The Geometry of \((t \mod q)\)-arcs

Sascha Kurz\(^1\), Ivan Landjev\(^2\), Francesco Pavese\(^3\), and Assia Rousseva\(^4\)

\(^1\)Mathematisches Institut, Universität Bayreuth, D-95440 Bayreuth, Germany, sascha.kurz@uni-bayreuth.de
\(^2\)New Bulgarian University, 21 Montevideo str., 1618 Sofia, Bulgaria and Bulgarian Academy of Sciences, Institute of Mathematics and Informatics, 8 Acad G. Bonchev str., 1113 Sofia, Bulgaria, i.landjev@nbu.bg
\(^3\)Dipartimento di Meccanica, Matematica e Management, Politecnico di Bari, Via Orabona 4, 70125, Bari, Italy, francesco.pavese@poliba.it
\(^4\)Faculty of Mathematics and Informatics, Sofia University, 5 J. Bourchier blvd., 1164 Sofia, Bulgaria, assia@fmi.uni-sofia.bg

Abstract

In this paper, we give a geometric construction of the three strong non-lifted \((3 \mod 5)\)-arcs in \(PG(3,5)\) of respective sizes 128, 143, and 168, and construct an infinite family of non-lifted, strong \((t \mod q)\)-arcs in \(PG(r,q)\) with \(t = (q+1)/2\) for all \(r \geq 3\) and all odd prime powers \(q\).

Keywords \((t \mod q)\)-arcs linear codes quadrics caps quasidivisible arcs sets of type \((m,n)\)

Mathematics Subject Classification (2000) 51E22 51E21 94B05

1 Introduction

The strong \((t \mod q)\)-arcs were introduced and investigated in \([2, 4, 6, 7]\) in connection with the extendability problem for Griesmer arcs. This problem is related in turn to the problem of the existence and extendability of arcs associated with Griesmer codes. In \([2]\) the classification of the strong \((3 \mod 5)\)-arcs was used to rule out the existence of the hypothetical \([104,4,82]_5\)-code, one of the four undecided cases for codes of dimension 4 over \(\mathbb{F}_5\). It turns out that apart from the many strong \((3 \mod 5)\)-arcs obtained from the canonical lifting construction, there exist three non-lifted strong \((3 \mod 5)\)-arcs of respective sizes 128, 143, and 168. This is a counterexample to the conjectured impossibility of strong \((3 \mod 5)\)-arcs in geometries over \(\mathbb{F}_5\) in dimensions larger than 2. The three arcs are constructed by a computer search, but display regularities which suggest a nice geometric structure.

In this paper, we give a geometric, computer-free construction of the three non-lifted strong \((3 \mod 5)\)-arcs in \(PG(3,5)\). Two of them are related to the non-degenerate quadrics of \(PG(3,5)\). Their construction can be generalized further to larger fields and larger dimensions.

2 Preliminaries

We define an arc in \(PG(r,q)\) as a mapping from the point set \(P\) of the geometry to the non-negative integers: \(K : P \to \mathbb{N}_0\). An arc \(K\) in \(PG(r,q)\) is called a \((t \mod q)\)-arc if \(K(L) \equiv t \pmod{q}\) for every
line $L$. It is immediate that $\mathcal{K}(S) \equiv t \mod q$ for every subspace $S$ with $\dim S \geq 1$. Increasing the multiplicity of an arbitrary point by $q$ preserves the property of being a $(t \mod q)$-arc. So, we can assume that the point multiplicities are integers contained in the interval $[0, q - 1]$. If the maximal point multiplicity is at most $t$ we call $\mathcal{K}$ a strong $(t \mod q)$-arc.

The extendability of the so-called $t$-quasidivisible arcs is related to structure properties of $(t \mod q)$-arcs. In particular, an $(n, s)$-arc $\mathcal{K}$ in $\text{PG}(r, q)$ with spectrum $(a_i)$ is called $t$-quasidivisible with divisor $\Delta$ if $s \equiv n + t \mod \Delta$ and $a_i = 0$ for all $i \neq n, n + 1, \ldots, n + t \mod \Delta$. It is quite common in coding theory that hypothetical Griesmer codes are associated with arcs that turn out to be $t$-quasidivisible with divisor $q$ for some $t$. The extendability of $t$-quasidivisible arcs is related to the structure of particular strong $(t \mod q)$-arcs associated with them.

Let $\mathcal{K}$ be an arc in $\text{PG}(r, q)$ and let $\sigma : \mathbb{N}_0 \to \mathbb{Q}$ be a function satisfying $\sigma(\mathcal{K}(H)) \in \mathbb{N}_0$ for every hyperplane $H$ in $\mathcal{H}$, where $\mathcal{H}$ is the set of all hyperplanes in $\text{PG}(r, q)$. The arc $\mathcal{K}_\sigma : \mathcal{H} \to \mathbb{N}_0$, $H \to \sigma(\mathcal{K}(H))$ is called the $\sigma$-dual of $\mathcal{K}$. For a $t$-quasidivisible arc with divisor $q$, we consider the $\sigma$-dual arc obtained for $\mathcal{K}_\sigma(H) = n + t - \mathcal{K}(H) \mod q$. It turns out that with this $\sigma$ the $\sigma$-dual to a $t$-quasidivisible arc $\mathcal{K}$ is a strong $(t \mod q)$-arc. Moreover, if $\mathcal{K}_\sigma$ contains a hyperplane in its support then $\mathcal{K}$ is extendable [4, 7].

There exist several straightforward constructions of $(t \mod q)$-arcs [4, 6, 7]. The first is the so-called sum-of-arcs construction.

**Theorem 1.** Let $\mathcal{K}$ and $\mathcal{K}'$ be a $(t_1 \mod q)$- and a $(t_2 \mod q)$-arc in $\text{PG}(r, q)$, respectively. Then $\mathcal{K} + \mathcal{K}'$ is a $(t \mod q)$-arc with $t \equiv t_1 + t_2 \mod q$. Similarly, $\alpha \mathcal{K}$, where $\alpha \in \{0, \ldots, p-1\}$ and $p$ is the characteristic of $\mathbb{F}_q$, is a $(t \mod q)$-arc with $t \equiv \alpha t_1 \mod q$.

For the special case of $t = 0$, and $q = p$ we have that the sum of two $(0 \mod p)$-arcs and the scalar multiple of a $(0 \mod p)$-arc are again $(0 \mod p)$-arcs. Hence the set of all $(0 \mod p)$-arcs is a vector space over $\mathbb{F}_p$, cf. [6].

The second construction is the so-called lifting construction, see [6, p. 230].

**Theorem 2.** Let $\mathcal{K}_0$ be a (strong) $(t \mod q)$-arc in a projective $s$-space $\Sigma$ of $\text{PG}(r, q)$, where $1 \leq s < r$. For a fixed projective $(r - s - 1)$-space $\Gamma$ of $\text{PG}(r, q)$, disjoint from $\Sigma$, let $\mathcal{K}$ be the arc in $\text{PG}(r, q)$ defined as follows:

- for each point $P$ of $\Gamma$, set $\mathcal{K}(P) = t$;
- for each point $Q \in \text{PG}(r, q) \setminus \Gamma$, set $\mathcal{K}(Q) = \mathcal{K}_0(R)$, where $R = \langle \Gamma, Q \rangle \cap \Sigma$.

Then $\mathcal{K}$ is a (strong) $(t \mod q)$-arc in $\text{PG}(r, q)$ of cardinality $q^{r-s} \cdot |\mathcal{K}_0| + t \frac{q^{r-s} - 1}{q-1}$.

Arcs obtained by the lifting construction are called lifted arcs. If $\Sigma$ is a point, then we speak of a lifting point. The iterative application of the lifting constructions gives the more general version stated above. In the other direction, in [6, Lemma 1] it was shown that the set of all lifting points forms a subspace.

The classification of strong $(t \mod q)$ arcs in $\text{PG}(2, q)$ is equivalent to that of certain plane blocking sets [5].

**Theorem 3.** A strong $(t \mod q)$-arc $\mathcal{K}$ in $\text{PG}(2, q)$ of cardinality $mq + t$ exists if and only if there exists an $((m - t)q + m, \geq m - t)$-blocking set $\mathcal{B}$ with line multiplicities contained in the set $\{m - t, m - t + 1, \ldots, m\}$.
The condition that the multiplicity of each point is at most $t$ turns out to be very strong. For $t = 0$, we have that the only strong $(0 \text{ mod } q)$-arc is the trivial zero-arc. For $t = 1$ the strong $(1 \text{ mod } q)$-arcs are the hyperplanes. For $t = 2$ all strong $(2 \text{ mod } q)$ arcs in $PG(r, q)$, for $r \geq 3, q \geq 5$, turn out to be lifted [6]. In $PG(2, q)$, all $(2 \text{ mod } q)$-arcs are also known (cf. [2, Lemma 3.7]). Apart from one sporadic example, all such arcs are again lifted. It was conjectured in [5] that all strong $(3 \text{ mod } 5)$-arcs in $PG(r, 5), r \geq 3$, are lifted. The computer classification reported in [2] shows that this conjecture is wrong: there exist $(3 \text{ mod } 5)$-arcs of respective sizes 128, 143, and 168 that are not lifted. In the next sections we give a geometric (compute-free) description of these arcs and define an infinite class of strong $(t \text{ mod } q)$-arcs in $PG(r, q), r \geq 3$, that are not lifted.

3 The arc of size 128

We shall need the classification of all strong $(3 \text{ mod } 5)$-arcs in $PG(2, 5)$ of sizes 18, 23, 28 and 33. It is obtained easily from Theorem 3 and can be found in [2, 6].

**Theorem 4.** Let $K$ be a strong $(3 \text{ mod } 5)$-arc in $PG(2, 5)$. Let $\lambda_i, i = 0, 1, 2, 3$, denote the number of $i$ points of $K$.

(a) If $|K| = 18$ then $K$ is the sum of three lines.

(b) If $|K| = 23$ then it has $\lambda_3 = 3, \lambda_2 = 4, \lambda_1 = 6$. The four 2-points form a quadrangle, the three 3-points are the diagonal points of the quadrangle, and the 1-points are the intersections of the diagonals with the sides of the quadrangle.

(c) If $|K| = 28$ then it has $\lambda_3 = 6, \lambda_1 = 10$. The 3-points form an oval, and the 1-points are the internal points to this oval.

(d) There exist ten non-isomorphic arcs with $|K| = 33$. These are:

   (i) the duals of the complements of the seven $(10, 3)$-arcs in $PG(2, 5)$ (cf. [3]);

   (ii) the dual of the multiset which is complement of the $(11, 3)$-arc with four external lines plus one point which is not on a 6-line ($\lambda_3 = 6, \lambda_2 = 5, \lambda_1 = 5$);

   (iii) the dual of a blocking set in which one double point forms an oval with five of the 0-points; the tangent to the oval in the 2-point is a 3-line ($\lambda_3 = 6, \lambda_2 = 5, \lambda_1 = 5$);

   (iv) the modulo 5 sum of three non-concurrent lines: two of them are lines of 3-points and one is a line of 2-points ($\lambda_3 = 8, \lambda_2 = 4, \lambda_1 = 1$).

Let us note that one of the strong $(3 \text{ mod } 5)$-arcs in case (d(i)) is obtained by taking as 3-points the points of an oval and as 1-points the external points to the oval.

Consider a $(3 \text{ mod } 5)$-arc $K$ in $PG(3, 5)$ which is of multiplicity 128. Let $\varphi$ be a projection from an arbitrary 0-point $P$ to a plane $\pi$ not incident with $P$:

$$\varphi: \begin{cases} \mathcal{P} \setminus \{P\} & \rightarrow \pi \\ Q & \rightarrow \pi \cap \langle P, Q \rangle. \end{cases}$$

(1)

Here $\mathcal{P}$ is again the set of points of $PG(3, 5)$. Note that $\varphi$ maps the lines through $P$ into points from $\pi$, and the planes through $P$ into lines in $\pi$. For every set of points $\mathcal{F} \subset \pi$, define the induced
It is clear that $P$ is incident with 3- and 8-lines, only. If there exists a 13-line $L$ through $P$ then all planes through $L$ have multiplicity at least 33 (Theorem 4) and $|K| \geq 6 \cdot 33 - 5 \cdot 13 = 133$, a contradiction.

An 8-line through $P$ is either of type $(3,3,1,1,0,0)$ (type $(\alpha)$), or of type $(3,2,2,1,0,0)$ (type $(\beta)$). Other types for an 8-line are impossible by the same counting argument as above: a plane through such a line has to be of multiplicity at least 33 (18-planes are impossible since $P$ is a 0-point), and we get a contradiction by the same counting argument as above. A 3-line through $P$ is of type $(\gamma_1)$ $(3,0,0,0,0,0)$, $(\gamma_2)$ $(2,1,0,0,0,0)$, or $(\gamma_3)$ $(1,1,1,0,0,0)$. A point in the projection plane is said to be of type $(\alpha)$, $(\beta)$, or $(\gamma_i)$ if it is the image of a line of the same type. Let us note that type $(\alpha)$ and $(\beta)$ are the same as types $(B_2)$ and $(B_3)$ from [2]; similarly type $(\gamma_i)$ coincides with type $(A_i)$, $i = 1, 2, 3$.

By Theorem 4, if a line in the projection plane has one 8-point then it contains:
- one point of type $(\alpha)$, one point of type $(\gamma_1)$, and four points of type $(\gamma_2)$, or else
- one point of type $(\beta)$, two points of type $(\gamma_1)$, two points of type $(\gamma_2)$ and one point of type $(\gamma_3)$.

We are going to prove that if $K$ is a strong $(3 \mod 5)$-arc in $PG(3,5)$ of cardinality 128 then the induced arc $\mathcal{K}^\varphi$ in $PG(2,5)$ is unique (up to isomorphism). It consists of seven 8-points and 24 3-points. Three of the 8-points are of type $(\alpha)$, and four are of type $(\beta)$. The 3-points are: six of type $(\gamma_1)$, twelve of type $(\gamma_2)$, and six of type $(\gamma_3)$. The points of type $(\beta)$ form a quadrangle, and the points of type $(\alpha)$ are the diagonal points. The intersections of the lines defined by the diagonal points with the sides of the quadrangle are points of type $(\gamma_3)$; the six points on the lines defined by the diagonal points that are not on sides of the quadrangle are of type $(\gamma_1)$; all the remaining 3-points are of type $(\gamma_2)$. The induced arc $\mathcal{K}^\varphi$ is presented on the picture below.

**Lemma 5.** Let $K$ be a strong $(3 \mod 5)$-arc in $PG(3,5)$ of cardinality 128. Let $\varphi$ be the projection from an arbitrary 0-point in $PG(3,5)$ into a plane disjoint from that point. Then the arc $\mathcal{K}^\varphi$ is unique up to isomorphism and has the structure described above.
Proof. We have seen that 0-points are incident only with lines of multiplicity 3 and 8. Hence $K^c$ has seven 8-points and twenty-four 3-points. Assume that six of the 8-points are collinear. Clearly, every 8-point is of type $(\alpha)$ since it is on a line containing two 8-points (and hence the image of a 28-plane). Every other point in the projection plane is also on a line containing two 8-points; hence all 3-points in the plane are of type $(\gamma_1)$ or $(\gamma_3)$. But now a line with one 8-point cannot have points of type $(\gamma_2)$, which is a contradiction with the structure of the $(3 \mod 5)$-arc of size 23.

Assume that five of the 8-points are collinear. Let $L$ be the line that contains them. If the two 8-points off $L$ define a line meeting $L$ in a 3-point, the proof is completed as above. Otherwise, the points off $L$ are on four lines containing two 8-points. Now it is easily checked that there exists a line with exactly one 8-point which has at least four 3-points that are not of type $(\gamma_2)$. This is a contradiction with the structure of the $(3 \mod 5)$-arc of size 23.

A similar argument rules out the possibility of four collinear 8-points. In all cases these have to be points of type $(\alpha)$. So are the remaining three 8-points. Now for all possible configurations of these seven points we get a 23-line without enough points of type $(\gamma_2)$.

We are going to consider in full detail the case when at most three 8-points in the projection plane are collinear. Assume there exists an oval of 8-points, $X_1, \ldots, X_6$, say, and let $Y$ be the seventh 8-point. All 8-points are of type $(\alpha)$ and let $YX_1X_2$ be a secant to the oval through $Y$. The lines $X_1X_j$, $j = 3, 4, 5, 6$, are images of planes without 2-points. Now an external line to the oval through $Y$ is a 23-line and has at most one point of type $(\gamma_2)$, a contradiction. In a similar way, we rule out the case where there exist five 8-points no three of which are collinear. We have to consider the different possibilities for the line defined by the remaining two 8-points: secant, tangent, or external line to the oval formed by the former five points and one additional point which has to be a 3-point.

We have shown so far that there are at most three collinear 8-points. It is also clear that there exist at least two lines that contain three 8-points. We consider the case where these lines meet in a 3-point. Denote the 8-points by $X_i, Y_i$, $i = 1, 2, 3$, and $Z$. We also assume that $X_1, X_2, X_3$ are collinear and so are $Y_1, Y_2, Y_3$. Each of the lines $ZX_i$, $i = 1, 2, 3$, also contains three 8-points; otherwise there exist five 8-points no three of which are collinear. Without loss of generality, the triples $Z, X_i, Y_i$, $i = 1, 2, 3$, are collinear. Now it is clear that all the points $X_i, Y_i$ are of type $(\alpha)$. Moreover, neither of the lines $X_iY_j$, $i \neq j$, has 3-points of type $(\gamma_2)$. Now if we consider a line through $X_3$ that does not have other 8-points it should contain four points of type $(\gamma_2)$. On the other hand, it intersects $X_1Y_2$ and $X_1Y_3$ in points which are not of this type which gives a contradiction.

Now we are left with only one possibility for the 8-points subject to the conditions: (i) each line contains at most three 8-points, (ii) lines incident with three 8-points meet in an 8-point, (iii) every 5-tuple of 8-points contains a collinear triple. The 8-points are the vertices of a quadrangle plus the three diagonal points. Furthermore, the diagonal points have to be of type $(\alpha)$ while the vertices of the quadrangle are forced to be of type $(\beta)$. This is due to the fact that through each of the vertices of the quadrangle there is a line with a single 8-point which meets the three lines defined by the diagonal points of type $(\alpha)$ in three different 3-points that are not of type $(\gamma_2)$ (since a 28-plane does not have 2-points). Thus we get the picture below.
The fact that a 23-line through a point of type \((\alpha)\) contains four points of type \((\gamma_2)\) and one point of type \((\gamma_1)\) identifies the six points of type \((\gamma_1)\).

Furthermore, a line with two points of type \((\alpha)\) must contain also two points of type \((\gamma_1)\) and two points of type \((\gamma_3)\). This identifies the six 3-points of type \((\gamma_1)\). The remaining 3-points are all of type \((\gamma_2)\). This implies the suggested structure.

Lemma 5 implies that given a nonlifted, strong \((3 \mod 5)\)-arc \(\mathcal{K}\) of cardinality 128, every 0-point is incident with
- three 8-lines of type \((3, 3, 1, 1, 0, 0)\),
- four 8-lines of type \((3, 2, 2, 1, 0, 0)\),
- six 3-lines of type \((3, 0, 0, 0, 0, 0)\),
- twelve 3-lines of type \((2, 1, 0, 0, 0, 0)\),
- six 3-lines of type \((1, 1, 1, 0, 0, 0)\)

Now this implies that
- \#(3-points) = \(3 \cdot 2 + 4 \cdot 1 + 6 \cdot 1 = 16\),
- \#(2-points) = \(4 \cdot 2 + 12 \cdot 1 = 20\),
- \#(1-points) = \(3 \cdot 2 + 4 \cdot 1 + 12 \cdot 1 + 6 \cdot 3 = 40\),
- \#(0-points) = \(1 + 3 \cdot 1 + 4 \cdot 1 + 6 \cdot 4 + 12 \cdot 3 + 2 \cdot 6 = 80\).

Furthermore, each 0-point is incident with six 33-planes, three 28-planes eighteen 23-planes and four 18-planes. Moreover the number of zeros in a 33-plane is 12, in a 28-plane – 15, in a 23-plane
This makes it possible to compute the spectrum of $K$. We have

\begin{align*}
a_{33} &= \frac{80 \cdot 6}{12} = 40, \\
a_{28} &= \frac{80 \cdot 3}{15} = 16, \\
a_{23} &= \frac{80 \cdot 18}{18} = 80, \\
a_{18} &= \frac{80 \cdot 4}{16} = 20.
\end{align*}

Furthermore, every 33-, 28-, 23-plane in $K$ is unique up to isomorphism.

From the above considerations we can deduce that no three 2-points are collinear. In other words they form a 20-cap $C$. Moreover, this cap has spectrum: $a_6(C) = 40, a_4(C) = 80, a_3(C) = 20, a_0(C) = 16$. It is not extendable to the elliptic quadric; in such case it would have (at least 20) tangent planes. Thus, this cap is complete and isomorphic to one of the two caps $K_1$ and $K_2$ by Abatangelo, Korchamros and Larato [1]. It is not $K_2$ since it has a different spectrum (cf. [1]). Hence the 20-cap on the 2-points in $\text{PG}(3,5)$ is isomorphic to $K_1$.

Consider the complete cap $K_1$. The collineation group $G$ of $K_1$ is a semidirect product of an elementary abelian group of order 16 and a group isomorphic to $S_5$ [1]. Hence $|G| = 1920$. The action of $G$ on $\text{PG}(3,5)$ splits the point set of $\text{PG}(3,5)$ into four orbits, denoted by $O_1^P, \ldots, O_4^P$, and the set of lines into six orbits, denoted by $O_1^L, \ldots, O_6^L$. The respective sizes of these orbits are

\begin{align*}
|O_1^P| &= 40, |O_2^P| = 80, |O_3^P| = 20, |O_4^P| = 16; \\
|O_1^L| &= 160, |O_2^L| = 240, |O_3^L| = 30, |O_4^L| = 160, |O_5^L| = 120, |O_6^L| = 96.
\end{align*}

The corresponding point-by-line orbit matrix $A = (a_{ij})_{4 \times 6}$, where $a_{ij}$ is the number of the points from the $i$-th point orbit incident with any line from the $j$-th line orbit is the following

\[
A = \begin{pmatrix}
3 & 1 & 4 & 1 & 2 & 0 \\
3 & 4 & 0 & 2 & 2 & 5 \\
0 & 1 & 2 & 2 & 0 & 0 \\
0 & 0 & 0 & 1 & 2 & 1
\end{pmatrix}.
\]

Set $w = (w_1, w_2, w_3, w_4)$. We look for solutions of the equation $wA \equiv 3j \pmod{5}$, where $j$ is the all-one vector, subject to the conditions $w_i \leq 3$ for all $i = 1, 2, 3, 4$. The set of all solutions is given by

\[
\left\{ w = (w_1, w_2, w_3, w_4) \mid w_1 \{0, \ldots, 4\}, \quad w_2 \equiv 1 \pmod{5}, w_3 \equiv 4 - 2w_1 \pmod{5}, w_4 = 3 \right\}.
\]

There exist two solutions that satisfy $w_i \leq 3$: $w = (3, 3, 3, 3)$ and $w = (1, 0, 2, 3)$. The first one yields the trivial ($3 \pmod{5}$)-arc formed by three copies of the whole space. The second one gives the desired arc of size 128.

It should be noted that the weight vectors $(0, 3, 2, 4), (1, 2, 0, 4), (2, 1, 3, 4),$ and $(3, 0, 1, 4)$ yield strong ($4 \pmod{5}$)-arcs of cardinalities 344, 264, 284, and 204, respectively, that are not lifted.
4 Strong \(\left(\frac{q+1}{2}\right)\mod q\)-arcs from quadrics and the arcs of size 143 and 168

For an arbitrary odd prime power \(q\) and an integer \(r \geq 2\), let \(Q\) be a quadric of \(PG(r, q)\) and let \(F\) be the quadratic form defining \(Q\). This means that a point \(P(x_0, \ldots, x_r)\) of \(PG(r, q^2)\) belongs to \(Q\) whenever \(F(x_0, \ldots, x_r) = 0\). The points of \(PG(r, q)\) outside \(Q\) are partitioned into two point classes, say \(P_1\) and \(P_2\). Indeed, if \(P(x_0, \ldots, x_r)\) is a point of \(PG(r, q) \setminus Q\), then \(P\) belongs to \(P_1\) or \(P_2\), according as \(F(x_0, \ldots, x_r)\) is a non-square or a square in \(F_q\). Now we define the arcs \(K_1\) and \(K_2\) in the following way:

- \(K_1\): for a point \(P\) of \(PG(r, q)\) set
  \[
  K_1(P) = \begin{cases} 
  \frac{q+1}{2} & \text{if } P \in Q, \\
  1 & \text{if } P \in P_1, \\
  0 & \text{if } P \in P_2.
  \end{cases}
  \]

- \(K_2\): for a point \(P\) of \(PG(r, q)\) set
  \[
  K_2(P) = \begin{cases} 
  \frac{q+1}{2} & \text{if } P \in Q, \\
  0 & \text{if } P \in P_1, \\
  1 & \text{if } P \in P_2.
  \end{cases}
  \]

The following result is well-known.

**Proposition 6.** Let \(f(x) = ax^2 + bx + c\), where \(a, b, c, \in F_q\), \(a \neq 0\), \(q\) odd. If \(F_q = \{\alpha_0, \alpha_1, \ldots, \alpha_{q-1}\}\). Denote by \(S\) be the list of the following elements from \(F_q\):
\[
a, f(\alpha_0), f(\alpha_1), \ldots, f(\alpha_{q-1}).
\]

Then

(a) if \(f(x)\) has two distinct roots in \(F_q\) the list \(S\) contains two zeros, \((q-1)/2\) squares and \((q-1)/2\) non-squares;

(b) if \(f(x)\) has one double root in \(F_q\) then \(S\) contains a zero and \(q\) squares, or a zero and \(q\) non-squares;

(c) if \(f(x)\) is irreducible over \(F_q\) then \(S\) contains \((q+1)/2\) squares and \((q+1)/2\) non-squares.

**Theorem 7.** Let the \(K_1\) and \(K_2\) be the arcs defined in (2) and (3), respectively. Then \(K_i\) is a \(\left(\frac{q+1}{2}\right)\mod q\) arc of \(PG(r, q)\), \(i = 1, 2\). Moreover, if \(Q\) is non-degenerate, then both arcs are not lifted.

**Proof.** Let \(\ell\) be a line of \(PG(r, q)\), then \(Q \cap \ell\) is a quadric of \(\ell\). Then, from Proposition 6 it follows that
\[
K_i(\ell) = \begin{cases} 
  2 \cdot \frac{q+1}{2} + \frac{q-1}{2} & \text{if } |\ell \cap Q| = 2, \\
  \frac{q+1}{2} + q & \text{if } |\ell \cap Q| = 1 \text{ and } |\ell \cap P_1| = q, \\
  \frac{q+1}{2} & \text{if } |\ell \cap Q| = 1 \text{ and } |\ell \cap P_1| = 0, \\
  \frac{q+1}{2} & \text{if } |\ell \cap Q| = 0.
\end{cases}
\]
Therefore $K_i$ is a $\left(\frac{q+1}{2} \mod q\right)$ arc of $\text{PG}(r, q)$, $i = 1, 2$. If $Q$ is non-degenerate, then through every point of $\text{PG}(r, q)$ there exists a line $r$ that is secant to $Q$. By construction, the line $r$ has two $\frac{q+1}{2}$ points, $\frac{q+1}{2}$ 1-points and $\frac{q+1}{2}$ 0-points. Hence $K_i$ is not lifted. 

**Corollary 8.** If $r$ is odd, then

$$|K_i| = \begin{cases} 
\frac{q+1}{2} \cdot \frac{(q^r+1)(q^r+1)}{q-1} + \frac{q^r-q^r}{2} & \text{if } Q \text{ is elliptic,} \\
\frac{q+1}{2} \cdot \frac{(q^r+1)(q^r+1)}{q-1} + \frac{q^r-q^r}{2} & \text{if } Q \text{ is hyperbolic.}
\end{cases}$$

If $r$ is even, then

$$|K_1| = \frac{q+1}{2} \cdot \frac{(q^r-1)}{q-1} + \frac{q^r-q^r}{2},$$

$$|K_2| = \frac{q+1}{2} \cdot \frac{(q^r-1)}{q-1} + \frac{q^r+q^r}{2}.$$

**Remark 9.** In the case when the quadric $Q$ is degenerate, then it is not difficult to see that the arc $K_i$, $i = 1, 2$, is lifted. Let $Q$ be a non-degenerate quadric of $\text{PG}(r, q)$, then $K_1$ and $K_2$ are projectively equivalent if $r$ is odd, but they are not in the case when $r$ is even. On the other hand, if $r$ is odd, there are two distinct classes of non-degenerate quadrics, namely the hyperbolic quadric and the elliptic quadric. Therefore in all cases Theorem 7 gives rise to two distinct examples of non lifted $\left(\frac{q+1}{2} \mod q\right)$ arcs of $\text{PG}(r, q)$.

### 4.1 The arcs of size 143 and 168

In [2], the following two strong non-lifted $(3 \mod 5)$-arcs in $\text{PG}(3, 5)$ were constructed by a computer search. The respective spectra are:

$$|\mathcal{F}_1| = 143, \quad a_{18}(\mathcal{F}_1) = 26, a_{23}(\mathcal{F}_1) = 0, a_{28}(\mathcal{F}_1)_{28} = 65, a_{33}(\mathcal{F}_1) = 65;$$

$$\lambda_0(\mathcal{F}_1) = 65, \lambda_1(\mathcal{F}_1) = 65, \lambda_2(\mathcal{F}_1) = 0, \lambda_3(\mathcal{F}_1) = 26;$$

and

$$|\mathcal{F}_2| = 168, \quad a_{28}(\mathcal{F}_2) = 60, a_{33}(\mathcal{F}_2) = 60, a_{43}(\mathcal{F}_2)_{36};$$

$$\lambda_0(\mathcal{F}_2) = 60, \lambda_1(\mathcal{F}_2) = 60, \lambda_2(\mathcal{F}_2) = 0, \lambda_3(\mathcal{F}_2) = 36.$$}

In addition, $|\text{Aut}(\mathcal{F}_1)| = 62400$, and $|\text{Aut}(\mathcal{F}_2)| = 57600$.

These arcs can be recovered from Theorem 7. Indeed, if $Q$ is an elliptic quadric of $\text{PG}(3, 5)$, then $K_1$ is a non lifted $(3 \mod 5)$ arc of $\text{PG}(3, 5)$ of size 143, whereas if $Q$ is a hyperbolic quadric of $\text{PG}(3, 5)$, then $K_1$ is a non lifted $(3 \mod 5)$ arc of $\text{PG}(3, 5)$ of size 168.
5 Further examples \((t \mod q)-\text{arcs}\)

A set of type \((m, n)\) in \(\text{PG}(r, q)\) is a set \(S\) of points such that every line of \(\text{PG}(r, q)\) contains either \(m\) or \(n\) points of \(S\), \(m < n\), and both values occur. Assume \(m > 0\). Then the only sets of type \((m, n)\) that are known, exist in \(\text{PG}(2, q), q\) square, and are such that \(n = m + \sqrt{q}\). In particular, sets of type \((1, 1 + \sqrt{q})\) either contains \(q + \sqrt{q} + 1\) and are Baer subplanes or \(q\sqrt{q} + 1\) points and are known as \emph{unitals}. For more details on sets of type \((m, n)\) in \(\text{PG}(2, q)\) see [8] and references therein.

\begin{thm}
\label{thm:K}
\text{Theorem 11.} \ K \in H \text{ such that } K(P) = \sqrt{q}, \text{ if } P \in S \text{ and } K(P) = 0, \text{ if } P \notin S. \text{ Then } K \text{ is a } (\sqrt{q} \mod q)-\text{arc of } \text{PG}(r, q).
\end{thm}

\text{Proof.} \ Let \(\ell\) be a line of \(\text{PG}(r, q)\). \(K(\ell) = m\sqrt{q}\), whereas if \(|\ell \cap S| = m + \sqrt{q}\), then \(K(\ell) = m\sqrt{q} + q\). \(\blacksquare\)

\text{In PG}(r, q), q \text{ square, let } H \text{ be a Hermitian variety of } \text{PG}(r, q), \text{ i.e., the variety defined by a Hermitian form of } \text{PG}(r, q). \text{ It is well-known that a line of } \text{PG}(r, q) \text{ has } 1, \sqrt{q} + 1 \text{ or } q + 1 \text{ points in common with } H. \text{ Let } K' \text{ be the arc of } \text{PG}(r, q) \text{ such that } K'(P) = \sqrt{q}, \text{ if } P \in H \text{ and } K'(P) = 0, \text{ if } P \notin H. \text{ Moreover, if } H \text{ is non-degenerate, then } K' \text{ is not lifted.}

\begin{thm}
\text{Theorem 11.} \ K' \text{ is a } (\sqrt{q} \mod q)-\text{arc of } \text{PG}(r, q). \text{ Moreover, if } H \text{ is non-degenerate, then } K' \text{ is not lifted.}
\end{thm}

\text{Proof.} \ Let \(\ell\) be a line of \(\text{PG}(r, q)\). Then

\[ K'(\ell) = \begin{cases}
\sqrt{q} & \text{if } |\ell \cap H| = 1, \\
\sqrt{q} + q & \text{if } |\ell \cap H| = \sqrt{q} + 1, \\
\sqrt{q}(1 + q) & \text{if } |\ell \cap H| = q + 1.
\end{cases} \]

If \(H\) is non-degenerate, then through every point of \(\text{PG}(r, q)\) there exists a line \(r\) such that \(|H \cap r| = \sqrt{q} + 1\). By construction, the line \(r\) has \(\sqrt{q} + 1\) \(q\)-points and \(q - \sqrt{q}\) \(0\)-points. Hence \(K'\) is not lifted. \(\blacksquare\)

Acknowledgements

The research of the second author was supported by the Bulgarian National Science Research Fund under Contract KP-06-Russia/33. The research of the fourth author was supported by the Research Fund of Sofia University under Contract 80-10-52/10.05. 2022. All authors would like to thank the organizers of the Sixth Irsee Conference on Finite Geometries for their invitation. During that conference the main ideas for this paper emerged.

References

[1] V. Abatangelo, G. Korchmaros, B. Larato, Classification of maximal caps in PG(3, 5) different from elliptic quadrics, \textit{J. of Geometry} 57(1996), 9–19.
[2] S. Kurz, I. Landjev and A. Rousseva, Classification of \((3 \mod 5)\)-arcs in \(PG(3, 5)\), *Adv. Math. Comm.*, 2022, to appear. doi:10.3934/amc.2021066

[3] I. Landjev, The geometry of \((n, 3)\)-arcs in the projective plane of order 5, Proc. of the Sixth Workshop on ACCT, Sozopol, 1996, 170–175.

[4] I. Landjev and A. Rousseva, On the extendability of Griesmer arcs. *Ann. Sof. Univ. Fac. Math. Inf.* 101(2013), 183–192.

[5] I. Landjev and A. Rousseva, The nonexistence of \((104, 22; 3, 5)\)-arcs, *Adv. Math. Comm.*, 10(2016), 601–611.

[6] I. Landjev and A. Rousseva, Divisible arcs, divisible codes and the extension problem for arcs and codes, *Problems of Information Transmission* 55(2019), 226–240.

[7] I. Landjev, A. Rousseva and L. Storme, On the extendability of quasidivisible Griesmer arcs, *Des. Codes Cryptogr.* 79(2016), 535–547.

[8] T. Penttila and G.F. Royle, Sets of type \((m, n)\) in the affine and projective planes of order nine, *Des. Codes Cryptogr.*, 6(1995), no. 3, 229–245.

[9] A. Rousseva, On the structure of \((t \mod q)\)-arcs in finite projective geometries, *Ann. Sofia Univ., Fac. Math and Inf.* 103(2016), 5–22.

[10] M. Tallini Scafati, Calotte di tipo \((m, n)\) in uno spazio di Galois \(S_{r,q}\), *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat.*, (8) 53(1972), 71–81.