Velocity-force characteristics of an interface driven through a periodic potential

A.M. Ettouhami and Leo Radzihovsky

1 Department of Physics, University of Colorado, Boulder, CO 80309
2 Department of Physics, Harvard University, Cambridge, MA 02138

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We study creep dynamics of a two-dimensional interface driven through a periodic potential using dynamic renormalization group methods. We find that the nature of weak-drive transport depends qualitatively on whether the temperature $T$ is above or below the equilibrium roughening transition temperature $T_c$. Above $T_c$, the velocity-force characteristics is Ohmic, with linear mobility exhibiting a jump discontinuity across the transition. For $T \leq T_c$, the transport is highly nonlinear, exhibiting an interesting crossover in temperature and weak external force $F$. For intermediate drive, $F > F_c$, we find near $T_c$ a power-law velocity-force characteristics $v(F) \sim F^\alpha$, with $\alpha = 1 \propto \delta$, and well-below $T_c$, $v(F) \sim e^{-\delta(F/F)^2}$, with $\delta = (1 - T/T_c)$. In the limit of vanishing drive ($F \ll F_c$) the velocity-force characteristics crosses over to $v(F) \sim e^{-\mu F}$, and is controlled by soliton nucleation.

I. INTRODUCTION

The problem of elastic media pinned by an external potential provides a unifying framework for understanding a large number of condensed matter phenomena, such as for example surface growth[30,44], non-linear transport[3,14] anisotropic metals[2,19], dissipation in superconductors[11,12,6] Wigner crystals[39], earthquakes[20,23], and friction[24]. Randomly-pinned elastic systems are also toy models for considerably more complicated problems of glasses[1,12,3]. Much of the recent interest in such problems has been rekindled by the discovery of high-temperature superconductors (HTSC) and efforts to understand the nature of their $B - T$ phase diagram and dissipation controlled by statics and dynamics of elastic arrays of vortex lines[12]. A combination of thermal fluctuations, pinning, and external drive leads to a wide range of new and interesting collective phenomena that is common to many physical realizations of elastic media.

There has been considerable progress in understanding the static properties of pinned elastic media, particularly in the case of random pinning. Among these many challenging questions, the velocity ($v$) – force ($F$) characteristics (the IV curve, in the context of superconductors and charge density waves (CDW’s)) is the observable that is most directly accessible experimentally[44,36] and is therefore of considerable theoretical interest.

Despite considerable richness of many aspects of the driven state at $T = 0$, at large drives the velocity-force characteristics of a uniformly sliding medium approaches Ohmic form with deviations $\delta v$ that can be computed perturbatively in the ratio $\delta v / v$. At zero temperature, if elastic, the medium is pinned for drives smaller than a critical $T = 0$ value $F_c$, and undergoes a nonequilibrium depinning transition to a sliding state, with $v \sim |F - F_c|^\beta$ playing the role analogous to an order parameter distinguishing pinned and sliding phases[12,13]. Finite temperature rounds the depinning transition[25,17], allowing activated creep motion of the elastic solid even for drive far below $F_c$.

Much of the insight about this highly nontrivial creep regime that is the focus of our work comes from a scaling theory of depinned droplet nucleation[21,36,37]. This approach generically predicts collectively pinned elastic media to exhibit a highly nonlinear $v(F)$, with a vanishing linear mobility, corresponding to transport activated over barriers that diverge with system size and vanishing drive. Recently, in the case of random pinning, these scaling predictions have been put on firmer ground through a detailed dynamic renormalization group (DFRG) calculations of $v(F)$, which indeed predict $v(F) \sim e^{-1/F^\beta}$, with a universal $\mu$ exponent. However, in the case of random pinning a number of technical problems with DFRG remain, precluding a fully controlled analysis.

It turns out, however, that many problems of interest, such as surface growth[30,44,19], 2D colloidal crystals in periodic potentials[6,5,26], and vortices pinned by artificial dot arrays[12,1,13] or by intrinsic pinning in e.g., HTSC, involve the considerably simpler but still nontrivial problem of periodic pinning. In addition to addressing numerous interesting physical problems, study of motion in a periodic potential provides a nice laboratory to explore new calculational methods.

A driven sine-Gordon model is the simplest description of such periodic pinning problem, with an important simplifying feature of absence of topological defects such as dislocations or phase slips that can be important for understanding the dynamics of vortex arrays and CDW’s[4]. Directly applicable to crystal growth phenomena, this model has been extensively studied in the literature[22,26]. In equilibrium, among many other things, it describes the famous crystal surface roughen-
The universal roughening crossover in temperature and applied force (Fig. 1). At an intermediate drive $F > F_*(g, T)$, larger than the pinning ($g$)- and temperature-dependent strong-coupling crossover force

$$F_*(g, T) \sim \begin{cases} \frac{e^{-b_1/g^2}}{g^{1/4}}, & T \rightarrow T_c^- \quad T \ll T_c \\ \frac{1}{g^{1/2}}, & T \rightarrow T_c^+ \quad T \ll T_c \end{cases} \quad (1.1)$$

the velocity-force characteristics strongly depends on the level of proximity to $T_c$, with:

$$v(F) \sim \begin{cases} F^{(1 + b_2 \tilde{T})}, & T \rightarrow T_c^- \quad T \ll T_c \\ e^{-\left(F_c/F\right)^{2i}}, & T \rightarrow T_c^+ \quad T \ll T_c \end{cases} \quad (1.2)$$

where $\tilde{T} = (1 - T/T_c)$, and $b_1$ and $b_2$ are nonuniversal constants of order unity. For sufficiently low drive $F < F_*(g, T)$, the motion is instead always via activated soliton creep, with the velocity-force characteristics crossing over to

$$v(F) \sim e^{-F_0/F}, \quad F < F_*, \quad (1.3)$$

with $F_0$ another characteristic force that will be defined below, Eq. (5.17).

This paper is organized as follows. We introduce the driven sine-Gordon model in Section II and analyze it in Section III using simple perturbation theory in the

![FIG. 1. Typical velocity-force characteristics of a driven interface in a periodic potential. At zero temperature, the interface remains pinned ($v = 0$) until $F$ reaches the critical force $F_c = pg$. At finite temperatures, we find that the near-equilibrium response of the interface to a small ($F \ll F_c$) driving force depends on whether $T$ is above or below the roughening temperature $T_c$. For $T > T_c$, the velocity-force characteristics is Ohmic ($v(F) \sim F$) down to $F = 0$, while for $T < T_c$ and forces smaller than a characteristic force $F_*$, the characteristics is strongly non-linear, $v \sim \exp \left(-F_0/F\right)$, creep motion via activation over barriers that diverge in a vanishing drive limit.](image)
pinning potential strength. While for weak pinning this computation is convergent for $T > T_c$, it fails for arbitrarily weak pinning in the smooth phase. In section [V] we employ dynamic RG techniques to make sense of these divergences, and in Section [VI] we use these results to compute $v(F)$ through the roughening transition. We conclude in Section [VII] with a summary of the results and a discussion of open problems and future directions.

II. DRIVEN SINE-GORDON MODEL

In equilibrium a two-dimensional sine-Gordon model of an elastic interface is described by a Hamiltonian

$$H = \int dr \left( \frac{1}{2} K (\nabla h)^2 - g \cos (ph(r)) \right),$$

(2.1)

where $r$ is a two dimensional vector in the $(xy)$ plane, $h(r)$ is the height of the interface above the $(xy)$ plane (taken to be along the $z$ direction in the embedding space) at location $r$, $K$ is the interfacial surface tension, $g$ is the pinning strength and $d = 2\pi / \lambda$ is the period of the potential. In the context of a crystalline surface, with $H$ characterizing its equilibrium roughness, the periodic pinning potential softly encodes lattice periodicity of the bulk crystal, corresponding to the $h \to h + d$ a symmetry of the surface energy, with $d$ being the crystal lattice constant perpendicular to the interface.

In the absence of any additional conservation laws, long scale equilibrium dynamics can be described by a simple, relaxational (model A) Langevin equation

$$\gamma \partial_t h = -\frac{\delta H}{\delta h(r, t)} + \zeta(r, t),$$

(2.2)

where $\gamma$ is the microscopic friction coefficient, and $\zeta(r, t)$ a zero-mean, Gaussian thermal noise describing the interaction of the system with the surrounding heat bath at temperature $T$, with

$$\langle \zeta(r,t)\zeta(r',t') \rangle = 2\gamma T \delta(r-r') \delta(t-t'),$$

(2.3)

in equilibrium imposed by the fluctuation-dissipation theorem (FDT), forbidding independent renormalization of $T$.

The dynamic description of an interface driven by an external force $F$ (in the context of crystal growth proportional to the difference between the chemical potentials of the solid and vapor phases) is substantially modified. In addition to the obvious addition of the driving force $F$ on the right-hand side of Eq. (2.2), nonequilibrium dynamics permits the appearance of nonconservative forces (those not expressible as derivatives of $H$), the most important of which is the famous Kardar-Parisi-Zhang (KPZ) $(\nabla h)^2$ nonlinearity, allowed by the explicit breaking by the drive of the $z \to -z$ symmetry. An additional important effect of driving appears as the renormalization of “temperature” $T$, corresponding to the breakdown of the fluctuation-dissipation theorem, that is, the renormalization of the friction coefficient $\gamma$ is independent of that of the variance of the noise $\xi$. Even if these nonequilibrium effects are not recognized a priori, they appear upon coarse-graining of equation (2.2), as soon as the external drive $F$ is included. The resulting nonequilibrium equation of motion is given by

$$\gamma \partial_t h = K \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 - p g \sin (ph(r, t)) + F + \zeta(r, t).$$

(2.4)

Our goal here is to apply the machinery of the dynamic RG to compute the velocity-force characteristics for the above model, focusing on the nontrivial creep regime of the smooth phase, where naive perturbative expansion in the pinning potential $g$ fails.

III. DYNAMIC PERTURBATION THEORY

It is instructive to first study the velocity-force characteristics through a simple perturbative expansion in the pinning potential $g$. Starting from equation (2.4), it is convenient to shift $h(r, t) = v_0 t + u(r, t)$ with $v_0 = F / \gamma$ the unperturbed ($g = \lambda = 0$) expression of the velocity. Averaging (2.4) over thermal fluctuations, and ignoring the KPZ term, we find that the velocity $v$ of the moving interface is given by

$$v = \langle \partial_t h \rangle,$$

(3.1a)

$$= \frac{F}{\gamma} - \frac{pg}{\gamma} \langle \sin (pu(r, t) + pv_0) \rangle,$$

(3.1b)

where we used the fact that $\langle \zeta(r, t) \rangle = 0$. We now let

$$u(r, t) = u_0(r, t) + u_g(r, t),$$

(3.2)

where

$$u_0(r, t) = \int dr' dt' R_0(r - r', t - t') \zeta(r', t')$$

(3.3)

is the thermal (noninteracting) part of the interface displacement,

$$u_g(r, t) = pg \int dr' dt' R_0(r - r', t - t') \times \sin \left( \frac{pF(t')}{\gamma} + pu_0(r', t') \right)$$

(3.4)

is the correction to $u$ linear in the pinning potential strength $g$, and $R_0(r - r', t - t') = \delta(u_0(r, t), t') / \delta F(r', t')$ is the response function of the free interface. Expanding equation (3.1b) in $u_g$, and averaging over the thermal noise $\zeta$, we find (see also Appendix [VII B]).

3
\[ v = F - \frac{p^2 g^2}{2\gamma} \int dr\, dt' e^{-\frac{1}{2} p^2 C_0(r-r', t-t')} \times \]
\[ \times \sin \left[ \frac{pF}{\gamma} (t-t') \right] R_0(r-r', t-t'), \quad (3.5) \]

where \( C_0(r-r', t-t') = \langle [u_0(r,t) - u_0(r', t')]^2 \rangle \) is the connected correlation function of a free interface given by

\[ C_0(r, t) \approx \frac{T}{2\pi K} \ln \left[ 1 + \Lambda^2 (r^2 + \frac{Kt}{\gamma}) \right]. \quad (3.6) \]

The above velocity-force characteristics, equation\(^\dagger\), is most easily evaluated at zero temperature where \( C_0(r, t) = 0 \). In this limit, using

\[ R_0(r, t) = \frac{\theta(t)}{\gamma} \int_q e^{-\frac{\Lambda^2 q^2}{2}} e^{iq\cdot r}, \quad (3.7) \]

\((\theta(t)\) is Heaviside’s unit step function\) inside Eq.\((3.5)\) and integrating over the time variable \(t'\), we obtain

\[ v = F - \frac{p^2 g^2}{2\gamma} \int dr \int_q e^{-\frac{\Lambda^2 q^2}{2}} e^{iq\cdot r}. \quad (3.8) \]

In the above equations and throughout the rest of this paper we use a shorthand notation \( \int_q \) for \( \int dq/(2\pi)^2 \), and the superscript \( \Lambda = 2\pi/a \) is the ultra-violet cutoff set by the in-plane lattice constant \( a \), generically distinct from the period \( d = 2\pi/p \) perpendicular to the interface. Performing the integration over the space variable \( r \) in the last equation, and using the resulting Dirac delta-function \((2\pi)^2\delta(q)\) to complete the \( q \) integral, we find a \( T = 0 \), leading order (in pinning \( g \)) expression for the \( v(F) \) characteristic.

\[ v = F \left( 1 - \frac{1}{2} \left( \frac{F_c}{F} \right)^2 \right), \quad F \gg F_c, \quad (3.9) \]

where \( F_c = pg \) is the zero-temperature critical force, in agreement with the condition \( F_c = \max \partial V(h)/\partial h \) of disappearance of metastability \( (V(h) = -g \cos(ph) \) is the pinning potential\). As is clear from this result for \( v(F) \), even at \( T = 0 \), the perturbative corrections are small for sufficiently large applied force \( F \) relative to the pinning force \( F_c \) (equivalently, for sufficiently weak pinning \( g \) at fixed \( F \)). In this fast moving regime, the metastability is absent and pinning gives only a small correction to the motion with \( v(F) \) deviating only weakly from the pinning-free Ohmic response \( v_0(F) = F/\gamma \). It is reassuring to note that, since at \( T = 0 \), only the \( q = 0 \) mode contributes to the \( v(F) \). Eq.\((3.5)\) agrees with the high-drive limit of the exact \( T = 0 \) result\(^2\) for a single particle driven through a one-dimensional sinusoidal potential

\[ v(F) = F \sqrt{1 - \frac{\left( F_c/F \right)^2}{2}}, \quad F > F_c. \quad (3.10) \]

This suggests that the \( v(F) \) characteristics of a driven interface should also exhibit a square-root cusp with an infinite slope at \( F = F_c \). At \( T = 0 \) the interface is strictly pinned for \( F \leq F_c \).

In contrast, at any finite temperature the interface moves for arbitrarily weak force and hence there is no sharp depinning transition. The perturbative expression for \( v \), Eq.\((3.5)\) can be readily evaluated by using the fluctuation-dissipation relation

\[ \theta(t) \partial_t C_0(r, t) = 2TR_0(r, t) \quad (3.11) \]

obeyed by the equilibrium response and correlation functions. Using this relation to eliminate \( R_0(r, t) \) from the rhs of Eq.\((3.5)\) and integrating by parts over \( t' \) we find

\[ v = F \left[ 1 - \frac{p^2 g^2}{2\gamma T} \int dt \cos(pFT/\gamma) e^{-\frac{1}{2} p^2 C_0(r,t)} \right]. \quad (3.12) \]

Inserting into this last equation the expression of the correlation function \( C_0(r, t) \) of a harmonic interface given in Eq.\((3.10)\) leads to

\[ v = F \left( 1 - \frac{p^2 g^2}{2\gamma T} \int dt \frac{\cos(pFT/\gamma)}{\left[ 1 + \Lambda^2 (r^2 + Kt/\gamma) \right]} \right), \quad (3.13) \]

where we defined

\[ \eta = \frac{T p^2}{4\pi K}. \quad (3.14) \]

Taking the limit \( F \to 0 \) in the above expression, and performing the time integration, we obtain

\[ \lim_{F \to 0} v(F) = \frac{1}{\gamma} \left( 1 - \frac{\pi p^2 g^2}{KTA} \int \frac{dr}{\left[ 1 + \Lambda^2 (r^2 + Kt/\gamma) \right]^{2 (\eta - 3)/2}} \right). \quad (3.15) \]

We now observe that the integral on the rhs of equation\((3.15)\) behaves very differently depending on whether \( T \) is smaller or greater than

\[ T_c \equiv \frac{8\pi K}{p^2}. \quad (3.16) \]

For \( T > T_c \), i.e., \( \eta > 2 \), the integral in Eq.\((3.15)\) is convergent, and leads to a finite (and for weak pinning \( g \), to an arbitrarily small) correction to the linear friction coefficient \( \gamma(F = 0) = 1/\lim_{F \to 0} v(F) \). In strong contrast, for \( T < T_c \) \((\eta < 2)\) above integral diverges signalling the breakdown of the perturbation theory at small values of the external force \( F \).

Having established the breakdown of perturbation theory for \( T < T_c \) in the limit of vanishingly small forces, we now turn our attention to the full velocity-force characteristics at finite values of the external drive. Starting
from equation (3.13), and performing the integration over space variables, we obtain
\[ \gamma \frac{F}{\gamma} \left( 1 - \frac{p^4 g^2}{8 K^2 A^4} \eta^{-1} \right) \int_0^\infty d\tau \cos(2f\tau) (\tau + 1)^{\eta - 1} \] (3.17)
where the dimensionless force \( f \) is given by (henceforth, we shall use both \( F \) and \( f \) to designate the driving force on our interface)
\[ \gamma(f) = \gamma \left[ 1 - \frac{p^4 g^2}{8 K^2 A^4} \eta^{-1} \right] (2f)^{\eta - 2} \Gamma(2 - \eta) \sin \left( 2f + \frac{\pi}{2}(\eta - 1) \right) + \frac{1}{(\eta - 2)} f_2 \left( \frac{4 - \eta}{2}, \frac{3 - \eta}{2}, -f^2 \right) \right]^{-1}, \]
which has the following limiting behavior as \( f \to 0 \),
\[ \gamma(f \to 0) = \gamma \left[ 1 + \frac{p^4 g^2}{8 K^2 A^4} \right] \left( 1 - (2 - \eta) \Gamma(2 - \eta) (2f)^{\eta - 2} \sin \left( \frac{\pi}{2}(\eta - 1) \right) \right]^{-1}. \] (3.20)

As found above, inside the rough phase, \( T > T_c^0 \) (\( \eta > 2 \)) and for sufficiently weak pinning, the perturbation theory remains valid at arbitrary \( f \), simply displaying crossover from a freely moving interface with “bare” mobility \( \mu_\infty = 1/\gamma \) at high drives to that with \( \text{finitely suppressed} \) low-drive mobility (as illustrated in figure 2):
\[ \gamma(f \to 0) \approx \gamma \left( 1 + \frac{p^4 g^2}{8 \eta K^2 A^4} \right) = \gamma \left( 1 + \frac{\pi p^4 g^2}{2TKA^4} \right). \] (3.21)

On the other hand, in agreement with Ref. 3, we find that in the “smooth”, low temperature \( T < T_c^0 \) (\( \eta < 2 \)) phase, the behavior is strikingly different with the correction to \( \nu_0(F) = F/\gamma \), Eq. (3.20), diverging and the perturbative approach failing as \( f \) is reduced below a characteristic force
\[ f_*(g, T) \approx \frac{1}{2} \left[ \frac{p^4 g^2 \eta \Gamma(2 - \eta) \sin \left( \frac{\pi}{2}(\eta - 1) \right)}{1 + \frac{p^4 g^2}{8 K^2 A^4} (2 - \eta)(\eta - 1)} \right]^{1/(2 - \eta)}. \] (3.22)

As \( T \to T_c^0 \) (\( \eta \to 2^- \)),
\[ f_*(T_c^0) \approx \frac{1}{2} \exp \left( -\frac{4K^2 A^4}{p^4 g^2} \right), \]
showing that the regime of forces \( 0 < f < f_* \) where perturbation theory fails becomes exponentially small as \( T_c^0 \) is approached from below. The unbounded growth of the perturbative friction coefficient as the external drive \( f \) approaches \( f_* \) from above (see Fig. 3) suggests that the interface in the low temperature, smooth phase is characterized by a vanishing linear mobility. 4

Although we will study this in more detail in following sections, already at this stage we can see a physical interpretation of this divergence. Perturbation theory in the pinning potential fails because even for an arbitrarily weak pinning \( g \), on sufficiently long scales greater than \( \xi \) (computed in Section IV), the periodic potential (for small \( h \) acting like a “mass”, \( \frac{1}{2} kg^2 \dot{h}^2 \)) necessarily dominates over the elastic energy density \( \frac{1}{2} \kappa (\nabla h)^2 \). Since (as is quite clear from the equation of motion, Eq. (2.4)) the applied force \( F \) dominates the elastic force on scales longer than
\[ \xi_F = \left( \frac{2\pi K}{pF} \right)^{1/2}, \]

(3.24)
a sufficiently weak force, \( F < F_\ast \), probes the interface on length scales longer than \( \xi \) and thereby leads to the breakdown of perturbation theory about the harmonic interface. Hence, although quite instructive, the perturbation theory fails to make predictions for \( v(F) \) or any other dynamic quantity in the smooth phase at sufficiently low drive \( F < F_\ast \), and a nonperturbative approach is necessary.

### IV. Dynamic Renormalization Group

Armed with the above discussion, we are now well-equipped to use dynamic RG analysis to make physical sense of these perturbative divergences, with the main goal being the calculation of \( v(F) \) in the smooth \( T < T_c \) phase for weak drive \( F < F_\ast \). It is convenient to perform this analysis in the frame co-moving with the “bare” velocity \( v_\circ = F/\gamma \) corresponding to the change of the dynamic fields to \( u(r, t) \equiv h(r, t) - Ft/\gamma \), which obeys

\[ \gamma \partial_t u = K \nabla^2 u + \frac{\lambda}{2} \langle \nabla u \rangle^2 + pg \sin(pu + \frac{pFt}{\gamma}) + \zeta(r, t). \]

(4.1)

Taking the nonlinear terms in the above equation as a small perturbation, the equation of motion can be directly expanded in these nonlinearities, leading to renormalization group recursion relations for model parameters. An equivalent but more convenient formulation is the field-theoretic approach of Martin, Siggia and Rose (MSR). In this approach, the dynamic correlation and response functions

\[ C(r, t) = \langle u(r, t)u(0, 0) \rangle, \]
\[ = \int [du][d\bar{u}] \, u(r, t)u(0, 0)e^{-S[u, \bar{u}]}, \]

(4.2a)

\[ R(r, t) = \langle \bar{u}(r, t)u(0, 0) \rangle, \]
\[ = \int [du][d\bar{u}] \, \bar{u}(r, t)u(0, 0)e^{-S[u, \bar{u}]}, \]

(4.2b)

are computed directly by integrating over the phonon and response fields \( u \) and \( \bar{u} \), treated as independent stochastic fields with a statistical weight \( e^{-S[u, \bar{u}]} \) imposed by the equation of motion, after integrating over the thermal noise \( \zeta(r, t) \). The resulting effective “action” \( S \) is given by \( S = S_0 + S_1 \), where

\[ S_0[u, \bar{u}] = \int dr dt \left\{ \frac{1}{2} (2\gamma T)^2 \bar{u}^2(r, t) + i\bar{u}(r, t) \left[ \gamma \partial_t u - K \nabla^2 u \right] \right\} \]

(4.3)
is the action of a pinning-free (harmonic) interface, and where \( S_1 = S_g + S_\lambda \), with

\[ S_g[u, \bar{u}] = pg \int dr dt \, i\bar{u}(r, t) \sin \left( pu(r, t) + \frac{pF}{\gamma} t \right) \]

(4.4)
the contribution of the pinning potential and

\[ S_\lambda[u, \bar{u}] = -\frac{\lambda}{2} \int dr dt \, i\bar{u}(r, t) \left( \nabla u \right)^2 \]

(4.5)
the contribution of the KPZ term to the nonlinearities in \( S \). To study the renormalization of \( S[u, \bar{u}] \) it is sufficient to work with the dynamic “partition function”

\[ Z = \int [du][d\bar{u}] \, e^{-S[u, \bar{u}]} \]

(4.6)
required to remain fixed at unity under an RG coarse-graining procedure. The advantage of the MSR formalism is its close resemblance to the equilibrium statistical mechanics, that makes it a rather straightforward task to apply RG transformations and to derive recursion relations for the various parameters entering the equation of motion \( \lambda(r) \). Like the static momentum-shell RG, the dynamic RG procedure consists of three main steps:

(i) Thinning of the degrees of freedom, whereby modes \( u(q) \), with \( q \) in an infinitesimal shell \( \Lambda < q < \Lambda (b = e^{\phi}) \) are perturbatively (in \( S_1 \)) integrated out.

(ii) Rescaling of space variables according to \( r = br' \), so as to restore (for convenience) the ultraviolet cutoff to its original value \( \Lambda \), and rescaling time variable according to \( t = t'b^2 \).

(iii) Rescaling of fields, in order (for convenience) to keep the harmonic part of the action invariant under rescaling in (ii).

We define “slow” \( \{ u^-, \bar{u}^- \} \) and “fast” fields \( \{ u^+, \bar{u}^+ \} \)

\[ u(q, t) = u^-(q, t) + u^+(q, t), \]

(4.7)

\[ u(q, t) = u^-(q, t) + \bar{u}^+(q, t), \]

(4.8)
with momentum support in Fourier space in the intervals \( 0 < q < \Lambda/b < \Lambda \) \((b = e^{\phi}) \) and \( S^+, S^- \)

\[ \langle S^+ \rangle_{\circ} \equiv \int [du][d\bar{u}] \, e^{-S_{\circ} + \bar{u}^-} \langle e^{-S_\circ} \rangle_{\circ}, \]

(4.9)
where \( \langle \cdots \rangle_{\circ} \) denotes an average taken with the statistical weight \( S_{\circ} \), and where the superscript \( c \) in \( \langle S^c \rangle_{\circ} \) denotes a connected average. To first order in the pinning strength \( g \), there is only one term in \( \langle S_g \rangle_{\circ} \), which renormalizes the dynamic action, namely

\[ \langle S_g \rangle_{\circ} = pgb^{-T_F^{\ast}/4\pi K} \times \int dr dt \, i\bar{u}^-(r, t) \sin \left( pu^-(r, t) + \frac{pF}{\gamma} t \right), \]

(4.10)
which physically arises from the suppression (from \(g\) to \(g^2 T p^2/4\pi K\)) of the effective pinning strength due to short-scale thermal fluctuations averaging away the periodic potential. In the above and throughout we will use \(\equiv\) to indicate that only the leading term has been kept. Similarly, to first order in the KPZ coupling \(\lambda\), we have the following perturbative correction to the dynamic action \(S\),

\[
(S\lambda_0)^{<} \equiv -\int d\tau dt \, \langle \dot{u}(\mathbf{r}, \tau) \rangle \left[ \frac{\lambda T \Lambda^2}{4\pi K} \right] ,
\]

which quite clearly renormalizes the effective external force.

Rescaling the space and time variables

\[
\mathbf{r} = b \mathbf{r'}, \quad t = b^z t',
\]

as well as the conjugate field \(\tilde{u}(\mathbf{r}, t)\)

\[
\tilde{u}^{<}(\mathbf{r}, t) = b^\chi \tilde{u}(\mathbf{r}', t') ,
\]

while for convenience leaving \(u(\mathbf{r}, t)\) unchanged in order to preserve the periodicity \((2\pi/p)\) of the original problem: we obtain the following lowest-order recursion relations:

\[
\begin{align*}
(\gamma T)(b) &= 6^{2+z+2\chi} (\gamma T) , \\
\gamma(b) &= 6^{2+z+2\chi} \gamma , \\
K(b) &= 6^{2+z+2\chi} K , \\
g(b) &= 6^{2+z+2\chi - T p^2/4\pi K} g , \\
(F/\gamma)(b) &= 6^{2+z+2\chi} (F/\gamma) ,
\end{align*}
\]

The dynamic exponents \(z\) and \(\chi\) can be fixed by requiring that \(K\) and \(\gamma\) be unchanged, to linear order in \(g\), under the RG transformation. This leads to the following values

\[z = 2, \quad \chi = -2\]

and to the following recursion relations for \(g\) and \(F\):

\[
\begin{align*}
\frac{dg}{d\ell} &= (2 - T p^2/4\pi K) g , \\
\frac{dF}{d\ell} &= 2F + \frac{\lambda T \Lambda^2}{4\pi K} ,
\end{align*}
\]

while the remaining quantities, \(K, \gamma, \lambda,\) and temperature \(T\), remain unchanged and suffer no renormalization to first order in \(g\) and \(\lambda\). Similar considerations, with details given in Appendix B, lead to the following recursion relations to second order in \(g\) and \(\lambda\):

\[
\frac{d}{d\ell} (\gamma T) = -\left( \frac{\Lambda^2}{8\pi K^3} + \frac{T p^6 g^2}{16\pi K^3 \Lambda^2} \frac{1}{1 + f^2} \right) (\gamma T) ,
\]

\[
\frac{dg}{d\ell} = \frac{(2 - T p^2)}{4\pi K} \gamma ,
\]

\[
\frac{dK}{d\ell} = \frac{T p^6 g^2}{16\pi K^3 \Lambda^2} \left( \frac{1}{1 + f^2} \right) ,
\]

\[
\frac{dF}{d\ell} = \frac{F}{8\pi K^2 \Lambda^2} - \frac{T p^6 g^2}{8\pi K^2 \Lambda^2} f + f^2 ,
\]

where \(f = (pF/2\Lambda^2)\) is the dimensionless force of equation \((4.13)\). Note that, because of the lack of a FDT for the driven system, in strong contrast to the equilibrium case (\(\lambda = F = 0\)) Eqs.\((4.16)\) and \((4.16)\) imply that \(T(\ell)\) flows nontrivially according to

\[
\frac{dT}{d\ell} = \frac{T \Lambda^2}{8\pi K^3} + \frac{T p^6 g^2}{8\pi K^3 \Lambda^2} \left( \frac{f}{1 + f^2} \right) T .
\]

Hence, \(T(\ell)\) is simply a measure of the strength of the white-noise component of the random force on the driven interface and is not associated with any equilibrium bath at a well-defined thermodynamic temperature.

The recursion relations \((4.16a)-(4.16l)\) contain most (but not all, as discussed in Section V) of the information we need to investigate the properties of the system beyond the failing perturbative expansion of Section II. Before turning to their full analysis and to the study of the velocity-force characteristics, it is useful to see how the previously derived static and equilibrium dynamic results \([2, 11]\) are recovered. We do this in the following subsections.

**A. Analysis of the static limit**

The static model, Eq. \((2.1)\), is characterized by two parameters \(K\) and \(g\) with the RG recursion relations reducing to the familiar Kosterlitz-Thouless form (derived by these last authors in a dual, Coulomb gas form \([2]\))

\[
\begin{align*}
\frac{dg}{d\ell} &= (2 - T p^2) g , \\
\frac{dK}{d\ell} &= \frac{T p^6 g^2}{8\pi K^2 \Lambda^2} .
\end{align*}
\]

At small \(g\), \(K(\ell)\) flows slowly, and the recursion relation for \(g\) implies the existence a phase transition (called “roughening” in the context of crystal surface \([14]\) at \(T_{\infty} = 8\pi K/\rho^2\)) in the limit \(g \to 0\) between two phases distinguished by the long scale (\(\ell \to \infty\)) behavior of \(q(\ell)\). For \(T > T_{\infty}\) thermal fluctuations are strong enough to effectively average away the long-length scale effects of the periodic pinning potential, which is therefore qualitatively unimportant for most (but not all) physical properties.
of this so-called “rough” phase. At these high temperatures the surface is logarithmically rough and the effects of a weak periodic potential can be taken into account in a controlled perturbative expansion. In strong contrast, for \( T < T_c \), the effective strength of the periodic potential relative to that of the harmonic elastic energy grows on long length scales, leading to a breakdown of perturbation theory in \( g \), no matter how weak its bare value might be. As a result, at long scales, the interface is pinned in this “smooth” phase, with bounded \( \text{rms} \) height fluctuations.

It is instructive to recall some of the physics which follows from the above recursion relations. It is convenient to first rewrite the flow equations for dimensionless couplings \( \tilde{g} \) and \( \eta \)

\[
\tilde{g} = \frac{\sqrt{2} p^2 g}{K \Lambda^2}, \quad (4.19a)
\]

\[
\eta = \frac{Tp^2}{4\pi K}, \quad (4.19b)
\]

that satisfy

\[
\frac{d\tilde{g}}{d\ell} = (2 - \eta) \tilde{g}, \quad (4.20)
\]

\[
\frac{d\eta}{d\ell} = -\frac{1}{4} \eta^2 \tilde{g}^2. \quad (4.21)
\]

These show that in equilibrium, the quantity \( \eta \) which is the measure of the ratio of thermal \( (T) \) to elastic \( (K) \) energy, always flows to zero at long scales, indicating that the low-temperature smooth phase is controlled by a strong coupling zero-temperature fixed point. Near \( T_c \), it is convenient to use a reduced temperature measured relative to the (noninteracting) \( T_{c0} = 8\pi K/p^2 \),

\[
\tilde{\tau} \equiv \eta - 2 = 2(T/T_{c0} - 1), \quad (4.22a)
\]

\[
\tilde{T} = T_{c0} \left( 1 + \frac{p^2 g}{\sqrt{2} K \Lambda^2} \right), \quad (4.26)
\]

with the flow equations simplifying to

\[
\frac{d\tilde{g}}{d\ell} = -\tilde{\tau} \tilde{g}, \quad (4.23a)
\]

\[
\frac{d\tilde{T}}{d\ell} = -\tilde{g}^2. \quad (4.23b)
\]

These can be easily integrated by multiplying Eq. (4.23a) and (4.23b) by \( \tilde{g} \) and \( \tilde{\tau} \), respectively, and taking the difference of the two resulting equations. The result is that near \( T_{c0} \) the flows are a family of hyperbolae

\[
\tilde{g}^2 - \tilde{\tau}^2 = c, \quad (4.24)
\]

labeled by a constant of integration

\[
c = \left( \frac{\sqrt{2} p^2 g}{K \Lambda^2} \right)^2 - \left( \frac{Tp^2}{4\pi K} - 2 \right)^2, \quad (4.25)
\]
determined by the bare value of model parameters \( g \) and \( K \). The resulting flows are illustrated in Fig. 4, showing three distinct regions of behavior. In the high temperature region below the thick line \( (c < 0) \), pinning is irrelevant, and it therefore describes the rough phase, separated from the low-temperature smooth phase (the region above the thick line) by a critical line separatrix \( \tilde{\tau} = \tilde{g} \). The latter therefore defines a true critical temperature given by

\[
T_c = T_{c0} \left( 1 + \frac{p^2 g}{\sqrt{2} K \Lambda^2} \right), \quad (4.26)
\]

distinct from its \( g \to 0 \) limit of \( T_{c0} = 8\pi K/p^2 \). Changing \( T \) corresponds to the variation of the dimensionless bare parameters along the dashed horizontal line indicated in Fig. 4. Above \( T_c \), \( \tilde{g}(\ell) \) flows to zero and

\[
\tilde{\tau}_R \equiv \tilde{\tau}(\ell \to \infty), \quad \tilde{\tau}_R = \sqrt{|c|}, \quad (4.27a)
\]

\[
K_R \equiv K(\ell \to \infty), \quad (4.28a)
\]

corresponding to the long-scale renormalized elastic constant

\[
K_R \equiv K(\ell \to \infty), \quad (4.28a)
\]

\[
= K \frac{T}{T_{c0}} \left( 1 + \sqrt{|c|}/2 \right)^{-1}. \quad (4.28b)
\]

It is comforting to find (using Eq. (4.25)) that \( K_R \) reduces to its bare value \( K \) at high temperatures. Using the fact that near, but above \( T_c \),

\[
FIG. 4. Renormalization-group flow in the \((\tilde{\tau}, \tilde{g})\) plane. Temperature variation for an actual system occurs along the dashed line. On the high temperature side of the separatrix \( \tilde{\tau} = \tilde{g} \) (indicated as the thick line), the periodic pinning \( \tilde{g} \) renormalizes to zero and the interface is rough on long length scales. Below \( T_c \) (to the left of the critical separatrix) the RG flow runs off to strong coupling \( \tilde{g} \) describing an interface that is smooth on long length scales.

\[
\tilde{g}^2 - \tilde{\tau}^2 = c, \quad (4.24)
\]

labeled by a constant of integration

\[
c = \left( \frac{\sqrt{2} p^2 g}{K \Lambda^2} \right)^2 - \left( \frac{Tp^2}{4\pi K} - 2 \right)^2, \quad (4.25)
\]
determined by the bare value of model parameters \( g \) and \( K \). The resulting flows are illustrated in Fig. 4, showing three distinct regions of behavior. In the high temperature region below the thick line \( (c < 0) \), pinning is irrelevant, and it therefore describes the rough phase, separated from the low-temperature smooth phase (the region above the thick line) by a critical line separatrix \( \tilde{\tau} = \tilde{g} \). The latter therefore defines a true critical temperature given by

\[
T_c = T_{c0} \left( 1 + \frac{p^2 g}{\sqrt{2} K \Lambda^2} \right), \quad (4.26)
\]

distinct from its \( g \to 0 \) limit of \( T_{c0} = 8\pi K/p^2 \). Changing \( T \) corresponds to the variation of the dimensionless bare parameters along the dashed horizontal line indicated in Fig. 4. Above \( T_c \), \( \tilde{g}(\ell) \) flows to zero and

\[
\tilde{\tau}_R \equiv \tilde{\tau}(\ell \to \infty), \quad (4.27a)
\]

\[
\tilde{\tau}_R = \sqrt{|c|}, \quad (4.27b)
\]

corresponding to the long-scale renormalized elastic constant

\[
K_R \equiv K(\ell \to \infty), \quad (4.28a)
\]

\[
= K \frac{T}{T_{c0}} \left( 1 + \sqrt{|c|}/2 \right)^{-1}. \quad (4.28b)
\]

It is comforting to find (using Eq. (4.25)) that \( K_R \) reduces to its bare value \( K \) at high temperatures. Using the fact that near, but above \( T_c \),

\[
FIG. 4. Renormalization-group flow in the \((\tilde{\tau}, \tilde{g})\) plane. Temperature variation for an actual system occurs along the dashed line. On the high temperature side of the separatrix \( \tilde{\tau} = \tilde{g} \) (indicated as the thick line), the periodic pinning \( \tilde{g} \) renormalizes to zero and the interface is rough on long length scales. Below \( T_c \) (to the left of the critical separatrix) the RG flow runs off to strong coupling \( \tilde{g} \) describing an interface that is smooth on long length scales.

\[
\tilde{g}^2 - \tilde{\tau}^2 = c, \quad (4.24)
\]

labeled by a constant of integration

\[
c = \left( \frac{\sqrt{2} p^2 g}{K \Lambda^2} \right)^2 - \left( \frac{Tp^2}{4\pi K} - 2 \right)^2, \quad (4.25)
\]
determined by the bare value of model parameters \( g \) and \( K \). The resulting flows are illustrated in Fig. 4, showing three distinct regions of behavior. In the high temperature region below the thick line \( (c < 0) \), pinning is irrelevant, and it therefore describes the rough phase, separated from the low-temperature smooth phase (the region above the thick line) by a critical line separatrix \( \tilde{\tau} = \tilde{g} \). The latter therefore defines a true critical temperature given by

\[
T_c = T_{c0} \left( 1 + \frac{p^2 g}{\sqrt{2} K \Lambda^2} \right), \quad (4.26)
\]

distinct from its \( g \to 0 \) limit of \( T_{c0} = 8\pi K/p^2 \). Changing \( T \) corresponds to the variation of the dimensionless bare parameters along the dashed horizontal line indicated in Fig. 4. Above \( T_c \), \( \tilde{g}(\ell) \) flows to zero and

\[
\tilde{\tau}_R \equiv \tilde{\tau}(\ell \to \infty), \quad (4.27a)
\]

\[
\tilde{\tau}_R = \sqrt{|c|}, \quad (4.27b)
\]

corresponding to the long-scale renormalized elastic constant

\[
K_R \equiv K(\ell \to \infty), \quad (4.28a)
\]

\[
= K \frac{T}{T_{c0}} \left( 1 + \sqrt{|c|}/2 \right)^{-1}. \quad (4.28b)
\]

It is comforting to find (using Eq. (4.25)) that \( K_R \) reduces to its bare value \( K \) at high temperatures. Using the fact that near, but above \( T_c \),
it is easy to show that the RG “time” \( \ell_* \) to reach strong-coupling is given by

\[
\ell_* \approx \frac{2}{\sqrt{\tau}}, \quad (4.34a)
\]
\[
\approx \frac{2}{\sqrt{2\tau_c \tau}}. \quad (4.34b)
\]

Consequently the correlation length in this critical region is of familiar KT form

\[
\xi_c \approx a e^\xi, \quad (4.35a)
\]
\[
\approx a e^{\alpha/[1 - T/T_c]^{1/2}}, \quad (4.35b)
\]

diverging extremely fast as \( T \to T_c^- \), with \( \alpha = \sqrt{2/\tilde{g}} = (T_c/T_{c0} - 1)^{-1/2} \) a nonuniversal constant.

Outside this critical region, defined by \( \tau < -1 \), deep in the smooth phase, the flows are qualitatively different. At weak coupling \( g p^2 \ll K A^2 \) (the only regime where the perturbative RG analysis is valid) because \( \tilde{\tau}(\ell) \) grows weakly (additively)

\[
\tilde{g}(\ell) \approx \tilde{g} e^{(2-n)\ell}, \quad (4.36)
\]
grows exponentially fast, reaching strong-coupling at the low-T correlation length \( \xi_g \approx a e^{\xi} \) given by

\[
\xi_g \approx \xi_0 (\xi_0 \Lambda)^{(2-n)/2}, \quad (4.37a)
\]
\[
\approx \Lambda^{-1} (\xi_0 \Lambda)^{2/(2-n)}, \quad (4.37b)
\]
\[
\approx \Lambda^{-1} \left( K \Lambda / (p^2 g) \right)^{1/(2-n)}. \quad (4.37c)
\]

On scales longer than the roughness correlation length the interface is smooth and is characterized by a strongly downward renormalized value of the pinning strength \( g_R \) determined by the value of unrescaled coupling \( g(\ell) = log(\xi_0 \Lambda) \) at the scale of the correlation length. Near the transition

\[
g_R \approx g(\Lambda \xi)^{-2} \ll g, \quad T \to T_c^-, \quad (4.38a)
\]
\[
\approx g e^{-2\alpha/[1 - T/T_c]^{1/2}}. \quad (4.38b)
\]

Deep in the smooth phase, for weak pinning, we instead find

\[
g_R \approx g(\Lambda \xi)^{-n} \ll g, \quad T \ll T_c, \quad (4.39a)
\]
\[
\sim g^{2/(2-n)}, \quad (4.39b)
\]

which for weak \( g \) is also substantially reduced by thermal fluctuations.

For strong pinning, fluctuations are unimportant and the correlation length reduces to the substantially shorter strong-coupling value \( \xi_0 = (K/|gp|^2)^{1/2} \) determined by the bare model parameters.

---

**FIG. 5.** Effective interface stiffness as a function of \( T/T_{c0} \) for \( \tilde{g} = 0.1 \). In the smooth phase, \( K_R \) scales with the system size, and is effectively infinite. At \( T = T^+_c \), \( K_R \) takes the value

\[
K_R(T^+_c) = K(1 + p^2 \tilde{g}/\sqrt{2} K A^2) \quad \text{with a universal ratio} \quad p^2/8\pi
\]
to the transition temperature \( T_c \). Far above \( T_c \), \( K_R \) goes to its bare value \( K \). The dashed line indicates the location of \( T_c \), which here is given by \( T_c = 1.05 T_{c0} \).

\[
c = \tilde{g}^2 - (\tilde{\tau} + \tau)^2, \quad (4.29a)
\]
\[
\approx -2\tilde{\tau} \tau, \quad (4.29b)
\]

with the true reduced temperature relative to the true (finite \( g \)) \( T_c \) given by

\[
\tau = \left( \frac{2T_c}{T_{c0}} \right) \left( 2T_c/T_{c0} - 1 \right), \quad (4.30)
\]

and \( \tilde{\tau}_c = \tilde{g} = 2(T_c/T_{c0} - 1) \), we find that in the limit \( T \to T_c^+ \)

\[
K_R(T) = K \frac{T_c}{T_{c0}} \left( 1 - \sqrt{\tilde{g}}|T/T_c - 1|^{1/2} \right). \quad (4.31)
\]

This leads at \( T_c \) to a renormalized value of the elastic constant \( K_R(T^+_c) \) that is enhanced relative to the bare value \( K \) and with the universal ratio to \( T_c \) given by

\[
\frac{K_R(T^+_c)}{T_c} = \frac{p^2}{8\pi}, \quad (4.32)
\]
consistent with the analogous result first discovered in the context of the \( XY \)-model [[2]] related to our problem by duality.

Below \( T_c \), the relative pinning strength runs off to strong coupling and the interface is smooth on length scales longer than the correlation length that we calculate below. Because the RG flows are qualitatively very different near and away from the two separatrices \( \tilde{g} = \pm \tilde{\tau} \), the value of this important length scale, that enters the velocity-force characteristics depends crucially on the distance from \( T_c \). In the critical region, defined by values of the bare parameters such that the weak-coupling \( (g) \) flow is near and roughly along either separatrix,

\[
\tilde{g}(\ell) \approx \pm \tilde{\tau}(\ell), \quad (4.33a)
\]
\[
\approx \frac{\tilde{g}}{1 \pm \tilde{g} \ell}, \quad (4.33b)
\]
B. Analysis of the equilibrium dynamics

We now turn our attention to the equilibrium \((F = \lambda = 0)\) dynamics of the sine-Gordon interface, characterized by an additional model parameter, the friction coefficient \(\gamma\), with the RG flow given by

\[
\frac{d\gamma}{d\ell} = \frac{1}{8} g^2 \eta \gamma. \tag{4.40}
\]

Combining this with the recursion relation Eq. \((4.18a)\), we find that the renormalized surface stiffness \(K_R\) and friction coefficient \(\gamma_R\) are related by

\[
\gamma_R = \gamma \left(\frac{K_R}{K}\right)^{1/2}. \tag{4.41}
\]

This together with the results of the previous subsection, show that the macroscopic linear mobility \(\gamma_R^{-1}\) is finitely renormalized in the rough phase, \(T > T_c\), and displays a square-root cusp approach to \(\gamma_R^{-1}(T_c^+) = \gamma^{-1}(T_{c0}/T_c)^{1/2}\) as \(T \to T_c^+\)

\[
\gamma_R^{-1}(T) \approx \gamma_R^{-1}(T_c^+) \left(1 + \frac{1}{2} \sqrt{g} T/T_c - 1\right)^{1/2}, \tag{4.42}
\]

similar to the results of Petschek and Zippelius for the renormalized diffusion coefficient of the XY-model as \(T \to T_{KT}\).

The effective friction coefficient \(\gamma(\ell)\) at scale \(\ell\) can be obtained by integrating the flow equation \((4.40)\)

\[
\gamma(\ell) = \gamma \exp \left[ \frac{1}{8} \int_0^\ell d\ell' \, g^2(\ell') \eta(\ell') \right]. \tag{4.43}
\]

Since below \(T_c\), at weak coupling, \(\tilde{g}^2(\ell)\eta(\ell)\) grows with \(\ell\), we find that the effective friction coefficient runs off to infinity as \(\ell \to \infty\) suggesting a vanishing of the macroscopic linear mobility in the smooth phase. A more detailed analysis of the equilibrium weak-coupling flow equations for large \(\ell\) gives

\[
\gamma(\ell) \approx \gamma \begin{cases} 
\exp \left[ \frac{r f}{4 \eta} \right], & T \to T_c^- \\
\exp \left[ \frac{\eta^2 \gamma^2 (2 - \eta)}{10(2 - \eta)} \right], & T \ll T_c 
\end{cases} \tag{4.44}
\]

Such diverging friction coefficient can be physically interpreted as activated creep dynamics over a pinning energy barrier that asymptotically grows with length scale, logarithmically for \(T \to T_c^-\) and as a power-law for \(T \ll T_c\).

It is important to keep in mind that this growth of the friction coefficient \(\gamma(\ell)\) found in Eq. \((4.44)\) extends only up to the strong-coupling length scale \(\xi = a e^{-\gamma}(\xi_0\) for \(T \to T_c^-\), Eq. \((3.53)\), and \(\xi_0\) for \(T \ll T_c\), Eq. \((3.37)\)) since it was derived based on a renormalization group approach that is perturbative in \(\tilde{g}\). In Section V, we will look in more detail at the physics on scales longer than \(\xi\), but we can already say at this point that (as we show in Section V) even in this strong coupling regime the effective friction coefficient diverges. Consequently, we find that the interface linear (and in fact any order-\(n\)) mobility exhibits a nonuniversal jump discontinuity to zero across the roughening transition as illustrated in Fig. 6.

![Figure 6](image)

**FIG. 6.** Effective linear mobility \(\gamma^{-1}\) as a function of \(T/T_{c0}\) in equilibrium \((F = 0)\) for \(\tilde{g} = 0.1\). Below the roughening temperature \(T_c\), the mobility vanishes and the interface is pinned. \(\gamma^{-1}_R(T)\) shows a square root cusp as \(T \to T_c^+\), and goes to its bare value \(\gamma^{-1}\) for \(T \gg T_c\). The dashed line indicates the location of \(T_c\), which here is given by \(T_c = 1.05 T_{c0}\).

V. NON-EQUILIBRIUM DYNAMICS AND THE VELOCITY-FORCE CHARACTERISTICS

A. Weak-coupling regime

We now turn to the full nonequilibrium problem, with the aim of deriving the velocity-force characteristics of an interface driven through a weak periodic potential, going beyond the failing (for \(T < T_c\)) perturbative approach of section III. As long as the pinning remains weak, the long scale physics of the driven interface is contained in the renormalization group equations Eqs. 4.21-4.26, which when rewritten in terms of the dimensionless variables \(\tilde{g}, \eta, f\) and the new KPZ coupling

\[
\dot{\lambda} = \frac{\lambda}{pK} \tag{5.1}
\]

are given by

\[
\frac{d\gamma}{d\ell} = \frac{1}{8} \eta \tilde{g}^2 \left(1 - f^2\right), \tag{5.2}
\]

\[
\frac{d\tilde{g}}{d\ell} = (2 - \eta) \tilde{g}, \tag{5.3}
\]

\[
\frac{d\eta}{d\ell} = \frac{1}{2} \eta^2 \lambda^2 + \frac{1}{8} \eta \tilde{g}^2 - 2 + 5f^2 + 3f^4 \tag{5.4}
\]

\[
\frac{d\lambda}{d\ell} = \frac{1}{8} \eta \tilde{g}^2 f(f^2 + 5), \tag{5.5}
\]

\[
\frac{df}{d\ell} = 2f + \frac{1}{2} \eta \lambda - \frac{1}{8} \eta \tilde{g}^2 f(3 - f^2) \tag{5.6}
\]
The most striking effect of nonequilibrium dynamics is the breakdown of the FDT and as a result a nontrivial upward renormalization (flow) of the effective "temperature" \( T(\ell) \) driven by the external force and the KPZ nonlinearity, reminiscent of nonequilibrium "heating" in randomly-pinned systems. Consequently, even for \( T < T_c \), for sufficiently strong drive the parameter \( 2 - \eta(\ell) \) determining the long-scale behavior of the periodic potential is driven negative, leading to the irrelevance of the pinning potential. Hence, as discussed in the Introduction, a finite external drive removes the qualitative distinction between the rough and smooth phases, and therefore rounds the roughening transition.

Here, we instead focus on the creep regime, where these particular nonequilibrium effects are unimportant. In this weak driving creep regime, we can ignore the KPZ nonlinearity and the most important role of \( F \), as can be clearly seen even at the level of perturbation theory, Eq. (3.24), and from the equation of motion, is to introduce a new length scale \( \xi_F \sim 1/\sqrt{F} \) defined in Eq. (3.24). Beyond this nonequilibrium length scale the effects of the pinning potential and its ability to renormalize \( \gamma(\ell) \) and \( K(\ell) \) are suppressed, as it is averaged away on scales longer than \( \xi_F \) (see for example the RG flow equations above and analysis below). Hence, for weak external drive \( F \), the effective values of friction and interface stiffness parameters are given by \( \gamma(\ell_F) \) and \( K(\ell_F) \) renormalized by Gaussian equilibrium fluctuations up to length scale \( \xi_F = \epsilon^{\ell_F} \). This therefore translates the strong \( \ell \) dependence of \( \gamma(\ell) \) into strong \( F \) dependence of the macroscopic mobility \( \gamma^{-1}(F) \). Substituting \( \xi_F \), Eq. (3.24), inside our equilibrium flow, Eqs. (4.44), and using

\[
v(F) = F/\gamma(\ell_F),
\]

we immediately obtain the velocity-force characteristics, Eq. (1.2), quoted in the Introduction.

This prediction for \( v(F) \), Eq. (1.2), applies as long as the relevant \( F \) probes length scales \( \xi_F \) on which the equilibrium weak-coupling flow equations remain valid. As discussed in the previous section, these flows in fact break down due to strong coupling effects (with \( g \) itself cutting off thermal Gaussian fluctuations) for length scales greater than \( \xi \), Eqs. (3.35), (3.37). Hence, our predictions for \( v(F) \), Eq. (1.2), remain valid only as long as \( \xi_F < \xi \) (i.e., it is the external force and not the periodic potential itself that cuts off the Gaussian fluctuations), which translates into the condition \( F > F_c \), with the crossover force \( F_c \) given by equation (1.1) and in agreement with perturbation theory.

To see this weak-coupling phenomenology emerge directly from our full nonequilibrium flow equations, Eqs. (5.3)-(5.4), we integrate these equations, with \( \lambda = 0 \) and ignoring the nonequilibrium flow of \( T(\ell) \) (a valid approximation in the \( F \to 0 \) limit). We find for the renormalized friction coefficient the following intermediate result

\[
\gamma_R(f) = \gamma \exp \left[ \frac{1}{8} \int_0^\infty d\ell \eta(\ell) \tilde{g}^2(\ell) \frac{1 - f^2(\ell)}{(1 + f^2(\ell))^2} \right].
\]

Since at low drive and weak coupling, well-below \( T_c \), \( \eta(\ell), K(\ell), \) and \( T(\ell) \) grow slowly and \( f(\ell) \) and \( \tilde{g}(\ell) \) grow strongly according to

\[
\tilde{g}(\ell) = \tilde{g} e^{(2 - \eta)\ell},
\]

\[
f(\ell) \approx f e^2,
\]

it is quite clear from Eq. (5.8) that as long as the weak coupling flows remain valid, in the smooth phase the flows are automatically cut off when \( f(\ell) \) gets to be \( > 1 \) leading to \( \ell_F \) discussed above.

Substituting Eqs. (5.3) into the expression of \( \gamma_R(f) \), Eq. (5.8), and integrating the resulting expression, we find

\[
\gamma_R(f) = \gamma \exp \left( \frac{1}{8} \tilde{g}^2 \frac{x - f}{2} \right),
\]

with \(-\bar{\eta} = (2 - \eta)/2 = (1 - T/T_c)\), and \( A(x, f) \) is the function given by

\[
A(x, f) = \frac{1}{2 f^4} \left( \frac{1}{2(2-x)} \right)^2 F \left( 2,2-x,3-x,-f^{-2} \right)
\]

\[
-2 f^2(1-x) \right) \right)^2 F \left( 2,1-x,2-x,-\frac{1}{f^2} \right),
\]

we immediately obtain the velocity-force characteristics, Eq. (1.2), quoted in the Introduction.

This prediction for \( v(F) \), Eq. (1.2), applies as long as the relevant \( F \) probes length scales \( \xi_F \) on which the equilibrium weak-coupling flow equations remain valid. As discussed in the previous section, these flows in fact break down due to strong coupling effects (with \( g \) itself cutting off thermal Gaussian fluctuations) for length scales greater than \( \xi \), Eqs. (3.35), (3.37). Hence, our predictions for \( v(F) \), Eq. (1.2), remain valid only as long as \( \xi_F < \xi \) (i.e., it is the external force and not the periodic potential itself that cuts off the Gaussian fluctuations), which translates into the condition \( F > F_c \), with the crossover force \( F_c \) given by equation (1.1) and in agreement with perturbation theory.

To see this weak-coupling phenomenology emerge directly from our full nonequilibrium flow equations, Eqs. (5.3)-(5.4), we integrate these equations, with \( \lambda = 0 \) and ignoring the nonequilibrium flow of \( T(\ell) \) (a valid approximation in the \( F \to 0 \) limit). We find for the renormalized friction coefficient the following intermediate result

\[
\gamma_R(f) = \gamma \exp \left[ \frac{1}{8} \int_0^\infty d\ell \eta(\ell) \tilde{g}^2(\ell) \frac{1 - f^2(\ell)}{(1 + f^2(\ell))^2} \right].
\]
where $2F^1$ denotes a hypergeometric function. When $T < T_c$, (i.e., $\tilde{\gamma} > 0$), taking the limit of the function $A(-\tilde{\gamma}, f)$ of equation (5.10) when $f \to 0$ leads to the following expression for the long-scale inverse of nonlinear mobility $\gamma_R$

$$\gamma_R(f) = \gamma \exp \left((F_\nu/F)^{2(1-T/T_c)}\right), \quad (5.12)$$

with

$$F_\nu(\tilde{g}, T) \approx \frac{2K\Lambda^2}{\tilde{g}} \left(\frac{\eta^2}{16(2-\eta)}\right)^{\frac{1}{1-\eta}} \mu f, \quad (5.13)$$

in full agreement with earlier more qualitative discussion of the velocity-force characteristics in the intermediate regime of forces $F > F_\nu$, and $F_\nu$ consistent with the perturbative result (3.22) for $\tilde{\gamma} \ll 1$.

As $F$ is lowered below $F_\nu$, eventually the saturation of $\gamma(\ell)$ breaks down and the flow behavior changes dramatically as strong-coupling length scales (at which our weak-coupling RG solution is invalid) are probed. Studying the point at which this happens as a function of model parameters, allows us to extract the crossover value of $F_\nu$, which we plot in Fig. 8. We find that there is a qualitative agreement between the analytical prediction for $F_\nu$, Eq. (3.22), and our numerical analysis.

### B. Strong-coupling regime

The weak-coupling behavior found in the previous subsection only extends up to the scale $\xi$, Eqs (4.35b), (4.37). Beyond this strong-coupling length, in the equilibrium model, the growth of $\tilde{g}(\ell)$ and $\gamma(\ell)$ is cutoff by the pinning potential, and an approach nonperturbative in $\tilde{g}$, where pinning is treated on equal footing with the elastic energy, is required. In this strong-coupling regime Gaussian interface fluctuations, considered so far, are strongly suppressed by

![FIG. 8. Characteristic force $f_\nu(T)$ as obtained from the numerical solution of the dynamic RG recursion relations (solid line) and from the perturbative estimate of equation (3.22), (dashed line), for $\tilde{g} = 0.1$. The curve $f_\nu(T)$ delimits two very different physical regimes. Above this curve, the interface moves with uniform velocity. On the other hand, for $f < f_\nu(T)$, the interface moves through the nucleation of soliton excitations.](image)

![FIG. 9. Schematic representation of the motion of a driven interface past the periodic pinning potential. (a) When $T > T_c$, or $T < T_c$ and $f > f_\nu$, the large fluctuations of the interface wash out the pinning potential on large length scales and the interface moves with a uniform velocity. (b) On the other hand, for $T < T_c$ and $f < f_\nu$, the fluctuations of the interface are small; as a result, most of the interface is pinned at a given minimum of the pinning potential, and motion from one minimum to the next takes place through soliton excitations.](image)

the pinning barrier that scales like $L^2$ relative to the elastic energy.

Instead, at low temperature the fluctuations are dominated by nontrivial saddle-point solutions (solitons) of $H$, Eq. (2.1), with model parameters, $K_R$, $g_R$, $\gamma_R$ renormalized by Gaussian fluctuations on weak-coupling scales $L < \xi$. The dominant soliton excitation, illustrated in projection in Fig. 9, corresponds to a circular patch of radius $R > \xi$ of a nearly flat interface moving over to a neighboring minimum of the periodic potential, with an energy cost that clearly grows linearly with $R$

$$E_{\text{soliton}}(R) \approx pgn \xi R, \quad (5.14)$$

where $g_R$ (Eqs. (4.39), (4.39)) and $\xi$ (Eqs (4.35), (4.37)) strongly depend on the proximity to $T_c$. At zero-drive, the barrier to such solitonic motion simply diverges and linear mobility vanishes identically. A velocity-force characteristics in the weak drive $F < F_\nu$ (i.e., $\xi < \xi_R$) regime can be analyzed via scaling nucleation theory [1]. In this creep regime the interface is in near metastable equilibrium with $F$ introducing a contribution

$$E_F(R) \approx -F \pi R^2 d \quad (5.15)$$

to the effective free energy. Balancing $E_F(R)$ against the soliton energy $E_{\text{soliton}}(R)$ we find that solitons of size larger than a critical radius

$$R_c \approx \left(\frac{pgn \xi}{2\pi d}\right) \frac{1}{F} \quad (5.16)$$

are unstable. In the $F \to 0$ limit, thermal activation rate of solitons of size $R_c \sim 1/F$ is quite clearly the limiting
step for interface creep motion. We therefore find that the weak-coupling velocity-force characteristics, Eq. (1.2) crosses over, for $F < F_*$, to that given by Eq. (1.3) in the Introduction, with

$$F_0 \approx \frac{(pgR\xi)^2}{4\pi d}.$$ (5.17)

For vanishing temperature and strong bare pinning potential our asymptotic (for $F < F_*$) result for $v(F)$ reduces to that found in Refs. [38]. However, at large $T < T_c$ and weak bare pinning $g$, we predict a strong thermal renormalization of the characteristic pinning energy

$$p^2 g^2 a^2 \to p^2 g^2 R^2 \xi^2$$ (5.18)

by thermal fluctuations on scales smaller than $\xi$.

VI. DISCUSSION AND CONCLUSIONS

In this paper we have studied the creep dynamics of a two-dimensional interface driven through a periodic potential. Using dynamic renormalization group methods and matching to strong coupling, we have calculated the velocity-force characteristics across the interface roughening transition. Consistent with previous studies, we find a qualitative change across the transition in the weak-drive velocity-force characteristics, with Ohmic transport for $T > T_c$ and a jump discontinuity in mobility across the transition. For $T < T_c$, in the asymptotic creep regime $F \ll F_*(g,T)$ and for strong bare coupling, where transport is via soliton activation at all scales, we recover previously found results for the velocity-force characteristics $v(F)$. However, for weak bare coupling and strong thermal fluctuations, we predict an intermediate drive $F > F_*(g,T)$ nonlinear regime with a continuously varying (with $T$) exponent, which asymptotically crosses over to the strong coupling result with strongly thermally renormalized characteristic pinning barrier. Unfortunately, because the characteristic force $F_*(g,T)$ that delineates between the intermediate drive regime and the strong-coupling regime coincides with the force marking the breakdown of the perturbative high-velocity expansion, we expect it to be difficult to observe this intermediate drive regime.

The new physical picture which emerges from the present study complements previously made predictions, which were based on a more elementary perturbative approach, as well as known results for the mobility at zero external drive. On the experimental side, the above picture may shed some light on experiments such as those of Wolf et al., who found that the growth velocity $v$ of a surface of crystalline Helium 4 is strongly reduced at $T_c$ from an Ohmic behavior $v \sim F$ for $T > T_c$ to an extremely slow growth rate for $T < T_c$, a result which is usually explained in terms of an onset of creep motion via soliton like excitations.

An interesting and experimentally relevant generalization of our results is a study of creep dynamics of a two-dimensional solid, driven through a one- or two-dimensional periodic potential, with applications to driven 2D colloidal crystals and vortices in superconducting films. Despite of considerably different geometry, in equilibrium these systems display a pinned-to-floating solid transition closely related to the roughening transition of 2D interfaces. However, new interesting ingredients arise. Some of the most important ones are the nonequilibrium conventive-like terms in vector phonon displacement and concomitant possible importance of dislocations. Combined with the considerably interesting behavior of the scalar sine-Gordon model studied here, we expect these to lead to even richer phenomenology. We expect that studies of these will shed considerable light on numerous experiments and simulations.

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VII. APPENDIX : STATIC MOMENTUM SHELL RENORMALIZATION GROUP

In this Appendix, we present technical details on the derivation of the renormalization group recursion relations for the driven sine-Gordon model in 2 + 1 dimensions. For completeness, we shall begin in section VII A by showing how the standard momentum shell RG with hard cutoff can be applied to the static version of this problem before deriving the full dynamic equations at nonzero external drive in section VII B.

A. Static RG

We decompose the field $h(r)$ in the Hamiltonian (2.1) into high and low wavevector components

$$h(r) = h^\prec(r) + h^\succ(r)$$ (7.1)

such that

$$h^\prec(r) = \int_q < h(q) e^{i qr},$$ (7.2)

$$h^\succ(r) = \int_q > h(q) e^{i qr},$$ (7.3)
where $f^< = \int_{\Lambda/b}^{\Lambda/b} \frac{dq}{2\pi q}$ and $f^\geq = \int_{\Lambda/b}^{\Lambda/b} \frac{dq}{2\pi q}$ denote integration in momentum space over the ranges $0 < |q| < \Lambda/b$ and $\Lambda/b < |q| < \Lambda$ respectively. In terms of these high and low momentum fields, the equilibrium Hamiltonian $H_0[h] = \frac{1}{2} \int d\mathbf{r} K(\nabla h)^2$ can be written as the sum

$$H_0^0[h] = H_0[h^<] + H_0[h^>] .$$

We now want to integrate over the fast component $h^>(\mathbf{r})$. To this end, we rewrite the partition function $Z = \int [dh] \exp(-\beta H)$ in the form (here $\beta = 1/T$ is the inverse temperature)

$$Z = \int [dh^<] [dh^>] e^{-\beta H_0[h^<] - \beta H_0[h^>] - \beta H_1[h^< + h^>]} ,$$

$$= \int [dh^<] e^{-\beta H_0[h^<]} \int [dh^>] e^{-\beta H_0[h^>] - \beta H_1[h^< + h^>]} ,$$

$$= \int [dh^<] e^{-\beta H_0[h^<]} + \beta \ln Z_0 \langle e^{-\beta H_1[h^< + h^>]} \rangle_0 ,$$

(7.4)

where $Z_0^> = \int [dh^>] \exp(-\beta H_0[h^>])$, and where the subscript (0) means that the average with respect to $h^>$

$$\langle H_1 \rangle_0 = - g e^{-\frac{1}{2} g^2 G^>(0)} \int d\mathbf{r} \cos[ph^<(\mathbf{r})] ,$$

$$\langle H_1^2 \rangle_0 = \frac{1}{2} g^2 e^{-\frac{1}{2} g^2 G^>(0)} \int d\mathbf{r} d\mathbf{r}' \left[ e^{g^2 G^>(\mathbf{r} - \mathbf{r}')} - 1 \right] \left\{ \cos(p(h^<(\mathbf{r}) + h^<(\mathbf{r}'))) + \cos(p(h^<(\mathbf{r}) - h^<(\mathbf{r}'))) \right\} ,$$

(7.9)

where $G^>(\mathbf{r} - \mathbf{r}') = \langle h^>(\mathbf{r}) h^<(\mathbf{r}') \rangle_0$ is the elastic propagator for fast fields (here $J_0$ is the zeroth order Bessel function)

$$G^>(\mathbf{r} - \mathbf{r}') = T \int_{\Lambda/b}^{\Lambda/b} \frac{dq}{K q^2} = T \frac{d\ell}{2\pi K} J_0(\Lambda |\mathbf{r} - \mathbf{r}'|) .$$

(7.10)

Given that $G^>(0) = T d\ell/2\pi K$, we see that the first order cumulant $(7.2)$, after the rescalings $(7.13)-(7.14)$, leads straightforwardly to the recursion relation $(7.18)$ for the pinning strength $g$. On the other hand, since the “kernel”

$$K(\mathbf{r}) = \left[ e^{g^2 G^>(\mathbf{r})} - 1 \right]$$

(7.11)

takes appreciable values only for small values of its argument, we see that the first term inside the integral in equation $(7.3)$ will contribute higher harmonic terms ($\sim \cos(2ph(\mathbf{r}))$) to the effective Hamiltonian, and hence we shall discard this term as irrelevant. In the second term, we shall make the approximation

$$\cos \left[ p \left( h^<(\mathbf{r}) - h^<(\mathbf{r}') \right) \right] \simeq 1 - \frac{1}{2} p^2 \left( h^<(\mathbf{r}) - h^<(\mathbf{r}') \right)^2 ,$$

$$\simeq 1 - \frac{1}{2} \beta^2 (\mathbf{r} - \mathbf{r}')_\alpha (\mathbf{r} - \mathbf{r}')_\beta \partial_\alpha h^<(\mathbf{r}) \partial_\beta h^<(\mathbf{r}) ,$$

(7.12)

where, in going from the first to the second line, we made use of the Taylor expansion

$$h^<(\mathbf{r}) - h^<(\mathbf{r}') \simeq (\mathbf{r} - \mathbf{r}')_\alpha \partial_\alpha h^<(\mathbf{r}) .$$

Inserting $(7.12)$ back into Eq. $(7.4)$, we obtain the following approximation to the second cumulant (we here use the symbol $\equiv$ to indicate that we retain only the term correcting the stiffness $K$)

$$\langle H_1^2 \rangle_0 \equiv - \frac{1}{8} p^2 g^2 e^{-\frac{1}{2} g^2 G^>(0)} \int d\mathbf{r} (\nabla h^<(\mathbf{r}))^2 \times$$

$$\times \int d\mathbf{r}' (\mathbf{r} - \mathbf{r}')^2 K(\mathbf{r} - \mathbf{r}') .$$

(7.13)

Since $G^> \propto d\ell = \ln b$, we can expand the exponential in a Taylor series in $G^>$,

$$e^{g^2 G^>(\mathbf{r} - \mathbf{r}') - 1} \simeq p^2 G^>(\mathbf{r} - \mathbf{r}') + \frac{1}{2} p^4 (G(\mathbf{r} - \mathbf{r}')^2 .$$

(7.14)

Now, the renormalization of $K$ involves the integral

$$\int d\mathbf{r} \mathbf{r}^2 K(\mathbf{r}) = \int d\mathbf{r} \mathbf{r}^2 \left[ e^{g^2 G^>(\mathbf{r})} - 1 \right] .$$

(7.15)

Inserting the expansion $(7.14)$ into this last expression, the first term gives a contribution

$$\int d\mathbf{r} \mathbf{r}^2 G^>(\mathbf{r}) = - \nabla^2 q G^>(q) \big|_{q=0}$$

(7.16)

which vanishes identically, since $G^>(q)$ has support only on the shell $\Lambda/b < q < \Lambda$. The second term gives
Thus, we obtain for the second cumulant (7.13) the following expression
\[ -\frac{\langle H^2 \rangle_{\tilde{\gamma}}}{2T} = \frac{T^2 p^3 \ell}{8 \pi K^2 A^4} \int \frac{dr}{2} \left( \nabla h^<(r) \right)^2. \]  

We now perform the following rescalings
\[ \mathbf{r} = e^{\ell} \mathbf{r}', \quad h^<(r) = e^{\ell^2} h(r'), \]  
so as to restore the ultraviolet cut-off back to \( \Lambda \). Because the pinning potential is a periodic function, it is convenient (although not necessary) to set the arbitrary field dimension \( \chi \) to zero, thereby preserving the period \( 2\pi/p \) of the original problem under RG transformations. Under such a transformation, the resulting effective Hamiltonian can be cast into its original form with effective \( \ell \)-dependent parameters \( K(\ell) \) and \( g(\ell) \) such that
\[ g(\ell) = g e^{(2 - T p^2/4 \pi K)\ell}, \]  
\[ K(\ell) = K + \frac{T p^6 g^2}{8 \pi K^2 A^4} \ell, \]  
or, in differential form
\[ \frac{dg}{d\ell} = 2 - \frac{T p^2}{4 \pi K} g, \]  
\[ \frac{dK}{d\ell} = \frac{T p^6 g^2}{8 \pi K^2 A^4}. \]  

### B. Dynamic RG

We now turn our attention to the derivation of the dynamic RG flow equations (1.16a)-(1.16c) for the driven sine-Gordon model. As we did in the static case, we define the following low and high momentum components of \( h^<(r, t) \) and \( h^>(r, t) \):
\[ u^<(r, t) = \int_{-\infty}^{r} h(q, \omega) e^{i(q \cdot r - \omega t)} dq, \]  
\[ u^>(r, t) = \int_{-\infty}^{\infty} h(q, \omega) e^{i(q \cdot r - \omega t)} dq, \]  

where, here and in what follows, \( \int_{q, \omega} \) on integrals stands for \( \int \frac{d^4 q}{(2\pi)^4} \frac{d\omega}{2\pi} \), and the superscripts > and < indicate integration over the high \((A/b < q < \Lambda)\) and low \((0 < q < A)\) momentum regions, respectively. Using the fact that \( u(q, \omega) = u^<(q, \omega) + u^>(q, \omega) \), it is not difficult to verify that the free part \( S_0 \) of the action decomposes into two diagonal pieces \( S^<_0 \) and \( S^>_0 \) depending only on \( u^<(q, \omega) \) and \( u^>(q, \omega) \) respectively
\[ S_0[u, \tilde{u}] = S_0[u^<, \tilde{u}^<] + S_0[u^>, \tilde{u}^>] \]  

As we did in the static RG, in order to be able to integrate out the fast component of the field \( u(r, t) \), we rewrite the generating functional \( Z \) in the form
\[ Z = \int \langle du^< \rangle \langle d\tilde{u}^< \rangle e^{-S_0[u^<, \tilde{u}^<]} + \ln Z^>_0 \exp\{-S_0[u^>, \tilde{u}^>]\} \]  
where \( Z^>_0 = \int \langle du^> \rangle \langle d\tilde{u}^> \rangle \exp\{-S_0[u^>, \tilde{u}^>]\} \) and \( \langle \cdots \rangle \) denotes statistical averaging with statistical weight \( e^{-S_0[u^<, \tilde{u}^<]} \). The perturbative correction to the dynamic action can therefore be expressed in terms of a cumulant expansion
\[ \langle e^{-S_0} \rangle_{0>} = 1 - \langle S_1 \rangle_{0>} + \frac{1}{2} \langle S^2_1 \rangle_{0>} + \cdots \]  

Reexponentiation of this expansion allows us to define the effective action
\[ S_{eff}[u, \tilde{u}] = S_0 + \langle S_1 \rangle_{u^<} - \frac{1}{2} \left\{ \left[S^2_1 - (\langle S_1 \rangle_{0^<})^2\right]_{0>} + \cdots \right\} \]  

from which we can derive dynamic RG flows for the parameters of the original equation of motion. This procedure, to first order in the pinning strength \( g \), has already been shown in the text. Here we are therefore only going to consider the second order correction to the original action \( S \). In fact, it turns out\[4\] that the only perturbative corrections to \( S \) to second order in perturbation theory come from the cumulants \(-\frac{1}{2}\langle S^2_1 \rangle_{0>}\) and \(-\frac{1}{2}\langle S^3_1 \rangle_{0>}\), i.e. we need not consider the cross term \(-\langle S_2 S_1 \rangle_{0>}\) which does not provide any perturbative corrections to the action. In the following, we shall only show how we compute the perturbative corrections arising from the sine-Gordon perturbation \(-\frac{1}{2}\langle S^2_1 \rangle_{0>}\), the unique term arising from \(-\frac{1}{2}\langle S^3_1 \rangle_{0>}\)
\[ \Delta S_{\lambda}(\gamma T) = \int dr dt \tilde{u}^2(r, t) \left[ \frac{T \Lambda^2 d^4 \ell}{8 \pi K^2} \right] \]  
having been repeatedly derived in the literature\[3, 4\]. Taking the Gaussian averages in equation (7.25) leads to the following expression of the second cumulant
\[ \Delta S_{g}[\tilde{u}, u] = -\frac{1}{2} \langle S^2_1 \rangle_{0>}. \]  

\[ \Delta S_{g}[\tilde{u}^< u] = -\frac{1}{2} p^2 g^2 \int dr dt \int dr' dr'' \tilde{u}^<(r, t) \tilde{u}^<(r', t') \mathcal{K}(r - r', t - t') \cos \left[p(u^<(r, t) - u^<(r', t')) + \frac{p F}{\gamma}(t - t')\right] \]
\[ -\frac{1}{2} p^2 g^2 \int dr dt \int dr' dr'' i \tilde{u}^<(r, t) \mathcal{K}(r - r', t - t') \sin \left[p(u^<(r, t) - u^<(r', t')) + \frac{p F}{\gamma}(t - t')\right]. \]  

(7.28)
Here the dynamic kernels $\tilde{K}(r, t)$ and $K(r, t)$ are given by

$$\tilde{K}(r, t) = \frac{1}{2} \left[ 1 - \cosh \left( p^2 G_0^>(r, t) \right) \right]$$

$$- \sinh \left( p^2 G_0^>(r, t) \right), \quad (7.29)$$

$$K(r, t) = e^{-\frac{1}{2} p^2 C_0^>(r, t)} R_0^>(r, t), \quad (7.30)$$

where $R_0^>(r, t) = \int_{q_0}^{\infty} e^{-i(q\cdot r - \omega t)}/(i\gamma \omega + Kq^2)$ and

$$G_0^>(r, t) = \frac{2\gamma T}{\gamma^2 \omega^2 + K^2 q^4}.$$  

We now decompose the sine and cosine in the integrand on the rhs of equation (7.28) according to

$$\cos [p(u_\omega - u_\omega')] + \frac{pF}{\gamma} (t - t') = \sin [p(u_\omega - u_\omega')] \cos \left[ \frac{pF}{\gamma} (t - t') \right] + \cos [p(u_\omega - u_\omega')] \sin \left[ \frac{pF}{\gamma} (t - t') \right], \quad (7.32a)$$

$$\sin [p(u_\omega - u_\omega')] + \frac{pF}{\gamma} (t - t') = \sin [p(u_\omega - u_\omega')] \cos \left[ \frac{pF}{\gamma} (t - t') \right] + \cos [p(u_\omega - u_\omega')] \sin \left[ \frac{pF}{\gamma} (t - t') \right]. \quad (7.32b)$$

The kernels $\tilde{K}(r - r', t - t')$ and $K(r - r', t - t')$ being short ranged both in space and time, we see that the major contribution to the action (7.28) comes from the regions $r \approx r'$ and $t \approx t'$ where $[u_\omega(r, t) - u_\omega(r', t')]$ is small. We therefore shall approximate

$$\sin \left[ p(u_\omega(r, t) - u_\omega(r', t')) \right] \approx p \left( u_\omega(r, t) - u_\omega(r', t') \right), \quad (7.33)$$

$$\cos \left[ p(u_\omega(r, t) - u_\omega(r', t')) \right] \approx 1 - \frac{1}{2} p^2 \left( u_\omega(r, t) - u_\omega(r', t') \right)^2, \quad (7.34)$$

and

$$u_\omega(r, t) - u_\omega(r', t') = (t - t') \partial_t u_\omega + (r - r') \partial_r u_\omega + \frac{1}{2} (r - r') \partial_r \partial_t u_\omega, \quad (7.35)$$

upon which we obtain the following expression for the second cumulant $-\frac{1}{2} \langle S_i^2 \rangle_{0>}$:

$$\Delta S[u, u] = -\frac{1}{2} \langle S_i^2 \rangle_{0>}[\tilde{u}, \tilde{u}] = \Delta S(\gamma T) + \Delta S(\gamma) + \Delta S(K) + \Delta S(\lambda) + \Delta S(F), \quad (7.36)$$

where

$$\Delta S(\gamma T) = \Delta S_\lambda(\gamma T) + \frac{1}{2} p^2 g^2 \int \mathrm{d}r \mathrm{d}t \int \mathrm{d}r' \mathrm{d}t' \tilde{K}(r - r', t - t') \lambda \left[ \frac{pF}{\gamma} (t - t') \right], \quad (7.37a)$$

$$\Delta S(\gamma) = \frac{1}{2} p^4 g^2 \int \mathrm{d}r \mathrm{d}t \int \mathrm{d}r' \mathrm{d}t' \tilde{K}(r - r', t - t') \lambda \left[ \frac{pF}{\gamma} (t - t') \right], \quad (7.37b)$$

$$\Delta S(K) = \frac{1}{4} p^4 g^2 \int \mathrm{d}r \mathrm{d}t \int \mathrm{d}r' \mathrm{d}t' \lambda \left[ -\nabla^2 u_\omega(r, t) \right] \lambda \left[ \frac{pF}{\gamma} (t - t') \right], \quad (7.37c)$$

$$\Delta S(\lambda) = \frac{1}{4} p^4 g^2 \int \mathrm{d}r \mathrm{d}t \int \mathrm{d}r' \mathrm{d}t' \lambda \left[ -\lambda \left( \nabla u_\omega(r, t) \right)^2 \right] \lambda \left[ \frac{pF}{\gamma} (t - t') \right], \quad (7.37d)$$

$$\Delta S(F) = \frac{1}{2} p^3 g^2 \int \mathrm{d}r \mathrm{d}t \int \mathrm{d}r' \mathrm{d}t' \lambda \left[ \frac{pF}{\gamma} (t - t') \right], \quad (7.37e)$$

Here we pause a moment to indicate that if we use the complete expression of the kernel $K(r, t)$ and use equation (7.37e) and let $b \rightarrow \infty$, then we obtain from equation (7.37d) above the following expression for the friction force $F_{fr}$ due to the pinning potential to order $g^2$,

$$F_{fr} = \frac{1}{2} p^3 g^2 \int \mathrm{d}r' \mathrm{d}t' \lambda \left[ \frac{pF}{\gamma} (t - t') \right] \times R_0(r - r', t - t') \sin \left[ \frac{pF}{\gamma} (t - t') \right], \quad (7.39)$$

which leads directly to the perturbative result (7.5) of the text.

We now go back to our dynamic RG recursion relations, (7.37a)-(7.37e). In the dynamic kernels of equations (7.29)-(7.30), we expand

$$\tilde{K}(r, t) = -p^2 G_0^>(r, t) - \frac{1}{4} p^4 (G_0^>(r, t))^2, \quad (7.40a)$$

and
\[ K(r, t) = R_0^2(r, t) - \frac{1}{2} p^2 C_0^2(r, t) R_0^2(r, t) , \]  
(7.40b)

and keep only the second term on the \( \text{rhs} \) of the above equations (the first term gives a vanishing contribution, for reasons which are identical to those explained after equation (7.41) of appendix A). Now, from equations (7.40a)-(7.40b), we see that the perturbative corrections to the bare parameters of the theory are given by the flows

\[
\frac{d(\gamma T)}{dt} \big|_{\text{pert}} = \frac{T \lambda^2}{8 \pi K^3} + \frac{p^2 g^2}{4(1 + f^2)^2} \int \frac{dK}{K} \frac{dK}{dt} K(r, t) \cos \left( \frac{pF}{\gamma} t \right) ,
\]

\[
\frac{d\gamma}{dt} \big|_{\text{pert}} = \frac{1}{2} \frac{p^2 g^2}{\pi K^3} \int \frac{dK}{K} \frac{dK}{dt} K(r, t) \cos \left( \frac{pF}{\gamma} t \right) ,
\]

\[
\frac{dK}{dt} \big|_{\text{pert}} = \frac{1}{16 \pi K^2 \Lambda^4} \left[ \frac{1}{1 + (f^2)^3} \right] ,
\]

\[
\frac{d\lambda}{dt} \big|_{\text{pert}} = \frac{1}{4 \pi K} \left[ \frac{1}{8 \pi K^2 \Lambda^2} \right] f .
\]

Using equations (7.40a)-(7.40b), the above recursion relations become

\[
\frac{d(\gamma T)}{dt} = \frac{T \lambda^2}{8 \pi K^3} + \frac{T p^2 g^2}{16 \pi K^2 \Lambda^4} \left[ 1 + f^2 \right] \left( \gamma T \right) ,
\]

\[
\frac{d\gamma}{dt} = \frac{T p^2 g^2}{16 \pi K^3 \Lambda^4} \left[ 1 + f^2 \right] \gamma ,
\]

\[
\frac{dK}{dt} = \frac{T p^2 g^2}{16 \pi K^2 \Lambda^4} \left[ 2 - 3 f^2 - f^4 \right] ,
\]

\[
\frac{d\lambda}{dt} = \frac{T p^2 g^2}{16 \pi K^2 \Lambda^4} \left[ f(1 + f^2)^3 \right] ,
\]

\[
\frac{dF}{dt} = \frac{T p^2 g^2}{4 \pi K} \frac{f}{8 \pi K^2 \Lambda^2} .
\]

On the other hand, we know from equations (4.12a)-(4.13) that the rescaling of fields and space and time variables produces the recursion relations

\[
\frac{d(\gamma T)}{dt} \big|_{\text{resc}} = \frac{d\gamma}{dt} \big|_{\text{resc}} = \frac{dK}{dt} \big|_{\text{resc}} = \frac{d\lambda}{dt} \big|_{\text{resc}} = 0 ,
\]

\[
\frac{dF}{dt} \big|_{\text{resc}} = 2 F .
\]

Using the recursion relations above along with the fact that, in a renormalization group transformation,

\[
\frac{d}{dt} = \frac{d}{dt} \big|_{\text{pert}} + \frac{d}{dt} \big|_{\text{resc}} ,
\]

leads directly to equations (4.16a)-(4.16c) of the text.
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