Linearization instabilities of the massive nonsymmetric gravitational theory

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Abstract. The massive nonsymmetric gravitational theory is shown to possess a linearization instability at purely GR field configurations, disallowing the use of the linear approximation in these situations. It is also shown that arbitrarily small antisymmetric sector Cauchy data leads to singular evolution unless an ad hoc condition is imposed on the initial data hypersurface.

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1. Introduction

The Nonsymmetric Gravitational Theory (NGT) (Moffat 1979, 1984, 1991) grew out of a reinterpretation of Einstein’s unified field theory (Einstein 1945; Einstein and Straus 1946) as a theory of gravitation alone, thereby bypassing the problems associated with the interpretation of the additional structure as representing the electromagnetic field (Infeld 1950, Callaway 1953). There were indications of problems even with this reinterpretation (Kunstatter et al. 1984; Kelly 1991, 1992) which were clarified by Damour, Deser and McCarthy (1992, 1993) who showed that the wave solutions of the weak field equations did not decrease at large distances from the source along the forward light cone (discussed in more detail in Clayton (1996a)).

The theory was subsequently altered into what will be referred to as the massive Nonsymmetric Gravitational Theory (mNGT) (Légaré and Moffat 1995; Moffat 1995a, b) by requiring that the linearized field equations reduce to those of a massive Kalb-Ramond (1974) field, guaranteeing that the linearized fields are well-behaved asymptotically far from the source (Clayton 1996a). The action for the Massive theory is 

\[ S_{\text{ngt}} = \int d^4x \left[ -g^{AB} R_{AB}^{\text{ns}} - g^{AB} \nabla_{c[A}[W_{B]}] + l^A \Lambda_A \right. \\
+ \left. \frac{3}{4} g^{(AB)} W_A W_B + \frac{1}{4} m^2 g^{[AB]} {\mathcal{G}_{[AB]}}, \right] \]

where the first three terms are identical to the vacuum UFT action, the last term is the Bonnor (1954) term, and the remaining contribution is the new feature of mNGT designed to obtain the correct linearisation. (Note that the Lagrange multiplier \( l^A \) is not a new feature in the NGT action; if one uses the connection as defined by Moffat (1991), it would not appear.)

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The convention of this work is to define the inverse of the (in general non-symmetric: $g_{AB} \neq g_{BA}$) fundamental tensor by $g^{AB} g_{BC} = g^{CB} g^{BA} = \delta^A_C$. Densities with respect to the determinant of the fundamental tensor $g := \det[g_{AB}]$ will be indicated by boldface. $T^{AB} := \sqrt{-g} R^{AB}$, and the symmetric and antisymmetric part of a tensor will be indicated by $T_{(AB)} := (T_{AB} + T_{BA})/2$ and $T_{[AB]} := (T_{AB} - T_{BA})/2$ respectively. A torsion-free covariant derivative is employed, characterised by the connection coefficients that define the parallel transport of the frame by $\nabla_{e^A}[e]^B_B = \Gamma^C_{AB} e^C$ (which is not required to be compatible with any tensor), and what are normally treated as the antisymmetric components of the connection coefficients are considered as a separate tensor $\Lambda^A_{BC}$, the trace of which is defined by $\Lambda_A := \Lambda^A_{AB}$. The action (1) has been written in a general linear frame, where the basis vectors are related to a coordinate basis via a vierbein $e_A = E_A^\mu \partial_\mu$, generally leading to nontrivial structure constants $[e_A, e_B] = C_{AB}^C e_C$.

The NGT Ricci tensor that appears in the action can be split up into two contributions: $R^{\text{pos}}_{AB} = R_{AB} + R^A_{AB}$; the first is identified as the Ricci tensor (i.e., it reduces to the GR Ricci tensor in the limit of vanishing antisymmetric sector), and the second contains contributions from the antisymmetric tensor field $\Lambda^A_{BC}$

\[
R_{AB} = e_C [\Gamma^C_{BA}] - e_B [\Gamma^C_{CA}] - \frac{1}{2} e_A [\Gamma^C_{BC}] + \frac{1}{2} e_B [\Gamma^C_{AC}] + \Gamma^E_{BA} \Gamma^C_{CE} - \Gamma^E_{CB} \Gamma^C_{EA} + \Gamma^E_{[AB]} \Gamma^C_{EC},
\]

\[
R^A_{AB} = \nabla_{e^B} [\Lambda]^B_A + \nabla_{e^A} [\Lambda]_B + \Lambda^C_{AD} \Lambda^D_{BC}. \tag{2a}
\]

In addition to the vanishing of the torsion tensor

\[
T^A_{BC} := \Gamma^A_{BC} - \Gamma^A_{CB} - C^A_{BC} = 0, \tag{3a}
\]

the field equations of mNGT take on the form (Clayton 1996a):

\[
\Lambda_A = 0, \tag{3b}
\]

\[
\Gamma^A = \frac{1}{2} g^{(AB)} W_B, \tag{3c}
\]

\[
\nabla_{e^B} [g^{AB}] = \frac{1}{2} g^{AB} W_B, \tag{3d}
\]

\[
\nabla_{e^C} [g^{AB}] - g^{DB} A^A_{CD} - g^{AD} A^B_{DC} + \frac{1}{4} \epsilon^C_{DE} [\Lambda]^D_B g^{(DE)} W_E = 0, \tag{3e}
\]

\[
R^{\text{ns}}_{AB} + \nabla_{e^A} [W]_B - \frac{1}{8} W_A W_B - \frac{1}{4} m^2 M_{AB} = 0, \tag{3f}
\]

where the mass tensor appearing in the last of these is defined by

\[
M_{AB} = g_{[AB]} - g_{CA} g_{BD} g^{[CD]} + \frac{1}{2} g_{BA} g^{[CD]} g_{CD}. \tag{4}
\]

By requiring that the linearisation of the antisymmetric sector take on the form of a Kalb-Ramond field, the field equations have picked up three additional constraints at the linearized level, which have since been shown to be absent in the nonperturbative theory (Clayton 1995, 1996b). The purpose of the present work is to examine the form of the exact field equations (3) near initially GR field configurations, demonstrating the fact that the exact evolution of the system is not well approximated by the linearized system. It will therefore have been shown that mNGT is not linearization stable about GR spacetimes, and so the linearization which guided the construction of the action does not describe the theory even in an approximate sense. Furthermore, it will be shown that unless an ad hoc constraint is imposed on the initial data hypersurface, unboundedly large velocities occur for arbitrarily small antisymmetric sector fields. This indicates a Cauchy instability (as defined in Hawking and Ellis (1973)), since the
field configuration at an infinitesimally small time later will not depend smoothly on the initial data near a GR configuration.

The paper will consist of two main sections: the first will review the field equations of mNGT in 3 + 1 decomposed form, and the second will explore the above issues, making it clear how and why the naive linearization fails. The discussion of the surface decomposition of mNGT is fairly terse, and further details about the formalism are given in (Clayton 1995, 1996b). This construction is a straightforward extension of that given by Isenberg and Nestor (1980), although it must be admitted that the structure of NGT introduces a great deal of algebraic complication that necessitates the introduction of many auxiliary fields in order to present results in a reasonably compact form. In addition, in order to work consistently with a surface adapted basis in NGT, it was necessary to develop a ‘moving frame’ formalism for NGT (Clayton 1996a).

2. The Surface-Decomposed Field Equations

What will be given here is a summary of the space and time decomposition that was used to consider the Cauchy problem in NGT (Clayton 1995, 1996b). Spacetime has been foliated into spacelike hypersurfaces \((\mathcal{M}, g) \sim \{(\Sigma_t, \hat{g}_t) | t \in I \subset \mathbb{R}\}\), where the metric of spacetime has been operationally identified with the symmetric components of the inverse of the fundamental tensor of NGT \(g^{-1}(\ab)\). Configuration space has therefore been chosen to consist of \(g[a\perp] \) and \(g_{ab}\), where \(\{a, b, \ldots\} \in \{1, 2, 3\}\) indicate components of a tensor on \(\Sigma\) expanded in a coordinate basis, and the index ‘\(\perp\)’ indicates a component normal to the hypersurface as defined by \(g(\ab)\).

This surface compatible basis is designed so that the symmetric components of the inverse of the fundamental tensor take on the ADM form (Arnowitt et al 1959):

\[
g^{-1} = e_\perp \otimes e_\perp - \gamma(ab) e_a \otimes e_b,
\]

where the basis may be written in terms of the coordinate basis \((\partial_t, \partial_a)\) as: \(e_\perp = \frac{1}{N} \partial_t - \frac{N^a}{N} \partial_a\), \(e_a = \partial_a\), and the covector bases as: \(\theta^\perp = N dt\), \(\theta^a = dx^a + N^a dt\). The time vector is decomposed as \(t = Ne_\perp + \hat{N}\), where \((N, N^a)\) are the lapse function and shift vector respectively, and the notation \(\hat{N} := N^a e_a\) has been employed. The non-vanishing structure constants are easily found to be:

\[
C_{\perp a} = e_a [\ln(N)], \quad C_{\perp a}^b = \frac{1}{N} e_a [N^b],
\]

and the fundamental tensor and its inverse become:

\[
g = F \theta^\perp \otimes \theta^\perp - (\alpha_a - \beta_a) \theta^\perp \otimes \theta^a - (\alpha_a + \beta_a) \theta^a \otimes \theta^\perp - G_{ab} \theta^a \otimes \theta^b,
\]

\[
g^{-1} = e_\perp \otimes e_\perp + B^a e_\perp \otimes e_a - B^a e_a \otimes e_\perp - \gamma^{ab} e_a \otimes e_b,
\]

where:

\[
F := 1 + G_{(ab)} B^a B^b, \quad \alpha_a := G_{[ab]} B^b, \quad \beta_a := G_{(ab)} B^b,
\]

from which one finds that \(\alpha_a B^a = 0\). The spatial part of the fundamental tensor is given by

\[
(\gamma^{ac} - B^a B^c) G_{cb} = G_{bc} (\gamma^{ac} - B^c B^a) = \delta^a_b.
\]

Note that invertibility of the fundamental tensor in \((6a)\) is required in order that the volume element in the action be nondegenerate, which in turn requires that \(G_{ab}\) exist.
The spacetime covariant derivatives have been decomposed as usual (Isenberg and Nester 1980) into surface covariant derivatives (written as $\nabla^{(3)}_a$) whose action on basis vectors in $T\Sigma$ (the space of vectors tangent to $\Sigma$) is given by $\nabla^{(3)}_a e_b = \Gamma^{(3)}_{ab} e_c$, and contributions from derivatives off of $\Sigma$ which will be written in terms of the surface tensors:

$$\Gamma := \Gamma_{\perp\perp}, \quad e_a := \Gamma_{a\perp}, \quad a_a := \Gamma_{a\perp}, \quad \sigma^a := \Gamma^a_{\perp\perp},$$

(7a)

In GR, (7a) are related by the algebraic compatibility conditions: $\sigma^a = \gamma^{ab}a_b, e_a = 0, \Gamma = 0, \text{and } u^a_b = \gamma^{ac}k_{bc}$, as well as the conditions that enforce the vanishing of torsion: $a_a = C_{\perp a}, w^a_b = u^a_b + C_{\perp b}^a, \Gamma^a_{[bc]} = 0, \text{and } k_{[ab]} = 0$. The antisymmetric tensor $\Lambda^A_{BC}$ is decomposed as:

$$b_a := \Lambda_{a\perp}, \quad j_{ab} := \Lambda_{ab\perp}, \quad v^a_b := \Lambda^a_{b\perp}, \quad \lambda^a_{bc} := \Lambda^a_{bc},$$

(7b)

where by definition $j_{ab}$ is an antisymmetric surface tensor. The vector field $W_A$ that appears in (1) is decomposed as $W_A = (W, W_a)$, and the traceless part of $\lambda^a_{bc}$ is defined by $\Lambda^2_{Tbc} = \Lambda^a_{bc} - \delta^a_{[b} b_{c]}$. It is also useful to make the following definitions:

$$K^a_{ab} := k^a_{ab} + j_{ab},$$

(8a)

$$k^a_{b} := \frac{1}{2}(\gamma^{ac}K^b_{ac} + \gamma^{ca}K^b_{ca}) = \gamma^{[ac]}k^b_{bc} + \gamma^{[ac]}j^b_{bc},$$

(8b)

$$j^a_b := \frac{1}{2}(\gamma^{ac}K^b_{ac} - \gamma^{ca}K^b_{ca}) = \gamma^{[ac]}k^b_{bc} + \gamma^{[ac]}j^b_{bc},$$

the first of which is defined for notational convenience, and the remaining are symmetric and antisymmetric sector contributions from tensors that have had indices ‘raised’ by $\gamma^{ab}$.

In the Hamiltonian formalism of mNGT (Clayton 1995, 1996b) the canonical momenta are weakly equivalent to the densitized components of the fundamental tensor appearing in (6a), and the set of conjugate pairs of phase space coordinates are

$$\{(B^a, \overline{W}_a), (\gamma^{ab}, K_{ab})\},$$

(9)

where the field

$$\overline{W}_a := -W_a + 2h_a,$$

(10)

naturally appears as as the momentum conjugate to $B^a$. (In Clayton 1995, 1996b) there is an additional pair of symmetric sector coordinates that does not exist in typical Cauchy analyses of GR which is removable by making imposing the second class constraints: $p \approx \sqrt{-g}(B^a, \gamma^{ab}), \text{and } u \approx \gamma^{ab}K_{ab}$ strongly.)

The fact that the spacetime covariant derivative has been defined to be torsion-free (3a) as in GR results in $k_{[ab]} = \Gamma^c_{[ab]} = 0$, now combined with:

$$a_a \approx c_a + C_{\perp a}, \quad w^a_b := u^a_b + C_{\perp b}^a,$$

(11)

and equation (3b) gives:

$$v^a_a \approx 0, \quad b_a \approx -\lambda^b_{ab}.$$
The symmetric sector Lagrange multipliers that appear in (7) may be solved for in terms of the antisymmetric sector Lagrange multipliers and Cauchy data as:

\[
\Gamma \approx B^a b_a - \frac{1}{5} \gamma^{(ab)} \beta_a (\nabla_b - 2b), \\
\sigma^a \approx \gamma^{(ab)} a_b - B^b v_b^a - \gamma^{[ab]} b_b, \\
c_a \approx -j_{ab} B^b + \frac{1}{5} \beta_a W + \frac{1}{8} G_{[ab]} \gamma^{(bc)} (\nabla_c - 2b_c), \\
u_b^a \approx B^a b_b + B^c \lambda_{ab}^c + k_b^a,
\]

and using the solution for \( v_b^a \) given in (13c) one finds the surface compatibility condition that determines \( \Gamma^a_b \) in terms of Cauchy data:

\[
\nabla_c^{(3)} \left[ \gamma^{(ab)} \right] \approx \gamma^{(ab)} c_c + 2B^a v_b^a + \gamma^{[ad]} \lambda_{ad}^b + \gamma^{[db]} \lambda_{db}^a. 
\]

(13b)

(The Lagrange multipliers \( v_b^a \) must be replaced in (13b) in order to solve for the surface connection coefficients, since \( v_b^a \) in (13c) has explicit dependence on \( \Gamma^a_b \).) In the antisymmetric sector one can solve for:

\[
W \approx \frac{4}{3} \nabla_a^{(3)} [B]^a, \\
v_b^a \approx \nabla_b^{(3)} [B]^a + j^a - \frac{1}{3} \delta^a_b W,
\]

leaving

\[
\nabla_c^{(3)} \left[ \gamma^{[ab]} \right] \approx -2B^a u_c^b + \gamma^{[ab]} c_c + \gamma^{(ad)} \lambda_{ad}^b + \gamma^{[db]} \lambda_{db}^a + \frac{1}{2} \delta^a b \gamma^{[bc]} (\nabla_c - 2b_c).
\]

(13d)

will be necessary in order to demonstrate the result. It is this rather surprising ability to isolate a relation that determines \( b_a \) in terms of Cauchy data that allows the analysis in the next section to be performed in a fairly straightforward manner.

It will be useful to define the two tensors:

\[
O_{1ab}^b := \gamma^{(ab)} - B^a B^b, \quad O_{2ab}^b := B^a B^b - \frac{1}{4} \gamma^{[ac]} G_{[cd]} \gamma^{(db)},
\]

(15a)

the inverses of which will be denoted \( O_{1ab}^{-1} \) and \( O_{2ab}^{-1} \), respectively, given in general by:

\[
O_{1ab}^{-1} = S_{ab} + \frac{B_a B_b}{1 - B \cdot B}, \\
O_{2ab}^{-1} = \frac{1}{\gamma \cdot G} \left[ S_{ab} - \frac{B_a G_b + G_a B_b}{G \cdot B} + \frac{B \cdot B + \gamma \cdot G}{(G \cdot B)(G \cdot B)} G_a \gamma_b \right].
\]

(15b)

Indices have been ‘raised’ or ‘lowered’ by \( \gamma^{(ab)} \) and its inverse \( S_{ab} \) (for example, \( G_a := S_{ab} G^b \), and \( \gamma^{(ab)} S_{bc} = \delta^a_c \)), and the notation \( B \cdot G := B^a G_a \) has been employed. The 3-dimensional antisymmetric symbol \( \epsilon_{abc} \) and the surface volume element \( \sqrt{S} := \sqrt{\det S_{ab}} \) have been used to define:

\[
\gamma^{[ab]} = \frac{2}{\sqrt{S}} \epsilon^{abc} \gamma_c, \quad \gamma_a := \frac{1}{4} \sqrt{S} \epsilon_{abc} \gamma^{bc}, \\
G_{[ab]} = 2\sqrt{S} \epsilon_{abc} G^c, \quad G^a := \frac{1}{4\sqrt{S}} \epsilon^{abc} G_{bc},
\]

(16)
where \( \epsilon_{123} = \epsilon^{123} = +1 \). Clearly in the limit of vanishing antisymmetric sector, \( O^{-1}_{2ab} \) in (15b) is not in general well-behaved due to the presence of \( (\gamma \cdot G)^{-1} \) which is singular in that limit.

Using these and the definition
\[
\frac{1}{2} \sqrt{-g} \Xi := e_b [\gamma^{[ab]} + k^b_c B^a - k^a_c B^b + \gamma^{[ab]} j_{bc} B^c - \frac{1}{6} \gamma^{[ab]} \beta_b \nabla_c \gamma^{(3)} (B) |c|, \tag{17}
\]
(14) is written as
\[
O_{2b}^a b_b \approx \frac{1}{2} O_{2b}^a \nabla_b + \frac{1}{4} O_{2ab} \nabla_c + \frac{1}{2} \Xi^a, \tag{18a}
\]
the solution of which is (in terms of the phase space coordinates)
\[
b_a \approx \frac{1}{2} \nabla_a + \frac{1}{2} O_{2ab} \nabla_c + \frac{1}{4} O_{2ab} \Xi^b. \tag{18b}
\]
In terms of the configuration space variables, equation (14) becomes
\[
O_{2b}^a b_b = \frac{1}{2} O_{2b}^a W_b + \frac{1}{4} O_{2ab} W_c + \frac{1}{2} \Xi^a, \tag{18c}
\]
where \( W_a \) must be solved for in terms of time derivatives of the configuration space variables \( (B^a, \gamma^{ab}) \) in (19a) below. The solution of (18c) may then be determined to be
\[
b_a = \frac{1}{2} W_a + \frac{1}{2} O_{2ab} W_c + \frac{1}{4} O_{2ab} \Xi^b, \tag{18d}
\]
resulting in
\[
\nabla_a = -W_a + 2 b_a = O_{1ab} O_{2a} \Xi^b + O_{1ab} \Xi^b. \tag{18e}
\]

The evolution of the canonical momenta as determined from Hamilton’s equations or, equivalently, from the compatibility conditions in (3c) that involve time derivatives of the fundamental tensor, is given by:
\[
\dot{B}^a \approx \frac{1}{2} \nabla (\gamma^{[ab]} + \gamma^{[ab]} \alpha b) \nabla b + \frac{1}{2} \gamma^{(ab)} B (\nabla b - 2 b)), \tag{19a}
\]
\[
\dot{\gamma}^{(ab)} \approx \frac{1}{2} \nabla (\gamma^{(ab)} + \gamma^{(ab)} (\Gamma + u) - 2 (\gamma^{(c[ab]} u^c) - \gamma^{(c[ab]} u^c)), \tag{19b}
\]
\[
\dot{\gamma}^{[ab]} \approx \frac{1}{2} \nabla (\gamma^{[ab]} - N (B^a \sigma b - B^b \sigma a - \gamma^{[ab]} (\Gamma + u) - 2 (\gamma^{(c[ab]} u^c) - \gamma^{(c[ab]} u^c)), \tag{19c}
\]
where \( \dot{L}^{(3)} \) is the surface Lie derivative. Partial derivatives with respect to the time coordinate have been represented by an overdot (e.g. \( \dot{B}^a \)), and are equivalent to the properly defined Lie derivative off of the surface (Isenberg and Nester 1980; Clayton 1995, 1996b). The evolution of the coordinates (from Hamilton’s equations or (3f)) is determined to be:
\[
\nabla a \approx \frac{1}{2} \nabla [\nabla a - 2 N Z_{(a \perp)} + \frac{1}{2} m^2 N M_{(a \perp)}, \tag{20a}
\]
\[
K_{ab} \approx \frac{1}{2} \nabla [K_{ab} - N Z_{ab} + \frac{1}{2} m^2 N M_{ab}, \tag{20b}
\]
where \( Z_{AB} \) as determined from (3f) are given by:
\[
Z_{(ab)} := R_{(ab)} - \nabla_{(a} [a]_{b) + (\Gamma + u) k_{ab} - a_{a} a_{b} - k_{cb} u_{a} - k_{cb} u_{b} \]
\[
+ b_{a} b_{b} - j_{ab} v_{b} - j_{bc} v_{a} + \lambda_{ad} \lambda_{bc} - \frac{3}{8} (W_a - 2 b_a) (W_b - 2 b_b), \tag{21a}
\]
\[
Z_{[a \perp]} := \frac{1}{2} \nabla [W_a + \frac{1}{2} W (a_{a} - c_{a}) - \nabla [\gamma^{(ab)} \alpha_{a}], \tag{21b}
\]
\[
- j_{ab} v_{b} + (c_{b} - a_{b}) v_{a} + u_{a} b_{b} - u_{b} b_{b} - u_{b} c_{ab}, \tag{21b}
\]
\[
Z_{[ab]} := - \nabla [W a [b]_{b}] + 2 \nabla [W a [b]_{b}] + \nabla [\gamma^{(ab)} \alpha_{a}], \tag{21b}
\]
\[
+ a_{b} b_{a} - a_{a} b_{a} - j_{cb} u_{a} - j_{cb} v_{a} - k_{ca} v_{b} + k_{cb} v_{a}. \tag{21b}
\]
In (21a) the surface Ricci tensor is defined from (2a) to be
\[ R^{(3)}_{ab} = e_c [\Gamma^b_{ca}] - \frac{1}{2} (e_b [\Gamma^c_{ba}] + e_a [\Gamma^b_{ca}]) + \Gamma^c_{ba} \Gamma^d_{dc} - \Gamma^d_{da} \Gamma^c_{bc}, \] (22)
which is constructed from contractions of the intrinsic curvature of \( \nabla^{(3)} \) on \( \Sigma \).

The remaining field equations in (3f) are the Gauss relation or Hamiltonian constraint (Clayton 1995, 1996b)
\[ \mathcal{H} \approx -(w - w_a u^a_b + v^a_b v^b_a + \gamma^{(ab)} R^{(3)}_{ab}) + \frac{1}{2} M_{\perp} + \gamma^{ab} M_{ab} \]
\[ - \gamma^{(ab)} (\lambda_{\alpha \beta}^{(a)} \lambda_{\gamma \alpha}^{(b)} - \frac{2}{5} W^a W_b + \frac{3}{2} W^b b_b) + \frac{3}{10} \sqrt{-g} (\mathcal{W})^2 \]
\[ - \gamma^{[ab]} (-\nabla^{(3)} W)_b + 2 \nabla^{(3)} [B]_b + \nabla^{(3)} [\lambda]_b - 2 B B^a a_a \]
\[ + \nabla^{(3)} [B]_b + \gamma^{(ab)} b_b - c_{ab} (B B^a B^b + \gamma^{(ab)} b_b) \approx 0, \] \hspace{1cm} (23a)
and the Codacci relations or momentum constraints
\[ \mathcal{H}_a \approx B^b a_a [W_b] - e_b [B^b W_a] + \gamma^{bc} e_c [K_{bc}] + \sqrt{-g} e_a [u] - 2 e_b [k_b^a] \approx 0. \] \hspace{1cm} (23b)
Diffeomorphism invariance of the mNGT action guarantees that the Poisson brackets of the Hamiltonian and momentum constraints satisfy the same closing relations as GR (Teitelboim 1973, 1980; Isenberg and Nester 1980), where the fact that \( \gamma^{(ab)} \) appears in the Poisson bracket of the Hamiltonian constraint is a consequence of the choice of using \( g^{(AB)} \) to define the unit normal to \( \Sigma \), and is discussed in more detail in (Clayton 1996b).

One now has the exact evolution equations of mNGT, written in surface decomposed form that is suitable for considering the Cauchy problem in more detail. As a first-order system, one would first solve (13) for the Lagrange multipliers replacing them everywhere in Hamilton’s equations (19) and (20) and the constraints (23). There would then be four undetermined Lagrange multipliers (\( N \) and \( N^a \)) and four diffeomorphism constraints (23) on the Cauchy data \( \bar{0}^B \). To consider the system as second-order, one would go further and solve (19) for the canonical coordinates \( (\bar{W}_a, K_{ab}) \), replacing them in the constraints (23) as well as (20), the collection of which would then become the second-order Euler-Lagrange equations of mNGT.

### 3. Linearization Instability

Considering the complexity of the field equations, it would be somewhat too optimistic to attempt to draw definitive conclusions about global solutions of the mNGT field equations (see Moffat (1995c) for some results). Instead it will be sufficient to consider the instantaneous problem here, considering Cauchy data for which the antisymmetric sector will be chosen to be an arbitrarily small perturbation of the symmetric sector. Normally one would expect that that the perturbation of the initial data would lead to evolution of the system that may be considered (at least for small enough times) as a perturbation of GR evolution. All contributions from the antisymmetric sector to the symmetric sector should appear at second-order, and the evolution equations would not normally drive the perturbations to become large; indeed, this is precisely what the naive linearization of the system leads one to believe.

Instead one finds in general that the exact field equations give contributions to the symmetric sector that appear at background order (i.e., not as a small correction, but of the same order as GR effects), and the field equations that determine the evolution
of the perturbations result in arbitrarily large velocities. This not only invalidates the use of the linear approximation, but also explicitly shows a Cauchy instability in mNGT.

To begin, we will show how the results of the linearizations given by Moffat (1995a, b) and Clayton (1996a) (about a fixed GR background) may be retrieved by a naive (and incorrect) assumption on the Lagrange multiplier fields in the Hamiltonian picture, or by dropping acceleration terms in the antisymmetric sector field equations that appear at higher-order, thereby revealing a constraint that does not properly exist in the theory.

The relevant field configurations are those in which the antisymmetric components of the fundamental tensor may be considered as perturbations of dominant GR (symmetric sector) components on the initial data hypersurface \( \Sigma_0 \). Thus, an expansion in powers of the antisymmetric components \((B^a, \gamma^{[ab]}))\) about the symmetric components \(\gamma^{(ab)}\) will be made, in which the components of the fundamental tensor take on the form:

\[
\begin{align*}
\sqrt{-g} &\sim \sqrt{\tilde{g}}, \\
\beta_a &\sim B_a := S_{ab} \gamma^b, \\
\alpha_a &\sim -\gamma_{[ab]} B^b, \\
G_{(ab)} &\sim S_{ab}, \\
G_{[ab]} &\sim -\gamma_{[ab]} := -S_{ac} S_{bd} \gamma^{[cd]},
\end{align*}
\]

Terms of order \( n \) in \((B^a, \gamma^{[ab]}))\) will be indicated by \(\mathcal{O}(\text{skew}^n)\), spatial indices will be raised and lowered using \(\gamma^{(ab)}\) and its inverse \(S_{ab}\), and \(\sim\) will indicate the dominant contribution in powers of \((B^a, \gamma^{[ab]}))\).

The assumption is that all of the antisymmetric sector tensors \((B^a, \tilde{W}_a, \gamma^{[ab]}, j_{ab}, \tilde{W}_b, v^a_b, \lambda^a_{bc})\) are \(\mathcal{O}(\text{skew}^1)\), therefore leaving the dominant \(\mathcal{O}(\text{skew}^0)\) terms of the symmetric sector Lagrange multipliers identical to those of GR:

\[
\Gamma \sim 0, \quad \zeta_a \sim 0, \quad \sigma^a \sim \gamma^{(ab)} a_b, \quad u^a_b \sim k^a_b \sim \gamma^{(ac)} k_{bc}, \quad \nabla_\gamma^{(3)} \sim 0, \quad (25a)
\]

and those in the antisymmetric sector:

\[
\begin{align*}
W &\sim \frac{1}{2} \nabla^{(3)} [B]_a, \\
v^a_b &\sim \nabla^{(3)} [B]^a - \frac{1}{2} \delta^a_b \nabla^{(3)} [B]^c + j^a_b, \\
\lambda^{ab} &\sim -\frac{1}{2} \gamma^{(cd)} (\nabla^{(3)} [\gamma]_{[ad]} + \nabla^{(3)} [\gamma]_{[bd]} - \nabla^{(3)} [\gamma]_{[ab]}),
\end{align*}
\]

\[
(25b)
\]

The trace of the last of these results in the constraints:

\[
\chi^a_1 \approx \frac{1}{2} \gamma^{(ab)} \tilde{W}_b + \nabla^{(3)} [\gamma]_{[ab]} + k B^a - k^a_b B^b \approx \frac{1}{2} \gamma^{(ab)} \tilde{W}_b + \frac{1}{2} \tilde{\zeta}^a \sim 0,
\]

which are equivalent to (14) after dropping all contributions higher than \(\mathcal{O}(\text{skew}^1)\), and where the linear order contribution to \(\tilde{\zeta}^a\) in (17) has been identified as \((1/2) \tilde{\zeta}^a \sim \nabla^{(3)} [\gamma]_{[ab]} + k B^a - k^a_b B^b\).

Note that in dropping the contributions from the Lagrange multiplier \(b_a\), equation (14) has been required to play a very different role in the linearized system than in the nonperturbative case. Whereas in the exact treatment, the relation (14) is a condition determining the Lagrange multiplier \(b_a\), in this linearization it has become a constraint on the Cauchy data, and \(b_a\) left at this stage as an undetermined Lagrange multiplier. As \(\chi^a_1\) are constraints, they must be preserved in time (to this order) for the evolution of the system to be consistent. It is straightforward to show that all contributions to \(\chi^a_1\) involving \(b_a\) cancel, resulting in an additional three constraints \(\chi^a_2 := \chi^a_1 \sim 0\).
Using the constraint (24) to determine $\bar{W}_a$, the linear contribution from (19a) clearly depends on $b_a$, as does (20a), as it takes on the form

$$\bar{W}_a \approx \mathcal{L}^{(3)}_N [\bar{W}]_a - 2N^1 Z_{[a\perp]} - m^2 N B_a,$$

(27)

where the linear order contribution to (21b) is ($R_{ab}^{(3)}$ is the $O$(skew$^3$) GR Ricci tensor)

$$1Z_{[a\perp]} \sim - R_{ab}^{(3)} B^b + k_a b_b c B_c - B_a k_b c + k_a b^b \gamma^{[bc]} a_c - 2 f_{ab} a^b$$

\[+ \frac{1}{2} k_b^b \bar{\omega}_b - 3 k_b^b a b - (\nabla a)^{[b} B^c - \delta^b_a \nabla^{[b} [B] c) a_b \]

\[- \gamma^{(bc)} \nabla^{(3)} [j] a_c + \nabla^{(3)} [k b_b \gamma^{[bc]}] + k^{bc} \nabla^{(3)} [\gamma]_{[ca]}.\]

(28)

Considering (20) as second-order field equations for the configuration space variables, (24) determines $b_a$ as

$$b_a \sim -\frac{1}{2} W_a - \frac{1}{2} \Xi_a;$$

(29)

however, inserting this into (20a) yields the algebraic constraint (which is in fact equivalent to $\chi_{2a} \sim 0$)

$$\bar{W}_a \sim -\frac{1}{2} \Xi_a \sim N^1 Z_{[a\perp]} - 2N^1 Z_{[a\perp]} - m^2 N B_a,$$

(30)

instead of the expected evolution equation. (The field $W_a$ is easily determined in terms of $\bar{B}^a$ using the linear contribution from (19a), and the time derivatives on $k_{ab}$ that occur in $\Xi_a$ are removed using (20b).) The equations (31) then play a similar role as the time-space components of the field equations in the massive Kalb-Ramond theory: $\partial_a F^{[a\perp]} + m^2 h^{[0a]} = 0$, which do not involve acceleration terms and are therefore constraint equations.

One could take this system seriously and determine whether the constraint algebra closes properly by computing $\chi_a^2$, presumably leading to a determination of $b_a$ instead of further constraints. This would recover a system analogous to the massive Kalb-Ramond field, and, consequently, the linear approximation of Moffat (1995a, b) and Clayton (1996a). A more systematic description has not been given since, as we shall see, this system is not an accurate representation of weak antisymmetric sector dynamics. It will be shown that no additional constraints properly exist, and the majority of choices of arbitrarily small antisymmetric sector Cauchy data result in singular evolution. Avoiding such configurations is possible by a particular choice of initial data, but in that case the system does not evolve in a manner consistent with the above linearization.

Considering the exact second-order system, (19a) may be solved exactly for $W_a$ in terms of Cauchy data as

$$W_a = \frac{4}{3} \frac{1}{N \sqrt{-g}} S_{ab} (\bar{B}^b - \mathcal{L}^{(3)}_N [\bar{B}]^b + N e_c [\gamma]^{[bc]} + N [\gamma]^{[bc]} C_{\perp c} {\Lambda}^c),$$

(31)

which, combined with (18d), may be used in order to determine the exact form of the acceleration terms appearing in (20a) to be

$$\bar{W}_a \sim \frac{4}{3} \frac{1}{N \sqrt{-g}} O_{ab}^{[bc]} S_{cd} \bar{B}^d;$$

(32)

the time derivatives on $k_{ab}$ and $j_{ab}$ that appear are once again removed by making use of (20b). Closest to lowest order $O_1^{ab} \sim \gamma^{(ab)}$, and so its inverse is $O_{1ab}^{-1} \sim S_{ab}$. The operator $O_2^{ab}$ in (15a) on the other hand, is $O$(skew$^2$)

$$O_2^{ab} \sim B^a \bar{B}^b + \frac{1}{4} \gamma^{[ac]} \gamma_{[cd]} \gamma^{(db)} = B^a \bar{B}^b - \gamma^{(ab)} \gamma + \gamma^a \gamma^b,$$

(33a)
where (24) has been used in order to deduce that $G_a \sim -\gamma_a$ in (16), also leading to the form of the inverse from (15b)

$$O_{2ab}^{-1} \sim \frac{1}{\gamma \cdot \gamma} \bigg[ \gamma_{(ab)} - \frac{1}{\gamma \cdot B} (B_a \gamma_b + \gamma_a B_b) + \frac{B \cdot B - \gamma \cdot \gamma}{(\gamma \cdot B)^2} \gamma_a \gamma_b \bigg],$$  \hspace{1cm} (33b)

which is clearly $\mathcal{O}(\text{skew}^{-2})$. The presence of the operator $O_{2ab}^{2b}$ in (32) immediately shows how these acceleration terms disappear in the above linearized analysis, since it causes the acceleration terms in (23) to appear at third-order, leaving these field equations as constraints which do not truly exist in the theory.

Furthermore (32) will result in very poorly behaved accelerations near configurations where the antisymmetric sector Cauchy data is small unless an $ad~hoc$ condition is imposed on the initial data, leaving a nontrivial wave equation for $B^a$. To see this, (20b) may be written to $\mathcal{O}(\text{skew}^3)$ as:

$$\frac{4}{3} \frac{1}{N \sqrt{S}} O_{2ab} \ddot{B}^b_\perp + \dot{\mathcal{Z}}_a \sim \mathcal{L}^{(3)} \bigg| \mathcal{Z}_a \bigg| - 2N \frac{1}{2} \mathcal{Z}_{(a \perp)} - N m^2 B_a + 3 \Psi_a,$$  \hspace{1cm} (34)

where $^3\Psi_a$ are the remaining non-acceleration contributions to the field equations. The acceleration of $B^a$ is therefore given by

$$\ddot{B}^a \sim \frac{4}{3} N \sqrt{S} O_{2ab}^{-1} \dot{B}^b_\perp (-2 \chi_{2b} + ^3\Psi_b),$$  \hspace{1cm} (35)

where $\chi_{2a}$ has been identified from the time evolution of (26). Due to the presence of $O_{2}^{-1} a b$ in (32), the right hand side is generally $\mathcal{O}(\text{skew}^{-1})$, resulting in unboundedly large accelerations resulting from arbitrarily small antisymmetric sector Cauchy data on $\Sigma_0$. (Note that the sign of the right hand side of (35) is uncorrelated with that of $B^a$ in general, since the sign of $O_{2}^{-1} a b$ may be chosen by adjusting the relative magnitudes of $B^a$ and $\gamma_{(a b)}$ on $\Sigma_0$.) To avoid this, one could require that the initial data satisfy a condition that mimics the linearized result (26)

$$\chi_{2a} \sim \frac{3}{4} \Psi_a,$$  \hspace{1cm} (36)

effectively constraining the linear order contributions to (32) to vanish up to terms of higher-order. However even in this case, (35) becomes

$$\ddot{B}^a \sim \frac{4}{3} N \sqrt{S} O_{2}^{-1} a b \bigg( \frac{3}{4} \Psi_b + ^3\Psi_b \bigg),$$  \hspace{1cm} (37)

which is a nontrivial $\mathcal{O}(\text{skew})$ evolution equation for $B^a$ that is not reproduced by the naive linearization. (Note that the second order spatial derivatives that combine with the time derivatives of $B^a$ to make (20a) a hyperbolic wave equation have also dropped out of $Z_{(a \perp)}$ at linear order in (28), presumably reappearing at third order to give (37) the appropriate hyperbolic form.)

From the first-order point of view, the relation (18b) determines the Lagrange multiplier $b_a$ in all cases when the operator $O_2^{2b}$ is nondegenerate, including weak antisymmetric sector field configurations. (If $O_2^{2b}$ is degenerate then it may only be inverted on some subspace, determining some components of $b_a$ and leaving the rest undefined. The analysis of these cases would be rather more complicated and will not be discussed further.) Since, as can be readily seen from (15b), $O_2^{-1} a b$ is $\mathcal{O}(\text{skew}^{-2})$ and therefore generally becomes singular for vanishingly small antisymmetric sector, the presence of $b_a$ in (19a) would drive $B^a$ to evolve arbitrarily quickly as one considers
vanishingly small antisymmetric sector Cauchy data. Explicitly, the last term in \(19a\) may be replaced using \(18b\) leaving

\[
\dot{B}^a \approx \mathcal{L}^{(3)}_N[B]^a - N(\nabla^{(3)}[\gamma^{[ab]} + \gamma^{[ab]}C_{\perp b}]) + \frac{3}{4} N \gamma^{(ab)} O_{abc}^{-1} (O_{ab} W_c + \Xi^c). \tag{38}
\]

The presence of \(O_{abc}^{-1}\) in the last term indicates that unboundedly large velocities generally occur for infinitesimally small antisymmetric sector Cauchy data on \(\Sigma_0\), and it is easy to see that the presence of \(b_a\) in \(Z[B_a] - N(\nabla(b) + \gamma^{[ab]} C_{\perp b}) + 3 \gamma^{[ab]} O_{abc}^{-1} (O_{ab} W_c + \Xi^c)\).

Once again choosing initial data such that \(36\) is satisfied removes all of these effects, as then \(18b\) becomes (to \(O(\text{skew}^3)\))

\[
b_a \sim \frac{1}{2} W_a + \frac{1}{2} O_{2ab}^{-1} (2 \gamma^b - B^b B^c W_c + 3 \Xi^b), \tag{39}
\]

effectively resulting in \(b_a \sim O(\text{skew}^3)\). (Note that this alone does not guarantee that \(b_a\) vanishes smoothly with the vanishing of the antisymmetric sector, but it is straightforward to choose data such that it does; for example, one could choose \(O_{1ab} W_b + \Xi^a \approx O_{2b} f_b\), where \(f_b\) is an \(O(\text{skew})\) covector field.

Although we have been focusing on the dynamics of \((B^a, W_a)\), it is fairly straightforward to see that unless \(36\) is imposed on \(\Sigma_0\) guaranteeing that \(b_a \sim O(\text{skew}^3)\), the evolution of the symmetric sector will generally not resemble that of GR. The contributions from \(b_a\) to the Hamiltonian constraint and the symmetric sector evolution equations imply that the symmetric sector cannot in general be considered as perturbative corrections to GR dynamics, since there are contributions from the antisymmetric sector that show up at \(O(\text{skew}^0)\). Even if one imposed \(36\) on \(\Sigma_0\), one would have to check that it is preserved in time in order for the system to remain well-behaved in evolution. This calculation would depend strongly on the chosen form of \(3\psi\) and will not be pursued here. Nevertheless, it is clear that if one makes the choice \(36\), the resulting dynamics cannot be well approximated by the linearized system, simply because there are no additional constraints appearing. The evolution equation \(19a\) is not a constraint, and neither is \(20a\), showing that the full six degrees of freedom in the antisymmetric sector propagate even in the weak field regime, contradicting the linearized results. This behaviour in the antisymmetric sector indicates that the linearization given by Moffat (1995a, b) and Clayton (1996a) does not represent the weak-field evolution as determined from the full field equations, and therefore cannot be trusted: mNGT is not linearization stable about GR backgrounds.

It is clear that these instabilities occur whenever the Lagrange multiplier \(b_a\) becomes singular as the perturbations of the Cauchy data vanish. The general form of the solution of \((14)\) specialized to spherically symmetric systems is

\[
b_1 \approx \frac{1}{2} \frac{\gamma^1 W_1}{(B^1)^2} + 2 \frac{k^2}{B^1}, \tag{40}
\]

and one finds that in regions of spacetime where \(B^1\) is vanishingly small (for example, in an exact GR background or in the asymptotic region of an arbitrary asymptotically flat spacetime), perturbations generally cause \(b_1\) to become arbitrarily large, which in turn cause antisymmetric sector velocities to become large. Thus although the Wyman sector solution (Wyman 1950; Cornish 1994) (which is becoming the basis for much of the phenomenology of NGT (Moffat and Sokolov 1995a, b)) assumes that both \(B^1\) and \(W_1\) vanish globally, if one considers perturbations of these fields on \(\Sigma_0\), the above behaviour reappears. Therefore, although the instabilities have been proven to exist for configurations very close to purely GR spacetimes, one expects
that any asymptotically flat spacetime will also suffer them. In particular, since there is no asymptotically flat spacetime in which \((B^1, \mathbf{\Pi}_1)\) is asymptotically nonvanishing (Clayton 1996a), the unique asymptotically flat, static spherically symmetric solution with nontrivial antisymmetric sector (the Wyman solution) will also be unstable against perturbations.

This is essentially the same effect as was found in (Isenberg and Nester 1977), where the effect of gravitational dynamics on the constraints of various derivative coupled vector fields was studied. It was found that constraints on the vector field may be lost when GR is considered as evolving concurrently with the vector field (as opposed to the vector field evolving on a GR background). This manifested itself as an increase in the number of degrees of freedom in the vector field, and singular behaviour in the evolution equations when approaching asymptotically flat spacetimes. There one finds no evidence of this when considering the vector field dynamics on a fixed GR background, which is analogous to the linear approximation here.

Kuchař (1977) has noted that fields such as these that are derivatively coupled to GR (as NGT may be considered to be) may propagate off of the light cone as determined by the spacetime metric, and indeed, there are explicit examples of such behaviour (Buchdahl 1958, 1962; Cohen 1967; Velo and Zwanziger 1969; Aragone and Deser 1971). It is therefore not surprising that the field equations of UFT (and hence NGT) have been found to propagate information in a manner that is inconsistent with a simple causal structure as determined from a single spacetime metric; in fact Maurer-Tison (1959) has found that in UFT there are three such metrics that one must take into account. In UFT (or massless NGT), these metrics were consistent in the sense that there was locally a largest light cone (although the metric that defined this would change depending on the strength of the antisymmetric sector), and in the limit of vanishing antisymmetric sector all three degenerated to the single causal metric of GR (Maurer-Tison 1956). Although these metrics are not known for mNGT at this time, given the fact that the acceleration terms for \(B^a\) have no simple GR limit, it is reasonable to suspect that one of these light cones has been altered so as to destroy the above mentioned consistency (perhaps resulting in a light cone that is degenerate in the GR limit).

4. Conclusions

It has been shown that arbitrarily small antisymmetric sector Cauchy data leads to singular evolution for the majority of possible choices of perturbatively small antisymmetric sector initial data. The results followed from an examination of the exact field equations of the massive nonsymmetric gravitational theory near purely general relativistic field configurations, and has been demonstrated by considering both the first and second order points of view, showing how the naive linearization fails to accurately describe the system, even in those cases where the choice of initial data does not lead to singular evolution. In doing so, it has been shown the constraints that guaranteed good fall-off for the linearized fields do not properly exist even for weak fields. Given this, it seems that the criterion for choosing the form of the mNGT action (Clayton 1996a) has not truly been fulfilled, and it remains unclear whether one has truly made an improvement over the original formulation of NGT.

The failure of linearization stability has been noted in GR for closed spaces (Brill and Deser 1973). Here it has been found that not only can the linearized system not be trusted, but also that what appear to be benign perturbations of particular initial
data (i.e., as having very little effect on the evolution of the system as a whole) result in very different evolution, in which some of the antisymmetric sector fields are given arbitrarily large velocities. This is clearly interpretable as a Cauchy instability in the usual sense, since the evolution of generic configurations that are arbitrarily ‘close’ to a GR spacetime (or part thereof) does not smoothly depend on the initial data, and makes it difficult to physically interpret such spacetimes. Denying the physical importance of these configurations amounts to labelling as unphysical purely GR spacetimes and asymptotically flat spacetimes, and a Newtonian limit would instead have to be recovered in some (presumably stable) region of spacetime in which the antisymmetric sector is not small compared to the symmetric sector. Clearly if one does not allow asymptotically flat (or nearly so) spacetimes, nor regions of spacetime where the symmetric sector dominates to the point where one essentially recovers GR physics locally, then these instabilities are avoided.

Despite the fact that these results indicate that the relevant (Wyman) solution would be unstable, there are some encouraging phenomenological results on galaxy dynamics (Moffat and Sokolov 1995a), as well as some optimism that the collapse of spherically symmetric matter would be nonsingular (Moffat and Sokolov 1995b). It is possible to further modify the dynamics of NGT in order to remove these instabilities and thereby recover this phenomenology by guaranteeing that three of the field equations appear as constraints rigorously, either imposed via Lagrange multipliers in the action (Moffat 1996), or leading to generalizations of the model introduced by Damour et al (1993) that possess ‘gauge invariant kinetic terms’. Alternatively, given that the antisymmetric sector should appear as a Kalb-Ramond field (Kalb and Ramond 1974), it is reasonable to expect that there should be some tie between NGT and string theory (Moffat 1995d). It may then be the case that the fundamentally nonlocal nature of strings (which does not have a Cauchy initial value formulation) would cause the NGT field equations to be an inadequate description of the system in some situations.

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