QUOTIENTS OF DEL PEZZO SURFACES OF HIGH DEGREE

ANDREY S. TREPALIN

Abstract. In this paper we study quotients of del Pezzo surface of degree 4 and more over arbitrary field \( k \) of characteristic zero by finite groups of automorphisms. We show that if del Pezzo surface contains a point defined over the ground field and degree is five or greater then the quotient is always \( k \)-rational. If degree is equal to four then the quotient can be non-\( k \)-rational only if the order of group is 1, 2 or 4.

1. Introduction

This paper is a following the author’s previous papers [Tr14] and [Tr13].

Let \( k \) be any field of characteristic zero. We study when quotients of \( k \)-rational surfaces by a finite group are not \( k \)-rational. From Minimal Model Program we know that any quotient of \( k \)-rational surface is birationally equivalent to a quotient of a conic bundle or a del Pezzo surface by the same group.

In the paper [Tr13] we construct infinite-dimensional series of non-\( k \)-rational quotients examples. But all these examples are quotients of conic bundles.

In the paper [Tr14] it is showed that any quotient of projective plane (which is a del Pezzo surface of degree 9) is \( k \)-rational. In this paper we show that quotient of a del Pezzo surface can be non-\( k \)-rational. The main result of the paper is the following.

**Theorem 1.0.1.** Let \( k \) be a field of characteristic zero, \( X \) be a del Pezzo surface over \( k \), \( X(\mathbb{Q}) \neq \emptyset \) and \( G \) be a finite subgroup of automorphisms of \( X \). If \( K^2_X \geq 5 \) then the quotient variety \( X/G \) is \( k \)-rational. If \( K^2_X = 4 \) and the order of \( G \) is not equal to 1, 2 or 4 then \( X/G \) is \( k \)-rational.

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To prove Theorem 1.0.1 we consider case-by-case a del Pezzo surface of certain degree and study its quotients by finite groups. The cases of degree 9 and 6 are considered in the paper [Tr14] (see Theorems 2.4.19 and 2.4.20). The cases of degree 8, 5 and 4 are considered in Propositions 3.1.1, 3.2.7 and 3.3.11 respectively.

The plan of this paper as follows.

In Section 2 we review some notions and facts about minimal rational surfaces, groups, singularities and quotients.

In Section 3 to prove Theorem 1.0.1 we study quotients of del Pezzo surfaces of certain degree case-by-case. In Subsection 3.3 it is also showed that the quotient of a del Pezzo surface of degree 4 by groups of order 2 and 4 can be non-$k$-rational. In 3.3.12, 3.3.14 and 3.3.16 there are constructed examples of action of group such that quotient is not $k$-rational. In all other cases it is proved that the quotient is $k$-rational.

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Convention. Throughout this paper $k$ is any field of characteristic zero, $\overline{k}$ is its algebraic closure. For a surface $X$ the surface $\overline{X} = X \otimes \overline{k}$ is its algebraic closure. If two surfaces $X$ and $Y$ are $k$-birationally equivalent then we write $X \approx Y$. If two divisors $A$ and $B$ are linearly equivalent then we write $A \sim B$.

2. Preliminaries

2.1. $G$-minimal rational surfaces. In this subsection we review main notions and results of $G$-equivariant minimal model program following the papers [Man67], [Isk79], [DI09a], [DI09b].

Definition 2.1.1. A rational variety $X$ is a variety over $k$ such that $\overline{X} = X \otimes \overline{k}$ is birationally equivalent to $\mathbb{P}^n_{\overline{k}}$.

A $k$-rational variety $X$ is a variety over $k$ such that $X$ is birationally equivalent to $\mathbb{P}^n_k$.

A variety $X$ over $k$ is a $k$-unirational variety if there exists a $k$-rational variety $Y$ and a dominant rational map $\varphi: Y \dashrightarrow X$.

Definition 2.1.2. A $G$-surface is a pair $(X, G)$ where $X$ is a projective surface over $k$ and $G$ is a finite subgroup of $\text{Aut}_k(X)$. A morphism of surfaces $f : X \to X'$ is called a $G$-morphism if for each $g \in G$ one has $fg = gf$.

A smooth $G$-surface $(X, G)$ is called $G$-minimal if any birational morphism of smooth $G$-surfaces $(X, G) \to (X', G)$ is an isomorphism.
Let \((X, G)\) be a smooth \(G\)-surface. A \(G\)-minimal surface \((Y, G)\) is called a \textit{minimal model} of \((X, G)\) if there exists a birational \(G\)-morphism \(X \rightarrow Y\).

The following theorem is the classical result about \(G\)-equivariant minimal model program.

**Theorem 2.1.3.** Any \(G\)-morphism \(f : X \rightarrow Y\) can be factorized in the following way:

\[ X = X_0 \rightarrow X_1 \rightarrow \ldots \rightarrow X_{n-1} \rightarrow X_n = Y, \]

where each \(f_i\) is a contraction of a set of disjoint \((-1)\)-curves \(\Sigma_i\) on \(X_i\), \(\Sigma_i\) is defined over \(k\) and \(G\)-invariant. In particular,

\[ K_Y^2 - K_X^2 \geq \rho(X)^G - \rho(Y)^G. \]

The classification of \(G\)-minimal rational surfaces is well-known due to V. Iskovskikh and Yu. Manin (see [Isk79] and [Man67]). We introduce some important notions before surveying it.

**Definition 2.1.4.** A smooth rational \(G\)-surface \((X, G)\) admits a conic bundle structure if there exists a \(G\)-morphism \(\phi : X \rightarrow B\) such that any scheme fibre is isomorphic to a reduced conic in \(P^2_k\) and \(B\) is a smooth curve.

**Definition 2.1.5.** A del Pezzo surface is a smooth projective surface \(X\) such that the anticanonical divisor \(-K_X\) is ample.

A singular del Pezzo surface is a normal projective surface \(X\) such that the anticanonical divisor \(-K_X\) is ample and all singularities of \(X\) are Du Val singularities.

The number \(d = K^2_X\) is called the degree of a (singular) del Pezzo surface \(X\).

A del Pezzo surface \(\overline{X}\) over \(\overline{k}\) is isomorphic to \(\mathbb{P}^2_{\overline{k}}, \mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}\) or the blowup of \(\mathbb{P}^2_{\overline{k}}\) at up to 8 points in general position (see [Man74, Theorem 2.5]). Configuration of \((-1)\)-curves on a del Pezzo surface plays an important role in geometry of del Pezzo surfaces. Throughout this paper we will use notation from the following remark.

**Remark 2.1.6.** Let \(\overline{X}\) be a del Pezzo surface of degree \(d\), \(3 \leq d \leq 7\). Then \(\overline{X}\) isomorphic to the blowup \(f : \overline{X} \rightarrow \mathbb{P}^2_{\overline{k}}\) at \(n = 9 - d\) points \(p_1, \ldots, p_n\) in general position. Put \(E_i = f^{-1}(p_i)\) and \(L = f^*(l)\), where \(l\) is the class of a line on \(\mathbb{P}^2_{\overline{k}}\). One has

\[ -K_{\overline{X}} \sim 3L - \sum_{i=1}^{n} E_i. \]
The set of $(-1)$-curves on $X$ consists of $E_i$, proper transforms $L_{ij} \sim L - E_i - E_j$ of the lines passing through a pair of points $p_i$ and $p_j$ and proper transforms
\[ Q_{i_1\ldots i_{n-5}} \sim 2L - \sum_{j \notin \{i_1,\ldots,i_{n-5}\}} E_j \]
of the conics passing through five points of the blowup.

In this notation one has:
\[
\begin{align*}
E_i \cdot E_j &= 0; & E_i \cdot L_{ij} &= 1; & E_i \cdot L_{jk} &= 0; \\
E_i \cdot Q_{i_1\ldots i_{n-5}} &= 1 \text{ if } i \neq i_j; & E_i \cdot Q_{i_1\ldots i_{n-5}} &= 0 \text{ if } i = i_j; \\
L_{ij} \cdot L_{ik} &= 0; & L_{ij} \cdot L_{kl} &= 1; \\
L_{ij} \cdot Q_{i_1\ldots i_{n-5}} &= 0 \text{ if } i \neq i_k \text{ and } j \neq i_k; & L_{ij} \cdot Q_{i_1\ldots i_{n-5}} &= 1 \text{ if } i = i_k \text{ or } j = i_k.
\end{align*}
\]

**Theorem 2.1.7** ([Isk79, Theorem 1]). Let $X$ be a $G$-minimal rational $G$-surface. Then either $X$ admits a $G$-equivariant conic bundle structure with $\text{Pic}(X)^G \cong \mathbb{Z}_2$, or $X$ is a del Pezzo surface with $\text{Pic}(X)^G \cong \mathbb{Z}$.

**Theorem 2.1.8** (cf. [Isk79, Theorem 4], [Isk79, Theorem 5]). Let $X$ admit a $G$-equivariant conic bundle structure, $G \subset \text{Aut}(X)$. Then:
\[
\begin{align*}
\text{(i)} & \text{ If } K_X^2 = 3, 5, 6, 7 \text{ or } X \cong \mathbb{F}_1 \text{ then } X \text{ is not } G\text{-minimal.} \\
\text{(ii)} & \text{ If } K_X^2 = 8 \text{ then } X \text{ is isomorphic to } \mathbb{F}_n \text{ and } G\text{-minimal if } n \neq 1. \\
\text{(iii)} & \text{ If } K_X^2 \neq 3, 5, 6, 7, 8 \text{ and } \rho(X)^G = 2 \text{ then } X \text{ is } G\text{-minimal.}
\end{align*}
\]

The following theorem is an important criterion of $k$-rationality over an arbitrary perfect field $k$.

**Theorem 2.1.9** ([Isk96, Chapter 4]). A minimal rational surface $X$ over a perfect field $k$ is $k$-rational if and only if the following two conditions are satisfied:
\[
\begin{align*}
\text{(i)} & \text{ } X(k) \neq \emptyset; \\
\text{(ii)} & \text{ } K_X^2 \geq 5.
\end{align*}
\]

An important class of rational surfaces is the class of toric surfaces.

**Definition 2.1.10.** A toric variety is a normal variety over $k$ containing an algebraic torus as a Zariski dense subset, such that the action of the torus on itself by left multiplication extends to the whole variety.

A variety $X$ is called a $k$-form of a toric variety if $X$ is toric.

Obviously, a $k$-form of a toric variety is $k$-rational.

The following lemma is well-known.
Lemma 2.1.11. Let $X$ be a $G$-minimal rational surface such that $X(\Bbbk) \neq \emptyset$. The following are equivalent:

(i) $X$ is a $\Bbbk$-form of a toric surface;

(ii) $K_X^2 \geq 6$;

(iii) $X$ is isomorphic to $\Bbb{P}^2_\Bbbk$, a smooth quadric $Q \subset \Bbb{P}^3_\Bbbk$, a del Pezzo surface of degree 6 or a minimal rational ruled surface $\Bbb{F}_n$ $(n \geq 2)$.

Corollary 2.1.12. Let $X$ be a rational $G$-surface such that $X(\Bbbk) \neq \emptyset$ and $\rho(X)^G + K_X^2 \geq 7$. Then there exists a $G$-minimal model $Y$ of $X$ such that $Y$ is a $\Bbbk$-form of toric surface. In particular, $X$ is $\Bbbk$-rational.

Proof. By Theorem 2.1.7 there exists a birational $G$-morphism $f : X \to Z$ such that $\rho(Z)^G \leq 2$. By Theorem 2.1.3 one has $K_Z^2 \geq K_X^2 + \rho(X)^G - \rho(Z)^G \geq 7 - \rho(Z)^G$.

If $\rho(Z)^2 = 1$ then $K_Z^2 \geq 6$ and $Z$ is a $\Bbbk$-form of a toric surface by Lemma 2.1.11. If $\rho(Z)^G = 2$ and $K_Z^2 = 5$ then $Z$ is not $G$-minimal by Theorem 2.1.8. Therefore there exists a minimal model $Y$ of $Z$ such that $K_Y^2 \geq 6$ and $Y$ is a $\Bbbk$-form of toric surface by Lemma 2.1.11.

The set $X(\Bbbk)$ is not empty so $Y(\Bbbk) \neq \emptyset$ and $X \approx Y$ is $\Bbbk$-rational by Theorem 2.1.9.

2.2. Groups. In this subsection we collect some results and notation used in this paper about groups.

In this paper we use the following notation:

- $\mathfrak{C}_n$ denotes the cyclic group of order $n$.
- $\mathfrak{D}_{2n}$ denotes the dihedral group of order $2n$.
- $\mathfrak{S}_n$ denotes the symmetric group of degree $n$.
- $\mathfrak{A}_n$ denotes the alternating group of degree $n$.
- $(i_1i_2\ldots i_j)$ denotes an element of $\mathfrak{S}_n$ which moves $i_1$ to $i_2$, $i_2$ to $i_3$, ..., $i_j$ to $i_1$.
- $\mathfrak{V}_4$ denotes Klein group isomorphic to $\mathfrak{C}_2^2$.
- $A \triangle B$ denotes a diagonal product of $A$ and $B$ over their common homomorphic image $D$, i.e. the subgroup of $A \times B$ of pairs $(a; b)$ such that $\alpha(a) = \beta(b)$ for some surjections $\alpha : A \to D$, $\beta : B \to D$.
- $\operatorname{diag}(a, b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.
- $\omega = e^{2\pi i}$.

To find fixed points of groups acting on a del Pezzo surface of degree 8 we apply the following well-known lemma. For the proof see, for example, [Tr13, Lemma 3.4].
Lemma 2.2.13. Elements \( g_1, g_2 \in \text{PGL}_2(\mathbb{K}) \) such that the group \( H = \langle g_1, g_2 \rangle \) is finite have the same pair of fixed points on \( \mathbb{P}^1_{\mathbb{K}} \) if and only if the group \( H \) is cyclic. Otherwise the elements \( g_1 \) and \( g_2 \) do not have a common fixed point.

The group \( \mathfrak{S}_5 \) often appears as a group of automorphisms of a rational surface. Therefore it is important to know its subgroups and normal subgroups of these subgroups. The following lemma is an easy exercise.

Lemma 2.2.14. Any nontrivial subgroup \( G \subset \mathfrak{S}_5 \) contains a nontrivial normal subgroup \( N \) conjugate to one of the following groups:

- \( C_2 \cong \langle (12) \rangle \),
- \( C_2 \cong \langle (12)(34) \rangle \),
- \( C_3 \cong \langle (123) \rangle \),
- \( \mathfrak{S}_4 \cong \langle (12)(34), (13)(24) \rangle \),
- \( C_5 \cong \langle (12345) \rangle \),
- \( A_5 \).

2.3. Singularities. All singularities appearing in this paper are toric singularities. These singularities are locally isomorphic to the quotient of \( \mathbb{A}^2 \) by a cyclic group generated by \( \text{diag}(\xi_m, \xi_q_m) \), where \( \xi_m = e^{2\pi i/m} \).

Such a singularity is called \( \frac{1}{m}(1, q) \). If \( \gcd(m, q) > 1 \) then the group \( \langle \text{diag}(\xi_m, \xi_q_m) \rangle \) contains a reflection and the quotient singularity is isomorphic to a quotient singularity with smaller \( m \).

A toric singularity can be resolved by some weighted blowups. Therefore it is easy to describe some numerical properties of a quotient singularity. We list here these properties for singularities appearing in this paper.

Remark 2.3.15. Let the group \( \mathfrak{C}_m \) act on a smooth surface \( X \) and \( f : X \to S \) be a quotient map. Let \( p \) be a singular point on \( S \) of type \( \frac{1}{m}(1, q) \), \( C \) and \( D \) be curves passing through \( p \) such that \( f^{-1}(C) \) and \( f^{-1}(D) \) are \( \mathfrak{C}_m \)-invariant and tangent vectors of these curves at the point \( f^{-1}(p) \) are eigenvectors of \( \mathfrak{C}_m \) action (\( C \) corresponds to the eigenvalue 1 and \( D \) corresponds to the eigenvalue \( q \)).

Let \( \pi : \tilde{S} \to S \) be a resolution of the singular point \( p \). Then the exceptional divisor is a chain of negative curves whose selfintersection is written in the last column of the following table. The curves \( \pi^{-1}(C) \) and \( \pi^{-1}(D) \) transversally intersect at a point only the first and the last of these curves respectively.

The following table presents changing of some numerical properties of \( \tilde{S} \) and \( S \) for the singularities with \( m \leq 5 \).
2.4. Quotients. In this subsection we collect some results about quotients of rational surfaces.

The following lemma is well-known, see, e.g., [Tr14, Lemma 4.2].

**Lemma 2.4.16.** Let \( \overline{X} \) be an \( n \)-dimensional toric variety over a field \( \mathbb{k} \) and let \( G \) be a finite subgroup in \( \text{Aut}(\overline{X}) \) conjugate to a subgroup of \( n \)-dimensional torus \( \mathbb{T}^n \subset \overline{X} \) acting on \( \overline{X} \). Then the quotient \( \overline{X}/G \) is a toric variety.

In particular, if \( G \) is a finite cyclic subgroup of the connected component of the identity \( \text{Aut}^0(\overline{X}) \subset \text{Aut}(\overline{X}) \), then the quotient \( \overline{X}/G \) is a toric variety.

We use the following definition for convenience.

**Definition 2.4.17.** Let \( X \) be a \( G \)-surface, \( \widetilde{X} \to X \) be its (\( G \)-equivariant) minimal resolution of singularities, and \( Y \) be a \( G \)-equivariant minimal model of \( \widetilde{X} \). We call the surface \( Y \) a \( G \)-MMP-reduction of \( X \).

The del Pezzo surface of degree 8 considered in this paper is a toric surface. Thus the following proposition is very useful.

**Proposition 2.4.18 ([Tr14, Proposition 4.4]).** Let a group \( G \) contain a normal subgroup \( \mathfrak{C}_p \), where \( p \) is prime. If \( X \) is a \( G \)-minimal \( \mathbb{k} \)-unirational \( \mathbb{k} \)-form of a toric surface then there exists a \( G/\mathfrak{C}_p \)-MMP-reduction \( Y \) of \( X/\mathfrak{C}_p \) such that \( Y \) is a \( \mathbb{k} \)-form of a toric surface. In particular, \( X/\mathfrak{C}_p \) is \( \mathbb{k} \)-rational.

| \( m \) | \( q \) | \( K^2_S - K^2_S \) | \( \pi^{-1}_s(C)^2 - C^2 \) | \( \pi^{-1}_s(D)^2 - D^2 \) | Exceptional divisor |
|------|------|----------------|----------------|----------------|------------------|
| 2    | 1    | 0              | 1              | 0              | 2                |
| 3    | 1    | 1              | 1              | 3              | 3                |
| 3    | 2    | 0              | 3              | 3              | 2, 2             |
| 4    | 1    | 1              | 4              | 4              | 4                |
| 4    | 3    | 0              | 4              | 4              | 2, 2             |
| 5    | 1    | 1              | 3              | 5              | 5                |
| 5    | 2    | 1              | 3              | 5              | 3                |
| 5    | 3    | 1              | 3              | 5              | 2, 3             |
| 5    | 4    | 0              | 5              | 5              | 2, 2, 2, 2       |

This is a table summarizing some calculations related to the theory of toric varieties and quotients. The entries in the table represent various invariants and their relationships, which are crucial in understanding the structure of these geometric objects.
The quotients of del Pezzo surfaces of degree 9 and 6 and conic bundles with $K_X^2 \geq 5$ were considered in the authors papers [Tr14] and [Tr13].

**Theorem 2.4.19 ([Tr14 Theorem 1.3]).** Let $G \subset \text{PGL}_3(k)$ be a finite subgroup. Then $\mathbb{P}^2_k/G$ is $k$-rational.

**Theorem 2.4.20 ([Tr14 Corollary 1.4]).** Let $X$ be a del Pezzo surface of degree 6 over $k$, $X(k) \not= \emptyset$ and $G$ be a finite subgroup of automorphisms of $X$. The quotient variety $X/G$ is $k$-rational.

**Theorem 2.4.21 ([Tr13 Proposition 1.6]).** Let $X$ be a conic bundle such that $K_X^2 \geq 5$, $G$ be a finite subgroup of $\text{Aut}_k(X)$ and $X(k) \not= \emptyset$. Then $X/G$ is $k$-rational.

### 3. Proof of Theorem 1.0.1

#### 3.1. Del Pezzo surface of degree 8

In this section we prove the following proposition.

**Proposition 3.1.1.** Let $X$ be a del Pezzo surface of degree 8, $G$ be a finite subgroup of $\text{Aut}_k(X)$ and $X(k) \not= \emptyset$. Then $X/G$ is $k$-rational.

**Lemma 3.1.2.** Let $X$ be a del Pezzo surface of degree 8, $G$ be a finite subgroup of $\text{Aut}_k(X)$, $\rho(X)^G = 1$. Then $X$ is isomorphic to a smooth quadric $Q \subset \mathbb{P}^4_k$ and the group $G$ is isomorphic to $A \Delta_D A$ or $(A \Delta_D A) \times \mathcal{C}_2$ where $A$ is one of the following groups: $\mathcal{C}_n$, $\mathcal{D}_2n$, $\mathcal{A}_4$, $\mathcal{S}_4$ or $\mathcal{A}_5$.

**Proof.** Since $X$ is minimal then $\overline{X}$ is isomorphic to $\mathbb{P}^1_k \times \mathbb{P}^1_k$ and

$$\text{Aut} \left( \overline{X} \right) \cong (\text{PGL}_2(k) \times \text{PGL}_2(k)) \rtimes \mathcal{C}_2.$$

Let $\pi_1 : \overline{X} \to \mathbb{P}^1_k$ and $\pi_2 : \overline{X} \to \mathbb{P}^1_k$ be projections on the first and the second factors of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ respectively. The group $\text{Pic} (\overline{X}) \cong \mathbb{Z}^2$ is generated by $a = \pi_1^{-1}(p)$ and $b = \pi_2^{-1}(q)$, where $p$ and $q$ are points on the first and the second factors respectively.

The group

$$G_0 = G \cap (\text{PGL}_2(k) \times \text{PGL}_2(k))$$

naturally acts on the factors of $\mathbb{P}^1_k \times \mathbb{P}^1_k$. Let $A \subset \text{PGL}_2(k)$ and $B \subset \text{PGL}_2(k)$ be the images of $G_0$ under the projections of $\text{PGL}_2(k) \times \text{PGL}_2(k)$ onto its factors. If the groups $A$ and $B$ are not isomorphic then any element $g \in \text{Gal} (k/k) \times G$ preserves the factors of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ and one has $ga \sim a$, $gb \sim b$ and $\rho(X)^G = 2$. Thus $A \cong B$ and the group $G$ is $A \Delta_D A$ or $(A \Delta_D A) \times \mathcal{C}_2$ where $A$ is a finite subgroup of $\text{PGL}_2(k)$, i.e. $\mathcal{C}_n$, $\mathcal{D}_2n$, $\mathcal{A}_4$, $\mathcal{S}_4$ or $\mathcal{A}_5$. \qed
In this subsection we use the notation as above.

**Lemma 3.1.3.** If a group \( G \cong A \times B \) acts on a smooth quadric \( X \subset \mathbb{P}^3_k \) (the group \( A \) trivially acts on \( \pi_2(X) \) and the group \( B \) trivially acts on \( \pi_1(X) \)) then \( X/G \) is isomorphic to a smooth quadric in \( \mathbb{P}^3_k \).

*Proof.* One has
\[
\overline{X}/G \cong \left( \mathbb{P}^1_k \times \mathbb{P}^1_k \right) / (A \times B) = \left( \mathbb{P}^1_k/A \right) \times \left( \mathbb{P}^1_k/B \right) \cong \mathbb{P}^1_k \times \mathbb{P}^1_k.
\]
Thus \( X/G \) is isomorphic to a smooth quadric in \( \mathbb{P}^3_k \). \( \square \)

**Lemma 3.1.4.** If a group \( G \cong C_n \Delta D_m \) acts on a smooth quadric \( X \subset \mathbb{P}^3_k \) then \( X/G \) is a \( k \)-form of a toric surface.

*Proof.* The groups \( C_n \) and \( C_m \) are subgroups of tori \( T_1 \subset \text{Aut}(\pi_1(X)) \) and \( T_2 \subset \text{Aut}(\pi_2(X)) \) respectively. Thus the group \( G \cong C_n \Delta D_m \) is a subgroup of the torus \( T_1 \times T_2 \subset \text{Aut}(X) \). Therefore \( X/G \) is a \( k \)-form of a toric surface by Lemma 2.4.16. \( \square \)

**Lemma 3.1.5.** Let a finite group \( G \) act on a smooth quadric \( X \subset \mathbb{P}^3_k \) and
\[
N \cong \mathcal{U}_4 \cong \mathcal{U}_4 \Delta \mathcal{U}_4 \mathcal{U}_4
\]
be a normal subgroup in \( G \). Then there exists a \( G/N \)-MMP-reduction \( Y \) of \( X/N \) such that \( Y \) is a \( k \)-form of a toric surface.

*Proof.* The group \( N \) faithfully acts on \( \pi_1(X) \) and \( \pi_2(X) \). By Lemma 2.2.13 there are six points on \( \pi_1(X) \) and six points on \( \pi_2(X) \) each of which is fixed by a non-trivial element of \( N \). Therefore there are 12 points on \( X \) each of which is fixed by a non-trivial element of \( N \). Thus the surface \( X/N \) is a singular del Pezzo surface of degree 2 with six \( A_1 \) singularities.

For any del Pezzo surface \( V \) of degree 2 with at worst Du Val singularities the linear system \( | - K_V | \) is base point free and defines a double cover
\[
f : V \to \mathbb{P}^2_k
\]
branched over a reduced quartic \( B \subset \mathbb{P}^2_k \). In our case \( V = X/N \) from the local equations one can obtain that \( B \) has six nodes. Consider a conic \( D \) on \( \mathbb{P}^2_k \) passing through 5 of these nodes. Therefore either \( D \cdot B \geq 10 \) that is impossible or \( D \) and \( B \) have a common irreducible component. Thus \( B \) consists of 4 lines \( l_1, l_2, l_3 \) and \( l_4 \) no three passing through a point. The preimage \( f^{-1}(l_i) \) is a rational curve passing through three singular points. From the Hurwitz formula one has
\[
f^{-1}(l_i) \cdot f^{-1}(l_j) = \frac{1}{2}.
\]
If
\[ \pi : \tilde{X}/N \to X/N \]
is the minimal \( G/N \)-equivariant resolution of singularities then the proper transform \( \pi_* f^{-1}(l_1 + l_2 + l_3 + l_4) \) consists of four disjoint \((-1)\)-curves, defined over \( k \). We can \( G/N \)-equivariantly contract these four curves and get a surface \( Y \) with
\[ K_Y^2 = K_{\tilde{X}/N}^2 + 4 = K_{X/N}^2 + 4 = \frac{1}{4}K_X^2 + 4 = 6. \]
So \( Y \) is a \( k \)-form of a toric surface by Lemma 2.1.11. □

**Lemma 3.1.6.** Let a finite group \( G \) act on a smooth quadric \( X \subset \mathbb{P}^3_k \) and
\[ N \cong \mathfrak{A}_5 \cong \mathfrak{A}_5 \triangle \mathfrak{A}_5 \]
be a normal subgroup in \( G \). Then there exists a \( G/N \)-MMP-reduction \( Y \) of \( X/N \) such that \( Y \) is \( k \)-form of a toric surface.

**Proof.** The group \( \mathfrak{A}_5 \triangle \mathfrak{A}_5 \) can act on \( \mathbb{P}^1_k \times \mathbb{P}^1_k \) in two different ways since the group \( \mathfrak{A}_5 \) has two different representations in \( \text{PGL}_2(k) \).

By Lemma 2.2.13 each cyclic subgroup \( \mathfrak{C}_p \subset N \) has exactly two fixed points on \( \pi_1(X) \) and on \( \pi_2(X) \) which are not fixed by any other element of the group \( N \). Thus such a subgroup has exactly four fixed points on \( X \). Thus the set of singular points of \( \tilde{X}/N \) is the following: two \( A_1 \) points, one \( A_2 \) point, one \( \frac{1}{2}(1,1) \) point and either one \( A_4 \) point and one \( \frac{1}{5}(1,1) \) point or two \( \frac{1}{5}(1,2) \) points. Non-trivial elements of the group \( N \) have only isolated fixed points. Thus
\[ K_{\tilde{X}/N}^2 = \frac{K_X^2}{60} = \frac{2}{15}, \quad \rho(\tilde{X}/N)^{G/N} = \rho(X)^G = 1. \]

Let \( f : X \to X/N \) be the quotient morphism and
\[ \pi : \tilde{X}/N \to X/N \]
be the minimal resolution of singularities, \( F_1 \) be a \( \mathfrak{C}_5 \)-invariant fibre of the projection \( \pi_1 \) and \( F_2 \) be a \( \mathfrak{C}_5 \)-invariant fibre of the projection \( \pi_2 \). Note that the group \( G \times \text{Gal}(k/k) \) permutes \( F_1 \) and \( F_2 \) since \( \rho(X)^G = 1 \).

In the first case (singularities of type \( A_4 \) and \( \frac{1}{5}(1,1) \)) one has
\[ K_{\tilde{X}/N}^2 = K_{X/N}^2 - \frac{1}{3} - \frac{9}{5} = -2, \quad \rho(\tilde{X}/N)^{G/N} \geq \rho(X/N)^{G/N} + 6 = 7. \]
Moreover the curves \( \pi_*^{-1} f(F_1) \) and \( \pi_*^{-1} f(F_2) \) are two disjoint curves on \( \tilde{X}/N \) with self-intersection number \(-1\). One can \( G/N \)-equivariantly contract these pair of curves and then \( G/N \)-equivariantly contract the
transforms of two $(-2)$-curves being the ends of the chain of negative curves $\pi^{-1}(A_4)$. We obtain a surface $Z$ such that $K_Z^2 = 2$ and $\rho(Z)^{G/N} \geq 5$. By Corollary 2.1.12 there exists a $G/N$-minimal model $Y$ of $Z$ such that $Y$ is a $k$-form of a toric surface.

In the second case (two $\frac{1}{5}(1,2)$ singularities) one has

$$K_{X/N}^2 = K_{X/N}^2 - \frac{1}{3} - 2 \cdot \frac{2}{5} = -1, \quad \rho(X/N)^{G/N} \geq \rho(X/N)^{G/N} + 5 = 6.$$ 

Moreover the curves $\pi^{-1}_* f(F_1)$ and $\pi^{-1}_* f(F_2)$ are two disjoint curves on $\overline{X}/N$ with selfintersection number $-1$. One can $G/N$-equivariantly contract these pair of curves and then $G/N$-equivariantly contract the transforms of two $(-2)$-curves which are irreducible components of $\pi^{-1}(\frac{1}{5}(1,2))$. We obtain a surface $Z$ such that $K_Z^2 = 3$ and $\rho(Z)^{G/N} \geq 4$. By Corollary 2.1.12 there exists a $G/N$-minimal model $Y$ of $Z$ such that $Y$ is a $k$-form of a toric surface.

\[\square\]

**Proof of Proposition 3.1.1** If $\rho(X)^G = 2$ then $X$ admits a $G$-equivariant conic bundle structure and $X/G$ is $k$-rational by Corollary 2.4.21. Otherwise $\rho(X)^G = 1$ and $X \simeq \mathbb{P}^1_k \times \mathbb{P}^1_k$.

Let $f_1 : G \to \text{Aut}(\mathbb{P}^1_k)$ and $f_2 : G \to \text{Aut}(\mathbb{P}^1_k)$ be homomorphisms to the groups of automorphisms of the first and the second factor of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ respectively. Then the group $K = \text{Ker} f_1 \times \text{Ker} f_2$ is a normal subgroup of $G$. Then by Lemma 3.1.3 the surface $X/K$ is a del Pezzo surface of degree 8 and $(X/K)/(G/K) = X/G$. So we can change $X$ to $X/K$ and from now assume that $K$ is trivial.

If $K$ is trivial then by Lemma 3.1.2 the group $G$ is isomorphic to $A \Delta A A$ or $(A \Delta A A) \times C_2$ where $A$ is one of the following groups: $C_n$, $D_n$, $A_4$, $S_4$ or $A_5$. For each of these groups we found a normal subgroup $N \triangleleft G$ such that there exists a $G/N$-MMP-reduction $Y$ of $X/N$ such that $Y$ is a $k$-form of a toric surface.

- If $G$ is isomorphic to $C_2$ then any MMP-reduction of $X/C_2$ is a $k$-form of a toric surface by Proposition 2.4.18.
- If $G$ is isomorphic to $C_n \Delta C_n C_n$, $(C_n \Delta C_n C_n) \times C_2$, $D_{2n} \Delta D_{2n} \Delta D_{2n}$ or $(D_{2n} \Delta D_{2n} \Delta D_{2n}) \times C_2$ then $N$ is $C_n \Delta C_n C_n$. Any $G/N$-MMP-reduction of $X/N$ is a $k$-form of a toric surface by Lemma 3.1.4.
- If $G$ is isomorphic to $A_4 \Delta A_4 A_4$, $(A_4 \Delta A_4 A_4) \times C_2$, $S_4 \Delta S_4 S_4$, or $(S_4 \Delta S_4 S_4) \times C_2$ then $N$ is $A_4 \Delta A_4 A_4$. There exists a $G/N$-MMP-reduction $Y$ of $X/N$ such that $Y$ is a $k$-form of a toric surface by Lemma 3.1.5.
• If $G$ is isomorphic to $A_5 \triangle A_5$ or $(A_5 \triangle A_5) \rtimes C_2$ then $N$ is $A_5 \triangle A_5$. There exists a $G/N$-MMP-reduction $Y$ of $X/N$ such that $Y$ is a $k$-form of a toric surface by Lemma 3.1.6.

One has $Y(k) \neq \emptyset$ since $X(k) \neq \emptyset$.

If the surface $Y$ is $\mathbb{P}^2_k$, $\mathbb{F}_n$ or a del Pezzo surface of degree 6 then $Y/(G/N) \cong X/G$ is $k$-rational by Theorems 2.4.19, 2.4.21 and 2.4.20 respectively. If the surface $Y$ is $\mathbb{P}^1_k \times \mathbb{P}^1_k$ we apply the procedure above with smaller group $G/N$. As a result we obtain that $X/G$ is $k$-rational.

3.2. Del Pezzo surface of degree 5. In this section we prove the following proposition.

**Proposition 3.2.7.** Let $X$ be a del Pezzo surface of degree 5, $G$ be a finite subgroup of $\text{Aut}_k(X)$ and $X(k) \neq \emptyset$. Then $X/G$ is $k$-rational.

The group $\text{Aut}(X)$ is isomorphic to $W(A_4) \cong S_5$. This group is generated by a subgroup $S_4$ and the element $(12345)$. In the notation of Remark 2.1.6 for any $\sigma \in S_4$ one has $\sigma(E_i) = E_{\sigma(i)}$ and $\sigma(L_{ij}) = L_{\sigma(i)\sigma(j)}$.

**Lemma 3.2.8.** Let a finite group $G$ act on a del Pezzo surface $X$ of degree 5 and $N$ be a nontrivial normal subgroup in $G$. If the group $N$ is isomorphic to $C_2$, $C_3$, $V_4$ then $X$ is not $G$-minimal.

**Proof.** If $N \cong C_2$ then it is conjugate to $\langle(12)\rangle$ or $\langle(12)(34)\rangle$. In the first case there are exactly four $N$-invariant $(-1)$-curves: $E_3$, $E_4$, $L_{12}$ and $L_{34}$. But only the curve $L_{34}$ intersects each other $N$-invariant $(-1)$-curve. Thus $L_{34}$ is $G$-invariant and defined over $k$ so it can be contracted. In the second case there are exactly two orbits consisting of disjoint $(-1)$-curves: $E_1$ and $E_2$, $E_3$ and $E_4$. Thus this fourtuple is $G$-invariant and defined over $k$ so it can be contracted.

If $N \cong C_3$ then it is conjugate to $\langle(123)\rangle$. There is exactly one $N$-invariant $(-1)$-curve. Thus this curve is $G$-invariant and defined over $k$ so it can be contracted.

If $N \cong V_4$ then it is conjugate to $\langle(12)(34),(13)(24)\rangle$. There is exactly one orbit of $N$ consisting of four disjoint $(-1)$-curves: $E_1$, $E_2$, $E_3$ and $E_4$. Thus this fourtuple is $G$-invariant and defined over $k$ so it can be contracted.

**Lemma 3.2.9.** Let a finite group $G$ act on a del Pezzo surface $X$ of degree 5 and $N \cong C_5$ be a normal subgroup in $G$. Then there exists a $G/N$-MMP-reduction $Y$ of $X/N$ such that $Y \cong \mathbb{P}^2_k$. 

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Proof. The element of order 5 acting on $X$ corresponds to the Cremona transformation

$$(x : y : z) \mapsto ((y - x)z : (z - x)y : yz)$$

of $\mathbb{P}^2_k$. This element has two fixed points: $(\sqrt{5} - 1 : 2 : \sqrt{5} + 1)$ and $(\sqrt{5} + 1 : -2 : \sqrt{5} - 1)$. The group acts on the tangent spaces of the fixed points as $\text{diag}(\xi_5, \xi_5^4)$. Thus on the quotient $X/N$ there are two $A_4$ singularities, $-K_{X/N}$ is ample and $X/N$ is a singular del Pezzo surface.

Let

$$\pi : \tilde{X}/N \to X/N$$

be the minimal resolution of singularities. The dual graph of negative curves on $X/N$ is well-known (see [AN88]).

Let us equivariantly contract four disjoint $(-1)$-curves, then equivariantly contract four transforms of the ends of the exceptional divisor of $\pi$ and get a surface $Y$. One has

$$K_Y^2 = K_{\tilde{X}/N}^2 + 8 = K_{X/N}^2 + 8 = \frac{1}{5}K_X^2 + 8 = 9.$$ 

Therefore $Y$ is isomorphic to $\mathbb{P}^2_k$.

Lemma 3.2.10. Let a finite group $G$ act on a del Pezzo surface $X$ of degree 5 and $N \cong \mathfrak{A}_5$ be a normal subgroup in $G$. Then a $G/N$-MMP-reduction $Y$ of $X/N$ is isomorphic to $\mathbb{F}_3$.

Proof. Let us consider fixed points of elements of $N$. The stabilizer of such a point is a subgroup of $\mathfrak{A}_5$ having faithful representation in $\text{GL}_2(\mathbb{F}_5)$. Any subgroup of $\mathfrak{A}_5$ is isomorphic to $\mathfrak{S}_2, \mathfrak{S}_3, \mathfrak{S}_4, \mathfrak{S}_3, \mathfrak{A}_4,$
\[ C_5, D_{10} \text{ or } A_5. \] The groups \( A_4 \) and \( A_5 \) do not have 2-dimensional representations and for the groups \( \Omega_4, S_3 \) and \( D_{10} \) two-dimensional representations are generated by reflections. All other groups are cyclic groups of prime order.

An element of order 5 has exactly two fixed points and this element acts on the tangent spaces of the fixed points as \( \text{diag}(\xi^5, \xi^4) \) (see the proof of Lemma 3.2.9).

Each element of order 3 in \( A_5 \) is conjugate to \( (123) \). The unique invariant \((-1)\)-curve for \( (123) \) is \( E_4 \). Let us \((123)\)-equivariantly contract the four \((-1)\)-curves \( E_i \). The group \( \langle (123) \rangle \) acts on \( \mathbb{P}^2_X \) and has no curves of fixed points since the line passing through \( p_1 \) and \( p_2 \) does not contain fixed points. Therefore the action of \( \langle (123) \rangle \) on \( \mathbb{P}^2_X \) is conjugate to \( \text{diag}(1, \omega, \omega^2) \). Thus it has 3 fixed points one of which is \( p_4 \). In the tangent space of the other points the group \( \langle (123) \rangle \) acts as \( \text{diag}(\omega, \omega^2) \). On the \((-1)\)-curve \( E_4 \) the group \( \langle (123) \rangle \) has two fixed points. This group acts on the tangent space of these points as \( \text{diag}(\omega, \omega) \).

There are 20 elements of order 3 in \( A_5 \) and they have 20 fixed points on \((-1)\)-curves. The stabilizer of such a point is \( C_3 \) so all these point are permuted by the group \( A_5 \).

Consider a group
\[ \Omega_4 = \langle (12)(34), (13)(24) \rangle. \]

One can \( \Omega_4\)-equivariantly contract the four \((-1)\)-curves \( E_i \). The group acts on \( \mathbb{P}^2_X \) and each nontrivial element has a line of fixed points not passing through the points \( p_i \). Thus these elements have a curve on fixed points in \( \overline{X} \) whose class in \( \text{Pic}(\overline{X}) \) is \( L \).

The images of \( L \) in \( \text{Pic}(\overline{X}) \) under the action of \( C_5 \) are \( 2L - E_1 - E_2 - E_3, \ 2L - E_1 - E_2 - E_4, \ 2L - E_1 - E_3 - E_4 \) and \( 2L - E_2 - E_3 - E_4 \). Thus the ramification divisor of the quotient morphism \( f : X \to X/\mathcal{A}_5 \) is \(-9K_X\). By the Hurvitz formula
\[ K^2_{X/\mathcal{A}_5} = \frac{1}{60}(K_X + 9K_X)^2 = \frac{25}{3}. \]

Moreover, there is one \( \frac{1}{3}(1,1) \) singularity and may be some Du Val singularities on \( \overline{X}/\mathcal{A}_5 \). Let \( \pi : Y \to X/\mathcal{A}_5 \) be the minimal resolution of singularities. One has
\[ K^2_Y = K^2_{X/\mathcal{A}_5} - \frac{1}{3} = 8, \quad \rho(Y) = 10 - K^2_Y = 2. \]

Therefore the only singularity on \( \overline{X}/\mathcal{A}_5 \) is \( \frac{1}{3}(1,1) \) and \( Y \) is \( \mathbb{F}_3 \).

**Proof of Proposition 3.2.7.** By Lemma 2.2.14 each group \( G \subset S_5 \) has a normal subgroup \( N \) isomorphic to \( C_2, C_3, \Omega_4, C_5 \) or \( A_5 \).
If \( N \) is isomorphic to \( \mathfrak{S}_2, \mathfrak{S}_3 \) or \( \mathfrak{S}_4 \) then \( X \) is not \( G \)-minimal by Lemma 3.2.8 and \( X/G \) is \( k \)-rational by Theorems 2.4.19, 2.4.21, 3.1.1 and 2.4.20.

If \( N \) is isomorphic to \( \mathfrak{S}_5 \) then there exists a \( G/N \)-MMP-reduction \( Y \) of \( X/N \) such that \( Y \) is isomorphic to \( \mathbb{P}^2_k \) by Lemma 3.2.9.

If \( N \) is isomorphic to \( \mathfrak{A}_5 \) then any \( G/N \)-MMP-reduction \( Y \) of \( X/N \) is isomorphic to \( \mathbb{F}_3 \) by Lemma 3.2.10.

In the both last cases \( Y(\mathbb{k}) \neq \emptyset \) since \( X(\mathbb{k}) \neq \emptyset \). Thus \( Y/(G/N) \cong X/G \) is \( k \)-rational by Theorem 2.4.19 and 2.4.21 respectively. \( \square \)

### 3.3. Del Pezzo surface of degree 4

Let \( X \) be a del Pezzo surface of degree 4. The group \( \text{Aut}(X) \) is a subgroup of the group
\[
W(D_5) \cong \mathfrak{S}_2 \times \mathfrak{S}_5.
\]

This group is generated by subgroups \( \mathfrak{S}_5 \) and \( \mathfrak{S}_4^2 \). In the notation of Remark 2.1.6 for any \( \sigma \in \mathfrak{S}_5 \) one has \( \sigma(E_i) = E_{\sigma(i)}, \sigma(L_{ij}) = L_{\sigma(i)\sigma(j)} \) and \( \sigma(Q) = Q \).

The surface \( \overline{X} \) is isomorphic to a surface of degree 4 in \( \mathbb{P}^4_k \) given by equations
\[
\sum_{i=1}^{5} x_i^2 = 0, \quad \sum_{i=1}^{5} a_i x_i^2 = 0.
\]

The group \( \mathfrak{S}_4^2 \) acts on \( \mathbb{P}^4_k \) and \( \overline{X} \) as a diagonal subgroup of \( \text{PGL}_5(\mathbb{k}) \). There are involutions of two kinds in such diagonal group: \( \iota_{ijkl} \) and \( \iota_{ij} \). These involutions switch sign of coordinates \( x_i, x_j, x_k, x_l \) or \( x_i, x_j \) respectively.

In this section we prove the following proposition.

**Proposition 3.3.11.** Let \( X \) be a del Pezzo surface of degree 4, \( G \) be a finite subgroup of \( \text{Aut}_k(X) \) and \( X(k) \neq \emptyset \). Then \( X/G \) is \( k \)-rational if \( G \) is not conjugate to any of the following groups \( \langle id \rangle, \langle \iota_{12} \rangle, \langle \iota_{12}, \iota_{13} \rangle \) or \( \langle (12)(34)\iota_{15} \rangle \).

Let us show that in these three cases the quotient can be non-\( k \)-rational.

**Example 3.3.12.** Suppose that field \( \mathbb{k} \) does not contain \( i, \sqrt{2} \) and \( \sqrt{3} \). Consider a surface \( X \) in \( \mathbb{P}^4_k \) given by equations
\[
x_1^2 + x_3^2 - x_4^2 - x_5^2 = 0, \quad -x_2^2 + 2x_3^2 - x_4^2 - 4x_5^2 = 0.
\]

Note that \( X(\mathbb{k}) \neq \emptyset \) since a \( k \)-point \( (0 : 1 : 1 : 1 : 0) \) lies on \( X \). The group \( G \cong \langle \iota_{12} \rangle \) acts on \( \mathbb{P}^4_k \) and switches sign of coordinates \( x_1 \) and \( x_2 \). We show that the quotient \( X/G \) is not \( k \)-rational.
The group $G$ has four fixed points on $X$: $(0:0:\pm\sqrt{3}:\pm\sqrt{2}:1)$ permuted by the Galois group. The quotient $X/G$ is a singular del Pezzo surface of degree 2 with four $A_1$ singularities. Let $f : X \rightarrow X/G$ be a quotient morphism and $\pi : \widetilde{X/G} \rightarrow X/G$ be the minimal resolution of singularities.

The linear system $| -K_{X/G} |$ is base point free and defines a double cover:

$$\varphi |_{-K_{X/G}} : X/G \rightarrow \mathbb{P}^2_k$$

branched over a reduced quartic $B \subset \mathbb{P}^2_k$. In our case from the local equations one can obtain that $B$ has four nodes. Thus $B$ is a union of two conics $C_1$ and $C_2$. Consider a linear system $|C|$ spanned by $C_1$ and $C_2$. This linear system defines a rational map

$$\varphi |_C : \mathbb{P}^2_k \rightarrow \mathbb{P}^1_k$$

whose general fibre is a reduced conic. This map has three singular fibres. The interdeterminacy points are points of intersection of $C_1$ and $C_2$ which are images of $A_1$ singularities. Thus the map

$$\varphi |_C \circ \varphi |_{-K_{X/G}} \circ \pi : \widetilde{X/G} \rightarrow \mathbb{P}^1_k$$

is regular and its general fibre is a pair of disjoint reduced conics. So $\widetilde{X/G}$ admits a structure of conic bundle with 6 singular fibres over double cover of $\mathbb{P}^1_k$. This cover is branched in two points which correspond to $C_1$ and $C_2$ in linear system $|C|$. Thus $\widetilde{X/G}$ is a conic bundle over $\mathbb{P}^1_k$.

These singular fibres are proper transforms of pairs of lines passing through 4 points $C_1 \cap C_2$ on $\mathbb{P}^2_k$. The morphism

$$\varphi |_{-K_{X/G}} \circ f : X \rightarrow \mathbb{P}^2_k$$

maps a point $(x_1 : x_2 : x_3 : x_4 : x_5)$ to $(x_3 : x_4 : x_5)$. Thus the points of $C_1 \cap C_2$ have coordinates $(\pm\sqrt{3} : \pm\sqrt{2} : 1)$ and pairs of lines passing through them have equations $x_3 = \pm\sqrt{3}x_5$, $x_4 = \pm\sqrt{2}x_5$ and $\sqrt{2}x_3 = \pm\sqrt{3}x_4$. The same equations in $\mathbb{P}^4_k$ give hyperplanes cutting out from $X$ the transforms of components of singular fibres on $\widetilde{X/G}$.

Consider any pair of such hyperplanes, for example, $x_3 = \pm\sqrt{3}x_5$. Then $X$ intersects each of these hyperplanes by a curve given by equations

$$x_1^2 - x_4^2 + 2x_3^2 = 0, \quad -x_2^2 - x_4^2 + 2x_5^2 = 0.$$ 

Therefore for each point of this curve $x_1^2 + x_2^2 = 0$. It means that the curve consist of two irreducible components (for one $x_1 = ix_2$ and for the other $x_1 = -ix_2$) and these components are permuted by the
Galois group. Moreover, two hyperplanes in the pair are permuted by the Galois group too. Thus all four components of two singular fibres on $\tilde{X}/G$ are transitively permuted by the Galois group.

In the same way one can show that for two other pairs of singular fibres on $\tilde{X}/G$ their components are permuted by the Galois group. It means that $X/G$ is relatively minimal. One has $K^2_{\tilde{X}/G} = 2$, so $\tilde{X}/G$ is minimal by Theorem 2.1.8. Thus by Theorem 2.1.9 the surfaces $\tilde{X}/G$ and $X/G$ are not $k$-rational.

Remark 3.3.13. Note that, in the previous example if one of the $\iota_{12}$-fixed points $p$ is defined over $k$ then its preimage $\pi^{-1}(f(p))$ on $\tilde{X}/G$ is a $(-2)$-section defined over $k$. Therefore one can contract 6 components of singular fibres meeting this section and get a $k$-rational surface of degree 8.

The next example show that the quotient $X$ by a group $\mathfrak{U}_4$ can be non-$k$-rational.

**Example 3.3.14.** Let us consider the surface $X$ from Example 3.3.12. The surface $\tilde{X}/\langle \iota_{12} \rangle$ is relatively minimal conic bundle. Therefore $\rho(\tilde{X}/\langle \iota_{12} \rangle) = 2$, $\rho (X/\langle \iota_{12} \rangle) = 1$ and $\rho (X)_{\langle \iota_{12} \rangle} = 1$.

Now consider the action of the group $G \cong \langle \iota_{12}, \iota_{13} \rangle$. Each non-trivial element of $G$ has 4 fixed points permuted by the Galois group. The hyperplane sections $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ cut out from $X$ elliptic curves $C_1$, $C_2$ and $C_3$ defined over $k$. Each of these curves contains 8 points fixed by non-trivial elements of $G$.

Let $f : X \to X/G$ be the quotient map and $\pi : \tilde{X}/G \to X/G$ be the minimal resolution of singularities. Then $f(C_i)$ is a rational curve containing four $A_1$ singularities. One has

$$f(C_i) \cdot f(C_j) = \frac{1}{4} C_i \cdot C_j = \frac{1}{4} K^2_X = 1.$$ 

Thus $\pi^{-1}_* f(C_1)$, $\pi^{-1}_* f(C_2)$ and $\pi^{-1}_* f(C_3)$ are three disjoint $(-1)$-curves defined over $k$. We can contract these three curves and get a surface $Y$ such that

$$\rho(Y) = \rho(\tilde{X}/G) - 3 = \rho (X/G) = \rho(X)^G \leq \rho(X)_{\langle \iota_{12} \rangle} = 1,$$

$$K^2_Y = K^2_{\tilde{X}/G} + 3 = K^2_{X/G} + 3 = \frac{1}{4} K^2_X + 3 = 4.$$ 

Thus $Y$ is a minimal del Pezzo surface of degree 4. The surfaces $Y$ and $X/G$ are not $k$-rational by Theorem 2.1.9.
In the previous example in fact we have proved the following lemma.

**Lemma 3.3.15.** Let a finite group $G$ act on a del Pezzo surface $X$ of degree 4 and $N \cong \mathfrak{U}_4 = \langle t_{12}, t_{13} \rangle$ be a normal subgroup in $G$. Then the surface $X/N$ is $G/N$-birationally equivalent to a del Pezzo surface of degree 4.

Now we show that the quotient $X$ by a group $\mathfrak{C}_4$ can be non-$k$-rational.

**Example 3.3.16.** Suppose that a field $k$ does not contain $i$ and $\sqrt{2}$.

Consider a surface $X$ in $\mathbb{P}^4_k$ given by equations

$$x_1^2 + x_2^2 + x_3x_4 + x_5^2 = 0, \quad x_1^2 - x_2^2 + x_3x_5 - x_4x_5 = 0.$$  

The group $G \cong \langle (12)(34)i_{15} \rangle$ acts on $\mathbb{P}^4_k$ and contains a normal subgroup $\langle i_{12} \rangle$. Denote the generator of the group $G$ by $g$. We show that the quotient $X/G$ is not $k$-rational.

We use the following notation:

$$p_1 = (0 : 0 : i : i : 1), \quad p_2 = (0 : 0 : -i : -i : 1),$$

$$p_3 = (0 : 0 : 1 : 0 : 0), \quad p_4 = (0 : 0 : 0 : 1 : 0)$$

are points fixed by $g^2$. Denote by $C_{13}, C_{14}, C_{23}, C_{24}, C_{34}, S_1$ and $S_2$ sections cut out from $X$ by hyperplanes $x_4 = ix_5, x_3 = ix_5, x_3 = -ix_5, x_4 = -ix_5, x_5 = 0, x_1 = 0$ and $x_2 = 0$ respectively. Note that as in Example 3.3.12 each section $C_{ij}$ consist of two irreducible components. It is easy to check that the group

$$\text{Gal} \left( k \left( \frac{1 + i}{\sqrt{2}} \right)/k \right) \cong \mathfrak{C}_4$$

permutes four irreducible components of $C_{13} \cup C_{24}$ and $C_{14} \cup C_{23}$ respectively.

Let $f : X \to X/G$ be the quotient map and $\pi : \widetilde{X/G} \to X/G$ be the minimal resolution of singularities. The points $f(p_1)$ and $f(p_2)$ are $A_3$ singularities. Exceptional divisors of their resolutions are chains consisting of three $(-2)$-curves each. The point $f(q_1) = f(q_2)$ is a $A_1$ singularity. The curves $\pi_1^{-1}f(S_1), \pi_1^{-1}f(S_2)$ and $\pi_1^{-1}f(C_{34})$ are three disjoint $(-1)$-curves.

Let $\sigma : X \to Y$ be the contraction of these curves. Then

$$K_Y^2 = K_{\widetilde{X/G}}^2 + 3 = K_{X/G}^2 + 3 = \frac{1}{4}K_X^2 + 3 = 4$$

and $Y$ is an exceptional conic bundle with two $(-2)$-sections permuted by the Galois group which are transforms of the central $(-2)$-curves of $\pi_1^{-1}f(p_i)$. Two singular fibres of $Y$ consist of the transforms of four ends of $\pi_1^{-1}f(p_i)$. These fibres contain $k$-points $\sigma \pi_1^{-1}f(S_i)$ thus the
components of each of these singular fibres are permuted by the Galois group. Two other singular fibres consist of \( \sigma \pi_{\ast}^{-1} f (C_{13}) = \sigma \pi_{\ast}^{-1} f (C_{14}) \) and \( \sigma \pi_{\ast}^{-1} f (C_{23}) = \sigma \pi_{\ast}^{-1} f (C_{24}) \). All their components are permuted by the Galois group. Therefore \( Y \) is relatively minimal. One has \( K^2_Y = 4 \), so \( Y \) is minimal by Theorem 2.1.8. Thus by Theorem 2.1.9 the surfaces \( Y \) and \( X/G \) are not \( k \)-rational.

Now we show that in all other cases the quotient of \( X \) is \( k \)-rational.

**Lemma 3.3.17.** Let a finite group \( G \) act on a del Pezzo surface \( X \) of degree 4 and \( h : \text{Aut}(X) \rightarrow S_5 \) be a natural map. Then the group \( h(G) \) do not contain subgroups conjugate to \( C_2 = \langle (12) \rangle \) and \( V_4 = \langle (12)(34), (13)(24) \rangle \).

**Proof.** The group \( C_2 \) acts on \( X \). Thus it is sufficient to prove that there are not subgroups in \( G \) conjugate to \( C_2 = \langle (12) \rangle \) and \( V_4 = \langle (12)(34), (13)(24) \rangle \).

Let the group \( C_2 \) act on \( X \). One can \( C_2 \)-equivariant contract five \((-1)\)-curves \( E_1, E_2, E_3, E_4 \) and \( E_5 \) and get a \( \mathbb{P}^2 \) with the action of \( C_2 \). The group of order 2 has a unique isolated fixed point on \( \mathbb{P}^2 \) and a unique line of fixed points. The points \( p_3, p_4 \) and \( p_5 \) are fixed by the group \( C_2 \). This three points do not lie on a line so one of these points is the isolated fixed point. The points \( p_1 \) and \( p_2 \) are permuted by the group \( C_2 \) thus the line passing through these two points contains the isolated fixed point of \( C_2 \). The proper transform of this line on \( X \) is a \((-2)\)-curve. Therefore the group \( C_2 = \langle (12) \rangle \) can not act on \( X \).

Let the group \( V_4 = \langle (12)(34), (13)(24) \rangle \) act on \( X \). One can \( V_4 \)-equivariant contract five \((-1)\)-curves \( E_1, E_2, E_3, E_4 \) and \( E_5 \) and get a \( \mathbb{P}^2 \) with the action of \( V_4 \). The point \( p_5 \) is fixed by the group \( V_4 \) thus this point is the unique isolated fixed point of an element of \( V_4 \). Therefore as in the previous paragraph three of points \( p_1, p_2, p_3, p_4 \) and \( p_5 \) lie on a line and the group \( V_4 \) can not act on \( X \). □

**Lemma 3.3.18.** Let a finite group \( G \) act on a del Pezzo surface \( X \) of degree 4 and \( N \cong \mathfrak{C}_2 = \langle (12)(34) \rangle \) be a normal subgroup in \( G \). Then there exists a \( G/N \)-MMP-reduction \( Y \) of \( X/N \) such that \( Y \) is a \( k \)-form of a toric surface.

**Proof.** Let the group \( \mathfrak{C}_2 = \langle (12)(34) \rangle \) act on \( X \). Let us \( \mathfrak{C}_2 \)-equivariantly contract five \((-1)\)-curves \( E_1, E_2, E_3, E_4 \) and \( E_5 \) on \( X \) and get a \( \mathbb{P}^2 \) with the action of \( \mathfrak{C}_2 \). The group \( \mathfrak{C}_2 \) has a unique isolated fixed point on \( \mathbb{P}^2 \) and a unique line of fixed points. As in the proof of Lemma 3.3.17 the point \( p_5 \) lie on this line. Thus on the surface \( X \) the group \( \mathfrak{C}_2 \) has...
2 isolated fixed points \( L_{12} \cap L_{34}, Q \cap E_5 \) and a curve of fixed points whose class in \( \text{Pic}(X) \) is \( L - E_5 \).

Let \( f : X \to X/N \) be the quotient morphism and
\[
\pi : X/N \to X/N
\]
be the minimal resolution of singularities. By the Hurwitz formula
\[
K_{X/N}^2 = \frac{1}{2} (K_X - L + E_5)^2 = 4.
\]
There are exactly two \( A_1 \) singularities on \( X/N \). The transforms \( \pi^{-1}_* f(L_{12}), \pi^{-1}_* f(L_{34}), \pi^{-1}_* f(Q) \) and \( \pi^{-1}_* f(E_5) \) are four disjoint \( G/N \)-invariant \((-1)\)-curves defined over \( k \). One can \( G/N \)-equivariantly contract these fourtuple and get a surface \( Y \) such that \( K_Y^2 = 8 \). Thus there exists a \( G/N \)-MMP-reduction \( Y \) of \( X/N \) such that \( Y \) is a \( k \)-form of a toric surface by Lemma 2.1.11. \( \Box \)

**Lemma 3.3.19.** Let a finite group \( G \) act on a del Pezzo surface \( X \) of degree 4 and \( N \cong \mathfrak{c}_3 = \langle \langle 123 \rangle \rangle \) be a normal subgroup in \( G \). Then there exists a \( G/N \)-MMP-reduction \( Y \) of \( X/N \) such that \( Y \) is a \( k \)-form of a toric surface.

**Proof.** Let the group \( \mathfrak{c}_3 = \langle \langle 123 \rangle \rangle \) act on \( X \). Let \( \sigma : \overline{X} \to \mathbb{P}^2_k \) be the \( \mathfrak{c}_3 \)-equivariant contraction of the five \((-1)\)-curves \( E_1, E_2, E_3, E_4 \) and \( E_5 \). The group \( \mathfrak{c}_3 \) has three isolated fixed points on \( \mathbb{P}^2_k \) two of which are \( p_4 \) and \( p_5 \). Denote the third fixed point by \( p \). The group \( \mathfrak{c}_3 \) acts on the tangent space of these fixed points as \( \text{diag}(\omega, \omega^2) \). Therefore there are five fixed points on \( \overline{X} \): \( E_4 \cap L_{45}, E_5 \cap L_{45}, E_5 \cap Q, E_4 \cap Q \) and \( \sigma^{-1}(p) \). The group \( \mathfrak{c}_3 \) acts on the tangent space of the point \( \sigma^{-1}(p) \) as \( \text{diag}(\omega, \omega^2) \) and on the tangent spaces of the other points as \( \text{diag}(\omega, \omega) \).

Let \( C_1, C_2, C_3 \) and \( C_4 \) be \( \mathfrak{c}_3 \)-invariant curves with classes
\[
2L - E_1 - E_2 - E_3 - E_4, 2L - E_1 - E_2 - E_3 - E_5, L - E_5, L - E_4
\]
passing through \( \sigma^{-1} p \) and an other fixed point (these curves are proper transforms of lines passing through \( p \) and \( p_4 \) or \( p_5 \) and conics passing through \( p, p_1, p_2, p_3 \) and \( p_4 \) or \( p_5 \)). One can check that there are no other rational curves with selfintersection number 0 passing through a pair of \( \mathfrak{c}_3 \)-fixed points.

Let \( f : X \to X/N \) be the quotient morphism and \( \pi : X/N \to X/N \) be the minimal resolution of singularities. Then there are four \( \frac{1}{3}(1,1) \) singularities \( f(E_4 \cap L_{45}), f(E_5 \cap L_{45}), f(E_5 \cap Q), f(E_4 \cap Q) \) and one \( A_2 \) singularity \( f(\sigma^{-1}(p)) \) on \( \overline{X}/N \). Consider 8 curves \( f(C_1), f(C_2), f(C_3), f(C_4), f(E_4), f(L_{45}), f(E_5) \) and \( f(Q) \). This eightuple is \( G/N \)-equivariant and defined over \( k \). The intersection number of any pair of
these curves is less than 1. Thus their proper transforms on $\widetilde{X}/N$ are eight disjoint $(-1)$-curves. One can $G/N$-equivariantly contract these curves and get a surface $Y$. Then

$$K_Y^2 = K_{\widetilde{X}/N}^2 + 8 = K_{X/N}^2 + \frac{28}{3} = \frac{1}{3}K_X^2 + \frac{28}{3} = 8.$$ 

Thus there exists a $G/N$-MMP-reduction $Y$ of $X/N$ such that $Y$ is a $k$-form of a toric surface by Lemma 2.1.11.

Lemma 3.3.20. Let a finite group $G$ act on a del Pezzo surface $X$ of degree 4 and $N \cong C_5 = \langle (12345) \rangle$ be a normal subgroup in $G$. Then $X$ is not $G$-minimal.

Proof. The group $N$ has a unique invariant $(-1)$-curve $Q$. Thus this curve is $G$-invariant and defined over $k$ so it can be contracted. □

Lemma 3.3.21. Let a finite group $G$ act on a del Pezzo surface $X$ of degree 4 and contain an element conjugate to $\iota_{1234}$. Then there exists a normal subgroup $N \triangleleft G$ such that a $G/N$-MMP-reduction $Y$ of $X/N$ is a $k$-form of a toric surface.

Proof. Note that the set of fixed points of an element conjugate to $\iota_{1234}$ is a hyperplane section of $X$ in $\mathbb{P}_k^4$.

Let $N$ be a subgroup of $G$ generated by elements conjugate to $\iota_{1234}$. Then the group $N$ is normal and one of the following possibilities holds:

- If $N$ is generated by one element conjugate to $\iota_{1234}$ then $N \cong C_2$ and by the Hurvitz formula $K_{X/N}^2 = \frac{1}{2}(2K_X)^2 = 8$.
- If $N$ is generated by two elements conjugate to $\iota_{1234}$ then $N \cong C_3^2$ and by the Hurvitz formula $K_{X/N}^2 = \frac{1}{4}(3K_X)^2 = 9$.
- If $N$ is generated by three elements conjugate to $\iota_{1234}$ then $N \cong C_5^2$ and by the Hurvitz formula $K_{X/N}^2 = \frac{1}{5}(4K_X)^2 = 8$.
- If $N$ is generated by four elements conjugate to $\iota_{1234}$ then $N$ contains the fifth element conjugate to $\iota_{1234}$, $N \cong C_4^2$ and by the Hurvitz formula $K_{X/N}^2 = \frac{1}{16}(6K_X)^2 = 9$.

The surface $X/N$ has at worst Du Val singularities then for any $G/N$-MMP-reduction $Y$ of $X/N$ one has $K_Y^2 \geq 8$. Thus $Y$ is a $k$-form of a toric surface by Lemma 2.1.11.

Proof of Proposition 3.3.11. If the group $G$ contains an element conjugate to $\iota_{1234}$ then by Lemma 3.3.21 there exists a normal subgroup $N \triangleleft G$ such that a $G/N$-MMP-reduction $Y$ of $X/N$ is a $k$-form of a toric surface.
If the group \( G \cap \mathfrak{C}_4^2 \) is conjugate to the group \( \mathfrak{V}_4 = \langle \iota_{12}, \iota_{13} \rangle \) then the group \( G \) is conjugate to a subgroup of
\[ \mathfrak{V}_4 \rtimes (\mathfrak{S}_3 \times \mathfrak{C}_2) = \langle \iota_{12}, \iota_{13}, (123), (12), (45) \rangle. \]
Such a group can not contain subgroup conjugate to \( \mathfrak{C}_2 = \langle \iota_{12} \rangle \) by Lemma 3.3.17. If the group \( G \) does not contain an element of order 3 then either \( G = \mathfrak{V}_4 \) or \( G \) contains a normal subgroup conjugate to \( N = \langle (12)(45) \rangle \) and there exists a \( G/N \)-MMP-reduction \( Y \) of \( X/N \) such that \( Y \) is a \( k \)-form of a toric surface by Lemma 3.3.18. Otherwise by Lemma 3.3.15 the quotient \( X/N \) is \( G/N \)-birationally equivalent to a del Pezzo surface \( Z \) of degree 4 and the group \( G/N \) contains an element of order 3. So we can replace \( X \) by \( Z \), \( G \) by \( G/N \) and start the proof from the beginning with the smaller group.

If the group \( G \cap \mathfrak{C}_4^2 \) is conjugate to the group \( \mathfrak{C}_2 = \langle \iota_{12} \rangle \) then the group \( G \) is conjugate to a subgroup of
\[ \mathfrak{C}_2 \times (\mathfrak{C}_2 \times \mathfrak{S}_3) = \langle \iota_{12}, (12), (345), (34) \rangle. \]
Such a group can not contain subgroup conjugate to \( \mathfrak{C}_2 = \langle (12) \rangle \) by Lemma 3.3.17. If such a group contains a subgroup conjugate to \( N = \langle (345) \rangle \) then this group is normal and there exists a \( G/N \)-MMP-reduction \( Y \) of \( X/N \) such that \( Y \) is a \( k \)-form of a toric surface by Lemma 3.3.19. Otherwise either group \( G \) is conjugate to \( \langle \iota_{12} \rangle \) or \( \langle \iota_{15}(12)(34) \rangle \) or the group \( G \) contains a normal subgroup conjugate to \( N = \langle (12)(34) \rangle \) and there exists a \( G/N \)-MMP-reduction \( Y \) of \( X/N \) such that \( Y \) is a \( k \)-form of a toric surface.

If the group \( G \cap \mathfrak{C}_4^2 \) is trivial then \( G \) is isomorphic to a subgroup of \( \mathfrak{S}_5 \). By Lemma 3.3.17 the group \( G \) can not contain subgroups conjugate to \( \mathfrak{C}_2 = \langle (12) \rangle \), \( \mathfrak{V}_4 \) and \( \mathfrak{A}_5 \). Thus by Lemma 2.2.14 the group \( G \) contains a normal subgroup \( N \) conjugate to \( \mathfrak{C}_2 = \langle (12)(34) \rangle \), \( \mathfrak{C}_3 = \langle (123) \rangle \) or \( \mathfrak{C}_5 = \langle (12345) \rangle \). In the last case the surface \( X \) is not \( G \)-minimal by Lemma 3.3.20 and the quotient \( X/G \) is \( k \)-rational by Theorems 2.4.19 and 3.2.7. In the two other cases there exist a \( G/N \)-MMP-reduction \( Y \) of \( X/N \) such that \( Y \) is a \( k \)-form of a toric surface by Lemmas 3.3.18 and 3.3.19. In all cases \( Y(k) \neq \emptyset \) since \( X(k) \neq \emptyset \). Thus
\[ Y/(G/N) \approx X/G \]
is \( k \)-rational by Theorems 2.4.19, 2.4.21, 3.1.1 and 2.4.20. □

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LABORATORY OF ALGEBRAIC GEOMETRY, GU-HSE, 7 VAVILOVA STR., MOSCOW 117312, RUSSIA

INDEPENDENT UNIVERSITY OF MOSCOW, MOSCOW 119002, RUSSIA

E-mail address: trepalin@mccme.ru