Recollements for Dualizing $k$-Varieties and Auslander’s Formulas

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Abstract
Given the pair of a dualizing $k$-variety and its functorially finite subcategory, we show that there exists a recollement consisting of their functor categories of finitely presented objects. We provide several applications for Auslander’s formulas: the first one realizes a module category as a Serre quotient of a suitable functor category. The second one shows a close connection between Auslander–Bridger sequences and recollements. The third one gives a new proof of the higher defect formula which includes the higher Auslander–Reiten duality as a special case.

Keywords Recollements · Dualizing $k$-varieties · Auslander–Bridger sequences · Auslander–Reiten theory

Mathematics Subject Classification Primary 18A25; Secondary 16G70

Introduction
The notion of recollements introduced in [7] provides an effective tool for categorical study of algebras. A recollement of abelian categories is a special case of Serre quotients where both the inclusion and the quotient functor admit left and right adjoints. Such a recollement situation is denoted throughout the paper by a diagram of the form below (see Definition 1.6 for details):

$$
\begin{array}{c}
B \\ \downarrow \\
\downarrow \\
A \\ \downarrow \\
\downarrow \\
C \\
\end{array}
$$

It provides a tool for deconstructing the middle category $A$ into “smaller ones” $B$ and $C$. There are a lot of recent work on recollements of abelian categories [11,26,27]. One of the most...
studied example is as follows. Let \( \Lambda \) be an associative ring (with unit) and \( e \) its idempotent. Then there exists a recollement:

\[
\begin{array}{ccc}
\text{Mod} \Lambda / \Lambda e & \longrightarrow & \text{Mod} \Lambda \\
& \uparrow & \\
\text{Mod} e \Lambda & \longrightarrow & \text{Mod} e \Lambda e.
\end{array}
\] (0.1)

If \( \Lambda \) is noetherian, it restricts to a recollement consisting of the categories of finitely generated modules. In fact, recollements of this type appeared in many contexts in representation theory, e.g. [9,10,23].

Our aim is to extend this recollement to functor categories over dualizing \( k \)-varieties. A dualizing \( k \)-variety can be considered as an analog of the category of finitely generated projective modules over a finite dimensional algebra, but with possibly infinitely many indecomposable objects up to isomorphism [3]. It is a Krull–Schmidt Hom-finite \( k \)-linear category \( A \) where the standard \( k \)-duality \( \text{Hom}_k(\mathbf{−}, k) \) induces the duality between \( \text{mod}\, A \) and \( \text{mod}(A^{\text{op}}) \).

**Theorem** (Theorem 2.5) Let \((A, B)\) be the pair of a dualizing \( k \)-variety \( A \) and its functorially finite subcategory \( B \). Then we have the following recollement:

\[
\begin{array}{ccc}
\text{mod}(A/[B]) & \longrightarrow & \text{mod} A \\
& \uparrow & \\
\text{mod} B & \longrightarrow & \text{mod} B.
\end{array}
\]

In Sect. 3, we show that in the functor category of a suitable dualizing \( k \)-variety, Auslander–Bridger sequences are nothing other than right-defining exact sequences of a recollement (Theorem 3.2).

In Sect. 4, we approach to higher Auslander–Reiten theory from a viewpoint of dualizing \( k \)-varieties. Auslander–Reiten theory is a fundamental tool for studying representation theory of Artin algebras, see [4,5]. In the late 2000s, higher Auslander–Reiten theory was introduced by Iyama [15,16]. In this section, we give a higher analog of Auslander’s defect formula for an \( n \)-cluster tilting subcategory \( B \) of \( \text{mod}\, A \), where \( A \) is a dualizing \( k \)-variety (Theorem 4.8). As an application, we give an equivalence

\[
\sigma_n : B \sim \overline{B}
\]

and bifunctorial isomorphisms

\[
\overline{B}(\sigma_n^{-1} y, x) \cong D \text{Ext}^n_A(x, y) \cong \overline{B}(y, \sigma_n x). \tag{0.2}
\]

In particular, \( \sigma_n \) coincides with the \( n \)-Auslander–Reiten translation \( \tau_n \) and (0.2) gives an \( n \)-Auslander–Reiten duality (Theorem 4.11).

A similar approach to higher Auslander–Reiten theory was given by Jasso and Kvaamme [20] independently, but our approach is slightly different since we do not use an explicit form of \( \tau_n \).

**Notation and Convention**

Throughout this paper we fix a commutative field \( k \). The symbol \( \Lambda \) always denotes a finite dimensional algebra over the field \( k \). The category of finite dimensional right \( \Lambda \)-modules and its full subcategory of projective (resp. injective) \( \Lambda \)-modules will be denoted by \( \text{mod}\, \Lambda \) and \( \text{proj}\, \Lambda \) (resp. \( \text{inj}\, \Lambda \)), respectively. The projectively (resp. injectively) stable category of \( \text{mod}\, \Lambda \) will be denoted by \( \text{mod}\, \Lambda \) (resp. \( \overline{\text{mod}}\, \Lambda \)).

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The symbols $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ always denote additive categories, and the set of morphisms $a \to b$ in $\mathcal{A}$ is denoted by $\mathcal{A}(a, b)$. We consider only additive functors between additive categories; that is functors $F$ which satisfy $F(f + g) = F(f) + F(g)$ whenever $f + g$ is defined. For a given category $\mathcal{A}$, we denote its opposite category by $\mathcal{A}^{\text{op}}$. For a functor $F : \mathcal{A} \to \mathcal{C}$, its image and kernel are defined as the full subcategories of $\mathcal{A}$

$$\text{Im } F := \{ y \in \mathcal{C} \mid \exists x \in \mathcal{A}, \ F(x) \cong y \} \quad \text{and} \quad \text{Ker } F := \{ x \in \mathcal{A} \mid F(x) = 0 \},$$

respectively. Let $\mathcal{B}$ be a subcategory of $\mathcal{A}$. We denote by $\mathcal{A} / [\mathcal{B}]$ the ideal quotient category of $\mathcal{A}$ modulo the (two-sided) ideal $[\mathcal{B}]$ in $\mathcal{A}$ consisting of all morphisms having a factorization through an object in $\mathcal{B}$. If there exists a fully faithful functor $\mathcal{B} \hookrightarrow \mathcal{A}$, we often regard $\mathcal{B}$ as a full subcategory of $\mathcal{A}$.

The additive category is said to be $k$-linear if each morphism-space $\mathcal{A}(x, y)$ is a $k$-module and the composition $\mathcal{A}(y, z) \times \mathcal{A}(x, y) \to \mathcal{A}(x, z)$ is $k$-bilinear. The additive functor is said to be $k$-linear if it gives a $k$-linear maps between morphism-spaces. In the case that given categories are $k$-linear, we consider only additive $k$-linear functors.

## 1 Preliminaries

In this section we recall the notion of dualizing $k$-varieties introduced by Auslander and Reiten [3]. We also recall the definition of recollements of abelian categories, as well as some basic properties which are needed in this paper.

### 1.1 Dualizing $k$-Varieties

We recall from [3] some basic facts about dualizing $k$-varieties. We denote by $\text{Ab}$ the category of abelian groups. For an essentially small category $\mathcal{A}$, a (right) $\mathcal{A}$-module is defined to be a contravariant functor $\mathcal{A} \to \text{Ab}$ and a morphism $X \to Y$ between $\mathcal{A}$-modules $X$ and $Y$ is a natural transformation. Thus we define an abelian category $\text{Mod}_\mathcal{A}$ of $\mathcal{A}$-modules, where we call this the functor category of $\mathcal{A}$. In the functor category $\text{Mod}_\mathcal{A}$, the morphism-space $(\text{Mod}_\mathcal{A})(X, Y)$ is usually denoted by $\text{Hom}_\mathcal{A}(X, Y)$. In the case that a given category $\mathcal{A}$ is $k$-linear, it is natural to consider, instead of $\text{Mod}_\mathcal{A}$; the equivalent category of $k$-linear functors from $\mathcal{A}$ to $\text{Mod}_k$, which is denoted by the same symbol.

An $\mathcal{A}$-module $X$ is finitely generated if there exists an epimorphism $\mathcal{A}(-, a) \to X$ for some $a \in \mathcal{A}$. An $\mathcal{A}$-module $X$ is said to be finitely presented if there exists an exact sequence

$$\mathcal{A}(-, b) \to \mathcal{A}(-, a) \to X \to 0$$

for some $a, b \in \mathcal{A}$. We denote by $\text{mod}\mathcal{A}$ the full subcategory of finitely presented $\mathcal{A}$-modules.

For an arbitrary category $\mathcal{A}$, the subcategory $\text{mod}\mathcal{A}$ is closed under cokernels and extensions in $\text{Mod}_\mathcal{A}$. However it is not always abelian since it is not necessarily closed under kernels. Let $f : b \to a$ be a morphism in $\mathcal{A}$. We call a morphism $g : c \to b$ a weak-kernel for $f$ if the induced sequence

$$\mathcal{A}(-, c) \xrightarrow{g \circ -} \mathcal{A}(-, b) \xrightarrow{f \circ -} \mathcal{A}(-, a)$$

is exact. We say $\mathcal{A}$ admits weak-kernels if every morphism in $\mathcal{A}$ has a weak-kernel. The notion of weak-cokernel is defined dually. We recall the following well-known fact.

**Lemma 1.1** [12, Theorem 1.4] The following are equivalent for a category $\mathcal{A}$.
(i) The category \( A \) admits weak-kernels.
(ii) The full subcategory \( \text{mod}\ A \) is an exact abelian subcategory in \( \text{Mod}\ A \), that is, it is abelian
and the canonical inclusion \( \text{mod}\ A \hookrightarrow \text{Mod}\ A \) is exact.

Like the case for module categories, \( \text{proj}\ A \) (resp. \( \text{inj}\ A \))
denotes the full subcategory of projective (resp. injective) \( A \)-modules in \( \text{mod}\ A \), and the projectively (resp. injectively) stable
category will be denoted by \( \text{mod}\ A := (\text{mod}\ A)/[\text{proj}\ A] \) (resp. \( \text{mod}^{-}\ A := (\text{mod}\ A)/[\text{inj}\ A] \)).
For a full subcategory \( B \subseteq \text{mod}\ A \) which contains \( \text{proj}\ A \) (resp. \( \text{inj}\ A \)), we also use the symbol \( B := B/[\text{proj}\ A] \) (resp. \( \overline{B} := B/[\text{inj}\ A] \)).

In the rest of this subsection, let \( A \) be a Krull–Schmidt \( k \)-linear category, that is, each object \( x \in A \) admits a decomposition \( x \cong \prod_{i=1}^{n} x_{i} \) with \( A(x_{i}, x_{i}) \) a local \( k \)-algebra for any \( i \in \{1, \ldots, n\} \). We also assume that \( A \) is \( \text{Hom-finite} \), that is, each morphism-space \( A(x, y) \)
is a finite dimensional \( k \)-module. We denote by \( D := \text{Hom}_{k}(−, k) : \text{mod}\ k \rightarrow \text{mod}\ k \)
the standard \( k \)-duality.

**Definition 1.2** [3, Section 2] A Krull–Schmidt \( k \)-linear category \( A \) is a dualizing \( k \)-variety if the standard \( k \)-duality \( D : \text{Mod}\ A \rightarrow \text{Mod}(A^{\text{op}}) \).
\( X \mapsto D \circ X \) induces a duality \( D : \text{mod}\ A \overset{\sim}{\rightarrow} \text{mod}(A^{\text{op}}) \).

It is obvious that \( A \) is a dualizing \( k \)-variety if and only if so is \( A^{\text{op}} \). If \( A \) is a dualizing \( k \)-variety, due to the duality between \( \text{mod}\ A \) and \( \text{mod}(A^{\text{op}}) \), \( \text{mod}\ A \) is closed under kernels in \( \text{Mod}\ A \).
Thus \( \text{mod}\ A \) is an exact abelian subcategory in \( \text{Mod}\ A \). By Lemma 1.1, \( A \) admits weak-kernels and weak-cokernels. The following proposition gives us basic examples of dualizing \( k \)-varieties.

**Proposition 1.3** [3, Props. 2.6, Prop. 3.4] Suppose \( A \) is a dualizing \( k \)-variety. Then \( \text{mod}\ A \)
is a dualizing \( k \)-variety. Moreover, \( \text{mod}\ A \) admits injective hulls and projective covers.

Next we recall the definition of functorially finite subcategories. The symbol \( X|_{B} \) denotes
the restricted functor of an \( A \)-module \( X \) onto a subcategory \( B \). Especially, for a functor
category \( \text{mod}\ A \) and its full subcategory \( B \) we also write \( \text{Ext}_{A}^{i}(B, x) := \text{Ext}_{A}^{i}(−, x)|_{B} \),
where \( x \in \text{mod}\ A \) and \( i \in \mathbb{Z}_{\geq 0} \).

**Definition 1.4** Let \( A \) be an arbitrary category and \( B \) a full subcategory in \( A \).

(i) The full subcategory \( B \) is contravariantly finite if the functor \( A(−, x)|_{B} \) is a finitely
generated \( B \)-module for each \( x \in A \).
(ii) The full subcategory \( B \) is covariantly finite if the functor \( A(x, −)|_{B} \) is a finitely generated
\( B^{\text{op}} \)-module for each \( x \in A \).
(iii) We call \( B \) a functorially finite if it is contravariantly finite and covariantly finite.

If a \( B \)-module \( A(−, x)|_{B} \) is finitely generated, then there exists an epimorphism

\[
\begin{array}{ccc}
B(−, b) & \xrightarrow{\alpha} & A(−, x)|_{B} \\
\end{array}
\]

in \( \text{Mod}\ B \) for some \( b \in B \). Then we call the induced map \( \alpha : b \rightarrow x \) a \textit{right \( B \)-approximation}
of \( x \). Dually we define the notion of \textit{left \( B \)-approximation}.

It is easy to verify that the subcategories \( \text{proj}\ A \) and \( \text{inj}\ A \) are functorially finite in \( \text{mod}\ A \)
if \( A \) is a dualizing \( k \)-variety. The following result gives a criterion for a given subcategory to be a dualizing \( k \)-variety.

**Proposition 1.5** [6, Theorem 2.3][15, Prop. 1.2] Let \( B \) be a functorially finite subcategory in
a dualizing \( k \)-variety \( A \). Then \( B \) is a dualizing \( k \)-variety.
1.2 Recollements of Abelian Categories

In this subsection we recall some basic facts on recollements of abelian categories. Let us start with introducing basic terminology. A pair of functors $L : A \to B$ and $R : B \to A$ is said to be an adjoint pair if there exists a bifunctorial isomorphism $A(a, Rb) \cong B(La, b)$ in $a \in A$ and $b \in B$. We simply denote this adjoint pair by $(L \dashv R)$. For a functor $F : A \to B$, we often denote its right (resp. left) adjoint by $F_\rho$ (resp. $F_\lambda$). If $F$ admits a right adjoint $F_\rho$ as well as a left adjoint $F_\lambda$, we denote this situation by $F_\rho F_\lambda$.

Throughout the rest of this subsection $A$ is always assumed to be an abelian category. To begin with, we recall the definition of recollement, following [11,26] (see also [25, Ch. 4]).

Definition 1.6 Let $A$, $B$ and $C$ be abelian categories. A recollement of $A$ relative to $B$ and $C$ is given by six functors $B \xrightarrow{e} A \xleftarrow{q} C$ such that

\begin{enumerate}
\item[(R1)] They form four adjoint pairs $(e_\lambda \dashv e)$, $(e \dashv e_\rho)$, $(q_\lambda \dashv q)$ and $(q \dashv q_\rho)$.
\item[(R2)] The functors $q_\lambda$, $q_\rho$ and $e$ are fully faithful.
\item[(R3)] $\text{Im } e = \text{Ker } q$.
\end{enumerate}

We denote this recollement by $(B, A, C)$ for short.

Notice that the functors $q$ and $e$ are exact, since each of them admits a right adjoint and a left adjoint. The following proposition shows that a recollement is a special case of Serre quotients.

Proposition 1.7 [25, Theorem 4.9] Let $q : A \to C$ be an exact functor. If it admits a fully faithful right adjoint $q_\rho$ (resp. left adjoint $q_\lambda$), the functor $q$ induces an equivalence between $C$ and the Serre quotient $A/\text{Ker } q$ of $A$ with respect to the Serre subcategory $\text{Ker } q$.

The following notions play a central role in Sect. 3.

Proposition 1.8 [26, Prop. 2.6] For a given recollement $(B, A, C)$ and an object $x \in A$,

\begin{enumerate}
\item[(i)] we have an exact sequence $0 \to ee_\rho(x) \xrightarrow{\eta} x \xrightarrow{\varepsilon} q_\rho q(x) \to y \to 0$, where $\eta$ and $\varepsilon$ are the counit and the unit of the adjoint pairs, respectively. We call it the right-defining exact sequence of $x$.
\item[(ii)] we have an exact sequence $0 \to y' \to q_\lambda q(x) \xrightarrow{\eta'} x \xrightarrow{\varepsilon'} ee_\lambda(x) \to 0$, where $\eta'$ and $\varepsilon'$ are the counit and unit of the adjoint pairs, respectively. We call it the left-defining exact sequence of $x$.
\end{enumerate}
Moreover, if there exists an exact sequence \( 0 \to x' \to x \to x'' \to x''' \to 0 \) with \( x', x'' \in \text{Im } e \) and \( x'' \in \text{Im } q_\rho \), then it is isomorphic to the right-defining exact sequence of \( x \). The dual statement holds for the left-defining exact sequences.

**Proof** We only prove the latter statement, that is, an exact sequence \( 0 \to x' \xrightarrow{f} x \xrightarrow{g} x'' \to 0 \) with \( x', x'' \in \text{Im } e \) and \( x'' \in \text{Im } q_\rho \) is isomorphic to the right-defining exact sequence of \( x \). By considering the right-defining exact sequences of \( x \) and \( x' \), we show that a given morphism \( x' \xrightarrow{f} x \) induces the following commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \to & ee_\rho(x) & \to & x & \to & q_\rho q(x) & \to & y & \to & 0 \\
& & ee_\rho(f) & \uparrow & f & & \rho(q_\rho(f)) & & \downarrow & & \rho\left(q_\rho(q(x))\right) \\
0 & \to & ee_\rho(x') & \to & x' & \to & q_\rho q(x') & \to & y' & \to & 0.
\end{array}
\]

Since \( x' \in \text{Im } e \), the counit \( ee_\rho(x') \xrightarrow{\sim} x' \) is an isomorphism. Since \( x'' \in \text{Im } q_\rho \) and \( e_\rho \) is left exact, the induced morphism \( e_\rho(f) : e_\rho(x') \xrightarrow{\sim} e_\rho(x) \) is an isomorphism. Thus we deduce the commutative diagram below:

\[
\begin{array}{ccccccccc}
e e_\rho(x) & \xrightarrow{\eta} & x \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
e e_\rho(f) & \xrightarrow{\cong} & f \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
e e_\rho(x') & \xrightarrow{\cong} & x'
\end{array}
\]

Hence the morphism \( x' \xrightarrow{f} x \) is isomorphic to the counit \( ee_\rho(x) \xrightarrow{\eta} x \).

Similarly we can show that a given morphism \( x \xrightarrow{g} x'' \) is isomorphic to the unit \( x \xrightarrow{\varepsilon} q_\rho q(x) \) under isomorphisms \( x'' \xrightarrow{\sim} q_\rho q(x'') \) and \( q_\rho q(g) : q_\rho q(x) \xrightarrow{\sim} q_\rho q(x'') \). A natural isomorphism \( y \to x'' \) is induced from the universality of cokernels. We have thus obtained desired isomorphisms. \( \square \)

### 2 Recollements Arising from Dualizing \( k \)-Varieties

We start with introducing basic terminology. Let us consider the notion of the tensor product \( \mathcal{B} \otimes \mathcal{A} \) of two additive categories. The objects of \( \mathcal{B} \otimes \mathcal{A} \) are the pairs \( (b, a) \) with \( b \in \mathcal{B} \) and \( a \in \mathcal{A} \) and the morphisms from \( (b, a) \) to \( (b', a') \) is the tensor product of abelian groups \( \mathcal{B}(b, b') \otimes \mathcal{A}(a, a') \). In the case that given categories \( \mathcal{A} \) and \( \mathcal{B} \) are \( k \)-linear, the morphism space \( (\mathcal{B} \otimes \mathcal{A})(b, a), (b', a') \) is defined to be the tensor product of \( k \)-modules \( \mathcal{B}(b, b') \otimes_k \mathcal{A}(a, a') \). We define \( \mathcal{B}-\mathcal{A} \)-bimodule to be a contravariant additive functor from \( \mathcal{B}^{\text{op}} \otimes \mathcal{A} \) to \( \text{Ab} \). Given a \( \mathcal{B}-\mathcal{A} \)-bimodule \( X \), we regard it as a contravariant functor from \( \mathcal{A} \) to \( \text{Mod}\mathcal{B} \) as follows: For each \( a \in \mathcal{A} \), we define a covariant functor \( X(-, a) : \mathcal{B} \to \text{Mod}k \) by setting \( X(g^{\text{op}}, a) : X(g^{\text{op}} \otimes 1_a) : \mathcal{B}(b^{\text{op}}, a) \to \mathcal{B}(b, a) \) for a morphism \( g : b' \to b \in \mathcal{B} \). Let \( f : a \to a' \) be a morphism in \( \mathcal{A} \). We define a natural transformation \( X(-, f) : X(-, a') \to X(-, a) \) by setting \( X(b^{\text{op}}, f) := X(1_{b^{\text{op}}} \otimes f) \) for each \( b \in \mathcal{B} \). These assignments give rise to a contravariant functor from \( \mathcal{A} \) to \( \text{Mod}\mathcal{B} \). Similarly, we regard a \( \mathcal{B}-\mathcal{A} \)-bimodule \( X \) as a covariant functor from \( \mathcal{B} \) to \( \text{Mod}\mathcal{A} \).
For later use, we recall in Proposition 2.2 that a $\mathcal{B}$-$\mathcal{A}$-bimodule $X$ induces a Hom-tensor adjunctions: Given a $\mathcal{B}$-$\mathcal{A}$-bimodule $X$, we define a covariant functor $\text{Hom}_A(X, -) : \text{Mod}_A \to \text{Mod}_B$ which sends $Y \in \text{Mod}_A$ to the functor $\text{Hom}_A(X, Y) : \mathcal{B} \to \text{Mod}_k$ given by $b \mapsto \text{Hom}_A(X(b^{\text{op}}, -), Y)$ for $b \in \mathcal{B}$. In the next lemma, we define a covariant functor $- \otimes_{\mathcal{B}} X : \text{Mod}_B \to \text{Mod}_A$.

**Lemma 2.1** Let $X$ be a $\mathcal{B}$-$\mathcal{A}$-bimodule. Then, there exists a unique right-exact functor $X^* : \text{Mod}_B \to \text{Mod}_A$ up to isomorphism which preserves coproducts and makes the following diagram commutative up to isomorphism

$$
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{X} & \text{Mod}_A \\
\downarrow & & \downarrow \\
\text{Mod}_B & \xrightarrow{X^*} & \\
\end{array}
$$

where we regard $X$ as a covariant functor from $\mathcal{B}$ to $\text{Mod}_A$ and $\mathcal{Y}$ denotes the Yoneda functor sending $b \in \mathcal{B}$ to $\mathcal{B}(-, b) \in \text{Mod}_B$. We write $- \otimes_{\mathcal{B}} X$ instead of $X^*$.

**Proof** This is well-known for experts but we recall a construction of $X^*$ for the convenience of the reader (e.g. [21, Section 2] for details). Any $\mathcal{B}$-module $M$ admits a projective presentation

$$
\bigoplus_{j \in J} B(-, b_j) \to \bigoplus_{i \in I} B(-, b_i) \to M \to 0.
$$

Thanks to the Yoneda Lemma, we obtain a set of morphisms $\{\beta_{ji} : b_j \to b_i\}_{j, i \in I}$ in $\mathcal{B}$. The induced set of morphisms $\{X(b^{\text{op}}_j, -) : X(b^{\text{op}}_j, -) \to X(b^{\text{op}}_i, -)\}_{j, i \in I}$ in $\text{Mod}_A$ gives a canonical morphism $f : \bigoplus_{j \in J} X(b^{\text{op}}_j, -) \to \bigoplus_{i \in I} X(b^{\text{op}}_i, -)$. Put $X^*(M) := \text{Cok } f$. We omit a remaining part of the proof. 

**Proposition 2.2** Let $X$ be a $\mathcal{B}$-$\mathcal{A}$-bimodule. Then, the induced functors $\text{Hom}_A(X, -) : \text{Mod}_A \to \text{Mod}_B$ and $- \otimes_{\mathcal{B}} X : \text{Mod}_B \to \text{Mod}_A$ form an adjoint pair $(- \otimes_{\mathcal{B}} X \dashv \text{Hom}_A(X, -))$.

**Proof** We shall show the bifunctorial isomorphism $\text{Hom}_A(M \otimes_{\mathcal{B}} X, Y) \cong \text{Hom}_B(M, \text{Hom}_A(X, Y))$ in $M \in \text{Mod}_B$ and $Y \in \text{Mod}_A$. Take a projective presentation

$$
\bigoplus_{j \in J} B(-, b_j) \to \bigoplus_{i \in I} B(-, b_i) \to M \to 0 \quad (2.1)
$$

of $M \in \text{Mod}_B$. By Lemma 2.1, applying $- \otimes_{\mathcal{B}} X$ to the above yields an exact sequence

$$
\bigoplus_{j \in J} X(b^{\text{op}}_j, -) \to \bigoplus_{i \in I} X(b^{\text{op}}_i, -) \to M \otimes_{\mathcal{B}} X \to 0
$$

in $\text{Mod}_A$. Since $\text{Hom}_A(-, Y)$ is left-exact and sends coproducts to products, we have an exact sequence

$$
0 \to \text{Hom}_A(M \otimes_{\mathcal{B}} X, Y) \to \prod_{i \in I} \text{Hom}_A(X(b^{\text{op}}_i, -), Y) \to \prod_{j \in J} \text{Hom}_A(X(b^{\text{op}}_j, -), Y)
$$

(2.2)
in \text{Mod}_k. On the other hand, applying \( \text{Hom}_B(\cdot, \text{Hom}_A(X, Y)) \) to the above presentation (2.1), we have an exact sequence

\[
0 \to \text{Hom}_B(M, \text{Hom}_A(X, Y)) \to \prod_{i \in I} \text{Hom}_A(X(b^\text{op}_i, \cdot), Y) \to \prod_{j \in J} \text{Hom}_A(X(b^\text{op}_j, \cdot), Y)
\]

(2.3)

in \text{Mod}_k. By comparing (2.2) and (2.3), we have a desired isomorphism. 

We often regard \( A \) as an \( A \)-\( A \)-bimodule by the following way

\[
A_A := A(\cdot, +) : A^{\text{op}} \otimes A \to \text{Ab}, \quad (a^{\text{op}}, a') \mapsto A(a', a).
\]

Consider a full subcategory \( B \) in \( A \). The canonical inclusion \( i : B \hookrightarrow A \) gives a natural \( A \)-\( B \)-bimodule structure on \( A \) by

\[
A_A_B := A(i(\cdot), +) : A^{\text{op}} \otimes B \to \text{Ab}, \quad (a^{\text{op}}, b) \mapsto A(i(b), a) = A(b, a).
\]

Similarly, we define a \( B \)-\( A \)-bimodule \( B_A_A := A(\cdot, i(\cdot)) \). The first step is to show the following elementary proposition, which is a categorical analog of the recollement (0.1) and well-known for experts. We denote by \( i^* : \text{Mod}_A \to \text{Mod}_B \), \( X \mapsto X|_B \) the natural restriction functor induced from \( i : B \to A \) and also denote by \( p^* : \text{Mod}(A/[B]) \to \text{Mod}_A \), \( X \mapsto X \circ p \) the natural restriction functor induced from \( p : A \to A/[B] \).

**Proposition 2.3** [26, Example 2.13] Let \((A, B)\) be the pair of an additive category \( A \) and its full subcategory \( B \). Then we have the following recollement:

\[
\begin{CD}
\text{Mod}(A/[B]) @> p^* >> \text{Mod}_A @> i^* >> \text{Mod}_B.
\end{CD}
\]

**Proof** (i) We shall construct the adjoint pairs on the right side in (2.4). Note that there exist isomorphisms \( i^* \cong \text{Hom}_A(B_A, A, \cdot) \cong - \otimes_A (A_A B) \). Thus it admits a left adjoint \( i_\lambda^* := - \otimes_B (B_A | A) \) and a right adjoint \( i_\rho^* := \text{Hom}_B(A_A | A_B, -) \). By easy calculation, we show that \( i_\lambda^* \) and \( i_\rho^* \) are fully faithful. In fact, we have the following isomorphisms:

\[
\begin{align*}
\text{id}_{\text{Mod}_B} & \cong \text{id}_{\text{Mod}_B}, \\
i^* \circ i_\lambda^* & \cong - \otimes_B (B_A | A) \otimes_A (A_A B) \cong - \otimes_B (A_A B) \\
i^* \circ i_\rho^* & \cong \text{Hom}_A(B_A | A), \text{Hom}_B(A_A | A_B, -) \\
& \cong \text{Hom}_B(B, -) \\
& \cong \text{id}_{\text{Mod}_B}.
\end{align*}
\]

Thus we have constructed the right side of (2.4).

(ii) We shall construct the adjoint pairs on the left side in (2.4). By a similar argument to the above, the restriction functor \( p^* \) is a fully faithful exact functor which admits a left adjoint \( p_\lambda^* \) and a right adjoint \( p_\rho^* \). Thus we have obtained the left side of (2.4).

(iii) It remains to show that \( \text{Im} p^* = \text{Ker} i^* \). This follows from the following obvious lemma.

**Lemma 2.4** Let \( X \) be an object in \( \text{Mod}_A \). Then \( X \) belongs to \( \text{Im} p^* \) if and only if \( X \) vanishes on objects in \( B \). In particular, we have \( \text{Im} p^* = \text{Ker} i^* \).
We have thus proved Proposition 2.3. \qed

We now give our main theorem.

**Theorem 2.5** Let \((\mathcal{A}, \mathcal{B})\) be the pair of a dualizing \(k\)-variety \(\mathcal{A}\) and its functorially finite subcategory \(\mathcal{B}\). Then the recollement (2.4) restricts to the following one:

\[
\begin{align*}
\mod(\mathcal{A}/[\mathcal{B}]) & \longrightarrow \mod \mathcal{A} \longrightarrow \mod \mathcal{B}.
\end{align*}
\] (2.5)

In particular, we have an equivalence \(\frac{\mod \mathcal{A}}{\mod(\mathcal{A}/[\mathcal{B}])} \simeq \mod \mathcal{B}\).

We call this the recollement arising from the pair \((\mathcal{A}, \mathcal{B})\) of a dualizing \(k\)-variety \(\mathcal{A}\) and a functorially finite subcategory \(\mathcal{B}\) in \(\mathcal{A}\).

In the rest of this section, we give a proof of Theorem 2.5. Let \(\mathcal{A}\) and \(\mathcal{B}\) be Krull–Schmidt Hom-finite \(k\)-linear categories.

First we consider the right part of the recollement (2.4). The functor \(i_*\) preserves indecomposable projectives because \(i_*\)\((\mathcal{B}(\cdot, b)) = \mathcal{B}(\cdot, b) \otimes \mathcal{A}\mathcal{A} \cong \mathcal{A}(\cdot, b)\). Since \(i_*\) is right-exact, we have the restricted functor \(i_* : \mod \mathcal{B} \rightarrow \mod \mathcal{A}\), which is denoted by the same symbol. In general \(i^*\) and \(i_*\) do not restrict to the subcategories of finitely presented functors. However, if \((\mathcal{A}, \mathcal{B})\) is a pair of a dualizing \(k\)-variety \(\mathcal{A}\) and its functorially finite subcategory \(\mathcal{B}\), \(i^*\) and \(i_*\) restrict to the subcategories.

Second we consider the left part of the recollement (2.4). Like the case for the canonical inclusion \(i\), although the left adjoint \(p^*_*\) preserves finitely presented functors, \(p^*\) and \(p^*_p\) do not necessarily preserve finitely presented functors.

The next lemma shows a necessary and sufficient condition so that \(i^*\) and \(p^*\) preserves finitely presented functors, see [8, Prop. 3.9] for the equivalence (i) and (iii) below.

**Lemma 2.6** Let \(\mathcal{A}\) be a category with weak-kernels and \(\mathcal{B}\) a full subcategory in \(\mathcal{A}\). Then the following are equivalent for the recollement (2.4):

(i) The category \(\mathcal{B}\) is contravariantly finite.
(ii) The functor \(i^*\) restricts to the functor \(i^* : \mod \mathcal{A} \rightarrow \mod \mathcal{B}\).
(iii) The functor \(p^*\) restricts to the functor \(p^* : \mod(\mathcal{A}/[\mathcal{B}]) \rightarrow \mod \mathcal{A}\).

Moreover, under the above equivalent conditions, we have the restricted adjoint pairs

\[
\begin{align*}
\mod(\mathcal{A}/[\mathcal{B}]) & \longrightarrow p^* \longrightarrow \mod \mathcal{A} \quad \text{and} \quad \mod \mathcal{A} \longrightarrow i^* \longrightarrow \mod \mathcal{B}.
\end{align*}
\]

**Proof** (i) \(\Rightarrow\) (ii): Since \(i^*\) is exact, we have only to show that \(i^*(\mathcal{A}(\cdot, x))\) is finitely presented for any \(x \in \mathcal{A}\). Since \(\mathcal{B}\) is contravariantly finite, there exists a right \(\mathcal{B}\)-approximation \(\alpha_0 : b_0 \rightarrow x\). The morphism \(\alpha_0\) induces an epimorphism \(\mathcal{B}(\cdot, b_0) \xrightarrow{\alpha_0^0} \mathcal{A}(\cdot, x)\mid_{\mathcal{B}} \rightarrow 0\), that is, the \(\mathcal{B}\)-module \(\mathcal{A}(\cdot, x)\mid_{\mathcal{B}}\) is finitely generated. Since \(\mod \mathcal{A}\) is abelian, we have the kernel-sequence \(0 \rightarrow X \rightarrow \mathcal{A}(\cdot, b_0) \xrightarrow{\alpha_0^0} \mathcal{A}(\cdot, x)\mid_{\mathcal{B}} \rightarrow 0\), that is, \(\mathcal{A}(\cdot, x)\mid_{\mathcal{B}}\) is finitely generated. Since \(X \in \mod \mathcal{A}\), there exists an epimorphism \(\mathcal{A}(\cdot, x') \rightarrow X \rightarrow 0\) and thus we have an exact sequence

\[
\mathcal{A}(\cdot, x')\mid_{\mathcal{B}} \rightarrow \mathcal{B}(\cdot, b_0) \rightarrow \mathcal{A}(\cdot, x)\mid_{\mathcal{B}} \rightarrow 0
\]
in \( \text{mod} \mathcal{B} \). The fact that \( \mathcal{A}(-, x')|_{\mathcal{B}} \) is finitely generated shows that \( \mathcal{A}(-, x)|_{\mathcal{B}} \) is finitely presented.

(iii) \( \Rightarrow \) (i): For any \( x \in \mathcal{A} \), the functor \( i^*(\mathcal{A}(-, x)) = \mathcal{A}(-, x)|_{\mathcal{B}} \) is finitely presented. This shows that \( \mathcal{B} \) is contravariantly finite by definition.

If \( \mathcal{A} \) is a dualizing \( k \)-variety and \( \mathcal{B} \) is functorially finite in \( \mathcal{A} \), then the restriction functor \( i^* : \text{mod} \mathcal{A} \to \text{mod} \mathcal{B} \) admits a right adjoint.

**Proposition 2.7** There exist the following adjoint pairs for the pair \((\mathcal{A}, \mathcal{B})\) of a dualizing \( k \)-variety \( \mathcal{A} \) and its contravariantly finite full subcategory \( \mathcal{B} \):

\[
\begin{array}{ccc}
\text{mod} \mathcal{A} & \xrightarrow{q} & \text{mod} \mathcal{B} \\
\downarrow & & \downarrow \\
\text{mod} (\mathcal{A}^{\text{op}}) & \xrightarrow{q'} & \text{mod} (\mathcal{B}^{\text{op}})
\end{array}
\]

where \( q := i^* \) is the restriction functor induced by the canonical inclusion \( i : \mathcal{B} \hookrightarrow \mathcal{A} \). Moreover, we have isomorphisms \( q_{\rho} \cong \text{Hom}_{\mathcal{B}}(\mathcal{A} \mathcal{A}, -) \) and \( q_{\lambda} \cong - \otimes_{\mathcal{B}} (\mathcal{B} \mathcal{A}) \).

**Proof** Recall that every dualizing \( k \)-variety admits weak-kernels and weak-cokernels. As we have seen in Lemma 2.6, there exists an adjoint pair \((q_{\lambda}, q_{\rho})\) between \( \text{mod} \mathcal{A} \) and \( \text{mod} \mathcal{B} \). Since \( \mathcal{A}^{\text{op}} \) is also a dualizing \( k \)-variety and \( \mathcal{B}^{\text{op}} \) is functorially finite in \( \mathcal{A}^{\text{op}} \), by Lemma 2.6, we have an adjoint pair \((q'_{\lambda}, q'_{\rho})\) between \( \text{mod} (\mathcal{A}^{\text{op}}) \) and \( \text{mod} (\mathcal{B}^{\text{op}}) \), where \( q'_{\lambda} \) is the restriction functor induced by the inclusion \( \mathcal{B}^{\text{op}} \hookrightarrow \mathcal{A}^{\text{op}} \). Since \( \mathcal{A} \) and \( \mathcal{B} \) are dualizing \( k \)-varieties, we have the following functors:

\[
\begin{array}{ccc}
\text{mod} \mathcal{A} & \xrightarrow{q_{\lambda}} & \text{mod} \mathcal{B} \\
\downarrow & & \downarrow \\
\text{mod} (\mathcal{A}^{\text{op}}) & \xrightarrow{q'_{\lambda}} & \text{mod} (\mathcal{B}^{\text{op}})
\end{array}
\]

First we notice that \( q \cong D q' D \) holds by definition. Put \( q_{\rho} := D q'_{\lambda} D : \text{mod} \mathcal{B} \to \text{mod} \mathcal{A} \). It is easy to check that \( q \) and \( q_{\rho} \) form an adjoint pair \((q, q_{\rho})\).

In the remaining part of the proof, we shall verify the latter statement, namely, an isomorphism \( q_{\rho} \cong \text{Hom}_{\mathcal{B}}(\mathcal{A} \mathcal{A}, -) \). This can be verified by the following calculations. Since \( q_{\rho} \) is left-exact and preserves injective objects, we have only to check the values of \( q_{\rho} \) on injective \( \mathcal{B} \)-modules. Due to the duality \( D : \text{mod} \mathcal{B} \to \text{mod} (\mathcal{B}^{\text{op}}) \), each injective \( \mathcal{B} \)-module is isomorphic to \( DB(x, -) \) for some \( x \in \mathcal{B} \).

\[
q_{\rho}(DB(x, -)) = D q'_{\lambda} D(DB(x, -))
\]

\[
\cong D((\mathcal{A} \mathcal{A}) \otimes_{\mathcal{B}} DB(x, -))
\]

\[
\cong \text{Hom}_{\mathcal{B}^{\text{op}}}(B(x, -), D_{\mathcal{A} \mathcal{A}})
\]

\[
\cong \text{Hom}_{\mathcal{B}}(\mathcal{A} \mathcal{A}, DB(x, -)).
\]

Therefore \( q_{\rho} \cong \text{Hom}_{\mathcal{B}}(\mathcal{A} \mathcal{A}, -) \) on \( \text{mod} \mathcal{B} \), and hence it is fully faithful.

By the discussion so far, we constructed the right part of the recollement \((2.5)\). Next, we shall construct the left part of \((2.5)\). We keep the assumption that \( \mathcal{A} \) is a dualizing \( k \)-variety and \( \mathcal{B} \) is functorially finite. Let us begin with a “finitely presented version” of Lemma 2.4.
Lemma 2.8  The following hold.
(i) A finitely presented $\mathcal{A}$-module $X$ belongs to $\text{mod}(\mathcal{A}/[\mathcal{B}])$ if and only if $X$ vanishes on objects in $\mathcal{B}$. In particular, we have $\text{mod}(\mathcal{A}/[\mathcal{B}]) = \text{Ker } q$.
(ii) The ideal quotient category $\mathcal{A}/[\mathcal{B}]$ algebra and $\text{mod}_q \text{Ker } X$ a functor. We call this generalized Auslander’s formula.

Proof (i) We only prove the “if” part. If $X(b) = 0$ for any $b \in \mathcal{B}$, there uniquely exists a functor $X' : \mathcal{A}/[\mathcal{B}] \to \text{Mod}_k$ such that $X \cong p^*X'$. We have only to show that $X' \in \text{mod}(\mathcal{A}/[\mathcal{B}])$. Applying $p^*_A$ to the isomorphism $X \cong p^*X'$ yields $p^*_AX \cong p^*_Ap^*X' \cong X'$. Since $p^*_A$ preserves finitely presented functors and $X \in \text{mod} \mathcal{A}$, we conclude $X' \in \text{mod}(\mathcal{A}/[\mathcal{B}])$.

(ii) Let $X \in \text{mod}(\mathcal{A}/[\mathcal{B}])$. Then $DX$ can be regarded as a finitely presented $\mathcal{A}^\text{op}$-module which vanishes on $\mathcal{B}$. Hence $DX \in \text{mod}(\mathcal{A}/[\mathcal{B}])^\text{op}$. Conversely, we can show that $DX' \in \text{mod} \mathcal{A}/[\mathcal{B}]$ for any $X' \in \text{mod}(\mathcal{A}/[\mathcal{B}])^\text{op}$.

By a similar argument in the proof of Proposition 2.7, we obtain the following.

Proposition 2.9  There exist the following adjoint pairs for the pair $(\mathcal{A}, \mathcal{B})$ of a dualizing $k$-variety $\mathcal{A}$ and its full subcategory $\mathcal{B}$:

\[
\begin{array}{ccc}
\text{mod}(\mathcal{A}/[\mathcal{B}]) & \xrightarrow{e_\lambda} & \text{mod} \mathcal{A}, \\
\text{mod}(\mathcal{A}/[\mathcal{B}]) & \xleftarrow{e_\rho} & \text{mod} \mathcal{A},
\end{array}
\]

where $e : p^*$ is the restriction functor induced by the canonical projection $p : \mathcal{A} \to \mathcal{A}/[\mathcal{B}]$. Moreover, we have isomorphisms $e_\rho \cong \text{Hom}_{\mathcal{A}}(\mathcal{A}/[\mathcal{B}](\mathcal{A}/[\mathcal{B}]), -)$ and $e_\lambda \cong - \otimes_\mathcal{A} (\mathcal{A}/[\mathcal{B}], \mathcal{A}/[\mathcal{B}])$.

Now we are ready to prove Theorem 2.5.

Proof of Theorem 2.5  By Proposition 2.7 and Proposition 2.9, we have four adjoint pairs $(q_\lambda \dashv q), (q \dashv q_\rho), (e_\lambda \dashv e)$ and $(e \dashv e_\rho)$ with $q_\rho, q_\lambda$ and $e$ fully faithful. By definition, $\text{Ker } q$ is a full subcategory in $\text{mod} \mathcal{A}$ consisting of functors which vanish on $\mathcal{B}$. Due to Lemma 2.8, we have $\text{Ker } q = \text{mod}(\mathcal{A}/[\mathcal{B}])$. Hence they form a recollement.

Now we apply Theorem 2.5 to the following special setting. Let $\Lambda$ be a finite dimensional $k$-algebra and $\mathcal{B}$ a functorially finite subcategory of $\text{mod} \Lambda$ containing $\Lambda$. Applying Theorem 2.5 to the pair $(\mathcal{B}, \text{proj} \Lambda)$ yields the following recollement:

\[
\begin{array}{ccc}
\text{mod} \mathcal{B} & \xrightarrow{e_\lambda} & \text{mod} \mathcal{B}, \\
\text{mod} \mathcal{B} & \xleftarrow{e_\rho} & \text{mod} \mathcal{B},
\end{array}
\]

where we identify $\text{mod}(\text{proj} \Lambda)$ with $\text{mod} \Lambda$ via the equivalence $\text{mod}(\text{proj} \Lambda) \xrightarrow{\sim} \text{mod} \Lambda, X \mapsto X(\Lambda)$. By Proposition 1.7, this recollement induces the following.

Corollary 2.10  Under the above assumption, there exists an equivalence

\[
\text{mod}(\text{mod} \mathcal{B}) \xrightarrow{\sim} \text{mod} \Lambda.
\]

We call this generalized Auslander’s formula.

By setting $\mathcal{B} = \text{mod} \Lambda$, we can recover classical Auslander’s formula $\text{mod}(\text{mod} \mathcal{B}) \xrightarrow{\sim} \text{mod} \Lambda$ (see [1, p. 205] and [24, p. 1] for definition).
3 Application to the Auslander–Bridger Sequences

The aim of this section is to show a close relationship between recollements and Auslander–Bridger sequences. Throughout this section, we fix a dualizing $k$-variety $A$. Let $\mathcal{B}$ be a functorially finite subcategory in mod $A$ which contains proj $A$ and inj $A$.

Firstly, we recall the definition of Auslander–Bridger sequence, following [18, Prop. 2.7]. For the category $\mathcal{B}$, we denote the $\mathcal{B}$-duality by $(-)^*$ := Hom$_{\mathcal{B}}(-, B)$. Note that the $\mathcal{B}$-duality yields a duality $(-)^*$ : proj $\mathcal{B}$ $\sim$ proj $($ $\mathcal{B}^{\text{op}})$, $\mathcal{B}(-, b) \mapsto \mathcal{B}(b, -)$. Let $X \in \text{mod} \mathcal{B}$ with a minimal projective presentation $\mathcal{B}(-, b_1) \xrightarrow{\alpha} \mathcal{B}(-, b_0) \rightarrow X \rightarrow 0$ and set

$$\text{Tr}X := \text{Cok} \alpha^*$$

in mod $(\mathcal{B}^{\text{op}})$, see [2].

**Definition-Proposition 3.1** For each object $X \in \text{mod} \mathcal{B}$, there exists an exact sequence

$$0 \rightarrow \text{Ext}^1_{\mathcal{B}^{\text{op}}}(\text{Tr}X, \mathcal{B}^{\text{op}}) \rightarrow X \xrightarrow{\epsilon} X^{**} \rightarrow \text{Ext}^2_{\mathcal{B}^{\text{op}}}(\text{Tr}X, \mathcal{B}^{\text{op}}) \rightarrow 0,$$

which is called the Auslander–Bridger sequence of $X$.

For the convenience of the reader, we recall from [1, Prop. 6.3] and [18, Prop. 2.7] the construction of the Auslander–Bridger sequence of $X$. Take a minimal projective presentation $\mathcal{B}(-, b_0) \rightarrow \mathcal{B}(-, b_1) \rightarrow X \rightarrow 0$ of $X$. Taking a left $\mathcal{B}$-approximation of a cokernel of $b_0 \rightarrow b_1$ yields an exact sequence $b_0 \rightarrow b_1 \rightarrow b_2$. Taking a left approximation of a cokernel of $b_1 \rightarrow b_2$ yields an exact sequence

$$b_0 \rightarrow b_1 \rightarrow b_2 \rightarrow b_3 \quad (3.1)$$

in mod $A$. By the construction, the sequence (3.1) induces the exact sequence

$$\mathcal{B}(b_3, -) \xrightarrow{\frac{\alpha}{g}} \mathcal{B}(b_2, -) \xrightarrow{\frac{\beta}{f}} \mathcal{B}(b_1, -) \xrightarrow{\frac{\gamma}{h}} \mathcal{B}(b_0, -) \quad (3.2)$$

in mod $(\mathcal{B}^{\text{op}})$. Note that $X^{**} = \text{Ker} f$ and Tr $X = \text{Cok} f$. Taking the $\mathcal{B}^{\text{op}}$-duality of (3.2) yields the following sequence

$$0 \rightarrow \text{Hom}_{\mathcal{B}^{\text{op}}}(\text{Tr}X, \mathcal{B}^{\text{op}}) \rightarrow \mathcal{B}(-, b_0) \xrightarrow{f^*} \mathcal{B}(-, b_1) \xrightarrow{g^*} \mathcal{B}(-, b_2) \xrightarrow{h^*} \mathcal{B}(-, b_3) \quad (3.3)$$

Note that $\text{Cok} f^* \cong X$ and $\text{Ker} h^* = X^{**}$. Since (3.3) is a complex, we have a canonical inclusion $i : \text{Im} g^* \hookrightarrow \text{Ker} h^*$ and a unique canonical epimorphism $\epsilon' : X \rightarrow \text{Im} g^*$. It is readily verified that there exists a commutative diagram with exact rows.

$$\begin{array}{ccc}
0 & \longrightarrow & \text{Im} f^* \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \text{Ker} g^* \\
& & \downarrow \\
& & \text{Im} g^* \longrightarrow 0 \\
& & \downarrow \\
X^{**} & \longrightarrow & \\
& & \downarrow \\
& & \text{Im} g^* \longrightarrow 0
\end{array}$$

We set $\epsilon := i \circ \epsilon'$. By the Snake Lemma, we have Ker $\epsilon \cong \text{Ker} \epsilon' \cong \text{Ker} g^*/\text{Im} f^*$. It is easy to verify that Cok $\epsilon \cong \text{Cok} i = \text{Ker} h^*/\text{Im} g^*$. Since (3.3) is the $\mathcal{B}^{\text{op}}$-duality of the projective resolution of Tr $X$, we have the isomorphisms Ker $g^*/\text{Im} f^* \cong \text{Ext}^1_{\mathcal{B}^{\text{op}}}(\text{Tr}X, \mathcal{B}^{\text{op}})$ and...
Ker $h^* / \text{Im } g^* \cong \text{Ext}^2_{\mathcal{B}^\text{op}}(\text{Tr} X, \mathcal{B}^\text{op})$. We have thus obtained the Auslander–Bridger sequence

$$0 \rightarrow \text{Ext}^1_{\mathcal{B}^\text{op}}(\text{Tr} X, \mathcal{B}^\text{op}) \rightarrow X \xrightarrow{\epsilon} X^{**} \rightarrow \text{Ext}^2_{\mathcal{B}^\text{op}}(\text{Tr} X, \mathcal{B}^\text{op}) \rightarrow 0.$$ 

We give an interpretation of the Auslander–Bridger sequences via the recollement appearing below. Due to Theorem 2.5, the pair $(\mathcal{B}, \text{proj} \mathcal{A})$ induces the following recollement:

$$\begin{align*}
\text{mod} \mathcal{B} & \xrightarrow{e_\rho} \text{mod} \mathcal{B} & \xrightarrow{q_\rho} \text{mod} \mathcal{A}, \\
\text{mod} \mathcal{B} & \xrightarrow{e_\rho} \text{mod} \mathcal{A},
\end{align*}$$

(3.4)

where we identify $\text{mod}(\text{proj} \mathcal{A})$ with $\text{mod} \mathcal{A}$ via the equivalence $\text{mod}(\text{proj} \mathcal{A}) \cong \text{mod} \mathcal{A}$.

**Theorem 3.2** Let $(\text{mod} \mathcal{B}, \text{mod} \mathcal{B}, \text{mod} \mathcal{A})$ be a recollement (3.4). Then the right-defining exact sequence

$$0 \rightarrow (ee_\rho) X \rightarrow X \rightarrow (q_\rho q) X \rightarrow X' \rightarrow 0$$

of $X \in \text{mod} \mathcal{B}$ is isomorphic to the Auslander–Bridger sequence of $X$.

In the rest, we give a proof of Theorem 3.2. By Lemma 2.8(i), $\text{mod} \mathcal{B}$ is a full subcategory in $\text{mod} \mathcal{B}$ consisting of objects $X$ which admits a projective presentation

$$\mathcal{B}(-, b_1) \rightarrow \mathcal{B}(-, b_0) \rightarrow X \rightarrow 0$$

with $b_1 \rightarrow b_0$ an epimorphism in $\mathcal{B}$. Proposition 2.7 gives an explicit description of the functor $q_\rho$.

**Lemma 3.3** The functor $q_\rho : \text{mod} \mathcal{A} \rightarrow \text{mod} \mathcal{B}$ sends $x$ to $\text{Hom}_\mathcal{A}(\mathcal{B}, x)$.

We define the 2nd syzygy category $\Omega^2(\text{mod} \mathcal{B})$ of $\text{mod} \mathcal{B}$ to be the full subcategory of $\text{mod} \mathcal{B}$ consisting of objects $X$ which admits an exact sequence $0 \rightarrow X \rightarrow \mathcal{B}(-, b_0) \rightarrow \mathcal{B}(-, b_1)$ for some $b_0, b_1 \in \mathcal{B}$.

**Lemma 3.4** We have the equality $\text{Im } q_\rho = \Omega^2(\text{mod} \mathcal{B})$.

**Proof** We show that $\text{Im } q_\rho \subseteq \Omega^2(\text{mod} \mathcal{B})$. Let $x \in \text{mod} \mathcal{A}$. Due to $\text{inj} \mathcal{A} \subseteq \mathcal{B}$, there exists an exact sequence $0 \rightarrow x \rightarrow b_0 \rightarrow b_1$ in $\text{mod} \mathcal{A}$ with $b_0, b_1 \in \mathcal{B}$ which is obtained by taking injective copresentation of $x$. Applying $q_\rho$ to the above exact sequence gives an exact sequence $0 \rightarrow q_\rho x \rightarrow q_\rho b_0 \rightarrow q_\rho b_1$. By Lemma 3.3, $q_\rho(b_i) \cong \mathcal{B}(-, b_i)$ for $i = 0, 1$. This implies that $q_\rho x \in \Omega^2(\text{mod} \mathcal{B})$.

To show the converse, take an object $X \in \Omega^2(\text{mod} \mathcal{B})$ with an exact sequence

$$0 \rightarrow X \rightarrow \mathcal{B}(-, b_1) \rightarrow \mathcal{B}(-, b_0).$$

Taking the kernel of $b_1 \rightarrow b_0$ yields an exact sequence $0 \rightarrow x \rightarrow b_1 \rightarrow b_0$ in $\text{mod} \mathcal{A}$. By applying $q_\rho$ to this, we have an exact sequence

$$0 \rightarrow q_\rho x \rightarrow \mathcal{B}(-, b_1) \rightarrow \mathcal{B}(-, b_0)$$

in $\text{mod} \mathcal{B}$. Thus we have $X \cong q_\rho x \in \text{Im } q_\rho$. This finishes the proof. □

Now we are ready to prove Theorem 3.2.
Each epimorphism \( b \) define the following concepts, which were introduced by Jasso and Kvamme independently.

(i) The fact that \( X^{*\ast} \in \Omega^2(\mod B) \) follows from the exact sequence. Actually, we can find the exact sequence \( 0 \to X^{*\ast} \to B(-, b_2) \to B(-, b_3) \) in (3.3).

(ii) Since \( q \) is a restriction functor with respect to the subcategory \( \proj \mathcal{A} \), we evaluate the sequence (3.3) on \( p \in \proj \mathcal{A} \). This yields an exact sequence

\[
B(p, b_0) \to B(p, b_1) \to B(p, b_2) \to B(p, b_3),
\]

since the sequence \( b_0 \to b_1 \to b_2 \to b_3 \) is exact. Therefore \( \epsilon(p) \) is an isomorphism, equivalently \( q(\epsilon) \) is an isomorphism. \( \square \)

4 Application to the \( n \)-Auslander–Reiten Duality

Throughout this section let \( \mathcal{A} \) be a dualizing \( k \)-variety and \( n \) a positive integer. We recall the notion of \( n \)-cluster tilting subcategory in \( \mod \mathcal{A} \). Let \( \mathcal{B} \) be a subcategory of \( \mod \mathcal{A} \). For convenience, we define the full subcategories \( \mathcal{B}_n \) and \( \mathcal{B}_n^\perp \) by

\[
\mathcal{B}_n := \{ x \in \mod \mathcal{A} \mid i \in \{1, \ldots, n\} \operatorname{Ext}^i_A(B, x) = 0 \},
\]

\[
\mathcal{B}_n^\perp := \{ x \in \mod \mathcal{A} \mid i \in \{1, \ldots, n\} \operatorname{Ext}^i_A(x, B) = 0 \}.
\]

Definition 4.1 [16, Def. 2.2] A functorially finite subcategory \( \mathcal{B} \) in \( \mod \mathcal{A} \) together with \( n \in \mathbb{N} \) is said to be \( n \)-cluster-tilting if the equalities \( \mathcal{B} = \mathcal{B}_n^{-1} \mathcal{B} = \mathcal{B}_n^{\perp -1} \mathcal{B} \) hold.

Note that 1-cluster tilting subcategory is nothing other than \( \mod \mathcal{A} \). It is obvious that every \( n \)-cluster tilting subcategory contains \( \proj \mathcal{A} \) and \( \inj \mathcal{A} \), since \( \operatorname{Ext}_A^n(\proj \mathcal{A}, -) \) and \( \operatorname{Ext}_A^n(-, \inj \mathcal{A}) \) is zero for any \( i > 0 \). This fact forces each right \( \mathcal{B} \)-approximation \( b \to x \) of \( x \) to be an epimorphism in \( \mod \mathcal{A} \), for every \( x \in \mod \mathcal{A} \). Dually each left \( \mathcal{B} \)-approximation is a monomorphism.

Throughout this section, \( \mathcal{B} \) always denotes an \( n \)-cluster-tilting subcategory in \( \mod \mathcal{A} \). We collect some facts for later use. The following notion is instrumental in this section.

Definition 4.2 [19, Def. 2.4] Let \( \mathcal{B} \) be an \( n \)-cluster-tilting subcategory in \( \mod \mathcal{A} \). A complex \( \delta : 0 \to b_{n+1} \to b_n \to \cdots \to b_0 \to 0 \) in \( \mathcal{B} \) is said to be \( n \)-exact if the induced sequences

\[
0 \to \mathcal{B}(-, b_{n+1}) \to \mathcal{B}(-, b_n) \to \cdots \to \mathcal{B}(-, b_0),
\]

\[
0 \to \mathcal{B}(b_0, -) \to \cdots \to \mathcal{B}(b_n, -) \to \mathcal{B}(b_{n+1}, -)
\]

are exact in \( \mod \mathcal{B} \) and \( \mod (\mathcal{B}^{\mathsf{op}}) \), respectively.

Lemma 4.3 [19, Prop. 3.17] The following hold for an \( n \)-cluster-tilting subcategory \( \mathcal{B} \).

(i) Each monomorphism \( b_{n+1} \to b_n \) in \( \mathcal{B} \) can be embedded in an \( n \)-exact sequence \( \delta : 0 \to b_{n+1} \to b_n \to \cdots \to b_0 \to 0 \). Moreover, \( \delta \) is uniquely determined up to homotopy.

(ii) Each epimorphism \( b_1 \to b_0 \) in \( \mathcal{B} \) can be embedded in an \( n \)-exact sequence \( \delta : 0 \to b_{n+1} \to \cdots \to b_1 \to b_0 \to 0 \). Moreover, \( \delta \) is uniquely determined up to homotopy.

As a generalization of Auslander’s defect introduced in [1] (see also Section IV. 4 in [4]), we define the following concepts, which were introduced by Jasso and Kvamme independently.
Definition 4.4 [20] Let $\delta : 0 \to b_{n+1} \to b_n \to \cdots \to b_0 \to 0$ be an $n$-exact sequence in $\mathcal{B}$. The contravariant $n$-defect $\delta^{*n}$ and the covariant $n$-defect $\delta_{*n}$ are defined by the exactness of the following sequences:

$$
0 \to B(-, b_{n+1}) \to B(-, b_n) \to \cdots \to B(-, b_0) \to \delta^{*n} \to 0,
$$

$$
0 \to B(b_0, -) \to \cdots \to B(b_n, -) \to B(b_{n+1}, -) \to \delta_{*n} \to 0.
$$

We give the following characterization of $n$-defects.

Proposition 4.5 The full subcategory of contravariant $n$-defects equals to $\text{mod} \mathcal{B}$ in $\text{mod} \mathcal{B}$. Dually the full subcategory of covariant $n$-defects equals to $\text{mod} (\mathcal{B}^{\text{op}})$ in $\text{mod} (\mathcal{B}^{\text{op}})$.

Proof We only prove the former statement. Consider $X \in \text{mod} \mathcal{B}$ with a projective presentation $\mathcal{B}(-, b_1) \xrightarrow{f_0} \mathcal{B}(-, b_0) \to X \to 0$. Assume that $X$ belongs to $\text{mod} \mathcal{B}$. Since $X$ vanishes on $p \in \text{proj} \mathcal{A}$, the map $f_0$ is an epimorphism in $\text{mod} \mathcal{A}$. By Lemma 4.3, the map $f_0$ can be embedded in an $n$-exact sequence $\delta : 0 \to b_{n+1} \to \cdots \to b_1 \to b_0 \to 0$. Hence $\delta^{*n} \cong X$.

Conversely, we shall show that contravariant $n$-defect $\delta^{*n}$ belongs to $\text{mod} \mathcal{B}$. Let $\delta : 0 \to b_{n+1} \to \cdots \to b_1 \to b_0 \to 0$ be the corresponding $n$-exact sequence and $p$ an object in $\text{proj} \mathcal{A}$. Then we have the right exact sequence

$$
\text{Hom} \mathcal{A}(p, b_1) \to \text{Hom} \mathcal{A}(p, b_0) \to \delta^{*n}(p) \to 0
$$

by definition. Since $b_1 \to b_0$ is epic in $\text{mod} \mathcal{A}$, this concludes that $\delta^{*n}(p) = 0$. Hence $\delta^{*n} \in \text{mod} \mathcal{B}$. $\square$

Next we shall show that there exists a duality between $\text{mod} \mathcal{B}$ and $\text{mod} (\mathcal{B}^{\text{op}})$. We denote $C(\mathcal{B})$ the category of complexes in $\mathcal{B}$. For convenience, we consider the homotopy category $K(\mathcal{B})$ of $C(\mathcal{B})$ and its full subcategory $K^{n\text{-ex}}(\mathcal{B})$ consisting of $n$-exact sequences $\delta : 0 \to b_{n+1} \to \cdots \to b_1 \to b_0 \to 0$ with the degree of $b_0$ being zero.

Proposition 4.6 For $n$-exact sequences $\delta : 0 \to b_{n+1} \to \cdots \to b_1 \to b_0 \to 0$ and $\delta' : 0 \to b'_{n+1} \to \cdots \to b'_{1} \to b'_0 \to 0$, the following are equivalent.

(i) The sequence $\delta$ is homotopy equivalent to $\delta'$.
(ii) There exists an isomorphism $\delta^{*n} \cong \delta'^{*n}$.
(iii) There exists an isomorphism $\delta_{*n} \cong \delta'_{*n}$.

Moreover, we have a duality $\Phi : \text{mod} (\mathcal{B}^{\text{op}}) \simeq \text{mod} \mathcal{B}$ sending $\delta_{*n}$ to $\delta^{*n}$.

Proof (i)⇔(ii): We assume that $\delta$ is homotopy equivalent to $\delta'$, that is, there exists chain maps $\phi : \delta \to \delta'$ and $\psi : \delta' \to \delta$:

$$
0 \longrightarrow b_{n+1} \overset{\beta_{n+1}}{\longrightarrow} \cdots \overset{\beta_1}{\longrightarrow} b_0 \overset{\beta_0}{\longrightarrow} 0
$$

$$
0 \longrightarrow b'_{n+1} \overset{\beta'_{n+1}}{\longrightarrow} \cdots \overset{\beta'_1}{\longrightarrow} b'_0 \overset{\beta'_0}{\longrightarrow} 0
$$

$$
0 \longrightarrow b_{n+1} \overset{\psi_{n+1}}{\longrightarrow} \cdots \overset{\psi_1}{\longrightarrow} b_1 \overset{\psi_0}{\longrightarrow} 0
$$

$$
0 \longrightarrow b'_{n+1} \overset{\psi'_{n+1}}{\longrightarrow} \cdots \overset{\psi'_1}{\longrightarrow} b'_1 \overset{\psi'_0}{\longrightarrow} 0
$$

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with $1 - \psi \phi$ and $1 - \phi \psi$ null-homotopic. By a standard argument on the above diagram, we have an isomorphism $\delta^{an} \cong \delta^{an}$. The converse is obvious. The implications (i) $\iff$ (iii) can be proved by a dual argument of the above.

We shall show the later statement. By Proposition 4.5, the functor $\text{Cok} : K^{n-\text{ex}}(B) \to \text{mod}B$ sending $\delta$ to $\text{Cok} \text{Hom}_B(\cdot, \beta_1) = \delta^{an}$ is full and dense. To show it is faithful, we take a morphism $\phi : \delta \to \delta'$ in $K^{n-\text{ex}}(B)$ such that $\text{Cok} \phi = 0$. The condition $\text{Cok} \phi = 0$ forces that $\phi$ is null-homotopic as follows: Via the Yoneda embedding $B \to \text{mod}B$ the morphism $\phi$ induces a chain map $\text{Hom}_B(\cdot, \phi) : \text{Hom}_B(\cdot, \delta) \to \text{Hom}_B(\cdot, \delta')$ of complexes in $\text{mod}B$. Since $\text{Cok} \text{Hom}_B(\cdot, \phi) = 0$, it follows that $\text{Hom}_B(\cdot, \phi)$ is zero in homology. Since $\text{Hom}_B(\cdot, \delta)$ and $\text{Hom}_B(\cdot, \delta')$ are complexes with projective components, we get that $\text{Hom}_B(\cdot, \phi)$ must be null-homotopic. Hence, since the Yoneda embedding $B \to \text{mod}B$ is full and faithful, it follows that $\phi$ is null-homotopic. Therefore $\text{Cok}$ gives an equivalence. Dually we have a duality $K^{n-\text{ex}}(B) \to \text{mod}(B^{op})$ sending $\delta$ to $\text{Cok} \text{Hom}_B(\beta_{n+1}, \cdot) = \delta^{an}$. It is obvious that the composed functor

$$\Phi : \text{mod}(B^{op}) \overset{\sim}{\to} K^{n-\text{ex}}(B) \overset{\sim}{\to} \text{mod}B, \quad \delta^{an} \mapsto \delta \mapsto \delta^{an}$$

gives a desired duality. $\square$

In the rest we shall construct the $n$-Auslander–Reiten duality from a viewpoint of dualizing $k$-variety. As we have seen in Lemma 2.8, the category $B$ is a dualizing $k$-variety and thus we have the duality $D : \text{mod}B \overset{\sim}{\to} \text{mod}(B^{op})$. By composing the duality $\Phi$ in Proposition 4.6 with the duality $D$, we have the following equivalence.

**Proposition 4.7** There exists an equivalence $\sigma_n : B \overset{\sim}{\to} B$ which makes the following diagram commutative up to isomorphism:

$$\begin{array}{ccc}
\text{mod}B & \overset{\Phi}{\longrightarrow} & \text{mod}(B^{op}) \\
D \downarrow & & \downarrow -\circ \sigma_n \\
\text{mod}(B^{op}) & \overset{-\circ \sigma_n}{\longrightarrow} & \text{proj}(B^{op})
\end{array}$$

**Proof** It is clear that $D \circ \Phi$ gives the equivalence from $\text{mod}(B^{op})$ to $\text{mod}(B^{op})$. We restrict this onto their projective objects, that is, $\text{proj}(B^{op}) \cong \text{proj}(B^{op})$. Thus we have the equivalence $\sigma_n : B \overset{\sim}{\to} B$ which makes the above diagram commutative up to isomorphisms. $\square$

By the dual argument, we have the equivalence $\sigma_n^- : B \to B$ which makes the following diagram commutative up to isomorphisms:

$$\begin{array}{ccc}
\text{mod}B & \overset{\Phi^{-1}}{\longrightarrow} & \text{mod}(B^{op}) \\
-\circ \sigma_n \downarrow & & \downarrow D \\
\text{mod}(B^{op}) & \overset{-\circ \sigma_n}{\longrightarrow} & \text{mod}B
\end{array}$$

As an immediate consequence of the above diagrams, we have the higher defect formula. Moreover, as a special case of the higher defect formula we obtain the higher Auslander–Reiten duality by using a modification of Krause’s proof of the classical formula (see [22]).
Theorem 4.8 There exist the following formulas:

(i) (Higher defect formula) functorial isomorphisms $D\delta^{sn} \cong \delta_{sn} \circ \sigma_n$ and $D\delta_{sn} \cong \delta^{sn} \circ \sigma_n^{-1}$;

(ii) (Higher Auslander–Reiten duality) bifunctorial isomorphisms $\mathcal{B}(\sigma_n^{-1}y, x) \cong D\text{Ext}^{1}_{\mathcal{A}}(x, y) \cong \mathcal{B}(y, \sigma_n x)$ in $x, y \in \mathcal{B}$.

Proof (i) It directly follows from Proposition 4.7 the fact that the duality $\Phi: \text{mod}(\mathcal{B}^{op}) \to \text{mod}\mathcal{B}$ sends $\delta_{sn}$ to $\delta^{sn}$ (Proposition 4.6).

(ii) We only prove the second isomorphism. Fix an object $y \in \mathcal{B}$. Let $y \hookrightarrow I(y)$ be an injective hull of $y$ in $\text{mod}\mathcal{A}$. Complete the $n$-exact sequence $\delta: 0 \to y \hookrightarrow I(y) \to b_{n-1} \to \cdots \to b_0 \to 0$. By Proposition 4.7, we have the isomorphisms $D\delta^{sn} \cong D\Phi(\delta_{sn}) \cong \delta_{sn} \circ \sigma_n$.

By [17, Lem. 3.5], we have the exact sequence

$$0 \to \mathcal{B}(-, y) \to \mathcal{B}(-, I(y)) \to \mathcal{B}(-, b_{n-1}) \to \cdots \to \mathcal{B}(-, b_0)$$

$$\to \text{Ext}^{n}_{\mathcal{A}}(-, y) \to \text{Ext}^{n}_{\mathcal{A}}(-, I(y))$$

on $\mathcal{B}$. Since $\text{Ext}^{n}_{\mathcal{A}}(-, I(y)) = 0$, we conclude $\delta^{sn} \cong \text{Ext}^{n}_{\mathcal{A}}(-, y)$. Since $y \hookrightarrow I(y)$ is an injective hull, the exact sequence

$$0 \to \mathcal{B}(b_0, -) \to \cdots \to \mathcal{B}(b_{n-1}, -) \to \mathcal{B}(I(y), -) \to \mathcal{B}(y, -) \to \delta_{sn} \to 0$$

shows the isomorphism $\delta_{sn} \cong \mathcal{B}(y, -)$. Therefore we obtain the desired isomorphism $D\text{Ext}^{n}_{\mathcal{A}}(-, y) \cong \mathcal{B}(y, \sigma_n(-))$. □

The isomorphisms in Theorem 4.8(ii) are nothing other than $n$-Auslander–Reiten duality. In particular, the functor $\sigma_n$ (resp. $\sigma_n^{-1}$) coincides with the $n$-Auslander–Reiten translation $\tau_n$ (resp. $\tau_n^{-1}$).

We recall the notion of the $n$-Auslander–Reiten duality. Let

$$\tau: \text{mod}\mathcal{A} \to \text{mod}\mathcal{A} \text{ and } \tau^{-}: \text{mod}\mathcal{A} \to \text{mod}\mathcal{A}$$

be the Auslander–Reiten translations. As a higher version of the Auslander–Reiten translation, the notion of $n$-Auslander–Reiten translation is defined as follows. We denote the $n$-th syzygy (resp. $n$-th cosyzygy) functor by $\Omega^n: \text{mod}\mathcal{A} \to \text{mod}\mathcal{A}$ (resp. $\Omega^{-n}: \text{mod}\mathcal{A} \to \text{mod}\mathcal{A}$).

Definition-Theorem 4.9 [16, Theorem 1.4.1] The $n$-Auslander–Reiten translations are defined to be the functors

$$\tau_n := \tau \Omega^{n-1}: \text{mod}\mathcal{A} \xrightarrow{\Omega^{n-1}} \text{mod}\mathcal{A} \xrightarrow{\tau} \text{mod}\mathcal{A},$$

$$\tau_n^{-1} := \tau^{-} \Omega^{-n-1}: \text{mod}\mathcal{A} \xrightarrow{\Omega^{-n-1}} \text{mod}\mathcal{A} \xrightarrow{\tau^{-}} \text{mod}\mathcal{A}.$$

These functors induce mutually quasi-inverse equivalences

$$\tau_n: \mathcal{B} \to \mathcal{B} \text{ and } \tau_n^{-1}: \mathcal{B} \to \mathcal{B}.$$

We have the following analog of Theorem 4.8(ii).

Proposition 4.10 [16, Theorem 1.5] There exist bifunctorial isomorphisms $\mathcal{B}(\tau_n^{-1}y, x) \cong D\text{Ext}^{1}_{\mathcal{A}}(x, y) \cong \mathcal{B}(y, \tau_n x)$ in $x, y \in \mathcal{B}$.

Combining results above, we obtain the following explicit form of $\sigma_n$ and $\sigma_n^{-1}$.  

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Theorem 4.11 The functor $\sigma_n$ and $\sigma_{n^{-}}$ are isomorphic to the $n$-Auslander–Reiten translations $\tau_n$ and $\tau_{n^{-}}$, respectively.

Proof Theorems 4.8 and 4.10 gives an isomorphism $B(y, \sigma_n x) \cong B(y, \tau_n x)$. By Yoneda Lemma, we have an isomorphism $\sigma_n \cong \tau_n$.

Note that Theorem 4.8 is independently obtained by Jasso and Kvamme in [20, Theorem 3.7, Corollary 3.8]. The proof is different since we proved the higher defect formula in Theorem 4.8 without using the explicit form of $\tau_n$.

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