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Decision Support

Spatial multi-attribute decision analysis: Axiomatic foundations and incomplete preference information

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This paper advances the theoretical foundations and the methodology of spatial decision analysis in which multi-attribute consequences of decision alternatives vary over a spatial region. First, we introduce necessary and sufficient conditions for representing the decision maker’s preferences among such decision alternatives with an additive spatial value function. This new axiomatization allows for the representation of preferences when the spatial region consists of an infinite number of locations, which is often the case in practical applications. Moreover, we show that spatial value functions suggested in the existing literature can be interpreted as special cases of our additive spatial value function. Second, motivated by the high effort required to elicit preferences in spatial decision problems, we develop a method for utilizing the additive spatial value function with incomplete preference information about spatial weights describing the importance of locations and attribute weights. This method provides defensible decision recommendations through the use of dominance concepts and decision rules. The applicability of the developed value function and analysis method is illustrated with a real-life application in air defense planning.

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1. Introduction

Many decisions have consequences that vary across a spatial or geographical region. For instance, consider the allocation of rescue helicopters among bases with the objective of providing rescue services throughout a country. Each alternative allocation implies an average response time for each location and region of the country. Some locations can be considered more important than others due to, e.g., high population or limited reach of land-based rescue services. These types of spatial decision problems in which the decision maker (DM) has to consider trade-offs among decision alternatives’ (multi-attribute) consequences across some spatial region are also common in, e.g., urban, environmental and transportation planning as well as in waste management, hydrology, agriculture, and forestry (see, e.g., Ferretti & Montibeller 2016; 2017; Keisler & Sundell 1997; Malczewski & Rinner 2015).

Technological advances in the collection and processing of spatial data with, e.g., geographic information systems, have led to an increased interest in developing decision analysis models, methods, and practices to tackle spatial decision problems (see, e.g., Wallenius et al. 2008). For instance, Malczewski & Rinner (2015) provide an overview on the application of multi-attribute decision analysis in the spatial context, while Ferretti & Montibeller (2016) identify key challenges in designing spatial decision support systems. Simon, Kirlwood, & Keller (2014) in turn lay the decision theoretic foundations of spatial preference models. In particular, for problems with a spatial region consisting of a finite number of locations, they establish necessary and sufficient conditions for representing the DM’s preferences among spatial decision alternatives with an additive homogeneous value function. That is, the same value function is used at each location to map the location-specific consequences onto a value scale. An alternative’s overall value is then obtained as a weighted average of these location-specific values, where the weights capture the importance of locations. Moreover, Simon et al. (2014) extend this model to multi-attribute problems by providing necessary and sufficient conditions for using an additive multi-attribute value function to aggregate the multi-attribute consequences at each location. Recently, Keller & Simon (2017) have developed non-additive

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preference models for problems in which spatial regions consist of a finite number of locations.

Existing spatial decision analysis models are not, however, well-suited for dealing with spatial regions consisting of a large or even infinite number of locations, which is often the case in practical applications. First, location weights are key parameters of spatial preference models as they represent the trade-offs the DM is willing to accept among consequences of different locations. In the existing models, the weight of each location must be specified. The effort required by the DM to assess all of these weights far exceeds that of traditional decision analysis applications, where the specification of dozens of attribute weights would be considered laborious. Second, the current literature does not offer an axiomatization for preference models in spatial decision problems where the spatial region consists of an infinite number of locations. Simon et al. (2014) conjecture that under reasonable assumptions, preferences could be represented with a value function in which an alternative’s value is obtained by integrating the product of location-specific weights and consequence values over the region. Nevertheless, the methodology of decision analysis lacks a solid axiomatic foundation for tackling problems with an infinite spatial region.

This paper addresses both of these gaps in the existing literature on spatial decision analysis. We provide an axiomatic basis for representing the DM’s preferences with an additive spatial value function when the spatial region consists of an infinite number of locations. This function captures preferences with two components: a consequence value function that maps the location-specific consequences of decision alternatives to a value scale, and a spatial weighting function that assigns a spatial weight to each subregion, i.e., subset of the spatial region. The spatial weights of subregions can be interpreted as their relative importance. The consequence value function can be either single- or multi-attribute. Hence, the additive spatial value function can also be deployed in problems where consequences are measured by several attributes. The spatial region can be of virtually any form including, but not limited to, a two-dimensional geographical area, a three-dimensional airspace, a one-dimensional road, or a time interval. The presented axioms are both necessary and sufficient. That is, any preference relation among the alternatives satisfying these axioms can be represented with an additive spatial value function, and the preference relation implied by any additive spatial value function satisfies these axioms. Moreover, the spatial value functions proposed by Simon et al. (2014) in their conjectures are obtained as special cases of the additive spatial value function provided by our axiomatization.

Our second contribution is a novel analysis method for deriving defensible decision recommendations based on incomplete spatial preference information. Importantly, the method does not assume that the spatial region consists of discrete subregions in which the alternatives yield constant consequences. Moreover, the method does not require specifying exact weights for all locations, but the DM may, e.g., provide an ordinal importance ranking of some subregions, or state her preference between two hypothetical decision alternatives. Incomplete preference information is represented as constraints on the spatial weights that the spatial weighting function can assign to subregions. Furthermore, the method includes computational approaches for (i) comparing the values of decision alternatives across all weighting functions satisfying these constraints, and (ii) providing decision recommendations based on dominance relations and decision rules. The method is also able to handle incomplete preference information on attribute weights when an additive multi-attribute consequence value function is used to aggregate multi-attribute consequences at each location.

The applicability of the new spatial value function and analysis method is illustrated with a realistic application to air defense planning. The air force commander of a country considers the possibility of air bases with the objective of maximizing air defense capability. The consequences of alternatives are computed with an air mission planning software and real attributes are used to measure air defense capability. The application illustrates how the additive spatial value function and the spatial analysis method (i) capture incomplete preference information through sets of feasible spatial weights of subregions and attribute weights, (ii) aid the iterative exploration of the DM’s preferences, and (iii) produce decision recommendations.

Beyond the spatial decision analysis literature discussed above, this paper intersects two other strands of research worth acknowledging. First, our axiomatization of the additive spatial value function makes use of the mathematical results underlying the Subjective Expected Utility theory, which was established by Savage (1954) and later refined by Fishburn (1970). Second, the modeling of incomplete preference information in spatial problems is motivated by its successful use in traditional areas of decision analysis, such as multi-attribute value models (e.g., Argyris, Morton, & Figueira 2014; Dias & Climaco 2000; Kadziński, Greco, & Słowiński 2012; Punkka & Salo 2013; Salo & Hämäläinen 1992; Weber 1985; 1987) and project portfolio selection (Filedner & Liesiö 2016; Liesiö, Mild, & Salo 2007; 2008; Liesiö & Salo 2012; Lourenço, Morton, & Bana e Costa 2012; Podinovski 2010; Tervonen, Liesiö, & Salo 2017; Villkumaa, Liesiö, Salo, & Ilmona-Shippeard 2018). Similar approaches are also employed in, e.g., data envelopment analysis (Salo & Punkka 2011) and simulation based decision making (Mattila & Virtanen 2015). It is worth highlighting that these earlier approaches cannot be applied to spatial problems, in which the generation of decision recommendations under incomplete preference information requires solving optimization problems with infinite-dimensional decision variable spaces.

The rest of the paper is structured as follows. Section 2 develops the additive spatial value model by providing the underlying axiomatization and the resulting representation theorem. It also discusses the assessment and interpretation of components of the additive spatial value function. Section 3 introduces the spatial analysis method for representing incomplete preference information, identifying dominance relations among decision alternatives and ranking the alternatives based on decision rules. Section 4 presents the application in the context of air defense planning. Section 5 discusses the relationship of our work to existing literature and its implications to decision support. Section 6 presents concluding remarks.

2. The additive spatial value model

2.1. Preliminaries

The spatial region under consideration is represented by the set $S$. Each location $s$ in the region corresponds to an element of this set, i.e., $s \in S$. Subsets $S' \subseteq S$ of the region are referred to as subregions. Decision alternatives correspond to mappings $z : S \rightarrow C$ that assign a consequence $c \in C$ to each location $s \in S$, where $C$ is the set of all possible consequences. The set of all such mappings $z$, i.e., the set of all possible decision alternatives, is denoted by $Z = \{z \mid z : S \rightarrow C\}$. Note that we use the term ‘subregion’ to refer to an arbitrary subset of the spatial region, not to a specific subset over which alternatives yield a constant consequence (cf. Simon et al. 2014). The set $S$ is not assumed to have any special structure (e.g., $S \subseteq \mathbb{R}^n$ is not assumed), although the axioms presented in Section 2.2 imply that $S$ contains an infinite number of locations. No assumptions are made regarding the set $C$ either: it can consist of a finite or infinite number of single- or multi-attribute consequences.

To illustrate the notation, consider the problem of selecting the position of a rescue helicopter base with the aim of providing res-
cue services across a country as quickly as possible. The region $S$ corresponds to the territory of the country and is here represented by the set $S = [0, 1] \times [0, 1]$. Thus, each geographical position within the country corresponds to a location $s = (s_x, s_y) \in S$ with the coordinates $s_x, s_y \in [0, 1]$. Subregions $S' \subseteq S$ are parts of the country, such as cities or counties. The objective is to minimize the time it takes for a helicopter to reach each location on average. Hence, the set of consequences $C = [0, \infty)$ represents the range of possible response times. Each potential position of the base i.e., each decision alternative is represented by a function $z \in Z$ that assigns a response time $z(s) \in C$ to each location $s \in S$. Suppose that the base can be built at either the coordinates $(0.63, 0.10)$ or $(0.28, 0.55)$. Assuming that the response time is proportional to a location’s distance from the base, these two decision alternatives correspond to the functions $z^1(s) = \sqrt{(s_x - 0.63)^2 + (s_y - 0.10)^2}$ and $z^2(s) = \sqrt{(s_x - 0.28)^2 + (s_y - 0.55)^2}$. For a visual representation of this example, see Appendix C1.

As a second example, suppose that a new factory is being built in the same country. When choosing the site for this factory, the resulting pollution needs to be considered. The severity of the pollution is described by the average concentration of two types of pollutants in the air at each location. The concentrations are obviously non-negative, and thus the set of consequences is $C = [0, \infty)$. Thus, $c \in C$ is always a vector consisting of two elements. Each potential site for the factory, i.e., each decision alternative, is represented by a function $z \in Z$ that assigns a vector of concentrations $z(s) = [z_1(s), z_2(s)] \in C$ to each location $s \in S$. Since the propagation of pollutants depends on various factors, including weather conditions, these functions cannot be defined through analytical formulas. Instead, they would likely be estimated by computing the concentrations with a simulation model at a representative set of locations for each decision alternative, and then interpolating the results to cover the entire region.

### 2.2. Preference axioms

In this section, we develop the axiomatic basis for representing preferences with an additive spatial value function. In particular, we are not claiming that any rational DM always must follow these axioms, but rather our goal is to provide sufficient and necessary preference conditions under which the additive spatial value function is an appropriate tool for decision support. Formally, the DM’s preferences among all possible decision alternatives $z \in Z$ are captured by a binary relation $\succ$, and $z^1 \succ z^2$ denotes that the alternative $z^1$ is at least as preferred as the alternative $z^2$. Strict preference $\succ$ and indifference $\sim$ are defined in the usual manner based on $\succ$.

The relation $\succ$ is assumed to form a weak order on the alternatives $Z$, i.e., the relation is transitive ($z^1 \succ z^2$ and $z^2 \succ z^3$ imply that $z^1 \succ z^3$) and complete ($z^1 \succ z^2$ or $z^2 \succ z^1$ holds for any $z^1, z^2 \in Z$). Furthermore, it is required that there exists strict preference between at least two alternatives. These standard assumptions are formalized by the following two axioms.

**Axiom 1 (Weak order).** The relation $\succ$ is a weak order on $Z$.

**Axiom 2 (Non-trivial preferences).** There exist $z^1, z^2 \in Z$ such that $z^1 \succ z^2$.

The remaining axioms concern alternatives that are defined as combinations of two other alternatives. The following notation is used to make the representation of such alternatives more compact.

**Definition 1.** For any alternative $z^1, z^2 \in Z$ and subregion $S' \subseteq S$, $(z^1 \cup z^2, S') \in Z$ denotes an alternative whose consequences are identical to those of $z^1$, except for locations in $S'$, where the consequences are identical to those of $z^2$. That is,

$$
\langle z^1 \cup z^2, S' \rangle(s) =
\begin{cases}
    z^1(s), & s \notin S' \\
    z^2(s), & s \in S'.
\end{cases}
$$

The third axiom states that if two alternatives have identical consequences in some subregion, then preference between these alternatives remains the same even if the identical consequences in the subregion are altered.

**Axiom 3 (Spatial preference independence).** For any $S' \subseteq S$ and $z^1, z^2, z^3, z^4 \in Z$,

$$
\langle z^1 \cup z^2, S' \rangle \succ \langle z^2 \cup z^3, S' \rangle \quad \text{if} \quad \langle z^1 \cup z^3, S \rangle \succ \langle z^2 \cup z^4, S \rangle.
$$

**Axiom 4 (Consequence consistency).** For any $c^1, c^2 \in C$, $z \in Z$, and not full $S' \subseteq S$,

$$
\langle c^1, c^2 \rangle \succ \langle c^1 \cup z, S' \rangle \succ \langle c^2, z, S' \rangle.
$$

This axiom provides an interpretation for preferences among the consequences $C$. Specifically, the consequence $c^1$ is preferred to the consequence $c^2$ if the constant alternative $c^1$ is preferred to the constant alternative $c^2$.

The fifth axiom considers comparisons between two alternatives that only assign two different consequences, one of which is strictly preferred to the other. In particular, the first alternative yields the less preferred consequence across a subregion $S'$ and the more preferred consequence across its complement $S\setminus S'$. The second alternative, in turn, yields the less preferred consequence across a subregion $S''$ and the more preferred one across its complement $S\setminus S''$. The axiom states that preference between the two alternatives does not depend on what the two consequences are.

**Axiom 5 (Spatial consistency).** For any $S', S'' \subseteq S$ and $c^1, c^2, c^3, c^4 \in C$ such that $c^3 \succ c^2$ and $c^3 > c^4$,

$$
\langle c^1, c^2, S' \rangle \succ \langle c^1 \cup c^4, S'' \rangle \quad \text{if} \quad \langle c^1 \cup c^4, S' \rangle \succ \langle c^3, c^4, S'' \rangle.
$$

The sixth axiom states that strict preference between two alternatives is not affected by changes in the alternatives’ consequences as long as these changes are limited to a sufficiently small subregion. Specifically, for any consequence $c$, there exists a finite partition of $S$ into mutually exclusive and collectively exhaustive subregions such that modifying either alternative to assign $c$ across any
single subregion does not alter the strict preference between the alternatives.

**Axiom 6** (Divisibility of subregions). For any \( z^1, z^2 \in Z \) such that \( z^1 > z^2 \) and \( c \in C \), there exists a partition of \( S \) into a finite number of subregions \( S^1, \ldots, S^n \) such that
\[
(z^1; c, S^i) > z^2 \quad \text{and} \quad z^1 > (z^2; c, S^i) \quad \forall \ i \in \{1, \ldots, n\}.
\]

The last axiom considers preference between an alternative \( z^1 \) and an alternative that is obtained by changing the consequences of \( z^1 \) in some subregion \( S \). Specifically, the alternative \( z^1 \) is modified across the subregion \( S^i \) to yield the consequence that another alternative \( z^2 \in Z \) assigns for some specific location \( c \in C \). This modified alternative can be expressed as \( (z^1; z^2(s), S^i) \), where \( z^2(s) \in Z \) is a constant alternative that yields the consequence \( z^2(s) \in C \) at every location. In case the original alternative \( z^1 \) is strictly preferred to the modified alternative \( (z^1; z^2(s), S^i) \) for every choice of \( s \in S^i \), the axiom requires that \( z^1 \) is at least as preferred as an alternative in which the consequences of \( z^1 \) are replaced by those of the alternative \( z^2 \) in the entire subregion \( S^i \), i.e., \( z^1 \succeq (z^1; z^2(S^i), S^i) \). This means that the DM's preferences are not transitive, for instance, she is indifferent between alternatives \( z^2 \) and \( z^3 \) and alternatives \( z^2 \) and \( z^1 \), but strictly prefers \( z^2 \) to \( z^3 \).

**Axiom 7** (Monotonicity). For any \( z^1, z^2 \in Z \) and \( S' \subseteq S \),
\[
(z^1; z^2(s), S') \succeq (z^1; z^2(S'), S') \quad \forall s \in S' \Rightarrow (z^1; z^2(s), S') \succeq (z^1; z^2(S'), S') \quad \forall s \in S' \Rightarrow (z^1; z^2(s), S') \succeq z^1.
\]

Table 1 provides a summary of the seven axioms. While each of these axioms enforces conditions on the preference relation \( \succeq \), they have different roles from the viewpoint of practical applications. **Axioms 1 and 2** ensure a complete and transitive ranking among all possible decision alternatives, and thus form a reasonable basis for any prescriptive decision model. **Axioms 6 and 7** serve more technical purposes as they enforce the continuity and monotonicity of preferences with respect to specific changes in the alternatives’ consequences. Both of these axioms seem acceptable in practical applications in which the spatial region \( S \) can be partitioned into arbitrarily small subregions.

The remaining **Axioms 3–5** play a key role in establishing the functional form of the additive spatial value function, and hence the acceptability of these axioms in a particular application context should be evaluated. Fortunately, these axioms are relatively intuitive, and it is straightforward to construct verification procedures directly from their definitions. For instance, spatial preference independence (**Axiom 3**) can be tested by asking if the DM’s preference between two alternatives would change as a result of varying the common consequence the alternatives share in some subregion. In turn, **Axiom 5** can be checked by asking the DM’s preference between two bi-consequence alternatives, and then testing if the preference can be reversed by changing these two consequences. A similar procedure can be used to test if **Axiom 4** holds, although in practice it seems unlikely that the axiom would be rejected if **Axiom 5** has been found acceptable.

### 2.3. Additive spatial value function

**Axioms 1–7** are necessary and sufficient for representing the preference relation \( \succeq \) with the additive spatial value function. This is formally stated by the following theorem.

**Theorem 1.** The preference relation \( \succeq \) satisfies Axioms 1–7 if and only if there exists a non-atomic finitely additive measure \( \alpha \) on the region \( S \) and a non-constant bounded function \( v : C \to R \) such that for any \( z^1, z^2 \in Z \)
\[
z^1 \succeq z^2 \iff \int_S v(z^1(s))d\alpha(s) \geq \int_S v(z^2(s))d\alpha(s).
\]

Furthermore, \( \alpha \) is unique up to multiplication with a positive constant and \( v \) is unique up to a positive affine transformation.

Detailed proofs are presented in Appendix A, but Table 1 provides an overview of the role of each axiom.
Theorem 1 implies that if the preference relation ≥ satisfies Axioms 1–7, then the value of a decision alternative z is given by the function

\[ V(z) = \int v(z(s)) \, d\alpha(s), \tag{1} \]

This value function is specified by two components: (i) the consequence value function \( v: C \to \mathbb{R} \) that assigns a value to each consequence \( c \in C \), and (ii) the spatial weighting function \( \alpha: \mathbb{R}^2 \to [0, \infty) \) that specifies the spatial weight \( \alpha(S') \) for each subregion \( S' \subseteq S \) (\( \mathbb{R}^2 \) is the set of all subsets of \( S \)). Since the value of an alternative is obtained as a 'weighted sum' of the consequence values of all locations, we refer to \( V(z) \) as the additive spatial value function.

The consequence value function \( v \) is cardinal and bounded. Cardinality here means that it is unique up to positive affine transformations, i.e., if the consequence value functions \( v' \) and \( v'' \) imply exactly the same preferences \( \simeq \) among all decision alternatives \( z \in Z \), then \( v' = tv + \lambda \) for some real-valued constants \( t > 0 \) and \( \lambda \). Cardinality and boundedness allow the scaling of the consequence value function onto a suitable interval. Thus, without loss of generality, it is assumed throughout the rest of the paper that \( v(c) \in [0, 1] \) for all \( c \in C \). Notably, the consequence value function does not depend on the location at which the consequence is obtained. In case of multi-attribute consequences, \( v \) is a multi-attribute value function that maps the vector-valued consequences \( c = (c_1, \ldots, c_m) \) onto real numbers.

The spatial weighting function \( \alpha \) is a finitely additive non-atomic measure on \( S \). Like all measures, it is non-negative, i.e., the spatial weight \( \alpha(S') \) of any subregion \( S' \subseteq S \) is non-negative. Finite additivity means that the spatial weight of the union of two non-overlapping subregions is equal to the sum of the subregions' spatial weights, i.e.,

\[ \alpha(S^1 \cup S^2) = \alpha(S^1) + \alpha(S^2), \quad \text{for all } S^1, S^2 \subseteq S, \quad \text{s.t. } S^1 \cap S^2 = \emptyset, \tag{2} \]

and implies that the spatial weight of an empty subregion is zero \( \alpha(\emptyset) = 0 \). Note that since \( \alpha \) is finitely additive, every set in \( S^2 \) is measurable. This would not be possible if \( \alpha \) was required to be countably additive. Finally, a measure is said to be non-atomic if for any \( \delta \in [0, 1] \) and \( S' \subseteq S \) there exists a subregion \( S'' \subseteq S' \) such that \( \alpha(S'') = \delta \alpha(S') \). Since \( \alpha \) is non-atomic, the spatial weight of a single location is always zero \( \alpha(s) = 0 \) for any \( s \in S \). This in turn means that \( S \) must consist of an infinite number of locations, since otherwise \( \alpha(S) = 0 \) for any \( S' \subseteq S \). All these properties of the spatial weighting function clearly hold even if \( \alpha \) is multiplied by a positive constant \( \kappa \). Moreover, using either of the spatial weighting functions \( \alpha \) or \( \alpha' = \kappa \alpha \) in the additive spatial value function \( (1) \) implies exactly the same preferences \( \simeq \) among the decision alternatives \( z \in Z \). Thus, the spatial weighting function can be scaled such that \( \alpha(S) = 1 \), which motivates the following definition for the set of all spatial weighting functions.

**Definition 3.** The set of all spatial weighting functions, denoted by \( \mathcal{A} \), consists of all finitely additive measures \( \alpha \) on the region \( S \) which are non-atomic and satisfy \( \alpha(S) = 1 \).

The selected scalings for the spatial weighting functions \( \alpha \) and the consequence value function \( v \) imply that \( V(z) \in [0, 1] \) for any decision alternative \( z \in Z \). Note that limiting \( \alpha \) to non-atomic measures is not restrictive in practice. For instance, a city, no matter how small its area, corresponds to a subregion and not an individual location.

### 2.4. Assessment of the consequence value function

The consequence value function \( v \) can be assessed using standard MAVT procedures (for a recent overview see, e.g., Eisenführ, Weber, & Langer 2010). The choice of a suitable procedure depends on the type of consequences that the set \( C \) contains as well as the application context. For instance, if the set of consequences \( C \) corresponds to a continuous unidimensional measurement scale (e.g., monetary value, response time), the DM can be asked to specify a sequence of equally preferred improvements on this scale which can be used to interpolate a piecewise linear value function (see, e.g., von Winterfeldt & Edwards 1986, Chapter 7). The use of a pre-specified functional form, e.g., linear or exponential, for the value function can in some cases be appropriate (see, e.g., Kirkwood 1997, Chapter 4). Direct scoring of each consequence can be applicable when the set \( C \) consists of small number of verbal ratings such as ‘poor’, ‘good’, and ‘excellent’ (see, e.g., Clemens 1996, Chapter 15).

In applications with multi-attribute consequences, it is often possible to apply the above techniques for each attribute separately to obtain the attribute-specific value functions. The attribute-specific values can then be aggregated using, for instance, an additive, multiplicative, or nonlinear multi-attribute value function (see, e.g., Dyer & Sarin 1979). These functions differ with regard to the underlying preference assumptions (e.g., mutual preference independence of attributes), as well as the number of parameters that need to be assessed by the DM. For instance, the additive multi-attribute value function requires specifying only the weight of each attribute, for which there exist several well-established techniques such as the SWING-method (see, e.g., von Winterfeldt & Edwards 1986). The nonlinear value function, in turn, makes less restrictive assumptions about the attributes, but contains substantially more parameters to be specified.

### 2.5. Assessment of the spatial weights

The assessment of spatial weights is based on the comparison of hypothetical decision alternatives by the DM and thus resembles the assessment of attribute weights in standard MAVT. Different types of assessment questions can be structured by varying the consequences that these hypothetical alternatives yield for different locations. Answers to these questions provide information on the spatial weights \( \alpha(S') \) of given subregions \( S' \subseteq S \), thus implying constraints that the spatial weighting function \( \alpha \) must satisfy. In what follows, we present two possible assessment approaches.

The first approach is based on comparing the spatial weights of two subregions \( S^1, S^2 \subseteq S \). This is done by considering two alternatives that assign only two consequences \( c^+ \) and \( c^- \), where the former is strictly preferred to the latter, i.e., \( c^+ > c^- \). The first alternative \( (c^-; c^+; S^1) \) assigns \( c^- \) in the subregion \( S^1 \) and \( c^+ \) in its complement. The second alternative \( (c^-; c^+; S^2) \) assigns \( c^+ \) in the subregion \( S^2 \) and \( c^- \) in its complement. Preference for the first alternative over the second one implies

\[
\begin{align*}
V((c^-; c^+; S^1)) &\geq V((c^-; c^+; S^2)) \\
\iff &\int_{S^1} v(c^+) \, d\alpha(s) + \int_{S^2} v(c^-) \, d\alpha(s) \geq \int_{S^1} v(c^-) \, d\alpha(s) + \int_{S^2} v(c^+) \, d\alpha(s) \\
&\iff \alpha(S^1)v(c^+) + (1 - \alpha(S^1))v(c^-) \geq \alpha(S^2)v(c^-) + (1 - \alpha(S^2))v(c^+) \\
&\iff v(c^+) + (v(c^+) - v(c^-))\alpha(S^1) \geq v(c^-) + (v(c^+) - v(c^-))\alpha(S^2) \\
&\iff \alpha(S^1) \geq \alpha(S^2). 
\end{align*}
\tag{3}
\]

Based on (3), it is easy to verify that indifference between the alternatives implies that the spatial weights for the two subregions are equal, i.e., \( \alpha(S^1) = \alpha(S^2) \).

Eq. (3) formally links the importance of subregions to preferences between decision alternatives. Eq. (3) is valid regardless of what the two consequences \( c^+ \) and \( c^- \) are, as long as the former is preferred to the latter. Hence, it provides an unambiguous interpretation for the statement “Subregion \( S^1 \) is at least as important as subregion \( S^2 \).” Furthermore, it may be more convenient for the
DM to state her preference by directly comparing the importance of subregions rather than by explicitly comparing decision alternatives.

Eq. (3) also provides an interpretation for statements of the form “Subregion $S^1$ is at least twice as important as subregion $S^2$.” Assume that the subregion $S^1$ can be partitioned into two subregions $S^{1a}$ and $S^{1b}$, both of which are at least as important as the subregion $S^2$. This implies that

$$\alpha(S^1) = \alpha(S^{1a} \cup S^{1b}) = \alpha(S^{1a}) + \alpha(S^{1b}) \geq \alpha(S^2) + \alpha(S^2) = 2\alpha(S^2) \Rightarrow \alpha(S^1) \geq 2\alpha(S^2).$$

We derive similarly the interpretation for preference statements of the form “Subregion $S^1$ is at least $\frac{k_1}{k_2}$ times as important as subregion $S^2$”, where $k_1$ and $k_2$ are positive integers. Formally, this statement holds if $S^1$ and $S^2$ can be partitioned into $k_1$ and $k_2$ subregions, respectively, such that each of the former subregions is at least as important as any of the latter. In this case

$$\alpha(S^1) = \alpha \left( \bigcup_{i=1}^{k_1} S_i \right) \geq \sum_{i=1}^{k_1} \alpha(S_i) \geq k_1 \min_i \alpha(S_i) \geq k_1 \max_i \alpha(S_i),$$

where

$$\Rightarrow \alpha(S^1) \geq \frac{k_1}{k_2} \alpha(S^2).$$

In the second approach, the spatial weight of a subregion is assessed directly with the help of the consequence value function $v$. Consider three consequence $c^1$, $c^2$, and $c^3$ such that $c^1 \geq c^2 \geq c^3$ and $c^1 > c^3$, and a subregion $S' \subseteq S$. The alternative $(c^3; c^1, S')$ is compared to the constant alternative $c^2$. Preference for the first alternative results in a lower bound for the spatial weight $\alpha(S')$, since

$$(c^3; c^1, S') \geq c^2 \Leftrightarrow v(c^3)\alpha(S') + v(c^1)\alpha(S') \geq v(c^2) \Leftrightarrow v(c^3)(1 - \alpha(S')) + v(c^1)\alpha(S') \geq v(c^2) \Leftrightarrow v(c^1)\alpha(S') - v(c^3)\alpha(S') \geq v(c^2) - v(c^1) \Leftrightarrow \alpha(S') \geq \frac{v(c^2) - v(c^1)}{v(c^1) - v(c^3)}.$$  

(6)

The last step relies on $c^1 > c^3$, since this implies that $v(c^1) - v(c^3) > 0$. Selecting $c^1$ and $c^3$ such that $v(c^1) = 1$ and $v(c^3) = 0$ simplifies the result to $k = v(c^3)$. This approach can also be applied to elicit the spatial weight of a given subregion $S'$ by asking the DM to provide a consequence $c^2$ such that $(c^2; c^1, S') \sim c^2$. If such a consequence exists, it determines the spatial weight of the subregion precisely, i.e., $\alpha(S') = k$.

The exact weight of any subregion can be determined based on the above approaches. The spatial weights of smaller subregions within this subregion, however, are not known unless separately determined. Since the region $S$ consists of an infinite number of locations, and therefore an infinite number of subregions, an infinite number of comparisons would be required to precisely determine the weighting function $\alpha$. A finite number of comparisons results at best in a finite partition $S^1, \ldots, S^n$ of $S$ such that the spatial weight of each subregion $S^i$ is fixed, i.e., $\alpha(S^i) = \omega_i$. This information is insufficient for evaluating $V(z)$ for the decision alternative $z$, since $\alpha(S^i) = \omega_i$. Therefore, additional information is necessary to determine $\alpha$. One option is to assume that the spatial weight is uniformly allocated within each subregion $S^i$ in the sense of some natural measure relevant for the application context, e.g., distance, area, or volume. Such an assumption may, however, be difficult to justify. Fortunately, decision recommendations can be derived without specifying an exact spatial weighting function with the method developed in the following section.

3. Modeling incomplete preference information

In this section, we relax the requirement of a unique spatial weighting function. In particular, we present a spatial analysis method for attaining decision recommendations based on a set of feasible spatial weighting functions, while the consequence value function $v$ is still assumed to be known. The method builds on the DM providing information on her preferences with the approaches developed in Section 2.5. We also extend the method to cover spatial multi-attribute decision problems where the consequence value function is only partially specified. It is assumed to be an additive multi-attribute value function with known attribute-specific value functions, but incompletely defined attribute weights.

3.1. Set of feasible spatial weighting functions

We define spatial preference statements as linear constraints on spatial weights. Formally, the preference statement $\pi \in \mathbb{N}$ restricts the spatial weights assigned to the non-overlapping subregions $S_{(\pi, 1)}, S_{(\pi, 2)}, \ldots$ by requiring that

$$\sum_k p_{(\pi, k)} \omega(S_{(\pi, k)}) \leq 0,$$

(7)

where $p_{(\pi, 1)}, p_{(\pi, 2)}, \ldots$ are real-valued coefficients. It is easy to see that the preference assessment approaches developed in Section 2.5 (i.e., Eqs. (3)-(6)) provide preference statements of the form (7). Note that we do not assume that preferences are assessed based on any pre-specified set of subregions. Rather, each preference statement may concern a different set, i.e., $\{S_{(\pi, 1)}, S_{(\pi, 2)}, \ldots\} \neq \{S_{(\pi', 1)}, S_{(\pi', 2)}, \ldots\}$. The set of feasible spatial weighting functions is defined based on preference statements as follows.

Definition 4. A spatial weighting function $\alpha \in \mathbb{A}$ is feasible if it satisfies each preference statement (7) given by the DM. The set of all feasible spatial weighting functions is denoted by $\mathbb{A}$.

Fig. 1 illustrates the feasibility of spatial weighting functions on the unidimensional spatial region $S = [0, 1]$ (the horizontal axis). The first statement $\alpha(S_1) \geq \alpha(S_2)$ concerns the subregions $S_1 = [0, 0.4]$ and $S_2 = [0.6, 1]$, and the second statement $\alpha(S_2) \geq \alpha(S_3)$ concerns the subregions $S_2$ and $S_3 = [0.3, 0.6]$. Together the two statements define the set of feasible spatial weighting functions $\mathbb{A}$. The vertical axis shows the derivatives of three weighting functions $\alpha^1, \alpha^2, \alpha^3 \in \mathbb{A}$. Hence, the spatial weight each function assigns to a specific subregion $(\alpha^i(S_j))$ corresponds to the area between its derivative and the horizontal axis. The weighting function $\alpha^1$ is not feasible $(\alpha^1 \not\in \mathbb{A})$ since $\alpha^1(S_1) = 0.33 < \alpha^1(S_2) = 0.49$. In turn, the weighting functions $\alpha^2$ and $\alpha^3$ are feasible $(\alpha^2, \alpha^3 \in \mathbb{A})$ since $\alpha^2(S_1) = 0.43 > \alpha^2(S_2) = 0.24$ and $\alpha^3(S_1) = 0.67 > \alpha^3(S_2) = 0.27 > \alpha^3(S_3) = 0.13$. Moreover, the set $\mathbb{A}$ includes an infinite number of spatial weighting functions besides $\alpha^2$ and $\alpha^3$. This example also demonstrates that a preference statement concerning given subregions does not fix the exact shape of the feasible weighting functions inside these subregions.

By definition, each preference statement can make some weighting functions $\alpha \in \mathbb{A}$ infeasible, but the inclusion of additional preference statements can never make an infeasible weighting function feasible. The introduction of each new preference statement thus leads to the replacement of the set of feasible weighting functions $\mathbb{A}$ by its subset $\mathbb{A}' \subseteq \mathbb{A}$.
We use the term incomplete spatial preference information to describe the situation in which spatial weights are incompletely defined and there is a non-singleton set of feasible weighting functions $A \subseteq A^0$. Without any preference statements, the set of feasible weighting functions is $A = A^0$. A singleton set of feasible weighting functions $A = \{\alpha\}$ would in turn correspond to complete spatial preference information. Even though each preference statement can render some weighting functions infeasible, the set $A$ cannot be reduced to a singleton set by a finite number of preference statements, as was discussed at the end of Section 2.5.

3.2. Comparison of alternatives under incomplete spatial preference information

The value of each alternative $\alpha$ depends on which feasible spatial weighting function $\alpha \in A$ is used in the evaluation of the additive spatial value function $V(z)$. It is therefore more suitable to describe alternatives’ values as intervals when incomplete spatial preference information is used. Reducing the set of feasible weighting functions can narrow the value intervals, but will never widen them.

Decision analysis methods commonly utilize two concepts to provide decision recommendations under incomplete preference information, namely dominance and potential optimality (see, e.g., Weber 1985; Hazen 1986; Athanassopoulos & Podinovski 1997). In particular, an alternative dominates another if it has (i) a greater or equal value for all weighting functions $\alpha \in A$ and (ii) a strictly greater value for some weighting function. Thus, the DM can focus on non-dominated alternatives that are not dominated by any other alternative. In turn, an alternative is said to be potentially optimal if for some weighting function $\alpha \in A$ it has the highest value among all alternatives. In this paper, we focus on the use of dominance to provide decision recommendations. This choice is motivated by computational considerations: it turns out that checking whether or not an alternative dominates another can be established by solving a series of standard optimization problems even though the set $A$ is of infinite dimension. However, similar approaches cannot be utilized to establish if an alternative is potentially optimal.

Note that our choice of focusing on non-dominated rather than potential optimal alternatives does not involve the risk of discarding some alternatives that would be defensible choices. Specifically, consider that there is a potentially optimal alternative that is dominated. Clearly, such an alternative cannot be recommended to the DM because the dominating alternative (i) is also potentially optimal, (ii) yields a greater or equal value for all weighting functions, and (iii) has a strictly greater value for some weighting function. Moreover, it is well-known that the set of non-dominated alternatives can include alternatives that are not potentially optimal, i.e., they do not obtain the highest value for any weighting function. However, in some cases selecting such an alternative can be justified. For instance, a non-dominated alternative that yields a constant value across all weighting functions $\alpha \in A$ can be seen as a robust choice even if it is not potentially optimal. Hence, dominance offers a conservative approach for screening alternatives. The following definition formalizes this concept.

**Definition 5.** The alternative $z^1$ dominates the alternative $z^2$ with regard to the set of feasible spatial weighting functions $A$, denoted as $z^1 \triangleright z^2$, if

$$V(z^1) \geq V(z^2), \text{ for all } \alpha \in A.$$  
$$V(z^1) > V(z^2), \text{ for some } \alpha \in A. \quad (8)$$

Technically, $D_A$ is a binary relation on $Z$ that is transitive ($z^1 \triangleright z^2$ and $z^2 \triangleright z^3$ imply $z^1 \triangleright z^3$) and asymmetric ($z^1 \triangleright z^2$ and $z^2 \triangleright z^1$ cannot both hold). Furthermore, if the decision alternative $z^1$ dominates $z^2$ with regard to the set $A$, then $z^2$ cannot dominate $z^1$ with regard to any $A' \subset A$. If the dominating alternative $z^1$ has a strictly higher value than $z^2$ for every $\alpha \in A$, then it dominates $z^2$ with regard to any $A' \subset A$.

To illustrate the concepts of value intervals and dominance, we revisit the example from Fig. 1 and introduce decision alternatives $z^1$ and $z^2$, whose consequence values across the spatial region $S = [0, 1]$ are shown in Fig. 2. The highest value for both $z^1$ and $z^2$ is obtained with a spatial weighting function $\alpha$ that assigns a unit spatial weight for an arbitrarily small subregion around the location $s = 0.2 \in S_1$. This weighting function is clearly feasible ($\alpha \in A$), since $\alpha(S_1) = 1 > \alpha(S_2) = \alpha(S_3) = 0$. Thus, the upper bounds of the value intervals are $V(z^1) < 0.7$ and $V(z^2) < 0.5$. The lower bound $V(z^2) > 0.1$ for the decision alternative $z^2$ is similarly obtained by assigning a unit weight near the location $s = 0 \in S_1$. However, finding the lower bound for the value of the alternative $z^1$ is not as
straightforward. The minimum consequence value \( \min V(z(1)) = 0.4 \) is obtained at the location \( s = 0.9 \), which is within the subregion \( S_2 \), but not within \( S_1 \). Since the spatial weight of \( S_2 \) must not be greater than that of \( S_1 \), no spatial weighting function can assign all of the spatial weight at this location. Indeed, it turns out that the lower bound for \( z^1 \) is found when half of the spatial weight is around the location \( s = 0 \) in \( S_1 \) and half around \( s = 0.9 \) in \( S_2 \). Then, the lower bound is \( V(z(1)) > 0.5v(z(1)(0)) + 0.5v(z(1)(0.9)) = 0.5 \cdot 0.4 + 0.5 \cdot 0.3 = 0.35 \).

Although the value intervals of \( z^1 \) and \( z^2 \) overlap, the alternative \( z^1 \) still dominates \( z^2 \) with regard to the set of feasible spatial weighting functions \( A \). The dominance can be established by finding a weighting function that minimizes the value difference \( V(z(1)) - V(z(2)) \). The consequence value difference \( v(z(1)) - v(z(2)) \) is at its lowest at the location \( s = 0.9 \), but, as discussed above, no feasible spatial weighting function can concentrate all of the spatial weight at this location. The lower bound of the value difference is obtained by concentrating the spatial weight evenly around the two locations \( s = 0.2 \) in \( S_1 \) and \( s = 0.9 \) in \( S_2 \) (see the horizontal dotted lines in Fig. 2), in which case \( V(z(1)) - V(z(2)) > 0.5(\text{max}_v(z(0.2)) - \text{var}(z(0.2))) + 0.5(\text{max}_v(z(0.9)) - \text{var}(z(0.9))) = 0.5 \cdot (0.7 - 0.5) + 0.5 \cdot (0.3 - 0.4) = 0.05 > 0 \). This implies that \( V(z(1)) > V(z(2)) \) for any feasible weighting function \( \alpha \in A \), and therefore \( z^1 D_A z^2 \).

When analyzing decision alternatives based on some set of feasible spatial weighting functions \( A \), the DM should not select a dominated alternative, as the alternative that dominates it would yield greater or equal value for any feasible weighting function. The decision alternatives under consideration can therefore be limited to the set of non-dominated alternatives \( N_D(\alpha) \), i.e., those that are not dominated by any other alternative. Since the dominance relation \( D_\alpha \) that determines \( N_D(\alpha) \) is contingent on the set of feasible weighting functions \( A \), so is the set \( N_D(\alpha) \). Additional preference statements generally reduce the set of non-dominated alternatives. Ideally, the preference statements would reduce \( Z_N(\alpha) \) to a singleton set, as its only element would then be unambiguously the most preferred decision alternative.

In case enough preference information to identify a single non-dominated alternative cannot be obtained, the maximin and minimax-regret decision rules can be used to rank decision alternatives (see, e.g., Salo and Hamäläinen 2001). However, these heuristic rules are not guaranteed to identify the most preferred alternative. The maximin rule ranks the alternatives according to their minimum value (i.e., \( \min_{\alpha \in \Omega} V(z) \)), and thus the highest ranked alternative is the one with the greatest minimum value, i.e., \( z^* = \arg \max_{\alpha \in \Omega} \min_{z \in \Omega} V(z) \). The maximum regret when selecting the alternative \( z^* \) is the highest value difference between \( z^* \) and the most valuable alternative across all feasible weighting functions, i.e., \( \max_{\alpha \in \Omega} \max_{z^* \in \Omega} \min_{z \in \Omega} V(z) \). The maximin-regret rule ranks the alternatives according to their maximum regret, and thus the highest ranked alternative is the one with the lowest maximum regret, i.e., \( z^* = \arg \min_{\alpha \in \Omega} \max_{z^* \in \Omega} \min_{z \in \Omega} V(z) \). It is important here to highlight that the decision rules should only be used to rank the remaining non-dominated alternatives after all available preference information has been utilized. If the decision rules are applied to the set of all alternatives, it is possible that a dominated alternative receives a higher rank than a non-dominated alternative.

3.3. Computational approach for establishing dominance

Whether or not the alternative \( z^1 \) dominates \( z^2 \) can be established by finding the lower and upper bounds for the value difference \( V(z(1)) - V(z(2)) \) across all feasible spatial weighting functions \( \alpha \in A \). Finding these bounds is computationally challenging, as it requires solving optimization problems with an infinite-dimensional decision variable space consisting of the feasible weighting functions. The problems can be solved, though, if the alternatives are assumed to be piecewise continuous.

**Definition 6.** A function \( f : S \rightarrow \mathbb{R} \) is piecewise continuous in the subregion \( S' \subseteq S \), \( \alpha \neq 0 \), if for any location \( s \in S' \) and \( \epsilon > 0 \), there exists a subregion \( S'' \subseteq S \) and a spatial weighting function \( \alpha \in A \) such that \( |f(s') - f(s'')| < \epsilon \) for all locations \( s'' \in S'' \).

This definition allows the exclusion of alternatives \( z \in Z \) that are computationally challenging but of little relevance in practical applications. For instance, consider the spatial region \( S = [0, 1] \) and a non-piecewise continuous decision alternative \( z \) such that \( v(z(s)) = 1 \) if \( s \in \mathbb{Q} \) and \( v(z(s)) = 0 \) elsewhere. Such an alternative is unlikely to have practical significance, but leads to computationally intractable cases.

The following lemma shows that the set of feasible spatial weighting functions \( A \) can always be represented using a system of linear constraints on a finite number of variables corresponding to spatial weights of subregions.

**Lemma 1.** Let \( A \subseteq A \) be the set of feasible spatial weighting functions (Definition 4) defined by the preference statements \( \pi \in \{1, \ldots, r\} \). Then, there exists \( \alpha \in \mathbb{R}^n \), a partition \( S^1, \ldots, S^n \) of the region \( S \), and a matrix \( P \in \mathbb{R}^{n \times n} \) such that

\[
A = \left\{ \alpha \in A \left| \sum_{i=1}^{n} P_i \pi(S^i) \leq 0, \forall \pi \in \{1, \ldots, r\} \right. \right\}.
\] (9)

Computationally tractable conditions for establishing dominance between two decision alternatives with regard to the set of feasible spatial weighting functions are provided by the following theorem.

**Theorem 2.** Assume the set of feasible spatial weighting functions is given by \( A = \{\alpha \in A : \sum_{i=1}^{n} P_i \pi(S^i) \leq 0, \forall \pi \in \{1, \ldots, r\} \} \). Then, \( z^1, z^2 \in Z \) be decision alternatives such that \( v(z(s)) - v(z(s)) \) is piecewise continuous in each subregion \( S^i \). Then, \( z^1 D_A z^2 \) if and only if

\[
\min_{\alpha \in \Omega} \sum_{i=1}^{n} \omega_i \inf_{z^i \in S^i} (v(z^1(s)) - v(z^2(s))) \geq 0 \quad \text{and} \quad \max_{\alpha \in \Omega} \sum_{i=1}^{n} \omega_i \sup_{z^i \in S^i} (v(z^1(s)) - v(z^2(s))) > 0.
\] (10)

(11)

where \( \Omega = \{\alpha \in [0, 1] : P_\Omega \leq 0, \sum_{i=1}^{n} \omega_i = 1\} \).

The theorem suggests a two-stage approach for establishing whether the alternative \( z^1 \) dominates \( z^2 \). The first stage consists of finding the infimum and supremum bounds of the difference between the consequence values of the two alternatives in each subregion \( S^i \). The selection of a suitable technique to obtain these bounds depends on the problem type. For instance, if the alternatives are defined through analytical formulas, then standard non-linear optimization techniques can be employed. On the other hand, if the alternatives’ consequences are stored in a data base, then an exhaustive search across the subregion may be necessary to obtain these bounds. In the second stage of the approach, the bounds are used as the objective function coefficients of the two linear programming problems in (10) and (11). The decision variables \( \omega \in \Omega \) correspond to the spatial weights of the subregions in the partition of \( S \). The solutions to these problems give the lower and upper bounds for the value difference between the alternatives \( z^1 \) and \( z^2 \). In case the lower bound is non-negative and the upper bound is strictly positive, the alternative \( z^1 \) dominates \( z^2 \). Conversely, \( z^2 \) dominates \( z^1 \) if the lower bound is strictly negative.
and the upper bound is non-positive. The set of non-dominated alternatives is identified by repeating this approach for each pair of alternatives.

Alternatives’ value intervals and ranks according to decision rules are also determined by solving the linear programming problems in (10) and (11). In particular, by setting \( v(z^2(s)) = 0 \) for all \( s \in S \), the problem in (10) gives the lower bound of the value interval of the alternative \( z^1 \). Since the problem in (11) provides the maximal value difference between the alternatives \( z^1 \) and \( z^2 \), computing it for all alternatives \( z^1 \) determines the maximum regret of \( z^2 \).

3.4. Extension to incomplete preference information on attribute weights

Thus far, the consequence value function \( v \) has been assumed to be precisely defined. In this section, we relax this assumption and consider the joint use of (i) incomplete spatial preference information, and (ii) incomplete preference information on attribute weights \( b_j \) of an additive multi-attribute consequence value function

\[
v(c) = \sum_{j=1}^{m} b_j v_j(c_j),
\]

where \( v_1, \ldots, v_m \) are the attribute-specific value functions. In particular, instead of an exact weight vector \( b \), we assume a set of feasible attribute weights

\[
B \subseteq B^0 = \left\{ b \in [0, 1]^m : \sum_{j=1}^{m} b_j = 1 \right\},
\]

where \( B \) is defined by linear constraints and \( B^0 \) is the set of all possible attribute weights. Several techniques for specifying incomplete preference information on attribute weights produce such a feasible set (see, e.g., Kirkwood & Sarin 1985; Salo & Hämäläinen 1992; Weber 1987). For instance, with \( m = 2 \) attributes, stating that a consequence with the first attribute at the most preferred level and the second at the least preferred level is preferable to a consequence with the first attribute at the least preferred level and the second attribute at the most preferred level results in the set \( B = \{ b \in B^0 | b_1 \geq b_2 \} \) of feasible attribute weights.

Incomplete preference information regarding the spatial weighting function and attribute weights is captured by the pair of sets \((A, B)\), where \( A \subseteq A^0 \) (Definition 4) and \( B \subseteq B^0 \) (Eq. (13)). The value of each alternative \( z \) is contingent on both the spatial weighting function \( \alpha \in A \) and the attribute weights \( b \in B \) that are used to evaluate \( V(z) \). An interval of possible values is associated with each alternative, and reducing \( A \) or \( B \) can narrow these intervals, but never widen them. The concept of dominance (Definition 5) readily extends to a setting that also includes a set of feasible attribute weights.

Definition 7. The alternative \( z^1 \) dominates the alternative \( z^2 \) with regard to the sets of feasible spatial weighting functions and attribute weights \((A, B)\), denoted by \( z^1 \triangleright D_{(A, B)} z^2 \), if

\[
V(z^1) \geq V(z^2), \text{ for all } \alpha \in A, b \in B.
\]

\[
V(z^1) > V(z^2), \text{ for some } \alpha \in A, b \in B.
\]

One alternative dominates another if its value is at least as great for all feasible spatial weighting functions and all feasible attribute weights, and strictly greater for at least some feasible spatial weighting function and some feasible attribute weights. The dominance relation \( D_{(A, B)} \) has similar properties as \( D_1 \). It is a transitive and asymmetric binary relation. If the alternative \( z^1 \) dominates \( z^2 \) with regard to the sets \((A, B)\), then \( z^2 \) cannot dominate \( z^1 \) even if additional preference information is provided. Furthermore, if the dominating alternative \( z^1 \) has a strictly higher value than \( z^2 \) for all \( \alpha \in A, b \in B \), then it dominates \( z^2 \) even if additional preference information is provided. These two properties apply whether the additional preference information concerns spatial weights, attribute weights, or both.

Whether or not the alternative \( z^1 \) dominates \( z^2 \) can be established by finding the lower and upper bounds for the value difference \( V(z^1) - V(z^2) \) over the feasible weighting functions \( \alpha \in A \) and the attribute weights \( b \in B \). The following theorem provides computationally tractable conditions for establishing whether dominance holds between a pair of alternatives.

Theorem 3. Let \( B \) be the set of feasible attribute weights and assume that the set of feasible spatial weighting functions is given by \( A = \{ \alpha \in A^0 \mid \sum_{i=1}^{n} P_i \alpha(S^i) \leq 0, \forall k \in \{1, \ldots, m\} \} \), where \( S^1, \ldots, S^n \) is a partition of the region \( S \). Furthermore, let \( z^1, z^2 \in Z \) be decision alternatives such that \( v(z^1(s)) - v(z^2(s)) \) is piecewise continuous in each subregion \( S \) and for every \( b \in B \). Then, \( z^1 \triangleright D_{(A, B)} z^2 \) if and only if

\[
\min_{\alpha \in \Omega, b \in B} \sum_{i=1}^{n} \omega_i \inf_{\alpha \in \Omega} \sum_{j=1}^{m} b_j (v_j(z^1_i(s)) - v_j(z^2_i(s))) \geq 0 \quad (15)
\]

\[
\max_{\alpha \in \Omega, b \in B} \sum_{i=1}^{n} \omega_i \sup_{\alpha \in \Omega} \sum_{j=1}^{m} b_j (v_j(z^1_i(s)) - v_j(z^2_i(s))) > 0, \quad (16)
\]

where \( \Omega = \{ \alpha \in [0, 1]^n | P_0 \leq 0, \sum_{i=1}^{n} \omega_i = 1 \} \).

The optimization problems in (15) and (16) are linear in both the spatial weights \( \omega \in \Omega \) and the attribute weights \( b \in B \). The optimal solutions are therefore found at combinations of the extreme points of the set of feasible attribute weights \( B \) and the extreme points of the set of feasible spatial weights \( \Omega \). The problems do include products of \( \omega \) and \( b \), though, and therefore cannot be solved directly by linear programming. One method for solving the problems is to first enumerate the extreme points of \( B \) using standard techniques (see, e.g., Mathiess & Rubin 1980; Taha 2003). With fixed attribute weights \( b \), the optimization problems in (15) and (16) correspond to those in (10) and (11), respectively. They can therefore be solved for each extreme point of \( B \) by applying the two-stage approach presented in Section 3.3. The resulting optimal value differences across the extreme points are sufficient for establishing if the alternative \( z^1 \) dominates \( z^2 \).

Alternatives’ value intervals and ranks according to decision rules are also determined by solving the optimization problems (15) and (16). In particular, the value interval of the alternative \( z^1 \) is computed by replacing \( z^2 \) in (15) and (16) with a constant alternative such that \( v_j(z^2_i(s)) = 0 \) for all \( s \in S, j \in \{1, \ldots, m\} \). The ranking according to the maximin decision rule is determined by the lower bounds of these value intervals. The maximum regrets are obtained by solving the problem (16) for each pair of alternatives.

4. Application to air defense planning

In this section, we illustrate the use of the additive spatial value function and the spatial analysis method with an air defense planning application. It is based on a real-life spatial decision problem, but the spatial region, the decision alternatives, and the DM’s preferences are designed specifically for the purpose of this illustration.

The air force commander of a fictional country decides on the positions of air bases with the objective of maximizing air defense capability across the country. A map of the country, including the potential positions for the air bases, is presented in Fig. 3. The bases serve two distinct functions. First, they provide maintenance such as refueling, rearming and repairing aircraft between
missions. Second, aircraft on alert and ready to move out on short notice are also stationed at the bases. The positioning of the bases thus affects multiple aspects of air defense capability in their surroundings. Determining the most appropriate positions requires making judgments on the importance of the different aspects and also the importance of different parts of the country with respect to air defense capability.

4.1. Decision alternatives

The decision alternatives correspond to combinations of base positions. Two types of air bases are considered: Main bases can simultaneously maintain up to five assets, i.e., groups of aircraft, while secondary bases can only service one asset at a time. Three position candidates have been identified for main bases (labeled A–C, see Fig. 3) and five for secondary bases (labeled 1–5). Due to limited availability of maintenance resources, the air force can operate only two main bases and three secondary bases simultaneously. This implies that the total number of decision alternatives is \( \binom{3}{2} \cdot \binom{5}{3} = 30 \) and they are denoted by \( z^1, \ldots, z^{30} \). The spatial region \( S \) across which the decision alternatives are considered is the land area of the country, i.e., bodies of water are not taken into account.

4.2. Attributes

The consequences \( c_j \) of decision alternatives are evaluated based on four attributes \( j \in \{1, 2, 3, 4\} \): 'force fulfillment', 'force sustainability', 'southern engagement frontier', and 'western engagement frontier'. These attributes measure different aspects of air defense capability across the country. The first two attributes \( j \in \{1, 2\} \) describe the capability to carry out multiple consecutive missions over a longer period of time by measuring the available fighter presence at each location. The target level of fighter presence is set at four assets. The 'force fulfillment' attribute \( c_1 = z^1_s \) is the average share of the target level that can be achieved at the location \( s \) when the decision alternative \( z^6 \) is selected. The 'force sustainability' attribute describes how well fighter presence can be maintained despite losing a secondary base. Specifically, \( c_2 = z^2_s \) is the average share of the target level that can be achieved at the location \( s \) in a situation where the most important secondary base for that particular location is not available.

The remaining two attributes \( j \in \{3, 4\} \) describe the capability to counter the first enemy attack using assets on alert and stationed at the bases. These attributes are based on the concept of an engagement frontier, which is the frontier where approaching hostile aircraft are first intercepted. The 'southern engagement frontier' attribute \( c_3 = z^3_s \) measures the distance of the location \( s \) from the closest point on the engagement frontier when the enemy approaches from the south and the decision alternative \( z^6 \) is selected. Correspondingly, the 'western engagement frontier' attribute \( c_4 = z^4_s \) measures the distance of the location \( s \) from the closest point on the engagement frontier when the enemy approaches from the west and the decision alternative \( z^6 \) is selected. The neighboring countries to the north and to the east are allies, so attacks from these directions are not taken into consideration.

For each alternative \( z^k \) \((k \in \{1, \ldots, 30\})\), the multi-attribute consequences across the spatial region, i.e., the functions \( z^1_k, \ldots, z^{30}_k \), are estimated with a dedicated air mission planning software. The software takes into account the positions and capacities of air bases, including turnaround, refueling and rearming times, as well as alert, taxi and scramble delays. For assets, it considers properties such as fuel consumption, weapons consumption, and flight speed during missions. The output of the software is illustrated in Appendix C2, where all four attributes are shown for three decision alternatives.

4.3. Consequence value function

The attribute-specific value functions of the additive value function (12) are presented in Table 2. A linear attribute-specific value function is applied for the force fulfillment and force sustainability attributes \((j \in \{1, 2\})\). Regarding the engagement frontier attributes, a greater distance to the frontiers is preferred in order to minimize the impact of enemy air attacks. The commander indicates, however, that increasing the distance has less impact the further away the frontier is. Hence, a strictly increasing concave value function \( v(c_j) = c_j/(c_j + \gamma_j) \) is used. The commander also specifies the level \( \gamma_j \) such that an increase in the distance from zero to this level is equally preferred to an increase from this level to infinity. The commander states that the level is 250 kilometers.

The alternatives are first compared without any preference information, i.e., \( A = A^0 \) and \( B = B^0 \). It turns out that none of the 30 decision alternatives dominate each other, and hence the commander provides preference statements. When assessing attribute weights, the commander states that the engagement frontier attributes measuring the ability to counter the first enemy attack \((j \in \{3, 4\})\) are more important than the force fulfillment and sustainability attributes measuring the capability for long term fighter presence \((j \in \{1, 2\})\). Regarding the engagement frontier attributes, an enemy air attack is more likely to originate from the west and hence the western engagement frontier attribute \((j = 4)\) is more important than the southern one \((j = 3)\). The force fulfillment attribute \((j = 1)\) is the main measure of fighter presence, while the force sustainability attribute \((j = 2)\) has a supporting role in measuring this capability in case one of the bases is destroyed. Hence, the force fulfillment attribute is seen as more important than the force sustainability attribute. Finally, the commander states that the force fulfillment and force sustainability attributes are together more important than the western engagement attribute. This ensures that force fulfillment cannot be fully neglected. The statements on the attributes’ importance result in the set of feasible attribute weights

\[
B = \{ b \in B^0 \ | \ b_4 \geq b_3 \geq b_1 \geq b_2, b_1 + b_2 \geq b_4 \}. \tag{17}
\]
Table 2
Attributes and their value functions.

| Attribute | Description | Unit | Value function |
|-----------|-------------|------|----------------|
| $c_1$     | Force fulfillment | %    | $v_1(c_1) = \frac{c_1}{100}$ |
| $c_2$     | Force sustainability | %    | $v_2(c_2) = \frac{c_2}{100}$ |
| $c_3$     | Distance to southern engagement frontier | kilometer | $v_3(c_3) = \frac{c_3}{c_3 + 250km}$ |
| $c_4$     | Distance to western engagement frontier | kilometer | $v_4(c_4) = \frac{c_4}{c_4 + 250km}$ |

4.4. Spatial preference information

The commander provides spatial preference information through statements of the form (7) concerning the importance of different geographical areas of the country. The country (region $S$) is divided into nine areas (subregions $S^1, \ldots, S^9$) that are shown on the map in Fig. 4. This division is based mainly on geographical features, but nuclear power plants as well as cities with a large population are treated as separate areas.

The country relies on the energy produced at the nuclear power plants, and hence the area $S^1$ is seen by the commander as the most important one. The second most important area is the capital area $S^2$, since it is the administrative center of the country and its defense has symbolic value. The other major cities are important hubs for industry and trade, which is why the area $S^3$ is the third most important. The remaining areas ($S^4, \ldots, S^9$) have been numbered in decreasing order of population, which is used as a proxy measure for the relative importance of these areas. These preference statements are of the form (3) and correspond to the constraints

$$\alpha(S^1) \geq \alpha(S^2) \geq \cdots \geq \alpha(S^9).$$

(18)

These constraints allow for a wide range of feasible spatial weighting functions. At one extreme, all of the spatial weight is assigned to the nuclear plants, i.e., $\alpha(S^1) = 1$. At the other extreme, the spatial weight of each area is the same, i.e., $\alpha(S^i) = 1/9, i \in \{1, \ldots, 9\}$. In order to narrow down the range of feasible weighting functions, the commander is asked to consider the relative importance of consecutive areas by comparing the spatial weight of $S^i$ to that of $S^{i+1}$ for each $i \in \{1, \ldots, 8\}$. Specifically, the commander is asked if the area $S^i$ is at least 3/2 times as important as the area $S^{i+1}$. The commander judges that this statement holds for most areas, which results in constraints of the form (5). For instance, the capital area ($S^2$) is indeed more than 3/2 times as important as the major cities ($S^3$). However, the commander states that the areas $S^4$, $S^7$, and $S^8$ are less than 3/2 times as important as the areas $S^2$, $S^5$, and $S^9$, respectively. These preference statements correspond to the constraints

$$\alpha(S^i) \leq \frac{3}{2} \alpha(S^{i+1}), i \in \{1, 7, 8\},$$

(19)

$$\alpha(S^i) \geq \frac{3}{2} \alpha(S^{i+1}), i \in \{2, 3, 4, 5, 6\}.$$  

(20)

Together, the spatial preference statements given by the commander determine the set of feasible weighting functions

$$A = \{\alpha \in A^0 \mid \alpha \text{ satisfies (18) – (20)}\}.$$  

(21)

4.5. Non-dominated decision alternatives

Fig. 5 shows the value interval for each decision alternative based on the sets of feasible spatial weighting functions $A$ and attribute weights $B$ (see Appendix B for computational details). All 30 value intervals overlap, whereby no definite conclusions regarding preferences among the alternatives can be drawn from these intervals. Dominances between the alternatives are illustrated by the dominance graph in Fig. 6. There are 14 dominated decision alternatives that are ruled out from further consideration. Moreover, position candidate $B$ is included in all 16 non-dominated alternatives $z \in ND_{A,B}$. Thus, regardless of which feasible spatial weighting function and attribute weights actually represent the commander’s preferences, one of the two main bases should always be placed at position candidate $B$. Even though the value intervals of all 30 decision alternatives overlap, nearly half of the alternatives are dominated based on the sets $A$ and $B$, which are defined by rather loose preference statements.

4.6. Additional spatial preference information

To obtain more conclusive decision recommendations about which of the 16 non-dominated alternatives should be selected, the commander provides additional spatial preference information. To do this, she further divides most of the areas ($S^i$) into smaller subareas ($S^i_j \subset S^i$) that are listed in Table 3. Then, the commander assesses the relative importance of the subareas within each area.

For the nuclear power plants $S^1$, the commander determines that the NW plant $S^1_{11}$ is more important than the NE plant $S^1_{12}$, but at most 3/2 times as important. Similarly, she judges that the SW city $S^2_{11}$ is more important than the NW city $S^2_{12}$, but at most 3/2 times as important. The preference statements correspond to the constraints (cf. Eqs. (3) and (5))

$$\alpha(S^1_{11}) \leq \alpha(S^1_{12}) \leq \frac{3}{2} \alpha(S^1_{12}), i \in \{1, 3\}.$$  

(22)

The capital city $S^3_{11}$ is considerably more important than its suburb $S^3_{12}$. The commander states that it is between 2 and 3 times as
important. By applying Eqs. (4) and (5), this preference statement results in the constraints
\[ 2\alpha(S_i^j) \leq \alpha(S_i^k) \leq 3\alpha(S_i^j). \] (23)

For the central area \( S^6 \) consisting of three subareas, the commander considers the west inland \( S^6_2 \) to be at least as important as the other two subareas \( S^6_1 \) and \( S^6_3 \) combined. She also states that the east inland \( S^6_1 \) is at least half as important as the other two subareas \( S^6_2 \) and \( S^6_3 \) combined. No lower bound is set for the importance of the mountains \( S^6_3 \), because they are mostly uninhabited. These preference statements result in the constraints
\[ \alpha(S^6_1) \geq \alpha(S^6_2) + \alpha(S^6_3), \] (24)
\[ \alpha(S^6_2) \geq \frac{1}{2} (\alpha(S^6_1) + \alpha(S^6_3)). \] (25)

Finally, for each of the remaining areas consisting of two subareas, the commander determines that the more important subarea is at most twice as important as the less important one. For instance, the west coast \( S^6_2 \) is more important than the SW coast \( S^6_3 \), but no more than twice as important, i.e.,
\[ \alpha(S^6_2) \leq 2\alpha(S^6_3), \] (26)
\[
\alpha(S^6_1) \leq \alpha(S^6_2) \leq 2\alpha(S^6_3), \] i.e., \( i \in \{4, 5, 7, 8\} \).

The additional spatial preference statements given by the commander restrict the set of feasible spatial weighting functions further. This new set is defined as
\[ A' = \{ \alpha \in A | \alpha \text{ satisfies (22)-(26)} \}, \] (27)

where \( A \) is given by Eq. (21).

4.7. Analysis of decision alternatives based on additional spatial preference information

The value intervals of all 30 original decision alternatives corresponding to the sets of feasible weights \( \{A, B\} \) and \( \{A', B\} \) are presented in Fig. 7. The intervals are narrower with the updated set \( A' \) of feasible spatial weighting functions, but they still overlap. The number of non-dominated alternatives, however, is reduced to just three, which is shown in the updated dominance graph of the 16 previously non-dominated alternatives depicted in Fig. 8. The non-dominated alternatives are now BC123, BC234, and BC235. Because candidates B and C are included in all three non-dominated alternatives, they are chosen as the positions for the main bases. Similarly, all non-dominated alternatives assign secondary bases to positions 2 and 3, and hence only the position of the final secondary base remains to be determined.

Since only three decision alternatives are non-dominated and they have similar value intervals, no additional preference statements are elicited from the commander. She is instead briefed on the rankings offered by the decision rules as well as on the consequences of the non-dominated alternatives. The rankings based on the maximin and minimin regret rules are BC123, BC235, BC234.
and BC235, BC234, BC123, respectively. The alternatives’ attribute-specific consequences are illustrated using visualizations, such as the maps presented in Appendix C.2. Based on all the available information, the commander can readily assess the non-dominated alternatives in a holistic manner and select the one that provides the best air defense capability.

5. Discussion

5.1. Special Cases of the Additive Spatial Value Function

The spatial value functions presented by Simon et al. (2014) are obtained as special cases of our additive spatial value function. First, if the spatial weight \( \alpha(S') \) of a subregion \( S' \) can be expressed as the integral of a function \( \alpha: S \rightarrow [0, \infty) \), i.e., \( \alpha(S') = \int_S \alpha(s)ds \), then the additive spatial value function (1) becomes

\[
V(z) = \int_S v(z(s))d\alpha(s) = \int_S \alpha(s)v(z(s))ds.
\]  
(28)

which corresponds to the value function proposed by Simon et al. (2014) for nondiscrete cases. In particular, they consider an Euclidean spatial region \( S \subseteq \mathbb{R}^2 \) and a closed interval of possible consequences, and present a conjecture on plausible conditions for representing preferences with the value function (28). In this value function, the importance of locations is captured by the function \( \alpha \). As Simon et al. point out, however, there is no established theory for determining a suitable functional form for \( \alpha \). A major insight of the value model developed in Section 2 is that it is more appropriate to assign spatial weights to subregions, as the spatial weighting function \( \alpha \) does, rather than to locations.

The second special case arises when the spatial region consists of a finite number of locations, i.e., \( S = \{s_1, \ldots, s_n\} \). There are then \( n \) subregions consisting of a single location and their spatial weights are denoted by \( \alpha_i = \alpha(\{s_i\}) \). The spatial weight of any subregion \( S' \) is given by \( \alpha(S') = \sum_{i \in S'} \alpha(\{s_i\}) \). Thus, the value function (1) becomes

\[
V(z) = \int_S v(z(s))d\alpha(s) = \sum_{i=1}^{n} \alpha_i v(z(s_i)) = \sum_{i=1}^{n} \alpha(z(s_i)).
\]  
(29)

which is equivalent to the value function proposed by Simon et al. (2014) for discrete cases. This value function is not compatible with the axiomatization introduced in Section 2.2, since the axioms imply an infinite number of locations. However, Simon et al. (2014) provide necessary and sufficient conditions for representing the DM’s preferences with the value function (29).

The spatial analysis method developed in Section 3 for incorporating incomplete preference information and providing decision recommendations can be readily deployed in applications utilizing the spatial value function (29). In particular, with a finite number of locations, the set of spatial weighting functions \( \mathcal{A}(S) \) includes all additive measures \( \alpha \) on the region \( S \) such that \( \alpha(S) = 1 \). With this set, the requirement of piecewise continuity (Definition 6) always holds, and the continuity assumption in Theorems 2 and 3 is always satisfied. Therefore, the computational approaches for obtaining the alternatives’ value intervals, identifying dominances among the alternatives and ranking the alternatives based on decision rules (see Sections 3.3 and 3.4) are readily applicable in spatial decision problems with a finite number of locations.

5.2. Implications for decision support

The analysis method developed in this paper and demonstrated in the air defense application helps tackle some of the challenges in the elicitation of spatial weights identified by Ferretti & Montibeller (2016). In particular, the method allows preference state-
ments between any subregions, and hence the DM can choose familiar ‘units of analysis’ based on the application context (e.g., provinces) rather than being forced to work with model artifacts (e.g., grid squares). Furthermore, the method avoids the estimation of precise weights for numerous locations, which can drastically alleviate the preference elicitation burden. Finally, the spatial weighting function is not restricted to any pre-specified functional form. This is particularly convenient, since there is no existing theory that would help in choosing the correct functional form as noted by Simon et al. (2014).

The spatial analysis method is computationally efficient enough to be used in interactive spatial decision support systems (see Appendix B for computational details of the air defense planning application). Such systems would enable the iterative exploration of the DM’s preferences and the examination of the effects that each iteration has on the dominance graph, the value intervals, and the rankings produced by decision rules. This type of an interactive process can aid the DM in defining further preference statements. If the DM regrets some of the previous statements, they can be removed from the next iteration. It is not necessary to settle on any fixed set of subregions in the interactive process. At each stage, the DM is allowed to define new subregions with regard to which preference statements are given. Each iteration typically narrows the value intervals, increases the number of dominances, and thus decreases the number of non-dominated decision alternatives, unless previous preference statements are being retracted. For instance, in the air defense planning application, the first iteration identified 14 of the 30 decision alternatives as dominated, while the second iteration reduced the number of non-dominated alternatives to just three despite the relatively loose preference statements.

Although the spatial analysis method was developed to produce defensible decision recommendations based on preference statements provided by the DM, it can also be utilized to conduct global sensitivity analysis for existing spatial decision models that require exact spatial weights. In particular, the robustness of these models can be analyzed by constructing a set of feasible spatial weighting functions that contains the weighting function implied by the original spatial weights. The resulting dominance graph and value intervals help analyze how sensitive the original ranking of decision alternatives is to variations in the spatial weights of locations. Hence, the method developed in this paper addresses the concerns raised by Ferretti & Montibeller (2016) on the challenges of conducting rigorous sensitivity analysis in spatial decision models.

The spatial analysis method can also be used to support spatial group decision making involving multiple DMs. The set of feasible weighting functions can be based on preference statements that each DM in the group is willing to accept. The resulting set of non-dominated decision alternatives contains the alternative that is most preferred by each individual DM, as well as those alternatives that represent justifiable compromises among the group. Such compromises are often sought in, e.g., environmental studies (see, e.g., Huang, Keisler, & Linkov 2011), where stakeholders commonly have conflicting objectives. Our spatial analysis method allows for the extension of this kind of studies to the spatial context, which is often relevant in environmental decision making.

6. Conclusions

This paper advances the theory and the methodology for analyzing spatial decision problems in which consequences of decision alternatives vary over a spatial region that can be of any shape and dimension. In particular, we provide the axiomatization for the additive spatial value function consisting of two components: (i) a consequence value function that maps the alternatives’ location-specific consequences onto a value scale, and (ii) a spatial weighting function representing the importance of each sub-region. The overall value of an alternative is obtained by integrating its location-specific value with respect to the spatial weighting function across the spatial region. Moreover, we develop the novel spatial analysis method for reducing the DM’s workload in assessing the importance of subregions, the number of which is often large or even infinite in practical spatial decision analysis applications. The method uses dominance concepts and decision rules to identify defensible decision recommendations without requiring the DM to specify a unique spatial weighting function. The reported real-life application in air defense planning illustrates the advantages of these new spatial decision analysis practices.

Our contributions open up several avenues for future research. First, the axiomatization of the additive spatial value function offers a platform to develop more complex spatial value functions by relaxing some of the seven axioms presented in this paper. For instance, relaxing Axiom 3 would allow for preference models where alternatives are viewed more holistically rather than considering each location separately. Relaxing Axiom 5, in turn, would permit preference models in which the value of a consequence depends on the location it is obtained. Second, our work aids in developing an axiomatic basis for additive spatial utility functions which also take into account uncertainties in location-specific consequences. Third, different preference assessment procedures for eliciting spatial weights should be developed and tested. This would require behavioral experiments on how consistently human subjects are able to answer different types of assessment questions, but also simulation studies on the discriminatory power of the assessment procedures in providing conclusive decision recommendations. Moreover, such simulation models could be used during an interactive decision support process to identify the types of spatial preference statements that are most effective in reducing the number of non-dominant alternatives. Fourth, the identification of all potentially optimal alternatives rather than all non-dominated ones could allow more alternatives to be discarded based on the same preference statements. The major challenge here is the development of an efficient computational approach for identifying a weighting function for which a particular alternative yields the highest value across all alternatives, or determining that no such function exists. Finally, we believe that the spatial value model and analysis method for handling incomplete spatial preference information and providing useful decision recommendations developed in this paper can benefit applied research in several areas. In particular, huge growth in the amount of spatial data being collected and stored is likely to result in an increase in spatial decision analysis applications. These applications in turn require models, methods, and practices for providing defensible decision recommendations without the exact specification of the DM’s preferences.

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Supplementary material

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