Anderson localization for random magnetic Laplacian on $\mathbb{Z}^2$

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Abstract

We consider a two dimensional magnetic Schrödinger operator on a square lattice with a spatially stationary random magnetic field. We prove Anderson localization near the spectral edges. We use a new approach to establish a Wegner estimate that does not rely on the monotonicity of the energy on the random parameters.

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1 Introduction

We consider a spinless quantum particle hopping on the two dimensional lattice $\mathbb{Z}^2$ and subject to a random magnetic field. This model is the magnetic analogue of the standard Anderson model with a random on-site potential but here the magnetic field carries the randomness in the system. The fluxes through the plaquets are i.i.d. random variables. For simplicity we assume that there is no external potential.

The main result of this paper is a Wegner estimate for the averaged density of states for the Hamiltonian restricted to a finite box $\Lambda$. More precisely, we show that the expected number of eigenvalues in a small spectral interval of length $\eta$ is bounded by $C\eta|\Lambda|^4$. This estimate exhibits the optimal (first) power of $\eta$, but its volume dependence is not optimal. Nevertheless, it can be used to prove spectral and dynamical localization via the standard multiscale argument.

A similar result has been obtained earlier by Klopp et al. in [9], but under a quite restrictive condition, namely that in a fixed domino tiling of $\mathbb{Z}^2$ the flux on each domino is deterministically zero. This condition was essential for the method of [9] to work since it ensured that the magnetic field was generated by a stationary vector potential that could be expressed in terms of independent gauges on each domino. The variation of the local flux thus influenced the quantum state only on a few sites. Diagonalization of a finite matrix then showed that near the spectral edges the energy is a strictly monotone function of the flux. This monotonicity observation provided the key input for the Wegner estimate in [9].

If the zero flux condition is removed and we consider independent fluxes on each plaquets, then apparently there is no direct monotone relation between the eigenvalues and the fluxes. Prior to our recent work [3], some form of monotonicity has always been used in the proofs of the Wegner type estimates in context of random Schrödinger operators (discrete or continuum). In case of random external potential with a definite sign,

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the monotonicity of an eigenvalue as a function of the random coupling constants is a direct consequence of first order perturbation theory. For sign indefinite potentials [5], for the random displacement model [8] and for random vector potentials [5, 4, 12] the monotonicity could still be extracted using some special structure of these models but similar ideas do not seem to apply for random magnetic fields. Note that i.i.d. (or stationary) random vector potentials and i.i.d. random magnetic fields represent different physical models; typically a stationary random magnetic field cannot be generated by a stationary random vector potential. We refer to Section 4 of [3] for an overview and further references.

In a recent paper [3] we have developed a new method to prove a Wegner estimate without any monotonicity mechanism. We considered the corresponding continuous model, i.e., the Schrödinger operator in \( \mathbb{R}^2 \) with a stationary random magnetic field. We proved a Wegner estimate for all energies and Anderson localization at the bottom of the spectrum. The magnetic field was required to have a positive lower bound to ensure that the current did not vanish. Furthermore, the random field had to contain modes on arbitrary small scales to control the large momentum regime of the current.

In the current work we extend this approach to the discrete case, where many technical complications of the continuum model are absent and we can thus consider very general random magnetic fields. Apart from independence, the only essential condition on the fluxes is that the fluxes should be separated away from 0 and \( \pi \), i.e. away from the “minimal” and “maximal” fluxes on each plaquet. This condition is analogous to the condition of positive lower bound on the magnetic field in the continuous model. The proof presented here is very simple and it highlights the essence of our new approach introduced in [3].

## 2 Model and Statements of Results

For any subset \( \Lambda \subset \mathbb{Z}^2 \) we introduce the Hilbert space \( \ell^2(\Lambda) \), with inner product

\[
\langle \varphi_1, \varphi_2 \rangle = \sum_{x \in \mathbb{Z}^2} \varphi_1(x) \varphi_2(x).
\]

The discrete magnetic Schrödinger operator will be defined on \( \mathcal{H} := \ell^2(\mathbb{Z}^2) \). For the precise definitions we will follow the notations in [9], see also [10].

Let \( \mathcal{E} \) be the set of directed edges (arrows) in \( \mathbb{Z}^2 \), i.e.,

\[
\mathcal{E} = \{(x, y) : x, y \in \mathbb{Z}^2, |x - y| = 1\}.
\]

For \( a = (x, y) \in \mathcal{E} \), we write \( \overline{a} = (y, x) \). For \( a \in \mathcal{E} \) we will denote by \( a_i \) and \( a_t \) the first respectively second entry of \( a \), i.e., \( a = (a_i, a_t) \). Let \( \mathcal{F} \) be the set of unit squares (plaquets) in \( \mathbb{Z}^2 \), i.e.,

\[
\mathcal{F} = \{(x_1, x_1 + 1) \times (x_2, x_2 + 1) : (x_1, x_2) \in \mathbb{Z}^2\}.
\]

We label the elements of \( \mathcal{F} \) by the lattice point in the lower left corner of the unit square, i.e., for \( x = (x_1, x_2) \in \mathbb{Z}^2 \) we define \( f_x := \{x_1, x_1 + 1\} \times \{x_2, x_2 + 1\} \in \mathcal{F} \). For any \( f_x \in \mathcal{F} \) we define the oriented boundary

\[
\partial f_x = \{(x, x + e_1), (x + e_1, x + e_1 + e_2), (x + e_1 + e_2, x + e_2), (x + e_2), x\} \subset \mathcal{E},
\]

where \( e_1 = (1, 0) \), and \( e_2 = (0, 1) \).

Let \( \mathbb{T} := \mathbb{R}/(2\pi \mathbb{Z}) \). A function \( A : \mathcal{E} \to \mathbb{T} \) will be called vector potential or gauge if it satisfies

\[
A(a) = -A(\overline{a}), \quad \forall a \in \mathcal{E}.
\]  

(2.1)

Let

\[
\mathcal{G} := \{A \in \mathbb{T}^\mathcal{E} : (2.1) \text{ holds}\}
\]

be the space of vector potentials. For \( (x, y) \in \mathcal{E} \) we will also use the notation \( A(x, y) := A((x, y)) \). For a given vector potential \( A \in \mathcal{G} \), the define the differential “curl” by

\[
dA(f) := \sum_{a \in \partial f} A(a), \quad f \in \mathcal{F},
\]
which is a function on $\mathcal{F}$ with values in $\mathbb{T}$. We will call $dA$ the magnetic field generated by $A$.

The discrete magnetic Schrödinger operator $H$ on $\mathbb{Z}^2$ with a vector potential $A \in \mathcal{G}$ is defined by

$$ (H(A)\psi)(x) = \sum_{y \in \mathbb{Z}^2 : |x-y|=1} \left( \psi(x) - e^{iA(x,y)}\psi(y) \right), \quad \psi \in \ell^2(\mathbb{Z}^2). $$

In view of the definition of $\mathcal{G}$ the operator $H(A)$ is self-adjoint and its quadratic form is given by

$$ (\psi, H\psi) = \frac{1}{2} \sum_{a \in \mathcal{E}} |\psi(a) - e^{iA(a)}\psi(a)|^2. $$

We have the bound $0 \leq H(A) \leq 8$, [10], and it is well known that $\sigma(H(0)) = [0, 8]$.

**Remark.** A function $\lambda : \mathbb{Z}^2 \to \mathbb{T}$ defines a gauge transformation. $U_\lambda \psi(x) := e^{i\lambda(x)}\psi(x)$, in the sense that $H(A_\lambda) = U_\lambda H(A)U_\lambda^*$ with $A_\lambda(a) := A(a) + d\lambda(a)$ and $d\lambda(a) := \lambda(a) - \lambda(a_i)$. It is well-known that if $A, \tilde{A} \in \mathcal{G}$ generate the same magnetic field, $dA = d\tilde{A}$, then $H(A)$ and $H(\tilde{A})$ are unitarily equivalent by means of a gauge transformation. Thus spectral properties of $H(A)$ only depend on the underlying magnetic field.

Next we will introduce the random magnetic field. Let $B_\omega = (\omega_f)_{f \in \mathcal{F}}$ be a family of independent, not necessarily identically distributed random variables taking values in $\mathbb{T}$. We assume that the distribution of $\omega_f$ is absolutely continuous and we denote by $v_f$ its density function defined on $\mathbb{T}$. We will denote the probability space by $\Omega := \mathbb{T}^\mathcal{F}$ and the expectation w.r.t. this probability measure by $E$. By $A_\omega$ we shall denote a vector potential such that $dA_\omega = B_\omega$.

The Wegner estimate will hold under the assumption that $v_f$ is a twice continuously differentiable function with uniformly bounded second derivative. Moreover, we will assume that $v_f$ is supported away from integer multiples of $\pi$. Note that not only the “minimal” flux $\omega_f \approx 0$ is excluded, but also the “maximal” flux $\omega_f \approx \pi$. To formulate this precisely, for any $b \in (0, \pi/2)$ we introduce the following subset of the torus

$$ \mathbb{T}_b := \mathbb{T} \setminus \{(b, b) \cup (\pi - b, \pi + b)\}. \quad (2.2) $$

**Assumption $A(b, D)$.** Let $v_f \in C^2(\mathbb{T})$ with $\|v_f\|_{C^2} \leq D$ and $\text{supp } v_f \subset \mathbb{T}_b$.

For a rectangle $\Lambda \subset \mathbb{Z}^2$, we consider the Hamiltonian $H_\Lambda(A)$ restricted to $\ell^2(\Lambda)$ as follows: for $\psi \in \ell^2(\Lambda)$ and $x \in \Lambda$ we set

$$ [H_\Lambda(A)\psi](x) := 4\psi(x) - \sum_{y \in \Lambda : |x-y|=1} e^{iA(x,y)}\psi(y). $$

This choice of boundary conditions is referred to as simple boundary conditions, see [7]. With respect to these boundary conditions we state the Wegner estimate. We remark that the proof of the localization presented in [7] uses this choice of boundary conditions for the Wegner estimate and for the box Hamiltonians in the multiscale analysis.

We introduce the cube centered at the origin of length $L \in \mathbb{N}$ by

$$ \Lambda_L := \{x \in \mathbb{Z}^2 : \max\{|x_1|, |x_2|\} \leq L\}. $$

We shall write $H_L = H_{\Lambda_L}$. By $\chi_{E,E}$ we will denote the characteristic function of the closed interval $[E - \eta/2, E + \eta/2]$. The Wegner estimate holds in an energy interval up to

$$ E_{\text{crit}} := 4 - \sqrt{8} = 1.1715... $$

which is the the maximal possible value for the bottom of the spectrum, see (2.3). We now state our main result on the Wegner estimate.
Theorem 2.1 Suppose Assumption A(b, D) holds for some $b \in (0, \pi/2)$ and a finite constant $D$. Then for any $E^* < E_{\text{crit}}$ there exists a finite constant $C = C(b, D, E^*)$ such that for any $E \geq 0$ and $\eta \geq 0$ with $E + \eta/2 \leq E^*$, we have

$$E \text{Tr}_E \eta(H_L(A_\omega)) \leq C \eta L^8.$$  

Remark. By using the unitary transformation $\psi(x) \rightarrow (-1)^{x_1 + x_2} \psi(x)$, we have the unitary equivalence $H(A) \cong 8 - H(A)$. Therefore the Wegner estimate holds likewise for $E - \eta/2 \geq 8 - E^*$ and the same symmetry applies for the result about localization (Theorem 2.2 below). The restriction $E^* < E_{\text{crit}}$ originates from the proof of Lemma 4.2. Since the Lifshitz asymptotics (2.4) only holds for small energies, a Wegner estimate for energies below $E_{\text{crit}}$ is sufficient for the Anderson localization near the band edge which is our main application of Theorem 2.1. Nevertheless it is an interesting question on its own to establish the largest possible upper threshold $E^*$ of the range of energies $E$ for which one could extend Lemma 4.2 and hence the Wegner estimate.

We now state the result on localization. For this part, we assume that the random fluxes $\omega_f$ are not only independent but also identically distributed with density $v$. Under this condition, $H(A_\omega)$ is an ergodic operator, and its spectrum $\sigma(H(A_\omega)) = \Sigma$ is actually independent of $\omega$ a.s., see [1] (Proposition V.2.4.).

For $E \in \mathbb{R}$, the integrated density of states is defined by

$$k(E) = \lim_{L \to \infty} \frac{1}{|A_L|} \# \{\text{eigenvalues of } H_{A_L}(A_\omega) \leq E \},$$

where the limit exists $\omega$ a.s. and is independent of the choice of sample $\omega$, see Appendix C of [10]. The density of states is independent of the choice of boundary conditions, which can be seen using the min-max principle [10]. Lifshitz asymptotics is shown in [9] under the assumptions that $\text{supp } v \subset \mathbb{T} \setminus (-c, c)$, $\pm c \in \text{supp } v$ for some $0 < c < \pi$ and $v$ is Lipschitz continuous on $\mathbb{T} \setminus (-c, c)$. Under these assumptions

$$\Sigma = [E_0, 8 - E_0], \quad \text{with } E_0 = E_0(c) := 4(1 - \cos(c/4)), \quad (2.3)$$

[10, 9]. In [9] it is shown (Theorem 1.2) that

$$\limsup_{E \downarrow E_0} \frac{\log(-\log(k(E)))}{\log(E - E_0)} \leq -1. \quad (2.4)$$

This result implies, roughly speaking,

$$k(E) \preceq e^{-(E - E_0)^{-1 + \delta}} \quad \text{as } E \downarrow E_0,$$

for any $\delta > 0$. Lifshitz tail estimate and Theorem 2.1 imply localization at the bottom of the spectrum using standard arguments, see [2, 11, 7] for details. Here we only state the result noting that the assumptions ensure that $E_0 < E_{\text{crit}}$ and thus a Wegner estimate always holds at the bottom of the spectrum.

Theorem 2.2 Suppose Assumption A(b, D) holds for some $b \in (0, \pi/2)$ and a finite constant $D$. Assume that the random fluxes $\omega_f$ are independent, identically distributed and $\pm b \in \text{supp } v$. Then Anderson localization holds near the bottom of the spectrum. Namely, there exists an $E_{\text{loc}} > E_0(b)$ such that $H(A_\omega)$ has dense pure point spectrum on $[E_0(b), E_{\text{loc}}]$ almost surely, and each eigenfunction associated to an energy in this interval decays exponentially.

3 Proof of the Wegner estimate: Theorem 2.1

Let $\alpha_f$, denote a vector potential of a magnetic field of flux 1 through the square $f$. I.e., $d\alpha_f = \delta_f$ where $\delta_f$ is the function on $\mathcal{F}$ which is one at $f$ and zero otherwise. Later in Section 4 we will choose specific gauges, see (4.2) below. In the current section we only need that the absolute value of $\alpha_f$ is bounded by one.
Let $\Lambda = \Lambda_L$ for some $L \in \mathbb{N}$. Define $\mathcal{F}_\Lambda := \{ f \in \mathcal{F} : f \subset \Lambda \}$ and $\mathcal{E}_\Lambda := \{ a \in \mathcal{E} : a \in \Lambda \times \Lambda \}$. Henceforth we restrict the functions $\alpha_f$ to $\mathcal{E}_\Lambda$. We define the vector potential

$$A_\omega = \sum_{f \in \mathcal{F}_\Lambda} \omega_f \alpha_f,$$

then $dA_\omega = B_\omega$. Moreover we will drop the subscript $\omega$ in the notation and set $H = H_L(A)$ in this section.

We will use perturbation theory, we abbreviate $Y_f = \frac{\partial}{\partial \omega_f}$, and for any state $\psi$ we introduce the expectation of the $Y_f$-derivative of the energy in state $\psi$:

$$\langle Y_f H \rangle_\psi := -\sum_{(x,y) \in \Lambda^2, |x-y|=1} \overline{\psi(x)} \alpha_f(x,y) e^{iA(x,y)} \psi(y).$$

Let $\lambda$ be a non-degenerate eigenvalue of $H$ with a normalized eigenfunction $\psi$. In that case the eigenvalue $\lambda$ is a function of the random variables $\{\omega_f\}$. For each $f \in \mathcal{F}$, we have by the Hellmann-Feynman theorem and a straightforward calculation that

$$Y_f \lambda = \frac{\partial \lambda}{\partial \omega_f} = \langle Y_f H \rangle_\psi. \quad (3.1)$$

To give a physical meaning to the right hand side we introduce the current of a wavefunction in the Hilbert space. For $\varphi \in l^2(\Lambda)$, we define the current $J_\varphi$ as the function on $\mathcal{E}_\Lambda$ given by

$$J_\varphi(a) := -2 \text{Re} \varphi(a^*) e^{iA(a)} \varphi(a^*).$$

The current does not depend on the chosen gauge and observe that

$$J_\varphi(a) = -J_\varphi(a^*). \quad (3.2)$$

If $\varphi$ is an eigenvector then it is a straightforward calculation to show that the “divergence” of $J_\varphi$ vanishes, namely for all $x \in \Lambda$

$$\sum_{e: |e|=1} J_\varphi(x, x+e) = 0. \quad (3.3)$$

We can write the right hand side of (3.1) as

$$\langle Y_f H \rangle_\psi = \frac{1}{2} \sum_{a \in \mathcal{E}_\Lambda} \alpha_f(a) J_\psi(a). \quad (3.4)$$

Notice, since we sum over directed edges we have a factor $\frac{1}{2}$. We also remark that the left hand side is gauge invariant, while the right hand side seems to depend on the gauge $\alpha_f$. The gauge independence of the right hand side follows from the fact that the divergence of $J_\psi$ is zero (3.3).

In Section 4 we will prove the following key technical estimate:

**Proposition 3.1** Let $b \in (0, \pi/2)$ and $E^* < E_{\text{crit}}$. Then there exists a constant $C = C(b, E^*)$ such that

$$\sum_{f \in \mathcal{F}_\Lambda} \langle Y_f H \rangle_\psi^2 \geq C^{-1} L^{-4} \quad (3.5)$$

for any normalized eigenfunction $\psi$ with eigenvalue in $(-\infty, E^*)$ and for any collection of fluxes $\omega_f \in T_b$. 

5
Proof of Theorem 2.1. Let $\lambda_1, \lambda_2, \ldots$ denote the eigenvalues of $H$. First we assume that they are simple. We fix $E^* < E_{\text{crit}}$. In the sequel we set $\chi = \chi_{E, \eta}$, with $E + \eta/2 \leq E^*$. From Proposition 3.1 and (3.1) it easily follows that

$$\text{Tr} \chi(H) = \sum_{\ell} \chi(\lambda_\ell) \leq C_{14} \sum_{\ell} \sum_{f \in F_\lambda} (Y_f \lambda_\ell)^2 \chi(\lambda_\ell), \quad (3.6)$$

To estimate the right hand side we introduce the following functions,

$$F(x) := \int_{-\infty}^x \chi(t)dt, \quad G(y) := \int_{-\infty}^y F(x)dx.$$ 

For any $Y = Y_f$ and any $\ell$ we have by the chain rule and Leibniz

$$(Y_\ell \lambda_\ell)^2 \chi(\lambda_\ell) = Y_f^2 G(\lambda_\ell) - (Y_f^2 \lambda_\ell) F(\lambda_\ell). \quad (3.7)$$

To estimate the sum of the second term over all eigenvalues $\lambda_\ell$, we will use the following Lemma, for a proof see [3].

**Lemma 3.2** We have for any $Y = Y_f$

$$\text{Tr} \left( Y^2 H \right) F(H) \leq \sum_{\ell} (Y^2 \lambda_\ell) F(\lambda_\ell). \quad (3.8)$$

Thus combining (3.6), (3.7) and (3.8), we have

$$\text{Tr} \chi(H) \leq C_{14} \sum_{f \in F_\lambda} \left( \text{Tr} Y_f^2 G(H) - \text{Tr} (Y_f^2 H) F(H) \right). \quad (3.9)$$

To estimate the second term we use that $\|F\|_\infty \leq C\eta$. Moreover, we use that $\|Y_f^2 H\| \leq 4$, which can be seen using that for any $\varphi \in \ell^2(\Lambda)$ we have

$$(Y_f^2 H \varphi)(x) = - \sum_{|x-y|=1} \alpha^2_f(x, y) e^{iA(x, y)} \varphi(y).$$

To estimate the first term in (3.9) we integrate by parts after taking expectation,

$$\text{ETr} Y_f^2 G(H) = \int \prod_{\zeta \in F_\lambda} v_\zeta(\omega_\zeta) d\omega_\zeta \frac{\partial^2}{\partial \omega_f^2} \text{Tr} G(H) = \int \prod_{\zeta \neq f} v_\zeta(\omega_\zeta) d\omega_\zeta \int v_f'(\omega_f) d\omega_f \text{Tr} G(H).$$

Using that $G(y) \leq C\eta y$ we can estimate $|\text{Tr} G(H)| \leq C L^2 \eta$. Thus we obtain

$$\text{ETr} \chi(H) \leq C\eta L^8, \quad (3.10)$$

and the Theorem follows in the non-degenerate case.

In the argument above we used that $\lambda = \lambda_\ell$ is a simple eigenvalue since we implicitly assumed that $\lambda_\ell$ is differentiable w.r.t. $\omega_f$. Differentiability of the eigenvalues may be ensured even in case of degeneracy by choosing an appropriate branch, but this choice may depend on $f$ hence a more careful treatment is needed which we explain now.

Suppose that $\lambda$ is a degenerate eigenvalue with degeneracy $m_\lambda$ for some fixed $\omega$. Let $P_\lambda$ denote the eigenprojection and we fix $\{\varphi_{h, \lambda}\}_{h = 1, \ldots, m_\lambda}$ an orthonormal basis of the corresponding eigenspace. Moreover, for each fixed $f \in F_\lambda$ we can also choose orthonormal eigenfunctions $\psi_{f, h, \lambda}$, with $h = 1, \ldots, m_\lambda$, which are real analytic functions of $\omega_f$ (if the other variables of $\omega$ are kept fixed) and their eigenvalues $E_{f, h, \lambda}$ are real.
analytic in $\omega_f$ as well, see [6] (Theorem 1.10. Section II). By analytic perturbation theory, the derivatives $Y_f E_{f, \lambda, h} (h = 1, 2, \ldots, m_\lambda)$ are the eigenvalues of the matrix

$$T := P_E (Y_f H) P_E | \text{ran} P_E.$$ 

By Jensen’s inequality, we have

$$\sum_{h=1}^{m_\lambda} (Y_f E_{f, \lambda, h})^2 = \text{Tr} T^2 = \sum_{h=1}^{m_\lambda} (\varphi_{\lambda, h}, T^2 \varphi_{\lambda, h})^2 \geq \sum_{h=1}^{m_\lambda} (\varphi_{\lambda, h}, T \varphi_{\lambda, h})^2 = \sum_{h=1}^{m_\lambda} (Y_f H)^2 \varphi_{\lambda, h}. \quad (3.11)$$

Summing (3.11) over $f$, we find, using Proposition 3.1, that for $\lambda \leq E^*$,

$$\sum_{f \in F} \sum_{h=1}^{m_\lambda} (Y_f E_{f, \lambda, h})^2 \geq m_\lambda C^{-1} L^{-4}. \quad (3.12)$$

We can now proceed as before. Using (3.12) and (3.7), we find

$$\text{Tr} \chi(H) = \sum_{\lambda \in \sigma(H)} m_\lambda \chi(\lambda) \leq \sum_{\lambda \in \sigma(H)} C L^4 \sum_{f \in F} \sum_{h=1}^{m_\lambda} (Y_f E_{f, \lambda, h})^2 \chi(E_{f, \lambda, h}) = C L^4 \sum_{f \in F} \sum_{\lambda \in \sigma(H)} \sum_{h=1}^{m_\lambda} (Y_f^2 G(E_{f, \lambda, h}) - (Y_f^2 E_{f, \lambda, h}) F(E_{f, \lambda, h})) \leq C L^4 \sum_{f \in F} \left( \text{Tr} Y_f^2 G(H) - \text{Tr} (Y_f^2 H) F(H) \right),$$

where we used Lemma 3.2 in the last step. The rest of the proof is the same as in the case of non-degenerate eigenvalues and we thus proved Theorem 2.1.

We remark that the possible degeneracy of the eigenvalues may also be treated by standard perturbation theory for almost all $\omega$, using the fact that the interior of the set of $\omega$’s where there is no eigenvalue crossing has full measure.

4 Proof of Proposition 3.1

We recall that by (3.4) we have

$$\langle Y_f H \rangle_\psi = \frac{1}{2} \sum_{a \in E_\Lambda} \alpha_f(a) J_\psi(\psi).$$

The following lemma inverts this linear relation and expresses the current in terms of $\langle Y_f H \rangle_\psi$.

Lemma 4.1 For $a \in E$, let $f_a$ be the unique square in $F$ such that $a \in \partial f$. Then for $a \in E_\Lambda$, we have

$$J_\psi(a) = c_a \langle Y_{fa} H \rangle_\psi - c_{\pi} \langle Y_{faH} \rangle_\psi, \quad (4.1)$$

where $c_a = 1$ if $f_a \in F_\Lambda$ and $c_a = 0$ otherwise.
Proof of Lemma 4.1. We will introduce four different gauges, \( \alpha^{(\tau)} \), \( \tau = 1, 2, 3, 4 \), for the magnetic field \( \delta_f \) whose flux is 1 through the square \( f \) and zero elsewhere. For \( f = f_x \in \mathcal{F} \) with \( x = (x_1, x_2) \) we set

\[
\begin{align*}
\alpha_f^{(1)}(y, y + e_2) &= 0, & \alpha_f^{(1)}(y, y + e_1) &= \begin{cases} -1, & \text{if } y_2 > x_2, y_1 = x_1 \\ 0, & \text{otherwise} \end{cases}, \\
\alpha_f^{(2)}(y, y + e_1) &= 0, & \alpha_f^{(2)}(y, y + e_2) &= \begin{cases} 1, & \text{if } y_1 > x_1, y_2 = x_2 \\ 0, & \text{otherwise} \end{cases}, \\
\alpha_f^{(3)}(y, y + e_2) &= 0, & \alpha_f^{(3)}(y, y + e_1) &= \begin{cases} 1, & \text{if } y_2 \leq x_2, y_1 = x_1 \\ 0, & \text{otherwise} \end{cases}, \\
\alpha_f^{(4)}(y, y + e_1) &= 0, & \alpha_f^{(4)}(y, y + e_2) &= \begin{cases} -1, & \text{if } y_1 \leq x_1, y_2 = x_2 \\ 0, & \text{otherwise} \end{cases},
\end{align*}
\]

and extend the definition to \( \mathcal{E} \) by (2.1).

Now we can prove Lemma 4.1. Let \( a = (y, y + e_1) \in \mathcal{E}_\Lambda \) be a horizontal edge. For \( -L < y_2 \), the gauge \( \tau = 1 \) gives (4.1). For the boundary case \( y_2 = -L \) we can use the gauge \( \tau = 3 \). The identity for the edge \( \tau \) follows using (3.2). The vertical edges follow similarly using the gauge \( \tau = 2, 4 \).

\[\square\]

The next lemma gives a lower bound on the current of an eigenfunction. The proof will be given in Section 5.

Lemma 4.2 Let \( b \in (0, \pi/2) \) and \( E^* < E_{\text{crit}} \). Then there exists a constant \( C = C(b, E^*) \) such that

\[
\sum_{a \in \mathcal{E}_\Lambda} |J_\psi(a)|^2 \geq CL^{-4}
\]

for all normalized eigenfunctions \( \psi \) of \( H_\Lambda(A) \), with energy \( E \leq E^* \), and for any vector potential \( A \) with \( dA \in \mathbb{T}_b \) (see (2.2)).

Note that the flux is required to be separated away from 0 and \( \pi \), i.e. not only the zero flux is excluded but also the “maximal” flux. The first exclusion is obvious since if \( dA = 0 \), then the eigenfunction can be chosen real and then clearly \( J_\psi = 0 \). If \( dA = \pi \) on each plaquet, then one can choose a gauge \( A \) with \( A(x, x + e_1) = 0 \) and \( A(x, x + e_2) = \pi x_1 \) and extend it by (2.1). In that case the Hamiltonian \( H_\Lambda(A) \) is real. In particular, the eigenfunctions are real for the maximal flux, hence the current vanishes in this case as well.

Proof of Proposition 3.1. Using Lemma 4.1 we have

\[
|J_\psi(a)|^2 \leq 2c_\omega (Y_{f_x}H)^2_\psi + 2c_\pi (Y_{f_x}H)^2_\psi
\]

Observing that each square gives rise to 8 directed edges, after summing up this inequality for all \( a \in \mathcal{E}_\Lambda \), we find

\[
32 \sum_{f \in \mathcal{F}_\Lambda} (Y_f H)^2_\psi \geq \sum_{a \in \mathcal{E}_\Lambda} |J_\psi(a)|^2.
\]

The proposition now follows from Lemma 4.2.

\[\square\]

5 Proof of the regularity lemma: Lemma 4.2

Let \( \psi \in \ell^2(\Lambda) \) be an eigenfunction of \( H_\Lambda(A) \) with energy \( E \). If \( \psi(x) \neq 0 \), we write

\[
\psi(x) = e^{i\lambda(x)} |\psi(x)| \quad (5.1)
\]
for some real function \( \lambda(x) \in \mathbb{T} \). Then the current is

\[
J_{\psi}(a) = 2|\psi(a)| |\psi(a)| \sin(\varphi_a),
\]

(5.2)

with \( \varphi_a := A(a) + \lambda(a_t) - \lambda(a_i) \). The goal will be to find a unit square \( Q \in \mathcal{F}_\Lambda \) such that \( \inf_{x \in Q} |\psi(x)| \) is bounded from below. Using that the magnetic flux through \( Q \) is separated away from zero and \( \pi \), one can then show that there is a nonvanishing current along the boundary of that square.

To this end let \( x_0 \in \Lambda \) be a point where \( \psi \) attains its maximum absolute value,

\[
|\psi(x_0)| = M := \sup_{x \in \Lambda} |\psi(x)|.
\]

(5.3)

In this section we use the convention that \( \psi(x) = 0 \) if \( x \notin \Lambda \).

**Lemma 5.1** Let the energy \( E \) of the eigenfunction \( \psi \) satisfy \( 0 \leq E \leq 4 \). Then the following statements hold.

(a) We have

\[
\max_{y \in \Lambda} \frac{|\psi(y)|}{|y-x_0|=1} \geq (1 - E/4)M.
\]

(b) Suppose \( \tilde{y}_0 \in \Lambda \) is a nearest neighbor of \( x_0 \) with \( |\psi(\tilde{y}_0)| \leq \epsilon M \) for some \( \epsilon \geq 0 \). Then

\[
\min_{y \in \Lambda \setminus \{\tilde{y}_0\}} \frac{|\psi(y)|}{|y-x|=1} \geq (2 - \epsilon)M.
\]

(c) Suppose \( y_0 \in \Lambda \) is a nearest neighbor of \( x_0 \) with \( |\psi(y_0)| \geq \kappa M \) for some \( \kappa \geq 0 \). Then

\[
\{(4 - E)\kappa - 2\} M \leq |\psi(y_0 + t)| + |\psi(y_0 - t)|,
\]

where \( t \) is a unit vector orthogonal to \( y_0 - x_0 \).

**Proof.** Adjusting the phase of the eigenvector we can assume that \( \psi(x_0) = M \). Since the statements of the lemma are gauge invariant, for notational simplicity we may choose for (a) and (b) a gauge, such that \( A(x_0, y) = 0 \) for all \( y \) with \( |x_0 - y| = 1 \). The statement (a) follows from the eigenvalue equation

\[
\sum_{y \in \Lambda} \frac{\psi(y)}{|y-x_0|=1} = (4 - E)\psi(x_0).
\]

and by taking real parts

\[
4 \max_{|y-x_0|=1} \text{Re} \psi(y) \geq (4 - E)\psi(x_0).
\]

For part (b), we fix \( y \in \Lambda \setminus \{\tilde{y}_0\} \) such that \( |y - x_0| = 1 \). Then by the eigenvalue equation

\[
\psi(y) = (2 - E)\psi(x_0) - \psi(\tilde{y}_0) + 2\psi(x_0) - \sum_{z \in \Lambda \setminus \{\tilde{y}_0\}} \sum_{|x_0-z|=1} \psi(z).
\]

(5.4)

Now taking the real part of both sides we find using that \( \text{Re}[\psi(x_0) - \psi(z)] \geq 0 \),

\[
\text{Re} \psi(y) \geq (2 - E)\psi(x_0) - \epsilon M.
\]

This yields (b). To prove (c), we choose a gauge such that \( A(y_0, y) = 0 \) for all \( |y_0 - y| = 1 \). By the eigenvalue equation at \( y_0 \)

\[
(4 - E)\psi(y_0) = \psi(x_0) + \psi(2y_0 - x_0) + \psi(y_0 + t) + \psi(y_0 - t).
\]
Hence by the triangle inequality

\[(4 - E)|\psi(y_0)| \leq 2M + |\psi(y_0 - t)| + |\psi(y_0 + t)|,
\]

and (c) now follows. \(\square\)

With the above lemma we can show the following proposition.

**Proposition 5.2** Let \(E^* < E_{\text{crit}} = 4 - \sqrt{8}\). There exists a positive number \(c = c(E^*)\) such that for any eigenfunction \(\psi\) with eigenvalue \(E \leq E^*\) there exists a cube \(Q \in F_\Lambda\) such that

\[
\min_{x \in Q}|\psi(x)| \geq cM.
\]  

**Proof.** Let \(\epsilon = 1/10\) and recall the definition of \(M\) and \(x_0\) from (5.3). We consider the following two cases.

**Case 1:** \(\min_{y \in \Lambda: |y - x_0| = 1} |\psi(y)| > \epsilon M\).

By Lemma 5.1 (a) there exists a nearest neighbor \(y_0 \in \Lambda\) of \(x_0\) such that

\[
|\psi(y_0)| \geq (1 - E^*/4)M > 0.
\]

Then by Lemma 5.1 (c) and using the notation introduced there there exists a \(\sigma \in \{-1,1\}\) such that

\[
|\psi(y_0 + \sigma t)| \geq \frac{1}{2} \{(4 - E^*)(1 - E^*/4) - 2\} M.
\]

Thus choosing the square \(Q = \{x_0, x_0 + \sigma t, y_0, y_0 + \sigma t\}\), the estimate (5.5) follows from the choice of \(E^* < 4 - \sqrt{8}\).

**Case 2:** \(\min_{y \in \Lambda: |y - x_0| = 1} |\psi(y)| \leq \epsilon M\).

Let \(\tilde{y}_0 \in \Lambda\) be the nearest neighbor of \(x_0\) with \(|\psi(\tilde{y}_0)| = \min_{y \in \Lambda: |y - x_0| = 1} |\psi(y)|\). Let \(y_0 = 2x_0 - \tilde{y}_0\). Then by Lemma 5.1 (b) and (c) there exists a \(\sigma \in \{-1,1\}\) such that

\[
|\psi(y_0 + \sigma t)| \geq \frac{1}{2} \{(4 - E^*)(2 - E^* - \epsilon) - 2\} M.
\]

Thus choosing the square \(Q = \{x_0, x_0 + \sigma t, y_0, y_0 + \sigma t\}\), the estimate (5.5) follows from Lemma 5.1 (b) and the choice of \(E^*\) and \(\epsilon\). \(\square\)

With the above Proposition we can show Lemma 4.2 using that a nonzero flux produces a current.

**Proof of Lemma 4.2.** Since \(\psi\) is \(l^2\)-normalized, clearly, \(|\psi(x_0)| \geq 1/L\), where \(x_0\) is defined by (5.3). By Proposition 5.2 there exists a unit square \(Q \in F_\Lambda\) such that for a constant \(c\) (depending on \(E^*\)), we have

\[
\inf_{x,y \in Q} |\psi(x)\psi(y)| \geq cL^{-2}.
\]

With respect to the parameterization (5.1) we have by (5.2)

\[
\sum_{a \in \partial Q} |J_\psi(a)|^2 \geq c^2 L^{-4} \sum_{a \in \partial Q} \sin^2(\varphi_a).
\]

This will imply the lemma using that for some positive constant \(c\) (depending on \(b\))

\[
\sum_{a \in \partial Q} \sin^2(\varphi_a) \geq c. \tag{5.6}
\]

To show this, we first note that \(\omega_0 = \sum_{a \in \partial Q} \varphi_a \in \mathbb{T}_b\). We will show that this implies that there is an \(a \in \partial Q\) such that \(\varphi_a \in \mathbb{T}_{b/8}\). The estimate (5.6) will then follow since \(\sin^2(\cdot)\) is bounded from below on \(\mathbb{T}_{b/8}\) by a positive constant (depending on \(b\)). Suppose that there would not be an \(a \in \partial Q\) such that \(\varphi_a \in \mathbb{T}_{b/8}\). Then \(\varphi_a = n_a \pi + b_a\) with \(n_a \in \mathbb{Z}\) and \(|b_a| \leq b/8\). In that case \(\omega_0 = n \pi + \hat{b}\), with \(n \in \mathbb{Z}\) and \(|\hat{b}| \leq b/2\). This implies \(\omega_0 \notin \mathbb{T}_b\), which is a contradiction. \(\square\)
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