A GENERALIZED APPROACH TO SPARSE AND STABLE PORTFOLIO OPTIMIZATION PROBLEM

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ABSTRACT. In this paper, we firstly examine the relation between the portfolio weights norm constraints method and the objective function regularization method in portfolio selection problems. We find that the portfolio weights norm constrained method mainly tries to obtain stable portfolios, however, the objective function regularization method mainly aims at obtaining sparse portfolios. Then, we propose some general sparse and stable portfolio models by imposing both portfolio weights norm constraints and objective function $L_1$ regularization term. Finally, three empirical studies show that the proposed strategies have better out-of-sample performance and lower turnover than many other strategies for tested datasets.

1. Introduction. Markowitz [24] laid down the groundbreaking work on the return-risk analysis which constituted a milestone in modern finance. In Markowitz return-risk analysis model, only expected returns and covariance matrix are two inputs. If the two input parameters are accurately known to investors, the investors can obtain the optimal portfolio positions by solving Markowitz mean-variance analysis model. However, since these two parameters are unknown which have to be estimated from a finite sample of historical data, estimation errors and parameter uncertainty have a great impact on the out-of-sample performance of Markowitz mean-variance model.

In portfolio optimization literature, it has long been recognized that Markowitz mean-variance model used with the sample mean and the sample covariance matrix is suboptimal, and usually delivers extremely poor out-of-sample performance (See, for example, Black and Litterman [2], Chopra and Ziemba [6], Green and Hollifield [17], Dai et al. [9]). Being aware of the importance of estimation errors and parameter uncertainty, various efforts have been made to modify the Markowitz mean-variance optimization model to obtain stable portfolio which depends less sensitively on

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the inputs. Ledoit and Wolf ([21], [22]) proposed replacing the sample covariance matrix with a weighted average of the sample covariance matrix and a low-variance target estimator matrix, $\hat{\Sigma}_{\text{target}}$. Jagannathan and Ma [20] imposed a no-short-sales constraint to minimum-variance model and gave some insightful explanations and demonstrations why the “wrong” constraint helps to find solution with better out-of-sample performance.

Recently, DeMiguel et al. [12] provided a general framework for finding stable portfolios which relies on solving the traditional minimum-variance portfolio problem with an additional norm of the portfolio-weight vector constraint. Fan et al. [13] proposed the large portfolio selection with a gross-exposure constraint on the portfolio allocation vector and showed that the estimation risk can be bounded by the $L_1$ norm of portfolio weights constraint. Behr et al. [4] imposed flexible upper and lower bounds on each portfolio weight and showed that incorporating these constraints into the portfolio optimization process can trade off the reduction of sampling error and loss of sample information.

Another approach is to directly add a regularization term to the objective function of a portfolio selection problem. Brodie et al. [3] reformulated the classical Markowitz mean-variance model as a constrained least-squares regression problem. Then they added a $L_1$-regularization term to the objective function. Both theoretical analysis and empirical studies show this penalty regularizes the optimization problem, and encourages sparse portfolios (i.e. portfolios with only few active positions). Xu et al. [32] presented sparse optimal portfolio selection models by imposing an $L_{1/2}$ regularization on the objective function of the traditional mean-variance portfolio selection problem and proposed a fast and efficient penalty half thresholding algorithm for the solution of proposed sparse portfolio selection model. Xing et al. [31] add an additional $L_{\infty}$-norm constraint minimum-variance portfolio (MVP) problem. The $L_{\infty}$ constraint can control the largest absolute component of the weight vector in MVP optimization. Chen et al. [7] proposed $L_p$-norm ($0 < p < 1$) regularized Markowitz mean-variance model to seek near-optimal sparse portfolios. Fastrich et al. [14] proposed new regularized minimum-variance problems by adding four types of non-convex penalty to the objective function, and provided a financial interpretation of their effect on asset weights.

Constructing the sparse portfolio, one can also add the cardinality constraint to portfolio selection models which have been investigated in the literature by many researchers (see [25], [23], [30], [5], [1], [27], [28], [29], [8], [10]). Unfortunately, the cardinality constrained problem is a NP-hard problem generally and computationally intractable. To deal with the hard cardinality constraint, many interesting methods have been proposed to solve the cardinality constrained problem alternatively. Frangioni and Gentile [15]) proposed a novel perspective reformulation for quadratic programs with semi-continuous variables. Recently, by relaxing the objective function as some separable functions, Gao and Li [16] obtain a cardinality constrained relaxation with closed-form solution. Based on this geometric approach, a branch-and-bound method is then developed in Gao and Li [16] for solving cardinality-constrained portfolio selection problems. A semidefinite program approach was presented in Zheng et al. [33] to improve the performance of MIQP solvers for quadratic programs with cardinality and minimum threshold constraints. Zheng et al. [33] applied it to solve cardinality-constrained portfolio selection problems. Computational results show that the proposed SDP approach can be advantageous for improving the performance of MIQP solvers.
The main objective of our paper is to propose a novel and non-cardinality-constraints portfolio strategy. Since, the portfolio weights norm constraints and the objective function regularization in portfolio selection problems are two main methods to make the resulting allocation depend less sensitively on the input vectors. A natural question follows: do the portfolio weights norm constraints and the objective function regularization term play different roles in reducing the undesired impact of parameter uncertainty and estimation errors? Can we find better portfolio selection model to reduce the undesired impact of parameter uncertainty and estimation errors? The answers to these questions clearly have important bearings on the debate about portfolio selection problems. Hence, our objective is to address these questions.

Our paper makes two contributions to the literature on portfolio selection in the presence of estimation error. Firstly, we examine the relation between the portfolio weights constraints method and the objective function regularization method in portfolio selection problem. We find that the portfolio weights constraints method mainly tries to obtain a stable portfolio, however, the objective function regularization method mainly aims at obtaining sparse portfolio. Secondly, we propose some general stable and sparse portfolio models by adding both portfolio weights constraints and objective function regularization.

The rest of this paper is organized as follows. In the next section, we introduce some existing approaches and illustrate the differences and connections between the portfolio weights norm constraints method and the objective function regularization method. In Section 3, we propose some general stable and sparse portfolio models. In Section 4, we report some empirical studies to test the proposed models.

2. The related portfolio weights constraints models and regularization models. Financial literature has largely shown that estimation errors in the expected return estimates are much larger than those in the covariance matrix estimates. For example, Jagannathan and Ma [20], report: The estimation error in the sample mean is so large nothing much is lost in ignoring the mean. Hence, in this paper, we also focus on the minimum-variance portfolio (MVP), which relies solely on the covariance structure and neglects the estimation of expected returns.

Suppose we have $n$ assets with returns $r_1, \ldots, r_n$ to be managed. Let $\mathbf{r}$ be the return vector, $\Sigma$ be its associated covariance matrix, and $\mathbf{w}$ be its portfolio allocation vector, satisfying $\mathbf{w}^T \mathbf{1}_n = 1$. Then, the variance of the portfolio return $\mathbf{w}^T \mathbf{r}$ is given by $\mathbf{w}^T \Sigma \mathbf{w}$. The minimum-variance portfolio (MVP) is the solution of the following quadratic programming problem:

$$\min_{\mathbf{w} \in \mathbb{R}^n} \frac{1}{2} \mathbf{w}^T \Sigma \mathbf{w}$$

$$s.t. \quad \mathbf{w}^T \mathbf{1}_n = 1, \quad (1)$$

where $\mathbf{1}_n = (1, 1, \ldots, 1)^T$ is a $n \times 1$ vector.

Jagannathan and Ma [20] proposed the following no-short-sales constrained minimum-variance portfolio model

$$\min_{\mathbf{w} \in \mathbb{R}^n} \frac{1}{2} \mathbf{w}^T \Sigma \mathbf{w}$$

$$s.t. \quad \mathbf{w}^T \mathbf{1}_n = 1, \quad \mathbf{w} \succeq 0. \quad (2)$$
Jagannathan and Ma [20] showed that constructing a no-short-sales constrained global minimum-variance portfolio from $\Sigma$ is equivalent to constructing a (unconstrained) global minimum-variance portfolio from

$$\bar{\Sigma} = \Sigma - (\lambda_1 \mathbf{1}_n + \mathbf{1}_n \lambda').$$

Here $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ are the lagrange multipliers for the nonnegativity constraints (2). In fact, the shortsale-constraint has an effect on reducing the estimation error of covariance matrix $\Sigma$. For example, whenever the nonnegativity constraint is binding for asset $i$, its covariances with other assets, $\sigma_{ij}, j \neq i$ are reduced by $\lambda_i + \lambda_j$, a positive quantity, and its variance $\sigma_i$ is reduced by $2\lambda$. Therefore, the new covariance matrix estimate, $\bar{\Sigma}$, is constructed by shrinking the large covariances toward the average covariances. As we know, the largest covariance estimates are more likely caused by upward-biased estimation error. Hence, imposing the no-short-sale constraint can reduce the estimation error of covariance matrix estimate. Empirical analysis also shows that minimum-variance portfolio model with no-short-sale constraint can obtain stable portfolio weights.

DeMiguel et al. [12] provided a general framework for finding portfolios that perform well out-of-sample in the presence of estimation error by imposing the $L_1$ or $L_A$-norm-constrained to minimum-variance portfolio model (1).

$$\begin{align*}
\min_{w \in \mathbb{R}^n} & \quad \frac{1}{2} w^T \Sigma w \\
\text{s.t.} & \quad w^T \mathbf{1}_n = 1, \\
& \quad ||w||_l \leq \tilde{\delta}.
\end{align*}$$

While $l = 1, A$, that is, $||w||_1 = \sum_{i=1}^n |w_i| \leq \tilde{\delta}$ and $||w||_A = \sqrt{w^T A w} \leq \tilde{\delta}.$

Note that for the special case where $A = I$, the A-norm is simply the 2-norm, $||w||_2 = \sqrt{\sum_{i=1}^n w_i^2} \leq \tilde{\delta}$, and therefore we refer to these as the 2-norm-constrained minimum variance portfolios. For the special case where $A = \tilde{\Sigma}_F$, and $||w||_F = \sqrt{w^T \tilde{\Sigma}_F w} \leq \tilde{\delta}$. That is, imposing a constraint on the $\tilde{\Sigma}_F$-norm of the portfolio is equivalent to imposing a constraint on the portfolio variance under the covariance estimator obtained from a 1-factor model. Therefore we refer to these as the $F$-norm-constrained minimum variance portfolios.

Because the squared 2-norm and $F$-norm are easier to analyze than the 2-norm and $F$-norm, in this paper we can also instead impose the equivalent constraints as follows $||w||_2^2 = \sum_{i=1}^n w_i^2 \leq \delta$ and $||w||_F^2 = w^T \tilde{\Sigma}_F w \leq \delta$, where $\delta = \tilde{\delta}^2$. For discussion convenience, we use squared 2-norm and $F$-norm from now on.

The squared A-norm-constrained minimum-variance portfolios models in DeMiguel et al. [12] can also be interpreted as portfolios that result from shrinking some of the elements of the sample covariance matrix. For example, the proposition 2 in DeMiguel et al. [12] shows the relation between the A-norm-constrained portfolios and the shrinkage portfolios proposed by Ledoit and Wolf ([21], [22]), that is, the solution to the minimum-variance problem in (2.1), with the sample covariance matrix, $\Sigma$, replaced by

$$\tilde{\Sigma}_{LW} = \frac{1}{1 + \nu} \Sigma + \frac{\nu}{1 + \nu} A,$$

coincides with the solution to the A-norm constrained minimum-variance portfolio problem (3). In particular, if we choose the matrix $A$ equal to the identity matrix, $I$, then there is a one-to-one correspondence between the A-norm-constrained portfolios and the shrinkage portfolio proposed in Ledoit and Wolf [22]. On the other
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hand, if A equals the 1-factor covariance matrix, \( \hat{\Sigma}_F \), then there is a one-to-one correspondence with the shrinkage portfolio in Ledoit and Wolf [21].

As we known, shrinkage estimators have been a popular method for reducing estimation error ever since they were introduced by James and Stein [19]. The idea behind shrinkage estimators is that shrinking an unbiased estimator toward a lower variance target which has a positive effect that it reduces the variance of the estimator. Out-of-sample performance of the proposed portfolios shows the norm constraints on the portfolio weights help to reduce the estimation error of covariance matrix estimate and obtain stable portfolio weights.

In what following, we will introduce a regularization minimum-variance portfolio problem and give some insightful explanations and demonstrations about its effects. Similarly in Brodie et al. [3], we add an \( L^1 \)-regularization term to the objective function of (1), to obtain the following \( L^1 \)-regularization minimum-variance portfolio model

\[
\min_{w \in \mathbb{R}^n} \frac{1}{2} w^T \Sigma w + \tau \|w\|_1
\]

s.t. \( w^T \mathbf{1}_n = 1 \),

(4)

where \( \|w\|_1 = \sum_{i=1}^n |w_i| \), \( \tau > 0 \) is a regularization parameter.

The lagrangian corresponding to the optimization problem stated in (4) is

\[
L(w, \gamma) = \frac{1}{2} w^T \Sigma w + \tau \|w\|_1 - \lambda (w^T \mathbf{1}_n - 1)
\]

(5)

Let \( \sigma_{ij} \) denote the \((i,j)\)-th off-diagonal term of \( \Sigma \), \( i, j = 1, \ldots, n \), and \( i \neq j \), and \( \sigma_{ii} = \sigma_i^2 \), \( i \) denote the \(i\)-th diagonal term, \( i = 1, \ldots, n \). The KKT conditions for the Lagrangian (5) are as following

\[
w_i \sigma_i^2 + \sum_{j \neq i} w_j \sigma_{ij} - \lambda = -\tau \quad \text{if } w_i > 0,
\]

(6)

\[
w_i \sigma_i^2 + \sum_{j \neq i} w_j \sigma_{ij} - \lambda = \tau \quad \text{if } w_i < 0,
\]

(7)

\[
\left| \sum_{j \neq i} w_j \sigma_{ij} - \lambda \right| \leq \tau \quad \text{if } w_i = 0,
\]

(8)

\[
w^T \mathbf{1}_n - 1 = 0.
\]

(9)

From KKT conditions for the Lagrangian (5), the \( L^1 \)-regularization minimum-variance portfolio optimization (4) can be viewed as a decision process in which the investor assigns the penalty parameter \( \tau \) to decide whether to include an asset in the portfolio or not. For example, for \( w_i = 0 \), from (8), we have

\[
\lambda - \tau \leq \frac{\partial w_i^T \Sigma w}{\partial w_i} = \sum_{j \neq i} w_j \sigma_{ij} \leq \lambda + \tau.
\]

(10)

If the \( L^1 \)-regularization term coefficient \( \tau \) becomes large, it is more likely that \( \frac{\partial w_i^T \Sigma w}{\partial w_i} \) will fall into the interval \([\lambda - \tau, \lambda + \tau]\), and then more assets will be excluded in the optimal portfolio. Mean-while, it is less likely that (7) will still hold as \( \tau \) increases, since \( \lambda + \tau \) will also increase. Hence, less assets with negative weights will be included in the optimal portfolio. When \( \tau \) becomes more larger, only a small number of assets will be included in the portfolio.
Similarly in Brodie et al. [3], we also have an interesting consequences for the $L_1$-regularization minimum-variance portfolio model with the following mathematical observations. Suppose that the two weight vectors $w_{\tau_1}$ and $w_{\tau_2}$ are minimizers for the objective function in (4), corresponding to the values $\tau_1$ and $\tau_2$, respectively, and both satisfy the constraint $w_{\tau_1}^T 1_n = 1$. By using the respective minimization properties of $w_{\tau_1}$ and $w_{\tau_2}$, we can obtain

\[
\frac{1}{2} w_{\tau_1}^T \Sigma w_{\tau_1} + \tau_1 \|w_{\tau_1}\|_1 \leq \frac{1}{2} w_{\tau_2}^T \Sigma w_{\tau_2} + \tau_1 \|w_{\tau_2}\|_1 + \tau_1 (\tau_1 - \tau_2) \|w_{\tau_1}\|_1 \quad \text{(11)}
\]

which implies that

\[
(\tau_1 - \tau_2)(\|w_{\tau_2}\|_1 - \|w_{\tau_1}\|_1) \geq 0. \quad \text{(12)}
\]

If $\tau_1 \geq \tau_2$, we have from (12) that $\|w_{\tau_2}\|_1 \geq \|w_{\tau_1}\|_1$. If all the $w_{\tau_1}^{T, i}$ are nonnegative, but some of the $w_{\tau_2}^{T, i}$ are negative, then we have $\|w_{\tau_2}\|_1 \geq \|w_{\tau_1}\|_1 = 1$. It implies that the optimal portfolio with nonnegative entries obtained by minimization procedure corresponds to the largest values of $\tau$, and thus typically to the sparsest solution. Since the penalty term, promoting sparsity, is weighted more heavily.

To achieve sparse portfolio, Gao and Li [16] also considered the following cardinality-constrained minimum variance portfolio problem (CC-Minvar)

\[
\min_{w \in \mathbb{R}^n} \quad \frac{1}{2} w^T \Sigma w
\]

\[
s.t. \quad w^T 1_n = 1, \quad \|w\|_0 \leq K, \quad \text{(13)}
\]

where $\|w\|_0$ represents the number of the nonzero entries of $w$ and $K$ is the chosen limit of stocks to be managed in the portfolio. The cardinality constraint has a negative and a positive effect. The negative effect is that the inherent combinatorial property makes the cardinality constrained problem NP-hard generally and hence computationally intractable, whereas the positive effect is that it can obtain any sparse portfolio.

From above discussion, the minimum-variance portfolio with the norm constraints on the portfolio weights shrinks some of the elements of the sample covariance matrix and tries to obtain stable portfolio weights. On the other hand, the $L_1$-regularization term in minimum-variance portfolio model’s objective function mainly has good effects on obtaining sparse portfolio weights. Thus, the portfolio weights norm constraints and the objective function regularization term play different roles in reducing the undesired impact of parameter uncertainty and estimation errors. A natural question follows: can we find better models to reduce the undesired impact of parameter uncertainty and estimation errors? The answers to this question clearly have important bearings on the debate about portfolio selection problems.

3. Sparse and stable portfolio. Recent study in Brodie et al. [3] showed that imposing an $L_1$-norm penalty on the objective function not only regularizes the problem, but also allows to automatically select a subset of assets to invest in
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(promoting sparsity). For \( p = 1 \) the amended minimization problem remains convex, and thus algorithmically more tractable. On the other hand, although \( L_p \) penalties with \( 0 < p < 1 \) may encourage more sparsity, due to the non-convexity of the \( L_p \)-norm \( (0 < p < 1) \), solving the resulting optimization problem, which exhibits multiple (local) optima, is a challenging task for which, to the best of our knowledge, no standard software solution exists. Therefore, in our optimization models, we focus on the case \( p = 1 \), which has both desirable features.

The above discussions motivate us how to achieve both sparsity and stability in a portfolio choice problem. We propose some new sparse and stable portfolio optimization models, which combine the portfolio weights \( L_2 \)-norm or \( L_F \)-norm constraints method and \( L_1 \)-regularization objective function method. The proposed models are as follows.

**Model 1**: \( L_1 \)-regularization and the portfolio weights \( L_2 \)-norm constraints minimum-variance portfolio model

\[
\min_{w \in \mathbb{R}^n} \frac{1}{2} w^T \Sigma w + \tau_0 \|w\|_1
\]

s.t. \( w^T \mathbf{1}_n = 1 \), \( \|w\|_2^2 \leq \delta \),

\[
\text{(14)}
\]

where \( \|w\|_2^2 = \sum_{i=1}^n w_i^2 \leq \delta \).

In what following, we will show some nice properties of Model 1. Firstly, the penalty on the \( L_1 \)-norm of the asset weights vector (i.e. \( L_1 \)-regularization) not only regularizes the problem, but also allows to automatically select a subset of assets to invest in promoting sparsity.

The lagrangian corresponding to the optimization problem stated in (14) is

\[
L(w, \gamma) = \frac{1}{2} w^T \Sigma w + \tau_0 \|w\|_1^1 - \lambda(w^T \mathbf{1}_n - 1) + \frac{\nu}{2} (\|w\|_2^2 - \delta)
\]

Let \( \sigma_{ij} \) denote the \((i, j)\)-th off-diagonal term of \( \Sigma \), \( i, j = 1, \ldots, n \), and \( i \neq j \), and \( \sigma_{ii} = \sigma_i^2 \), \( i \) denote the \( i \)-th diagonal term, \( i = 1, \ldots, n \). The KKT conditions for the Lagrangian (15) are as following

\[
w_i \sigma_i^2 + \sum_{j \neq i}^{n} w_j \sigma_{ij} - \lambda + \nu w_i = -\tau_0 \quad \text{if } w_i > 0,
\]

\[
w_i \sigma_i^2 + \sum_{j \neq i}^{n} w_j \sigma_{ij} - \lambda + \nu w_i = \tau_0 \quad \text{if } w_i < 0,
\]

\[
|\sum_{j \neq i}^{n} w_j \sigma_{ij} - \lambda| \leq \tau_0 \quad \text{if } w_i = 0,
\]

\[
w^T \mathbf{1}_n - 1 = 0,
\]

\[
\nu \geq 0, \quad \|w\|_2^2 - \delta \leq 0, \quad \nu(\|w\|_2^2 - \delta) = 0.
\]

From KKT conditions for the Lagrangian (15), the \( L_1 \)-regularization and the portfolio weights \( L_2 \)-norm constraints minimum-variance portfolio model (14) also can be viewed as a decision process in which the investor assigns the penalty parameter \( \tau_0 \) to decide whether to include an asset in the portfolio or not. For example,
for \( w_i = 0 \), from (18), we have
\[
\lambda - \tau_0 \leq \frac{\partial w^T \Sigma w}{\partial w_i} = n \sum_{j \neq i} w_j \sigma_{ij} \leq \lambda + \tau_0.
\] (21)

Following the similar analysis as (10), we can obtain that if \( \tau_0 \) becomes more larger, only a small number of assets will be included in the portfolio.

The \( L_2 \)-norm constraints on the portfolio weights can shrink some of the elements of the sample covariance matrix and tries to obtain stable portfolio weights. The specific reasons are as following which are familiar with DeMiguel et al. [12]. The following proposition shows the relation between the 2-norm-constrained portfolios and the shrinkage portfolios proposed by Ledoit and Wolf [22].

**Proposition 1.** Provided \( \Sigma \) is nonsingular and positive definite, for each \( \nu \geq 0 \), there exist \( \delta \) and \( \tau_0 \) such that the solution to the \( L_1 \)-regularization minimum-variance problem in (2.4), with the sample covariance matrix, \( \hat{\Sigma} \), replaced by
\[
\tilde{\Sigma} = \frac{1}{1 + \nu} \Sigma + \frac{\nu}{1 + \nu} I,
\] (22)

coincides with the solution to the \( L_1 \)-regularization and portfolio weights \( L_2 \)-norm constraints minimum-variance portfolio model (14).

**Proof of proposition 1.** If the sample covariance matrix, \( \Sigma \), replaced by (22), the lagrangian corresponding to the \( L_1 \)-regularization minimum-variance portfolio model (4) is
\[
L(w, \lambda) = \frac{1}{2} w^T \left( \frac{1}{1 + \nu} \Sigma + \frac{\nu}{1 + \nu} I \right) w + \tau \| w \|_1^1 - \lambda (w^T 1_n - 1).
\] (23)

On the other hand, the lagrangian corresponding to the optimization problem stated in (14) is
\[
L(w, \lambda, \nu)
\]
\[
= \frac{1}{2} w^T \Sigma w + \tau_0 \| w \|_1^1 - \lambda (w^T 1_n - 1) + \frac{\nu}{2} (\| w \|_2^2 - \delta)
\]
\[
= \frac{1}{2} w^T (\Sigma + \nu I) w + \tau_0 \| w \|_1^1 - \lambda (w^T 1_n - 1) - \frac{\nu}{2} \delta
\]
\[
= \frac{1}{2} w^T \left( \frac{1}{1 + \nu} \Sigma + \frac{\nu}{1 + \nu} I \right) w + \tau_0 \| w \|_1^1 - \frac{\lambda}{1 + \nu} (w^T 1_n - 1) - \frac{\nu}{2(1 + \nu)} \delta.
\] (24)

In addition, because \( \Sigma \) is nonsingular and positive definite, \( I \) is positive definite, and \( \nu \geq 0 \), we know that the matrix \( \tilde{\Sigma} \) in (22) is positive definite, due to the convexity of the \( L_1 \)-norm, and the first-order optimality conditions for the minimum-variance problem with \( \tilde{\Sigma} \) are then also sufficient for optimality. Then there is a unique global minimizer to the problem (4) with the sample covariance matrix, \( \Sigma \), replaced by (22).

Comparing the lagrangian function (23) and (24), if \( \tau = \frac{\tau_0}{1 + \nu} \), the \( L_1 \)-regularization minimum-variance portfolio model (4) with the sample covariance matrix, \( \Sigma \), replaced by (22) has the same solution as the model (14). Then we can obtain Proposition 1.

Moreover, from \( \tilde{\Sigma} \) in (22), it is obvious that there is a one-to-one correspondence between the \( L_1 \)-regularization and portfolio weights \( L_2 \)-norm constraints minimum-variance portfolio model (14) and the shrinkage portfolio proposed in Ledoit and Wolf [22].
The $L_2$-norm constraint can be reformulated equivalently as follows

$$
\sum_{i=1}^{n} (w_i^2 - \frac{1}{n})^2 = \sum_{i=1}^{n} w_i^2 + \sum_{i=1}^{n} \left(\frac{1}{n}\right)^2 - \sum_{i=1}^{n} \frac{2w_i}{n} = \sum_{i=1}^{n} w_i^2 - \frac{1}{n} \leq (\delta - \frac{1}{n}). \quad (25)
$$

If $\delta = \frac{1}{n}$, from (25), we can obtain $w = \frac{1}{n}$. This means that no short sales are allowed. Note also that the $1/N$ portfolio is a special case of the 2-norm constrained portfolio with $\delta = \frac{1}{n}$.

If $1 < \delta < +\infty$, from proposition 1, the $L_2$-norm-constrained minimum-variance portfolios can also be interpreted as portfolios that result from shrinking some of the elements of the sample covariance matrix. That is

$$
\tilde{\Sigma} = \frac{1}{1 + \nu} \Sigma + \frac{\nu}{1 + \nu} I. \quad (26)
$$

Then, Model 1 can be rewritten as following

$$
\min_{w \in \mathbb{R}^n} \frac{1}{2} w^T \tilde{\Sigma} w + \tau_0 \|w\|_1,
\text{s.t.} \quad w^T \mathbf{1}_n = 1. \quad (27)
$$

It is obvious that the model (27) will try to obtain a sparse and stable portfolio.

**Remark.** As the $\nu$ in (26) is factor of lagrange multiplier which is undecided before solving Model 1. For the purpose of applications, Model 1 enjoys some advantages comparing to the model (27).

**Model 2**: $L_1$-regularization and the portfolio weights $L_F$-norm constraints minimum-variance portfolio model

$$
\min_{w \in \mathbb{R}^n} \frac{1}{2} w^T \Sigma w + \tau_0 \|w\|_1
\text{s.t.} \quad w^T \mathbf{1}_n = 1, \quad \|w\|_F^2 = w^T \Sigma_F w \leq \delta. \quad (28)
$$

Based on the similar analysis as in Model 1, Model 2 also enjoys some nice properties. Firstly, the penalty on the $L_1$-norm of the asset weights vector (i.e. $L_1$-regularization) not only regularizes the problem, but also allows to automatically select a subset of assets to invest in promoting sparsity.

Secondly, the $F$-norm constraints on the portfolio weights shrinking some of the elements of the sample covariance matrix and try to obtain stable portfolio weights. If $0 < \delta < +\infty$, from Proposition 1, the $F$-norm-constrained minimum-variance portfolios can also be interpreted as portfolios that result from shrinking some of the elements of the sample covariance matrix. That is

$$
\tilde{\Sigma} = \frac{1}{1 + \nu} \Sigma + \frac{\nu}{1 + \nu} \Sigma_F. \quad (29)
$$

In particular, if $A$ equals the 1-factor covariance matrix $\Sigma_F$, $\tilde{\Sigma}$ in (29), then there is a one-to-one correspondence with the shrinkage portfolio in Ledoit and Wolf [21].

Following the similar analysis as Model 1, it is obvious that the Model 2 will also try to obtain a sparse and stable portfolio.

From the above analysis, we can expect the proposed models can take advantage of the attractive features of the portfolio weights constraints method and $L_1$-regularization method, thereby improving the out-of-sample performance. Moreover, for the purpose of applications, Model 1 and Model 2 enjoy some advantages. On the other hand, due to the convexity of the $L_p$-norm ($p \geq 1$), we can utilize
standard software, for example, optimization package cvx (Grant and Boyd [18]) to solve.

4. Empirical studies. In this section, we apply the models described above to construct optimal portfolios and evaluate their out-of-sample performance by employing real market data.

However, since our interest is to compare the out-of-sample performance of proposed portfolio models, we focus on the out-of-sample portfolio variance, out-of-sample portfolio Sharpe ratio, and portfolio turnover (trading volume), not the sparsity of obtained portfolios. For the sparsity, the cardinality-constrained minimum variance problem (CC-Minvar) can obtain any sparse portfolio, but the $L_1$-regularization method can obtain limited sparse portfolios. In this paper, we consider all portfolio models in a frictionless world. Under the assumption of frictionless market, we find the moderate sparsity perform better in the empirical studies with rolling-horizon procedure as in DeMiguel et al. [12]. Hence, for the cardinality-constrained global minimum variance problem (CC-Minvar), we select $K$ from $[10\%n, 20\%n]$.

4.1. Data and models. In our empirical studies, the tested portfolio models have the following meanings:

- “1/N” stands for equally-weighted (1/N) portfolio (DeMiguel et al.[11]).
- “MINC” stands for minimum-variance portfolio with shortsales constrained (Jagannathan and Ma [20]).
- “CC-Minvar” stands for the cardinality-constrained minimum variance problem (13) (Gao and Li [16]).
- “NC1V” stands for portfolio weights $L_1$-norm-constrained minimum-variance portfolio with $\delta$ calibrated using cross-validation over portfolio variance (DeMiguel et al. [12]).
- “NC2V” stands for portfolio weights $L_2$-norm-constrained minimum-variance portfolio with $\delta$ calibrated using cross-validation over portfolio variance (DeMiguel et al. [12]).
- “NCFV” stands for portfolio weights $L_F$-norm-constrained minimum-variance portfolio with $\delta$ calibrated using cross-validation over portfolio variance (DeMiguel et al. [12]).
- “Model 1” stands for $L_1$-regularization plus portfolio weights $L_2$-norm-constrained minimum-variance portfolio with $\tau_0$ and $\delta$ calibrated using cross-validation over portfolio variance.
- “Model 2” stands for $L_1$-regularization plus portfolio weights $L_F$-norm-constrained minimum-variance portfolio with $\tau_0$ and $\delta$ calibrated using cross-validation over portfolio variance.

We compute the optimal solutions of the above models by optimization package cvx (Grant and Boyd [18]) except for the CC-Minvar. The CC-Minvar is a cardinality-constrained minimum variance portfolio problem which is NP-hard generally. We compute the optimal solutions of CC-Minvar by the semidefinite program approach presented in Zheng et al. [33].

In our empirical studies, we use five historical returns of exchange-traded assets to test the out-of-sample performance of proposed models. The list of the data used in our empirical studies is described in Table 1.

Table 1 lists five different datasets used for the evaluation of the portfolio performance, their abbreviations, the number of assets that each dataset comprises,
the time period over which we use data from each particular dataset, and the data sources.

The one hundred and forty-eight Fama French portfolios, FF-100 and FF-48, are taken from Ken French’s website http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html and represent different cuts of the US stock market. The data sets of 500CRSP and 100CRSP are constructed in a way that is similar to Jagannathan and Ma [20], and DeMiguel et al.[12] with monthly rebalancing. The data set of S&P 500 we use is daily stock return data of S&P 500 index which spans from 31/12/2007 to 31/12/2014. We use 250 daily stock return data for estimation, 21 days for prediction, monthly rebalancing.

4.2. Performance measures. We compare the performance of the proposed portfolio models to the portfolio models in the literature using the following three criteria:(1) Out-of-sample portfolio variance; (2) Out of-sample portfolio Sharpe ratio; (3) Portfolio turnover (trading volume).

We utilize the following rolling-horizon procedure for the comparison that is similar to DeMiguel et al.[12].

- Firstly, we choose the length of the estimation time window by \( \tau < T \) to perform the estimation of covariance matrix, where \( T \) is the total number of returns in the data set. For the FF-100, FF-48, 500 CRSP and 100 CRSP in our experiments, we use an estimation window of \( \tau = 120 \) data points, which for monthly data corresponds to 10 years. For the S&P500 in our experiments, we use an estimation window of \( \tau = 250 \) data points (corresponding to one year of market data).
- Secondly, we compute the different portfolios by using the return data over the estimation time window, \( \tau \).
- Thirdly, we repeat this rolling-window procedure for the next month by including the (return) data for the next month and dropping out the (return) data for the earliest month. We will continue doing this until the end of the data set is reached.

After holding the portfolios \( w^i_t \) unchanged for 1 month, we give the out-of-sample return at time \( t + 1 : r^i_{t+1} = w^i_t r_{t+1} \), where \( r_{t+1} \) denotes the asset returns. The time window is moved forward, so that the formerly out-of-sample days become part of the in-sample window and the oldest observations drop out. At the end of this process, we have generated \( T - \tau \) portfolio weights \( w^i_t, t = \tau, \tau + 1, \ldots, T - 1 \), for each portfolio optimization strategy \( i \).

We utilize the generated returns and weights for each strategy to compute the out-of-sample variance(\( \hat{\sigma} \)), Sharpe ratio(SR), and turnover(TR):

\[
(\hat{\sigma}^i) = \frac{1}{T - \tau - 1} \sum_{t=\tau}^{T-1} (w^i_t r_{t+1} - \hat{\mu}^i)^2,
\]

(30)
with \((\hat{\mu}^i) = \frac{1}{T-\tau} \sum_{t=\tau}^{T-1} (w^i_t r_{t+1}),\) \((\hat{SR}^i) = \frac{\hat{\mu}^i}{\hat{\sigma}^i},\) \((\hat{TR}^i) = \frac{1}{T-\tau-1} \sum_{t=\tau}^{T-1} \sum_{j=1}^{N} |w^i_{j,t+1} - w^i_{j,t+1}|,\)

In the definition of the out-of-sample variance(\(\hat{\sigma}\)), sharpe ratio(\(SR\)), and turnover (\(TR\)), some portfolio weights have following meanings:

- \(w^i_{j,t}\) denotes the portfolio weight in asset \(j\) chosen at time \(t\) under strategy \(i\),
- \(w^i_{j,t+}\) the portfolio weight before rebalancing but at \(t+1\),
- \(w^i_{j,t+1}\) the desired portfolio weight at time \(t+1\) (after rebalancing),

4.3. **Out-of-Sample evaluation of the proposed portfolios.**

4.3.1. **Discussion of the out-of-sample variances.** Table 2 shows the out-of-sample variances of all considered portfolio strategies with five different datasets.

| Dataset | 500 CRSP | 100 CRSP | S&P 500 | FF-100 | FF48 |
|---------|---------|---------|---------|--------|------|
| Model 1 | 0.00049 | 0.00109 | 0.00060 | 0.00098 | 0.00119 |
| Model 2 | 0.00042 | 0.00102 | 0.00052 | 0.00091 | 0.00111 |
| 1/N     | 0.00172 | 0.00168 | 0.00179 | 0.00208 | 0.00232 |
| MINC    | 0.00094 | 0.00139 | 0.00108 | 0.00142 | 0.00141 |
| NC1V    | 0.00081 | 0.00119 | 0.00078 | 0.00121 | 0.00129 |
| NC2V    | 0.00073 | 0.00121 | 0.00084 | 0.00125 | 0.00132 |
| NCFV    | 0.00070 | 0.00123 | 0.00087 | 0.00120 | 0.00125 |
| CC-Minvar | 0.00072 | 0.00113 | 0.00082 | 0.00102 | 0.00120 |

Assessing the variances within the tested strategies, we find that imposing both \(L_1\)-regularization and portfolio weights norm-constrained, or solely portfolio weights norm-constrained typically have lower out-of-sample variances than the other portfolios. For instance, the \(1/N\), and minimum-variance portfolio with short-sales constrained (MINC) portfolios always achieve out-of-sample variances that are higher than the other portfolios, and the differences are statistically significant.

Assessing the variances within the group of our developed strategies (Model 1 and Model 2), we find that Model 1 and Model 2 do not result in a significantly different out-of-sample variance. Here we find, that Model 2 yields a sight lower variance than Model 1.

Comparing the variances of our developed strategies (Model 1 and Model 2) from the literature proposed strategies, we see that Model 1 and Model 2 always achieve out-of-sample variances that are lower than those of the NC1V, NC2V, NCFV, and the differences are more statistically significant for big dataset, for example, 500 CRSP and S&P 500.

However, the out-of-sample variances of the cardinality-constrained minimum variance portfolio problem(CC-Minvar) is not always lower than those of the DeMiguel et al. [12] portfolios (NC1V, NC2V, NCFV). For instance, the NCFV portfolio attains lower variances than the CC-Minvar portfolio for the 500 CRSP
data set. On the other hand, the NC1V portfolio has a higher variance than the CC-Minvar portfolio for the S&P 500 data set, and the difference is not statistically significant.

4.3.2. Discussion of the out-of-sample Sharpe ratios. Table 3 reports the out-of-sample Sharpe ratios for the different portfolios with five different data sets.

Table 3. Out-of-sample Sharpe ratio of the portfolio strategies.

| Dataset  | 500 CRSP | 100 CRSP | S&P 500 | FF-100 | FF48 |
|----------|----------|----------|---------|--------|------|
| Model 1  | 0.4278   | 0.4460   | 0.4314  | 0.3998 | 0.3146|
| Model 2  | 0.4636   | 0.4548   | 0.4568  | 0.4024 | 0.3220|
| 1/N      | 0.3102   | 0.3358   | 0.3586  | 0.2808 | 0.2524|
| MINC     | 0.3882   | 0.3649   | 0.3723  | 0.3142 | 0.2712|
| NC1V     | 0.4001   | 0.4122   | 0.4055  | 0.3224 | 0.2922|
| NC2V     | 0.4151   | 0.4212   | 0.4186  | 0.3522 | 0.2916|
| NCFV     | 0.4063   | 0.4136   | 0.4028  | 0.3458 | 0.2806|
| CC-Minvar| 0.4042   | 0.4326   | 0.4101  | 0.3875 | 0.3206|

Comparing the Sharpe ratios within the tested strategies, we note that Model 1 and Model 2 have higher Sharpe ratios than both the equally weighted (1/N) and minimum-variance portfolio with short sales constrained portfolio (MINC) for all data sets, and the difference is substantial and significant on all considered data sets.

Assessing the Sharpe ratios within the group of our developed strategies (Model 1 and Model 2), we see that Model 2 almost always attains slight higher Sharpe ratios than Model 1.

We also note that the imposing both $L_1$-regularization and portfolio weights norm-constrained portfolios (Model 1 and Model 2) have higher Sharpe ratios than the solely portfolio weights norm-constrained portfolios (NC1V, NC2V, NCFV) for most considered data sets.

Comparing the Sharpe ratios of the cardinality-constrained minimum variance portfolio problem (CC-Minvar) to our developed strategies (Model 1 and Model 2), we can also see that the CC-Minvar has lower Sharpe ratios than the Model 1 and Model 2 for all data sets except for FF48 data set.

4.3.3. Discussion of turnover. Table 4 shows the turnover of each portfolio strategy with five different data sets.

Table 4. Turnover of the portfolio strategies.

| Dataset  | 500 CRSP | 100 CRSP | S&P 500 | FF-100 | FF48 |
|----------|----------|----------|---------|--------|------|
| Model 1  | 0.4028   | 0.3124   | 0.4120  | 0.3068 | 0.2580|
| Model 2  | 0.4145   | 0.3182   | 0.4166  | 0.3022 | 0.2528|
| 1/N      | 0.0625   | 0.0445   | 0.0586  | 0.0508 | 0.0324|
| MINC     | 0.3125   | 0.2025   | 0.4030  | 0.2221 | 0.1822|
| NC1V     | 0.6654   | 0.4670   | 0.6208  | 0.5308 | 0.2822|
| NC2V     | 0.6022   | 0.4249   | 0.6030  | 0.5421 | 0.3168|
| NCFV     | 0.5948   | 0.4132   | 0.5870  | 0.5134 | 0.2762|
| CC-Minvar| 0.3542   | 0.2802   | 0.3980  | 0.2632 | 0.2356|
Unsurprisingly, the long only portfolio strategies $1/N$ and MINC exhibit the lowest turnover of all portfolio strategies for all considered datasets. The cardinality-constrained global minimum variance problem (CC-Minvar) can also obtain lower turnover. The turnover of Model 1 and Model 2 is comparatively similar. Model 1 has slight lower turnover than Model 2.

Comparing the turnover of the imposing both $L_1$-regularization and portfolio weights norm-constrained portfolios to those of the portfolios from solely norm-constrained portfolios, we note that Model 1 and Model 2 have lower turnover than solely norm-constrained portfolios (NC1V, NC2V, NCFV).

5. Conclusion. In this paper, we show that the performance of the minimum-variance portfolio can be substantially improved by combining regularization methods with norm constraints on the portfolio weights. Our main contributions and findings can be summarized as follows.

Firstly, we find the minimum variance portfolio with norm constraints on the portfolio weights shrinking some of the elements of the sample covariance matrix and try to obtain stable portfolio weights. On the other hand, the $L_1$-regularization term in the objective function of minimum-variance portfolio models mainly have good effects on obtaining sparse portfolios.

Secondly, we propose some general stable and sparse portfolio models by adding the portfolio weights constraints and the objective function regularization. Moreover, we present insights into these new approaches as well as their connections to alternative strategies in literature. We show that the proposed portfolio models enjoy not only nice stable property of solely portfolio weights norm-constrained portfolio models in DeMiguel et al.[12], but also sparse property of solely $L_1$-regularization portfolio model in Brodie et al.[3].

Finally, three empirical studies show that the proposed strategies have better out-of-sample performance and lower turnover than other strategies. It turned out that these approaches perform well when dealing with large data sets, where they not only outperformed the naïve $1/N$ portfolio but also the norm-constrained portfolio models in DeMiguel et al.[12].

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