I-theorem and self-organization in the van der Pol generator

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Abstract. Based on the S- and I-theorems of evolution of q- entropy and q- discrimination
information, the process of self-organization developing in the van der Pol generator with
variation of the feedback parameter at induced transitions between stationary states in an open
nonextensive system is studied.

To date, many properties of nonextensive (nonadditive) statistical systems that are described by non-
Gaussian and non-Gibbs distributions in various fields of physics have been studied. A new area of
research - nonextensive statistical mechanics and thermodynamics - is being developed. A detailed
review of related works is given in [1-3].

For additive systems, self-organization in the generator was considered in [4], where the S-
theorem on decrease in the renormalized Boltzmann–Gibbs–Shannon entropy (this entropy corresponds to the
equal values of the average energy) was formulated for the first time. The I-theorem states that the
renormalized Kullback discrimination information increases during self-organization of open systems
[5, 6].

The van der Pol generator is described by the Langevin equations
\[
\frac{dx}{dt} = v, \quad \frac{dv}{dt} + (-a + bE)v + ax = \sqrt{D}y(t),
\]
where \(a = a_f - \gamma\), \(a_f\) is the feedback coefficient (control parameter), \(\gamma\) and \(b\) are the coefficients of
linear and nonlinear friction, \(E = m\left(\dot{v}^2 + \omega_0^2x^2\right)/2\) is the oscillations energy, and \(D\) is the noise
intensity. Moments \(y(t)\) of the Langevin source are determined by the expressions \(\langle y(t) \rangle = 0\) and
\(\langle y(t)y(t') \rangle = 2\delta(t-t')\). In additive statistical thermodynamics, induced transitions between states of
an open system for \(\gamma\), \(|a|\), \(b\langle E \rangle < \omega_0\) are described by the Fokker–Planck equation for the
distribution [4],
\[
\frac{\partial p}{\partial t} = \frac{\partial}{\partial E}\left[(-a + bE)Ep + D(E)E\frac{\partial p}{\partial E}\right].
\]
Its solution for a stationary state with given noise \(D(E) = D\) has the form of the “canonical” Gibbs
distribution
with effective “free energy” $F$, effective Hamilton function $H$ and $k = 1$. The control parameter here is feedback coefficient $a_f$.

In the case of nonextensive systems, equation (2) can be generalized so that the dependence of the diffusion coefficient on the distribution is taken into account. This yields the quasi-parabolic equation [7]

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial E} \left[ (a + bE) E p + D_q(E, p) E \frac{\partial p}{\partial E} \right],$$

which describes the Brownian motion. The properties of this equation are of particular interest. In the stationary state at $q = 1$, the diffusion coefficient is $D_q(E, p) = D$; at $q \neq 1$, determination of the diffusion coefficient yields a solution in the known form of non-Gibbs equilibrium distribution with $D = \text{const}$. These conditions are satisfied by function $D_q(E, p) = Dp^{1-q}$. As a result, for the stationary state, we have the distribution

$$p_0 = \left[ 1 - (1 - q)D^{-1}(-aE + bE^2/2) \right]^{(1-q)\Gamma(1-q)} Z^{-1}_{\gamma},$$

which depends on control parameter $a_f$.

Equation (4) is nonlinear with respect to distribution, and its solution describes the Brownian motion, which will be used to reveal self-organization in the van der Pol generator at induced transitions between states.

In accordance with [4], we determine two values of the control parameter, apply approximations that lead to two distributions and consider induced transitions between the corresponding stationary states.

First, we set the initial condition $a_f = 0$ for the state of “physical chaos” [4] and assume that $b(E)/\gamma \sim Db/\gamma^2 << 1$. Then, equation (5) yields the “equilibrium” distribution

$$p_0 = \left[ 1 - (1 - q)D^{-1}(-aE + bE^2/2) \right]^{(1-q)\Gamma(1-q)} Z^{-1}_{\gamma},$$

where $\Gamma(\xi)$ is the gamma function and $1 < q < 2$.

Entropy Havrda-Charvat-Daroczy [8, 9] and the mean energy have the form

$$S_q \left( p_0 \right) = \frac{1 - \int p_0^q dE}{q-1} = \frac{1}{q-1} \left\{ \left[ \frac{D}{\gamma(q-1)} \right] \Gamma \left( \frac{1}{q-1} \right) \Gamma^{-1} \left( \frac{q}{q-1} \right) Z^{-q}_{\gamma} \right\},$$

$$E_q \left( p_0 \right) = \int E p_0^q dE = \left[ \frac{D}{\gamma(q-1)} \right]^2 \Gamma \left( \frac{2-q}{q-1} \right) \Gamma^{-1} \left( \frac{q}{q-1} \right) Z^{-q}_{\gamma}.$$
The entropy and mean energy is obtained in the same fashion,

\[
S_q\left(p_{(2)}\right) = \frac{1}{q-1} \left[ 1 - \left( \frac{\pi D}{2b(q-1)} \right)^q \Gamma\left( \frac{q+1}{2(q-1)} \right) \Gamma^{-1}\left( \frac{q}{q-1} \right) \right] \left[ \frac{q}{q-1} \right]^{\gamma_q} Z_{\phi(2)}^{-q},
\]

\[
E_q\left(p_{(2)}\right) = \left[ \frac{D}{b(q-1)} \right] \Gamma\left( \frac{1}{q-1} \right) \Gamma^{-1}\left( \frac{q}{q-1} \right) Z_{\phi(2)}^{-q}.
\]

At high energies, equations (6) and (8) yield asymptotic power-type distributions \(p_{(1)} : E^{-\gamma_{q(1)}}\) and \(p_{(2)} : E^{-\gamma_{q(2)}}\), which are typical of fractal systems. If \(q=1\), we obtain the respective well-known distributions \(p_{(1)} = (\gamma/D) \exp(-\gamma E/D)\) and \(p_{(2)} = \sqrt{2b/\pi D} \exp(-b^2 E/2D)\), which are used in additive statistical thermodynamics [4]. The limiting value \(q=2\) yields the powerfunction distributions \(p_{(1)} = C_1 E^{-1}\) and \(p_{(2)} = C_2 E^{-2}\), which are also valid for fractal systems at low energies.

To find the degree of order of microstates in the generator, consider the renormalized distribution

\[
\tilde{p}_{(i)} = \left[ 1 - (1-q)D^{-1}(a_i) \right] \gamma E \left[ \frac{q}{q-1} \right]^{\gamma_q} Z_{\phi(2)}^{-1},
\]

\[
\tilde{Z}_{\phi(2)} = \left[ \frac{D}{b(q-1)} \right] \Gamma\left( \frac{1}{q-1} \right) \Gamma^{-1}\left( \frac{q}{q-1} \right) Z_{\phi(2)}^{-1},
\]

where \(\tilde{D}(a_i) = D\) and \(\tilde{Z}_{\phi(2)} = Z_{\phi(2)}\) at \(a_i = 0\). The entropy and the mean energy take the form

\[
S_q\left(\tilde{p}_{(i)}\right) = \frac{1}{q-1} \left[ 1 - \left( \frac{\tilde{D}(a_i)}{\gamma(q-1)} \right) \Gamma\left( \frac{1}{q-1} \right) \Gamma^{-1}\left( \frac{q}{q-1} \right) \right] \left[ \frac{q}{q-1} \right]^{\gamma_q} \tilde{Z}_{\phi(2)}^{-q},
\]

\[
E_q\left(\tilde{p}_{(i)}\right) = \left[ \frac{\tilde{D}(a_i)}{\gamma(q-1)} \right] \Gamma\left( \frac{2-q}{q-1} \right) \Gamma^{-1}\left( \frac{q}{q-1} \right) \tilde{Z}_{\phi(2)}^{-q}.
\]

Analyzing the transition between \(p_{(2)}\) and \(\tilde{p}_{(i)}\), we find Rathie-Kannappan discrimination information [10]

\[
I_q\left(p_{(2)} : \tilde{p}_{(i)}\right) = \frac{1}{q-1} \left[ 1 - \int p_{(2)}^{\phi(i)} dE \right] = \tilde{Z}_{\phi(2)}^{-1} \left[ S_q\left(p_{(2)}\right) - S_q\left(\tilde{p}_{(i)}\right) \right] + \tilde{D}^{-1}(a_i) \left[ E_{q(2)} - E_{q(i)} \right].
\]

Now we require that the mean energies be equal,

\[
\int E p_{(2)}^{\phi}\ dE = \int E \tilde{p}_{(i)}^{\phi}\ dE.
\]
This equality follows from the additivity condition [2]
\[ I_q\left(p_{(2)} : \tilde{p}_{(1)}\right)Z_{q_{(1)}}^{i-q} = \left[I_q\left(p_{(2)} : p_{(1)}\right) - I_q\left(p_{(2)} : \tilde{p}_{(1)}\right)\right]Z_{q_{(1)}}^{i-q}. \]  
(14)

whatever the value of the control parameter. Then, we find a dependence between the renormalized and given values of noise,
\[
\frac{\tilde{D}(a_{r})}{\gamma(q-1)} \Gamma\left(\frac{2-q}{q-1}\right) \Gamma^{-q}\left(\frac{2-q}{q-1}\right) = 
\frac{D}{b(q-1)} \left[\frac{2b(q-1)}{\pi D}\right]^{\frac{q}{2}} \Gamma\left(\frac{1}{q-1}\right) \Gamma^{-q}\left(\frac{3-q}{2(q-1)}\right), \tilde{D}(a_{r}) > D.
\]  
(15)

As a result, we prove the validity of the S- and I- theorems,
\[
I_q\left(p_{(2)} : \tilde{p}_{(1)}\right)Z_{q_{(1)}}^{i-q} = \left[I_q\left(p_{(2)} : p_{(1)}\right) - I_q\left(p_{(2)} : \tilde{p}_{(1)}\right)\right]Z_{q_{(1)}}^{i-q} = -\left[S_q\left(p_{(2)}\right) - S_q\left(\tilde{p}_{(1)}\right)\right] = 
\left[\frac{\pi D}{2b(q-1)}\right]^\frac{q}{2} \Gamma\left(\frac{q+1}{2(q-1)}\right) \Gamma^{-1}\left(\frac{q}{q-1}\right)Z_{q_{(1)}}^{i-q} - 
\left[\frac{\tilde{D}(a_{r})}{\gamma(q-1)}\right] \Gamma\left(\frac{1}{q-1}\right) \Gamma^{-1}\left(\frac{q}{q-1}\right)Z_{q_{(1)}}^{i-q} > 0.
\]  
(16)

Thus, it follows from equations (16) that, as feedback parameter \( a_{r} \) increases from zero to \( \gamma \), renormalized Havrda-Charvat-Daroczy entropy \( S_q\left(\tilde{p}_{(1)}\right) < S_q\left(p_{(2)}\right) \) decreases (S- theorem) and simultaneously renormalized Rathie-Kannappan discrimination information \( I_q\left(p_{(2)} : p_{(1)}\right) > I_q\left(p_{(2)} : \tilde{p}_{(1)}\right) \) increases (I-theorem). This is an indication of self-organization in the van der Pol generator. The same result follows from analysis of the third stationary state with developed generation and corresponding renormalization relative to the state with \( a_{r} = 0 \). In this case, a change in entropy \( D\left[S_q\left(p_{(1)}\right) - S_q\left(\tilde{p}_{(1)}\right)\right] = R_{min} \) [5, 6] (where \( k^{-1}D \) plays the role of the effective temperature) is due to work on ordering in the generator.

Calculations lead to the inequality \( d\left[S_q\left(\tilde{p}_{(1)}\right)\right]/da_{r} < 0 \), which proves that the state of the generator with the distribution \( \tilde{p}_{(1)} \) is stable. Note that equality (14) and, accordingly, condition (13) need not be fulfilled in the general case. Then, inequality (16) does not hold and one cannot argue that a given process of generation is self-organizing.

Thus, it has been shown that an increase in the feedback parameter to the generation threshold in the Van der Pol generator for the case of nonextensive systems with \( 1 < q < 2 \) causes self-organization with an increase in the extent of order and decrease in the extent of disorder. At \( q = 1 \), the results coincide with those known for additive systems [4]. In this work, only two stationary states and the transition between them have been considered. The results obtained here can be applied in analysis of fractal electrical oscillatory systems for the presence of self-organization.

Only one work is known [11] in which the van der Pol generator is also considered using Havrda-Charvat-Daroczy entropy and Bregman’s measure [12].
\[ D(p : p_0) = \int \left[ f(p) - f(p_0) - (p - p_0) \frac{\partial f(p)}{\partial p_0} \right] dX \]  

(17)

as the discrimination information. Functional (17) was first introduced in [12] as a convex function in problems of linear and convex programming. Function \( f(p) = k(q-1)^{\frac{1}{q}} (p^q - p) \) follows from the definition of entropy in the form \( S_q = -\int f(p) dX \); consequently, equation (17) yields the measure [11, 13]

\[ D(p : p_0) = -\left[ S_q(p) - S_q(p_0) \right] - \int (p - p_0) \frac{\partial f(p_0)}{\partial p_0} dX \]

\[ = -\frac{k}{q-1} \int p^q dX + k \int p_0 q dX - \frac{kq}{q-1} \int p_0^{q+1} dX, \]

(18)

used here. At \( q = 1 \), equation (18) gives the Kullback discrimination information borrowed from additive statistical mechanics.

We present some remarks. A disadvantage of Bregman’s measure (17) is that it does not provide any composition law for independent systems. Note that the author of [11] applies distributions borrowed from [4] that characterize the stationary states of additive systems described by equation (2). In addition, he considers transitions between distributions \( p_{(i)} \) and \( p_{(2)}^{q} \int p_{(2)}^{q} dX \) from various families, which is inconsistent with the approach used in [14, 15]. That only such an averaging procedure over unnormalized distributions of \( p^q \) is valid [8, 9] also follows from the condition of thermodynamic equilibrium for nonextensive systems [16]. Also in work [11] value of a derivative \( \frac{d}{da_f} \) is not determined. Since the equations use unnormalized distributions \( p^q \) and \( p_0^q \) from the same family, the new functional [7]

\[ D_q(p : p_0) = \int \left[ f(p) - f(p_0) - (p^q - p_0^q) \frac{\partial f(p_0)}{\partial (p_0^q)} \right] dX \]

(19)

is a natural generalization of measure (17) for nonextensive systems to determine the discrimination information. If the above function \( f(p) \) is used as the Havrda-Charvat-Daroczy entropy, this functional takes the form

\[ D_q(p : p_0) = -\left[ S_q(p) - S_q(p_0) \right] - \int (p^q - p_0^q) \frac{\partial f(p_0)}{\partial (p_0^q)} dX = \frac{1}{1-q} \left( 1 \int p^q p_0^{q+1} dX \right), \]

(20)

which expectedly coincides with the expression for Rathie-Kannappan discrimination information [10]. This allows one to find the respective discrimination information in the case of other expressions for the nonextensive entropy and use it to detect the self-organization process in open systems.

References

[1] Tsallis C 2009 Introduction to Nonextensive Statistical Mechanics. Approaching a Complex World (New York: Springer) p 382

[2] Zaripov R G 2010 Principles of Nonextensive Statistical Mechanics and Geometry of Measures of the Disorder and the Order (Kazan: Kazan A.N. Tupolev State Technical University Press) p 404 (in Russian)
[3] Naudts Jan 2011 Generalized Thermostatistics (London: Springer) p 201
[4] Klimontovich Yu L 1983 Soviet Technical Physics Letters 9 606
[5] Zaripov R G 1988 Soviet Technical Physics 33 1366
[6] Zaripov R G 1999 Discrimination information and disorder-order transitions (Kazan: Kazan A.N. Tupolev State Technical University Press) p 155 (in Russian)
[7] Zaripov R G 2009 Technical Physics 54 6
[8] Havrda J and Charvat F 1967 Kybernetika 3 30
[9] Daroczy Z 1970 Inform. Control. 16 36
[10] Rathie P N and Kannappan Pl 1972 Inform. Control. 20 38
[11] Bagci G B 2007 Arxiv preprint arXiv: cond-mat/0705.2053
[12] Bregman L M 1967 USSR Comput. Math. Math. Phys. 7 200
[13] Abe S and Bagci G B 2005 Phys. Rev. E 71 016139
[14] Zaripov R G 2001 Russian Phys. J. 44 1159
[15] Zaripov R G 2004 Russian Phys. J. 47 647
[16] Zaripov R G 2006 Technical Physics 51 1393