Cantor families of periodic solutions for completely resonant nonlinear wave equations

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Abstract: We prove existence of small amplitude, $2\pi/\omega$-periodic in time solutions of completely resonant nonlinear wave equations with Dirichlet boundary conditions, for any frequency $\omega$ belonging to a Cantor-like set of positive measure and for a new set of nonlinearities. The proof relies on a suitable Lyapunov-Schmidt decomposition and a variant of the Nash-Moser Implicit Function Theorem. In spite of the complete resonance of the equation we show that we can still reduce the problem to a finite dimensional bifurcation equation. Moreover, a new simple approach for the inversion of the linearized operators required by the Nash-Moser scheme is developed. It allows to deal also with nonlinearities which are not odd and with finite spatial regularity.

Keywords: Nonlinear Wave Equation, Infinite Dimensional Hamiltonian Systems, Periodic Solutions, Variational Methods, Lyapunov-Schmidt reduction, small divisors, Nash-Moser Theorem.

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1 Introduction and main result

We consider the completely resonant nonlinear wave equation

\[
\begin{cases}
  u_{tt} - u_{xx} + f(x,u) = 0 \\
  u(t,0) = u(t,\pi) = 0
\end{cases}
\]

where the nonlinearity $f(x,u) = a_p(x)u^p + O(u^{p+1})$ with $p \geq 2$ is analytic with respect to $u$ but has only finite regularity with respect to $x$. More precisely, we assume

(H) There is $\rho > 0$ such that $\forall (x,u) \in (0,\pi) \times (-\rho,\rho)$, $f(x,u) = \sum_{k=p}^{\infty} a_k(x)u^k$, $p \geq 2$, where $a_k(x) \in H^1((0,\pi), \mathbb{R})$ and $\sum_{k=p}^{\infty} ||a_k||_{H^1}r^k < \infty$ for any $r \in (0,\rho)$.

We look for small amplitude, $2\pi/\omega$-periodic in time solutions of equation (1) for all frequency $\omega$ close to 1 and in some set of positive measure.

Equation (1) is an infinite dimensional Hamiltonian system possessing an elliptic equilibrium at $u = 0$. The frequencies of the linear oscillations at $u = 0$ are $\omega_j = j$, $\forall j = 1,2,\ldots$, and satisfy infinitely many resonance relations. Any solution $v = \sum_{j\geq 1} a_j \cos(jt + \theta_j) \sin(jx)$ of the linearized equation at $u = 0$,

\[
\begin{cases}
  u_{tt} - u_{xx} = 0 \\
  u(t,0) = u(t,\pi) = 0
\end{cases}
\]

is $2\pi$-periodic in time. For this reason equation (1) is called a completely resonant Hamiltonian PDE.

Existence of periodic solutions close to a completely resonant elliptic equilibrium for finite dimensional Hamiltonian systems has been proved by Weinstein [25], Moser [21] and Fadell-Rabinowitz [15]. The proofs are based on the classical Lyapunov-Schmidt decomposition which splits the problem into
two equations: the range equation, solved through the standard Implicit Function Theorem, and the bifurcation equation, solved via variational arguments.

For proving existence of small amplitude periodic solutions of completely resonant Hamiltonian PDEs like (1) two main difficulties must be overcome:

(i) a “small denominators” problem which arises when solving the range equation;

(ii) the presence of an infinite dimensional bifurcation equation: which solutions $v$ of the linearized equation (2) can be continued to solutions of the nonlinear equation (1)?

The appearance of the small denominators problem (i) is easily explained: the eigenvalues of the operator $\partial u - \partial_{xx}$ in the spaces of functions $u(t, x)$, $2\pi/\omega$-periodic in time and such that, say, $u(t, .) \in H^3_0(0, \pi)$ for all $t$, are $-\omega^2 l^2 + j^2$, $l \in \mathbb{Z}$, $j \geq 1$. Therefore, for almost every $\omega \in \mathbb{R}$, the eigenvalues accumulate to 0. As a consequence, for most $\omega$, the inverse operator of $\partial u - \partial_{xx}$ is unbounded and the standard Implicit Function Theorem is not applicable.

The first existence results for small amplitude periodic solutions of (1) have been obtained in (10), for the nonlinearity $f(x, u) = u^3$, and in (2) for $f(x, u) = u^3 + O(u^5)$, imposing on the frequency $\omega$ the “strongly non-resonance” condition $|\omega l - j| \geq \gamma/l$, $\forall l \neq j$. For $\gamma > 0$ small enough, the frequencies $\omega$ satisfying such condition accumulate to $\omega = 1$ but form a set $W_\omega$ of zero measure. For such $\omega$’s the spectrum of $\partial u - \partial_{xx}$ does not accumulate to 0 and so the small divisor problem (i) is by-passed. Next, problem (ii) is solved by means of the Implicit Function Theorem, observing that the $0^{th}$-order bifurcation equation (which is an approximation of the exact bifurcation equation) possesses, for $f(x, u) = u^3$, non-degenerate periodic solutions, see (3).

In (3)-(5), for the same set $W_\omega$ of strongly non-resonant frequencies, existence and multiplicity of periodic solutions has been proved for any nonlinearity $f(u)$. The novelty of (4)-(5) was to solve the bifurcation equation via a variational principle at fixed frequency which, jointly with min-max arguments, enables to find solutions of (1) as critical points of the Lagrangian action functional.

Unlike (2)-(4)-(5), a new feature of the results of this paper is that the set of frequencies $\omega$ for which we prove existence of $2\pi/\omega$-periodic in time solutions of (1) has positive measure, actually has full density at $\omega = 1$.

The existence of periodic solutions for a set of frequencies of positive measure has been proved in (9) in the case of periodic boundary conditions in $x$ and for the nonlinearity $f(x, u) = u^3 + \sum_{4 \leq j \leq \delta} a_j(x)u^j$ where the $a_j(x)$ are trigonometric cosine polynomials in $x$. The nonlinear equation $u_{tt} - u_{xx} + u^3 = 0$ possesses a continuum of small amplitude, analytic and non-degenerate periodic solutions in the form of travelling waves $u(t, x) = \delta p_0(\omega t + x)$ where $\omega^2 = 1 + \delta^2$ and $p_0$ is a non-trivial $2\pi$-periodic solution of the ordinary differential equation $p_0'' = -p_0^3$. With these properties at hand the small divisors problem (i) is solved via a Nash-Moser implicit function Theorem adapting the estimates of Craig-Wayne (11).

Recently, the existence of periodic solutions of (1) for frequencies $\omega$ in a set of positive measure has been proved in (16) using the Lindstedt series method for odd analytic nonlinearities $f(u) = au^3 + O(u^5)$ with $a \neq 0$. The reason for which $f(u)$ must be odd is that the solutions are obtained as analytic sine-series in $x$, see comments at the end of the section.

We also quote the recent papers (17)-(18) on the standing wave problem for a perfect fluid under gravity and with infinite depth which leads to a nonlinear and completely resonant second order equation.

In this paper we prove the existence of periodic solutions of the completely resonant wave equation (1) with Dirichlet boundary conditions for a set of frequencies $\omega$’s of positive measure and with full density at $\omega = 1$ and for a new set of nonlinearities $f(x, u)$ satisfying (H) (including for example $f(x, u) = u^3$); we do not require that $f(x, u)$ can be extended on $(-\pi, \pi) \times \mathbb{R}$ to an analytic function $g(x, u)$ satisfying the oddness assumption $g(-x, -u) = -g(x, u)$, and we assume only finite regularity in the spatial variable $x$, see Theorem (17).

Let us describe accurately our result. Normalizing the period to $2\pi$, we look for solutions $u(t, x)$,
2π-periodic in time, of the equation

\[
\begin{cases}
\omega^2 u_{tt} - u_{xx} + f(x, u) = 0 \\
u(t, 0) = u(t, \pi) = 0
\end{cases}
\]  

(3)

in the real Hilbert space

\[
X_{\sigma,s} := \left\{ u(t, x) = \sum_{l \in \mathbb{Z}} \exp(ilt) u_l(x) \mid u_l \in H_0^1((0, \pi), C), \quad u_l = u_{-l}(x) \forall l \in \mathbb{Z}, \right\}
\]

and

\[
||u||_{\sigma,s}^2 := \sum_{l \in \mathbb{Z}} \exp(2\sigma|l|)(l^2 + 1)||u_l||_{H^1}^2 < +\infty.
\]

For \( \sigma > 0, s \geq 0 \), the space \( X_{\sigma,s} \) is the space of all 2π-periodic in time functions with values in \( H_0^1((0, \pi), \mathbb{R}) \), which have a bounded analytic extension in the complex strip \( |\text{Im} \; t| < \sigma \) with trace function on \( |\text{Im} \; t| = \sigma \) belonging to \( H^s(T, H_0^1((0, \pi), C)) \).

Note that if \( u \in X_{\sigma,s} \) is a solution of (3) in a weak sense then the map \( x \mapsto u_{xx}(t, x) = \omega^2 u_{tt}(t, x) - f(x, u(t, x)) \) belongs to \( H_0^1(0, \pi) \) for all \( t \in T \); hence \( u(t, \cdot) \in H^s(0, \pi) \subset C^2((0, \pi)) \) and it is easy to justify that \( u \) is a classical solution.

For \( 2s > 1 \), \( X_{\sigma,s} \) is a Banach algebra with respect to multiplication of functions, namely\(^3\)

\[
u_1, u_2 \in X_{\sigma,s} \implies u_1 u_2 \in X_{\sigma,s} \quad \text{and} \quad ||u_1 u_2||_{\sigma,s} \leq C ||u_1||_{\sigma,s}||u_2||_{\sigma,s}.
\]

The space of the solutions of the linear equation \( v_{tt} - v_{xx} = 0 \) that belong to \( X_{\sigma,s} \) is\(^4\)

\[
V := \left\{ v(t, x) = \sum_{l \in \mathbb{Z}} \exp(ilt) \sin(lx) \mid u_l \in C, \quad u_l = -u_{-l}, \quad \sum_{l \in \mathbb{Z}} \exp(2\sigma|l|)(l^2 + 1)||u_l||^2 < +\infty \right\}.
\]

Let \( \varepsilon = \frac{\omega^2 - 1}{2} \). Instead of looking for solutions of (3) in a shrinking neighborhood of 0 it is a convenient devise to perform the rescaling \( u \rightarrow \delta u \) with \( \delta := |\varepsilon|^{1/p-1} \) (in most cases, see however subsection \[\text{[**]]}\), obtaining

\[
\begin{cases}
\omega^2 u_{tt} - u_{xx} + \varepsilon g(\delta, x, u) = 0 \\
u(t, 0) = u(t, \pi) = 0
\end{cases}
\]

(4)

where

\[
g(\delta, x, u) := s^* \frac{f(x, \delta u)}{\delta p} = s^* \left( a_p(x) u^p + \delta a_{p+1}(x) u^{p+1} + \ldots \right)
\]

and \( s^* := \text{sign}(\varepsilon) \), namely \( s^* = 1 \) if \( \omega > 1 \) and \( s^* = -1 \) if \( \omega < 1 \).

The main result of this paper is:

Theorem 1.1 Consider the completely resonant nonlinear wave equation (1) where the nonlinearity \( f(x, u) = a_p(x) u^p + O(u^{p+1}), \; p \geq 2, \) satisfies assumption (H).

1) There exists an open set \( A_p \in H^1((0, \pi), \mathbb{R}) \) such that, for all \( a_p \in A_p \), there is \( \sigma > 0 \) and a \( C^\infty \)-curve \( [0, \delta_0) \supset \delta \mapsto u(\delta)(t, x) \in X_{\sigma,s} \) with the following properties:

- (i) There exists \( s^* \in \{-1, 1\} \) and a Cantor set \( C_{a_p} \subset [0, \delta_0) \) satisfying

\[
\lim_{\eta \to 0^+} \frac{\text{meas}(C_{a_p} \cap (0, \eta))}{\eta} = 1
\]

(5)

such that, for all \( \delta \in C_{a_p}, \; u(\delta) \) is a 2π/ω-periodic in time classical solution of (1) with \( \omega = \sqrt{2s^*\delta^{p-1} + 1} \);

\(^3\)The proof is as in [23] recalling that \( H_0^1((0, \pi), C) \) is a Banach algebra with respect to multiplication of functions.

\(^4\)\( V \) can also be written as

\[
V := \left\{ v(t, x) = \eta(t + x) - \eta(t - x) \mid \eta \in C^\infty_{\sigma,s+1}(T, \mathbb{R}) \right\}
\]

where \( C^\infty_{\sigma,s+1}(T, \mathbb{R}) \) denotes the space of all 2π-periodic functions with have a bounded analytic extension in the complex strip \( |\text{Im} \; t| < \sigma \) with trace function on \( |\text{Im} \; t| = \sigma \) belonging to \( H^{s+1}(T, \mathbb{C}) \).
\begin{itemize}
  \item (ii) \( \|\tilde{w}(\delta) - \delta u_0\|_{\sigma,s} = O(\delta^2) \) for some \( u_0 \in V \setminus \{0\} \), where \( \tilde{w}(\delta)(t,x) = u(\delta)(t/\omega, x) \).
\end{itemize}

All \( a_3(x) \in H^1(0,\pi) \) such that \( \langle a_3 \rangle := (1/\pi) \int_0^\pi a_3(x)dx \neq 0 \) is in \( A_3 \). Hence (i) and (ii) hold true for any nonlinearity like \( f(x,u) = a_3(x)u^3 + \sum_{k \geq 4} a_k(x)u^k, (a_3) \neq 0 \), with \( s^* = \text{sign}(\langle a_3 \rangle) \).

2) In the case \( f(x,u) = a_2u^2 + \sum_{k \geq 4} a_k(x)u^k, a_2 \neq 0 \), conclusions (i) and (ii) still hold true with \( \omega = \sqrt{-23} \).\( ^{3} \)

Remark 1.1

(i) Since equation (7) is autonomous, any non-trivial time periodic solution \( u(t,x) \) of (7) generates a circle of solutions, i.e., for any \( \theta \in \mathbb{R} \), \( u_\theta := u(t - \theta, x) \) is a solution too.

(ii) Multiplicity of periodic solutions \( u(t,x) \) of (7) with increasing norm could be found, as in [5].

Sketch of the proof.

We require first, for simplicity of exposition (see note 5), that
\[
\exists v \in V \quad \text{such that} \quad \int_\Omega a_p(x)v^{p+1}(t,x) \, dt dx \neq 0, \quad \Omega := T \times (0, \pi). \tag{6}
\]

To fix the ideas, we assume there is \( v \in V \) such that \( \int_\Omega a_p(x)v^{p+1} > 0 \) and we look for periodic solutions with frequency \( \omega > 1 \), so that \( s^* = 1 \) and \( \omega = \sqrt{23} \) (if \( \int_\Omega a_p(x)v^{p+1} < 0 \) for some \( v \in V \), then we can look for solutions of frequency \( \omega < 1 \), so that \( s^* = -1 \) and \( \omega = \sqrt{-23} \)).

Condition (6) is verified for \( p \) odd, if \( a(x) \) is not antisymmetric w.r.t. to \( x = \pi/2 \), and for \( p \) even, if \( a(x) \) is not symmetric w.r.t. to \( x = \pi/2 \), see Lemma 6.1 in the Appendix.

In order to find solutions of (7), we try to implement the usual Lyapunov-Schmidt reduction according to the decomposition \( X_{\sigma,s} = V \oplus W \) where
\[
W := \left\{ w = \sum_{l \in \mathbb{Z}} \exp(ilt) \, w_l(x) \in X_{\sigma,s} \mid w_{-l} = \overline{w_l} \right\} \quad \text{and} \quad \int_0^\pi w_l(x) \sin(lx) \, dx = 0, \forall l \in \mathbb{Z}. \tag{7}
\]

Looking for solutions \( u = v + w \) with \( v \in V \), \( w \in W \) and we are led to solve the bifurcation equation (called the \( (Q) \)-equation) and the range equation (called the \( (P) \)-equation)
\[
\begin{cases}
-\Delta v = \Pi_V g(\delta, x, v + w) \\
L_\omega w = \varepsilon \Pi_W g(\delta, x, v + w)
\end{cases}
\tag{8}
\]

where
\[
\Delta v := v_{xx} + v_{tt}, \quad L_\omega := -\omega^2 \partial_{tt} + \partial_{xx}
\]
and \( \Pi_V : X_{\sigma,s} \to V \), \( \Pi_W : X_{\sigma,s} \to W \) denote the projectors respectively on \( V \) and \( W \).

Since \( V \) is infinite dimensional a difficulty arises in the application of the method of [11] in presence of small divisors: if \( v \in V \cap X_{\sigma_0,s} \) then the solution \( w(\delta, v) \) of the range equation, obtained with any Nash-Moser iteration scheme will have a lower regularity, e.g. \( w(\delta, v) \in X_{\sigma_0/2,s} \). Therefore in solving next the bifurcation equation for \( v \in V \), the best estimate we can obtain is \( v \in V \cap X_{\sigma_0/2,s+2} \), which makes the scheme incoherent. Moreover we have to ensure that the \( 0^{th} \)-order bifurcation equation\(^5\), i.e. the \( (Q) \)-equation for \( \delta = 0 \),
\[
-\Delta v = \Pi_V \left\{ a_p(x)v^p \right\}
\tag{9}
\]
has solutions \( v \in V \) which are analytic, a necessary property to initiate an analytic Nash-Moser scheme (in [11, 12] this problem does not arise since, dealing with non-resonant or partially resonant Hamiltonian PDEs like \( u_{tt} - u_{xx} + a_1(x)u = f(x,u) \), the bifurcation equation is finite dimensional).

We overcome this difficulty thanks to a reduction to a \textit{finite dimensional} bifurcation equation (on a subspace of \( V \) of dimension \( N \) independent of \( \omega \)). This reduction can be implemented, in spite of the complete resonance of equation (11), thanks to the compactness of the operator \( (\Delta)^{-1} \).

\(^5\)The right hand side \( \Pi_V (a_p(x)v^p) \) is not identically equal to 0 in \( V \) iff condition (2) holds. If not verified, as for \( f(x,u) = u^2 \), the \( 0^{th} \)-order non-trivial bifurcation equation will involve higher order nonlinear terms, see [11] and subsection (3).
We introduce the decomposition \( V = V_1 \oplus V_2 \) where
\[
\begin{align*}
V_1 & := \{ v \in V \mid v(t,x) = \sum_{l=1}^{N} \left( \exp (ilt)u_l + \exp (-ilt)\overline{u_l} \right) \sin(lx), \ u_l \in \mathbb{C} \} \\
V_2 & := \{ v \in V \mid v(t,x) = \sum_{l=N+1}^{\infty} \left( \exp (ilt)u_l + \exp (-ilt)\overline{u_l} \right) \sin(lx), \ u_l \in \mathbb{C} \}
\end{align*}
\]
Setting \( v := v_1 + v_2 \), with \( v_1 \in V_1, v_2 \in V_2 \), is equivalent to
\[
\begin{align*}
-\Delta v_1 & = \Pi V_1 g(\delta,x,v_1 + v_2 + w) & (Q_1) \\
-\Delta v_2 & = \Pi V_2 g(\delta,x,v_1 + v_2 + w) & (Q_2) \\
L_w w & = \varepsilon \Pi W g(\delta,x,v_1 + v_2 + w) & (P)
\end{align*}
\]
where \( \Pi V_i : X_{\sigma,s} \to V_i (i = 1, 2) \), denote the orthogonal projectors on \( V_i \) \( (i = 1, 2) \). We specify that all the norms \( || \cdot ||_{\sigma,s} \) are equivalent on \( V_1 \). In the sequel \( B(\rho, V_1) \) will denote \( B(\rho, V_1) := \{ v_1 \in V_1 \mid ||v_1||_{0,s} < \rho \} \).

Our strategy to find solutions of system \((10)\) - and hence to prove Theorem 1.4 - is the following. We solve first the \((Q_2)\)-equation obtaining \( v_2 = v_2(\delta,v_1,w) \in V_2 \cap X_{\sigma,s}^0 \) by a standard Implicit Function Theorem provided we have chosen \( N \) large enough and \( 0 < \sigma < \sigma^* \) small enough - depending on the nonlinearity \( f \) but independent of \( \delta \), see section 3.

Next we solve the \((P)\)-equation, obtaining \( w = w(\delta,v_1) \in W \cap X_{\sigma/2,s}^0 \) by means of a Nash-Moser Implicit Function Theorem for \((\delta,v_1)\) belonging to some Cantor-like set of parameters, see section 3.

A major role is played by the inversion of the linearized operators obtained at any stage of the Nash-Moser iteration. As usual, the main difficulty in controlling such inverse operators is due to the fact the diagonal elements may be arbitrarily small.

Our approach -presented in section 4- is different from the one in [11]-[12]-[13]-[14] which is based on the Fröhlich-Spencer technique [14]. It allows to deal with nonlinearities \( f(x,u) \) with finite regularity in the spatial variable \( x \) and which are not the restriction to \((0,\pi) \times \mathbb{R} \) of a smooth odd function.

We first develop \( u(t,\cdot) \in H^1_0((0,\pi),\mathbb{R}) \) in time-Fourier expansion only and we distinguish the “diagonal part” \( D = \text{diag}\{D_k\}_{k \in \mathbb{Z}} \) of the operator that we want to invert. Next, using Sturm-Liouville theory (see Lemma 1.1), we diagonalize each \( D_k \) in a suitable basis of \( H^1_0((0,\pi),\mathbb{R}) \) (close, but different from \((\sin jx)_{j \geq 1}\)). Assuming a “first order Melnikov non resonance condition” (Definition 3.3) we prove that its eigenvalues are polynomially bounded away from 0 and so we invert \( D \) with sufficiently good estimates (Corollary 1.2). The presence of the “off-diagonal” Toeplitz operators requires to analyze the “small divisors”: for our method it is sufficient to prove that the product of two “small divisors” is larger than any constant if the corresponding “singular sites” are close enough, see Lemma 1.4.

If the nonlinearity \( f(x,u) \) can be extended to an analytic (in both variables) odd function then the Dirichlet problem on \([0,\pi]\) is actually equivalent to the \(2\pi\)-periodic problem within the space of all odd functions and a natural configuration space to use is \( Y := \{ u(x) = \sum_{j \geq 1} u_j \sin(jx) \mid \sum_j \exp(2ajj^2u_j^2) < +\infty \} \). On the other hand, for (still analytic) non odd nonlinearities \( f \), it is not possible in general to find a smooth periodic solution \( u(t,) \) that belongs to \( Y \) for all \( t \). For example, if \( f(x,u) = u^2 \) then it is easy to see (deriving twice the \( \delta \)-equation w.r.t. \( x \) and using that \( u_{xx}(t,0) = 0 \)) that any smooth solution \( u \) must satisfy \(-u_{xxxx}(t,0) + 2u_x^2(t,0) = 0 \). Moreover, if \( u \) is not trivial then \( u_x(.,0) \) is not identically 0. Hence \( u_{xxxx}(.,0) \) does not vanish anywhere, which implies that \( u(t,) \) is not in \( Y \) for all \( t \). For these reasons we shall consider as configuration space \( H^1_0((0,\pi),\mathbb{R}) \) (and the solutions that we shall find satisfy \( u(t,.) \in H^1_0(0,\pi) \cap H^2(0,\pi) \).

Finally (section 5) we solve the finite dimensional \((Q_1)\)-equation for a new set of nonlinearities : for these nonlinearities, we can define a smooth curve \( (\delta \mapsto v_1(\delta) \in V_1) \) such that, if \( \delta \) belongs to some “large” set, \( (\delta,v_1(\delta)) \) gives rise to an exact solution of equation \((1)\).

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Notations: \( B(R;X) \) denotes the open ball of radius \( R \) in the space \( X \) centered at \( 0 \). \( f(z) = O(g(z)) \) means that there is a universal constant \( C \) such that \( |f(z)| \leq C|g(z)| \). \( \overline{z} \) is the complex conjugated of \( z \).
2 Solution of the \((Q_2)\)-equation

The 0\(^{th}\)-order bifurcation equation \((1)\) is the Euler-Lagrange equation of the functional \(\Phi_0 : V \to \mathbb{R}\), defined by

\[
\Phi_0(v) = \frac{||v||_{H^1}^2}{2} - \int_\Omega a_p(x) \frac{|v|^{p+1}}{p+1} \, dx dt, \quad \Omega = \mathbb{T} \times (0, \pi),
\]

where \(||v||_{H^1}^2 = \int_\Omega v_x^2 + v_z^2 \, dx \, dt\). By the Mountain-pass Theorem \((1)\) and condition \((H)\) (recall that we assume \(\int_\Omega a_p(x)|v|^{p+1} > 0\) for some \(v \in V\), the critical set

\[
K_{0,c} := \left\{ v \in V \mid \Phi_0(v) = 0, \Phi_0(v) \leq c \right\}
\]

is non-trivial (i.e. \(\neq \{0\}\)) for \(c > 0\) large enough and is compact for the \(H^1\)-topology, see \((1)\).

In fact, by a direct bootstrap argument, \(K_{0,c}\) is a compact subset of \(V \cap H^k(\Omega)\), for any \(k \geq 0\). Therefore \(K_{0,c} \subset V \cap C^\infty(\Omega)\) (even if \(a_p(x) \in H^l((0,\pi), \mathbb{R})\) only, because the projection \(\Pi_V\) has a regularizing effect in the variable \(x\)) and we have \((a\, priori\, estimate)\)

\[
\sup_{v \in K_{0,c}} ||v||_{0,s} < R < +\infty.
\]

By the analyticity assumption \((H)\) on the nonlinearity \(f\) and the Banach algebra property of \(X_{\sigma,s}\), the Nemitsky operator 

\[
X_{\sigma,s} \ni u \to g(\delta, x, u) \in X_{\sigma,s}
\]

is in \(C^\infty(\mathcal{U}_0, X_{\sigma,s})\), where \(\mathcal{U}_0 = \{ u \in X_{\sigma,s} \mid ||\delta||_{\|\cdot\|_{\sigma,s}} \leq \delta \}\) and \(\delta\) depends on the radius of convergence \(\rho\) of the power series that defines \(f(x,u)\).

Lemma 2.1 (Solution of the \((Q_2)\)-equation) There exists \(\sigma > 0, N \in \mathbb{N}_+, \delta_0 > 0\) such that, \(\forall 0 \leq \sigma < \sigma_0, \forall ||v_1||_{0,s} \leq 2R, \forall ||w||_{\sigma,s} \leq 1, \forall \delta \in [0, \delta_0]\), there exists a unique \(v_2 = v_2(\delta, v_1, w) \in X_{\sigma,s}\) with

\[
||v_2(\delta, v_1, w)||_{\sigma,s} \leq 1
\]

which solves the \((Q_2)\)-equation.

Moreover, for any \(v \in K_{0,c}, v_2(0, \Pi_V v, 0) = \Pi_V v\).

Furthermore \(v_2(\delta, v_1, w) \in X_{\sigma,s+2}, v_2(\cdot, \cdot, \cdot) \in C^\infty(\{ 0, \delta_0 \} \times B(2R; V_1) \times B(1; W \cap X_{\sigma,s}), V_2 \cap X_{\sigma,s+2})\) and \(Dv_2, D^2 v_2\) are bounded on \((0, \delta_0) \times B(2R; V_1) \times B(1; W \cap X_{\sigma,s})\).

Finally, if in addition \(||w||_{\sigma,s'} < +\infty\) for some \(s' \geq s\), then \((provided \delta_0 has been chosen small enough)\)

\[
||v_2(\delta, v_1, w)||_{\sigma,s'} \leq K(s', ||w||_{\sigma,s'})
\]

Proof. Fixed points of the nonlinear operator \(N(\delta, v_1, w, \cdot) : V_2 \to V_2\) defined by

\[
N(\delta, v_1, w, v_2) := (-\Delta)^{-1}\Pi_V g(\delta, x, v_1 + w + v_2)
\]

are solutions of equation \((Q_2)\). For \(w \in W \cap X_{\sigma,s}, v_2 \in V_2 \cap X_{\sigma,s}\) we conclude that \(N(\delta, v_1, w, v_2) \in V_2 \cap X_{\sigma,s+2}\) since \(g(\delta, x, v_1 + w + v_2) \in X_{\sigma,s}\) and because of the regularizing property of the operator \((-\Delta)^{-1}\Pi_V : X_{\sigma,s} \to V_2 \cap X_{\sigma,s+2}\).

Let \(B := \{ v_2 \in V_2 \cap X_{\sigma,s} \mid ||v_2||_{\sigma,s} \leq 1 \}\). We claim that there exists \(N \in \mathbb{N}, \sigma > 0\) and \(\delta_0 > 0\), such that for any \(0 \leq \sigma < \sigma_0\), \(||v_1||_{0,s} \leq 2\), \(||w||_{\sigma,s} \leq 1, \delta \in [0, \delta_0]\) the operator \(v_2 \to N(\delta, v_1, w, v_2)\) is a contraction in \(B\), more precisely

\[
\begin{align*}
& (i) \quad ||v_2||_{\sigma,s} \leq 1 \Rightarrow ||N(\delta, v_1, w, v_2)||_{\sigma,s} \leq 1; \\
& (ii) \quad \forall v_2, \tilde{v}_2 \in B \Rightarrow ||N(\delta, v_1, w, v_2) - N(\delta, v_1, w, \tilde{v}_2)||_{\sigma,s} \leq (1/2)||v_2 - \tilde{v}_2||_{\sigma,s}.
\end{align*}
\]

Let us prove \((i)\). \(\forall u \in X_{\sigma,s}, ||(-\Delta)^{-1}\Pi_V u||_{\sigma,s} \leq (C/(N+1)^2)||u||_{\sigma,s}\) and so \(\forall ||w||_{\sigma,s} \leq 1, ||v_1||_{0,s} \leq 2R, \delta \in [0, \delta_0]\),

\[
||N(\delta, v_1, w, v_2)||_{\sigma,s} \leq \frac{C}{(N+1)^2}||g(\delta, x, v_1 + w + v_2)||_{\sigma,s} \leq \frac{C'}{(N+1)^2} \left( ||v_1||_{\sigma,s}^p + ||v_2||_{\sigma,s}^p + ||w||_{\sigma,s}^p \right)
\]

\[
\leq \frac{C'}{(N+1)^2} \left( \exp(\sigma p N)||v_1||_{0,s}^p + ||v_2||_{\sigma,s}^p + 1 \right) \leq \frac{C'}{(N+1)^2} \left( (4R)^p + ||v_2||_{\sigma,s}^p + 1 \right)
\]
for $\exp(\sigma N) \leq 2$, where we have used that $||v_1||_{s,r} \leq \exp(\sigma N)||v_1||_{0,s} \leq 4R$. For $N$ large enough (depending on $R$) and $\sigma = \ln 2/N$, (i) follows. Property (ii) can be proved similarly and the existence of a unique solution $v_2(\delta, v_1, w) \in V_2 \cap X_{\sigma,s}$ follows.

Since $K_{\sigma,c}$ is compact in $V \cap H^s(\Omega)$, we may assume that $N$ has been chosen so large that for all $v \in K_{\sigma,c}, ||\Pi_{V_2} v||_{0,s} \leq 1/2$. Any $v \in K_{\sigma,c}$ solves $[9]$, hence $\Pi_{V_2} v$ solves the $(Q_2)$-equation associated with $(0, \Pi_{V_1} v, 0)$. Hence $\Pi_{V_2} v \in B$ and $\Pi_{V_2} v = \mathcal{N}(0, \Pi_{V_1} v, 0)$, which implies that $\Pi_{V_2} v = v_2(0, \Pi_{V_1} v, 0)$.

Because the map $(\delta, v_1, w, v_2) \mapsto \mathcal{N}(\delta, v_1, w, v_2)$ is $C^\infty$, by the Implicit Function Theorem $v_2 : \{ (\delta, v_1, w) \mid \delta \in [0, \delta_0], ||v_1||_{0,s} \leq 2R, ||w||_{s,s} \leq 1 \} \to V_2 \cap X_{\sigma,s}$ is a $C^\infty$ map. Finally, since $(-\Delta)^{-1}\Pi_{V_2}$ is a continuous linear operator from $X_{\sigma,s}$ to $V_2 \cap X_{\sigma,s+2}$ and

$$v_2(\delta, v_1, w) = (-\Delta)^{-1}\Pi_{V_2} \left( g(\delta, x, v_1 + w + v_2(\delta, v_1, w)) \right),$$

by the regularity of the Nemitsky operator induced by $g$, $v_2(\cdot, \cdot, \cdot) \in C^\infty([0, \delta_0] \times B(2R; V_1) \times B(1; W \cap X_{\sigma,s}), V_2 \cap X_{\sigma,s+2})$. The estimates for the derivatives can be obtained similarly.

Finally, assume that $||w||_{s,s'} < +\infty$ for some $s' \geq s$. We have

$$v_2(\delta, v_1, w) = (-\Delta)^{-1}\Pi_{V_2} g(\delta, x, v_1 + w + v_2(\delta, v_1, w)).$$

Using the regularizing properties of the operator $(-\Delta)^{-1}\Pi_{V_2}$ and the fact that $||v_1||_{s,r} < +\infty$ for all $r \geq s$, we can derive by a simple bootstrap argument a bound on $||v_2(\delta, v_1, w)||_{s,s'}$. It is useful here to observe that $^6 ||a_1(x)(u')||_{s,r} \leq C||a_1||_{H^1}||u||_{s,s}||u||_{s,r}$ for $r \geq s, \delta, \sigma, s, s$. Therefore, if $u \in X_{\sigma,r}$ and $\delta ||u||_{s,s}$ is small enough, then $g(\delta, x, u) \in X_{\sigma,r}$. ■

**Remark 2.1** Lemma 2.1 implies, in particular, that any solution $v \in K_{\sigma,c}$ of the $0$th order bifurcation equation [4] is not only in $V \cap C^\infty(\Omega)$ but actually belongs to $V \cap X_{\sigma,s+2}$ and therefore is analytic in $t$ and hence in $x$.

We conclude this section with a Lemma which is a standard consequence of the Lyapunov-Schmidt decomposition dealing with variational equations.

Let us define the “reduced” functional $\Psi_0 : B(2R, V_1) = \{ v_1 \in V_1 \mid ||v_1||_{0,0} < 2R \} \to \mathbb{R}$ by

$$\Psi_0(v_1) := \Phi_0(v_1 + v_2(0, v_1, 0)).$$

**Lemma 2.2** If $\overline{v}_1$ is a critical point of $\Psi_0$, then $\overline{v} := \overline{v}_1 + v_2(0, \overline{v}_1, 0)$ is a critical point of $\Phi_0 : V \to \mathbb{R}$, i.e. $\overline{v} \in V$ is a solution of the $0$th order bifurcation equation [4]. Conversely, if $\overline{v}$ is a critical point of $\Phi_0$ of critical value $\leq c$ then there is a critical point $\overline{v}_1 \in B(2R, V_1)$ of $\Psi_0$ such that $\overline{v} = \overline{v}_1 + v_2(0, \overline{v}_1, 0)$.

**Proof.** Since $v_2(0, v_1, 0)$ solves the $(Q_2)$-equation (for $0 = 0, w = 0$, $d\Phi_0(v_1 + v_2(0, v_1, 0))[k] = 0, \forall k \in V_2$. Moreover, since $\forall v_1 \in V_1 v_2(0, v_1, 0) \in V_2, \partial_{v_1} v_2(0, v_1, 0)[h] \in V_2, \forall h \in V_1$. Therefore

$$d\Psi_0(v_1)[h] = d\Phi_0(v_1 + v_2(0, v_1, 0))[h + \partial_{v_1} v_2(0, v_1, 0)[h]] = d\Phi_0(v_1 + v_2(0, v_1, 0))[h]$$

$$= \int_\Omega \left( -\Delta v_1 - \Pi_{V_1} \left( a_p(x)(v_1 + v_2(0, v_1, 0)) \right) \right) h \, dx \, dt. \quad (14)$$

Therefore if $\overline{v}_1$ is a critical point of $\Psi_0$, then $\overline{v} := \overline{v}_1 + v_2(0, \overline{v}_1, 0)$ is a solution of equation [4]. Conversely, assume that $\overline{v}$ is a critical point of $\Phi_0$, with $\Phi_0(\overline{v}) \leq c$ and let $\overline{v}_1 = \Pi_1 \overline{v}$. By Lemma 2.1

$\Pi_{V_2} \overline{v} = v_2(0, \overline{v}_1, 0)$ and it is clear that $\overline{v}_1$ is a critical point of $\Psi_0$. ■

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6This inequality can be obtained considering the extension of the maps to the complex strip of width $\sigma$, using that $H^l(0, \pi)$ is a Banach algebra, and the inequality $||u||_{H^{l,per}} \leq K||u||_{H^{l,per}}^4 + ||u||_{H^{l,per}}^4$ for $2\pi$-periodic functions. The last inequality follows from $||u||_{H^{l,per}} \leq C(1)||u||_{H^{l,per}}^4 + ||u||_{H^{l,per}}^4$ which is related to the Gagliardo-Nirenberg inequalities.
3 Solution of the \((P)\)-equation

By the previous section we are reduced to solve the \((P)\)-equation with \(v_2 = v_2(\delta, v_1, w)\), namely

\[
L_\omega w = \varepsilon \Pi W \Gamma(\delta, v_1, w)
\]

where

\[
\Gamma(\delta, v_1, w)(t, x) := g\left(\delta, x, v_1(t, x) + w(t, x) + v_2(\delta, v_1, w)(t, x)\right).
\]

The solution \(w = w(\delta, v_1)\) of the \((P)\)-equation (13) is obtained by means of a Nash-Moser Implicit Function Theorem for \((\delta, v_1)\) belonging to a Cantor-like set of parameters.

We consider the orthogonal splitting \(W = W^{(n)} \oplus W^{(n)\perp}\) where

\[
W^{(n)} = \{ w \in W \mid w = \sum_{l=-L_n}^{L_n} \exp(ilt) w_l(x) \} \quad \text{and} \quad W^{(n)\perp} = \{ w \in W \mid w = \sum_{|l| > L_n} \exp(ilt) w_l(x) \},
\]

and \(L_n\) are integer numbers (we will choose \(L_n = L_0 2^n\) with \(L_0 \in \mathbb{N}\) large enough). We denote by

\[
P_n : W \rightarrow W^{(n)} \quad \text{and} \quad P_n^\perp : W \rightarrow W^{(n)\perp}
\]

the orthogonal projectors onto \(W^{(n)}\) and \(W^{(n)\perp}\).

The convergence of the recursive scheme is based on properties (P1)-(P2)-(P3) below.

- **(P1) (Regularity)** \(\Gamma(\cdot, \cdot, \cdot) \in C^\infty([0, \delta_0) \times B(2R; V_1) \times B(1; W \cap X_{\sigma,s}, X_{\sigma,s})\). Moreover, $$D\Gamma$$ and \(D^2\Gamma\) are bounded on \([0, \delta_0) \times B(2R; V_1) \times B(1; W \cap X_{\sigma,s})\).

(P1) is a consequence of the \(C^\infty\)-regularity of the Nemistky operator induced by \(g(\delta, x, u)\) on \(X_{\sigma,s}\) and of the properties of the map \(v_2\) stated in Lemma 2.1.

- **(P2) (Smoothing estimate)** \(\forall w \in W^{(n)\perp} \cap X_{\sigma,s}\) and \(\forall 0 \leq \sigma' \leq \sigma, ||w||_{\sigma',s} \leq \exp(-L_n(\sigma-\sigma')) ||w||_{\sigma,s}\).

The next property (P3) is an invertibility property of the linearized operator \(L_n(\delta, v_1, w) : W^{(n)} \rightarrow W^{(n)}\) defined by

\[
L_n(\delta, v_1, w)[h] := L_\omega h - \varepsilon P_n \Pi W D_n \Gamma(\delta, v_1, w)[h]
\]

where \(w\) is the approximate solution obtained at a given step of the Nash-Moser iteration.

The invertibility of \(L_n(\delta, v_1, w)\) is obtained excising the set of parameters \((\delta, v_1)\) for which 0 is an eigenvalue of \(L_n(\delta, v_1, w)\). Moreover, in order to have bounds for the norm of the inverse operator \(L_n^{-1}(\delta, v_1, w)\) which are sufficiently good for the recursive scheme, we also excise the parameters \((\delta, v_1)\) for which the eigenvalues of \(L_n(\delta, v_1, w)\) are too small. We prefix some definitions.

**Definition 3.1 (Mean value)** For \(\Omega := \mathbf{T} \times (0, \pi)\) we define

\[
M(\delta, v_1, w) := \frac{1}{|\Omega|} \int_{\Omega} \partial_u g(\delta, x, v_1(t, x) + w(t, x) + v_2(\delta, v_1, w)(t, x)) \, dx \, dt.
\]

**Definition 3.2** We define for \(1 < \tau < 2\)

\[
[w]_{\sigma,s} := \inf \left\{ \sum_{i=0}^{q} \frac{||h_i||_{\sigma_i,s}}{\alpha_i^{\beta - 1}} : q \geq 1, \ 0 \geq \sigma_i > \sigma, \ h_i \in W^{(i)} \text{ with } w = \sum_{i=0}^{q} h_i \right\}
\]

where \(\beta := \frac{2 \tau}{\tau - 1}\) and we set \([w]_{\sigma,s} := \infty\) if the above set is empty.
Definition 3.3 (First order Melnikov non-resonance condition) Let $0 < \gamma < 1$ and $1 < \tau < 2$. We define (recall that $\omega = \sqrt{2\delta^{p-1} + 1}$ and $\varepsilon = \delta^{p-1}$)

$$\Delta_n^{\gamma,\tau}(v_1, w) := \left\{ \delta \in [0, \delta_0) \mid |\omega k - j| \geq \frac{\gamma}{(k + j)^{\tau}}, \left| \omega k - j - \varepsilon \frac{M(\delta, v_1, w)}{2j} \right| \geq \frac{\gamma}{(k + j)^{\tau}} \right\}$$

for all $k \in \mathbb{N}, j \geq 1, k \neq j, \frac{1}{3|\varepsilon|} < k, k \leq L_n, j \leq 2L_n$. We claim that:

- **(P3)** (Invertibility of $L_n$) There exist positive constants $\mu$, $\delta_0$ and $C$ such that, if $|w|_{s,s} \leq \mu$, $||v_1||_{0,s} \leq 2R$ and $\delta \in \Delta_n^{\gamma,\tau}(v_1, w) \cap [0, \delta_0)$ for some $0 < \gamma < 1$, $1 < \tau < 2$, then $L_n(n, v_1, w)$ is invertible and the inverse operator $L_n^{-1}(\delta, v_1, w) : W^{(n)} \rightarrow W^{(n)}$ satisfies

$$\left\| L_n^{-1}(\delta, v_1, w)[h] \right\|_{s,s} \leq \frac{C}{\gamma} (L_n)^{\tau-1} ||h||_{s,s}.$$ (18)

Property (P3) is the core of the convergence proof and where the analysis of the small divisors enters into play. Property (P3) is proved in section [4].

3.1 The Nash-Moser scheme

We are going to define recursively a sequence $\{w_n\}_{n \geq 0}$ with $w_n = w_n(\delta, v_1) \in W^{(n)}$, $w_0 = 0$, defined on smaller and smaller sets of “non-resonant” parameters $(\delta, v_1)$, $A_0 \subseteq A_{n-1} \subseteq \ldots \subseteq A_1 \subseteq A_0 := \{(\delta, v_1) \mid \delta \in [0, \delta_0), ||v_1||_{0,s} \leq 2R\}$. The sequence $(w_n(\delta, v_1))$ will converge to a solution $w(\delta, v_1)$ of the (P)-equation (15) for $(\delta, v_1) \in A_\infty := \cap_{n \geq 1} A_n$. The main goal of the construction is to show that, at the end of the recurrence, the set of parameters $A_\infty := \cap_{n \geq 1} A_n$ for which we have the solution $w(\delta, v_1)$, remains sufficiently large.

We define inductively the sequence $\{w_n\}_{n \geq 0}$. Define the “loss of analyticity” $\gamma_n$ by

$$\gamma_n := \frac{\gamma_0}{n^2 + 1}, \quad \sigma_0 = \sigma, \quad \sigma_{n+1} = \sigma_n - \gamma_n, \quad \forall \ n \geq 0,$$

where we choose $\gamma_0 > 0$ small such that the “total loss of analyticity” $\sum_{n \geq 0} \gamma_n = \gamma_0 \sum_{n \geq 0} 1/(n^2 + 1) \leq \sigma/2$, i.e. $\lim_{n \rightarrow +\infty} \sigma_n \geq \sigma/2 > 0$. We also assume

$$L_n := L_0 2^n, \quad \forall \ n \geq 0,$$

for some large integer $L_0$ specified in the next proposition.

Proposition 3.1 (Induction) Let $w_0 = 0$ and $A_0 := \{(\delta, v_1) \mid \delta \in [0, \delta_0), ||v_1||_{0,s} \leq 2R\}$. There exists $\varepsilon_0 := \varepsilon_0(\gamma, \tau)$, $L_0 := L_0(\gamma, \tau) > 0$ such that $\forall |\varepsilon| \gamma^{-1} < \varepsilon_0$, there exists a sequence $\{w_n\}_{n \geq 0}$, $w_n = w_n(\delta, v_1) \in W^{(n)}$, of solutions of the equation

$$(P_n) \quad L_0 w_n - \varepsilon P_{\varepsilon}^0 \Pi W \Gamma(\delta, v_1, w_n) = 0,$$

defined for $(\delta, v_1) \in A_n \subseteq A_{n-1} \subseteq \ldots \subseteq A_1 \subseteq A_0$, with $w_n(\delta, v_1) = \sum_{i=0}^{n} h_i(\delta, v_1)$, $h_0 := w_0 = 0$, $h_i = h_i(\delta, v_1) \in W^{(i)}$ satisfying $||h_i||_{\sigma_i,s} \leq |\varepsilon| \gamma^{-1} \exp(-\chi^i)$ for some $1 < \chi < 2$, $\forall i = 0, \ldots, n$.

Proof. Fix some $\chi \in (1, 2)$. We assume $\varepsilon_0 \gamma^{-1} > 0$ small enough such that

$$\sum_{i \geq 0} \frac{\varepsilon_0}{\gamma^i} \exp(-\chi^i)(1 + i^2)^{\frac{2(i+1)}{\gamma_0}} < \mu.$$ (19)

The proof proceeds by induction. We recall that $\beta := (2 - \tau)/2$ and that $\mu$ is defined in property (P3).
Suppose we have already defined a solution \( w_n = w_n(\delta, v_1) \in W^{(n)} \) of equation \((P_n)\) satisfying the properties stated in the proposition. We want to define

\[
w_{n+1} = w_{n+1}(\delta, v_1) := w_n(\delta, v_1) + h_{n+1}(\delta, v_1), \quad h_{n+1}(\delta, v_1) \in W^{(n+1)}\]

as an exact solution of the equation

\[
(P_{n+1}) \quad L_\omega w_{n+1} - \varepsilon P_{n+1} \Pi W \Gamma(\delta, v_1, w_{n+1}) = 0.
\]

In order to find a solution \( w_{n+1} = w_n + h_{n+1} \) of equation \((P_{n+1})\) we write, for \( h \in W^{(n+1)} \),

\[
L_\omega (w_n + h) - \varepsilon P_{n+1} \Pi W \Gamma(\delta, v_1, w_n + h) = L_\omega w_n - \varepsilon P_{n+1} \Pi W \Gamma(\delta, v_1, w_n) + L_\omega h - \varepsilon P_{n+1} \Pi W D_\omega \Gamma(\delta, v_1, w_n)[h] + R(h) = r_n + L_{n+1}(\delta, v_1, w_n)[h] + R(h),
\]

where, since \( w_n \) solves equation \((P_n)\),

\[
\begin{cases}
  r_n := L_\omega w_n - \varepsilon P_{n+1} \Pi W \Gamma(\delta, v_1, w_n) = -\varepsilon P_{n+1} \Pi W \Gamma(\delta, v_1, w_n) \\
  R(h) := -\varepsilon P_{n+1} \Pi W \left(\Gamma(\delta, v_1, w_n + h) - \Gamma(\delta, v_1, w_n) - D_\omega \Gamma(\delta, v_1, w_n)[h]\right)
\end{cases}
\]

satisfy, by properties \((P1)\) and \((P2)\),

\[
||r_n||_{\sigma_{n+1, s}} \leq |\varepsilon| C \exp (-L_n \gamma_n) \left||P_{n+1} \Pi W \Gamma(\delta, v_1, w_n)\right||_{\sigma_{n, s}} \leq |\varepsilon| C' \exp (-L_n \gamma_n) \quad (22)
\]

and, by property \((P1)\), \( \forall h, h' \in W^{(n+1)} \) with \( ||h||_{\sigma_{n+1, s}} \), \( ||h'||_{\sigma_{n+1, s}} \) small enough

\[
\left\{ \begin{array}{l}
  ||R(h)||_{\sigma_{n+1, s}} \leq C |\varepsilon| ||h||_{\sigma_{n+1, s}}^2 \\
  ||R(h) - R(h')||_{\sigma_{n+1, s}} \leq C |\varepsilon| (||h||_{\sigma_{n+1, s}} + ||h'||_{\sigma_{n+1, s}}) ||h - h'||_{\sigma_{n+1, s}}.
\end{array} \right. \quad (23)
\]

Since \( ||h_i||_{\sigma_{i, s}} \leq |\varepsilon| \gamma^{-1} \exp (-\chi^i), \forall i = 0, \ldots, n \), by \((12)\),

\[
[w_n]_{\sigma_{n+1, s}} \leq \sum_{i=0}^{n} \frac{||h_i||_{\sigma_{i, s}}}{\sigma_i - \sigma_{n+1}} \leq \frac{|\varepsilon|}{\gamma} \sum_{i=0}^{n} \frac{\exp(-\chi^i)}{2(\gamma^i-1)/\gamma} \leq \frac{|\varepsilon|}{\gamma} \sum_{i=0}^{n} \exp(-\chi^i) \left(1 + i^2 \frac{2^{i+1}}{\gamma_0}ight) < \mu. \quad (24)
\]

Hence by property \((P3)\), the linear operator \( L_{n+1}(\delta, v_1, w_n) : W^{(n+1)} \to W^{(n+1)} \) is invertible for \( (\delta, v_1) \) restricted to the set of parameters

\[
A_{n+1} := \left\{ (\delta, v_1) \in A_n \mid \delta \in \Delta_{n+1}(v_1, w_n) \right\} \subseteq A_n.
\]

Moreover

\[
\left||L_{n+1}(\delta, v_1, w_n)^{-1}\right||_{\sigma_{n+1, s}} \leq \frac{C}{\gamma} (L_{n+1})^{(n+1)} \quad \forall (\delta, v_1) \in A_{n+1}. \quad (25)
\]

By \((20)\) equation \((P_n)\) is equivalent to the fixed point problem

\[
w_{n+1} = w_n + h, \quad G(\delta, v_1, w_n, h) = h, \quad (27)
\]

for the nonlinear operator \( G(\delta, v_1, w_n, \cdot) : W^{(n+1)} \to W^{(n+1)} \), defined by

\[
G(\delta, v_1, w_n, h) := -L_{n+1}(\delta, v_1, w_n)^{-1}(r_n + R(h)).
\]

To complete the proof of the proposition we need the following Lemma.

**Lemma 3.1 (Contraction)** \( G(\delta, v_1, w_n, \cdot) \) maps the ball \( B(\rho_{n+1}; W^{(n+1)}) := \{ w \in W^{(n+1)} \mid ||w||_{\sigma_{n+1, s}} \leq \rho_{n+1} \} \) of radius \( \rho_{n+1} := |\varepsilon| \gamma^{-1} \exp (-\chi^{n+1}) \) into itself and is a contraction in this ball.
Proof. By (28), (24), and (28),

\[ ||G(\delta, v_1, w_n, h)||_{s_{n+1}} = ||L_{n+1}(\delta, v_1, w_n)^{-1}(r_n + R(h))||_{s_{n+1}} \leq \frac{C}{\gamma}(L_{n+1})^{-1}\left(||r_n||_{s_{n+1}} + ||R(h)||_{s_{n+1}}\right) \leq \frac{C}{\gamma}(L_{n+1})^{-1}\left(||\varepsilon\exp(-L_n\gamma_n) + |\varepsilon| |h|^2_{s_{n+1}}\right). \]  \hspace{1cm} (28)

By (28), if \(||h||_{s_{n+1}} \leq \rho_{n+1}\) then \(||G(\delta, v_1, w_n, h)||_{s_{n+1}} \leq C'(L_{n+1})^{-1}\gamma^{-1}|\varepsilon|(|\exp(-L_n\gamma_n) + \rho_{n+1}^2) \leq \rho_{n+1}\), provided that

\[ \frac{C}{\gamma}(L_{n+1})^{-1}|\varepsilon| \exp(-L_n\gamma_n) \leq \frac{\rho_{n+1}}{2} \quad \text{and} \quad \frac{C}{\gamma}(L_{n+1})^{-1}|\varepsilon|\rho_{n+1} \leq \frac{1}{2}. \]  \hspace{1cm} (29)

It is easy to check that for \(L_{n+1} := L_02^{n+1}\) and \(\rho_{n+1} := |\varepsilon|\gamma^{-1} \exp(-\chi^{-1})\) both inequalities in (29) are satisfied \(\forall n \geq 0\), choosing \(L_0\) large enough and \(|\varepsilon|\gamma^{-1} \leq \varepsilon_0\) small enough. With similar estimates, using (28), we can prove that \(\forall n, h' \in B(\rho_{n+1}; W^{(n+1)})\), \(||G(\delta, v_1, w_n, h') - G(\delta, v_1, w_n, h)||_{s_{n+1}} \leq \exp(-\chi^{-1})|h - h'|_{s_{n+1}}\) again for \(L_0\) large enough and \(|\varepsilon|\gamma^{-1} \leq \varepsilon_0\) small enough, uniformly in \(n\), and we conclude that \(G(\delta, v_1, w_n, \cdot)\) is a contraction on \(B(\rho_{n+1}; W^{(n+1)})\). \(\blacksquare\)

By the standard Contraction Mapping Theorem we deduce the existence, for \(L_0\) large enough and \(\varepsilon\) small enough, of a unique \(h_{n+1} \in W^{(n+1)}\) solving (27) and satisfying

\[ ||h_{n+1}||_{s_{n+1}} \leq \rho_{n+1} = \frac{|\varepsilon|}{\gamma} \exp(-\chi^{-1}). \]

Summarizing, \(w_{n+1}(\delta, v_1) = w_n(\delta, v_1) + h_{n+1}(\delta, v_1)\) is a solution in \(W^{(n+1)}\) of equation \((P_{n+1})\), defined for \((\delta, v_1) \in A_{n+1} \subseteq A_n \subseteq \ldots \subseteq A_1 \subseteq A_0\), and \(w_{n+1}(\delta, v_1) = \sum_{i=1}^{n+1} h_i(\delta, v_1), h_i = h_i(\delta, v_1) \in W^{(i)}\) satisfying \(||h_i||_{s_i} \leq |\varepsilon|\gamma^{-1} \exp(-\chi^{-1})\) for some \(\chi \in (1, 2), \forall i = 0, \ldots, n + 1\). \(\blacksquare\)

Remark 3.1 A difference with respect to the usual “quadratic” Nash-Moser scheme, is that \(h_n(\delta, v_1)\) is found as an exact solution of equation \((P_n)\). It appears to be more convenient to prove the regularity of \(h_n(\delta, v_1)\) with respect to the parameters \((\delta, v_1)\), see Lemma 3.2.

Corollary 3.1 (Solution of the \((P)\)-equation) For \((\delta, v_1) \in A_\infty := \cap_{n \geq 0} A_n, \sum_{i=0}^{\infty} h_i(\delta, v_1)\) converges normally in \(X_{\overline{\gamma}/2,s}\) to a solution \(w(\delta, v_1) \in W \cap X_{\overline{\gamma}/2,s}\) of equation (17), with \(||w(\delta, v_1)||_{\overline{\gamma}/2,s} \leq C|\varepsilon|\gamma^{-1}.\)

Proof. By proposition 3.1 for \((\delta, v_1) \in A_\infty := \cap_{n \geq 0} A_n, \sum_{i=0}^{\infty} ||h_i(\delta, v_1)||_{\overline{\gamma}/2,s} < \infty\). Hence \(\sum_{i \geq 0} h_i(\delta, v_1)\) converges normally in \(X_{\overline{\gamma}/2,s}\) to some \(w(\delta, v_1) \in W \cap X_{\overline{\gamma}/2,s}\), and we have

\[ ||w(\delta, v_1)||_{\overline{\gamma}/2,s} \leq \sum_{i \geq 0} ||h_i(\delta, v_1)||_{\overline{\gamma}/2,s} \leq \sum_{i \geq 0} ||h_i(\delta, v_1)||_{s_{i},s} \leq \sum_{i \geq 0} |\varepsilon|\gamma^{-1} \exp(-\chi^{-1}) = O(|\varepsilon|\gamma^{-1}). \]  \hspace{1cm} (30)

Let us justify that \(L_\omega w = \varepsilon \Pi W \Gamma(\delta, v_1, w)\). Since \(w_n\) solves equation \((P_n)\),

\[ L_\omega w_n = \varepsilon P_n \Pi W \Gamma(\delta, v_1, w_n) = \varepsilon \Pi W \Gamma(\delta, v_1, w_n) - \varepsilon P_n^\perp \Pi W \Gamma(\delta, v_1, w_n). \]  \hspace{1cm} (31)

We have

\[ P_n^\perp \Pi W \Gamma(\delta, v_1, w_n) \leq C \exp(-L_n(\sigma_n - (\overline{\sigma}/2))) \leq C \exp(-\gamma_0 L_02^{n}/(n^2 + 1)). \]

Hence, by (P1), the right hand side in (31) converges in \(X_{\overline{\gamma}/2,s}\) to \(\Gamma(\delta, v_1, w)\). Moreover \((L_\omega w_n) \to L_\omega w\) in the sense of distributions. Hence \(L_\omega w = \varepsilon \Pi W \Gamma(\delta, v_1, w)\). \(\blacksquare\)

Before proving the key property (P3) on the linearized operator we prove a “Whitney-differentiability” property for \(w(\delta, v_1)\) extending \(w(\cdot, \cdot)\) in a smooth way on the whole \(A_0\).

For this, some bound on the derivatives of \(h_n = w_n - w_{n-1}\) is required.
Lemma 3.2 (Estimates for the derivatives of \( h_n \) and \( w_n \)) For \( |\varepsilon|\gamma^{-1} \) small enough and \( L_0 \) large enough, \( \forall n \geq 0 \), the function \( (\delta, v_1) \rightarrow h_n(\delta, v_1) \) is in \( C^\infty(A_n, W^{(n)}) \) and the \( k^{th} \)-derivative \( D^kh_n(\delta, v_1) \) satisfies

\[
\left|D^kh_n(\delta, v_1)\right|_{\sigma_n,s} \leq \frac{|\varepsilon|}{\gamma} K_1(k, \chi) \exp(-\chi n),
\]

for any \( \chi \in (0, \chi) \) and a suitable positive constant \( K_1(k, \chi) \).

As a consequence, the function \( (\delta, v_1) \rightarrow w_n(\delta, v_1) = \sum_{i=1}^{n} h_n(\delta, v_1) \) is in \( C^\infty(A_n, W^{(n)}) \) and the \( k^{th} \)-derivative \( D^kw_n(\delta, v_1) \) satisfies

\[
\left|D^kw_n(\delta, v_1)\right|_{\sigma_n,s} \leq \frac{|\varepsilon|}{\gamma} K_2(k),
\]

for a suitable positive constant \( K_2(k) \).

**Proof.** First, \( ||\partial_\lambda h_0||_{\sigma_n,s} = ||\partial_\lambda w_0||_{\sigma_n,s} = 0 \) (we denote \( \lambda := (\delta, v_1) \)). Next, assume, by induction, that \( h_n = h_n(\delta, v_1) \) is a \( C^\infty \) map defined in \( A_n \). We shall prove that \( h_{n+1} = h_{n+1}(\delta, v_1) \) is \( C^\infty \) too. First recall that \( h_{n+1} = h_{n+1}(\delta, v_1) \) is defined, in Proposition 3.1, for \( (\delta, v_1) \in A_{n+1} \) as a solution of \( W^{(n+1)} \) of equation (34), namely

\[
(P_{n+1}) \quad U_{n+1}(\delta, v_1, h_{n+1}(\delta, v_1)) = 0,
\]

where

\[
U_{n+1}(\delta, v_1, h) := L_\omega(w_n + h) - \varepsilon P_{n+1} \Pi W \Gamma(\delta, v_1, w_n + h).
\]

The map \( U_{n+1} : [0, \delta_0] \times V_1 \times W^{(n+1)} \rightarrow W^{(n+1)} \) is \( C^\infty \), and we claim that \( D_\lambda U_{n+1}(\delta, v_1, h_{n+1}) = L_{n+1}(\delta, v_1, w_{n+1}) \) is invertible and that

\[
\left|L_{n+1}(\delta, v_1, w_{n+1})^{-1}\right|_{\sigma_{n+1,s}} \leq \frac{C'}{\gamma} (L_{n+1})^{-1}.
\]

As a consequence, by the Implicit Function Theorem, the map \( (\delta, v_1) \rightarrow h_{n+1}(\delta, v_1) \) is in \( C^\infty(A_{n+1}, W^{(n+1)}) \).

Let us prove (34). By Proposition 3.1 \( ||h_{n+1}||_{\sigma_{n+1,s}} \leq C|\varepsilon|\gamma^{-1} \exp(-\chi n) \). Hence, by (P1),

\[
\left|L_{n+1}(\delta, v_1, w_{n+1}) - L_{n+1}(\delta, v_1, w_n)\right|_{\sigma_{n+1,s}} \leq C|\varepsilon| \left| h_{n+1} \right|_{\sigma_{n+1,s}} \leq C \frac{\gamma^2}{\gamma} \exp(-\chi n+1).
\]

Recalling (26), \( L_{n+1}(\delta, v_1, w_n) \) is invertible and \( ||L_{n+1}(\delta, v_1, w_n)^{-1}||_{\sigma_{n+1,s}} \leq (L_{n+1})^{-1} \). Hence, provided that \( \varepsilon \gamma^{-1} \) is small enough (note that \( (L_{n+1})^{-1} = (L_0 2^n)^{-1} << \exp(\chi n) \) for \( n \) large), \( L_{n+1}(\delta, v_1, w_{n+1}) \) is invertible and (34) holds.

We now prove in detail estimate (32) for \( k = 1 \). Deriving equation \( (P_{n+1}) \) with respect to some coordinate \( \lambda \) of \( (\delta, v_1) \in A_{n+1}, \) we obtain

\[
(P'_{n+1}) \quad L_{n+1}(\delta, v_1, w_{n+1}) \left[ \partial_\lambda h_{n+1}(\delta, v_1) \right] = -\left( \partial_\lambda U_{n+1} \right)(\delta, v_1, h_{n+1}(\delta, v_1)).
\]

Hence, by (34), \( \partial_\lambda h_{n+1} \) satisfies the estimate

\[
\left|\partial_\lambda h_{n+1}\right|_{\sigma_{n+1,s}} \leq \frac{C}{\gamma} (L_{n+1})^{-1} \left| \left( \partial_\lambda U_{n+1} \right)(\delta, v_1, h_{n+1}) \right|_{\sigma_{n+1,s}}.
\]

There holds

\[
(\partial_\lambda U_{n+1})(\delta, v_1, h) = L_\omega \partial_\lambda w_n - \varepsilon P_{n+1} \Pi W \left[ (\partial_\lambda \Gamma)(\delta, v_1, w_n + h) + \partial_\lambda \Gamma(\delta, v_1, w_n + h)[\partial_\lambda w_n] \right]
\]

that we can write as

\[
(\partial_\lambda U_{n+1})(\delta, v_1, h) = (\partial_\lambda U_{n+1})(\delta, v_1, 0) + r(\delta, v_1, h),
\]

where
where \( r(\delta, v_1, h) := (\partial_\lambda U_{n+1})(\delta, v_1, h) - (\partial_\lambda U_{n+1})(\delta, v_1, 0) \) satisfies, by (P1),

\[
\left\| r(\delta, v_1, h) \right\|_{\sigma_{n+1}, s} \leq C|\varepsilon| \left\| h \right\|_{\sigma_{n+1}, s} \left( 1 + \left\| \partial_\lambda w_n \right\|_{\sigma_{n+1}, s} \right).
\]

(38)

Now, since \( w_n = w_n(\delta, v_1) \in W^{(n)} \) solves equation \((P_n)\),

\[
P_n U_{n+1}(\delta, v_1, 0) = L_w w_n - \varepsilon P_n \Pi W \Gamma(\delta, v_1, w_n) = 0, \forall (\delta, v_1) \in A_n.
\]

Hence differentiating w.r.t. \( \lambda \) we get \( P_n (\partial_\lambda U_{n+1})(\delta, v_1, 0) = 0 \), and so

\[
(\partial_\lambda U_{n+1})(\delta, v_1, 0) = P_n^+ (\partial_\lambda U_{n+1})(\delta, v_1, 0) = P_n^+ L_w \partial_\lambda w_n - \varepsilon P_n^+ P_{n+1} \Pi W \tilde{\Gamma}(\delta, v_1) = -\varepsilon P_n^+ P_{n+1} \Pi W \tilde{\Gamma}(\delta, v_1),
\]

(39)

where, by \((30)\), \( \tilde{\Gamma}(\delta, v_1) := (\partial_\lambda \Gamma)(\delta, v_1, w_n) + \partial_\lambda \Gamma(\delta, v_1, w_n) [\partial_\lambda w_n] \). By \((39), (P2)\), (P1)

\[
\left\| (\partial_\lambda U_{n+1})(\delta, v_1, 0) \right\|_{\sigma_{n+1}, s} \leq \varepsilon \left\| \exp(-L_n \gamma_n) \right\|_{\Pi W \tilde{\Gamma}(\delta, v_1)} \left\| \sigma_{n,s} \right\| \leq C|\varepsilon| \left\| \exp(-L_n \gamma_n) \right\| \left( 1 + \left\| \partial_\lambda w_n \right\|_{\sigma_{n,s}} \right).
\]

(40)

Combining \((32)\), \((37), (38)\), and the bound \( \|h_{n+1}\|_{\sigma_{n+1}, s} \leq |\varepsilon| \gamma^{-\lambda - 1} \exp(-\chi n^{\lambda - 1}) \), we get

\[
\left\| \partial_\lambda h_{n+1} \right\|_{\sigma_{n+1}, s} \leq \frac{C}{\gamma} (L_{n+1})^{-2 \lambda - 1} \left( \varepsilon \left\| \Gamma(\delta, v_1, 0) \right\|_{\Pi W \Gamma(\delta, v_1)} \left( 1 + \left\| \partial_\lambda w_n \right\|_{\sigma_{n,s}} \right) \right) \leq C(\chi) \frac{|\varepsilon|}{\gamma} \left\| \exp(-\chi n^{\lambda - 1}) \right\| \left( 1 + \sum_{i=0}^{n} \left\| \partial_\lambda h_i \right\|_{\sigma_{n,s}} \right)
\]

for any \( \chi \in (1, \chi), \varepsilon \gamma^{-\lambda} \) small and \( L_0 \) large. Hence, setting \( a_n := \left\| \partial_\lambda h_n \right\|_{\sigma_{n,s}} \) we get

\[
a_0 = 0 \quad \text{and} \quad a_{n+1} \leq C(\chi) \frac{|\varepsilon|}{\gamma} \left\| \exp(-\chi n^{\lambda - 1}) \right\| \left( 1 + a_0 + \ldots + a_n \right)
\]

which implies \( \left\| \partial_\lambda h_n \right\|_{\sigma_{n,s}} = a_n \leq K(\chi) |\varepsilon| \gamma^{-\lambda - 1} \exp(-\chi n^{\lambda - 1}) \), \forall n \geq 0, for a suitable positive constant \( K(\chi) \). We can prove in the same way the general estimate \((32)\) from which \((33)\) follows.

Since, by \((32)\), \( h_n(\delta, v_1) = O(\varepsilon \gamma^{-\lambda - 1} \exp(-\chi n^{\lambda - 1})) \) with all its derivatives, and the “non-resonant” set \( A_n \) is obtained at each step deleting strips of size \( O(\gamma/L_0^\lambda) \), we can define (by interpolation, say) a \( C^\infty \)-extension \( \bar{w}(\delta, v_1) \) of \( w(\delta, v_1) \) for all \( (\delta, v_1) \in A_0 \). More precisely we can prove:

**Lemma 3.3 (Smooth Extension \( \bar{w} \) of \( w \) on \( A_0 \))** Given \( \nu > 0 \), there exists a function \( \bar{w} \in C^\infty(A_0, W \cap X_{\sigma, 2}) \) such that, if \( (\delta, v_1) \in A_\infty := \cap_{n \geq 0} A_n \) and \( \dist((\delta, v_1), \partial A_n) \geq 2\nu/L_0^\lambda \), \forall n \geq 0, then \( \bar{w}(\delta, v_1) \) solves the (P) equation \((32)\).

More precisely \( \bar{w}(\delta, v_1) \) satisfies, \forall \( k \in \mathbb{N} \),

\[
\left\| D^k \bar{w}(\delta, v_1) \right\|_{\sigma/2, s} \leq \frac{|\varepsilon| C(k)}{\nu^k}, \quad \forall (\delta, v_1) \in A_0
\]

(41)

for suitable constants \( C(k) > 0 \). Moreover \( \bar{w}(\delta, v_1) := \lim_{n \to \infty} \bar{w}_n(\delta, v_1) \) where \( \bar{w}_n \) is in \( C^\infty(A_0, W^{(n)}) \), and the sequence \( (\bar{w}_n) \) converges uniformly in \( A_0 \) to \( \bar{w} \) for the norm \( \| \sigma/2, s \) more precisely

\[
\forall (\delta, v_1) \in A_0, \quad \left\| \bar{w}(\delta, v_1) - \bar{w}_n(\delta, v_1) \right\|_{\sigma/2, s} \leq C \frac{|\varepsilon|}{\gamma^k} \exp(-\chi n^{\lambda - 1}).
\]

(42)
Proof. First we endow \( \mathbf{R} \times V_1 \) with the Borelian positive measure defined by \( \mu(E) = m(L^{-1}(E)) \), where \( L \) is some automorphism from \( \mathbf{R}^{N+1} \) to \( \mathbf{R} \times V_1 \) and \( m \) is the Lebesgue measure in \( \mathbf{R}^{N+1} \). Let \( \varphi: \mathbf{R} \times V_1 \to \mathbf{R}_+ \) be a \( C^\infty \)-map supported in the open ball of radius 1 centered at 0 with \( \int \varphi \, d\mu = 1 \). Let

\[
\tilde{A}_n := \left\{ \lambda = (\delta, v_1) \in A_n \mid \text{dist}(\lambda, \partial A_n) \geq \frac{\nu}{L_n^3} \right\} \subset A_n,
\]

where \( \nu \) is some small constant to be specified later. We define \( \varphi_n, \psi_n: \mathbf{R} \times V_1 \to \mathbf{R}_+ \) as

\[
\varphi_n(\lambda) := \left( \frac{L_n^3}{\nu} \right)^{N+1} \varphi \left( \frac{L_n^3}{\nu} \lambda \right), \quad \psi_n(\lambda) := \int_{\tilde{A}_n} \varphi_n(\lambda - \eta) \, d\mu(\eta) = \left( \varphi_n \ast \delta_{\tilde{A}_n} \right)(\lambda),
\]

where \( \delta_{\tilde{A}_n} \) is the characteristic function of the set \( \tilde{A}_n \), namely, \( \delta_{\tilde{A}_n}(\lambda) := 1 \) if \( \lambda \in \tilde{A}_n \) and \( \delta_{\tilde{A}_n}(\lambda) := 0 \) if \( \lambda \notin \tilde{A}_n \). Clearly \( \varphi_n \) is a \( C^\infty \) map supported in the open ball of radius \( \nu/L_n^3 \) centered at 0 and \( \int \varphi_n \, d\mu = 1 \).

It follows that \( 0 \leq \psi_n(\lambda) \leq 1 \), \( \text{supp} \psi_n \subset \text{int} A_n \) and \( \psi_n(\lambda) = 1 \) if \( \lambda \in \tilde{A}_n \) satisfies \( \text{dist}(\lambda, \partial A_n) \geq 2\nu/L_n^3 \). Moreover \( \psi_n \) is \( C^\infty \) and it is easy to check that \( |D^k \psi_n(\lambda)| \leq C(k)(L_n^3/\nu)^k, \forall k \in \mathbf{N} \), \( \forall \lambda \in \mathbf{R} \times V_1 \) for a suitable positive constant \( C(k) \).

Now we can define \( \tilde{w}_n: A_0 \to W(n) \) by

\[
\tilde{w}_n(\lambda) := w_0(\lambda) = 0, \quad \tilde{w}_{n+1}(\lambda) := \tilde{w}_n(\lambda) + \tilde{h}_{n+1}(\lambda)
\]

where \( \tilde{h}_{n+1}(\lambda) := \psi_{n+1}(\lambda)h_{n+1}(\lambda) \) if \( \lambda \in A_{n+1} \) and \( \tilde{h}_{n+1}(\lambda) := 0 \) if \( \lambda \notin A_{n+1} \). (note that \( h_{n+1} \) is \( C^\infty \) because \( \text{supp} \psi_{n+1} \subset \text{int} A_{n+1} \)). We define \( \tilde{w}_n \in C^\infty(A_0, W(n)) \) by \( \tilde{w}_n(\lambda) := \sum_{i=1}^{n} \tilde{h}_i(\lambda) \).

By the bound \( |D^k \psi_n(\lambda)| \leq C(k)(L_n^3/\nu)^k \), \forall k \in \mathbf{N}, \forall \lambda \in A_0, \forall n \geq 0 \).

As a consequence, the sequence \((\tilde{w}_n)\) (and all its derivatives) converges uniformly in \( A_0 \) for the norm \( \| \|_{2,\gamma,s} \) on \( W_n \), to some function \( \tilde{w}(\delta, v_1) \in C^\infty(A_0, W \cap X_{2,\gamma,s}) \) which satisfies \( (11) \) and \( (12) \).

Note that if \( \lambda \notin A_{\infty} := \cap_{n \geq 0} A_n \) then the series \( \tilde{w}(\lambda) = \sum_{n \geq 1} \tilde{h}_n(\lambda) \) is a finite sum. On the other hand, if \( \lambda \in A_{\infty} \) and \( \text{dist}(\lambda, \partial A_n) \geq 2\nu/L_n^3, \forall n \geq 0 \), then \( \tilde{w}(\lambda) = \tilde{w}(\lambda) \) solves the \( (P) \)-equation \( (15) \).

We complete this part with the following Lemma which will be used in section \( 5 \) Define

\[
B_n := \left\{ (\delta, v_1) \in A_0 \mid \delta \in \Delta_{n+1}^{2\gamma/\tau}(v_1, \tilde{w}(\delta, v_1)) \right\}
\]

where we have replaced \( \gamma \) with \( 2\gamma \) in the definition of \( \Delta_{n+1}^{2\gamma/\tau} \), see Definition \( 3.3 \).

Lemma 3.4 If \( \nu \gamma^{-1} > 0 \) and \( \gamma^{-1} \) are small enough, then \( B_n \subset \{ (\delta, v_1) \in A_n \mid \text{dist}(\delta, v_1, \partial A_n) \geq 2\nu/L_n^3 \}, \forall n \geq 1 \) and hence, if \( (\delta, v_1) \in B_{\infty} := \cap_{n \geq 1} B_n \), then \( \tilde{w}(\delta, v_1) \) solves the \( (P) \)-equation \( (15) \).

Proof. This is a consequence of \( (11), (12) \) and the previous Lemma (we use that \( L^0_n = o(\exp(\tilde{\lambda}^n)) \) and \( L^0_n = o(L^n_0) \) as \( n \to \infty \)).

Remark 3.2 Up to now, we have not justified that \( A_{\infty}, B_{\infty} \) are not empty. It can be proved as in Proposition \( 5.1 \) that for any \( v_1 \in B(2R, V_1) \) the set \( B_{v_1} := \{ \delta \in [0, \delta_0] \mid (\delta, v_1) \in B_{\infty} \} \) satisfies \( \lim_{\eta \to 0^+} \text{meas}(0, \eta) \cap B_{v_1}/\eta = 0 \). Hence \( A_{\infty} \neq \emptyset \).

4 Analysis of the linearized problem: proof of \( (P3) \)

We prove in this section the key property \( (P3) \) on the inversion of the linear operator \( L_n(\delta, v_1, w) : W(n) \to W(n) \) defined in \( (17) \). Let

\[
a(t, x) := \partial_n g \left( \delta, x, v_1(t, x) + w(t, x) + v_2(\delta, v_1, w)(t, x) \right)
\]
and define the linear operators \( D, M_1, M_2 : W^{(n)} \to W^{(n)} \) by

\[
\begin{align*}
Dh & := L_\omega h - \varepsilon P_n \Pi_W (a_0(x) h) \\
M_1 h & := \varepsilon P_n \Pi_W (\varpi(t,x) h) \\
M_2 h & := \varepsilon P_n \Pi_W (a(t,x) \partial_w v_2 [h])
\end{align*}
\]

where

\[
\begin{align*}
\{ a_0(x) & := (1/2\pi) \int_0^{2\pi} a(t,x) \ dt \\
\varpi(t,x) & := a(t,x) - a_0(x).
\end{align*}
\]

\[\mathcal{L}_n(\delta, v, w) \text{ can be written as}
\]

\[
\mathcal{L}_n(\delta, v, w)[h] := L_\omega h - \varepsilon P_n \Pi_W D_w \Gamma(\delta, v_1, w)[h]
\]

\[= L_\omega h - \varepsilon P_n \Pi_W \left( \left( \partial_u g(\delta, x, v_1 + w + v_2(\delta, v_1, w)) \right) \left( h + \partial_w v_2(\delta, v_1, w)[h] \right) \right)
\]

\[= L_\omega h - \varepsilon P_n \Pi_W \left( a(t,x) h \right) - \varepsilon P_n \Pi_W \left( a(t,x) \partial_w v_2(\delta, v_1, w)[h] \right)
\]

\[= Dh - M_1 h = M_2 h.
\]

First (Step 1) we prove that, assuming the “first order Melnikov non-resonance condition \( \delta \in \Delta_2^\gamma(\varepsilon, \varpi) \) (see Definition 3.3) the linear operator \( D \) is invertible, see Corollary 4.2. Next (Step 2) we prove that \( M_1, M_2 \) are small enough with respect to \( D \), yielding the invertibility of the whole \( \mathcal{L}_n \).

Through this section we shall use the notations \( F_k := \{ f \in H_0^1((0,\pi); \mathbb{C}) \mid \int_0^{\pi} f(x) \sin(kx) \ dx = 0 \} \) whence the space \( W \), defined in (44), and its corresponding projector \( \Pi_W : X_{\sigma,s} \to W \), are written, for any \( h = \sum_{k \in \mathbb{Z}} \exp(ikt)h_k \)

\[
W = \left\{ h \in X_{\sigma,s} \mid h_k \in F_k \ \forall k \in \mathbb{Z} \right\}, \quad \Pi_W h(t,x) = \sum_{k \in \mathbb{Z}} \exp(ikt)(\pi_k h_k)(x),
\]

where \( \pi_k : H_0^1((0,\pi); \mathbb{C}) \to F_k \) is the \( L^2 \)-orthogonal projector onto \( F_k \) (note that \( \pi_{-k} = \pi_k \) and \( \pi_k \varpi = \varpi \pi_k \)), hence \( \pi_{-k} h_{-k} = \pi_k h_k \).

**Step 1: Inversion of \( D : W^{(n)} \to W^{(n)} \).**

In term of time-Fourier series, \( D \) is defined by \( \forall h \in W^{(n)} \),

\[(Dh)_k = D_k h_k \quad \forall |k| \leq L_n,
\]

where \( D_k : D(D_k) \subset F_k \to F_k \) is the linear operator

\[D_k u = \omega^2 k^2 u - S_k u \quad \text{and} \quad S_k u := -\partial_{xx} u + \varepsilon \pi_k (a_0(x) u).
\]

Note that \( S_k = S_{-k} \).

We now analyze the spectral properties of the Sturm-Liouville type operator \( S_k \). We shall assume that \( |\varepsilon| a_0 |\infty < 1 \), so that

\[
(u, v)_\varepsilon := \int_0^{\pi} u_x \varpi_x + \varepsilon a_0(x) u \varpi \ dx
\]

defines a scalar product on \( H_0^1((0,\pi); \mathbb{C}) \), hence on \( F_k \), and its associated norm is equivalent to the standard \( H^1 \)-norm. More precisely \(^7\)

\[
||u||_\varepsilon = ||u||_{H^1 \left( 1 + O(|\varepsilon| a_0 |\infty \right))} \quad \forall u \in F_k.
\]

\(^7\forall u \in H_0^1((0,\pi), \int_0^{\pi} u(x)^2 \ dx \leq \int_0^{\pi} u_x^2(x) \ dx \) since the least eigenvalue of \( -\partial_{xx} \) with Dirichelet B.C. on \((0,\pi)\) is 1.
Lemma 4.1 (Sturm-Liouville) The operator $S_k : D(S_k) \subset F_k \rightarrow F_k$ possesses a $(\cdot,\cdot)_\varepsilon$-orthonormal basis $(v_{k,j})_{j \geq 1, j \neq |k|}$ of real eigenvectors with real eigenvalues $(\lambda_{k,j})_{j \geq 1, j \neq |k|}$, i.e., $S_k v_{k,j} = \lambda_{k,j} v_{k,j}$, $\lambda_{k,j} \in \mathbb{R}$, and $\lambda_{k,j} = \lambda_{-k,j}$, $v_{-k,j} = v_{k,j}$.

Moreover, $(v_{k,j})_{j \geq 1, j \neq |k|}$ is an orthogonal basis also for the $L^2$-scalar product in $F_k$. Defining $\varphi_{k,j} = v_{k,j}/\|v_{k,j}\|_{L^2}$, $\lambda_{k,j}$ and $\varphi_{k,j}$ have the asymptotic expansion as $j \rightarrow +\infty$

$$\lambda_{k,j} = \lambda_{k,j}(\delta, v_1, w) = j^2 + \varepsilon M(\delta, v_1, w) + O\left(\varepsilon |a_0|_{H^s}^2 / j\right), \quad |\varphi_{k,j} - \sqrt{2 \varepsilon \sin(jx)}|_{L^2} = O\left(\varepsilon |a_0|_{H^s}^2 / j\right) \quad (45)$$

where $M(\delta, v_1, w)$, introduced in Definition 3.1, is the mean value of $a_0(x)$ on $(0, \pi)$.

Proof. In the Appendix. ■

By Lemma 4.1, the linear operator $D_k : D(D_k) \subset F_k \rightarrow F_k$ possesses a $(\cdot,\cdot)_\varepsilon$-orthonormal basis $(v_{k,j})_{j \geq 1, j \neq |k|}$ of real eigenvectors with real eigenvalues $(\omega^2 k^2 - \lambda_{k,j})_{j \geq 1, j \neq |k|}$. As a consequence we derive

Corollary 4.1 (Diagonalization of $D$) The operator $D : W^{(n)} \rightarrow W^{(n)}$ is the diagonal operator $\text{diag}\{\omega^2 k^2 - \lambda_{k,j}\}$ in the basis $\{\varphi_{0,j} : j \geq 1\} \cup \{(j,k)_{k \neq |k|}\}$ of real eigenvectors with real eigenvalues $(\omega^2 k^2 - \lambda_{k,j})_{j \geq 1, j \neq |k|}$.

Noting that $\min_{|k| \leq L_n} |\omega^2 k^2 - \lambda_{k,j}| \rightarrow \infty$ as $j \rightarrow +\infty$, we deduce from Corollary 4.1 that the linear operator $D : W^{(n)} \rightarrow W^{(n)}$ is invertible if all its eigenvalues \{(\omega^2 k^2 - \lambda_{k,j} (\delta, v_1, w) |k| \leq L_n, j \geq 1, j \neq |k|\} are different from zero. If this holds, we can define $D^{-1}$ as well as $|D|^{-1/2} : W^{(n)} \rightarrow W^{(n)}$ by

$$W^{(n)} \ni h = \sum_{k=-L_n}^{L_n} \exp(ikt) h_k \mapsto |D|^{-1/2} h := \sum_{k=-L_n}^{L_n} \exp(ikt) |D_k|^{-1/2} h_k \in W^{(n)}$$

where $|D_k|^{-1/2} : F_k \rightarrow F_k$ is the compact operator defined by

$$|D_k|^{-1/2} \varphi_{k,j} := \varphi_{k,j} / \sqrt{|\omega^2 k^2 - \lambda_{k,j}|}, \quad \forall j \geq 1, j \neq |k|.$$  

The “small divisor problem” ($i$) is that some of the eigenvalues of $D, \omega^2 k^2 - \lambda_{k,j}$, can become arbitrarily large for $(k, j) \in \mathbb{Z}^2$ sufficiently large and therefore the norm of $|D|^{-1/2}$ can become arbitrarily large as $n \rightarrow \infty$.

In order to quantify this phenomenon we define for all $k$

$$\alpha_k := \min_{j \neq |k|} |\omega^2 k^2 - \lambda_{k,j}|. \quad (46)$$

Note that $\alpha_{-k} = \alpha_k$. If $\forall |k| \leq L_n$, $\alpha_k \neq 0$, then $D$ is invertible and, since $(\varphi_{k,j})_{k \neq j}$ is an orthogonal basis for the $\langle \cdot, \cdot \rangle_\varepsilon$ scalar product, $\| |D_k|^{-1/2} u \|_\varepsilon \leq \alpha_k^{-1/2} \| u \|_\varepsilon$. Hence, by (44),

$$\left\| |D_k|^{-1/2} u \right\|_{H^1} \leq \frac{C}{\sqrt{\alpha_k}} \| u \|_{H^1}, \quad \forall k \in \mathbb{Z}. \quad (47)$$

The condition $\forall |k| \leq L_n$, $\alpha_k \neq 0$, depends very sensitively on the parameters $(\delta, v_1)$. Assuming the “first order Melnikov non-resonance condition” $\delta \in \Delta_{\gamma_+} \cap |0, \delta_0|$ (see Definition 3.8), we obtain, in Lemma 4.2, a lower bound of the form $c/|k|^{\gamma-1}$ for the moduli of the eigenvalues of $D_k$ and, therefore, in Corollary 4.2 sufficiently good estimates for the inverse of $D$.

Lemma 4.2 (Lower bound for the eigenvalues of $D$) If $\delta \in \Delta_{\gamma_+} \cap |0, \delta_0|$ and $\delta_0$ is small enough (depending on $\gamma$), then (recall that $1 < \tau < 2$)

$$\alpha_k := \min_{j \geq 1, j \neq |k|} |\omega^2 k^2 - \lambda_{k,j}| \geq \frac{\gamma}{|k|^{\gamma-1}} > 0, \quad \forall |k| \leq L_n. \quad (48)$$
PROOF. Since \( \alpha_{-k} = \alpha_k \) it is sufficient to consider \( k \geq 0 \). By the asymptotic expansion \( \lambda_k \) for the eigenvalues \( \lambda_{k,j} \), using that \( ||a_0)||_{H^1}, |M(\delta, v_1, w)| \leq C; \\
|\omega^2 k^2 - \lambda_{k,j}| = |\omega^2 k^2 - j^2 - \varepsilon M(\delta, v_1, w) + O(\varepsilon ||a_0||_{H^1})| \\
= (\omega k - \sqrt{j^2 + \varepsilon M(\delta, v_1, w)})(\omega k + \sqrt{j^2 + \varepsilon M(\delta, v_1, w)}) + O(\varepsilon^2 \frac{j^2}{k}) \\
\geq |\omega k - \sqrt{j^2 + \varepsilon M(\delta, v_1, w)}| \sqrt{j^2 + \varepsilon M(\delta, v_1, w)} + O(\varepsilon^2 \frac{j^2}{k}) \\
\geq |\omega k - j - \varepsilon \frac{M(\delta, v_1, w)}{2j}| \frac{k}{2} - C(\epsilon^2 \frac{j^2}{k^3} + \frac{\varepsilon}{j}) \geq C\gamma \frac{k}{k^3} - C(\epsilon^2 \frac{j^2}{k^3} + \frac{\varepsilon}{j}), \\
(49)
since \( \delta \in \Delta^2, \tau(v_1, w) \). If \( \alpha_k := \min_{j \geq 1, j \neq k} |\omega^2 k^2 - \lambda_{k,j}| \) is attained at \( j = j(k) \), i.e. \( \alpha_k = |\omega^2 k^2 - \lambda_{k,j}| \) then, \( j \leq 2k \) and therefore, by \( (19) \) and since \( 1 < \tau \leq 2, \) we obtain \( (18) \). \( \blacksquare \)

**Corollary 4.2 (Estimate of \(|D|^{-1/2}\)** If \( \delta \in \Delta^2, \tau(v_1, w) \cap [0, \delta_0) \) and \( \delta_0 \) is small enough, then \( D : W(n) \to W(n) \) is invertible and \( \forall s \geq 0 \)
\[
\left| \left| |D|^{-1/2} h \right| \right|_{\sigma, s} \leq C \frac{1}{\sqrt{|\gamma|}} \left| \left| h \right| \right|_{\sigma, s + \frac{1}{2} - 1} \quad \forall h \in W(n). 
(50)
\]

**PROOF.** Use \( (17) \) and \( (18) \). \( \blacksquare \)

**Step 2: Inversion of \( \mathcal{L}_n : W^{(n)} \to W^{(n)} \).**

In order to show the invertibility of \( \mathcal{L}_n : W^{(n)} \to W^{(n)} \) it is a convenient devise to write 
\[
\mathcal{L}_n = D - M_1 - M_2 = |D|^{1/2} (U - R_1 - R_2) |D|^{1/2}
\]
where
\[
U := |D|^{-1/2} D |D|^{-1/2} = |D|^{-1} D \quad \text{and} \quad R_i := |D|^{-1/2} M_i |D|^{-1/2}, \quad i = 1, 2.
\]
We shall prove the invertibility of \( U - R_1 - R_2 : W^{(n)} \to W^{(n)} \) showing that, for \( \varepsilon \) small enough, \( R_1 \) and \( R_2 \) are small perturbations of \( U \).

**Lemma 4.3 (Estimate of \(|U^{-1}|\) |U : W^{(n)} \to W^{(n)} is an invertible operator and its inverse \( U^{-1} \) satisfies, \( \forall s \geq 0 \),
\[
\left| \left| U^{-1} h \right| \right|_{\sigma, s} = \left| \left| h \right| \right|_{\sigma, s} \left( 1 + O(\varepsilon ||a_0||_{H^1}) \right) \quad \forall h \in W(n).
(51)
\]

**PROOF.** \( U_k := |D|^{-1/2} D_k : F_k \to F_k \) being orthogonal for the \( (\ , \ )_\varepsilon \) scalar product, it is invertible and \( \forall u \in F_k, ||U_k^{-1} u||_\varepsilon = ||u||_\varepsilon \). Hence, by \( (14) \), there is \( C \geq 0 \) such that
\[
\forall u \in F_k, \quad ||U_k^{-1} u||_{H^1} \leq ||u||_\varepsilon (1 + C\varepsilon ||a_0||_{H^1})
\]
Therefore, \( U = |D|^{-1} D \), being defined by \( (U h)_k = U_k h_k \), \( \forall |k| \leq L_n, U \) is invertible, \( (U^{-1} h)_k = U_k^{-1} h_k \) and \( (51) \) holds. \( \blacksquare \)

For proving the smallness of \( R_1 \) and \( R_2 \) we need the following preliminary Lemma.

**Lemma 4.4 There are \( \mu > 0, \delta_0 > 0 \) and \( C > 0 \) with the following property : if \( ||v_1||_{0, s} \leq 2R, [w]_{\sigma, s} \leq \mu \) and \( \delta \in [0, \delta_0) \), then \( ||a||_{\sigma, s + \frac{1}{2} - 1} \leq C \).**
An elementary calculus shows that
\[ \sum_{i=0}^{q} \frac{\|h_i\|_{\sigma,s + 2(\tau-1)}}{(\sigma_i - \sigma)\tau_1} \leq 2\mu. \] (52)

An elementary calculus shows that
\[ \|h_i\|_{\sigma,s + 2(\tau-1)} \leq C(\tau) \frac{\|h_i\|_{\sigma,s}}{\min(\sigma_i - \sigma, \tau_1)}. \] (53)

Hence, by Lemma 3.1 provided \( \mu \) is small enough, \( \nu_2(\delta, v_1, w) \) is well defined and \( \|v_2(\delta, v_1, w)\|_{\sigma,s + 2(\tau-1)} \leq C' \). Hence, by the algebra property of the norm \( \|\| \) and the analyticity of \( f \), \( \|a\|_{\sigma,s + 2(\tau-1)} \) is bounded by some constant, provided \( \delta_0 \) has been chosen small enough. \( \square \)

The “smallness” of \( R_2 : W^{(n)} \rightarrow W^{(n)} \) is just a consequence of the regularizing property of \( \partial_w \nu_2 : X_{\sigma,s} \rightarrow X_{\sigma,s+2} \) proved in Lemma 2.1 and Lemma 4.4.

**Lemma 4.5 (Estimate of \( R_2 \))** Under the hypotheses of (P3), there exists a constant \( C > 0 \) depending on \( \mu \) such that
\[ \|R_2h\|_{\sigma,s + \frac{\tau-1}{2}} \leq C \frac{|\epsilon|}{\gamma} \|h\|_{\sigma,s + \frac{\tau-1}{2}} \quad \forall h \in W^{(n)}. \]

**Proof.** Using (40) and the regularizing estimates \( \|\partial_w \nu_2[u]\|_{\sigma,s+2} \leq C\|u\|_{\sigma,s} \) of Lemma 2.1 we get
\[ \|R_2h\|_{\sigma,s + \frac{\tau-1}{2}} \leq \frac{C}{\sqrt{\gamma}} \left\| M_2 |D|^{-1/2}h \right\|_{\sigma,s+\tau-1} = \frac{C}{\sqrt{\gamma}} \left\| P_n \Pi_W \left( a \partial_w \nu_2 \left[ |D|^{-1/2}h \right] \right) \right\|_{\sigma,s+\tau-1} \]
\[ \leq \frac{C}{\sqrt{\gamma}} \|a\|_{\sigma,s+\tau-1} \left\| \partial_w \nu_2 \left[ |D|^{-1/2}h \right] \right\|_{\sigma,s+\tau-1} \]
\[ \leq \frac{C}{\sqrt{\gamma}} \|a\|_{\sigma,s+\tau-1} \left\| D^{-1/2}h \right\|_{\sigma,s+2} \]
\[ \leq \frac{C}{\sqrt{\gamma}} \|a\|_{\sigma,s+\tau-1} \left\| D^{-1/2}h \right\|_{\sigma,s+2} \leq C \frac{|\epsilon|}{\gamma} \|h\|_{\sigma,s + \frac{\tau-1}{2}} \]

since \( 1 < \tau < 2 \) and by Lemma 4.4 \( \|a\|_{\sigma,s+\tau-1} \leq \|a\|_{\sigma,s+2(\tau-1)} \leq C. \) \( \square \)

The estimate of the “off-diagonal” operator \( R_1 : W^{(n)} \rightarrow W^{(n)} \) requires, on the contrary, a careful analysis of the “small divisors” and the use of the “first order Melnikov non-resonance condition” \( \delta \in \Delta_k^{(\gamma,\tau)} \). For clarity, we enunciate such property separately.

**Lemma 4.6 (Analysis of the Small Divisors)** Let \( \delta \in \Delta_k^{(\gamma,\tau)} \cap [0, \delta_0] \), with \( \delta_0 \) small. There exists \( C > 0 \) such that, \( \forall l \neq k, \)
\[ \frac{1}{\alpha_k \alpha_l} \leq C \frac{|k - l|^{2\tau - 1}}{\gamma^2 |\epsilon|^{\tau - 1}} \quad \text{where} \quad \beta := \frac{2 - \tau}{\tau}. \] (54)

**Proof.** To obtain (54) we distinguish different cases.

---

*Using that \( \max_{k \geq 1} k^n \exp\left\{-|\sigma_i - \sigma|k\right\} \leq C(\alpha)/(\sigma_i - \sigma)^n \)
• First case: \(|k - l| \geq \max(|k|, |l|)\). Then \((\alpha_k \alpha_l)^{-1} \leq C|k - l|^{2 - \frac{1}{2}} / \gamma^2\).

Indeed we estimate both \(\alpha_k, \alpha_l\) with the lower bound \(\llbracket k \rrbracket \), \(\alpha_k \geq C\gamma / |k|^\tau - 1\), \(\alpha_l \geq C\gamma / |l|^\tau - 1\) and therefore
\[
\frac{1}{\alpha_k \alpha_l} \leq C \frac{|k|^{\tau - 1} |l|^{\tau - 1}}{\gamma^2} \leq C \frac{[\max(|k|, |l|)]^{2(\tau - 1)}}{\gamma^2} \leq C \frac{|k - l|^{2 - \frac{1}{2}}}{\gamma^2}.
\]

• Second case: \(|k - l| < \max(|k|, |l|)\) and \((|k| \leq 1/3|\varepsilon| \text{ or } |l| \leq 1/3|\varepsilon|)\). Then \((\alpha_k \alpha_l)^{-1} \leq C / \gamma\).

Note that, in this case, \(\text{sign}(l) = \text{sign}(k)\) and, to fix the ideas, we assume in the sequel that \(l, k \geq 0\) (the estimate for \(k, l < 0\) is the same, since \(\alpha_k \alpha_l = \alpha_{-k} \alpha_{-l}\)).

Suppose, for example, that \(0 \leq k \leq 1/3|\varepsilon|\). We claim that if \(\varepsilon\) is small enough, then \(\alpha_k \geq (k + 1)/8\). Indeed, \(\forall j \neq k\),
\[
|\omega k - j| = |\omega k - k + k - j| \geq |k - j| - |\omega - 1| \text{ if } k \geq 1 \iff |k| \geq \frac{1}{3}.
\]

Therefore \(\forall 1 \leq k < 1/3|\varepsilon|, \forall j \neq k, j \geq 1, |\omega^2 k^2 - j^2| = |\omega k - j| \geq (\omega k + 1)/3 \geq (k + 1)/6\) and so \(\alpha_k := \min_{j \neq k, j \geq 1} |\omega^2 k^2 - \lambda_{k,j}| = \min_{j \neq k, j \geq 1} |\omega^2 k^2 - \lambda_{k,j}|\). Analogously let \(i = i(k) \geq 1\) be the unique integer such that \(\alpha_l = |\omega^2 l^2 - \lambda_{k,i}|\).

• Third case: \(0 \leq |k - l| < \max(|k|, |l|)\) and both \(|k|, |l| > 1/3|\varepsilon|\). We have to distinguish two sub-cases. For this, \(\forall k \in \mathbb{Z}, \text{ let } j = j(k) \geq 1\) be the unique integer such that \(\alpha_k := \min_{j \neq k} |\omega^2 k^2 - \lambda_{k,n}| = |\omega^2 k^2 - \lambda_{k,j}|\). Analogously let \(i = i(k) \geq 1\) be the unique integer such that \(\alpha_l = |\omega^2 l^2 - \lambda_{i,l}|\).

• Fourth case: \(0 < |k - l| < \max(|k|, |l|)\) and \(|k - l| \neq |j - i|\). Then \((\alpha_k \alpha_l)^{-1} \leq C / \gamma |\varepsilon|^{-1}\).

Indeed \(\|\omega k - j\| - |\omega l - i\| = |\omega k - l| - (j - i)| \geq |\omega - 1| |k - l| \geq |\varepsilon| / 2\) and therefore or \(|\omega k - j| > |\varepsilon| / 4\) or \(|\omega l - i| \geq |\varepsilon| / 4\). It follows that \(|\omega^2 k^2 - j^2| = |\omega k - j| \geq |\omega k + j| \geq |\omega k| / 2 \geq |\varepsilon| k / 3\) and so, for \(\varepsilon\) small enough, \(|\alpha_k| \geq |\varepsilon| k / 4\), or \(|\alpha_l| \geq |\varepsilon| l / 4\). Hence, since \(l \leq 2k\) and \(k > 1/3\),
\[
\frac{1}{\alpha_k \alpha_l} \leq C \frac{|k|^\tau - 1}{\gamma |\varepsilon| k} \leq C \frac{\gamma^2 |\varepsilon|^\tau}{\gamma |\varepsilon|^{\tau - 1}} \leq C \frac{\gamma^2 |\varepsilon|^\tau}{\gamma |\varepsilon|^{\tau - 1}}.
\]

\(\bullet\) Fourth case: \(0 < |k - l| < \max(|k|, |l|)\) and \(|k - l| \neq |j - i|\). Then \((\alpha_k \alpha_l)^{-1} \leq C / \gamma^2\).

Using that \(\omega\) is \(\gamma - \tau\)-Diophantine, that \(|k - l| < \max(|k|, |l|)\) and so \(l \geq 2k\),
\[
\|\omega k - j\| - |\omega l - i\| = |\omega (k - l) - (j - i)| \geq \frac{\gamma}{|k - l|^\tau} \geq \frac{\gamma}{\max(|k|, |l|)^{\beta \tau}} \geq C \frac{\gamma^2}{k^{2 \beta \tau}}
\]
such that \(|\omega k - j| \geq C / 2k^{\beta \tau}\) or \(|\omega l - i| \geq C / 2l^{\beta \tau}\). Therefore \(|\omega^2 k^2 - j^2| \geq C \gamma k^{1 - \beta \tau}\). We estimate \(\alpha_l\) with the worst possible lower bound and so, using also \(l \leq 2k\),
\[
\frac{1}{\alpha_k \alpha_l} \leq C \frac{|l|^\tau - 1}{\gamma^2 k^{1 - \beta \tau}} \leq C \frac{\gamma^2 |\varepsilon|^\tau}{\gamma^2} \leq C \frac{\gamma^2}{\gamma^2},
\]
since \(\beta := (2 - \tau) / \tau\). Collecting the estimates of all the previous cases, \(\llbracket k \rrbracket\) follows. $^9$

$^9$Indeed \(|k - l| \leq (\max(|k|, |l|)\) and so \(l \leq k\) or \(l \leq k + l^\beta\) and so, \(l / 2 < k\), since \(l \geq 1 / 3|\varepsilon|\).
Lemma 4.7 (Bound of an off-diagonal operator) Assume that $\delta \in \Delta_{x}^{n}(v_1, v_2) \cap [0, \delta_0]$ and let, for some $s' \geq s$, $b(t, x) \in X_{s, s' + \frac{\pi}{2}}$ satisfy $b_0(x) = 0$, i.e. $\int_{0}^{2\pi} b(t, x) \, dt \equiv 0$, $\forall x \in (0, \pi)$. Defining the operator $T_n : W^{(n)} \to W^{(n)}$ by

$$T_n h := |D|^{-1/2} P_n \Pi_W \left( b(t, x) |D|^{-1/2} h \right),$$

there is a constant $\widetilde{C}$, independent of $b(x, t)$ and of $p$, such that

$$\left\| T_n h \right\|_{\sigma, s'} \leq \frac{\widetilde{C}}{|\gamma|} \left\| b \right\|_{\sigma, s' + \frac{\pi}{2}} \left\| h \right\|_{\sigma, s'} \quad \forall h \in W^{(n)}.$$

**Proof.** For $h \in W^{(n)}$, we have $(T_n h)(t, x) = \sum_{|k| \leq L_n} (T_n h)_k(x) \exp(ikt)$, with

$$
(T_n h)_k = |D_k|^{-1/2}\pi_k \left( b |D|^{-1/2} h \right)_k
= |D_k|^{-1/2}\pi_k \left[ \sum_{|l| \leq L_n} b_{k-l}|D_l|^{-1/2}h \right]
$$

(since $T_{k-1}(x) = b_{-k}(x)$, the linear operator $T_n$ is represented by a self-adjoint Toeplitz matrix in $\text{Mat}(2L_n \times 2L_n, H^1((0, \pi), \mathbb{C})$ which is zero on the diagonal, as $b_0(x) = 0$). Abbreviating $B_m := \left\| b_m(x) \right\|_{H^1}$, we get from (55) and (57), using that $B_0 := \left\| b_0(x) \right\|_{H^1} = 0$,

$$\left\| (T_n h)_k \right\|_{H^1} \leq C \sum_{|l| \leq L_n, l \neq k} \frac{B_{k-l}}{\sqrt{\alpha_k \alpha_l}} \left\| h_l \right\|_{H^1}. \quad (56)$$

Hence, by (57),

$$\left\| (T_n h)_k \right\|_{H^1} \leq \frac{C}{|\gamma| \left\| \frac{2\pi}{\bar{\sigma}} \right\|} s_k \quad \text{where} \quad s_k := \sum_{|l| \leq L_n} B_{k-l}|l^{-\frac{1}{2}} \left\| h_l \right\|_{H^1}. \quad (57)$$

By (57), setting $\bar{s}(t) := \sum_{|k| \leq L_n} s_k \exp(ikt)$,

$$\left\| T_n h \right\|_{\sigma, s'}^2 = \sum_{|k| \leq L_n} \exp(2\sigma|k|)(k^{2s'} + 1) \left\| (T_n h)_k \right\|_{H^1}^2 \leq \frac{C^2}{\gamma^2 |\alpha|^{\sigma-1}} \sum_{|k| \leq L_n} \exp(2\sigma|k|)(k^{2s'} + 1)s_k^2 = \frac{C^2}{\gamma^2 |\alpha|^{\sigma-1}} \left\| \bar{s} \right\|_{\sigma, s'}^2 \quad (58)$$

It turns out that $\bar{s} = P_n(\vec{b}c)$ where $\vec{b}(t) := \sum_{t \in \mathbb{Z}} |l|^{\sigma-1} B_l \exp(ilt)$ and $\vec{c}(t) := \sum_{|l| \leq L_n} \left| h_l \right|_{H^1} \exp(ilt)$. Therefore, by (58),

$$\left\| T_n h \right\|_{\sigma, s'} \leq \frac{C}{\gamma |\alpha|^{\sigma-1}} \left\| \vec{b} \right\|_{\sigma, s'} \leq \frac{C}{\gamma |\alpha|^{\sigma-1}} \left\| \vec{b} \right\|_{\sigma, s'} \left\| \vec{c} \right\|_{\sigma, s'} \leq \frac{C}{\gamma |\alpha|^{\sigma-1}} \left\| b \right\|_{\sigma, s' + \frac{\pi}{2}} \left\| h \right\|_{\sigma, s'}$$

since $\left\| \vec{b} \right\|_{\sigma, s'} \leq \left\| b \right\|_{\sigma, s' + \frac{\pi}{2}}$ and $\left\| \vec{c} \right\|_{\sigma, s'} = \left\| h \right\|_{\sigma, s'}$. ■

Lemma 4.8 (Estimate of $R_1$) Under the hypotheses of (P3), there exists a constant $C > 0$ depending on $\mu$ such that

$$\left\| R_1 h \right\|_{\sigma, s + \frac{\pi}{2}} \leq \left\| \frac{1}{\gamma} \right\|_{\sigma, s + \frac{\pi}{2}} C \| h \|_{\sigma, s + \frac{\pi}{2}} \quad \forall h \in W^{(n)}.$$
PROOF. Recalling the definition of \( R_1 := |D|^{-1/2} M_1 |D|^{-1/2} \) and using Lemma \( \text{L} \), since \( \pi(t, x) \) has zero time-average,

\[
\left\| R_1 h \right\|_{\sigma,s+\frac{1}{p}} \leq C \left\| |D|^{-1/2} M_1 |D|^{-1/2} h \right\|_{\sigma,s+\frac{1}{p}} \leq C \left\| |D|^{-1/2} P_n \Pi_W (|D|^{-1/2} h) \right\|_{\sigma,s+\frac{1}{p}}
\]

\[
\leq |\epsilon| \frac{C}{|w|^{\gamma}} |\pi|_{\sigma,s+\frac{1}{p}+\frac{1}{p}} ||h||_{\sigma,s+\frac{1}{p}} \leq |\epsilon| \frac{C}{|w|^{\gamma}} |\pi|_{\sigma,s+\frac{1}{p}+\frac{1}{p}} \left\| |D|^{-1/2} P_n \Pi_W (|D|^{-1/2} h) \right\|_{\sigma,s+\frac{1}{p}}
\]

since \( 0 < \beta < 1 \) and, by Lemma \( \text{L} \), \( |\pi|_{\sigma,s+\frac{2(\gamma-1)}{p}} \leq |a|_{\sigma,s+\frac{2(\gamma-1)}{p}} \leq C \).

**Proof of Property (P3).** Under the hypothesis of (P3), the linear operator \( U : W^{(n)} \to W^{(n)} \) is invertible by Lemma \( \text{L} \) and, by Lemmas \( \text{L} \) and \( \text{L} \) provided \( \delta \) is small enough, \( \|U^{-1} R_1\|_{\sigma,s+\frac{1}{p}} \) and \( \|U^{-1} R_2\|_{\sigma,s+\frac{1}{p}} < 1/4 \). Therefore also the linear operator \( U - R_1 - R_2 : W^{(n)} \to W^{(n)} \) is invertible and its inverse satisfies

\[
\| (U - R_1 - R_2)^{-1} h \|_{\sigma,s+\frac{1}{p}} = \| (I - U^{-1} R_1 - U^{-1} R_2)^{-1} U^{-1} h \|_{\sigma,s+\frac{1}{p}} \leq 2 \| U^{-1} h \|_{\sigma,s+\frac{1}{p}} \leq C \| h \|_{\sigma,s+\frac{1}{p}} \quad \forall h \in W^{(n)}.
\]

Hence \( \mathcal{L}_n \) is invertible, \( \mathcal{L}_n^{-1} = |D|^{-1/2} (U - R_1 - R_2)^{-1} |D|^{-1/2} : W^{(n)} \to W^{(n)} \), and by \( \text{L} \), \( \text{L} \),

\[
\| \mathcal{L}_n^{-1} h \|_{\sigma,s} = \| |D|^{-1/2} (U - R_1 - R_2)^{-1} |D|^{-1/2} h \|_{\sigma,s} \leq \frac{C}{|\omega|^{\gamma}} \| (U - R_1 - R_2)^{-1} |D|^{-1/2} h \|_{\sigma,s+\frac{1}{p}} \leq \frac{C'}{|\omega|^{\gamma}} \| |D|^{-1/2} h \|_{\sigma,s+\frac{1}{p}} \leq \frac{C''}{\gamma} \| h \|_{\sigma,s+\tau-1} \leq \frac{C''}{\gamma} (L_n)^{-1} \| h \|_{\sigma,s}
\]

which completes the proof of property (P3).

## 5 Solution of the \((Q_1)\)-equation

Finally, we have to solve the finite dimensional \((Q_1)\)-equation

\[
- \Delta v_1 = \Pi V_1 G(\delta, v_1)
\]

where

\[
G(\delta, v_1)(t, x) := g(\delta, x, v_1(t, x) + \bar{w}(\delta, v_1)(t, x) + v_2(\delta, v_1, \bar{w}(\delta, v_1))(t, x))
\]

and we have to ensure that there are solutions \((\delta, v_1) \in B_\infty\) for a set\(^{10}\) of \(\delta\)'s of positive.

### 5.1 The set \( \mathcal{A}_p \)

By Lemma \( \text{L} \), the \( 0^{th} \)-order \((Q_1)\)-equation

\[
- \Delta v_1 = \Pi V_1 G(0, v_1)
\]

is the Euler-Lagrange equation of the “reduced” functional \( \Psi_0 : B(2R, V_1) \to \mathbb{R} \) defined in \( \text{L} \) (note that \( G(0, v_1) = a_p(x)(v_1 + v_2(0, v_1, 0))^p \) and see formula \( \text{L} \)).

\(^{10}\)It is rather easy to see that for \( \omega \) close to \( 1 \) and \( \delta = |\omega^2 - 1|/2 \) \( \{ \delta \} \) contains the whole ball \( B(2R, V_1) \) only if \( \omega \) is strongly nonresonant, i.e. belongs to the zero-measure set \( W_0 := \{ \omega \in \mathbb{R} \mid |\omega l - j| \geq \gamma/l, \forall j \neq k, l \geq 0, j \geq 1 \} \) for some \( \gamma > 0 \). If \( \omega \) is strongly nonresonant, the existence of \( 2\pi/\omega \)-periodic solutions can be proved \( \text{L} \) for any nonlinearity \( f \). For more general frequencies, the set \( E_\omega \) has gaps, which makes the analysis of the \((Q1)\) equation more delicate.
For any \( a_p(x) \in H^1((0, \pi), \mathbb{R}) \) such that condition (1) is verified, \( \Psi_0 \), or the functional \( \tilde{\Psi}_0 \) obtained replacing \( a_p(x) \) by \(-a_p(x)\), possesses, by the Mountain-pass Theorem (11), a non-trivial critical point \( \overline{v}_1 \in V_1 \) with \( \| \overline{v}_1 \|_{0,s} \leq R \) (\( R \) depending on \( a_p \)). More precisely, due to time translation invariance, we have a circle of critical points. In fact the functional \( \Psi_0 \) is invariant under the action of \( \mathbb{R}/2\pi \mathbb{Z} \) on \( V_1 \) defined by
\[
(\theta * v_1)(t, x) := v_1(t - \theta, x).
\]
We shall say that a circle of critical points \( [v_1] := \{ \theta * v_1 ; \theta \in \mathbb{R}/2\pi \mathbb{Z} \} \) is non degenerate when \( \ker \Psi_0(v_1) \) is spanned by \( (\partial / \partial \theta)(\theta * v_1)_{|\theta = 0} = \partial_{t}v_1 \).

We recall that, by Lemma 6.1 condition (4) holds for any \( a_p(x) \in \mathcal{O} \subset H^1((0, \pi), \mathbb{R}) \) where
\[
\mathcal{O} := \{ a(x) \in H^1((0, \pi), \mathbb{R}) \mid \begin{cases} a(\pi - x) = -a(x), & \text{for some } x \in [0, \pi], \text{ if } p \text{ is odd} \\ a(\pi - x) = a(x), & \text{for some } x \in [0, \pi], \text{ if } p \text{ is even} \end{cases} \}.
\]
We can define similarly the nondegenerate critical circles of \( \Phi_0 \) in \( V \).

**Remark 5.1** \([v_1] \) is a non-degenerate critical circle of \( \Psi_0 : V_1 \to \mathbb{R} \) iff \( [\overline{v}] = [v_1 + v_2(0, \overline{v}_1, 0)] \) is a non-degenerate critical circle of \( \Phi_0 : V \to \mathbb{R} \), i.e. iff \( \overline{v} \in V \) is, up to time translations, a non-degenerate solution of the \( 0^d \)-order bifurcation equation \( (7) \). Moreover, if \( [\overline{v}] \) is a nondegenerate critical circle of \( \Phi_0 \) of critical value \( c \) then \( [\Pi_{1}, \overline{v}] \) is a nondegenerate critical circle of \( \Psi_0 \).

In \( \mathcal{B} \) it is proved that, for the nonlinearity \( f(x, u) = u^3 \), the critical points of \( \Phi_0 \) are non-degenerate up to time translations.

Let us define
\[
\mathcal{A}_p^1 := \{ a_p(x) \in \mathcal{O} \cap H^1((0, \pi), \mathbb{R}) \mid \text{there is a non-degenerate critical circle } [\overline{v}] \subset V \setminus \{0\} \text{ of } \Phi_0 \text{ or } \tilde{\Phi}_0 \},
\]
where \( \tilde{\Phi}_0 \) is obtained from \( \Phi_0 \) replacing \( a_p \) with \(-a_p \).

For any \( a_p \in \mathcal{A}_p^1 \), the \( 0^d \)-order bifurcation equation \( (7) \) (or the one obtained substituting \(-a_p \) for \( a_p \)) possesses a non-trivial, non-degenerate solution (up to time translations) \( \overline{v} \in V \). We can always assume that we have chosen \( c \) large enough in \( (12) \), so that, by Lemma 2.2, \( \overline{v} = v_1 + v_2(0, v_1, 0) \) for some non degenerate (up to time translations) solution \( v_1 \in B(2R, V_1) \) of equation \( (12) \). By the Implicit function Theorem, there exists a \( C^\infty \)-curve of solutions of the \( (Q_1) \)-equation \( (13) \)
\[
v_1(\cdot) : [0, \delta_0) \to V_1 \quad \text{with} \quad v_1(0) = \overline{v}_1.
\]
Let us define the Cantor-like set
\[
\mathcal{C}_{a_p, \overline{v}_1} := \{ \delta \in [0, \delta_0) \mid (\delta, v_1(\delta)) \in B_\infty \}.
\]
The smoothness of \( v_1(\cdot) \) implies that the Cantor set \( \mathcal{C}_{a_p, \overline{v}_1} \) has full density at the origin, i.e. satisfies the measure estimate \( (5) \) of Theorem 1.1 (i).

**Proposition 5.1** (Measure estimate of \( \mathcal{C}_{a_p, \overline{v}_1} \)) \( \forall a_p(x) \in \mathcal{A}_p^1, \lim_{\eta \to 0^+} \text{ meas } (\mathcal{C}_{a_p, \overline{v}_1} \cap (0, \eta))/\eta = 1. \)

**Proof.** Recall that
\[
B_\infty := \cap_{n \geq 1} B_n \quad = \quad \{ (\delta, v_1) \in A_0 : \left| \omega(\delta)l - j - \delta^{p-1} \frac{M(\delta, v_1, \overline{w}(\delta, v_1))}{2j} \right| \geq \frac{2\gamma}{(l + j)^r}, \ \left| \omega(\delta)l - j \right| \geq \frac{2\gamma}{(l + j)^r}, \ \forall l, j \geq \frac{1}{3\beta^{p-1}}, \ l \neq j \}
\]
where \( \omega(\delta) = \sqrt{1 + 2\delta^{p-1}} \) (or \( \omega(\delta) = \sqrt{1 - 2\delta^{p-1}} \)), \( M(\delta, v_1, w) \) is defined in Definition 3.1 and \( \overline{w}(\delta, v_1) \) in Lemma 3.3.
Let $0 < \eta < \delta_0$. The complementary set of $C_{\subseteq, \tau_1}$ in $(0, \eta)$ is

$$C^c_{a_{\subseteq, \tau_1}} := \{ \delta \in (0, \eta) \mid \left| \omega(\delta)l - j - \frac{\delta^{p-1}m(\delta)}{2j} \right| < \frac{2\gamma}{(l+j)^\tau} \text{ or } \left| \omega(\delta)l - j \right| < \frac{2\gamma}{(l+j)^\tau} \}$$

for some $l, j > \frac{1}{3\delta^{p-1}}, l \neq j$

where $m(\delta) := M(\delta, v_1(\delta), \bar{\omega}(\delta, v_1(\delta)))$ is a function in $C^\infty([0, \delta_0], \mathbb{R})$ since $\delta \mapsto v_1(\delta)$ is $C^\infty$ and $\bar{\omega}(\delta, v_1)$ is, by Lemma 3.3 in $C^\infty(A_0, W \cap X_{\tau/2, s})$. This implies, in particular,

$$|m(\delta)| + |m'(\delta)| \leq C, \quad \forall \delta \in [0, \delta_0/2] \tag{63}$$

for some positive constant $C$.

We claim that, for any interval $[\delta_1/2, \delta_1] \subset (0, \eta) \subset [0, \delta_0/2]$ the following measure estimate holds:

$$\text{meas}(C^c_{a_{\subseteq, \tau_1}} \cap \left[ \frac{\delta_1}{2}, \delta_1 \right]) \leq K_1(\tau) \gamma \eta^{(p-1)(\tau-1)} \text{meas}\left( \left[ \frac{\delta_1}{2}, \delta_1 \right] \right) \tag{64}$$

for some constant $K_1(\tau) > 0$.

Before proving (64), we show how to conclude the proof of the Lemma. Writing $(0, \eta) = \cup_{n \geq 0} [\eta/2^n + 1, \eta/2^n]$ and applying the measure estimate (64) to any interval $[\delta_1/2, \delta_1] = [\eta/2^{n+1}, \eta/2^n]$, we get

$$\text{meas}(C^c_{a_{\subseteq, \tau_1}} \cap [0, \eta]) \leq K_1(\tau) \gamma \eta^{(p-1)(\tau-1)} \eta,$$

whence $\lim_{\eta \to 0^+} \text{meas}(C_{a_{\subseteq, \tau_1}} \cap (0, \eta))/\eta = 1$, proving the Lemma.

We now prove (64). We have

$$C^c_{a_{\subseteq, \tau_1}} \cap \left[ \frac{\delta_1}{2}, \delta_1 \right] \subset \bigcup_{(l, j) \in I_R} \mathcal{R}_{l, j}(\delta_1) \tag{65}$$

where

$$\mathcal{R}_{l, j}(\delta_1) := \left\{ \delta \in \left[ \frac{\delta_1}{2}, \delta_1 \right] \mid \left| \omega(\delta)l - j - \frac{\delta^{p-1}m(\delta)}{2j} \right| < \frac{2\gamma}{(l+j)^\tau} \text{ or } \left| \omega(\delta)l - j \right| < \frac{2\gamma}{(l+j)^\tau} \right\}$$

and

$$I_R := \left\{ (l, j) \mid l > \frac{1}{3\delta^{p-1}}, l \neq j; \quad j \in [1 - c_0\delta^{p-1}, 1 + c_0\delta^{p-1}]) \right\}$$

(note indeed that $\mathcal{R}_{j,l}(\delta_1) = \emptyset$ unless $j/l \in [1 - c_0\delta^{p-1}, 1 + c_0\delta^{p-1}]$ for some constant $c_0 > 0$ large enough).

Next, let us prove that

$$\text{meas}(\mathcal{R}_{l, j}(\delta_1)) = O\left( \frac{\gamma}{(l+1)^{\tau+1}} \right). \tag{66}$$

Define $f_{l, j}(\delta) := \omega(\delta)l - j - (\delta^{p-1}m(\delta)/2j)$ and $S_{j,l}(\delta_1) := \{ \delta \in [\delta_1/2, \delta_1] \mid |f_{l, j}(\delta)| < 2\gamma/(l+j)^\tau \}$. Provided $\delta_0$ has been chosen small enough (recall that $j, l \geq 1/3\delta_0^{p-1}$),

$$f'_{l, j}(\delta) = \frac{(p-1)\delta^{p-2}}{\sqrt{1 + 2\delta^{p-1}}} - \frac{(p-1)\delta^{p-2}m(\delta)}{2j} - \frac{\delta^{p-1}m'(\delta)}{2j} \geq \frac{(p-1)\delta^{p-2}}{2} \left( l - \frac{C}{l} \right) \geq \frac{(p-1)\delta^{p-2}l}{4}$$

and therefore $f'_{l, j}(\delta) \geq (p-1)\delta^{p-2}l/2^p$ for any $\delta \in [\delta_1/2, \delta_1]$. This implies

$$\text{meas}(S_{j,l}(\delta_1)) \leq \frac{4\gamma}{(l+j)^\tau} \times \left( \min_{\delta \in [\delta_1/2, \delta_1]} f'_{l, j}(\delta) \right)^{-1} \leq \frac{4\gamma}{(l+j)^\tau} \times \frac{2^p}{(p-1)l\delta^{p-2}} = O\left( \frac{\gamma}{(l+1)^{\tau+1}} \right).$$

Similarly we can prove

$$\text{meas}\left( \left\{ \delta \in \left[ \frac{\delta_1}{2}, \delta_1 \right] \mid |\omega(\delta)l - j| < \frac{2\gamma}{(l+j)^\tau} \right\} \right) = O\left( \frac{\gamma}{(l+1)^{\tau+1}} \right).$$
and the measure estimate (60) follows.

Now, by (59), (60) and since, for a given \( l \), the number of \( j \) for which \( (l, j) \in I_R \) is \( O(\delta_1^{-l}) \),

\[
\text{meas}(C_{\delta_2, \mathcal{R}_l} \cap \left[ \frac{\delta_1}{2}, \delta_1 \right]) \leq \sum_{(l, j) \in I_R} \text{meas}(R_{j, l}(\delta_1)) \leq C \sum_{l \geq 1/3\delta_1^{-1}} \delta_1^{p-1}l \times \frac{\gamma}{l^{r+1} \delta_1^{p-2}} \leq K_2(\tau) \gamma \delta_1^{1+(p-1)(r-1)}
\]

whence we obtain (61) since \( 0 < \delta_1 < \eta \).

Now for \( \sigma, s, n \), formula

\[
\bar{u}(\delta) := \delta \left[ v_1(\delta) + \bar{w}(\delta, v_1(\delta)) + v_2(\delta, v_1(\delta), \bar{w}(\delta, v_1(\delta))) \right]
\]

defines a smooth path \( \bar{u} : [0, \delta_0) \to X_{\pi/2, \pi} \), and \( \bar{u}(\delta) = \delta u_0 + O(\delta^2) \) with \( u_0 = \bar{v}_1 + v_2(0, \bar{v}_1, 0) \in V \).

By Lemma 5.3 \( \bar{u} \) is a solution of (67) for \( \delta \in C_{\delta_2, \mathcal{R}_l} \) and conclusions (i) and (ii) of Theorem 1.1 hold by Proposition 5.1.

Now we can look for \( 2\pi/(n \omega) \) time-periodic solutions of (11) (i.e. \( 2\pi/n \) time-periodic solutions of (40)) as well. Let

\[
X_{\sigma, s, n} := \left\{ u \in X_{\sigma, s} \mid u \text{ is } 2\pi/n \text{ time-periodic} \right\}.
\]

Replacing \( X_{\sigma, s} \) with \( X_{\sigma, s, n} \), we can develop similarly the arguments of sections 2 and 3. Define the linear map \( \mathcal{H}_n : V \to V \) by:

\[
(\mathcal{H}_n u)(t, x) := \eta(n(t + x)) - \eta(n(t - x))
\]

and denote by \( V_n := \mathcal{H}_n V \) (resp. \( W_n \)) the subspace of \( V \) (resp. \( W \)) formed by the functions \( 2\pi/n \)-periodic in \( t \).

Using the decomposition \( X_{\sigma, s, n} = V_n \oplus W_n \) and introducing an appropriate finite dimensional subspace \( V_{1, n} \) of \( V_n \), we obtain associated \( (Q1), (Q2), (P) \)-equations (like in (10)), which can be solved exactly as in the case \( n = 1 \).

The \( 0^{th} \)-order bifurcation equation is the same (but in \( V_n \)) and the corresponding functional is just the restriction of \( \Phi_0 \) to \( V_n \). As \( \Phi_0, \Phi_{0|V_n} \) (or \( \bar{\Phi}_0|V_n \)) possesses “mountain pass” critical circles. Let, for \( n \geq 2 \),

\[
A_p^n := \left\{ a_p(x) \in \mathcal{O} \subset H^1(0, \pi) \mid \text{there is a non-degenerate critical circle } [\pi] \subset V \text{ of } \Phi_{0|V_n} \text{ or } \bar{\Phi}_{0|V_n} \right\},
\]

and \( A_p = \cup_{n=1}^{\infty} A_p^n \). By the implicit function theorem, \( A_p \) is an open subset of \( H^1(0, \pi) \).

By the arguments of Proposition 5.1 we obtain that if \( a_p \in A_p \), then conclusions (i) and (ii) of Theorem 1.1 hold (since \( 2\pi/(n \omega) \) time-periodic solutions are just peculiar \( 2\pi/\omega \) time-periodic solutions).

This proves the first part of Theorem 1.1.

### 5.2 Case \( f(x, u) = a_3(x)u^3 + O(u^4) \)

We assume here that \( f(x, u) = a_3(x)u^3 + O(u^4) \), where

\[
\frac{1}{\pi} \int_0^\pi a_3(x) \, dx := \langle a_3 \rangle \neq 0.
\]

(67)

To fix the ideas, we deal with the case \( \langle a_3 \rangle > 0 \).

Note that assumption (64) also implies condition (60), i.e. that there is \( v \in V \) such that \( \int_\Omega a_3(x)v^4 \neq 0 \).

Indeed we know that \( \int_\Omega a_3(x)v^4 = 0 \), \( \forall v \in V \) iff \( a_3(\pi - x) = -a_3(x), \forall x \in [0, \pi] \). But in this case \( \langle a_3 \rangle = 0 \).

Therefore the \( 0^{th} \)-order bifurcation equation is (49) (with \( p = 3 \)), i.e. the Euler-Lagrange equation of

\[
\Phi_0(v) = \frac{\|v\|^2_{H^1}}{2} - \int_\Omega a_3(x)v^4 \frac{4}{4}
\]

(68)
The functional $\Phi_n(v) = \Phi_0(\mathcal{H}_n v)$ has the following development: for $v(t, x) = \eta(t + x) - \eta(t - x) \in V$ we obtain, using that 
\[
\int_\Omega v^4 = \int_\Omega (\mathcal{H}_n v)^4,
\]
\[
\Phi_n(v) = 2\pi n^2 \int_\Omega \eta^2(t)dt - \langle a_3 \rangle \int_\Omega \frac{v^4}{4} - \int_\Omega \left( a_3(x) - \langle a_3 \rangle \right) (\mathcal{H}_n v)^4.
\]
Hence
\[
\Phi_n\left( \frac{\sqrt{2\pi} n}{\sqrt{\langle a_3 \rangle}} v \right) = \frac{8\pi n^4}{\langle a_3 \rangle} \left\{ \frac{1}{2} \int_\Omega \eta^2(s)ds - \frac{1}{8\pi} \int_\Omega v^4 + \frac{1}{8\pi} \int_\Omega \left( \frac{a_3(x)}{\langle a_3 \rangle} - 1 \right) (\mathcal{H}_n v)^4 dt \right\}
\]
where
\[
\Psi(\eta) := \frac{1}{2} \int_\Omega \eta^2(s)ds - \frac{1}{4} \int_\Omega \eta^4(s)ds - \frac{3}{8\pi} \int_\Omega \eta^2(s)ds^2,
\]
\[
R_n(v) := \frac{1}{8\pi} \int_\Omega b(x)(\mathcal{H}_n v)^4 dt dx, \quad b(x) := \frac{a_3(x)}{\langle a_3 \rangle} - 1.
\]

To get that $a_3 \in A_3$, it is enough to prove that $\Psi$ has a non-degenerate critical circle and that $R_n$ is a small perturbation for large $n$, more precisely that $D^2 R_n \to 0$, $D R_n \to 0$ uniformly on bounded sets as $n \to +\infty$. Then, by the implicit function theorem, for $n$ large enough, $\Phi_n$ too (hence $\Phi_{0|V_n}$) has a non-degenerate critical circle, which implies that $a_3 \in A_3$.

The critical points of $\Psi$ in $E := \{ \eta \in H^1(T) \mid \int_T \eta = 0 \}$ are the $2\pi$-periodic solutions with zero mean value of
\[
\dot{\eta} + \eta^3 + 3\langle \eta^2 \rangle \eta = C, \quad C \in \mathbb{R}
\]
By [BP] it is known that there exists a solution to this problem (with $C = 0$) which is a non-degenerate (up to time translations) critical point of $\Psi$ in $E$.

**Lemma 5.1** There holds
\[
||D R_n(v)||, \ |D^2 R_n(v)|| \to 0 \quad \text{as} \quad n \to +\infty
\]
uniformly for $v$ in bounded sets.

**Proof.** We shall prove the estimate only for $D^2 R_n$. We have
\[
|D^2 R_n(v)[h, k]| = \frac{3}{2\pi} \int_\Omega b(x)(\mathcal{H}_n v)^2 (\mathcal{H}_n h) (\mathcal{H}_n k)
\]
\[
= \frac{3}{2\pi} \int_0^\pi b(x)g(nx) dx
\]
where $g(y)$ is the $\pi$-periodic function defined by
\[
g(y) := \int_T (\eta(t + y) - \eta(t - y))^2 (\beta(t + y) - \beta(t - y)) (\gamma(t + y) - \gamma(t - y)) dt,
\]
$\beta$ and $\gamma$ being associated with $h$ and $k$ as $\eta$ is with $v$. Developing in Fourier series $g(y) = \sum_{l \in \mathbb{Z}} g_l \exp(i2ly)$ we have $g(nx) = \sum_{l \in \mathbb{Z}} g_l \exp(i2lnx)$. Extending $b(x)$ to a $2\pi$-periodic function, we also write $b(x) = \sum_{l \in \mathbb{Z}} b_l \exp(i2lx)$, with $b_0 = \langle b \rangle = 0$.

Therefore
\[
|D^2 R_n(v_n)[h, k]| = \frac{3}{2} \left| \sum_{l \neq 0} g_l b_{-ln} \right| \leq \frac{3}{2} \left( \sum_{l \neq 0} g_l^2 \right)^{1/2} \left( \sum_{l \neq 0} b_{ln}^2 \right)^{1/2}
\]
\[
\leq \frac{3}{2} ||g||_{L^2(0, \pi)} \left( \sum_{l \neq 0} b_{ln}^2 \right)^{1/2}
\]
\[
\leq C ||v_0||_\infty^2 ||\beta||_\infty ||\gamma||_\infty \left( \sum_{l \neq 0} b_{ln}^2 \right)^{1/2}.
\]
Since \( \sum_{l \neq 0} b_{l}^{2} )^{1/2} \rightarrow 0 \) as \( n \rightarrow \infty \) it proves (ii). With a similar calculus we can prove that \( DR_{n}(v) \rightarrow 0 \) as \( n \rightarrow +\infty \).

This completes the proof of part 1) of Theorem 1.1

5.3 Case \( f(x, u) = a_{2}u^{2} + O(u^{4}) \)

We now prove part 2) of Theorem 1.1

In this case condition (i) is violated. It turns out (see [3]) that we must take \( \omega < 1 \) (i.e. \( \varepsilon < 0 \)), and \( \delta = |\varepsilon|^{1/2} \) in the rescaling. By the computations of [3], the 0th-order bifurcation equation is the Euler Lagrange equation of the functional \( \Phi \) defined by

\[
\Phi(v) = \frac{||v||_{H}^{2}}{2} + \frac{a_{2}^{2}}{2} \int_{\Omega} v^{2}L^{-1}v^{2},
\]

where \( L^{-1} : W \rightarrow W \) is the inverse operator of \( -\partial_{tt} + \partial_{xx} \).

We can still use the same arguments to solve the (Q2) and (P) equations. As explained in subsection 5.4 if the functional \( \Phi_{n} : V \rightarrow \mathbb{R} \), defined by \( \Phi_{n}(v) = \Phi(\mathcal{H}_{n}v) \) possesses a nondegenerate critical circle, then (i) and (ii) hold in Theorem 1.1 (with \( \omega = \sqrt{1 - 2\delta^{2}} \)).

\( \Phi_{n} \) admits the following development (Lemmae 3.7 and 3.8 in [4]): for \( v(t, x) = \eta(t + x) - \eta(t - x) \)

\[
\Phi_{n}(v) = 2\pi n^{2} \int_{T} \dot{\eta}^{2}(t)dt - \frac{\pi a_{2}^{2}}{12} \left( \int_{T} \eta^{2}(t) dt \right)^{2} + \frac{a_{2}^{2}}{2n^{2}} \left( \int_{\Omega} v L^{-1}v^{2} + \frac{\pi^{2}}{6} \left( \int_{T} \eta^{2}(t) dt \right)^{2} \right) .
\]

Hence we can write

\[
\Phi_{n}(\sqrt{\frac{12n}{\pi a_{2}}} \eta) = \frac{48n^{4}}{a_{2}^{2}} \int_{T} \dot{\eta}^{2}(s) ds - \frac{1}{4} \left( \int_{T} \eta^{2}(s) ds \right)^{2} + \frac{1}{n^{2}} R(\eta) \right] = \frac{48n^{4}}{a_{2}^{2}} \left[ \Psi(\eta) + \frac{1}{n^{2}} R(\eta) \right] \quad (72)
\]

where

\[
\Psi(\eta) = \frac{1}{2} \int_{T} \dot{\eta}^{2}(s) ds - \frac{1}{4} \left( \int_{T} \eta^{2}(s) ds \right)^{2} \quad (73)
\]

and \( R : E \rightarrow \mathbb{R} \) is a smooth map defined on \( E := \{ \eta \in H^{1}(T) \mid \int_{T} \eta = 0 \} \). In order to prove that \( \Phi_{n} \) has a non-degenerate critical circle for \( n \) large enough, it is enough to prove that:

**Lemma 5.2** \( \Psi : E \rightarrow \mathbb{R} \) possesses a critical point which is non degenerate, up to time translations.

**Proof.** The critical points of \( \Psi \) in \( E \) are the 2\( \pi \)-periodic solutions of zero mean value of the equation

\[
\ddot{\eta} + \left( \int_{T} \eta^{2}(t) dt \right) \dot{\eta} = C, \quad C \in \mathbb{R} .
\]

Since \( \ddot{\eta} \) and \( \eta \) have both zero mean value, any solution of (74) must satisfy \( C = 0 \). Equation (74) (with \( C = 0 \)) has a 2\( \pi \)-periodic solution of the form \( \dot{\eta}(t) = (1/\sqrt{\pi}) \sin t \).

We claim that \( \dot{\eta} \) is non-degenerate, up to time-translations, i.e. \( \text{Ker}(D^{2} \Psi)(\dot{\eta}) = \langle \dot{\eta} \rangle \).

The linearized equation of (74) at \( \dot{\eta} \) is

\[
\ddot{h} + \left( \int_{T} \eta^{2}(t) dt \right) h + 2 \left( \int_{T} \eta(t)h(t) dt \right) \dot{\eta} = c,
\]

and again \( c \) must be equal to 0. We get

\[
\ddot{h} + \frac{2}{\pi} \left( \int_{T} \sin t h(t) dt \right) \sin t = 0 . \quad (75)
\]

Developing in time-Fourier series

\[
h(t) = \sum_{k \geq 1} \left( a_{k} \sin kt + b_{k} \cos kt \right)
\]
we find out that any solution of the linearized equation \( \text{eq} \) satisfies
\[
-k^2 b_k + b_k = 0, \quad \forall k \geq 1, \quad -k^2 a_k + a_k = 0, \quad \forall k \geq 2, \quad a_1 = 0
\]
and therefore \( h \in \langle \cos t \rangle = \langle \dot{h} \rangle \).

By the previous Lemma, for \( n \) large enough there is a non-degenerate critical circle of \( \Phi_n \) in \( V \). This completes the proof of Theorem \[8\].

\section{Appendix}

\textbf{Lemma 6.1} If \( q \) is an even integer, then
\[
\int_\Omega a(x)v^q(t,x) \, dt \, dx = 0, \quad \forall v \in V \iff \left\{ a(\pi - x) = -a(x), \quad \forall x \in [0, \pi] \right\}.
\]

If \( q \geq 3 \) is an odd integer, then
\[
\int_\Omega a(x)v^q(t,x) \, dt \, dx = 0, \quad \forall v \in V \iff \left\{ a(\pi - x) = a(x), \quad \forall x \in [0, \pi] \right\}.
\]

\textbf{Proof.} We first assume that \( q = 2s \) is even. If \( a(\pi - x) = -a(x) \) \( \forall x \in (0, \pi) \), then, for all \( v \in V \),
\[
\int_\Omega a(x)v^{2s}(t,x) \, dt \, dx = \int_\Omega a(\pi - x)v^{2s}(t,\pi - x) \, dt \, dx
= \int_\Omega -a(x)(-v(t + \pi,x))^{2s} \, dt \, dx
= -\int_\Omega a(x)v^{2s}(t,x) \, dt \, dx
\]
and so \( \int_\Omega a(x)v^{2s}(t,x) \, dt \, dx = 0 \).

Now assume that \( \Sigma(v) := \int_\Omega a(x)v^{2s}(t,x) \, dt \, dx = 0 \) \( \forall v \in V \). Writing that \( D^{2s}\Sigma = 0 \), we get
\[
\int_\Omega a(x)v_1(t,x) \ldots v_2s(t,x) \, dt \, dx = 0, \quad \forall (v_1, \ldots, v_{2s}) \in V^{2s}.
\]
Choosing \( v_{2s}(t,x) = v_{2s-1}(t,x) = \cos (lt) \sin(lx) \), we obtain
\[
\frac{1}{4} \int_\Omega a(x)v_1(t,x) \ldots v_{2(s-1)}(t,x)(\cos(2lt) + 1)(1 - \cos(2lx)) \, dt \, dx = 0
\]
Taking limits as \( l \to \infty \), there results \( \int_\Omega a(x)v_1(t,x) \ldots v_{2(s-1)}(t,x) \, dt \, dx = 0 \) \( \forall (v_1, \ldots, v_{2(s-1)}) \in V^{2(s-1)} \).

Iterating this operation, we finally get
\[
\forall (v_1, v_2) \in V^2 \quad \int_\Omega a(x)v_1(t,x)v_2(t,x) \, dt \, dx = 0, \quad \text{and} \quad \int_0^\pi a(x) \, dx = 0.
\]
Choosing \( v_1(t,x) = v_2(t,x) = \cos(lt)\sin(lx) \) in the first equality, we derive that \( \int_0^\pi a(x)\sin^2(lx) \, dx = 0 \).

Hence
\[
\forall l \in \mathbb{N} \quad \int_0^\pi a(x)\cos(2lx) \, dx = 0.
\]
This implies that \( a \) is orthogonal in \( L^2(0, \pi) \) to
\[
F = \left\{ b \in L^2(0, \pi) \mid b(\pi - x) = b(x) \text{ a.e.} \right\}.
\]
Hence \( a(\pi - x) = -a(x) \) a.e., and, since \( a \) is continuous, the identity holds everywhere.
We next assume that \( q = 2s + 1 \) is odd, \( q \geq 3 \). The first implication is derived in a similar way. Now assume that \( \int_{\Omega} a(x) v^\alpha(t, x) \, dt \, dx = 0 \quad \forall v \in V \). We can prove exactly as in the first part that

\[
\forall (v_1, v_2, v_3) \in V^3 \int_{\Omega} a(x) v_1(t, x) v_2(t, x) v_3(t, x) \, dt \, dx = 0.
\]

Choosing \( v_1(t, x) = \cos(l_1 t) \sin(l_1 x) \), \( v_2(t, x) = \cos(l_2 t) \sin(l_2 x) \), \( v_3(t, x) = \cos((l_1 + l_2) t) \sin((l_1 + l_2) x) \) and using the fact that \( \int_{0}^{2\pi} \cos(l_1 t) \cos(l_2 t) \cos((l_1 + l_2) t) \, dt \neq 0 \), we obtain

\[
\int_{0}^{\pi} a(x) \left[ \sin^2(l_1 x) \cos(l_2 x) + \sin^2(l_2 x) \sin(l_1 x) \cos(l_1 x) \right] \, dx = \int_{0}^{\pi} a(x) \sin(l_1 x) \sin(l_2 x) \sin((l_1 + l_2) x) \, dx = 0.
\]

(76)

Letting \( l_2 \) go to infinity and taking limits, (76) yields \( \int_{0}^{\pi} (1/2) a(x) \sin(l_1 x) \cos(l_1 x) \, dx = 0 \). Hence

\[
\forall l > 0 \int_{0}^{\pi} a(x) \sin(2lx) = 0.
\]

This implies that, in \( L^2(0, \pi) \), \( a \) is orthogonal to \( G = \{ b \in L^2(0, \pi) \mid b(\pi - x) = -b(x) \text{ a.e.} \} \). Hence \( a(\pi - x) = a(x) \forall x \in (0, \pi) \). \( \square \)

**Proof of Lemma 4.1** Let \( K_k(\varepsilon) = S_k^{-1}(\varepsilon) \) be the selfadjoint compact operator of \( F_k \) defined by

\[
(K_k(\varepsilon) u, v)_\varepsilon = (u, v)_{L^2}, \quad \forall u, v \in F_k
\]

(in other words \( K_k(\varepsilon) u \) is the unique weak solution \( z \in F_k \) of \( S_k z := u \)).

Note that \( K_k(\varepsilon) \) is a positive operator, i.e., \( K_k(\varepsilon) u, u \rangle_\varepsilon > 0, \forall u \neq 0 \), and that \( K_k(\varepsilon) \) is also selfadjoint for the \( L^2 \)-scalar product.

Let \( F_{k,r} = \{ u \in F_k \mid u((0, \pi)) \subset \mathbb{R} \} \). We have \( S_k(F_{k,r} \subset F_{k,r}, \) and by the spectral theory of compact selfadjoint operators in Hilbert spaces, there is a \( \langle \cdot, \cdot \rangle_{\varepsilon} \)-orthonormal basis \((v_{k,j})_{j \geq 1, j \neq k}\) of \( F_k \), such that \( v_{k,j} \in F_{k,r} \) is an eigenvector of \( K_k(\varepsilon) \) associated to a positive eigenvalue \( \lambda_{k,j}(\varepsilon) \), the sequence \( (\lambda_{k,j}(\varepsilon))_j \) is non-increasing and tends to 0 as \( j \to +\infty \). Each \( v_{k,j}(\varepsilon) \) belongs to \( D(S_k) \) and is an eigenvector of \( S_k \) with associated eigenvalue \( \lambda_{k,j}(\varepsilon) = 1/\nu_{k,j}(\varepsilon) \). \( (\lambda_{k,j}(\varepsilon))_{j \geq 1} \) is a sequence non decreasing and tending to \( +\infty \) as \( j \to +\infty \).

The map \( \varepsilon \mapsto K_k(\varepsilon) \in L(F_k, F_k) \) is differentiable and \( K_k'(\varepsilon) = -K_k(\varepsilon) M K_k(\varepsilon) \), where \( M u := \pi_k(a_0 u) \).

For \( u = \sum_{j \neq k} \alpha_j v_{k,j}(\varepsilon) \in F_k \),

\[
\langle u, u \rangle_\varepsilon = \sum_{j \neq k} |\alpha_j|^2 \quad \text{and} \quad (u, u)_{L^2} = \sum_{j \neq k} \frac{|\alpha_j|^2}{\lambda_{k,j}(\varepsilon)}.
\]

As a consequence,

\[
\lambda_{k,j}(\varepsilon) = \min \left\{ \max_{u \in F_k \mid \|u\|_{L^2} = 1} \langle u, u \rangle_\varepsilon \mid F \text{ subspace of } F_k \text{ of dimension } j \text{ (if } j < k \), } j - 1 \text{ (if } j > k \} \right\}. \quad (77)
\]

It is clear by inspection that \( \lambda_{k,j}(0) = j^2 \) and that we can choose \( v_{k,j}(0) = \sqrt{2/\pi} \sin(jx)/j \). Hence, by \( (14) \), \( |\lambda_{k,j}(\varepsilon) - j^2|^2 \leq |\varepsilon| \| a_0 \|_\infty \leq 1 \), from which we derive

\[
\forall l \neq j \quad |\lambda_{k,l}(\varepsilon) - \lambda_{k,j}(\varepsilon)| \geq (l + j) - 2 \geq 2 \min(l, j) - 1 \quad (\geq 1).
\]

(78)

In particular, the eigenvalues \( \lambda_{k,j}(\varepsilon) \) \( (v_{k,j}(\varepsilon)) \) are simple. By the variational characterization \( (17) \) we also see that \( \lambda_{k,j}(\varepsilon) \) depends continuously on \( \varepsilon \), and we can assume without loss of generality that \( \varepsilon \mapsto v_{k,j}(\varepsilon) \) is a continuous map to \( F_k \).
Let $\varphi_{k,j}(\varepsilon) := \sqrt{\lambda_{k,j}(\varepsilon)} v_{k,j}(\varepsilon)$. $(\varphi_{k,j}(\varepsilon))_{j \neq k}$ is a $L^2$-orthogonal family in $F_k$ and

$$\forall \varepsilon \left\{ \begin{array}{l}
K_k(\varepsilon)\varphi_{k,j}(\varepsilon) = \nu_{k,j}(\varepsilon)\varphi_{k,j}(\varepsilon) \\
(\varphi_{k,j}(\varepsilon), \varphi_{k,j}(\varepsilon))_{L^2} = 1
\end{array} \right.$$

We observe that the $L^2$-orthogonality w.r.t. $\varphi_{k,j}(\varepsilon)$ is equivalent to the $(\cdot, \cdot)$-orthogonality w.r.t. $\varphi_{k,j}(\varepsilon)$, and that $E_{k,j}(\varepsilon) := [\varphi_{k,j}(\varepsilon)]^\perp$ is invariant under $K_k(\varepsilon)$. Using that $L_{k,j} := (K_k(\varepsilon) - \nu_{k,j}(\varepsilon)I)|_{E_{k,j}(\varepsilon)}$ is invertible, it is easy to derive from the Implicit Function Theorem that the maps $(\varepsilon \mapsto \nu_{k,j}(\varepsilon))$ and $(\varepsilon \mapsto \varphi_{k,j}(\varepsilon))$ are differentiable.

Denoting by $P$ the orthogonal projector onto $E_{k,j}(\varepsilon)$, we have

$$\varphi'_{k,j}(\varepsilon) = L^{-1}(-PK'_k(\varepsilon)\varphi_{k,j}(\varepsilon)) = L^{-1}(PK_kMK_k\varphi_{k,j}(\varepsilon)) = \nu_{k,j}(\varepsilon)L^{-1}K_kPM\varphi_{k,j}(\varepsilon),$$

$$\nu'_{k,j}(\varepsilon) = \left( K_k'_{j}(\varepsilon)\varphi_{k,j}(\varepsilon), \varphi_{k,j}(\varepsilon) \right)_{L^2} = -\left( K_kMK_k\varphi_{k,j}(\varepsilon), \varphi_{k,j}(\varepsilon) \right)_{L^2} = -\nu_{k,j}(\varepsilon)\left( M\varphi_{k,j}(\varepsilon), \varphi_{k,j}(\varepsilon) \right)_{L^2}. \quad (79)$$

We have

$$\nu_{k,j}L^{-1}K_k\left( \sum_{l \neq j} \alpha_l v_{k,l} \right) = \sum_{l \neq j} \frac{\nu_{k,j}v_{k,l}}{\nu_{k,l} - \nu_{k,j}}\alpha_l v_{k,l} = \sum_{l \neq j} \frac{\alpha_l}{\lambda_{k,j} - \lambda_{k,l}}v_{k,l}. \quad \text{Hence, by (78), } |\nu_{k,j}L^{-1}K_kPu|_{L^2} \leq |u|_{L^2}/j. \quad \text{We obtain } |\varphi'_{k,j}(\varepsilon)|_{L^2} = O(|a_0|_{\infty}/j). \quad \text{Hence}

$$|\varphi_{k,j}(\varepsilon) - \sqrt{\frac{2}{\pi}}\sin(jx)|_{L^2} = O\left( \frac{|a_0|_{\infty}}{j} \right).$$

Hence, by (78),

$$\lambda'_{k,j}(\varepsilon) = \left( M\varphi_{k,j}(\varepsilon), \varphi_{k,j}(\varepsilon) \right)_{L^2} = \int_0^\pi a_0(x)(\varphi_{k,j}(\varepsilon))^2 \, dx$$

$$= \frac{2}{\pi} \int_0^\pi a_0(x)(\sin(jx))^2 \, dx + O\left( \frac{|a_0|_{\infty}^2}{j} \right).$$

Writing $\sin^2(jx) = (1 - \cos(2jx))/2$, and since $\int_0^\pi a_0(x)\cos(2jx) \, dx = \int_0^\pi (a_0)_x(x)\sin(2jx)/2j \, dx$, we get

$$\lambda'_{k,j}(\varepsilon) = \frac{1}{\pi} \int_0^\pi a_0(x) \, dx + O\left( \frac{|a_0|_{H^1}}{j} \right) = M(\delta, v_1, w) + O(\varepsilon\|a_0\|_{H^1}/j),$$

which is the first estimate in (78).

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