Quantum dynamical semigroups for diffusion models with Hartree interaction

A. Arnold\textsuperscript{1} and C. Sparber\textsuperscript{2}

Abstract

We consider a class of evolution equations in Lindblad form, which model the dynamics of dissipative quantum mechanical systems with mean-field interaction. Particularly, this class includes the so-called Quantum Fokker-Planck-Poisson model. The existence and uniqueness of global-in-time, mass preserving solutions is proved, thus establishing the existence of a nonlinear conservative quantum dynamical semigroup. The mathematical difficulties stem from combining an unbounded Lindblad generator with the Hartree nonlinearity.

Key words: open quantum system, Lindblad operators, quantum dynamical semigroup, dissipative operators, density matrix, Hartree equation

AMS (2000) classification: 81Q99, 82C10, 47H06, 47H20

1 Introduction

This paper is concerned with quantum mechanical multi-particle systems coupled to an external reservoir, i.e. so called open quantum systems [Da, BrPe]. The dynamics of such systems can often be approximately described by kinetic equations in the mean-field limit. Such self-consistent models appear in a wide range of physical applications, both quantum mechanical and classical, for example in gas dynamics, stellar dynamics, plasma physics, and electron transport. The corresponding nonlinear evolution equations are obtained as approximations to the underlying (linear) many-particle models, and there exists a vast body of literature on their mathematically rigorous derivation: the classical Vlasov-Poisson system in [BrHe, Ba]; the Hartree equation from the $N$-body Schrödinger equation in the mean-field limit in [ErYa]; the Hartree-Fock equation in [LaMa]. All of these models have in common that they fall into the class of Markovian approximation for the underlying dynamics and we refer to [Sp] for an extended overview of such derivations for a variety of kinetic equations.

\textsuperscript{1}Institut für Numerische Mathematik, Universität Münster, Einsteinstr. 62, D-48149 Münster, Germany, e-mail: anton.arnold@math.uni-muenster.de,

\textsuperscript{2}Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Vienna, Austria, e-mail: christof.sparber@univie.ac.at.
In addition to a self-consistent Coulomb field we shall here be interested in quantum systems which in addition have a **dissipative** interaction with their environment. In many (practical) applications of such open quantum systems the interaction with a reservoir is described in a rather simple phenomenological manner, often using diffusion operators, quantum-BGK or relaxation-type terms \([\text{CaLe, DeRi, Ar1}]\) when considered in a kinetic formalism. A prominent example of a linear open quantum system is the so called **quantum optical master equation** and its variants \([\text{GaZo, Va1}]\). However nonlinear mean-field models for open quantum systems also play an important role e.g. in laser physics (cf. \([\text{HeLi}]\) and \([\text{Sp}]\) for the Lieb-Hepp and the Dicke-Haken-Lax laser model, resp). In this work we shall be interested in a particular class of models which are frequently used in quantum optics \([\text{DHR, OC, Va1}]\) and the simulation of nano-scale semiconductor devices \([\text{FMR, JuTa}]\), namely the quantum kinetic Wigner-Fokker-Planck equation (WFP)

\[
\partial_t w + \xi \cdot \nabla_x w + \Theta[V]w = Qw, \quad x, \xi \in \mathbb{R}^d, t > 0, \tag{1.1}
\]

which governs the time evolution of the Wigner function \(w(x, \xi, t)\) in (position-velocity) **phase-space** under the action of the potential \(V(x, t)\). In (1.1) the pseudo-differential operator \(\Theta[V]\) is defined by

\[
\Theta[V]w(x, \xi, t) := \frac{i}{(2\pi)^d} \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \left[ V\left( x + \frac{y}{2}, t \right) - V\left( x - \frac{y}{2}, t \right) \right] w(x, \xi', t) e^{iy \cdot (\xi - \xi')} d\xi' dy. \tag{1.2}
\]

\(Q\) denotes the following diffusion operator

\[
Qw(x, \xi) := D_{pp} \Delta_\xi w + 2\eta \text{div}_\xi (\xi w) + D_{qq} \Delta_x w + 2D_{pq} \text{div}_x (\nabla_\xi w), \tag{1.3}
\]

with diffusion constants \(D\) (cf. \([2.17]\) below) and the friction constant \(\eta \geq 0\). Here and in the sequel we set the physical constants \(\hbar = m = e = 1\), for simplicity. In semiconductor applications \(w(t, x, \xi)\) is the quasi-distribution of the electron gas and \(Q\) models (phenomenologically) its interaction with a phonon bath. In our mean-field model the **Hartree-type nonlinearity** then stems from the repulsive Coulomb and \(Q\) models (phenomenologically) its interaction with a phonon bath. Hence, (1.1) is coupled to the **Poisson equation**

\[
\Delta V = -n, \tag{1.4}
\]

where \(n = \int w \, d\xi\) is the **particle density** of the electrons.

Moreover, such **Quantum Fokker-Planck** (QFP) type equation are the most prominent model in the description of quantum Brownian motion, where a (massive) quantum particle interacts with a heat bath and a possible external potential, see e.g. \([\text{CaLe, Di, Li1, OC}]\) and \([\text{HuMa}]\), where this setting is proposed as a description of **decoherence**. Indeed most of these equations can be traced back to an early work by Feynman and Vernon \([\text{FeVe}]\). While formal derivations of QFP equations were given in \([\text{CaLe, Di, Va}]\), a rigorous derivation from many-body quantum mechanics is still missing, at least for the general class of models considered here. To the authors’ knowledge, the only results in this direction are \([\text{CEFM, FMR}]\), where special cases of the QFP equation arise, resp., in a **space-time scaling limit** and a **weak coupling limit** for a particle interacting with an infinite heat bath of harmonic oscillators, i.e. phonons.
In this paper we shall investigate well-posedness of QFP type equations with a mean-field Coulomb potential – the above mentioned Wigner-Poisson-Fokker-Planck equation (WPFP) \( \text{(1.1)-(1.4)} \) being one typical example. Specifically, we establish existence and uniqueness of global-in-time solutions to the Cauchy problem. Many of the analytical tools developed in the sequel will, however, directly apply to other open quantum systems in mean-field approximation (e.g. to the Dicke-Haken-Lax laser model). First analytical results on the WFP and WPFP equations \( \text{(1.1)} \) were obtained in \([\text{SCDM}]\) (well-posedness of the linear equation, convergence to the unique steady state with an exponential rate), in \([\text{ALMS}]\) (local-in-time solution for the mean-field model in 3D), and in \([\text{ACD}]\) (global-in-time solution for the mean-field model in 1D).

In the mathematical analysis of mean-field QFP equations several parallel problems have to be coped with: the Wigner framework often used in applications seems inappropriate since the particle density \( n = \int w \, d\xi \) is not naturally defined in this setup (typically, \( w \in L^2(\mathbb{R}^d \times \mathbb{R}^d) \); cf. \([\text{Ar, ALMS}]\) for more details). We are hence led to study the equivalent evolution of the density matrix \( \rho(t) \) in the space of positive trace class operators \( J_1 \). Moreover, in order to deal with the Hartree nonlinearity, an appropriate energy-space \( E \subset J_1 \) needs to be introduced, which is a generalization of the one used in \([\text{BDF}]\). In \( J_1 \) the evolution of the quantum system is then governed by a so called Markovian master equation,

\[
\begin{align*}
\frac{d}{dt} \rho &= \mathcal{L}(\rho), \quad t > 0, \\
\rho|_{t=0} &= \rho_0 \in J_1.
\end{align*}
\]

The considered Liouvillian \( \mathcal{L} \) is obtained as a generalization of the one given by an inverse Wigner transformation of \( \text{(1.1)} \) and will be stated in \( \text{(2.5)} \) below. Since \( \mathcal{L} \) (and in particular the included Lindblad operators \( \mathcal{L} \)) are unbounded, this can be difficult even for linear equations and may lead to non unique and non conservative solutions. E.B. Davies showed in \([\text{Da1}]\) that it is possible to construct, for a quite general class of unbounded Lindblad generators \( \mathcal{L} \), a so called minimal solution to the above master equation. However, this construction is in general not unique, i.e. \( \mathcal{L} \) does not uniquely determine a corresponding quantum dynamical semigroup (QDS) \( \Phi_t(\rho_0) = e^{\mathcal{L}t} \rho_0 \). In particular, this implies that the minimal solution may not be conservative, i.e. trace preserving (cf. example 3.3 in \([\text{Da1}]\)), which would be inappropriate for the above mentioned applications.

While linear QDS have been studied intensively in the last three decades \([\text{FaRe, Al, AlFa}]\), the literature on nonlinear QDS is no so abundant, see e.g. \([\text{Al, AlMe, BDF}]\). By now, various sufficient conditions for the conservativity of linear QDS can be found in \([\text{ChFa, CGQ, Ho}]\). For many concrete examples, however, these conditions are rather difficult to verify, as we shall discuss in more detail at the end of section \( [9] \). Moreover the assumptions on the nonlinearity introduced in \([\text{AlMe}]\) seem too strong for most physical applications.

In this perspective, the present work establishes the existence and uniqueness of a conservative QDS for a concrete family of unbounded Lindblad generators \( \mathcal{L} \) (including the WPFP model) with Hartree interaction. We shall consider Lindblad operators (representing the coupling to the reservoir) which are linear combinations of the position and momentum operators, i.e. so called quasifree dynamical semigroups \([\text{Li1}]\).
We briefly remark that the classical counterpart of WPFP, i.e. the Vlasov-Poisson-Fokker-Planck system (and its linear version, the classical kinetic Fokker-Planck or Kramers equation [13])

\[ \partial_t f + \xi \cdot \nabla_x f - \nabla_x V \cdot \nabla_\xi f = D \Delta_\xi f + 2\eta \text{div}_\xi (\xi f), \quad x, \xi \in \mathbb{R}^d, t > 0 \tag{1.5} \]

allows for a much easier mathematical analysis. This is due to a natural

\[ L^1(\mathbb{R}^d \times \mathbb{R}^d) \] framework for (1.5) and to the positivity of the phase-space density \( f(t, x, \xi) \), cf. [Bo] for the well-posedness analysis, [Dr] for existence of a unique steady state, and [DeVr] for convergence results to the steady state for the linear model.

This paper is organized as follows:

After introducing the model in section 2 we will prove in section 3 existence and uniqueness of a global, mass preserving solution to the linear equation, i.e. the existence of a conservative QDS. A crucial analytical tool towards this end is a new density lemma (relating minimal and maximal operator realizations) for Lindblad generators \( \mathcal{L} \) that are quadratic in the position and momentum operator. The mean field will then be included in section 4 (we shall restrict ourselves for simplicity to the case of \( d = 3 \) spatial dimensions). We prove that the self-consistent potential is a locally Lipschitz perturbation of the free evolution in an appropriate “energy space”, and this yields a local-in-time existence and uniqueness result. Finally, we shall prove global existence of a conservative QDS in section 5 by establishing a-priori estimates for the mass and total energy of the system.

2 The model equation

In the sequel we shall use the following standard notations:

**Definition 2.1.** \( \mathcal{J}_1 \) is the space of trace class operators on \( L^2(\mathbb{R}^d) \) with the norm \( \| A \|_1 := \text{Tr} |A| \), where \( \text{Tr} \) denotes the usual operator trace on \( \mathcal{B}(L^2(\mathbb{R}^d)) \).

\( \mathcal{J}^*_1 \subset \mathcal{J}_1 \) denotes the subspace of self-adjoint trace class operators. Similarly, \( \mathcal{J}_2 \) is the space of Hilbert-Schmidt operators with the norm \( \| A \|_2 := (\text{Tr} |A|^2)^{1/2} \) and \( \| \cdot \|_\infty \) denotes the operator norm in \( \mathcal{B}(L^2(\mathbb{R}^d)) \). \( \| \cdot \|_p, 1 \leq p \leq \infty \) is the norm of \( L^p(\mathbb{R}^d) \)-functions.

We consider open quantum systems of massive, spin-less particles within an effective single-particle approximation, as it has been derived for example in [CEFM]. Hence, at every time \( t \in \mathbb{R} \) a physically relevant, mixed state of our system is uniquely given by a positive operator \( \rho(t) \in \mathcal{J}^*_1 \), in the sequel called density matrix operator. Since \( \rho \) is also Hilbert-Schmidt it can be represented by an integral operator \( \rho(t) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \), i.e.

\[ (\rho(t)f)(x) := \int_{\mathbb{R}^d} \rho(x, y, t)f(y)dy. \tag{2.1} \]

Its kernel \( \rho(\cdot, \cdot, t) \in L^2(\mathbb{R}^{2d}) \) is then called the density matrix function of the state \( \rho \) and it satisfies \( \| \rho(t) \|_2 = \| \rho(\cdot, \cdot, t) \|_2 \). By abuse of notation we shall identify from now on the operator \( \rho \in \mathcal{J}^*_1 \) with its kernel \( \rho(\cdot, \cdot) \in L^2(\mathbb{R}^{2d}) \). It is well known that we can decompose the kernel in the following form

\[ \rho(x, y) = \sum_{j \in \mathbb{N}} \lambda_j \psi_j(x)\overline{\psi_j(y)}, \quad \lambda_j \geq 0, \tag{2.2} \]
where \( \{ \lambda_j \} \in l^1(\mathbb{N}) \) and the complete o.n.s. \( \{ \psi_j \} \subset L^2(\mathbb{R}^d) \) are the eigenvalues and eigenfunctions of \( \rho \). Using equation (2.2) one can define the particle density \( n[\rho] \) by setting \( x = y \), to obtain

\[
  n[\rho](x) := \sum_{j \in \mathbb{N}} \lambda_j |\psi_j(x)|^2, \quad x \in \mathbb{R}^d.
\]  

(2.3)

However, since \( \{ x = y \} \subset \mathbb{R}^{2d} \) is a set of measure zero, this is not a mathematically rigorous procedure for a kernel \( \rho(x,y) \) that is merely in \( L^2(\mathbb{R}^{2d}) \). On the other hand, if \( \rho(x,y) \) is indeed the kernel of an operator \( \rho \in J^s_1 \) it is known, cf. [Ar], [LiPa], that the particle density can be rigorously defined by

\[
  n[\rho](x) := \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \rho \left( x + \frac{\eta}{\varepsilon}, x - \frac{\eta}{\varepsilon} \right) e^{-|\eta|^2/2\varepsilon} \frac{d\eta}{(2\pi\varepsilon)^d} \in L^1_+_1(\mathbb{R}^d).
\]  

(2.4)

And it satisfies \( \| n \|_1 = \text{Tr}(\rho) \) for \( \rho \geq 0 \). This issue of rigorously defining \( n[\rho] \) is one of the mathematical motivations for analyzing our mean field evolution equations as an abstract evolution problem for the operator \( \rho \) on the Banach space \( J^s_1 \).

**Remark 2.2.** Note that we cannot use the decomposition (2.2) in order to pass to a PDE problem for the \( \psi_j \), since the considered dissipative evolution equation in general does not conserve the occupation probabilities \( \lambda_j \). This is in sharp contrast to unitary dynamical maps generated by the von Neumann equation of standard quantum mechanics.

We consider the following (nonlinear) dissipative equation modeling the motion of particles, interacting with each other and with their environment

\[
\begin{aligned}
  \frac{d}{dt}\rho &= \mathcal{L}(\rho) := -i[H,\rho] + A(\rho), \quad t > 0, \\
  \rho|_{t=0} &= \rho_0 \in J^s_1.
\end{aligned}
\]  

(2.5)

Here, \([\cdot, \cdot]\) is the commutator bracket, \( H \) and \( A(\rho) \) are formally self-adjoint and of Lindblad class. More precisely, we consider the Hamiltonian operator

\[
  H := -\frac{\Delta}{2} + V[\rho](x,t) - i\mu [x, \nabla]_+, \quad \mu \in \mathbb{R},
\]  

(2.6)

denoting by \([\cdot, \cdot]_+\) the anti-commutator. The operators \( x \) and \( \nabla \) are, respectively, the multiplication and gradient operator on \( \mathbb{R}^d \), i.e. \( [x, \nabla]_+ = x \cdot \nabla + \nabla \cdot x = 2x \cdot \nabla + d \).

**Remark 2.3.** The operator \( H \) is sometimes called adjusted Hamiltonian, due to the appearance of the \( [x, \nabla]_+ \) term. Depending on the particular model, such a term may or may not be present, see e.g. [De] [Di1]. Nevertheless it is included here, in order to keep our presentation as general as possible.

The (real-valued) potential \( V \) is assumed to be of the form

\[
  V[\rho](x,t) := \frac{|x|^2}{2} + V_1(x) + \phi[\rho](x,t), \quad x \in \mathbb{R}^d,
\]  

(2.7)

where the first term of the r.h.s. denotes a possible confinement potential and \( V_1 \in L^\infty(\mathbb{R}^d) \) is a bounded perturbation of it. We point out that the quadratic
Quantum Dynamical Semigroups

Confinement potential is not necessary for the subsequent mathematical analysis, it is just an option. \( \phi \) is the Hartree- or mean field-potential, obtained from the self-consistent coupling to the Poisson equation

\[-\Delta \phi(\rho) = n(\rho). \quad (2.8)\]

For \( d = 3 \), we therefore get the usual Hartree-term:

\[ \phi(\rho)(x, t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{n(\rho)(y, t)}{|x - y|} \, dy, \quad x, y \in \mathbb{R}^3, \quad (2.9) \]

where \( n \) is computed from \( \rho \) by (2.4). This mean field approximation describes the (repulsive) Coulombian interaction of the particles with each other.

The non-Hamiltonian part is defined as

\[ A(\rho) := \sum_{j=1}^{m} L_j \rho L_j^* - \frac{1}{2} [L_j^* L_j, \rho], \quad m \in \mathbb{N}, \quad (2.10) \]

or equivalently

\[ A(\rho) = \sum_{j=1}^{m} \frac{1}{2} [L_j \rho, L_j^*] + \frac{1}{2} [L_j, \rho L_j^*], \quad (2.11) \]

where the linear operators \( L_j \) (Lindblad operators) are assumed to be of the form

\[ L_j := \alpha_j \cdot x + \beta_j \cdot \nabla + \gamma_j, \quad \alpha_j, \beta_j, \gamma_j \in \mathbb{C}^d. \quad (2.12) \]

Its adjoint is \( L_j^* = \bar{\alpha}_j \cdot x - \bar{\beta}_j \cdot \nabla + \bar{\gamma}_j \), and in the following we shall use the notation

\[ L := \sum_{j=1}^{m} L_j^* L_j. \quad (2.13) \]

**Remark 2.4.** Linear models with Hamiltonians that are quadratic in the position and momentum operator and with Lindblad operators of the form (2.12) give rise to so called quasifree QDS, and they are explicitly solvable in terms of Greens functions [Li1, SCDM]. In order to deal with nonlinear problems (in a “finite energy subspace” of \( \mathcal{F}_1 \)) we shall, however, not use this representation, which moreover can not be generalized to higher order models, cf. remark 2.7.

**Remark 2.5.** In the framework of second quantization and in \( d = 1 \), the space \( L^2(\mathbb{R}) \) is unitarily mapped onto \( \mathcal{F}_s(\mathbb{C}) \), the symmetric or bosonic Fock space over \( \mathbb{C} \). This space is frequently used, for example in quantum optics, in order to describe two-level bosonic systems, cf. [AlFa], [GZ].

Assuming \( \gamma = 0, \beta = 1 \) and \( \alpha = 1/2 \), the Lindblad operators \( L, L^* \), become then the usual bosonic creation- and annihilation-operators

\[ a f(x) := (\frac{x}{2} + \partial_x) f(x), \quad a^* f(x) := (\frac{x}{2} - \partial_x) f(x), \quad (2.14) \]

which, in contrast to the corresponding fermionic creation- and annihilation-operators, are unbounded. Of course, all results in our work can be equivalently interpreted in this framework of second quantization.
Example 2.6. A particularly interesting example in the above class is the Quantum Fokker-Planck equation (QFP). As a PDE for the kernel $\rho(t,x,y) \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ it reads

$$\left\{ \begin{array}{l}
i \partial_t \rho = \frac{1}{\hbar} \left[ -\frac{\Delta}{2} + V(t,x), \rho \right] + i A(\rho), \quad t > 0, \\
\rho|_{t=0} = \rho_0(x,y) \in L^2(\mathbb{R}^d \times \mathbb{R}^d),
\end{array} \right. \quad (2.15)$$

where

$$A(\rho) := -\gamma(x-y) \cdot (\nabla_x - \nabla_y) \rho + D_{qq} \nabla_x + \nabla_y \rho + \frac{D_{pp}}{\hbar^2} |x-y|^2 \rho + \frac{2iD_{pq}}{\hbar} (x-y) \cdot (\nabla_x + \nabla_y) \rho. \quad (2.16)$$

This model can be written in the form (2.5), (2.10), iff the conditions

$$D_{pp} D_{qq} - D_{pq}^2 \geq \frac{\eta^2}{4}, \quad D_{pp}, D_{qq} \geq 0, \quad (2.17)$$

hold (see [Li1, ALMS] for more details and a particular choice of the parameters $\mu_j, \alpha_j, \beta_j, \gamma_j$). Using the Wigner transform [Wi, LiPa]:

$$w(x,\xi,t) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \rho \left( x + \frac{y}{2}, x - \frac{y}{2}, t \right) e^{i\xi \cdot y} dy. \quad (2.18)$$

the QFP equation (2.15) can be transformed into the kinetic Wigner-Fokker-Planck equation (1.1). In physical units $D_{qq}, D_{pq} \sim \mathcal{O}(\hbar^2)$, cf. [De, Va], and hence we indeed obtain, at least formally, the kinetic Fokker-Planck equation (1.5) in the (semi-)classical limit $\hbar \to 0$. Note that for $\eta > 0$, condition (2.17) implies that the diffusion operator $Q$ from (1.3) is uniformly elliptic, which disqualifies the classical FP diffusion operator (i.e. $\theta_{qq} = \theta_{pq} = 0$) [Ri] as an appropriate quantum mechanical equation. Nevertheless, this Caldeira-Leggett master equation [CaLe] is sometimes used in applications as a phenomenological quantum model, cf. [St].

Remark 2.7. To close this section we mention an interesting model from quantum optics which is not yet covered by our present analysis. The Jaynes-Cumming model with phase damping reads

$$\frac{d}{dt} \rho = -i [H, \rho] + \kappa [H[H, \rho]], \quad (2.19)$$

where $\kappa \in \mathbb{R}_+$ denotes the damping constant, cf. [Lo]. Since it involves Lindblad operators $L_j$ that are quadratic polynomials of the position and momentum operators, it will be the focus of future research to (hopefully) extend the lemma 3.7 (below) to such cases.

3 Existence of a conservative QDS for the linear problem

We consider the linear evolution problem on $\mathcal{J}_s^a(L^2(\mathbb{R}^d))$

$$\left\{ \begin{array}{l}
\frac{d}{dt} \rho = L(\rho), \quad t > 0, \\
\rho|_{t=0} = \rho_0 \in \mathcal{J}_1.
\end{array} \right. \quad (3.1)$$
Here, $\mathcal{L}(\rho) := -i[H, \rho] + A(\rho)$ is the formal generator of a QDS on $\mathcal{J}^*_1$, with

$$H = -\frac{\Delta}{2} + \frac{|x|^2}{2} + V_1(x) - i\mu[x, \nabla]_+.$$  

(3.2)

**Definition 3.1.** Given any Hilbert space $\mathcal{H}$, one defines a *conservative quantum dynamical semigroup* (QDS) as a one parameter $C^0$ - semigroup of bounded operators $\Phi_t: \mathcal{J}_1(\mathcal{H}) \rightarrow \mathcal{J}_1(\mathcal{H})$, (3.3)

which in addition satisfies:

(a) The dual map $\Phi^*_t: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$, defined by

$$\text{Tr}(A\Phi_t(\rho)) = \text{Tr}(\Phi^*_t(A)\rho),$$

for all $\rho \in \mathcal{J}_1(\mathcal{H})$, $A \in \mathcal{B}(\mathcal{H})$, is completely positive. This means that the map

$$\Phi^*_t \otimes I_n : \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}_n) \rightarrow \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}_n)$$

is positive (i.e. positivity preserving) for all $n \in \mathbb{N}$. Here $\mathcal{H}_n$ denotes a finite dimensional Hilbert space and $I_n$ is the $n - \text{dimensional}$ unit matrix.

(b) $\Phi_t$ is trace preserving, i.e. conservative (or unital).

**Remark 3.2.** The notion QDS is sometimes reserved for the dual semigroup $\Phi^*_t$. Physically speaking, this corresponds to the *Heisenberg picture*. The appropriate continuity is then

$$\lim_{t \rightarrow 0} \text{Tr}(\rho(\Phi^*_t(A) - A)) = 0,$$

for all $\rho \in \mathcal{J}_1(\mathcal{H})$, $A \in \mathcal{B}(\mathcal{H})$, i.e. ultraweak continuity. Complete positivity can be defined also for operators on general $C^\ast$-Algebras $\mathcal{A}$ and it is known that complete positivity and positivity are equivalent only if $\mathcal{A}$ is commutative. (Counter-examples can be found already for $2 \times 2$ complex valued matrices, see e.g. [AlFa].) Again, from a physical point of view, complete positivity can be interpreted as preservation of positivity under entanglement.

Following the classical work of Davies [Da1] we shall start to investigate the properties of the operator

$$Y := -iH - \frac{1}{2}L.$$  

(3.7)

First we need the following technical lemma, the proof of which introduces some important notations used throughout this work.

**Lemma 3.3.** Let $P := p_2(x, -i\nabla)$ be a linear operator on $L^2(\mathbb{R}^d)$ over the field $\mathbb{C}$, where $p_2$ is a complex valued, quadratic polynomial and specify its domain by

$$\mathcal{D}(P) := \{f: \text{Re } f, \text{Im } f \in C^\infty_0(\mathbb{R}^d)\}.$$  

(3.8)

Then $\overline{P}$ is the maximal extension of $P$ in the sense that

$$\mathcal{D}(\overline{P}) = \{f \in L^2(\mathbb{R}^d) : \text{the distribution } Pf \in L^2(\mathbb{R}^d)\}.$$  

(3.9)
Proof. (sketch) We define a mollifying delta sequence by
\[ \varphi_n(x) := n^d \varphi(nx), \quad x \in \mathbb{R}^d, n \in \mathbb{N}, \]  
with \( \varphi \in C_0^\infty \) and \( \varphi \geq 0, \) \( \varphi(x) = \varphi(-x), \) \( \int_{\mathbb{R}^d} \varphi(x)dx = 1, \) supp \( \varphi \subset \{ |x| < 1 \}. \)

Also, a sequence of radially symmetric cutoff function is defined by
\[ \chi_n(x) := \chi \left( \frac{|x|}{n} \right), \quad x \in \mathbb{R}^d, n \in \mathbb{N}, \]  
with \( \chi_n \in C_0^\infty, \) \( 0 \leq \chi \leq 1, \) supp \( \chi \subset [0, 1], \) \( \chi|_{[0, \frac{1}{n}]} \equiv 1. \)

For \( f \in L^2(\mathbb{R}^d) \) we define an approximating sequence in \( \mathcal{D}(P) \) by
\[ f_n(x) := \chi_n(x)(f * \varphi_n)(x), \quad n \in \mathbb{N}. \]  
We have to prove that for all \( f \in L^2(\mathbb{R}^d), \) with \( Pf \in L^2(\mathbb{R}^d), \) \( f_n \to f \) in the graph norm \( \|f\|_P := \|f\|_2 + \|Pf\|_2. \) We clearly have
\[ f_n \xrightarrow{n \to \infty} f \text{ in } L^2(\mathbb{R}^d), \] (3.13)
and it remains to prove \( Pf_n \to Pf \) in \( L^2(\mathbb{R}^d). \) This is now analogous to the proof of lemma 2.2 in \[ACD\], when extended to complex valued functions \( f. \) A similar strategy is used again in the proof of lemma 3.4 below.

Remark 3.4. Lemma 3.3 asserts that the minimal and maximal operators defined by the expression \( P = p_{2}(x, -i \nabla) \) coincide. This fact is closely related to the essential self-adjointness of Schrödinger operators. The lemma provides an elementary proof of the well known fact that the Hamiltonian \( H = -\Delta - |x|^2 \) is essentially self-adjoint on \( C_0^\infty(\mathbb{R}^d), \) cf. corollary to theorem X.38 in \[Resi2\], – just apply the lemma to \( H \) with \( \mathcal{D}(H) = C_0^\infty(\mathbb{R}^d) \) and to \( H^*|_{\mathcal{D}(H)}. \) On the other hand, it is well known that \( H = -\Delta + x^2 - x^4 \) is not essentially self-adjoint on \( C_0^\infty(\mathbb{R}), \) cf. example 1 of X.5 in \[Resi2\]. Therefore, lemma 3.3 can, in general, not be extended to higher order polynomials \( p(x, -i \nabla). \)

With the above lemma we can now prove that the main technical assumption on the operator \( Y \) (imposed in \[D1\], \[ChFe\]) is fulfilled.

Proposition 3.5. Let \( V_1 = 0 \) and let the operator \( Y \) be defined on
\[ \mathcal{D}(Y) := \{ f \in L^2(\mathbb{R}^d) : \Delta f, |x|^2 f \in L^2(\mathbb{R}^d) \}. \]  
(a) Then its closure \( \bar{Y} \) is the infinitesimal generator of a \( C_0 \) - contraction semigroup on \( L^2(\mathbb{R}^d). \)
(b) Further, the operators \( L_j, L_j^* : \mathcal{D}(\bar{Y}) \to L^2(\mathbb{R}^d) \) satisfy
\[ \langle Yf, g \rangle + \langle f, Yg \rangle + \sum_{j=1}^{m} \langle L_j f, L_j g \rangle = 0, \quad \forall f, g \in \mathcal{D}(\bar{Y}), \] (3.15)
where \( \langle \cdot, \cdot \rangle \) denotes the standard scalar product on \( L^2(\mathbb{R}^d). \)
Proof. First note that for \( f \in \mathcal{D}(Y) \) the term \( x \cdot \nabla f \), which appears in \( Yf \), is also in \( L^2(\mathbb{R}^d) \). This can be obtained by an interpolation argument. Further, \( \mathcal{D}(Y) \) is dense in \( L^2(\mathbb{R}^d) \), since \( C_0^\infty(\mathbb{R}^d) \) is. By Lemma 3.3 we have
\[
\mathcal{D}(Y) = \{ f \in L^2(\mathbb{R}^d) : Yf \in L^2(\mathbb{R}^d) \}.
\]

Part (a): The proof proceeds in several steps:

Step 1: We study the dissipativity of \( Y \), which in our case is defined by
\[
\text{Re} \langle Yf, f \rangle \leq 0, \quad \forall f \in \mathcal{D}(Y).
\]
Since \( H \) from (3.7) is symmetric we obtain
\[
\text{Re} \langle iHf, f \rangle = 0, \quad \forall f \in \mathcal{D}(Y).
\]
Also we get
\[
-\text{Re} \langle L^*_j L_j f, f \rangle = -\langle L_j f, L^*_j f \rangle \leq 0, \quad \forall f \in \mathcal{D}(Y).
\]
Thus \( Y \) is dissipative and by theorem 1.4.5b of [Pa] also its closure \( \overline{Y} \) is.

Step 2: Its adjoint is \( Y^* = iH - \frac{1}{2}L \), with domain of definition \( \mathcal{D}(Y^*) \). We have \( \mathcal{D}(Y^*) \supseteq \mathcal{D}(Y) \), since
\[
\langle Yf, g \rangle = \langle f, Y^* g \rangle, \quad \forall f, g \in \mathcal{D}(Y).
\]
As in step 1 we conclude that \( Y^* \big|_{\mathcal{D}(Y)} \) is dissipative. We can now apply lemma 3.3 to \( P = Y^* \big|_{\mathcal{D}(P)} \) with \( \mathcal{D}(P) \) defined in (3.8). Then \( P \) is dissipative on \( \mathcal{D}(P) \subseteq \mathcal{D}(Y) \subseteq \mathcal{D}(Y^*) \). Since \( Y^* \) is closed, we have \( \mathcal{D}(Y^*) = \overline{\mathcal{D}(P)} \), the domain of the maximal extension. Thus \( Y^* \) is dissipative on all of \( \mathcal{D}(Y^*) \).

Step 3: Application of the Lumer-Phillips theorem (corollary 1.4.4 in [Pa]) to \( \overline{Y} \) (with \( (\overline{Y})^* = Y^* \)) implies the assertion.

Part (b): We need to show: If \( f, Yf \in L^2(\mathbb{R}^d) \), then \( L_j f, L^*_j f \in L^2(\mathbb{R}^d) \) follows. This can be easily seen from the fact that
\[
\frac{1}{2} \sum_j \langle L_j f, L_j f \rangle = -\text{Re} \langle Yf, f \rangle < \infty.
\]
Equation (3.15) is then obtained by a simple computation.

With these properties of \( \overline{Y} \) (as stated in proposition 3.3), theorem 3.1 of [Da1] asserts that (3.1) has a so called minimal solution:

Proposition 3.6. [Davies '77] There exists a positive \( C_0 \) - semigroup of contractions \( \Phi_t \) on \( J_1^* \). Its infinitesimal generator is the evolution operator \( L \), defined on a sufficiently large domain \( \mathcal{D}(L) \), such that \( J_1^* \supseteq \mathcal{D}(L) \supseteq \mathcal{D}(Z) \).

Here, \( Z : \mathcal{D}(Z) \to J_1^* \) is the maximally extended operator with domain
\[
\mathcal{D}(Z) = \{ \rho \in J_1^* (L^2(\mathbb{R}^d)) : Z(\rho) := Y \rho + \rho Y^* \in J_1^* (L^2(\mathbb{R}^d)) \}.
\]
From the above proposition we learn that the formal generator \( L \), in general, does not unambiguously define a solution of the corresponding master equation,
Quantum Dynamical Semigroups

in the sense of semigroups. Also, it is well known, that the obtained minimal solution need not be trace preserving (for nonconservative examples see e.g. [Da1, Ho]).

On the other hand, if the semigroup corresponding to the minimal solution preserves the trace, it is the unique conservative QDS associated to the abstract evolution problem [CQ, ChFa, FaRe, Ho]. We are going to prove now that in our case the minimal solution is indeed the unique QDS. To this end, we need to introduce some more notation:

From now on we denote by

\[(M(g)f)(x) := g(x)f(x), \quad (C(g)f)(x) := (g * f)(x), \quad g \in C_0^\infty(\mathbb{R}^d),\]

a family of multiplication and convolution operators on \(L^2(\mathbb{R}^d)\), where "\(*" is the usual convolution w.r.t. \(x\). Further we define, for \(n \in \mathbb{N}\), a family of sets \(D_n \subset J_1^s(L^2(\mathbb{R}^d))\) by

\[D_n := \{\sigma_n \in J_1^s : \exists \rho \in J_1^s \text{ s.t. } \sigma_n = M(\chi_n)C(\varphi_n)\rho C(\varphi_n)M(\chi_n)\}, \quad (3.17)\]

where \(\chi_n, \varphi_n\) are the cutoff resp. mollifying functions defined in the proof of lemma 3.3 above. For an operator \(\rho \geq 0\) with kernel (2.2), the operator \(\sigma_n\) has an integral kernel given by

\[\sigma_n(x, y) = \chi_n(x)\varphi_n(x)\rho(x, y)\varphi_n(y)\chi_n(y) = \sum_{j \in \mathbb{N}} \lambda_j \varphi_{j,n}(x) \varphi_{j,n}(y), \quad (3.18)\]

where \(\varphi_{j,n}(x) := \chi_n(x)(\varphi_n * \psi_j)(x) \in C_0^\infty(\mathbb{R}^d)\) and \(\|\varphi_{j,n}\|_2 \leq \|\psi_j\|_2 = 1\). Since \(\sigma_n \geq 0\) we get

\[\|\sigma_n\|_1 = \text{Tr} \sigma_n = \sum_{j \in \mathbb{N}} \lambda_j \|\varphi_{j,n}\|_2^2 \leq \sum_{j \in \mathbb{N}} \lambda_j = \|\rho\|_1. \quad (3.19)\]

The union of all sets \(D_n\) will be denoted by

\[D_\infty := \bigcup_{n \in \mathbb{N}} D_n, \quad (3.20)\]

Also we shall write for the graph norm corresponding to \(L\)

\[\|\rho\|_L := \|\rho\|_1 + \|L(\rho)\|_1. \quad (3.21)\]

Then the following technical result, which is a key point in the existence and uniqueness analysis, holds.

**Lemma 3.7.** Let \(V_1 = 0\). Then:

(a) The set \(D_\infty\) is dense in \(J_1^s\).

(b) \(D_\infty \subset D(Z) \subset D(L)\).

(c) The operator \(L \big|_{D_\infty}\) is the maximal extension of \(L\), in the sense that for each \(\rho \in J_1^s\), with \(L(\rho) \in J_1^s\), there exists a sequence \(\{\sigma_n\}_{n \in \mathbb{N}} \subset D_\infty\), such that

\[\lim_{n \to \infty} \|\rho - \sigma_n\|_L = 0. \quad (3.22)\]
Proof. The proof is deferred to the appendix.

Remark 3.8. For all $\rho \in J_t^1$, $L(\rho)$ can be defined (at least) as an operator $L(\rho) : C_c^\infty(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$, the space of distributions. For $L(\rho) \in J_t^1$ to hold, first of all an appropriate extension has to exist, such that $L(\rho) \in \mathcal{B}(L^2(\mathbb{R}^d))$.

We are now in the position to state our first main theorem:

Theorem 3.9. Let $V_1 = 0$. The evolution operator $L$ generates on $J_t^1$ a conservative quantum dynamical semigroup of contractions $\Phi_t(\rho) = e^{Lt}\rho$. This QDS yields the unique mild solution, in the sense of semigroups, for the abstract evolution problem (3.1).

Proof. Existence of $\Phi_t(\rho) = e^{Lt}\rho$ is guaranteed by proposition 3.6. As a semigroup generator $L$ is closed, and by lemma A.1 it is the maximally extended evolution operator. This implies uniqueness of the semigroup. Complete positivity then follows from Stinespring’s theorem [Sti, AlFa].

It remains to prove the conservativity for the obtained QDS. This will be done by using a similar argument as in the proof of theorem 3.2 in [Da1].

Step 1: For the special case $\rho_0 \in D(L)$ the trajectory $\Phi_t(\rho_0)$ is a classical solution (in the sense of semigroups, cf. [Pa]), i.e. $\Phi_t(\rho_0) \in C^1([0, \infty), J_t(L^2(\mathbb{R}^d)))$ and $\Phi_t(\rho_0) \in D(L), \forall t \geq 0$. Hence $\text{Tr} \Phi_t(\rho_0) \in C^1([0, \infty), \mathbb{R})$ and we calculate for $t \geq 0$:

$$
\frac{d}{dt} \text{Tr} \Phi_t(\rho_0) = \text{Tr} \frac{d}{dt} \Phi_t(\rho_0) = \text{Tr} L(\Phi_t(\rho_0)) = 0. \tag{3.23}
$$

To justify the last equality we note that $D_\infty$ is $\| \cdot \|_\mathcal{L}$ - dense in $D(L)$, by lemma A.1 (c). Thus we can approximate $\Phi_t(\rho_0)$, for every fixed $t \geq 0$, by an appropriate sequence $\{\sigma_n\} \subseteq D_\infty$. Since $D_\infty$ is included in the domain of each “term” $A.1$ of the operator $L$ (as the proof of lemma A.1 (b) shows), the cyclicity of the trace yields $\text{Tr} L(\Phi_t(\rho_0)) = 0$. Equation (3.23) then implies

$$
\text{Tr} \Phi_t(\rho_0) = \text{Tr} \rho_0 = 0, \quad \forall \rho_0 \in D(L), t \geq 0.
$$

Step 2: The general case $\rho_0 \in J_t^1(L^2(\mathbb{R}^d))$ (i.e. $\Phi_t(\rho_0)$ is a mild solution) follows from step 1 and the fact that $D(L)$ is dense in $J_t^1(L^2(\mathbb{R}^d))$. 

From the above theorem, we obtain the the following corollary:

Corollary 3.10. For $\rho \in D(L)$ let

$$
\tilde{L}(\rho) := L(\rho) + L_p(\rho), \tag{3.24}
$$

where

$$
L_p(\rho) := -i[V_1, \rho] + \sum_{j=m+1}^{\infty} L_j \rho L_j^* - \frac{1}{2} [L_j^* L_j, \rho]_+, \tag{3.25}
$$

with $V_1 \in L^\infty(\mathbb{R}^d)$, $L_j \in \mathcal{B}(L^2(\mathbb{R}^d))$ and the sum converges in $\mathcal{B}(J_t^1(L^2(\mathbb{R}^d)))$. Then the perturbed operator $\tilde{L}$ again uniquely defines a conservative QDS of contractions.
Proof. Existence and uniqueness of the $C_0$-semigroup follows from standard perturbation results, cf. [Pa]. To prove conservativity of the perturbed QDS, let $\rho(t)$ denote the solution of

$$\frac{d}{dt}\rho = \tilde{L}(\rho), \quad \rho(0) = \rho_0.$$ 

The conservativity then follows from Duhamel’s representation

$$\rho(t) = \Phi_t(\rho_0) + \int_0^t \Phi_{t-s}(\mathcal{L}_p(\rho(s))) \, ds,$$  

by noting that $\text{Tr}(\mathcal{L}_p(\rho)) = 0$. All other properties can be established by the same procedure as in theorem 1 of [AlMe] or by a Picard iteration. \qed

Remark 3.11. An alternative approach to prove theorem 3.9 could be to verify the sufficient conditions of [ChFa]. In fact their assumptions A1 and A2 are simple consequences of our lemma 3.3 and proposition 3.5. For their third condition A3 however, one would need to prove that $C_\infty^0(\mathbb{R}^d)$ is a core for $Y^2$, defined on

$$D(Y^2) := \{f \in D(\nabla) : \nabla f \in D(\nabla)\}.$$ 

With considerable more effort, the proof should be possible by extending the strategy of lemma 3.3. However, one can expect quite cumbersome calculations.

4 Local-in-time existence of the mean field QDS

We shall now prove existence and uniqueness of local-in-time solutions for the nonlinear evolution problem

$$\begin{cases}
  \frac{d}{dt}\rho = \mathcal{L}(\rho), & t > 0 \\
  \rho(0) = \rho_0 \in \mathcal{J}^s.
\end{cases}$$  

(4.1)

Here, the nonlinear map $\mathcal{L}$ is given by

$$\mathcal{L}(\rho) := -i \left[ -\frac{\Delta}{2} + V[\rho] - i\mu [x, \nabla]_+ \rho \right] + A(\rho),$$  

(4.2)

where the self-consistent potential $V[\rho]$ is given as in (2.7) and $A(\rho)$ is the Lindblad operator defined by (2.11) and (2.12).

To this end, we shall prove that the linear evolution problem (3.1) not only defines a $C_0$-semigroup in $\mathcal{J}^s$ (guaranteed by theorem 3.9) but also in an appropriate energy space. This is a parallel procedure (apart from severe technical difficulties) to solving the Schrödinger-Poisson equation in $H^1(\mathbb{R}^d)$, cf. [GiVe].

Note that Davies’ construction of a minimal QDS is valid only in $\mathcal{J}_1$. Hence, the required additional regularity of $\Phi_t(\rho_0)$ has to be established explicitly. Also, one has to prove separately that this nonlinear model conserves the positivity and the trace of $\rho$.

In the following, we shall restrict ourselves to the physical most important case of $d = 3$ spatial dimensions.

Let us start by introducing the following definitions:
Definition 4.1. The kinetic energy of a density matrix operator \( \rho \in J_s^1 \) is defined by

\[
E^{\text{kin}}[\rho] := \frac{1}{2} \text{Tr}(\sqrt{-\Delta} \rho \sqrt{-\Delta}),
\]

where \( \sqrt{-\Delta} \) denotes a pseudo-differential operator with symbol \( |\xi| \), \( \xi \in \mathbb{R}^d \), i.e.

\[
\sqrt{-\Delta} f(x) := \left( \frac{2\pi}{d} \right)^d \int_{\mathbb{R}^d} |\xi| (\mathcal{F}f)(\xi) e^{i\xi \cdot x} d\xi, \quad \forall f \in H^1(\mathbb{R}^d).
\]

Further, we define the external and the self-consistent potential energy of \( \rho \in J_s^1 \) by

\[
E^{\text{ext}}[\rho] := \frac{1}{2} \text{Tr}(|x| \rho |x|), \quad E^{\text{sc}}[\rho] := \frac{1}{2} \text{Tr}(\phi[\rho] \rho).
\]

The total energy will be denoted by

\[
E^{\text{tot}}[\rho] := E^{\text{kin}}[\rho] + E^{\text{ext}}[\rho] + E^{\text{sc}}[\rho].
\]

In the sequel we shall work in the following energy space \( \mathcal{E} \):

\[
\mathcal{E} := \{ \rho \in J_s^1 : \sqrt{1-\Delta + |x|^2} \rho \sqrt{1-\Delta + |x|^2} \in J_s^1 \},
\]

equipped with the norm

\[
\|\rho\|_{\mathcal{E}} := \| \sqrt{1-\Delta + |x|^2} \rho \sqrt{1-\Delta + |x|^2} \|_1
\]

This energy norm is a generalization of the one defined in [BDF]. In case \( \rho \) is indeed a physical state, i.e. \( \rho \geq 0 \), and if in addition \( \rho \in \mathcal{D}_\infty \), one easily gets

\[
\|\rho\|_{\mathcal{E}} = \|\rho\|_1 + \| \sqrt{-\Delta} \rho \sqrt{-\Delta} \|_1 + \| |x| \rho |x| \|_1, \quad \forall \rho \in \mathcal{D}_\infty, \rho \geq 0.
\]

Hence, a density argument, similar to lemma [4.71 (c)], implies for all \( \rho \geq 0 \) that \( \rho \in \mathcal{E} \) is equivalent to \( \rho \in J_s^1 \) and \( E^{\text{kin}}[\rho] + E^{\text{ext}}[\rho] < \infty \).

We further remark that in the above definitions we neglected the term \(-i\mu[x, \nabla]_+\), which appears in the generalized (or adjusted) Hamiltonian operator \( (2.6) \) of our system. Thus, even in the linear case, we have \( E^{\text{tot}}[\rho] \neq \text{Tr}(H \rho) \). The latter term would be the more common definition for the energy of the system. We note that we shall use \( E^{\text{tot}}[\rho] \) only for deriving a-priori estimates and towards this end \( E^{\text{tot}}[\rho] \) is the more convenient expression.

Remark 4.2. Using the cyclicity of the trace, one formally obtains the more common expression for the kinetic energy of a physical state \( \rho \geq 0 \):

\[
E^{\text{kin}}[\rho] := \frac{1}{2} \text{Tr}(\sqrt{-\Delta} \rho \sqrt{-\Delta}) = \frac{1}{2} \text{Tr}(-\Delta \rho) \geq 0.
\]

However, these two expressions for \( E^{\text{kin}}[\rho] \) are not fully equivalent, since \( \Delta \rho \in J_s^1 \) requires more regularity on \( \rho \) than just requiring \( \sqrt{-\Delta} \rho \sqrt{-\Delta} \in J_s^1 \). (For more details see e.g. [A] and the references given therein.) We further remark that if the kernel of \( \rho \) is given as in [2.2] the kinetic energy reads

\[
E^{\text{kin}}[\rho] = \frac{1}{2} \sum_{j \in \mathbb{N}} \lambda_j \| \nabla \psi_j \|_2^2 \geq 0.
\]
Similarly we get that for physical states $\rho \geq 0$ it holds $E^{\text{ext}}[\rho] \geq 0$, as well as $E^{\text{sc}}[\rho] \geq 0$, since $\rho \geq 0$ implies $\eta[\rho] \geq 0$ and hence $\phi[\rho] \geq 0$, by \textcolor{red}{[284]}.

Finally, note the additional factor $1/2$ in front of the term $E^{\text{sc}}[\rho]$, which does not appear in the Hamiltonian \textcolor{red}{[280], [281]}. It is due to the self-consistent nonlinearity, cf. \textcolor{red}{[284]}.

Using these definitions, we will now prove that the sum of kinetic and (external) potential energy is continuous in time during the linear evolution.

**Lemma 4.3.** Let $V_1 = 0$ and $\rho_0 \in \mathcal{E}$, then

$$
(E^{\text{kin}} + E^{\text{ext}})[\rho(t)] \in C([0, \infty); \mathbb{R}),
$$

where $\rho(t) := \Phi_t(\rho_0) \in C([0, \infty), \mathcal{F}^+_1)$ denotes the unique QDS for the linear evolution problem, given by \textcolor{red}{[284]}.

**Proof.** First, we note that each $\rho \in \mathcal{E} \subset \mathcal{J}_1^+$ can be uniquely decomposed into:

$$
\rho_{1,2} := \Lambda^{-1}(\Lambda \rho \Lambda)^\pm \Lambda^{-1}, \quad \Lambda := \sqrt{1 - \Delta + |x|^2},
$$

and $(\Lambda \rho \Lambda)^\pm$ denotes the positive resp. negative part of $(\Lambda \rho \Lambda) \in \mathcal{J}_1^+$. It holds: $\rho_{1,2} \geq 0$, as well as $\rho_{1,2} \in \mathcal{E}$.

Using this decomposition for the initial data $\rho_0 \in \mathcal{E}$ and since $\Phi_t$ preserves positivity, we can restrict ourselves in the following to the case $\rho_0 \geq 0$, hence $\rho(t) \geq 0$.

The idea is now to derive a differential inequality for $E^{\text{kin}} + E^{\text{ext}}$ from (3.1).

Let us define some energy functionals for positive $\rho \in \mathcal{J}_1^+$:

$$
E^{\text{kin}}_{k,l}[\rho] := -\frac{1}{2} \text{Tr}(\partial_k \rho \partial_l), \quad E^{\text{ext}}_{k,l}[\rho] := \frac{1}{2} \text{Tr}(x_k \rho x_l),
$$

with $k, l = 1, \ldots, d$. For $\rho \in \mathcal{D}_\infty$, the cyclicity of the trace implies

$$
E^{\text{kin}}[\rho] = \sum_{k=1}^d E^{\text{kin}}_{k,k}[\rho], \quad E^{\text{ext}}[\rho] = \sum_{k=1}^d E^{\text{ext}}_{k,k}[\rho]
$$

and, by a density argument, the formulas \textcolor{red}{(4.15)} also hold for $\rho \in \mathcal{E}$.

**Step 1:** We apply the operators $x_k$, $\partial_k$ (from left and right) to \textcolor{red}{(3.1)} and take traces. A lengthy but straightforward calculation, using the cyclicity of the trace and setting w.r.o.g. $\text{Tr}(\rho(t)) = 1$, yields for the kinetic energy:

$$
\sum_{k=1}^d \frac{d}{dt} E^{\text{kin}}_{k,k} = \frac{1}{2} \sum_{k=1}^d \sum_{j=1}^m |\alpha_{j,k}|^2 - 4\mu \sum_{k=1}^d E^{\text{kin}}_{k,k} - 2 \sum_{k,l=1}^d \sum_{j=1}^m \text{Re}(\alpha_{j,k} \overline{\beta_{j,l}}) E^{\text{kin}}_{k,l} - \sum_{k,l=1}^d \sum_{j=1}^m \text{Im}(\alpha_{j,k} \overline{\alpha_{j,l}}) \text{Tr}(\partial_k \rho x_l) + \text{Im}(\alpha_{j,k} \overline{\gamma_{j,l}}) \text{Tr}(\rho \partial_k)
$$

$$
+ i \left( \frac{d}{2} + \sum_{k=1}^d \text{Tr}(\partial_k \rho x_k) \right).
$$

\textcolor{red}{(4.16)}
Quantum Dynamical Semigroups

For the external energy we obtain:

\[
\sum_{k=1}^{d} \frac{d}{dt} E_{k,k}^{ext} = -\frac{1}{2} \sum_{k=1}^{d} \sum_{k,j=1}^{m} |\beta_{j,k}|^2 + 4\mu \sum_{k=1}^{d} E_{k,k}^{ext} + 2 \sum_{k,l=1}^{d} \sum_{j=1}^{m} Re(\alpha_{j,k}\overline{\beta_{j,l}}) E_{k,l}^{ext} \\
+ i \sum_{k,l=1}^{d} \sum_{j=1}^{m} Im(\beta_{j,k}\overline{\beta_{j,l}}) Tr(\partial_k \rho x_l) + Im(\beta_{j,k}\overline{\gamma_j}) Tr(\rho x_k) \\
- i \left( \frac{d}{2} + \sum_{k=1}^{d} Tr(\partial_k \rho x_k) \right).
\]  (4.17)

**Step 2:** These equations are not closed in \(E^{kin}\) and \(E^{ext}\). To circumvent this problem, we shall use interpolation arguments: First, note that \((\partial_k \rho \partial_k) \in \mathcal{J}_1\), iff \((\partial_k \sqrt{\rho}) \in \mathcal{J}_2\), cf. \[ReSi1\]. Thus we can estimate

\[
\|\partial_k \rho \| \leq \|\sqrt{\rho}\|_2 \|\sqrt{\rho} \partial_k\|_2 = \|\rho\|_1 \|\partial_k \rho \partial_k\|_1.
\]

Likewise, we get

\[
\|\partial_k \rho x_l\|_2^2 \leq \|\partial_k \sqrt{\rho}\|_2 \|\sqrt{\rho} x_l\|_2 = \|\partial_k \rho \partial_k\|_1 \|x_l \rho x_l\|_1
\]

and one easily derives analogous estimates for the off-diagonal energy-terms \(E_{k,l}^{kin}\). Hence, estimating term-by-term in (4.16), (4.17), we finally obtain

\[
\left| \frac{d}{dt} \sum_{k=1}^{d} (E_{k,k}^{kin} + E_{k,k}^{ext})[\rho(t)] \right| \leq K \sum_{k=1}^{d} (E_{k,k}^{kin} + E_{k,k}^{ext})[\rho(t)],
\]

with some generic constant \(K \geq 0\). Applying Gronwall’s lemma then gives the desired result. \(\square\)

This lemma directly leads to our next proposition:

**Proposition 4.4.** Assume that \(\rho_0 \in \mathcal{E}\) and \(V_1 \in L^\infty(\mathbb{R}^d)\) s.t. additionally \(\nabla V_1 \in L^q(\mathbb{R}^d)\), for some \(3 \leq q \leq \infty\). Then

\[
\Phi_t(\rho_0) \in C([0, \infty), \mathcal{E}),
\]  (4.18)

where \(\Phi_t(\rho_0)\) denotes the unique linear QDS corresponding to \(\mathcal{J}_1\).

**Proof.** The proof is based on a generalization of Grömm’s theorem. As described in the proof of lemma 4.3 above, we only need to consider, w.r.o.g., the case \(\rho(t) \geq 0\).

**Step 1:** At first, one proves that for all \(f, g \in L^2(\mathbb{R}^d)\) and \(s \geq 0\),

\[
\lim_{t \to s} \langle f, \Lambda \rho(t) \Lambda g \rangle = \langle f, \Lambda \rho(s) \Lambda g \rangle,
\]  (4.19)

where \(\langle \cdot, \cdot \rangle\) denotes the standard \(L^2(\mathbb{R}^d)\) scalar product. Choosing two sequences \(\{f_n\}, \{g_n\} \subset C_0^\infty(\mathbb{R}^d)\), s.t. \(f_n \to f, g_n \to g\) in \(L^2(\mathbb{R}^d)\) the assertion then follows from a fairly standard approximation procedure.

**Step 2:** Let \(V_1 = 0\) first. By theorem 2.20 in \[Si\] (a generalization of Grömm’s theorem), step 1 and the continuity of

\[
\|\rho(t)\|_1 + 2(E^{kin} + E^{ext})[\rho(t)] = \|\Lambda \rho(t)\Lambda\|_1
\]
Lemma 4.5. Let \( \rho \in \mathcal{E} \) and \( d = 3 \), then \( \phi[\rho] \in L^\infty(\mathbb{R}^3) \). Moreover, the operator \( [\phi[\rho], \rho] \) is a local Lipschitz map from \( \mathcal{E} \) into itself.

Proof. Once again we decompose \( \rho = \rho_1 - \rho_2 \) s.t. \( \rho_{1,2} \geq 0 \) and \( \rho_{1,2} \in \mathcal{E} \), as given in Lemma 3.11 in [Ar]. In \( d = 3 \), we explicitly get from (2.9)

\[
\phi[\rho_j] = -\frac{1}{4\pi|x|} * n[\rho_j], \quad \nabla \phi[\rho_j] = \frac{x}{4\pi|x|^3} * n[\rho_j], \quad j = 1, 2.
\]

Therefore, the Hardy-Littlewood-Sobolev inequality and the generalized Young inequality, cf. [ReSi2], imply for \( j = 1, 2 \):

\[
\phi[\rho_j] \in L^3_{w}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3), \quad 3 < p < \infty,
\]

as well as

\[
\nabla \phi[\rho_j] \in L^{3/2}_{w}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3), \quad 3/2 < p < \infty.
\]

Here, \( L^p_w \) denotes the weak \( L^p \)-spaces, cf. [ReSi2]. Hence, by a Sobolev imbedding, we obtain \( \phi[\rho] \in L^\infty(\mathbb{R}^d) \). Similar arguments as given in the proof of lemma 3.11 in [Ar] then imply that \( [\phi[\rho], \rho] \) is a local Lipschitz map in the energy space \( \mathcal{E} \). To this end we first estimate

\[
\|\Lambda \phi[\rho] \rho \Lambda\|_1 \leq \|\Lambda \phi[\rho] \Lambda^{-1}\|_\infty \|\Lambda \rho \Lambda\|_1
\]

and use the assumption \( \Lambda \rho \Lambda \in \mathcal{J}_1 \). For the first factor on the r.h.s. one calculates for \( f \in C_0^\infty(\mathbb{R}^3) \):

\[
\|\Lambda \phi[\rho] \Lambda^{-1} f\|_2^2 = \|\nabla (\phi[\rho] \Lambda^{-1} f)\|_2^2 + \|\sqrt{1 + |x|^2} \phi[\rho] \Lambda^{-1} f\|_2^2
\]

We rewrite the operator of the first term on the r.h.s. as

\[
\nabla (\phi[\rho] \Lambda^{-1}) = \left[ (\nabla \phi[\rho]) + \phi[\rho] \nabla (1 - \Delta)^{-1/2} \right] \left[ (1 - \Delta)^{1/2} \Lambda^{-1} \right],
\]

where both factors are in \( B(L^2(\mathbb{R}^3)) \). The first factor is bounded since \( \nabla \phi[\rho] \in L^1(\mathbb{R}^3) \) and since \( (1 - \Delta)^{-1/2} \) is a bounded map from \( L^2(\mathbb{R}^3) \) into \( H^1(\mathbb{R}^3) \rightarrow L^6(\mathbb{R}^3) \), due to a Sobolev imbedding.

Summarizing we obtain

\[
\|\phi[\rho], \rho\|_\mathcal{E} \leq C \|\rho\|_\mathcal{E}^2, \quad \forall \rho \in \mathcal{E},
\]

and the Lipschitz continuity then follows in a straightforward way.
We remark that the nonlinear map \( \rho \mapsto [\phi(\rho), \rho] \) is continuous in \( \mathcal{E} \), but not in \( \mathcal{J}^* \). However, the linear evolution problem \( \mathbf{4.1} \) in general does not generate a contractive QDS on \( \mathcal{E} \subset \mathcal{J} \), except in the case of a unitary dynamic (i.e. \( L_j = 0 \)). Hence, in order to obtain a global-in-time (nonlinear) existence and uniqueness result, we can not apply the results of [AlMe], which would require contractivity of the linear QDS in \( \mathcal{E} \).

In the nonlinear evolution problem \( \mathbf{4.4} \) the situation is even worse. Already in the case of a unitary time-evolution only \( E^{\text{tot}} \) is conserved (for \( \mu = 0 \)), whereas \( \|\rho(t)\|_\rho \) is not, due to the possible energy exchange between the potential and the kinetic parts. Hence a unitary but self-consistent evolution problem does not generate a contractive semigroup in \( \mathcal{E} \) either.

With the above results, we are able to state the following local-in-time result:

**Theorem 4.6.** Let \( \rho_0 \in \mathcal{E}, \ d = 3 \) and \( V_1 \in L^\infty(\mathbb{R}^3) \) s.t. \( \nabla V_1 \in L^q(\mathbb{R}^3) \), for some \( 3 \leq q \leq \infty \), then:

(a) Locally in time, the nonlinear evolution problem \( \mathbf{4.4} \) has a unique mild solution \( \tilde{\Phi}_t(\rho_0) \in C([0, T), \mathcal{E}) \), where \( \tilde{\Phi}_t(\cdot) \) denotes the nonlinear semigroup obtained by perturbing the linear QDS with the Hartree potential. This self-consistent potential satisfies: \( \phi \in C([0, T); C_0(\mathbb{R}^3)) \). The map \( \rho_0 \mapsto \tilde{\Phi}_t(\rho_0) \) is Lipschitz continuous on some (small enough) ball \( \{\|\rho - \rho_0\|_\rho < \varepsilon\} \subset \mathcal{E}, \) uniformly for \( 0 \leq t \leq T_1 < T \). Further, if the maximum time of existence \( T > 0 \) is finite, we have

\[
\lim_{t \to T} \|\tilde{\Phi}_t(\rho_0)\|_\varepsilon = \infty. \tag{4.20}
\]

(b) For \( L(\rho_0) \in \mathcal{E} \) we obtain a classical solution \( \tilde{\Phi}_t(\rho_0) \in C^1([0, T), \mathcal{E}) \).

(c) The semigroup \( \tilde{\Phi}_t \) is conservative.

(d) The semigroup \( \tilde{\Phi}_t \) is positivity preserving and contractive on \( \mathcal{J}^*(L^2(\mathbb{R}^3)) \). Hence, it furnishes a nonlinear QDS: \( \tilde{\Phi}_t : \mathcal{E} \to \mathcal{E} \subset \mathcal{J}^* \).

**Proof.** Part (a, b): By proposition \( \mathbf{4.4} \) the unique conservative QDS \( \Phi_t \), obtained from theorem \( \mathbf{3.3} \) also maps the energy space \( \mathcal{E} \) into itself. Lemma \( \mathbf{4.6} \) and a standard perturbation result (cf. theorem 6.1.4 in [Pa]) then yield the local-in-time existence of a solution for the nonlinear, i.e. mean field problem. The continuity of \( \phi \) follows from the proof of lemma \( \mathbf{4.6} \) using \( \Phi_t(\rho_0) \in C([0, T); \mathcal{E}) \). The local Lipschitz continuity of the map \( \rho_0 \mapsto \Phi_t(\rho_0) \) follows from theorem 6.1.2 in [Pa] and the uniform lower bound for the existence time of trajectories \( \Phi_t(\rho) \) that start in the neighborhood of \( \rho_0 \) (cf. proof of theorem 6.1.4 in [Pa]).

Part (c): The proof follows from Duhamel’s representation, analogous to \( \mathbf{3.20} \).

Part (d): Having in mind the result of part (a), we consider the nonlinear evolution problem \( \mathbf{4.1} \) as a linear evolution problem with time-dependent Hamiltonian and write it in the following form:

\[
\begin{align*}
\frac{d}{dt} \rho &= -i [H, \rho] + A(\rho) - i[\phi(t), \rho], \quad t > 0, \\
\rho(0) &= \rho_0 \geq 0.
\end{align*}
\tag{4.21}
\]

Here, \( \phi \in C([0, T); C_0(\mathbb{R}^3)) \) is the self-consistent potential \( \phi(\rho) \). To prove the assertions of part (d), we shall approximate \( \phi(t) \) on \([0, T_1], \ T_1 < T \), by the
piecewise constant potential:

\[ \vartheta(t) := \phi(t_n), \quad t_n \leq t < t_{n+1}, \quad 0 \leq n \leq N - 1, \]

with the uniform grid points: \( t_n = n \Delta t, \Delta t = T_1/N \). Hence, \( \rho(t), t \in [0, T_1] \) is approximated by \( \varsigma_N \in C([0, T_1]; J_s^t(L^2(\mathbb{R}^3))) \), solving

\[
\begin{align*}
\frac{d}{dt} \varsigma_N &= -i[H, \varsigma_N] + A(\varsigma_N) - i[\vartheta(t), \varsigma_N], \quad t > 0, \\
\varsigma_N(0) &= \rho_0 \geq 0.
\end{align*}
\] (4.22)

Since \( \vartheta(t) \in C_b(\mathbb{R}^3) \), corollary 3.10 applies to the generator in (4.22) on each time-interval \([t_n, t_{n+1}]\). In summary we have the following facts:

\( \phi \) is uniformly continuous on \([0, T_1]\) w.r.t. \( \| \cdot \|_\infty \), the solutions of (4.21) satisfies:

\[ \| \rho(t) \|_1 \leq K, \quad 0 \leq t \leq T_1, \]

and the propagator corresponding to (4.22) is contractive on \( J_s^t(L^2(\mathbb{R}^3)) \).

With these ingredients it is standard to verify that

\[ \lim_{N \to \infty} \varsigma_N = \rho, \quad \text{in} \quad C([0, T_1]; J_s^t(L^2(\mathbb{R}^3))), \]

\( \text{cf.} \) the proof of theorem 1 in [AlMe] e.g. Hence, the positivity of \( \rho(t) = \tilde{\Phi}_t(\rho_0) \) follows from the positivity of \( \varsigma_N(t) \).

Analogously, the contractivity of the propagator corresponding to (4.22) implies the contractivity of \( \tilde{\Phi}_t(\rho_0) \) in \( J_s^t(L^2(\mathbb{R}^3)) \).

**Remark 4.7.** If no confinement potential is present and \( \text{Im}(\alpha_{j,k}) = 0 \), \( \forall j, k, l \), then theorem 4.6 also holds in the kinetic energy space \( E_{\text{kin}} \). In particular, this is true for the QFP equation, where one can derive an exact ODE for the kinetic energy, \( \text{cf.} \) [ALMS].

In the next section we shall derive a-priori estimates on \( \tilde{\Phi}_t(\rho) \) to prove the global-in-time existence of a conservative QDS for the mean field problem.

## 5 A-priori estimates and global existence of the mean field QDS

From theorem 4.6 we already know that \( \| \rho(t) \|_1 = \| \rho_0 \|_1 \), for \( 0 \leq t < T \). It remains to prove an a-priori estimate on the energy of the nonlinear system. As a preliminary step, we introduce a generalized version of the Lieb-Thirring inequality:

**Lemma 5.1.** Assume \( d = 3 \) and let \( \rho \in J_s^t \), \( \rho \geq 0 \) be s.t. \( E_{\text{kin}}[\rho] < \infty \). Then the following estimate holds:

\[ \| n[\rho] \|_p \leq K_p \| \rho \|_1^\theta E_{\text{kin}}[\rho]^{1-\theta}, \quad 1 \leq p \leq 3, \] (5.1)

with

\[ \theta := \frac{3-p}{2p}. \] (5.2)

**Proof.** The proof is given in the appendix of [Ar], \( \text{cf.} \) also [LiPa].
In the sequel this estimate will be used to derive an a-priori bound for the total energy.

**Proposition 5.2.** Assume $\rho_0 \in \mathcal{E}$, $\rho_0 \geq 0$ and $d = 3$. Then there exists a $K > 0$ such that

$$E^{\text{tot}}[\rho(t)] \leq e^{Kt} E^{\text{tot}}[\rho_0], \quad 0 \leq t < T,$$

where $\rho(t) := \hat{\Phi}_t(\rho_0)$, denotes the unique local-in-time solution of the nonlinear evolution problem \[4.7\].

**Proof.** Since $\hat{\Phi}_t$ is positivity preserving, we assume w.r.o.g. $\rho_0 \geq 0$ and hence have $\rho(t) \geq 0$, for all $0 \leq t < T$. The idea is again to derive a differential inequality for $E^{\text{tot}}$. We first consider a classical solution $\hat{\Phi}_1(\rho_0) \in C^1([0, T], \mathcal{E})$ obtained from an initial condition with $\mathcal{L}(\rho_0) \in \mathcal{E}$.

**Step 1:** We calculate the time derivative of the total energy, using the short notation $\dot{\rho} \equiv \frac{d}{dt}\rho$.

$$\frac{d}{dt} E^{\text{tot}}[\rho] = \frac{d}{dt} \text{Tr} \left( -\frac{1}{2} \Delta \rho \phi + \frac{1}{2} |x| \rho \frac{\partial \rho}{\partial x} + \phi \rho \right) - \frac{1}{2} \frac{d}{dt} \text{Tr}(\phi \rho)$$

$$= \text{Tr} \left( -\frac{1}{2} \Delta \dot{\rho} \phi + \frac{1}{2} |x| \dot{\rho} \frac{\partial \rho}{\partial x} + \phi \dot{\rho} \right) + \text{Tr}(\dot{\phi} \rho)$$

$$- \frac{1}{2} \frac{d}{dt} \text{Tr}(\phi \rho). \quad (5.4)$$

For our classical solution $\rho(t)$ the calculation \[5.4\] is rigorous since $||\rho||_\mathcal{E} \in C^1([0, T])$ and the self-consistent potential satisfies $\Phi \in C^1([0, T]; C_b(\mathbb{R}))$.

In order to simplify the last term on the r.h.s. of \[5.4\] we evaluate the trace in the eigenbasis of $\rho$ (cf. \[2.2\]). This gives

$$\frac{1}{2} \frac{d}{dt} \text{Tr}(\dot{\rho} \rho) = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \dot{\phi}(x) n(x) dx.$$

We now proceed as in \[Ar\]: Integrating by parts several times and using the Poisson equation \[2.8\], we obtain

$$\frac{1}{2} \frac{d}{dt} \text{Tr}(\dot{\rho} \rho) = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla \phi||^2 dx = - \int_{\mathbb{R}^3} \dot{\phi} \rho (x) \Delta \phi \rho (x) dx$$

$$= \int_{\mathbb{R}^3} \dot{\phi} \rho (x) n(x) dx = \text{Tr}(\dot{\phi} \rho).$$

Inserting this into \[5.4\], we get

$$\frac{d}{dt} E^{\text{tot}}[\rho] = \text{Tr} \left( -\frac{1}{2} \Delta \dot{\rho} \phi + \frac{1}{2} |x| \dot{\rho} \frac{\partial \rho}{\partial x} + \phi \dot{\rho} \right)$$

$$= \text{Tr} \left( -\frac{1}{2} \Delta \mathcal{L}(\rho) \phi + \frac{1}{2} |x| \mathcal{L}(\rho) |x| + \phi \rho \mathcal{L}(\rho) \right). \quad (5.5)$$

In the following, we shall derive a differential inequality for $E^{\text{tot}}[\rho]$ from \[5.5\]. This expression is now considerably easier to deal with, since the self-consistent potential enters as if it was an additional external field (note that the factor $1/2$ in front of $\phi \rho$ has been eliminated).
Step 2: Similarly to the proof of lemma 4.3, we introduce an energy-functional

\[ E_{k,l}^{\text{tot}}[\rho] := E_{k,l}^{\text{kin}}[\rho] + E_{k,l}^{\text{ext}}[\rho] + \frac{1}{3} E_{k,l}^{\text{sc}}[\rho], \quad k, l = 1, 2, 3, \]

where \( E_{k,l}^{\text{kin}} \), \( E_{k,l}^{\text{ext}} \) are defined as in (4.14). Again, for all \( \rho \in \mathcal{D}_\infty \), we have

\[ E_{k,l}^{\text{tot}}[\rho] = \sum_{k=1}^{3} E_{k,k}^{\text{tot}}[\rho] \]

and, by a density argument, this carries over to \( \rho \in \mathcal{E} \). After some lengthy, but straightforward calculations (with extensive use of the cyclicity of the trace), we get from (5.5), the following equation:

\[
\frac{d}{dt} \sum_{k=1}^{3} E_{k,k}^{\text{tot}} = \left( \frac{d}{dt} \sum_{k=1}^{3} E_{k,k}^{\text{kin}} - i \sum_{k=1}^{3} \Tr((\partial_k^2 \phi[\rho])\rho + (\partial_k \phi[\rho])(\partial_k \rho)) \right) \\
+ \frac{d}{dt} \sum_{k=1}^{3} E_{k,k}^{\text{ext}} + 2i \sum_{k=1}^{3} \Tr(x_k \rho(\partial_k \phi[\rho])) \\
- i \sum_{k,l=1}^{m} \Im(\overline{\gamma_{j,k}} \beta_{j,l}) \Tr(x_k \rho(\partial_l \phi[\rho])) \\
- i \sum_{k=1}^{3} \sum_{l=1}^{m} \Im(\overline{\gamma_{j,k}}) \Tr(\rho(\partial_l \phi[\rho])).
\]  

(5.6)

Note that the first term of the r.h.s. of (5.6) – in big brackets – equals the time derivative of \( E_{k,k}^{\text{kin}} \) under the linear time-evolution. It is given by (4.16). On the other hand, one easily checks that the time derivative of \( E_{k,k}^{\text{ext}} \) under the nonlinear time-evolution is equal to the linear one, hence given by (4.17). Since these kinetic and the external (potential) energy terms can be treated (by interpolation arguments) as in the proof of lemma 4.3, it remains to estimate the last three terms on the r.h.s. of (5.6).

Keep in mind, that we want to use a Gronwall lemma in the end. Hence, we need to find appropriate linear bounds for the r.h.s. of (5.6). (In the following we shall denote by \( K \) positive, not necessarily equal, constants.)

Step 3: We first consider the term \( \Tr(\rho(\partial_k \phi[\rho])) \). In order to calculate the trace, we need to guarantee that \( \rho(\partial_k \phi[\rho]) \in \mathcal{J}_1 \). Using the Sobolev inequality we estimate for \( \varphi \in L^2(\mathbb{R}^3) \):

\[ \| (\sqrt{-\Delta} + I)^{-1} \varphi \|_6 \leq K \| (\sqrt{-\Delta} + I)^{-1} \varphi \|_{H^1} \leq K \| \varphi \|_2, \]

since \( \| (\sqrt{-\Delta} + I) \cdot \|_2 \) is an equivalent norm to \( \| \cdot \|_{H^1} \). Hölder’s inequality and the bounds obtained in the proof of lemma 4.5 then imply

\[ \| (\partial_k \phi[\rho]) (\sqrt{-\Delta} + I)^{-1} \varphi \|_2 \leq \| \partial_k \phi[\rho] \|_{L^3} \| (\sqrt{-\Delta} + I)^{-1} \varphi \|_6 \leq K \| \partial_k \phi[\rho] \|_3 \| \varphi \|_2. \]

In other words, \( (\partial_k \phi[\rho]) (\sqrt{-\Delta} + I)^{-1} \) is a bounded operator on \( L^2(\mathbb{R}^3) \) and we get

\[ \| \rho(\partial_k \phi[\rho]) \|_1 \leq \| (\partial_k \phi[\rho]) (\sqrt{-\Delta} + I)^{-1} \|_{L^\infty} \| (\sqrt{-\Delta} + I) \rho \|_1 \leq K \| \partial_k \phi[\rho] \|_3 (E_{k,k}^{\text{kin}}[\rho] + \| \rho \|_1). \]
Thus $\rho(\partial_k \phi[\rho]) \in J_1$, so we can calculate its trace in the eigenbasis of $\rho$ and estimate it:

$$|\text{Tr}(\rho(\partial_k \phi[\rho]))| = \int_{\mathbb{R}^3} \partial_k \phi[\rho](x) n[\rho](x) dx \leq \|\nabla \phi[\rho]\|_2 \|n[\rho]\|_2.$$  

The generalized Young inequality and the Lieb-Thirring inequality (5.1) imply

$$\|\nabla \phi[\rho]\|_2 \leq K\|n[\rho]\|_{6/5} \leq K\|\rho\|_1^{3/4} E^{\text{kin}}[\rho]^{1/4}. \quad (5.7)$$

Further, using again (5.1), we have

$$\|n[\rho]\|_2 \leq K\|\rho\|_1^{1/4} E^{\text{kin}}[\rho]^{3/4}. \quad (5.8)$$

Hence, we obtain the following estimate:

$$|\text{Tr}(\rho(\partial_k \phi[\rho]))| \leq K\|\rho\|_1 E^{\text{kin}}[\rho], \quad (5.9)$$

which is suitable for our purpose, due to the linear dependence on $E^{\text{kin}}[\rho]$.

**Step 4:** Next, we need to estimate the term

$$\sum_{k,l=1}^{3} \xi_{k,l} \text{Tr}(x_k \rho(\partial_l \phi[\rho])),$$

with the short-hand $\xi_{k,l} := \text{Im}(\alpha_{j,k} \beta_{j,l})$. To guarantee that $x_k \rho(\partial_l \phi[\rho]) \in J_1$, we only need to show $\sqrt{\rho}(\partial_l \phi[\rho]) \in J_2$, since we already know $x_k \sqrt{\rho} \in J_2$.

This can be done as in step 3 above by noting that $\sqrt{\rho}(\sqrt{-\Delta} + I) \in J_2$ and $(\sqrt{-\Delta} + I)^{-1} \partial \phi[\rho] \in B(L^2(\mathbb{R}^3))$.

Hence, we can again calculate $\text{Tr}(x_k \rho(\partial_l \phi[\rho]))$ in the eigenbasis of $\rho$:

$$\sum_{k,l=1}^{3} \xi_{k,l} \text{Tr}(x_k \rho(\partial_l \phi[\rho])) = \sum_{k,l=1}^{3} \xi_{k,l} \int_{\mathbb{R}^3} x_k \partial_l \phi[\rho](x) n[\rho](x) dx$$

$$= - \sum_{k,l,m=1}^{3} \xi_{k,l} \int_{\mathbb{R}^3} x_k \partial_l \phi[\rho](x) \partial^2_{m,m} \phi[\rho](x) dx, \quad (5.10)$$

where we have used the Poisson equation (2.8) for the last equality. Integration by parts gives

$$\sum_{k,l=1}^{3} \xi_{k,l} \text{Tr}(x_k \rho(\partial_l \phi[\rho])) = \sum_{k,l=1}^{3} \xi_{k,l} \int_{\mathbb{R}^3} \partial_l \phi[\rho](x) \partial_k \phi[\rho](x) dx$$

$$+ \sum_{k,l,m=1}^{3} \xi_{k,l} \int_{\mathbb{R}^3} x_k \partial^2_{l,m} \phi[\rho](x) \partial_m \phi[\rho](x) dx.$$
Adding the equations (5.10) and (5.9) yields, after another integration by parts:

\[
2 \sum_{k,l=1}^{3} \xi_{k,l} \text{Tr}(x_{k} \rho (\partial_{l} \phi[\rho])) = \sum_{k,l=1}^{3} \xi_{k,l} \int_{\mathbb{R}^3} \partial_{l} \phi \partial_{k} \phi \, dx \sum_{k,l,m=1}^{3} \left[ x_{k} \partial_{m} \phi \partial_{l,m}^{2} \phi - x_{k} \partial_{l} \phi \partial_{m,m}^{2} \phi \right] \, dx
\]

\[
= \sum_{k,l=1}^{3} \xi_{k,l} \int_{\mathbb{R}^3} \partial_{l} \phi \partial_{k} \phi \, dx - \sum_{k,l,m=1}^{3} \xi_{k,l} \int_{\mathbb{R}^3} \left[ \delta_{k,m} \partial_{l,m}^{2} \phi + x_{k} \partial_{l,m,m}^{3} \phi - \delta_{k,l} \partial_{m,m}^{2} \phi - x_{k} \partial_{l,m}^{3} \phi \right] \, dx
\]

\[
= 2 \sum_{k,l=1}^{3} \xi_{k,l} \int_{\mathbb{R}^3} \partial_{l} \phi \partial_{k} \phi \, dx - \sum_{k,m=1}^{3} \xi_{k,k} \int_{\mathbb{R}^3} |\partial_{m} \phi|^{2} \, dx,
\]

(5.11)

where we write \(\phi \equiv \phi[\rho]\) for simplicity and denote by \(\delta_{k,l}\) the Kronecker symbol.

Therefore we can estimate

\[
\left| \sum_{k,l=1}^{3} \xi_{k,l} \text{Tr}(x_{k} \rho (\partial_{l} \phi[\rho])) \right| \leq K \|\nabla \phi[\rho]\|_{2}^{2},
\]

where \(K\) depends on the coefficients \(\xi_{k,l}\). Hence, using the same estimates as in (5.7), we have

\[
\left| \sum_{k,l=1}^{3} \xi_{k,l} \text{Tr}(x_{k} \rho (\partial_{l} \phi[\rho])) \right| \leq K \|\rho\|_{1}^{3/2} E_{\text{kin}}^{1/2}[\rho]^{1/2}
\]

\[
\leq K \|\rho\|_{1} \left( \|\rho\|_{1} + E_{\text{kin}}^{1/2}[\rho] \right),
\]

which is the desired linear bound.

The third term in (5.6) can be treated analogously to the previous case.

**Step 5:** The steps 1-4, together with the estimates obtained in the proof of lemma 4.3, imply

\[
\frac{d}{dt} E_{\text{tot}}^{\rho}(t) \leq K E_{\text{tot}}^{\rho}(t), \quad 0 \leq t < T,
\]

(5.12)

with some generic constant \(K \geq 0\). Applying Gronwall’s lemma then proves the assertion.

Strictly speaking, all the calculations of steps 2 – 5 first have to be done for an approximating sequence \(\{\sigma_{n}\} \subseteq D_{\infty}\) such that \(\sigma_{n} \to \rho(t)\) in \(E\) for each fixed \(t \in [0, T)\) (cf. the proof of theorem 5.9). The estimate (5.12) then also holds for the limit \(\rho(t)\) since the constant \(K\) is independent of \(\{\sigma_{n}\}\).

**Step 6:** So far we have proved (5.8) for classical solutions. By theorem 4.6(a) any mild solution (i.e. \(\Phi(t, \rho_{0}) \in C([0, T), E)\) can be approximated in \(E\) uniformly on \(0 \leq t \leq T_{1} < T\) by classical solutions. Hence (5.8) carries over to all initial conditions \(\rho_{0} \in E\) with \(\rho_{0} \geq 0\).

In view of (4.1), and since \(\|\rho(t)\|_{E} \leq E_{\text{tot}}^{\rho}(t)\) we conclude from the above proposition that \(T = \infty\) and obtain our main result:
Theorem 5.3. Let $\rho_0 \in \mathcal{E}$, $d = 3$ and $V_1 \in L^\infty(\mathbb{R}^3)$ s.t. $\nabla V_1 \in L^q(\mathbb{R}^3)$, for some $3 \leq q \leq \infty$:
Then, the nonlinear evolution problem (4.7) admits a unique mild solution, i.e. it generates a nonlinear conservative QDS: $\Phi_t(\rho_0) \in C([0,\infty), \mathcal{E})$.

6 Appendix: Proof of Lemma 3.7

Without loss of generality we can assume that $\rho$ is a nonnegative operator. (Otherwise one can split $\rho$ into its positive and negative part and prove the result separately for each one.) Its eigenvalues are $\lambda_j \geq 0$ and the eigenvectors $\psi_j$ are orthonormal.

Part (a): For each $\rho \in \mathcal{J}_1^q$ with finite rank $N \in \mathbb{N}$ we shall show that the approximation sequence $\{\sigma_n\} \subset \mathcal{D}^\infty$, defined in (3.17), satisfies $\sigma_n \to \rho$ in $\mathcal{J}_1$.

Part (b): The inclusion $\mathcal{D}(Z) \subset \mathcal{D}(\mathcal{L})$ is already clear from proposition 3.10. Thus it remains to show that for each $\sigma_n \in \mathcal{D}_n \subset \mathcal{D}^\infty$, with some fixed $n \in \mathbb{N}$, we have $Z(\sigma_n) \in \mathcal{J}_1^q$: First, note that $Z(\sigma_n) := Y\sigma_n + \sigma_n Y^*$ is a linear combination of the following terms (and their adjoints)

$$x_k\sigma_n x_l, \partial_k \sigma_n \partial_l, \partial_k \sigma_n x_l, x_k x_l \sigma_n, \partial_k \partial_l \sigma_n, x_k \partial_l \sigma_n, x_k \sigma_n, \partial_k \sigma_n, \quad (A.1)$$

where $1 \leq k, l \leq d$ and $\partial_k := \partial_{x_k}$. (Indeed not all of these terms really appear in the expression of $Z$, but since the same argument for $\mathcal{L}$ is needed in the proof of theorem 3.9 we shall consider this more general case.)

Since $\sigma_n$ has a representation given by $\sigma_n = M(\chi_n)C(\varphi_n)\rho C(\varphi_n)M(\chi_n)$, for some $\rho \in \mathcal{J}_1^q$, we have to prove that the operator compositions $x^a \nabla^b M_n C_n$ are in $\mathcal{B}(L^2(\mathbb{R}^d))$. Here the multi-indices $a, b \in \mathbb{N}_0^d$ are such that $|a| + |b| \leq 2$. As an example we consider the operator $x_k \partial_l$ and write for $f \in L^2(\mathbb{R}^d)$:

$$x_k\partial_l M_n C_n f(x) = x_k \partial_l (\chi_n(x)(\varphi_n \ast f))(x) = x_k [\partial_l \chi_n(x)(\varphi_n \ast f)(x) + \chi_n(x)(\partial_l \varphi_n \ast f)(x)].$$

Since $\varphi, \chi \in C_0^\infty$ (see the proof of lemma 3.3) we have that

$$\|x_k \partial_l M_n C_n f\|_2 \leq K_{k,l,n} \|f\|_2$$

and thus $x_k \partial_l M_n C_n \in \mathcal{B}(L^2(\mathbb{R}^d))$. Hence $x_k \partial_l \sigma_n = x_k \partial_l M_n C_n \rho C_n M_n \in \mathcal{J}_1^q$.

The other terms in (A.1) can then be handled in a similar way.

Part (c): After the proof of part (a) it remains to show that for all $\rho \in \mathcal{J}_1^q$ with $\mathcal{L}(\rho) \in \mathcal{J}_1^q$, the following statement holds:

$$\lim_{n \to \infty} \|\mathcal{L}(\sigma_n) - \mathcal{L}(\rho)\|_1 = 0.$$
one. This simplification is possible since no cancellation occurs between the individual terms of $K\rho$. To simplify the notation further, we shall from now on write $v := x_1$, $\partial := \partial_{x_1}$. We choose $K$ in the form

$$K\rho = K_1\rho + K_1^*,$$

where

$$K_1\rho = v\rho v + \partial \rho \partial + \partial v\rho + v^2\rho + \partial^2\rho + v\partial\rho + v\rho + \partial \rho.$$

The general ($d$-dimensional) case $L\rho = -i[H,\rho] + A(\rho)$ described above is then a straightforward extension. The proof now follows again in several steps:

**Step 1:** We write

$$K(\sigma_n) = K(M(\chi_n)C(\varphi_n)\rho C(\varphi_n)M(\chi_n))$$

$$= M(\chi_n)C(\varphi_n)K\rho C(\varphi_n)M(\chi_n) + R_n(\rho) + R_n(\rho)^*.$$

Since $K\rho \in J_1^*$, we can decompose it into $K\rho = K_+(\rho) - K_-(\rho)$, $K_\pm(\rho) \geq 0$. Applying part (a) of this lemma then yields

$$\lim_{n \to \infty} \| M(\chi_n)C(\varphi_n)K\rho M(\chi_n)C(\varphi_n) - K\rho \|_1 = 0.$$

It remains to prove that $R_n(\rho) \to 0$ in $J_1$, as $n \to \infty$, which also implies $R_n(\rho)^* \to 0$ in $J_1$. For technical reasons (which will become clear in step 3) we split this remainder term into two parts: $R_n(\rho) = R^1_n(\rho) + R^2_n(\rho)$, and treat each of them separately.

**Step 2:** After some lengthy calculations, $R^1_n(\rho)$ can be written as

$$R^1_n(\rho) = M(\partial\chi_n)C(\varphi_n)\rho C(\varphi_n)M(\partial\chi_n)$$

$$+ M(\partial^2\chi_n)C(\varphi_n)\rho C(\varphi_n)M(\chi_n) + M(\chi_n)C(v\varphi_n)\rho C(\varphi_n)M(\chi_n)$$

$$+ M(\partial\chi_n)C(\varphi_n)\rho C(\varphi_n)M(\chi_n) - M(\chi_n)C(v^2\varphi_n)\rho C(\varphi_n)M(\chi_n)$$

$$- 2M(\chi_n)C(v\varphi_n)\rho C(\varphi_n)M(\chi_n),$$

where, on the level of the kernels, we have used several times the basic identity $v(f * g) = vf *g + f * vg$. Now we calculate for $f \in L^2(\mathbb{R}^d)$ (remember $v = x_1$)

$$\langle C(x_1\varphi_n)f, (x) \rangle := \int_{\mathbb{R}^d} (x_1 - y_1) \varphi_n(x - y) f(y) dy$$

$$= \frac{1}{n} \int_{\mathbb{R}^d} n^{d+1}(x_1 - y_1) \varphi(n(x - y)) f(y) dy = O\left(n^{-1}\right).$$

Thus we have $\|C(v\varphi_n)\|_\infty = O\left(n^{-1}\right)$ and similarly we obtain

$$\|C(\varphi_n)\|_\infty = \|M(\chi_n)\|_\infty = O(1),$$

$$\|M(\partial\chi_n)\|_\infty = O\left(n^{-1}\right),$$

$$\|C(v^2\varphi_n)\|_\infty = \|M(\partial^2\chi_n)\|_\infty = O\left(n^{-2}\right).$$
With these relations we can estimate
\[
\| R_n^1(\rho) \|_1 \leq \| \rho \|_1 \left( \| M(\chi_n) \|_\infty^2 \| C(v\varphi_n) \|_{\infty}^2 + \| \rho \|_1 \| M(\chi_n) \|_\infty^2 \| C(\varphi_n) \|_{\infty} \right) + \| \rho \|_1 \| M(\chi_n) \|_\infty^2 \| C(\varphi_n) \|_{\infty} \right) + \| \rho \|_1 \| M(\chi_n) \|_\infty^2 \| M(\partial \chi_n) \|_{\infty}^2 + \| \rho \|_1 \| M(\chi_n) \|_{\infty} \| M(\varphi_n) \|_{\infty} \right) + \| \rho \|_1 \| C(\varphi_n) \|_{\infty} \| M(\chi_n) \|_\infty^2 \| M(\varphi_n) \|_{\infty} \right) + \| \rho \|_1 \| C(\varphi_n) \|_{\infty} \| C(\varphi_n) \|_{\infty} \| M(\partial \chi_n) \|_{\infty}^2 \right) = O \left( n^{-1} \right).
\]
Thus \( R_n^1(\rho) \to 0 \) uniformly in \( J_1 \), as \( n \to \infty \).

**Step 3:** Again a lengthy, but straightforward calculation shows that the second part of the remainder can be written in the form
\[
R_n^2(\rho) = M(n\partial \chi_n)C(\varphi_n)\rho C(\frac{\partial \varphi_n}{n})M(\chi_n)
\]
\[
+ M(\chi_n)C(\partial(v\varphi_n))\rho C(\varphi_n)M(\chi_n) + M(\chi_n)C(\frac{\partial \varphi_n}{n})\rho C(\varphi_n)M(n\partial \chi_n)
\]
\[
+ M(n\partial \chi_n)C(\varphi_n)\rho C(\varphi_n)M(\frac{v}{n}\chi_n) + M(\chi_n)C(\frac{\partial \varphi_n}{n})\rho C(nv\varphi_n)M(\chi_n)
\]
\[
+ M(\frac{v}{n}\chi_n)C(\varphi_n)\rho C(nv\varphi_n)M(\chi_n) + M(\chi_n)C(nv\varphi_n)\rho C(\varphi_n)M(\frac{v}{n}\chi_n)
\]
\[
+ 2M(\frac{v}{n}\chi_n)C(nv\varphi_n)\rho C(\varphi_n)M(\chi_n).
\]

In contrast to step 2 these terms do not converge to zero uniformly in \( J_1 \), hence we shall proceed differently:

As an example we consider the ninth term on the right hand side and write
\[
M(\chi_n)C(nv\varphi_n)\rho C(\varphi_n)M(\frac{v}{n}\chi_n) = M(\chi_n)C(nv\varphi_n)\rho^N C(\varphi_n)M(\frac{v}{n}\chi_n)
\]
\[
+ M(\chi_n)C(nv\varphi_n)(\rho - \rho^N) C(\varphi_n)M(\frac{v}{n}\chi_n),
\]
where \( \rho^N \) is the trace class operator \( \rho \) “cut” at finite rank \( N \in \mathbb{N} \), such that \( \| \rho - \rho^N \|_1 \leq \varepsilon, \varepsilon \in \mathbb{R}_+ \). Direct calculations, similar to the one in step 2, imply
\[
\| C(nv\varphi_n) \|_{\infty} \leq K, \| M(n^{-1}v\chi_n) \|_{\infty} \leq K, K \in \mathbb{R},
\]
with \( K \) independent of \( n \in \mathbb{N} \). Thus we can estimate
\[
\| M(\chi_n)C(nv\varphi_n)(\rho - \rho^N) C(\varphi_n)M(\frac{v}{n}\chi_n) \|_1 \leq \varepsilon K^2.
\]
Define \( \Pi \) to be the projector on \( \text{ran}(\rho^N) \). Then \( \rho^N = \Pi \rho^N \) and
\[
\| C(nv\varphi_n)\rho^N \|_1 \leq \| C(nv\varphi_n)\Pi \|_{\infty} \| \rho^N \|_1.
\]
Now, since \( \dim(\text{ran}(\rho^N)) < \infty \) and since strong convergence equals uniform convergence on finite dimensional spaces [Riesz], we get
\[
\lim_{n \to \infty} \| C(nv\varphi_n)\Pi \|_{\infty} = 0.
\]
Combining (A.2) - (A.5) we thus have

$$\lim_{n \to \infty} \| M(\chi_n) \, C(nv\varphi_n) \rho^N \, C(\varphi_n) M\left(\frac{v}{n}\chi_n\right)\|_1 = 0.$$  \hfill (A.6)

Combining (A.3) and (A.6) shows that

$$\| M(\chi_n) \, C(nv\varphi_n) \rho \, C(\varphi_n) M\left(\frac{v}{n}\chi_n\right)\|_1$$

can be made arbitrarily small for $N$ sufficiently large. All other terms appearing in the expression of $R^2_n$ can now be treated in the same way.

In summary we have proved in steps 1 to 3 the assertion of the lemma.

Acknowledgement:
This work has been supported by the Austrian Science Foundation FWF through grant no. W8 and the Wittgenstein Award 2000 of Peter Markowich. Further support has been given by the European Union research network HYKE, by the DFG-project AR277/3-2 and by the DFG-Graduiertenkolleg: Nichtlineare kontinuierliche Systeme und deren Untersuchung mit numerischen, qualitativen und experimentellen Methoden.

References

[Al] R. Alicki, Invitation to quantum dynamical semigroups, in: P. Garbaczewski, R. Olkiewicz (eds.), Dynamics of Dissipation, Lecture Notes in Physics 597, Springer (2002).

[AlFa] R. Alicki, M. Fannes, Quantum dynamical systems, Oxford University Press 2001.

[AlMe] R. Alicki, J. Messer, Nonlinear quantum dynamical semigroups for many-body open systems, J. Stat. Phys. 32 (1983), no. 3, 299-312.

[ACD] A. Arnold, J. A. Carrillo, E. Dhamo, On the periodic Wigner-Poisson-Fokker-Planck system, J. Math. Anal. Appl. 275 (2002), 263-276.

[Ar] A. Arnold, Self-Consistent Relaxation-Time Models in Quantum Mechanics, Comm. PDE 21 (1996), no. 3/4, 473-506.

[Ar1] A. Arnold, The relaxation-time von Neumann-Poisson equation, in: Proceedings of ICIAM 95, Hamburg (1995), Oskar Mahrenholtz, Reinhard Mennicken (eds.), ZAMM 76 S2 (1996), 293-296.

[ALMS] A. Arnold, J. L. Lopez, P. A. Markowich, J. Soler, Analysis of Quantum Fokker-Planck Models: A Wigner Function Approach, to appear in Rev. Mat. Iberoam. (2004).

[BaMa] C. Bardos, N. Mauser, The weak coupling limit for systems of $N \to \infty$ quantum particles. State of the art and applications, to appear in: Proceedings Congrès National d’Analyse Numérique (2003).
Quantum Dynamical Semigroups

[Ba] J. Batt, *N-particle approximation to the nonlinear Vlasov-Poisson system*, Nonlinear Anal. 47 (2001), no. 3, 1445-1456.

[Bo] F. Bouchut, *Existence and uniqueness of a global smooth solution for the Vlasov-Poisson-Fokker-Planck system in three dimensions*, J. Funct. Anal. 111 (1993), no. 1, 239-258.

[BDF] A. Bove, G. Da Prato, G. Fano, *On the Hartree-Fock time-dependent problem*, Comm. Math. Phys. 49 (1976), 25-33.

[BrHe] W. Braun, K. Hepp, *The Vlasov dynamics and its fluctuations in the 1/N limit of interacting classical particles*, Comm. Math. Phys. 56 (1977), no. 2, 101-113.

[BrPe] H. P. Breuer, F. Petruccione, *Concepts and methods in the theory of open quantum systems*, in: F. Benatti, R. Floreanini (eds.), *Irreversible Quantum Dynamics*, Lecture Notes in Physics 622, Springer (2003).

[CaLe] A. O. Caldeira, A. J. Leggett, *Path integral approach to quantum Brownian motion*, Physica A 121 (1983), 587-616.

[CEFM] F. Castella, L. Erdös, F. Frommlet, P. Markowich, *Fokker-Planck equations as Scaling Limit of Reversible Quantum Systems*, J. Stat. Physics 100 (2000), no. 3/4, 543-601.

[CGQ] A. M. Chebotarev, J. C. Garcia, R. B. Quezada, *Interaction representation method for Markov master equations in quantum optics*, ANESTOC, Proc. of the 4th int. workshop, Trends in Math., Stochastic Analysis and Math. Physics, Birkhäuser 2001.

[ChFa] A. M. Chebotarev, F. Fagnola, *Sufficient Conditions for Conservativity of Quantum Dynamical Semigroups*, J. Funct. Anal. 118 (1993), 131-153.

[Da] E. B. Davies, *Quantum Theory of Open Systems*, Academic Press (1976).

[Da1] E. B. Davies, *Quantum dynamical semigroups and the neutron diffusion equation*, Rep. Math. Phys. 11 (1977), no. 2, 169-188.

[De] H. Dekker, *Quantization of the linearly damped harmonic oscillator*, Phys. Rev. A 16-5 (1977), 2126-2134.

[DeVi] L. Desvillettes, C. Villani, *On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: the linear Fokker-Planck equation*, Comm. Pure Appl. Math. 54 (2001), no. 1, 1-42.

[DeRi] P. Degond, C. Ringhofer, *Quantum moment hydrodynamics and the entropy principle*, J. Stat. Phys. 112(3) (2003) 587-628.

[Di] L. Diósi, *On high-temperature Markovian equations for quantum Brownian motion*, Europhys. Lett. 22 (1993), 1-3.

[Di1] L. Diósi, *Caldeira-Leggett master equation and medium temperatures*, Physica A 199 (1993), 517-526.
Quantum Dynamical Semigroups

[DHR] P. Domokos, P. Horak, H. Ritsch, *Semiclassical theory of cavity-assisted atom cooling*, J. Phys. B 34 (2001), 187-201.

[Dr] K. Dressler, *Steady states in plasma physics—the Vlasov-Fokker-Planck equation*, Math. Methods Appl. Sci. 12 (1990), no. 6, 471-487.

[ErYa] L. Erdös, H.-T. Yau, *Derivation of the nonlinear Schrödinger equation from a many body Coulomb system*, Adv. Theor. Math. Phys. 5 (2001), no. 6, 1169-1205.

[FaRe] F. Fagnola, R. Rebolledo, *Lectures on the qualitative analysis of Quantum Markov Semigroups*, Quantum Probab. White Noise Anal. 14 (2002), 197-239.

[FeVe] R. Feynman, F. L. Vernon, *The theory of a general quantum system interacting with a linear dissipative system*, Ann. Physics 24 (1963), 118-173.

[FMR] F. Frommlet, P. Markowich, C. Ringhofer, *A Wigner Function Approach to Phonon Scattering*, VLSI Design 9 (1999), no. 4, 339-350.

[GaZo] C. W. Gardiner, P. Zoller, *Quantum Noise*, Springer (2000).

[GiVe] J. Ginibre, G. Velo, *On a class of non linear Schrödinger equations with non local interaction*, Math. Z. 170 (1980), 109-136.

[HeLi] K. Hepp, E. H. Lieb, *The laser: a reversible quantum dynamical system with irreversible classical macroscopic motion*. Dynamical systems, theory and applications, Lecture Notes in Phys. 38, Springer (1975) 178-207.

[Ho] A. S. Holevo, *Covariant quantum dynamical semigroups: unbounded generators*, in: A. Bohm, H. D. Doebner, P. Kielanowski (eds.), *Irreversibility and Causality*, Lecture Notes in Physics 504, Springer (1998).

[HuMa] B. L. Hu, A. Matacz, *Quantum Brownian Motion in a Bath of Parametric Oscillators: A model for system-field interactions*, Phys. Rev. D 49 (1994), 6612-6635.

[JuTa] A. Jüngel, S. Tang, *Numerical approximation of the viscous quantum hydrodynamic model for semiconductors*, preprint (2004), available at: [http://numerik.mathematik.uni-mainz.de/~juengel/publications/juengel.html](http://numerik.mathematik.uni-mainz.de/~juengel/publications/juengel.html)

[Li] G. Lindblad, *On the generators of quantum mechanical semigroups*, Comm. Math. Phys. 48 (1976), 119-130.

[Li1] G. Lindblad, *Brownian motion of a quantum harmonic oscillator*, Rep. Math. Phys. 10 (1976), 393-406.

[LiPa] P. L. Lions, T. Paul, *Sur les measures de Wigner*, Rev. Math. Iberoamericana 9 (1993) 553-618.

[Lo] W. Louisell, *Quantum statistical properties of radiation*, John Wiley (1973).
[OC] R. F. O’Connell, *Wigner distribution function approach to dissipative problems in quantum mechanics with emphasis on decoherence and measurement theory*, J. Opt. B: Quantum Semiclass. Opt. 5 3 (2003), 349-359.

[Pa] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer (1983).

[ReSi1] M. Reed, B. Simon, *Methods of Modern Mathematical Physics Vol. 1*, Academic Press (1972).

[ReSi2] M. Reed, B. Simon, *Methods of Modern Mathematical Physics Vol. 2*, Academic Press (1975).

[Ri] H. Risken, *The Fokker-Planck Equation*, Springer Series on Synergetics, Springer (1989).

[Si] B. Simon, *Trace ideals and their applications*, Cambridge Univ. Press (1979).

[SCDM] C. Sparber, J. A. Carrillo, J. Dolbeault, P. Markowich, *On the Long Time behavior of the Quantum Fokker-Planck Equation*, to appear in Monatsh. f. Math. (2004).

[Sp] H. Spohn, *Kinetic equations from Hamiltonian dynamics: Markovian limits*, Rev. Modern Phys. 52 (1980) no. 3, 569-615.

[Sti] W. F. Stinespring, *Positive functions on C*-Algebras*, Proc. AMS 6 (1955), 211-216.

[St] M. A. Stroscio, *Moment-equation representation of the dissipative quantum Liouville equation*, Supperlattices and Microstructures 2 (1986), 83-87.

[Va] B. Vacchini, *Translation-covariant Markovian master equation for a test particle in a quantum fluid*, J. Math. Phys. 42 (2001), 4291-4312.

[Va1] B. Vacchini, *Quantum optical versus quantum Brownian motion master-equation in terms of covariance and equilibrium properties*, J. Math. Phys. 43 (2002), 5446-5458.

[Wi] E. Wigner, *On the quantum correction for the thermodynamical equilibrium*, Phys. Rev. 40 (1932), 742-759.