A Study of Wall-Crossing:
Flavored Kinks in $D = 2$ QED

Sungjay Lee$^1$ and Piljin Yi$^2$

School of Physics, Korea Institute for Advanced Study, Seoul 130-722, Korea

Abstract

We study spectrum of $D = 2\,\mathcal{N} = (2,2)$ QED with $N + 1$ massive charged chiral multiplets, with care given to precise supermultiplet countings. In the infrared the theory flows to $\mathbb{CP}^N$ model with twisted masses, where we construct generic flavored kink solitons for the large mass regime, and study their quantum degeneracies. These kinks are qualitatively different and far more numerous than those of small mass regime, with features reminiscent of multi-pronged $(p,q)$ string web, complete with the wall-crossing behavior. It has been also conjectured that spectrum of this theory is equivalent to the hypermultiplet spectrum of a certain $D = 4$ Seiberg-Witten theory. We find that the correspondence actually extends beyond hypermultiplets in $D = 4$, and that many of the relevant indices match. However, a $D = 2$ BPS state is typically mapped to several different kind of dyons whose individual supermultiplets are rather complicated; the match of index comes about only after summing over indices of these different dyons. We note general wall-crossing behavior of flavored BPS kink states, and compare it to those of $D = 4$ dyons.

$^1$sjlee@kias.re.kr
$^2$piljin@kias.re.kr
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1 Introduction

The wall-crossing in four-dimensional supersymmetric theory [1–3] has been a subject of interests to many string theorists and mathematicians. This phenomenon of discontinuity in the BPS spectrum across walls of marginal stability, as one changes either parameters or vacuum expectation value of a theory, has been a source of enormous difficulty in understanding the detailed structure of theories like $\mathcal{N} = 2$ Seiberg-Witten theories and Calabi-Yau compactified type II string theories.

For BPS states in four-dimensional theories, this phenomenon has been understood in various physical viewpoints, such as from geometric realization of BPS states in string theory [5,6], from solitonic dynamics [7,8] and quantum bound states thereof [9,10], from a classical soliton picture of the low energy effective theory [11,12], and also later from supergravity attractor flow [13]. From the spacetime viewpoint, the wall-crossing occurs simply because the wavefunction of the BPS state in question becomes so large (as one approaches a wall of marginal stability) that the state in question cannot be regarded as a one-particle state anymore [9, 10, 13]. Despite this simple and compelling physical picture, a systematic and practical approach to the wall-crossing phenomenon which can cover all part of the moduli space had not been available.

Recently, there appeared a new remarkable development in this regard. It states that such discontinuities of spectrum across walls of marginal stability is actually necessary for the continuity of the vacuum moduli space metric. According to Gaiotto, Moore and Neitzke (GMN) [14,15], the continuity of the vacuum moduli space metric of $S^1$-compactified Seiberg-Witten theory implies the so-called Kontsevich-Soibelman relations [16] among BPS dyons across any given wall of marginal stability, which in turn tells us how the BPS spectra would change across such walls. Cecotti and Vafa [17] has recently suggested another interesting explanation of Kontsevich-Soibelman’s formulae with spin refinement [18], using the partition function of A-model topological string.

While the derivation by GMN was intended for $\mathcal{N} = 2$ Seiberg-Witten theory, the idea itself must be applicable to all wall-crossing phenomena. This new machinery is also important in that for the first time we have a systematic and local prescription for computing BPS spectrum. Although there were powerful methods which allowed explicit construction/counting of BPS states in certain regions of the moduli space [9,10,19,20], this new wall-crossing formula is far more comprehensive in its potential applications.

This observation that discontinuity of BPS spectra is related to continuity of some

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\footnote{See Ref. [4] for a review in the field theory side.
physical quantity has, on the other hand, a previously known analog in the context of two-dimensional $\mathcal{N} = (2, 2)$ theories. Cecotti and Vafa [21, 22] noted some time ago that if one assumes continuity of a twisted partition function

$$\mathcal{F}(\beta; m^i) = \text{tr}(-1)^R \text{Re}^{-\beta H}$$

throughout parameter space of the theory, this necessarily implies (dis-)appearance of BPS topological kinks across walls of marginal stability. Here $R$ is the fermion number, and $m^i$’s are the parameters of the theory. The above twisted partition is in turn related to the natural metric in the parameter space, and obeys the so-called $tt^*$ equation [23]. In fact, GMN also noted that some of mathematical structures of $tt^*$ equation is very closely mirrored by those that appear in their formulation of the four-dimensional wall-crossing.

Independent of this, another interesting similarity between $D = 4 \mathcal{N} = 2$ and $D = 2 \mathcal{N} = (2, 2)$ theories was noted in the literature: It has been conjectured [24, 25] that two-dimensional $\mathcal{N} = 2$ QED with $N + 1$ massive chiral multiplets possesses a BPS spectrum which is related to that of $SU(N+1)$ Seiberg-Witten theory with $N+1$ massive flavors at the root of the baryonic branch. In view of the new development in the Seiberg-Witten theory concerning the wall-crossing, and given its analog in $tt^*$ system, it is of some interest to clarify the precise correspondence and potential differences. In this article, we aim to study the two-dimensional theory with care given to precise BPS multiplet countings, and compare their wall-crossing phenomena against that of the Seiberg-Witten theory.

$\mathcal{N} = (2, 2)$ QED with $N+1$ chiral multiplets with twisted masses has been studied much previously. Initial studies by Hanany and Hori [26] and also by Dorey [24, 25] concentrated on implications of effective superpotential of the gauge-multiplet and its similarity to certain Seiberg-Witten spectral curve of $D = 4$ theory. Later works [27–31] refined this relationship further by giving physical reasonings, if somewhat sketchy, for the correspondence and also looked at $D = 2$ spectrum more closely by considering massive excitations of simple kink solutions.

In this paper, we expand on these existing works and solve for all possible flavored kinks. We give precise criteria for existence of such flavored kink states, set up the low energy dynamics of kinks, count their degeneracies, and provide wall-crossing formula. This allows a more refined look at the proposed “equivalence” of the spectra. We also hope that it will provide a playground for understanding wall-crossing phenomena in $D = 2$ when conserved charged other than the topological ones are present.

In section 2 and 3, we review the theory and search for all possible kink soliton solutions. Although kinks are simple and well-known objects, global charge allows the variety of kink solutions to increase greatly. Apart from simple “dyonic” kinks whose flavor charge is proportional to the topological charge, there are much more
flavored kinks whose central charges and stability criteria mimics those of the \((p, q)\) open string webs \([6, 7]\). In section 4, we quantize these solitons, elevate them to quantum BPS states, and count their degeneracy. These BPS states exhibit wall-crossing behavior, just as open string web does, which we put in the context of general \(D = 2\) and \(\mathcal{N} = (2, 2)\) theories following Cecotti and Vafa’s results. In section 5, we compare this spectra to its conjectured counterpart in \(D = 4\) Seiberg-Witten theory. Although, the two sides have some common features, essentially due to the open string web analogy, absence of “angular momentum” in the \(D = 2\) theory leads to quantitatively different spectra. However, a set of distinct dyons with different quark contents are mapped to a single type of favored kink; interestingly, if one sum over the relevant indices of the former, the result matches precisely with the degeneracy of the flavored kink. We rely on the four-dimensional wall-crossing formula to reach this conclusion. We close with conclusion.

2 \(\mathbb{C}P^N\) with twisted masses

Let us first summarize basic properties of \(\mathcal{N} = (2, 2)\) supersymmetric theories in two dimensions.\(^\text{#2}\) In particular we discuss the massive representation of \(\mathcal{N} = (2, 2)\) SUSY algebra and the CFIV index \([21]\) which effectively counts the short multiplets only.

**supersymmetry algebra** The \(\mathcal{N} = (2, 2)\) superalgebra can read off from the four-dimensional \(\mathcal{N} = 1\) superalgebra via trivial dimensional reduction as

\[
\begin{align*}
\{ Q_+, \bar{Q}_+ \} &= 2Z, & \{ Q_+, \bar{Q}_- \} &= -2(P_0 - P_3), \\
\{ Q_- , \bar{Q}_- \} &= 2\bar{Z}, & \{ Q_-, \bar{Q}_+ \} &= -2(P_0 + P_3),
\end{align*}
\]

(2.1)

where the central charge \(Z\) is

\[
Z = P_1 - iP_2 .
\]

(2.2)

For later convenience, let us summarize the \(U(1)_R \times U(1)_A\) charges of supersymmetric generators

\[
\begin{array}{c|cccc}
U(1)_R & Q_+ & Q_- & \bar{Q}_+ & \bar{Q}_- \\
\hline
U(1)_A & +1 & +1 & -1 & -1 \\
\hline
+1 & -1 & +1 & -1
\end{array}
\]

(2.3)

Here \(U(1)_A\) symmetry comes from the rotational symmetry \(SO(2)\) in four dimensions.

\(^\text{#2}\)Please see Appendix A for further notations and conventions.
In massive theories, one of the two $U(1)$ symmetries are explicitly broken, and suppose we choose the following basis that preserve $U(1)_R$

$$
\mathcal{A} = \frac{1}{\sqrt{2}}(Q_+ + Q_-), \quad \mathcal{B} = \frac{1}{\sqrt{2}}(Q_+ - Q_-).
$$

(2.4)

Making the central charge $Z$ real via a suitable $U(1)_A$ rotation, the supersymmetry algebra can be recast as

$$
\{\mathcal{A}, \mathcal{A}^\dagger\} = -2(M - Z), \quad \{\mathcal{B}, \mathcal{B}^\dagger\} = -2(M + Z), \quad \{\mathcal{A}, \mathcal{B}^\dagger\} = 0.
$$

(2.5)

One can therefore conclude that, for massive BPS multiplets, the algebra eventually is reduced to that of a single fermion oscillator.

**CFIV index**  With this, the index that count BPS multiplets is

$$
\Omega = \text{tr} \left[ (-1)^{R} R \right].
$$

(2.6)

This is a proper index since for long multiplets in Fock vacuum of R-charge $f$

$$
[f] \otimes ([1] \oplus [0])^2 \implies [f + 2] \oplus [f + 1] \oplus [f],
$$

the index $\Omega$ identically vanishes

$$
\Omega = 0.
$$

(2.7)

On the other hand, for generic BPS multiplets

$$
[f] \otimes ([1] \oplus [0]) \implies [f + 1] \oplus [f],
$$

one can have non-vanishing $\Omega$

$$
\Omega = (-1)^{f+1}.
$$

(2.8)

The simplicity of $D = 2$ theory is such that we have only two types of BPS multiplets, labeled by this sign, which is because of the small supersymmetry compounded by absence of spin.\(^\#3\)

\(^\#3\)The mirror symmetry, or t-duality in two-dimensional supersymmetric theory, exchanges those two R-symmetries

$$
U(1)_R \leftrightarrow U(1)_A, \quad Q_+ \leftrightarrow \bar{Q}_+.
$$

(2.9)

In the mirror-symmetric dual, the proper index now in turn is defined with $U(1)_A$ charge,

$$
\Omega = \text{tr} \left[ (-1)^A A \right].
$$
2.1 review on massive $\mathbb{CP}^N$-model

We consider a two-dimensional supersymmetric QED which flows down to a massive $\mathbb{CP}^N$-model with twisted masses. It is well-known that the massless $\mathbb{CP}^N$-model can be easily understood as IR limit of a gauged linear sigma model (GLSM) with a photon field $V$ and $N + 1$ chiral matter fields $\phi^i$ of unit charge. Introducing the Fayet-Iliopoulos (FI) parameter $r$ together with theta-angle $\theta$, the Lagrangian takes the following form

$$L = \int d^4\theta \left[ \phi_i^+ e^{-2V} \phi^i - \frac{1}{4e^2} \Sigma \Sigma \right] - \text{Im} \left[ \tau \int d^2\hat{\theta} \Sigma \right], \quad \tau = -ir + \frac{\theta}{2\pi}, \quad (2.10)$$

where $i$ run from $0, 1, ..., N$. Again, the notations and conventions used here are introduced in appendix A. For a positive FI parameter $r > 0$, the supersymmetric vacuum can be described by

$$\sum_i |\phi^i|^2 = r, \quad \sigma = 0, \quad (2.11)$$

which defines a projective space $\mathbb{CP}^N$. On the generic point of vacuum moduli space, the $U(1)$ vector multiplet and chiral mode orthogonal to $\mathbb{CP}^N$ are combined to a long multiplet of mass $\sqrt{r}e$ by the Higgs mechanism. In the IR limit where $e^2$ diverges, these modes become very heavy so that they decouple from the low-energy dynamics of the theory. It leads to a $\mathcal{N} = (2, 2)$ $\mathbb{CP}^N$ model.

We will present a simple way to obtain the effective Lagrangian for the above low-energy theory, $\mathcal{N} = (2, 2)$ $\mathbb{CP}^N$ model. For simplicity, let us first turn off the theta-angle $\theta = 0$ for a while. Note that we can then rewrite the Fayet-Iliopoulos (FI) term as

$$L_{\text{FI}} = 2r \int d^4\theta \ V. \quad (2.12)$$

The decoupling phenomenon of massive modes in the Higgs phase can be realized effectively as the vanishing Maxwell term in the limit of $e^2 \to \infty$. The low-energy theory at IR is now governed by the following Lagrangian

$$L \simeq \int d^4\theta \left[ \phi_i^+ e^{-2V} \phi^i + 2rV \right]. \quad (2.13)$$

Here the vector multiplet becomes an auxiliary fields that one can solve out:

$$\delta V : \quad r = \phi_i^+ e^{-2V} \phi^i \Rightarrow \ V = -\frac{1}{2} \log \left( \frac{r}{\phi_i^+ \phi^i} \right). \quad (2.14)$$
Componentwise, the gauge field, for example, is determined by
\[ A_\mu = \frac{1}{2i\phi_i^\dagger \phi^i} \left( \phi_i^\dagger \partial_\mu \phi^i - \partial_\mu \phi_i^\dagger \phi^i - i\bar{\psi}_i \bar{\sigma}_\mu \psi^i \right), \tag{2.15} \]
which implies that above procedure can be understood as supersymmetric version of solving the Gauss law in GLSM. Inserting the result back into the Lagrangian, one can finally obtain
\[ \mathcal{L}^{\text{IR}} = r \int d^4 \theta \left[ \log \left( \sum_i \phi_i^\dagger \phi^i \right) \right]. \tag{2.16} \]
Assuming one of matter fields, say \( \phi^0 \), does not vanish, one can rewrite the above Lagrangian as
\[ \mathcal{L}^{\text{IR}} = r \int d^4 \theta \left[ \log (\phi^0_0 \phi^0) + \log (1 + Z_m^\dagger Z^m) \right] = r \int d^4 \theta \left[ \log (1 + Z_m^\dagger Z^m) \right], \tag{2.17} \]
where we used for the last equality the chirality of \( \phi^0 \). (2.17) is precisely the lagrangian for the \( \mathcal{N} = (2, 2) \) supersymmetric non-linear sigma model with target space \( \mathbb{C} P^N \).

Here chiral superfields \( z^m \ (m = 1, 2, ..., N) \) are defined as
\[ Z^m = \frac{\phi^m}{\phi^0}, \tag{2.18} \]
from which one can identify it bosonic and fermionic part as
\[ z^m = \frac{\phi^m}{\phi^0}, \quad \chi^m = \frac{1}{(\phi^0)^2} (\phi^0 \psi^m - \psi^0 \phi^m). \tag{2.19} \]

The model we are eventually interested in is a massive version of this theory. The so-called twisted masses can be introduced by gauging the flavor symmetry \( U(N+1) \) and give expectation values to the corresponding twisted chiral field \( \hat{\Sigma} \) as
\[ \langle \hat{\Sigma} \rangle = \text{diag} (\langle \hat{\Sigma}_0 \rangle, \langle \hat{\Sigma}_1 \rangle, ..., \langle \hat{\Sigma}_N \rangle) = \begin{pmatrix} m_0 \\ m_1 \\ \vdots \\ m_n \end{pmatrix}. \tag{2.20} \]
These vev acts as mass terms for the chiral multiplets, and can be incorporated into the Lagrangian as
\[ \mathcal{L} = \int d^4 \theta \left[ \phi_i^\dagger e^{-2V} \phi^i e^{2V} - \frac{1}{4e^2 \Sigma \Sigma} \right] - \text{Im} \left[ \tau \int d^2 \hat{\theta} \hat{\Sigma} \right]. \tag{2.21} \]
With these twisted masses, there are \( N + 1 \) classical discrete vacua in this theory. They correspond to

\[
\sigma = m_i , \quad |\phi^i|^2 = r \quad \text{and} \quad \phi^k = 0 , \quad k \neq i
\]  

for each \( i = 0,1,\ldots,N \). With such discrete set of vacua, various topological kink solitons are present, which are the objects of our interest. One can show that this massive theory flows down to

\[
\mathcal{L}^\text{IR}_\text{mass} = r \int d^4 \theta \left[ \log \left( 1 + z_m^\dagger e^{2 \langle \hat{V}_m \rangle - 2 \langle \hat{V}_0 \rangle} z^m \right) \right].
\]  

In this article, we will be classifying and counting BPS multiplets of this theory, with a care given to quantum degeneracy and wall-crossing in weak coupling regime \( r \gg 1 \) of the sigma model.

The FI parameter \( r \) indeed receives the quantum correction at one-loop level, which leads to the RG running of renormalized FI parameter \( r(\mu) \)

\[
\mu \frac{\partial}{\partial \mu} r(\mu) = - \frac{N + 1}{2\pi} \rightarrow r(\mu) \simeq \frac{N + 1}{2\pi} \log \left[ \frac{\mu}{\Lambda_\sigma} \right],
\]  

where \( \Lambda_\sigma \) denotes the RG-invariant dynamical scale where the perturbative analysis breaks down. In order to rely on our analysis in the article, we therefore have to introduce sufficiently large twisted masses \( m^i \)

\[
e \gg |m^i - m^j| \gg \Lambda_\sigma,
\]  
such that the renormalized coupling \( r(\mu) \) are frozen in the weak-coupling regime. On the other hand, the low-energy theory of (2.10) in another interesting parameter region \( e \ll \Lambda \) have been explored in [24, 26] to study the BPS states in \( \mathbb{C}P^N \) model at strong coupling, which will be briefly discussed in section 5. It has been shown that there is the discrepancy between BPS spectra at weak and strong coupling of the theory, which strongly implies the existence of curves of marginal stability somewhere at strong coupling region. Quantum aspects of central charges and strong/weak coupling marginal stability walls were also recently investigated in Ref. [32, 33].

As emphasized again, we will explore the curves of marginal stability and wall-crossing phenomena not in strong-coupling regime but in weak-coupling regime.

**conserved charges** For later convenience, we summarize some conserved charges. The bosonic part of energy functional of this theory takes the following simple form

\[
\mathcal{E} = \int dx^3 \sum_i \left[ |D_0 \phi^i|^2 + |D_3 \phi^i|^2 + |\sigma - m_i|^2 |\phi^i|^2 \right].
\]  

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In the infrared, one can express the energy functional in terms of sigma model variables as

\[
E = r \int d^3x \left[ \frac{(1 + \bar{z} \cdot z) \delta_m^n - \bar{z}_n z^m}{(1 + \bar{z} \cdot z)^2} (\dot{\bar{z}}_m \dot{z}_n + \partial_3 \bar{z}_m \partial_3 z^n) \right] + \frac{1}{(1 + \bar{z} \cdot z)^2} \sum_n (m_n - m_0)^2 |z_n|^2
\]

\[
+ \frac{1}{(1 + \bar{z} \cdot z)^3} \sum_{n < p} (m_n - m_p)^2 |z_n|^2 |z_p|^2 (1 + |z_n|^2 + |z_p|^2)
\]

\[
+ \frac{1}{(1 + \bar{z} \cdot z)^3} \sum_{n \neq p \neq q} (m_n - m_p)(m_n - m_q)|z_n|^2 |z_p|^2 |z_q|^2 .
\]  

Introducing the twisted mass terms, flavor symmetry group \( SU(N + 1) \) of \( \mathbb{CP}^N \) model is spontaneously broken down to \( U(1)^N \). Those charges are defined by following: \( N \) \( U(1) \) charges can be parameterized by a following \( N + 1 \)-vector

\[
\vec{Q} = (Q_0, Q_1, \ldots, Q_N) ,
\]  

where each component is given by

\[
Q_0 = -i \int d^3x \phi^1 \partial_0 \phi^0 + \text{c.c.}
\]

\[
= r \int d^3x \sum_m (\bar{z}_m \partial_0 \bar{z}_m - \partial_0 \bar{z}_m \bar{z}_m)^2 ,
\]  

\[
Q_n = -i \int d^3x \phi^1 \partial_0 \phi^n + \text{c.c.}
\]

\[
= r \int d^3x \sum_m (\bar{z}_m \partial_0 \bar{z}_m - \partial_0 \bar{z}_m \bar{z}_m) + \bar{z}_n \partial_0 \bar{z}_n + i \sum_m (\bar{z}_m \partial_0 \bar{z}_m - \partial_0 \bar{z}_m \bar{z}_m)
\]

\[
\frac{1}{1 + \sum_m \bar{z}_m \bar{z}_m} \frac{1}{(1 + \sum_m \bar{z}_m \bar{z}_m)^2} .
\]  

Note that the charge components \( Q_i \ (i = 0, 1, \ldots, N) \) always satisfy the traceless condition

\[
Q_0 + Q_1 + \ldots + Q_N = 0 .
\]  

**central charge** Finally let us recall the expression of central charge \( Z \) for \( \mathcal{N} = (2, 2) \) massive \( \mathbb{CP}^N \) model. Based on the two-dimensional Witten effect and simple BPS spectra of (2.23) such as fundamental excitations and kink solutions, central charge \( Z \) takes the following form at weak coupling limit \( r \gg 1 \)

\[
Z = \sum_i m^i (Q_i + i\tau T^i) , \quad \tau = \frac{\theta}{2\pi} - ir ,
\]  

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as discussed in [24]. Here $T$ denotes the topological charge associated with kinks. Because the theory possesses $N + 1$ discrete vacua, $T$ naturally live in the $SU(N + 1)$ root lattice. For a topological kink from vacuum $j$ to vacuum $i$, our convention is such that $T_j = -1$, $T_i = 1$, and $T_k = 0$ for $k \neq j, i$.

The exact expression for central charge $Z$ has also proposed in [26] as

$$Z = \sum_i \left( m^i Q_i + m^i_D T_i \right), \quad m^i_D = \mathcal{W}(e_i),$$  \hspace{1cm} (2.31)

where $e_i$ are determined by roots of the polynomial equation

$$\prod_i (x - m_i) - \Lambda^{N+1}_\sigma = \prod_i (x - e_i) = 0,$$  \hspace{1cm} (2.32)

and $\mathcal{W}(e_i)$ are given by

$$\mathcal{W}(e_i) = \frac{N + 1}{2\pi} e_i + \sum_i \frac{m_i}{2\pi \log \left[ \frac{e_i - m_i}{\mu} \right]}.$$  \hspace{1cm} (2.33)

We will discuss in Section 5 an interesting implication of the exact expression of central charge $Z$ in relation to four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories.

2.2 BPS equations

The supersymmetry transformation for $z^m$ can be read off from those of GLSM fields: for examples, the variation rules for fermions $\chi^m$ are given by

$$\delta \chi^m = \frac{1}{(\phi^0)^2} \left( \phi^0 \delta \psi^m - \delta \psi^0 \phi^m \right) + \cdots,$$  \hspace{1cm} (2.34)

where we suppressed the irrelevant terms in our discussion. The transformation rules for GLSM fermion fields $\psi^i$ are given by

$$\delta \psi^i = \tau^3 \epsilon D_3 \phi^i + \epsilon D_0 \phi^i - i\tau^I \epsilon (\sigma_I \phi^i - \phi^i m^i_I),$$  \hspace{1cm} (2.35)

where $I$ run form 1, 2. Here we substitute (2.15) for the GLSM gauge fields:

$$A_\mu = \frac{\bar{z}_m \partial_\mu z^m - \partial_\mu \bar{z}_m \cdot z^m}{2i(1 + \bar{z}_m z^m)} + \cdots,$$  \hspace{1cm} (2.36)

and also substitute the following for the GLSM vector scalar $\sigma$

$$\sigma = \frac{m_0 + m_n \bar{z}_n z^n}{1 + \bar{z}_m z^m} + \cdots,$$  \hspace{1cm} (2.37)
with \( m_i \equiv (m^i)_1 - i(m^i)_2 \). We dropped again the fermion contribution here, which are irrelevant in our discussion below.

Inserting the above results (2.35) back into (2.34), BPS solitons of \( \mathbb{C}P^N \)-model should satisfy the following condition

\[
\phi^0 (\tau^3 e^3 D_3 \phi^n + e D_0 \phi^n + i \tilde{\tau}_{m_n} \phi^n) - \phi^n (\tau^3 e^3 D_3 \phi^0 + e D_0 \phi^0) = 0 .
\]

where \( \tilde{\tau}_{m_n} \) is defined as

\[
\tilde{\tau}_{m_n} \equiv \tau^I (m^I - m^0) = \left( \begin{array}{c} m_{n0} \\ m_{n0} \end{array} \right), \quad m_{n0} = m_n - m_0 .
\]

### 2.3 BPS (multi-)kinks

**simple BPS kinks** Let us first review BPS kinks solutions. Since they are static particle, the BPS equation (2.38) can be simplifies as

\[
\tau^3 (D_3 \phi^n - z^n D_3 \phi^0 + i \tau^3 \tilde{\tau}_{m_n} \phi^n) \epsilon = 0 .
\]

As referred to appendix for detailed computation, one can show that

\[
D_3 \phi^n - z^n D_3 \phi^0 = r \frac{\partial_3 z^n}{\sqrt{1 + z_n z^n}} ,
\]

from which one can massage the above BPS equation into

\[
\left[ \frac{\partial_3 z^n}{\sqrt{1 + z_n z^n}} + i \tau^3 \tilde{\tau}_{m_n} \frac{z^n}{\sqrt{1 + z_n z^n}} \right] \epsilon = 0 .
\]

Since \((\tau^3 \tilde{\tau}_{m_n})^2 = -|m_{n0}|^2\), the BPS equation is finally given by

\[
\partial_3 z^n \pm |m_{n0}| z^n = 0 , \quad z^m = 0 \quad \text{for} \quad m \neq n ,
\]

provided that \( m_n \neq m_n \). The solutions are therefore given by

\[
z^n = \exp \left[ \pm |m_{n0}| (x^3 - x_0) \right] .
\]

The energy of this configuration saturate a topological energy bound since

\[
\mathcal{E} = r \int dx^3 \left[ \frac{1}{(1 + z_n z_n)^2} |\partial_3 z^n \mp |m_{n0}| z^n|^2 \pm \frac{|m_{n0}|}{(1 + z_n z_n)^2} \partial_3 (\bar{z}_n z_n) \right] \geq -r|m_{n0}| \left[ \frac{1}{1 + z_n z_n} \right]_{x^3 = -\infty} \left. \right|_{x^3 = +\infty} = r|m_{n0}| .
\]
**composite kinks**  Let us denote a BPS kink which interpolates from \(m\)th vacuum to \(n\)th vacuum as \(nm\)-kink. Suppose that the phases of two mass-parameters \(m_{10}\) and \(m_{20}\) are aligned as parallel. Without loss of generality, one can set \(|m_{20}| > |m_{10}|\). Then, the 20-kink can be also understood as a bound state of a 10-kink and a 21-kink: the BPS equations for 20-kink are

\[
\partial_3 z^1 + i \tau^3 \hat{\tau} m_{10} z^1 = 0, \quad \partial_3 z^1 + i \tau^3 \hat{\tau} m_{20} z^1 = 0, \quad [\tau^3 \hat{\tau} m_{10}, \tau^3 \hat{\tau} m_{20}] = 0, \tag{2.46}
\]
or equivalently

\[
\partial_3 z^1 \mp |m_{10}| z^1 = 0, \quad \partial_3 z^2 \mp |m_{20}| z^2 = 0. \tag{2.47}
\]
The solution then turns out to be

\[
z^1 = \exp \left[ \pm |m_{10}| x^3 \right], \quad z^2 = \exp \left[ \pm |m_{20}| (x^3 - x_0) \right], \tag{2.48}
\]
after a suitable choice of the origin. Here \(x_0\) parameterizes the relative distance between constituent BPS kinks. Note that the phase factor of each \(z^m\) describes one-parameter degeneracy of such kink solutions, so in fact we can have an arbitrary complex number multiplying each of \(z^{1,2}\)s. The fact that they have the same energy can be directly checked. See Appendix A.

Obviously, this can be repeated for other \(z_m\)’s straightforwardly. When all \(m_{n0}\)’s are aligned in the complex plane, the general solution is

\[
z^m = \zeta^m \exp \left[ \pm |m_{m0}| x^3 \right] \tag{2.49}
\]
with arbitrary complex numbers \(\zeta^m\)’s which are moduli coordinates of the soliton.

The GLSM \(\sigma\) field (2.37) is useful for describing the general behavior of the kink solution, which is depicted in Figure 2.1 for this solution. For a finite \(x_0\), \(\sigma\) starts with the vacuum \(\sigma = m_0\), approaches the vacuum \(\sigma = m_1\) (never touches it), and eventually goes to the vacuum \(\sigma = m_2\) as \(x^3\) increases. This shows that the solution indeed a sequential sum of 10-kink and 21-kink.

**2.4 zero modes**

Here we briefly dwell on details of fermion zero mode counting. Bosonic ones were already noted in previous section: there is one complex bosonic collective coordinate for each \(z^n\) kink, provided that all masses \(m_{n0}\) are of the same phase. We will find below that for each \(z_m\) kink there is also one complex fermionic zero modes. The linearized fermion equations of motion are given by

\[
\bar{\sigma}^M \left( D_M \chi^n + D_M z^m \Gamma^m_{ml} \chi^l \right) = 0, \tag{2.50}
\]
Figure 2.1: Configuration of the GLSM field $\sigma$. It implies that the system is placed in $\sigma = m_1$ vacuum at $x^3 = -\infty$, and in $\sigma = m_2$ vacuum at $x^3 = +\infty$. The size of the plateau near $m_1$ is determined by how far 10-kink and 21-kink are separated, which is in turn determined by certain ratio between $\zeta^1$ and $\zeta^2$.

with

$$\Gamma^m_{ml} = -\frac{\delta^m_l z_m + \delta^m_l \bar{z}_l}{1 + \bar{z} \cdot z}, \quad \bar{\Gamma}^m_{\bar{m}l} = -\frac{\delta^{\bar{m}}_{\bar{l}} \bar{z}_{\bar{m}} + \delta^{\bar{m}}_{\bar{l}} \bar{z}_{\bar{l}}}{1 + \bar{z} \cdot z},$$ (2.51)

where the covariant derivatives are defined as

$$D_M \chi^n = \partial_M \chi^n + i(\hat{A}^n_M - \hat{A}^0_M)\chi^n.$$ (2.52)

Here $M$ run from 0, 1, 2, 3. The twisted mass terms are written as if it is gauge field along 2, 3 directions, and contributes

$$\bar{\sigma}^M(\hat{A}^n_M - \hat{A}^0_M) = -\hat{\tau}_{mn0} = -\left(\begin{array}{cc} 0 & m_n - m_0 \\ \bar{m}_n - \bar{m}_0 & 0 \end{array}\right).$$ (2.53)

Clearly the derivative $\partial_M$ runs only for $M = 0, 1$.

For simplicity let us again take the example of a double-kink with aligned masses $|m_{20}| > |m_{10}| > 0$. The BPS solution in this case was

$$z^1 = \zeta^1 \exp\left[|m_{10}|x^3\right],$$

$$z^2 = \zeta^2 \exp\left[|m_{20}|x^3\right].$$

Recall that, despite its deceptively simple appearance, the solution should be viewed as a combination of two kinks, one from 0 to 1 and another from 1 to 2, which will...
interact with each other when one begins to move them around. The fermionic zero modes in this background are equally simple and deceptive. There are exactly one zero mode for each \( \chi \), and we find (in the limit of \( \zeta^1 = 0 \))

\[
\chi_0^1 = e^{i m_{10} x^3} \epsilon_0 \\
\chi_0^2 = e^{i m_{20} x^3} \epsilon_0 .
\]  

(2.54)

with the constant spinor obeying \( i \tau_3 \hat{\tau}_{m20} \epsilon_0 = -|m_{20}| \epsilon_0 \).

The Goldstino mode, in the limit \( |\zeta^1| \ll 1 \), is the combination \( \chi_1^1, \chi_2^2 = \zeta^1, \zeta^2 e^{i m_{10,20} x^3} \epsilon_0 \), quantization of which endows the soliton with the basic BPS multiplet structure. The other combination is more interesting. This is a superpartner to the nontrivial bosonic moduli of the kinks that encodes relative separation and mutual interaction of 10-kink and 21-kink. We will come back to them later when we search for quantum spectrum of flavored kinks.

## 3 Flavored kink solitons and marginal stability

Since the theory has \( U(1)^N \) flavor charges, BPS objects may carry both topological and flavor charges. A kink with generic flavor charge will be called flavored kinks. We present in this section the explicit construction of flavored kink solitons together with preliminary discussion on their marginal stability behavior. An important fact here is that these generic flavored kinks appears only when the mass parameters of the theory is misaligned, i.e., when they are no longer lined up in the complex plain. This is analogous to (dis-)appearance of generic dyons in \( D = 4, \mathcal{N} = 2 \) SYM and also of 1/4 BPS dyons in \( D = 4, \mathcal{N} = 4 \) SYM, depending on how the vacuum expectation values of adjoint scalar fields are aligned or misaligned.

In order to investigate the dyonic spectrum of the two-dimensional \( \mathbb{CP}^N \) model, let us introduce the time-dependence on the phase factor of sigma model fields \( z^m \). Then, the BPS equation (2.38) can be rewritten as

\[
\tau^3 (D_3 \phi^n - z^n D_3 \phi^0 + i \tau^3 \tilde{\tau}_{m,n} \phi^n) \epsilon + (D_0 \phi^0 - z^n D_0 \phi^n) \epsilon = 0 .
\]  

(3.1)

Inserting (2.36) into the above equation, one can show that flavored kinks should satisfy the following

\[
\left[ \frac{\partial z^n}{\sqrt{1 + \sum_m \bar{z}_m z^n}} + i \tilde{\tau}_{m,n} \frac{z^n}{\sqrt{1 + \sum_m \bar{z}_m z^n}} + \frac{\partial_0 z^n}{\sqrt{1 + \sum_m \bar{z}_m z^n}} \right] \epsilon = 0 .
\]  

(3.2)
3.1 simple flavored kinks

Let us again review simple flavored kink solutions whose topological charge and flavor charge are parallel [24]. In this case, without loss of generality, one can turn off all complex field $z^n$ expect one, say $z^1$.

Then, the above BPS equations (3.2) can be simplified as

$$
(\partial_0 z^1 + i\hat{\tau}_E)\epsilon + \tau^3 (\partial_3 z^1 + i\tau^3 \hat{\tau}_M)\epsilon = 0,
$$

(3.3)

where $\hat{\tau}_{E,M}$ are defined by

$$
\hat{\tau}_E + \hat{\tau}_M = \hat{\tau}_{m_{10}}.
$$

(3.4)

In order to have solutions to this equation, we have to demand the projectors $\hat{\tau}_{E,M}$ to satisfy the following compatibility condition

$$
[\hat{\tau}_E, \tau^3 \hat{\tau}_M] = 0.
$$

(3.5)

One can easily find a family of solution, parameterized by

$$
\hat{\tau}_E = \vec{\tau} \cdot \vec{m}_E, \quad \hat{\tau}_M = \vec{\tau} \cdot \vec{m}_M,
$$

(3.6)

where vectors $\vec{m}_E$ and $\vec{m}_M$ are orthogonal decomposition of $\vec{m}_{10}$ as depicted in figure 3.1.

For the case (a), the flavored kink solution is

$$
z^1 = \exp\left[\pm |m_M|x^3 \pm i|m_E|t\right],
$$

(3.7)
For the case (b), the flavor kink solution is instead given by
\[ z^1 = \exp \left[ \pm |m_M| x^3 \mp i|m_E| t \right]. \] (3.8)

Without loss of generality, let us concentrate on the case (a). Some conserved charges of the simple flavored kink solutions are in order.

**flavor charge** For a simple flavored kink, the nonvanishing flavor charges (2.28) are
\[ Q_1 = -Q_0 = \pm r \int_{-\infty}^{+\infty} dx^3 \frac{|m_E|}{2\cosh^2(|m_M|x^3)} = r \frac{|m_E|}{|m_M|}. \] (3.9)

**energy** For the simple flavored kinks, the energy functional (2.26) can be massaged into a sum of complete squares like
\[
\mathcal{E} = r \int dx^3 \frac{1}{(1 + |z^1|^2)^2} \left[ |\partial_3 z^1 \mp |m_M| z^1|^2 + |\partial_0 z^1 \mp i|m_E| z^1|^2 \right.
\]
\[ \left. \mp |m_E|i(z_1 \partial_0 z^1 - \partial_0 \bar{z}_1 z^1) \pm |m_M| \partial_3 (\bar{z}_n z^n) \right] \]
\[ \geq \mp |m_E| |Q_0 \mp r|m_M| \frac{1}{1 + \bar{z}_n z^n} \left. \right|_{x^3 = -\infty}^{x^3 = +\infty} = \pm \frac{r|m_{10}|^2}{|m_M|}, \] (3.10)

where we used \(|m_M|^2 + |m_E|^2 = |m_{10}|^2\). Since
\[ Z = -m_{10} Q_0 + irm_{10} = r \frac{|m_{10}|}{|m_M|} e^{i\phi_{m_{10}}} (-|m_E| + i|m_M|) = \frac{r|m_{10}|^2}{|m_M|} e^{i\phi_{m_E}}, \] (3.11)
the solutions are indeed BPS with \(\mathcal{E} = |Z|\).

### 3.2 composite flavored kinks and marginal stability

It has been noted previously that the solitonic sector of this \(D = 2\) QED has some features reminiscent of certain \(D = 4\) Seiberg-Witten theory, where the topological charge and the flavor charges are mapped to the magnetic charge and the electric charges, respectively. On the other hand, dyonic solitons in the \(N = 2\) supersymmetric gauge theories in four dimensions are such that magnetic and electric charges are generically not parallel [6,7]. This is in turn related to existence of multi-pronged strings in string theory.
These class of $D = 4$ BPS states are useful in that one can study the issue of marginal stability in weakly-coupled regime of the theory. In this subsection, we will look for their analog in $D = 2$ theory, considering flavored kinks whose topological and flavor charge are not parallel misaligned, and discuss their marginal stability briefly. In section 4, their quantum spectrum and wall-crossing phenomena will be explored in more details.

For simplicity, let us first assume that

$$z^1 = z^1(x^3, t), \quad z^2 = z^2(x^3), \quad z^m = 0 \quad \text{for} \quad m \neq 1, 2.$$  \hspace{1cm} (3.12)

For this ansatz, the BPS equation (3.2) can be rewritten as

$$\left[ \partial_3 z^2 + i \tau^3 \hat{x}_{m0} z^2 \right] \epsilon = 0,$$

$$\left[ \tau^3 \partial_3 z^1 + \partial_0 z^1 + i \hat{x}_{m0} z^1 \right] \epsilon = 0.$$  \hspace{1cm} (3.13)

Guided by the previous example of simple flavored kink, let us rewrite the second equation into the following form

$$\tau^3 \left[ \partial_3 z^1 + i \tau^3 \hat{x}_{mE} z^1 \right] \epsilon + \left[ \partial_0 z^1 + i \hat{x}_{mE} z^1 \right] \epsilon = 0, \quad \hat{x}_E + \hat{x}_M = \hat{x}_{m10}.$$  \hspace{1cm} (3.14)

In order to find out half-BPS solutions, we therefore have to demand three projectors to commute to each other

$$[\tau^3 \hat{x}_{m20}, \tau^3 \hat{x}_{m3}] = 0, \quad [\tau^3 \hat{x}_{mM}, \hat{x}_{mE}] = 0, \quad [\tau^3 \hat{x}_{m20}, \hat{x}_{mE}] = 0.$$  \hspace{1cm} (3.15)

One can again easily parameterize the solutions of the above relations as

$$\hat{x}_E = \vec{\tau} \cdot \vec{m}_E, \quad \hat{x}_M = \vec{\tau} \cdot \vec{m}_M,$$  \hspace{1cm} (3.16)

where vectors $\vec{m}_E$ and $\vec{m}_M$ are depicted in figure 3.2. The BPS solutions of interests are

$$z^1 = \exp \left[ |m_M| x^3 + i |m_E| t \right], \quad z^2 = \exp \left[ |m_{20}| (x^3 - x_0) \right],$$  \hspace{1cm} (3.17)

after a suitable choice of origin of $x^3$.

**flavor charge and marginal stability** For the above solution, the flavor charges (2.28) are

$$Q_0 = -2r |m_E| \int_{-\infty}^{+\infty} dx^3 \frac{|z^1|^2}{(1 + |z^1|^2 + |z^2|^2)^2},$$

$$Q_2 = -2r |m_E| \int_{-\infty}^{+\infty} dx^3 \frac{|z^1|^2 |z^2|^2}{(1 + |z^1|^2 + |z^2|^2)^2},$$

$$Q_1 = -Q_0 - Q_2.$$  \hspace{1cm} (3.18)
When we place the mass parameter $m_{10}$ on a so-called wall of marginal stability, as depicted in figure 3.2 (b), the relative distance $x_0$ diverges such that the 20-flavored kink decays into two constituent 10- and 21-flavored kinks. This is an underlying physical reason for the phenomenon of wall-crossing. At wall-crossing, one can easily show that the GLSM field $\sigma$ actually turns touches the vacuum $\sigma = m_1$, as described in figure 3.2 (b).

For classical soliton whose flavored charges are not quantized, this can be viewed backward as a process where the flavor charges are increased until the kink solution decompose into two. This “maximal” or “critical” flavor charge can can be read off from the solution as

$$Q_0^{cr} \simeq -r \tan \theta , \quad Q_2^{cr} \simeq -r \tan \tilde{\theta} , \quad Q_1^{cr} \simeq +r (\tan \theta + \tan \tilde{\theta}) .$$

With quantized (and thus fixed) flavor charges, we can use this formula to determine
the critical values of $\theta$ and $\tilde{\theta}$, which in turn determine the marginal stability wall for breaking this soliton to a simple flavored 10-kink and a simple flavored 21-kink.

**central charge**  As discussed before, the central charge of the present model can take the following form

$$Z = \sum_n m^n (Q_n + \tau T_n), \quad \tau = \frac{\theta}{2\pi} - ir.$$  

(3.20)

For the composite flavored kinks, the central charge $Z_{20}$ can be decomposed into those of constituent particles, say

$$Z_{20} = Z_{10} + Z_{21}, \quad Z_{10} = -m_{10} Q_0 + \tau m_{10}, \quad Z_{21} = +m_{21} Q_2 + \tau m_{21}.$$  

(3.21)

On the wall of marginal stability where the flavor charges take their critical values $\vec{Q}^{cr}$, the central charges of constituent particles become

$$Z_{10} = m_{10} (\tan \theta - i), \quad Z_{21} = m_{21} (-\tan \tilde{\theta} - i).$$  

(3.22)

Note that, on the wall of marginal stability, the phases of two mass-parameters satisfy the relations below

$$\theta + \tilde{\theta} = \varphi_{m_{21}} - \varphi_{m_{10}},$$

from which one can conclude that phase difference between $Z_{10}$ and $Z_{21}$ is

$$\arg(Z_{21}) - \arg(Z_{10}) = -\tilde{\theta} - \theta + \varphi_{m_{21}} - \varphi_{m_{10}} = 0!$$  

(3.23)

As expected, we find that phases of the two central charges $Z_{10}, Z_{21}$ coincides at the marginal stability wall.

## 4 Quantum BPS states and wall-crossing

### 4.1 low energy interactions of kinks

In this section, we construct and count quantum BPS states of topological kinks with flavor charges, by studying the low energy interactions of simple kinks. When $m_{i0}$ are all of same phase, each kink carries one complex bosonic moduli, and their moduli
space is naturally Kähler. The holomorphic coordinates $\zeta^i$'s are defined in terms of the soliton solution as

$$z^i = e^{m_{i0}x^3} e^{m_{i0}x^i + i\theta^i} \equiv e^{m_{i0}x^3} \zeta^i.$$  

(4.1)

The moduli space dynamics is obtained by taking time-dependence of the form $\zeta^i(t)$ with small velocity as usual. The Kähler potential is found by integrating the field theory kinetic term as [38]

$$K(\bar{\zeta}, \zeta) = \int d^3x \ K(\bar{z}, z) = r \int d^3x \log \left[ 1 + \sum_i e^{2m_{i0}x^3} \bar{\zeta}_i \zeta^i \right],$$

(4.2)

from which the moduli space metric follows

$$g_{ij}(\zeta^i, \bar{\zeta}_i) = r \int d^3x \left[ \frac{e^{2m_{i0}x^3} \delta^i_j}{1 + \sum_k e^{2m_{k0}x^3} \bar{\zeta}_k \zeta^k} - \frac{e^{2(m_{i0} + m_{j0})x^3} \bar{\zeta}_i \zeta^j}{(1 + \sum_k e^{2m_{k0}x^3} \bar{\zeta}_k \zeta^k)^2} \right].$$

(4.3)

Here let us first concentrate on $\mathbb{CP}^2$ model, from which we can read off the indices of all BPS states following an argument of type found in Ref. [19].

For the moment, let us further assume $m_{20} = 2m_{10}$. This causes two different restrictions on the mass parameters for our purpose. One is the special ratio between the two absolute values, which is harmless in counting supersymmetric states. The other, namely alignment of the two phases, pose a physical restriction to the spectrum. We will shortly abandon the latter.

The moduli space metric is then compactly written as

$$g = g_{\text{com}} + g_{\text{rel}}, \quad g_{\text{com}} = \frac{r}{4m} \left| d\log \zeta_2 \right|^2, \quad g_{\text{rel}} = \frac{r}{4m} F(|\zeta_1|^4 / |\zeta^2|^2) \left| d\zeta_1^2 / \zeta_1^2 \right|^2,$$

with

$$F(1/w) = \frac{1}{w(1-4w)} + \frac{2}{(1-4w)^{3/2}} \log \left( \frac{1 - \sqrt{1-4w}}{1 + \sqrt{1-4w}} \right),$$

(4.4)

for $4w < 1$ and

$$F(1/w) = -\frac{1}{w(4w-1)} + \frac{4}{(4w-1)^{3/2}} \tan^{-1} \left( \sqrt{4w-1} \right),$$

(4.5)

for $4w > 1$. This shows that $\zeta^2$ plays the role of the center of mass coordinates, while

$$\zeta_{\text{rel}} \equiv \zeta^1 / \sqrt{\zeta^2}$$

20
plays the role of the relative coordinate. It is important for a later purpose to note that in the limit of $|\zeta_{\text{rel}}| \to \infty$ $g_{\text{rel}}$ is reduced simply to

$$g_{\text{rel}} \simeq \frac{r}{m} \left| d\zeta_{\text{rel}}/\zeta_{\text{rel}} \right|^2. \quad (4.6)$$

On the other hand, in the limit of $\zeta_{\text{rel}} \to 0$, we have

$$g_{\text{rel}} \sim \left| d\zeta_{\text{rel}} \right|^2. \quad (4.7)$$

so $\zeta_{\text{rel}}$ is itself a good coordinate near origin where the two kinks coincides in real space.

The phases $\theta_{1,2}$ of $\zeta^{1,2}$ are each $2\pi$-periodic and turning on their (integral) momenta corresponds to turning on $U(1)$ flavor charges of type $q^{i0} = q^i - q^0$; $q^i$ is the charge of $i$-th diagonal unbroken favor group. Defining the phase of $\zeta_{\text{cm}}$ as $\theta_{\text{cm}}$ and $\zeta_{\text{rel}}$ as $\varphi$, we find

$$\theta_{\text{cm}} = \theta^2, \quad \varphi = \theta^1 - \frac{\theta^2}{2}, \quad (4.8)$$

and thus

$$q^{10} = q, \quad q^{20} = q_{\text{cm}} - \frac{q}{2}, \quad (4.9)$$

where $q_{\text{cm}}$ and $q$ are conjugate momenta of $\theta_{\text{cm}}$ and $\varphi$. The actual flavor charge for these are

$$(q^0, q^1, q^2, \ldots) = (q_{\text{cm}} - q/2, q, -q_{\text{cm}} - q/2, 0, 0, \ldots). \quad (4.10)$$

Note that $q$ is integral while $q_{\text{cm}}$ should be integral or half-integral depending on whether $q$ is even or odd. Such a correlation between relative and center of mass charges is common, and here due to the identification

$$(\theta_{\text{cm}}, \varphi) \sim (\theta_{\text{cm}} + 2\pi, \varphi - \pi). \quad (4.11)$$

The total moduli space has the form

$$\mathbb{R} \times \left[0, 4\pi \right] \times \frac{\mathcal{M}_2}{\mathbb{Z}_2}, \quad (4.12)$$

where the relative moduli space $\mathcal{M}_2$ has a topology of $R^2$ and where $\mathbb{Z}_2$ acts as (4.11). The center of mass phase and the quotient action depends on the masses of individual kinks, in general.

Such a charge state, say with $q_{\text{cm}} = 0$, precisely corresponds to the classical solution we find in the previous section with $q = Q_E$. As we saw there, however, a flavored kink states of this kind do not appear unless some of the twisted masses are misaligned in the complex plane. On the other hand, with such misaligned masses, the
Figure 4.1: The profiles of attractive scalar potential in the moduli space dynamics of two-kinks system, induced by tension of composite kinks.

A composite kink for which we obtained the moduli dynamics is no longer a solution to the equation of motion unless $\zeta_{rel} = 0$. With $m_{20} = 2m_M > 0$ and $m_{10} = m_M + im_E$, the relative moduli space makes sense only if $m_E = 0$ while the flavored kinks appears only if $m_E \neq 0$.

These two issues are in fact tied together. Whenever $m_E \neq 0$, unflavored 20-kink configuration costs more energy than the central charge bound and this extra energy,

$$\Delta \mathcal{E} = r|m_E|^2 \int dx^2 \frac{|z_1|^2(1 + |z_2|^2)}{(1 + |z_1|^2 + |z_2|^2)^2},$$

should be interpreted as a potential in the two-kink moduli space dynamics.

With $m_{20} = 2m_M \equiv 2m$, we find

$$\Delta \mathcal{E} = \frac{r|m_E|^2}{m} \frac{|\zeta_2|^2}{|\zeta_1|^4} F(|\zeta_1|^4/|z_2|^2) = \frac{m_E^2}{2} g_{rel} \left( \frac{\partial}{\partial \varphi}, \frac{\partial}{\partial \varphi} \right)$$

Thus, the bosonic part of relative moduli space dynamics must be modified to

$$L_{rel} = \frac{1}{2}(g_{rel})_{\mu\nu} \dot{y}^{\mu} \dot{y}^{\nu} - \frac{1}{2}m_E^2 (g_{rel})_{\mu\nu} K^{\mu} K^{\nu}$$

where

$$K = \frac{\partial}{\partial \varphi}.$$
This potential energy on the moduli space $\Delta \mathcal{E}$, depicted in figure 4.1, shows physical separations between simple kinks are no longer a moduli degree of freedom, since it generates an attractive force between the two kinks. On the other hand, when the conjugate momentum $q$ of $\varphi$ is turned on, this induces a repulsive angular momentum barrier between the two kinks. For finite relative charge $q$, then, one can generically expect flavored two kinks states with the relative position determined by the balance of these two forces. The amount of the flavor charge, the mass parameter $m_E$, and the size of $\zeta_{rel}$ are all interrelated, which was shown implicitly in the classical analysis of section 3.

More generally, we may consider $L_{0}$-kink dynamics, regarded as a collection of 10-kink, 21-kink, 32-kink etc, with

$$m_{p0} = m_{M}^{(p)} + im_{E}^{(p)}$$

for $p = 1, 2, ..., L - 1$ and $0 < m_{M}^{(1)} < m_{M}^{(2)} < \cdots < m_{M}^{(L-1)} < m_{L0}$. The above Lagrangian generalizes to

$$L_{rel} = \frac{1}{2} (g_{rel})_{\mu\nu} \dot{y}^\mu \dot{y}^\nu - \frac{1}{2} (g_{rel})_{\mu\nu} (m_{E}^{(p)} K_{p}^{\mu})(m_{E}^{(q)} K_{q}^{\nu}), \quad (4.17)$$

where $K_{p}$'s are linear combinations of holomorphic Killing vector fields, induced by flavor $U(1)$ rotations on the soliton.

### 4.2 counting generic BPS states

This form of moduli dynamics with potential has well-known supersymmetric extensions, provided that $K$ is a Killing vector field. Such massive nonlinear sigma-model mechanics first appeared with complex supersymmetry in a work by Freedman and Alvarez-Gaume [39], while the form of relevance for us was found more recently in the context of BPS dyons of the Seiberg-Witten theory [4,10,40]. In this subsection, let us outline this modified moduli dynamics and solve for flavored BPS multi-kink states explicitly. See appendix B for a short review.

Without the potential, the moduli dynamics of the kinks would be the ordinary nonlinear sigma model where the real fermions match 1-1 with real bosons. Therefore the supercharge in question can be understood geometrically as the spinorial Dirac operator on the moduli space,

$$Q = i\Gamma^{I}\nabla_{I}, \quad (4.18)$$

where $\nabla_{I}$'s are the covariant derivative with ordinary spin connection and $\Gamma^{I}$'s the Dirac matrices.
The addition potential energy shifts this supercharge. With general \(L\)-kink case, the supercharge is shifted as

\[
Q = \Gamma^I \left( i \nabla_I + \sum_p m_E^{(p)} K_I^p \right).
\]

Taking square of this supercharge, one finds

\[
\{ Q, Q \} = H - Z,
\]

where the central charge (to be distinguished from \(Z\) of the field theory) is defined via Lie-derivatives

\[
Z = -i \sum_p m_E^{(p)} L_{K^p},
\]

with respect to the Killing vectors, whose action is part of the global \(U(1)^N\) flavor rotations acting the kinks.

Since the BPS state must saturate the bound \(H - Z = 0\), the search for BPS states in any given kink sector boils down to finding zero modes of \(Q\) on the moduli space. This task is in principle very complicated. However, one can reduce counting problem to that of two-body problems, at least for the index of such quantum mechanics.

With \(m_E \neq 0\), the operator \(H - Z\) has a massgap which separate the continuum from the ground state. Such operators are called Fredholm operators, for which usual index theorem applies; one simply choose to scale up the values of \(a_p\)'s, thus increasing the mass gap indefinitely, while keeping the index unaffected. This localizes the index computation to the fixed points of the vector fields \(K^a\)'s. Once this happens, the counting problem becomes that of harmonic oscillators and factorizes into minimal units with two bosonic and two fermionic coordinates [19]. The latter is a two-kink problem, so it suffices to count BPS bound states in a two-kink problem in order to compute index for arbitrary multi-kink states.

For flavored 20-kink state problem, we have seen that the supercharge reduces to

\[
Q = \Gamma^I \left( i \nabla_I + m_E K_I \right),
\]

when \(m_{20} = 2m\) and \(m_{10} = m + im_E\) with real \(m\) and real \(m_E\). The Hamiltonian is nonnegative and has the general form

\[
H = \frac{1}{2} (g_{\text{rel}})_{\mu\nu} \left( \pi^\mu \pi^\nu + m_E^2 K^\mu K^\nu \right) + \cdots,
\]

where the ellipsis denotes terms involving fermions and \(\pi^\mu\)'s are the canonical conjugate momenta of the moduli coordinates \(y\)'s.
With $\zeta_{rel} = e^{\rho+i\varphi}$, the metric for $\mathcal{M}_2$ is

$$g_{rel} = f(\rho)^2 \left( d\rho^2 + d\varphi^2 \right),$$  \hspace{1cm} (4.24)

where

$$f(\rho)^2 = \frac{2r}{m} e^{-4\rho} F(e^{4\rho}).$$  \hspace{1cm} (4.25)

In the relevant orthonormal frame,

$$e^\rho = f(\rho) d\rho, \quad e^{\varphi} = f(\rho) d\varphi, \quad \omega^\rho_\varphi = \frac{\partial_\rho f(\rho)}{f(\rho)} d\varphi,$$  \hspace{1cm} (4.26)

the supercharge reduces to

$$Q = \Gamma^\rho \frac{1}{f(\rho)} \left[ \partial_\rho + \frac{1}{2} \frac{\partial_\rho f(\rho)}{f(\rho)} + i \Gamma^{\rho \varphi} \left( q - m_E f(\rho)^2 \right) \right],$$  \hspace{1cm} (4.27)

in the charge $q$ sector, that is, when $-i\partial/\partial\varphi \to q$. A supersymmetric state in this sector has the central charge $q m_E$, which must be saturated by the nonnegative Hamiltonian. Thus, a BPS bound state is possible only if $q m_E \geq 0$.

Denoting two chiral components of $\Psi$ under $i \Gamma^{\rho \varphi}$ by $u_\pm$, the zero-mode solves

$$\partial_\rho [\sqrt{f(\rho)} u_\pm] \pm \left( q - m_E f(\rho)^2 \right) [\sqrt{f(\rho)} u_\pm] = 0,$$  \hspace{1cm} (4.28)

from which one can obtain one and only one normalizable solution

$$u_- = \frac{u_0}{\sqrt{f(\rho)}} e^{iq\varphi} \exp \left[ \int_{\rho_0}^\rho d\rho' \left( q - m_E f(\rho)^2 \right) \right],$$  \hspace{1cm} (4.29)

whenever

$$0 \leq q < q_{cr} = m_E f(\infty)^2 = 2r \frac{|m_E|}{m}.$$  \hspace{1cm} (4.30)

The upper bound comes from the asymptotic normalizability while the lower bound is required by normalizability at origin ($\rho \to -\infty$),

$$u_- \simeq u_0 \left( \frac{8m}{r\pi} \right)^{\frac{q}{4}} e^{(q-1/2)\rho + i q \varphi} \exp \left[ - \frac{r|m_E|}{16m} e^{2\rho} \right].$$  \hspace{1cm} (4.31)

Although $q = 0$ wavefunction is mildly singular at origin, it is still normalizable.\footnote{Note that the upper bound on the electric charge is precisely the critical charge obtained from the classical construction of flavored composite dyons in the section 3.}

$$q_{cr} = Q_1 = -(Q_0 + Q_2) = r \left( \tan \theta + \tan \tilde{\theta} \right) \simeq 2r \frac{|m_E|}{m}. \hspace{1cm} (4.32)$$
In summary, we found exactly one flavored bound state of the 10-kink and 21-kink for each integral relative charge \( q \) from 0 up to \( q_{cr} = 2r|m_{E}|/m \) and for arbitrary half-integral (odd \( q \)) or integral (even \( q \)) \( q_{cm} \). Each of such bound states complete into a BPS multiplet, thanks to the Goldstino mode. These flavored kinks become unstable against decay to a pair of simple flavored kinks (10- and 21) when the mass parameters are changed such that the critical relative charge \( q_{cr} \) becomes smaller or equal to \( q \).

Index computation for more general flavored multi-kink states follows immediately. As argued above, the problem factorizes into several two-body problems. We consider general flavored \( L0 \)-kink, viewed as bound state of 10-kink, 21-kink, 32-kink, etc. For the \( p \)-th pair, there is one “relative” flavor charge \( q^{(p)} \). When this charge obeys the conditions,

\[
0 \leq |q^{(p)}| < q^{(p)}(m_{i0}) \quad \text{and} \quad 0 < m_{E}^{(p)} q^{(p)},
\]

the above two-body result tells us that the index is unit. The total index for this \( L \)-body problem is a product of all such two-body indices, so we learn finally that

\[
\Omega = (-1)^f,
\]

where \( f \) is the \( R \)-charge of the soliton, provided that (4.33) is satisfied for all \( p = 1, 2, \ldots, L - 1 \). Otherwise

\[
\Omega = 0,
\]

which we will take as an evidence that the corresponding BPS does not exist.

### 4.3 wall-crossing

After lengthy computations, we finally arrive at wall-crossing issues at large mass limit of this massive \( D = 2 \) QED. Since \( q_{cr}^{(p)} \sim r m_{E}^{(p)}/m_{L0} \), there is a wall of marginal stability for these flavored kink at \( r m_{E}^{(p)}/m_{L0} \sim q^{(p)} \), details of which would follow once we compute the metric and the potential on the moduli space. This is a tedious but straightforward exercise. For us, it suffices to know that these walls of marginal stability are determined by \( r \) and \( q \)'s, and they extend to the asymptotic region of large \( r \). Across any such a wall, the flavored multi-kink states break into a pair of smaller flavored multi-kink states, such as \( L0 \)-kink interpolating between 0 vacuum and \( L \) vacuum breaking up into a flavored \( K0 \)-kink and a flavored \( LK \)-kink. The latter two objects exist on both side of this particular wall, so the jump in the spectrum is only for the bound state, and we have the simple jumping formula

\[
|\Delta \Omega| = 1.
\]
As we saw in section 2, the marginal stability wall is, as always, defined by the phase alignment of the two central charges of the flavored $K^0$-kink and the flavored $LK$-kink.

In fact, this simple wall-crossing formula is a special case of general wall-crossing where we are considering bound states of two BPS particles with unit degeneracy. For this, let us review a result from [21]. They defined a twisted partition function of $D = 2$ field theories as

$$F(\beta; m^i) = \lim_{l \to \infty} \frac{i\beta}{l} \text{tr}(-1)^R \text{Re}^{-\beta H},$$

(4.37)

where $l$ is the regulated size of the spatial line. Alternatively this may be thought of as expectation value of $R$ when the theory is defined on $S^1 \times \mathbb{R}^1$ with Euclidean signature and periodic boundary condition on $S^1$. A single-particle BPS state, $Z$, contributes

$$F_Z = i\beta(-1)^f \int \frac{dp}{\pi} e^{-\beta \sqrt{p^2 + |Z|^2}} = \frac{i(-1)^f}{\pi} \int d\mu |Z| \cosh \mu e^{-\beta |Z| \cosh \mu},$$

(4.38)

with the rapidity $\mu = \sinh^{-1}(p/|Z|)$. Note that, as we vary the parameters of the theory, wall-crossing will occur somewhere and this contribution from single particle BPS states will have to be disappear in a discontinuous manner.

On the other hand, $\hat{\Omega}$ also receives contributions from many particle sectors. In particular, with the decomposition of the central charge as, $Z = Z_1 + Z_2$, the two-particle contribution is of some interests. Following Cecotti et al., we also finds that, when the pair of BPS states $Z_{1,2}$ backscatter,#5 there is a contribution from the two-particle sector of the type

$$F_{Z_1+Z_2} =$$

$$d_1 d_2 \frac{i(-1)^{f_1+f_2}}{4\pi^2} \int d\mu_1 d\mu_2 \beta (|Z_1| \cosh \mu_1 + |Z_2| \cosh \mu_2) e^{-\beta (|Z_1| \cosh \mu_1 + |Z_1| \cosh \mu_1)}$$

$$\times \frac{\partial}{\partial \mu_1} \log \left( \frac{\sinh(\mu_2 - \mu_1 + i\epsilon)}{\sinh(\mu_1 - \mu_2 + i\epsilon)} \right),$$

(4.39)

where $2\epsilon = \text{Im} \log(Z_2/Z_1)$ and $d_{1,2}$ are the number of such BPS supermultiplets of central charge $Z_{1,2}$.

---

#5 Even in $D = 2$ what one means by forward-scattering and backward-scattering can be somewhat ambiguous when particles can change species. However, we are mostly interested in situations when two particles in question are clearly distinct, with different masses for example, so that the particles are unambiguously labeled. In this context, backscattering means the sign flip of the relative rapidity before and after.
Recall that the wall of marginal stability would be at $\epsilon = 0$ where the two central charges line up in the complex plane. Because of the logarithm, the two-particle expression $F_{Z_1+Z_2}$ also has a discontinuous imaginary part, and in fact

$$\lim_{\epsilon \to 0^\pm} F_{Z_1+Z_2} = \pm d_1 d_2 \frac{F_Z}{2}, \quad (4.40)$$

so that

$$\lim_{\epsilon \to 0^+} F_{Z_1+Z_2} - \lim_{\epsilon \to 0^-} F_{Z_1+Z_2} = d_1 d_2 F_Z. \quad (4.41)$$

Although individual contributions are discontinuous, the twisted partition function $\hat{\Omega}$ itself can be continuous provided that $Z$ state exists as a one-particle BPS state only on the $\epsilon < 0$ side. The continuity of the twisted partition function seems reasonable, and this would then imply a rather general wall-crossing behavior. Assuming such a continuity of $F$, and since $\Omega(Z_{1,2}) = (-1)^{h_1 h_2} d_1 d_2$, we then find the general wall-crossing formula across $Z \to Z_1 + Z_2$ walls of marginal stability,

$$\Delta \Omega(Z) = \pm \Omega(Z_1) \Omega(Z_2). \quad (4.42)$$

For flavored domain walls in the massive $\mathbb{C}P^N$ theory, we found $|\Delta \Omega(Z)| = 1$, which is easily explained by this wall-crossing formula, since elementary excitations and simple kinks all have unit index, $|\Omega| = 1$. Building more complicated flavored kinks out of them can only generate flavored kinks with $|\Omega| = 1$ because the wall-crossing formula (4.42) is so simple.

Wall-crossing in $D = 2$ was originally studied by Cecotti and Vafa for purely topological kinks [22]. For this case, the central charges simplifies as differences of “canonical coordinates” which in our case are simply the masses $m_i \sim \tau m^i$, and $F$ can be explicitly solved using the $tt^*$ equations [23]. Introduction of flavor charges to the kink should modify the latter approach somewhat, if not drastically, which will appear elsewhere.

5 $D = 4 \mathcal{N} = 2$ $SU(N + 1)$ with flavors

This two-dimensional QED shows certain features reminiscent of the Seiberg-Witten theory of four dimensions. This was first noted by Hanany and Hori [26] who found that the renormalization of the FI parameters $\tau = -ir + \theta/2\pi$ and the asymptotic form of the four-dimensional $\tau_{SW}$ have a close resemblance. This was taken up later more seriously by Dorey [24] who argued that the spectrum of this theory is related to that of $SU(N + 1)$ Seiberg-Witten theory with $N + 1$ flavors of masses $m^i$. The correspondence was supposed to be precise at the root of the baryonic branch where
the vacuum expectation values of the Seiberg-Witten scalars match with the quark masses. This conjecture was further extended by Dorey, Hollowood, and Tong [25].

The most compelling reason for this conjecture comes from the exact central charge (2.31) of the BPS states, obtained from effective superpotential $\mathcal{W}(\Sigma)$ after integrating over all chiral multiplets of (2.21) in the parameter region $\epsilon \ll \Lambda_\sigma$. In [26], it has been pointed out that the periods $m^i_D - m^j_D$ (2.33) are in perfect matching with those of the Seiberg-Witten curve at baryonic root of the corresponding $\mathcal{N} = 2$ $SU(N + 1)$ gauge theory with massive $N + 1$ quarks.

This latter observation, strictly speaking, tells us only that the set of central charges in the two theories may coincides, not necessarily the actual particle content. Nor does not say much about degeneracies of general BPS states on the two sides. Yet, one may go a bit further and hope that at least hypermultiplets of Seiberg-Witten theory may match against $D = 2$ spectra, since these can be potentially massless somewhere in the moduli space (or parameter space for $D = 2$) and can be associated with singular structure of the latter. This is precisely the conjecture of Dorey and his collaborators.

Now that we found a very rich spectrum of flavored kinks, counted their degeneracy, and found the wall-crossing behavior, let us come back to this conjecture and see how it lives up to its promise. In generic Seiberg-Witten theory of rank large than one, typical BPS dyons are not in the hypermultiplet. Rather they come with large angular momentum which is already evident in the classical soliton solutions. As we will see below, under the proposed correspondence between $D = 2$ QED and the Seiberg-Witten theory, a typical flavored kink we found would be mapped to such dyons with high angular momenta. Let us explore to what extent and in what sense there might be an “equivalence” of BPS spectra of the two theories.

Recall the central charge of Seiberg-Witten theory,

$$Z_{SW} = \tilde{a}_D \cdot \tilde{Q}_m + \tilde{a} \cdot \tilde{Q}_e + \sum_f m^f S_f.$$  \hspace{1cm} (5.1)

In the asymptotic region, we have $\tilde{a}_D = \tau_{4D} \tilde{a}$. For $SU(N + 1)$ theory with $N + 1$ fundamental hypermultiplets, we have a special point where $a^i = m^f = i$, where the central charge simplifies to

$$Z_{SW} = \tau_{4D} \tilde{m} \cdot \tilde{Q}_m + \tilde{m} \cdot \tilde{Q}_e^{adj} + \sum (m^i - m^j) \tilde{Q}_{ij}.$$  \hspace{1cm} (5.2)

$Q_e^{adj}$ denotes electric charges in the adjoint root lattice and the combined contribution from the matter multiplet

$$\tilde{Q} = S + Q_e^{matter}.$$  \hspace{1cm} (5.3)
effectively lives in a $SU(N+1)$ root lattice, which explains why we wrote the last term in Eq. (5.2) as mass differences. For “unit” magnetic charges, we have the following mapping from $D = 4$ theories,

$$Q_m \rightarrow T,$$
$$Q_{a_{adj}} + \bar{Q} \rightarrow Q,$$
$$\tau_{4D} \rightarrow \tau = \frac{\theta}{2\pi} - ir,$$
$$(\bar{a}, \bar{a}_D) \rightarrow (\bar{m}, \bar{m}_D),$$

(5.4)
to $D = 2$. Note that $Q$'s we found are always in the root lattice which is achieved on the left hand side by mixing of $SU(N+1)$ color weights and $SU(N+1)$ favor weights at this special point in the Seiberg-Witten moduli space. This map forms the basis of the conjectured equivalence of BPS spectra on the two sides. Writing the root system of $SU(N+1)$ as collection of $e_i - e_j$ with $0 \leq j < i \leq N$, and mapping the $D = 2$ central charge to this, we see that the $k_i$-kink corresponds to a magnetic root of $e_k - e_i$ whereas $j_l$ flavor charge maps to either a $(e_l - e_j)$-vector meson, or an $e_j$ colored quark of $l$-th flavor (or vice versa).

Finally, the relevant index for $D = 4 \mathcal{N} = 2$ theory is the second helicity trace.

$$\Omega_{SW} = -2 \text{tr}(-1)^F J_3^2.$$

(5.5)

which counts various BPS multiplets with some weights. Actual values are

$$\Omega_{SW}([s]_{\text{spin}} \otimes [\text{half Hypermultiplet}]) = (-1)^{2s}(2s + 1),$$

(5.6)

where the first factor denotes the angular momentum multiplet under the $SO(3)$ little group, denoted by its spin. For example, a charged vector gives $-2$.

### 5.1 BPS dyons in pure $SU(N+1)$ and wall-crossing

What are known in literature about such a large-rank Seiberg-Witten theory come from weak coupling analysis, that is, in the limit of large vacuum expectation values. In this regime, the low energy dynamics of monopoles are easily set up and reliable for general $\mathcal{N} = 2$ theories. In particular, dyons in pure $SU(N+1)$ theory whose magnetic charge is a (dual) root, as opposed to arbitrary linear combinations thereof, are completely classified and counted by Stern and Yi [19]. Let us summarize their result first.

As in $D = 2$, an ordering is possible when the adjoint vacuum expectation values $a^i = m^i$ almost line up in the complex plane. By overall $U(1)$ rotation, we can take them to be almost real, such that

$$\text{Re } m^0 < \text{Re } m^1 < \cdots \text{Re } m^N,$$

(5.7)
as we did in the previous sections for $D = 2$ theory. Without loss of generality, take dyons of magnetic charge $e_L - e_0$. With the above ordering of vev’s, electric charges of dyons are restricted as

$$-\left(\frac{k + \sum n^{(p)}}{2}\right)e_0 + n^{(1)}e_1 + n^{(2)}e_2 + \cdots + n^{(L-1)}e_{L-1} + \left(\frac{k - \sum n^{(p)}}{2}\right)e_L,$$

(5.8)

with integers $k$ and $n^{(p)}$’s correlated such that the coefficients of $e_{L,0}$ are also integral. For a BPS dyon of such a charge to exist, the charges must obey the inequalities

$$n^{(1)} \times \text{Im} m^1 > 0, \quad n^{(2)} \times \text{Im} m^2 > 0, \quad \ldots, \quad n^{(L-1)} \times \text{Im} m^{L-1} > 0,$$

(5.9)

and also that the individual electric charge does not exceed the critical value, which goes as

$$|n^{(p)}| < \frac{8\pi^2}{e^3} \sum_{q} \mu^{-1}_{pq} \text{Im} m^q,$$

(5.10)

where the matrix $\mu$ is a reduced mass matrix defined in terms of Re $m^q$’s. See Ref. [4, 19]

When these conditions are satisfied, the degeneracy is known [19]. Furthermore, the angular momentum content is also not difficult to find, and the end result is that the dyon is in the following multiplet,

$$\left(\bigotimes_p \left[\frac{|n^{(p)}| - 1}{2}\right]\right) \otimes \text{[half Hypermultiplet]}.$$

(5.11)

Note that the dyon appears not as a single supermultiplet but rather as a sum of many supermultiplets with spins up to $(\sum |n^{(p)}| - L + 1)/2$. The index $\Omega_2$ of such a dyon is

$$\Omega_{SW} = (-1)^{\sum n^{(p)} - L + 1} \prod_p |n^{(p)}|.$$

(5.12)

In fact, the computation of BPS bound states for kinks of previous section is modeled after the computation here. This result was later reproduced by Denef from more stringy viewpoint [20].

Recently a startling proposal by Kontsevich and Soibelman (KS) [16] was given for all wall-crossing behavior of $D = 4 \mathcal{N} = 2$ theories, which seems to fit all known examples of wall-crossings of these theories. For our purpose, we will not really need the full power of KS proposal but a corollary for the so-called semi-primitive cases. One considers BPS bound states of the form $\gamma(s) = \gamma_1 + s\gamma_2$, where $\gamma$’s denote electromagnetic charges of the states and we assume that $\gamma_{1,2}$ are primitive, namely

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they are not integer multiple of other charge vector. Denoting \( \Omega_{t,s} \equiv \Omega_{SW}(t\gamma_1 + s\gamma_2) \), we have the wall-crossing formula for \( \Omega_{1,s} \) as a consequence of KS formula:

\[
\Omega_{1,0} + \sum_{s \geq 1} \Delta \Omega_{1,s} y^s = \Omega_{1,0} \prod_{s' \geq 1} \left( 1 - (-1)^{s'(\gamma_1,\gamma_2)} y^{s'} \right)^{\pm s'(\gamma_1,\gamma_2)\Omega_{0,s'}}.
\]  

(5.13)

The Schwinger product of the charges \( \langle \gamma_1, \gamma_2 \rangle \) enters the exponents everywhere. When only \( \Omega_{t,0} \) and \( \Omega_{0,s} \) are nonzero on one side of the wall, this would determine \( \Omega_{1,s} = \Delta \Omega_{1,s} \) completely on the other side of the wall.

This was first suggested by Denef and Moore [41] as a phenomenological formula. It can also be derived from the KS formula, which shows how to fix the sign in the last exponent in terms of the sign of the relative phase of the two central charges \( Z_1 \) and \( Z_2 \) on the side of the wall. We left the sign ambiguous since we will presently fit this formula to the known spectrum where the correct sign appears quite obviously.

A further simplification results if we take \( \Omega_{0,s} = 0 \) for all but \( s = 1 \). As far as we know, in all \( D = 4 \mathcal{N} = 2 \) field theories, no non-primitive charge state has ever been found as one particle states.\(^{\#6}\) Then we have,

\[
\Omega_{1,0} + \sum_{s \geq 1} \Omega_{1,s} y^s = \Omega_{1,0} \left( 1 - (-1)^{(\gamma_1,\gamma_2)} y \right)^{\pm (\gamma_1,\gamma_2)\Omega_{1,0}}.
\]  

(5.14)

Let us see how this fits with the known spectrum of dyons we discussed above. Take for example the simplest \( L = 2 \). We will write the charge vectors as \( \gamma_1 = (e_2 - e_0; e_1 - e_0) \) and \( \gamma_2 = (0; e_1 - e_0) \) so that

\[
\gamma(s) = (e_2 - e_0; (s + 1)e_1 - (s + 1)e_0).
\]  

(5.15)

In terms of dyons whose degeneracy we saw earlier, this corresponds to \( L = 2 \), \( n^{(1)} = k = s + 1 \). One may be tempted to take \( \gamma_1 = (e_2 - e_0; 0) \) but this state is absent in this corner of moduli space and cannot be used as \( \gamma_1 \).

From the knowledge of \( \Omega_{1,0} = 1 \) and \( \Omega_{0,1} = -2 \) (because it is a vector multiplet), we find

\[
\sum_{n \geq 0} y^n \Omega((e_2 - e_0; (s + 1)e_1 - (s + 1)e_0)) = (1 + y)^{\pm 2}.
\]  

(5.16)

With the negative sign in the exponent (which is something that can be checked independently), we find

\[
\Omega_{SW}((e_2 - e_0; n^{(1)}e_1 - n^{(1)}e_0)) = (-1)^{n^{(1)}-1} n^{(1)},
\]  

(5.17)

\(^{\#6}\)This is one notable difference from the supergravity countings, despite many other similarities. We do not know of an explicit proof of this statement, although there were examples where this absence was shown in some cases.
after putting \( n^{(1)} = s + 1 \) in the expression. It is clear that this procedure can be repeated for more complicated dyons with \( L > 3 \) by taking \( \gamma_2 = (e_p - e_0) \) for all \( p = 1, \ldots, L - 1 \), which results in

\[
\Omega_{SW}((e_L - e_0; \sum_{p=1}^{L-1} n^{(p)} e_p - \sum_{p=1}^{L-1} n^{(p)} e_0)) = (-1)^{\sum (n^{(p)} - 1)} \prod_p n^{(p)},
\]

in precise accordance with the general index formulae computed in the low energy dynamics approach. Now that we have some confidence in how wall-crossing formula reproduce known spectra, let us move on to the flavored cases.

5.2 flavored dyons from wall-crossing formula

The actual dyons whose spectra was proposed to be equivalent to that of \( D = 2 \) theory are those that appear in \( SU(N + 1) \) Seiberg-Witten theory with \( N + 1 \) fundamental hypermultiplets with masses \( m_i \)'s. Furthermore, the comparison can be made only at the root of the baryonic branch. Recall that well inside the baryonic branch, where electric charges are screened, the vector mesons and massive hypermultiplets together form a long multiplet. Let us denote them as

\[
W_{ij}, \quad q^{(j)}, \quad \tilde{q}^{(i)},
\]

where \( q, \tilde{q} \) are the two chiral multiplets of the hypermultiplets and are, respectively, in the representations \((N + 1, N + 1)\) and \((\bar{N} + 1, N + 1)\) under \( SU(N + 1)_{gauge} \times SU(N + 1)_{flavor} \). Given the map (5.4), the correspondence between the flavored kinks and \( D = 4 \) dyons are easy to see.

Let us first consider the simplest nontrivial case with \( L = 2 \). The kinks of topological and flavor charge\(^7\)

\[
(T, Q) = (e_2 - e_0; n(e_1 - e_0)),
\]

can be mapped to a monopole of charge \((e_2 - e_0)\), which we denote by \( M_{20} \), bound with \( n \) electrically charged particles which can be either \( W_{10} \) or \( \tilde{q}_{0}^{(1)} \). The other quark, \( \tilde{q}_{1}^{(0)} \) cannot bind to this monopole since it does not have the right dynamical charge. Thus we find the following map,

\[
(T, Q) = (e_2 - e_0; n(e_1 - e_0)) \rightarrow M_{20} + nW_{10} \text{ or } M_{20} + (n - 1)W_{10} + \tilde{q}_{0}^{(1)}. \quad (5.21)
\]

\(^7\)Although general flavored kink in this simple example would be more like

\[
(T, Q) = (e_2 - e_0; k'(e_2 - e_0) + n(e_1 - e_0))
\]

for any integer \( k' \), we set \( k' = 0 \) because it affects neither the marginal stability nor degeneracy, at least in the leading order in \( 1/r \). The same goes for \( L0 \)-kink cases we later consider.
The quark cannot bind more than once due to the Pauli exclusion principle, although this can also be deduced from the wall-crossing formula. See below.

In figuring out degeneracies of these dyons, one crucial information missing is with what minimal electric charge the dyon actually exist as a hypermultiplet. In this asymptotic corner and in the pure $SU(N+1)$ case, we saw that $M_{20} + W_{10}$ is the first such hypermultiplet. With flavors present, this need not be true anymore. In fact the original conjecture on equivalence of $D=2$ and $D=4$ spectra relied heavily on the fact that the two theories share the same spectral curve, suggesting that at least hypermultiplet content of $D=4$ theory should be faithfully reflected in $D=2$ theories. This leads us to guess that the first hypermultiplet is the purely magnetic bound state, $M_{20}$, namely a magnetic monopole of charge $e_2 - e_0$. Our objective here is to reproduce the rest of BPS spectra from this single assumption.

We may naively repeat the analysis of the pure case. From the wall-crossing formula, we deduce that
\[ \sum y^s \Omega_{SW}(M_{20} + s\tilde{q}^{(1)}_0) = 1 + y, \]  
which, as promised, shows that quarks can bind to a monopole at most once. Using the wall-crossing formula one more time, we find
\[ \Omega_{SW}(M_{20} + nW_{10}) = (-1)^n(n + 1), \]
\[ \Omega_{SW}(M_{20} + (n - 1)W_{10} + \tilde{q}^{(1)}_0) = (-1)^{n-1}n. \]
Note that individual spectra of these dyons are rather nontrivial and come with high angular momentum content. However, it is intriguing that the sum of these two indices is rather simple
\[ \Omega_{SW}(M_{20} + nW_{10}) + \Omega_{SW}(M_{20} + (n - 1)W_{10} + \tilde{q}^{(1)}_0) = (-1)^n, \]  
and actually coincides with the $D=2$ counting of flavored kinks, up to a sign.

More generally, for dyons with magnetic charge $e_L - e_0$, the relevant indices are
\[ \Omega_{SW}(M_{L0} + \sum_{p=1}^{L-1} l^{(p)}W_{p0} + \sum_{p'} \tilde{q}^{(p')}_{0}) = (-1)^{\sum l^{(p)}} \prod (l^{(p)} + 1), \]  
where $\{p'\}$ is a subset of $\{1, 2, \ldots, L - 1\}$. The map to $D=2$ flavored kink follows the same rule as before; These dyons are mapped to flavored $L0$-kinks with $p0$-flavor charges $q^{(p)}$ being equal to either $n^{(p)} = l^{(p)}$ (when $p \neq p'$) or $n^{(p')} = l^{(p')} + 1$. Summing over the indices for fixed $q^{(p)} = n^{(p)'}$’s, we find
\[ \sum_{\{p'\}} \left( \prod_{p=1, p \neq p'}^{L-1} (-1)^{n^{(p)}}(n^{(p)} + 1) \prod_{p'}(-1)^{n^{(p')}-1}(n^{(p')}) \right), \]  
(5.25)

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which is the same as

\[ (-1)^{\sum_n n(p)} \prod_{p=1}^{L-1} ((n^{(p)} + 1) - n^{(p)}) = (-1)^{\sum n^{(p)}}. \]  

(5.26)

We thus find that under the proposed map (5.4), \( D = 2 \) indices equal precisely to the sum of \( D = 4 \) indices of all corresponding dyons, possibly up to a sign.

Note that this cancellation among \( D = 4 \) indices, and the resulting match against \( D = 2 \) index, is possible only upon very fine-tuned relationships among these dyons with different quark contents.

6 Conclusion

In this paper, we reviewed \( D = 2 \mathcal{N} = (2, 2) \) QED with twisted masses, with emphasis on BPS spectra in the large mass limit. With \( N + 1 \) chiral matter fields, one finds BPS kink solutions endowed with \( U(1)^N \) flavor charges, whose stability criteria mimics those of \( D = 4 \mathcal{N} = 2 \) dyons. In the classical limit, this also coincides with that of open string web, or equivalently 1/4 BPS dyons of \( \mathcal{N} = 4 \) Yang-Mills theory, giving us a pictorial way to determine the marginal stability walls. We quantized these solitons to obtain degeneracies, which turned out to be unit for all such solitons. This result is consistent with general wall-crossing behavior expected in \( D = 2 \mathcal{N} = (2, 2) \) theories, namely,

\[ \Delta \Omega(Z_1 + Z_2) = \pm \Omega(Z_1)\Omega(Z_2). \]

Wall-crossing of \( D = 2 \) topological kinks has been studied in depth where \( tt^* \) equation makes a prominent appearance. It would be very interesting to explore further how this could be refined to situations with conserved charges (such as flavor charges) other than topological charges.

We also compared the spectrum to the conjectured \( D = 4 \) counterpart, i.e., that of the \( SU(N+1) \) Seiberg-Witten theories with \( N + 1 \) massive fundamental hypermultiplets, at the root of the baryonic branch. Due to the special nature of this point in the moduli space, where the gauge symmetry and the flavor symmetry are locked, one type of flavored kink is mapped to several different kind of dyons with different quark contents. The degeneracies of the latter, as counted by the second helicity trace, can be complicated and large unlike those of the kinks. However, this difference is remedied miraculously once we sum over the indices of all the corresponding dyons with different quark content, which gives at the end,

\[ |\Omega| = 1 = \left| \sum_{\text{dyons}} \Omega_{\text{SW}} \right|. \]

(6.1)
for each flavored kink that exists on the left hand side and for all the corresponding
dyons on the right hand side.

One cannot really say that spectra of the two theories are equivalent, since various
dyons that are mapped to one type of flavored kink will generally carry mutually
different electric and flavor charges. Note also that in this map only a subset of
$D = 4$ BPS dyons participate. A topological charge of a kink is always mapped to a
dual root of the gauge group; since general dyons may carry more general (magnetic)
weight that lie in the dual root lattice, there must be dyons that do not fit in this
correspondence. Given such obvious differences, the agreement (6.1) is all the more
remarkable.

The question of whether and how wall-crossing behaviors and indices of $D = 2$
theories and those of $D = 4$ theories might be related deserves further study. $D = 4$
wall-crossing received much attention lately, as we noted already, and some of math-
ematical tools there have uncanny resemblance to those of $tt^*$ equations. Whether
such a mathematical resemblance has anything to do with the present example is
unclear, but it still begs for a clarification. In particular, the partial agreement (6.1)
of $D = 2$ and $D = 4$ indices, despite vastly different BPS spectra with their different-
looking individual indices, needs to be understood better. In a recent study [42],
Gaiotto pointed out a relationship between surface operators in $D = 4$ $\mathcal{N} = 2$ gauge
theories and $D = 2$ sigma model whose UV theory is $\mathcal{N} = (2, 2)$ QED with massive
chiral matters. It would be interesting to see what are the implications in the present
context.

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Appendix

A Miscellany

notations and conventions  One convenient way to describe two-dimensional supersymmetric theories is to use the four-dimensional superspace formalism of Wess and Bagger followed by a suitable dimensional reduction: let us compactify the four-dimensional theories along $x^1, x^2$ directions so that chiral and anti-chiral spinors $\psi_\alpha, \bar{\psi}_{\dot{\alpha}}$ reduce to two-dimensional complex spinors $(\psi_1, \psi_2) \equiv (\psi_+, \psi_-)$, $(\bar{\psi}_1, \bar{\psi}_2) \equiv (\bar{\psi}_-, \bar{\psi}_+)$.

Here $\pm$ denote the charges under $U(1)_A$ R-symmetry, arising from the spatial rotation in the compactified dimensions.

In addition to usual superfields with four supercharges such as vector and chiral superfields, it is well-known that two-dimensional theories allow a so-called twisted chiral superfield. The twisted chiral superfield $\hat{\Phi}$ is defined as

$$D_+ \hat{\Phi} = D_+ \hat{\Phi} = 0 .$$

Defining twisted fermionic coordinates $\hat{\theta}_\alpha = (\theta_+, -\bar{\theta}_+)$, the twisted chiral superfield has the following component field expansion

$$\hat{\Phi} = \hat{\phi} + \sqrt{2} \hat{\theta} \hat{\bar{\psi}} + \hat{\theta} \hat{\bar{\theta}} \hat{F} .$$

As a comment, the chiral/twisted chiral-multiplets are indeed in a mirror pair.

One peculiar example of such twisted chiral superfields is of the form

$$\Sigma = D_+ \bar{D}_+ V ,$$

where $V$ denote the vector multiplet. The component field expansions of the above superfield $\Sigma$ read

$$\Sigma = (A_1 - iA_2) + 2i\bar{\theta}_+ \lambda_+ + 2i\theta_+ \bar{\lambda}_+ + 2\theta_+ \bar{\theta}_+ (D + iF_{03}) + \cdots$$

$$= \hat{\phi} + \sqrt{2} \hat{\theta} \hat{\bar{\psi}} + \hat{\theta} \hat{\bar{\theta}} \hat{F} .$$

with

$$\hat{\phi} = A_1 - iA_2, \quad \hat{\psi}_\alpha = -\sqrt{2} i(\lambda_+, \bar{\lambda}_+), \quad \hat{F} = D + iF_{03} .$$

Using $\Sigma$, the Fayet-Iliopoulos term and topological $\theta$-term can be combined as

$$L_{FI} + L_\theta = -\text{Im} \left[ \tau \int d^2 \hat{\theta} \Sigma \right] = rD - \frac{\theta}{2\pi} F_{03} ,$$

where $\tau = -ir + \frac{\theta}{2\pi}$. 

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covariant derivative  Using the inhomogeneous parameterization \( z^m \) of \( \mathbb{CP}^N \), the GLSM scalar fields can be expressed up to overall \( U(1) \) phase as

\[
\phi^0 = \sqrt{\frac{r}{1 + \bar{z}_m z^m}}, \quad \phi^n = \sqrt{\frac{r}{1 + \bar{z}_m z^m}}.
\]

The \( U(1) \) gauge field \( A_\mu \) (2.15) now in turn becomes

\[
A_\mu = \frac{\bar{z}_m \partial_\mu z^m - \partial_\mu \bar{z}_m z^m}{2i(1 + \bar{z}_m z^m)}.
\]

The various covariant derivatives are then given by

\[
D_\mu \phi^0 = -\sqrt{\frac{r}{1 + \bar{z}_m z^m}} \frac{\bar{z}^m \partial_\mu z^m}{1 + \bar{z}^m z_m},
\]

\[
D_\mu \phi^n = +\sqrt{\frac{r}{1 + \bar{z}_m z^m}} \left[ \partial_\mu z^n - \frac{\bar{z}^m (\bar{z}_m \partial_\mu z^m)}{1 + \bar{z}^m z_m} \right].
\]

Inserting the above results back into the BPS equation (2.38), one can obtain (3.2).

energy for composite kinks  For the composite kink solution, it needs much elaboration to massage the energy functional to sum of complete squares and boundary terms. Since two mass parameters \( m_{10} \) and \( m_{20} \) are now parallel, let us set them to be purely real without loss of generality. From the general expression of energy functional (2.26), one can obtain

\[
\mathcal{E} = \int dx^3 \frac{r}{1 + |z^1|^2 + |z^2|^2} \left[ \left( 1 + |z^1|^2 + |z^2|^2 \right) |\bar{z}_2 \partial_3 z_1 - \bar{z}_1 \partial_3 z_2|^2 \right] + \frac{|\bar{z}_1 \partial_3 z^1 + \bar{z}_2 \partial_3 z^2|^2}{|z^1|^2 + |z^2|^2}
\]

\[
+ m_{10}^2 |z^1|^2 + m_{20}^2 |z^2|^2 + m_{12}^2 |z^1|^2 |z^2|^2
\]

\[
= \int dx^3 \frac{r}{1 + |z^1|^2 + |z^2|^2} \left[ |\bar{z}_1 (\partial_3 z^1 - m_{10} z^1) + \bar{z}_2 (\partial_3 z^2 - m_{20} z^2)|^2 \right] + \frac{|\bar{z}_2 (\partial_3 z^1 - m_{10} z^1) - \bar{z}_1 (\partial_3 z^2 - m_{20} z^2)|^2}{|z^1|^2 + |z^2|^2}
\]

\[
+ \left( m_{10} + m_{12} |z^2|^2 \right) \partial_3 |z^1|^2 + \left( m_{20} - m_{12} |z^1|^2 \right) \partial_3 |z^2|^2
\]

\[
\geq \frac{r}{1 + |z^1|^2 + |z^2|^2} \left( m_0 + m_1 |z^1|^2 + m_2 |z^2|^2 \right)^{x^3 = +\infty} = rm_{20}.
\]

It implies that the composite kink saturating the bound has the same mass as the simple (20)-kink solution.
B  Low energy dynamics of kinks

B.1  fermion zero mode counting with aligned masses

We begin by clarifying the number of fermionic zero modes in the simple kink background. Under the \( (20) \)-kink background, one can naturally define inner products of \( \chi^{1,2} \) as

\[
\langle \tilde{\chi}^1 | \chi^1 \rangle = \int d^3 x \frac{1}{1 + e^{2|m_{20}|x^3}} \tilde{\chi}^1 \chi^1 , \\
\langle \tilde{\chi}^2 | \chi^2 \rangle = \int d^3 x \frac{1}{(1 + e^{2|m_{20}|x^3})^2} \tilde{\chi}^2 \chi^2 ,
\]

(B.1)

from which the adjoints of \( \mathcal{D}^{1,2} \) becomes

\[
\langle \mathcal{D}^{(1,2)} | \tilde{\chi}^{1,2} \chi^{1,2} \rangle = \langle \tilde{\chi}^{1,2} | \mathcal{D}^{(1,2)} \chi^{1,2} \rangle .
\]

(B.2)

It will be shown that the redefined fermion fields \( \eta^{1,2} \)

\[
\eta^1 = \frac{1}{\sqrt{1 + e^{2|m_{20}|x^3}}} \chi^1 , \quad \eta^2 = \frac{1}{1 + e^{2|m_{20}|x^3}} \chi^2 ,
\]

(B.3)

are convenient to study their zero-modes in manifest normalizability. Then, one can rewrite the fermion quadratic pieces in the sigma-model Lagrangian as

\[
\langle \chi^{1,2} | \mathcal{D}^{(1,2)} \chi^{1,2} \rangle = \int d^3 x \, \eta^\dagger \mathcal{D}^{(1,2)} \eta^{1,2} .
\]

(B.4)

One finds that the equations of motions for \( \eta^{1,2} \) can be simplified as

\[
\omega \eta^1 \equiv D^{(1)} \eta^1 = \left[ i \tau^3 \partial_3 - \hat{\tau}_{m_{10}} + \hat{\tau}_{m_{20}} \left( \frac{|z|^2}{1 + |z|^2} \right) \right] \eta^1 \\
\omega \eta^2 \equiv D^{(2)} \eta^2 = \left[ i \tau^3 \partial_3 - \hat{\tau}_{m_{20}} \left( 1 - \frac{2|z|^2}{1 + |z|^2} \right) \right] \eta^2 .
\]

(B.5)

Inserting the explicit configuration of the kink solution, the above differential operators can be reduced to

\[
D^{(1)} = i \tau^3 \partial_3 - \hat{\tau}_{m_{10}} + \hat{\tau}_{m_{20}} f(x^3) , \\
D^{(2)} = i \tau^3 \partial_3 - \hat{\tau}_{m_{20}} \left( 1 - 2f(x^3) \right)
\]

(B.6)

with

\[
f(x^3) = \frac{e^{2|m_{20}|x^3}}{1 + e^{2|m_{20}|x^3}} , \quad \partial_3 f = 2|m_{20}|f(1-f) \geq 0 .
\]

(B.7)
Assuming the alignment of phases of $m_{10}$ and $m_{20}$, it is then easy to show that, for $\eta^1$, 

\[
D^{(1)\dagger} D^{(1)} = D^{(1)} D^{(1)\dagger} = \left[-\partial_3^2 + |m_{10}|^2 - 2|m_{10}||m_{20}|f + |m_{20}|^2 f^2 \right]1_4 + i\partial_3 f \tau^3 \tilde{\tau}_{m_{20}} \\
\equiv -\partial_3^2 + V^{(\pm)}(x^3),
\]

where the effective potentials are given by 

\[
V^{(1)\pm} = (|m_{10}| - |m_{20}|f)^2 \pm 2|m_{20}|^2 f(1 - f), \quad i\tau^3 \tilde{\tau}_{m_{20}} \pm \mp|m_{20}|.
\]

By definition, $V^{(1)\pm}_+ \geq V^{(1)\pm}_-$ always. The profile of the effective potentials $V^{(1)\pm}(x^3)$ is depicted in figure B.1 (a), where you can see their extremum and asymptotic values are given by 

\[
V^{(1)\pm}\left|_{\text{min}} = \left(|m_{20}| - |m_{10}|\right)^2 + |m_{10}|^2 \right. , \quad \left. \begin{cases} 
V^{(1)\pm}_+(x^3 \to -\infty) = |m_{10}|^2 \\
V^{(1)\pm}_+(x^3 \to +\infty) = (|m_{20}| - |m_{10}|)^2
\end{cases} \right. \quad (B.10)
\]

\[
V^{(1)\pm}\left|_{\text{max}} = -\frac{2}{3}|m_{20}||m_{20}| - |m_{10}| - \frac{1}{3}|m_{20}|^2 \right. , \quad \left. \begin{cases} 
V^{(1)\pm}_+(x^3 \to -\infty) = |m_{10}|^2 \\
V^{(1)\pm}_+(x^3 \to +\infty) = (|m_{20}| - |m_{10}|)^2
\end{cases} \right.
\]

from which one can show that $D^{(1)\dagger} D^{(1)}$, $D^{(1)\dagger} D^{(1)}$ with $i\tau^3 \tilde{\tau}_{m_{20}} = |m_{20}|$ becomes manifestly positive definite. It implies that there is no normalizable zero-modes for the above chirality. For another chirality $i\tau^3 \tilde{\tau}_{m_{20}} = -|m_{20}|$, one can have a normalizable zero-mode $n^{(1)}_0$ 

\[
n^{(1)}_0 = e^{|m_{10}|x^3} \left(1 + e^{2|m_{20}|x^3}\epsilon_0 \right) , \quad i\tau^3 \tilde{\tau}_{m_{20}} \epsilon_0 = -|m_{20}| \epsilon_0 \Rightarrow \lambda^{(1)}_0 = e^{|m_{10}|x^3} ,
\]

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provided that \(|m_{20}| \geq |m_{10}|\).

Let us now in turn consider the Dirac operator for \(\eta^2\). One can again easily show that

\[
D^{(2)\dagger} D^{(2)} = D^{(2)} D^{(2)\dagger} = \left[ -\partial_3^2 + |m_{20}|^2 (1 - 2f)^2 \right] 14 + \left[ 2|m_{20}|f(1 - f) \right] i\tau^3 \hat{\tau}_{m_{20}}
\]

\[
= \begin{cases} 
-\partial_3^2 + |m_{20}|^2 & \text{for } i\tau^3 \hat{\tau}_{m_{20}} = +|m_{20}| \\
-\partial_3^2 + |m_{20}|^2 (1 - 8f(1 - f)) & \text{for } i\tau^3 \hat{\tau}_{m_{20}} = -|m_{20}|
\end{cases}
\]

\[
= \begin{cases} 
-\partial_3^2 + |m_{20}|^2 & \geq 0 \\
-\partial_3^2 + V^{(2)}(x^3) &
\end{cases}
\]

(B.12)

which implies that there is no normalizable zero-modes for the former chirality \(i\tau^3 \hat{\tau}_{m_{20}} = |m_{20}|\). On the other hand, the effective potential \(V^{(2)}(x^3)\), depicted in figure B.1 (b), has its minimum and asymptotic values like

\[
V^{(2)}_{\text{min}} = -|m_{20}|^2, \quad V^{(2)} \rightarrow |m_{20}|^2 \text{ as } x^3 \rightarrow \pm \infty,
\]

(B.13)

from which one can expect a normalizable zero-mode \(\eta^2_0\) of chirality \(i\tau^3 \hat{\tau}_{m_{20}} = -|m_{20}|\) whose the explicit expression becomes

\[
\eta^2_0 = \frac{1}{\cosh \left[ |m_{20}| x^3 \right]} \Rightarrow \chi^2_0 = e^{\frac{|m_{20}|}{x^3}} \epsilon_0.
\]

(B.14)

\[\text{B.2 \ the two-kink moduli space metric}\]

As discussed in literatures, a general kink can decompose into several fundamental kinks. Each of fundamental kink has two obvious collective coordinates, position and phase. It implies that the moduli space of kinks is therefore toric Kähler manifold.

For computational simplicity and concreteness, let us consider the present model with \(m_{20} = 2m_{10} = 2m\). From (4.3), the metric components can read

\[
g_{11} = \left| \frac{\zeta^2}{\zeta^1} \right|^2 \frac{r}{4m} F \left( |\zeta^1|^4 / |\zeta^2|^2 \right)
\]

\[
g_{22} = \frac{r}{4m} \left| \frac{\zeta^1}{\zeta^2} \right|^2 + \left| \frac{\zeta^1}{\zeta^2} \right|^6 \left[ \frac{r}{4m} F \left( |\zeta^1|^4 / |\zeta^2|^2 \right) \right]
\]

\[
g_{12} = -2 \frac{\bar{\zeta} \zeta^2}{|\zeta^1|^6} \left[ \frac{r}{4m} F \left( |\zeta^1|^4 / |\zeta^2|^2 \right) \right],
\]

(B.15)
where \( F(x) \) is defined in Eqs. (4.4,4.5). Bosonic kinetic terms of interacting multi-kinks therefore take the following form

\[
L_{\text{kin}}^{\text{boson}} = L_{\text{com}} + L_{\text{rel}},
\]

where

\[
L_{\text{com}} = \frac{r}{4m} \left| d\log \zeta \right|^2, \quad L_{\text{rel}} = \frac{r}{4m} F\left( |\zeta_1|^4/|\zeta_2|^2 \right) \left| d\frac{\zeta_2}{\zeta_1^2} \right|^2.
\] (B.16)

In the limit of \( |\zeta_2|/|\zeta_1| \to \infty \), \( L_{\text{rel}} \) is asymptotic to

\[
L_{\text{rel}} \simeq \frac{r}{4m} \cdot \frac{\pi}{4} \left| d\frac{\zeta_1}{\sqrt{\zeta_2}} \right|^2.
\] (B.17)

Note that the moduli space metric of interacting two-kinks (or multi-kinks in four-dimensional \( \mathcal{N} = 2 \) SQED) has been explored by David Tong [38], although our result appears slightly different from his.

## B.3 supersymmetric low energy dynamics with potential

For completeness, we present in this section a short review on supersymmetric non-linear sigma-model quantum mechanics with potential. Let us begin by the Lagrangian which takes the following form

\[
L_{\text{kin}} = \frac{1}{2} g_{IJ} \left[ \partial_0 \Phi^I \partial_0 \Phi^J + i \Psi^I D_0 \Psi^J \right],
\] (B.18)

where the covariant derivatives are

\[
D_0 \Psi^I = \partial_0 \Psi^I + \partial_0 \Phi^K \Gamma_{JK}^I \Phi^K.
\] (B.19)

and the fermions are real. Since the kink solitons possess equal number of bosonic and fermionic collective coordinate, this quantum mechanics is appropriate for the

The above Lagrangian has a real supersymmetry whose Nöther charge is given by

\[
Q = i \sqrt{2} g_{IJ} \partial_0 \Phi^I \partial_0 \Phi^J.
\] (B.20)

Once we quantize the system, the real fermion fields \( \Psi^I \) cab be represented as gamma matrices \( \Gamma^I \)

\[
\{ \Psi^I, \Psi^J \} = \delta^{IJ} \rightarrow \Psi^I = \frac{1}{\sqrt{2}} \Gamma^I.
\] (B.21)
It implies that the supercharge can be represented on the Hilbert space as the spinorial Dirac operator

\[ Q = i \gamma^I \nabla_I = i \gamma^I \left( \partial_I + \frac{1}{4} \omega_{IAB} \Gamma^{AB} \right). \tag{B.22} \]

When the geometry has a restricted holonomy, the supersymmetry is enhanced. In particular, for a Kähler space such as our multi-kink moduli space, the supersymmetry is enhanced to \( \mathcal{N} = 2 \).

One may introduce to the above model a supersymmetry-preserving deformation of the form

\[ L_{\text{def}} = -\frac{1}{2} \left[ g_{IJ} G^I G^J + i \nabla_I G_J \Psi^I \Psi^J \right]. \tag{B.23} \]

One can show that the total Lagrangian \( L = L_{\text{kin}} + L_{\text{def}} \) is invariant under a supersymmetry whose Nöther charge is deformed as

\[ Q = \sqrt{2} \Psi^I \left[ ig_{IJ} \dot{\Phi}^J + G_I \right]. \tag{B.24} \]

After canonical quantization, demanding the Jacobi identity for the deformed supercharge tells us that \( G^I \) in fact turns out to be a Killing vector field

\[ [Q, \{Q, Q\}] = 0 \rightarrow \nabla_I G_J + \nabla_J G_I = 0. \tag{B.25} \]

When the manifold is Kähler with the complex structure \( J \), \( \mathcal{N} = 2 \) supersymmetry remain consistent with introduction of \( G \) provided that \( G \) is not only Killing but also holomorphic,

\[ L_{GJ} = 0. \tag{B.26} \]

One can split \( \{Q, Q\} \) into two conserved quantities as

\[ \{Q, Q\} = 4(\mathcal{H} - \mathcal{Z}) \tag{B.27} \]

where \( \mathcal{H} \) and \( \mathcal{Z} \) denote Hamiltonian and central charge

\[ \mathcal{H} = \frac{1}{2} g_{IJ} \left[ \partial_0 \Phi^I \partial_0 \Phi^J + G^I G^J \right] + \frac{i}{2} \nabla_I G_J \Psi^I \Psi^J, \]

\[ \mathcal{Z} = G_I \partial_0 \Phi^I - \frac{i}{2} \nabla_I G_J \Psi^I \Psi^J. \tag{B.28} \]

Note here that the positive energy BPS states of real supersymmetry then preserve all the supercharges of the moduli space dynamics. As a final comment, the deformed supercharge now in turn can be represented as

\[ Q \doteq \Gamma^I (i \nabla_I + G_I). \tag{B.29} \]

since we may view the wavefunctions as sections of the spinor bundle over the moduli space.
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