Partial Dictionary Based Off-Grid DOA Estimation Using Combined Coprime and Nested Array

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A partial dictionary based direction of arrival (DOA) estimation method which addresses the off-grid problem and exploits combined coprime and nested array (CCNA) is proposed. Compared to general coprime array, CCNA yields two sparse coprime subarrays in the coarray domain by adding a third subarray in the physical-array domain. To ensure the DOA estimation performance, the subarray with larger aperture is chosen, and the cyclic phase ambiguity caused by the sparse subarray allows partial dictionary covering arbitrary cycle to represent the whole atoms, and then, the off-grid sparse reconstruction method is developed to amend the grid mismatch. After the sparse recovery and off-grid compensation, ambiguous DOA estimations can be eliminated by substituting the estimations into the whole virtual array. Multiple simulations verify that the proposed algorithm outperforms the other state-of-the-art methods in terms of DOA estimation accuracy and angular resolution.

1. Introduction

Direction of arrival (DOA) estimation using antenna array is an important issue in many systems, e.g., radar, sonar, and wireless communication [1–5]. Compared to conventional subspace based methods, sparse representation based DOA estimation methods have been attractive since they can provide higher resolution and require fewer samples [6], and many effective sparse representation based methods have been proposed. The greedy methods [7, 8] require the prior information of source number and are sensitive to the noise, and the $l_1$-norm based algorithms, such as the $l_1$-norm singular value decomposition ($l_1$-SVD) method [9], sparse recovery using weighted subspace fitting (SRWSF) method [10], sparse representation of array covariance vector (SRACV) method [11], and sparse iterative covariance-based estimation (SPICE) method [12], can reduce the sensitivity to noise and estimate the angles via convex optimization. However, these methods discretize the whole spatial range into a grid, which will result in performance degradation when the sources are not exactly located on the grid, i.e., the grid mismatch problem [13]. In [14], the off-grid sources were considered and estimated by introducing grid offsets in the sparse Bayesian inference (SBI). Based on the joint sparsity between original signal and the grid mismatch variables, joint sparse recovery method was proposed in [15]. Meanwhile, the grid-less methods are developed to directly recover the covariance matrix based on atomic norm or nuclear norm minimization [16, 17]. However, these methods only concentrate on the physical array model, which has limited degree of freedom (DOF).

Sparse array design has been developed to increase the virtual DOF in the difference coarray domain. Nested array was proposed in [18], which can generate $O(N^2)$ DOF in the difference coarray domain with only $O(N)$ physical antennas [19, 20]. Nested array can also be applied in radar system to increase the virtual DOF and enhance the spatial resolution [21]. However, the nested array has a dense subarray, which suffers from the mutual coupling problem. Coprime array [22], another well-known sparse array, was proposed to reduce the mutual coupling influence. With $O(M + N)$ sparsely spaced physical antennas, coprime array can achieve $O(MN)$ DOF [23], which is generally
nonuniform, and the coarray has more holes compared to that of nested array. Therefore, many works are developed to modify the coprime array to generate more continuous virtual elements in the coarray domain, such as the augmented coprime array (ACA) [24], generalized coprime array (GCA) [25], and thinned coprime array (TCA) [26]. To deal with the one-snapshot situation in the coarray domain, sparse representation based methods have also been introduced for sparse array [27–29]. For off-grid sources, a joint reconstruction method named joint LASSO (JLASSO) was proposed in [30], which can exploit the large DOF in the coarray domain of coprime array and amend the grid mismatch via joint sparse recovery. However, the computation complexity is very high due to the dictionary covering whole spatial range.

There is also another way to utilize coprime array, i.e., the separate processing of the two subarrays of coprime array, and the unique estimation is determined from the coincident results from the two subarrays, such as the combined multiple signal classification (MUSIC) method [31], partial search MUSIC method [32], root MUSIC method [33], and combined estimation of signal parameters via rotational invariance technique (ESPRIT) based method [34]. However, as the subarrays are processed separately in the physical-array domain, the exploited DOF is limited, and an alternative way is to transform the coprime relationship into the coarray domain [35]. A combined coprime and nested array (CCNA) geometry, which is obtained by adding a third subarray nested to both of the two subarrays, was proposed in [36], where the coprime subarrays are transformed into the coarray domain to achieve large aperture and DOF. However, the utilized MUSIC method results in aperture loss.

In this paper, we propose an off-grid DOA estimation method, which requires only partial dictionary based on CCNA. Due to the nested relationship within the subarrays, the two virtual subarrays are still sparse but uniform after the vectorization of the covariance matrices. Different from conventional schemes, we adopt the subarray with larger aperture for DOA estimation to avoid the negative effect brought by the smaller subarray. The sparsity of the virtual array enables partial dictionary covering partial spatial range to represent the whole-range atoms. Meanwhile, the off-grid sparse reconstruction method is developed to amend the grid mismatch. Finally, ambiguous DOA estimations can be eliminated based on coprime-ness by substituting the estimations into the whole virtual array. Numerical simulations show that the proposed algorithm outperforms the ACA method [24], partial search (PS) MUSIC [32], root MUSIC method [33], and CCNA with root MUSIC [36] in terms of estimation accuracy and angular resolution.

**Notation:** $(\cdot)^T$, $(\cdot)^*$, $(\cdot)^H$, and $(\cdot)^\dagger$ denote transposition, conjugation, conjugate-transposition, and pseudo-inversion, respectively. $E[\cdot]$ and vec($\cdot$) denote the operations of expectation and vectorization, respectively. diag($\bf{a}$) is a diagonal matrix with vector $\bf{a}$ being the diagonal elements, and $I_p$ is a $p \times p$ identity matrix. $\| \cdot \|_2$ means $L_2$ norm, and $\angle(a)$ means the phase of $a$. $\otimes$, $\circ$, and $\otimes$ denote kronecker product, Khatri_rao product, and element-wise division, respectively.

**2. Data Model**

Figure 1 shows the structure of CCNA, which is composed of three subarrays. Subarray 1 and subarray 2 form the original coprime array, where $M$ and $N$ are coprime integers. The third subarray is arranged along the negative side with $L$ elements and interelement spacing being $MNd$, where $d$ is the unit spacing, which is generally set as half-wavelength. It is also indicated that subarray 1 and subarray 3 form a nested array with the minimum interelement spacing being $Nd$, and subarray 1 and subarray 3 form another nested array with the minimum interelement spacing being $Md$. The total antenna number of CCNA is $M + N + L - 2$ as the subarrays share the same element in the origin.

Assume that there are $K$ plane waves impinging upon the array with DOAs being $\theta_k$, $k = 1, \cdots, K$, which is angle between the wave line and $Z$ axis. Then, the outputs of the subarrays are expressed as

$$
\begin{align*}
\bf{x}_1(t) &= \bf{A}_1 \bf{s}(t) + \bf{n}_1(t), \\
\bf{x}_2(t) &= \bf{A}_2 \bf{s}(t) + \bf{n}_2(t), \\
\bf{x}_3(t) &= \bf{A}_3 \bf{s}(t) + \bf{n}_3(t),
\end{align*}
$$

where $\bf{s}(t) = [s_1(t), \cdots, s_K(t)]^T \in \mathbb{C}^{K \times 1}$ is the signal vector. $\bf{n}_1(t)$, $\bf{n}_2(t)$, and $\bf{n}_3(t)$ are the additive white Gaussian noise (AWGN) vectors with the same noise power $\sigma^2$. $\bf{A}_1 = [\bf{a}_1(\theta_1), \cdots, \bf{a}_1(\theta_K)]$, $\bf{A}_2 = [\bf{a}_2(\theta_1), \cdots, \bf{a}_2(\theta_K)]$, and $\bf{A}_3 = [\bf{a}_3(\theta_1), \cdots, \bf{a}_3(\theta_K)]$ denote the direction matrices of subarray 1, subarray 2, and subarray 3, respectively. The columns are the corresponding steering vectors, which are expressed as

$$
\begin{align*}
\bf{a}_1(\theta_k) &= [1, e^{-j\pi \sin \theta_1}, \cdots, e^{-j(M-1)\pi \sin \theta_1}]^T, \ k = 1, \cdots, K, \\
\bf{a}_2(\theta_k) &= [1, e^{-jM\pi \sin \theta_1}, \cdots, e^{-j(N-1)M\pi \sin \theta_1}]^T, \ k = 1, \cdots, K, \\
\bf{a}_3(\theta_k) &= [1, e^{-jM\pi \sin \theta_1}, \cdots, e^{j(L-1)M\pi \sin \theta_1}]^T, \ k = 1, \cdots, K.
\end{align*}
$$
3. Partial Dictionary Based Off-Grid DOA Estimation Method

3.1. Sparse Representation Using Partial Dictionary. Combine the outputs of subarray 1 and subarray 3 to form the first nested array \( y_1(t) = [x_1^T(t), x_3^T(t)]^T \), whose covariance matrix is

\[
R_1 = E[y_1(t)y_1^H(t)] = A_{n1}R_1A_{n1}^H + \sigma^2I_{M+L-1},
\]

where \( A_{n1}(t) = [A_1^T(t), A_3^T(t)]^T \) is the combined direction matrix and \( R_1 = E[s(t)s^H(t)] = \text{diag}(\sigma_1^2, \cdots, \sigma_K^2) \) is a diagonal matrix containing signal powers. To obtain the virtual whole spatial range as a grid loss caused by the spatial smoothing \([24]\). Discretize the ray 2 and subarray 3, which can be expressed as

\[
\text{Obtain another virtual array from the overall output of subarray 2 and subarray 3, which can be expressed as}
\]

\[
r_2 = A_{n2}p + \sigma^2e_s,
\]

where \( e_s \) denotes a column vector after the same selecting operation and \( A_{n2} = [a_{n2}(\theta_1), a_{n2}(\theta_2), \cdots, a_{n2}(\theta_K)] \) is the direction matrix corresponding to an \((2LN-1)\)-element array located from \(-(LM-1)Md\) to \((LM-1)Md\) with interelement spacing being \(Md\). The steering vector is

\[
a_{n2}(\theta_k) = \left[ e^{-j(\pi\sin \theta_k)}e^{-j(\pi\sin \theta_k)} \cdots e^{-j(\pi\sin \theta_k)}1, e^{-j(\pi\sin \theta_k)}e^{-j(\pi\sin \theta_k)} \cdots e^{-j(\pi\sin \theta_k)} \right]^T, k = 1, \cdots, K.
\]

Now, the two virtual coprime subarrays in Eq. (5) and Eq. (7) are obtained, and the large interelement spacing will result in parameter estimation ambiguity problem. However, our method will in turn exploit the phase ambiguity to reduce the complexity and then eliminate the ambiguity based on the coprime-ness between \(M\) and \(N\). Suppose \(M > N\), then the first subarray in Eq. (5) achieves larger aperture than that in Eq. (7), so we choose \(r_1\) to estimate the DOA for better estimation performance.

As the virtual output has only one snapshot, sparse representation framework will be established to avoid the aperture loss caused by the spatial smoothing \([24]\). Discretize the whole spatial range as a grid \(\theta_1, \theta_2, \cdots, \theta_p\) \((P \gg K)\), and suppose that all the true DOAs fall in the grid, i.e., the dictionary \(\Omega = [a_1(\theta_1), \cdots, a_1(\theta_p), e_s] \) contains the columns of \(A_{n1}\), then Eq. (5) can be rewritten in a sparse form as

\[
r_1 = \Omega p,
\]

where \( p \in \mathbb{C}^{(P+1)\times 1} \) is a sparse vector, whose elements corresponding to the true DOAs keep the same with those in \( p \) and last element is noise power \(\sigma^2\). After sparse recovery, the positions of nonzero elements (except the last element) in \( \Omega \) will give the estimations of the DOAs. However, as the interelement spacing of the virtual array is \(Nd\), which is larger than half-wavelength, then there are phase ambiguities in \(\Omega\). To clearly elaborate this problem, let \( z_p = e^{-j\pi\sin \tilde{\theta}_p} \); then, \( z_p \) determines the uniqueness of \(a_{n1}(\tilde{\theta}_p)\) due to the Vandermonde structure, i.e., if \( z_p = z_{p_0}\) then \(a_{n1}(\tilde{\theta}_p) = a_{n1}(\tilde{\theta}_{p_0})\).

As \(N > 1\), there is a cyclic phase ambiguity in \(z_p\). Except for \(\tilde{\theta}_p\), there are other \((N-1)\) angles \(\tilde{\theta}_{p,n}, n = 2, \cdots, N\) satisfying

\[
e^{-j\pi\sin \tilde{\theta}_{p,n}} = z_p, n = 2, \cdots, N.
\]

It can be derived from Eq. (10) that the relationship between \(\tilde{\theta}_p\) and \(\tilde{\theta}_{p,n}, n = 2, \cdots, N\) is

\[
r_1 = \Omega(p) = (A_{n1}^* \circ A_{n1})p + \sigma^2 \vec{e}(I_{M+L-1}),
\]

where \( p = [\sigma_1^2, \cdots, \sigma_K^2]^T \). Due to the nested relationship, there are \(2LM - 1\) continuous elements located from \(-(LM-1)Nd\) to \((LM-1)Nd\) in the virtual array with interelement spacing being \(Nd\) \([36]\). After selecting continuous elements from \(r_1\), then, we obtain

\[
r_{n1} = W_{n1}r_1 = A_{n1}p + \sigma^2e_s,
\]
\[
\sin \tilde{\theta}_{p,n} = \sin \tilde{\theta}_p - \frac{2m}{N}, \quad n = 2, \cdots, N,
\]

where \( m \) is an integer making \( \sin \tilde{\theta}_{p,n} \) locate at the range \([-1, 1] \). If angles \( \tilde{\theta}_{p,n}, n = 2, \cdots, N \) are also located in the grid \( \tilde{\theta}_1, \tilde{\theta}_2, \cdots, \tilde{\theta}_p \), then \( \tilde{\theta}_p \) and \( \tilde{\theta}_{p,n}, n = 2, \cdots N \) will provide \( N \) identical columns in the dictionary matrix \( \Omega \) due to Eq. (10). This will not only cause the estimation ambiguity but may also make the sparse recovery fail. However, we can in turn exploit the ambiguity to reduce the size of the dictionary and then reduce the complexity of sparse recovery accordingly.

From Eq. (11), it is shown that the \( N \) solutions \( \sin \tilde{\theta}_p \) and \( \sin \tilde{\theta}_{p,n}, n = 2, \cdots, N \) are uniformly distributed among the range \([-1, 1] \) following a circle \( 2/N \). An example is shown in Figure 2, where \( \sin \tilde{\theta}_p = 0.5 \) and \( N = 3 \), then other two solutions are \( \sin \tilde{\theta}_{p,2} = -1/6 \) and \( \sin \tilde{\theta}_{p,3} = -5/6 \), respectively. It is also shown by the dashed lines in Figure 2 that if we divide the whole range into \( N \) cycles with width being \( 2/N \), then there is only one solution in one cycle based on Eq. (11). As these solutions provide identical atoms in the dictionary, we can choose one cycle as a representative to construct the dictionary. Without loss of generality, we choose range \([-1/N, 1/N] \), and the angle range is \([\arcsin(-1/N), \arcsin(1/N)] \), whose corresponding dictionary is denoted by \( \Omega_{\text{sub}} \), then, the sparse form in Eq. (9) becomes

\[
r_{i1} = \Omega_{\text{sub}} \rho,
\]

where \( \rho \in \mathbb{C}^{(P+1) \times 1} \) is a \( K \)-sparse vector, whose elements corresponding to the true steering vectors (maybe not the true DOAs) keep the same with those in \( \rho \), and the others are zero (except the last element). For example, as shown in Figure 2, the true solution is 0.5, and the representative solution in the partial dictionary \( \Omega_{\text{sub}} \) is -1/6.

Due to Eq. (12), the phase ambiguity in the dictionary can be avoided now. Meanwhile, as now the size of the dictionary \( \Omega_{\text{sub}} \) is only \( 1/N \) of its original size, the computation complexity of sparse recovery can be reduced.

### 3.2. Off-Grid Sparse Representation Framework

Now a partial dictionary based sparse representation framework is established, but it is built based on the assumption that the true DOAs or their representative angles are located in the grid. However, the angles are very likely to lie off the discretized grid, no matter how fine the grid is defined. Off-grid sources will bring in grid mismatch problem and degrade the sparse recovery performance significantly. In this section, we take the off-grid problem into account and reformulate the sparse representation to enhance the robustness to grid mismatch.

Within the range \([\arcsin(-1/N), \arcsin(1/N)] \), we denote the uniformly sampled grid as \( \tilde{\theta}_1, \tilde{\theta}_2, \cdots, \tilde{\theta}_N \) with adjacent interval being \( \theta \). Then, the true DOA or its representative angle \( \tilde{\theta}_k \) can be represented by a nearest grid \( \tilde{\theta}_{q,k} \) plus an offset \( \alpha_k \), which is among the range \([-g/2, g/2] \). Based on first order Taylor expansion around the grid [14], the true steering vector can be approximately expressed as

\[
\sin \tilde{\theta}_{p,n} = \sin \tilde{\theta}_p - \frac{2m}{N} + \frac{2 \alpha_k}{\pi} \frac{2m}{N} + \frac{2 \alpha_k}{\pi} \frac{2m}{N} \cdots + \frac{2 \alpha_k}{\pi} \frac{2m}{N}, \quad n = 2, \cdots, N,
\]

where \( \alpha_k = \tilde{\theta}_k - \tilde{\theta}_{q,k} \). Then, Eq. (12) is revised as

\[
r_{i1} = (\Omega_{\text{sub}} + \Omega'_{\text{sub}} \Lambda) \rho,
\]

where \( \Omega'_{\text{sub}} = [\partial a_{i1}(\tilde{\theta}_1)/\partial \theta_1, \cdots, \partial a_{i1}(\tilde{\theta}_Q)/\partial \theta_Q, e_1] \), \( \Lambda = \text{diag}(\beta) \), and

\[
\beta(q) = \begin{cases} 
\alpha_k, & \text{if } \tilde{\theta}_{q,k} = \tilde{\theta}_q, \quad k = 1, \cdots, K, \quad q = 1, \cdots, Q. \\
0, & \text{others }
\end{cases}
\]

Let \( \omega = \Lambda \rho \), then it is easy to verify that \( \omega \) and \( \rho \) are joint sparse [30]. So the off-grid sparse formulation can be expressed as

\[
\min ||h||_{2,1} \\
\text{s.t. } r_{i1} = \Omega_{\text{sub}} \rho + \Omega'_{\text{sub}} \omega \\
-\frac{1}{2} \rho \leq \omega \leq \frac{1}{2} \rho,
\]

where \( h = [\rho^T, \omega^T]^T \), and \( ||h||_{2,1} = \sum_{i=1}^{Q} \sqrt{\rho_i^2 + \omega_i^2} \), where \( \rho_i \) means the \( i \)-th element of \( \rho \). It should be noted that the covariance matrix in Eq. (3) can only be estimated via finite snapshots

\[
\tilde{R}_t = \left( \frac{1}{T} \right) \sum_{t=1}^{T} y_1(t) y_1^H(t),
\]

where \( T \) denotes the snapshot number. So the sparse form in Eq. (16) is not robust due to the residual error. Use \( \Delta r_1 \) to denote the deviation of \( r_1 \) in Eq. (6), then according to [37], \( \Delta r_1 \) follows asymptotic normal distribution with zero mean and covariance matrix being \( 1/T (R_t^2 \otimes R_t) \)

\[
\Delta r_1 \sim \text{AsN} \left( 0, \frac{1}{T} (R_t^2 \otimes R_t) \right).
\]
After selecting operation

$$\Delta \mathbf{r}_1 \sim \text{AsN} \left( 0, \frac{1}{T} \mathbf{W}_1 \left( \mathbf{R}_1^T \otimes \mathbf{R}_1 \right) \mathbf{W}_1^H \right).$$  \hspace{1cm} (19)$$

Define weight matrix \( \mathbf{W} = (1/T)\mathbf{W}_1 \left( \mathbf{R}_1^T \otimes \mathbf{R}_1 \right) \mathbf{W}_1^H \), then

$$\mathbf{W}^{-1/2} \Delta \mathbf{r}_1 \sim \text{AsN}(0, \mathbf{I}_{2LM-1}).$$  \hspace{1cm} (20)

From Eq. (20), \( \| \mathbf{W}^{-1/2} \Delta \mathbf{r}_1 \|^2 \) follows an asymptotic chi-square distribution with \( 2LM-1 \) DOF. Consequently, the enhanced sparse recovery problem can be formulated as

$$\begin{align*}
\text{min} & \| \mathbf{h} \|_{2,1} \\
\text{s.t.} & \| \mathbf{W}^{-1/2} \left( \mathbf{r}_1 - \mathbf{\Omega}_{\text{sub}} \hat{\mathbf{\rho}} - \mathbf{\Omega}'_{\text{sub}} \hat{\mathbf{\omega}} \right) \|_2 \leq \xi \\
& -\frac{a}{2} \hat{\mathbf{\rho}} \leq \hat{\mathbf{\omega}} \leq \frac{a}{2} \hat{\mathbf{\rho}},
\end{align*}$$  \hspace{1cm} (21)

where \( \mathbf{W} = (1/T)\mathbf{W}_1 \left( \mathbf{R}_1^T \otimes \mathbf{R}_1 \right) \mathbf{W}_1^H \) is the approximate weight matrix; \( \xi \) is the up bound of the fitting error, which can be set as \( \xi = \sqrt{\text{chi2inv}(1-p, 2LM-1)} \) \([11]\), where chi2inv\((1-p, 2LM-1)\) denotes the inverse cumulative distribution function that makes the inequality holds with a probability \((1-p)\). Generally, it is enough to set \( p = 0.001 \) to make it nearly a sure event.

After solving Eq. (21) via CVX \([38, 39]\), we can obtain the estimations of \( \hat{\mathbf{\rho}} \) and \( \hat{\mathbf{\omega}} \), which are denoted as \( \hat{\mathbf{\rho}} \) and \( \hat{\mathbf{\omega}} \), respectively.

3.3. Ambiguity Elimination. The positions of nonzero elements in the first \( Q \) elements of \( \hat{\mathbf{\rho}} \) and \( \hat{\mathbf{\omega}} \) give the initial DOA estimations \( \hat{\theta}_{q,k}, k = 1, \cdots, K \), which are the grids nearest to the true DOAs or their representative angles. Besides, the offset vector can be obtained via

$$\hat{\mathbf{\beta}} = \text{diag} \left( \hat{\rho}/\hat{\omega} \right),$$  \hspace{1cm} (22)

where ./ means element-wise division. Then, the angles are obtained via

$$\hat{\theta}_k = \hat{\theta}_{q,k} + \alpha_k, k = 1, \cdots, K,$$  \hspace{1cm} (23)

where the offsets \( \alpha_k, k = 1, \cdots, K \) are obtained from the first \( Q \) elements of \( \hat{\mathbf{\beta}} \) in Eq. (22).

Now, the angles are estimated with offsets being compensated, but the angles in Eq. (23) may be true DOAs and also may be representative angles. As been discussed in Eq. (10) and Eq. (11), there are totally \( N \) angles including \( \hat{\theta}_k \) sharing the same atom, and their relationship is

$$\sin \hat{\theta}_{k,n} = \sin \hat{\theta}_k - \frac{2m}{N}, n = 2, \cdots, N,$$  \hspace{1cm} (24)

where \( m \) is an integer making \( \sin \hat{\theta}_{k,n} \) locate at the range \([-1, 1]\).

To determine the unique DOA without ambiguity, we substitute the \( N \) angles in Eq. (24) into the whole virtual array

$$\max_{\mathbf{a}_n} \mathbf{a}_n^H \mathbf{r},$$  \hspace{1cm} (25)

where \( \mathbf{r} = [\mathbf{r}_{11}^T, \mathbf{r}_{12}^T]^T \) and \( \mathbf{a}_n = [\mathbf{a}_{11}^T(\hat{\theta}_{k,n}), \mathbf{a}_{22}^T(\hat{\theta}_{k,n})]^T \). Due to the coprime relationship between the two subarrays, the unique DOA can be determined from the coincide results from the two subarrays. Consequently, if the whole array containing both two subarrays is exploited, unique angle is determined by finding the maximum value in Eq. (25).
For the complexity, the proposed method only requires partial dictionary, and the main complexity lies in the construction of two covariance matrices and sparse recovery. The total number of complex multiplications is about 
\[
\left( (M + L - 1)^2 + (N + L - 1)^2 \right) T + (2LM - 1)n^3 + N(2LM + 2 LN - 2),
\]
where \( n \) denotes the dictionary size. The proposed method has lower complexity than peak search method [24] and other sparse representation methods that require whole dictionary [27–30]. Compared to DOA estimation methods with closed-form solution, e.g., ESPRIT and root-MUSIC, the proposed method costs more but achieves better estimation performance, which will be verified in the simulation section below.

### 4. Simulation Results

In the simulations, the CCNA is configured with \( M = 4, N = 3, \) and \( L = 3. \) \( T = 500 \) snapshots are collected to estimate the covariance matrix, and the root mean square error (RMSE) is defined below to measure the DOA estimation performance.
\[
\text{RMSE} = \frac{1}{K} \sum_{k=1}^{K} \sqrt{\frac{1}{D} \sum_{i=1}^{D} (\hat{\theta}_{k,i} - \theta_{k,i})^2},
\]

where \(\hat{\theta}_{k,i}\) denotes the estimations of \(\theta_{k,i}\) of the \(i\)-th Monte Carlo trial and \(D = 200\) trials are carried out.

With the measurement of RMSE, Figures 3 and 4 present the angle estimation accuracy comparisons between the proposed algorithm and other methods versus SNR and snapshot number, respectively. The ACA method [24], PS-MUSIC using prototype coprime array [32], root-MUSIC method using prototype coprime array [33], and root-MUSIC using CCNA [36] all adopt the same number of physical antennas with the proposed method for fair comparison. It is indicated from Figures 3 and 4 that the proposed algorithm outperforms the other methods, and the main reasons include (1) the virtual subarray with larger aperture is chosen to avoid the negative effect from the smaller subarray, and (2) the off-grid sparse representation is established to amend the grid mismatch problem. The PS-MUSIC has the worst performance, especially with low SNR, as it has the additional pairing problem, and it utilizes the data from the physical array, which has limited DOF.

To test the resolution performance of the algorithms, we choose two closely spaced sources with angles being \(\theta_1 = 10^\circ\) and \(\theta_2 = 12^\circ\), respectively. Figure 5 shows the estimation results of the algorithms over 20 trials with SNR = 0 dB. It is indicated that the proposed method can always clearly identify the two sources, while the other methods have big deviations. Consequently, the proposed method achieves the best angular resolution.

5. Conclusions

An off-grid DOA estimation method exploiting CCNA is proposed. Based on the nested relationships within the three subarrays, two virtual coprime subarrays are obtained firstly in the coarray domain. Thereafter, subarray with larger aperture is chosen for enhanced estimation performance, and cyclic phase ambiguity is exploited to reduce the size of the dictionary. Meanwhile, off-grid sparse reconstruction method is established to amend the grid mismatch. Finally, DOA is uniquely determined by substituting the ambiguous into the whole array. Compared to other methods with simulations, the proposed approach is verified that it has better DOA estimation performance and angular resolution.

Data Availability

Data are available in the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

Acknowledgments

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