INFORMATION IN STOCK PRICES
AND
SOME CONSEQUENCES

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Summary

Stock prices will rarely follow the assumed model but, when stock’s transaction times are dense in the interval \([t_0, T]\), they determine risk neutral probability (ies) \(P^*\) for the stock price at time \(T\). The remaining available risk neutral probabilities at \(T\) correspond to stock prices with different jumps-variability. The findings indicate that \(P^*\) may be a mixture. The necessary and sufficient condition used to obtain \(P^*\) is related with the flow of information and concepts in Market Manipulation, \(ii\) contributes in understanding the relation between market informational efficiency and the arbitrage-free option pricing methodology. \(P^*-\)price \(C\) for the stock’s European call option expiring at \(T\) is also obtained. For “calm” stock prices, \(C\) coincides with the Black-Scholes-Merton price and confirms its universal validity without stock price modeling assumptions. Additional results for calm stock: \(a\) show that volatility’s role is fundamental in the call’s transaction, \(b\) clarify the behaviors of the trader and the call’s buyer and \(c\) confirm quantitatively that the buyer’s price carries an exponentially increasing volatility premium.

Key words and phrases: Calm stock, contiguity and information, European option, infinitely divisible distribution, market manipulation, risk neutral probability, statistical experiment, stock price-density, stock price information, the third fundamental theorem of asset pricing, volatility

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1 Introduction

In reality stock prices do not follow the assumed model and research has been devoted in the past and recently to discover the information the prices hide; see, e.g., Borovička, Hansen and Scheinkman (2014) and references therein. Without model assumptions, issues that could be resolved with the prices’ information include: 1) the explanation for the frequent use of the Black-Scholes-Merton (B-S-M) price (Black and Scholes, 1973, Merton 1973), 2) the determination of risk neutral probability (-ies) $P^*$ to be used at time $T$ for this stock’s price, and 3) the relation of $P^*$ with available information from the stock prices and the consequences. This work sheds light on 1)-3) among other obtained results.

Mean-adjusted stock prices and Le Cam’s theory of statistical experiments are used to determine one or more $P^*$ without price modeling assumptions and $P$ unknown. Each $P^*$ is obtained from a different probability $Q$ with Lévy triple $[\mu Q, \sigma^2 Q, L_Q]$. $Q$ is supported by the stock prices since the triple’s components are determined either by the sequence of sums of successive, adjusted prices’ jumps, or from one of its subsequences. $P^*$-price $C$ of the European call option is obtained. For “calm” stock, with small jumps forcing $P^*$’s Lévy measure to concentrate at zero, $C$ confirms the universal validity of B-S-M price. When the sequence of jumps’ sums has subsequences converging weakly to different limits, $C$ can be obtained from $P^*$-mixtures determined by the cluster points of this sequence. The remaining available risk neutral probabilities at $T$ correspond to stock prices with different jumps-variability and should be used to price only those stocks’ derivatives.

Adopting the statistical experiments’ motivation from the 2-players’ game, the European option’s transaction is seen as game between a trader and a buyer. A relation of the approach with the Kullback-Leibler relative entropy is presented. The connection of $Q$ and the obtained $P^*$ with the flow of information from the stock prices is revealed, and is related with the notion of information used in the Market Manipulation literature; see, e.g., Cherian and Jarrow (1995). $Q$ is determined by sequences of beliefs-probabilities $\{P_{tr,n}\}$ and $\{P_{bu,n}\}$, respectively, of the trader and of the buyer, $n$ determines the number of stock prices providing information and $P_{tr,n}$ is equivalent to $P_{bu,n}$ for every $n \geq 1$. The obtained $P^*$ is risk neutral, if and only if, the limit experiment of the beliefs-probabilities, with information from countably infinite stock prices, consists also of two equivalent probabilities. Thus, neither the trader nor the buyer have private information on the stock’s values. This results complements those in Jarrow (2013) which demonstrate the intimate relationship between an informationally efficient market and option pricing theory. It also confirms the Third Fundamental Theorem of Asset Pricing by providing sufficient conditions under which the family of stock price returns is an information set that makes the market efficient (Jarrow and Larsson, 2012, Jarrow, 2012, Jarrow, 2013, Corollary (Market Efficiency) p. 91).

For calm stock: i) the importance of volatility’s role in the European calls’ transaction is confirmed, ii) it is shown quantitatively that the probability $S_T$ exceeds strike price $X$ is larger for the buyer than for the trader, and iii) it is shown that the price the buyer expects to pay for the option includes a premium increasing exponentially with volatility.

The foregoing results appear in sections 4, 5. In section 2, a 2-step method is proposed to determine $P^*$ via probability $Q$ which motivates the embedding in the statistical experiments framework. The tools in section 3 include mean-adjusted price $p_t (= S_t/E_P S_t)$ which is density on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$; $t$ denotes time, $t_0 \leq t \leq T$. Beliefs-probabilities $P_{tr,n}$ and $P_{bu,n}$ are $p_t$-products at various trading times $t$ in the interval $[t_0, T]$. 
For the reader who feels uncomfortable because \( P_{tr,n} \) and \( P_{bu,n} \) are product of prices-densities and indicate independence of price returns, it should be reminded that Fama’s weak Efficient Market Hypothesis implies either independence or slight dependence of the stock price returns (Fama, 1965, p. 90, 1970, pp. 386, 414). Modeling “slight” dependence with weak dependence is acceptable in Finance (Duffie, 2010, personal communication). Thus, the limiting laws obtained under \( P_{tr,n} \) and \( P_{bu,n} \) remain valid under weak dependence and the obtained results hold.

Janssen and Tietje (2013) use statistical experiments to discuss “the connection between mathematical finance and statistical modelling” for \( d \)-dimensional price processes. Some of the differences in their work are: a) The price process is not standardized and \( P^* \) is assumed to exist. b) Convergence of the likelihood ratios to a normal experiment is obtained under the contiguity assumption. c) The relation between \( P^* \) and contiguity is not revealed. d) There are no results explaining the behaviors of the trader and of the buyer. e) There is no proof of the universal validity of B-S-M formula without stock price modeling assumptions.

The theory of statistical experiments used is in Le Cam (1969, Chapters 1 and 2, 1986, Chapters 10 and 16), Le Cam and Yang (1990, Chapters 1-4, 2000, Chapters 1-5) and in Roussas (1972, Chapter 1). Theory of option pricing can be found, e.g., in Musiela and Rutkowski (1997). A concise and very informative presentation of Lévy processes theory can be found in Kyprianou (2006). Proofs and some Lemmas are in the Appendix.

2 The approach to obtain \( P^* \) via \( Q \)

\( P^* \) to be used at \( T \) is equivalent to the physical probability \( P \) and satisfies the equation:

\[
E_{P^*}(\frac{S_T}{S_t}|\mathcal{F}_t) = e^{r(T-t)}; \tag{1}
\]

stock prices \( \{S_t, t > 0\} \) are defined on the probability space \((\Omega, \mathcal{F}, P)\), \( \{\mathcal{F}_t\} \) is the natural filtration and \( t \) denotes time. We consider a re-expression of (1),

\[
E_{P^*}(\frac{S_T}{S_t}|\mathcal{F}_t) = E_{Q^*}[\exp\{V_{T-t} + \ln a_{[t,T]}\}|\mathcal{F}_t] = e^{r(T-t)}, \tag{2}
\]

\[
V_{T-t} = \ln \frac{S_T/ES_T}{S_t/ES_t}, \tag{3}
\]

\[
a_{[t,T]} = \frac{ES_T}{ES_t}; \tag{4}
\]

\( ES_t \) denotes \( E_PS_t \) for every \( t \)-value. \( Q^* \) is the cumulative distribution function (c.d.f.) of \( V_{T-t} \) under \( P^* \) and will be obtained in two steps. Unless needed, domains of integration are omitted since they are determined by the c.d.f.s./probabilities.

**Step 1:** Determine \( Q \) for \( V_{T-t} \) under which \( \{S_t/ES_t\} \) is a martingale, i.e.

\[
E_Q(\exp\{V_{T-t}\}|\mathcal{F}_t) = \int e^v dQ(v|\mathcal{F}_t) = 1. \tag{5}
\]

There is no involvement of the interest (i.e. of \( r \)) in Step 1.
Step 2: $Q^*$ is the translated $Q$, 

$$Q^*(v|\mathcal{F}_t) = Q[v + \ln a_{[t,T]} - r(T - t)|\mathcal{F}_t].$$

(6)

Remark 2.1 $Q$ is the key element that allows to obtain $\mathcal{P}^*$ and reveals its relation with the flow of information. When $Q$ is determined, (2) holds under $Q^*$:

$$E_{Q^*}[e^{V_{T-t}+\ln a_{[t,T]}|\mathcal{F}_t}] = \int e^{v+\ln a_{[t,T]}dQ[v + \ln a_{[t,T]} - r(T - t)|\mathcal{F}_t]} = \int e^{w+rd(T-t)}dQ(w|\mathcal{F}_t) = e^{r(T-t)}.

(5)$$

For Geometric Brownian motion the 2-step approach allows to obtain $Q$.

Example 2.1 Let $S_t$ be a geometric Brownian motion,

$$S_t = s_0\exp\{(\mu - \frac{\sigma^2}{2})t + \sigma B_t\}$$

with $B_t$ standard Brownian motion, $t > 0$ and $s_0$ the price at $t = 0$. Since $ES_t = s_0\exp\{\mu t\}$ and for $t < T$

$$V_{T-t} = -\frac{\sigma^2}{2}(T - t) + \sigma(B_T - B_t),$$

(3) holds under $\mathcal{P}(= Q)$ and (2) holds under $Q^*$.

In Example 2.1 $Q$, $\mathcal{P}^*$ and $Q^*$ are easily obtained because the distribution of $V_{T-t}$ is normal. Can one similarly obtain $Q$ (and therefore $Q^*, \mathcal{P}^*$) in other situations? Without stock price modeling assumptions only the stock prices provide information and can determine $Q$ via a sequence $Q_n$; $n$ increases with the flow of information, i.e., the number of stock prices in $(t,T)$. This is supported by Cox, Ross and Rubinstein (1979) where, for the binomial price model, B-S-M price is obtained as limiting price. Also, by the terms $\pm 0.5\sigma^2$ in the standard normal c.d.fs. of B-S-M price, indicating these c.d.fs. are limits of expected values under contiguous sequences of probabilities.

When there are “many” transactions in $[t,T]$ an embedding in Le Cam’s statistical experiments, where contiguity was introduced, allows to determine $Q$.

Remark 2.2 When $r$ is not fixed in $[t,T]$, let $B_{[t,T]} = e^{\int_t^T r_sds}; r_s$ determines the interest at time $s$, as $r$ determined interest $i$. Then in (1) and (2), $B_{[t_0,T]}$ is replacing $e^{r(T-t_0)}$ and in (6), $\ln B_{[t_0,T]}$ is replacing $r(T - t_0)$. All subsequent results herein still hold with these changes.

3 The Tools and the Embedding

A binary statistical experiment $\mathcal{E}$ consists of probabilities $\{Q_1, Q_2\}$ on $(\hat{\Omega}, \hat{\mathcal{F}})$ (Blackwell, 1951). Le Cam (see, e.g., 1986) introduced a distance $\Delta$ between experiments and proved
that \( \Delta \)-convergence of binary experiments \( \mathcal{E}_n = \{Q_{1,n}, Q_{2,n}\}, n \geq 1 \), to \( \mathcal{E} \) is equivalent to weak convergence of likelihood ratios \( \frac{dQ_{2,n}}{dQ_{1,n}} \) under \( Q_{1,n} \) (resp. \( Q_{2,n} \)) to the distribution of \( \frac{dQ_2}{dQ_1} \) under \( Q_1 \) (resp. \( Q_2 \)).

Consider the process of prices-densities

\[
\{p_t = \frac{S_t}{ES_t}, \quad t \in [0, T]\}.
\]  (7)

Embed the stock prices in \([t_0, T]\) in the statistical experiments’ framework by re-expressing \( V_{T-t_0} \) in (3) using the intermediate prices-densities,

\[
V_{T-t_0} = \ln \frac{S_T}{ES_T} = \ln \left( \frac{S_T}{ES_{t_0}} \cdots \frac{S_{t_0}}{ES_{t_0}} \cdots \frac{S_{t_{k_n}^{n-1}}}{ES_{t_{k_n}^{n-1}}} \cdots \frac{S_{t_0}}{ES_{t_0}} \right) = \ln \left( \prod_{j=1}^{k_n} p_{t_{j}^{n}} \right) = \ln \left( \prod_{j=1}^{k_n} \frac{p_{t_{j}^{n}}}{p_{t_{j-1}^{n}}} \right) = \Lambda_{k_n}. \]  (8)

The products \( \Pi_{j=1}^{k_n} p_{t_{j}^{n}} \) and \( \Pi_{j=1}^{k_n} \frac{p_{t_{j}^{n}}}{p_{t_{j-1}^{n}}} \) determine, respectively, beliefs-probabilities \( P_{tr,n} \) and \( P_{bu,n} \) in \((\Omega^{k_n}, \mathcal{F}^{k_n})\) and the statistical experiment

\( \mathcal{E}_n = \{P_{tr,n}, P_{bu,n}\}; \)

\( t_{0}^{n} = t_0 \) and \( t_{k_n}^{n} = T \) for each \( n \). \( P_{tr,n} \) and \( P_{bu,n} \) are both unknown but when \( n \) and so \( k_n \) increase to infinity, with mild assumptions, the theory provides the asymptotic distributions of \( \ln \frac{dp_{bu,n}}{dp_{tr,n}} \) under both \( P_{tr,n} \) and \( P_{bu,n} \).

From several equivalent definitions of contiguity we present one that will reveal the relation between \( \mathcal{P}^* \) and the stock prices’ information.

**Definition 3.1** (see, e.g. Le Cam, 1986, p. 87, Definition 5) Let \( \mathcal{E}_n = \{Q_{1,n}, Q_{2,n}\}, n \geq 1 \), be a sequence of statistical experiments. Then, the sequence \( \{Q_{1,n}\} \) is contiguous to the sequence \( \{Q_{2,n}\} \) if in all cluster points \( \mathcal{E} = \{Q_1, Q_2\} \) of the sequence \( \{\mathcal{E}_n\} \) (for \( \Delta \)-convergence), \( Q_1 \) is dominated by \( Q_2 \).

If, in addition, \( \{Q_{2,n}\} \) is contiguous to sequence \( \{Q_{1,n}\} \) then the sequences \( \{Q_{1,n}\}, \{Q_{2,n}\} \) are contiguous.

From Definition 3.1 for contiguous sequences of probabilities \( \{Q_{1,n}\}, \{Q_{2,n}\} \) forming a sequence of binary experiments \( \mathcal{E}_n \), each cluster point experiment of \( \{\mathcal{E}_n\} \) consists of mutually absolutely continuous probabilities.

4 Modeling \( V_{T-t_0} \)

It is shown that when the trading times are dense in \([t_0, T]\) the distributions of \( \log \frac{p_T}{p_{t_0}} \) under \( P_{tr,n} \) and \( P_{bu,n} \) are infinitely divisible, normal in particular for calm stock, and \( Q \) is obtained (Propositions 4.1, 4.2 and Corollary 4.1). When \( t_0 \) is the present and \( S_{t_0} = s_{t_0} \), \( \frac{p_T}{p_{t_0}} \) becomes \( p_T \) and \( \mathcal{P}^* \) obtained via \( Q \) is practically \( S_T \)'s probability (Proposition 4.3).
4.1 The Assumptions

Let $Y_{n,j}$ be a fluctuation measure of \( \frac{p_{t_{j}}}{p_{t_{j-1}}} \) from unity,
\[
Y_{n,j} = \sqrt{\frac{p_{t_{j}}}{p_{t_{j-1}}}} - 1, \ j = 1, \ldots, k_n.
\] (9)

Assume

(A1) \( S_t > 0 \) and \( ES_t < \infty \) for every \( t \in [t_0, T] \),
(A2) a countable number of stock’s transactions in any open interval of \([t_0, T]\),
(A3) for the prices-densities \( p_{t_0}, p_{t_1}, \ldots, p_{t_n} \) with mesh size \( \delta_n = \sup\{t^n_j - t^n_{j-1}; j = 1, \ldots, k_n\}, \ k_n = k_n(\delta_n), t^n_0 = t_0, t^n_{k_n} = T, \)

(i) \( \lim_{\delta_n \to 0} \sup_{\delta_n} E\left[ p_{t_{j+1}} Y^2_{n,j} \right], \ j = 1, \ldots, k_n \) = 0,

(ii) there is positive \( b \) : \( \sup_{n} \sum_{j=1}^{k_n} E\left[ p_{t_{j+1}} Y^2_{n,j} \right] \leq b < \infty. \)

(A4) Under \( \mathcal{P} \), \( \lim_{n \to \infty} S_{t^n} = s_{t_0} \) in probability, \( \lim_{n \to \infty} ES_{t^n} = s_{t_0}. \)

Assumption A1 allows the passage from stock prices to prices-densities and \(-1 < Y_{n,j} < \infty, j = 1, \ldots, k_n\). Assumptions A1 and A2 provide sequences of probabilities with \( n \)-th terms, \( P_{tr,n} \) and \( P_{bu,n} \), mutually absolutely continuous. Assumption A3(i) indicates that the contribution of the ratio \( \frac{p_{t_{j+1}}}{p_{t_{j}}} \) does not affect the distribution of \( V_{T-t_0} \), \( j = 1, \ldots, k_n \).

Assumption A3(ii) implies that \( V_{T-t_0} \)’s variance is finite. Assumption A4 allows, along with \( Q \), to change the conditional expectation in (11) to expected value. Most important, A4 allows to show that when \( t_0 \) is the present and \( S_{t_0} = s_{t_0} \) then \( V_{T-t_0} = \ln(S_T/ES_T) \) and \( Q \) determines \( P^* \) for \( S_T \).

4.2 The \( Q \)-distribution(s) of \( V_{T-t_0} \)

Assumption A3 implies also that under \( \{P_{tr,n}\} \) and \( \{P_{bu,n}\} \) the sequences of distributions of \( \sum_{j=1}^{k_n} Y_{n,j} \) are relatively compact. Thus, we can choose a subsequence \( \{k_n\} \), for which \( \sum_{j=1}^{k_{n'}} Y_{n',j} \) converges weakly, respectively, under \( P_{tr,n'} \) and \( P_{bu,n'} \), to infinitely divisible distributions. Without loss of generality we use \( \{n\} \) and \( \{k_n\} \) instead of \( \{n'\} \) and \( \{k_{n'}\} \). If there are two or more subsequences with different weak limits, the stock prices support more than one risk neutral probabilities and \( \mathcal{P}^* \) can be modeled as mixture of infinitely divisible distributions.

\( V_{T-t_0} \) is approximated in probability by a linear function of \( \sum_{j=1}^{k_n} Y_{n,j} \) and has under \( P_{tr,n} \) as limit the infinitely divisible distribution \( Q_0. \) A translation of \( Q_0 \) is usually needed to obtain \( Q \) satisfying (3).

Proposition 4.1 (Le Cam, 1986, Proposition 2, p. 462) Assume that A1 – A3 hold and that \( \sum_{j=1}^{k_n} Y_{n,j} \) has under \( P_{tr,n} \) a weak limit with \( \text{\Lévy triple } [\mu, \sigma^2, L_{tr}] \). Under \( P_{tr,n} \), \( \Lambda_{k_n} = \)
\[ \ln \Pi_{j=1}^{k_n} \frac{p_{n,j}}{p_{n,j-1}} \] converges in distribution to \( \Lambda \) with c.d.f. \( Q_0 \) and for every \( s \in (0, 1) \) its moment generating function,

\[
\psi_{Q_0}(s) = \ln E_{Q_0}e^{sY} = \mu_{[t_0,T]}s + \frac{\sigma_{[t_0,T]}^2}{2}s^2 + \int_{[-1,0] \cup (0,\infty)} [(1 + y)^{2s} - 1 - 2sy]L_{tr}(dy); \quad (10)
\]

\[
\mu_{[t_0,T]} = 2\mu - \sigma^2 < 0, \quad \sigma_{[t_0,T]}^2 = 4\sigma^2,
\]

and \( \mu, \sigma^2, L_{tr} \) are all determined in Remark 4.1.

**Remark 4.1** Under A1, in (11) \( L_{tr}(-1) = 0 \). The model parameters

\[
\mu_{[t_0,T]} = 2\mu - \sigma^2 = (2\mu_1 - \sigma_1^2)(T - t_0), \quad \sigma_{[t_0,T]}^2 = 4\sigma^2 = 4\sigma_1^2(T - t_0);
\]

\( \mu_1 \) and \( \sigma_1^2 \) are determined from an interval of length unity,

\[
\mu_1 = \lim_{n \to \infty} \sum_{j=1}^{k_n} E_{P_{n,j-1}} Y_{n,j} < 0, \quad \sigma_1^2 = \lim_{n \to \infty} \lim_{\tau \to 0} \sum_{j=1}^{k_n} E_{P_{n,j-1}} Y_{n,j}^2 I(|Y_{n,j}| \leq \tau). \quad (11)
\]

The Lévy measure in an interval of length unity is

\[
L_{tr}(y) = \lim_{n \to \infty} \sum_{j=1}^{k_n} E_{P_{n,j-1}} Y_{n,j}^2 I(Y_{n,j} \leq y). \quad (12)
\]

From Lévy-Khintchine theorem (see, e.g., Kyprianou, 2006, Theorem 2.1, p. 35), \( V_{T-t} \) can be seen as Lévy process thus the conditional expected value in (2) is an expected value; see also Lemma 6.1 in the Appendix.

The next proposition provides \( Q \) and \( Q^* \) (via \( Q_0 \)) and the necessary and sufficient condition (13) to obtain \( Q. I \) denotes indicator function.

**Proposition 4.2** a) \( Q_0 \) in Proposition 4.1 satisfies (2), if and only if,

\[
\mu_{[t_0,T]} + .5\sigma_{[t_0,T]}^2 + E_{Ltr} Y^2 I(Y \neq 0) = 2\mu + \sigma^2 + E_{Ltr} Y^2 I(Y \neq 0) = 0. \quad (13)
\]

b) \( Q \) satisfying (3) has Lévy triple \([- .5\sigma_{[t_0,T]}^2 - E_{Ltr} Y^2 I(Y \neq 0), \sigma_{[t_0,T]}^2, L_{tr}] \},

\[
Q(v) = \int \Phi(\frac{v + .5\sigma_{[t_0,T]}^2 + E_{Ltr} Y^2 I(Y \neq 0) - y}{\sigma_{[t_0,T]}})L_{tr,Pois}(dy); \quad (14)
\]

\( \Phi \) denotes the c.d.f. of a standard normal random variable, \( L_{tr,Pois} \) is the probability of the Poissonian component.

c) Let \( Q_{bu} \) be the limit distribution of \( \Lambda_{k_n} \) under \( P_{n,bu} \). \( Q_0 \) in Proposition 4.1 is mutually absolutely continuous with \( Q_{bu} \), if and only if, (13) holds. Then, \( Q_{bu} \) has Lévy triple \([.5\sigma_{[t_0,T]}^2 + E_{Ltr} Y^2 I(Y \neq 0), \sigma_{[t_0,T]}^2, L_{bu}] \},

\[
Q_{bu}(v) = \int \Phi(\frac{v - .5\sigma_{[t_0,T]}^2 - E_{Ltr} Y^2 I(Y \neq 0) - y}{\sigma_{[t_0,T]}})L_{bu,Pois}(dy); \quad (15)
\]

\( Q_{bu} \) converges in distribution to \( Q_{bu} \).
\( L_{bu, \text{Pois}} \) is the probability of the Poissonian component.

d) For \( Q \) in b),
\[
dQ_{bu}(v) = e^v dQ,
\]
\[
\int_{\{v : v > x\}} e^v dQ(v) = 1 - Q_{bu}(x), \quad \forall x \in R.
\]
e) \( Q^* \) is obtained using \( Q \) in b) and (6).

**Remark 4.2** From (13), \( Q_0 \) is risk neutral when the drift \( \mu \) equals \(-0.5 \sigma^2 - 0.02 E_L Y^2 I(Y \neq 0)\). There are several possible \( \tilde{\sigma} \) values and \( \text{Lévy measures} \) \( \tilde{L}_{tr} \) for which (13) holds, but those supported by the stock prices (at unit time length) are given by (11) and (12). The remaining \( \tilde{L}_{tr} \) and \( \tilde{\sigma} \) correspond to stock prices with different jumps-variability.

The result allowing to obtain \( \mathcal{P}^* \) for \( S_T \) via \( Q \) follows.

**Proposition 4.3** Assume that \( A_1 - A_4 \) hold for stock prices \( \{S_t, t_0 < t \leq T\} \). When \( t_0 \) is the present, \( S_{t_0} = s_{t_0} \), Proposition 4.2 holds for \( V_{T-t_0} = \ln(S_T/ES_T) \) and \( Q, Q^*, \mathcal{P}^* \) are obtained.

A result connecting directly \( \mathcal{P}^* \) for \( S_T \) and \( Q \) follows also.

**Lemma 4.1** When \( t \) is the present. \( S_t = s_t \) and
\[
\mathcal{P}^*(S_T \leq u) = Q(\ln \frac{u}{s_t} - r(T-t)).
\]

## 4.3 The \( \mathcal{Q} \)-distribution(s) of \( V_{T-t_0} \) for Calm Stock

Calm stock has prices-densities \( p_{t+\delta}, \; p_t \) (see (11)) that do not differ much with respect to \( P \) for small \( \delta \)-values, thus excluding the case of unusual jumps for \( \frac{p_{t+\delta}}{p_t} \).

**Definition 4.1** Let \( t_1^a < \ldots < t_{k_n}^a \) be a partition of \( (t_0, t, T = t_{k_n}^a) \), with mesh size \( \delta_n = \sup\{t_j^a - t_{j-1}^a, \; j = 1, \ldots, k_n\} \). Stock \( \{S_t\} \) is calm in \( [t_0, T] \) if for any \( \epsilon(>0) \) and any partition
\[
\lim_{\delta_n \to 0} \sum_{j=1}^{k_n} E_{P_{t_j}} (\sqrt{\frac{P_{t_j}}{P_{t_{j-1}}}} - 1)^2 I(|\sqrt{\frac{P_{t_j}}{P_{t_{j-1}}}} - 1| > \epsilon) = \lim_{\delta_n \to 0} \sum_{j=1}^{k_n} E_{P_{t_j}} Y_{n,j}^2 I(|Y_{n,j}| > \epsilon) = 0;
\]
\( I \) is the indicator function, \( dP_t = p_t dP \).

Thus, for calm stock random variables \( Y_{n,j}, \; j = 1, \ldots, k_n \) satisfy Lindeberg’s condition, \( \sum_{j=1}^{k_n} Y_{n,j} \) has asymptotically normal distribution and the same holds for
\[
\Lambda_{k_n} = 2 \sum_{j=1}^{k_n} \ln(1 + Y_{n,j}),
\]
i.e., for \( V_{T-t_0} \).
Corollary 4.1 When $A_1, A_2,$ and $A_3(ii)$ hold for a calm stock in $[0, T]$, for every convergent subsequence $\sum_{j=1}^{k_n} Y_{n,j}$ there is $\sigma_{[t_0,T]} > 0$ such that

\[(i) \quad Q(v) = \Phi\left(v + \frac{\sigma_{[t_0,T]}^2}{2}\sigma_{[t_0,T]}\right),\]  

\[(ii) \quad Q_{bu}(v) = \Phi\left(v - \frac{\sigma_{[t_0,T]}^2}{2}\sigma_{[t_0,T]}\right).\]  

When, in addition, $\sum_{j=1}^{k_n} E_{P_{j-1}} Y_{n,j}^2$ has a limit as $n$ increases to infinity then $Q$ and so $P^*$ are uniquely determined.

5 Applications and Consequences

5.1 Option pricing of a European call

Make the usual assumption:

\((A_5)\) The market consists of the stock $S$ and a risk-less bond that appreciates at fixed rate $r$ and there are no dividends or transaction costs. The option is European. The buyer prefers to pay less than more.

$P^*$-price of a European call is obtained from a weakly convergent subsequence of $\sum_{j=1}^{k_n} Y_{n,j}$. When $\sum_{j=1}^{k_n} Y_{n,j}$ has several cluster points, the fair price is a weighted sum of the corresponding $P^*$-prices.

Proposition 5.1 Assume $A_1 – A_5$ hold. The $P^*$-price $C$ of the European call option at $t_0$, with strike price $X$ at expiration $T$, is

\[C = s_{t_0} R_{bu} - X e^{-r(T-t_0)} R_{tr},\]  

\[R_{bu} = 1 - Q_{bu}\left[\ln\left(\frac{X}{s_{t_0}}\right) - r(T-t_0)\right]\]  

\[= \int \Phi\left(\frac{\ln(s_{t_0}/X) + r(T-t_0) + 5\sigma_{[t_0,T]}^2 + E_{L_{tr}} Y^2 I(Y \neq 0) + \sigma_{[t_0,T]}^2}{\sigma_{[t_0,T]}^2}\right) L_{bu, Pois}(dy)\]  

\[R_{tr} = \int \Phi\left(\frac{\ln(s_{t_0}/X) + r(T-t_0) - 5\sigma_{[t_0,T]}^2 - E_{L_{tr}} Y^2 I(Y \neq 0) - \sigma_{[t_0,T]}^2}{\sigma_{[t_0,T]}^2}\right) L_{tr, Pois}(dy).\]

For calm stock, B-S-M price is obtained without model assumptions thus justifying its universality and frequent use.

Corollary 5.1 For calm stock, under the assumptions $A_1, A_2, A_3(ii), A_4, A_5$, the coefficients $R_{bu}$ and $R_{tr}$ in (22) are

\[R_{bu} = \Phi\left(\frac{\ln(s_{t_0}/X) + r(T-t_0) + 5\sigma_{[t_0,T]}^2}{\sigma_{[t_0,T]}^2}\right),\]  

\[R_{tr} = \Phi\left(\frac{\ln(s_{t_0}/X) + r(T-t_0) - 5\sigma_{[t_0,T]}^2}{\sigma_{[t_0,T]}^2}\right).\]
5.2 Binary Statistical Experiments, Information and $\mathcal{P}^*$

We relate the approach in this work with notions of information.

**Definition 5.1** (see, e.g., Cover and Thomas, 2005, p. 19) The relative entropy, or Kullback-Leibler distance, between two densities $f$ and $g$ is

$$D(f||g) = -E_f \ln \frac{g(X)}{f(X)}$$

Herein we use the distribution of $\ln g(X)$ under $f$ rather than its expected value (24). In our notation $f$, $g$ are, respectively, either $P_{tr,n}, P_{bu,n}$ or $Q, Q_{bu}$.

In each of the binary experiments $\mathcal{E}_n = \{P_{tr,n}, P_{bu,n}\}$ and $\{Q, Q_{bu}\}$, the beliefs-probabilities for $S_T$’s distribution are, respectively, those of the trader and the buyer. $P^*$ is determined via $Q$ that satisfies (5) if and only if (13) holds. The latter equation is equivalent to contiguity of the sequences $\{P_{tr,n}\}$ and $\{P_{bu,n}\}$; see the proof of Proposition 4.2 c). Thus, the obtained $\mathcal{P}^*$ is risk neutral, if and only if, $\{P_{tr,n}\}$ and $\{P_{bu,n}\}$ are contiguous. To see what this means in terms of information observe that for each $n$, $P_{tr,n}$ and $P_{bu,n}$ are mutually absolutely continuous and are based on information from $k_n$ stock prices before $T$ for determining $S_T$’s distribution. Therefore, the corresponding induced probabilities for $\Lambda_{k_n}, P_{tr,n} \circ \Lambda^{-1}_{k_n}$ and $P_{bu,n} \circ \Lambda^{-1}_{k_n}$, are also mutually absolutely continuous. Thus, neither the trader nor the buyer have private information on $S_T$’s values any time in $(t_0, T)$. This can be seen as market efficiency, thus the results are a compagnon of the Third Fundamental Theorem of Asset Pricing (Jarrow, 2012) which characterizes the conditions under which an equivalent martingale probability measure exists in the economy. Recall that the above hold for each convergence subsequence $\{\mathcal{E}_n\}$.

Mutual absolute continuity of traders’ beliefs-probabilities is used in Financial Economics, e.g., the area of Market Manipulation. Cherian and Jarrow (1995, p. 616, Assumption 3), provide two conditions to avoid arbitrage due to “manipulator’s” information. In the context of a trader and a buyer and with our notation these conditions are:

i) trader’s $Q$ and buyer’s $Q_{bu}$ are mutually absolutely continuous, and

ii) there exists a risk neutral probability equivalent to $Q$.

When the trader-manipulator uses the obtained $\mathcal{P}^*$ in this work, it has been shown that i) and ii) are equivalent.

5.3 Binary Statistical Experiments and Calm stock

By adopting the statistical experiment model in option pricing, empirical findings are confirmed quantitatively and new information is obtained for calm stock.

a) The theory of statistical experiments provides an explanation for volatility’s role in the transaction.

Under the assumptions of Proposition 4.1 from (20) and (21) it follows that the binary experiment $\mathcal{E}_n = \{P_{tr,n}, P_{bu,n}\}$ converges to the Gaussian experiment $\mathcal{G} = \{P_0 = N(0,1), P_T = N(\sigma_{[t_0,T]}, 1)\}$ when $n \to \infty$. From $\mathcal{G}$’s form it is clear that volatility, $\sigma_{[t_0,T]}$, is the determining factor in the transaction.
b) From (20) and (21), \( Q(v) \) is larger than \( Q_{bu}(v) \) for any \( v \) and therefore the event \( \{S_T > X\} \) has higher probability for the buyer than for the trader.

c) Let \( P_{bu}^* \) be the belief-probability of the buyer corresponding to \( P^* \), obtained with the same translations on \( Q_{bu} \), as for \( P^* \) via \( Q \). The buyer's price has indeed a volatility premium, with the coefficient \( s_{t_0} \) in the trader's price's (22) replaced by \( s_0 e^{\sigma_s^2 t_0} \):

\[
E_{P_{bu}^*} e^{-r(T-t_0)} (S_T - X) I(S_T > X)
= s_0 e^{\sigma_s^2 t_0} \Phi \left( \frac{\ln(s_0/X) + r(T-t_0) + 1.5 \sigma_s^2}{\sigma_s} \right) - X e^{-r(T-t_0)} \Phi \left( \frac{\ln(s_0/X) + r(T-t_0) + .5 \sigma_s^2}{\sigma_s} \right)
\]

6 Appendix: Proofs and a Lemma

**Lemma 6.1** Under assumptions \( \mathcal{A}1 - \mathcal{A}4 \),

\[
E_{Q^*} \left[ \exp \{V_{T-t} + \ln a_{[t,T]} \} | \mathcal{F}_t \right] = E_{Q^*} \exp \{V_{T-t} + \ln a_{[t,T]} \}. \tag{25}
\]

**Proof of Lemma 6.1:** \( \mathcal{F}_t \) is determined by the countably many transactions in \((0,t)\). For every time sequence \( 0 < t_{m_n^1} < t_{m_n^2} < \ldots < t_{m_n^n} < t \), that becomes dense in \([0,t] \) as \( n \) increases to infinity, the corresponding prices

\[
S_{t_1}, S_{t_{m_n^1}}, S_{t_{m_n^2}}, \ldots, S_{t_{m_n^n}}
\]

provide the same information as

\[
\frac{S_t}{ES_t}, \frac{S_{t_{m_n^1}}}{ES_{t_{m_n^1}}}, \frac{S_{t_{m_n^2}}}{ES_{t_{m_n^2}}}, \ldots, \frac{S_{t_{m_n^n}}}{ES_{t_{m_n^n}}}
\]
or,

\[
\frac{S_t}{S_{t_{m_n^1}}}, \frac{S_{t_{m_n^1}}}{S_{t_{m_n^2}}}, \frac{S_{t_{m_n^2}}}{S_{t_{m_n^3}}}, \ldots, \frac{S_{t_{m_n^n}}}{S_{t_{m_n^{n-1}}}}
\]

that coincides with

\[
\frac{S_t}{S_{t_{m_n^1}}}, \frac{S_{t_{m_n^1}}}{S_{t_{m_n^2}}}, \frac{S_{t_{m_n^2}}}{S_{t_{m_n^3}}}, \ldots, \frac{S_{t_{m_n^n}}}{S_{t_{m_n^{n-1}}}}, \frac{S_{t_{m_n^{n-1}}}}{S_{t_{m_n^n}}}
\]
or, by taking logarithms

\[
V_{T-t_{m_n^1}} - V_{T-t}, V_{T-t_{m_n^2}} - V_{T-t_{m_n^1}}, \ldots, V_{T-t_{m_n^n}} - V_{T-t_{m_n^{n-1}}}, S_{t_{m_n^n}}/ES_{t_{m_n^n}}. \tag{26}
\]

When \( n \) is large, the last term in (26) approaches unity since both numerator and denominator converge to \( s_0 \) (from \( \mathcal{A}4 \)) and the remaining terms are increments of a Lévy process and are therefore independent of \( T - T^* \).

**Proof of Proposition 4.2:**

a) In Proposition 4.1 the moment generating function \( \psi_{Q_0}(s) \) is determined for \( s \in (0,1) \). The integrand in \( \psi_{Q_0}(s) \) (see (10)),

\[
(1 + y)^{2s} - 1 - 2sy
\]
is bounded by “some” multiple of $y^2$ (Le Cam, 1986, p. 465, lines 18-22) and $E_{L_{tr}}Y^2$ is finite from assumption A3. From dominated convergence theorem,

$$\lim_{s \to 1} \ln \psi_{Q_0}(s) = \ln \psi_{Q_0}(1) = \ln E_{Q_0}e^{V_{T-t_0}} = \mu_{[t_0,T]} + 0.5\sigma_{[t_0,T]}^2 + E_{L_{tr}}Y^2 I(Y \neq 0). \quad (27)$$

(27) holds for $Q_0$ if and only if $\ln \psi_{Q_0}(1)$ in (27) vanishes.

b) Follows from a).

c) Since i) the Lévy measure $L_{tr}$ has no mass at -1 (by A1) and ii) Proposition 5.1 shows $\Delta$-convergence of (a subsequence of) the experiments $E_n = \{P_{tr,n}, P_{bu,n}\}$ to the experiment $\{Q_0, Q_{bu}\}$, it follows that $\{P_{tr,n}\}$ and $\{P_{bu,n}\}$ are contiguous, i.e. $Q_0$ and $Q_{bu}$ are mutually absolutely continuous, if and only if,

$$\lim_{s \to 1} \psi_{Q_0}(s) = \psi_{Q_0}(1) = 1, \quad (28)$$

which from (27) holds, if and only if, (13) holds.

d) Due to contiguity, (16) holds (see, e.g. Le Cam and Yang, 1990, p. 22, the Proposition).

Proof of Proposition 4.3: It is enough to show that $A1 - A4$ hold for $t_0 \leq t \leq T$. When $S_{t_0} = s_{t_0}$ assumptions $A1, A2, A4$ still hold. For $A3$ observe that since $S_{t_0} = s_{t_0},$

$$0 \leq E_{P_{t_0}} Y_{n,1}^2 = \int (\sqrt{P_t^n} - 1)^2 dP = 2(1 - \int \sqrt{P_t^n} dP) \leq 2. \quad (29)$$

From (29) it follows that $A3(ii)$ holds.

For $A3(i)$ to hold it is enough to show that

$$\lim_{n \to \infty} E_{P_{t_0}} Y_{n,1}^2 = 0$$

or from (29) that

$$\lim_{n \to \infty} E_P|\sqrt{P_t^n} - 1| = 0. \quad (30)$$

Since

$$E_P|\sqrt{P_t^n} - 1| \leq E_P|P_t^n - 1|$$

for (30) to hold it is enough that

$$\lim_{n \to \infty} E_P|P_t^n - 1| = 0. \quad (31)$$

(31) follows from:

Lemma 6.2 (Roussas, 2005, Lemma 3, p. 138, 2014, Lemma 3, p. 109) Assume $X_n \geq 0$, $EX_n < \infty$, $n \geq 1$. Then

$$\lim_{n \to \infty} E|X_n - X| = 0 \iff \text{plim}_{n \to \infty} X_n = X \text{ and } \lim_{n \to \infty} EX_n = EX < \infty.$$  

with $X_n = P_{t^n_1}$, $X = 1$ since $E_PP_t^n = 1$ $\forall$ $n \geq 1$ and $A4$ holds. □
Proof of Lemma 4.1:

\[ P^*(S_T \leq u) = P^*(\ln \frac{S_T}{s_t} \leq \ln \frac{u}{s_t}) = Q^*(\ln \frac{u}{s_t} - \ln a_{[t,T]}) = Q(\ln \frac{u}{s_t} - r(T - t)) ; \]

the last equality follows from (6).

Proof of Corollary 4.1: Follows from (14) and (15) for Lévy measures \( L_{tr} \) and \( L_{bu} \) concentrated at \( y = 0 \).

Proof of Proposition 5.1: The option’s price,

\[ E_{P^*} e^{-r(T-t_0)}(S_T - X)I(S_T > X), \tag{32} \]

is obtained via \( Q \) and Propositions 4.2, 4.3.

We calculate separately each of the two expected values in (32) excluding constants.

\[ E_{P^*} I(S_T > X) = P^*(\ln \frac{S_T}{s_{t_0}} > \ln \frac{X}{s_{t_0}}) = 1 - Q(\ln \frac{X}{s_{t_0}} - r(T - t_0)) \]

\[ = 1 - \int \Phi(\frac{\ln \frac{X}{s_{t_0}} - r(T - t_0) + .5\sigma_{[t_0,T]}^2L_{tr,Y^2I(Y \neq 0) - y}{\sigma_{[t_0,T]}}}{\sigma_{[t_0,T]}}) L_{tr,Pois}(dy), \tag{33} \]

with the penultimate and the last equalities obtained using, respectively, (6) and (14).

\[ E_{P^*} S_T I(S_T > X) = s_{t_0} E_{Q^*} e^{V_{t_0}} e^{v + \ln a_{[t_0,T]}I(V_{t_0} > \ln \frac{X}{s_{t_0}} - \ln a_{[t_0,T]})} \]

\[ = s_{t_0} \int_{\{w > \ln(X/s_{t_0}) - \ln a_{[t_0,T]}\}} e^v dQ(v + \ln a_{[t_0,T]} - r(T - t_0)) \]

\[ = s_{t_0} e^{r(T-t_0)} \int_{\{w > \ln(X/s_{t_0}) - r(T - t_0)\}} e^w dQ(w) \]

\[ = s_{t_0} e^{r(T-t_0)} [1 - Q_{bu}(\ln(X/s_{t_0}) - r(T - t_0))] \]

\[ = s_{t_0} e^{r(T-t_0)} [1 - \Phi(\frac{\ln(X/s_{t_0}) - r(T - t_0) - .5\sigma_{[t_0,T]}^2L_{tr,Y^2I(Y \neq 0) - y}{\sigma_{[t_0,T]}})}{\sigma_{[t_0,T]}}) L_{bu,Pois}(dy)], \tag{34} \]

with the second, the penultimate and the last equalities obtained using, respectively, (16), (16) and (15).

Replacing (33), (34) in (32), \( P^* \)-price (22) is obtained.

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