Determination of Reynolds shear stress from the turbulent mean velocity profile and spectral characteristics of Orr-Sommerfeld -Squire equations

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Abstract. Theoretically, based on a waveguide model, the expression of the tangential stress is formulated for steady, two-dimensional incompressible fluid flow over a flat plate in turbulent boundary layer. It is dependent on some factors, one of them, the behaviour of the last damping mode eigenvalues, and eigenfunctions, that are deduced from solution Orr-Sommerfeld equation by spectral Chebyshev collocation Method. Verification of the latter method is investigated by comparison the deduced formula of turbulent tangential stress with experimental data. In addition to, weight factors in this expression are connected to define the condition of dynamical system solution for multiple 3-wave resonance. This system is solved numerically, and the dynamic invariant is normalized to obtain the time average of the square modulus harmonic, and sub harmonics amplitudes by theorem Birkhoff-Khinchin. Comparison is made between the time-averaged and the phase average for the square modulus of harmonic, and sub harmonic amplitudes that defined on the unit sphere, in the state of multiple 3-wave resonance.

Keywords: Incompressible fluid flow – Orr-Sommerfeld eigenfunctions –Turbulent tangential stress

1. Introduction

In [1], based on Navier-Stokes equations, the validity of presentation of a developed turbulent field of fluctuations in the form of superposition of field coherent structures and stochastic component is considered. However, it is not explicitly shown that the mean field of the longitudinal velocity component has a known behaviour with the logarithmic part of the dependence on the transverse coordinate. In the presence of a given dependence of the tangential stresses on \( y \), the problem is reduced to solve the Reynolds equations. Last works [2-7], studied the average longitudinal velocity profile, using different asymptotic methods with various auxiliary data borrowed from the experiment. Especially, in [3], an explicit expression for the average velocity profile as a function of the Reynolds number over the boundary layer thickness is obtained, which is heuristically compiled from the law of the wall behaviour and the behaviour in the region of the velocity defect. The models based on a dynamic waveguide representation by pulsation are considered for a turbulent boundary layer [8]. The solution of the Reynolds equations for a developed turbulent boundary layer on a plate in an incompressible fluid in the presence of specified tangential stresses is obtained by the method of asymptotic expansions [9]. In [10], an analytical method is presented for determining the Reynolds shear stress profile in steady, two-dimensional wall-bounded flows using the mean stream wise velocity. Since the near-wall region can generally be subdivided into viscous sub layer, buffer layer and fully turbulent layer for turbulent flow over a flat plate. Therefore, the meshed elements need to be carefully arranged to capture the velocity and turbulence in different layers. Moreover, in [11], the
turbulent shear flow is generally divided into three layers (inner, meso, outer) with two overlap regions. The meso layer is the intermediate layer in between inner (Prandtl) limit and outer (Karman) limit in a turbulent shear flow which covers domain of buffer layer and traditional overlap region. In order to explain the theory of dynamic system more accurate, more exact numerical methods are achieved, which are requiring considerable expertise in modern mathematics [12]. The formula used in [13] can be used to formulate the well-known theorem of Birkhoff-Khinchin. In detail, more knowledge that under some conditions is proved that, the average time of a function is almost anywhere, and that is connected to the averaged space. The average time for nearly all initial points is the same, as a special class of Ergodic system [14]. Ergodic definitions and assumptions are fundamental to ergodic theory implementations. The underlying concept is that the average time of their properties for certain structures is equal to the average space [15]. Menyuk et al. [16] discussed ergodicity to some extent, clarifying the dependence on the values of the interaction coefficients, through the case of two or more triads that can lead to chaotic behavior. In this paper, we consider a model composed of incompressible fluid over a flat plate in the turbulent boundary layer. The objective of the present work is to compare the time-averaged, and the phase average for the squares modulus of harmonic, and sub harmonic amplitudes, that defined on the unit sphere, in the state of multiple 3-wave resonance. We compute theoretically the tangential stress expression, based on a waveguide model of turbulent boundary layer. The plan of this work is as follows: In the next section, a dynamical system of finite mode model is investigated. In the third section, the expression of the Reynolds shear stress is formulated to describe its behaviour. Finally, the salient results of our analysis are discussed in the conclusion.

2. Statement of the problem
To gain insight into the randomness of the turbulence, we consider the existence of random solutions for differential equations. We consider finite model consist of the main harmonic, and five sub harmonics, as a dynamical system for describing the coherent structure [17], for two-dimensional incompressible flow, with zero pressure gradient, flowing over a flat plate.

$$\frac{da_1}{dt} = \sum_{s=1}^{n} \Lambda_1^s a_1^s a_2^s, \quad \frac{da_2^s}{dt} = \Lambda_2^s a_1 a_2^s, \quad \frac{da_3^s}{dt} = \Lambda_3^s a_1 a_2^s, \quad s = 1, 2, \cdots, n $$ (1)

The boundedness of system solution (1) is controlled by the positivity of the quantities $q_2^s, q_3^s$ which based on the determination of coefficients $\Lambda_1^s, \Lambda_2^s, \Lambda_3^s$. The solution of the above system of differential equations is formulated with statistical initial conditions, which are distributed on the surface 6-dimensional ellipsoid. But a change of variables [1], in the model version, by $a_i(t) = \phi(t) \bar{a}_i (\phi(t)), \phi(t) = \sqrt{3}/\Phi$ are introduced here to capture solutions of the dynamical system (1), on the unit sphere, and so on, the theory of Birkhoff-Khinchin is applied [13]. Explicit solutions for amplitudes are also found in classical textbooks [18], moreover, solutions for amplitudes and phases are found in modern form [19]. Based on the above explanation, solutions of the dynamical system (1) are represented in the form of dynamical invariant [17] as follow:

$$I^{(s)}(t_i) = |\bar{a}_1|^2 + \sum_s q_{1s}^2 |\bar{a}_2|^2 + q_{3s}^2 |\bar{a}_3|^2 = 1, q_{12}^s = \frac{q_2}{q_1}, q_{13}^s = \frac{q_3}{q_1}, s = 1, 2, \ldots, n.$$ (2)

Here the sum is formulated by index $s$, for all sub harmonics in the case of multiple three wave resonance. It is found that, the dynamic invariant is defined on the unit sphere, and it is formulated by a quadratic form of the principal harmonic, and sub-harmonics amplitudes as illustrated in Eq. (2). Ergodicity of the dynamical system (1) is defined by the following relation [15]:

$$\lim_{T \to \infty} \frac{1}{\sigma(S)} \int_S f(x) d\sigma = \frac{1}{\sigma(S)} \int_S f(x) d\sigma,$$ (3)

where $\sigma(S)$ is the area of the surface. It is clear that, as outlined in table 1, the time average over a single sub harmonic almost equal space overage over the appropriate sphere surface, for almost every sub harmonic.
3. Formulation of tangential stress

In this section, a wave guide theory of a developed turbulent boundary layer is presented, to relate the dynamics of Tollmien Schlichting waves to the turbulent tangential stresses. This expression is defined in terms of the last damping mode eigenvalue, and their corresponding eigenfunction, that are deduced from solving the spectral problem Orr-Sommerfeld equation by Chebyshev collocation method for a given mean velocity profile $U(y)$ [3]. Introducing the Fourier form of continuity, and the normal component of the vorticity equation, then the formula of tangential stresses, is defined by

$$u = \frac{i}{k} \left( \frac{d v}{d y} - \beta \eta \right), \quad \eta = -\frac{\beta v}{-\omega(k) + \alpha U(y)} \frac{d U}{d y} \quad (4)$$

Here, $\alpha, \beta$ are the wave numbers of the oscillation in the stream-wise and span-wise directions respectively, and $\omega(k)$ is the dispersion relation of the spectral problem Orr-Sommerfeld equation, and $v, \eta$ are the vertical components of velocity and vorticity that deduced from Orr-Sommerfeld, Squire, equations respectively. Substituting representation (4), in $\tau = \langle u v \rangle$, thus we have the following formula:

$$\tau = -\langle u v \rangle = \frac{i}{k^2} \left( \frac{d v}{d y} - \beta \delta \left( \omega - \omega_{T-S} \right)(k) \right) e^{ikx - i\omega_{T-S} t} \phi_0(k, y),$$

$$A_{\alpha, k}(t_i) = a_1(t_i) \delta(k - k_1) + a_1^*(t_i) \delta(k + k_1) + \sum a_2(t_i) \delta(k - k_2) + a_2^*(t_i) \delta(k + k_2) + \sum a_3(t_i) \delta(k - k_3) + a_3^*(t_i) \delta(k + k_3),$$

$$\langle u v \rangle = \lim_{t_i \to \infty} \left( \int_{t_i}^{t_{i+1}} \int_{t_0}^{b} u v d t_0 \right) d t_i$$

Here $\delta(k)$ is Delta-Dirac function, and $A_{\alpha, k}$ is the amplitude of the vertical velocity component in a developed turbulent boundary layer, summation is carried out for all sub harmonics of the lowest harmonic allowed by the dispersion equation and the condition of positive certainty of the wave invariant in the state of multiple three-wave resonance [17]. Averaging this expression over $t_s, t_i$ gives an expression for the turbulent tangential stress in the discrete representation of the coherent structure. Here, the index summation $s, \tau$ is simulated for the main harmonic, and five sub-harmonics, where $\langle u v \rangle_{t_i}$ is the time average for harmonic or sub-harmonic amplitude.

The value of parameter $\varepsilon = \varepsilon(Re, \tau) = 0.033$ that deduced from Reynolds number, $Re_{\tau} = 7.57 \times 10^4$ is provided in [9], and also $\phi = 0.417133$, in a stationary case [1] is calculated from formula, that defined above in section 2. Values of $q_{12}^\tau$ and $q_{13}^\tau$ are determined, to define the boundedness for solving the dynamical system (1), that

| Number of sub harmonic | $\lim_{\sigma \to \infty} f(x)^\tau$ | $\frac{1}{\sigma} \int f(x) d\sigma$ |
|------------------------|----------------------------------|----------------------------------|
| first sub harmonic     | 0.0825247                       | 0.0817011                        |
| second sub harmonic    | 0.0199271                       | 0.0190730                        |
| third sub harmonic     | 0.0930313                       | 0.0961254                        |
| four sub harmonic      | 0.0338027                       | 0.0333333                        |
| five sub harmonic      | 0.0287763                       | 0.0286095                        |
based on coefficients \( \left( \Lambda_1^*, \Lambda_2^*, \Lambda_3^* \right) \). It is clear that the mathematical expression of (7) represents structure of least damping mode eigenfunction \( \varphi_0 (k, y) \) of the Orr-Sommerfeld equation, for different combination of the wave vectors. These vectors are especially located on the curve of three wave resonance, as displayed in Fig.1.

\[
\tau = \frac{e^2 \phi^2}{(2 \pi)^4} \left[ f_1(k_1, y) + 2 \sum_{j=1}^{s} \sum_{l=1}^{s} \left( \frac{|\tilde{\alpha}_j|}{K_j} \right) \left( f_2(k_j', y) + f_3(k_l', y) \frac{\partial U(y)}{\partial y} \right) \right]
\]

(7)

\[
\langle...\rangle = \frac{1}{t_i} \int_{t_i}^{t_i + d t_i} ... \quad q_{s_i} = q_{s_i}
\]

\[
f_1(k_1, y) = \frac{\alpha_i \mathrm{Im} \left[ \varphi_0^* (k_i) \frac{\partial \varphi_0 (k_1)}{\partial y} \right]}{|k_i|^2};
\]

\[
f_2(k_j', y) = \alpha_j \mathrm{Im} \left[ \varphi_0^* (k_j') \frac{\partial \varphi_0 (k_j')}{\partial y} \right];
\]

\[
f_3(k_l', y) = \frac{\left( \beta_i^2 \right) \varphi_0 (k_l') \omega_i (k_l')}{\omega_i^2 (k_l') + \left( \omega_i^2 (k_l') - \alpha_i U(y) \right)^2}.
\]

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**Fig.1** Curve three wave resonance, which defined the least damping mode of Musker Profile \([3]\) for \( \text{Re} = 7.57 \times 10^4 \).

As outlined in formula (7), the left side represent the tangential stress. The terms on the right hand side depend on some factors, like, parameter values \( \left( e, \phi, q_{12}, q_{13} \right) \), time averages of harmonic, and sub harmonics amplitudes \( \left( |\tilde{\alpha}_j| \right) \), the mean velocity profile Musker \( U(y) \) \([3]\), eigenvalues, and eigenfunctions for the least damping.
mode of different wave vectors \( \mathbf{k}^s_i = (\alpha^s_i, \beta^s_i) \). The real and imaginary parts of leading mode eigenvalues are outlined by \( \omega = \text{Re}[\omega] \) and \( \omega = \text{Im}[\omega] \) respectively. The asterisk for \( \phi^s_{\omega}(\mathbf{k}) \) denotes a complex conjugate of the least damping mode eigenfunction \( \phi_{\omega}(\mathbf{k}) \). The index sum \( s \) is considered for two wave vectors \( \mathbf{k}_2, \mathbf{k}_1 \), which are satisfied the triad resonance with the main wave vector. The first part in the right, concerns only the leading mode eigenfunction, for the main wave harmonic \( \mathbf{k}_1 = (1, 0) \), as shown in Fig.2 with index number 1.

| Number of sub harmonic | \( q_{12}^s \)   | \( q_{13}^s \)   |
|------------------------|----------------|----------------|
| \( s = 1, \alpha_1 = 1.685, \beta_1 = 13.38 \) | 0.156531 | 0.102806 |
| \( s = 2, \alpha_2 = 2.313, \beta_2 = 27.84 \) | 0.258746 | 0.22306 |
| \( s = 3, \alpha_3 = 2.832, \beta_3 = 40.38 \) | 0.296102 | 0.170738 |
| \( s = 4, \alpha_4 = 3.308, \beta_4 = 54.84 \) | 0.136389 | 0.07947605 |
| \( s = 5, \alpha_5 = 3.633, \beta_5 = 69.78 \) | 0.0849696 | 0.0216714 |

The second and third terms are represented leading mode eigenfunctions for five sub harmonics amplitudes, which are corresponding to five wave vectors \( \mathbf{k}_2, \mathbf{k}_3 \) respectively. In addition, these terms are multiplied by 2, due to the symmetry of the wave vector component \( \beta \).

The wave vector \( \mathbf{k}^s_i, s = 1, \ldots, 5 \), are indicated in Fig.2 by index numbers, \( \left( 2^1, 2^2, 2^3, 2^4, 2^5 \right) \), where the power is referred to the number of sub harmonic, that are corresponding to five multipliers values
(\bar{q}_{1}^{1}, \bar{q}_{1}^{2}, \bar{q}_{1}^{3}, \bar{q}_{1}^{4}, \bar{q}_{1}^{5})$, respectively as outlined above in table 2, and the similar situation is also defined for $k_{s}^{s}, s=1, \ldots, 5$, for $\bar{q}_{1}^{1}, \bar{q}_{1}^{2}, \bar{q}_{1}^{3}, \bar{q}_{1}^{4}, \bar{q}_{1}^{5}$. Generally, the flow field with the normal coordinate $y$ is divided into three sub domains, the inner, intermediate, and outer $y \sim \Delta$, $y \sim 1$, $y \sim 1/e$ respectively as outlined in [9]. The behaviour of tangential stress with the normal coordinate $y$ is completely defined in Fig.3. In addition, Fig. 4 is formulated to determine the behaviour of tangential stress, only in the external domain of the boundary layer. So that, a shift is applied to the thickness of the boundary layer, to focus the behaviour study of tangential stress on the outer region. It is found that, the behaviour of tangential stress has a qualitative and quantitative agreement with experimental data Klebanoff [20].

4. Conclusions

In this paper, we determine the expression of the tangential stress, in terms of the waveguide model in the turbulent boundary layer, for given mean velocity profile Mucker. Their behavior depends mainly on the spectral characteristic of eigenvalues, and eigenfunction Orr-Sommerfeld equation, that obtained by using the spectral Chebyshev collocation method. Generation of eigenfunctions, for different combination of the wave vectors $k_{1}^{s}, k_{s}^{s}$ in the deduced formula of tangential stress are related to the structure for a set of sub harmonic amplitudes, in the case of multiple three wave resonance. Numerical solutions of the dynamical system are contributed to obtain agreement between the time, and space averages for the square of main harmonic, and five sub harmonics amplitudes. It is found that, the behaviour of turbulent tangential stress, with the shift in the normal coordinate $y$, has a good agreement with the experimental data Klebanoff.

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