Three-dimensional simulations of the magnetic stress in a neutron star crust

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Abstract

We present the first fully self-consistent three-dimensional model of a neutron star’s magnetic field, generated by electric currents in the star’s crust via the Hall effect. We find that the global-scale field converges to a Hall-attractor state, as seen in recent axisymmetric models, but that small-scale features in the magnetic field survive even on much longer timescales. These small-scale features propagate toward the dipole equator, where the crustal electric currents organize themselves into a strong equatorial jet. By calculating the distribution of magnetic stresses in the crust, we predict that neutron stars with fields stronger than $10^{14}$ G can still be subject to starquakes more than $10^5$ yr after their formation.

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I. INTRODUCTION

Neutron stars are of interest not only for the exotic states of matter they contain, but also for their magnetic fields, which are the strongest in the universe. In the case of pulsars, the magnetic field produces beams of synchrotron radiation that can be detected thousands of light-years away. The field can also affect the dynamics of the star itself, through the magnetic stress it exerts on the star’s solid outer layer — known as the crust. In neutron stars with especially strong magnetic fields (the so-called magnetars) the magnetic stress can be strong enough to fracture the crust, producing a starquake [1]. This is believed to be the mechanism behind the X-ray outbursts and gamma-ray flares detected in these objects [2].

The magnetic fields of pulsars, inferred from spindown measurements, can be anywhere in the range $10^8$–$10^{15}$ G. However, the spindown rate depends only on the large-scale component of the field at the magnetic poles, so these measurements may underestimate the actual field strength in the star. There is observational evidence that at least some neutron stars have stronger magnetic features on smaller scales [3–5]. To interpret observations of neutron stars it is therefore necessary to develop a self-consistent, three-dimensional (3D) model of their magnetic fields.

The external field of a neutron star is generated by electric currents flowing within its crust. Because the ions in the crust are locked together in a rigid Coulomb lattice, the currents arise purely through the flow of electrons, whose dynamics depend primarily on the Hall effect [6]; this situation is commonly referred to as electron magneto-hydrodynamics (EMHD). Recently there have been numerous studies of EMHD in neutron star crusts [7–15]. However, because of difficulties solving the EMHD equations computationally in 3D, in all of these studies the magnetic field was assumed to be axisymmetric. Until now, the only three-dimensional models of EMHD have used a simplified periodic-box geometry, with uniform electron density and electrical conductivity [16–18]. These models demonstrate that the magnetic field can have a complex three-dimensional structure, but they cannot be used to predict the global-scale field morphology in a real neutron star. Currently, therefore, the most reliable predictions regarding the nature of the global-scale field come from axisymmetric models.

Using one such axisymmetric model, Gourgouliatos and Cumming [14] found that an
initially dipolar magnetic field evolves towards a quasi-steady configuration that they called a “Hall attractor”. After \( \sim 10^5 \) yr the electric currents in the crust are concentrated in a narrow jet around the dipole equator, producing a strong belt of poloidal field. However, because their model was only axisymmetric, it is unknown whether such a state is stable to 3D perturbations in a real neutron star.

Here, we present for the first time global 3D numerical simulations of the magnetic field in a neutron star crust. We use a pseudo-spectral code that is well established for solving MHD problems, and that we have adapted to solve the EMHD equations. This code allows us to perform simulations that are not only fully 3D, but also higher resolution than any of those previously presented even in 2D. We describe how the Hall attractor is modified in 3D, and discuss the magnitude and distribution of magnetic stresses within the crust.

II. THE MODEL

We work in a reference frame that corotates with the neutron star’s crust. The macroscopic electromagnetic fields obey Maxwell’s equations

\[
\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E},
\]

\[
\mathbf{J} = \frac{c}{4\pi} \nabla \times \mathbf{B},
\]

which here are written in Gaussian cgs units, with \( \mathbf{E} \) and \( \mathbf{B} \) the electric and magnetic fields, \( \mathbf{J} \) the electric current, and \( c \) the speed of light. In Equation (2) we have omitted the displacement current, which is entirely negligible on the typical timescale (\( \gtrsim 10^4 \) yr) on which the magnetic field evolves.

Because the ions are fixed within the crust, the electric current, \( \mathbf{J} \), depends only on the electron fluid velocity, \( \mathbf{v} \), with

\[
\mathbf{J} = -en \mathbf{v},
\]

where \( n \) is the electron number density and \( e = |e| \) is the elementary charge. We close the equations using the generalized Ohm’s Law

\[
\frac{1}{\sigma} \mathbf{J} = \frac{1}{en} \nabla P_e + \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right),
\]

which describes the balance of forces on the electron gas. Here, \( P_e \) is the electron pressure, and \( \sigma \) is the electrical conductivity. If the crust is either sufficiently cool (i.e., well below
the electron degeneracy temperature) or close to isothermal, then the pressure term can be approximated as the gradient of the electron chemical potential, and combined with the electric potential [6]. The complete system of equations can then be combined into a single equation describing the evolution of the magnetic field:

\[ \frac{\partial B}{\partial t} = \frac{c}{4\pi e} \nabla \times \left( \frac{1}{n} B \times (\nabla \times B) \right) - \nabla \times (\eta \nabla \times B) , \]

where the electron density \( n \) and magnetic diffusivity \( \eta = \frac{c^2}{4\pi \sigma} \) are both fixed functions of radius \( r \). The two terms on the right-hand side of Equation (5) represent the Hall effect and Ohmic dissipation, respectively.

In this study we approximate the structure of the crust as fixed and spherically symmetric. We do not therefore take account of any deformations in the crust arising from magnetic stresses. However, we can use our results to calculate the magnetic stress and determine whether it would be large enough to induce fractures in the crust. We take the top and bottom of the crust to be the spherical surfaces \( r = R \) and \( r = 0.9R \) respectively, where \( R = 10 \text{ km} \) is the radius of a typical neutron star. For \( n(r) \) and \( \eta(r) \) we adopt the same analytical profiles used by Gourgouliatos and Cumming [14]. Specifically,

\[ n = n_0 \left( 1 + \frac{1 - r/R}{0.0463} \right)^4 \]

\[ \eta = \eta_0 \left( 1 + \frac{1 - r/R}{0.0463} \right)^{-8/3} , \]

where \( n_0 = 2.5 \times 10^{34} \text{ cm}^{-3} \) and \( \eta_0 = 4.0 \times 10^{-4} \text{ cm}^2\text{s}^{-1} \) are the values at \( r = R \). These profiles are only rough approximations to the (highly uncertain) profiles in real neutron stars, but fortunately our results are not sensitive to the specific profiles used.

Finally, we must impose boundary conditions on the magnetic field at the top and bottom boundaries of the crust. At the top we impose vacuum boundary conditions, matching the magnetic field to a current-free field outside the star. At the bottom we impose either the idealized boundary conditions used by Gourgouliatos and Cumming [14], which are \( B_r = 0 \) and \( J_r = 0 \), or the more realistic boundary conditions used by Hollerbach and Rüdiger [8], which correspond to \( E_\theta = 0 \) and \( E_\phi = 0 \) in spherical polar coordinates. From here on we refer to these two sets of conditions as GC and HR respectively. The HR boundary conditions model the core of the star as a type-I superconductor, from which any magnetic flux is expelled by the Meissner effect. In order to implement the HR boundary conditions in the
model, it is convenient to make a small modification to the electron density profile \( n(r) \), as described by Hollerbach and Rüdiger [8], to make \( 1/n \) vanish at the bottom of the crust. We therefore use a slightly modified density profile \( \tilde{n}(r) \), defined as \( 1/\tilde{n}(r) = 1/n(r) - 1/n(0.9R) \).

The relative importance of the Hall effect and Ohmic dissipation terms in Equation (5) depends on the strength of the magnetic field. Assuming that the field has a characteristic strength \( B_0 = 10^{14} \text{G} \), as is observed for magnetars, the typical ratio of these terms is given by the Hall parameter,

\[
H \equiv \frac{cB_0}{4\pi en_0\eta_0} \simeq 50,
\]

implying that the Hall effect dominates the dynamics of the magnetic field. In that case we expect the field to evolve on the Hall timescale,

\[
t_{\text{Hall}} \equiv \frac{4\pi en_0R^2}{cB_0} \simeq 1.6 \text{Myr},
\]

where we have assumed that the characteristic lengthscale for the magnetic field is \( R = 10 \text{km} \). If the field has strong, small-scale features, then these will evolve on a shorter timescale.

The code we use to solve the EMHD equation (5) is adapted from the 3D MHD code PARODY, developed by Dormy et al. [19] and Aubert et al. [20]. We have made significant modifications to the code in order to solve the EMHD problem. The code is pseudo-spectral, and uses spherical harmonic expansions in latitude and longitude, and a discrete grid in radius, making it perfectly suited to solving problems in spherical-shell geometry. The results that we present here have a resolution of 128 grid-points in radius, and spherical harmonics up to degree \( l = 100 \). We have benchmarked the code against the previously published axisymmetric results of Hollerbach and Rüdiger [8, 21] and Gourgouliatos and Cumming [14] and find excellent agreement in all cases.

III. RESULTS

A. Robustness of the Hall attractor

One of the primary motivations for this work is to determine whether the “Hall attractor” seen in earlier axisymmetric simulations is robust against 3D perturbations. We have therefore repeated one simulation of Gourgouliatos and Cumming [14], using the same boundary
conditions and “Ohmic eigenmode” initial condition for the axisymmetric component of the magnetic field. This initial condition represents the slowest-decaying solution of Equation (5) in the absence of the Hall term, and is normalized so that the strength of the magnetic field where the dipole axis intersects the surface of the star is $0.48 \times 10^{14}$ G. To this initial axisymmetric poloidal field we add a low amplitude, small scale 3D perturbation with both poloidal and toroidal components. The results are presented in Figure 1 demonstrating that the magnetic field evolves towards a state broadly similar to the axisymmetric Hall attractor seen by Gourgouliatos and Cumming [14]. However, significant non-axisymmetric features persist in the simulation even on long timescales (comparable to the global Hall timescale $t_{\text{Hall}}$). These features would rapidly decay by the action of Ohmic dissipation alone, and so their longevity can be attributed to the Hall effect. In fact it has previously been shown that, in the presence of a strong density gradient, the Hall effect can sustain or even amplify small-scale features in the magnetic field [22].

The most notable feature of the magnetic field is the strong concentration of poloidal flux in a “belt” around the equator of the magnetic dipole. As emphasized by Gourgouliatos and Cumming [14], this belt corresponds to a strong equatorial jet of electrons within the crust that circles the dipole axis.

The magnetic field is primarily poloidal, with a weaker toroidal component in the inner crust. However, the GC boundary conditions used in this simulation preclude radial electric currents, $J_r$, at either the top or the bottom of the crust, implying that the toroidal field must also vanish there. In a real neutron star, currents are able to flow across the interface between the crust and core, generating significant toroidal fields. We have therefore repeated this same simulation using the more realistic HR boundary conditions, which allow for finite toroidal field at the lower boundary. The results are illustrated in Figure 2 which compares the azimuthally averaged poloidal and toroidal components of the magnetic field at the same time in both simulations. We see that the geometry of the poloidal field is modified, but that a similar Hall attractor state exists. The toroidal field in the second simulation is much stronger, as we would expect, but still significantly weaker than the poloidal field. This toroidal field is associated with a meridional flow of electrons that is equatorward at the surface of the star, and drags the poloidal magnetic field lines. As a result, the equatorial belt of poloidal flux, and the corresponding electron jet, is even stronger in this simulation than in the simulation with GC boundary conditions. The jet is also pushed slightly deeper
into the crust by the meridional flow of electrons, which turns inward near the equator. The effect on the magnetic field is even more conspicuous in Figure 3, which shows the radial and azimuthal components of the magnetic field at the top and bottom of the crust, respectively, at several times during the simulation. The plots show how the small-scale initial perturbations to the magnetic field are organised by the large-scale flow of electrons, during its convergence to the Hall attractor. The most notable feature is again the equatorial belt, which manifests as bands of positive and negative $B_r$ on either side of the equator at the star’s surface.

B. The effect of field strength

Spin-down measurements of neutron stars tell us only the strength of the large-scale magnetic field near the poles. However, it is clear from Figure 3 that the field strength elsewhere in the crust can greatly exceed this observed value. At the time plotted in Figure 2b, for example, the strength of the surface magnetic field varies between $\simeq 5 \times 10^{12}$ G at the dipole axis and $\simeq 3 \times 10^{13}$ G near the equator. Near the bottom of the crust, the local field strength is larger by a further order of magnitude than at the surface. This suggests that the Hall effect may be even more significant in neutron stars than previously thought. We have therefore repeated the same simulation shown in Figure 3 but with a stronger initial magnetic field (i.e., a larger Hall parameter $H$). Because the evolution of the global-scale magnetic field occurs on the Hall timescale $t_{\text{Hall}}$, we expect that a stronger field will converge more rapidly to the Hall attractor, before the small-scale magnetic features have been dissipated by resistivity. This is confirmed in Figure 4 which shows the surface radial field in three simulations with increasing magnetic field strengths. In each case the field is plotted at time $t = 0.6 t_{\text{Hall}}$, which corresponds to $t \simeq 1$, 0.5, and 0.25 Myr respectively, and an equatorial jet of electrons has already formed. In the case with the strongest magnetic field, the jet is broader and more spatially disordered; at later times in the same simulation, the jet becomes increasingly laminar, but remains broader than in the other simulations.
C. The crustal magnetic stress

If the magnetic shear stress within the crust exceeds the breaking stress of the Coulomb lattice then it will induce a crustal fracture. Molecular dynamics models of the Coulomb lattice indicate that the breaking stress is approximately $5 \times 10^{-3}$ of the electron pressure, $P_e [23]$. At each point within the crust, the strongest magnetic shear stress is exerted on a surface that makes an angle of $45^\circ$ to the local magnetic field direction, and this stress is exactly equal to the magnetic pressure, $P_m$. The condition for fracturing can therefore be expressed as $P_m \gtrsim 5 \times 10^{-3} P_e$, where

$$P_m = \frac{|B|^2}{8\pi} \quad \text{and} \quad P_e \simeq \frac{\hbar c}{12\pi^2} (3\pi^2 n)^{4/3}.$$  

In general, fractures are most likely to occur near to the surface of the crust, where the density and pressure are lowest, and the energy released in a near-surface fracture can directly power outbursts and flares. In Figure 5 we plot the ratio of magnetic pressure to breaking stress at the surface of the crust in the same simulation shown in Figure 4c, revealing that patches around the dipole equator are susceptible to crustal fracturing. These patches are localized in both latitude and longitude, as a consequence of the three-dimensionality of the magnetic field. They persist on long timescales, of order $t_{\text{Hall}}$, and propagate azimuthally in the direction of the electron jet.

IV. DISCUSSION

Our results indicate that neutron starquakes are most common in the vicinity of the electron jet within the crust, which typically forms a ring around the dipole axis. This jet forms on the Hall timescale given by Equation (9), which is dependent on the overall strength of the magnetic field, and persists throughout the subsequent evolution. At earlier times, the structure of the magnetic field is much more dependent on the initial conditions for the proto-neutron star, which unfortunately are not well known. The surface field is strongest in localized patches well away from the dipole axis, and these patches become more intense if electric currents are permitted to flow between the crust and the superconducting core. The field in these patches is typically an order of magnitude stronger than that at the poles.

We find no evidence in our results for magnetic spots close the the dipole axis, which have been invoked as an explanation for plasma pair creation in radio pulsars [5] and for
the X-ray spectrum of SGR 0418+5729 [24]. In our simulations, the surface features migrate equatorward as a result of the toroidal magnetic field in the crust and the associated meridional flow of electrons. In order to reverse this migration, a strong toroidal field of the opposite sign must be present within the crust, so that the associated meridional flow of electrons is poleward at the star’s surface [25]. There is no obvious process that can generate such a strong toroidal field within the crust, so it must be either a remnant from the star’s formation [26] or the result of toroidal flux expulsion from the core [27]. In fact, there is some indirect observational evidence that the crust can harbour a strong toroidal field [28, 29]. To fully describe the processes that can impart a strong toroidal field to the crust, it will be necessary to couple our crust model to a model of the star’s superconducting core.

In an isolated neutron star, heat is transported through the crust primarily by electron diffusion. Because electrons are deflected by the Lorentz force, the heat flux is greatly inhibited across magnetic field lines. The strong concentration of poloidal magnetic flux in the equatorial belt will therefore trap heat within the electron jet while the rest of the star cools [30]. Molecular dynamics simulations of the crust demonstrate that the breaking stress is very sensitive to temperature [23], and so the equatorial region of the crust may be even more susceptible to fracturing than we have found here. We are currently extending our model to describe the coupled thermal–magnetic evolution of the crust, and its coupling to the superconducting core.

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[1] M. Ruderman, Astrophys. J. 382, 587 (1991).
[2] C. Thompson and R. C. Duncan, Astrophys. J. 561, 980 (2001) astro-ph/0110675.
[3] T. Güver, E. Göğüş, and F. Özel, Mon. Not. R. Astron. Soc. 418, 2773 (2011).
[4] E. V. Gotthelf, J. P. Halpern, and J. Alford, Astrophys. J. 765, 58 (2013).
[5] U. Geppert, J. Gil, and G. Melikidze, Mon. Not. R. Astron. Soc. 435, 3262 (2013).
[6] P. Goldreich and A. Reisenegger, Astrophys. J. 395, 250 (1992).
[7] D. A. Shalybkov and V. A. Urpin, A&A 321, 685 (1997).
[8] R. Hollerbach and G. Rüdiger, Mon. Not. R. Astron. Soc. 347, 1273 (2004).
[9] J. A. Pons and U. Geppert, A&A 470, 303 (2007).
[10] R. Perna and J. A. Pons, Astrophys. J. 727, L51 (2011).
[11] Y. Kojima and S. Kisaka, Mon. Not. R. Astron. Soc. 421, 2722 (2012).
[12] D. Viganò, J. A. Pons, and J. A. Miralles, Computer Physics Communications 183, 2042 (2012).
[13] J. A. Pons, D. Viganò, and N. Rea, Nature Physics 9, 431 (2013).
[14] K. N. Gourgouliatos and A. Cumming, Mon. Not. R. Astron. Soc. 438, 1618 (2014).
[15] P. Marchant, A. Reisenegger, J. Alejandro Valdivia, and J. H. Hoyos, Astrophys. J. 796, 94 (2014).
[16] D. Biskamp, E. Schwarz, A. Zeiler, A. Celani, and J. F. Drake, Phys. Plasmas 6, 751 (1999).
[17] J. Cho and A. Lazarian, Astrophys. J. 615, L41 (2004).
[18] C. J. Wareing and R. Hollerbach, J. Plasma Phys. 76, 117 (2010).
[19] E. Dormy, P. Cardin, and D. Jault, Earth Planet. Sci. Lett. 160, 15 (1998).
[20] J. Aubert, J. Aurnou, and J. Wicht, Geophys. J. Int. 172, 945 (2008).
[21] R. Hollerbach and G. Rüdiger, Mon. Not. R. Astron. Soc. 337, 216 (2002).
[22] T. S. Wood, R. Hollerbach, and M. Lyutikov, Phys. Plasmas 21, 052110 (2014).
[23] A. I. Chugunov and C. J. Horowitz, Mon. Not. R. Astron. Soc. 407, L54 (2010).
[24] A. Tiengo, P. Esposito, S. Mereghetti, R. Turolla, L. Nobili, F. Gastaldello, D. Götz, G. L. Israel, N. Rea, L. Stella, S. Zane, and G. F. Bignami, Nature 500, 312 (2013).
[25] U. Geppert and D. Viganò, Mon. Not. R. Astron. Soc. 444, 3198 (2014).
[26] C. Thompson and R. C. Duncan, Astrophys. J. 408, 194 (1993).
[27] S. K. Lander, Mon. Not. R. Astron. Soc. 437, 424 (2014).
[28] N. Shabaltas and D. Lai, Astrophys. J. 748, 148 (2012).
[29] K. N. Gourgouliatos and A. Cumming, Mon. Not. R. Astron. Soc. 446, 1121 (2015).
[30] D. Viganò, N. Rea, J. A. Pons, R. Perna, D. N. Aguilera, and J. A. Miralles, Mon. Not. R. Astron. Soc. 434, 123 (2013).
FIG. 1. Upper panels: lines of the azimuthally averaged poloidal field at successive times, illustrating convergence to the Hall attractor. The dashed lines indicate the boundaries of the crust. Lower panels: projections of the radial component of the magnetic field at the surface at early (left) and late (right) times in the same simulation. Even after convergence of the global-scale field, significant non-axisymmetric features remain.

FIG. 2. Poloidal fieldlines and contours of toroidal field for the axisymmetric component after 1.2 Myr. The left and right panels show the cases with GC and HR boundary conditions, respectively.
FIG. 3. The magnetic field at various times in the simulation with HR boundary conditions. Top row: $B_r$ at $r = R$. Bottom row: $B_\phi$ at $r = 0.9R$. Colorbars are adjusted to the minimum and maximum values in each plot.

FIG. 4. The surface radial magnetic field in three simulations with $B_0 = 10^{14}$ G, $2 \times 10^{14}$ G, $4 \times 10^{14}$ G (i.e., $H = 50, 100, 200$). Each plot shows a snapshot at $t = 0.6 t_{\text{Hall}}$. 
FIG. 5. Lines of the magnetic field generated by currents in the crust. The coloring indicates the ratio of magnetic pressure to breaking stress at the surface of the crust. A portion of the surface has been cut away to show fieldlines inside the crust, whose lower boundary is the gray sphere.