THE RATE OF GROWTH OF THE MINIMUM CLIQUE SIZE OF
GRAPHS OF GIVEN ORDER AND CHROMATIC NUMBER

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ABSTRACT. Let \( Q(n, c) \) denote the minimum clique number over graphs with \( n \) vertices and chromatic number \( c \). We determine the rate of growth of the sequence \( \{Q(n, \lfloor rn \rfloor)\}_{n=1}^{\infty} \) for any fixed \( 0 < r \leq 1 \). We also give a better upper bound for \( Q(n, \lfloor rn \rfloor) \).

1. Introduction

Let \( \omega(G) \), \( \alpha(G) \), and \( \chi(G) \) denote the clique number, independence number, and chromatic number, respectively, of a graph \( G \). We will also use \( |G| \) to denote the number of vertices and \( \|G\| \) to denote the number of edges of \( G \). Furthermore, let \( \omega(n, k) = \min \{ \omega(G) : |G| = n \text{ and } \alpha(G) \leq k \} \) the inverse Ramsey number. Define
\[
Q(n, c) = \min \{ \omega(G) : |G| = n \text{ and } \chi(G) = c \}.
\]
The goal of this research is to determine \( Q(n, c) \) as exactly as possible.

Biró, Füredi, and Jahanbekam [1] gave an exact formula for \( Q(n, c) \) for the case when \( c \geq (n+3)/2 \) in terms of inverse Ramsey numbers. They proved the following.

Theorem 1. For \( n \geq 2k + 3 \)
\[
Q(n, n-k) = n - 2k + q(k)
\]
where
\[
q(k) = \min \sum_{i=1}^{s} (\omega(2k_i + 1, 2) - 1)
\]
where the minimum is taken over positive integers \( k_1, \ldots, k_s \) with \( k_1 + \cdots + k_s = k \), and \( s \leq 3 \).

Liu [2] determined the rate of growth of \( Q(n, \lfloor n/k \rfloor) \) for \( k \) fixed positive integer, still, in terms of inverse Ramsey numbers. He proved that \( Q(n, \lfloor n/k \rfloor) = \Theta(\omega(n, k)) \) for \( k \) positive integer. The natural question (also specifically posed by Liu) remained to determine the rate of growth of the sequence in cases when \( k \) is not an integer. In this paper we provide the answer to this question proving the following theorem.

Theorem 2. Fix \( 0 < r \leq 1 \) and let \( k = \lfloor 1/r \rfloor \). Then there exists \( 0 < d_r \leq 1 \) such that for \( n \) large enough
\[
d_r \omega(n, k) \leq Q(n, \lfloor rn \rfloor) \leq \omega(n, k).
\]

We go beyond these bounds in Section 3: we provide a stronger upper bound for \( Q(n, \lfloor rn \rfloor) \). We hope that the improved bound is close to the actual value, in fact it is plausible to believe that it is asymptotically correct.
A related line of research was done in [2], in which the authors study the chromatic gap: \( \text{gap}(G) = \max\{\chi(G) - \omega(G) : |V(G)| = n\} \). The obvious relationship \( \text{gap}(n) = \max\{c - Q(n, c)\} \) makes our questions slightly more general.

## 2. Proof of the main theorem

In the following proof, we generalize some of Liu’s ideas to make it work for arbitrary (non-integer) positive real numbers, though at the end the proof is substantially different. Still, it is very interesting to note that the jumps in the rate of growth happens when \( r \) is a reciprocal of a positive integer.

We will need the following simple lemma.

### Lemma 3

For all \( 0 < r \leq 1 \), and \( n, k \) positive integers, if \( rn \geq k \), then

\[
\omega([rn], k) \geq \frac{1}{r} \omega(n, k).
\]

**Proof.** Observe that \( \omega \) is monotone and sub-additive in its first variable. Therefore

\[
\left\lfloor \frac{1}{r} \right\rfloor \omega([rn], k) \geq \omega \left( \left\lfloor \frac{1}{r} \right\rfloor [rn], k \right) \geq \omega(n, k).
\]

\( \square \)

Now we will prove that \( Q(n, [rn]) \leq \omega(n, k) \): we do this by exhibiting a graph with \( n \) vertices, chromatic number \( [rn] \), and clique number at most \( \omega(n, k) \). Let \( G \) be a Ramsey graph with \( |G| = n \), \( \alpha(G) = k \), and \( \omega(G) = \omega(n, k) \). Then

\[
\chi(G) \geq \left\lceil \frac{n}{k} \right\rceil = \left\lceil \frac{n}{[1/r]} \right\rceil \geq \left\lceil \frac{n}{1/r} \right\rceil = [rn].
\]

Drop edges from \( G \) until we get a subgraph \( G' \) with \( \chi(G') = [rn] \). Then \( |G'| = n \), and \( \omega(G') \leq \omega(n, k) \).

Now we will prove the existence of the constant \( d_r \).

Let \( G \) be a graph with \( |G| = n \) and \( \chi(G) = [rn] \). In the first step, we will show that there exists a constant \( c_r \) (that only depends on \( r \)), and an \( H \) subgraph of \( G \), such that \( |H| \geq c_r n \), and \( \alpha(H) \leq k \). We will construct \( H \) from \( G \) by removing independent sets of size \( k + 1 \), as many as possible. In other words, Let \( S \) be a largest collection of disjoint independent sets of size \( k + 1 \) in \( G \), and let \( H = G - S \).

From the maximality of \( S \), it is clear that \( \alpha(H) \leq k \). Let \( t = |S| \). Since \( \chi(G) \leq t + |H| \), and \( |H| = n - t(k + 1) \), we have

\[
[rn] = \chi(G) \leq t + n - t(k + 1) = n - tk.
\]

This implies \( tk \leq n - [rn] \leq n - rn = (1 - r)n \), so \( t \leq (1 - r)n/k \). It follows that

\[
|H| \geq n - \frac{(1 - r)n}{k}(k + 1) = \left( \frac{k + 1}{k} r - \frac{1}{k} \right) n
\]

Let \( c_r = \frac{k + 1}{k} r - \frac{1}{k} \). Recall that \( k = [1/r] \), so \( c_r \) is determined by \( r \), and \( r > 1/(k + 1) \); therefore also \( c_r > 0 \).

We established the existence of a subgraph \( H \) with \( |H| \geq c_r \) and \( \alpha(H) \leq k \).

Then for large \( n \),

\[
\omega(G) \geq \omega(H) \geq \omega([c_r n], k) \geq \frac{1}{c_r} \omega(n, k),
\]
where the last inequality follows from Lemma 3. Hence we may choose \( d_r = 1/[1/c_r] \).

Our constants \( d_r \) provide improvements on Liu’s constants in case \( r \) is the reciprocal of an integer. Indeed if \( r = 1/k \) for a \( k \) integer and \( k \to \infty \), Liu’s constants will exponentially converge to zero, while \( c_r = 1/k^2 = r^2 \).

It is also very interesting to note that as \( r \) approaches the reciprocal of integer from above, \( c_r \to 0 \). We tend to believe that this is just an artifact of the proof, but it would be very interesting to see this question settled one way or the other.

### 3. Better Upper Bound

In the previous section we only proved a weak bound for \( Q(n, [rn]) \), because that was all we needed to establish the rate of growth. But that bound is certainly not optimal. In the following, we show how to get better bounds in case \( r \) is not a reciprocal of an integer.

**Theorem 4.** Let \( 0 < r \leq 1 \) such that \( 1/r \) is not an integer. Let \( k = \lceil 1/r \rceil \), and let \( m = n - k\lfloor rn \rfloor \), \( l = (k+1)\lfloor rn \rfloor - n \). Let \( q(\beta, \alpha) = \min \sum \omega(\alpha \beta_i, \alpha) \) where the minimum is taken over sums \( \sum \beta_i = \beta \) with \( \beta_i > 0 \) integers. Then for large enough \( n \),

\[
Q(n, [rn]) \leq q(l, k) + q(m, k+1).
\]

Before the proof, let us comment on the requirement on \( 1/r \). Notice that for large \( n \), we have \( m > 0 \); in other words, for all \( r \) that is not a reciprocal of an integer there exists \( N \) such that \( n > N \) implies \( n - k\lfloor rn \rfloor > 0 \). Therefore, the quantity \( q(m, k+1) \) in the statement is well-defined. If we do not set the requirement on \( r \), the statement breaks down at reciprocals of integers due to some rounding problems. Note that, on the other hand, \( l > 0 \) is always true, because the rounding in that case works in our favor.

It may seem that the requirement on \( r \) takes away from the power of the theorem, but in fact if \( r \) is close to the reciprocal of an integer, \( m \) will get close to zero, and then the statement of the theorem is hardly an improvement on Theorem 2. In fact, it is expected that this approach would not prove any better bounds for exact reciprocals of integers.

The motivation of the theorem is that we do not believe that the jumps in the rate of growth proven in Theorem 2 show the whole picture. Between these jumps, far from reciprocals of integers, the upper bound can be improved, as it is demonstrated by the theorem.

**Proof of Theorem 4.** We will exhibit a graph on \( n \) vertices (for large \( n \)) with chromatic number \( \lceil rn \rceil \) and clique number at most \( q(l, k) + q(m, k+1) \). To do this, let \( t_1, \ldots, t_a \) be the numbers that minimize \( q(l, k) \), and let \( m_1, \ldots, m_b \) be the numbers that minimize \( q(m, k+1) \). Let \( L_1, \ldots, L_a \) be Ramsey graphs with \( |L_i| = kl_i \), \( \alpha(L_i) \leq k \), and \( \omega(L_i) = \omega(kl_i, k) \). Similarly, let \( M_1, \ldots, M_b \) Ramsey graphs with \( |M_i| = (k+1)m_i \), \( \alpha(M_i) \leq k+1 \), and \( \omega(M_i) = \omega((k+1)m_i, k+1) \). Now construct \( G \) by taking the disjoint union of \( L_1, \ldots, L_a, M_1, \ldots, M_b \), and add every edge between any two of these components. Then clearly, \( |G| = kl + (k+1)m = n \), and

\[
\chi(G) = \sum_{i=1}^{a} \chi(L_i) + \sum_{j=1}^{b} \chi(M_j) \geq \sum_{i=1}^{a} \frac{|L_i|}{k} + \sum_{j=1}^{b} \frac{|M_j|}{k+1} = l + m = \lfloor rn \rfloor.
\]
Furthermore

$$\omega(G) = \sum_{i=1}^{a} \omega(L_i) + \sum_{j=1}^{b} \omega(M_j) = q(l, k) + q(m, k + 1).$$

Now apply the usual trick of dropping edges until the chromatic number is down to $\lceil rn \rceil$ to get the example graphs. \hfill $\square$

**Corollary 5.** Let $0 < r \leq 1$ such that $1/r$ is not an integer, and $k, l, m$ defined as in Theorem 4. Then

$$Q(n, \lceil rn \rceil) \leq \omega(kl, k) + \omega((k + 1)m, k + 1).$$

**Proof.** By the definition of the function $q(\beta, \alpha)$ from Theorem 4 we have $q(\beta, \alpha) \leq \omega(\beta, \alpha)$, and the statement follows. \hfill $\square$

Note that $Q(n, \lceil rn \rceil) \leq \omega(kl, k) + \omega((k + 1)m, k)$ is a direct consequence of Theorem 2 and sub-additivity of $\omega$ in the first variable. But the corollary does provide actual improvements over Theorem 2 because the function $\omega(\cdot, \cdot)$ is monotone decreasing in the second variable.

The corollary may be weaker than Theorem 4, but it has the advantage that it expresses the upper bound as the sum of only two inverse Ramsey numbers, as opposed to a minimum over sums of inverse Ramsey numbers, like the theorem does.

### 4. Final notes

The bound provided by Theorem 4 is almost certainly not exact, because one can probably improve on it by just choosing sizes more carefully for the Ramsey graphs $L_i$ and $M_j$. But perhaps the more interesting problem that is left open is to establish an asymptotically correct formula for the sequence $Q(n, \lceil rn \rceil)$. As we mentioned above, we believe that the bounds in Section 3 have a good chance to be asymptotically correct, but proving it would probably require a good understanding of certain restricted clique packings of graphs.

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**References**

[1] Csaba Biró, Zoltán Füredi, and Sogol Jahanbekam, *Large chromatic number and Ramsey graphs*, Graphs Combin. **29** (2013), no. 5, 1183–1191.

[2] András Gyárfás, András Sebő, and Nicolas Trotignon, *The chromatic gap and its extremes*, J. Combin. Theory Ser. B **102** (2012), no. 5, 1155–1178.

[3] Gaku Liu, *Minimum clique number, chromatic number, and Ramsey numbers*, Electron. J. Combin. **19** (2012), Research Paper 55, 10 pp. (electronic).

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