Fermion Interactions in the Wilson Yukawa Approach for Lattice Chiral Gauge Theories

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Abstract

We consider fermion-gauge couplings in the Wilson-Yukawa approach for lattice chiral gauge theories. At the leading order of a fermionic hopping parameter expansion we find that the fermion-gauge coupling has a chiral and tree-like structure. We argue that this fermion-gauge coupling remains non-zero in the continuum limit taken in the Higgs phase. Possible fermion-scalar couplings in this approach are considered. We also evaluate the fermion interaction with an external gauge field in the slightly modified model and show that it has a chiral structure in general.
1 Introduction

Construction of chiral gauge theories on the lattice is an important task. The prime motivation for this effort is that one would like to have full understanding of the standard SU(2) × U(1)$_Y$ electroweak theory, which subsumes, but is not restricted to, perturbation expansions. Indeed, although the standard electroweak model has been quite successful phenomenologically, it leaves a number of fundamental questions unanswered, including the nature of the symmetry breaking and, in particular, why the chiral gauge symmetry SU(2) × U(1)$_Y$ is realized in a "spontaneously broken" mode, whereas the vectorial gauge invariances of SU(3)$_{\text{color}}$ and U(1)$_{\text{e.m.}}$ are realized symmetrically (in confined and deconfined modes, respectively). The conventional Higgs mechanism does not explain this since, among other things, no \textit{ab initio} reason is given for choosing the symmetry-breaking sign of the coefficient of $\phi^\dagger \phi$ in $\mathcal{L}_{\text{ew}}$. Partially in response to this inadequacy and the problems of the quadratic mass shift and necessity of fine-tuning in the Higgs sector, appealing alternative mechanisms have been proposed involving dynamical symmetry breaking without any fundamental scalars. However, one knows very little about the properties of such pure chiral gauge theories. A better understanding of these might well yield a deeper insight into how the electroweak symmetry is realized in nature. Moreover, since the top quark is now known to have a mass greater than the $W$ boson, the corresponding physical Yukawa coupling in the standard model is of order unity, so that in studying the physics of the top quark, it is desirable to use a method which is capable of nonperturbative calculations.

The lattice approach provides a potentially useful tool for all of these purposes since it deals with the full path integral and is not limited to perturbative expansions in couplings. However, there has been a serious difficulty in applying this method: because of fermion doubling, each lattice fermion field yields $2^d = 16$ fermion modes, half of one chirality and half of the other, so that the lattice theory is nonchiral (e.g. Ref. [1]). There are several lattice formulations of the chiral gauge theories. Among these formulations the Wilson-Yukawa approach\cite{4,5,6,7} has been extensively studied during the past few years\cite{8,9,10,11}. It is now established that the decoupling of fermion doublers can be made in the strong Wilson-Yukawa coupling region. Therefore it seems that this approach successfully describes
lattice chiral gauge theories. However base on a recent numerical study\cite{10} which suggests that only gauge singlet fermions become light in the symmetric phase of the fermion-scalar system in this approach, it is claimed\cite{11} that the gauge-fermion interaction vanishes in the continuum limit in the order of $a^2$ where $a$ is a lattice spacing, and therefore that the Wilson-Yukawa approach fails to describe the chiral gauge interaction. In this paper we consider this problem in detail to see whether the Wilson-Yukawa approach indeed fails or not. We calculate the gauge-fermion coupling using a fermionic hopping parameter expansion and find that it is likely that the fermion-gauge coupling in the Higgs phase remains non-zero in the continuum limit. The Wilson-Yukawa approach is still alive.

In the next section, we define a simple model in this approach where a right-handed part of the fermion is a gauge singlet. We briefly summarize how the fermion doublers are decoupled in the continuum limit. In sect. III we calculate the gauge fermion interaction\cite{12} using the hopping parameter expansion and find that the fermion-gauge coupling has a chiral and tree-like structure. We argue that this fermion-gauge coupling remains non-zero in the continuum limit taken in the Higgs phase. There we discuss a possible explanation why the gauge interaction in the standard model is perturbative. A Yukawa coupling is also considered. In sect. IV we calculate a charged fermion interaction with an external gauge field in the modified model given in ref.\cite{11} where it is claimed that this coupling becomes vector-like. We show that this coupling appears to be chiral in general. A conclusion is given in sect. V and a more complicated model where both left- and right-handed fermions are gauge non-singlet are considered.

2 The Model and the Decoupling of Fermion Doublers

We define a lattice chiral gauge theory with a right-handed fermion being gauge singlet in the Wilson-Yukawa approach. The action is given by

$$S = S_G + S_H + S_F + S_Y + S_{WY}$$  \hspace{1cm} (2.1)
where $S_G$ is the action for the gauge link variable $U_{n,\mu} \in U(1)$ or $SU(2)$,

$$S_H = \beta_h \sum_{n,\mu} r_n r_{n+\mu} \text{Re}[d_N \text{Tr}(g_n U_{n,\mu} g_n^\dagger)] - \lambda \sum_n [(r_n^2 - 1)^2 + r_n^2] \quad (2.2)$$

is the action for a Higgs field $\phi_n = r_n g_n$ with $r_n \in R$ defined by $r_n^2 = \phi_n^\dagger \phi_n$, $g_n \in U(1)$ or $SU(2)$ satisfying $g_n^\dagger g_n = 1$, and $d_N = 1$ for the $U(1)$ case and $d_N = 1/2$ for the $SU(2)$ case. $S_F + S_Y + S_{WY}$ is the fermionic part of the action where

$$S_F = \frac{1}{2} \sum_{n,\mu} \bar{\psi}_n [(U_{n,\mu} P_L + P_R) \psi_{n+\mu} - (U_{n,-\mu} P_L + P_R) \psi_{n-\mu}] \quad (2.3)$$

is the naive fermion action with $P_{L/R} = \frac{1 \pm \gamma_5}{2}$,

$$S_Y = y \sum_n r_n \bar{\psi}_n (g_n P_L + g_n^\dagger P_R) \psi_n \quad (2.4)$$

is the Yukawa interaction with a Yukawa coupling $y$, and

$$S_{WY} = -\frac{\bar{r}}{2} \sum_{n,\mu} \bar{\psi}_n [P_L (g_n \psi_{n+\mu} + g_n^\dagger \psi_{n-\mu} - 2g_n \psi_n) + P_R g_n^\dagger (\psi_{n+\mu} + \psi_{n-\mu} - 2\psi_n)] \quad (2.5)$$

is the Wilson-Yukawa term with the Wilson-Yukawa coupling $r$.

There are few remarks for the above action.

1. Only $g$ field instead of $\phi$ appears in the Wilson-Yukawa term. If $\phi$ is introduced in the Wilson-Yukawa term, we have to tune the Wilson parameter $r$ in order to decouple the fermion doublers in the continuum limit. Although this is possible, it is not so easy in this case to formulate the Hopping parameter expansion. Therefore we do not introduce $\phi$ in the Wilson-Yukawa term in this paper.

2. The lattice Higgs action $S_H$ is obtained from the usual continuum (cn) parametrization\cite{13}

$$S_{H,\text{cn}} = -\int d^4 x [D_\mu \varphi_{\text{cn}}]^2 - \lambda_{\text{cn}} (|\varphi_{\text{cn}}|^2 - v_{\text{cn}}^2)^2 \quad (2.6)$$

under the substitutions

$$\varphi_{\text{cn}} \equiv \sqrt{\frac{\beta_h}{2}} \phi$$
\[ v_{cn}^2 a^2 = \frac{\beta_h}{2} \left[ 1 + \frac{(\beta_h d - 1)}{2\lambda} \right] \]
\[ \lambda_{cn} = \frac{4}{\beta_h^2} \lambda \]

where \( d \) is the space-time dimension and \( a \) is the lattice spacing. The \( \lambda \to \infty \) limit is usually taken in the Wilson-Yukawa approach so that the Higgs field becomes non-linear \( \sigma \) model ( : \( \forall n, r_n = 1 \) ). The shift symmetry of the above action\[14\] prohibits Yukawa interactions between the gauge singlet fermion and the non-linear Higgs field in this limit. In this paper we use the linear Higgs field instead in order to allow the Yukawa interaction. See also ref.\[15\].

3. It is also noted that the vacuum expectation value of \( r_n \) is not equal to the vacuum expectation valu of \( \phi_n : \langle \phi_n \rangle = \langle r_n g_n \rangle \neq \langle r_n \rangle \). This \( \langle r_n \rangle \) is always non-zero and

\[
\lim_{\lambda \to \infty} \langle r_n \rangle = 1.
\]

We therefore define \( \rho_n \equiv r_n - \langle r_n \rangle \) as the dynamical variable.

The gauge singlet fermion field \( \chi \) is defined by

\[
\chi_n = \frac{1}{\sqrt{2K}} (g_n P_L + P_R) \psi_n, \quad \bar{\chi}_n = \frac{1}{\sqrt{2K}} \bar{\psi}_n (g_n^{\dagger} P_R + P_L)
\]

where \( K = \frac{1}{2(\langle r_n \rangle y + dr)} \) is the hopping parameter. Since the gauge singlet fermion field is expected to appear as a particle in the strong Wilson-Yukawa coupling region, we rewrite the fermionic part of the action in terms of \( \chi \) and obtain

\[
S_F + S_{WY} + S_Y = \sum_{n,\mu} \bar{\chi}_n [(1 + 2K \rho_n) \chi_n - K (T_{n,\mu} \chi_{n+\mu} + T_{n,-\mu} \chi_{n-\mu})]
\]

where

\[
T_{n,\mu} = r - \gamma_\mu (U_{n,\mu}^g P_L + P_R)
\]
\[
T_{n,-\mu} = r + \gamma_\mu (U_{n,-\mu}^g P_L + P_R)
\]

and \( U_{n,\mu}^g \equiv g_n U_{n,\mu} g_{n+\mu}^\dagger \) is the gauge singlet link variable. If \( K \) is small, we can expand fermionic quantities in terms of \( K \) and we call this expansion
the hopping parameter expansion (HPE)\textsuperscript{[3, 8]}. It is noted that the large $r$ expansion\textsuperscript{[16]} is also applicable to our simple action. We check that both expansions give the same results for the quantities we calculate in this paper.

In order to calculate the fermion propagator, we write down a recursion relation for the fermion:

\begin{equation}
\langle \chi_n \bar{\chi}_m \rangle = -\delta_{nm} - 2K\delta_{nm} \langle \rho_n \chi_n \bar{\chi}_m \rangle + K \sum_{\pm \mu} \langle T_{n,\mu} \chi_{n+\mu} \bar{\chi}_m \rangle.
\end{equation}

(2.10)

Using the truncation of the recursion relation and the fact that $\langle \rho_n \rangle = 0$ we obtain

\begin{equation}
\langle \chi_n \bar{\chi}_m \rangle = -\delta_{nm} + K \sum_{\pm \mu} \langle T_{n,\mu} \rangle \langle \chi_{n+\mu} \bar{\chi}_m \rangle.
\end{equation}

(2.11)

The fermion propagator is calculated as

\begin{equation}
G_F(n - m) \equiv \langle \chi_n \bar{\chi}_m \rangle = -\left[ 1 - K \sum_{\pm \mu} T_{n,\mu} \nabla_{\mu} \right]^{-1}
\end{equation}

where $T_{\pm \mu} = r \mp \gamma_{\mu}(L P_L + P_R)$, $L = \langle U_{\bar{\alpha},\alpha}^{g} \rangle$ and $[\nabla_{\pm}]_{nm} = \delta_{m,n} \pm \mu$. In the momentum space it becomes

\begin{equation}
G_F(p)^{-1} = -2K \left[ \sum_{\mu} \gamma_{\mu} \sin \mu (L P_L + P_R) + y \langle r_n \rangle + r \sum_{\mu} (1 - \cos \mu) \right].
\end{equation}

(2.13)

The physical fermion mass in lattice unit is given by

\begin{equation}
m_{phy} a = y \langle r_n \rangle Z_F^{1/2}
\end{equation}

(2.14)

while the mass of the fermion doubler becomes

\begin{equation}
m_d a = (y \langle r_n \rangle + 2 rl) Z_F^{1/2}
\end{equation}

(2.15)

where $Z_F = \frac{1}{L}$ and $l$ denotes the number of $\pi$’s for the doubler momenta. (We call such doublers as $l$-th doublers.) From the above formula, it is easy to see the following results\textsuperscript{[1, 3, 8, 9].}

1. We have to tune the Yukawa coupling $y$ such that $\lim_{a \to 0} y = 0$ in order to get the finite $m_{phy}$ in the continuum limit since $\lim_{a \to 0} \langle r_n \rangle \cdot Z_F^{1/2} \neq 0$. This also shows that the fermion mass stays non-zero even in the symmetric phase of the scalar model where the vacuum
expectation value of the Higgs field, $v_R = \langle \phi_n^R \rangle$, vanishes. Therefore, the perturbative relation $m_{\text{phy}} = Y_R v_R$ does not hold in the strong Wilson-Yukawa coupling region, where $Y_R$ is the renormalized Yukawa coupling.

2. With this tuning of $y$, the fermion doublers are decoupled in the continuum limit since $\lim_{a \to 0} m_d a = 2lr \times Z_F \neq 0$.

In the next section we calculate fermion couplings\textsuperscript{12} using the HPE.

### 3 Fermion Couplings

In the action (2.9) the gauge singlet fermions $\chi, \bar{\chi}$ couple to the singlet link variable $U_{n,\mu}^g$. Since naively

$$U_{n,\mu}^g \simeq U_{n,\mu}^T(\text{transverse mode}) + g_n g_{n+\mu} (\text{longitudinal mode}),$$

the action describes an interaction between the singlet fermion and a massive vector field. This is the type of interactions needed for the electro-weak interaction. We have to study more carefully, however, whether this is true or not since the quantum fluctuation of $g$ field may invalidate the naive expectation. In the scaling region of the lattice gauge-Higgs system, the operator $U_{n,\mu}^g$ is expected to interpolate the Higgs fields as well as the gauge field such that

$$U_{n,\mu}^g = < U_{n,\mu}^g > + H(n) + iA_\mu^a(n)T^a + \cdots$$

(3.2)

where $H$ is the spin-0 interpolation field for the Higgs field, $A_\mu$ is the spin-1 interpolation field for the gauge field, and $T^a$ which satisfies $Tr(T^a T^b) = \delta_{ab}$ is the generator of the gauge group in the representation of the left-handed fermion. Other excited states, which are denoted by $\cdots$ above, may exist, but we neglect them hereafter. Since these operators are dimensionless on the lattice, the two points functions of these operators in the scaling region become

$$G^H(p) = \sum_n < H(0) H(n) > e^{ipa-n} \simeq \frac{Z_H}{p^2 a^2 + m_H^2 a^2}$$

(3.3)

$$\delta_{ab} G^{A}_{\mu\nu}(p) = \sum_n < A_\mu^a(0) A_\nu^b (n) > e^{ipa-n} \simeq \delta_{ab} \frac{Z_A}{p^2 a^2 + m_G^2 a^2} \left( \delta_{\mu\nu} + \frac{p_\mu p_\nu}{m_G^2} \right)$$

(3.4)
in the momentum space, where \( m_G \) is the dimensionful mass of gauge (Higgs) field. The residue \( Z_G \) becomes the wave-function renormalization constant for the operator \( A_\mu \) and it gives the overlapping between this operator and the possible asymptotic state for the gauge (Higgs) field. If \( Z_G \) remains non-zero in the continuum limit, the asymptotic gauge (Higgs) field exists in this theory and it couples to \( A_\mu \). This also shows how a linear Higgs field can appear from the non-linear field \( g \). We hope that the appearance of the linear Higgs field leads to a renormalizable theory in the scaling region even if the original lattice action is not perturbatively renormalizable due to the non-linearity of \( g \) in the Wilson-Yukawa term.

Since the operator \( \rho \) can also interpolate the Higgs fields, we can define

\[
G^\rho(p) = \sum_n <\rho_0 \rho_n> \cdot e^{ipa \cdot n} \simeq \frac{Z_\rho}{p^2a^2 + m_H^2a^2}. \tag{3.5}
\]

From eq. 3.2 and eq. 3.5, we also obtain

\[
G^{H\rho}(p) = \sum_n <H(n) \cdot \rho_n> \cdot e^{ipa \cdot n} \simeq \frac{\sqrt{Z_H Z_\rho}}{p^2a^2 + m_H^2a^2}. \tag{3.6}
\]

Now we demonstrate how the fermion coupling with the gauge (Higgs) field is calculated in the hopping parameter expansion\[12\]. First we introduce source terms for the gauge field and the Higgs field in the action:

\[
S + \sum_n J^n_H H(n) + \sum_{n,\mu} J^a_{n,\mu} A^a_\mu(n). \tag{3.7}
\]

Next we write down the previous recursion relation for the fermion propagator in the presence of the source terms. Using the same truncation of the previous calculation we obtain

\[
\langle \chi_n \bar{\chi}_m \rangle_J = -\delta_{nm} - 2K\delta_{nm}\langle \rho_n \rangle_J \langle \chi_n \bar{\chi}_m \rangle_J + K \sum_{\pm \mu} \langle T_{n,\mu} \rangle_J \langle \chi_{n+\mu} \bar{\chi}_m \rangle_J \tag{3.8}
\]

where \( \langle \cdot \rangle_J \) is the vacuum expectation value in the presence of the source \( J \)'s. Taking the derivative with respect to the sources and putting them zero we obtain the following three points functions

\[
\langle \chi_n \bar{\chi}_m A^a_\mu(l) \rangle = iK \sum_{s,\nu} G_F(n - s)\gamma_\nu T^a [G_F(s + \nu - m)G^a_{\nu\mu}(s + \nu/2 - l) + G_F(s - \nu - m)G^A_{\nu\mu}(s - \nu/2 - l)] \tag{3.9}
\]
\[
\langle \chi_n \bar{\chi}_m H(l) \rangle = K \sum_{s, \nu} G_F(n - s) \gamma_\nu G_F(s + \nu - m) G^H(s + \nu/2 - l) \\
- G_F(s - \nu - m) G^H(s - \nu/2 - l) \\
+ 2K y \sum_s G_F(n - s) G_H(s - l) G_F(s - m). \tag{3.10}
\]

From the above expression we obtain the renormalized fermion-gauge vertex denoted as \( \Gamma^a_{\mu}(p, q) \) in momentum space;

\[
\Gamma^a_{\mu}(p, q) = i \gamma_\mu P_L T^a \cos \left( \frac{(p + q)_\mu a}{2} \right) Z^1/2 / Z_F \tag{3.11}
\]

where \( p \) and \( q \) are fermion momenta with \( p - q \) being the momentum transfer to the gauge field. This vertex has the correct chiral structure (\( \gamma_\mu P_L \)) and gauge structure (\( T^a \)). In the continuum limit the coupling constant \( g^r_F \) becomes

\[
g^r_F = \lim_{a \to 0} \frac{Z^1/2}{L}. \tag{3.12}
\]

It is generally believed that the continuum limit of the gauge-Higgs system can be taken at \(( \beta_g, \beta_h ) = ( \infty, \beta^c_h )\), where \( \beta_g \) is the inverse gauge coupling, and \( \beta^c_h \) is the second order phase transition point of the Higgs system without gauge field. At this continuum limit, it is easy to see that \( 0 < L < 1 \). If the continuum limit taken in the broken phase gives a non-zero \( Z_G \);

\[
\lim_{\beta_g \to \infty, \beta_h \to \beta^c_h} Z_G \neq 0, \tag{3.13}
\]

the fermion-gauge interaction remains non-zero in the continuum limit. If this is the case, the Wilson-Yukawa approach can describes the fermion interaction with the massive gauge field. Therefore it is very crucial and important for the Wilson-Yukawa approach to investigate the behaviour of \( Z_G \) near the continuum limit. For example, the perturbative expansion in the unitary gauge \(( \forall g_n = 1 \) gives \( L = 1 + O(g_r^2) \) and \( Z_G = g_r^2 + O(g_r^4) \) where \( g_r \) is the renormalized coupling of this perturbative expansion, and the formula for \( g^r_F \) give the consistent relation \( g^r_F = g_r \). The perturbative calculation, however, may not be reliable due to the large quantum fluctuation of \( g \) field near the scaling region even in the broken phase. In order to obtain the behaviour of \( Z_G \) more reliably we have to calculate it numerically. This numerical calculation should be done.

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\(^2\) It is noted that the lattice gauge field \( A_\mu \) is related to the continuum gauge field \( A^{con}_\mu \) such that \( A_\mu = a g_r A^{con}_\mu \).
Assuming that the Wilson-Yukawa approach gives the non-zero coupling, we may explain why the electroweak interaction is well described by the perturbative expansion of the coupling constant. The argument is the following. Since only gauge singlet fields such that $\chi$, $\bar{\chi}$ and $U^g$ can appear as the asymptotic states in the strong Wilson-Yukawa coupling region where the fermion doubler is removed, interactions among these neutral fields is expected to be weak. Indeed the hopping parameter expansion which is valid in the strong Wilson-Yukawa region show that the fermion-gauge vertex is dominated by the tree-like diagram. Furthermore higher order corrections to the vertex in the hopping parameter expansion correspond to those of the perturbative expansion of the gauge coupling constant. This is the reason why the perturbative theory successfully describes the fermion-gauge interaction in the region where the hopping parameter expansion is reliable.

Recently it is claimed that the fermion-gauge coupling vanishes as $O(a^2)$ in the continuum limit taken in the symmetric phase. This is equivalent to the behaviour $Z_G = O(a^4)$ near the scaling region. This behaviour may be true in the symmetric phase, since the symmetric phase may be similar to the confining theory and only glue balls, not the gauge field, can appear as asymptotic states in the confining theory. The Wilson-Yukawa approach may fail to describe the symmetric phase of chiral gauge theories. Even if this is the case, it does not mean that the Wilson-Yukawa approach fails to describe the standard model in the broken phase. Since we expect that the massive gauge field appear in the broken phase, it is likely that $\lim_{a \to 0} Z_G \neq 0$ in the broken phase.

Finally the renormalized fermion-Higgs coupling $\Gamma_H(p, q)$ is calculated as

$$\Gamma_H(p, q) = i \sum_\mu \gamma_\mu P_L \sin \frac{(p+q) a}{2} Z_{HF}^{1/2} Z_F + y Z_{F}^{1/2} Z_F$$

(3.14)

The first term, which has been calculated for the case of the non-linear Higgs field, is the derivative coupling with dimension 5 and this coupling vanishes as $O(a)$ in the continuum limit [12]. The second term is the ordinary Yukawa coupling which becomes

$$y \lim_{a \to 0} Z_{\rho}^{1/2} Z_F$$

(3.15)

in the continuum limit [13]. From the triviality of the Yukawa interaction, however, we expect that the Yukawa coupling logarithmically goes to zero.
in the continuum limit. Therefore $Z_F^{1/2}Z_F$ should logarithmically go to zero according to the prediction of the renormalization group equation.

4 Charged Fermion Couplings with the External Gauge Field

In this section we consider fermion couplings with the external gauge field in the slightly modified Wilson-Yukawa term [11]:

$$S_{WY} = -\frac{1}{2}\bar{\psi}_n [P_Lg_n(U_{n,\mu}\psi_{n+\mu} + U_{n,-\mu}\psi_{n-\mu} - 2\psi_n) + P_R(U_{n,\mu}g_n^\dagger\psi_{n+\mu} + U_{n,-\mu}g_n^\dagger\psi_{n-\mu} - 2g_n^\dagger\psi_n)].$$ (4.1)

In terms of the vectorially charged fermion field $C_n$ defined as

$$C_n = \frac{1}{\sqrt{2K}}(P_L + g_n^\dagger P_R)\psi_n = g_n^\dagger N_n, \quad \bar{C}_n = \frac{1}{\sqrt{2K}}\bar{\psi}_n(P_R + g_n P_L) = \bar{N}_ng_n,$$

(4.2)

the fermionic part of the action becomes

$$S_F + S_Y + S_{WY} = \sum_n \bar{C}_n[C_n - K\sum_\mu (T_{n,\mu}^c C_{n+\mu} + T_{n,\mu}^c C_{n-\mu})]$$ (4.3)

where

$$T_{n,\mu}^c = U_{n,\mu}(r - \gamma_\mu P_L) - R_{n,\mu}\gamma_\mu P_R$$
$$T_{n,-\mu}^c = U_{n,-\mu}(r + \gamma_\mu P_L) + R_{n,-\mu}\gamma_\mu P_R$$ (4.4)

and $R_{n,\mu} = g_n^\dagger g_{n+\mu}$. For simplicity we take the $\lambda \to 0$ limit here ($\langle r_n \rangle = 1$ and $\rho_n = 0$). It is noted that the charged fermion propagator in the global limit ($\forall U_{n,\mu} = 1$) of this model is reliably calculated by the hopping parameter expansion, and we obtain

$$G_F^{c}(p) = \sum_n \langle C_0\bar{C}_n \rangle e^{ip\cdot n} = -(2K)^{-1}[i\sum_\mu \gamma_\mu \sin p_\mu (P_L + Z_{R}^{-1}P_R) + M(p)]^{-1}$$ (4.5)

where $Z_{R}^{-1} = R \equiv \langle R_{n,\mu} \rangle$. This is the reason why we use the modified action instead of the one in the previous two sections.

Introducing the external gauge field $V_\mu^{a}(n)T^a$ such that

$$U_{n,\mu} = \exp[iV_\mu^{a}(n)T^a]$$ (4.6)
we define the current $J^a_\mu$ to which the external gauge field $V^a_\mu$ couples:

$$J^a_\mu(n) = \left[ \frac{\delta S}{i \delta V^a_\mu(n)} \right]_{V^a_\mu=0}.$$  \hfill (4.7)

The gauge current becomes

$$J^a_\mu(n) = \beta_h d_N \text{tr}(g_n T^a g^\dagger_{n+\mu} - g_{n+\mu} T^a g^\dagger_n) + K[C_n T^a(\gamma_\mu P_L - r)C_{n+\mu} + C_{n+\mu} T^a(\gamma_\mu P_L + r)C_n].$$  \hfill (4.8)

We calculate the 3 points function using the hopping parameter expansion. By a method similar to that in the previous sections, we obtain

$$\langle C^i_n \bar{C}_m J^a_\mu(l) \rangle = K (T^a)^{ij} [d_N \beta_h \sum_{s,\nu} G_F(n-s) \{ G^{B}_{\mu\nu}(l-s) \gamma_\nu P_R \times G_F(s+\nu-m) + G^{B}_{\mu\nu}(l-s+\nu) \gamma_\nu P_R G_F(s-\nu-m) \} + G_F(n-l) \times(\gamma_\nu P_L - r) G_F(l+\mu-m) + G_F(n-l-\mu)(\gamma_\mu P_L + r) G_F(l-m)]$$  \hfill (4.9)

where

$$\delta_{ab} G^{B}_{\mu\nu}(l-s) = \langle B^a_\mu(l) B^b_\nu(s) \rangle.$$  \hfill (4.10)

and $B^a_\mu(n) = \text{Im}[\text{tr}(g_{n+\mu} T^a g^\dagger_n)]$. After the renormalization for the fermion field such that

$$C_n = \frac{1}{\sqrt{2K}} (Z^{1/2}_R P_R + P_L) C^r_n, \quad \bar{C}_n = \frac{1}{\sqrt{2K}} \bar{C}^r_n (Z^{1/2}_R P_L + P_R)$$  \hfill (4.11)

and

$$G_F(n) = \frac{1}{2K} (Z^{1/2}_R P_R + P_L) G^r_F(n) (Z^{1/2}_R P_L + P_R)$$  \hfill (4.12)

(Here $r$ stands for the renormalized.), we obtain

$$\langle C^i_n \bar{C}_m J^a_\mu(l) \rangle = \frac{T^a}{2} [d_N \beta_h Z_R \sum_{s,\nu} G^r_F(n-s) \{ G^{B}_{\mu\nu}(l-s) \gamma_\nu P_R G^r_F(s+\nu-m) + G^{B}_{\mu\nu}(l-s+\nu) \gamma_\nu P_R G^r_F(s-\nu-m) \} + G^r_F(n-l)(\gamma_\mu P_L - Z^{1/2}_R) \times G^r_F(l+\mu-m) + G^r_F(n-l-\mu)(\gamma_\mu P_L + Z^{1/2}_R) G^r_F(l-m)].$$  \hfill (4.13)

This result shows that the charged fermion interaction with the external gauge field is chirally asymmetric: the left-handed fermion and the right-handed fermion have different couplings with the external gauge field.
In the limit that $\beta_h \to 0$ of the above result, we can reproduce the result in ref.\cite{11}. Since in this limit
\begin{equation}
G_{\mu\nu}(l - s)^B = \frac{1}{2} \delta_{\mu,\nu} \delta_{l,s} \tag{4.14}
\end{equation}
and $Z_R^{-1} \equiv R = \frac{dN}{2} \beta_h + O(\beta_h^2)$, we obtain
\begin{equation}
\lim_{\beta_h \to 0} \langle \bar{C}^r_m \bar{C}^r_n J^a_{\mu}(l) \rangle = \frac{T^a}{2} [+G_F^r(n - l)(\gamma_{\mu} - Z_{1/2}^1 r)G_F^r(l + \mu - m)
+G_F^r(n - l - \mu)(\gamma_{\mu} + Z_{1/2}^1 r)G_F^r(l - m)]. \tag{4.15}
\end{equation}
This is the result of ref.\cite{11}. Their claim that the charged fermion coupling with the external gauge field becomes vector-like is only true in the special limit that $\beta_h \to 0$ and the coupling in general is chirally asymmetric.

If we include the effect of the fermion determinant by the HPE\cite{6, 8}, we have to replace $\beta_h$ with $\beta_{\text{eff}}^h = \beta_h + 2d/2K^2$ in all results of this paper.

5 Conclusion and Discussion

In this paper we calculate the fermion couplings using the hopping parameter expansion. We obtain the following results.

1. The interaction between the gauge singlet fermion and the massive gauge field has a chiral structure and it likely stays non-zero in the continuum limit taken within the Higgs phase. Since we can take the unitary gauge such that $\forall g_n = 1$ in the Higgs phase, there is no essential difference between the singlet fermion and the charged fermion. Therefore it is concluded that the Wilson-Yukawa approach can correctly describe the fermion-gauge interaction of the standard model. Since the fermion-gauge interaction in the HPE is dominated by the tree diagram of the ordinary perturbative expansion of the coupling, the success of the perturbative expansion in the standard model is understood by the goodness of the HPE in the strong Wilson-Yukawa coupling region.
2. By introducing the linear part of the Higgs fields, $r_n$, in the action, the non-zero Yukawa coupling is calculated in the Wilson-Yukawa approach. We show that the Yukawa coupling at the leading order of the HPE is identical to the perturbative Yukawa coupling of the standard model. The Wilson-Yukawa approach can correctly describe the Yukawa interaction of the standard model.

3. In the slightly modified model where the charged fermion is reliably treated by the HPE, the charged fermion interaction with the external gauge field is calculated. We show that the interaction in general is chirally asymmetric. Therefore, it is not so obvious that the symmetric phase of this theory becomes a vector-like theory as claimed in ref. [11].

The above conclusion can easily be extended to the more complicated models where both the left-handed part and the right-handed part of fermion are gauge non-singlet. (The hypercharge sector of the standard model is an example of such models.) For example, the fermion-gauge coupling is calculated as follows. The covariant derivative part of the fermion action is given by

$$ S_F = \frac{1}{2} \sum_{n, \mu} \bar{\psi}_n \left[ (D^L(U_{n, \mu})P_L + D^R(U_{n, \mu})P_R)\psi_{n+\mu} - (D^L(U_{n, -\mu})P_L + D^R(U_{n, -\mu})P_R)\psi_{n-\mu} \right] $$

(5.1)

where $D^{L(R)}$ is the representation of the gauge field for the left(right)-handed fermion. By the method of sect. III, we obtain the following renormalized gauge-fermion vertex in the continuum limit.

$$ \Gamma^a_{\mu}(p,q) = i\gamma_{\mu}(g^a_L D^L(T^a)P_L + g^a_R D^R(T^a)P_R) $$

(5.2)

where the renormalized couplings are given by

$$ g^a_L = (Z^L_A)^{1/2} Z^L_F, \quad g^a_R = (Z^R_A)^{1/2} Z^R_F. $$

(5.3)

Here $Z^L_F$ and $Z^R_F$ are the wave function renormalizations for the fermion fields and the HPE gives $Z^L_F = \langle D^L(U^g_\mu) \rangle$ and $Z^R_F = \langle D^R(U^g_\mu) \rangle$ at the leading order. The wave function renormalizations for the bosonic fields are defined by

$$ \sum_n \langle L^{a}_{\mu}(0)L^{b}_{\nu}(n) \rangle \sim \delta_{ab} \frac{Z^{L}_{A}}{p^2 a^2 + m^2 G a^2} \left( \delta_{\mu\nu} + \frac{p_{\mu} p_{\nu}}{m^2 G} \right) $$

(5.4)
and

\[ \sum_n \langle R^a_\mu(0) R^b_\mu(n) \rangle e^{ip_a n} \simeq \delta_{ab} \frac{Z^R_A}{p^2 a^2 + m^2_G a^2} \left( \delta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2_G} \right). \quad (5.5) \]

where \( L^a_\mu = D^L(A^a_\mu) \) and \( R^a_\mu = D^R(A^a_\mu) \). From the consistency among \( A_\mu, \)
\( L_\mu \) and \( R_\mu \), we obtain

\[ \sum_n \langle L^a_\mu(0) A^b_\nu(n) \rangle e^{ip_a n} \simeq \delta_{ab} \frac{Z^L_A \cdot Z_A}{p^2 a^2 + m^2_G a^2} \left( \delta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2_G} \right) \quad (5.6) \]

and

\[ \sum_n \langle R^a_\mu(0) A^b_\nu(n) \rangle e^{ip_a n} \simeq \delta_{ab} \frac{Z^R_A \cdot Z_A}{p^2 a^2 + m^2_G a^2} \left( \delta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2_G} \right). \quad (5.7) \]

The renormalized fermion gauge couplings \( g^L \) and \( g^R \) are both non-zero and
the left-handed coupling \( g^L \) is different from the right-handed coupling \( g^R \) if
\( D^L \neq D^R \). In the hypercharge sector of the standard model, we expect that
\( g^L \propto Y_L \) and \( g^R \propto Y_R \) in the scaling region. Here \( Y_L (Y_R) \) is the hypercharge
of the left-handed (right-handed) fermion; \( D^{L(R)}(U) = U^{Y_L(Y_R)} \).

Final remark is the following. The hopping parameter expansion we use
relates the fermionic quantities such as the propagator or the 3-points function
to the bosonic quantities such as \( L, Z_A \) or \( Z_H \), which, however, can not
be calculated by this expansion. We have to calculate these quantities by
other methods such as a mean-field calculation, a 1/d expansion or numerical simulations. In particular, a reliable evaluation for \( Z_A \) is crucial for the
Wilson-Yukawa approach, as mentioned before.

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