Self-dual Tensors in

Six-Dimensional Supergravity

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Abstract

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Self-dual Tensors in Six-Dimensional Supergravity

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We review some properties of the field equations of six-dimensional (1,0) supergravity coupled to tensor and vector multiplets, and in particular their relation to covariant and consistent anomalies and a peculiar Noether identity for the energy-momentum tensor. We also describe a lagrangian formulation for this system, obtained applying the Pasti-Sorokin-Tonin prescription.

1 Introduction

Six-dimensional (1,0) supergravity is built out of four types of multiplets. Aside from the gravitational multiplet \((e^a_{\mu}, \psi_{L,\mu}, B^a_{\mu \nu})\), one has the option to add tensor multiplets \((B^{\mu \nu}, \chi_{R}, \phi)\), vector multiplets \((A_{\mu}, \lambda_{L})\) and hypermultiplets. This talk is devoted to some notable properties of the low-energy couplings between the first three types of multiplets, and is meant to complement our previous short review. Couplings between several tensor multiplets have entered the stage relatively recently, since they emerge rather naturally only in perturbative type-I vacua. On the other hand, perturbative heterotic vacua always involve a single tensor multiplet, that is responsible for the corresponding Green-Schwarz mechanism.

The couplings between vector and tensor multiplets are rather unconventional, since they are “classically anomalous” and conspire to implement a generalized Green-Schwarz mechanism. Limitedly to the gauge anomalies, in six dimensions the Green-Schwarz mechanism is visible in the low-energy field equations, and brings about a number of oddities, most notably some singularities in the moduli space of tensor multiplets. These signal an important new phenomenon, a phase transition related to strings of vanishing tension, likely to be of quite some interest in the coming years.

In formulating the low-energy couplings between tensor and vector multiplets, one has two natural options. The first is related to covariant field equations and to the corresponding covariant anomalies. It has the virtue of respecting gauge covariance and supersymmetry, but the resulting field equations are not integrable. The second is related to consistent, and thus integrable, field equations. These may be derived from an action principle that satisfies Wess-Zumino consistency conditions, and as a result embody a supersymmetry anomaly. The complete field equations, first obtained in this framework, are...
not unique. A quartic coupling for the gauginos, proportional to \((\bar{\lambda} \gamma^\mu \lambda)^2\), is undetermined, while the gauge algebra contains an extension that makes the construction consistent for any choice of it. We would like to stress that all these features are determined by local couplings in these “classical” field equations, that are thus a remarkable laboratory for current algebra. Indeed, a closer scrutiny of their properties brought about one further surprise: in a theory with gauge and supersymmetry anomalies, gravitational anomalies are not directly related to the divergence of the energy-momentum tensor. Thus, in a theory without gravitational anomalies the energy-momentum tensor need not be divergenceless.

We shall return to the energy-momentum tensor in the next Section, where we shall also review our recent work on the completion of the covariant equations. Aside from this general lesson, there are also a number of subtleties related to the rigid limit of the coupled equations. These are not considered here, but have been discussed at length in ref. Actually, in mentioning an action principle, we have been somewhat cavalier about the presence of (anti)self-dual antisymmetric tensors, that bring about a number of traditional difficulties. The long-standing problem of giving a covariant action principle to this type of fields has recently been given a compact solution, in terms of a single auxiliary scalar mode, by Pasti, Sorokin and Tonin (PST). The last Section is devoted to the completion of the consistent action principle of ref. using their method. This new result, not contained in the original presentation, is included here for completeness.

## 2 Consistent and covariant field equations and their anomalies

In this Section we would like to review some basic properties of six-dimensional \((1, 0)\) supergravity coupled to vector and tensor multiplets. This model is rather peculiar, since it embodies gauge anomalies induced by tensor couplings to be disposed of by fermion loops that one is actually not accounting for. Consequently, in formulating the field equations, one is faced with the familiar choice between covariant and consistent anomalies. The former originate from a set of covariant non-integrable equations invariant under local supersymmetry, while the latter, generally regarded as the most interesting ones, originate from a set of integrable equations, with a consistent supersymmetry anomaly related to the consistent gauge anomaly by Wess-Zumino consistency conditions.

To lowest order in the fermi fields, in the conventions of ref. the

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\[ a \] A number of fermionic couplings, also to hypermultiplets, with additional singularities related to those of ref. were anticipated in ref.
fermionic equations are

\[ \gamma^{\mu
\nu\rho} D_{\nu} \Psi_{\rho} + v_{r} H^{r\mu
\rho\gamma}_{\nu} \Psi_{\rho} - \frac{i}{2} x^{m}_{r} H^{r\mu
\rho\gamma}_{\nu} \chi^{m}_{\rho} \]

\[ + \frac{i}{2} x^{m}_{r} \partial_{\nu} v_{r} \gamma^{\nu} \gamma^{\mu} \lambda^{m} - \frac{1}{\sqrt{2}} v_{r} c e^{r z} t r_{z} (F_{\sigma\tau} \gamma^{\rho} \gamma^{\mu} \lambda) = 0 \quad , \]  

\[ (1) \]

\[ \gamma^{\mu} D_{\mu} \chi^{m} - \frac{1}{12} v_{r} H^{r\mu\nu\rho\gamma}_{\mu\nu} \chi^{m} - \frac{i}{2} x^{m}_{r} H^{r\mu\nu\rho\gamma}_{\mu\nu} \Psi_{\rho} \]

\[ - \frac{i}{2} x^{m}_{r} \partial_{\nu} v_{r} \gamma^{\mu} \gamma^{\nu} \Psi_{\mu} - \frac{i}{\sqrt{2}} x^{m}_{r} c e^{r z} t r_{z} (F_{\mu\nu} \gamma^{\mu} \lambda) = 0 \quad , \]  

\[ (2) \]

and

\[ v_{r} e^{r z} \gamma^{\mu} D_{\mu} \lambda + \frac{1}{2} \left( \partial_{\mu} v_{r} \right) e^{r z} \gamma^{\mu} \lambda + \frac{1}{2 \sqrt{2}} v_{r} e^{r z} F_{\alpha\beta} \gamma^{\mu} \gamma^{\alpha\beta} \Psi_{\mu} \]

\[ + \frac{i}{2 \sqrt{2}} x^{m}_{r} c e^{r z} F_{\mu\nu} \gamma^{\mu\nu} \chi^{m} - \frac{1}{12} c e^{r z} H^{r\mu\nu\rho} \gamma^{\mu\nu} \lambda = 0 \quad , \]  

\[ (3) \]

while the consistent bosonic equations are

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \partial_{\mu} v^{r} \partial_{\nu} v_{r} - \frac{1}{2} g_{\mu\nu} \partial_{\alpha} v^{r} \partial^{\alpha} v_{r} - G_{r\alpha\beta} H^{r}_{\mu\nu} \gamma^{\alpha\beta} \]

\[ + 4 v_{r} c e^{r z} t r_{z} (F_{\alpha\mu} F_{\alpha\nu} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F_{\alpha\beta}) = 0 \quad , \]  

\[ (4) \]

\[ x^{m}_{r} D_{\mu} (\partial^{\mu} v^{r}) + \frac{2}{3} x^{m}_{r} v_{s} H^{r}_{\alpha\beta\gamma} H^{s\alpha\beta\gamma} - x^{m}_{r} c e^{r z} t r_{z} (F_{\alpha\beta} F_{\alpha\beta}) = 0 \quad , \]  

\[ (5) \]

and

\[ D_{\mu} (v_{r} e^{r z} F^{\mu\nu}) - e^{r z} G_{r s} H^{s\nu\rho\sigma} F_{\rho\sigma} - \frac{1}{8 e^{r z}} \epsilon^{\mu\rho\alpha\beta\gamma\delta} c_{r} A_{\rho} c e^{r z} t r_{z}' (F_{\alpha\beta} F_{\gamma\delta}) \]

\[ - \frac{1}{12 e^{r z}} \epsilon^{\mu\rho\alpha\beta\gamma\delta} c_{r} F_{\rho\alpha} c e^{r z} c e^{r z}' \omega_{3\beta\gamma\delta} = 0 \quad . \]  

\[ (6) \]

Moreover, the tensor fields satisfy (anti)self-duality conditions, conveniently summarized as

\[ G_{r s} H^{s\mu\nu} = \frac{1}{6 e^{r z}} \epsilon^{\mu\rho\alpha\beta\gamma} H_{r\alpha\beta\gamma} \quad , \]  

\[ (7) \]

where \( G_{r s} = v_{r} v_{s} + x^{m}_{r} x^{m}_{s} \).

In this talk, as in refs. \[6, 9, 10, 8\], we are confining our attention to residual anomalies, that correspond to reducible traces. In type-I vacua, these are the contributions left over by tadpole conditions. In String Theory, all residual
anomalies are absent \textit{a priori}, since they would draw their origin from the non-planar one-loop amplitude, that is regulated by the momentum flow. On the other hand, in Field Theory the residual anomalies draw their origin from fermion loops, and are disposed of by the Green-Schwarz couplings that we are actually discussing. In particular, eqs. (1)-(6) embody a reducible gauge anomaly
\begin{equation}
A_\Lambda = -\frac{1}{4} c^\mu c^\nu c^\rho c^\sigma tr_z(\Lambda \partial_\mu A_\nu)tr_{z'}(F_{\alpha\beta} F_{\gamma\delta})
\end{equation}
and a corresponding supersymmetry anomaly
\begin{equation}
A_\epsilon = -c^\epsilon c^\epsilon' [tr_z(\delta_\epsilon A A)tr_{z'}(F^2) - 2tr_z(\delta_\epsilon A F)\omega^z_3]
\end{equation}
determined by Wess-Zumino consistency conditions.

The complete field equations were obtained from the commutator of two supersymmetry transformations on the fermi fields, in the spirit of refs. However, in this case one is actually solving Wess-Zumino consistency conditions, and these do not fix a cubic contribution to the gaugino equation and a related quartic contribution to the vielbein equation. This lack of uniqueness, a rather surprising phenomenon in supergravity, reflects a familiar property of anomalies, that are defined up to the variation of a local functional. It would be interesting to investigate further this issue directly in String Theory.

The gauge anomaly $A_\Lambda = \delta_\Lambda \mathcal{L}$ naturally satisfies the condition
\begin{equation}
A_\Lambda = -tr(\Lambda D_{\mu} J^\mu)
\end{equation}
where $J^\mu = 0$ is the complete field equation of the vector field. One can similarly show that the supersymmetry anomaly is related to the field equation of the gravitino, that we write succinctly $\bar{J}^\mu$, according to
\begin{equation}
A_\epsilon = -(\epsilon D_{\mu} \bar{J}^\mu)
\end{equation}

Moreover, these equations embody an amusing Noether identity for the energy-momentum tensor, a general result in Field Theory, that came as a little surprise to us. In a theory with gauge and supersymmetry anomalies, the gravitational anomaly is \textit{not} simply related to the divergence of the energy-momentum tensor, since
\begin{equation}
A_\xi = \delta_\xi \mathcal{L} = 2\xi_{\nu} D_{\mu} T^{\mu\nu} + \xi_{\nu} tr(A^\nu D_{\mu} J^\mu) + \xi_{\nu} (\bar{\Psi}^\nu D_{\mu} \bar{J}^\mu)
\end{equation}

In particular, in our case we are not accounting for gravitational anomalies, that would result in higher-derivative couplings, and indeed one can verify that
the divergence of the energy-momentum tensor does not vanish, but satisfies
the relation

\[ D_\mu T^{\mu\nu} = -\frac{1}{2} tr(A^\nu D_\mu J^\mu) - \frac{1}{2} (\bar{\Psi}^\nu D_\mu \bar{J}^\mu) . \]  

(13)

Thus, the lack of conservation of matter currents feeds an algebraic inconsistency in the Einstein equations. We are not aware of a previous discussion of this simple but neat result in the literature.

Consistent and covariant gauge anomalies are related by the divergence of a local functional. In six dimensions, to lowest order in the fermi fields, the residual covariant gauge anomaly is

\[ A^{\text{cov}}_\Lambda = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta\gamma\delta} c^z c^{z'} tr_z (\Lambda F_{\mu\nu}) tr_{z'} (F_{\alpha\beta} F_{\gamma\delta}) , \]

(14)

and is related to the consistent anomaly according to

\[ A^{\text{cons}}_\Lambda + tr[\Delta D_\mu f^\mu] = A^{\text{cov}}_\Lambda , \]

(15)

where

\[ f^\mu = -c^z c^{z'} \epsilon^{\mu\nu\alpha\beta\gamma\delta} \left\{ \frac{1}{4} A_\nu tr_{z'} (F'_{\alpha\beta} F'_{\gamma\delta}) + \frac{1}{6} F_{\nu\alpha} \omega'_{\beta\gamma\delta} \right\} . \]

(16)

Comparing eq. (14) and eq. (10), one can see that

\[ A_e = tr(\delta_e A_\mu f^\mu) , \]

(17)

and this implies that the transition from consistent to covariant anomalies turns a model with a supersymmetry anomaly into one without any. Indeed, six-dimensional supergravity coupled to vector and tensor multiplets was originally formulated in this fashion in ref. to lowest order in the fermi fields, extending the work of Romans.

This observation can be generalized to all orders in the fermi fields. The complete supersymmetry anomaly has the form

\[ A_e = tr(\delta_e A_\mu f^\mu) + \delta_e e_\mu a g^{\mu a} , \]

(18)

where

\[
\begin{align*}
    f^\mu &= c^z c^{z'} tr_{z'} \left( -\frac{1}{4} \epsilon^{\mu\nu\alpha\beta\gamma\delta} A_\nu F'_{\alpha\beta} F'_{\gamma\delta} - \frac{1}{6} \epsilon^{\mu\nu\alpha\beta\gamma\delta} F_{\nu\alpha} \omega'_{\beta\gamma\delta} \\
    &+ \frac{ie}{2} F_{\nu\rho} (\bar{\lambda} \gamma^{\mu\rho} \lambda') + \frac{ie}{2} (\bar{\lambda} \gamma^{\mu\rho} \lambda') F_{\nu\rho} + ie (\bar{\lambda} \gamma_\nu \lambda') F^{\mu\nu} \\
    &- \frac{e}{2\sqrt{2}} (\bar{\lambda} \gamma^{\mu\nu} \gamma^\rho \lambda')(\bar{\lambda} \gamma_\nu \Psi_\rho) + \frac{e a m \epsilon^{\mu} c^{z'} z c^{z'} v t}{2\sqrt{2}} (-\frac{3i}{2\sqrt{2}} (\bar{\lambda} \gamma^\mu \lambda'(\bar{\lambda} \gamma^\mu \lambda'(\bar{\lambda} \gamma^\mu \lambda'))
\end{align*}
\]
It should be appreciated that both expressions depend on the but no longer integrable. The divergence of the complete covariant vector 

\[ \frac{i}{4\sqrt{2}}(\bar{\lambda}_\gamma \gamma^{\mu\nu\rho} \lambda')(\bar{\lambda}_\gamma\gamma^{\mu\nu} \lambda') \] 

\[ - \frac{i}{2\sqrt{2}}(\bar{\lambda}_\gamma \gamma^{\mu\nu\rho} \lambda')(\bar{\lambda}_\gamma\gamma^{\mu\nu} \lambda') \]

\[ + e[-i\alpha \bar{F}_{\nu\rho}(\bar{\lambda}_\gamma \gamma^{\mu\nu\rho} \lambda') + 2i\alpha(\bar{\lambda}_\gamma \gamma^{\mu\nu\rho} \lambda')\bar{F}_{\nu\rho} - 6i\alpha(\bar{\lambda}_\gamma \gamma^\nu\lambda')\bar{F}^\nu] \]

\[ + \frac{ex^m e^sz}{v t e^z}[-i\alpha \sqrt{2}(\bar{\lambda}_\gamma \gamma^\mu \lambda')(\bar{\lambda}_\gamma \gamma^\mu \lambda') + \frac{i\alpha}{2\sqrt{2}}(\bar{\lambda}_\gamma \gamma^{\mu\nu\rho} \lambda')(\bar{\lambda}_\gamma \gamma^\mu \lambda')] \]

\[ + \frac{ex^m e^sz}{v t e^z} \frac{i\alpha}{\sqrt{2}}(\bar{\lambda}_\gamma \gamma^\mu \lambda')(\bar{\lambda}_\gamma \gamma^\mu \lambda') - \frac{i\alpha}{2\sqrt{2}}(\bar{\lambda}_\gamma \gamma^{\mu\nu\rho} \lambda')(\bar{\lambda}_\gamma \gamma^{\mu\nu} \lambda') \]

\[ + \frac{i\alpha}{\sqrt{2}}(\bar{\lambda}_\gamma \gamma^\mu \lambda')(\bar{\lambda}_\gamma \gamma^\mu \lambda')] \]

differs from the expression in eq. (19) by the addition of higher-order fermionic terms, while

\[ g^\mu_\alpha = c^z e^{r^z} tr_{r,z} \left\{ \frac{e^{r^z}}{2}(\bar{\lambda}_\gamma \gamma^{\mu\nu\rho} \lambda')(\bar{\lambda}_\gamma \gamma^\mu \lambda') + \frac{ae^\rho}{2}(\bar{\lambda}_\gamma \gamma^\nu \lambda')(\bar{\lambda}_\gamma \gamma^\nu \lambda') \right\} \]

\[ (eq. A^\mu)_{(cov)} = J^\mu_{(cov)} = \frac{\delta L}{\delta A^\mu} - f^\mu \]

and similarly for the Einstein equation, the resulting theory is supersymmetric but no longer integrable. The divergence of the complete covariant vector equation satisfies

\[ tr(\Lambda D_\mu f^\mu_{(cov)}) = -A^\mu_{(cov)} \]

where

\[ A^\mu_{(cov)} = c^z e^{r^z} tr_{r,z} \left\{ \frac{e^{r^z}}{2}(\bar{\lambda}_\gamma \gamma^{\mu\nu\rho} \lambda')(\bar{\lambda}_\gamma \gamma^\mu \lambda') \right\} \]

\[ + \frac{ie}{2\sqrt{2}}(\bar{\lambda}_\gamma \gamma^{\mu\nu\rho} \lambda') F^\nu_{\rho} + e\Lambda D_\mu \{ i(\bar{\lambda}_\gamma \gamma^\mu \lambda') F^\mu_{\nu} - i\alpha \bar{F}_{\nu\rho}(\bar{\lambda}_\gamma \gamma^{\mu\nu\rho} \lambda') \}
\]

\[ + i\alpha(\bar{\lambda}_\gamma \gamma^\mu \lambda') \bar{F}^\nu_{\rho} - 6i\alpha(\bar{\lambda}_\gamma \gamma^\nu \lambda') \bar{F}^\mu_{\nu} \]

\[ - \frac{1}{2\sqrt{2}}(\bar{\lambda}_\gamma \gamma^{\mu\nu\rho} \lambda') \]

\[ - \frac{i}{2\sqrt{2}}(\bar{\lambda}_\gamma \gamma^\mu \lambda')(\bar{\lambda}_\gamma \gamma^{\mu\nu\rho} \lambda') + \frac{i\alpha}{2\sqrt{2}}(\bar{\lambda}_\gamma \gamma^{\mu\nu\rho} \lambda') \]

\[ - \frac{i}{2\sqrt{2}}(\bar{\lambda}_\gamma \gamma^{\mu\nu\rho} \lambda')(\bar{\lambda}_\gamma \gamma^{\mu\nu} \lambda') \]

\[ + \frac{ex^m e^sz}{v t e^z} \left\{ \frac{e^{r^z}}{2}(\bar{\lambda}_\gamma \gamma^\mu \lambda')(\bar{\lambda}_\gamma \gamma^\mu \lambda') \right\} \]

\[ + \frac{i\alpha}{\sqrt{2}}(\bar{\lambda}_\gamma \gamma^\nu \lambda')(\bar{\lambda}_\gamma \gamma^\nu \lambda') \]

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\[ + \frac{i\alpha}{\sqrt{2}}(\bar{\lambda}_\gamma \gamma^\nu \lambda')(\bar{\lambda}_\gamma \gamma^\nu \lambda') \]
now contains higher-order fermi terms. Finally, one can study the divergence of the Rarita-Schwinger and Einstein equations in the covariant model. With the covariant equations obtained from the consistent ones by the redefinition of eq. (21) and by

\[(eq. \ e^\mu_a)_{(\text{cov})} = \frac{\delta L}{\delta e^\mu_a} - g^\mu_a, \quad (24)\]

where \(g^\mu_a\) is defined in eq. (20), the usual procedure shows that the divergence of the Rarita-Schwinger equation vanishes for any value of the parameter \(\alpha\). On the other hand, the divergence of the energy-momentum tensor presents further subtleties that we would like to address. In particular, it vanishes to lowest order, while it gives a covariant non-vanishing result if all fermion couplings are taken into account. The subtleties have to do with the transformation of the vector under general coordinate transformations,

\[\delta_\xi A_\mu = -\xi^\alpha \partial_\alpha A_\mu - \partial_\mu \xi^\alpha A_\alpha \quad (25)\]

and with the corresponding (off-shell) form of the identity of eq. (12). Starting again from the consistent equations, one finds

\[A_\xi = 2\xi_\nu D_\mu T^{\mu \nu} + \xi_\nu tr(A^\nu D_\mu J_\mu) + \xi_\nu tr(F^{\mu \nu} J_\mu) + \xi_\nu (\bar{\Psi}^\nu D_\mu \tilde{J}^\mu) . \quad (26)\]

Reverting to the covariant form then eliminates the divergence of the Rarita-Schwinger equation and alters the vector equation, so that the third, “off-shell” term, has to be retained. The final result,

\[D_\mu T^{\mu \nu}_{(\text{cov})} = -\frac{1}{2} tr(A^\nu D_\mu J^{\mu \nu}_{(\text{cov})} + f_\mu F^{\mu \nu} + A^\nu D_\mu f_\mu) - \frac{1}{2} \varepsilon^{\alpha \nu \alpha} D_\mu g^\mu_a , \quad (27)\]

is nicely verified by our equations. As anticipated, this implies that the divergence of \(T^{\mu \nu}_{(\text{cov})}\) vanishes to lowest order in the fermi couplings.

### 3 PST construction and covariant Lagrangian formulation

In the previous Section we have reviewed a number of properties of 6d (1,0) supergravity coupled to vector and tensor multiplets. We have always confined our attention to the field equations, thus evading the traditional difficulties met with the action principles for (anti)self-dual tensor fields. In this Section we would like to complete our discussion and present an action principle for the consistent field equations. What follows is an application of a general method introduced by Pasti, Sorokin and Tonin (PST), that have shown how
to obtain Lorentz-covariant Lagrangians for (anti)self-dual tensors with a single auxiliary field. Alternative constructions, some of which preceded the work of PST, need an infinite number of auxiliary fields, and bear a closer relationship to the BRST formulation of closed-string spectra. This method has already been applied to a number of systems, including (1, 0) six-dimensional supergravity coupled to tensor multiplets and type IIB ten-dimensional supergravity, whose (local) gravitational anomaly has been shown to reproduce the well-known results of Alvarez-Gaumé and Witten. Still, an action principle for the consistent equations reviewed in the previous Section is of some interest since, as we have seen, these six-dimensional models have a number of unfamiliar properties.

Let us begin by considering a single 2-form with a self-dual field strength in six-dimensional Minkowski space. The PST lagrangian

$$\mathcal{L} = \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{4(\partial\phi)^2} \partial^\mu \phi H^{-\sigma\mu\rho} H^{-\sigma\nu\rho} \partial_\sigma \phi \ ,$$

(28)

where $H = dB$ and $H^{-} = H - *H$, is invariant under the gauge transformation $\delta B = d\Lambda$, as well as under the additional gauge transformations

$$\delta B = (d\phi)\Lambda' \ ,$$

(29)

and

$$\delta \phi = \Lambda \ ,$$

$$\delta B_{\mu\nu} = \frac{\Lambda}{(\partial\phi)^2} H^{-\sigma\mu\rho} \partial_\sigma \phi \ .$$

(30)

The last two types of gauge transformations can be used to recover the usual field equation of a self-dual 2-form. Indeed, the scalar equation results from the tensor equation contracted with

$$\frac{H_{\mu\nu\rho} \partial^\rho \phi}{(\partial\phi)^2} \ ,$$

(31)

and consequently does not introduce any additional degrees of freedom. The invariance of eq. (30) can then be used to eliminate the scalar field. This field, however, cannot be set to zero, since this choice would clearly make the Lagrangian of eq. (29) inconsistent. With this proviso, using eq. (29) one can see that the only solution of the tensor-field equation is precisely the self-duality condition for its field strength.
We now want to apply this construction to six-dimensional supergravity coupled to vector and tensor multiplets. In the conventions of ref. 10, the theory describes a single self-dual 2-form

\[ \hat{H}_{\mu\nu\rho} = v_r \hat{H}^r_{\mu\nu\rho} - i \frac{1}{8} (\hat{\chi}^m \gamma_{\mu\nu\rho} \chi^m) \] (32)

and \( n \) antself-dual 2-forms

\[ \hat{H}^m_{\mu\nu\rho} = x^m_r \hat{H}^r_{\mu\nu\rho} + i \frac{1}{4} x^m_r c^{rz} tr_z (\bar{\lambda} \gamma_{\mu\nu\rho} \lambda) \] . (33)

The complete Lagrangian is obtained adding the term

\[ -\frac{1}{4} \frac{\partial^\mu \phi \partial^\sigma \phi}{(\partial \phi)^2} [\hat{H}_{\mu\nu\rho} \hat{H}_{\sigma\nu\rho} + \hat{H}^m_{\mu\nu\rho} \hat{H}^m_{\sigma\nu\rho}] \] (34)

to eq. (2.6) of ref. 8. It can be shown\(^\text{19}\) that the 3-form

\[ \hat{K}_{\mu\nu\rho} = \hat{H}_{\mu\nu\rho} - 3 \frac{\partial_{\mu \phi} \partial^\sigma \phi}{(\partial \phi)^2} \hat{H}^\sigma_{\nu\rho} \] (35)

is identically self-dual, while the 3-forms

\[ \hat{K}^m_{\mu\nu\rho} = \hat{H}^m_{\mu\nu\rho} - 3 \frac{\partial_{\mu \phi} \partial^\sigma \phi}{(\partial \phi)^2} \hat{H}^m_{\nu\rho} \] (36)

are identically antself-dual. With these definitions, we can display rather simply the complete supersymmetry transformations of the fields. Actually, only the transformations of the gravitino and of the tensorinos are affected, and become

\[ \delta \Psi_{\mu} = \hat{D}_{\mu} \epsilon + \frac{1}{4} \hat{K}_{\mu\nu\rho} \gamma^{\nu\rho} \epsilon + \frac{i}{32} (\hat{\chi}^m \gamma_{\mu\nu\rho} \chi^m) \gamma^{\nu\rho} \epsilon - \frac{3i}{8} (\hat{\chi}^m \gamma_{\mu} \chi^m) - \frac{i}{8} (\hat{\epsilon} \gamma_{\mu\nu\rho} \chi^m) \gamma^{\nu\rho} \epsilon - \frac{9i}{8} v_r c^{\nu\rho} tr_z [\hat{\epsilon} \gamma_{\mu\nu\rho} \lambda] - \frac{i}{8} v_r c^{\nu\rho} tr_z [\hat{\epsilon} \gamma_{\mu\nu\rho} \lambda] \] - \frac{i}{16} v_r c^{\nu\rho} tr_z [\hat{\epsilon} \gamma_{\mu\nu\rho} \lambda] \] + \frac{i}{16} v_r c^{\nu\rho} tr_z [\hat{\epsilon} \gamma_{\mu\nu\rho} \lambda] \] + \frac{i}{16} v_r c^{\nu\rho} tr_z [\hat{\epsilon} \gamma_{\mu\nu\rho} \lambda] \]

\[ \delta \chi^m = i \frac{x^m_r}{x^m_r} \partial_{\alpha} v^{\rho} \gamma^{\alpha} \epsilon + i \frac{\hat{K}^m_{\alpha\beta\gamma}}{12} \gamma^{\alpha\beta\gamma} \epsilon + \frac{i}{2} x^m_r c^{\nu\rho} tr_z [\hat{\epsilon} \gamma_{\alpha} \lambda] \] . (37)

while the scalar field \( \phi \) is invariant under supersymmetry.\(^\text{19}\) It can be shown that the complete lagrangian transforms under supersymmetry as dictated by the Wess-Zumino consistency conditions.
We now turn to describe the corresponding modifications of the supersymmetry algebra. In addition to general coordinate, gauge and supersymmetry transformations, the commutator of two supersymmetry transformations on $B_{\mu\nu}^r$ now generates two local PST transformations with parameters

$$\Lambda'_{\tau\mu} = \frac{\partial^\sigma}{(d\phi)^2} (v_{\tau} \dot{\sigma}_{\mu\alpha} - x_r^m \dot{H}_{\sigma\mu\alpha}^m) \xi^\alpha, \quad \Lambda = \xi^\alpha \partial_\alpha \phi.$$  \hspace{1cm} (38)

The transformation of eq. (38) on the scalar field $\phi$ is opposite to its coordinate transformation, and this gives an interpretation of the corresponding commutator

$$[\delta_1, \delta_2] \phi = \delta_{gct} \phi + \delta_{PST} \phi = 0,$$  \hspace{1cm} (39)

that vanishes consistently with the invariance of $\phi$ under supersymmetry. Finally, the commutator on the vielbein determines the parameter of the local Lorentz transformation, that is now

$$\Omega_{ab} = -\xi^\nu (\omega_{ab} - \dot{\sigma}_{ab} - \frac{i}{8} (\chi^m \gamma_{ab} \chi^m))$$

$$+ \frac{1}{2} (\chi^m \epsilon_1) (\chi^m \gamma_{ab} \epsilon_2) - \frac{1}{2} (\chi^m \epsilon_2) (\chi^m \gamma_{ab} \epsilon_1)$$

$$+ v_r c^{rz} tr_z [(\tilde{e}_1 \gamma_\lambda) (\tilde{e}_2 \gamma_\lambda) - (\tilde{e}_2 \gamma_\lambda) (\tilde{e}_1 \gamma_\lambda)].$$  \hspace{1cm} (40)

All other parameters remain unchanged while, aside from the extension, the algebra closes on-shell on the modified field equations of the fermi fields. Of course, the resulting field equations reduce to those of previous Section once one fixes the PST gauge invariances in order to recover the conventional equations for (anti)self-dual tensor fields.

For completeness, we conclude by displaying the lagrangian of six-dimensional supergravity coupled to vector and tensor multiplets,

$$e^{-1} \mathcal{L} = \frac{1}{4} R + \frac{1}{12} G_{rs} H^{\tau\mu\rho\lambda} H_{\mu\rho\lambda}^s - \frac{1}{4} \partial_{\nu} v^r \partial^\mu v_r - \frac{1}{2} v_r c^{rz} tr_z (F_{\mu\lambda} F^{\mu\lambda})$$

$$- \frac{1}{8} \epsilon^{\mu\nu\alpha\beta\gamma\delta} c_{\tau} B_{\mu\nu} tr_z (F_{\alpha\beta} F_{\gamma\delta}) - i \frac{1}{2} (\bar{\Psi}_{\mu} \gamma^{\mu\nu\rho} D_{\nu} [\frac{1}{2} (\omega + \dot{\omega})]) \Psi_{\nu}$$

$$- i \frac{1}{8} v_r [H + \dot{H}] tr_z (\bar{\Psi}_{\mu} \gamma_{\nu} \Psi_{\nu}) + i \frac{1}{48} v_r [H + \dot{H}] [\bar{\Psi}_{\mu} \gamma_{\nu} \Psi_{\nu}]$$

$$+ \frac{i}{2} (\chi^m \gamma_{\mu} D_{\mu} (\tilde{\omega}) \chi^m) - \frac{i}{24} v_r \dot{H}_{\mu\nu\rho} (\chi^m \gamma_{\mu\nu\rho} \chi^m)$$

$$+ \frac{1}{4} x_r [\partial_{\nu} v^r + \partial_{\lambda} v^r] (\bar{\Psi}_{\mu} \gamma_{\nu} \gamma_{\lambda} \chi^m) - \frac{1}{8} x_r [H + \dot{H}] tr_z (\bar{\Psi}_{\mu} \gamma_{\nu} \gamma_{\lambda} \chi^m)$$

$$+ \frac{1}{24} x_r [H + \dot{H}] tr_z (\bar{\Psi}_{\mu} \gamma_{\nu} \gamma_{\lambda} \chi^m) + iv_r c^{rz} tr_z (\bar{\chi} \gamma_{\mu} D_{\mu} \chi^m).$$
where $\alpha$ is the (undetermined) coefficient of the quartic coupling for the gauginos, and the corresponding supersymmetry transformations

$$
\delta\epsilon^{\mu} = -i(\tilde{c}^{\mu}a\Psi_{\mu}) ,
$$

$$
\delta B_{\mu\nu} = iv^{\nu}(\bar{\Psi}_{\mu}[\gamma_{\nu}]c) + \frac{1}{2}x^{m}\gamma_{m}\gamma_{\nu}\epsilon - 2c^{\nu}tr_{z}(A_{\mu}/\delta A_{\nu}) ,
$$

$$
\delta\nu_{\nu} = x^{m}(\bar{\chi}_{\nu}c) ,
$$

$$
\delta A_{\mu} = -\frac{i}{\sqrt{2}}(\tilde{c}_{\mu\nu}) ,
$$

$$
\delta\Psi_{\mu} = \tilde{D}_{\mu}\epsilon + \frac{1}{4}K_{\mu\rho\gamma\nu}\gamma^{\rho}\epsilon + \frac{i}{32}(\bar{\chi}_{\mu}\gamma_{\nu}\gamma^{m}\gamma_{\rho}c\gamma_{\nu}) - \frac{3i}{8}(\bar{\chi}_{\mu}c\gamma_{\nu})\gamma_{\rho}c\gamma_{\nu},
$$

$$
\delta\lambda = \frac{i}{2}x^{m}\partial_{\mu}v_{\nu}c^{\mu}\epsilon + \frac{i}{12}F_{\mu\rho\gamma}c^{\mu}c^{\nu}c^{\rho}\epsilon + \frac{1}{8}x^{m}c^{\nu}tr_{z}(\bar{c}\gamma_{\mu}\gamma_{\nu}) ,
$$

$$
\delta\lambda = \frac{1}{2\sqrt{2}}F_{\mu\nu}c^{\mu}c^{\nu} - \frac{1}{2}v_{\nu}c^{m}c^{\nu}(\bar{\chi}^{m})c - \frac{1}{4}v_{\nu}c^{m}c^{\nu}(\bar{\chi}^{m})c
$$

(41)
\[ + \frac{1}{8} \frac{\epsilon^{rz}}{v_5} (\bar{\psi} \gamma_{\mu\nu} \epsilon) \gamma^{\mu\nu} \lambda \].

(42)

One further comment is in order. Kavalov and Mkrtchyan obtained long ago a complete action for pure d=6 (1,0) supergravity in terms of a single tensor auxiliary field. Their work may be connected to this special case of our result via an ansatz relating their tensor to the PST scalar. Still, the PST formulation has the virtue of simplicity and makes it manifest that the extra degrees of freedom may be locally eliminated via additional gauge transformations.

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