LOCAL MOVES ON KNOTS AND PRODUCTS OF KNOTS II

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Abstract. Let $S_n$ be the set of simple $n$-knots in $S^{n+2}$, where $n \in \mathbb{N}$ and $n \geq 3$. For $K \in S_{2m}$, take the knot product $K \otimes \text{Hopf}$, where Hopf denotes the Hopf link. Then this defines a bijective map $S_{2m} \to S_{2m+4}$ if $m \in \mathbb{N}$ and $2m \geq 8$.

Let $J$ and $K$ be 1-links in $S^3$. Suppose that $J$ is obtained from $K$ by a single pass-move, which is a local move on 1-links. Then the $(4k+1)$-dimensional submanifold $J \otimes^k \text{Hopf} \subset S^{4k+3}$ is obtained from $K \otimes^k \text{Hopf}$ by a single $(2k+1, 2k+1)$-pass-move ($k \in \mathbb{N}$), which is a local move on $(4k+1)$-submanifolds $\subset S^{4k+3}$.

Let $a, b, a', b'$ be natural numbers. If the $(a, b)$ torus link is pass-move equivalent to the $(a', b')$ torus link, then the Brieskorn manifold $\Sigma(a, b, 2, ..., 2)$, where there are an even number of 2’s, is diffeomorphic to $\Sigma(a', b', 2, ..., 2)$.

Let $J$ and $K$ be (not necessarily connected) 2-dimensional closed oriented submanifolds in $S^4$. Suppose that $J$ is obtained from $K$ by a single ribbon-move, which is a local move on 2-dimensional submanifolds $\subset S^4$. Then the $(4k+2)$-submanifold $J \otimes^k \text{Hopf} \subset S^{4k+4}$ is obtained from $K \otimes^k \text{Hopf}$ by a single $(2k+1, 2k+2)$-pass-move ($k \geq 2, k \in \mathbb{N}$), which is a local move on $(4k+2)$-dimensional submanifolds $\subset S^{4k+4}$.

1. Introduction

In [10] we began to research relations between knot products and local moves on knots. This paper is a sequel to [10]. Knot products, or products of knots, were defined and have been researched in [7, 9, 10]. Local moves on high dimensional knots were defined and have been researched in [10, 15, 16, 17, 18, 19, 20, 21].

Our main results are Theorem 2.1, 4.3 and 4.1.

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Keywords: local moves on 1-knots, local moves on high dimensional knots, crossing-changes on 1-links, pass-moves on 1-links, products of knots, pass-moves on high dimensional links, twist-moves on high dimensional links, branched cyclic covering spaces, Seifert hypersurfaces, Seifert matrices.
2. A MAIN RESULT ON PRODUCTS OF KNOTS AND EVEN DIMENSIONAL SIMPLE KNOTS

Let $f : C^n \to C$ be a (complex) polynomial mapping with an isolated singularity at the origin of $C^n$. That is, $f(0) = 0$ and the complex gradient $df$ has an isolated zero at the origin. The link of this singularity is defined by the formula $L(f) = V(f) \cap S^{2n-1}$. Here the symbol $V(f)$ denotes the variety of $f$, and $S^{2n-1}$ is a sufficiently small sphere about the origin of $C^n$. Given another polynomial $g : C^n \to C$, form $f + g$ with domain $C^{n+m} = C^n \times C^m$ and consider $L(f + g) \subset S^{2n+2m+1}$.

We use a topological construction for $L(f + g) \subset S^{2n+2m+1}$ in terms of $L(f) \subset S^{2n+1}$ and $L(g) \subset S^{2m+1}$. The construction generalizes the algebraic situation. Given nice codimension-two embeddings $K \subset S^n$ and $L \subset S^m$, we form a product $K \otimes L \subset S^{n+m+1}$. Then $L(f) \otimes L(-g) \simeq L(f + g)$.

Let $K \subset S^n$ and $L \subset S^m$ be (not necessarily connected) codimension two oriented closed submanifolds. Take smooth maps

$$\begin{cases}
  f : D^{n+1} \to D^2 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\} \\
  g : D^{m+1} \to D^2
\end{cases}$$

such that

$$\begin{cases}
  f^{-1}((0, 0)) \cap \partial D^{n+1} \\
  g^{-1}((0, 0)) \cap \partial D^{m+1}
\end{cases}$$

are

$$K \subset S^n, \quad L \subset S^m.$$  

Let $f + g$ be a smooth map

$$D^{n+1} \times D^{m+1} \to D^2$$

$$(x, y) \mapsto f(x) - g(y).$$

We define $K \otimes L$ to be $(f + g)^{-1}((0, 0)) \cap \partial(D^{n+1} \times D^{m+1})$ in $\partial(D^{n+1} \times D^{m+1})$. If $S^n - K$ or $S^m - L$ is the total space of a $S^1$ fiber bundle as in [7, 9], $K \otimes L$ is a smooth codimension two closed submanifold $\subset S^{n+m+1}$.

Let $S_n$ be the set of simple $n$-knots in $S^{n+2}$, where $n \in \mathbb{N}$ and $n \geq 3$. See [1, 2, 8, 11, 14] for simple knots. For $K \in S_n$, take the knot product $K \otimes \text{Hopf}$, where Hopf is the Hopf link. By [7, 9], $K \otimes \text{Hopf} \in S_{n+4}$. Thus we define a map $S_n \to S_{n+4}$, and call it $\otimes \text{Hopf}$. In Theorem 12.8 of [10] we proved that $\otimes \text{Hopf} : S_{2m+1} \to S_{2m+5}$ is a one-to-one map $(2m + 1 \neq 1, 3$ and $m \in \mathbb{N})$. See [3] for detail. In this paper we prove the following theorem.

**Theorem 2.1.** $\otimes \text{Hopf} : S_{2m} \to S_{2m+4}$ is a one-to-one map $(2m \geq 8$ and $m \in \mathbb{N})$. 

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3. Review of local moves on knots

Two 1-links are \textit{pass-move-equivalent} if one is obtained from the other by a sequence of pass-moves. See the following figure for an illustration of the pass-move. Each of four arcs in the 3-ball may belong to different components of the 1-link.

If \( K \) and \( J \) are pass-move-equivalent and if \( K \) and \( K' \) are equivalent, then we also say that \( K' \) and \( J \) are pass-move-equivalent. See \([8]\) for detail. (We will review the word ‘equivalent’ near Figure 1.2.)

In the 1-link case local moves are very useful for research. It is well known that in the 2-link case ribbon 2-links are changed into the trivial link by the following local move, which we call the ribbon-move. See Figure 1.1. We review ribbon-moves on smooth closed oriented 2-dimensional submanifolds \( \subset S^4 \). See \([17]\) for detail. Before that, we review ribbon 2-links.

A 2-link \( L = (K_1, \ldots, K_m) \) is called a \textit{ribbon} 2-link if \( L \) satisfies the following properties.

1. There is a self-transverse immersion \( f : D^3_1 \amalg \cdots \amalg D^3_m \to S^4 \) such that \( f(\partial D^3_i) = K_i \).
2. The singular point set \( C (\subset S^4) \) of \( f \) consists of double points. \( C \) is a disjoint union of 2-discs \( D^2_i (i = 1, \ldots, k) \).
3. Put \( f^{-1}(D^3_j) = D^3_{jB} \amalg D^3_{jS} \). The 2-disc \( D^3_{jS} \) is trivially embedded in the interior \( \text{Int} D^3_\alpha \) of a 3-disc component \( D^3_\alpha \). The circle \( \partial D^2_{jB} \) is trivially embedded in the boundary \( \partial D^3_\beta \) of a 3-disc component \( D^3_\beta \). The 2-disc \( D^2_{jB} \) is trivially embedded in the 3-disc component \( D^3_\beta \). (Note that there are two cases, \( \alpha = \beta \) and \( \alpha \neq \beta \).)

Let \( K_1 \) and \( K_2 \) be smooth closed oriented 2-dimensional submanifolds \( \subset S^4 \). We say that \( K_2 \) is obtained from \( K_1 \) by one \textit{ribbon-move} if there is a 4-ball \( B \) embedded in \( S^4 \) with the following properties.

1. \( K_1 \) coincides with \( K_2 \) in \( S^4 - \overline{B} \). This identity map from \( \overline{K_1 - B} \) to \( \overline{K_2 - B} \) is orientation preserving.
2. \( B \cap K_1 \) is drawn as in Figure 1.1.(1). \( B \cap K_2 \) is drawn as in Figure 1.1.(2).
Figure 1.1.(1): Ribbon-move

Figure 1.1.(2): Ribbon-move
We regard $B$ as (a closed 2-disc)$\times[0, 1] \times \{t\} - 1 \leq t \leq 1$. We put $B_t =$ (a closed 2-disc)$\times[0, 1] \times \{t\}$. Then $B = \cup B_t$. In Figure 1.1.1 and 1.1.2, we draw $B_{-0.5}$, $B_0$, $B_{0.5}$ to $B$. We draw $K_1$ and $K_2$ by the bold line. The fine line denotes $\partial B_t$.

$B \cap K_1$ (resp. $B \cap K_2$) is diffeomorphic to $D^2 \Pi (S^1 \times [0, 1])$, where $\Pi$ denotes the disjoint union.

$B \cap K_1$ has the following properties: $B_t \cap K_1$ is empty for $-1 \leq t < 0$ and $0.5 < t \leq 1$. $B_0 \cap K_1$ is diffeomorphic to $D^2 \Pi (S^1 \times [0, 0.3]) \Pi (S^1 \times [0.7, 1])$. $B_{0.5} \cap K_1$ is diffeomorphic to $(S^1 \times [0.3, 0.7])$. $B_t \cap K_1$ is diffeomorphic to $S^1 \Pi S^1$ for $0 < t < 0.5$. (Here we draw $S^1 \times [0, 1]$ to have the corner in $B_0$ and in $B_{0.5}$. Strictly to say, $B \cap K_1$ in $B$ is a smooth embedding which is obtained by making the corner smooth naturally.)

$B \cap K_2$ has the following properties: $B_t \cap K_2$ is empty for $-1 \leq t < -0.5$ and $0 < t \leq 1$. $B_0 \cap K_2$ is diffeomorphic to $D^2 \Pi (S^1 \times [0, 0.3]) \Pi (S^1 \times [0.7, 1])$. $B_{-0.5} \cap K_2$ is diffeomorphic to $(S^1 \times [0.3, 0.7])$. $B_t \cap K_2$ is diffeomorphic to $S^1 \Pi S^1$ for $-0.5 < t < 0$. (Here we draw $S^1 \times [0, 1]$ to have the corner in $B_0$ and in $B_{-0.5}$. Strictly to say, $B \cap K_1$ in $B$ is a smooth embedding which is obtained by making the corner smooth naturally.)

In Figure 1.1.1 and 1.1.2 there are an oriented cylinder $S^1 \times [0, 1]$ and an oriented disc $D^2$ as we stated above. We do not make any assumption about the orientation of the cylinder and the disc. The orientation of $B \cap K_1$ (resp. $B \cap K_2$) coincides with that of the cylinder and that of the disc.

Suppose that $K_2$ is obtained from $K_1$ by one ribbon-move and that $K'_2$ is equivalent to $K_2$. Then we also say that $K'_2$ is obtained from $K_1$ by one ribbon-move. If $K_1$ is obtained from $K_2$ by one ribbon-move, then we also say that $K_2$ is obtained from $K_1$ by one ribbon-move.

Two 2-knots $K_1$ and $K_2$ are said to be ribbon-move equivalent if there are 2-knots $K_1 = \tilde{K}_1, \tilde{K}_2, ..., \tilde{K}_{r-1}, \tilde{K}_r = K_2$ ($r \in \mathbb{N}, p \geq 2$) such that $\tilde{K}_i$ is obtained from $\tilde{K}_{i-1}$ ($1 < i \leq r$) by one ribbon-move.

It is natural to ask the following questions:

(1) Are all 2-links ribbon-move-equivalent to the trivial link?

(2) Is there a nonribbon 2-link which is ribbon-move-equivalent to the trivial link?

In [17] it was proved that (1) has a negative answer and that (2) has an affirmative answer. Next it is very natural to consider the following problem:

Classify 2-links (resp. 2-knots) up to ribbon-move equivalence.

In [17, 18, 20] many results are obtained. However this problem remains open.

Thus it is very natural to consider local moves on high dimensional links. We must begin by considering what kind of local moves are fruitful to research high dimensional links. Ribbon-moves on 2-links and $(p, q)$-pass-moves on $(p + q - 1)$-links are natural ones. In this paper we discuss them.
We review \((p,q)\)-pass-moves on \(n\)-knots \((p,q \in \mathbb{N}, \ p + q = n + 1)\). See [10] for detail. We work in the smooth category.

An \((oriented)\)\((ordered)\ \(m\)-component \(n\)-\(\text{(dimensional)}\) \(\text{link}\) is a smooth, oriented submanifold \(L = (L_1, \ldots, L_m) \subset S^{n+2}\), which is the ordered disjoint union of \(m\) manifolds, each PL homeomorphic to the \(n\)-sphere. If \(m = 1\), then \(L\) is called a \(\text{knot}\).

Let \(id : S^{n+2} \to S^{n+2}\) be the identity map. We say that \(n\)-links \(L\) and \(L'\) are \(\text{identical}\) if \(id(L) = L'\) and \(id|_L : L \to L'\) is an orientation and order preserving diffeomorphism map.

We say that \(n\)-links \(L\) and \(L'\) are \(\text{equivalent}\) if there exists an orientation preserving diffeomorphism \(f : S^{n+2} \to S^{n+2}\) such that \(f(L) = L'\) and \(f|_L : L \to L'\) is an orientation and order preserving diffeomorphism.

An \(m\)-component \(n\)-\(\text{link}\) \(L = (L_1, \ldots, L_m)\) is called a \(\text{trivial} \ (n-)\ \text{link}\) if each \(L_i\) bounds an \((n + 1)\)-ball \(B_i\) trivially embedded in \(S^{n+2}\) and if \(B_i \cap B_j = \phi(i \neq j)\). If \(L\) is a \(\text{knot}\) (i.e., \(m = 1\)), then \(L\) is called a \(\text{trivial} \ (n-)\ \text{knot}\).

Let \(K_+, K_-\) be \(n\)-dimensional closed oriented submanifolds \(\subset S^{n+2} \ (n \in \mathbb{N})\). Let \(B\) be an \((n + 2)\)-ball trivially embedded in \(S^{n+2}\). Suppose that \(K_+\) coincides with \(K_-\) in \(S^{n+2} - B\).

Take an \((n + 1)\)-dimensional \(p\)-\(\text{handle}\) \(h^n_\ast\ (\ast = +, -)\) and an \((n + 1)\)-\(\text{dimensional}\) \((n + 1 - p)\)-\(\text{handle}\) \(h^{n+1-p}_\ast\) in \(B\) with the following properties.

1. \(h^n_\ast \cap \partial B\) is the attaching part of \(h^n_\ast\). \(h^{n+1-p}_\ast \cap \partial B\) is the attaching part of \(h^{n+1-p}_\ast\).
2. \(h^n_\ast\) (resp. \(h^{n+1-p}_\ast\)) is embedded trivially in \(B\).
3. \(h^n_\ast \cap h^{n+1-p}_\ast = \phi.\)
4. The attaching part of \(h^n_+\) coincides with that of \(h^n_-\). The linking number (in \(B\)) of \([h^n_+ \cup (-h^n_-)]\) and \([h^{n+1-p}_\ast\) whose attaching part is fixed in \(\partial B\)]
   is one if an orientation is given.

Let \(K_*(\ast = +, -)\) satisfy that

\[ K_* \cap \text{Int}B = (\partial h^n_\ast - \partial B) \cup (\partial h^{n+1-p}_\ast - \partial B). \]

We say that \(K_+\) and \(K_-\) differ by one \((p, n + 1 - p)\)-\(\text{pass-move}\) in \(B\).

Then there is a Seifert hypersurface \(V_*\) for \(K_* \ (\ast = +, -)\) with the following properties:

\(V_+ = V_0 \cup h^n_0 \cup h^{n+1-p}_0\ (\ast = +, -)\). \(V_- \cap B = h^n_0 \cup h^{n+1-p}_0\). (The idea of the proof is the Thom-Pontrjagin construction.) We say that \(V_-\) (resp. \(V_+\)) is obtained from \(V_+\) (resp. \(V_-\)) by a \((p, n + 1 - p)\)-\(\text{pass-move}\) in \(B\).

Figure [I2] is a diagram of a \((p,q)\)-\(\text{pass-move}\). Figure [I3], which consists of the two figures (1) (2), is a diagram of a \((p,q)\)-\(\text{pass-move}\).

Note that the \((1,1)\)-\(\text{pass-move}\) is defined on \(1\)-\(\text{links}\) and is the same as the \(\text{pass-move}\) on \(1\)-\(\text{links}\).
A \((p, n + 1 - p)\)-pass-move on an \(n\)-dimensional closed submanifold \(\subset S^{n+2}\). Note \(B = B^{n+2} = D^{n+2} \subset S^{n+2}\). This figure includes \(h^n_p\) and \(h^{n+1-p}\).

In [17] are proved the following results: Let \(K\) and \(K'\) be 2-dimensional closed oriented submanifolds \(\subset S^4\). Then the following conditions (1) and (2) are equivalent.

(1) \(K\) is (1,2)-pass-move-equivalent to \(K'\).
(2) \(K\) is ribbon-move-equivalent to \(K'\).

Furthermore if \(K\) is obtained from \(K'\) by one ribbon-move, then \(K\) is obtained from \(K'\) by one (1,2)-pass-move.

We draw a (1,2)-pass-move (the \(p = 1\) and \(n = 2\) case) in Figure 1.4, which consists of the two figures (1) (2). Since \(K_+\) and \(K_-\) are related by a (1,2)-pass-move in \(B\), \(B\) has the following properties. We regard \(B\) as (2-disc) \(\times [0, 1] \times \{t\mid -1 \leq t \leq 1\}\).

(i) \(K_+ \cap B, K_- \cap B, K_0 \cap B\) coincide each other.
(ii) \(B \cap K_+, B \cap K_-\) are shown in Figure 1.4.

In Figure 1.4 we draw \(B_{-0.5} \cap K_+, B_0 \cap K_+, B_{0.5} \cap K_+\), where \(B_{t_0} = \text{(2-disc)} \times [0, 1] \times \{t\mid t = t_0\}\). We suppose that each vector \(\overrightarrow{x}, \overrightarrow{y}\) in Figure 1.4 is a tangent vector of each disc at a point. (Note that we use \(\overrightarrow{x}\) (resp. \(\overrightarrow{y}\)) for different vectors.) The orientation of each disc in Figure 1.4 is determined by the each set \(\{\overrightarrow{x}, \overrightarrow{y}\}\). In [17], near Figure 4.1 and 4.2, more explanation of the figure of \(B \cap K_+\) and that of \(B \cap K_-\) are given.
This cube is $B = D^{n+2} = D^1 \times D^p \times D^{n+1-p}$

$B \cap K_+$

Figure 3.1: $(p, n + 1 - p)$-pass-move

$B \cap K_-$

Figure 3.2: $(p, n + 1 - p)$-pass-move
Figure 4.1: (1,2)-pass-move

Figure 4.2: (1,2)-pass-move
It is very natural to consider the following problem: If codimension two closed oriented submanifolds \( K \) and \( J \) in \( S^l \) are equivalent by a local move, then for a codimension two closed oriented submanifold \( A \) in \( S^w \), are \( K \otimes A \) and \( J \otimes A \) equivalent by another local move? In [10] we obtained many results. In this paper we further prove the theorems in the following section.

4. Main results on local moves on knots and product of knots

**Theorem 4.1.** Let \( J \) and \( K \) be 1-links in \( S^3 \). Suppose that \( J \) is obtained from \( K \) by a single pass-move. Then \( J \otimes^k \text{Hopf} \) is obtained from \( K \otimes^k \text{Hopf} \) by a single \((2k+1, 2k+1)\)-pass-move (\( k \in \mathbb{N} \)).

Theorem 4.1 implies Theorem 4.2 (We prove it in §8).

**Theorem 4.2.** (1) Let \( a, b, a', b' \) be natural numbers. If the \((a, b)\) torus link is pass-move equivalent to the \((a', b')\) torus link, then the Brieskorn manifold \( \Sigma(a, b, 2, ..., 2) \), where there are an even number of \( 2 \)'s, is diffeomorphic to \( \Sigma(a', b', 2, ..., 2) \).

(2) The converse of (1) is false in general.

Another one of our main results is as follows.

**Theorem 4.3.** Let \( J \) and \( K \) be (not necessarily connected) 2-dimensional closed oriented submanifolds in \( S^4 \). Suppose that \( J \) is obtained from \( K \) by a single ribbon-move. Then \( J \otimes^k \text{Hopf} \) is obtained from \( K \otimes^k \text{Hopf} \) by a single \((2k+1, 2k+2)\)-pass-move (\( k \geq 2, k \in \mathbb{N} \)).

Theorems 2.1, 4.3, and 4.1 connect the classification problem of \( n \)-knots by a local-move equivalence with that of \((n + \alpha)\)-dimensional stable knots by another local-move equivalence, where \( \alpha \in \mathbb{N} \) is sufficiently large. (If \( n = 1, 2 \), ‘stable’ is replaced by ‘simple’.) See [1, 2, 3, 4, 5] for stable knots. Thus we can consider lower and higher dimensional knots together.

Local moves on high dimensional submanifolds are exciting ways of explicit construction of high dimensional figures. They are also a generalization of local moves on 1-links as we saw in this section. They are useful to research link cobordism, knot cobordism, and the intersection of submanifolds (see [15]). There remain many exciting problems. Some of them are proper in high dimension and others are analogous to 1-dimensional cases. For example, we do not know a local move on high dimensional knots which is an unknotted operation.

**Note.** We abbreviate ‘manifold-with-boundary’ (resp. ‘submanifold-with-boundary’) to ‘manifold’ (resp. ‘submanifold’) when this is clear from the context.
5. Proof of Theorem 2.1

Let $\mathcal{E}_n$ be the set of (not necessarily connected) $n$-dimensional closed oriented submanifolds in $S^{n+2}$. We abbreviate $A \otimes$ (the Hopf link) to $A \otimes$ Hopf. Let $K \in \mathcal{E}_n$. By [7, 9], $K \otimes$ Hopf $\in \mathcal{E}_{n+4}$. Thus we obtain a map

$$\otimes \text{Hopf} : \mathcal{E}_n \to \mathcal{E}_{n+4}.$$ 

Let $A$ be a subset of $\mathcal{E}_n$. We abbreviate a map $\otimes \text{Hopf}|_A$ to $\otimes \text{Hopf}$. Let $\otimes \text{Hopf}(A)$ be denoted by $A \otimes$ Hopf. In [7, 9, 10] we proved the following theorems.

**Note.** If $A \otimes B$ is a knot product, then $A$ (resp. $B$) is not necessarily PL homeomorphic to the standard sphere by the definition. So in the literature on knot products we sometimes say that if $K$ is a knot, $K$ is a spherical knot although a knot is PL homeomorphic to the standard sphere by the definition. (Note: Some papers define a spherical knot to be homeomorphic to the standard sphere.)

**Theorem 5.1.** ([7, 9, 10]) Let $K_n$ be the set of $n$-dimensional spherical knots in $S^{n+2}$, where $n \in \mathbb{N}$. Then we have the following.

1. $K_n \otimes \text{Hopf} \subset K_{n+4}$.
2. Let $K, J \in K_n$. Suppose $K$ is knot-cobordant to $J$. Then $K \otimes \text{Hopf}$ is cobordant to $J \otimes \text{Hopf}$. Thus we can define a homomorphism

$$\otimes \text{Hopf} : C_n \to C_{n+4},$$

where $C_n$ is the $n$-dimensional knot cobordism group.
3. $\otimes \text{Hopf} : C_n \to C_{n+4}$ is an isomorphism if $n \neq 1, 3$.
   - $\otimes \text{Hopf} : C_1 \to C_5$ is injective and not surjective.
   - $\otimes \text{Hopf} : C_3 \to C_7$ is surjective and not injective.

Let $S_n$ be the set of $n$-dimensional spherical simple knots in $S^{n+2}$, where $n \in \mathbb{N}$ and $n \geq 3$.

4. $S_n \otimes \text{Hopf} \subset S_{n+4}$.

**Note.** See [11, 13] for knot cobordism. See [1, 2, 3, 4, 5, 14] for simple knots.

**Theorem 5.2.** (Theorem 12.8 of [10]) We have the following.

- $\otimes \text{Hopf} : S_{2m+1} \to S_{2m+5}$ is a bijective map if $2m + 1 \neq 1, 3$ and $m \in \mathbb{N}$.
- $\otimes \text{Hopf} : S_3 \to S_7$ is injective and not surjective.
- $\otimes \text{Hopf} : S_1 \to S_5$ is surjective and not injective.

It is natural to ask the following question.

**Question 5.3.** What kind of map is $\otimes \text{Hopf} : S_{2m} \to S_{2m+4}$ ($m \in \mathbb{N}$, $2m \geq 4$)?

In this paper we prove the following theorem.
Lemma 5.4. Let $B$ be an element in the intersection is denoted by $(x, p)$ where $x \in \mathbb{R}$ and $(x, p)$ is denoted by $\mathbb{R}$. Suppose $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^{n+2}$, and $A \cap B = \phi$. Take a continuous map
\[
\alpha : A \times B \rightarrow \mathbb{R}^{n+2} - \{0\}
\]
\[(a, b) \mapsto a - b.
\]
Here we regard $a$ and $b$ as elements of $\mathbb{R}^{n+2}$. Hence we can define $a - b$ consistently. Thus we can define a continuous map
\[
\tau(\alpha) : A \times B \rightarrow S^{n+1}
\]
\[(a, b) \mapsto \frac{a - b}{|a - b|},
\]
where $|a - b|$ is the distance between $a$ and $b$. We make $\tau(\alpha)$ into a continuous map
\[
\theta(\alpha) : A \land B \rightarrow S^{n+1},
\]
where $\land$ denotes the smash product.

Let $\mathbb{R}^{n+3} = \mathbb{R} \times \mathbb{R}^{n+2}$. Suppose that an element in $\mathbb{R}^{n+3} = \mathbb{R} \times \mathbb{R}^{n+2}$ is represented by $(x, p)$, where $x \in \mathbb{R}$ and $p \in \mathbb{R}^{n+2}$. Construct the suspension $\Sigma A$ from $A$, $(1, 0)$ and $(-1, 0)$, where $A \subset \mathbb{R}^{n+2}$, $1 \in \mathbb{R}$, and $-1 \in \mathbb{R}$, $0 \in \mathbb{R}^{n+2}$. Of course $\Sigma A$ is not in $\{0\} \times \mathbb{R}^{n+2}$ but in $\mathbb{R}^{n+3}$. Define a continuous map $\beta_1(\alpha) : \Sigma A \times B \rightarrow \mathbb{R}^{n+3} - \{0\}$ as follows. Note that $B \subset \{0\} \times \mathbb{R}^{n+2} \subset \mathbb{R}^{n+3}$. We use the following lemma.

Lemma 5.4. Let $0 \leq t \leq 1$. Take an element in the intersection of $\Sigma A \subset \mathbb{R} \times \mathbb{R}^{n+2}$ and $\{x|x = t\} \times \mathbb{R}^{n+2} \subset \mathbb{R} \times \mathbb{R}^{n+2}$. Then for an element $(0, a) \in A \subset \{x|x = 0\} \times \mathbb{R}^{n+2}$, the element in the intersection is denoted by $(t, (1 - t)a) \in \{x|x \in \mathbb{R}\} \times \mathbb{R}^{n+2} = \mathbb{R} \times \mathbb{R}^{n+2}$.

Define a continuous map $\beta_{1,u}(\alpha)$ to be
\[
\beta_{1,u}(\alpha) : (\Sigma A \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}^{n+2})) \times B \rightarrow \mathbb{R}^{n+3} - \{0\}
\]
\[(p, q) \mapsto p - q.
\]
Recall $B \subset \{x|x = 0\} \times \mathbb{R}^{n+2}$. Hence this continuous map is represented by
\[
((t, (1 - t)a), (0, b)) \mapsto (t, (1 - t)a - b)
\]
\[= (t, a - at - b).
\]
in another way. We use the following lemma.

Lemma 5.5. Let $-1 \leq t \leq 0$. Take an element in the intersection of $\Sigma A$ and $\{x|x = t\} \times \mathbb{R}^{n+2}$. Then for an element $(0, a) \in A$, the element in the intersection is denoted by $(t, (1 + t)a)$.
Define a continuous map \( \beta_{1,d}(\alpha) \) to be
\[
\beta_{1,d}(\alpha) : (\Sigma A \cap (\mathbb{R}_{\leq 0} \times \mathbb{R}^{n+2})) \times B \rightarrow \mathbb{R}^{n+3} - \{0\}
\]
\[
(p, q) \mapsto p - q.
\]

Hence this continuous map is represented by
\[
((t, (1 + t)a), (0, b)) \mapsto (t, (1 + t)a - b)
\]

in another way. The continuous map \( \gamma_0 \) is defined by using \( \beta_{1,u}(\alpha) \) and \( \beta_{1,d}(\alpha) \).

We can define a continuous map \( \beta_2(\alpha) : A \times \Sigma B \rightarrow \mathbb{R}^{n+3} - \{0\} \) in the same way as we define \( \beta_1(\alpha) \). Let \( \beta(\alpha) = \beta_1(\beta_2(\alpha)) : \Sigma A \times \Sigma B \rightarrow \mathbb{R}^{n+4} - \{0\} \). We will use \( \beta(\beta(\alpha)) : \Sigma^2 A \times \Sigma^2 B \rightarrow \mathbb{R}^{n+6} - \{0\} \). Let \( \beta^2(\alpha) \) denote \( \beta(\beta(\alpha)) \).

Define a continuous map \( \gamma_1(\alpha) : \Sigma A \times B \rightarrow \mathbb{R}^{n+3} - \{0\} \) as follows. Let \( 0 \leq t \leq 1 \). Define a continuous map \( \gamma_{1,u}(\alpha) \) to be
\[
\gamma_{1,u}(\alpha) : (\Sigma A \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}^{n+2})) \times B \rightarrow \mathbb{R}^{n+3} - \{0\}
\]
\[
((t, (1 - t)a), (0, b)) \mapsto (t, (1 - t)(a - b)).
\]

Let \( -1 \leq t \leq 0 \). Define a continuous map \( \gamma_{1,d}(\alpha) \) to be
\[
\gamma_{1,d}(\alpha) : (\Sigma A \cap (\mathbb{R}_{\leq 0} \times \mathbb{R}^{n+2})) \times B \rightarrow \mathbb{R}^{n+3} - \{0\}
\]
\[
((t, (1 + t)a), (0, b)) \mapsto (t, (1 + t)(a - b)).
\]

The continuous map \( \gamma_1(\alpha) \) is defined by using \( \gamma_{1,u}(\alpha) \) and \( \gamma_{1,d}(\alpha) \).

We can define a continuous map \( \gamma_2(\alpha) : A \times \Sigma B \rightarrow \mathbb{R}^{n+3} - \{0\} \) in the same way as we make \( \gamma_1(\alpha) \). Let \( \gamma(\alpha) = \gamma_1(\gamma_2(\alpha)) : \Sigma A \times \Sigma B \rightarrow \mathbb{R}^{n+4} - \{0\} \). We will use \( \gamma(\gamma(\alpha)) : \Sigma^2 A \times \Sigma^2 B \rightarrow \mathbb{R}^{n+6} - \{0\} \). Let \( \gamma^2(\alpha) \) denote \( \gamma(\gamma(\alpha)) \).

**Lemma 5.6.** The above continuous maps \( \beta_1(\alpha) \) and \( \gamma_1(\alpha) \) are homotopic.

**Proof of Lemma 5.6.** We define a homotopy \( \xi_s : \Sigma A \times B \rightarrow \mathbb{R}^{n+3} - \{0\} \), where \( 0 \leq s \leq 1 \) such that \( \xi_0 = \beta \) and that \( \xi_1 = \gamma \) as follows. Let \( 0 \leq t \leq 1 \). Define a continuous map \( \xi_{s,u} \) to be
\[
\xi_{s,u} : \Sigma A \times B \rightarrow \mathbb{R}^{n+3} - \{0\}
\]
\[
((t, (1 - t)a), (0, b)) \mapsto (t, s(1 - t)(a - b) + (1 - s)(a - b - ta)) = (t, (1 - t)a - (1 - st)b).
\]

Let \( -1 \leq t \leq 0 \). Define a continuous map \( \xi_{s,d} \) to be
\[
\xi_{s,d} : \Sigma A \times B \rightarrow \mathbb{R}^{n+3} - \{0\}
\]
\[
((t, (1 + t)a), (0, b)) \mapsto (t, s(1 + t)(a - b) + (1 - s)(a - b + ta))
\]
The continuous map \( \xi_s \) is defined by using \( \xi_{s,u} \) and \( \xi_{s,d} \).

Let \( p \in \Sigma A \times B \). Let \( \xi(p) = (t, \kappa) \). If \( t \neq 0 \), \( \xi_s(p) \in \mathbb{R}^{n+3} - \{0\} \). If \( t = 0 \), \( \kappa = a - b \).

Hence \( \kappa \neq 0 \). Hence \( \xi_s(p) \in \mathbb{R}^{n+3} - \{0\} \). Therefore \( \xi_s \) is well-defined.

We can prove the following lemma in the same way as we prove Lemma 5.6.

**Lemma 5.7.** The above continuous maps \( \beta^2(\alpha) \) and \( \gamma^2(\alpha) \) are homotopic.

We review the classification of even dimensional simple knots by Farber in [1, 2, 3, 4, 5]. In particular, see §1 and 2 of [1] for detail.

Let \( K \) be a \( 2m \)-dimensional simple knot \( \subset S^{2m+2} = \mathbb{R}^{2m+2} \cup \{\ast\} \). Let \( V \) be an \((m - 1)\)-connected Seifert hypersurface for \( K \). Push off \( V \) to the positive direction of the normal bundle of \( V \) in \( \mathbb{R}^{n+2} \), and call it \( V_+ \). Let \( A = V \) and \( B = V_+ \) in the definition of the above continuous map \( \theta(\alpha) \). We abbreviate \( \theta(\alpha) : V \land V_+ \rightarrow S^{2m+1} \) to \( \theta : V \land V \rightarrow S^{2m+1} \).

By using this \( \theta \), define an \( S \)-map \( \theta : V \land V \rightarrow S^{2m+1} \), where \( \rightarrow \) denotes an \( S \)-map, and call it the Seifert homotopy pairing for \( V \). Here we let \( f : X \rightarrow Y \) denote an \( S \)-map made from a continuous map \( f : X \rightarrow Y \). Note that we use the notation \( f \) for both. The Seifert homotopy pairing defines a spherical pairing \((V, \theta)\). The \( n \)-isometry \( (V, u, z) \) corresponding to this \( n \)-pairing will be called the \( n \)-isometry of the manifold \( V \). See §1 and 2 of [1] for spherical pairings and \( n \)-isometries. Spherical \( n \)-pairings and \( n \)-isometries are in one-to-one correspondence by the correspondence written in §2.4 of [1]. Furthermore Lemma 2.8 of [1] shows that spherical \( n \)-pairings are contiguous if and only if the corresponding \( n \)-isometries are contiguous.

**Theorem 5.8.** (Theorem 2.6 of [1].) The \( R \)-equivalence class of the \( 2m \)-isometry \((V, u, z)\) does not depend on the choice of the Seifert hypersurface \( V \) and is well-defined by the type of the given \( 2m \)-dimensional simple knot \( K \). If \( 2m \geq 8 \), then the map, sending each \( 2m \)-dimensional simple knot to the \( R \)-equivalence class of the \( 2m \)-isometry of some \((m - 1)\)-connected Seifert hypersurface of this knot, is a bijection of the set of isotopy types of \( 2m \)-dimensional simple knots to the set of \( R \)-equivalence classes of \( 2m \)-isometries given on all virtual complexes of length \( \leq 2 \).

**Note.** See §1 and 2 of [1] for ‘\( R \)-equivalence classes’ and ‘\( n \)-isometries given on all virtual complexes of length \( \leq 2 \)’.

We state another theorem which was proved in [1, 2, 3, 4, 5] after some preparations (see Theorem 5.12). Take the suspension \( \Sigma X \) of \( X \). \( \Sigma X \) is made from \( X \) and \([-1, 1]\) as usual. An element of \( \Sigma X \) is represented by \((t, x)\) by using \( t \in [-1, 1] \) and \( x \in X \) as usual. This way of representation is different from the previous way that we used in order to define \( \beta(\alpha) \) and \( \gamma(\alpha) \).
Take a continuous map
\[ f : X \to Y \]
\[ x \mapsto y. \]
Define a continuous map to be
\[ \Sigma f : \Sigma X \to \Sigma Y \]
\[ (t, x) \mapsto (t, y), \]
and call it the suspension of \( f \). Note that \((1, x)\) and \((1, x')\) represent a same element even if \( x \neq x' \).
Take a continuous map
\[ f : A \times B \to Y \]
\[ (a, b) \mapsto y. \]
Define a continuous map to be
\[ \Sigma^{1,0} f : (\Sigma A) \times B \to \Sigma Y \]
\[ ((t, a), b) \mapsto (t, y), \]
and call it the \((1, 0)\)-suspension of \( f \). Define a continuous map to be
\[ \Sigma^{0,1} f : A \times (\Sigma B) \to \Sigma Y \]
\[ (a, (t, b)) \mapsto (t, y), \]
and call it the \((0, 1)\)-suspension of \( f \). We can define \( \Sigma^{\nu, \mu} f \) in the same manner. We have the following.

**Lemma 5.9.** Take \( \alpha, \tau(\alpha) \) as above. Then \( \Sigma^{2,2} \tau(\alpha) : \Sigma^2 A \times \Sigma^2 B \to \Sigma^4 S^{n+1} \) is homotopic to \( \tau(\gamma^2(\alpha)) \). \( \text{Note: } \Sigma^4 S^{n+1} \text{ is regarded as } S^{n+5}. \)

Furthermore we have the following.

**Lemma 5.10.** \( \Sigma^\psi A \land \Sigma^\nu - \psi B \) is homotopy type equivalent to \( \Sigma^\phi A \land \Sigma^\nu - \phi B \).

Make
\[ \Sigma^{\psi, \nu - \psi} f : \Sigma^\psi A \times \Sigma^\nu - \psi B \to Y \]
into a continuous map
\[ \Sigma^\psi A \land \Sigma^\nu - \psi B \to Y. \]
Make
\[ \Sigma^\psi, \nu - \phi f : \Sigma^\phi A \times \Sigma^\nu - \phi B \to Y \]
into a continuous map
\[ \Sigma^\phi A \land \Sigma^\nu - \phi B \to Y. \]
By Lemma 5.10 these two new continuous maps are regarded as the same one, and call it \( \Sigma^\nu f \). Hence Lemma 5.9 implies the following.
Lemma 5.11. \( \Sigma^4 \theta (\alpha) : \Sigma^2 A \wedge \Sigma^2 B \to S^{n+5} \) is homotopic to \( \theta (\gamma^2 (\alpha)) : \Sigma^2 A \wedge \Sigma^2 B \to S^{n+5} \).

Note. Recall that for finite CW complexes, the reduced suspension of \( X \) is homotopy type equivalent to the ordinary suspension. The suspensions are used in §1 and 2 of [4] and is also used in the following Theorem 5.12.

§1 and 2 of of [4] imply the following.

Theorem 5.12. (Theorem 2.6 of [4].) (1) Let \( K^{2m} \) be a simple \( 2m \)-knot. Let \( V^{2m+1} \) be an \( (m-1) \)-connected Seifert hypersurface for \( K^{2m} \). Let \( \theta : V^{2m+1} \wedge V^{2m+1} \to S^{2m+1} \) be a Seifert homotopy pairing. Make an \( S \)-map \( \Sigma^4 \theta : \Sigma^2 V^{2m+1} \wedge \Sigma^2 V^{2m+1} \to S^{2m+5} \) by using the suspensions. Then there is a simple \((2m+4)\)-knot \( J^{2m+4} \subset S^{2m-6} (2m \geq 8) \) with the following property: Let \( U^{2m+5} \) be an \((m+1)\)-connected Seifert hypersurface for \( J^{2m+4} \). Let \( \rho : U^{2m+5} \wedge U^{2m+5} \to S^{2m+5} \) be a Seifert homotopy pairing. Then \( \Sigma^4 \theta \) is \( R \)-equivalent to \( \rho \).

(2) The operation sending \( K^{2m} \) to \( J^{2m+4} \) in (1) gives a one-to-one map \( S_{2m} \to S_{2m+4} (2m \geq 8 \text{ and } m \in \mathbb{N}) \).

Lemma 5.11 implies the following.

Lemma 5.13. We add the following to Theorem 5.12: suppose \( \theta \) is made from a continuous map

\[ \alpha_V : V^{2m+1} \times V^{2m+1}_+ \to \mathbb{R}^{2m+2} - \{0\}. \]

\[ (a, b) \mapsto a - b. \]

That is, \( \theta \) is an \( S \)-map made from \( \theta (\alpha_V) \). (This \( S \)-map is also called \( \theta (\alpha_V) \).) Then \( \theta (\gamma^2 (\alpha_V)) \) is \( R \)-equivalent to \( \rho \) and is a Seifert homotopy pairing for \( J^{2m+4} \). The correspondence \( \theta (\alpha_V) \to \theta (\gamma^2 (\alpha_V)) \) gives the bijection \( S_{2m} \to S_{2m+4} (2m \geq 8 \text{ and } m \in \mathbb{N}) \).

Lemma 6.1 of [9] implies the following.

Theorem 5.14. (Lemma 6.1 of [9].) Let \( K^{2m} \) be a simple \( 2m \)-knot in \( S^{2m+2} = \mathbb{R}^{2m+2} \cup \{\ast\} \) \((m \in \mathbb{N} - \{0, 1\})\). Suppose \( K^{2m} \subset \mathbb{R}^{2m+2} \). Let \( V^{2m+1} \) be an \((m-1)\)-connected Seifert hypersurface \( \subset \mathbb{R}^{2m+2} \) for \( K^{2m} \). Make \( V^{2m+1} \) from \( V^{2m+1}_+ \) as usual. Take a continuous map

\[ \alpha_V : V^{2m+1} \times V^{2m+1}_+ \to \mathbb{R}^{2m+2} - \{0\} \]

\[ (a, b) \mapsto a - b. \]

Take \( \beta^2 (\alpha_V) \) as above. Make \( K^{2m} \otimes \text{Hopf} \) in \( S^{2m+6} = \mathbb{R}^{2m+6} \cup \{\ast\} \). Then there is an \((m+1)\)-connected Seifert hypersurface \( W^{2m+5} \) for \( K^{2m} \otimes \text{Hopf} \) with the following properties.
(1) $W^{2m+5}$ is homotopy type equivalent to $\Sigma^2 V^{2m+1}$ by continuous maps 
$\zeta: \Sigma^2 V^{2m+1} \to W^{2m+5}$ and $\zeta': W^{2m+5} \to \Sigma^2 V^{2m+1}$.
Make $W^{2m+5}$ from $W^{2m+5}$ as usual. Then $W^{2m+5}$ is homotopy type equivalent to $\Sigma^2 V^{2m}$
by continuous maps $\zeta_+: \Sigma^2 V^{2m+1} \to W^{2m+5}_+$ and $\zeta': W^{2m+5}_+ \to \Sigma^2 V^{2m+1}_+$.

(2) Take a continuous map
$\alpha_W: W^{2m+5} \times W^{2m+5} \to \mathbb{R}^{2m+6} - \{0\}$

$(a,b) \mapsto a - b$.

Take the continuous map $\zeta \times \zeta_+: \Sigma^2 V^{2m+1} \times \Sigma^2 V^{2m+1} \to W^{2m+5} \times W^{2m+5}$. Then the
continuous map $\alpha_W \circ (\zeta \times \zeta_+)$ is homotopic to the continuous map $\beta^2(\alpha_V)$.

The $S$-map made from $\theta(\alpha_W \circ (\zeta \times \zeta_+))$ is a Seifert homotopy pairing for $K^{2m} \otimes \text{Hopf}$.

By Theorem 5.8 we have the following.

**Lemma 5.15.** We add the following to Theorem 5.14: the $S$-map made from $\theta(\beta^2(\alpha_V))$
is a Seifert homotopy pairing for $K^{2m} \otimes \text{Hopf}$.

Lemma 5.14 implies the following.

**Lemma 5.16.** We add the following to Theorem 5.14: the $S$-map made from $\theta(\gamma^2(\alpha_V))$
is a Seifert homotopy pairing for $K^{2m} \otimes \text{Hopf}$.

Lemma 5.13 and 5.16 imply Theorem 2.1. This completes the proof of Theorem 2.1.

**Note.** We could extend Theorem 2.1 to the case of the stable knots corresponding to
the set of $R$-equivalence classes of $2m$-isometries given on all virtual complexes of a given
length.

6. **Lemmas for Proof of Theorem 4.3**

**Theorem 4.3.** Let $J$ and $K$ be (not necessarily connected) 2-dimensional closed oriented
submanifolds in $S^4$. Suppose that $J$ is obtained from $K$ by a single ribbon-move. Then
$J \otimes^k \text{Hopf}$ is obtained from $K \otimes^k \text{Hopf}$ by a single $(2k+1, 2k+2)$-pass-move ($k \geq 2, k \in \mathbb{N}$).

**Note.** (1) By Proposition 4.3 of [17], $J$ is ribbon-move equivalent to $K$ if and only if $J$
is (1,2)-pass-move to $K$.
(2) The converse of Theorem 4.3 is false by Theorem 9.13 of [10].

We need the following lemmas in order to prove Theorem 4.3. In this section and the
following one we prove these lemmas. In the following section we complete the proof of
Theorem 4.3.
By the assumption, $J$ and $K$ differ only by a single (1,2)-pass-move in a 4-ball $B^4$ trivially embedded in $S^4$ as shown in Figure 6.1. Let $V_J$ (resp. $V_K$) be a Seifert hypersurface for $J$ (resp. $K$). By the Pontrjagin-Thom construction we can suppose that $V_J$ and $V_K$ differ only by a single (1,2)-pass-move in the 4-ball $B^4$ as shown in Figure 6.2. That is, we have the following. See [17] for detail.

**Lemma 6.1.** (1) $V_J \cap B^4$ (resp. $V_K \cap B^4$) is a disjoint union of a 3-dimensional 1-handle and a 3-dimensional 2-handle which are attached to $V_J-\text{Int}B^4$ (resp. $V_K-\text{Int}B^4$).
The core of the attached part of the 3-dimensional 1-handle (resp. 2-handle) is embedded trivially in $\partial B^4$. The handles are attached to $\partial B^4$ with the trivial framing.

The 3-dimensional 1-handle (resp. 2-handle) is embedded trivially in $B^4$ so that the attached part satisfies the above condition (2).

**Note.** If we attach an $n$-dimensional $p$-handle $h^p$ to the standard sphere $S^m$ ($n \leq m$), the phrase ‘attaching $h^p$ to $S^m$ with the trivial framing’ makes sense as usual.

**Remark on Handle Notation.** Take a handle decomposition of $V_J$ (resp. $V_K$) which consists of a single 3-dimensional 0-handle $h_0^J$ (resp. $h_0^K$), the above 3-dimensional 1-handle, the above 3-dimensional 2-handle and other handles. Call this 3-dimensional 1-handle $h_1^J$ (resp. $h_1^K$). Call this 3-dimensional 2-handle $h_2^J$ (resp. $h_2^K$). We abbreviate by removing the subscript $J$ (resp. $K$) when this is clear from the context.

We prove the following lemma in the following section.

**Lemma 6.2.** Under the assumption of Lemma 6.1, we can suppose that the attached part of the 3-dimensional 1-handle $h^1$ (resp. 2-handle $h^2$) is embedded in the boundary of the 3-dimensional 0-handle $h^0$.

**Note.** Take a handle decomposition of a connected compact manifold. It does not hold in general that the attached part of each $i$-handle $\subset$ a 0-handle ($i > 0$).

Let $[2]$ be the empty knot of degree two. Recall that the Hopf link $\leq [2] \otimes [2]$. See [7, 9] for the empty knot.

**Lemma 6.3.** ([7, 9].) Let $K$ be a (not necessarily connected) $n$-dimensional closed oriented submanifold $\subset S^{n+2} = \partial D^{n+3} \subset D^{n+3}$ ($n \in \mathbb{N}$). Let $V$ be a Seifert hypersurface for $K$. Then there is a Seifert hypersurface $W$ for $K \otimes [2]$ with the following property: Take the tubular neighborhood $N(V)$ of $V$ in $S^{n+2}$. Take two copies of $D^{n+3}$. Attach the two copies $D^{n+3}$ by identifying two $N(V)$.

**Note.** We use $B^*$ for the ball where the local moves are carried out. So we use $D^*$ for the above ball.

Take $B^4$ and $V_Z$ in Lemma 6.1 and 6.2 and call $B^4$, $B_Z$ ($Z = J, K$). Take $D^5$ in Lemma 6.3 and call it $D_Z^5$. Make a Seifert hypersurface $V_{Z \otimes [2]}$ for $Z$ from the two copies of $D_Z^5$. By [6, 7, 9, 24, 25], Lemma 6.1, 6.2, and 6.3, we have the following.

**Lemma 6.4.** There is a Seifert hypersurface $V_{Z \otimes [2]}$ for $Z \otimes [2]$ ($Z = J, K$) $\subset S^6$ and a 6-ball $B^6$ trivially embedded in $S^6$ with the following properties: (See the remark on handle notation prior to Lemma 6.3)
(1) There is the orientation preserving identity map
\[ \iota : S^6 \to S^6 \] such that \( \iota(V_{J \otimes [2]} - \text{Int}B_{J \otimes [2]}) = V_K \otimes [2] - \text{Int}B_K \otimes [2] \). Here \( B^6 \) for \( Z \otimes [2] \) is called \( B_{Z \otimes [2]} \) for the convenience.

(2) \( V_{Z \otimes [2]} \cap \text{Int}B_{Z \otimes [2]} \) is a disjoint union of a 5-dimensional 2-handle \( h^2_{Z \otimes [2]} \) — the attaching part and a 5-dimensional 3-handle \( h^3_{Z \otimes [2]} \) — the attaching part.

(3) \( V_{Z \otimes m[2]} \cap \partial B_{Z \otimes m[2]} \) is a 4-dimensional compact connected manifold-with-boundary \( P_{Z \otimes [2]} \).

\[ \iota(P_{J \otimes [2]}) = P_{K \otimes [2]} \]

\( P_{Z \otimes [2]} \) is made from \( B_J(\subset D^5_J) \) and \( B_K(\subset D^5_K) \) by identifying the tubular neighborhood of the attached part of \( h^1_J \) (resp. \( h^2_J \)) of \( \partial B_J \) with that of \( h^1_K \) (resp. \( h^2_K \)) of \( \partial B_K \).

(4) The attached part of \( h^2_{Z \otimes [2]} \) (resp. \( h^2_{Z \otimes [2]} \)) is embedded in \( P_{Z \otimes [2]} \). The core of the attached part is embedded trivially in \( \partial B_{Z \otimes [2]} \). The handle is attached to \( \partial B_{Z \otimes [2]} \) with the trivial framing.

The core of the attached part of \( h^{1+\epsilon}_{Z \otimes [2]} \), where \( \epsilon = 1, 2 \), is an \( \epsilon \)-sphere, and call it \( S^{e}_{Z \otimes [2]} \).

Recall that \( S^{1}_{Z \otimes [2]} \otimes \# S^{2}_{Z \otimes [2]} \) is embedded in \( P_{Z \otimes [2]} \). By [7, 9], we can suppose that \( \iota(S^e_{J \otimes [2]}) \) is isotopic to \( S^e_{K \otimes [2]} \) in \( P_{K \otimes [2]} \). By the uniqueness of the tubular neighborhood we can suppose the following fact (\( \Theta \)): \( \iota(\text{the attached part of } h^{1+\epsilon}_{J \otimes [2]}) = \text{the attached part of } h^{1+\epsilon}_{K \otimes [2]} \).

Note that \( \iota(S^{1}_{J \otimes [2]} \otimes \# S^{2}_{J \otimes [2]} \) is not necessarily isotopic to \( S^{1}_{K \otimes m[2]} \otimes \# S^{2}_{K \otimes [2]} \) in \( P_{K \otimes [2]} \). By [7, 9] we have the following.

**Lemma 6.5.** The diffeomorphism map of the attaching part of \( h^{1+\epsilon}_{J \otimes [2]} \), where \( \epsilon = 1, 2 \), to the attached part can be regarded as the same one as that of \( h^{1+\epsilon}_{K \otimes [2]} \) by using the natural identity map \( h^{1+\epsilon}_{J \otimes [2]} \) to \( h^{1+\epsilon}_{K \otimes [2]} \) and the fact (\( \Theta \)) a few lines above here.

We prove the following lemma in the following section.

**Lemma 6.6.** Take \( J \otimes m[2] \) and \( K \otimes m[2] \) in \( S^{2m+4} \) \((m \geq 3)\). Then there is a Seifert hypersurface \( V_{Z \otimes m[2]} \) for \( Z \otimes m[2] \) \((Z = J, K)\) and a \((2m+4)\)-ball \( B^{2m+4} \) trivially embedded in \( S^{2m+4} \) with the following properties:

(1) There is the orientation preserving identity map
\[ \iota : S^{2m+4} \to S^{2m+4} \] such that \( \iota(V_{J \otimes m[2]} - \text{Int}B_{J \otimes m[2]}) = V_K \otimes m[2] - \text{Int}B_K \otimes m[2] \). Here \( B^{2m+4} \) for \( Z \otimes m[2] \) is called \( B_{Z \otimes m[2]} \) for the convenience.

(2) \( V_{Z \otimes m[2]} \cap \text{Int}B_{Z \otimes m[2]} \) is the disjoint union of a \((2m+3)\)-dimensional \((m+1)\)-handle \( h^{m+1}_{Z \otimes m[2]} \) — the attaching part and a \((2m+3)\)-dimensional \((m+2)\)-handle \( h^{m+2}_{Z \otimes m[2]} \) — the attaching part.
Lemma 6.7. \( V_{Z \otimes m} \cap \partial B_{Z \otimes m} \) is a \((2m + 2)\)-dimensional compact connected \((m - 1)\)-connected manifold-with-boundary \( P_{Z \otimes m} \). Hence \( \iota(P_{J \otimes m}) = P_{K \otimes m} \).

(4) The attached part of \( h_{Z \otimes m}^{m+\varepsilon} \), where \( \varepsilon = 1, 2 \), is embedded in \( P_{Z \otimes m} \). The core of the attached part is embedded trivially in \( \partial B_{Z \otimes m} \). The handle is attached to \( \partial B_{Z \otimes m} \) with the trivial framing.

(5) The diffeomorphism map of the attaching part of \( h_{J \otimes m}^{m+\varepsilon} \), where \( \varepsilon = 1, 2 \), to the attached part can be regarded as the same one as that of \( h_{K \otimes m}^{m+\varepsilon} \) in the way explained in Lemma 6.8.

The core of the attached part of \( h_{Z \otimes m}^{m+\varepsilon} \), where \( \varepsilon = 1, 2 \), is an \((m + \varepsilon - 1)\)-sphere, and call it \( \tilde{S}_{Z \otimes m}^{m+\varepsilon-1} \). Recall that \( \tilde{S}_{Z \otimes m}^{m+1} \) is embedded in \( P_{Z \otimes m} \). If \( m \geq 2 \), by the Mayer-Vietoris exact sequence \( H_0(P_{Z \otimes m}; Z) \cong Z \), \( H_m(P_{Z \otimes m}; Z) \cong Z \), \( H_{m+1}(P_{Z \otimes m}; Z) \cong Z \), and \( H_i(P_{Z \otimes m}; Z) \cong 0 \) (\( i \neq 0, m, m + 1 \)). By the van Kampen theorem \( \pi_1 P_{Z \otimes m} \cong 1 \). By Hurewicz’s theorem \( \pi_m P_{Z \otimes m} \cong Z \). By [7, 9], \( S_{Z \otimes m}^{m+\varepsilon-1} \) is the generator of \( H_{m+\varepsilon-1}(P_{Z \otimes m}; Z) \cong Z \). Recall \( \iota(P_{J \otimes m}) = P_{K \otimes m} \). \( \iota(S_{J \otimes m}^{m+\varepsilon-1}) \) is homotopic to \( \tilde{S}_{K \otimes m}^{m+\varepsilon-1} \) in \( P_{K \otimes m} \). Note that \( \iota(S_{J \otimes m}^{m+\varepsilon-1}) \) is not necessarily isotopic to \( \tilde{S}_{K \otimes m}^{m+\varepsilon-1} \) in \( P_{K \otimes m} \). Furthermore by [6, 24, 25] we have the following.

**Lemma 6.7.** Let \( m \geq 3 \).

1. By using an isotopy of \( \tilde{S}_{K \otimes m}^{m+\varepsilon-1} \) in \( P_{K \otimes m} \), we can let \( \iota(\tilde{S}_{K \otimes m}^{m+\varepsilon-1}) = \tilde{S}_{K \otimes m}^{m+\varepsilon-1} \) in \( P_{K \otimes m} \).

2. Take \( \iota \) as in (1). If \( \iota(\tilde{S}_{J \otimes m}^{m+\varepsilon-1}) \) and \( \tilde{S}_{K \otimes m}^{m+\varepsilon-1} \) represent the same cycle in \( P_{K \otimes m} - \tilde{S}_{K \otimes m}^{m+\varepsilon-1} \), then we can let \( \iota(S_{J \otimes m}^{m+\varepsilon-1}) \) be \( S_{K \otimes m}^{m+\varepsilon-1} \) in \( P_{K \otimes m} \) by using an isotopy of \( S_{K \otimes m}^{m+\varepsilon-1} \) in \( P_{K \otimes m} \).

By Lemma 6.7(1) and the uniqueness of the tubular neighborhood, we can suppose the following fact (\( \Theta \)) if \( m \geq 3 \):

\( \iota \) the attached part of \( h_{J \otimes m}^{m+\varepsilon} \) is the attached part of \( h_{K \otimes m}^{m+\varepsilon} \)(\( \varepsilon = 1, 2 \)).

By [7, 9] we have the following.

**Lemma 6.8.** The diffeomorphism map of the attaching part of \( h_{J \otimes m}^{m+\varepsilon} \), where \( \varepsilon = 1, 2 \), to the attached part can be regarded as the same one as that of \( h_{K \otimes m}^{m+\varepsilon} \) by using the natural identity map \( h_{J \otimes m}^{m+\varepsilon} \) to \( h_{K \otimes m}^{m+\varepsilon} \) and the fact (\( \Theta \)) a few lines above here.
Proof of Lemma 6.2. Take the tubular neighborhood $N(B^4)$ of $B^4$ in $S^4$. We can suppose that there exists an orientation preserving diffeomorphism map $\nu : N(B^4) \to B^4$ such that $\nu|_{V \cap N(B^4)} : V \cap N(B^4) \to V \cap B^4$ is an orientation preserving diffeomorphism map. Recall $V$ is a Seifert hypersurface. Take a 4-dimensional 3-handle $H^3$ embedded in $(\text{Int} N(B^4)) - B^4$ and attach it to $h^2(\subset V)$. See Figure 6.3. This surgery makes a new Seifert hypersurface, and call it $V$ again.

Take a handle decomposition of this new $V$ and move $B^4$ by isotopy with the following properties: There are $h_1$ and $h_2$ in Lemma 6.1. There is a new 3-dimensional 0-handle in $(\partial(H^3))$-(the attached part) such that the attached part of $h^2$ is embedded in the new 3-dimensional 0-handle and that the attached part is also embedded in $\partial B^4$ trivially. We can do that without changing the submanifold type of the new $V$ in $S^4$. There are 3-dimsensional 1-handles which are not $h_1$ and which connect with all 3-dimensional 0-handles. Make these 0-handles into a new 3-dimensional 0-handle, and call it $h^0$.

Move $B^4$ and $h^0$ by isotopy so that ‘the attached part of $h_1$ II that of $h_2$’ is embedded in $h^0$ and that the disjoint union is also embedded in $\partial B^4$ trivially. We can do that without changing the submanifold type of the new $V$ in $S^4$.

Thus we obtain a handle decomposition that satisfies the condition of Lemma 6.2. □

Proof of Lemma 6.6. Lemma 6.6 (1) is proved in the same way as in the corresponding part of Proof of Theorem 8.1 in [10].

Repeating the operation of gluing the two copies of $D^{2m+3}$ as in Lemma 6.3 on $m$, we obtain a new Seifert hypersurface $V_{Z^0} \oplus m[2]$ ($m \in \mathbb{N}$) as in Lemma 6.3. By [7, 9] and Lemma...
By this fact and Lemma 6.7, we can suppose that $ι$ of Lemma 7.1.

By Lemma 6.8, we have Lemma 6.6 (5). This completes the proof of Lemma 6.6.

**Proof of Theorem 4.3.** Let $Q$ be a compact manifold with a handle decomposition. Let $h^p$ be a $p$-handle in the handles of the handle decomposition. If the core of the attached part of $h^p$ is null-homologous in the $(p - 1)$-skelton of the handle decomposition, $h^p$ represents an element $∈ H_p(Q; Z)$, and let $[h^p]$ denote the element.

Let $ξ$ and $ζ$ be cycles in a compact oriented manifold $R$. Let $ξ \cdot ζ$ in $R$ denote the intersection product of $ξ$ and $ζ$ in $R$. We sometimes delete ‘in $R$’ when this is clear from the context.

By Lemma 6.2, $[h^2_ξ]$ and $[h^2_ζ]$ make sense. By Lemma 6.2 and [7, 9], the attached part of $h^{m+\varepsilon}_{Z\otimes m[2]}$, where $ε = 1, 2$, is embedded in a $(2m + 3)$-dimensional 0-handle in $V_{Z\otimes m[2]}$. Hence $[h^{m+1}_{Z\otimes m[2]}]$ and $[h^{m+2}_{Z\otimes m[2]}]$ make sense. By the assumption of Theorem 4.3,

$$[h^1_J] \cdot [h^2_J] \text{ in } V^3_J = [h^1_K] \cdot [h^2_K] \text{ in } V_K.$$

By [7, 9] we have the following lemma.

**Lemma 7.1.** Let $m = 2k (k ∈ \mathbb{N})$. Let $Z = J, K$. We have

$$[h^{m+1}_{Z\otimes m[2]}] \cdot [h^{m+2}_{Z\otimes m[2]}] \text{ in } V_{Z\otimes m[2]} = [h^1_Z] \cdot [h^2_Z] \text{ in } V_Z.$$

**Note.** Recall $Z \otimes 2k [2] = Z \otimes k$ Hopf.

By this lemma

$$[h^{m+1}_{J\otimes m[2]}] \cdot [h^{m+2}_{J\otimes m[2]}] \text{ in } V_{J\otimes m[2]} = [h^{m+1}_{K\otimes m[2]}] \cdot [h^{m+2}_{K\otimes m[2]}] \text{ in } V_{K\otimes m[2]}.$$

By this fact and Lemma 6.7, we can suppose that $ι (S_{J\otimes m[2]}^{m+1} \Pi S_{J\otimes m[2]}^{m+1})$ is $S_{K\otimes m[2]}^{m+1} \Pi S_{K\otimes m[2]}^{m+1}$ if $m = 2k$ and $k ≥ 2$. By this fact and Lemma 6.6 (5), $V_{J\otimes m[2]}$ is diffeomorphic to $V_{K\otimes m[2]}$ if $m = 2k$ and $k ≥ 2$.

By [6, 24, 25] and the uniqueness of the tubular neighborhood, we have Lemma 7.2 and 7.3.

**Lemma 7.2.** Let $m ∈ \mathbb{N}$ and $m ≥ 2$. Take the two copies of the $(2m + 4)$-ball, and call them $B_{J\otimes m[2]}$ and $B_{K\otimes m[2]}$. Take the orientation preserving identity map $ι : B_{J\otimes m[2]} → B_{K\otimes m[2]}$. Take a $(2m + 3)$-dimensional $(m + 1)$-handle $h^{m+\varepsilon}_{Z\otimes m[2]}$ ($ε = 1, 2$, $Z = J, K$) with the following properties:

1. $h^{m+1}_{Z\otimes m[2]}$ (resp. $h^{m+2}_{Z\otimes m[2]}$) is embedded in $B_{Z\otimes m[2]}$ trivially.
(2) The core of the attached part of \( h_{m+1}^{m+2} \) (resp. \( h_{m+2}^{m+2} \)) is embedded in \( \partial B_{m+2} \) trivially. The handle is attached to \( \partial B_{m+2} \) with the trivial framing.

(3) \( \iota \) (the attached part of \( h_{m+2}^{m+2} \)) = the attached part of \( h_{m+2}^{m+2} \).

Then we have the following:

(i) By using an isotopy of \( h_{m+1}^{m+2} \) with keeping the boundary in \( B_{m+2} \), we can suppose that \( \iota(h_{m+1}^{m+2}) = h_{m+1}^{m+2} \).

(ii) By using an isotopy of \( h_{m+2}^{m+2} \) with keeping the boundary in \( B_{m+2} \), we can suppose that \( \iota(h_{m+2}^{m+2}) = h_{m+2}^{m+2} \).

**Note.** There is not an isotopy of \( h_{m+1}^{m+2} \) with keeping the boundary in \( B_{m+2} \) in general such that \( \iota(h_{m+1}^{m+2}) = h_{m+1}^{m+2} \).

**Lemma 7.3.** Under the assumption (1)(2)(3) of Lemma 7.2, suppose \( \iota(h_{J_{m+2}}^{m+2}) = h_{K_{m+2}}^{m+2} \). Since \( \iota \) (the attached part of \( h_{J_{m+2}}^{m+2} \)) = the attached part of \( h_{K_{m+2}}^{m+2} \), \( \iota \) (the core of \( h_{J_{m+2}}^{m+2} \)) and the core of \( h_{K_{m+2}}^{m+2} \) are made into an \((m+1)\)-sphere in \( B_{m+2} \) of \( h_{K_{m+2}}^{m+2} \), where we give appropriate orientations. Suppose that the \((m+1)\)-sphere is a vanishing cycle in \( B_{m+2} \). Then we have

\[
\iota(h_{J_{m+2}}^{m+2}) \Pi h_{J_{m+2}}^{m+2} = h_{K_{m+2}}^{m+2} \Pi h_{K_{m+2}}^{m+2}
\]

by an isotopy of \( h_{K_{m+2}}^{m+2} \) with keeping the boundary in \( B_{m+2} \).

By [7, 9] we have the following lemma.

**Lemma 7.4.** Let \( m = 2k(k \in \mathbb{N}) \). The \( \mathbb{Z} \)-Seifert pairing for \( h_{m+1}^{m+2} \) and \( h_{m+2}^{m+2} \) associated with the Seifert hypersurface \( V_{m+2} \) is equal to

\((-1)^k \times \) (that of \( h_{m+1}^{m+2} \) and \( h_{m+2}^{m+2} \) associated with \( V_{m+2} \)).

By Lemma 6.3(4), 7.3 and 7.4 \( V_{J_{m+2}} \) and \( V_{K_{m+2}} \) differ by a single \((m+1, m+2)\)-pass-move if \( m = 2k \), \( k \in \mathbb{N} \) and \( k \geq 2 \). Hence \( J \otimes k \) Hopf is obtained from \( K \otimes k \) Hopf by a single \((2k+1, 2k+2)\)-pass-move \((k \in \mathbb{N}, k \geq 2) \). This completes the proof of Theorem 4.3.

**Note.** By [7, 9] we have Lemma 7.3 and 7.6.

**Lemma 7.5.** If \( m = 2k + 1(k \in \mathbb{N} \cup \{0\}) \),

\( [h_{m+1}^{m+2} \cdot h_{m+2}^{m+2}] \in V_{m+2} \neq [h_{m+1}^{m+2} \cdot h_{m+2}^{m+2}] \in V_{m+2} \) (\( Z = J, K \)) in general.

Note the difference between Lemma 7.1 and Lemma 7.5.
Lemma 7.6. \( J \otimes 2k+1 [2] \) and \( K \otimes 2k+1 [2] \) are not diffeomorphic or homeomorphic in general \((k \in \mathbb{N})\).

By this lemma \( J \otimes 2k+1 [2] \) and \( K \otimes 2k+1 [2] \) are not \((2k+2, 2k+3)\)-pass-move equivalent in general \((k \in \mathbb{N})\).

8. Lemmas for Proof of Theorem 4.1

Theorem 4.1. Let \( J \) and \( K \) be 1-links in \( S^3 \). Suppose that \( J \) is obtained from \( K \) by a single pass-move. Then \( J \otimes^k \text{Hopf} \) is obtained from \( K \otimes^k \text{Hopf} \) by a single \((2k+1, 2k+1)\)-pass-move \((k \in \mathbb{N})\).

Note. (1) Theorem 9.1 of [10] proved the case where \( J \) is a 1-component 1-link.
(2) The converse of Theorem 4.1 is false by Theorem 8.4.4 of [10].
(3) In Theorem 4.1 if we replace ‘1-links in \( S^3 \)’ with ‘\(2\nu+1\)-dimensional closed oriented submanifolds in \( S^{2\nu+3}(\nu \in \mathbb{N})\)’, and ‘\(2k+1, 2k+1\)’ with ‘\(2k+\nu+1, 2k+\nu+1\)’, we could prove it in the same way as the proof although we prove only that case for the convenience.

Furthermore, in Theorem 4.3 if we replace ‘2-dimensional closed oriented submanifolds in \( S^4 \)’ with ‘\(2+2\nu\)-dimensional closed oriented submanifolds in \( S^{2\nu+4}(\nu \in \mathbb{N})\)’, and ‘\((2k+1, 2k+2)\)’ with ‘\((2k+1+\nu, 2k+2+\nu)\)’, and if we remove ‘\(k \geq 2\)’, we could prove it in the same way as the proof although we prove only that case for the convenience.

Theorem 4.2. (1) Let \( a, b, a', b' \) be natural numbers. If the \((a, b)\) torus link is pass-move equivalent to the \((a', b')\) torus link, then the Brieskorn manifold \( \Sigma(a, b, 2, ..., 2) \), where there are an even number of 2’s, is diffeomorphic to \( \Sigma(a', b', 2, ..., 2) \).
(2) The converse of (1) is false in general.

We prove Theorem 4.2 by using Theorem 4.1 before proving Theorem 4.1.

Proof of Theorem 4.2. We first prove Theorem 4.2 (1). Let \( K_{(a,b)} \) denote the \((a, b)\) torus link. Let \((a, b, 2, ..., 2)\) in \( \Sigma(a, b, 2, ..., 2) \) be a \((2k+2)\)-tuple of natural numbers. By Theorem 4.1 \( K_{(a,b)} \otimes^k \text{Hopf} \) is \((2k+1, 2k+1)\)-pass-move equivalent to \( K_{(a',b')} \otimes^k \text{Hopf} \). Hence \( K_{(a,b)} \otimes^k \text{Hopf} \) is diffeomorphic to \( K_{(a',b')} \otimes^k \text{Hopf} \). By [7, 9] \( K_{(a,b)} \otimes^k \text{Hopf} \) (resp. \( K_{(a',b')} \otimes^k \text{Hopf} \)) is diffeomorphic to \( \Sigma(a, b, 2, ..., 2) \) (resp. \( \Sigma(a', b', 2, ..., 2) \)). Hence \( \Sigma(a, b, 2, ..., 2) \) is diffeomorphic to \( \Sigma(a', b', 2, ..., 2) \).

We next prove Theorem 4.2 (2). By Exercise 8 in p.177-178 of [22] and Theorem 10.4 in p.261 of [8], there are the torus knots, \( K_{(a,b)} \) and \( K_{(a',b')} \), such that the Arf invariant of \( K_{(a,b)} \) is different from that of \( K_{(a',b')} \). Hence \( K_{(a,b)} \) is not pass-move equivalent to \( K_{(a',b')} \) by [8]. However by [7, 9, 12, 23] \( \Sigma(a, b, 2, 2) \) (resp. \( \Sigma(a', b', 2, 2) \)) is diffeomorphic to the standard sphere. Hence \( \Sigma(a, b, 2, 2) \) is diffeomorphic to \( \Sigma(a', b', 2, 2) \). \( \square \)
Let $V_J$ (resp. $V_K$) be a Seifert hypersurface for $J$ (resp. $K$). We can suppose that $V_J$ and $V_K$ differ only in the 3-ball $B^3$ as shown in Figure 8.1. We have the following by the definition of the pass-move.

**Lemma 8.1.** There is a handle decomposition of $V_J$ (resp. $V_K$) with the following properties:

1. $V_J \cap B^3$ (resp. $V_K \cap B^3$) is a disjoint union of two 2-dimensional 1-handles as shown in Figure 8.1. Call one of these 2-dimensional 1-handles $h_{\alpha,J}$ and the other $h_{\beta,K}$. The handle $h_{\alpha,J}$ (resp. $h_{\beta,K}$) is embedded trivially in $B^3$.

2. There is a single 2-dimensional 0-handle $h_0^J$ (resp. $h_0^K$) such that the attached part of $h_{\alpha,J}$ (resp. $h_{\beta,K}$) is embedded in $\partial(h_0^J)$ (resp. $\partial(h_0^K)$), and furthermore is embedded in $\partial B^3$ as shown in Figure 8.1.

Take $B^3$ and $V_Z$ in Lemma 8.1 and call $B^3$, $B_Z$ ($Z = J, K$). Take $D^4$ in Lemma 6.3 and call it $D_Z^4$. Make a Seifert hypersurface $V_{Z \otimes [2]}$ for $Z$ from the two copies of $D_Z^4$. By [6, 7, 9, 24, 25] and Lemma 6.3 we have the following Lemma 8.2.

**Lemma 8.2.** There is a Seifert hypersurface $V_{Z \otimes [2]}$ for $Z \otimes [2] (Z = J, K) \subset S^5$ and a 5-ball $B^5$ trivially embedded in $S^5$ with the following properties:
Lemma 8.3. The diffeomorphism map of the attaching part of $h_{\ast,J}\otimes[2]$, where $\ast = \alpha, \beta$, to the attached part can be regarded as the same one as that of $h_{\alpha,K}\otimes[2]$ by using the natural identity map $h_{\ast,J}\otimes[2] \to h_{\ast,K}\otimes[2]$ and the fact (Θ) a few lines above here.

We prove the following lemma in the following section.

Lemma 8.4. Take $J \otimes[m] [2]$ and $K \otimes[m] [2]$ in $S^{2m+3}(m \in \mathbb{N})$. Then there is a Seifert hypersurface $V_{Z\otimes[m]}$ for $Z\otimes[m] [2](Z = J, K)$ and a $(2m+3)$-ball $B^{2m+3}$ trivially embedded in $S^{2m+3}$ with the following properties:

(1) There is the orientation preserving identity map $\iota : S^{2m+3} \to S^{2m+3}$ such that $\iota(V_{J\otimes[m]} - \text{Int}B_{J\otimes[m]}) = V_{K\otimes[m]} - \text{Int}B_{K\otimes[m]}$. Here $B^{2m+3}$ for $Z\otimes[m] [2]$ is called $B_{Z\otimes[m]}$ for the convenience.

(2) $V_{Z\otimes[m]} \cap \text{Int}B_{Z\otimes[m]}$ is the disjoint union of a $(2m+2)$-dimensional $(m+1)$-handle $h_{\alpha,Z\otimes[m]}$ - the attaching part and a $(2m+2)$-dimensional $(m+1)$-handle $h_{\beta,Z\otimes[m]}$ - the attaching part.

(3) $V_{Z\otimes[m]} \cap \partial B_{Z\otimes[m]}$ is a $(2m+1)$-dimensional compact connected manifold-with-boundary $P_{Z\otimes[m]}$. Hence $\iota(P_{J\otimes[m]}) = P_{K\otimes[m]}$. 

Recall that $\iota^{\circ} S_{\alpha,Z\otimes[2]}$ is a $(2m+3)$-dimensional compact connected manifold-with-boundary $P_{Z\otimes[2]}$. 

$\iota(P_{J\otimes[2]}) = P_{K\otimes[2]}$. $P_{Z\otimes[2]}$ is made from $B_{J}(\subset D_{J}^{4})$ and $B_{K}(\subset D_{K}^{4})$ by identifying the tubular neighborhood of the attached part of $h_{\alpha,J}$ (resp. $h_{\beta,J}$) of $\partial B_{J}$ with that of $h_{\alpha,K}$ (resp. $h_{\beta,K}$) of $\partial B_{K}$.

The core of the attached part of $h_{\ast,Z\otimes[2]}$, where $\ast = \alpha, \beta$, is a circle, and call it $S_{\ast,Z\otimes[2]}^{1}$. 

Note that $\iota(S_{\ast,J\otimes[2]}^{1})$ is isotopic to $S_{\ast,K\otimes[2]}^{1}$ in $P_{K\otimes[2]}$. By the uniqueness of the tubular neighborhood we can suppose the following fact (Θ):

$\iota$ (the attached part of $h_{\ast,J\otimes[2]}^{1}$) = the attached part of $h_{\ast,K\otimes[2]}^{1}$.

Note that $\iota(S_{\ast,J\otimes[2]}^{1})$ is not necessarily isotopic to $S_{\ast,K\otimes[2]}^{1}$ in $P_{K\otimes[2]}$. By [7, 9] we have the following.

Lemma 8.3. The diffeomorphism map of the attaching part of $h_{\ast,J\otimes[2]}$, where $\ast = \alpha, \beta$, to the attached part can be regarded as the same one as that of $h_{\alpha,K\otimes[2]}$ by using the natural identity map $h_{\ast,J\otimes[2]} \to h_{\ast,K\otimes[2]}$ and the fact (Θ) a few lines above here.

We prove the following lemma in the following section.
(4) The attached part of $h_{\alpha,Z \otimes m[2]}$ (resp. $h_{\beta,Z \otimes m[2]}$) is embedded in $P_{Z \otimes m[2]}$. The core of the attached part is embedded trivially in $\partial B_{Z \otimes m[2]}$. The handle is attached to $\partial B_{Z \otimes m[2]}$ with trivial framing.

(5) The diffeomorphism map of the attaching part of $h_{*,J \otimes m[2]}$, where $* = \alpha, \beta$, to the attached part can be regarded as the same one as that of $h_{\alpha,K \otimes m[2]}$ in the way explained in Lemma 8.3 and 8.6.

The core of the attached part of $h_{*,Z \otimes m[2]}$, where $* = \alpha, \beta$, is an $m$-sphere, and call it $S_{*,Z \otimes m[2]}^m$. Recall that $S_{\alpha,Z \otimes m[2]}^m$ is embedded in $P_{Z \otimes m[2]}$. If $m \geq 2$, by the Mayer-Vietoris exact sequence $H_0(P_{Z \otimes m[2]}; \mathbb{Z}) \cong \mathbb{Z}$, $H_m(P_{Z \otimes m[2]}; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$, and $H_i(P_{Z \otimes m[2]}; \mathbb{Z}) \cong 0$ ($\neq 0, m$). By the van Kampen theorem $\pi_1 P_{Z \otimes m[2]} \cong 1$. By Hurewicz's theorem $\pi_m P_{Z \otimes m[2]} \cong \mathbb{Z} \oplus \mathbb{Z}$. By [7, 9], $S_{\alpha,Z \otimes m[2]}^m$ and $S_{\beta,Z \otimes m[2]}^m$ generate $H_m(P_{Z \otimes m[2]}; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Recall $\iota(P_{J \otimes m[2]}) = P_{K \otimes m[2]}$. We can suppose that $\iota(S_{*,J \otimes m[2]}^m)$ is homotopic to $S_{*,K \otimes m[2]}^m$ in $P_{K \otimes m[2]}$. Note that $\iota(S_{\alpha,J \otimes m[2]}^m) \cong S_{\beta,J \otimes m[2]}^m$ is not necessarily isotopic to $S_{\alpha,K \otimes m[2]}^m$ or $S_{\beta,K \otimes m[2]}^m$ in $P_{K \otimes m[2]}$. Furthermore by [3, 24, 25] we have the following.

**Lemma 8.5.** Let $m \geq 2$.

1. By using an isotopy of $S_{\alpha,K \otimes m[2]}^m$ in $P_{K \otimes m[2]}$, we can let $\iota(S_{\alpha,J \otimes m[2]}^m) = S_{\alpha,K \otimes m[2]}^m$ in $P_{K \otimes m[2]}$.

2. Take $\iota$ as in (1). If $\iota(S_{\alpha,J \otimes m[2]}^m) \cong S_{\alpha,K \otimes m[2]}^m$ represent the same cycle in $P_{K \otimes m[2]} - S_{\alpha,K \otimes m[2]}^m$, then we can let $\iota(S_{\alpha,J \otimes m[2]}^m) \cong S_{\alpha,K \otimes m[2]}^m \cong S_{\beta,K \otimes m[2]}^m$ in $P_{K \otimes m[2]}$ by using an isotopy of $S_{\alpha,K \otimes m[2]}^m$ or $S_{\beta,K \otimes m[2]}^m$ in $P_{K \otimes m[2]}$.

By Lemma 8.5(1) and the uniqueness of the tubular neighborhood, we can suppose the following (\Theta) if $m \geq 2$:

$\iota(\text{the attached part of } h_{*,J \otimes m[2]}) = \text{the attached part of } h_{*,K \otimes m[2]}(\ast = \alpha, \beta)$.

By [7, 9] we have the following.

**Lemma 8.6.** Let $m \geq 2$. The diffeomorphism map of the attaching part of $h_{*,J \otimes m[2]}$, where $* = \alpha, \beta$, to the attached part can be regarded as the same one as that of $h_{*,K \otimes m[2]}$ by using the natural identity map $h_{*,J \otimes m[2]} \rightarrow h_{*,K \otimes m[2]}$ and the fact (\Theta) a few lines above here.

9. **Proof of a lemma in the previous section and that of Theorem 4.1**

**Proof of Lemma 8.4.** Lemma 8.4(1) is proved in the same way as in the corresponding part of Proof of Theorem 8.1 in [10].
Repeating the operation of gluing the two copies of $D^{2m+2}$ as in Lemma 6.3 on $m$, we obtain a new Sifert hypersurface $V_{Z \otimes^m [2]}$ ($m \in \mathbb{N}$) as in Lemma 6.3. By [7, 9] and Lemma 8.2, we have Lemma 8.4 (2). By [6, 24, 25] and Lemma 8.2, we have Lemma 8.4 (4). By using van Kampen’s theorem and the Mayer-Vietoris exact sequence, we have Lemma 8.4 (3).

By Lemma 8.3 and 8.6, we have 8.4 (5). This completes the proof of Lemma 8.4.

□

Proof of Theorem 4.1. By Lemma 8.1, $[h_*, Z]$ makes sense ($* = \alpha, \beta$). By Lemma 8.1 and [7, 9], the attached part of $h_*, Z \otimes^m [2]$ is embedded in a $(2m+2)$-dimensional 0-handle in $V_{Z \otimes^m [2]}$ ($* = \alpha, \beta$). Hence $[h_*, Z \otimes^m [2]]$ makes sense. By the assumption of Theorem 4.1 $[h_{\alpha, J} \otimes^m [2]]$ in $V_J = [h_{\alpha, K} \otimes^m [2]]$ in $V_K$.

By [7, 9] we have the following lemma.

Lemma 9.1. Let $m = 2k (k \in \mathbb{N})$. Let $Z = J, K$. We have

$$[h_{\alpha, Z \otimes^m [2]}] \cdot [h_{\beta, Z \otimes^m [2]}] \text{ in } V_{Z \otimes^m [2]} = [h_{Z}] \cdot [h_{Z}] \text{ in } V_{Z}.$$ 

Note. Recall $Z \otimes^{2k} [2] = Z \otimes^k \text{Hopf}$.

By this lemma

$$[h_{\alpha, J \otimes^m [2]}] \cdot [h_{\beta, J \otimes^m [2]}] \text{ in } V_{J \otimes^m [2]} = [h_{\alpha, K \otimes^m [2]}] \cdot [h_{\beta, K \otimes^m [2]}] \text{ in } V_{K \otimes^m [2]}.$$ 

By this fact and Lemma 8.3 we can suppose that $\iota(S_{\alpha, J \otimes^m [2]} \square S_{\beta, J \otimes^m [2]})$ is $S_{\alpha, K \otimes^m [2]} \square S_{\beta, K \otimes^m [2]}$ if $m = 2k$ and $k \geq 1$. By this fact and Lemma 8.4 (5), $V_{J \otimes^m [2]}$ is diffeomorphic to $V_{K \otimes^m [2]}$ if $m = 2k$ and $k \geq 1$.

By [6, 24, 25] and the uniqueness of the tubular neighborhood, we have Lemma 9.2 and 9.3.

Lemma 9.2. Let $m \in \mathbb{N}$. Take the two copies of the $(2m+3)$-ball, and call them $B_{J \otimes^m [2]}$ and $B_{K \otimes^m [2]}$. Take the orientation preserving identity map $\iota : B_{J \otimes^m [2]} \to B_{K \otimes^m [2]}$. Take a $(2m+2)$-dimensional $(m+1)$-handle $h_*, Z \otimes^m [2] (\ast = \alpha, \beta. \ Z = J, K)$ with the following properties:

(1) $h_{\alpha, Z \otimes^m [2]}$ (resp. $h_{\beta, Z \otimes^m [2]}$) is embedded in $B_{Z \otimes^m [2]}$.

(2) The core of the attached part of $h_{\alpha, Z \otimes^m [2]}$ (resp. $h_{\beta, Z \otimes^m [2]}$) is embedded in $\partial B_{Z \otimes^m [2]}$ trivially. The handle is attached to $\partial B_{Z \otimes^m [2]}$ with the trivial framing.

(3) $\iota(\text{the attached part of } h_*, J \otimes^m [2]) = \text{the attached part of } h_*, K \otimes^m [2]$.
Then we have the following:

(i) By using an isotopy of $h_{a,K\otimes^m[2]}$ with keeping the boundary in $B_{K\otimes^m[2]}$, we can suppose that $\iota(h_{a,J\otimes^m[2]}) = h_{a,K\otimes^m[2]}$.

(ii) By using an isotopy of $h_{\beta,K\otimes^m[2]}$ with keeping the boundary in $B_{K\otimes^m[2]}$, we can suppose that $\iota(h_{\beta,J\otimes^m[2]}) = h_{\beta,K\otimes^m[2]}$.

Note. There is not an isotopy of $h_{a,K\otimes^m[2]} \sqcup h_{\beta,K\otimes^m[2]}$ with keeping the boundary in $B_{K\otimes^m[2]}$ in general such that $\iota(h_{a,J\otimes^m[2]} \sqcup h_{\beta,J\otimes^m[2]}) = h_{a,K\otimes^m[2]} \sqcup h_{\beta,K\otimes^m[2]}$.

**Lemma 9.3.** Under the assumption (1)(2)(3) of Lemma 9.2 suppose $\iota(h_{a,J\otimes^m[2]}) = h_{a,K\otimes^m[2]}$. Since $\iota$ (the attached part of $h_{a,J\otimes^m[2]}$) = the attached part of $h_{a,K\otimes^m[2]}$, $\iota$ (the core of $h_{\beta,J\otimes^m[2]}$) and the core of $h_{\beta,K\otimes^m[2]}$ are made into an $(m + 1)$-sphere in $B_{K\otimes^m[2]} - h_{a,K\otimes^m[2]}$, where we give appropriate orientations. Suppose that the $(m + 1)$-sphere is a vanishing cycle in $B_{K\otimes^m[2]} - h_{a,K\otimes^m[2]}$. Then we have

$$\iota(h_{a,J\otimes^m[2]} \sqcup h_{\beta,J\otimes^m[2]}) = h_{a,K\otimes^m[2]} \sqcup h_{\beta,K\otimes^m[2]}$$

by an isotopy of $h_{a,K\otimes^m[2]} \sqcup h_{\beta,K\otimes^m[2]}$ with keeping the boundary in $B_{K\otimes^m[2]}$.

By [7] 9 we have the following lemma.

**Lemma 9.4.** Let $m = 2k (k \in \mathbb{N})$. The $\mathbb{Z}$-Seifert pairing for $[h_{a,Z\otimes^m[2]}]$ and $[h_{\beta,Z\otimes^m[2]}]$ associated with the Seifert hypersurface $V_{Z\otimes^m[2]}^{2m+2}$ is equal to $(-1)^k \cdot (\text{that of } [h_Z] \text{ and } [h_Z])$ associated with $V_Z$.

By Lemma 8.4(4), 9.3 and 9.4 $V_{J\otimes^m[2]}$ and $V_{K\otimes^m[2]}$ differ by a single $(m + 1, m + 1)$-pass-move if $m = 2k$, $k \in \mathbb{N}$. Hence $J \otimes^k$ Hopf is obtained from $K \otimes^k$ Hopf by a single $(2k + 1, 2k + 1)$-pass-move ($k \in \mathbb{N}$). This completes the proof of Theorem 4.1.

Note. By [7] 9 we have Lemma 9.3 and 9.6.

**Lemma 9.5.** If $m = 2k + 1 (k \in \mathbb{N} \cup \{0\})$,

$$[h_{a,Z\otimes^m[2]}] \cdot [h_{\beta,Z\otimes^m[2]}] \in V_{Z\otimes^m[2]} \neq [h_{a,Z}] \cdot [h_{\beta,Z}] \in V_Z \quad (Z = J, K) \quad \text{in general.}$$

Note the difference between Lemma 9.1 and Lemma 9.5.

**Lemma 9.6.** $J \otimes^{2k+1} [2]$ and $K \otimes^{2k+1} [2]$ are not diffeomorphic or homeomorphic in general ($k \in \mathbb{N}$).

By this lemma $J \otimes^{2k+1} [2]$ and $K \otimes^{2k+1} [2]$ are not $(2k + 2, 2k + 2)$-pass-move equivalent in general ($k \in \mathbb{N}$) as we stated it in [10].
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