HAMILTONIANS FOR COMPACT HOMOGENEOUS UNIVERSES

Masayuki TANIMOTO

Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto, Japan

Tatsuhiko KOIKE

Department of Physics, Keio University, Kanagawa, Japan

Akio HOSOYA

Department of Physics, Tokyo Institute of Technology, Tokyo, Japan

We briefly show how we can obtain Hamiltonians for spatially compact locally homogeneous vacuum spacetimes. The dynamical variables are categorized into the curvature parameters and the Teichmüller parameters. While the Teichmüller parameters usually parameterise the covering group of the spatial sections, we utilise another suitable parameterization where the universal cover metric carries all the dynamical variables and with this we reduce the Hamiltonians. For our models, all dynamical variables possess their clear geometrical meaning, in contrast to the conventional open models.

1 Introduction

Spatially homogeneous cosmological models e.g., give good prototypes for many theoretical models in gravity and cosmology. Some controversies arise, however, when considering Hamiltonian structures of them, which are of large interest, especially, for an application to quantum gravity. For example, it is well known that the models known as Bianchi class B do not possess a natural Hamiltonian reduced from the full Hamiltonian. Even for the class A models, a sort of discrepancies of dynamical degrees of freedom is pointed out by Ashtekar and Samuel. For example, the Kasner solution, which is the vacuum solution of Bianchi I, has only one dynamical degree of freedom, i.e., there is only one free parameter which can be specified freely at an initial Cauchy surface. Note, however, that an odd number of dynamical degrees of freedom cannot come out from a Hamiltonian system. So, when one wants a Hamiltonian, one usually works with the so-called diagonal model, which has three dynamical variables and gives four dynamical degrees of freedom in the Hamiltonian view. If we work with the full, nondiagonal model, which may be the most natural in the Hamiltonian view, we have ten dynamical degrees of freedom with six dynamical variables. Thus, we have obtained three possible numbers (i.e., 1, 4 and 10) of dynamical degrees of freedom for Bianchi I!

These discrepancies for the spatially open model are responsible for the ambiguous specification of the dynamical variables. Note that a (group invariant) spatial section of the spatially open Bianchi I model is a three dimensional Euclid space, which has no free parameters specifying the intrinsic geometry, since any Euclid space is isometric to the one with the standard metric $dx^2 + dy^2 +dz^2$. This proves that the open Bianchi I model possesses no dynamical variables, though it has
one “dynamical degrees of freedom” as in the Kasner solution). When employing
the diagonal or full model, the metric components are treated as if they are true
dynamical variables, but in this case they lose their geometrical nature. As a result,
the open Bianchi I model cannot admit a consistent Hamiltonian structure.

How about spatially compact Bianchi models? For example, we can compactify
the Euclid space and make a torus $T^3$ by first fixing three independent vectors $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$ in the standard metric and then identifying each two points $\vec{p}$ and $\vec{q}$ such that
$\vec{p} - \vec{q} = k\vec{a}_1 + l\vec{a}_2 + m\vec{a}_3$, $k, l, m \in \mathbb{Z}$. If we smoothly vary the three vectors $\vec{a}_i$, then the quotient manifold in general smoothly varies nonisometrically. More precisely, six independent parameters in $\vec{a}_i$ can induce nonisometric deformations of the quotient. Such parameters, denoted collectively as $\tau$, are called Teichmüller parameters $3, 4$.

We may regard the Teichmüller parameters as (part of the) dynamical variables of
a spatially compactified locally homogeneous spacetime. As for the $T^3$ model on
Bianchi I, the dynamical variables are the six Teichmüller parameters only, and we
can prove for this system there exists a consistent Hamiltonian structure.

In this article, we show a skeleton of our method $5, 6, 7$ of obtaining consistent
Hamiltonians for spatially compact locally homogeneous (SCH) spacetimes. (See
also the paper by Kodama $8$, where a somewhat different approach is presented.)

2 Method for reducing the Hamiltonian for a compact homogeneous
universe

As mentioned above, a flat torus is locally isometric to the standard Euclid space
$(\mathbb{R}^3, \eta_{ab})$, where $\eta_{ab}$ is the standard Euclid metric. The six Teichmüller parameters $\tau = \{a_i\}$ are the independent parameters in the covering group $A_\tau$, represented by

$$A_\tau = \{\vec{a}_1, \vec{a}_2, \vec{a}_3\} = \left\{ \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \\ 0 & a_{33} \end{pmatrix} \right\}. \quad (1)$$

The flat torus is then represented by $(\mathbb{R}^3, \eta_{ab}) / A_\tau$. Similarly, any compact and
locally homogeneous manifold can be represented by

$$(\tilde{M}, \tilde{h}_{ab}^{\text{std}}[r]) / A_\tau,$$

where $(\tilde{M}, \tilde{h}_{ab}^{\text{std}}[r])$ is a homogeneous manifold which is free from diffeomorphisms,
$r$ is a set of parameters in a standard metric $\tilde{h}_{ab}^{\text{std}}$, and $\tau$ is a set of Teichmüller parameters. A proper action of $A_\tau$ on $\tilde{M}$ is understood. The parameters $r, \tau$ are our dynamical variables. (For the flat torus case, $r = \emptyset$.)

We then define a diffeomorphism $\phi_\tau : \tilde{M} \to \tilde{M}$ such that

$$A_\tau = \phi_\tau \circ A_0 \circ \phi_\tau^{-1}, \quad (3)$$

where $A_0$ is the covering group for a set of fixed Teichmüller parameters $\tau = \tau_0$.
With this, we obtain another parameterization

$$(\tilde{M}, \tilde{h}_{ab}^{\text{dyn}}[r, \tau]) / A_0, \quad (4)$$
where
\[ \tilde{h}_{ab}^{\text{dyn}}[r, \tau] \equiv \phi_{\tau} \ast \tilde{h}_{ab}^{\text{std}}[r]. \]  
(5)

We shall refer to \( \phi_{\tau} \) as a Teichmüller diffeomorphism (TD). TDs are not unique. We refer to a TD implemented in the HPDs as an HPTD.

If we consider a spacetime metric \( \tilde{g}_{ab}^{\text{dyn}}(r, \tau) \) whose spatial part is given by Eq.(5), where \( r \) and \( \tau \) are free functions of time \( t \), then this gives a possible ansatz of a SCH spacetime. We do this with the HPTDs and the synchronous gauge, since the ansatz obtained by doing so, an ordinary Bianchi type spacetime metric, gives a dynamically consistent one, since the extrinsic curvature and the metric contains the same transitive symmetry (isometry) group.

As for \( T^3 \) on Bianchi I, the HPTD is given by the following linear transformation
\[
\phi_{\tau}: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} a_1^1 & a_2^1 & a_3^1 \\ 0 & a_2^2 & a_3^2 \\ 0 & 0 & a_3^3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},
\]
(6)

We can calculate the spatial metric according to Eq.(5) by inducing the standard Euclid metric, and finally obtain the spacetime metric with the synchronous gauge with \( a_i^j \) being functions of time. The Hamiltonian can be reduced with this spacetime metric ansatz.

3 Concluding remark

We have very shortly shown how we obtain Hamiltonians for spatially compact locally homogeneous spacetimes. The points are the use of the parameterization (4) and the use of the HPTDs. The spacetime metric ansatz thereby obtained enables us to reduce the Hamiltonian for many spatially compact locally homogeneous spacetime models. We remark, however, that in some cases the HPTDs exist only for part of the Teichmüller deformations, or do not exist at all. In these cases, the dynamics of the Teichmüller deformations degenerates or freezes, respectively.

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