Saturated Feedback Control to Improve Ride Comfort for Uncertain Nonlinear Macpherson Active Suspension System With Input Delay

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ABSTRACT A robust saturation control approach is developed for input-time delay Macpherson active suspensions, subject to dynamical uncertainties, exogenous disturbances, and road excitations. The proposed control method comprises of a linear combination of two smooth saturation functions of a filtered signal and a regulation error, hence the control law is smooth and bounded by a known and adjustable constant bound. An auxiliary signal involving a finite integral over the delayed time interval of past control values is exploited to convert the delayed system into a delay-free system, and Lyapunov–Krasovskii (LK) functionals are constructed to eliminate the residual delayed terms in a Lyapunov-based analysis. The vertical displacement and velocity of the sprung mass are proven to uniformly ultimately bounded regulating to improve the ride comfort, despite model uncertainties, additive disturbances and the input delay. Several simulations are performed to verify the improvement in the ride comfort under different road profiles, while the tire deflection and suspension deflection are within an admissible limitation in comparison with two other suspensions.

INDEX TERMS Active suspension system, actuator saturation, input-time delay, nonlinear uncertain systems, robust control.

I. INTRODUCTION

Due to its important roles in vehicle performance, vehicle suspension control has been an interested subject in research literature. Ride comfort, road holding and suspension deflection are three critical performance requirements for controlling vehicle suspensions. However, these criteria are usually contrary, thus a compromise of the criteria must be attained. The control design solution for active suspension is a potential technical approach to enhance ride comfort, while holding the suspension deflection and tire deflection in an admissible level [1].

Various control techniques for active suspensions aiming at promoting the ride comfort have been presented in literature. For example, a non-fragile \(H_\infty\) output-feedback control in [2] was constructed to increase the ride comfort inside the intent frequency span and also assure the hard requirements in the time-domain. However, the optimal control approach is not probably an appropriate solution for vehicle suspensions consisting of dynamics uncertainties and exogenous disturbances; because common methodology of \(H_\infty\) control approach for vehicle suspensions is all constraints weighted and formed into an unique cost function to be minimized to acquire an optimal control gain [3], [4], thus this control approach develops based on the linearization approximation of the suspension dynamics and requires the plenty knowledge of the system dynamics which is sometimes impossible to satisfy in the practice. The nonlinear nature in both kinematics and dynamics behaviors of the Macpherson suspension was indicated in [5]–[11], hence the application of the control methods constructing based on the dynamics linearization approximation for Macpherson active suspensions will lead to degrade control performance.

Several robust nonlinear control strategies for active suspension systems have been introduced recently. For examples, the method in [12] presented a robust nonlinear suspension control system developed using the combination of fuzzy logic, neural network control, and sliding mode
control (SMC) methodologies; Chen et. al. introduced an improved SMC for nonlinear active suspensions to accomplish the nominal optimal performance and better robustness in [13]; Taghavifar et. al. presented an adaptive SMC based indirect fuzzy neural network system for a nonlinear suspension subject to uncertain parameters and road excitations in [14], or Wang et. al. developed an Active Disturbance Rejection Control combining with a fuzzy SMC to improve the ride comfort of full car suspension systems, whereas unmodeled dynamics and external disturbances are estimated by an extended state observer in [15]. However, these approaches are discontinuous feedback methods with infinite control bandwidth and chattering limitations. Besides, the time delay and amplitude limitation issues of actuators have not been investigated thoroughly in controlling for uncertain nonlinear active suspensions.

Time delay is inevitable and unfortunately, a resource of instability and attenuation problems of the system performance. Time delay in practical systems can be caused by many reasons, for instance, the control torque created by an internal combustion engine can be postponed caused by fuel-air mixing, ignition delays, cylinder pressure force propagation, or communication delays exist in remote control applications (such as, master-slave teleoperation of robot, haptic systems) whereas time is irresistibly demanded to feedback the control information. Hence, the time delay particularly in actuators is also an important issue that needs careful consideration in active control of vehicle suspension systems. There are some results about the active suspension control with actuator input delay, such as [1], [16], [17]. However, these control strategies were developed based on the conventional model of the suspension with only the parameter uncertainties, so without allowing dynamics uncertainties and/or exogenous disturbances, the suspension dynamics is required to be linearization.

Moreover, the fact that control inputs are a function depending on the system states, thus large initial conditions and/or unmodeled disturbances may evoke the controller to exceed physical limitations. Specially, control errors can add up over the delay interval for systems with input delays, also leading to large actuator requirements, aggravating potential problems with actuator saturation [18]. Due to the control performance degradation and the potential control failure risk, control schemes for active suspension systems ensuring performance within the actuator limitations are motivated. A saturated adaptive robust control strategy has been introduced to address the control problem of uncertain active suspension systems with saturated inputs in [19]. However, to the best of author’s knowledge, a control method for uncertain Macpherson active suspensions considering all saturation limit, input delay, dynamics uncertainties and external disturbances is still an open problem.

The problem of $H_{\infty}$ state-feedback controller for semi-active seat suspension systems with both time-varying input delay and actuator saturation in [31], however, this control method develops for the conventional suspension systems with parametric uncertainties only, and a set of linear matrix inequalities needs to be solved approximately by numerical methods. Recently, Dinh et. al. presented a robust saturated RISE feedback control for uncertain nonlinear Macpherson active suspension systems in [29], and this saturated controller is developed based on the nonlinear dynamics model of the Macpherson suspension, but didn’t consider about the input delay issue of the system. The contribution of this paper is that the both time delay and actuator saturation issues of the control input are taken into account for the nonlinear Macpherson suspension without transforming the system via the linearization step, whereas the saturated control design can predict/compensate for known input delays in active suspensions with nonlinear uncertainties and exogenous disturbances. The technical challenges for this control method are that to develop the stability analysis for the underactuated system to obtain the delay-free control input, and handle with the remaining delayed cross terms. A predictor term containing a finite integral of past control values over the delay time interval is utilized to inject a delay free control input into the stability analysis, and LK functionals are exploited in the design and stability analysis. The continuous saturated controller is developed with the bound on the control to be known and adjustable by changing the feedback gains. The control objective is to achieve uniformly ultimately bounded regulation of the vertical displacement and velocity of the sprung mass to improve ride comfort, which is proven by Lyapunov stability analysis. The performance of the proposed control method is examined by numerical simulations in comparison with two other suspensions in the improvement of the ride comfort while the suspension deflection and tire deflection within acceptable level.

II. SYSTEM MODEL AND OBJECTIVES

Different suspension dynamic models have been considered for analyzing the suspension oscillating behavior. Dynamic models of the Macpherson suspension system were discussed in [5]–[11]. In the following development, the nonlinear dynamics of the active Macpherson suspension subjected to the control input delay can be expressed via the following state space representation with the vertical displacement $z_i$ of the sprung mass and the rotation angle $\theta$ of the control arm chosen as the generalized coordinates [29]

$$\dot{x}(t) = F(x) + G(x_3)u(t - \tau) + H(x_3)z_i(t) + d(x, t). \quad (1)$$

The model figure of a quarter-car Macpherson suspension is referred to Fig. 1 in [29] for details. In (1), $x = [x_1 \; x_2 \; x_3 \; x_4]^T \triangleq [\xi \; \dot{\xi} \; \theta \; \dot{\theta}]^T$ denotes the state vector with a finite initial condition $x(0) = x_0$, $z_i \in \mathbb{R}$ is the road excitation, $u(t - \tau) \in \mathbb{R}$ represents the delayed active control force, where $\tau \in \mathbb{R}^+$ is a known constant time delay. Throughout this paper, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ is the set of strictly positive real numbers, $\mathbb{R}^n$ is the n-dimensional Euclidean space, and a time-dependent delayed function denoted as $\xi(t - \tau)$ or $\zeta(t)$ is defined as $\zeta_t \triangleq$
The essential objectives of the active suspension design is to expeditiously regulate the vertical car body displacement for ride comfort. The contribution of the control method in this paper is the construction of an amplitude-limited and continuous controller which ensures the vertical displacement, velocity $z_s$, $\dot{z}_s$ of the chassis are uniformly ultimately bounded regulation, despite unmatched uncertainties, nonlinear disturbances and the delay of the control input in the system. To quantify the state regulation objective, a regulation error $e \in R$ and a filtered regulation error $r \in R$ are defined as
\[
e \triangleq z_s, \quad r \triangleq \dot{e} + \alpha \tanh(e) + \tanh(e_f) + e_c,
\]
where $e_f \in R$ is an auxiliary signal whose dynamics are given by
\[
\dot{e}_f \triangleq \cosh^2(e_f) \left\{-kr + \tanh(e) - \gamma\tanh(e_f)\right\},
\]
and $k, \alpha, \gamma \in R$ denote constant positive control gains. Based on Assumption 2, the regulation errors $e, r$ are measurable, whereas the measurable term $e_c \in R$ is subsequently defined. The motivation for the formation of the regulation errors and auxiliary signals is the need of adding and subtracting intermediate terms in the stability analysis step.

**Assumption 3:** The system in (1) is assumed that if the control force is limited below a priori limit (i.e. $|u| \leq \overline{u}$, where $\overline{u}$ is a known positive constant), the unmeasurable states $x_u(t) = \left[\theta \ \dot{\theta}\right]^T$ in (1) are bounded in terms of the measurable regulation errors such that the following condition holds:
\[
\|x_u\| \leq c_1 + c_2 \|z\|
\]
where $z(t) \in R^4$ is defined as
\[
z \triangleq \left[e \ \tanh(e_f) \ r \ e_c\right]^T,
\]
and $c_1, c_2 \in R$ are known nonnegative bounding constants. This assumption is equivalent to assuming that the rotation of the control arm is bounded in a stable limit if the controller is saturated in a priori bound. The aim of this active suspension control design is to guarantee the system stability under the existence of the control input delay and to improve ride comfort, while suspension deflection and tire deflection are assumed within an acceptable level by applying saturated control input in order to make sure the car safety.
III. CONTROL DEVELOPMENT

Utilizing the regulation error in (4) and substituting the filtered error in (5) yield the open-loop dynamics as follows

\[
\dot{r} = C \dot{x} - kr + \alpha \left( \cosh^{-2}(e) - 1 \right) \dot{e} + \tanh(e) - y \tanh(\epsilon(t)) + \dot{e}_z,
\]

where \( C \triangleq \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \) is a known constant vector, \( e_z(t) \) introduced in (4) is defined as the finite integral over the delay-time interval \([t - \tau, t]\) of the past control inputs based on the Leibniz-Newton formula as

\[
e_z = \int_{t-\tau}^{t} \nu(\xi)d\xi.
\]

In (9), the term \( \nu(t) \in \mathbb{R} \) is the fixed proportion of the control input \( u(t) \), which is designed based on the loop of the control development and stability analysis as follow

\[
\nu \triangleq \hat{\Omega} u = ktanh(\epsilon(t)) + \gamma tanh(e(t)),
\]

where a constant feedforward estimate \( \hat{\Omega} \in \mathbb{R} \) is defined as \( \hat{\Omega} \triangleq CG \), and the constant estimate \( \hat{G} \in \mathbb{R}^3 \) is an invertible best-guess vector of the uncertain input matrix \( G \). A feature of the controller in (10) needed to be emphasized is that this control law is upper bounded by the adjustable control gain \( k \), as \(|u| \leq \hat{\Omega}^{-1} \sqrt{k + 2} \leq \hat{\Omega} \). Hence, this control amplitude can be limited below a known bound. To facilitate the subsequent development, the auxiliary function \( \Omega(x_3) \in \mathbb{R} \) is defined as

\[
\hat{\Omega} \triangleq \hat{\Omega} \hat{\Omega}^{-1},
\]

which can be separated into components as

\[
\hat{\Omega} = 1 + \Delta(x_3).
\]

In (11), the unknown function \( \Omega(x_3) \in \mathbb{R} \) is defined as \( \Omega(x_3) \triangleq CG(x_3) \).

Assumption 4: The subsequent development is based on the assumption that the constant estimate \( \hat{\Omega} \) introduced in (10) can be selected such that

\[
|\Delta| \leq \varepsilon,
\]

where \( \varepsilon \in \mathbb{R} \) is a known positive constant.

To facilitate the subsequent analysis, (1), the time derivative of \( r \) can be extracted by using (4), (5), (8), and (9) as

\[
\dot{r} = C \dot{x} - kr + \alpha \left( \cosh^{-2}(e) - 1 \right) \dot{e} + \tanh(e) - y \tanh(\epsilon(t)) + \nu - \nu_t.
\]

Utilizing (10) - (12), the closed-loop error system is abbreviated as

\[
\dot{r} = N_d + \tilde{N} - kr + \nu_k(\epsilon(t)) - tanh(\epsilon(t)) + \Delta v_t,
\]

where the two auxiliary functions \( N_d(t), \tilde{N}(x, \omega) \in \mathbb{R} \) are defined as

\[
N_d \triangleq CH \dot{x} + C d, \quad \tilde{N} \triangleq \alpha \left( \cosh^{-2}(e) - 1 \right) \left( r - \alpha \tanh(e) - \tanh(e_t) - e_z \right) - \tanh(\epsilon(t)) + \nu (\epsilon(t)) + CF.
\]

According to the Macpherson suspension dynamics provided details in [29], the unknown functions \( g_1(x_3), g_2(x_3), h_1(x_3), h_2(x_3) \) are assessed to be upper bounded by known constants, and the functions \( f_1(x), f_2(x) \) can be upper bounded by state-dependent terms. Hence, by using the suspension dynamics in [29], Assumption 3, the properties of hyperbolic tangent function in (3), the error definitions in (4) and (5), the expression in (16) can be evaluated to be bounded by an upper bound as

\[
\|N_d\| \leq \rho (\|\|) \|z\| + \varepsilon_1,
\]

where the function \( \rho (\|\|) \in \mathbb{R} \) is a globally invertible, nondecreasing positive function. Using Assumption 1 and the suspension dynamics in [29], the following bound can be developed

\[
\|N_d\| \leq \varepsilon_2,
\]

where \( \varepsilon_1, \varepsilon_2, i = 1, 2 \) are determinable positive constants.

In the subsequent stability analysis, \( D \subseteq \mathbb{R}^n \) is the open and connected set defined as \( D \triangleq \left\{ y \in \mathbb{R}^n \right\} \|y\| \leq \max(1, \sqrt{\frac{1}{\varepsilon}}) \right\} \right\} \), where \( \varepsilon \) is introduced in (13), \( \lambda, k_2 \) are subsequently defined. Let \( y \in \mathbb{R}^5 \) be defined as

\[
y \triangleq \left[ e \tanh(\epsilon(t)) \right] r \sqrt{\hat{\Omega}} \sqrt{P}.
\]

where \( Q, P \in \mathbb{R}^+ \) are the auxiliary Lyapunov-Krasovskii functionals defined as [24]

\[
Q \triangleq \frac{\varepsilon}{2} \int_{t-\tau}^{t} |u(\xi)|^2 d\xi,
\]

\[
P \triangleq \omega \int_{t-\tau}^{t} \left( \int_{\xi}^{t} |u(\xi)|^2 d\xi \right) ds,
\]

where the constant \( \omega \in \mathbb{R}^+ \) is known and positive.

IV. STABILITY ANALYSIS

Theorem 1: Given the input- delayed nonlinear Macpherson active suspension system in (1), the saturated controller given in (10) guarantees semi-globally uniformly ultimately bounded regulation of the vertical displacement of the chassis \( z_s \), provided the adjustable control gains \( \alpha, \gamma, k \) are selected according to the following sufficient conditions

\[
k_1 > \frac{\varepsilon}{2}, \quad \alpha > \frac{\gamma^2}{4} + 2(k + 2)(\omega + \frac{\varepsilon}{2}),
\]

\[
\gamma > k + 2(\omega + \frac{\varepsilon}{2}), \quad \omega \phi^2 > 2 \tau,
\]

where the positive constant \( \psi \in \mathbb{R}^+ \) is known and adjustable.

Proof: Let consider the Lyapunov function candidate function \( V_L : D \rightarrow \mathbb{R}, \) defined as

\[
V_L \triangleq \ln(\cosh(\epsilon(t))) + \frac{1}{2} \tau^2 + \frac{1}{2} \tanh^2(\epsilon(t)) + Q + P.
\]

The Lyapunov function candidate in (24) is a Lipschitz continuous positive definite function, which can be bounded using (3) as:

\[
\phi_1 (\|y\|) \leq V_L \leq \phi_2 (\|y\|).
\]
Based on (3) and (24), \( \phi_1, \phi_2 : \mathcal{D} \to \mathbb{R} \) in (25) are the continuous, positive-definite, strictly increasing functions, defined as \( \phi_1(y) \triangleq \frac{1}{2} \ln (\cosh (||y||)) \), \( \phi_2(y) \triangleq ||y||^2 \).

Using (4), (5), (14), (21), (20) and taking the time derivative of (24) yield
\[
\dot{V}_L = \tanh(e) \left( r - \alpha \tanh(e) - \tanh(e_f) - r_e \right) + r \left( N_d + \tilde{N} - kr + k \tanh(e_f) - \tanh(e) + \Delta v r \right) \\
+ \tanh(e_f) \left( - kr + \tanh(e_f) - \gamma \tanh(e_f) \right) + \alpha \tau ||v||^2 \\
- \omega \int_{\tau}^{t} ||v(\xi)||^2 d\xi + \frac{\epsilon}{2} \left( ||v||^2 - ||v_r||^2 \right) .
\]
\[
(26)
\]
Cancelling common terms and using (17), (18), the time derivative in (26) can be assessed as
\[
\dot{V}_L \leq - \alpha \tanh(e)^2 - \gamma \tanh(e_f)^2 + \tanh(e)||e||_2 \\
- k ||r||^2 + r ||\rho ||(||z||)||z|| + \xi_1 + \xi_2 + \Delta ||v_r|| + \omega \tau ||v||^2 \\
- \omega \int_{\tau}^{t} ||v(\xi)||^2 d\xi + \frac{\epsilon}{2} \left( ||v||^2 - ||v_r||^2 \right) .
\]
\[
(27)
\]
Several cross terms in (27) can be upper bounded by using Assumption 4, Young’s inequality and (10) as
\[
||\Delta|| \cdot ||r|| \cdot ||v||_2 \leq \frac{\epsilon}{2} ||r||^2 + \frac{\epsilon}{2} ||v||_2^2 , \\
|\tanh(e)||e||_2 \leq \frac{\epsilon}{4} \tanh(e)^2 + \frac{1}{\epsilon^2} ||e||_2^2 ,
\]
where the constant \( \epsilon \) is known and
\[
||v||^2 \leq k^2 ||\tanh(e_f)||^2 + 4 ||\tanh(e)||^2 \\
+ 4k ||\tanh(e_f)|| ||\tanh(e)|| \\
\leq \left( k^2 + 2k \right) ||\tanh(e_f)||^2 + \left( 2k + 4 \right) ||\tanh(e)||^2 .
\]
\[
(29)
\]
Furthermore, the following integral can be upper bounded by using the Cauchy-Schwarz inequality, as (18), (20)
\[
||e||_2^2 \leq \tau \int_{\tau}^{t} ||v(\xi)||^2 d\xi .
\]
\[
(30)
\]
The time derivative expression in (27) can be further upper bounded by using (29), (30) and completing the squares as
\[
\dot{V}_L \leq - \left( \alpha - \frac{\epsilon^2}{4} - 2 \left( k + 2 \right) \left( \omega \tau + \frac{\epsilon}{2} \right) \right) ||\tanh(e)||^2 \\
- \left( \gamma - k \left( k + 2 \right) \left( \omega \tau + \frac{\epsilon}{2} \right) \right) ||\tanh(e_f)||^2 \\
- \left( k_1 - \frac{\epsilon}{2} \right) ||r||^2 - \left( \frac{\epsilon}{2 \tau} - \frac{1}{\epsilon^2} \right) ||e||_2^2 \\
- \frac{\epsilon}{2} \int_{\tau}^{t} ||v(\xi)||^2 d\xi + \frac{\epsilon^2 ||z||^2 + \xi_1 + \xi_2}{4k} + \frac{\xi_1 + \xi_2}{4k} ,
\]
where the control gain \( k \) introduced in (5) is separated into the summation of adjustable constants \( k_1, k_2, k_3 \in \mathbb{R}^+ \), as
\[
k = k_1 + k_2 + k_3 .
\]
The inequality (18), (20)
\[
\int_{\tau}^{t} \left( \int_{\tau}^{t} ||v(\xi)||^2 d\xi \right) ds \leq \tau \sup_{s \in [\tau, t]} \left[ \int_{s}^{t} ||v(\xi)||^2 d\xi \right] \\
= \tau \int_{\tau}^{t} ||v(\xi)||^2 d\xi
\]
can be exploited to upper bound the derivative (31) as
\[
\dot{V}_L \leq - \left( \alpha - \frac{\epsilon^2}{4} - 2 \left( k + 2 \right) \left( \omega \tau + \frac{\epsilon}{2} \right) \right) ||\tanh(e)||^2 \\
- \left( \gamma - k \left( k + 2 \right) \left( \omega \tau + \frac{\epsilon}{2} \right) \right) ||\tanh(e_f)||^2 \\
- \left( k_1 - \frac{\epsilon}{2} \right) ||r||^2 - \left( \frac{\epsilon}{2 \tau} - \frac{1}{\epsilon^2} \right) ||e||_2^2 \\
- \frac{\epsilon}{2} \int_{\tau}^{t} \left( \int_{\tau}^{t} ||v(\xi)||^2 d\xi \right) ds - \frac{\epsilon}{4} \int_{\tau}^{t} ||v(\xi)||^2 d\xi \\
+ \frac{\epsilon^2 ||z||^2 + \xi_1 + \xi_2}{4k} + \frac{\xi_1 + \xi_2}{4k} .
\]
\[
(32)
\]
Let \( \chi(e, \gamma, r, e, Q, P) \in \mathbb{R}^6 \) be defined as \( \chi \triangleq \left[ \tanh(e) \tanh(e_f) r e \right] \sqrt{Q} \sqrt{P} \right)^T \), and the inequalities \( ||\chi|| \geq ||z|| \), then (32) can be upper bounded as
\[
\dot{V}_L \leq - \frac{\lambda}{2} ||\chi||^2 - \left( \frac{\lambda}{2} - \frac{\rho^2 (||z||)}{4k} \right) ||\chi||^2 + \frac{\xi_1 + \xi_2}{4k} ,
\]
\[
(33)
\]
where the definition of the auxiliary constant \( \lambda \in \mathbb{R}^+ \) as is
\[
\lambda \triangleq \min \left\{ k_1 - \frac{\epsilon}{2}, \alpha - \frac{\epsilon^2}{4} - 2 \left( k + 2 \right) \left( \omega \tau + \frac{\epsilon}{2} \right), \frac{\omega}{4} \right\} .
\]
If the sufficient conditions in (22) are satisfied, then \( \lambda > 0 \). The conditions in (22) are solvable for a sufficiently small \( \tau \). Using (20), (30) and the definitions of \( z, y \) in (7), (19) to obtain the inequality \( ||z|| \leq \max \left( 1, \sqrt{\frac{\epsilon}{4 \tau}} \right) ||y|| \), provided \( \gamma \in \mathcal{D} \forall \gamma \in [\tau, t] \), and utilizing the fact that \( ||\chi||^2 \geq \tanh^2 (||y||) \), the expression (33) can be rewritten as
\[
\dot{V}_L \leq - \phi_3 (||y||) + \frac{\xi_1 + \xi_2}{4k} ,
\]
\[
(34)
\]
where \( \phi_3 : \mathcal{D} \to \mathbb{R} \) is a strictly increasing nonnegative function, defined as \( \phi_3 (||y||) \triangleq \frac{\epsilon^2}{4} \tanh^2 (||y||) \). Given (25) and (34), \( y(\cdot) \), (via standard linear analysis and the definition of \( y(\cdot) \)), \( e(\cdot) \) and \( r(\cdot) \) are uniformly ultimately bounded (25) in the sense that
\[
|e(\tau)| \leq ||y(\tau)|| < \Upsilon, \quad \forall \tau \geq T, \quad \forall \|y(0)\| .
\]
(35)
provided the sufficient conditions in (22) and (23) are satisfied. In (35), \( \Upsilon \) denotes the radius of a ball enclosing the
vertical displacement $z_r$, selected according to Corless and Leitmann (1981)

$$
\gamma > \left( \phi_1^{-1} \circ \phi_2 \right) \left( \phi_3^{-1} \left( \frac{\zeta_1 + \zeta_2}{4k_3} \right) \right).
$$

(36)

The dimension of the ultimate bound in (36) can be forced smaller by choosing $k_3$ bigger, where $T \left( T, ||y(0)|| \right) \in \mathbb{R}$ is a positive constant that expresses the ultimate time to approach the ball (25]).

\section{V. SIMULATION RESULTS}

Several Matlab numerical simulations are executed to examine the controller in (10) for a quarter-car model with dynamics provided in [29], and the Macpherson suspension dynamic and kinematic parameters indicated in [30, Table 1]. In addition, the friction disturbance $d$ is assumed applying to (1), as $d = \left[ 0 \ d_1(t) \ 0 \ d_2(t) \right]$ where $d_1 = 5.3x_2 + 8.45\tanh(x_2)$ and $d_2 = 1.14x_4 + 2.35\tanh(x_4)$ represent the summation of the static and the dynamic frictions. The performance of the proposed controller executed by Matlab numerical simulations which is taken into account both the delay and limited bound issues of the actuators is compared with both a passive suspension system and an active suspension with PID controller. The performance of the proposed controller is evaluated by considering both time and frequency domain responses with various delay time values varying from 1 ms to 200 ms, when the wheel is disturbed by two types of road profiles: bump and sinusoidal excitations.

\subsection{A. TIME RESPONSE}

The simulations in the time domain consider the vehicle driving at a steady horizontal speed of $V = 50mph$, excited by a road bump, as $z_r = |z_r| \left[ 1 - \cos(\omega_r(t - 0.5)) \right]$ if $0.5 \leq t \leq T + 0.5$ and $z_r = 0$ otherwise, where $|z_r| = 5cm$ or $|z_r| = 7.5cm$ is the half bump height, $\omega_r = 2\pi V/D_r = 2\pi /T$ is the frequency of road excitation, $D_r$ is the width of the bump $D_r = 10m$.

The active control force is selected to be limited as $|u_a| \leq \pi = 2500N$, which is also the maximum damping force in [17]. The gains of the proposed controller are selected as $\alpha = 20$, $k = 250$, $\gamma = 15$ and the estimate of the input matrix $\hat{G} = \left[ 0.1 \ 0.3 \ 0.2 \ 0.04 \right]^T$. Assumption 3 indicates that the constant estimate $\hat{G}$ must be picked sufficiently close to $G(x_3)$ to satisfy the condition (13). Simulation results exhibit the robustness of the introduced controller with respect to the difference between $\hat{G}$ and $G(x_3)$, specially with the above overestimated value $\hat{G}$. The initial conditions of the suspension and controller were chosen as follows, $x(0) = 0$, $e_f(0) = 0$ and $e_r(0) = 0$. The Lyapunov stability analysis presents conservative sufficient gain conditions. The control gains were tuned by the trial and error approach.

The performance of the proposed active suspension is compared against the corresponding passive suspension, and the active suspension controlled by a PID controller, whereas the gains of the PID controller were tuned to acquire the best possible performance in the given saturation bounds. Finally, the control gains of the PID controller are $K_p = 2400$, $K_d = 1550$. $K_i = 100$.

The criteria of evaluation for the suspensions consist of the body displacement and acceleration expressing for ride comfort and the suspension stroke, tire deflection for road holding, whereas the suspension stroke and tire deflection are determined as $y_{sd}(t) = \sqrt{l_0^2 + l_b^2 - 2l_bl_d\cos(\alpha' - \theta)} - \sqrt{l_0^2 + l_b^2 - 2l_bl_d\cos(\alpha')} + y_{sd} = z_s + l_c \cdot (\sin(\theta - \theta_0) - \sin(\theta_r))$.

Several simulation results were obtained using various time delays and various bump heights. The control performances are examined for the short and long input delay durations. The time responses of the chassis displacement and acceleration of three simulated suspensions under two driving conditions depicted in Figures 1 and 2: case 1: subjecting to a 10-cm-high bump excitation ($|z_r| = 5cm$) and 5ms input delay ($\tau = 5ms$) or case 2: subjecting to a 15-cm-high bump excitation ($|z_r| = 7.5cm$) and 200ms input delay ($\tau = 200ms$). Both active suspensions (using proposed method and PID control methods) are superior over the passive one in ride comfort under the same driving conditions. The rotation angular of control arm,
the suspension stroke, and tire deflection are also presented respectively in Figs. 3, 4, and 5 to evaluate the road holding criteria, and the control inputs of two controlled suspensions are depicted in Fig. 6. The results show that the tire deflection in the suspension with saturated controller is considerably similar in comparison with the passive suspension and the active one with PID controller. The proposed saturated controller for the active suspension significantly improves the ride quality, but still keeps the rotation angular of control arm, the deterioration of the suspension deflection and tire deflection within an acceptable level in comparison with the passive one and the active one with PID control. However, to obtain more accurate comparison, further investigation in frequency domain is needed.

B. FREQUENCY RESPONSE
In the frequency domain simulations, the vertical acceleration of the vehicle body, the suspension and tire deflection responses to different road disturbances having the frequency varying from 2Hz to 30Hz are determined, then the variance gains of the corresponding measure of interest are computed.
FIGURE 6. Control input of active suspension using proposed method and active suspension using PID, subject to a 10-cm-high bump excitation and 5ms input delay (in left side) or a 15-cm-high bump excitation and 200ms input delay (in right side).

FIGURE 7. Variance gain of the transfer function from the sinusoid excitation $z_r$ to the chassis displacement $z_s$ over the considered frequency range 2Hz - 30Hz, subject to a 10-cm-high bump excitation and 5ms input delay (in left side) or a 15-cm-high bump excitation and 200ms input delay (in right side).

using the definition given in [27], [28] as

$$G_z(j\omega) = \sqrt{\int_0^{2\pi N/\omega} \int_0^{2\pi N/\omega} \frac{z_s^2 dt}{z_r^2 dt}},$$

where $z$ represents the performance measure of interest which is the vertical chassis acceleration $\ddot{z}_s$, the suspension deflection $y_{sd}$ or tire deflection $y_{td}$, respectively. In meanwhile, the suspension is subjected to various constant input delays, where $\tau = 5 ms$ for short delay duration and $\tau = 200 ms$ for long delay duration, and the road excitation is selected as the sinusoid $z_r = |z_r| \sin(\omega t)$ with $t \in [0, 2\pi N/\omega]$ where $N$ is a big enough integer for the system to reach the steady state, practically $N$ can be chosen as $N = 15$, and $\omega = 2\pi f$ with $f \in [2, 30] Hz$; $|z_r|$ is selected to be 10cm for the medium amplitude and 15cm for the high amplitude. With each frequency $f$ and each input delay case, the corresponding output signals are recorded to calculate the variance gains. The vertical displacement $z_s$, the chassis acceleration $\ddot{z}_s$, the suspension deflection $y_{sd}$ and the tire deflection $y_{td}$ response to the road disturbance $z_r$ in the frequency range 2Hz - 30Hz and two different input delays representing the ride comfort, rattle space and road-holding performances showed in Figs. 7, 8, 9 and 10, respectively.

It's worthy to note that in all simulations, the system is stable under both short and long input delays. Moreover, with regard to the ride comfort aspect, the transfer function from the road profile $z_r$ to the body displacement and acceleration caused by the proposed method have variance gains smaller than the corresponding gains resulted from the remaining suspensions. In comparison between graphs of variance gains in Figs. 7 and 8, the active suspension applying the saturated control approach has better ride comfort (i.e. smaller gains) than the passive suspension at almost all frequencies, and than the active suspension with PID control at almost all frequencies including the human sensitive range (4Hz-8Hz). The requirement of road-holding performance was not considered directly in control design step, but is testified in simulation section by comparing the peak values of the transfer functions from the road excitation to the suspension stroke and tire deflection through various simulation results depicted in Figs. 9 and 10, respectively. The peak value of the tire deflection for the active suspension applying the proposed saturated control is little bigger than the peak value in the passive one and the active one with PID controller for both cases of input delays. A similar comparison is also completed for the suspension stroke. Hence, comparing variance gains of the suspension stroke and tire deflection with regards to the rattle space and road-holding performances, the proposed
approach shows the equivalent achievement throughout the examined frequency span. In summary, the proposed saturated control approach effectively enhances the ride comfort performance, while the suspension deflection and tire deflection are retained at an admissible level to ensure the rattle space limit and the car safety, even with different delay times.

VI. CONCLUSION
A continuous saturated controller is constructed for input delay Macpherson active suspension systems, consisting of nonlinear uncertainties, additive bounded disturbances and subject to constant input delay. The bound on the control input is guaranteed by using the hyperbolic tangent functions and can be adjusted by adjusting the control gains, which is chosen based on the sufficient gain conditions. Lyapunov stability analysis exploiting LK functionals is used to prove the saturated controller guaranteeing uniformly ultimately bounded regulation of the vertical displacement and velocity of the vehicle body despite uncertainties in the dynamics, exogenous disturbances and the input delay issue. Simulation results exhibit the advantage.
in the ride comfort of the proposed active suspension system.

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