ON NON-RATIONAL FIBERS OF DEL PEZZO FIBRATIONS OVER CURVES

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Abstract. We consider threefold del Pezzo fibrations over a curve germ whose central fiber is non-rational. Under the additional assumption that the singularities of the total space are at worst ordinary double points, we apply a suitable base change and show that there is a 1-to-1 correspondence between such fibrations and certain non-singular del Pezzo fibrations equipped with a cyclic group action.

Introduction

It is classically known that a cubic del Pezzo surface can degenerate into a cone over an elliptic curve in a non-singular family. We investigate when a del Pezzo surface can degenerate into a non-rational surface in a “reasonably good” family. By such family we mean a del Pezzo fibration in the sense of the Minimal Model Program (the MMP for short), see Definition 1.1. In particular, the total space of the fibration should have at worst terminal singularities. The main invariant of such fibrations is the degree $K_{X_0}^2$ of its general fiber. Since the general fiber is non-singular, $1 \leq K_{X_0}^2 \leq 9$. Our question is local, so we consider fibrations over curve germs.

The motivation for the problem comes from the three-dimensional MMP. If we apply the MMP to a (non-singular) rationally connected threefold $U$ over the field of complex numbers, we obtain a variety $X$ birational to $U$ such that it admits a Mori fiber space structure. That is, there is a morphism $\pi : X \to B$ with connected fibers, $\pi$-ample anti-canonical class $-K_X$ and $\dim B < \dim X$. If $\dim B = 0$ then $X$ is a Fano variety. The rationality problem for (singular) Fano threefolds is far from complete solution, although much is known in the non-singular case, see [IP99, Chapter 12]. If $\dim B = 2$ then $\pi$ is called a $\mathbb{Q}$-conic bundle. Its fibers are trees of rational curves. In this case the rationality problem for the fibers of $\pi$ is trivial. We work with the case $\dim B = 1$ which is called a del Pezzo fibration. Its general fiber is rational. But a special fiber can be non-rational. It is easy to show that such fiber is a surface which is birationally ruled over a curve $C$ of genus $g(C) > 0$.

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In this paper we show that the properties of such del Pezzo fibrations that contain a non-rational fiber, for example the value of \( g(C) \), depend on \( K^2_{X_n} \) and on singularities of \( X \). In Proposition 1.3 we prove that if \( X \) is non-singular (respectively, terminal Gorenstein) then \( K^2_{X_n} \leq 3 \) (resp., \( \leq 4 \)) and the non-rational fiber is a cone over an elliptic curve. This fact is rather elementary and follows from the classification of Gorenstein del Pezzo surfaces [HW81]. As mentioned in Remark 1.2 in the terminal Gorenstein case any fiber is reduced and irreducible, and moreover, a non-rational fiber is necessarily normal. On the other hand, in the non-Gorenstein terminal case, multiple fibers are possible. However, their multiplicity is bounded by 6 as shown in [MP09].

In Theorem 2.4 we use the base change construction to show that in the non-singular case such del Pezzo fibrations with a non-rational fiber are in 1-to-1 correspondence with non-singular \( \mu_n \)-del Pezzo fibrations with certain properties.

This shows that the non-rational fibers of terminal Gorenstein del Pezzo fibrations form a very restricted class. On the other hand, if we allow \( X \) to have worse than terminal singularities then the non-rational fibers are not bounded, see Example 1.7. We also give examples of terminal fibrations whose special fiber is birationally ruled over a curve \( C \) of genus \( g(C) = 2, 3, 4 \). It is not known whether one can achieve \( g(C) > 4 \) in this setting, see Question 1.6.

Then we consider the fibrations with very mild singularities, the ordinary double points. Using the base change construction, we classify such fibrations with non-rational central fiber in terms of certain \( \mu_n \)-del Pezzo fibrations, see Theorem 3.4. It appears that in this case \( K^2_{X_n} = 1 \) or 4.

For other results on rationality in families see [KT17], [T16], [P17] and references therein. For the classification of non-rational del Pezzo surfaces see [HW81], [F95].

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1. Preliminaries

We work over the field of complex numbers. We use terminology and notation of the Minimal Model Program (e.g., [Ma02], [KMM87]).

**Definition 1.1.** Let \( X \) be a three-dimensional normal projective variety with at worst terminal \( \mathbb{Q} \)-factorial singularities and let \( B \) be a non-singular curve. Then \( \pi: X \longrightarrow B \) is called a **del Pezzo fibration** (resp., a **weak del Pezzo fibration**) if the following conditions hold:

(i) \( \pi \) is projective and has connected fibers;

(ii) \(-K_X\) is \( \pi \)-ample (resp., \( \pi \)-nef and \( \pi \)-big);
(iii) $\pi$ is an extremal contraction, that is $\rho(X/B) = 1$.

The degree of a (weak) del Pezzo fibration is the degree of its general fiber $X_\eta$. Since $X$ is terminal $X_\eta$ is a non-singular del Pezzo surface.

We say that a del Pezzo fibration $\pi: X \to B$ is non-singular (resp., Gorenstein) if so its total space $X$. If in the above definition $X$ is an complex analytic space and $\pi$ is a proper map, we call $\pi: X \to B$ an analytic del Pezzo fibration. When we consider $X$ as a germ over $o \in B$ we use the notation $\pi: X \to B \ni o$.

Let $G$ be a group. Then one can define a $G$-del Pezzo fibration as in Definition 1.1 with the following modifications: we require $X$ to be $G$-factorial (that is, every $G$-invariant Weil divisor is $Q$-Cartier) and have $\rho^G(X/B) = 1$. In this paper we will work with $\mu_n$-del Pezzo fibrations where $\mu_n$ is the cyclic group of order $n$. We fix a primitive root of unity of degree $n$ and denote denote it by $\zeta_n$.

Remark 1.2. Let $\pi: X \to B \ni o$ be a Gorenstein del Pezzo fibration. Consider the fiber $F = \pi^{-1}(o)$. Since $\rho(X/B) = 1$ the fiber $F$ is irreducible. Since $X$ is Gorenstein $F$ is reduced [Ka88, 5.1]. Assume that $F$ is non-rational. Then $F$ is normal [R94, AF03].

Proposition 1.3. Let $\pi: X \to B \ni o$ be a Gorenstein del Pezzo fibration such that the fiber $F = \pi^{-1}(o)$ is non-rational. Then $F$ is a generalised cone over an elliptic curve and $K^2_F \leq 4$. Moreover, if $X$ is non-singular then $K^2_F \leq 3$.

Proof. The first claim follows from the classification of Gorenstein del Pezzo surfaces, see for example [HW81]. Notice that $F$ has only one simple elliptic singularity $x_0$. Let $\phi: T \to F$ be the minimal resolution. We have $K_T = \phi^*K_F - E_0$ where $E_0$ is an elliptic curve. Thus, $K^2_T = K^2_F + E_0^2 = d + E_0^2$. By the Noether formula $K^2_T + \chi_{\text{top}}(T) = 12\chi(O_T) = 0$, and $\chi_{\text{top}}(T) = 0$ since $T$ is a ruled surface over an elliptic curve, so $d = -E_0^2$. On the other hand, by [KM98, 4.57] the dimension of the tangent space at $x_0$ to $F$ is equal to $\max(3, -E_0^2)$. If $X$ is Gorenstein it has hypersurface singularities, hence $-E_0^2 = d \leq 4$. If $X$ is non-singular, $-E_0^2 = d \leq 3$, so we are done. $\square$

The next example shows that the case $d = 4$ occurs.

Example 1.4. Let $X$ be given by the equations

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + tx_5^2 = 0,$$

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 + tx_5^2 = 0$$

in $\mathbb{P}^4 \times \mathbb{A}_t^1$ where $a_i \in \mathbb{C}$. One checks that for a general choice of $a_i$ the threefold $X$ has one $cA_1$ singularity, and the fiber $F$ over $0 \in \mathbb{A}_t^1$ is a cone over an elliptic curve.
There are examples of non-Gorenstein fibrations with a non-rational fiber that is birationally ruled over the curve $C$ with $g(C) > 1$.

**Example 1.5.**

(i) $X = (f_6(x, y, w) + tz^3 = 0) \subset \mathbb{P}(1, 1, 2, 3) \times \mathbb{A}^1_t$ where $(x, y, z, w)$ have the weights $(1, 1, 2, 3)$, the polynomial $f_6$ has degree 6 and is general. The morphism $\pi : X \rightarrow B = \mathbb{A}^1_t$ is induced by the projection to the second factor. Notice that $X$ has one terminal singularity of type $\frac{1}{2}(1, 1)$. A general fiber is a degree 1 del Pezzo surface. The central fiber $F$ is a cone over a hyperelliptic curve $C$ of genus 2.

(ii) $X = (f_4(x, y, z) + tw^2 = 0) \subset \mathbb{P}(1, 1, 1, 2) \times \mathbb{A}^1_t$ where $(x, y, z, w)$ have the weights $(1, 1, 1, 2)$. Notice that $X$ has one terminal singularity of type $\frac{1}{2}(1, 1, 1)$. A general fiber is a degree 2 del Pezzo surface. The central fiber $F = \pi^{-1}(0)$ is a cone over a plane quartic curve $C$, so $g(C) = 3$.

(iii) $X = (f_6(x, y, z) + tw^2 = 0) \subset \mathbb{P}(1, 1, 2, 3) \times \mathbb{A}^1_t$ where $(x, y, z, w)$ have the weights $(1, 1, 2, 3)$. Notice that $X$ has one terminal singularity of type $\frac{1}{3}(1, 1, 2)$. A general fiber is a degree 1 del Pezzo surface. The central fiber $F$ is a cone over a trigonal curve $C$ of genus 4.

In the above examples $F$ is normal. However, if we take a special polynomial $f_i$ we can get a non-normal and non-rational fiber. This contrasts with the Gorenstein case. The following natural question was posed by J. Blanc:

**Question 1.6.** Is there a del Pezzo fibration $\pi : X \rightarrow B$ such that its fiber is birationally ruled over a curve $C$ with $g(C) > 4$?

At the moment, the answer to this question is not known. Terminal singularities is an important restriction as the following example shows.

**Example 1.7.** For a moment we consider a fibration that has worse than terminal singularities. Define $\pi : X \rightarrow B$ as follows:

$$X = (f_n(x, y, z) + tw = 0) \subset \mathbb{P}(1, 1, 1, n) \times \mathbb{A}^1_t$$

where the coordinates $x, y, z, w$ have the weights $(1, 1, 1, n)$ and the polynomial $f_n$ is general and has degree $n$. Clearly, $X$ has one singular point of type $\frac{1}{n}(1, 1, 1)$. In particular, $X$ is log terminal. A general fiber is isomorphic to $\mathbb{P}^2$. The fiber over $t = 0$ is a cone over a plane curve of degree $n$. One can construct similar (log terminal) degenerations to a cone over a curve of arbitrarily large genus in del Pezzo fibrations of any degree $1 \leq d \leq 9$, see [K13, 3.9].
2. NON-SINGULAR FIBRATIONS

Let \( \pi : X \longrightarrow B \ni o \) be a non-singular del Pezzo fibration such that the fiber \( F = \pi^{-1}(o) \) is non-rational. Then \( K_F^2 \leq 3 \) by Proposition 1.3. We start with the description of the base change construction.

**Construction 2.1.** Let \( x_0 \) be the (simple elliptic) singularity of \( F \). By [KM98, 4.57] there exists a weighted blow-up \( \psi : Z \longrightarrow X \) of \( x_0 \in X \) with the weights \((c_1, c_2, c_3)\) for some \( c_i \) such that \( F_Z = \psi^{-1}_* F \) is the minimal resolution of \( F \). We have \( K_Z = \psi_\ast F - E |_{F_Z} \). In this case \( E |_{F_Z} \) is reduced irreducible non-singular elliptic curve, call it \( C \). Notice that \( F_Z = \psi_* \pi^{-1}_* F \) for \( n \geq 2 \), and \( E \simeq \mathbb{P}(c_1, c_2, c_3) \). Then

\[
K_Z = \psi_* \pi_\ast K_X + (n - 1)E, \quad n = c_1 + c_2 + c_3.
\]

After the blow-up \( \psi \) the threefold \( Z \) may obtain some number of cyclic quotient singularities. However, \( F_Z \) does not pass through them. Indeed, let \( z_0 \) be a singular point on \( Z \) and suppose that \( z_0 \in F_Z \). Since \( z_0 \) is a cyclic quotient singularity, \( C^3 \) covers an analytic neighbourhood \( U \) of \( z_0 \). This covering induces an unramified covering of \( F_Z \cap U - \{z_0\} \). But \( F_Z \) is non-singular, hence \( \pi_1(F_Z \cap U - \{z_0\}) = 0 \). This is a contradiction.

Now we make a base change. Pick a local coordinate \( t \) at the point \( o \in B \) and consider the following commutative diagram:

\[
\begin{array}{ccc}
W & \longrightarrow & Z \\
\downarrow_{\pi_W} & & \downarrow_{\pi} \\
B' & \longrightarrow & B
\end{array}
\]

where \( B' \simeq B, \alpha : t \mapsto t^n \) and \( W \) is the normalization of \( Z \times_B B' \). At a general point of \( E \) the threefold \( Z \) is isomorphic to

\[
\text{Spec } \mathbb{C}[x, y, z, t] / (t - z^n),
\]

and the fiber \( \pi^{-1}_Z(0) \) is given by \( (t = 0) \). After the base change we have

\[
\text{Spec } \mathbb{C}[x, y, z, t] / (t^n - z^n)
\]

which is singular in codimension 1. After the normalization we see that \( h \) is étale in the neighbourhood of a general point of \( E_W := h^{-1}(E) \). Similarly, one can check that the morphism \( h \) is ramified along \( F_W := h^{-1}(F_Z) \) and at all the singular points of \( Z \).

The fiber \( \pi^{-1}_W(0) \) is reduced. However, it is reducible: \( \pi^{-1}_W(0) = F_W + E_W \), where \( E_W \) covers \( E \), and \( F_W \) is isomorphic to \( F_Z \) via \( h \). More precisely, \( h|_{E_W} \) is totally ramified at \( E_W \cap F_W =: C_W \). It follows that \( F_W \) is non-singular. Moreover, \( F_W \) and \( E_W \) intersect transversally. The Galois group \( \mathbb{G}_n \) of \( h \) acts on \( W \) preserving the central fiber. We make a \( \mathbb{G}_n \)-equivariant contraction of \( F_W \) (see computation below) and get a
µₙ-del Pezzo fibration \( \pi_V : V \to B \ni o \) with a rational central fiber.
All these maps are shown in the following diagram:

\[
\begin{array}{ccc}
F_W + E_W \subset W & \xrightarrow{h} & Z \supset F_Z + nE \\
\downarrow \tau & & \downarrow \psi \\
E_V \subset V & \xrightarrow{\pi_V} & X \supset F \\
\downarrow \pi_V & & \downarrow \pi \\
B' & \xrightarrow{\alpha} & B
\end{array}
\]

(2.2)

**Computation 2.3.** As before, \( \phi : T \to F \) is the minimal resolution. Denote by \( f_T \) a ruling of \( T \), and by \( f_Z \) a ruling of \( F_Z \), put \( f := \psi(f_Z) \).

We need the following formulas.

\[
K_F \cdot f = \phi^* K_F \cdot f = \phi^* K_f \\
= (K_T + E_0) \cdot f_T = -2 + 1 = -1,
\]

\[
K_Z \cdot f_Z = (\psi^* K_X + (n - 1)E) \cdot f_Z \\
= K_X \cdot f + n - 1 \\
= K_F \cdot f + n - 1 = n - 2.
\]

We want to contract \( F_W \). We calculate \( K_W \cdot f_W \) where \( f_W \) is a ruling of \( F_W \cong F_Z \). Since \( h \) is totally ramified along \( F_W \) by the Hurwitz formula we have

\[
K_W = h^* K_Z + (n - 1)F_W.
\]

Since \( (F_W + E_W) \equiv 0 \) over \( B \) we get

\[
K_W \cdot f_W = (h^* K_Z + (n - 1)F_W) \cdot f_W \\
= K_Z \cdot f_Z - (n - 1)E_W \cdot f_W \\
= n - 2 - (n - 1) = -1.
\]

Thus \( F_W \) can be contracted to a non-singular curve. We get a contraction morphism \( \tau : W \to V \). By the Hurwitz formula for \( h|_{E_W} \) we have

\[
K_E = h|_{E_W}^* \left( K_E + \frac{n-1}{n} R \right), \quad K_E = -(c_1 + c_2 + c_3)H = -nH
\]

where \( R \sim bH \) is the ramification divisor, \( H \) is the positive generator of \( \text{Cl} \ E \cong \mathbb{Z} \), and \( b \in \mathbb{Z}_{\geq 1} \).

Now we go in the other direction. We start from a \( \mu_n \)-del Pezzo fibration \( \pi_V : V \to B \ni o \) with the following conditions: the central fiber \( E_V = \pi_V^{-1}(o) \) is \( \mu_n \)-invariant and has a fixed elliptic curve \( C_V \) such that the \( \mu_n \)-action on the projectivization of the normal bundle \( \mathbb{P}(N_{C/V}) \) is trivial. We blow-up \( C_V \) and obtain a \( \mu_n \)-del Pezzo fibration
\(\pi_W : W \rightarrow B \ni o\) with the central fiber \(E_W + F_W\). Denote the contraction morphism by \(\tau : W \rightarrow V\). By assumption, \(\mathbb{g}_n\) fixes \(F_W\) pointwise. We take the quotient \(h : W \rightarrow Z\) by the \(\mathbb{g}_n\)-action. Notice that \(h\) is ramified along \(F_W\), and \(E_W\) is a degree \(n\) cover of \(h(E_W) =: E\). Now we show that \(E\) can be contracted. One checks that any curve in \(E\) is \(K_Z\)-negative. It follows that there is a contraction morphism \(\psi : Z \rightarrow X\) to a terminal del Pezzo fibration \(\pi : X \rightarrow B \ni o\). We claim that the point \(x_0 := \psi(E)\) is non-singular on \(X\). We consider three cases.

(i) \(d = 3\). One checks that \(E_W/\mathbb{g}_3 \simeq \mathbb{P}^2\), and \(f\) is the blow-down to a non-singular point.

(ii) \(d = 2\). One checks that \(E_W/\mathbb{g}_4 \simeq \mathbb{P}(1,1,2)\), and \(f\) is the inverse of a weighted blow-up with the weights \((1,1,2)\) of a non-singular point.

(iii) \(d = 1\). One checks that \(E_W/\mathbb{g}_6 \simeq \mathbb{P}(1,2,3)\), and \(f\) is the inverse of a weighted blow-up with the weights \((1,2,3)\) of a non-singular point.

We are ready to prove the following theorem.

**Theorem 2.4.** Let \(\pi : X \rightarrow B \ni o\) be a non-singular del Pezzo fibration such that the fiber \(F = \pi^{-1}(o)\) is non-rational. Then there is 1-to-1 correspondence between such \(\pi\) and \(\mathbb{g}_n\)-del Pezzo fibrations \(\pi_V : V \rightarrow B \ni o\) with the following properties:

- the central fiber \(E_V = \pi_V^{-1}(o)\) is a non-singular \(\mathbb{g}_n\)-minimal del Pezzo surface of degree \(d\),
- the locus of fixed points of \(\mathbb{g}_n\) is an elliptic curve \(C \subset E_V\),
- the action of \(\mathbb{g}_n\) on \(\mathbb{P}(N_{C/V})\) is trivial.

There are only three possible cases (here \(d = K_F^2\)):

(i) \(d = 3\), \(n = 3\),
\[
E_V \simeq (w^3 = q_3(x,y,z)) \subset \mathbb{P}^3,
\mathbb{g}_3 : w \mapsto \zeta_3 w,
F \simeq (0 = q_3(x,y,z)) \subset \mathbb{P}^3;
\]

(ii) \(d = 2\), \(n = 4\),
\[
E_V \simeq (w^2 = q_4(x,y) + z^4) \subset \mathbb{P}(1,1,1,2),
\mathbb{g}_4 : z \mapsto \sqrt{-1} z,
F \simeq (w^2 = q_4(x,y)) \subset \mathbb{P}(1,1,1,2);
\]

(iii) \(d = 1\), \(n = 6\),
\[
E_V \simeq (w^2 = z^3 + \alpha x^4 z + \beta x^6 + y^6) \subset \mathbb{P}(1,1,2,3),
\mathbb{g}_6 : y \mapsto \zeta_6 y, \quad \alpha, \beta \in \mathbb{C},
F \simeq (w^2 = z^3 + \alpha x^4 z + \beta x^6) \subset \mathbb{P}(1,1,2,3).
\]
Proof. By Proposition 1.3 we have $d \leq 3$. We consider three cases: $d = -E^3 = 1, 2, 3$. According to [KM98, 4.57], $\text{mult}_{x_0} F = 3, 2, 2$, respectively. We apply the general construction described above.

**Case** $d = 3$. In this case we can take $\psi$ to be the standard blow-up of $x_0$. We have

$$K_Z = \psi^* K_X + 2E, \quad F_Z = \psi^* F - 3E$$

and $E \simeq \mathbb{P}^2$. By adjunction $K_{F_Z} = \psi^|_{F_Z} K_F - E|_{F_Z}$, and $F_Z$ is non-singular. By Construction we get a non-singular fibration into cubic surfaces $\pi_V : V \rightarrow B \ni o$ with the non-singular fiber $E_V = \pi^{-1}(o)$. Moreover, the group $\mu_3$ acts on $\pi_V$, and the fixed curve of this action is a non-singular elliptic curve $C_V$. Since $E_V$ is non-singular del Pezzo surface with the action of $\mu_3$, we may apply the classification of [DI10] and get the case (i) of the theorem.

**Case** $d = 2$. By [KM98, 4.57] up to an analytic change of coordinates in the neighbourhood of $x_0$ the fiber $F \subset X$ is given by the equation

$$q_4(x, y) + w^2 = 0,$$

and $\text{mult}_{x_0} q_4 = 4$. Blow up $x_0 \in X$ with the weights $(1, 1, 2)$ in $x, y, z$. Notice that the blow-up with the weights $(1, 1, 1)$ leads to a non-normal surface $F_Z$. We get

$$K_Z = \psi^* K_X + 3E, \quad F_Z = \psi^* F - 4E$$

where $E \simeq \mathbb{P}(1, 1, 2)$ is the exceptional divisor and $F_Z = \psi^{-1} F$. Notice that $F_Z$ is non-singular, and $Z$ has one singular point $p$ of type $\frac{1}{3}(1, 1, 1)$ which corresponds to the unique singular point $p$ of $E$. Put $C = E \cap F_Z$. The curve $C$ does not pass through $p$.

We apply Construction 2.1. Locally one checks that $h$ is ramified at two points $q_1, q_2 \in W$ such that $\{q_1, q_2\} = h^{-1}(p)$, and that $W$ is non-singular. Using the classification of [DI10] we get the case (ii) of the theorem.

**Case** $d = 1$. By [KM98, 4.57] up to an analytic change of coordinates in the neighbourhood of $x_0$ the fiber $F \subset X$ is given by the equation

$$w^2 + z^3 + zq_4(x) + q_6(x) = 0$$

where $\text{mult}_{x_0} q_i \geq i$. We blow-up $x_0 \in X$ with the weights $(1, 2, 3)$ in $x, y, z$. Denote the blow-up morphism by $\psi : Z \rightarrow X$. We get

$$K_Z = \psi^* K_X + 5E, \quad F_Z = \psi^* F - 6E$$

where $E \simeq \mathbb{P}(1, 2, 3)$. Notice that $F_Z$ is non-singular.

It is easy to see that $Z$ has two singular points $p_1$ and $p_2$ of types $\frac{1}{2}(1, 1, 1)$ and $\frac{1}{3}(1, 1, 2)$ which correspond to the singular points of $E$. Put $C = E \cap F_Z$. The curve $C$ does not pass through $p_1$, $p_2$. Locally
one checks that $h$ is ramified at the preimages of $p_1$ and $p_2$, and that $W$ is non-singular. Using the classification of [DIII] we get the case [iii] of the theorem.

3. Ordinary double points

Suppose that $\pi : X \longrightarrow B \ni o$ is a del Pezzo fibration with singularities that are analytically isomorphic to $(xy + zt = 0) \subset \mathbb{C}^4$. Such singularities are called ordinary double points. By Remark 1.2 the non-rational fiber $F = \pi^{-1}(o)$ is a reduced irreducible normal Gorenstein surface with a unique simple elliptic singularity $x_0 \in F$.

**Proposition 3.1.** Let $\pi : X \longrightarrow B \ni o$ be a del Pezzo fibration with at worst ordinary double points. Suppose that the central fiber $F = \pi^{-1}(o)$ is non-rational and $X$ has at least one singular point on $F$. Then $F$ is a generalised cone over an elliptic curve and its degree $d = K_F^2$ is equal to either 1 or 4.

**Proof.** The first claim again follows from the classification [HWSI]. Since $F$ is Cartier, the point $x_0$ is the only singularity of $X$ on $F$. It corresponds to the vertex of the cone. Consider the standard resolution $\psi : Z \longrightarrow X$ of $x_0$. The exceptional divisor $E$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. We have

$$K_Z = \psi^* K_X + E,$$

$$F_Z = \psi^* F - nE,$$

$$K_{F_Z} = \psi|_{F_Z}^* K_F - (n - 1)E|_{F_Z}$$

where $n \geq 1$. We consider two cases: $n \geq 2$ and $n = 1$.

**Case** $n \geq 2$. We show that $n = 2$ and $F_Z$ is non-singular. Notice that the exceptional divisors of $\psi|_{F_Z}$ have negative integral discrepancies. Consider the normalization $\nu : \overline{F_Z} \longrightarrow F_Z$. The resulting discrepancies of $\nu \circ \psi|_{F_Z}$ are also negative and integral. Since $F$ has a simple elliptic singularity, any divisor on $\overline{F_Z}$ with negative discrepancy should appear on the minimal resolution $\phi : T \longrightarrow F$. Recall that there is only one $\phi$-exceptional divisor $E_0$, and its discrepancy is $-1$. Hence there is only one $\nu \circ \psi|_{F_Z}$-exceptional prime divisor on $\overline{F_Z}$, and $\nu$ is crepant. Thus $F_Z$ is normal, $E|_{F_Z}$ is reduced, and $F_Z$ is dominated by $T$. Thus $F_Z$ is non-singular and $n = 2$. Moreover, $E \cap F_Z$ is a non-singular elliptic curve $C$. On $E$ it is given by a divisor of bidegree $(2, 2)$.

**Case** $n = 1$. Then $F_Z = \psi^* F - E$, and $F_Z|_E = -E|_E$, so $E \cap F_Z$ is a divisor of bidegree $(1, 1)$ on $E$. In particular, it is reduced. Hence $F_Z$ is normal. Moreover, $E \cap F_Z$ cannot be irreducible: in this case $F_Z$
would be non-singular, but any resolution of $F$ should contain a non-rational exceptional curve. Hence $E \cap F_Z$ is a union of two intersecting lines $L_1$ and $L_2$. The point $p$ of their intersection is singular on $F_Z$. The morphism $\psi|_{F_Z}$ is crepant: $K_{F_Z} = \psi|_{F_Z}^* K_F$. Consider the minimal resolution $\chi: \tilde{F} \to F_Z$ and the commutative diagram

$$
\begin{array}{ccc}
F_Z & \xrightarrow{\chi} & \tilde{F} \\
\downarrow{\psi|_{F_Z}} & & \downarrow\eta \\
F & \xrightarrow{\phi} & T
\end{array}
$$

(3.2)

The morphism $\eta$ exists since $T$ is the minimal resolution of $F$.

\begin{lemma}
The point $p$ is a simple elliptic singularity on $F_Z$, and $\eta$ is the blow-down of two $(-1)$-curves $\chi^{-1} L_1$ and $\chi^{-1} L_2$.
\end{lemma}

\begin{proof}
Suppose that there exists a $\chi$-exceptional curve $E'$ such that $E' \neq \tilde{E}_0 := \eta^{-1} E_0$. Since $\chi^{-1}(p)$ is connected we may assume that $E'$ intersects $\tilde{E}_0$ (because $T$ contains only one $\phi$-exceptional curve $E_0$ with negative discrepancy). Since $K_{F}$ is $\chi$-nef we have

$$0 \leq K_{\tilde{F}} \cdot E' = (\chi^* \psi|_{F_Z}^* K_F - \tilde{E}_0) \cdot E' = -\tilde{E}_0 \cdot E' \leq 0.$$

Thus $E'$ does not intersect $\tilde{E}_0$ which is a contradiction. Thus $\tilde{E}_0$ is the unique $\chi$-exceptional curve. It is a non-singular elliptic curve since it dominates $E_0 \subset T$. Clearly, $\chi^{-1} L_1$ and $\chi^{-1} L_2$ are disjoint $(-1)$-curves. \qed

We have $K_{\tilde{F}} = \chi^* \psi|_{F_Z}^* K_F - \tilde{E}_0$. Thus $K_{\tilde{F}}^2 = d + \tilde{E}_0^2$. By the Noether formula we get $K_{\tilde{F}}^2 + \chi_{\text{top}}(\tilde{F}) = 0$. Here $\chi_{\text{top}}(\tilde{F}) = 2$ since $\tilde{F}$ is a blow-up of two points on the ruled surface $T$. Thus $K_{\tilde{F}}^2 = -2$, and $-\tilde{E}_0^2 = d + 2$.

On the other hand, by [KM98, 4.57] we have $-\tilde{E}_0^2 \leq \dim T_{p,Z} = 3$ (recall that $Z$ is non-singular). Hence $d + 2 \leq 3$, thus $d = 1$ and $E_0^2 = -1$. \qed

We are ready to prove

\begin{theorem}
Let $\pi : X \to B \ni o$ be a del Pezzo fibration with at worst ordinary double points. Suppose that the fiber $F = \pi^{-1}(o)$ is non-rational and $X$ has at least one singular point on $F$. Then there is 1-to-1 correspondence between such $\pi$ and (weak and analytic in the case \textbf{(ii)} below) $\mu_n$-del Pezzo fibrations $\pi_V : V \to B \ni o$ with the following conditions:

- the central fiber $E_V = \pi_V^{-1}(o)$ is a non-singular (weak in the case \textbf{(ii)} below) del Pezzo surface of degree $d$ with $\rho^2(E_V) = 2$,
- the locus of fixed points of $\mu_n$ is an elliptic curve $C \subset E_V$,
\end{theorem}
the action of $\mu_n$ on $\mathbb{P}(N_{C/V})$ is trivial.

There are only two possible cases (here $d = K^2_F$):

(i) $d = 4$, $n = 2$, $E_V$ has two $\mu_2$-conic bundle structures,
(ii) $d = 1$, $n = 4$, $E_V$ has two $\mu_4$-invariant $(-1)$-curves.

Proof. By Proposition 3.1 there are two cases to consider: $d = 1$ or 4.

Case $d = 4$. We are in the setting of the first case of Proposition 3.1. We make the base change. We will construct the following diagram:

$$
\begin{array}{c}
F_W + E_W \subset W \xrightarrow{h} Z \supset F_Z + 2E \\
\downarrow \tau \hspace{1cm} \downarrow \psi \\
E_V \subset V \hspace{1cm} X \supset F \\
\downarrow \pi_V \hspace{1cm} \downarrow \pi \\
B' \xrightarrow{\alpha} B
\end{array}
$$

(3.5)

here $B' \simeq B$, $\alpha : t \mapsto t^2$, and $W$ is the normalization of $Z \times_B B'$. As in Construction 2.1, one checks that $W$ is non-singular, $h$ is ramified along $F_W := h^{-1}(F_Z)$, and the covering map

$$
h|_{E_W} : h^{-1}(E) \twoheadrightarrow E
$$

is ramified along a non-singular elliptic curve $E \cap F_Z$. The Galois group $\mu_2$ of $h$ acts on $W$. By the Hurwitz formula $E_W$ is a quartic del Pezzo surface. One checks that $F_W$ can be contracted to a non-singular elliptic curve, so we obtain a $\mu_2$-equivariant morphism $\tau : W \rightarrow V$. Hence $\pi_V : V \rightarrow B \ni o$ is a fibration into quartic del Pezzo surfaces with a non-singular central fiber $E_V$. Notice that $\rho^2(E_V) = 2$ since $E_V$ admits two $\mu_2$-equivariant conic bundle structures.

If we start from a $\mu_2$-del Pezzo fibration $\pi_V : V \rightarrow B \ni o$ of degree 4 with the properties as in the theorem, one checks that we can go along the diagram in the other direction and get a del Pezzo fibration $\pi : X \rightarrow B \ni o$ with a non-rational central fiber and an ordinary double point.

Case $d = 1$. Let $\psi : Z \rightarrow X$ be a small resolution of $x_0 \in X$. That is, the exceptional locus of $\psi$ is a curve $L \simeq \mathbb{P}^1$ and $Z$ is a non-singular complex manifold. Notice that $F_Z := \psi^{-1}_* F$ is a singular complex surface. As in Lemma 3.3 one checks that $F_Z$ has one simple elliptic singularity, say $z_0 \in F_Z \subset Z$. Arguing as in Lemma 3.3 we see that the self-intersection of the exceptional elliptic curve equals $-2$. Let $\psi' : Z' \rightarrow Z$ be the blow-up with the weights $(1, 1, 2)$. From [KM98 4.57]
it follows that $F_{Z'} = \psi_{*}^{-1}F_Z$ is the minimal resolution of $F_Z$. We have

$$
K_{Z'} = \psi^{*}K_Z + 3E',
$$
$$
F_{Z'} = \psi^{*}F_Z - 4E',
$$
$$
K_{F_{Z'}} = \psi'|_{F_{Z'}}^*K_F - E'|_{F_{Z'}}
$$

where $E' \cong \mathbb{P}(1,1,2)$ and $F_{Z'} = \psi_{*}^{-1}F_Z$. Notice that $Z'$ has one singular point of type $\frac{1}{2}(1,1,1)$, and $F_{Z'}$ has one reducible fiber. We will construct the following diagram

$$
\begin{array}{ccc}
F_{W'} + E_{W'} & \subset W' & \psi' \downarrow \\
\tau \downarrow & Z' & \subset F_{Z'} + 2E_{Z'} \subset W
\end{array}
$$

(3.6)

where $B' \cong B$, $\alpha : t \mapsto t^4$, and $W$ is the normalization of $Z \times_B B'$. As in the previous case $W$ is non-singular, $h$ is ramified along $F_W := h^{-1}(F_Z)$, and $h|_{E_W}$ is ramified along a non-singular elliptic curve $E_W \cap F_W$ where $E_W := h^{-1}(E_{Z'})$. The Galois group $\mu_4$ of $h$ acts on $W$, and the fiber $\pi_W^{-1}(o) = F_W + E_W$ is reduced. By the Hurwitz formula $E_W$ is a degree 2 del Pezzo surface. One checks that $E_W$ is non-singular. Notice that $F_W \cong F_{Z'}$ has one reducible fiber $f_W' = f_1 + f_2$. Both $f_1$ and $f_2$ are $(-1)$-curves on $F_W$. Without loss of generality, assume that $f_1$ intersects the elliptic curve $C_W := F_W \cap E_W$.

We make a flop $h'$ in the curve $f_1$. It is the simplest Atiyah-Kulikov flop, see [Ku77, 4.2]. We obtain a threefold $W'$ with the central fiber $E_{W'} + F_{W'}$ where $E_{W'}$ and $F_{W'}$ are the strict transforms of $E_W$ and $F_W$, $E_{W'}$ is the blow-up of a point in $E_W$, and $F_{W'}$ is the blow-down of $f_1$. Thus, $E_{W'}$ is a non-singular weak (that is $-K_{E_{W'}}$ is nef and big) del Pezzo surface of degree 1. Then $F_{W'}$ is a ruled surface that can be contracted onto a curve, and we get a degree 1 del Pezzo fibration $\pi_V : V \to B \ni o$.

If we start from a $\mu_4$-del Pezzo fibration $\pi_V : V \to B \ni o$ of degree 1 with the properties as in the theorem, one checks that we can go along the diagram in the other direction and get a del Pezzo fibration $\pi : X \to B \ni o$ with a non-rational central fiber and an ordinary double point. □

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