RESEARCH PAPER

Homogenous balance method for solving exact solutions of the nonlinear Benny -luke equation and Vakhnenko-Parkes equation

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A B S T R A C T:

In this article, We apply the homogeneous balance method to construct the many families of exact solution of travelling wave solution of nonlinear equations, the Benny -Luke equation and vakhnenko-parkes equation. As the result, many solitary wave solution are obtained from the solution by hyperbolic function when the parameters were taken at special values and we compared the result with the solution of F-expansion method and (κ) -expansion method, we obtained the same results by certain hypotheses. Also we drew a 3D graph of exact solution for the special Benny -luke equation and vakhnenko-parkes equation by help of the maple.

KEY WORDS: Application, Homogeneous balance method ,The Benny-Luke equation, The Vakhnenko-Parkes equation

1. INTRODUCTION:

The study of nonlinear partial differential equations (PDEs) happened anywhere in applied mathematics and physics including engineering sciences and biological sciences. Exact solutions for nonlinear partial differential equations essential role in many phenomena in such as fluid mechanics, hydrodynamics, optics, plasma physics and so on (Zayed and Arnous, 2012). The homogeneous balance method, which is an important and suitable algebraic method for the computation of exact travelling wave solution, was first introduced by (Wang et al., 1996; Wang M, 1995). This method was further developed by many authors (Elwakil et al., 2004, 2003, 2007; Fan, 2003, 2000; M.Khalfallah, 2009a, 2009b; Zhao X et al., 2006).

Akbar et al (Akbar et al., 2017) used the F-expansion method to obtained the exact solution of the Benney –Luke equation in the following form

\[ u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} + u_{tt}u_{xx} + 2u_{x}u_{xt} = 0 \]

(1.1)

Where a and b nonzero constant, and one of solution obtained by F-expansion method is

\[ u(x,t) = \alpha_0 \pm \frac{4(a-b)}{\sqrt{(4ak+1)(4bk+1)}} \left( m - \sqrt{-k} \tanh(\sqrt{-k} \eta) \right) \]

(1.2)

where \( \eta = \frac{x \pm \left( \frac{2k-1}{4k-1} \right) t}{2} \)

Also, Abazari (Abazari, 2010) studies the Vakhnenko-parkes equation
\[
\frac{uu_{xt} - u_xu_{xt} + u^2u_t}{\xi} = 0 \quad (1.3)
\]

by applying the \( \left( \frac{G'}{G} \right)^n \)-expansion method, he obtained the exact solution of this equation

\[
u(x, t) = \frac{3k^2(\lambda^2 - 4\mu)(c_0^2 - c_1^2)}{2(c_0 \sinh(\frac{\lambda^2 - 4\mu}{2} \xi) + c_1 \cosh(\frac{\lambda^2 - 4\mu}{2} \xi))^2}
\]

(1.4)

In this paper, by using the homogenous balance method, we prove that the travelling wave solution of nonlinear equation can be expressed by a polynomial in \( \sum_{i=0}^{m} q_i \phi^i \), \( u = u(\xi) \), such that \( \xi = x - \beta t \), the degree of this polynomial can be obtained from considering the homogeneous balance method between the highest order derivatives and the nonlinear term appearing in the given equation, the coefficient of this polynomial can be got from solving a set of algebraic equation (Wang et al., 2008). The power of these methods, will make the determination solitary type of solution is easy and we used the method to get more general exact solution of the nonlinear equations of the Benny–luke equation and vakhnenko-parkes equation.

2-Application of homogeneous balance method

2.1- The Benny –luke equation

In this section, the homogenous balance method is used to calculate the exact solution for the Benny –Luke equation (1.1).

We use wave transformation to reduce equation (1.1) to the following ODE;

\[
(x, t) = u(\xi), \quad \xi = x - \beta t
\]

(2.1.1)

Where \( \beta \) is pace of the travelling wave Eq. (2.1.1) transforms Eq. (1.1) into the following ODEs

\[
(\beta^2 - 1)u'' + (a - b\beta^2)u'' - 3\beta u'u'' = 0
\]

(2.1.2)

Integrating Eq. (2.1.2) with respect \( \xi \) once and setting the constant of integration to zero, we obtained

\[
(\beta^2 - 1)u' + (a - b\beta^2)u'' - \frac{3}{2} \beta (u')^2 = 0
\]

(2.1.3)

Balancing \( (u''') \) with \( (u')^2 \) yields \( m=1 \) then results;

\[
u = q_0 + q_1 \phi
\]

(2.1.4)

Where \( q_0 \) and \( q_1 \) are constant and we can determined the derivatives of \( u(x, t) \) in the Eq. (2.1.3) such that \( q_1 \neq 0 \) from equation (2.1.4). Substituting \( u, u', u'' \) in Eq.(2.1.3) and then equating all the coefficient of power of \( \phi \) to zero results the following:

\[
(\phi)^0: q_1 b\beta^2 - q_1 \beta^2 - aq_1 + q_1 = 0
\]

\[
(\phi)^1: -\frac{3}{2} \beta q_1^2 - q_1 - 7q_1 b\beta^2 + 7aq_1 + q_1 \beta^2 = 0
\]

\[
(\phi)^2: 12q_1 b\beta^2 + 3bq_1^2 - 12aq_1 = 0
\]

\[
(\phi)^3: 6aq_1 - \frac{3}{2} \beta q_1^2 - 6bq_1 \beta^2
\]

= 0

Solving the equations by using the maple, we obtain the value of

\[
\beta = \pm \sqrt{\frac{a-1}{b-1}}, \quad q_1 = \pm \frac{2(a-b)}{\sqrt{(a-1)(b-1)}}
\]

(2.1.5)

\[
u(x, t) = q_0 \pm \frac{2(a-b)}{\sqrt{(a-1)(b-1)}} \left( 1 - \tanh \left( \frac{\frac{a-1}{b-1}}{2} \right) \right)
\]

(2.1.6)

Table 1: Comparison of proposed solution with Solution of F-expansion method

| Solution of F-expansion method | proposed homogeneous balance solution |
|--------------------------------|--------------------------------------|
| If \( k = \frac{-1}{4}, m = \frac{1}{2} \) then Eq.(1.2) become | If \( q_0 \neq 0 \) Eq.(2.1.5) becomes |
| \( u(x, t) = a_0 \) | \( u(x, t) = q_0 \pm \) |
| \( \pm \frac{2(a-b)}{\sqrt{(a-1)(b-1)}} \left( 1 - \tanh \left( \frac{\frac{a-1}{b-1}}{2} \right) \right) \) | \( \pm \frac{2(a-b)}{\sqrt{(a-1)(b-1)}} \left( 1 - \tanh \left( \frac{\frac{a-1}{b-1}}{2} \right) \right) \) |

2.2- Vakhnenko –parkes equation:
We would like to apply the proposed method to obtain a travelling wave solution of (1.3) from equation (2.2.1)

\[ u = u(\xi), \quad \xi = kx + wt \]  

(2.2.1)

Where \( w \) and \( k \) are constants. We substitute Eq. (2.2.1) into Eq. (1.3) to get the nonlinear ordinary differential equation

\[ k^2 w u'''' - k^2 w u'' + w u' + w^2 u' = 0 \]  

(2.2.2)

Integrating Eq. (2.2.2) once with respect to \( \xi \) and setting the integrating constant as zero yields

\[ 3k^2 u'''' - 3k^2 (u')^2 + u^3 = 0 \]  

(2.2.3)

We balancing between \( u^3 \) and \((u')^2\). We get \( m = 2 \). We then suppose that Eq. (2.2.3) has the following formal solution:

\[ u(\xi) = q_0 + q_1 \varphi + q_2 \varphi^2 \]  

(2.2.4)

Where \( q_0 \), \( q_1 \) and \( q_2 \) are constant which are unknown to be determined such that \( q_2 \neq 0 \) from (2.2.4)

Substituting \( u, u' \) and \( u'' \) in (2.2.3) and equating all the coefficients of power of \( \varphi \) to zero

\[ (\varphi)^0: q_0^3 = 0 \]

\[ (\varphi)^1: 3k^2 q_0 q_1 + 3q_0^2 q_1 = 0 \]

\[ (\varphi)^2: -9k^2 q_0 q_1 + 12k^2 q_0 q_2 + 3q_0^2 q_2 + 3q_0 q_1^2 = 0 \]

\[ (\varphi)^3: 6k^2 q_0 q_1 - 30k^2 q_0 q_2 - 3k^2 q_1^2 - 3k^2 q_1 q_2 + 6q_0 q_2 + 3q_0^2 q_1 + q_1^3 = 0 \]

\[ (\varphi)^4: 18k^2 q_0 q_2 + 3k^2 q_1^2 - 15k^2 q_1 q_2 + 3q_0 q_2^2 + 3q_0^2 q_2 = 0 \]

\[ (\varphi)^5: 12k^2 q_1 q_2 - 6k^2 q_1^2 + 3q_1 q_2^2 = 0 \]

\[ (\varphi)^6: 6k^2 q_2^2 + q_2^3 = 0 \]

Solving the set of the algebraic equation by using maple software, we obtained the following result

\[ q_0 = 0, \quad q_1 = 6k^2, q_2 = -6k^2 \]

\[ u(x,t) = \frac{3}{2} k^2 \text{sech}^2 \left( \frac{kx + wt}{2} \right) \]  

(2.2.5)

Table 2: Comparison of proposed solution with Solution of \( \left( \frac{C_c}{\varphi} \right) \)-expansion method

| Method | proposed homogeneous balance solution |
|--------|----------------------------------------|
| If \( \lambda = 1, m = 0, c_2 = 0 \) and \( c_1 = 1 \) then Eq. (1.4) becomes \( u(x,t) = \frac{3k^2}{2} \text{sech}^2 \left( \frac{kx + wt}{2} \right) \) |

3- 3D graph of exact solution for Benny -Luke equation and Vakhnenko-Parkes equation

Fig. 1 The solitary wave 3D graphics of Eq. (2.1.5) for \( a = 2, b = 3, x = -10, ..., 10 \) and \( t = 0, ..., 10 \)
Fig. 2: The solitary wave 3D graphics of Eq.(2.2.5) for $k = 2, w = 2, x = -10, ..., 10$ and $t = 0, ..., 10$.

4-Remark of homogeneous balance method:

In this research paper, the homogeneous balance method is proposed to find the exact solution for nonlinear equations, the Benny-Luke equation and Vakhnenko-Parkes equation for the first time. A comparison was made between this method and other methods under special circumstances and we found that this method is simpler than other methods and this solution will be very useful in difficult application. The advantage of this method is clear through the ability to apply it to a different selected equation, and hopefully it will be used for future studies in applied sciences.

The proposed method has high efficiency and practicality in finding exact solutions, also it is shown that the proposed method is direct, effective and can be applied to many other nonlinear evolution equations in mathematical physics.

We now summarize the mechanism of homogeneous balance method, by six steps as follows

\textbf{Step 1.} Consider we have a nonlinear partial differential equations in the form

$$F(u, u_t, u_x, u_{xt}, u_{xx}, ...) = 0$$

(4.1)

where $F$ is a polynomial in $u(x, t)$ and its PDEs.

\textbf{Step 2.} Using the wave transformation (4.2)

$$u(x, t) = u(\xi), \, \xi = x - \beta t,$$

(4.2)

to reduce equation (4.1) to the following ODE

$$P(u, u', u'', ...) = 0$$

(4.3)

where $P$ is a polynomial in $u(\xi)$.

\textbf{Step 3.} Suppose that equation (4.3) has the formal solution

$$u(\xi) = \sum_{i=0}^{m} q_i \varphi^i$$

(4.4)

where $q_i (i = 0, 1, \ldots, m)$ are constants to be determined, such that $q_m \neq 0$,

and $\varphi(\xi)$ is the solution of the equation (4.5)

$$\varphi'(\xi) = \varphi^2(\xi) - \varphi(\xi).$$

(4.5)

The equation (4.5) has the solution

$$\varphi(\xi) = \frac{1}{1 + \varepsilon^\xi}$$

(4.6)

\textbf{Step 4.} We calculate all the derivatives of $u, u', u''$, ... for the polynomial $u(\xi)$ such that, we get

$$u = q_0 + q_1 \varphi + q_2 \varphi^2 + \ldots$$

$$u' = (\varphi - 1) \varphi[q_1 + 2q_2 \varphi + \ldots]$$

$$u'' = (\varphi - 1) \varphi[(-1 - 2\varphi)q_1 + 2\varphi(-2 + 3\varphi)q_2 + \ldots]$$
\[ u''' = (\varphi - 1)\varphi[(1 - 6\varphi + 6\varphi^2)q_1 + 2\varphi(4 - 15\varphi + 12\varphi^2)q_2 + \cdots] \]

\[ u^{(4)} = (\varphi - 1)\varphi[(-1 + 14\varphi - 36\varphi^2 + 24\varphi^3)q_1 + 2\varphi(-8 + 57\varphi - 108\varphi^2 + 60\varphi^3)q_2 + \cdots] \]

\[ u^{(5)} = (\varphi - 1)\varphi[(-1 - 30\varphi + 150\varphi^2 - 240\varphi^3 + 120\varphi^4)q_1 + 2\varphi(16 - 165\varphi + 660\varphi^2 - 840\varphi^3 + 360\varphi^4)q_2 + \cdots] \]

**Step 5.** Determine the positive integer \( m \) in equation (4.4) by considering the homogeneous balance between the highest order derivatives and the nonlinear terms in equation (4.3).

**Step 6.** Substitute equation (4.4) into equation (4.3), we calculate all the necessary derivatives \( u', u'', \ldots \) of the equation \( u(\xi) \). As a result of this substitution, we obtain a polynomial of \( (\varphi)^j, (i = 0, 1, 2, \ldots) \). In this polynomial we gather all terms of same powers and equating them to zero, we obtain a system of algebraic equations which can be solved by the Maple to get the unknown parameters \( q_m(n = 0, 1, \ldots, m), \beta \).

Consequently, obtain the exact solutions of equation (4.1)

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