ON STABILITY FOR IMPULSIVE DELAY DIFFERENTIAL EQUATIONS AND APPLICATION TO A PERIODIC LASOTA-WAZEWSKA MODEL

TERESA FARIA
Departamento de Matemática and CMAF-CIO
Faculdade de Ciências, Universidade de Lisboa
Campus Grande, 1749-016 Lisboa, Portugal

José J. Oliveira*
CMAT and Departamento de Matemática e Aplicações
Escola de Ciências, Universidade do Minho
Campus de Gualtar, 4710-057 Braga, Portugal

(Communicated by José A. Langa)

Abstract. We consider a class of scalar delay differential equations with impulses and satisfying an Yorke-type condition, for which some criteria for the global stability of the zero solution are established. Here, the usual requirements about the impulses are relaxed. The results can be applied to study the stability of other solutions, such as periodic solutions. As an illustration, a very general periodic Lasota-Wazewska model with impulses and multiple time-dependent delays is addressed, and the global attractivity of its positive periodic solution analysed. Our results are discussed within the context of recent literature.

1. Introduction. The high impact of differential equations with impulses in terms of their application in population dynamics, disease modelling and other fields has lead to an increasing interest in the theory of impulsive systems. Recently, the theoretical analysis of existence and regularity of solutions to impulsive systems with delays, as well as the study of concrete impulsive models used in mathematical biology, have become an important area of research.

In this paper, we study a class of scalar impulsive differential equations with an instantaneous negative feedback term and (possibly unbounded) delays.

Let \( \tau : [0, \infty) \to [0, \infty) \) be a continuous function such that \( \lim_{t \to \infty} (t - \tau(t)) = \infty \).

Without loss of generality, we suppose that \( t \mapsto t - \tau(t) \) is non-decreasing; otherwise, whenever necessary, we can replace \( t \mapsto t - \tau(t) \) by \( d(t) := \inf_{s \geq t} (s - \tau(s)) \), which is non-decreasing and satisfies the above conditions for \( t - \tau(t) \).

For \( t \geq 0 \), denote by \( PC(t) = PC([-\tau(t), 0]; \mathbb{R}) \) the space of real functions that are piecewise continuous functions on \([-\tau(t), 0]\) and left continuous on \((-\tau(t), 0]\), with the norm \( \|\phi\|_t := \sup_{\theta \in [-\tau(t), 0]} |\phi(\theta)| \) for \( \phi \in PC(t) \).

2010 Mathematics Subject Classification. 34K45, 34K25, 92D25.
Key words and phrases. Delay differential equation, impulses, Yorke condition, global attractivity, Lasota-Wazewska model, periodic solution.

*Corresponding author: Tel: +351253604084, e-mail: jjoliveira@math.uminho.pt.
We consider a scalar impulsive delay differential equation (DDE) of the form
\[ \begin{align*}
x'(t) + a(t)x(t) &= f(t, x_t), \quad 0 \leq t \neq t_k, \\
\Delta(x(t_k)) &= x(t_k^+) - x(t_k^-) = I_k(x(t_k)), \quad k = 1, 2, \ldots,
\end{align*} \tag{1.1} \]
where: \(x'(t)\) is the left-hand derivative of \(x(t)\), \((t_k)_k\) is an increasing sequence of positive numbers with \(t_k \to \infty\), \(a : [0, \infty) \to [0, \infty)\) and \(I_k : \mathbb{R} \to \mathbb{R}\) are continuous functions; \(x_t\) denotes the restriction of \(x(t)\) to the interval \([t - \tau(t), t]\), with \(x_t \in \text{PC}(t)\) given by
\[ x_t(\theta) = x(t + \theta) \quad \text{for} \quad -\tau(t) \leq \theta \leq 0; \]
\(f(t, \varphi)\) is a functional defined for \(t \geq 0\) and \(\varphi \in \text{PC}(t)\) with some regularity discussed below. We also assume that \(f(t, 0) = 0\) for \(t \geq 0\) and \(I_k(0) = 0\) for \(k \in \mathbb{N}\), thus \(x \equiv 0\) is a solution of (1.1).

The particular case of (1.1) with \(a(t) \equiv 0\) reads as
\[ \begin{align*}
x'(t) &= f(t, x_t), \quad 0 \leq t \neq t_k, \\
\Delta(x(t_k)) &= I_k(x(t_k)), \quad k = 1, 2, \ldots,
\end{align*} \tag{1.2} \]
and has been studied by many authors (see e.g. [2, 18, 20, 21, 22, 23, 24]). In the present study, the aim however is to take full advantage of the negative instantaneous feedback given by the term \(a(t)x(t)\) on the left-hand side of (1.1).

A rigorous abstract formulation of the existence of solutions problem for (1.1), or for more general impulsive DDEs, has been well established in the literature, and will be omitted here (see e.g., [1, 7, 8, 10, 18, 19] and references therein for more details). We need however to set some properties for \(f\), as well as clarify the space of initial conditions.

For a compact interval \([\alpha, \beta] \subset \mathbb{R}\), denote by \(B([\alpha, \beta]; \mathbb{R})\) the space of bounded functions from \([\alpha, \beta]\) to \(\mathbb{R}\) equipped with the supremum norm, and \(\text{PC}([\alpha, \beta]; \mathbb{R})\) the subspace of \(B([\alpha, \beta]; \mathbb{R})\) of functions that are piecewise continuous on \([\alpha, \beta]\) and left continuous on \((\alpha, \beta]\). Now, define the space \(\text{PC} = \text{PC}((\alpha, 0]; \mathbb{R})\) as the space of functions from \((\alpha, 0]\) to \(\mathbb{R}\) for which the restriction to each compact interval \([\alpha, \beta] \subset (\alpha, 0]\) is in the closure of \(\text{PC}([\alpha, \beta]; \mathbb{R})\) in \(B([\alpha, \beta]; \mathbb{R})\). A function \(\varphi \in \text{PC}\) is continuous everywhere except at most for an enumerable number of points \(s\) for which \(\varphi(s^-), \varphi(s^+)\) exist and \(\varphi(s^-) = \varphi(s)\). Denote by \(\text{BPC}\) the subspace of all bounded functions in \(\text{PC}\), \(\text{BPC} = \{ \varphi \in \text{PC} : \varphi\) is bounded on \((\alpha, 0]\)\}, with the supremum norm \(\|\varphi\| = \sup_{s \leq 0} |\varphi(s)|\). Clearly, each space \(\text{PC}(t)\) can be taken as a subspace of \(\text{BPC}\). For equations without impulses, the subspaces of continuous functions \(C, BC\) and \(C(t)\) of \(\text{PC}, \text{BPC}\) and \(\text{PC}(t)\), respectively, will be considered.

For \(t \geq 0, \varphi \in \text{BPC}\), we write \(f(t, \varphi_0) = f(t, L(t, \varphi)) =: F(t, \varphi), \) where \(L(t, \varphi) = \varphi_0, \) i.e., \(L(t, \varphi) = \varphi|_{(-\tau(t), 0]}\). For the impulsive DDE (1.1), we consider initial conditions of the form
\[ x(t_0 + s) = \varphi(s), \quad s \leq 0, \tag{1.3} \]
with \(t_0 \geq 0, \varphi \in \text{BPC}\). In view of our purposes, we may suppose that the extension \(F : [0, \infty) \times \text{BPC} \to \mathbb{R}\) of \(f\) is continuous or piecewise continuous (for simplicity, we abuse the language and refer to \(f\) as being continuous or piecewise continuous as well), but more general frameworks are allowed. For \(f\) piecewise continuous, under very general conditions (which are satisfied with the hypotheses we shall imposed in Section 2), from [1, 10, 17, 19] it follows that the initial value problem (1.1)-(1.3) has a unique solution \(x(t)\) defined on \([t_0, \infty)\). This solution will be denoted by \(x(t, t_0, \varphi)\).
As an important example of a one-dimensional DDE appearing in mathematical biology, we refer to the well-known Lasota-Wazewska equation
\[ N'(t) = -aN(t) + be^{-\beta N(t-\tau)}, \quad t \geq 0 \]
\((a, b, \beta, \tau > 0)\) introduced in \[15\] to model the survival of read blood cells. Generalisations of this equation with periodic coefficients and multiple delays have received much attention from mathematicians and other researchers (see \[3, 6\] and references therein). More recently (\[5, 9, 16\]), impulses have been added to such models, as in
\[ N'(t) + a(t)N(t) = \sum_{i=1}^{n} b_i(t)e^{-\beta_i(t)N(t-\tau_i(t))}, \quad 0 \leq t \neq t_k, \]
\(\Delta N(t_k) := N(t_k^+) - N(t_k) = I_k(N(t_k)), \quad k = 1, 2, \ldots,\)
where all the coefficients and delays are periodic functions with a common period \(\omega > 0\) and \(0 < t_1 < t_2 < \ldots\) with \(t_{k+p} = t_k + \omega, \quad k = 1, 2, \ldots\) for some positive integer \(p\).

The investigation in this paper was partially inspired by the works of Tang \[13\], Yan \[17\] and Zhang \[23\]. Another strong motivation to study the stability of impulsive models of the form (1.1) was to apply criteria of stability for such models to the Lasota-Wazewska impulsive system (1.4), and obtain generalisations of Liu and Takeuchi’s result in \[9\].

Often the entire space \(BPC\) is not a suitable set of initial conditions, and more restrictive sets should be considered. A set \(S \subseteq BPC\) is called an *admissible set of initial conditions* if
\[ \varphi \in S \Rightarrow x_t(\cdot, t_0, \varphi) \in S, \quad t \geq t_0.\]
For models from mathematical biology as (1.4), clearly only positive solutions are meaningful, and therefore admissible. In this paper we establish sufficient conditions for the stability of the zero solution of the impulse DDE (1.1). These results can be applied to study the stability of other solutions, such as periodic solutions. We recall here some stability definitions for an admissible set of initial conditions \(S \subseteq BPC\).

**Definition 1.1.** Let \(f(t, 0) = 0\) for \(t \geq 0\) and \(I_k(0) = 0, \quad k \in \mathbb{N}\). We say that the solution \(x \equiv 0\) of (1.1) is **stable** in \(S\) if for any \(\varepsilon > 0\) and \(t_0 \geq 0\), there exists \(\delta = \delta(t_0, \varepsilon) > 0\) such that
\[ ||\varphi|| < \delta \Rightarrow |x(t, t_0, \varphi)| < \varepsilon, \quad \text{for} \quad t \geq t_0, \quad \varphi \in S.\]
The solution \(x \equiv 0\) of (1.1) is said to be **asymptotically stable** in \(S\) if it is stable and for any \(t_0 \geq 0\), there exists \(\delta = \delta(t_0) > 0\) such that
\[ \varphi \in S, \quad ||\varphi|| < \delta \Rightarrow |x(t, t_0, \varphi)| \to 0 \quad \text{as} \quad t \to \infty.\]
The solution \(x \equiv 0\) of (1.1) is said to be **globally attractive** in \(S\) if all solutions of (1.1) with initial conditions in \(S\) tend to zero as \(t \to \infty\). Finally, the solution \(x \equiv 0\) of (1.1) is **globally asymptotically stable** if it is stable and global attractive. If it is well understood which set \(S\) we are dealing with, we omit the reference to it.

The remainder of this paper is organized in two sections. Section 2 deals with the stability and global asymptotic stability of the zero solution of the impulsive DDE (1.1). First, a main set of assumptions for (1.1) is introduced, and a brief comparison with other hypotheses considered in the literature is presented. Sufficient conditions for the global attractivity of zero are established by treating separately non-oscillatory and oscillatory solutions of (1.1). In Section 3, the global asymptotic
stability of a positive ω-periodic solution to (1.4) is studied by using the results in Section 2. The particular case of constant delays \( \tau_i(t) = m_i \omega \) for \( m_i \) positive integers \( 1 \leq i \leq n \), is further analysed; in this situation, better results are obtained when the impulses in (1.4) are given by linear functions \( I_k(u) = b_k u \). A comparison of our criteria with recent results in the literature is also included.

2. Stability. In this section, we address the stability and global attractivity of the trivial solution of (1.1) in \( BPC \), but another admissible set of initial conditions \( S \subseteq BPC \) can be chosen.

The main assumptions that will be imposed are taken from the ones described below. Hypothesis (H3) will be chosen in alternative to (H2). Occasionally, weaker versions of these assumptions will be considered.

(H1) there exist positive sequences \((a_k)\) and \((b_k)\) such that
\[
 b_k x^2 \leq x[x + I_k(x)] \leq a_k x^2, \quad x \in \mathbb{R}, \ k \in \mathbb{N}; \tag{2.1}
\]
(H2) (i) the sequence \( P_n = \prod_{k=1}^{n} a_k \) is bounded; (ii) \( \int_0^{\infty} a(u) \, du = \infty \);

(H3) (i) the sequence \( P_n = \prod_{k=1}^{n} a_k \) is convergent;

(ii) if \( w : [0, \infty) \to \mathbb{R} \) is a bounded, non-oscillatory and piecewise differentiable function with \( w'(t)w(t) \leq 0 \) on \( (t_k, t_k + 1), k \in \mathbb{N} \), and \( \lim_{t \to \infty} w(t) = c \neq 0 \), then
\[
 \int_0^{\infty} f(s, w_s) \, ds = -\text{sgn}(c)\infty;
\]

(H4) there exist piecewise continuous functions \( \lambda_1, \lambda_2 : [0, \infty) \to [0, \infty) \) such that
\[
 -\lambda_1(t)M_t(\varphi) \leq f(t, \varphi) \leq \lambda_2(t)M_t(-\varphi), \quad t \geq 0, \ \varphi \in PC(t), \tag{2.2}
\]
where \( M_t(\varphi) := \max \left\{ 0, \sup_{\theta \in [-\tau(t), 0]} \varphi(\theta) \right\} \) is the Yorke’s functional on \( PC(t) \);

(H5) there exists \( T > 0 \) with \( T - \tau(T) \geq 0 \) such that
\[
 \alpha_1 \alpha_2 < 1,
\]
where the coefficients \( \alpha_i := \alpha_i(T) \) are given by
\[
 \alpha_i = \sup_{t \geq T} \int_{t-\tau(t)}^{t} \lambda_i(s)e^{-\int_{t-\tau(t)}^{s} a(u) \, du} B(s) \, ds, \quad i = 1, 2, \tag{2.3}
\]
with \( B(t) := \max_{\theta \in [-\tau(t), 0]} \left( \prod_{k:t+\theta \leq t_k < t} b_k^{-1} \right) \).

We observe that the hypotheses (H1) and (H4) imply \( I_k(0) = 0 \) and \( f(t,0) = 0 \) for \( k \in \mathbb{N}, t \geq 0 \), thus \( x = 0 \) is an equilibrium point of (1.1). In (H5) above, we make use of the standard convention that the product \( B(t) \) is equal to one if the number of factors is zero.

In the remainder of this paper, we shall use the notation
\[
 A(t) = \int_0^{t} a(u) \, du, \quad t \geq 0. \tag{2.4}
\]
We shall consider either the set of conditions (H2), or alternatively the requirements in (H3). Condition (H2)(ii) translates as \( \lim_{t \to \infty} A(t) = \infty \) and is fulfilled in
many interesting models in the literature; in this case, instead of (H3)(i) it will be sufficient to impose (H2)(i). However, it is useful to consider the alternative hypothesis (H3)(ii), which in particular allows dealing with (1.2). The constraint on the impulses given by (H1) implies in particular that \( x(t_k^-) x(t_k^+) > 0 \) if \( x(t_k^-) \neq 0 \), with lower and upper bounds \( b_k \leq x(t_k^+)/x(t_k^-) \leq a_k \) for \( k = 1, 2, \ldots \).

We now compare our hypotheses with the ones in the literature, in particular in references [17, 23], two major sources of inspiration for the analysis in this section. As far as we know, the hypotheses on the impulses, (H1) and either (H2)(i) or (H3)(i), are novel and strongly relax the usual requirements in the literature on stability for impulsive DDEs under Yorke-type conditions. In fact, typically either \( I_k \) are supposed to be linear functions, or condition (2.1) is assumed with \( \lambda_1(t) = \lambda_2(t) =: \lambda(t) \) and an extra condition to deal with non-oscillatory solutions was added; moreover, instead of hypothesis (H5) or (2.6), for the global attractivity of the trivial solution of (1.1) Yan imposed the restriction

\[
\sigma := \sup_{t \geq 0} \int_{t-\tau(t)}^{t} \lambda(s) x^{\int_{t-\tau(t)}^{s} a(u) du} B(s) ds < \frac{3}{2},
\]

where \( \alpha_1, \alpha_2 \) are as in (2.3) with \( a(t) \equiv 0 \). In [17], Yan considered (1.1) with a set of more restrictive assumptions: again, the impulsive functions \( I_k \) were subject to condition (2.5), the Yorke condition (H4) was imposed with \( \lambda_1(t) = \lambda_2(t) =: \lambda(t) \) and an extra condition to deal with non-oscillatory solutions was added; moreover, instead of hypothesis (H5) or (2.6), for the global attractivity of the trivial solution of (1.1) Yan imposed the restriction

\[
\alpha_1 \alpha_2 < (3/2)^2
\]

where \( \alpha_1, \alpha_2 \) are as in (2.3) with \( a(t) \equiv 0 \). In [17], Yan considered (1.1) with a set of more restrictive assumptions: again, the impulsive functions \( I_k \) were subject to condition (2.5), the Yorke condition (H4) was imposed with \( \lambda_1(t) = \lambda_2(t) =: \lambda(t) \) and an extra condition to deal with non-oscillatory solutions was added; moreover, instead of hypothesis (H5) or (2.6), for the global attractivity of the trivial solution of (1.1) Yan imposed the restriction

\[
\sigma := \sup_{t \geq 0} \int_{t-\tau(t)}^{t} \lambda(s) x^{\int_{t-\tau(t)}^{s} a(u) du} B(s) ds < \frac{3}{2},
\]

where \( B(t) \) is defined as in (H5). In the case \( \lambda_1(t) = \lambda_2(t) = \lambda(t) \), it is clear that \( \alpha_1 = \alpha_2 \leq \sigma \) for \( \alpha_1, \alpha_2 \) given by (2.3), with \( \alpha_1 = \alpha_2 < \sigma \) if \( a(t) \neq 0 \); there are however positive functions \( a(t) \) for which the condition \( \sigma < \frac{3}{2} \) is less restrictive than \( \alpha_1 \alpha_2 < 1 \). In this situation, it would be convenient to achieve stability results under hypotheses similar to or less restrictive than (2.7). This will be the subject of a forthcoming paper. Nonetheless we should emphasise that the main idea here was to take full advantage of the negative feedback term \( a(t)x(t) \), rather than working with a \( \frac{3}{2} \)-type condition. This kind of approach has also been taken for non-impulsive DDEs: we shall refer later in this section to the work of Tang [13] (see also [6, 14] for some alternative criteria), where the DDE \( x'(t) + ca(t)x(t) = f(t, x_t) \), with the functions \( a(t), \tau(t), f(t, x_t) \) as in (1.1) and \( c \) a positive constant, was studied assuming (H2)(ii) and the following Yorke condition:

\[-a(t)M_1(\varphi) \leq f(t, \varphi) \leq a(t)M_1(-\varphi), \quad t \geq 0, \varphi \in C(t).\]

We start our analysis with an auxiliary result from [17]. Let \( x(t) \) be a solution of (1.1) on \([0, \infty)\) and define \( y(t) \) by

\[ y(t) = \prod_{k:0 \leq t_k < t} J_k(x(t_k))x(t), \quad (2.8) \]
Lemma 2.1. [17] If $x(t)$ is a solution of (1.1) on $[0, \infty)$, then the function $y(t)$ defined by (2.8) is a continuous function satisfying
\[
y'(t) + a(t)y(t) = \prod_{k:0 \leq t_k < t} J_k(x(t_k)) f(t, x_t), \quad t \geq 0, \ t \neq t_k.
\]

To prove the global asymptotic stability of the trivial solution, we consider separately oscillatory and non-oscillatory solutions to (1.1). We recall that a solution $x(t)$ is oscillatory if it is not eventually zero and has arbitrarily large zeros; otherwise, $x(t)$ is non-oscillatory. First, we establish criteria about the asymptotic behaviour of all non-oscillatory solutions.

Lemma 2.2. Assume (H1), (H2)(i) and

\[(H4^*) \text{ for } t \geq 0 \text{ and } \varphi \in PC(t), f(t, \varphi) \leq 0 \text{ if } \varphi \geq 0 \text{ and } f(t, \varphi) \geq 0 \text{ if } \varphi \leq 0.
\]

Then, all non-oscillatory solutions of (1.1) are bounded. If in addition (H2)(ii) holds, then all non-oscillatory solutions of (1.1) converge to zero as $t \to \infty$.

Proof. From (H1), we have
\[
a_k^{-1} \leq J_k(u) \leq b_k^{-1} \quad \text{for } u \neq 0, \ k \in \mathbb{N}. \tag{2.10}
\]

Take a solution $x(t)$ of (1.1) and let $y(t)$ be defined by (2.8). From (2.10), we have
\[
|y(t)| \prod_{k:0 \leq t_k < t} J_k^{-1}(x(t_k)) \leq |y(t)| \prod_{k:0 \leq t_k < t} a_k, \tag{2.11}
\]

For a non-oscillatory solution $x(t)$, assume that $x(t) > 0$ for $t \gg 0$ (the situation is analogous if $x(t) < 0$ for $t \gg 0$). Then $y(t) > 0$ for large $t$ and, from (2.9) and (H4*), $y'(t) \leq y(t) + a(t)y(t) \leq 0$ for $t \gg 0, t \neq t_k$. In particular, there are $c_0, w \geq 0$ such that $y(t) \searrow c_0$ and $e^{A(t)}y(t) \searrow w$ as $t \to \infty$, where $A(t)$ is as in (2.4). On the other hand, from (H2)(ii) and (2.11) we have $0 < x(t) \leq My(t)$ for $t \to \infty$, where $M > 0$ is such that $P_k = \prod_{i=1}^k a_i \leq M$ for $k \in \mathbb{N}$. Consequently $x(t)$ is bounded. Moreover, if (H2)(ii) holds we deduce that $\lim_{t \to \infty} x(t) = \lim_{t \to \infty} y(t) = 0$. \hfill \square

When (H2)(ii) is not satisfied, the convergence to zero of non-oscillatory solutions is obtained if (H3)(ii) is imposed and, rather than having $P_k$ simply bounded, $P_k$ is required to be convergent.

Lemma 2.3. Assume (H1), (H4*), and either (H2) or (H3). Then, all non-oscillatory solutions of (1.1) converge to zero as $t \to \infty$.

Proof. As above and without loss of generality, suppose that $x(t)$ is an eventually positive solution of (1.1), and let $y(t)$ be defined by (2.8). From the proof of Lemma 2.2, $y(t) \searrow c_0, e^{A(t)}y(t) \searrow w$ as $t \to \infty$, with $c_0 \geq 0$. If $\lim_{t \to \infty} A(t) = \infty$, by Lemma 2.2 we have $\lim_{t \to \infty} x(t) = 0$, hence we may suppose that $\lim_{t \to \infty} A(t) = A_0 \in [0, \infty)$, and that (H3) is satisfied. Let $\lim P_k = \gamma$.

For $t \gg 0$, $0 < x(t) \leq \left( \prod_{k:0 \leq t_k < t} a_k \right) y(t)$, and by (H3)(i)
\[
c_+ := \limsup_{t \to \infty} x(t) \leq \gamma c_0, \quad c_- := \liminf_{t \to \infty} x(t) \geq 0.
\]

We now show that $c_+ = c_- = 0$.\hfill \square
Lemma 2.4. Assume (H1) and (H4) and (H5) we have \( x'(t) \leq 0 \) on \((t_k, t_{k+1})\) for large \(k\), which implies that there exist subsequences \((t_{n_1(k)})\) and \((t_{n_2(k)})\) of \((t_k)\), with \(n_2(k) \leq n_1(k)\), such that \( x(t_{n_1(k)}) \to c_+ \) and \( x(t_{n_2(k)}) \to c_- \). From (1.1) and (H1), we have
\[
x(t_{n_1(k)}) = x(t_{n_2(k)}) + I_{n_1(k)}(x(t_{n_1(k)})) \leq a_{n_1(k)}x(t_{n_1(k)}) \leq a_{n_1(k)}x(t_{n_1(k)} - 1),
\]
and, by iteration,
\[
x(t_{n_1(k)}) \leq \left( \prod_{i=n_2(k)}^{n_1(k)} a_i \right) x(t_{n_2(k)}).
\]
Since
\[
\prod_{i=n_2(k)}^{n_1(k)} a_i = P_{n_1(k)}^{-1} \to \gamma^{-1} = 1 \quad \text{as} \quad n \to \infty,
\]
from (2.12) it follows that \( c_+ \leq c_- \), and consequently \( c_+ = c_- = c \). Hence \( x(t) \to c \geq 0 \) as \( t \to \infty \). If \( c > 0 \), define \( \varepsilon := \inf_{n \in \mathbb{N}} \left( \prod_{i=1}^{n} a_i^{-1} \right) > 0 \). From (2.9) and (2.10), we obtain
\[
y(t)e^{A(t)} = y(0) + \int_{0}^{t} e^{A(s)} \left( \prod_{k:0 \leq t_k < s} J_k(x(t_k)) \right) f(s, x_s) \, ds
\]
\[
\leq y(0) + \varepsilon \int_{0}^{t} e^{A(s)} f(s, x_s) \, ds.
\]
But by (H3)(ii), \( \int_{0}^{\infty} f(s, x_s) \, ds = -\infty \), hence \( \int_{0}^{\infty} e^{A(s)} f(s, x_s) \, ds = -\infty \) as well, and from (2.13) we get \( w = -\infty \), which is a contradiction. Thus \( c = 0 \), and the proof is complete.

Remark 2.1. It is easy to verify from the above proof that the assumption (H3)(ii) is not needed at all if (H3)(i) holds with \( \lim_{n} \left( \prod_{k=1}^{n} a_k \right) = 0 \).

The goal now is to show that (H1), (H4), (H5) are sufficient conditions for the trivial equilibrium to be a global attractor of the oscillatory solutions of (1.1). A first auxiliary result is crucial to establish upper and lower bounds for oscillatory solutions, and was inspired in arguments of [23].

Lemma 2.4. Assume (H1), (H4), and
\[
\alpha_1 \alpha_2 \leq 1
\]
for some \( \alpha_1 = \alpha_1(T), \alpha_2 = \alpha_2(T) \) as in (2.3). Let \( y(t) \) be a solution of (2.9) on \([0, \infty)\) and \( t_0 \geq T \) such that \( y(t_0) = 0 \). Then, for any \( \eta > 0 \), the following conditions hold:

(i) If \( -\eta \leq y(t) \leq \eta \) for \( t \in [t_0 - \tau(t_0), t_0] \), then \( -\eta \leq y(t) \leq \eta \alpha_2 \) for all \( t > t_0 \);

(ii) If \( -\eta \alpha_1 \leq y(t) \leq \eta \) for \( t \in [t_0 - \tau(t_0), t_0] \), then \( -\eta \alpha_1 \leq y(t) \leq \eta \) for all \( t > t_0 \).
Proof. We shall prove (i), the proof of (ii) being similar. If the assertion (i) is false, there exists $T_0 > t_0$ such that either $y(T_0) > \alpha_2 \eta$ or $y(T_0) < -\eta$. We consider these two situations separately.

**Case 1.** Suppose that $y(T_0) > \alpha_2 \eta$ for some $T_0 > t_0$, with $-\eta \leq y(t) < y(T_0)$ for $t \in [t_0, T_0]$.

We first prove that there is $\xi_0 \in [T_0 - \tau(T_0), T_0]$ such that $y(\xi_0) = 0$. Otherwise, we obtain necessarily that $y(t) > 0$ for $t \in [T_0 - \delta - \tau(T_0 - \delta), T_0]$ and some small $\delta > 0$ (recall that $y(t)$ is continuous), and from (2.9) and (H4) it follows that

$$y'(t) \leq -a(t)y(t) + \prod_{k:0 \leq t_k < t} J_k(x(t_k)) \lambda_2(t)M_k(-x_t) \leq 0, \quad t \in [T_0 - \delta, T_0],$$

implying that $y(T_0 - \delta) \geq y(T_0)$, which contradicts the definition of $T_0$.

Choose $\xi_0 \in [T_0 - \tau(T_0), T_0]$ such that $y(\xi_0) = 0$. We may suppose that $y(t) > 0$ for $\xi_0 < t < T_0$, thus $t_0 \leq \xi_0$. By (2.2), (2.8), (2.9), and (2.10), for $s \in [\xi_0, T_0] \setminus \{t_k\}$ we obtain

$$\left( e^{A(s)}y(s) \right)' = \prod_{k:0 \leq t_k < s} J_k(x(t_k)) e^{A(s)}f(s, x_s)$$

$$\leq e^{A(s)}\lambda_2(s) \prod_{k:0 \leq t_k < s} J_k(x(t_k))M_s(-x_s)$$

$$= e^{A(s)}\lambda_2(s) \prod_{k:0 \leq t_k < s} J_k(x(t_k)).$$

$$\cdot \max \left\{ 0, \sup_{\theta \in [-\tau(s), 0]} \left( -y(s + \theta) \prod_{k:0 \leq t_k < \theta} J_k(x(t_k))^{-1} \right) \right\}$$

$$= e^{A(s)}\lambda_2(s) \max \left\{ 0, \sup_{\theta \in [-\tau(s), 0]} \left( -y(s + \theta) \prod_{k:0 \leq t_k < \theta} J_k(x(t_k)) \right) \right\}$$

$$\leq e^{A(s)}\lambda_2(s)B(s)M_s(-x_s),$$

with $B(s)$ defined as in (H5). Now, for $s \in [\xi_0, T_0], s \neq t_k$, we have $y_s(\theta) \geq -\eta$ for $s \in [\xi_0, T_0], \theta \in [s - \tau(s), s]$, thus

$$\left( e^{A(s)}y(s) \right)' \leq \eta e^{A(s)}\lambda_2(s)B(s).$$

Integrating over $[\xi_0, T_0]$, we get

$$y(T_0) \leq \eta e^{-A(T_0)} \int_{\xi_0}^{T_0} e^{A(s)}\lambda_2(s)B(s) \, ds$$

$$= \eta \int_{\xi_0}^{T_0} e^{-\int_{\xi_0}^{s} a(u) \, du} \lambda_2(s)B(s) \, ds \leq \alpha_2 \eta,$$

which contradicts the definition of $T_0$.

**Case 2.** Suppose that $y(T_0) < -\eta$ for some $T_0 > t_0$, with $y(T_0) < y(t) \leq \alpha_2 \eta$ for $t \in [t_0, T_0]$.

Reasoning as above, we deduce that there is $\xi_0 \in [t_0, T_0) \cap [T_0 - \tau(T_0), T_0)$ such that $y(\xi_0) = 0$ and $y(t) < 0$ for $\xi_0 < t < T_0$. Since $y(\theta) \leq \alpha_2 \eta$ for $t \in [\xi_0, T_0], t \neq t_k$,
\( \theta \in [t - \tau(t), t] \), we now obtain
\[
\left( e^{A(t)y(t)} \right)' \geq -e^{A(t)} \lambda_1(t) B(t) \alpha_2 \eta.
\]
Integrating over \([\xi_0, T_0]\), and using the inequality \( \alpha_1 \alpha_2 \leq 1 \), we get
\[
y(T_0) \geq -\alpha_2 \eta e^{-A(T_0)} \int_{\xi_0}^{T_0} e^{A(s)} \lambda_1(s) B(s) \, ds
\]
\[
= -\alpha_2 \eta \int_{\xi_0}^{T_0} e^{-\int_s^T a(u) \, du} \lambda_1(s) B(s) \, ds \geq -\alpha_1 \alpha_2 \eta \geq -\eta,
\]
which is not possible.

Sufficient conditions for the boundedness of all solutions of (1.1) follow immediately from Lemmas 2.2, 2.3 and 2.4.

**Theorem 2.1.** Assume (H1), (H2)(i), (H3) and (H4) and (H5). Then the zero solution of (1.1) is (uniformly) stable.

We are now in the position to prove the main result in this section.

**Theorem 2.2.** Assume (H1), either (H2) or (H3), (H4) and (H5). Then the zero solution of (1.1) is globally asymptotically stable.

**Proof.** By virtue of Lemma 2.3 and (2.11), we only need to show that zero attracts all oscillatory solutions of (2.9). Take an oscillatory solution \( y(t) \) of (2.9), and set
\[
u := \limsup_{t \to \infty} y(t), \quad -v := \liminf_{t \to \infty} y(t).
\]
Clearly \( 0 \leq u, v < \infty \), because \( y(t) \) is bounded. For \( t \geq 0 \), denote \( d(t) = t - \tau(t) \) and \( d^2(t) = d(d(t)) \). Fix \( \varepsilon > 0 \) and consider \( T_0 > 0 \) as in (H5) with \( d^2(T_0) > 0 \) and such that
\[
-v_{\varepsilon} := -(v + \varepsilon) < y(t) < u + \varepsilon := u_{\varepsilon}, \quad \text{for} \quad t \geq d^2(T_0).
\]
As \( y(t) \) is continuous, there exists a sequence \( s_n \to \infty \) with \( s_n \geq T_0 \) such that \( y(s_n) > 0 \), \( y(s_n) \) are local maxima, and \( y(s_n) \to u \) as \( n \to \infty \). We may assume that \( y(s) < y(s_n) \) for \( s_n - s > 0 \) small. As in the proof of Lemma 2.4, by the Yorke condition (H4) we deduce that for each \( n \in \mathbb{N} \) there exists \( \xi_n \in [s_n - \tau(s_n), s_n] \) such that \( y(\xi_n) = 0 \) and \( y(s) > 0 \) for \( s \in (\xi_n, s_n] \). By (2.16), we have \( y(s) > -v_{\varepsilon} \) for \( s \in [s_n - \tau(\xi_n), s_n] \). Arguing as in the proof of Case 1 of Lemma 2.4(i), we conclude that \( y(s_n) \leq \alpha_2 v_{\varepsilon} \). Letting \( n \to \infty \) and \( \varepsilon \to 0^+ \), we obtain
\[
u \leq \alpha_2 v.
\]
Similar arguments lead to
\[
u \leq \alpha_1 u.
\]
From (2.17), (2.18), we have
\[
u \leq \alpha_1 \alpha_2 u, \quad v \leq \alpha_1 \alpha_2 v.
\]
Under the constraint \( \alpha_1 \alpha_2 < 1 \), (2.19) is possible only if \( u = 0 \) and \( v = 0 \), thus \( y(t) \to 0 \) as \( t \to \infty \).

For the situation without impulses we obtain the following criterion:
Corollary 2.1. For \( a(t), \tau(t), f(t, \varphi) \) as in (1.1), consider the scalar DDE
\[
x'(t) + a(t)x(t) = f(t, x_t), \quad t \geq 0,
\]
and assume either (H2)(ii) or (H3)(ii), (H4) and \( \alpha_1 \alpha_2 < 1 \), where
\[
\alpha_i := \alpha_i(T) = \sup_{t \geq T} \int_{t-\tau(t)}^t \lambda_i(s)e^{-\int_s^t a(u)du}ds, \quad i = 1, 2,
\]
for some \( T > 0 \). Then the zero solution of (2.20) is globally asymptotically stable.

Even for the situation without impulses (2.20), when \( a(t) \neq 0 \) but \( \int_0^\infty a(t)dt < \infty \), the additional requirement (H3)(ii) must be imposed together with (H4), otherwise zero need not attract the eventually monotone solutions, as shown by the next counter-example.

Example 2.1. Consider the scalar DDE without impulses
\[
x'(t) + \frac{1}{(t+1)^2}x(t) = g(x(t-\tau)), \quad t \geq 0,
\]
where \( \tau > 0 \) and \( g : \mathbb{R} \to \mathbb{R} \) is defined by \( g(x) = -x \) for \( x \leq 0 \), \( g(x) = 0 \) for \( x > 0 \). It is apparent that (2.2) is satisfied with \( \lambda_1(t) \equiv 0, \lambda_2(t) \equiv 1 \). For \( \alpha_i = \alpha_i(T), i = 1, 2 \), defined in (2.21), we get \( \alpha_1 = 0 \) and \( \alpha_2 < \tau < \infty \). Therefore, (H5) holds and zero is an attractor of all oscillatory solutions. Note however that both (H2)(ii) and (H3)(ii) fail, due to
\[
\int_0^\infty \frac{1}{(t+1)^2}dt = 1 < \infty \quad \text{and} \quad \int_0^\infty |g(x(t-\tau))|dt < \infty
\]
if \( \lim_{t \to \infty} x(t) = c > 0 \). On the other hand, observe that zero is not globally asymptotically stable for (2.22), since all functions \( x(t) = c \exp(\frac{1}{t+1}) \) with \( c > 0 \) are solutions of (2.22).

Theorem 2.2 can be slightly improved for systems (1.1) with \( f(t, x_t) \) of the form
\[
f(t, x_t) = \sum_{i=1}^n f_i(t, x_t^i),
\]
where \( f_i(t, x_t^i) = f_i(t, x_{\lfloor t-\tau_i(t) \rfloor}) \) with \( \tau_i(t) \) satisfying the above conditions for \( \tau(t) \), as follows. Let \( \tau(t) = \max_{1 \leq i \leq n} \tau_i(t) \). Suppose now that each \( f_i, i = 1, \ldots, n, \) satisfies the Yorke condition (H4), i.e., there exist piecewise continuous functions \( \lambda_{1,i}, \lambda_{2,i} : [0, \infty) \to [0, \infty) \) such that
\[
-\lambda_{1,i}(t)M_i(\varphi) \leq f_i(t, \varphi_{\lfloor t-\tau_i(t) \rfloor}) \leq \lambda_{2,i}(t)M_i(-\varphi), \quad t \geq 0, 1 \leq i \leq n, \quad \varphi \in PC^i(t),
\]
for \( PC^i(t) = PC([-\tau_i(t), 0]; \mathbb{R}) \) and \( M_i(\varphi) = \max\{0, \sup_{\theta \in [-\tau_i(t), 0]} \varphi(\theta)\} \). A careful reading of the proof of Lemma 2.4 (see formula (2.14)) shows the validity of the statement below.

Theorem 2.3. For (1.1) with \( f(t, x_t) \) of the form (2.23), assume (H1) and either (H2) or (H3). Suppose also that the Yorke conditions (2.24) hold and that there is \( T > 0 \) such that \( \alpha_1 \alpha_2 < 1 \), where \( \alpha_j := \alpha_j(T) \) are given by
\[
\alpha_j(T) := \sup_{t \geq T} \int_{t-\tau(t)}^t \sum_{i=1}^n \lambda_{j,i}(s)e^{-\int_s^t a(u)du}B_j(s)ds, \quad j = 1, 2,
\]
and \( B_i(t) := \max_{\theta \in [-\tau_i(t), 0]} \left( \prod_{k=t+\theta \leq t_k < t} b_k^{-1} \right), \) \( i = 1, \ldots, n. \) Then the zero solution of (1.1) is globally asymptotically stable.

The next criterion is an important particular case of Theorem 2.3.

**Corollary 2.2.** Consider (1.1) with \( f(t, x_t) \) given by (2.23). Assume (H1), either (H2) or (H3), and that conditions (2.24) hold with

\[
\sum_{i=1}^{n} \lambda_{j,i}(t) B_i(t) \leq c_j a(t), \quad j = 1, 2,
\]

where \( c_1, c_2 \) are positive constants. If either

\[
A := \limsup_{t \to 0} \int_{t-\tau(t)}^{t} a(u) \, du < \infty
\]

with \( \sqrt{c_1 c_2} (1 - e^{-A}) < 1, \) or \( A = \infty \) with \( c_1 c_2 < 1, \) then the zero solution of (1.1) is globally asymptotically stable. In particular this is the case if conditions (2.24) and (2.26) are satisfied with

\[
\sum_{i=1}^{n} \lambda_{j,i}(t) B_i(t) \leq a(t), \quad j = 1, 2.
\]

**Proof.** Suppose \( A < \infty. \) In this scenario, we may take \( \alpha_j \) in (2.25) given by

\[
\alpha_j = \alpha_j(T) = c_j \sup_{t \geq T} \int_{t-\tau(t)}^{t} a(s) e^{-f_j^*(-u)} \, ds = c_j \sup_{t \geq T} \left( 1 - e^{-f_j^*(u)} \right),
\]

therefore, for any \( \varepsilon > 0 \) there is \( T > 0 \) such that \( \alpha_j \leq c_j (1 - e^{-(A+\varepsilon)}), \) \( j = 1, 2. \) If \( A = \infty, \) instead of (2.27) we obtain \( \alpha_j = c_j, j = 1, 2. \)

As a particular case, a criterion given by Tang [13] and stated below is obtained by considering the DDE without impulses (2.20) and taking \( c_1 = c_2 = 1 \) in Corollary 2.2.

**Corollary 2.3.** For \( a(t), \tau(t), f(t, \varphi) \) as in (1.1), assume (H2)(ii), (2.26) and

\[
-a(t) \mathcal{M}_t(\varphi) \leq f(t, \varphi) \leq a(t) \mathcal{M}_t(-\varphi), \quad t \geq 0, \quad \varphi \in C(t),
\]

where \( \mathcal{M}_t(\varphi) \) is as in (2.2). Then the zero solution of (2.20) is globally asymptotically stable.

3. A periodic Lasota-Wazewska model with impulses. In this section, we study a periodic Lasota-Wazewska model with impulses (see e.g. [5, 9, 16]):

\[
N'(t) + a(t) N(t) = \sum_{i=1}^{n} b_i(t) e^{-\beta_i(t) N(t-\tau_i(t))}, \quad 0 \leq t \neq t_k,
\]

\[
\Delta N(t_k) := N(t_k^+) - N(t_k) = I_k(N(t_k)), \quad k = 1, 2, \ldots,
\]

where \( 0 < t_1 < t_2 < \cdots < t_k < \cdots, t_k \to \infty, \) and

(i) the functions \( a(t), b_i(t), \beta_i(t), \tau_i(t) \) are continuous, positive and \( \omega \)-periodic,

(ii) the functions \( I_k : [0, \infty) \to \mathbb{R} \) are continuous with \( u + I_k(u) > 0 \) for \( u > 0, \)

\( k \in \mathbb{N}; \) moreover, there is a positive integer \( p \) such that

\[
t_{k+p} = t_k + \omega, \quad I_{k+p}(u) = I_k(u), \quad k \in \mathbb{N}, u > 0.
\]
Special attention will be given to the particular case of (3.1) with \( \omega \)-periodic constant delays \( \tau_i(t) = m_i \omega \) for \( m_i \) positive integers, \( i = 1, \ldots, n \). For some related models, see also [4, 11, 12].

Without loss of generality, we may suppose that there are exactly \( p \) instants of impulses on the interval \([0, \omega], t_1, t_2, \ldots, t_p\). With minimal changes, we can also consider a more general framework, with \( a(t), b_i(t), \beta_i(t), \tau_i(t) \) piecewise continuous functions. Due to the biological applications, we only consider positive solutions of (3.1), corresponding to initial conditions \( N_0(\cdot) \). For some related system (3.14) addressed later in this section), for which the existence of a positive \( \omega \)-periodic solution was proven under a very general condition.

Assume now that there exists a positive \( \omega \)-periodic solution \( N^*(t) \), and that \( I_k(u) \geq 0 \) for \( u \geq 0, k \in \mathbb{N} \). If

\[
CT^\infty < 1,
\]

where \( I^\infty = \limsup_{u \to \infty} \sum_{k=1}^{p} \frac{I_k(u)}{u} \) and \( C = \frac{e^{\int_0^\omega a(u) \, du}}{e^{\int_0^\omega a(u) \, du} - 1} \), then (3.1) has at least one \( \omega \)-periodic positive solution \( N^*(t) \).

Some criteria for the existence of an \( \omega \)-periodic solution to (3.1) have been established. Namely, the following result is a consequence of Theorem 2.3 in Li et al. [5]:

**Lemma 3.1.** [5] For (3.1), assume \((f_0), (i_0)\), and that \( I_k(u) \geq 0 \) for \( u \geq 0, k \in \mathbb{Ninterpretedtext})) \).

Let us however mention that the assumption \( I_k(u) \geq 0 \) for all \( u \geq 0, k \in \mathbb{N} \) in [5] is quite strong, since it requires that the impulses are always positive. In view of the biological meaning of the model, the natural constraint is only that \( I_k(u) + u > 0 \) for \( u > 0 \). On the other hand, as we shall see, Liu and Takeuchi [9] considered the particular case of (3.1) with \( \omega \)-periodic constant delays \( \tau_i(t) = m_i \omega \) for \( m_i \) positive integers, \( i = 1, \ldots, n \), and linear impulses \( I_k(u) = b_k u \) with constants \( b_k > -1 \) (see system (3.14) addressed later in this section), for which the existence of a positive \( \omega \)-periodic solution was proven under a very general condition.

For system (3.1) with \((f_0), (i_0)\) fulfilled, we now impose the following additional hypotheses:

(i) there exist constants \( a_1, \ldots, a_p \) and \( b_1, \ldots, b_p \), with \( b_k > -1, \) and such that

\[
b_k \leq \frac{I_k(x) - I_k(y)}{x - y} \leq a_k, \quad x, y \geq 0, x \neq y, k = 1, \ldots, p;
\]

(ii) \( \prod_{k=1}^{p} (1 + a_k) \leq 1 \).

Assume now that there exists a positive \( \omega \)-periodic solution \( N^*(t) \), and effect the change of variables \( x(t) = N(t) - N^*(t) \). Eq. (3.1) is transformed into

\[
x'(t) + a(t)x(t) = f(t, x_t), \quad 0 \leq t \neq t_k,
\]

\[
\Delta x(t_k) = \tilde{I}_k(x(t_k)), \quad k \in \mathbb{N},
\]

where

\[
f(t, \varphi) = \sum_{i=1}^{n} b_i(t) e^{-c_i(t)} \left[ e^{-\beta_i(t) \varphi(-\tau_i(t))} - 1 \right],
\]

\[
c_i(t) = \beta_i(t) N^*(t - \tau_i(t)), \quad i = 1, \ldots, n,
\]

\[
\tilde{I}_k(u) = I_k(N^*(t_k) + u) - I_k(N^*(t_k)), \quad k = 1, \ldots, p.
\]

C. B. S. and S. L. J. are supported by FAPESP, Brazil.
Now, we insert this transformed system into the framework of the previous sections: (3.2) has the form (1.1), where the function \( f(t, x_t) \) may have jump discontinuities at the points \( t \) such that \( t - \tau_i(t) = t_k \) for some \( 1 \leq i \leq n, k \in \mathbb{N} \). Recall that \( x(t) + N^*(t) > 0 \) for \( t \geq 0 \), for any solution \( x(t) \) of (3.2). Naturally, \( S = \{ \phi \in PC(\mathbb{R}) : \phi(\theta) \geq -N^*(\theta) \text{ for } -\bar{\tau} < \theta < 0, \phi(0) > -N^*(0) \} \) is taken as the set of admissible initial conditions for (3.2), and the spaces \( PC^i(t) \) in (2.24) are replaced by \( S^i(t) = \{ \phi \in PC^i(t) : \phi(\theta) \geq -N^*(t - \theta) \text{ for } -\tau_i(t) \leq \theta \leq 0 \} \).

**Lemma 3.2.** Under \((f_0), (i_0)-(i_2)\), system (3.2) satisfies (H1) and (H2).

**Proof.** Let \( f \) and \( \hat{I}_k \) be defined by (3.3). It is apparent that (H1) and (H2)(i) hold for (3.2), since

\[
\tilde{b}_k \leq \frac{u + \hat{I}_k(u)}{u} \leq \tilde{a}_k, \quad k \in \mathbb{N}, u \neq 0, u > -N^*(t_k),
\]

with

\[
\tilde{b}_k = b_k + 1 > 0, \quad \tilde{a}_k = a_k + 1,
\]

and the positive sequence \( (P_k)_{k \in \mathbb{N}} \) defined by \( P_k = \prod_{i=1}^{k} \tilde{a}_i \) is bounded, with

\[
P_k \leq \max \{ \prod_{i=1}^{j} \tilde{a}_i : j = 1, 2, \ldots, p \}.
\]

Since \( a(t) \) is positive and \( \omega \)-periodic, we have \( \int_{0}^{\infty} a(t) \, dt = \infty \). \( \square \)

We now choose adequate \( \lambda_1, \lambda_2 \) for the Yorke condition to be satisfied.

**Lemma 3.3.** Assume \((f_0), (i_0)-(i_2)\). If \( N^*(t) \) is a positive \( \omega \)-periodic solution of system (3.1), then \( (3.2) \) satisfies (2.24) with \( \lambda_{1,i}, \lambda_{2,i} : [0, \infty) \rightarrow [0, \infty) \) given by

\[
\lambda_{1,i}(t) = \beta_i(t)b_i(t)e^{-\beta_i(t)N^*(t-\tau_i(t))}, \quad \lambda_{2,i}(t) = \beta_i(t)b_i(t), \quad t \geq 0, 1 \leq i \leq n. \quad (3.4)
\]

**Proof.** Write \( c_i(t), f_i(t, \varphi) \) as in (3.3). Take \( t \geq 0, i \in \{1, \ldots, n\} \) and \( \varphi \in S^i(t) \). We have

\[
e^{-\beta_i(t)(-\tau_i(t))} - 1 \geq e^{-\beta_i(t)M^i(\varphi)} - 1 \geq -\beta_i(t)M^i(\varphi),
\]

thus \( f_i(t, \varphi) \geq -\beta_i(t)b_i(t) \varphi(-\tau_i(t))e^{-d_i(t)} \) for some \( d_i(t) \) between \( c_i(t) \) and \( c_i(t) + \beta_i(t) \varphi(-\tau_i(t)) > 0 \), hence \( f_i(t, \varphi) \leq \beta_i(t)b_i(t)M^i(\varphi) \). \( \square \)

The next criterion follows as an immediate consequence of Theorem 2.3.

**Theorem 3.1.** Assume \((f_0), (i_0)-(i_2)\), and that there is a positive \( \omega \)-periodic solution \( N^*(t) \) of system (3.1). If there is \( T > \bar{\tau} \) such that \( \alpha_1, \alpha_2 < 1 \) with

\[
\alpha_1 := \sup_{t \geq T} \int_{t-\tau(t)}^{t} \sum_{i=1}^{n} \beta_i(s)b_i(s)e^{-\beta_i(s)N^*(s-\tau_i(s))} B_i(s)e^{-\int_{s}^{t} a(u) \, du} \, ds,
\]

\[
\alpha_2 := \sup_{t \geq T} \int_{t-\tau(t)}^{t} \sum_{i=1}^{n} \beta_i(s)b_i(s)B_i(s)e^{-\int_{s}^{t} a(u) \, du} \, ds,
\]

where \( B_i(t) = \max_{\theta \in [-\tau_i(t), 0]} \left( \prod_{k+i+\theta \leq t < k} (1 + b_k)^{-1} \right) \), then \( N^*(t) \) is globally asymptotically stable, in the sense that it is stable and any positive solution \( N(t) \) of (3.1) satisfies \( \lim_{t \to \infty} (N(t) - N^*(t)) = 0 \).
Since \( a(t), \tau(t) \) are uniformly bounded, condition (2.26) is trivially satisfied. Taking \( \lambda_{j,i}(s) = \beta_i(t)b_i(t), j = 1, 2, i = 1, \ldots, n \), Corollary 2.2 provides an alternative criterion:

**Theorem 3.2.** Assume \( (f_0), (i_0)-(i_2) \), and that there is a positive \( \omega \)-periodic solution \( N^*(t) \) of system (3.1). Then \( N^*(t) \) is globally attractive if

\[
\sum_{i=1}^{n} \beta_i(t)b_i(t) \max_{\theta \in [-\tau_i(t), 0]} \prod_{k: \tau_k \in [t+\theta, t+h]} (1 + b_k)^{-1} \leq a(t), \quad t \geq 0. \tag{3.5}
\]

In what follows, we use the notations \( \beta(t) = \max_{1 \leq i \leq n} \beta_i(t) \) and, for an \( \omega \)-periodic function \( g \),

\[
\overline{\beta} = \max_{0 \leq t \leq \omega} g(t).
\]

Namely, \( \overline{N} = \max_{0 \leq t \leq \omega} N^*(t) \) and \( \overline{\beta N^*} = \max_{0 \leq t \leq \omega \leq 1 \leq i \leq n} \beta_i(t)N^*(t) \).

**Remark 3.1.** For Theorem 3.2, we used conditions (2.24) with \( \lambda_{1,i}(t) = \lambda_{2,i}(t) = \beta_i(t)b_i(t), 1 \leq i \leq n \); the choice of \( \lambda_{1,i}(t) \) in (3.4) however provides a better estimate. Alternatively, instead of \( \lambda_{2,i}(t) = b_i(t)\beta_i(t) \), one can choose

\[
\lambda_{2,i}(t) = \frac{1}{N^*}(e^{\overline{\beta N^*}} - 1)b_i(t)e^{-\beta_i(t)N^*(t-\tau_i(t))}, \quad t \geq 0, 1 \leq i \leq n. \tag{3.6}
\]

In fact, for \( t \geq 0, i \in \{1, \ldots, n\} \) and \( \varphi \in S^i(t) \), we obtain the inequality

\[
e^{-\beta_i(t)\varphi(-\tau_i(t))} - 1 \leq e^{\beta_i(t)M^i_{\omega}(-\varphi)} - 1 \leq \frac{e^{\overline{\beta N^*}} - 1}{N^*} M^i_{\omega}(-\varphi),
\]

which yields \( f_i(t, \varphi) \leq \lambda_{2,i}(t)M^i_{\omega}(-\varphi) \) for \( \lambda_{2,i}(t) \) as in (3.6). Although the choice of \( \lambda_{2,i}(t) \) in (3.4) is always better than the one above (because with \( x = \overline{\beta N^*} \) and \( c_i(t) = \beta_i(t)N^*(t-\tau_i(t)) \leq x \), we have \( (e^x - 1)e^{-c_i(t)} - x \geq 0 \) for \( x \geq 0 \)), the use of \( \lambda_{2,i}(t) \) as in (3.6) turns out to be effective for the special case of (3.1) analysed below.

We now proceed with a deeper analysis of the case of (3.1) with time independent delays multiple of the period,

\[
\tau_i(t) = m_i\omega,
\]

where \( m_i \) are positive integers, \( i = 1, \ldots, n \). In this situation, (3.2) becomes

\[
x'(t) + a(t)x(t) = \sum_{i=1}^{n} b_i(t)e^{-\beta_i(t)N^*(t)} \left[e^{-\beta_i(t)x(t-m_i\omega)} - 1\right], \quad 0 \leq t \neq t_k, \tag{3.7}
\]

\[
\Delta x(t_k) = \tilde{I}_k(x(t_k)), \quad k \in \mathbb{N},
\]

where \( \tilde{I}_k(u) = I_k(N^*(t_k) + u) - I_k(N^*(t_k)) \) are as in (3.3).

In what follows we establish upper estimates for \( \alpha_1, \alpha_2 \) defined by (2.25) which, although not as sharp as the ones in Theorem 3.1, are easier to handle. To simplify the exposition, assume \( I_k(0) = 0 \) for \( k \in \mathbb{N} \) – which is natural from a biological point of view –, so that \( I_k(N^*(t_k)) \geq b_kN^*(t_k) \); otherwise, \( I_k(N^*(t_k)) \geq I_k(0) + b_kN^*(t_k), 1 \leq k \leq p \), and straightforward changes should be introduced in the computations below.
Theorem 3.3. Consider (3.1) with \( \tau_i(t) = m_i \omega \) \((m_i \in \mathbb{N})\) and denote \( \bar{m} = \max_{1 \leq i \leq n} m_i \). Assume \((f_0), (i_0)-(i_2)\) with \( I_k(0) = 0 \) for \( k \in \mathbb{N} \), and that there is a positive \( \omega \)-periodic solution \( N^*(t) \) of system (3.1). If

\[
B_{\bar{m}} \left( \frac{e^{\bar{m}N^*}}{\beta N} - 1 \right)^{\frac{1}{2}} \left( 1 - e^{-\bar{m}f_{0}^{-} a(u)} \right) \cdot \left[ 1 - \left( 1 - e^{-\int_{0}^{-} a(u)} \right)^{-1} \sum_{k=1}^{n} \min(b_k, 0) \right] < 1,
\]

where \( B = \max_{1 \leq i,j \leq p} \prod_{k=1}^{j} (1 + b_{i+k})^{-1} \), then \( N^*(t) \) attracts any positive solution \( N(t) \) of (3.1).

Proof. Condition \((i_2)\) implies \( B \geq 1 \) for \( B \) defined above, hence for \( t \geq 0 \) and \( 1 \leq i \leq n \)

\[
B_i(t) := \max_{\theta \in [-m_i \omega, 0]} \left( \prod_{k:t+\theta \leq t_{k}} (b_k + 1)^{-1} \right) \leq B_{m_i} \leq B_{\bar{m}}.
\]

For the sake of simplicity, in what follows we suppose that the coefficients \( b_k \) in \((i_1)\) satisfy \( b_k \in (-1, 0) \) for \( 1 \leq k \leq p \); otherwise we may replace \( b_k \) by \( \min(0, b_k) \).

Now, choose \( \lambda_{1,i} \) as in (3.4) and \( \lambda_{2,i} \) as in (3.6). For (3.7) we obtain

\[
\lambda_{1,i}(t) = \beta_i(t) b_i(t) e^{-\beta_i(t) N^*(t)}, \quad \lambda_{2,i}(t) = \frac{1}{N^*} (e^{\bar{m}N^*} - 1)b_i(t) e^{-\beta_i(t) N^*(t)}, \quad 0 \leq t \neq t_k.
\]

Since \( N^*(t) \) is an \( \omega \)-periodic solution of (3.1), for \( t > 0, t \neq t_k \)

\[
\sum_{i=1}^{n} b_i(t)e^{-\beta_i(t) N^*(t)} = (N^*)'(t) + a(t)N^*(t), \quad (3.9)
\]

with \( N^*(t) \) having possible jumps at the points \( t_k \). From (3.9), for \( t > 0 \) we derive

\[
\alpha_1(t) := \int_{t}^{t_{k}} \sum_{i=1}^{n} \lambda_{1,i}(s) B_i(s) e^{-\int_{s}^{t} a(u)} du \ ds
\]

\[
\leq \tilde{\beta} B_{\bar{m}} \int_{t}^{t_{k}} \frac{d}{ds} \left[ N^*(s) e^{-\int_{s}^{t} a(u)} du \right] \ ds
\]

\[
= \tilde{\beta} B_{\bar{m}} \left[ N^*(t) \left( 1 - e^{-\bar{m}f_{0}^{-} a(u)} \right) - \sum_{k:t \in [t-\bar{m} \omega, t]} I_k(N^*(t_k)) e^{-\int_{s}^{t} a(u)} du \right]
\]

\[
\leq \tilde{\beta} B_{\bar{m}} \left[ N^*(t) \left( 1 - e^{-\bar{m}f_{0}^{-} a(u)} \right) - \sum_{k:t \in [t-\bar{m} \omega, t]} b_k N^*(t_k) e^{-\int_{t_k}^{t} a(u)} du \right].
\]

(3.10)

With \( z_k = b_k N^*(t_k) \), we have

\[
\sum_{k:t \in [t-\bar{m} \omega, t]} z_k e^{-\int_{t_k}^{t} a(u)} du = \left( \sum_{k:t \in [t-\omega, t]} z_k e^{-\int_{t_k}^{t} a(u)} du \right),
\]
\[
\cdot \left(1 + e^{-\int_0^\omega a(u) \, du} + \cdots + e^{-\left(m-1\right)\int_0^\omega a(u) \, du}\right)
\geq \left(\sum_{k=1}^p z_k\right) \frac{1 - e^{-m\int_0^\omega a(u) \, du}}{1 - e^{-\int_0^\omega a(u) \, du}}.
\]

The estimates (3.10) and (3.11) yield
\[
\alpha_1(t) \leq B^m \beta N^* \left(1 - e^{-m\int_0^\omega a(u) \, du}\right) \left[1 - \left(1 - e^{-\int_0^\omega a(u) \, du}\right)^{-1}\sum_{k=1}^p b_k\right] =: \sigma_1.
\]

In a similar way, we obtain
\[
\alpha_2(t) := \int_{t-m\omega}^{t} \sum_{i=1}^{n} \lambda_2, i(s) B_i(s)e^{-\int_0^s a(u) \, du} ds
\leq B^m \left(e^{\beta N^*} - 1\right) \left(1 - e^{-m\int_0^\omega a(u) \, du}\right) \left[1 - \left(1 - e^{-\int_0^\omega a(u) \, du}\right)^{-1}\sum_{k=1}^p b_k\right] =: \sigma_2.
\]

Clearly, condition \(\sigma_1 \sigma_2 < 1\) is equivalent to (3.8). \(\Box\)

As a by-product, we obtain some results for DDEs without impulses by setting \(b_k = a_k = 0\) for \(1 \leq k \leq p\) in the above theorems.

**Corollary 3.1.** Consider
\[
N'(t) + a(t)N(t) = \sum_{i=1}^{n} b_i(t)e^{-\beta_i(t)N(t-m_i \omega)}, \quad t \geq 0,
\]
where \(\omega > 0, m_i \in \mathbb{N}\) and the coefficient functions satisfy \((f_0)\). Let \(m = \max_{1 \leq i \leq n} m_i\). Then, there is a positive \(\omega\)-periodic solution \(N^*(t)\), which is a global attractor of all positive solutions if one of the following conditions holds:

(i) \(\alpha_1 \alpha_2 < 1\) for
\[
\alpha_1 = \sup_{t \in [0, \omega]} \int_{t-m\omega}^{t} \sum_{i=1}^{n} \beta_i(s) b_i(s)e^{-\beta_i(s)N^*(s)} e^{-\int_0^s a(u) \, du} ds
\]
\[
\alpha_2 = \sup_{t \in [0, \omega]} \int_{t-m\omega}^{t} \sum_{i=1}^{n} \beta_i(s) b_i(s)e^{-\int_0^s a(u) \, du} ds;
\]

(ii) \(\sum_{i=1}^{n} \beta_i(t)b_i(t) \leq a(t), \quad t \geq 0;\)

(iii) \(\beta N^*(e^{\beta N^*} - 1)^{\frac{1}{2}} \left(1 - e^{-m\int_0^\omega a(u) \, du}\right) < 1.\)

**Corollary 3.2.** For the DDE
\[
N'(t) + a(t)N(t) = b(t)e^{-N(t-m \omega)}, \quad t \geq 0,
\]
where \(\omega > 0, m \in \mathbb{N}\) and \(a(t), b(t)\) are positive, \(\omega\)-periodic and continuous functions, there is a positive \(\omega\)-periodic solution \(N^*(t)\), which is a global attractor of all positive solutions if
\[
N^* \left(1 - e^{-m\int_0^\omega a(u) \, du}\right) e^{-m\int_0^\omega a(u) \, du} \sup_{t \in [0, \omega]} \int_{t}^{t+m\omega} b(t+s)e^{\int_0^s a(u) \, du} ds < 1.
\]
Proof. Here, we use Lemma 3.3 and Corollary 2.1 directly. The Yorke condition (2.2) is satisfied with \( \lambda_1(t) = b(t)e^{-N^*(t)} \) and \( \lambda_2(t) = b(t) \). For \( \alpha_1(t), \alpha_2(t) \) given by (2.21) we obtain

\[
\alpha_1 := \alpha_1(t) = \sup_{t \in [0, \omega]} \int_{t-m\omega}^{t} b(s)e^{-N^*(s)} e^{-\int_{s}^{t} a(u) \, du} \, ds = \sup_{t \in [0, \omega]} \int_{t-m\omega}^{t} \frac{d}{ds} \left[ N^*(s)e^{-\int_{s}^{t} a(u) \, du} \right] \, ds = \sup_{t \in [0, \omega]} N^*(t) \left( 1 - e^{-m \int_{0}^{\alpha} a(u) \, du} \right) = \overline{N^*} \left( 1 - e^{-m \int_{0}^{\alpha} a(u) \, du} \right)
\]

and

\[
\alpha_2 := \alpha_2(t) = \sup_{t \in [0, \omega]} \int_{t-m\omega}^{t} b(s)e^{\int_{s}^{t} a(u) \, du} \, ds = e^{-m \int_{0}^{\alpha} a(u) \, du} \sup_{t \in [0, \omega]} \int_{0}^{m\omega} b(t+s)e^{\int_{t+s}^{t} a(u) \, du} \, ds.
\]

\[\square\]

Remark 3.2. Corollary 3.2 is easily extended to (3.12), however our interest here is to compare the statement in Corollary 3.2 with [3]. Using an iterative technique, Graef et al. [3], showed that the \( \omega \)-periodic positive solution \( N^*(t) \) of (3.13) is globally attractive if

\[
\sigma := \int_{0}^{m\omega} b(s)e^{-N^*(s)} \, ds \leq 1.
\]

With the notations of the above proof, clearly \( \alpha_1 < \sigma \). On the other hand, observe that \( \sigma = \int_{0}^{m\omega} a(s)N^*(s) \, ds \); whether \( \alpha_2 \leq \sigma \) or not depends on the relative sizes of the functions \( a(t), b(t) \) and the delay \( m\omega \). Therefore, the criteria in Corollary 3.2 and in [3] are not comparable without further information on the coefficient functions and delay.

We now study the particular case of (3.1) with \( \tau_i(t) \equiv m_i\omega \) (\( m_i \) positive integers) and the impulses given by \( I_k(u) = b_ku \):

\[
N'(t) + a(t)N(t) = \sum_{i=1}^{n} b_i(t) e^{-\beta_i(t)N(t-m_i\omega)}, \quad 0 \leq t \neq t_k,
\]

\[
\Delta N(t_k) = b_kN(t_k), \quad k = 1, 2, \ldots.
\]

(3.14)

As before, we assume that \( a(t), b_i(t), \beta_i(t) \) are as in (f_0), and that (i_0) holds, i.e., \( b_k > -1 \) for \( k \in \mathbb{N}, \ 0 < t_1 < t_2 < \cdots < t_p < \omega \) and

\[
t_{k+p} = t_k + \omega, \quad b_{k+p} = b_k, \quad \forall k \in \mathbb{N},
\]

(3.15)

where \( p \) is some positive integer. In this setting, assumption (i_2) translates as

\[
\prod_{i=1}^{p} (1 + b_k) \leq 1.
\]

(3.16)

Under these assumptions, the existence of a positive \( \omega \)-periodic solution \( N^*(t) \) of (3.14) follows from [9]. The next theorem recovers the criterion for its global asymptotic stability established in [9].
Theorem 3.4. Consider system (3.14) with $b_k > -1$ for all $k$, and assume $(f_0)$, (3.15) and (3.16). For $N^\ast(t)$ a positive $\omega$-periodic solution, whose existence is given in [9], $N^\ast(t)$ is globally attractive if
\[
\sum_{i=1}^{n} \left( \prod_{k=1}^{p} (1 + b_k)^{-m_i} \right) b_i(t) \beta_i(t) \leq a(t), \quad t \geq 0. \tag{3.17}
\]

Proof. After translating the $\omega$-periodic solution $N^\ast(t)$ to the origin, we obtain system (3.7) with $I_k(x(t_k)) = b_kx(t_k)$, $k = 1, 2, \ldots$. We now effect the change of variables in (2.8), which in this situation reads as
\[
y(t) = \prod_{k:0 \leq t_k < t} (1 + b_k)^{-1}x(t), \tag{3.18}
\]
and obtain an equivalent DDE with piecewise coefficients but no impulses, given by
\[
y'(t) + a(t)y(t) = \sum_{i=1}^{n} \bar{b}_i(t)e^{-\hat{\beta}_i(t)}N^\ast(t) \left( e^{-\hat{\beta}_i(t)}y(t-m_i,\omega) - 1 \right), \tag{3.19}
\]
where
\[
\bar{b}_i(t) = b_i(t) \prod_{k:0 \leq t_k < t} (1 + b_k)^{-1}, \quad \hat{\beta}_i(t) = \beta_i(t) \prod_{k:0 \leq t_k < t-m_i,\omega} (1 + b_k), \quad i \leq i \leq n.
\]
Reasoning as in Lemma 3.3, we deduce that for (3.19) the Yorke condition (2.2) holds with $\lambda_1(t) = \lambda_2(t) = \sum_{i=1}^{n} \bar{b}_i(t)\hat{\beta}_i(t)$. Next, we observe that
\[
\sum_{i=1}^{n} \bar{b}_i(t)\hat{\beta}_i(t) = \sum_{i=1}^{n} \left( \prod_{k=1}^{p} (1 + b_k)^{-m_i} \right) b_i(t)\beta_i(t),
\]
and apply Corollary 2.2 to (3.19) to obtain the result. \hfill \Box

Remark 3.3. Yan [16] considered model (3.14) with the following additional constraint: the function
\[
t \mapsto \Theta(t) := \prod_{0 \leq t_k < t} (1 + b_k)
\]
is $\omega$-periodic. As pointed out by Liu and Takeuchi [9], the condition of $\Theta(t)$ being $\omega$-periodic is too restrictive: it implies (3.15) and that $\prod_{i=1}^{p}(1 + b_k) = 1$. Under these assumptions, Yan [16] gave additional sufficient conditions for the existence and global attractivity of a unique $\omega$-periodic solution of (3.14). Nevertheless, Liu and Takeuchi remarked that Yan’s proof was not complete. In turn, they proved themselves the existence of a positive $\omega$-periodic solution $N^\ast(t)$ to (3.14) if
\[
\prod_{i=1}^{p}(1 + b_k)e^{-\int_{0}^{\infty} a(s) \, ds} < 1,
\]
which is always true if (3.16) is satisfied, and showed that $N^\ast(t)$ is globally attractive if, in addition to (3.16), the above condition (3.17) was imposed. The technique in [9] is quite different from the approach in [16]: the latter adapts the method in [3], whereas the main idea in [9] is to assume a Yorke-type condition of the form (2.24), and use [13]. Although this scenario is a particular situation of the general situation treated in our Corollary 2.2, the work in [9] was a strong motivation for the investigation carried out in this section.
For the particular case of system (3.14) under conditions \((f_0), (i_0)\) and (3.16), let \(N^*(t)\) be a positive \(\omega\)-periodic solution. Now, rather than (3.18), we use an alternative change of variables:

\[
y(t) = \frac{N(t)}{N^*(t)} - 1.
\]

This leads to the equivalent system

\[
y'(t) = -r(t)y(t) + \frac{1}{N^*(t)} \sum_{i=1}^{n} b_i(t)e^{-\beta_i(t)N^*(t)} \left[ e^{-\beta_i(t)N^*(t)}y(t-m_i\omega) - 1 \right], \quad 0 \leq t \neq t_k,
\]

with \(\Delta y(t_k) = 0\) for \(k \in \mathbb{N}\), where \(r(t)\) is a piecewise continuous function defined by

\[
r(t) = \frac{1}{N^*(t)} \sum_{i=1}^{n} b_i(t)e^{-\beta_i(t)N^*(t)}.
\]

In other words, (3.21) is a DDE with piecewise continuous coefficients and no impulses. As before, let \(\bar{m} = \max_{1 \leq i \leq n} m_i\). In this situation, the set of admissible initial conditions for (3.21) is \(S_1 = \{ \varphi \in C([-\bar{m}\omega, 0]) : \varphi(0) \geq -1 \text{ for } -\bar{m}\omega \leq \theta < 0, \varphi(0) > -1 \}\).

**Remark 3.4.** For the situation of (3.1) with \(\tau_i(t) \equiv m_i\omega\) and impulsive functions \(I_k\) satisfying \((i_1)\) with \(b_k < a_k\) for some \(k \in \{1, \ldots, p\}\), the change of variables (3.20) is not suitable for our purposes, since it transforms (3.1) into (3.21) together with the impulsive conditions

\[
\Delta y(t_k) = \hat{I}_k(y(t_k)), \quad k \in \mathbb{N},
\]

where \(\hat{I}_k\) satisfy

\[
\frac{1 + b_k}{1 + a_k} x^2 \leq x[\hat{x} + \hat{I}_k(\hat{x})] \leq \frac{1 + a_k}{1 + b_k} x^2, \quad x \in \mathbb{R}, \quad k \in \mathbb{N}.
\]

Therefore (H2)(i) is never satisfied, and the results in Section 2 cannot be invoked.

We now apply Corollary 2.1 to the non-impulsive equation (3.21).

**Theorem 3.5.** Consider system (3.14) with \(b_k > -1\) for all \(k\), assume \((f_0), (3.15), (3.16)\), and let \(N^*(t)\) be an \(\omega\)-periodic positive solution. If \(\alpha_1 \alpha_2 < 1\), where

\[
\alpha_1 := \sup_{t \in [0, \omega]} \frac{1}{N^*(t)} \int_{-\bar{m}\omega}^{t} \sum_{i=1}^{n} b_i(s)\beta_i(s)N^*(s)e^{-\beta_i(s)N^*(s)}f'(s) a(s) du \prod_{k : s \leq t_k < t} (1 + b_k) ds
\]

\[
\alpha_2 := \sup_{t \in [0, \omega]} \frac{1}{N^*(t)} \int_{-\bar{m}\omega}^{t} \sum_{i=1}^{n} b_i(s)\beta_i(s)N^*(s)e^{-\beta_i(s)N^*(s)}f''(s) a(s) du \prod_{k : s \leq t_k < t} (1 + b_k) ds,
\]

where \(\bar{m} = \max_{1 \leq i \leq p} m_i\), then \(\lim_{t \to \infty} \left( N(t) - N^*(t) \right) = 0 \) for any positive solution \(N(t)\) of (3.14).

**Proof.** First, we observe that for \(t > 0, t \neq t_k\),

\[
r(t) = \frac{1}{N^*(t)} \left[ (N^*)'(t) + a(t)N^*(t) \right].
\]
For $t > \bar{m}\omega$ and $t - \bar{m}\omega \leq s < t$, we get
\[
\int_s^t r(u) \, du = \int_s^t a(u) \, du + \int_s^t \left( N^*(u) \right) \frac{(N^*)'(u)}{N^*(u)} \, du \\
= \int_s^t a(u) \, du + \log \left( \frac{N^*(t)}{N^*(s)} \right) + \sum_{k:s < t_k < t} \log \left( \frac{N^*(t_k)}{N^*(t_k')} \right) \\
= \int_s^t a(u) \, du + \log \left( \frac{N^*(t)}{N^*(s)} \right) \prod_{k:s \leq t_k < t} (1 + b_k)^{-1},
\]
and hence
\[
e^{-\int_s^t r(u) \, du} = e^{-\int_s^t a(u) \, du} \frac{N^*(s)}{N^*(t)} \prod_{k:s \leq t_k < t} (1 + b_k). \tag{3.23}
\]
For the DDE without impulses (3.21), the Yorke hypothesis (H4) is satisfied with
\[
\lambda_1(s) = \sum_{i=1}^n b_i(s)\beta_i(s)e^{-\beta_i(s)N^*(s)}, \quad \lambda_2(s) = \sum_{i=1}^n b_i(s)\beta_i(s).
\]
Now, let $\alpha_j(t) = \int_{t-k\omega}^{t} \lambda_j(s)e^{-\int_s^t r(u) \, du} \, ds$ for $j = 1, 2, t > 0$. The $\omega$-periodic functions $\alpha_j(t)$ satisfy $\alpha_j(t) \leq \alpha_j$ for $\alpha_j, j = 1, 2$, as in (3.22), and the result follows from Corollary 2.1.

Following the approach in Theorem 3.3, we now get estimates for $\alpha_1, \alpha_2$ which are easier to verify, although they are not as refined as in (3.22).

**Theorem 3.6.** Consider system (3.14) with $b_k > -1$ for all $k$ and denote $\bar{m} = \max_{1 \leq i \leq p} m_i$. Assume $f_0$, (3.15), (3.16), and let $N^*(t)$ be an $\omega$-periodic positive solution. If
\[
\sigma := \left( \max_{s < \bar{m}N^*} \left( e^{\beta(s)N^*} - 1 \right) \right)^\frac{1}{2} \left[ 1 - \left( e^{-\int_0^\omega a(u) \, du} \prod_{i=1}^p (1 + b_k) \right)^{\bar{m}} \right] < 1, \tag{3.24}
\]
then $\lim_{t \to \infty} (N(t) - N^*(t)) = 0$ for any positive solution $N(t)$ of (3.14).

**Proof.** We first observe that, for any fixed $t > 0$, the function $h(s) := e^{-\int_s^t r(u) \, du}$ defined for $s \in [t - \bar{m}\omega, t]$ is continuous. Applying Theorem 3.3 to (3.14), we deduce the global asymptotic stability of $N^*(t)$ if $\sigma < 1$, where $\sigma$ is given by (3.8) with $b_k = 0$ for all $k$ and $a(t)$ is replaced by $r(t)$. Thus $\sigma$ is as in (3.24), since (3.23) implies
\[
e^{-\bar{m}\omega} \int_0^\omega r(u) \, du = e^{-\bar{m}\omega} \int_0^\omega a(u) \, du \prod_{k=1}^p (1 + b_k)^{\bar{m}}.
\]

**Remark 3.5.** Under stronger constraints, including that $\prod_{i=1}^p (1 + b_k) = 1$, Yan [16] claimed that the condition
\[
\sum_{i=1}^n \eta_i \int_0^\omega b_i(s) \left( \prod_{k:0 < t_k < s} (1 + b_k) \right) e^{-\beta_i(s)N^*(s)} \, ds \leq 1,
\]
with $\eta_i = \sup_{s \geq 0} \left( \prod_{k:0 < t_k < s} (1 + b_k) \right) \beta_i(t)$, implies the global attractivity of the positive $\omega$-periodic solution $N^*(t)$ of (3.14) (see also Remark 3.3, for a comment in [9]). In
any case, the results in Theorems 3.5 and 3.6 are not easily comparable with the claim in [16].

Acknowledgments. This work was supported by Fundação para a Ciência e a Tecnologia, under projects UID/MAT/04561/2013 (T. Faria) and PEstOE/MAT/UI0013/2014 (J.J. Oliveira).

REFERENCES

[1] T. Faria, M. C. Gadotti and J. J. Oliveira, Stability results for impulsive functional differential equations with infinite delay, *Nonlinear Anal.*, **75** (2012), 6570–6587.

[2] K. Gopalsamy and B. G. Zhang, On delay differential equations with impulses, *J. Math. Anal. Appl.*, **139** (1989), 110–122.

[3] J. R. Graef, C. Qian and P. W. Spikes, Oscillation and global attractivity in a periodic delay equation, *Canad. Math. Bull.*, **39** (1996), 275–283.

[4] H.-F. Huo, W.-T. Li and X. Liu, Existence and global attractivity of positive periodic solution of an impulsive delay differential equation, *Appl. Anal.*, **83** (2004), 1279–1290.

[5] X. Li, X. Liu, D. Jiang and X. Zhang, Existence and multiplicity of positive periodic solutions to functional differential equations with impulse effects, *Nonlinear Anal.*, **62** (2005), 683–701.

[6] G. Liu, A. Zhao and J. Yan, Existence and global attractivity of unique positive periodic solution for a Lasota-Wazewska model, *Nonlinear Anal.*, **64** (2006), 1737–1746.

[7] X. Liu and G. Ballinger, Uniform asymptotic stability of impulsive delay differential equations, *Computers Math. Appl.*, **41** (2001), 903–915.

[8] X. Liu and G. Ballinger, Existence and continuability of solutions for differential equations with delays and state-dependent impulses, *Nonlinear Anal.*, **51** (2002), 633–647.

[9] X. Liu and Y. Takeuchi, Periodicity and global dynamics of an impulsive delay Lasota-Wazewska model, *J. Math. Anal. Appl.*, **41** (2001), 903–915.

[10] A. Ouahab, Existence and uniqueness results for impulsive functional differential equations with scalar multiple delay and infinite delay, *Nonlinear Anal.*, **67** (2007), 1027–1041.

[11] S. H. Saker and J. O. Alzabut, On the impulsive delay hematopoiesis model with periodic coefficients, *Rocky Mountain J. Math.*, **39** (2009), 1657–1688.

[12] S. H. Saker and J. O. Alzabut, Existence of periodic solutions, global attractivity and oscillation of impulsive delay population model, *Nonlinear Anal. RWA*, **8** (2007), 1029–1039.

[13] X. H. Tang, Asymptotic behavior of delay differential equations with instantaneously terms, *J. Math. Anal. Appl.*, **302** (2005), 342–359.

[14] X. H. Tang and X. Zou, Stability of scalar delay differential equations with dominant delayed terms, *Proc. Roy. Soc. Edinburgh Sect. A*, **133** (2003), 951–968.

[15] M. Wazewska-Czyzewska and A. Lasota, Mathematical problems of the dynamics of red blood cells system, *(Polish) Mat. Stos. (3)*, **6** (1976), 23–40.

[16] J. Yan, Existence and global attractivity of positive periodic solution for an impulsive Lasota-Wazewska model, *J. Math. Anal. Appl.*, **279** (2003), 111–120.

[17] J. Yan, Stability for impulsive delay differential equations, *Nonlinear Anal.*, **63** (2005), 66–80.

[18] J. Yan, A. Zhao and J. J. Nieto, Existence and global attractivity of positive periodic solution of periodic single-species impulsive Lotka-Volterra systems, *Math. Comput. Modelling*, **40** (2004), 509–518.

[19] R. Ye, Existence of solutions for impulsive partial neutral functional differential equation with infinite delay, *Nonlinear Anal.*, **73** (2010), 155–162.

[20] J. S. Yu, Explicit conditions for stability of nonlinear scalar delay differential equations with impulses, *Nonlinear Anal.*, **46** (2001), 53–67.

[21] J. S. Yu and B. G. Zhang, Stability theorem for delay differential equations with impuls, *J. Math. Anal. Appl.*, **199** (1996), 162–175.

[22] H. Zhang, L. Chen and J. J. Nieto, A delayed epidemic model with stage-structure and pulses for pest management strategy, *Nonlinear Anal. RWA*, **9** (2008), 1714–1726.
[23] X. Zhang, Stability on nonlinear delay differential equations with impulses, Nonlinear Anal., 67 (2007), 3003–3012.

[24] A. Zhao and J. Yan, Asymptotic behavior of solutions of impulsive delay differential equations, J. Math. Anal. Appl., 201 (1996), 943–954.

Received July 2015; revised June 2016.

E-mail address: teresa.faria@fc.ul.pt
E-mail address: jjoliveira@math.uminho.pt