Comparative Study of Pontryagin Maximum Principle and Direct Method in Aircraft Optimal Control Design

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Abstract

This paper deals with the problem of open-loop control design for a dynamic object which is described by a nonlinear system of differential equations. To solve this problem a direct method is used, in which the target functional is minimized by population-based algorithm. The method is applied to a test problem which consists of finding optimal control for spatial motion of maneuverable aircraft. Proposed technique is compared with two classical solutions to considered problem. One of them is based on equating to zero the partial control derivatives of the Hamilton function, whereas the other on the Hamilton function maximum over controls (Pontryagin maximum principle). The paper confirms high degree of similarity between solutions obtained by all considered methods of selecting target functional. However, classical algorithms show slightly worse accuracy and higher sensitivity to the quality of initial approximation. In addition, proposed direct method permits to evade the necessity to solve a two–point boundary value problem required in classical algorithms.

Keywords: optimal open-loop control, Pontryagin maximum principle, population-based numerical optimization algorithm

1. Introduction

A number of ways exist for solving the open-loop control problem [1], starting from the classical results of the calculus of variations and the Pontryagin maximum principle and continuing with relatively new developments such as methods of inverse dynamic problems [2], decomposition [3] and embedded systems.

In recent decades considerable attention have been attracted to the direct methods [4, 5]. These techniques are based on the assumption that possible control could be parameterized and, thus, reduce the control search to a single-criterion unconditional multi-parameter optimization problem. In such formulation the problem appears much simpler than traditional two-point boundary value problem. Since the use of many well-proven gradient methods (including the Newton method with its modifications) is difficult for a large number of parameters, turn to another type of optimization algorithms, namely, genetic and population-based ones looks reasonable. These algorithms have shown encouraging results for multidimensional optimization problems, therefore we use the particle swarm method [6, 7], belonging to the population-based algorithms.

The paper proposes a direct method for the formation of open-loop control of an aircraft and compares it with the classical results of optimal control theory — solution to the Lagrange problem obtained on the basis of the calculus of variations and the Pontryagin maximum principle.
2. Specification of problem

Let us assume that nonlinear dynamical object model takes the form

$$\dot{x} = f(x, u, t),$$

where $x$ and $\dot{x}$ — state or output vector and its time derivatives, $u$ — vector of control or input vector, $f(\cdot)$ — vector function of vector argument. Initial conditions $x(t_0) = x_0$ are considered known.

Targeted functional

$$J(x, u) = \int_{t_i}^{t_f} F(x, u, t) \, dt$$

is determined on time interval $t \in [t_i, T]$.

The task is to find such control that provides for the considered time interval the minimum deviations of the output signals from the given ones. Then in target functional integrand can be written as follows

$$F(x, u, t) = F(x - \bar{x}, u, t),$$

where $\bar{x}$ — vector of given output signals.

Expressions (1)-(2) describe the well-known Lagrange problem with a free end point, since the initial conditions at the left end are given and the boundaries of the time interval are fixed. The traditional recommendation to find the optimal control is the transition to a two-point boundary value problem. To do this, the Hamilton function is written

$$H = (\lambda, \Gamma) - F,$$

where conjugate vector function is obtained from equation

$$\dot{\lambda} = -F^T \lambda + F_u, \quad \lambda(T) = 0,$$

Optimal control is found from necessary optimality condition

$$\frac{\partial H}{\partial \bar{u}} = \lambda F_u - F_u = 0,$$

$\bar{u}$ here means optimal control.

The two-point boundary-value problem arises, since the initial conditions for the object model (1) are given at the beginning of the interval, and for the conjugate function (4) at its end. Search for solution to this problem is often fraught with significant difficulties. The form of condition (5) corresponds to the case when restrictions are not imposed on the control.

If there are restrictions, the necessary minimum condition is determined by the Pontryagin maximum principle

$$H(\bar{x}, \bar{u}, \psi, \tau) = \max_{u \in G_u} H(\bar{x}, u, \psi, \tau),$$

where $\bar{u}$ is chosen from set of feasible controls $u \in G_u$.

Let us now consider the proposed direct method. It consists of the minimization of the target functional in the form (3).

To do this the control signals, which are functions of time, are represented on the considered interval in the form of a cubic Hermite spline. The spline parameters are a priori unknown. To find them, the problem of multidimensional parametric optimization is solved, i.e. the parameters that deliver the minimum to the functional (3) are found. We find a numerical solution using one of the population optimization methods — the particle swarm algorithm.

Functional (3) in its physical sense reaches a minimum when the model output signals corresponding to optimal control are closest to the given output signals. Therefore, the theoretical basis for this approach is the well-known theorem of the existence and uniqueness of the solution for a finite-dimensional system of nonlinear differential equations. From a practical point of view, the advantage of the direct method is that it does not require the solution of a two-point boundary value problem.

3. Control object

The model of the object takes the form presented below

$$\dot{x} = \omega - \frac{qS}{mV} c_w(\alpha) - \frac{P}{m} \sin(\alpha) +$$

$$+ \frac{g}{V} (\sin(\alpha) \sin(\nu) + \cos(\alpha) \cos(\nu) \cos(\gamma)), \quad$$

$$\dot{V} = -\frac{qS}{m} c_w(\alpha) + \frac{P}{m} \cos(\alpha) +$$

$$+ \frac{g}{V} (\cos(\alpha) \sin(\nu) + \sin(\alpha) \cos(\nu) \cos(\gamma)), \quad$$

$$\dot{\psi} = \frac{1}{\cos(\nu)} \omega \sin(\nu),$$

where $\alpha, \beta$ — angles of attack and sideslip, rad; $\omega$ — angular velocity with respect to the axis $Ox$, rad/s; $\nu, \gamma$ — angles of pitch, roll and yaw, rad; $V$ — airspeed, m/s; $\psi$ — flight altitude, m; $c_w(\alpha)$ — drag coefficient; $c_l(\alpha)$ — lift coefficient; $m$ — aircraft mass, kg; $S$ — wing surface area, $m^2$; $P_t$ — projection of thrust force on axis $Ox$ of body-fixed coordinate system, N; $g$ — gravity acceleration, m/s$^2$; $q$ — dynamic air pressure, Pa.

The system of differential equations is also supplemented by an algebraic equation for $\omega$,

$$\omega = \dot{\psi} \cos(\nu) - \dot{\psi} \cos(\nu) \sin(\gamma),$$

where $\dot{\psi} = \text{required value of yaw angle}$.

For the given model, we choose pitch and roll angles as control signals. They have the advantage that, firstly, they are quite smooth and therefore can be well approximated by splines with a small number of nodes, and secondly, by definition, they have restrictions on the range of values $(\pm 90^\circ)$ in pitch angle, $\pm 180^\circ$ in roll angle). The practical implementation of these angles also does not cause
fundamental difficulties, since the methods for designing tracking contours for given pitch and roll angles are currently developed quite well.

Direct methods for solving the problem require expansion of control signals in the basis of linearly independent functions. In this case, it is necessary to strive for a compromise between the accuracy of approximation of the signal, which leads to an increase in the number of parameters, and the computational complexity of solving the parametric optimization problem. In this work, we used Hermite splines of the third order \([8]\) with a distance between nodes of about 8 s.

To test the proposed method, a specially formulated problem was solved. Known signals were set as an input. They were used to model the movement of the object according to the system \((7, 8)\). Resulted output signals were taken as given output \(\hat{x}\). It was further assumed that the model of the object is known, and the task was set to search for control that minimizes target functional

\[
J(x, u) = \sum_{i=1}^{N} \left( k_1 (\alpha(t_i) - \hat{\alpha}(t_i))^2 + k_2 (V(t_i) - \hat{V}(t_i))^2 + k_3 (\phi(t_i) - \hat{\phi}(t_i))^2 \right),
\]

where \(k_1, k_2, k_3, k_4\) — weight coefficients, \(\hat{\alpha}, \hat{V}, \hat{\phi}\) — given values of angle of attack, airspeed, flight altitude and angle of yaw.

Target functional for Lagrange problem was formulated differently. Provided that condition of equation of control derivatives to zero is hardly obtainable for numerical calculation, proximity measure of control derivatives to zero was established on considered interval

\[
J(x, u) = \sum_{i=0}^{N} \left( k_1 \frac{\partial H}{\partial u_1} + k_2 \frac{\partial H}{\partial u_2} \right),
\]

4. Results

Main results concerning operation of direct method and the classic two-point boundary value problem solution in test examples where control signals do not come close to their limit values are presented in \([9]\). This corresponds to the Lagrange problem \((1, 2)\). If there are restrictions on control, the classical theory recommends applying the maximum principle, where the target functional is determined by expression \((6)\).

So, using a problem with restrictions as example, we compare three methods: the direct method, the solutions based on the equality of the derivatives of the Hamilton function to zero, and the solution by the maximum principle, i.e., by the maximum condition of the Hamilton function \((6)\).

Examples are chosen in such way that the desired control signals reach their limits. We introduce a limit of ±40° in pitch angle and ±90° in roll angle.

We conduct the numerical experiment described in Section 3, taking into account the fact that the control signals cannot go beyond the above limits. As can be seen in figure 1, the application of the direct method with functional \((9)\) allows us to achieve the high degree of correspondence in input signals.

![Figure 1. Comparison of given input signals (blue line) with input signals obtained using the direct method (violet line).](image1)

Using the functional of the derivatives of the Hamilton control function \((10)\) (figure 2) leads to results close to those obtained previously (figure 1). The accuracy of the estimates, as in the case without limitations, is quite high, although somewhat inferior to the direct method.

![Figure 2. Comparison of given input signals (blue line) with input signals obtained using the partial control derivatives of the Hamilton function (violet line), functional (10).](image2)
following approach. Suppose that at some stage of the operation of the numerical optimization algorithm $M$ control options are obtained, each of them has its own Hamilton function $H_i(t_j), i = 1, M$, calculated for $N$ discrete time instants $t_j, j = 1, N$. Among these functions, we consider the maximum one that at any moment of time $t_j$, or at most of such moments, surpasses all others. Consider the moment of discrete time $t_j$ for which $M$ values of the Hamilton function are determined. We assign a rank $r_j'$ to each of these values according to the following rule: the maximum value corresponds to a rank of 1, the next to a rank of 2, and so on. Then each function $H_i(t_j)$ and, accordingly, each control $u_i$, can be associated with a generalized rank, calculated by the formula

$$J(u_i) = (\sum_{j=1}^{N} r_j' - N)^2. \quad (11)$$

Let us explain the meaning of the generalized rank (11). Suppose that for some control the Hamiltonian function is the largest at all time moments, that is, at each instant it has rank 1. Then the generalized rank (11) is equal to zero and has the minimum possible value. This allows us to reduce the problem of finding the maximum Hamilton function to the problem of minimizing functional (11).

The results of processing the test example, obtained when parameter values are ±20% of actual value, are presented in figure 3.

**Figure 3.** Comparison of given input signals (blue line) with input signals obtained using an algorithm based on Hamilton function maximum (violet line), functional (11).

Note that for a wide search area (±90° in pitch, ±180° in roll), the convergence of the algorithm to the true values is not guaranteed. Figure 3 shows that the obtained accuracy with respect to the input signals approximately corresponds to the above-considered results of the direct method (figure 1) and the method based on the derivatives of the Hamilton function (figure 2), although it is somewhat inferior to them.

5. Conclusions

A direct method for finding the optimal open-loop control based on the parameterization of the control signal and the direct minimization of the functional using the population algorithm is proposed. The efficiency of the method is confirmed by mathematical modeling.

The paper also compares the direct method with two classical approaches based on equating to zero the partial control derivatives of the Hamilton function and the maximum of the Hamiltonian control function (maximum principle). Comparison is performed using test problem describing the spatial motion of a maneuverable aircraft. In the framework of the considered example, all methods showed similar results. At the same time, the accuracy of classical algorithms appears to be slightly worse, and they showed a higher sensitivity to the quality of the initial approximation.

Thus, the practical significance of the results obtained is that, in comparison with classical algorithms, the application of the direct method is significantly simpler, at least for the considered class of control problems, when the controls are sufficiently smooth functions and the right-hand sides of the differential equations of the object are continuous and have continuous first derivatives. In the direct method, for example, cumbersome mathematical calculations necessary for obtaining the conjugate vector function are not required.

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