Generalizations of Quantum Mechanics

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Abstract

We review realistic models that reproduce quantum theory in some limit and yield potentially new physics outside that limit. In particular, we consider deterministic hidden-variables theories (such as the pilot-wave model) and their extension to “quantum nonequilibrium,” and we consider the continuous spontaneous localization model of wave function collapse. Other models are briefly discussed.

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1 Introduction

According to the so-called “Copenhagen Interpretation,” standard quantum theory is limited to describing experimental situations. It is at once remarkably successful in its predictions, and remarkably ill-defined in its conceptual structure: what is an experiment? what physical objects do or do not require quantization? how are the states realized in nature to be characterized? how and when is the wave-function “collapse postulate” to be invoked? Because of its success, one may suspect that quantum theory can be promoted from a theory of measurement to a theory of reality. But, that requires there to be an unambiguous specification (S) of the possible real states of Nature and their probabilities of being realized.

There are several approaches that attempt to achieve S. The more conservative approaches (consistent histories, environmental decoherence, many worlds) do not produce any predictions that differ from the standard ones because they do not tamper with the usual basic mathematical formalism. Rather, they utilize structures compatible with standard quantum theory to elucidate S. These approaches, which will not be discussed in this article, have arguably been less successful so far at achieving S than approaches that introduce significant alterations to quantum theory.

This article will largely deal with the two most well-developed realistic models that reproduce quantum theory in some limit and yield potentially new and testable physics outside that limit. First, the pilot-wave model, which will be discussed in the broader context of “hidden-variables theories.” Second, the continuous spontaneous localization (CSL) model, which describes wave function collapse as a physical process. Other related models will also be discussed briefly.

Due to bibliographic space limitations, this article contains a number of uncited references, of the form “(author) in (year).” Those in the next section can be found in Valentini (2002b, 2004a,b) or at www.arxiv.org. Those in subsequent sections can be found in Adler (2004), Bassi and Ghirardi (2003), Pearle (1999) (or in subsequent papers by these authors, or directly, at www.arxiv.org), and in Wallstrom (1994).

2 Hidden Variables and Quantum Nonequilibrium

A deterministic hidden-variables theory defines a mapping \( \omega = \omega(M, \lambda) \) from initial hidden parameters \( \lambda \) (defined, e.g., at the time of preparation of a quantum state) to final outcomes \( \omega \) of quantum measurements. The mapping depends on macroscopic experimental settings \( M \), and fixes the outcome for each run of the experiment. Bell’s theorem of 1964 shows that, for entangled quantum states of widely separated systems, the mapping must be nonlocal: some outcomes for (at least) one system must depend on the setting for another distant system.

In a viable theory, the statistics of quantum measurement outcomes – over an ensemble of experimental trials with fixed settings \( M \) – will agree with quantum theory for some special distribution \( \rho_{QT}(\lambda) \) of hidden variables. For example, expectation values will coincide with the predictions of the Born rule

\[
\langle \omega \rangle_{QT} = \int d\lambda \, \rho_{QT}(\lambda) \omega(M, \lambda) = \text{Tr}(\hat{\rho} \hat{\Omega})
\]

for an appropriate density operator \( \hat{\rho} \) and Hermitian observable \( \hat{\Omega} \). (As is customary in this context, \( \int d\lambda \) is to be understood as a generalised sum.) However, given the mapping \( \omega = \omega(M, \lambda) \) for individual trials, one may, in principle, consider nonstandard distributions \( \rho(\lambda) \neq \rho_{QT}(\lambda) \) that yield statistics outside the domain of ordinary quantum theory (Valentini 1991, 2002a). We may say that such distributions correspond to a state of quantum nonequilibrium.

Quantum nonequilibrium is characterised by the breakdown of a number of basic quantum constraints. In particular, nonlocal signals appear at the statistical level. We shall first illustrate this for the hidden-variables model of de Broglie and Bohm. Then we shall generalize the discussion to all (deterministic) hidden-variables theories.
At present there is no experimental evidence for quantum nonequilibrium in nature. How-
however, from a hidden-variables perspective, it is natural to explore the theoretical properties of
nonequilibrium distributions, and to search experimentally for the statistical anomalies associ-
ated with them.

From this point of view, quantum theory is a special case of a wider physics, much as thermal
physics is a special case of a wider (nonequilibrium) physics. (The special distribution \( \rho_{QT}(\lambda) \)
is analogous to, say, Maxwell’s distribution of molecular speeds.) Quantum physics may be
compared with the physics of global thermal equilibrium, which is characterised by constraints –
such as the impossibility of converting heat into work (in the absence of temperature differences)
– that are not fundamental but contingent on the state. Similarly, quantum constraints such as
statistical locality (the impossibility of converting entanglement into a practical signal) are seen
as contingencies of \( \rho_{QT}(\lambda) \).

2.1 Pilot-Wave Theory

The de Broglie-Bohm “pilot-wave theory” – as it was originally called by de Broglie, who first
presented it at the Fifth Solvay Congress in 1927 – is the classic example of a deterministic
hidden-variables theory of broad scope (Bohm 1952; Bell 1987; Holland 1993). We shall use it
to illustrate the above ideas. Later, the discussion will be generalised to arbitrary theories.

In pilot-wave dynamics, an individual closed system with (configuration-space) wave function
\( \Psi(X, t) \) satisfying the Schrödinger equation
\[
\frac{i\hbar}{\partial t} \frac{\partial \Psi}{\partial t} = \hat{H} \Psi
\]
has an actual configuration \( X(t) \) with velocity
\[
\dot{X}(t) = \frac{J(X, t)}{|\Psi(X, t)|^2}
\]
where \( J = J[\Psi] = J(X, t) \) satisfies the continuity equation
\[
\frac{\partial |\Psi|^2}{\partial t} + \nabla \cdot J = 0
\]
(which follows from (1)). In quantum theory, \( J \) is the “probability current”. In pilot-wave theory,
\( \Psi \) is an objective physical field (on configuration space) guiding the motion of an individual
system.

Here, the objective state (or ontology) for a closed system is given by \( \Psi \) and \( X \). A probability
distribution for \( X \) – discussed below – completes an unambiguous specification \( S \) (as mentioned
in the introduction).

Pilot-wave dynamics may be applied to any quantum system with a locally conserved current
in configuration space. Thus, \( X \) may represent a many-body system, or the configuration of a
continuous field, or perhaps some other entity.

For example, at low energies, for a system of \( N \) particles with positions \( x_i(t) \) and masses \( m_i \),
\( i = 1, 2, \ldots, N \), with an external potential \( V \), the \( (1) \) (with \( X \equiv (x_1, x_2, \ldots, x_N) \)) reads
\[
\frac{i\hbar}{\partial t} \frac{\partial \Psi}{\partial t} = \sum_{i=1}^{N} -\frac{\hbar^2}{2m_i} \nabla^2_i \Psi + V \Psi
\]
while \( (2) \) has components
\[
\frac{dx_i}{dt} = \frac{\hbar}{m_i} \text{Im} \left( \frac{\nabla_i \Psi}{\Psi} \right) = \frac{\nabla_i S}{m_i}
\]
(where \( \Psi = |\Psi| e^{(i/\hbar)S} \)).
In general, (1) and (2) determine $X(t)$ for an individual system, given the initial conditions $X(0), \Psi(X,0)$ at $t = 0$. For an arbitrary initial distribution $P(X,0)$, over an ensemble with the same wave function $\Psi(X,0)$, the evolution $P(X,t)$ of the distribution is given by the continuity equation

$$\frac{\partial P}{\partial t} + \nabla \cdot (P\dot{X}) = 0$$

(6)

The outcome of an experiment is determined by $X(0), \Psi(X,0)$, which may be identified with $\lambda$. For an ensemble with the same $\Psi(X,0)$, we have $\lambda = X(0)$.

2.1.1 Quantum Equilibrium

From (3) and (6), if we assume $P(X,0) = |\Psi(X,0)|^2$ at $t = 0$, we obtain $P(X,t) = |\Psi(X,t)|^2$ – the Born-rule distribution of configurations – at all times $t$.

Quantum measurements are, like any other process, described and explained in terms of evolving configurations. For measurement devices whose pointer readings reduce to configurations, the distribution of outcomes of quantum measurements will match the statistical predictions of quantum theory (Bohm 1952; Bell 1987; Dürr, Goldstein and Zanghì 2003). Thus quantum theory emerges phenomenologically for a “quantum equilibrium” ensemble with distribution $P(X,t) = |\Psi(X,t)|^2$ (or $\rho(\lambda) = \rho_{QT}(\lambda)$).

2.1.2 Quantum Nonequilibrium

In principle, as we saw for general hidden-variables theories, we may consider a nonequilibrium distribution $P(X,0) \neq |\Psi(X,0)|^2$ of initial configurations while retaining the same deterministic dynamics (1), (2) for individual systems (Valentini 1991). The time evolution of $P(X,t)$ will be determined by (6).

As we shall see, in appropriate circumstances (with a sufficiently complicated velocity field $\dot{X}$), (6) generates relaxation $P \to |\Psi|^2$ on a coarse-grained level, much as the analogous classical evolution on phase space generates thermal relaxation. But for as long as the ensemble is in nonequilibrium, the statistics of outcomes of quantum measurements will disagree with quantum theory.

Quantum nonequilibrium may have existed in the very early universe, with relaxation to equilibrium occurring soon after the big bang. Thus, a hidden-variables analogue of the classical thermodynamic “heat death of the universe” may have actually taken place (Valentini 1991). Even so, relic cosmological particles that decoupled sufficiently early could still be in nonequilibrium today, as suggested by Valentini in 1996 and 2001. It has also been speculated that nonequilibrium could be generated in systems entangled with degrees of freedom behind a black-hole event horizon (Valentini 2004a).

Experimental searches for nonequilibrium have been proposed. Nonequilibrium could be detected by the statistical analysis of random samples of particles taken from a parent population of (for example) relics from the early universe. Once the parent distribution is known, the rest of the population could be used as a resource, to perform tasks that are currently impossible (Valentini 2002b).

2.2 $H$-Theorem: Relaxation to Equilibrium

Before discussing the potential uses of nonequilibrium, we should first explain why all systems probed so far have been found in the equilibrium state $P = |\Psi|^2$. This distribution may be accounted for along the lines of classical statistical mechanics, noting that all currently-accessible systems have had a long and violent astrophysical history.

Dividing configuration space into small cells, and introducing coarse-grained quantities $\bar{P}, |\Psi|^2$, a general argument for relaxation $\bar{P} \to |\Psi|^2$ is based on an analog of the classical coarse-
graining $H$-function. The coarse-grained $H$-function

$$H = \int dX \, P \ln(\hat{P}/|\Psi|^2)$$

(minus the relative entropy of $\hat{P}$ with respect to $|\Psi|^2$) obeys the $H$-theorem (Valentini 1991)

$$\dot{H}(t) \leq H(0)$$

(assuming no initial fine-grained microstructure in $P$ and $|\Psi|^2$). Here, $\dot{H} \geq 0$ for all $\hat{P}$, $|\Psi|^2$ and $\dot{H} = 0$ if and only if $\hat{P} = |\Psi|^2$ everywhere.

The $H$-theorem expresses the fact that $P$ and $|\Psi|^2$ behave like two “fluids” that are “stirred” by the same velocity field $X$, so that $P$ and $|\Psi|^2$ tend to become indistinguishable on a coarse-grained level. Like its classical analog, the theorem provides a general understanding of how equilibrium is approached, while not proving that equilibrium is actually reached. (And of course, for some simple systems – such as a particle in the ground state of a box, for which the velocity field $\nabla S/m$ vanishes – there is no relaxation at all.) A strict decrease of $\dot{H}(t)$ immediately after $t = 0$ is guaranteed if $\dot{X}_0 \cdot \nabla (P_0/|\Psi_0|^2)$ has non-zero spatial variance over a coarse-graining cell, as shown by Valentini in 1992 and 2001.

A relaxation timescale $\tau$ may be defined by $1/\tau^2 \equiv - (d^2H/dt^2)/\bar{H}_0$. For a single particle with quantum energy spread $\Delta E$, a crude estimate given by Valentini in 2001 yields $\tau \sim (1/\varepsilon)\hbar^2/m^{1/2}(\Delta E)^{3/2}$, where $\varepsilon$ is the coarse-graining length. For wave functions that are superpositions of many energy eigenfunctions, the velocity field (generally) varies rapidly, and detailed numerical simulations (in two dimensions) show that relaxation occurs with an approximately exponential decay $\dot{H}(t) \approx \bar{H}_0 e^{-t/\tau}$, with a time constant $\tau_c$ of order $\tau$ (Valentini and Westman 2005).

Equilibrium is then to be expected for particles emerging from the violence of the big bang. The possibility is still open that relics from very early times may not have reached equilibrium before decoupling.

### 2.3 Nonlocal Signaling

We now show how nonequilibrium, if it were ever discovered, could be used for nonlocal signaling.

Pilot-wave dynamics is nonlocal. For a pair of particles $A$, $B$ with entangled wave function $\Psi(x_A, x_B, t)$, the velocity $\dot{x}_A(t) = \nabla_A S(x_A, x_B, t/m_A$ of $A$ depends instantaneously on $x_B$, and local operations at $B$ – such as switching on a potential – instantaneously affect the motion of $A$. For an ensemble $P(x_A, x_B, t) = |\Psi(x_A, x_B, t)|^2$, local operations at $B$ have no statistical effect at $A$: the individual nonlocal effects vanish upon averaging over an equilibrium ensemble.

Nonlocality is (generally) hidden by statistical noise only in quantum equilibrium. If instead $P(x_A, x_B, 0) \neq |\Psi(x_A, x_B, 0)|^2$, a local change in the Hamiltonian at $B$ generally induces an instantaneous change in the marginal $p_A(x_A, t) \equiv \int d^3x_B \, P(x_A, x_B, t)$ at $A$. For example, in one dimension a sudden change $\dot{H}_B \rightarrow \dot{H}'_B$ in the Hamiltonian at $B$ induces a change $\Delta p_A \equiv p_A(x_A, t) - p_A(x_A, 0)$ (for small $t$) (Valentini 1991),

$$\Delta p_A = -\frac{\tau^2}{4m} \frac{\partial}{\partial x_A} \left( a(x_A) \int d^3x_B \, b(x_B) \, P(x_A, x_B, 0) - |\Psi(x_A, x_B, 0)|^2 \right)$$

(8)

(Here $m_A = m_B = m$, $a(x_A)$ depends on $\Psi(x_A, x_B, 0)$, while $b(x_B)$ also depends on $\dot{H}_B$ and vanishes if $\dot{H}_B = \bar{H}_B$). The signal is generally nonzero if $P_0 \neq |\Psi_0|^2$.

Nonlocal signals do not lead to causal paradoxes if, at the hidden-variable level, there is a preferred foliation of spacetime with a time parameter that defines a fundamental causal sequence. Such signals, if they were observed, would define an absolute simultaneity as discussed by Valentini in 1992 and 2005. Note that in pilot-wave field theory, Lorentz invariance emerges as a phenomenological symmetry of the equilibrium state, conditional on the structure of the field-theoretical Hamiltonian (as discussed by Bohm and Hiley in 1984, Bohm, Hiley and Kaloyerou in 1987, and Valentini in 1992 and 1996).
2.4 Subquantum Measurement

In principle, nonequilibrium particles could also be used to perform “subquantum measurements” on ordinary, equilibrium systems. We illustrate this with an exactly solvable one-dimensional model (Valentini 2002b).

Consider an apparatus “pointer” coordinate $y$, with known wave function $g_0(y)$ and known (ensemble) distribution $\pi_0(y) \neq |g_0(y)|^2$, where $\pi_0(y)$ has been deduced by statistical analysis of random samples from a parent population with known wave function $g_0(y)$. (We assume that relaxation may be neglected: for example, if $g_0$ is a box ground state, $\dot{y} = 0$ and $\pi_0(y)$ is static.) Consider also a “system” coordinate $x$ with known wave function $\psi_0(x)$ and known distribution $\rho_0(x) = |\psi_0(x)|^2$. If $\pi_0(y)$ is arbitrarily narrow, $x_0$ can be measured without disturbing $\psi_0(x)$, to arbitrary accuracy (violating the uncertainty principle).

To do this, at $t = 0$ we switch on an interaction Hamiltonian $\hat{H} = a\hat{x}\hat{p}_y$, where $a$ is a constant and $p_y$ is canonically conjugate to $y$. For relatively large $a$, we may neglect the Hamiltonians of $x$ and $y$. For $\Psi = \Psi(x,y,t)$, we then have $\partial \Psi / \partial t = -a x \partial \Psi / \partial y$. For $|\Psi|^2$ we have the continuity equation $\partial |\Psi|^2 / \partial t = -a x \partial |\Psi|^2 / \partial y$, which implies the hidden-variable velocity fields $\dot{x} = 0$, $\dot{y} = ax$ and trajectories $x(t) = x_0$, $y(t) = y_0 + ax_0 t$.

The initial product $\Psi_0(x,y) = \psi_0(x)g_0(y)$ evolves into $\Psi(x,y,t) = \psi_0(x)g_0(y - ax t)$. For $at \to 0$ (with a large but fixed), $\Psi(x,y,t) \to \psi_0(x)g_0(y)$ and $\psi_0(x)$ is undisturbed: for small $at$, a standard quantum pointer with the coordinate $y$ would yield negligible information about $x_0$. Yet, for arbitrarily small $at$, the hidden-variable pointer coordinate $y(t) = y_0 + ax_0 t$ does contain complete information about $x_0$ (and $x(t) = x_0$). This “subquantum” information will be visible to us if $\pi_0(y)$ is sufficiently narrow.

For, over an ensemble of similar experiments, with initial joint distribution $P_0(x,y) = |\psi_0(x)|^2 \pi_0(y)$ (equilibrium for $x$ and nonequilibrium for $y$), the continuity equation $\partial P / \partial t = -a x \partial P / \partial y$ implies that $P(x,y,t) = |\psi_0(x)|^2 \pi_0(y - ax t)$. If $\pi_0(y)$ is localised around $y = 0$ ($\pi_0(y) = 0$ for $|y| > w/2$), then a standard (faithful) measurement of $y$ with result $y_{\text{meas}}$ will imply that $x$ lies in the interval $(y_{\text{meas}}/at - w/2at, y_{\text{meas}}/at + w/2at)$ (so that $P(x,y,t) \neq 0$). Taking the simultaneous limits $at \to 0$, $w \to 0$, with $w/at \to 0$, the midpoint $y_{\text{meas}}/at \to x_0$ (since $y_{\text{meas}} = y_0 + ax_0 t$ and $|y_0| \leq w/2$), while the error $w/2at \to 0$. If $w$ is arbitrarily small, a sequence of such measurements will determine the hidden trajectory $x(t)$ without disturbing $\psi(x,t)$, to arbitrary accuracy.

2.5 Subquantum Information and Computation

From a hidden-variables perspective, immense physical resources are hidden from us by equilibrium statistical noise. Quantum nonequilibrium would probably be as useful technologically as thermal or chemical nonequilibrium.

2.5.1 Distinguishing Nonorthogonal States

In quantum theory, nonorthogonal states $|\psi_1\rangle$, $|\psi_2\rangle$ ($\langle \psi_1 | \psi_2 \rangle \neq 0$) cannot be distinguished without disturbing them. This theorem breaks down in quantum nonequilibrium (Valentini 2002b). For example, if $|\psi_1\rangle$, $|\psi_2\rangle$ are distinct states of a single spinless particle, then the associated de Broglie-Bohm velocity fields will in general be different, even if $\langle \psi_1 | \psi_2 \rangle \neq 0$, and so will the hidden-variable trajectories. Subquantum measurement of the trajectories could then distinguish the states $|\psi_1\rangle$, $|\psi_2\rangle$.

2.5.2 Breaking Quantum Cryptography

The security of standard protocols for quantum key distribution depends on the validity of the laws of quantum theory. These protocols would become insecure given the availability of nonequilibrium systems (Valentini 2002b).
The protocols known as BB84 and B92 depend on the impossibility of distinguishing nonorthogonal quantum states without disturbing them. An eavesdropper in possession of nonequilibrium particles could distinguish the nonorthogonal states being transmitted between two parties, and so read the supposedly secret key. Further, if subquantum measurements allow an eavesdropper to predict quantum measurement outcomes at each “wing” of a (bipartite) entangled state, then the EPR (Einstein-Podolsky-Rosen) protocol also becomes insecure.

2.5.3 Subquantum Computation

It has been suggested that nonequilibrium physics would be computationally more powerful than quantum theory, because of the ability to distinguish nonorthogonal states (Valentini 2002b). However, this ability depends on the (less-than-quantum) dispersion $w$ of the nonequilibrium ensemble. A well-defined model of computational complexity requires that the resources be quantified in some way. Here, a key question is how the required $w$ scales with the size of the computational task. So far, no rigorous results are known.

2.6 Extension to All Deterministic Hidden-Variables Theories

Let us now discuss arbitrary (deterministic) theories.

2.6.1 Nonlocal Signaling

Consider a pair of two-state quantum systems $A$ and $B$, which are widely separated and in the singlet state. Quantum measurements of observables $\hat{\sigma}_A \equiv \mathbf{m}_A \cdot \hat{\sigma}_A$, $\hat{\sigma}_B \equiv \mathbf{m}_B \cdot \hat{\sigma}_B$ (where $\mathbf{m}_A, \mathbf{m}_B$ are unit vectors in Bloch space and $\hat{\sigma}_A, \hat{\sigma}_B$ are Pauli spin operators) yield outcomes $\sigma_A, \sigma_B = \pm 1$, in the ratio $1:1$ at each wing, with a correlation $\langle \hat{\sigma}_A \hat{\sigma}_B \rangle = -\mathbf{m}_A \cdot \mathbf{m}_B$. Bell’s theorem shows that for a hidden-variables theory to reproduce this correlation – upon averaging over an equilibrium ensemble with distribution $\rho_{QB}(\lambda)$ – it must take the nonlocal form

$$\sigma_A = \sigma_A(\mathbf{m}_A, \mathbf{m}_B, \lambda), \quad \sigma_B = \sigma_B(\mathbf{m}_A, \mathbf{m}_B, \lambda) \quad (9)$$

More precisely, to obtain $\langle \sigma_A \sigma_B \rangle_{QT} = -\mathbf{m}_A \cdot \mathbf{m}_B$ (where $\langle \sigma_A \sigma_B \rangle_{QT} \equiv \int d\lambda \rho_{QT}(\lambda) \sigma_A \sigma_B$), at least one of $\sigma_A, \sigma_B$ must depend on the measurement setting at the distant wing. Without loss of generality, we assume that $\sigma_A$ depends on $\mathbf{m}_B$.

For an arbitrary nonequilibrium ensemble with distribution $\rho(\lambda) \neq \rho_{QT}(\lambda)$, in general $\langle \sigma_A \sigma_B \rangle \equiv \int d\lambda \rho(\lambda) \sigma_A \sigma_B$ differs from $-\mathbf{m}_A \cdot \mathbf{m}_B$, and the outcomes $\sigma_A, \sigma_B = \pm 1$ occur in a ratio different from $1:1$. Further, a change of setting $\mathbf{m}_B \rightarrow \mathbf{m}'_B$ at $B$ will generally induce a change in the outcome statistics at $A$, yielding a nonlocal signal at the statistical level. To see this, note that, in a nonlocal theory, the “transition sets”

$$T_A(-,+), \equiv \{ \lambda | \sigma_A(\mathbf{m}_A, \mathbf{m}_B, \lambda) = -1, \sigma_A(\mathbf{m}_A, \mathbf{m}'_B, \lambda) = +1 \}$$

$$T_A(+,-), \equiv \{ \lambda | \sigma_A(\mathbf{m}_A, \mathbf{m}_B, \lambda) = +1, \sigma_A(\mathbf{m}_A, \mathbf{m}'_B, \lambda) = -1 \}$$

cannot be empty for arbitrary settings. Yet, in quantum equilibrium, the outcomes $\sigma_A = \pm 1$ occur in the ratio $1:1$ for all settings, so the transition sets must have equal equilibrium measure, $\mu_{QT}[T_A(-,+)] = \mu_{QT}[T_A(+,-)]$ ($d\mu_{QT} \equiv \rho_{QT}(\lambda) d\lambda$). That is, the fraction of the equilibrium ensemble making the transition $\sigma_A = -1 \rightarrow \sigma_A = +1$ under $\mathbf{m}_B \rightarrow \mathbf{m}'_B$ must equal the fraction making the reverse transition $\sigma_A = +1 \rightarrow \sigma_A = -1$. (This “detailed balancing” is analogous to the principle of detailed balance in statistical mechanics.) Since $T_A(-,+), T_A(+,-)$ are fixed by the deterministic mapping, they are independent of the ensemble distribution $\rho(\lambda)$.

Thus, for $\rho(\lambda) \neq \rho_{QT}(\lambda)$, in general $\mu[T_A(-,+)] \neq \mu[T_A(+,-)]$ ($d\mu \equiv \rho(\lambda) d\lambda$): the fraction of the nonequilibrium ensemble making the transition $\sigma_A = -1 \rightarrow \sigma_A = +1$ will not in general balance the fraction making the reverse transition. The outcome ratio at $A$ will then change under $\mathbf{m}_B \rightarrow \mathbf{m}'_B$ and there will be an instantaneous signal at the statistical level from $B$ to $A$ (Valentini 2002a).
Thus, in any deterministic hidden-variables theory, nonequilibrium distributions \( \rho(\lambda) \neq \rho_{QT}(\lambda) \) generally allow entanglement to be used for nonlocal signaling (just as, in ordinary statistical physics, differences of temperature make it possible to convert heat into work).

### 2.6.2 Experimental Signature of Nonequilibrium

Quantum expectations are additive, \( \langle c_1 \hat{\Omega}_1 + c_2 \hat{\Omega}_2 \rangle = c_1 \langle \hat{\Omega}_1 \rangle + c_2 \langle \hat{\Omega}_2 \rangle \), even for noncommuting observables \( \{\hat{\Omega}_1, \hat{\Omega}_2\} \neq 0 \), with \( c_1, c_2 \) real. As emphasised by Bell in 1966, this seemingly trivial consequence of the (linearity of the) Born rule \( \langle \hat{\Omega} \rangle = \text{Tr}(\hat{\rho}\hat{\Omega}) \) is remarkable because it relates statistics from distinct, “incompatible” experiments. In nonequilibrium, such additivity generically breaks down (Valentini 2004b).

Further, for a two-state system with observables \( \mathbf{m} \cdot \hat{\sigma} \), the “dot-product” structure of the quantum expectation \( \langle \mathbf{m} \cdot \hat{\sigma} \rangle = \text{Tr}(\hat{\rho}\mathbf{m} \cdot \hat{\sigma}) = \mathbf{m} \cdot \mathbf{P} \) (for some Bloch vector \( \mathbf{P} \)) is equivalent to expectation additivity (Valentini 2004b). Nonadditive expectations then provide a convenient signature of nonequilibrium for any two-state system. For example, the sinusoidal modulation of the quantum transmission probability for a single photon through a polariser

\[
\rho_{QT}^+(\Theta) = \frac{1}{2} (1 + \langle \mathbf{m} \cdot \hat{\sigma} \rangle) = \frac{1}{2} (1 + \mathbf{P} \cos 2\Theta)
\]  

(where an angle \( \theta \) on the Bloch sphere corresponds to a physical angle \( \Theta = \theta/2 \)) will generically break down in nonequilibrium. Deviations from (10) would provide an unambiguous violation of quantum theory (Valentini 2004b).

Such deviations were searched for by Papaliolios in 1967, using laboratory photons and successive polarisation measurements over very short times, to test a hidden-variables theory (distinct from pilot-wave theory) due to Bohm and Bub (1966), in which quantum measurements generate nonequilibrium for short times. Experimentally, successive measurements over timescales \( \sim 10^{-13} \text{ sec} \) agreed with the (quantum) sinusoidal modulation \( \cos^2 \Theta \) to \( \lesssim 1\% \). Similar tests might be performed with photons of a more exotic origin.

### 3 Continuous Spontaneous Localization Model (CSL)

The basic postulate of CSL is that the state vector \( |\psi, t\rangle \) represents reality. Since, for example, in describing a measurement, the usual Schrödinger evolution readily takes a real state into a non-real state, that is, into a superposition of real states (such as apparatus states describing different experimental outcomes), CSL requires a modification of Schrödinger’s evolution. To the Hamiltonian is added a term which depends upon a classical randomly fluctuating field \( w(x,t) \) and a mass-density operator \( \hat{A}(x,t) \). This term acts to collapse a superposition of states, which differ in their spatial distribution of mass density, to one of these states. The rate of collapse is very slow for a superposition involving a few particles, but very fast for a superposition of macroscopically different states. Thus, very rapidly, what you see (in nature) is what you get (from the theory). Each state vector evolving under each \( w(x,t) \) corresponds to a realizable state, and a rule is given for how to associate a probability with each. In this way, an unambiguous specification \( S \) as mentioned in the introduction is achieved.

#### 3.1 Requirements for Stochastic Collapse Dynamics

Consider a normalized state vector \( |\psi, t\rangle = \sum_n \alpha_n(t) |a_n\rangle \) \((\langle a_n | a_{n'} \rangle = \delta_{nn'})\) which undergoes a stochastic dynamical collapse process. This means that, starting from the initial superposition at \( t = 0 \), for each run of the process, the squared amplitudes \( x_n(t) \equiv |\alpha_n(t)|^2 \) fluctuate until all but one vanish, that is, \( x_m(\infty) = 1, (x_{\neq m}(\infty) = 0) \) with probability \( x_m(0) \).

This may be achieved simply, assuming negligible effect of the usual Schrödinger evolution,
if the stochastic process enjoys the following properties (Pearle 1979):

\[ \sum_n x_n(t) = 1 \]  \hspace{1cm} (11a)

\[ x_n(t) = x_n(0) \]  \hspace{1cm} (11b)

\[ x_n(\infty)x_m(\infty) = 0 \quad \text{for} \quad m \neq n, \]  \hspace{1cm} (11c)

where the overbar indicates the ensemble average at the indicated time. The only way that a sum of products of non-negative terms can vanish is for at least one term in each product to vanish. Thus, according to (11a), for each run, at least one of each pair \{x_n(\infty), x_m(\infty)\} (n \neq m) must vanish. This means that at most one \(x_n(\infty)\) might not vanish and, by (11a), applied at \(t = \infty\), one \(x_n(\infty)\) must not vanish and, in fact, must equal 1: hence, each run produces collapse. Now, let the probability of the outcome \(\{x_n(\infty) = 1, x_{\neq n}(\infty) = 0\}\) be denoted \(P_n\).

Since \(x_n(\infty) = 1 \cdot P_n + \sum_{m \neq n} 0 \cdot P_m = P_n\) then, according to the Martingale property (11b), applied at \(t = \infty\), \(P_n = x_n(0)\); hence the ensemble of runs produces the probability postulated by the usual “collapse rule” of standard quantum theory.

A (nonquantum) stochastic process which obeys these equations is the gambler’s ruin game. Suppose one gambler initially possesses the fraction \(x\). Suppose one gambler initially possesses the fraction \(x\) and the other has the fraction \(x'\). They toss a coin: heads, a dollar goes from gambler 1 to gambler 2, tails the dollar goes the other way. (11a) is satisfied since the sum of money in the game remains constant, (11b) holds because it is a fair game, and (11c) holds because each game eventually ends. Thus, gambler \(i\) wins all the money with probability \(x_i(0)\).

### 3.2 CSL in Essence

Consider the (non-unitary) Schrödinger picture evolution equation

\[ |\psi, t\rangle_w = T e^{-\int_0^t dt' [i\hat{H} + (4\lambda)^{-1} |w(t') - 2\lambda \hat{A}|^2]} |\psi, 0\rangle, \]  \hspace{1cm} (12)

where \(\hat{H}\) is the usual Hamiltonian, \(w(t')\) is an arbitrary function of white noise class, \(\hat{A}\) is a Hermitian operator \((\hat{A} | a_n \rangle = a_n | a_n \rangle)\), \(\lambda\) is a collapse rate parameter, \(T\) is the time-ordering operator and \(\hbar = 1\). Associated with this, the probability rule

\[ P_t(w) Dw \equiv_w \langle \psi, t| \psi, t\rangle_w \frac{t/dt}{(2\pi \lambda/dt)^{1/2}} \]  \hspace{1cm} (13)

is defined, which gives the probability that nature chooses a noise which lies in the range \(\{w(t'), w(t') + dw(t')\}\) (for calculational purposes, time is discretized, with \(t_0 = 0\)).

Equations (12) and (13) contain the essential features of CSL, and are all that is needed to discuss the simplest collapse behavior. Set \(\hat{H} = 0\), so there is no competition between collapse and the usual Schrödinger evolution, and let the initial statevector be \(|\psi, 0\rangle = \sum_n a_n |a_n\rangle\).

Equations (12) and (13) become

\[ |\psi, t\rangle_w = \sum_n |a_n\rangle e^{-((4\lambda)^{-1} \int_0^t dt' |w(t') - 2\lambda a_n|^2} \]  \hspace{1cm} (14a)

\[ P_t(w) = \sum_n |a_n|^2 e^{-((2\lambda)^{-1} \int_0^t dt' |w(t') - 2\lambda a_n|^2} \]  \hspace{1cm} (14b)

When the un-normalized state vector in (14a) is divided by \(P_t^{1/2}(w)\) and so normalized, the squared amplitudes are

\[ x_n(t) = |a_n|^2 e^{-((2\lambda)^{-1} \int_0^t dt' |w(t') - 2\lambda a_n|^2} / P_t(w), \]

which are readily shown to satisfy (11a), (11b), and (11c) in the form \(x_n^{1/2}(\infty)x_m^{1/2}(\infty) = 0\) \((m \neq n)\) (which does not change the argument in section 3.1, but makes for an easier calculation). Thus, (14a), (14b) describe collapse dynamics.
To describe collapse to a joint eigenstate of a set of mutually commuting operators $\hat{A}^r$, replace $(4\lambda)^{-1}|w(t') - 2\lambda\hat{A}|^2$ in the exponent of $(13)$ by $\sum_i (4\lambda)^{-1}|w^r(t') - 2\lambda\hat{A}^r|^2$. The interaction picture state vector in this case is $(12)$ multiplied by $\exp(i\hat{H}t)$:

$$|\psi, t)_w = \mathcal{T} e^{-(4\lambda)^{-1} \int_{0}^{t} dt' \sum_i |w^r(t') - 2\lambda\hat{A}^r(t')|^2} |\psi, 0),$$

where $\hat{A}^r(t') \equiv \exp(i\hat{H}t')\hat{A}^r \exp(-i\hat{H}t')$. The density matrix follows from $(15)$, $(13)$:

$$\hat{\rho}(t) = \int P_t(w) D(w|\psi, t) w \langle \psi, t|/P_t(w) = \mathcal{T} e^{-\lambda/2 \int_{0}^{t} dt' \sum_i |\hat{A}_L(t') - \hat{A}_R(t')|^2} \hat{\rho}(0)$$

where $\hat{A}_L(t') (\hat{A}_R(t'))$ appears to the left (right) of $\hat{\rho}(0)$, and is time-ordered (time reverse-ordered). In the example described by $(14)$, the density matrix $(16)$ is

$$\hat{\rho}(t) = \sum_{n, m} e^{-\lambda t/2(a_n - a_m)^2} a_n a_m^* |a_n\rangle\langle a_m|,$$

which encapsulates the ensemble’s collapse behavior.

### 3.3 CSL

The CSL proposal (Pearle 1989) is that collapse is engendered by distinctions between states at each point of space, so the index $r$ of $\hat{A}^r$ in $(14)$ becomes $x$,

$$|\psi, t)_w = \mathcal{T} e^{-\lambda x} \int_{0}^{t} dt' dx' |w(x', t') - 2\lambda\hat{A}(x', t')|^2 |\psi, 0),$$

and the distinction looked at is mass density. However, one cannot make the choice $\hat{A}(x, 0) = \hat{M}(x)$, where $\hat{M}(x) = \sum m_i \hat{\xi}^i(x) \hat{\xi}_i(x)$ is the mass density operator ($m_i$ is the mass of the $i$th type of particle, so $m_e$, $m_p$, $m_n$... are the masses of electrons, protons, neutrons..., and $\hat{\xi}_i(x)$ is the creation operator for such a particle at location $x$), because this entails an infinite rate of energy increase of particles (23) with $a = 0$. Instead, adapting a “gaussian smearing” idea from the Ghirardi et al. (1986) spontaneous localization (SL) model (see section 3.6), choose $\hat{A}^x$ as, essentially, proportional to the mass in a sphere of radius $a$ about $x$:

$$\hat{A}(x, t) \equiv e^{i\hat{H}t} \frac{1}{(\pi a^2)^{3/4}} \int d\tau \frac{\hat{M}(\tau)}{m_p} e^{-(2a^2)^{-1}(x - \tau)^2} e^{-i\hat{H}t}$$

The parameter value choices of SL, $\lambda \approx 10^{-10} \text{sec}^{-1}$ (according to $(17)$, the collapse rate for protons) and $a \approx 10^{-7} \text{cm}$ are, so far, consistent with experiment (see section 3.4), and will be adopted here.

The density matrix associated with $(17)$ is, as in $(16)$,

$$\hat{\rho}(t) = \mathcal{T} e^{-\lambda t/2 \int_{0}^{t} dt' dx' |\hat{A}_L(x', t') - \hat{A}_R(x', t')|^2} \hat{\rho}(0),$$

which satisfies the differential equation

$$\frac{d\hat{\rho}(t)}{dt} = -\lambda \int dx' [\hat{A}(x', t), [\hat{A}(x', t), \hat{\rho}(t)]]$$

of Lindblad-Kossakowski form.

### 3.4 Consequences of CSL

Since the state vector dynamics of CSL is different from that of standard quantum theory, there are phenomena for which the two make different predictions, allowing for experimental tests. Consider an $N$-particle system with position operators $\hat{X}_i$ ($\hat{X}_i | x_i \rangle = x_i | x_i \rangle$). Substitution of
\[ \hat{A}(x') \] from (18) in the Schrödinger picture version of (20), integration over \( x' \), and utilization of
\[ f(z)M(z|x) = \sum_{i=1}^{N} m_i f(\hat{X}_i) \delta(z - \hat{X}_i)|x\]
results in
\[
\frac{d\hat{\rho}(t)}{dt} = -i[\hat{\rho}(t), \hat{H}] - \frac{\lambda}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} m_i m_j \rho_{ij} \exp\left[-(4a^2)^{-1}(\hat{X}_i - \hat{X}_j)^2\right] \approx \frac{\lambda N^2}{2} [\hat{\rho}(t)]^2.
\]

which is a useful form for calculations first suggested by Pearle and Squires in 1994.

### 3.4.1 Interference
Consider the collapse rate of an initial state \( \phi = \alpha_1 |1\rangle + \alpha_2 |2\rangle \), where \( |1\rangle \), \( |2\rangle \) describe a clump of matter, of size \( \ll a \), at different locations with separation \( \gg a \). Electrons may be neglected because of their small collapse rate compared to the much more massive nucleons, and the nucleon mass difference may be neglected. In using (21) to calculate \( d[1] \frac{d\rho(1)}{dt} \), since \( \exp[-(4a^2)^{-1}(\hat{X}_i - \hat{X}_j)^2] \approx 1 \) when acting on state \( |1\rangle \) or \( |2\rangle \), and \( \approx 0 \) when \( \hat{X}_i \) acts on \( |1\rangle \) and \( \hat{X}_j \) acts on \( |2\rangle \), (21) yields, for \( N \) nucleons, the collapse rate \( \lambda N^2 \):
\[
\frac{d[1]}{dt} \frac{d\rho(t)|2\rangle}{dt} = -i[1[\hat{\rho}(t), \hat{H}]|2\rangle - \lambda N^2 (1|\hat{\rho}(t)|2\rangle).
\]

If the clump undergoes a two-slit interference experiment, where the size and separation conditions above are satisfied for \( \Delta T \text{sec} \), and if the result agrees with the standard quantum theory prediction to 1%, it also agrees with CSL provided \( \lambda^{-1} > 100N^2\Delta T \). So far, interference experiments with \( N \) as large as \( \approx 10^4 \) have been performed by Nairz, Arndt and Zeilinger in 2000. The SL value of \( \lambda \) would be testable, that is, the quantum predicted interference pattern would be “washed out” to 1% accuracy, if the clump were an \( \approx 10^{-6}\text{cm} \) radius sphere of mercury, which contains \( N \approx 10^8 \) nucleons, interfered for \( \Delta T = .01\text{sec} \). Currently envisioned but not yet performed experiments (e.g., by Marshall, Simon, Penrose and Bouwmeester in 2003) have been analyzed (e.g., by Bassi, Ippoliti and Adler in 2004 and by Adler in 2005) involving a superposition of a larger clump of matter in slightly displaced positions, entangled with a photon whose interference pattern is measured: these proposed experiments are still too crude to detect the SL value of \( \lambda \), or the gravitationally-based collapse rate proposed by Penrose in 1996 (see section 4 and papers by Christian in 1999 and 2005).

### 3.4.2 Bound State Excitation
Collapse narrows wave packets, thereby imparting energy to particles. If \( \hat{H} = \sum_{i=1}^{N} \hat{P}_i^2/(2m_i) + V(x_1,...,x_N) \), it is straightforward to calculate from (21) that
\[
\frac{d}{dt}(\hat{H}) = \frac{d}{dt} \text{Tr}[\hat{H}\hat{\rho}(t)] = \sum_{i=1}^{N} \frac{3\hbar}{4m_i a^2}.
\]

For a nucleon, the mean rate of energy increase is quite small, \( \approx 3 \times 10^{-25}\text{eV/sec} \). However, deviations from the mean can be significantly greater.

For, (21) predicts excitation of atoms and nuclei. Let \( |E_0\rangle \) be an initial bound energy eigenstate. Expanding (21) in a power series in \( \text{bound state size}/a^2 \), the excitation rate of state \( |E\rangle \) is
\[
\Gamma \equiv \frac{d(E|\hat{\rho}(t)|E)}{dt}\bigg|_{t=0} = \frac{\lambda}{2a^2} \sum_{i=1}^{N} \frac{m_i}{m_p} \hat{X}_i|E_0\rangle \langle E_0|\langle E_0|E_0\rangle \sum_{i=1}^{N} \frac{m_i}{m_p} \hat{X}_i|E\rangle + \text{O(size/a)}^4.
\]
Since $|E_0\rangle$, $|E\rangle$ are eigenstates of the center of mass operator $\sum_{i=1}^{N} m_i \hat{X}_i / \sum_{i=1}^{N} m_i$ with eigenvalue 0, the dipole contribution explicitly given in (24) vanishes identically. This leaves the quadrupole contribution as the leading term, which is too small to be measured at present.

However, the choice of $\hat{A}(x)$ as mass density operator was made only after experimental indication. Let $g_i$ replace $m_i/m_p$ in (24), so that $\lambda g_i^2$ is the collapse rate for the $i$th particle. Then, experiments looking for the radiation expected from “spontaneously” excited atoms and nuclei, in large amounts of matter for a long time, as shown by Collett, Pearle, Avignone and Nussinov in 1995, Pearle, Ring, Collar and Avignone in 1999 and Jones, Pearle and Ring in 2004, have placed the following limits:

$$|g_e/g_p - m_e/m_p| < 12 m_e/m_p, \quad |g_n/g_p - m_n/m_p| < 3(m_n - m_p)/m_p.$$  

### 3.4.3 Random Walk

According to (17), (13), the center of mass wave packet, of a piece of matter of size $\approx a$ or smaller, containing $N$ nucleons, achieves equilibrium size $s$ in a characteristic time $\tau_s$, and undergoes a random walk through a root mean square distance $\Delta Q$:

$$s \approx \left[ \frac{a^2 \hbar}{\lambda m_p N^3} \right]^{1/4}, \quad \tau_s \approx \frac{N m_p a^2}{\hbar}, \quad \Delta Q \approx \frac{\hbar \lambda^{1/2} s^{3/2}}{m_p a}. \quad (25)$$

The results in (25) were obtained by Collett and Pearle in 2003. These quantitative results can be qualitatively understood as follows.

In time $\Delta t$, the usual Schrödinger equation expands a wave packet of size $s$ to $\approx s + (\hbar/N m_p s) \Delta t$. CSL collapse, by itself, narrows the wave packet to $\approx s[1 - \lambda N^2 (s/a)^2 \Delta t]$. The condition of no change in $s$ is the result quoted above. $\tau_s$ is the time it takes the Schrödinger evolution to expand a wavepacket near size $s$ to size $s$: $(\hbar/N m_p s) \tau_s \approx s$. The $t^{3/2}$ dependence of $\Delta Q$ arises because this is a random walk without damping (unlike Brownian motion, where $\Delta Q \sim t^{1/2}$). The mean energy increase $\approx \lambda N \hbar^2 a^{-2} t$ of (25) implies the root-mean-square velocity increase $\approx (\lambda N \hbar^2 a^{-2} t)^{1/2}$, whose product with $t$ is $\Delta Q$.

For example, a sphere of density 1gm/cc and radius $10^{-5}$cm has $s \approx 4 \times 10^{-7}$cm, $\tau_s \approx 0.6$sec and $\Delta Q \approx 5$[t in days]$^{3/2}$/cm. At the reported achieved low pressure of $5 \times 10^{-17}$Torr at 4.2$^\circ$K reported by Gabrielse’s group in 1990, the mean collision time with gas molecules is $\approx 80$min, over which $\Delta Q \approx 0.7$mm. Thus, observation of this effect should be feasible.

### 3.5 Further Remarks

It is possible to define energy for the $w(x,t)$ field so that total energy is conserved: as the particles gain energy, the $w$-field loses energy, as shown by Pearle in 2005.

Attempts to construct a special relativistic CSL-type model have not yet succeeded although Pearle in 1990, 1992, 1999, Ghirardi, Grassi and Pearle in 1990 and Nicrosini and Rimini in 2003 have made valiant attempts. The problem is that the white noise field $w(x,t)$ contains all wavelengths and frequencies, exciting the vacuum in lowest order in $\lambda$ to produce particles at the unacceptable rate of infinite energy/sec-cc. Collapse models which utilize a colored noise field $w$ have a similar problem in higher order. In 2005, Pearle suggested a “quasi-relativistic” model which reduces to CSL in the low speed limit.

CSL is a phenomenological model which describes dynamical collapse so as to achieve S. Besides needing decisive experimental verification, it needs identification of the $w(x,t)$ field with a physical entity.

Other collapse models which have been investigated are briefly described below.

### 3.6 Spontaneous Localization Model (SL)

The SL model of Ghirardi et al. (1986), although superseded by CSL, is historically important and conceptually valuable. Let $\hat{H} = 0$ for simplicity, and consider a single particle whose wave
function at time $t$ is $\psi(x,t)$. Over the next interval $dt$, with probability $1 - \lambda dt$, it does not change. With probability $\lambda dt$ it does change, by being “spontaneously localized” or “hit.” A hit means that the new (unnormalized) wavefunction suddenly becomes

$$\psi(x, t + dt) = \psi(x, t)(\pi a^2)^{-3/4} e^{-(2a^2)^{-1}(x-z)^2}$$

with probability $\lambda dt dz \int dx |\psi(x, t + dt)|^2$.

Thus $z$, the “center” of the hit, is most likely to be located where the wavefunction is large. For a single particle in the superposition described in section 3.4.1, a single hit is overwhelmingly likely to reduce the wave function to one or the other location, with total probability $|\alpha_i|^2$, at the rate $\lambda$.

For an $N$ particle clump, it is considered that each particle has the same independent probability, $\lambda dt$, of being hit. But, for the example in section 3.4.1, a single hit on any particle in one location of the clump has the effect of multiplying the wave function part describing the clump in the other location by the tail of the gaussian, thereby collapsing the wave function at the rate $\lambda N$.

By use of the gaussian hit rather than a delta function hit, SL solves the problem of giving too much energy to particles as mentioned in section 3.3. It also solves the problem of achieving a slow collapse rate for a superposition of small objects and a fast collapse rate for a superposition of large objects. However, the SL hits on individual particles destroys the (anti-) symmetry of wave functions. The CSL collapse toward mass density eigenstates removes that problem. Also, while SL modifies the Schrödinger evolution of a wave function, it involves discontinuous dynamics and so is not described by a modified Schrödinger equation as is CSL.

4 Other Models

For a single (low-energy) particle, the polar decomposition $\Psi = Re^{i/\hbar}S$ of the Schrödinger equation implies two real equations,

$$\frac{\partial R^2}{\partial t} + \nabla \cdot \left( R^2 \frac{\nabla S}{m} \right) = 0 \quad (26)$$

(the continuity equation for $R^2 = |\Psi|^2$) and

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V + Q = 0 \quad (27)$$

where $Q \equiv -(\hbar^2/2m)\nabla^2 R/R$ is the “quantum potential”. (These equations have an obvious generalisation to higher-dimensional configuration space.) In 1926, Madelung proposed that one should start from (26) and (27) – regarded as hydrodynamical equations for a classical charged fluid with mass density $mR^2$ and fluid velocity $\nabla S/m$ – and construct $\Psi = Re^{i/\hbar}S$ from the solutions.

This “hydrodynamical” interpretation suffers from many difficulties, especially for many-body systems. In any case, a criticism by Wallstrom (1994) seems decisive: (26) and (27) (and their higher-dimensional analogs) are not, in fact, equivalent to the Schrödinger equation. For, as usually understood, the quantum wave function $\Psi$ is a single-valued and continuous complex field, which typically possesses nodes ($\Psi = 0$), in the neighborhood of which the phase $S$ is multivalued, with values differing by integral multiples of $2\pi \hbar$. If one allows $S$ in (26), (27) to be multi-valued, there is no reason why the allowed values should differ by integral multiples of $2\pi \hbar$, and in general $\Psi$ will not be single-valued. On the other hand, if one restricts $S$ in (26), (27) to be single-valued, one will exclude wave functions – such as those of nonzero angular momentum – with a multivalued phase. (This problem does not exist in pilot-wave theory as we have presented it here, where $\Psi$ is regarded as a basic entity.)

Stochastic mechanics, introduced by Fényes in 1952 and Nelson (1966), has particle trajectories $x(t)$ obeying a “forward” stochastic differential equation $dx(t) = b(x(t), t)dt + dw(t)$, where
\(b\) is a drift (equal to the mean forward velocity) and \(w\) a Wiener process, and also a similar “backward” equation. Defining the “current velocity” \(v = \frac{1}{2}(b + b^*),\) where \(b^*\) is the mean backward velocity, and using an appropriate time-symmetric definition of mean acceleration, one may impose a stochastic version of Newton’s second law. If one assumes, in addition, that \(v\) is a gradient (\(v = \nabla S/m\) for some \(S\)), then one obtains (26), (27) with \(R \equiv \sqrt{\rho},\) where \(\rho\) is the particle density. Defining \(\Psi \equiv \sqrt{\rho}e^{i\phi/m}\), it appears that one recovers the Schrödinger equation for the derived quantity \(\Psi\). However, again, there is no reason why \(S\) should have the specific multivalued structure required for the phase of a single-valued complex field. It then seems that, despite appearances, quantum theory cannot in fact be recovered from stochastic mechanics (Wallstrom 1994). The same problem occurs in models that use stochastic mechanics as an intermediate step (e.g., Markopoulou and Smolin in 2004): the Schrödinger equation is obtained only for exceptional, nodeless wave functions.

Bohm and Bub (1966) first proposed dynamical wave function collapse through deterministic evolution. Their collapse outcome is determined by the value of a Wiener-Siegel hidden variable (a variable distributed uniformly over the unit hypersphere in a Hilbert space identical to that of the statevector). In 1976, Pearle proposed dynamical wave function collapse equations where the collapse outcome is determined by a random variable, and suggested (Pearle 1979) that the modified Schrödinger equation be formulated as an Itô stochastic differential equation, a suggestion which has been widely followed. (The equation for the state vector given here, which is physically more transparent, has its time derivative equivalent to a Stratonovich stochastic differential equation, which is readily converted to the Itô form.) The importance of having the density matrix describing collapse be of the Lindblad-Kossakowski form was emphasized by Gisin in 1984 and Diosi in 1988. The stochastic differential Schrödinger equation which achieves this was found independently by Diosi in 1988 and by Belavkin, Gisin and Pearle in separate papers in 1989 (see also Ghirardi et al. 1990).

A gravitationally motivated stochastic collapse dynamics was proposed by Diosi in 1989 (and somewhat corrected by Ghirardi et al. in 1990). Penrose emphasized in 1996 that a quantum state, such as that describing a mass in a superposition of two places, puts the associated space-time geometry also in a superposition, and has argued that this should lead to wave function collapse. He suggests that the collapse time should be \(\sim \hbar/\Delta E,\) where \(\Delta E\) is the gravitational potential energy change obtained by actually displacing two such masses: for example, the collapse time \(\approx \hbar/(Gm^2/R),\) where the mass is \(m,\) its size is \(R,\) and the displacement is \(\approx R\) or larger. No specific dynamics is offered, just the vision that this will be a property of a correct future quantum theory of gravity.

Collapse to energy eigenstates was first proposed by Bedford and Wang in 1975 and 1977 and, in the context of stochastic collapse (e.g., (11) with \(\hat{A} = \hat{H}\)), by Milburn in 1991 and Hughston in 1996, but it has been argued by Finkelstein in 1993 and Pearle in 2004 that such energy-driven collapse cannot give a satisfactory picture of the macroscopic world. Percival in 1995 and in a 1998 book, and Fivel in 1997 have discussed energy-driven collapse for microscopic situations.

Adler (2004) has presented a classical theory (a hidden-variables theory) from which it is argued that quantum theory “emerges” at the ensemble level. The classical variables are \(N \times N\) matrix field amplitudes at points of space. They obey appropriate classical Hamiltonian dynamical equations which he calls “trace dynamics,” since the Hamiltonian, Lagrangian, Poisson bracket, etc. expressions have the form of the trace of products of matrices and their sums with constant coefficients. Using classical statistical mechanics, canonical ensemble averages of (suitably projected) products of fields are analyzed and it is argued that they obey all the properties associated with Wightman functions, from which quantum field theory, and its non-relativistic limit quantum mechanics, may be derived. As well as obtaining the algebra of quantum theory in this way, it is argued that statistical fluctuations around the canonical ensemble can give rise to wave function collapse behavior, of the kind discussed here, both energy-driven and CSL-type mass-density-driven collapse. The Hamiltonian needed for this theory to work is not provided but, as the argument progresses, its necessary features are delimited.
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