Interpolation inequality and some applications

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Abstract
We investigate explicit universal estimate of finite Morse index solutions to polyharmonic equations. Differently to previous works [3, 7, 8, 14], propose here a direct proof using a new interpolation inequality and a delicate bootstrap argument under large superlinear and subcritical growth conditions to show that the universal constant grows as a power function of the Morse index. Also, our interpolation inequality allows us to provide local \( L^p - W^{2,r,p} \) estimate.

Keywords: Interpolation inequality, Universal estimate, Morse index, Pohozaev identity, Bootstrap argument.

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1. Introduction

1.1. Interpolation inequalities.

Let \( n, r \geq 2 \) be two integer numbers and \( p \geq 2 \) a real number. We designate by \( \Omega \) an open subset of \( \mathbb{R}^n \) and \( B_R \) the ball of radius \( R > 0 \) centered at the origin. Let \( j = (j_1, j_2, \ldots, j_n) \) be a multi index, the weak \( j^{th} \) partial derivative and the magnitude of the \( q^{th} \) gradient of \( u \in W^{r,p}_{\text{loc}}(\Omega) \) are respectively defined in \( \Omega \) by

\[
D^j u = \frac{\partial^{|j|} u}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}, \quad 1 \leq |j| \leq r \quad \text{and} \quad |\nabla^q u| = \left( \sum_{|j|=q} |D^j u|^p \right)^{\frac{1}{p}}, \quad 1 \leq q \leq r.
\]

(1.1)

Let \( \varepsilon \in (0, 1) \) and \( 1 \leq q \leq r-1 \). From an obvious dilation argument, the standard interpolation inequality \([1]\) implies

\[
R^{(q-r)} \int_{B_R} |\nabla^q v|^p \leq \varepsilon \int_{B_R} |\nabla^r v|^p + C_\varepsilon R^{q-p} \int_{B_R} |v|^p, \quad v \in W^{r,p}(B_R).
\]

(1.2)

where \( C = C(n, p, r) \) is a positive constant. According to (1.2), one can establish the following weighted interpolation inequality (see \([14, 15, 20]\))

\[
R^{(q-r)} \Phi_q'(v) \leq \varepsilon \Phi_r'(v) + C_\varepsilon R^{q-p} \int_{B_R} |v|^p,
\]

(1.3)

where \( \Phi_q \) is a family of weighted semi-norms defined by

\[
\Phi_q(v) = \left( \sup_{0 < \alpha < 1} (1 - \alpha)^q \int_{B_R} |\nabla^q v|^p \right)^{\frac{1}{p}}, \quad 0 \leq q \leq r.
\]

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Inequality (1.3) together with the following cut-off function \( \psi = \psi_{a,R} \in C_c^1(\mathbb{R}^n), a \in (0, 1) \)
\[
\psi(x) = \exp\left(\frac{|x|^2}{aR} - \frac{x}{R} - \alpha'\right)
\]
if \( aR < |x| < \alpha'R \), \( \psi \equiv 1 \) if \( |x| \leq aR \) and \( \psi \equiv 0 \) if \( |x| \geq \alpha'R \) where \( \alpha' = \frac{1 + \alpha}{2} \).

are quite useful to provide the energy estimate which is essential to classify stable at infinity weak solution of the \( p \)-polyharmonic equations [15] (see also [14, 20] for \( p = 2 \)). The reader may consults [1, 11, 10] for further applications of (1.3). When \( p \geq 2 \) we introduce a new interpolation inequality which will be more relevant in providing integral estimates in various contexts. In particular it will be helpful to establish explicit universal estimate and local \( L^p - W^{2/p} \)-estimate (see Appendix C). Moreover, our inequality relies on a more general cut-off function related to two bounded open subset \( \omega \) and \( \omega' \) such that \( \overline{\omega} \subset \omega' \subset \overline{\omega'} \subset \Omega \). Precisely, denote \( d = \text{dist}(\omega, \Omega \cap \omega') \), we have

**Lemma 1.1.** There exist \( \psi \in C_c^\infty(\omega') \) and a positive constant \( C \) depending only on \( n, p, k, m \) such that
\[
\begin{cases}
0 \leq \psi \leq 1 \text{ and } \psi \equiv 1 \text{ if } x \in \omega;
|\nabla^k \psi(x)|^p \leq Cd^{-k}p, \forall x \in \omega' \text{ and } k \in \mathbb{N}.
\end{cases}
\tag{1.4}
\]

Moreover, we have
\[
|\nabla^k \psi^m| \leq Cd^{-k}p^{m-k}, \forall x \in \omega' \text{ and } m > k.
\tag{1.5}
\]

As usual, we used the power function \( \psi^m, m > r \) as a cut-off function (see [6, 21, 14, 15]). Let \( (q, k) \in \mathbb{N}^+ \times \mathbb{N}^+ \), \( q + k = r \), our main first result reads as follows.

**Lemma 1.2.** For every \( 0 < \varepsilon < 1 \), there exists a positive constant \( C = C(n, r, p, m) \) such that for any \( u \in W^{r,p}_{\text{loc}}(\Omega) \), we have
\[
\int_\omega |\nabla^q u|^p |\nabla^k (\psi u)|^p \leq Cd^{-kp} \int_\omega |\nabla^q u|^p |\nabla^k (\psi u)|^p \leq \varepsilon \int_\omega |\nabla^q u|^p |\nabla^k (\psi u)|^p + C\varepsilon^{1-p'} d^{-pr} \int_\omega |u|^p |\nabla^{r-m} u|^r.
\tag{1.6}
\]

Consequently,
\[
\int_\omega |\nabla^q u|^p |\nabla^k u|^p \leq 2 \varepsilon \int_\omega |\nabla^q u|^p |\nabla^k u|^p + C\varepsilon^{1-p'} d^{-pr} \int_\omega |u|^p |\nabla^{r-m} u|^r.
\tag{1.7}
\]

and
\[
\int_\omega |\nabla^q u|^p |\nabla^k \psi u|^p \leq Cd^{-kp} \int_\omega |\nabla^q u|^p |\nabla^k \psi u|^p \leq C\varepsilon^{1-p'} d^{-pr} \int_\omega |u|^p |\nabla^{r-m} u|^r.
\tag{1.8}
\]

1.2. Explicit universal estimate.

Consider the following polyharmonic problem:
\[
(-\Delta)^j u = f(x, u), \text{ in } \Omega.
\tag{1.9}
\]

Here, \( \Omega \) is a proper domain of \( \mathbb{R}^n, u \in C^{2j}(\Omega) \), \( f \) and \( f' = \frac{\partial f}{\partial s} \) belong to \( C(\Omega \times \mathbb{R}) \). The associated quadratic form of (1.9) is defined by
\[
Q_u(h) = \int_\Omega |D_j h|^2 - \int_\Omega f'(x, u) h^2, \text{ } h \in C^1_c(\Omega),
\tag{1.10}
\]
where \( D_j h = \nabla \Delta^{j-1} h, |D_j h|^2 = |\nabla \Delta^{j-1} h|^2 \) if \( r = 2j - 1 \) and \( D_j h = \Delta^j h, |D_j h|^2 = (\Delta h)^2 \) if \( r = 2j, j \in \mathbb{N}^+ \). The Morse index of \( u \), denoted by \( \text{ind}(u) \) is defined as the maximal dimension of all subspaces \( V \) of \( C^1_c(\Omega) \) such that \( Q_u(h) < 0, \forall h \in V \setminus \{0\} \). In previous works [7, 8, 14], universal estimate has been established from blow-up technique and some available Liouville-type theorems classifying finite Morse index solutions (see also the case of positive solutions in [9, 19, 21, 22]). However, this procedure fails to derive explicit estimate and requires a restrictive asymptotic
Thanks to Lemma 1.2, we establish explicit universal estimate under the following large superlinear and subcritical growth conditions:

There exist \( s_0 > 0, c_1 > 1 \) and \( 1 < p_1 \leq p_2 < \frac{n + 2r}{n - 2r} \) such that for all \( (x, s) \in \Omega \times \mathbb{R} \setminus [-s_0, s_0], \)

\[(h_1) \quad \text{(Super-linearity)} \quad f'(x, s)s^2 \geq p_1 f(x, s)s; \]

\[(h_2) \quad \text{(Subcritical growth)} \quad (p_2 + 1)F(x, s) \geq f(x, s)s, \text{ where } F(x, s) = \int_{s_0}^s f(x, t)dt; \]

\[(h_3) \quad |(\nabla, F)(x, s)| \leq c_1(F(x, s))^s, \text{ for all } (x, s) \in \Omega \times \mathbb{R}; \]

\[(h_4) \quad |f'(x, s)| \leq c_1, \text{ for all } (x, s) \in \Omega \times [-s_0, s_0], \quad |f(x, 0)| \leq c_1 \text{ and } \pm f(x, \pm s_0) \geq \frac{1}{c_1} \text{ for all } x \in \Omega. \]

When \( f(x, s) = f(s) \) the above assumptions are reduced to \((h_1)-(h_2)\) (with \( f(\pm s_0) > 0 \)) and obviously are weaker than \((h_0)\). Let \( K \in C^1(\Omega) \) be a positive function such that \( K^{-1} |\nabla K| \in L^\infty(\Omega) \), and \( 1 < p_1 < p_2 < \frac{n + 2r}{n - 2r} \) and denote \( s_\pm = \max(s, 0), s_\mp = \max(-s, 0) \). The nonlinearity \( f(x, s) = K(x)(s_\pm^{p_1} - s_\mp^{p_2}) \) satisfies \((h_1)-(h_2)\) but violates \((h_0)\).

Let \( \alpha \in (0, 1) \), \( y \in \Omega \). Denote \( \delta_1 = \text{dist}(y, \partial \Omega), \delta_2 = \inf(\alpha, \delta_1) \). We have

**Theorem 1.1.** Assume that \( f \) satisfies \((h_1)-(h_4)\). Then, there exist \( \alpha_0 \in (0, 1), \gamma_1 > 0, \gamma_2 > 0 \) and a positive constant \( C = C(\alpha_0, n, r, p_1, p_2, s_0, c_1) \) independent of \( \Omega \) such that for any finite Morse index solution \( u \) of \((1.9)\) and for every \( \alpha \in (0, \alpha_0) \), we have

\[
\sum_{j=0}^{2r-1} d_j^{|\nabla^j u(y)|} \leq C(1 + \delta_1)^{\gamma_1} d_1^{\gamma_1}, \quad \forall y \in \Omega. \tag{1.11}
\]

Precisely, if \( \frac{p_2 + 1}{p_2} < \frac{n}{2r} \) then \( \gamma_1 = \frac{4r^2(p_1 + 1)p_2}{(p_1 - 1)(2r(p_2 + 1) - n(p_2 - 1))} \) and \( \gamma_2 = \gamma_1 + \frac{2r(p_2 + 1)}{2r(p_2 + 1) - n(p_2 - 1)}. \)

**Remark 1.1.** Denote \( \Omega_\alpha = \{ y \in \Omega, \delta_1 \geq \alpha \}, \) \( \alpha \in (0, \alpha_0) \). As a direct consequence of \((1.11)\), we have

\[
||u||_{C^{2r-1}(\Omega_\alpha)} \leq C\alpha^{1-2r-\gamma_1}(1 + \delta_1)^{\gamma_1} \text{ and if } y \in \Omega \setminus \Omega_\alpha, \text{ then } \sum_{j=0}^{2r-1} |\nabla^j u(y)| \leq C(1 + \delta_1)^{\gamma_1} d_1^{1-2r-\gamma_1}. \]

To prove Theorem 1.1, we make use of Lemmas 1.1 and 1.2 to obtain a first integral estimate on a ring around \( y \) (see Section 3). By virtue of a variant of the Pohozaev identity, we extend this estimate to a ball centered at \( y \) as follows

\[
d_1^{n-1} \int_{B(y, d_1)} |f(x, u)|^{\frac{2r+1}{r}} \leq C \left(\frac{1 + \delta_1}{d_1}\right)^{\frac{2r+1}{n+1}} d_1^{\gamma_1+1}, \quad \forall y \in \Omega. \tag{1.12}
\]

As \( p_2 \) is subcritical, we used a delicate boot strap argument to end the proof of Theorem 1.1. Note that estimate \((1.12)\) holds when \( \frac{n + 2r}{n - 2r} < p_1 \leq p_2 \), but it is not clear which procedure would be helpful to derive \((1.11)\). Also, inequality \((1.12)\) could be extended to solutions of the \( p \)-polyharmonic equation. However, we do not dispose to any \( L^p \)-regularity result to start the boot strap procedure. Regarding the case of bounded domain, explicit \( L^p \)-bounds of

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1 If \( \Omega \) is an unbounded domain we assume in addition that \( K(x) \geq c_0 > 0 \) for all \( x \in \Omega. \)

2 In the statement of Theorem 1.1 we used \((1.1)\) with \( p = 1. \)

3 Note that the boot strap argument requires a subcritical growth.
finite Morse index solutions of the second order Dirichlet boundary-value problem has been obtained in \cite{12, 13, 22} under similar assumptions of (h1)-(h4) which improve the a priori $L^\infty$-estimates stated in \cite{3, 17}. Also, in \cite{16} the authors examined the influence of the type boundary conditions involving the biharmonic and triharmonic problems to provide similar explicit $L^\infty$-bounds. The general higher order case $r \geq 4$, is more difficult since some needed local interior estimates near the boundary are so hard to achieve.

This paper is organized as follows: Section 2 is devoted to the proofs of Lemmas \ref{lemma1} and \ref{lemma2}. In section 3, we give the proof of Theorem \ref{theorem1}. In appendix C, we provide the proof of local $L^p$-$W^{2,p}$ estimate.

In the following, $C$ (respectively $C_r$) denotes always generic positive constants depending only on $(n, p, r, k, m)$ (respectively on $(\ell, n, p, r, m)$) which could be changed from one line to another.

2. Proofs of Lemmas \ref{lemma1} and \ref{lemma2}

Proof of Lemma \ref{lemma1} Set $\omega_d = \{x \in \Omega, \text{dist}(x, \omega) < \frac{d}{4}\}$, where $d = \text{dist}(\omega, \Omega \setminus \omega')$, we have $\omega \subset \omega_d \subset \omega'$. Let $h = \chi_{\omega_d}$ be the indicator function of $\omega_d$ and $g \in C^\infty_c(\mathbb{R}^n)$ a nonnegative function such that $\text{supp}(g) \subset B_1$ and $\int_{\mathbb{R}^n} g(x)dx = 1$. Set

$$g_d(x) = \left(\frac{8}{d}\right)^n g \left(\frac{8x}{d}\right)$$

and $\psi(x) = \int_{\mathbb{R}^n} g_d(y)h(x-y)dy = \int_{B_8^c} g_d(y)h(x-y)dy$.

We have $0 \leq \psi \leq 1$ and $\text{supp}(\psi) \subset \omega_d + B_{\frac{d}{4}} \subset \omega'$ (see proposition 4.18 in \cite{4}). Since $\omega + B_{\frac{d}{4}} \subset \omega_d$, then $\psi(x) = 1$ if $x \in \omega$. Also, $\psi \in C^\infty_c(\mathbb{R}^n)$ with $D^i\psi(x) = \int_{B_8^c} D^i g_d(y)h(x-y)dy$ (see proposition 4.20 in \cite{4}). Therefore,

$$|D^i\psi(x)| \leq \int_{B_8} |D^j g_d|dy \leq \left(\frac{8}{d}\right)^{|j|} \int_{B_8} |D^j g(y)|dy \leq Cd^{-|j|}.$$

Now, from (1.1), one can see that $|\nabla^l \psi(x)|^p \leq Cd^{-lp}$, $\forall x \in \omega' \setminus \omega$ and $k \in \mathbb{N}$, where is $C = C(n, k, p) > 0$.

Proof of (1.5). The proof will be done by working inductively with respect $k \geq 1$. Observe that (1.5) is an immediate consequence of (1.4) if $k = 1$. Assume now that the following inequality holds for all $1 \leq l \leq k$ and $m > 1$

$$|\nabla^l \psi^m| \leq Cd^{-l}\psi^{m-1}, \quad \forall x \in \omega'. \quad (2.1)$$

Let $m > k + 1$, fix $j = (j_1, j_2, ..., j_n)$ such that $2 \leq |j| \leq k + 1$ and $i_0 \in \{1, 2, ..., n\}$ such that $j_{i_0} \neq 0$ and denote $j_\ast = (j_1, ..., j_{i_0} - 1, ..., j_n)$. According to Leibnitz’s formula, we have

$$D^j \psi^m = mD^k \left(\psi^{m-1} \frac{\partial \psi}{\partial x_{i_0}}\right) = m\psi^{m-1} D^j \psi + m \sum_{s+t+|j_{i_0}|=j_\ast \neq j_0} a_{j_\ast} D^s \psi^{m-1} D^t \frac{\partial \psi}{\partial x_{i_0}}, \quad \text{where } |s| + |t| = k, a_{j_\ast} \in \mathbb{R}.$$

From (1.4), we derive

$$|\nabla^j \psi^m| \leq C \left(d^{-k} \psi^{m-1} + \sum_{1 \leq s \leq k} d^{k-1-s} |\nabla^l \psi^{m-1}|\right), \quad \forall x \in \omega'. \quad (2.4)$$

According to our assumption (2.1), $m - 1 > k$ and the above inequality, we derive that (1.5) holds for $k + 1$. This achieves the proof of Lemma \ref{lemma1}. \hfill \Box

2.1. Proof of Lemma \ref{lemma2}

We will use the following elementary inequalities. For $p \geq 2$, $\varepsilon \in (0, 1)$, $a$, $b$ and $c$ positive real numbers, we have

$$b^p \leq 2a^p + C|a - b|^p, \quad ab^{p-2}c \leq \frac{1}{p}a^{1-p}b^p + \frac{p-2}{p}c^p + \frac{1}{p}b^p. \quad (2.2)$$
Let \( \psi \) the cut-off function defined in Lemma 1 and \( m > r \). Inequality (1.8) is an immediate consequence of (1.6) and (1.7). Also, inequality (1.7) follows from (1.6). In fact, from (1.3), we have \( |\nabla u|^p \psi^m = \sum_{|\alpha|=r} |D^\alpha u|^p \psi^m \). Thus, the first inequality of (2.2) (with \( a = |D^r (u \psi^m)| \) and \( b = |D^r (u \psi^m)| \) and Leibnitz’s formula [1] imply
\[
|\nabla u|^p \psi^m \leq 2|\nabla (u \psi^m)|^p + C \sum_{|\alpha|=r} |D^\alpha (u \psi^m) - D^\alpha (u) \psi^m|^p.
\]
In view of (1.5), we get \( \int_{\omega} |\nabla u|^p \psi^m \leq 2 \int_{\omega} |\nabla (u \psi^m)|^p + C \sum_{q+k=r} d^{-pk} \int_{\omega} |\nabla^q u|^p \psi^p(m-k). \) Hence, inequality (1.7) follows from (1.6).

**Proof of (1.6).** Set \( I_q = d^{-pk} \int_{\omega} |\nabla^q u|^p \psi^p(m-k). \) From (1.5), we have \( \int_{\omega} |\nabla^q u|^p \psi^p(m-k) \leq C I_q. \) Thus, to provide (1.6), we have only to prove the following inequality:
\[
I_q \leq \varepsilon I_r + C \varepsilon^{1-p} I_{r-1}, \hspace{1cm} 1 \leq q \leq r - 1.
\]
We divide the proof of (2.3) into two steps.

**Step 1.** We establish the following first-order interpolation inequality:
\[
I_q \leq \varepsilon I_{q+1} + C \varepsilon^{1-p} I_{q-1}, \hspace{1cm} 1 \leq q \leq r - 1.
\]
Recall that \( \psi \in C^\infty_c (\omega') \) and denote \( u_{\omega'} \) the restriction of \( u \) on \( \omega' \). Observe that by virtue of Meyers-Serrin’s density theorem [1] and using Lebesgue’s dominated convergence theorem [2], one can reduce the proof of (2.3) to \( u_{\omega'} \) belonging to \( C^\infty (\omega') \cap W^{r,p} (\omega') \). Let \( j = (j_1, j_2, ..., j_n) \) be a multi index with \( |j| = q \leq r - 1 \) and \( i_0 \in \{1, 2, ..., n\} \) such that \( j_{i_0} \neq 0 \). Set \( j_- = (j_1, ..., j_{i_0} - 1, ..., j_n), \hspace{1cm} |j_-| = q - 1 \) and \( j_+ = (j_1, ..., j_{i_0} + 1, ..., j_n), \hspace{1cm} |j_+| = q + 1 \). As \( p \geq 2 \) and \( |j| \leq r - 1 \), we have
\[
|D^j u|^p \psi^p(m-k) \in C^1 (\omega') \hspace{1cm} \text{and} \hspace{1cm} \frac{\partial (|D^j u|^p \psi^p(m-k))}{\partial x_{i_0}} = (p-1)|D^j u|^p D^j u.
\]
From (1.5) on has \( |\nabla \psi| \leq C d^{-1} \), then integration by parts yields
\[
d^{-pk} \int_{\omega} |D^j u|^p \psi^p(m-k) = d^{-pk} \int_{\omega} |D^j u|^p D^j u \frac{\partial |D^j u|^p \psi^p(m-k)}{\partial x_{i_0}} = -(p-1)d^{-pk} \int_{\omega} |D^j u|^p D^j u \frac{\partial |D^j u|^p \psi^p(m-k)}{\partial x_{i_0}}.
\]
Thus,
\[
d^{-pk} \int_{\omega} |D^j u|^p \psi^p(m-k) \leq Cd^{-pk} \int_{\omega} |D^j u|^p |\nabla^j u|^p |D^j u|^p \psi^p(m-k)\hspace{1cm} \text{and} \hspace{1cm} C d^{-pk} \int_{\omega} |D^j u|^p |\nabla^j u|^p |D^j u|^p \psi^p(m-k)\).
\]
Taking into account that \( I_q = \sum_{|\alpha|=r} d^{-pk} \int_{\omega} |D^\alpha u|^p \psi^p(m-k) \) with \( k = r - q \), so inequality (2.4) implies
\[
I_q \leq C(I_1 + I_2) \hspace{1cm} \text{where} \hspace{1cm} I_1 = d^{-pk} \int_{\omega} |D^j u|^p |\nabla^j u|^p |D^j u|^p \psi^p(m-k) \) and \( I_2 = d^{-(p+1)} \int_{\omega} |D^j u|^p |\nabla^j u|^p |D^j u|^p \psi^p(m-k)\).
\]
with \( a = d^{-(k+1)}|\nabla^{q-1}u|g^{m-(k+1)} \), \( b = d^{-(k+1)}|\nabla u|g^{r-k} \) and \( c = d^{-(k+1)}|\nabla^{q+1}u|g^{m-(k-1)} \) (respectively \( c = d^{-(k+1)}|\nabla u|g^{r-k} \)), implies

\[
J_1 \leq \frac{1}{p}e^{1-p}I_{q-1} + \frac{p-2}{p}\varepsilon I_q + \frac{1}{p}\varepsilon I_{q+1}, \quad \text{and} \quad J_2 \leq \frac{1}{p}e^{1-p}I_{q-1} + \frac{p-1}{p}\varepsilon I_q.
\]

Combining the above inequalities with (2.7), we deduce \((1-2C\varepsilon)I_q \leq C\varepsilon I_{q-1} + C\varepsilon I_{q+1} \). Hence, the inequality (2.4) follows by replacing \( \varepsilon \) by \( \frac{\varepsilon}{4(1 + C)} \).

**Step 2. End of the proof of (2.5).** The case \( r = 2 \), or \( r \geq 3 \) and \( q = 1 \) is an immediate consequence of (2.4). Let \( r \geq 3, 2 \leq q \leq r-1 \) and \( 2 \leq t \leq q \) and set \( S_t = \sum_{i=2}^{t} I_i \). We apply (2.4) where one substitutes \( q \) by \( t - i \) and \( \varepsilon \) by \( \varepsilon^{r-i} \), we derive \( C\varepsilon^{r-i} I_{r-t} \leq C^{r+1}\varepsilon^{r-i} I_{r-t+1} + C\varepsilon I_{r-t+1} \). Since \( S_t \leq S_q \) and \( 0 < \varepsilon < 1 \), the summation of the above inequalities from \( i = 0 \) to \( i = t-1 \) yields

\[
I_t \leq C\varepsilon^{r-i} I_0 + \varepsilon I_{r+1} + C\varepsilon S_q \quad \text{if} \quad 2 \leq t \leq q.
\]

Summing now (2.8) from \( t = 2 \) to \( t = q \) and substituting \( \varepsilon \) by \( \frac{\varepsilon}{2(1 + C)} \), we arrive at \( S_q \leq C\varepsilon^{r-i} I_0 + \varepsilon I_{r+1} \), for all \( 1 \leq q \leq r-1 \). Combining (2.8) with \( t = q \) and the last inequality, we obtain

\[
I_q \leq C\varepsilon^{r-i} I_0 + \varepsilon I_{r+1}, \quad 1 \leq q \leq r-1.
\]

To end the proof of (2.5), we iterate (2.9) as follows

\[
\begin{align*}
I_q & \leq C\varepsilon^{r-i} I_0 + I_{r+1}, \\
I_{q+1} & \leq C\varepsilon^{r-i} I_0 + I_{r+1}, \\
& \vdots \\
I_{r-1} & \leq C\varepsilon^{r-i} I_0 + \varepsilon I_r.
\end{align*}
\]

Hence, the summation of the above inequalities yields \( I_q \leq C\varepsilon^{r-i} I_0 + \varepsilon I_r \), which is the desired inequality (2.5). The proof of Lemma 1.2 is completed. \( \square \)

3. **Proof of Theorem 1.1**

3.1. **Preliminary results.**

\( B(y, \lambda) \) stands for the ball of radius \( \lambda > 0 \) centered at \( y \in \mathbb{R}^n \). Let \( \psi \) be the cut-off function defined in Lemma 1.1 related to two open subset \( \omega \) and \( \omega' \) of \( B(y, \lambda) \). Thanks to Lemma 1.2 with \( p = 2 \), we establish the following technical lemma:

**Lemma 3.1.** For every \( 0 < \varepsilon < 1 \), there exists a positive constant \( C_\varepsilon = C(n, m, r, \varepsilon) \) such that, for all \( u \in H'(B(y, \lambda)) \), we have

\[
\int_{B(y, \lambda)} |D\psi|^2 \leq C_\varepsilon \left( \int_{B(y, \lambda)} |D\psi|^2 + d^{-2r} \int_{B(y, \lambda)} |u|^2 \psi^{2(m-r)} \right);
\]

\[
\int_{B(y, \lambda)} |D\psi|^{2m} \leq C_\varepsilon \left( \int_{B(y, \lambda)} |D\psi|^{2m} + d^{-2r} \int_{B(y, \lambda)} |u|^2 \psi^{2(m-r)} \right);
\]

\[
\int_{B(y, \lambda)} |\nabla\psi|^2 \leq C_\varepsilon \left( \int_{B(y, \lambda)} |D\psi|^{2m} + d^{-2r} \int_{B(y, \lambda)} |u|^2 \psi^{2(m-r)} \right);
\]

\[
\int_{B(y, \lambda)} |\nabla\psi|^2 \leq C_\varepsilon \left( \int_{B(y, \lambda)} |D\psi|^{2m} + d^{-2r} \int_{B(y, \lambda)} |u|^2 \psi^{2(m-r)} \right);
\]
Collecting now, inequalities (3.6), (3.7), (1.7) and (1.8), we have

\[
\int_{B(y, \lambda)} |D_t u \cdot D_t (\nabla u \cdot (x-y))\psi^{2m} - D_t u \cdot D_t (\nabla u \cdot (x-y))\psi^{2m}| \leq C(1 + \frac{\lambda}{d}) \left( \sum_{1 \leq q \leq r} d^{2(r-q)} \int_{\omega \setminus \omega'} |\nabla^q u|^2 + d^{-2r} \int_{\omega'} |u|^2 \right).
\]

(3.4)

Proofs of Lemma 3.1
For \( v \in H'(B(y, \lambda)) \) and \( \eta \in C_c^\infty(B(y, \lambda)) \), set \( A(\eta, v) := D_t (\eta v) - \eta D_t v \). A simple computations yield

\[
|D_t (u \eta)|^2 - \eta^2 |D_t u|^2 = 2 \eta D_t u \cdot A(\eta, u) + |A(\eta, u)|^2, \quad \eta^2 D_t u \cdot D_t v - D_t u \cdot D_t (\eta v^2) = -D_t u \cdot A(\eta^2, v),
\]

and \( |A(\eta, v)| \leq C \sum_{q+k=r, \eta \neq r} |\nabla^q v| |\nabla^r \eta| \). Therefore,

\[
|D_t (u \eta)|^2 - \eta^2 |D_t u|^2 \leq C \sum_{q+k=r, \eta \neq r} \left( |D_t u| |\nabla^q u| |\nabla^r \eta| + |\nabla^q \eta| \right) + |\nabla^q u| |\nabla^r \eta|).
\]

Choosing now \( \eta = \psi^m \) and using (1.5), we obtain

\[
\int_{B(y, \lambda)} \left| \left( \psi^{2m} D_t u \cdot D_t v - D_t u \cdot D_t (\psi^{2m}) \right) \right| \leq CS_1 (u, v); \quad (3.5)
\]

\[
\int_{B(y, \lambda)} \left| D_t (u \psi^m) \right|^2 - \psi^{2m} |D_t u|^2 \leq C(S_1 (u, u) + S_2 (u)); \quad (3.6)
\]

where \( S_1 (u, v) = \int_{\omega \setminus \omega'} |\nabla^r u| \left( \sum_{0 \leq q \leq r-1} d^{-q} |\nabla^q u| |\psi^{2m+q-r}| \right) \) and \( S_2 (u) = \int_{\omega \setminus \omega'} \left( \sum_{0 \leq q \leq r-1} d^{-q} |\nabla^q u|^2 |\psi^{2m+2-2(r-q)}| \right). \)

We invoke Cauchy-Schwarz’s inequality:

\[
\left| \sum_{q=0}^{r-1} a_q \right|^2 \leq e |a_q|^2 + C_e \sum_{q=0}^{r-1} |a_q|^2, \quad (a_0, a_1, ..., a_e) \in \mathbb{R}^{r+1},
\]

with \( a_e = |\nabla^r u| |\psi^m| \) and \( a_q = d^{-q-1} |\nabla^q u| |\psi^{(m+q-r)}| \) if \( q = 0, 1, ..., r-1 \). As \( pm - (r-q) = (p-1)m + (m-r-q) \), we arrive at

\[
S_1 (u, u) + S_2 (u) \leq e \int_{B(y, \lambda)} |\nabla^r u|^2 |\psi^{2m}| + C_e \sum_{q+k=r, \eta \neq r} d^{-2k} \int_{B(y, \lambda)} |\nabla^q u|^2 |\psi^{2m+k-2(r-q)}|.
\]

(3.7)

Hence, inequality (3.1) follows from (3.5) (with \( v = u \), (3.7) and inequalities (1.7), (1.8) of Lemma 1.2.

Collecting now, inequalities (3.6), (3.7), (1.7) and (1.8), we obtain

\[
\int_{B(y, \lambda)} \left| D_t u \psi^{2m} - D_t (u \psi^m) \right|^2 - \leq e \int_{B(y, \lambda)} |\nabla^r (u \psi^m)|^2 + C_e d^{-2r} \int_{B(y, \lambda)} |u|^2 |\psi^{2m-2(r-q)}|.
\]

(3.8)

\[^4\text{Both } D_t \text{ and } A(\eta, v) \text{ are respectively scalar operators if } r \text{ is even, and } n\text{-vectorial operators if } r \text{ is odd.}\]

\[^5\text{Observe that } |D_t u| \leq C |\nabla^r u|.\]
Thus, the proof of (3.4) follows by collecting (3.5), (3.11). This ends the proofs of Lemma 3.1.

Fix now where the above Cauchy-Schwarz’s inequality, yields

\[ S_1(u, v) = \int_{\Omega} |\nabla u| \sum_{0 \leq q < 1} d^{-q} |\nabla^q v| \rho^{2m+q-r} \]

the above Cauchy-Schwarz’s inequality, yields

\[ S_1(u, v) \leq \int_{\omega} |\nabla u|^2 + C \sum_{0 \leq q \leq r} d^{-q} \int_{\omega} |\nabla^q v|^2. \]  

(3.10)

Fix now \( v = \nabla u \cdot (x-y) \) and taking into account that

\[ |\nabla^q (\nabla u \cdot (x-y))|^2 \leq C (|x|^{q+1} |u|^2 + |\nabla u|^2) \leq C (\frac{d^2}{d}) |\nabla^{q+1} u|^2 + |\nabla u|^2, \]

we deduce that

\[ S_1(u, v) \leq C (1 + (\frac{d^2}{d}) \sum_{0 \leq q \leq r} d^{-q} \int_{\omega} |\nabla^q u|^2 + d^{-2} \int_{\Omega} |u|^2. \]  

(3.11)

Thus, the proof of (3.4) follows by collecting (3.5), (3.11). This ends the proofs of Lemma 3.1. □

At last, in view of assumptions (h1)-(h3), we have (see the proof in Appendix A):

**Lemma 3.2.** Let \( t > 1 \) and set \( q_1 = \frac{p_2 + 1}{p_2} \). There exists a positive constant \( C = C(s_0, p_1, p_2, c_1) \) such that for all \((x, s) \in \Omega \times \mathbb{R}, \) we have

- [1] \( f'(x, s)s^2 \geq p_1 f(x, s) - C; \)
- [2] \((p_2 + 1)F(x, s) \geq f(x, s)s - C; \)
- [3] \(|s|^{p_1 + 1} \leq C f(x, s)|s| + 1, \) \(|f(x, s)s| \leq f(x, s)s + C \) and \( F(x, s) \leq C f(x, s) + 1; \)
- [4] \(|f(x, s)|^2 \leq C f(x, s) + 1 \) and \( |f(x, s)| \leq C |s|^{1}; \)
- [5] For all \( \epsilon \in (0, 1), 0 \leq a \leq 1 \) and \( b > 0 \) we have \( as^b \leq C + \epsilon |f(x, s)| |s|^{\frac{p_1}{2}} + e^{\frac{\epsilon}{|s|}} b^{\frac{p_1}{2}}. \)

### 3.2. End of the proof of Theorem 1.1

Recall that \( d_j = \inf(\alpha, \delta_j), \) where \( \delta_j = \text{dist}(y, \partial \Omega), y \in \Omega \) and \( \alpha \in (0, 1). \) For \( j = 1, 2, \cdots, i(u) + 1, \) set

\[ A_j := \{ x \in \mathbb{R}^n : a_j < |x - y| < b_j \}, \quad \alpha_j = \frac{2(j + i(u))}{4(i(u) + 1)} d_j, \quad b_j = \frac{2(j + i(u)) + 1}{4(i(u) + 1)} d_j, \]  

and

\[ A_j' := \{ x \in \mathbb{R}^n : a_j' < |x - y| < b_j' \}, \quad \alpha_j' = \frac{2(j + i(u)) - 1}{4(i(u) + 1)} d_j, \quad b_j' = \frac{2(j + i(u)) + 1}{4(i(u) + 1)} d_j. \]

Observe that \( \mathbb{A} \subset A_j' \subset A_j \subset B(y, d_j) \) and let \( \psi_j \in C^\infty(B(y, d_j) \) be the cut-off function defined in Lemma 1.1 with \( \omega = A_j \) and \( \omega' = A_j' \) and satisfying \( \supp(\psi_j) \subset A_j', 0 \leq \psi_j \leq 1 \) if \( x \in A_j' \) and \( \psi_j = 1 \) if \( x \in A_j. \) Moreover, we have

\[ |\nabla^k (\psi_j^2)(x)|^2 \leq C \psi_j^{2m-k} \left( \frac{1 + i(u)}{d_j} \right)^{2k}, \]  

(3.12)
From inequality (1.8) we derive
\[
\sum_{1 \leq q \leq r} d^{-p(r-q)} \int_{\lambda_i} |\nabla^q u|^p \leq e \int_{B(y,d_y)} |\nabla' (u^m_j)|^2 + C_{s} \left( \frac{1 + i(u)}{d_y} \right)^{2r} \int_{B(y,d_y)} u^2 \psi_j^{(m-r)}.
\] (3.13)

In the sequel we choose \( m = \frac{(p_1 + 1)r}{2} \) so that \( m > r \) and \( \frac{(p_1 + 1)(m - r)}{p_1 - 1} = m \). Thus, point 5 of Lemma 3.2 with \( s = u, a = \psi_j^{(m-r)} \) and \( b = \left( \frac{1 + i(u)}{d_y} \right)^{-2} \), yields
\[
\left( \frac{1 + i(u)}{d_y} \right)^{-2r} \int_{B(y,d_y)} u^2 \psi_j^{(m-r)} \leq C d_y^m + e \int_{B(y,d_y)} |f(x,u)u| \psi_j^{2m} + C tersuch that
\[
C_{s} \left( \frac{1 + i(u)}{d_y} \right)^{2r} \int_{B(y,d_y)} u^2 \psi_j^{(m-r)}.
\] (3.14)

Next observe that \( \text{supp}(u_j^m) \cap \text{supp}(u_j^m) = \emptyset, \forall 1 \leq l \neq j \leq 1 + i(u) \), then according to the definition of the quadratic form (3.15) we derive
\[
Q_a \left( \sum_{1 \leq j \leq m} \lambda_j u^2 \right) = \sum_{1 \leq j \leq m} \lambda_j^2 Q_a (u^2).
\]
in view of the definition of \( i(u) \), there exists \( j_0 \in \{1, 2, ..., 1 + i(u)\} \) such that \( Q_a (u^2)_{j_0} \geq 0 \). Therefore, point 1 of Lemma 3.2 implies
\[
p_1 \int_{B(y,d_y)} f(x,u)u \psi_j^{2m} - C d_y^m \leq \int_{B(y,d_y)} f' (x,u)^2 \psi_j^{2m} \leq \int_{B(y,d_y)} |D_j (u \psi_j^{2m})|^2.
\] (3.15)

We divide the proof into three steps.

**Step 1.** We shall prove the following estimate
\[
\sum_{1 \leq q \leq r} d^{-p(r-q)} \int_{\lambda_i} |\nabla^q u|^p \leq e \int_{B(y,d_y)} |\nabla' (u^m_j)|^2 + C_{s} \left( \frac{1 + i(u)}{d_y} \right)^{2r} \int_{B(y,d_y)} u^2 \psi_j^{(m-r)}.
\] (3.16)

Multiplying equation (1.9) by \( -\frac{1 + p_1}{2} u \psi_j^{2m} \), integrating by parts, we obtain
\[
- \frac{1 + p_1}{2} \int_{B(y,d_y)} f(x,u)u \psi_j^{2m} = - \frac{1 + p_1}{2} \int_{B(y,d_y)} D_j u D_j (u \psi_j^{2m}).
\]

We combine the last equality with (3.15) and point 3 of Lemma 3.2 yields
\[
p_1 \int_{B(y,d_y)} f(x,u)u \psi_j^{2m} + \int_{B(y,d_y)} |D_j (u \psi_j^{2m})|^2 \leq C d_y^m + \frac{p_1 - 1}{2} \int_{B(y,d_y)} |D_j (u \psi_j^{2m})|^2 - D_j u D_j (u \psi_j^{2m}).
\]

It follows from (3.1) and (3.9) that
\[
\int_{B(y,d_y)} |\nabla' (u^m_j)|^2 + \int_{B(y,d_y)} |f(x,u)u| \psi_j^{2m} \leq C d_y^m + \epsilon \int_{B(y,d_y)} |\nabla' (u^m_j)|^2 + C_{s} \left( \frac{1 + i(u)}{d_y} \right)^{2r} \int_{B(y,d_y)} u^2 \psi_j^{(m-r)}.
\]

Collecting the last inequalities with (3.13) and (3.14), we get\(^6\)
\[
\sum_{1 \leq q \leq r} d^{-p(r-q)} \int_{\lambda_i} |\nabla^q u|^p + \int_{B(y,d_y)} |f(x,u)u| \psi_j^{2m} \leq C d_y^m \left( \frac{1 + i(u)}{d_y} \right)^{2r} \int_{B(y,d_y)} u^2 \psi_j^{(m-r)}.
\]

\(^6\)Observe that \( d_y^m \leq d_y^m \left( \frac{1 + i(u)}{d_y} \right)^{2r} \) as \( d_y = \text{inf}(\alpha, \delta_y) < 1 \).
Therefore, inequality (3.16) follows as \( \psi_j(x) = 1 \) if \( x \in A_{j_0} \).

**Step 2.** We shall use the following identity (see the proof in appendix B):

\[
D_i u D_i (\nabla u \cdot (x - y)) = \frac{1}{2} \nabla|D_i u|^2 \cdot (x - y) + r|D_i u|^2, 
\]

(3.17)

to establish a variant of the Pohozaev identity and we exploit (3.16) to prove that:

\[
d^2 \int_{B(y, \frac{d}{2})} |f(x, u)|^p \leq C \left( \frac{1 + i(u)}{d^\gamma} \right)^{\frac{2n+1}{p^2-1}}. 
\]

(3.18)

Recall that \( A_{j_0} = \{ x \in \mathbb{R}^n; \ a_{j_0} < |x-y| < b_{j_0} \} \). We invoke again Lemma 3.1 with \( \omega = B(y, a_{j_0}) \), \( \omega' = B(y, b_{j_0}) \) and let \( \psi \in C_0^\infty(B(y, b_{j_0})) \) such that

\[
\psi \equiv 1 \text{ for all } x \in B(y, a_{j_0}) \text{ and } |\nabla^2 \psi| \leq C \left( \frac{1 + i(u)}{d^\gamma} \right)^{k} \psi^{2m-k} \forall x \in A_{j_0} \text{ and } k = 1, \ldots, r.
\]

(3.19)

Multiplying equation (1.9) by \( u \psi^{2m} \) (respectively by \( (\nabla u \cdot (x - y)) \psi^{2m} \)) and integrating by parts, we get

\[
\int_{B(y, d)} D_i u D_i (u \psi^{2m}) = \int_{B(y, d)} f(x, u)u \psi^{2m} \text{ respectively } \int_{B(y, d)} D_i u D_i (u \nabla u \cdot (x - y) \psi^{2m}) = \int_{B(y, d)} f(x, u) \nabla u \cdot (x - y) \psi^{2m}. 
\]

According to identity (3.2) (respectively by (3.4), (3.19) and (3.16)), we derive

\[
\int_{B(y, d)} |D_i u|^2 \psi^{2m} - \int_{B(y, d)} f(x, u)u \psi^{2m} \leq C \int_{B(y, d)} |D_i \psi|^2 + C \left( \frac{1 + i(u)}{d^\gamma} \right)^{2r} \int_{B(y, d)} u^2 \psi^{2(m-r)}; 
\]

(3.20)

\[
\int_{B(y, d)} \psi^{2m} D_i u D_i (u \nabla u \cdot (x - y)) \leq \int_{B(y, d)} f(x, u) \nabla u \cdot (x - y) \psi^{2m} + Cd_1 \left( \frac{1 + i(u)}{d^\gamma} \right)^{2r} + C \left( \frac{1 + i(u)}{d^\gamma} \right)^{-2r} \int_{A_{j_0}} u^2 \psi^{2(m-r)}.
\]

As above, using point 5 of Lemma 3.2 and (3.16), there holds that

\[
C \left( \frac{1 + i(u)}{d^\gamma} \right)^{-2r} \int_{A_{j_0}} u^2 \psi^{2(m-r)} \leq C d_1 + \frac{1}{2} \int_{A_{j_0}} |f(x, u)u \psi^{2m} + C d_1 \left( \frac{1 + i(u)}{d^\gamma} \right)^{\frac{2n+1}{p^2-1}} \leq C d_1 \left( \frac{1 + i(u)}{d^\gamma} \right)^{\frac{2n+1}{p^2-1}}. 
\]

(3.21)

Combining these inequalities we get

\[
\int_{B(y, d)} \psi^{2m} D_i u D_i (u \nabla u \cdot (x - y)) \leq \int_{B(y, d)} f(x, u) \nabla u \cdot (x - y) \psi^{2m} + C d_1 \left( \frac{1 + i(u)}{d^\gamma} \right)^{\frac{2n+1}{p^2-1}}. 
\]

(3.22)

In one hand, integration by parts of the first term of the right hand-side, gives

\[
\int_{B(y, d)} f(x, u) \nabla u \cdot (x - y) \psi^{2m} = -n \int_{B(y, d)} F(x, u) \psi^{2m} - \int_{A_{j_0}} F(x, u)(\nabla \psi^{2m} \cdot (x - y)) + \int_{B(y, d)} (\nabla \cdot F)(x, u) \cdot (x - y) \psi^{2m};
\]

Invoking now assumption (h3) with points 2-3 of Lemma 3.2, 3.19 and using again (3.16), it follows that imply

\[
\int_{B(y, d)} f(x, u) \nabla u \cdot (x - y) \psi^{2m} = (C \alpha + \frac{n}{p^2 + 1}) \int_{B(y, d)} f(x, u)u \psi^{2m} + C d_1 \left( \frac{1 + i(u)}{d^\gamma} \right)^{\frac{2n+1}{p^2-1}}. 
\]

(3.23)
On the other hand, using (3.17) and integrating by parts we derive
\[
\int_{B(y,d_\lambda)} \psi^{2m} D_{\alpha} u D_{\alpha} (\nabla u \cdot (x-y)) = \frac{2r-n}{2} \int_{B(y,d_\lambda)} |D_{\alpha} u|^2 \psi^{2m} - \frac{1}{2} \int_{B(y,d_\lambda)} |D_{\alpha} u|^2 (\nabla u \cdot (x-y)).
\]
As $|D_{\alpha} u|^2 \leq |\nabla u|^2$ and $|x-y| \leq 1$, it follows from (3.19) and (3.16) that
\[
\int_{B(y,d_\lambda)} \psi^{2m} D_{\alpha} u D_{\alpha} (\nabla u \cdot (x-y)) = \frac{2r-n}{2} \int_{B(y,d_\lambda)} |D_{\alpha} u|^2 \psi^{2m} + C d_\lambda^n \left( \frac{1 + i(u)}{d_y} \right)^{\frac{2n}{(p+1)(n-2r)} + 1}.
\]
Collecting inequalities (3.22), (3.23) and the last equality we arrive at
\[
\left( \frac{2n}{(p+1)(n-2r)} - \frac{C\alpha}{(n-2r)} \right) \int_{B(y,d_\lambda)} f(x,u) u \psi^{2m} - \frac{1}{2} \int_{B(y,d_\lambda)} |D_{\alpha} u|^2 \psi^{2m} \leq C d_\lambda^n \left( \frac{1 + i(u)}{d_y} \right)^{\frac{2n}{(p+1)(n-2r)} + 1}.
\]
(3.24)
We choose $\alpha = \alpha_0 \in (0,1)$ small enough so that
\[
\frac{2n}{(p+1)(n-2r)} - \frac{C\alpha_0}{(n-2r)} > 1
\]
and we combine the above inequality with (3.20) and (3.3), we deduce that
\[
\int_{B(y,d_\lambda)} |\nabla u|^2 \psi^{2m} + \int_{B(y,d_\lambda)} f(x,u) u \psi^{2m} \leq C \int_{B(y,d_\lambda)} |\nabla(u \psi^m)|^2 + C \left( \frac{1 + i(u)}{d_y} \right)^{\frac{2n}{(p+1)(n-2r)} + 1}.
\]
(3.25)
Inequality (3.21) and points 3 of Lemma 3.2 imply
\[
\int_{B(y,d_\lambda)} |\nabla(u \psi^m)|^2 + \int_{B(y,d_\lambda)} |f(x,u) u | \psi^{2m} \leq C d_\lambda^n \left( \frac{1 + i(u)}{d_y} \right)^{\frac{2n}{(p+1)(n-2r)} + 1}.
\]
(3.26)
Observe now that $\psi \equiv 1$ for all $x \in B(y,\frac{d_y}{2}) \subset B(y,a_y)$, so estimate (3.18) follows from the above inequality and point 4 of Lemma 3.2.

**Step 3. Boot-strap procedure.** Set $\lambda = \frac{d_y}{2} < 1$, $u_\lambda(x) = u(y + \lambda x)$ and $g_\lambda(x) = f(y + \lambda x, u(y + \lambda x))$, $x \in B_1$, then $u_\lambda$ satisfies
\[
(-\Delta u_\lambda)^{\frac{1}{2}} = \lambda^{\frac{1}{2}} g_\lambda \text{ in } B_1.
\]
By virtue of (3.18), we have
\[
\int_{B_1} |g_\lambda|^{p_1} = 2^n d_\lambda^n \int_{B(y,\frac{d_y}{2})} |f(x,u)|^{p_1} \leq C \left( \frac{1 + i(u)}{d_y} \right)^{\frac{2n}{(p+1)(n-2r)} + 1}.
\]
(3.26)
We invoke local $L^p$-$W^{2,p}$ estimate (see Corollary 3.1 in the Appendix C) and Rellich-Kondrachov’s theorem [11]. Let $q > 1$, then point 3 of Lemma 3.2 implies 8
\[
|u_\lambda|_{L^q(B_\frac{d_y}{2})} \leq C |u_\lambda|_{W^{2,p}(B_1)} \leq C (\|g_\lambda\|_{L^q(B_1)} + |u_\lambda|_{L^q(B_1)}) \leq C (\|g_\lambda\|_{L^q(B_1)} + 1),
\]
where $q^* = \frac{qn}{n - 2rq}$ if $2rq < n$ and for all $q^* > 1$ if $q = \frac{n}{2r}$.
(3.27)

---

8 Recall that $\frac{2n}{(p+1)(n-2r)} > 1$.

9 Observe that $\lambda = \frac{d_y}{2} < 1$. 

and
\[ \|u_i\|_{C^{\alpha}(eta_2)} \leq C\|u_i\|_{W^{1,\alpha}(eta_2)} \leq C(\|g_i\|_{L^1(B_1)} + \|u_i\|_{L^1(B_1)}) \leq C(\|g_i\|_{L^1(B_1)} + 1), \text{ if } 2rq > n. \]  
\[ (3.28) \]

So, inequality (3.27) and point 4 of Lemma 3.2 give
\[ \|g_i\|_{L^1(B_2)} \leq C(\|g_i\|_{L^1(B_1)} + 1)^{p_2}, \text{ if } 2rq \leq n. \]
\[ (3.29) \]

If \( 2rq_1 \geq n \) (respectively \( 2rq_1 = n \)) the desired estimate (1.11) follows from (3.26) and (3.28) (with \( q = q_1 \)) (respectively (3.26), (3.27) (with \( q = q_1 \)) and (3.28), (3.29) (with \( q = p_2 \)). The case \( 2rq_1 < n \) needs more involving analysis. As \( q_1 = \frac{p_2 + 1}{p_2} \) and \( 1 < p_2 < \frac{n + 2r}{n - 2r} \), we have
\[ q_1^n = \frac{(p_2 + 1)n}{n - 2r} > \frac{(p_2 + 1)n}{n - 2r} > \frac{1}{q_1} - \frac{2rp_2}{(n - 1)} < 0. \]

Set \( q_2 = \frac{q_1^n}{p_2} \) and \( q_{k+1} = \frac{q_{k+1}^n}{p_2} \). We claim that there exists \( k_0 \in \mathbb{N}^+ \) such that
\[ 2rq_{k+1} > n \text{ and } 2rq_{k_0} < n. \]
\[ (3.30) \]

Suppose by contradiction that \( 2rq_k < n \) for all \( k \in \mathbb{N}^+ \). Then, \( \frac{1}{q_{k+1}} = \frac{p_2}{q_k} - \frac{2rp_2}{n} \) and therefore
\[ \frac{1}{q_{k+1}} = \frac{p_2}{q_k} - \frac{2rp_2}{n} - p_{k+1} = p_2 \left( \frac{1}{q_1} - \frac{2rp_2}{n} \right) + \frac{2rp_2}{n(p - 1)}. \]
\[ (3.31) \]

We reach a contradiction since \( \frac{1}{q_k} \to -\infty \). Set now
\[ \beta = \frac{2rp_2}{n(p - 1)} \left( \frac{2rp_2}{n(p - 1)} - \frac{1}{q_1} \right) ^{-1} = \frac{2r(p + 1)}{2r(p + 1) - n(p - 1)}. \]
From (3.31), we have \( p_{k_0} < \beta \) and \( p_{k+1} > \beta \). Hence, iterating (3.29), we obtain
\[ \|g_i\|_{L^1(B_3)} \leq C(\|g_i\|_{L^1(B_1)} + 1)^{\beta}. \]

Set \( \gamma_1 = \frac{(p_1 + 1)\beta}{q_1} = \frac{2r(p_1 + 1)p_2}{2r(p_2 + 1) - n(p_2 - 1)} \) and \( \gamma_2 = \beta + \frac{2r}{p_1 - 1} \gamma_1 \). As \( rq_{k_0} > n \), the last inequality with (3.28) and (3.26) imply
\[ \|u_i\|_{C^{\alpha}([s_0, s_0])} \leq C(1 + i(u)^{2})^{\frac{2r}{p_1 - 1} \gamma_1}. \]

According to the definition of \( u_i \), we get
\[ \sum_{j=0}^{2r-1} d_j^i \|\nabla u_i\| (-\frac{\beta}{2r}) \leq C(1 + i(u)^{2})^{\frac{2r}{p_1 - 1} \gamma_1}. \] This achieves the proof of Theorem 1.1.

Appendix A: Proof of Lemma 3.2
In the following, \( C \) denotes generic positive constant depending only on the parameters \( (s_0, p_1, p_2) \) and the constant \( c_1 \) of assumptions \((h_1)-(h_6)\). The following inequalities are an immediate consequence of \((h_4)\):
\[ |F(x,s)|, |f(x,s)| \leq C, \forall (x,s) \in \Omega \times [-s_0, s_0]. \]
\[ (3.32) \]
Hence, points 1 and 2 follow from (h1)-(h2). Also, in view of (3.32) and the fact that the nonlinearity \(-f(x,s)\) satisfies (h1)-(h4), we need only to prove points 3 and 4 for all \((x, s) \in \Omega \times [s_0, \infty)\).

**Proof of point 3.** According to (h1), we have

\[
f''(x, s)s \geq p_1f(x, s), \forall (x, s) \in \Omega \times [s_0, \infty)
\]

which implies \(\left(\frac{f(x, s)}{s^{p_1}}\right)'' \geq 0\). As \(f(x, s_0) \geq \frac{1}{c_1}\) for all \(x \in \Omega\) (see (h4)), we derive

\[
f(x, s) \geq \frac{s^{p_1}}{c_1s_0^{p_1}} \text{ and } f(x, s) \geq \frac{s^{p_1+1}}{c_1s_0}, \forall (x, s) \in \Omega \times [s_0, \infty),
\]

which imply the first inequality of point 3. Integrating now over \([s_0, s]\) and using (h2), we derive \(\frac{f(x, s)s}{p_2 + 1} \leq F(x, s) \leq \frac{f(x, s)s}{p_1 + 1} + C, (x, s) \in \Omega \times [s_0, \infty)\) which pmlies the second and third inequalities of point 3.

**Proof of point 4.** According to (h2), we have \(\left(\frac{F(x, s)}{s^{p_1+1}}\right)'' \leq 0, \forall (x, s) \in \Omega \times [s_0, \infty)\) which with (3.32) imply \(F(x, s) \leq C s^{p_1+1} \forall (x, s) \in \Omega \times [s_0, \infty)\). Hence, from (h2) and point 2, we get \(|f(x, s)|^{\frac{1}{2}} \leq C|x|, \forall (x, s) \in \Omega \times [s_0, \infty)\). Consequently, for \(t > 0\) and \(q_1 = \frac{p_2 + 1}{p_1}\), we derive

\[|f(x, s)|^{\frac{1}{2}} \leq C|f(x, s)| s \text{ and } |f(x, s)|^{\frac{1}{2}} \leq C|x|, (x, s) \in \Omega \times [s_0, \infty).\]

**Proof of point 5.** In view of Young’s inequality, we obtain \(as^2b \leq \varepsilon s^{p_1+1}a \frac{\alpha_1}{\alpha_2} + \varepsilon \frac{1}{\alpha_2} b \frac{\alpha_1}{\alpha_2}\). Recall that \(0 \leq a \leq 1\) and using point 3, we derive point 5. This end the proof of Lemma 3.2. \(\square\)

**Appendix C: Proof of (3.17).** Noticing that (3.17) is trivial for \(r = 1\). Let \(k \in \mathbb{N}^+\). If \(r = 2k\), i.e. \(D_r = \Delta^k\), apply Leibnitz’s formula, we have

\[
\Delta^k(\nabla u \cdot (x - y)) = \nabla(\Delta^k u)(x - y) + 2k\Delta^k u.
\]

Multiplying (3.35) by \(\Delta^k u\) and taking into account that \(\Delta^k u \nabla(\Delta^k u) \cdot (x - y) = \frac{1}{2} \nabla((\Delta^k u)^2) \cdot (x - y)\), This achieves the proof of (3.17).

If \(r = 2k + 1\), that is \(D_r = \nabla \Delta^k\). According to (3.35) we derive

\[
D_r u \cdot D_r(\nabla u \cdot (x - y)) = \nabla \Delta^k \nabla(\Delta^k u)(x - y) + (r - 1)|D_r u|^2.
\]

Therefore, (3.17) follows as \(\nabla w \cdot \nabla(\nabla w \cdot (x - y)) = \frac{1}{2} \nabla((\nabla w)^2) \cdot (x - y) + |\nabla w|^2\), \(\forall w \in C^1(\mathbb{R}^n)\).

**Appendix C: Local \(L^p\)–\(W^{2,p}\)-estimate, \(t \in \mathbb{N}^+, p \geq 2\).** Consider the linear higher order elliptic problem of the form

\[
Lu = g \text{ in } \Omega.
\]

Here \(\Omega\) is a domain of \(\mathbb{R}^n\) and

\[
L = \left(- \sum_{i,k=1}^{n} a_{ik}(x) \frac{\partial^2}{\partial x_i \partial x_k} \right) + \sum_{|\beta| \leq r-1} b_{\beta}(x) D^\beta
\]

where \(b_{\beta} \in L^\infty(\Omega), a_{ik} \in C^{2-r}(\Omega)\) and \(L\) is a local uniformly elliptic operator, that is for all bounded open subset \(S\) of \(\Omega\) there exists a constant \(\lambda_0 > 0\) with \(\lambda_0 |g|^2 \leq \sum_{i,k=1}^{n} a_{ik}(x) \xi_i \xi_k \leq \lambda_0 |g|^2\) for all \(\xi \in \mathbb{R}^n, x \in \Omega\). Let \(A\) and \(A'\) be two bounded open subset of \(\Omega\) such that \(\overline{A} \subseteq A' \subseteq \overline{A'} \subseteq \Omega\) and \(\omega'\). When \(p \geq 2\) by virtue of Lemma 3.2 we establish local analogue of the celebrated \(L^p\)–\(W^{2,p}\) estimate of Agmon-Douglas-Nirenberg. Set \(d = \text{dist}(A, \Omega \setminus A')\).
Applying again (1.8) with $r = \|\omega\|$ and depending only on $\omega$, in the following we insert the above inequality in the right-hand side of (3.37) and we choose $C_{\text{Corollary 3.1}}$. This achieves the proof of Corollary 3.1.

**Proof of Corollary 3.1.**
In the following $C$ denotes a generic positive constant which depends on the parameters stated in Corollary 3.1. As $A$ is a compact subset of $A'$, we can find $x_i \in A$, $i = 1, 2, \ldots, i_0$ such that for any $u \in W_{\text{loc}}^{2,p}(\Omega)$ a weak solution of (3.37), we have

$$\|u\|_{W^{2,p}(A)} \leq C \left( \|g\|_{L^p(A)} + \|u\|_{L^p(A')} \right).$$

Using now inequality (1.8) with $r = \|\omega\|$ and $\omega' = B(x_i, \frac{d}{2})$. A simple computations give

$$L(u\psi^m) = u\psi^m + uL(\psi^m) + b_0 u\psi^m + \sum_{1 \leq i < j \leq p} c_{i,j} D^i u D^j(\psi^m),$$

where $c_{i,j} \in L^\infty(A')$. As $u\psi^m \in W^{2,p}(\Omega') \cap W_0^{2,p}(\Omega')$ with $\omega'$ is of class $C^{2r}$, Agmon-Douglis-Nirenberg's global estimate [2] and (1.8) imply

$$\sum_{1 \leq i \leq 2r} \int_{\Omega'} |\nabla^i (u\psi^m)|^p \leq C \left( \|g\|_{L^p(\Omega')}^p + \|u\|_{L^p(\Omega')}^p \right) + \sum_{1 \leq i \leq 2r-1} \sum_{1 \leq q \leq s} \int_{\Omega'} |\nabla^i u|^p |\nabla^{s-q} \psi^m|^p. \tag{3.37}$$

Using now inequality (1.8) with $r = s$, we obtain

$$\sum_{1 \leq i \leq 2r} \int_{\Omega'} |\nabla^i u|^p |\nabla^{s-q} \psi^m|^p \leq C \int_{\Omega'} |\nabla^s (u\psi^m)|^p + C_{\text{Corollary 3.1}} \int_{\Omega'} |u|^p |\psi^{(m-s)}|^p.$$

Applying again (1.8) with $r = s + 1$ and replacing $\epsilon$ by $\frac{d}{2}$, we obtain

$$\int_{\Omega'} |\nabla^s u|^p |\nabla^{s-q} \psi^m|^p \leq \int_{\Omega'} |\nabla^s u|^p |\nabla^{(m-s)}|^p \leq C \int_{\Omega'} |\nabla^s (u\psi^m)|^p + C_{\text{Corollary 3.1}} \int_{\Omega'} |u|^p |\psi^{(m-s-1)}|^p.$$

Collecting the two last inequalities, we derive

$$\sum_{1 \leq i \leq 2r-1} \sum_{1 \leq q \leq s} \int_{\Omega'} |\nabla^i u|^p |\nabla^{s-q} \psi^m|^p \leq C \int_{\Omega'} |\nabla^s (u\psi^m)|^p + C_{\text{Corollary 3.1}} \int_{\Omega'} |u|^p.$$

We insert the above inequality in the right-hand side of (3.37) and we choose $\epsilon = \frac{1}{2C}$, it follows that

$$\|u\psi^m\|_{W^{2,p}(\Omega')} \leq C \left( \|g\|_{L^p(\Omega')} + \|u\|_{L^p(\Omega')} \right).$$

Since $\psi(x) = 1$ if $x \in \omega$, we obtain

$$\|u\|_{W^{2,p}(B(x_i, \frac{d}{2}))}^p \leq C \left( \|g\|_{L^p(\Omega')}^p + \|u\|_{L^p(\Omega')}^p \right) \leq C \left( \|g\|_{L^p(A')}^p + \|u\|_{L^p(A')}^p \right).$$

As $A \subset \bigcup_{1 \leq i \leq i_0} B(x_i, \frac{d}{2})$ and $i_0$ depends only on $A$ and $d$, we derive

$$\|u\|_{W^{2,p}(A')}^p \leq C_{i_0} \left( \|g\|_{L^p(A')}^p + \|u\|_{L^p(A')}^p \right).$$

This achieves the proof of Corollary 3.1. □
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