A multi-level data-driven finite element method in polymorphic uncertainty quantification

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This contribution explores the ability to conduct polymorphic uncertainty computations within the data-driven finite element framework. The constitutive equation is replaced by data sets that determine the material behavior, and a fuzzy load is used for the polymorphic approach. To conduct many simulations with the discretized variables and to increase numerical efficiency, the numerical cost is reduced by a multi-level method.

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1 Introduction

The data-driven finite element method is a new approach to boundary value problems where the constitutive equations are replaced by data sets. These point sets are directly taken from the experiment thus avoiding empirical simplifications and parameter fitting altogether.

The initial idea was introduced by Kirchdoerfer and Ortiz [1] and has been extended in several publications since, see [2] and reference therein. The data-driven framework applies to boundary value problems from all fields, see e.g. [4], but most applications focus on mechanics and the constitutive behavior of the solid material. Then the data sets consist of strain-stress pairs, \((\epsilon, \sigma)\), measured in typical tension or shear tests.

Typically, in polymorphic uncertainty approaches stochastic and fuzzy variables are used together. For a full polymorphic simulation both variables are discretized and many single simulations are performed with the discretized values. Since a nested loop runs over the individual values, this is often very costly and time-consuming. The scheme in Fig. 1 shows the input, the differences between the two variables, and the output of a typical polymorphic approach. Due to the complexity, sometimes only parameters of the stochastic and fuzzy distributions can be calculated, such as the mean and the variance of the stochastic distribution and the belief function (lower bound), and the plausibility function (upper bound) of the fuzzy variable.

Fig. 1: The input variables are treated differently inside of a polymorphic computation depending on the type of uncertainty

Fuzzy and stochastic models often have the same problem as constitutive models - the model and its parameters need to be chosen and fitted. In contrast to material modeling, there is usually significantly less information available on the variability of the parameter. Stochastic distributions and fuzzy models have to be determined before the simulations, and so mostly elementary assumptions, like a normal distribution or a triangular fuzzy number of the uncertain quantity, are made. The data-driven framework enables us to implement all knowledge about the material which is at hand. Therefore we want to

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Fig. 2: Computation of a clamped plate with the data-driven finite element method using $30^3$ (left) and $150^3$ (right) data points

exemplarily combine the material uncertainty described by the data-driven approach with a load uncertainty represented by a fuzzy variable.

2 Governing equations

The aim of the data-driven finite element method is to find the data out of a set $\mathcal{D}$ that are closest to the real state. On every material point of the finite element mesh, a $(\epsilon, \sigma)$ pair is assigned describing the material state there. We define a distance function

$$
\Psi(\epsilon, \sigma) = \min_{(\epsilon', \sigma') \in \mathcal{D}} \left( \Psi^e(\epsilon - \epsilon') + \Psi^\sigma(\sigma - \sigma') \right)
$$

(1)

which measures the distance of a strain-stress state $(\epsilon, \sigma)$ to the data points $(\epsilon', \sigma') \in \mathcal{D}$ of the data set. The functions which weight the strains and stresses are the elastic energy densities

$$
\Psi^e(\epsilon) = \frac{1}{2} \epsilon : C^e : \epsilon \\
\Psi^\sigma(\sigma) = \frac{1}{2} \sigma : D^\sigma : \sigma
$$

(2)

with $C^e$ and $D^\sigma$ being numerical weights that calibrate units and magnitude of the strains and stresses. Here we set $D^\sigma = (C^e)^{-1}$ in matrix notation. The aim of the data-driven boundary value problem is then to minimize the global penalty function

$$
W = \int_\Omega \Psi(\epsilon, \sigma) \, d\Omega
$$

(3)

with respect to the kinematic strain-displacement relation and the static balance equation with body force $b$,

$$
\epsilon(u) = \frac{1}{2} \left( \nabla u + \nabla^T u \right), \quad 0 = \text{div}(\sigma) + b.
$$

(4)

Enforcing the balance equation by Lagrange parameters $\lambda$ and using a standard finite element approximation with ansatz functions $N_i$ for the displacement $u = \sum_i N_i \hat{u}_i$ and the lagrangian field $\lambda = \sum_i N_i \hat{\lambda}_i$ we get two finite element systems to solve

$$
K \hat{u} = \int_\Omega B^T C^e \epsilon^* \, d\Omega \\
K \hat{\lambda} = f - \int_\Omega B^T \sigma^* \, d\Omega
$$

(5)

where $B$ is the matrix of differentials, $K = \bigcup_e K_e$ is the element stiffness matrix, with $K_e = \int_{\Omega_e} B^{eT} C_e B^e \, d\Omega$, and the values $\epsilon^*, \sigma^*$ denote the (still unknown) optimal data pairs of set $\mathcal{D}$ at all material points. For a more detailed derivation of the system we refer to [2]. The optimal data pairs are found in an alternating projection algorithm. Starting with a random initialization, the prior systems of equations can be solved. Then, with the displacement and the lagrangian field, new trial points are computed

$$
\epsilon^{(j)} = B \hat{u}^{(j)} \\
\sigma^{(j)} = \sigma^{*(j)} + C^\sigma B \lambda^{*(j)}.
$$

(6)

The data points of the next iteration are the closest to the trial points in the phase space. In Fig. 2 we exemplary show two data-driven computations of a clamped plate in a plane-stress state, where the effect of a different data set can be seen.

3 Multi-level approach

Polymorphic uncertainty computations are rather costly due to the two nested loops of the discretization of the stochastic and the fuzzy variable. The data-driven finite element method adds additional computational effort because the data-driven approach itself is more costly than a typical finite element computation. Another system of equations has to be solved, and the
search for the closest data points needs to be performed in every step of the solution. Therefore, we use a numerical speed-up method introduced in [3]. Instead of the whole data set, we only use a subset covering the phase space as good as possible. In a first data-driven simulation, data points are assigned which are closest to the actual states. Around these assigned points, the set is expanded by data points from the original data set. The next data-driven simulation has a locally refined set $D$ and is more accurate, cf. [3].

Fig. 3: Illustration of the multi-level approach for two elements

The procedure is illustrated in Fig. 3: In the first computation, the black data set is used. After the simulation, two data points are found (black crosses), and the next data set is expanded around those points (green). In the next iteration step, two data points are found (green crosses), and the red data is the input for the next iteration. This refinement can be repeated as often as needed to achieve the wanted accuracy.

4 Numerical example

Our computational example is a clamped plate of length $l$ and height $h = l/4$, which is loaded from atop by a distributed load $p_0$. The simulation presumes plane-stress conditions. In order to conduct a polymorphic uncertainty computation a triangle fuzzy variable is used for $p_0$. Further variables and the geometry of the cantilever beam-like structure are displayed in the table below.

| Parameter      | Value          |
|----------------|----------------|
| length $l$     | 2m             |
| height $h$     | 0.5m           |
| width $b$      | 0.01m          |
| load $p_0$     | $\Delta$-Fuzzy [58, 63, 68] Pa |
| Young’s modulus $E$ | 25 data sets sampled with mean $3 \cdot 10^6$ Pa |
| data points per set | $75^3$ |
| Poisson’s ratio $\nu$ | 0.3 |
| mesh size      | $25 \times 10 \times 4$ |
| int. points per elem. | 8 |

The data-driven approach opens the possibility to account for uncertain material parameters directly in the data. We assume here that tensile tests have been performed to characterize the material. In order to validate measurements, several (virtual) tests are carried out, and their results are used as input for the data-driven method.

The ability to include the randomness, fluctuations, and variation of the material behavior without a model can be very advantageous because it skips all additional assumptions. The determination of stochastic or other uncertainty models is even more capricious than fitting a constitutive law. By direct use of the data, everything which is known is implemented into the simulation. Nonetheless, a data-driven simulation only provides a deterministic simulation, here with 25 data sets randomly assigned to the finite elements.

For comparison, in a stochastic finite element method, the material’s elasticity follows a stochastic field. Such stochastic fields are typically used to model uncertainties in field quantities with a broad range of specific applications. Due to the
difficulties of justifying a particular choice of the stochastic field, typically, a normal distribution $\mathcal{N}$ is assumed. Another, more straightforward approach to material uncertainty sets the elastic modulus of the structure as a random variable. In that sense we compare three different approaches here:

1. a fuzzy-data-driven approach (fuzzy-DD) where the load is fuzzy and the material is described by 25 data sets; the $\epsilon - \sigma$ data are generated and influenced by a normal random variable with mean $E$ and variance $(0.1E)^2$.

2. a fuzzy-stochastic finite element method approach (fuzzy-SFEM) where the load is fuzzy and the Young’s modulus is a stochastic field, $E(x) \sim \mathcal{N}_E(0,1E)^2$, i.e., in every finite element $E$ is a realization of this field.

3. a fuzzy-stochastic approach (fuzzy-stoch.) where the load is fuzzy and the Young’s modulus is a stochastic variable, i.e., the whole structure has a modulus $E \sim \mathcal{N}_E(0,1E)^2$.

In Fig. 4, the results of our simulations are shown. Displayed are the distribution functions of the displacement $u(0, h/2)$ for 1000 samples combined with $\alpha$ cuts at presumption levels $\{0, 0.2, 0.5, 1\}$. The linear kinematic does not modify the uncertainty propagation through the model; the triangular profile remains. Because the fuzzy-stoch. approach is the only one that afflicts the whole plate by one variable, it differs from the results of the other two approaches. The fuzzy-DD and the fuzzy-SFEM coincide nearly, which shows their similarity. Generally speaking, the data-driven method converges against the classical solution of the boundary value problem for increasing and reasonable data points. The data points of the data-driven sets are “measured” points from the random field and, therefore, the results shown here are similar for both an ad-hoc stochastic field assumption with classical FEM and the data-driven FEM with varying data sets.

5 Conclusion

Here we demonstrate that data-supported polymorphic uncertainty computations are feasible. Therefore, we used the data-driven finite element method in its simplest form. Data-driven methods can be crucial if only low information is at hand.

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