Interpolating and sampling sequences in finite Riemann surfaces

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Abstract

We provide a description of the interpolating and sampling sequences on a space of holomorphic functions on a finite Riemann surface, where a uniform growth restriction is imposed on the holomorphic functions.

1. Introduction and statement of the results

Let $S$ be an open finite Riemann surface endowed with the Poincaré (hyperbolic) metric. We will study some properties of holomorphic functions on the Riemann surface with uniform growth control. Namely, we will deal with the Banach space $A_\phi(S)$ of holomorphic functions $H(S)$ on the surface such that $\|f\| := \sup S |f| e^{-\phi} < \infty$, where $\phi$ is a given subharmonic function that controls the growth of the functions.

An open finite Riemann surface is the interior of a smooth bordered compact Riemann surface. It is automatically hyperbolic and, therefore, it has a Green function. The fact that $\phi$ is subharmonic is a natural assumption on the weight that limits the growth. Any other growth control given by a weight $\psi$, $\|f\|_* = \sup S |f| e^{-\psi}$ can be replaced by an equivalent subharmonic function because $\phi = \sup \|f\|_* \log |f|$ is a subharmonic function and $A_\phi(S) = A_\psi(S)$ with equality of norms, $\sup S |f| e^{-\psi} = \sup S |f| e^{-\phi}$.

We have fixed a metric. It is then natural to restrict the possible weights $\phi$, in such a way that the functions in $A_\phi$ oscillate in a controlled way when the points are nearby in the Poincaré metric. This is achieved, for instance, by assuming that $\phi$ has bounded Laplacian (the Laplace–Beltrami operator with respect to the hyperbolic measure). That is, if in a local coordinate chart the Poincaré metric is of the form $ds^2 = e^{2\nu(z)}|dz|^2$, then we assume that $\Delta \phi = 4e^{-2\nu(z)}(\partial^2 \phi / \partial z \partial \bar{z})$ satisfies $C^{-1} \leq \Delta \phi \leq C$. If we want to deal with other weights, then it is possible to introduce a natural metric associated to the weight as it is done in the plane in [7]. In this work, we will only consider the Poincaré metric and bounded Laplacian since it already covers many interesting cases and is technically simpler.

The problems that we will consider are the following.

(A) The description of the interpolating sequences for $A_\phi(S)$: that is, the sequences $\Lambda \subset S$ such that it is always possible to find an $f \in A_\phi(S)$ such that $f(\lambda) = v_\lambda$ for all $\lambda \in \Lambda$ whenever the data $\{v_\lambda\}_\Lambda$ satisfy the compatibility condition $\sup_\Lambda |v_\lambda| e^{-\phi(\lambda)} < +\infty$.

(B) The description of sampling sets for $A_\phi(S)$: that is, the sets $E \subset S$ such that there is a constant $C > 0$ that satisfies

$$
\sup_E |f| e^{-\phi} \leq C \sup_S |f| e^{-\phi}, \quad \forall f \in A_\phi(S).
$$

In the solution of these problems the Poincaré distance and the potential theory on the surface play a key role. This has already been observed by Schuster and Varolin in [12], where they provided sufficient conditions for a sequence to be interpolating/sampling for functions in

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a slightly different context where the weighted uniform control of the growth of the functions is replaced by a weighted $L^2$ control. Their condition basically coincides with the description that we reach, so our work can be considered as the counterpart of their theorems, although we will give a different proof of their results as well. We will rely on the well-known case of the disk and some simplifying properties of finite Riemann surfaces. Their method of proof looks more promising if one wants to extend the result to Riemann surfaces with more complicated topology.

When the surface is a disk, which will be our model situation, the corresponding problems have been solved in [1, 9] and in a different way in [14]. Of course, the more basic problem of describing the interpolating sequences for bounded holomorphic functions in open finite Riemann surfaces (in our notation $\phi \equiv 0$) has been known for a long time; see [15].

We now introduce some definitions that will be needed to state our results. For any point $z \in S$ and any $r > 0$ we denote by $D(z, r)$ the domain in the surface $S$ that consists of points at hyperbolic distance from $z$ less than $r$. For any given $r > 0$ the domains are topological disks of the radius $r$, if the center $z$ is outside a big enough compact of $S$, as we will see in Subsection 2.1.

A sequence $\Lambda$ of points in $S$ is uniformly separated if there is an $\varepsilon > 0$ such that the domains $\{D(\lambda, \varepsilon)\}_{\lambda \in \Lambda}$ are pairwise disjoint.

Let $g_r(z, w)$ be the Green function associated to the surface $D(z, r)$ with pole at the ‘center’ $z$ and let $g(z, w) = g_\infty(z, w)$ be the Green function associated to the surface $S$. We define the densities as

$$D^+_{\phi}(\Lambda) := \limsup_{r \to \infty} \sup_{z \in S} \frac{\sum_{1/2<d(z,\lambda)<r} g_r(z, \lambda)}{\int_{D(z,r)} g_r(z, w) i \partial \bar{\partial} \phi(w)},$$

and

$$D^-_{\phi}(\Lambda) := \liminf_{r \to \infty} \inf_{z \in S} \frac{\sum_{1/2<d(z,\lambda)<r} g_r(z, \lambda)}{\int_{D(z,r)} g_r(z, w) i \partial \bar{\partial} \phi(w)}. \quad (1.1)$$

The following theorem is the main result of this paper.

**Theorem 1.** Let $S$ be an open finite Riemann surface and let $\phi$ be a subharmonic function with bounded Laplacian.

(A) A sequence $\Lambda \subset S$ is an interpolating sequence for $A\phi(S)$ if and only if it is uniformly separated and $D^+_{\phi}(\Lambda) < 1$.

(B) A set $E \subset S$ is a sampling set for $A\phi(S)$ if and only if it contains a uniformly separated sequence $\Lambda \subset E$ such that $D^-_{\phi}(\Lambda) > 1$.

In Section 2 we will prove some key properties of open finite Riemann surfaces. In particular, we need to study the behavior of the hyperbolic metric as we approach the boundary of the surface. We will also prove some weighted uniform estimates for the inhomogeneous Cauchy–Riemann equation on the surface, Theorem 6, that is of interest by itself.

In Section 3, we use the tools and lemmas proved in Section 2 to reduce the interpolating and sampling problem in $S$ to a problem near the boundary that can be reduced to the known case of the disk.

Finally, in Section 4 we show how our results can be extended to other Banach spaces of holomorphic functions where the uniform growth is replaced by weighted $L^p$ spaces.

A final word on notation: By $f \lesssim g$ we mean that there is a constant $C$ independent of the relevant variables such that $f \leq C g$, and by $f \simeq g$ we mean that $f \lesssim g$ and $g \lesssim f$. 
2. Basic properties of open finite Riemann surfaces

We start with the definition and then we collect some properties of $S$ that follow from the restrictions that we are assuming on the topology of $S$.

**Definition 2.** An open finite Riemann surface is the interior of a smooth bordered compact Riemann surface.

Our surface is an open Riemann surface and it is in fact an open subset of a compact surface (the double, see [11]). See Figure 1 for a typical representation. Observe that the genus is finite and the boundary of the surface consists of a finite number of smooth closed Jordan curves. In most of what follows, the particular case of a smooth finitely connected open set in $\mathbb{C}$ has all the difficulties of the general case.

The following claim follows, for instance, from [10, Propositions 7.1–7.4].

**Lemma 3.** For any $(0,1)$-form $\omega$, there is a solution $u$ to the inhomogeneous Cauchy–Riemann equation $\bar{\partial}u = \omega$. Moreover, since $S$ has an essential extension to a compact Riemann surface, if the data are smooth forms with compact support $K$ in $S$ then there is a bounded linear solution $u = T[w]$ with the bound $|u| \leq C_K \sup_K \langle \omega \rangle$.

In this statement and in the following, $\langle \omega \rangle$ is the Poincaré length of the $(0,1)$-form $\omega$.

In the disk we have Blaschke factors that are very useful to divide out zeros of holomorphic functions without essentially changing the norm. The analogous functions that provide us with the same property in the case of finite Riemann surfaces are given by the next proposition.

**Proposition 4.** There is a constant $C = C(S) > 0$ such that for any point $z \in S$ there is a function $h_z \in \mathcal{H}(S)$ with

$$\sup_{w \in S} |\log |h_z(w)|| - g(z, w)| < C,$$

where $g(z, w)$ is the Green function of $S$. In particular, $h_z(w)$ is a bounded holomorphic function that vanishes only on the point $z$ and for any $\varepsilon > 0$, $K > |h_z(w)| > C(\varepsilon)$ if $d(z, w) > \varepsilon$.

**Proof.** The obstruction for a harmonic function $u$ to have a harmonic conjugate is that for a set of generators $\{\gamma_i\}_{i=1}^m$ of the homology we have $\int_{\gamma_i} *du = 0$, $i = 1, \ldots, m$. If we want $u = \log |f|$ for an $f \in \mathcal{H}(S)$, then we just need that $\int_{\gamma_i} *du \in \mathbb{Z}$.
In a finite Riemann surface there are \( \{h_j\}_{j=1}^n \) functions in the algebra of \( S \) without zeros such that \( \int_{\gamma_i} d\log|h_j| = \delta_{ij} \); see [16, Lemma 1]. We define the function

\[
v(z) = u(z) - \sum_i \left( \int_{\gamma_i} du \right) \log|h_i(z)| + \sum_i \left[ \int_{\gamma_i} du \right] \log|h_i(z)|,
\]

where \([x]\) denotes the integer part of \( x \). As \( \int_{\gamma_i} dv \in \mathbb{Z} \), then \( v \) is the logarithm of a holomorphic function \( \log|f| = v \). Therefore, there is a constant \( C \) such that any harmonic function \( u \) in \( S \) admits a holomorphic function \( f \) with \( |u - \log|f|| < C \). Take a point \( z \in S \) and any holomorphic function \( k_z \in \mathcal{H}(S) \) that vanishes only on \( z \). Then \( g(z,w) - \log|k_z(w)| \) is harmonic in \( S \), and therefore there is a holomorphic function \( f_z \) such that \( |g(z,w) - \log|k_z(w)|| - \log|f_z| < C \). Thus we may define \( h_z(w) = f_z(w)k_z(w) \) and it has the estimate \( |g(z,w) - \log|h_z|| < C \). The estimate \( |g(z,w)| > C(\varepsilon) \) when \( d(z,w) > \varepsilon \) holds in finite Riemann surfaces; see, for instance, [4, Theorem 5.5].

2.1. The hyperbolic metric on an open finite Riemann surface

The open ends of the Riemann surface can be parametrized as follows: the boundary of the Riemann surface \( S \) is a finite union of smooth closed curves \( \tilde{\gamma}_i \), \( i = 1, \ldots, n \). Near each \( \tilde{\gamma}_i \) there is a closed geodesic \( \gamma_i \) that is homotopic to \( \tilde{\gamma}_i \). The subdomain of \( S \) bounded by \( \gamma_i \) and \( \tilde{\gamma}_i \) is called a ‘funnel’, following the terminology of [5, 6].

We need to be more precise about the hyperbolic metric in the funnel. There are nice coordinates in the funnel that provide good estimates. These are given by the collar theorem. Let \( \mathbb{D} \) be the universal holomorphic cover of \( S \), and let \( T_\gamma \in \text{Aut}(\mathbb{D}) \) be the deck transformation corresponding to the closed loop \( \gamma \). Consider the surface \( Y = \mathbb{D}/\{T_\gamma^n\}_{n \in \mathbb{Z}} \). This is an annulus since \( \pi_1(Y) = \mathbb{Z} \). If we take the quotient of \( Y \) by the rest of the deck transformations of the universal cover, we get a holomorphic covering map \( \pi_\gamma \) from \( Y \to S \) which is a local isometry (in \( Y \) and \( S \) we consider the Poincaré metric inherited from \( \mathbb{D} \)). In fact, \( Y = \{e^{-R} < |z| < e^R\} \), where \( R = \pi^2/\text{Length}(\gamma) \), and \( \pi_\gamma \) maps the unit circle isometrically to \( \gamma \). Moreover, \( \pi_\gamma \) is an isometric injection of the outer part of the annulus \( \{1 < |z| < e^R\} \) onto the funnel. These will be called the standard coordinates of the funnel (Figure 2). See [3, 5] for details.

The Poincaré metric in the funnel is explicit in the standard coordinates and is comparable to the hyperbolic metric on the disk in the coordinate disk \( |z| < e^R \), when restricted to \( |z| > 1 \).

We denote by \( A_i, i = 1, \ldots, n \), the funnels of \( S \) bounded by \( \gamma_i \) and \( \tilde{\gamma}_i \).

2.2. The inhomogeneous Cauchy–Riemann equation on the surface

We want to solve the inhomogeneous Cauchy–Riemann equation on \( S \) with weighted uniform estimates. In order to get good estimates, it is useful to find functions \( f \in \mathcal{H}(S) \) with precise
size control; that is, $|f| \simeq e^\phi$ outside a neighborhood of the zero set of $f$. With this function, we can later modify an integral formula to obtain a bounded solution to the $\bar{\partial}$-equation when the data have compact support. The following lemma provides such a function that in other contexts has been termed a ‘multiplier’.

**Lemma 5.** Let $S$ be a finite Riemann surface and let $\phi$ be a subharmonic function with bounded Laplacian. Then there is a function $f$ with uniformly separated zero set $\Sigma$ such that $|f| \simeq e^\phi$ whenever $d(z, \Sigma) > \varepsilon$. Moreover, if we fix any compact $K$ in $S$, then it is possible to find $f$ with the above properties and without zeros in $K$.

**Proof.** In any of the funnels $A_i$, we transfer the subharmonic weight $\phi$ to the standard coordinate chart $1 < |z| < e^{R_i}$. We define a weight $\phi_i$ on the disk $|z| < e^{R_i}$ in such a way that $\phi_i$ has bounded invariant Laplacian and, moreover, $|\phi - \phi_i| < C$ on the region $1 < |z| < e^{R_i}$. One way to do so is the following: we assume from the very beginning that $\phi$ is smooth (this is no restriction since otherwise it can be approximated by a smooth function). Define

$$\phi_i(z) = \phi(z)\chi(z) + M_i||z||^2,$$

(2.1)

where $\chi$ is a cutoff function such that $\chi \equiv 1$ in $e^{R_i/2} < |z| < e^{R_i}$, $\chi \equiv 0$ in $|z| < 1$ and $M_i$ is taken big enough such that $\phi_i$ is subharmonic and the invariant Laplacian of $\phi_i$ is bounded above and below.

We are under the hypothesis of the result from [13] which states that there is a holomorphic function in the disk $f_i$ with separated zero set $Z(f_i)$ (in the hyperbolic metric of the disk) such that $|f_i| \simeq e^{\phi_i}$ whenever $d(z, Z(f_i)) > \varepsilon$. Since the hyperbolic metric of the disk is comparable to the hyperbolic metric in the funnel, we have found a function $f_i \in \mathcal{H}(A_i)$ with separated zero set such that $|f_i(z)| \simeq e^{\phi_i(z)}$ if $d(z, Z(f_i)) > \varepsilon$. Moreover, dividing out $f_i$ by a finite Blaschke product, we can assume that $f_i$ is zero-free in any prefixed compact of the disk.

We consider the ‘core’ of $S$ to be $S \setminus \tilde{A}_i$, where $\tilde{A}_i$ are the outer part of the funnels mapped by $e^{S_i} < |z| < e^{R_i}$. Further, $S_i$ is assigned such large values as to make sure that the compact $K$ in the hypothesis of the lemma is contained in the core of $S$. We adjust the $f_i$, $i = 1, \ldots, n$, as mentioned before, to make sure that they are zero-free in the inner part of the funnels $1 < |z| < e^{S_i}$. We finally define $f_0 \equiv 1$ in the core of $S$.

To patch the different $f_i$ together, we will need to solve a Cousin II problem with bounds. Our data are $f_i$ defined on the inner parts of the funnels mapped by $1 < |z| < e^{S_i}$. The data are bounded above and below in the inner parts of the funnels (because $\phi$ is bounded above and below in any compact of $S$ and $f_i$ have no zeros there). We want to find functions $g_i \in \mathcal{H}(A_i)$ and $g_0$ holomorphic on the core of $S$ such that $f_i = g_0/g_i$ in the inner part of the funnel. If, moreover, $g_i$ and $g_0$ are bounded (above and below) then the function $f$ defined as $f_i g_i$ in each of the funnels $A_i$ and $g_0$ on the core of $S$ is holomorphic on $S$ and has the desired growth properties. To find the functions $g_i$, observe that since the intersection of the funnel $A_i$ with the core of $S$ strictly separates the outer part of the funnel from the inner part of the core, we can reduce the Cousin II problem to solving a $\bar{\partial}$-equation with bounded estimates of the solution on $S$ when the data are bounded and with compact support (the support is in the inner part of the funnels). This can be achieved by Lemma 3.

With this function, we can then obtain the following result which is interesting by itself.

**Theorem 6.** Let $S$ be a finite Riemann surface and let $\phi$ be a subharmonic function with a bounded Laplacian. There is a constant $C > 0$ such that for any $(0, 1)$-form $\omega$ on $S$ there is
a solution \( u \) to the inhomogeneous Cauchy–Riemann equation \( \bar{\partial}u = \omega \) in \( S \) with the estimate
\[
\sup_{z \in S} |u(z)| e^{-\phi(z)} \leq C \sup_{z \in S} (\omega(z)) e^{-\phi(z)},
\]
whenever the right-hand side is finite.

Recall that the notation \( \langle \omega(z) \rangle \) means the hyperbolic norm of \( \omega \) at the point \( z \).

**Proof.** Let \( w_i \) be the form \( w \) restricted to the funnel \( A_i \). We take a standard coordinate chart and we may think of \( w_i \) as a \((0,1)\)-form defined on the disk \( |z| < e^{R_i} \) and with support in \( 1 < |z| < e^{R_i} \). Consider, as in the proof of Lemma 5, a subharmonic function \( \phi_i \) in the disk with bounded Laplacian and such that \( |\phi - \phi_i| < C \) if \( 1 < |z| < e^{R_i} \).

By the results in [8, Theorem 2], there is a solution \( u_i \) to the problem \( \bar{\partial}u_i = w_i \) in the disk \( |z| < e^{R_i} \) with the estimate
\[
\sup_{|z| < e^{R_i}} |u_i(z)| e^{-\phi_i} \leq C_i \sup_{1 < |z| < e^{R_i}} (w_i) e^{-\phi}.
\]
Observe that the hyperbolic metric of the disk and of the surface \( S \) in the funnel are equivalent. We consider that \( \tilde{u}_i = u_i \chi_i \), where \( \chi_i \) is a cutoff function with support in \( 1 < |z| < e^{R_i} \) and such that \( |\chi_i| \equiv 1 \) if \( |z| > e^{R_i}/2 \). The function \( \tilde{u}_i \) is extended by 0 to \( S \), and it has the global estimate \( \sup_{S} |\tilde{u}_i(z)| e^{-\phi} \leq C_i \sup_{S} (w_i) e^{-\phi} \). Now \( \partial \tilde{u}_i \) coincides with \( w \) on the outer part of the funnel \( A_i \). Thus the \((0,1)\)-form \( w_k = w - \sum_i \partial \tilde{u}_i \) has compact support in \( S \) and it satisfies \( \sup_{S} (w_k) e^{-\phi} \leq \sup_{S} (w) e^{-\phi} \). The desired solution is then \( u = \sum_i \tilde{u}_i + v \), where \( v \) is such that \( \partial v = w_k \). We must then solve \( \bar{\partial}v = w_k \) with weighted uniform estimates but with the advantage that \( w_k \) has compact support \( K \).

Let \( T(w_k) \) be a solution operator for \( \bar{\partial}u = w_k \). We take the operator \( T \) given by Lemma 3; the estimate \( \sup_{S} |T[w_k](z)| \leq C_K \sup_{K} (w_k) \) holds. Take \( f \) with \( |f| \simeq e^\phi \) and without zeros in \( K \) as given in Lemma 5. Then we define \( R \) as
\[
R[w_k](z) = f(z)T[w_k/f](z),
\]
which solves \( \bar{\partial}R[w_k] = w_k \) with the estimate
\[
\sup_{S} |R[w_k] e^{-\phi} | \leq C_K \sup_{K} (w_k) e^{-\phi}.
\]
The solution is thus \( v = R[w_k] \).

3. **The main results**

**Proposition 7.** A separated sequence \( \Lambda \subset S \) is interpolating for \( A_\phi(S) \) if and only if the sequences \( \Lambda_i = \Lambda \cap A_i \) are interpolating in \( A_\phi(A_i) \).

**Proof.** We only need to prove that we can pass from the local to the global interpolation property. We split the proof into two steps.

(i) From a funnel \( A_i \) to global \( S \); we need to prove that there are finite sets \( F_i \subset \Lambda_i \) such that \( \bigcup_{i=1}^n (\Lambda_i \setminus F_i) \) is interpolating globally.

(ii) Filling up the remainder; we shall prove that by adding a finite number of points to an interpolating sequence we still get an interpolating sequence. Thus \( \Lambda \) is interpolating if \( (\Lambda_1 \setminus F_1) \cup \ldots \cup (\Lambda_n \setminus F_n) \) is interpolating.

Let \( \tilde{\gamma} \) be one of the closed curves on the boundary. Take a funnel \( A \) with outer end curve in \( \tilde{\gamma} \) and inner end curve in \( \gamma \). The constant of interpolation in the funnel \( A \) is \( K > 0 \). Take a cutoff
function $\chi_\varepsilon$ with support in the funnel such that $\langle \partial \chi_\varepsilon \rangle < \varepsilon/(KC)$ (where $C$ is the constant in Theorem 6), the support is in a thick annulus of hyperbolic thickness $M = M(\varepsilon, K, C)$. We consider a smaller funnel where $\chi_\varepsilon \equiv 1$. The sequence $\Lambda$ in this smaller funnel still has at most interpolation constant $K$. We can interpolate arbitrary values on $\Lambda$ being small near the inner curve $\gamma$ of $A$ in the following way. Take some values $v_\lambda$ with norm one. Take a function in the funnel $f$ with norm at most $K$ that solves the interpolation problem. We are going to approximate it by a function in $A$ that is small near $\gamma$. Cut it off by $\chi_\varepsilon$ and correct via the inhomogeneous Cauchy–Riemann equation

$$\bar{\partial}u = f\partial\chi.$$ 

The function $h = u - f\chi$ is holomorphic. By using Theorem 6 it is possible to solve the equation with a solution $u$ such that $\sup |u|e^{-\phi} \leq \varepsilon$. The function $h$ does not solve the problem directly, but it almost does. We reiterate the procedure. Interpolating the error $v_\lambda - h(\lambda)$ and with a convergent series we finally get a function $g$ such that $h(\lambda) = v_\lambda$, $\sup_{\bar{A}} |h|e^{-\phi} \leq 2$ and, moreover, in the inner half of the funnel that we denote by $A$, we have $\sup_{\bar{A}} |h|e^{-\phi} \leq \varepsilon$.

Now it is easier to make it global. Take a new cutoff function $\chi$ with support in the funnel $A$ and that is one on the outer part of it (that is, $A \setminus \bar{A}$). Then we need to solve

$$\bar{\partial}u = h\partial\chi,$$

with good global estimates in $S$. These are given by Theorem 6. We have solved the interpolation problem when the sequence lies in the funnels. For the general situation, we only need to add a finite number of points. The existence of ‘Blaschke’-type factors $h_\lambda(z)$ provided by Proposition 4 shows that $\Lambda \cup \lambda$ is interpolating if $\Lambda$ is interpolating (it is immediate to build functions in the space such that $f|_{\Lambda} = 0$ and $f(\lambda) \neq 0$). □

For the sampling part, we need the following definition.

**Definition 8.** Given the pair $(S, \phi)$ of a finite Riemann surface and a subharmonic function defined on it, we associate to it the pairs $(D_i, \phi_i)_{i=1,\ldots,n}$ of disks $D_i$ and subharmonic functions $\phi_i$ defined on the disks as follows: if $A_i = \{1 < |z| < e^{R_i}\}$, $i = 1, \ldots, n$, are the standard charts of the funnels of $S$, then we define $D_i = \{1 < |z| < e^{R_i}\}$ and $\phi_i$ is any subharmonic function in $D_i$ such that $|\phi_i - \phi| < C$ in the region $1 < |z| < e^{R_i/2}$, $\Delta \phi_i = \Delta \phi$ in $e^{R_i/2} < |z| < e^{R_i}$ and $\Delta \phi_i \simeq 1$ in $|z| < e^{R_i/2}$. They can be defined similarly as in (2.1), but to make sure that $\Delta \phi_i = \Delta \phi$ we may take instead

$$\phi_i(z) = \phi(z)\chi(z) + M_i \psi_i(z),$$

where $\psi_i$ is any bounded subharmonic function in $D_i$ such that $\Delta \psi_i(z) = 1$ if $|z| < e^{R_i/2}$ and $\Delta \psi_i(z) = 0$ elsewhere. We can take, for instance, $\psi_i(z) = \int_{|w| < e^{R_i/2}} (\log |z - w|/2\pi) \, dm(w)$.

The funnels $A_i$ can be considered funnels of $S$ and they are subdomains of $D_i$ too. We will exploit this double nature in the following theorem.

**Theorem 9.** Let $S$ be a finite Riemann surface and let $\phi$ be a subharmonic function with bounded Laplacian. A separated sequence $\Lambda$ is sampling for $A_\phi(S)$ if and only if all the sequences in the funnels $\Lambda_i = \Lambda_i \cap A_i \subset D_i$ are sampling sequences for $A_{\phi_i}(D_i)$, where $(D_i, \phi_i)$ are the associated pairs to $S$ given by Definition 8.

Thus theorem and Proposition 7 show that the properties of sampling and interpolation depend only on the behavior of the sequence and the weight near the boundary pieces.

To prove Theorem 9 we need some earlier results.
LEMMA 10. Let $S$ be a finite Riemann surface and let $\phi$ be a subharmonic function with bounded Laplacian. A sequence $\Lambda \subset S$ is a uniqueness sequence for $A_\phi(S)$ if and only if all the sequences in the funnels $\Lambda_i = \Lambda_i \cap A_i \subset D_i$ are uniqueness sequences for $A_{\phi_i}(D_i)$, where $(D_i, \phi_i)$ are the associated pairs to $S$ given by Definition 8.

**Proof.** It is easier to deal with this proof by negation. Let $\Lambda$ be contained in the zero set of a function $f \in A_\phi(S)$. Therefore, $\Lambda$ is in the zero set of $f \in A_\phi(A_i)$. We factor out a finite number of zeros $E_i$ from the function $f$ and obtain a new function $g \in A_\phi(A_i)$ without zeros in $1 < |z| \leq e^{R_i/2}$ and such that $A_i \setminus E_i \subset \mathbb{Z}(g)$. Take the disk $D_i$ and consider the cover by two open sets $|z| > 1$ and $|z| < e^{R_i/2}$. On the first set we have the function $g$ and on the second the function $1$. The quotient is bounded above and below in the intersection of the sets. This defines a bounded Cousin II in the disk $D_i$ problem that can be solved with bounded data. We get a new function $h \in A_{\phi_i}(D_i)$ that vanishes in $Z(g)$. We can now add the finite number of zeros $E_i$ without harm. The reciprocal implication follows with the same argument. \qed

The next result is inspired by a result of Beurling [2, pp. 351–365] that relates the property of sampling sequence to that of uniqueness for all weak limits of the sequence. In the context of the Bernstein space (in the original work by Beurling), the space was fixed, it was $\mathbb{C}$, the space of functions, the Bernstein class, was fixed, and Beurling considered translates and limits in the sampling sequence. Here we need to move and take limits of the sequence (by zooming on appropriate portions of it), but we also need to change the support space (portions of $S$ near the funnel that look like the unit disk) and we will also move the space of functions by changing the weights. We need the following definitions.

**DEFINITION 11.** We consider triplets $(D_n, \phi_n, \Lambda_n)$, where $D_n$ are disks $D_n = D(0, r_n) \subset \mathbb{D}$, $\phi_n$ are subharmonic functions defined in a neighborhood of $D_n$ and $\Lambda_n$ is a finite collection of points in $D_n$. We say that $(D_n, \phi_n, \Lambda_n)$ converges weakly to $(\mathbb{D}, \phi, \Lambda)$ (where $\mathbb{D}$ is the unit disk, $\phi$ a subharmonic function in $\mathbb{D}$ and $\Lambda$ a discrete sequence in $\mathbb{D}$) if the following conditions are fulfilled:

(i) The domains $D_n$ tend to $\mathbb{D}$, that is: $r_n \rightarrow 1$.
(ii) The weights $\phi_n$ tend to the weight $\phi$ in the sense that $\Delta \phi_n$ as measures converge weakly to $\Delta \phi$.
(iii) The sequences $\Lambda_n$ converge weakly to $\Lambda$: that is, the measure $\sum_{\lambda \in \Lambda_n} \delta_\lambda$ converges weakly to the measure $\sum_{\lambda \in \Lambda} \delta_\lambda$.

Let us fix a point $p \in S$. If a sequence of points $z_n \in S$ goes to $\infty$, that is, $d(z_n, p) \rightarrow \infty$, from a point $n_0$ on, it will eventually belong to the union of the funnels $A_1 \cup \ldots \cup A_n$. If we take the set of points $D_n = \{ z \in S; d(z, z_n) < d(z_n, p)/2 \}$ then $D_n$ is an hyperbolic disk contained in the funnels if $n$ is big enough. In each of the $D_n$ we consider the function $\phi_n = \phi|_{D_n}$ and $\Lambda_n = \Lambda \cap D_n$. Thus for any sequence of points $z_n$ with $d(p, z_n) \rightarrow \infty$, we build a triplet $(D_n, \phi_n, \Lambda_n)$ for $n$ big enough.

**DEFINITION 12.** Let $W(S, \phi, \Lambda)$ be the set of all triplets $(\mathbb{D}, \phi^*, \Lambda^*)$ that are weak limits of triplets $(D_n, \phi_n, \Lambda_n)$ associated to any sequence $z_n$ such that $d(p, z_n) \rightarrow \infty$.

The following is the theorem of Beurling in our context.

**THEOREM 13.** Let $S$ be a Riemann surface of finite type and let $\phi$ be a subharmonic function with bounded Laplacian. A separated sequence $\Lambda$ is sampling for $A_\phi(S)$ if and only if
the following hold.

(i) The sequence $\Lambda$ is a uniqueness set for $A_\phi(S)$.

(ii) For any triplet $(\mathbb{D}, \phi^*, \Lambda^*) \in W(S, \phi, \Lambda)$, the sequence $\Lambda^*$ is a uniqueness set for $A_{\phi^*}(\mathbb{D})$.

Proof. Let us prove that the uniqueness conditions imply that $\Lambda$ is a sampling sequence. If it were not, there would be a sequence of functions $f_n \in A_\phi(S)$ such that $\sup_\Lambda |f_n e^{-\phi}| \leq 1/n$ and $\sup_S |f_n e^{-\phi}| = 1$. Take a sequence of points $z_n$ with $|f_n(z_n)| e^{-\phi} \geq 1/2$. If $z_n$ are bounded we can take a subsequence of points that we still denote by $z_n$, convergent to $z^* \in S$, and by a normal family argument there is a subsequence of $f_n$ convergent to $f \in A_\phi$, such that $f|_\Lambda \equiv 0$, $f(z^*) \neq 0$. However, this is not possible. Thus $z_n$ must be unbounded. Then we take the triplets $(D_n, \phi_n, \Lambda_n)$ associated to $z_n$ and $D_n \to \mathbb{D}$ because $z_n \to \infty$ and the hyperbolic radius of $D_n$ is $d(z_n, p)/2$. Since $\phi_n$ has bounded Laplacian, the mass of $\Delta \phi_n$ restricted to any compact $K$ in $\mathbb{D}$ is bounded, and thus we can take a subsequence that converges weakly to a positive measure $\mu$ in $\mathbb{D}$ which satisfies $(1 - |z|)^2 \mu \simeq 1$ because all the measures $\Delta \phi_n$ satisfy this inequality with uniform constants. Let $\phi$ be such that $\Delta \phi^* = \mu$. Since $\Lambda_n = D_n \cap \Lambda$ are all separated with uniform bound, there is a weak limit $\Lambda^*$. The functions $f_n$ in the disks can be modified by a factor $e^{\phi_n}$ in such a way that $h_n = f_n e^{\phi_n}$ satisfies $h_n(0) = 1$ and $|h_n| \leq e^{\phi_n + \text{Re}(\phi_n)}$, if $n$ is big enough and $\sup_{\Lambda_n} |h_n e^{-\phi_n + \text{Re}(\phi_n)}| \leq 1/n$. We can add an harmonic function $v$ to $\phi^*$ in such a way that $\phi_n + \text{Re}(\phi_n) \to v + \phi^*$ uniformly on compact sets. Thus $h_n$ has a subsequence convergent to $h \in A_{\phi^*}$, $h(0) = 1$ and $h|_{\Lambda^*} \equiv 0$, which, by assumption, is not possible.

In the other direction, we assume that $\Lambda$ is a sampling sequence for $A_\phi(S)$, and that $(\mathbb{D}, \phi^*, \Lambda^*) \in W(S, \phi, \Lambda)$. We want to prove that any $f \in A_{\phi^*}(\mathbb{D})$ that vanishes in $\Lambda^*$ is identically 0. Take a sequence of points $z_n$ that escapes to infinity and $(D_n, \phi_n, \Lambda_n)$ the associated triple that converges weakly to $(\mathbb{D}, \phi^*, \Lambda^*)$. As $\phi_n \to \phi^*$ and $\Lambda_n \to \Lambda^*$ uniformly on compact sets we can take a sequence of radii $s_n$ such that

\[ d(\Lambda \cap D(z_n, s_n), \Lambda^* \cap D(0, s_n)) < 1/n, \ D(z_n, s_n) \subset D(z_n, r_n) \quad \text{and} \quad |\phi_n - \phi^*| \leq 1/n. \]

If $f$ vanishes in $\Lambda^*$, this means that $f$ is very small in $D(z_n, s_n) \cap \Lambda$. Assume that $f(0) = 1$. Take a cutoff function $\chi_n$ such that $\chi_n \equiv 0$ outside $D(z_n, s_n)$, $\chi(z_n) = 1$, and $\langle d\chi \rangle < \varepsilon_n$. Define $g_n = f\chi_n - u_n$, where $\partial\bar{u} = f\partial\bar{\chi}_n$ are the solution estimates by Theorem 6. Clearly $g_n$ is small in all points of $\Sigma$ and it has at least norm 1. Thus we are contradicting the fact that $\Lambda$ is sampling.

Observe that one particular instance of finite Riemann surface, where we can apply the result, are the disks $D_i$ associated to the tunnels with the metric $\phi_i$. The final piece for the proof of Theorem 9 is then the following.

**Lemma 14.** If $S$ is a finite Riemann surface, $\phi$ a subharmonic function with bounded Laplacian and $\Lambda$ is a uniformly separated sequence, then all possible weak limits coincide with the weak limits of the disks associated to the surface, that is,

\[ W(S, \phi, \Lambda) = W(D_1, \phi_1, \Lambda_1) \cup \ldots \cup W(D_n, \phi_n, \Lambda_n). \]

Proof. The proof amounts to the observation that the metric in $D_i$ converges uniformly to the metric in $S$ as $z \to \partial D_i$, and in the definition of weak limits we only consider uniform convergence over compacts.

Theorem 9 follows now immediately from Theorem 13 and Lemmas 10 and 14.

Now Theorems 7 and 9 show that the property of being a sampling/interpolating sequence is determined by the behavior near the boundary, more precisely in the associated disks. In these
disks there is a precise description of the interpolating and sampling sequences (see [1, 9]) that can be transported to the surface. If we rewrite it we get the density conditions of Theorem 1, but the disks are not hyperbolic disks on the surface, and they correspond to hyperbolic disks in disks $D_i$, but since the condition is only relevant near the boundary, the disks in both metrics look more and more similar. Moreover the difference between the corresponding Green functions converges to 0 uniformly as we approach the boundary. Finally, as the sequence is uniformly discrete and the Laplacian of the weight is bounded above and below, the small difference is absorbed by the fact that the inequalities are strict, and this proves Theorem 1. In fact it is possible to replace the Green function $g_r$ of $D(z,r)$ by the Green function $g$ of $S$ in the definition of the density, (1.1), because, as before, $\sup_{w \in D(z,r)} |g_r(z,w) - g(z,w)| \to 0$ as $z$ approaches the boundary.

4. Some $L^p$-variants

We have considered until now pointwise growth restrictions. It is possible to obtain other results from our theorem in different Banach spaces of holomorphic functions. Consider, for instance, the weighted Bergman spaces

$$A^p_\phi(S) = \left\{ f \in \mathcal{H}(S); \int_S |f|^p e^{-\phi} \, dA < +\infty \right\},$$

where $dA$ is the hyperbolic area measure in $S$ and $p \in [1, \infty)$. The natural problem in this context is the following.

**Definition 15.** Let $S$ be a finite Riemann surface, and let $\phi$ be a subharmonic function with bounded Laplacian bigger than one, that is, $1 + \varepsilon < \Delta \phi < M$.

- A sequence $\Lambda \subset S$ is interpolating for $A^p_\phi(S)$ if for any value $v_\lambda$ such that

$$\sum_{\lambda \in \Lambda} |v_\lambda|^p e^{-\phi(\lambda)} < \infty,$$

there is a function $f \in A^p_\phi(S)$ such that $f(\lambda) = v_\lambda$.

The spaces $A^p_\phi$ can be empty if we only ask $\phi$ to be with positive bounded Laplacian. It is then natural to require that the Laplacian is strictly bigger than one so that the Laplacian plus the curvature of the metric in the manifold is strictly positive and there are functions in the space (consider the case of the disk $S = \mathbb{D}$, for instance).

Let $\phi_0$ be a subharmonic function in $S$ such that $\Delta \phi_0 = 1$. The corresponding theorem will be the following.

**Theorem 16.** Let $S$ be a finite Riemann surface, and let $\phi$ be a subharmonic function with bounded Laplacian strictly bigger than one. Let $p \in [1, +\infty)$ and $\Lambda$ be a separated sequence. The sequence is interpolating for $A^p_\phi(S)$ if and only if $D^*_\phi(\phi - \phi_0)(\Lambda) < 1/p$.

In the case of the unit disk $dA(z) = (1 - |z|)^{-2}$ this description is well known; see, for instance, [14, Theorems 2 and 3].

**Proof.** The proof of the theorem is the same *mutatis-mutandi* as in the $L^\infty$ setting. The basic tool that allows us to glue the pieces together is the next theorem, which is the generalization of Theorem 6 and is proved in the same way.
Theorem 17. Let \( S \) be a finite Riemann surface, let \( \phi \) be a subharmonic function with a bounded Laplacian strictly bigger than one and let \( p \in [1, \infty) \). There is a constant \( C = C(p, S) > 0 \) such that for any \((0, 1)\)-form \( \omega \) on \( S \) there is a solution \( u \) to the inhomogeneous Cauchy–Riemann equation \( \bar{\partial}u = \omega \) in \( S \) with the estimate
\[
\int_S |u(z)|^p e^{-\phi(z)} dA(z) \leq C \int_S \langle \omega(z) \rangle^p e^{-\phi(z)} dA(z),
\]
whenever the right-hand side is finite.

Proof. The proof of this result is again the same as that of Theorem 6. We can separately solve the Cauchy–Riemann equation in each funnel using [8, Theorem 2], and glue them together with a Cauchy–Riemann equation with data that have compact support and can be solved using the operator (2.2).

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