Quantum Time Inversion to Prevent DDoS Attacks:
A Potentially Realizable TENET Technology

Del Rajan
New Zealand

(Dated: \TeX-ed November 24, 2021)

An effect known as time inversion was introduced in Christopher Nolan’s film TENET, and here we demonstrate that such an effect exists for quantum information. We do this by investigating the von Neumann entropy of experimentally realized photons that are entangled in time. Using these findings, we design a technology called a Turnstile and apply it to detect Distributed Denial-of-Service (DDoS) attacks in quantum networks. Pragmatically, our work can be viewed as formulating the quantum analogue of the Shannon entropic DDoS detection systems used in classical networks.

Introduction.—Quantum information has the astonishing ability to perform information tasks that would be impossible or very difficult to do with only classical information [1]. For instance, the property of superposition is used by quantum computers to drastically outperform the best classical supercomputers on certain tasks [2, 3]. Another example is the realization of secure quantum cryptographic protocols which are predicated on the impossibility to copy quantum information [4]. Perhaps the most radical departure from mere classical information is that quantum information can be teleported [5], recently to distances exceeding a 1000km [6]. Beyond this established work, a central aim of quantum information science is to explore novel aspects of this information and design useful technologies with it. Our main results are to demonstrate progress in this aim.

An effect known as time inversion was introduced in Christopher Nolan’s contemporary science fiction film TENET [7], and we will show that such a temporal effect exists for quantum information. Specifically, that it can be uncovered in an experiment that generated photons which were entangled in time [8]. These findings also further the role that temporal entanglement can play within quantum information science [9].

We proceed to discuss Distributed Denial-of-Service (DDoS) attacks in both classical and quantum networks. Using the time inversion effect for quantum information, we design a DDoS detection technology for quantum networks. On a pragmatic level, one can view this work as solely formulating the quantum analogue of classical DDoS detection systems. This advances the applicability as expressed through mapping

\[ t' = n\tau \rightarrow t = n\tau, \]
\[ t' = (n + e)\tau \rightarrow t = (n + e)\tau, \]  

(1)

\[ n \in \mathbb{Z} \text{ is arbitrarily chosen and } e \in \mathbb{Z}^+. \]

To characterize time inversion, three conditions need to be fulfilled. For terminology, one refers to the system that is experiencing time inversion as inverted, and the technology that can invert a system as a Turnstile.

The first condition is that when a system is inverted, it experiences the environment going backwards in time in descending consecutive values. The second condition is that the inverted system itself is moving forward in its own time in ascending consecutive values. If time inversion is activated at \( t' = t = n\tau \), then the two conditions can be jointly expressed as

\[ t' = n\tau \rightarrow t = n\tau, \]
\[ t' = (n + e)\tau \rightarrow t = (n + e)\tau. \]  

(2)

The third condition, as conveyed in the film, is the manifestation of this temporal effect is directly related to a property that the inverted system’s “entropy runs backwards.” Hence time inversion is also referred to as inverting a system’s entropy.

From these three conditions and the use of multiple Turnstiles, one can deduce the various film sequences regarding time inversion presented in TENET.

Entanglement in time.—Our results will utilize the experimental work in [8], which we review in the Supplemental material. Their experiment performed a temporal version of entanglement swapping for polarized photons. It could generate three types of Bell states. The first were spatial Bell states such as \( |\psi^{+}_{0,0}\rangle_{a,b} \) where subscripts indicate spatial modes and superscripts represent the temporal labels of the photons, which in this case for both are \( t = 0 \). The second were delayed Bell states such as \( |\psi^{-}_{0,\tau}\rangle_{a,b} \) where one photon was immediately measured at \( t = 0 \), while the other photon goes through a delay line to be absorbed at time \( t = \tau \). Lastly were the temporal Bell states such as \( |\psi^{0,2\tau}_{a,b}\rangle \) where one photon was generated and measured at \( t = 0 \), while the other photon was
generated later at \( t = \tau \) and delayed till \( t = 2\tau \). For this latter case, we have an entanglement in time between photons that do not coexist.

Quantum time inversion.—We abstract the previous experiment as a quantum information system, and show that its information-theoretic properties satisfy the three conditions of time inversion. This uncovers a time inversion effect for quantum information, and allows for a potentially realizable quantum Turnstile. Specifically the system we want to perform inversion on, at some arbitrarily chosen time \( t = n\tau \), is a part of the quantum information in a spatial Bell state generated at that time \( t = n\tau \),

\[
|\beta_{xy}\rangle^{n\tau,n\tau} = \left( \frac{|0\rangle^{n\tau} \otimes |y\rangle^{n\tau} + (-1)^{x} |1\rangle^{n\tau} \otimes |\bar{y}\rangle^{n\tau}}{\sqrt{2}} \right). \tag{3}
\]

Here we have a choice between \( xy = 00, 01, 10, \) or \( 11 \), for the system. The bar denotes negation, and superscripts are the values of the time \( t \) associated to the environment. The notation can also refer to a Bell state generated at an earlier time but which gets inverted at \( t = n\tau \).

With respect to the computational basis states, the quantum information in (3) takes the form

\[
\left( \sqrt{2}, \sqrt{2}, \frac{(-1)^{x}y}{\sqrt{2}}, \frac{(-1)^{x}\bar{y}}{\sqrt{2}} \right)^{n\tau,n\tau}. \tag{4}
\]

This is the input to the quantum Turnstile and only part of this information will be inverted.

To satisfy the first condition as a proof of principle, we consider the simplest case of a history of length one. Therefore, the desired output of the Turnstile at \( t = (n + 1)\tau \) should be temporal Bell state \( |\beta_{xy}\rangle^{(n-1)\tau,(n+1)\tau} \) which can be expanded as

\[
\left( \frac{|0\rangle^{(n-1)\tau} \otimes |y\rangle^{(n+1)\tau} + (-1)^{x} |1\rangle^{(n-1)\tau} \otimes |\bar{y}\rangle^{(n+1)\tau}}{\sqrt{2}} \right). \tag{5}
\]

This is equivalent to

\[
\left( \sqrt{2}, \sqrt{2}, \frac{(-1)^{x}y}{\sqrt{2}}, \frac{(-1)^{x}\bar{y}}{\sqrt{2}} \right)^{(n-1)\tau,(n+1)\tau}. \tag{6}
\]

where part of the information in (4) is experiencing the environment going backwards in time to \( t = (n-1)\tau \) (while the non-inverted part is moving forward in time to \( t = (n + 1)\tau \).

The quantum Turnstile that accomplishes this task starts with a spatial Bell state \( |\beta_{00}\rangle^{(n-1)\tau,(n-1)\tau} \) which it modifies through a delay line to a delayed Bell state \( |\beta_{xy}\rangle^{(n-1)\tau,n\tau} \) where subsystem \( A \) is measured at \( t = (n-1)\tau \) and subsystem \( B \) is delayed to \( t = n\tau \). It connects this to the input (3) which is labelled \( C \) and \( D \), with the Turnstile applying the same delay afterwards to subsystem \( D \). The resulting state can be written as

\[
|\text{Turnstile} \rangle \otimes |\text{Input} \rangle \equiv |\beta_{00}\rangle^{(n-1)\tau,n\tau} \otimes |\beta_{xy}\rangle^{n\tau,(n+1)\tau} = \frac{1}{2} \left( |\beta_{xy}\rangle^{(n-1)\tau,(n+1)\tau} \otimes |\beta_{00}\rangle^{n\tau,n\tau} \right)
\]

\[
+ |\beta_{xy}\rangle^{(n-1)\tau,(n+1)\tau} \otimes |\beta_{00}\rangle^{n\tau,n\tau}
\]

\[
+ (-1)^{x} |\beta_{xy}\rangle^{(n-1)\tau,(n+1)\tau} \otimes |\beta_{01}\rangle^{n\tau,n\tau}
\]

\[
+ (-1)^{x} |\beta_{xy}\rangle^{(n-1)\tau,(n+1)\tau} \otimes |\beta_{11}\rangle^{n\tau,n\tau}. \tag{7}
\]

The Turnstile proceeds to perform a Bell state projection on \( BC \) at \( t = n\tau \). This returns one of four possible outcomes with consequences for \( AD \)

\[
|\beta_{00}\rangle^{n\tau,n\tau} \rightarrow |\beta_{xy}\rangle^{(n-1)\tau,(n+1)\tau},
\]

\[
|\beta_{01}\rangle^{n\tau,n\tau} \rightarrow (-1)^{x} |\beta_{xy}\rangle^{(n-1)\tau,(n+1)\tau},
\]

\[
|\beta_{10}\rangle^{n\tau,n\tau} \rightarrow |\beta_{xy}\rangle^{(n-1)\tau,(n+1)\tau},
\]

\[
|\beta_{11}\rangle^{n\tau,n\tau} \rightarrow (-1)^{x} |\beta_{xy}\rangle^{(n-1)\tau,(n+1)\tau}. \tag{8}
\]

Depending on the outcome, the Turnstile applies a particular unitary operator to subsystem \( D \) (which is the part moving forward in time to \( t = (n + 1)\tau \)),

\[
(I \otimes I) |\beta_{xy}\rangle^{(n-1)\tau,(n+1)\tau},
\]

\[
(I \otimes (-1)^{x} \hat{\sigma}_{1}) |\beta_{xy}\rangle^{(n-1)\tau,(n+1)\tau},
\]

\[
(I \otimes (-1)^{x} \hat{\sigma}_{3}) |\beta_{xy}\rangle^{(n-1)\tau,(n+1)\tau},
\]

\[
(I \otimes (-1)^{x+y} \hat{\sigma}_{3} \hat{\sigma}_{1}) |\beta_{xy}\rangle^{(n-1)\tau,(n+1)\tau}. \tag{9}
\]

where \( \hat{\sigma}_{1} = |0\rangle \langle 1| + |1\rangle \langle 0|, \hat{\sigma}_{2} = -i |0\rangle \langle 1| + i |1\rangle \langle 0| \) and \( \hat{\sigma}_{3} = |0\rangle \langle 0| - |1\rangle \langle 1| \). This results in the desired output \( |\beta_{xy}\rangle^{(n-1)\tau,(n+1)\tau} \) at \( t = (n + 1)\tau \).

For the second condition, we need to show that the inverted quantum information is moving forward in its own time, meaning that \( |\beta_{xy}\rangle^{(n-1)\tau,(n+1)\tau} \) is equivalent to \( |\beta_{xy}\rangle^{(n-1)\tau,(n+1)\tau} \) where \( t' \) is the time as experienced by the inverted information. To show this, we assume heuristically that if an inverted system goes into the past and its choices in that past are inherently indeterministic, then the inverted system is moving forward according to its own time.

In the Turnstile at the time of inversion at \( t = n\tau \), there is an intrinsic uncertainty on the outcomes (8). This also encapsulates the indeterminism on the state that the inverted information would take in the environment’s past at \( t = (n-1)\tau \). The Born rule assigns a probability of 1/4 to each outcome. One can compute the Shannon entropy of this distribution

\[
H(X) \equiv - \sum_{i} p_{i} \log_{2} p_{i}, \tag{10}
\]

where \( p_{i} \) are the probabilities associated to random variable \( X \) and logarithms are taken to base 2. This results in
H(X) = 2 at the time of inversion, and captures the non-zero indeterminism of the choice the inverted information will make in the past. Thus fulfilling the condition.

Finally to satisfy the last condition, we refer to the von Neumann quantum entropy \([1]\) of a density operator \(\rho\) which is defined as

\[
S(\rho) = -\text{Tr}(\rho \log \rho).
\]  

(11)

The entropy is zero if and only if it is a pure state. For an arbitrary composite system with subsystems \(A\) and \(B\), the joint and conditional entropy are respectively

\[
S(\rho_{AB}) = -\text{Tr}(\rho_{AB} \log \rho_{AB}),
\]

(12)

\[
S(\rho_A | \rho_B) \equiv S(\rho_{AB}) - S(\rho_B),
\]

(13)

where \(\rho_A = \text{Tr}_B(\rho_{AB})\) and \(\rho_B = \text{Tr}_A(\rho_{AB})\). Suppose \(\rho_{AB}\) is a pure state, then \(\rho_{AB}\) is entangled if and only if

\[
S(\rho_A | \rho_B) < 0.
\]

(14)

In this case the entropy of either subsystem, \(S(\rho_A)\) or \(S(\rho_B)\), is referred to as the entanglement entropy.

In the Turnstile, the density operator for the input (3) is

\[
\rho^{nt,nt}_{C,D} = |\beta_{xy}^{nt}|_{B,C} \langle \beta_{xy}^{nt}|_{B,C},
\]

(15)

and the associated conditional entropy at \(t = nt\) is

\[
S(\rho^{nt}_{C,D} | \rho^{nt}_{D}) = S(\rho^{nt}_{C,D}) - S(\rho^{nt}_{D}) = 0 - 1 = -1,
\]

(16)

which indicates entanglement (14) as expected. Note that at that time before projection, subsystem \(A\) and subsystem \(D\) are not entangled

\[
S(\rho^{(n-1)\tau}_{A,D} | \rho^{nt}_{D}) = S(\rho^{(n-1)\tau}_{A,D}) - S(\rho^{nt}_{D}) = 2 - 1 = 1.
\]

(17)

For the output (6), the density operator takes the form

\[
\rho^{(n-1)\tau,(n+1)\tau}_{A,D} = |\beta_{xy}^{(n-1)\tau,(n+1)\tau}|_{A,D} \langle \beta_{xy}^{(n-1)\tau,(n+1)\tau}|_{A,D},
\]

(18)

and the associated conditional entropy at \(t = (n + 1)\tau\) is

\[
S(\rho^{(n-1)\tau}_{A,D} | \rho^{(n+1)\tau}_{D}) = S(\rho^{(n-1)\tau}_{A,D}) - S(\rho^{(n+1)\tau}_{D}) = 0 - 1 = -1,
\]

(19)

signifying entanglement, with a loss of it in the input

\[
S(\rho^{nt}_{C,D} | \rho^{(n+1)\tau}_{D}) = S(\rho^{nt}_{C,D}) - S(\rho^{(n+1)\tau}_{D}) = 2 - 1 = 1.
\]

(20)

We see that part of the conditional entropy of \(CD\) has moved to \(AD\). Given \(A\) only exists in the past, that entropy can be said to have ‘run backwards’ in time.

In terms of the joint entropy, just before inversion (16) we see that \(S(\rho^{nt}_{C,D}) = 0\) which specifies a pure state. After inversion (20) we have that \(S(\rho^{nt,nt}_{C,D}) = 2\) which signifies missing information in \(CD\). Part of that

missing information has disappeared by going into past and existing there, as described by the updated entropy \(S(\rho^{(n-1)\tau,(n+1)\tau}_{A,D}) = 0\) in (19).

We call \(S(\rho^{(n-1)\tau}_{A,D} | \rho^{(n+1)\tau}_{D})\) and \(S(\rho^{(n-1)\tau,(n+1)\tau}_{A,D})\) in (19) respectively as the inverted conditional entropy and the inverted joint entropy, and the term \(S(\rho^{(n-1)\tau}_{A,D})\) in those quantities as the inverted entanglement entropy.

**Classical DDoS.**—In classical networks, Distributed Denial-of-Service (DDoS) attacks are rather frequent events \([10, 11]\) and their reach can extend into the critical infrastructure of a nation, as witnessed in Estonia \([12]\) and New Zealand \([13]\). In such networks, information is transmitted in the form of data packets and the role of directing this traffic is performed by routers. To initiate a DDoS flooding attack, many routers would direct packets from multiple attack nodes to a victim node. The intent behind this flood of traffic is to overload the victim node so that it becomes unresponsive to legitimate traffic.

Preventing a DDoS attack requires most essentially the ability to identify the attack traffic as early as possible \([14]\). To achieve this temporal capability, detection systems have been designed using the Shannon entropy (10) and reviews of such entropic approaches can be found in \([11, 14, 15]\). We briefly outline one of these methods \([16]\).

A flow at a router is group of packets categorized as

\[
f_{ij}(u_i, d_j, t) \equiv \{ u_i, d_j, t > |u_i \in U, d_j \in D \},
\]

(21)

where \(i, j \in \mathbb{Z}^+, U\) is the set of the upstream routers, \(D\) denotes the set of destination addresses from the router, and \(t\) is the time stamp. Let \(| f_{ij}(u_i, d_j, t)|\) represent the number of packets of flow \(f_{ij}\) at time \(t\). For a given time interval \(\Delta T\), the variation of the number of packets for a given flow is defined as

\[
N_{ij}(u_i, d_j, t + \Delta t) \equiv |f_{ij}(u_i, d_j, t + \Delta T)| - |f_{ij}(u_i, d_j, t)|.
\]

(22)

If \(| f_{ij}(u_i, d_j, t) | = 0\), then \(N_{ij}(u_i, d_j, t + \Delta T)\) is the number of packets of flow \(f_{ij}\) that went through the router during time interval \(\Delta T\). The quantity

\[
p_{ij}(u_i, d_j, t + \Delta T) = \frac{N_{ij}(u_i, d_j, t + \Delta T)}{\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} N_{ij}(u_i, d_j, t + \Delta T)}
\]

(23)

gives the probability of the flow \(f_{ij}\) over all flows at the router with

\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} p_{ij}(u_i, d_j, t + \Delta T) = 1.
\]

(24)

The computation of the Shannon entropy (10) at the router is obtained through

\[
H(F) = - \sum_{i,j} p_{ij}(u_i, d_j, t + \Delta T) \log p_{ij}(u_i, d_j, t + \Delta T),
\]

(25)
where $F$ is the associated random variable with respect to flows during $\Delta T$. If the total number of flows is constrained to $N$, then (25) is rather simply

$$H(F) = H(p_1, p_2, \ldots, p_N) = -\sum_{i=1}^{N} p_i \log p_i,$$

with $0 \leq H(F) \leq \log N$. The lower bound occurs when there is only one flow.

In order to model a DDoS attack, a number of assumptions are made. These are that there is no extraordinary change of traffic in a very short time for the non-attack case, the number of attack packets is at least an order of magnitude higher than that of normal flows, there is only one attack ongoing at a time, and that the number of flows is stable for both non-attack and attack cases.

Suppose attack flows start passing through the router at $t = (n+1)\tau$, hence $t = n\tau$ signifies the time just before the attack. The respective distributions are

$$\{p_1^{(n+1)\tau}, p_2^{(n+1)\tau}, \ldots, p_N^{(n+1)\tau}\},$$

and

$$\{p_1^{n\tau}, p_2^{n\tau}, \ldots, p_N^{n\tau}\}.$$  

A consequence of the assumptions is that $p_k^{n\tau} \ll p_k^{(n+1)\tau}$ for some $k$. Further reasoning with Jensen’s inequality leads to

$$- \sum_{i=1}^{N} p_i^{n\tau} \log p_i^{n\tau} \gg - \sum_{i=1}^{N} p_i^{(n+1)\tau} \log p_i^{(n+1)\tau}.$$  

Expressing this in terms of the entropy (26) gives

$$H(F^{n\tau}) \gg H(F^{(n+1)\tau}).$$

The entropy at the router drops dramatically as soon as attack flows are passing through, thus allowing for an ability to detect an attack as early as possible.

Quantum DDoS.—Quantum networks [17] generate entanglement over long distances. These entanglement flows are routed through devices known as repeaters [18, 19] which perform entanglement swapping to connect two spatially entangled links into a longer entangled link. A simple routing scenario for this involves spatial Bell pairs generated at both the request node and the repeater node. One qubit of the request pair reaches the repeater to be Bell projected with a qubit at the repeater. Afterwards, the non-projected qubit of the repeater pair leaves towards the destination node.

Quantum networks can also experience DDoS attacks [20], notably resulting in an in-principle weakness to perform quantum key distribution [21]. Given that repeaters have a maximum session capacity [22], we consider DDoS attacks where requests exceed that capacity.

In this setting it is ideal to model the repeater as a Turnstile. The first reason is that both utilize entanglement swapping. The second reason is that in the routing scenario both the qubit reaching the repeater and the qubit leaving the repeater take time to travel, thus we can map these to delayed subsystems $B$ and $D$ in the Turnstile. Third is that to minimize memory, the repeater should generate the Bell pair at the same time the request is received, which maps to the generation of $CD$ given delayed $B$. Finally, in a DDoS scenario the attack node would avoid risk of traceback thereby immediately destroying the qubit that remains at its node, which maps to $A$ in the Turnstile.

As a consequence of this model, the inverted entanglement entropy will be used to formulate a DDoS detection system analogous to the classical case. To derive this, we will use and modify some material from [23].

Our model starts with the request node generating $|\beta_0\rangle_{A,B}$ at arbitrary time $t = (n-1)\tau$. The qubit $B$ is sent to the repeater which takes time $\tau$, while qubit $A$ is destroyed. This is simply the Turnstile state. The repeater generates $|\beta_0\rangle_{C,D}$ at time $t = n\tau$, which can be viewed as an instantiation of the input (3).

We consider the quantities at $t = n\tau$ before the projection. The total system is $\rho_{A,B,C,D}^{(n-1)\tau}$, which denotes $|\beta_0\rangle_{A,B}^{(n-1)\tau}\langle\beta_0|_{A,B}^{(n-1)\tau}$. The subsystem held at the repeater is $\rho_{B,C,D}^{n\tau}$ as it excludes $\rho_{A}^{(n-1)\tau}$. Given qubits $C$ and $D$ are jointly in a pure state, we have that

$$S(\rho_{B,C,D}^{n\tau}) = S(\rho_{B}^{n\tau}) = 1.$$  

The qubit $B$ is maximally mixed since it is entangled with qubit $A$ in a delayed Bell state. Thus the entanglement entropy of qubit $A$ before the projection equates to

$$S(\rho_{A}^{(n-1)\tau}) = S(\rho_{B,C,D}^{n\tau}).$$

We take the partial trace to obtain the density operator for qubit $D$

$$\rho_{D}^{n\tau} = \text{Tr}_{A,B,C,D}(\rho_{A,B,C,D}^{(n-1)\tau}).$$

We have that $S(\rho_{D}^{n\tau}) = 1$ since it is entangled with $\rho_{C}^{n\tau}$.

The inverted entanglement entropy forms a crucial role for a repeater session. A successful session occurs when the entanglement is swapped (8). The swapping is successful only if the repeater uses some rank 1 orthogonal projectors $\Pi_i$ such that no matter what outcome occurs at the repeater on qubits $B$ and $C$, the value of the entanglement entropy of $\rho_{A}^{(n-1)\tau}$ must equal the value of the inverted entanglement entropy of $\rho_{D}^{(n-1)\tau}$. Stated in terms of the inverted information, the representation of $\rho_{A}^{(n-1)\tau}$ must be the same before and after projection.

We proceed to examine the quantities after the projection at $t = n\tau$. If the repeater obtains outcome $i$, then the density operator of $D$ at $t = n\tau$ is

$$\rho_{D_i} = \frac{1}{p_i} \text{Tr}_{A,B,C}(\Pi_i \rho_{A,B,C,D}^{(n-1)\tau}).$$  

(35)
where \( p_i \) is the associated Born probability. Given
\[
\sum_i \Pi_i = \mathbb{I}, \tag{36}
\]
we have that
\[
\rho_D^{n\tau} = \sum_i p_i \rho_{D_i}. \tag{37}
\]
After measurement, qubits \( A \) and \( D \) are in a pure entangled state. The entanglement entropy of \( D \) equates to the inverted entanglement entropy of \( A \) (the entropy is inverted as this is the entropy in the past after the projection)
\[
S(\rho_{D_i}) = S(\rho_A^{(n-1)\tau}). \tag{38}
\]
Using the crucial condition that a successful session requires the entropy and inverted entropy of \( \rho_A^{(n-1)\tau} \) to equate, we can formulate relationships between the quantities before and after projection. Specifically using (33) and (38), we obtain
\[
S(\rho_{D_i}) = S(\rho_{B,C,D}^{n\tau,n\tau,n\tau}), \tag{39}
\]
and furthermore
\[
S(\rho_{B,C,D}^{n\tau,n\tau,n\tau}) = S(\rho_{D_i}) = \sum_i p_i S(\rho_{D_i}). \tag{40}
\]
Applying the concavity inequality [23] to (37) results to
\[
S(\rho_D^{n\tau}) \geq \sum_i p_i S(\rho_{D_i}). \tag{41}
\]
Combining (40) and (41) gives
\[
S(\rho_{B,C,D}^{n\tau,n\tau,n\tau}) = \sum_i p_i S(\rho_{D_i}) \leq S(\rho_D^{n\tau}). \tag{42}
\]
Therefore, before the projection one can predict that a successful session is possible when and only when
\[
S(\rho_{B,C,D}^{n\tau,n\tau} | \rho_D^{n\tau}) = S(\rho_{B,C,D}^{n\tau,n\tau}) - S(\rho_D^{n\tau}) \leq 0. \tag{43}
\]
A failed session will occur when \( S(\rho_{B,C,D}^{n\tau,n\tau} | \rho_D^{n\tau}) > 0 \). For our specific case (31), we have that \( S(\rho_{B,C,D}^{n\tau,n\tau} | \rho_D^{n\tau}) = 0 \) which implies a successful session ahead.

The capacity of the repeater is defined as the maximum number of sessions it can facilitate simultaneously and this is directly related to the number of Bell pairs that can be stored in memory [22]. In our case at \( t = n\tau \), this would be the maximum number of copies of \( \rho_{C,D}^{n\tau,n\tau} \) generated to keep the service at full capacity. After that time, any unused Bell pairs get destroyed and the system regenerates to full capacity at the next time point.

With respect to capacity, we modify our previous analysis to \( N \) copies of system \( \rho_{A,B,C,D}^{(n-1)\tau,n\tau,n\tau,n\tau} \) for large \( N \). The repeater would make a complete projective measurement on \( (\rho_{B,C}^{n\tau,n\tau}) \otimes N \). In this case all the entropies are multiplied by \( N \). Hence the condition (43) for sessions is maintained and we can interpret it in terms of capacity.

If \( S(\rho_{B,C}^{n\tau,n\tau} | \rho_D^{n\tau}) < 0 \) before the projection, then
\[
-S(\rho_{B,C}^{n\tau,n\tau} | \rho_D^{n\tau}) \text{ is the number of Bell pairs left afterwards in the memory. It quantifies the unused capacity after the session requests have been fulfilled.}
\]
Whereas if \( S(\rho_{B,C}^{n\tau,n\tau} | \rho_D^{n\tau}) > 0 \), it not possible to carry out a session to begin with, and predicts an unresponsive service due to requests exceeding the capacity. Therefore we can use (43) to model a quantum DDoS attack.

Suppose a flood of attack requests reaches a repeater at \( t = (n+1)\tau \), and the traffic behaviour follow the assumptions as in the classical case. The entropy of the requests at the repeater is quantified as
\[
S(\rho_B^{n\tau}) \ll S(\sigma_B^{(n+1)\tau}), \tag{44}
\]
where \( \rho \) is used to label the systems involved prior to attack, and \( \sigma \) denotes the systems involved in the attack. The repeater generates the same full capacity at each time point hence
\[
S(\rho_{C,D}^{n\tau,n\tau}) = S(\sigma_{C,D}^{(n+1)\tau,(n+1)\tau}), \tag{45}
\]
and with (44) we obtain
\[
S(\rho_{B,C,D}^{n\tau,n\tau,n\tau}) \ll S(\sigma_{B,C,D}^{(n+1)\tau,(n+1)\tau,(n+1)\tau}). \tag{46}
\]
From (45) we have that \( S(\rho_D^{n\tau}) = S(\sigma_D^{(n+1)\tau}) \). Combining this with (46) produces an entropic DDoS detection formula at the repeater
\[
S(\rho_{B,C}^{n\tau,n\tau,n\tau} | \rho_D^{n\tau}) \ll S(\sigma_{B,C}^{n\tau,n\tau} | \sigma_D^{n\tau}). \tag{47}
\]
The conditional entropies in (47) encode both the requests and the capacity. In an attack the entropy increases dramatically at the repeater, signifying a drastic reduction in capacity. This provides an early detection system that is comparable to the classical case (30).

Conclusion.—In TENET time inversion was used to prevent an attack on the world, and in an analogous manner we have shown that quantum time inversion can be used to prevent a DDoS attack. Given that our analysis is based on the experimental results in [8], it allows for the possibility of a realizable technology. We have left the speculative aspects of our work with further analysis in the Supplemental material.

Lastly, the tenet known as the Copenhagen interpretation was co-founded by Niels Bohr who is attributed to saying, “Anyone who is not shocked by quantum theory has not understood it.” In many ways, this is not so different to the scientist in the film describing time inversion as, “Don’t try to understand it. Feel it.”
[1] M. A. Nielsen and I. L. Chuang, *Quantum computation and quantum information*. (Cambridge University Press, 2010).

[2] F. Arute et al., “Quantum supremacy using a programmable superconducting processor”, Nature, 574, 505–510 (2019).

[3] Y. Wu et al., “Strong quantum computational advantage using a superconducting quantum processor”, Physical Review Letters, 127, 180501 (2021).

[4] F. Xu et al., “Secure quantum key distribution with realistic devices”, Reviews of Modern Physics, 92, 025002 (2020).

[5] C.H. Bennett et al., “Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels”, Physical Review Letters, 70, 1895–1899 (1993).

[6] J. Ren et al., “Ground-to-satellite quantum teleportation”, Nature, 549, 70–73 (2017).

[7] C. Nolan (director and screenwriter), “TENET”, Warner Bros, (2020).

[8] E. Megidish et al., “Entanglement Swapping between Photons that have Never Coexisted”, Physical Review Letters, 110, 210403 (2013).

[9] D. Rajan, “Quantum Entanglement in Time”, PhD thesis Victoria University of Wellington, (2020).

[10] E. Osterweil et al., “21 Years of Distributed Denial-of-Service: A Call to Action”, Computer, 53, 94–99 (2020).

[11] T. Mahjabin et al., “A survey of distributed denial-of-service attack, prevention, and mitigation techniques”, International Journal of Distributed Sensor Networks, 13, 1550147717741463 (2017).

[12] M. Lesk, “The new front line: Estonia under cyberassault”, IEEE Security & Privacy, 5, 76–79 (2007).

[13] F.M.A., “Market Operator Obligations Targeted Review – NZX”, Financial Markets Authority New Zealand, (2021).

[14] A. Koay, “Detecting High and Low Intensity Distributed Denial of Service (DDoS) Attacks”, PhD thesis Victoria University of Wellington, (2019).

[15] İ. Özçelik, and R.R. Brooks, “Deceiving entropy based DoS detection”, Computers & Security, 48, 234–245 (2015).

[16] S. Yu et al., “Traceback of DDoS attacks using entropy variations”, IEEE transactions on parallel and distributed systems, 22, 412–425 (2010).

[17] S. Wehner et al., “Quantum internet: A vision for the road ahead”, Science, 362, 6412 (2018).

[18] M. Pant et al., “Routing entanglement in the quantum internet”, npj Quantum Information, 5, 1–9 (2019).

[19] Y. Lee et al., “A quantum router architecture for high-fidelity entanglement flows in multi-user quantum networks”, arXiv:2005.01852 [quant-ph] (2020).

[20] T. Satoh et al., “Attacking the quantum internet”, IEEE Transactions on Quantum Engineering, 2, 1–17 (2021).

[21] R. Clark et al., “The impact of quantum technologies on secure communications”, Australian Strategic Policy Institute, (2021).

[22] J. Rabbie et al., , “Designing quantum networks using preexisting infrastructure”, arXiv:2005.14715 [quant-ph] (2020).

[23] E. Witten, “A mini-introduction to information theory”, La Rivista del Nuovo Cimento, 43, 187–227 (2020).
Here we present supplemental material that complements the main text. It is structured as follows: The first section provides a review of the experiment on entanglement in time that our theoretical work is utilizing. The second section adds comments on the time inversion effect. Thirdly we provide the derivation of some equations in the main text. Fourth, we briefly discuss the process of DDoS mitigation. Next we examine the different possibilities for the inverted quantum entropy from the perspective of the past, and in particular construct a quantum analogue of the classical transfer entropy. The last section explores a speculative direction in that we look at the conditions needed to receive quantum information from the future. As a result, we design a temporal version of the quantum Petz recovery channel which is able to detect DDoS attacks that have occurred in the future.

I. ENTANGLEMENT IN TIME

Central to our work is the utilization of the experimental results in [1] which in this section we provide a brief overview of. This experiment can be viewed as a temporal version of entanglement swapping, which produced an entanglement between photons that do not coexist. The procedure used polarized photons which can be in any of the four spatial Bell states

\[ |\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|h_a0\rangle \pm |v_{a,b}\rangle), \]

\[ |\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|h_av_b\rangle \pm |v_{a,b}\rangle), \]  \hspace{1cm} (1)

where \( h_a(v_b) \) represents a horizontal (vertical) polarized photon in spatial mode \( a(b) \). The experiment generated two pairs of photons separated by a well-defined time interval \( \tau \). The four photon state was described as

\[ |\psi^-\rangle_{a,b} \otimes |\psi^-\rangle_{a,b} = \frac{1}{2}(|h^0a0\rangle - |v^0_{a,b}\rangle \otimes (|h^0v^0_b\rangle - |v^0_{a,b}\rangle), \]  \hspace{1cm} (2)

where the subscripts refer to spatial modes and the superscripts are the time labels. One photon from each pair was delayed by \( \tau \) in a delay line. The quantum state was reordered and written as

\[ |\psi^-\rangle_{a,b} \otimes |\psi^-\rangle_{a,b} = \frac{1}{2}(|\psi^+\rangle_{a,b} |\psi^+\rangle_{a,b} - |\psi^-\rangle_{a,b} |\psi^-\rangle_{a,b} - |\phi^+\rangle_{a,b} |\phi^+\rangle_{a,b} + |\phi^0\rangle_{a,b} |\phi^0\rangle_{a,b}), \]  \hspace{1cm} (3)

where the states on the l.h.s of equation (3) can be referred as delayed Bell states.

The non-delayed photon from the first pair was measured at \( t = 0 \), immediately after it is generated. We refer to it as the first photon. The delayed photon from the first pair and the non-delayed photon from the second pair underwent a Bell state projective measurement at \( t = \tau \). This caused the first photon and the delayed photon from the second pair, which we refer to as the last photon, to collapse into the same state. Thus entanglement was ‘swapped.’ The last photon was then measured at \( t = 2\tau \). The first and last photon which never coexisted were entangled in one Bell states on the r.h.s of equation (3), such as in \( |\psi^-\rangle_{a,b} \), which we refer to as temporal Bell states.

The experimentalists constructed the density operator of the first and last photon for analyzing the correlations to establish entanglement. Crucial in their findings was that if the projection was not carried out adequately at \( t = \tau \), then the first and last photons do not become entangled in time. Additionally, the temporal Bell states were interpreted in [1] as: “...measuring the last photon affects the physical description of the first photon in the past, before it has even been measured. Thus, the “spooky action” is steering the system’s past.” This propagation of the quantum state into the past can be seen far more concretely in that prior to the projection of the photons at \( t = \tau \), the measurement outcome is unknown. Using linear optical elements, the possible outcomes are \( |\phi^+\rangle_{a,b} \) and \( |\phi^-\rangle_{a,b} \). This implies that the first and last photon collectively collapse to are either \( |\phi^+\rangle_{a,b} \) or \( |\phi^-\rangle_{a,b} \). However this is determined only much after the measurement of the first photon.
II. COMMENTS ON TIME INVERSION

In this section, we add some further comments about time inversion. In particular, we examine some differences between time inversion as presented *TENET* and the concept of time travel as depicted in other films.

The first condition for time inversion is that the system moves backwards in the environment’s time in descending consecutive values. This is in contrast to the time travel conveyed in most other films. In those narratives, the system of interest can move almost immediately to the environment’s distant past, without traversing the time points in between. Another difference is that after the system reaches a temporal point in the environment’s past, the system is then constrained to only move forward in that environment’s time.

The second condition for time inversion is that the inverted system itself is moving forward in its own time in ascending consecutive values. This implies that from the perspective of the inverted system going into the environment’s past, it still has ‘free will.’ This concept is conveyed and discussed by the characters in *TENET*. The literature on quantum foundations often entails discussions on free will, most notably through the Free Will Theorem [2]. There, the notion of free will is ascribed to spatially entangled particles due to the indeterminism of the responses under measurement. In our main text regarding quantum time inversion, we had quantified an indeterminism of the choice the inverted information will make in the past. We speculate that one can apply the Free Will Theorem to that indeterminism as it is also predicated on an entanglement of particles, albeit in time rather than space. This alludes to the notion that the inverted information itself has ‘free will.’

The third condition states that the effect of time inversion is directly related to modifying system’s entropy. This is in contrast to general relativistic time machines which require the manipulation of the spacetime metric itself [3, 4]. Such exotic geometries can be visualized in Christopher Nolan’s other film *Interstellar* [5].

In terms of our work on quantum time inversion, several extensions can be explored. One is to investigate the use of temporal multi-entangled states in order to extend the inversion history. This could then be applied to the quantum DDoS scenario for a traceback capability [6]. The second would be to use the ontological aspects of quantum information [7] and the thermodynamic meaning of negative conditional entropies [8] to identify a time inversion for the physical aspects of quantum systems.

III. DERIVATION OF THE QUANTUM TURNSTILE EQUATIONS

For the quantum Turnstile in the main text, we had the equation for $|\text{Turnstile}\rangle \otimes |\text{Input}\rangle$ as

\[
\begin{align*}
|\beta_{00}\rangle_{A,B}^{(n-1)\tau,n\tau} \otimes & |\beta_{xy}\rangle_{C,D}^{n\tau,(n+1)\tau} = \frac{1}{2} (|\beta_{xy}\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} \otimes |\beta_{00}\rangle_{B,C} + |\beta_{xy}\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} \otimes |\beta_{10}\rangle_{B,C}^ {n\tau,n\tau} \\
& + (-1)^x |\beta_{xy}\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} \otimes |\beta_{01}\rangle_{B,C}^{n\tau,n\tau} + (-1)^x |\beta_{xy}\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} \otimes |\beta_{11}\rangle_{B,C}^{n\tau,n\tau} ).
\end{align*}
\]
It can be derived in the following manner:

\[
|\beta_{00}\rangle_{A,B}^{(n-1)\tau,n\tau} |\beta_{xy}\rangle_{C,D,}^{n\tau,(n+1)\tau} = \left( \frac{|00\rangle_{A,B}^{(n-1)\tau,n\tau} + |11\rangle_{A,B}^{(n-1)\tau,n\tau}}{\sqrt{2}} \right) \otimes \left( \frac{|0y\rangle_{C,D}^{n\tau,(n+1)\tau} + (-1)^x |1y\rangle_{C,D}^{n\tau,(n+1)\tau}}{\sqrt{2}} \right)
\]

\[
= \frac{1}{2} \left( |00\rangle_{A,B}^{(n-1)\tau,n\tau} |0y\rangle_{C,D}^{n\tau,(n+1)\tau} + (-1)^x |00\rangle_{A,B}^{(n-1)\tau,n\tau} |1y\rangle_{C,D}^{n\tau,(n+1)\tau} \right.
\]

\[
+ |11\rangle_{A,B}^{(n-1)\tau,n\tau} |0y\rangle_{C,D}^{n\tau,(n+1)\tau} + (-1)^x |11\rangle_{A,B}^{(n-1)\tau,n\tau} |1y\rangle_{C,D}^{n\tau,(n+1)\tau} \right)
\]

\[
= \frac{1}{2} \left( |0y\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} [00]_{B,C}^{n\tau,n\tau} + (-1)^x |0y\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} [01]_{B,C}^{n\tau,n\tau} \right.
\]

\[
+ |1y\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} [10]_{B,C}^{n\tau,n\tau} + (-1)^x |1y\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} [11]_{B,C}^{n\tau,n\tau} \right)
\]

\[
= \frac{1}{4} \left[ 2 |0y\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} [00]_{B,C}^{n\tau,n\tau} + 2(-1)^x |1y\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} [11]_{B,C}^{n\tau,n\tau} \right]
\]

\[
+ 2(-1)^x |0y\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} [01]_{B,C}^{n\tau,n\tau} + 2 |1y\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} [10]_{B,C}^{n\tau,n\tau} \right]
\]

\[
+ \frac{1}{2} \left[ |0y\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} [00]_{B,C}^{n\tau,n\tau} - |0y\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} [01]_{B,C}^{n\tau,n\tau} \right.
\]

\[
+ (-1)^x |0y\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} [01]_{B,C}^{n\tau,n\tau} - |0y\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} [01]_{B,C}^{n\tau,n\tau} \right)
\]

\[
+ \frac{1}{2} \left[ (-1)^x |0y\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} [01]_{B,C}^{n\tau,n\tau} + (-1)^x |0y\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} [01]_{B,C}^{n\tau,n\tau} \right.
\]

\[
+ \frac{1}{2} \left[ (-1)^x |0y\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} [01]_{B,C}^{n\tau,n\tau} - (-1)^x |0y\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} [01]_{B,C}^{n\tau,n\tau} \right.
\]

\[
+ \frac{1}{2} \left[ (-1)^x |0y\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} [01]_{B,C}^{n\tau,n\tau} + (-1)^x |0y\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} [01]_{B,C}^{n\tau,n\tau} \right)
\]

\[
+ \frac{1}{2} \left[ (-1)^x |0y\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} [01]_{B,C}^{n\tau,n\tau} - (-1)^x |0y\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} [01]_{B,C}^{n\tau,n\tau} \right)
\]

\[
= \frac{1}{2} \left[ |00\rangle_{A,B}^{(n-1)\tau,n\tau} + |11\rangle_{A,B}^{(n-1)\tau,n\tau} \right]
\]

\[
+ \frac{1}{2} \left[ |0y\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} + (-1)^x |1y\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} \right)
\]

\[
+ \left( \frac{|0y\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} + (-1)^x |1y\rangle_{A,D}^{(n-1)\tau,(n+1)\tau}}{\sqrt{2}} \right)
\]

\[
+ \left( \frac{(-1)^x |0y\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} + (-1)^x |1y\rangle_{A,D}^{(n-1)\tau,(n+1)\tau}}{\sqrt{2}} \right)
\]

\[
= \frac{1}{2} (|\beta_{xy}\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} \otimes |\beta_{00}\rangle_{B,C}^{n\tau,n\tau} + |\beta_{xy}\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} \otimes |\beta_{01}\rangle_{B,C}^{n\tau,n\tau} + (-1)^x |\beta_{xy}\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} \otimes |\beta_{11}\rangle_{B,C}^{n\tau,n\tau})
\]

We also applied the following operators to the respective states to obtain the outcome $|\beta_{xy}\rangle_{A,D}^{(n-1)\tau,(n+1)\tau}$:

\[
(\mathbb{I} \otimes \mathbb{I}) |\beta_{xy}\rangle_{A,D}^{(n-1)\tau,(n+1)\tau},
\]

\[
(\mathbb{I} \otimes (-1)^x \sigma_1) (-1)^x |\beta_{xy}\rangle_{A,D}^{(n-1)\tau,(n+1)\tau},
\]

\[
(\mathbb{I} \otimes (-1)^x \sigma_2) |\beta_{xy}\rangle_{A,D}^{(n-1)\tau,(n+1)\tau},
\]

\[
(\mathbb{I} \otimes (-1)^x \sigma_3) |\beta_{xy}\rangle_{A,D}^{(n-1)\tau,(n+1)\tau},
\]

\[
(\mathbb{I} \otimes (-1)^x \sigma_3 \sigma_1) (-1)^x |\beta_{xy}\rangle_{A,D}^{(n-1)\tau,(n+1)\tau}.
\]

Let us show that this indeed computes the desired result:
\[(I \otimes (-1)^x \hat{\sigma}_1)(-1)^x |\beta_{xy}\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} = (I \otimes (-1)^x \hat{\sigma}_1) \left( \frac{(-1)^x |0\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} + (-1)^x |1\rangle_{A,D}^{(n-1)\tau,(n+1)\tau}}{\sqrt{2}} \right) \] (26)

\[= \frac{1}{\sqrt{2}} (|0\rangle_A^{(n-1)\tau} \otimes (-1)^x (-1)^{x-1} \hat{\sigma}_1 |\beta_{xy}\rangle_{D}^{(n+1)\tau}) \] (27)

\[+ \frac{1}{\sqrt{2}} (|1\rangle_A^{(n-1)\tau} \otimes (-1)^x \hat{\sigma}_1 |\beta_{xy}\rangle_{D}^{(n+1)\tau}) \] (28)

\[= \frac{1}{\sqrt{2}} (|0\rangle_A^{(n-1)\tau} \otimes (-1)^{x+1} |\beta_{xy}\rangle_{D}^{(n+1)\tau}) \] (29)

\[+ \frac{1}{\sqrt{2}} (|1\rangle_A^{(n-1)\tau} \otimes (-1)^x |\beta_{xy}\rangle_{D}^{(n+1)\tau}) \] (30)

\[= \frac{1}{\sqrt{2}} (|0\rangle_A^{(n-1)\tau} \otimes |\beta_{xy}\rangle_{D}^{(n+1)\tau}) \] (31)

\[+ \frac{1}{\sqrt{2}} ((-1)^x |1\rangle_A^{(n-1)\tau} \otimes |\beta_{xy}\rangle_{D}^{(n+1)\tau}) \] (32)

\[= \left( |0\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} + (-1)^x |1\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} \right) \] (33)

\[= |\beta_{xy}\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} \] (34)

\[(I \otimes (-1)^y \hat{\sigma}_3) |\beta_{xy}\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} = (I \otimes (-1)^y \hat{\sigma}_3) \left( \frac{|0\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} + (-1)^x |1\rangle_{A,D}^{(n-1)\tau,(n+1)\tau}}{\sqrt{2}} \right) \] (35)

\[= \frac{1}{\sqrt{2}} ((0)_A^{(n-1)\tau} \otimes (-1)^y \hat{\sigma}_3 |\beta_{xy}\rangle_{D}^{(n+1)\tau}) \] (36)

\[+ \frac{1}{\sqrt{2}} (|1\rangle_A^{(n-1)\tau} \otimes (-1)^y (-1)^{x-1} \hat{\sigma}_3 |\beta_{xy}\rangle_{D}^{(n+1)\tau}) \] (37)

\[= \frac{1}{\sqrt{2}} ((0)_A^{(n-1)\tau} \otimes (-1)^y (-1)^y |\beta_{xy}\rangle_{D}^{(n+1)\tau}) \] (38)

\[+ \frac{1}{\sqrt{2}} (|1\rangle_A^{(n-1)\tau} \otimes (-1)^y (-1)^y |\beta_{xy}\rangle_{D}^{(n+1)\tau}) \] (39)

\[= \frac{1}{\sqrt{2}} ((0)_A^{(n-1)\tau} \otimes (-1)^y |\beta_{xy}\rangle_{D}^{(n+1)\tau}) \] (40)

\[+ \frac{1}{\sqrt{2}} (|1\rangle_A^{(n-1)\tau} \otimes (-1)^y (-1)^y |\beta_{xy}\rangle_{D}^{(n+1)\tau}) \] (41)

\[= \frac{1}{\sqrt{2}} ((0)_A^{(n-1)\tau} \otimes |\beta_{xy}\rangle_{D}^{(n+1)\tau}) \] (42)

\[+ \frac{1}{\sqrt{2}} (|1\rangle_A^{(n-1)\tau} \otimes (-1)^y |\beta_{xy}\rangle_{D}^{(n+1)\tau}) \] (43)

\[= \frac{1}{\sqrt{2}} ((0)_A^{(n-1)\tau} \otimes |\beta_{xy}\rangle_{D}^{(n+1)\tau}) \] (44)

\[+ \frac{1}{\sqrt{2}} (|1\rangle_A^{(n-1)\tau} \otimes |\beta_{xy}\rangle_{D}^{(n+1)\tau}) \] (45)

\[= \left( |0\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} + (-1)^x |1\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} \right) \] (46)

\[= |\beta_{xy}\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} \] (47)
\[(I \otimes (-1)^{x+y} \sigma_3 \sigma_1) (1)^x |\beta\rangle_{A,D} = (I \otimes (-1)^{x+y} \sigma_3 \sigma_1) (1)^x |\beta\rangle_{A,D} \]

\[\begin{align}
&= \frac{1}{\sqrt{2}} \left( |0\rangle^{(n-1)\tau} \otimes (-1)^x |0\rangle^{(n+1)\tau} \right) \\
&= |1\rangle^{(n-1)\tau} \otimes (-1)^x |y\rangle^{(n+1)\tau} \\
&= \frac{1}{\sqrt{2}} \left( |0\rangle^{(n-1)\tau} \otimes (-1)^x |y\rangle^{(n+1)\tau} \right) \\
&= |1\rangle^{(n-1)\tau} \otimes (-1)^x |y\rangle^{(n+1)\tau} \\
&= \frac{1}{\sqrt{2}} \left( |0\rangle^{(n-1)\tau} \otimes |y\rangle^{(n+1)\tau} \right) \\
&= (-1)^x |1\rangle^{(n-1)\tau} \otimes |y\rangle^{(n+1)\tau} \\
&= \left( |0\rangle^{(n-1)\tau} + (-1)^x |1\rangle^{(n-1)\tau} \right) \\
&= \sqrt{2} \left( |\beta\rangle_{A,D}^{(n-1)\tau} \right) \\
\end{align}\]

IV. DDOS MITIGATION: ‘BLACK HOLE’ ROUTING

In classical networks, after detecting a DDoS attack it is common to employ a mitigation method \[9\]. An example of such a countermeasure is known as ‘black hole’ routing \[10\]. In such a scenario, the attack packets are forwarded to a null logical interface to be discarded or dropped. As a result, the service capacity is unaffected, leaving service available to legitimate traffic.

For the quantum DDoS case presented in the main text, we can also involve a mitigation method after detection. Analogous to the classical case, we can introduce a ‘black hole’ routing strategy using the Turnstile model of a repeater.

We illustrate this with a simple example. Suppose we have attack delayed Bell states \(|\beta_0\rangle^{(n-1)\tau, n\tau}_{A_1B_1} \) and \(|\beta_0\rangle^{(n-1)\tau, n\tau}_{A_2B_2}\). Here labels \(A_1\) and \(A_2\) refer to the qubits destroyed at the respective attack nodes at \(t = (n-1)\tau\), and labels \(B_1\) and \(B_2\) are the qubits that reach the repeater at \(t = n\tau\). Under normal conditions, the repeater would perform a projective measurement with the Bell pairs generated at the repeater. Given that the attack has been detected, the repeater rather performs a joint projective measurement on the attack qubits themselves, \(B_1\) and \(B_2\) at \(t = n\tau\). We can write this as

\[\begin{align}
|\beta_0\rangle^{(n-1)\tau, n\tau}_{A_1B_1} \otimes |\beta_0\rangle^{(n-1)\tau, n\tau}_{A_2B_2} &= \frac{1}{2} \left(|\beta_0\rangle^{(n-1)\tau, (n-1)\tau}_{A_1A_2} \otimes |\beta_0\rangle^{(n-1)\tau, (n-1)\tau}_{B_1B_2} + |\beta_0\rangle^{(n-1)\tau, (n-1)\tau}_{A_1A_2} \otimes |\beta_0\rangle^{(n-1)\tau, (n-1)\tau}_{B_1B_2} \right) \\
&\quad + |\beta_0\rangle^{(n-1)\tau, (n-1)\tau}_{A_1A_2} \otimes |\beta_0\rangle^{(n-1)\tau, (n-1)\tau}_{B_1B_2} + |\beta_0\rangle^{(n-1)\tau, (n-1)\tau}_{A_1A_2} \otimes |\beta_0\rangle^{(n-1)\tau, (n-1)\tau}_{B_1B_2} \\
\end{align}\]

This joint measurement would swap the entanglement to qubits \(A_1\) and \(A_2\) in the past at \(t = (n-1)\tau\), which no longer exist. Consequently, the Bell pairs generated at the repeater have not been used, thereby leaving the repeater capacity available for legitimate traffic. Thus in this situation the concept of the black hole routing refers to discarding or dropping the quantum information into the past.

An interesting speculation is whether such a routing exists for physical black holes with respect to their puzzling information loss \[11\]. More specifically, it would be of interest to explore whether an inverted entanglement entropy can assist in fully computing the unitary Page curve, without invoking additional geometric constructs.
V. INVERTED ENTROPY FROM THE PERSPECTIVE OF THE PAST

In the temporal entanglement experiment [1] that our work is based on, they generated spatial Bell states such as $|\psi^-\rangle_{a,b}$, delayed Bell states such as $|\psi^-\rangle_{a,b}^{0,\tau}$, and finally temporal Bell states such as $|\psi^-\rangle_{a,b}^{0,2\tau}$. For the latter, it was said that the last photon at $t = 2\tau$ is entangled to the first photon at $t = 0$ in the past which does not exist anymore. The mathematical pure state description also indicates that from the perspective of the first photon at $t = 0$, it is entangled to the last photon at $t = 2\tau$. In [1] this observation is expressed directly as: “Another point of view that one can take is that the measurement of the first photon is immediately steering the future physical description of the last photon. In this case, the action is on the future of a part of the system that has not yet been created.” This suggests the label $t = 0$ for mode $a$ is identical in both the spatial Bell state $|\psi^-\rangle_{a,b}^{0,0}$ and temporal Bell state $|\psi^-\rangle_{a,b}^{0,2\tau}$. As a result we have a perplexing situation in that from perspective of the first photon at $t = 0$, it is entangled to both the last photon at $t = 2\tau$ as well as its spatial Bell partner at $t = 0$.

We can translate this ambiguous situation into the setting of the quantum Turnstile. The density operator of the Turnstile output is a temporal Bell state

$$\rho_{A,D}^{(n-1)\tau,(n+1)\tau} = |\beta_{xy}\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} \langle \beta_{xy}|_{A,D}^{(n-1)\tau,(n+1)\tau},$$

where subsystem $A$ and $D$ do not coexist. From the perspective of $D$, it is entangled to $A$ in the past. This can be equivalently expressed through the inverted conditional entropy at $t = (n+1)\tau$

$$S(\rho_A^{(n-1)\tau} | \rho_D^{(n+1)\tau}) = S(\rho_A^{(n-1)\tau,(n+1)\tau}) - S(\rho_D^{(n+1)\tau}) = 0 - 1 = -1.$$  

Using the reasoning above, it may also be said that from the perspective of $A$ at $t = (n-1)\tau$ it is entangled to $D$ at $t = (n+1)\tau$ which has not yet been created. Hence the value of inverted conditional entropy (63) from the perspective of $t = (n-1)\tau$ remains the same. We write this as

$$S_P(\rho_A^{(n-1)\tau} | \rho_D^{(n+1)\tau}) = -1,$$

where we use subscript $P$ to signify the inverted conditional entropy from the perspective of the system in the past, at $t = (n-1)\tau$. This opposed to (63) where the same quantity is viewed from $t = (n+1)\tau$. We will only use subscript $P$ when referring to inverted entropies as there is an implicit ambiguity in which time point to view it from.

Expanding on this, in our quantum Turnstile scenario if the projection is carried out at $t = n\tau$ then we have

$$|\beta_{xy}\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} \otimes |\beta_{xy}\rangle_{B,C}^{n\tau,n\tau},$$

where $x, y$ can take any binary values depending on the outcome of the projection. In all those cases, we would have value (64) at $t = (n-1)\tau$. If projection does not occur then we are simply left with delayed Bell states

$$|\beta_{00}\rangle_{A,B}^{(n-1)\tau,n\tau} \otimes |\beta_{00}\rangle_{C,D}^{n\tau,(n+1)\tau},$$

as there is only a delay line with absorption. In this case, the inverted conditional entropy at $t = (n-1)\tau$ takes trivial value

$$S_P(\rho_A^{(n-1)\tau} | \rho_D^{(n+1)\tau}) = 2 - 1 = 1,$$

which means inversion (projection) did not take place in the future. This is equivalent to a trivial product state between temporal systems. Analogous to the ambiguous situation in the experiment, we also have that $A$ is spatially entangled to $B$ at $t = (n-1)\tau$ through the generation of initial state $|\beta_{00}\rangle_{A,B}^{(n-1)\tau,(n-1)\tau}$ which can be quantified as

$$S(\rho_A^{(n-1)\tau} | \rho_B^{(n-1)\tau}) = -1.$$ 

We have removed the subscript $P$ to indicate this is not an inverted conditional entropy between temporal systems but rather the the standard conditional entropy since both subsystems exist at $t = (n-1)\tau$.

In this section, we want to clarify this ambiguous situation, and in particular explore the ways the quantity (64) can (or cannot) exist at $t = (n-1)\tau$. There are in fact three possibilities, and by considering these we hope to gain a deeper understanding of underlying entanglement in time.
A. First possibility: Quantum transfer entropy

The first possibility as viewed through the experiment [1] is that the first photon at \( t = 0 \) is indeed entangled, in one of the temporal Bell states such as \( |\psi^+\rangle_{a,b} \), to the last photon in the future which has not yet been created and furthermore also spatially entangled to another photon as part of the initially generated spatial Bell pair \( |\psi^+\rangle_{a,b} \).

We believe this simultaneous entanglement to different particles, one across time and one across space, is the key characteristic of entanglement in time, as opposed to entanglement in space. This is in contrast to a delayed Bell pair where a photon in that state is only ever entangled to the same other photon, first across space and then across time.

Viewing this in terms of the Turnstile, this first possibility means that at \( t = (n-1)\tau \) both the conditional entropy \( S(\rho_A^{(n-1)\tau} | \rho_B^{(n-1)\tau}) \) and the inverted conditional entropy \( S_P(\rho_A^{(n-1)\tau} | \rho_D^{(n+1)\tau}) \) have the same non-trivial value

\[
S(\rho_A^{(n-1)\tau} | \rho_B^{(n-1)\tau}) = -1, \quad S_P(\rho_A^{(n-1)\tau} | \rho_D^{(n+1)\tau}) = -1. \tag{69}
\]

The former indicates the spatial entanglement between \( A \) and \( B \) at \( t = (n-1)\tau \), while the latter is the temporal entanglement between \( A \) and \( D \) at \( t = (n-1)\tau \). Combining these entropic quantities gives

\[
S(\rho_A^{(n-1)\tau} | \rho_B^{(n-1)\tau}) + S_P(\rho_A^{(n-1)\tau} | \rho_D^{(n+1)\tau}) < 0, \tag{70}
\]

which is a violation of the monogamy of entanglement [12] for system \( A \) at \( t = (n-1)\tau \).

We will analyze this first possibility further by harnessing some recent developments in classical information theory regarding temporal systems [13]. The classical conditional mutual information is given by

\[
I(X;Y|Z) = H(X,Z) + H(Y,Z) - H(X,Y,Z) - H(Z), \tag{71}
\]

which measures the reduction in the uncertainty about \( X \) that we obtain from \( Y \) given the value of \( Z \). One can then define the classical transfer entropy [13, 14] as

\[
T_{Y\rightarrow X} = I(Y_n;X_{n+1}|X_n), \tag{72}
\]

which measures how much information about the next observation of \( X \) can be found in observation \( Y_n \) in the context of the past state \( X_n \). The subscripts refer to the time points. The transfer entropy has been applied to the study of classical physical systems and the arrow of time [15].

A related quantity of interest is the active information storage

\[
A_X = I(X_n;X_{n+1}), \tag{73}
\]

which measures how much information about the next observation of \( X \) can be found in the past state \( X_n \).

The transfer entropy can be interpreted more clearly as information transfer through a distributed information processing task. More precisely

\[
H(X_{n+1}) = I(X_n;X_{n+1}) + I(Y_n;X_{n+1}|X_n) + H(X_{n+1}|X_n,Y_n), \tag{74}
\]

which demonstrates how the computation of the next value of \( X \) is composed of the stored information through the active information storage, the transferred information through the transfer entropy, and the uncertainty left in \( X \) given the past of \( X \) and \( Y \).

We aim to construct the quantum analogue of the classical transfer entropy so to better analyze the inverted quantum entropy from the perspective of the past. To carry this task out, we note that the quantum conditional mutual information for a tripartite system is given by

\[
I(A;D|B) = S(BA) + S(BD) - S(BAD) - S(B). \tag{75}
\]

We define the quantum transfer entropy for a temporal tripartite system as

\[
T = I(B^{n\tau}; D^{(n+1)\tau}; A^{(n-1)\tau}). \tag{76}
\]

However, we modify this further by rather conditioning it on the future and specifying it for the Turnstile system

\[
T^{(n-1)\tau}_P = I_P^{(n-1)\tau}(\rho_A^{(n-1)\tau}; \rho_B^{(n+1)\tau}; \rho_D^{(n+1)\tau}), \tag{77}
\]

where we use subscript \( P \) with superscript \((n-1)\tau\) to signify this quantity as viewed at \((n-1)\tau\). The notion of conditioning on the future can found in the classical backwards transfer entropy [16] which quantifies the time-reversal directed information flow from the future to the past.
We expand (77) as
\[
S_p^{(n-1)\tau}(\rho_{A,D}^{(n-1)\tau},(n+1)\tau) + S_p^{(n-1)\tau}(\rho_{B,D}^{n\tau},(n+1)\tau) - S_p^{(n-1)\tau}(\rho_{A,B,D}^{(n-1)\tau},n\tau,(n+1)\tau) - S_p^{(n-1)\tau}(\rho_{D}^{(n+1)\tau}).
\] (78)
where these quantities are viewed from \(t = (n - 1)\tau\).

In our Turnstile situation, there are only two possibilities ahead regarding states. Either projection will occur (65) or it will not (66). This constraint on the possible states allows us to calculate at \(t = (n - 1)\tau\) the entropic values about the future. Regardless of which case occurs, the last three terms in (78) have values
\[
S_p^{(n-1)\tau}(\rho_{B,D}^{(n+1)\tau}) = 2, \quad S_p^{(n-1)\tau}(\rho_{A,B,D}^{(n-1)\tau},(n+1)\tau) = 1, \quad S_p^{(n-1)\tau}(\rho_{D}^{(n+1)\tau}) = 1.
\] (79)
Therefore, we can write the quantum transfer entropy (77) in terms of only the inverted conditional entropy
\[
T_p^{(n-1)\tau} = I_p^{(n-1)\tau}(\rho_A^{(n-1)\tau},\rho_B^{n\tau}|\rho_D^{(n+1)\tau}) = S_p(\rho_A^{(n-1)\tau}|\rho_D^{(n+1)\tau}) + 1.
\] (80)
If projection occurs in the future (65) then \(S_p(\rho_A^{(n-1)\tau}|\rho_D^{(n+1)\tau}) = -1\) which implies
\[
T_p^{(n-1)\tau} = I_p^{(n-1)\tau}(\rho_A^{(n-1)\tau},\rho_B^{n\tau}|\rho_D^{(n+1)\tau}) = 0.
\] (81)
If projection fails in the future (66), then \(S_p(\rho_A^{(n-1)\tau}|\rho_D^{(n+1)\tau}) = 1\), which means
\[
T_p^{(n-1)\tau} = I_p^{(n-1)\tau}(\rho_A^{(n-1)\tau},\rho_B^{n\tau}|\rho_D^{(n+1)\tau}) = 2.
\] (82)
In this subsection, we have assumed the existence of the inverted conditional entropy in the past at \(t = (n - 1)\tau\), and using (80) we also have the existence of the quantum transfer entropy in the past at \(t = (n - 1)\tau\). Both \(S_p(\rho_A^{(n-1)\tau}|\rho_D^{(n+1)\tau})\) and \(I_p^{(n-1)\tau}\) contain information about the future, or more precisely the inverted quantum information from the future. However, the quantum transfer entropy is a better tool to quantify this information transfer given the classical analogue [16] represents the time-reversal directed information flow from the future to the past.

B. Second possibility: Grandfather paradox

The second possibility is that the first photon from its perspective at \(t = 0\) is only temporally entangled to the last photon in the future such as in \(|\psi^{-}\rangle_{a,b}^{0.2\tau}\), and is in fact not entangled in the spatial Bell state that it was generated in. The reason for this is that the projection performed in the future at \(t = \tau\) has already influenced the past with its entanglement swapping, thereby erasing from history the initial spatial entanglement \(|\psi^{-}\rangle_{a,b}^{0.2\tau}\) that resided at \(t = 0\).

In terms of the Turnstile, this possibility means that at \(t = (n - 1)\tau\) we have that
\[
S(\rho_A^{(n-1)\tau}|\rho_B^{(n-1)\tau}) = 1, \quad S_p(\rho_A^{(n-1)\tau}|\rho_D^{(n+1)\tau}) = -1.
\] (83)
One can also attribute the reasoning of this possibility to the enforcement of the monogamy of entanglement since
\[
S_p(\rho_A^{(n-1)\tau}|\rho_D^{(n+1)\tau}) + S_p(\rho_A^{(n-1)\tau}|\rho_B^{(n+1)\tau}) \geq 0.
\] (84)
However this possibility raises a paradox: At \(t = (n - 1)\tau\) we have an entanglement between \(A\) and \(D\), which in this possibility means the initial spatial entanglement between \(A\) and \(B\) at \(t = (n - 1)\tau\) was ‘killed.’ However the entanglement between \(A\) and \(D\) emerged from the entanglement between \(A\) and \(B\) through the projection with its entanglement swapping. Given that the latter entanglement no longer exists at \(t = (n - 1)\tau\) (and thereby never reaches the projection at \(t = n\tau\)), this means the former entanglement should also not exist at \(t = (n - 1)\tau\). Stated in another way, we have a ‘grandfather paradox’ [4] for entanglement.

C. Third possibility: Many past worlds

The third possibility is that the first photon is only entangled in the spatial Bell state it was generated in \(|\psi^{-}\rangle_{a,b}^{0.0}\), and from its perspective at \(t = 0\) is not temporally entangled to the last photon in the future through pure state \(|\psi^{-}\rangle_{a,b}^{0.2\tau}\). This mathematical pure state description (after the projection outcome) should only be interpreted from
time $t = 2\tau$ and not from time $t = 0$. In terms of the Turnstile this possibility means that at $t = (n-1)\tau$, we have a trivial value for the inverted entropy

$$S(\rho_A^{(n-1)\tau} | \rho_B^{(n-1)\tau}) = -1, \quad Sp(\rho_A^{(n-1)\tau} | \rho_D^{(n+1)\tau}) = 1. \quad (85)$$

This conservative possibility however presents us with a very puzzling situation. We express this through the setting of the experiment: If the projection is carried out at $t = \tau$ and part an entanglement does move into the past to $t = 0$ as indicated through temporal Bell state $|\psi{-0.2}\tau, \beta_{a,b}\rangle$, then why is this part of the entanglement not identified from the perspective of the past at $t = 0$?

A solution can be suggested through the many worlds interpretation of quantum theory. Rather than simply considering the creation of parallel worlds into the future, one can extend the interpretation to also consider the creation of parallel worlds into the past. In this way the entanglement in the experiment does not go into the past from which it is derived but rather goes into the past of another timeline which is created from its present. Therefore the first index $t = 0$ in pure state $|\psi{-0.2}\tau, \beta_{a,b}\rangle$ and in pure state $|\psi{-0.0}\beta_{a,b}\rangle$ are not the same. They are part of two different timelines representing two different past worlds.

VI. DETECTING A DDoS EVENT BEFORE IT OCCURS

We emphasize this section undertakes a speculative direction. However this is necessary in order to explore the potential capabilities of the work in the main text. Alternatively one can take the view that the following are merely toy models to facilitate an understanding of the limits on the resources that temporal entanglement can offer.

In our main text, we considered a quantum DDoS scenario where the repeater service remained available while the attack was active. Here we consider an extreme attack case where the distinction can only made between the service running and the service failing where the latter signifies a successful attack. This may be attributed to cases where the initial density of attack requests are overwhelmingly concentrated that they immediately exceed the capacity of the repeater. We observe that when the service has failed, the state is written as

$$|\beta_{xy}\rangle_{A,D}^{(n-1)\tau,n\tau} \otimes |\beta_{00}\rangle_{C,D}^{n\tau,(n+1)\tau}, \quad (86)$$

given that entanglement swapping did not occur at $t = n\tau$ due to the DDoS attack. However if the service was operating, then we have

$$|\beta_{xy}\rangle_{A,D}^{(n-1)\tau,(n+1)\tau} \otimes |\beta_{00}\rangle_{B,C}^{n\tau,n\tau}, \quad (87)$$

given that entanglement swapping occurred at $t = n\tau$.

In this section, we want to detect the attack event before the event occurs. If it occurs at $t = n\tau$, our aim is to detect this at $t = (n-1)\tau$. This is in contrast to the system in the main text which can at best detect an attack in real-time. Stated more astonishingly, we want the ability to detect DDoS attacks that have occurred in the future.

To accomplish this, we assume the first (69) of the three possibilities considered in the previous section. Specifically we assume that inverted entropies can take non-trivial values at $t = (n-1)\tau$. We will proceed to compute the quantum transfer entropy and use it operationally to design a technology which we refer to as a temporal Petz recovery channel. Such a system has the ability to indirectly receive the inverted quantum information from the future, and we will apply it to detect a DDoS event before it occurs. This capability is analogous to the temporal pincer in TENET where an advantage is gained by information about the future obtained through inversion.

Finally we proceed to show the relationship of this technology to the Turnstile, and briefly outline a speculative potential realization of this system using quantum computing circuits.

A. Temporal Petz recovery channel

We consider the quantum transfer entropy (77) for our scenario at $t = (n-1)\tau$, with the conditioning on the past (76) so that

$$T_P^{(n-1)\tau} = I_P^{(n-1)\tau}(\rho_D^{(n+1)\tau}; \rho_B^{n\tau} | \rho_A^{(n-1)\tau}). \quad (88)$$

This can be expanded into

$$S_P^{(n-1)\tau}(\rho_{A,D}^{(n-1)\tau,(n+1)\tau}) + S_P^{(n-1)\tau}(\rho_{A,B}^{(n-1)\tau,n\tau}) - S_P^{(n-1)\tau}(\rho_{A,B,D}^{(n-1)\tau,n\tau,(n+1)\tau}) - S_P^{(n-1)\tau}(\rho_A^{(n-1)\tau}). \quad (89)$$
This situation is constrained in that we only consider the possibility of a DDoS attack (86) or the service operating (87). Hence the above expression can be re-written as
\[
S_P^{(n-1)\tau}(\rho_A^{(n-1)\tau}|\rho_D^{(n+1)\tau}) + S_P^{(n-1)\tau}(\rho_{AB}^{(n-1)\tau,n\tau}) - S_P^{(n-1)\tau}(\rho_{AB,D}^{(n-1)\tau,n\tau,(n+1)\tau}),
\]
with the first term signifying the inverted conditional entropy at \(t = (n-1)\tau\) between \(A\) and \(D\) (69). If a DDoS attack occurs in the future (86), then
\[
S_P^{(n-1)\tau}(\rho_A^{(n-1)\tau}|\rho_D^{(n+1)\tau}) = 1, \quad S_P^{(n-1)\tau}(\rho_{AB}^{(n-1)\tau,n\tau}) = 0, \quad S_P^{(n-1)\tau}(\rho_{AB,D}^{(n-1)\tau,n\tau,(n+1)\tau}) = 1
\]
and if the service is operating in the future (87) we have
\[
S_P^{(n-1)\tau}(\rho_A^{(n-1)\tau}|\rho_D^{(n+1)\tau}) = -1, \quad S_P^{(n-1)\tau}(\rho_{AB,D}^{(n-1)\tau,n\tau,(n+1)\tau}) = 1.
\]
Note that \(S_P^{(n-1)\tau}(\rho_{AB,D}^{(n-1)\tau,n\tau,(n+1)\tau})\) takes the same value, and the value of \(S_P^{(n-1)\tau}(\rho_{AB}^{(n-1)\tau,n\tau})\) depends on the value of \(S_P^{(n-1)\tau}(\rho_{D}^{(n-1)\tau})\). Therefore, for either case (86) or (87), the expression (90) equates to zero. In other words, we always have
\[
T_P^{(n-1)\tau} = I_P^{(n-1)\tau}(\rho_D^{(n+1)\tau};\rho_B^{(n-1)\tau}|\rho_A^{(n-1)\tau}) = 0.
\]
at \(t = (n-1)\tau\).

There exists an operational definition of the quantum conditional mutual information (75) \([17, 18]\). It has been found that a zero quantum conditional mutual information corresponds to states \(\rho_{KL,M}\) whose system can be reconstructed just by acting on \(L\). More precisely, the theorem states that the following are equivalent:

1. \(I(K;M|L) = 0\),
2. There exists a quantum map \(T_{L \rightarrow LM}(\rho_{KL}) = \rho_{KL,M}.\) Such a map can be taken to be
\[
T_{L \rightarrow LM}[X_L] = \frac{1}{2}\rho_{LM}^{1/2}X_L \rho_L^{1/2} \otimes \mathbb{I}_M)\rho_{LM}^{1/2},
\]
which is known as the Petz recovery channel \([19]\).

The quantum transfer entropy (93) is a form of quantum conditional mutual information. Therefore applying this theorem to (93) means that there exists a channel at \(t = (n-1)\tau\) that performs
\[
T_P^{(n-1)\tau}(\rho_{AB}^{(n-1)\tau,(n+1)\tau}) = (\rho_{AB,D}^{(n-1)\tau,(n-1)\tau,(n+1)\tau}),
\]
where
\[
T_P^{(n-1)\tau}[X_A^{(n-1)\tau}] = (\rho_{AB,D}^{(n-1)\tau,(n+1)\tau})^{1/2}(\rho_A^{(n-1)\tau})^{-1/2}X_A^{(n-1)\tau}(\rho_A^{(n-1)\tau})^{-1/2} \otimes \mathbb{I}_D)\rho_{AB,D}^{(n-1)\tau,(n+1)\tau})^{1/2},
\]
which we refer to as the temporal Petz recovery channel. This channel reconstructs the state \((\rho_{AB,D}^{(n-1)\tau,(n-1)\tau,(n+1)\tau})\) at time \(t = (n-1)\tau\). One can think of this channel as receiving the inverted information from the future.

Using (96), we can design an entropic DDoS detection system which can at \(t = (n-1)\tau\) detect an attack that will occur in the future at \(t = n\tau\). In this design, the detection system operates as a network node which we refer to as the detection node. This node generates state \(|\beta_0\rangle_{A,B}^{(n-1)\tau,(n-1)\tau}\) at \(t = (n-1)\tau\), and will send qubit \(B\) so that it reaches the repeater node at \(t = n\tau\). If projection is successful ahead, then we have state (87). Alternatively if a DDoS attack will occur at the repeater at \(t = n\tau\) due to attack traffic from other nodes, then the state of our detection node with the repeater will be (86). However prior to sending qubit \(B\) to the repeater, the detection node will first send its state \(|\beta_0\rangle_{A,B}^{(n-1)\tau,(n-1)\tau}\) as an input to the temporal Petz recovery channel (96) at time \(t = (n-1)\tau\).

Notice that the channel (96) does not require physical access to system \(D\), which has not yet been created. Rather it needs access to the quantum state \(\rho_{AB,D}^{(n-1)\tau,(n+1)\tau}\) at \(t = (n-1)\tau\) which it has through the temporal entanglement and our assumption of the first possibility (69). Furthermore, the detection node only has knowledge only its qubits \(AB\) and has no knowledge of state \(\rho_{AB,D}^{(n-1)\tau,(n+1)\tau}\). There is a separation in knowledge between the detection node and the channel. Rather the node can only input its state \(|\beta_0\rangle_{A,B}^{(n-1)\tau,(n-1)\tau}\) into the channel (96) to receive an output at \(t = (n-1)\tau\). Furthermore, in the output we assume the detection node can only access the quantum state of its
qubits $AB$. Using this limited output from the channel at $t = (n - 1)\tau$, we will show that the detection node can
detect if an attack occurred in the future at $t = n\tau$.

To start this calculation, we form a density operator of initial state of the detection node $|\beta_{00}\rangle_{A,B}^{(n-1)\tau,(n-1)\tau}$

$$
\rho_{B,A}^{(n-1)\tau,(n-1)\tau} = |\beta_{00}\rangle_{B,A}^{(n-1)\tau,(n-1)\tau}(\beta_{00})_B^{(n-1)\tau,(n-1)\tau}
$$

$$
= \frac{(00)_{B,A}^{(n-1)\tau,(n-1)\tau} + |11\rangle_{B,A}^{(n-1)\tau,(n-1)\tau}}{\sqrt{2}} \frac{(00)_{B,A}^{(n-1)\tau,(n-1)\tau} + |11\rangle_{B,A}^{(n-1)\tau,(n-1)\tau}}{\sqrt{2}}
$$

$$
= \frac{1}{2}[|0\rangle_B^{(n-1)\tau} \langle 0|_B^{(n-1)\tau} \otimes |0\rangle_A^{(n-1)\tau} \langle 0|_A^{(n-1)\tau} + |0\rangle_B^{(n-1)\tau} \langle 1|_B^{(n-1)\tau} \otimes |0\rangle_A^{(n-1)\tau} \langle 1|_A^{(n-1)\tau}]
$$

$$
+ |1\rangle_B^{(n-1)\tau} \langle 0|_B^{(n-1)\tau} \otimes |1\rangle_A^{(n-1)\tau} \langle 0|_A^{(n-1)\tau} + |1\rangle_B^{(n-1)\tau} \langle 1|_B^{(n-1)\tau} \otimes |1\rangle_A^{(n-1)\tau} \langle 1|_A^{(n-1)\tau}]
$$

The conditional entropy of the initial state is

$$
S(\rho_A^{(n-1)\tau}|\rho_B^{(n-1)\tau}) = -1.
$$

We input this state into the temporal Petz recovery channel (95)

$$
\mathbb{I}_B \otimes T_P^{(n-1)\tau}(\rho_{B,A}^{(n-1)\tau,(n-1)\tau})
$$

$$
= \frac{1}{2}[|0\rangle_B^{(n-1)\tau} \langle 0|_B^{(n-1)\tau} \otimes T_P^{(n-1)\tau}(|0\rangle_A^{(n-1)\tau} \langle 0|_A^{(n-1)\tau} + |0\rangle_B^{(n-1)\tau} \langle 1|_B^{(n-1)\tau} \otimes |0\rangle_A^{(n-1)\tau} \langle 1|_A^{(n-1)\tau}]
$$

$$
+ |1\rangle_B^{(n-1)\tau} \langle 0|_B^{(n-1)\tau} \otimes T_P^{(n-1)\tau}(|1\rangle_A^{(n-1)\tau} \langle 0|_A^{(n-1)\tau} + |1\rangle_B^{(n-1)\tau} \langle 1|_B^{(n-1)\tau} \otimes |1\rangle_A^{(n-1)\tau} \langle 1|_A^{(n-1)\tau}]).
$$

There are two possible outputs which depends on the form the map $T_P^{(n-1)\tau}$ takes, which in turns depends on whether
or not a DDoS attack occurred in the future at $t = n\tau$. We will carry out the calculation to construct the channel
(96) for the two scenarios. In order to make the presentation tidy, we will drop the superscripts as the subscripts
alone can be used to infer the temporal label.

We first consider the case where the DDoS attack occurred in the future (86). The density operator can be written as

$$
\rho_{A,B,C,D} = |\beta_{00}\rangle_{A,B}^{(n-1)\tau,n\tau} \otimes T_P^{(n-1)\tau,n\tau}(|\beta_{00}\rangle_{A,B}^{(n-1)\tau,n\tau} \otimes \beta_{00}_{C,D}^{(n+1)\tau,n\tau} \langle \beta_{00}|_{C,D}^{(n+1)\tau,n\tau})
$$

By expanding this expression out and taking the partial trace, we find that

$$
\rho_{A,D} = \text{Tr}_{BC}(\rho_{A,B,C,D})
$$

$$
= \frac{1}{4}(|0\rangle_{A} \langle 0|_{A} \otimes |0\rangle_{D} \langle 0|_{D} + |0\rangle_{A} \langle 0|_{A} \otimes |1\rangle_{D} \langle 1|_{D} + |1\rangle_{A} \langle 1|_{A} \otimes |0\rangle_{D} \langle 0|_{D} + |1\rangle_{A} \langle 1|_{A} \otimes |1\rangle_{D} \langle 1|_{D}).
$$

One can also verify that the square root of it equates to

$$
\rho_{A,D}^{-\frac{1}{2}} = \frac{1}{2}(|0\rangle_{A} \langle 0|_{A} \otimes |0\rangle_{D} \langle 0|_{D} + |0\rangle_{A} \langle 0|_{A} \otimes |1\rangle_{D} \langle 1|_{D} + |1\rangle_{A} \langle 1|_{A} \otimes |0\rangle_{D} \langle 0|_{D} + |1\rangle_{A} \langle 1|_{A} \otimes |1\rangle_{D} \langle 1|_{D}).
$$

Furthermore we look at the reduced density operator of $A$ in (106) which gives

$$
\rho_A = \text{Tr}_D(\rho_{A,D}) = \frac{1}{2}(|0\rangle_{A} \langle 0|_{A} + |1\rangle_{A} \langle 1|_{A}),
$$

and moreover

$$
\rho_A^{-\frac{1}{2}} = \sqrt{2} |0\rangle_{A} \langle 0|_{A} + \sqrt{2} |1\rangle_{A} \langle 1|_{A}.
$$

Using (108) and (110) we construct the temporal Petz recovery channel (96) to compute the relevant terms in (102)
for this DDoS attack case:

$$
T_P(|0\rangle_{A} \langle 0|_{A}) = \rho_{A,D}^{-\frac{1}{2}}(|0\rangle_{A} \langle 0|_{A})\rho_{A,D}^{-\frac{1}{2}} \otimes \mathbb{I}_D \rho_{A,D}^{-\frac{1}{2}} = \frac{1}{2} (|00\rangle_{AD} \langle 00|_{AD} + \frac{1}{2} |01\rangle_{AD} \langle 01|_{AD})
$$

$$
T_P(|1\rangle_{A} \langle 0|_{A}) = \rho_{A,D}^{-\frac{1}{2}}(|1\rangle_{A} \langle 1|_{A})\rho_{A,D}^{-\frac{1}{2}} \otimes \mathbb{I}_D \rho_{A,D}^{-\frac{1}{2}} = \frac{1}{2} (|10\rangle_{AD} \langle 10|_{AD} + \frac{1}{2} |11\rangle_{AD} \langle 11|_{AD})
$$

$$
T_P(|1\rangle_{A} \langle 1|_{A}) = \rho_{A,D}^{-\frac{1}{2}}(|1\rangle_{A} \langle 1|_{A})\rho_{A,D}^{-\frac{1}{2}} \otimes \mathbb{I}_D \rho_{A,D}^{-\frac{1}{2}} = \frac{1}{2} (|10\rangle_{AD} \langle 10|_{AD} + \frac{1}{2} |11\rangle_{AD} \langle 11|_{AD})
$$
Thus the output of the temporal Petz recovery channel (102) equates to

\[
\mathbb{I}_B \otimes \mathcal{T}_P^{(n-1)\tau} (\rho_{B,A}^{(n-1)\tau,(n-1)\tau})
\]

\[
= \frac{1}{2} \left( |0\rangle_B \langle 0|_B \otimes \left( \frac{1}{2} |00\rangle_{AD} \langle 00|_{AD} + \frac{1}{2} |01\rangle_{AD} \langle 01|_{AD} \right) + |0\rangle_B \langle 0|_B \otimes \left( \frac{1}{2} |00\rangle_{AD} \langle 10|_{AD} + \frac{1}{2} |01\rangle_{AD} \langle 11|_{AD} \right) 
\]

\[
+ |1\rangle_B \langle 0|_B \otimes \left( \frac{1}{2} |10\rangle_{AD} \langle 00|_{AD} + \frac{1}{2} |11\rangle_{AD} \langle 01|_{AD} \right) + |1\rangle_B \langle 0|_B \otimes \left( \frac{1}{2} |10\rangle_{AD} \langle 10|_{AD} + \frac{1}{2} |11\rangle_{AD} \langle 11|_{AD} \right) \right)
\]

\[
= \frac{1}{4} \left( |000\rangle_{BAD} \langle 000|_{BAD} + |001\rangle_{BAD} \langle 001|_{BAD} + |000\rangle_{BAD} \langle 001|_{BAD} + |000\rangle_{BAD} \langle 001|_{BAD} + |100\rangle_{BAD} \langle 011|_{BAD} + |001\rangle_{BAD} \langle 111|_{BAD} 
\]

\[
+ |110\rangle_{BAD} \langle 000|_{BAD} + |111\rangle_{BAD} \langle 001|_{BAD} + |110\rangle_{BAD} \langle 110|_{BAD} + |111\rangle_{BAD} \langle 111|_{BAD} \right).
\]

Taking the partial trace over D and collecting the terms (and putting the superscripts back in the last line) results in

\[
\rho_{AB}^{(n-1)\tau,(n-1)\tau} = \frac{1}{2} \left( |00\rangle_{AB} \langle 00|_{AB} + |01\rangle_{AB} \langle 11|_{AB} + |11\rangle_{AB} \langle 00|_{AB} + |11\rangle_{AB} \langle 11|_{AB} \right)
\]

\[
= \left( \frac{1}{\sqrt{2}} |00\rangle_{AB} + \frac{1}{\sqrt{2}} |11\rangle_{AB} \right) \left( \frac{1}{\sqrt{2}} |00\rangle_{AB} + \frac{1}{\sqrt{2}} |11\rangle_{AB} \right)
\]

\[
= \beta_{00}^{(n-1)\tau,(n-1)\tau} \beta_{00}^{(n-1)\tau,(n-1)\tau}. \tag{120}
\]

This is the output state of qubits AB from the temporal Petz recovery channel at the detection node at \( t = (n - 1)\tau \). The conditional entropy for this output at \( t = (n - 1)\tau \) results in

\[
S(\rho_A^{(n-1)\tau} | \rho_B^{(n-1)\tau}) = -1. \tag{123}
\]

which is the same value of the input (101).

Next we consider the non-DDoS case meaning that the service is operating normally in the future (87). The density operator for AD in this case is

\[
\rho_{AD} = \beta_{xy}^{(n-1)\tau,(n-1)\tau} \beta_{xy}^{(n-1)\tau,(n-1)\tau}, \tag{124}
\]

which equates to (where we drop the superscripts again for presentation clarity)

\[
\rho_{AD} = \frac{1}{2} ( |0\rangle_A \langle 0|_A |0\rangle_A \langle 0|_A + ( - 1)^x |0\rangle_A \langle 1|_A |1\rangle_A \langle 0|_A + ( - 1)^x |1\rangle_A \langle 1|_A |0\rangle_A \langle 0|_A + |1\rangle_A \langle 0|_A |0\rangle_A \langle 1|_A |1\rangle_A \langle 0|_A |1\rangle_A \langle A,D). \tag{125}
\]

One can verify that

\[
\rho_{A,D}^{\frac{1}{2}} = \frac{1}{2} ( |0\rangle_A \langle 0|_A |0\rangle_A \langle 0|_A + ( - 1)^x |0\rangle_A \langle 1|_A |1\rangle_A \langle 0|_A + ( - 1)^x |1\rangle_A \langle 1|_A |0\rangle_A \langle 0|_A + |1\rangle_A \langle 0|_A |0\rangle_A \langle 1|_A |1\rangle_A \langle 0|_A |1\rangle_A \langle A,D). \tag{126}
\]

Looking at the reduced density operator of A in (125) gives

\[
\rho_A = \text{Tr}_D (\rho_{A,D}) = \frac{1}{2} ( |0\rangle_A \langle 0|_A + |1\rangle_A \langle 1|_A), \tag{127}
\]

and moreover

\[
\rho_A^{-\frac{1}{2}} = \sqrt{2} |0\rangle_A \langle 0|_A + \sqrt{2} |1\rangle_A \langle 1|_A. \tag{128}
\]
Using these quantities we can calculate the relevant terms in (102) for this non-attack case. The final results are:

\[ T_P(|0\rangle_A \langle 0|_A) = \frac{\gamma}{2} A,D (\rho_{A,D}^{-\frac{1}{2}} |0\rangle_A \langle 0|_A \rho_{A,D}^{-\frac{1}{2}} \otimes I_D) \rho_{A,D}^{-\frac{1}{2}} \]

\[ = \frac{1}{2} |0\rangle_A \langle 0|_A \rho_{A,D}^{\frac{1}{2}} |0\rangle_A \langle 0|_A \rho_{A,D}^{\frac{1}{2}} \otimes I_D) \rho_{A,D}^{\frac{1}{2}} \]

\[ + \frac{1}{2} |1\rangle_A \langle 1|_A \rho_{A,D}^{\frac{1}{2}} |1\rangle_A \langle 1|_A \rho_{A,D}^{\frac{1}{2}} \otimes I_D) \rho_{A,D}^{\frac{1}{2}} \]

\[ T_P(|1\rangle_A \langle 1|_A) = \frac{1}{2} (\rho_{A,D}^{-\frac{1}{2}} |0\rangle_A \langle 0|_A \rho_{A,D}^{-\frac{1}{2}} \otimes I_D) \rho_{A,D}^{-\frac{1}{2}} \]

\[ = 0 \]

\[ T_P(|1\rangle_A \langle 1|_A) = \frac{1}{2} (\rho_{A,D}^{-\frac{1}{2}} |1\rangle_A \langle 1|_A \rho_{A,D}^{-\frac{1}{2}} \otimes I_D) \rho_{A,D}^{-\frac{1}{2}} \]

\[ = 0 \]

Then the output of the temporal Petz recovery channel (102) equates to

\[ \mathbb{I}_B \otimes T_P^{(n-1)\tau} \rho^{(n-1)\tau,(n-1)\tau}_{B,A} \]

\[ = \frac{1}{2} \left( |0\rangle_B \langle 0| \otimes \frac{1}{2} |0\rangle_A \langle 0|_A \rho_{A,D}^{\frac{1}{2}} |0\rangle_A \langle 0|_A \rho_{A,D}^{\frac{1}{2}} \otimes I_D) \rho_{A,D}^{\frac{1}{2}} \right) \]

\[ + \frac{1}{2} |1\rangle_B \langle 1| \otimes \frac{1}{2} |0\rangle_A \langle 0|_A \rho_{A,D}^{\frac{1}{2}} |1\rangle_A \langle 1|_A \rho_{A,D}^{\frac{1}{2}} \otimes I_D) \rho_{A,D}^{\frac{1}{2}} \]

\[ = \frac{1}{4} \left( |00\rangle_B \langle 00| \otimes |0\rangle_A \langle 0|_A \rho_{A,D}^{\frac{1}{2}} |0\rangle_A \langle 0|_A \rho_{A,D}^{\frac{1}{2}} \otimes I_D) \rho_{A,D}^{\frac{1}{2}} \right) \]

\[ + \frac{1}{4} |10\rangle_B \langle 10| \otimes |0\rangle_A \langle 0|_A \rho_{A,D}^{\frac{1}{2}} |1\rangle_A \langle 1|_A \rho_{A,D}^{\frac{1}{2}} \otimes I_D) \rho_{A,D}^{\frac{1}{2}} \]

Taking the partial trace over \( D \) and simplifying the terms gives

\[ \rho^{(n-1)\tau,(n-1)\tau}_{A,B} = \frac{1}{2} \left( |0\rangle_B \langle 0| \otimes |0\rangle_A \langle 0|_A + |1\rangle_B \langle 1| \otimes |0\rangle_A \langle 0|_A + |0\rangle_B \langle 0| \otimes |1\rangle_A \langle 1|_A + |1\rangle_B \langle 1| \otimes |1\rangle_A \langle 1|_A \right) \]

For the non-attack case, this is the output state of qubits \( AB \) from the temporal Petz recovery channel at \( t = (n - 1)\tau \). The associated conditional entropy is

\[ S(\rho^{(n-1)\tau}_{A}, \rho^{(n-1)\tau}_{B}) = 1, \]

which is a different value to the attack case (123).

To summarize, if a DDoS attack event takes place in the future at \( t = n\tau \), then the conditional entropy of the output \( AB \) at \( t = (n - 1)\tau \) will be \(-1\). However if the service was operating as normal in the future, then the conditional entropy of the output \( AB \) at \( t = (n - 1)\tau \) would take value 1. Thus, we have designed an entropic DDoS detection system that can detect an attack event before the event even occurs.

### B. Relationship to Turnstile

The Turnstile transmits inverted information from the future to the past. Conversely, the temporal Petz recovery channel in the past receives that inverted information from the future. We want to express the recovery channel

\[ T_P^{(n-1)\tau} [X_{A}^{(n-1)\tau}] = (\rho_{A,D}^{(n-1)\tau),(n+1)\tau})^{\frac{1}{2}} (\rho_{A}^{(n-1)\tau})^{\frac{1}{2}} X_{A}^{(n-1)\tau} (\rho_{A,D}^{(n-1)\tau})^{\frac{1}{2}} \otimes I_D) (\rho_{A,D}^{(n-1)\tau),(n+1)\tau})^{\frac{1}{2}}, \]

in terms of the Turnstile. To do this, let the channel \( \mathcal{N} \) capture the Turnstile process which transmits the inverted information to the past

\[ \mathcal{N}(\rho_{A,D}^{(n-1)\tau,(n+1)\tau}) = \rho_{A}^{(n-1)\tau}. \]
We can then express the temporal Petz recovery channel (145) as a composition of three channels
\[
(\cdot) \rightarrow \left( \rho^{(n-1)\tau}_A \right)^{-\frac{1}{2}} \cdot \left( \rho^{(n-1)\tau}_A \right)^{-\frac{1}{2}} = [\mathcal{N}(\rho^{(n-1)\tau,(n+1)\tau}_{A,D})]^{-\frac{1}{2}} \cdot \left( \mathcal{N}(\rho^{(n-1)\tau,(n+1)\tau}_{A,D}) \right)^{-\frac{1}{2}}, \tag{147}
\]
\[
(\cdot) \rightarrow (\cdot) \otimes I_D = \mathcal{N}^\dagger (\cdot), \tag{148}
\]
\[
(\cdot) \rightarrow \left( \rho^{(n-1)\tau,(n+1)\tau}_{A,D} \right)^{\frac{1}{2}} \cdot \left( \rho^{(n-1)\tau,(n+1)\tau}_{A,D} \right)^{\frac{1}{2}} \tag{149}
\]
where $\mathcal{N}^\dagger$ is the Hilbert-Schmidt adjoint of the Turnstile channel $\mathcal{N}$.

C. Technological design

Recently a quantum algorithm was created to implement the Petz recovery channel [20]. We apply this into our setting to design a temporal Petz recovery channel that may be realized using quantum computing circuits.

We assume we have some access to the inverted quantum information $\rho^{(n-1)\tau,(n+1)\tau}_{A,D}$ and $\mathcal{N}(\rho^{(n-1)\tau,(n+1)\tau}_{A,D})$. We also assume we have the following circuit $U_{EAD\rightarrow EA}^N$ which is a unitary extension of channel (146). The exact means of obtaining these quantities is unknown at this point of exploration.

Let
\[
\Gamma_{E\tilde{E}} \equiv |\Gamma\rangle \langle \Gamma|_{E\tilde{E}}, \tag{150}
\]
de note an operator proportional to maximally entangled state $E$ and reference system $\tilde{E}$ where
\[
|\Gamma\rangle_{E\tilde{E}} \equiv \sum_{i=0}^{d_E-1} |i\rangle_E |i\rangle_{\tilde{E}}, \tag{151}
\]
with $d_E$ denoting the dimension of system $E$.

The quantum algorithm for this temporal Petz recovery channel can be written as
\[
V^T_{A\rightarrow E\tilde{D}} \equiv (|0\rangle_{E'} \otimes \mathbb{I}_{EAD})(\rho^{(n-1)\tau,(n+1)\tau}_{A,D})^{\frac{1}{2}} (U_{EAD\rightarrow EA}^N)^{\dagger} (\mathcal{N}(\rho^{(n-1)\tau,(n+1)\tau}_{A,D}))^{-\frac{1}{2}} (|\Gamma\rangle_{E\tilde{E}} \otimes \mathbb{I}_A). \tag{152}
\]
The first step of the algorithm as shown on the right is to tensor the system into this maximally entangled state. Then we apply maps (147), (148), and (149) sequentially with the first and third steps employing the quantum singular value transformation. The last step is to collapse into the environment state, with a trace over the environment $\tilde{E}$ to implement channel $T$.
[15] R.E. Spinney et al., “Transfer entropy in physical systems and the arrow of time”, Physical Review E, 94, 022135 (2016).
[16] S. Ito, “Backward transfer entropy: Informational measure for detecting hidden Markov models and its interpretations in thermodynamics, gambling and causality”, Scientific reports, 6, 1–10 (2016).
[17] O. Fawzi, and R. Renner, “Quantum conditional mutual information and approximate Markov chains”, Communications in Mathematical Physics, 340, 575–611 (2015).
[18] P. Hayden et al., “Structure of states which satisfy strong subadditivity of quantum entropy with equality”, Communications in Mathematical Physics, 246, 359–374 (2004).
[19] D. Petz, “Sufficiency of channels over von Neumann algebras”, The Quarterly Journal of Mathematics, 39, 97–108 (1988).
[20] A. Gilyén et al., “Quantum algorithm for Petz recovery channels and pretty good measurements”, arXiv:2006.16924 [quant-ph] (2020).