Minimum survival probabilities in a two-dimensional risk model perturbed by diffusion

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Abstract

In this paper we consider the finite time minimum survival probability and ultimate minimum survival probability in a two-dimensional risk model perturbed by diffusion. Using some properties of the minimum survival probability we obtain the equation of the finite time minimum survival probability and ultimate minimum survival probability that they are satisfied and, the explicit expressions for ultimate minimum survival probability are given in a special case.

Keywords: ruin probability, minimum survival probability, two-dimensional risk process

1 Introduction

In recent years a lot of work has been done with study of dependent classes of insurance business. Among other [1](2003) considered most first, the ruin probability in the two-dimensional risk model no perturbed by diffusion, and the problem to solve of sum ruin probability. And [2](2008) studied the express of equation satisfying the ruin probability and a method to solve of the ruin probability in a special two-dimensional risk model no perturbed by diffusion. [3](2007) considered the problem to obtain of Lundbergs lower bound for maximum ruin probability and studied the asymptotic estimation for finite time maximum ruin probability in a two-dimensional risk model perturbed by diffusion. Consider the a two-dimensional risk process that is perturbed by diffusion:
\[ dR(t) = Cdt + \sigma dW(t) + \int_{|z|<1} \alpha(z)(\mu - \nu)(dt, dz) \]
\[ + \int_{|z|>1} \alpha(z)\mu(dt, dz) \]  
(1)

\[ R(0) = u \]

where \( u = (u_1, u_2)' \) is the initial reserve of two insurance business and \( W(t) = (W_1(t), \ldots, W_{d_1}(t))' \) is independent and \( d_1 \)-dimensional Wiener process vector, \( \mu(dt, dz) = (\mu_1(dt, dz), \ldots, \mu_{d_2}(dt, dz))' \) is independent and Poisson measure vector and that \( E(\mu_i(dt, dZ)) = \nu_i(dt, dZ) = \nu_i(dZ)dt \), and \( R(t) = (R_1(t), R_2(t))' \) is the surplus reserve of two insurance business. Then process \( R(t) = (R_1(t), R_2(t))' \) that is the surplus of two insurance company is homogeneous Markov process.

We can rewrite \( R(t) \)

\[ R(t) = u + Ct + dW(t) + \int_0^1 \int_{|Z|<1} \alpha(Z)(\mu - \nu)(ds, dZ) \]
\[ + \int_0^1 \int_{|Z|>1} \alpha(Z)\mu(ds, dZ) \]  
(2)

Now one assume that \( \int_{|z|>1} |\alpha_k(z)|\nu_k(Z) < +\infty \) for any \( k \in 1, \ldots, d_2 \), \( i = 1, 2 \)

Then (1) can rewriter

\[ dR(t) = \tilde{C}(t) + \sigma dW(t) + \int_{R_0^2} \alpha(Z)(\mu - \nu)(dt, dZ) \]  
(1)'

where \( \tilde{C} = C + \int_{|z|>1} \alpha(Z)\mu(dZ) \)

For the predicable three characteristics \( (C, \sigma, \nu) \) of two-dimensional random process \( \{R(t)\}_{t \in \mathbb{R}} \) and \( \lambda = (\lambda_1, \lambda_2)' \in \mathbb{R}^2 \),

\[ \ln \varphi(\lambda) = \{i(C, \lambda) + \frac{1}{2}(\lambda', \sigma \lambda) + \int_{R^2}[e^{i\ln(Z)} - 1 - i\lambda'(Z) + I(|Z| < 1)]\nu(dZ)\}t \]

We consider a \( \sigma- \) algebra \( \mathcal{G}_t \) that following way:

\[ \mathcal{G}_t = \sigma\{R(s) : s \leq t\} \cup N \]

where \( N = \sigma\{A_\alpha ; P(A_\alpha) = 0\} \)
Definition 1 For $0 \leq s, x \in R^2_+$

$$T_{s,x} = \inf \{ t \geq s : R_1(t) < 0 \text{or} R_2(t) < 0 | R(s) = x \}$$  \(3\)

is called minimum ruin time of state $x$ at time $s$.

Then $T_{s,x}$ is $(\mathcal{H}_t)$- stop time.

Definition 2 For $0 < s < T < +\infty, x \in R^2_+$

$$\phi_{\min}(s, x, T) = p(T_{s,x} < T)$$
$$\Phi_{\min}(s, x, T) = 1 - \phi_{\min}(s, x, T)$$  \(4\)

is called respectively finite time minimum ruin probability and finite time minimum survival probability of state at time $s$.

And when $T = +\infty$, i.e

$$\phi_{\min}(s, x) = \lim_{T \to +\infty} \phi_{\min}(s, x; T)$$
$$\Phi_{\min}(s, x) = \lim_{T \to +\infty} \Phi_{\min}(s, x; T)$$  \(5\)

is called respectively ultimate minimum ruin probability and ultimate minimum survival probability of state $x$ at time $s$. We leads to the equations that satisfied for finite time and ultimate minimum survival probability, and consider a method of solution for finite time minimum survival probability.

2 Ultimate minimum survival probability equation.

Theorem 1 For any $0 \leq s < t < T \phi_{\min}(t, R(t); T)$ is $(\mathcal{H}_t)$ martingale, and

$$\Phi_{\min}(t, R(t); +\infty) = \Phi_{\min}(R(t))$$

proof Let $A_{s,t} = \{ \omega : \min_{s \leq t < T} \{ R_1(t), R_2(t) \} > 0 \}$ and

$$I_{A_{s,t}}(\omega) = \begin{cases} 1 & \omega \in A_{s,t} \\ 0 & \omega \notin A_{s,t} \end{cases}$$

Then can write $\Phi_{\min}(s, x; T) = E[I_{A_{s,T}}(\omega) | R(s) = x]$

This can represent $\Phi(t, R(s); T) = E[I_{A_{s,T}}(\omega) | R(s)]$.

For any $t(s < t < T)$ since $\{ R(t) \}_t \in R^+$

$$E\{ \Phi_{\min}(t, R(t); T) | \mathcal{H}_t \} = E\{ E[I_{A_T} | R(s)] | R(s) \} = E[I_{A_T} | R(s)] =$$

$$= \Phi_{\min}(s, R(s); T)$$

i.e. $\Phi_{\min}(t, R(t); T)_{t \in [0,T]}$ is martingale , and taking into account that risk process $(2)$ is time homogeneous Markov process for $s < t$, we obtain that
Thus for $s, t < t$
\[\Phi_{\text{min}}(t, x) = \Phi_{\text{min}}(s, x) = \Phi_{\text{min}}(x).\]
This implies theorem 1.

Now let \(\Phi_{\text{min}}(s, x; T) = \Phi_{\text{min}}(s, x), \Phi_{\text{min}}(s, x; +\infty) = \Phi_{\text{min}}(x).\)

**Theorem 2** If \(\Phi_{\text{min}}(s, x) \in C^{1,2}([0, T] \times \mathbb{R}^2_+)\) \((\Phi_{\text{min}}(x) \in C^{1,2}(\mathbb{R}^2_+))\) for any \((s, x) \in [0, T] \times \mathbb{R}^2_+\) then

\[
\begin{align*}
\frac{\partial}{\partial t} \Phi_{\text{min}}(s, x) + \tilde{L} \Phi_{\text{min}}(s, x) &= 0 \\
\Phi_{\text{min}}(s, (x_1, 0)) &= 0, \Phi_{\text{min}}(s, (x_2, 0)) = 0, \Phi_{\text{min}}(T, (x_1, x_2)) &= 1 \\
L \Phi_{\text{min}}(x) &= 0 \\
\Phi_{\text{min}}(x_1, 0) &= 0, \Phi_{\text{min}}(0, x_2) = 0
\end{align*}
\]

(6)

(7)

where operation \(\tilde{L}\) is

\[
\tilde{L} \Phi_{\text{min}}(s, x) = \sum_{i=1}^{2} C_i \frac{\partial \Phi_{\text{min}}(s, x)}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{d_1} \sigma_{ik} \sigma_{jk} \cdot
\]

\[
\cdot \frac{\partial^2 \Phi_{\text{min}}(s, x)}{\partial x_i \partial x_j} + \sum_{k=1}^{d_2} \int_{\mathbb{R}^2} \left( \Phi_{\text{min}}(s, x) - \alpha_k(z) \right) - \Phi_{\text{min}}(s, x) -
\]

\[- I_{|z| \leq 1} \sum_{i=1}^{2} \alpha_{ik}(z) \frac{\partial \Phi_{\text{min}}(s, x)}{\partial x_i}(z) \nu_k(dz) \]

(8)

**proof** From theorem 1 since \(\{\Phi_{\text{min}}(t, R(t))\}_{t \in [0, T]}\) is martingale, applying ZhoZhuGyong transformation formula [1].

\[\Phi_{\text{min}}(t, R(t)) = \Phi_{\text{min}}(s, R(s)) + \int_s^t \left\{ \frac{\partial \Phi_{\text{min}}(u, R(u))}{\partial u} + \sum_{i=1}^{2} C_i \frac{\partial \Phi_{\text{min}}(u, R(u))}{\partial x_i} \right\} du + \int_s^t \sum_{i=1}^{2} \frac{\partial^2 \Phi_{\text{min}}(u, R(u))}{\partial x_i \partial x_j} \sigma_{ik} \sigma_{jk} du + \int_s^t \sum_{i=1}^{2} \frac{\partial \Phi_{\text{min}}(u, R(u))}{\partial x_i} \cdot
\]

\[
\cdot \sum_{j=1}^{d_1} \sigma_{ij} \omega_j(u) + \sum_{k=1}^{d_2} \int_s^t \int_{|z| < 1} \left[ \Phi_{\text{min}}(u, R(u) + \alpha_k(z)) - \Phi_{\text{min}}(u, R(u)) \right] - \Phi_{\text{min}}(u, R(u)) \right) - \sum_{i=1}^{2} \alpha_{ik}(z) \frac{\partial \Phi_{\text{min}}(u, R(u))}{\partial x_i}(z) \nu_k(du, dz) +
\]

\[
+ \sum_{k=1}^{d_2} \int_s^t \int_{|z| < 1} \left[ \Phi_{\text{min}}(u, R(u) + \alpha_k(z)) - \Phi_{\text{min}}(u, R(u)) \right] \mu_k - \nu_k \right)
\]

(9)
where
\[ \alpha(z) = \begin{pmatrix} \alpha_1(z) \\ \alpha_2(z) \end{pmatrix} \]

Thus from this theorem condition and result of theorem 1, applying conditional expectation of \( \mathbb{I}_s \), to (9), we obtain
\[
E\{\Phi_{\min}(t, R(t))|\mathbb{I}_s\} = \Phi_{\min}(s, R(s)) + E\{\int_s^t E\left(\frac{\partial}{\partial u}\Phi_{\min}(u, R(u)) + \right. \]
\[ + \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \Phi_{\min}(u, R(u)) \cdot C_i + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} (\sum_{k=1}^{d_1} \sigma_{ik} \sigma_{jk}) \Phi_{\min}(u, R(u)) + \]
\[ \left. + \int_{R^2} \sum_{k=1}^{d_2} [\Phi_{\min}(u, R(u) + \alpha_k(z)) - \Phi_{\min}(u, R(u)) - I_{\{|z| \leq 1\}} \cdot \sum_{i=1}^{2} \alpha_{ik}(z) \frac{\partial}{\partial x_i} \Phi_{\min}(u, R(u)) \nu_k(dz)] du \right\} \]

Therefore \( \frac{d}{dt}\Phi_{\min}(u, x) + \tilde{L}\Phi_{\min}(u, x) = 0 \), where \( \tilde{L} \) is same to (8) and boundary conditions obtain from theorem 2 for finite time minimum survival probability. In similar way applying same variation we obtain equation (7) for \( \Phi_{\min}(R(t)) \).

In case consider for Equation (1). One obtain from result of theorem 2.
\[
\frac{\partial}{\partial s}\Phi_{\min}(s, x) + \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \Phi_{\min}(s, x) \cdot \tilde{C}_i + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{d_1} \sigma_{ik} \sigma_{jk} \Phi_{\min}(s, x) + \]
\[ + \int_{R^2} \sum_{k=1}^{d_2} [\Phi_{\min}(s, x + \alpha_k(z)) - \Phi_{\min}(s, x) \cdot \sum_{i=1}^{2} \alpha_{ik}(z) \frac{\partial}{\partial x_i} \Phi_{\min}(s, x)] \nu_k(dz) = 0 \] (10)

And taking into account KimZhuGyong [2] that the random measure \( \nu(dz) \) represent \( \nu_k(dz) = \lambda_k p_k(z) \) for hyper density function \( p_k(z) \) and \( \alpha(z) = z \) in case \( \alpha(z) = z \) we obtain from (10)
\[
\frac{\partial}{\partial s}\Phi_{\min}(s, x) + \sum_{i=1}^{2} \tilde{C}_i \frac{\partial}{\partial x_i} \Phi_{\min}(s, x) + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij} \frac{\partial^2 \Phi_{\min}(s, x)}{\partial x_i \partial x_j} -
\[ - a_0 \Phi_{\min}(s, x) + \int_{R^2} \Phi_{\min}(s, z) \sum_{k=1}^{d} \lambda_k p_k(z - x) dz = 0 \]
where

$$\tilde{C}_i = C_i - \sum_{k=1}^{d_2} \lambda_k a_k, i = 1, 2$$

$$a_0 = \sum_{k=1}^{d_1} \lambda_k, a_{ij} = \sum_{k=1}^{d_1} \sigma_{ik} \sigma_{jk}, i = 1, 2, j = 1, 2$$

$$a_k = \int_{R^+} z p_k(z) dz, k = 1, d_2$$

3 Finite time minimum survival probability

We consider two compound Poisson process

$$x_1(t) = \sum_{k=1}^{M_1(t)} z_{1k}, x_2(t) = \sum_{k=1}^{M_2(t)} z_{2k}$$

where \( \{z_{1k}\}, \{z_{2k}\} \) are independent and have same distributions \( F_1(z), F_2(z) \), respectively. Then for two process we construct two-dimensional random measures in the following way:

1. Case \( M_1(t) = M_2(t) = N(t) \) and \( N(t)_{t \in R_k} \) is time homogeneous Poisson process that \( EN(t) = \lambda t \) for any \( t \in R_+ \) and \( \Gamma \in B^2 \) we define the measure \( \mu \) that

$$\mu((0, t], \Gamma) = \sum_{k=1}^{N(t)} I_{\Gamma}(z_{1k} z_{2k})$$

Then measure \( \mu \) have the following property

Lemma 1

1. \( \mu((0, t], \Gamma) \) is integral random measure on \( B^2 \)
2. \( \mu \) is non-decreasing function and independent increment process of \( t \)
3. \( \nu((0, t], \Gamma) = E(\mu((0, t], \Gamma)) = \lambda t \int_{\Gamma} dF_1(z_1) dF_2(z_2) \)

2. Case \( M_1(t) = N_1(t) + N_3(t), M_2(t) = N_2(t) + N_3(t) \) and any \( \{N_i(t)\}, i = 1, 2, 3 \) are independent and have Poisson distribution with parameter \( \lambda_i (i = 1, 2, 3) \) we define the measures that
\[ \mu_1((0,t], \Gamma') = \sum_{k=1}^{N_1(t)} I_{\Gamma}(z_{1k}, z_{3k}) \]
\[ \mu_2((0,t], \Gamma') = \sum_{k=1}^{N_1(t)} I_{\Gamma}(z_{4k}, z_{2k}) \]  \( (12) \)
\[ \mu_3((0,t], \Gamma') = \sum_{k=1}^{N_1(t)} I_{\Gamma}(z_{1k}, z_{2k}) \]

Then the measures \( \mu_1, \mu_2, \mu_3 \) are independent and satisfy the results of Lemma 1. Let \( d_1 = 2, d_2 = 3 \) in model (1) and (2), and using integral random measure (13) we can represent model (1) in the following way:

\[
\begin{cases}
    dR(t) = C dt + \sigma dW(t) - \int_{R^2_+} z^{(1)}(dt, dz) \mu_1 (dt, dz) - \\
    \quad - \int_{R^2_+} z^{(2)}(dt, dz) \mu_2 (dt, dz) - \int_{R^2_+} z \mu_3 (dt, dz)
\end{cases}
\]

(13)

where \( R(t) = (R_1(t), R_2(t))' \), \( C = (C_1, C_2)' \)

\[
\omega = \begin{pmatrix} \omega_1(t) \\ \omega_2(t) \end{pmatrix}', \quad \sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix},
\]
\[
z^{(1)} = (z_1, 0)', \quad z^{(2)} = (0, z_2)', \quad z = (z_1, z_2)', \quad u = (u_1, u_2)'
\]

Then taking into account the equation of Theorem 2 which satisfied for model (14) we obtain the following equation

\[
L \Phi_{\text{min}}(t, x) + \lambda_1 \int_0^\infty \Phi_{\text{min}}(t, x - z^{(1)}) p_1(z_1) dz_1 + \\
+ \lambda_2 \int_0^\infty \Phi_{\text{min}}(t, x - z^{(2)}) p_2(z_2) dz_2 + \\
+ \lambda_3 \int_0^\infty \int_0^\infty \Phi_{\text{min}}(t, x - z) p_1(z_1) p_2(z_2) dz_1 dz_2 = 0
\]

(14)

\[
\Phi_{\text{min}}(t, (x_1, 0)') = \Phi_{\text{min}}(t, (0, x_2)') = 0, \Phi_{\text{min}}(T, (x_1, x_2)') = 1
\]

where operation \( L \) is

\[
L = \frac{\partial}{\partial t} + \sum_{i=1}^2 C_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i=1}^2 \frac{\partial^2}{\partial z_i^2} - \lambda
\]

(\( \lambda = \lambda_1 + \lambda_2 + \lambda_3 \))
Now we consider new part derivative operation

\[ L^* = -\frac{\partial}{\partial t} - \sum_{i=1}^{2} C_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{2} \frac{\partial^2}{\partial x_i^2} - \lambda \]  
(\lambda = \lambda_1 + \lambda_2 + \lambda_3)  

(16)

**Theorem 3** For \( t(\tau \leq t \leq T^2) \), \( x \in R_+^2 \), the solution of a part derivative equation

\[
\begin{align*}
L^* k(t, x, z, \xi) &= 0 \\
k(t, (0, x_2)') &= 0, \quad k(t, (x, 0)') = 0 \\
k(\tau, x; \tau, \xi) &= \delta(\xi - x)
\end{align*}
\]  
(17)

is obtained by the following formula:

\[
k(t, x; \tau, \xi) = \exp \{ \beta(t - z) + \langle \alpha, x - \xi \rangle \} \cdot \prod_{i=1}^{2} \frac{1}{\sqrt{2\pi \sigma_i^2(t - \tau)}} \left\{ \exp \left\{ -\frac{(x_i - \xi_i)^2}{2\sigma_i^2(t - \tau)} \right\} - \exp \left\{ -\frac{(x_i + \xi_i)^2}{2\sigma_i^2(t - \tau)} \right\} \right\} \]  
(18)

where \( \alpha = (\alpha_1, \alpha_2)' \), \( \alpha_i = \frac{C_i}{\sigma_i} \), \( \beta = -\left( \sum_{i=1}^{2} \frac{C_i}{\sigma_i^2} + \lambda \right) \), \( \xi = (\xi_1, \xi_2)' \), and \( \langle ., . \rangle \) is the symbol of scalar product.

Denote the solution of (18) by \( k(t, x; \tau, \xi) \). Taking into following transformation:

\[ s(t, x; \tau, \xi) = e^{-\langle \alpha, x \rangle} k(t, x; \tau, \xi) \]  
(19)

then the equation (18) can represent

\[
\frac{\partial s(t, x; \tau, \xi)}{\partial t} - \left\{ \langle C, \alpha \rangle + \frac{1}{2} \alpha' \sigma^{-1} \alpha - \lambda \right\} s(t, x; \tau, \xi) + \\
\left\{ [-C_1 + \sigma_1^2 \alpha_1] \frac{\partial s(t, x; \tau, \xi)}{\partial x_1} + [-C_2 + \sigma_2^2 \alpha_2] \frac{\partial s(t, x; \tau, \xi)}{\partial x_2} + \\
\frac{1}{2} \left[ \sigma_1^2 \frac{\partial^2 s(t, x; \tau, \xi)}{\partial x_1^2} + \sigma_2^2 \frac{\partial^2 s(t, x; \tau, \xi)}{\partial x_2^2} \right] \right\} = 0
\]  
(20)

Denoting \( \alpha_i = \frac{C_i}{\sigma_i} \), \( i = 1, 2 \), \( \beta = -\left( \lambda + \frac{C_1^2}{2\sigma_1^2} + \frac{C_2^2}{2\sigma_2^2} \right) \), then for \( t(\tau \leq t \leq T) \) and \( x, \xi \in R_+^2 \) the equation (21) can represent the following way:

\[
\left\{ \left\{ \frac{-\beta - \frac{1}{2} \left[ \sigma_1^2 \frac{\partial^2}{\partial x_1^2} + \sigma_2^2 \frac{\partial^2}{\partial x_2^2} \right] }{\sigma_1} + \beta \right\} s(t, 0; \tau, \xi) = 0 \\
s(t, x; \tau, \xi) = 0 \\
s(\tau, x; \tau, \xi) = e^{-\langle \alpha, x \rangle} \delta(x - \xi)
\]  
(21)
If $\Gamma(t, x; \tau, \xi)$ is an extended solution of (21) in definite domain $[\tau, T] \times \mathbb{R}^2$, we can represent

$$
\Gamma(t, x; \tau, \xi) =
\begin{cases}
  s(t, (x_1, x_2); \tau, \xi), & x_1 \geq 0, x_2 \geq 0 \\
  -s(t, (-x_1, x_2); \tau, \xi), & x_1 > 0, x_2 \geq 0 \\
  -s(t, (x_1, -x_2); \tau, \xi), & x_1 \geq 0, x_2 < 0 \\
  s(t, (-x_1, -x_2); \tau, \xi), & x_1 < 0, x_2 < 0
\end{cases}
$$

Then initial condition of (22) can represent.

$$
\Gamma(\tau, x; \tau, \xi) =
\begin{cases}
  e^{<\alpha, x>} \delta(x_1 - \xi_1, x_2 - \xi_2), & x_1 \geq 0, x_2 \geq 0 \\
  -e^{\alpha_1 x_1 - \alpha_2 x_2} \delta(x_1 + \xi_1, x_2 + \xi_2), & x_1 > 0, x_2 < 0 \\
  -e^{-\alpha_1 x_1 + \alpha_2 x_2} \delta(x_1 + \xi_1, x_2 - \xi_2), & x_1 < 0, x_2 > 0 \\
  e^{<\alpha, x>} \delta(x_1 + \xi_1, x_2 + \xi_2), & x_1 < 0, x_2 < 0
\end{cases}
$$

Thus the boundary value problem of (21) leads to the initial value problem of

$$
\Gamma(t, x; z, \xi) = 0 = \varphi(x, \xi)
$$

From the solving method of Kolmogorov-Feller equation

$$
\Gamma(t, x; z, \xi) = \varphi(x, \xi) * \left( \frac{1}{2\pi} \right)^2 \int \int_{\mathbb{R}^2} e^{-i<\theta, X>}
\cdot \exp \left\{ \frac{1}{2} \left( \sigma_1^2 \theta_1^2 + \sigma_2^2 \theta_2^2 \right) (t - \tau) + \beta(t - \tau) \right\} \, d\theta_1 \theta_2
= \varphi(x, \xi) * \prod_{i=1}^{2} \frac{1}{\sqrt{2\pi \sigma_i^2(t - \tau)}} \exp \left\{ -\frac{x_i^2}{2\sigma_i^2(t - \tau)} \right\} \exp \{ \beta(t - \tau) \}$$
where \( \ast \) is convolution symbol and \( \theta = (\theta_1, \theta_2)' \). Thus,

\[
\Gamma(t, x; z, \xi) = \int_{\mathbb{R}^2} \varphi(x, y, \xi) \prod_{i=1}^{2} \frac{1}{\sqrt{2\pi\sigma_i(t - \tau)}} \cdot \exp \left\{ - \frac{y_i^2}{2\sigma_i(t - \tau)} \right\} \exp\{\beta(t - \tau)\} dy_1y_2 = \\
eq \exp\{- < \alpha, \xi >\} \frac{1}{\sqrt{2\pi(t - \tau)\sigma_1\sigma_2}} \cdot \{\exp \left\{ - \frac{(x_1 - \xi_1)^2}{\sigma_1^2} + \frac{(x_2 - \xi_2)^2}{\sigma_2^2} }{2(t - \tau)} \right\} - \\
- \exp \left\{ - \frac{(x_1 - \xi_1)^2}{\sigma_1^2} + \frac{(x_2 + \xi_2)^2}{\sigma_2^2} }{2(t - \tau)} \right\} - \\
- \left\{ \exp \left\{ - \frac{(x_1 - \xi_1)^2}{\sigma_1^2} + \frac{(x_2 - \xi_2)^2}{\sigma_2^2} }{2(t - \tau)} \right\} + \\
+ \left\{ \exp \left\{ - \frac{(x_1 - \xi_1)^2}{\sigma_1^2} + \frac{(x_2 + \xi_2)^2}{\sigma_2^2} }{2(t - \tau)} \right\} \right\} \right\} \exp\{\beta(t - \tau)\}
\]

Taking into account he (20) and (23), the solution of equation (18) is

\[
K(t, x; z, \xi) = e^{<\alpha, X>} S(t, x; z, \xi) = \\
= \exp\{< x, x - \xi > + \beta(t - \tau)\} \frac{1}{2\pi(t - \tau)\sigma_1\sigma_2} \cdot \prod_{i=1}^{2} \left\{ \exp \left\{ - \frac{(x_i - \xi_i)^2}{2(t - \tau)\sigma_i^2} \right\} - \exp \left\{ - \frac{(x_i + \xi_i)^2}{2(t - \tau)\sigma_i^2} \right\} \right\}
\]

**Theorem 4** A part derivative equation (15) is equivalent to the following integral equation

\[
\Phi_{\min}(\tau, \xi) = \int_{\tau}^{T} \int_{\mathbb{R}^2} \Phi_{\min}(t, x) G(t, x; \tau, \xi) dxdt + F(\tau, \xi) \quad (24)
\]
where

\[ F(\tau, \xi) = \int \int_{R^2_+} K(T, x; \tau, \xi) dx \]

\[ G(t, x; \tau, \xi) = -\lambda_1 \int_{x_1}^{+\infty} K(t, (z_1, z_2); \tau, \xi) P_1(x_1 - z_1) dz_1 - \]
\[ -\lambda_2 \int_{x_2}^{+\infty} K(t, (x_1, z_2); \tau, \xi) P_2(x_2 - z_2) dz_2 - \]
\[ -\lambda_3 \int_{x_1}^{+\infty} \int_{x_2}^{+\infty} K(t, (z_1, z_2); \tau, \xi) P_1(x_1 - z_1) P_2(x_2 - z_2) dz_1 dz_2 \]

**Theorem 5** Integral equation (25) in domain \( \Sigma = \{(t, \xi) \in [\tau, T] \times R^2_+\} \) has a unique solution

\[ \Phi_{\min}(\tau, \xi) = F(\tau, \xi) + \sum_{k=1}^{\infty} \int_{\tau}^{T} \int \int_{R^2_+} G^k(t, x; z, \xi) F(t, x) dx dt \tag{25} \]

where

\[ G^{(1)}(t, x; z, \xi) = G(t, x; z, \xi) \]
\[ G^{(k)}(t, x; z, \xi) = \int_{\tau}^{T} \int \int_{R^2_+} G^{k-1}(t, s; z) dz ds \]

**References**

[1] Chan W.S, Some results on ruin probabilities in a two-dimensional risk model, *Insurance Mathematics and Economics*, 33 (2003), 345-358

[2] Florin Avrana, et al, A two-dimensional ruin problem on the positive quadrant, *Insurance Mathematics and Economics*, 42 (2008), 227-234

[3] Junhai Li, et al, On the ruin probabilities of a bidimensional perturbed risk model, *Insurance Mathematics and Economics*, 41 (2007), 185-195

[4] Kim Zhu Gyong, A solving method of Kolmogorov-Feller equation, *Science Kim Il Sung University*, (2006), 6-8

[5] Zho Zhu Gyong, Stochastic integral equation, *Academical publication*, (1963)