Skew Symmetric Bundle Maps on Space-Time

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Abstract. We study the "Lie Algebra" of the group of Gauge Transformations of Space-time. We obtain topological invariants arising from this Lie Algebra. Our methods give us fresh mathematical points of view on Lorentz Transformations, orientation conventions, the Doppler shift, Pauli matrices, Electro-Magnetic Duality Rotation, Poynting vectors, and the Energy Momentum Tensor $T$.

1. Introduction

Let $M$ be a space-time and $T(M)$ its tangent bundle. Thus $M$ is a 4-dimensional manifold with a nondegenerate inner product $\langle \cdot, \cdot \rangle$ on $T(M)$ of index $-++$. We study the space of bundle maps $F: T(M) \to T(M)$ which are skew symmetric with respect to the metric, i.e., $\langle Fv, v \rangle = 0$ for all $v \in T_x(M)$ and all $x \in M$.

A skew symmetric $F$ has invariant planes and eigenvector lines in each $T_x(M)$. We give necessary and sufficient conditions as to when these plane systems and line systems form subbundles in Theorem 7.3. Also we determine the space of those $F$ which give the same underlying structure. This is done by introducing the bundle map $T_F = F \circ F - \frac{1}{4}(tr F^2)I : T(M) \to T(M)$. Then the space of skew symmetric $F$ which give rise to the same $T$ is homeomorphic to $\text{Map}(M, S^1)$, the space of maps of $M$ into the circle $S^1$. (See Theorem 7.11.)

We also show that the space of skew symmetric $F$ has a natural complexification. (See Propositions 2.2 and 2.3) This leads to an equivalence between the $F$ and vector fields on the complexified tangent bundle $T(M) \otimes \mathbb{C}$. The complexified study leads to several beautiful relations which link our subject matter to Clifford Algebras and Quaternions. (See Corollaries 4.6 and 4.7 and Theorem 4.8.) We naturally find many points of contact with Physics, especially classical electromagnetism. These considerations frequently govern our choice of notation. The physical motivations and remarks will be explored in the Scholia; and the mathematical motivations and links will be found in the Remarks.

Scholium 1.1. Physical connections.

a) Each skew symmetric $F$ corresponds to a two-form $\hat{F}$. The electro-magnetic tensor is a two form. In the classical theory it satisfies Maxwell’s equations. The symmetric bundle map $T_F$ corresponds to the energy-momentum tensor of the electro-magnetic field. The homotopy invariants arising from the existence of subbundles must give physical information if there is any physical content in Classical...
Electro-Magnetism. We show that the invariants distinguish the two main cases; a classical free electron and a classical electron in a magnetic field.

b) We give formulas in terms of $E$ and $B$ for the eigenvectors of $F$. Changing observers gives the same eigenvector multiplied by a factor. For “radiative” $F$, this factor reduces to the Doppler shift. One wonders if the more general shift for non-radiative $F$ has any physical meaning.

c) The space of skew symmetric $F$ has a canonical splitting of space and time. It is mapped isomorphically onto $T(M) \otimes \mathbb{C}$ by a choice of a field of observers. Thus any complex tangent vector field corresponds to a skew-symmetric $F$. So, for example, if the solutions of the Dirac equation have any physical content, then the homotopy invariants of the corresponding $F$ must have physical import.

**Remark 1.2. Mathematical Motivation.**

The mathematical point of view of this work stems from the author’s study of the space of bundle equivalences in $[G_1]$, $[G_2]$, $[G_3]$. These bundle equivalences form spaces which later became popular known as groups of gauge transformations. The main result of these papers is that the classifying space of these groups of gauge transformations is the space of maps of the base space into the classifying space of the fibration in question.

This theorem has played an important role, at least in the mathematical part of of Gauge Theory. It entered into the theory via Proposition 2.4 of [AB]. But the point of view of these works concerned spaces of connections, instead of spaces of bundle equivalences. The original point of view was furthered in papers by Booth, Heath and Piccinini among others, see for example [BP].

In this present work, we study other types of bundle maps. The “Lie Algebras” of “Gauge Transformation Groups” seems to be a natural class to study. The skew-symmetric bundle maps of space-time are the “Lie Algebra” of the group of isometries on $T(M)$, i.e. bundle maps $Q : T(M) \to T(M)$ so that $\langle Qv, Qw \rangle = \langle v, w \rangle$.

**Scholium 1.3. Physical Point of View.**

Galileo’s famous quote that the Laws of Nature are written in the language of geometry should be revised in view of the development of Topology in this century. As topology underlies geometry, one would expect that some Laws of Nature would be expressed in terms of the elementary homotopy invariants of topology. Among these are the degrees of maps and the index of vector fields.

Our method for discovering these laws follows Galileo. To the argument that no one had seen an object travel at a constant velocity forever along a straight line, Galileo replies: Let us assume it is true, derive its mathematical consequences, and see if they relate to what is observed. Thus we begin by studying infinitesimal rigid motions $F$ on space-time $M$, and observe connections with electromagnetism, etc. The idea of separating the physical from the mathematical arguments via Scholia is borrowed from Newton’s Principia.

**Remark 1.4. Levels of notation.**

We proceed by adding layers of notation to our space-time. We descend one level for every choice we estimate we make. We begin at Level −1 with the inner product and continue by choosing an orientation at Level −2. By Level −10 we have chosen
an orthonormal basis for the tangent space of $M$. We eventually end at Level $-16$, which are the standard coordinates for Minkowski Space.

This approach permits us to understand that choosing an orientation is like taking a complex conjugate. It also allows a clear view of Lorentz Transformations at the various levels. The major technique of computation in this paper is given by a Level $-10$ block matrix which allows Level $-10$ calculations to produce Level $-2$ statements.

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I have had very productive conversations with Barrett O’Neill, Stephen Parrot, and Solomon Gartenhaus. Gartenhaus gave me a key example which forced me to think more deeply at the beginning of this work. Barrett O’Neill gave me many ideas. The best one is the definition of the complexification map $c$. Barrett O’Neill’s book Semi-Riemannian Manifolds [O] exposes space-time in a rigorous mathematical manner. Stephen Parrott’s book [P] provided great stimulation and guidance.

2. Notation and Preliminaries

A space-time $M$ is a smooth 4-dimensional orientable manifold with a Lorentzian metric $\langle \cdot, \cdot \rangle$ defined on the tangent bundle $T(M)$ and a nonzero future pointing time-like vector field. If $x \in M$, then $T_x = T_x(M)$ will denote the 4-dimensional tangent space over $x$. The space of vectors orthogonal to a vector $u \in T_x$ will be denoted by $T_u x$.

A skew symmetric bundle map is a map $F : T(M) \to T(M)$ which covers the identity on the base, is a vector bundle map, and is skew symmetric that is,

$$F(\alpha v_x + \beta w_x) = \alpha F(v_x) + \beta F(w_x) \in T_x$$

and

$$\langle F(v_x), w_x \rangle = \langle v_x, -F(w_x) \rangle.$$

Let $\ell$ be the vector bundle over $M$ whose fibre $\ell_x$ is the vector space of skew symmetric linear transformation $F_x : T_x \to T_x$. Then the space of cross-sections $\Gamma(\ell)$ to $\ell$ corresponds to the space of bundle maps in the usual manner. Let $\Lambda^2(M)$ be the bundle of two forms over $M$. Thus the fibre $\Lambda^2(M)_x$ are bilinear antisymmetric maps $\hat{F}_x : T_x \times T_x \to \mathbb{R}$. Any two-form is a cross-section to $\Lambda^2(M)$.

Now $\ell$ is bundle equivalent to $\Lambda^2(M)$. Let $\rho : \ell \to \Lambda^2(M)$ so that $\rho(F_x) = \hat{F}_x$ where

$$\hat{F}_x(v_x, w_x) = \langle v_x, F_x(w_x) \rangle.$$

The non-degeneracy of $\langle \cdot, \cdot \rangle$ implies that $\rho$ is an isomorphism on each fibre, thus $\ell$ sets up a bijection between two-forms and bundle maps.

Level $-1$. Lorentz Inner Product.

Notation plays an important role in Mathematics and Physics. It is a powerful aid to calculation. But notation can blur distinctions and confuse reasoning. For that reason we will introduce notation in Levels. Each improvement of notation is based on more and more choices. The above notation is called Level $-1$. As we add choices of frame fields and coordinates we descend eventually to Level $-16$, which is the canonical coordinates of Minkowski Space-time. The number describing the
Levels approximates the number of choices made to introduce the notation. We have already made one choice in Level −1 by assuming that $\langle \cdot, \cdot \rangle$ has signature $-++$, we could have assumed signature $+-+$. Level 0 then has inner product $\epsilon \langle \cdot, \cdot \rangle$ where $\epsilon$ is $\pm 1$. The geometry does not change with the change of $\epsilon$. The geodesics remain the same and skew-symmetric bundle maps remain the same so the choice $-++$ does not affect our work. But in comparing our results with other authors, be aware that the electro-dynamicists usually choose $++$. Thus S. Parrott [8] chooses $++$, we could have assumed signature $-+-$. Level $-1$ has signature $-+-$. Level 0 then has inner product $\epsilon \langle \cdot, \cdot \rangle$ where $\epsilon$ is $\pm 1$. The geometry does not change with the change of $\epsilon$. The geodesics remain the same and skew-symmetric bundle maps remain the same so the choice $-++$ does not affect our work.

LEVEL −2. ORIENTATION.

Since $M$ is orientable, there is a volume form $\Omega \in \Lambda^4(M)$. There are two choices consistent with the metric, $\pm \Omega$. We choose $\Omega$ as the orientation. We could have chosen $-\Omega$. Now the Hodge dual is an isomorphism defined on $\Lambda^2(M)$, satisfying $\ast(\ast \eta) = -\eta$ for $\eta \in \Lambda^2(M)$. Under $\rho : \ell \rightarrow \Lambda^2(M)$ the Hodge dual corresponds to an operator $\ast$ on $\Gamma(\ell)$. It satisfies

$$ (aF)^* = aF^* \quad \text{and} \quad (F + G)^* = F^* + G^* \quad \text{and} \quad F^{**} = -F. \quad (4) $$

Let $u \in T_x(M)$ be an observer. That is $u$ is a future pointing time-like vector such that $\langle u, u \rangle = -1$. Then we define

$$ E_u = Fu \quad \text{and} \quad B_u = -F^* u. \quad (5) $$

Note that $E_u$ and $B_u \in T^* s$. If we change the orientation, we obtain a new $\ast'$. This is related to the old $\ast$ by $F^{\ast'} = -F^\ast$. Thus for change of orientation, $E_u$ remains the same, but $B_u$ becomes $-B_u$.

If $v$ and $w$ are space-like vectors in $T_x$, they span a space-like plane if and only if they are linearly independent and

$$ v^2 w^2 - \langle v, w \rangle^2 > 0. \quad (6a) $$

If

$$ v^2 w^2 - \langle v, w \rangle^2 = 0, \quad (6b) $$

they span a light-like plane and if

$$ v^2 w^2 - \langle v, w \rangle^2 < 0 \quad (6c) $$

they span a space-like plane.

Let $u$ be an observer. We define the dot product and cross product on $T^*_m$.

**Definition 2.1.** Let $v$ and $w \in T^*_m$. Define $v \cdot w = \langle v, w \rangle$ and $v \times w = vw \cos \theta$ where $\theta$ is defined to be the angle between $v$ and $w$.

Now we define $v \times w = v \times u w$ is the unique vector orthogonal to $v$ and $w$ in $T^*_m$ of length $|vw \sin \theta|$ so that $\Omega(v, u, v, w, v \times w) \geq 0$.

This cross product satisfies the usual relations:

\[
\begin{align*}
\mathbf{v} \times (\alpha \mathbf{w} + \beta \mathbf{x}) &= \alpha (\mathbf{v} \times \mathbf{w}) + \beta (\mathbf{v} \times \mathbf{x}) \\
\mathbf{v} \times \mathbf{w} &= 0 \quad \text{if and only if} \quad \alpha \mathbf{v} = \beta \mathbf{w} \\
(u \times v) \times \mathbf{w} &= (\mathbf{w} \cdot u)\mathbf{v} - (\mathbf{w} \cdot v)u \\
u \cdot (v \times w) &= -v \cdot (u \times w) \\
v \cdot (v \times w) &= 0
\end{align*}
\]
We use $F^*$ to impose a complex structure on $\ell$. Define
\[ e^{i\theta}F = \cos \theta F + \sin \theta F^*. \] (9)

**Proposition 2.2.** The action $e^{i\theta}$ on $\Gamma(\ell)_x$ induces a complex structure.

**Proof.** Any complex number $z = ae^{i\theta}$, so $z \cdot F = e^{i\theta}(aF)$. We check that $e^{i\theta'}(e^{i\theta}F) = e^{i(\theta + \theta')}F$ and $e^{i\theta} \cdot (F + F') = e^{i\theta} \cdot F + e^{i\theta} \cdot F'$. □

Consider $T(M) \otimes \mathbb{C}$. We define the innerproduct $\langle \cdot , \cdot \rangle_\mathbb{C}$ on $T_x(M) \otimes \mathbb{C}$ by
\[ \langle iu, v \rangle_\mathbb{C} = \langle u, iv \rangle_\mathbb{C} = i \langle u, v \rangle \] when $u, v \in T_x(M)$. (10)

If $a$ and $b, c$ and $d$ are in $T^u_x$, we define
\[ (a + ib) \times (c + id) = (a \times c - b \times d) + i(b \times c + a \times d). \] (11)

Let $\ell_\mathbb{C}$ be the bundle of linear maps $F : T_x \otimes \mathbb{C} \to T_x \otimes \mathbb{C}$ skew symmetric with respect to $\langle \cdot , \cdot \rangle_\mathbb{C}$. Let $F_x \in \ell_\mathbb{C}$ act on $T_x \otimes \mathbb{C}$ by
\[ F(a + ib) = F(a) + iF(b). \] (12)

Define $c : \ell \to \ell_\mathbb{C}$ and $\overline{c} : \ell \to \ell_\mathbb{C}$ by
\[ cF = F - iF^* \quad \text{and} \quad \overline{c}F = F + iF^*. \] (13)

Note that changing the orientation means replacing $F^*$ by $F^* := -F^*$. Hence the complex structure is changed so that $c$ becomes $\overline{c} = c'$.

**Proposition 2.3.** $c$ is a complex bundle map.

**Proof.** $cF_x$ is skew symmetric on $T_x \otimes \mathbb{C}$. Also $c$ commutes with addition and multiplication. It is complex because
\[ c(e^{i\theta} \cdot F) = e^{i\theta}(cF). \] (14)

This follows because
\[ e^{i\theta}(cF) = (\cos \theta + i \sin \theta)(F - iF^*) \]
\[ = \cos \theta F + \sin \theta F^* + i(\sin \theta F - \cos \theta F^*) \]
\[ = e^{i\theta} \cdot F - i(e^{i\theta} \cdot F^*) = e^{i\theta} \cdot F - i(e^{i\theta} F)^* \]
\[ = c(e^{i\theta} F). \]

□

We will show presently that $c$ is injective.

**Scholium 2.4.** Maxwell’s equations and Lorentz’ Law.

a) We chose $\rho : \ell \to \Lambda^2$ to be given by (3), $\bar{F}(v, w) = \langle v, F(w) \rangle$, in order to agree with the standard index conventions of tensor analysis. Parrott’s otherwise careful book makes the opposite choice, $\bar{F}(v, w) = (F(v), w)$, and is thus inconsistent with
his index conventions. This has little import for his book, since he deals mostly
with forms, but it could cause confusion if one is using skew symmetric operators.

b) Electro-magnetic tensors are two-forms. Classically they satisfy Maxwell’s
equations:
\[ d\hat{F} = 0, \quad d * \hat{F} = J. \]  \hspace{1cm} (15)

We can write Maxwell’s equations in terms of skew symmetric bundle maps as
follows.
\[ \text{div} \; F = j, \quad \text{div} \; F^* = 0 \]  \hspace{1cm} (16)
where \( j \) is a one form. We may reduce this to one equation by extending \( \text{div} \) to the
complex case by \( \text{div}(iF) = i \text{div}(F) \). Then \( F \) satisfies Maxwell’s equation if and
only if \( \text{div}(cF) \) is real.

c) The Lorentz Law: Suppose a particle with charge \( q \) is moving in an electromag-
netic field \( \hat{F} \) with 4–velocity \( u \). Then its acceleration is \( a = qFu \) where \( \rho(F) = \hat{F} \).
This is the reason we chose the symbol \( E \) to equal \( Fu \). The charge is motionless
with respect to the \( u \) observer, hence its acceleration is given by the electric field
\( E \) as seen by that observer. Also \( B = -F^*u \) corresponds to the magnetic field, as
will be seen shortly.

**Level −9. Orthonormal Bases.** Choose orthonormal vector fields \( e_0, e_1, e_2, e_3 \), so
\[ \langle e_0, e_0 \rangle = -1 \quad \text{and} \quad \langle e_i, e_j \rangle = \delta_{ij}. \]  \hspace{1cm} (17)

Already this notation restricts the topology of the \( M \). It must be parallelizable
for such a basis to exist. Fortunately we can find local regions which admits these
orthogonal frame fields. Now \( F(e_i) = \sum F_{ij} e_j \). So \( \langle F(e_i), e_j \rangle = F_{ij} \langle e_j, e_j \rangle \). Hence
\( F \) is skew symmetric if and only if \( F_{ji} \langle e_i, e_i \rangle = -F_{ij} \langle e_j, e_j \rangle \). So we can represent
\( F \) by a matrix of the form
\[ F = \begin{pmatrix}
0 & E_1 & E_2 & E_3 \\
E_1 & 0 & B_3 & -B_2 \\
E_2 & -B_3 & 0 & B_1 \\
E_3 & B_2 & -B_1 & 0
\end{pmatrix} \]
where \( F_{0i} = F_{i0} = E_i \) and \( F_{ij} = -F_{ji} = B_k \). \hspace{1cm} (18)

We find it convenient to partition this matrix into blocks. So
\[ F = \begin{pmatrix}
0 & \text{E}^T \\
\text{E} & \times \text{B}
\end{pmatrix} \]  \hspace{1cm} (19)

where \( \text{E} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \) and \( \text{B} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \). Here the notation \( \times \text{B} \) means
\[ (\times \text{B}) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = (v_1 e_1 + v_2 e_2 + v_3 e_3) \times (B_1 e_1 + B_2 e_2 + B_3 e_3), \]
or
\[ (\times \text{B}) \text{v} = \text{v} \times \text{B} \]  \hspace{1cm} (20)
for short. This assumes that $e_1 \times e_2 = e_3$. If $e_1 \times e_2 = -e_3$, then $(\times B)v = B \times v$.

**Level –10: Oriented Orthonormal Bases.** Same as in Level –9, but here we require $e_1 \times e_2 = e_3$.

Now in Level –9 we have

$$F = \begin{pmatrix} 0 & E^T \\ E & \times B \end{pmatrix} \quad \text{and} \quad F^* = \begin{pmatrix} 0 & -B^T \\ -B & \times E \end{pmatrix}.$$  \hfill (21)

Then

$$cF = \begin{pmatrix} 0 & A^T \\ A \times (-iA) \end{pmatrix} \quad \text{where} \quad A = E + iB$$  \hfill (22)

Note that any matrix of the form $\begin{pmatrix} 0 & E^T \\ E & \times B \end{pmatrix}$ represents a skew symmetric linear map.

**Scholium 2.5. Lorentz transformation at level –2.**

Let $u$ and $u'$ be observers. Then

$$u' = \frac{1}{\sqrt{1 - w^2}} (u + w)$$  \hfill (23)

where $w$ is space-like in $T_u^x$. We call $w$ the velocity of $u'$ relative to $u$. There is a symmetric formula

$$u = \frac{1}{\sqrt{1 - w'^2}} (u' + w')$$

But note that $w'$ does not lie in the same subspace as $w$. However $w = w'$ and $w$ and $w'$ both lie in the $u$, $u'$ plane. Now if a particle moves along $u'$ as seen by $u$, then

$$a = qF u' = \frac{q}{\sqrt{1 - w^2}} \begin{pmatrix} 0 & E^T \\ E & \times B \end{pmatrix} \begin{pmatrix} 1 \\ w \end{pmatrix} = q [(E \cdot w)u + E + w \times B] / \sqrt{1 - w^2}.$$  \hfill (24)

This is a more familiar form of the Lorentz Law.

The block matrix of Level –10 gives a very effective way of discovering facts about $F$. Most of the time we will use Level –2 proofs or Level –10 proofs. But what are definitely superior are Level –2 statements.

Now from the block matrices of Level –10 we quickly find several facts.

**Proposition 2.6.**  

- $\dim_{\mathbb{R}} \ell_x = 6$, so $\dim_{\mathbb{C}} \ell_x = 3$.
- For a given observer field $u$, there is an $F$ for every pair of vector fields $E$ and $B$ in $T_u^x$.
- The map $c : \ell \to \ell_{\mathbb{C}}$ is injective, since the map $\phi_u : \ell \to T_u^x \otimes \mathbb{C}$ is a vector bundle equivalence where

$$\phi_u(F) = cF u = F u - iF^* u = E + iB$$  \hfill (25)
3. Key Relations

Using the notation of Level −10 we obtain the following facts by straightforward calculation.

**Lemma 3.1.** Let the commutator be denoted by \([x, y] = xy - yx\).

a) \((xA)(xB) = AB^T - (A \cdot B)I\)
where \(I\) is the 3×3 identity matrix and all vectors are column vectors.

b) \([([x]B'), ([x]B)] = [x(B' \times B)]\).

c) \([E'E^T, EE^T] = [x(E' \times E)]\).

d) \([B + B'] = [xB + xB']\).

e) \([x(B)]^T = [-(B)] = -(xB)\)

f) \(v^T(xB) = (B \times v)^T\)

**Proof of f.**

\[
v^T(xB) = [(xB)^Tv]^T = [(-B)v]^T = [v \times (-B)]^T = [B \times v]^T.
\]

\[\square\]

A key result is the following

**Theorem 3.2.**

\[FF^* = F^*F = -(E \cdot B)I.\]

**Proof.** Use (21) and multiply out using Lemma 3.1a.

**Corollary 3.3.** \(\langle Fv, F^*v \rangle = (E \cdot B)(v, v)\) for any \(v \in T(M)\). Hence \(E_u \cdot B_u = E_{u'} \cdot B_{u'}\) for any two observers.

**Proof.**

\[
\langle Fv, F^*v \rangle = -\langle F^*Fv, v \rangle = -(E \cdot B)v, v \rangle = E \cdot B(v, v).
\]

Thus \(E_{u'} \cdot (-B_{u'}) = E \cdot B(-1).\) \[\square\]

**Corollary 3.4.** \(-E \cdot B = \lambda_F \cdot \lambda_{F^*}\), where \(\lambda_F\) is the eigenvalue for an eigenvector \(s\) of \(F\) and \(\lambda_{F^*}\) is the eigenvalue of \(s\) for \(F^*\).

**Proof.** Since \(F\) and \(F^*\) commute, they have a common eigenvector \(s\). Then

\[\lambda_F \cdot \lambda_F s = F^*F s = -(E \cdot B)s.\]

\[\square\]

**Corollary 3.5.** \(F^2 - F^*2 = (E^2 - B^2)I.\)

**Proof.** Apply Theorem 3.2 to \((F + F^*)(F + F^*)^*\). So

\[
(F + F^*)(F + F^*)^* = -\langle(F + F^*)u, -(F + F^*)^*u\rangle I
-(F^2 - F^*2) = -(E - B) \cdot (B + E)I
F^2 - F^*2 = (E^2 - B^2)I.
\]

The second equation follows from (4) and the definition of \(E\) and \(B\). \[\square\]
Corollary 3.6. \( E^2_a - B^2_a = E^2_{\overline{a}} - B^2_{\overline{a}} \).

Corollary 3.7. \( \lambda^2_F - \lambda^2_{\overline{F}} = E^2 - B^2 \).

Definition 3.8. Let \( T_F = \frac{1}{2}(F^2 + F^*2) \). Thus \( T_F \) is a bundle map which is symmetric with respect to \( \langle \cdot , \cdot \rangle \).

Proposition 3.9. \( T_F = F^2 - \frac{E^2 - B^2}{2} I \).

Proof. Use Corollary 3.5.

Proposition 3.10. \( T_F = \begin{bmatrix} \frac{E^2 + B^2}{2} & - (E \times B)^T \\ EE^T + BB^T - \frac{E^2 + B^2}{2} I \end{bmatrix} \).

Proof. Compare [P], p.117, equation (28). Use equations Lemma 3.1a and Proposition 3.9. □

Corollary 3.11. Trace \( (T_F) = 0 \).

Corollary 3.12. Trace \( (F^2) = 2(E^2 - B^2) \), hence

\[ T_F = F^2 - \frac{1}{4} \text{tr}(F^2) I. \]

Proof. Use Corollary 3.11 and Proposition 3.9.

Scholium 3.13. Energy-Momentum tensor.

a) Physically \( T_F \) is a multiple of the energy-momentum tensor. See [P], p.116, equation (20).

b) The Poynting 4-vector as seen by observer \( u \) is

\[ Tu = \frac{E^2 + B^2}{2} u + E \times B. \]  \hspace{1cm} (26)

Thus \( \frac{E^2 + B^2}{2} \) is interpreted as the energy of the electromagnetic field \( F \), and \( E \times B \) is interpreted as the 3-momentum per unit volume of the field \( F \).

4. The Complex Structure and Commutators

Using the commutator relations Lemma 3.1b and c and matrix multiplication, we obtain the following key result for commutators \([F_1,F_2] = F_1F_2 - F_2F_1\).

Theorem 4.1.

\[ [F_1,F_2] = \begin{bmatrix} 0 \\ (-E_1 \times B_2 - B_1 \times E_2) \times (E_1 \times B_2 - B_1 \times E_2) \end{bmatrix} \]

In other words

\[ [F',F]u = -E' \times B - B' \times E \]

\[ -[F',F]^*u = E' \times E - B' \times B. \]

We remark that this result also holds for complex \( F_1 \) and \( F_2 \) since the argument is just formal.
Corollary 4.2.
\[ [F_1, F_2]^* = [F_1^*, F_2^*] = [F_1^*, F_2]. \]  
Hence
\[ e^{i(\theta + \phi)} \cdot [F_1, F_2] = [e^{i\theta} \cdot F_1, e^{i\phi} \cdot F_2]. \]  
Proof. (27) follows from (Theorem 4.1) and (28) follows from (27).

Hence the complexification of \( \Gamma(\ell) \) commutes with the Lie algebra structure of \( \Gamma(\ell) \).

Theorem 4.3.
\[ [cF, cG] = 2c([F, G]). \]  
Proof.
\[ [cF, cG] = (F - iF^*)(G - iG^*) - (G - iG^*)(F - iF^*) \]
\[ = FG - F^*G^* - i(FG^* + F^*G) - GF + G^*F - i(-GF^* - G^*F) \]
\[ = [F, G] + [G^*, F^*] - i([F, G^*] + [F^*, G]) \]
\[ = [F, G] + [G, F]^* - i([F, G] + [F, G]^*) \]
\[ = 2([F, G] - i[F, G]^*) = 2c[F, G]. \]

where the last equality comes from the definition of \( c \), and the previous two equalities come from (27) and (4).

Corollary 4.4.
\[ (c[F_1, F_2])u = i(E_1 + B_1) \times (E_2 + iB_2) \]
for observer \( u \).

Proof. \( c[F_1, F_2] = \frac{1}{2}[cF_1, cF_2] \) by Theorem 4.3. Now \( cF_1 = \begin{pmatrix} 0 & A_1^T \\ A_1 \times (-iA_1) \end{pmatrix} \)
where \( A_1 = E_1 + iB_1 \) and similarly for \( cF_2 \). Now by Theorem 4.1 for complex \( F \), we have
\[ [cF_1, cF_2]u = -A_1 \times (-iA_2) - (-i(A_1) \times A_2) \]
\[ = 2iA_1 \times A_2 = 2i(E_1 + iB_1) \times (E_2 + iB_2). \]

\[ \Box \]

Theorem 4.5. Let \( \mathbb{F} = \begin{pmatrix} 0 & A^T \\ A \times C \end{pmatrix} \) where \( A \) and \( C \) are complex 3-vectors. Then
\( \mathbb{F}^2 = kI \) if and only if \( k = A \cdot A \) and \( C = \pm iA \).

Proof. Assume \( \mathbb{F}^2 = kI \). Then \( \mathbb{F}^2w = kw \) for any vector \( w \). For \( w = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \) we get
\( A \cdot A = k \) and \( A \times C = 0 \). Thus \( C = sA \) for some \( s \in \mathbb{C} \). Hence \( \mathbb{F} = \begin{pmatrix} 0 & A^T \\ A \times (sA) \end{pmatrix} \)
and \( \mathbb{F}^2 = (A \cdot A)I \). Apply \( \begin{pmatrix} 0 \\ v \end{pmatrix} \) to this last equation. We obtain
\( (v \cdot A)A + (v \times sA) \times (sA) = (A \cdot A)v. \)
Using the third and fourth equations of (8) and rearranging terms we get

\[(1 + s^2)(A \cdot v)A = (1 + s^2)(A \cdot A)v\]

for arbitrary v. Thus \(1 + s^2 = 0\). Hence \(s = \pm i\).

For the converse, first suppose \(s = -i\), so \(F = \left( \begin{array}{cc} 0 & A^T \\ A & i(-A) \end{array} \right) \). Then \(F = F - iF^* = cF\) where \(cFu = \mathbf{E} + i\mathbf{B} = \mathbf{A}\). Then

\[F^2 = (cF)^2 = (F - iF^*)^2 = F^2 - F^*2 - i(FF^* + F^*F)\]
\[= (E^2 - B^2)I - 2i(-E \cdot B)I\]
\[= (E^2 - B^2 + 2E \cdot B)iI\]
\[= (E + iB) \cdot (E + iB)I = (A \cdot A)I.\]

Similarly if \(s = i\), \(\overline{F^2} = (A \cdot A)I\) implies \(F^2 = (A \cdot A)I\). \(\square\)

**Corollary 4.6.**

\[(cF)^2 = (A \cdot A)I = \lambda_{cF}^2 I\]
\[(\overline{cF})^2 = \lambda_{\overline{cF}}^2 I.\]

**Proof.** The inner equality of the first line was proved above. Apply this equation to an eigenvector \(s\) to get the last equality of the first line. The second equality follows from complex conjugation.

**Corollary 4.7.** \(cF_1cF_2 + cF_2cF_1 = 2(A_1 \cdot A_2)I\).

**Proof.**

\[cF_1cF_2 + cF_2cF_1 = (cF_1 + cF_2)^2 - cF_1^2 - cF_2^2\]
\[= [(A_1 + A_2) \cdot (A_1 + A_2) - (A_1 \cdot A_1 - A_1 \cdot A_2 \cdot A_2)]I = 2(A_1 \cdot A_2)I.\]

\(\square\)

**Theorem 4.8.** \(cF_1\overline{cF_2} = \overline{cF_2}cF_1\).

**Proof.** We must show that \([cF_1, cF_2] = 0\). Apply theorem 4.1 where \(E_j = A_j\) and \(B_j = (-1)^jiA_j\) for \(j = 1, 2\). Then all cross products must be zero in Theorem 4.1 and we obtain the desired result. \(\square\)

**Remark 4.9. Clifford Algebras.**

According to the first proposition of [LM], Corollary 4.6 is a clue that \(c : \ell_x \to \mathbb{C}(4)\) involves representations of Clifford modules. Here \(\mathbb{C}(4)\) is the space of linear maps on \(T_x \otimes \mathbb{C}\), a 16 dimensional space which is a complex Clifford Algebra. The image of \(c\) in \(\mathbb{C}(4)\) generates the Quaternions tensored with \(\mathbb{C}\). The complex conjugate \(\overline{c}\) generates another complex representation of the Quaternions in \(\mathbb{C}(4)\). The two representations commute, and they generate all of \(\mathbb{C}(4)\) under composition. This probably has something to do with the fact that \(so(4) \simeq so(3) \times so(3)\)? But it might be that this particular representation by means of \(F - iF^*\) is new.
Scholium 4.10. Pauli Matrices.

The Pauli matrices of physics play an important role in quantum mechanics. The relations among their products are their key features, the actual form of the matrices is not important. Thus we have \( \sigma_x, \sigma_y, \sigma_z \) so that \( \{ \sigma_i, \sigma_j \} = \delta_{ij}I \) and \( [\sigma_x, \sigma_y] = 2i\sigma_z \), [F, III, 11-4]. We get the same relations using \( cF \) as follows. Let \( E_x, E_y, E_z \) be the \( F \) with zero \( B \) field and with unit \( E \) fields pointing along the \( x, y, z \) axes, respectively, of Minkowski space. So for example \( E_x = \begin{pmatrix} 0 & e_x \\ e_x & 0 \end{pmatrix} \).

Denote \( \sigma_x = cE_x, \sigma_y = cE_y \) and \( \sigma_z = cE_z \). Then \( \sigma_x, \sigma_y, \sigma_z \) satisfy the Pauli matrix relations. In addition, \( \sigma_x, \sigma_y, \sigma_z \) commute with the \( \sigma \)'s and satisfy the Pauli relations among themselves except that \( \sigma_x \sigma_y = -i \sigma_z \). Also \( \sigma_x, \sigma_y, \sigma_x, \sigma_y \) generate the Clifford algebra \( \mathbb{C}(4) \). This can be shown by brute force.

5. Eigenvectors

Recall our notation in which \( \pi F = \overline{cF} \) and \( \lambda_{cF} = \overline{\lambda_{cF}} \).

Proposition 5.1. \( cF \circ \overline{cF} = 2T_F \). Hence \( \lambda_{cF} \lambda_{cF} = 2\lambda_T \).

Proof. \( cF \circ \overline{cF} = (F - iF^*)(F + iF^*) = F^2 + F^{*2} \) since \( FF^* = F^{*2}F \). Now apply the definition of \( T_F \), (Definition 3.8).

Corollary 5.2. \( T_{e^{i\theta}F} = T_F \).

Proof.

\[
T_{e^{i\theta}F} = \frac{1}{2}(e^{i\theta} \circ \overline{cF} \circ e^{-i\theta} \circ \overline{cF}) = \frac{1}{2}(e^{i\theta}cF) \circ e^{-i\theta}cF = \frac{1}{2}cF \overline{cF} = T_F.
\]

\[\square\]

Corollary 5.3. \( T^2 = \lambda_T^2I \) where \( \lambda_T \) is an eigenvalue of \( T \).

Proof. \( T^2 = \frac{1}{4}((cF)(\overline{cF}))^2 = \frac{1}{4}(cF)^2(\overline{cF})^2 = \frac{1}{4}\lambda_{cF}^2 \overline{\lambda_{cF}}^2I \) by Theorem 4.8. So \( T^2 = \lambda_T^2I \).

Theorem 5.4. Let \( F \in \Gamma(\ell) \) and let \( \lambda_F \) be an eigenvalue of \( F \) and \( \lambda_T \) be an eigenvalue of \( T_F \).

a) \( \lambda_T = \sqrt{\frac{(E^2 - B^2)}{2} + (E \cdot B)^2} \)

b) \( \lambda_F \pm \sqrt{\frac{(E^2 - B^2)}{2}}, \quad \lambda_{F^*} \pm \sqrt{\lambda_T - \frac{(E^2 - B^2)}{2}} \).

c) \( \lambda^4 - (E^2 - B^2)\lambda^2 - (E \cdot B)^2 \), or equivalently, \( \lambda^4 - (\lambda_T^2 - \lambda_F^2)\lambda^2 - (\lambda_F \lambda_{F^*})^2 \), is the characteristic polynomial of \( F \).
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Proof. Corollaries 3.4 and 3.7 gives the equations \( \lambda_F \lambda_F^* = -E \cdot B \) (Corollary 3.4) and \( \lambda_F^2 - \lambda_{F^*}^2 = E^2 - B^2 \) (Corollary 3.7). Eliminating \( \lambda_{F^*} \) from (Corollary 3.4) and (Corollary 3.7) gives \( \lambda_F^2 - (E^2 - B^2)\lambda_F^2 - (E \cdot B)^2 = 0 \). Solving gives b). Then a) follows from \( \lambda_T = \lambda_F^2 - \frac{(E^2 - B^2)}{2} \) which follows from Corollary 3.7. To be absolutely certain that c) is the characteristic polynomial, one must calculate \( \det(F - \lambda I) \) for \( F \) represented as a matrix in (18). □

**Proposition 5.5.** If \( s \) is an eigenvector of \( F \in \ell_x \), then \( \lambda_F(s, s) = 0 \). So if \( \lambda_F \neq 0 \), then \( s \) is a null vector. Both \( \lambda_F \) and \( \lambda_{F^*} \) are zero if and only if \( \lambda_T = 0 \). In that case \( s \) is a multiple of \( \frac{E^2 + B^2}{2} u + E \times B \), which is null.

**Proof.** \( \lambda_F(s, s) = \langle \lambda_F s, s \rangle = \langle Fs, s \rangle = -\langle s, F_s \rangle = -\lambda_F(s, s) \). The same argument holds for the complex \( cF \), so \( \lambda_{cF}(s, s) = 0 \). Since \( \lambda_T = \frac{1}{2} \lambda_F \overline{\lambda_{cF}} \), we get the second sentence. Now \( \lambda_T = 0 \) if and only if \( E = B \) and \( E \cdot B = 0 \). Under those conditions, use (19) to show that \( \frac{E^2 + B^2}{2} u + E \times B \) is an eigenvector and is a null vector. □

**Scholium 5.6.** The Null and non null cases.

The null and non-null cases are when \( \lambda_T = 0 \) and \( \lambda_T \neq 0 \) respectively. If \( \lambda_T = 0 \) then \( E = B \) and \( E \cdot B = 0 \). This is called the null case mathematically. Physicists identify an electro-magnetic field with \( E = B \) and \( E \cdot B = 0 \) as the radiative or wave-like case. In the null case \( \lambda_F = \lambda_{F^*} = \lambda_T = \lambda_{cF} = 0 \). The characteristic polynomial is \( \lambda^4, T = F^2, F^2 u \) is the eigenvector of \( F^2 \).

**Proof.** \( F^2(F^2 u) = F^4 u = T^2 u = 0 \). So \( s = F^2 u = T u = \frac{E^2 + B^2}{2} u + E \times B \). (The Poynting 4-vector). Now \( s \) is null, i.e. \( \langle s, s \rangle = 0 \). Since \( \langle s, s \rangle = \langle Ts, Ts \rangle = \langle T^2 s, s \rangle = (0, s) = 0 \). So image \( (T^2) = \text{span } s \). Then \( \text{dim ker } T = 3 \). □

Now consider the non-null case. Then \( \lambda_T \neq 0 \). Hence \( \lambda_{cF} \neq 0 \), so one of \( \lambda_F \) or \( \lambda_{F^*} \) is not zero. Hence there are two real null eigenvectors of \( cF \), \( s \) for \( \lambda_{cF} \) and \( s_- \) for \( -\lambda_{cF} \). Both \( s \) and \( s_- \) are linearly independent. Since \( T = \frac{1}{2} cFcF, s \) and \( s_- \) are both eigenvectors of \( T \) with eigenvalue \( \lambda_T > 0 \).

Let \( \Pi_+ \) be the space of eigenvectors of \( T \) in \( T_x(M) \) corresponding to \( \lambda_T \) and let \( \Pi_- \) be the space of eigenvectors corresponding to \( -\lambda_T \). Then \( \Pi_+ = \text{image } (\Phi_+) \) and \( \Pi_- = \text{image } (\Phi_-) \) where \( \Phi_+ = \lambda_T I + T \) and \( \Phi_- = -\lambda_T I + T \). Now \( \Phi_\pm \) are symmetric with respect to \( \langle \cdot, \cdot \rangle \). Note that \( \Phi_\pm^2 = \pm 2\lambda_T \Phi_\pm \) and \( \Phi_+ \Phi_- = \Phi_- \Phi_+ = 0 \), all because of the fact that \( T^2 = \lambda_T^2 I \). From this we obtain:

**Proposition 5.7.** Let \( F \) be non null.

a) \( \Pi_+ \) is orthogonal to \( \Pi_- \).

b) \( \Pi_+ \) is time-like and \( \Pi_- \) is space like.

c) \( \text{dim } \Pi_+ = \text{dim } \Pi_- = 2 \).

d) \( F(\Pi_\pm) \subset (\Pi_{\pm}) \), i.e. \( \Pi_\pm \) are invariant subspaces of \( F \).

**Proof.** The following two lemmas prove a), b) and c). And d) follows since for \( v \in \Pi_\pm \) we have \( \pm \lambda_T F(v) = F(\pm \lambda_T v) = F(T(v)) = T(F(v)) \). So \( F(v) \in \Pi_\pm \). □

**Lemma 5.8.** Suppose \( Q : T_x \to T_x \) is symmetric with respect to \( \langle \cdot, \cdot \rangle \). If \( Q \) has a time-like eigenvector, then \( Q \) has an orthonormal frame of eigenvectors.
Proof. Let $u$ be a time-like eigenvector of $Q$. We may assume that $\langle u, u \rangle = -1$. Consider $T_u^x$, the space of vectors orthogonal to $u$. Then $Q : T_u^x \to T_u^x$ since $\langle u, Qu \rangle = \langle Qu, v \rangle = \lambda_Q \langle u, v \rangle = 0$ if $v \in T_u^x$. Hence $Q : T_u^x \to T_u^x$. But $T_u^x$ is space-like and $\langle , \rangle$ on $T_u^x$ is positive definite and $Q$ is symmetric. Hence there is an orthonormal set of eigenvectors on $T_u^x$ by a famous theorem. Call them $e_1, e_2, e_3$. Then $u, e_1, e_2, e_3$ is the desired frame. \qed

Lemma 5.9. Let $Q : T_x \to T_x$ be a linear map which is
a) symmetric with respect to $\langle , \rangle$, i.e. $\langle Qu, v \rangle = \langle v, Qu \rangle$.
b) $Q^2 = \lambda^2 I$.
c) $\text{Trace } (Q) = 0$.
d) $\langle u, Qu \rangle < 0$ for some future timelike $u$.

Then if $\lambda = 0$, there is a null eigenvector $s$ so that image $(Q) = \text{span } s$. If $\lambda \neq 0$, then the set of all eigenvectors corresponding to $\pm \lambda$ form two 2 dimensional subspaces $\Pi_{\pm}$, and $\Pi_+$ is orthogonal to $\Pi_-$, and $\Pi_+$ is time-like and $\Pi_-$ is space like.

Proof. Suppose $\lambda = 0$. Then $\langle Qu, v \rangle = \langle Q^2v, v \rangle = 0$ for all $v \in T_x$. So the image of $Q$ consists of null-vectors. Since $\langle u, Qu \rangle < 0$ for some time-like $u$, we see that $Qu \neq 0$ and that $Q(Qu) = 0$. So $Qu$ is the desired $s$.

Suppose $\lambda \neq 0$. Let $\langle u, Qu \rangle < 0$ for observer $u$. Consider $\lambda u + Qu$. Then $\langle \lambda u + Qu, \lambda u + Qu \rangle = -2\lambda^2 + 2\lambda \langle u, Qu \rangle < 0$. So $\lambda u + Qu$ is time-like. But $Q(\lambda u + Qu) = \lambda Qu + Q^2u = \lambda(\lambda u + Qu)$. So $\lambda u + Qu$ is a time-like eigenvector. Thus by Lemma 5.8, there is an orthonormal eigenvector frame. Since trace $(Q) = 0$, two of the vectors of the frame correspond to $\lambda$ and generate a time-like plane $\Pi_+$ and the orthogonal two generate $\Pi_-$ and are space-like. \qed

Corollary 5.10. If $Q : T_x \to T_x$ is as in the theorem above, there is an antisymmetric $F : T_x \to T_x$ so that $T_F = Q$.

Proof. If $\lambda \neq 0$, $\Pi_+$ intersects the light cone in two null-subspaces generated by, say, $s_+$ and $s_-$ respectively. Let $\lambda_F = \sqrt{2\lambda}$. Define $F s_+ = \lambda_F s_+$ and $F s_- = -\lambda_F s_-$. Let $F(v) = 0$ for all $v$ in $\Pi_-$ so we are defining $\lambda_F = 0$. Then there is a unique linear map which satisfies these conditions and $Q = F^2 - \lambda_F^2 I$. Note $F$ is antisymmetric on $\Pi_-$ since it is trivial and on $\Pi_+$ since

$$\langle \alpha s_+ + \beta s_-, F(\alpha s_+ + \beta s_-) \rangle = \langle \alpha s_+ + \beta s_-, \alpha \lambda_F s_+ - \beta \lambda_F s_- \rangle = 0,$$

since $s_+$ and $s_-$ are null.

If $\lambda = 0$, choose observer $u$ and let $s = Qu$. Choose $E$ and $B \in T_u$ so that $s, E, B$ are in the kernel of $Q$ and are mutually orthogonal and of sufficient length so that $s = E^2 u + E \times E$ where $B = E$. Then let $Fu = E$, $F(B) = 0$, $F(s) = 0$ and $F(E) = s$. Then $F$ is determined and $F^2 = Q$. \qed

Remark 5.11.

The question is, given $Q$ over $TM$, does there exist an $F$ so that $Q = T_F$?
6. Complex Eigenvectors

Let \( \phi_+ = \lambda_c F I + cF \) and \( \phi_- = -\lambda_c F I + cF \). Let \( \bar{\phi}_+ = \overline{\lambda_c F I + cF} \) and \( \bar{\phi}_- = \overline{-\lambda_c F I + cF} \). Since \( \lambda F^2 I + F \overline{cF} = \overline{cF} cF \), we obtain the following facts.

**Theorem 6.1.** Let \( cF : T_x \otimes \mathbb{C} \to T_x \otimes \mathbb{C} \) and \( cF \neq 0 \).

a) The image of \( \langle \phi_\pm \rangle \) equals the \( \pm \lambda_c F \) eigenspace of \( cF \) and the image of \( \langle \bar{\phi}_\pm \rangle \) equals the \( \pm \lambda_c F \) eigenspace of \( cF \).

b) The kernel of \( \langle \phi_\pm \rangle \) equals the \( \pm \lambda_c F \) eigenspace of \( cF \), and the kernel of \( \langle \bar{\phi}_\pm \rangle \) equals the \( \pm \lambda_c F \) eigenspace of \( \overline{cF} \).

c) The eigenspaces of \( cF \) and \( \overline{cF} \) consist of null vectors.

d) The eigenspaces of \( cF \) and \( \overline{cF} \) have dimension 2.

**Proof.**

We easily see that

\[
cF\phi_\pm = \pm \lambda_c F \phi_\pm, \quad cF\bar{\phi}_\pm = \pm \overline{\lambda_c F} \bar{\phi}_\pm
\]  

(29)

\[
\phi_\pm \bar{\phi}_\mp = 0, \quad \bar{\phi}_\pm \bar{\phi}_\mp = 0
\]  

(30)

\[
\langle \phi_\pm v, w \rangle = \langle v, \bar{\phi}_\mp w \rangle
\]  

(31)

a) follows from (29)

b) follows from (30) and a)

c) follows from a) and (31).

d) For an observer \( u \), the vectors \( \phi_+ \bar{\phi}_- u \) and \( \phi_+ \bar{\phi}_- u \) are eigenvectors of \( cF \) by (29).

Now \( \phi_+ \overline{\phi}_- u \) is an eigenvector of \( cF \) corresponding to \( \lambda_c F \) as well as an eigenvector of \( \overline{cF} \) corresponding to \( \overline{\lambda_c F} \). On the other hand \( \phi_+ \overline{\phi}_- u \) is an eigenvector of \( cF \) corresponding to \( \lambda_c F \) and also an eigenvector of \( \overline{cF} \) corresponding to \( -\overline{\lambda_c F} \). If \( \phi_+ \overline{\phi}_- u \) is linear dependent on \( \phi_+ \overline{\phi}_- u \), then \( -\overline{\lambda_c F} = \overline{\lambda_c F} \), hence \( \lambda_c F = 0 \), hence \( F \) is null. Thus if \( F \) is nonnull, \( \phi_+ \overline{\phi}_- u \) and \( \phi_+ \overline{\phi}_- u \) are linearly independent eigenvectors. If \( F \) is null, then \( F^2 u = E^2 u + E \times B \) and \( cF u = E + iB \) are linearly independent eigenvectors of the eigenspace. Hence \( \dim(\text{image}(\phi_+)) \geq 2 \) and similarly \( \dim(\text{ker}(\phi_+)) \geq 2 \). Therefore d) is proved. \( \Box \)

**Lemma 6.2.** Suppose \( a \) and \( b \) are real vectors in \( T_x \). Then \( a + ib \in T_x \otimes \mathbb{C} \) is null if and only if either \( a \) or \( b \) are linear dependent null vectors, or \( a \) and \( b \) are both space–like and have the same length and are orthogonal.

**Proof.** Let \( v = a + ib \). Now \( \langle v, v \rangle = 0 \) if and only if \( \langle a, a \rangle = \langle b, b \rangle \) and \( \langle a, b \rangle = 0 \). If \( a \) or \( b \) is null, so is the other. Since they are orthogonal null vectors, they must be linearly dependent.

On the other hand, if one of \( a \) or \( b \) is space–like, so is the other and they have equal lengths and are orthogonal. Neither \( a \) or \( b \) can be time–like, since if one were, they both would be. But no two time–like vectors are orthogonal. \( \Box \)

**Lemma 6.3.** Let \( a \) and \( b \) be space–like in \( T_x \). Then \( a \) and \( b \) span a space–like plane if and only if \( a^2 \) \( b^2 - \langle a, b \rangle^2 > 0 \). Thus if \( a \) and \( b \) are orthogonal and space–like, they span a space–like plane.
Proof. $\langle \alpha a + \beta b, \ \alpha b + \beta b \rangle$ is greater than zero if and only if the determinant of
\[
\begin{pmatrix}
a^2 & \langle a, b \rangle \\
\langle a, b \rangle & b^2
\end{pmatrix}
\]
is greater than zero.

Lemma 6.4. Any null subspace of $T_x \otimes \mathbb{C}$ has a degenerate inner product. That is any two vectors in a subspace of null vectors are orthogonal.

Proof. Suppose $s$ and $s'$ are null vectors in a null subspace $V$. Then $s + s'$ is in $V$. Hence $\langle s + s', s + s' \rangle = 0$. Expanding the left side yields $\langle s, s' \rangle = 0$.

Remark 6.5. Null planes.

Suppose $E + iB \in T^*_x \otimes \mathbb{C}$ is a null vector in the rest space of an observer $u$. Then $s = E^2 u + E \times u B$ is a real null vector. Now $s$ and $E + iB$ span a null plane $V$, which is the image of $cF$ where $F$ is a null skew symmetric operator with $Fu = E$ and $F^*u = -B$.

Also $s_- = E^2 u - E \times B$ is a real null vector. Again $s_-$ and $E + iB$ span a null plane $V'$, which is the image of $\overline{cG}$ for a null $G$ so that $Gu = E$ and $G^*u = B$.

Thus we have two kinds of null planes, those which are the images of null $cF$ and those which are the images of null $\overline{cF}$.

We can think of these null planes from a geometric point of view. Suppose $E + iB$ is a space–like null vector. Then $E$ and $B$ span a space–like plane $\Pi_s \subset T_x$, by Lemma 6.3. Let $\Pi_t$ be the time–like plane orthogonal to $\Pi_s$. Then $\Pi_t$ intersects the light cone in two one–dimensional null lines. One of these null lines and $E + iB$ spans a null plane and the other line and $E + iB$ spans the “conjugate” null plane containing $E + iB$.

Thus given a space–like null vector $v$, there are exactly two null planes containing $v$. We say these two planes are $\ast$–conjugate with respect to $v$. If $V$ is a null plane and contains a light–like null vector $v$, then we say that $\mathbf{V}$ is $\ast$–conjugate to $V$ with respect to $v$. The planes which are the image of a null $cF$ are called $\ast$–consistent null planes and those which are the image of a null $\overline{cF}$ are called $\ast$–inconsistent.

Lemma 6.6. In $T_x \otimes \mathbb{C}$

a) Every null plane contains a real null vector

b) The eigenspaces of $cF$ are $\ast$–consistent planes. The eigenspaces of $\overline{cF}$ are $\ast$–inconsistent planes.

c) The intersection of a $\ast$–consistent and a $\ast$–inconsistent plane is one dimensional.

Proof. a) Choose an appropriate basis and use analysis to obtain the conditions for a null–plane.

b) By continuity and connectivity of $\ell_x \oplus \mathbb{C}$.

c) Let $V$ be the null plane spanned by $A = E + iB$ and $s = E^2 u + E \times B$ for $u$ an observer orthogonal to $E$ and $B$. Then $V$ is both the image and kernel of a null $cF$ such that $cFu = A$, since $cF^2 = 0$. Let $\overline{W}$ be a $\ast$–inconsistent null plane. It is the image of some null $\overline{cG}$. Now $\overline{W} \neq V$ since $\overline{W}$ is $\ast$–inconsistent, so $cF(\overline{W}) \neq 0$. Since $cF$ and $cG$ commute by Theorem 4.8, we see that $\overline{W} \cap V \neq 0$. So $W \cap V$ is one dimensional. □
Theorem 6.7. Let $F$ and $G$ be skew symmetric bundle maps. Let $\phi = \lambda_cF I + cF$ and $\gamma = \lambda_cG I + cG$. Note that the choice of which of the two eigenvectors $\pm \lambda_cF$ is not reflected in the notation.

a) $\phi \gamma = \gamma \phi$.
b) The image of $(\phi \gamma)$ is one dimensional and is generated by a null vector which is an eigenvector of both $cF$ and $cG$ with associated eigenvalues $\lambda_cF$ and $\lambda_cG$ respectively.
c) The image of $(\phi \phi)$ is generated by a real null vector $s$ which is an eigenvector of $cF$ corresponding to $\lambda_cF$.

Proof.
b) From (a), the image of $\phi \gamma$ is the one dimensional sub space $(\text{image } (\phi)) \cap (\text{image } (\gamma))$.
c) Let $\gamma = \phi$ and apply (b).

Corollary 6.8. The eigenvector $s$ for a skew symmetric bundle map satisfies the following equation in terms of $E_u$ and $B_u$,

$$s = 2(\lambda_T u + \frac{E^2 + B^2}{2} + E \times B + \lambda_F E - \lambda_F^* B).$$

Proof. Recall $s = \phi \phi u$ where $\phi \gamma = \gamma \phi$. Expand that equation and use equations (5), (13), and (26) and Proposition 5.1. $\Box$

Theorem 6.9. $\Psi_v : \ell \oplus C \rightarrow T(M) \otimes C$ is a bundle equivalence if $v$ is a non null vector field.

Proof. Both bundles are 4 dimensional and $\Psi_v$ is a bundle map, so we only need to show that $\Psi_v$ has zero kernel. So assume $(\alpha I + cF)v = 0$. Then $v$ is an eigenvector of $cF$, hence by Theorem 6.1d we have, in contradiction to the hypothesis, that $v$ is a null vector.

7. Eigenbundles

Given a skew symmetric $F \in \Gamma(\ell)$, we define a map $\psi_F : M \rightarrow C$ by setting

$$\psi_F(m) = \lambda_{cF_m}^2 = (E^2 - B^2) + 2i(E \cdot B)$$

evaluated at $m$.

We define a sequence of open submanifolds $M \supset M_0 \supset M_1$ based on the given $F$.

$$M_0 = \{ m \in M \mid F_m \text{ is defined and not identically zero}\}$$
$$M_1 = \{ m \in M \mid F_m \text{ is not null}\}$$

Since $F_m$ is null if and only if $\lambda_{cF} = 0$, we see that

$$\psi_F^{-1}(C - 0) = M_1.$$

Skew Symmetric Bundles Maps
Definition 7.1. We define the degree of $F$, denoted $\deg F$, to be the degree of $\psi_F : M_1 \to \mathbb{C} - 0$. We define the degree of $\psi : M_1 \to \mathbb{C} - 0$ to be the integer which corresponds to the generator of the subgroup $(\text{image } (\psi)) \subset H_1(\mathbb{C} - 0) \cong \mathbb{Z}$.

Remark 7.2. The degree of $\psi$ in Definition 7.1 is related to the usual Brouwer degree of Algebraic Topology. This can be seen in [G4]. Note, the definition of $\deg \psi$ yields a non-negative integer, in contrast to the usual Brouwer degree.

Theorem 7.3. The following are equivalent:

a) $\deg \psi$ is even
b) there is a line bundle of eigenvectors of $F$ over $M_1$
c) the invariant plane bundle $\Pi_+$ is an orientable 2-plane bundle over $M_1$

Proof. Consider $\tilde{M}_1$, the set of pairs $(m, \alpha)$ where $m \in M_1$, and $\alpha \in \mathbb{C} - 0$ is equal to either one of the two eigenvalues $\pm \lambda_{TF_m}$. Then $\tilde{M}_1$ is a double covering space of $M_1$. If $\tilde{M}_1$ is not connected, then it is possible to choose one $\alpha$ at each $m$ in a continuous way over $M_1$. The choice of the eigenvector corresponding to $\alpha(m)$ gives the line bundle of eigenvectors. Conversely, a line bundle of eigenvectors over $M_1$ will select a continuous choice of corresponding eigenvalues, so $\tilde{M}_1$ will be disconnected. Now we have a commutative diagram

$$
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\tilde{\psi}} & \mathbb{C} - 0 \\
\downarrow{p} & & \downarrow{\text{sq}} \\
M & \xrightarrow{\psi} & \mathbb{C} - 0
\end{array}
$$

where $p(m, \alpha) = m$ and $\tilde{\psi}(m, \alpha) = \alpha$ and $\text{sq}(z) = z^2$. If $\tilde{M}$ is not connected, there is a cross-section $s$ to $p$. Then $\deg \psi = \deg (\text{sq} \circ \tilde{\psi} \circ s)$ is even since the degree of $\text{sq}$ is 2. This proves that (a) and (b) are equivalent.

For (c), the plane bundle $\Pi_+$ of time-like invariant planes of $F$ is also the eigenbundle of $T_F$ corresponding to $\lambda_T > 0$. Now we can always choose a nonzero time-like vector field $u$ over $M$. Then $\Phi_+(u) = \lambda_T u + Tu$ is a non zero vector field of eigenvectors of $T$. Hence there is a trivial line sub-bundle $\varepsilon$ in $\Pi_+$. Hence $\Pi_+ = \varepsilon \oplus \nu$, where $\nu$ is the orthogonal line bundle. If $\Pi_+$ were orientable, then $\nu$ would be trivial and we could use the direction in $\nu$ to choose at each $m$ one of the two null eigenvector subspaces in $(\Pi_+)_m$. Hence we would get a line bundle $\eta$ of eigenvectors of $F$. Conversely, if the eigenbundle $\eta$ existed, then $\Pi_+ = \varepsilon \oplus \eta$. Since $\eta$ is a trivial line bundle (because $M$ is time-oriented) this implies that $\Pi_+$ is orientable. □

Scholium 7.4. The Phase of $\psi$.

We may write $\psi_F(m) = \lambda_{TF_m}^2 = 2\lambda_T m e^{i\alpha}$ for some angle $\alpha$, which we will call the phase of $\psi$. Suppose we have two paths in space-time from $A$ to $B$ which do not pass over radiation. If we measure the difference of the phase after having traveled from point $A$ to point $B$ along the two paths, we will find that they differ by a multiple $n$ of $2\pi$. If $n$ is not zero, then the two paths linked wave-like regions. If $n$ is even, then the continuous extension of the same eigenvector at $A$ along the two paths result in the same eigenvector at $B$.
Corollary 7.5. Let $F$ be a skew symmetric bundle map. There is a plane sub-
eigen–bundle $\eta$ of $T(M_0) \otimes \mathbb{C}$ if and only if deg $\psi_F$ is even.

Proof. If deg $\psi_F$ is even, then we can choose continuously one $\lambda_{\psi_{F_m}}$ out of the two possible. Thus $\phi = \lambda_{\psi_{F}}I + cF$ is a well–defined bundle map since there is no ambiguity with $\lambda_{\psi_{F}}$. Now over $M_0$, the image of $(\phi_m)$ is always a two plane by Theorem 6.1d. The unambiguous choice of $\lambda_F$ gives a bundle map $\phi$ whose image is a plane bundle $\eta$. Conversely, if $\eta$ is a plane eigenbundle, it selects the eigenvalue $\lambda_{\psi_{F_m}}$ at each $m$ which correspond to the plane $\eta_m$.

Remark 7.6. Chern Classes.

Complex plane bundles are classified by their Chern classes. For any skew symmetric $F$, there is a plane bundle over $M_0 - M_1$ given by the image of $(cF)$. Let $\alpha \in H^2(M_0 - M_1; \mathbb{Z})$ be its Chern class. Then if deg $\psi$ is even, there is a $\beta \in H^2(M; \mathbb{Z})$ such that $i^*(\beta) = \alpha$ where $i$ is the inclusion map.

We may do even better. Given $F \in \Gamma(\ell)$, let $\hat{\mathcal{M}} = \{ (m, z) \in M_0 \times \mathbb{C} \mid z = \pm \lambda_{\psi_{F_m}} \}$. Let $\hat{T}$ be the pullback bundle of $T(M_0)$ by $p$. Then the plane subbundle of $\hat{T}$ defined by the image of $(\phi)$ is defined. This bundle has a Chern class $\beta \in H^2(\hat{\mathcal{M}}; \mathbb{Z})$ and $i^*(\beta) = \alpha$ always. Note that $\hat{\mathcal{M}}$ is not always a manifold.

Corollary 7.7. Let $F \in \Gamma(\ell)$ and suppose that $E \cdot B = 0$ for all $m \in M_0$. Then deg $\psi_F = 0$ and ker$(F)$ is a plane sub–bundle of $T(M_0)$.

Corollary 7.8. If $0 \in \mathbb{C}$ is a regular value of $\psi_F$, then deg $\psi_F = 1$.

Proof. If $0$ is a regular value of $\psi_F$ we can find a small circle about $0$ which lifts to $M_1$. Thus $1 \in \mathbb{Z} \cong H_1(\mathbb{C} - 0; \mathbb{Z})$ is the image of $\psi_s$.

Scholium 7.9. Electrons.

a) A classical free electron at rest in Minkowski space $M$ can be represented by an $F$ such that $E(r, t) = -\frac{r}{r^3}$ and $B(r, t) = 0$. Thus $M_0 = M_1 = M$– (the time axis). The deg of the free electron is zero by Corollary 7.7.

b) A classical electron at rest in a constant magnetic field will be represented by an $F$ such that $E(r, t) = -\frac{r}{r^3}$ and $B = e_x$. Then $M_0 = M$– (the time axis) and $M_1 = M_0 - S$ where $S$ in each space slice is a circle of radius $1$ in the $yz$ plane centered on the electron. The deg of the electron in a constant magnetic field is $1$ by Corollary 7.8.

Scholium 7.10. Electro–Magnetic Duality Rotation.

Equation (9) is called the Electro–Magnetic Duality Rotation by Physicists. We noted that $T_{\psi \cdot F} = T_F$ in Corollary 5.2. Thus for any map $\varphi : M_0 \rightarrow S^1$, the skew symmetric bundle map $\varphi : F$ defined by $(\varphi \cdot F)_m = \varphi(m)F_m$ gives rise to the same $T$ as does $F$.

On the other hand, suppose $F' = \varphi F$. Then $\psi_{F'} = \varphi^2 \psi_F$. So $\psi_{s'} = (2\varphi_s + \psi_s)$ on the first homology groups. Thus deg $\psi' = \deg \psi + 2k$ for some $k$. So deg $(\varphi F)$ has the same parity as deg $(F)$.

Theorem 7.11. The space of skew symmetric operators over $M_0$ which gives rise to the same $T$ is homeomorphic to the space of maps $\varphi : M_0 \rightarrow S^1$. The path components of map $(M_0, S^1)$ correspond to the elements of $H^1(M_0; \mathbb{Z})$. 

Proof. We only need show that given $F'_m$ and $F_m$, with the same $T$, there is a $\theta$ such that $F'_m = e^{i\theta}F_m$. Now $F'_m$ and $F_m$ must have the same invariant planes and the same eigenvectors. Also $\lambda_F^2 + \lambda_{F'}^2 = 2\lambda_T$ and the same holds for $F'$. So we can rotate $F$ until $\lambda_F = \lambda_{F'}$ and $\lambda_F = \lambda_{F'}$. So $F$ and $F'$ agree on $\Pi_+$. Similarly they agree on $\Pi_-$. For null $F$ and $F'$, the $E$ and $B$ must have the same length and same $E \times B$, so one can rotate into the other.

Scholium 7.12. Electron “States” for the same Energy Momentum.

a) For the free electron $F$ of 7.10, $H^1(M^0;\mathbb{Z}) \cong 0$. So all the “states” $e^{i\theta}F$ are homotopic to one another. All of them have eigenvector bundles.

b) For the electron in a constant magnetic field, $F$, there are infinitely many homotopy classes of “states” giving rise to the same energy momentum $T_F$. Since $H^1(M_1;\mathbb{Z}) \cong \mathbb{Z}$, these states correspond to the integers. Each state has odd degree. Thus there is no eigenvector line bundle over $M_1$ for any state.

8. Lorentz Transformations

Lorentz Transformations play an important role in Physics. They are an artifact of Level $-16$, the standard coordinates of Minkowski space. As we move up through the levels of notation they seem to dwindle in importance. That is because one of their main functions, relating different choices of systems of notation, is eliminated as the choices are eliminated. What remains are two things, changes of observers in Level $-2$ as mentioned in Scholium 2.5, and the Gauge group of bundle isometries of Level 0. At these levels we obtain a fresh perception of the Lorentz Transformation.

Level $-16$. Minkowski Space–time.

At Level $-10$ we have coordinates for the tangent space, but not for the manifold. A choice of 4 functions coordinatizes $M$. We need to tie in our bases of the tangent bundle with the gradients of the coordinate functions. We can use the gradients as a basis, but usually they will not be orthonormal. Or, we can use the Gram Schmidt process on them to get a more complicated orthonormal basis. The best thing would be to find coordinates whose gradients are orthonormal. That is what is done for Minkowski space.

So let $M = \mathbb{R}^4$. Put coordinates $t, x, y, z$ on $M$ with orthonormal gradients $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. We could choose another such coordinate system $t', x', y', z'$. The formulas relating them are called the Lorentz transformation. See [F], page I - 15 - 3.

Scholium 8.1. Lorentz Transformations of Electro-Magnetism.

Feynman in [F], II, 26.2 carries out the Lorentz transformation in Level $-16$, and then tries to express the results in notation at Level $-2$. Calling $E'$ and $B'$ the transformed version of the original $E$ and $B$, he relates them by the formula

\begin{align*}
E'_\parallel &= E_\parallel \\
E'_\perp &= \frac{(E + w \times B)_\perp}{\sqrt{1 - w^2}} \\
B'_\parallel &= B_\parallel \\
B'_\perp &= \frac{(B - w \times B)_\perp}{\sqrt{1 - w^2}}.
\end{align*}

(8.1)

Here $E'_\parallel$ means the component of $E'$ parallel to the relative velocity $w$ of the two coordinate frames and $E'_\perp$ means the component of $E'$ orthogonal to $w$. This formula is both correct and meaningless.
Let us give a Level \(-2\) derivation of (8.1). Let \(E = Fu\) and \(B = -F^*u\). Let \(u' = \frac{1}{\sqrt{1-w^2}}(u+w)\). Then \(E' = Fu'\) and \(B' = -F^*u'\). Substituting (21) and (23) into these formulas results in

\[
E' = \frac{E \cdot w}{\sqrt{1-w^2}} (u + \frac{w}{w^2}) + \frac{1}{\sqrt{1-w^2}} \left( E - \frac{E \cdot w}{w^2} w + w \times_u B \right)
\]

(8.2)

\[
B' = \frac{B \cdot w}{\sqrt{1-w^2}} (u + \frac{w}{w^2}) + \frac{1}{\sqrt{1-w^2}} \left( B - \frac{B \cdot w}{w^2} w + w \times_u E \right)
\]

(8.3)

Each of the four terms on the right hand sides of the above equations are orthogonal to \(u'\) and hence lie in \(T_{u'}\). The last terms in each equation are orthogonal to \(w\) and lies in a plane orthogonal to the \(u,u'\) plane. These are \(E'_{\perp}\) and \(B'_{\perp}\) respectively.

The first terms in each equation are the parallel components

\[
E'_{\parallel} = \left( \frac{E \cdot w}{\sqrt{1-w^2}} \right) (u + \frac{w}{w^2}) \quad \text{and} \quad B'_{\parallel} = \left( \frac{B \cdot w}{\sqrt{1-w^2}} \right) (u + \frac{w}{w^2})
\]

(8.4)

But \(E_{\parallel} = \frac{E \cdot w}{w} w\) and \(B_{\parallel} = \frac{B \cdot w}{w} w\). So \(E'_{\parallel} \neq E_{\parallel}\) and \(B'_{\parallel} \neq B_{\parallel}\) contrary to the assertion in (8.1). However they are both in the \(u,w\) plane and \(E'_{\parallel} = E_{\parallel}\) and \(B'_{\parallel} = B_{\parallel}\).

**Proof.**

\[
E'_{\parallel} \cdot E'_{\parallel} = \left( \frac{E \cdot w}{1-w^2} \right)^2 \left( u + \frac{w}{w^2} \right) \cdot \left( u + \frac{w}{w^2} \right)
\]

\[
= \left( \frac{E \cdot w}{1-w^2} \right)^2 \left( -1 + \frac{w^2}{w^2} \right)
\]

\[
= \left( \frac{E \cdot w}{w} \right)^2 = E^2_{\parallel}
\]

Similarly for \(B'_{\parallel} = B_{\parallel}\).

**Scholium 8.2. The Doppler Shift.**

Let \(s_u\) be an eigenvector of \(F\) corresponding to \(\lambda_F\) as seen by an observer \(u\). Suppose

\[
u' = \frac{1}{\sqrt{1-w^2}}(u+w)
\]

(8.5)

is another observer. Then \(u'\) sees a different eigenvector \(s_{u'}\). But \(s_{u'}\) must be a multiple of \(s_u\) since they are eigenvectors. So the question is, what is the multiple in terms of \(E, B\) and \(w\)? The answer is:

\[
s_{u'} = \frac{1}{\sqrt{1-w^2}} \left[ 1 + \frac{(E \times B) \cdot w + \lambda_F E \cdot w - \lambda_F E \cdot B \cdot w}{\lambda_T + \frac{E^2 + B^2}{2}} \right] s_u.
\]

(8.6)
Proof. Define
\[ \phi(v) = \left( \frac{\langle v, s \rangle}{\langle u, s \rangle} \right) s_u \] (8.7)
where \( s \) is an eigenvector corresponding to \( -\lambda_F \). Then \( \phi \) is a linear map whose image is the span of \( s_u \) and whose kernel is the space of vectors orthogonal to \( s \).

Now \( \phi(u) = s_u \).

Now \( \Phi := (\lambda c F I + c F) \circ (\lambda c F I + c F) \) has the same properties and let \( \Phi(u) := s_u \).

Then \( \Phi = \phi \). Let \( s_- = \Phi^{-1}(u) = (\lambda c F I + c F) \circ (\lambda c F I + c F) u \). Now
\[ s_u = 2 \left( \lambda_T + \frac{E^2 + B^2}{2} u + E \times B + \lambda_F E - \lambda_F B \right) \] (8.8)
and
\[ s_- = 2 \left( \lambda_T + \frac{E^2 + B^2}{2} u + E \times B - \lambda_F E + \lambda_F B \right) \] (8.9)
from Corollary 6.8 and \( s_- \) is the same with the signs changed on \( \lambda_F \) and \( \lambda_F^* \).

Now \( s_u' = \phi(u') = \left( \frac{\langle u', s \rangle}{\langle u, s \rangle} \right) s_u \). Substituting (8.5) into this equation yields
\[ s_u' = \frac{1}{\sqrt{1 - w^2}} \left( 1 + \frac{\langle w, s \rangle}{\langle u, s \rangle} \right) s_u. \] (8.10)

Now
\[ \langle u, s \rangle = -2 \left( \lambda_T + \frac{E^2 + B^2}{2} \right) \] (8.11)
using (8.9). Then using (8.9) to calculate \( \langle w, s \rangle \) and substituting this into (8.10) we obtain (8.6).

Now (8.6) holds for all \( F \in \Gamma(\ell) \). If we restrict to null \( F \) we should see (8.6) reduce to a simpler form. In the null case \( \lambda_F = \lambda_F^* = 0 \) and \( E = B \). So equation (8.6) reduces to
\[ s_u' = \frac{1}{\sqrt{1 - w^2}} \left( 1 - w \cdot \frac{E \times B}{E^2} \right) s_u. \] (8.12)
Now \( w \cdot \frac{E \times B}{E^2} \) is the component along the \( E \times B \) direction. If we assume that \( w = w_r \), that is \( w \) is pointing in the radial direction, then
\[ s_u' = \frac{1 - w_r}{1 + w_r} s_u. \] (8.13)
Here \( \frac{1 - w_r}{1 + w_r} \) is the Doppler shift ratio. This suggests that null \( F \) propagate along null geodesics by parallel translation.

Scholium 8.3. Eliminating \( E \times B \).

In the non–null case there is a Lorentz transformation so that \( E' \times B' = 0 \). We may see this clearly using Level –2 methods. Suppose \( u' \) is an eigenvector of \( T_F \).

Then
\[ T_F u' = \lambda_T u' = \frac{E_{u'}^2 + B_{u'}^2}{2} u' + E_{u'} \times B_{u'}. \] (8.14)
The second equality shows that \( E_u' \times B_u' = 0 \) and \( \frac{E_{u'}^2 + B_{u'}^2}{2} = \lambda T \). Now we can always find an eigenvector \( u' \) by setting \( u' = (\lambda T + T)u \). Thus the relative velocity is

\[
 w = \frac{(E_u \times B_u)}{\lambda T + \frac{E_{u'}^2 + B_{u'}^2}{2}}. \tag{8.15}
\]

At Level \(-10\), the Lorentz transformations become equations relating the choice of orthonormal bases \( e_0, e_1, e_2, e_3 \) and \( e_0', e_1', e_2', e_3' \). In the block matrices formalism, the Lorentz transformation becomes an invertible matrix \( \Lambda \) so that

\[
\begin{pmatrix}
0 \\
E' \\
E' \times B'
\end{pmatrix} = \Lambda^{-1}\begin{pmatrix}
0 \\
E \\
E \times B
\end{pmatrix} \Lambda. \tag{8.16}
\]

Although we used many Level \(-10\) arguments in this paper, our statements were usually Level \(-2\). The only choices necessary were of different observers. The algebraic component of the Lorentz Transformations \( \Lambda \) becomes the bundle isometries of \( T(M) \), that is the group of Gauge Transformations. These can be thought of at Level 0.

**Remark 8.4. The exponential map \( e^F \).**

The exponential map maps the “Lie Algebra” \( \Gamma(\ell) \) onto the group of bundle isometries \( \mathcal{G} \) of \( T(M) \). This exponential map is a diffeomorphism near the identity. It has a beautiful representation using the \( e^F \) notation.

\[
e^F := I + F + \frac{1}{2!}F^2 + \frac{1}{3!}F^3 + \ldots \tag{8.17}
\]

where \( F^n \) means \( F \) composed with itself \( n \)–times. For \( F \) a bundle map, \( e^F \) satisfies several properties.

a) \( e^F \) is a well–defined bundle map

b) \((e^F)^{-1} = e^{-F} \) if \( F \) is skew symmetric

c) \( \langle e^F v, w \rangle = \langle v, e^{-F} w \rangle \) if \( F \) is skew symmetric

d) \( e^{F + F'} = e^F \circ e^{F'} \) if \( FF' = F'F \)

e) \( \left. \frac{d}{dt} e^F \right|_{t=0} = F \)

f) Every isometry \( Q \) can be written as \( Q = e^F \) for a skew symmetric \( F \), at least locally.

g) If \( s \) is an eigenvector of \( F \) corresponding to \( \lambda_F \), then \( s \) is an eigenvector of \( e^F \) corresponding to \( e^{\lambda_F} \).

h) If \( F \) is skew symmetric and null, then since \( F^3 = 0 \) we have \( e^F = I + F + \frac{1}{2}F^2 \).

Now these properties also hold for \( \mathbb{F} \in \Gamma(\ell \otimes \mathbb{C}) \) and \( e^\mathbb{F} \). So the fact that \( \langle cF \rangle^2 = \lambda_F^2 \) gives us the following striking result.

**Theorem 8.5.** \( e^{cF} = \cosh(\lambda_F)I + \frac{\sinh(\lambda_F)}{\lambda_F} (cF) \)

where \( \cosh(x) = \frac{e^x + e^{-x}}{2} \)

and \( \sinh(x) = \frac{e^x - e^{-x}}{2} \).
Corollary 8.6. \( e^F = \left( \cosh \left( \frac{\lambda}{2} \right) I + \frac{\sinh \left( \frac{\lambda}{2} \right)}{\lambda} cF \right) \circ \left( \cosh \left( \frac{\lambda}{2} \right) I + \frac{\sinh \left( \frac{\lambda}{2} \right)}{\lambda} cF \right) \)

where \( \lambda = \lambda_{cF} \).

**Proof.** \( e^F = e^{(cF+\tau F)/2} = e^{cF/2} e^{\tau F/2} \), this last by Remark 8.4d. Then apply Theorem 8.5.

We leave it as an exercise to the reader to expand Corollary 8.6 and obtain an equation involving only real quantities.

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