Fock–Sobolev spaces and their Carleson measures

Hong Rae Cho a,1, Kehe Zhu b,*

a Department of Mathematics, Pusan National University, Pusan 609-735, Republic of Korea
b Department of Mathematics and Statistics, State University of New York at Albany, Albany, NY 12222, USA

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Abstract

We study a class of holomorphic spaces \( F^{p,m} \) consisting of entire functions \( f \) on \( \mathbb{C}^n \) such that \( \partial^\alpha f \) is in the Fock space \( F^p \) for all multi-indices \( \alpha \) with \( |\alpha| \leq m \). We prove a useful Fourier characterization, namely, \( f \in F^{p,m} \) if and only if \( z^\alpha f(z) \) is in \( F^p \) for all \( \alpha \) with \( |\alpha| = m \). We obtain duality and interpolation results for these spaces, including the interesting fact that, for \( 0 < p \leq 1 \), \( (F^{p,m})^* = F^{\infty,m} \). We also characterize Carleson measures for \( F^{p,m} \) in terms of simple polynomial growth conditions.

Keywords: Fock space; Fock–Sobolev space; Carleson measure; Gaussian measure; Reproducing kernel

1. Introduction

Let \( \mathbb{C}^n \) be the complex \( n \)-space and \( dv \) be the ordinary volume measure on \( \mathbb{C}^n \). If \( z = (z_1, \ldots, z_n) \) and \( w = (w_1, \ldots, w_n) \) are points in \( \mathbb{C}^n \), we write

\[
 z \cdot \bar{w} = \sum_{j=1}^{n} z_j \bar{w}_j, \quad |z| = (z \cdot \bar{z})^{1/2}.
\]
For any $0 < p \leq \infty$ we let $L^p_g$ denote the space of Lebesgue measurable functions $f$ on $\mathbb{C}^n$ such that the function $f(z)e^{-\frac{1}{2}|z|^2}$ is in $L^p(\mathbb{C}^n, dv)$.

When $0 < p < \infty$, it is clear that

$$L^p_g = L^p\left(\mathbb{C}^n, e^{-\frac{p}{2}|z|^2} dv(z)\right).$$

In this case, we norm the space $L^p_g$ by

$$\|f\|_p = \left[\left(\frac{p}{2\pi}\right)^n \int_{\mathbb{C}^n} |f(z)e^{-\frac{1}{2}|z|^2}|^p dv(z)\right]^{\frac{1}{p}}.$$

For $p = \infty$ the norm in $L^\infty_g$ is defined by

$$\|f\|_\infty = \text{esssup}\{ |f(z)|e^{-\frac{1}{2}|z|^2} : z \in \mathbb{C}^n \}.$$ 

Strictly speaking, $\|f\|_p$ is not a norm when $0 < p < 1$, but we believe the reader would have no objection to our usage of the term “norm” for all $0 < p \leq \infty$.

Let $F^p$ denote the space of entire functions in $L^p_g$. Then $F^2$ is a closed subspace of the Hilbert space $L^2_g$ with inner product

$$\langle f, g \rangle = \frac{1}{\pi^n} \int_{\mathbb{C}^n} f(z)\overline{g(z)}e^{-|z|^2} dv(z).$$

The orthogonal projection $P : L^2_g \rightarrow F^2$ is given by

$$Pf(z) = \frac{1}{\pi^n} \int_{\mathbb{C}^n} f(w)K(z,w)e^{-|w|^2} dv(w),$$

where $K(z, w) = e^{z \cdot \overline{w}}$ is the reproducing kernel of $F^2$. It is well known that the Fock projection $P$ above is a bounded projection from $L^p_g$ onto $F^p$ for $1 \leq p \leq \infty$. See [12] for example.

The spaces $F^p$, especially $F^2$, have had a long history in mathematics and mathematical physics and have been given a wide variety of appellations, including many combinations and permutations of the names Bargmann, Fischer, Fock, and Segal. See [1–4,7,8,12–14,17]. In this paper we are going to call them Fock spaces, for no particular reason other than personal tradition. We refer the reader to [17,19] for more recent and systematic treatment of Fock spaces.

In what follows we use the conventional multi-index notation. Thus for an $n$-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers we write

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n!, \quad \partial^\alpha = \partial_{\alpha_1} \cdots \partial_{\alpha_n},$$

where $\partial_j$ denotes partial differentiation with respect to the $j$-th component. If $z = (z_1, \ldots, z_n)$, then $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$.

For any non-negative integer $m$ and $0 < p \leq \infty$ we consider the space $F^{p,m}$ consisting of entire functions $f$ on $\mathbb{C}^n$ such that
\[ \| f \|_{F^p,m} := \sum_{|\alpha| \leq m} \| \partial^\alpha f \|_p < \infty, \]

where \( \| \cdot \|_p \) is the norm in \( F^p \). Because of the similarity to the way the classical Sobolev spaces are defined, we are going to call \( F^{p,m} \) Fock–Sobolev spaces. On the other hand, some readers may find it more appropriate to call \( F^{p,m} \) Besov–Fock spaces. See [10,16] for other similar Sobolev spaces.

We can now state the main results of the paper.

**Theorem A.** Suppose \( 0 < p \leq \infty \), \( m \) is a non-negative integer, and \( f \) is an entire function on \( \mathbb{C}^n \). Then \( f \in F^{p,m} \) if and only if the function \( z^\alpha f(z) \) is in \( F^p \) for all multi-indices \( \alpha \) with \( |\alpha| = m \). Moreover, \( \| f \|_{F^{p,m}} \) is comparable to the norm of the function \( |z|^m f(z) \) in \( L^p \).

This is a useful Fourier characterization of the Fock–Sobolev spaces. It allows us to introduce the following norm on \( F^{p,m} \) when \( 0 < p < \infty \):

\[ \| f \|_p^{p,m} = \omega_{n,p,m} \int_{\mathbb{C}^n} |z|^m f(z) e^{-\frac{1}{2} |z|^2} |z|^m \, dv(w), \]

where

\[ \omega_{n,p,m} = \left( \frac{p}{2} \right)^{(mp/2)+n} (n-1)! \frac{(n+1)!}{\pi^n \Gamma((mp)/2+n)} \]

is a normalizing constant so that the constant function 1 has norm 1 in \( F^{p,m} \). When \( p = \infty \), we define

\[ \| f \|_{\infty,m} = \sup_{z \in \mathbb{C}^n} [ |z|^m |f(z)| e^{-\frac{1}{2} |z|^2}]. \]

Let \( L^{p,m}_g \) denote the space of Lebesgue measurable functions \( f \) on \( \mathbb{C}^n \) such that the function \( |z|^m f(z) \) is in \( L^p_g \). As usual, it follows easily from the mean-value property of subharmonic functions (see [18,19] for example) that \( F^{p,m} \) is a closed subspace of \( L^{p,m}_g \).

**Theorem B.** Let \( m \) be a non-negative integer.

(a) The orthogonal projection \( P_m : L^{2,m}_g \to F^{2,m} \) is a bounded projection from \( L^{p,m}_g \) onto \( F^{p,m} \) when \( 1 \leq p \leq \infty \).

(b) If \( 1 < p < \infty \) and \( 1/p + 1/q = 1 \), then the Banach dual of \( F^{p,m} \) can be identified with \( F^{q,m} \) under the integral pairing

\[ \langle f, g \rangle_m = \omega_{n,2,m} \int_{\mathbb{C}} f(z) \overline{g(z)} e^{-|z|^2} |z|^{2m} \, dv(z). \]

(c) If \( 0 < p \leq 1 \), then the Banach dual of \( F^{p,m} \) can be identified with \( F^{\infty,m} \) under the integral pairing above.
(d) If \( 1 \leq p_0 < p_1 \leq \infty \) and \( \theta \in (0, 1) \), then

\[
\left[ F^{p_0,m}_p, F^{p_1,m}_p \right]_\theta = F^{p,m}_p,
\]

where \( p \) is determined by

\[
\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}.
\]

When \( m = 0 \), the above result (except part (c)) can be found in [12]. The more recent paper [9] determines the exact duality and projection constants in this case.

Let \( 0 < p < \infty \). A non-negative Borel measure \( \mu \) on \( \mathbb{C}^n \) is called a Carleson measure for \( F^{p,m}_p \) if there exists a constant \( C \) such that

\[
\int_{\mathbb{C}^n} \left| f(z) e^{-\frac{1}{2}|z|^2} \right|^p d\mu(z) \leq C \| f \|_{p,m}^p
\]

for all \( f \in F^{p,m}_p \). We call \( \mu \) a vanishing Carleson measure for \( F^{p,m}_p \) if

\[
\lim_{k \to \infty} \int_{\mathbb{C}^n} \left| f_k(z) e^{-\frac{1}{2}|z|^2} \right|^p d\mu(z) = 0
\]

whenever \( \{f_k\} \) is a bounded sequence in \( F^{p,m}_p \) that converges to 0 uniformly on compact subsets of \( \mathbb{C}^n \).

It was of course Carleson (see [5]) who first studied positive Borel measures \( \mu \) on the unit disk \( \mathbb{D} \) that satisfy the condition

\[
\int_{\mathbb{D}} |f(z)|^p d\mu(z) \leq C \int_0^{2\pi} |f(e^{it})|^p dt
\]

for functions \( f \) in the Hardy space \( H^p \). The above condition is clearly equivalent to the continuous inclusion of \( H^p \) in \( L^p(\mathbb{D}, d\mu) \). Carleson’s original characterization of such measures was given geometrically as \( \mu(S_{h,t}) \leq Ch \), where

\[
S_{h,t} = \{ r e^{i\theta} : 1 - h < r < 1, |\theta - t| < h \}
\]

are “Carleson squares” near the unit circle and \( C \) is independent of \( h \) and \( t \). These measures were subsequently called Carleson measures and the notion has since been extended to and studied in many different contexts in which one is interested in the continuous or compact inclusion of a Banach space \( X \) of analytic functions in certain \( L^p(\mathbb{D}, d\mu) \). See [18] for the study of Carleson measures in the context of Bergman spaces and [11,19] in the context of ordinary Fock spaces. The geometric characterization of Carleson measures for the spaces \( F^{p,m}_p \) is the following.

**Theorem C.** Suppose \( m \) is a non-negative integer, \( 0 < p < \infty \), and \( r \) is a positive radius. Let \( \mu \) be any positive Borel measure on \( \mathbb{C}^n \). Then
(a) \( \mu \) is a Carleson measure for \( F_{p,m} \) if and only if there exists a positive constant \( C \) such that
\[
\mu(B(z,r)) \leq C(1 + |z|)^{mp}
\]
for all \( z \in \mathbb{C}^n \), where \( B(z,r) = \{ w \in \mathbb{C}^n : |w - z| < r \} \) is the Euclidean ball centered at \( z \) with radius \( r \).

(b) \( \mu \) is a vanishing Carleson measure for \( F_{p,m} \) if and only if
\[
\lim_{|z| \to \infty} \frac{\mu(B(z,r))}{(1 + |z|)^{mp}} = 0.
\]

Throughout the paper we write \( X \lesssim Y \) or \( Y \gtrsim X \) for non-negative quantities \( X \) and \( Y \) whenever there is a constant \( C > 0 \) (independent of the parameters in \( X \) and \( Y \)) such that \( X \leq CY \). Similarly, we write \( X \approx Y \) if \( X \lesssim Y \) and \( Y \lesssim X \).

2. A Fourier characterization of Fock–Sobolev spaces

The purpose of this section is to prove Theorem A. More specifically, for an entire function \( f \) and a non-negative integer \( m \), the condition \( \partial^{\alpha} f \in F_p \) for all \( |\alpha| \leq m \) is equivalent to the condition \( z^{\alpha} f \in F_p \) for all \( |\alpha| = m \).

We begin with a few technical lemmas that are needed in the proof of the result mentioned above. The first two lemmas incorporate the extra factor \( |z|^a \) into various integrals.

**Lemma 1.** Let \( 0 < p < \infty \), \( 0 < b < \infty \), and \( 0 \leq a < \infty \). Then there is a constant \( C = C(a,b) > 0 \) such that
\[
\int_{\mathbb{C}^n} |f(w)|^p e^{-b|w|} d\mu(w) \leq C \int_{\mathbb{C}^n} |f(w)|^p |w_j|^a e^{-b|w|} d\mu(w)
\]
for all entire functions \( f \) on \( \mathbb{C}^n \) and \( j = 1, \ldots, n \).

**Proof.** By the subharmonicity of \( |f(z)|^p \) and the maximum principle, there is a positive constant \( C = C(a) \) such that
\[
\int_{|w_j| \leq 1} |f(w)|^p dA(w_j) \leq C \int_{|w_j| \leq 1} |f(w)|^p |w_j|^a dA(w_j)
\]
for all entire \( f \) and all \( 0 < p < \infty \), where \( dA \) is the area measure on \( \mathbb{C} \) (see [18]). Since \( e^{-b|w_j|^2} \) is both bounded above and bounded below on \( |w_j| \leq 1 \), we deduce that
\[
\int_{\mathbb{C}} |f(w)|^p e^{-b|w_j|^2} dA(w_j) \leq C' \int_{\mathbb{C}} |f(w)|^p |w_j|^a e^{-b|w_j|^2} dA(w_j)
\]
for some constant \( C' = C(a,b) > 0 \). The desired estimate then follows from iterated integration. \( \square \)
Lemma 2. Let $0 < p < \infty$, $0 < b < \infty$, and $0 \leq a < \infty$. Then there is a constant $C = C(a, b) > 0$ such that

$$
\int_{\mathbb{C}^n} |f(w)|^p |w|^a e^{-b|w|^2} \, dv(w) \leq C \int_{|w| \geq 1} |f(w)|^p |w|^a e^{-b|w|^2} \, dv(w)
$$

for all entire functions $f$ on $\mathbb{C}^n$.

**Proof.** This follows easily from the subharmonicity of $|f(z)|^p |z|^a$ and integration in polar coordinates. □

The next lemma gives a pointwise estimate for entire functions in terms of certain $L^p$-integrals. In particular, this will give us optimal pointwise estimates for functions in the spaces $F^{p,m}$; see (4) below which is valid for all $p$.

Lemma 3. Let $0 < p < \infty$, $0 < b < \infty$, $0 < t < \infty$, and $0 \leq a < \infty$. Then there is a constant $C = C(a, b, t) > 0$ such that

$$
|f(z)|^p |z|^a e^{-b|z|^2} \leq C \int_{|w-z| < t} |f(w)|^p (1 + |w|)^a e^{-b|w|^2} \, dv(w) \quad (3)
$$

for all entire functions $f$ on $\mathbb{C}^n$ and all $z \in \mathbb{C}^n$.

**Proof.** Given $z \in \mathbb{C}^n$, the subharmonicity of the function

$$
w \mapsto |f(z+w)e^{-awz/p}|^p
$$

yields

$$
|f(z)|^p \lesssim \int_{|w| < t} |f(z+w)e^{-2bwz/p}|^p e^{-b|w|^2} \, dv(w)
$$

$$
= e^{b|z|^2} \int_{|w-z| < t} |f(w)|^p e^{-b|w|^2} \, dv(w).
$$

Now, since $|z| < t + |w|$ for $|w-z| < t$, we have

$$
|f(z)|^p |z|^a e^{-b|z|^2} \lesssim \int_{|w-z| < t} |f(w)|^p (t + |w|)^a e^{-b|w|^2} \, dv(w)
$$

$$
\lesssim \int_{|w-z| < t} |f(w)|^p (1 + |w|)^a e^{-b|w|^2} \, dv(w),
$$

where the suppressed constants depend only on $a$, $b$, and $t$. □
In the case $m = 0$, it is well known that $F^p \subset F^q$ whenever $0 < p \leq q \leq \infty$; see [12] for example. It follows from this and the definition of $F^{p,m}$ (in terms of partial derivatives) that $F^{p,m} \subset F^{q,m}$ whenever $0 < p \leq q \leq \infty$. If Theorem A had already been proved, the following lemma would have been automatic, at least in the case when $a$ is a positive integer $m$, as it would simply be stating that $F^{p,m} \subset F^{1,m}$ for $0 < p < 1$. But we are still in the process of proving Theorem A, and this Minkowski-like estimate is the key to our subsequent analysis in the case $0 < p < 1$.

**Lemma 4.** Let $0 < p \leq 1$, $a \geq 0$, and $b > 0$. There exists a constant $C = C(a, b, p) > 0$ such that

$$\left[ \int_{\mathbb{C}^n} |f(z)|^a e^{-b|z|^2} |dz| \right]^p \leq C \int_{\mathbb{C}^n} |f(z)|^a e^{-b|z|^2} |dz|^p$$

for all entire functions $f$ on $\mathbb{C}^n$.

**Proof.** By Lemma 3,

$$|f(z)|^a e^{-b|z|^2} \leq C \left[ \int_{\mathbb{C}^n} |f(w)|^a e^{-b|w|^2} |dw|^{1/p} \right]^{1/p}.$$ 

Since $(1 + |w|)^{pa} \leq C(1 + |w|)^{pa}$, it follows from Lemma 1 that

$$|f(z)|^a e^{-b|z|^2} \leq C' \left[ \int_{\mathbb{C}^n} |f(w)| |w|^a e^{-b|w|^2} |dw|^{1/p} \right]^{1/p}. \quad (4)$$

The desired inequality then follows easily by writing

$$|f(z)|^a e^{-b|z|^2} = |f(z)|^a e^{-b|z|^2} |f(z)||z|^a e^{-b|z|^2} |1 - p$$

and estimating the factor $|f(z)|^a e^{-b|z|^2} |1 - p$ using (4). \( \square \)

**Proposition 5.** Suppose $0 < p \leq \infty$, $\alpha$ is a multi-index of non-negative integers, and $f$ is an entire function on $\mathbb{C}^n$. If the function $z^\alpha f(z)$ is in $F^p$, then so is $\partial^\alpha f$. Moreover, there exists a positive constant $C = C(p, \alpha)$ such that $\|\partial^\alpha f\|_p \leq C\|z^{\alpha} f\|_p$ for all $f$.

**Proof.** If the function $w^{\alpha} f(w)$ is in $F^p$, then by repeated use of Lemma 1 we have $f \in F^p$, so that

$$f(z) = \frac{1}{\pi^n} \int_{\mathbb{C}^n} e^{z \cdot \overline{w}} f(w) e^{-|w|^2} dv(w), \quad z \in \mathbb{C}^n.$$ 

Note that the formula above holds when $p = 2$ because of the reproducing property of the kernel function $e^{z \cdot \overline{w}}$. For other values of $p$, this follows from standard pointwise estimates and a limit argument. In fact, the case $p \geq 1$ was discussed in [12], and the case $0 < p < 2$ follows from the
Hilbert space case and the embedding $F^p \subset F^2$. Now differentiating under the integral sign, we obtain
\[
\partial^\alpha f(z) = \frac{1}{\pi^n} \int_{\mathbb{C}^n} e^{z \cdot \overline{w}} w^\alpha f(w) e^{-|w|^2} \, dv(w). \tag{5}
\]

The convergence of the integrals above follows from standard pointwise estimates for functions in Fock spaces.

We rewrite (5) as $\partial^\alpha f = P g$, where $g(z) = z^\alpha f(z)$. Since $g \in L^p_g$ and $P$ is a bounded projection of $L^p_g$ onto $F^p$ for $1 \leq p \leq \infty$, the desired result is clear when $1 \leq p \leq \infty$. Moreover, if the sharp bound for $P$ calculated in [9] is used here, we can obtain a quantitative estimate for the constant $C$ in the proposition.

When $0 < p < 1$, we note from (5) that
\[
|\partial^\alpha f(z)| \leq \frac{1}{\pi^n} \int_{\mathbb{C}^n} |w^\alpha f(w) e^{z \cdot w} e^{-|w|^2}| \, dv(w).
\]

By Lemma 4, there exists a positive constant $C$, independent of $f$ and $z$, such that
\[
|\partial^\alpha f(z)|^p \leq C \int_{\mathbb{C}^n} |w^\alpha f(w) e^{z \cdot w} e^{-|w|^2}|^p \, dv(w).
\]

It follows from this and Fubini’s theorem that we can estimate the integral
\[
I = \int_{\mathbb{C}^n} |\partial^\alpha f(z) e^{-\frac{1}{2}|z|^2}|^p \, dv(z)
\]
as follows:
\[
I \lesssim \int_{\mathbb{C}^n} e^{-\frac{p}{2}|z|^2} \, dv(z) \int_{\mathbb{C}^n} |w^\alpha f(w)|^p e^{-p|w|^2} |e^{z \cdot w}|^p \, dv(w)
= \int_{\mathbb{C}^n} |w^\alpha f(w)|^p e^{-p|w|^2} \, dv(w) \int_{\mathbb{C}^n} |e^{z \cdot w}|^2 e^{-\frac{p}{2}|z|^2} \, dv(z)
\approx \int_{\mathbb{C}^n} |w^\alpha f(w) e^{-\frac{1}{2}|w|^2}|^p \, dv(w).
\]

This shows that $\partial^\alpha f \in F^p$ and completes the proof of the proposition. □

**Corollary 6.** Let $0 < p \leq \infty$ and $m$ be a non-negative integer. Then there is a constant $C = C(m) > 0$ such that
\[
\sum_{|\alpha| \leq m} \|\partial^\alpha f\|_p \leq C \|z|^m f\|_p
\]
for all entire functions $f$ on $\mathbb{C}^n$. 
Proof. This follows from Proposition 5 and Lemma 1. □

We have now finished the proof of one half of Theorem A. To prove the other half, we will need more detailed information about the reproducing kernel of $F^{2,m}$. To this end, we introduce a class of radial differential operators acting on entire functions.

Thus for an entire function $f$ we let

$$f(z) = \sum_{k=0}^{\infty} f_k(z)$$

be its homogeneous expansion, where $f_k$ is a homogeneous polynomial of degree $k$ (the collection of terms of degree $k$ in the Taylor expansion of $f$). For any non-negative integer $m$, we define a differential operator $D^m$ as follows:

$$D^m f(z) = \sum_{k=0}^{\infty} \frac{(n+k+m-1)!}{(n+k-1)!} f_k(z). \quad (6)$$

The operator $D^m$ is clearly invertible with its inverse given by

$$D_m f(z) = \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n+k+m-1)!} f_k(z). \quad (7)$$

Let

$$h_m(\lambda) = \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n+k+m-1)!} \lambda^k = \frac{d^{n-1}}{d\lambda^{n-1}} \left[ \sum_{k=0}^{\infty} \frac{\lambda^{n-1+k}}{(n+k+m-1)!} \right].$$

Note that

$$\sum_{k=0}^{\infty} \frac{\lambda^{n-1+k}}{(n+k+m-1)!} = \sum_{k=n+m-1}^{\infty} \frac{\lambda^{k-m}}{k!} = \frac{e^\lambda - q_{n+m-1}(\lambda)}{\lambda^m},$$

where $q_0 = 0$ and $q_k$ is the Taylor polynomial of $e^\lambda$ of order $k - 1$ for $k \geq 1$. For $k \geq 1$ we have

$$\frac{d}{d\lambda} \left[ e^\lambda - q_k(\lambda) \right] = e^\lambda - q_{k-1}(\lambda).$$

It follows that there are constants $c_k$ (with $c_m = 1$) such that

$$h_m(\lambda) = \sum_{k=m}^{m+n-1} c_k \frac{e^\lambda - q_k(\lambda)}{\lambda^k}. \quad (8)$$
Moreover, for \( k \geq 1 \) it is easy to check that
\[
\frac{e^{\lambda} - q_k(\lambda)}{\lambda^k} = \sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{(k + \ell)!} = \frac{1}{(k - 1)!} \int_0^1 (1-t)^{k-1} e^{t\lambda} \, dt.
\] (9)

Recall from the study of circular domains in \( \mathbb{C}^n \) that the radial derivative \( \mathcal{R} \) is defined by
\[
\mathcal{R} f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z).
\]

In terms of the homogeneous expansion of \( f \), we have
\[
\mathcal{R} f(z) = \sum_{k=1}^{\infty} kf_k(z).
\]

In the context of Fock spaces, the operator \( \mathcal{R} \) is often called the Euler operator, which is the infinitesimal generator of dilations.

The following calculations show that the operator \( \mathcal{D}^m \) is an \( m \)-th degree polynomial of \( \mathcal{R} \) with constant coefficients:
\[
\mathcal{D}^m f(z) = \sum_{k=0}^{\infty} \frac{(n + k - 1)!}{(n + k + m - 1)!} (z \cdot w)^k f_k(z)
\]
\[
= \sum_{k=0}^{\infty} \left( k^m + c_{n,0} k^{m-1} + \cdots + c_{n,m} \right) f_k(z)
\]
\[
= (\mathcal{R}^m + c_{n,0} \mathcal{R}^{m-1} + \cdots + c_{n,m}) f(z),
\]
where \( c_{n,j} \) are positive integers. It follows that
\[
(1 + |z|)^{-m} |\mathcal{D}^m f(z)| \lesssim (1 + |z|)^{-m} (|z|^m + \cdots + |z| + 1) \sum_{|\alpha| \leq m} |\partial^\alpha f(z)|
\]
\[
\lesssim \sum_{|\alpha| \leq m} |\partial^\alpha f(z)|
\] (10)
for all entire functions \( f \) on \( \mathbb{C}^n \).

Lemma 7. Let \( m \) be a non-negative integer. Then \( \mathcal{D}_m K(z, w) = h_m(z \cdot \overline{w}) \), where \( K(z, w) \) is the reproducing kernel of \( F^2 \) and \( \mathcal{D}_m \) is taken with respect to \( z \).

Proof. Direct calculation from the definition shows that
\[
\mathcal{D}_m K(z, w) = \sum_{k=0}^{\infty} \frac{(n + k - 1)!}{(n + k + m - 1)!} \frac{(z \cdot \overline{w})^k}{k!} = h_m(z \cdot \overline{w}). \] \( \square \)
The following result gives an upper estimate for $D_m K(z, w)$.

**Proposition 8.** Let $m$ be a non-negative integer. Then we have

$$|D_m K(z, w)| \lesssim \frac{e^{\frac{1}{2}|z|^2 + \frac{1}{2}|w|^2 - \frac{1}{8}|z - w|^2}}{(1 + |z||w|)^m}, \quad z, w \in \mathbb{C}^n.$$ 

**Proof.** The case $\text{Re} \lambda = \text{Re}(z \cdot \overline{w}) \leq 1$ follows from (9). In fact,

$$|h_m(\lambda)| \lesssim \sum_{k=m}^{m+n-1} \int_0^1 (1 - t)^{k-1} e^{t(\text{Re} \lambda)} \, dt \lesssim 1.$$ 

For the case $\text{Re} \lambda = \text{Re}(z \cdot \overline{w}) \geq 1$, we write $\text{Re}(z \cdot \overline{w}) = |z||w| \cos \theta$, where $\theta$ is the angle between $z$ and $w$ identified as real vectors in $\mathbb{R}^{2n}$, and $\delta = \cos^{-1}(\frac{1}{4})$. If $|\theta| \leq \delta$, then

$$|\text{Re}(z \cdot \overline{w})| \approx |z \cdot \overline{w}| \approx 1 + |z||w|.$$ 

By (8), we have

$$|h_m(\lambda)| \lesssim \frac{|e^\lambda|}{|\lambda|^m} \lesssim \frac{e^{\text{Re}(z \cdot \overline{w})}}{(1 + |z||w|)^m} \leq \frac{e^{\frac{1}{2}|z|^2 + \frac{1}{2}|w|^2 - \frac{1}{8}|z - w|^2}}{(1 + |z||w|)^m}.$$ 

If $\delta < \theta \leq \frac{\pi}{2}$, then

$$\text{Re} \lambda = \text{Re}(z \cdot \overline{w}) = |z||w| \cos \theta < \frac{1}{4} |z||w|.$$ 

By (8), we have

$$|h_m(\lambda)| \lesssim \frac{|e^\lambda|}{|\lambda|^m} \lesssim \frac{e^{\text{Re}(z \cdot \overline{w})}}{|\lambda|^m} \leq \frac{e^{\frac{1}{2}|z||w|}}{(1 + |z||w|)^m} \leq \frac{e^{\frac{1}{2}|z||w|}}{(1 + |z||w|)^m} \leq \frac{e^{\frac{1}{2}|z||w|}}{(1 + |z||w|)^m}.$$ 

This completes the proof of the proposition. \(\square\)

**Lemma 9.** Suppose $1 \leq p \leq \infty$ and $L(z, w)$ is an integral kernel satisfying the condition

$$|L(z, w)| \leq C e^{\frac{1}{2}|z|^2 + \frac{1}{2}|w|^2 - \frac{1}{8}|z - w|^2}, \quad z, w \in \mathbb{C}^n.$$ 

Then the integral operator $T$ induced by $L(z, w)$,

$$Th(z) = \int_{\mathbb{C}^n} h(w)L(z, w)e^{-|w|^2} \, dv(w),$$

is bounded on $L^p_g$. 
Proof. Fubini’s theorem and the size estimate for $L(z, w)$ show that
\[
\|Th\|_1 \leq C \|h\|_1 \int_{\mathbb{C}^n} e^{-\frac{1}{8}|z-w|^2} \, dv(w) = (8\pi)^n C \|h\|_1
\]
and
\[
\|Th\|_\infty \leq C \|h\|_\infty \int_{\mathbb{C}^n} e^{-\frac{1}{8}|z-w|^2} \, dv(w) = (8\pi)^n C \|h\|_\infty.
\]
The result then follows from the Stein–Weiss interpolation theorem in [15].

We now prove the converse of Proposition 5.

Proposition 10. Suppose $0 < p \leq \infty$ and $m$ is a non-negative integer. Then there is a constant $C > 0$ such that
\[
\|z|^m f\|_p \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_p
\]
for all $f \in F^{p,m}$.

Proof. By (10), it is enough to prove that
\[
\|z|^m f\|_p \leq C \|(1 + |z|)^{-m} D^m f\|_p.
\]
From the reproducing formula for $D^m f$ we obtain
\[
f(z) = D_m D^m f(z) = \frac{1}{\pi^n} \int_{\mathbb{C}^n} D^m f(w) D_m K(z, w)e^{-|w|^2} \, dv(w).
\]
Note that commuting $D_m$ with the integral follows from term-by-term expansion of the kernel function and the uniform convergence on compact sets. This together with Lemma 2 shows that
\[
|z|^m |f(z)| \lesssim |z|^m \int_{|w| \geq 1} |D^m f(w) D_m K(z, w)e^{-|w|^2}| \, dv(w)
\]
\[
= \int_{C^n} (1 + |w|)^{-m} |D^m f(w)| \|L(z, w)|e^{-|w|^2} \, dv(w),
\]
where
\[
L(z, w) = |z|^m (1 + |w|)^m \chi_{|w| \geq 1} (w) D^m K(z, w).
\]
It is easy to see from Lemma 8 that
\[ |L(z, w)| \lesssim e^{\frac{1}{2}|z|^2 + \frac{1}{2}|w|^2 - \frac{1}{8}|z-w|^2}. \]
So the desired result follows from Lemma 9 when \( 1 \leq p \leq \infty \).

When \( 0 < p < 1 \), it follows from Lemma 4, Lemma 2, and Lemma 7 that
\[ |f(z)|^p \lesssim \left| \int_{\mathbb{C}^n} \mathcal{D}^m f(w) \mathcal{D}_m K(z, w) e^{-|w|^2} d\nu(w) \right|^p. \]
Fubini’s theorem shows that the integral
\[ I = \int_{|z| \geq 1} |z|^m f(z) e^{-\frac{1}{2}|z|^2} |^p d\nu(z) \]
satisfies the following estimates:
\[ I \lesssim \int_{|w| \geq 1} (1 + |w|)^{-mp} |\mathcal{D}^m f(w)|^p e^{-\frac{1}{2}|w|^2} d\nu(w) \int_{|z| \geq 1} e^{-\frac{1}{2}|z-w|^2} d\nu(z) \]
\[ \lesssim \int_{|w| \geq 1} \left| \left(1 + |w| \right)^{-m} \mathcal{D}^m f(w) e^{-\frac{1}{2}|w|^2} \right|^p d\nu(w). \]
Another application of Lemma 2 then proves the desired result. \( \square \)

Combining Propositions 5 and 10, we have proved the following theorem, the main result of this section.

**Theorem 11.** Suppose \( 0 < p \leq \infty \), \( m \) is a non-negative integer, and \( f \) is an entire function on \( \mathbb{C}^n \). Then \( f \in F^{p,m} \) if and only if every function \( z^\alpha f(z) \) is in \( F^p \), where \( |\alpha| = m \). Moreover, there is a positive constant \( C \) such that
\[ C^{-1} \|z|^m f\|_p \leq \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_p \leq C \|z|^m f\|_p \]
for all \( f \in F^{p,m} \).
3. Duality and complex interpolation

There are several ways of defining inner products that induce equivalent norms on \( F^{2,m} \). For example, the inner product

\[(f, g) \mapsto \sum_{|\alpha| \leq m} \langle \partial^\alpha f, \partial^\alpha g \rangle \]  \tag{11}\]

is quite natural. However, it is difficult for us to calculate the reproducing kernel of \( F^{2,m} \) using the natural inner product in (11). To get a better handle on the reproducing kernel, we are going to use the following inner product on \( F^{2,m} \):

\[(f, g)_m = \omega_{n,2,m} \int_{\mathbb{C}^n} f(z) \overline{g(z)} |z|^{2m} e^{-|z|^2} dv(z). \]  \tag{12}\]

By Lemma 3, there exists a positive constant \( C \), independent of \( f \) and \( z \), such that

\[ |f(z)| \leq C \|f\|_{2,m} \frac{e^{\frac{1}{2} |z|^2}}{(1 + |z|)^m} \]

for all \( f \in F^{2,m} \) and \( z \in \mathbb{C}^n \). This shows that each point evaluation is a bounded linear functional on \( F^{2,m} \). Consequently, each \( z \in \mathbb{C}^n \) gives rise to a function \( K^m_z \in F^{2,m} \) such that

\[ f(z) = \langle f, K^m_z \rangle, \quad f \in F^{2,m}. \]

As is well known, we have

\[ K_m(z, w) := K^m_z(w) = \sum_{\alpha} e_\alpha(z) \overline{e_\alpha(w)}, \]  \tag{13}\]

where \( \{e_\alpha\} \) is any orthonormal basis for \( F^{2,m} \). Note that monomials are mutually orthogonal, so the normalized monomials \( \left\{ \frac{z^\alpha}{\|z^\alpha\|_{2,m}} \right\} \) form an orthonormal basis for \( F^{2,m} \).

An exact formula for the reproducing kernel \( K_m(z, w) \) was obtained in [6]. For the reader’s convenience, we include the formula and its proof here.

**Theorem 12.** Let \( m \) be a non-negative integer. Then

\[ K_m(z, w) = \frac{(n + m - 1)!}{(n - 1)!} D_m K(z, w) \]

for \( z, w \in \mathbb{C}^n \).

**Proof.** An elementary computation yields

\[ \|z^\alpha\|_{2,m}^2 = \frac{(n - 1)! \alpha!(n - 1 + m + |\alpha|)!}{(n + m - 1)(n - 1 + |\alpha|)!}. \]
for each multi-index $\alpha$. Thus, setting $e_\alpha = z^\alpha/\|z^\alpha\|_{2,m}$, we have

$$e_\alpha(z) = \frac{(n + m - 1)!(n - 1 + |\alpha|)!}{(n - 1)!\alpha!(n - 1 + m + |\alpha|)!} z^\alpha.$$

Given $z, w \in \mathbb{C}^n$, some manipulations with the help of (13) yield

$$K_m(z, w) = \frac{(n + m - 1)!}{(n - 1)!} h_m(z \cdot \overline{w}) = \frac{(n + m - 1)!}{(n - 1)!} D_m K(z, w).$$

**Corollary 13.** Let $m$ be a non-negative integer. Then

$$\left| K_m(z, w) \right| \lesssim e^{\frac{1}{2}|z|^2 + \frac{1}{2}|w|^2 - \frac{1}{8}|z-w|^2}, \quad z, w \in \mathbb{C}^n.$$

**Corollary 14.** Let $m$ be a non-negative integer and $0 < p \leq \infty$. Then $K_m(\cdot, w) \in F^{p,m}$ for each $w \in \mathbb{C}^n$ and there is a constant $C > 0$ such that

$$\left\| K_m(\cdot, w) \right\|_{p,m} \leq C e^{\frac{1}{2}|w|^2/|w|^m}, \quad w \in \mathbb{C}^n.$$

Since $F^{2,m}$ is a closed subspace of the Hilbert space $L^{2,m}_g$, there is an orthogonal projection $P_m : L^{2,m}_g \to F^{2,m}$, and the projection is given by

$$P_m f(z) = \omega_{n,2,m} \int_{\mathbb{C}^n} f(w) K_m(z, w)|w|^{2m} e^{-|w|^2} d\nu(w).$$

**Proposition 15.** For each $1 \leq p \leq \infty$ and non-negative integer $m$ the reproducing operator $P_m$ is bounded from $L^{p,m}_g$ onto $F^{p,m}$. 

**Proof.** We have

$$|z|^m \left| P_m f(z) \right| \lesssim \int_{\mathbb{C}^n} |w|^m \left| f(w) \right| \left| L_m(z, w) \right| e^{-|w|^2} d\nu(w),$$

where $L_m(z, w) = |z|^m |w|^m K_m(z, w)$. By Corollary 13,

$$\left| L_m(z, w) \right| \lesssim (1 + |z||w|)^m \left| K_m(z, w) \right| \lesssim e^{\frac{1}{2}|z|^2 + \frac{1}{2}|w|^2 - \frac{1}{8}|z-w|^2}.$$

The desired result then follows from Lemma 9 and Theorem 11. 

**Theorem 16.** Suppose $1 < p < \infty$ and $1/p + 1/q = 1$. Then $(F^{p,m})^*$, the Banach dual of $F^{p,m}$, can be identified with $F^{q,m}$ under the integral pairing in (12).
Proof. That every function in $F^{q,m}$ induces a bounded linear functional on $F^{p,m}$ via the integral pairing in (12) follows from Hölder’s inequality.

On the other hand, if $F$ is a bounded linear functional on $F^{p,m}$, then according to the Hahn–Banach extension theorem, $F$ can be extended (without increasing its norm) to a bounded linear functional on $L^{p,m}_g$. By the usual duality of $L^{p,m}_g$ spaces, there exists some $h \in L^{q,m}_g$ such that $F(f) = \langle f, h \rangle_m$ for all $f \in F^{p,m}$. By Proposition 15, $P_m$ is a bounded projection from $L^{p,m}_g$ onto $F^{p,m}$. So if we let $g = P_m(h)$, then $g \in F^{p,m}$ and

$$F(f) = \langle f, h \rangle_m = \langle P_m(f), h \rangle_m = \langle f, P_m(h) \rangle_m = \langle f, g \rangle_m$$

for all $f \in F^{p,m}$. This completes the proof of the theorem. \qed

A similar result also holds for $0 < p \leq 1$. To prove it, we require the following density lemma whose proof in the $p \geq 1$ case is usually given as a consequence of standard duality theorems.

**Lemma 17.** Let $S$ be the set of all finite linear combinations of kernel functions, namely, functions of the form

$$f(z) = \sum_{k=1}^{N} c_k K_m(z, z_k).$$

Then $S$ is dense in $F^{p,m}$ for any $0 < p < \infty$.

**Proof.** This result was proved in [19] in the case $n = 1$ and $m = 0$. The proof for our more general case here is the same and we sketch the main steps here. More specifically, for any positive parameter $t$ we consider the weighted Gaussian measure $e^{-t|z|^2} d\nu(z)$, the associated Fock spaces $F^p_t$, and the corresponding Fock–Sobolev spaces $F^{p,m}_t$. The reproducing kernel for $F^2_t$ is $K_t(z, w) = e^{tz \cdot \bar{w}}$, and the reproducing kernel for $F^{2,m}_t$ is given by $K_{m,t}(z, w) = h_m(tz \cdot \bar{w})$, where $h_m$ is the function from Lemma 7.

As was in the case $n = 1$ and $m = 0$, we can find a positive constant $C$ and a positive parameter $t$ such that $\|f\|_{p,m} \leq C \|f\|_{2,m,t}$ for all entire functions $f$. Now if $g$ is any polynomial and

$$f(z) = \sum_{k=1}^{N} c_k K_m(z, z_k),$$

then

$$\|g - f\|_{p,m} \leq C \left\| g - \sum_{k=1}^{N} c_k K_m(\cdot, z_k) \right\|_{2,m,t}$$

$$= C \left\| g - \sum_{k=1}^{N} c_k K_{m,t}(\cdot, z_k/t) \right\|_{2,m,t}.$$
Since the set of polynomials is dense in each $F^{p,m}_t$, and the set of finite linear combinations of kernel functions $K_{m,t}(z, w)$ is obviously dense in the Hilbert space $F^{2,m}_t$, we conclude that every function in $F^{p,m}_t$ can be approximated in norm by elements of $S$. □

We can now prove the duality theorem for $F^{p,m}_t$ when $0 < p \leq 1$.

**Theorem 18.** Suppose $0 < p \leq 1$ and $m$ is a non-negative integer. Then the dual space of $F^{p,m}_t$ can be identified with $F^{\infty,m}_{\infty}$ under the integral pairing in (12).

**Proof.** First suppose that $g \in F^{\infty,m}_{\infty}$ and

$$F(f) = \omega_{n,2,m} \int_{\mathbb{C}^n} f(z) \overline{g(z)} e^{-|z|^2} |z|^{2m} dv(z), \quad f \in F^{p,m}_t.$$  

Then

$$|F(f)| \lesssim \|g\|_{\infty,m} \int_{\mathbb{C}^n} |z|^m |f(z) e^{-|z|^2/2}| dv(z),$$

and an application of Lemma 4 yields

$$|F(f)| \lesssim C \|g\|_{\infty,m} \|f\|_{p,m}, \quad f \in F^{p,m}_t,$$

where $C$ is a positive constant independent of $f$. This shows that $F$ defines a bounded linear functional on $F^{p,m}_t$.

Conversely, suppose that $F$ is a bounded linear functional on $F^{p,m}_t$. We consider the function $g$ defined on $\mathbb{C}^n$ by

$$g(w) = \overline{F(K_m(\cdot, w))}.$$  

From Corollary 14 we see that $K_m(\cdot, w)$ is in $F^{p,m}_t$, so $g$ is well defined. The function $g$ is also entire on $\mathbb{C}^n$. In fact, a direct computation using the appropriate version of Hölder’s inequality shows that the power series expansion

$$K_m(z, w) = \sum_{k=0}^{\infty} e_k(z) e_k(w),$$

where each $e_k$ is a monomial, converges in the norm topology of $F^{p,m}_t$ whenever $w$ is restricted to a compact subset of the complex plane. Thus

$$g(w) = \sum_{k=0}^{\infty} \overline{F(e_k)} e_k(w),$$

where the series converges uniformly on compacta, which clearly shows that $g$ is holomorphic in $w$.  

It follows from the boundedness of $F$ on $F^{p,m}$ that

$$|w|^m |g(w)| \lesssim \|F\| \left[ \int_{\mathbb{C}^n} |z|^m |w|^m |K_m(z, w)| e^{\frac{1}{2} |z|^2} |e|^p |d\nu(z)\right]^{1/p}$$

$$\lesssim \|F\| \left[ \int_{\mathbb{C}^n} e^{\frac{1}{2} |w|^2} e^{-\frac{p}{8} |z-w|^2} |e|^p |d\nu(z)\right]^{1/p}$$

$$\lesssim e^{\frac{1}{2} |w|^2} \|F\|.$$  

This shows that $g \in F^{\infty,m}$ and $\|g\|_{\infty,m} \lesssim \|F\|$.

Fix any $\zeta \in \mathbb{C}^n$ and consider the function $f(z) = K_m(z, \zeta)$ in $F^{p,m}$. Using the reproducing property of the reproducing kernel $K_m(z, w)$, we have

$$\langle f, g \rangle_m = \omega_{n,2,m} \int_{\mathbb{C}^n} K_m(w, \zeta) F(K_m(\cdot, w)) e^{-|w|^2} d\nu(w)$$

$$= F(K_m(\cdot, \zeta)) = F(f).$$

It follows that $F(f) = \langle f, g \rangle_m$ for all functions $f$ of the form

$$f(z) = \sum_{k=1}^{N} c_k K_m(z, \zeta_k).$$

The proof of the theorem is now complete in view of Lemma 17. □

It is well known that the family of spaces $L^{p,m}_g$ interpolates in the usual way:

$$[L^{p_0,m}_g, L^{p_1,m}_g]_{\theta} = L^{p,m}_g,$$

where $1 \leq p_0 < p_1 \leq \infty$, $\theta \in (0, 1)$, and

$$\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}.$$  

See [15,19] for example.

**Theorem 19.** Suppose $1 \leq p_0 < p_1 \leq \infty$, $\theta \in (0, 1)$, and

$$\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}.$$  

Then $[F^{p_0,m}, F^{p_1,m}]_{\theta}$, the complex interpolation space between $F^{p_0,m}$ and $F^{p_1,m}$, can be identified with $F^{p,m}$.
Proof. That $[F_{p_0,m}^{p}, F_{p_1,m}^{p}]_\theta \subset F_{p,m}^{p}$ follows from the definition of complex interpolation, the fact that each $F_{p_k,m}^{p}$ is a closed subspace of $L_{g}^{p_k,m}$, and the fact that $[L_{g}^{p_0,m}, L_{g}^{p_1,m}]_\theta = L_{g}^{p,m}$.

Conversely, if $f \in F_{p,m}^{p} \subset L_{g}^{p,m}$, then it follows from the fact that $[L_{g}^{p_0,m}, L_{g}^{p_1,m}]_\theta = L_{g}^{p,m}$ that there exists a function $F(z, w)$, where $z \in \mathbb{C}^n$ and $0 \leq \text{Re}(w) \leq 1$, and a positive constant $C$, such that

$\text{(a)} \quad F(z, \theta) = f(z)$ for all $z \in \mathbb{C}^n$.

$\text{(b)} \quad \| F(\cdot, w) \|_{p_0,m} \leq C$ for all $\text{Re}(w) = 0$.

$\text{(c)} \quad \| F(\cdot, w) \|_{p_1,m} \leq C$ for all $\text{Re}(w) = 1$.

Let $G(z, w) = P_m F(z, w)$, where the projection $P_m$ is applied with respect to the variable $z$. Then for any fixed $w$ the function $G(z, w)$ is entire in $z$, and the boundedness of the projection $P_m$ on $L_{g}^{p_k,m}$ shows that there is another positive constant $C$ such that

$\text{(a)} \quad G(z, \theta) = f(z)$ for all $z \in \mathbb{C}^n$.

$\text{(b)} \quad \| G(\cdot, w) \|_{p_0,m} \leq C$ for all $\text{Re}(w) = 0$.

$\text{(c)} \quad \| G(\cdot, w) \|_{p_0,m} \leq C$ for all $\text{Re}(w) = 1$.

This shows that $f \in [F_{p_0,m}^{p}, F_{p_1,m}^{p}]_\theta$, and completes the proof of the theorem. □

Note that we have only proved the interpolation identities with equivalent norms. Exactly how these norms compare is not known.

4. Carleson measures

In this section we characterize positive Borel measures on $\mathbb{C}^n$ that satisfy the Carleson-type condition introduced in Section 1.

Lemma 20. Let $0 < p < \infty$ and $0 \leq a < \infty$. Then we have

$$\int_{\mathbb{C}^n} |e^{z \cdot \overline{w}}|^p (1 + |w|)^a e^{-\frac{p}{2} |w|^2} d\nu(w) \lesssim (1 + |z|)^a e^{\frac{p}{2} |z|^2}.$$ 

Proof. For any $z, w \in \mathbb{C}^n$ we have

$$\frac{1 + |w|}{1 + |z|} \leq 1 + |z - w|.$$ 

It follows that

$$\int_{\mathbb{C}^n} |e^{z \cdot \overline{w}}|^p (1 + |w|)^a e^{-\frac{p}{2} |w|^2} d\nu(w)$$

$$= \int_{\mathbb{C}^n} e^{-\frac{p}{2} |z|^2 + p \text{Re}(z \cdot \overline{w}) - \frac{p}{2} |w|^2} (1 + |w|)^a e^{\frac{p}{2} |z|^2} d\nu(w)$$
\[ e^{\frac{p}{2}|z|^2} \int_{\mathbb{C}^n} e^{-\frac{p}{2}|z-w|^2} (1 + |w|)^a \, dv(w) \leq (1 + |z|)^a e^{\frac{p}{2}|z|^2} \int_{\mathbb{C}^n} e^{-\frac{p}{2}|z-w|^2} (1 + |z - w|)^a \, dv(w) \leq (1 + |z|)^a e^{\frac{p}{2}|z|^2}. \]

**Theorem 21.** Suppose \( 0 < p < \infty \), \( m \) is a non-negative integer, and \( r \) is a positive radius. Then the following two conditions are equivalent for any positive Borel measure \( \mu \) on \( \mathbb{C}^n \).

(a) \( \mu \) is a Carleson measure for \( F_{p,m} \).

(b) There exists a positive constant \( C \) such that

\[
\mu(B(z, r)) \leq C(1 + |z|)^{mp} \quad \text{for all } z \in \mathbb{C}^n. \tag{14}
\]

**Proof.** First assume that \( \mu \) is a Carleson measure for \( F_{p,m} \). Taking \( f = 1 \) in (1) shows that \( \mu(S) < \infty \) for any compact set \( S \).

Fix any \( a \in \mathbb{C}^n \) and let \( f(z) = e^{z-a} \) in (1). By Lemma 20, there exists another constant \( C > 0 \), independent of \( a \), such that

\[
\int_{\mathbb{C}^n} |e^{z-a} e^{-\frac{p}{2}|z|^2}|^p \, d\mu(z) \leq C(1 + |a|)^{mp} e^{\frac{p}{2}|a|^2}.
\]

In particular,

\[
\int_{|z-a| < r} |e^{z-a} e^{-\frac{p}{2}|z|^2}|^p \, d\mu(z) \leq C(1 + |a|)^{mp} e^{\frac{p}{2}|a|^2}
\]

for all \( a \in \mathbb{C}^n \). Completing a square in the exponent, we can rewrite the inequality above as

\[
\int_{|z-a| < r} e^{-\frac{p}{2}|z-a|^2} \, d\mu(z) \leq C(1 + |a|)^{mp} e^{\frac{p}{2}|a|^2},
\]

from which we deduce that

\[
\mu(B(a, r)) \leq Ce^{\frac{p}{2}r^2}(1 + |a|)^{mp}
\]

for all \( a \in \mathbb{C}^n \). This shows that condition (a) implies condition (b).

Next we assume that there exists a constant \( C > 0 \) such that (14) holds for all \( z \in \mathbb{C}^n \). We proceed to estimate the integral

\[
I(f) = \int_{\mathbb{C}^n} |f(z)e^{-\frac{1}{2}|z|^2}|^p \, d\mu(z)
\]

for any function \( f \in F_{p,m} \).
For any positive $\rho$ let $Q_\rho$ denote the following rectangular box in $\mathbb{C}^n$:

$$Q_\rho = \{z = (z_1, \ldots, z_n): z_j = x_j + iy_j, \ 0 < x_j \leq \rho, \ 0 < y_j \leq \rho, \ j = 1, \ldots, n\}.$$  

Let $Z$ denote the set of all integers and consider the lattice

$$Z^{2n}_\rho = \{(\ell_1 \rho + i k_1 \rho, \ldots, \ell_n \rho + i k_n \rho): \ \ell_j \in Z, \ k_j \in Z, \ j = 1, \ldots, n\}.$$  

It is clear that

$$\mathbb{C}^n = \bigcup \{Q_\rho + a: a \in Z^{2n}_\rho\}$$

is a decomposition of $\mathbb{C}^n$ into disjoint rectangles of side length $\rho$. Thus

$$I(f) = \sum_{a \in Z^{2n}_\rho} \int_{Q_\rho + a} \left|f(z)e^{-\frac{1}{2}|z|^2}\right|^p \, d\mu(z).$$  

We fix positive numbers $\rho$ and $t$ such that $t + \sqrt{2n}\rho = r$. By (3), there exists a constant $C$ such that

$$|f(z)|^p e^{-\frac{p}{2}|z|^2} \leq C \int_{|w-z|<t} \left|f(w)e^{-\frac{1}{2}|w|^2}\right|^p \, dv(w)$$

for all $z \in \mathbb{C}^n$. From this we easily deduce that

$$\left|f(z)e^{-\frac{1}{2}|z|^2}\right|^p \leq \frac{C}{(1+|z|)^{mp}} \int_{|w-z|<t} \left|(1+|w|)^m f(w)e^{-\frac{1}{2}|w|^2}\right|^p \, dv(w)$$

for all $z \in \mathbb{C}^n$, where $C$ is another positive constant. Now if $z \in Q_\rho + a$, where $a \in Z^{2n}_\rho$, then $B(z, t) \subset B(a, r)$ by the triangle inequality, and $1 + |z|$ is comparable to $1 + |a|$. It follows that

$$\left|f(z)e^{-\frac{1}{2}|z|^2}\right|^p \leq \frac{C}{(1+|a|)^{mp}} \int_{|w-a|<r} \left|(1+|w|)^m f(w)e^{-\frac{1}{2}|w|^2}\right|^p \, dv(w),$$

where $C$ is another positive constant. Therefore,

$$I(f) \leq C \sum_{a \in Z^{2n}_\rho} \frac{\mu(B(a, r))}{(1+|a|)^{mp}} \int_{B(a, r)} \left|(1+|w|)^m f(w)e^{-\frac{1}{2}|w|^2}\right|^p \, dv(w).$$

Combining this with the assumption in (14), we find another positive constant $C$ such that

$$I(f) \leq C \sum_{a \in Z^{2n}_\rho} \int_{B(a, r)} \left|(1+|w|)^m f(w)e^{-\frac{1}{2}|w|^2}\right|^p \, dv(w).$$
It is clear that there exists a positive integer $N$ such that each point in $\mathbb{C}^n$ belongs to at most $N$ of the balls $B(a, r)$, where $a \in \mathbb{Z}^{2n}_\rho$. It follows that

$$I(f) \leq CN \int_{\mathbb{C}^n} \left| \left(1 + |w|\right)^m f(w) e^{-\frac{1}{2} |w|^2} \right|^p dv(w).$$

Since $C$ and $N$ are independent of $f$, this and (2) show that condition (b) implies condition (a). \qed

A similar characterization of vanishing Carleson measures for $F_{p,m}$ holds.

**Theorem 22.** Suppose $0 < p < \infty$, $m$ is a non-negative integer, $r$ is a positive radius, and $\mu$ is a positive Borel measure on $\mathbb{C}^n$. Then the following are equivalent:

(a) $\mu$ is a vanishing Carleson measure for $F_{p,m}$.
(b) $\mu(B(a, r))/(1 + |a|)^{mp} \to 0$ as $|a| \to \infty$.

**Proof.** Suppose $\mu$ is a Carleson measure for $F_{p,m}$ and satisfies condition (b). Let $\{f_j\}$ be a sequence in $F_{p,m}$ such that $\|f_j\|_{p,m} < M$ for some constant $M$ and all $j$, and $\{f_j(z)\}$ converges to 0 uniformly on every compact subset of $\mathbb{C}^n$. We proceed to show that

$$I(f_j) := \int_{\mathbb{C}^n} \left| f_j(z)e^{-\frac{1}{2} |z|^2} \right|^p d\mu(z) \to 0$$

as $j \to \infty$.

Let $\varepsilon > 0$ be given. We choose a sufficiently large $R > 0$ so that $\mu(B(z, r)) < (1 + |z|)^{mp}\varepsilon$ whenever $|z| > R/2$. Since $f_j \to 0$ uniformly on compact sets, there exists $j_0$ such that

$$\int_{|z| < R} \left| f_j(z)e^{-\frac{1}{2} |z|^2} \right|^p d\mu(z) < \varepsilon, \quad j > j_0.$$

We denote by $\{a_k\}$ a rearrangement of the lattice set $\mathbb{Z}^{2n}_\rho$ such that $|a_k| \leq R/2$ for $1 \leq k < k_0$ and $|a_k| > R/2$ for $k \geq k_0$, where $\rho$ is any fixed positive number. In particular, $\{Q_\rho + a_k: k \geq k_0\}$ is a covering of $\{z: |z| > R\}$. If $j > j_0$, then

$$I(f_j) = \int_{|z| < R} \left| f_j(z)e^{-\frac{1}{2} |z|^2} \right|^p d\mu(z) + \int_{|z| > R} \left| f_j(z)e^{-\frac{1}{2} |z|^2} \right|^p d\mu(z) \leq \varepsilon + \sum_{k=k_0}^{\infty} \int_{Q_\rho + a_k} \left| f_j(z)e^{-\frac{1}{2} |z|^2} \right|^p d\mu(z).$$
\[ \leq \varepsilon + C \sum_{k=k_0}^{\infty} \frac{\mu(B(a_k,r))}{(1 + |a_k|)^mp} \int_{B(a_k,r)} \left| \left(1 + |w|\right)^m f_j(w) e^{-\frac{1}{2}|w|^2}\right|^p d\mu(w) \]
\[ \leq \varepsilon + C\varepsilon \sum_{k=k_0}^{\infty} \int_{B(a_k,r)} \left| \left(1 + |w|\right)^m f_j(w) e^{-\frac{1}{2}|w|^2}\right|^p d\mu(w). \]

From the local finiteness of the covering \{B(a_k,r)\}, there exists a positive integer \(N\) such that
\[ \sum_{k=k_0}^{\infty} \int_{B(a_k,r)} \left| \left(1 + |w|\right)^m f_j(w) e^{-\frac{1}{2}|w|^2}\right|^p d\mu(w) \leq N \|f_j\|_{p,m}^p \leq N M^p. \]

Therefore, we have \( I(f_j) \leq \varepsilon + \varepsilon C N M^p \) whenever \( j > j_0 \). Since \( \varepsilon \) is arbitrary, this shows that \( I(f_j) \to 0 \) as \( j \to \infty \).

For the converse, consider the functions
\[ f_a(w) = \frac{e^{w\cdot a} - \frac{1}{2}|a|^2}{(1 + |a|)^m}, \quad a \in \mathbb{C}^n, \quad w \in \mathbb{C}^n. \]

By Lemma 20, there is a positive constant \( C \) such that \( \|f_a\|_{p,m} \leq C \) for all \( a \in \mathbb{C}^n \). It is clear that \( f_a(w) \to 0 \) as \( |a| \to \infty \) and the convergence is uniform on compact subsets of \( \mathbb{C}^n \). If \( \mu \) is a vanishing Carleson measure for \( F_{p,m} \), then
\[ \lim_{|a| \to \infty} \int_{\mathbb{C}^n} \left| f_a(z) e^{-\frac{1}{2}|z|^2}\right|^p d\mu(z) = 0. \]

In particular,
\[ \lim_{|a| \to \infty} \int_{B(a,r)} \left| f_a(z) e^{-\frac{1}{2}|z|^2}\right|^p d\mu(z) = 0, \]

which can be rewritten as
\[ \lim_{|a| \to \infty} \frac{1}{(1 + |a|)^mp} \int_{B(a,r)} e^{-\frac{1}{2}|z-a|^2} d\mu(z) = 0. \]

Since
\[ \mu(B(a,r)) \leq e^{r^2/2} \int_{B(a,r)} e^{-\frac{1}{2}|z-a|^2} d\mu(z), \]

and the constant \( e^{r^2/2} \) is independent of \( a \), we have \( \mu(B(a,r))/(1 + |a|)^mp \to 0 \) as \( |a| \to \infty \). \( \square \)
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