ANDREEV REFLECTION
AND THE SEMICLASSICAL BOGOLIUBOV-DE GENNES HAMILTONIAN:
RESONANT STATES

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Abstract: We present a semi-classical analysis of the opening of superchannels in gated mesoscopic SNS junctions. For perfect junctions (i.e. hard-wall potential), this was considered by [ChLeBl] in the framework of scattering matrices. Here we allow for imperfections in the junction, so that the complex order parameter continues as a smooth function, which is a constant in the superconducting banks, and vanishes rapidly inside the lead. We obtain quantization rules for resonant Andreev states near energy $E$ close to the Fermi level, including the determination of the resonance width.

0. Introduction.

Bogoliubov-de Gennes Hamiltonian is a $2 \times 2$ matrix $\mathcal{P}(x, \xi)$ defined for $(x, \xi) \in T^* \mathbb{R}$, which describes the dynamics of a pair of quasi-particles (hole/electron) in a 1-D metallic lead connecting 2 superconducting contacts. Diagonal terms are of the form $\pm (\xi^2 - \mu(x))$, where $\mu(x)$ stands for the chemical potential, while the off-diagonal interaction with the superconducting bulk is modeled through a complex potential, or superconducting gap, $\Delta_0 e^{i\phi/2}$ at the boundary ; due to the finite range of the junction, we may consider that the interaction continues to a smooth function $x \mapsto \Delta(x) e^{i\phi(x)/2}$ on a neighborhood of the lead (say, the interval $[-L, L]$). So we assume that $\Delta(x)$, we will call henceforth the “gap function”, is a smooth positive function, increasing on $x > 0$, with $\Delta(x) = 0$, $|x| \leq x_1 < L$ and $\Delta(x) = \Delta_0$, $|x| \geq x_2 > L$ (ignoring the fact that $\Delta$ shows typically isolated zeroes (vortices) in the superconducting bank). In the same way, we will assume that $\phi(x) = \text{sgn}(x) \phi$ takes only 2 values. The chemical potential $\mu(x)$ will be extended also to a smooth positive function on a neighborhood of $[-L, L]$, constant and $\Delta_0$ for $|x| \geq x_2$. As is usual for a metal, we assume that $\mu$ and $\Delta$ are even in $x$, which provides this model with the CPT symmetry.

We introduce a “Planck constant” $h > 0$, which stands for the ratio of $L$ to the characteristic de Broglie wave-length, and take usual $h$-Weyl quantization,

$$\mathcal{P}(x, hD_x) = \begin{pmatrix} (hD_x)^2 - \mu(x) & \Delta(x)e^{i\phi(x)/2} \\ \Delta(x)e^{-i\phi(x)/2} & -(hD_x)^2 + \mu(x) \end{pmatrix}$$

An electron $e^-$ moving in the metallic lead with energy $0 < E \leq \Delta$ (measured with respect to Fermi level $E_F$) and kinetic energy $K_+(x) = \mu(x) + \sqrt{E^2 - \Delta(x)^2}$ is reflected back from the superconductor as a hole $e^+$, with kinetic energy $K_-(x) = \mu(x) - \sqrt{E^2 - \Delta(x)^2}$, injecting a Cooper pair into the bulk. When $\inf_{[-L, L]} \mu(x) \geq E$, and $\phi \neq 0$, this process yields so called phase-sensitive Andreev states, carrying supercurrents proportional to the $\phi$-derivative of the eigen-energies $E_k(h) \text{ of } \mathcal{P}(x, hD_x)$. Since

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\( \mathcal{P}(x, hD_x) \) is self-adjoint, there is of course also an electron moving to the left, and a hole moving to the right (in fact, \( \mathcal{P}(x, hD_x) \) is the Hamiltonian for 2 pairs of quasi-particles), for no net transfer of charge can occur through the lead in absence of thermalisation. So we stress that Bogoliubov-de Gennes Hamiltonian is only a simplified model for superconductivity, and that a more thorough treatment should also take into account the self-consistency relations coupling the quasi-particle with the gap function \( \Delta(x) \) and the phase \( \phi(x) \), that we treat here as “effective potentials” (see [KeSo]).

In the case where \( \Delta(x) \) is a “hard-wall” potential, this was studied in [ChLeBl], [CaMo] in the framework of scattering matrices. In [BeIfaRo], we derived semi-classical quantization rules for Andreev states near energy \( E \), from a microlocal study of the Hamiltonian in the “inner region” \( \Delta(x) \leq E \) alone. For simplicity, we assumed that \( \Delta(x) \) varies linearly near \( E \), namely if \( x_0 \in ]x_1, x_2[ \) is such that \( \Delta(x_0) = E \), then \( \mu(x) = \mu = \text{Const.} \) and \( \Delta(x) = E + \alpha(x - x_0) \) near \( x_0 \).

Here we want to take also into account the “outer region” \( \Delta(x) \geq E \) (i.e. \( |x| \geq x_0 \)) of the junction, entering the superconducting bulk. As a matter of fact, the microlocal solutions, purely oscillating in \( \Delta(x) \leq E \), acquire a complex phase in \( \Delta(x) \geq E \), which is of course related to phase-space tunneling. We make the assumption that the junction is extended, in such a way that the quasi-particle turns into a resonant state before creating a new Cooper pair, its dynamics still being governed by Bogoliubov-de Gennes Hamiltonian. So we assume that \( \mu(x) \) and \( \Delta(x) \) are defined on the entire real line, taking constant values for \( |x| \geq x_2 > L \), so that (0.1) can be defined as a self-adjoint operator on \( L^2(\mathbb{R}) \otimes \mathbb{C}^2 \). We will translate the usual theory of analytic dilations [ReSi] in the context of CPT symmetry, and find semi-classical resonances near a “scattering” Andreev level, i.e. complex correction to the real eigen-energies \( E_k(h) \) of \( \mathcal{P}(x, hD_x) \).

1) The real part of the resonances.

The bicharacteristic set in \( \{ \xi > 0 \} \) at energy \( E \), of the form \( \det \mathcal{P}(x, \xi) - E = 0, \) or \( \xi^2 = K(x) \), consists of : (1) two real curves \( \rho^\pm \) over \( [-x_0, x_0] \), joining smoothly to a close curve at the “branching points” \( a' = (-x_0, \xi_0) \) and \( a = (x_0, \xi_0) \) (so to make \( \rho_+ \cup \rho_- \) diffeomorphic to \( S^1 \)) ; (2) complex branches \( \rho^\pm \) over \( ]-\infty, -x_0[ \), and \( \rho^\pm \) over \( ]x_0, +\infty[ \) respectively. They all have a vertical tangent at \( a, a' \). We complete this picture by reflection on the \( x \) axis, denoting the corresponding branching points by \( b', b \).

a) Microlocal solutions supported on \( \rho^\pm \)

First we recall from [3] the construction of distributions microlocalized on the Lagrangians \( \rho^\pm \), and verifying the PT symmetries of the problem. We denote the parity operator by \( \gamma : u(x) \rightarrow u(-x) \), and the time reversal operator by \( \mathcal{T} : u(x) \rightarrow \overline{u(x)} \).

**Definition 1.1:** We call “admissible \( C^2 \)-valued Lagrangian distribution” an oscillatory integral

\[
I(S, \varphi)(x, h) = (2\pi h)^{-d/2} \int_{\mathbb{R}^d} e^{i \varphi(x, \Theta; h)/h} S(x, \Theta; h) d\Theta
\]

with the following properties : (1) \( \varphi(x, \Theta, h) \) denotes a non degenerate phase-function, and

\[
S(x, \Theta; h) = S_0(x, \Theta; h) + hS_1(x, \Theta; h) + \cdots
\]
a $C^2$-valued amplitude (i.e. a classical symbol in $h$), $S_0 = (e^{i\phi/2}X)$ possibly depending on $h$ (with the property that $\phi(x) = \text{sgn}(x)\phi$); (2) The symbols $X = X(x, \Theta, h)Y = Y(x, \Theta, h)$ have their principal part $(X'_{0}) = \lambda(x, \Theta; h)(X'_{0})$, $\lambda \in \mathbb{C}$, proportional to a real vector $(X'_{0})$, depending also on $(x, \Theta; h)$.

Of course, all these functions may depend on additional parameters. One of the main problem consists in finding microlocal solutions near the branching points $a, a'$. Due to PT symmetry, it suffices to focus on $a = (x_0, \zeta_0)$. In $h$-Fourier representation, the Hamiltonian takes the form

$$\mathcal{P}^\alpha(-hD_\xi, \xi) = \begin{pmatrix} \xi^2 - \mu & e^{i\phi/2}(E - \alpha hD_\xi - \alpha x_0) \\ e^{i\phi/2}(E - \alpha hD_\xi - \alpha x_0) & -\xi^2 + \mu \end{pmatrix}$$

where $\mu = \xi_0^2$ is a constant, equal to the value of the chemical potential at $x_0$. Consider the equation $(\mathcal{P}^\alpha(-hD_\xi, \xi) - E)\tilde{U} = 0$, where $\tilde{U} = \begin{pmatrix} \tilde{\varphi}_1 \\ \tilde{\varphi}_2 \end{pmatrix}$. Clearly, the system decouples, and to account for time-reversal symmetry, it is convenient to introduce the scaling parameter $\beta = \sqrt{\alpha/(2\xi_0)^{3/2}}$, together with the changes of variables $\xi = \xi_0(\pm 2\beta\xi' + 1)$. The functions $\tilde{u}_{\pm, \beta}(\xi') = (\xi^2 - \mu - E)^{-1/2}e^{-i(E - \alpha x_0)\xi/\alpha h}\tilde{\varphi}_2$ satisfy a second order ODE of the form

$$(\tilde{P}_{\pm, \beta}(-hD_{\xi'}, \xi', h) - \frac{E_1^2}{\beta^2})\tilde{u}_{\pm, \beta}(\xi') = 0$$

with $E_1 = (2\xi_0)^{-2}E$, and

$$\tilde{P}_{\pm, \beta}(-hD_{\xi'}, \xi', h) = (hD_{\xi'})^2 + (\xi' + \beta\xi)^2$$

$$+ h^2(2\xi_0)^{-2}\beta^2(2\beta^2\xi'^2 \pm 2\beta\xi' + \frac{3}{4} + E_1)(\beta^2\xi'^2 + \beta\xi' - E_1)^{-2}$$

Operators $\tilde{P}_{\beta}$ and $\tilde{P}_{-\beta}$ are unitarily equivalent, and so have the same spectrum. Up to the $O(h^2)$ term, $\tilde{P}_{\pm, \beta}(-hD_{\xi'}, \xi', h)$ have the structure of an “anharmonic oscillator”, with “potential wells” at $\xi' = 0, \pm 1/\beta$ separated by a “barrier” at $\xi' = \mp 1/(2\beta)$. It is also well known [HeSj] that, viewed as a $h$-PDO of order 0, microlocally defined near $(x', \xi') = 0$, $\tilde{P} = \tilde{P}_{\pm, \beta}$ can be taken to the normal form of a harmonic oscillator, away from the “barrier”. More precisely, there exists a real-valued analytic symbol $F(t, h) = F_{\pm, \beta}(t, h) \sim \sum_{j=0}^{\infty} F_j(t)h^j$, defined for $t \in \text{neigh}(0)$, $F_0(0) = 0$, $F_0'(0) = \frac{1}{2}$, $F_1(t) = \text{Const}$., and (formally) unitary FIO’s $A = A_{\pm, \beta}$ whose canonical transformations $\kappa_A$ defined in a neighborhood of $(0,0)$, are close to identity and map this point onto itself, such that

$$A^* F(\tilde{P}, h)A = P_0 = \frac{1}{2}((hD_\eta)^2 + \eta^2 - h)$$

Define the large parameter $\nu$ by $F(e^{\nu h_{\xi, \alpha}}, h) = \nu h$. So $\tilde{u} = \tilde{u}_{\pm}$ solves (1.2) microlocally near $(0,0)$ iff $\nu = \nu^* \tilde{u}$ solves Weber equation $(P_0 - h \nu)\nu = 0$ microlocally near $(0,0)$, when $h \nu \sim \frac{E^2}{h_{\xi, \alpha}}$ is small enough. The well known parabolic cylinder functions $D_{\nu}$ and $D_{-\nu-1}$, provide with a basis of solutions of $\frac{1}{2}((hD_\eta)^2 + \eta^2 - h)\nu = \nu \nu$. We shall use $D_{-\nu-1}$, and write

$$v = \nu^* \tilde{u}_{\pm, \beta} = \sum_{\varepsilon = \pm 1} \alpha^{(-\nu-1)^2} D_{-\nu-1}(i\varepsilon(h/2)^{-1/2}\eta)$$

$\alpha_{\varepsilon, \pm, \beta} = \begin{pmatrix} a \pm \beta \nu \\ a \pm \beta \nu \end{pmatrix}$
for complex constants \( \alpha^{(-\nu-1)}_{\varepsilon,\pm\beta} \).

These microlocal solutions can be expressed in the spatial representation by taking inverse \( \hbar \)-Fourier transformation; expanding integrals of the type (1.4) by stationary phase, both pieces of bicharacteristics \( \rho_{\pm} \) contribute to \( U_{\varepsilon,\pm\beta} \) near \( a \). Microlocal solutions near \( a' \) are deduced by PT symmetry.

Once microlocal solutions \( U_{\varepsilon,\beta}^{a,-\nu-1} \) have been obtained that way near the branching point \( a \), it is standard to extend them up to \( a' \) as WKB solutions \( (U_{\varepsilon,\beta}^{a,-\nu-1})_{\text{ext}} \), taking advantage that \( \mathcal{P} \) has simple characteristics away from \( a, a' \). When \( \Delta(x) \equiv 0 \), i.e. for \( -x_1 \leq x \leq x_1 \), they are completely decoupled, which means that the solution is either a pure electronic state, i.e. colinear to the vector \( (1) \) or pure hole state, i.e. colinear to \( (0) \). Otherwise, they are a superposition of electronic/hole states. We summarize these constructions in the :

**Proposition 1.2:** For \( x < x_0 \) near \( x_0 \), there are 2 basis of oscillating microlocal solutions of \((\mathcal{P}^a - E)U = 0\) indexed by \( \varepsilon = \pm 1 \):

\[
\sum_{\rho = \pm 1} U_{\rho,\varepsilon,\pm\beta}^{a,\nu}(x,h'), \sum_{\rho = \pm 1} U_{\rho,\varepsilon,\pm\beta}^{a,-\nu-1}(x,h')
\]

Here the branch with \( \rho = \rho_{\pm} = \pm 1 \) is microlocalized on \( \rho_{\pm} \), i.e. the part on \( \rho_{+} \) (\( \xi > \xi_0 \) near \( a \)), belongs to the electron state, while the part \( \rho_{-} \) (\( \xi < \xi_0 \) near \( a \)) belongs to the hole state; they satisfy, for \( \rho = \pm \):

\[
U_{\rho,\varepsilon,\beta}^{a,-\nu-1} = U_{\rho,\varepsilon,\beta}^{a,-\nu-1} + O(h)
\]

and

\[
U_{\rho,\varepsilon,\beta}^{a,\nu-1} = U_{\rho,\varepsilon,\beta}^{a,\nu-1}
\]

Each of these solutions is an admissible \( \mathcal{C}^2 \)-valued lagrangian distribution in the sense of Definition 1.1. Divide all microlocal solutions by the trivial factor \( e^{ix/4}e^{iE_0\xi_0/h'} \), \( E_0 = E - \alpha x_0 \). Then with the notations of (1.4) the general solution of \((\mathcal{P}^a - E)U = 0\) is of the form

\[
U = \sum_{\rho,\varepsilon} \mathcal{O}^{(-\nu-1)}_{\varepsilon,\pm\beta} U_{\rho,\varepsilon,\pm\beta}^{a,-\nu-1}
\]

The solutions near \( a' \) are given by symmetry, e.g. \( U_{\rho,\varepsilon,\beta}^{a',\varepsilon,\beta} = \mathcal{J}U_{\rho,\varepsilon,\beta}^{a,-\nu-1} \). Moreover both microlocal families can be extended as WKB solutions (satisfying Definition 1.1) along the bicharacteristics.

Note also that in this region where \( \mu(x) \) is a constant, \( U_{\rho,\varepsilon,\pm\beta} = e^{ix\xi_0/h}U_{\rho,\varepsilon,\pm\beta\varepsilon h'} \) with \( U_{\rho,\varepsilon,\pm\beta\varepsilon h'} \) oscillating on a frequency scale \( 1/h' = 1/(\alpha h) \), so if we think of the slope \( \alpha \) to be large, \( U_{\rho,\varepsilon,\pm\beta} \) behaves as a plane wave \( e^{ix\xi_0/h} \), modulated by a slowly varying function.

**b) Real holonomy and approximate Bohr-Sommerfeld quantization condition.**

The microlocal kernel \( K_{\hbar}(E) \) of \( \mathcal{P} - E \) on \( ] - x_0, x_0 [ \times \mathbb{R}_+ \) can be viewed as a 4-D fibre vector bundle \( \mathcal{F}_{\hbar}(E) \) of admissible Lagrangian distributions over \( \mathbb{S}^1 \). We characterize the real part of the
For each microlocal solution $K_a$ orthonormal basis of the contributions of The Lagrangian distributions $V(1.7)$ such that $(U, V)$ for $W$ right, and due to symmetry, the independent, modulo error terms $O_M$ tension of the normalized microlocal solutions $\rho$ given by action integrals along $\chi$ introduced in [HeSj], [Ro]. Namely, let $\chi = \chi^a$ be a smooth cut-off supported on a sufficiently small neighborhood of $a$, equal to 1 near $a$, $\omega_{\pm} = \omega_{\pm}^a$ a small neighborhood of $\rho_{\pm} \cap \text{supp}[P, \chi^a]$, and $\chi_{\omega_{\pm}} = \chi_{\omega_{\pm}}^a$ a cut-off equal to 1 near $\omega_{\pm}$. We take Weyl $h$-quantization of these symbols, and for $U, V \in K_h(E)$, we call

$$W_{\omega_{\pm}}(U, V) = (\chi_{\omega_{\pm}} \frac{i}{h}[P, \chi]|U|V) = (\chi_{\omega_{\pm}} \frac{i}{h}[P, \chi]|\hat{U}|\hat{V})$$

the microlocal Wronskian of $(U, V)$ in $\omega_{\pm}$. This is a sesquilinear form on $K_h(E)$, and $W_{\omega_{\pm}}(U, U)$ is independent, modulo error terms $O(h^\infty)$, of the choices of $\chi^a$ and $\chi_{\omega_{\pm}}^a$ as above. Taking into account both contributions of $\rho_{\pm}$ we define also

$$W(U, V) = W_{\omega_{+}}(U, V) + W_{\omega_{-}}(U, V)$$

For each microlocal solution $\hat{U} = \hat{U}_{\epsilon, \pm, \beta}^{a, -\nu - 1}$, it turns out that $W(U, V)$ have asymptotic expansions in $h'$, of the form $w_0(E, \beta) + h'w_1(E, \beta) + \cdots$, with $w_0(E, \beta) > 0$.

Given $\chi = \chi^a$, let now $\bar{\chi} = \bar{\chi}^a$ be a new cut-off equal to 1 on the support of $\chi^a$, and to 0 outside a slightly larger set. For $U, V \in K_h(E)$ we set $(U|V)_{\bar{\chi}} = (\bar{\chi}U|V)$. Then it is easy to see that there is an orthonormal basis of $K_h(E)$ for the “scalar product” $(U|V)_{\bar{\chi}}$, which is at the same time orthogonal for $W(U, V)$ (everything being defined modulo $O(h^\infty)$.) This allows to find $V_{\epsilon} = V_{\epsilon, \beta}^{a, -\nu - 1}$ of the form (1.7) such that $(V_{\epsilon}|V_{\epsilon'})_{\bar{\chi}} = \delta_{\epsilon, \epsilon'}$, $(\epsilon, \epsilon' = \pm 1)$, $W(V_{\pm}, V_{\pm}) > 0$, and $W(V_{\pm}, V_{\mp}) = 0$. Of course, by the symmetry $\nabla TP(x, hD_x) = P(x, hD_x)\nabla'$, such normalized microlocal solutions exist as well near $a'$. The Lagrangian distributions

$$F_{\epsilon, \beta}^{a, -\nu - 1} = \chi_{\omega_{\pm}} \frac{i}{h}[P, \chi^a]|U_{\epsilon, \beta}^{a, -\nu - 1}$$

and similarly $F_{\epsilon, \beta}^{a, -\nu - 1}$ span the microlocal co-kernel $K_h^*(E)$ of $P - E$ in $]-x_0, x_0[\times \mathbb{R}_+$, as $\epsilon = \pm 1$.

The same holds for or $G_{\epsilon, \beta}^{a, -\nu - 1}$ obtained by replacing $U_{\epsilon, \beta}^{a, -\nu - 1}$ by the “orthonormal basis” $V_{\epsilon, \beta}^{a, -\nu - 1}$ as above.

Because of Proposition 1.2, the normalized microlocal solutions $V_{\epsilon, \beta}^{a, -\nu - 1}$ are related to the extension of the normalized microlocal solutions $V_{\epsilon, \beta}^{a, -\nu - 1}$ along the bicharacteristics by a monodromy matrix $M^{a, a'} = \left(\begin{array}{cc}d_{11} & d_{12} \\ d_{21} & d_{22}\end{array}\right) \in U(2)$. Similarly, we obtain $M^{a', a}$ by extending from the left to the right, and due to symmetry, $M^{a', a} = (M^{a, a'})^{-1} = (M^{a, a'})^*$. Diagonal entries of these matrices are given by action integrals along $\rho_{\pm}$ (see e.g. [Ro]). Off-diagonal terms are $O(h')$ and can be computed with the help of the Wronskian (in the ordinary sense) associated with the system $(P - E)U = 0$ (see e.g. [Ba]).

The quantization condition is satisfied, precisely when the rank of that system drops of one unit (actually, because of degeneracy, of 2 units), i.e. when dim $K_h(E) = \text{dim } K_h^*(E) = 2$. This amounts to set to zero the determinant of some Gram matrix $\text{Gram}(E, h)$ expressed in the basis
(V_{a,-\nu}^{\nu} \cdot V_{a',-\beta}^{\nu'}, \cdot \cdot \cdot \cdot \cdot (G_{e,\beta}^{\nu}, \cdot \cdot \cdot \cdot \cdot G_{e',-\beta}^{\nu'}). So E = E_k(h) is an eigenvalue, modulo $O(h^\infty)$, of $P(x, hD_x)$, corresponding to an Andreev state, iff det Gram(E, h) = 0.

Here we note the sensitivity of the energy levels $E_k(h)$ with respect to $\phi$. In the “hard-wall” limit $\alpha \to \infty$, we recover the quasi-particle spectrum, of the form $\cos \phi = \cos\left(\frac{g(E_k(h))}{h} - 2 \arccos\left(\frac{E_k(h)}{\Delta_0}\right)\right)$ for some smooth function $g$ (see [CayMon], [ChLesBl]).

2) The imaginary part of the resonances.

The considerations above are not sufficient to account for exponentially small corrections to $E_k(h)$. Further information will be extracted from a Grusin problem.

a) Microlocal solutions with complex phase.

Microlocal solutions, computed in the real phase space, are purely oscillating in the metallic lead $[-x_0, x_0]$. To get information in the “superconducting part of the junction”, we need use “infinitesimal” invariance by time reversal and conjugation of charge. The substitutions $\xi' \mapsto \pm i\xi'$, or equivalently, $\beta \mapsto \pm i\beta$, leave invariant equation (1.1), with a new operator $\tilde{P}_{\pm i\beta}$ in Fourier-Laplace representation. Microlocal solutions of $(\tilde{P}_{\pm i\beta} - E)\tilde{u} = 0$ are constructed similarly and, on the real domain, independently of those of $(\tilde{P}_{\mp i\beta} - E)\tilde{u} = 0$.

Thus the fibre bundle of microlocal solutions on $R \times R_+$ (i.e. microlocal kernel of $P - E$) splits as $F^<_h(E) \oplus F_h(E) \oplus F^>_h(E)$, where we recall $F_h(E)$ from Sect.1, and $F^<_h(E)$ are 2-D (trivial) fibre bundles over $R$.

Nevertheless, taking advantage that the coefficients are analytic near $a, a'$, there is a way to couple $F^<_h(E)$ with $F_h(E)$ in the complex domain. This, together with the assignment that the global section be “outgoing” at infinity, accounts for complex holonomy.

First we investigate complex holonomy near $a$, and consider the family of operators, obtained by extending $\tilde{P}_{\pm \beta}$ along a path $\{e^{i\gamma}\beta, 0 \leq \gamma \leq 2\pi\}$ in the complex plane; similarly, we consider the family of Lagrangian distributions obtained by extending $\tilde{u}_\beta(\xi')$ along that path. They will solve (1.1) iff $\gamma = 0, \pm \pi/2, \pi$.

These are related through their Lagrangian manifolds as follows : consider (for simplicity) the principal part of $\tilde{P}_{\pm \beta}$ and $\tilde{P}_{\pm i\beta}$, namely $\tilde{Q}_\beta(-hD_{\xi'}, \xi') = (hD_{\xi'})^2 + (\xi' + \beta \xi'^2)^2$ and $\tilde{Q}_{i\beta}(-hD_{\xi'}, \xi') = (hD_{\xi'})^2 + (\xi' + i\beta \xi'^2)^2$. The potentials being equal for $\xi' = 0$ and $\xi' = -2/(1 + i)\beta$, the real Lagrangian manifold $\rho_+$ near $a$ extends analytically along the loop $\{e^{i\gamma}\beta : \gamma \in [0, 2\pi]\}$ in the complex domain, so that it intersects $\rho_+^2$ at $-2/(1 + i)\beta$ for $\gamma = \pi/2$. We can argue similarly for the other branches. Actually, both $\rho_\pm$ and $\rho_+^2$ are branches of a single 2-sheeted Riemann surface, with complex “turning points”.

We can assign to this analytic manifold microlocal solutions for $\tilde{P}_{e^{i\gamma}\beta}$ as in (1.2) with complex phase, which yields in turn solutions of $(P - E)U = 0$ for relevant values $0, \pm \frac{\pi}{2}, \pi$ of the parameter $\gamma$ ; these solutions are very similar to the $U_{\varepsilon, \pm \beta}$’s given in Proposition 1.2. The monodromy operator, acting on microlocal solutions, is known as connection isomorphism, see [DeDiPh] and references therein, and also [Fe], or [Ro,Sect.4,g] in the case of a system. This connection isomorphism is given by a matrix $N^a \in U(2)$, whose entries are expressed in term of exponentials of action integrals computed along Stokes lines between the complex turning points.
Let us consider next the conditions at infinity: for $|x| > x_2$, $P$ has constant coefficients, so we make an analytic dilation of the form $x \mapsto \exp[(\text{sgn} x)\vartheta] x$, $\vartheta > 0$. Plane waves with positive momentum have the phase $\exp[i x (\xi_1 + i \xi_2)/h]$ where $\xi_1 \pm i \xi_2 = (\mu_0 \pm i \sqrt{\Delta_0^2 - E^2})^{1/2}$, according to the choice of $\rho_{<,>}$. Analytic distortion is turned on for $|x|$ large enough, and $\vartheta$ in the complex upper-half plane. We denote by $P_\vartheta$ the distorted operator. So for all $\text{Im} \vartheta \geq 0$ small enough, we can make the “electronic state” (resp. “hole state”) exponentially decaying at $+\infty$ (resp. $-\infty$), which models the scattering process $e^+ \to e^−$, and similarly for the scattering process $e^− \to e^+$, thus preserving conservation of charge.

b) A Grusin problem and the width of resonances.

Following a classical procedure in Fredholm theory, we can translate the original eigenvalue problem for $P$ into a finite dimensional problem via the Grusin operator [HeSj3, Sect 4]; this is essentially the isomorphism $(H^2(\mathbb{R}) \otimes C^2)/\tilde{K}_h(E) \to \text{Ran}(P - E) \subset L^2(\mathbb{R}) \otimes C^2$. Here $\tilde{K}_h(E)$ denotes the 6-D microlocal kernel of $P - E$ in $\mathbb{R} \times \mathbb{R}_+$, restricted to the set of outgoing functions defined above. For $P = P_\vartheta$, we consider $G(E) = G(\vartheta, E)$ of the form:

$$G(E) = \left( \begin{array}{cc} P - E & R_- \\ R_+ & 0 \end{array} \right) : (H^2(\mathbb{R}) \otimes C^2) \times C^6 \to (L^2(\mathbb{R}) \otimes C^2) \times C^6$$

(2.1)

$$R_-(x_1, \cdots, x_6) = \sum_{j=1}^6 x_j G_j, \quad R_+ U = ((U\vert G_j))_{1 \leq j \leq 6}$$

where the $G_j$’s range over the basis of co-kernel $\tilde{K}_h^*(E)$ consisting of $G_{a,\beta}^\alpha, G_{a, -\beta}^\alpha, G_{a, i\beta}^\alpha, G_{a, -i\beta}^\alpha$ (or their analytic continuation at the branching points).

At this point we make the following remark: Since our Grusin operator (2.1) involves only positive frequencies, it cannot be associated with the self-adjoint operator $P_\vartheta$ (for real $\vartheta$). But resonances are due precisely to a breaking of time-reversal symmetry, and their imaginary part is computed by introducing a $h$-Pseudo-differential cutoff $\Phi(x, hD_x)$ supported in $\{\xi > 0\}$. Because negative frequencies will be eventually removed, we may best think of (2.1) as a short-hand notation for the “full” Grusin operator $G(E)$, that would take into account the negative frequencies as well.

For all $h > 0$ small enough, $G(E)$ is bijective, with bounded inverse

$$E(E) = \left( \begin{array}{cc} E_0(E) & E_+(E) \\ E_-(E) & E_-(E) \end{array} \right)$$

and has the property, that $E$ is an eigenvalue of $P$ iff $\det E_-(E) = 0$. The construction of $E(E)$ is carried as in [HeSj], [Ro], selecting solutions according to the prescriptions above. Matrix $E_-(E)$ decouples modulo $O(h^\infty)$, with a $4 \times 4$ block conjugated to Gram($E$); the interaction with the “incoming hole” and “outgoing electron” occurs through the “turning points” in the complex domain, involving the connection isomorphisms $N^\alpha, N^\alpha'$.

For complex $(\vartheta, E)$, we note that $(P_\vartheta - E)^* = P_{\overline{\vartheta}} - \overline{E}$. Applying distortion to the Grusin operator as well, we get:

$$G(\vartheta, E) = \left( \begin{array}{cc} P_\vartheta - E & R_-(\vartheta, E) \\ R_+(\vartheta, E) & 0 \end{array} \right), \quad E(\vartheta, E) = \left( \begin{array}{cc} E_0(\vartheta, E) & E_+(\vartheta, E) \\ E_-(\vartheta, E) & E_-(\vartheta, E) \end{array} \right)$$
We can prove that $G(\vartheta, E)$ is well-posed for all $\vartheta \in \mathbb{C}$ small enough, with inverse $E(\vartheta, E)$. Recall from [Ro,Prop.7.1] the following identity:

**Proposition 2.1:** Let $\Phi \in C_0^\infty(\mathbb{R}^2; \mathbb{R})$. With the notations above

\[
\begin{align*}
[R_-(\vartheta, E)^*\Phi E_+(\vartheta, E)]^*E_-(\vartheta, E) - ([R_-(\vartheta, E)^*\Phi E_+(\vartheta, E)]^*E_-(\vartheta, E))^* \\
= E_+(\vartheta, E)^*[P_{\vartheta}, \Phi]E_+(\vartheta, E)
\end{align*}
\]

In the self-adjoint case, the corresponding statement would be \( (R_+^{*} E_+) E_- \) is self-adjoint. The determination of the width of resonances then goes as in [Ro], though it is somewhat more complicated due to the structure of $E_-(\vartheta, E)$. Take $W(\vartheta, E) \in \text{Ker} E_- (\vartheta, E)$, and set $A(\vartheta, E) = [R_-(\vartheta, E)^*\Phi E_+(\vartheta, E)]^*$. From (2.2) and the identity

\[
(W(\vartheta, E)|A(\vartheta, E)E_-(\vartheta, E)W(\vartheta, E)) - (A(\vartheta, E)E_-(\vartheta, E)W(\vartheta, E)|W(\vartheta, E)) = 0
\]

we get

\[
(A(\vartheta, E)E_-(\vartheta, E)W(\vartheta, E)|W(\vartheta, E)) - (A(\vartheta, E)E_-(\vartheta, E)W(\vartheta, E)|W(\vartheta, E))
\]

Evaluating both members of this equality gives an implicit equation for the imaginary part of the resonance, showing that behaves like $\exp[-2\int_\tau \xi dx/h']$, where $\tau \subset \mathbb{C}$ is a path connecting the complex branching points in $\rho_- \cap \rho_+$.

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