Amortized Constant Round Atomic Snapshot in Message Passing Systems

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Abstract  
We study the lattice agreement (LA) and atomic snapshot problems in asynchronous message-passing systems where up to $f$ nodes may crash. Our main result is a crash-tolerant atomic snapshot algorithm with amortized constant round complexity. To the best of our knowledge, the best prior result is given by Delporte et al. [TPDS, 18] with amortized $O(n)$ complexity if there are more scans than updates. Our algorithm achieves amortized constant round if there are $\Omega(\sqrt{k})$ operations, where $k$ is the number of actual failures in an execution and is bounded by $f$. Moreover, when there is no failure, our algorithm has $O(1)$ round complexity unconditionally.

To achieve amortized constant round complexity, we devise a simple early-stopping lattice agreement algorithm and use it to “order” the update and scan operations for our snapshot object. Our LA algorithm has $O(\sqrt{k})$ round complexity. It is the first early-stopping LA algorithm in asynchronous systems.

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Introduction

The lattice agreement (LA) problem [9] and atomic snapshot object (ASO) problem [11] are two closely related problems in the literature. In the LA problem, given input values from a lattice, nodes have to decide output values that lie on a chain of the input lattice and satisfy some non-trivial validity property. The atomic snapshot object is a concurrent object well studied in shared memory, e.g., [1][2][5]. An atomic snapshot object is partitioned into segments. Node $i$ can either update the $i$-th segment (single-writer model), or instantaneously scan all segments of the object. In shared memory, Attiya et al. [9] showed how to apply algorithms for one problem to solve the another problem.

Both LA algorithms and atomic snapshot objects have a wide spectrum of applications. For example, LA algorithms can be used to implement an update-query state machines [19] and linearizable conflict-free replicated data types (CRDT) [29]. Atomic snapshot objects can be used for solving approximate agreement [11], randomized consensus [3][4], and implementing wait-free data structures in shared memory [21][4]. In message-passing systems, atomic snapshot objects can be used for creating self stabilizing memory, and detecting
stable properties to debug distributed programs. In essence, ASO simplify the design and verification of many distributed and concurrent algorithms (example applications can be found in [30], [27] and [13]). Recently, Guerraoui et al. [20] also demonstrated a mechanism to use ASO for cryptocurrency.

**Contribution:** Our main contribution and closely-related works in message-passing networks are summarized in the table below. All of our algorithms are proven correct when up to \( f < \frac{n}{2} \) nodes may crash. Our LA algorithm is *early-stopping* in the sense that the round complexity depends only on \( k \) (\( k \leq f \)), the actual number of failures in an execution. We also present a general transformation to implement ASO from any LA algorithm. Combined with the \( O(\log n) \) LA algorithm in [33], we obtain an ASO implementation that takes \( O(\log n) \) rounds for both *Update* and *Scan*. Our primary contribution is an ASO algorithm which has amortized constant round complexity when there are \( \Omega(\sqrt{k}) \) operations for each node and incurs only constant message size overhead. As a byproduct, we obtain a linearizable update-query state machines [19] implementation that takes amortized \( O(1) \) rounds for each update and query operation and \( O(1) \) message size overhead.

| Problem | Reference | Round Complexity |
|---------|-----------|-------------------|
| LA      | [33]      | \( O(\log f) \) |
| this paper |           | \( O(\sqrt{k}) \) |
| ASO     | [16]      | \( O(1) \) for *Update*, \( O(n) \) for *Scan* |
| LA [33] + transformation [this paper] |           | \( O(\log n) \) for both *Update* and *Scan* |
| this paper |           | amortized \( O(1) \) for both *Update* and *Scan* |

**Related Work: Lattice Agreement** The lattice agreement (LA) problem is well studied both in synchronous (e.g., [9, 25, 34]) and asynchronous (e.g., [34, 34, 19]) message-passing systems with crash failures. Mavronicolasa et al. [25] give an early-stopping algorithm with round complexity of \( O(\min\{h, \sqrt{f}\}) \), where \( h \) is the height of the input lattice. This is the only early-stopping LA algorithm that we know before our work. In asynchronous systems, the lattice agreement problem cannot be solved when \( f \geq \frac{n}{2} \). All existing work assume that \( f < \frac{n}{2} \). Faleiro et al. [19] give the first algorithm for this problem which takes \( O(n) \) rounds. Xiong et al [33] present an algorithm with round complexity of \( O(\log f) \). LA in the Byzantine fault model is also studied recently. Algorithms for both synchronous systems and asynchronous have been proposed [32, 31, 17, 18]. The equivalence quorum technique for our LA algorithm is quite different from the techniques in these papers.

**Related Work: Atomic Snapshot** ASO is well studied in shared memory, e.g., [1, 9, 12, 23]. Due to space constraint, we focus our discussion in message-passing networks. In message-passing systems, there are many algorithms for implementing atomic read/write registers in the presence of crash faults [7, 24, 13, 6]. A simple way to implement an atomic snapshot object is to first build \( n \) SWMR (single-writer/multi-reader) atomic registers, and then use a shared-memory ASO algorithm, e.g., [1, 9, 12].

Delporte et al. [16] present the first algorithm for directly implementing an ASO in crash-prone asynchronous message-passing systems. In their implementation, each *Update* operation takes two rounds and \( O(n) \) messages, and each *Scan* operation takes \( O(n) \) rounds and \( O(n^2) \) messages. A recent preprint by Attiya et al. [10] implement a store-collect object in dynamic networks with continuous churn. They also show how to use the store-collect object to build an ASO. These two algorithms and the read/write-register-based algorithms have worse round complexity than our ASO algorithm. In terms of techniques, our ASO algorithm is inspired by [9]. We will discuss in more details in Section 3.

**System Model:** We consider an asynchronous message-passing system composed of
n nodes with unique identifiers from \{1, 2, ..., n\}. Nodes do not have clocks and cannot determine the current time nor directly measure how much time has elapsed since some event. Each node has exactly one server thread and at most one client thread. Client threads invoke SCAN or UPDATE operations. Each client thread can have at most one SCAN or UPDATE operation at any time, i.e., each process is sequential. Server threads handle incoming messages (i.e., event-driven message handlers). Local computation is negligible compared to the message delay (or network latency). At most f nodes may fail by crashing in the system. We use k, where k ≤ f, to denote the actual number of failures in a given execution.

Each pair of nodes can communicate with each other by sending messages along point-to-point channels. Channels are reliable and FIFO (First-In, First-Out). “Reliable” means that a message m sent by node i to node j is eventually received by node j if j has not already crashed. That is, once the command “send m to j” is completed at node i, then the network layer is responsible for delivering m to j. The delivery will occur even if node i crashes after completing the “send” command. Such a channel can be implemented by a reliable broadcast primitive in practical networks [14]. FIFO means that if message m1 is sent before message m2 by node i to node j, then m1 is delivered before m2 at node j.

**Lattice Agreement (LA):** Let \((X, \leq, \sqcup)\) be a finite join semi-lattice with a partial order \(\leq\) and join \(\sqcup\). Two values \(u\) and \(v\) in \(X\) are comparable iff \(u \leq v\) or \(v \leq u\). The join of \(u\) and \(v\) is denoted as \(\sqcup\{u, v\}\). \(X\) is a join semi-lattice if a join exists for every non-empty finite subset of \(X\). In this paper, we use the term lattice instead of join semi-lattice for simplicity. More background on join semi-lattices can be found in [15].

In the lattice agreement problem [9], each node \(i\) proposes a value \(x_i \in X\) and must decide on some output \(y_i \in X\) such that the following properties are satisfied:

- **Downward-Validity:** For all \(i \in [1..n]\), \(x_i \leq y_i\).
- **Upward-Validity:** For all \(i \in [1..n]\), \(y_i \leq \sqcup\{x_1, ..., x_n\}\).
- **Comparability:** For all \(i \in [1..n]\) and \(j \in [1..n]\), either \(y_i \leq y_j\) or \(y_j \leq y_i\).

**Atomic Snapshot Object (ASO):** The snapshot object is made up of \(n\) segments (one per node), and provides two operations: UPDATE and SCAN. Node \(i\) invokes \(UPDATE(v)\) to write value \(v\) into the \(i\)-th segment of the snapshot object. We adopt the single-writer semantics, i.e., only \(i\) can write to the \(i\)-th segment. The SCAN operation allows a node to obtain an instantaneous view of the snapshot object. The SCAN returns a vector \(\text{Snap}\), where \(\text{Snap}[i]\) is a value of the \(i\)-th segment.

Intuitively, a snapshot object is atomic (or linearizable) [22] if every operation appears to happen instantaneously at some point in time between its invocation and response events. More formally, for each execution, there exists a sequence or ordering \(\sigma\) that contains all SCAN and UPDATE operations in the execution and satisfies the following properties:

- **Real-time order:** If operation \(Op_1\) completes before operation \(Op_2\) starts in the execution, then \(Op_1\) appears before \(Op_2\) in \(\sigma\).
- **Sequential specification:** If a SCAN operation returns the vector \(\text{Snap}\), then for every \(i \in \{1, \ldots, n\}\), \(\text{Snap}[i]\) is the value written by the UPDATE operation by node \(i\) that precedes the SCAN operation in \(\sigma\) or the initial value if no such UPDATE exists.

## 2 Early-Stopping Lattice Agreement Algorithm

We first present ELA (Early-stopping Lattice Agreement) in Algorithm [1]. Our constant amortized round atomic snapshot implementation uses a variation of ELA to “order” SCAN and
Amortized Constant Round Atomic Snapshot in Message-Passing Systems

Update, which will be discussed in the next section. The ELA algorithm is inspired by the stable vector algorithm by Attiya et al. [8] and Mendes et al. [26]. One of our key contributions is the formal abstraction of the equivalence quorum condition and its application to lattice agreement and atomic snapshot objects implementations.

Each node \( i \) is given an input \( x \), and at all times, \( i \) maintains a vector of sets, \( V_i[1] \cdots V_i[n] \), where \( V_i[j] (j \neq i) \) stores the set of values received from node \( j \). We denote that a variable \( v \) belongs to node \( i \) by attaching to it the subscript \( i \), for example \( v_i \). When the node identity is clear from the context, we often omit the subscript.

The ELA algorithm has two main parts: exchange all values known so far, and determine when it is “safe” to output a value using a decision rule. One key challenge is to identify the decision rule to enable the early-stopping property. Our decision rule is based on the existence of an equivalence quorum. Let \( V \) be a vector of size \( n \) and \( i \in \{1, \cdots, n\} \). We define the predicate \( EQ(V, i) \) as follows.

**Definition 1 (Predicate \( EQ(V, i) \)).** \( EQ(V, i) \) is true iff \( \exists Q \subseteq \{1, \cdots, n\} \) s.t. \( |Q| \geq n - f \land V[j] = V[i], \forall j \in Q \). When the predicate is true, we call \( Q \) as the equivalence quorum.

In ELA, node \( i \) decides when the predicate \( EQ(V_i, i) \) becomes true for the first time. Intuitively, \( EQ(V_i, i) \) becomes true when node \( i \) learns about \( n - f \) nodes (including \( i \)) with identical sets of values. Then, node \( i \) decides on the join of all values in \( V_i[j] \).

**Algorithm 1** ELA (Early-stopping Lattice Agreement): Code for node \( i \)

| Local Variables: /* These variables can be accessed and modified by any thread at \( i \). */ |
|---|---|
| \( x \) &gt; input at node \( i \) |
| \( V[1, \cdots, n] \) &gt; vector of sets at node \( i \), initially, \( V[j] = \emptyset \forall j \neq i, V[i] = \{x\} \)

**When Lattice Agreement is invoked:** /* Event handler: executing atomically in background, even after node \( i \) decides */

1: Send \( (x, i) \) to all
2: Wait until \( EQ(V, i) = true \)
3: \( V^* \leftarrow \) the vector \( V \) that satisfies \( EQ(V, i) \) for the first time
4: Decide \( y \leftarrow \bigcup\{v \mid v \in V^*[i]\} \)

**Correctness of ELA:** Consider any execution of Algorithm 1. We show that the outputs of correct nodes satisfy the three properties defined in Section 1. Due to space constraint, proofs are presented in Appendix B. Downward-validity and upward-validity are straightforward from the code. Lemmas 2 is key for proving comparability in Lemma 3. For any two sets \( U \) and \( V \), we say \( U \) and \( V \) are comparable if either \( U \subseteq V \) or \( V \subseteq U \).

**Lemma 2.** For any two nodes \( i \) and \( j \), fix time \( t \) and \( t' \), and then the set \( V_i[s] \) at time \( t \) and the set \( V_j[s] \) at time \( t' \) are comparable for each node \( s \).

By applying Lemma 2 and the decision rule, we have the following lemma.

**Lemma 3.** [Comparability] For any two nodes \( i \) and \( j \), \( y_i \) and \( y_j \) are comparable.

**Round Complexity:** Given an execution of ELA, let \( D \) be the maximum message delay. That is, if both the sender and the receiver are nonfaulty, then the sender’s message will be received by the receiver within time \( D \). We divide time into intervals of length \( D \) and each interval is called a round. For simplicity, we assume that every node initiates the ELA algorithm at the same time. The analysis can be generalized to the case when nodes invoke ELA within constant number of rounds. We begin with a useful definition.
Definition 4 (Exposed value in an interval). We say a value $v$ is an exposed value in interval $[t, t + D)$ if some nonfaulty node receives $v$ in interval $[t, t + D)$, and no nonfaulty node has received $v$ before time $t$.

Note that by definition, any exposed value for $t > D$ must be the input of some faulty node. We have the following lemma, which guarantees the termination of our algorithm.

Lemma 5. [Termination] For an arbitrary interval $[t, t + 2D)$. If there does not exist any exposed value in this interval, then all undecided nonfaulty nodes decide by time $t + 2D$.

Lemma 5 follows from the observation that if there is no exposed value in the $[t, t + 2D)$ interval, then at the end of the interval, for each node $i$, we must have $V_i[j] = V_i[i]$ for each nonfaulty $j$. Now we introduce the notion of failure chain of an exposed value.

Definition 6 (Failure chain of an exposed value). A sequence of nodes $p_1, p_2, ..., p_m$ is said to form a failure chain of an exposed value $v$ if (i) $p_1, p_2, ..., p_{m-1}$ are faulty, and $p_m$ is correct; (ii) the input value of $p_1$ is $v$; (iii) $p_i$ receives value $v$ from $p_{i-1}$; and (iv) For $1 \leq i < m - 1$, $p_i$ crashes while sending $(v, p_i)$ to other nodes, i.e., $p_1$ crashes when executing line 1 and $p_2, ..., p_{m-2}$ crash when executing line 7.

Lemma 7. If value $v$ is an exposed value in interval $[t, t + D)$, then value $v$ has a failure chain with length at least $tD + 1$.

The following lemma can be derived from condition (iv) of Definition 6.

Lemma 8. For any two exposed values $v$ and $u$ with failure chain $P_v$ and $P_u$ respectively. Then, the first $|P_v| - 2$ nodes in $P_v$ and the first $|P_u| - 2$ nodes in $P_u$ are disjoint.

Lemma 9. If an execution has $k \leq f$ crash failures, then ELA takes at most $2\sqrt{k}$ rounds.

Proof Sketch. Since there are at most $k$ failures in the execution, and an exposed value in interval $[t, t + D)$ is associated with $\geq \frac{t}{D} - 1$ unique faulty nodes by Lemma 7 and 8, we cannot have $> 2\sqrt{k}$ distinct intervals with exposed values. Lemma 5 then implies that ELA takes at most $2\sqrt{k}$ rounds.

3 Atomic Snapshot Object

In this section, we present two algorithms for implementing an atomic snapshot object in crash-prone asynchronous message-passing systems with $f < \frac{n}{2}$.

3.1 General Transformation

Attiya et al. [9] gave an elegant algorithm that transforms any wait-free lattice agreement algorithm to a wait-free atomic snapshot object in the shared memory systems. Their key idea is to invoke a sequence of lattice agreement instances to obtain comparable snapshots.

To adapt the algorithm in [9] for message-passing systems, we need to make two main modifications: (i) Replace each read or write step in shared memory by sending a read or write message to all nodes and waiting for $n - f$ acknowledgements; and (ii) Add another write step (sending the input to at least $n - f$ nodes) before invoking a lattice agreement instance.

Since the algorithm is similar to the algorithm in [9] except for these two changes, we present the algorithm, TS-ASO, and its proof in Appendix C. Using the $O(\log n)$-round lattice
Amortized Constant Round Atomic Snapshot in Message-Passing Systems

agreement algorithm by Xiong et al. [33] in our transformation gives an implementation of atomic snapshot objects that take $O(\log n)$ rounds for both Update and Scan operations. One drawback of our transformation is that it does not necessarily “preserve” the round complexity of the lattice agreement algorithm. This is because the round complexity analysis of some lattice agreement algorithms depends on the assumption that each node starts around the same time and different nodes might participate in the same lattice agreement instance at different times in TS-ASO. Therefore, directly using our ELA algorithm in the transformation gives a round complexity of $O(n)$.

To address this issue, we propose our second atomic snapshot algorithm.

3.2 Algorithm with Constant Amortized Round Complexity

Our second atomic snapshot algorithm, AC-ASO (amortized constant atomic snapshot object), uses the equivalence quorum technique and a novel mechanism of invoking lattice agreement instances to ensure amortized round complexity. In addition, TS-ASO requires message size overhead of $O(n)$, because each node needs to collect the states of at least a quorum of nodes before participating in a particular lattice agreement instance. AC-ASO only incurs $O(1)$ message size overhead. As a byproduct, we obtain a linearizable update-query state machines [19] that take amortized $O(1)$ rounds for each update and query command and $O(1)$ message size overhead, shown in Appendix A.

3.2.1 Main Techniques of AC-ASO

Our algorithm AC-ASO is inspired by [9], i.e., invoking a sequence of lattice agreement instances to implement atomic snapshot. The key technical contribution is to identify how to tightly glue different components together to obtain amortized constant round complexity. We first discuss two goals that need to be achieved for correctness. Then we introduce a new mechanism of invoking lattice operations, namely LatticeRenewal(), and discuss how we achieve the desired round complexity. Finally, we compare TS-ASO and AC-ASO.

In our discussion below, we call an instance of lattice agreement a lattice operation for brevity. Following [9], we will use a tag (or a logical timestamp) to distinguish different lattice operations in AC-ASO. Hence by “nodes participate in lattice operation with the same tag,” we mean that these nodes are in the same instance of the lattice agreement algorithm. Due to the property of lattice agreement, these nodes are guaranteed to obtain comparable outputs. Each value written by the Update is also assigned a tag as well. Later in Definition 12, we formally define the tag for operations and values.

Goals for ensuring correctness: In our design, when a Scan or Update operation completes, it obtains a “view.” Roughly speaking, a view represents a set of values that are observed by the operation and are “safe” to return (to be introduced formally later in Definition 14). Inspired by [9], we want to achieve the following two goals in our algorithm: (G1) views obtained by all Scan and Update operations are comparable; and (G2) once an Update completes, its written value is “visible” to any subsequent operations. These two goals allow us to use a natural mechanism to construct a linearization for a given execution.

1 Note that the algorithm in [33] actually has round complexity $O(\log f)$; however, if we plug in the original version, our transformation becomes $O(n)$ rounds. We need to make a simple modification of the algorithm in [33] to get $O(\log n)$ round complexity. Please refer to Appendix C for more details.

2 The round complexity guarantee of the $O(\log n)$ rounds algorithm in [33] does not depend on the assumption that all nodes start the algorithm around the same time.
It is simple to achieve goal (G2). AC-ASO ensures that once an UPDATE completes, a quorum of nodes have seen the update. For goal (G1), we require each node to participate in lattice operation(s) to complete its SCAN and UPDATE operations. Intuitively, an operation is completed if it obtains a view, which could be an output of a lattice operation invoked by this operation, or an output borrowed from another lattice operation invoked by some other operation. AC-ASO achieves goal (G1) by maintaining the following invariant:

**Invariant 10.** In AC-ASO, the views returned by lattice operations are comparable.

**Lattice Renewal:** We introduce the \texttt{LatticeRenewal()} procedure to guarantee Invariant 10. We stress that even though the usage of borrowed view is not new, we are not aware of any prior work that achieves amortized constant rounds for atomic snapshot objects. \texttt{LatticeRenewal()} is a mechanism to invoke a sequence of lattice operations to provide the following desirable properties:

(P1) \texttt{LatticeRenewal()} invokes at most three lattice operations in a row.

(P2) If any of the lattice operations does not observe a higher tag, then \texttt{LatticeRenewal()} returns the view obtained by that particular lattice operation, namely direct view.

(P3) If all three lattice operations observe a larger tag, then \texttt{LatticeRenewal()} fails to find a direct view. It will then wait to borrow a view from a lattice operation invoked by other nodes, namely indirect view.

(P4) Views returned by \texttt{LatticeRenewal()} are comparable with each other.

(P1) is mainly for correctness and improved round complexity as we will explain next. (P2) to (P4) jointly guarantee Invariant 10. Due to the properties of lattice agreement, views returned by the lattice operation with the same tag are comparable. For lattice operations with different tags, we rely on (P2) and (P3). (P2) implies that a lattice operation returns a view iff it does not observe a lattice operation with a higher tag. This together with our approach of obtaining tags ensure that the view returned by a lattice operation with a smaller tag must be known by a lattice operation with a larger tag. Therefore, when the lattice operation with a larger tag starts, its view is at least as large as the view of any lattice operation with a smaller tag. This allows later lattice operations to learn older views.

Due to message delays and concurrent UPDATES, it is possible that all three lattice operations fail to return a view. In this case, we rely on (P3) to ensure that \texttt{LatticeRenewal()} is able to obtain an indirect view. Moreover, our design guarantees that such an indirect view can be borrowed within a constant amount of rounds. In Lemma 21, we formally prove that properties (P2) to (P4) are enough to maintain Invariant 10.

In AC-ASO, a SCAN and UPDATE operation invokes \texttt{LatticeRenewal()} (after some preprocessing) and the operation is completed when \texttt{LatticeRenewal()} obtains a view. Invariant 10 can be used to prove (P4), which then guarantees goal (G1) – views obtained by all SCAN and UPDATE operations are comparable (as formally proved in Lemma 22). Later in our correctness proof, this allows us to construct a linearization of operations.

**Round Complexity:** AC-ASO ensures amortized constant round complexity, and each operation takes $O(\sqrt{k})$ rounds in the worst case. On a high level, AC-ASO uses the equivalence quorum technique to implement the underlying lattice operation. Worst case round complexity roughly follows the analysis for ELA (Algorithm 1) as presented in Section 2. For amortized constant round, the main property we rely on is the early-stopping property which ensures that if no node fails, then the lattice operation completes in a constant number of rounds. By assumption, a crashed node does not participate in the algorithm anymore;
hence, if we have enough number of \texttt{SCAN} and \texttt{UPDATE} operations, AC-ASO achieves amortized constant round complexity.

Recall that the round complexity analysis of ELA depends on the notion of \textit{exposed values} (Definition \ref{def:exposed}) and the time interval these values appear. Unlike the (single-shot) lattice agreement problem, atomic snapshot is long-living; thus, it is possible that \texttt{UPDATE}s attempt to write values with the same tag consecutively in a way that these values are all treated as the input to a particular lattice operation, which eventually “slow down” the progress of that lattice operation, and hence the \texttt{UPDATE} operation. This is also the reason that if we plug ELA into our transformation algorithm in Appendix \ref{appendix:C} we get $O(n)$ round complexity.

Our idea to address this issue is in fact simple: \textit{increment tags and invoke lattice operation(s) in a way that “late” \texttt{UPDATE} does not prevent the progress of existing lattice operations}. More concretely, consider a lattice operation with tag $T$, say $\text{Lattice}(T)$, starts at time $t$. Fix a constant $D$ (which will become clear in Lemma \ref{lemma:tsaso}). We need to ensure that (i) All values from \texttt{UPDATE} operations that start after time $t + D$ must have tag strictly greater than $T$; and (ii) “slow writers” that participate in $\text{Lattice}(T)$ after time $t + D$ do not introduce any exposed value with tag $T$. These two properties ensure that the lattice operation in AC-ASO completes in $O(\sqrt{k})$ rounds in the worst case, and has a constant amortized round complexity. This observation together with the design that $\text{LatticeRenewal}()$ invokes at most three lattice operations give the desired round complexity.

\textbf{TS-ASO vs AC-ASO:} Recall that TS-ASO is our general transformation algorithm adapted from \cite{9} (presented in Appendix \ref{appendix:C}). We present high-level comparison between AC-ASO and TS-ASO here, and details in Appendix \ref{appendix:D}.

\begin{enumerate}
  \item[(D1)] TS-ASO participates in the first lattice operation using the largest tag read from a quorum, whereas AC-ASO adds 1 to obtain a new tag for the first lattice operation.
  \item[(D2)] AC-ASO has an initial lattice operation in addition to the ones in $\text{LatticeRenewal}()$.
  \item[(D3)] TS-ASO collects the states of at least a quorum and writes the join of the states collected to at least a quorum, and participates in at most two lattice operations to obtain a view. AC-ASO, instead, directly uses at most three lattice operations to obtain a view.
\end{enumerate}

Roughly speaking, (D1) allows nodes to invoke a lattice operation with the same tag around the same time. (D2) ensures property (P3) of $\text{LatticeRenewal}()$, particularly, some node without obtaining a direct view can always borrow a view. (D3) ensures property (P4) of $\text{LatticeRenewal}()$ and constant message size overhead. The details of the necessity of three lattice operations are presented in Lemma \ref{lemma:three} and the round complexity analysis can be found in Lemma \ref{lemma:round}.

\subsection{3.2.2 AC-ASO Description and Pseudocode}

The pseudocode of AC-ASO is presented in Algorithm \ref{algorithm:acaso}. We first describe key variables used, followed by the procedures and message handlers.

\textbf{Variables:} Each value (written by an \texttt{UPDATE} operation) is associated with a timestamp of the form $(r, j)$, where $r$ is the tag and $j$ is the ID of the writer who initiates the \texttt{UPDATE}. The exact value of the tag in the timestamp is determined in the \texttt{UPDATE} operation. For brevity, we often use value to denote a \textit{value-timestamp pair}. For a set of values $H$, we use $H \leq r$ to denote the set of values with tag at most $r$.

At all time, each node $i$ keeps track of a vector $V_i$ of size $n$, which represents the vector of “view” at node $i$. Formally, for $j \in [n]$, $V_i[j]$ is the set of written and/or forwarded values that $i$ has received from node $j$. In our design, each node $i$ needs to forward a
Algorithm 2 ASO: Code for node $i$

Local Variables: /* These variables can be accessed and modified by any thread at p. */

- $V[1\cdots n]$: vector of "views". $V[j]$ is the set of values received from $j$
- $\maxTag$: integer, largest tag ever seen via "writeTag", "echoTag" messages.
- $D[1\cdots n]$: vector of views from good lattice operations.

Derived Variable:

- $V^{\leq r} \leftarrow [V[1]^{\leq r}, V[2]^{\leq r}, \ldots, V[n]^{\leq r}]$: vector of "views" w/ tag at most $r$

Initialization:

1. $V \leftarrow [\emptyset, \emptyset, \ldots, \emptyset]$
2. $D \leftarrow [\emptyset, \emptyset, \ldots, \emptyset]$
3. $\maxTag \leftarrow 0$

When $\text{UPDATE}(v)$ is invoked:

4. $r \leftarrow \text{readTag}()$
5. $ts \leftarrow \langle r + 1, i \rangle$
6. Send ("value", (v, ts)) to all
7. $\text{Lattice}(r)$ : Phase 0
8. $r' \leftarrow \max\{r + 1, \maxTag\}$
9. $\text{updateView} \leftarrow \text{LatticeRenewal}(r')$
10. Return ACK

When $\text{SCAN}()$ is invoked:

11. $r \leftarrow \text{readTag}()$
12. $\text{scanView} \leftarrow \text{LatticeRenewal}(r)$
13. Return $\text{extract}(\text{scanView})$

Procedure $\text{Lattice}(r)$:

14. $\text{writeTag}(r)$
15. $\text{Wait until } \text{EQ}(V^{\leq r}, i) = \text{True}$
   /* Execute lines 16 to line 21 atomically */
16. $V^* \leftarrow$ the vector $V^{\leq r}$ that satisfies
   $\text{EQ}(V^{\leq r}, i)$ for the first time
17. if $\maxTag \leq r$ then
18. Send ("goodLA", $r$) to all
19. Return (true, $V^*[i]$)
20. else
21. Return (false, $\emptyset$)

Procedure $\text{LatticeRenewal}(r)$:

22. for phase $\leftarrow 1$ to 3 do
23. (status, view) $\leftarrow \text{Lattice}(r)$
24. if status $= \text{true}$ then
25. Return view : Direct View
26. else if phase $= 3$ then
27. Break
28. $r \leftarrow \maxTag$
29. $\text{Wait until }$ receiving ("goodLA", $r$) from some node $j$
30. Return $D[j]$: Indirect View

Procedure extract($S$):

31. $\text{Snap} \leftarrow [1, \cdots, n]$
32. for $j = 1$ to $n$ do
33. $\text{Snap}[j] \leftarrow v$, where $\langle v, \langle t', j \rangle \rangle \in S$, and $t'$ is the largest tag of $j$'s values in $S$
34. Return $\text{Snap}$

Procedure $\text{readTag}()$:

35. Send ("readTag") to all
36. $\text{Wait until}$ receiving $\geq n - f$ ("readAck", *) msgs
37. Return largest tag contained in readAck msgs

Procedure $\text{writeTag}(tag)$:

38. Send ("writeTag", tag) to all
39. $\text{Wait until}$ receiving $\geq n - f$ ("writeAck", tag) msgs

/* Event handlers: executing in background */
/* All event handlers executed atomically */

Upon receiving ("value", (u, ts)) from $j$:

40. Add (u, ts) into $V[j]$, $V[i]$
41. if (u, ts) has not been seen before then
42. Send ("value", (u, ts)) to all

Upon receiving ("writeTag", tag) from $j$:

43. $\maxTag \leftarrow \max\{\maxTag, tag\}$
44. Send ("echoTag", tag) to all
45. Send ("writeAck", tag) to $j$

Upon receiving ("echoTag", tag) from $j$:

46. $\maxTag \leftarrow \max\{\maxTag, tag\}$

Upon receiving ("readTag") from $j$:

47. Send ("readAck", $\maxTag$) to $j$

Upon receiving ("goodLA", $r$) from $j$:

48. $D[j] \leftarrow V[j]^{\leq r}$: borrow $j$'s view

/* NOTE: All our event handlers are atomic; hence, when receiving a ("goodLA", *) Line 48 will be executed before Line 29 if there is a pending LatticeRenewal() */
value it receives for the first time. In this case, we say a value is forwarded by \( i \). Two other variables are related to \( V_i \): (i) \( V_i^{\leq r} \) is the vector of views with tag at most \( r \), i.e., \( V_i^{\leq r} = [V_i[1]^{\leq r}, V_i[2]^{\leq r}, \ldots, V_i[n]^{\leq r}] \); and (ii) \( D_i[j] \) is a \textit{particular view borrowed from node} \( j \) that can be “safely” returned. The meaning of “safe” will become clear when we discuss the lattice operation.

Each node also keeps track of a variable \( \text{maxTag} \), which represents the largest tag it has ever received via “\text{writeTag}” messages or “\text{echoTag}” messages. Note that it is possible that there are some values with tag larger than \( \text{maxTag} \) in \( V_i \).

**Procedures:** We explain two helper procedures, \( \text{Lattice}(r) \) and \( \text{LatticeRenewal}(r) \), and two interface procedures, \( \text{Update}(v) \) and \( \text{SCAN}() \). Other procedures are fairly straightforward from the pseudocode.

\( \text{Lattice}(r) \): Each node uses the \( \text{Lattice}(r) \) procedure to run the \( r \)-th instance of lattice agreement, or in our terminology, lattice operation with tag \( r \). The goal of a lattice operation is to solve lattice agreement, except that it is associated with an input tag \( r \) and the termination condition depends on \( r \) and the messages received, especially those with tag \( \leq r \).

Consider a \( \text{Lattice}(r) \) invocation at node \( i \). Node \( i \) first writes the input tag \( r \) to at least \( n - f \) nodes. Then it waits until the \textit{equivalence quorum} predicate (Definition 1) becomes true for the first time. After that if the \( \text{maxTag} \) value is strictly larger than \( r \), then the lattice operation returns \( (\text{false}, \emptyset) \). Otherwise, it returns \( (\text{true}, V^* \) \) where \( V^* = V_i^{\leq r} \) is the vector that satisfies the equivalence quorum predicate. In this case, \( \text{Lattice}(r) \) is said to be a good lattice operation, as defined next. An important design choice here is that line 16 to 21 are executed \textit{atomically}. Therefore, once \( V^* \) satisfies the equivalence quorum predicate for the first time, \textit{no other value is added to} \( V^* \).

\[ \text{Definition 11 (Lattice Operation).} \text{ We call each execution of the } \text{Lattice}(r) \text{ procedure as a lattice operation with tag} \ r. \text{ A lattice operation is good if it returns true at line 19.} \]

\( \text{LatticeRenewal}(r) \): The \( \text{LatticeRenewal}(r) \) procedure is also given a parameter \( r \). It contains at most three lattice operations. If some lattice operation is good, it returns the view obtained by the good lattice operation, i.e., \textit{direct view}. If the first two lattice operation are \textit{not} good, then by definition, it means that node \( i \) has observed a larger tag, i.e., condition at line 17 returns \textit{false}. Therefore, node \( i \) initiates the next lattice operation with tag equal to \textit{maxTag}. If the third lattice operation is also not good, then node \( i \) waits for a “\textit{goodLA}” message from some other node \( j \) to obtain a view from \( j \)'s good lattice operation. In this case, the view is called an \textit{indirect view} or a \textit{borrowed view}.

It is fairly straightforward to see that our design satisfies the (P1) to (P3) stated earlier in Section 3.2.1 (P4) and the round complexity are less obvious, and depend on how we combine different components together. Both \( \text{Update} \) and \( \text{SCAN} \) operations use the \( \text{LatticeRenewal} \) procedure to obtain a view. This approach works owing to (P4). Intuitively, the moment that \( \text{Update} \) and \( \text{SCAN} \) obtains a view is the synchronization point.

\( \text{Update}(v) \): To write value \( v \), node \( i \) first obtains a tag by reading from at least \( n - f \) nodes. Let \( r \) denote the largest tag in the received \textit{readAck} messages. Then, \( i \) constructs the timestamp of value \( v \) as the \((r + 1, i)\) tuple. It sends value \( v \) with its timestamp to all nodes. Then, a lattice operation with tag \( r \) is invoked. This step is called the \textit{phase 0} lattice operation of the \( \text{Update} \) operation. After the phase 0 lattice operation, the \( \text{Update} \) obtains a new tag \( r' \) and executes \( \text{LatticeRenewal}(r') \). The view returned by \( \text{LatticeRenewal}(r') \) is \textit{not} used; and hence discarded. Node \( i \) returns the \textit{ACK} to complete the \textit{Update}.

In addition to the \( \text{LatticeRenewal} \) procedure, another subtle point is to execute the phase 0 lattice operation \textit{before} invoking \( \text{LatticeRenewal} \). The way we devise them ensures
that for each tag, there is a good lattice operation. Recall that a lattice operation is good if it returns true at line 17. Lemma 16 presented later explains this statement in more details.

Scan(): The code for a Scan operation is quite simple. It first obtains a tag \( r \) similarly. Then, it executes the \( \text{LatticeRenewal}(r) \). After \( \text{LatticeRenewal}(r) \) returns a view \( \text{scanView} \), the node takes the most recent value by each node in \( \text{scanView} \) by executing the \( \text{extract}(\text{scanView}) \) procedure.

Message Handlers: All the handlers execute in the background; hence, even if a node does not have a pending Update or Scan operation, it still processes messages. Moreover, all the handlers are executed atomically, i.e., during the period that a handler is executing, no other part of the code can take step. All the handlers should be clear from the code. One subtle part to note is that a node does not update its \( \maxTag \) variable when it receives a “value” message. The \( \maxTag \) variable is only updated when a node receives a “writeTag” message or “echoTag” message. This design helps AC-ASO achieve the desired round complexity. Especially, we rely on it to prove Lemma 16. For completeness, we also present detailed description of message handlers is in Appendix E.1.

3.3 Proof of Correctness

For correctness, we need to prove termination and construct a linearizable sequence of Update and Scan operations for any execution. We first discuss important definitions and properties of our algorithm to facilitate the proof. Then, we prove termination and show the linearization construction.

Useful Definitions: Tags and Views: We say an Update or Scan is direct if its \( \text{LatticeRenewal}() \) procedure returns at line 25; otherwise, it is indirect. Intuitively, an operation is direct if it there is a good lattice operation during \( \text{LatticeRenewal}() \). We define the tag of an operation and value as following.

- **Definition 12 (Tag of Update or Scan):** The tag of an Update or Scan operation is the tag of its last lattice operation.

- **Definition 13 (Timestamp/Tag of a value):** The timestamp of a value is the \( \langle r + 1, i \rangle \) (tag-ID tuple) at line 5 in the \( \text{Update}(v) \) procedure. The tag of a value is defined as the tag contained in its timestamp. For value \( v \), we use \( ts_v \) to denote its timestamp.

Now we introduce an important concept, view, that is used throughout our proof.

- **Definition 14 (View):** We define the views for a node and operations as below:
  - For a node \( i \), its view is defined as the set \( V_i[i] \).
  - For a good lattice operation with tag \( T \) (\( \text{Lattice}(T) \)) at node \( i \), its view is defined as the set of values with tag at most \( T \) in \( V_i[i] \) right after completing line 15, i.e., \( V_i[i] \leq T \).
  - For an Update or Scan operation, its view is defined as the set returned by its \( \text{LatticeRenewal}() \) procedure.

We present two lemmas on properties of the tags. Lemma 15 follows directly from line 8.

- **Lemma 15.** The tags of values are non-skipping, i.e., if there is a value with tag \( T \geq 1 \), then there is also a value with tag \( T - 1 \).

The proof of the following lemma explains why we need the phase 0 lattice operation.

- **Lemma 16.** If the largest tag in the system is \( T \) at time \( t \), i.e., \( \max_{i \in [n]} \maxTag_i = T \) at time \( t \), then for each \( 1 \leq z \leq T - 1 \), there is a good lattice operation with tag \( z \) before time \( t \).
Proof. To prove the lemma, we first prove the following claim.

Claim 17. If the largest tag in the system is $T$ at time $t$, then there exists a good lattice operation with tag $T - 1$ that completes before $t$.

Proof of Claim 17. Observe that a value with tag $T$ must be sent by some node, since we consider only crash failures. Let node $i$ be the first node that sends tag $T$ to all other nodes inside the writeTag procedure in operation $Op$. Since $Op$ is the first operation to send tag $T$, $Op$ must be an Update operation. Let $L_0$ denote the phase 0 lattice operation inside $Op$. When $Op$ executes line 8 we have $\maxTag_i < T$, and $r_i = T$ when line 8 completes. This implies that $r_i = T - 1$ at line 9. Thus, the phase 0 lattice operation $L_0$ of $Op$ at line 7 must have tag $T - 1$. That is, node $i$ invokes $\text{Lattice}(T - 1)$ as the phase 0 lattice operation.

Since by assumption, $Op$ is the first operation that proposes tag $T$ and it proposes tag $T$ after line 8 (through the writeTag step at line 14), no node has proposed tag $T$ before the execution of line 8. Hence, during the execution of the phase 0 lattice operation $L_0$ at line 7 we have $\maxTag_i \leq T - 1$. Recall that the tag of $L_0$ is $T - 1$. This implies that at line 14 inside $L_0$, we have $\maxTag_i \leq T - 1 = r_i$. Thus, $L_0$ is a good lattice operation. Moreover, by assumption, $L_0$ completes before time $t$. This proves Claim 17.

Since tags are non-skipping by Lemma 15, applying Claim 17 inductively gives us that for each $1 \leq z \leq T - 1$, there exists a good lattice operation with tag $z$ before time $z$.

Termination: We show that each operation eventually terminates if each lattice operation terminates. We will show that each lattice operation takes $O(\sqrt{k})$ rounds later.

Lemma 18. [Termination] If each lattice operation eventually terminates, then Update and Scan operations in Algorithm 2 eventually terminate.

Proof Sketch. We show that each LatticeRenewal() eventually terminates. The only blocking part is line 23. Lemma 16 implies that if a node observes a tag $T$, then it must be able to borrow a good view for some tag smaller than $T$.

Useful Lemmas: Comparable Views: Next we prove Invariant 10 which is formally stated in Lemma 21.

Lemma 19. The view of each Update or Scan operation is the same as the view of some good lattice operation.

Similar to Lemma 1, we obtain the following lemma on the views (bounded by tag $T$) at node $i$ and $j$ due to our assumption on FIFO channel. Its proof is in Appendix E.4.

Lemma 20. For any two nodes $i$ and $j$ and tag $T$, fix any time $t$ and $t'$, the set $V_i[s] \leq T$ at time $t$ and the set $V_j[s] \leq T$ at time $t'$ are comparable for each node $s$.

Next, we prove an important lemma which shows our key usage of lattice operation and the equivalence quorum technique. Lemma 21 is a formal statement of Invariant 10. We put its full proof in Appendix E.5.

Lemma 21. The views returned by all good lattice operations are comparable.

Proof Sketch. For any two lattice operations $Op_i$ and $Op_j$ with tag $T_i$ and $T_j$, if $T_i = T_j$, then Lemma 20 and the equivalence quorum predicate imply that their views must be comparable. Otherwise, assume w.l.o.g $T_i < T_j$. Our algorithm guarantees that the view of $Op_i$ must be a subset of the view of $Op_j$. Intuitively, the fact that $Op_i$ does not observe $T_j$ at line 17 implies that $Op_j$ must complete its line 14 step after $Op_i$ has completed. This ensures that $Op_j$ must have received all values in the view of $Op_i$ when $Op_j$ starts line 15.
Lemma 22 immediately follows from Lemma 19 and 21. Lemma 22 allows us to construct a linearization of SCAN and Update operations later.

Lemma 22. The views returned by all Update and Scan operations are comparable.

Useful Lemma: Visible Views: To respect the atomicity semantics, we also need to ensure that (i) once an Update is completed, then its value is visible to subsequent Scan’s; and (ii) once a Scan reads certain set of values, these values are also visible to subsequent Scan’s. We prove these two through the usage of views and tags.

The lemma below is straightforward from the code. Refer Definition 14 for view definition.

Lemma 23. For a good lattice operation Op by node i with tag T, let H denote the view of node i right before Op execute line 13 and HOp denote the view of Op. Then, H^{ST} \subseteq HOp.

The next lemma is the main reason that we need to have three lattice operations in the LatticeRenewal() procedure.

Lemma 24. Let Op be an Update or Scan operation by node i with tag T. Let HOp denote its view. Let H denote the view of node i right before Op executes its LatticeRenewal() invocation. Then, H^{ST} \subseteq HOp.

Proof. We assume that Op is an Update operation. The proof for the other case is similar. If Op is an Update with a direct view, the claim follows from Lemma 23 since by definition of a direct view, Op obtains a view from its good lattice operation. Now, consider the case when Op completes with an indirect view. By construction, Op must continue to phase 3. Let R1, R2 and R3 denote the tags for each of the three lattice operations in Op’s invocation of LatticeRenewal(), respectively. Then, by Definition 12 T = R3, the tag of the last lattice operation in Op. Moreover, Op must have received (“goodLA”, R3) message from some other node j. Let Lj denote this particular lattice operation by node j. By construction, Lj is a good lattice operation with tag R3. Then, we prove the following claim.

Claim 25. Tag R3 was not known by node i during its phase 1 lattice operation.

Proof of Claim 25. First observe that R1 < R2 < R3, since none of the lattice operation in i’s LatticeRenewal() is good. The fact that Op obtains tag R0 such that R2 < R3 during phase 2 implies that when Op completes its phase 1, it has not learned R3; otherwise, it would not proceed to phase 2 with tag R2, since R2 < R3.

Let Li denote the lattice operation by Op in phase 1 at node i. Consider the writeTag procedure in Li and Lj. Let Qi and Qj denote the set of nodes that sent the “writeAck” messages in responding to the “writeTag” message of Li and Lj, respectively. Since both set of nodes are of size at least n - f, there exists a nonfaulty node s ∈ Qi ∩ Qj such that node s must have received the “writeTag” message from Lj after sending “writeAck” message in responding to the “writeTag” message of Li. Otherwise, Op would obtain tag R3 for phase 2, a contradiction to Claim 25.

Since communication is reliable and FIFO, and node i sends all values in H before sending the “writeTag” message in lattice operation Li, node s must receive all values in H and sends out to all other nodes before sending the “writeAck” message in responding to the “writeTag” message of Lj. Thus, node j must have received all values in H before it completes line 14 of Lj. Let Hj denote the view of lattice operation Lj. Since Lj is a good lattice operation, Lemma 23 implies that H^{JR3} \subseteq Hj.

By assumption, node i borrows j’s view after it has received the (“goodLA”, R3) at line 29. Thus, Hj \subseteq HOp. Therefore, H^{JR3} = H^{ST} \subseteq Hj \subseteq HOp.
Lemma 26. For any two operations $O_{pi}$ with tag $T_i$ and $O_{pj}$ with tag $T_j$, respectively. If $O_{pi} \to O_{pj}$, then $T_i \leq T_j$.

Now we prove the important lemma on views being “visible” to subsequent operations.

Lemma 27. For any two operations $O_{pi}$ of node $i$ and $O_{pj}$ of node $j$ with views $H_i$ and $H_j$, respectively. If $O_{pi} \to O_{pj}$, then $H_i \subseteq H_j$.

Proof. Let $T_i$ and $T_j$ denote the tag of $O_{pi}$ and $O_{pj}$, respectively. Consider the following two cases. Let $H$ denote the view of node $j$ right before $O_{pj}$ invokes $LatticeRenewal()$. Lemma 26 implies that $H^{\leq T_i} \subseteq H_j$. To prove the lemma, we need to show that $H_i \subseteq H^{\leq T_j}$.

Case 1: $O_{pi}$ obtains a direct view.
Consider $O_{pi}$’s last lattice operation $L_i$. Then $L_i$ is a good lattice operation with tag $T_i$. By definition, $H_i$ is also the view of $L_i$. Let $Q_i$ denote the equivalence quorum for $L_i$. Then, we have $H_i = V_i[w]^{\leq T_i} \subseteq V_w[w]^{\leq T_i}$ for each node $w \in Q_i$ when $L_i$ completes. Let $Q_j$ denote the set of nodes which send “readAck” for the first “readTag” message of $O_{pj}$. Since $O_{pj}$ starts after $O_{pi}$ completes, there exists a nonfaulty node $s \in Q_i \cap Q_j$ such that node $s$ sends out all the values in its current view, which must include all the values in $H_i$, to all the other nodes before sending the “readAck” message for $O_{pj}$’s first “readTag” message. By FIFO channels, node $j$ must have received all the values in $H_i$ before it completes the $writeTag$ procedure of $O_{pj}$. Lemma 26 implies that $T_j \geq T_i$. This together with the observation that the largest tag in $H_i$ is $T_i$, we have $H_i \subseteq H^{\leq T_j}$.

Case 2: $O_{pi}$ obtains an indirect view.
The view $H_i$ of $O_{pi}$ is the same as the view of some good lattice operation $L$ and $O_{pi}$ must have received the “goodLA” message sent by $L$. Thus, when $O_{pi}$ completes, $L$ have completed its execution of line 18. Then, by a similar argument in case 1, $H_i \subseteq H^{\leq T_j}$.

Construction of a linearization: For a given execution, we construct a sequence $\sigma$ of all $Update$ and $Scan$ operations in the execution such that $\sigma$ preserves the semantics of atomic snapshot object. The construction is similar to one from [9], and presented below.

Insert $Scan$ operations: First, we construct a sequence $\sigma'$ which includes all $Scan$ operations. The $Scan$ operations are ordered in $\sigma'$ according to the order of their associated views. Specifically, for any two $Scan$ operations $Sc_i$ and $Sc_j$ that have view $H_i$ and $H_j$, respectively, if $H_i < H_j$, then $Sc_i$ appears before $Sc_j$ in $\sigma'$. If $H_i = H_j$ and $Sc_i \to Sc_j$, then $Sc_i$ appears before $Sc_j$ in $\sigma'$. Otherwise, $Sc_i$ and $Sc_j$ are ordered arbitrarily.

Insert $Update$ operations: Second, we insert all $Update$ operations into $\sigma'$. Consider an $Update$ operation $Op$ that writes $v$ with timestamp $ts_v$. We insert $Op$ after all $Scan$ operations whose view do not include $(v, ts_v)$ and before all $Scan$ operations whose view contains $(v, ts)$. That is, $Op$ is inserted just before the first $Scan$ operation in $\sigma'$ such that its view contains $(v, ts)$. For any two $Update$ operations $Op_1$ and $Op_2$ that fit between the same pair of $Scan$ operations. If $Op_1 \to Op_2$, then we insert $Op_1$ before $Op_2$ in sequence $\sigma$. Otherwise, $Op_1$ and $Op_2$ are ordered arbitrarily.
 Similar to the proof given in [9], the proof of the following theorem uses Lemma 22 and 27 to show that σ is a linearizable sequence. We put it in Appendix E.6.

Theorem 28. AC-ASO (Algorithm 2) implements an atomic snapshot object.

3.4 Round Complexity

Now we analyze the round complexity of our algorithm. We assume the local computation time is negligible compared with the message delay. We show that each lattice operation takes $O(\sqrt{k})$ rounds. The proofs of the two lemmas below are in Appendix E.7 and E.8.

Lemma 29. Suppose there exists a lattice operation that starts at time $t$ with tag $T$, then any Update operation starting after time $t + D$ must assign a tag $> T$ for its value. Thus, all values with tags at most $T$ must have been sent out by time $t + D$.

Lemma 30. Let $Op$ denote $\text{Update}(v)$ operation. If $Op$ completes before time $t$, then for each nonfaulty node $i$, $(v, ts_v) \in V_i[j]$ for each nonfaulty node $j$ by the end of time $t + 2D$.

Recall that the exposed value is introduced in Definition 4.

Lemma 31. Each lattice operation takes $O(\sqrt{k})$ message delays in the worst case.

Proof. Let $L_i$ be a lattice operation at node $i$. Suppose $L_i$ starts at time $t$ with tag $T$. According to the condition at line 15, the termination of $L_i$ only depends on values with tags at most $T$. Thus, we do not need to consider the values from Update operations that start after time $t + D$ by Lemma 29. That is, for the termination of $L_i$, we only need to consider Update operations that start before time $t + D$. Now, we prove an important claim.

Claim 32. There are at most $k$ exposed values with tag $\leq T$ in intervals after time $t + 4D$.

Proof of Claim 32. By Lemma 30, all values with tags at most $T$ from Update operations that have completed before time $t + D$ must be contained in $V_i[j]$ for each pair of nonfaulty nodes $i$ and $j$ by time $t + 3D$, i.e., known by all nonfaulty nodes. Thus, by definition of the exposed values, we have that values from Update operations that have completed before time $t + D$ cannot be exposed values in intervals after time $t + 3D$. Since the values from Update operations that start after time $t + D$ must have tag greater than $T$, only values from Update operations that start before time $t + D$ and have not completed by time $t + D$ can be exposed values in intervals after time $t + 3D$. Let $U$ denote the set of these values. Moreover, by definition, there can at most one such Update operation per node.

Consider an arbitrary value $v \in U$ and the $\text{Update}(v)$ operation. We show that if $\text{Update}(v)$ is from a nonfaulty node, then value $v$ cannot be an exposed value for intervals after time $t + 3D$. Consider lines 4 to 6 of $\text{Update}(v)$, since local computation takes negligible time, and line 4 takes at most $2D$ time, value $v$ must be sent to all other nodes at line 6 before time $t + 3D$. Thus, by time $t + 4D$, value $v$ must be known by all nodes that have not crashed at this time, including all the nonfaulty nodes. Therefore, $v$ cannot be an exposed value in intervals after time $t + 4D$. Thus, a value in $v \in U$ can be an exposed value for intervals after time $t + 4D$ iff $\text{Update}(v)$ is from a faulty node. This proves the claim.

The above Claim implies that after time $t + 4D$, if we only consider values with tag at most $T$, Lemmas 5, 7, and 8 still hold. Thus, similar to the proof in Lemma 9, $l_p$ must terminate in $O(\sqrt{k})$ rounds.
The proof of Claim 32 also explains why we need to put line 6 before the initial lattice operation at line 7. If we switch the order of line 6 and line 7, then we cannot guarantee that value $v$ is sent to all the other nodes before time $t + 3D$, even though $\text{Update}(v)$ starts before time $t + D$.

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**Linearizable Update-Query State Machines**

In this section, we show how to implement a linearizable update-query state machine using our atomic snapshot algorithm. A update-query state machine only supports two types of operations: update and query. It does not support update and query mixed operations. It also assume that all updates are commutable, so the order of updates does not matter. Many data structures such as sets, sequences, certain types of key-value tables, and graphs [28] can be designed with commuting updates.

The implementation, shown in Algorithm 3, is almost same as the atomic snapshot implementation in Algorithm 2, except that we let the Scan operation return its view, i.e., the set obtained at line 12. The view is a set of update commands from clients. Each element in the vector \( V_p \) is a command. For an update command \( up \) from a client, node \( p \) invokes Update\( (up) \). When receiving a query command from a client, node \( p \) invokes the modified Scan() to return a set of commands and then apply these commands and return responses accordingly.

**Algorithm 3 Linearizable UQ State Machines**

Upon receiving update \( up \):
1: \( \text{Update}(up) \)
2: Respond ok to client

Upon receiving query:
3: \( \text{view} := \text{Scan}() \)
4: \( \text{reply} := \text{Apply}(\text{view}) \)
5: Respond \( \text{reply} \) to client

The following theorem implies that each command takes \( O(1) \) rounds if there is no crash fault. \( O(1) \) message size overhead means that if we assume that the size of each command is \( O(1) \), then each message in the implementation has size \( O(1) \).

**Theorem 33.** There exists an implementation of linearizable update/query state machines such that each command takes \( O(\sqrt{k}) \) message delays and \( O(1) \) message size overhead, where \( k \) is the actual number of crash failures in the system.

**B Proofs for the ELA algorithm**

**B.1 Proof of Lemma 2**

**Lemma 2.** For any two nodes \( i \) and \( j \), fix time \( t \) and \( t' \), and then the set \( V_i[s] \) at time \( t \) and the set \( V_j[s] \) at time \( t' \) are comparable for each node \( s \).

**Proof.** The value of set \( V_i[s] \) is modified only when \( i \) receives a message from node \( s \). Since \( s \) is a non-faulty node and the communication is FIFO, the set \( V_i[s] \) at time \( t \) must be the same as the set \( V_s[s] \) at some time \( t_i \) and the set \( V_j[s] \) at time \( t' \) must be the same as the set \( V_s[s] \) at some time \( t_j \). The set \( V_s[s] \) is non-decreasing. Thus, \( V_i[s] \) at time \( t \) must be comparable with \( V_j[s] \) at time \( t' \).

**B.2 Proof of Lemma 3**

**Lemma 3.** [Comparability] For any two nodes \( i \) and \( j \), \( y_i \) and \( y_j \) are comparable.

**Proof.** Let \( V_i \) denote the vector at node \( i \) and \( V_j \) denote the vector at node \( j \) when nodes \( i \) and \( j \) decide. The statement of the lemma is proved if we show that \( V_i[i] \) and \( V_j[j] \) are comparable.
The decision condition on line 2 states that there exists a set $Q_i$ of size at least $n - f$ such that $V_i[t] = V_i[s]$ for each $s \in Q_i$ and a set $Q_j$ of size at least $n - f$ such that $V_j[j] = V_j[s]$ for each $s \in Q_j$. Since $f < \frac{n}{2}$, there exists a correct process $s \in Q_i \cap Q_j$. Lemma 2 implies that $V_i[s]$ and $V_j[s]$ are comparable. This leads to the conclusion that $V_i[s]$ (which is equal to $V_i[s]$) is comparable to $V_j[j]$ (which is equal to $V_j[s]$)

B.3 Proof of Lemma 5

Lemma 5. [Termination] For an arbitrary interval $[t, t + 2D)$. If there does not exist any exposed value in this interval, then all undecided nonfaulty nodes decide by time $t + 2D$.

Proof. Let $C$ denote the set of correct nodes in an execution. Let node $i$ be an undecided node at time $t$ that does not crash by $t + 2D$. We show that by time $t + 2D$, we have $V_i[j] = V_i[i]$ for each $j \in C$. As a result, the predicate EQ at line number 2 becomes true, and node $i$ decides at Line 4.

Proof by contradiction. Suppose there exists a value $v \in V_i[i] - V_i[j]$ at time $t + 2D$. Since by assumption, value $v$ is not an exposed value in this interval, it must be received by some correct node $s$ (or it is the input value of node $s$) at some time $t_s < t$. Thus, value $v$ must be sent to all by node $s$ at time $t_s$ and received by all correct nodes by time $t_s + D < t + D$, including node $j$. By the algorithm, node $j$ must send value $v$ to all the other nodes before time $t + D$. Thus, node $i$ must receive value $v$ from node $j$ by time $t + 2D$, and add $v$ into $V_i[i]$ and $V_i[j]$, a contradiction to the assumption that $v \in V_i[i] - V_i[j]$.

B.4 Proof of Lemma 7

Lemma 7. If value $v$ is an exposed value in interval $[t, t + D)$, then value $v$ has a failure chain with length at least $\frac{f}{2} + 1$.

Proof. Recall the definition of an exposed value $v$ occurring in an interval $(t, t + D)$: there has to be a failure chain ending with a correct process that receives value $v$ in interval $(t, t + D)$. Let $p_1, \ldots, p_{m-1}, p_m$ denote such a failure chain for value $v$, where $p_1, \ldots, p_{m-1}$ are faulty and node $p_m$ is non-faulty.

Assume by contradiction that the length of this failure chain is $l \leq \frac{f}{2}$. Since $D$ is the message delay in the execution, node $p_1$ (in the failure chain) hears about $v$ at time $t_1 \leq t \cdot D$. Thus if $l \leq \frac{f}{2}$, the correct node $p_m$ hears about $v$ at time $t_m \leq \frac{f}{2} \cdot D = t$ making $v$ an exposed value occurring in an interval prior to $(t, t + D)$. This contradicts the assumption in the statement of the lemma that $v$ is an exposed value in interval $(t, t + D)$.

B.5 Proof of Lemma 8

Lemma 8. For any two exposed values $v$ and $u$ with failure chain $P_v$ and $P_u$ respectively. Then, the first $|P_v| - 2$ nodes in $P_v$ and the first $|P_u| - 2$ nodes in $P_u$ are disjoint.

Proof. Let $V$ and $U$ denote the set of the first $|P_v| - 2$ nodes in $P_v$ and the first $|P_u| - 2$ nodes in $P_u$. Suppose node $i \in V \cap U$ for contradiction. By condition (iv) of Definition 6, node $i$ crashes while sending $v$ to other nodes on line 7 of ELA. Since lines 5 to 7 of ELA are executed atomically, node $i$ cannot crash while sending value $u$ to other nodes at line 7, a contradiction.
B.6 Proof of Lemma 9

Lemma 9. If an execution has \( k \leq f \) crash failures, then ELA takes at most \( 2\sqrt{k} + 1 \) rounds.

Proof. Let us assume that the algorithm takes \( 2\sqrt{k} + 1 \) rounds for contradiction. By Lemma 5 we know that to prevent the algorithm from terminating, there has to be at least one exposed value every two rounds. Lemma 7 gives us the length of any failure chain and Lemma 8 states that a faulty node (except for the last 2 nodes in a failure chain) can be a part of only one failure chain. Thus if the algorithm terminates in round \( 2\sqrt{k} + 1 \), the number of faulty nodes must be at least \( 1 + 3 + \cdots + 2\sqrt{k} - 1 > k \), leading to a contradiction. ◀

C General Transformation in Message Passing Systems

In this section, we show how to adapt the transformation given by Attiya et al. in [9] for shared memory systems to work in message passing systems. The transformation algorithm, TS-ASO, is shown in Algorithm 4.

In the algorithm, each node \( p \) keeps track of a vector \( \text{Snap} \) of size \( n \), which is the local view of the shared object, i.e., \( \text{Snap}[q] \) stores the most recent value written by node \( q \) known by node \( p \). The variable \( V \) is a map from tag number to snapshot. \( V[r] \) is the snapshot vector obtained for tag \( r \). Variable \( r \) denotes the tag number of the last lattice agreement instance that node \( p \) has completed. \( \maxTag \) keeps track of the largest tag ever seen by a node. \( ts \) is a sequence number for values, which is increased by one when a new value needs to be written. Each value is associated with a timestamp. For any value \( v \), we use \( ts_v \) to denote its associated timestamp. If a variable belongs to node \( i \), we use the subscript \( i \) to denote it. For example, \( \maxTag_i \) denotes the value of variable \( \maxTag \) at node \( i \).

Scan() operation: a SCAN operation invokes at most two lattice operations to obtain a view. We call the two for loops as its two phases. At each phase, it first decides which lattice agreement instance to run by reading the largest tag from at least \( n - f \) nodes at line 2. Then, it writes the tag obtained to at least \( n - f \) nodes at line 4. At line 5, it reads the local state from at least \( n - f \) nodes and the join of all states received will be used as input for its lattice agreement. At line 6, it writes the join of all states read at line 5 to at least \( n - f \) nodes. Then, it invokes the lattice agreement instance with the tag obtained at line 2 and the vector obtained at line 5 as input parameters. After completion of the lattice agreement, if it does not observe a higher tag number, then it directly returns the view obtained from its lattice agreement invocation. If it does not return a view after lattice agreement invocation, it borrows a view from some other node, which is guaranteed to exist.

Update(\( v \)) operation: To write value \( v \), node \( p \) increases \( ts_p \) by one and assign it to be the timestamp of \( v \). It writes value \( v \) with its timestamp to at least \( n - f \) nodes. Then, it executes a SCAN operation. The view returned by the SCAN operation is not used. The SCAN operation acts as a synchronization point.

C.1 Proof of Correctness

In this section, we show that atomic object implementation is linearizable by explicitly constructing a linearization of all update and scan operations. We can first obtain the following lemma regarding to the views returned by all operations. We say a view returned by an operation is a direct view if this view is returned in the first phase of the scan procedure. Otherwise, we call this view as an indirect view. We can readily see that a direct view of \( p_i \) is obtained from an execution of lattice agreement of \( p_i \). An indirect view of node \( i \) is
Algorithm 4 TS-ASO: code for node p.

Local Variables:
- \( \text{Snap} \) ▷ vector of size \( n \), local view of the shared object.
- \( V \) ▷ snapshots obtained. \( V[r] \) is the snapshot obtained for tag \( r \)
- \( r \) ▷ tag number for lattice agreement
- \( \text{maxTag} \) ▷ Integer, largest tag ever seen
- \( ts \) ▷ timestamp for value

Procedure \( \text{Scan}() \):
1: for phase := 1 to 2 do
2: \( \text{readTag()} \)
3: \( r \leftarrow \max(\text{maxTag}, r + 1) \)
4: \( \text{writeTag}(r) \)
5: input := \( \text{readState}() \)
6: \( \text{writeState}(\text{input}) \)
7: output := \( \text{LA}(r, \text{input}) \)
8: \( \text{readTag()} \)
9: if \( \text{maxTag} \leq r \) then
10: \( V[r] := \text{output} \)
11: \( \text{writeView}(\text{output}, r) \)
12: return \( V[r] \)
13: else if phase = 2 then
14: wait until \( V[r] \neq \emptyset \)
15: return \( V[r] \)

Procedure \( \text{Update}(v) \):
16: \( ts \leftarrow ts + 1 \)
17: \( \text{writeValue}(v, ts) \)
18: return \( \text{Scan}() \)

Procedure \( \text{readState}() \):
26: Send (“\text{readState}”) to all
27: Wait until receiving
28: \( \geq n – f \) (“\text{readStateAck}”, +) msgs
29: \( \text{Let} \ S_j \) denote the state vector received from \( j \)
30: \( \text{return} \bigcup_j \ S_j \)

Procedure \( \text{writeState}(S) \):
31: Send (“\text{writeState}”, \( S \)) to all
32: Wait until receiving
33: \( \geq n – f \) (“\text{writeStateAck}”) msgs

Procedure \( \text{writeView}(v, r) \):
34: Send (“\text{writeView}”, \( v, r \)) to all
35: Wait until receiving
36: \( \geq n – f \) (“\text{viewAck}”) msgs

/* Event handlers: executing in background */
/* All event handlers executed atomically */
Upon receiving (“\text{value}”, \( u, ts \)) from \( q \):
34: \( \text{Snap}[q] \leftarrow \max(\text{Snap}[q], (u, ts)) \)

Upon receiving (“\text{writeTag}”, tag) from \( q \):
35: \( \text{maxTag} \leftarrow \max(\text{maxTag}, \text{tag}) \)
36: Send (“\text{writeTagAck}”, \( q \)) to \( q \)

Upon receiving (“\text{readTag}”) from \( q \):
37: Send (“\text{readTagAck}”, \text{maxTag}) to \( q \)

Upon receiving (“\text{readState}”) from \( q \):
38: Send (“\text{readStateAck}, Snap)) to \( q \)

Upon receiving (“\text{writeState}”, \( S \)) from \( q \):
39: \( \text{Snap} \leftarrow \text{Snap} \uplus S \)
40: Send (“\text{writeStateAck}”) to \( q \)

Upon receiving (“\text{writeView}, U, r) from \( q \):
41: \( V[r] \leftarrow V[r] \uplus U \)
direct view of some other node \(j\). We call an invocation of a lattice agreement with tag \(r\) as a lattice operation with tag \(r\).

**Lemma 34.** Consider a lattice operation \(op\) by node \(p\) with tag \(r\), suppose \(op\) returns \(V_{op}\) at time \(t\). Then, for each \(j\), there exists a set of nodes \(Q_j\) with size at least \(n - f\) such that \(V_{op}[j] \leq Snap_i[j]\) for each \(i \in Q_j\) at time \(t\).

**Proof.** Let \(P\) denote the set of nodes which invokes the lattice operation \(op\) before or at time \(t\). By Upward-Validity, we have that \(V_{op}[j] = input_p[j]\) for some node \(p \in P\). By line 5, there exists a set of nodes \(Q_j\) with size at least \(n - f\) such that \(input_p \leq Snap_q\) for each \(i \in Q_j\). Therefore, \(V_{op}[j] = input_p[j] \leq Snap_i[j]\) for any \(i \in Q_j\). \(\diamondsuit\)

**Lemma 35.** If two operations return \(view_i\) and \(view_j\), then \(view_i\) and \(view_j\) are comparable.

**Proof.** We only need to show that \(view_i\) and \(view_j\) are comparable if they are direct views. Let \(op_i\) and \(op_j\) denote the two operations that return \(view_i\) and \(view_j\), respectively. We have the following cases.

Case 1. \(view_i\) and \(view_j\) are obtained from the same lattice operation. By comparability of lattice agreement, \(view_i\) and \(view_j\) are comparable.

Case 2. \(view_i\) are obtained from lattice operation with tag \(r_1\) and \(view_j\) is obtained from lattice operation with round \(r_2\). Assume that \(r_2 > r_1\), w.l.o.g. Assume that \(op_i\) obtains \(view_i\) in **first phase**. The case that \(op_i\) returns \(view_i\) in **second phase** is symmetric. Then, in line 5, \(op_i\) finds no nodes with tag number greater than \(r_1\). Therefore, \(op_i\) obtains \(view_i\) before \(op_j\) complete line 5. Then, when \(op_j\) starts to read states of at least \(n - f\) nodes at line 5, it must be able to read \(view_i[j]\) for each \(j\). By Lemma 34 and quorum intersection. Thus, \(view_i\) is less than or equal to the input of \(op_j\) for the lattice operation with tag \(r_j\). By Downward-Validity of lattice agreement, we have \(view_i \leq view_j\). \(\diamondsuit\)

Now, we associate an view with the beginning of an operation. For operation \(op\), the view associated with the beginning of \(op\) is \(view\) such that \(view'[j]\) is the largest value written by \(p_j\) which is contained in the local state of at least \(n - f\) nodes.

**Lemma 36.** Assume operation \(op\) returns \(view\) and let \(view'\) denote the view associated with the beginning of \(op\), then \(view' \leq view\).

**Proof.** Consider the following two cases.

Case 1. \(op\) return \(view\) directly from lattice operation with tag \(r\). W.l.o.g, assume that \(op\) returns \(view\) in the **first phase**. The case where \(op\) returns \(view\) in the **second phase** is symmetric. Let \(input\) be the input of \(op\) for the lattice operation with tag \(r\) at line 7. By definition of \(view'\), when \(op\) executes line 5, it must be able to read all values in \(view'\). Since nodes write increasing values to the snapshot object, \(view' \leq input\). By Downward-Validity of lattice agreement, we have \(input \leq view\). Thus, \(view' \leq view\).

Case 2. \(op\) returns \(view\) indirectly. Then, \(op\) must continue to phase 2. Let \(r_1\) and \(r_2\) denote the tag number \(op\) obtains at line 5 of phase 1 and phase 2, respectively. We have \(r_1 < r_2\). Consider the second phase, the condition at line 5 is satisfied and \(op\) borrows a \(view\) of some other nodes for tag \(r_2\). \(view\) must be a direct view of some operation \(op'\) for tag \(r_2\). W.l.o.g, assume that \(op'\) returns \(view\) in the **first phase**. The case where \(op'\) returns \(view\) in the **second phase** is symmetric. Since \(r_1 < r_2\), \(op'\) must start line 5 after \(op\) starts. Otherwise, \(op\) would obtain tag number \(r_2\) instead of \(r_1\) for its first phase. By the definition of the view associated with an operation, \(op'\) must be able to read all values in \(view'\) at line 5. Downward-Validity of lattice agreement implies that \(view' \leq view\). \(\diamondsuit\)
Lemma 37. Consider two operations $op_i$ and $op_j$ that return $view_i$ and $view_j$, respectively. If $op_i \rightarrow op_j$, then $view_i \leq view_j$.

Proof. Let $view$ be the view associated with the beginning of $op_j$. By Lemma 36, $view \leq view_j$. Since $op_i$ obtains $view_i$ before the beginning of $op_j$, Lemma 34 implies that $view_i \leq view$. Thus, $view_i \leq view_j$.

Lemma 38. Let $up$ be an Update operation by node $p$ that writes value $v$, and returns $view_p$. Then, $(v, ts_v) \leq view_p[p]$.

Proof. Let $view$ be the view associated with the beginning of the Scan operation embedded in $up$. Since $up$ writes $(v, ts_v)$ to at least $n - f$ nodes before its embedded Scan operation, then $(v, ts) \leq view[p]$. By Lemma 36 we have $view \leq view_p$. Thus, $(v, ts_v) \leq view_p[p]$.

The linearization sequence of the scan and update operations is constructed in the same way as the one given in [9]. First, we construct a sequence $\sigma'$ which only includes all scan operations. The sequence also includes the scan operations embedded in the update operations. The scan operations are ordered in $\sigma'$ according to the order of the views returned by them. Specifically, for any two scan operations $sc_i$ and $sc_j$ that return $view_i$ and $view_j$, respectively, if $view_i < view_j$, then $sc_i$ appears before $sc_j$ in $\sigma'$. If $view_i = view_j$ and $sc_i \rightarrow sc_j$, then $sc_i$ appears before $sc_j$ in $\sigma'$. Otherwise, $sc_i$ and $sc_j$ are ordered arbitrarily.

Now we create a linearization sequence $\sigma$ from $\sigma'$ by inserting all update operations into $\sigma'$. Consider an operation $op_i$ that writes value $v$. We insert $op$ after all scan operations that return a value strictly smaller than $v$ and before all scan operations that return a value greater than or equal to $v$. That is, $op$ is inserted just before the first scan operation that returns a view which contains $v$. For any two operations $op_1$ and $op_2$ that fit between the same pair of scan operations. If $op_1 \rightarrow op_2$, then we put $op_1$ before $op_2$ in sequence $\sigma$. Otherwise, $op_1$ and $op_2$ are ordered arbitrarily.

As long as we have the above lemmas, the proof which shows that $\sigma$ is a linearization is the same as the proof in [9].

Theorem 39. There exists an atomic snapshot object implementation in asynchronous crash-prone message passing systems, which requires $O(\log n)$ message delays per update or scan operation, where $f < \frac{n}{2}$ is the maximum number of crash failures in the system.

Proof. The paper [33] presents an $O(\log f)$ rounds algorithm for the lattice agreement problem in asynchronous crash-prone message passing systems. Directly plugging in their algorithm into our transformation result in $O(n)$ rounds complexity for update and scan operations, since their algorithm requires all nodes to start around the same time. Their algorithm can be simply modified to run in $O(\log n)$ rounds even if nodes start at different times. ▷

D Comparison between TS-ASO and AC-ASO

Let TS-ASO denote the general transformation in Appendix C. We list the two primary differences between TS-ASO and AC-ASO here.

1) In both TS-ASO and AC-ASO, to write a new value (in an Update operation), a node needs to first read the largest tag from a quorum of nodes. Let $r$ denote the tag obtained. In TS-ASO, a node directly participates in the lattice operation with tag $r$ (This design ensures that there is good lattice operation for each tag). This design is also the main reason why TS-ASO cannot preserve the round complexity of our ELA algorithm, since the round
complexity of our ELA algorithm depends on the assumption that each node starts the algorithm around the same time and nodes can join the lattice operation with the same tag at quite different times. To solve this problem, our idea is to let a node participate a lattice operation with a strictly greater tag than the tag it reads from a quorum. That is, if a node observes a tag \( r \), then it participates the lattice operation with tag \( r + 1 \). This ensures that all nodes participate the lattice operation with same tag around the same time (at most constant round apart). If we only have the above modification, we cannot guarantee that there exists a good lattice operation for each tag, then when some lattice operation needs to borrow a view from a good lattice operation, the existence of such a good lattice operation is not guaranteed. Thus, to tackle this problem, our idea is to use a dummy lattice operation whose only purpose is to ensure the existence of a good lattice operation for each tag. Specifically, we let each \texttt{Update} operation executes an initial lattice operation with tag \( r \) but without introducing a new value with tag \( r \). This initial lattice operation guarantees the existence of a good lattice operation for each tag but does not prevent the termination of existing lattice operations with the same tag due to the reason below. Our lattice operation has the following properties: the termination of a lattice operation with tag \( T \) depends on only the values with tags at most \( T \). Since the initial lattice operation with tag \( r \) does not introduce a new value with tag \( r \), it does not prevent progress of other existing lattice operations with tag at most \( r \). This is also the reason why our design is not a general transformation that preserves the round complexity of any lattice agreement algorithm.

2) In TS-ASO, before participating a lattice operation, a node needs to collect the states of at least a quorum of nodes and use their join as input for the lattice operation. In TS-ASO, such a read step is important in ensuring the correctness. (First, it ensures that a lattice operation with a bigger tag must be able to read the view obtained by a lattice operation with a smaller tag. Second, it ensures that a later \texttt{Update} or \texttt{Scan} operation must be able to read the view obtained by previous (completed) \texttt{Update} or \texttt{Scan} operation.) Each state read from other nodes is a vector of \( n \) values. Thus, each message in the lattice operation has \( O(n) \) overhead in size (it contains at least \( n \) values). In our design, we would like to remove such overhead in message size. Note that our ELA algorithm has constant message size overhead. In AC-ASO, when a node participates a lattice operation, it does not collect the states of at least a quorum of nodes and use that as input for lattice operation. Without the reading step, two lattice operations are not sufficient to guarantee correctness. To ensure correctness, we will show that three lattice operations are sufficient.

### E Proofs for the AC-ASO Algorithm

#### E.1 Message Handlers of AC-ASO

- Upon receiving a “\texttt{writeTag}” message: node \( i \) updates its \texttt{maxTag} to be the maximum of its current \texttt{maxTag} and the received tag. Then, it responds a \texttt{writeAck}.
- Upon receiving a “\texttt{echoTag}” message: node \( i \) updates its \texttt{maxTag} to be the maximum of its current \texttt{maxTag} and the received tag.
- Upon receiving a “\texttt{value}” message from node \( j \): node \( i \) adds the value into \( V_i[i] \) and \( V_i[j] \). It then forwards this value to all other nodes if it has never done so before. It is important to note that a node does not update its \texttt{maxTag} variable when it receives a value with a larger tag from a “\texttt{value}” message. The \texttt{maxTag} variable is only updated when a node receives a “\texttt{writeTag}” message or “\texttt{echoTag}” message.
- Upon receiving a “\texttt{readTag}” message: node \( i \) responds a \texttt{readAck} message along with the largest tag it has ever seen via \texttt{writeTag} messages.
Upon receiving a “goodLA” message with tag \( r \) from node \( j \): node \( i \) borrows the view from node \( j \) by recording \( V_i[j] \leq r \). Our design ensures that the borrowed view is identical to the view from \( j \)'s good lattice operation. By assumption the communication is FIFO, and node \( j \) sends the message (“goodLA”, \( r \)) right after its view satisfies the equivalence predicate at line \( 15 \) thus, \( V_i[j] \leq r \) must be the same as the view of the particular good lattice operation at node \( j \). We formally prove this claim in Lemma 19.

E.2 Proof of Lemma 18

Lemma 18. [Termination] If each lattice operation eventually terminates, then Update and Scan operations in Algorithm 3 eventually terminate.

Proof. We show that each \texttt{LatticeRenewal()} invocation terminates. Let \( Op \) denote the \texttt{LatticeRenewal()} procedure at node \( i \). We only need to show that the condition at line \( 29 \) is eventually satisfied if \( Op \) has not returned earlier, since the condition is the only blocking code inside \( Op \). Consider the phase 3 lattice operation \( L_3 \). Since \( Op \) continues to line \( 29 \) with phase = 3, \( L_3 \) is not a good lattice operation. This means that the \texttt{for} loop of \( Op \) breaks at line \( 27 \) hence, \( r_i \) at line \( 29 \) is equal to the tag used by \( L_3 \). Since \( L_3 \) is not a good lattice operation, and it returns \texttt{false}, we have \( \text{maxTag}_i > r_i \) at line \( 17 \). In other words, at this point of time, the largest tag in the system is at least \( \text{maxTag}_i \). Lemma 16 implies that a good lattice operation with tag \( r_i \) must be completed before \( L_3 \) executes line \( 17 \). By assumption of the reliable communication channel, node \( i \) is able to receive a (“completed”, \( r_i \)) message from the good lattice operation. After receiving the message, condition at line \( 29 \) is satisfied, and hence, \texttt{LatticeRenewal()} terminates.

E.3 Proof of Lemma 19

Lemma 19. The view of each Update or Scan operation is the same as the view of some good lattice operation.

Proof. If the operation is direct, then by definition, its view is the view of its final lattice operation in \texttt{LatticeRenewal()} procedure. Otherwise, the operation borrows the view from some other node \( j \) at line \( 29 \) which must be the view of \( j \)'s good lattice operation. This is because only a good lattice operation sends a “goodLA” message.

E.4 Proof of Lemma 20

Lemma 20. For any two nodes \( i \) and \( j \) and tag \( T \), fix any time \( t \) and \( t' \), the set \( V_i[s] \leq T \) at time \( t \) and the set \( V_j[s] \leq T \) at time \( t' \) are comparable for each node \( s \).

Proof. The value of set \( V_i[s] \leq T \) is modified only when \( i \) receives a new value with tag \( \leq T \) from node \( s \). Since the communication is FIFO, the set \( V_i[s] \leq T \) at time \( t \) must be the same as the set \( V_j[s] \leq T \) at some time \( t_i \) and the set \( V_j[s] \leq T \) at time \( t' \) must be the same as the set \( V_j[s] \leq T \) at some time \( t_j \). The set \( V_i[s] \leq T \) is non-decreasing. Thus, \( V_i[s] \leq T \) at time \( t \) must be comparable with \( V_j[s] \leq T \) at time \( t' \).

E.5 Proof of Lemma 21

Lemma 21. The views returned by all good lattice operations are comparable.
Proof. Consider two good lattice operations $Op_i$ with tag $T_i$ by node $i$ and $Op_j$ with tag $T_j$ by node $j$. Let $H_i$ and $H_j$ denote node $i$ and $j$’s view, respectively, after they complete line 15. Recall that by Definition 14, $H_i = V_i^* = V_i[i]^{≤T_i}$ and $H_j = V_j^* = V_j[j]^{≤T_j}$ right after the equivalence quorum predicate is satisfied.

To prove the lemma, we need to show that either $H_i \subseteq H_j$ or $H_j \subseteq H_i$. Assume without loss of generality $T_i \leq T_j$. Then consider two following cases.

- **Case 1:** $T_i = T_j = T$. Intuitively, both nodes participate in the same instance of lattice agreement, and thus, they will obtain comparable outputs (views).
  Formally, let $W_i$ and $W_j$ denote the equivalence quorum of lattice operation $Op_i$ and $Op_j$, respectively. Thus, there exists a nonfaulty node $s \in W_i \cap W_j$ such that $H_i = V_i[i]^{≤T} = V_i[s]^{≤T}$ and $H_j = V_j[j]^{≤T} = V_j[s]^{≤T}$. Lemma 20 implies that $V_i[s]^{≤T}$ and $V_j[s]^{≤T}$ are comparable. Thus, $H_i$ must be comparable with $H_j$, since to satisfy the equivalence quorum predicate, $H_i = V_i[i]^{≤T} = V_i[s]^{≤T}$ and $H_j = V_j[j]^{≤T} = V_j[s]^{≤T}$.

- **Case 2:** $T_i < T_j$. In this case, we show that $H_i \subseteq H_j$. Roughly speaking, we want to show that lattice operation with a larger tag start with a view that is at least as large as the view of any lattice operation with a smaller tag. We rely on Property 2 of LatticeRenewal() and the way we update maxTag’s to prove the claim.

We make the following observations:

- **Obs. 1:** the tag $T_j$ is not known by node $i$ when $Op_i$ completes line 15. This is because (i) $Op_i$ is a good lattice operation, and the condition $maxTag_i \leq r_i$ at line 14 of $Op_i$ must return true; and (ii) line 15 and 17 are executed atomically, and hence $V_j^* = V_j[i]^{≤T}$ does not change during this block of code.

- **Obs. 2:** There exists a node $s$ such that $H_i \subseteq V_i[s]^{≤T_i}$ and $V_i[s]^{≤T_i} \subseteq V_j[j]^{≤T_j}$. Let $W_i$ denote the equivalence quorum of $Op_i$. Let $Q_i$ denote the set of at least $n - f$ nodes that sent the “writeAck” messages in responding to the “writeTag” message of $Op_j$. Since both set are of size at least $n - f$, there exists a node $s \in W_i \cap Q_i$ such that (i) $H_i = V_i^* = V_i[s]^{≤T_i}$ (due to the equivalence predicate); and (ii) node $s \in Q_i$. By Obs. 1, and the assumption of FIFO communication, node $s$ must have received all values in $V_i[s]^{≤T_i}$ before receiving the “writeTag” message of $Op_j$. Otherwise, $Op_j$ would observe tag $T_j$ at line 17. Thus, node $s$ must send out all values in $V_i[s]^{≤T_i}$ to all before sending writeAck for the “writeTag” message of $op_j$. Then, when node $j$ receives the writeAck from node $s$, it must also receive all values in $V_i[s]^{≤T_i}$, i.e., the view of node $j$ contains all values in $V_i[s]^{≤T_i}$ after line 14 of $op_j$ completes. Since $H_j$ is the set of values with tag at most $T_j > T_i$ in the history of node $j$ when $op_j$ completes line 15, $V_i[s]^{≤T_i} \subseteq H_j$. Therefore, $H_i \subseteq H_j$.

| E.6 Proof of Theorem 28 |

We first show the following two lemmas.

- **Lemma 40.** The sequence $σ$ preserves the semantics of atomic snapshot object, i.e., for any SCAN operation which returns $Snap$, for any $i$, $Snap[i]$ must be the value written by the latest Update operation of node $i$ that appears before the SCAN operation in $σ$.

**Proof.** Assume $Snap[i] = v$, and let $op$ be the Update($v$) operation by node $i$. By the construction of $σ$, $op$ appears before $sc$ in $σ$. First, we have that any Update operation by node $i$ that writes a value strictly greater than $v$ is ordered after $sc$ in $σ$. Furthermore, any
UPDATE operation by node $i$ that writes a value strictly smaller than $v$ is ordered before $op$ in $\sigma$. Therefore, $op$ is the last UPDATE operation by node $i$ that is ordered before SCAN.

The following lemma is implied by Lemma 24.

► **Lemma 41.** Let $Op$ be an UPDATE operation by node $i$ that writes value $v$ with timestamp $ts$ and has view $H_i$. Then, $\langle v, ts \rangle \in H_i$.

► **Lemma 42.** The sequence $\sigma$ respects the real-time order of operations, i.e., for any two operations $Op_i$ and $Op_j$, if $Op_i \rightarrow Op_j$, then $Op_i$ appears before $Op_j$ in $\sigma$.

**Proof.** Let $H_i$ and $H_j$ be the views of $op_i$ and $op_j$, respectively. Let node $i$ and node $j$ denote the nodes where $op_i$ and $op_j$ take place, respectively. Note that $i$ may be equal to $j$.

By Lemma 27, we have $H_i \subseteq H_j$. We consider the following cases.

- $op_i = \text{UPDATE}(v)$ and $op_j = \text{UPDATE}(u)$: If $op_i$ and $op_j$ are placed between the same pair of SCAN operations, then they are ordered according to $\rightarrow$. Hence, $op_i$ appears before $op_j$ in $\sigma$. Otherwise, there exists a SCAN operation $sc$ with view $H_{sc}$ between $op_i$ and $op_j$ in $\sigma$. Suppose, by way of contradiction, that $op_j$ is ordered before $op_i$. In other words, $sc$ appears after $op_j$ and appears before $op_i$ in $\sigma$. Then $\langle v, ts_v \rangle \not\in H_{sc}$ and $\langle u, ts_u \rangle \in H_{sc}$. Lemma 41 implies that $\langle v, ts_v \rangle \in H_i$. On the other hand, since $op_i \rightarrow op_j$, we have $\langle u, ts_u \rangle \not\in H_i$. Thus, $H_{sc} \cap H_i = \emptyset$. Moreover, $H_{sc}$ and $H_i$ are incomparable, a contradiction to Lemma 22.

- $op_i = \text{UPDATE}(v)$ and $op_j = \text{SCAN}$: If $H_i \neq H_j$, which means $H_i \subset H_j$. In this case, $op_i$ is ordered before $op_j$ by construction. Otherwise, since two SCAN operations that return the same view are ordered according to $\rightarrow$, $op_i$ is ordered before $op_j$ in $\sigma$.

- $op_i = \text{SCAN}$ and $op_j = \text{UPDATE}(v)$: Clearly, $\langle v, ts_v \rangle \not\in H_i$. Since $op_j$ is ordered after all SCAN operations whose view does not contain $\langle v, ts_v \rangle$, it follows that $op_i$ appears before $op_j$ in $\sigma$.

► **Theorem 28.** AC-ASO (Algorithm 2) implements an atomic snapshot object.

**Proof.** Immediately follows from Lemma 40 and 42.

E.7 Proof of Lemma 29

► **Lemma 29.** Suppose there exists a lattice operation that starts at time $t$ with tag $T$, then any UPDATE operation starting after time $t + D$ must assign a tag $> T$ for its value. Thus, all values with tags at most $T$ must have been sent out by time $t + D$.

**Proof.** Let operation $op$ denote such a lattice operation. Since $op$ sends its tag $T$ to all in the writeTag function at line 28, by time $t + D$, each correct node must have received tag $T$. Thus, any UPDATE operation that starts after time $t + D$ must obtain a tag greater than $T$ for its value. Thus, all values with tags at most $T$ must come from UPDATE operations that start before time $t + D$. Since local computation does not take time, all such values must be sent out at line 6 before time $t + D$.
E.8 Proof of Lemma 30

Lemma 30. Let $Op$ denote $\text{UPDATE}(v)$ operation. If $Op$ completes before time $t$, then for each nonfaulty node $i$, $(v, ts_i) \in V_i[j]$ for each nonfaulty node $j$ by the end of time $t + 2D$.

Proof. Since $op$ completes before time $t$, $(v, ts_i)$ must be sent to node $q$ before time $t$ and must be received by node $q$ by time $t + D$. Thus, node $p$ must receive $(v, ts_i)$ from node $q$ by the end of $t + 2D$. ▶