A multivariate Lévy process model with linear correlation

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In this paper, we develop a multivariate risk-neutral Lévy process model and discuss its applicability in the context of the volatility smile of multiple assets. Our formulation is based upon a linear combination of independent univariate Lévy processes and can easily be calibrated to a set of one-dimensional marginal distributions and a given linear correlation matrix. We derive conditions for our formulation and the associated calibration procedure to be well-defined and provide some examples associated with particular Lévy processes permitting a closed-form characteristic function. Numerical results of the option premiums on three currencies are presented to illustrate the effectiveness of our formulation with different linear correlation structures.

Keywords: Volatility modelling; Stochastic jumps; Non-Gaussian option pricing; Non-Gaussian distributions; Multivariate volatility; Model calibration; Mathematical finance; Implementation of pricing

1. Introduction

Pure-jump Lévy processes and their marginal infinite divisibility have attracted considerable attention amongst practitioners and academics for the prime reason that the flexibility of their distributions, such as heavy tails and asymmetry, suits very well various practical subjects in, for example, finance, telecommunications, turbulence, to mention just a few. In particular, in mathematical finance, there exists a vast literature on Lévy process modeling for a single asset (see, for example, Carr et al. 2002, Kou 2002, Madan et al. 1998, Prause 1999, Rydberg 1997, and Schoutens and Teugels 1998). In the recent structured products market, it has become quite normal for the coupons to be determined by more than a single index, for example two FX rates USDJPY and AUDJPY. However, once the pricing model for multi-asset exotics is required to go beyond the Black–Scholes framework, there is still no market consensus on how to model both volatility smiles and an intended correlation structure. In the literature, there has recently been increasing interest in multivariate Lévy process modeling. For example, the Lévy copula models of Kallsen and Tankov (2006) and Tankov (2005) completely characterize the law of a multivariate Lévy process, and are applied in pricing basket options, while Luciano and Schoutens (2006) and Moosbrucker (2006) propose the production of some dependence among components in the variance gamma process frameworks by setting a (fully or partially) common time-changing stochastic process for every component. A general construction of multi-factor Lévy models from Lévy models on rays is studied by Boyarchenko and Levendorskiı˘ (2002). Among other examples of the multivariate model related to Lévy processes and infinitely divisible distributions are the multidimensional structure model of discount factors of Bakshi et al. (2007), and the various credit derivatives models of Baxter (2007), Hull and White (2004), O’Kane and Schloegl (2003) and Kalemanova et al. (2007).

In this paper we propose a multivariate risk-neutral Lévy process model based on a linear combination of independent univariate Lévy processes that can easily be calibrated to a set of one-dimensional marginal distributions and a given linear correlation matrix. Our formulation has three contributing features. First, applying Lévy processes permitting a parametric form of the characteristic function, we can easily derive the conditions for both our formulation and the Carr–Madan method (Carr and Madan 1999) to be well-defined.
(Proposition 3.1). Second, our model is built so as to precisely satisfy a given linear correlation matrix. Regardless of the increasing interest in the tail dependence in extreme value theory, from the viewpoint of financial practitioners it is usually more important to make sure of attaining an intended linear correlation. Although, as claimed in the literature, the linear correlation may be too simple, it is in fact sufficiently complicated in the sense that its change is unhedgable, even in the Black–Scholes framework. Third, and perhaps most importantly, compared with existing models, our formulation provides a considerably easier way to simulate the resultant multivariate Lévy process as a linear combination of independent univariate Lévy processes.

The rest of the paper is organized as follows. Section 2 gives some general notation and motivates our model development. Section 3 formulates the model and derives closed-form characteristic functions. Section 4 presents numerical results for the option premiums for three currencies, USDJPY, EURJPY and AUDJPY, with different linear correlation structures and also considers the extreme value dependence. Finally, section 5 concludes.

2. Preliminaries

Let us begin with some notation and definitions that will be used in the following. \( \mathbb{R}^d \) is the d-dimensional Euclidean space with norm \( \| \cdot \| \), \( \mathbb{R}_0^d := \mathbb{R}^d \setminus \{ 0 \} \), and \( \mathcal{B}(\mathbb{R}_0^d) \) is the Borel σ-field of \( \mathbb{R}_0^d \). \( \mathcal{L}(X) \) denotes the law of the random vector \( X \), while \( = \) is the equality in law. We denote by \( \mathbf{G} \in \mathbb{R}^{d \times d} \) the linear correlation matrix for a d-dimensional random vector of the form

\[
\mathbf{G} := \begin{pmatrix}
1 & \rho_{1,2} & \rho_{1,3} & \cdots & \rho_{1,d} \\
\rho_{2,1} & 1 & \rho_{2,3} & \cdots & \rho_{2,d} \\
\rho_{3,1} & \rho_{3,2} & 1 & \cdots & \rho_{3,d} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\rho_{d,1} & \rho_{d,2} & \rho_{d,3} & \cdots & 1
\end{pmatrix},
\]

which is symmetric positive definite and where for each \( i, j \), \( |\rho_{i,j}| \leq 1 \). A stochastic process \{ \( X_t : t \geq 0 \) \} in \( \mathbb{R}^d \) is called a Lévy process if it has stationary and independent increments, if it is stochastically continuous, and if \( X_0 = 0 \) almost surely. It is well-known that the marginal distributions of a Lévy process are infinitely divisible, that is, by the Lévy–Khintchine representation, their characteristic function is uniquely characterized as

\[
\mathbb{E}[e^{i(y,X_t)}] = \exp \left[ i\left( \langle y, \gamma \rangle - \frac{1}{2} \langle y, \Sigma \rangle \right) + \int_{\mathbb{R}^d} (e^{i(y,z)} - 1 - i(y,z)1_{\{|z| \leq 1\}}) \nu(dz) \right],
\]

where \( \gamma \in \mathbb{R}^d \), \( \Sigma \in \mathbb{R}^{d \times d} \) is symmetric non-negative definite, and \( \nu \) is a Lévy measure, that is, a σ-finite measure on \( \mathbb{R}_0^d \), satisfying \( \int_{\mathbb{R}_0^d} (|z| \wedge 1) \nu(dz) < +\infty \). In this paper, we will only consider Lévy processes with a finite second moment. The above characteristic function can then be rewritten as

\[
\mathbb{E}[e^{i(y,X_t)}] = \exp \left[ i\left( \frac{1}{2} \langle y, \Sigma \rangle + \int_{\mathbb{R}^d} (e^{i(y,z)} - 1 - i(y,z)1_{\{|z| \leq 1\}}) \nu(dz) \right) \right],
\]

where the Lévy measure \( \nu \) is now in \( L^2(\mathbb{R}_0^d) \), that is, \( \int_{\mathbb{R}_0^d} |z|^2 \nu(dz) < +\infty \). It also holds that every infinitely divisible distribution with a finite second moment is uniquely characterized by the three parameters \( (\gamma, \Sigma, \nu) \) in the representation (1). We denote by \( \varphi \) the distinguished logarithm of the characteristic function (1), that is, \( \varphi(y) := \ln \mathbb{E}[e^{i(y,X_0)}] \). For convenience, we will write \( \varphi_X \) for the distinguished logarithm for the random vector \( X \).

Consider a non-trivial \( d \)-dimensional random vector

\[
Z := (Z_1, \ldots, Z_d)',
\]

whose law is known componentwise and such that \( \mathbb{E}[|Z|^2] < +\infty \). Define also the following notation:

(i) the mean vector \( \mu := (\mu_1, \ldots, \mu_d)' := \mathbb{E}[Z] \); and
(ii) the standard-deviation vector \( \sigma := (\sigma_1, \ldots, \sigma_d)' := [\mathbb{E}[(Z - \mu)^2]]^{1/2} \).

The main scope of our model is to ease the calibration procedure in finding a set of \( (\Sigma, X) \), where \( C \in \mathbb{R}^{d \times d} \) and \( X := (X_1, \ldots, X_d)' \) is an infinitely divisible random vector in \( \mathbb{R}^d \) with independent components, in the following sense:

(i) the law of \( CX + \mu \approx \mathbb{E}[Z] \); and
(ii) the correlation matrix of \( CX \) is approximately or precisely \( \mathbf{G} \), which we give as an exogenous factor.

To motivate our discussion, let us begin with the simplest, yet very illustrative, example. Suppose the random vector \( Z \) is purely Gaussian. Let \( X \) be a \( d \)-dimensional standard normal random vector (with independent components) and let \( C \) be the lower-triangular matrix obtained via the Cholesky decomposition of the variance–covariance matrix \( \text{diag}(\sigma) \mathbf{G} \text{diag}(\sigma) \). It is then evident that the law of \( CX + \mu \) is identical to (and not only an approximation of) that of \( Z \), since the normal random vector is characterized by the mean vector and the variance–covariance matrix. In most cases, the matrix \( C \) is taken as the lower-triangular Cholesky matrix, since then \( CX \) does not require full matrix multiplication due to the lower-triangular structure of \( C \). Note, however, that \( C \) can also be any matrix satisfying \( CC' = A \). For example, it can be a matrix obtained via singular value decomposition.

Let us now apply the same approach to the case when \( Z \) is infinitely divisible without a Gaussian component. For simplicity, we assume that the random vector \( Z \) is in \( \mathbb{R}^2 \).
and $C \in \mathbb{R}^{2 \times 2}$ is the lower-triangular Cholesky matrix of $	ext{diag}(\sigma)G = \text{diag}(\sigma)$, whose $(i,j)$-element is denoted by $c_{i,j}$. For each component $k$, we denote by $\nu_k$ the Lévy measure of $Z_k$, that is

$$E[e^{i\beta Z_k}] = \exp \left[ i\mu_k + \int_{\mathbb{R}} (e^{i\beta z} - 1 - i\beta z) \nu_k(\mathrm{d}z) \right].$$

We wish to find a random vector $X$ that gives

$$Z_1 - \mu_1 \leq c_{1,1} X_1,$$

$$Z_2 - \mu_2 \leq c_{2,1} X_1 + c_{2,2} X_2.$$

The law of the first component can easily be recovered by setting the characteristic function

$$E[e^{i \beta X_1}] \sim \psi_1(\beta) := E \left\{ \exp \left[ i \frac{Z_1 - \mu_1}{\sigma_1} \right] \right\}.$$ (2)

Let $T_\beta$ be a transformation defined by $(T_\beta \nu)(B) := \nu(B^{-1} \beta)$ for $B \in \mathcal{B}(\mathbb{R}^d)$. For the second component, using the independence between $X_1$ and $X_2$, we set

$$E[e^{i \beta X_2}] \sim \psi_2(\beta) := E \left\{ \exp \left[ i \frac{Z_2 - \mu_2}{\sigma_2} \right] \right\}$$

$$= \exp \left[ i \frac{c_{2,1}}{\sigma_2} \mu_1 - \frac{c_{2,2}}{\sigma_2} \mu_2 \right]$$

$$+ \int_{\mathbb{R}} (e^{i\beta z} - 1 - i\beta z)$$

$$\times \left( T_{-\frac{c_{2,1}}{\sigma_2},\frac{c_{2,1}}{\sigma_2}} \nu_2 - T_{\frac{c_{2,2}}{\sigma_2},\frac{c_{2,2}}{\sigma_2}} \nu_1 \right)(\mathrm{d}z),$$ (3)

where the division in (3) is well-defined since the characteristic function of infinitely divisible distributions is necessarily non-zero. A crucial drawback here is that the above measure

$$T_{-\frac{c_{2,1}}{\sigma_2},\frac{c_{2,1}}{\sigma_2}} \nu_2 - T_{\frac{c_{2,2}}{\sigma_2},\frac{c_{2,2}}{\sigma_2}} \nu_1$$

might not be a Lévy measure. More precisely, this is clearly in $L^1(\mathbb{R}^d)$, while it might no longer be a non-negative measure. If that is the case, then $\psi_2$ is not necessarily a characteristic function of any distribution.

**Example 2.1:** The Meixner distribution of Schoutens and Teugels (1998) is infinitely divisible and its Lévy measure is characterized by three parameters $(a,b,d)$ in the form

$$\nu(\mathrm{d}z) = d \frac{\exp(bz/a)}{z \sinh(za/a)} \mathrm{d}z, \quad z \in \mathbb{R},$$

where $a > 0, b \in (-\pi, \pi)$, and $d > 0$. Now, let $Z_1$ and $Z_2$ be identically distributed Meixner random variables whose common Lévy measure is set with the parameters $(a, b) = (1, -2)$ and $d = 2(\cos(b/2)/a)^2$. This formulation guarantees $E[Z_1] = 0$ and $\text{Var}(Z_1) = 1$. Following the aforementioned procedure, we find independent random variables $X_1$ and $X_2$, for example, with the correlation matrix

$$G = \begin{pmatrix}
1 & -0.6 \\
-0.6 & 1
\end{pmatrix}.$$ 

A numerical experiment shows that the measure (4) falls below zero approximately for $z > 0.27$. Such a measure certainly cannot be a Lévy measure.

### 3. Model formulation

Let us now proceed to our model formulation. To maintain the full generality of the discussion, we drop the assumption of the lower-triangular structure of the matrix $C \in \mathbb{R}^{d \times d}$.

Let $\{X_t : t \geq 0\}$ be a Lévy process in $\mathbb{R}^d$ without a Gaussian component and with independent components. Moreover, we mean by $\{X_{t,k} : t \geq 0\}$ the $k$th component of $\{X_t : t \geq 0\}$, and denote by $\nu_k$ its Lévy measure on $\mathbb{R}$. Due to the component independence, the Lévy process can be characterized componentwise in full. We denote by $\psi_k$ the distinguished logarithm of the $k$th component, that is, $e^{\psi_k(\beta)} := E[e^{i\beta X_t,k}]$. Let us then put the following assumption in force:

$$\text{Var}(X_{t,1}) = \cdots = \text{Var}(X_{t,d}) = t \xi_0^2,$$ (5)

for some $\xi > 0$.

Next, let $\{Z_t : t \geq 0\}$ be a stochastic process in $\mathbb{R}^d$, with its $k$th component written as $\{Z_{t,k} : t \geq 0\}$ and defined by

$$Z_{t,k} := \sum_{l=1}^d (c_{k,l} X_{t,l} - t \psi_l(-ic_{k,l})),$$ (6)

provided that, for all $k$ and $l$,

$$|\psi_l(-ic_{k,l})| < +\infty.$$ (7)

By assumption (5), this formulation guarantees that the given linear correlation matrix $G$ is attained, that is, for each $t \geq 0$ and for $k_1, k_2$,

$$\text{Corr}(Z_{t,k_1}, Z_{t,k_2}) = \frac{\text{Cov}(Z_{t,k_1}, Z_{t,k_2})}{\sqrt{\text{Var}(Z_{t,k_1}) \text{Var}(Z_{t,k_2})}} = \sum_{l=1}^d c_{k_1,l} c_{k_2,l} = \rho_{k_1, k_2}.$$ (8)

**Remark 1:** The independence of the components implies that the Lévy measure $\nu$ on $\mathbb{R}^d$ of $\{X_t : t \geq 0\}$ is supported on the axes of $\mathbb{R}^d$ and can be written as

$$\nu(\mathrm{d}z_1, \ldots, \mathrm{d}z_d) = \sum_{k=1}^d \delta_0(\mathrm{d}z_k) \nu_k(\mathrm{d}z_k) \delta_0(\mathrm{d}z_{k+1}) \cdots \delta_0(\mathrm{d}z_d),$$

where $\delta_0$ is the Dirac delta measure at zero. After the linear transformation by the matrix $C$, the Lévy measure on $\mathbb{R}^d$ of $\{Z_t : t \geq 0\}$ is given by $(TC)(B) := \nu(C^{-1}B)$ for $B \in \mathcal{B}(\mathbb{R}^d)$. Note that it is supported on $d$ lines in $\mathbb{R}^d$. 
\{a(c_{1,k}, \ldots, c_{d,k})' : a \in \mathbb{R}\}_{k=1, \ldots, d}$. In particular, if the matrix $C$ is that obtained via singular value decomposition, its support is the union of all the eigenspaces of the correlation matrix $G$.

Now let $r$ be a non-negative (constant) risk-free rate and let $\{S_t : t \geq 0\}$ be a stochastic process in $\mathbb{R}^d$, again with its $k$th component written as $\{S_{t,k} : t \geq 0\}$ and defined by

$$S_{t,k} := e^{\alpha t} S_{0,k} \exp(Z_{1,k}), \quad t \geq 0,$$

where $S_{0,k}$ is a positive constant. It is immediate that, for each $k$, the discounted process of $\{S_{t,k} : t \geq 0\}$ is a martingale with respect to the natural filtration generated by the Lévy process $\{X_t : t \geq 0\}$, since, for each $t > 0$,

$$\exp \left[ \sum_{i=1}^{d} \left( c_{i,k} X_{i,t} - t \psi_i(-ic_{i,k}) \right) \right] = \frac{e^{\sum_{i=1}^{d} c_{i,k} X_{i,t}}}{\mathbb{E}[e^{\sum_{i=1}^{d} c_{i,k} X_{i,t}}]},$$

where the last equality holds by the component independence of $\{X_i : t \geq 0\}$.

Next, fix $T > 0$ and let $\tilde{\mu}_k$ be the characteristic function of the logarithm of the $k$th component of $S_T$, that is

$$\tilde{\mu}_k(y) := \mathbb{E}[e^{iy \ln S_{T,k}}] = e^{iy(T+\ln S_{0,k})} \mathbb{E}[e^{iyZ_{1,k}}] = e^{iy(T+\ln S_{0,k})} \prod_{i=1}^{d} e^{ip_i(c_{i,k})y}.$$  \hspace{1cm} (8)

When its closed form is available and when the maturity $T$ is not very small, the vanilla call option price with strike $K > 0$, that is

$$C(K) := e^{-rT} \mathbb{E}[(S_{T,k} - K)_+],$$

can be efficiently computed using the well-known method of Carr and Madan (1999),

$$C(K) = \frac{e^{-rT-\alpha \ln K}}{\pi} \int_0^\infty \text{Re} \left[ e^{-iy \ln K} \tilde{\mu}_k(y - (\alpha + 1)i) \right] dy,$$  \hspace{1cm} (9)

where $\alpha$ is a positive constant satisfying

$$\mathbb{E}[S_{T,k}^{\alpha+1}] < +\infty.$$  \hspace{1cm} (10)

Noting that, for each $k$,

$$\mathbb{E}[S_{T,k}^{\alpha+1}] = (e^{rT} S_{0,k})^{\alpha+1} \prod_{i=1}^{d} \mathbb{E} \left[ e^{[(\alpha+1)c_{i,k} X_{i,T} - (\alpha+1)T \psi_i(-ic_{i,k})]} \right]$$

$$= (e^{rT} S_{0,k})^{\alpha+1} \prod_{i=1}^{d} e^{T \psi_i(-ic_{i,k})},$$

in order for both (7) and (10) to hold we need to have, for every $k$ and $l$,

$$|\psi_i(-ic_{i,k})| \lor |\psi_i(-i(\alpha + 1)c_{i,k})| < +\infty.$$  \hspace{1cm} (11)

The following rewrites conditions (7) and (11) as integrability conditions of the underlying Lévy measure.

**Proposition 3.1:** The condition (7) holds if and only if for each $l$,

$$\int_{-\infty}^{+\infty} e^{\max(c_{l,k})} \psi_l(dz) < +\infty,$$

and

$$\int_{-\infty}^{+\infty} e^{\min(c_{l,k})} \psi_l(dz) < +\infty.$$  

Moreover, condition (11) holds if and only if for each $l$,

$$\int_{|z| > 1} e^{\max(|o|)\max(c_{l,k})} \psi_l(dz) < +\infty,$$

and

$$\int_{|z| > 1} e^{\min(|o|)\min(c_{l,k})} \psi_l(dz) < +\infty.$$  

**Proof:** By Theorem 25.17(ii) of Sato (1999), $\varphi(-ic)$ is well-defined if and only if $\int_{|z| > 1} e^{\varphi(z)} dz < +\infty$. Therefore, equation (7) holds if and only if

$$\int_{|z| > 1} e^{\max(c_{l,k})} \psi_l(dz) < +\infty,$$

while (11) holds if and only if

$$\int_{|z| > 1} e^{\max(|o|)\max(c_{l,k})} \psi_l(dz) < +\infty,$$

for every $l$. With $a > 0$, the rest is straightforward. \quad \square

**Remark 2:** In view of (6) and (8), an addition of the drift to each component $\{X_{t,k} : t \geq 0\}$ cancels out. We may thus define the distinguished logarithm at our convenience, such as

$$\varphi_k(y) = \int_{\mathbb{R}_0} (e^{iyz - 1} - 1) \psi_k(dz),$$

$$\varphi_0(y) = \int_{\mathbb{R}_0} (e^{iyz - 1} - 1 - iyz) \psi_0(dz),$$

or

$$\varphi_k(y) = \int_{\mathbb{R}_0} (e^{iyz - 1} - 1 - iyz1_{[0,\alpha]}(|z|)) \psi_k(dz),$$

for some $c > 0$, whenever those integrals are well-defined.

**Example 3.2** (Meixner process): Suppose that the $l$th component $\{X_{t,l} : t \geq 0\}$ is a Meixner process with parameter $(a, b, d)$, as described in example 2.1. Its variance is then given in closed form:

$$\int_{\mathbb{R}_0} z^2 \psi(dz) = \frac{d}{2} \left( \frac{a}{\cos(b/2)} \right)^2,$$

while the distinguished logarithm is as simple as

$$\varphi(y) = 2d \left[ \ln \left( \frac{\cos \left( \frac{b}{2} \right)}{2} \right) - \ln \left( \cosh \left( \frac{ay - ib}{2} \right) \right) \right].$$
We can show that the condition (11) reduces to
\[
\frac{\pi - b}{a} > (\alpha + 1) \max \left(0, \max_k (c_{k,i}) \right),
\]
and
\[
\frac{\pi + b}{a} > -(\alpha + 1) \min \left(0, \min_k (c_{k,i}) \right).
\]
Note that the above domain of \((a, b)\) is not quite so simple for the calibration procedure, in the sense that the parameters \(a\) and \(b\) restrict each other.

**Example 3.3** (variance gamma process): The variance gamma process of Madan et al. (1998) is a Lévy process obtained by evaluating Brownian motion with drift at a random time given by a gamma process:
\[
X_t = \theta Y_t + \sigma W_{Y_t},
\]
where \(\{W_t: t \geq 0\}\) is a standard Brownian motion and where \(\{Y_t: t \geq 0\}\) is a gamma process whose marginal is characterized by \(\mathbb{E}[e^{i\eta Y_t}] = (1-i\eta \gamma)^{-\eta} \) (see also Boyarchenko and Levendorskiı́ (2000)). Its Lévy measure is given in the form
\[
v(dz) = \exp\left(\frac{\theta}{\sigma^2}z - \left(\frac{\theta^2}{2\sigma^2} + \frac{\theta^2}{2\sigma^2} \right)\right) \frac{dz}{\eta |z|}, \quad z \in \mathbb{R}_0,
\]
where \(\eta > 0, \theta \in \mathbb{R},\) and \(\sigma > 0\). This also implies that the variance gamma process can be viewed as the difference of two independent gamma processes. Its variance is given in closed form:
\[
\mathbb{E}[\sigma^2] = \frac{\theta^2}{\sigma^2} + \frac{\theta^2}{\sigma^2} \approx (\alpha + 1) \max \left(0, \max_k (c_{k,i}) \right),
\]
and
\[
\text{variance gamma process}
\]

\[
\mathbb{E}[\sigma^2] = \frac{\theta^2}{\sigma^2} + \frac{\theta^2}{\sigma^2} \approx -(\alpha + 1) \min \left(0, \min_k (c_{k,i}) \right).
\]
We observe that, as in example 3.2, the parameters \(\theta, \sigma,\) and \(\eta\) restrict each other in a complex manner.

**Example 3.4** (CGMY process): The CGMY process proposed by Carr et al. (2002, 2003) is also a Lévy process obtained by tempering the Lévy measure of the non-Gaussian stable distribution, and its Lévy measure is given in the form
\[
v(dz) = \left[ C_e^{G(z)} I(z < 0) + G_e^{-Mz} I(z > 0) \right] \frac{dz}{\mathbb{R}_0},
\]
where \(C_e, C_p, G, M > 0\) and \(Y_n, Y_p \in (-\infty, 2)\). Its variance is given in closed form:
\[
\int_{\mathbb{R}_0} z^2 v(dz) = C_e^{\Gamma(2 - Y_e)} + C_p^{\Gamma(2 - Y_p)}
\]
while the distinguished logarithm is as simple as
\[
\varphi(y) = C_e^{\Gamma(-Y_e)\left[ (G + iy) Y_e - G Y_e \right]} + C_p^{\Gamma(-Y_p)\left[ (M - iy) Y_p - M Y_p \right]},
\]
provided that \(Y_e \neq 1\) and \(Y_p \neq 1\). We can show that condition (11) is satisfied when
\[
G \begin{cases} \geq -(\alpha + 1) \min(0, \min_k (c_{k,i})), & \text{if } Y_e \in (0, 2), \\ > -(\alpha + 1) \min(0, \min_k (c_{k,i})), & \text{otherwise}, \end{cases}
\]
and
\[
M \begin{cases} \geq (\alpha + 1) \max(0, \max_k (c_{k,i})), & \text{if } Y_p \in (0, 2), \\ > (\alpha + 1) \max(0, \max_k (c_{k,i})), & \text{otherwise}.
\end{cases}
\]
Unlike the case of the Meixner and the variance gamma distributions, each condition consists of a single parameter, and can thus easily be set in the calibration algorithm.

### 4. Numerical illustration: multi-currency

We consider the 1-year currency options on the three foreign exchanges USDJPY, EURJPY, and AUDJPY, and model their mean-correcting forward forex dynamics by the following stochastic process:
\[
F_t = \frac{e^{-r_t}}{\mathbb{E}[e^{-r}]} F_0 + \mathcal{E}_{Z_1 Z_2 Z_3},
\]
where \(r_t\) and \(F_t\) denote the continuously compounded domestic and foreign risk-free rates, respectively. In our case, ‘domestic’ means ‘JPY’, while ‘foreign’ corresponds to either one of ‘USD’, ‘EUR’, or ‘AUD’. We assume for simplicity that both risk-free rates are deterministic. \(F_0\) is the spot FX rate, while \(e^{-r_t} F_0 = \mathbb{E}[F_t]\) is the so-called forward FX rate at time \(t\). For notational convenience, we henceforth replace the time indices of \(\{F_t: t \geq 0\}\) and \(\{Z_t: t \geq 0\}\) by FX indices, that is, \(F_1, F_2, F_3\) denote the USDJPY, EURJPY, and AUDJPY rates at 1 year, respectively, while \(Z_1, Z_2, Z_3\) are defined analogously.

#### 4.1. Data description

In general, the market quotes for currency options are available in the form of Delta-Neutral straddle implied volatility (iv_{DN}), 10- and 25-delta risk reversals (RR10 and RR25), and 10- and 25-delta strangle margins (SM10 and SM25). They are related to implied volatilities as follows:
\[
\begin{align*}
RR10 &= iv_{10C} - iv_{10P}, & SM10 &= (iv_{10C} + iv_{10P})/2 - iv_{DN}, \\
RR25 &= iv_{25C} - iv_{25P}, & SM25 &= (iv_{25C} + iv_{25P})/2 - iv_{DN}.
\end{align*}
\]
Table 1 gives the 1-year implied volatilities for 2006/10/25.

The strikes for each implied volatility are derived from the equations

\[
0.10 = \frac{K_{10C}}{\mathbb{E}[F]} \mathcal{N}(d(E[F], K_{10C}, iv_{10C})),
\]

\[
0.25 = \frac{K_{25C}}{\mathbb{E}[F]} \mathcal{N}(d(E[F], K_{25C}, iv_{25C})),
\]

\[
0.10 = \frac{K_{10P}}{\mathbb{E}[F]} [1 - \mathcal{N}(d(E[F], K_{10P}, iv_{10P}))],
\]

\[
0.25 = \frac{K_{25P}}{\mathbb{E}[F]} [1 - \mathcal{N}(d(E[F], K_{25P}, iv_{25P}))],
\]

where \( \mathcal{N} \) is the cumulative standard normal distribution and \( d(F, K, iv) := \ln(F/K) / iv \). Finally, we have \( \mathbb{E}[F] = \mathbb{E}[F] \exp(-iv_{10C}^2/2) \), which implies that the delta-neutral strike is not identical to the ATM strike, but always smaller, \( \mathbb{E}[F] < \mathbb{E}[F] \). We give in table 2 the strikes, together with spot FX rates, forward FX rates \( \mathbb{E}[F] \), and JPYcdf, the discount factor compounding currency basis cost on the same date. With the information in table 2, the call premiums are computed using JPYcdf \( \times \mathbb{E}[(F - K)_+] \) and are reported in table 3.

### 4.2. Underlying Lévy process

For our numerical experiments, we employ a Lévy process whose \( l \)th component has a Lévy measure given by

\[
\nu_l(dz) = \frac{1}{\eta_l|z|} \left[ e^{-\eta_l|z|} I(z < 0) + e^{-\eta_l|z|} I(z > 0) \right] dz, \quad z \in \mathbb{R}_0,
\]

where \( \eta_l > 0 \), \( \tau_l > 0 \), and \( \kappa_l > 0 \). We set the characteristic function as

\[
\mathbb{E}[e^{ivX_t}] := \exp \left[ t \int_{\mathbb{R}_0} (e^{-ivz} - 1) \nu_l(dz) \right].
\]

From its gamma structure, it immediately follows that

\[
e^{iv_{10}(s)} = \left( 1 + \frac{iv}{\tau} \right)^{-1/\tau} \left( 1 - \frac{iv}{\kappa} \right)^{-1/\kappa}
\]

\[
e^{iv_{25}(s)} = \left( 1 + \frac{iv}{\tau} \left( \frac{1}{\tau} - \frac{1}{\kappa} \right) \right)^{-1/\tau},
\]

and

\[
\text{Var}(X_t) = \frac{\tau}{\eta} \left( \sqrt{\theta^2 + \frac{\sigma^2}{\eta}} - \theta \right)^{-1},
\]

\[
\kappa = \frac{\tau}{\eta} \left( \sqrt{\theta^2 + \frac{2\sigma^2}{\eta}} + \theta \right)^{-1},
\]

where the parameters \( (\theta, \sigma) \) are as defined in example 3.3. Here, by unfastening the above dependence between \( \tau \) and \( \kappa \), we not only make the Lévy measure more flexible, but also simplify the conditions for (11), compared with those derived in example 3.3, to

\[
\begin{align*}
\tau_l &> \left( \alpha + 1 \right) \min(0, \min_{(c_l, u)}) , \\
\kappa_l &> \left( \alpha + 1 \right) \max(0, \max_{(c_l, u)}),
\end{align*}
\]

that is, each condition consists of a single parameter. Yet our Lévy process exhibits the features of the variance gamma process: (i) its characteristic function is given in closed form; and (ii) it can be viewed as the difference of two independent gamma processes. One thing our Lévy process does not inherit from the variance gamma process is the property of ‘time-changed Brownian motion’ of (12), since parameters \( \tau \) and \( \kappa \) are no longer dependent as in (15) and (16). From a practical point of view, however, this property is not very important, as long as our Lévy process can still be simulated by two independent gamma processes.

### Table 1. One-year implied volatilities for 2006/10/25 in percentage points.

|          | iv_{10P} | iv_{25P} | iv_{10C} | iv_{25C} |
|----------|----------|----------|----------|----------|
| USDJPY   | 9.44     | 8.37     | 7.70     | 7.44     | 7.66     |
| EURJPY   | 8.40     | 7.59     | 7.05     | 6.86     | 7.07     |
| AUDJPY   | 9.70     | 8.51     | 7.60     | 7.15     | 7.27     |

### Table 2. Call premiums.

|          | K_{10P} | K_{25P} | K_{10C} | K_{25C} | Spot | JPYcdf |
|----------|---------|---------|---------|---------|------|--------|
| USDJPY   | 100.91  | 107.48  | 113.39  | 119.63  | 125.70 | 119.25 | 113.73 | 0.9931 |
| EURJPY   | 130.51  | 137.93  | 144.82  | 152.11  | 159.22 | 149.85 | 145.18 | 0.9933 |
| AUDJPY   | 76.01   | 80.88   | 85.41   | 89.93   | 94.19 | 90.77  | 85.66  | 0.9936 |

### Table 3. Strikes, spot FX, forward FX, and discount factor.

|          | K_{10P} | K_{25P} | K_{10C} | K_{25C} | Spot | JPYcdf |
|----------|---------|---------|---------|---------|------|--------|
| USDJPY   | 13.22   | 7.575   | 3.635   | 1.274   | 0.408|
| EURJPY   | 15.13   | 8.791   | 4.231   | 1.497   | 0.480|
| AUDJPY   | 15.13   | 8.791   | 4.231   | 1.497   | 0.480|
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For condition (5) to be satisfied, with the help of the closed-form variance (14), we fix the parameter \( \eta \) for each \( l \),

\[
\eta_l = \frac{1}{\xi^2} \left( \frac{1}{\tau_l^2} + \frac{1}{\kappa_l^2} \right).
\]  

(18)

Hence, in the calibration procedure, we will only control the remaining two parameters \((\tau_l, \kappa_l)\) for the \( l \)th component. Note that, under condition (18), the realized linear correlation matrix of the three FXs is assumed to be precisely \( \mathbf{G} \), whatever the calibration result.

4.3. Calibration with given correlation

We test for the following correlation structures:

\[
\begin{align*}
\mathbf{G}_1 := & \begin{bmatrix} 1 & 0.25 & 0.50 \\
0.25 & 1 & 0.60 \\
0.50 & 0.60 & 1 \end{bmatrix}, \\
\mathbf{G}_2 := & \begin{bmatrix} 1 & 0.80 & 0.75 \\
0.80 & 1 & 0.85 \\
0.75 & 0.85 & 1 \end{bmatrix}, \\
\mathbf{G}_3 := & \begin{bmatrix} 1 & 0.65 & 0.75 \\
0.65 & 1 & 0.20 \\
0.75 & 0.20 & 1 \end{bmatrix}, \\
\mathbf{G}_4 := & \begin{bmatrix} 1 & -0.25 & 0.50 \\
-0.25 & 1 & 0.60 \\
0.50 & 0.60 & 1 \end{bmatrix},
\end{align*}
\]

where \( \mathbf{G}_1, \mathbf{G}_2 \) and \( \mathbf{G}_3 \) are approximate historical correlations among the three FXs (USDJPY, EURJPY, and AUDJPY from the first component to the third) for 200 days (2006/1/16–2006/10/25), 100 days (2006/6/5–2006/10/25), and 25 days (2006/9/19–2006/10/25), respectively. We also prepare \( \mathbf{G}_4 \), which is an artificial modification of \( \mathbf{G}_1 \), to test negative correlation. Here, we obtain the matrix \( \mathbf{C} \) such that \( C^C = \mathbf{G} \) via singular value decomposition, and these are given by, respectively,

\[
\begin{align*}
\mathbf{C}_1 := & \begin{bmatrix} 0.2084 & 0.6782 & 0.7047 \\
0.3121 & -0.5398 & 0.7818 \\
-0.4352 & -0.0623 & 0.8982 \end{bmatrix}, \\
\mathbf{C}_2 := & \begin{bmatrix} 0.2948 & -0.1024 & 0.9501 \\
-0.2194 & -0.2902 & 0.3135 \\
-0.2355 & 0.0279 & 0.9715 \end{bmatrix}, \\
\mathbf{C}_3 := & \begin{bmatrix} 0.1332 & -0.6801 & 0.7209 \\
0.1669 & 0.5821 & 0.7958 \\
-0.1488 & 0.8707 & 0.4686 \end{bmatrix}, \\
\mathbf{C}_4 := & \begin{bmatrix} 0.1651 & -0.6930 & 0.7018 \\
0.1897 & 0.0800 & 0.9786 \end{bmatrix},
\end{align*}
\]

We perform the calibration via the Nelder–Mead direct search method to minimize the difference between market premiums (market-pr) and model premiums (model-pr) in the sense of the root mean square error (rmse), defined as

\[
\text{rmse} := \left[ \frac{1}{\#(pr)} \sum \{(\text{market-pr}) - (\text{model-pr})\}^2 \right]^{1/2}.
\]

In our example, \(#(pr) = 15\) (that is, five premiums for each of the three currencies) and the calibration is a minimization problem with six parameters \((\tau_1, \kappa_1, \ldots, \tau_3, \kappa_3)\), due to the common variance condition (5), where those parameters can take values in the domain (17). During the calibration procedure, we compute the model premiums using the Carr–Madan method (9).

It is well-known that the calibration of infinitely divisible marginals is an ill-posed inverse problem (see, for example, Cont and Tankov (2006)), and therefore local minima are always a problem. We thus perform the minimization problem many times, with different common variance \( \xi^2 \) in (5), and from different randomly chosen starting points. The calibration results in table 4 are the best in each case. As can be seen from figure 1, our formulation is capable of fitting market premiums very well with different linear correlation structures.

| \( \tau \) | \( \kappa \) | \( \eta \) | \( \tau \) | \( \kappa \) |
|---|---|---|---|---|
| \( \mathbf{G}_{1}^T \| \| \mathbf{G} \| = 0.075 \) | 9.775 | 9.415 | 3.866 | X_1 | 6.962 | 6.685 | 7.646 |
| \( \mathbf{G}_{2}^T \| \| \mathbf{G} \| = 0.075 \) | 16.06 | 15.12 | 1.467 | \( X_2 \) | 9.684 | 9.074 | 4.055 |
| \( \mathbf{G}_{3}^T \| \| \mathbf{G} \| = 0.075 \) | 18.38 | 19.96 | 0.9723 | \( X_3 \) | 18.49 | 21.10 | 0.9190 |
| \( \mathbf{G}_{4}^T \| \| \mathbf{G} \| = 0.075 \) | 17.44 | 16.37 | 1.248 | \( X_4 \) | 20.65 | 22.26 | 0.7757 |

| \( \mathbf{G}_{5}^T \| \| \mathbf{G} \| = 0.075 \) | 22.37 | 24.81 | 0.6442 | \( X_5 \) | 35.57 | 46.00 | 0.2245 |

| \( \mathbf{G}_{6}^T \| \| \mathbf{G} \| = 0.075 \) | 6.189 | 6.380 | 9.008 | \( X_6 \) | 5.143 | 5.029 | 13.75 |

| \( \mathbf{G}_{7}^T \| \| \mathbf{G} \| = 0.075 \) | 20.13 | 19.05 | 3.107 | \( X_8 \) | 20.84 | 21.45 | 2.984 |

Scatter plots of 500 points for currency pairs are given in figure 2. We observe that, compared with the ordinary Gaussian framework, our Lévy process model can express a dependence in more-negative values. These realizations can easily be generated based on a linear combination of three random variables, \( X_1, X_2 \) and \( X_3 \). In particular, thanks to the gamma structure of the underlying Lévy measure (13), each component \( X_t \) is simulated as the difference of independent gamma random variables, for which a sample generation routine is included with most standard mathematical and statistical software. This simplicity in implementation should be a great advantage compared with existing multivariate Lévy process models. For example, the stochastic volatility formulation of Luciano and Schoutens (2006) is based on a time-changed Brownian motion, and usually requires a Euler-type discretization of sample paths for simulation, while the Lévy copula model (Tankov 2005) resorts to the series representation of Lévy processes, which is well-known to be computationally very expensive for most simulations.

4.4. Extreme value dependence

Our formulation can be used to approximate a set of one-dimensional marginal distributions with a given linear correlation structure, and is not modeled to have control over the extreme value dependence. It is worth checking how our multivariate model behaves with extreme values. To this end, for a fixed \( T \), and for each \( k \), let \( F^{-1}_k(u) \) be the inverse cumulative distribution function of \( Z_{T,k} \), that is,
Figure 1. Market premiums (○) and model premiums (——). The x axis indicates the moneyness of the strikes, $K/E[F]$. Recall that $K_{DN} < E[F]$.

Figure 2. Scatter plots of 500 points of each currency pair, with the correlation matrix $G_t$. Each dotted-line indicates a corresponding forward FX rate $E[F]$. 
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5. Concluding remarks

In this paper we have formulated a multivariate risk-neutral Lévy process model that can easily be calibrated to a set of one-dimensional marginal distributions and a given linear correlation matrix. Our model is built on a linear combination of independent univariate Lévy processes so as to precisely realize the given linear correlation matrix. Applying Lévy processes permitting closed-form characteristic functions, our formulation provides a very simple calibration procedure in the Carr–Madan framework. The resultant multivariate Lévy processes can easily be generated for simulation use as a linear combination of independent univariate Lévy processes. Numerical results indicate that our model is capable of simultaneously fitting three currency options, with different linear correlation structures, and strongly correlated downside jumps can also be well expressed.

As future research, it would be interesting to conduct further empirical analyses on different sets of currencies and different types of financial assets, for example a set of equity indices, and with different Lévy processes, such as the CGMY process. Moreover, an extension to the multi-maturity context should certainly be pursued. This may pave the way to the evaluation of popular, but very intricate, financial exotics in the Lévy process framework, a subject that will be studied by Kawai (2009).

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