On the derivatives $\partial^2 P_\nu(z)/\partial \nu^2$ and $\partial Q_\nu(z)/\partial \nu$
of the Legendre functions with respect to their degrees

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Abstract

We provide closed-form expressions for the degree-derivatives $\partial^2 P_\nu(z)/\partial \nu^2$ and $\partial Q_\nu(z)/\partial \nu$ of the Legendre functions with respect to their degrees. For $\partial^2 P_\nu(z)/\partial \nu^2$, we find that

$$\left. \frac{\partial^2 P_\nu(z)}{\partial \nu^2} \right|_{\nu=n} = -2P_n(z) \text{Li}_2 \left( \frac{1-z}{2} \right) + B_n(z) \ln \frac{z+1}{2} + C_n(z),$$

where $\text{Li}_2((1-z)/2)$ is the dilogarithm function, $P_n(z)$ is the Legendre polynomial, while $B_n(z)$ and $C_n(z)$ are certain polynomials in $z$ of degree $n$. For $\partial Q_\nu(z)/\partial \nu$ and $z \in \mathbb{C} \setminus [-1,1]$, we derive

$$\left. \frac{\partial Q_\nu(z)}{\partial \nu} \right|_{\nu=n} = -P_n(z) \text{Li}_2 \left( \frac{1-z}{2} \right) - \frac{1}{2} P_n(z) \ln \frac{z-1}{2} - \frac{1}{4} B_n(z) \ln \frac{z-1}{2} + \frac{1}{4} B_n(z) \ln \frac{z+1}{2}$$

$$- \frac{(-1)^n}{4} B_n(-z) \ln \frac{z-1}{2} - \frac{\pi^2}{6} P_n(z) + \frac{1}{4} C_n(z) - \frac{(-1)^n}{4} C_n(-z).$$

A counterpart expression for $\partial Q_\nu(x)/\partial \nu|_{\nu=n}$, applicable when $x \in (-1,1)$, is also presented. Explicit representations of the polynomials $B_n(z)$ and $C_n(z)$ as linear combinations of the Legendre polynomials are given.

Key words: Legendre functions; parameter derivatives; dilogarithm

MSC2010: 33C05, 33B30

1 Introduction

Over the past 20 years or so, a growth of interest in parameter derivatives of various special functions has been observed. The research done on the subject is documented in a number of papers reporting diverse methods for finding such derivatives for orthogonal polynomials in one [1–3] and two [4–6] variables, for Bessel functions [7–10], for Legendre and allied functions [11–18], and also for various types of hypergeometric functions [1, 19–22].

In Refs. [11, 12], we presented results of our investigations on the first-order derivative of the Legendre function of the first kind with respect to its degree. We showed that $\left[ \partial P_\nu(z)/\partial \nu \right]_{\nu=n}$,
with \( n \in \mathbb{N}_0 \), is of the form
\[
\left. \frac{\partial P_n(z)}{\partial \nu} \right|_{\nu = n} = P_n(z) \ln \frac{z + 1}{2} + R_n(z),
\]
where \( P_n(z) \) is the Legendre polynomial of degree \( n \) and \( R_n(z) \) is another polynomial in \( z \) of the same degree. We investigated properties of the polynomials \( R_n(z) \) and arrived at their several explicit representations, including the following one:
\[
R_n(z) = 2[\psi(2n + 1) - \psi(n + 1)]P_n(z) + 2 \sum_{k=0}^{n-1} (-1)^{n+k} \frac{2k + 1}{(n-k)(n+k+1)} P_k(z),
\]
where \( \psi(z) = d \ln \Gamma(z)/dz \) is the digamma function.

In the year 2012, Dr. George P. Schramkowski kindly informed the present author that in the course of doing research on a certain problem in theoretical hydrodynamics, he had come across higher-order derivatives \( [\partial^k P_n(z)/\partial \nu^k]_{\nu = n} \), with \( n \in \mathbb{N}_0 \) and \( k \geq 2 \). Using Mathematica, Schramkowski found that
\[
\left. \frac{\partial^2 P_n(z)}{\partial \nu^2} \right|_{\nu = 0} = -2 \text{Li}_2 \frac{1 - z}{2},
\]
where
\[
\text{Li}_2(z) = -\int_0^z \frac{\ln(1-t)}{t} \, dt
\]
is the dilogarithm function [23, 24]. In Ref. [25], we gave an analytical proof of the result displayed in Eq. (1.3), and also we derived a closed-form formula for the third-order derivative \( [\partial^3 P_n(z)/\partial \nu^3]_{\nu = 0} \).

That work was then extended by Laurenzi [26], who found an expression for the fourth-order derivative \( [\partial^4 P_n(z)/\partial \nu^4]_{\nu = 0} \).

The primary purpose of the present work is to pursue further the research initiated by Schramkowski and continued by us in Ref. [25]. We shall show that for arbitrary \( n \in \mathbb{N}_0 \) the second-order derivative \( [\partial^2 P_n(z)/\partial \nu^2]_{\nu = n} \) may be expressed in the form
\[
\left. \frac{\partial^2 P_n(z)}{\partial \nu^2} \right|_{\nu = n} = -2P_n(z) \text{Li}_2 \frac{1 - z}{2} + B_n(z) \ln \frac{z + 1}{2} + C_n(z),
\]
where the polynomials \( B_n(z) \) and \( C_n(z) \) have the following representations in terms of the Legendre polynomials:
\[
B_n(z) = 4[\psi(2n + 1) - \psi(n + 1)]P_n(z) + 4 \sum_{k=0}^{n-1} \frac{2k + 1}{(n-k)(n+k+1)} P_k(z)
\]
and
\[
C_n(z) = \left\{ -\frac{\pi^2}{3} + 4[\psi(2n + 1) - \psi(n + 1)]^2 + 4\psi(2n + 1) - 2\psi(n + 1) \right\} P_n(z)
+ 4 \sum_{k=0}^{n-1} (-1)^{n+k} \frac{2k + 1}{(n-k)(n+k+1)} \left\{ 2 \left[ \psi(n+k+1) - \psi(n-k+1) \right]
- \psi \left( \frac{n+k}{2} + 1 \right) + \psi \left( \frac{n-k}{2} + 1 \right) \right\}
- (-1)^{n+k} \frac{2k + 1}{(n-k)(n+k+1)} - \frac{2n+1}{(n-k)(n+k+1)} P_k(z),
\]
with \( \psi(z) \) being already defined under Eq. (1.2), \( \psi_1(z) = d \psi(z)/dz \), and \( \lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x \} \) standing for the integer part of \( x \). In fact, the above result for \( [\partial^2 P_n(z)/\partial \nu^2]_{\nu = n} \), valid for \( n \in \mathbb{N}_0 \), may be easily extended to any \( n \in \mathbb{Z} \), since with the use of the well-known identity
\[
P_{-\nu-1}(z) = P_{\nu}(z)
\]
one immediately finds that

$$\frac{\partial^2 P_{\nu}(z)}{\partial \nu^2} \bigg|_{\nu=-n-1} = \frac{\partial^2 P_{\nu}(z)}{\partial \nu^2} \bigg|_{\nu=n} \quad (n \in \mathbb{N}_0). \quad (1.9)$$

In addition to the above summarized study on the second-order degree-derivative of the Legendre function of the first kind, which will be presented in detail in Sec. 2 below, later in Sec. 3 we shall also prove that if \( z \in \mathbb{C} \setminus [-1, 1] \) and \( n \in \mathbb{N}_0 \), then the first-order derivative \( \frac{\partial Q_{\nu}(z)}{\partial \nu} \bigg|_{\nu=n} \), where \( Q_{\nu}(z) \) is the Legendre function of the second kind, is given by

$$\frac{\partial Q_{\nu}(z)}{\partial \nu} \bigg|_{\nu=n} = - P_n(z) \text{Li}_2 \left( \frac{1-z}{2} \right) - \frac{1}{2} P_n(z) \ln \frac{z+1}{2} - \frac{1}{4} B_n(z) \ln \frac{z+1}{2} - ( -1)^n B_n(-z) \ln \frac{z-1}{2} - \frac{\pi^2}{6} P_n(z) + \frac{1}{4} C_n(z) - \frac{( -1)^n}{4} C_n(-z). \quad (1.10)$$

A counterpart expression for \( \frac{\partial Q_{\nu}(x)}{\partial \nu} \bigg|_{\nu=n} \), applicable when \( x \in (-1, 1) \), will also be derived.

## 2 The derivatives \( [\frac{\partial^2 P_{\nu}(z)}{\partial \nu^2}]_{\nu=n} \)

### 2.1 The general form of \( [\frac{\partial^2 P_{\nu}(z)}{\partial \nu^2}]_{\nu=n} \)

Our point of departure is the well-known recurrence relation

\((\nu + 1) P_{\nu+1}(z) - (2\nu + 1) z P_{\nu}(z) + \nu P_{\nu-1}(z) = 0\) \quad (2.1)

obeyed by the Legendre functions of the first kind. Differentiating it twice with respect to \( \nu \) and setting then \( \nu = n \) yields

\[(n + 1) \frac{\partial^2 P_{\nu}(z)}{\partial \nu^2} \bigg|_{\nu=n+1} - (2n + 1) z \frac{\partial^2 P_{\nu}(z)}{\partial \nu^2} \bigg|_{\nu=n} + n \frac{\partial^2 P_{\nu}(z)}{\partial \nu^2} \bigg|_{\nu=n-1} = -2 \left[ \frac{\partial P_{\nu}(z)}{\partial \nu} \bigg|_{\nu=n+1} - 2z \frac{\partial P_{\nu}(z)}{\partial \nu} \bigg|_{\nu=n} + \frac{\partial P_{\nu}(z)}{\partial \nu} \bigg|_{\nu=n-1} \right]. \quad (2.2)\]

If we replace the first-order derivatives on the right-hand side with expressions following from Eq. (1.1), this furnishes

\[(n + 1) \frac{\partial^2 P_{\nu}(z)}{\partial \nu^2} \bigg|_{\nu=n+1} - (2n + 1) z \frac{\partial^2 P_{\nu}(z)}{\partial \nu^2} \bigg|_{\nu=n} + n \frac{\partial^2 P_{\nu}(z)}{\partial \nu^2} \bigg|_{\nu=n-1} = -2 \left[ P_{n+1}(z) - 2z P_n(z) + P_{n-1}(z) \right] \ln \frac{z+1}{2} - 2 \left[ R_{n+1}(z) - 2z R_n(z) + R_{n-1}(z) \right]. \quad (2.3)\]

From the formal point of view, one may look at Eq. (2.3) as a second-order difference equation and then two additional conditions are necessary to single out the sequence \( [\frac{\partial^2 P_{\nu}(z)}{\partial \nu^2}]_{\nu=n} \) from its general solution. Such conditions may be chosen in a variety of ways but for our purposes it is most convenient to take the explicit expression for \( [\frac{\partial^2 P_{\nu}(z)}{\partial \nu^2}]_{\nu=0} \), given in Eq. (1.1), as the first one. The second suitable condition follows from Eq. (2.3) after one lets \( n = 0 \). Then, with the use of the identities

\[P_{-1}(z) = 1, \quad P_0(z) = 1, \quad P_1(z) = z\] \quad (2.4)

and [11, Sec. 5.2]

\[R_{-1}(z) = -2 \ln \frac{z+1}{2}, \quad R_0(z) = 0, \quad R_1(z) = z - 1, \quad (2.5)\]
Eq. (2.3) reduces to the form
\[
\left. \frac{\partial^2 P_\nu(z)}{\partial \nu^2} \right|_{\nu=1} - z \left. \frac{\partial^2 P_\nu(z)}{\partial \nu^2} \right|_{\nu=0} = 2(z+1) \ln \frac{z+1}{2} - 2(z-1). \tag{2.6}
\]

On combining Eq. (2.6) with Eq. (1.1), one finds that
\[
\left. \frac{\partial^2 P_\nu(z)}{\partial \nu^2} \right|_{\nu=1} = -2z \text{Li}_2 \frac{1-z}{2} + 2(z+1) \ln \frac{z+1}{2} - 2(z-1). \tag{2.7}
\]

If necessary, Eq. (2.3) may be applied recursively, with Eqs. (1.3) and (2.7) used as initial conditions, to generate the derivative in question for any particular \( n \geq 2 \). However, as we shall show below, it is also possible to obtain a closed-form representation for \( \left[ \frac{\partial^2 P_\nu(z)}{\partial \nu^2} \right]_{\nu=n} \).

As the first step towards that goal, we observe that the structure of Eq. (2.3), together with explicit expressions for the derivatives \( \left[ \frac{\partial^2 P_\nu(z)}{\partial \nu^2} \right]_{\nu=0} \) and \( \left[ \frac{\partial^2 P_\nu(z)}{\partial \nu^2} \right]_{\nu=1} \) displayed in Eqs. (1.3) and (2.7), fix the form of \( \left[ \frac{\partial^2 P_\nu(z)}{\partial \nu^2} \right]_{\nu=n} \) to be
\[
\left. \frac{\partial^2 P_\nu(z)}{\partial \nu^2} \right|_{\nu=n} = A_n(z) \text{Li}_2 \frac{1-z}{2} + B_n(z) \ln \frac{z+1}{2} + C_n(z), \tag{2.8}
\]

where \( A_n(z) \), \( B_n(z) \) and \( C_n(z) \) are polynomials in \( z \) of degree \( n \). Since the right-hand side of Eq. (2.3) does not contain the dilogarithm function, the polynomial \( A_n(z) \) solves the homogeneous recurrence
\[
(n+1)A_{n+1}(z) - (2n + 1)zA_n(z) + nA_n(z) = 0 \tag{2.9}
\]
subject to the initial conditions
\[
A_0(z) = -2 = -2P_0(z), \quad A_1(z) = -2z = -2P_1(z), \tag{2.10}
\]

which follow from Eqs. (1.1) and (2.8). Hence, we deduce the following expression for \( A_n(z) \) in terms of the Legendre polynomial \( P_n(z) \):
\[
A_n(z) = -2P_n(z). \tag{2.11}
\]

Consequently, Eq. (2.8) becomes
\[
\left. \frac{\partial^2 P_\nu(z)}{\partial \nu^2} \right|_{\nu=n} = -2P_n(z) \text{Li}_2 \frac{1-z}{2} + B_n(z) \ln \frac{z+1}{2} + C_n(z). \tag{2.12}
\]

The representations of the polynomials \( B_n(z) \) and \( C_n(z) \) remain to be established.

### 2.2 Differential equations for the polynomials \( B_n(z) \) and \( C_n(z) \)

It is known that the Legendre function \( P_\nu(z) \) obeys the differential identity
\[
\left[ \frac{d}{dz} (1-z^2) \frac{d}{dz} + \nu(\nu+1) \right] P_\nu(z) = 0. \tag{2.13}
\]

If we differentiate it twice with respect to \( \nu \) and then put \( \nu = n \), this gives
\[
\left[ \frac{d}{dz} (1-z^2) \frac{d}{dz} + n(n+1) \right] \left. \frac{\partial^2 P_\nu(z)}{\partial \nu^2} \right|_{\nu=n} = -2(2n+1) \left. \frac{\partial P_\nu(z)}{\partial \nu} \right|_{\nu=n} - 2P_n(z) \tag{2.14}
\]

and further, after Eq. (1.1) is plugged into the first term on the right-hand side,
\[
\left[ \frac{d}{dz} (1-z^2) \frac{d}{dz} + n(n+1) \right] \left. \frac{\partial^2 P_\nu(z)}{\partial \nu^2} \right|_{\nu=n} = -2(2n+1)P_n(z) \ln \frac{z+1}{2} - 2(2n+1)R_n(z) - 2P_n(z). \tag{2.15}
\]
Next, we insert Eq. (2.12) into the left-hand side of Eq. (2.15) and equate those terms appearing on both sides which involve the logarithmic factor. This yields the following inhomogeneous differential equation for $B_n(z)$:

$$\left[ \frac{d}{dz}(1 - z^2)\frac{d}{dz} + n(n + 1) \right] B_n(z) = 4 \left( z + 1 \right) \frac{dP_n(z)}{dz} - n P_n(z). \tag{2.16}$$

Similarly, after equating polynomial expressions on both sides, we arrive at the inhomogeneous equation for $C_n(z)$:

$$\left[ \frac{d}{dz}(1 - z^2)\frac{d}{dz} + n(n + 1) \right] C_n(z) = 2(z - 1) \frac{dB_n(z)}{dz} + B_n(z) - 2(2n + 1)R_n(z). \tag{2.17}$$

Consider Eq. (2.16). It is evident that it does not possess a unique polynomial solution, since to any particular solution of that form one may add an arbitrary multiple of the Legendre polynomial $P_n(z)$, which results in another polynomial solution. To make the polynomial solution unique, we thus need an additional constraint. The latter follows from the limiting relation [27]

$$P_\nu(z) \xrightarrow{z \to -1} \frac{\sin(\pi \nu)}{\pi} \ln \frac{z + 1}{2} + O(1), \tag{2.18}$$

from which we find

$$\lim_{\nu \to \nu \to 1} \frac{\partial^2 P_\nu(z)}{\partial \nu^2} = 0. \tag{2.19}$$

Hence, the left-hand side of Eq. (2.12) remains finite for $z \to -1$ and to make the right-hand side also finite in that limit, we are forced to put

$$B_n(-1) = 0. \tag{2.20}$$

If, in turn, we wish to make the polynomial solution to Eq. (2.17) unique, we use the identity

$$P_{\nu}(1) = 1. \tag{2.21}$$

Differentiating twice with respect to $\nu$, we obtain

$$\lim_{\nu \to \nu \to 1} \frac{\partial^2 P_\nu(1)}{\partial \nu^2} = 0. \tag{2.22}$$

Since $\text{Li}_2 0 = 0$ and $\ln 1 = 0$, we deduce that $C_n(z)$ is constrained to obey

$$C_n(1) = 0. \tag{2.23}$$

Below we shall exploit Eqs. (2.16), (2.20), (2.17) and (2.23) to determine the polynomials $B_n(z)$ and $C_n(z)$.

### 2.3 Construction of the polynomials $B_n(z)$

A general form of the polynomials $B_n(z)$ may be obtained with ease. In Ref. [11, Sec. 5.2.2], we have found that the polynomials $R_n(z)$ obey the differential relation

$$\left[ \frac{d}{dz}(1 - z^2)\frac{d}{dz} + n(n + 1) \right] R_n(z) = 2 \left( z - 1 \right) \frac{dP_n(z)}{dz} - n P_n(z). \tag{2.24}$$

Hence, with the use of the well-known identity

$$P_n(-z) = (-1)^n P_n(z), \tag{2.25}$$

we deduce that

$$\left[ \frac{d}{dz}(1 - z^2)\frac{d}{dz} + n(n + 1) \right] R_n(-z) = (-1)^n 2 \left( z + 1 \right) \frac{dP_n(z)}{dz} - n P_n(z). \tag{2.26}$$
Comparison of Eqs. (2.16) and (2.26) shows that the polynomial \( B_n(z) \) must be of the form
\[
B_n(z) = (-1)^n 2R_n(-z) + b_n P_n(z). \tag{2.27}
\]
To determine the constant \( b_n \), we put \( z = -1 \) in Eq. (2.27). By virtue of the constraint (2.20), with the use of the relations
\[
P_n(-1) = (-1)^n \tag{2.28}
\]
and (cf. Ref. [11, Eq. (5.10)])
\[
R_n(1) = 0, \tag{2.29}
\]
we infer that
\[
b_n = 0, \tag{2.30}
\]
and thus finally we arrive at
\[
B_n(z) = (-1)^n 2R_n(-z). \tag{2.31}
\]
On combining Eq. (2.31) with Eq. (1.2), we have the following explicit representation of \( B_n(z) \):
\[
B_n(z) = 4\left[ \psi(2n + 1) - \psi(n + 1) \right] P_n(z) + 4 \sum_{k=0}^{n-1} \frac{2k + 1}{(n-k)(n+k+1)} P_k(z). \tag{2.32}
\]
Further expressions for \( B_n(z) \) may be obtained if one combines Eq. (2.31) with Eqs. (2.1), (2.2), (2.4) and (5.90) from Ref. [11] or with Eqs. (11) and (12) from Ref. [12].

### 2.4 Construction of the polynomials \( C_n(z) \)

We shall seek a representation of \( C_n(z) \) in the form of a linear combination of Legendre polynomials:
\[
C_n(z) = \sum_{k=0}^{n} c_{nk} P_k(z). \tag{2.33}
\]

Action on both sides of Eq. (2.33) with the Legendre differential operator appearing on the left-hand side of Eq. (2.17) gives
\[
\left[ \frac{d}{dz}(1 - z^2) \frac{d}{dz} + n(n + 1) \right] C_n(z) = \sum_{k=0}^{n-1} (n-k)(n+k+1)c_{nk} P_k(z). \tag{2.34}
\]
On the other side, with the aid of Eqs. (1.2) and (2.32), and of the identity
\[
(z - 1) \frac{dP_n(z)}{dz} = n P_n(z) + \sum_{k=0}^{n-1} (-1)^{n+k} (2k + 1) P_k(z), \tag{2.35}
\]
after some algebra we find that the expression on the right-hand side of Eq. (2.17) may be written as
\[
2(z - 1) \frac{dB_n(z)}{dz} + B_n(z) - 2(2n + 1)R_n(z)
= \sum_{k=0}^{n-1} \left\{ (-1)^{n+k} 8(2k + 1) [\psi(2n + 1) - \psi(n + 1)]
+ 8(2k + 1) \sum_{m=k}^{n-1} (-1)^{m+k} \frac{2m + 1}{(n-m)(n+m+1)}
- \frac{4(2k + 1)^2}{(n-k)(n+k+1)} - (-1)^{n+k} \frac{4(2n + 1)(2k + 1)}{(n-k)(n+k+1)} \right\} P_k(z). \tag{2.36}
\]
Equations (2.17), (2.34) and (2.36) yield

\[ c_{nk} = (-1)^{n+k} \frac{\psi(2n+1) - \psi(n+1)}{(n-k)(n+k+1)} \frac{8(2k+1)}{} \]
\[ + \frac{8(2k+1)}{(n-k)(n+k+1)} \sum_{m=k}^{n-1} (-1)^{m+k} \frac{2m+1}{(n-m)(n+m+1)} \]
\[ - \frac{4(2k+1)^2}{(n-k)^2(n+k+1)^2} - (-1)^{n+k} \frac{4(2n+1)(2k+1)}{(n-k)^2(n+k+1)^2} \quad (0 \leq k \leq n-1). \]  
(2.37)

It is proven in Appendix A.1 that

\[ \sum_{m=k}^{n-1} (-1)^{n+m} \frac{2m+1}{(n-m)(n+m+1)} = -\psi(2n+1) + \psi(n+k+1) + \psi(n+1) - \psi(n-k+1) \]
\[ - \psi \left( \left\lfloor \frac{n+k}{2} \right\rfloor + 1 \right) + \psi \left( \left\lfloor \frac{n-k}{2} \right\rfloor + 1 \right) \quad (0 \leq k \leq n-1), \]  
(2.38)

where \( [x] = \max\{n \in \mathbb{Z} : n \leq x \} \). Use of Eq. (2.38) casts Eq. (2.37) into the final form

\[ c_{nk} = (-1)^{n+k} \frac{8(2k+1)}{(n-k)(n+k+1)} \]
\[ \times \left\{ \psi(n+k+1) - \psi(n-k+1) - \psi \left( \left\lfloor \frac{n+k}{2} \right\rfloor + 1 \right) + \psi \left( \left\lfloor \frac{n-k}{2} \right\rfloor + 1 \right) \right\} \]
\[ - \frac{4(2k+1)^2}{(n-k)^2(n+k+1)^2} - (-1)^{n+k} \frac{4(2n+1)(2k+1)}{(n-k)^2(n+k+1)^2} \quad (0 \leq k \leq n-1). \]  
(2.39)

Equation (2.39) says nothing about the coefficient \( c_{nn} \). But from Eqs. (2.23), (2.21) and (2.33) it can be deduced that \( c_{nn} \) may be expressed as

\[ c_{nn} = -\sum_{k=0}^{n-1} c_{nk}. \]  
(2.40)

This implies that the polynomial \( C_n(z) \) may be written as

\[ C_n(z) = \sum_{k=0}^{n-1} c_{nk}[P_k(z) - P_n(z)], \]  
(2.41)

or explicitly, if the result in Eq. (2.39) is used, as

\[ C_n(z) = 4 \sum_{k=0}^{n-1} (-1)^{n+k} \frac{2k+1}{(n-k)(n+k+1)} \]
\[ \times \left\{ 2 \left[ \psi(n+k+1) - \psi(n-k+1) - \psi \left( \left\lfloor \frac{n+k}{2} \right\rfloor + 1 \right) + \psi \left( \left\lfloor \frac{n-k}{2} \right\rfloor + 1 \right) \right] \]
\[ - (-1)^{n+k} \frac{2k+1}{(n-k)(n+k+1)} - \frac{2n+1}{(n-k)(n+k+1)} \right\} [P_k(z) - P_n(z)]. \]  
(2.42)

A bit different formula for \( C_n(z) \) is obtained if the coefficient \( c_{nn} \) is expressed in a closed form.
To find the latter, we combine Eqs. (2.37) and (2.40) and write

\[ c_{nn} = -8[\psi(2n + 1) - \psi(n + 1)] \sum_{k=0}^{n-1} (-1)^{n+k} \frac{2k + 1}{(n-k)(n+k+1)} \]

\[ -8 \sum_{k=0}^{n-1} (-1)^k \frac{2k + 1}{(n-k)(n+k+1)} \sum_{m=k}^{n-1} (-1)^m \frac{2m + 1}{(n-m)(n+m+1)} \]

\[ + 4 \sum_{k=0}^{n-1} \frac{(2k + 1)^2}{(n-k)^2(n+k+1)^2} + 4 \sum_{k=0}^{n-1} (-1)^{n+k} \frac{(2n + 1)(2k + 1)}{(n-k)^2(n+k+1)^2}. \] (2.43)

The sums appearing in Eq. (2.43) are evaluated in individual subsections of the appendix, where it is found that

\[ \sum_{k=0}^{n-1} (-1)^{n+k} \frac{2k + 1}{(n-k)(n+k+1)} = -\psi(2n + 1) + \psi(n + 1), \] (2.44)

\[ \sum_{k=0}^{n-1} (-1)^k \frac{2k + 1}{(n-k)(n+k+1)} \sum_{m=k}^{n-1} (-1)^m \frac{2m + 1}{(n-m)(n+m+1)} \]

\[ = \frac{\pi^2}{12} - \frac{\gamma}{2n + 1} + \frac{1}{2}[\psi(2n + 1) - \psi(n + 1)]^2 - \frac{1}{2n + 1} \psi(2n + 1) - \frac{1}{2} \psi_1(2n + 1), \] (2.45)

\[ \sum_{k=0}^{n-1} \frac{(2k + 1)^2}{(n-k)^2(n+k+1)^2} = \frac{\pi^2}{6} - \frac{2\gamma}{2n + 1} - \frac{2}{2n + 1} \psi(2n + 1) - \psi_1(2n + 1) \] (2.46)

and

\[ \sum_{k=0}^{n-1} (-1)^{n+k} \frac{(2n + 1)(2k + 1)}{(n-k)^2(n+k+1)^2} = -\frac{\pi^2}{12} + \psi_1(2n + 1) - \frac{1}{2} \psi_1(n + 1), \] (2.47)

with \( \gamma \) standing for the Euler–Mascheroni constant and with \( \psi_1(\zeta) = d\psi(\zeta)/d\zeta \) being the trigamma function. Plugging the results (2.44)–(2.47) into the right-hand side of Eq. (2.43) furnishes the coefficient \( c_{nn} \) in the compact form

\[ c_{nn} = -\frac{\pi^2}{3} + 4[\psi(2n + 1) - \psi(n + 1)]^2 + 4\psi_1(2n + 1) - 2\psi_1(n + 1). \] (2.48)

Hence, by virtue of Eqs. (2.33), (2.39) and (2.48), we eventually arrive at the sought formula

\[ C_n(z) = \left\{ -\frac{\pi^2}{3} + 4[\psi(2n + 1) - \psi(n + 1)]^2 + 4\psi_1(2n + 1) - 2\psi_1(n + 1) \right\} P_n(z) \]

\[ + 4 \sum_{k=0}^{n-1} (-1)^{n+k} \frac{2k + 1}{(n-k)(n+k+1)} \left\{ 2\left[\psi(n + k + 1) - \psi(n - k + 1) \right] \right. \]

\[ - \psi\left(\left\lfloor \frac{n + k}{2} \right\rfloor + 1 + \psi\left(\left\lfloor \frac{n - k}{2} \right\rfloor + 1 \right) \right) \right. \]

\[ - (-1)^{n+k} \frac{2k + 1}{(n-k)(n+k+1)} - \frac{2n + 1}{(n-k)(n+k+1)} \right\} P_k(z), \] (2.49)

alternative to the one in Eq. (2.42).
2.5 Explicit expressions for $[\partial^2 P_\nu(z)/\partial \nu^2]_{\nu=n}$ with $0 \leq n \leq 3$

It may be of interest to see how the derivatives $[\partial^2 P_\nu(z)/\partial \nu^2]_{\nu=n}$ look explicitly for several lowest values of $n$. From Eqs. (2.12), (2.32) and (2.49), for $0 \leq n \leq 3$ we find that

\[
\frac{\partial^2 P_\nu(z)}{\partial \nu^2}
\bigg|_{\nu=0} = -2 \operatorname{Li}_2 \left( \frac{1-z}{2} \right), \tag{2.50a}
\]

\[
\frac{\partial^2 P_\nu(z)}{\partial \nu^2}
\bigg|_{\nu=1} = -2z \operatorname{Li}_2 \left( \frac{1-z}{2} \right) + (2z+2) \ln \frac{z+1}{2} - 2z + 2, \tag{2.50b}
\]

\[
\frac{\partial^2 P_\nu(z)}{\partial \nu^2}
\bigg|_{\nu=2} = (-3z^2 + 1) \operatorname{Li}_2 \left( \frac{1-z}{2} \right) + \left( \frac{7}{2}z^2 + 3z - \frac{1}{2} \right) \ln \frac{z+1}{2} - \frac{11}{4}z^2 + \frac{5}{2}z + \frac{1}{4}, \tag{2.50c}
\]

\[
\frac{\partial^2 P_\nu(z)}{\partial \nu^2}
\bigg|_{\nu=3} = (-5z^3 + 3z) \operatorname{Li}_2 \left( \frac{1-z}{2} \right) + \left( \frac{37}{6}z^3 + 5z^2 - \frac{5}{2}z - \frac{4}{3} \right) \ln \frac{z+1}{2} - \frac{155}{36}z^3 + \frac{23}{6}z^2 + \frac{19}{12}z - \frac{10}{9}. \tag{2.50d}
\]

3 The derivatives $[\partial Q_\nu(z)/\partial \nu]_{\nu=n}$ and $[\partial Q_\nu(x)/\partial \nu]_{\nu=n}$

In Refs. [11, 12], we exploited representations of the first-order derivatives $[\partial P_\nu(z)/\partial \nu]_{\nu=n}$ found therein to obtain expressions for the Legendre functions of the second kind $Q_\nu(z)$, with $n \in \mathbb{N}_0$, both for $z \in \mathbb{C} \setminus [-1, 1]$ and for $z = x \in (-1, 1)$. Below we shall show that the knowledge of the second-order derivatives $[\partial^2 P_\nu(z)/\partial \nu^2]_{\nu=n}$ allows one to obtain explicit formulas for the first-order derivatives $[\partial Q_\nu(z)/\partial \nu]_{\nu=n}$ and $[\partial Q_\nu(x)/\partial \nu]_{\nu=n}$, again with $n \in \mathbb{N}_0$.

3.1 The derivatives $[\partial Q_\nu(z)/\partial \nu]_{\nu=n}$ for $z \in \mathbb{C} \setminus [-1, 1]$

3.1.1 The general form of $[\partial Q_\nu(z)/\partial \nu]_{\nu=n}$

The Legendre function of the second kind, $Q_\nu(z)$, may be defined in terms of the function of the first kind through the formula

\[
Q_\nu(z) = \frac{\pi}{2} \frac{e^{\pi i \nu} P_\nu(z) - P_\nu(-z)}{\sin(\pi \nu)} \quad \text{(Im } z \geq 0). \tag{3.1}
\]

Hence, it follows that

\[
\frac{\partial Q_\nu(z)}{\partial \nu} = \frac{\pi}{2 \sin^2(\pi \nu)} \left\{ -\pi[P_\nu(z) - P_\nu(-z) \cos(\pi \nu)] + \left[ e^{\pi i \nu} \frac{\partial P_\nu(z)}{\partial \nu} - \frac{\partial P_\nu(-z)}{\partial \nu} \right] \sin(\pi \nu) \right\} \quad \text{(Im } z \geq 0). \tag{3.2}
\]

From this, for $n \in \mathbb{N}_0$, with the use of the L’Hospital rule, we obtain

\[
\frac{\partial Q_\nu(z)}{\partial \nu}
\bigg|_{\nu=n} = -\frac{\pi^2}{4} P_n(z) \mp \frac{i\pi}{2} \frac{\partial P_n(z)}{\partial \nu} \bigg|_{\nu=n} + \frac{1}{4} \frac{\partial^2 P_n(z)}{\partial \nu^2} \bigg|_{\nu=n} + \frac{(1-n)^2}{4} \frac{\partial^2 P_n(-z)}{\partial \nu^2} \bigg|_{\nu=n} \quad \text{(Im } z \geq 0). \tag{3.3}
\]

If in the above formula the second-order derivatives $[\partial^2 P_\nu(z)/\partial \nu^2]_{\nu=n}$ are substituted with expressions following from Eq. (2.12) and the first-order derivative $[\partial P_\nu(z)/\partial \nu]_{\nu=n}$ is replaced by the right-hand side of Eq. (1.1), this yields $[\partial Q_\nu(z)/\partial \nu]_{\nu=n}$ in the form

\[
\frac{\partial Q_\nu(z)}{\partial \nu}
\bigg|_{\nu=n} = \frac{1}{2} P_n(z) \left( \operatorname{Li}_2 \left( \frac{z+1}{2} \right) - \frac{1-z}{2} \right) + \frac{1}{4} B_n(z) \mp \frac{i\pi}{2} P_n(z) \ln \frac{z+1}{2} - \frac{(1-n)^2}{4} \frac{\partial^2 P_n(-z)}{\partial \nu^2} \bigg|_{\nu=n} \quad \text{(Im } z \geq 0). \tag{3.4}
\]
A more elegant expression for \( \partial Q_\nu(z)/\partial \nu \rvert_{\nu=n} \) follows if the dilogarithm \( \text{Li}_2[(1+z)/2] \) is eliminated from Eq. (3.4) with the aid of the Euler’s identity [23, Eq. (1.11)]

\[
\text{Li}_2 z + \text{Li}_2(1-z) = \frac{\pi^2}{6} - \ln z \ln(1-z),
\]

(3.5)

the relation

\[
1 - z = e^{\pi i \nu}(z - 1) \quad (\text{Im } z \geq 0)
\]

(3.6)

and the result in Eq. (2.31). Proceeding in that way, one eventually finds that

\[
\frac{\partial Q_\nu(z)}{\partial \nu} \bigg|_{\nu=n} = -P_n(z) \text{Li}_2 \frac{1-z}{2} - \frac{1}{2}P_n(z) \ln \frac{z+1}{2} \ln \frac{z-1}{2} + \frac{1}{4}B_n(z) \ln \frac{z+1}{2} - \frac{(-1)^n}{4}B_n(-z) \ln \frac{z-1}{2} - \frac{\pi^2}{6}P_n(z) + \frac{1}{4}C_n(z) - \frac{(-1)^n}{4}C_n(-z) \quad (n \in \mathbb{N}_0).
\]

(3.7)

### 3.1.2 Explicit expressions for \( \partial Q_\nu(z)/\partial \nu \rvert_{\nu=n} \) with \( 0 \leq n \leq 3 \)

Explicit forms of the derivatives \( \partial Q_\nu(z)/\partial \nu \rvert_{\nu=n} \) with \( 0 \leq n \leq 3 \), obtained from Eq. (3.7) with the use of Eqs. (3.22) and (2.49), are

\[
\frac{\partial Q_\nu(z)}{\partial \nu} \bigg|_{\nu=0} = -\text{Li}_2 \frac{1-z}{2} - \frac{1}{2} \ln \frac{z+1}{2} \ln \frac{z-1}{2} - \frac{\pi^2}{6},
\]

(3.8a)

\[
\frac{\partial Q_\nu(z)}{\partial \nu} \bigg|_{\nu=1} = -z \text{Li}_2 \frac{1-z}{2} - \frac{1}{2} \ln \frac{z+1}{2} \ln \frac{z-1}{2} + \left( \frac{1}{2} + \frac{1}{2} \ln \frac{z+1}{2} \right) \ln \frac{z-1}{2} - \frac{\pi^2}{6} z + 1,
\]

(3.8b)

\[
\frac{\partial Q_\nu(z)}{\partial \nu} \bigg|_{\nu=2} = -\left( \frac{3}{4} z^2 + \frac{1}{2} \right) \text{Li}_2 \frac{1-z}{2} + \left( \frac{3}{4} z^2 + \frac{1}{4} \right) \ln \frac{z+1}{2} \ln \frac{z-1}{2} + \left( \frac{7}{8} z^2 + \frac{3}{4} z - \frac{1}{8} \right) \ln \frac{z+1}{2} + \left( -\frac{7}{8} z^2 + \frac{3}{4} z + \frac{1}{8} \right) \ln \frac{z-1}{2} - \frac{\pi^2}{4} z^2 + \frac{5}{4} z + \frac{\pi^2}{12},
\]

(3.8c)

\[
\frac{\partial Q_\nu(z)}{\partial \nu} \bigg|_{\nu=3} = -\left( \frac{5}{2} z^3 + \frac{3}{2} z^2 \right) \text{Li}_2 \frac{1-z}{2} + \left( \frac{5}{4} z^3 + \frac{3}{4} z^2 \right) \ln \frac{z+1}{2} \ln \frac{z-1}{2} + \left( \frac{37}{24} z^3 + \frac{5}{4} z^2 - \frac{5}{8} z - \frac{1}{3} \right) \ln \frac{z+1}{2} + \left( -\frac{37}{24} z^3 + \frac{5}{4} z^2 + \frac{5}{8} z - \frac{1}{3} \right) \ln \frac{z-1}{2} - \frac{5 \pi^2}{12} z^3 + \frac{23}{12} z^2 + \frac{\pi^2}{4} z - \frac{5}{9}.
\]

(3.8d)

We find it remarkable that coefficients in the polynomial part of \( \partial Q_\nu(z)/\partial \nu \rvert_{\nu=n} \) are alternately irrational and rational.

### 3.2 The derivatives \( \partial Q_\nu(x)/\partial \nu \rvert_{\nu=n} \) for \( -1 < x < 1 \)

On the real interval \(-1 < x < 1\), the Legendre function of the second kind, \( Q_\nu(x) \), is defined as the average of the limits \( Q_\nu(x+i0) \) and \( Q_\nu(x-i0) \) resulting when \( z \) approaches \( x \) from the upper (\( \text{Im } z > 0 \)) and lower (\( \text{Im } z < 0 \)) half-planes, respectively. One has

\[
Q_\nu(x) = \frac{1}{2} \left[ Q_\nu(x+i0) + Q_\nu(x-i0) \right] \quad (-1 < x < 1),
\]

(3.9)

and consequently

\[
\frac{\partial Q_\nu(x)}{\partial \nu} \bigg|_{\nu=n} = \frac{1}{2} \left( \frac{\partial Q_\nu(x+i0)}{\partial \nu} \bigg|_{\nu=n} + \frac{1}{2} \frac{\partial Q_\nu(x-i0)}{\partial \nu} \bigg|_{\nu=n} \right).
\]

(3.10)
From this, with the use of Eq. (3.7) and the identity
\[ x - 1 \pm 0 = e^{\pm i\pi} (1 - x) \quad (-1 < x < 1), \]  
(3.11)

one finds that
\[
\partial Q_{\nu}(x) \bigg|_{\nu=n} = -P_n(x) \operatorname{Li}_2 \left(\frac{1 - x}{2}\right) - \frac{1}{2} P_n(x) \ln \frac{1 + x}{2} - \frac{1}{2} \ln \frac{1 - x}{2} + \frac{1}{4} B_n(x) \ln \frac{1 + x}{2}
\]
\[ - \frac{(-1)^n}{4} B_n(-x) \ln \frac{1 - x}{2} - \frac{\pi^2}{6} P_n(x) + \frac{1}{4} C_n(x) - \frac{(-1)^n}{4} C_n(-x) \quad (n \in \mathbb{N}_0). \]
(3.12)

There is no need to provide here explicit representations for the derivatives \( \partial Q_{\nu}(x) / \partial \nu \big|_{\nu=n} \) for several lowest non-negative values of \( n \). As it is seen from Eqs. (3.7) and (3.12), such representations for \( 0 \leq n \leq 3 \) may be immediately deduced from Eqs. (3.8a)–(3.8d) after the replacement of \( z \) with \( x \) is made everywhere in the latter set of equations, except for the logarithm \( \ln[(z - 1)/2] \), which is to be substituted with \( \ln[(1 - x)/2] \).

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**A Appendix: Proofs of summation formulas used in Sec. 2.4**

**A.1 The summation formulas (2.38) and (2.44)**

We denote
\[
S_1 = \sum_{k=m}^{n-1} (-1)^{n+k} \frac{2k + 1}{(n - k)(n + k + 1)} \quad (0 \leq m \leq n - 1).
\]
(A.1.1)

We have
\[
S_1 = \sum_{k=m}^{n-1} (-1)^{n+k} \frac{n-k}{n} - \sum_{k=m}^{n-1} (-1)^{n+k} \frac{n+k+1}{n+k+1}
\]
(A.1.2)

and further
\[
S_1 = \sum_{k=1}^{n+m} \frac{(-1)^k}{k} + \sum_{k=n+m+1}^{2n} \frac{(-1)^k}{k} = \sum_{k=1}^{n-m} \frac{(-1)^k}{k} + \sum_{k=1}^{2n} \frac{(-1)^k}{k} - \sum_{k=1}^{n+m} \frac{(-1)^k}{k}.
\]
(A.1.3)

However, it is easy to show that
\[
\sum_{k=1}^{N} \frac{(-1)^k}{k} = -\sum_{k=1}^{N} \frac{1}{k} + \sum_{k=1}^{\lfloor N/2 \rfloor} \frac{1}{k} \quad (N \in \mathbb{N}_0),
\]
(A.1.4)

where \( \lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\} \) stands for the integer part of \( x \). Since
\[
\sum_{k=1}^{N} \frac{1}{k} = \psi(N + 1) - \psi(1) \quad (N \in \mathbb{N}_0),
\]
(A.1.5)

with \( \psi(z) = d \ln \Gamma(z)/dz \) being the digamma function, Eq. (A.1.4) may be rewritten in the form
\[
\sum_{k=1}^{N} \frac{(-1)^k}{k} = -\psi(N + 1) + \psi \left( \left\lfloor \frac{N}{2} \right\rfloor + 1 \right) \quad (N \in \mathbb{N}_0).
\]
(A.1.6)
Application of the result (A.1.6) to each of the three sums on the extreme right-hand side of Eq. (A.1.3) gives finally
\[
\sum_{k=m}^{n-1} (-1)^{n+k} \frac{2k + 1}{(n - k)(n + k + 1)} = -\psi(2n + 1) + \psi(n + m + 1) + \psi(n + 1) - \psi(n - m + 1) - \psi \left( \left\lfloor \frac{n + m}{2} \right\rfloor + 1 \right) + \psi \left( \left\lfloor \frac{n - m}{2} \right\rfloor + 1 \right) \quad (0 \leq m \leq n - 1). \tag{A.1.7}
\]
After \( k \) is interchanged with \( m \), Eq. (A.1.7) becomes identical with Eq. (2.38).

For \( m = 0 \), Eq. (A.1.7) becomes
\[
\sum_{k=0}^{n-1} (-1)^{n+k} \frac{2k + 1}{(n - k)(n + k + 1)} = -\psi(2n + 1) + \psi(n + 1), \tag{A.1.8}
\]
which is Eq. (2.44).

### A.2 The summation formula (2.45)

We denote
\[
S_2 = \sum_{k=0}^{n-1} (-1)^{k} \frac{2k + 1}{(n - k)(n + k + 1)} \sum_{m=k}^{n-1} (-1)^{m} \frac{2m + 1}{(n - m)(n + m + 1)}. \tag{A.2.1}
\]
Application of the identity
\[
\sum_{k=0}^{N_2} f_k \sum_{m=0}^{N_2} f_m = \frac{1}{2} \left( \sum_{k=0}^{N_2} f_k \right)^2 + \frac{1}{2} \sum_{k=0}^{N_2} f_k^2 \quad (N_1 \leq N_2) \tag{A.2.2}
\]
transforms Eq. (A.2.1) into
\[
S_2 = \frac{1}{2} \sum_{k=0}^{n-1} (-1)^{k} \frac{2k + 1}{(n - k)(n + k + 1)} \right)^2 + \frac{1}{2} \sum_{k=0}^{n-1} \frac{(2k + 1)^2}{(n - k)^2(n + k + 1)^2}. \tag{A.2.3}
\]
from which, with the help of Eqs. (A.1.8) and (A.3.7), we obtain
\[
\sum_{k=0}^{n-1} (-1)^{k} \frac{2k + 1}{(n - k)(n + k + 1)} \sum_{m=k}^{n-1} (-1)^{m} \frac{2m + 1}{(n - m)(n + m + 1)} = \frac{\pi^2}{12} - \frac{\gamma}{2n + 1} + \frac{1}{2} \psi'(2n + 1) - \psi(n + 1) \left| 2n + 1 \right| - \frac{1}{2} \psi(2n + 1), \tag{A.2.4}
\]
which is Eq. (2.45).

To prove the identity (A.2.2), we write the obvious chain of equalities (\( N_1 \leq N_2 \) is assumed)
\[
\left( \sum_{k=N_1}^{N_2} f_k \right)^2 = \sum_{k=N_1}^{N_2} f_k \sum_{m=N_1}^{N_2} f_m = \sum_{k=N_1}^{N_2} f_k \sum_{m=N_1}^{k} f_m + \sum_{k=N_1}^{N_2} f_k \sum_{m=k}^{N_2} f_m - \sum_{k=N_1}^{N_2} f_k^2. \tag{A.2.5}
\]
Manipulating with the first term on the extreme right-hand side of Eq. (A.2.5), we have
\[
\sum_{k=N_1}^{N_2} f_k \sum_{m=N_1}^{k} f_m = \sum_{k=N_1}^{N_2} f_k \sum_{m=k}^{N_2} f_m = \sum_{k=N_1}^{N_2} f_k \sum_{m=k}^{N_2} f_m. \tag{A.2.6}
\]
Plugging the result (A.2.6) into Eq. (A.2.5), we obtain

\[
\left( \sum_{k=N_1}^{N_2} f_k \right)^2 = 2 \sum_{k=N_1}^{N_2} f_k \sum_{m=k}^{N_2} f_m - \sum_{k=N_1}^{N_2} f_k^2, \tag{A.2.7}
\]

from which the identity in Eq. (A.2.2) follows immediately.

**A.3 The summation formula (2.46)**

We denote

\[
S_3 = \sum_{k=0}^{n-1} \frac{(2k+1)^2}{(n-k)^2(n+k+1)^2}. \tag{A.3.1}
\]

If we carry out the partial fraction decomposition of the summand, we have

\[
S_3 = \sum_{k=0}^{n-1} \frac{1}{(n-k)^2} + \sum_{k=0}^{n-1} \frac{1}{(n+k+1)^2} = \frac{1}{2n+1} \sum_{k=0}^{n-1} \frac{1}{n-k} - \frac{2}{2n+1} \sum_{k=0}^{n-1} \frac{1}{n+k+1}, \tag{A.3.2}
\]

and further, after obvious rearrangements,

\[
S_3 = \sum_{k=0}^{2n-1} \frac{1}{(k+1)^2} - \frac{2}{2n+1} \sum_{k=1}^{2n-1} \frac{1}{k}. \tag{A.3.3}
\]

Now, it holds that

\[
\sum_{k=0}^{N-1} \frac{1}{(k+1)^2} = \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} - \sum_{k=0}^{\infty} \frac{1}{(k+N+1)^2} = \psi_1(1) - \psi_1(N+1) \quad (N \in \mathbb{N}_0), \tag{A.3.4}
\]

where \( \psi_1(z) = d\psi(z)/dz \) is the trigamma function. On employing Eqs. (A.3.4) and (A.1.5) in Eq. (A.3.3), after using the well-known relations

\[
\psi(1) = -\gamma \quad \tag{A.3.5}
\]

(here and below \( \gamma \) stands for the Euler–Mascheroni constant) and

\[
\psi_1(1) = \frac{\pi^2}{6}, \tag{A.3.6}
\]

we finally obtain

\[
\sum_{k=0}^{n-1} \frac{(2k+1)^2}{(n-k)^2(n+k+1)^2} = \frac{\pi^2}{6} - \frac{2\gamma}{2n+1} - \frac{2}{2n+1} \psi(2n+1) - \psi_1(2n+1), \tag{A.3.7}
\]

which is Eq. (2.46).

**A.4 The summation formula (2.47)**

We denote

\[
S_4 = \sum_{k=0}^{n-1} (-1)^{n+k} \frac{(2n+1)(2k+1)}{(n-k)^2(n+k+1)^2}. \tag{A.4.1}
\]

A partial-fraction decomposition of the summand gives

\[
S_4 = \sum_{k=0}^{n-1} (-1)^{n+k} \frac{1}{(n-k)^2} - \sum_{k=0}^{n-1} (-1)^{n+k} \frac{1}{(n+k+1)^2}. \tag{A.4.2}
\]
With a little bit of algebra on the right-hand side of Eq. (A.4.2), we obtain

$$S_4 = \sum_{k=0}^{2n-1} \frac{(-1)^{k+1}}{(k+1)^2}$$

(A.4.3)

and further

$$S_4 = -\sum_{k=0}^{2n-1} \frac{1}{(k+1)^2} + \frac{1}{2} \sum_{k=0}^{n-1} \frac{1}{(k+1)^2}.$$  

(A.4.4)

From this, with reference to Eqs. (A.3.4) and (A.3.6), we eventually arrive at

$$\sum_{k=0}^{n-1} (-1)^{n+k} \frac{(2n+1)(2k+1)}{(n-k)(n+k+1)^2} = -\frac{\pi^2}{12} + \psi_1(2n+1) - \frac{1}{2}\psi_1(n+1),$$

(A.4.5)

which is Eq. (2.47).

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