CONJUGACY CLASSES IN LOOP GROUPS AND G-BUNDLES ON ELLIPTIC CURVES

VLADIMIR BARANOVSKY AND VICTOR GINZBURG

1. Introduction

Let \( \mathbb{C}[[z]] \) be the ring of formal power series and \( \mathbb{C}((z)) \) the field of formal Laurent power series, the field of fractions of \( \mathbb{C}[[z]] \). Given a complex algebraic group \( G \), we will write \( G((z)) \) for the group of \( \mathbb{C}((z)) \)-rational points of \( G \), thought of as a formal “loop group”, and \( a(z) \) for an element of \( G((z)) \). Let \( q \) be a fixed non-zero complex number. Define a “twisted” conjugation action of \( G((z)) \) on itself by the formula

\[
(a(z) \mapsto g(a(z)) = g(q \cdot z) \cdot a(z) \cdot g(z)^{-1}).
\]

We are concerned with the problem of classifying the orbits of the twisted conjugation action. If \( q = 1 \) twisted conjugation becomes the ordinary conjugation, and the problem reduces to the classification of conjugacy classes in \( G((z)) \).

In this paper we will be interested in the case \( |q| < 1 \). Let \( G[[z]] \subset G((z)) \) be the subgroup of \( \mathbb{C}[[z]] \)-points of \( G \). A twisted conjugacy class in \( G((z)) \) is called integral if it contains an element of \( G[[z]] \).

Introduce the elliptic curve \( \mathcal{E} = \mathbb{C}^*/q\mathbb{Z} \). Our main result is the following

**Theorem 1.2.** Let \( G \) be a complex connected semisimple algebraic group. Then there is a natural bijection between the set of integral twisted conjugacy classes in \( G((z)) \) and the set of isomorphism classes of semi-stable holomorphic principal \( G \)-bundles on \( \mathcal{E} \).

The main reason we are interested in this result is that twisted conjugacy classes in \( G((z)) \) may be interpreted as ordinary conjugacy classes in a larger group. Specifically, the group \( \mathbb{C}^* \) acts on \( \mathbb{C}((z)) \) by field automorphisms rescaling the variable \( z \), i.e., \( t \in \mathbb{C}^* \) acts by \( a(z) \mapsto a(t \cdot z) \). This gives a \( \mathbb{C}^* \)-action on the group \( G((z)) \) called “rotation of the loop”. Write \( \mathbb{C}^* \rtimes G((z)) \) for the corresponding semidirect product. Then, for any \( (q, a), (1, g) \in \mathbb{C}^* \rtimes G((z)) \), we have \( (1, g) \cdot (q, a) \cdot (1, g)^{-1} = (q, qa) \). Thus, twisted conjugacy
classes in $G((z))$ is essentially the same thing as ordinary conjugacy classes in a Kac-Moody group (=standard central extension of $\mathbb{C}^* \ltimes G((z))$).

We arrived at theorem 1.2 while trying to find an algebraic version of the following unpublished analytic result due to Looijenga (cf. [EFK]). Let $G$ be a connected complex Lie group, $G(\mathbb{C}^*)_{hol}$ the group of all holomorphic maps $a : \mathbb{C}^* \rightarrow G$ (possibly with an essential singularity at $z = 0$), and let $q$ be a fixed non-zero complex number such that $|q| < 1$. Then Looijenga showed that

**Proposition 1.3.** There is a natural bijection between the set of all twisted-conjugacy classes in $G(\mathbb{C}^*)_{hol}$ and the set of isomorphism classes of arbitrary holomorphic $G$-bundles on $E$.

**Proof.** Observe that the pull-back via the projection $\pi : \mathbb{C}^* \rightarrow \mathbb{C}^*/q\mathbb{Z} = E$ establishes an equivalence between the category of $G$-bundles on $E$ and the category of $q^{\mathbb{Z}}$-equivariant holomorphic $G$-bundles on $\mathbb{C}^*$. We associate to $a \in G(\mathbb{C}^*)_{hol}$ the trivial holomorphic $G$-bundle $\mathbb{C}^* \times G \rightarrow \mathbb{C}^*$ on $\mathbb{C}^*$ with $q^{\mathbb{Z}}$-equivariant structure given by the action $q : (z,g) \mapsto (q \cdot z, a(z) \cdot g)$. The corresponding $G$-bundle on $E$ will be referred to as the $G$-bundle with multiplier $a$. It is easy to see that two $G$-bundles on $E$ associated to two different multipliers are isomorphic if and only if the multipliers are twisted conjugate. Conversely, it is known that any holomorphic $G$-bundle on $\mathbb{C}^*$ is trivial. The action of the element $q$ on such a trivial bundle has to be of the form $q : (z,g) \mapsto (q \cdot z, a(z) \cdot g)$, where $a : \mathbb{C}^* \rightarrow G$ is a holomorphic map (changing trivialization has the effect of replacing $a$ by a twisted conjugate map). Hence, every $q^{\mathbb{Z}}$-equivariant holomorphic $G$-bundle on $\mathbb{C}^*$ can be obtained via the above construction. \[\square\]

Although motivation for theorem 1.2 came from loop groups, the result itself is most adequately understood in the framework of $q$-difference equations. To explain this assume, for simplicity, that $G = GL_n$. Given $q \in \mathbb{C}^*$ and $a(z) \in GL_n((z))$, we consider a difference equation

$$x(q \cdot z) = a(z) \cdot x(z), \quad (1.4)$$

where $x(z) \in \mathbb{C}^n((z))$ is the unknown $\mathbb{C}^n$-valued formal power series. It is clear that if $x(z)$ is a solution to (1.4) and $g(z) \in GL_n((z))$, then $\tilde{x}(z) := g(z)x(z) \in \mathbb{C}^n((z))$ is a solution to a similar equation with $a(z)$ being replaced by $\tilde{a}(z) = g(qz) \cdot a(z) \cdot g(z)^{-1}$, a twisted conjugate loop. Therefore classification of equations (1.4) modulo transformations $x(z) \mapsto \tilde{x}(z)$ reduces to the classification of the twisted conjugacy classes in $GL_n((z))$. 


Equation (1.4) should be regarded as a $q$-analogue of the first order differential equation
\[ \frac{dx}{dz} = a(z) \cdot x(z), \tag{1.5} \]
and twisted conjugation (1.1) should be regarded as a $q$-analogue of the gauge transformation: $a(z) \mapsto g(z) \cdot a(z) \cdot g(z)^{-1} + z \frac{dg}{dz}g(z)^{-1}$. It is well-known that the classification of equivalence classes of equations like (1.5) depends in an essential way on the type of functions $a$ and $g$ one is considering. If one puts himself into analytic framework, then $a$ and $g$ are taken to be elements of $\mathfrak{gl}_n(\mathbb{C}^*)_{hol}$ and $GL_n(\mathbb{C}^*)_{hol}$, respectively. It is well-known and easy to prove that in this case the differential equation is completely determined (up to equivalence) by the monodromy of its fundamental solution. Thus, there is a natural bijection between the set of equivalence classes of differential equations of type (1.5) and the set of conjugacy classes in $G$. This is a differential equation analogue of proposition 1.3.

The situation changes drastically if $\mathfrak{gl}_n(\mathbb{C}^*)_{hol}$ and $GL_n(\mathbb{C}^*)_{hol}$ are replaced by formal loops $\mathfrak{gl}_n((z))$ and $GL_n((z))$, respectively. The classical theory says that for the equation to be determined by its monodromy it should have regular singularity at $z = 0$. This is a differential analogue of the “integrality” condition in theorem 1.2. Thus, the $G$-bundle in theorem 1.2 should be thought of as a $q$-analogue of the monodromy of a differential equation.

Classification of $q$-difference equations (1.4) is equivalent to the classification of $D_q$-modules, where by a $D_q$-module we mean a finite rank $\mathbb{C}((z))$-module $M$ equipped with an invertible $\mathbb{C}$-linear operator $q : M \to M$ such that
\[ q(f(z) \cdot m) = f(qz) \cdot (qm), \quad \forall f \in \mathbb{C}((z)), \, m \in M. \]
A $D_q$-module is a module over a smash product of the group algebra of the group $q\mathbb{Z}$ with $\mathbb{C}((z))$. A $D_q$-module $M$ is said to be integral if there exists a free $\mathbb{C}[[z]]$-submodule $L \subset M$ of maximal rank such that $q(L) \subset L$ and $q^{-1}(L) \subset L$. Integral $D_q$-modules form an abelian category, $\mathcal{M}_q$. It is easy to verify that tensor product over $\mathbb{C}((z))$ makes $\mathcal{M}_q$ into a tensor category. On the other hand, tensor product of vector bundles makes the category of degree zero semistable holomorphic vector bundles on $\mathcal{E}$ into a tensor category, $Vect_{ass,\mathcal{E}}(\mathcal{E})$, with $\text{Hom}$’s given by vector bundle morphisms. The following result is a natural strengthenning of theorem 1.2 in the $G = GL$-case.
Theorem 1.6. The tensor category $\mathcal{M}_q$ is equivalent to $\text{Vect}_{ss,0}(\mathcal{E})$.

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2. From loop groups to $G$-bundles on $\mathcal{E}$.

The ring homomorphism $\mathbb{C}[[z]] \to \mathbb{C}$, $f \mapsto f(0)$ induces, for any algebraic group $H$, a natural group homomorphism $H[[z]] \to H$. Let $H_1[[z]]$ denote the kernel of this homomorphism, a “congruence subgroup”. We use the notation $H[z]$ and $H[z, z^{-1}]$ for the groups of $\mathbb{C}[z]$- and $\mathbb{C}[z, z^{-1}]$-points of $H$, respectively. Thus, $H[z] \subset H[[z]]$ and $H[z, z^{-1}] \subset H((z))$. Elements of $H[z, z^{-1}]$ will be referred to as polynomial loops.

From now on we fix a complex connected semisimple algebraic group $G$ with Lie algebra $\mathfrak{g}$, and $q \in \mathbb{C}^*$ such that $|q| < 1$.

Our proof of theorem 1.2 consists of several steps. We first assign to an integral element $a \in G((z))$ a $G$-bundle on $\mathcal{E}$. The naive idea of using $a$ as a multiplier (cf. proof of proposition 1.3) can not be applied here directly, for $a$ is only a formal loop, hence, does not give a holomorphic map in general. To overcome this difficulty, we prove the following result.

Proposition 2.1. For any $a \in G[[z]]$, there exists a Borel subgroup $B \subset G$ with unipotent radical $U$, such that $a$ is twisted conjugate to a polynomial loop of the form $a_0 \cdot a_1(z)$ where $a_0 \in B$ and $a_1 \in U[[z]]$.

To prove the proposition we need some preparations. Recall that for a semisimple element $s \in G$, the adjoint action of $s$ on $\mathfrak{g}$ has a weight space decomposition $\mathfrak{g} = \bigoplus_{\lambda} \mathfrak{g}_{\lambda}$ where $\mathfrak{g}_{\lambda}$ is the eigenspace corresponding to an eigenvalue $\lambda \in \mathbb{C}^*$.

Let $a(z) = a_0 \cdot a_1(z) \in G[[z]]$, where $a_0 \in G$ is a constant loop and $a_1(z) \in G_1[[z]]$. Write $a_0^{ss} \in G$ for the the semisimple part in the Jordan decomposition of $a_0$, and let $\mathfrak{g} = \bigoplus_{\lambda} \mathfrak{g}_{\lambda}$ be the weight space decomposition with respect to the adjoint action of $a_0^{ss}$.

Definition. The element $a(z) = a_0 \cdot a_1(z)$ is called aligned if it can be written as a product $a_0 \exp(x_1 z) \exp(x_2 z^2) \cdots$, where $x_i \in \mathfrak{g}_{q^i}$.

Note that the product above is finite and gives an element of $G[z]$. Hence any aligned element is a polynomial loop.

Lemma 2.2. For any $a \in G[[z]]$, one can find $g \in G_1[[z]]$ such that $g a$ is aligned.
Proof. Following [BV, pp.31, 68], we will construct a sequence of elements \( x_i \in \mathfrak{g} \) and \( y_i \in \mathfrak{g}_{q^i} \), as follows. Note that the exponential map gives a bijection \( z \cdot \mathfrak{g}[[z]] \xrightarrow{\sim} G_1[[z]] \). Therefore we can write \( a = a_0 \exp(a_1 z) \exp(a' z^2) \), where \( a_1 \in \mathfrak{g} \) and \( a' \in \mathfrak{g}[[z]] \).

Since the operator \( (q \cdot \text{Ad}_{a_0^{-1}} - \text{Id}) \) is invertible on \( \bigoplus_{\lambda \neq q} \mathfrak{g}_\lambda \), there are uniquely determined elements \( x_1 \in \bigoplus_{\lambda \neq q} \mathfrak{g}_\lambda \) and \( y_1 \in \mathfrak{g}_q \) such that

\[
(q \cdot \text{Ad}_{a_0^{-1}} - \text{Id})(x_1) + a_1 = y_1.
\]

We next find \( y_2 \). To that end, set \( g_1 = \exp(x_1 z) \). Then the above equation implies \( q a = a_0 \exp(y_1 z) \exp(a_2 z^2) \exp(a' z^3) \), where \( a_2 \in \mathfrak{g} \) and \( a' \in \mathfrak{g}[[z]] \). Hence there exist uniquely determined elements \( x_2 \in \bigoplus_{\lambda \neq q^2} \mathfrak{g}_\lambda \) and \( y_2 \in \mathfrak{g}_{q^2} \) such that

\[
(q^2 \cdot \text{Ad}_{a_0^{-1}} - \text{Id})(x_2) + a_2 = y_2.
\]

Set \( g_2 = \exp(x_2 z^2) \exp(x_1 z) \). Then the above equation insures that \( g a = a_0 \exp(y_1 z) \exp(y_2 z^2) \exp(a_3 z^3) \exp(a'' z^4) \), where \( a_3 \in \mathfrak{g} \) and \( a'' \in \mathfrak{g}[[z]] \). Iterating this process we construct a sequence \( \{ x_i \in \mathfrak{g}, i = 1, 2, \ldots \} \), such that setting \( g_k := \exp(x_k z^k) \exp(x_{k-1} z^{k-1}) \ldots \exp(x_1 z) \) we get

\[
g_k a = a_0 \cdot \exp(y_1 z) \exp(y_2 z^2) \ldots \exp(y_k z^k) \exp(y z^{k+1}), \tag{2.3}
\]

where \( y_i \in \mathfrak{g}_{q^i} \) and \( y \in \mathfrak{g}[[z]] \). Then the product \( g := \lim g_k = \ldots \exp(x_k z^k) \cdot \exp(x_{k-1} z^{k-1}) \ldots \ldots \exp(x_1 z) \) stabilizes since \( \mathfrak{g}_{q^k} = 0 \) for all \( k >> 0 \). Equation (2.3) shows that \( q a \) is aligned. \( \square \)

Proof of Proposition 2.1. Choose a maximal torus \( T \subset G \) containing \( a_0^{ss} \). Let \( R \subset \text{Hom}(T, \mathbb{C}^*) \) be the set of roots of \((G, T)\). The subset consisting of the roots \( \gamma \in R \) such that \( |\gamma(a_0^{ss})| \leq 1 \) defines a parabolic \( P \subset G \). Then, for any \( i > 0 \), the subspace \( \mathfrak{g}_{q^i} \) is contained in the nilradical of \( \text{Lie}P \), for \( |q| < 1 \).

Further, we may choose a Borel subgroup \( B \subset P \) that contains the unipotent part of \( a_0 \). Let \( U \) denote the unipotent radical of \( B \). Then the element \( \exp(y_1 z) \exp(y_2 z^2) \ldots \exp(y_k z^k) \) constructed in the proof of lemma 2.2 belongs to \( U[[z]] \), and the proposition follows. \( \square \)

Lemma 2.5. Let \( B \) and \( \bar{B} \) be two Borel subgroups with unipotent radicals \( U, \bar{U} \). Let \( a = a_0 \cdot a_1 \), \((a_0 \in B, a_1 \in U[[z]])\), and \( \bar{a} = \bar{a}_0 \cdot \bar{a}_1 \), \((\bar{a}_0 \in B, \bar{a}_1 \in \bar{U}[\mathfrak{g}]]\)
We prove by descending induction on $(z)$. Then any element \( g \in G((z)) \) such that \( ga = \tilde{a} \) is a polynomial loop, i.e., \( g \in G[z, z^{-1}] \).

**Proof.** Multiplying \( g \) by an element of \( G \) we may assume that \( B = \tilde{B} \).

Further, we find a faithful rational representation \( \rho : G \to SL_n(\mathbb{C}) \) such that \( B \) is the inverse image of the subgroup of upper triangular matrices in \( SL_n(\mathbb{C}) \). Thus, applying \( \rho \), we are reduced to proving the lemma in the case \( G = SL_n(\mathbb{C}) \) and \( B = \) upper triangular matrices. From now on, \( a \) and \( \tilde{a} \) are are assumed to be upper triangular polynomial matrices. Set \( M = \max(\deg a, \deg \tilde{a}) \), the maximum of the degrees of the corresponding matrix-valued polynomials.

Let \( ga = \tilde{a} \). We can write \( g(z) = \sum_{k \geq k_0} g(k) z^k \), where \( g(k) \) are complex \( n \times n \)-matrices. Computing the bottom left corner matrix entry of each side of the equation \( g(qz)a(z) = \tilde{a}(z)g(z) \) yields:

\[
g(k)_{n,1} \cdot (q^k a_{1,1} - \tilde{a}_{n,n}) = 0.
\]

It follows, since the diagonal entries of \( a, \tilde{a} \) are nonzero, that there exists at most one \( k \), say \( k = K_1 \), such that \( g(k)_{n,1} \neq 0 \). Using this, we now compute the two matrix entries standing on \((n-1) \times 1\) and \( n \times 2 \) places of each side of the equation \( g(qz)a(z) = \tilde{a}(z)g(z) \). We find that, for any \( k > K_1 + M \):

\[
g(k)_{n-1,1} \cdot (q^k a_{1,1} - \tilde{a}_{n-1,n-1}) = 0, \quad g(k)_{n,2} \cdot (q^k a_{2,2} - \tilde{a}_{n,n}) = 0.
\]

We deduce, as before, that there exists \( K_2 >> 0 \) such that for all \( k \geq K_2 \), we have \( g(k)_{n-1,1} = g(k)_{n,2} = 0 \).

Continuing the process of computing the entries of each side of the equation \( g(qz)a(z) = \tilde{a}(z)g(z) \) along the diagonals (moving from bottom left corner to top right corner) we prove by descending induction on \((i - j)\) that \( g(k)_{i,j} = 0 \), for all \( k \gg 0 \). \( \square \)

We define a map from integral twisted conjugacy classes in \( G((z)) \) to \( G \)-bundles on the elliptic curve \( \mathcal{E} = \mathbb{C}^*/q^\mathbb{Z} \) as follows. Given an element \( a \in G((z)) \) in an integral twisted conjugacy class, we find (proposition 2.1) an aligned element \( f \in G((z)) \) which is twisted conjugate to \( a \). The loop \( f \) being polynomial, it gives a well-defined holomorphic map \( f : \mathbb{C}^* \to G \). Hence, we can associate to \( f \) the holomorphic \( G \)-bundle on \( \mathcal{E} \) with multiplier \( f \), see proof of proposition 1.3. If \( f' \) is another aligned element which is twisted conjugate to \( a \), then by lemma 2.5, \( f \) and \( f' \) are twisted conjugate to each other via a Laurent polynomial, hence a holomorphic loop. It follows
that the $G$-bundles with multipliers $f$ and $f'$ are isomorphic. Thus, we have associated to $a$ a well-defined isomorphism class of $G$-bundles on $\mathcal{E}$.

**Remark 2.6.** Note that if $a$ is a polynomial loop, the above map does not necessarily takes the twisted conjugacy class of $a$ (in $G((z))$) to the holomorphic $G$-bundle on $\mathcal{E}$ with multiplier $a$, even though the latter is defined. We do not claim that if $a$ and $a'$ are two polynomial loops that are twisted conjugate in $G((z))$ then the $G$-bundles on $\mathcal{E}$ with multipliers $a$ and $a'$ are isomorphic. Moreover, the following polynomial loops

$$a = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, \quad a' = \begin{pmatrix} z & z^{-1} \\ z & 0 \end{pmatrix}$$

are twisted conjugate in $\text{SL}_2((z))$ by a *divergent* element, while the holomorphic $\text{SL}_2$-bundles on $\mathcal{E}$ with multipliers $a$ and $a'$ are not isomorphic. This is an obstacle for trying to extend the correspondence of theorem 1.2 beyond the set of integral twisted conjugacy classes on the one hand, and semi-stable $G$-bundles on the other.

### 3. Going to a finite covering.

Recall that for any positive integer $m$ the field imbedding $\mathbb{C}((z)) \hookrightarrow \mathbb{C}((w))$, $z \mapsto w^m$, makes $\mathbb{C}((w))$ a Galois extension of $\mathbb{C}((z))$ with the Galois group $\mathbb{Z}/m\mathbb{Z}$. From now on we will write $z^{1/m}$ instead of $w$, so that $(z^{1/m})^m = z$, and the generator of the Galois group acts as $\omega : z^{1/m} \mapsto e^{2\pi i/m} z^{1/m}$. Let $G((z^{1/m}))$ denote the group of $\mathbb{C}((z^{1/m}))$-rational points of $G$. We view $G((z))$ as the subgroup of $\omega$-fixed points in $G((z^{1/m}))$. We will sometimes write $a = a(z^{1/m})$ for an element of $G((z^{1/m}))$.

Further, we fix $\tau$ in the upper half-plane, $\text{Im}\tau > 0$, such that $q = e^{2\pi i \tau}$. The automorphism $f(z) \mapsto f(q \cdot z)$ of the field $\mathbb{C}((z))$ can be extended to an automorphism of $\mathbb{C}((z^{1/m}))$ via the assignment $z^{1/m} \mapsto e^{2\pi i \tau/m} z^{1/m}$. This gives rise to a twisted conjugation action $g : a \mapsto g a$ on $G((z^{1/m}))$ that extends the one on $G((z))$.

**Definition.** An element $s \in G$ is said to be *reduced* if, for any finite dimensional rational representation $\rho : G \to GL(V)$, and any eigenvalue $\lambda$ of the operator $\rho(s)$, the equation $\lambda^k = q^l$, (for some $k, l \in \mathbb{Z} \setminus \{0\}$) implies $\lambda = 1$, i.e., if there are no eigenvalues of the form $\lambda = e^{2\pi i \tau \cdot r}$, $r \in \mathbb{Q}$, except $\lambda = 1$.

View $G$ as the subgroup of “constant loops” in $G((z^{1/m}))$. 

THEOREM 3.1. Let $\alpha(z) \in G[[z]]$ be an aligned element. Then one can find a positive integer $m$ and $g \in G((z^{1/m}))$ such that $\%a$ is a constant loop and moreover the element $\%a \in G$ is reduced.

To prove the theorem, we fix a maximal torus $T \subset G$, and let $X^*(T) = \text{Hom}_{\text{alg}}(T, \mathbb{C}^*)$ and $X_*(T) = \text{Hom}_{\text{alg}}(\mathbb{C}^*, T)$ denote the weight and co-weight lattices, respectively. There is a canonical perfect pairing $\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$. We first prove

**Lemma 3.2.** For any $s \in T$ there exists $\phi \in X_*(T)$ and an integer $m \neq 0$ such that the following holds:

(i) $s = \phi(e^{2\pi i \tau/m}) \cdot s_{\text{red}}$ where $s_{\text{red}}$ is reduced;

(ii) Let $\alpha \in X^*(T)$. If $\alpha(s) = q^l$ for some $l \in \mathbb{Z}$, then $\langle \alpha, \phi \rangle / m = l$ and $\alpha(s_{\text{red}}) = 1$.

**Proof of Lemma.** In $\mathbb{C}^*$ consider the subgroup

$$\Gamma = \{ z \in \mathbb{C}^* \mid \exists k, l \in \mathbb{Z} \text{ such that } z^k = q^l \}.$$  

Let $L$ be the subgroup of the weights $\alpha \in X^*(T)$ such that $\alpha(s) \in \Gamma$. Clearly, if $\alpha \in X^*(T)$ and $m \cdot \alpha \in \Gamma$ for some integer $m \neq 0$, then $\alpha \in \Gamma$. Hence, by the well known structure theorem about subgroups in $\mathbb{Z}^n$ we deduce that $L$ splits off as a direct summand in $X^*(T)$. Therefore, there is another lattice $L_{\text{red}} \subset X^*(T)$ such that $X^*(T) = L \oplus L_{\text{red}}$. This direct sum decomposition of lattices must be induced by a direct product decomposition $T = T_1 \times T_{\text{red}}$, where $T_1$ and $T_{\text{red}}$ are subtori in $T$ such that $X^*(T_1) = L$ and $X^*(T_{\text{red}}) = L_{\text{red}}$. Thus, we have $s = s_1 \cdot s'_{\text{red}}$, where $s_1 \in T_1$ and $s'_{\text{red}} \in T_{\text{red}}$.

For any $\alpha \in X^*(T)$, we have by construction $\alpha(s_1) \in \Gamma$. Furthermore, $\alpha(s) \in \Gamma$ implies $\alpha \in L$, hence $\alpha(T_{\text{red}}) = 1$. Therefore, for $\alpha \in X^*(T)$ such that $\alpha(s'_{\text{red}}) \in \Gamma$ we have $\alpha(s_1 \cdot s'_{\text{red}}) \in \Gamma$, hence $\alpha \in L$, hence $\alpha(s'_{\text{red}}) = 1$. Thus, $s'_{\text{red}}$ is reduced.

View the groups $X^*(T_1)$ and $X_*(T_1)$ as lattices in $(\text{Lie}T_1)^*$ and $\text{Lie}T_1$, respectively, so that $X_*(T_1)$ is the kernel of the exponential map. Write $s_1 = \exp(h)$, where $h \in \text{Lie}T_1$. Since $\alpha(s_1) \in \Gamma$ for any $\alpha \in X^*(T)$, and elements of $\Gamma$ have the form $z = e^{2\pi i (\tau + r)}$, $r, r' \in \mathbb{Q}$, it follows that $\alpha(h) \in \mathbb{Q} \cdot \tau + \mathbb{Q}$. Hence, $h \in \tau \cdot \mathbb{Q} \otimes_{\mathbb{Z}} X_*(T_1) + \mathbb{Q} \otimes_{\mathbb{Z}} X_*(T_1)$. Therefore, there exist $\phi, \psi \in X_*(T_1)$ and an integer $m$ such that $h = \frac{1}{m} \phi + \frac{1}{m} \psi$. Thus, $s_1 = \exp(h) = \epsilon \cdot \phi(e^{2\pi i \tau/m})$, where $\epsilon = \psi(e^{2\pi i /m})$ is an element of order $m$. We put $s_{\text{red}} = \epsilon \cdot s'_{\text{red}}$. Clearly, $s_{\text{red}}$ is reduced, and $s = s_1 \cdot s'_{\text{red}} = \phi(e^{2\pi i \tau/m}) \cdot s_{\text{red}}$. 

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To prove part (ii), let $\alpha \in X^*(T)$ be such that $\alpha(s) = q^l$ for some $l \in \mathbb{Z}$. Then $\alpha \in L$, hence $\alpha(s_{red}) = 1$. Furthermore, the equation

$$e^{2\pi i \cdot \tau \cdot l} = q^l = \alpha(s) = e^{2\pi i \cdot \tau \cdot (\langle \alpha, \phi \rangle/m + 2\pi i \cdot \langle \alpha, \psi \rangle/m}$$

yields $\tau \cdot (l - \langle \alpha, \phi \rangle/m) + \langle \alpha, \psi \rangle/m \in \mathbb{Z}$. It follows, since $\langle \alpha, \phi \rangle$ and $\langle \alpha, \psi \rangle$ are integers, that $l = \langle \alpha, \phi \rangle/m$ and that $\langle \alpha, \psi \rangle/m \in \mathbb{Z}$. Hence $\alpha(\epsilon) = \alpha(\epsilon) \cdot \alpha(s_{red}) = 1$, and (ii) follows. \( \square \)

**Proof of theorem 3.1.** We choose the Borel subgroup $B = T \cdot U$ as constructed in the proof of proposition 2.1. Thus we have $a(z) = a_0 \exp(x_1 z) \exp(x_2 z^2) \cdots \exp(x_k z^k)$, where $a_0^{ss} \in T$ and $x_i \in g_q \subset \text{Lie} B$, where $g_q$ stands for the $q^i$-eigenspace of $\text{Ad} a_0^{ss}$.

Applying lemma 3.2 to $s = a_0^{ss}$, we find an integer $m$ and an algebraic group homomorphism $\phi : C^* \to T$ such that $a_0^{ss} = \phi(e^{2\pi i \cdot \tau \cdot m}) \cdot s_{red}$.

For any integer $i \geq 1$ we can write $x_i = \sum_{\alpha} x_{\alpha, i}$, where $\alpha$ is a positive root of $(G, T)$ such that $\alpha(a_0^{ss}) = q^i$ and $x_{\alpha, i}$ is a non-zero root vector corresponding to $\alpha$. For such an $\alpha$, part (ii) of lemma 3.2 yields $\alpha(\phi(z^{1/m})) = z^{\langle \alpha, \phi \rangle/m} = z^i$. We set $g = \phi(z^{1/m})$, a well-defined element of the group $G((z^{1/m}))$. Then, we obtain

$$(\text{Ad} g)(x_{\alpha, i}) = \alpha(\phi(z^{1/m})) \cdot x_{\alpha, i} = z^i \cdot x_{\alpha, i}.$$  

It follows that a similar equation holds for $x_i$ instead of $x_{\alpha, i}$. From this we deduce

$$g^{-1} \cdot \exp(x_i z^i) \cdot g = \exp(x_i). \quad (3.2.1)$$

Further, let $u$ be the unipotent part of the Jordan decomposition of $a_0$. Write $u = \exp(y)$. We have $y = \sum_{\alpha} y_{\alpha}$, where $y_{\alpha} \in \text{Lie} U$ are root vectors. Since $a_0^{ss}$ commutes with $y$, we deduce similarly, using lemma 3.2(ii), that $\alpha(\phi) = 0$ for any root $\alpha$ such that $y_{\alpha} \neq 0$. It follows that $g^{-1} \cdot u \cdot g = u$. From this and (3.2.1) we obtain

$$q_a = \phi(e^{2\pi i \cdot \tau \cdot m} z^{1/m})^{-1} \cdot a_0^{ss} \cdot u \cdot \exp(x_1 z) \exp(x_2 z^2) \cdots \exp(x_k z^k) \cdot \phi(z^{1/m})$$

$$= \phi(e^{2\pi i \cdot \tau \cdot m})^{-1} \cdot a_0^{ss} \cdot u \cdot \exp(x_1) \cdot \exp(x_2) \cdots \exp(x_k).$$

Using lemma 3.2(ii) we see that $\phi(e^{2\pi i \cdot \tau \cdot m})^{-1} \cdot a_0^{ss} \cdot u = s_{red} \cdot u$. This element is reduced, and the theorem follows. \( \square \)

**Lemma 3.3.** Let $s \in G$ be reduced. Then any element $g \in G((z))$ such that $q_s = s$ is a constant loop.
Proof. Consider the adjoint representation \( \rho : G \to GL(g) \). We choose a basis in \( g \) such that \( \rho(s) \) is an upper-triangular matrix. Given \( g \) such that \( \partial s = s \), we write \( \rho(g) = \sum_{k \geq k_0} g(k) z^k \), where \( g(k) \) are complex \( n \times n \) matrices. The same process as in the proof of lemma 2.5 gives equations of the type

\[
g(k)_{m,n} \cdot (q^k s_{n,n} - s_{m,n}) = 0 \quad , \quad k \in \mathbb{Z}.
\]

Since \( s \) is reduced, this implies \( g(k)_{m,n} = 0 \) for all \( k \neq 0 \). Hence the image of \( g \) in \( GL(g)((z)) \) is constant. It follows that \( g \) is itself constant, for the kernel of the adjoint representation \( G \to GL(g) \) is finite. □

Corollary 3.4. Let \( a \in G[[z]] \) be aligned and \( s \in G \) be reduced. Assume \( g \in G((z^{1/m})) \) is such that \( \partial a = s \). Then \( g \in G([z^{1/m}, z^{-1/m}]) \) is a Laurent polynomial loop in \( z^{1/m} \). Furthermore, \( \theta = g(e^{2\pi i \tau/m} z^{1/m}) g(z^{1/m})^{-1} \) is a constant loop, and \( \theta^m = 1 \).

Proof. The first claim follows from lemma 2.5. To prove the second claim, recall the Galois automorphism \( \omega : f(z^{1/m}) \mapsto f(e^{2\pi i \tau/m} z^{1/m}) \) on \( \mathbb{C}((z^{1/m})) \). We apply the induced automorphism of \( G((z^{1/m})) \) to the equation \( \partial a = s \). The RHS being independent of \( z \), and \( a \) being fixed by \( \omega \), we get \( \omega^s = s \). This equation together with the original one, \( \partial a = s \), yields \( \theta^s = s \), where \( \theta = g(e^{2\pi i \tau/m} z^{1/m}) g(z^{1/m})^{-1} \). Hence, \( \theta \) is a constant loop, by lemma 3.3. Further, applying the automorphism \( \omega \) several times to the first equation below we get a sequence of equations

\[
\theta = (\omega g) \cdot g^{-1} \quad , \quad \theta = (\omega^2 g) \cdot (\omega g)^{-1} \quad , \quad \ldots \quad , \quad \theta = (\omega^m g) \cdot (\omega^{m-1} g)^{-1}.
\]

Since \( \omega^m = 1 \), taking the product of all these equations yields \( \theta^m = 1 \). □

We fix two generators, an “a-cycle” and a “b-cycle”, of the fundamental group \( \pi_1(\mathcal{E}) \) as follows. The a-cycle is defined to be the image of a generator of \( \pi_1(\mathbb{C}^*) = \mathbb{Z} \) under the imbedding \( \pi_1(\mathbb{C}^*) \hookrightarrow \pi_1(\mathcal{E}) \) induced by the projection \( \mathbb{C}^* \to \mathbb{C}^*/q^\mathbb{Z} = \mathcal{E} \). The b-cycle is the image of the segment \( [1, q] \subset \mathbb{C}^* \) under the projection.

Given an integer \( m \neq 0 \), set \( m\mathcal{E} = \mathbb{C}^*/q^m \), and let \( m\pi : m\mathcal{E} \to \mathcal{E} \), \( z \mapsto z^m \) be the natural projection. Thus, \( m\mathcal{E} \) is an elliptic curve and the map \( m\pi \) is an \( m \)-sheeted Galois covering with the Galois group \( \mathbb{Z}/m\mathbb{Z} \) acting as monodromy around the a-cycle.

Proposition 3.5. Let \( a \in G[z] \) be an aligned element and \( P \) the principal \( G \)-bundle on \( \mathcal{E} \) with multiplier \( a \). Then there exists a positive integer \( m \) such that

\[
\]
(i) The bundle $\pi^* P$ is isomorphic to the holomorphic $G$-bundle on $\pi\mathcal{E}$ with a reduced constant multiplier $s \in G$;

(ii) Let $\tilde{\nabla}$ be the holomorphic connection on $\pi^* P$ transported via the isomorphism (i) from the trivial connection $d$ on the $G$-bundle with multiplier $s$. Then $\nabla$ descends to a well-defined holomorphic connection $\nabla$ on $P$. The latter has finite monodromy around $a$-cycle and a reduced monodromy around $b$-cycle.

PROOF. By theorem 3.1 there exists an element $g \in G(z^{1/m})$ such that $g a = s$ is a constant loop, and $s \in G$ is reduced. By corollary 3.4, $g = g(z^{1/m})$ is a Laurent polynomial in $z^{1/m}$. Hence, $g$ may be viewed as a well-defined $G$-valued regular function on the $m$-fold covering of $\mathbb{C}^*$. Let $P$ be the $G$-bundle on $\mathcal{E}$ with multiplier $a$. It follows that the pull-back, $\pi^* P$, has a multiplier which is a twisted conjugate of $s$. This proves part (i).

To prove (ii), recall that any $G$-bundle with a constant multiplier $s$ has a natural flat holomorphic connection which is given (in the trivialization on $\mathbb{C}^*$ corresponding to $s$) by the deRham differential $d$. We transport this connection to $\pi^* P$ via the isomorphism given by the loop $g$. The connection $\tilde{\nabla} = g^{-1} \circ d \circ g$ thus obtained descends to a connection on $P$ if and only if it is invariant under the Galois action of $\mathbb{Z}/m\mathbb{Z}$. But by corollary 3.4 we have $\omega g = \theta \cdot g$, hence we get

$$\omega(\tilde{\nabla}) = (\theta \cdot g)^{-1} \circ d \circ (\theta \cdot g) = g^{-1} \cdot (\theta^{-1} \circ d \circ \theta) \cdot g = g^{-1} \circ d \circ g = \tilde{\nabla},$$

since $\theta$ commutes with $d$. Thus, $\tilde{\nabla}$ is fixed by the Galois action.

To compute the monodromy, note that $g^{-1}$ is a flat section of the connection $\tilde{\nabla}$. Hence the monodromy of $\nabla$ around $b$-cycle equals $g(e^{2\pi i \tau / m} z^{1/m})$. $g(z^{1/m})^{-1} = \theta$. Since the covering $\pi^* : \pi \mathcal{E} \to \mathcal{E}$ has no monodromy around $b$-cycle and has finite monodromy around $a$-cycle, it follows that $\nabla$ also has monodromy $\theta$ around $b$-cycle and has finite monodromy around $a$-cycle. \qed

Given a finite dimensional rational $G$-module $V$, write $V_P$ for the associated vector bundle on $\mathcal{E}$ corresponding to a principal $G$-bundle $P$.

**Lemma 3.6.** Let $P$ be the $G$-bundle with an aligned multiplier, and $\nabla$ the connection on $P$ constructed in proposition 3.5. Then, for any rational representation $\phi : G \to GL(V)$, every holomorphic section of the associated vector bundle $V_P$ is flat with respect to the induced connection on $V_P$.

**Proof.** Since the connection $\nabla$ constructed in proposition 3.5 was obtained from a connection $\tilde{\nabla}$ on $\pi^* P$, the claim for $V_P$ is equivalent to a similar
claim for the vector bundle $m^\pi_* V_p$. This vector bundle is isomorphic to the vector bundle $V$ on $m^\pi \mathcal{E}$ with multiplier $\phi(s)$, so that the connection $\nabla$ is isomorphic to the trivial connection $d$. Thus, proving the claim amounts to showing that any holomorphic section of the vector bundle $V$ with multiplier $\phi(s)$ is constant.

To that end, write the matrix $\phi(s)$ in Jordan form $\phi(s) = \bigoplus J(\lambda_i, n_i)$, where $J(\lambda_i, n_i)$ is the $(n_i \times n_i)$ Jordan block with eigenvalue $\lambda_i$. This gives the corresponding vector bundle decomposition $V = \bigoplus V_i$ where $V_i$ is the vector bundle with multiplier $J(\lambda_i, n_i)$. If $L_i$ denotes the line bundle with multiplier $\lambda_i$, then there is a canonical vector bundle imbedding $L_i \hookrightarrow V_i$. Furthermore, one can prove (using, e.g., the Fourier-Mukai transform) that the imbedding induces an isomorphism $\Gamma(m^\pi L_i) \cong \Gamma(m^\pi V_i)$ of the spaces of global sections. Hence, any holomorphic section of $V_i$ comes from a holomorphic section of $L_i$. But $L_i$ is a degree zero line bundle, hence has a non-zero section only if it is the trivial bundle, i.e. if $\lambda_i = q^m$. Observe now that $\lambda_i$ is an eigen-value of the matrix $\phi(s)$. Since $s \in G$ is reduced, equation $\lambda_i = q^m$ implies $\lambda_i = 1$. But then the only holomorphic section of $L_i$ is a constant section. The latter is annihilated by the deRham differential $d$, and the lemma is proved. □

**Proposition 3.7.** Let $a, a_1 \in G((z))$ be two aligned elements. If the $G$-bundle on $E$ with multiplier $a$ is isomorphic to the $G$-bundle on $E$ with multiplier $a_1$, then $a$ is twisted conjugate to $a_1$ via a polynomial loop.

**Proof.** By theorem 3.1, there exist an integer $m \geq 1$ and elements $g, g_1 \in G((z^{1/m}))$ such that

$$g^a = s, \quad g^a_1 = s_1 \quad \text{where } s, s_1 \in G \text{ are reduced.} \tag{3.7.1}$$

Let $\nabla, \nabla_1$ be the holomorphic connections on the $G$-bundles on $E$ with multipliers $a$ and $a_1$, respectively, constructed in proposition 3.5. The monodromies of the connections around $a$-cycle, are equal to $s$ and $s_1$, respectively, and the monodromies around $b$-cycle are equal to $\theta$ and $\theta_1$, respectively. By proposition 3.5 we have $\theta^m = \theta_1^m = 1$. If the $G$-bundles with multipliers $a$ and $a_1$ are isomorphic, then we may view $\nabla_1$ as another holomorphic connection on the $G$-bundle $P$ with multiplier $a$.

Since the cotangent bundle on $E$ is trivial the difference $X = \nabla_1 - \nabla$ may be viewed as a holomorphic section, $X$, of the adjoint bundle $g_p$. Since $s$ is reduced the section $X$ is flat with respect to $\nabla$, by lemma 3.6. Let $p : \hat{E} \to E$ be a universal cover of $E$. The bundle $p^* P$ on $\hat{E}$ has a horizontal
Finally, using (7.3.1) we calculate Thus, the notation of (7.3.1), and put \( \varphi \) the first formula in (7.3.2) we deduce \( \varphi/m \), and \( \phi \) completes the proof. 1. Thus, we may find a maximal torus \( T \) containing \( \theta, \theta_1 \), and \( \phi \in X \), such that \( x = \phi/m \) (cf. proof of lemma 3.2).

Clearly, \( \phi(z^{1/m}) \) is a well defined element of \( G((z^{1/m})) \), and from the first formula in (7.3.2) we deduce \( \phi(e^{2\pi i \cdot \tau/m}z^{1/m}) \cdot s \cdot \phi(z^{-1/m}) = s_1 \). Recall the notation of (7.3.1), and put \( f(z^{1/m}) = g_1(z^{1/m})^{-1} \cdot \phi(z^{1/m}) \cdot g(z^{1/m}) \in G((z^{1/m})). \) We claim that \( f \in G((z)) \). To prove this, it suffices to show that \( f(e^{2\pi i \cdot \tau/m}z^{1/m}) = f(z^{1/m}). \) The latter follows from the chain of equalities:

\[
\begin{align*}
f(e^{2\pi i \cdot \tau/m}z^{1/m}) &= g_1^{-1}(e^{2\pi i \cdot \tau/m}z^{1/m}) \cdot \phi(e^{2\pi i \cdot \tau/m}z^{1/m}) \cdot g(e^{2\pi i \cdot \tau/m}z^{1/m}) = \\
g_1^{-1}(z^{1/m}) \cdot \theta_1^{-1} \cdot \exp(x) \cdot \phi(z^{1/m}) \cdot \theta \cdot g(z^{1/m}) = \\
g_1^{-1}(z^{1/m}) \cdot \theta_1^{-1} \cdot \exp(x) \cdot \theta \cdot \phi(z^{1/m}) \cdot g(z^{1/m}) = \\
g_1^{-1}(z^{1/m}) \cdot \phi(z^{1/m}) \cdot g(z^{1/m}) = f(z^{1/m}).
\end{align*}
\]

Finally, using (7.3.1) we calculate

\[
f_a = g_1^{-1} \phi g_a = g_1^{-1} \phi s = g_1^{-1} s_1 = a_1.
\]

Thus, \( a \) and \( a_1 \) are twisted conjugate by an element of \( G((z)) \). Lemma 2.5 completes the proof. \( \square \)

4. Semistable \( G \)-bundles and holomorphic connections.

Recall that \( G \) is a complex connected semisimple group. For the definition and properties of semistable holomorphic \( G \)-bundles on an elliptic curve we refer to [R] and [RR].

Proposition 4.1. A holomorphic principal \( G \)-bundle over an elliptic curve is semistable if and only if it has a holomorphic connection (necessarily flat).
The "if" part is a corollary of the main result of [B]. The "only if" part follows from theorem 4.2 below. □

**Theorem 4.2.** For any semistable $G$-bundle $P$ on $E$, there exists a holomorphic connection on $P$ with finite order monodromy around a cycle such that, for any rational $G$-module $V$, every holomorphic section of the associated vector bundle $V_P$ is flat with respect to the induced connection on $V_P$.

**Proof.** We choose and fix a faithful rational representation $G \to GL(V)$. By a theorem of Ramanan and Ramanathan [RR], semistability of $P$ implies semistability of $V_P$. By the classification of semistable vector bundles on $E$, due to Atiyah [A], any semistable vector bundle is isomorphic to the vector bundle with a constant multiplier. Hence, the bundle $V_P$ has constant multiplier $a \in GL(V)$. View $a$ as an element of the semisimple group $PGL = PGL(V)$, and let $P_a$ be the principal $PGL$-bundle with (constant) multiplier $a$. By construction, the $PGL(V)$-bundle $P_a$ is induced from the $G$-bundle $P$ via the composite homomorphism $\rho : G \to GL(V) \to PGL(V)$.

We may regard the element $a \in PGL$ as a constant aligned loop in $PGL((z))$. Applying proposition 3.5, we see that there is an integer $m \geq 1$ and a reduced element $s \in PGL$ such that the bundle $m\pi_*P$ on $mE$ is isomorphic to the $PGL$-bundle on $mE$ with multiplier $s \in PGL$. Let $\nabla$ be the connection on $P_a$ constructed in proposition 3.5.

We claim that the connection $\nabla$ on $P_a$ arises from a holomorphic connection on the $G$-bundle $P$ via the composite homomorphism $\rho : G \to GL(V) \to PGL(V)$. Note that this composition has finite kernel, so that the induced canonical map $i : P \to P_a$ is an immersion. Let $TP_a$ be the tangent bundle on $P_a$. Our claim is equivalent to saying that the distribution in $TP_a$ formed by the horizontal subspaces of the connection $\nabla$ is tangent to the immersed submanifold $i(P) \subset P_a$. Observe that the canonical map $i : P \to P_a$ gives rise to a holomorphic section $\nu : E = P/G \to P_a/\rho(G)$. The horizontal distribution is tangent to $i(P)$ if and only if $\nu$ is a horizontal section.

To show the latter, we apply Chevalley’s theorem [S, Theorem 5.1.3] to the algebraic subgroup $\rho(G) \subset PGL$. The theorem says that we can find a rational representation $\phi : PGL \to GL(E)$ and a 1-dimensional subspace $1 \subset E$ such that $\rho(G) = \{g \in PGL \mid \phi(g)(1) = 1\}$. Notice, that since $G$ is semisimple, it stabilises a vector $l \in 1$. Hence, the assignment $g \mapsto g(l)$ gives rise to an imbedding $PGL/\rho(G) \hookrightarrow E$. Now let $E_{P_a}$ be the associated vector
bundle corresponding to $E$, equipped with the connectioned induced by $\nabla$. The imbedding $PGL/\rho(G) \hookrightarrow E$ gives rise to an imbedding $P_a/\rho(G) \hookrightarrow E_{Pa}$ compatible with the connections. To show that $\nu$ is horizontal, it suffices to show that its image under the above imbedding is a flat section. But this image is a holomorphic section of $E_{Pa}$. By lemma 3.6, any holomorphic section of the vector bundle $E_{Pa}$ is flat with respect to the connection on $E_{Pa}$ induced by $\nabla$. This proves that $\nu$ is horizontal, so that the horisontal distribution on $TP_a$ is tangent to $i(P)$ and the connection $\nabla$ comes from a holomorphic $G$-connection on $P$.

Observe further that the connection $\nabla$ on $P_a$ has finite monodromy around $a$-cycle. The map $i : P \to P_a$ being an immersion with finite fibers, it follows that the $G$-connection on $P$ also has finite monodromy around $a$-cycle.

Finally, it remains to show that there exists a holomorphic connection on $P$ with finite order monodromy around $a$-cycle such that, for any rational $G$-module $V$, every holomorphic section of the associated vector bundle $V_p$ is flat with respect to the induced connection on $V_p$. We do not claim that the connection we have constructed has this property. Instead we proceed as follows. We first use the connection that we have constructed above to prove that any semistable $G$-bundle on $E$ is isomorphic to a $G$-bundle with an aligned multiplier. This will be done in the proof of theorem 4.3 below. We can then apply proposition 3.5(ii) and lemma 3.6 to get a connection on $P$ with all the required properties. □

**Theorem 4.3.** A $G$-bundle on $E$ is semistable if and only if it is isomorphic to the $G$-bundle with an aligned multiplier $a \in G[z]$.

**Proof.** By proposition 3.5(ii), any $G$-bundle $P$ with an aligned multiplier has a holomorphic connection. Then, the “if” part of proposition 4.1 (due to Biswas) implies that $P$ is semistable.

Conversely, let $P$ be a semistable $G$-bundle. By theorem 4.2, we can equip $P$ with a holomorphic connection that has monodromies $\theta, b \in G$ around the $a$- and $b$-cycles respectively, such that $\theta^m = 1$ for some integer $m \geq 1$. Observe that the elements $\theta$ and $b$ commute, for $\pi_1(E)$ is an abelian group. Hence, there is a maximal torus $T \subset G$ such that $\theta, b^{ss} \in T$. As in the proof of proposition 2.1, we choose a Borel subgroup $B \supset T$ such that $b \in B$ and $|\alpha(b^{ss})| \leq 1$, for any positive (with respect to $B$) root $\alpha$.

Further, since $\theta^m = 1$ there exists $\phi \in X_*(T)$ such that $\theta = \phi(e^{2\pi i/m})$. Set $g = \phi(z^{1/m})^{-1}$, a well-defined polynomial loop in $G((z^{1/m}))$. We put $a = \phi(z^{1/m})^{-1}$ and $b = g^{-1}$.

**Theorem 4.4.** A $G$-bundle on $E$ is semistable if and only if it is isomorphic to the $G$-bundle with an aligned multiplier $a \in G[z]$.
$\Phi \in G((z^{1/m}))$. We have $g(e^{2\pi i/m}z^{1/m}) = \theta^{-1}g(z^{1/m})$. Since $\theta$ commutes with $b$, we deduce that $a(e^{2\pi i/m}z^{1/m}) = a(z^{1/m})$. It follows that $a$ is fixed by the Galois group, hence, $a \in G((z))$.

Let $U$ be the unipotent radical of $B$. We have $b = b^{ss} \cdot u$ where $u \in U$. Hence, the condition $|\alpha(b^{ss})| \leq 1$ for any positive root $\alpha$, insures that $a = \Phi = b^{ss} \cdot a_1$ where $a_1 \in U_1[[z]]$. Moreover, since $g$ is a polynomial loop we have $a_1 \in U_1[z]$. By proposition 1.3, the element $a$ is twisted conjugate in $G((z))$ to an aligned element $a'$. Using lemma 2.5 and the fact that $a \in B \cdot U[z]$, we see that $a$ is twisted conjugate to $a'$ via a polynomial loop. Thus, there is an element $f \in G[z^{1/m}, z^{-1/m}]$ such that

$$fa' = b, \quad f(e^{2\pi i/m}z^{1/m})f(z^{1/m})^{-1} = \theta.$$  

These equations show (see proof of proposition 3.5) that the $G$-bundle $P'$ with multiplier $a'$ has a holomorphic connection with the monodromies $\theta$ and $b \in G$ around $a$- and $b$-cycle, respectively. Thus $P$ and $P'$ are two $G$-bundles with connections that have the same monodromy. Since a holomorphic $G$-bundle with connection is determined, up to isomorphism, by the monodromy representation, we deduce that $P \simeq P'$, and the theorem follows. $\square$

**Proof of theorem 1.2.** Proposition 3.5 shows that the $G$-bundle associated to any integral twisted conjugacy class in $G((z))$ via the procedure described at the end of §2 has a holomorphic connection, hence is semistable, due to proposition 4.1. Theorem 4.3 insures that the map \{integral twisted conjugacy classes\} $\rightarrow$ \{isomorphism classes of semistable $G$-bundles\} is surjective. Injectivity of the map follows from proposition 3.7. $\square$

**Sketch of proof of theorem 1.6.** We note first that although $GL_n$ is not a semisimple group, lemma 3.2 and all other results of sections 3 - 4 hold for $G = GL_n$. In particular, any holomorphic degree zero semistable vector bundle on $E$ has a holomorphic connection with finite order monodromy around $a$-cycle and a reduced monodromy around $b$-cycle (since any semistable vector bundle is isomorphic to the one with a constant multiplier $[A]$, it has a holomorphic connection without monodromy around $a$-cycle at all. Moreover, it is easy to achieve that the monodromy matrix around $b$-cycle has no eigenvalues of the form $q^k$, $k \in \mathbb{Z} \setminus \{0\}$. But in order to ensure that the monodromy around $b$-cycle is reduced one still has to allow a finite monodromy around $a$-cycle). In this way, one shows that an analogue of theorem 1.2 holds for $G = GL_n$. 

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Now let $V$ be a rank $n$ and degree zero semistable vector bundle on $E$, and $\nabla$ a holomorphic connection on $V$ with order $m$ monodromy around $a$-cycle. Let $\pi : \mathbb{C}^* \to E$ and $m^p : \mathbb{C}^* \to \mathbb{C}^*, \ z \mapsto z^m$ denote the projections. We know that the pull-back $m^*(\pi^*V)$ can be trivialized by means of a flat (with respect to the pull-back connection) frame. If $\nabla'$ is another holomorphic connection on $V$ with finite order monodromy around $a$-cycle, then we can choose a large enough integer $m$ and two trivializations of $m^p*(\pi^*V)$ that are flat with respect to $m^p\nabla$ and $m^p\nabla'$, respectively. We claim that the transition matrix between any two such trivializations is given by a holomorphic map $a : \mathbb{C}^* \to GL_n$ with a pole at $z = 0$. To prove the claim, we may choose and fix one holomorphic connection $\nabla^\circ$ on $V$ that has finite order monodromy around $a$-cycle and, moreover, a reduced monodromy around $b$-cycle. Using an argument similar to that in the proof of proposition 3.7 one shows that the transition matrix between the trivialization corresponding to a flat frame with respect to $\nabla^\circ$ and the trivialization corresponding to a flat frame with respect to any other connection with finite monodromy around $a$-cycle is given by a holomorphic loop with a pole at $z = 0$. This proves the claim.

Next, we define a class of “moderate” holomorphic sections of $\pi^*V$ as follows. Write $\mathcal{R}$ for the ring of holomorphic functions on $\mathbb{C}^*$ with a pole at $z = 0$. Let $s$ be a holomorphic section of $\pi^*V$, let $\nabla$ be a holomorphic connection on $V$ with finite order monodromy around $a$-cycle, and let $s_1, \ldots, s_n$ be a frame of flat sections of $m^p*(\pi^*V)$, for some $m$. We say that $s$ is moderate with respect to $\nabla$ if $m^p* s$ is expressed as a linear combination of the sections $s_1, \ldots, s_n$ with coefficients in $\mathcal{R}$. It is clear that this definition does not depend on the choice of a flat frame and the choice of finite covering $m^p : \mathbb{C}^* \to \mathbb{C}^*$. Moreover, the claim proved in the previous paragraph insures that $s$ is moderate with respect to $\nabla$ if and only if it is moderate with respect to $\nabla'$ provided both connections have finite order monodromy around $a$-cycle. Thus, there is a well defined notion of a moderate holomorphic section of $\pi^*V$. Clearly, moderate sections form an $\mathcal{R}$-module, to be denoted $V(\mathcal{R})$.

In the previous setting, assume that $V$ is the vector bundle with a constant multiplier $a \in GL_n(\mathbb{C})$, and $\nabla = d$ is the trivial connection on $V$. Then the tautological trivialization of $\pi^*V$ is flat. Furthermore, it is clear that a section of $\pi^*V$ is moderate if and only if all its coordinates (relative to the trivialization) belong to $\mathcal{R}$. It follows that $V(\mathcal{R}) \simeq \mathcal{R}^n$ is a free $\mathcal{R}$-module. The $q^{\mathbb{Z}}$-equivariant structure on $\pi^*V$, see proof of proposition 1.3,
provides an operator \( q : V(\mathcal{R}) \to V(\mathcal{R}) \). Viewed as an operator \( \mathcal{R}^n \to \mathcal{R}^n \) via the isomorphism \( V(\mathcal{R}) \cong \mathcal{R}^n \), this operator has the form

\[
q : f(z) \mapsto a^{-1} \cdot f(q \cdot z) \quad \text{where} \quad a = \text{multiplier}. \tag{4.4}
\]

Let \( \text{Vect}_{ss,0}(\mathcal{E}) \) be the tensor category of degree zero holomorphic semistable vector bundles on \( \mathcal{E} \). We define a functor \( F : \text{Vect}_{ss,0}(\mathcal{E}) \to \mathcal{M}_q \) by the assignment

\[
V \mapsto F(V) = \mathbb{C}((z)) \otimes_{\mathcal{R}} V(\mathcal{R}),
\]

where the operator \( q : F(V) \to F(V) \) extends the one introduced above. Formula (4.4) shows that \( q \) and \( q^{-1} \) preserve the standard lattice \( L = \mathbb{C}[z]^n \subset \mathbb{C}((z))^n \). Thus, \( F(V) \in \mathcal{M}_q \).

Let \( V, V' \in \text{Vect}_{ss,0}(\mathcal{E}) \). It is clear from construction that there is a natural map \( V(\mathcal{R}) \otimes_{\mathcal{R}} V'(\mathcal{R}) \to (V \otimes V')(\mathcal{R}) \). This map is in effect an isomorphism. To see this, choose trivializations of \( \pi^*V \) and \( \pi^*V' \) corresponding to constant multipliers. Then the tensor product trivialization on \( \pi^*(V \otimes V') \) corresponds to the tensor product of the multipliers. We have seen that in these trivializations one has \( V(\mathcal{R}) \cong \mathcal{R}^{rkV}, V'(\mathcal{R}) \cong \mathcal{R}^{rkV'} \) and \( (V \otimes V')(\mathcal{R}) \cong \mathcal{R}^{rkV \cdot rkV'} \). It follows that the natural map above becomes the standard isomorphism \( \mathcal{R}^{rkV} \otimes_{\mathcal{R}} \mathcal{R}^{rkV'} \xrightarrow{\sim} \mathcal{R}^{rkV \cdot rkV'} \). Applying \( \mathbb{C}((z) \times_{\mathcal{R}} (\bullet)) \) we deduce that \( F \) is a tensor functor.

Recall now that both \( \mathcal{M}_q \) and \( \text{Vect}_{ss,0}(\mathcal{E}) \) are rigid tensor categories, cf. [DM], (in particular, \( \text{Vect}_{ss,0}(\mathcal{E}) \) is an abelian \( \mathbb{C} \)-category with finite dimensional \( \text{Hom} \)'s). To prove that a tensor functor between two rigid tensor categories is an equivalence, it suffices to show it is fully faithful. The analogue of theorem 1.2 for \( G = GL_n \) insures that the functor \( F \) is full.

It remains to show that, for any \( V, V' \in \text{Vect}_{ss,0}(\mathcal{E}) \), the functor \( F \) induces an isomorphism \( \text{Hom}_{\text{Vect}_{ss,0}(\mathcal{E})}(V', V) = \text{Hom}_{\mathcal{M}_q}(F(V'), F(V)) \). Using the duality functor on rigid tensor categories one reduces to the case \( V' = 1_{\mathcal{E}} \), the unit of the tensor category \( \text{Vect}_{ss,0}(\mathcal{E}) \), i.e. the trivial one-dimensional vector bundle. In that case we have

\[
\text{Hom}_{\text{Vect}_{ss,0}(\mathcal{E})}(1_{\mathcal{E}}, V) = \Gamma(\mathcal{E}, V) = \Gamma_{\text{hol}}(\mathbb{C}^*, \pi^*V)^q
\]

is the fixpoint subspace of the \( q^Z \)-action on the space of all holomorphic sections of \( \pi^*V \). On the other hand, we have \( \text{Hom}(F(1_{\mathcal{E}}), F(V)) = F(V)^q \), is the \( q \)-fixpoint space of the operator \( q \). In the trivialization corresponding
to a constant multiplier $a$ the operator $q$ is given by formula (4.4). Therefore, we find

$$F(V)^q = \{ f(z) \in \mathbb{C}((z))^n \mid f(q \cdot z) = af(z) \}$$

Expanding $f$ as a formal Laurent series $f(z) = \sum_{k \geq k_0} f_k z^k$, shows that $f \in F(V)^q$ if and only if, for any $k$, the coefficient $f_k \in \mathbb{C}^n$ is an eigenvalue of the operator $a$ with eigenvalue $q^k$. It follows that $f_k$ must vanish for all $k \gg 0$. Hence $f$ is a Laurent polynomial section, $f \in V(\mathbb{R})^q$. Similarly, any element of $\Gamma_{hol}(\mathbb{C}^*, \pi^*V)^q$ is a Laurent polynomial. We see that the natural imbeddings

$$F(V)^q \hookrightarrow V(\mathbb{R})^q \hookrightarrow \Gamma_{hol}(\mathbb{C}^*, \pi^*V)^q = \Gamma(E, V)$$

are both bijections. The theorem follows. □

References

[A] Atiyah M.F., Vector bundles over an elliptic curve Proc. London. Math. Soc. (3), 7, 1957, 414-452.

[B] Biswas I. Principal bundles admitting a holomorphic connection. preprint [alg-geom/9601019].

[BV] Babbit D.G., V.S.Varadarajan. Formal reduction theory of meromorphic differential equations: a group theoretic view. Pac. Jour. Math. 109 (1983), no.1, 1-80.

[DM] Deligne P., Milne J. Tannakian Categories, LN in Math, 900 (1982) 101–228.

[EFK] Etingof P., Frenkel I., Kirillov A, Jr. Spherical functions on affine Lie groups. Duke Math. J., 80 (1995), 59-90.

[E] Etingof P. Galois groups and connection matrices for difference equations. Electronic Research Announcements 1 (1995).

[R] Ramanathan A. Stable principal bundles on a compact Riemann surface. Math. Ann. 213 (1975) 129-152.

[RR] Ramanan S., Ramanathan A. Some remarks on the instability flag. Tohoku Math. J. 36 (1984) 269-291.

[S] Springer T.A. Linear algebraic groups. Progress in Mathematics vol.9, Boston: Birkhäuser, 1981.