A continuous topological phase transition between 1D anti-ferromagnetic spin-1 boson superfluids

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Spin-1 bosons on a 1-dimensional chain, with anti-ferromagnetic spin interaction between neighboring bosons, may form a spin-1 boson condensed state that contains both gapless charge and spin excitations. We argue that the gapless spin excitations are unstable in the spin-1 boson condensed state, and the latter will become one of two new superfluids by opening a spin gap. One superfluid must have spin-1 ground state on a ring if it contains an odd number of bosons and has no degenerate states at the chain end. The other superfluid has spin-0 ground state on a ring for any numbers of bosons and has a spin-1/2 degeneracy at the chain end. The two superfluids have the same symmetry and only differ by a spin-1 symmetry protected topological order. Although Landau theory forbids continuous phase transition between two phases with the same symmetry, the phase transition between the two superfluids can be generically continuous, which is described by a conformal field theory (CFT) $su_2 \oplus u_1 \oplus su_2 \oplus u_1$. Such a CFT has a spin fractionalization: a spin-1 excitation can decay into a spin-1/2 right mover and a spin-1/2 left mover.

Introduction: 1+1D spin-1 bosons may form a super-fluid via boson condensation that has gapless charge and spin excitations. However, if the spins on two nearby bosons have an anti-ferromagnetic interaction, we argue that the above spin-1 boson condensed state is unstable. The spin excitations will open a finite energy gap and the spin-1 bosons will form a superfluid state where the spin excitations are fully gapped.\(^1\) (For spin-1 boson systems with integer bosons per site, see Ref. 2.) In fact there are two such superfluid phases: (1) two spin-1 bosons form a spin singlet bound state and then the spin-0 boson pair form a superfluid. (2) The bosons do not form pairs but the spin-1’s on the bosons form a gapped state, just like an anti-ferromagnetic spin-1 chain.\(^3,4\) The above two superfluid phases have the same symmetry, but the second one also have degenerate spin-1/2 states at one end of the chain. So the second superfluid is a topological superfluid, while the boson pair superfluid is a non-topological superfluid.

Although Landau theory forbids two phases with the same symmetry to have a continuous phase transition between them, in this paper, we will show that there is a continuous phase transition between the topological and the non-topological superfluids. (Some examples of topological phase transitions that do not change the symmetry can be found in Ref. 5–11.) Both phases are symmetric under $SO(3) \times U(1) \times trn$, where $trn$ stands for the translational symmetry.

The above result is obtained by first constructing a critical theory with only one relevant operator. The low energy excitations of the critical point are described by a conformal field theory (CFT) given by current (Kac-Moody) algebras\(^12\) $su_2 \oplus u_1 \oplus su_2 \oplus u_1$. Here $su_N$ denotes the level-$k$ $su(N)$ Kac-Moody algebra and the CFT built from it. The $su_N$ CFT has a central charge $c = \frac{k(N^2-1)}{k+N}$. Likewise $u_1$ denotes the $u(1)$ current algebra, and the associated CFT, whose central charge is $c = 1$. The absence or the presence of bar indicates right-moving or left-moving sectors. Our method is to construct a modular invariant partition function (explained later) of the critical point, which turns out to be stringent enough to uniquely determine the critical theory.

We then show that, when we perturb the critical point with its only symmetric relevant operator, the system will flow to the above two superfluid phases, depending the sign of the perturbation. We argue that such unstable critical point actually correspond to the spin-1 boson condensed state with anti-ferromagnetic interaction.

We would like to mention that the two spin-gapped superfluids for spin-1 bosons and their phase transition $su_2 \oplus u_1 \oplus su_2 \oplus u_1$ can be realized by spin-1/2 electrons on a ladder. Such a critical point and its neighboring spin-gapped states on the ladder has been studied in Ref. 13.

The construction of critical state: We construct the $su_2 \oplus u_1 \oplus su_2 \oplus u_1$ critical state, starting from a free fermionic model. This allows us to easily obtain the crystal momentum and other quantum numbers of the local operators. Later we show that in an interacting phase, the fermionic excitations are fully gapped and the low energy theory is composed of purely spin-1 bosons. The Hamiltonian of a spin-1/2 charge-1 electron model on 1D lattice is

$$H_0 = \sum_i (-t_{\alpha a,i+1}^c c_{\alpha a,i} + h.c.) - \mu c_{\alpha a,i}^d c_{\alpha a,i}$$  \hspace{1cm} (1)$$

where $\alpha = \uparrow, \downarrow$ is the spin index and $a = 1, 2$ is the flavor index. The chemical potential $\mu$ is chosen such that the fermions are at incommensurate filling. At low energy, the above model has two right-moving spin-1/2 fermions with crystal momentum $k_F$ and two left-moving spin-1/2 fermions with crystal momentum $-k_F$. The two right-moving spin-1/2 fermions are described by $\psi_{\alpha a}$ where $\alpha$ and $a$ are the spin and flavor index as well. Similarly, the two left-moving spin-1/2 fermions are described by $\overline{\psi}_{\alpha a}$. The model is translational invariant and the role of crystal momenta here is analogous to $U(1)$ charges that are conserved mod 2$\pi$. The low energy effective
Hamiltonian density is given by
\[ \mathcal{H}_0 = \psi_{aa}^\dagger(x) i \nu_0 \partial_x \psi_{aa}(x) - \bar{\psi}_{aa}^\dagger(x) i \nu_0 \partial_x \bar{\psi}_{aa}(x) \] (2)

At low energy, generically the spin, flavor and charge degrees of freedom may have different velocities, the model has an emergent \([SU_s(2) \times SU_f(2) \times U(1)]_\eta \times [SU_s(2) \times SU_f(2) \times U(1)]_\eta\) symmetry for right and left movers. Correspondingly, the right movers of the above system are described by a CFT with total central charge \(c = 4\),
\[ su_2^s \oplus su_2^f \oplus u1_c, \] (3)

where the excitations in \(su_2^s\) carry \(SU_s(2)\) spin quantum numbers, the excitations in \(su_2^f\) carry \(SU_f(2)\) flavor quantum numbers, and the excitations in \(u1\) carry the \(U(1)\) charges. Similarly, the left movers of the above system are described by a CFT with central charge \(\tau = 4\),
\[ \bar{su}_2^s \oplus \bar{su}_2^f \oplus u1_t. \] (4)

The local operators in the theory are powers of the fermion operators \(\psi_{aa}, \bar{\psi}_{aa}\). The fermion operators can be represented in terms of the primary fields of the above CFTs\(^{12}\),
\[ \psi_{aa} \sim e^{i \frac{x}{u1} } \sigma_+ e^{x \chi_2} \sigma_f e^{x \frac{i}{2} } = e^{i \frac{x}{u1} } V^{su_2^2} \chi_2^{su_2^f} V_1^{su_2^f} \] \(\bar{\psi}_{aa} \sim e^{i \frac{x}{u1} } \sigma_- e^{x \chi_2} \sigma_f e^{x \frac{i}{2} } = e^{i \frac{x}{u1} } V^{\bar{su}_{2}^2} \chi_2^{\bar{su}_{2}^f} V_1^{\bar{su}_{2}^f} \) (5)

Here, for right movers, which are functions of \(z = \tau + ix\) (\(\tau\) and \(x\) are imaginary time and space coordinate),
\(1\) \(\varphi_e\) is the bosonic field in \(u1_c\) CFT,
\(2\) \(\eta_s, \sigma_s\) and \(\phi_s\) are the Ising CFT fields and the bosonic field to represent \(su_2^s\) primary fields,
\(3\) \(\eta_f, \sigma_f\) and \(\phi_f\) are the Ising CFT fields and the bosonic field to represent \(su_2^f\) primary fields,
\(4\) \(V^{su_2^2} \chi_2^{su_2^f} V_1^{su_2^f}\) is the primary field of \(su_2^s\) that is also a spin-\(\frac{1}{2}\) doublet. And similarly for \(V^{\bar{su}_2^2} \chi_2^{\bar{su}_2^f} V_1^{\bar{su}_2^f}\).

The fields for the left movers, which are functions of \(\bar{z} = \bar{\tau} - ix\), are analogously defined. (In this paper, all the bosonic fields are normalized such that \(\langle \phi(z_1) \phi(z_2) \rangle = -\ln(z_1 - z_2)\), i.e. the scaling dimension of \(e^{i \phi}\) is \(\frac{1}{2}\).) As expected, the fermionic operators are charge-1 spin doublet, as well as flavor doublet. Their scaling dimensions are \(\frac{1}{4}\), which is in fact, what we use to determine the \(u1\) part of \(5\).

Next, we obtain the desired critical state by simply gapping the flavor sector \(su_2^f \oplus \bar{su}_2^f\). This can be achieved dynamically by adding a strong repulsive interaction for the flavor charges, which can gap out all the flavor fluctuations. The resulting critical state is what we want, described by CFT
\[ su_2^s \oplus u1 \oplus \bar{su}_2^f \oplus u\Gamma. \] (6)

The total central charge is \(c = \frac{3}{2} + 1\) for right movers and \(\tau = \frac{3}{2} + 1\) for left movers.

Note that in the theory, all fermionic operators, shown in \(5\), carry half-integer flavor of \(SU_f(2)\), while the low energy CFT \(6\) contains only flavor-singlet excitations. It means all low energy excitations are bosonic, and must carry even charges and integer spins. The low energy theory is therefore a bosonic theory. Furthermore, the critical state should be viewed as a state of charge-2 spin-1 bosons (i.e. electron pairs). It is interesting that such a “superconducting” state can be induced by a repulsive interaction in the flavor density channel.

We like to argue that the above critical state, \(su_2^s \oplus u1 \oplus \bar{su}_2^f \oplus u\Gamma\), actually corresponds to the spin-1 boson condensed state for anti-ferromagnetic spin interaction. In the spin-1 boson condensed state, both charge \(U(1)\) symmetry and spin \(SO(3)\) symmetry temp to be spontaneously broken. But in 1+1D, a continuous symmetry cannot really be broken. Here we assume the spins of the bosons form an algebraic anti-ferromagnetic order, i.e. the neighboring bosons have opposite spins. (For ferromagnetic interaction, the spins can have a true long range order that spontaneously break the spin rotation symmetry.) Thus the spin of the bosons can develop only an algebraic long range order. We believe that the gapless spin excitations in such a state are described by the conserved \(su_2\) spin currents. And the gapless charge excitations are described by the conserved \(u1\) charge currents. Thus the critical state \(su_2^s \oplus u1 \oplus \bar{su}_2^f \oplus u\Gamma\) should describe such a spin-1 boson condensed state where the charge operators and spin operators both have algebraic correlations.

**The local operators and partition function:** The physical properties of the critical state (i.e. the spin-1 boson condensed state) are encapsulated in the local operators and their correlation functions. If we know the partition function of the CFT \(6\), we can read out the allowed local operators. Reversely, a legitimate partition function should only contain allowed local operators. For a CFT to be a bosonic low energy theory, its partition function must be modular invariant. This condition puts strong restriction on allowed local operators.\(^{14}\)

To see the modular invariance condition, it is convenient to consider the theory on the space-time torus, specified by two independent lattice vector \(\omega_1\) and \(\omega_2\) on a complex plane. The partition function of a conformal invariant theory depends solely on \(\tau = \omega_2/\omega_1\), assuming all the low energy modes, including right and left movers, has the same velocity that is set to be 1. In the Hamiltonian formalism, the partition function for a bosonic system is given by
\[ Z(\tau) = \text{Tr} e^{-\text{Im}(\tau) H - i\text{Re}(\tau) K} \] (7)
where \(H\) is the Hamiltonian of the system and \(K\) is the total momentum operator of the system. The modular invariance means that
\[ T : Z(\tau) \to Z(\tau + 1) = Z(\tau) \] \[ S : Z(\tau) \to Z(\tau) \left(\frac{-1}{\tau}\right) = Z(\tau) \] (8)
Eqn. (7) can be expanded as

$$Z(\tau) = q^{-\tau^*}(q^*)^{-\tau} \sum_{(h,\overline{h})} N_{h,\overline{h}} q^h (q^*)^{\overline{h}}$$

(9)

where $c$, $\tau$ are the central charge for right and left movers, $q = e^{-i\pi \tau_2}$, and $L$ is the size of the 1D system. The summation $\sum_{(h,\overline{h})}$ is over a set of pairs $(h, \overline{h})$, which gives rise to the spectrum of scaling dimensions of local operators. The expansion coefficients $N_{h,\overline{h}}$ must be positive integers.

For CFTs described by current algebras, $Z(\tau)$ can be reorganized into a finite summation, $Z(\tau) = \sum_{\mu,\nu} M_{\mu,\nu} x_{\mu} x_{\nu}$, where $\mu(\nu)$ runs over the labels of primary fields of current algebras for right (left) movers. The multiplicity $M_{\mu,\nu}$ must be a non-negative integer. $x_{\mu}$ or $x_{\nu}$ is the so-called character associated with the primary field of current algebras, labeled by $\mu$ or $\nu$.

The character $x^{u_{1N}}_m$ of $u_{1N}$ CFT is given by

$$x^{u_{1N}}_m(\tau) = \eta^{-1}(q) \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(m+nR)^2}$$

(10)

where $\eta(q) = q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1 - q^n)$, $0 \leq m < N$, and $R^2 = N$. It contains primary fields of $u_{1N}$ current algebra,

$$e^{i(\frac{\tau}{2} + nR)\phi}$$

(11)

When $N = 1$, under modular transformation, $x^{u_{1N}}_m$ transform as

$$x^{u_{1N}}_m(-\frac{1}{\tau}) = \sum_{j} S_{ij} x^{u_{1N}}_j(\tau), \quad S_{ij} = \frac{1}{N} e^{-i\frac{2\pi i j}{N}}$$

(12)

The character $x^{su_2}_j(\tau)$ of $su_2$ CFT is given by

$$x^{su_2}_j(\tau) = \frac{\eta(q)\eta(q^4)}{[\eta(q)]^5} \cdot \sum_{n \in \mathbb{Z}} (2j + 1 + 2n(k+2)) q^{n(2j+1+(k+2)n)}$$

(13)

where $j \in \mathcal{P} = \{0, \frac{1}{2}, \cdots, \frac{k}{2}\}$. It corresponds to the primary field of $su_2$ CFT, $V_{j,m}$, $m = -j, -j+1, \cdots, j$. The modular transformations are

$$x^{su_2}_j(-1/\tau) = \sum_{l \in \mathcal{P}} S_{jl} x^{su_2}_l(\tau),$$

$$S_{jl} = \sqrt{\frac{2}{k+2}} \sin \left[ \frac{\pi(2j + 1)(2l + 1)}{k+2} \right]$$

(14)

$$x^{su_2}_j(\tau + 1) = e^{-i2\pi \frac{2k+1}{k+2}} e^{i2\pi \frac{2j+1+(k+1)n}{k+2}} x^{su_2}_j(\tau).$$

We can use the above $u_{1N}$ and $su_2$ characters to construct the modular invariant partition function. Before the construction, that low energy local operators must be only flavor-singlet further constraints the allowed operators. In particular, one allowed type is given by

$$| \psi_{\alpha\alpha} (i\sigma^2)_{\alpha\beta} (i\sigma^2)_{ab} \psi_{\beta b} |^2 \sim e^{i2\pi e} + \cdots ,$$

$$| \psi_{\alpha\alpha} (i\sigma^2)_{\alpha\beta} (i\sigma^2)_{ab} \psi_{\beta b} |^2 \sim e^{i2\pi e} + \cdots .$$

(15)

They are charge-4 local bosonic operators with scaling dimensions $(h,\overline{h}) = (2,0)$ and $(\overline{h},h) = (0,2)$. They are also purely chiral operators as they contain only right movers or only left movers. Next we examine the charge-2 spin-0 local bosonic operators

$$\psi_{\alpha\alpha} (i\sigma^2)_{\alpha\beta} (i\sigma^2)_{ab} \psi_{\beta b} , \quad \overline{\psi}_{\alpha\alpha} (i\sigma^2)_{\alpha\beta} (i\sigma^2)_{ab} \overline{\psi}_{\beta b} .$$

(16)

However, in the low energy sector, the only purely chiral charge-2 spin-0 operators built from eqn. (5) are $e^{i\pi e}$ and $e^{i\pi e}$. They have scaling dimensions $(h,\overline{h}) = (\frac{1}{2},0)$ and $(\overline{h},h) = (0,\frac{1}{2})$ with half-integral conformal spin $\frac{1}{2}$ and are not local bosonic operators. This implies that the charge-2 spin-0 local bosonic operators in eqn. (16) do not belong to the gapless sector, and are gapped operators.

Another type of purely chiral low energy local operators is given by

$$\psi_{\alpha\alpha} (i\sigma^2)_{\alpha\beta} (i\sigma^2)_{ab} \psi_{\beta b} \sim e^{i\pi e} \psi_{1,1} + \cdots ,$$

$$\overline{\psi}_{\alpha\alpha} (i\sigma^2)_{\alpha\beta} (i\sigma^2)_{ab} \overline{\psi}_{\beta b} \sim e^{i\pi e} \overline{\psi}_{1,1} + \cdots .$$

(17)

These charge-2 spin-1 local bosonic operators with scaling dimensions $(h,\overline{h}) = (1,0)$ and $(\overline{h},h) = (0,1)$ also belong to the gapless sector.

Since the chiral charged spin-0 local operators in the $u_{1N}$ sector are generated by $e^{i\pi e}$, but not $e^{i\pi e}$, from eqn. (11), we deduce that $R = 2$ and $N = 4$, and the $u_{1N}$ CFT is more precisely $u_{14}$ CFT.

We are ready to construct the partition functions for the critical state. The partition function must contain a term $\chi^{u_{14}}_0 \chi^{su_2}_0 \chi^{u_{14}}_0 \chi^{su_2}_0$, which corresponds to the presence of a vacuum state. It must also contain $\chi^{u_{14}}_0 \chi^{su_2}_2 \chi^{u_{14}}_0 \chi^{su_2}_2$, and $\chi^{u_{14}}_1 \chi^{su_2}_1 \chi^{u_{14}}_1 \chi^{su_2}_1$, corresponding to the presence of the local operators in (17) and their product. Starting from these spin-integer bosonic terms

$$\chi^{u_{14}}_0 \chi^{su_2}_2 \chi^{u_{14}}_0 \chi^{su_2}_2 \chi^{u_{14}}_1 \chi^{su_2}_1$$

(18)

there is a single solution of modular invariant partition function

$$Z(\tau) = | \chi^{u_{14}}_0 \chi^{su_2}_2 + \chi^{u_{14}}_1 \chi^{su_2}_1 |^2$$

(19)

$$+ | \chi^{u_{14}}_0 \chi^{su_2}_2 + \chi^{u_{14}}_2 \chi^{su_2}_2 |^2 + | \chi^{u_{14}}_1 \chi^{su_2}_2 + \chi^{u_{14}}_2 \chi^{su_2}_2 |^2$$

The above partition function is non-diagonal, and is distinct from that of the direct sum of $su_2$ and $u_{14}$ CFTs. In particular, it implies that the following operators (which are not purely chiral) are also low energy local operators:

$$e^{i\pi e} \psi_{1,1} \psi_{1,1}, \quad \overline{\psi}_{1,1} \psi_{1,1}, \quad \psi_{1,1} \psi_{1,1} \psi_{1,1} \psi_{1,1}, \quad \overline{\psi}_{1,1} \psi_{1,1} \psi_{1,1} \psi_{1,1}.$$
where \( l = -1, 0, 1 \). The first four operators have scaling dimensions \((h, \bar{h}) = (\frac{1}{2}, \frac{1}{2})\), and the last one has \((h, \bar{h}) = (\frac{5}{16}, \frac{5}{16})\). Table I summarizes the local operators in the critical state \( su_2^2 \oplus u_4 \oplus su_2^2 \oplus u_4 \), and all their quantum numbers. Note that the local operators \( e^{\pm i (\varphi_c - \varphi_r)} \) that have total integral spin are actually composed of left and right spin-1/2 operators. When such a spin-1 excitation is created locally, it will actually fractionalize into a spin-1/2 right-mover and a spin-1/2 left-mover.

As shown in Table I, when \( k_F \) corresponds to incommensurate filling, there is only one local operator \( \sum_l V_{1,l}^{su^2_2} V_{1,l}^{u^2_4} \) that carries trivial quantum numbers. It has a scaling dimension \((h, \bar{h}) = (\frac{1}{2}, \frac{1}{2})\), and thus represents a relevant perturbation. We would like to show that this perturbation drives the critical state to a superfluid state where all spin excitations are gapped. The observation is that the spin sector \( su_2^2 \oplus su_2^2 \) with central charge \( c + \bar{c} = \frac{1}{2} + \frac{1}{2} \) can be described by three right-moving and three left-moving Majorana fermions that form the spin-1 representation of \( SU^*(2) \). The relevant local operator \( \sum_l V_{1,l}^{su^2_2} V_{1,l}^{su^2_2} \) corresponds to \( SU^*(2) \) symmetric mass term for the Majorana fermions, which also has a scaling dimension \((h, \bar{h}) = (\frac{1}{2}, \frac{1}{2})\). Adding such a mass term can gap out all the spin excitations.

We see that the critical state \( su_2^2 \oplus u_4 \oplus su_2^2 \oplus u_4 \), i.e., the spin-1 boson condensed state, is unstable against the gap opening for spin excitations. It is actually a critical point that describes a quantum phase transition between two spin-gapped superfluid states of charge-2 spin-1 bosons (electron pairs).

**Continuous phase transition between two spin-1 bosonic superfluids:** There are two stable phases where the charge operators have algebraic correlations, and all spin operators have exponential decay correlations. One of the spin gapped state is induced by a pairing of charge-2 spin-1 bosons to form charge-1 spin-0 boson pairs (or electron quadruplets). We refer this state as a non-topological superfluid. The other spin gapped state corresponds to the spin-1 on the bosons forming a spin disordered state of spin-1 anti-ferromagnetic chain (which is the AKLT state\(^4\)) or an \( SO(3) \) SPT state\(^1\). We refer such a state as a topological superfluid.

For a system on a ring with \( 2N \) charge-2 spin-1 bosons, both the non-topological superfluid and the topological superfluid has a spin-0 ground state. However, for a system on a ring with \( 2N + 1 \) bosons, the non-topological superfluid must have spin-1 ground state, while the topological superfluid still has a spin-0 ground state. For a system on an open chain with \( 2N \) bosons, the topological superfluid will have emergent spin-1/2 at the two chain ends, while the non-topological superfluid does not have such emergent spin-1/2. In the following, we like to argue that the \( su_2^2 \oplus u_4 \oplus su_2^2 \oplus u_4 \) CFT describes a continuous phase transition between the above two superfluids, despite they have the same symmetry.

Our argument is based on the known results for the antiferromagnetic spin-1 chain (which corresponds to a commensurate filling for our spin-1 boson system). The critical point between the AKLT phase and the dimerized phases that spontaneously breaks translational symmetry are believed to be described by \( su_2^2 \oplus su_2^2 \) CFT\(^1\), which is confirmed by DMRG calculation\(^1\). In our proposed continuous phase transition, the only modification is that spins are charged. The gapless spins at the critical point are described by the same \( su_2^2 \oplus su_2^2 \) CFT. Also, the spin-1’s in the topological superfluid form the AKLT state, and the spin-1’s in the non-topological superfluid form the paired singlets. This comparison suggests that \( su_2^2 \oplus u_4 \oplus su_2^2 \oplus u_4 \) describes a continuous phase transition between the topological and the non-topological superfluids (see Fig. 1).

This research is partially supported by NSF grant DMR-1506475 and DMS-1664412.

| operators | spin | charge | \( k \) | \( h, \bar{h} \) |
|-----------|------|--------|------|------------|
| \( e^{\pm i \varphi_c} V_{1,l}^{su^2_2} \) | 1 | ±2 | ±2\( k_F \) | 1, 0 |
| \( e^{\pm i \varphi_r} V_{1,l}^{su^2_2} \) | 1 | ±2 | ±2\( k_F \) | 0, 1 |
| \( e^{\pm i (\varphi_c - \varphi_r)} \) | 0 | ±4 | \( \frac{1}{2}, \frac{1}{2} \) |
| \( e^{\pm i (\varphi_c - \varphi_r)} \) | 0 | 0 | ±4\( k_F \) | \( \frac{1}{2}, \frac{1}{2} \) |
| \( V_{1,l}^{su^2_2} \) | 0, 1, 2 | 0 | 0 | \( \frac{1}{2}, \frac{1}{2} \) |
| \( e^{\pm i (\varphi_c - \varphi_r)} \) | 0, 1 | 0 | ±2\( k_F \) | \( \frac{3}{16}, \frac{3}{16} \) |

| TABLE I. Quantum numbers of local operators in the critical state \( su^2_2 \oplus u_4 \oplus su^2_2 \oplus u_4 \). Here \( k \) is the crystal momentum, which is obtained by noticing that a right (left) moving charge-1 operator carries a crystal momentum \( k_F \) (−\( k_F \)). |

**FIG. 1.** Phase diagram of anti-ferromagnetic spin-1 bosons with symmetry \( SO(3) \times U(1) \times trn \). The horizontal axis is \( g \) in the interaction term \( g \sum V_{1,l}^{su^2_2} V_{1,l}^{su^2_2} \), and vertical axis is the the boson chemical potential \( \mu_B \).
1 F. H. L. Essler, G. V. Shlyapnikov, and A. M. Tsvelik, Journal of Statistical Mechanics: Theory and Experiment 2, 02027 (2009), arXiv:0811.0844.
2 F. Zhou and M. Snoek, Annals of Physics 308, 692 (2003), cond-mat/0306697.
3 F. D. M. Haldane, Physics Letters A 93, 464 (1983).
4 I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Commun. Math. Phys. 115, 477 (1988).
5 X.-G. Wen and Y.-S. Wu, Phys. Rev. Lett. 70, 1501 (1993).
6 W. Chen, M. P. A. Fisher, and Y.-S. Wu, Phys. Rev. B 48, 13749 (1993).
7 X.-G. Wen, Phys. Rev. Lett. 84, 3950 (2000), cond-mat/9908394.
8 T. Senthil, A. Vishwanath, L. Balents, S. Sachdev, and M. P. A. Fisher, Science 303, 1490 (2004).
9 T. Senthil and M. P. A. Fisher, (2005), cond-mat/0510459.
10 Y. Ran and X.-G. Wen, (2006), cond-mat/0609620.
11 S. Sachdev and X. Yin, ArXiv e-prints (2008), arXiv:0808.0191.
12 P. Francesco, P. Mathieu, and D. Sénéchal, Conformal field theory (Springer Science & Business Media, 2012).
13 D. Controzzi and A. M. Tsvelik, Phys. Rev. B 72, 035110 (2005), cond-mat/0503050.
14 J. L. Cardy, Nucl. Phys. B 270, 186 (1986).
15 Z.-C. Gu and X.-G. Wen, Phys. Rev. B 80, 155131 (2009), arXiv:0903.1069.
16 I. Affleck and F. Haldane, Physical Review B 36, 5291 (1987).
17 F. Michaud, F. Vernay, S. R. Manmana, and F. Mila, Physical review letters 108, 127202 (2012).