Holomorphic Sectional Curvature Tensors of Complex Finsler Manifolds

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Abstract In this article, we examine the behavior of holomorphic sectional curvature tensors of a strongly pseudoconvex complex Finsler manifold $(M, F)$. We prove that holomorphic sectional curvature tensors of the canonical connection are equal those of the Chern-Finsler connection if and only if $F$ is a Kähler Finsler metric. In addition, we also prove that holomorphic sectional curvatures of the canonical connection coincide with those of the Chern Finsler connection if and only if $F$ is a weakly Kähler Finsler metric. At last, we generalize Bismut connection into the complex Finsler geometry.

Key words holomorphic sectional curvature tensor, Kähler Finsler metric, Bismut connection

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1 Introduction

Complex Finsler manifolds are complex manifolds endowed with complex Finsler metrics, which are more general than Hermitian manifolds. Recently, complex Finsler geometry has attracted a interest because a number of complex Finsler metrics play an important role in geometric function theory of holomorphic mappings.

As is well known, a Kähler metric which is also a Riemannian metric, the Chern connection coincides with the Levi-Civita connection, being equivalent to the Chern connection is torsion-free. In the complex Finsler geometry, the Chern Finsler connection associated to a strongly pseudoconvex Finsler metric enjoys some beautiful features as the Chern connection associated to a Hermitian metric. According to the vanishing of some parts of the torsion of the Chern Finsler connection, there are three kinds of metric notions in the complex Finsler geometry, which are the extension of the Kähler metrics. These three kinds of metrics are called respectively the
strongly Kähler, Kähler and weakly Kähler [1]. In fact, Kähler Finsler metrics are actually strongly Kähler [4]. Hence, there are only two kinds of Kähler Finsler metrics with respect to the Chern-Finsler connection in the complex Finsler geometry.

Let $F$ be a strongly pseudoconvex complex Finsler metric on a complex manifold $M$. Denote the Chern-Finsler connection associated to $F$ by $D$, and the Chern-Finsler connection $D$ can introduce a canonical connection $\nabla$. Denote holomorphic sectional curvature tensors of the canonical connection and the Chern-Finsler connection by $R_{\alpha\bar{\beta}\mu\bar{\nu}}$ and $\Omega_{\alpha\bar{\beta};\mu\bar{\nu}}$, respectively, where lowercase Greek indices run from 1 to $n = \dim \mathbb{C}(M)$. When $F$ is a Kähler Finsler metric, one has $R_{\alpha\bar{\beta}\mu\bar{\nu}} = \Omega_{\alpha\bar{\beta};\mu\bar{\nu}}$. A naive question is, when a strongly pseudoconvex complex Finsler metric $F$ satisfies $R_{\alpha\bar{\beta}\mu\bar{\nu}} = \Omega_{\alpha\bar{\beta};\mu\bar{\nu}}$, must it be Kähler Finsler? In this paper, we give a positive answer to this question. We have the following:

**Theorem 1.1.** Let $(M, F)$ be a strongly pseudoconvex complex Finsler manifold. Then

$$R_{\alpha\bar{\beta}\mu\bar{\nu}} = \Omega_{\alpha\bar{\beta};\mu\bar{\nu}},$$

if and only if $F$ is a Kähler Finsler metric.

We also study strongly pseudoconvex complex Finsler metrics satisfies that holomorphic sectional curvatures of their canonical connections and Chern-Finsler connections coincide.

**Theorem 1.2.** Let $(M, F)$ be a strongly pseudoconvex complex Finsler manifold. Then

$$\overline{c}_F(v) = K_F(v),$$

if and only if $F$ is a weakly Kähler Finsler metric, where $\overline{c}_F(v)$ and $K_F(v)$ are the holomorphic sectional curvature of the canonical connection and the Chern Finsler connection along a holomorphic vector $v$, respectively.

The rest of this paper is arranged as follows. In Section 2, we provide a brief overview of the complex Finsler geometry. In Section 3, the canonical connection is introduced. In Section 4, we prove Theorem 1.1 and Theorem 1.2. In Section 5, we generalize Bismut connection on a strongly pseudoconvex complex Finsler manifold.

## 2 Preliminaries

Let $M$ be a $n$-dimensional complex manifold with $n \geq 2$. Denote the holomorphic tangent bundle of $M$ by $\pi : T^{1,0}M \to M$. Suppose that $z = (z^1, \cdots, z^n)$ is a local complex coordinate
system on $M$, then locally an element of the holomorphic tangent bundle $T^{1,0}M$ is written as

$$v = v^\alpha \frac{\partial}{\partial z^\alpha}.$$ 

So $(z, v)$ is a local complex coordinate system on $T^{1,0}M$. Denote the slit holomorphic tangent bundle by $\hat{M} = T^{1,0}M \setminus \{0\}$. Then $\{\partial_\alpha, \hat{\partial}_\beta\}$ gives a local holomorphic frame on the holomorphic tangent bundle $T^{1,0}\hat{M}$ of $\hat{M}$, where $\partial_\alpha = \frac{\partial}{\partial z^\alpha}$, $\hat{\partial}_\beta = \frac{\partial}{\partial \bar{v}^\beta}$.

**Definition 2.1.** \[1\] A complex Finsler metric on a complex manifold $M$ is a continuous function $F : T^{1,0}M \to [0, +\infty)$ satisfying:

(a) $G = F^2$ is smooth on $\hat{M}$;
(b) $F(z, v) > 0$ for all $(z, v) \in \hat{M}$;
(c) $F(z, \zeta v) = |\zeta|F(z, v)$ for all $(z, v) \in T^{1,0}M$ and $\zeta \in \mathbb{C}$.

For simplicity, we denote $\bar{z}^\beta, \bar{v}^\beta$ by $\bar{z}^\beta, \bar{v}^\beta$, the derivatives of $G$ respective to $v$ by

$$G_\alpha = \hat{\partial}_\alpha G = \frac{\partial G}{\partial v^\alpha}, \quad G_\alpha = \hat{\partial}_\alpha G = \frac{\partial G}{\partial \bar{v}^\alpha}, \quad G_{\alpha\bar{\beta}} = \hat{\partial}_\alpha \hat{\partial}_\beta G = \frac{\partial^2 G}{\partial v^\alpha \partial \bar{v}^\beta},$$

and the derivatives of $G$ respective to $z$ by indexes after a semicolon, for instances

$$G_{;\alpha} = \partial_{;\alpha} G = \frac{\partial G}{\partial z^\alpha}, \quad G_{;\alpha\bar{\beta}} = \hat{\partial}_{;\alpha} G = \frac{\partial^2 G}{\partial \bar{z}^\alpha \partial \bar{v}^\beta}, \quad G_{;\alpha\bar{\beta}} = \hat{\partial}_{;\alpha} \hat{\partial}_{;\beta} G = \frac{\partial^2 G}{\partial \bar{z}^\alpha \partial \bar{v}^\beta}.$$

**Definition 2.2.** \[1\] A complex Finsler metric $F$ is called strongly pseudoconvex if the Levi matrix $(G_{\alpha\bar{\beta}})$ is positive definite on $\hat{M}$.

Let $(M, F)$ be a strongly pseudoconvex complex Finsler manifold. Denote the Christoffel symbols of the Chern-Finsler nonlinear connection by

$$\Gamma^\lambda_{\mu\nu} = G^{\lambda\alpha} G_{\lambda\mu\nu}, \quad (2.1)$$

where $(G^{\lambda\alpha}) = (G_{\alpha\lambda})^{-1}$. Set

$$\delta_\mu = \partial_\mu - \Gamma^\alpha_{\mu\nu} \hat{\partial}_\alpha, \quad \delta v^\alpha = dv^\alpha + \Gamma^\alpha_{;\beta} dz^\beta \quad (2.2)$$

then the holomorphic tangent bundle $T^{1,0}\hat{M}$ of $\hat{M}$ can be decomposed the sum of the horizontal bundle $\mathcal{H}$ spanned by $\{\delta_\mu\}$ and the vertical bundle $\mathcal{V}$ spanned by $\{\hat{\partial}_\alpha\}$. The dual frame of $\{\delta_\mu, \hat{\partial}_\alpha\}$ is $\{dz^\mu, dv^\alpha\}$. In other words, we have

$$T^{1,0}\hat{M} = \mathcal{H} \oplus \mathcal{V} = \text{span}\{\delta_\mu\} \oplus \text{span}\{\hat{\partial}_\alpha\},$$

$$T^{*1,0}\hat{M} = \mathcal{H}^* \oplus \mathcal{V}^* = \text{span}\{dz^\mu\} \oplus \text{span}\{\delta v^\alpha\}. \quad (3)$$
The strongly pseudoconvex complex Finsler metric $F$ introduces a Hermitian metric $\langle , \rangle$ on the vertical bundle $\mathcal{V}$. For $\forall v \in \tilde{M}_z$, we set

$$\langle Z, W \rangle_v = G_{\alpha\beta}(z, v)Z^\alpha \bar{W}^\beta, \quad Z = Z^\alpha \dot{\partial}_\alpha, W = W^\alpha \dot{\partial}_\alpha \in \mathcal{V}_v.$$  

There exists a unique complex vertical connection $D : \mathcal{X}(\mathcal{V}) \to \mathcal{X}(T^*_{\mathbb{C}}\tilde{M} \otimes \mathcal{V})$ such that

$$X \langle Z, W \rangle = \langle \nabla_X Z, W \rangle + \langle Z, \nabla_{\bar{X}} W \rangle$$

for all $X \in T^{1,0}\tilde{M}$ and $Z, W \in \mathcal{X}(\mathcal{V})$. Furthermore, this connection is good. The unique good complex vertical connection $D$ is called the Chern-Finsler connection, which was first introduced in [6] and systemically studied in [1]. Its connection 1-forms are given by

$$\theta^\alpha_\beta = G^{\lambda\alpha} \partial_{\beta\lambda} = \Gamma^\alpha_{\beta\mu} dz^\mu + \Gamma^\alpha_{\beta\gamma} \delta^\gamma, \quad (2.3)$$

where

$$\Gamma^\alpha_{\beta\mu} = G^{\lambda\alpha} \delta^\mu (G_{\beta\lambda}), \quad \Gamma^\alpha_{\beta\gamma} = G^{\lambda\alpha} G_{\beta\lambda\gamma}. \quad (2.4)$$

Let $\Theta : \mathcal{V} \to \mathcal{H}$ be a complex horizontal map locally defined by $\Theta(\dot{\partial}_\alpha) = \delta_\alpha$. We can transfer the Hermitian structure $\langle , \rangle$ on $\mathcal{H}$ just by setting

$$\langle H, K \rangle_v = \langle \Theta^{-1}(H), \Theta^{-1}(K) \rangle_v, \quad \forall H, K \in \mathcal{H}_v.$$  

Then we can define a Hermitian structure on the whole $T^{1,0}\tilde{M}$, just by requiring $\mathcal{H}$ to be orthogonal to $\mathcal{V}$. We can also extend $D$ to a complex linear connection on $\mathcal{H}$ by setting

$$D_X H = \Theta(D_X(\Theta^{-1} H)), \quad H \in \mathcal{X}(\mathcal{H}), \quad X \in T^*_{\mathbb{C}}\tilde{M}. \quad (2.5)$$

The $(2,0)$-torsion and $(1,1)$-torsion of the Chern-Finsler connection are denoted by $\theta = \theta^\mu \otimes \delta_\mu$ and $\tau = \tau^\alpha \otimes \dot{\partial}_\alpha$ respectively, where

$$\theta^\mu = \frac{1}{2} \left( \Gamma^\mu_{\nu,\sigma} - \Gamma^\mu_{\sigma,\nu} \right) dz^\sigma \wedge dz^\nu + \Gamma^\mu_{\nu,\gamma} \delta^\gamma \wedge dz^\nu, \quad \tau^\alpha = -\delta_\nu \left( \Gamma^\alpha_{\mu} \right) dz^\mu \wedge \bar{d}z^\nu - \dot{\partial}_\beta \left( \Gamma^\alpha_{\mu} \right) dz^\mu \wedge \delta^\nu. \quad (2.6)$$

\textbf{Definition 2.3.} [1] We call a strongly pseudoconvex complex Finsler metric $F$ strongly Kähler if $\theta(H, K) = 0, \forall H, K \in \mathcal{H}$; Kähler if $\theta(H, \chi) = 0, \forall H \in \mathcal{H}$; weakly Kähler if $\langle \theta(H, \chi), \chi \rangle = 0, \forall H \in \mathcal{H}$.  

\text{4}
Hence, a strongly pseudoconvex complex Finsler metric \( F \) is strongly Kähler iff
\[
\Gamma^\alpha_{\beta\mu} = \Gamma^\alpha_{\mu\beta} \quad \text{equivalently} \quad \delta_\mu (G^\beta_{\lambda\overline{\lambda}}) = \delta_\beta (G^\mu_{\lambda\overline{\lambda}});
\]
is Kähler iff \( (\Gamma^\alpha_{\beta\mu} - \Gamma^\alpha_{\mu\beta})v^\beta = 0 \); is weakly Kähler iff \( G_\alpha (\Gamma^\alpha_{\beta\mu} - \Gamma^\alpha_{\mu\beta})v^\beta = 0 \). By [4], Kähler Finsler metrics are actually strongly Kähler.

The curvature operator of the Chern-Finsler connection is given by
\[
\Omega = \Omega^\beta_{\beta\mu} \otimes (d\overline{z}^\beta \otimes \delta_\alpha + \delta v^\beta \otimes \dot{\delta}_\alpha),
\]
where \( \Omega^\beta_{\beta\mu} = \partial \theta^\beta_{\beta\mu} \). In local coordinates, \( \Omega^\beta_{\beta\mu} \) can be decomposed as
\[
\Omega^\beta_{\beta\mu} = \Omega^\beta_{\beta\mu\nu} dz^\mu \wedge d\overline{z}^\nu + \Omega^\beta_{\beta\mu\nu} d\overline{z}^\mu \wedge dv^\nu + \Omega^\beta_{\beta\mu\nu} dv^\mu \wedge \delta v^\nu,
\]
where
\[
\begin{align*}
\Omega^\beta_{\beta\mu\nu} &= -\delta_\nu (\Gamma^\beta_{\beta\mu}) - \Gamma^\beta_{\beta\gamma} \delta_\nu (\Gamma^\gamma_{\mu}) \quad \text{and} \quad \Omega^\beta_{\beta\mu\nu} = -\delta_\nu (\Gamma^{\overline{\beta}}_{\overline{\beta}\mu}), \\
\Omega^\beta_{\beta\mu\nu} &= \dot{\delta}_\nu (\Gamma^\beta_{\beta\mu}) - \Gamma^\beta_{\beta\gamma} \dot{\delta}_\nu (\Gamma^\gamma_{\mu}) \quad \text{and} \quad \Omega^\beta_{\beta\mu\nu} = -\dot{\delta}_\nu (\Gamma^{\overline{\beta}}_{\overline{\beta}\mu}).
\end{align*}
\]
Denote the holomorphic sectional curvature tensor by \( \Omega_{\alpha\beta;\mu\nu} = G_{\gamma\overline{\beta}} \Omega^\gamma_{\alpha;\mu\nu} \). Locally,
\[
\Omega_{\alpha\beta;\mu\nu} = -\delta_\nu \delta_\mu (G_{\alpha\beta}) + \delta_\mu (G_{\alpha\lambda}) \tilde{G}^{\lambda\overline{\beta}} \delta_\nu (G_{\kappa\beta}) - G_{\alpha\beta;\gamma} \delta_\nu (\Gamma^{\overline{\gamma}}_{\overline{\gamma};\mu}).
\]
The horizontal holomorphic flag curvature \( K_F(H) \) of \( F \) along a horizontal vector \( H = H^\alpha \delta_\alpha \in \mathcal{H}_v \) is given by
\[
K_F(H) = \frac{\langle \Omega(H, H) H, H \rangle_v}{\langle H, H \rangle_v}. \tag{2.11}
\]
The holomorphic sectional curvature \( K_F(v) \) of \( F \) along \( v \) is given by
\[
K_F(v) = \frac{\langle \Omega(\chi, \overline{\chi}) \chi, \chi \rangle_v}{G(v)^2}. \tag{2.12}
\]

### 3 The canonical connection

An \( n \)-dimensional complex manifold \( M \) is also a \( 2n \)-dimensional real manifold with the canonical complex structure \( J \). We set
\[
x^\alpha = \text{Re} \, (z^\alpha), \quad x^{\alpha+n} = \text{Im} \, (z^\alpha), \quad u^\alpha = \text{Re} \, (v^\alpha), \quad u^{\alpha+n} = \text{Im} \, (v^\alpha).
\]
Hence \( x = (x^1, \cdots, x^{2n}) \) is a local real coordinate system on \( M \), \( (x, u) = (x^1, \cdots, x^{2n}, u^1, \cdots, u^{2n}) \) is a local coordinate system on the real tangent bundle \( TM \). Assume that lowercase Latin indices run from 1 to \( 2n \). We still denote \( u = u^\alpha \frac{\partial}{\partial x^\alpha} \in T_xM \), then \( u = v + \overline{v} \). Let \( \nu_{\mathbb{R}} \) be the
real vertical bundle spanned by \( \{ \frac{\partial}{\partial x^i} \} \). In addition, there is a nature real horizontal bundle \( \mathcal{H}_R \) spanned by \( \{ \frac{\partial}{\partial x^n} \} \), where
\[
\frac{\delta}{\delta x^a} = \delta_a + \delta_n, \quad \frac{\delta}{\delta x^{a+n}} = i(\delta_a - \delta_n). \tag{3.1}
\]

The dual frame of \( \{ \frac{\delta}{\delta x^a}, \frac{\partial}{\partial x^i} \} \) is \( \{ dx^i, \delta u^i \} \), where
\[
\delta u^a = \frac{1}{2} (\delta v^a + \delta \alpha^a), \quad \delta u^{a+n} = -\frac{i}{2} (\delta v^a - \delta \alpha^a). \tag{3.2}
\]

We denote \( \frac{\partial}{\partial z} = \left( \frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n} \right) \), \( \frac{\partial}{\partial v} = \left( \frac{\partial}{\partial v_1}, \cdots, \frac{\partial}{\partial v_n} \right) \), \( dx = (dx^1, \cdots, dx^{2n}) \), \( dz = (dz^1, \cdots, dz^n) \), \( du = (du^1, \cdots, du^{2n}) \), \( \delta u = (\delta u^1, \cdots, \delta u^{2n}) \), \( \delta v = (\delta v^1, \cdots, \delta v^n) \). Then
\[
\frac{\delta}{\delta x} = \left( \frac{\delta}{\delta z}, \frac{\delta}{\delta z} \right)^T, \quad \delta u = (\delta v, \delta \bar{v}) (T^{-1})^t, \tag{3.3}
\]

where
\[
T = \left( \begin{array}{c} I \\ iI \end{array} \right), \quad T^{-1} = \frac{1}{2} \left( \begin{array}{c} I \\ -iI \\ iI \end{array} \right).
\]

Define \( N = \left( N^j_i \right) \) such that \( \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial v^j} \), \( \delta u^i = du^i + N^j_i dx^j \). Set \( \Gamma = \left( \Gamma_{\alpha}^\beta \right) \). We have
\[
N = T^t \text{diag} \{ \Gamma, \bar{\Gamma} \} (T^{-1})^t. \tag{3.4}
\]

We define an isomorphism \( \circ : T^{1,0}M \rightarrow TM \) by
\[
v^o = v + \bar{v}, \quad \forall v \in T^{1,0}M,
\]

with the inverse \( \circ : TM \rightarrow T^{1,0}M \) defined by
\[
u_o = \frac{1}{2} (u - iJu), \quad \forall u \in TM.
\]

From the above isomorphism, we denote the slit real tangent bundle by \( TM_0 = TM \setminus \{0\} \). Hence, we can extend \( \circ \) on \( T^{1,0}\bar{M} \) and \( \circ \) on \( TT\bar{M}_0 \).

There is a nature Riemannian inner product \( (,)_R \) on \( \mathcal{V}_R \), which is given by
\[
(X,Y) = \text{Re} \langle X_o, Y_o \rangle, \quad \forall X,Y \in \mathcal{V}_R.
\]

Let \( \hat{\Theta} : \mathcal{V}_R \rightarrow \mathcal{H}_R \) be a real horizontal map locally defined by \( \hat{\Theta} \left( \frac{\partial}{\partial u^i} \right) = \frac{\delta}{\delta x^i} \). We can transfer the Riemannian inner product \( (,)_R \) on \( \mathcal{H}_R \) just by setting
\[
(H,K) = \left( \hat{\Theta}^{-1}(H), \hat{\Theta}^{-1}(K) \right), \quad \forall H,K \in \mathcal{H}_R.
\]
Then we can define a Riemannian inner product $(\cdot, \cdot)$ on the whole $TTM_0 = H_R \oplus V_R$, just by requiring $H_R$ to be orthogonal to $V_R$. Setting
\[
(g_{ij}) = \begin{pmatrix}
\text{Re} (G_{\alpha \bar{\beta}}) & \text{Im} (G_{\alpha \bar{\beta}}) \\
-\text{Im} (G_{\alpha \bar{\beta}}) & \text{Re} (G_{\alpha \bar{\beta}})
\end{pmatrix},
\]
(3.5)
then $(X, Y) = g_{ij}X^i Y^j$ for $\forall X, Y \in V_R$. In addition,
\[
(g^{ij}) := (g_{ij})^{-1} = \begin{pmatrix}
\text{Re} (G_{\alpha \bar{\beta}})^{-1} & \text{Im} (G_{\alpha \bar{\beta}})^{-1} \\
-\text{Im} (G_{\alpha \bar{\beta}})^{-1} & \text{Re} (G_{\alpha \bar{\beta}})^{-1}
\end{pmatrix}.
\]
(3.6)
If we denote $G_0 = (g_{ij})$, $G_1 = (G_{\alpha \bar{\beta}})$,
then
\[
G_0 = T^* \text{diag} \{ G_1, \bar{G}_1 \} (T^{-1})^t,
\]
(3.7)
\[
(G_0)^{-1} = T^* \text{diag} \{ (G_1)^{-1}, (\bar{G}_1)^{-1} \} (T^{-1})^t.
\]
(3.8)
We define the real horizontal radial vector field $\hat{\chi} := \chi^o = \chi + \bar{\chi}$, the real vertical radial vector field $\hat{i} := \iota^o = \iota + \bar{\iota}$. In local coordinates, $\hat{\chi} = u^i \frac{\delta}{\delta x^i}$, $\hat{i} = u^i \frac{\partial}{\partial u^i}$. Let $\hat{\nabla} : \mathcal{X}(V_R) \to \mathcal{X}(T^* T M_0 \otimes V_R)$ be a real vertical connection with 1-forms
\[
\omega^i_j = \gamma^i_{jk} dx^k + \gamma^i_{jk} \delta u^k.
\]
Setting a global real canonical form $\hat{\eta} = dx^i \otimes \frac{\delta}{\delta x^i} + \delta u^i \otimes \frac{\partial}{\partial u^i}$, and the torsion $\hat{\theta} = \nabla \hat{\eta}$. We can check $\hat{\eta} = \eta + \bar{\eta}$, where $\eta = dz^\alpha \otimes \delta_\alpha + \delta u^\alpha \otimes \partial_\alpha$. In addition,
\[
\hat{\theta} = \dot{\theta}^i \otimes \frac{\delta}{\delta x^i} + \dot{i}^i \otimes \frac{\partial}{\partial u^i},
\]
where
\[
\dot{\theta}^i = \frac{1}{2} \left( \gamma^i_{jk} - \gamma^i_{kj} \right) dx^k \wedge dx^j + \gamma^i_{jk} \delta u^k \wedge dx^j,
\]
(3.9)
\[
\dot{i}^i = \frac{1}{2} \left( \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j} \right) dx^k \wedge dx^j + \frac{1}{2} \left( \frac{\partial N^i_j}{\partial u^k} - \gamma^i_{kj} \right) \delta u^k \wedge dx^j
\]
\[
+ \frac{1}{2} \left( \gamma^i_{jk} - \gamma^i_{kj} \right) \delta u^k \wedge \delta u^j.
\]
(3.10)
**Theorem 3.1.** Let $(M, F)$ be a strongly pseudoconvex complex Finsler manifold. There is a unique vertical connection $\nabla$ such that
for any $X \in TTM_0$, and $V, W \in V_\mathbb{R}$ one has

$$X(V, W) = (\nabla_X V, W) + (V, \nabla_X W); \quad (3.11)$$

(2) $\hat{\theta}(Y, Z) = 0$ for all $Y, Z \in V_\mathbb{R}$;

(3) $\hat{\theta}(H, K) \in V$ for all $H, K \in H_\mathbb{R}$.

**Proof.** Assume such a connection $\nabla$ exists. By (3.9) and (3.10), (3) and (4) yield $\gamma_{jk}^i = \gamma_{kj}^i$ and $\gamma_{jk}^i = \gamma_{ki}^j$, respectively. The same as Levi-Civita connection in Riemannian geometry, (2) yields

$$\gamma_{jk}^i = \frac{1}{2} g^{il} \left( \frac{\partial g_{jl}}{\partial u^k} + \frac{\partial g_{kl}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^l} \right), \quad (3.12)$$

$$\gamma_{jk}^i = \frac{1}{2} g^{il} \left( \frac{\partial \delta g_{jl}}{\partial u^k} + \frac{\partial \delta g_{kl}}{\partial u^j} - \frac{\partial \delta g_{jk}}{\partial u^l} \right). \quad (3.13)$$

Let $\{U_A, (z_A^\alpha, v_A^\beta)\}$ and $\{U_B, (z_B^\beta, v_B^\alpha)\}$ be two local holomorphic coordinate charts of $(z, v) \in U_A \cap U_B \subset T^{1,0}M$, the transformation matrix be $K$, where $K_{\beta}^\alpha = \frac{\partial z^\alpha_B}{\partial z^\beta_A}$. We have

$$dz_B = (dz_A)K, \quad \delta v_B = (\delta v_A)K, \quad \frac{\delta}{\delta z_B} = \frac{\delta}{\delta z_A}(K^{-1})^t, \quad \frac{\partial}{\partial v_B} = \frac{\partial}{\partial v_A}(K^{-1})^t,$$

$$dx_B = (dx_A)\tilde{K}, \quad \delta u_B = (\delta u_A)\tilde{K}, \quad \frac{\delta}{\delta x_B} = \frac{\delta}{\delta x_A}(\tilde{K}^{-1})^t, \quad \frac{\partial}{\partial u_B} = \frac{\partial}{\partial u_A}(\tilde{K}^{-1})^t,$$

where

$$\tilde{K} = T^t \text{diag} \left\{ K, \tilde{K} \right\} (T^{-1})^t, \quad (3.14)$$

i.e., $\tilde{K}_{ij}^k = \frac{\partial x_B^k}{\partial x_A^i}$. In addition,

$$(G_1)_A = K(G_1)_B \tilde{K}^t, \quad (G_0)_A = \tilde{K}(G_1)_B \tilde{K}^t.$$  

Since $v_B^\alpha = v_A^\beta K_{\beta}^\alpha$, we have $dv_B = (dv_A)K + (dz_A)H$, where $H_{\alpha}^\beta = \frac{\partial^2 z_B^\beta}{\partial z_A^\alpha \partial z_A^\gamma} v_A^\gamma$. By $\delta v = dv + (dz)\Gamma$, we have

$$(dv_A)K + (dz_A)H + (dz_A)K \Gamma_B = dv_B + (dz_B)\Gamma_B = \delta v_B = (\delta v_A)K = ((dv_A) + (dz_A)\Gamma_A) K,$$

which yields that

$$\Gamma_B = K^{-1} ((\Gamma_A)K - H). \quad (3.15)$$

By (3.4) and (3.15), we have

$$N_B = \tilde{K}^{-1} \left((N_A)\tilde{K} - \tilde{H}\right), \quad (3.16)$$

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where
\[ \dot{H} = T^4 \text{diag} \left\{ H, \dot{H} \right\} (T^{-1})^4. \] (3.17)

Note that
\[ \dot{H}_\beta = \frac{1}{2} (H^\alpha + \bar{H}_\beta) \frac{\partial z^\beta}{\partial x^\alpha} + \frac{\partial z^\beta}{\partial x^\alpha} \left( \frac{\partial \bar{z}^\beta}{\partial x^\alpha} + \frac{\partial \bar{z}^\beta}{\partial x^\alpha} \right) = \frac{\partial^2 x_B^\alpha}{\partial x_A^\beta} u^A, \]
and \[ \dot{H}_\beta^{\alpha+n} = \frac{\partial^2 x_B^\alpha}{\partial x_A^\beta} u^A, \] where \[ \dot{H}_\beta^{\alpha+n} = \frac{\partial^2 x_B^\alpha}{\partial x_A^\beta} u^A. \] Direct computation shows that
\[
\left( \frac{\delta g_{ij}}{\delta x^k} g^{il} \right)_B = (\bar{K}^{-1})^p \frac{\delta }{\delta x^A} \left( (\bar{K}^{-1})^s_j (g_{st})_A (\bar{K}^{-1})^l_k \right) \tilde{K}_m^l (g^{mq})_A \tilde{K}_q^i
\]
\[ = (\bar{K}^{-1})^p \left( \bar{K}^{-1} \right)^s_j \left[ \frac{\delta (g_{st})_A}{\delta x^A} (g^{aq})_A \tilde{K}_q^i - \frac{\partial^2 x_B^i}{\partial x_A^a \partial x^A} \right]
\]
\[ - (\bar{K}^{-1})^p \left( \bar{K}^{-1} \right)^s_j (g_{st})_A \frac{\partial^2 x_B^i}{\partial x_A^a \partial x^A} \left( \bar{K}^{-1} \right)^t_b (g^{aq})_A \tilde{K}_q^i
\]
\[ \left( \frac{\delta g_{kl}}{\delta x^j} g^{il} \right)_B = (\bar{K}^{-1})^p \left( \bar{K}^{-1} \right)^s_j \left[ \frac{\delta (g_{pt})_A}{\delta x^A} (g^{aq})_A \tilde{K}_q^i - \frac{\partial^2 x_B^i}{\partial x_A^a \partial x^A} \right]
\]
\[ - (\bar{K}^{-1})^p \left( \bar{K}^{-1} \right)^s_j (g_{pt})_A \frac{\partial^2 x_B^i}{\partial x_A^a \partial x^A} \left( \bar{K}^{-1} \right)^t_b (g^{aq})_A \tilde{K}_q^i
\]

we have
\[ \left( \gamma_{jk}^i \right)_B = (\bar{K}^{-1})^p \left( \bar{K}^{-1} \right)^s_j \left[ (\gamma_{sp})_A \tilde{K}_q^i - \frac{\partial^2 x_B^i}{\partial x_A^a \partial x^A} \right]. \] (3.18)

Similarly,
\[ \left( \gamma_{jk}^i \right)_B = (\bar{K}^{-1})^p \left( \bar{K}^{-1} \right)^s_j \left( \gamma_{sp} A \tilde{K}_q^i. \right. \] (3.19)
Hence,
\[
(\omega^j)_B = \left(\tilde{K}^{-1}\right)^* (\omega^j)_A \tilde{K}^i_q - \left(\tilde{K}^{-1}\right)^* d \left(\tilde{K}^i_q\right).
\] (3.20)

Therefore the forms
\[
\omega^i_j = \gamma^i_{jk} d x^k + \gamma^i_{jk} \delta u^k
\]
are connections of a vertical connection such that (1)-(3).

Furthermore, one can transfer \(\nabla\) on \(\mathcal{H}_\mathbb{R}\) defined by
\[
\nabla_{\delta} \delta x^i = \omega^j_i \otimes \delta \delta x^j.
\]
Now we extend the Riemannian inner product \((, )\) to the complexified bundle
\[
T_C TM_0 = TT M_0 \otimes \mathbb{C} = \mathcal{H} \oplus \tilde{\mathcal{H}} \oplus \mathcal{V} \oplus \tilde{\mathcal{V}}.
\]

Then
\[
(\delta_\alpha, \delta_\beta) = (\delta_\bar{\alpha}, \delta_\bar{\beta}) = \left(\hat{\partial}_\alpha, \hat{\partial}_\beta\right) = \left(\hat{\partial}_{\bar{\alpha}}, \hat{\partial}_{\bar{\beta}}\right) = 0, \\
(\delta_\alpha, \delta_{\bar{\beta}}) = \overline{(\delta_\bar{\alpha}, \delta_\beta)} = \left(\hat{\partial}_\alpha, \hat{\partial}_{\bar{\beta}}\right) = \left(\hat{\partial}_{\bar{\alpha}}, \hat{\partial}_\beta\right) = \frac{1}{2}G_{\alpha\bar{\beta}}.
\]

We complexize the connection \(\nabla\) also denoted by \(\hat{\nabla}\). For simplicity, we denote
\[
h_{\alpha\beta} = h_{\bar{\alpha}\bar{\beta}} = 0, \quad h_{\alpha\bar{\beta}} = \frac{1}{2}G_{\alpha\bar{\beta}}.
\]

Then the complexified connection coefficients are given by
\[
L^k_{ij} = \frac{1}{2} h^{kl} \left[\delta_j \left(h_{ul}\right) + \delta_i \left(h_{jl}\right) - \delta_l \left(h_{ij}\right)\right], \\
C^k_{ij} = \frac{1}{2} h^{kl} \left[\hat{\partial}_j \left(h_{ul}\right) + \hat{\partial}_i \left(h_{jl}\right) - \hat{\partial}_l \left(h_{ij}\right)\right],
\] (3.21)
(3.22)

where \(i, j, k, l \in \{1, \cdots, n, \bar{1}, \cdots, \bar{n}\}\), \(\delta_i = \delta_\alpha\) and \(\hat{\partial}_i = \hat{\partial}_\alpha\) if \(i = \alpha\), \(\delta_i = \delta_{\bar{\alpha}}\) and \(\hat{\partial}_i = \hat{\partial}_{\bar{\alpha}}\) if \(i = \bar{\alpha}\).

That is,
\[
L^{\alpha}_{\beta\mu} = \frac{1}{2} G^{\lambda\bar{\alpha}} \left(\delta_\mu (G_{\beta\bar{\lambda}}) + \delta_{\bar{\lambda}} (G_{\mu\bar{\beta}})\right), \quad C^{\alpha}_{\beta\mu} = \Gamma^{\alpha}_{\beta\mu}, \\
L^{\bar{\alpha}}_{\bar{\beta}\mu} = \frac{1}{2} G^{\lambda\bar{\alpha}} \left(\delta_{\bar{\mu}} (G_{\beta\bar{\lambda}}) - \delta_{\bar{\lambda}} (G_{\beta\bar{\mu}})\right), \quad L^{\bar{\alpha}}_{\bar{\beta}\mu} = C^{\bar{\alpha}}_{\bar{\beta}\mu} = C^{\alpha}_{\beta\mu} = 0,
\] (3.23)

and \(L^{\alpha}_{\bar{\beta}\mu} = L^{\alpha}_{\bar{\mu}\bar{\beta}} = L^{\bar{\alpha}}_{\bar{\mu}\beta} = L^{\bar{\alpha}}_{\beta\mu}\). Its connection 1-forms are given by
\[
\omega^\alpha_{\beta\mu} = L^{\alpha}_{\beta\mu} d z^\mu + L^{\bar{\alpha}}_{\bar{\beta}\mu} d z^\mu + C^{\alpha}_{\beta\mu} \delta u^\mu, \quad \omega^\bar{\alpha}_{\bar{\beta}\mu} = L^{\bar{\alpha}}_{\bar{\beta}\mu} d z^\mu.
\] (3.24)

We call \(\hat{\nabla}\) the canonical connection induced by the Chern Finsler connection \(D\). We define
\(\tilde{\nabla} = \nabla|_{\mathcal{T}_{1.0}\mathcal{M}}\). Munteanu \[7\] called the connection \(\tilde{\nabla}\) the canonical connection. In addition, the canonical connection \(\hat{\nabla}\) and the Chern Finsler connection \(D\) coincide if and only if \(F\) is Kähler Finsler \[7\].
Remark 3.2. The connection \( \hat{\nabla} \) is a natural generalization of the Levi-Civita connection on the holomorphic tangent bundle of a Hermitian manifold [5].

4 Proofs of Theorem 1.1 and Theorem 1.2

The holomorphic sectional curvature tensor associated with \( \nabla \) is defined by

\[
R_{\alpha\beta\mu\nu} = 2 \left( \nabla_{\delta\alpha} \nabla_{\delta\beta} \delta_{\gamma} - \nabla_{\delta\alpha} \nabla_{\delta\beta} \delta_{\gamma} - \nabla_{[\delta\alpha,\delta\beta]} \delta_{\gamma} \right). \tag{4.1}
\]

The horizontal holomorphic flag curvature \( \hat{c}_F(H) \) associated with \( \nabla \) along a horizontal vector \( H = H^\alpha \delta_\alpha \in \mathcal{H}_v \) is defined by

\[
\hat{c}_F(H) = \frac{R_{\alpha\delta\mu\nu}(v) H^\alpha \hat{H}^\beta H^\mu \hat{H}^\nu}{(G_{\alpha\beta}(v) H^\alpha \hat{H}^\beta)^2}. \tag{4.2}
\]

The holomorphic sectional curvature \( \hat{c}_F(v) \) associated with \( \nabla \) along \( v \) is defined by

\[
\hat{c}_F(v) = \frac{R_{\alpha\delta\mu\nu}(v) v^\alpha \hat{v}^\beta v^\mu \hat{v}^\nu}{G(v)^2}. \tag{4.3}
\]

As (4.1)-(4.3), we can define the holomorphic sectional curvature tensor \( \hat{R}_{\alpha\beta\mu\nu} \), the horizontal holomorphic flag curvature \( \hat{K}_F(H) \) and the holomorphic sectional curvature \( \hat{K}_F(v) \) associated with \( \hat{\nabla} \).

Lemma 4.1. Let \((M,F)\) be a strongly pseudoconvex complex Finsler manifold. Then

\[
R_{\alpha\beta\mu\nu} = -\frac{1}{2} \left[ \delta_\nu \delta_\alpha (G_{\mu\beta}) + \delta_\beta \delta_\mu (G_{\alpha\nu}) + \delta_\nu (G^\sigma_{\mu\beta}) G_{\alpha\beta\sigma} + \delta_\beta (G^\sigma_{\mu\beta}) G_{\alpha\beta\sigma} \right.
+ \frac{1}{4} \left[ \delta_\nu (G_{\alpha\lambda}) + \delta_\lambda (G_{\alpha\nu}) \right] G^{\lambda\gamma} \left[ \delta_\nu (G_{\gamma\beta}) + \delta_\beta (G_{\gamma\nu}) \right]
- \frac{1}{4} \left[ \delta_\nu (G_{\alpha\lambda}) - \delta_\lambda (G_{\alpha\nu}) \right] G^{\lambda\gamma} \left[ \delta_\nu (G_{\gamma\beta}) - \delta_\beta (G_{\gamma\nu}) \right]
- \frac{1}{4} \left[ \delta_\beta (G_{\mu\lambda}) - \delta_\lambda (G_{\mu\nu}) \right] G^{\lambda\gamma} \left[ \delta_\nu (G_{\gamma\beta}) - \delta_\beta (G_{\gamma\nu}) \right]
+ \frac{1}{2} \left[ \delta_\nu (G^\gamma_{\mu\beta}) G_{\alpha\beta\gamma} - \delta_\mu (G^\gamma_{\beta\mu}) G_{\alpha\beta\gamma} \right]. \tag{4.4}
\]

Proof. Direct computation shows that

\[
\nabla_{\delta_\mu} \nabla_{\delta_\nu} \delta_\alpha = \left[ \delta_\nu (L^\alpha_{\delta\mu}) + L^\alpha_{\delta\nu} L^\gamma_{\delta\mu} + L^\sigma_{\delta\nu} L^\gamma_{\delta\mu} \right] \delta_\gamma
+ \left[ \delta_\mu (L^\gamma_{\delta\nu}) + L^\gamma_{\delta\mu} L^\alpha_{\delta\nu} \right] \delta_\gamma.
\]

\[
\nabla_{\delta_\mu} \nabla_{\delta_\nu} \delta_\alpha = \left[ \delta_\nu (L^\alpha_{\delta\mu}) + L^\sigma_{\delta\nu} L^\gamma_{\delta\mu} \right] \delta_\gamma + \left[ \delta_\mu (L^\gamma_{\delta\nu}) + L^\nu_{\delta\mu} L^\gamma_{\delta\nu} \delta_\gamma,
\]

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we have
\[ 2 (\nabla_{\delta_\nu} \nabla_{\delta_\nu} \delta_\alpha - \nabla_{\delta_\nu} \nabla_{\delta_\nu} \delta_\alpha, \delta_\beta)_v = \left( \left( \delta_\mu \left( L^\gamma_{\alpha\beta} \right) + L^\sigma_{\alpha\beta} L^\gamma_{\sigma\mu} + L^\sigma_{\alpha\beta} L^\gamma_{\sigma\mu} \right) - \left( \delta_\nu \left( L^\gamma_{\alpha\mu} \right) + L^\sigma_{\alpha\mu} L^\gamma_{\sigma\nu} \right) \right) G_{\gamma\beta}. \]

In addition,
\[ (\delta_\mu \left( L^\gamma_{\alpha\beta} \right) + L^\sigma_{\alpha\beta} L^\gamma_{\sigma\mu} ) G_{\gamma\beta} = \delta_\mu \left( L^\gamma_{\alpha\beta} G_{\gamma\beta} \right) + L^\gamma_{\alpha\mu} \left[ -\delta_\mu (G_{\gamma\beta}) + L^\gamma_{\mu\beta} G_{\sigma\beta} \right] \]
\[ = \frac{1}{2} \left[ \delta_\mu \delta_\nu (G_{\alpha\beta}) - \delta_\mu \delta_\beta (G_{\alpha\nu}) + \frac{1}{2} L^\gamma_{\alpha\mu} \left[ \delta_\gamma (G_{\gamma\beta}) - \delta_\mu (G_{\gamma\beta}) \right] \right] \]
\[ = \frac{1}{2} \left[ \delta_\mu \delta_\nu (G_{\alpha\beta}) - \delta_\beta \delta_\mu (G_{\alpha\nu}) + \frac{1}{2} L^\gamma_{\alpha\mu} \left[ \delta_\gamma (G_{\gamma\beta}) - \delta_\beta (G_{\gamma\nu}) \right] \right] \]
\[ = - \frac{1}{4} \left[ (\delta_{\alpha\beta}) \delta_\gamma (G_{\gamma\beta}) - \delta_\alpha (G_{\gamma\beta}) \right] G^{\alpha\gamma} \left[ \delta_\alpha (G_{\gamma\beta}) - \delta_\beta (G_{\gamma\alpha}) \right], \]
\[ L^\gamma_{\alpha\beta} L^\gamma_{\mu\beta} G_{\gamma\beta} = - \frac{1}{4} \left[ \delta_\beta (G_{\mu\beta}) - \delta_\mu (G_{\mu\beta}) \right] G^{\alpha\gamma} \left[ \delta_\alpha (G_{\gamma\beta}) - \delta_\beta (G_{\gamma\alpha}) \right], \]
\[ 2 (\nabla_{\delta_\alpha} \delta_\nu \delta_\alpha, \delta_\beta)_v = 2 (\delta_\nu (G^\alpha_{\gamma\mu}) G^\alpha_{\gamma\nu} \delta_\sigma, \delta_\beta)_v = G_{\alpha\beta} \delta_\nu (G^\alpha_{\gamma\mu}), \]

Since
\[ [\delta_\mu, \delta_\nu] = \delta_\nu \left( \Gamma^\gamma_{\mu\alpha} \right) \delta_\sigma - \delta_\mu \left( \Gamma^\gamma_{\nu\alpha} \right) \delta_\tau, \]
we have
\[ \frac{1}{2} \left[ [\delta_\mu, \delta_\nu] (G_{\alpha\beta}) - [\delta_\mu, \delta_\beta] (G_{\alpha\nu}) \right] G_{\alpha\beta} \delta_\nu (G^\gamma_{\mu\alpha}) \]
\[ = \frac{1}{2} \left[ -\delta_\nu \left( \Gamma^\gamma_{\mu\beta} \right) G_{\alpha\beta} - \delta_\mu \left( \Gamma^\gamma_{\nu\beta} \right) G_{\alpha\beta} - \delta_\beta \left( \Gamma^\gamma_{\mu\nu} \right) G_{\alpha\beta} \right]. \]

Therefore, we complete the proof. \(\square\)

**Remark 4.2.** We can check \(\overline{R_{\alpha\beta\mu\nu}} = R_{\beta\alpha\mu\nu} \) but \(R_{\alpha\beta\mu\nu} \neq R_{\mu\nu\alpha\beta}.\)

We can also directly check \(R_{\alpha\beta\mu\nu} = \Omega_{\alpha\beta\mu\nu} \) if \(F\) is a Kähler Finsler metric. Next we consider an inverse problem. When a strongly pseudoconvex complex Finsler metric \(F\) satisfies \(R_{\alpha\beta\mu\nu} = \Omega_{\alpha\beta\mu\nu},\) must it be Kähler Finsler?
Proof of Theorem 1.1 If $R_{\alpha\beta\mu
u} = \Omega_{\alpha\beta\mu
u}$, then

$$K_F(H) = K_F(H), \quad \forall H = H^\alpha \delta_\alpha \in \mathcal{H}_v.$$ 

By (4.4), we have

$$R_{\alpha\beta\mu\nu} H^\alpha \tilde{H}^\beta H^\mu \tilde{H}^\nu = -\left[\delta_\mu \delta_\nu (G_{\alpha\beta}) + \delta_\nu (\Gamma^\sigma_\mu) G_{\alpha\beta\sigma}\right] H^\alpha \tilde{H}^\beta H^\mu \tilde{H}^\nu + \delta_\mu (G_{\alpha\beta}) \tilde{G}^{\lambda\gamma} \delta_\nu (G_{\gamma\beta}) H^\alpha \tilde{H}^\beta H^\mu \tilde{H}^\nu - \frac{1}{2} H^\alpha \tilde{H}^\nu \left[\delta_\nu (G_{\alpha\lambda}) - \delta_\lambda (G_{\alpha\nu})\right] \tilde{G}^{\lambda\gamma} \left[\delta_\mu (G_{\gamma\beta}) - \delta_\gamma (G_{\mu\beta})\right] \tilde{H}^\beta H^\mu,$$

i.e.,

$$R_{\alpha\beta\mu\nu} H^\alpha \tilde{H}^\beta H^\mu \tilde{H}^\nu - \Omega_{\alpha\beta\mu\nu} H^\alpha \tilde{H}^\beta H^\mu \tilde{H}^\nu = - \frac{1}{2} H^\alpha \tilde{H}^\nu \left[\delta_\nu (G_{\alpha\lambda}) - \delta_\lambda (G_{\alpha\nu})\right] \tilde{G}^{\lambda\gamma} \left[\delta_\mu (G_{\gamma\beta}) - \delta_\gamma (G_{\mu\beta})\right] \tilde{H}^\beta H^\mu. \quad (4.5)$$

Since $(G^{\lambda\gamma})$ is positive definite, then

$$R_{\alpha\beta\mu\nu} H^\alpha \tilde{H}^\beta H^\mu \tilde{H}^\nu = \Omega_{\alpha\beta\mu\nu} H^\alpha \tilde{H}^\beta H^\mu \tilde{H}^\nu$$

if and only if

$$\left[\delta_\mu (G_{\gamma\beta}) - \delta_\gamma (G_{\mu\beta})\right] \tilde{H}^\beta H^\mu = 0 \quad (4.6)$$

holds for any horizontal vector $H = H^\alpha \delta_\alpha \in \mathcal{H}_v$. Take $H = \delta_1 \in \mathcal{H}_v$, then $\delta_1 (G_{\gamma\beta}) = \delta_\gamma (G_{\beta\beta})$, $1 \leq \gamma \leq n$. Take $H = \delta_2 \in \mathcal{H}_v$, then $\delta_2 (G_{\gamma\beta}) = \delta_\beta (G_{\beta\beta})$, $1 \leq \beta \leq n$. Take $H = \delta_n \in \mathcal{H}_v$, then $\delta_n (G_{\gamma\beta}) = \delta_\gamma (G_{\mu\beta})$, $1 \leq \gamma \leq n$. Hence, if $R_{\alpha\beta\mu\nu} = \Omega_{\alpha\beta\mu\nu}$, then $\delta_\mu (G_{\gamma\beta}) = \delta_\gamma (G_{\mu\beta})$ for $\forall 1 \leq \mu, \beta, \gamma \leq n$, that is $F$ is a Kähler Finsler metric.

Proof of Theorem 1.2 By (4.5), we have

$$R_{\alpha\beta\mu\nu} v^\alpha \tilde{v}^\beta v^\mu \tilde{v}^\nu - \Omega_{\alpha\beta\mu\nu} v^\alpha \tilde{v}^\beta v^\mu \tilde{v}^\nu = - \frac{1}{2} \left[\chi (G_{\lambda\beta}) - G_{\lambda\beta}\right] \tilde{G}^{\lambda\gamma} [\chi (G_{\gamma\gamma}) - G_{\gamma\gamma}]. \quad (4.7)$$

Hence,

$$R_{\alpha\beta\mu\nu} v^\alpha \tilde{v}^\beta v^\mu \tilde{v}^\nu = \Omega_{\alpha\beta\mu\nu} v^\alpha \tilde{v}^\beta v^\mu \tilde{v}^\nu$$

if and only if $\chi (G_{\gamma\gamma}) - G_{\gamma\gamma} = 0$ for $\forall v \in T^{1,0} M$. Note that

$$G_\alpha (\Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta}) v^\beta = G_{\gamma\gamma} - \chi (G_{\gamma\gamma}),$$
therefore
\[ R_{\alpha\bar{\beta}\mu\bar{\nu}} v^\alpha \bar{v}^\beta v^\mu \bar{v}^\nu = \Omega_{\alpha\bar{\beta}\mu\bar{\nu}} v^\alpha \bar{v}^\beta v^\mu \bar{v}^\nu \]
if and only if \( F \) is weakly Kähler Finsler. We complete the proof.

It follows the proof of Lemma 4.1 that the following result is obtained.

**Proposition 4.3.** Let \((M, F)\) be a strongly pseudoconvex complex Finsler manifold. Then
\[ \hat{R}_{\alpha\bar{\beta}\mu\bar{\nu}} = R_{\alpha\bar{\beta}\mu\bar{\nu}} + \frac{1}{4} [\delta_{\bar{\beta}} (G_{\mu\bar{\lambda}}) - \delta_{\lambda} (G_{\mu\bar{\beta}})] G^{\bar{\lambda}} \gamma [\delta_{\alpha} (G_{\gamma\bar{\nu}}) - \delta_{\gamma} (G_{\alpha\bar{\nu}})]. \] (4.8)

Hence, we have
\[ (R_{\alpha\bar{\beta}\mu\bar{\nu}} - \hat{R}_{\alpha\bar{\beta}\mu\bar{\nu}}) H^\alpha \bar{H}^{\beta} K^{\mu} \bar{K}^{\nu} \leq 0, \quad \forall H, K \in \mathcal{H}. \] (4.9)

**5 A new connection**

Set \( \omega_H = \sqrt{-1} G_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta, \partial_H = dz^\alpha \otimes \delta_{\alpha}, \bar{\partial}_H = d\bar{z}^\alpha \otimes \delta_{\alpha}. \) Now we introduce a new connection on \( \tilde{M}. \) We define a new connection on the horizontal bundle \( \mathcal{H} \) by
\[ \tilde{D} = \hat{\nabla} + S \] (5.1)
where \( S \) is a 1-form defined by
\[ \langle S(X)H, K \rangle = 2(S(X)H, \bar{K}) = \sqrt{-1}(\partial_H - \bar{\partial}_H) \omega_H(X, H, \bar{K}) \] (5.2)
for any \( X \in T_C \tilde{M}, H, K \in \mathcal{H}. \) We can also extend \( \tilde{D} \) to a complex linear connection on \( \mathcal{V} \) by setting
\[ \tilde{D}_X V = \Theta^{-1}(\tilde{D}_X (\Theta(V)), \quad V \in \mathcal{X}(V), \quad X \in T_C \tilde{M}. \] (5.3)

**Lemma 5.1.** Connection 1-forms of \( \tilde{D} \) are given by
\[ \tilde{\theta}_\beta = \Gamma^\alpha_{\mu\beta} dz^\mu + 2L^A_{\beta\mu} d\bar{z}^\mu + \Gamma^\alpha_{\beta\mu} \delta v^\mu. \] (5.4)

**Proof.** Denote connection 1-forms of \( \tilde{D} \) by
\[ \tilde{\theta}_\beta = \tilde{\Gamma}^\alpha_{\beta\mu} dz^\mu + \tilde{\Gamma}^\alpha_{\beta\mu} d\bar{z}^\mu + \tilde{\Gamma}^\alpha_{\beta\mu} \delta v^\mu + \tilde{\Gamma}^\alpha_{\beta\mu} \delta \bar{v}^\mu. \]
Let \( X = \delta_{\mu}, H = \delta_{\beta}, K = \delta_{\lambda}, \) then
\[ \sqrt{-1}(\partial_H - \bar{\partial}_H) \omega_H(\delta_{\mu}, \delta_{\beta}, \delta_{\lambda}) = -\frac{1}{2} [\delta_{\mu} (G_{\beta\lambda}) - \delta_{\beta} (G_{\mu\lambda})]. \]
\[ \langle \nabla_{\delta_{\mu}} \delta_{\beta}, \delta_{\lambda} \rangle = L^A_{\beta\mu} G_{A\lambda}, \quad \langle \tilde{D}_{\delta_{\mu}} \delta_{\beta}, \delta_{\lambda} \rangle = \tilde{\Gamma}^\alpha_{\beta\mu} G_{A\lambda}. \]
By the definition of $\hat{D}$, we have
\[
\hat{\Gamma}^{\alpha}_{\beta\mu} G_{\alpha\lambda} = L^\alpha_{\beta\mu} G_{\alpha\lambda} - \frac{1}{2} \left[ \delta_\mu \left( G_{\beta\lambda} \right) - \delta_\lambda \left( G_{\mu\beta} \right) \right].
\]
Hence,
\[
\hat{\Gamma}^{\alpha}_{\beta\mu} = G_{\lambda\mu} \delta_\lambda \left( G_{\mu\beta} \right) = \Gamma^\alpha_{\mu\beta}.
\]
The proof of the other three is similar. □

**Remark 5.2.** The connection $\hat{D}$ is a natural generalization of the Bismut connection in Hermitian geometry [3]. We can directly check
\[
\hat{D} = \hat{\nabla} + \left( \hat{\nabla} - D \right).
\]  

The holomorphic sectional curvature tensor associated with $\hat{D}$ is defined by
\[
\hat{R}^{\alpha}_{\beta\mu\nu} = \left\langle \hat{D}_{\delta_\mu} \hat{D}_{\delta_\nu} \delta_\alpha - \hat{D}_{\delta_\nu} \hat{D}_{\delta_\mu} \delta_\alpha - \hat{D}_{[\delta_\mu, \delta_\nu]} \delta_\alpha, \delta_\beta \right\rangle_v.  
\]  

In addition, we can also define the horizontal holomorphic flag curvature $\hat{K}_F(v)$ and the holomorphic sectional curvature $\hat{K}_F(v)$ associated with $\hat{D}$.

Being similar to Lemma 4.1, we can obtain the following result.

**Lemma 5.3.** Let $F$ be a strongly pseudoconvex complex Finsler metric on $M$. Then
\[
\hat{R}^{\alpha}_{\beta\mu\nu} = \Omega^{\alpha}_{\beta\mu\nu},
\]
if and only if $F$ is a Kähler Finsler metric. In addition,
\[
\hat{K}_F(v) = K_F(v),
\]
if and only if $F$ is a weakly Kähler Finsler metric.

*Theorem 5.4.* Let $(M, F)$ be a strongly pseudoconvex complex Finsler manifold. We have
\[
\hat{R}^{\alpha}_{\beta\mu\nu} = \Omega^{\alpha}_{\beta\mu\nu},
\]
if and only if $F$ is a Kähler Finsler metric. In addition,
\[
\hat{K}_F(v) = K_F(v),
\]
if and only if $F$ is a weakly Kähler Finsler metric.
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