Hamilton cycles
in line graphs of 3-hypergraphs

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Abstract
We prove that every 52-connected line graph of a rank 3 hypergraph is Hamiltonian. This is the first result of this type for hypergraphs of bounded rank other than ordinary graphs.

1 Introduction

We refer the reader to Section 2 for any definitions not included in this introduction.

It is easy to see that there are graphs of arbitrarily high vertex-connectivity that do not admit a Hamilton cycle. On the other hand, in some classes of graphs, sufficient connectivity implies Hamiltonicity. One example is the class of planar graphs (4-connected planar graphs are Hamiltonian by a classic result of Tutte [17]). For claw-free graphs, a conjecture of Matthews and Sumner [13] states that vertex-connectivity greater than or equal to 4 is sufficient as well.

Conjecture 1.1. Every 4-connected claw-free graph is Hamiltonian.

Conjecture 1.1 is open, with the following being currently the best general result of this form.

Theorem 1.2 ([10]). All 6-connected claw-free graphs are Hamiltonian.

If we restrict Conjecture 1.1 to line graphs (which form a subclass of the class of claw-free graphs), we obtain the following conjecture of Thomassen [16]. (See also [3] for an extensive account of problems related to Conjectures 1.1 and 1.3.)

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**Conjecture 1.3.** Every 4-connected line graph is Hamiltonian.

Ryjáček [14] proved that Conjectures 1.1 and 1.3 are in fact equivalent. He introduced a closure technique which shows that for any positive integer \( k \), all \( k \)-connected claw-free graphs are Hamiltonian if and only if all \( k \)-connected line graphs are.

It is natural to ask if an analogue of Theorem 1.2 could be proved for \( K_{1,r+1} \)-free graphs, where \( r \geq 3 \). This is not known, although the question has been around for quite some time. Jackson and Wormald [9, p. 142] asked whether every \((r + 1)\)-connected \( K_{1,r+1} \)-free graph is Hamiltonian, where \( r \geq 3 \). Chen and Schelp [4] noted that a conjecture of Chvátal [5] would imply that every \( 2r \)-connected \( K_{1,r+1} \)-free graph is Hamiltonian; however, this particular conjecture of Chvátal (‘every 2-tough graph is Hamiltonian’) has since been disproved [1].

A positive result on a weaker version of the problem for \( r = 3 \) is established in [15]: 6-connected \( K_{1,4} \)-free graphs which, in addition, contain no induced copy of \( K_{1,4} + e \) (the simple graph with degree sequence 1, 1, 2, 2, 4), are Hamiltonian.

By analogy with claw-free graphs, one might guess that the problem of Hamiltonicity of \( K_{1,r+1} \)-free graphs could be reduced to the special case of line graphs of hypergraphs of rank \( r \). This may be so, but no extension of Ryjáček’s technique that would accomplish this task is known. Still, line graphs of hypergraphs of rank \( r \) are a natural starting point for an investigation of \( K_{1,r+1} \)-free graphs.

Even for this class of graphs, no analogue of Theorem 1.2 is known. The following conjecture has recently been proposed in [7]:

**Conjecture 1.4.** For any \( r \geq 2 \), there is an integer \( \phi(r) \) such that every \( \phi(r) \)-connected line graph of a rank \( r \) hypergraph is Hamiltonian.

A stronger form of the conjecture in [7] includes the statement that \( \phi(r) = 2r \) works.

Li et al. [12] recently found a close relation between line graphs of rank 3 hypergraphs and Conjecture 1.3. They showed the latter conjecture to be equivalent to the statement that every 4-connected line graph of a rank 3 hypergraph is Hamilton-connected (that is, for any two vertices \( u, v \), it has a Hamilton path joining \( u \) to \( v \)).

In this note, we establish Conjecture 1.4 in the first open case, \( r = 3 \). We use a result of DeVos et al. [6] on disjoint \( T \)-connectors as our main tool.

**Theorem 1.5.** If \( G \) is the line graph of a rank 3 hypergraph and \( G \) is 52-connected, then \( G \) is Hamiltonian.
The method easily extends to Hamilton-connectedness, at the price of a slight increase in the constant (from 52 to 54). To keep our notation and terminology simpler, we prove Theorem 1.5 just for Hamiltonicity.

2 Preliminaries

Our terminology mostly follows Bondy and Murty [2]. Graphs may contain parallel edges but no loops.

Given a graph $H$, we say that a graph $G$ is $H$-free if $G$ contains no induced copy of $H$. Claw-free is used as a synonym for $K_{1,3}$-free.

A hypergraph consists of a vertex set $V$ and a multiset $E$ of hyperedges, each of which is a nonempty subset of $V$. The rank of a hypergraph $\mathcal{H}$ is the maximum cardinality of a hyperedge of $\mathcal{H}$. A hypergraph of rank $r$ is also referred to as an $r$-hypergraph.

The line graph $L(\mathcal{H})$ of a hypergraph $\mathcal{H} = (V, E)$ has $E$ as its vertex set, with an edge linking $e$ and $f$ ($e, f \in E$) whenever $e$ and $f$ intersect. Observe that if $\mathcal{H}$ has rank $r$, then $L(\mathcal{H})$ is $K_{1,r+1}$-free.

3 Tools

3.1 $T$-connectors

Let $T$ be an arbitrary set of vertices of a graph $G$. We say that $T$ is $k$-edge-connected in $G$ if for any $s_1, s_2 \in T$, $G$ contains $k$ edge-disjoint paths from $s_1$ to $s_2$. By Menger’s Theorem, $T$ is $k$-edge-connected if and only if $G$ contains no edge-cut $X$ such that $|X| < k$ and at least two components of $G - X$ contain vertices of $T$.

Let $P$ be a path in $G$. Following [18], we define the operation of short-cutting $P$ as deleting all edges of $P$ and then adding an edge joining the end vertices of $P$. A path in $G$ is a $T$-path if its end vertices belong to $T$ and none of its other vertices are in $T$. A $T$-connector in $G$ is the union of a family of edge-disjoint $T$-paths in $G$ such that short-cutting them one by one, we obtain a graph whose induced subgraph on $T$ is connected. Observe that all the vertices of a $T$-connector $C$ whose degree in $C$ is odd belong to $T$.

DeVos et al. [6, Theorem 1.6] proved the following result on edge-disjoint $T$-connectors (see also [11, 18]).

Theorem 3.1 ([6]). For $k \geq 1$, if $T \subseteq V(G)$ is $(6k + 6)$-edge-connected in $G$, then $G$ contains $k$ edge-disjoint $T$-connectors.
3.2 Hamiltonicity of line graphs of 3-hypergraphs

A well-known result of Harary and Nash-Williams [8] characterises graphs whose line graph is Hamiltonian. We use an extension of this result to 3-hypergraphs, given in [12]. (A more general extension, valid for all hypergraphs, was found in [7].)

Let \( H \) be a 3-hypergraph. The incidence graph \( IG(H) \) of \( H \) is the bipartite graph with vertex set \( V(H) \cup E(H) \) and edges of the form \((v,e)\), where \( v \in V(H) \), \( e \in E(H) \) and \( v \in e \). The vertices of \( IG(H) \) belonging to \( E(H) \) are called white, the other vertices are black. Note that each white vertex of \( IG(H) \) has degree 2 or 3.

A closed walk \( Q \) in a graph \( G \) is a sequence \( v_0, e_0, v_1, \ldots, e_{k-1}, v_k \), such that \( e_i \) is an edge of \( G \) with end vertices \( v_i \) and \( v_{i+1} \) \((0 \leq i \leq k-1)\) and \( v_k = v_0 \). Each of the vertices \( v_i \) is said to be visited by \( Q \) (as many times as it appears in \( Q \)); similarly, an edge \( e_i \) is said to be traversed by \( Q \) (again with possible multiplicity). A closed trail is a closed walk visiting each edge at most once.

Let \( v_i \) be a vertex visited once by the above walk. The predecessor edge of \( v_i \) is defined to be \( e_{i-1} \) (with subtraction modulo \( k \)). Similarly, the successor edge of \( v_i \) is \( e_i \) if \( i < k \), and \( e_0 \) otherwise.

Given an arbitrary set \( W \) of vertices of degree 2 or 3 in \( G \), a closed \( W \)-quasitrail in \( G \) is a closed walk which traverses each edge at most twice, and if an edge \( e \) is traversed twice, then it has an end vertex \( w \in W \) such that \( w \) is visited once and \( e \) is both the predecessor edge and the successor edge of \( w \). A closed \( W \)-quasitrail in \( G \) is dominating if it visits at least one vertex in every edge of \( G \).

We will use the following characterisation of 3-hypergraphs with Hamiltonian line graphs, which follows from [12, Corollary 7].

**Theorem 3.2 ([12])**. Let \( H \) be a 3-hypergraph and let \( W \) be the set of white vertices of its incidence graph \( IG(H) \). The line graph \( L(H) \) of \( H \) is Hamiltonian if and only if \( IG(H) \) contains a dominating closed \( W \)-quasitrail.

As remarked above, hypergraphs of arbitrary rank whose line graph is Hamiltonian were recently characterised in [7].

4 Proof of Theorem 1.5

Let \( L(H) \) be the line graph of a 3-hypergraph \( H \). Suppose that \( L(H) \) is 52-connected; in fact, it is enough if \( L(H) \) is 18-connected and its minimum

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1This graph is denoted by \( Gr(H) \) in [12]. Since the same symbol is used in [13] with a slightly different meaning, we opt for the alternative \( IG(H) \).
degree is at least 52. We prove that $L(H)$ is Hamiltonian.

Consider the graph $IG(H)$ and recall that every vertex of $H$ is a black vertex of $IG(H)$. Let us call such a vertex of $IG(H)$ heavy if its degree is at least 18. Since the minimum degree of $L(H)$ is at least $52 > 3 \cdot 17$, every hyperedge of $H$ contains a heavy vertex. Therefore, every white vertex of $IG(H)$ is adjacent to a heavy vertex. Let $T$ be the set of heavy vertices of $IG(H)$.

**Lemma 4.1.** The set $T$ is 18-edge-connected in $IG(H)$.

**Proof.** For the sake of a contradiction, consider an edge-cut $X$ in $IG(H)$ of size less than 18 that separates two vertices $s_1, s_2 \in T$. Each edge $e$ of $IG(H)$ corresponds to a hyperedge $e'$ of $H$; let $X' \subseteq E(H)$ be the set of the (fewer than 18) corresponding hyperedges for the edges in $X$. Removing the hyperedges in $X'$, we separate $s_1$ from $s_2$ in $H$. We claim that $X'$ is a vertex cut in $L(H)$; to prove this, we need to show that at least two components of $H - X'$ contain at least one hyperedge each.

But this is not hard. Since $s_1$ is heavy, it is incident with at least 18 hyperedges in $H$, and at most 17 of these hyperedges can be in $X'$. Thus, at least one hyperedge $e_1$ containing $s_1$ is a hyperedge of $H - X'$. Similarly, there is a hyperedge $e_2 \notin X'$ containing $s_2$. Then, in $L(H)$, $e_1$ and $e_2$ are two vertices separated by the vertex cut $X'$ of size less than 18, so $L(H)$ is not 18-connected contrary to the assumption. \[\square\]

**Lemma 4.2.** The graph $IG(H)$ contains a closed trail visiting every vertex in $T$.

**Proof.** By Lemma 4.1 and Theorem 3.1, $IG(H)$ contains two edge-disjoint $T$-connectors, say $A_1$ and $A_2$. It is a standard observation that $A_1 \cup A_2$ contains a connected subgraph $C$ with all degrees even such that $C$ covers all vertices in $T$. To prove it, let $B$ be the set of vertices of $A_1$ with odd degree in $A_1$. Then $|B|$ is even, and by the definition of $T$-connector, $B \subseteq T$. We partition $B$ in pairs arbitrarily, and join each of the pairs by a path in $A_2$. The symmetric difference $D$ of all these paths is a subgraph of $A_2$, and by a simple parity argument, the set of odd degree vertices of $D$ is precisely $B$. Now $A_1 \cup D$ is the desired subgraph $C$. Since every vertex of $IG(H)$ has even degree in $C$ and $C$ is connected, there is a closed trail traversing precisely the edges in $C$. \[\square\]

Let $R$ be a closed trail obtained from Lemma 4.2. We aim to use Theorem 3.2 to prove that $R$ gives rise to a Hamilton cycle in $L(H)$. Let $W$ be the set of white vertices of $IG(H)$. We now construct a closed dominating $W$-quasitrail from $R$. 

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Consider a white vertex \( w \) of \( IG(H) \) not visited by \( R \). The vertex \( w \) is adjacent to a heavy vertex \( v \), and every heavy vertex is traversed by \( R \). Let us insert in \( R \) a detour to \( w \) immediately following the visit to \( v \). That is, an occurrence of \( v \) in \( R \) will be changed to \( v, vw, w, wv, v \). Repeating this operation for each unvisited white vertex (choosing one heavy neighbour arbitrarily if there are more than one), we obtain a closed walk visiting each white vertex, and therefore dominating all edges of \( IG(H) \). In fact, the resulting walk is a closed dominating \( W \)-quasitrail, so Theorem 3.2 implies that \( L(H) \) is Hamiltonian. The proof is complete.

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