CENTRALLY EXTENDED $W_{1+\infty}$
AND THE KP HIERARCHY

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ABSTRACT

It is well known that the centerless $W_{1+\infty}$ algebra provides a hamiltonian structure for the KP hierarchy. In this letter we address the question whether the centerful version plays a similar rôle in any related integrable system. We find that, surprisingly enough, the centrally extended $W_{1+\infty}$ algebra yields yet another Poisson structure for the same standard KP hierarchy. This is proven by explicit construction of the infinitely many new hamiltonians in closed form.

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\[ W_{1+\infty} \] is an ubiquitous mathematical structure. It appears in totally different contexts. Most of them are intrinsically 2 dimensional: Quantum Hall effect [1], 2-D quantum gravity [2], 2-D fluid dynamics [3], large N QCD [4] etc. But also in different approaches to four dimensional quantum gravity this algebra seems to play a relevant rôle [5][6].

The content of this letter is mainly concerned with the KP hierarchy. Precisely the KP phase-space has proven to be the natural arena in the construction of \( W \) type algebras. It is an infinite dimensional phase-space endowed with a (bi-)hamiltonian structure (see [7] for a master’s review), i.e. there is a pair of coordinated Poisson structures, where the so called “first” is linear and the “second” is non-linear. The former one was identified in [8] as the centerless \( W_{1+\infty} \) algebra.

Inspired by this result there have been some attempts to see what integrable system would arise from the centrally extended version of this algebra [9], the natural conjecture being that the central extension should parameterize some kind of (“quantum”) integrable deformation of the KP hierarchy and, thereafter, of the KP equation. However, the hard part of the job, namely: the construction of the infinite tower of hamiltonians in involution with respect of these Poisson brackets was, to our knowledge, not solved. Hence the conjecture remained unproven.

In [10] it was shown that one need not restrict the KP phase space to the ring of pseudodifferential operators of the form \( \partial^q + u_1 \partial^{q-1} + u_2 \partial^{q-2} + \ldots \) with \( q \in \mathbb{Z} \). With due care many structures admit an analytic continuation to complex values of \( q \). This proved to be the case for the second Gelfand-Dickey hamiltonian structure, and the Poisson-bracket algebra that one obtains received the name of \( W_{KP}^{(q)} \).

Interestingly enough, this construction showed how to recover the centrally extended \( W_{1+\infty} \) algebra as a particular contraction \( q \to 0 \) of the algebra \( W_{KP}^{(q)} \), thus supporting evidence that this algebra could provide again a hamiltonian structure for the standard KP hierarchy. In this letter we prove that this is indeed the case by completing the analysis of [10] when \( q \to 0 \) (\( q \in \mathbb{R}_+ \)). After suitably isolating some infinities that appear in the limiting procedure, we manage to obtain all the hamiltonians in closed form.
The KP hierarchy is defined as the infinite system of equations given in Lax form by
\[
\frac{\partial L}{\partial t_n} = [(L^n)_+, L], \quad n = 1, 2, 3, \ldots,
\] (2.1)
where \( L \) is the pseudodifferential operator \( L = \partial + u_2\partial^{-1} + u_3\partial^{-3} + \ldots \), and \( L_+ \) and \( L_- \) are the usual projections onto differential and integral parts. This system of equations is bi-hamiltonian, i.e. it admits the form of Hamilton’s equations with respect to two different sets of Poisson brackets
\[
\frac{\partial L}{\partial t_n} = \{H_{n+1}, L\}_1 = \{H_n, L\}_2,
\] (2.2)

The infinite set of hamiltonians can be expressed in closed form as follows
\[
H_n = \frac{1}{n} \text{Tr} L^n = \frac{1}{n} \int \text{Res} L^n
\] (2.3)
where the residue \( \text{Res} \) picks the coefficient of \( \partial^{-1} \) in any \( \Psi DO \). The two set of Poisson brackets which have been labeled by 1 and 2 correspond to the centerless \( W_{1+\infty} \) and \( W_{KP} \) algebras respectively [8][11].

The basic observation made in [10] is that one may implement the KP hierarchy on the space \( S_q \) of pseudodifferential operators (\( \Psi DO’s \)) of the form
\[
\Lambda_{\alpha,q} = \alpha \partial^q + \sum_{j=1}^{\infty} u_j \partial^{q-j}
\] (2.4)
where \( \alpha \) and \( q \) are complex numbers. The use of non-integer powers of the derivative operator deserves some explanation. From the operational point of view, the only relevant piece of information is contained in the composition law that generalizes the Leibnitz rule:
\[
\partial^q f = \sum_{j=0}^{\infty} \left[ \begin{array}{c} q \\ j \end{array} \right] f^{(j)} \partial^{q-j}
\] (2.5)

involving the generalized binomial coefficients
\[
\left[ \begin{array}{c} q \\ j \end{array} \right] \equiv \frac{q(q-1) \cdots (q-j+1)}{j!} \quad q \in \mathbb{R}.
\] (2.6)

Furthermore, we shall need to make sense of objects like \( \log \partial \), which we will use later on. We choose to do so by thinking about this operator as \( \log \partial = \)
\[
\lim_{q \to 0} \frac{1}{q} (\partial^q - 1), \text{ and use this limiting expression to extract the corresponding composition law from (2.5)}
\]

\[
(\log \partial) f(x) = f(x) \log \partial - \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} f^{(j)}(x) \partial^{-j} \quad (2.7)
\]

Notice that for any \( \Psi DO A \in S_q \), the commutator \([\log \partial, A]\) is a \( \Psi DO \) in \( S_{q-1} \), i.e. the commutator with \( \log \partial \) lowers by one the order of \( A \).

The Lax equations (2.1) can be implemented on \( S_q \) with due care. In order to do so we first need to define the \( q \)'th root of \( \Lambda_{\alpha,q} \). \((\Lambda_{\alpha,q})^{1/q}\) can be obtained from its generic expression with \( q \) being an integer by formally allowing the parameter \( q \) to become an arbitrary complex number. After a somewhat tedious calculation one obtains

\[
(\Lambda_{\alpha,q})^{1/q} = \alpha^{1/q} \left[ (\partial + \frac{1}{c} u_1 + \frac{1}{c} (u_2 - \frac{q-1}{2}(u_1^2 + \frac{1}{c} u_1^2)))\partial^{-1}
\right.
\]

\[
\frac{1}{c}(u_3 - \frac{q-1}{2} u_2') + \frac{q-1}{12} u_1'' - \frac{q-1}{c} u_1 u_2 + \frac{q(q-1)}{6c^2} u_1 u_1' + \frac{q-1}{6c} (2q-1) u_1^3 \partial^{-2}
\]

\[
\frac{1}{c}(u_4 - \frac{q-1}{2} u_3') + \frac{q-1}{12} u_2'' - \frac{q-1}{24} u_2''
\]

\[
- \frac{(q-1)(2q-1)(q+5)}{24c} u_1'u_1' + \frac{(q-1)(q+1)}{12} u_1 u_1'' + \frac{q-1}{2c} u_1' u_2
\]

\[
+ \frac{q-1}{c} u_1' u_2' - \frac{q-1}{2} u_1 u_3 + \frac{q-1}{2c} u_2 - \frac{(q-1)(2q^2+q-1)}{4c^2} u_1 u_1^3
\]

\[
+ \frac{(q-1)(2q-1)}{2c^2} u_2 u_1 u_2 - \frac{(q-1)(6q^2-5q+1)}{24c^2} u_1^4 \partial^{-3} + \ldots
\]

(2.8)

where \( c \) stands for the product \( \alpha q \).

Now the KP hierarchy is defined on the space \( S_q \) through the following system

\[
\frac{\partial \Lambda_{\alpha,q}}{\partial t_n} = \left[ ((\Lambda_{\alpha,q})^{n/q})^+, \Lambda_{\alpha,q} \right] = \left[ \Lambda_{\alpha,q}, ((\Lambda_{\alpha,q})^{n/q})^- \right], \quad n \in \mathbb{Z} \quad (2.9)
\]

From the second form of these equations, it is evident that the field \( u_1 \) does not evolve. Therefore, it is customary to choose as initial conditions \( u_1(x) = 0 \).

Also from (2.8) it is obvious that the \( n \)-th equation is proportional to \( \alpha^{n/q} \). Therefore we may renormalize all the times by defining \( \tilde{t}_n \equiv \alpha^{n/q} t_n \), so as to
absorb this factor. With these changes, the first few equations are given by

\[
\begin{align*}
\frac{\partial u_i}{\partial \tilde{t}_1} &= u_i' & i = 2, 3, \ldots \\
\frac{\partial u_2}{\partial \tilde{t}_2} &= 2u_3' - (q - 2)u_2'' \\
\frac{\partial u_2}{\partial \tilde{t}_3} &= 3u_4' + \frac{3(3-q)}{2}u_3'' + \frac{(q-3)^2}{4}u_2''' + \frac{3(3-q)}{\alpha}u_2u_2' \\
\frac{\partial u_3}{\partial \tilde{t}_2} &= 2u_4' + u_3'' - \frac{(q-1)(q-2)}{3}u_2''' - \frac{2(q-2)}{\alpha}u_2u_2'
\end{align*}
\]

(2.10)

Notice that after rescaling the times, the factors \(q\) and \(\alpha\) always appear in the combination \(c = \alpha q\). This is an essential fact for the rest of the discussion, and it can be seen to hold for the whole hierarchy. Moreover the actual value of \(c\) is irrelevant and it can be made to disappear by rescaling \(u_i \to cu_i\). Nevertheless, we prefer to keep track of this factor in what follows.

Following the usual steps, one may solve for the KP equation,

\[
\frac{3}{4}rac{\partial^2 u_2}{\partial \tilde{t}_2^2} = \frac{\partial}{\partial x} \left( \frac{\partial u_2}{\partial \tilde{t}_3} - \frac{1}{4}u_2''' - \frac{3}{\alpha}u_2u_2' \right) \quad (x \equiv \tilde{t}_1)
\]

(2.11)

It is a main result of [10] that for arbitrary values of \(q \neq 0\), the equations of motion in (2.9) admit the form of Hamilton’s equations,

\[
\frac{\partial \Lambda_{\alpha,q}}{\partial \tilde{t}_n} = \left\{ \tilde{H}_n, \Lambda_{\alpha,q} \right\}_{2,q}
\]

(2.12)

the Hamilton’s functions being given by the general expression

\[
\tilde{H}_n = \frac{c}{n} \alpha^{-n/q} \text{Tr}(\Lambda_{\alpha,q})^{n/q}.
\]

(2.13)

As an example we write down explicitly the first three cases:

\[
\begin{align*}
\tilde{H}_1^{(q)} &= \int (u_2 - \frac{q-1}{2c}u_1^2) \\
\tilde{H}_2^{(q)} &= \int (u_3 - \frac{q-2}{c}u_1u_2 + \frac{(q-1)(q-2)}{3c}u_1^3) \\
\tilde{H}_3^{(q)} &= \int (u_4 - \frac{q-3}{c}u_1u_2' + \frac{(q-1)(q-3)}{8c}u_1u_1'' + \frac{2q^2-9q+9}{2c^2}u_2^2 - \frac{q-3}{c}u_1u_3 - \frac{q-3}{2c}u_2^2 - \frac{(q-1)(6q^2-27q+27)}{24c^3}u_1^4)
\end{align*}
\]

(2.14)

The brackets in (2.12) are a generalization of the (second) Gelfand-Dickey brackets written in (2.2) to the space \(S_q\) [10]. In terms of the basis functions


\[ u_i, \, i = 1, 2, \ldots \text{ these Poisson brackets define a non-linear algebra named } W_{KP}^{(q)}. \]

Its first few brackets look as follows:

\[
\begin{align*}
\{ u_1(x), \, u_1(y) \}_{q, 2} &= c \partial_x \cdot \delta(x-y), \\
\{ u_1(x), \, u_2(y) \}_{q, 2} &= -c^2 \delta^2 + (q-1) \partial u_1 \cdot \delta(x-y), \\
\{ u_1(x), \, u_3(y) \}_{q, 2} &= c^2 (2q-1)(q-1) \partial^3 - \frac{(q-1)(q-2)}{2} u_1 \partial^2 u_1 + (q-2) \partial u_2 \cdot \delta(x-y), \\
\{ u_2(x), \, u_2(y) \}_{q, 2} &= -c^2 \frac{(q-1)(2q-1)}{6} \partial^3 - u_2 \partial - u_1 \partial \frac{(q-1)(q-2)}{2c} \partial^3 u_1 + \frac{c^2}{2} \partial^2 u_1 - u_1 \partial^2 u_2, \\
\{ u_2(x), \, u_3(y) \}_{q, 2} &= c^2 \frac{(q-1)(q-2)(3q-1)}{24} \partial^4 + \frac{q(q-1)(q-2)}{6} u_1 \partial^2 u_1 - \frac{(q-2)(q-3)}{3c} u_1 \partial^2 u_1 \\
&+ \frac{c^2}{2} u_1 \partial^2 u_2 \cdot \delta(x-y), \\
\{ u_2(x), \, u_4(y) \}_{q, 2} &= \left( \frac{q^2}{60} - \frac{c}{2} \right) u_1 \partial^2 u_2 + \frac{c}{2} u_1 \partial^2 u_3 \cdot \delta(x-y), \\
\{ u_3(x), \, u_3(y) \}_{q, 2} &= -c^2 \frac{(q-1)(q-2)(3q-2)}{120} \partial^5 - \left( \partial^2 u_3 - u_3 \partial^2 \right)^2 \\
&- 2(u_3 \partial + \partial u_4) + \frac{q(q-1)(q-2)}{6} \partial^3 u_2 + \frac{(q-2)(q-3)}{3c} \partial^3 u_1 \\
&+ \frac{q(q-1)(q-2)(3q-5)}{120} \partial^4 u_1 - u_1 \partial^4 - \frac{q(q-1)(q-2)(3q-2)}{120} \partial^3 u_1 \\
&- \frac{c}{6} \left( u_1 \partial^3 u_3 + u_3 \partial u_1 \right) \cdot \delta(x-y), \\
&
\end{align*}
\]

\((2.15)\)

The reader may verify that with the information contained in (2.14) and in (2.15), the expression (2.12) yields equations (2.10).

A word of caution here: the Poisson brackets (2.15) that span the \( W_{KP}^{(q)} \) algebra do not stabilize the initial condition \( u_1(x) = 0 \) except for the Hamiltonian flows generated by (2.14) that yield precisely the KP-evolution equations (2.10). Therefore in order to reproduce these equations correctly one has to maintain the field \( u_1 \) throughout the calculation, and only at the very end set it to zero. One can however neglect terms with more that one \( u_1 \) in (2.14), whose contribution to the equations of motion will be proportional to this field. Alternatively, one may set \( u_1 = 0 \) from the start, but then the Poisson brackets need to be reduced consistently via the Dirac procedure. The resulting non-linear algebra is named \( \hat{W}_\infty^{(q)} \) \([10]\) (or, for \( q = 1 \), \( \hat{W}_\infty \) \([12]\)).
§3 Full $W_{1+\infty}$ as a Hamiltonian Structure for KP at $q = 0$

Let us have a closer look at equations (2.10). They become ill defined in the limit $q \to 0$. However, due to the fact that in the denominator $q$ always enters in the combination $c = \alpha q$ we may define a more interesting “scaling” limit where $q \to 0$ and $\alpha \to \infty$ such that $c = \alpha q$ is held constant.

No less important is the fact that this limit may be taken directly at the level of the Lax equations (2.9); the Lax pair involving, on one side, the following operator

$$ \Lambda_c = \lim_{\substack{q \to 0 \\alpha = c/q}} \Lambda_{\alpha,q} = c \log \partial + u_2 \partial^{-2} + ... \quad (3.1) $$

and, on the other, integer powers of its “infinitesimal root” (c.f.(2.8)),

$$ (\Lambda_c)^\infty \equiv \lim_{\substack{q \to 0 \\alpha = c/q}} \alpha^{-1/q}(\Lambda_{\alpha,q})^{1/q} $$

$$ = \partial + \frac{1}{c} u_2 \partial^{-1} + \frac{1}{c}(u_3 - \frac{1}{2} u'_2) \partial^{-2} + ... $$

Namely, the Lax equation

$$ \frac{\partial}{\partial t_n} \Lambda_c = [((\Lambda_c)^\infty)_+^n, \Lambda_c] \quad (3.2) $$

automatically encodes all the limiting expressions obtained from the flows in (2.10). The term $c \log \partial$ on the right hand side arises from $\lim_{q \to 0} \alpha \partial^q$ and shows that the Lax operator is peculiar when we induce the KP hierarchy on the space $S_0$. Nevertheless, we should consider the Lax pair as an auxiliary device and care only about the consistency of the system it defines. In what concerns the KP equation (2.11), it survives this limit intact.

A comment is in order. The relevant fact that the hierarchy defined by the Lax system (3.2) is no other than KP (yet in a peculiar basis), can be proven using the formalism of Sato. In this language, the KP flows rather live on the Volterra group of operators of the form $\Phi = 1 + a_1 \partial^{-1} + a_2 \partial^{-2} + ...$ defined as

$$ \frac{\partial}{\partial t_n} \Phi = -(\Phi \partial^n \Phi^{-1})_+ \Phi \quad (3.3) $$

Their commutativity follows as the result of a straightforward computation [7]. These flows can be induced on $S_q$ by means of the dressing transformation $\Lambda_{\alpha,q} = \Phi \alpha \partial^q \Phi^{-1}$. On $S_0$ we may as well represent the flows if we dress instead log $\partial$:

$$ \Lambda_c = \Phi(c \log \partial) \Phi^{-1} = c \log \partial + [\Phi, c \log \partial] \Phi^{-1} \quad (3.4) $$

From this expression, and using (3.3), the Lax equation (3.2) is recovered.
One may wonder if the Hamiltonian formulation of the KP flows expressed in (2.12) is as robust as the Lax formulation in the limit $q \to 0$. There are two separate pieces that we must check: the Poisson brackets (2.15) and the Hamiltonians (2.14).

Concerning the first piece, we again have to refer to [10] where this limit has been shown to yield the famous centrally extended linear $W_{1+\infty}$ algebra; in short: $\lim_{q \to 0} \{c/q, \} = \{c, \}$ where

$$
\{u_i(x), u_j(y)\}_{1+\infty} = (c(-1)^{i+1}(i+1)!j(j-1)!/((i+j-1)!)^{i+j-1}
- \sum_{l=1}^{j-1} \left[ j - 1 \right] \partial^l u_{i+j-l-1} + \sum_{l=1}^{i-1} \left[ i - 1 \right] u_{i+j-l-1} (-\partial)^{i+j-l-1}(x-y) + \ldots
$$

The first few particular cases are easily recovered taking the limit directly in (2.15). It is important to note the role of $c$ that here parameterizes the central extension of the algebra.

The question about the fate of the Hamiltonian equations (2.12) in this limit can be recasted in a form that leads us back to the original motivation of this work: using the Poisson brackets given by the centrally extended algebra $W_{1+\infty}$, can we find related Hamiltonians for the KP hierarchy?

The answer looks trivially positive, as a glance at (2.14) reveals no pathologies in the desired limit. More generally, using the freedom to rescale $c = 1$ (i.e. $\alpha = 1/q$), the limiting definitions (cf. (2.13))

$$
\tilde{H}^{(0)}_n = \lim_{q \to 0} \frac{q^n}{n} \text{Tr}(\Lambda q)^{n/q}.
$$

yield well defined expressions for all $n$.

$$
\begin{align*}
\tilde{H}^{(0)}_1 &= \int u_2 \\
\tilde{H}^{(0)}_2 &= \int (u_3 + 2u_1u_2) \\
\tilde{H}^{(0)}_3 &= \int (u_4 + \frac{3}{2}u_1u_2' + 3u_1u_3 + \frac{3}{2}u_2^2) \\
&\vdots
\end{align*}
$$

where we have discarded terms with higher powers of the field $u_1$ since, eventually, they will not contribute to the equations of motion when we set $u_1 = 0$.

As the flows generated by (3.6) commute, involution i.e. $\{H_i, H_j\}_{1+\infty} = 0$, follows automatically.
§4 Conclusions

We would like to single out three concluding remarks:

1.- The centrally extended $W_{1+\infty}$ algebra provides a hamiltonian structure for the KP hierarchy. The possibility of consistently defining the KP flows at $q = 0$ relies on the fact that all the singularities that appear can be absorbed in an infinite renormalization of the times $t_n \rightarrow \tilde{t}_n = \alpha^n/\alpha t_n$ and the hamiltonians $H_n \rightarrow \tilde{H}_n$.

2.- Contrary to some claims, we showed that the $c$ in the central extension of the algebra does not parameterize an integrable deformation of the KP hierarchy.

3.- A natural question to ask is: how about the centrally extended $W_\infty$? i.e. for what hierarchy does this algebra provide a Poisson structure? The centerless case is, of course, not problematic since it yields the first hamiltonian structure of KP at $q = 1$ [8]. Concerning the centerful $W_\infty$ one may wish to start from the centrally extended $W_{1+\infty}$ and set $u_1 = 0$, but then the Poisson structure has to be consistently reduced via Dirac brackets. The reduced algebra is not the linear $W_\infty$ but instead a nonlinear algebra named $W^\#_\infty$ [10](else, first reducing $u_1 = 0$ for $q \neq 0$ brings us from $W_{KP}^{(q)}$ to $\tilde{W}_\infty^{(q)}$, and the subsequent contraction $q \rightarrow 0$ yields back the same algebra). It turns out that the centerful $W_\infty$ algebra can be also produced out of $W_{KP}^{(q)}$ as a different contraction, namely: $q \rightarrow 1$ and $\alpha \rightarrow \infty$ with $c' = \alpha(q - 1)$ kept finite; the central extension being proportional to $c'$ in $W_\infty$. However in this limit $c = q\alpha \rightarrow \infty$, and a glance at the equations of motion (2.10) reveals that the KP flows collapse since all the nonlinear terms vanish. Of course, the KP equation (2.11) linearizes as well. That is, within the present scheme, the associated hierarchy is not KP but a linear truncation thereof. Nevertheless this does not rule that the centrally extended $W_\infty$ algebra could still be a hamiltonian structure for KP, yet the construction of the hamiltonians claims for a different approach and remains for the moment an open question.

Summarizing, we have shown how to express the KP hierarchy in hamiltonian form from the centrally extended $W_{1+\infty}$ algebra; in particular, we have provided a closed expression for the hamiltonian functions. The method of analytic continuation in the parameter $q$ has shown to be a powerful tool in our analysis. Notice, for example, that standard methods for obtaining the conserved charges from Lax equations, like taking the traces of powers of the Lax operator, are not even defined for operators of the form $c \log \partial + ......$. Hence our hamiltonians can only be defined via the limiting procedure expressed in (3.6). The relevance of these hamiltonians in systems where the centrally extended $W_{1+\infty}$ algebra plays a dynamical rôle, such as the Quantum Hall Effect or
D=2 non-critical strings, is currently under investigation. It is also an appealing challenge to search in them for a physical counterpart of the deformation parameter $q$.

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REFERENCES

[1] A. Capelli, C. Trugenberger and G. Zemba, Nucl. Phys. B396 (1993) 465
S. Iso, D. Karabali and B. Sakita, Phys. Lett. 296B (1992) 143.

[2] M. Fukuma, H. Kawai, R. Nakayama, Comm. Math. Phys. 143 (1991) 371;
H. Itoyama and Y. Matsuo, Phys. Lett. 262B (1991) 233.

[3] S. Nojiri, M. Kawamura, A Sugamoto, Mod. Phys. Lett. A9 (1994) 1159, and hep-th/9409164

[4] D. Gross and W. Taylor, Nucl. Phys. B356 (1991) 208

[5] K. Takasaki, Comm. Math. Phys. 94 (1984) 35;
Q.-H. Park, Phys. Lett. 238B (1991) 287

[6] H.L. Hu, Phys. Lett. 324B (1994) 293

[7] L. A. Dickey, Soliton equations and Hamiltonian systems, Advanced Series in Mathematical Physics Vol.12, World Scientific Publ. Co..

[8] K. Yamagishi, Phys. Lett. 259B (1991) 436;
F. Yu and Y.-S. Wu, Phys. Lett. 236B (1991) 220

[9] J. Gawrylczyk and J. Lukierski, University of Wroclaw preprint, February 1993.

[10] J. M. Figueroa-O’Farrill, J. Mas, and E. Ramos Comm. Math. Phys. 158 (1993) 17.

[11] J. M. Figueroa-O’Farrill, J. Mas, and E. Ramos, Phys. Lett. 266B (1991) 298;
L. A. Dickey, Annals NY Acad. Sci. 491(1987) 131

[12] F. Yu and Y.-S. Wu, Nucl. Phys. B373 (1992) 713.