GENERALIZED CALABI’S CORRESPONDENCE AND COMPLETE SPACELIKE SURFACES

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Abstract. Extending Calabi’s correspondence between minimal graphs in the Euclidean space \( \mathbb{R}^3 \) and maximal graphs in the Lorentz-Minkowski spacetime \( \mathbb{L}^3 \) to a wide class of 3-manifolds carrying a unit Killing vector field, we construct a twin correspondence between graphs with prescribed mean curvature \( H \) in the Riemannian Generalized Bianchi-Cartan-Vranceanu (GBCV) space \( \mathbb{E}^3(M, \tau) \) and spacelike graphs with prescribed mean curvature \( \tau \) in the GBCV spacetime \( \mathbb{L}^3(M, H) \). For instance, the prescribed mean curvature equation in \( \mathbb{L}^3 \) can be transformed into the minimal surface equation in the generalized Heisenberg space with prescribed bundle curvature. We present several applications of the twin correspondence and study the moduli space of complete spacelike surfaces in the GBCV spacetimes.

1. Motivation and main results

1.1. History of Calabi problems. The significant role played by maximal and constant mean curvature submanifolds in spacetimes is widely recognized. Maximal submanifolds immersed in a Lorentzian manifold are spacelike submanifolds with vanishing mean curvature, and naturally arise as solutions of the variational problem of locally maximizing the induced volume functional within the class of spacelike submanifolds.

Among various interesting applications, the issue of the existence of maximal submanifolds appears in the Schoen-Yau proof of the Positive Mass Theorem and also in the analysis of solutions of the Einstein-Yang-Mills equation. Very recently, Bonsante and Schlenker used the geometry of maximal surfaces in the anti de Sitter spacetime to establish a variant of Schoen’s conjecture on the universal Teichmüller space.

Determining the existence, uniqueness, and regularity of maximal and constant mean curvature submanifolds in various spacetimes is one of the important problems in both mathematical relativity and differential geometry. Indeed, such submanifolds are Riemannian manifolds reflecting some of the Riemannian features of their ambient spacetimes.

In 1970, Calabi presented a remarkable uniqueness result on the global behavior of maximal surfaces. He proved that, in the Lorentz-Minkowski space \( \mathbb{L}^3 = (\mathbb{R}^3, dx^2 + dy^2 - dz^2) \), the only entire maximal graphs defined

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over the whole xy-plane are spacelike planes, which implies that spacelike planes are the only complete maximal surfaces in \( \mathbb{L}^3 \). On the other hand, the beautiful theorem of Bernstein states that the only entire minimal graphs in the Euclidean space \( \mathbb{E}^3 = (\mathbb{R}^3, dx^2 + dy^2 + dz^2) \) are planes. However, it is well-known that \( \mathbb{E}^3 \) admits different complete non-planar minimal surfaces (e.g., catenoids, Costa’s surface, etc).

Cheng and Yau \cite{ChengYau} extended Calabi’s Theorem to higher dimension by proving that the only complete maximal hypersurface in Lorentz-Minkowski space \( \mathbb{L}^{n+1}\geq3 \) are the spacelike hyperplanes. Nonetheless, Bombieri, De Giorgi and Giusti \cite{BombieriDeGiorgiGiusti} disproved the Riemannian analog by finding entire non-planar minimal hypersurfaces in the Euclidean space \( \mathbb{E}^{n+1}\geq9 \).

During the last few decades, special attention has been given to the existence, uniqueness, and non-existence of constant mean curvature surfaces in more general ambient spaces. Here, we sketch some of known results.

- Fernández and Mira \cite{FernandezMira} Theorem 3.1] constructed a large family of complete maximal surfaces in the static Robertson-Walker (RW) space \( \mathbb{H}^2 \times \mathbb{R} \) endowed with the Lorentzian product metric \( \langle \cdot, \cdot \rangle_{\mathbb{H}^2} - dz^2 \). They are entire graphs over the hyperbolic base \( \mathbb{H}^2 \), and so they induce foliations of the ambient space. On the other hand, Albujer \cite{Albujer} Examples 3.1 and 3.3] proved that \( \mathbb{H}^2 \times \mathbb{R} \) also admits non-complete entire maximal graphs.

- Albujer andAlias \cite{AlbujerAlias} Theorem 3.3] proved that, given an arbitrary Riemannian surface \( \mathcal{M} \) with Gaussian curvature \( K_\mathcal{M} \geq 0 \), complete maximal surfaces in the product space \( \mathcal{M} \times \mathbb{R} \) (endowed with the Lorentzian product metric \( \langle \cdot, \cdot \rangle_{\mathcal{M}} - dz^2 \) are totally geodesic. Furthermore, if \( K_\mathcal{M} \) does not identically vanish, then they show that \( \Sigma \) must be a spacelike horizontal slice \( \mathcal{M} \times \{t_0\} \) for some \( t_0 \in \mathbb{R} \).

- Calabi type problems in the Generalized Robertson-Walker (GRW) spacetimes are topics of increasing interest in recent years \cite{Montiel31, Montiel33}. Given a smooth warping function \( \phi : \mathbb{R} \rightarrow (0, +\infty) \) and a base Riemannian manifold \( \mathcal{M} \), the GRW spacetime \( \mathcal{M} \times \phi \mathbb{R} \) is defined as \( \mathcal{M} \times \mathbb{R} \) equipped with the Lorentzian warped metric \( \phi(z)^2 \langle \cdot, \cdot \rangle_{\mathcal{M}} - dz^2 \), and carries a global timelike conformal vector field \( \phi(z) \partial_z \).

GRW spacetimes are particular cases of Conformally Stationary (CS) spacetimes which are time-orientable spacetimes admitting a timelike conformal field. For instance, see Montiel’s observation \cite{Montiel31}, Section 3]. Several existence results of spacelike immersions in CS spacetimes are also known \cite{Montiel22, Montiel24}.

Calabi’s original proof \cite{Calabi} of the fact that spacelike planes are the only entire maximal graphs in \( \mathbb{L}^3 \) is motivated by his interesting duality between minimal graphs in \( \mathbb{E}^3 \) and maximal graphs in \( \mathbb{L}^3 \). This twin correspondence is extended to a correspondence between graphs of constant mean curvature \( H \) in the Riemannian Bianchi–Cartan–Vranceanu (BCV) space \( \mathbb{E}^3(\kappa, \tau) \).
and spacelike graphs of constant mean curvature $\tau$ in the BCV spacetime $L^3(\kappa, H)$ [29]. In the particular case $\tau = H = 0$, the twin correspondence reduces to the Albujer-Alías duality [2].

Since the twin correspondence in BCV spaces sends entire graphs to entire graphs, it becomes a natural and useful tool to study Bernstein-Calabi type problems. For instance, entire spacelike graphs of constant mean curvature $\frac{1}{2}$ in $L^3 = L^3(\mathbb{E}^2, 0)$ are the twin surfaces of entire minimal graphs in the Riemannian Heisenberg space $\text{Nil}^3 = \mathbb{E}^3(\mathbb{E}^2, \frac{1}{2})$. Since the moduli space of entire spacelike graphs of constant mean curvature $\frac{1}{2}$ in $L^3$ is large (Treibergs [35, Theorem 2] showed that such spacelike graphs can be asymptotic to an arbitrary $C^2$ perturbation of the light cone in $L^3$), it follows that there exist lots of entire minimal graphs in $\text{Nil}^3$.

Calabi’s duality can also be extended to higher codimension [27].

1.2. Sketch of main results. The key point of this article is to study prescribed mean curvature surfaces in a manifold via their twin surfaces in different ambient manifolds.

The ambient manifolds we will deal with are introduced in Section 2 and will be called Riemannian and Lorentzian Generalized Bianchi–Cartan–Vranceanu (GBCV in the sequel) spaces, which will be denoted by $E^3(M, \tau)$ and $L^3(M, \tau)$, respectively, where $M$ will be a Riemannian surface, and $\tau$ will be any smooth function on $M$. These GBCV spaces are locally characterized by admitting a Riemannian submersion over $M$ with bundle curvature $\tau$, and whose fibers are the integral curves of a unit Killing vector field (timelike in the Lorentzian case), so GBCV spaces are often called Killing submersions [17, 30]. The main technical tool in this section will be the concept of Calabi potential, which yields an explicit way of recovering the metric of the GBCV spaces $E^3(M, \tau)$ and $L^3(M, \tau)$ in terms of $M$ and $\tau$. Moreover, Killing submersions are spaces where graphs arise in a natural way as smooth sections of the aforementioned Riemannian submersion, and their mean curvatures admit divergence-form expressions (see Lemma 3.5).

GBCV spaces can be understood as twisted product spaces since if the bundle curvature is identically zero, then the GBCV spaces recover the Riemannian and Lorentzian product spaces $M \times \mathbb{R}$. If $M$ is the simply-connected constant curvature $\kappa$ surface and the bundle curvature $\tau$ is also constant, they reduce to the classical BCV spaces $E^3(\kappa, \tau)$ and $L^3(\kappa, \tau)$.

Section 3 is devoted to establish our Calabi type twin correspondence, and to show that twin surfaces admit simultaneous conformal coordinates. More explicitly, graphs with prescribed mean curvature $H$ in the Riemannian GBCV space $E^3(M, \tau)$ correspond to spacelike graphs with prescribed mean curvature $\tau$ in the Lorentzian GBCV space $L^3(M, H)$. In other words, the twin correspondence swaps the mean and bundle curvatures.
As a particular case, if the common base $M$ is the Euclidean plane $\mathbb{E}^2$ and $H = \tau = 0$, the twin correspondence reduces to the classical Calabi correspondence [2]. When the curvature of the base, the bundle curvature and the mean curvature are all constants, it reduces to the twin correspondence in the classical BCV spaces [27, Theorem 2].

In Section 4, we will make use of the twin correspondences to investigate complete spacelike surfaces in the GBCV spacetimes. We recall that completeness of entire spacelike graphs in Lorentzian manifolds is not guaranteed in general [1].

We establish that complete spacelike surfaces in the GBCV spacetime $\mathbb{L}^3(M, \tau)$ with simply-connected base $M$ are entire graphs. In particular, the study of complete spacelike surfaces in the GBCV spacetime reduces to that of the entire spacelike solutions of the corresponding prescribed mean curvature equation, where classical PDE techniques can be applied. It is important to remark that complete spacelike surfaces in a GBCV spacetime are in correspondence with non-trivial spacelike foliations of their ambient spaces modulo the 1-parameter group of isometries associated to the unit Killing vector field.

We find that there exists a large family of GBCV spacetimes carrying no complete spacelike surfaces, and this is achieved by obtaining a sharp bound on the mean curvature function of an entire graph in the Riemannian GBCV spaces. More explicitly, we prove that if the bundle curvature $\tau$ of the GBCV spacetime $\mathbb{L}^3(M, \tau)$ satisfies $\inf_M |\tau| > \frac{1}{2} \sqrt{-\kappa}$, where $\text{Ch}(M)$ denotes the Cheeger constant of the non-compact simply-connected base $M$, then $\mathbb{L}^3(M, \tau)$ admits no complete spacelike surfaces. As a consequence, if we denote by $c \leq 0$ the infimum of the Gaussian curvature of $M$ and the estimate $\inf_M |\tau| > \frac{1}{2} \sqrt{-\kappa}$ holds, then it is possible to conclude that $\mathbb{L}^3(M, \tau)$ does not admit complete spacelike surfaces, either. This is due to the non-existence theorem above and a comparison result for Cheeger constants. It is important to notice that there is no assumption on the mean curvature of the complete spacelike surface.

As a particular case, we deduce that there do not exist complete spacelike surfaces in the classical BCV spacetime $\mathbb{L}^3(\kappa, \tau)$ with constant base curvature $\kappa \leq 0$ and constant bundle curvature $|\tau| > \frac{1}{2} \sqrt{-\kappa}$. Moreover, we show that the lower bound $\frac{1}{2} \sqrt{-\kappa}$ is sharp, for the moduli space of complete maximal surfaces in the anti de Sitter spacetime $\mathbb{L}^3(\kappa, \frac{1}{2} \sqrt{-\kappa})$ is large. In fact, we prove that a maximal surface in $\mathbb{L}^3(\kappa, \frac{1}{2} \sqrt{-\kappa}) = \mathbb{L}^3(\mathbb{H}^2(\kappa), \frac{1}{2} \sqrt{-\kappa})$ is complete if and only if it is an entire spacelike graph defined over the whole hyperbolic base $\mathbb{H}^2(\kappa)$. This extends the Cheng-Yau result [11] that entire constant mean curvature spacelike graphs in $\mathbb{L}^3 = \mathbb{L}^3(\mathbb{E}^2, 0)$ are complete.

As a final consequence of the non-existence result, we prove that the GBCV spacetime $\mathbb{L}^3(M, \tau)$ is not distinguishing (in the sense of causality) when $\inf_M |\tau| > \frac{1}{2} \text{Ch}(M)$.
2. Construction of GBCV spaces

We will begin by defining models for the Riemannian and Lorentzian Generalized Bianchi–Cartan–Vranceanu spaces (GBCV in the sequel) in terms of the bundle curvature and a conformal parametrization of the base surface as follows.

**Definition 2.1 (GBCV spaces)** \( E^3(M, \tau) \) and \( L^3(M, \tau) \). Let \( M = (\Omega, g = \delta(x, y)^{-2}(dx^2 + dy^2)) \) denote a Riemannian surface, where \( \Omega \subseteq \mathbb{R}^2 \) is open and star-shaped with respect to the origin, and \( \delta \in C^\infty(\Omega) \) is positive. Given \( \tau \in C^2(\Omega) \), we will call the Calabi potential the function \( C_{\delta, \tau} \in C^2(\Omega) \) defined by

\[
C_{\delta, \tau}(x, y) = 2 \int_0^1 t \frac{\tau(tx, ty)}{\delta(tx, ty)^2} dt,
\]

Thus the Calabi potential \( C_{\delta, \tau} \) satisfies the divergence-form equation

\[
\frac{\partial}{\partial x} (x C_{\delta, \tau}) + \frac{\partial}{\partial y} (y C_{\delta, \tau}) = 2\tau,
\]

and allows us to define two 3-dimensional manifolds as follows:

(a) The Riemannian GBCV space \( E^3(M, \tau) \) is the product 3-manifold \( \Omega \times \mathbb{R} \) endowed with the Riemannian metric

\[
\frac{1}{\delta(x, y)^2} (dx^2 + dy^2) + (dz + C_{\delta, \tau}(x, y)(ydx - xdy))^2.
\]

(b) The GBCV spacetime \( L^3(M, \tau) \) is the 3-manifold \( \Omega \times \mathbb{R} \) endowed with the Lorentzian metric

\[
\frac{1}{\delta(x, y)^2} (dx^2 + dy^2) - (dz - C_{\delta, \tau}(x, y)(ydx - xdy))^2.
\]

We will call the 2-manifold \( M = (\Omega, g) \) the base surface. The real-valued function \( \tau \) will be called the bundle curvature function of the ambient GBCV spaces \( E^3(M, \tau) \) and \( L^3(M, \tau) \). This is motivated by Remark 2.2 below.

**Remark 2.2 (GBCV spaces as Killing submersions).** The projection map \( \pi : \Omega \times \mathbb{R} \rightarrow \Omega \) given by \( \pi(x, y, z) = (x, y) \) is a Riemannian submersion from the ambient GBCV spaces to the base surface, and its fibers are the integral curves the unit Killing vector field \( \partial_z \). Spaces satisfying these conditions are known as Killing submersions \([17]\). When \( M \) is simply-connected, the Riemannian GBCV spaces as defined above are characterized as the only simply-connected 3-manifolds which admit a structure of Killing submersion over \( M \) with bundle curvature function \( \tau \) \([30]\). Hence the Calabi potential can be thought as a way of recovering the metric of the GBCV space in terms of its bundle curvature.

**Remark 2.3 (GBCV spaces as twisted product spaces).** If the bundle curvature function \( \tau \) identically vanishes, GBCV spaces reduce to the product
spaces $\mathbb{E}^3(M, 0) = M \times \mathbb{R}$ with the Riemannian product metric $\langle \cdot, \cdot \rangle_M + dz^2$, and $\mathbb{L}^3(M, 0) = M \times \mathbb{R}$ with the Lorentzian product metric $\langle \cdot, \cdot \rangle_M - dz^2$.

On the other hand, if both the curvature of the base $M$ and the bundle curvature $\tau$ are constant, geometric meanings of the bundle curvature $\tau$ are revealed in [14, Section 2.1] and [16, Proposition 1.6.2], and their proofs naturally extend to the GBCV spaces. Let $\mathcal{K}^3$ denote the GBCV space $\mathbb{E}^3(M, \tau)$ or $\mathbb{L}^3(M, \tau)$ carrying the unitary Killing field $\xi = \partial_z$ and let $\nabla$ be its associated Levi-Civita connection. Then the bundle curvature function $\tau$ satisfies the geometric identity

$$\nabla_X \xi = \tau X \wedge \xi, \quad \text{for all } X \in \mathfrak{X}(\mathcal{K}^3).$$

We note that the vector product $\wedge$ in $\mathcal{K}^3$ is determined by an oriented orthonormal frame, which can be explicitly chosen by using the Calabi potential. More precisely, the global orthonormal frame $\{E_1, E_2, E_3\}$ in $\mathbb{E}^3(M, \tau)$ will be assumed positively oriented, where

$$E_1 = \delta (\partial_x - y \, C_{\delta, \tau} \partial_z), \quad E_2 = \delta (\partial_x + x \, C_{\delta, \tau} \partial_z), \quad E_3 = \partial_z.$$ 

Likewise, the orientation in the Lorentzian case will be given by the orthonormal frame $\{L_1, L_2, L_3\}$ in $\mathbb{L}^3(M, \tau)$, where

$$L_1 = \delta (\partial_x + y \, C_{\delta, \tau} \partial_z), \quad L_2 = \delta (\partial_x - x \, C_{\delta, \tau} \partial_z), \quad L_3 = \partial_z.$$ 

Another geometric interpretation of the bundle curvature function of the Killing submersion $\pi : \mathcal{K}^3 \to M$ can be found in [30, Proposition 3.3]. In the Riemannian case, the proof of (2.3) is available in [17, Proposition 2.6].

Remark 2.4 (GBCV spaces as extensions of BCV spaces $\mathbb{E}^3(\kappa, \tau)$ and $\mathbb{L}^3(\kappa, \tau)$). Classical BCV spaces have constant base curvature and constant bundle curvature. Given a constant $\kappa \in \mathbb{R}$, we consider $\delta(x, y) = 1 + \frac{1}{4}(x^2 + y^2)$, which is positive in $\Omega = \mathbb{R}^2$ for $\kappa \geq 0$ or $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < \frac{4}{\kappa}\}$ for $\kappa < 0$.

Then the metric $g = \delta(x, y)^{-2}(dx^2 + dy^2)$ on $\Omega$ has constant curvature $\kappa$, and $M = (\Omega, g)$ is isometric to $\mathbb{E}^2$ (for $\kappa = 0$), $\mathbb{H}^2(\kappa)$ (for $\kappa < 0$), or $\mathbb{S}^2(\kappa)$ minus a point (for $\kappa > 0$). When the bundle curvature function $\tau(x, y)$ is constant, the associated Calabi potential is nothing but $C_{\delta, \tau}(x, y) = \tau \delta(x, y)^{-1}$, so we get the classical BCV spaces $\mathbb{E}^3(\kappa, \tau)$ and $\mathbb{L}^3(\kappa, \tau)$, see [14]. Hence, in this notation we obtain $\mathbb{E}^3(\kappa, \tau) = \mathbb{E}^3(M^2(\kappa), \tau)$ and $\mathbb{L}^3(\kappa, \tau) = \mathbb{L}^3(M^2(\kappa), \tau)$, where $M^2(\kappa)$ denotes the aforementioned surfaces of constant curvature $\kappa$.

3. Twin correspondences in GBCV spaces

We will now derive formulas for the mean curvature of vertical graphs in the GBCV spaces. Keeping the notation of Definition 2.1, the graph of a function $u \in C^2(\Omega')$, where $\Omega' \subseteq \Omega$ is an open subset, is the surface

$$\{(x, y, z) \in \Omega' \times \mathbb{R} : z = u(x, y)\}.$$
In other words, we are considering graphs over the global zero-section \( z = 0 \). In the Lorentzian case, the graph is said to be spacelike if the induced metric from the ambient manifold is Riemannian. The following lemma can be easily deduced.

**Lemma 3.5 (Mean curvature of graphs in GBCV spaces).** Let \( u \in C^2(\Omega') \) denote the height function of a graph defined on some open subset \( \Omega' \subseteq \Omega \).

1. The mean curvature function \( H \) of the graph of \( u \) in the Riemannian GBCV space \( \mathbb{R}^3(M, \tau) \) satisfies

\[
2H = \delta^2 \left( \frac{\partial}{\partial x} \left( \frac{\alpha}{\omega} \right) + \frac{\partial}{\partial y} \left( \frac{\beta}{\omega} \right) \right) = \text{div}_M \left( \frac{G}{\sqrt{1 + ||G||_M^2}} \right),
\]

where \( \alpha = u_x + y C_{\delta, \tau}, \beta = u_y - x C_{\delta, \tau}, \omega = \sqrt{1 + \delta^2(\alpha^2 + \beta^2)} \), and the vector field \( G \) on \( \Omega' \subseteq M \) is given by \( G = \delta^2(\alpha \partial_x + \beta \partial_y) \).

2. If the graph of \( u \) is spacelike in the Lorentzian BCV space \( \mathbb{L}^3(M, \tau) \), then its mean curvature function \( \tilde{H} \) satisfies

\[
2\tilde{H} = \delta^2 \left( \frac{\partial}{\partial x} \left( \frac{\tilde{\alpha}}{\tilde{\omega}} \right) + \frac{\partial}{\partial y} \left( \frac{\tilde{\beta}}{\tilde{\omega}} \right) \right) = \text{div}_M \left( \frac{G}{\sqrt{1 - ||G||_M^2}} \right),
\]

where \( \tilde{\alpha} = u_x - y C_{\delta, \tau}, \tilde{\beta} = u_y + x C_{\delta, \tau}, \tilde{\omega} = \sqrt{1 - \delta^2(\tilde{\alpha}^2 + \tilde{\beta}^2)} \), and the vector field \( \tilde{G} \) on \( \Omega' \subseteq M \) is given by \( \tilde{G} = \delta^2(\tilde{\alpha} \partial_x + \tilde{\beta} \partial_y) \). We notice that the graph of \( u \) is spacelike in \( \mathbb{L}^3(M, \tau) \) if and only if

\[
1 - ||G||_M^2 = 1 - \delta^2(\tilde{\alpha}^2 + \tilde{\beta}^2) > 0.
\]

**Example 3.6 (Helicoids in GBCV spaces with rotational symmetry).** Vertical translations (i.e., the elements of the 1-parameter group of isometries associated to the Killing vector field \( \partial_z \)) are generically the only isometries in the GBCV spaces. If we additionally assume that both the conformal factor and the bundle curvature are radial with respect to the origin (i.e., \( \delta(x, y) \) and \( \tau(x, y) \) are functions of \( x^2 + y^2 \)), then the induced Calabi potential \( C_{\delta, \tau} \) is also radial, and rotations about the axis \( x = y = 0 \) are also isometries. It is easy to check that, given \( \mu_1, \mu_2 \in \mathbb{R} \), the helicoid

\[
\mathcal{H}_{\mu_1, \mu_2} = \{(\rho \cos(\theta), \rho \sin(\theta), \mu_1 \theta + \mu_2) : \rho, \theta \in \mathbb{R}\}
\]

has zero mean curvature in the GBCV spaces when radial symmetry is assumed, though in the Lorentzian case, the spacelike condition is not guaranteed. Note that \( \mathcal{H}_{\mu_1, \mu_2} \) is a horizontal plane for \( \mu_1 = 0 \) and converges to a vertical plane when \( \mu_1 \to \infty \). In fact, each level curve \( l_{z_0} = \mathcal{H}_{\mu_1, \mu_2} \cap \{z = z_0\} \) is an ambient geodesic, so \( \mathcal{H}_{\mu_1, \mu_2} \) is also a ruled surface. Moreover, except
at the axis \( x = y = 0 \), they can be expressed locally as the graphs

\[
z = \mu_1 \arctan \left( \frac{y}{x} \right) + \mu_2.
\]

It is worth to mention that the only ruled minimal surfaces in the Heisenberg space \( \text{Nil}_3 = \mathbb{E}^3(\mathbb{E}^2, \frac{1}{2}) \) are (part of) planes, helicoids, and hyperbolic paraboloids up to ambient isometries [127, Theorem 2.3]. It would be interesting to extend this characterization to GBCV spaces with rotational symmetry.

We will construct a Calabi-type correspondence between graphs with mean curvature function \( H \) in the Riemannian GBCV space \( \mathbb{E}^3(M, \tau) \) and spacelike graphs with mean curvature function \( \tau \) in the Lorentzian GBCV space \( \mathbb{E}^3(M, H) \). More explicitly, we obtain:

**Theorem 3.7 (Twin correspondence).** Let \( M \) be an open domain \( \Omega \subseteq \mathbb{R}^2 \), star-shaped with respect to the origin, endowed with the metric \( \delta \) and \( \tau \) are positive. Let \( \Omega \rightarrow \Omega \) be a simply-connected open domain such that the graph of \( g \) is spacelike in \( \mathbb{E}^3(M, \tau) \) and has mean curvature function \( \tau \).

Let \( \Omega \subseteq \mathbb{R}^2 \) and \( \Omega \subseteq \mathbb{R}^2 \) be such that the graph of \( g \) is spacelike in \( \mathbb{E}^3(M, \tau) \) and has mean curvature function \( \tau \).

The graphs in (a) and (b) can be chosen to satisfy the twin relations:

\[
(\tilde{\alpha}, \tilde{\beta}) = \left( -\frac{\beta}{\omega'}, \frac{\alpha}{\omega'} \right), \quad \text{or equivalently,} \quad (\alpha, \beta) = \left( \frac{\tilde{\beta}}{\tilde{\omega}'} - \frac{\tilde{\alpha}}{\tilde{\omega}'} \right),
\]

where

\[
(\alpha, \beta) = \left( f_x + y C_{\delta \tau}, f_y - x C_{\delta \tau} \right), \quad \omega = \sqrt{1 + \delta^2 (\alpha^2 + \beta^2)},
\]

\[
(\tilde{\alpha}, \tilde{\beta}) = \left( g_x - y C_{\delta H}, g_y + x C_{\delta H} \right), \quad \tilde{\omega} = \sqrt{1 - \delta^2 (\tilde{\alpha}^2 + \tilde{\beta}^2)}.
\]

Moreover, let us parametrize the graphs as \( F(x, y) = (x, y, f(x, y)) \) and \( G(x, y) = (x, y, g(x, y)) \), \( (x, y) \in \Omega' \). Then, the mapping \( F(x, y) \rightarrow G(x, y) \) is a conformal diffeomorphism between the two graphs. If \( I \) and \( I' \) denote the induced metrics on the graphs \( z = f(x, y) \) in \( \mathbb{E}^3(M, \tau) \) and \( z = g(x, y) \) in \( \mathbb{L}^3(M, H) \), respectively, then

\[
I'_{G(x,y)} = \frac{1}{\omega(x, y)^2} I_{F(x,y)}.
\]

In the conditions above, the graph of \( f \) in \( \mathbb{E}^3(M, \tau) \) and the graph of \( g \) in \( \mathbb{L}^3(M, H) \) are called twin surfaces.
Proof. We will begin by proving (a) and the proof of (b) will be analogous. Lemma 3.5 shows that the mean curvature function $H$ of the graph of $f$ over $\Omega'$ in the GBCV space $E^3(M, \tau)$ is the divergence-form equation

$$2H = \frac{\partial}{\partial x} \left( \frac{\alpha}{\omega} \right) + \frac{\partial}{\partial y} \left( \frac{\beta}{\omega} \right), \quad (3.4)$$

where $(\alpha, \beta) = (f_x + yC_{\delta,\tau}, f_y - xC_{\delta,\tau})$ and $\omega = \sqrt{1 + \delta^2(\alpha^2 + \beta^2)}$. Equation (2.2) tells us that the Calabi potential $C_{\delta,\tau}(x, y)$ satisfies the divergence-form equation

$$2\tau = \frac{\partial}{\partial x} (xC_{\delta,\tau}) + \frac{\partial}{\partial y} (yC_{\delta,\tau}). \quad (3.5)$$

Likewise, the Calabi potential $C_{\delta,H}(x, y)$ satisfies

$$2H = \frac{\partial}{\partial x} (xC_{\delta,H}) + \frac{\partial}{\partial y} (yC_{\delta,H}). \quad (3.6)$$

Hence, by using (3.6), we are able to rewrite the mean curvature equation (3.4) as the zero-divergence equation

$$0 = \frac{\partial}{\partial x} \left( \frac{\alpha}{\omega} - xC_{\delta,H} \right) + \frac{\partial}{\partial y} \left( \frac{\beta}{\omega} - yC_{\delta,H} \right).$$

Since the domain $\Omega'$ is simply connected, Poincaré’s Lemma yields the existence of a function $g \in C^2(\Omega')$ such that

$$(g_x, g_y) = \left( -\frac{\beta}{\omega} + yC_{\delta,h}, \frac{\alpha}{\omega} - xC_{\delta,H} \right).$$

Setting $(\tilde{\alpha}, \tilde{\beta}) = (g_x - yC_{\delta,H}, g_y + xC_{\delta,H})$, we obtain the spacelike condition

$$1 - \delta^2 (\tilde{\alpha}^2 + \tilde{\beta}^2) = \frac{1}{\omega^2} > 0,$$

so it makes sense to introduce $\tilde{\omega} = \sqrt{1 - \delta^2(\alpha^2 + \beta^2)}$, from where the twin relations (3.3) easily follow. Finally, we employ the twin relations, the integrability condition $\frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial y}(f_x)$ and (3.5) to deduce that

$$\frac{\partial}{\partial x} \left( \frac{\tilde{\alpha}}{\tilde{\omega}} \right) + \frac{\partial}{\partial y} \left( \frac{\tilde{\beta}}{\tilde{\omega}} \right) = \frac{\partial}{\partial x} (-\tilde{\beta}) + \frac{\partial}{\partial y} (\tilde{\alpha})$$

$$= \frac{\partial}{\partial x} (-f_y + xC_{\delta,\tau}) + \frac{\partial}{\partial y} (f_x + yC_{\delta,\tau})$$

$$= \frac{\partial}{\partial x} (xC_{\delta,\tau}) + \frac{\partial}{\partial y} (yC_{\delta,\tau}) = \frac{2\tau}{\delta^2}.$$

This means that the spacelike graph $z = g(x, y)$ over the same domain $\Omega'$ in $\mathbb{L}^3(M, \tau)$ has mean curvature function $\tau$, so the correspondence holds.
We will now deal with the last paragraph in the statement. Let $\zeta = \xi + i\eta$ be a local complex conformal coordinate on the graph $z = f(x, y)$ and let us consider the coordinate transformation $\phi : (x, y) \mapsto (\xi, \eta)$ to obtain the local isothermal parametrization of the graph $z = f(x, y)$ in $\mathbb{E}^3(M, \tau)$:

$$\hat{F} = F \circ \phi^{-1} : (\xi, \eta) \mapsto (x, y) \mapsto (x, y, f(x, y)).$$

Here, $\phi^{-1}$ denotes the local inverse coordinate transformation $(\xi, \eta) \mapsto (x, y) = (p(\xi, \eta), q(\xi, \eta))$, so our goal is to prove that

$$\hat{G} = G \circ \phi^{-1} : (\xi, \eta) \mapsto (x, y) \mapsto (x, y, g(x, y))$$

is an isothermal parametrization of the twin graph $z = g(x, y)$ in $\mathbb{L}^3(M, H)$ and that the arising conformal factor is $\frac{1}{\omega^2}$. Taking the complexified operator $\hat{\frac{D}{\zeta}} = \frac{1}{2} (\frac{D}{\zeta} - i \frac{D}{\bar{\zeta}})$, we will be done by proving that

$$\left\{ \frac{\partial \hat{G}}{\partial \zeta}, \frac{\partial \hat{G}}{\partial \bar{\zeta}} \right\}_{\mathbb{L}^3(M, H)} = 0 \quad \text{and} \quad \left\{ \frac{\partial \hat{G}}{\partial \zeta}, \frac{\partial \hat{G}}{\partial \bar{\zeta}} \right\}_{\mathbb{E}^3(M, \tau)} = \frac{1}{\omega^2} \left\{ \frac{\partial \hat{F}}{\partial \zeta}, \frac{\partial \hat{F}}{\partial \bar{\zeta}} \right\}_{\mathbb{E}^3(M, \tau)}.$$

We consider the global orthonormal frames $\{E_1, E_2, E_3\}$ and $\{L_1, L_2, L_3\}$ in $\mathbb{E}^3(M, \tau)$ and $\mathbb{L}^3(M, H)$, respectively, given by

$$E_1 = \delta (\partial_x - y C_{\delta, \tau} \partial_z), \quad E_2 = \delta (\partial_x + x C_{\delta, \tau} \partial_z), \quad E_3 = \partial_z, \quad L_1 = \delta (\partial_x + y C_{\delta, H} \partial_z), \quad L_2 = \delta (\partial_x - x C_{\delta, H} \partial_z), \quad L_3 = \partial_z.
$$

The chain rule gives

$$\frac{\partial \hat{F}}{\partial \zeta} = \frac{\partial p}{\partial \zeta} \frac{\partial F}{\partial x} + \frac{\partial p}{\partial \bar{\zeta}} \frac{\partial F}{\partial y} = \frac{\partial p}{\partial \zeta} \left( \frac{\partial}{\partial x} + \frac{\partial f}{\partial x} \partial z \right) + \frac{\partial p}{\partial \bar{\zeta}} \left( \frac{\partial}{\partial y} + \frac{\partial f}{\partial y} \partial z \right) = \frac{1}{\delta} \frac{\partial p}{\partial \zeta} E_1 + \frac{1}{\delta} \frac{\partial q}{\partial \zeta} E_2 + \left( \frac{\alpha}{\delta} \frac{\partial p}{\partial \zeta} + \frac{\beta}{\delta} \frac{\partial q}{\partial \zeta} \right) E_3,$$

and, analogously,

$$\frac{\partial \hat{G}}{\partial \zeta} = \frac{1}{\delta} \frac{\partial p}{\partial \zeta} L_1 + \frac{1}{\delta} \frac{\partial q}{\partial \zeta} L_2 + \left( \frac{\alpha}{\delta} \frac{\partial p}{\partial \zeta} + \frac{\beta}{\delta} \frac{\partial q}{\partial \zeta} \right) L_3.$$

Using the twin relations $(\tilde{\alpha}, \tilde{\beta}) = (-\frac{\beta}{\omega'}, \frac{\alpha}{\omega'})$, it is easy to deduce following two equalities that will come in handy for the next computations:

$$\frac{1 - \tilde{\alpha}^2 \delta^2}{\delta^2} = \frac{1}{\omega^2} \left( \frac{1}{\delta^2} + \alpha^2 \right) \quad \text{and} \quad \frac{1 - \tilde{\beta}^2 \delta^2}{\delta^2} = \frac{1}{\omega^2} \left( \frac{1}{\delta^2} + \beta^2 \right).$$
Finally, we get
\[
\left( \frac{\partial \tilde{G}}{\partial \zeta}, \frac{\partial \tilde{G}}{\partial \zeta} \right)_{L^3(M,H)} = \left( \frac{1}{\delta} \frac{\partial p}{\partial \zeta} \right)^2 + \left( \frac{1}{\delta} \frac{\partial q}{\partial \zeta} \right)^2 - \left( \alpha \frac{\partial p}{\partial \zeta} + \beta \frac{\partial q}{\partial \zeta} \right)^2 \]
\[
= \left( \frac{1 - \bar{\alpha}^2 \delta^2}{\delta^2} \right) \left( \frac{\partial p}{\partial \zeta} \right)^2 + \left( \frac{1 - \bar{\beta}^2 \delta^2}{\delta^2} \right) \left( \frac{\partial q}{\partial \zeta} \right)^2 - 2 \bar{\alpha} \bar{\beta} \left( \frac{\partial p}{\partial \zeta} \frac{\partial q}{\partial \zeta} + \frac{\partial q}{\partial \zeta} \frac{\partial p}{\partial \zeta} \right) \]
\[
= \left( \frac{1 + \alpha^2 \delta^2}{\alpha^2 \delta^2} \right) \left( \frac{\partial p}{\partial \zeta} \right)^2 + \left( \frac{1 + \beta^2 \delta^2}{\beta^2 \delta^2} \right) \left( \frac{\partial q}{\partial \zeta} \right)^2 - 2 \left( \frac{\alpha \beta}{\alpha^2} \right) \left( \frac{\partial p}{\partial \zeta} \frac{\partial q}{\partial \zeta} + \frac{\partial q}{\partial \zeta} \frac{\partial p}{\partial \zeta} \right) \]
\[
= \frac{1}{\omega^2} \left( \frac{\partial \tilde{E}}{\partial \zeta} \right)_{L^3(M,H)} \]
\]
where, in the last line we used that \( \zeta \) is conformal in \( z = f(x, y) \), and
\[
\left( \frac{\partial \tilde{G}}{\partial \zeta}, \frac{\partial \tilde{G}}{\partial \zeta} \right)_{L^3(M,H)} = \left| \frac{1}{\delta} \frac{\partial p}{\partial \zeta} \right|^2 + \left| \frac{1}{\delta} \frac{\partial q}{\partial \zeta} \right|^2 - \left( \alpha \frac{\partial p}{\partial \zeta} + \beta \frac{\partial q}{\partial \zeta} \right) \left( \frac{\partial p}{\partial \zeta} \right)^2 + \left( \alpha \frac{\partial p}{\partial \zeta} + \beta \frac{\partial q}{\partial \zeta} \right) \left( \frac{\partial q}{\partial \zeta} \right)^2 \]
\[
= \left( \frac{1 - \bar{\alpha}^2 \delta^2}{\delta^2} \right) \left( \frac{\partial p}{\partial \zeta} \right)^2 + \left( \frac{1 - \bar{\beta}^2 \delta^2}{\delta^2} \right) \left( \frac{\partial q}{\partial \zeta} \right)^2 - \bar{\alpha} \bar{\beta} \left( \frac{\partial p}{\partial \zeta} \frac{\partial q}{\partial \zeta} + \frac{\partial q}{\partial \zeta} \frac{\partial p}{\partial \zeta} \right) \]
\[
= \left( \frac{1 + \alpha^2 \delta^2}{\alpha^2 \delta^2} \right) \left( \frac{\partial p}{\partial \zeta} \right)^2 + \left( \frac{1 + \beta^2 \delta^2}{\beta^2 \delta^2} \right) \left( \frac{\partial q}{\partial \zeta} \right)^2 - 2 \left( \frac{\alpha \beta}{\alpha^2} \right) \left( \frac{\partial p}{\partial \zeta} \frac{\partial q}{\partial \zeta} + \frac{\partial q}{\partial \zeta} \frac{\partial p}{\partial \zeta} \right) \]
\[
= \frac{1}{\omega^2} \left( \frac{\partial \tilde{E}}{\partial \zeta} \right)_{L^3(M,H)} \]
\]

Remark 3.8 (Angle functions). The functions \( \omega \) and \( \tilde{\omega} \) in the twin correspondence have an interesting geometric meaning. Note that the upward-pointing unit normal vector fields \( N_f \) of the graph \( z = f(x, y) \) in \( \mathbb{E}^3(M, \tau) \), and \( N_g \) of the graph \( z = g(x, y) \) in \( L^3(M, H) \) read as
\[
N_f = -\frac{\alpha \delta}{\omega} E_1 - \frac{\beta \delta}{\omega} E_2 + \frac{1}{\omega} E_3, \quad N_g = -\frac{\tilde{\alpha} \delta}{\omega} L_1 - \frac{\tilde{\beta} \delta}{\omega} L_2 + \frac{1}{\omega} L_3.
\]
Thus, the so-called angle functions \( u = \left< N_f, E_3 \right>_{\mathbb{E}^3(M, \tau)} \) and \( \tilde{u} = \left< N_g, L_3 \right>_{L^3(M, H)} \) are nothing but \( u = \frac{1}{\omega} \) and \( \tilde{u} = \frac{1}{\omega} \). In particular, the twin surfaces have inverse angle functions.

Corollary 3.9 (Prescribed mean curvature as zero mean curvature). Let \( \phi : \mathbb{R}^2 \to \mathbb{R} \) be a smooth function, and \( \Omega \subset \mathbb{R}^2 \) simply-connected.
There exists a twin correspondence between graphs defined over the domain \(\Omega\) with prescribed mean curvature function \(\phi\) in Euclidean space \(\mathbb{R}^3\) and maximal graphs defined over the same domain \(\Omega\) in the generalized Heisenberg spacetime \(\text{Nil}^3(\phi) = L^3(\mathbb{E}^2, \phi)\).

There exists a twin correspondence between spacelike graphs defined over the domain \(\Omega\) with prescribed mean curvature function \(\phi\) in Lorentz-Minkowski space \(L^3\) and minimal graphs defined over the same domain \(\Omega\) in the generalized Heisenberg space \(\text{Nil}^3(\phi) = E^3(\mathbb{E}^2, \phi)\).

**Corollary 3.10 (Twin correspondence in the BCV spaces, [27, Theorem 2]).** Given constants \(\kappa, \tau, H \in \mathbb{R}\), there exists a twin correspondence between graphs with mean curvature \(H\) in the Riemannian BCV space \(E^3(\kappa, \tau)\) and spacelike graphs with mean curvature \(\tau\) in the BCV spacetime \(L^3(\kappa, H)\).

**Proof.** This is the twin correspondence when the curvature of the base, the bundle curvature, and the mean curvature are all constant. \(\square\)

**Example 3.11 (Twin surfaces of helicoidal surfaces with constant mean curvature \(\tau\) in \(L^3\) are catenoids in \(\text{Nil}^3(\tau)\)).** Let \(\lambda \geq 0\) and consider the spacelike helicoidal surface with constant mean curvature \(\tau\) in \(L^3 = L^3(\mathbb{E}^2, 0)\)

\[
z = g(x, y) = \lambda \arctan \left( \frac{y}{x} \right) + \tau h \left( \sqrt{x^2 + y^2} \right),
\]

where the function \(h : (\lambda, \infty) \rightarrow \mathbb{R}\) satisfies the ODE

\[
\frac{d}{dt} h(t) = \frac{\tau^2 - \lambda^2}{\tau^2 t^2 + 1}.
\]

Its twin surface in \(\text{Nil}^3(\tau) = E^3(\mathbb{E}^2, \tau)\) induces the half catenoid \(z = f(x, y)\) defined over the domain \(\sqrt{x^2 + y^2} > \lambda\). It is a rotationally invariant minimal surface of the form

\[
z = f(x, y) = \lambda \rho \left( \sqrt{x^2 + y^2} \right),
\]

where the one-variable function \(\rho : (\lambda, \infty) \rightarrow \mathbb{R}\) satisfies the ODE

\[
\frac{d}{dt} \rho(t) = \sqrt{\frac{\tau^2 t^2 + 1}{t^2 - \lambda^2}}.
\]

4. Complete spacelike surfaces in GBCV spacetimes

Entire graphs in a Riemannian GBCV space \(E(M, \tau)\) are complete when the base surface \(M\) is complete. This assertion fails to be true in the general Lorentzian case, as examples constructed by Albujer [11] in the Robertson-Walker spacetime \(H^2(-1) \times \mathbb{R}\) with its product Lorentzian metric show.

We will begin by proving that complete spacelike surfaces in a GBCV spacetime \(L^3(M, \tau)\) are entire graphs. In particular, the study of complete maximal surfaces in the GBCV spacetimes reduces to that of the entire spacelike solutions of the corresponding maximal equation. Moreover,
this implies that the existence of complete constant mean curvature surfaces in the GBCV spacetime ensures the existence of non-trivial foliations. Throughout this section, we will assume that the base surface is given by

\[ M = (\Omega, g = \delta^{-2}(dx^2 + dy^2)) \]

for some \( \delta \in C^\infty(\Omega) \) positive, where \( \Omega \subseteq \mathbb{R}^2 \) is open and star-shaped with respect to the origin.

**Lemma 4.12** *(Covering map lemma, [26] Ch. VIII, Lemma 8.1)]* Let \( \phi \) be a map from a connected complete Riemannian manifold \( \mathcal{R}_1 \) onto another connected Riemannian manifold \( \mathcal{R}_2 \) of the same dimension. If the map \( \phi \) is the distance non-decreasing map, then \( \phi \) is a covering map, and \( \mathcal{R}_2 \) is also complete.

**Lemma 4.13** *(Complete spacelike surfaces in \( L^3(M, \tau) \) are entire graphs)*. Let us assume that there exists a complete spacelike surface \( \Sigma \) in the GBCV space \( L^3(M, \tau) \). Then \( \Sigma \) is an entire graph and \( M \) is also complete.

*Proof.* The same ideas in [2, Lemma 3.1] and [3, Proposition 3.3] work here. We begin with a complete spacelike surface \( \Sigma \subset L^3(M, \tau) \). From Definition 2.1, we get that \( L^3(M, \tau) \) is \( \Omega \times \mathbb{R} \) endowed with the Lorentzian metric

\[
\frac{1}{\delta^2}(dx^2 + dy^2) - (dz - C_{\delta, \tau}(x, y)(ydx - xdy))^2 \leq \frac{1}{\delta^2}(dx^2 + dy^2).
\]

This inequality implies that the projection \( \pi : (x, y, z) \in \Omega \times \mathbb{R} \to (x, y) \in \Omega \) from the ambient space \( L^3(M, \tau) \) to the base \( M = (\Omega, \delta^{-2}(dx^2 + dy^2)) \) is a distance non-decreasing map.

Since the induced metric in the spacelike surface \( \Sigma \) is complete, Lemma 4.12 applied to \( \pi|_{\Sigma} : \Sigma \to \Omega \) ensures that \( \pi|_{\Sigma} \) is a covering map and \( \Sigma \) is complete. As the domain \( \Omega \) is simply-connected, the covering map \( \pi|_{\Sigma} : \Sigma \to \Omega \) must be a global diffeomorphism, so \( \Sigma \) is an entire graph. \( \square \)

In 1976, Cheng and Yau [11] proved the remarkable result that any entire spacelike graph with constant mean curvature in the Lorentz-Minkowski spacetime \( L^3 = L^3(\mathbb{E}^2, 0) \) defined on the whole Euclidean plane \( \mathbb{E}^2 \) is complete. We will now show that maximal surfaces in the three dimensional Lorentzian hyperbolic space also admit a Cheng-Yau type property.

**Theorem 4.14** *(Complete maximal surfaces in anti de Sitter spacetime)*. Given a constant \( \kappa < 0 \), let \( \Sigma \) be a maximal surface in the anti de Sitter spacetime \( L^3(\mathbb{H}^2(\kappa), \frac{1}{2} \sqrt{-\kappa}) \), where \( \mathbb{H}^2(\kappa) \) denotes the hyperbolic plane of constant curvature \( \kappa \). The following two statements are equivalent:

(a) The surface \( \Sigma \) is complete.

(b) The surface \( \Sigma \) is an entire graph over the whole base \( \mathbb{H}^2(\kappa) \).

Moreover, the moduli space of complete maximal surfaces in \( L^3(\kappa, \frac{1}{2} \sqrt{-\kappa}) \) is large.

*Proof.* From Lemma 4.13 it is clear that \( (a) \Rightarrow (b) \), so we will focus on proving \( (b) \Rightarrow (a) \). By homothetically rescaling the metric of \( L^3(\mathbb{H}^2(\kappa), \frac{1}{2} \sqrt{-\kappa}) \), we will suppose that \( \kappa = -1 \) without losing generality.
Let $\Sigma$ be an entire maximal graph in $L^3(\mathbb{H}^2(-1), \frac{1}{2})$, and consider its twin entire graph $\Sigma^*$ with constant mean curvature $\frac{1}{2}$ in $\mathbb{H}^2 \times \mathbb{R} = E^3(\mathbb{H}^2(-1), 0)$. Since $\Sigma^*$ is simply-connected, we can take its sister minimal surface $\hat{\Sigma}^*$ in the Heisenberg group $\text{Nil}_3 = E^3(\mathbb{E}^2, \frac{1}{2})$ via the Daniel correspondence [14]. From [15, Corollary 3.3] we deduce that $\hat{\Sigma}^*$ is also an entire graph over $E^2$, so we can employ the twin correspondence again to associate an entire spacelike graph $\hat{\Sigma}$ in $L^3(\mathbb{E}^2, 0)$ with constant mean curvature $\frac{1}{2}$:

$$
\Sigma \subset L^3(\mathbb{H}^2, \frac{1}{2}) \leftarrow \Sigma^* \subset E^3(\mathbb{H}^2, 0) \leftarrow \hat{\Sigma} \subset L^3(\mathbb{E}^2, 0) \leftarrow \hat{\Sigma}^* \subset E^3(\mathbb{E}^2, \frac{1}{2})
$$

Daniel correspondence yields the existence of an isometry between $\Sigma^*$ and $\hat{\Sigma}^*$ preserving the angle function. In view of Theorem [37] the metrics of the Lorentzian graphs are conformal to the ones of the corresponding Riemannian with conformal factor the square of the angle function, so we deduce that $\Sigma$ and $\hat{\Sigma}$ are isometric surfaces. Since $\hat{\Sigma}$ is complete by Cheng-Yau result [11], then so is $\Sigma$.

To prove the last assertion in the statement, Fernández-Mira [20, Theorem 1 and Proposition 14] or Cartier-Hauswirth [10, Theorem 3.9] show that there are many entire graphs exist in the product space $\mathbb{H}^2 \times \mathbb{R} = E^3(\mathbb{H}^2(-1), 0)$ with constant mean curvature $\frac{1}{2}$. By rescaling the metric, this means that there are many entire graphs of mean curvature $\frac{1}{2} \sqrt{-\kappa}$ in the product space $\mathbb{H}^2(\kappa) \times \mathbb{R} = E^3(\mathbb{H}^2(\kappa), 0)$. By the twin correspondence, these surfaces correspond to entire maximal graphs in $L^3(\mathbb{H}^2(\kappa), \frac{1}{2} \sqrt{-\kappa})$, which are complete by the equivalence between (a) and (b). □

**Remark 4.15.** Bonsante and Schlenker [8] used the geometry of maximal surfaces in the anti de Sitter spacetime to give a variant of Schoen’s conjecture on the universal Teichmüller space. The proof of Theorem [4,14] gives a geometrical equivalence of the following entire graphs in different spaces:

(a) entire maximal spacelike graphs (defined over the hyperbolic plane $\mathbb{H}^2$) in the anti de Sitter spacetime $L^3(\mathbb{H}^2, \frac{1}{2})$.
(b) entire mean curvature $\frac{1}{2}$ graphs (defined over the hyperbolic plane $\mathbb{H}^2$) in the Riemannian product space $\mathbb{H}^2 \times \mathbb{R}$.
(c) entire mean curvature $\frac{1}{2}$ spacelike graphs (defined over the Euclidean plane $\mathbb{E}^2$) in the Lorentz-Minkowski space $L^3$.
(d) entire minimal graphs (defined over the Euclidean plane $\mathbb{E}^2$) in the Heisenberg space $\text{Nil}_3(\frac{1}{2}) = E^3(\mathbb{E}^2, \frac{1}{2})$.

In order to give a sharp non-existence result for complete spacelike surfaces in GBCV spacetimes, we introduce Cheeger’s isoperimetric constant.
Definition 4.16. The Cheeger constant of a non-compact Riemannian surface $M$ without boundary is defined as

\begin{equation}
\text{Ch}(M) = \inf \left\{ \frac{\text{Length}(\partial D)}{\text{Area}(D)} : D \subset M \text{ open and regular} \right\} \geq 0.
\end{equation}

Here, an open subset $D \subset M$ is said regular if it is relatively compact and its boundary is a smooth curve so the quotient in (4.1) makes sense.

Theorem 4.17. Let $M$ be a non-compact simply-connected surface.

(a) Given $H \in C^\infty(M)$ such that $\inf_M |H| > \frac{1}{2} \text{Ch}(M)$, the space $\mathbb{E}^3(M, \tau)$ admits no entire graphs with mean curvature $H$ for any $\tau \in C^\infty(M)$.

(b) Given $\tau \in C^\infty(M)$ such that $\inf_M |\tau| > \frac{1}{2} \text{Ch}(M)$, the spacetime $\mathbb{L}^3(M, \tau)$ admits neither complete spacelike surfaces nor entire spacelike graphs.

Proof. We will use a classical argument [18, 23, 32, 34] originally due to Heinz to obtain item (a). Let us argue by contradiction supposing that such an entire graph exists, and assume first that $H > 0$ (the case $H < 0$ will be treated later). Its mean curvature function $H$ admits the expression

\begin{equation}
2H = \text{div}_M \left( \frac{G}{\sqrt{1 + \|G\|^2_M}} \right),
\end{equation}

for some vector field $G$ on $M$ as in (3.1). Letting $H_0 = \inf_M (H)$ and integrating (4.2) over an open regular domain $D \subset M$, we get

\[ 2H_0 \text{Area}(D) \leq \int_D \text{div}_M \left( \frac{G}{\sqrt{1 + \|G\|^2_M}} \right) = \int_{\partial D} \frac{\langle G, \eta \rangle}{\sqrt{1 + \|G\|^2_M}} \leq \text{Length}(\partial D), \]

where $\eta$ denotes a outer unit conormal vector field to $D$ along its boundary and we used the divergence formula and Cauchy-Schwarz inequality. As this is valid for all open regular domains, we deduce that

\[ H_0 = \inf_M (H) = \inf_M |H| < \frac{1}{2} \text{Ch}(M), \]

contradicting the hypothesis in the statement. If $H < 0$, then the argument above can be adapted by changing $G$ into $-G$, to get that $-2H_0 \text{Area}(D) \leq \text{Length}(\partial D)$, so $-H_0 = \inf_M |H| < \frac{1}{2} \text{Ch}(M)$ and we also get a contradiction.

In order to prove item (b), we will reason by contradiction again: if there existed such a complete spacelike surface $\Sigma$, then $\Sigma$ would be an entire graph by Lemma 4.13 so its twin surface $\widetilde{\Sigma}$ would be an entire graph in $\mathbb{E}^3(M, H)$, where $H$ denotes the mean curvature of $\Sigma$. The mean curvature of $\widetilde{\Sigma}$ would be $\tau$, satisfying $\inf_M |\tau| > \frac{1}{2} \text{Ch}(M)$ and contradicting item (a). \qed

Corollary 4.18. Let $M$ be a complete non-compact simply-connected surface and let $c = \inf \{K(p) : p \in M\} \leq 0$, where $K$ denotes the Gaussian curvature of $M$. 

(a) Given $H \in C^\infty(M)$ such that $\inf_M |H| > \frac{1}{2} \sqrt{-c}$, the space $E^3(M, \tau)$ admits no entire graphs with mean curvature function $H$ for any $\tau \in C^\infty(M)$.

(b) Given $\tau \in C^\infty(M)$ such that $\inf_M |\tau| > \frac{1}{2} \sqrt{-c}$, the spacetime $L^3(M, \tau)$ admits neither complete spacelike surfaces nor entire spacelike graphs.

Proof. The estimation in [18, Lemma 4.1] gives $\text{Ch}(M) \leq \sqrt{-c}$. Then, the statements (a) and (b) immediately follow from Theorem 4.17. □

Observe that this result is a generalization of the non-existence of entire maximal graphs in Lorentzian Heisenberg space with non-zero constant bundle curvature [28, Theorem 4.1]. Note also that Theorem 4.14 (for $c < 0$) and the classical classification result by Calabi [9] (for $c = 0$) show that the constant lower bound $\frac{1}{2} \sqrt{-c}$ in Corollary 4.18 is sharp. We would like to point out that there are numerous existence results of spacelike submanifolds with constant mean curvature in different spacetimes (e.g., see [4,5,13,21]).

As a last consequence, Theorem 4.17 also gives information about causality in $L^3(M, \tau)$ spaces when we look at them as spacetimes. A spacetime $L$ is said distinguishing when two different points $p, q \in L$ have different future or past cones. Equivalently, if for any $p \in L$ and any neighborhood $U$ of $p$, there exists a neighborhood $V \subset U$ of $p$, such that causal (i.e., non-spacelike) curves starting at $p$ and leaving $V$ never enter again in $V$.

Corollary 4.19. The GBCV spacetime $L^3(M, \tau)$ is not distinguishing when the bundle curvature $\tau$ satisfies $\inf_M |\tau| > \frac{1}{2} \text{Ch}(M)$.

Proof. It follows from the fact that distinguishing spacetimes with a complete timelike Killing vector field admit a Riemannian submersion structure whose fibers are the integral curves of the Killing vector field, and having complete spacelike surfaces [24]. □

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