Repeating optimizations required to compute parameter estimates and bootstrap standard errors of complex models can be computationally burdensome. In Forneron and Ng (2020), we design a resampled Newton-Raphson algorithm (rNR) that provides consistent estimates and valid standard errors in one run of the optimizer. The key insight is that the algorithm serves as a resampling device to produce a Markov chain of iterates with desirable properties. In this paper, we illustrate that rNR can speed up BLP estimation from almost five hours using standard (n out of n) bootstrap to just over an hour and can be further reduced to fifteen minutes using a resampled quasi-Newton (rQN) algorithm that does not directly compute the Hessian. A Monte-Carlo exercise using Probit IV regressions shows that rNR and rQN provide accurate estimates and coverage. The appeal of the proposed approach goes beyond faster computation. A re-sampling based indirect inference estimator not only produces standard errors easily, but is also more efficient than one obtained by classical optimization. This is illustrated by a dynamic panel model example.

I. The Setup

Many economic applications entail minimizing a sample objective function \( Q_n(\theta) \) with respect to a vector of parameters \( \theta \) to obtain an estimate \( \hat{\theta}_n = \arg\min_{\theta} Q_n(\theta) \). Under regularity conditions, \( \hat{\theta}_n \sim \sqrt{n} \) consistent for the true value \( \theta^* \) and \( \mathbb{V}^{1/2}\mathbb{V}(\hat{\theta}_n - \theta^*) \sim d N(0, I_d) \).

The sandwich variance \( \mathbb{V} \) required for inference depends on both the gradient and the Hessian which are often analytically intractable. Bootstrap inference approximates the asymptotic distribution but requires repeated optimization each time a batch of data of size \( n \) is resampled. Alternatives are available to speed up computation but they still necessitate a preliminary estimate \( \hat{\theta}_n \).

The Newton-Raphson algorithm computes \( \hat{\theta}_n \) by iterating until convergence:

\[
\theta_{k+1} = \theta_k - \gamma_k [H_n(\theta_k)]^{-1} G_n(\theta_k),
\]

where \( \gamma_k \) is a learning rate, \( G_n(\theta_k) \) is the gradient, and the conditioning matrix is set to the inverse of the Hessian \( H_n(\theta_k) \) so that \( [H_n(\theta_k)]^{-1} G_n(\theta_k) \) determines the direction of the update. In Forneron and Ng (2020), we propose a novel resampled Newton-Raphson algorithm (rNR) that produces an estimate of \( \theta \) and its standard errors in one run of the optimizer.

**Algorithm rNR**

1. **Inputs:** (a) initial guess \( \theta_0 \); (b) bootstrap sample size \( B \) and burn-in period \( \text{burn} \); (c) batch size \( m \leq n \), and (d) fixed learning rate \( \gamma \in (0, 1] \).

2. **Resample:** For \( b = 1, \ldots, \text{burn} + B \)
   a. Resample a \((b+1)\)-th batch of data of size \( m \),
   b. Update \( H_b = H_{m}^{b+1}(\theta_b) \) and \( G_b = G_{m}^{b+1}(\theta_b) \),
   c. Update \( \theta_{b+1} = \theta_b - \gamma H_b^{-1} G_b \).

3. **Outputs:** Discard the first \( \text{burn} \) draws. Let \( \phi(\gamma) = \frac{\gamma^2}{1 - (1 - \gamma)^2} \) and output
   a. \( \bar{G}_{rNR} = \frac{1}{B} \sum_{b=1}^{B} \theta_b \),
   b. \( \bar{V}_{rNR} = \frac{m}{\phi(\gamma)} \text{var}(\theta_b) \) where \( \text{var}(\theta_b) = \frac{1}{B} \sum_{b=1}^{B} (\theta_b - \bar{G}_{rNR})(\theta_b - \bar{G}_{rNR})' \).
The main idea of rNR is to combine estimation with inference by exploiting the randomness due to re-sampling within the optimizer. Each \( b+1 \)-th sample consists of \( m \) observations drawn randomly from the original data. Then \( Q_m^{(b+1)}(\theta) \) is evaluated, its gradient \( G_m^{(b+1)}(\theta_b) \) and Hessian \( H_m^{(b+1)}(\theta_b) \) are used to update \( \theta_n \) to \( \theta_{b+1} \). The estimator is computed by taking the mean over draws after discarding the first \( \text{BURN} \) iterates to reduce the impact of the initial guess \( \theta_0 \). Standard errors are obtained from the draws after a sample size and scale adjustment of \( \sqrt{\frac{m}{n(\ell)}} \). Like MCMC, inference is sampling-based but the approach is fully frequentist. Unlike other bootstrap shortcuts, our approach does not require a preliminary estimate \( \hat{\theta}_n \).

Evaluating the direction of change using small batches of data is in the spirit of stochastic optimization, but there are two important differences. First, while the learning rate \( \gamma_b \) in stochastic optimization declines with each \( b \), our \( \gamma \in (0,1] \) is constant. This allows us to analytically establish that the draws \( \{\theta_b\}_{b=1}^B \) form a Markov chain with stationary ergodic properties. Second, whereas stochastic optimization uses \( m \) fixed (as small as one) with efficient computation as a goal, we also have inference in mind which necessitate \( m \) to increase faster than \( \sqrt{n} \). Statistical and computational efficiency are conflicting goals in this context.

Under certain conditions in Forneron and Ng (2020), the rNR draws have two properties:

\[
\sqrt{n} (\bar{\theta}_{\text{rNR}} - \bar{\theta}_n) = o_p(1), \quad \text{(Estimation)}
\]
\[
V_{\text{rNR}}^{-1/2} \sqrt{m} (\theta_b - \bar{\theta}_n) \xrightarrow{d} N(0, I_d), \quad \text{(Inference)}
\]

where \( V_{\text{rNR}} \) is defined in Algorithm above. The first (consistency) result states that the mean estimator \( \bar{\theta}_{\text{rNR}} \) is first order equivalent to the classical estimator \( \bar{\theta}_n \). The second (inference) result states that the distribution of the draws is first-order equivalent to that of \( \theta_{b,m} \), the bootstrap distribution. A sketch of the argument is as follows. For the same resampling scheme, it is known that the standard bootstrap yields valid inference. We show that when \( m \) and \( \gamma \in (0,1] \) are appropriately chosen, the distribution of the rNR draws is close to that of the standard bootstrap up to scale, and by implication, close to the limiting distribution of \( \theta_n \). But unlike the standard bootstrap which needs repeated optimizations, rNR produces standard errors in the same optimization that produces estimates \( \bar{\theta}_{\text{rNR}} \). This means that upon completion of that single run, a \((1 - \alpha/2)\% \) confidence interval for the \( j \)-th coefficient can be immediately constructed as:

\[
(\bar{\theta}_{\text{rNR},j} + q_{\alpha/2}, \bar{\theta}_{\text{rNR},j} + q_{1 - \alpha/2})
\]

where \( q_{\alpha/2} \) is the \( \alpha/2 \) quantile of \( \sqrt{\frac{m}{n(\ell)}} (\theta_{b,j} - \bar{\theta}_{\text{rNR},j}) \). Wald statistics can also be computed using \( V_{\text{rNR}} \) as a plug in of \( V^1 \).

The two results above also hold for a faster resampled quasi-Newton algorithm, called rQN, which approximates the Hessian by a least-squares interpolation scheme, it is described in Forneron and Ng (2020). This scheme ensures the conditioning matrix is both symmetric and positive definite which is required for inference. Though the consistency result also holds for many conditioning matrices, the inferential result only holds for conditioning matrices that approximate the inverse Hessian sufficiently well because the sandwich variance structure cannot be replicated otherwise. Thus, resampled gradient descent which uses an identity matrix for conditioning will give incorrect standard errors but valid estimates.

Algorithms rNR and rQN are especially useful when the model is costly to optimize. But they also have statistical appeals:- the draws are immediately available for post estimation diagnostics, and in the case of simulation estimation, \( \bar{\theta}_{\text{rNR}} \) can even be more efficient than an estimate obtained from classical optimization. We now illustrate some of these properties.

### II. Example 1: Demand for Cereal

We consider the BLP model of Berry, Levinsohn and Pakes (1995) for the cereal data generated in Nevo (2000). The data consists of market shares \( s_{gj} \) in market
Table 1—: Demand for Cereal: Estimates and Standard Errors (Random Coefficients)

|       | Estimates | Standard Errors |
|-------|-----------|-----------------|
|       | $\hat{\theta}_n$ | rNR | rQN | BOO | DMK | rNR | rQN |
| stdy  |           | 0.284 | 0.263 | 0.273 | 0.129 | 0.127 | 0.123 | 0.120 |
|       | price     | 2.032 | 2.188 | 1.983 | 1.198 | 1.026 | 0.975 | 0.950 |
|       | sugar     | -0.008 | -0.006 | 0.006 | 0.017 | 0.012 | 0.012 | 0.012 |
|       | mushy     | -0.077 | -0.055 | -0.044 | 0.177 | 0.168 | 0.166 | 0.167 |
| income| const.    | 3.581 | 3.464 | 3.646 | 0.666 | 0.738 | 0.714 | 0.662 |
|       | price     | 0.467 | 1.335 | 0.111 | 3.829 | 4.275 | 4.040 | 3.569 |
|       | sugar     | -0.172 | -0.171 | -0.174 | 0.028 | 0.028 | 0.027 | 0.031 |
|       | mushy     | 0.690 | 0.647 | 0.694 | 0.345 | 0.346 | 0.339 | 0.333 |
|       | time      | 4h36m | 1h1m | 58m | 15m |

$g \in \{1, \ldots, 94\}$ for product $j \in \{1, \ldots, 24\}$. Parameters on terms that enter linearly are projected out by 2SLS. We then drop interaction terms that seem difficult to identify. This leaves us with $d = 8$ parameters that enter the moment conditions $g$ non-linearly. Evaluation of the objective and its gradient is costly because fixed-point iterations are needed to invert market shares. We perform $m$ out of $n$ resampling at the market level. This level of clustering controls for possible correlations in the unobservables at the market level. That is, for each $b = 1, \ldots, B$ we draw markets $g_1^{(b)}, \ldots, g_{94}^{(b)}$ from $\{1, \ldots, 94\}$ with replacement, taking the associated shares and characteristics $\{s_{g_1^{(b)}}, X_{g_1^{(b)}}\}_{b=1}^{24}$ as observations within each market. We set $\gamma = 0.2$ and $\text{BURN} = 10$ draws. Since the number of clusters is relatively small, we set $m = n = 94$.

Table 1 indicates that the rNR estimates are similar to $\hat{\theta}_n$ obtained from classical optimization. The standard errors are similar across methods but the rNR ones are nearly 5x faster to compute than the bootstrap and are comparable to Davidson and MacKinnon (1999), denoted as DMK, even excluding the time used to get the preliminary estimate. The rQN further reduces computation time over rNR by a factor of 4.

The estimates based on classical optimization reported above use only 20 integration draws as in Nevo (2000). More accurate estimates will require more draws, so the gains in using rNR and rQN are conservative. Besides inference on the parameters, the rNR draws can also be useful in post-estimation analysis. For instance, in more involved counterfactuals such as merger analyses, the delta-method can be challenging to apply while re-evaluating counterfactuals on bootstrap draws is straightforward.

III. Example 2: Probit IV Regression

The second example uses simulations to evaluate the finite sample properties of rNR and rQN. We consider a probit instrumental variable regression model specified as

$$y_{1i} = 1\{\alpha y_{2i} + \beta_0 + \beta_1 x_i + \rho v_i + u_i\},$$

$$y_{2i} = \xi_0 + \xi_1 x_i + \pi z_i + v_i,$$

where $x_i, z_i$ are independent and exponentially distributed with rate 1; $v_i, u_i$ are independent standard normal; $\theta^i = (\xi_0, \xi_1, \pi, \alpha, \beta_0, \beta_1, \rho) = (0, 1, 1, 1, 0, 1, 1)$. These seven coefficients are jointly estimated in a just-identified GMM system using the sample vector of moments:

$$\bar{g}_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} \left( r_{1i}(\theta) \otimes (1, x_i, z_i, r_{2i})' \right),$$

where $\otimes$ is the kronecker product, $r_{2i}(\theta) = y_{2i} - (\xi_0 + \xi_1 x_i + \pi z_i)$ and $r_{1i}(\theta) = y_{1i} - \Phi(\alpha y_{2i} + \beta_0 + \beta_1 x_i + \rho r_{2i})$. We set $n = 500$ in each of the 1000 Monte-Carlo replications. The estimates $\theta_n$ are computed using the BFGS routine in R.
Table 2—: Probit IV: finite sample properties in estimation and inference

| m   | Average Estimate rNR | Standard Deviation rNR | Rejection Rates | γ = 0.2 | γ = 0.1 |
|-----|----------------------|------------------------|-----------------|---------|---------|
| 500 | 1.033                | 1.037                  | 0.211           | 0.212   | 0.210   |
| 100 | 1.022                | 1.042                  | 0.218           | 0.219   | 0.219   |
| 50  | 1.003                | 1.072                  | 0.217           | 0.459   | 0.210   |

For rNR, rQN we use $B = 2000$ and consider $\gamma \in \{0.2, 0.1\}$, $m \in \{50, 100, 200\}$. For BFGS, rNR and rQN $\theta_0 = (0, \ldots, 0)$. For the bootstrap, optimization is initialized at $\hat{\theta}_n$ and we only use $B = 500$ as is common practice, but even this is slower than rNR and rQN with $B = 2000$.

Table 2 compares the properties of the estimates and quantile-based confidence intervals for $\alpha$, the coefficient on the endogenous regressor $y_{2i}$, which is a parameter of interest. The average estimates and standard errors with classical estimation $\hat{\theta}_n$ are 1.034 and 0.210 which are generally comparable to rNR and rQN reported in the table. The exception is the $m = 50$, $\gamma = 0.2$ case which can be attributed to three replications for which BURN = 50 appears to be too small. For coverage, the usual $\hat{\alpha}_n \pm 1.96 \cdot \text{se}(\hat{\alpha}_n)$ confidence interval is very close to the 95% level with a rejection rate of 0.055. Using $m = 500$ and $\gamma = 0.2$, the coverage of rNR, rQN is comparable to that of the bootstrap. For $m < n$ the accuracy of the bootstrap declines while rNR and rQN are less affected. For $\gamma = 0.1$, coverage is closer to the nominal 95% confidence level for the entire range of $m$ and $\gamma$ values.

IV. Example 3: Simulation-based Estimation

The third example highlights the statistical gains of rNR/rQN for simulation-based estimation. Consider the linear dynamic panel model:

$$ y_{it} = \rho y_{i,t-1} + \beta x_{it} + \alpha_i + \sigma e_{it}, $$

where $t \in \{1, \ldots, T\}, i \in \{1, \ldots, n\}, \theta = (\rho, \beta, \sigma)$. The least-squares dummy variable (LSDV) estimator is inconsistent as $n \to \infty$ with $T$ fixed. Gourieroux, Phillips and Yu (2010) consider simulation estimation of $\theta$ using $\hat{\psi}_n = \hat{\theta}_n^{\text{LSDV}}$ as auxiliary statistics. Given draws $e_{it}^* \sim \text{N}(0, 1)$ and a value of $\theta$, simulate $S$ panels of $y_{it}$ of size $(n, T)$ using (1), compute the simulated moments $\hat{\psi}_n^*(\theta) = \hat{\theta}_n^{\text{LSDV}}$. The indirect inference (IND) estimator $\hat{\theta}_{n,\text{IND}} = \arg\min_{\theta} \| \hat{\psi}_n - \frac{1}{2} \sum_s \hat{\psi}_n^*(\theta) \|$ has an automatic bias correction property and is consistent as $n \to \infty$ even if $T$ is fixed, but its variance is inflated by a factor $(1 + \frac{1}{S})$ due to simulation noise. The $n$ out of $n$ bootstrap is often used to obtain standard errors of indirect inference estimates. Throughout the estimation above, the covariates $x_{it}$ and the simulation draws $e_{it}^*$ are fixed while the optimizer solves for $\hat{\theta}_{n,\text{IND}}$.

In contrast, rNR and rQN resample $m$ out of $n$ individual paths of $(x_{it})_{t=1, \ldots, T}$ and simulate new draws $e_{it}^{*,b}$ at each iteration $b$. This has two advantages. First, as in the examples above, it is faster than the conventional bootstrap in producing standard errors. Second, the simulation noise across $b$ averages out, and as a consequence, rNR/rQN achieve the same asymptotic variance as an IND estimator that uses $S = \infty$ simulations. This statistical efficiency gain comes for free since we only use finitely many $S$ simulated samples at each iteration $b$, and with $m$ possibly less than $n$. The only proviso is that a second chain of draws is needed to produce correct standard errors and confidence intervals, as shown in Forneron and Ng (2020).
Table 3—: Dynamic Panel: finite sample properties in estimation and inference

|      | Average Estimate | Standard Deviation | Rejection Rates |
|------|------------------|--------------------|-----------------|
|      | \( m \) | \( \hat{r} \) | \( \hat{r}_{qN} \) | \( \hat{r}_{qr} \) | \( \hat{r}_{qN} \) | \( \hat{r}_{qr} \) | \( \hat{r}_{qN} \) | \( \hat{r}_{qr} \) |
|      | \( n = 500 \) | 0.599 | 0.599 | 0.023 | 0.023 | 0.049 | 0.049 | 0.049 | 0.049 |
|      | \( n = 100 \) | 0.598 | 0.598 | 0.023 | 0.023 | 0.049 | 0.048 | 0.049 | 0.049 |

To illustrate, data are simulated with \((\rho, \beta, \sigma) = (0.6, 1, 1)\), and \( x_{it}, e_{it} \) are iid standard normal with \( n = 500 \) and \( T = 5 \). We use \( m \in \{100, 500\} \), \( B = 2000 \), \( S \in \{1, 10\} \), \( \text{burn} = 45 \), and \( \gamma = 0.1 \) for \( \hat{r}_{nr} \) and \( \hat{r}_{qN} \). For the bootstrap we only use \( B = 500 \) as is common practice. The LSDV estimate \( \hat{\rho}_{LS} \) is 0.306 on average with standard deviation 0.017, exhibiting significant downward bias from the true \( \rho = 0.6 \). IND removes the downward bias almost entirely with an average estimate of 0.599 and 0.601 for \( S = 1, 10 \), respectively. The standard deviation of the IND estimates is 0.032 and 0.024 for \( S = 1, 10 \). Table 3 shows that \( \hat{r}_{nr} \) and \( \hat{r}_{qN} \) preserve this bias correction and have smaller standard deviations even with \( S = 1 \) and \( m < n \), as predicted by theory. Coverage is close to the nominal 95% level for the usual \( \hat{\theta}_n \pm 1.96 \cdot \text{se}(\hat{\theta}_n) \) confidence interval, with rejection rates of 0.049 and 0.046 for \( S = 1, 10 \). Bootstrap, \( \hat{r}_{nr} \), and \( \hat{r}_{qN} \) have similar coverage. Increasing \( S \) has little effect on \( \hat{r}_{nr} \), \( \hat{r}_{qN} \) but improves the accuracy of IND.

REFERENCES

Andrews, D. W. K. 2002. “Higher-Order Improvements of a Computationally Attractive k-Step Bootstrap for Extremum Estimators.” \textit{Econometrica}, 70:1: 119–162.

Berry, Steven, James Levinsohn, and Ariel Pakes. 1995. “Automobile Prices in Market Equilibrium.” \textit{Econometrica}, 63(4): 841.

Brunner, Daniel, Florian Heiss, André Romahn, and Constantin Weiser. 2017. Reliable estimation of random coefficient logit demand models. DICE Discussion Paper.

Davidson, Russell, and James G. MacKinnon. 1999. “Bootstrap Testing in Nonlinear Models.” \textit{International Economic Review}, 40(2): 487–508.

Forneron, J., and S. Ng. 2020. “Estimation and Inference by Stochastic Optimization.” arXiv:2004.09627.

Gouriéroux, Christian, Peter C.B. Phillips, and Jun Yu. 2010. “Indirect inference for dynamic panel models.” \textit{Journal of Econometrics}, 157(1): 68–77.

Honoré, Bo E., and Luojia Hu. 2017. “Poor (Wo)man’s Bootstrap.” \textit{Econometrica}, 85(4): 1277–1301.

Kline, Patrick, and Andres Santos. 2012. “A Score Based Approach to Wild Bootstrap Inference.” \textit{Journal of Econometric Methods}, 1(1).

Nevo, A. 2000. “A Practitioner’s Guidel to Estimation of Random-Coellicients Logit Models of demand.” \textit{Journal of Economics and Management Strategy}, 9:4: 513–548.