A DYNAMICAL SYSTEM OF TEMPERATURE-DEPENDENT SEX LINKED INHERITANCE

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Abstract. Recently, R.Varro introduced a gonosomal algebra of the temperature-dependent sex determination system which is controlled by three temperature ranges. In this paper we study dynamical systems which are given by quadratic evolution operators of the gonosomal algebras of sex-linked populations. We show that this evolution operator can be reduced to an evolution operator of free population. Then using behavior of the free population we describe the set of limit points for trajectories of several evolution operators of the sex-linked populations.

1. Introduction

It is known that some species of reptiles, including alligators, some turtles, and the tuatara, sex is determined by the temperature at which the egg is incubated during a temperature-sensitive period. In reptiles (snakes, crocodiles, turtles, lizards) there are two types of sex determination, either a genotypic determination controlled according to the species by XY- or ZW-system, either a determination depending on the incubation temperature of eggs.

Following [10] we recall that temperature-dependent sex determination is controlled by three temperature ranges, the eggs subject to feminizing temperatures (resp. masculinizing) give rise to 100 percentage or a majority of females and those subject to transition temperatures provide 50 percentage females and 50 percentage males.

In [10] the following algebraic model is considered: let \( A \) be a linear space with basis \((e_i, \tilde{e}_i)_{1 \leq i \leq n}\) where \(e_i\) (resp. \(\tilde{e}_i\)) are female (resp. male) genetic types present in a population and subject to temperature-dependent sex determination. Denote by \(\tau_1, \tau_2\) and \(\tau_3\) the probability that eggs are incubated at feminizing temperatures, masculinizing temperatures and transition temperatures respectively, thus

\[ \tau_1, \tau_2, \tau_3 \geq 0, \quad \tau_1 + \tau_2 + \tau_3 = 1. \]

For each \(r = 1, 2\) \((r = 1\) for feminizing temperatures, \(r = 2\) for masculinizing temperatures\) denote by \(\mu_r\) and \(\tilde{\mu}_r\) respectively the proportions of females and males arising from eggs placed in the environment \(r\), thus we have

\[ \mu_r + \tilde{\mu}_r = 1, \quad \mu_1 \geq \tilde{\mu}_1 \geq 0, \quad 0 \leq \mu_2 \leq \tilde{\mu}_2. \]

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1 see https://en.wikipedia.org/wiki/Sex-determination_system
For all $1 \leq i, p \leq n$ denote by $\theta_{ipk}$ the egg proportion of $e_k$ type in the laying of a female $e_i$ crossed with a male $\tilde{e}_p$, thus

$$\theta_{ipk} \geq 0, \; \sum_{k=1}^{n} \theta_{ipk} = 1. \tag{1.1}$$

Then the space $A$ equipped with the following algebra structure:

$$e_i e_j = \tilde{e}_p \tilde{e}_q = 0,$$

$$e_i \tilde{e}_p = (\mu_1 \tau_1 + \mu_2 \tau_2 + \frac{1}{2} \tau_3) \sum_{k=1}^{n} \theta_{ipk} e_k + (\tilde{\mu}_1 \tau_1 + \tilde{\mu}_2 \tau_2 + \frac{1}{2} \tau_3) \sum_{k=1}^{n} \theta_{ipk} \tilde{e}_k. \tag{1.2}$$

This algebra $A$ is called a gonosomal algebra (see [10]) and the product $e_i \tilde{e}_p$ gives the genetic distribution of progeny of a female $e_i$ with a male $\tilde{e}_p$.

Denote

$$a = \mu_1 \tau_1 + \mu_2 \tau_2 + \frac{1}{2} \tau_3, \; \; b = \tilde{\mu}_1 \tau_1 + \tilde{\mu}_2 \tau_2 + \frac{1}{2} \tau_3.$$

Note that $a \geq 0, b \geq 0$ and $a + b = 1$.

Consider the following set

$$S = \{ z = (x_1, ..., x_n, y_1, ..., y_n) \in R^{2n},$$

$$x_i \geq 0, y_j \geq 0, \sum_{i=1}^{n} x_i \neq 0, \sum_{j=1}^{n} y_j \neq 0, \sum_{i=1}^{n} x_i + \sum_{j=1}^{n} y_j = 1 \}.$$

We call the partition into types hereditary if for each possible state $z \in S$ describing the current generation, the state $z' \in S$ is uniquely defined describing the next generation. This means that the association $z \rightarrow z'$ defines a map $W : S \rightarrow S$ called the evolution operator [5].

For any point $z^{(0)} \in S$ the sequence $z^{(t)} = W(z^{(t-1)}), t = 1, 2, ...$ is called the trajectory of $z^{(0)}$. Denote by $\omega(z^{(0)})$ the set of limit points of the trajectory. Since $\{z^{(n)}\} \subset S$ and $S$ is compact, it follows that $\omega(z^{(0)}) \neq \emptyset$. Obviously, if $\omega(z^{(0)})$ consists of a single point, then the trajectory converges, and $\omega(z^{(0)})$ is a fixed point. However, looking ahead, we remark that convergence of the trajectories is not the typical case for the dynamical systems.

If $z' = (x'_1, ..., x'_n, y'_1, ..., y'_n)$ is a state of the system gens in the next generation then by the rule (1.2) we get the evolution operator $W_a : S \rightarrow S$ defined by

$$W_a : \left\{ \begin{array}{l}
   x'_k = \frac{\sum_{i=1}^{n} \sum_{p=1}^{n} \theta_{ipk} x_i y_p}{\sum_{i=1}^{n} x_i} \frac{\sum_{p=1}^{n} y_p}{},
   y'_k = \frac{(1-a) \sum_{i=1}^{n} \sum_{p=1}^{n} \theta_{ipk} x_i y_p}{\sum_{i=1}^{n} x_i} \frac{\sum_{p=1}^{n} y_p}{},
\end{array} \right. \tag{1.3}$$

where $k = 1, ..., n$.

In this paper our goal is to study dynamical systems generated by operator (1.3).

\footnote{This set is a subset of $(2n - 1)$-dimensional simplex.}
2. Dynamics systems generated by the operator (1.3)

For \( a \in (0,1) \) we denote

\[
S_a \equiv S_a^{2n-2} = \left\{ z = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in S : \sum_{i=1}^{n} x_i = a, \sum_{j=1}^{n} y_j = 1 - a \right\}.
\]

The following lemma is useful

**Lemma 1.** For any fixed \( a \in (0,1) \) we have

1. for any \( z = (x, y) \in S \) the following holds

\[
z' = (x', y') = W_a(z) \in S_a, \quad \text{i.e.,} \quad W_a(S) \subset S_a;
\]

2. the set \( S_a \) is invariant with respect to \( W_a \), i.e., \( W_a(S_a) \subset S_a \).

**Proof.** The proof follows from the following easily checked equality

\[
\sum_{k=1}^{n} x'_k = a, \quad \sum_{k=1}^{n} y'_k = 1 - a.
\]

\[\Box\]

For each fixed \( a \in (0,1) \), by this Lemma \( \Box \) the investigation of the sequence \( z^{(t)} = W_a(z^{(t-1)}) \), \( t = 1, 2, \ldots \), for each point \( z^{(0)} \in S \) is reduced to the case \( z^{(0)} \in S_a \). Therefore we are interested to the following dynamical system:

\[
z^{(0)} \in S_a, \quad z^{(1)} = W_a(z^{(0)}), \quad z^{(2)} = W_a(z^{(1)}), \ldots
\]

the main problem is to study the limit

\[
\lim_{m \to \infty} z^{(m)} = \lim_{m \to \infty} W_a^m(z^{(0)}).
\]

The restriction on \( S_a \) of the operator \( W_a \), denoted simply by \( W \), has the form

\[
W : \left\{ \begin{array}{l}
x'_k = (1 - a)^{-1} \sum_{i=1}^{n} \sum_{p=1}^{n} \theta_{ipk} x_i y_p, \\
y'_k = a^{-1} \sum_{i=1}^{n} \sum_{p=1}^{n} \theta_{ipk} x_i y_p,
\end{array} \right.
\]

where \( k = 1, \ldots, n \).

Denote \( \beta = (1 - a)/a \). A point \( z \in S_a \) is called a fixed point of \( W \) if \( W(z) = z \).

**Lemma 2.** If \( z = (x, y) \in S_a \) is a fixed point of \( W \) then

\[
y_k = \beta x_k, \quad k = 1, \ldots, n.
\]

**Proof.** Straightforward. \( \Box \)

By this lemma the problem of finding fixed points of \( W_a \) is reduced to solution of the following system:

\[
x_k = a^{-1} \sum_{i=1}^{n} \sum_{p=1}^{n} \theta_{ipk} x_i x_p, \quad k = 1, \ldots, n.
\]
Lemma 3. If \( z^{(0)} = (x_1^{(0)}, \ldots, x_n^{(0)}; y_1^{(0)}, \ldots, y_n^{(0)}) \in S_a \) is an initial point and \( z^{(m)} = (x_1^{(m)}, \ldots, x_n^{(m)}; y_1^{(m)}, \ldots, y_n^{(m)}) = W_a^{(m)}(z^{(0)}) \) then
\[
y_k^{(m)} = \beta x_k^{(m)}, \quad \text{for any} \quad k = 1, \ldots, n, \quad m \geq 1.
\]

Proof. Follows from the following equality
\[
\begin{align*}
x_k^{(m)} &= (1 - a)^{-1} \sum_{i=1}^{n} \sum_{p=1}^{n} \theta_{ipk} x_i^{(m-1)} y_p^{(m-1)}, \quad k = 1, \ldots, n, \quad m \geq 1. \\
y_k^{(m)} &= a^{-1} \sum_{i=1}^{n} \sum_{p=1}^{n} \theta_{ipk} x_i^{(m-1)} y_p^{(m-1)}, \quad k = 1, \ldots, n, \quad m \geq 1.
\end{align*}
\] (2.5)

By this lemma to investigate \( z^{(m)} \) it suffices to study \( x^{(m)} = (x_1^{(m)}, \ldots, x_n^{(m)}) \) given by \( x^{(m)} = V(x^{(m-1)}), \quad m \geq 1, \) where
\[
V : x_k' = a^{-1} \sum_{i=1}^{n} \sum_{p=1}^{n} \theta_{ipk} x_i x_p, \quad k = 1, \ldots, n.
\] (2.6)

Introduce new variables \( u_i = x_i/a, \quad i = 1, 2, \ldots, m \) then from (2.6) we obtain the operator \( V \) given by
\[
V : u_k' = \sum_{i=1}^{n} \sum_{p=1}^{n} \theta_{ipk} u_i u_p, \quad k = 1, \ldots, n.
\] (2.7)

where \( \theta_{ipk} \) satisfies (1.1).

Note that dynamical systems generated by the operator (2.7) were studied in many papers (see [5]-[9], and [1] for a review, [4] and references therein for recent results).

Thus we have the following

Corollary 1. The dynamical system given by (2.3) is equivalent to the dynamical system given by (2.7).

To illustrate the above-mentioned results for investigation of (2.3) we consider the following three classes of operators:

C1. Assume \( \theta_{ipk} = 0 \) if \( k \notin \{i, j\} \), and \( \theta_{ipk} \neq \frac{1}{2} \), for each \( i, p, k = 1, \ldots, n \).
C2. Divide the set \( E = \{1, 2, \ldots, n\} \) into three parts:
\[
\{1\}, F = \{2, \ldots, m\}, M = \{m + 1, \ldots, n\},
\]
where \( 2 \leq m \leq n - 1 \).

Now define coefficients \( \theta_{ijk} \) as follows
\[
\theta_{ijk} = \begin{cases} 
1, & \text{if} \quad k = 1, i, j \in F \cup \{1\} \quad \text{or} \quad i, j \in M \cup \{1\}; \\
0, & \text{if} \quad k \neq 1, i, j \in F \cup \{1\} \quad \text{or} \quad i, j \in M \cup \{1\}; \\
\geq 0, & \text{if} \quad i \in F, j \in M, \forall k.
\end{cases}
\] (2.8)

C3. Now we give an example of operator (2.3) which has periodic orbits. Consider \( n = 3 \) and coefficients as the following
\[
\theta_{113} = \theta_{123} = \theta_{213} = \theta_{223} = \theta_{312} = \theta_{231} = 1, \quad \theta_{331} = c, \theta_{332} = d,
\] (2.9)
where \( c + d = 1 \) and the remaining \( \theta_{ijk} = 0 \). The corresponding operator (2.7) has the form
\[
\begin{align*}
x_1' &= cx_2^2 + 2x_2x_3 \\
x_2' &= dx_3^2 + 2x_1x_3 \\
x_3' &= (x_1 + x_2)^2.
\end{align*}
\]

Denote
\[
\text{int}\, S_a = \left\{ z = (x_1, ..., x_n, y_1, ..., y_n) \in S_a : \prod_{i=1}^{n} x_i y_i \neq 0 \right\}.
\]
\[
\partial S_a = S_a \setminus \text{int}\, S_a.
\]

**Theorem 1.** If condition C1 is satisfied then

1) If \( z(0) \in \text{int}\, S_{m-1} \) is not a fixed point, then \( \omega(z(0)) \subset \partial S_a \).

2) The set \( \omega(z(0)) \) either consists of a single point or is infinite.

3) If the operator (2.3) has an isolated fixed point \( z^* \in \text{int}\, S_a \) then for any initial point \( z(0) \in \text{int}\, S_a \setminus \{ z^* \} \) the trajectory \( \{ z(n) \} \) does not converge.

If condition C2 is satisfied then

4) Let \( \theta_{ijk} \) be as in (2.8) then operator (2.3) has a unique fixed point \((a, 0, ..., 0), (1 - a, 0, ..., 0)\). For any \( z_0 \in S_a \), the trajectory \( \{ z(n) \} \) tends to this fixed point exponentially rapidly.

In case C3 we have

5) If (2.9) satisfied then corresponding operator (2.3) has unique fixed point
\[
z^* = (ax_1^*, ax_2^*, ax_3^*, (1 - a)x_1^*, (1 - a)x_2^*, (1 - a)x_3^*),
\]
where
\[
x_1^* = \frac{(7 - 3\sqrt{5})c + 4\sqrt{5} - 8}{2(4 - \sqrt{5})}; \quad x_2^* = \frac{(3\sqrt{5} - 7)c + \sqrt{5} - 1}{2(4 - \sqrt{5})}; \quad x_3^* = \frac{3 - \sqrt{5}}{2}.
\]

There is unique cyclic theory \( \{ (ac, ad, 0, 1 - a, 0, 0), (a, 0, 0, 1 - a, 0, 0) \} \). For any initial point \( z(0) \in S_a \) the \( \omega \)-limit set has the following form
\[
\omega(z(0)) = \begin{cases} 
\{ (ac, ad, 0, 1 - a, 0, 0), (a, 0, 0, 1 - a, 0, 0) \}, & \text{if } z_3(0) \neq ax_3^*, \\
\{ z^* \}, & \text{if } z_3(0) = ax_3^*.
\end{cases}
\]

**Proof.** 1)-3) follow from Theorem 2.4 of [1] by Corollary 1.

Part 4) follows from Theorem 2.24 of [1] (see also [6]).

Part 5) is the consequence of Corollary 1 and Theorem 2 of [7].

**Remark 1.** It follows from Theorem 1 that the operator (2.3) has a trajectory which converges; or does not converge with a finite set of limit points; or does not converge with an infinite set of limit points. Note that these are all possible cases which one expects for a given sequence of real numbers. Therefore, we can choose parameters of the dynamical system generated by the operator (2.3) to have as rich behavior as needed.
3. Full analysis in two dimensional case

In general it is difficult to solve the system (2.4). Let us solve it for \( n = 2 \).

Case \( n = 2 \): If this case using \( x_2 = a - x_1 \) we can reduce the first equation to a quadratic equation with four parameters, which has the following two solutions

\[
\begin{align*}
x_{1,1} &= \frac{a}{2} \cdot \frac{2 \theta_3 - \theta_2 + 1 + \sqrt{(\theta_2 - 1)^2 + 4 \theta_3 (1 - \theta_1)}}{\theta_1 + \theta_3 - \theta_2}, \\
x_{1,2} &= \frac{a}{2} \cdot \frac{2 \theta_3 - \theta_2 + 1 - \sqrt{(\theta_2 - 1)^2 + 4 \theta_3 (1 - \theta_1)}}{\theta_1 + \theta_3 - \theta_2},
\end{align*}
\]

(3.1)

\( \theta_1 = \theta_{111}, \quad \theta_2 = \theta_{121} + \theta_{211}, \quad \theta_3 = \theta_{221}. \)

Thus if the parameters satisfy the following condition

\[
(\theta_2 - 1)^2 + 4 \theta_3 (1 - \theta_1) \geq 0
\]

(3.2)

then the fixed points are

\[
z_j = (x_{1,j}, x_{2,j}, y_{1,j}, y_{2,j}), j = 1, 2,
\]

where \( x_{2,j} = a - x_{1,j} \) and \( y_{i,j} = \beta x_{i,j} \). Below we find conditions on parameters of the operator (2.6) to guarantee \( z_j \in S_a \).

For \( n = 2 \), using \( x_1 + x_2 = a \), from (2.6) we get

\[
T : x'_1 = a^{-1}(\theta_1 + \theta_3 - \theta_2)x_1^2 + (\theta_2 - 2 \theta_3)x_1 + \theta_3 a
\]

(3.3)

Note that (3.3) maps \( S_a^1 = [0, a] \) to itself, i.e., \( T : S_a^1 \mapsto S_a^1 \).

Lemma 4.  

a. Uniqueness of fixed point:

1) If \( \theta_1 \in [0; 1], \theta_2 \in [0; 1], \theta_3 = 0 \) then (3.3) has a unique fixed point \( x_1 = 0 \);
2) If \( \theta_1 = 0, \theta_2 = 1, \theta_3 = 1 \) then (3.3) has a unique fixed point \( x_1 = \frac{a}{2} \);
3) If \( \theta_1 = 1, \theta_2 \in [1; 2], \theta_3 \in (0; 1) \) then the fixed point of (3.3) is \( x_1 = a \);
4) If \( \theta_1 \in (0; 1), \theta_2 \in [0; 2], \theta_3 \in (0; 1) \) then the fixed point of (3.3) is

\[
x_1 = \frac{a}{2} \cdot \frac{2 \theta_3 - \theta_2 + 1 + \sqrt{(\theta_2 - 1)^2 + 4 \theta_3 (1 - \theta_1)}}{\theta_1 + \theta_3 - \theta_2}.
\]

b. Two fixed points:

1) If \( \theta_1 = 1, \theta_2 \in [0; 1], \theta_3 \in [0; 1] \) then the mapping (3.3) has two fixed points

\[
x_{1,1} = a, \quad x_{1,2} = \frac{\theta_3 a}{\theta_3 - \theta_2 + 1};
\]

2) If \( \theta_1 = 1, \theta_2 \in [0; 1) \cup (1; 2), \theta_3 = 0 \) then (3.3) has two fixed points

\[
x_{1,1} = 0, \quad x_{1,2} = a.
\]

3) If \( \theta_1 \in [0; 1], \theta_2 \in (1; 2], \theta_3 = 0 \) then mapping (3.3) has fixed points

\[
x_{1,1} = 0, \quad x_{1,2} = \frac{(1 - \theta_2) a}{\theta_1 - \theta_2}.
\]

c. If \( \theta_1 = \theta_2 = 1, \theta_3 = 0 \) then the set \( S_a^1 \) is the set of fixed points of (3.3).
Proof. The proof consists detailed analysis of the quadratic equation
\[ x = a^{-1}(\theta_1 + \theta_3 - \theta_2)x^2 + (\theta_2 - 2\theta_3)x + \theta_3a. \] (3.4)
This equation has solutions \( \boxed{3.1} \). Carefully checking of \( x_{1,1}, x_{1,2} \in S_0^1 \) completes the proof.
\( \square \)

For the mapping \( (3.3) \) define type of fixed points.

**Definition 1.** Suppose \( x_0 \) is a fixed point for \( T \). Then \( x_0 \) is an attracting fixed point if \( |T'(x_0)| < 1 \). The point \( x_0 \) is a repelling fixed point if \( |T'(x_0)| > 1 \). Finally, if \( |T'(x_0)| = 1 \), the fixed point is called neutral or saddle \( [2] \).

The following results are very simple to prove:

**Lemma 5.** 1) The type of the unique fixed points for \( (3.3) \) are as follows:

\[
\begin{align*}
x_1 &= \begin{cases} 
0 & \text{is attracting if } \theta_1 \in [0; 1), \theta_2 \in [0; 1), \theta_3 = 0, \\
0 & \text{is saddle if } \theta_1 \in [0; 1), \theta_2 = 1, \theta_3 = 0, \\
\frac{a}{2} & \text{is saddle if } \theta_1 = 0, \theta_2 = 1, \theta_3 = 1, \\
a & \text{is attracting if } \theta_1 = 1, \theta_2 \in (1; 2), \theta_3 \in (0; 1], \\
a & \text{is saddle if } \theta_1 = 1, \theta_2 \in \{1; 2\}, \theta_3 \in (0; 1], \\
x^* & \text{is attracting if } (\theta_2 - 1)^2 + 4\theta_3(1 - \theta_1) < 4, \\
x^* & \text{is repeller if } (\theta_2 - 1)^2 + 4\theta_3(1 - \theta_1) > 4, \\
x^* & \text{is saddle if } (\theta_2 - 1)^2 + 4\theta_3(1 - \theta_1) = 4,
\end{cases}
\end{align*}
\]

where
\[
x^* = \frac{a}{2} \cdot \frac{2\theta_3 - \theta_2 + 1 - \sqrt{(\theta_2 - 1)^2 + 4\theta_3(1 - \theta_1)}}{\theta_1 + \theta_3 - \theta_2}.
\]

2) The type of two fixed point of \( (3.3) \)

\[
\begin{align*}
x_{1,1} &= \begin{cases} 
a & \text{is repelling if } \theta_1 = 1, \theta_2 \in [0; 1), \theta_3 \in (0; 1], \\
0 & \text{is attracting if } \theta_1 = 1, \theta_2 \in [0; 1), \theta_3 = 0, \\
0 & \text{is repelling if } \theta_1 = 1, \theta_2 \in (1; 2), \theta_3 = 0, \\
or & \theta_1 \in [0; 1], \theta_2 \in (1; 2], \theta_3 = 0,
\end{cases}
\end{align*}
\]

\[
x_{1,2} &= \begin{cases} 
\frac{\theta_2a}{\theta_2 - \theta_1} & \text{is attracting if } \theta_1 = 1, \theta_2 \in [0; 1), \theta_3 \in (0; 1], \\
\theta_1 & \text{is repelling if } \theta_1 = 1, \theta_2 \in [0; 1), \theta_3 = 0, \\
\theta_1 & \text{is attracting if } \theta_1 = 1, \theta_2 \in (1; 2), \theta_3 = 0, \\
\frac{(1-\theta_2)a}{\theta_1 - \theta_2} & \text{is attracting if } \theta_1 \in [0; 1], \theta_2 \in (1; 2], \theta_3 = 0.
\end{cases}
\]

**Proposition 1.** Let \( x^{(0)} \in S_0^1 \) be an initial point then
Proof. Follows from Lemma 1, Lemma 2 and Proposition 1.

1) \[
\lim_{m \to \infty} T^{(m)}(x_0) = \lim_{m \to \infty} x^{(m)} = \begin{cases}
0, & \text{if } \theta_1 \in [0; 1), \theta_2 \in [0; 1], \theta_3 = 0, \\
a, & \text{if } \theta_1 = 1, \theta_2 \in [1; 2], \theta_3 \in (0; 1), \\
x^*, & \text{if } \theta_1 \in [0; 1), \theta_2 \in [0; 2], \theta_3 \in (0; 1), \\
\frac{\theta_3 a}{\theta_3 - \theta_2 + 1}, & \text{if } \theta_1 = 1, \theta_2 \in [0; 1), \theta_3 \in (0; 1), \\
0, & \text{if } \theta_1 = 1, \theta_2 \in [0; 1), \theta_3 = 0, \\
a, & \text{if } \theta_1 = 1, \theta_2 \in (1; 2), \theta_3 = 0, \\
\frac{(1-\theta_3) a}{\theta_1 - \theta_2}, & \text{if } \theta_1 \in [0; 1], \theta_2 \in (1; 2], \theta_3 = 0,
\end{cases}
\]

2) \[
\lim_{k \to \infty} T^{(2k)}(x_0) = \lim_{k \to \infty} x^{(2k)} = x^{(0)}, \quad \text{if } \theta_1 = 0, \theta_2 = 1, \theta_3 = 1.
\]

3) \[
\lim_{k \to \infty} T^{(2k+1)}(x_0) = \lim_{k \to \infty} x^{(2k+1)} = a - x^{(0)}, \quad \text{if } \theta_1 = 0, \theta_2 = 1, \theta_3 = 1.
\]

Proof. Follows from Lemma 2 taking into account the graph of the mapping \( S^3 \). \qed

Theorem 2. Let \( z^0 = (x_1^0, x_2^0, y_1^0, y_2^0) \in S^2 \) be an initial point then

1) \[
\lim_{m \to \infty} W^{(m)}(z_0) = \begin{cases}
(0, a, 0, 1-a), & \text{if } \theta_1 \in [0; 1), \theta_2 \in [0; 1], \theta_3 = 0, \\
(a, 0, 1-a, 0), & \text{if } \theta_1 = 1, \theta_2 \in [1; 2], \theta_3 \in (0; 1), \\
(x^*, a-x^*, \beta x^*, \beta(a-x^*)), & \text{if } \theta_1 \in [0; 1), \theta_2 \in [0; 2], \theta_3 \in (0; 1), \\
\frac{(1-\theta_3) a}{\theta_3 - \theta_2 + 1}, & \text{if } \theta_1 = 1, \theta_2 \in [0; 1), \theta_3 \in (0; 1), \\
(0, a, 0, 1-a), & \text{if } \theta_1 = 1, \theta_2 \in [0; 1), \theta_3 = 0, \\
(a, 0, 1-a, 0), & \text{if } \theta_1 = 1, \theta_2 \in (1; 2), \theta_3 = 0, \\
\frac{(1-\theta_3) a}{\theta_1 - \theta_2}, & \text{if } \theta_1 \in [0; 1], \theta_2 \in (1; 2], \theta_3 = 0,
\end{cases}
\]

2) \[
\lim_{k \to \infty} W^{(2k)}(z^0) = (x_1^0, a-x_1^0, \beta x_1^0, \beta(a-x_1^0)), \quad \text{if } \theta_1 = 0, \theta_2 = 1, \theta_3 = 1, \\
\lim_{k \to \infty} W^{(2k+1)}(z^0) = (a-x_1^0, x_1^0, \beta(a-x_1^0), \beta x_1^0), \quad \text{if } \theta_1 = 0, \theta_2 = 1, \theta_3 = 1.
\]

Proof. Follows from Lemma 1, Lemma 2 and Proposition 1. \qed

4. An Example of an \( n \)-Dimensional Case

Now we consider the following constraint on heredity coefficients \( x(i) \)

\[
\theta_{ipk} = \begin{cases}
c_{ik}, & \text{if } p = j, \\
1, & \text{if } p \neq j, k = l, \quad j, l = 1, \ldots, n, \\
0, & \text{if } p \neq j, k \neq l,
\end{cases}
\]
Then operator defined by (2.6) has the following form

\[ U_a : \begin{cases} 
  x'_k = a^{-1} \sum_{i=1}^{n} c_{ik} x_i x_j, & k = 1, \ldots, n, \ k \neq l \\
  x'_l = a^{-1}(\sum_{i=1}^{n} c_{il} x_i x_j + \sum_{j=1}^{n} \sum_{j \neq p=1}^{n} x_i x_p).
\end{cases} \tag{4.1} \]

We will consider the following case

\[ c_{ik} = \begin{cases} 
  1, & i = k, \\
  0, & i \neq k,
\end{cases} \]

then the operator defined by (4.1) has the following form

\[ U : \begin{cases} 
  x'_k = a^{-1} x_k x_j, & k = 1, \ldots, n, \ k \neq l \\
  x'_l = a^{-1}(x_l x_j + \sum_{i=1}^{n} \sum_{j \neq p=1}^{n} x_i x_p). \end{cases} \tag{4.2} \]

Let us find all fixed points of \( U \) given by (4.2), i.e. we solve the following system of equations

\[ \begin{cases} 
  x_k = a^{-1} x_k x_j, & k = 1, \ldots, n, \ k \neq l \\
  x_l = a^{-1}(x_l x_j + \sum_{i=1}^{n} \sum_{j \neq p=1}^{n} x_i x_p), \end{cases} \tag{4.3} \]

From the first equation of the system (4.3) by \( k = j \) and \( k \neq l \) we get

\[ x_j, 1 = 0, \quad x_j, 2 = a. \]

Consequently, since \( \sum_{i=1}^{n} x_k = a \) we obtain

\[ x_{l,1} = a, \quad x_{l,2} = 0. \]

Thus we have proved the following

**Proposition 2.**

1) If \( j = l \) then the operator (4.2) has unique fixed point

\[ x = (0, \ldots, 0, a, 0, \ldots, 0); \]

2) If \( j \neq l \) then (4.2) has two fixed points

\[ x_{1,l} = (0, \ldots, 0, a, 0, \ldots, 0), \quad x_{2,j} = (0, \ldots, 0, a, 0, \ldots, 0). \]

**Proposition 3.** Let \( x^{(0)} = (x^{(0)}_1, \ldots, x^{(0)}_n) \in S^{n-1} \) be an initial point.

1) If \( j = l \) then

\[ \lim_{m \to \infty} U^m(x^{(0)}) = \lim_{m \to \infty} x^{(m)} = x = (0, \ldots, 0, a, 0, \ldots, 0); \]

2) If \( j \neq l \) then

\[ \lim_{m \to \infty} U^m(x^{(0)}) = \lim_{m \to \infty} x^{(m)} = \begin{cases} 
  x_{1,l}, & \text{if } x^{(0)}_j \neq a, \\
  x_{2,j}, & \text{if } x^{(0)}_j = a.
\end{cases} \]
Proof. 1) Let \( j = l \). From (4.2) we get

\[
U^{(m+1)}(x^{(0)}) : \begin{cases}
  x_k^{(m+1)} = a^{-1} x_k^{(m)} x_l^{(m)}, & k = 1, \ldots, n, \quad k \neq l \\
  x_l^{(m+1)} = a^{-1} x_l^{(m)} + \sum_{i=1}^{n} x_i^{(m)} x_p^{(m)}. 
\end{cases}
\]  

(4.4)

Since \( \sum_{i=1}^{n} x_i^{(m)} = a, \quad x_l^{(m)} \in [0, a] \) we have

\[
x_l^{(m+1)} = a^{-1} x_l^{(m)} + \sum_{i=1}^{n} \sum_{i \neq p=1}^{n} x_i^{(m)} x_p^{(m)} =
\]

\[
a - a^{-1} x_l^{(m)} (a - x_l^{(m)}) \geq a - a^{-1} a(a - x_l^{(m)}) = x_l^{(m)},
\]

\[
x_k^{(m+1)} = a^{-1} x_k^{(m)} x_l^{(m)} \leq a^{-1} x_k^{(m)} a = x_k^{(m)}, \quad k = 1, \ldots, n, \quad k \neq l.
\]

Thus \( x_l^{(m)} \) is a non-decreasing sequence, which bounded from above by \( a \) the sequence \( x_k^{(m)}, k = 1, \ldots, n, k \neq l \) is a non-increasing and with lower bound 0. Consequently, each \( x_i^{(m)} \) has a limit say \( \alpha_i, \quad i = 1, 2, \ldots, n. \)

From equations of (4.4) for limit values \( \alpha_1 \) we get the following equations

\[
\begin{align*}
  \alpha_l &= a - a^{-1} \alpha_l (a - \alpha_l), \\
  \alpha_k &= a^{-1} \alpha_k \alpha_l, \quad k = 1, \ldots, n, \quad k \neq l, \\
  \sum_{i=1}^{n} \alpha_i &= a
\end{align*}
\]

(4.5)

It is easy to see that the system (4.5) has the following solution:

\[
\alpha_l = a, \quad \alpha_k = 0, \quad k = 1, \ldots, n, \quad k \neq l.
\]

2) Let \( j \neq l \). From (4.2) we get

\[
U^{(m+1)}(x^{(0)}) : \begin{cases}
  x_k^{(m+1)} = a (a^{-2} x_k^{(0)} x_j^{(0)}) 2^m, & k = 1, \ldots, n, \quad k \neq l \\
  x_l^{(m+1)} = a (1 - (a^{-1} x_j^{(0)}) 2^m (a - x_j^{(0)}) x_j^{(0)}).
\end{cases}
\]  

(4.6)

It is easy to see that the system (4.6) has the following limit:

if \( x_j^{(0)} = a \) then

\[
\lim_{m \to \infty} U^{(m+1)}(x^{(0)}) = \lim_{m \to \infty} x_k^{(m+1)} = \begin{cases}
  a, & \text{if } k = j, \\
  0, & \text{if } k = 1, \ldots, n, \quad k \neq j
\end{cases}
\]

if \( x_j^{(0)} \neq a \) then

\[
\lim_{m \to \infty} U^{(m+1)}(x^{(0)}) = \lim_{m \to \infty} x_k^{(m+1)} = \begin{cases}
  a, & \text{if } k = l, \\
  0, & \text{if } k = 1, \ldots, n, \quad k \neq l
\end{cases}
\]

\( \Box \)

Summarizing we obtain the following
Theorem 3. Let $z^{(0)} = (x_1^{(0)}, ..., x_n^{(0)}; y_1^{(0)}, ..., y_n^{(0)}) \in S^{2n-2}_a$ be an initial point.

1) If $j = l$ then
$$
\lim_{m \to \infty} W^{(m)}(x^{(0)}) = \lim_{m \to \infty} z^{(m)} = (0, ..., 0, a, 0, 0, 0, 0, 1 - a, 0, ..., 0)
$$

2) If $j \neq l$ then
$$
\lim_{m \to \infty} W^{(m)}(x^{(0)}) = \lim_{m \to \infty} z^{(m)} =
\begin{cases}
  z_{1,l} = (0, ..., 0, a, 0, 0, 0, 0, l, n-l, l, n-l, 0, 0, 0), & \text{if } x_j^{(0)} \neq a, \\
  z_{2,j} = (0, ..., 0, a, 0, 0, 0, 0, l, n-l, l, n-l, 0, 0, 0), & \text{if } x_j^{(0)} = a,
\end{cases}
$$

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