A Graded Möbius Transform and its Harmonic Interpretation

Samy Abbes
University Paris Diderot – Paris 7
CNRS Laboratory PPS (UMR 7126)
Paris, France
samy.abbes@univ-paris-diderot.fr

Abstract

We give a graded version of the Möbius inversion formula in the framework of trace monoids. The formula is based on a graded version of the Möbius transform, related to the notion of height deriving from the Cartier-Foata normal form of the elements of a trace monoid.

Using the notion of Bernoulli measures on the boundary of a trace monoid developed recently, we study a probabilistic interpretation of the graded inversion formula. We introduce Möbius harmonic functions for trace monoids and obtain an integral representation formula for them, analogous to the Poisson formula for harmonic functions associated to random walks on trees.

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Introduction

A random walk on a regular oriented tree of finite degree, which we identify with a free monoid $\Sigma^*$, is specified by a probability distribution $f$ on the finite set $\Sigma$ of generators. In turn, the law of the trajectories of the random walk is a Bernoulli measure on the space of $\Sigma$-valued infinite sequences, which identifies with the boundary at infinity $\partial \Sigma^*$ of the tree. Recall that, if we denote by $\uparrow x$, for $x$ ranging over $\Sigma^*$, the set of infinite sequences of which $x$ is a prefix, then Bernoulli measures on $\partial \Sigma^*$ are characterized by the multiplicative property:

$$P(\uparrow xy) = P(\uparrow x)P(\uparrow y),$$

valid for all $x, y \in \Sigma^*$.

Let $P$ denote the Markov operator acting on real valued functions defined on $\Sigma^*$, and such that $P\lambda(x) = \sum_{a \in \Sigma} f(a)\lambda(xa)$ for all $x \in \Sigma^*$ and for all functions $\lambda : \Sigma^* \to \mathbb{R}$. Harmonic functions relative to the pair $(\Sigma^*, P)$ are those functions $\lambda : \Sigma^* \to \mathbb{R}$ such that $P\lambda = \lambda$, hence in the kernel of the discrete Laplace operator $\Delta = I - P$. It is well known that bounded harmonic functions are in a linear and isometric one-to-one correspondence with measurable and essentially bounded real valued functions defined on $\partial \Sigma^*$, through the Poisson formula:

$$\forall x \in \Sigma^* \quad \lambda(x) = \frac{1}{P(\uparrow x)} \int_{\uparrow x} \varphi(\xi) dP(\xi), \quad \varphi \in L^\infty(\partial \Sigma^*). \quad (1)$$

If $\lambda$ is bounded harmonic, then $\varphi$ is obtained as the $P$-a.s. limit of the bounded martingale $(\lambda(X_n), \mathcal{F}_n)_{n \geq 1}$, where $(X_n)_{n \geq 1}$ is the random walk on $\Sigma^*$ and $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.
In this paper, we study the notion of harmonicity in a slightly different framework. First, instead of considering a free monoid $\Sigma^*$, we allow some generators to commute with each other, and consider thus free partially commutative monoids, usually called trace monoids [6], and referred to in the literature—as also in the case of groups—as to heap monoids [18], locally free monoids with respect to a graph [13, 16], partially commutative monoids [4], graph monoids [7]. Hence a trace monoid $\mathcal{M}$ is a finitely presented monoid of the form $\mathcal{M} = \Sigma^*/\mathcal{R}$, where $\mathcal{R}$ is the congruence relation generated by pairs of the form $(ab, ba)$, for $(a, b)$ ranging over a given symmetric and irreflexive relation $I$ on $\Sigma$, called an independence relation. The elements of a trace monoids are called traces. Much of the above framework for free monoids can be transposed to trace monoids. In particular, there is a natural notion of infinite trace—which corresponds to a simplified version of the stable normal form for infinite words in the sense of [17]. The boundary at infinity $\partial \mathcal{M}$ of the monoid $\mathcal{M}$ is defined as the set of infinite traces. An elementary cylinder $\uparrow u$, for $u \in \mathcal{M}$, is defined as the subset of those infinite traces of which $u$ is a prefix: $\uparrow u = \{ \xi \in \partial \mathcal{M} : u \leq \xi \}$, and $\mathfrak{F} = \sigma(\uparrow u : u \in \mathcal{M})$ is the $\sigma$-algebra that equips $\partial \mathcal{M}$.

Second, instead of considering random walks on a trace monoid, we directly consider Bernoulli measures on the boundary of the monoid. A Bernoulli measure on a trace monoid is defined as a probability measure $P$ on $(\partial \mathcal{M}, \mathfrak{F})$, such that the following multiplicative property holds:

$$\forall u, v \in \mathcal{M} \quad P(\uparrow (u \cdot v)) = P(\uparrow u)P(\uparrow v).$$

(2)

In a recent work with J. Mairesse [2], we have conducted a thorough study of Bernoulli measures for trace monoids, by showing how to characterize them through probabilistic parameters and by giving an explicit construction of them by means of the combinatorial structure of the monoid. Connecting them with more familiar objects usually found in this journal, Bernoulli measures can be seen as weighted Patterson-Sullivan measures. However, they are not given as the law of entrance of a random walk into the boundary at infinity of the monoid, except if the trace monoid reduces to a free monoid, which corresponds to the empty independence relation $I = \emptyset$. Hence, the framework found in [13, 16, 17] for instance does not apply for Bernoulli measures.

The main topic of this paper is the notion of harmonicity in the framework of trace monoids equipped with Bernoulli measures. In order to obtain a dual representation between bounded functions on the boundary $\partial \mathcal{M}$ on the one hand, and functions defined on $\mathcal{M}$ invariant with respect to a certain linear operator, consider a pair $(\mathcal{M}, P)$ where $\mathcal{M}$ is a trace monoid and $P$ is a Bernoulli measure on $\partial \mathcal{M}$, and let $\varphi \in L^\infty(\partial \mathcal{M})$. We define an associated function $\lambda : \mathcal{M} \to \mathbb{R}$ by:

$$\forall u \in \mathcal{M} \quad \lambda(u) = \frac{1}{P(\uparrow u)} \int_{\uparrow u} \varphi(\xi) \, dP(\xi).$$

(3)

Then we observe that the function $\lambda$ satisfies the following relation:

$$\forall u \in \mathcal{M} \quad \sum_{c \in \mathcal{C}} (-1)^{|c|} f(c) \lambda(u \cdot c) = 0,$$

(4)

where $\mathcal{C}$ denotes the set of cliques of the finite graph $(\Sigma, I)$, and $f : \mathcal{M} \to \mathbb{R}$ is the multiplicative function on $\mathcal{M}$ defined by $f(u) = P(\uparrow u)$. Because of
the deep relationship between the expression and the Möbius polynomial of the pair $(\Sigma, I)$, defined by $\mu_M(X) = \sum_{c \in \mathcal{C}} (-1)^{|c|} X^{|c|}$, we call the operator $\Delta$ acting on functions $\lambda : M \to \mathbb{R}$ by:

$$\forall u \in M \quad \Delta \lambda(u) = \sum_{c \in \mathcal{C}} (-1)^{|c|} \lambda(u \cdot c),$$

(5)

the Möbius-Laplace operator on $M$; functions in the kernel of $\Delta$, we call Möbius harmonic.

Decomposing cliques according to their size, $\Delta$ writes as:

$$\Delta = I - P, \quad P \lambda(u) = \sum_{a \in \Sigma} f(a) \lambda(u \cdot a) - \sum_{c \in \mathcal{C} : |c| \geq 2} (-1)^{|c|} f(c) \lambda(u \cdot c).$$

(6)

Hence, as it turns out, Möbius harmonic functions do not have an obvious interpretation as invariant functions with respect to a Markov operator; for $P = I - \Delta$ is not a positive operator, unless the trace monoid reduces to the free monoid $\Sigma^*$. This contrasts with the case of random walk on trees [3, 14], but also on more general hyperbolic structures [9, 10].

Nevertheless, the correspondence through the Poisson formula still holds: the main result of this paper is the existence, for every bounded Möbius harmonic function $\lambda : M \to \mathbb{R}$, of a unique essentially bounded function $\varphi \in L^\infty(\partial M)$ on the boundary, such that formula (3) holds for the pair $(\lambda, \varphi)$.

Our technique of proof resembles to some extent the usual technique, for trees for instance. However, starting from a bounded Möbius harmonic function $\lambda : M \to \mathbb{R}$, obtaining the martingale $(Y_n, \mathcal{F}_n)_{n \geq 1}$ which converges $\mathbb{P}$-a.s. towards the adequate function $\varphi \in L^\infty(\partial M)$ is more involved than usual. In order to put in motion the martingale machinery, we rely on a generalization of the Möbius transform, as popularized by G.-C. Rota [15]. The original Möbius inversion formula, first formulated for integers, was shown by Rota to be a particular case of a formula best formulated in the incidence algebra associated to a general class of partial orders. For a trace monoid, the Möbius inversion formula writes as follows, for any function $F : \mathcal{C} \to \mathbb{R}$ defined on the set of cliques of the graph $(\Sigma, I)$:

$$\forall c \in \mathcal{C} \quad F(c) = \sum_{c' \in \mathcal{C} : c' \geq c} H(c'), \quad H(c) = \sum_{c' \in \mathcal{C} : c' \geq c} (-1)^{|c'| - |c|} F(c').$$

The extended Möbius inversion formula that we prove in this paper holds for functions $F : M \to \mathbb{R}$ defined on $M$ rather than on $\mathcal{C}$ only. We show that it is an adequate tool to obtain the integral representation formula for bounded Möbius harmonic functions.

The extended Möbius transform on which the new inversion formula is based, makes use of the natural graded structure attached to elements of a trace monoid. Indeed, traces can be put in a normal form—the Cartier-Foata normal form. The graded structure of the trace monoid $M$ is the partition of $M$ into traces with a fixed number of elements in their Cartier-Foata normal form. This number is called the height of a trace. Observe that the height does not correspond to the geodesic distance between an element of the monoid and the identity of the monoid.

Without any doubt, the graded Möbius inversion formula that we state should be valid for more general “graded partial orders”, of which braid monoids
should typically be an instance. We felt however that the trace monoid case was already non trivial, yet it allows for a thorough presentation of the main ideas.

Organization of the paper. In Section 1, we recall elements on the Combinatorics of trace monoids, following [4], and we provide essential information on the boundary of trace monoids and on Bernoulli measures, following [2]. The contributions of the paper appear in Sections 2–4. In Section 2, we introduce the graded M"obius transform and we prove the associated inversion formula. Bounded M"obius harmonic functions are the topic of Section 3. Section 4 deals with examples of non-negative and unbounded M"obius harmonic functions and introduces the analogous of the Green and the Martin kernel. Finally, Section 5 concludes the paper.

1—Trace Monoids and Bernoulli Measures on their Boundary

This section collects material on trace monoids, introduces the boundary of a trace monoid and associated Bernoulli measures.

Trace monoids. Let $\Sigma$ be a finite set, referred to as to the alphabet. Elements of $\Sigma$ are called letters. By convention, we only consider throughout the paper alphabets of cardinality $> 1$. An independence relation $I$ on $\Sigma$ is a binary relation on $\Sigma$, symmetric and irreflexive. Let $\Sigma^*$ be the free monoid on $\Sigma$, and let $R$ be the congruence relation on $\Sigma^*$ generated by $\{(ab, ba) : (a, b) \in I\}$. The quotient monoid $M(\Sigma, I) = \Sigma^*/R$ is called a trace monoid, and its elements are called traces. Classical references on trace monoids are [4, 6, 18].

The dependence relation associated to an independence relation $I$ on $\Sigma$ is defined by $D = (\Sigma \times \Sigma) \setminus I$. The trace monoid $M(\Sigma, I)$ is said to be irreducible whenever the dependence relation $D$ makes the graph $(\Sigma, D)$ connected.

Put $M = M(\Sigma, I)$. For any trace $u \in M$, any two representative words of $u$ have the same length, which defines the length $|u|$ of $u$. Let $0$ denote the empty trace, image in $M$ of the empty word, and let $\cdot$ denote the concatenation of traces. The prefix relation on $M$, denoted $\leq$, is defined by:

$$\forall u, v \in M \quad u \leq v \iff \exists w \in M \quad v = u \cdot w.$$ 

This is a partial order relation on $M$.

Trace monoids are known to be right and left cancellative, meaning:

$$\forall x, x' \in M \quad \forall y, z \in M \quad y \cdot x \cdot z = y \cdot x' \cdot z \implies x = x'.$$ 

(7)

Boundary and elementary cylinders. Bernoulli measures. Let $H$ denote the set of non-decreasing sequences $(x_n)_{n \geq 0}$ in $M$, those sequences such that $x_n \leq x_{n+1}$ for all integers $n \geq 0$. We identify any two sequences $x = (x_n)_{n \geq 0}$ and $y = (y_n)_{n \geq 0}$ in $H$ such that $x \preceq y$ and $y \preceq x$, where we have defined relation $\preceq$ as follows:

$$\forall x, y \in H \quad x \preceq y \iff \forall n \geq 0 \quad \exists m \geq 0 \quad x_n \leq y_m.$$ 

Let $\overline{M}$ denote the quotient set $H/\equiv$, with $x \equiv y \iff x \preceq y \land y \preceq x$. Then $\overline{M}$ is just the collapse of the pre-ordered set $(H, \preceq)$, and as such, is is equipped
with a partial ordering relation \( \leq \). There is a canonical injection \( \mathcal{M} \to \overline{\mathcal{M}} \) which respects the ordering, and which maps each trace \( u \in \mathcal{M} \) to the image in \( \overline{\mathcal{M}} \) of the constant sequence \( (u, u, \cdots) \). This justifies that elements of \( \overline{\mathcal{M}} \) are called \emph{generalized traces}. Elements of \( \partial \mathcal{M} = \overline{\mathcal{M}} \setminus \mathcal{M} \), identifying \( \mathcal{M} \) with its image in \( \overline{\mathcal{M}} \), are called \emph{infinite traces}. The set \( \partial \mathcal{M} \) is called the \emph{boundary} of \( \mathcal{M} \) \cite{1, 2}. By construction, any sequence \((x_n)_{n \geq 1}\) of elements of \( \overline{\mathcal{M}} \) which is non decreasing has a least upper bound in \( \overline{\mathcal{M}} \), denoted \( \bigvee_{n \geq 1} x_n \).

For each trace \( u \in \mathcal{M} \), the \emph{elementary cylinder of base} \( u \) is the following subset of \( \partial \mathcal{M} \):
\[
\uparrow u = \{ \xi \in \partial \mathcal{M} : u \leq \xi \}. \tag{8}
\]

We denote by \( \mathcal{F} \) the \( \sigma \)-algebra on \( \partial \mathcal{M} \) generated by the countable collection of elementary cylinders. We say that a probability measure \( \mathbb{P} \) on \( (\partial \mathcal{M}, \mathcal{F}) \) is a \emph{Bernoulli measure} \cite{2} if \( \mathbb{P}(\uparrow u) > 0 \) for all \( u \in \mathcal{M} \), and if:
\[
\forall u, v \in \mathcal{M} \quad \mathbb{P}(\uparrow (u \cdot v)) = \mathbb{P}(\uparrow u)\mathbb{P}(\uparrow v). \tag{9}
\]

\textbf{Cliques and Cartier-Foata decomposition.} An \emph{independence clique}, or a \emph{clique} for short, of a pair \((\Sigma, I)\), is defined as a subset \( c \subseteq \Sigma \) of the alphabet, such that any two distinct letters \( a, b \in c \) satisfy \( (a, b) \notin I \). If \( c = \{a_1, \ldots, a_n\} \) is a clique, then the product \( a_1 \cdots a_n \in \mathcal{M} \) is independent of the chosen enumeration of \( c \). Therefore cliques identify with their images in \( \mathcal{M} \). The order of cliques in \( \mathcal{M} \) corresponds to the inclusion ordering on subsets of \( \Sigma \). We denote by \( \mathcal{C} \) the set of cliques associated to \( \mathcal{M} \), and by \( \mathcal{C}^0 = \mathcal{C} \setminus \{0\} \) the set of non empty cliques.

Let \( c, c' \) be two cliques. We say that \( (c, c') \) is \emph{Cartier-Foata admissible}, denoted by \( c \to c' \), if for every letter \( b \in c' \), there exists a letter \( a \in c \) such that \( (a, b) \notin I \). In the heap of pieces interpretation of Viennot \cite{13}, this corresponds to the letter \( b \) being blocked from below by the letter \( a \). It is known that for every non empty trace \( u \in \mathcal{M} \), there exists a unique integer \( n \geq 1 \) and a unique sequence of non empty cliques \((c_i)_{1 \leq i \leq n}\) such that:
\[
\forall i \in \{1, \ldots, n - 1\} \quad c_i \to c_{i+1}, \quad \quad u = c_1 \cdots c_n. \tag{10}
\]

This sequence \((c_i)_{1 \leq i \leq n}\) is called the \emph{Cartier-Foata decomposition} of \( u \) \cite{3}. The integer \( n \) is called the \emph{height} of \( u \), we denote it by \( n = \tau(u) \).

The Cartier-Foata decomposition extends to infinite traces: for every infinite trace \( \xi \in \partial \mathcal{M} \), there exists a unique infinite sequence \((c_i)_{i \geq 1}\) of non empty cliques \cite{2} Lemma 8.4] such that:
\[
\forall i \geq 1 \quad c_i \to c_{i+1}, \quad \quad \xi = \bigvee_{n \geq 1} \{c_1 \cdots c_n\}. \tag{10}
\]

The infinite sequence \((c_i)_{i \geq 1}\) is called the \emph{Cartier-Foata decomposition} of \( \xi \).

\textbf{Möbius transform.} Möbius polynomial. Characterization of Bernoulli measures. The Möbius polynomial of \( \mathcal{M} \) is \( \mu_{\mathcal{M}}(X) \in \mathbb{Z}[X] \) defined by:
\[
\mu_{\mathcal{M}}(X) = \sum_{c \in \mathcal{C}} (-1)^{|c|} X^{|c|}. \tag{10}
\]
It is also referred to in the literature as to the clique polynomial of \((\Sigma, I)\), and coincides up to a change of variable with the independence polynomial \([12]\) of the graph \((\Sigma, D')\), where \(D' = (\Sigma \times \Sigma) \setminus \{(x, x) : x \in \Sigma\}\).

It is known that \(\mu_M(X)\) has a unique root \(p_0\) of smallest modulus, and that \(p_0 \in (0, 1)\) \([5, 8, 11]\). If \(M\) is irreducible, then \([2]\) Th. 5.1 there is a unique Bernoulli measure \(P\) on \((\partial M, \mathcal{F})\) such that \(P(\uparrow u) = p_0^{\mid u\mid}\) for all \(u \in M\).

More generally, let \(f : M \to \mathbb{R}\) be a function. We say that \(f\) is a valuation if \(f(u \cdot v) = f(u)f(v)\) holds for all \(u, v \in M\). Let \(f\) be a positive valuation on \(M\), and assume that \(M\) is irreducible. Let \(h : \mathcal{C} \to \mathbb{R}\) be the Möbius transform of the restriction \(f|_{\mathcal{C}}\), defined by:

\[
\forall c \in \mathcal{C} \quad h(c) = \sum_{c' \in \mathcal{E} : c' \geq c} (-1)^{|c'|-|c|} f(c') .
\] (11)

Then \(f(u) = P(\uparrow u)\) for some Bernoulli measure \(P\) if and only if the following two conditions \([2]\) Th. 3.3] are satisfied:

\[
h(0) = 0 , \quad \forall c \in \mathcal{C} \quad h(c) > 0 .
\] (12)

The conditions in \([12]\) consist in a polynomial equality, and several polynomial inequalities, involving only a finite number of parameters, namely the numbers \(f(a)\) for \(a\) ranging over \(\Sigma\), and that characterize the valuation \(f\).

Note that the Möbius transform \(h : \mathcal{C} \to \mathbb{R}\) defined in \([11]\) makes sense for any function \(f : \mathcal{C} \to M\), and not only for the restriction to \(\mathcal{C}\) of a valuation defined on \(M\).

Let \(\mathbb{P}\) be a Bernoulli measure on \(\partial M\), and let \(f(u) = \mathbb{P}(\uparrow u)\) be the associated valuation. The sequence of non empty cliques \((C_i(\xi))_{i \geq 1}\) which appear in the Cartier-Foata decomposition of an infinite trace \(\xi \in \partial M\), forms a sequence of random variables. We know \([2]\) Th. 4.1] that, under the measure \(\mathbb{P}\), the sequence \((C_i)_{i \geq 1}\) is a time-homogeneous Markov chain, which satisfies the following property, for every finite sequence of non empty cliques \(c_1 \to \ldots \to c_n\):

\[
\mathbb{P}(C_1 = c_1, \ldots, C_n = c_n) = f(c_1) \cdots f(c_{n-1})h(c_n) .
\] (13)

The law of \(C_1\), which is the initial distribution of the chain, coincides with the restriction \(h|_{\mathcal{C}}\). The transition matrix \(P = (P_{c,c'})_{(c,c') \in \mathcal{E} \times \mathcal{E}}\) is given by:

\[
P_{c,c'} = \begin{cases} 0, & \text{if } \neg(c \to c') , \\ h(c')/g(c), & \text{if } c \to c' , \end{cases}
\]

\[
g(c) = \sum_{c' \in \mathcal{E} : c \to c'} h(c') .
\] (14)

Furthermore, as a consequence of the assumption \(h(0) = 0\) stated in \([12]\), one has \([2]\) Prop. 10.3:

\[
\forall c \in \mathcal{C} \quad h(c) = f(c)g(c) .
\] (15)

**Ordering and Cartier-Foata decomposition.** We recall some results related to the Cartier-Foata decomposition of traces and of infinite traces.

We still denote by \(C_n(\xi)\) the \(n\)th clique in the Cartier-Foata decomposition of an infinite trace \(\xi \in \partial M\), dropping the dependency with respect to \(\xi\) when seeing \(C_n\) as a random variable defined on \(\partial M\). Let \(u \in M\) be a trace, of
Cartier-Foata decomposition $c_1 \rightarrow \ldots \rightarrow c_n$. Then one has [2 Prop. 8.5] the following equalities of subsets of $\partial M$, putting $v = c_1 \cdot \ldots \cdot c_{n-1}$:

\[ \uparrow u = \{ \xi \in \partial M : C_1 \cdot \ldots \cdot C_n \geq u \} \]
\[ \{ \xi \in \partial M : C_1 = c_1, \ldots, C_n = c_n \} = \uparrow u \setminus \bigcup_{c \in \mathcal{C} : c > c_n} \uparrow (v \cdot c) \]

Finally, define two cliques $c$ and $c'$ to be parallel whenever $c \times c' \subseteq I$, denoted by $c \parallel c'$. If $u, v \in \mathcal{M}$ are two traces, with $u = c_1 \rightarrow \ldots \rightarrow c_n$ and $v = d_1 \rightarrow \ldots \rightarrow d_p$ their Cartier-Foata decompositions, then [2 Lemma 8.1] $u \leq v$ if and only if $n \leq p$, and there exists cliques $\gamma_1, \ldots, \gamma_n$ such that:

\[ d_i = c_i \cdot \gamma_i \text{ for } i \in \{1, \ldots, n\} \text{; and} \]
\[ \gamma_i \parallel c_j \text{ for all } i, j \in \{1, \ldots, n\} \text{ with } i \leq j. \] (19)

The sequence of cliques $(\gamma_i)_{1 \leq i \leq n}$ as above is unique. An illustration is given in Figure 1 in next section.

### 2—The Graded Möbius Transform

The Möbius inversion formula, which holds for general classes of partial orders [15], takes the following form for trace monoids: for any function $f : \mathcal{C} \rightarrow \mathbb{R}$, with Möbius transform $h : \mathcal{C} \rightarrow \mathbb{R}$ defined as in (11), the function $f$ can be retrieved from its transform through the formula (see [2 Prop. 10.1] for a justification):

\[ \forall c \in \mathcal{C}, \quad f(c) = \sum_{c' \in \mathcal{C} : c' \geq c} h(c'). \] (20)

In this section, we give a generalization of (20). For this, we introduce the graded Möbius transform of functions with domain $\mathcal{M}$, instead of $\mathcal{C}$ only. The graded Möbius transform uses the partition of $\mathcal{M}$ according to the height of traces.

The probabilistic interpretation of the corresponding inversion formula will be the topic of next section.

**Definition 2.1**—Let $\mathcal{M}$ be a trace monoid, and let $F : \mathcal{M} \rightarrow \mathbb{R}$ be a function. The graded Möbius transform of $F$ is the function $H : \mathcal{M} \rightarrow \mathbb{R}$ defined as follows. For $u \in \mathcal{M}$ a generic non empty trace, denote by $c$ the last clique of the Cartier-Foata decomposition of $u$. Let also $v$ be the unique trace such that $u = v \cdot c$. Then define $H(u)$ by:

\[ H(u) = \sum_{c' \in \mathcal{C} : c' \geq c} (-1)^{|c'| - |c|} F(v \cdot c'). \] (21)

Define also $H(0) = \sum_{c \in \mathcal{C}} (-1)^{|c|} F(c)$.

How to retrieve $F$ from its graded Möbius transform $H$ is stated in next result. Recall that the height $\tau(u)$ of a trace $u \in \mathcal{M}$ is the number of cliques in its Cartier-Foata decomposition, with the convention $\tau(0) = 0$. For each trace $u \in \mathcal{M}$, we put:

for $u \neq 0$: $\mathcal{M}(u) = \{ x \in \mathcal{M} : \tau(x) = \tau(u) \land u \leq x \}$,

$\mathcal{M}(0) = \mathcal{C}$.
See an illustration in Figure 1.

- Theorem 2.2—Let $F : \mathcal{M} \to \mathbb{R}$ be a function, and let $H : \mathcal{M} \to \mathbb{R}$ be the graded Möbius transform of $F$. Then:

$$\forall u \in \mathcal{M} \quad F(u) = \sum_{x \in \mathcal{M}(u)} H(x). \quad (22)$$

Remark. 1. If $\tau(u) = 1$, then $u = c$ is a non empty clique. Hence $H(u)$ coincides with the value $h(u)$, where $h : \mathcal{G} \to \mathbb{R}$ is the Möbius transform of the restriction $F|_{\mathcal{G}}$. Formula $(22)$ writes as: $\sum_{c' \in \mathcal{G} : c' \geq c} h(c') = F(c)$, which is the standard Möbius inversion formula $(20)$ for $F|_{\mathcal{G}}$.

2. Both the definition of the graded Möbius transform and the inversion formula $(22)$ are valid for functions taking values in any commutative group instead of $\mathbb{R}$.

Proof of Theorem 2.2. We first give an alternative formulation for the graded Möbius transform of $F$, still denoting by $c$ the last clique in the Cartier-Foata decomposition of $u$:

$$H(u) = \sum_{\delta \in \mathcal{C} : \delta \parallel c} (-1)^{|\delta|} F(u \cdot \delta), \quad (23)$$

resulting from the change of variable $c' = c \cdot \delta$ in $(21)$.

We now come to the proof of the identity $(22)$. If $u = 0$, then the identity follows from the standard Möbius inversion formula $(20)$.

Hence, let $u \in \mathcal{M}$ be a non empty trace, and let $c_1 \to \ldots \to c_n$ be the Cartier-Foata decomposition of $u$. According to the results recalled in § 1 in [13, 19], the Cartier-Foata decomposition of a generic $x \in \mathcal{M}(u)$ is of the form $d_1 \to \ldots \to d_n$ with $d_i = c_i \cdot \gamma_i$, where $(\gamma_1, \ldots, \gamma_n)$ is a sequence of cliques uniquely determined by $x$, and such that (1) $\gamma_i \parallel c_i, \ldots, c_n$ for all $i \in \{1, \ldots, n\}$, and (2) $c_1 \cdot \gamma_1 \to \ldots \to c_n \cdot \gamma_n$ holds. Consequently, using $(23)$ above, the computation goes as follows:

$$\sum_{x \in \mathcal{M}(u)} H(x) = \sum_{x \in \mathcal{M}(u)} \sum_{\delta \in \mathcal{C} : \delta \parallel c_n \cdot \gamma_n} (-1)^{|\delta|} F(x \cdot \delta)$$

$$= \sum_{\gamma_1, \ldots, \gamma_{n-1} \in \mathcal{C} : \gamma_i \parallel c_i, \ldots, c_n \text{ for } 1 \leq i \leq n-1 \text{ and } c_1 \cdot \gamma_1 \to \ldots \to c_{n-1} \cdot \gamma_{n-1}} R(\gamma_1, \ldots, \gamma_{n-1}) \quad (24)$$
with
\[
R(\gamma_1, \ldots, \gamma_{n-1}) = \sum_{\gamma_n \in \mathcal{C}} \sum_{\delta \in \mathcal{C} : \delta \in \gamma_n} (-1)^{|\delta|} F(c_1 \cdot \gamma_1 \cdot \ldots \cdot c_{n-1} \cdot \gamma_{n-1} \cdot \gamma \cdot \delta)
\]

The range of the cliques \(\gamma_n \in \mathcal{C}\) in the scope of the above sum is identical to \(\gamma_n \| c_n\) and \(c_{n-1} \cdot \gamma_{n-1} \rightarrow \gamma_{n}\), since \(c_{n-1} \rightarrow c_n\) holds by hypothesis. Using the change of variable \(\gamma = \gamma_n \cdot \delta\) yields:
\[
R(\gamma_1, \ldots, \gamma_{n-1}) = \sum_{\gamma_n \in \mathcal{C}} \sum_{\gamma \in \mathcal{C} : \gamma \geq \gamma_n} (-1)^{|\gamma| - |\gamma_n|} F(c_1 \cdot \gamma_1 \cdot \ldots \cdot c_{n-1} \cdot \gamma_{n-1} \cdot c_n \cdot \gamma) K(\gamma)
\]

with
\[
K(\gamma) = \sum_{\gamma_n \in \mathcal{C} : \gamma_n \| c_n, \quad c_{n-1} \cdot \gamma_{n-1} \rightarrow \gamma_{n}} (-1)^{|\gamma_n|} 1_{\{\gamma \| c_{n-1} \cdot \gamma_{n-1}\}} \quad \text{by the binomial formula.}
\]

Returning to (24), we obtain thus:
\[
\sum_{x \in \mathcal{M}(u)} H(x) = \sum_{\gamma_1, \ldots, \gamma_{n-1} \in \mathcal{C}} \sum_{\gamma \in \mathcal{C} : \gamma \geq \gamma_n} (-1)^{|\gamma|} F(c_1 \cdot \gamma_1 \cdot \ldots \cdot c_{n-1} \cdot \gamma_{n-1} \cdot c_n \cdot \delta)
\]

Applying recursively the same transformation eventually yields:
\[
\sum_{x \in \mathcal{M}(u)} H(x) = \sum_{\gamma_1, \ldots, c_n} (-1)^{|\delta|} F(c_1 \cdot \gamma_1 \cdot \ldots \cdot c_n \cdot \delta)
\]
\[
= \sum_{\gamma \in \mathcal{C} : \gamma \leq \delta} (-1)^{|\gamma|} F(c_1 \cdot \delta \cdot \ldots \cdot c_n)
\]
\[
= F(c_1 \cdot \ldots \cdot c_n) = F(u).
\]

The proof is complete.

3—Möbius Harmonic Functions and their Integral Representation

We introduce notations that will be used throughout this section and the next one. We assume that \(\mathcal{M} = \mathcal{M}(\Sigma, I)\) is an irreducible trace monoid, equipped with a Bernoulli measure \(\mathbb{P}\) on \((\delta \mathcal{M}, \mathcal{F})\). We define the functions \(f, h : \mathcal{M} \rightarrow \mathbb{R}\) by letting \(f(u) = \mathbb{P}(\uparrow u)\) for \(u \in \mathcal{M}\), and \(h\) is the graded Möbius transform of \(f\) (see Definition 2.1).

By definition of a Bernoulli measure, the function \(f\) is multiplicative over \(\mathcal{M}\): \(f(u \cdot v) = f(u) f(v)\). Therefore, if \(u \in \mathcal{M}\) is such that \(u = v \cdot c\), with \(c\) the last
clique in the Cartier-Foata decomposition of \( u \), it follows from Definition 2.1 that \( h(u) = f(v)h(c) \). Hence, according to (13), if \( c_1 \rightarrow \ldots \rightarrow c_n \) are \( n \geq 1 \) non empty cliques and \( u = c_1 \cdot \ldots \cdot c_n \), one has:

\[
P(C_1 = c_1, \ldots , C_n = c_n) = h(u).
\] (25)

- **Definition 3.1**—A Möbius harmonic function, relative to a pair \((\mathcal{M}, \mathcal{P})\) as above, is a function \( \lambda : \mathcal{M} \rightarrow \mathbb{R} \) such that:

\[
\forall u \in \mathcal{M} \quad \sum_{c \in \mathcal{C}} (-1)^{|c|} f(c) \lambda(u \cdot c) = 0.
\] (26)

Obviously, Möbius harmonic functions form a real vector space. The first example of Möbius harmonic functions are constant functions. Indeed, if \( \lambda = 1 \) identically on \( \mathcal{M} \), then (26) reduces to:

\[
\sum_{c \in \mathcal{C}} (-1)^{|c|} f(c) = 0,
\]

which holds since we recognize the Möbius transform \( h \) evaluated at 0 in the above expression, and \( h(0) = 0 \) by (12).

Another way to obtain bounded Möbius harmonic functions is given by the next result. We denote by \( L^\infty(\partial \mathcal{M}) \) the space of functions \( \varphi : \partial \mathcal{M} \rightarrow \mathbb{R} \) bounded \( P \)-modulo 0.

- **Proposition 3.2**—For every \( \varphi \in L^\infty(\partial \mathcal{M}) \), the function \( \lambda : \mathcal{M} \rightarrow \mathbb{R} \) defined by:

\[
\forall u \in \mathcal{M} \quad \lambda(u) = \frac{1}{f(u)} \int\limits_{\uparrow u} \varphi(\xi) dP(\xi)
\] (27)

is Möbius harmonic and bounded on \( \mathcal{M} \).

**Proof.** It is obvious that \( \lambda \) thus defined is bounded on \( \mathcal{M} \) by \( \| \varphi \|_\infty \). Let \( u \in \mathcal{M} \) be a trace, and consider the following non disjoint union:

\[
\uparrow u = \bigcup_{a \in \Sigma} \uparrow (u \cdot a) = \bigcup_{i=1}^k \uparrow (u \cdot a_i),
\] (28)

where \( \Sigma = \{ a_1, \ldots , a_k \} \) is an enumeration of \( \Sigma \). We decompose the integral in (27) with respect to the union (28), and using Poincaré inclusion-exclusion principle:

\[
f(u) \lambda(u) = \sum_{r=1}^k (-1)^{r+1} \sum_{1 \leq i_1 < \ldots < i_r \leq k} \int\limits_{\uparrow (u \cdot a_{i_1}) \cap \ldots \cap \uparrow (u \cdot a_{i_r})} \varphi dP.
\]

An intersection \( \uparrow (u \cdot a_{i_1}) \cap \ldots \cap \uparrow (u \cdot a_{i_r}) \) is empty unless \( \{ a_{i_1}, \ldots , a_{i_r} \} \) is a clique, in which case the intersection coincides with \( \uparrow (u \cdot a_{i_1} \cdot \ldots \cdot a_{i_r}) \). Henceforth the above sum evaluates as:

\[
f(u) \lambda(u) = \sum_{c \in \mathcal{C}} (-1)^{|c|+1} \int\limits_{\uparrow (u \cdot c)} \varphi dP,
\]

with \( \mathcal{C} = \mathcal{C} \setminus \{ 0 \} \).
Introducing $f(u \cdot c)$ in the above sum in order to recognize $\lambda(u \cdot c)$ yields:

$$f(u \lambda(u) + \sum_{c \in \mathcal{C}} (-1)^{|c|} f(u \cdot c) \lambda(u \cdot c) = 0.$$ 

Since $f$ is multiplicative and positive on $\mathcal{M}$, we simplify by $f(u)$, and recognize in $\lambda(u)$ the missing term of the sum for $c = 0$, yielding:

$$\sum_{c \in \mathcal{C}} (-1)^{|c|} f(c) \lambda(u \cdot c) = 0,$$

which was to be proved.

Our goal is now to prove a converse for Proposition 3.2; hence, starting from a bounded Möbius harmonic function $\lambda : \mathcal{M} \to \mathbb{R}$, to find $\varphi \in L^\infty(\partial \mathcal{M})$ such that (27) holds. The remaining of the section is devoted to the proof of this result, stated in Theorem 3.4 below.

From now on, we fix a bounded Möbius harmonic function $\lambda : \mathcal{M} \to \mathbb{R}$. Our method of proof to find the adequate $\varphi \in L^\infty(\partial \mathcal{M})$ follows loosely the same line of proof than for harmonic functions on trees for instance [14]. However, the issue here is to find an adequate martingale, the expression of which is not obvious a priori.

Reasoning by analysis, assume first that $\varphi$ exists. Recall that we have defined in §1 the sequence of random variables $(C_n)_{n \geq 1}$, given by the non empty cliques of the Cartier-Foata decomposition of a generic element $\xi \in \partial \mathcal{M}$. For each integer $n \geq 1$, consider the sub-$\sigma$-algebra of $\mathbb{F}$ generated by $(C_1, \ldots, C_n)$:

$$\mathcal{F}_n = \sigma(C_1, \ldots, C_n).$$

Obviously, $(\mathcal{F}_n)_{n \geq 1}$ forms a filtration of $\mathbb{F}$, and therefore the sequence of conditional expectations $(E(\varphi|\mathcal{F}_n), \mathcal{F}_n)_{n \geq 1}$ is a martingale. Putting $Y_n = E(\varphi|\mathcal{F}_n)$, the problem that we face is to express $Y_n$ using $\lambda$ only. Since $Y_n$ is $\mathcal{F}_n$-measurable, $Y_n$ can be seen as a function of the $n$ first cliques $C_1, \ldots, C_n$. Some computations, the details of which will be given in the proof of Theorem 3.4 below, lead to the following potential form for $Y_n$:

$$Y_n = \frac{1}{h(C_n)} \sum_{c \in \mathcal{C} : c \geq C_n} (-1)^{|c|-|C_n|} f(c) \lambda(V \cdot c), \text{ with } V = C_1 \cdot \ldots \cdot C_{n-1}. \quad (29)$$

Based on the above analysis, we are naturally brought to prove the following result.

**Lemma 3.3**—Let $\lambda : \mathcal{M} \to \mathbb{R}$ be a bounded Möbius harmonic function. For each integer $n \geq 1$, let $Y_n$ be the $\mathcal{F}_n$-measurable random variable defined by (29). Then $(Y_n, \mathcal{F}_n)_{n \geq 1}$ is a bounded martingale.

**Proof.** It is obvious that $Y_n$ is $\mathcal{F}_n$-measurable, bounded and thus integrable. Hence we only have to show that $E(Y_n|\mathcal{F}_{n-1}) = Y_{n-1}$ holds for all $n \geq 2$.

Putting $Z_n = E(Y_n|\mathcal{F}_{n-1})$, we evaluate $Z_n$ on the atom $\{C_1 = c_1, \ldots, C_{n-1} = c_{n-1}\}$ with $c_1 \to \ldots \to c_{n-1}$, by computing as follows, and putting $v =
\[ c_1 \cdot \ldots \cdot c_{n-1} : \]

\[ Z_n = \sum_{c \in \mathcal{C}_n : c_{n-1} \rightarrow c} \mathbb{P}(C_n = c|C_1 = c_1, \ldots, C_{n-1} = c_{n-1}) Y_n(c_1, \ldots, c_{n-1}, c) \]

\[ = \sum_{c \in \mathcal{C}_n : c_{n-1} \rightarrow c} \frac{h(c)}{g(c_{n-1})} \frac{1}{h(c)} \sum_{c' \in \mathcal{C} : c' \geq c} (-1)^{|c'| - |c|} f(c') \lambda(v \cdot c') \]

where \( g \) is the normalization factor defined in (14),

\[ = \frac{1}{g(c_{n-1})} \sum_{c' \in \mathcal{C}} (-1)^{|c'|} f(c') \lambda(v \cdot c') \sum_{c \in \mathcal{C} : c \leq c' \land c_{n-1} \rightarrow c} (-1)^{|c|} \]

Using the binomial formula, we have for any two cliques \( d \) and \( d' \):

\[ \sum_{\delta \in \mathcal{C} : \delta \leq d \land d \rightarrow \delta} (-1)^{|\delta|} = \begin{cases} 1, & \text{if there is no } \delta \leq d' \text{ but } 0 \text{ such that } d \rightarrow \delta \\ \Leftrightarrow d \parallel d' \\ 0, & \text{otherwise} \end{cases} \]

Hence:

\[ K(c', c_{n-1}) = -1 + \sum_{c \in \mathcal{C} : c \leq c' \land c_{n-1} \rightarrow c} (-1)^{|c|} = -1 \{-c' \setminus c_{n-1}\}. \]

All put together, this yields:

\[ Z_n = \frac{-1}{g(c_{n-1})} \sum_{c \in \mathcal{C} : \neg c \setminus c_{n-1}} (-1)^{|c|} f(c) \lambda(v \cdot c). \tag{30} \]

According to the M"obius harmonicity of \( \lambda \) at \( v \), one has:

\[ \sum_{c \in \mathcal{C}} (-1)^c f(c) \lambda(v \cdot c) = 0. \]

Decomposing \( \mathcal{C} \) into those \( c \in \mathcal{C} \) such that \( c \parallel c_{n-1} \) and those \( c \in \mathcal{C} \) such that \( \neg(c \parallel c_{n-1}) \), and re-injecting in (30) yields:

\[ Z_n = \frac{1}{g(c_{n-1})} \sum_{c \in \mathcal{C} : c \parallel c_{n-1}} (-1)^{|c|} f(c) \lambda(v \cdot c), \]

and with the change of variable \( \delta = c \cdot c_{n-1} \):

\[ Z_n = \frac{1}{g(c_{n-1})} \sum_{\delta \in \mathcal{C} : \delta \geq c_{n-1}} (-1)^{|\delta| - |c_{n-1}|} \frac{f(\delta)}{f(c_{n-1})} \lambda(w \cdot \delta), \tag{31} \]

where \( w = c_1 \cdot \ldots \cdot c_{n-2} \). But \( g(c_{n-1}) f(c_{n-1}) = h(c_{n-1}) \), as recalled in (10). Henceforth, Equation (31) writes as \( Z_n = Y_{n-1} \), which was to be shown. \( \square \)
Since the sequence \((Y_n, \mathcal{F}_n)_{n \geq 1}\) is a bounded martingale, it converges \(\mathbb{P}\)-a.s. and in the space \(L^1(\partial M)\) to a limit \(\phi \in L^\infty(\partial M)\). It is natural to expect that this limit is the adequate candidate for the integral representation of \(\lambda\). This is true indeed, and the proof of this fact is based on the inversion formula for graded Möbius transforms proved in §2 above. Therefore Theorem 3.4 below provides a probabilistic interpretation of the inversion formula.

**Theorem 3.4**—For every bounded Möbius harmonic function \(\lambda : M \to \mathbb{R}\), there exists a unique \(\phi \in L^\infty(\partial M)\) such that:

\[
\forall u \in M \quad \lambda(u) = \frac{1}{\mathbb{P}(\uparrow u)} \int_{\uparrow u} \phi \, d\mathbb{P}.
\]

(32)

The above formula establishes a bijective and isometric linear correspondence between \(L^\infty(\partial M)\) and the space of bounded Möbius harmonic functions on \(M\). Both this correspondence and its inverse are positive operators.

**Proof.** Let \(\lambda : M \to \mathbb{R}\) be a bounded Möbius harmonic function. Let \(\phi \in L^\infty(\partial M)\) be the limit, \(\mathbb{P}\)-a.s. and in \(L^1(\partial M)\), of the martingale \((Y_n, \mathcal{F}_n)_{n \geq 1}\) defined as in Lemma 3.3. We prove that (32) holds for this function \(\phi\).

Let \(u \in M\) be a trace. To compute the integral of \(\phi\) over \(\uparrow u\), we rely on the description (16) of \(\uparrow u\) stated in §1. Let \(n = \tau(n)\) be the height of \(u\).

Then, by (16), and denoting as in the proof of Theorem 2.2:

\[M(u) = \{ x \in M : \tau(x) = n \land u \leq x \}, \]

one has:

\[\uparrow u = \{ \xi \in \partial M : C_1 \cdot \ldots \cdot C_n \geq u \} = \bigcup_{x \in M(u)} \{ \xi \in \partial M : C_1 \cdot \ldots \cdot C_n = x \},\]

the last union being disjoint. Accordingly, and using formula (25) found above, one has:

\[\int_{\uparrow u} \phi \, d\mathbb{P} = \sum_{x \in M(u)} \int_{C_1 \cdot \ldots \cdot C_n = x} \phi \, d\mathbb{P} = \sum_{x \in M(u)} h(x)\mathbb{E}(\phi | C_1 \cdot \ldots \cdot C_n = x).\]

By definition of the conditional expectation \(Y_n = \mathbb{E}(\phi | \mathcal{F}_n)\), using the expression (29) and since \(\mathcal{F}_n = \sigma(C_1, \ldots, C_n)\), we deduce:

\[\int_{\uparrow u} \phi \, d\mathbb{P} = \sum_{x \in M(u)} f(y) \sum_{c \in \mathcal{F} : c \geq \gamma_n} (-1)^{|c| - |\gamma_n|} f(c) \lambda(y \cdot c),\]

(33)

where \(x = y \cdot \gamma_n\) is the decomposition of a generic element \(x \in M(u)\) such that \(\gamma_n\) is the last clique in the Cartier-Foata decomposition of \(x\).

Let \(F : M \to \mathbb{R}\) be the function defined by \(F(u) = f(u)\lambda(u)\), and let \(H\) be the graded Möbius transform of \(F\) (see Definition 2.1). Since \(f\) is multiplicative, Equation (33) writes as:

\[\int_{\uparrow u} \phi \, d\mathbb{P} = \sum_{x \in M(u)} H(x) = F(u) \quad \text{by Theorem 2.2},\]

where \(H(x) = f(u)\lambda(u)\).
This shows formula (32).

We have shown the existence of \( \varphi \), and we now focus on its uniqueness. It is enough to show that \( \varphi = \lim_{n \to \infty} Y_n \) necessarily holds \( \mathbb{P} \)-a.s. if \( \varphi \in L^\infty(\partial \mathcal{M}) \) satisfies (32). Hence, consider \( \varphi \in L^\infty(\partial \mathcal{M}) \). The limit \( \varphi = \lim_{n \to \infty} \mathbb{E}(\varphi | \mathcal{F}_n) \) holds \( \mathbb{P} \)-a.s., since \( \forall n \geq 1 \), \( \mathcal{F}_n = \mathcal{F} \), as attested by (10). We are thus bound to prove \( \mathbb{E}(\varphi | \mathcal{F}_n) = Y_n \). For this, we compute as follows, considering a sequence \( c_1 \to \ldots \to c_n \) of non empty cliques, and putting \( u = c_1 \cdot \ldots \cdot c_n \) and \( v = c_1 \cdot \ldots \cdot c_{n-1} \) (this is the computation we promised just above Lemma 3.3):

\[
\mathbb{E}(\varphi | C_1 = c_1, \ldots, C_n = c_n) = \frac{1}{h(u)} \int_{C_1 = c_1, \ldots, C_n = c_n} \varphi \, d\mathbb{P} = \frac{1}{h(u)} \left( \int_{\uparrow u} \varphi \, d\mathbb{P} - \int_{\bigcup_{i \in \varnothing^+ : c_i > c_n} \uparrow (v \cdot c)} \varphi \, d\mathbb{P} \right),
\]

the later equality according to (17).

Let \( \{a_1, \ldots, a_k\} \) be an enumeration of those letters \( a \in \varnothing \) such that \( a \parallel c_n \). Then it is obvious that:

\[
\bigcup_{c \in \varnothing^+ : c > c_n} \uparrow (v \cdot c) = \bigcup_{i=1}^k \uparrow (v \cdot c_n \cdot a_i).
\]

Applying Poincaré inclusion-exclusion principle as in the proof of Proposition 3.2 we deduce the following expression for \( L \) defined in (34) above:

\[
L = \sum_{i=1}^k (-1)^{i+1} \sum_{1 \leq i_1 < \ldots < i_r \leq k} \int_{\uparrow (v \cdot c_n \cdot a_{i_1} \cap \ldots \cap \uparrow (v \cdot c_n \cdot a_{i_r})} \varphi \, d\mathbb{P} = \sum_{c \in \varnothing^+ : c > c_n} (-1)^{|c|+1} \int_{\uparrow (v \cdot c)} \varphi \, d\mathbb{P}.
\]

Returning to (34), we obtain:

\[
\mathbb{E}(\varphi | C_1 = c_1, \ldots, C_n = c_n) = \frac{1}{h(u)} \sum_{c \in \varnothing^+ : c \geq c_n} (-1)^{|c|-|c_n|} f(v \cdot c) \lambda(v \cdot c).
\]

But \( h(u) = f(v)h(c_n) \) and \( f(v \cdot c) = f(v)f(c) \), hence we obtain the expected expression (29) defining \( Y_n \) for \( \mathbb{E}(\varphi | \mathcal{F}_n) \). This proves the uniqueness of \( \varphi \).

Let \( \text{MH}^\infty(\mathcal{M}) \) denote the linear space of bounded M"obius harmonic functions of \( \mathcal{M} \), and let \( \Psi : L^\infty(\partial \mathcal{M}) \to \text{MH}^\infty(\mathcal{M}) \) be the transformation defined by (32). It is obvious that \( \Psi \) is linear, and we have shown that \( \Psi \) is bijective: it remains only to show that \( \Psi \) and \( \Psi^{-1} \) are positive and isometric.

It is obvious on the expression (32) that \( \Psi \) is a positive operator (\( \varphi \geq 0 \implies \lambda \geq 0 \)). And since \( f = \mathbb{P}(\uparrow \cdot) > 0 \) on \( \mathcal{M} \) by assumption, the fact that \( \Psi^{-1} \) is also positive follows from Lemma 3.3 below. Since \( \Psi \) and \( \Psi^{-1} \) are positive operators, and since \( \Psi(1) = 1 \), it is an easy consequence that they are isometric.

\( \square \)
Lemma 3.5—If $\varphi \in L^\infty(\partial M)$ is such that $\int_{\uparrow u} \varphi \, d\mathbb{P} \geq 0$ holds for all $u \in M$, then $\varphi \geq 0$ holds $\mathbb{P}$-a.s. on $\partial M$.

Proof. The collection of elementary cylinders, to which is added the empty set, is stable by finite intersections. Hence the lemma is an application of the Monotone Class theorem, since $\mathcal{F}$ is the $\sigma$-algebra generated by all elementary cylinders $\uparrow u$, for $u \in M$.

Corollary 3.6—Let $\lambda : M \to \mathbb{R}$ be a bounded and non negative Möbius harmonic function. Then for any trace $u \in M$, if $c$ is the last clique in the Cartier-Foata decomposition of $u$, one has:

$$\sum_{\delta \in \mathcal{W} : \delta \subseteq c} (-1)^{|\delta|} f(\delta) \lambda(u \cdot \delta) \geq 0. \quad (35)$$

Remark. The result of the corollary is not obvious, because of the presence of negative terms in the sum. We shall see in §4 below an example where (35) does not hold for a non negative unbounded Möbius harmonic function.

Proof of Corollary 3.6. Let $u \in M$ and $\lambda : M \to \mathbb{R}$ be as in the statement, and let $\varphi \in L^\infty(\partial M)$ be associated to $\lambda$ as in Theorem 3.4. Let $c_1 \to \ldots \to c_n$ be the Cartier-Foata decomposition of $u$, and put $v = c_1 \cdot \ldots \cdot c_{n-1}$.

Then Theorem 3.4 states that $\varphi \geq 0$ holds $\mathbb{P}$-a.s. on $\partial M$. Therefore $Y_n = \mathbb{E}(\varphi | \mathcal{F}_n)$ is $\mathbb{P}$-a.s. non negative on $\partial M$. Evaluating $Y_n$ on the atom $\{C_1 = c_1, \ldots, C_n = c_n\}$ of $\mathcal{F}_n$ according to the expression (29) for $Y_n$, which was derived in the course of the proof of Theorem 3.4 yields:

$$\frac{1}{h(c_n)} \sum_{c \in \mathcal{W} : c \geq c_n} (-1)^{|c|-|c_n|} f(c) \lambda(v \cdot c_n) \geq 0.$$ 

Since $h > 0$ on $\mathcal{C}$ by (12), and since $f$ is multiplicative, the result follows from the change of variable $c = c_n \cdot \delta$ in the above sum.

4—Additional Remarks

In this section, we examine some examples of unbounded Möbius harmonic functions that arise naturally. We consider as above a pair $(M, \mathbb{P})$, where $M$ is an irreducible trace monoid, and $\mathbb{P}$ is a Bernoulli measure on $(\partial M, \mathcal{F})$. As usual we put $f(u) = \mathbb{P}(\uparrow u)$ for $u \in M$, and $h : M \to \mathbb{R}$ defined as the graded Möbius transform of $f$.

The first observation is that, if $\nu$ is any finite measure on $(\partial M, \mathcal{F})$, then the function $\lambda : M \to \mathbb{R}$ defined by:

$$\forall u \in M \quad \lambda(u) = \frac{1}{\mathbb{P}(\uparrow u)} \nu(\uparrow u), \quad (36)$$

is Möbius harmonic. The proof is similar to the proof of Proposition 3.2. It is also a reformulation of Proposition 2.1 of [1]. Note that applying (36) to the measure $d\nu = \varphi \, d\mathbb{P}$ brings back the result of Proposition 3.2 on the Möbius harmonicity of $\lambda$. Contrary to the result of Proposition 3.2 however, in general the function $\lambda$ defined in (36) is unbounded.
The Green kernel of \((M, P)\) is defined by:
\[
\forall x, y \in M \quad G(x, y) = \begin{cases} f(y)/f(x), & \text{if } x \leq y, \\ 0, & \text{otherwise.} \end{cases}
\]
\(37\)

The fact that our basic object \(M\) is a semi-group rather than a group makes that \(G\) is not positive on \(M \times M\). For \(y \in M\), put \(G_y = G(\cdot, y)\), and define \(\Delta G_y : M \to \mathbb{R}\) by:
\[
\forall x \in M \quad \Delta G_y(x) = \sum_{c \in \mathcal{C}} (-1)^{|c|} f(c) G_y(x \cdot c).
\]
\(38\)

Easy calculations show that:
\[
\forall x, y \in M \quad \Delta G_y(x) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{otherwise.} \end{cases}
\]
\(39\)

In other words, extending in the obvious way Definition 2.1 to functions Möbius harmonic on a subset of \(M\), \(G_y\) is Möbius harmonic on \(M \setminus \{y\}\). Therefore \(y\) appears as the unique singularity of \(G_y\). The standard idea from Martin theory is to “send the singularity at infinity”. We define the Martin kernels \(K_y\), for \(y\) ranging over \(M\), as follows:
\[
\forall y \in M \quad K_y = \frac{G(\cdot, y)}{G(0, y)} = \frac{1}{f(\cdot)} 1_{\{\cdot \leq y\}}.
\]
\(40\)

Sending \(y\) to infinity consists in taking a limit along a sequence \((y_n)_{n \geq 1}\) of traces converging to a point \(\xi \in \partial M\) of the boundary. The topological framework on \(\overline{M} = M \cup \partial M\) does not present any particular difficulty; the easiest way is to simply identify generalized traces with their Cartier-Foata decomposition, and to use the standard metric constructions on sequences, either finite or infinite, taking values in the finite set \(\mathcal{C}\). Hence we use this topological framework without stating more formal definitions. The space of Möbius harmonic functions is then endowed with the pointwise convergence; any limit of Möbius harmonic functions is Möbius harmonic.

Within this framework, it is visible on \((40)\) that, if \(y_n \to \xi \in \partial M\), then \((K_{y_n})_{n \geq 1}\) converges to \(K_\xi : M \to \mathbb{R}\), defined by:
\[
\forall \xi \in \partial M \quad \forall x \in M \quad K_\xi(x) = \frac{1}{f(x)} 1_{\{x \leq \xi\}},
\]
\(41\)

which is Möbius harmonic, this time on \(M\). The Martin kernel \(K_\xi\) thus defined corresponds to the harmonic function defined as in \((36)\) with respect to the Dirac measure \(\delta_\xi\) concentrated on \(\xi\). It is obviously unbounded.

In general, if \(\nu\) is a finite measure on \(\partial M\) such that the associated Möbius harmonic function \(\lambda\) is bounded on \(\partial M\), then \(\nu\) is regular with respect to \(P\) and \(\varphi \in L^\infty(\partial M)\) associated to \(\lambda\) by Theorem 3.4 coincides \(P\)-a.s. with the Radon-Nykodim derivative \(d\nu/dP\). This is obviously not the case for the Dirac measures \(\delta_\xi\).

Does every non negative Möbius harmonic function \(\lambda : M \to \mathbb{R}\) originate from a—necessarily finite—measure \(\nu\) on \(\partial M\) as in \((36)\)? The answer is negative, as the following example reveals.
Assume that $\mathbb{P}$ is the uniform Bernoulli measure on $\partial M$, which is defined by $\mathbb{P}(\uparrow u) = p_0^{[u]}$ for every trace $u \in M$, where $p_0$ is the unique root of smallest modulus of the Möbius polynomial $\mu_M(X)$ (see §1). Let $p$ be another non negative root of $\mu_M(X)$, if it exists, and define $\lambda : M \to \mathbb{R}$ by:

$$\forall u \in M \quad \lambda(u) = \left( \frac{p}{p_0} \right)^{[u]}.$$  \hspace{1cm} (42)

Then $\lambda$ is Möbius harmonic, and it is clearly unbounded since $p > p_0$. We claim that there exists no finite measure $\nu$ on $(\partial M, \mathcal{F})$ such that (36) would hold for $\lambda$ and $\nu$.

By contradiction, assume that $\nu$ exists. Then we would have $\nu(\uparrow u) = p_0^{[u]}$ for all $u \in M$, and in particular $\nu$ would be a probability measure assigning an equal probability to all cylinders $\uparrow u$ for $u$ ranging over traces of a fixed length. But then, it follows from the—rather difficult—result of [2, Th. 5.1 point 2] that $\nu(\uparrow u) = p_0^{[u]}$ for all $u \in M$, a contradiction. The claim is proved.

It remains only to check that there exists irreducible trace monoids with different real non negative roots of their Möbius polynomials. It is easy to find such examples. Consider for instance the trace monoid generated by $(\Sigma, I)$ depicted on Figure 2—it was already worked out in [2, §6]. Then $\mu_M(X) = 1 - 5X + 5X^2$ has the two roots $p_0 = \frac{1}{2} - \frac{\sqrt{5}}{10}$ and $p_1 = \frac{1}{2} + \frac{\sqrt{5}}{10}$. The uniform Bernoulli measure on $\partial M$ is characterized by $\mathbb{P}(\uparrow u) = p_0^{[u]}$, for all $u \in M$. This example is specially interesting since the second root $p_1$ lies itself in the interval $(0, 1)$. Henceforth the function $u \in M \mapsto p_1^{[u]}$ satisfies $u \leq v \implies p_1^{[u]} \leq p_1^{[v]}$; whereas for values $p > 1$, this is the reverse ordering, which disqualifies $p^{[u]}$ at once for being represented as $p^{[u]} = \nu(\uparrow u)$ for any measure $\nu$.

Typically, this example provides a non negative Möbius harmonic function $\lambda(u) = (p_1/p_0)^{[u]}$ such that (35) does not hold. Indeed, evaluating the left hand member of (35) at trace $u = a_1$ for $\lambda(u) = (p_1/p_0)^{[u]}$ yields:

$$\lambda(a_1) - f(a_3)\lambda(a_1 \cdot a_3) - f(a_4)\lambda(a_1 \cdot a_4) = \frac{p_1}{p_0}(1 - 2p_1) < 0.$$  

5—Conclusion

This work suggests extensions in different directions. First, the graded Möbius transform is likely to extend to finitely presented monoids with an adequate
normal form for their elements, typically finite type Coxeter monoids, including braid monoids. The extension to these monoids of the probabilistic framework of Bernoulli measures and of Möbius harmonicity is a reasonable target. Dealing with groups rather than monoids is a non-trivial extension, since the partial order structure collapses.

Second, pursuing the elements of a potential theory for Bernoulli measures, either in the framework of trace monoids or in a more general framework, is also natural. In particular, the notion of super-Möbius harmonic functions has a natural definition. A Green representation of super-Möbius harmonic functions seems to arise naturally.

Finally, establishing a bridge with the theory of Poisson-Furstenberg boundary seems to be an interesting task, despite the first obstruction mentioned in the Introduction: Möbius harmonic functions are not invariant with respect to an obvious Markov operator.

Bibliography

[1] S. Abbes. On countable completions of quotient ordered semigroups. *Semigroup Forum*, 3(77):482–499, 2008.

[2] S. Abbes and J. Mairesse. Uniform and Bernoulli measures on the boundary of trace monoids. *arXiv* 1407.5879, July 2014. [http://arxiv.org/abs/1407.5879](http://arxiv.org/abs/1407.5879).

[3] P. Cartier. Fonctions harmoniques sur un arbre. In *Symposia Mathematica*, volume IX, pages 203–270. Academic Press, 1972.

[4] P. Cartier and D. Foata. *Problèmes combinatoires de commutation et réarrangements*, volume 85 of Lecture Notes in Mathematics. Springer, 1969.

[5] P. Csikvári. Note on the smallest root of the independence polynomial. *Combinatorics, Probability and Computing*, 22(1):1–8, 2013.

[6] V. Diekert. *Combinatorics on Traces*, volume 454 of Lecture Notes in Computer Science. Springer, 1990.

[7] D.C. Fisher. The number of words of length $n$ in a graph monoid. *The American Mathematical Monthly*, 96(7):610–614, 1989.

[8] M. Goldwurm and M. Santini. Clique polynomials have a unique root of smallest modulus. *Information Processing Letters*, 75(3):127–132, 2000.

[9] V.A Kaimanovich. Boundaries of invariant Markov operators: the identification problem. In M. Pollicott and K. Schmidt, editors, *Ergodic Theory of $Z^d$-actions, Proceedings of Warwick Symposium 1993–94*, volume 228 of London Math. Soc. Lecture Note Series, pages 127–176. Cambridge University Press, 1996.

[10] V.A. Kaimanovich. The Poisson formula for groups with hyperbolic properties. *Annals of Mathematics. Second Series*, 152(3):659–692, 2000.

[11] D. Krob, J. Mairesse, and I. Michos. Computing the average parallelism in trace monoids. *Discrete Mathematics*, 273:131–162, 2003.

[12] V.E. Levit and E. Mandrescu. The independence polynomial of a graph – a survey. In *Proceedings of the First International Conference on Algebraic Informatics*, pages 233–254. Aristotle University of Thessaloniki, 2005.
[13] A.V. Malyutin. The Poisson-Furstenberg boundary of a locally free group. In Representation theory, dynamical systems, combinatorial and algorithmic methods. Part IX, volume 301 of Zap. Nauchn. Sem. POMI, pages 195–211. POMI, St. Petersburg, 2003. English transl.: Journal of Mathematical Sciences 129(2): 3787–3795, 2005.

[14] F. Mouton. Comportement asymptotique des fonctions harmoniques sur les arbres. In Séminaire de Probabilités, volume XXXIV, pages 353–373. Université de Strasbourg, 2000.

[15] G.-C. Rota. On the foundations of combinatorial theory I. Theory of Möbius functions. Z. Wahrscheinlichkeitstheorie, 2:340–368, 1964.

[16] A. Vershik, S. Nechaev, and R. Bikbov. Statistical properties of locally free groups with applications to braid groups and growth of random heaps. Communications in Mathematical Physics, 212(2):469–501, 2000.

[17] A.M. Vershik. Dynamic theory of growth in groups: entropy, boundaries, examples. Uspekhi Mat. Nauk, 55(4):59–128, 2000. English transl.: Russian Math. Surveys 55(4):667–733, 2000.

[18] X. Viennot. Heaps of pieces, I : basic definitions and combinatorial lemmas. In Combinatoire énumérative, volume 1234 of Lecture Notes in Mathematics, pages 321–350. Springer, 1986.