K-theory for the maximal Roe algebra of certain expanders
Hervé Oyono-Oyono, Guoliang Yu

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Abstract. We study in this paper the maximal version of the coarse Baum-Connes assembly map for families of expanding graphs arising from residually finite groups. Unlike for the usual Roe algebra, we show that this assembly map is closely related to the (maximal) Baum-Connes assembly map for the group and is an isomorphism for a class of expanders. We also introduce a quantitative Baum-Connes assembly map and discuss its connections to K-theory of (maximal) Roe algebras.

Keywords: Baum-Connes Conjecture, Coarse Geometry, Expanders, Novikov Conjecture, Operator Algebra K-theory, Roe Algebras

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1. Introduction

In this paper, we study K-theory of (maximal) Roe algebras for a class of expanders. The Roe algebra was introduced by John Roe in his study of higher index theory of elliptic operators on noncompact spaces [12]. The K-theory of Roe algebra is the receptacle for the higher indices of elliptic operators. If a space is coarsely embeddable into Hilbert space, then K-theory of Roe algebra and higher indices of elliptic operators are computable [15]. Gromov discovered that expanders do not admit coarse embedding into Hilbert space [5]. The purpose of this paper is to completely or partially compute K-theory of the (maximal) Roe algebras associated to certain expanders. In particular, we prove the maximal version of the coarse Baum-Connes conjecture for a special class of expanders. The coarse Baum-Connes conjecture is a geometric analogue of the Baum-Connes conjecture [1] and provides an algorithm of computing K-theory of Roe algebras and higher indices of elliptic operators. We also prove the (maximal) coarse Novikov conjecture for a class of expanders. The coarse Novikov conjecture gives a partial computation of K-theory of Roe algebras and an algorithm to determine non-vanishing of higher indices for elliptic operators. Our results on the coarse Novikov conjecture are more general than results obtained in [3, 4, 6]. The question of computing K-theory of (maximal) Roe algebras associated to general expanders remains open. We show that this question is closely related to certain quantitative Novikov conjecture and the quantitative Baum-Connes conjecture for the K-theory of (maximal) Roe algebras. We explore this connection to prove the quantitative Novikov conjecture and the quantitative Baum-Connes conjecture in some cases.

The class of expanders under examination in this paper is those associated to a finitely generated and residually finite group $\Gamma$ with respect to a family $\Gamma_0 \supset \Gamma_1 \supset \ldots \supset \Gamma_n \supset \ldots$ of finite index normal subgroups. The behaviour of the Baum-Connes assembly map for $\Gamma$ and of the coarse Baum-Connes assembly map for the metric space $X(\Gamma) = \bigsqcup_{i \in \mathbb{N}} \Gamma_i/\Gamma$ can differ quite substantially: if $\Gamma$ has the property $\tau$ with respect to the family $(\Gamma_i)_{i \in \mathbb{N}}$, then $(\Gamma_i/\Gamma)_{i \in \mathbb{N}}$ is a family of expanders and the coarse assembly map for $X(\Gamma)$ fails to be an isomorphism, although for example for $\Gamma = SL_2(\mathbb{Z})$, the assembly maps is an isomorphism. In [4] was introduced the maximal Roe algebra of a coarse space and a maximal coarse assembly map with value in this algebra was defined. As we shall see, the behaviour of this maximal coarse assembly map differs quite substantially from the behaviour of the coarse assembly map for $X(\Gamma)$. Indeed, we shall see that the maximal coarse assembly map for $X(\Gamma)$ and the coarse assembly map for $X(\Gamma)$ fail to be equivalent. In particular, as a consequence of [14], if $\Gamma$ satisfies the strong Baum-Connes conjecture, then the maximal coarse assembly map for $X(\Gamma)$ is an isomorphism. As a spin-off we also obtain the injectivity of the coarse assembly map when $\Gamma$ coarsely embeds in a Hilbert Space.

This suggests that the properties of the maximal coarse assembly map for $X(\Gamma)$ is closely related to some asymptotic properties of the maximal Baum-Connes assembly maps for $\Gamma$ with coefficients in the family $\{C(\Gamma_i)\}_{i \in \mathbb{N}}$. For this purpose, we define quantitative assembly maps that take into account the propagation in the crossed product $\{C(\Gamma_i)\times_{\text{max}} \Gamma_i\}_{i \in \mathbb{N}}$. Notice that although $\{C(\Gamma_i)\times_{\text{max}} \Gamma_i\}$ and $\{C(\Gamma_i)\}$ are Morita equivalent, the imprimitivity bimodule between these two algebras introduces some distortion in the propagation and the relevant propagation is...
the one coming from $\Gamma$. In this setting, we show that injectivity and bijectivity of the maximal coarse assembly map are equivalent to some asymptotic statements for these quantitative assembly maps. For surjectivity, and up to a slight modification in the sequence of normal subgroups, we also obtain similar results.

The paper is organised as follows. In section 2, we review results from [4] and [8] concerning maximal Roe algebras and coarse assembly maps. We also show the existence of a short exact sequence (see section 2.2)

$$0 \to K(\ell^2(X(\Gamma)) \otimes H) \to C^*_{\text{max}}(X(\Gamma)) \to \text{A}_\Gamma \rtimes_{\text{max}} \Gamma \to 0.$$  

In section 3, we collect results about Baum-Connes assembly maps that we use later on. In section 4, we state for the left hand side of the Baum-Connes assembly map an analogue of the exact sequence of equation 1.1. We show that assembly maps intertwines this exact sequence with the one induced in $K$-theory by the exact sequence 1.1, and obtain injectivity and bijectivity results for the maximal coarse assembly map for $X(\Gamma)$. In section 5.3 we set asymptotic statements concerning some quantitative assembly maps and we discuss examples of groups that satisfy these statements.

2. K-theory for maximal Roe algebras

2.1. Maximal Roe algebra of a locally compact metric space. In this section, we collect from [4] results concerning the maximal Roe algebra of a locally compact metric space that we will need in this paper.

2.1.1. The case of a discrete space. Let $\Sigma$ be a discrete metric space equipped with a proper distance $d$. Let us denote by $C[\Sigma]$ the algebra of locally compact operators with finite propagation of $\ell^2(\Sigma) \otimes H$, where $H$ is a separable Hilbert space, i.e (bounded) operators $T$ of $\ell^2(\Sigma) \otimes H$ such that when written as a family $(T_{x,y})_{(x,y) \in \Sigma^2}$ of operators on $H$, then

- $T_{x,y}$ is compact for all $x$ and $y$ in $\Sigma$;
- there exists a real $r$ such that $d(x,y) > r$ implies that $T_{x,y} = 0$ ($T$ is said to have propagation less than $r$).

For any real $r$, we define $C_r[\Sigma]$ as the set of elements of $C[\Sigma]$ with propagation less than $r$. It is straightforward to check that $C[\Sigma]$ is a $*$-algebra. The (usual) Roe algebra $C^*(\Sigma)$ is the closure of $C[\Sigma]$ viewed as a subalgebra of operator of $L(\ell^2(\Sigma) \otimes H)$. The next lemma, proved in [4], shows that if $\Sigma$ has bounded geometry, then $C[\Sigma]$ admits an enveloping algebra.

**Lemma 2.1.** Let $\Sigma$ be a discrete metric space with bounded geometry. For any positive number $r$, there exists a real $c_r$ such that for any $*$-representation $\phi$ of $C[\Sigma]$ on a Hilbert space $H_\phi$ and any $T$ in $C_r[\Sigma]$, then $\|\phi(T)\|_{L(H_\phi)} \leq c_r\|T\|_{\ell^2(\Sigma) \otimes H}$.

This enveloping algebra is then

**Definition 2.2.** [4] The maximal Roe algebra of a discrete metric space $\Sigma$ with bounded geometry, denoted by $C^*_{\text{max}}(\Sigma)$, is the completion of $C[\Sigma]$ with respect to the $*$-norm

$$\|\phi(T)\| = \sup_{(\phi, H_\phi)} \|\phi(T)\|_{L(H_\phi)},$$

when $(\phi, H_\phi)$ runs through representations $\phi$ of $C[\Sigma]$ on a Hilbert space $H_\phi$. 


The general case.} Let $X$ be a locally compact space, equipped with a metric $d$. A $X$-module is a Hilbert space $H_X$ together with a $\ast$-representation $\rho_X$ of $C_0(X)$ in $H_X$. We shall often write $f$ instead of $\rho_X(f)$ for the action of $f$ on $H_X$. If the representation is non-degenerate, the $X$-module is said to be non-degenerate. A $X$-module is called standard if no non-zero function of $C_0(X)$ acts as a compact operator on $H_X$. In the literature, the terminology $C_0(X)$-ample is also used for such a representation [4].

**Definition 2.3.** Let $H_X$ be a standard non-degenerate $X$-module and let $T$ be a bounded operator on $H_X$.

(i) The support of $T$ is the complement of the open subset of $X \times X$

$\{(x, y) \in X \times X \text{ s.t. there exist } f \text{ and } g \text{ in } C_0(X) \text{ satisfying } f(x) \neq 0, g(y) \neq 0 \text{ and } f \cdot T \cdot g = 0\}$.

(ii) If there exists a real $r$ such that for any $x$ and $y$ in $X$ such that $d(x, y) > r$, then $(x, y)$ is not in the support of $T$, then the operator $T$ is said to have finite propagation (in this case propagation less than $r$).

(iii) The operator $T$ is said to be locally compact if $f \cdot T$ and $T \cdot f$ are compact for any $f$ in $C_0(X)$. We then define $C[X]$ as the set of locally compact and finite propagation bounded operators of $H_X$.

(iv) The operator $T$ is said to pseudo-local if $[f, T]$ is compact for all $f$ in $C_0(X)$.

It is straightforward to check that $C[X]$ is a $\ast$-algebra and that for a discrete space, this definition coincides with the previous one. Moreover, up to (non-canonical) isomorphism, $C[X]$ does not depend on the choice of $H_X$. The Roe algebra $C^\ast(X)$ is then the norm closure of $C[X]$ in the algebra $\mathcal{L}(H_X)$ of bounded operators on $H_X$. Although $C^\ast(X)$ is not canonically defined, we shall see later on that up to canonical isomorphism, its $K$-theory does not depend on the choice a non-degenerated standard $X$-module.

**Definition 2.4.** A net in a locally compact space $X$ is a countable subset $\Sigma$ such that there exists numbers $\varepsilon$ and $r$ satisfying

- $d(y, y') \geq \varepsilon$ for any distinct elements $y$ and $y'$ of $\Sigma$;
- For any $x$ in $X$, there exists $y$ in $\Sigma$ such that $d(x, y) \leq r$.

The following result was proved in [4].

**Lemma 2.5.** If a locally compact space $X$ contains a net with bounded geometry, then with notation of definition 2.4, there exists a unitary map $\Psi : H_\Sigma \to H_X$ that fulfills the following conditions:

(i) The homomorphism $\mathcal{L}(H_\Sigma) \to \mathcal{L}(H_X); T \mapsto \Psi \ast T \ast \Psi$ restricts to an algebra $\ast$-isomorphism $C[\Sigma] \to C[X]$;

(ii) There exists a number $r$ such that for every $x$ in $X$ and $y$ in $\Sigma$ with $d(x, y) < r$, then there exists $f$ in $C_0(X)$ and $g$ in $C_0(\Sigma)$ which satisfy $f(x) \neq 0$, $g(y) \neq 0$ and $f \cdot \Psi \cdot g = 0$ (i.e $\Psi$ has propagation less than $r$).

Then, if $\Psi : H_\Sigma \to H_X$ is a unitary map as in lemma 2.5, the $\ast$-isomorphism $C[\Sigma] \to C[X]; T \mapsto \Psi \ast T \ast \Psi$ extends to an isomorphism $Ad\Psi : C^\ast(\Sigma) \to C^\ast(X)$.

As a consequence of lemma 2.5, lemma 5.3 admits the following generalisation to spaces that contain a net with bounded geometry.
Lemma 2.6. Let $X$ be a locally compact metric that contains a net with bounded geometry and let $H_X$ be a standard non-degenerate $X$-module. Then for any positive number $r$, there exists a real $c_r$ such that for any $*$-representation $\phi$ of $C[X]$ on a Hilbert space $H_\phi$ and any $T$ in $C[X]$ with propagation less than $r$, then $\|\phi(T)\|_{L(H_\phi)} \leq c_r \|T\|_{H_X}$.

This allowed to define for $X$ the maximal Roe algebra as in the discrete case.

Definition 2.7. Let $X$ be a locally compact metric space that contains a net with bounded geometry. The maximal Roe algebra of $X$, denoted by $C^\text{max}(X)$, is the completion of $C[X]$ with respect to the $*$-norm

$$\|\phi(T)\| = \sup_{(\phi, H_\phi)} \|\phi(T)\|_{L(H_\phi)},$$

when $(\phi, H_\phi)$ runs through representation $\phi$ on of $C[X]$ a Hilbert space $H_\phi$.

2.2. Maximal Roe algebra associated to a residually finite group. Let $\Gamma$ be a residually finite group, finitely generated. Let $\Gamma_0 \supset \Gamma_1 \supset \ldots \Gamma_n \supset \ldots$ be a decreasing sequence of finite index subgroups of $\Gamma$ such that $\bigcap_{i \in \mathbb{N}} \Gamma_i = \{e\}$. Let $d$ be a left invariant metric associated to a given finite set of generators. Let us endow $\Gamma / \Gamma_i$ with the metric $d(a\Gamma_i, b\Gamma_i) = \min\{d(a\gamma_1, b\gamma_2), \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_i\}$. We set $X(\Gamma) = \prod_{i \in \mathbb{N}} \Gamma / \Gamma_i$ and we equip $X(\Gamma)$ with a metric $d$ such that,

- on $\Gamma / \Gamma_i$, then $d$ is the metric defined above;
- $d(\Gamma / \Gamma_i, \Gamma / \Gamma_j) \geq i + j$ if $i \neq j$.
- the group $\Gamma$ acts on $X(\Gamma)$ by isometries.

Let us define by $K(H)$ the algebra of compact operators of $H$. Then the $C^*$-algebra $\ell^\infty(X(\Gamma), K(H))$ acts on $\ell^2(X(\Gamma)) \otimes H$ by pointwise action of $K(H)$. This action is clearly by propagation zero locally compact operators. The group $\Gamma$ acts diagonally on the Hilbert space $\ell^2(X(\Gamma)) \otimes H$ by finite propagation operators, the action being on $\ell^2(X(\Gamma))$ induced by the action on $X(\Gamma)$ and trivial on $H$. From this, we get a covariant representation of $(\ell^\infty(X(\Gamma), K(H)), \Gamma)$ on $\ell^2(X(\Gamma))$, where the action of $\Gamma$ on $\ell^\infty(X(\Gamma), K(H))$ is induced by the action of $\Gamma$ on $X(\Gamma)$ by translations. This yields to a $*$-homomorphism $C_0(\Gamma, \ell^\infty(X(\Gamma), K(H))) \longrightarrow C^\text{max}(X(\Gamma))$ (where $C_0(\Gamma, \ell^\infty(X(\Gamma), K(H)))$ is equipped with the convolution product) and thus, setting $B_\Gamma = \ell^\infty(X(\Gamma), K(H))$, to a $*$-homomorphism

$$B_\Gamma \otimes_{\max} \Gamma \longrightarrow C^\text{max}(X(\Gamma)).$$

Under this map, the image of $B_{\Gamma, 0} \overset{\text{def}}{=} C_0(X(\Gamma), K(H)) \otimes_{\max} \Gamma$ is contained in $K(\ell^2(X(\Gamma)) \otimes H)$. Thus if we set $A_{\Gamma} = B_\Gamma / B_{\Gamma, 0}$, then we finally get a $*$-homomorphism

$$\Phi_{\Gamma} : A_{\Gamma} \otimes_{\max} \Gamma \longrightarrow C^\text{max}_{\text{max}}(X(\Gamma)) / K(\ell^2(X(\Gamma)) \otimes H).$$

Proposition 2.8. $\Phi_{\Gamma}$ is a $*$-isomorphism, i.e we have an short exact sequence

$$0 \longrightarrow K(\ell^2(X(\Gamma)) \otimes H) \longrightarrow C^\text{max}_{\text{max}}(X(\Gamma)) \longrightarrow A_{\Gamma} \otimes_{\max} \Gamma \longrightarrow 0.$$
and the identity on $H$. For any integer $i$, we have a unique decomposition $T''_i = \sum_{i=1}^p b_k L_{\gamma_k \Gamma_i}$, where $\gamma_k \Gamma_i$ belongs to $B_{H/\Gamma_i}(\Gamma_i, r)$ and $b_k$ belongs to $C(\Gamma/\Gamma_i, K(H))$ and is viewed as an operator acting on $\ell^2(\Gamma/\Gamma_i) \otimes H$ by pointwise action of operators of $K(H)$. Since $B_{T}(e, r) \cap \Gamma_i = \{ e \}$ for $i \geq n$, the element $\gamma_k \Gamma_i$ lifts to a unique element of $B_{\Gamma}(e, r_i)$. If $B_{T}(e, r_i) = \{ g_1, \ldots, g_m \}$, then there is a unique decomposition $T''_i = \sum_{i=1}^m f_k L_{g_k \Gamma_i}$, with $f_k \in C(\Gamma/\Gamma_i, K(H))$. Let us denote for $k = 1, \ldots, m$ by $\phi_k(T)$ the image of $(f_k^i)_{i \geq n}$ under the projection

$$
\prod_{i \geq n} C(\Gamma/\Gamma_i, K(H)) \longrightarrow \prod_{i \geq n} C(\Gamma/\Gamma_i, K(H))/\bigoplus_{i \geq n} C(\Gamma/\Gamma_i, K(H)) \cong A_{\Gamma}.
$$

It is then straightforward to check that we obtain in this way a well defined map

$$
\Lambda_T : C_{\tau}[X(\Gamma)] \longrightarrow C_{\tau}(\Gamma, A_{\Gamma}); T \mapsto \sum_{k=1}^m \phi_k(T) \delta_k,
$$

where $\delta_k$ is the Dirac function at an element $g$ of $\Gamma$. Moreover, if $\tau' \geq \tau$, then $\Lambda_{\tau'}$ restricts to $\Lambda_{\tau}$ on $C_{\tau}[X(\Gamma)]$ and the maps $\Lambda_{\tau'}$ extends to a $*$-homomorphism $C[X(\Gamma)] \longrightarrow C_{\tau}(\Gamma, A_{\Gamma})$ and thus to a $*$-homomorphism $C_{\max}[X(\Gamma)] \longrightarrow A_{\Gamma} \rtimes \max \Gamma$. This homomorphism clearly factorizes through a $*$-homomorphism

$$
C_{\max}^*[X(\Gamma)]/K(\ell^2(\Gamma) \otimes H) \longrightarrow A_{\Gamma} \rtimes \max \Gamma
$$

which provides an inverse for $\Phi_T$.

$L e m m a 2.9$. There exists a unique homomorphism

$$
\Psi_{A_{\Gamma}, \Gamma, \text{red}} : C_{\max}^*[X(\Gamma)] \longrightarrow A_{\Gamma} \rtimes \text{red} \Gamma
$$

such that $\lambda_{X(\Gamma)} \circ \Psi_{A_{\Gamma}, \Gamma, \text{red}} = \Psi_{A_{\Gamma}, \Gamma, \max} \circ A_{\Gamma, \text{red}}$.

Proof. If such an homomorphism exists, it is clearly unique. Let us prove the existence. Let $T$ be an element of $C_{\tau}(X(\Gamma))$ such that $\|T\|_{C_{\tau}(X(\Gamma))} = 1$ and let us set $x_T = \lambda_{X(\Gamma)} \circ \Psi_{A_{\Gamma}, \Gamma, \max}(T)$. Let us view $A_{\Gamma} \rtimes \text{red} \Gamma$ as an algebra of adjointable operator on the right $A_{\Gamma}$-Hilbert module $A_{\Gamma} \otimes \ell^2(\Gamma)$. For any positive real $\varepsilon$, let $\xi$ be an element of $A_{\Gamma} \otimes \ell^2(\Gamma)$ such that $\|\xi\|_{A_{\Gamma} \otimes \ell^2(\Gamma)} = 1$ and

$$
\|x_T - \xi\|_{A_{\Gamma} \otimes \ell^2(\Gamma)} \geq \|x_T\|_{A_{\Gamma} \rtimes \text{red} \Gamma} - \varepsilon.
$$

We can assume without loss of generality that $\xi$ lies indeed in $C_{\varepsilon}(\Gamma, A_{\Gamma})$ and is supported in some $B_{\varepsilon}(e, s)$ for $s$ positive real. Let us fix an integer $k$ such that $B_{\varepsilon}(e, 2r + 2s) \cap \Gamma_k = \{ e \}$. There is a decomposition $T = T' + T''$ with $T'$ in $K(\ell^2(\Gamma_{k-1} \Gamma_{\Gamma_1}) \otimes H)$ and $T''$ in $\prod_{i \geq k} K(\ell^2(\Gamma_{k-1} \Gamma_{\Gamma_1}) \otimes H) \times \Gamma/\Gamma_{k-1}$. Since this decomposition is diagonal, we get $\|T''\| \leq 1$. Actually $T''$ can be viewed as an adjointable operator on the right $\ell^\infty(\bigcup_{i \geq k} H/\Gamma_i, K(H))$-Hilbert module $\prod_{i \geq k} C(\Gamma/\Gamma_i, K(H)) \otimes \ell^2(\Gamma_{k-1} \Gamma_{\Gamma_1})$. Let us chose a lift $\xi'$ in $C_{\varepsilon}(\Gamma, B_{\varepsilon}(e, r))$ of $\xi$ under the
map induced by the canonical projection $B_{T} \to A_{T}$ such that $\|\xi\|_{B_{T} \otimes \ell^{2}(\Gamma)} \leq 1 + \varepsilon$ and $\xi'$ is an element of $\prod_{i \geq k} C(\Gamma / \Gamma_{i}, \mathcal{K}(H))$, we can write $\xi' = (\xi'_{i})_{i \geq k}$. Since $B_{T}(e, 2r) \cap \Gamma_{i} = \{e\}$ for $i \geq k$, the map $B_{T}(e, r) \to B_{T/\Gamma_{i}}(\Gamma_{i}, r); \gamma \mapsto \gamma \Gamma_{i}$ is bijective. Hence, if we define $\xi''_{i}$ in $C_{c}(\Gamma / \Gamma_{i}, \mathcal{K}(H))$ with support in $B_{T/\Gamma_{i}}(\Gamma_{i}, r)$ by $\xi''_{i}(\gamma) = \xi'_{i}(\gamma)$ for any integer $i \geq k$ and $\gamma$ in $B_{T}(e, r)$, then $\xi''_{i} = (\xi''_{i})_{i \geq k}$ is an element of $\prod_{i \geq k} C(\Gamma / \Gamma_{i}, \ell^{2}(\Gamma_{i}))$ such that $\|\xi''\| = \|\xi'\| \leq 1 + \varepsilon$. Let us now define $\eta''_{i} = (\eta''_{i})_{i \geq k}$ in $\prod_{i \geq k} C(\Gamma / \Gamma_{i}, \mathcal{K}(H))$ by $\eta''_{i} = T'' \cdot \xi''_{i}$. Then $\eta''_{i} \in C(\Gamma / \Gamma_{i}, C(\Gamma / \Gamma_{i}, \mathcal{K}(H)))$ has support in $B_{T/\Gamma_{i}}(\Gamma_{i}, r + s)$. Let us define $\eta'_{i} = C(\Gamma, C(\Gamma / \Gamma_{i}, \mathcal{K}(H)))$ with support in $B_{T}(e, r + s)$ by $\eta'_{i}(\gamma) = \eta''_{i}(\gamma \Gamma_{i})$ for $i \geq k$ and $\gamma$ in $B_{T}(e, r + s)$. Since $B_{T}(e, 2r + 2s) \cap \Gamma_{i} = \{e\}$ for $i \geq k$, the map $B_{T}(e, r + s) \to B_{T/\Gamma_{i}}(\Gamma_{i}, r + s); \gamma \mapsto \gamma \Gamma_{i}$ is bijective and thus $\|\eta''\| = \|\eta'\|$.

It is then straightforward to check that the image of $\eta'$ under the map $C_{c}(\Gamma, B_{T}) \to C_{c}(\Gamma, A_{T})$ induced by the canonical projection $B_{T} \to A_{T}$ is precisely $x_{T} \cdot \xi$. The result is then a consequence of the following:

$$
\|x_{T} \cdot \xi\| \leq \|\eta''\| \leq \|\xi''\| \leq 1 + \varepsilon
$$

\[\square\]

**Proposition 2.10.** The inclusion $\mathcal{K}(\ell^{2}(X(\Gamma)) \otimes H) \hookrightarrow C_{\max}(X(\Gamma))$ induced an injection $\mathcal{Z} \hookrightarrow K_{0}(C_{\max}(X(\Gamma)))$.

**Proof.** Let $p$ be a projector in $\mathcal{K}(\ell^{2}(X(\Gamma)) \otimes H)$ such that $[p] = 0$ in $K_{0}(C_{\max}(X(\Gamma)))$. We can assume without loss of generality that $p \in \mathcal{K}(\ell^{2}(\prod_{i=1}^{n} \Gamma / \Gamma_{i}) \otimes H)$ for some $n$. This means that

$$
\begin{pmatrix}
0 & 0 & 0 \\
0 & I_{k} & 0 \\
0 & 0 & 0
\end{pmatrix}
$$

is homotopic to

$$
\begin{pmatrix}
I_{k} & 0 & 0 \\
0 & I_{k} & 0 \\
0 & 0 & I_{k}
\end{pmatrix}
$$

in some $M_{N}(\mathcal{K}(C_{\max}(X(\Gamma))))$ by a homotopy of projectors. Hence for every positive number $\varepsilon < 1/4$ there exists a real $r$ such that

$$
\begin{pmatrix}
0 & 0 & 0 \\
0 & I_{k} & 0 \\
0 & 0 & 0
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
I_{k} & 0 & 0 \\
0 & I_{k} & 0 \\
0 & 0 & I_{k}
\end{pmatrix}
$$

are connected by a homotopy $(p_{t})_{t \in [0,1]}$ of $\varepsilon$-almost projectors of $M_{N}(\mathcal{K}(\ell^{2}(\prod_{i=1}^{k-1} \Gamma / \Gamma_{i}) \otimes H))$ (i.e selfadjoint elements satisfying $\|p_{t}^{2} - p_{t}\| \leq \varepsilon$ such that $p_{t}$ has propagation less than $r$ for all $t$ in $[0,1]$. Let us fix an integer $k \geq \max\{n, r\}$. Then for every $t \in [0,1]$ we can write $p_{t} = p'_{t} + p''_{t}$, where $(p'_{t})_{t \in [0,1]}$ is a homotopy of selfadjoint elements in

$$
M_{N}(\mathcal{K}(\ell^{2}(\prod_{i=1}^{k-1} \Gamma / \Gamma_{i}) \otimes H)) + CId_{\mathcal{L}(\prod_{i=1}^{k-1} \Gamma / \Gamma_{i}) \otimes H})}
$$
and \((p_t')_{t \in [0,1]}\) is a homotopy of selfadjoint elements in

\[
M_N \left( \prod_{i \geq k} \mathcal{K}(\ell^2(\Gamma_i)) \otimes H \right) + \mathbb{C}Id_{\prod_{i \geq k} \gamma_i} \otimes H.
\]

Moreover, since \(p_t\) can be written diagonally as \(p_t' \oplus p_t''\) in the decomposition \(\ell^2(X(\Gamma)) \otimes H = \ell^2(\prod_{i=1}^{k-1} \Gamma_i) \otimes H \bigoplus \ell^2(\prod_{i \geq k} \Gamma_i) \otimes H\), then \(p_t''\) is also a \(\varepsilon\)-projectors with propagation less than \(r\). Let \(\varphi : \mathbb{R} \to \mathbb{R}\) be a continuous function such that \(\varphi(s) = 0\) if \(s \leq 1/2\) and \(\varphi(s) = 1\) if \(\frac{\sqrt{s - \frac{1}{2}}}{2} \leq s\). Then \((\varphi(p_t'))_{t \in [0,1]}\) is a homotopy of projectors in \(M_N(\mathcal{K}(\prod_{i=1}^{k-1} \Gamma_i) \otimes H) + \mathbb{C}Id_{\prod_{i=1}^{k-1} \gamma_i} \otimes H\) between

\[
\begin{pmatrix}
p & 0 & 0 \\
0 & I_{k-1} & 0 \\
0 & 0 & 0
\end{pmatrix}

\text{and}

\begin{pmatrix}
I_k & 0 & 0 \\
0 & I_{k-1} & 0 \\
0 & 0 & 0
\end{pmatrix}

\text{and thus } p = 0. \quad \square

In conclusion, we get

**Corollary 2.11.** With previous notations, we have

(i) a short exact sequence

\[
0 \to Z \to K_0(C^*_{max}(X(\Gamma))) \xrightarrow{\Psi_{\Gamma,F,max,+}} K_0(A_{F} \rtimes_{max} \Gamma) \to 0,
\]

where the copy of \(Z\) in \(K_0(C^*_{max}(X(\Gamma)))\) is generated by the class of a rank one projector on \(\ell^2(X) \otimes H\).

(ii) an isomorphism

\[
K_1(C^*_{max}(X(\Gamma))) \xrightarrow{\Psi_{\Gamma,F,max,+}} K_1(A_{F} \rtimes_{max} \Gamma)
\]

**Remark 2.12.** The same proof as for proposition 2.11 applies to show that the injection \(K_0(\mathcal{K}(\ell^2(X(\Gamma))) \otimes H) \hookrightarrow C^*(X(\Gamma))\) induces an injection \(Z \hookrightarrow K_0(C^*(X(\Gamma)))\). But when the group \(\Gamma\) has the property \(\tau\) with respect to the family \((\Gamma_i)_{i \in \mathbb{N}}\), then \(X(\Gamma)\) is a family of expanders and it was proved in [8] that the composition

\[
Z \to K_0(C^*(X(\Gamma))) \xrightarrow{\Psi_{\Gamma,F,max,+}} K_0(A_{F} \rtimes_{max} \Gamma)
\]

(and thus the composition \(K_0(\mathcal{K}(\ell^2(X(\Gamma))) \otimes H) \hookrightarrow C^*(X(\Gamma)) \xrightarrow{\Psi_{\Gamma,F,max,+}} K_0(A_{F} \rtimes_{max} \Gamma)\) is not exact in the middle)

### 2.3 Assembly map for the Maximal Roe algebra.

Let \(X\) be a locally compact metric space, then according to a result of [8] that we shall recall below, the \(K\)-theory group \(K_*^\text{red}(C^*(X))\) up to canonical isomorphism does not depend on the chosen non-degenerated standard \(X\)-module defining \(C^*(X)\). In [8] was defined an assembly map for the Roe algebra \(\hat{\mu}_X : K_*^\text{red}(C^*(X)) \to K_*^\text{red}(C^*(X))\) in the following way. Let \(H_X\) be a standard \(X\)-Hilbert module with respect to a non-degenerated representation \(\rho_X : C_0(X) \to \mathcal{L}(H_X)\). Let us define the following subalgebras of \(\mathcal{L}(H_X)\)

\[
\begin{align*}
D(X) &= \{ T \in \mathcal{L}(H_X) \text{ such that } [f,T] \in \mathcal{K}(H_X) \text{ for all } f \in C_0(X) \}, \\
C(X) &= \{ T \in \mathcal{L}(H_X) \text{ such that } f : T \in \mathcal{K}(H_X) \text{ and } T \cdot f \text{ for all } f \in C_0(X) \}, \\
D^*(X) &= \{ T \in D(X) \text{ and } T \text{ is in the closure of finite propagation operators} \}.
\end{align*}
\]
Then every element in $K_*(X)$ can be represented by a $K$-cycle $(\rho_X, H_X, T)$, with $T \in D(X)$. This operator then defines a class $[T]$ in $K_{*+1}(D(X)/C(X))$ and we get in this way an isomorphism called the Paschke duality:

$$K_*(X) \cong K_{*+1}(D(X)/C(X)); ([\rho_X, H_X, T]) \mapsto [T].$$

According to [8, Lemma 12.3.2], the $C^*$-algebras inclusions $C^*(X) \hookrightarrow C(X)$ and $D^*(X) \hookrightarrow D(X)$ induce an isomorphism

$$(2.1) \quad D^*(X)/C^*(X) \cong D(X)/C(X).$$

Using the inverse of this isomorphism, we get finally an isomorphism

$$K_*(X) \cong K_{*+1}(D^*(X)/C^*(X))$$

which when composed with the boundary map in $K$-theory associated to the short exact sequence

$$0 \to C^*(X) \to D^*(X) \to D^*(X)/C^*(X) \to 0$$

gives rise to the assembly map

$$\tilde{\mu}_{X,*} : K_*(X) \to K_*(C^*(X)).$$

**Remark 2.13.** Every element $x$ in $K_*(X)$ can be indeed, represented by a $K$-cycle $(\rho_X, H_X, T)$, with $T \in D(X)$. In this case, $(T, C^*(X))$ is $K$-cycle for $K_*(C^*(X)) = KK(C, C^*(X))$ and thus defines an element of $K_*(C^*(X))$ we shall denote by $\text{Ind}_X T$. It is then straightforward to check that $\tilde{\mu}_{X,*}(x) = \text{Ind}_X T$.

In order to define the coarse Baum-Connes assembly maps, we shall recall some functoriality results of the Roe algebras under coarse maps.

Let $\phi : X \to Y$ be a coarse map between locally compact metric spaces. Let $H_X$ (resp. $H_Y$) be a non-degenerated standard $X$-Hilbert module (resp. $Y$-Hilbert module) with respect to a representation $\rho_X$ (resp. $\rho_Y$). Recall from [8] that there is an isometry $V : H_X \to H_Y$ that covers $\phi$, i.e., there exists a real $s$ such that for any $x$ and $y$ in $X$ with $d(\phi(x), y) > s$, we can find $f$ in $C_0(Y)$ and $g$ in $C_0(X)$ that satisfy $f(y) \neq 0$, $g(x) \neq 0$ and $\rho_Y(f)V\rho_X(g) = g$. The map $\mathcal{L}(H_X) \to \mathcal{L}(H_Y); T \mapsto VTV^*$ then restricts to a *-homomorphism $C[H_X] \to C[H_Y]$ and thus to a homomorphism $\text{Ad} V : C^*(X) \to C^*(Y)$. The crucial point, due to [8], is that the homomorphism $\text{Ad} V : K_*(C^*(X)) \to K_*(C^*(Y))$ induced in $K$-theory by $\text{Ad} V$ does not depend on the choice of the isometry $V$ covering $\phi$. Hence, we define $\phi_* = \text{Ad} V : K_*(C^*(X)) \to K_*(C^*(Y))$, where $V : H_X \to H_Y$ is any isometry covering $\phi$.

**Remark 2.14.**

(i) If $\phi : X \to X$ is a coarse map such that for some real $C$, we have $d(x, \phi(x)) < C$ for all $x$ in $X$, then $\text{Id}_{H_X}$ covers $\phi$ and hence $\phi_* = \text{Id}_{K_*(C^*(X))}$.

(ii) If $\phi : X \to Y$ and $\psi : Y \to Z$ are two coarse maps, then $(\phi \circ \psi)_* = \phi_* \circ \psi_*$. 

(iii) In consequence if $\phi : X \to Y$ is a coarse equivalence, then $\phi_*$ is an isomorphism. Moreover, if $\phi' : X \to Y$ is another coarse equivalence, then $\phi_* = \phi'_*$. 

(iv) In particular, by choosing for two non-degenerated standard $X$-modules $H_X$ and $H_X'$ an isometry $V : H_X \to H_X'$ that covers $\text{Id}_X$, we see that up to canonical isomorphisms, the $K$-group $K_*(C^*(X))$ does not depend on the choice of a non-degenerated standard $X$-module.
The previous construction can be extended to maximal Roe algebras by using the following lemma.

**Lemma 2.15.** With above notations, assume that $X$ and $Y$ a locally compact metric spaces both containing nets with bounded geometry. For any isometry $V : H_X \to H_Y$ covering a coarse map $\phi : X \to Y$, we have

(i) $C[\phi X]_1 \to C[Y]_1; T \mapsto VTV^*$ extends to a homomorphism $C[\phi X]_1 \to C[Y]_1$ and thus to a homomorphism $\text{Ad}_{\max} V : C^*_\max(\phi X) \to C^*_\max(Y)$.

(ii) The homomorphism $\text{Ad}_{\max,*} V : K_*(C^*_\max(\phi X)) \to K_*(C^*_\max(Y))$ induced by $\text{Ad}_{\max} V$ in $K$-theory does not depend on the choice of the isometry $V : H_X \to H_Y$ covering $\phi$.

**Proof.** The first item is just a consequence of the universal properties for $C^*_\max(X)$ and $C^*_\max(Y)$. For the second point, assume that $V' : H_X \to H_Y$ is another isometry covering $\phi$ and let us set $W = V'V^*$. Then $W$ is a partial isometry of $H_Y$ with finite propagation. Then $Wx$ and $(\text{Id}_{H_X} - W^*W)x$ are in $C[Y]$ for any $x$ in $C[Y]$, and since

$$(x^*x = x^*W^*Wx + x^*(\text{Id}_{H_X} - W^*W)^2x = (Wx)^*Wx + (x(\text{Id}_{H_X} - W^*W))(\text{Id}_{H_X} - W^*W)x,$$

we get that $(Wx)^*Wx \leq x^*x$ in $C^*_\max(Y)$ and thus $\|Wx\| \leq \|x\|$. Hence $C[Y] \to C[\phi X]_1; x \mapsto W \cdot x$ extends to a bounded linear map $C^*_\max(\phi X) \to C^*_\max(X)$ we shall denote again by $W$. But $W^*$ is also a partial isometry of $H_Y$ with finite support and it is straightforward to check that for any $x$ and $y$ in $C^*_\max(Y)$ then $W^*(x)y = xW(y)$ and thus $W$ is a multiplier of $C^*_\max(Y)$. Since $\text{Ad}_{\max,*} V = W \cdot \text{Ad}_{\max,*} V \cdot W^*$, we get the result by using [8, Lemma 4.6.2].

With above notation, this allowed to define for a coarse map $\phi : X \to Y$ with $X$ and $Y$ both containing nets with bounded geometry,

$$\phi_{\max,*} = \text{Ad}_{\max,*} V : K_*(C^*_\max(\phi X)) \to K_*(C^*_\max(Y)),$$

where $V : H_X \to H_Y$ is any isometry covering $\phi$. Notice that the remark 2.14 obviously admits a analogous formulation for maximal Roe algebra. In view of remark 2.13 and in order to define the maximal assembly map, we will need the following result.

**Lemma 2.16.** Let $H_X$ a standard non-degenerated $X$-module. Then, for any pseudo-local operator $T$ on $H_X$ with finite propagation, the map

$$C[\phi X] \to C[\phi X]; x \mapsto Tx$$

extends in a unique way to a multiplier of $C^*_\max(\phi X)$ we shall again denote by $T$. Moreover, for any positive real $r$, there exists a real constant $c_r$ such that if $T$ has propagation less than $r$, then

$$\|T\|_{M(C^*_\max(\phi X))} \leq c_r\|T\|_{L(H_X)};$$

where $M(C^*_\max(\phi X))$ stands for the algebra of multiplier of $C^*_\max(\phi X)$.

**Proof.** We can assume without loss of generality that $T$ as an operator on $H_X$ is norm 1.
Let \((f_i)_{i \in I}\) be a partition of unit, whose supports have uniformly bounded diameters. Since \(T - \sum_{i \in I} f_i^{1/2} T f_i^{1/2}\) is in \(C[X]\) and has norm operator for \(H_X\) less than 2, then according to lemma 2.6 it is enough to prove the result for \(\sum_{i \in I} f_i^{1/2} T f_i^{1/2}\) instead of \(T\). For an element \(x\) of \(C[X]\), let us set

\[
x' = \sum_{i \in I} f_i^{1/2} T f_i^{1/2} x,
\]

and

\[
y = (\sum_{i \in I} f_i^{1/2} A f_i^{1/2})^* (\sum_{i \in I} f_i^{1/2} A f_i^{1/2}) - \sum_{i \in I} f_i^{1/2} A^* A f_i^{1/2}.
\]

Then \(A\) is a positive pseudo-local operator of \(H_X\) and \(y\) is a self-adjoint element in \(C[X]\) and

\[
x'^* x' - x^* x = \sum_{i \in I} x^* f_i^{1/2} (T^* T - \Id_{H_X}) f_i^{1/2} x + x^* y x
\]

\[
= - (\sum_{i \in I} f_i^{1/2} A f_i^{1/2})^* (\sum_{i \in I} f_i^{1/2} A f_i^{1/2}) + x^* (y + y') x,
\]

where \(y' = (\sum_{i \in I} f_i^{1/2} A f_i^{1/2})^2 - \sum_{i \in I} f_i^{1/2} A^2 f_i^{1/2}\) lies in \(C[X]\). According to Lemma 2.6, since \(y\) and \(y'\) has operator norm on \(H_X\) bounded by 2 and have propagation less than \(r\), there exists a real constant \(c_r\), depending only on \(H_X\) and \(r\) such that \(\|y + y'\| \leq c_r\), in \(C^*_{\max}(X)\). Hence \(x'^* x' - x^* x \leq c_r x^* x\) and hence \(\|x'\| \leq (1 + c_r)^{1/2}\|x\|\) in \(C^*_{\max}(X)\). In consequence, the map \(C[X] \to C[X]; \ x \mapsto \sum_{i \in I} f_i^{1/2} T f_i^{1/2} x\), is bounded for the norm of \(C^*_{\max}(X)\) and thus extends to a bounded linear map \(C^*_{\max}(X) \to C^*_{\max}(X)\).

Applying the preceding point also to \(T^*\), we get that \((T^* x)' y = x(T y)\) for all \(x\) and \(y\) in \(C^*_{\max}(X)\) (check it on \(C[X]\)) and thus \(T\) is a multiplier for \(C^*_{\max}(X)\).

Remark 2.17. The set of pseudo-local operator of \(H_X\) of finite propagation is a \(\ast\)-subalgebra of \(\mathcal{L}(H_X)\) which contains \(C[X]\) as an ideal. From preceding lemma, we get then a \(\ast\)-homomorphism from the algebra of pseudo-local operator of \(H_X\) of finite propagation to the multiplier of \(C^*_{\max}(X)\) whose restriction to \(C[X]\) is just the inclusion \(\ast\)-homomorphism \(C[X] \hookrightarrow C^*_{\max}(X)\).

Corollary 2.18.

\begin{itemize}
\item If \((\rho_X, H_X, T)\) is a \(K\)-cycle for \(K_s(X)\) with \(T\) of finite propagation. Then \((T, C^*_{\max}(X))\) is a \(K\)-cycle for \(K_s(C^*_{\max}(X)) = KK(\mathbb{C}, C^*_{\max}(X))\) we shall denote by \(\Ind_{X,\max} T\).
\item \((\rho_X, H_X, T) \mapsto \Ind_{X,\max} T\) gives rise to a homomorphism

\[
\hat{\mu}_{X,\max} : K_s(X) \to K_s(C^*_{\max}(X)).
\]
\end{itemize}

Proof. We only have to check that the definition of \(\Ind_{X,\max} T\) only depends on the class of \((\rho_X, H_X, T)\) in \(K_s(X)\). But if \((\rho_X, H_X, T)\) and \((\rho_X, H_X, T')\) are two \(K\)-cycle for \(K_s(X)\) with \(T\) and \(T'\) of finite propagation then

\begin{itemize}
\item if \(f(T - T')\) is compact for all \(f\) in \(C_0(X)\), then \(\Ind_{\max} T = \Ind_{\max} T'\);
\end{itemize}
if $T$ and $T'$ are connected by a homotopy of operators $(T_s)_{s \in [0,1]}$ such that $(\rho_X, H_X, T_s)$ is a K-cycle for all $s$ in $[0,1]$, then according to the preceding point, we can replace $(T_s)_{s \in [0,1]}$ by $(\sum_{i \in \mathbb{N}} f_i^{1/2} I_i^1 T_i^1 f_i^{1/2} s \in [0,1])$, where $(f_i)_{i \in \mathbb{N}}$ is a partition of unit with support of uniformly bounded diameter.

The result is then a consequence of the second item of lemma 2.16.

**Remark 2.19.** Let $x$ be an element of $K_0(X)$ represented by an even K-cycle $(\rho_X, H_X, T)$ as in the previous corollary. Let us set

$$W = \begin{pmatrix} \text{Id}_{H_X} & T \\ 0 & \text{Id}_{H_X} \end{pmatrix} \begin{pmatrix} \text{Id}_{H_X} & 0 \\ -T & \text{Id}_{H_X} \end{pmatrix} \begin{pmatrix} \text{Id}_{H_X} & T \\ 0 & \text{Id}_{H_X} \end{pmatrix} \begin{pmatrix} 0 & -\text{Id}_{H_X} \\ \text{Id}_{H_X} & 0 \end{pmatrix} - 1.$$  

Then

$$W \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} W^{-1} = \begin{pmatrix} 0 & 0 \\ \text{Id}_{H_X} & 0 \end{pmatrix}$$

defines an element in $K_0(C^{\text{\text{max}}}_{\text{\text{max}}}(X))$ which is precisely $\hat{\mu}_{\Gamma, \text{\text{max}}}(x)$.

We are now in position to define the coarse Baum-Connes assembly maps. Recall that for a proper metric set $\Sigma$ and a real $r$, the Rips complex of order $r$ is the set $P_r(\Sigma)$ of probability measures on $\Sigma$ with support of diameter less than $r$. Recall that $P_r(\Sigma)$ is a locally finite simplicial complex that can be provided with a proper metric extending the euclidian metric on each simplex. Moreover, by viewing an element of $\Sigma$ as a Dirac measure, we get an inclusion $\Sigma \hookrightarrow P_r(\Sigma)$, which turns out to be a coarse equivalence. If we fix for each real $r$ a coarse equivalence $\phi_r : P_r(\Sigma) \rightarrow \Sigma$, then the collections of homomorphisms given by the compositions

$$K_\ast(P_r(\Sigma)) \xrightarrow{\hat{\mu}_{\Gamma, \text{\text{max}}}(\cdot)} K_\ast(C^\ast(P_r(\Sigma))) \xrightarrow{\phi_r \ast} K_\ast(C^\ast(\Sigma))$$

and

$$K_\ast(P_r(\Sigma)) \xrightarrow{\hat{\mu}_{\Gamma, \text{\text{max}}}(\cdot)} K_\ast(C^\ast(P_r(\Sigma))) \xrightarrow{\phi_r \ast} K_\ast(C^\ast(\Sigma))$$

give rise respectively to the the Baum-connes coarse assembly map

$$\mu_{\Sigma, \ast} : \lim_{r \rightarrow 0} K_\ast(P_r(\Sigma)) \rightarrow K_\ast(C^\ast(\Sigma))$$

and to the maximal Baum-Connes assembly map

$$\mu_{\Sigma, \ast, \text{\text{max}}} : \lim_{r \rightarrow 0} K_\ast(P_r(\Sigma)) \rightarrow K_\ast(C^\ast(\Sigma)).$$

Moreover, if $z$ is in $\lim_{r \rightarrow 0} K_\ast(P_r(\Sigma))$ comes from a K-cycle $(\rho_{P_r(\Sigma)}, H_{P_r(\Sigma)}, T)$ for some $K_\ast(P_r(\Sigma))$, where $T$ is a finite propagation operator on the non-degenerated standard $P_r(\Sigma)$-module $H_{P_r(\Sigma)}$ then

$$\mu_{\Sigma, \ast}(z) = \phi_r \ast \text{Ind}_{P_r(\Sigma)} T$$

and

$$\mu_{\Sigma, \text{\text{max}}, \ast}(z) = \phi_r \ast \text{Ind}_{P_r(\Sigma)} T.$$

**Remark 2.20.** Let $\lambda_\Sigma : C^\ast_{\text{\text{max}}}(\Sigma) \rightarrow C^\ast(\Sigma)$ be the homomorphism induced from the representation $C[X] \hookrightarrow B(H_\Sigma)$. Then $\mu_{\Sigma, \ast} = \mu_{\Sigma, \text{\text{max}}, \ast} \circ \lambda_\Sigma$.

### 3. The Baum-Connes Assembly Map

We gather this section with result we will need later on concerning the Baum-Connes assembly map and its left hand side. For a proper $\Gamma$-space $X$ and a $\Gamma$-algebra $A$, we shall denote for short $KK^\ast_G(X, A)$ instead of $KK^\ast_G(C_0(X), A)$.
3.1. Definition of the maximal assembly map. Let $X$ be a locally compact proper and cocompact $\Gamma$-space and let $(\rho, \mathcal{E}, T)$ be a $K$-cycle for $KK^\Gamma_r(X, A)$. Up to averaging with a cut-off function, we can assume that the operator $T$ is $\Gamma$-equivariant. Let $\mathcal{E}_T$ be the separated completion of $C_c(X) \cdot \mathcal{E}$ with respect to the $A \times_{\text{max}} \Gamma$-valued scalar product defined by $\langle e/e' \rangle_{\mathcal{E}_T}(\gamma) = \langle \xi/\gamma(\xi') \rangle_{\mathcal{E}}$ for $\xi$ and $\xi'$ in $C_c(X) \cdot \mathcal{E}$ and $\gamma$ in $\Gamma$ (recall that the separated completion is obtained by first divide out by the submodule of vanishing elements for the pseudo-norm associated to the inner product and then by completion of the quotient with respect to the induced norm). Up to replace $T$ by $\sum_{\gamma \in \Gamma} \gamma(f^{1/2})T\gamma(f^{1/2})$, for $f \in C_c(X, [0, 1])$ a cut-off function with respect to the action of $\Gamma$ of $X$, the map $C_c(X) \cdot \mathcal{E} \to C_c(X) \cdot \mathcal{E}$; $\xi \mapsto T\xi$ extends to an adjointable operator $T_{\Gamma} : \mathcal{E}_T \to \mathcal{E}_T$. Then we can check that $(T_{\Gamma}, T_{\mathcal{E}})$ is a $K$-cycle for $KK_*(\mathbb{C}, A \times_{\text{max}} \Gamma) = K_*(A \times_{\text{max}} \Gamma)$ whose class $\text{Ind}_{\Gamma, A, \text{max}} T$ only depends on the class of $(\rho, \mathcal{E}, T)$ in $KK^\Gamma_r(X, A)$. The left hand side of the maximal assembly map is then the topological $K$-theory for $\Gamma$ with coefficients in $A$

$$K^\text{top}_*(\Gamma, A) = \lim_{r \to 0} KK^\Gamma_r(P_r(\Gamma), A)$$

and the assembly map

$$\mu_{\Gamma, A, \text{max}} : K^\text{top}_*(\Gamma, A) \to K_*(A \times_{\text{max}} \Gamma)$$

is defined for an element $x$ in $KK^\Gamma_r(\Gamma, A)$ coming from the class of a $K$-cycle $(\rho, \mathcal{E}, T)$ for $KK^\Gamma_r(P_r(\Gamma), A)$ by $\mu_{\Gamma, A, \text{max}}(x) = \text{Ind}_{\Gamma, A, \text{max}} T$.

3.2. Induction. We recall now from [11] the description of induction to a group, from the action of one of its subgroup on a $C^\ast$-algebra, and the behaviour of the left-hand side of the Baum-Connes assembly map under this transformation.

Let $\Gamma'$ be a subgroup of a discrete group $\Gamma$, and let $A$ be a $\Gamma'$-$C^\ast$-algebra. Define $\Gamma_c\Gamma' A = \{ f : \Gamma \to A \text{ such that } \gamma' \cdot f(\gamma \gamma') = f(\gamma) \text{ for all } \gamma, \gamma' \in \Gamma' \}

$$

and $\gamma \Gamma' \mapsto \| f(\gamma) \|$ is in $C_0(\Gamma/\Gamma')$}.

Then $\Gamma$ acts on $\Gamma_c\Gamma' A$ by left translation and it is a standard fact that the dynamical systems $(\Gamma_c\Gamma', A, \Gamma)$ and $(A, \Gamma')$ have equivalent covariant representations. In particular, the $C^\ast$-algebras $A \times_{\text{max}} \Gamma'$ and $\Gamma_c\Gamma' A \times_{\text{max}} \Gamma$ are Morita equivalent (the same holds for reduced crossed products). Notice that if the action of $\Gamma'$ on $A$ is indeed the restriction of an action of $\Gamma$, then $\Gamma_c\Gamma' A = C_0(\Gamma/\Gamma', A); f \mapsto [\gamma \Gamma' \mapsto \gamma \cdot f(\gamma)]$ is a $\Gamma$-equivariant isomorphism, where $C_0(\Gamma/\Gamma', A) \cong C_0(\Gamma/\Gamma') \otimes A$ is provided with the diagonal action of $\Gamma$. In [11] was defined an induction homomorphism

$$\Gamma_c, \Gamma' A : K^\text{top}_\Gamma(\Gamma', A) \to K^\text{top}_\Gamma(\Gamma, A),$$

which turned out to be an isomorphism. If $\Gamma'$ has finite index in $\Gamma$, then $\Gamma_c, \Gamma' A$ can be described quite easily as follows. Recall first that if $\Gamma'$ is a subgroup of $\Gamma$ with finite index, then the family of inclusions $C_0(P_r(\Gamma')) \hookrightarrow C_0(P_r(\Gamma))$ gives rise to an isomorphism

$$\lim_r KK^\Gamma_r(P_r(\Gamma'), A) = K^\text{top}_\Gamma(\Gamma', A) \cong \lim_r KK^\Gamma_r(P_r(\Gamma), A),$$

with $\Gamma$ the maximal group of bounded cocycles on $\Gamma$. The left hand side of the Baum-Connes assembly map under this transformation.
and under this identification, the assembly map is defined as before: if \( x \) in an element in \( K^*_\text{top}(\Gamma, A) \) coming from the class of a \( K \)-cycle \((\rho, \mathcal{E}, T)\) for \( KK^\Gamma_*((P, \Gamma), A)\), then \( \mu_{\Gamma, A, \text{max},*}(x) = \text{Ind}_{\Gamma, A, \text{max}} T \).

Now let \((\rho, \mathcal{E}, T)\) be a \( K \)-cycle for some \( KK^\Gamma_*'(X, A)\), where \( X \) is a proper and cocompact \( \Gamma'\)-space. Let us define

\[
\Gamma^\prime_\text{c}, \mathcal{E} = \{ \xi : \Gamma \to A, \gamma' \cdot (\xi \gamma') = \xi(\gamma) \text{ for all } \gamma \in \Gamma \}.
\]

The pointwise right \( A \)-Hilbert module structure provides a right \( \Gamma^\prime_\text{c}, \mathcal{E} \) which is covariant for the action of \( \Gamma \) by left translations. If \( \Gamma \) is chosen \( \Gamma'\)-equivariant, then \( \Gamma \to \mathcal{E}; \gamma \mapsto T, \xi \in \Gamma^\prime_\text{c}, \mathcal{E} \) and we get in this way a \( \Gamma \)-equivariant and adjointable operator \( \Gamma^\prime_\text{c}, T : \Gamma^\prime_\text{c}, \mathcal{E} \to \Gamma^\prime_\text{c}, \mathcal{E} \). Finally, for \( f \in C_0(X) \), pointwise left multiplication by \( \gamma \mapsto \gamma'(\rho(f)) \) defines a covariant representation \( \Gamma^\prime_\text{c}, \rho \) of \( C_0(X) \) on the Hilbert right \( \Gamma^\prime_\text{c}, A \)-Hilbert module \( \Gamma^\prime_\text{c}, \mathcal{E} \). It is straightforward to check that \((\Gamma^\prime_\text{c}, \rho, \Gamma^\prime_\text{c}, \mathcal{E}, \Gamma^\prime_\text{c}, T)\) is a \( K \)-cycle for \( KK^\Gamma_*'(X, \Gamma^\prime_\text{c}, A) \) whose class only depends on the class of \((\rho, \mathcal{E}, T)\) in \( KK^\Gamma_*'(X, A) \). On the other hand, if \((\rho', \mathcal{E}', T')\) is a \( K \)-cycle for \( KK^\Gamma_*'(X, \Gamma^\prime_\text{c}, A) \), and if we consider the \( \Gamma'\)-equivariant homomorphism \( \psi : \Gamma^\prime_\text{c}, A \to A; f \mapsto f(e) \), then \( \mathcal{E} = \mathcal{E}' \otimes_{\psi, A} \mathcal{E} \) is a \( \Gamma'\)-covariant right \( A \)-Hilbert module. Let us set \( T = T' \otimes_{\psi, A} \text{Id}_A \) and \( \rho(f) = \rho'(f) \otimes_{\psi, A} \text{Id}_A \) for all \( f \in C_0(X) \). Then \((\rho, \mathcal{E}, T)\) is a \( K \)-cycle for \( KK^\Gamma_*'(X, A) \) and we can check that \((\Gamma^\prime_\text{c}, \rho, \Gamma^\prime_\text{c}, \mathcal{E}, \Gamma^\prime_\text{c}, T)\) is a \( K \)-cycle unitary equivalent to \((\rho', \mathcal{E}', T')\). Finally, we get an isomorphism

\[
\Gamma^\prime_\text{c}, X, A \ast_* \colon KK^\Gamma_*'(X, A) \xrightarrow{\cong} KK^\Gamma_*'(X, \Gamma^\prime_\text{c}, A)
\]

which maps the class of a \( K \)-cycle \((\rho, \mathcal{E}, T)\) for \( KK^\Gamma_*'(X, A) \) to the class of the \( K \)-cycle \((\Gamma^\prime_\text{c}, \rho, \Gamma^\prime_\text{c}, \mathcal{E}, \Gamma^\prime_\text{c}, T)\) in \( KK^\Gamma_*'(X, \Gamma^\prime_\text{c}, A) \). Under the identification of equation \(3.1\), the family of isomorphisms \((\Gamma^\prime_\text{c}, X, A \ast_{*\ast} \Gamma^\prime_\text{c}, A)_{\Gamma > 0}\) gives rise to an isomorphism

\[
\Gamma^\prime_\text{c}, X, A \ast_{\ast\ast} \colon K^\ast_{\text{top}}(\Gamma, A) \xrightarrow{\cong} K^\ast_{\text{top}}(\Gamma, \Gamma^\prime_\text{c}, A).
\]

Moreover, up to the identification \( K_{\ast}(A \times_{\text{max}} \Gamma') \cong K_{\ast}(\Gamma^\prime_\text{c}, A \times_{\text{max}} \Gamma) \) induced by the Morita equivalence, we have

\[
\mu_{\Gamma', A, \text{max}} = \mu_{\Gamma', \Gamma^\prime_\text{c}, A, \Gamma, \text{max}} \circ \Gamma^\prime_\text{c}, X, A \ast_{\ast\ast}.
\]

**Remark 3.1.** Let \( A \) be a \( \Gamma \)-algebra, let \( \Gamma' \) be a subgroup of \( \Gamma \) with finite index and let \((\rho, \mathcal{E}, T)\) be a \( K \)-cycle for \( KK^\Gamma_*'(X, A) \), such that the action of \( \Gamma' \) on \( \mathcal{E} \) is indeed the restriction of a covariant action of \( \Gamma \). Under the identification \( \Gamma^\prime_\text{c}, A \cong C(\Gamma/\Gamma', A) \) we have seen before, then

\[
\Gamma^\prime_\text{c}, \mathcal{E} \to C(\Gamma/\Gamma', \mathcal{E}); \xi \mapsto [\gamma \mapsto \gamma \cdot \xi(\gamma)]
\]

is a \( \Gamma \)-equivariant isomorphism of right \( C(\Gamma/\Gamma', A) \)-Hilbert module, where we have equipped \( C(\Gamma/\Gamma', A) \cong C(\Gamma/\Gamma') \otimes \mathcal{E} \) with the diagonal action of \( \Gamma \). Moreover, under this identification, \( \Gamma^\prime_\text{c}, \rho \) is given pointwise by the representation \( \rho \) and \( \Gamma^\prime_\text{c}, T \) is the pointwise multiplication by \( \gamma \mapsto \gamma(T) \).

### 3.3. The left hand side for product of stable algebras

As it was proved in \(2\), the topological \( K \)-theory for a group is a functor with respect to the coefficients which commutes with direct sums, i.e \( K^\ast_{\text{top}}(G, \bigoplus_{\ell \in \ell} A_\ell) = \bigoplus_{\ell \in \ell} K^\ast_{\text{top}}(G, A_\ell) \) for every locally compact group \( G \) and every family \((A_\ell)_{\ell \in \ell}\) of \( C^\ast\)-algebras \( A_\ell \), equipped with
Lemma 3.2. Let $\mathcal{A} = \{A_i\}_{i \in I}$ be a family of unital $C^*$-algebras. Let
\[
\Theta_{\mathcal{A}} \colon K_*(\prod_{i \in I} (A_i \otimes \mathcal{K}(H))) \to \prod_{i \in I} K_*(A_i)
\]
be the homomorphism induced on the $j$-th factor by the projection
\[
\prod_{i \in I} (A_i \otimes \mathcal{K}(H)) \to A_j \otimes \mathcal{K}(H).
\]
Then $\Theta_{\mathcal{A}}^*$ is an isomorphism.

Proof. Is clear that $\Theta_{\mathcal{A}}^*$ is onto. The injectivity of $\Theta_{\mathcal{A}}^*$ is then a consequence of the next lemma. \hfill \square

Lemma 3.3. There exists a map $\phi : [0, +\infty] \to [0, +\infty]$ such that for any unital $C^*$-algebra $A$, the following properties hold:

(i) If $p$ and $q$ are projectors in some $M_n(A)$ connected by a homotopy of projectors. Then there exists integers $k$ and $N$ with $n + k \leq N$ and a homotopy of projectors $(p_t)_{t \in [0,1]}$ in $M_N(A)$ connecting $\operatorname{diag}(p, 0, \ldots, 0)$ and $\operatorname{diag}(q, I_k, 0)$ and such that for any positive real $\varepsilon$ and any $s$ and $t$ in $[0,1]$ with $|s - t| \leq \phi(\varepsilon)$, then $\|p_s - p_t\| \leq \varepsilon$.

(ii) If $u$ and $v$ are homotopic unitaries in $U_n(A)$, then there exists an integer $k$ and a homotopy $(w_t)_{t \in [0,1]}$ in $U_{n+k}(A)$ connecting $\operatorname{diag}(u, 0, \ldots, 0)$ and $\operatorname{diag}(v, I_k)$ such that for any positive real $\varepsilon$ and any $s$ and $t$ in $[0,1]$ with $|s - t| \leq \phi(\varepsilon)$, then $\|w_s - w_t\| \leq \varepsilon$.

Proof. We can assume without loss of generality that $n = 1$.

* Let us notice first that using [14, Proposition 5.2.6, page 90], then there exists a positive real $\alpha$ such that for any unital $C^*$-algebra $A$ and any projectors $p$ and $q$ in $A$ such that $\|p - q\| \leq \alpha$, then $q = u \cdot p \cdot u^*$ for some unitary $u$ of $A$ with $\|u - 1\| \leq 1/2$. Hence there is a self-adjoint element $h$ of $A$ with $\|h\| \leq 1$ in $2$ such that $u = \exp(ih)$. Considering the homotopy of projectors $(\exp(th) \cdot p \cdot \exp(-th))_{t \in [0,1]}$, we see that there exists a map $\phi_1 : [0, +\infty] \to [0, +\infty]$ such that for any $C^*$-algebra $A$ and any projectors $p$ and $q$ in $A$ such that $\|p - q\| \leq \alpha$, then $p$ and $q$ are connected by a homotopy of projectors $(p_t)_{t \in [0,1]}$ and such that for any positive real $\varepsilon$ and any $s$ and $t$ in $[0,1]$ with $|s - t| \leq \phi_1(\varepsilon)$, then $\|p_s - p_t\| \leq \varepsilon$.

* By considering for a projector $p$ in $A$ the homotopy of projectors
\[
\begin{pmatrix}
\cos^2 \pi t/2 \cdot p & \sin \pi t/2 \cdot \cos \pi t/2 \cdot p \\
\sin \pi t/2 \cdot \cos \pi t/2 \cdot p & \sin^2 \pi t/2 \cdot p + 1 - p
\end{pmatrix}_{t \in [0,1]}
\]
in $M_2(A)$, we also get that there exists a map $\phi_2 : [0, +\infty] \to [0, +\infty]$ such that for any $C^*$-algebra $A$ and any projector $p$ in $A$, then $\operatorname{diag}(1,0)$ and $\operatorname{diag}(p, 1 - p)$ are connected by a homotopy of projectors $(q_t)_{t \in [0,1]}$ such that for any positive real $\varepsilon$ and any $s$ and $t$ in $[0,1]$ with $|s - t| \leq \phi_2(\varepsilon)$, then $\|q_s - q_t\| \leq \varepsilon$. \hfill \square
• To prove the general case, let $p$ and $q$ be two homotopic projectors in a $C^*$-algebra $A$, and let $p = p_0, p_1, \cdots, p_m = q$ be $m + 1$ projectors in $A$ such that $\|p_{i+1} - p_i\| \leq \alpha$ for $i = 0, \cdots, m - 1$. Let us consider the following projectors in $M_{2m+1}(A)$:

$$
q_0 = \text{diag}(p_0, I_{2k-1}, 0) \\
q_1 = \text{diag}(p_0, 1, 0, \cdots, 1, 0) \\
q_2 = \text{diag}(p_0, 1 - p_1, p_1, \cdots, 1 - p_m, p_m) \\
q_3 = \text{diag}(p_0, 1 - p_0, p_1, \cdots, p_m - 1, 1 - p_m - 1, p_m) \\
q_4 = \text{diag}(1, 0, \cdots, 1, 0, p_m) \\
q_5 = \text{diag}(0, I_{2k-1}, p_m) \\
q_6 = \text{diag}(p_m, I_{2k-1}, 0)
$$

Since $\|q_3 - q_2\| \leq \alpha$, if we set $\phi = \min\{\phi_1, \phi_2\}$ and if we use the previous cases, we get for every $l$ in $\{0, 5\}$ homotopies $(q_i^H)_{t \in [l, l+1]}$ between $q_i$ and $q_i+1$ such that for any positive real $\varepsilon$ and any $s$ and $t$ in $[0, 1]$ with $|s - t| \leq \phi(\varepsilon)$, then $\|q_i^t - q_i^s\| \leq \varepsilon$. Hence, by considering the total homotopy, we get the result.

The proof for unitaries is similar.

\[\square\]

**Proposition 3.4.** Let $\Gamma$ be a discrete group. Let $A = (A_i)_{i \in \mathbb{N}}$ be a family of $C^*$-algebras equipped with an action of $\Gamma$ by automorphisms. Let us equip $A_i \otimes \mathcal{K}(H)$ with the diagonal action, the action of $\Gamma$ on $\mathcal{K}(H)$ being trivial and let us then consider the induced action on $\prod_{i \in I}(A_i \otimes \mathcal{K}(H))$. Let

$$
\Theta^{\Gamma, A}_*: KK_{\Gamma}(P_t(\Gamma), A_i) \rightarrow \prod_{i \in I} KK_{\Gamma}(P_r(\Gamma), A_i)
$$

be the homomorphism induced on the $k$-th factor by the projection

$$
\prod_{i \in I}(A_i \otimes \mathcal{K}(H)) \rightarrow A_k \otimes \mathcal{K}(H).
$$

Then $\Theta^{\Gamma, A}_*$ is an isomorphism.

**Proof.** Let us set $B_i = A_i \otimes \mathcal{K}(H)$ for $i$ in $I$. We can define an analogous morphism

$$
\Theta^{X}_*: KK^X(X, B_i) \rightarrow \prod_{i \in I} KK^X(X, B_i)
$$

for any locally compact space $X$ equipped with an action of $\Gamma$. Let us denote by $\Theta^{X, k}_*: KK^X(X, B_i) \rightarrow KK^X(X, B_k)$ the homomorphism induced by the projection on the $k$-th factor. Up to take a barycentric subdivision of $P_r(\Gamma)$, we can assume that $P_r(\Gamma)$ is a locally finite and finite dimension typed simplicial complex, equipped with a simplicial and type preserving action of $\Gamma$. Let $Z_0, \cdots, Z_n$ be the skeleton decomposition of $P_r(\Gamma)$. Then $Z_j$ is a simplicial complex of dimension $j$, locally finite and equipped with a proper, cocompact and type preserving simplicial action of $\Gamma$. Let us prove by induction on $i$ that $\Theta^{Z_i}_*$ is an isomorphism. The 0-skeleton $Z_0$ is a finite union of orbits and thus, for $j = 0$, it is enough to prove that $\Theta^{\Gamma, F}_*$ is an isomorphism when $F$ is a finite subgroup of $\Gamma$. Let us recall from [11] that for every $C^*$-algebra $B$ equipped with an action of $\Gamma$, there is a natural
restriction isomorphism \( \text{Res}^B_{F,T} : KK^t(\Gamma/F, B) \to KK^t(\mathbb{C}, B) \cong K_t(B \times F) \). We get by naturality the following commutative diagram

\[
\begin{array}{ccc}
KK^t(\Gamma/F, \Pi \in I B_i) & \xrightarrow{\Theta^t_{\Gamma/F}} & KK^t(\Gamma/F, B_k) \\
\text{Res}^{\Pi \in I B_i} & \downarrow & \text{Res}^{B_k} \\
K_t((\Pi \in I B_i) \times F) & \to & K_t(B_k \times F)
\end{array}
\]

where the bottom row is induced by the homomorphism \( \Pi \in I (B_i \times F) \to B_k \times F \) arising from the projection on the \( k \)-th factor \( \Pi \in I B_i \to B_k \). Since \( F \) is finite, \( \Pi \in I (B_i \times F) \) is naturally isomorphic to \((\Pi \in I B_i) \times F\), and under this identification, the bottom row homomorphism induces by Lemma 3.2 an isomorphism

\[
K_t((\Pi \in I B_i) \times F) \to \prod_{i \in I} K_t(B_i \times F).
\]

Hence \( \Theta^t_{\Gamma/F} \) is an isomorphism.

Let us assume that we have proved that \( \Theta^t_{\gamma^{-1}} \) is an isomorphism. Then the short exact sequence

\[
0 \to C_0(Z_j \setminus Z_{j-1}) \to C_0(Z_j) \to C_0(Z_{j-1}) \to 0
\]

gives rise to an natural long exact sequence

\[
\to KK^t(Z_{j-1}, \bullet) \to KK^t(Z_j, \bullet) \to KK^t(Z_j \setminus Z_{j-1}, \bullet) \to KK^t_{j+1}(Z_{j-1}, \bullet)
\]

and thus by naturallity, we get a diagram

\[
\begin{array}{cccc}
n_i \in I \quad KK^t(Z_j \setminus Z_{j-1}, B_i) & \xrightarrow{\Theta^t_{\gamma^{-1}}} & KK^t(Z_j, B_i) & \xrightarrow{\Theta^t_{\gamma^{-1}}} & KK^t_{j+1}(Z_{j-1}, B_i) \\
\| & \| & \| & \| & \|
\end{array}
\]

Let \( \hat{\sigma}_j \) be the interior of the standard \( j \)-simplex. Since the action of \( \Gamma \) is type preserving, then \( Z_j \setminus Z_{j-1} \) is equivariantly homeomorphic to \( \hat{\sigma}_j \times \Sigma_j \), where \( \Sigma_j \) is the set of center of \( j \)-simplices of \( Z_j \), and where \( \Gamma \) acts trivially on \( \hat{\sigma}_j \). This identification, together with Bott periodicity provides a commutative diagram

\[
\begin{array}{ccc}
KK^t(Z_j \setminus Z_{j-1}, \Pi \in I B_i) & \xrightarrow{\Theta^t_{\gamma^{-1}}} & KK^t_{j+1}(\Sigma_j, \Pi \in I B_i) \\
\| & \| & \\
\Pi \in I \quad KK^t(Z_j \setminus Z_{j-1}, B_i) & \xrightarrow{\Theta^t_{\gamma^{-1}}} & \Pi \in I \quad KK^t_{j+1}(\Sigma_j, B_i)
\end{array}
\]

By the first step of induction, \( \Theta^t_{\gamma} \) is an isomorphism, and hence \( \Theta^t_{\gamma^{-1}} \) is an isomorphism. Using the induction hypothesis and the five lemma, we get then that \( \Theta^t_{\gamma} \) is an isomorphism.

Let \( (A_i)_{i \in \mathbb{N}} \) be a family of \( \Gamma \)-algebras, let \( H \) be an Hilbert space and let \( x \) be an element of \( KK^t(P_r(\Gamma), \prod_{i \in \mathbb{N}} A_i \otimes K(H)) \) (the action of \( \Gamma \) on \( H \) being trivial) represented by a \( K \)-cycle \((\phi, E, T)\). Let \( p_k : \prod_{i \in \mathbb{N}} A_i \otimes K(H) \to A_k \otimes K(H) \) be the canonical projection on the \( k \)-th factor, and let us set \( E_k = E \otimes p_k A_k \otimes K(H) \), \( T_k = \)
Moreover, we can assume that the cocompact simplicial complex (equipped with a simplicial action of $\Gamma$) is a $\Gamma$-equivariant right $\prod_{i \in \mathbb{N}} A_\ast \otimes K(H)$-Hilbert module. Moreover, if $S$ is a compact operator on $E_i$, then for every $\varepsilon > 0$, there exists a finite rank operator $S'$ on $E_i$ such that $\|S - S'\| \leq \varepsilon$. Hence $(S_i)_{i \in \mathbb{N}}$ provides a finite rank operator on $\prod_{i \in \mathbb{N}} E_i$ such that $\|S_i - S_i'\| \leq \varepsilon$ for all integer $i$. Consequently, $(\langle \phi_i \rangle)_{i \in \mathbb{N}}, \prod_{i \in \mathbb{N}} E_i, (T_i)_{i \in \mathbb{N}}$ is a $K$-cycle for $KK^\ast(\Pi_k, (\prod_{i \in \mathbb{N}} A_i \otimes K(H)))$ which in view of the isomorphism of proposition [4] represents also $x$. Using the imprimitivity bimodule implementing the Morita equivalence between $A_i$ and $A_i \otimes K(H)$, we can actually replace $E_i$ by $K(A_i \otimes H, H_i)$ where $H_i = E_i \otimes_{A_i \otimes K(H)} A_i$. Hence we obtain that every element $x$ in $KK^\ast(\Pi_k, (\prod_{i \in \mathbb{N}} A_i \otimes K(H)))$ can be represented by a $K$-cycle $((\phi_i))_{i \in \mathbb{N}}, \prod_{i \in \mathbb{N}} K_A(A_i \otimes H, H_i), (T_i)_{i \in \mathbb{N}}$ such that for every integer $i$,

- $\mathcal{H}_i$ is a $\Gamma$-equivariant right $A_i$-Hilbert module;
- $\phi_i$ is a $\Gamma$-equivariant representation of $C_0(\Pi_k, (\Gamma))$ on $\mathcal{H}_i$;
- $T_i$ is a $\Gamma$-equivariant operator on $\mathcal{H}_i$;
- the action of $T_i$ and of $\phi_i(f)$ for $f$ in $C_0(\Pi_k, (\Gamma))$ on $K_A(A_i \otimes H, H_i)$ being by left composition.

Moreover, we can assume that $\|T_i\| \leq 1$ for all positive integer $i$.

As a consequence of proposition [4], we get

**Corollary 3.5.** If $\Gamma$ admits a universal example which is a finite dimension and cocompact simplicial complex (equipped with a simplicial action of $\Gamma$), then we have an isomorphism

$$KK^\ast(\Pi_k, (\prod_{i \in \mathbb{N}} A_i \otimes K(H))) \rightarrow \prod_{i \in \mathbb{N}} K^\ast(\Gamma, A_i \otimes K(H)) \cong \prod_{i \in \mathbb{N}} K^\ast(\Gamma, A_i)$$

induced on the $k$-th factor by the projection

$$\Pi_k (\prod_{i \in \mathbb{N}} A_i \otimes K(H)) \rightarrow (\prod_{i \in \mathbb{N}} A_i \otimes K(H)).$$

**3.4. The case of coverings.** Recall from [10] that for a cocompact covering $\tilde{X} \rightarrow X$ of group $\Gamma$, we have a natural isomorphism

$$T^\Gamma_{\tilde{X} \rightarrow X} : KK^\ast(\tilde{X}, \mathbb{C}) \rightarrow K^\ast(X)$$

which can be described as follows. Let $(\rho, H, T)$ be a $K$-cycle for $KK^\ast(\tilde{X}, \mathbb{C})$. We can assume without loss of generality that the representation $\rho : C_0(\tilde{X}) \rightarrow L(H)$ is non-degenerated. We can also assume that the operator $T$ of $L(H)$ is $\Gamma$-equivariant and that

$$T \cdot C^\ast(X) \cdot H \subset C^\ast(X) \cdot H.$$

If $\langle \cdot, \cdot \rangle$ is the scalar product on the Hilbert space $H$, then we can define on $C^\ast(X) \cdot H$ the inner product

$$\langle \xi, \eta \rangle = \sum_{\gamma \in \Gamma} \langle \xi, \gamma(\eta) \rangle.$$

Then $\langle \langle \cdot, \cdot \rangle \rangle$ is positive and thus by taking the separated completion of $C^\ast(X) \cdot H$, we get a Hilbert space $\hat{H}$. The operator $T$ being equivariant, its restriction to $C^\ast(X) \cdot H$ extends to a continuous operator $\hat{T}$ on $\hat{H}$. Since $\rho$ is non-degenerated, it extends to a representation of $C(X)$ (viewed as an algebra of multiplier for $C_0(\tilde{X})$) on $H$ by equivariant operator. Moreover, since this representation preserves...
Thus we get
\[ \tilde{\rho}(\tilde{\mathcal{K}}) \cdot H, \]
induces a representation \( \tilde{\rho} \) of \( C(X) \) on \( \tilde{H} \). It is straightforward to check that \( (\tilde{\rho}, \tilde{H}, \tilde{T}) \) is a \( \mathcal{K} \)-cycle for \( \mathcal{K}(X) \) and we get this way a homomorphism
\[ \Upsilon_{\mathcal{T}, X}^\Gamma : \mathcal{K} \mathcal{K}(C_b(\tilde{X}), \mathbb{C}) \longrightarrow \mathcal{K}(X) \]
which maps the class of \( (\rho, H, T) \) to the class of \( (\tilde{\rho}, \tilde{H}, \tilde{T}) \).

**Theorem 3.6.** \( \Upsilon_{\mathcal{T}, X}^\Gamma \) is a isomorphism.

We want now to study how the propagation behave under the above transformation. So assume that \( \tilde{X} \) is a locally compact metric space equipped with a free, proper, isometric and cocompact action of \( \Gamma \). Let \( \eta \) be a \( \Gamma \)-invariant measure, and let \( \hat{\eta} \) be the measure induced on \( \tilde{X} = \tilde{X}/\Gamma \). Let us set \( \tilde{H}_{\tilde{X}} = L^2(\hat{\eta}) \otimes H \) and \( H_X = L^2(\eta) \otimes H \). We can view \( C(X) \) as the algebra of \( \Gamma \)-invariant continuous and bounded functions on \( \tilde{X} \) and according to this, for any continuous and compactly supported function \( f : \tilde{X} \rightarrow \mathbb{C} \), then \( \hat{f} = \sum_{\gamma \in \Gamma} \gamma(f) \) belongs to \( H_{\tilde{X}} \). It is straightforward to check that \( f \mapsto \hat{f} \) extends to a unitary map \( \hat{H}_{\tilde{X}} \rightarrow H_X \).

**Lemma 3.7.** If \( T \) is a locally compact equivariant operator on \( H_{\tilde{X}} \) with propagation less than \( r \). Then, under the above identification between \( H_{\tilde{X}} \) and \( H_X \), the operator \( \hat{T} \) is a compact operator with propagation less than \( r \).

**Proof.** Since \( T \) is equivariant and since \( \tilde{X} \) is cocompact, the operator \( T \) is given by a kernel \( K : \tilde{X} \times \tilde{X} \rightarrow \mathcal{K}(H) \) such that \( K(\gamma x, \gamma y) = K(x, y) \) for almost all \((x, y)\) in \( \tilde{X} \times \tilde{X}\) and with cocompact support (for the diagonal action of \( \Gamma \) on \( \tilde{X} \times \tilde{X} \)) of diameter less than \( r \). Under the above identification between \( \hat{H}_{\tilde{X}} \) and \( H_X \), then for any continuous and compactly supported function \( f : \tilde{X} \rightarrow \mathbb{C} \), we have
\[ \hat{T} \cdot \hat{f} = \sum_{\gamma \in \Gamma} \gamma(T \cdot f) = \sum_{\gamma \in \Gamma} T \cdot \gamma(f). \]

By viewing \( X \) as a borelian fundamental domain for the action of \( \Gamma \) on \( \tilde{X} \), we get
\[ \hat{T} \cdot \hat{f}(x) = \sum_{\gamma \in \Gamma} \int_{\tilde{X}} K(x, y)f(\gamma y)d\hat{\eta}(y) \]
\[ = \sum_{(\gamma, \gamma') \in \Gamma^2} \int_{\tilde{X}} K(x, \gamma' y)f(\gamma y)d\hat{\eta}(y) \]
\[ = \sum_{(\gamma, \gamma') \in \Gamma^2} \int_{X} K(\gamma^{-1}x, y)f(\gamma y)d\eta(y) \]
\[ = \int_{X} \sum_{\gamma' \in \Gamma} K(\gamma^{-1}x, \gamma' y)d\eta(y) \]
\[ = \int_{X} F(x, y)\hat{f}(y)d\eta(y), \]
with
\[ F : \tilde{X} \times \tilde{X} ; (x, y) \mapsto \sum_{\gamma \in \Gamma} K(\gamma x, y). \]

The kernel \( F \) is \( \Gamma \times \Gamma \)-invariant and thus can be viewed as a kernel on \( X \times X \) and thus we get \( \hat{T} \cdot \hat{f}(x) = \int_{X} F(x, y)\hat{f}(y)d\eta(y) \). Hence \( \hat{T} \) is a compact operator and
since \( F(x, y) = 0 \) for almost every \((x, y)\) in \( X \times X \) such that \( d(x, y) \geq r \), we see that \( \hat{T} \) as propagation less than \( r \). □

The previous lemma can be extended to pseudo-local operators on \( H^\infty_X \) with finite propagation. Recall from [3] that if \( T \) is a pseudo-local equivariant operator on \( H^\infty_X \) with finite propagation, then \( \hat{T} \) is a pseudo-local operator on \( \hat{H}^\infty_X \cong H_X \)

**Lemma 3.8.** With notation of lemma 3.7, if \( T \) is pseudo-local \( \Gamma \)-equivariant operator on \( H^\infty_X \) with propagation less than \( r \). Then, the operator \( \hat{T} \) is a pseudo-local operator with propagation less than \( r \)

**Proof.** Let \( f_1, \ldots, f_n \) be a partition of unit for \( X \) with support of diameter less than \( r \). Let us set \( T' = \sum_{i=1}^n f_i^{1/2} \circ q \cdot \hat{T} \cdot f_i^{1/2} \circ q \), where \( q : \hat{X} \to X \) is the projection map of the covering. Then \( T' - T \) is an equivariant locally compact on \( H^\infty_X \) with propagation less than \( r \) and thus, according to lemma 3.7, we get that \( \hat{T} - \hat{T}' \) is compact and has propagation less than \( r \). Since \( \hat{T}' = \sum_{i=1}^n f_i^{1/2} \cdot \hat{T} \cdot f_i^{1/2} \), then \( T' \) is a pseudo-local operator of propagation less than \( r \) and hence we get the result. □

4. **The left hand side for the coarse space associated to a residually finite group**

The aim of this section is to state for the sources of the assembly maps the analogous of proposition 2.8, i.e the existence of a group homomorphism

\[
\Psi_{X(\Gamma),*} : \lim_r KK^\Gamma(P_r(X(\Gamma)), \mathbb{C}) \to K^\top(\Gamma, A_r),
\]

such that

\[
(4.1) \quad \Psi_{\Gamma, A_r,*} \circ \mu_{X(\Gamma), \text{max,*}} = \mu_{\Gamma, A_r, \text{max,*}} \circ \Psi_{X(\Gamma),*}.
\]

4.1. **Rips complexes associated to a residually finite group.** Let \( \Gamma \) be a residually finite group, finitely generated. Let \( \Gamma_0 \supset \Gamma_1 \supset \ldots \supset \Gamma_n \supset \ldots \) be a decreasing sequence of normal finite index subgroups of \( \Gamma \) such that \( \bigcap_{i \in \mathbb{N}} \Gamma_i = \{e\} \). Recall notations of section 2.2, if \( d \) be a left invariant metric associated to any finite set of generators for \( \Gamma_i \), then we endow \( \Gamma_i \) with the metric \( d(a\Gamma_i, b\Gamma_i) = \min\{d(a\gamma_1, b\gamma_2), \gamma_1 \text{ and } \gamma_2 \text{ in } \Gamma_i\} \). We set \( X(\Gamma) = \prod_{i \in \mathbb{N}} \Gamma_i \) and we equip \( X(\Gamma) \) with a metric \( d \) such that on \( \Gamma_i \), then \( d \) is the metric defined above and \( d(\Gamma_i \Gamma_j, \Gamma_i \Gamma_j) \geq i + j \) if \( i \neq j \).

For every integer \( n \) such that \( r > n \), then

\[
P_r(X(\Gamma)) = P_r(\prod_{i=1}^{n-1} \Gamma_i / \Gamma_i) \bigcap \left( \prod_{i \geq n} P_r(\Gamma_i / \Gamma_i) \right),
\]

where \( P_r(\prod_{i=1}^{n-1} \Gamma_i / \Gamma_i) \) and \( \prod_{i \geq n} P_r(\Gamma_i / \Gamma_i) \) can be viewed as distinct open subsets of \( P_r(X(\Gamma)) \). Hence we have a splitting

\[
K_*(P_r(X(\Gamma))) \cong K_*(P_r(\prod_{i=1}^{n-1} \Gamma_i / \Gamma_i)) \bigoplus K_*(\prod_{i \geq n} P_r(\Gamma_i / \Gamma_i))
\]

(4.2)

\[
\cong K_*(P_r(\prod_{i=1}^{n-1} \Gamma_i / \Gamma_i)) \bigoplus K_*(P_r(\Gamma_i / \Gamma_i))
\]
Lemma 4.1. \( K \) that the support of \( K \) be the homomorphism induced on the \( W \) have 

\[ f : \Gamma \Gamma_i \to [0,1]; \ \gamma \Gamma_i \mapsto \sum_{g \in \Gamma_i} f(\gamma g). \]

Then \( \tilde{f} \) is a probability on \( \Gamma \Gamma_i \). Let \( \gamma \) and \( \gamma' \) be elements of \( \Gamma \) such that \( \tilde{f}(\gamma \Gamma_i) \neq 0 \) and \( \tilde{f}(\gamma \Gamma_i) \neq 0 \). Then there exists \( g \) and \( g' \) in \( \Gamma_i \) such that \( f(\gamma g) \neq 0 \) and \( f(\gamma' g) \neq 0 \). Since \( f \) is in \( P_r(\Gamma) \), we get that \( d(\gamma g, \gamma' g) \leq \gamma r \) and hence \( d(\gamma \Gamma_i, \gamma' \Gamma_i) \leq \gamma r \). Thus \( \tilde{f} \) belongs to \( P_r(\Gamma_i) \), and since \( \gamma \tilde{f} = \tilde{f} \) for any \( \gamma \) in \( \Gamma_i \), we finally obtain a continuous map \( \nu_{r,i} : P_r(\Gamma_i) \to P_r(\Gamma / \Gamma_i); \tilde{f} \mapsto \tilde{f} \), where \( \tilde{f} \) is the class in \( P_r(\Gamma) / \Gamma_i \) of \( f \) in \( P_r(\Gamma) \). For a positive real \( r \) and an integer \( n \), let 

\[ A_{*,r,n} : \prod_{k \geq n} K_r(\Gamma) \to P_r(\Gamma / \Gamma_k) \] 

be the homomorphism induced on the \( k \)-th factor by the map 

\[ P_r(\Gamma / \Gamma_k) \to P_r(\Gamma / \Gamma_k); \tilde{f} \mapsto \tilde{f}. \]

Lemma 4.1. Let \( i \) be an integer such that \( B_{\Gamma}(e, 2r) \cap \Gamma_i = \{ e \} \). Let \( \{ \gamma_1, \ldots, \gamma_n \} \) and \( \{ \gamma'_1, \ldots, \gamma'_n \} \) be subsets of \( \Gamma \) of diameter less than \( r \) and such that \( \gamma_j \gamma'_j^{-1} \) is in \( \Gamma_i \) for all \( j \) in \( \{ 1, \ldots, n \} \), then \( \gamma_j \gamma'_j^{-1} = \gamma_k \gamma'_k^{-1} \) for all \( j \) and \( k \) in \( \{ 1, \ldots, n \} \).

Proof. We have \( d(\gamma_1, \gamma_j) \leq \gamma r \) and \( d(\gamma'_1, \gamma'_j) \leq \gamma r \) for all \( j \) in \( \{ 1, \ldots, n \} \). Let us set \( g = \gamma_1 \gamma'_1^{-1} \). Then 

\[ d(\gamma_j, g \gamma_j') \leq d(\gamma_j, \gamma_1) + d(\gamma_1, g \gamma'_1) \leq d(\gamma_j, \gamma_1) + d(g \gamma'_1, g \gamma'_j) \leq d(\gamma_j, \gamma_1) + d(g \gamma'_1, g \gamma'_j) \leq 2r. \]

Hence, since \( \Gamma_i \) is normal, \( \gamma_j^{-1} g \gamma_j' = (\gamma_j^{-1} g \gamma_j) \gamma_j^{-1} (\gamma'_j \gamma_j^{-1}) \gamma_j \) belongs to \( B_{\Gamma}(e, 2r) \cap \Gamma_i \) and thus \( \gamma_j = g \gamma'_j \). \( \square \)

Let \( i \) be an integer such that \( B_{\Gamma}(e, 4r) \cap \Gamma_i = \{ e \} \). Let \( h \) be an element of \( P_r(\Gamma / \Gamma_i) \). We can choose a finite subset \( \{ \gamma_1, \ldots, \gamma_n \} \) of diameter less than \( 2r \) such that the support of \( h \) lies in \( \{ \gamma_1 \Gamma_i, \ldots, \gamma_n \Gamma_i \} \). According to lemma 4.1 a such subset is unique up to left translations by an element of \( \Gamma_i \). Let us define \( h \) in \( P_{2r}(\Gamma / \Gamma_i) \) as the class of the probability of \( P_r(\Gamma) \) with support in \( \{ \gamma_1, \ldots, \gamma_n \} \) with value on an element \( \gamma \) in that set \( \tilde{h}(\gamma) = h(\gamma \Gamma_i) \). It is straightforward to check that if \( h \) is in \( P_r(\Gamma / \Gamma_i) \), then \( \tilde{h} \) is the image of \( h \) under the inclusion map \( P_r(\Gamma / \Gamma_i) \to P_{2r}(\Gamma / \Gamma_i) \). If \( f \) is an element of \( P_r(\Gamma) \), then since \( B_{\Gamma}(e, r) \cap \Gamma_i = \{ e \} \), the intersection of the support of \( f \) with any \( \gamma \Gamma_i \) for \( \gamma \) in \( \Gamma \) has at most one element. Hence, according to lemma 4.1, \( f \) is the image of the class of \( f \) in \( P_r(\Gamma / \Gamma_i) \) under the inclusion \( P_r(\Gamma / \Gamma_i) \to P_{2r}(\Gamma / \Gamma_i) \).

Lemma 4.2. Let \( r \) be a positive real and let \( i \) be an integer such that \( B_{\Gamma}(e, r) \cap \Gamma_i = \{ e \} \). Then the action of \( \Gamma_i \) on \( P(\Gamma) \) is free.
Proof. Let \( f \) be an element of \( P_r(\Gamma) \). If \( \gamma \cdot f = f \) with \( \gamma \in \Gamma_i \), then the support of \( f \) is invariant under the action of \( \gamma \). In particular, \( \gamma = g_1 \cdot g_2^{-1} \) with \( g_1 \) and \( g_2 \) in the support of \( f \). Hence \( \gamma \) is in \( B_r(e,r) \cap \Gamma_i \) and thus \( \gamma = e \). \( \square \)

In consequence, with condition of the lemma above, \( P_r(\Gamma) \to P_r(\Gamma)/\Gamma_i \) is a covering map and since \( \Gamma_i \) has finite index in \( \Gamma \), this covering is cocompact.

4.2. Construction of \( \Psi_{X(\Gamma)} \). For a positive real \( r \) and an integer \( n \), such that \( r \geq n \) and \( B_r(e,4r) \cap \Gamma_n = \{ e \} \) let us define

- \( \Psi^{1}_{*,r,n} : K_*(P_r(X(\Gamma))) \to \prod_{i \geq n} K_*(P_r(\Gamma/\Gamma_i)) \) the projection homomorphism corresponding to the decomposition in equation (4.2) of section 4.1.

- \( \Psi^{2}_{*,r,n} : \prod_{k \geq n} K_*(P_r(\Gamma/\Gamma_k)) \to \prod_{k \geq n} K_*(P_{2r}(\Gamma/\Gamma_k)) \) the homomorphism induced on the \( k \)-th factor by the maps \( P_r(\Gamma/\Gamma_k) \to P_{2r}(\Gamma/\Gamma_k) \); \( h \mapsto h \).

- \( \Psi^{3}_{*,r,n} : \prod_{k \geq n} K_*(P_r(\Gamma/\Gamma_k)) \to \prod_{k \geq n} K_*(P_r(\Gamma,\mathbb{C})) \) the homomorphism given on the \( k \)-th factor by the inverse of the isomorphism \( \Psi_{P_{2r}(\Gamma,i)} \) (see section 3.4).

- \( \Psi^{4}_{*,r,n} : \prod_{k \geq n} K_*(P_r(\Gamma,\mathbb{C})) \to \prod_{k \geq n} K_*(P_r(\Gamma,\mathbb{C})) \) the homomorphism given on the \( k \)-th factor by the induction homomorphism \( \Gamma_{i} \to \Gamma_{i} \) (see section 3.4).

- \( \Psi^{5}_{*,r,n} : \prod_{k \geq n} K_*(P_r(\Gamma,\mathbb{C})) \to \prod_{k \geq n} K_*(P_r(\Gamma,\mathbb{C})) \) the inverse of the isomorphism \( \Theta_{\Gamma,i}^{-1} \) of proposition 3.4 applied to the family \( \mathcal{A} = (C(\Gamma/\Gamma_i))_{i \in \mathbb{N}} \).

- \( \Psi^{6}_{*,r,n} : K_*(P_r(\Gamma,\mathbb{C})) \to K_*(P_r(\Gamma,\mathbb{C})) \) the homomorphism induced by the \( \Gamma \)-equivariant epimorphism

\[
\ell^\infty(\Upsilon_{\Gamma,i}, K(\mathbb{H}))) \to \mathcal{A}_r.
\]

Remark 4.3. (i) Let us also define for any real \( r \) and \( r' \) such that \( 0 \leq r \leq r' \) the homomorphisms

\[
k_{*,r,r'}^{k \geq n} : \prod_{k \geq n} K_*(P_r(\Gamma/\Gamma_k)) \to \prod_{k \geq n} K_*(P_{r'}(\Gamma/\Gamma_k))
\]

and

\[
l_{*,r,r'}^{k \geq n} : \prod_{k \geq n} K_*(P_r(\Gamma/\Gamma_k)) \to \prod_{k \geq n} K_*(P_{r'}(\Gamma/\Gamma_k))
\]

respectively induced on the \( k \)-th factor by the inclusions \( P_r(\Gamma/\Gamma_k) \to P_{r'}(\Gamma/\Gamma_k) \) and \( P_{r'}(\Gamma/\Gamma_k) \to P_{r'}(\Gamma/\Gamma_k) \). According to the discussion that follows lemma 4.3 and with notations of section 4.4, if \( n \) is chosen such that \( n \leq r \) and \( B_r(e,4n) \cap \Gamma_n = \{ e \} \), then

\[
\Psi^2_{*,r,n} \circ \Lambda_{*,r,n} = l_{*,r,2r}^{k \geq n}
\]

and

\[
\Lambda_{*,2r,n} \circ \Psi^2_{*,r,n} = l_{*,r,2r}^{k \geq n}.
\]
According to remark 4.3 and to proposition 3.4, we have a splitting and an isomorphism
\[ \bigoplus_{i \geq n} KK^T_i(P_r(\Gamma), C(\Gamma/\Gamma_i)) \xrightarrow{=} KK^T_n(P_r(\Gamma), C_0(\sqcup_{i \geq n} \Gamma/\Gamma_i, \mathcal{K}(H))). \]
According to the next lemma, \( \Psi_{\ast r,n}^6 \) is an epimorphism.

**Proposition 4.4.** The equivariant short exact sequence
\[
0 \longrightarrow C_0(\sqcup_{i \geq n} \Gamma/\Gamma_i, \mathcal{K}(H)) \longrightarrow \ell^\infty(\sqcup_{i \geq n} \Gamma/\Gamma_i, \mathcal{K}(H)) \longrightarrow \Lambda \longrightarrow 0
\]
gives rise to a short exact sequence
\[
0 \longrightarrow KK^T_n(P_r(\Gamma), C_0(\sqcup_{i \geq n} \Gamma/\Gamma_i, \mathcal{K}(H))) \longrightarrow KK^T(n, \ell^\infty(\sqcup_{i \geq n} \Gamma/\Gamma_i, \mathcal{K}(H))) \longrightarrow KK^T(P_r(\Gamma), A_{\Gamma}) \longrightarrow 0.
\]

**Proof.** Using the six-term exact sequence associated to an equivariant short exact sequence of \( C^* \)-algebras, this amounts to show that the inclusion
\[ \iota : C_0(\sqcup_{i \geq n} \Gamma/\Gamma_i, \mathcal{K}(H)) \hookrightarrow \ell^\infty(\sqcup_{i \geq n} \Gamma/\Gamma_i, \mathcal{K}(H)) \]
induces a monomorphism
\[ \iota_* : KK^T_n(P_r(\Gamma), C_0(\sqcup_{i \geq n} \Gamma/\Gamma_i, \mathcal{K}(H))) \hookrightarrow KK^T_n(P_r(\Gamma), \ell^\infty(\sqcup_{i \geq n} \Gamma/\Gamma_i, \mathcal{K}(H))). \]
According to remark 4.3 and to proposition 3.4, we have a splitting
\[ KK^T_n(P_r(\Gamma), C_0(\sqcup_{i \geq n} \Gamma/\Gamma_i, \mathcal{K}(H))) \cong \bigoplus_{i \geq n} KK^T_n(P_r(\Gamma), C(\Gamma/\Gamma_i)) \]
and an isomorphism
\[ KK^T_n(P_r(\Gamma), \ell^\infty(\sqcup_{i \geq n} \Gamma/\Gamma_i, \mathcal{K}(H))) \cong \prod_{i \geq n} KK^T_n(P_r(\Gamma), C(\Gamma/\Gamma_i)). \]
Up to these identifications, \( \iota_* \) is the inclusion
\[ \bigoplus_{i \geq n} KK^T_n(P_r(\Gamma), C(\Gamma/\Gamma_i)) \hookrightarrow \prod_{i \geq n} KK^T_n(P_r(\Gamma), C(\Gamma/\Gamma_i)). \]

**Remark 4.5.** Taking the inductive limit over all the \( P_r(\Gamma) \), we get a short exact sequence
\[
0 \longrightarrow KK^T_{\ast r}(\Gamma, C_0(\sqcup_{i \geq n} \Gamma/\Gamma_i, \mathcal{K}(H))) \longrightarrow KK^T_{\ast r}(\Gamma, \ell^\infty(\sqcup_{i \geq n} \Gamma/\Gamma_i, \mathcal{K}(H))) \longrightarrow KK^T_{\ast r}(\Gamma, A_{\Gamma}) \longrightarrow 0
\]
In the same way, since the composition
\[ \bigoplus_{i \geq n} K_* (C(\Gamma/\Gamma_i, \mathcal{K}(H)) \times_{\max \Gamma} \Gamma) \rightarrow \bigoplus_{i \geq n} K_* (C(\Gamma/\Gamma_i, \mathcal{K}(H)) \times_{\max \Gamma} \Gamma) \rightarrow \prod_{i \geq n} K_* (C(\Gamma/\Gamma_i, \mathcal{K}(H)) \times_{\max \Gamma} \Gamma) \]
is injective, where the second map is induced on the \( k \)-th factor by the projection \( \prod_{i \geq n} C(\Gamma/\Gamma_i, \mathcal{K}(H)) \rightarrow C(\Gamma/\Gamma_{k+1}, \mathcal{K}(H)) \), the exact sequence for maximal cross product
\[
0 \longrightarrow C_0(\sqcup_{i \geq n} \Gamma/\Gamma_i, \mathcal{K}(H)) \times_{\max \Gamma} \rightarrow \ell^\infty(\sqcup_{i \geq n} \Gamma/\Gamma_i, \mathcal{K}(H)) \times_{\max \Gamma} \rightarrow A_{\Gamma} \times_{\max \Gamma} \rightarrow 0
\]
Lemma 4.7. The inclusion induces a homomorphism

\[ 0 \rightarrow \oplus_{i \geq n} K_\ast(C_0(\Gamma_i, K(H))) \rightarrow K_\ast(\mathbb{Z} \rightarrow \Gamma_i, K(H))) \rightarrow K_\ast(\mathbb{Z} \rightarrow \Gamma_i, K(H)) \rightarrow 0 \]

and moreover, the assembly maps intertwine the corresponding above exact sequences.

Let \( r \) be a positive real and let \( n \) be an integer such that \( n \leq r \) and \( B_r(e, 4r) \cap \Gamma_n = \{e\} \). Let us define

\[ \Psi_{\ast, r, n} : KK_\ast(P_r(X(\Gamma)), \mathbb{C}) \rightarrow KK_\ast(P_2r(\Gamma), A_\Gamma) \]

by

\[ \Psi_{\ast, r, n} = \Psi_{\ast, 2r, n}^6 \circ \Psi_{\ast, 2r, n}^4 \circ \Psi_{\ast, 2r, n}^3 \circ \Psi_{\ast, r, n}^2 \circ \Psi_{\ast, r, n}^1 \].

Notice that \( \Psi_{\ast, r, n} \) does not depend on the choice of the integer \( n \) such that \( n \leq r \) and \( B_r(e, 4r) \cap \Gamma_n = \{e\} \).

For any positive real \( r \) and \( r' \) such that \( r \leq r' \), let

\[ \Gamma_{\ast, r, n} : KK_\ast(P_r(\Gamma), A_\Gamma) \rightarrow KK_\ast(P_{2r}(\Gamma), A_\Gamma) \]

be the homomorphism induced by the inclusion \( P_r(\Gamma) \subset P_{2r}(\Gamma) \).

Lemma 4.6. For every element \( y \) in \( KK_\ast(P_r(\Gamma), A_\Gamma) \), there exists an element \( x \) in \( KK_\ast(P_r(X(\Gamma)), \mathbb{C}) \) such that \( \Psi_{\ast, r, n}(x) = \Gamma_{\ast, r, n}(y) \).

Proof. According to proposition 4.4, the homomorphism \( \Psi_{\ast, 2r, n}^6 \) is onto. Since \( \Psi_{\ast, 2r, n}^6, \Psi_{\ast, 2r, n}^4 \) and \( \Psi_{\ast, 2r, n}^3 \) are isomorphisms, there exists a \( z \) in \( \prod_{i \geq n} K_\ast(P_i(\Gamma)/\Gamma_i) \) such that \( y = \Psi_{\ast, r, n}^6 \circ \Psi_{\ast, r, n}^4 \circ \Psi_{\ast, r, n}^3(z) \). Using equation 4.3, we have

\[ \Psi_{\ast, r, n}^2 \circ \Lambda_{\ast, r, n}(z) = \Gamma_{\ast, r, n}(z) \]

and since \( \Psi_{\ast, r, n}^1 \) is onto, there exists an element \( x \) in \( K_\ast(P_r(X(\Gamma))) \) such that \( \Lambda_{\ast, r, n}(z) = \Psi_{\ast, r, n}^1(x) \). The lemma is then a consequence of the equality

\[ \psi_{\ast, 2r, n}^6 \circ \psi_{\ast, 2r, n}^4 \circ \psi_{\ast, 2r, n}^3 \circ \psi_{\ast, r, n}^1 = \Gamma_{\ast, r, n} \circ \psi_{\ast, 2r, n}^6 \circ \psi_{\ast, 2r, n}^4 \circ \psi_{\ast, 2r, n}^3 \]

\[ \square \]

Let us denote for a pair of real \( r \) and \( r' \) such that \( 0 \leq r \leq r' \) by \( X^{(r)} : KK_\ast(P_r(X(\Gamma))) \rightarrow KK_\ast(P_{2r}(X(\Gamma))) \) the morphism induced by the inclusion \( P_r(X(\Gamma)) \subset P_{2r}(X(\Gamma)) \). The class \( \Gamma_0 \) of \( \Gamma_0 \) of \( \Gamma_0 \) can be viewed as an element of \( P_r(X(\Gamma)) \) and this inclusion induces a homomorphism

\[ \kappa_{\ast, r} : Z \cong KK_\ast(\{[\Gamma_0]\}) \rightarrow KK_\ast(P_r(X(\Gamma))). \]

Lemma 4.7. Let \( x \) be an element of \( K_\ast(P_r(X(\Gamma))) \) such that \( \Psi_{\ast, r}(x) = 0 \), then there exists a real \( r' \) such that \( r \leq r' \) and \( \Psi_{\ast, r'}(x) \) is in the range of \( \kappa_{\ast, r'} \).

Proof. Let us fix an integer \( n \) such that \( n \geq r \) and \( B_r(e, 4r) \cap \Gamma_n = \{e\} \). According to proposition 4.3, we get that

\[ \psi_{\ast, r, n}^4 \circ \psi_{\ast, r, n}^3 \circ \psi_{\ast, r, n}^2 \circ \psi_{\ast, r, n}^1(x) \in \bigoplus_{k \geq n} KK_\ast(P_{2r}(\Gamma), C(\Gamma/\Gamma_k)). \]
Since $Ψ^4_{2r,n}$ and $Ψ^3_{2r,n}$ restricts on direct summands to isomorphisms, then we get that $Ψ^4_{2r,n} \circ Ψ^1_{2r,n}(x)$ lies in $\bigoplus_{k \geq n} KK_* (P_{2r}(Γ)/T_k, C)$. According to equation 4.4,

$$k \geq n, \quad ι^1_{*, r, 2r} \circ Ψ^1_{*, r, n}(x) = Ψ^1_{*, 2r, n} \circ ι^4_{*, r, 2r}(x) \in \bigoplus_{k \geq n} KK_* (P_{2r}(Γ)/T_k, C).$$

But then $ι^X_{*, r, 2r}(x)$ lies in a finite sum of summands of $\bigoplus_{k \geq n} KK_* (P_{2r}(Γ)/T_k, C)$ and thus we get that for some integer $m$ and some real $s$ with $s \geq 2r$ and $m \geq \sup\{n, s\}$, then $ι^X_{*, r, s}(x)$ belongs to $KK_* (P_{s} (\sqcup_{k=0}^{m} Γ/k), C)$. But since $\prod_{0 \leq k \leq m} Γ/k$ is finite, $P_{s} (\sqcup_{k=0}^{m} Γ/k)$ is compact and up to choose a bigger $s$ is also convex. Hence, $ι^X_{*, r, r'}(x)$ lies in the range of $κ_{*, r'}$ for $r'$ big enough. □

It is straightforward to check that $Ψ^X_{*, r, r'} \circ ι^X_{*, r, r'} = ι^X_{*, r, r'} \circ Ψ^X_{*, r}$ and thus the family of homomorphism $(Ψ^X_{*, r})_{r \leq 0}$ gives rise to a homomorphism

$$Ψ^X_{(Γ),*} : \lim_{r \to r_0} KK_* (P_r(Γ), C) \longrightarrow K^*_a (Γ, AΓ).$$

Let $x_0$ be the image of (any) $κ_{*, r}(1)$ in $\lim_{r} KK_* (P_r(Γ), C)$. As a consequence of lemmas 4.4 and 4.5, we get

\textbf{Theorem 4.8.}

- $Ψ^X_{(Γ),*}$ is onto;
- In odd degree, $Ψ^X_{(Γ),*}$ is an isomorphism;
- In even degree, $\ker Ψ^X_{(Γ),*}$ is the infinite cyclic group generated by $x_0$;

\subsection*{Compatibility of $Ψ^X_{(Γ),*}$ with the assembly maps.}

The proof of equation 4.3 require some preliminary work. For a locally compact and proper $Γ$-space $X$, the notion of standard-$X$-module, was extended to the equivariant case in [15] as follows.

\textbf{Definition 4.9.} Let $X$ be a locally compact and proper $Γ$-space, let $H$ be a $Γ$-Hilbert space. A non-degenerated $Γ$-equivariant representation $ρ : C_0(X) \to \mathcal{L}(H)$ is called $X$-$Γ$-ample if when extended to $C_0(X) \times Γ$, then $ρ(C_0(X) \times Γ) \cap \mathcal{K}(H) = \{0\}$.

\textbf{Example 4.10.} If $η$ is a $Γ$-invariant measure on $P_τ(Γ)$ fully supported i.e with support $P_τ(Γ)$ and if $H$ is a separable Hilbert space, then $L^2(η) \otimes H$ equipped with the diagonal action of $Γ$, trivial on $H$ together with the representation

$$ρ : C_0(P_τ(Γ)) \to \mathcal{L}(L^2(η) \otimes H); \quad f \mapsto f \otimes 1_H$$

is an $X$-$Γ$-ample representation. The reason is that $P_τ(Γ)$ contains as a $Γ$-space a copy of $Γ \times Y$, where $Y$ is an open subset of $P_τ(Γ)$, and where $Γ$ acts diagonally, by left translations on $Γ$ and trivially on $Y$.

\textbf{Lemma 4.11.} Let $X$ be locally compact and proper $Γ$-space, let $H_0$ and $H_1$ be two $Γ$-Hilbert spaces and let $ρ_i : C_0(X) \to \mathcal{L}(H_i)$ for $i = 0, 1$ be two non-degenerated and $Γ$-equivariant representations. Assume that $ρ_0$ is $X$-$Γ$-ample. Then there exists

- $H_2$ a $Γ$-Hilbert space;
- $ρ_2 : C_0(X) \to \mathcal{L}(H_2)$ a non-degenerated $Γ$-equivariant representation;
- $U : H_1 \oplus H_2 \to H_0$ a unitary

such that for every $f$ in $C_0(X)$,

$$U \cdot (ρ_1 \oplus ρ_2)(f) - ρ_0(f) \cdot U \in \mathcal{K}(H_1 \oplus H_2, H_0).$$
Proof. Up to replace \( \rho_1 \) by \( \rho_0 \oplus \rho_1 \), we can assume without loss of generality that \( \rho_1 \) is also \( X\Gamma \)-ample. Then, according to [13, Theorem 3.4.6], there exists a \( \Gamma \)-equivariant isometry \( W : H_1 \to H_0 \) such that \( W \cdot \rho_1(f) = \rho_0(f) \cdot W \) is in \( \mathcal{K}(H_1, H_0) \) for every \( f \in C_0(X) \). Let us set \( P = \text{Id}_{H_0} - W \cdot W^* \). Now, by using the completely positive map

\[
C_0(X) \to \mathcal{L}(P \cdot H_0); \quad f \mapsto P \cdot \rho_0(f) \cdot P,
\]

we can use the proof of [8, Theorem 3.4.6] to conclude. \( \square \)

From this and by using next lemma, we can prove that if \( X \) is a locally compact and proper \( \Gamma \)-space, then every element in \( KK^\Gamma_1(X, \mathbb{C}) \) can be represented by a \( \mathcal{K} \)-cycle supported on a prescribed \( X\Gamma \)-ample and non-degenerated representation.

**Lemma 4.12.** Let \( G \) be a locally compact group and let \( A \) and \( B \) be two \( G \)-algebras. Let \((\rho, \mathcal{E}, T)\) be a \( \mathcal{K} \)-cycle for \( KK^\mathcal{G}_1(A, B) \) and let \( \rho' \) be an equivariant representation of \( A \) on the right \( B \)-Hilbert module \( \mathcal{E} \) such that \( \rho(a) - \rho'(a) \) is compact for all \( a \) in \( A \). Then \((\rho', \mathcal{E}, T)\) is a \( \mathcal{K} \)-cycle for \( KK^\mathcal{G}_1(A, B) \) equivalent to \((\rho, \mathcal{E}, T)\).

**Proof.** It is clear that \((\rho', \mathcal{E}, T)\) is a \( \mathcal{K} \)-cycle for \( KK^\mathcal{G}_1(A, B) \). Then

\[
\left( \frac{\cos \pi/2 - \sin \pi/2}{\sin \pi/2 \cos \pi/2} \right) \cdot \left( \begin{array}{c} 0 \quad \text{Id}_\mathcal{E} \end{array} \right) \cdot \left( \frac{\cos \pi/2 \sin \pi/2}{-\sin \pi/2 \cos \pi/2} \right)_{t \in [0, 1]}
\]

provides a homotopy between the \( \mathcal{K} \)-cycles \((\rho \oplus \rho', \mathcal{E} \oplus \mathcal{E}, T \oplus \text{Id}_\mathcal{E})\) and \((\rho \oplus \rho', \mathcal{E} \oplus \mathcal{E}, \text{Id}_\mathcal{E} \oplus T)\). \( \square \)

**Corollary 4.13.** Let \( X \) be a locally compact and proper \( \Gamma \)-space and let \( \rho_X \) be a \( X\Gamma \)-ample representation of \( C_0(X) \) on a \( \Gamma \)-Hilbert space \( H_X \). Then every element of \( KK^\Gamma_1(X, \mathbb{C}) \) can be represented by a \( \mathcal{K} \)-cycle \((\rho_X, H_X, T)\) where \( T \) is a \( \Gamma \)-equivariant operator on \( H_X \).

We fix once for all a separable Hilbert space \( H \) and for each real \( r \) a \( \Gamma \)-invariant measure \( \eta_r \) on \( P_\tau(\Gamma) \) fully supported. Let us consider \( H_{P_\tau(\Gamma)} = L^2(\eta_r) \otimes H \) with the \( X\Gamma \)-ample representation \( \rho_r \) defined in example [10]. Define \( \Psi^\Gamma_r(H_{P_\tau(\Gamma)}) \) as the *-algebra of pseudo-local, \( \Gamma \)-equivariant and finite propagation operators on \( H_{P_\tau(\Gamma)} \). An element of \( \Psi^\Gamma_r(H_{P_\tau(\Gamma)}) \) is called a \( \mathcal{K} \)-cycle if it satisfies the \( \mathcal{K} \)-cycle condition with respect to \( \rho_r \).

**Lemma 4.14.** Let \( x \) be an element of \( KK^\Gamma_1(P_\tau(\Gamma), \ell^\infty(X(\Gamma), \mathcal{K}(H(\cdot))) \). Then there exists a real \( s \) and a family \((T_i)_{i \in \mathbb{N}}\) of bounded operators on \( H_{P_\tau(\Gamma)} \) such that

(i) \( T_i \) is a \( \mathcal{K} \)-cycle of \( \Psi^\Gamma_r(P_\tau(\Gamma)) \) of propagation less than \( s \) and \( \|T_i\| \leq 1 \) for every integer \( i \);

(ii) Under the identification

\[
\ell^\infty(X, \mathcal{K}(H, H_{P_\tau(\Gamma)})) \cong \prod_{i \in \mathbb{N}} \mathcal{C}(\mathbb{R}/\mathbb{Z}, \mathcal{K}(H, H_{P_\tau(\Gamma)})) \cong \prod_{i \in \mathbb{N}} \mathcal{K}_1^\Gamma(H, H_{P_\tau(\Gamma)}),
\]

then

\[
((\mathcal{K}_1^\Gamma, \rho_r), \ell^\infty(X, \mathcal{K}(H, H_{P_\tau(\Gamma)})), (\mathcal{K}_1^\Gamma, T_i)_{i \in \mathbb{N}})
\]

is a \( \mathcal{K} \)-cycle that represents \( x \).

(iii) If \( x_i \) is the class of \((\rho_r, H_{P_\tau(\Gamma)}), T_i) \) in \( KK^\Gamma_1(P_\tau(\Gamma), \mathbb{C}) \), then \( \Theta^\Gamma_{\mathcal{A}, \delta}(x) = (\mathcal{K}_1^\Gamma, x_i)_{i \in \mathbb{N}} \), with \( \mathcal{A} = (\mathcal{C}(\mathbb{R}/\mathbb{Z}, H_{P_\tau(\Gamma)}))_{i \in \mathbb{N}} \).
Proof. Item (iii) is a consequence of item (ii) together with proposition 3.4. According to the discussion following proposition 3.4, we can assume that $x$ is represented by a $K$-cycle $(\psi_i)_{i \in \mathbb{N}} = \prod_{i \in \mathbb{N}} C_{C(\Gamma \setminus \Gamma_i)}(C(\Gamma \setminus \Gamma_i), \mathcal{E}_i), (T_i)_{i \in \mathbb{N}}$ such that for all integer $i$

- $E_i$ is a $\Gamma$-equivariant $C(\Gamma \setminus \Gamma_i)$-Hilbert module;
- $T_i$ is a $\Gamma$-equivariant adjointable operator on $E_i$ with $\|T_i\| \leq 1$;
- $\phi_i$ is a $\Gamma$-equivariant representation of $C_0(P_r(\Gamma))$ on $E_i$;
- $C_0(P_r(\Gamma))$ and $T_i$ act then on $K_{C(\Gamma \setminus \Gamma_i)}(H, \mathcal{E}_i)$ by left composition.

But for every integer $i$, there exist a $\Gamma$-Hilbert space $H_i$, a $\Gamma$-equivariant representation $\psi_i$ of $C_0(P_r(\Gamma))$ on $H_i$ and a $\Gamma$-equivariant bounded operator $F_i$ on $H_i$ such that $E_i = \Delta_{F_i} H_i$, $\phi_i = \Delta_{F_i} \psi_i$ and $T_i = \Delta_{F_i} F_i$. Up to replace $H_i$ by the $\Gamma$-Hilbert space induced by the inclusion $\Gamma_i \rightarrow \Gamma$, we can assume that $H_i$ is a $\Gamma$-Hilbert which is up to add the degenerated $K$-cycle $(\rho_r, H_{P_r(\Gamma)}, \text{Id}_{H_{P_r(\Gamma)}})$ can be chosen $X$-$\Gamma$-ample. By adding the degenerated $K$-cycle

$$\left( \bigoplus_{k \in \mathbb{N}, k \neq i} \psi_k; \bigoplus_{k \in \mathbb{N}, k \neq i} H_k; \bigoplus_{k \in \mathbb{N}, k \neq i} \text{Id}_{H_k}\right),$$

we can assume that $H_i = H_0$ and $\psi_i = \psi_0$. According to lemma 4.11 by taking an unitary equivalence of $K$-cycle, we can assume that $H_0 = H_{P_r(\Gamma)}$ and that $\psi_0(f) - \rho_r(f) \in H(K_{P_r(\Gamma)})$ for every integer $f$ in $C_0(P_r(\Gamma))$. It is straightforward to check that

$$(\Delta_{F_i} \psi_0)_{i \in \mathbb{N}}, (\rho_r, H_{P_r(\Gamma)}), (\Delta_{F_i} F_i)_{i \in \mathbb{N}}$$

is a $K$-cycle for $KK^G_r(P_r(\Gamma), \ell^\infty(X(\Gamma), K(H)))$, which is by proposition 3.4 and lemma 4.12 equivalent to

$$(\Delta_{F_i} \psi_0)_{i \in \mathbb{N}}, (\ell^\infty(X(\Gamma), K(H)), (\Delta_{F_i} F_i)_{i \in \mathbb{N}}).$$

If $f \in C_c(P_r(\Gamma), [0, 1])$ is a cut-off function for the action of $\Gamma$ on $P_r(\Gamma)$, then up to replace $F = (\Delta_{F_i} F_i)_{i \in \mathbb{N}}$ by $\sum_{\gamma \in \Gamma} \gamma(f) F\gamma(f)$, we can assume that there exists a real $s$ such that for all integer $i$ the operator $F_i$ has propagation less than $s$. \qed

Set $\zeta = \ell^\infty(X(\Gamma), K(H, \ell^2(\Gamma \setminus H)))$ and let $\gamma_\Gamma$ be the right $\ell^\infty(X(\Gamma), K(H)) \rtimes_{\text{max}} \Gamma$-Hilbert module constructed from $\zeta$ in section 3.3. Viewing $\ell^2(\Gamma) \otimes \ell^\infty(X(\Gamma), K(H))$ as a right $\ell^\infty(X(\Gamma), K(H))$-Hilbert submodule of $\ell^\infty(X(\Gamma), K(H, \ell^2(\Gamma \setminus H)))$, we see that

$$C_c(\Gamma) \cdot \ell^\infty(X(\Gamma), K(H, \ell^2(\Gamma \setminus H))) = C_c(\Gamma) \cdot (\ell^2(\Gamma) \otimes \ell^\infty(X(\Gamma), K(H)))$$

and under this identification, we get that

$$C_c(\Gamma) \otimes \ell^\infty(X(\Gamma), K(H)) \rightarrow \ell^\infty(X(\Gamma), K(H)) \rtimes_{\text{max}} \Gamma; \delta \gamma \otimes a \mapsto \gamma^{-1}(a) \delta \gamma^{-1}$$

extends to isomorphism of right $\ell^\infty(X(\Gamma), K(H)) \rtimes_{\text{max}} \Gamma$-Hilbert module

$$(4.5) \quad \zeta_\Gamma \cong \ell^\infty(X(\Gamma), K(H)) \rtimes_{\text{max}} \Gamma.$$
Lemma 4.15. Let \((S_i)_{i \in \mathbb{N}}\) be a family in \(\prod_{i \in \mathbb{N}} \Psi^\Gamma_i(\Gamma)\) uniformly bounded and with propagation uniformly bounded by a real \(s\). Then under the identification
\[
\zeta \equiv \prod_{i \in \mathbb{N}} C(\Gamma/\Gamma_i, K(H, \ell^2(\Gamma) \otimes H)) \cong \prod_{i \in \mathbb{N}} \ell^2(\Gamma_i, K(H, \ell^2(\Gamma) \otimes H)),
\]

(i) there exists a unique multiplier \(\lambda_\Gamma(S_i)_{i \in \mathbb{N}}\) of \(\ell^\infty(\chi(\Gamma), K(H)) \rtimes_{\max} \Gamma\) which under the identification \(\zeta \equiv \ell^\infty(\chi(\Gamma), K(H)) \rtimes_{\max} \Gamma\) restricts to \((\ell^2(\Gamma_i, S_i)_{i \in \mathbb{N}})\) on \(C_\mathcal{C}(\Gamma) \cdot \zeta\).
(ii) The multiplier image of \(\lambda_\Gamma(S_i)_{i \in \mathbb{N}}\) under the canonical projection
\[
\ell^\infty(\chi(\Gamma), K(H)) \rtimes_{\max} \Gamma \to \chi(\Gamma) \rtimes_{\max} \Gamma
\]
and the multiplier image of \(\bigoplus_{i \in \mathbb{N}} \hat{S}_i\) under the map
\[
C^*(\chi(\Gamma)) \xrightarrow{\Phi_{\Gamma}} \chi(\Gamma) \rtimes_{\max} \Gamma
\]
coincide.

Proof. Let us prove first the lemma for a family \((S_i)_{i \in \mathbb{N}}\) of locally compact operators. Since such families are algebraically generated by families

- \((f_i)_{i \in \mathbb{N}}\) uniformly bounded with \(f_i \in C(\Gamma/\Gamma_i, K(H))\) acting by pointwise multiplication;
- \((R_\gamma)_{\gamma \in \Gamma}\), for \(\gamma \in \Gamma\), where \(R_\gamma\) is induced by the right regular representation on \(\ell^2(\Gamma) \otimes H\).

this amounts to prove the lemma for these families.

Then the elements of \(\ell^\infty(\chi(\Gamma), K(H)) \rtimes_{\max} \Gamma\) given by

- the image of \((f_i)_{i \in \mathbb{N}}\) viewed as element of \(\ell^\infty(\chi(\Gamma), K(H))\) under the inclusion \(\ell^\infty(\chi(\Gamma), K(H)) \hookrightarrow \ell^\infty(\chi(\Gamma), K(H)) \rtimes_{\max} \Gamma\) for the first case.
- the element \(\delta_\gamma \in \ell^\infty(\chi(\Gamma), K(H)) \rtimes_{\max} \Gamma\) for the family \((R_\gamma)_{\gamma \in \Gamma}\), where \(\delta_\gamma\) be the Dirac function at \(\gamma\).

satisfy the required property. For a family \((S_i)_{i \in \mathbb{N}}\) of pseudo-local operators, let us set for \(i\) integer \(S_i' = \sum_{\gamma \in \Gamma} \delta_\gamma S_i \delta_\gamma\). Then \(S_i' - S_i\) is \(\Gamma_i\)-equivariant and locally compact for all integer \(i\). Moreover, as already mentioned in subsection 3.1, \((\ell^2(\Gamma_i, S'_i)\) extends to an adjointable operator of \(\zeta_\Gamma\) and thereby to a multiplier \(\lambda_\Gamma(S'_i)_{i \in \mathbb{N}}\) of \(\ell^\infty(\chi(\Gamma), K(H)) \rtimes_{\max} \Gamma\). We then set \(\lambda_\Gamma(S_i)_{i \in \mathbb{N}} = \lambda_\Gamma(S'_i)_{i \in \mathbb{N}} + \lambda_\Gamma(S_i - S'_i)_{i \in \mathbb{N}}\). Unicity is quite obvious. Since pseudo-local operator on \(\ell^2(\Gamma) \otimes H\) are multiplier for locally compact operator, we get item

(ii) by multiplicativity of \((S_i)_{i \in \mathbb{N}}\) \(\lambda_\Gamma(S_i)_{i \in \mathbb{N}}\) and of \((S_i)_{i \in \mathbb{N}}\) \(\bigoplus_{i \in \mathbb{N}} \hat{S}_i\) \(\square\)

we are now in position to prove the main theorem of the section.

Theorem 4.16. Let \(\Gamma\) be a residually finite and finitely generated discrete group with respect to a sequence \(\Gamma_0 \subset \cdots \Gamma_n \subset \cdots\) of finite index normal subgroups. Then if we set \(X(\Gamma) = \prod_{n \in \mathbb{N}} \Gamma/\Gamma_i\), we have
\[
\Psi_{\Gamma, A_r, **} \circ \mu_{X(\Gamma), \text{max}, * \ast} = \mu_{\Gamma, A_r, \text{max}, * \ast} \circ \Psi_{\Gamma(\Gamma), **}\n\]

Proof. Let \(r\) be a real and let \(n\) be any integer such that \(2r \leq n\) and \(B(c, 4r) \cap \Gamma_n = \{e\}\). Then
\[
K_*(P_r(X(\Gamma))) = K_*(P_r(\prod_{i = 1}^{n-1} \Gamma/\Gamma_i)) \oplus \prod_{i \geq n} K_*(P_r(\Gamma/\Gamma_i)).
\]
If under this identification, $x$ comes from $K_*(P_\gamma(\bigoplus_{i=1}^{n-1} \Gamma_i))$, then $\Psi_X(\gamma_r)(x) = 0$ and using the naturality of the assembly map, we can see that $\mu_{X(\gamma_r),\max,r}$ lies in $K_*(C^*_{\max}(\bigoplus_{i=1}^{n-1} \Gamma_i)) \subset K_*(C^*(X(\gamma_r)))$ and hence $\Psi_{\gamma_r} \circ \mu_{X(\gamma_r),\max,r}(x) = 0$. Thereby, we have to prove that $\Psi_{\gamma_r} \circ \mu_{X(\gamma_r),\max,r}(x) = \mu_{\gamma_r,\max,r} \circ \Psi_X(\gamma_r)$, for element $x$ coming under the identification of equation \eqref{eq:identification}. According to equation \eqref{eq:identification}, and up to replace $r$ by $2r$, we can assume that $x' = \Lambda_{\gamma_r,\min}(y)$, with $y \in \bigoplus_{i=0}^{n} K_*(P_\gamma(\Gamma_i))$. We can assume indeed without loss of generality that $n = 0$ and that the action of $\Gamma_i$ on $P_\gamma(\Gamma_i)$ is free for all integer $i$. From now on, we will write $\Lambda_r$ (resp. $\Psi_r, i = 1, \ldots, 6$) instead of $\Lambda_{\gamma_r,\min}$ (resp. $\Psi_{\gamma_r,\min}, i = 1, \ldots, 6$). Let us set $z = \Psi_k(y)$ in $\bigoplus_{k \in K} K^\Gamma_k(C_0(P_\gamma(\Gamma)), \mathbb{C})$. The proof of the theorem is divided in the following steps.

**First Step:** Assume that $z$ is given by a family of K-cycles $(\rho_{P_\gamma(\Gamma)}(\gamma_r), H_{P_\gamma(\Gamma)}, T_i)_{i \in \mathbb{N}}$ such that for a real $s$, then for all integer $i$ the operator $T_i = H_\gamma(\Gamma_i)$-equivariant with $\|T_i\| \leq 1$ and has propagation less than $s$. Let us set $H_{P_\gamma(\Gamma)} = L^2(\eta_{r,i}) \otimes H$ and $\rho_{P_\gamma(\Gamma)} : C_0(P_\gamma(\Gamma)) \to \mathcal{L}(L^2(\eta_{r,i}) \otimes H) ; f \mapsto f \otimes Id_H$ where for all integer $i$, the measure $\eta_{r,i}$ is induced by $\eta_r$ on $P_\gamma(\Gamma_i)$. Let us choose a $\Gamma$-equivariant coarse map $\phi_r : P_\gamma(\Gamma) \to \Gamma$. Then $\phi_r$ is a coarse equivalence and induces a coarse equivalence

$$
\phi_r : \bigoplus_{i \in \mathbb{N}} P_\gamma(\Gamma)/\Gamma_i \to \bigoplus_{i \in \mathbb{N}} \Gamma/\Gamma_i = X(\gamma).
$$

Let us show that with notations of section \ref{sec:coarse_equivalence},

$$
\mu_{X(\gamma_r),\max,r}(x) = \phi_{r,\max,r} \ \text{Ind}_{\max_r} \ X(\gamma) \oplus \bigoplus_{k \in \mathbb{N}} T_k
$$

where $\bigoplus_{k \in \mathbb{N}} T_k$ is viewed as an operator on the non-degenerated standard $\bigoplus_{k \in \mathbb{N}} P_\gamma(\Gamma)/\Gamma_k$-module $\bigoplus_{k \in \mathbb{N}} H_{P_\gamma(\Gamma)/\Gamma_k}$ (for the representation $\bigoplus_{i \in \mathbb{N}} \rho_{H_{P_\gamma(\Gamma)/\Gamma_i}}$).

Let $\nu_{r,k} : P_\gamma(\Gamma)/\Gamma_k \to P_\gamma(\Gamma/\Gamma_k) ; h \mapsto \tilde{h}$ be the map defined in section \ref{sec:map}. Notice that the family $(\nu_{r,k})_{k \in \mathbb{N}}$ induces a coarse equivalence $\nu_r : \bigoplus_{k \in \mathbb{N}} P_\gamma(\Gamma)/\Gamma_k \to \bigoplus_{k \in \mathbb{N}} P_\gamma(\Gamma/\Gamma_k)$. Moreover, if we set

$$
\phi_k : C_0(P_\gamma(\Gamma/\Gamma_k)) \to \mathcal{L}(H_{P_\gamma(\Gamma)/\Gamma_k}) ; f \mapsto \rho_{P_\gamma(\Gamma)/\Gamma_k}(f \circ \nu_{r,k})
$$

then $x' = \Lambda_\gamma(y)$ is the class of the K-cycle

$$
\left( \bigoplus_{k \in \mathbb{N}} \phi_k, \bigoplus_{k \in \mathbb{N}} H_{P_\gamma(\Gamma)/\Gamma_k}, \bigoplus_{k \in \mathbb{N}} T_k \right)
$$

in $K_*(\bigoplus_{k \in \mathbb{N}} (P_\gamma(\Gamma/\Gamma_k))$. For any non-degenerated standard $P_\gamma(\Gamma/\Gamma_k)$-module $H_k$ given by a representation $\rho_k$, then $\phi_k \oplus \rho_k$ also provides a non-degenerated standard $P_\gamma(\Gamma/\Gamma_k)$-Hilbert module structure for $H_{P_\gamma(\Gamma/\Gamma_k)} \oplus H_k$. Since the K-cycles

$$
\left( \bigoplus_{k \in \mathbb{N}} \phi_k, \bigoplus_{k \in \mathbb{N}} H_{P_\gamma(\Gamma/\Gamma_k)} \oplus H_k, \bigoplus_{k \in \mathbb{N}} T_k \oplus Id_{H_k} \right)
$$

and

$$
\left( \bigoplus_{k \in \mathbb{N}} \phi_k \oplus \rho_k, \bigoplus_{k \in \mathbb{N}} H_{P_\gamma(\Gamma/\Gamma_k)} \oplus H_k, \bigoplus_{k \in \mathbb{N}} T_k \oplus Id_{H_k} \right)
$$
are equivalent, we get that

\[ \mu_{X(\Gamma), \text{max}, \ast}(x) = \nu_{r, \text{max}, \ast} \text{Ind}_{k \in \mathbb{N}} \prod_{k \in \mathbb{N}} P_r(\Gamma / T_k) \bigoplus_{k \in \mathbb{N}} \hat{T}_k \otimes \text{Id}_{H_k}, \]

where \( \psi_r : \prod_{k \in \mathbb{N}} P_r(\Gamma / T_k) \rightarrow X(\Gamma) \) is any coarse equivalence. Since the inclusion \( \bigoplus_{k \in \mathbb{N}} H_{P_r(\Gamma / T_k)} \hookrightarrow \bigoplus_{k \in \mathbb{N}} H_{P_r(\Gamma / T_k)} \otimes H_k \) covers the coarse map \( v_r : \prod_{k \in \mathbb{N}} P_r(\Gamma / T_k) \rightarrow \prod_{k \in \mathbb{N}} P_r(\Gamma / T_k) \), we get that

\[ \text{Ind}_{k \in \mathbb{N}} P_r(\Gamma / T_k) \left( \bigoplus_{k \in \mathbb{N}} \hat{T}_k \otimes \text{Id}_{H_k} \right) = \nu_{r, \text{max}, \ast} \text{Ind}_{k \in \mathbb{N}} P_r(\Gamma / T_k) \left( \bigoplus_{k \in \mathbb{N}} \hat{T}_k \right). \]

Notice that since \( \phi_r \) and \( \psi_r \circ v_r \) are both coarse equivalence between \( \prod_{k \in \mathbb{N}} P_r(\Gamma / T_k) \) and \( X(\Gamma) \), then \( \phi_r = \psi_r \circ v_r \) and hence we get the equality of equation 4.7.

**Second step:** According to [17], there exists a \( \Gamma \)-equivariant isometrie \( W_r : H_{P_r(\Gamma)} \rightarrow \ell^2(\Gamma) \otimes H \) that covers \( \phi_r : P_r(\Gamma) \rightarrow \Gamma \). Then, if \( W_{r,k} : H_{P_r(\Gamma / T_k)} \rightarrow \ell^2(\Gamma / T_k) \otimes H \) stands for the isometry induced by \( W_r \) for all integer \( k \), then

\[ \bigoplus_{k \in \mathbb{N}} W_{r,k} : \bigoplus_{k \in \mathbb{N}} H_{P_r(\Gamma / T_k)} \rightarrow \bigoplus_{k \in \mathbb{N}} \ell^2(\Gamma / T_k) \otimes H \]

is an isometrie that covers \( \phi_r \) and thus

\[ \phi_{r, \text{max}, \ast} \text{Ind}_{k \in \mathbb{N}} P_r(\Gamma / T_k) \bigoplus_{k \in \mathbb{N}} \hat{T}_k = \text{Ind}_{k \in \mathbb{N}} P_r(\Gamma / T_k) \bigoplus_{k \in \mathbb{N}} W_{r,k} \hat{T}_k W_{r,k}^* + \text{Id}_{\ell^2(\Gamma / T_k) \otimes H} \cdot W_{r,k} W_{r,k}^*. \]

Finally we get that

\[ \mu_{X(\Gamma), \text{max}, \ast}(x) = \text{Ind}_{k \in \mathbb{N}} P_r(\Gamma / T_k) \bigoplus_{k \in \mathbb{N}} W_{r,k} \hat{T}_k W_{r,k} + \text{Id}_{\ell^2(\Gamma / T_k) \otimes H} \cdot W_{r,k} W_{r,k}^*. \]

**Third step:** By naturality of the assembly map, we get that

\[ \mu_{\Gamma, A \Gamma, \text{max}, \ast} \circ \Psi_\ast(x) = \Psi_{\Gamma, \ast} \circ \mu_{\Gamma, \ell^\infty(X(\Gamma), \mathcal{K}(H)), \text{max}, \ast}(z'), \]

where

- \( z' \) is the element in \( K^{\text{top}}_{\ell^\infty}(\Gamma, \ell^\infty(X(\Gamma), \mathcal{K}(H))) \) coming from \( \Psi_{\Gamma} \circ \Psi_{\ell^\infty}(z) \in KK^{\ell^\infty}(\Gamma, \ell^\infty(X(\Gamma), \mathcal{K}(H))));
- \( \Psi_{\ell^\infty} : \ell^\infty(X(\Gamma), \mathcal{K}(H)) \otimes_{\text{max}} \Gamma \rightarrow A_{\Gamma} \otimes_{\text{max}} \Gamma \) is induced by the projection \( \Psi_{\ell^\infty} : \ell^\infty(X(\Gamma), \mathcal{K}(H)) \rightarrow A_{\Gamma} \).

Let us compute \( \mu_{\Gamma, \ell^\infty(X(\Gamma), \mathcal{K}(H)), \text{max}, \ast}(z'). \)

According to lemma [11] under the identification

\[ \ell^\infty(X(\Gamma), \mathcal{K}(H, H_{P_r(\Gamma)})) \cong \prod_{i \in \mathbb{N}} C(\Gamma / T_i, \mathcal{K}(H, H_{P_r(\Gamma)})) \cong \prod_{i \in \mathbb{N}} \mathbb{F}_{\ell^\infty}(\mathcal{K}(H, H_{P_r(\Gamma)})), \]

the element \( \Psi_{\ell^\infty} \circ \Psi_{\ell^\infty}(z) \) of \( KK^{\ell^\infty}(\Gamma, \ell^\infty(X(\Gamma), \mathcal{K}(H))) \) can be represented by a K-cycle

\[ (\mathbb{F}_{\ell^\infty}(\mathcal{K}(H, H_{P_r(\Gamma)})), (\mathbb{F}_{\ell^\infty}(\mathcal{K}(H, H_{P_r(\Gamma)})), (\mathbb{F}_{\ell^\infty}(\mathcal{K}(H, H_{P_r(\Gamma)})), (\mathbb{F}_{\ell^\infty}(\mathcal{K}(H, H_{P_r(\Gamma)})))) \right) \]
bijectivity of the maximal coarse Baum-Connes assembly map. Let $\Gamma$ be a residually

Applications.

\[
\begin{array}{c}
\sum_n \pi_n \\
\downarrow \\
\cdots
\end{array}
\]

We end this section with application concerning injectivity and

\[\begin{align*}
\ell \Gamma \in C_\ast(\Gamma) & \rightarrow C_\ast(\Gamma) \cdot f \mapsto W_r \circ f,
\end{align*}\]

which extends to an isometric $W_r : \mathcal{E}_\Gamma \rightarrow \mathcal{E}_\Gamma$. As we have seen before, $\mathcal{E}_\Gamma$ is a right-\(\ell^\infty(X(\Gamma), \mathcal{K}(H))\times_{\max} \Gamma\) module isomorphic to \(\ell^\infty(X(\Gamma), \mathcal{K}(H))\times_{\max} \Gamma\) and in view of this, $\mathcal{E}_\Gamma$ is a direct factor of \(\ell^\infty(X(\Gamma), \mathcal{K}(H))\times_{\max} \Gamma\). Moreover, if $F_r$ is the operator of $\mathcal{E}_\Gamma$ extending $C_\ast(\Gamma) \cdot f \mapsto C_\ast(\Gamma) \cdot f$; $f \mapsto T \circ f$, then we get with notations of lemma 4.15, that

\[
\lambda_r(W_r F_r W_r^\ast + \text{Id}_{\mathcal{E}_\Gamma})_{\otimes H} = W_r F_r \cdot W_r^\ast + \text{Id}_{\mathcal{E}_\Gamma} - W_r W_r^\ast.
\]

Hence $\mu_{r, \ell^\infty(X(\Gamma), \mathcal{K}(H)), \Gamma}(\cdot')$ is the class in $K_\ast(\ell^\infty(X(\Gamma), \mathcal{K}(H))\times_{\max} \Gamma)$ of the K-cycle \(\ell^\infty(X(\Gamma), \mathcal{K}(H))\times_{\max} \Gamma, \lambda_r(W_r F_r W_r^\ast + \text{Id}_{\mathcal{E}_\Gamma})_{\otimes H} = W_r W_r^\ast)_{\otimes H}\). The theorem is then a consequence of lemma 4.15.

\[\square\]

4.4. Applications. We end this section with application concerning injectivity and bijectivity of the maximal coarse Baum-Connes assembly map. Let $\Gamma$ be a residually finite and finitely generated discrete group with respect to a fixed sequence $\Gamma_0 \subset \cdots \Gamma_n \subset \cdots$ of finite index normal subgroups. Recall that we have defined $X(\Gamma) = \varprojlim_{\Gamma \rightarrow \Gamma_n}$ and $A_r = \ell^\infty(X(\Gamma), \mathcal{K}(H))/C_0(X(\Gamma), \mathcal{K}(H))$. We can formulate corollary 2.11 theorem 4.8 and theorem 4.16 together as follows: We have a commutative diagram

\[
\begin{array}{cccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & K_0(\mathcal{P}_{\ast}(\Gamma)) & \xrightarrow{\Phi_{\ast}(\Gamma), \ast} & K_0^{\text{top}}(\Gamma, A_r) & \longrightarrow & 0 \\
& & & \downarrow \mu_{\ast}(\Gamma, \ast) & \downarrow \mu_{r, A_r, \ast} & \downarrow \mu_{\gamma, A_r, \ast} \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & K_0(C_{\ast}^{\text{max}}(\Gamma)) & \xrightarrow{\Phi_{A_r, \ast}} & K_0(A_r \times_{\max} \Gamma) & \longrightarrow & 0
\end{array}
\]

with exact rows and a commutative diagram

\[
\begin{array}{c}
\lim_r KK_1(\mathcal{P}_{\ast}(\Gamma)) \xrightarrow{\Phi_{\ast}(\Gamma), \ast \otimes} K_1^{\text{top}}(\Gamma, A_r) \\
\downarrow \mu_{\ast}(\Gamma, \ast) & \downarrow \mu_{r, A_r, \ast} & \downarrow \mu_{\gamma, A_r, \ast} \\
K_1(C_{\ast}^{\text{max}}(\Gamma)) & \xrightarrow{\Phi_{A_r, \ast}} & K_1(A_r \times_{\max} \Gamma)
\end{array}
\]

K-theory for the maximal Roe algebra of certain expanders 31
From this commutative diagram, we can deduce the following series of results concerning injectivity and bijectivity of assembly maps.

**Theorem 4.17.** The following assertions are equivalent:

(i) The maximal coarse assembly map

\[ \mu_{X(\Gamma),\text{max},*} : \lim_{r} K_{*}(P_{r}(X(\Gamma)), \mathbb{C}) \to K_{*}(C_{\text{max}}^{*}(X(\Gamma))) \]

is an isomorphism.

(ii) the maximal assembly map

\[ \mu_{\Gamma,A_{r},\text{max}} : K_{*}^{\text{top}}(\Gamma, A_{r}) \to K_{*}(A_{r} \rtimes_{\text{max}} \Gamma) \]

is an isomorphism.

Example of groups that satisfies item (ii) of the theorem are provided by groups that satisfy the so called strong Baum-Connes conjecture. Recall first that a \( \Gamma \)-algebra \( D \) is said to be a proper \( \Gamma \)-algebra, if \( D \) is a \( C_{0}(Z) \)-algebra for some proper \( \Gamma \)-space \( Z \) in such a way that the structure map \( \Phi : C_{0}(Z) \to ZM(D) \) is \( \Gamma \)-equivariant. A group \( \Gamma \) satisfies the strong Baum-Connes conjecture if there exist, a proper \( \Gamma \)-algebra \( D \), an element \( \alpha \) in \( KK^{\Gamma}_{*}(D, \mathbb{C}) \) and a element \( \beta \) in \( KK^{\Gamma}_{*}(\mathbb{C}, D) \) such that \( \beta \otimes_{D} \alpha \) is the unit of \( KK^{\Gamma}_{*}(\mathbb{C}, \mathbb{C}) \). It is well know (see [14] for instance) that if \( \Gamma \) satisfies the strong Baum-Connes conjecture, then \( \mu_{\Gamma,B,\text{max}} : K_{*}^{\text{top}}(\Gamma, B) \to K_{*}(B \rtimes_{\text{max}} \Gamma) \) is an isomorphism for every \( \Gamma \)-algebra \( B \). As a consequence, we get

**Corollary 4.18.** If \( \Gamma \) satisfies the strong Baum-Connes conjecture, then

\[ \mu_{X(\Gamma),\text{max},*} : \lim_{r} K_{*}(P_{r}(X(\Gamma)), \mathbb{C}) \to K_{*}(C_{\text{max}}^{*}(X(\Gamma))) \]

is an isomorphism.

As a particular case, we obtain

**Corollary 4.19.** If \( \Gamma \) is a group with the Haagerup property, then

\[ \mu_{X(\Gamma),\text{max},*} : \lim_{r} K_{*}(P_{r}(X(\Gamma)), \mathbb{C}) \to K_{*}(C_{\text{max}}^{*}(X(\Gamma))) \]

is an isomorphism.

**Example 4.20.** If \( \Gamma = SL_{2}(\mathbb{Z}) \) and \( \Gamma_{k} = \ker : SL_{2}(\mathbb{Z}) \to SL_{2}(\mathbb{Z}/k\mathbb{Z}) \), then \( \mu_{X(\Gamma),\text{max},*} \) is an isomorphism while \( \mu_{X(\Gamma),*} \) is not surjective.

**Remark 4.21.** If the group \( \Gamma \) has the Kazdhan property \((T)\), then the family of projectors corresponding to the \( i \)-eigenvalue of the Laplacians of the family \( \{\Gamma/\Gamma_{i}\}_{i \in \mathbb{N}} \) provides a projector \( p \) in \( C_{\text{max}}^{*}(X(\Gamma)) \). We know from [14] that the image in \( K_{0}(C^{*}(X(\Gamma))) \) of the class of \( p \) under the homomorphism \( \lambda_{X(\Gamma),*} : K_{0}(C_{\text{max}}^{*}(X(\Gamma))) \to K_{0}(C^{*}(X(\Gamma))) \) is not in the range of the coarse assembly map

\[ \mu_{X(\Gamma),*} : \lim_{r} K_{*}(P_{r}(X(\Gamma)), \mathbb{C}) \to K_{*}(C^{*}(X(\Gamma))). \]

Hence, according to remark 2.29, the assembly map \( \mu_{X(\Gamma),\text{max},*} \) is not surjective.

Recall from [14] that if the group \( \Gamma \) satisfies the strong Baum-Connes conjecture, then \( \Gamma \) is \( K \)-amenable and in particular, the \( K \)-theory of reduced and maximal crossed product coincide. This allows to get explicit computation for \( K_{*}(C_{\text{max}}^{*}(X(\Gamma))) \) in the following situation.
Corollary 4.22. If $\Gamma$ satisfies the strong Baum-Connes conjecture and admits a universal example for proper action which is simplicial, with simplicial and cocompact action of $\Gamma$, then we have a short exact sequence

$$0 \to \mathbb{Z} \to K_0(C^*_{\text{max}}(X(\Gamma))) \to \prod_{i \in \mathbb{N}} K_0(C^*_{\text{red}}(\Gamma_i)) \big/ \bigoplus_{i \in \mathbb{N}} K_0(C^*_{\text{red}}(\Gamma_i)) \to 0$$

and an isomorphism

$$K_1(C^*_{\text{max}}(X(\Gamma))) \xrightarrow{\cong} \prod_{i \in \mathbb{N}} K_1(C^*_{\text{red}}(\Gamma_i)) \big/ \bigoplus_{i \in \mathbb{N}} K_1(C^*_{\text{red}}(\Gamma_i)).$$

Proof. First notice that since $\Gamma$ is $K$-amenable, then

$$\lambda_{\Gamma, A_{\Gamma},*}: K_* (A_{\Gamma} \rtimes_{\text{max}} \Gamma) \to K_* (A_{\Gamma} \rtimes_{\text{red}} \Gamma).$$

Let us show that we have an isomorphism

$$K_* (A_{\Gamma} \rtimes_{\text{red}} \Gamma) \xrightarrow{\cong} \prod_{i \in \mathbb{N}} K_* (C^*_{\text{red}}(\Gamma_i)) \big/ \bigoplus_{i \in \mathbb{N}} K_* (C^*_{\text{red}}(\Gamma_i)).$$

Let us consider the following commutative diagram

$$\begin{array}{ccc}
K^*_{\text{top}}(\Gamma, \prod_{i \in \mathbb{N}} C(\Gamma/\Gamma_i, K(H))) & \xrightarrow{\mu_{\Gamma, \prod_{i \in \mathbb{N}} C(\Gamma/\Gamma_i, K(H))}} & K_* ((\prod_{i \in \mathbb{N}} C(\Gamma/\Gamma_i, K(H))) \rtimes_{\text{red}} \Gamma) \\
\downarrow & & \downarrow \\
\prod_{i \in \mathbb{N}} K^*_{\text{top}}(\Gamma, C(\Gamma/\Gamma_i)) & \xrightarrow{\prod_{i \in \mathbb{N}} \mu_{\Gamma, C(\Gamma/\Gamma_i, K(H))}} & \prod_{i \in \mathbb{N}} K_* (C(\Gamma/\Gamma_i, K(H))) \rtimes_{\text{red}} \Gamma
\end{array}$$

where the vertical arrow are induced on the $k$-th factor by the projection

$$\prod_{i \in \mathbb{N}} C(\Gamma/\Gamma_i, K(H)) \to C(\Gamma/\Gamma_k, K(H)).$$

But since the group $\Gamma$ admits a universal example for proper action which is simplicial, with simplicial and cocompact action of $\Gamma$, then the left vertical arrow is an isomorphism. Since $\Gamma$ satisfies the Baum-Connes conjecture, then the horizontal map are also isomorphism. Hence the right vertical map is also an isomorphism and hence the result is a consequence of remark and of the Morita equivalence between $C(\Gamma/\Gamma_i) \rtimes_{\text{red}} \Gamma$ and $C^*_{\text{red}}(\Gamma_i)$. \qed

Regarding injectivity, we have similar results.

Theorem 4.23. The following assertions are equivalent:

(i) The maximal coarse assembly map

$$\mu_{X(\Gamma), \text{max}*}: \lim_{\Gamma} K_* (P_{\Gamma}(X(\Gamma)), C) \to K_* (C^*_{\text{max}}(X(\Gamma)))$$

is injective.

(ii) the maximal assembly map

$$\mu_{\Gamma, A_{\Gamma}, \text{max}*}: K^*_{\text{top}}(\Gamma, A_{\Gamma}) \to K_* (A_{\Gamma} \rtimes_{\text{max}} \Gamma)$$

is injective.

We can also deduce the following result concerning the (usual) coarse Baum-Connes conjecture.
Theorem 4.24. Assume that the assembly map
$$\mu_{\Gamma,A_r,\text{red},*}: K^*_{\Gamma}(\Gamma, A_r) \to K_*(A_r \rtimes_{\text{red}} \Gamma)$$
is injective. Then the coarse assembly map
$$\mu_{X(\Gamma),*}: \lim_f K_*(P_f(X(\Gamma)), \mathbb{C}) \to K_*(C^*(X(\Gamma)))$$
is also injective.

Proof. In the even case, let us consider the following diagram
$$
\begin{array}{ccc}
0 & \longrightarrow & Z \\
& \longrightarrow & \lim\mathcal{X}_r K^0_0(P_r(X(\Gamma)), \mathbb{C}) \xrightarrow{\Phi_{X(\Gamma),*}} K^0_{\text{top}}(\Gamma, A_r) \xrightarrow{\psi} 0 \\
& \longrightarrow & \\
0 & \longrightarrow & K_0(C^*_{\text{max}}(X(\Gamma))) \xrightarrow{\lambda_{X(\Gamma),*}} K_0(A_r \rtimes_{\text{max}} \Gamma) \xrightarrow{\lambda_{\Gamma, A_r,*}} 0 \\
& \longrightarrow &
\end{array}
$$

where the bottom left corner horizontal arrow is induced by the inclusion
$$\mathcal{K}(\ell^2(X(\Gamma)) \otimes H) \hookrightarrow C^*(X(\Gamma))$$
and is according to remark 2.12 injective. Thereby, since the top row is exact, we get that injectivity of $$\mu_{\Gamma,A_r,\text{max},*} = \lambda_{\Gamma, A_r} \circ \mu_{\Gamma,A_r,\text{max},*}$$ implies injectivity of $$\mu_{X(\Gamma),*} = \lambda_{X(\Gamma)} \circ \mu_{X(\Gamma),\text{max},*}$$.

It was proved in [13] that for a group $$\Gamma$$ which embeds uniformly in a Hilbert space, then $$\mu_{\Gamma,B,*}$$ is injective for any $$\Gamma$$-algebra $$B$$. As a consequence we obtain

Corollary 4.25. Let $$\Gamma$$ be a group uniformly embeddable in a Hilbert space, then the coarse assembly map
$$\mu_{X(\Gamma),*}: \lim_f K_*(P_f(X(\Gamma)), \mathbb{C}) \to K_*(C^*(X(\Gamma)))$$
is injective.

The last application is to rational injectivity of $$\mu_{X(\Gamma),\text{max},*}$$. Theorem 4.24 admits an obvious rational version. This allowed to recover the following result of [4].

Theorem 4.26. Assume that $$\Gamma$$ admits a universal example for proper action which is simplicial and with simplicial and cocompact action of $$\Gamma$$. If $$\mu_{\Gamma,C,\text{max},*}$$ is rationally injective, then $$\mu_{X(\Gamma),\text{max},*}$$ is also rationally injective.

Proof. Since rational injectivity of $$\mu_{*,\mathbb{C},\text{max},*}$$ is inherited by finite index subgroups, we get under the hypothesis of the theorem that $$\mu_{\Gamma,C,\text{max},*}$$ is rationally injective for all integer $$i$$. Since assembly maps are compatible with induction, we get that $$\mu_{\Gamma,C(\Gamma_i),\text{max},*}$$ is rationally injective. According to corollary 3.5 and since we have the commutative diagram

(4.8)
$$
\begin{array}{ccc}
K^*_{\Gamma}(X(\Gamma), \mathcal{K}(H)) & \xrightarrow{\mu_{\Gamma,I^\infty(X(\Gamma), \mathcal{K}(H)),\text{max},*}} & K_*(I^\infty(X(\Gamma), \mathcal{K}(H)) \rtimes_{\text{max}} \Gamma) \\
\downarrow & & \downarrow \\
\Pi_{i\in\mathbb{N}} K^*_{\Gamma}(C(\Gamma_i)) & \xrightarrow{\mu_{\Gamma,C(\Gamma_i),\text{max},*}, i\in\mathbb{N}} & \Pi_{i\in\mathbb{N}} K_*(C(\Gamma_i) \rtimes_{\text{max}} \Gamma)
\end{array}
$$

Theorem 4.24 admits an obvious rational version. This allowed to recover the following result of [4].
where the vertical arrows are induced on the $k$-th factor up to Morita equivalence by the projection $\Pi_{i \in \mathbb{N}} \ell^\infty(X(\Gamma), K(H)) \to C(\Gamma/\Gamma_i, K(H))$, we see that

$$
\mu_{\Gamma, \ell^\infty(X(\Gamma), K(H)), \max,*} : K_*^{\text{top}}(\Gamma, \ell^\infty(X(\Gamma), K(H))) \to K_*(\ell^\infty(X(\Gamma), K(H)) \rtimes_{\max} \Gamma)
$$

is rationally injective. As we have already seen before, the assembly is also compatible with direct sum of coefficients. Hence we get that

$$
\mu_{\Gamma, C_0(X(\Gamma), K(H)), \max,*} : K_*^{\text{top}}(\Gamma, C_0(X(\Gamma), K(H))) \to K_*(C_0(X(\Gamma), K(H)) \rtimes_{\max} \Gamma)
$$

is also rationally injective. By using the maps induced for each integer $k$ by the $k$-th factor $\Pi_{i \in \mathbb{N}} \ell^\infty(X(\Gamma), K(H)) \to C(\Gamma/\Gamma_k, K(H))$, with see that the inclusion $C_0(X(\Gamma), K(H)) \hookrightarrow \ell^\infty(X(\Gamma), K(H))$ induces inclusions

$$
K_*^{\text{top}}(\Gamma, C_0(X(\Gamma), K(H))) \hookrightarrow K_*^{\text{top}}(\Gamma, \ell^\infty(X(\Gamma), K(H)))
$$

and

$$
K_*(C_0(X(\Gamma), K(H)) \rtimes_{\max} \Gamma) \hookrightarrow K_*(\ell^\infty(X(\Gamma), K(H)) \rtimes_{\max} \Gamma)
$$

and thus we get a commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & K_*^{\text{top}}(\Gamma, C_0(X(\Gamma), K(H))) & \rightarrow & K_*^{\text{top}}(\Gamma, \ell^\infty(X(\Gamma), K(H))) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & K_*(C_0(X(\Gamma), K(H)) \rtimes_{\max} \Gamma) & \rightarrow & K_*(\ell^\infty(X(\Gamma), K(H)) \rtimes_{\max} \Gamma) & \rightarrow & 0 \\
& & & & \downarrow & & \\
& & & & K_*(A_\Gamma \rtimes_{\max} \Gamma) & \rightarrow & 0
\end{array}
$$

with exact rows and where the vertical arrows are given by the assembly maps. Using once again the commutativity of diagram 4.8, we get that if $\mu_{\Gamma, A_\Gamma, \max,*}(x)$ comes rationally from an element in $K_*(C_0(X(\Gamma), K(H)) \rtimes_{\max} \Gamma)$, then $x$ comes rationally from an element in $K_*^{\text{top}}(\Gamma, C_0(X(\Gamma), K(H)))$. Hence

$$
\mu_{\Gamma, A_\Gamma, \max,*} : K_*^{\text{top}}(\Gamma, A_\Gamma) \rightarrow K_*(A_\Gamma \rtimes_{\max} \Gamma)
$$

is rationally injective.

5. Asymptotic quantitative Novikov/Baum-Connes conjecture

Corollary 4.22 suggest that the property of the coarse assembly map

$$
\mu_{X(\Gamma), *} : \lim_{\Gamma} K_*(P_\Gamma(X(\Gamma)), \mathbb{C}) \rightarrow K_*(C^*(X(\Gamma)))
$$

should be closely related to the family fo assembly maps

$$
(\mu_{\Gamma, C, \max,*} : K_*^{\text{top}}(\Gamma_i, \mathbb{C}) \rightarrow K_*(C^*_\text{max}(\Gamma_i)))_{i \in \mathbb{N}}.
$$

We this, we introduce some quantitative assembly maps which take into account the propagation. The relevant propagation here is indeed the one induced by $\Gamma$ under the Morita equivalence between $C^*_\text{max}(\Gamma_i)$ and $C(\Gamma/\Gamma_i) \rtimes_{\max} \Gamma$. We then give asymptotic statements for these quantitative assembly maps and give examples of group for which they are satisfied.
5.1. Almost projectors, almost unitaries and propagation.

Definition 5.1. Let $A$ be a unital C$^*$-algebra and let $\varepsilon \in (0, 1/4)$.

- An element $p$ in $A$ is called an $\varepsilon$-**projector** if $p = p^*$ and $\|p^2 - p\| < \varepsilon$.
- An element $u$ in $A$ is called an $\varepsilon$-**unitary** if $\|u^*u - 1\| < \varepsilon$ and $\|uu^* - 1\| < \varepsilon$.

Notice that if $p$ is an $\varepsilon$-projector of a C$^*$-algebra $A$ and $\phi_\varepsilon : \mathbb{R} \to \mathbb{R}$ is any continuous function such that $\phi_\varepsilon(t) = 0$ for $t < \frac{1 - \sqrt{1 - 4\varepsilon}}{2}$ and $\phi_\varepsilon(t) = 1$ for $t > \frac{1 + \sqrt{1 - 4\varepsilon}}{2}$, then $\phi_\varepsilon(p)$ is a projector. Moreover, we have $\|\phi_\varepsilon(p) - p\| \leq 2\varepsilon$.

Remark 5.2. If $A$ is a C$^*$-algebra, then for every $\varepsilon \in (0, 1/4)$

- if $p$ is an $\varepsilon$-projector of $A$, then any element $q$ in $A$ such that $\|p - q\| < \frac{\varepsilon - \|p^2 - p\|}{4}$ is an $\varepsilon$-projector. In this case $(tp + (1-t)q)_{t \in [0,1]}$ is a homotopy of $\varepsilon$-projectors between $p$ and $q$ and in consequence $\phi_\varepsilon(p)$ and $\phi_\varepsilon(q)$ are homotopic projectors.
- if $A$ is unital and if $u$ is an $\varepsilon$-unitary of $A$, then any element $v$ such that $\|u - v\| < \frac{\varepsilon - \max\{\|u^*u - 1\|, \|uu^* - 1\|\}}{3}$ is $\varepsilon$-unitary and $(tu + (1-t)v)_{t \in [0,1]}$ is a homotopy of $\varepsilon$-unitary connecting $u$ and $v$.

Definition 5.3. Let $A$ be a $\Gamma$-algebra. An element $x$ of $A_{\max} \Gamma$ is said to be of finite propagation if $x$ lies in $C_0(\Gamma, A)$. We say that $x$ has propagation less than $r$ if the support of $x$ as an element of $C_0(\Gamma, A)$ is in $B_r(\epsilon, s)$. These definitions have an obvious extension to $A_{\max} \Gamma$ by requiring the unit to be of propagation zero.

For $\varepsilon \in (0, 1/4)$, $A$ a unital $\Gamma$-algebra, and $p_0$ and $p_1$ two $\varepsilon$-projectors of $A_{\max} \Gamma \otimes K(H)$ with propagation less than $s$ and $n_0$ and $n_1$ positive integers, we write $(p_0, n_0) \sim_{s, \varepsilon} (p_1, n_1)$ if there is an integer $k$ and a $\varepsilon$-projector homotopy $(q_t)_{t \in [0,1]}$ in $C([0,1], (A_{\max} \Gamma) \otimes K(H))$ between $\left(\begin{smallmatrix} n_0 & 0 \\ 0 & I_{k+n_0} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} n_1 & 0 \\ 0 & I_{k+n_1} \end{smallmatrix}\right)$ such that $q_t$ has propagation less than $s$ for every $t$ in $[0,1]$. Similarly if $u_0$ and $u_1$ are $\varepsilon$-unitaries in $A_{\max} \Gamma \otimes K(H)$ with propagation less than $s$, we write $u_0 \sim_{s, \varepsilon} u_1$ if there is a $\varepsilon$-unitary homotopy $(v_t)_{t \in [0,1]}$ in $C([0,1], (A_{\max} \Gamma) \otimes K(H))$ between $u_0$ and $u_1$ such that $v_t$ has propagation less than $s$ for every $t$ in $[0,1]$.

Notice that if $p$ and $q$ are two $\varepsilon$-projectors in $A_{\max} \Gamma \otimes K(H)$ such that $p_0 \sim_{s, \varepsilon} p_1$ then $\phi_\varepsilon(p_0)$ and $\phi_\varepsilon(p_1)$ are homotopic projectors.

5.2. Propagation and assembly map. As before $\Gamma$ is a finitely generated group which is residually finite with respect to a family $\Gamma_0 \supset \Gamma_1 \supset \ldots \supset \Gamma_n \supset \ldots$ of normal finite index subgroups of $\Gamma$.

Recall that $\Psi^\Gamma_r(\Gamma)$ and $\Psi^\Gamma_r(P_r(\Gamma))$ are respectively the $*$-algebras of pseudo-local, $\Gamma$-equivariant and finite propagation operators on $l^2(\Gamma) \otimes H$ and $H_{P_r(\Gamma)}$. Recall from section 5.2 and corollary 4.11 that any element of $K_0(P_r(\Gamma), C(\Gamma/\Gamma_i))$ can be represented by K-cycle $(I^0_r, \rho_r, C(\Gamma/\Gamma_i, H_{P_r(\Gamma)}), I^0_r, F)$, where

- $\rho_r$ is the standard representation of $C_0(P_r(\Gamma))$ on $H_{P_r(\Gamma)} = L^2(P_r(\Gamma), \eta_r) \otimes H$;
- $F$ is a K-cycle of $\Psi^\Gamma_r(H_{P_r(\Gamma)})$.

We have identified $I^0_{\Gamma_i} H_{P_r(\Gamma)}$ with $C(\Gamma/\Gamma_i, H_{P_r(\Gamma)})$, $I^0_{\Gamma_i} C(\Gamma/\Gamma_i) \cong C(\Gamma/\Gamma_i) \otimes H_{P_r(\Gamma)}$ provided with the diagonal action of $\Gamma$;

- Under this identification, $I^0_{\Gamma_i} \rho_r$ is the pointwise representation $\rho_r$ on $H_{P_r(\Gamma)}$ and $I^0_{\Gamma_i} F$ is the pointwise action by $\Gamma/\Gamma_i \to B(H_{P_r(\Gamma)}); \gamma \mapsto \gamma(F)$.
For a K-cycle $F$ of $\Psi^\Gamma, (P_r(\Gamma))$, let us denote by $x_F$ the corresponding element in $K^{\top} (\Gamma, C(\Gamma/\Gamma_1))$ coming from the K-cycle $(I_r^\Gamma, \rho_r, C(\Gamma/\Gamma_1, H_{P_r(\Gamma)}), I_r^\Gamma, T_1)$.

Let us set $\zeta = I_r^\Gamma, \ell^2(\Gamma) \otimes H \cong C(\Gamma/\Gamma_1, \ell^2(\Gamma) \otimes H)$ and let $\zeta, \Gamma$ be the right $C(\Gamma/\Gamma_1) \rtimes_{\max} \Gamma$-Hilbert module constructed from $\zeta$ in section 5.3. Notice that $\zeta, \Gamma$ as a right $C(\Gamma/\Gamma_1) \rtimes_{\max} \Gamma$-Hilbert module is isomorphic to $H \otimes C(\Gamma/\Gamma_1) \rtimes_{\max} \Gamma$ (compare with isomorphism of equation [4.2]).

Proceeding as we did for proving lemma 4.15 and denoting the multiplier algebra of $C(\Gamma/\Gamma_1, K(H)) \rtimes_{\max} \Gamma$ by $M(C(\Gamma/\Gamma_1, K(H)) \rtimes_{\max} \Gamma)$, we get with notations of section 4.3.

**Lemma 5.4.** For every integer $i$, there is a $*$-homomorphism

$$\lambda_i : \Psi_i^\Gamma(\Gamma) \to M(C(\Gamma/\Gamma_1, K(H)) \rtimes_{\max} \Gamma)$$

such that

- Under the identification $\zeta, \Gamma \cong H \otimes C(\Gamma/\Gamma_1) \rtimes_{\max} \Gamma$, then $\lambda_i(S)$ restricts to $I_r^\Gamma, S$ on $C_c(\Gamma) \cdot \zeta, \Gamma$;
- For any $f$ in $C(\Gamma/\Gamma_1, K(H))$, viewed as a locally compact operator on $\ell^2(\Gamma) \otimes H$, then $\lambda_i(f)$ is the image of $f$ under the inclusion $C(\Gamma/\Gamma_1, K(H)) \hookrightarrow C(\Gamma/\Gamma_1, K(H)) \rtimes_{\max} \Gamma$;
- For every $\gamma$ in $\Gamma$, then $\lambda_i(R_\gamma)$ is the left multiplication by $\delta_\gamma \in C(\Gamma/\Gamma_1) \rtimes_{\max} \Gamma$ (viewed as a multiplier of $C(\Gamma/\Gamma_1, K(H)) \rtimes_{\max} \Gamma$).

**Remark 5.5.** Let us denote by $C[\Gamma]^{\Gamma, i}$ the set of $\Gamma_i$-equivariant operators of $C[\Gamma]$. Then $C[\Gamma]^{\Gamma, i}$ is a $*$-algebra isomorphic to $C_c(\Gamma, C(\Gamma/\Gamma_1))$ (equipped with convolution product) and thus $\lambda_i$ induces by restriction a homomorphism

$$C((\Gamma/\Gamma_1) \rtimes_{\max} \Gamma) \to M(C(\Gamma/\Gamma_1, K(H)) \rtimes_{\max} \Gamma),$$

which is in fact the natural inclusion. According to [4, lemma 4.13], lemma 2.4 can be generalised to the equivariant case and hence for every real $t$ and any integer $i$, there exists a positive real $C_{t, i}$ such that for any element $S$ of $C[\Gamma]^{\Gamma, i}$ with propagation less than $t$, then $\|\lambda_i(S)\|_{C(\Gamma/\Gamma_1, K(H)) \rtimes_{\max} \Gamma} \leq C_{t, i}\|S\|_{\ell^2(\Gamma) \otimes H}$.

Let us fix until the end of this subsection

- a $\Gamma$-equivariant coarse equivalence $\tilde{\phi}_r : P_r(\Gamma) \to \Gamma$;
- a isometry $W_r : H_r \to \ell^2(\Gamma) \otimes H$ that covers $\phi_r$.

By using the same argument as in the third step of the proof of theorem 4.16 we get the following result.

**Proposition 5.6.** Let $x$ be in $K^{\top} (\Gamma, C(\Gamma/\Gamma_1))$ coming from an element $x_F$ in $K K^\Gamma_r (P_r(\Gamma), C(\Gamma/\Gamma_1))$ for a K-cycle $F$ in some $\Psi^\Gamma, (P_r(\Gamma))$. Then $\mu_{r, C(\Gamma/\Gamma_1), \max, *}(x)$ is the class in $K_* (C(\Gamma/\Gamma_1) \rtimes_{\max} \Gamma)$ of the K-cycle

$$(H \otimes C(\Gamma/\Gamma_1) \rtimes_{\max} \Gamma, \lambda_i(W_r F W_r^* + \text{Id}_{\ell^2(\Gamma) \otimes H} - W_r W_r^*)) .$$

Define $F_{r, i} = \lambda_i(W_r F W_r^* + \text{Id}_{\ell^2(\Gamma) \otimes H} - W_r W_r^*)$ for $F$ a K-cycle of $\Psi^\Gamma, (P_r(\Gamma))$ and let us set in the even case

$$V_F = \begin{pmatrix} \text{Id}_H & F_{r, i} \cdot \text{Id}_H & 0 \\ 0 & \text{Id}_H & 0 \\ -F_{r, i} \cdot \text{Id}_H & 0 & \text{Id}_H \end{pmatrix} .$$
where $\text{Id}_H$ is viewed as the unit of $C(\Gamma/\Gamma_1, K(H))^{\times_{\max} \Gamma}$. Since $\lambda_i$ is a $*$-homomorphism, we see that the matrix

$$V_F \begin{pmatrix} \text{Id}_H & 0 \\ 0 & 0 \end{pmatrix} V_F^{-1} - \begin{pmatrix} \text{Id}_H & 0 \\ 0 & 0 \end{pmatrix}$$

has coefficients in $C(\Gamma/\Gamma_1, K(H))^{\times_{\max} \Gamma}$ and moreover we get

**Proposition 5.7.** With notations of proposition 5.4, if $x$ in $K^{op}_\Lambda (\Gamma, C(\Gamma/\Gamma_1))$ comes from an element $x_F$ in $KK_\Lambda^j(\Gamma, C(\Gamma/\Gamma_1))$ for a $K$-cycle $F$ in some $\Psi^j(\Gamma_1)$ and if we set $e_F = V_F \begin{pmatrix} \text{Id}_H & 0 \\ 0 & 0 \end{pmatrix} V_F^{-1}$, then

\[ \mu_{\Gamma, C(\Gamma/\Gamma_1), \text{max}, 0}(x) = [e_F] - \left[ \begin{pmatrix} \text{Id}_H & 0 \\ 0 & 0 \end{pmatrix} \right]. \]

The crucial point is that with notations of proposition 5.7, then the coefficients of the idempotent $e_F$ have indeed finite propagation depending only on the propagation of $F$. Since with notation of lemma 5.4, the algebra $C[\Gamma]^{\times_{\max}}$ is generated by $R_{\gamma}$ for $\gamma$ in $\Gamma$ an by functions $f$ in $C(\Gamma/\Gamma_1, K(H))$, it is straightforward to check that $\lambda_i$ is propagation preserving. Using this, we obtain for every positive real $r$ the existence of a non-decreasing function $h_r : \mathbb{R}^+ \to \mathbb{R}^+$ (which is in fact affine), independant on $i$, such that for every $s$ and every $K$-cycle $F$ in $\Psi^j(\Gamma)$ with propagation less than $s$, then with notation of lemma 5.4, the idempotent $e_F$ has propagation less than $h_r(s)$. Notice that $e_F$ has operator norm less than $\alpha_{r,i} = (1 + \|F\|r_i)^6$. Recall that if we set $e_F' = (1 + (2e_F - 1)(2e_F - 1))^{1/2} \cdot (1 + (2e_F - 1)(2e_F - 1))^{1/2}$, then $e_F'$ is a projector equivalent to $e_F$. Fix once for all two sequences of real polynomial functions $(P_j)_{j \in \mathbb{N}}$ and $(Q_j)_{j \in \mathbb{N}}$ such that $P_j$ and $Q_j$ have degree $j$ for all $j \in \mathbb{N}$ and on every compact subset of $\mathbb{R}^+$,

- $(P_j)_{j \in \mathbb{N}}$ converges uniformly to $t \mapsto \sqrt{1 + t}$;
- $(Q_j)_{j \in \mathbb{N}}$ converges uniformly to $t \mapsto \sqrt[4]{1 + t}$.

Let us define $\Psi^j_\Lambda(\Gamma_1)(P_j(\Gamma)) = \{ T \in \Psi^j(\Gamma_1) \}$ such that $\|T\| \leq 1$. For $F$ a $K$-cycle of $\Psi^j_\Lambda(\Gamma_1)$, a positive real $r$ and $\epsilon$ in $(0, 1/4)$ let $j_{\epsilon,i}$ be the smallest integer such that

$$|P_j(t) - \sqrt{1 + t}| \leq ((8 + 4\alpha_{r,i}))^{-2} \text{ and } |Q_j(t) - \frac{1}{\sqrt[4]{1 + t}}| \leq ((8 + 4\alpha_{r,i}))^{-2}$$

for all integer $j \geq j_{\epsilon,i}$ and all $t \in [0, 4\alpha_{r,i}]$. For $F$ a $K$-cycle of $\Psi^j_\Lambda(\Gamma_1)(P_j(\Gamma))$ with propagation less than $s$, let us set

$$\tilde{p}_{F,\epsilon} = \frac{1}{2} \tilde{Q}_{j_{\epsilon,i},i}(2e_F - 1)(2e_F - 1)(2(e_F + e_F')P_{j_{\epsilon,i},i}(2e_F - 1)(2e_F - 1)).$$

Then $\|e_F' - \tilde{p}_{F,\epsilon}\| < \epsilon/8$ and according to remark 5.2, then $\tilde{p}_{F,\epsilon}$ is a $\epsilon$-projection and has propagation less than $(2j_{\epsilon,i} + 1)h_r(s)$. Moreover, for any continuous function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(t) = 0$ for $t < \frac{1 - \sqrt{1 - \epsilon/2}}{2}$ and $\phi(t) = 1$ for $t > \frac{1 + \sqrt{1 - \epsilon/2}}{2}$, then $\phi_{\epsilon}(\tilde{p}_{F,\epsilon})$ is a projector equivalent to $e_F$. Now fix an identification between $K(H)$ and the closure of $\cup_{n \in \mathbb{N}} M_n(C)$ and consider $q_n$ the rank $2n$ projector of $M_2(C(\Gamma/\Gamma_1, K(H))^{\times_{\max} \Gamma})$ corresponding to the identity of $M_2(M_n(C))$. For a $K$-cycle $F$ of $\Psi^j_\Lambda(\Gamma_1)(P_j(\Gamma))$, let $n_{F,\epsilon}$ be the smaller integer $n$ such that $\|q_n \tilde{p}_{F,\epsilon} / q_n - \tilde{p}_{F,\epsilon}/2 + (\text{Id}_H - I_0) / 2\| < \epsilon/8$ and set $p_{F,\epsilon} = q_{n_{F,\epsilon}} \tilde{p}_{F,\epsilon} / 2 q_{n_{F,\epsilon}}$. Then $\|p_{F,\epsilon} + (\text{Id}_H - I_0) / 2\| < \epsilon/4$, and according to remark 5.2, $p_{F,\epsilon}$ is as a summand of $p_{F,\epsilon} + (\text{Id}_H - I_0) / 2$. 

\[ \text{Id}_H \]

\[ 0 \]

\[ 0 \]

\[ 0 \]
a $\varepsilon$-projector in $M_{2n,F,\varepsilon}(C(\Gamma/\Gamma_i)\rtimes_{\text{max}} \Gamma)$. Moreover, we have

$$\mu_{F,C(\Gamma/\Gamma_i),\text{max},0}(x_F) = [\phi_\varepsilon(p_{F,\varepsilon})] - [I_{n,F,\varepsilon}].$$

In the odd case, if $F$ is a K-cycle of $\Psi_1^P(P_i(\Gamma))$ with propagation less than $s$, let us set using the notations of the discussion following proposition 5.6 $q_F = 1/(2(p_{F,\varepsilon} + \text{Id}_H))$. For $\varepsilon$ in $(0, 1/4)$ and $r$ positive, let $l_{r,F,\varepsilon}$ be the smallest integer such that \(\sum_{t=l_{r,F,\varepsilon}+1}^{\infty} (\alpha_{r,i} + 2)^t/t! < \varepsilon/(3\alpha_{r,i} + 6)\). Let us define

$$u_{F,\varepsilon} = \sum_{l=0}^{l_{r,F,\varepsilon}} (2\pi q_F)^t/t! - q_F \sum_{t=1}^{l_{r,F,\varepsilon}} (2\pi)^t/t!.$$

It is straightforward to check that

- $u_{F,\varepsilon} - \text{Id}_H$ is indeed an element of $C(\Gamma/\Gamma_i, K(H))\rtimes_{\text{max}} \Gamma$ with propagation less than $l_{r,F,\varepsilon}h_r(s)$.
- $\|u_{F,\varepsilon} - e^{2\pi i q_F}\| \leq \varepsilon/3$.

In view of remark 5.3, $u_{F,\varepsilon}$ is a $\varepsilon$-unitary. Moreover, if $x$ in $K^{\text{top}}(\Gamma, C(\Gamma/\Gamma_i))$ comes from $x_F$ in $KK^1_c(P_i(\Gamma), C(\Gamma/\Gamma_i))$ for $F$ in some $\Psi_1^P(P_i(\Gamma))$, then $\mu_{F,C(\Gamma/\Gamma_i),\text{max},1}(x)$ is the class of $u_{F,\varepsilon}$ in $K_1(C(\Gamma/\Gamma_i)\rtimes_{\text{max}} \Gamma)$.

**Remark 5.8.** According to remark 5.3, for all $\varepsilon$ in $(0, 1/4)$, i positive integer and $s$ positive real, then the sets $\{j_{r,F,\varepsilon}: F$ K-cycle of $\Psi_1^P(P_i(\Gamma))$ of propagation less than $s$ and $l_{r,F,\varepsilon}$ $F$ K-cycle of $\Psi_1^P(P_i(\Gamma))$ of propagation less than $s$ are bounded. Thereby, if $J_{r,i,\varepsilon}$ and $L_{r,i,\varepsilon}$ are respectively their supremum, then for all K-cycle $F$ of $\Psi_1^P(P_i(\Gamma))$ with propagation less than $s$, we get that $\overline{p_{F,\varepsilon}}$ and $p_{F,\varepsilon}$ have propagation less than $(2J_{r,i,\varepsilon} + 1)h_r(s)$ and $u_{F,\varepsilon}$ has propagation less than $L_{r,i,\varepsilon}h_r(s)$.

With notations of lemma 2.2, let $x$ be an element of $K^{\text{top}}(\Gamma, B_T)$. Under the identification $KK^1_c(P_i(\Gamma), B_T) \cong \prod_{i\in\mathbb{N}} KK^1_c(P_i(\Gamma), C(\Gamma/\Gamma_i))$ of proposition 3.4, we can assume that $x$ comes from an element $(x_{F,\varepsilon})_{i\in\mathbb{N}}$, where

- $F_i$ is K-cycle of $\Psi_1^P(P_i(\Gamma))$ for every positive integer $i$;
- there exists a real $s$ such that $F_i$ has propagation less than $s$ for every positive integer $i$.

By viewing $B_T\rtimes_{\text{max}} \Gamma = \ell^\infty(\bigcup_{i\in\mathbb{N}} C(\Gamma/\Gamma_i, K(H))\rtimes_{\text{max}} \Gamma)$ as an algebra of multipliers of $\bigoplus_{i\in\mathbb{N}} C(\Gamma/\Gamma_i, K(H))\rtimes_{\text{max}} \Gamma$, we see that $B_T\rtimes_{\text{max}} \Gamma$ is indeed a closed $*$-subalgebra of $\prod_{i\in\mathbb{N}} C(\Gamma/\Gamma_i, K(H))\rtimes_{\text{max}} \Gamma$. In particular, with above notations, if $x$ is even and since $\|\lambda(S_i)\| \leq \|\lambda_T(S_i)\|_{i\in\mathbb{N}}$ for all uniformly bounded family $S_i$ in $\prod_{i\in\mathbb{N}} \Psi_1^P(\Gamma_i)$ with propagation uniformly bounded and all integer $j$, then the idempotent $(e_{F,\varepsilon})_{i\in\mathbb{N}}$ and hence the projector $(e_{F,\varepsilon})_{i\in\mathbb{N}}$ belong to $M_2(B_T\rtimes_{\text{max}} \Gamma)$ and moreover, $\mu_{F,\text{max},0}(x) = [(e_{F,\varepsilon})_{i\in\mathbb{N}}] - [0_{0,0}]$. Furthermore, the family of integers $(j_{r,F,\varepsilon})_{i\in\mathbb{N}}$ is bounded and hence $(p_{F,\varepsilon})_{i\in\mathbb{N}}$ and $(p_{F,\varepsilon})_{i\in\mathbb{N}}$ are $\varepsilon$-projector in $M_2(B_T\rtimes_{\text{max}} \Gamma)$. Since $\|\overline{p_{F,\varepsilon}} - e_{F,\varepsilon}\| < \varepsilon$ for all integer $i$, we finally get that $(\phi_\varepsilon(p_{F,\varepsilon}))_{i\in\mathbb{N}}$ is a projector of $M_2(B_T\rtimes_{\text{max}} \Gamma)$ homotopic to $(e_{F,\varepsilon})_{i\in\mathbb{N}}$ and hence

$$\mu_{F,\text{max},0}(x) = [(\phi_\varepsilon(p_{F,\varepsilon}))_{i\in\mathbb{N}}] - [(\text{Id}_H)_{i\in\mathbb{N}}] = [(\phi_\varepsilon(p_{F,\varepsilon}))_{i\in\mathbb{N}}] - [(I_{n,F,\varepsilon})_{i\in\mathbb{N}}].$$
In the same way, in the odd case we get that \((u_F)_i \in \mathbb{N}\) is a \(\varepsilon\)-unitary of \(B_{\Gamma,\{n,\}}\) and
\[ \mu_{\Gamma, B_{\Gamma,\{n,\}}} (x) = [(u_F)_i] \in \mathbb{N} \].

5.3. Asymptotic statements. For any integer \(i\) and any positive real \(r, r', s, s'\) and any \(\varepsilon \in (0, 1/2)\), we let us consider the following statements

\(Q_{1_0}(i, r, r', s, \varepsilon)\): for any (even) K-cycle \(F\) of \(\Psi_1^\Gamma (P_r (\Gamma))\), then \((p_{F, \varepsilon}, n_{F, \varepsilon}) \sim_{18s, s}\) \((0, 0)\) implies that \(x_F\) lies in the kernel of the homomorphism
\[ KK_0^\Gamma (P_r (\Gamma), C(\Gamma/T_i)) \longrightarrow KK_0^\Gamma (P_{r'} (\Gamma), C(\Gamma/T_i)) \]
induced by the inclusion \(P_r (\Gamma) \hookrightarrow P_{r'} (\Gamma)\).

\(Q_{1_1}(i, r, r', s, \varepsilon)\): for any (odd) K-cycle \(F\) of \(\Psi_1^\Gamma (P_r (\Gamma))\), then \(u_{F, \varepsilon} \sim_{\varepsilon, s} \text{Id}_H\) implies that \(x_F\) lies in the kernel of the homomorphism
\[ KK_1^\Gamma (P_r (\Gamma), C(\Gamma/T_i)) \longrightarrow KK_1^\Gamma (P_{r'} (\Gamma), C(\Gamma/T_i)) \]
induced by the inclusion \(P_r (\Gamma) \hookrightarrow P_{r'} (\Gamma)\).

\(Q_{5_0}(i, r, s, s', \varepsilon)\): For any \(\varepsilon\)-projector \(p\) in some \(M_k (C(\Gamma/T_i)) \cong_{\text{max} \Gamma}\) with propagation less than \(s\), and any integer \(n\), there exists a (even) K-cycle \(F\) of \(\Psi_1^\Gamma (P_r (\Gamma))\) such that \((p_{F, \varepsilon}, n_{F, \varepsilon}) \sim_{18s, s} (p, n)\).

\(Q_{5_1}(i, r, s, s', \varepsilon)\): For any \(\varepsilon\)-unitary \(u\) in \(C(\Gamma/T_i, K(H)) \cong_{\text{max} \Gamma}\) with propagation less than \(s\), there exists a (odd) K-cycle \(F\) of \(\Psi_1^\Gamma (P_r (\Gamma))\) such that \(u_{F, \varepsilon} \sim_{\varepsilon, s} u\).

Remark 5.9. It is straightforward to check that if two \(\varepsilon\)-projectors are \(\varepsilon\)-closed, then they are homotopic as \(18\varepsilon\)-projectors and hence conditions \(Q_{1_0}\) and \(Q_{5_0}\) do not depend on a particular choice of sequences of polynomial functions \((P_n)_n \in \mathbb{N}\) and \((Q_n)_n \in \mathbb{N}\) used in the definition of \(p_{F, \varepsilon}\). Moreover, replacing \(n_{F, \varepsilon}\) by any integer \(n\) with \(n \geq n_{F, \varepsilon}\) and \(p_{F, \varepsilon}\) by \(\varepsilon n_{F, \varepsilon}/2q_n\) does not either affect conditions \(Q_{1_0}\) and \(Q_{5_0}\).

Theorem 5.10. Let \(\Gamma\) be a finitely generated group residually finite with respect to a family \(\Gamma_0 \supset \Gamma_1 \supset \ldots \Gamma_n \supset \ldots\) of normal finite index subgroups and let \(l\) be in \(\{0, 1\}\). The following statements are equivalent:

(i) For any positive real \(r\) the following condition holds: there is an \(\varepsilon\) in \((0, 1/2)\) such that for any positive real \(s\), there exists an integer \(j\) and a positive real \(r'\) for which \(Q_{1_1}(i, r, r', s, \varepsilon)\) is true for all \(i \geq j\).

(ii) The maximal coarse assembly map \(\mu_{X(\Gamma), \{n,\}}\) is injective.

Proof. Let us give the proof in the even case, the odd one been quite similar (even simpler). In view of theorem 4.14b condition (ii) is equivalent to injectivity of \(\mu_{\Gamma, B_{\Gamma,\{n,\}}}\). Let us prove that condition (i) implies injectivity of \(\mu_{\Gamma, B_{\Gamma,\{n,\}}}\). According to remark 4.3, this amounts to prove that for any \(x\) in \(K_0^{\text{top}}(\Gamma, B_{\Gamma})\), then the condition \(\mu_{\Gamma, B_{\Gamma,\{n,\}}} (x) = 0\) implies that \(x\) belongs indeed to \(K_0^{\text{top}}(\Gamma, B_{\Gamma,\{n,\}})\). Up to replace \(\cup_{i \in \mathbb{N}} \Gamma_i / \Gamma_1\) by \(\cup_{i \geq i_0} \Gamma_i / \Gamma_1\) for some integer \(i_0\), we can actually assume that \(\mu_{\Gamma, B_{\Gamma,\{n,\}}} (x) = 0\). Suppose that \(x\) comes from an element \((x_{F_i})_i \in \mathbb{N}\) in some \(KK_0^\Gamma (P_r (\Gamma), B_{\Gamma}) \cong \bigoplus_{i \in \mathbb{N}} K K_0^\Gamma (P_r (\Gamma), C(\Gamma/T_i))\) for \(r\) positive real, where \((F_i)_i \in \mathbb{N}\) is a family of \(K\)-cycles in \(\bigoplus_{i \in \mathbb{N}} \Psi_1^\Gamma (P_r (\Gamma))\) with propagation uniformly bounded. Then there exist integers \(k\) and \(n\) and a projector homotopy in \(M_{n+k+2} (B_{\Gamma, \{n,\}})\) between \(\left(\phi_{\varepsilon, (p_{F_{i_0}}, \varepsilon)}\right)_i \in \mathbb{N}\) and \(\left(I_{n, (p_{F_{i_0}}, \varepsilon)}\right)_i \in \mathbb{N}\), where
and proceeding as we did in section 5.2 to obtain for any \( z \)

\[ \text{Proof.} \]

the even case, the odd case being quite similar. Recall that we have fixed an assumption of the theorem for \( s \) as above, then there exists an \( r' \) such that \( x_r \) lies in the kernel of \( KK_0^1(P, \Gamma, C(\Gamma/\Gamma_1)) \rightarrow KK_0^1(P, \Gamma, C(\Gamma/\Gamma_1)) \) for all integer \( i \geq j \). This implies that \( x \) comes indeed from an element in \( \bigoplus_{i=0}^{j-1} KK_0^1(P, \Gamma, C(\Gamma/\Gamma_1)) \) and hence belongs to \( \bigoplus_{i \in \mathbb{N}} K_0^{\text{top}}(\Gamma, C(\Gamma/\Gamma_1)) \).

Conversely, assume that for some positive real \( r \), then for any \( \varepsilon \) in \((0,1/72)\) there exists a positive real \( s \) such that for every integer \( j \) and positive real \( r' \), there exists an integer \( i \) with \( i \geq j \) for which \( QI_0(i, r, r', s, \varepsilon) \) does not hold. Let us prove that \( \mu_{\Gamma, P, r, \max, 0} \) is not injective. If \( r \) is as above, let us fix \( \varepsilon \) in \((0,1/72)\) and \( (r'_i)_{i \in \mathbb{N}} \) an increasing and unbounded sequence of positive reals. Then we can find an increasing sequence \( (j_i)_{i \in \mathbb{N}} \) of integers, and for each integer \( i \) a K-cycle \( F_i \) in \( \Psi_{1, i}^1(P, \Gamma) \) such that \( \left( P_{F_i, x, n_{F_i, x}} \right) \sim_{18\varepsilon, s} \left( 0,0 \right) \) and \( x_{F_i, x} \) does not belong to the kernel of \( KK_0^1(P, \Gamma, C(\Gamma/\Gamma_1)) \rightarrow KK_0^1(P, \Gamma, C(\Gamma/\Gamma_1)) \). By using a cut-off function for the action of \( \Gamma \) on \( P, \Gamma \) and in view of remark 5.3 we can actually assume that the family \( (F_{j, i})_{i \in \mathbb{N}} \) as propagation uniformly bounded. Define for any integer \( k \) the K-cycle \( F_k \) of \( \Psi_{1, i}^1(P, \Gamma) \) to be \( F_{j, k} \) if \( k = j \), for some integer \( i \) and \( \text{Id} \) otherwise. Let \( x \) be the element of \( K_0^{\text{top}}(\Gamma, B_r) \) arising from \( (x_{F_i})_{i \in \mathbb{N}} \). We clearly have \( \mu_{\Gamma, P, r, \max, 0}(x) = 0 \) and \( x \) does not sit in \( \bigoplus_{i \in \mathbb{N}} K_0^{\text{top}}(\Gamma, C(\Gamma/\Gamma_1)) \). Hence, the image of \( x \) under the epimorphism \( K_0^{\text{top}}(\Gamma, B_r) \rightarrow K_0^{\text{top}}(\Gamma, A_r) \) is a non vanishing element of the kernel of \( \mu_{\Gamma, P, r, \max, 0} \).

**Corollary 5.11.** If \( \Gamma \) is residually finite, finitely generated and uniformly embeddable into a Hilbert space, then \( \Gamma \) satisfies condition (i) of theorem 5.10.

**Remark 5.12.** As already mentioned, under the assumption of corollary 5.11, the reduced assembly map \( \mu_{\Gamma, P, \max, \ast} \) is injective. Moreover, the group \( \Gamma \) is \( K \)-exact and hence, in view of the proof of theorem 5.10 and if we consider conditions \( QI_0 \) and \( QI_1 \) with reduced assembly maps instead of maximal ones, we get in this setting an analogue of corollary 5.11 for \( \Gamma \).

**Theorem 5.13.** Let \( \Gamma \) be a finitely generated group residually finite with respect to a family \( \Gamma_0 \supset \Gamma_1 \supset \ldots \Gamma_n \supset \ldots \) of normal finite index subgroups and let \( l \) be in \( \{0,1\} \). Assume that there exists \( \varepsilon \) in \((0,1/72)\) such that the following condition holds: for every positive real \( s \), there exist positive real \( r \) and \( s' \) and an integer \( j \) such that \( QS_l(i, r, s, s', \varepsilon) \) is true for all integer \( i \geq j \). Then \( \mu_{\Gamma, P, r, \max, l} \) is surjective.

**Proof.** As for injectivity, it is enough in view of theorem 4.10 to prove that \( \mu_{\Gamma, P, r, \max, l} \) is surjective and according to remark 4.3, this amounts to prove that for any \( z \) in \( K_1(B_r, \max, \Gamma) \), there exists an element \( x \) in \( K_{1, r, \max, \ast}(\Gamma, B_r) \) such that \( \mu_{\Gamma, P, r, \max, l}(x) \sim z \) belongs to \( K_1(B_r, \max, \Gamma) \). As before, we give the proof in the even case, the odd case being quite similar. Recall that we have fixed an
identification $\mathcal{K}(H) \cong \bigcup_{n \in \mathbb{N}} M_n(\mathbb{C})$. It is then straightforward to check that every element in $K_0(B_F \rtimes_{\max} \Gamma)$ can be written down as the difference of the classes of projector that satisfies the strong Baum-Connes conjecture. Let $\epsilon$ be as in the assumption of the theorem. We can indeed assume that there exists a positive real $s$ and $\epsilon$-projector of $(\prod_{i \in \mathbb{N}} M_n_i(C(\Gamma_i))) \rtimes_{\max} \Gamma$ with propagation less than $s$ and such that $p = (\phi_\epsilon(q_i))_{i \in \mathbb{N}}$. Let $r$ and $s'$ be positive reals and let $j$ be a positive integer such that $QS_0(i, r, s, s', \epsilon)$ is true for every integer $i \geq j$, i.e. there exists a $K$-cycle $F_i$ in $\Psi^1(P_r(\Gamma))$ such that $(p_{F_i, x, s, s', \epsilon}) \sim_{\epsilon} (q_i, 0)$. By using a cut-off function for the action of $\Gamma$ on $P_r(\Gamma)$ and in view of remark 5.3 we can actually assume that the family $(F_i)_{i \in \mathbb{N}}$ as propagation uniformly bounded. If we set $F_i = I_{H_{F_i, x}}$ for every positive integer $i$ with $i \leq j - 1$ and then consider the element $x$ of $K_0^{top}(G, B_F)$ coming from $(x_{F_i})_{i \in \mathbb{N}} \in KK^1_0(P_r(\Gamma), B_F)$, we get that $\mu_{\Gamma, A_F, \max, 0}(x) - [p]$ belongs to $\bigoplus_{i \in \mathbb{N}} K_0(C(\Gamma_i)) \rtimes_{\max} \Gamma$. 

As we shall see, up to a slight modification in the sequence of finite index normal subgroups in $\Gamma$, we get a converse result for theorem 5.13. This allows in particular to deal at least with group that satisfies the strong Baum-Connes conjecture. Let us set for any $i$ integer $X_i(\Gamma) = \prod_{j \geq i} \Gamma_j / \Gamma_j$ and $X_\infty(\Gamma) = \prod_{i \in \mathbb{N}} X_i(\Gamma)$ provided with the action of $\Gamma$ inherited by the action on $X(\Gamma)$. Let us equip $X_\infty(\Gamma)$ with a $\Gamma$-invariant metric $d$ such that the restriction of $d$ to each $X_i(\Gamma)$ coincides with the metric on $X(\Gamma)$ and $d(X_i(\Gamma), X_j(\Gamma)) \leq i + j$ for every integer $i$ and $j$. Let us set $A_F^\infty = C^\infty(X_\infty(\Gamma), \mathcal{K}(H)) / C_0(X_\infty(\Gamma), \mathcal{K}(H))$. The space $X_\infty(\Gamma)$ is indeed construct in the same way as $X(\Gamma)$ by considering the sequence of finite index normal subgroups $\Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \Gamma_3 \supset \ldots$ and hence, according to theorem 4.13, we get

**Proposition 5.14.** The following assertions are equivalent

(i) The maximal coarse assembly map

$$\mu_{X_\infty(\Gamma), \max, \epsilon} : \lim \ K_*(P_r(X_\infty(\Gamma)), \mathbb{C}) \to K_*(C_{max}(X_\infty(\Gamma)))$$

is surjective.

(ii) the maximal assembly map

$$\mu_{\Gamma, A_F^\infty, \max, \epsilon} : K^*_*(\Gamma, A_F^\infty) \to K_*(A_F^\infty \rtimes_{\max} \Gamma)$$

is surjective.

We have of course analogous statements for injectivity and isomorphism. We are now in position to give a weak converse result for theorem 5.13

**Theorem 5.15.** Let $\Gamma$ be a finitely generated group, residually finite with respect to a family $\{0, 1\}$ of normal finite index subgroups and let $l$ be in $\{0, 1\}$. Assume that the maximal coarse assembly map

$$\mu_{X_\infty(\Gamma), \max, l} : \lim \ K_l(P_r(X_\infty(\Gamma)), \mathbb{C}) \to K_l(C_{max}(X_\infty(\Gamma)))$$

is onto. Then there exists $\epsilon$ in $(0, 1/72)$ such that the following condition is satisfied: for every positive reals $s$, there exist positive reals $r$ and $s'$ and an integer $j$ such that $QS_4(i, r, s, s', \epsilon)$ is true for all integer $i \geq j$.  


Proof. As before we give the prove for the even case. Assume that for all $\varepsilon$ in $(0, 1/72)$ there exists a positive real $s$ such that for all positive reals $r$ and integer $j$ there exists an integer $i$ with $i \geq j$ for which $QS_0(i, r, s, s', \varepsilon)$ does not holds.  

In view of proposition 5.14, let us show that $\varepsilon$-projection $q_{j,k}$ to $\Psi_{\Gamma}^T(P_k(\Gamma))$ for which $(x_F, s, n_F, s') \sim_{18\varepsilon,s'} (q_{j,k}, k, m_{j,k})$. For $j$ and $k$ integers such that $j \geq k$, define

- $q_{j,k}$ to be $q_{j,k}$ for some integer $i$ and $q_{j,k} = 0 = 0$ otherwise.
- $m_{j,k}$ to be $m_{j,k}$ if $j = j_k$ for some integer $i$ and $m_{j,k} = 0 = 0$ otherwise.

Let us set $B_{\Gamma}^\infty = \ell^\infty(X_\infty(\Gamma), \mathcal{K}(H))$ and $B_{\Gamma,0}^\infty = C_0(X_\infty(\Gamma), \mathcal{K}(H))$. As in the proof of theorem 5.13 surjectivity will fail if there is no $x$ in $K_{0}(X_\infty(\Gamma), \mathcal{K}(H))$ such that $\mu_{(\ell^\infty, \mathcal{K}(H))} = 0$ and $\mu_{(\ell^\infty, \mathcal{K}(H))} = 0$. Let us fix $\varepsilon$ in $(0, 1/72)$ and $(s')\in\mathbb{N}$ and $(r_t)\in\mathbb{N}$ increasing and unbounded sequences of positive reals. Then for each integer $k$, there exist an increasing sequence of elements $(j_k)\in\mathbb{N}$, an $\varepsilon$-projection $q_{j,k}$ with propagation less than $s$ in some $M_{j,k}(C(\Gamma/\Gamma_j) \times_{\text{max}} \Gamma)$ and an integer $m_{j,k}$ such that there is no K-cycle $F$ in $\Psi_{\Gamma}^T(P_k(\Gamma))$ for which $(x_F, s, n_F, s') \sim_{18\varepsilon,s'} (q_{j,k}, k, m_{j,k})$. For $j$ and $k$ integers such that $j \geq k$, define

- $q_{j,k}$ to be $q_{j,k}$ for some integer $i$ and $q_{j,k} = 0 = 0$ otherwise.
- $m_{j,k}$ to be $m_{j,k}$ if $j = j_k$ for some integer $i$ and $m_{j,k} = 0 = 0$ otherwise.

Let us set $B_{\Gamma}^\infty = \ell^\infty(X_\infty(\Gamma), \mathcal{K}(H))$ and $B_{\Gamma,0}^\infty = C_0(X_\infty(\Gamma), \mathcal{K}(H))$. As in the proof of theorem 5.13 surjectivity will fail if there is no $x$ in $K_{0}(X_\infty(\Gamma), \mathcal{K}(H))$ such that $\mu_{(\ell^\infty, \mathcal{K}(H))} = 0$ and $\mu_{(\ell^\infty, \mathcal{K}(H))} = 0$. Let us fix $\varepsilon$ in $(0, 1/72)$ and $(s')\in\mathbb{N}$ and $(r_t)\in\mathbb{N}$ increasing and unbounded sequences of positive reals. Then for each integer $k$, there exist an increasing sequence of elements $(j_k)\in\mathbb{N}$, an $\varepsilon$-projection $q_{j,k}$ with propagation less than $s$ in some $M_{j,k}(C(\Gamma/\Gamma_j) \times_{\text{max}} \Gamma)$ and an integer $m_{j,k}$ such that there is no K-cycle $F$ in $\Psi_{\Gamma}^T(P_k(\Gamma))$ for which $(x_F, s, n_F, s') \sim_{18\varepsilon,s'} (q_{j,k}, k, m_{j,k})$. For $j$ and $k$ integers such that $j \geq k$, define

- $q_{j,k}$ to be $q_{j,k}$ for some integer $i$ and $q_{j,k} = 0 = 0$ otherwise.
- $m_{j,k}$ to be $m_{j,k}$ if $j = j_k$ for some integer $i$ and $m_{j,k} = 0 = 0$ otherwise.

Let us set $B_{\Gamma}^\infty = \ell^\infty(X_\infty(\Gamma), \mathcal{K}(H))$ and $B_{\Gamma,0}^\infty = C_0(X_\infty(\Gamma), \mathcal{K}(H))$. As in the proof of theorem 5.13 surjectivity will fail if there is no $x$ in $K_{0}(X_\infty(\Gamma), \mathcal{K}(H))$ such that $\mu_{(\ell^\infty, \mathcal{K}(H))} = 0$ and $\mu_{(\ell^\infty, \mathcal{K}(H))} = 0$. Let us fix $\varepsilon$ in $(0, 1/72)$ and $(s')\in\mathbb{N}$ and $(r_t)\in\mathbb{N}$ increasing and unbounded sequences of positive reals. Then for each integer $k$, there exist an increasing sequence of elements $(j_k)\in\mathbb{N}$, an $\varepsilon$-projection $q_{j,k}$ with propagation less than $s$ in some $M_{j,k}(C(\Gamma/\Gamma_j) \times_{\text{max}} \Gamma)$ and an integer $m_{j,k}$ such that there is no K-cycle $F$ in $\Psi_{\Gamma}^T(P_k(\Gamma))$ for which $(x_F, s, n_F, s') \sim_{18\varepsilon,s'} (q_{j,k}, k, m_{j,k})$. For $j$ and $k$ integers such that $j \geq k$, define

- $q_{j,k}$ to be $q_{j,k}$ for some integer $i$ and $q_{j,k} = 0 = 0$ otherwise.
- $m_{j,k}$ to be $m_{j,k}$ if $j = j_k$ for some integer $i$ and $m_{j,k} = 0 = 0$ otherwise.



Corollary 5.16. Let $\Gamma$ be a finitely generated group residually finite with respect to a family $\Gamma_0 \supset \Gamma_1 \supset \ldots \supset \Gamma_n \supset \ldots$ of normal finite index subgroups and let $l$ be in
If $\Gamma$ satisfies the strong Baum-Connes conjecture (for example if $\Gamma$ has the Haagerup property) then there exists $\varepsilon$ in $(0, 1/72)$ such that the following condition is satisfied: for every positive reals $s$, there exist positive real $r$ and $s'$ and an integer $j$ such that $QS_l(i, r, s, s', \varepsilon)$ is true for all integer $i \geq j$.

Since group satisfying the strong Baum-Connes conjecture are $K$-amenable [14], the same result holds if we replace in the definition of conditions $QS_0$ and $QS_1$ maximal assembly maps by the reduced one. More generally, in this setting, the analogue of the hypothesis of theorem 5.15 implies the surjectivity of the reduced Baum-Connes assembly map $\mu_{\Gamma, B^\infty \Gamma} : K^*_l(\Gamma, B^\infty \Gamma) \to K^*_l(B^\infty \Gamma \rtimes_{\max} \Gamma)$ for $l$ in $\{0, 1\}$ for $K$-exact groups (in particular for groups that embed uniformly in a Hilbert space).

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Université Blaise Pascal & CNRS, Clermont-Ferrand, France and PIMS, University of Victoria, Canada
E-mail address: oyono@math.cnrs.fr

Vanderbilt University, Nashville, USA
E-mail address: guoliang.yu@vanderbilt.edu