PRINCIPAL SERIES OF HERMITIAN LIE GROUPS INDUCED FROM HEISENBERG PARABOLIC SUBGROUPS

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ABSTRACT. Let $G$ be an irreducible Hermitian Lie group and $D = G/K$ its bounded symmetric domain in $\mathbb{C}^d$ of rank $r$. Each $\gamma$ of the Harish-Chandra strongly orthogonal roots $\{\gamma_1, \ldots, \gamma_r\}$ defines a Heisenberg parabolic subgroup $P = MAN$ of $G$. We study the principal series representations $\text{Ind}_P^G(1 \otimes e^\nu \otimes 1)$ of $G$ induced from $P$. These representations can be realized as the $L^2$-space on the minimal $K$-orbit $S = Ke = K/L$ of a root vector $e$ of $\gamma$ in $\mathbb{C}^d$, and $S$ is a circle bundle over a compact Hermitian symmetric space $K/L_0$ of $K$ of rank one or two. We find the complementary series, reduction points, and unitary subrepresentations in this family of representations.

0. INTRODUCTION

In the present paper we shall study composition series and complementary series for degenerate principal series representations for an irreducible Hermitian Lie group $G$ induced from a Heisenberg parabolic subgroup. Let $D = G/K \subset \mathbb{C}^d$ be the bounded symmetric domain of $G$ in its Harish-Chandra realization in $\mathbb{C}^d = \mathfrak{p}^\ast +$. Any choice of a Harish-Chandra strongly orthogonal root determines a one-dimensional split subgroup $A = \mathbb{R}^+$ of $G$ and a Heisenberg parabolic subgroup $P = MAN$ of $G$. We study the induced representation $I(\nu) = \text{Ind}_P^G(1 \otimes e^\nu \otimes 1)$ of $G$ from $P$. We shall find its complementary series, the reduction points, explicit realization of certain finite dimensional representations, and unitarizable subrepresentations.

The representation $I(\nu)$ can be realized on the $L^2$-space on a homogeneous space $K/L$ of $K$, $L = M \cap K$, and $K/L$ is an orbit of $K$ in $\mathbb{C}^d$, with the Harish-Chandra root vector as a base point. It is a circle bundle $K/L \to K/L_0$ over its projectivization $K/L_0$. The space $K/L_0$ itself is a compact Hermitian symmetric space. We can find the irreducible decomposition of $L^2(K/L)$ by using the Cartan-Helgason theorem for line bundles over $K/L_0$. We study then the Lie algebra action of $\mathfrak{g}$ on $L^2(K/L)$.

The induced representations $\text{Ind}_P^G(\nu)$ from Heisenberg parabolic subgroup $P$ can be viewed as the counterpart of the representations $\text{Ind}_Q^G(\nu)$ from the Siegel parabolic subgroup $Q$, both $P$ and $Q$ being maximal parabolic subgroups. The nilpotent part in $P$ is a Heisenberg nilpotent group whereas it is abelian in $Q$. The representations $\text{Ind}_Q^G(\nu)$ and their analogues are well studied and are of considerable interests as they are closely related to the holomorphic discrete series $[15]$. The general case of maximal parabolic subgroups with abelian nilradical can be put in the setup of

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Koecher’s construction of Lie algebras and the corresponding induced representations have also been intensively studied; see e.g. [16, 21]. The analysis is done in [16] by using eigenvalues of intertwining differential operators and the tensor product structure of induced representations, and in [21] by computing differentiations and recursions of spherical polynomials along a torus in the symmetric space $K/L$. However in our case the nilpotent group $N$ is non-abelian and $K/L$ is not symmetric, there is no known construction of intertwining operators, so the method in [16] seems not possible, and we shall adapt the method in [21]. Now the manifold $K/L_0$ is a complex manifold and the differentiation involves vector fields in $T^{(1,0)}(K/L_0)$, we have to develop further the technique in loc. cit. We consider the differentiation along products of projective sphere $SU(2)/U(1)$ in $K/L_0$ and find the differentiation formulas by considering classical spherical polynomials of $SU(2)$. By using Weyl group symmetry of spherical functions we find then the required formulas for the Lie algebra actions.

The study of induced representations from Heisenberg parabolic subgroups can be put in a rather general context. It has been proved by Howe [5] that in most semisimple Lie groups $G$ there are Heisenberg parabolic subgroups, and these groups have been all classified. The introduction of them is closed related to Howe’s notion of the rank and minimal representations. Recently Frahm [2] has studied the intertwining differential operators for the induced representations for Heisenberg parabolic subgroups in some non-Hermitian Lie groups. Also spherical duals of classical groups have been studied extensively; see [1, 10, 11]. When $G = SU(p, q)$ the induced representations can be realized on a homogeneous cone in $\mathbb{C}^{p+q}$ and they have been studied by Howe-Tan [6] in greater details.

We mention that the space $K/L_0$ has an interesting geometry, it is the variety of minimal rational tangents in a fixed tangent space $V = T^{(1,0)}_0(D)$ of the symmetric space $D$; more precisely it is the projectivization $\mathbb{P}(V)$ of all tangent vectors $v \in V$ with maximal holomorphic sectional curvature. It is also projectivization of the space $K/L$ of minimal tripotents and plays important role in complex differential geometry; see [7, 13].

The paper is organized as follows. In Section 1 we introduce the parabolic subalgebra $p = m + a + n = m + \mathbb{R}\xi + n$ and the principal series $(I(\nu), \pi_\nu)$. In Section 2 we find the irreducible decomposition for $I(\nu) = L^2(G/P) = L^2(K/L)$ under $K$. The action of $\pi_\nu(\xi)$ on $I(\nu)$ is done in Section 3 and it is one of our main results. As consequences we find the complementary series, certain unitarizable subrepresentations, and also realization of certain finite dimensional representations of $\mathfrak{g}^\mathbb{C}$ in the induced representations in Section 4. In Section 5 we treat the case when $g = \mathfrak{su}(d, 1), \mathfrak{sp}(r, \mathbb{R})$ where $K/L$ is the sphere and $K/L_0$ is the complex projective space. We give a simpler proof of the classical result of Johnson-Wallach [8]. In Appendix A we compute certain recurrence formulas for spherical polynomials over the complex projective line $\mathbb{P}^1$; we need only the leading coefficients in the formulas which can be easily proved by other methods, but we
present the complete formulas as they might be of independent interests in special functions. The complete list of the spaces $G/K$ and $K/L_0$ are given in Appendix [B].

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**Notation.** Hermitian symmetric spaces have rather rich structure so is the notation. For the convenience of the reader we list the main symbols used in our paper.

1. $D = G/K$, bounded symmetric domain of $D$ of rank $r$ realized in the Jordan triple $V = \mathbb{C}^d$ with Jordan triple product $\{x,\bar{y},z\} = D(x,y)z$ and Jordan characteristic $(a,b)$ (or root multiplicities); $d = r + \frac{1}{2}r(r-1) + rb$, the dimension.
2. $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$, Cartan decomposition, $\mathfrak{p} = \{\xi(v) = v + Q(z)\bar{v}; v \in V\}$ as holomorphic vector fields on $D$.
3. $\mathfrak{g}^C = \mathfrak{p}^- + \mathfrak{t}^C + \mathfrak{p}^+$, Harish-Chandra decomposition respect to the center element $Z$ of $\mathfrak{t}$, $\text{Ad}(Z)|_{\mathfrak{p}^+} = \pm i; \mathfrak{p}^+ = V$.
4. $e = e_1$, a fixed minimal tripotent, $V = V_1 + V_1 + V_0 = V_1(e) + V_1(e) + V_0(e)$, the Peirce decomposition with respect to $e$, and $\{e,v_1,v,\bar{w}\}$ a Jordan quadrangle.
5. $\{e,e,D(e,e)\}$, standard $\mathfrak{sl}(2)$-triple; $H_0 = iD(e,e)$.
6. $\xi = \xi(e) = e + \bar{e} \in \mathfrak{p}, a = \mathbb{R}\xi, \mathfrak{g} = \mathfrak{n} - 2 + \mathfrak{n} - 1 + \mathfrak{m} + \mathfrak{a} + \mathfrak{n}_1 + \mathfrak{n}_2$, the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$, $\mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2$, a Heisenberg Lie algebra; $\rho_0 = 1 + (r-1)a + b$, half sum of positive roots.
7. $M, A, N$, the corresponding subgroups with Lie algebra $M, A, N, L = M \cap K = \{k \in K; ke = c\} \subseteq K$ with Lie algebra $\mathfrak{l}$.
8. $\mathfrak{k} = [\mathfrak{t},\mathfrak{t}]$, the semisimple part of $\mathfrak{t}$, and $K_1 \subset K$ the corresponding Lie group.
9. $S = K/L = G/P$, the manifold of rank one tripotents.
10. $S_1 = \mathbb{P}(S) = K/L_0 = K_1/L_1$, the projective space of $S$, also called the variety of minimal rational tangents; $L_0 = \{k \in K; ke = \chi(k)e, \chi(k) \in U(1)\}$ the subgroup of elements of $K$ fixing the line $Ce$ with Lie algebra $I_0 = \{X \in \mathfrak{t}; Xe = \chi(X)e, e \in \mathbb{C}\}$. $S_1 = K/L_0 = K_1/L_1$, semisimple Hermitian symmetric space of rank two of dimension $d_1 = \dim_\mathbb{C} V_1 = (r-1)a + b, (a_1, b_1)$, the Jordan characteristic of $S_1$;
11. $\chi_l(k) = \chi(k)^l$, character on $L_0, l \in \mathbb{Z}$.
12. $\mathfrak{t} = \mathfrak{q} + \mathfrak{l}_0, \mathfrak{t}_1 = \mathfrak{q} + \mathfrak{l}_1$ Cartan decomposition for the symmetric space $S_1 = K/L_0 = K_1/L_1; \mathfrak{q} = \{D(v,e) - D(e,v), v \in V_1\}; \mathfrak{l}_1 = \mathfrak{l}_0 \cap \mathfrak{t}_1$, the semisimple component of $\mathfrak{t}_0$.
13. $\mathfrak{t}^C = \mathfrak{q}^- + \mathfrak{l}_1^C + \mathfrak{q}^+, \mathfrak{t}_1^C = \mathfrak{q}^- + \mathfrak{l}_1^C + \mathfrak{q}^+$, the Harish-Chandra decomposition of $\mathfrak{t}^C, \mathfrak{t}_1^C$ for the Hermitian symmetric space $K/L_0 = K_1/L_1; \mathfrak{q}^+ = \{D(v,e); v \in V_1\}; \mathfrak{q}^- = \{D(e,v); v \in V_1\}$; the Jordan triple defined on $\mathfrak{q}^+ = \{D(v,e); v \in V_1\}$ is via the Lie bracket,

$$[[D(v,e),D(e,w)],D(v,e)] = D(D(v,w)u,e),$$
and is isomorphic to the Jordan triple \( V_1 \subset V \); \( q^+ \) is the the holomorphic tangent space of \( S_1 = K/L_0 = K_1/L_1 \) at \( L_0 \in K/L_0 \).

(14) \( \mathfrak{k}_1^* = \mathfrak{i}_1 + iq \), the non-compact dual of \( \mathfrak{k}_1 = \mathfrak{i}_1 + q \); \( \mathfrak{h}_{1\mathbb{R}} \subset iq \), Cartan subspace.

(15) \( \rho := \rho_1^* = \rho_1 \alpha_1 + \rho_2 \alpha_2 = (\rho_1, \rho_2), \) \( \rho_1 = 1 + a_1 + b_1, \rho_2 = 1 + b_1 \), half sum of positive restricted roots of \( \mathfrak{k}_1^* \) with respect to \( \mathfrak{h}_{1\mathbb{R}} \).

(16) \( \Phi_{\lambda,l} \), Harish-Chandra spherical function for the symmetric pair \( (\mathfrak{k}_1^*, \mathfrak{l}_1) \) with one-dimensional character \( \chi_l \), \( C(\lambda, l) = C(\lambda, -l) \). Harish-Chandra \( C \)-function for the symmetric space \( S_1 = K/L_0 = K_1/L_1 \) with character \( \chi_l \); \( \phi_{\mu,l} \) spherical polynomial.

1. Preliminaries

We shall use the Jordan triple description of Hermitian symmetric spaces; see [12] [19].

1.1. Hermitian symmetric space \( D = G/K \). Let \( D \) be an irreducible bounded symmetric domain of rank \( r \) in \( V = \mathbb{C}^d \). Let \( G \) be the group of bi-holomorphic automorphisms of \( D \), and \( K = \{ k \in G; k0 = 0 \} \) the maximal compact subgroup of \( G \), so that \( D = G/K \). The space \( V \) has the structure of an irreducible Jordan triple with triple product \( \{ x, y, z \} = D(x, y)z \) with the corresponding \( \text{End}(V, V) \)-valued quadratic form \( Q(x), Q(x) \overline{y} = \frac{1}{2} D(x, y)x \). Note that in [12] \( D(x, y) \) is written as \( D(x, y) \), and to ease notation we write it just as \( D(x, y) \) so it is conjugate linear in \( y \). Let \( (a, b) \) be the Jordan characteristic of \( V \), and \( b = 0 \) when \( D \) is of tube domain. The dimension \( d = r + \frac{q}{2}r(r - 1) + rb \).

Let \( \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \) be the Cartan decomposition of \( \mathfrak{g} \). Realized as holomorphic vector fields on \( D \), i.e., as \( V \)-valued functions on \( D \), the space \( \mathfrak{p} \) is

\[
\mathfrak{p} = \{ \xi(v) = v - Q(z)\overline{v}; v \in V \}.
\]

The adjoint action \( v \mapsto \text{ad}(k)v \) of \( K \) as well as \( \mathfrak{k} \) on \( \mathfrak{p} \) coincides with its defining action on \( D \) and will be written just as \( kv = \text{ad}(k)v, Xv = \text{Ad}(X)v, k \in K, X \in \mathfrak{k} \) when no confusion would arise.

Denote \( Z \in \mathfrak{k} \) the central element defining the complex structure of \( \mathfrak{p} \) and \( \mathfrak{g}^\mathbb{C} = \mathfrak{p}^+ + \mathfrak{t}^\mathbb{C} + \mathfrak{p}^- \) be the Harish-Chandra decomposition, \( Z|_{\mathfrak{p}^+} = \pm i \). The spaces \( \mathfrak{p}^+ \) is identified with \( V \) via the identification \( V \ni v = \frac{1}{2}(\xi v - i\xi w) \in \mathfrak{p}^+ \) and \( \overline{V} = \{ \overline{v}; v \in V \} \) with \( -Q(z)v = \mathfrak{p}^- \). The Lie algebra \( \mathfrak{t} = \mathfrak{k}_1 \oplus \mathbb{R}(iZ) \), where \( \mathfrak{k}_1 = [\mathfrak{k}, \mathfrak{k}] \) is the semisimple part of \( \mathfrak{k} \) with trivial center. Let \( K_1 \subset K \) be the corresponding semisimple subgroup of \( K \) with Lie algebra \( \mathfrak{k}_1 \).

We fix the Euclidean inner product on \( V \) so that a minimal tripotent has norm 1, and fix the corresponding normalization of the Killing form on \( \mathfrak{g}^\mathbb{C} \). All orthogonality in Lie algebra \( \mathfrak{g}^\mathbb{C} \) below is with respect to the Killing form.

1.2. Maximal parabolic subgroup \( P = MAN \) of \( G \) and induced representation \( \text{Ind}_{\mathfrak{p}}^G(\nu) \). We fix in the rest of the paper a minimal tripotent \( e = e_1 \) and denote

\[
\xi = \xi_e, \quad H_0 = i D(e, e), \quad a = i R \xi \in \mathfrak{p}.
\]
A Harish-Chandra strongly orthogonal root $\gamma_1$ for $g^\mathbb{C}$ can be chosen so that its co-root is $D(e,e)$, $\gamma_1(D(e,e)) = 2$. We shall only need $\gamma_1$ below.

The Peirce decomposition of $V = \mathbb{C}^d$ with respect to the tripotent $e$ is

\begin{equation}
V = V_2 + V_1 + V_0, \quad V_j = V_j(e) := \{ v \in V; D(e,e)v = jv \}, \quad j = 0, 1, 2,
\end{equation}

Furthermore $V_2 = \mathbb{C}e$ is one dimensional, $V_1$ is of dimension $d_1 = \dim \mathbb{C}V_1 = (r-1)a + b$, and $V_0$ is a Jordan triple of rank $r - 1$ and dimension $1 + \frac{r}{2}a(r-1)(r-2) + (r-1)b$. The Jordan rank of $V_1$ is

\begin{equation}
\text{rank} \ V_1 = \begin{cases} 
2, & g = \mathfrak{su}(d,1), \mathfrak{sp}(m,\mathbb{R}) \\
1, & g = \mathfrak{su}(d,1), \mathfrak{sp}(m,\mathbb{R}). 
\end{cases}
\end{equation}

Certain computations have to be done depending on the different cases.

We shall need the description for the root spaces of $g$ under $a = \mathbb{R}\xi$.

**Lemma 1.1.** The root space decomposition of $g$ under $a = \mathbb{R}\xi$ is

\begin{equation}
g = \mathfrak{n}_+ + (\mathfrak{m} + a) + \mathfrak{n}, \quad \mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2, \quad \mathfrak{n}_- = \mathfrak{n}_{-1} + \mathfrak{n}_{-2},
\end{equation}

where $\mathfrak{n}_{\pm 2}$, $\mathfrak{n}_{\pm 1}$, and $\mathfrak{m} + a$ are the root spaces of $\xi$ with roots $\pm 2, \pm 1, 0$, respectively. The subspaces are given by

\begin{equation}
\mathfrak{m} = \mathfrak{l} \oplus \{ \xi_v; v \in V_0 \}, \quad \mathfrak{l} = \mathfrak{m} \cap \mathfrak{k} = \{ X \in \mathfrak{k}; Xe = 0 \},
\end{equation}

\begin{equation}
\mathfrak{n}_1 = \{ \xi_v + (D(e,v) - D(v,e)); v \in V_1 \},
\end{equation}

and

\begin{equation}
\mathfrak{n}_2 = \mathbb{R}(\xi_i - H_0).
\end{equation}

The half sum of positive roots is

\begin{equation}
\rho_0 := 1 + (r-1)a + b = 1 + \dim \mathbb{C}V_1.
\end{equation}

**Proof.** The root spaces are described in [12, Lemma 9.14]. The root space $\mathfrak{n}_1$ has real dimension $2 \dim \mathbb{C}V_1$ and thus $\rho_0 = 1 + \dim \mathbb{C}V_1 = 1 + (r-1)a + b$. \qed

The nilpotent algebra $\mathfrak{n}$ is a Heisenberg Lie algebra. Their appearance in general semisimple Lie algebras has been classified in [5].

Let $M, A, N, L = M \cap K = \{ k \in K; ke = e \}$, be the corresponding Lie subgroups and $P = MAN$ the parabolic group of $G$.

A linear functional $\nu$ on $a^\mathbb{C}$ will be identified as $\nu \in \mathbb{C}, \nu = \nu(\xi)$. The main object of this paper is the induced representation

\begin{equation}
I(\nu) = \text{Ind}^G_P(\nu) := \text{Ind}^G_P(1 \otimes e^\nu \otimes 1)
\end{equation}

defined as the space of measurable functions on $G$ such that

\[ f(gme^{t\xi}n) = e^{-\nu} f(g), m \in M, n \in N, t \in \mathbb{R}, \]
and $f|_K \in L^2(K)$. Any $f \in I(\nu)$ as function on $K$ is right $L$-invariant, and thus $f \in L^2(K/L)$. The corresponding $(\mathfrak{g}^C, K)$-representation, $\mathfrak{g}^C$ being the complexification $\mathfrak{g}$, will also be denoted by $I(\nu)$. If $\nu = \rho_k + i\lambda, \lambda \in \mathbb{R}, \lambda \neq 0$, then $I(\nu)$ is a unitary irreducible representation of $G$ on $L^2(K/L)$; see e.g. [9].

2. Decomposition of $(L^2(K/L), K)$ and spherical polynomials

We assume throughout Sections 2 and 3 that the Hermitian symmetric symmetric domain $D = G/K$ is irreducible of rank $r \geq 2$ and is not the Siegel domain $Sp(r, \mathbb{R})/U(n)$; this case and the rank one domain $SU(d, 1)/U(d)$ will be treated in Section 5.

2.1. The homogeneous space $S = G/P = K/L$ as circle bundle over compact Hermitian symmetric space $S_1 = K/L_0 = K_1/L_1$. The homogeneous space $S := G/P = K/L$ of $K$ can be realized as the manifold of rank one tripotents in $V$, $S = Ke \subset V$. We shall fix this realization in the rest of the paper and use the global coordinates on $V$ for $S$ when needed.

To find the decomposition of $L^2(K/L)$ under $K$ we consider the projectivization

$$S \rightarrow S_1 = \mathbb{P}(S) = \{[v] := \mathbb{C}v \in \mathbb{P}(V) ; v \in S\}, \quad v \mapsto [v]$$

of $S$ in the projective space $\mathbb{P}(V)$ of $V$. Then $S_1 = K/L_0$, where $L_0 = \{k \in K ; ke \in \mathbb{C}e\}$, and $L \subset L_0$ is a normal subgroup with $L_0/L$ being the circle group $U(1)$. The natural map $S = K/L \rightarrow S_1 = \mathbb{P}(S) = K/L_0$ defines a fibration

$$S = K/L \rightarrow S_1 = K/L_0,$$

of $\mathbb{P}(S)$ with fiber the circle $U(1) = L_0/L$.

The space $S_1 = \mathbb{P}(S)$ is a compact Hermitian symmetric space of (non-simple) $K$. The complete list of $(G, K, L_0)$, $D = G/K$, $S = G/P = K/L \rightarrow S_1 = K/L_0$ is given in Tables [1][2]. The space $S_1 = K/L_0$ is also the variety of minimal rational tangents (VMRT) [7] of the symmetric space $D$.

The action of $L_0$ on $e$ defines a character of $\chi : L_0 \rightarrow U(1)$, namely

$$L_0 = \{k \in K ; ke \in \mathbb{C}e\} = \{k \in K ; ke = e(k)e\}.$$

The spaces $S$ and $S_1$ are also homogeneous space of the semisimple part $K_1 \subset K$. Let $L_1 = \{k \in K_1 ; ke = e(k)e\} \subset L_0$, then $S_1 = K/L_0 = K_1/L_1$, $S = K/L \cap L_1$. The fibration (2.1) above becomes a circle bundle $S = K/L \rightarrow S_1 = K/L_0 = K_1/L_1$ for $K_1$-homogeneous spaces with the fiber $U(1) = L_1/L \cap L_1$.

The element $exp(\pi H_0), H_0 = iD(e, e)$, defines a Cartan involution $Ad \exp(\pi H_0)$ on $\mathfrak{k}$ with the corresponding Cartan decomposition

$$\mathfrak{k} = \mathfrak{l}_0 + \mathfrak{q}.$$

The space $\mathfrak{q}$, similar to (1.1), is represented by

$$q = \{D(v, e) - D(e, v) ; v \in V_1\}. $$
We fix now a complex structure on \( q \) and the corresponding Harish-Chandra decomposition of \( \mathfrak{t}^C \).

**Lemma 2.1.** Define the \( K \)-invariant complex structure on \( S_1 = K/L_0 \) by the element \( iD(e, e) \in \mathfrak{l} \), i.e. by \( \frac{1}{2} \text{Ad}(iD(e, e)) \in \mathfrak{l} \) on \( \mathfrak{q} \mathfrak{q} = T[\mathfrak{c}](S_1) \), at the base point \([\mathfrak{c}] = \mathbb{C}eS_1\). In terms of elements in \( \mathfrak{l}'_1 \) in the semi-simple part \( \mathfrak{t}^C_1 \) of \( \mathfrak{t}_1^C \), it is the same as

\[
- \frac{1}{2} \text{Ad}(H_0)|_q = \frac{1}{2} \text{Ad}(Z')|_q, \quad Z' = \frac{i}{d}Z - H_0 \in \mathfrak{l}_1 \subset \mathfrak{t}_1.
\]

The Harish-Chandra decomposition of \( \mathfrak{t}^C \) is

\[
\mathfrak{t}^C = \mathfrak{q}^- + \mathfrak{l}'_0 + \mathfrak{q}^+
\]

with

\[
T^{(1,0)}[\mathfrak{c}] = \mathfrak{q}^+ = \{D(v, e); v \in V_1\}, \quad T^{(0,1)}[\mathfrak{c}] = \mathfrak{q}^- = \{-D(v, e); v \in V_1\}.
\]

**Proof.** The elements \( cZ, c \in \mathbb{R} \) is in the center of \( \mathfrak{t} \) so that \( \text{Ad}(X) = \text{Ad}(cZ + X) \) on \( \mathfrak{t} \) for any \( X \in \mathfrak{t} \). The semisimple component of \( X \) in \( \mathfrak{t}_1 \) is obtained as \( X - \frac{1}{2} \text{tr} \text{Ad}(X)_{p^+}, d = \dim V \). Thus we have \( (2.4). \) The Harish-Chandra decomposition is obtained by the commutator formula in Jordan triples \([12]\).

\[
[D(u, v), D(x, y)] = D(D(u, v)x, y) - D(x, D(v, u)y).
\]

Here the complex structure on \( q \) is chosen so that

\[
v \in V_1 \subset V = p^+ \rightarrow D(v, e) \in q^+
\]

is complex linear so that the complex structures in \( p \) and \( q \) match in this sense. The Lie algebra structure in \( \mathfrak{t}^C \) defines \( q^+ \) as a Jordan triple, and it is isomorphic to \( V_1 \). To avoid confusion we shall keep the notation \( q^+ \).

2.2. **Cartan subalgebra** \( \mathfrak{h}_{\mathfrak{q}} \subset i\mathfrak{q} \) and the restricted root system for the non-compact symmetric pair \((\mathfrak{t}^*, \mathfrak{l}_0) = (\mathfrak{l}_0 + i\mathfrak{q}, \mathfrak{l}_0)\). We construct now split Cartan subalgebra in \( i\mathfrak{q} \) for the symmetric pair \((\mathfrak{t}^*, \mathfrak{l}_0) = (\mathfrak{l}_0 + i\mathfrak{q}, \mathfrak{l}_0)\) and its semisimple part \((\mathfrak{t}^*_1, \mathfrak{l}_1) = (\mathfrak{l}_1 + i\mathfrak{q}, \mathfrak{l}_1)\), and find the corresponding root system. They will be used in the decomposition of \( L^2(K/L) \) and computation of Harish-Chandra \( c \)-function. We need the notion of a Jordan quadrangle \([14]\) p. 12, p.16).

An ordered quadruple \((u_0, u_1, u_2, u_3)\) of minimal tripotents is called Jordan quadrangle if the following three conditions are satisfied, for all \( i \) modulo 4,

1. \( u_i \) and \( u_{i+1} \) are in each other’s Peirce \( V_1 \)-space, \( u_i \in V_1(u_{i+1}), u_{i+1} \in V_1(u_i)\);
2. \( u_i \) and \( u_{i+2} \) are orthogonal as tripotents;
3. \( D(u_i, u_{i+1})u_{i+2} = u_{i+3} \).
Recall that we have assumed in this section that the domain $D \neq III_n = Sp(n, \mathbb{R})/U(n), D \neq I_{n,1} = SU(n, 1)/U(n)$ (the Type IV domain IV$_3$ = III$_2$ is also excluded). Then starting with the fixed minimal tripotent $e$ there are minimal tripotents $v_1, w, v_2$ such that $(u_0, u_1, u_2, u_3) = (e, v_1, w, v_2)$ is a Jordan quadrangle. This is implicitly in [14] where orthogonal bases (called grids) are constructed for Jordan triple systems, and we provide brief arguments. The Jordan triple system $V$ is of rank $r \geq 2$ so there exists a Jordan algebra $V'$ as a sub-triple of $V$ of rank two with $e_1 + e_2$ as identity element, where $e_1 = e, e_2$ are the Harish-Chandra strongly orthogonal root vectors; for

$$D = I_{r, r+4}, II_{2r}, II_{2r+1}, IV_n (n > 4), V, VI$$

the corresponding $D'$ is

$$D' = I_{2,2}, II_4, II_4, IV_4, IV_8, IV_{10}.$$ 

In all cases the Jordan algebra $M_{2,2}$ of square $2 \times 2$-matrices, $D = I_{2,2} = IV_4$, forms a Jordan sub-triple, since

$$I_{2,2} = IV_4 \subset II_4 = IV_6 \subset IV_8 \subset IV_{10}$$

in the sense of Jordan sub-triples. The following standard matrices

$$E_{11}, E_{12}, E_{22}, E_{21}$$

form a Jordan quadrangle in $M_{2,2}$ and in $V$.

Fix in the rest of the paper the Jordan quadrangle $\{e, v_1, w, v_2\}$. We have

$$(2.7) \quad H_0 v_1 = iD(e, e)v_1 = iv_1, H_0 v_2 = iD(e, e)v_2, D(v_1, v_1)e = D(v_2, v_2)e = e,$$

and

$$(2.8) \quad D(e, w) = D(v_1, v_2) = 0, D(v_1, e)v_1 = D(v_2, e)v_2 = 0, D(e, v_1)w = v_2, D(v_1, e)v_2 = w,$$

which we shall use below. See [14] p. 12, p.16] for further details.

The above construction results in the following two commuting copies of $sl(2, \mathbb{C})$-triples in $\mathfrak{t}_1^\mathbb{C}$,

$$(2.9) \quad E_j^+ = D(v_j, e) \in \mathfrak{q}^+, E_j^- = D(e, v_j) \in \mathfrak{q}^-, H_j = D(v_j, v_j) - D(e, e) \in \mathfrak{t}_1^\mathbb{C}, E_j := E_j^+ - E_j^- \in \mathfrak{q},$$

with the canonical relation

$$[H_j, E_j^\pm] = 2E_j^\pm, [E_j^+, E_j^-] = H_j, j = 1, 2.$$ 

Moreover $\mathbb{R}(iE_1) + \mathbb{R}(iE_2) \subset i\mathfrak{q}$ is maximal abelian.

**Definition 2.2.** Let

$$(2.10) \quad \mathfrak{h}_0 = \mathbb{R}E_1 + \mathbb{R}E_2 \subset \mathfrak{q}, \quad \mathfrak{h}_{i\mathfrak{q}} = \mathbb{R}(iE_1) + \mathbb{R}(iE_2) \subset i\mathfrak{q}, \quad \mathfrak{h}_0^\mathbb{C} = CE_1 + CE_2.$$ 

Extend the abelian subalgebra $CE_1 + CE_2$ of $\mathfrak{t}_1^\mathbb{C}$ to a Cartan subalgebra

$$\mathfrak{h}_0^\mathbb{C} := (CE_1 + CE_2) \circ \mathfrak{h}_+, \quad \mathfrak{h}_+ \subset l_1 \subset \mathfrak{t}_1^\mathbb{C},$$
of \( \mathfrak{k}^C \), so that
\[
\mathfrak{h}^C := (\mathbb{C} \mathfrak{Z} + \mathbb{C} \mathfrak{E}_1 + \mathbb{C} \mathfrak{E}_2) \oplus \mathfrak{h}_+
\]
is a Cartan subalgebra of \( \mathfrak{k}^C \) and \( \mathfrak{g}^C \). Define \( \{\alpha_0, \alpha_1, \alpha_2\} \) to be the dual basis vectors of \( \{i\mathfrak{Z}, i\mathfrak{E}_1, i\mathfrak{E}_2\} \) that are vanishing on \( \mathfrak{h}_+ \).

Now
\[
\tag{2.11}
(\mathbb{C} \mathfrak{Z} + \mathbb{C} \mathfrak{H}_1 + \mathbb{C} \mathfrak{H}_2) \oplus \mathfrak{h}_+ \subset \mathfrak{k}^C \subset \mathfrak{g}^C
\]
is a Cartan subalgebra of three algebras \( \mathfrak{l}_0^C, \mathfrak{k}^C \) and \( \mathfrak{g}^C \).

We shall need the Cartan-Helgason theorem in [17] for line bundles over \( K/L_0 \) defined by the characters \( \chi_l = \chi^l \); the character in [17] is defined using Cayley transform and Cartan subalgebras instead the geometric definitions here. The relevant Cayley transforms in our setup is
\[
c = c_l = \operatorname{ad} \left( \exp(-\frac{\pi i}{4}(E_1^+ + E_2^+ + E_1^- + E_2^-)) \right).
\]

**Lemma 2.3.**

1. The subspace \( \mathfrak{h}_{iq} \) is maximal abelian in \( \mathfrak{i}\mathfrak{q} \). If \( \mathfrak{g} = \mathfrak{su}(r + b, r), r > 1 \) then \( K/L_0 \) is an irreducible symmetric space and the restricted root system for the non-compact dual \( \mathfrak{k}^* = \mathfrak{l}_0 + i\mathfrak{q} \) with respect to \( \mathfrak{h}_{iq} \) is
\[
R(\mathfrak{k}^*, \mathfrak{h}_{iq}) = R(\mathfrak{k}_1, \mathfrak{h}_{iq}) := \{\pm 2\alpha_1, \pm 2\alpha_2\} \cup \{\pm \alpha_1 \pm \alpha_2\} \cup \{\pm \alpha_1, \pm \alpha_2\}
\]
with root multiplicities \( (1, a_1, 2b_1) \) for the three subsets of roots, \( a_1, b_1 \) being given in Tables I, II. The half-sum of the positive roots with respect to the ordering \( \alpha_1 > \alpha_2 > 0 \) is
\[
\rho := \rho_{\mathfrak{t}_1} = \rho_1 \alpha_1 + \rho_2 \alpha_2 = (\rho_1, \rho_2), \quad \rho_1 = 1 + a_1 + b_1, \rho_2 = 1 + b_1.
\]
The two linear functionals \( \rho_{\mathfrak{t}_1} \) and \( \rho_{\mathfrak{g}} \) are related by
\[
\rho_{\mathfrak{g}} = 1 + \rho_1 + \rho_2.
\]

If \( \mathfrak{g} = \mathfrak{su}(r + b, r), r > 1 \) then \( K/L_0 = K_1/L_1 \) is reducible, \( \mathfrak{t}_1 = \mathfrak{su}(r + b) + \mathfrak{su}(r), \mathfrak{t}_1^* = \mathfrak{su}(1, r + b - 1) + \mathfrak{su}(1, r - 1) \), and the restricted root system is
\[
R(\mathfrak{t}_1^*, \mathfrak{h}_{iq}) = R(\mathfrak{l}_1^*, \mathfrak{h}_{iq}) := \{\pm 2\alpha_1, \pm \alpha_1\} \cup \{\pm 2\alpha_2, \pm \alpha_2\}
\]
with root multiplicities \( (1, r + b - 1), (1, r - 1) \) respectively. The corresponding \( \rho \) is
\[
\rho_{\mathfrak{t}_1} = \rho_1 \alpha_1 + \rho_2 \alpha_2 = (\rho_1, \rho_2), \quad \rho_1 = r + b - 1, \rho_2 = r - 1.
\]
The relation \( \rho_{\mathfrak{g}} = 1 + \rho_1 + \rho_2 \) holds also in this case.

2. The Cayley transform \( c \) exchanges two Cartan subalgebras
\[
c : (\mathbb{C} \mathfrak{Z} + \mathbb{C} \mathfrak{H}_1 + \mathbb{C} \mathfrak{H}_2) \oplus \mathfrak{h}_+ \rightarrow \mathfrak{h}^C,
\]
\[
c : H_j \rightarrow i\mathfrak{E}_j
\]
and the pullbacks
\[
c^*(2\alpha_j), j = 1, 2, \text{ are the Harish-Chandra orthogonal roots for } (\mathfrak{t}_1^*, \mathfrak{l}_1).
\]
Proof. We have assumed that $D$ is not $SU(d,1)/U(d)$ nor $Sp(n,\mathbb{R})/U(n)$, so that $K/L_0 = K_1/L_1$ is of rank two; see the Tables [12]. The rest is a consequence of general results on root systems of non-compact Hermitian Lie algebra applied to the non-compact Helgason dual $\mathfrak{k}^* = \mathfrak{l}_0 + i\mathfrak{q}$ of the compact $\mathfrak{k} = \mathfrak{t}_0 + i\mathfrak{q}$. It follows also from [12, Lemma 9.14], the Lie algebra $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ there being replaced by our $\mathfrak{k}_1^* = i\mathfrak{q} + \mathfrak{l}_1$. The abelian subspace in $i\mathfrak{q}$ is obtained from the Harish-Chandra root vectors corresponding to a frame of minimal tripotents in $q^+$, and in the present case the frame in $q^+$ is $\{D(v_1,e), D(v_2,e)\}$. Finally the dimension of a general Jordan triple of characteristics $(r,a,b)$ is $r + \frac{1}{2}r(r-1)a + rb$. We have by Lemma [12], $\rho_0 = 1 + \dim\mathbb{C} V_1$ and $\dim\mathbb{C} V_1 = 2 + a_1 + 2b_1 = \rho_1 + \rho_2$ since $V_1$ is a Jordan triple with characteristic $(2, a_1, b_1)$, and $\rho_0 = 1 + \rho_1 + \rho_2$. The fact about the Cayley transform is well-known; see e.g. [17, 12, Proposition 10.6(3)].

We note that the evaluation of character $\chi$ on $H_0, H_1, H_2$ is given by

$$\chi(H_0) = 2\chi(Z) = 2i, \chi(H_1) = \chi(H_2) = -i,$$

since $H_0 e = 2Ze = 2ie, H_je = i(D(v_j, v_j) - D(e, e))e = -ie$. We observe and keep in mind that the geometrically natural choice of the complex structure of $S_1 = K/L_0 = K_1/L_1$ results in some discrepancy: For $H_o = iD(e,e), \text{ad}(-iH_0) = \text{ad}(D(e,e)), \text{has non-negative eigenvalues} 2, 1, 0$ on $p^+ = V = V_2 + V_1 + V_0$, whereas it has negative eigenvalue $-1$ on $q^+$,

$$[D(e,e), D(v,e)] = -D(v,e), \quad D(v,e) \in q^+.$$

2.3. Cartan-Helgason Theorem for $K/L_0$. Let $L^2(K, L_0, \chi_l)$ be the $L^2$-space of sections of the homogeneous line bundle $K \times_{(L_0, \chi_l)} \times \mathbb{C}$ defined by the $\chi_l^{-1}$ of $L_0$. The space $L^2(K, L_0, \chi_l)$ consists of $f \in L^2(K)$ such that

$$f(gh) = \chi(h)^l f(g), \quad g \in K, h \in L_0, \quad he = \chi(h)e.$$

It follows immediately from the definitions of $L_0$ and $\chi$ that

$$L^2(K/L) = \sum_{l=-\infty}^{\infty} L^2(K, L_0, \chi_l).$$

under the regular left action of $K$.

We shall treat extensively functions on $K$ that are transforming under $L_0$ as in (2.18) and it is convenient to give the following

**Definition 2.4.** An element $f \in L^2(K)$ is called $(l_1, l_2)$-spherical if

$$f(h_1kh_2) = \chi_{l_1}(h_1)\chi_{l_2}(h_2)f(k), \quad h_1, h_2 \in L_0.$$

**Lemma 2.5.** (1) Let $g = su(r + b, r), r > 1$. The space $L^2(K, L_0, \chi_l)$ is decomposed as a sum of irreducible representations of $K$,

$$L^2(K, L_0, \chi_l) = \sum_{\mu} W_{\mu, l},$$
where each \( W_{\mu,l} \) has highest weight given by

\[
(2.20) \quad l\alpha_0 + \mu, \quad \mu = (\mu_1, \mu_2) = \mu_1 \alpha_1 + \mu_2 \alpha_2, \quad \mu_1 \geq \mu_2 \geq |l|, \quad \mu_1 = \mu_2 = l, \text{mod } 2.
\]

Moreover each space \( W_{\mu,l} \) contains a unique vector \((l, l)\)-spherical element \( \phi_{\mu,l} \).

(2) Let \( \mathfrak{g} = \mathfrak{su}(r + b, r) \), \( r > 1 \). The space \( L^2(K, L_0, \chi_l) \) is decomposed as above with

\[
\mu_1, \mu_2 \geq |l| \quad \mu_1 = \mu_2 = l, \text{mod } 2.
\]

**The highest weight vector in \( W_{\mu,l} \) can be chosen as**

\[
f(x) = z_l^{l_0} w^{l_1'} w^{l_2'}, \quad x = zw^* \in S \subset M_{r+b}(\mathbb{C}),
\]

where we have written a rank-one projection \( x \in S \subset M_{r+b}(\mathbb{C}) \) as \( x = zw^* \), \( z \in \mathbb{C}^{r+b} \), \( w \in \mathbb{C}^{r} \), \( |z| = \|w\| = 1 \), and where \((p, q, p', q')\) are subject to the condition

\[
\mu_1 = p + q, \mu_2 = p' + q', l = p - q = p' - q'.
\]

The spherical polynomial \( \phi_{\mu,l} \) in this case is \( \phi_{\mu_1,-l}(zw^*) = \phi_{\mu_1,-l}(z) \phi_{\mu_2,l}(z) \) where \( \phi_{\alpha,-l} \) is \((l, l)\)-spherical polynomial on \( \mathbb{P}(\mathbb{C^n}) \).

**Proof.** (1) The statement for the decomposition of \( L^2(K, L_0, \chi_l) \) as representation of the semisimple group \( K_1 \) is in [17, Theorem 7.2]; our \((\mathfrak{t}_1, \mathfrak{l}_1)\) corresponds to \((\mathfrak{g}, \mathfrak{k})\) there. More precisely our character \( \chi_l \) is precisely the same as \( \chi_{-l} \) in [17]. The character \( \chi_{-l} \) on \( K \) in [17] for the Hermitian symmetric space \( D = G/K \subset \mathbb{C}^d = p^\perp \) is defined by

\[
X \in \mathfrak{t}^\mathbb{C} \mapsto \frac{l}{\text{tr Ad } D(e, e)|_{p^\perp}} \text{tr Ad } X|_{p^\perp};
\]
equivalently it is determined [17 (5.1)] by \( \chi_{-l} : D(e, e) \mapsto l \). For our symmetric pair \((\mathfrak{t}_1, \mathfrak{l}_1)\) the corresponding \( D(e, e) \) is \( H_1 = D(v_1, v_1) - D(e, e) \) described in Lemma 2.3 and \( H_1 e = -l \) and thus \( \chi_l(H_1) = -l \). (Alternatively we can also prove this by computing using the duality relation in Appendix B.) The results in [17] then determine the highest weights of \( W_{\mu,l} \) on \( \mathfrak{h}_q = \mathbb{R}(iE_1) + \mathbb{R}(iE_2) \) as \( \mu_1 \alpha_1 + \mu_2 \alpha_2 \) in our statement.

Finally it is trivial to find the weight of \( W_{\mu,l} \) on the central element \( Z \). The right action of \( \exp(sZ) \) on \( \phi \in W_{\mu,l} \) is, using \( Ze = ie \),

\[
(2.21) \quad \pi_\nu(\exp(sZ)\phi) (h) = \phi(\exp(-sZ)h) = \phi(h \exp(-sZ)) = \exp(-isl)\phi(h).
\]

Thus \( \pi_\nu(Z)\phi = -il\phi \), and \( \pi_\nu(iZ)\phi = l\phi \). Thus the highest weight of \( W_{\mu,l} \) is \( l\alpha_0 + \mu_1 \alpha_1 + \mu_2 \alpha_2 \).

(2) This part is well-known; see e.g. [6, 8].

\[
I(\nu) = L^2(K/L) = \sum_{l=\infty}^{\infty} \sum_{\mu} W_{\mu,l}
\]

Altogether we have now \( I(\nu) \) is
with \( \mu \) being specified above.

**Remark 2.6.** The exact formulas for the highest weight vectors above in the case \( \mathfrak{g} = \mathfrak{su}(r + b, r) \) are not needed in our paper However it is possible to prove Theorem 3.1 below by using the weight vectors instead of spherical vectors; see [6]. Note also that the parametrization of the spherical polynomials on \( \mathbb{P}(\mathbb{C}^n) \) generated by the \((p, q)\)-spherical harmonic polynomial \( z_1^{p-q} \) as \( \phi_{\mu_1, -l} \), \( \mu_1 = p + q, l = p - q \), is due to the our geometric definition of character \( \chi \). Recall that due to (2.16) \( \phi_{\mu, l} \) satisfies

(2.22) 
\[
\phi_{\mu, l}(k e^{iH}) = e^{-iH} \phi_{\mu, l}(k), \quad j = 1, 2,
\]

The same parametrization is used in Appendix A.

2.4. Harish-Chandra \( C \)-function and expansion of the \((l, l)\)-spherical polynomials. A major technical step in the proof of Theorem 3.1 below is to use the Harish-Chandra \( c \)-function to compute certain expansions and differentiations involving the spherical polynomials \( \phi_{\mu, l} \). We recall that the spherical polynomial \( \phi_{\mu, l}(h) = \phi_{\mu, l}(h) \) on \( K/L = K_1/L_1 \) is a special case of the Harish-Chandra spherical function \( \Phi_{\lambda, l}(h) \) for the non-compact pair \((\mathfrak{t}_1^\ast, \mathfrak{l}_1)\) corresponding to the one dimensional character \( \chi_l \) of \( L_1 \subset L_0 \); see [17, 18].

More precisely [4, 17, 18]

(2.23) 
\[
\phi_{\mu, l}(h) = \Phi_{-(\mu + \rho), l}(h), h \in K_1,
\]

where \( \rho = \rho_{\mathfrak{t}_1^\ast} \). The spherical function \( \Phi_{\lambda, l} \) is invariant with respect to the Weyl group \( W(\mathfrak{t}_1^\ast, \mathfrak{h}_q) \) of the root system \( R(\mathfrak{t}_1^\ast, \mathfrak{h}_q) \) in (2.12). Eventually we shall replace \( \lambda \) by \( -i(\mu + \rho) \) and use Weyl group symmetry in \( \mu + \rho \). The leading term of \( \Phi_{\lambda, l}(h) \) is given by the limit formula

(2.24) 
\[
\lim_{H \to \infty} e^{-i(\lambda - \rho)(H)} \Phi_{\lambda, l}(\exp(H)) = C(\lambda, l),
\]

for \( H \) in the positive Weyl Chamber of the root system (2.12), i.e., for \( H = x_1(iE_1) + x_2(iE_2) \in \mathfrak{h}_q, x_1 > x_2 > 0 \), and for \( \text{Re}(i\lambda) = y_1\alpha_1 + y_2\alpha_2, y_1 > y_2 > 0 \); see [4] Ch. IV, Theorem 6.14; Ch. V, Section 4], [18, Theorem 3.6]. In particular

(2.25) 
\[
\phi_{\mu, l}(\exp(H)) = \Phi_{\lambda, l}(\exp(H)) = C(\lambda, l)e^{i(\lambda - \rho)(H)} + L.O.T., H \in \mathfrak{h}_q^C
\]

as an expansion of trigonometric polynomial on the complexification \( \exp(\mathfrak{h}_q^C) \cdot [e] \) of the real torus \( \exp(\mathfrak{h}_q) \cdot [e] \subset K/L_0 = \mathbb{P}(K/L), [e] = \mathbb{C}e \), with lower order terms (L.O.T.) being trigonometric polynomials of lower order in the sense defined by the Weyl Chalmers. Here \( C(\lambda, l) = C(\lambda, -l) \) is the Harish-Chandra \( C \)-function, and in our case it is given by

\[
C(\lambda, l) = c_0 \prod_{\epsilon = \pm 1} \frac{\Gamma\left(\frac{1}{2}(\lambda_1 + \epsilon\lambda_2)\right)}{\Gamma\left(\frac{1}{2}(\lambda_1 + \epsilon\lambda_2)\right)} \prod_{j=1, 2} 2^{-\lambda_j} \Gamma(i\lambda_j)
\]

for \( \lambda = \lambda_1\alpha_1 + \lambda_2\alpha_2 \), where \( c_0 \) is normalized so that \( C(-i\rho, 0) = 1 \) for the Harish-Chandra \( C \)-function \( C(-i\rho, 0) \) with trivial line bundle, \( l = 0 \); see loc. cit.. We observe also that the \( C \)-function is positive \( C(\lambda, l) \) for \( \lambda = -i(\mu + \rho) \).
In particular we shall need the spherical polynomials \( \phi_{(1,1),k}(k) \) for \( \mu = (1,1) \). The corresponding representations space of \( K \) is \( p^+ = V \) or \( p^- = \bar{V} \), and the spherical polynomial is the matrix coefficient

\[
\phi_{(1,1),1}(k) = \langle ke, e \rangle, \quad \phi_{(1,1),-1}(k) = \langle e, ke \rangle.
\]

Indeed the space \( p^+ = V \) is a representation of \( \mathfrak{k} \) of highest weight \( \alpha_1 + \alpha_2 \) and representation of \( \mathfrak{k} \) of highest weight \( \alpha_0 + \alpha_1 + \alpha_2 \) since \( Z \) acts as \( i \), the corresponding highest weight vector is

\[
v_0 = \frac{1}{2} \left( (v_1 - ie) + (v_2 + iw) \right).
\]

In other words, recalling that \( e \) is the root vector of the Harish-Chandra strongly orthogonal root \( \gamma_1 \), we see that \( \alpha_1 + \alpha_2 \) is conjugated to \( \gamma_1 \).

3. LIE ALGEBRA \( g \)-ACTION ON \( I(\nu) = \text{Ind}_K^G(\nu) \)

We compute the Lie algebra action of \( g \) on \( I(\nu) \). For that purpose we denote the right differentiation of Lie algebra elements \( X \in g \) on functions \( f \) on \( G \) by \( Xf \),

\[
Xf(g) = \frac{d}{ds} f(g \exp(sX))|_{s=0}.
\]

Then \( X \) commutes with the left regular action

\[
X(f)(hx) = X(f(h))x,
\]

and intertwines the right action \( f(x) \rightarrow f(xh) = f_h(x) \) as

\[
(Xf)_h(x) = (Xf)(xh) = \left( (\text{ad} h(X))f_h \right)(x).
\]

First it follows from (1.2) and (2.2) that \( H_0e = 2ie, \chi_l(\exp(tH_0)) = e^{2it}, \chi_l(H_0) = 2il \). Thus any element \( f \in L^2(K, L_0, \chi_l) \) is an eigenfunction of the differentiation by \( H_0 \),

\[
H_0f = 2ilf.
\]

**Theorem 3.1.** Let \( g \) be a simple Hermitian Lie algebra of rank \( r \geq 2 \) and \( g \neq \mathfrak{sp}(r, \mathbb{R}) \). The action of \( \varpi_\nu(\xi) \) on \( \phi_{\mu,l} \) is given by

\[
2^3 \varpi_\nu(\xi) \phi_{\mu,l} = \sum_{\sigma=(\sigma_1,\sigma_2)=(\pm 1, \pm 1)} \left( \nu + \sigma_1(\mu_1 + \rho_1) + \sigma_2(\mu_2 + \rho_2) - (\rho_1 + \rho_2) \right)
\times \left( c_{\mu,l}(\mu + \sigma, l + 1) \phi_{\mu+\sigma,l+1} + c_{\mu,l}(\mu + \sigma, l - 1) \phi_{\mu+\sigma,l-1} \right),
\]

where the coefficients \( c_{\mu,l}(\mu + \sigma, l \pm 1) \) are given by

\[
c_{\mu,l}(\mu + \sigma, l + 1) = \frac{C(-i(s_\sigma(\mu + \rho)), l)}{C(-i(\sigma_1 \alpha_1 + \sigma_2 \alpha_2 + s_\sigma(\mu + \rho)), l + 1)},
\]

\[
c_{\mu,l}(\mu + \sigma, l - 1) = \frac{C(-i(s_\sigma(\mu + \rho)), l)}{C(-i(\sigma_1 \alpha_1 + \sigma_2 \alpha_2 + s_\sigma(\mu + \rho)), l - 1)}.
\]
and \( s_\sigma \) is in the Weyl group \( W \) of the root system (2.12) such that \( s_\sigma (\alpha_1 + \alpha_2) = \sigma_1 \alpha_1 + \sigma_2 \alpha_2 \). Moreover all the coefficients are positive.

It is understood here that the term \( \phi_{\mu+\sigma_{1,1}\pm 1} = \phi_{(\mu_1+\sigma_1,\mu_2+\sigma_2),\pm 1} \) will not appear in the RHS if \( (\mu_1 + \sigma_1) \alpha_1 + (\mu_2 + \sigma_2) \alpha_2 \) is not one of the highest weights specified in Lemma 2.5.

**Remark 3.2.** It is remarkable that all the coefficients of \( \phi_{\mu+\sigma_{1,1}\pm 1} \) have a rather uniform formula. Actually it is relatively easy to find the coefficient of the leading term \( \phi_{(\mu_1+1,1),\pm 1} \) and the other coefficients can be obtained from the Weyl group symmetry and by unitarity of \( \pi_\nu \) for \( \nu = \rho_\mathfrak{g} + ix, x \in \mathbb{R} \). We shall find all the coefficients independent of the unitarity, and we prove some recursion formulas for spherical polynomials which might be of independent interests [22].

**Proof.** We claim first that for any \( X \in \mathfrak{p}^\mathbb{C} \),

\[
\pi_\nu (X) W_{\mu,1} \subseteq \sum_{\sigma_1,\sigma_2=\pm 1} W_{\mu+\sigma_{1,1}\pm 1}.
\]

This follows by considering the tensor product \( \mathfrak{p}^\mathbb{C} \otimes W_{\mu,1} \) as representation of \( K \). Indeed let \( \mathfrak{g} = \mathfrak{su}(r+b,r) \). The adjoint action of the central element \( Z \in \mathfrak{k} \) on \( \mathfrak{p}^\mathbb{C} \) is \( \pm i \), and its right action on \( W_{\mu,1} \) is \( il \). Let \( X \in \mathfrak{p}^\mathbb{C} \), then \( \pi_\nu (X) W_{\mu,1} \) is of weight \( i(l+1) \) under \( Z \) for any \( X \in \mathfrak{p}^\mathbb{C} \). It is also a classical fact that the highest weights in the tensor product decomposition of \( W_{\mu,1} \otimes \mathfrak{p}^\mathbb{C} \) under \( \mathfrak{t}^\mathbb{C} \) are of the form \( \mu + l \alpha_0 + \nu' \) where \( \nu' \) is a weight appearing in \( \mathfrak{p}^\mathbb{C} \). The space \( \mathfrak{p}^\mathbb{C} \) is of highest weight \( (1,1) = \alpha_1 + \alpha_2 \) under the Cartan subalgebra \( (\mathfrak{h}_\mathfrak{q})^\mathbb{C} = \mathbb{C} E_1 + \mathbb{C} i E_2 \) of \( \mathfrak{h}_\mathfrak{q}^\mathbb{C} \) and the only weights in \( \mathfrak{p}^\mathbb{C} \) of the form \( c_1 \alpha_1 + c_2 \alpha_2 \) are \( \sigma_1 \alpha_1 + \sigma_2 \alpha_2, \sigma_1 \alpha_1, \sigma_2 \alpha_2 \). However by the Cartan-Helgason theorem, Lemma 2.5 we see that \( \sigma_1 \alpha_1, \sigma_2 \alpha_2 \) are not eligible. Thus \( \pi_\nu (X) W_{\mu,1} \) is of the claimed form.

When \( \mathfrak{g} = \mathfrak{su}(r+b,r) \) the Weyl group for the root system of \( (\mathfrak{t}^\mathbb{C}, \mathfrak{h}_\mathfrak{q}) \) is \( (\mathbb{Z}/2)^2 \) consisting of only sign changes instead of all signed permutation \( (\mathbb{Z}/2)^2 \rtimes S_2 \), but all the relevant weights \( \sigma_1 \alpha_1 + \sigma_2 \alpha_2 \) are still in the orbit of the Weyl group \( (\mathbb{Z}/2)^2 \) so the arguments are valid for \( \mathfrak{g} = \mathfrak{su}(r+b,r) \) as well.

Next, the element \( \xi = \xi_e \) is invariant under \( L \subset K \), thus \( \pi_\nu (\xi) \phi_{\mu,1} \) is a sum of the \( L \)-invariant vectors in \( \sum_{\sigma_1,\sigma_2=\pm 1} W_{\mu+\sigma_{1,1}\pm 1} \), and is further by Lemma 2.5 a linear combination of \( \phi_{\mu+\sigma_{1,1}\pm 1} \). The rest of the proof is to determine the coefficients.

Notice that each function in the linear combination is determined by its restriction on the complexification \( \exp (\mathfrak{h}_\mathfrak{q}^\mathbb{C}) \cdot [e] \) once the line parameter \( l \) is given, so it is enough to find the expansion restricted on the torus (after the differentiations) as the line bundle parameters of each term in the expansion are already fixed.

We have

\[
\pi_\nu (\xi) \phi_{\mu,1}(k) = \frac{d}{ds} \phi_{\mu,1}(\exp(-s\xi)k)|_{s=0} = \frac{d}{ds} \phi_{\mu,1}(kk^{-1}\exp(-s\xi)k)|_{s=0} = \frac{d}{ds} \phi_{\mu,1}(k \exp(-s \text{ad}(k^{-1}\xi)))|_{s=0}
\]

(3.7)

\[
= - \left( (\text{ad}(k^{-1}\xi)) \phi_{\mu,1} \right)(k), \ k \in K,
\]
where \((\text{ad}(k^{-1})\xi)\phi_{\mu,l})(k)\) is right differentiation of the Lie algebra valued vector field \(\text{ad}(k^{-1})\xi\) on \(\phi_{\mu,l}\) evaluated at \(k \in K\). The element \(\phi_{\mu,l}\) is in the induced representation any differentiation of \(\phi_{\mu,l}\) along the Lie algebra \(m + n\) is zero, and we need formulas for \(\text{ad}(k^{-1})\xi = \xi e^{-1}\) mod \(m + n\).

**Lemma 3.3.** Let \(V = V_2 + V_1 + V_0 = C e + V_1 + V_0\) be the Peirce decomposition with respect to the minimal tripotent \(e\) and \(P_2, P_1, P_0\) the corresponding projections. Any element \(\xi u \in p\) has the following decomposition according to \((1.5)\), mod \(m + n\),

\[
\xi u = \Re\langle u, e\rangle \xi + \Im\langle u, e\rangle H_0 + D(P_1(u), e) - D(e, P_1(u)).
\]

**Proof.** Write \(u = P_2 u + P_1 u + P_0 u = \langle u, e\rangle e + u_1 + u_0 = \Re\langle u, e\rangle e + i \Im\langle u, e\rangle e + u_1 + u_0\). Then

\[
\xi_{(u,e)e} = \Re\langle u, e\rangle \xi e + \Im\langle u, e\rangle \xi ie,
\]

with

\[
\xi ie = (\xi ie - H_0) + H_0 = H_0, \text{ mod } n
\]

by \((1.8)\). In view of \((1.7)\) we have

\[
\xi u_1 = \left(\xi u_1 + D(e, u_1) - D(u_1, e)\right) + \left(D(u_1, e) - D(e, u_1)\right) = D(u_1, e) - D(e, u_1), \text{ mod } n,
\]

and \(\xi u_0 \in m\). This proves \((3.8)\). \(\square\)

Using Lemma 3.3 and the formula \((2.26)\) we see that \(\text{ad}(k^{-1})\xi\), mod \(m^\C + n^\C\), is

\[
\text{ad}(k^{-1})\xi = \Re\langle k^{-1}e, e\rangle \xi + \Im(k^{-1}e, e)H_0 + D(P_1(k^{-1}e), e) - D(e, P_1(k^{-1}e)) = \Re\langle ke, e\rangle \xi - \Im\langle ke, e\rangle H_0 + D(P_1(k^{-1}e), e) - D(e, P_1(k^{-1}e)).
\]

Hence

\[
\pi_\nu(\xi)\phi_{\mu,l}(k) = I + II + III
\]

with

\[
I = -\Re\langle k^{-1}e, e\rangle \left(\xi \phi_{\mu,l}\right)(k),
\]

\[
II = \Im\langle ke, e\rangle \left(H_0 \phi_{\mu,l}\right)(k)
\]

and

\[
III = \left(\left[D(e, P_1(k^{-1}e)) - D(P_1(k^{-1}e), e)\right] \phi_{\mu,l}\right)(k).
\]
Using the definition (1.10) of the induced representation we have the right differentiation of $\xi$ on any element $f \in L^2(K/L) = \text{Ind}^G_H(\nu)$ has eigenvalue $-\nu$, $\xi \phi_{\mu,l} = -\nu \phi_{\mu,l}$, and the first term is

$$I = -\text{Re}(k^{-1}e,e)(\xi \phi_{\mu,l})(k)$$

$$= \nu \text{Re}(k^{-1}e,e)\phi_{\mu,l}(k)$$

$$= \frac{\nu}{2}(ke,e)\phi_{\mu,l}(k) + \frac{\nu}{2}(e,ke)\phi_{\mu,l}(k)$$

$$= I^+ + I^-$$

with

$$I^+ = \frac{\nu}{2}(ke,e)\phi_{\mu,l}(k) = \frac{\nu}{2}\phi_{(1,1),1}(k)\phi_{\mu,l}(k)$$

and

$$I^- = \frac{\nu}{2}\phi_{(1,1),-1}(k)\phi_{\mu,l}(k).$$

The second term $II$, in view of (3.4), is

$$II = \text{Im}(ke,e)(2il\phi_{\mu,l})(k)$$

$$= l(ke,e)\phi_{\mu,l}(k) - l(e,ke)\phi_{\mu,l}(k)$$

$$= l\phi_{(1,1),1}(k)\phi_{\mu,l}(k) - l\phi_{(1,1),-1}(k)\phi_{\mu,l}(k) = II^+ + II^-.$$

We proceed to find recursion formulas for $\phi_{(1,1),1}\phi_{\mu,l}$. For that purpose we find explicit coordinates for the complex torus $\exp(h^C) \cdot [e], \exp(h_A)e \in S = K/L \subset V$. We shall treat all functions as trigonometric functions on the compact homogeneous space $K/L$, the evaluation and extension to the complexification $\exp(h^C) \cdot [e]$ being done differentiations.

**Lemma 3.4.** If $k = \exp(x_1E_1 + x_2E_2), x_1, x_2 \in \mathbb{C}$, then

$$k^{-1}e = \cos x_1 \cos x_2 e - (\sin x_1 \cos x_2 v_1 + \sin x_2 \cos x_1 v_2) + \sin x_1 \sin x_2 w.$$

**Proof.** Using (2.8) we find

$$E_1e = (D(v_1,e) - D(e,v_1))e = v_1, E_1^2e = E_1v_1 = (D(v_1,e) - D(e,v_1))v_1 = -e$$

and generally

$$E_1^{2m}e = (-1)^m e, E_1^{2m+1}e = (-1)^m v_1.$$

Therefore

$$e^{-x_1E_1}e = \cos x_1 e - \sin x_1 v_1,$$

and also $e^{-x_2E_2}e = \cos x_2 e - \sin x_2 v_2$. We compute further

$$E_2v_1 = (D(v_2,e) - D(e,v_2))v_1 = v_1, E_2^2v_1 = (D(v_2,e) - D(e,v_2))w = -v_1,$$

and in general

$$E_2^{2m}v_1 = (-1)^m v_1, E_2^{2m+1}v_1 = (-1)^m w.$$

This implies that

$$e^{-x_2E_2} \cdot v_1 = \cos x_2 v_1 - \sin x_2 w.$$
We have then
\[ k^{-1}e = e^{-x_1E_1-x_2E_2}e = e^{-x_2E_2}(\cos x_1e - \sin x_1v_1) \]
\[ = (\cos x_1(\cos x_2e - \sin x_2v_2) + \sin x_1(\cos x_2v_1 - \sin x_2w) \]
\[ = \cos x_1 \cos x_2e - (\sin x_1 \cos x_2v_1 + \sin x_2 \cos x_1v_2) + \sin x_1 \sin x_2w. \]

\[ \Box \]

**Lemma 3.5.** The following recursion formulas hold,
\[ \phi_{(1,1),1} \phi_{\mu,l} = \frac{1}{4} \sum_{\sigma_1,\sigma_2=\pm 1} c_{\mu,l}(\mu + \sigma, l + 1) \phi_{\mu+\sigma,l+1}, \]
\[ \phi_{(1,1),-1} \phi_{\mu,l} = \frac{1}{4} \sum_{\sigma_1,\sigma_2=\pm 1} c_{\mu,l}(\mu + \sigma, l - 1) \phi_{\mu+\sigma,l-1}, \]
where \( c_{\mu,l}(\mu + \sigma, l \pm 1) \) are given in Theorem 3.1.

**Proof.** Observe again that by general tensor product arguments the product \( \phi_{(1,1),\pm 1} \phi_{\mu,l} \) is a sum of \( \phi_{\mu+\sigma,l-1}, \sigma = (\pm 1, \pm 1) \). We use the idea in [20] by considering the leading term of \( \phi_{(1,1),1} \Phi_{\lambda,l} \) by using the Harish-Chandra \( C \)-function; see also [21]. We recall (2.23) and consider the expansion
\[ \phi_{(1,1),1} \Phi_{\lambda,l} = A_{\lambda-i\sigma,l+1} \Phi_{\lambda+\sigma,l+1} + L.O.T.. \]
(Presumably L.O.T. will not appear for general \( \lambda \) but it will not concern us here.) We let \( h = \exp(H), H = x_1(iE_1) + x_2(iE_2) \). First it is clear from (2.26) and (3.11) that
\[ \phi_{(1,1),1}(h) = \langle he, e \rangle = \cosh x_1 \cosh x_2 = \frac{1}{4}(e^{x_1} + e^{-x_1})(e^{x_2} + e^{-x_2}) = \frac{1}{4} e^{x_1+x_2} + L.O.T.. \]
The coefficient \( \frac{1}{4} \) can also be obtained using the general formula (2.24); indeed the evaluation of Harish-Chandra \( C \)-function is
\[ C((1, 1), 1) = C(\alpha_1 + \alpha_2, 1) = \frac{1}{4}. \]
Using the limit formulas (2.24) again we see that the coefficient \( A_{\lambda-i((1,1),l+1} \) of the leading term \( \Phi_{\lambda-i((1,1),l+1} \) is
\[ A_{\lambda-i((1,1),l+1} = \frac{1}{4} \frac{C(\lambda, l)}{C(\lambda-i(\alpha_1 + \alpha_2), l+1)}. \]
We use then the Weyl group symmetry \( \Phi_{\lambda,l} = \Phi_{s(\lambda),l} \), \( s \in \mathbb{Z}_2 \times \mathbb{Z}_2 = \{\pm 1\} \times \{\pm 1\} \) to get
\[ A_{\lambda-i\sigma,l+1} = \frac{1}{4} \frac{C(s_{\sigma}\lambda, l)}{C(s_{\sigma}(\lambda-i(\alpha_1 + \alpha_2), l+1)}. \]
The coefficients in the expansion (3.12) are then \( A_{\lambda-i\sigma,l+1} \) with \( \lambda = -i(\mu + \rho) \). The second expansion (3.13) is proved by the same method.

\[ \Box \]
We can now apply the lemma to both terms $I$ and $II$,

\begin{equation}
I^+ = \frac{\nu}{2} \phi_{(1,1),1}(k) \phi_{\mu,l}(k) = \frac{\nu}{2^3} \sum \sigma_1,\sigma_2 = \pm 1 \ c_{\mu,l}(\mu + \sigma, l + 1) \phi_{\mu+\sigma,l+1}(k),
\end{equation}

\begin{equation}
I^- = \frac{\nu}{2} \phi_{(1,1),-1}(k) \phi_{\mu,l}(k) = \frac{\nu}{2^3} \sum \sigma_1,\sigma_2 = \pm 1 \ c_{\mu,l}(\mu + \sigma, l - 1) \phi_{\mu+\sigma,l-1}(k),
\end{equation}

\begin{equation}
II^+ = l \phi_{(1,1),1}(k) \phi_{\mu,l}(k) = \frac{1}{4} \sum \sigma_1,\sigma_2 = \pm 1 \ c_{\mu,l}(\mu + \sigma, l + 1) \phi_{\mu+\sigma,l+1},
\end{equation}

\begin{equation}
II^- = -l \phi_{(1,1),-1}(k) \phi_{\mu,l}(k) = -\frac{1}{4} \sum \sigma_1,\sigma_2 = \pm 1 \ c_{\mu,l}(\mu + \sigma, l - 1) \phi_{\mu+\sigma,l-1}.
\end{equation}

The third term $III$ is

\[
III = (D(e, P_1(k^{-1}e)) \phi_{\mu,l})(k) - (D(P_1(k^{-1}e), e) \phi_{\mu,l})(k) := III^+ + III^-
\]

**Lemma 3.6.** We have the following recurrence formula for the right differentiations of the vector fields $D(e, P_1(k^{-1}e))$ and $-D(P_1(k^{-1}e), e)$ on $\phi_{\mu,l}$,

\begin{equation}
III^+ = \left(D(e, P_1(k^{-1}e)) \phi_{\mu,l}\right)(k) = \sum_{\sigma_1,\sigma_2 = \pm 1} b_{\mu+\sigma,l+1} \phi_{\mu+\sigma,l+1}(k),
\end{equation}

\begin{equation}
III^- = -\left(D(P_1(k^{-1}e), e) \phi_{\mu,l}\right)(k) = \sum_{\sigma_1,\sigma_2 = \pm 1} b_{\mu+\sigma,l-1} \phi_{\mu+\sigma,l-1}(k),
\end{equation}

where the coefficients $b_{\mu+\sigma,l\pm1}$ are given by

\[
b_{\mu+\sigma,l+1} = \frac{1}{2^3} \left(\sigma_1(\mu + \rho_1) + \sigma_2(\mu_2 + \rho_2) - (\rho_1 + \rho_2) - 2l\right) c_{\mu,l}(\mu + \sigma, l + 1),
\]

\[
b_{\mu+\sigma,l-1} = \frac{1}{2^3} \left(\sigma_1(\mu + \rho_1) + \sigma_2(\mu_2 + \rho_2) - (\rho_1 + \rho_2) + 2l\right) c_{\mu,l}(\mu + \sigma, l - 1),
\]

**Proof.** Denote $X$ the vector field

\[
X(k) = D(e, P_1(k^{-1}e))
\]

acting on functions on $K$ by right differentiation, $f \rightarrow (X(k)f)(k)$. With slightly abuse of notation we abbreviate it sometimes as $(Xf)(k)$. $III^+ = (Xk)(\phi_{\mu,l})(k) = (X\phi_{\mu,l})(k)$. We prove first that $X\phi_{\mu,l}$ is $(l + 1, l + 1)$-spherical. Some care has to be taken as $X$ is vector field taking values in the Lie algebra of $\mathfrak{g}^\mathbb{C}$; the transformation rule of $X\phi_{\mu,l}$ under the center of $K$ in $L_0$ is easily checked but we have to prove it for all $L_0$. The space $V_1$ is invariant under the subgroup $L_0 \subset K$, and

\[
P_1((hk)^{-1}e) = P_1(k^{-1}h^{-1}e) = \chi(h)^{-1} P_1(k^{-1}e), \ hP_1((kh)^{-1}e) = P_1(k^{-1}e), h \in L_0.
\]

Also elements $h \in K$ act on $D(x, y)$ as Jordan triple automorphisms $\text{ad}(h)D(u, v) = D(hu, hv)$, $D(u, v)$ is conjugate linear in $v$, and $\overline{\chi(h)} = \chi^{-1}(h), h \in L_0$; elements $k \in L$ act as Jordan triple
isomorphism as \( Le = e \) and \( L_0 \) as isomorphism up to the character \( \chi \). Thus the vector field \( X(k) = D(e, P_1(k^{-1}e)) \) satisfies
\[
X(hk) = \chi(h)X(k), \quad \text{ad}(h)(X(kh)) = \chi(h)X(k), \quad h \in L_0.
\]
It follows by the chain rules (3.2) and (3.3),
\[
(X(hk)\phi_{\mu,l})(hk) = (X(hk)\phi_{\mu,l}(h))(k)
= (\chi(h)X(k)\chi_l(h)\phi_{\mu,l}(\cdot))(k)
= \chi_{l+1}(h)(X(k)\phi_{\mu,l})(k)
\]
and
\[
(X(k)\phi_{\mu,l})(kh) = (\text{ad}(h)X(kh))(\phi_{\mu,l}(\cdot))(k) = \chi(h)\chi_l(h)(X(k)\phi_{\mu,l})(k)
= \chi_{l+1}(h)(X(k)\phi_{\mu,l})(k).
\]
Thus \((X(k)\phi_{\mu,l})(k)\) must be of the form (3.20). To find the coefficients we consider the group \( SL(2, \mathbb{C})^2 = SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \) with the Lie algebra \( \mathfrak{s}l(2, \mathbb{C}) \oplus \mathfrak{s}l(2, \mathbb{C}) \) generated by (2.9) and the restriction of \( \phi_{\mu,l} \) on \( SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \) and its expansion in terms of spherical polynomial of \( SL(2, \mathbb{C}) \), namely we consider the branching of the representation \((K^c, W_{\mu,l})\) under \( SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \). The highest weight of the representation \( W_{\mu,l} \) restricted to \( SL(2) \times SL(2) \) is \( \mu = (\mu_1, \mu_2) = \mu_1 \alpha_1 + \mu_2 \alpha_2 \), and thus the representation \( \mathbb{O}^{\mu_1} \mathbb{C}^2 \otimes \mathbb{O}^{\mu_2} \mathbb{C}^2 \) of \( SL(2) \times SL(2) \) appears in \((K^c, \mu)\), and all other representations are of the form \((\mu_1', \mu_2')\) with \( \mu_1' < \mu_1 \). Let \( \psi_{m,l}(g) \) be the \((l, l)\)-spherical polynomials for the group \( SL(2, \mathbb{C}) \) in the Appendix A (A.3). Comparing the leading term of \( \phi_{\mu,l}(g_1, g_2) \) and \( \psi_{\mu_1,l}(g_1)\psi_{\mu_2,l}(g_2) \) we have then
\[
\phi_{\mu,l}(g_1, g_2) = C(-i(\mu + \rho), l)\psi_{\mu_1,l}(g_1)\psi_{\mu_2,l}(g_2) + L.O.T, \quad (g_1, g_2) \in SL(2, \mathbb{C})^2.
\]
The vector field \( X(k) = D(e, P_1(k^{-1}e)) \) restricted to \( k = (k_1, k_2) = (\exp(x_1E_1), \exp(x_2E_2)) \in SU(2) \times SU(2) \) is, by Lemma 3.3, of the form
\[
X(k) = -\sin x_1 \cos x_2 D(e, v_1) - \sin x_2 \cos x_1 D(e, v_2)
= -\sin x_1 \cos x_2 E_1^- - \sin x_2 \cos x_1 E_2^-.
\]
The vector field \( X(k) \) takes values also in the complexification of the Lie algebra \( \mathfrak{s}l(2, \mathbb{C}) \). Thus the restriction of
\[
\left( X\phi_{\mu,l} \right)_{SU(2) \times SU(2)} = X \left( \phi_{\mu,l} |_{SU(2) \times SU(2)} \right).
\]
Moreover the Lie algebra differentiations by \( D(e, v_1), D(e, v_2) \) clearly preserve the degree. Thus we have
\[
\left( X\phi_{\mu,l} \right)(k_1, k_2) = C(-i(\mu + \rho), l) \left( -\sin x_1 (E_1^- \psi_{\mu_1,l})(k_1) \right) \left( \cos x_2 \psi_{\mu_2,l}(k_2) \right)
+ C(-i(\mu + \rho), l) \left( -\sin x_2 (E_2^- \psi_{\mu_2,l})(k_2) \right) \left( \cos x_1 \psi_{\mu_1,l}(k_1) \right) + L.O.T..
We use now Lemma [A.1] (A.7), and obtain

\[ (3.23) \quad - \sin x_j (E_j \psi_{\mu_j,l})(k_j) = \frac{1}{4} (\mu_j - l) \psi_{\mu_j+1,l+1}(k_j) + L.O.T., \quad j = 1, 2 \]

The leading term of \( \cos x_2 \psi_{\mu_2,l}(k_2) \), \( k_2 = \exp(x_2 E_2) \), is clearly the same as \( \frac{1}{2} \psi_{\mu_2+1,l}(k_2) \). Thus

\[ \left( - \sin x_1 (E_1 \psi_{\mu_1,l})(k_1) \right) \left( \frac{1}{8} (\mu_1 - l) \psi_{\mu_1+1,l+1}(k_1) \psi_{\mu_2+1,l+1}(k_2) \right) \; \text{and obtain} \]

\[ \left( - \sin x_2 (E_2 \psi_{\mu_2,l})(k_2) \right) \left( \cos x_1 \psi_{\mu_1,l}(k_1) \right) \quad \text{similarly for} \quad \left( - \sin x_2 (E_2 \psi_{\mu_2,l})(k_2) \right) \left( \cos x_1 \psi_{\mu_1,l}(k_1) \right) \quad \text{We have then} \]

\[ \left( X \phi_{\mu,l} \right)(k_1, k_2) = \frac{1}{8} (\mu_1 - l + \mu_2 - l) C(-i(\mu + \rho), l) \psi_{\mu_1+1,l+1}(k_1) \psi_{\mu_2+1,l+1}(k_2) + L.O.T. \]

\[ = \frac{1}{8} (\mu_1 + \mu_2 - 2l) C(-i(\mu + \rho), l) \psi_{\mu_1+1,l+1}(k_1) \psi_{\mu_2+1,l+1}(k_2) + L.O.T. \]

On the other hand the leading term in RHS of (3.20) is

\[ b_{\mu+1,l+1,l+1} \phi_{\mu+1,l+1,l+1}(k_1, k_2) \]

\[ = b_{\mu+1,l+1,l+1} C(-i(\mu + 1, 1 + \rho), l + 1) \psi_{\mu_1+1,l+1}(k_1) \psi_{\mu_2+1,l+1}(k_2) + L.O.T. \]

It follows then that

\[ b_{\mu+1,l+1,l+1} = \frac{1}{8} (\mu_1 + \mu_2 - 2l) c_{\mu,l}(\mu + (1, 1), l + 1) \]

where \( c_{\mu,l}(\mu + (1, 1), l + 1) \) is given in (3.5). This proves the formula for the leading coefficient.

To find the other coefficients we write

\[ \mu_1 + \mu_2 - 2l = (\mu_1 + \rho_1) + (\mu_2 + \rho_2) - (\rho_1 + \rho_2) - 2l \]

and use the Weyl group symmetry as in the proof of Lemma above to get

\[ b_{\mu+\sigma,l+1} = \frac{1}{8} \left( \sigma_1 (\mu_1 + \rho_1) + \sigma_2 (\mu_2 + \rho_2) - (\rho_1 + \rho_2) - 2l \right) c_{\mu,l}(\mu + \sigma, l + 1), \]

for \( \sigma_1, \sigma_2 = \pm 1 \).

To prove (3.21) we consider the vector field \( Y(k) = -D(P_1(k^{-1} e), e) \phi_{\mu,l} \) and its restriction to \( SL(2, \mathbb{C})^2 \). We have

\[ Y = Y(k) = \sin x_1 \cos x_2 E_1^+ + \sin x_2 \cos x_1 E_2^+. \]

We use then (A.10) and (A.6) to find the leading term of the expansion \( Y \phi_{\mu,l} \) and obtain all the coefficients by Weyl group symmetry. \qed
III^- = \left( D(P_1(k^{-1}e), e)\phi_{\mu, l}\right)(k)

= \sum_{\sigma_1, \sigma_2 = \pm 1} \frac{1}{2^3} \left( \sigma_1(\mu_1 + \rho_1) + \sigma_2(\mu_2 + \rho_2) - (\rho_1 + \rho_2) + 2l \right) c_{\mu, l}(\mu + \sigma, l - 1, \phi_{\mu, \sigma, l - 1}(k).

Altogether we find 

\pi_{\nu}(\xi)\phi_{\mu, l} = (I^+ + II^+ + III^+) + (I^- + II^- + III^-),

\begin{align*}
(I^+ + II^+ + III^+) &= \frac{1}{2^3} \sum_{\sigma_1, \sigma_2 = \pm 1} \left( \nu - \sigma_1(\mu_1 + \rho_1) - \sigma_2(\mu_2 + \rho_2) - (\rho_1 + \rho_2) \right) \phi_{\mu, \sigma, l + 1}, \\
(I^- + II^- + III^-) &= \frac{1}{2^3} \sum_{\sigma_1, \sigma_2 = \pm 1} \left( \nu - (\sigma_1(\mu_1 + \rho_1) - \sigma_2(\mu_2 + \rho_2) - (\rho_1 + \rho_2) \right) \phi_{\mu, \sigma, l - 1}.
\end{align*}

This finished the proof.

\begin{remark}
It is remarkable that in the formula the line bundle parameter \( l \) disappear due to the cancellation of \( 2l \) in the sum \( II + III \). When \( g = \mathfrak{su}(r + b, r) \), \( r > 1 \), \( \mathfrak{t} = \mathfrak{s}(u(r + b) + u(r)) \), the action of \( \pi \) on \( I(\nu) \) has been studied in details in [6]. In this case the parameter \( l \) indeed does not appear in affine term \( \nu + \sigma_1(\mu_1 + \rho_1) + \sigma_2(\mu_2 + \rho_2) - (\rho_1 + \rho_2) \) for the action; also the coefficients \( -A^{\pm, \pm} \) in [6] Lemma 4.1 can be formulated, writing \( \mu_1 = (\mu_1 + \rho_1) - \rho_1, \mu_2 = (\mu_2 + \rho_2) - \rho_2, \) as

\begin{align*}
\nu \pm (\mu_1 + \rho_1) \pm (\mu_2 + \rho_2) - \rho_1 - \rho_2,
\end{align*}

with our \( \nu \) being their \(-a = -(\alpha + \beta)\), \( \mu_1 = m_1 + m_2, \mu_2 = n_1 + n_2, \rho_1 = p - 1, \rho_2 = q - 1, \) and our \( l \) their \( m_1 - m_2 = n_2 - n_1 \); see further [6] (4.10)-(4.11)]. The Weyl group symmetry is again manifest here.
\end{remark}

4. Reduction points, complementary and composition series

4.1. Reduction points and finite dimensional subrepresentations. We study now the existence of intertwining operators between representations \( I(\nu) \) and \( I(\nu') \), and we find certain finite-dimensional representations at the reduction point of \( I(\nu) \).

\begin{theorem}
Let \( g = \mathfrak{sp}(r, \mathbb{R}) \) be a simple Hermitian Lie algebra of rank \( r \geq 2 \).

1. There exists an intertwining operator between the induced irreducible representations \( I(\nu) \) and \( I(\nu') \) if and only if \( \nu = \nu' \) or \( \nu + \nu' = 2\rho_g \).
2. \( I(\nu) \) is reducible if and only if \( \nu \) is an even integer, \( \nu \geq 2\rho_1 + 2 \) or \( \nu \leq 2\rho_2 - 2 \). Moreover at the point \( \nu = -2k, k \geq 1 \), the the symmetric tensor product \( S^k(\mathfrak{g}^\mathbb{C}) \) is realized as (reducible, generally) finite-dimensional subrepresentation of \( I(\nu) \) via

\begin{equation}
(4.1) \quad T: S^k(\mathfrak{g}^\mathbb{C}) \to I(\nu), X \mapsto f(g) = (\mathfrak{g}^k \text{ad}(g)(E_0), X), g \in G,
\end{equation}

where \( E_0 = \xi_{0^e} - iH_0 \in \mathfrak{n}_2 \) is the basis vector of the center \( \nu_2 \) of the nilpotent algebra \( \mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2 \) and \( (\cdot, \cdot) \) is the Killing form in \( \mathfrak{g}^\mathbb{C} \) extended to \( S^k(\mathfrak{g}^\mathbb{C}) \).
\end{theorem}
Proof. The first part and the second part on reductions points are done similarly as in [6, 16, 21]. Now let \( \nu = -2k \) be an negative even integer. We prove that \( T \) above is an intertwining operator from \( S^k(g^C) \) into \( I(\nu), \nu = -2k \). The functions \( f = f_X \) transform under \( P = MAN \) as

\[
f(gm) = (\otimes^k \text{ad}(gm)(E_0), X) = (\otimes^k \text{ad}(g) \text{ad}(m)(E_0), X) = (\otimes^k \text{ad}(g), X) = f(g), \quad m \in M,
\]
since \( M \) centralizes \( E_0, f(gn) = f(g), n \in N \), as \( E_0 \) is in the center of \( N \), and

\[
f(g e^{t\xi}) = (\otimes^k \text{ad}(g) \text{ad}(e^{t\xi})(E_0), X) = (\otimes^k (e^{2t} \text{ad}(g)(E_0), X) = e^{2kt} f(g) = e^{-\nu t} f(g), \quad m \in N,
\]
since \( \text{Ad}(\xi)E_0 = 2E_0, \text{ad}(e^{t\xi})(E_0) = e^{2t}E_0. \)

Thus \( f \in I(\nu) \). The intertwining property of \( T \) is obvious by its definition. This completes the proof. \( \square \)

4.2. Complementary series. We determine the complementary series i.e, that case when \( \nu \) is real and the whole module \( (g^C, K) \)-module \( I(\nu) \) is unitary and irreducible.

**Theorem 4.2.** The complementary series \( I(\nu) \) appears precisely in the range \( \nu = \rho_\theta + \delta, |\delta| < \delta_0, \)

\[
\delta_0 = \begin{cases} 
1 + b, & g = su(r + b, r) \\
3, & g = so^*(2r) \\
3, & g = so(2, n), \quad n > 4. \\
5, & g = e_6(-14) \\
5, & g = e_7(-25)
\end{cases}
\]

Proof. The abstract arguments in determining the complementary series here is the same as in [6, 16, 21], so we will only present some brief computations. Let \( \nu \) be real. Suppose the \( (g^C, K) \)-module \( I(\nu) \) is irreducible with invariant Hermitian inner product \( \langle \cdot, \cdot \rangle_\nu \). By Schur lemma we have \( \langle f, f \rangle_\nu = S(\nu, \mu, l) \| f \|^2 \) for all \( f \in W_{\mu,l} \), where \( \| f \|^2 \) is the norm square in \( L^2(K/L) \) and \( S(\nu, \mu, l) \) is the Schur proportionality constant. Then \( \pi_\nu(\xi) \) is skew symmetric and in particular

\[
\langle \pi_\nu(\xi) \phi_{\mu,l}, \phi_{\mu+\sigma,l+1} \rangle_\nu = -\langle \phi_{\mu,l}, \pi_\nu(\xi) \phi_{\mu+\sigma,l+1} \rangle_\nu
\]

Write the expansion of \( \pi_\nu(\xi) \phi_{\mu,l} \) in Theorem 3.1 as

\[
\pi_\nu(\xi) \phi_{\mu,l} = \sum_{\sigma_1, \sigma_2 = \pm 1} A(\nu, \mu, l; \mu + \sigma, l + 1) \phi_{\mu+\sigma,l+1} + \sum_{\sigma_1, \sigma_2 = \pm 1} A(\nu, \mu, l; \mu + \sigma, l - 1) \phi_{\mu+\sigma,l-1}
\]

Thus the invariance of of the Hermitian form above becomes

\[
A(\nu, \mu, l; \mu + \sigma, l + 1) S(\mu + \sigma, l + 1) = -A(\nu, \mu + \sigma, l + 1; \mu, l) S(\mu + \sigma, l),
\]

\[
A(\nu, \mu, l; \mu + \sigma, l - 1) S(\mu + \sigma, l - 1) = -A(\nu, \mu + \sigma, l - 1; \mu, l) S(\mu, l).
\]

That \( I(\nu) \) is unitary and irreducible is equivalent to all Schur proportionality constants \( S(\mu, l) \) are positive. It implies that \( A(\nu, \mu, l; \mu + \sigma, l + 1) \) and \( A(\nu, \mu + \sigma, l + 1; \mu, l) \) have opposite signs, as well
as $A(\nu, \mu, l; \mu + \sigma, l - 1)$ and $A(\nu, \mu + \sigma, l - 1; \mu, l)$. This determines the range of $\nu$, given by the condition
\[
|\delta| < \min\{\rho, \rho - 2\rho_2, 2\rho_1 - \rho + 2\}.
\]
By case by case computations we get the range as claimed. \qed

**Remark 4.3.** The complementary series for $SU(p, q)$ has been found before in [10] (ii)(b), p. 49; 5.2, p. 69], [6, 4.4].

4.3. **Composition series and unitarizable subrepresentations**. The composition series for $I(\nu)$ at reducible points $\nu$ is a bit involved. We shall only determine the unitary subrepresentations at some typical reduction points. The proof of the following result is done by examining the signs of the coefficients $\nu + \sigma_1(\mu_1 + \rho_1) + \sigma_2(\mu_2 + \rho_2) - \rho_1 - \rho_2$ in Theorem 3.1.

**Theorem 4.4.** Suppose $\nu \leq 2\rho_2 - 2$ is an even integer. Then there are two unitarizable subrepresentations $S^+(\nu) \subset I(\nu)$ consisting of the $K$-types
\[
S^+(\nu) = \sum_{(l, \mu, \nu_1 - \nu_2 \geq \nu + 2\rho_1)} W_{\mu, l}, \quad S^-(\nu) = \sum_{(l, \mu, \nu_1 - \nu_2 \geq \nu + 2\rho_2)} W_{\mu, l}.
\]

5. **The cases of $g = \mathfrak{su}(d, 1)$ and $g = \mathfrak{sp}(r, \mathbb{R})$**

We treat the remaining cases when $g = \mathfrak{sp}(r, \mathbb{R}), \mathfrak{su}(d, 1)$ where the space $K/L_0$ is the complex projective space $\mathbb{P}^{d-1}$.

5.1. $g = \mathfrak{su}(d, 1)$. This case is already treated in [8] by using rather explicit computations of differentiating hypergeometric functions. We shall give somewhat easier proof of their results by using our method above; this avoids explicit computations involving special functions and gives conceptual expression for the action of $\pi_\nu(\xi)$ in terms of Harish-Chandra $c$-functions.

The Cartan decomposition is $g = \mathfrak{k} + \mathfrak{p} = \mathfrak{u}(d) + \mathbb{C}^d$, with $\mathfrak{t}^C = \mathfrak{gl}(d)$. The Jordan triple $V = \mathbb{C}^d$ with $\{x, y, z\} = D(x, y)z = (x, y)z + (z, y)x$. We fix the tripotent $e = e_1$, the standard basis vector of $\mathbb{C}^d$ and $\xi = \xi_e$. The half-sum $\rho_\mathfrak{g}$ is $\rho_\mathfrak{g} = d$. The space $S = K/L$ is the sphere $S$ in $\mathbb{C}^d$ with $L$ the isotropic subgroup of $e \in S$, and $S_1 = K/L_0$ the projective space $\mathbb{P}(\mathbb{C}^d)$ with $S \to S_1$ as a circle bundle over $\mathbb{P}(\mathbb{C}^d)$ by the defining map $z \mapsto [z]$. The tangent space of $S_1$ is realized as $T^{(1,0)}[e]S_1 = \{D(v, e) ; v \in V_1 = \mathbb{C}^{d-1}\}, T^{(0,1)}[e]S_1 = \{D(e, v) ; v \in V_1\}$. We fix an $\mathfrak{su}(2)$-subalgebra in $\mathfrak{k}$ and will perform the differentiation along tangent vectors $E^\pm$ in the complex totally geodesic submanifold $\mathbb{P}^1 \subset S_1$:
\[
E^+ = D(e_2, e), \quad E^- = D(e, e_2), \quad E = E^+ - E^- \in T^1[e](K/L_0).
\]
Put $H = [E^+, E^-] = D(e_2, e_2) - D(e, e)$. The element $iE$ generates a Cartan subalgebra for the non-compact dual $(\mathfrak{t}_{i1}^*, \mathfrak{l}_1)$, and positive roots are $\{2\alpha_1, \alpha_1\}$ with $\alpha_1$ the dual element of $iE$, and the half-sum is $\rho_{\mathfrak{t}_1} = d - 1$. 
The decomposition of $L^2(K/L)$ is well-known,

$$L^2(K/L) = \sum_{m \geq |l|, m \equiv l \mod 2} W_{m,l}.$$ 

Each space $W_{m,l}$ is generated by $z_1^p z_2^q$. In terms of the previous notation

$$p = \frac{m + l}{2}, q = \frac{m - l}{2}, \ m \geq |l|, m \equiv l \mod 2.$$

Now the coefficients in the expansion of $\pi_\nu(\xi) \phi_{m,l}$ can be written as $\left(\nu + \pm (m + d - 1) \pm c\right)$ for some constants $c$ as in Theorem 3.1. We write them explicitly.

**Theorem 5.1.**

(1) The action of $\pi_\nu(\xi)$ on $\phi_{m,l}$ is given by

$$2^2 \pi_\nu(\xi) \phi_{m,l} = (\nu + m + l) c_{m,l}(m + 1, l + 1) \phi_{m+1,l+1} + (\nu - m - 2d + 2 + l) c_{m,l}(m + 1, l + 1) \phi_{m-1,l+1} + (\nu + m - l) c_{m,l}(m + 1, l - 1) \phi_{m+1,l-1} + (\nu - m - l - 2d + 2) c_{m,l}(m - 1, l - 1) \phi_{m-1,l-1}.$$ 

(2) The complementary series is in the range $\nu = \rho_\delta + \delta, |\delta| < d = \rho_\delta$.

**Proof.** We follow the computations of $\pi_\nu(\xi) \phi_{m,l}$ in the proof of Theorem 3.1 and indicate the necessary changes. We find first the coefficient of $\phi_{m,l}$ and the other coefficients will be found by general arguments. We have $\pi_\nu(\xi) \phi_{m,l} = I + II + III, I = I^+ + I^-,$

$$I^+ = \frac{\nu}{2} \phi_{1,1}(k) \phi_{m,l}(k)$$

and the spherical polynomial $\phi_{1,1}(k)$ is

$$\phi_{1,1}(k) = \phi_{1,1}(z) = (ke, e) = z_1, \ z = ke \in S.$$ 

as function on the sphere $S = K/L$. For $k = \exp(tE)$ we have $\phi_{1,1}(k) = \cos x$ and its complexification is $\cosh x = \frac{1}{2}(e^x + e^{-x})$. The expansion (3.15) now has coefficient $\frac{1}{2}$. Thus the leading term in $I^+$ is

$$\frac{\nu}{4} c_{m,l}(m + 1, l + 1) \phi_{m+1,l+1}$$

where $c_{m,l}(m + 1, l + 1)$ is a quotient of two Harish-Chandra $c$-functions.

The term $II^+$ in (3.18) in the present case becomes $II^+ = l \phi_{1,1} \phi_{m,l}$ and has the leading term

$$\frac{l}{2} c_{m,l}(m + 1, l + 1) \phi_{m+1,l+1}.$$

The third term $III^+$ is treated similarly as in Lemma 3.6 and (3.23) gives that the leading term of $III^+$ is $m_l c_{m,l}(m + 1, l + 1) \phi_{m+1,l+1}$. Altogether terms involving $\phi_{m+1,l+1}$ in $\pi_\nu(\xi) \phi_{m,l}$ are

$$\frac{\nu + 2l + m - l}{4} c_{m,l}(m + 1, l + 1) \phi_{m+1,l+1} = \frac{\nu + m + l}{4} c_{m,l}(m + 1, l + 1) \phi_{m+1,l+1}$$

with

$$\frac{\nu + m + l}{4} = \frac{\nu + (m + \rho_1) - \rho_1 + l}{4}.$$
in terms of the Weyl group invariant parameter \( m + \rho_1^* \). Observe again that the highest weight with respect to \( E \) is \( m\alpha_1 \) so the Weyl group symmetric is with respect to \(( m + \rho_1^* )\alpha_1 \). The coefficient of \( \phi_{m+1,l+1} \) involving \( \nu \) is

\[
\nu + m + l = \nu + ( m + \rho_1^* ) - \rho_1^* + l.
\]

Thus \( \phi_{m-1,l+1} \) has coefficient

\[
\nu - ( m + \rho_1^* ) - \rho_1^* + l = \nu - m - 2\rho_1^* + l = \nu - m - 2d + 2l.
\]

The other two coefficient \( \phi_{m\pm 1,l-1} \) is found by the unitarity of \( \pi_\nu(\xi) \) at \( \nu = \rho_B + ix, x \in \mathbb{R} \) and by the Weyl group symmetry. This coincides then with our formula. As mentioned above in Section 1, in particular Lie algebra elements of Jordan triple as in Remark 5.2, it is enough to determine the leading coefficient \( a_{m+1,l+1}(\nu + m + l) \).

5.2. \( \mathfrak{g} = \mathfrak{sp}(r, \mathbb{R}), r \geq 2 \). The Jordan triple here is \( V = M_r^\mathbb{R} = \{ v \in M_r(\mathbb{C}); v = v^t \} \) of complex matrices with the triple product \( D(u, v)w = uv^*w + wv^*u \). This case is rather special and we provide all details. The normalization of Euclidean norm in \( p^+ \) is as before with minimal tripotents having norm 1. The group \( K = U(r) \) acts on \( V \) and by \( A \in K : Z \to AZA^t \). To avoid confusion with various realizations we recall that all Lie algebra elements are realized via Jordan triple as in Section 1, in particular Lie algebra elements of \( \mathfrak{p}^C = \mathfrak{gl}(r, \mathbb{C}) \) appear as \( D(u, v) : w \to D(u, v)w \); they are identified with usual matrices \( uv^* \) if we still want matrix realizations.

We fix the minimal tripotent the diagonal matrix \( e = \text{diag}(1, 0, \ldots, 0) \in V \) and \( \xi = \xi_{e} \in \mathfrak{p} \) in (1.2). The functional \( \rho_B \) is now \( \rho_B = r \). A subtle point here is that the group \( L = M \cap K \subset K \) is

\[
L = \mathbb{Z}_2 \times U(r-1) = \{ h = \text{diag}(h_0, h_1); h_0 = \pm 1, h \in U(r-1) \}.
\]

Let

\[
L_0 = U(1) \times U(r-1) = \{ h = \text{diag}(h_0, h_1); h_0 \in U(1), h \in U(r-1) \}.
\]

Thus \( S = K/L \subset V \) is the real projective space \( \mathbb{P}(\mathbb{R}^{2r}) \subset V \) and \( S_1 = K/L_0 \) is again the complex projective space \( \mathbb{P}(\mathbb{C}^r) \subset \mathbb{P}(V) \) of rank one realized in the projectivization of \( V \).

The Peirce decomposition \( V = V_2 + V_1 + V_0 \) with respect to \( e \) is the block \((1 + r - 1) \times (1 \times r - 1)\)-partition of \( V \). We fix

\[
v = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in V_1, \quad w = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in V_0,
\]
Proof. Following the earlier computation in Section 4 we have

where elements in 

Thus the leading term in

Theorem 5.3.

Then II is given by

(2) There is no complementary series in the family \(I(\nu)\) if \(r = 2\). If \(r > 2\) the complementary series are in the range \(\nu = r + \delta, |\delta| < r - 2\).

Proof. Following the earlier computation in Section 4 we have

The matrix coefficient \(\langle ke, e \rangle = \phi_{2,2}(k)\), and its restriction on the torus \(\exp(\mathbb{R}E)e\) is

Thus \(\phi_{2,2}(k) = (\cos x)^2\) and it has an expansion

Thus the leading term in \(I^+\) is

where \(c_{m,l}(m + 1, l + 1)\) is the quotient Harish-Chandra c-functions for \(\phi_{2m+2,2l+2}\) and \(\phi_{2m,2l}\).

In the second term II we have \(H_0e = 2ie\), and then

which has the leading term

\[ \frac{1}{4} c_{m,l}(m + 1, l + 1) \phi_{2m+2,2l+2}. \]
The vector field $X(k) = D(e, P_1(k^{-1}e))$ in (3.22) has restriction

$$X(\exp(xE)) = D(e, P_1(\exp(-xE)e)) = -\cos x \sin x D(e, v),$$

by (5.2). The leading term of the expansion $-\cos x \sin x E^{-\phi_{2m,2l}}$ is obtained, using the proof of Lemma A.1 in Appendix A as

$$-\cos x \sin x E^{-\phi_{2m,2l}}(\exp(xE)) = \frac{2m - 2l}{8} e^{i 2mx} + L.O.T..$$

Altogether we find

$$I^+ + II^+ + III^+ = \frac{\nu + 2l + 2m - 2l}{8} c_{m,l}(m + 1, l + 1) \phi_{2m+2l+2} + L.O.T.$$

The rest is done as in the proof of Theorem 3.1 above.

\[\square\]

**Remark 5.4.** We note the trivial representation corresponds to $\nu = r + \delta$, $\delta = r$, whereas the complementary serie range is $|\delta| < r - 2$, so the gap is 2 between the end of complementary series and the trivial representation, similar to the rank one case $Sp(n,1)$ (and there is no gap for $SU(n,1)$).

**Remark 5.5.** Even integers $\nu = -2k$ are reduction points for $I(\nu)$ in both cases above, $g = su(d,1), sp(r, \mathbb{R})$. The map (4.1) realizes $S^k(g^\mathbb{C})$ as a subrepresentations of $I(\nu)$.

There are some interesting non-spherical unitary principal series of $Sp(n, \mathbb{R})$ and it might be interesting to study them in details.

**Appendix A. Recursion formulas for differentiations of spherical polynomials**

Let

$$SU(2) = \left\{ g = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : |a|^2 + |b|^2 = 1 \right\}$$

and fix the following Lie algebra $sl(2, \mathbb{C})$ elements,

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} , \ E^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} , \ E^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} , \ E = E^+ + E^- \in i su(2).$$

Then the $sl(2)$-algebras $sl(2) = CH_j + CE^+_j + CE^-_j, j = 1, 2$, defined in (2.9) are isomorphic to the present $sl(2)$ via the identification

$$H_j \leftrightarrow H, \ E^+_j \leftrightarrow E^+;$$

as well as the identification of the compact $su(2)$-real forms

$$su(2) = \mathbb{R}i H_j + \mathbb{R} E_j + \mathbb{R}(i(E^+_j + E^-_j)) \leftrightarrow \mathbb{R}i H + \mathbb{R} E + \mathbb{R}(i(E^+ + E^-)).$$
Let

\[ U(1) = \left\{ u_\theta = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} = e^{i\theta H} \right\}. \]

Recall \( \chi_l(H_j) = -l \) and the transformation rule (2.22) of \( \phi_{m,l} \) under \( \exp(itH_j) \). Accordingly we let \( \chi_l(H) = -l \), and the spherical polynomials \( \phi_{m,l} \) in the present case satisfy \( \phi_{m,l}(e^{i\theta H} g) = \phi_{m,l}(g) \). Consider the representation of \( SL(2, \mathbb{C}) \) on the symmetric tensor \( S^m := \mathbb{C}^m \mathbb{C}^2 \) of the defining representation \( \mathbb{C}^2 \), and write the action simply as \( g \rightarrow gv, v \in S^m \mathbb{C}^2 \), as well as the Lie algebra action. Let \( -m \leq l \leq m, m = l \mod 2 \). The \( l \)-spherical polynomial is given by the matrix coefficient

(A.1) \[ \phi_{m,l}(g) = \binom{m}{k}(g(e_1^k e_2^{m-k}), e_1^k e_2^{m-k}), 0 \leq k \leq m, l = m - 2k, \]

where the tensor \( e_1^k e_2^{m-k} = e_1^k \otimes e_2^{m-k} \) is as usual viewed as polynomial on the dual space of \( \mathbb{C}^2 \). This is verified by

(A.2) \[ \phi_{m,l}(g u_\theta) = \phi_{m,l}(ge^{i\theta H}) = (ge^{i\theta H}(e_1^k e_2^{m-k}), e_1^k e_2^{m-k}) = e^{i(2k-m)\theta} \phi_{m,l}(g) = e^{-i\theta} \phi_{m,l}(g) = \chi_l(u_\theta) \phi_{m,l}(g), \]

also \( \phi_{m,l}(u_\theta g) = \chi_l(u_\theta) \phi_{m,l}(g) \) along with the normalization \( \phi_{m,l}(I) = \binom{m}{k} ||e_1^k e_2^{m-k}||^2 = 1 \). For our purpose in Section 3 it is more convenient to consider the spherical polynomial

(A.3) \[ \psi_{m,l}(g) = \frac{2^m}{\binom{m}{k}} \phi_{m,l}(g) = 2^m (g(e_1^k e_2^{m-k}), e_1^k e_2^{m-k}), l = m - 2k, \]

which has the normalization the leading term of \( \psi_{m,l}(\exp tE) \) being \( e^{int} \), i.e.

\[ \psi_{m,l}(\exp tE) = e^{int} + L.O.T. \]

where L.O.T. is a trigonometric polynomial of lower order. The following lemma is used in the proof of Lemma 3.6. Actually we need only to find the leading term of the trigonometric polynomials \( \langle ge_1, e_2 \rangle (E^\phi_{m,l})(g) \) and \( \langle ge_2, e_1 \rangle (E^\phi_{m,l})(g) \) for \( g = \exp(tE) \), which is elementary.

**Lemma A.1.** The following recurrence formulas hold,

(A.4) \[ \langle ge_1, e_2 \rangle (E^\phi_{m,l})(g) = \frac{1}{4} \frac{(m-l)(m+l+2)}{m+1} (\phi_{m+1,l+1} - \phi_{m-1,l+1}), \]

(A.5) \[ \langle ge_2, e_1 \rangle (E^\phi_{m,l})(g) = \frac{1}{4} \frac{(m-l)(m+l+2)}{m+1} (\phi_{m+1,l-1} - \phi_{m-1,l-1}). \]

When restricted to \( g = \exp(xE) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \) and written in terms of \( \psi_{m,l} \) they are

(A.6) \[ -\sin x (E^\psi_{m,l})(g) = \frac{1}{4} (m-l)\psi_{m+1,l+1} - \frac{1}{4} \frac{(m+l+2)(m-l)^2}{m(m+1)} \psi_{m-1,l+1}, \]

(A.7) \[ \sin x (E^\psi_{m,l})(g) = \frac{1}{4} (m+l)\psi_{m+1,l-1}(g) - \frac{1}{4} \frac{(m-l+2)(m+l)}{m(m+1)} \psi_{m-1,l-1}(g). \]
Proof. First we find the weight of \( f(g) = (ge_2, e_1)(E^+\psi_{m,l})(g) \) under the regular left and right actions of \( \exp(i\theta H) \). We have \( E^+ \) is of weight 2 under \( \text{ad}(H) \). Also the matrix coefficient \( \langle ge_2, e_1 \rangle \) transforms as \( \langle ge^{iH}e_2, e_1 \rangle = e^{-i\theta}(\langle ge_2, e_1 \rangle) \), thus it is of weight \(-1\) under the right regular action of \( H \). The character \( \chi_l \) is defined by \( \chi_l(H) = -l \) thus \( f(g) = (ge_2, e_1)(E^+\psi_{m,l})(g) \) is of weight \(-l+2-1 = -(l-1) = \chi_{l-1}(H) \) in the sense \( f(g \exp(i\theta H)) = \chi_{l-1}(\exp(i\theta H))f(g) \).

Similarly \( \exp(i\theta H)ge_2, e_1 = e^{i\theta}(ge_2, e_1) \) and the right differentiation by \( E^+ \) commutes with the left action. Thus \( f(\exp(i\theta H)g) = e^{-i(l-1)\theta}f(g), \) and \( f(g) \) is a linear combination of \( \phi_{m+1,l+1} \) and \( \phi_{m-1,l-1} \) as it is the matrix coefficient of \( \mathbb{C}^2 \otimes S^m = S^{m+1} \oplus S^{m-1} \mathbb{C}^2 \).

(A.8) \[
\begin{align*}
\langle e_2, e_1 \rangle (e^{k_1 e_2^{m-k}}, e^{k_1 e_2^{m-k}}) &= (e_2 \otimes e^{k_1 e_2^{m-k}}, e_1 \otimes e^{k_1 e_2^{m-k}}).
\end{align*}
\]

Both \( e_1 \otimes e^{k_1 e_2^{m-k}} \) and \( e_2 \otimes e^{k_1 e_2^{m-k}} \) are of weight \( i(1+2k-m) \) under \( H \), the corresponding weight vector in the space \( S^{m+1} \) respectively \( S^{m-1} \) is \( e^{k_1 e_2^{m-k}} \). In view of (A.1) the formula (A.5) becomes

\[
\begin{align*}
\langle e_2, e_1 \rangle (e^{k_1 e_2^{m-k}}, e^{k_1 e_2^{m-k}}) &= (e_2 \otimes e^{k_1 e_2^{m-k}}, e_1 \otimes e^{k_1 e_2^{m-k}})
\end{align*}
\]

(A.9) \[
\begin{align*}
A &= (m+1)(m-k)(m+1)\langle e^{k_1 e_2^{m-k}}, e^{k_1 e_2^{m-k}} \rangle + B(m+1)(m+1)\langle e^{k_1 e_2^{m-k}}, e^{k_1 e_2^{m-k}} \rangle
\end{align*}
\]

Evaluating at \( g = I \) we get \( B = -A. \) Next we specify the equality to the self-adjoint element

\[
\begin{align*}
g &= \begin{pmatrix} ch x & sh x \\
sh x & ch x \end{pmatrix} = \begin{pmatrix} x & x \\
x & x \end{pmatrix}^2 = h^2
\end{align*}
\]

and look for the coefficients of \( e^{(m+1)x} \). We have

\[
\begin{align*}
\langle e_2 \otimes e^{k_1 e_2^{m-k}}, e_2 \otimes e^{k_1 e_2^{m-k}} \rangle &= \langle h(e_1 \otimes e^{k_1 e_2^{m-k}}), h(e_2 \otimes e^{k_1 e_2^{m-k}}) \rangle,
\end{align*}
\]

and its leading term is

\[
\begin{align*}
e^{(m+1)x} 2^{2(m+1)}(e_1 + e_2)^{m+1}, (e_1 + e_2)^{m+1} &= e^{(m+1)x} 2^{2m+1},
\end{align*}
\]

and the LHS has leading term \( \left( \begin{smallmatrix} m+k \end{smallmatrix} \right) \left( \begin{smallmatrix} m-k \end{smallmatrix} \right) \left( \begin{smallmatrix} m+1 \end{smallmatrix} \right) \left( \begin{smallmatrix} m+1 \end{smallmatrix} \right) \). The term \( e^{(m+1)x} \) appears only in the first summand in the RHS which has leading term \( A(m+1)^2 \). Thus

\[
\begin{align*}
A &= \left( \begin{smallmatrix} m+k \end{smallmatrix} \right) \left( \begin{smallmatrix} m-k \end{smallmatrix} \right) (m+1)^2
\end{align*}
\]

This proves (A.5), and then (A.11) by using (A.3)

The formulas (A.4) and (A.10) are proved by the same methods.
Remark A.2. In terms of $\psi_{m,l}$ they become

\begin{equation}
\langle ge_1, e_2 \rangle (E^+ \psi_{m,l})(g) = \frac{1}{4} (m-l) \psi_{m+1,l+1} - \frac{1}{4} \frac{(m+l+2)(m+l)^2}{m(m+1)} \psi_{m-1,l+1}.
\end{equation}

\begin{equation}
\langle ge_2, e_1 \rangle (E^+ \psi_{m,l})(g) = \frac{1}{4} (m+l) \psi_{m+1,l-1} - \frac{1}{4} \frac{(m+2-l)(m+l)^2}{m(m+1)} \psi_{m+1,l-1};
\end{equation}

The spherical polynomial $\phi_{m,l}$ is special case of the spherical function $\Phi_{\lambda,l}$ with $\lambda = -i(m+\rho) = -i(m+1)$ in our case, and $\Phi_{\lambda,l}$ is invariant with respect to the Weyl group action $\lambda \rightarrow -\lambda$. Namely, the pair of the coefficients

$$\frac{\pm 1}{4} \frac{(m-l)(m+l+2)}{m+1} = \frac{\pm 1}{4} \frac{((m+1) - (l+1))(m+1) + (l+1)}{m+1}$$

in the lemma is invariant by the change $m+1 \rightarrow -(m+1)$ and this symmetry is indeed obvious here. These formulas are all classical trigonometric identities and can be obtained by other methods.

**Appendix B. Table of Hermitian symmetric spaces $G/K$ and their varieties of minimal rational tangents $K/L_0$. Duality relation for**

$(\dim(X), \text{genus}(X))$ for $X = G/K, K/L_0$

**B.1. Tables.** We give a list of $G/K$ and the corresponding projective spaces $S_1 = \mathbb{P}(S) = K/L_0 = K_1/L_1$ as compact Hermitian symmetric space; see [3, 12, 7]. The compact dual of a noncompact Hermitian symmetric space $D$ is denoted by $D^*$.

| $D = G/K$ | $G$ | $K$ | $(a,b)$ |
|-----------|-----|-----|--------|
| $I_{r+b,r}$ | $SU(r+b,r)$ | $S(U(r+b) \times U(r))$ | $(2,b)$ |
| $II_{2r}$ | $SO^*(4r)$ | $U(2r)$ | $(4,0)$ |
| $II_{2r+1}$ | $SO^*(4r+2)$ | $U(2r+1)$ | $(4,2)$ |
| $III_r$ | $Sp(r,\mathbb{H})$ | $U(r)$ | $(1,0)$ |
| $IV_{n>4}(r=2)$ | $SO(n,2)$ | $SO(n) \times SO(2)$ | $(n-2,0)$ |
| $V(r=2)$ | $E_{6(-14)}$ | $Spin(10) \times SO(2)$ | $(6,4)$ |
| $VI(r=3)$ | $E_{7(-25)}$ | $E_6 \times SO(2)$ | $(8,0)$ |

**Table 1.** Non-compact Hermitian symmetric space $D = G/K$

**B.2. Duality between $(d, p)$ and $(d_1, p_1)$ for $G/K$ and $K/L$.** Let $d = \dim \mathbb{C} D = r + \frac{1}{2} ar(r-1) + rb$, $p = 2 + a(r-1) + b$, the dimension and the genus of $D$. In terms of Lie algebra actions they are

$$(d, p) = \left( \text{tr} \ Ad(-iZ)|_{p^*}, \text{tr} \ Ad(D(e,e))|_{p^*} \right)$$

where $D(e,e)$ is the Harish-Chandra co-root of $\gamma_1, \gamma_1(D(e,e)) = 2$. Similarly let

$$(d_1, p_1) = (\dim(K_1^*/L), \text{genus}(K_1^*/L))$$
The compact Hermitian symmetric space \( \mathbb{P}(S) = K/L_0 = K'/L' \). For type I domain \( I_{r,b-r}, r \geq 2 \), \( \mathbb{P}(S) \) is a product \( \mathbb{P}^{r-1} \times \mathbb{P}^{r+b-1} \) of projective spaces with the corresponding \( (a_1, b_1) \) is \((0, r + b - 2), (0, r - 2)\) for each factor.

if \( D \neq SU(r, r + b)/S(U(r) \times U(r + b)) \). Put

\[
d' = \dim(\mathbb{P}^{r-1}) = r - 1, d' = r, d'' = \dim(\mathbb{P}^{r+b-1}) = r + b - 1, d'' = r + b
\]

when \( D = SU(r, r + b)/S(U(r) \times U(r + b)) \).

The following duality between the pairs \((\dim(D), \text{genus}(D))\) and \((\dim(K/L_0), \text{genus}(K/L_0))\) is mentioned in Lemma 2.5 and might be of independent interest. It can be proved by trace computations or by case-by-case computations of the tables above.

**Lemma B.1.**
1. Let \( D \) be of rank \( r \geq 2 \) and is one of the domains II, IV, V, VI. Then

\[
\frac{p}{d} + \frac{d_1}{p_1} = 2.
\]

2. Let \( D \) be of Type I with \( r \geq 2 \). Then

\[
\frac{p}{d} + \frac{d'}{p'} + \frac{d''}{p''} = 2.
\]

3. Let \( D \) be the Siegel domain II. Then

\[
\frac{p}{d} + 2 \frac{d_1}{p_1} = 2.
\]
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