Quantitative Diophantine approximation on affine subspaces

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Abstract
Recently, Adiceam et al. (Adv Math 302:231–279, 2016) proved a quantitative version of the convergence case of the Khintchine–Groshev theorem for nondegenerate manifolds, motivated by applications to interference alignment. In the present paper, we obtain analogues of their results for affine subspaces.

Keywords
Diophantine approximation on manifolds · Flows on homogeneous spaces · Khintchine–Groshev theorem · Quantitative Diophantine approximation · Interference alignment

Mathematics Subject Classification 11J83 · 11K60

1 Introduction

The theory of Diophantine approximation on manifolds has seen significant advances in recent years. This subject is mainly concerned with the question: under which conditions do proper subsets of \( \mathbb{R}^n \) inherit Diophantine properties which are generic for \( \mathbb{R}^n \) with respect to Lebesgue measure? A simple example of such a generic Diophantine property is provided by the classical Khintchine–Groshev theorem. For \( \mathbf{q} = (q_1, \ldots, q_n) \in \mathbb{Z}^n \), we denote

\[ \| \mathbf{q} \| := \max_{1 \leq i \leq n} |q_i|, \]

and we will use \( | | \) for both the Lebesgue measure of a measurable subset of \( \mathbb{R}^n \) as well as the absolute value of a real number. Let \( \psi : \mathbb{R}_+ \cup \{0\} \rightarrow \mathbb{R}_+ \cup \{0\} \) be a non-increasing function and consider the set of \( \psi \)-approximable vectors, namely \( x \in \mathbb{R}^n \) for which there exist infinitely many \( \mathbf{q} \in \mathbb{Z}^n \) such that

\[ |p + x \cdot \mathbf{q}| < \psi (\| \mathbf{q} \|^n) \]
for some $p \in \mathbb{Z}$. The Khintchine–Groshev theorem [6,12,13] states that the set of $\psi$-approximable vectors is a null (resp. co-null) set in terms of Lebesgue measure, according as the sum

$$\sum_{k=1}^{\infty} \psi(k)$$

(1.2)

converges or diverges. Let $U$ be an open subset of $\mathbb{R}^d$ and let $f : U \to \mathbb{R}^n$ be a differentiable map. Then $f$ is said to be nondegenerate at $x \in U$ if $\mathbb{R}^n$ is spanned by the partial derivatives of $f$ at $x$ of order up to $l$ for some $l$, and nondegenerate if it is nondegenerate at almost every point of $U$. Nondegenerate manifolds, i.e., manifolds parametrised by nondegenerate maps, inherit many generic Diophantine properties from ambient Euclidean space. For instance, in an influential paper Kleinbock and Margulis [16] resolved a long standing conjecture of Sprindžuk by showing that nondegenerate maps are extremal, i.e., almost every point on such a manifold is not very well approximable.

Subsequently, Bernik et al. [5] established the convergence case of Khintchine’s theorem for nondegenerate manifolds. This result was independently established by Beresnevich [2]. In fact, both [16] and [5] prove multiplicative versions of these results. In Adiceam et al. [1] have recently proved an interesting quantitative improvement of the convergence Khintchine theorem for nondegenerate manifolds. Their motivation comes from electronics, more precisely the study of interference alignment.

At the opposite end of the spectrum from nondegenerate manifolds lie affine subspaces. The study of Diophantine approximation on affine subspaces and their submanifolds was systematically initiated in the works [14,15] of Kleinbock and has seen recent progress. Since arbitrary affine subspaces cannot be expected to inherit generic Diophantine properties, the interesting question of finding necessary and sufficient conditions on affine subspaces to ensure inheritance of a given property plays a key role in investigations. We refer the reader to the recent survey [11] for a comprehensive discussion as well as references.

In this paper, we undertake the study of the refined, quantitative, version of the Khintchine–Groshev theorem from Adiceam et al. [1] in the context of affine subspaces. We provide a sufficient condition for an affine subspace to satisfy such a theorem. This condition is introduced in the next subsection after which we state the main result of the paper. In addition to the interest in this problem from the Diophantine point of view, it is possible that the result proved here could have applications in interference alignment. This is explained in Adiceam et al. [1], and indeed the example presented there, concerns a line!

### 1.1 Diophantine exponents of matrices

Let $\mathcal{H}$ be an $s$ dimensional affine subspace of $\mathbb{R}^n$. We can permute variables and assume that $\mathcal{H}$ is of the form $\{(x, xA' + \alpha_0) : x \in \mathbb{R}^s\}$ where $\alpha_0 \in \mathbb{R}^{n-s}$ and $A' \in \text{Mat}_{s \times n-s}(\mathbb{R})$.

Denoting the matrix $\begin{pmatrix} \alpha_0 \\ A' \end{pmatrix}$ by $A$, we can rewrite the parametrization as

$$x \mapsto (x, \bar{x}A) \quad \text{where} \quad \bar{x} = (1, x).$$

(1.3)

The Diophantine exponent $\omega(A)$ of a matrix $A \in \text{Mat}_{m \times n}(\mathbb{R})$ is defined to be the supremum of $v > 0$ for which there are infinitely many $q \in \mathbb{Z}^n$ such that

$$\|Aq + p\| < \|q\|^{-v}$$

(1.4)
for some \( p \in \mathbb{Z}^m \). It is well known that \( n/m \leq \omega(A) \leq \infty \) for all \( A \in \text{Mat}_{m \times n}(\mathbb{R}) \) and that \( \omega(A) = n/m \) for Lebesgue almost every \( A \). We now introduce the higher Diophantine exponents of \( A \) as defined by Kleinbock in [15]. For \( A \in \text{Mat}_{s+1 \times n-s}(\mathbb{R}) \), we set
\[
R_A = (\text{Id}_{s+1} A). \tag{1.5}
\]
Let \( e_0, \ldots, e_n \) denote the standard basis of \( \mathbb{R}^{n+1} \) and set
\[
W_{i \rightarrow j} = \text{span}\{e_i, \ldots, e_j\}. \tag{1.6}
\]
Let \( w \in \wedge^j(W_{0 \rightarrow n}) \) represent a discrete subgroup \( \Gamma \) of \( \mathbb{Z}^{n+1} \). Define the map \( c : \wedge^j(W_{0 \rightarrow n}) \rightarrow (\wedge^{j-1}(W_{1 \rightarrow n}))^{n+1} \) by
\[
c(w)_i = \sum_{J \subset \{1, \ldots, n\} \atop \#J = j-1} \langle e_i \wedge e_J, w \rangle e_J \tag{1.7}
\]
and let \( \pi_* \) denote the projection \( \wedge(W_{0 \rightarrow n}) \rightarrow \wedge(W_{s+1 \rightarrow n}) \). For each \( j = 1, \ldots, n-s \), define
\[
\omega_j(A) = \sup \left\{ v \left| \exists w \in \wedge^j(\mathbb{Z}^{n+1}) \text{ with arbitrary large } \|\pi_*(w)\| \text{ such that } \|R_A c(w)\| < \|\pi_*(w)\|^{-\frac{1}{j-1}} \right\}. \tag{1.8}
\]
It is shown in Lemma 5.3 of [15] that \( \omega_1(A) = \omega(A) \) thereby justifying the terminology.

### 1.2 Main theorem

Let \( \psi \) be an approximation function. Assume that
\[
\sum_{k=1}^{\infty} \psi(k) < \infty. \tag{1.9}
\]
Since \( \psi \) is assumed to be non-increasing, it is easy to see that condition (1.9) is equivalent to saying that
\[
\sum_{\psi} := \sum_{q \in \mathbb{Z}^{n} \setminus \{0\}} \psi(||q||^n) < \infty. \tag{1.10}
\]
We consider \( \psi \)-approximable points on affine subspaces, namely solutions to the inequality
\[
|\langle x, Ax \rangle - q + p| < \psi(||q||^n). \tag{1.11}
\]
As a corollary of (Theorem 1.2, [10]), we have that for any open ball \( U \) in \( \mathbb{R}^s \), the measure of the set
\[
\{x \in U : \text{for infinitely many } q \in \mathbb{Z}^n, \exists p \in \mathbb{Z} \text{ such that(1.11)holds}\} \tag{1.12}
\]
is zero, provided (1.10) holds and
\[
\omega_j(A) < n \text{ for every } j = 1, \ldots, n-s. \tag{1.13}
\]
Thus for almost all \( x \in U \), there exists a constant \( \kappa > 0 \) such that
\[
|\langle x, Ax \rangle q + p| \geq \kappa \psi(||q||^n) \text{ for all } p \in \mathbb{Z}, q \in \mathbb{Z}^n \setminus \{0\}. \tag{1.14}
\]
We want to get $\kappa$ independent of $x$ as far as possible, in a measure theoretic sense. To emphasize this, consider the set

$$B(U, \psi, \kappa) := \{x \in U : (1.14) \text{ holds}\}.$$  

(1.15)

Our aim is to investigate the dependence between $\kappa$ and the size of the set (1.15). Our main Theorem is

**Theorem 1.1** Let $\mathcal{H}$ be an $s$-dimensional affine subspace parametrized as in (1.3). Assume that

$$\omega_j(A) < n \text{ for every } j = 1, \ldots, n - s.$$  

(1.16)

Consider a non-increasing approximation function $\psi$ such that $\psi(k) \leq \frac{1}{k^s}$ for all $k \in \mathbb{N}$ and assume that (1.9) holds. Fix an open ball $U \in \mathbb{R}^s$ of radius $r$. Then there exist two explicitly computable constants $K_0$ and $K_1$, depending on $s$, $n$, $U$ and $A$ only, with the following property:

for any $\xi \in (0, 1)$,

$$|B(U, \psi, \kappa)| \geq (1 - \xi)|U|,$$  

(1.17)

holds with

$$\kappa < \min \left\{ 1, \frac{\xi}{2K(s)\sum_{\psi}}, \frac{r}{2^{n-\frac{3}{2}}\sqrt{ns}}, \left(\frac{\xi}{2K_0K_1}\right)^{(s(n+1))} \right\},$$  

(1.18)

where

$$K(s) := \frac{4^{2^e+1}s^{s/2}N_s}{V_s},$$

$V_s$ is the volume of the $s$-dimensional Euclidean unit ball and $N_s$ denotes the Besicovitch covering constant of $\mathbb{R}^s$.

**Remark** 1. Although we have not pursued it here, it is plausible that Theorem 1.1 is also true for nondegenerate submanifolds of affine subspaces under the same condition, i.e. (1.16).

2. We will follow the general strategy of [1] to prove Theorem 1.1, indeed this can be traced back to the work of Bernik, Kleinbock and Margulis [5]. The proof splits into two separate cases, the ‘big gradient’ and ‘small gradient’. Most of this paper is devoted to the latter case, and involves nondivergence estimates for polynomial like flows on the space of unimodular lattices.

3. It is worthwhile considering the case where $\mathcal{H}$ is a hyperplane namely an $n - 1$ dimensional subspace of $\mathbb{R}^n$. In this case, $A$ is an $n \times 1$ matrix and the condition (1.16) takes a particularly simple form, namely that for some $\delta > 0$,

$$\max_i |p_i + a_i q| > |q|^{1-n+\delta}$$

for every $p \in \mathbb{Z}^n$, and all but finitely many $q \in \mathbb{Z}$.

2 The gradient division

For $\kappa > 0$ and $q \in \mathbb{Z}^n \setminus \{0\}$, we define

$$\mathcal{L}(q) := \{x \in U : |p + (x, \tilde{x}A)q| < \kappa \psi(\|q\|^n) \text{ for some } p \in \mathbb{Z}\}.$$  

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As \( \bigcup_{\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}} \mathcal{L}(\mathbf{q}) = U \setminus B(U, \psi, \kappa) \), it suffices to prove that
\[
\left| \bigcup_{\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}} \mathcal{L}(\mathbf{q}) \right| < \xi |U|.
\]

It is traditional to approach Khintchine–Groshev type theorems by separately considering the case when we have a ‘large derivative’ and the case when we do not. We are thus interested in the cases where \( \nabla (\mathbf{x}, \tilde{\mathbf{x}} A) \cdot \mathbf{q} = [\text{Id}_s A']^s \mathbf{q} \), where \( A' \) is as introduced in the beginning of \( \S 1.1 \), gets big or small. Let
\[
\mathcal{L}_{\text{small}}(\mathbf{q}) = \left\{ \mathbf{x} \in \mathcal{L}(\mathbf{q}) : \| \nabla (\mathbf{x}, \tilde{\mathbf{x}} A) \cdot \mathbf{q} \| < \frac{\sqrt{n s \| \mathbf{q} \|^2}}{2r} \right\}
\]
(2.1)
where \( r \) is the radius of \( U \) and \( \mathcal{L}_{\text{large}}(\mathbf{q}) = \mathcal{L}(\mathbf{q}) \setminus \mathcal{L}_{\text{small}}(\mathbf{q}) \). We will prove that for \( \kappa \) given by (1.18),
\[
\sum_{\mathbf{q} \in \mathbb{Z}^n} |\mathcal{L}_{\text{large}}(\mathbf{q})| \leq \xi^2 |U|
\]
(2.2)
and
\[
\left| \bigcup_{\mathbf{q} \in \mathbb{Z}^n} \mathcal{L}_{\text{small}}(\mathbf{q}) \right| < \frac{\xi}{2} |U|.
\]
(2.3)

3 Estimating the measure of \( \mathcal{L}_{\text{large}}(\mathbf{q}) \)

In this section, we will establish (2.2). The proof of this follows immediately from Proposition 3.1 ([1], Theorem 4) Let \( U \subseteq \mathbb{R}^2 \) be a ball of radius \( r \) and \( f \in C^2(2U) \) where \( 2U \) is the ball with the same center as \( U \) and radius \( 2r \). Set
\[
L^* := \sup_{|\beta| = 2, \mathbf{x} \in 2U} \| \partial^\beta f(\mathbf{x}) \|
\]
(3.1)
and
\[
L := \max \left\{ L^*, \frac{1}{4r^2} \right\}.
\]
(3.2)
Then for every \( \delta' > 0 \) and every \( \mathbf{q} \in \mathbb{Z}^n \setminus \{0\} \), the set of all \( \mathbf{x} \in U \) such that \( |p + f(\mathbf{x}) \mathbf{q}| < \delta' \) for some \( p \in \mathbb{Z} \) and
\[
\| \nabla f(\mathbf{x}) \mathbf{q} \| \geq \sqrt{n s L \| \mathbf{q} \|}
\]
(3.3)
has measure at most \( K_s \delta' |U| \).

The proof of Proposition 3.1 is done by applying [5, Lemma 2.2] appropriately.

To prove (2.2) from Proposition 3.1, we take \( f(\mathbf{x}) = (\mathbf{x}, \tilde{\mathbf{x}} A) \) and \( \delta' = \kappa \psi(\| \mathbf{q} \|^n) \). Clearly \( L^* = 0 \) and \( L = \frac{1}{4r^2} \). Hence by Proposition 3.1, we get that
\[
|\mathcal{L}_{\text{large}}(\mathbf{q})| \leq K_s \kappa \psi(\| \mathbf{q} \|^n) |U|,
\]
and thus, taking \( \kappa \leq \frac{\xi}{2K_s \sum \psi} \),
\[
\sum_{\mathbf{q} \in \mathbb{Z}^n} |\mathcal{L}_{\text{large}}(\mathbf{q})| \leq K_s \kappa \sum \psi |U| \leq \frac{\xi}{2} |U|.
\]
□

To estimate \( |\mathcal{L}_{\text{small}}(\mathbf{q})| \), we shall employ dynamical tools. To begin with, we need to recall a few elementary properties of ‘good functions’ which will be discussed in the following section.
4 (\( C, \alpha \))-good functions

Let \( C \) and \( \alpha \) be positive numbers and \( V \) be a subset of \( \mathbb{R}^s \). A function \( f : V \to \mathbb{R} \) is said to be \((C, \alpha)\)-good on \( V \) if for any open ball \( B \subseteq V \), and for any \( \varepsilon > 0 \), one has:

\[
\left| \left\{ x \in B \mid |f(x)| < \varepsilon \right\} \right| \leq C \left( \frac{\varepsilon}{\sup_{x \in B} |f(x)|} \right)^\alpha |B|.
\]

The following elementary properties of \((C, \alpha)\)-good functions will be used.

(G1) If \( f \) is \((C, \alpha)\)-good on an open set \( V \), so is \( \lambda f \) \( \forall \lambda \in \mathbb{R} \);

(G2) If \( f_i, i \in I \) are \((C, \alpha)\)-good on \( V \), so is \( \sup_{i \in I} |f_i| \);

(G3) If \( f \) is \((C, \alpha)\)-good on \( V \) and for some \( c_1, c_2 > 0 \), \( c_1 \leq \frac{|f(x)|}{|g(x)|} \leq c_2 \) for all \( x \in V \), then \( g \) is \((C(c_2/c_1)^\alpha, \alpha)\)-good on \( V \).

(G4) If \( f \) is \((C, \alpha)\)-good on \( V \), it is \((C', \alpha')\)-good on \( V' \) for every \( C' \geq C, \alpha' \leq \alpha \) and \( V' \subset V \).

One can note that from (G2), it follows that the supremum norm of a vector valued function \( f \) is \((C, \alpha)\)-good whenever each of its components is \((C, \alpha)\)-good. Furthermore, in view of (G3), we can replace the norm by an equivalent one, only affecting \( C \) but not \( \alpha \).

The next Proposition provides the most important class of good functions.

Proposition 4.1 (Lemma 3.2 in [5]) Any polynomial \( f \in \mathbb{R}[x_1, ..., x_s] \) of degree not exceeding \( l \) is \((C_{s,l}, \frac{1}{s^l})\)-good on \( \mathbb{R}^s \), where \( C_{s,l} = \frac{2^{s+l}(s+l+1)}{V_s s^l} \). In particular, constant and linear polynomials are \((\frac{2^{s+2}}{V_s}, \frac{1}{s})\)-good on \( \mathbb{R}^s \).

5 Small gradients

For each \( t \in \mathbb{Z}^+ \), we define \( A_t \) as the set

\[
\left\{ x \in U : \exists p \in \mathbb{Z}, q \in \mathbb{Z}^n \text{ s.t. } \begin{cases} |p + (x, \bar{x}A)q| < \frac{k}{2^n} \\ \|\nabla(x, \bar{x}A) \cdot q\| < \frac{\sqrt{ns}}{\sqrt{2^t r^2/2}} \\ 2^t \leq \|q\| < 2^{t+1} \end{cases} \right\}.
\]

It is now immediate that

\[
\bigcup_{q \in \mathbb{Z}^n \setminus \{0\}} L_{\text{small}}(q) \subseteq \bigcup_{t=0}^\infty A_t,
\]

since \( \forall x \in \mathbb{R}, \psi(x) \leq 1/x \). It is therefore enough to show

\[
\sum_{t=0}^\infty |A_t| < \frac{\xi}{2} |U|.
\]

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For $\beta \in \left(0, \frac{1}{2(n+1)}\right)$, we set

$$\delta := \frac{\kappa}{2^m}, \quad K := \sqrt{\frac{ns}{2r^2}} \cdot 2^{1/2}, \quad T := 2^{t+1},$$

(5.2)

$$\epsilon' := (\delta K T^{n-1})^{1/\pi^2} = \left(\kappa 2^{n-1} \sqrt{\frac{ns}{2r^2}} \cdot 2^{1/2}\right)^{1/\pi^2} = \left(\frac{\kappa 2^{n-\frac{3}{2}} \sqrt{ns}}{r}\right)^{1/\pi^2} \frac{1}{2^{t/2(n+1)}},$$

(5.3)

and

$$\epsilon := 2^{\beta t} \epsilon' = \left(\frac{\kappa 2^{n-\frac{3}{2}} \sqrt{ns}}{r}\right)^{1/\pi^2} \frac{2^{\beta t}}{2^{t/2(n+1)}}.$$ (5.4)

Define

$$u_x := \begin{pmatrix} 1 & 0 & x & x A' + a_0 \\ 0 & I_s & I_s & A' \\ 0 & 0 & I_n \end{pmatrix}$$

(5.5)

and

$$g_t := \text{diag} \left( \frac{\epsilon}{\delta}, \frac{\epsilon}{K}, \ldots, \frac{\epsilon}{K}, \frac{\epsilon}{T}, \ldots, \frac{\epsilon}{T} \right).$$ (5.6)

Denote by $\Lambda$ the subgroup of $\mathbb{Z}^{1+s+n}$ consisting of vectors of the form:

$$\Lambda = \left\{ \begin{pmatrix} p \\ 0 \\ \vdots \\ 0 \\ q \end{pmatrix} \mid p \in \mathbb{Z}, \ q \in \mathbb{Z}^n \right\}.$$ (5.7)

It can be easily seen that

$$A_t \subseteq \tilde{A}_t := \{ x \in U : \| g_t u_x \lambda \| < \epsilon \text{ for some } \lambda \in \Lambda \setminus \{0\} \}. $$ (5.8)

We shall show that if $\kappa$ is taken to be not exceeding $\frac{r}{2^{n-\frac{3}{2}} \sqrt{ns}}$ and 1 then, depending on $A$, $\beta$ can be suitably chosen so that

$$|\tilde{A}_t| \leq K_0 \kappa^{\frac{1}{(n+1)}} \frac{1}{2^{\left(\frac{2^{n-\frac{3}{2}}}{s} - \beta\right) t}} |U|,$$ (5.9)

for some explicit constant $K_0$ depending on $s$, $n$, $U$ and $A$ only. One can then set

$$K_1 := \sum_{t=0}^{\infty} \frac{1}{2^{\left(\frac{2^{n-\frac{3}{2}}}{s} - \beta\right) t}}$$

(5.10)

and reduce $\kappa$ sufficiently to conclude

$$\sum_{t=0}^{\infty} |A_t| \leq K_0 K_1 \kappa^{\frac{1}{(n+1)}} |U| < \frac{\epsilon}{2} |U|;$$

which establishes (5.1).

The inequality (5.9) will be proved using the quantitative nondivergence estimate of Kleinbock and Margulis in the next section.
6 A quantitative nondivergence estimate

Let $W$ be a finite dimensional real vector space. For a discrete subgroup $\Gamma$ of $W$, we set $\Gamma_R$ to be the minimal linear subspace of $W$ containing $\Gamma$. A subgroup $\Gamma$ of $\Lambda$ is said to be primitive in $\Lambda$ if $\Gamma = \Gamma_R \cap \Lambda$. We denote the set of all nonzero primitive subgroups of $\Gamma$ by $\mathcal{L}(\Gamma)$. Let $j := \dim(\Gamma_R)$ be the rank of $\Gamma$. We say that $w \in \bigwedge^j(W)$ represents $\Gamma$ if

$$w = \begin{cases} 1 & \text{if } j = 0 \\ v_1 \land \cdots \land v_j & \text{if } j > 0 \text{ and } v_1, \ldots, v_j \text{ is a basis of } \Gamma. \end{cases}$$

In fact, one can easily see that such a representative of $\Gamma$ is always unique up to a sign.

A function $\nu : \bigwedge(W) \rightarrow \mathbb{R}_+$ is called submultiplicative if

(i) $\nu$ is continuous with respect to natural propology on $\bigwedge(W)$;
(ii) $\forall t \in \mathbb{R}$ and $w \in \bigwedge(W)$, $\nu(tw) = |t|\nu(w)$, i.e. it is homogeneous;
(iii) $\forall u, w \in \bigwedge(W)$, $\nu(u \land w) \leq \nu(u)\nu(w)$.

In view of property (ii) as given above, without any confusion, we can define $\nu(\Gamma) = \nu(w)$, where $w$ represents $\Gamma$.

Now we shall come to the “quantitative nondivergence estimate” which is a generalization of Theorem 5.2 of [16].

**Theorem 6.1** ([5], Theorem 6.2) Let $W$ be a finite dimensional real vector space, $\Lambda$ a discrete subgroup of $W$ of rank $k$, and a ball $B = B(x_0, r_0) \subset \mathbb{R}^s$ and a continuous map $H : \tilde{B} \rightarrow \text{GL}(W)$ be given, where $\tilde{B} = B(x_0, 3^k r_0)$. Take $C \geq 1, \alpha > 0$, $0 < \rho < 1$ and $\nu$ be a submultiplicative function $\bigwedge(W)$. Assume that for any $\Gamma \in \mathcal{L}(\Lambda)$,

(KM1) the function $x \mapsto \nu(H(x) \Gamma)$ is $(C, \alpha)$-good on $\tilde{B}$,
(KM2) $\sup_{x \in B} \nu(H(x) \Gamma) \geq \rho$ and
(KM3) $\forall x \in \tilde{B}, \#\{\Gamma \in \mathcal{L}(\Lambda) : \nu(H(x) \Gamma) < \rho\} < \infty$.

Then for every $\epsilon'' > 0$ one has :

$$|[x \in B : \nu(H(x))'' < \epsilon'' \text{ for some } \gamma \in \Lambda\backslash\{0\}]| < k(3^s N) C \left(\frac{\epsilon''}{\rho}\right)^\alpha |B|. \quad (6.1)$$

With the intention of using Theorem 6.1 to prove (5.8), we set $W = \mathbb{R}^{1+s+n}$ with basis $e_0, e_{a_1}, \ldots, e_{a_s}, e_1, \ldots, e_q$, $\Lambda$ as given in (5.7), $B = U$ and $H(x) = g_u x$. The submultiplicative function $\nu$ will be chosen, as introduced in [5, §7], in the following way:

Let $W_s$ be the subspace of $W$ spanned by $e_{a_1}, \ldots, e_{a_s}$. We shall identify $W_s$ with $\mathbb{R}^{a+1}$ canonically. Also let $W$ be the ideal of $\bigwedge(W)$ generated by $\bigwedge^2(W_s)$, $\pi_s$ is the orthogonal projection with kernel $W$ and $\|w\|_e$ be the Euclidean norm of $\pi_s(w)$. In simple words, if $w$ is written as a sum of exterior products of the base vectors $e_i$ and $e_{a_i}$, to compute $\nu(w)$, we ignore the components containing exterior products of type $e_{a_i} \land e_{a_j}$, $1 \leq i \neq j \leq s$, and consider the Euclidean norm of rest. It is immediate that $\nu|_W$ agrees with the Euclidean norm.

We now seek for proper $C, \alpha, \rho$ which make (KM1)-(KM3) true. The condition (KM3) can be established for any $\rho \leq 1$ exactly in the way it is done in [5, §7]. The following section is devoted to the verification of the remaining ones along with the search for the explicit constants.
7 Checking (KM1) and (KM2)

We begin with the explicit computation of $H(x)w$ for all $w \in \bigwedge^k (W^\perp_W)$ and $k = 1, \ldots, n + 1$. First writing $x = (x_1, \ldots, x_s)$ and $(x, \tilde{A}) = (f_1(x), \ldots, f_n(x))$, we see that

1. $H(x)e_0 = \frac{e}{\delta} e_0$
2. $H(x)e_i = \frac{e}{\delta} f_i(x)e_0 + \frac{e}{\delta} \sum_{j=1}^s \frac{\partial f_j(x)}{\partial x_i} e_{s+j} + \frac{e}{\delta} e_i$ for $1 \leq i \leq n$.

Note that each $f_i(x)$ is a polynomial $x_1, \ldots, x_s$ with degree at most 1 so that each $\frac{\partial f_j(x)}{\partial x_i}$ is constant.

7.1 Checking (KM1)

Since $\Lambda = \mathbb{Z}^{1+s+n} \cap W^\perp_W$, any representative $w \in \bigwedge^k (W)$ of any subgroup of $\Lambda$ of rank $k$, $1 \leq k \leq n + 1$, can be written as $\sum I a_I e_I$, where each $a_I \in \mathbb{Z}$ and $e_I = e_{i_1} \land \cdots \land e_{i_k}$ with $i_1, \ldots, i_k \in \{0, 1, \ldots, n\}$, $i_1 < \cdots < i_k$.

Since each component of $\pi_* (H(x)w)$ is a polynomial in $x_1, \ldots, x_s$ with degree at most 1 in view of (4.1), each of them is $\frac{1}{2^{s+2} V_s}$-good on $\tilde{U}$. This makes $\|\pi_* (H(x)w)\|$ good on $\tilde{U}$. As

$$\frac{1}{\delta KT^n - 1} \leq \frac{\|\pi_* (H(x)w)\|}{\nu(\pi_* (H(x)w))} \leq 1,$$

whence, from property (G4) of good functions, $\nu(\pi_* (H(x)w))$ is $(C, \alpha)$-good with

$$C := \max \left\{ \frac{2^{(s+2+\frac{1+s+n}{2s})}}{V_s}, 1 \right\} \quad \text{and} \quad \alpha := \frac{1}{s}. \quad (7.1)$$

This verifies (KM1). \hfill \square

7.2 Checking (KM2)

Let $\Gamma$ be a subgroup of $\Lambda$ with rank $k$ and $w \in \bigwedge^k (W^\perp_W)$ represent $\Gamma$. We first consider the case $k = n + 1$. So $w = w e_0 \land e_1 \land \cdots \land e_n$ where $w \in \mathbb{Z}\setminus\{0\}$. For any $x \in U$, the coefficient of $e_0 \land e_{s+1} \land e_2 \land \cdots \land e_n$ in $\pi_* (H(x)w)$ is clearly seen to be

$$\frac{w \epsilon^{n+1}}{\delta KT^{n+1}}.$$

Now looking at (5.2), we see that

$$\sup_{x \in U} \nu(H(x)\Gamma) = \sup_{x \in U} \nu(H(x)w) \geq \sup_{x \in U} \|\pi_* (H(x)w)\| \geq \left| w \epsilon^{n+1} \frac{\delta KT^{n-1}}{\delta KT^{n-1}} \right| \quad (7.2)$$

Assume now $1 \leq k \leq n$. To bound the norm of $\|\pi_* (H(x)w)\|$ from below, we will proceed along the lines of Ghosh [10, §5.3] using a technique from Kleinbock [15]. As observed in Ghosh [10, §5.3], for any $x \in U$, $\|\pi_* (H(x)w)\| \geq \|\tilde{g}_l, \tilde{u}_x w\|$ where
and 
\[ \tilde{u}_x = \begin{pmatrix} 1 & x \\ 0 & I_n \end{pmatrix} \tilde{x}A, \]

and 
\[ \tilde{g}_t = \text{diag} \left( \frac{\varepsilon}{\delta T^{-1}}, \ldots, \frac{\varepsilon}{T} \right). \]

This inspires us to bound \( \sup_{x \in U} \|\tilde{g}_t \tilde{u}_x w\| \) from below. It follows from (4.6) in Kleinbock [15] that
\[
\sup_{x \in U} \|\tilde{g}_t \tilde{u}_x w\| \geq \frac{1}{2^{n+\frac{r}{2}}} \max \left\{ \left( \frac{\varepsilon^k}{\delta T^{k-1}} \right) \sup_{x \in U} \| (x, \tilde{x}A) c(w) \|, \left( \frac{\varepsilon}{T} \right)^k \| \pi(w) \| \right\}
\]
where \( \pi \) is the projection from \( \bigwedge(W_\Lambda) \) to \( \bigwedge(W_{1-n}) \) and \( W_{1-n} \) stands for the span of \( e_1, \ldots, e_n \).

We recall that
\[
(x, \tilde{x}A) = \tilde{x}R_A
\]
where \( R_A \) is defined in (1.5). Because of this, we can replace in our norm calculations, \( \sup_{x \in U} \| (x, \tilde{x}A) c(w) \| \) by \( \sup_{x \in U} \| \tilde{x}R_A c(w) \| \). As the functions \( x_1, \ldots, x_s \) are linearly independent over \( \mathbb{R} \) on \( U \), the map \( v \mapsto \sup_{x \in U} \| \tilde{x}v \| \) defines a norm on \( \bigwedge(W_{1-n})^{s+1} \) which must be equivalent to the supremum norm on \( \bigwedge(W_{1-n})^{s+1} \), whence for a constant \( K_2 > 0 \) depending on \( s, n \) and \( U \), we have
\[
\sup_{x \in U} \| \tilde{x}R_A c(w) \| \geq K_2 \| R_A c(w) \|,
\]
and consequently
\[
\sup_{x \in U} \|\tilde{g}_t \tilde{u}_x w\| \geq \frac{1}{2^{n+\frac{r}{2}}} \max \left\{ \left( \frac{\varepsilon^k}{\delta T^{k-1}} \right) K_2 \| R_A c(w) \|, \left( \frac{\varepsilon}{T} \right)^k \| \pi(w) \| \right\}.
\]

We first note that from Lemma 5.1 in Kleinbock [15] we get that for any \( n - s < k \leq n \) and for all but finitely many \( w \in \bigwedge^k(\Lambda) \)
\[
\| R_A c(w) \| \geq 1.
\]

It therefore follows that for a constant \( K_3 > 0 \) depending alone on \( A \),
\[
\sup_{x \in U} \|\tilde{g}_t \tilde{u}_x w\| \geq \frac{K_2K_3}{2^{n+\frac{r}{2}}} \left( \frac{\varepsilon^k}{\delta T^{k-1}} \right).
\]

From (5.2) together with the choice \( \kappa \leq \frac{r}{2^{n+\frac{r}{2}}/\sqrt{8n}} \),
\[
\frac{\varepsilon^k}{\delta T^{k-1}} = \left( \frac{2^{n-\frac{3}{2}}/\sqrt{8n}}{r} \right)^{\frac{k}{2n}} \times \frac{1}{2^{2(2n+1)-\beta)}/2n} \times \frac{2^{2n}}{\kappa} \times \frac{1}{2^{2(n+1)(k-1)}} \leq \left( \frac{2^{n-\frac{3}{2}}/\sqrt{8n}}{r} \right)^{\frac{k}{2n}} \times \frac{1}{2^{2(2n+1)-\beta)}/2n} \times \frac{2^{2n}}{\kappa} \times \frac{1}{2^{2(n+1)(k-1)}} \leq \sqrt{\frac{\pi}{2\pi}} \times 2 \left( 1 - \frac{1}{2(n+1)(k-1)} \right).
\]

Picking \( \beta \in \left( 0, \frac{1}{2(n+1)} \right) \) appropriately, thus we get for all subgroups \( \Gamma \) of \( \Lambda \) with rank \( n - s + 1, \ldots, n \),
\[ \sup_{x \in U} \| \tilde{g}_t \tilde{u}_x \| \geq \frac{K_2 K_3 \sqrt{\eta s}}{2^{s+1} r} \quad (7.11) \]

holds true.

We will now show how to get analogous lower bounds for subgroups of lower ranks. Recall first that a straightforward consequence of (1.16) is that we can get constants \( \theta, K_4 > 0 \) that depend on \( A \) only, with the property: for every \( 1 \leq k \leq n - s \) and \( w \in \Lambda^k(\Lambda) \),

\[ \| R_A e(w) \| \geq K_4 \| \pi_\bullet(w) \|^{-\frac{(n-\theta)+1-k}{k}}. \quad (7.12) \]

Also for the purposes of obtaining bounds in the lower ranks, we can replace \( \pi(w) \) in (7.7) with \( \pi_\bullet(w) \) as \( \| \pi(w) \| \geq \| \pi_\bullet(w) \| \). Therefore, for given \( 1 \leq k \leq n - s \) and \( w \in \Lambda^k(\Lambda) \), it suffices to examine

\[ \max \left\{ \left( \frac{\varepsilon^k}{\delta T^{k-1}} \right) K_2 K_4 \| \pi_\bullet(w) \|^{-\frac{(n-\theta)+1-k}{k}}, \left( \frac{\varepsilon}{T} \right)^k \| \pi_\bullet(w) \| \right\}. \quad (7.13) \]

Hence we seek for the solution of equation

\[ \frac{K_2 K_4}{\delta T^{k-1}} y^{-(n-\theta)+1-k} = \frac{1}{T^k} \] \[ \quad (7.14) \]

which gives \( y = (K_2 K_4)^{\frac{k}{(n-\theta)+1}} \left( \frac{T}{\delta} \right)^{\frac{k}{(n-\theta)+1}} \). This yields that (7.13) is at least

\[
\begin{align*}
(K_2 K_4)^{\frac{k}{(n-\theta)+1}} \left( \frac{T}{\delta} \right)^{\frac{k}{(n-\theta)+1}} \left( \frac{\varepsilon}{T} \right)^k &= \left( K_2 K_4 \right)^{\frac{k}{(n-\theta)+1}} \left( \frac{2}{\kappa} \right)^{\frac{k}{(n-\theta)+1}} 2^{\frac{(n-\theta)+1}{(n-\theta)+1}} \left( \kappa \frac{2^{n-3}}{r} \sqrt{\eta s} \right)^{\frac{k}{(n-\theta)+1}} \frac{1}{2^{\frac{(n-\theta)+1}{(n-\theta)+1}}} \left( \frac{1}{\kappa} \right)^{\frac{k}{(n-\theta)+1}} \left( \frac{1}{\kappa} \right)^{1-(\frac{1}{\kappa})} k t \\
&= \left( K_2 K_4 \right)^{\frac{k}{(n-\theta)+1}} \left( \frac{2}{\kappa} \right)^{\frac{k}{(n-\theta)+1}} \frac{1}{2^{\frac{k}{2}}} \left( \kappa \frac{2^{n-3}}{r} \sqrt{\eta s} \right)^{\frac{k}{(n-\theta)+1}} \left( \frac{1}{\kappa} \right)^{\frac{k}{(n-\theta)+1}} \left( \frac{1}{\kappa} \right)^{1-(\frac{1}{\kappa})} k t \\
&= \left( K_2 K_4 \right)^{\frac{k}{(n-\theta)+1}} \left( \frac{2^{n-3}}{r} \sqrt{\eta s} \right)^{\frac{k}{(n-\theta)+1}} \left( \frac{1}{\kappa} \right)^{\frac{k}{(n-\theta)+1}} \left( \frac{1}{\kappa} \right)^{1-(\frac{1}{\kappa})} k t .
\end{align*}
\]

Write

\[ K_5 := \min_{1 \leq k \leq n-s} \left( K_2 K_4 \right)^{\frac{k}{(n-\theta)+1}} \left( \frac{2^{n-3}}{r} \sqrt{\eta s} \right)^{\frac{k}{(n-\theta)+1}} \left( \frac{1}{\kappa} \right)^{\frac{k}{(n-\theta)+1}} \left( \frac{1}{\kappa} \right)^{1-(\frac{1}{\kappa})} k t . \]

Clearly it depends on \( s, n, A \) and \( U \) only. As \( \kappa \leq 1 \), by further refining the choice of \( \beta \) if necessary, one can bound (7.13) from below by \( K_5 \); whence, in view of (7.7), one obtains for every \( 1 \leq k \leq n - s \) and \( w \in \Lambda^k(\Lambda) \),

\[ \sup_{x \in U} \| \tilde{g}_t \tilde{u}_x w \| \geq \frac{1}{2^{\frac{n}{2}+1}} K_5 . \quad (7.15) \]

Summarizing the observations (7.2), (7.11) and (7.15), we finally confirm (KM2) with the following explicit choice of \( \rho \)

\[
\min \left\{ \frac{1}{2}, \frac{K_2 K_3 \sqrt{\eta s}}{2^{\frac{n}{2}+1} r}, \frac{1}{2^{\frac{n}{2}+1}} K_5 \right\}. \quad (7.16)
\]
8 The proof of (5.9)

The first step towards the proof is the observation
\[
\tilde{A}_t \subseteq \{ x \in U : \nu(H(x)\lambda) < \sqrt{1 + s + n} \varepsilon \text{ for some } \lambda \in \Lambda \setminus \{0\} \}.
\]

This is clear since \(\nu|_W\) coincides with the Euclidean norm on \(W\). Now applying the quantitative nondivergence estimate given by Theorem 6.1 with \(\varepsilon'' = \sqrt{1 + s + n} \varepsilon\), \(C\), \(\alpha\) and \(\rho\) as given in (7.1) and (7.16), we have
\[
|\tilde{A}_t| \leq |\{ x \in U : \nu(H(x)\lambda) < \sqrt{1 + s + n} \varepsilon \text{ for some } \lambda \in \Lambda \setminus \{0\} \}|. \tag{8.1}
\]

\[
\leq (n + 1)(3^s N_s)^{n+1} C(1 + s + n) \frac{1}{2^7} \left( \frac{\varepsilon}{\rho} \right) \frac{1}{\kappa^{\frac{1}{2(n+1)}}} \frac{1}{\left( \frac{2^{n-\frac{3}{2}} \sqrt{ns}}{r} \right)^{\frac{1}{2(n+1)}}} |U|.
\]

Denoting
\[
(n + 1)(3^s N_s)^{n+1} C(1 + s + n) \frac{1}{2^7} \left( \frac{\varepsilon}{\rho} \right) \frac{1}{\kappa^{\frac{1}{2(n+1)}}} \frac{1}{\left( \frac{2^{n-\frac{3}{2}} \sqrt{ns}}{r} \right)^{\frac{1}{2(n+1)}}} \kappa \rho^\frac{\beta}{2} |U|.
\]

by \(K_0\) we hereby conclude (5.9).

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References

1. Adiceam, F., Beresnevich, V., Levesley, J., Velani, S., Zorin, E.: Diophantine approximation and applications in interference alignment. Adv. Math. 302, 231–279 (2016)
2. Beresnevich, V.: A Groshev type theorem for convergence on manifolds. Acta Math. Hung. 94(1–2), 99–130 (2002)
3. Beresnevich, V., Bernik, V., Dickinson, H., Dodson, M.M.: On linear manifolds for which the Khintchin approximation theorem holds. Vesti Acad Navuk Belarusi. Ser. Fiz. Mat. Navuk 2, 14–17 (2000) (Belorussian)
4. Beresnevich, V., Bernik, V., Kleinbock, D., Margulis, G.: Metric Diophantine approximation: the Khintchine-Groshev theorem for non-degenerate manifolds. Moscow Math. J. 2(2), 203–225 (2002)
5. Bernik, V., Kleinbock, D., Margulis, G.A.: Khintchine type theorems on manifolds: the convergence case for the standard and multiplicative versions. Int. Math. Res. Not. 9, 453–486 (2001)
6. Dodson, M.M.: Diophantine approximation, Khintchine’s theorem, torus geometry and Hausdorff dimension, dynamical systems and diophantine approximation. Smin. Congr., 19, Soc. Math. France, Paris, pp. 1–20 (2009)
7. Galagher, P.: Metric simultaneous diophantine approximation. J. Lond. Math. Soc. 37, 387–390 (1962)
8. Ghosh, A.: A Khintchine-type theorem for hyperplanes. J. Lond. Math. Soc. 72(2), 293–304 (2005)
9. Ghosh, A.: A Khintchine Groshev theorem for affine hyperplanes. Int. J. Number Theory 7(4), 1045–1064 (2011)
10. Ghosh, A.: Diophantine approximation and the Khintchine–Groshev theorem. Monatsh. Math. 163(3), 281–299 (2011)
11. Ghosh, A.: Diophantine approximation on subspaces of $\mathbb{R}^n$ and dynamics on homogeneous spaces. In: Ji, L., Papadopoulos, A., Yau, S.-T. (eds.) Handbook of Group Actions, ALM 41, vol. IV, Chap. 9, pp. 509–527
12. Groshev, A.: Une théorème sur les systèmes des formes linéaires. Dokl. Akad. Nauk SSSR 9, 151–152 (1938)
13. Khintchine, A.: Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen. Math. Ann. 92, 115–125 (1924)
14. Kleinbock, D.: Extremal subspaces and their submanifolds. Geom. Funct. Anal. 13(2), 437–466 (2003)
15. Kleinbock, D.: An extension of quantitative nondivergence and applications to diophantine exponents. Trans. Am. Math. Soc. 360(12), 6497–6523 (2008)
16. Kleinbock, D., Margulis, G.A.: Flows on homogeneous spaces and diophantine approximation on manifolds. Ann. Math. 148, 339–360 (1998)
17. Schmidt, W.: Metrische Sätze über simultane approximation abhängiger Grössen. Monatsch. Math. 68, 154–166 (1964)
18. Sprindžuk, V.G.: Achievements and problems in diophantine approximation theory. Russ. Math. Surv. 35, 1–80 (1980)
19. Sprindžuk, V.G.: Metric Theory of Diophantine Approximations. Wiley, New York (1979)