian of \(G\) can be characterized by

\[
L = H^T \text{blkdiag}([|A_k|])H,
\]

where the \(k\)-th \(d \times dn\) row block of \(H\) corresponds to the edge whose matrix weight is the \(k\)-th block in \(\text{blkdiag}([|A_k|])\).

**Proof.** The proof is straightforward thus is omitted. □

### 3. Problem Formulation and Motivation

Consider a multi-agent network consisted of \(n \in \mathbb{N}\) agents. The states of each agent \(i \in \mathcal{V}\) is denoted by \(x_i(t) \in \mathbb{R}^d\) where \(d \in \mathbb{N}\). The interaction protocol reads

\[
\dot{x}_i(t) = -\sum_{j \in \mathcal{N}_i} |A_{ij}|(x_i(t) - \text{sgn}(A_{ij})x_j(t)), i \in \mathcal{V},
\]

where \(A_{ij} \in \mathbb{R}^{d \times d}\) denotes the weight matrix on edge \((i, j)\). The collective dynamics of the multi-agent network \(\text{[1]}\) can be characterized by

\[
\dot{x}(t) = -Lx(t),
\]

where \(x(t) = [x_1^T(t), x_2^T(t), \ldots, x_n^T(t)]^T \in \mathbb{R}^{dn}\) and \(L\) is the matrix-valued graph Laplacian.

**Definition 3 (Bipartite Consensus).** The multi-agent network \(\text{[1]}\) is said to admit a bipartite consensus solution if there exists a solution \(x\) such that \(\lim_{t \to \infty} x_i(t) = \lim_{t \to \infty} x_j(t) \neq 0\) or \(\lim_{t \to \infty} x_i(t) = -\lim_{t \to \infty} x_j(t) \neq 0\) for any \(\{i, j\} \subset \mathcal{V}\). When \(\lim_{t \to \infty} x_i(t) = -\lim_{t \to \infty} x_j(t) = 0\) for all \(\{i, j\} \subset \mathcal{V}\), the network admits a trivial consensus.

We employ the following example to motivate our work in this paper.

**Example 1.** Consider the network \(G_1\) in Figure 1 which is structurally imbalanced with one negative circle. One may obtain a structurally balanced network from it by negating the sign of \(e_{23}\) or alternatively, by negating the sign of \(e_{34}\).

![Figure 1: A structurally imbalanced matrix-weighted network \(G_1\). The red solid (resp., dashed) lines denote edges weighted by positive definite (resp., semi-definite) matrices; the blue solid (resp., dashed) lines denote edges weighted by negative definite (resp., semi-definite) matrices.](image)

The edges are endowed with matrix-valued weights

\[
A_{23} = \begin{bmatrix}
-2 & 2 & 0 \\
2 & -2 & 0 \\
0 & 0 & -1
\end{bmatrix} \preceq 0,
\]

\[
A_{24} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} \succeq 0,
\]

\[
A_{12} = \begin{bmatrix}
-2 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix} \prec 0,
\]
and $A_{15} = -A_{23}, A_{45} = A_{12}, A_{34} = -A_{12}$. Note that \( \text{null}(A_{23}) = \text{null}(A_{15}) = \text{span}([1, 1, 0]^T) \) and \( \text{null}(A_{24}) = \text{span}([0, 1, 0]^T) \). The null space of the weight matrices of the remaining edges are trivially spanned by the zero vector.

Under the above selection of edge weights, we examine the evolution of multi-agent system (2) on $G_1$, yielding the state trajectories of each agent shown in Figure 2. Despite the fact that $G_1$ is structurally imbalanced, a bipartition of agents emerges by their steady-states, namely the agents of $V_a = \{2, 3, 4\}$ converge to \( [1.6525 \ 1.6525 \ 0]^T \), others of $V_b = \{1, 5\}$ converge to \(-1.6525 \ -1.6525 \ 0]^T\), and both of the steady states are spanned by \([1 \ 0 \ 1]^T\). What we have noticed is that \([1 \ 1 \ 0]^T\) happens to span the null space of $A_{23}$; interestingly, by negating the sign of $A_{23}$, the resulting network becomes structurally balanced and the structurally balanced partition of nodes is precisely $V_a = \{2, 3, 4\}$ and $V_b = \{1, 5\}$.

4. Balancing Set of Matrix-weighted Networks

A prominent structural feature of the edge we studied in Example 1 concerns its negation of the sign, by which the structurally imbalanced network is rendered structurally balanced. This method was studied in Katai and Iwai [4], Harary [5] which suggested that given any structurally balanced network, one can always transform it into a structurally balanced one with any preassigned node partition by negating the signs of the relevant edges. We refer to this approach in the literature and introduce the following concept to embed it in the context of the matrix-weighted multi-agent networks.

**Definition 4** (Balancing set). Let $(\mathcal{V}_1, \mathcal{V}_2)$ denote a bipartition of node set $\mathcal{V}$ in a network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. Define the $(\mathcal{V}_1, \mathcal{V}_2)$—balancing set $\mathcal{E}^b(\mathcal{V}_1, \mathcal{V}_2)$ as a set of edges such that a $(\mathcal{V}_1, \mathcal{V}_2)$—structurally balanced network can be obtained if the sign of each edge in $\mathcal{E}^b(\mathcal{V}_1, \mathcal{V}_2)$ is negated.

**Example 2.** In Figure 8 we have constructed a matrix-weighted network $\mathcal{G}$ that is structurally balanced. Given a node partition $(\mathcal{V}_1, \mathcal{V}_2)$ of $\mathcal{G}$, its balancing set $\mathcal{E}^b(\mathcal{V}_1, \mathcal{V}_2)$ is consisted of the negative connections within $\mathcal{V}_1$ and $\mathcal{V}_2$ and the positive connections between them (edges colored in red). By negating the signs of the edges in $\mathcal{E}^b(\mathcal{V}_1, \mathcal{V}_2)$, we derive a graph $\mathcal{G}'$ that is structurally balanced between $\mathcal{V}_1$ and $\mathcal{V}_2$.

**Remark 1.** The $(\mathcal{V}_1, \mathcal{V}_2)$—balancing set $\mathcal{E}^b(\mathcal{V}_1, \mathcal{V}_2)$ is empty if and only if the network is $(\mathcal{V}_1, \mathcal{V}_2)$—structurally balanced. For instance, the balancing set $\mathcal{E}^b(\mathcal{V}_1, \mathcal{V}_2)$ of the structurally balanced $\mathcal{G}'$ in Figure 8 is empty since there is no edge to be negated.
Note that the matrix-valued weight plays a role in shaping the null space of the matrix-valued graph Laplacian, thus more constraints on the balancing set are needed to complete the definition. We proceed to quantitatively characterize the contribution of matrix-valued weights to the null space of the matrix-valued graph Laplacian.

**Definition 5.** A set of matrices $A_i \in \mathbb{R}^{n \times n}, i \in \mathcal{I}$, are said to have non-trivial intersection of null spaces, or, to have non-trivial intersecting null space if

$$\bigcap_{i=1}^{l} \text{null}(A_i) \neq \{0\}.$$ 

**Definition 6 (Non-trivial balancing set).** A $(\mathcal{V}_1, \mathcal{V}_2)$-balancing set $\mathcal{E}^{nb}(\mathcal{V}_1, \mathcal{V}_2)$ of a matrix-weighted network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is a non-trivial $(\mathcal{V}_1, \mathcal{V}_2)$-balancing set (NBS), denoted by $\mathcal{E}^{nb}(\mathcal{V}_1, \mathcal{V}_2)$, if the weight matrices associated with edges in $\mathcal{E}^{nb}(\mathcal{V}_1, \mathcal{V}_2)$ have non-trivial intersection of null spaces.

**Remark 2.** The non-trivial balancing set $\mathcal{E}^{nb}(\mathcal{V}_1, \mathcal{V}_2) = \emptyset$ if and only if the corresponding balancing set $\mathcal{E}(\mathcal{V}_1, \mathcal{V}_2) = \emptyset$. In this case, define $W(\mathcal{E}^{nb}) = 0$ where $0$ is the $d \times d$ zero matrix.

A matrix-weighted network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ has a unique non-trivial balancing set if there is only one bipartition $(\mathcal{V}_1, \mathcal{V}_2)$ of $\mathcal{V}$ such that the corresponding $(\mathcal{V}_1, \mathcal{V}_2)$-balancing set meets

$$\bigcap_{e \in \mathcal{E}^{nb}(\mathcal{V}_1, \mathcal{V}_2)} \text{null}(W(e)) \neq \{0\}.$$ 

In this case, we shall denote

$$\text{null}(\mathcal{E}^{nb}) = \bigcap_{e \in \mathcal{E}^{nb}(\mathcal{V}_1, \mathcal{V}_2)} \text{null}(W(e))$$

for brevity.

**Remark 3.** It is noteworthy that to have only an empty non-trivial balancing set does not suggest there is no NBS in the graph; a graph without NBS is a structurally imbalanced graph for which any partition of nodes has a balancing set whose weight matrices share a trivial intersecting null space.

## 5. Main Results

### 5.1. General Networks

In this section, we set to examine the validity of the concept termed as the non-trivial balancing set through its correlation with the network steady-state, bearing in mind the question of to what extent is the non-trivial balancing set a satisfactory interpretation of the numerical solutions.

We shall recall some facts about the algebraic structure of the Laplacian null space when bipartite consensus is achieved on matrix-weighted networks. Also, technical preparations Lemma 3 and Lemma 4 are presented for the proof of Theorem 1 and Theorem 2, one is referred to the Appendix for their proofs.

**Lemma 2.** Su et al. [17] The matrix-weighted multi-agent network [2] achieves bipartite consensus if and only if there exists a gauge transformation $D$ such that $\text{null}(L) = S = \text{span}(D(1_n \otimes \Psi))$, where $D$ is a gauge transformation, $\Psi = [\psi_1, \psi_2, ..., \psi_{\mathcal{S}}]$, and $\psi_i, i \in \mathcal{S}, s \leq d$, are orthogonal basis vectors in $\mathbb{R}^d$.

**Remark 4.** The space $S$ in Lemma 2 is defined as the bipartite consensus subspace in Su et al. [17] and in Pan et al. [12], when $s = d$, $S$ is proved to be the bipartite consensus subspace of a structurally balanced matrix-weighted network.

**Lemma 3.** Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a matrix-weighted network with $n$ agents, each of dimension $d$. If $\mathcal{G}$ is structurally balanced with partition $(\mathcal{V}_1, \mathcal{V}_2)$, then there exists a gauge transformation $D \in \mathbb{R}^{nd \times nd}$ such that $\text{span}(D(1_n \otimes I_d)) \subseteq \text{null}(L(\mathcal{G})), \text{ and blk}_{ii}(D) = I_d$ for $i \in \mathcal{V}_1$ and $\text{blk}_{ii}(D) = -I_d$ for $i \in \mathcal{V}_2$, where $L(\mathcal{G})$ is the network Laplacian.

**Lemma 4.** For any nonzero $v_1, v_2, v \in \mathbb{R}^d$, the linear combination of $D_1(1_n \otimes v_1)$ and $D_2(1_n \otimes v_2)$ do not yield $D(1_n \otimes v)$ whether or not $v_1$ and $v_2$ are linearly independent, where $D$ is a gauge transformation, $D_1, D_2$ are gauge transformations with $D_1 \neq D_2$ and $D_1 \neq -D_2$.

In the Preliminary section, the gauge transformation is introduced as a block matrix $D = \text{blkdiag}(I_d, -I_d)$ for which the identity blocks $I_d, -I_d$ are sequenced on the diagonal with a certain pattern of signs. Since the gauge matrix $D$ has $N$ $Nd$-by-$d$ column blocks, the non-zero diagonal blocks could be considered to have a one-to-one correspondence with the well-indexed $N$ agents, reflecting either the relative positivity/negativity of their steady states, or their bipartition by structure. To exert this correspondence, consider a matrix-weighted network $\mathcal{G}$ for
which a partition \((V_1, V_2)\) of the node set \(V\) defines a balancing set \(E^b(V_1, V_2)\). We map this partition onto a gauge matrix \(D\) such that \(\text{blk}_1(D) = I_d\) for \(i \in V_1\) and \(\text{blk}_2(D) = -I_d\) for \(i \in V_2\), suppose the nodes are properly indexed. The node partition is then fully described by the pattern of the signs of the diagonal blocks, and we phrase \(E^b(V_1, V_2)\) as a balancing set with division \(D\) or of \(D\)-division to notify how the nodes are actually partitioned given the edge set \(E^0(V_1, V_2)\).

We are ready to establish how the non-trivial balancing set is related to the null space of the matrix-valued Laplacians.

**Theorem 1.** For a matrix-weighted network \(G\), the following properties are equivalent:

1) there exists a non-trivial balancing set \(E^{nb}\) in \(G\) with division \(D\), such that \(\text{span}\{\Xi\} \subseteq \text{null}(E^{nb})\),

2) \(\text{span}\{D(I_a \otimes \Xi)\} \subseteq \text{null}(L(G))\),

where \(D\) is a gauge transformation and \(\Xi = [\xi_1, \ldots, \xi_r]\) where \(\xi_i \in \mathbb{R}^d, i \in \mathcal{I}, 0 < r \leq d\) are linearly independent.

**Proof.** For any matrix-weighted network the graph Laplacian can be expressed as

\[
L = H^T \text{blkdiag}\{[A_k]\} H,
\]

where \(H\) is the signed incidence matrix and the blocks in \(\text{blkdiag}\{[A_k]\}\) are ordered the same as their appearances in \(H\). Thus, \(Lx = 0\) if and only if \(\text{blkdiag}\{[A_k]\} \nabla Hx = 0\), namely,

\[
[A_{ij}]^\nabla (x_i - \text{sgn}(A_{ij})x_j) = 0, \forall (i,j) \in E.
\]

Note that

\[
(x_i - \text{sgn}(A_{ij})x_j)^T |A_{ij}| (x_i - \text{sgn}(A_{ij})x_j) = ||A_{ij}||^2 (x_i - \text{sgn}(A_{ij})x_j) ||^2 = 0,
\]

then

\[
[A_{ij}]^\nabla (x_i - \text{sgn}(A_{ij})x_j) = 0
\]

if and only if

\[
A_{ij}(x_i - \text{sgn}(A_{ij})x_j) = 0, \tag{3}
\]

which implies

\[
Hx \in \text{null}(\text{blkdiag}\{[A_k]\})
\]

where the weight matrices \(A_{ij}\) are relabelled as \(A_k, k \in |E|\).

1) \(\rightarrow\) 2): Consider when the network has a non-trivial balancing set \(E^{nb}\) with division \(D\). Without loss of generality, we block the signed incidence matrix \(H\) as \(H = [H_1^T \ H_2^T]^T\) where \(H_2\) corresponds to edges in \(E^{nb}\) whose weights have intersecting null space that is non-trivial. We know from Definition 4 that the edges in \(H_1\) constructs a structurally balanced subgraph, and since \(E^{nb}(V_1, V_2)\) is with division \(D\), the blocks of \(D\) are assigned as \(\text{blk}_1(D) = I_d\) for \(i \in V_1\) and \(\text{blk}_2(D) = -I_d\) for \(i \in V_2\).

From Lemma 3 the gauge transformation \(D\) satisfies

\[
\text{span}\{D(I_a \otimes I_d)\} \subseteq \text{null}(H_1^T H_1)
\]

because \(H_1^T H_1\) is the Laplacian matrix of a structurally balanced network with the same topology as the \(H_1\) subgraph except the absolute weights are all identity matrices. It is then derived that \(\text{span}\{H_1(D(I_a \otimes I_d))\} = \{0\}\). Now consider \(H_2 D(1_a \otimes \xi_p)\) where

\[
\xi_p \in \text{span}\{\xi_1, \ldots, \xi_r\} \subseteq \text{null}(E^{nb}) = \bigcap_{A_j \in W(E^{nb})} \text{null}(A_j), p \in \mathcal{I}
\]

For any row block \(H_2^j \in \mathbb{R}^{d \times nd}\) of \(H_2\), it is composed of either \(\pm I_d\) or 0, and the corresponding weight matrix \(A_j\) breaks the structural balance of the \(H_1\) subgraph. That means, if \(H_2^j\) has two \(I_d\) matrices, the corresponding weight matrix is negative (semi-)definite and the \(H_1\) subgraph puts the connected vertices in the same partition (e.g., both are in \(V_1\)), therefore \(H_2^j D\) has two \(I_d\) matrices or two \(-I_d\) matrices. If \(H_2^j\) has \(I_d\) and \(-I_d\), the corresponding weight matrix is positive (semi-)definite and the \(H_1\) subgraph puts the connected vertices in different partitions (one in \(V_1\) and the other in \(V_2\)), then \(H_2^j D\) has two \(I_d\) matrices or two \(-I_d\) matrices. Thus

\[
H_2 D(1_a \otimes \xi_p) = 2[\pm e^T_p, \ldots, \pm e^T_p]^T \in \mathbb{R}^{n \times |E|}
\]

and since \(\text{span}\{1_a \otimes \xi_p\} \subseteq \text{span}\{1_a \otimes I_d\}\),

\[
H D(I_a \otimes \xi_p) = \begin{bmatrix} H_1 D(I_a \otimes \xi_p) & H_2 D(I_a \otimes \xi_p) \\ 0 |E| \in |E| & 2 & 0 \end{bmatrix}
\]

Block the matrix \(A\) as \(A = \text{blkdiag}\{[A_1, A_2]\}\), where matrix \(A_1\) has the weight matrices of the \(H_1\) subgraph as its diagonal blocks, and matrix \(A_2\), the \(H_2\) subgraph. Then we have

\[
\text{blkdiag}\{[A_1, A_2]\} H D(I_a \otimes \xi_p) = 0 |E| \text{ for } p \in \mathcal{I}
\]

Because the \(H_2\) subgraph contains edges in \(E^{nb}\) and \(\xi_p \in \bigcap_{A_j \in W(E^{nb})} \text{null}(A_j)\), therefore \(A_j H_2 D(1_a \otimes \xi_p) = 0\) holds, for \(p \in \mathcal{I}\). Hence we have proved that \(\text{span}\{D(1_a \otimes \Xi)\} \subseteq \text{null}(L)\).

2) \(\rightarrow\) 1): Consider when the Laplacian has \(D(1_a \otimes \Xi) \subseteq \text{null}(L)\) for \(p \in \mathcal{I}\), which means to assign the nodes with \(+\xi_p\) or \(-\xi_p\) according to the sign pattern of \(D\) satisfies \(A_{ij}(x_i - \text{sgn}(A_{ij})x_j) = 0, \forall (i,j) \in E\). The sign pattern of \(D\) corresponds to a partition \((V_1, V_2)\) which defines a balancing set \(E^b(V_1, V_2)\) on the graph. For any edge \(e_{lm} \in E^b(V_1, V_2)\), it either a) connects within \(V_1\) or \(V_2\) and has \(A_{lm} \prec 0(A_{lm} \preceq 0)\), or b) connects between \(V_1\) and \(V_2\) and has \(A_{lm} \preceq 0(A_{lm} \succeq 0)\). Note that those in \(V_1\) are assigned with \(+\xi_p\) by \(D(1_a \otimes \xi_p)\) while those in \(V_2\) are assigned with \(-\xi_p\). Therefore for a), eqn. \(\mathbf{3}\) gives \(A_{lm}(\xi_p + \xi_p) = 0\) and \(\xi_p \in \text{null}(A_{lm})\); for b), eqn. \(\mathbf{3}\) gives \(A_{lm}(\xi_p - \xi_p) = 0\) and \(\xi_p \in \text{null}(A_{lm})\).
by Corollary 1, formations $D \in E$ a corresponding non-trivial balancing set $D$ span $\Psi)$ then let $\bar{\Psi}$ partite consensus solution with a steady state $\bar{\Psi}$ if bipartite consensus is achieved on the matrix-weighted network, there is at least one NBS in the Theorem 1 has illustrated how the existence of a non-trivial balancing set $\bar{\Psi}$ whose partition follows the sign pattern of $\bar{\Psi}$, and the columns of $\bar{\Psi}$ are included in the non-trivially intersecting null space $\null(E^{nb})$. The correlation plays a central role in establishing the fact that when bipartite consensus is admitted, there is at least one NBS in the matrix-weighted network. We are now able to derive a necessary condition on the bipartite consensus for matrix-weighted networks in general.

**Theorem 2.** If the multi-agent system $G$ admits a bipartite consensus solution with a steady state $\bar{x} \neq \mathbf{0}$, then there exists a unique non-trivial balancing set $E^{nb}$ in $G$ such that $\bar{x} \in \null(E^{nb})$ for all $i \in V$.

**Proof.** If bipartite consensus is achieved on the matrix-weighted network $G$, then by Lemma 2 there is $\null(L) = \span\{D(1_n \otimes \Psi)\}$ where $\psi_1, ..., \psi_r$ are orthogonal vectors, and for the steady state we have $\bar{x} \in \span\{D(1_n \otimes \Psi)\}$, i.e., $\bar{x} \in \span\{\Psi\}$. According to Theorem 1 $\span\{D(1_n \otimes \Psi)\} \subset \null(L)$ implies that there exists a non-trivial balancing set $E^{nb}(V_1, V_2)$ with division $D$ such that $\span(\Psi) \subset \null(E^{nb})$. We will first show that actually, $\span(\Psi) = \null(E^{nb})$ by raising the fact that $\rank(\null(L)) = \rank(D(1_n \otimes \Psi)) = \rank(1_n \otimes \Psi) = \rank(1_n) \cdot \rank(\Psi) = \rank(\Psi)$. If there is $\psi^* \in \mathbb{R}^d$ for which $\psi^* \in \null(E^{nb})$ but $\psi^* \notin \span(\Psi)$, proposition 1) $\rightarrow$ 2) of Theorem 1 states that $D(1_n \otimes \psi^*) \in \null(L)$, then let $\Psi' = [\psi_1, ..., \psi_r, \psi^*]$, there is $\span\{D(1_n \otimes \Psi')\} \subset \null(L)$ and $\null(L)$ is raised by rank one. Therefore there exists $E^{nb}(V_1, V_2)$ with division $D$ such that $\null(L) = D(1_n \otimes \null(E^{nb}(V_1, V_2)))$.

Now suppose there exists another partition $(V'_1, V'_2)$ with a corresponding non-trivial balancing set $E^{nb}(V'_1, V'_2)$, then by Corollary 1

$$x^* \in D'(1_n \otimes \null(E^{nb}(V'_1, V'_2))) \subset \null(L),$$

while $D \neq D'$ and $D \neq -D'$. Meanwhile Lemma 1 states that since $\bar{x}$ and $x^*$ are composed of distinct gauge transformations $D$ and $D'$, their linear combination does not yield any vector of the form $D(1_n \otimes v)$ for $v \in \mathbb{R}^d$, which implies $k_1 \bar{x} + k_2 x^* \notin \span(D(1_n \otimes \Xi))$. Therefore one has

$$k_1 \bar{x} + k_2 x^* \notin \null(L),$$

for $k_1, k_2 \neq 0$, which is a contradiction. Thus $E^{nb}(V_1, V_2)$ is the only non-trivial balancing set in the network and satisfies $\null(E^{nb}) = \span(\Xi)$, then the agents converge to the non-trivially intersecting null space of the unique NBS $\bar{x} \in \null(E^{nb})$.

A non-trivial balancing set is defined by a partition of nodes that aims at achieving structural balance on itself; Theorem 2 states that when bipartite consensus is admitted on a matrix-weighted network, one is bound to find a non-trivial balancing set, a set of edges, as a third party that prevents the structural balance from happening, and is with a non-trivially intersecting null space. For any other grouping of agents, one would find their corresponding balancing set to have null spaces that intersect only trivially.

Looking back on the definition of the non-trivial balancing set, we see that when the NBS is somehow unique, the bipartition of the agents’ convergence states is mirrored in the particular grouping $(V_1, V_2)$ of this NBS (which is also encoded in $D$). Even more noteworthy is the intersecting null space $\null(E^{nb})$ that directly contributes to the Laplacian null space, as is indicated by $\null(L) = D(1_n \otimes \null(E^{nb}))$, which means the agents converge to a linear combination of the vectors that span $\null(E^{nb})$. Based on this impression, the definition of the NBS is rather a rephrasing of those vectors of the crucial form $D(1_n \otimes \Xi)$ whose role is immediately twofold: to split the agents into groups and to grant the convergence state of the network.

5.2. Networks with A Positive-negative Spanning Tree

In this subsection, we examine the matrix-weighted network with a positive-negative spanning tree. The following theorem is derived with respect to the non-trivial balancing set.

**Theorem 3.** For a matrix-weighted network $G$ with a positive-negative spanning tree, under protocol 1, we have:

1) bipartite consensus is admitted if and only if it has a unique non-trivial balancing set;
2) trivial consensus is admitted when no non-trivial balancing set is present in the graph.

**Proof.** First we provide the proof of part 1).

(Sufficiency) When the network has a unique non-trivial balancing set $E^{nb}$ with division $D$, Corollary 1 suggests that $D(1_n \otimes \null(E^{nb})) \subset \null(L)$. We proceed to show that in fact, in the presence of the positive-negative spanning tree, $D(1_n \otimes \null(E^{nb}))$ span the whole Laplacian null space. We know from the previous proof that to derive $\null(L)$ is to solve for $x$ that satisfies a series of equations

$$A_{ij}(x_i - \sgn(A_{ij}) x_j) = 0, \forall (i, j) \in E. \quad (5)$$
Because for any two nodes there is a path with edges whose weight matrices are all positive (negative) definite, solve equation 5 along the path, we can only derive \( x_p = x_q \) or \( x_p = -x_q \) for any pair of nodes \( p, q \in V \). Thus any solution in the Laplacian null space could be represented as \( D'(1_n \otimes w) \) for some \( w \in \mathbb{R}^n \) and a gauge matrix \( D' \).

Suppose there exists \( x^* = D'(1_n \otimes w) \in \text{null}(L) \) with \( D' \neq D \) and \( D' \neq -D \), then according to Theorem 1 there exists a non-trivial balancing set \( \mathcal{E}^{\text{nb}} \) with division \( D' \) in the network which contradicts our premise that \( \mathcal{E}^{\text{nb}} \) is unique. Now suppose there is \( x^* = D(1_n \otimes w) \in \text{null}(L) \) with \( w \notin \text{null}(\mathcal{E}^{\text{nb}}) \), then with equation (4) we have \( \text{blkdiag}(A_k)HD(1_n \otimes w) \neq 0 \), hence \( D(1_n \otimes w) \notin \text{null}(L) \) which is clearly a contradiction. Therefore the Laplacian null space is spanned by \( D(1_n \otimes \text{null}(\mathcal{E}^{\text{nb}})) \), and bipartite consensus is admitted.

(Necessity) This is readily verified with Theorem 2.

Now we proceed to prove part 2). We have established that in the presence of a positive-negative spanning tree, for any \( p, q \in V \) there is \( x_p = \pm x_q \) by solving 5 along the positive-negative definite path that connects them. The tree contains no circle, the relative positivity and negativity of \( x_p \) and \( x_q \) naturally gives a partition \( \{V_1, V_2\} \) of the node set which defines a balancing set \( \mathcal{E}^b(V_1, V_2) \). Note that the tree itself is structurally balanced with respect to this division, thus the edges of the tree are not included in \( \mathcal{E}^b(V_1, V_2) \); in other words, edges in \( \mathcal{E}^b(V_1, V_2) \) either has \( A_{ij} < 0(A_{ij} \leq 0) \) and connects within \( V_1 \) or \( V_2 \) (where the solution has been \( x_p = x_q \)), or has \( A_{ij} > 0(A_{ij} \geq 0) \) and connects between \( V_1 \) and \( V_2 \) (where the solution has been \( x_p = -x_q \)). When no non-trivial balancing set is found in the network, it means there exist two edges \( W(e_1) = W((i_k, i_{k+1})) = A_1 \) and \( W(e_2) = W((i_{k'}, i_{k'+1})) = A_2 \) in the balancing set \( \mathcal{E}^b(V_1, V_2) \) that have \( \text{null}(A_1) \cap \text{null}(A_2) = \{0\} \). Both edges satisfy

\[
\begin{align*}
 x_{k_1} - \text{sgn}(A_1)x_{k_1+1} &= 2x_{k_1} \in \text{null}(A_1) \\
 x_{k_2} - \text{sgn}(A_2)x_{k_2+1} &= 2x_{k_2} \in \text{null}(A_2),
\end{align*}
\]

and since \( \text{null}(A_1) \cap \text{null}(A_2) = \{0\} \), the fact that \( x_{k_1} = \pm x_{k_2} \) gives \( x_{k_1} = x_{k_2} = 0 \), therefore \( x_i = 0 \) for \( i \in V \) and a trivial consensus is admitted on the network.

One fact suggested by Graph Theory is that a network may be spanned by trees with different choices of edges; one could consider when there is another positive-negative spanning tree that gives a distinct partition by solving eqn. (3). Since no non-trivial balancing set exists in the network, i.e., every balancing set is trivial for all possible partitions of \( V \), the above reasoning applies for all positive-negative spanning trees and the conclusion stands.

In Theorem 2 the uniqueness of the non-trivial balancing set is proposed as a necessary condition for the bipartite consensus on matrix-weighted networks with general structural features. Therefore in Theorem 3 we confine ourselves to matrix-weighted networks with positive-negative spanning trees and have found the uniqueness of the NBS to be both necessary and sufficient, with the assistance of the intrinsic structural balance of the positive-negative spanning tree in the sufficient part. We have also established that to have at least one NBS is quite necessary for such networks to admit any steady-state other than the trivial consensus.

For the derivation of the theorems so far, we mention that though some of the discussions in the proofs have touched on the notion of structural balance, the idea itself is not engaged in the formulation of the theorems, where the steady-state behaviour is directly associated with the existence (or non-existence) of the NBS. It is safe to say the non-trivial balancing set has taken the place of the structural balance as a proper indication of the system behaviour, and the graph-theoretic correspondence is partly rebuilt.

Remark 5. Considering the necessary and sufficient condition for the bipartite consensus derived on the scalar-weighted network (Altalini 1), which is the structural balance property of the network, we are aware that this is well incorporated into the framework of Theorem 3 since all weights that are scalar are the \( 1 \times 1 \) positive/negative definite matrix weights, and the positive-negative spanning tree naturally exists.

5.3. A Counter Example and A Sufficient Condition

It is only natural, at this point, to ask if there is any possibility for the uniqueness of the NBS to be also conveniently sufficient even in the absence of a positive-negative spanning tree. However, we have come to a negative conclusion on this by raising the following counter-example.

Example 3. The matrix-weighted network \( G_{\text{counter}} \) is a 7-node structurally imbalanced network with the unique non-trivial balancing set being \( \mathcal{E}^{\text{nb}}(V_1, V_2) = \{e_{23}, e_{25}, e_{46}\} \), given by the illustration in Figure 4 with the following weight matrix arrangements:

![Figure 4: The matrix-weighted network \( G_{\text{counter}} \) for Example 3.](image-url)
somewhat trivial, under which a node outside $G(T)$ can merge with it through semi-definite paths. For now we denote the network with a positive-negative spanning tree and a non-trivial balancing set as $G^{nb}(T)$. According to Theorem 4, $G^{nb}(T)$ admits the bipartite consensus, so that part of the agents converge to $\zeta \in \mathbb{R}^d$, of which we assume the signs to be $\text{sgn}(x_i) = 1$, while other agents would converge to $-\zeta$ and their signs are written as $\text{sgn}(x_i) = -1$. A node can merge with $G^{nb}(T)$ if the expanded network obtains bipartite consensus altogether.

**Theorem 4.** Consider a matrix-weighted network $G^{nb}(T)$ with a positive-negative spanning tree and a non-trivial balancing set, which naturally has all its agents converging either to $\zeta \in \mathbb{R}^d$ or $-\zeta \in \mathbb{R}^d$ under (9). Suppose a vertex $i_r \notin G^{nb}(T)$ has $m$ paths $P_k = \{(i_r, i_{k+1}), \ldots, (i_{r|P_k|}, i_{k+1})\}, k \in m$, to reach $G^{nb}(T)$, each path has only its last vertex in $G^{nb}(T)$, which is $i_{|P_k|+1} \in G^{nb}(T), k \in m$. Then $i_r$ merge with $G^{nb}(T)$ if for any $k_1, k_2 \in m$, there is $\text{sgn}(P_{k_1})\text{sgn}(x^{i_{k_1}}_{|P_{k_1}|+1}) = \text{sgn}(P_{k_2})\text{sgn}(x^{i_{k_2}}_{|P_{k_2}|+1})$, and $\cap_{k=1}^m \text{null}(P_k) = 0$.

**Proof.** Suppose $P_1, P_2 \in \{P_k\}, i_1^{k_1|P_{k_1}|+1}, i_2^{k_2|P_{k_2}|+1} \in G^{nb}(T)$, $x^{1|P_{k_1}|+1}, x^{2|P_{k_2}|+1}$ denote their final states. Then there is

$$x_r - \text{sgn}(P_1)x^{1|P_{k_1}|+1} \in \text{null}(P_1),$$

$$x_r - \text{sgn}(P_2)x^{2|P_{k_2}|+1} \in \text{null}(P_2),$$

since $\text{sgn}(P_1)\text{sgn}(x^{1|P_{k_1}|+1}) = \text{sgn}(P_2)\text{sgn}(x^{2|P_{k_2}|+1})$. The left-hand sides are both about $x_r - \text{sgn}(P_1)\text{sgn}(x^{2|P_{k_2}|+1})$. With $\text{null}(P_1) \cap \text{null}(P_2) = \{0\}$, $x_r$ has a unique solution $\text{sgn}(P_1)x^{1|P_{k_1}|+1}$ thus is merged with $G^{nb}(T)$. \[\square\]

Theorem 4 extends our study of the spanning-tree case in Theorem 3 to when there exist positive/negative semi-definite paths between agents, the conclusion being a sufficient condition. We also have the inference that on a general matrix-weighted network, if the subgraphs spanned by positive/negative-definite trees do not have their separate non-trivial balancing sets in the first place, then the network as a whole is incapable of achieving bipartite consensus.

6. Simulation Example

This section provides numerical examples of the theorems we have derived, based on the network constructed in Figure 1. Now we can see that $G_1$ is structurally imbalanced with a unique non-trivial balancing set $\{e_2\}$, which yields a structurally balanced node partition $\{1, 5\}, \{2, 3, 4\}$. Under dynamics (1) the agents admit bipartite consensus as in Figure 2 and the final states are determined by the intersecting null space of the NBS $\text{span}([1 \ 0 \ 0]^T)$. 

\[\begin{align*}
A_{23} &= \begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \preceq 0, \quad v_a = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \\
A_{14} &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \succeq 0, \quad v_c = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \\
A_{17} &= \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix} \succeq 0, \quad v_d = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \\
A_{12} &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \succeq 0,
\end{align*}\]

and $A_{23} = A_{25} = A_{16}$, $A_{12} = A_{13} = A_{45} = A_{56} = A_{17}$. We have written down the vectors that span the null spaces of the semi-definite weight matrices. It is seen that the network $G_{\text{counter}}$ consists of four independent circles, three of which are negative and one is positive. The non-trivial balancing set must enclose edges that eliminate the negative circles simultaneously without generating any other one. The existence of the positive circle $\{1, 4, 7\}$ has restrained the NBS from including $e_{14}$ as a result, despite that $e_{23}, e_{14}, e_{46}$ share the same eigenvector $v_9$ for the zero eigenvalue. $E^{sd}(V_1, V_2) = \{e_{23}, e_{25}, e_{46}\}$ is then unique as a non-trivial balancing set; however, we could see from Figure 5 that the numerical solution suggests the network yields a non-trivial consensus solution, rather than a bipartite consensus solution, for the structurally imbalanced $G_{\text{counter}}$. Therefore the uniqueness of the NBS alone is not a sufficient condition in any strict sense for general matrix-weighted networks.

We would like to close the main study of this work with a sufficient condition on the bipartite consensus, albeit...
Now suppose the edge weight \( A_{34} \) is also semi-definite, and set \( A_{34} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix} \). Then \( G_1 \) has two non-trivial balancing sets that give different node partitions. While \( \{e_{23}\} \) still partition the agents into \( \{1,5\}, \{2,3,4\} \), \( \{e_{34}\} \) produces partition \( \{1,5,3\}, \{2,4\} \). Figure 6 shows that bipartite consensus is not achieved under this circumstance.

One could easily turn Figure 6 into a graph without any non-trivial balancing set by setting \( A_{23} = A_{12} \). In this case \( G_1 \) has a positive-negative spanning tree, and as expected, the agents have only admitted a trivial consensus since there is no NBS in the graph, refer to Figure 7.

7. Concluding Remarks

In this paper, we have established the significance of the non-trivial balancing set to the bipartite consensus of matrix-weighted networks. It is shown that the uniqueness of such a set is a necessary condition in admitting the bipartite consensus. Moreover, if bipartite consensus is indeed achieved, the final states of the agents are determined by none other than the intersecting null space of the non-trivial balancing set. The uniqueness of the NBS is specifically studied on networks with positive-negative spanning trees, which turns out to be both necessary and sufficient for the bipartite consensus. Based on this conclusion, we have given the condition to extend the tree with semi-definite matrix-weighted paths while preserving the bipartite consensus on the resulting network. However, we are aware that this condition is formulated in an algorithmic fashion that does not involve much structural attribute of the network; for future research, we would expect the establishment of a sufficient condition for the bipartite consensus with the concept of the NBS that is more structure-based and applicable for specific control problems.

Appendix

1. Proof for Lemma 3

**Proof.** If \( G \) is connected and \( (V_1, V_2) \) – structurally balanced, construct a gauge transformation matrix \( D \) such that \( \text{blk}_{ii}(D) = I_d \) for \( i \in V_1 \) and \( \text{blk}_{ii}(D) = -I_d \) for \( i \in V_2 \), let \( x \in \text{span}\{D(I_n \otimes I_d)\} \), then \( A_{ij}(x_i - \text{sgn}(A_{ij})x_j) = 0 \) holds for all \( (i, j) \in E \), because when \( x_i = x_j \), there is \( i, j \in V_1 \) or \( i, j \in V_2 \), and \( \text{sgn}(A_{ij}) > 0 \) \( (\text{sgn}(A_{ij}) \geq 0) \); when \( x_i = -x_j \), there is \( i \in V_1, j \in V_2 \) or \( i \in V_2, j \in V_1 \), and \( \text{sgn}(A_{ij}) < 0 \) \( (\text{sgn}(A_{ij}) \leq 0) \). Therefore we have \( \text{span}\{D(I_n \otimes I_d)\} \subset \text{null}(L(G)) \).

If \( G \) is disconnected, note that \( G \) is structurally balanced if and only if all its components are structurally balanced. Denote the components of \( G \) as \( G_i = (V_i, E_i, A_i) \) when \( i \in q \) and \( |V_i| = n_i \). Let \( L_i \) denote the matrix-valued Laplacian of \( G_i \) for all \( i \in q \). Then

\[
L(G) = \text{blkdiag}\{L_i\}.
\]

Again there exist \( D_i \in \mathbb{R}^{n_i \times n_i} \) such that \( \text{span}\{L_i^{T}D_i(I_n \otimes I_d)\} = \{0\} \) for all \( i \in q \). Therefore, one can choose \( D = \text{blkdiag}\{D_i\} \), then there is \( \text{span}\{D(I_n \otimes I_d)\} \subset \text{null}(L(G)) \) which completes the proof. \( \square \)

**Lemma 5.** For a set of linearly independent vectors \( v_1, ..., v_r \in \mathbb{R}^d, 2 \leq r \leq d \), with \( \forall k_i \neq 0, i \in \mathbb{R} \), the linear combination \( x = k_1D_1(I_n \otimes v_1) + ... + k_rD_r(I_n \otimes v_r) \neq D(I_n \otimes v) \) where \( v \in \mathbb{R}^d \) and \( D \) is a gauge transformation, if the sign patterns of the gauge transformations \( D_1, D_2, ..., D_r \) are distinct from each other, that is, there is no \( D_p, D_q \) with \( D_p = D_q \) or \( D_p = -D_q, p, q \in \mathbb{R} \).
Proof. Write $x$ in its block form as $x = \text{blk}(x_1^T x_2^T \ldots x_n^T)^T$, $x_k \in \mathbb{R}^d$. Suppose we use $D_1(1_n \otimes v_1), D_2(1_n \otimes v_2), \ldots, D_r(1_n \otimes v_r)$ for the linear combination, then

$$x = \sum_{j=1}^{r} k_j D_j (1_n \otimes v_j)$$  \hspace{2cm} (6)

The blocks of $x$ are written as

$$x_1 = z_{11} v_1 + \cdots + z_{1r} v_r,$$
$$x_2 = z_{21} v_1 + \cdots + z_{2r} v_r,$$
$$\ldots$$
$$x_l = z_{l1} v_1 + \cdots + z_{lr} v_r,$$
$$\ldots$$
$$x_n = z_{n1} v_1 + \cdots + z_{nr} v_r,$$

where $|z_{ij}| = |z_{ji}| = \ldots = |z_{nj}| = |k_j|$ for $j \in \mathbb{L}$. The sign pattern of a gauge transformation is the sequence of signs of the diagonal blocks, $\{1 +1 \quad 1 \quad -1 \}$ for $\text{blkdiag}(I_d, I_d, \ldots, I_d)$ for instance. We use $\text{sgn}(D^j)$ to denote the sign of the $i$th diagonal block of gauge transformation $D_j$, which can be either $+1$ or $-1$. Then $z_{ij} = \text{sgn}(D^j) k_j$.

Note that the gauge transformations $D_1$ and $D_r$ are of different sign patterns, therefore there exist two blocks of $x$, say, $x_1$ and $x_t$, so that

$$\text{sgn}(D^l_1) = \text{sgn}(D^r_1)$$  \hspace{2cm} (7)

and

$$\text{sgn}(D^l_1) = -\text{sgn}(D^r_1).$$  \hspace{2cm} (8)

Suppose $x = D(1_n \otimes v)$, then we should have $x_l = \pm x_t$. If $x_l = x_t$, then (a) suppose $\text{sgn}(D^l_1) = \text{sgn}(D^r_1)$, as a consequence of eqn. (7) and (8), there is $\text{sgn}(D^l_1) = -\text{sgn}(D^r_1)$, i.e., $z_{l1} = z_{r1}$ and $z_{lt} = -z_{rt}$. So when we equate $x_l$ and $x_t$, there is

$$(z_{r2} - z_{l2}) v_2 + \cdots + (z_{r_{t-1}} - z_{l_{t-1}}) v_{t-1} + 2 z_{lt} v_r = 0,$$

then $v_2, \ldots, v_t$ becomes linearly dependent since there is at least $z_{lt} \neq 0$, thus we have derived a contradiction; (b) suppose $\text{sgn}(D^l_1) = -\text{sgn}(D^r_1)$, then there is $\text{sgn}(D^l_1) = \text{sgn}(D^r_1)$, i.e., $z_{l1} = -z_{r1}$ and $z_{lr} = z_{rt}$, so when we equate $x_l$ and $x_t$, we have

$$2z_{l1} v_1 + (z_{l2} - z_{r2}) v_2 + \cdots + (z_{l_{t-1}} - z_{r_{t-1}}) v_{t-1} = 0,$$

which contradicts the fact that $v_1, \ldots, v_{t-1}$ are linearly independent.

For $x_l = -x_t$, the contradictions can be derived similarly by discussing (a) $\text{sgn}(D^l_1) = \text{sgn}(D^r_1)$ and (b) $\text{sgn}(D^l_1) = -\text{sgn}(D^r_1)$. \hfill $\square$

3. Proof for Lemma 5. When $v_2 = k_r v_1$, suppose $x = \alpha D_1 (1_n \otimes v_1) + \beta D_2 (1_n \otimes v_2) = D (1_n \otimes v), \alpha \neq 0, \beta \neq 0$, then the blocks of $x$ are written as $x_l = (\text{sgn}(D^1_1) \alpha + \text{sgn}(D^2_2) \beta) v_1, i = 1, \ldots, n$. Because we can find $x_p$ and $x_q$ with $\text{sgn}(D^p_1) = \text{sgn}(D^q_2)$ and $\text{sgn}(D^p_1) = -\text{sgn}(D^q_2)$, there is $x_p = \text{sgn}(D^p_1)(\alpha + \beta) v_1$, $x_q = \text{sgn}(D^q_2)(\alpha - \beta) v_1$. Let $x_p = x_q$ or $x_p = -x_q$ we can always derive $\alpha = 0$ or $\beta = 0$, thus is a contradiction. \hfill $\square$

References

[1] Altafini, C., 2012. Consensus problems on networks with antagonistic interactions. IEEE Transactions on Automatic Control 58, 935–946.

[2] de Badyn, M.H., Mesbahi, M., 2020. H2 performance of series-parallel networks: A compositional perspective. IEEE Transactions on Automatic Control.

[3] Barooah, P., Hespanha, J.P., 2008. Estimation from relative measurements: Electrical analogy and large graphs. IEEE Transactions on Signal Processing 56, 2181–2193.

[4] Foight, D.R., de Badyn, M.H., Mesbahi, M., 2020. Performance and design of consensus on matrix-weighted and time scaled graphs. arXiv preprint arXiv:2006.04617.

[5] Friedkin, N.E., Proskurnikov, A.V., Tempo, R., Parsegov, S.E., 2016. Network science on belief system dynamics under logic constraints. Science 354, 321–326.

[6] Godsil, C.D., Royle, G., Godsil, C., 2001. Algebraic Graph Theory. volume 207. Springer New York.

[7] Harary, F., 1959. On the measurement of structural balance. Behavioral Science 4, 316–323.

[8] Jadbabaie, A., Lin, J., Morse, A.S., 2003. Coordination of groups of mobile autonomous agents using nearest neighbor rules. IEEE Transactions on automatic control 48, 988–1001.

[9] Katali, O., Iwai, S., 1978. Studies on the balancing, the minimal balancing, and the minimum balancing processes for social groups with planar and nonplanar graph structures. Journal of Mathematical Psychology 18, 140–176.

[10] Mesbahi, M., Egerstedt, M., 2010. Graph theoretic methods in multiagent networks. Princeton University Press.

[11] Olfati-Saber, R., Murray, R.M., 2004. Consensus problems in networks of agents with switching topology and time-delays. IEEE Transactions on automatic control 49, 1520–1533.

[12] Pan, L., Shao, H., Mesbahi, M., Xi, Y., Li, D., 2019. Bipartite consensus on matrix-valued weighted networks. IEEE Transactions on Circuits and Systems II: Express Briefs 66, 1441–1445.

[13] Pan, L., Shao, H., Mesbahi, M., Xi, Y., Li, D., 2020. On the controllability of matrix-weighted networks. IEEE Control Systems Letters 4, 572–577.

[14] Pan, L., Shao, H., Xi, Y., Li, D., 2021. Bipartite consensus problem on matrix-valued weighted directed networks. Science China Information Sciences 64, 149204.

[15] Ren, W., Atkins, E., 2005. Second-order consensus protocols in multiple vehicle systems with local interactions. IEEE/CAA Journal of Automatica Sinica. 41, 415–419. doi:10.1016/j.j.automatica.2017.12.004
[19] Tuna, S.E., 2016. Synchronization under matrix-weighted laplacian. Automatica 73, 76–81.

[20] Tuna, S.E., 2017. Observability through a matrix-weighted graph. IEEE Transactions on Automatic Control 63, 2061–2074.

[21] Tuna, S.E., 2019. Synchronization of small oscillations. Automatica 107, 154–161.

[22] Ye, M., Trinh, M.H., Lim, Y.H., Anderson, B.D., Ahn, H.S., 2020. Continuous-time opinion dynamics on multiple interdependent topics. Automatica 115, 108884.

[23] Zhao, S., Zelazo, D., 2015. Translational and scaling formation maneuver control via a bearing-based approach. IEEE Transactions on Control of Network Systems 4, 429–438.