BI-ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE,
REGULAR SEMI-CLASSICAL WEIGHTS AND INTEGRABLE
SYSTEMS

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Abstract. The theory of bi-orthogonal polynomials on the unit circle is developed for a general class of weights leading to systems of recurrence relations and derivatives of the polynomials and their associated functions, and to functional-difference equations of certain coefficient functions appearing in the theory. A natural formulation of the Riemann-Hilbert problem is presented which has as its solution the above system of bi-orthogonal polynomials and associated functions. In particular for the case of regular semi-classical weights

\[ w(z) = \prod_{j=1}^{m} (z - z_j(t))^{\rho_j}, \]

consisting of \( m \in \mathbb{Z} > 0 \) finite singularities, difference equations with respect to the bi-orthogonal polynomial degree \( n \) (Laguerre-Freud equations or discrete analogs of the Schlesinger equations) and differential equations with respect to the deformation variables \( z_j(t) \) (Schlesinger equations) are derived completely characterising the system.

1. Introduction

The unitary group \( U(N) \) with Haar (uniform) measure has eigenvalue probability density function (see e.g. 23 Chapter 2)

\[ \frac{1}{(2\pi)^N N!} \prod_{1 \leq j < k \leq N} |z_k - z_j|^2, \quad z_l := e^{i\theta_l} \in \mathbb{T}, \quad \theta_l \in (-\pi, \pi], \]

where \( \mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \} \). Our interest is in averages over \( U \in U(N) \) of class functions \( w(U) \) (i.e. symmetric functions of the eigenvalues of \( U \) only) which have the factorization property \( \prod_{l=1}^{N} w(z_l) \) for \( \{ z_1, \ldots, z_N \} \in \text{Spec}(U) \). Introducing the Fourier components \( \{ w_l \}_{l \in \mathbb{Z}} \) of the weight \( w(z) = \sum_{l=-\infty}^{\infty} w_l z^l \), due to the well known identity 45

\[ \left\langle \prod_{l=1}^{N} w(z_l) \right\rangle_{U(N)} = \text{det}[w_{i-j}]_{i,j=1,\ldots,N}, \]

we are equivalently studying Toeplitz determinants. However we are interested in the situation where the weights are not necessarily positive or even real valued \( \bar{w}(z) \neq w(z) \) for \( z \in \mathbb{T} \), where the bar denotes the complex conjugate, and consequently the Toeplitz matrices are non-Hermitian \( \bar{w}_n \neq -w_n \). The motivations for studying these types of weights come from many diverse applications, in particular

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the gap probabilities and characteristic polynomial averages in random matrix theory, the spin-spin correlations of the square lattice Ising model, the density matrix of one-dimensional systems of impenetrable bosons and probability distributions for various classes of non-intersecting lattice path problems. All the above applications correspond to special cases of the weight

\[ w(z) = t^{-\mu} z^{-\omega} (1 + z)^{2\omega_1} (1 + tz)^{2\mu} \begin{cases} 1, & \theta \in (-\pi, \pi - \phi) \\ 1 - \xi, & \theta \in (\pi - \phi, \pi) \end{cases} , \]

where \( \mu, \omega = \omega_1 + i\omega_2 \) are complex parameters and \( \xi, t = e^{i\phi} \) are complex variables.

We have previously shown \[24\] that (1.2) with weight (1.3) can, as a function of \( t \), be characterised as a \( \tau \)-function for the Painlevé VI system, and thus be expressed in terms of a solution of the sixth Painlevé equation. Here we will develop theory which allows for a characterisation of a general class of Toeplitz determinants as a function of the discrete variable \( N \). In the particular case of the weight (1.3) the resulting difference equations in \( N \) will be studied in a companion paper \[25\] where it is shown they are equivalent to the discrete fifth Painlevé equation dPV associated with the degeneration of the rational surface \( D_4^{(1)} \rightarrow D_5^{(1)} \).

To characterise the unitary group averages (1.2) we have found it necessary to substantially develop the theory of bi-orthogonal polynomial systems on the unit circle \( \{ \phi_n(z), \phi_n^*(z) \}_{n=0}^\infty \), generalising the Szegő and Geronimus theory applying to orthogonal polynomial systems. They are defined by

\[ \int_T \frac{d\xi}{2\pi i\xi} w(\xi) \phi_m(\xi) \phi_n^*(\xi) = \delta_{m,n} . \]

Issues relating to the existence of such systems and their basic properties is taken up in Section 2. A study of bi-orthogonal polynomial systems on the unit circle was begun by Baxter \[11, 12\] where their elementary properties were elucidated and the connection between the absolute convergence of the trigonometric expansion of \( \log w(e^{i\theta}) \) and absolute convergence of certain coefficients of the bi-orthogonal polynomials was investigated. Sometime later other useful elements of the theory of orthogonal polynomial systems were added by Jones, Njástad and Thron in their study \[35\] of the trigonometric moment problem and related quadrature formulae on the unit circle, namely the associated polynomials and the Hilbert transform of the weight. The task of determining the functional, difference and differential equations for orthogonal polynomial systems on the unit circle was initiated in \[33\] but was not completed nor extended to cover the bi-orthogonal situation. In extending the theory in this direction we have sought to cast it in a way so that the appearance is as close to the Szegő-Geronimus theory as possible, and indeed we find that virtually all formulae from the older theory can be taken over but now the complex conjugated variables have to be re-interpreted as independent variables.

We wish to emphasise that the works described above, which are directly relevant
to the present study of certain classes of weights, constitute only a small fraction of existing literature on orthogonal polynomial systems on the unit circle. The themes taken up in this broader body of work, for example as recounted in [44], are really rather different problems and are formulated for the most general types of real, positive measures on the unit circle. Finally we note that to a large extent the task of characterising the analogous Hankel determinants has been completed for weights defined on the real line and this could be founded upon the standard theory of orthogonal polynomial systems. In particular we note the works of Bauldry [10], Bonan and Clark [16], Belmehdi and Ronveaux [13], and Magnus [38, 37, 36] where in the last three works the differential and difference structures have been clearly revealed and linked to isomonodromy preserving deformations, culminating in the identification of the simplest nontrivial case with the sixth Painlevé system.

In our formulation of the problem we start with a weight, possessing certain features to be made precise later, which we take as given and wish to calculate the Toeplitz determinants via auxiliary quantities arising in the bi-orthogonal polynomial theory. To this end we have derived closed systems of differential relations for the polynomials \( \{ \phi_n(z) \}_{n=0}^{\infty} \), their reciprocal polynomials \( \{ \phi_n^*(z) \}_{n=0}^{\infty} \), and associated functions \( \{ \epsilon_n(z) \}_{n=0}^{\infty} \), \( \{ \epsilon_n^*(z) \}_{n=0}^{\infty} \) in Proposition 2.6. In the notation of (2.69) and Corollary 2.3 let

\[
Y_n(z; t) := \begin{pmatrix} \phi_n(z) & \epsilon_n(z)/w(z) \\ \phi_n^*(z) & -\epsilon_n^*(z)/w(z) \end{pmatrix}
\]

We show

\[
\frac{d}{dz} Y_n := A_n Y_n,
\]

where the entries in the matrix \( A_n \) are parameterised by four coefficient functions \( \Omega_n(z) \), \( \Omega_n^*(z) \), \( \Theta_n(z) \), \( \Theta_n^*(z) \) in (2.71), and complete sets of difference and functional relations for these coefficient functions are given in Proposition 2.7 and Corollary 2.2. We also formulate a \( 2 \times 2 \) matrix Riemann-Hilbert problem in Proposition 2.8 for general classes of weights which parallels the case for orthogonal polynomials on the line [34, 21, 22, 18] and whose solution is simply related to \( Y_n \).

For our particular applications the weights are members of the regular semi-classical class

\[
w(z) = \prod_{j=1}^{m} (z - z_j(t))^\rho_j, \quad \rho_j \in \mathbb{C},
\]

with an arbitrary number \( m \) of isolated finite singularities located at \( z_j(t) \). These are also known as generalised Jacobi weights. Using the projection of the unit circle onto the interval \([-1, 1]\) a system of orthogonal polynomials on the unit circle is equivalent to two related systems of orthogonal polynomials on this interval. The generalised Jacobi orthogonal polynomial systems with support \([-1, 1]\) have been studied from a number of points of view. For example the Stieltjes electrostatic problem was generalised to include an arbitrary number of fixed charges in [27].
Uniform asymptotics of the orthogonal polynomials were first derived in [2], [3] and this was extended to their derivatives in [4], [48], [47] and [49]. Further properties of the polynomials, such as their upper bounds, were investigated in [19], [43] and estimates of the associated functions and polynomials were made in [17]. Interest in issues relating to quadrature problems were taken up in [20] where lower and upper bounds of the Christoffel function were found and quadrature inequalities were derived in [42]. The zeros of the orthogonal polynomials have also been studied, in particular the spacing of consecutive zeros in [20] and their asymptotic formulae in [40]. We wish to mention too a work similar in spirit to our own [26], where the orthogonal polynomial system
\[ \{p_n(t)\}_{n=0}^{\infty} \]
was defined by
\[ \int_C dt \omega(t)p_n(t)t^k = 0, \quad k = 0, \ldots, n-1. \]
In this work the weight had \( m = 3 \) distinct finite singularities of the regular semi-classical class and its support was a closed curve \( C \) enclosing all the singularities. Recurrence relations in \( n = \deg(p_n) \) for the three-term recurrence coefficients were found and heuristic arguments were given for the asymptotic behaviour of these as well as the polynomials themselves.

A key feature of regular semi-classical weights is that
\[ \frac{1}{w(z)} \frac{d}{dz} w(z) = \frac{2V(z)}{W(z)}, \]
where \( V \) and \( W \) are polynomials such that \( \deg V(z) < m, \deg W(z) = m \). The coefficient functions for regular semi-classical weights are polynomials of \( z \) with bounded degree, \( \deg \Theta_n(z) = \deg \Theta^*_n(z) = m - 1, \deg \Theta_n(z) = \deg \Theta^*_n(z) = m - 2 \) (see Proposition 3.1). In addition evaluations of these functions at the singular points satisfy bilinear relations (see Proposition 3.2) which lead directly to one of the pair of coupled discrete Painlevé equations. Deformation derivatives of the linear system of differential equations above with respect to arbitrary trajectories of the finite singularities are given in Proposition 3.3 which can summarised as
\[ \frac{d}{dt} Y_n := B_n Y_n = \left\{ B_{\infty} - \sum_{j=1}^{m} \frac{A_{nj}}{z - z_j} \frac{d}{dt} z_j \right\} Y_n, \quad \text{where} \quad A_n = \sum_{j=1}^{m} \frac{A_{nj}}{z - z_j}, \]
and consequently systems of Schlesinger equations for the elements of \( A_{nj} \) (or the coefficient functions evaluated at \( z_j \)) are given in (3.77-3.79). It is quite natural that systems governed by regular semi-classical weights preserve the monodromy data of the solutions \( Y_n \) about each singularity \( z_j \) with respect to arbitrary deformations.

We mention at this point that the full definition of a regular semi-classical weight given below in Definition 3.1 is restrictive, and has been generalised by relaxing some of the conditions in a series of works [41], [39], [40]. In these works the orthogonal polynomial systems were characterised by integral representations of semi-classical linear functionals with respect to certain paths in the complex plane. An irregular
semi-classical weight arising under less restrictive conditions can be recovered from a particular regular semi-classical weight through limiting processes involving the coalescence of singular points in the same way as the fifth to the first Painlevé systems are recovered from the sixth. The differential and difference structures of orthogonal and bi-orthogonal polynomial systems with polynomial log-derivatives of the weight \( \deg(W) = 0, \deg(V) > 2 \) defined on \( \mathbb{R} \) have been studied [14], [15] in the context of matrix models. Analogous structures for a bi-orthogonal polynomial system on the unit circle with a simple Laurent polynomial log-derivative for the weight were found in [31], motivated by applications to unitary matrix models.

In Section 2 we derive systems of differential-difference and functional relations for bi-orthogonal polynomials and associated functions on the unit circle for a general class of weights and formulate the Riemann-Hilbert problem. In Section 3 we specialise to regular semi-classical weights and derive bilinear difference equations. In addition we calculate the deformation derivatives of the bi-orthogonal polynomial system, derive a system of Schlesinger equations and show the deformations are of the isomonodromic type.

2. BI-ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE AND RIEMANN-HILBERT PROBLEM

We consider a complex function for our weight \( w(z) \), analytic in the cut complex plane and which possesses a Fourier expansion

\[
(2.1) \quad w(z) = \sum_{k=-\infty}^{\infty} w_k z^k, \quad w_k = \int_{\mathbb{T}} \frac{d\zeta}{2\pi i\zeta} w(\zeta)\zeta^{-k},
\]

where \( z \in D \subset \mathbb{C} \) and \( \mathbb{T} \) denotes the unit circle \( |\zeta| = 1 \) with \( \zeta = e^{i\theta}, \theta \in (-\pi, \pi] \). Hereafter we will assume that \( z^j w(z), z^j w'(z) \in L^1(\mathbb{T}) \) for all \( j \in \mathbb{Z} \). We will also assume that the trigonometric sum converges in an annulus \( D = \{ z \in \mathbb{C} : \Delta_1 < |z| < \Delta_2 \} \) and \( \mathbb{T} \subset D \). Until we arrive at our specific weights of interest in Section 3, namely the regular semi-classical class, we will formally assume convergence holds. The doubly infinite sequence \( \{w_k\}_{k=-\infty}^{\infty} \) are the trigonometric moments of the distribution \( w(e^{i\theta})d\theta/2\pi \) and define the trigonometric moment problem. Define the Toeplitz determinants

\[
I_n[w] := \det \left[ \int_{\mathbb{T}} \frac{d\zeta}{2\pi i\zeta} w(\zeta)\zeta^{\epsilon-j+k} \right]_{0 \leq j, k \leq n-1},
\]

\[
= \det \left[ w_{\epsilon+j-k} \right]_{0 \leq j, k \leq n-1},
\]

\[
(2.2) \quad = \frac{1}{n!} \int_{\mathbb{T}^n} \prod_{i=1}^{n} \frac{d\zeta_i}{2\pi i \zeta_i} w(\zeta_i) \prod_{1 \leq j < k \leq n} |\zeta_j - \zeta_k|^2,
\]

where \( \epsilon \) will take the integer values 0, \pm 1.
We define a system of bi-orthogonal polynomials \( \{ \phi_n(z), \bar{\phi}_n(z) \} \) with respect to the weight \( w(z) \) on the unit circle by the orthogonality relation

\[
\int_T \frac{d\zeta}{2\pi i\zeta} w(\zeta) \phi_m(\zeta) \bar{\phi}_n(\zeta) = \delta_{m,n}.
\]

This system is taken to be orthonormal and the coefficients in a monomial basis are defined by

\[
\phi_n(z) = \kappa_n z^n + l_n z^{n-1} + m_n z^{n-2} + \ldots + \phi_n(0) = \sum_{j=0}^{n} c_{n,j} z^j,
\]

\[
\bar{\phi}_n(z) = \bar{\kappa}_n z^n + \bar{l}_n z^{n-1} + \bar{m}_n z^{n-2} + \ldots + \bar{\phi}_n(0) = \sum_{j=0}^{n} \bar{c}_{n,j} z^j,
\]

where \( \bar{\kappa}_n \) is chosen to be equal to \( \kappa_n \) without loss of generality (this has the effect of rendering many results formally identical to the pre-existing theory of orthogonal polynomials). Notwithstanding the notation \( \bar{c}_{n,j} \) in general is not equal to the complex conjugate of \( c_{n,j} \) and is independent of it. We also define the reciprocal polynomial by

\[
\phi_n^*(z) := z^n \bar{\phi}_n(1/z) = \sum_{j=0}^{n} \bar{c}_{n,j} z^{n-j}.
\]

The bi-orthogonal polynomials can be defined up to an overall factor by the orthogonality with respect to the monomials

\[
\int_T \frac{d\zeta}{2\pi i\zeta} w(\zeta) \phi_n(\zeta) \bar{\zeta}^j = 0 \quad 0 \leq j \leq n-1,
\]

whereas their reciprocal polynomials can be similarly defined by

\[
\int_T \frac{d\zeta}{2\pi i\zeta} w(\zeta) \phi_n^*(\zeta) \bar{\zeta}^j = 0 \quad 1 \leq j \leq n.
\]

The linear system of equations for the coefficients \( c_{n,j}, \bar{c}_{n,j} \) arising from

\[
c_{n,n} \int_T \frac{d\zeta}{2\pi i\zeta} w(\zeta) \bar{\zeta}^m \bar{\phi}_n(\zeta) = \begin{cases} 0 & 0 \leq m \leq n-1 \\ 1 & m = n \end{cases},
\]

\[
\bar{c}_{n,n} \int_T \frac{d\zeta}{2\pi i\zeta} w(\zeta) \bar{\zeta}^m \phi_n(\zeta) = \begin{cases} 0 & 0 \leq m \leq n-1 \\ 1 & m = n \end{cases},
\]
has the solution

\[
c_{nj} = \frac{1}{c_{n,n}} \det \begin{pmatrix} w_0 & \ldots & 0 & \ldots & w_{-n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{n-1} & \ldots & 0 & \ldots & w_{-1} \\ w_n & \ldots & 1 & \ldots & w_0 \end{pmatrix},
\]

(2.11)

\[
\bar{c}_{nj} = \frac{1}{c_{n,n}} \det \begin{pmatrix} w_0 & \ldots & w_{-n} \\ \vdots & \vdots & \vdots \\ w_n & \ldots & w_0 \end{pmatrix}.
\]

(2.12)

and in particular one has the following results.

**Proposition 2.1** ([11]). The leading and trailing coefficients of the polynomials \( \phi_n(z) \), \( \bar{\phi}_n(z) \) are

\[
c_{nn} = \bar{c}_{nn} = \kappa_n = \frac{1}{\kappa_n \frac{I_n^0}{I_{n+1}^0}},
\]

(2.13)

\[
c_{n0} = \phi_n(0) = (-1)^n \frac{1}{\kappa_n \frac{I_n^1}{I_{n+1}^0}}, \quad \bar{c}_{n0} = \bar{\phi}_n(0) = (-1)^n \frac{1}{\kappa_n \frac{I_n^{-1}}{I_{n+1}^0}}.
\]

(2.14)

**Proposition 2.2** ([11]). The bi-orthogonal system \( \{\phi_n(z), \phi_n^*(z)\}_{n=0}^{\infty} \) exists if and only if \( I_n^0 \neq 0 \) for all \( n \geq 1 \).

**Remark 2.1.** We shall see that a failure of this condition can and generically must occur in the case of regular semi-classical weights which contain deformation parameters \( z_1, \ldots, z_m \) and this is precisely the condition for a movable singularity (in this case a pole) in the dynamics with respect to the \( z_j \).

A consequence of the above solutions are the following determinantal and integral representations for the polynomials,

\[
\phi_n(z) = \frac{\kappa_n}{I_n^0} \det \begin{pmatrix} w_0 & \ldots & w_j & \ldots & w_{-n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{n-1} & \ldots & w_{n-j-1} & \ldots & w_{-1} \\ 1 & \ldots & z^j & \ldots & z^{n} \end{pmatrix},
\]

(2.15)

\[
= (-1)^n \kappa_n \frac{I_n^0 [w(\zeta)(\zeta - z)]}{I_n^0 [w(\zeta)]},
\]

(2.16)
\[ \phi_n^*(z) = \kappa_n \det \begin{pmatrix} w_0 & \ldots & w_{-n+1} & z^n \\ \vdots & \vdots & \vdots & \vdots \\ w_{n-j} & \ldots & w_{-j+1} & z^j \\ \vdots & \vdots & \vdots & \vdots \\ w_n & \ldots & w_1 & 1 \end{pmatrix} \]

(2.18)

\[ = \kappa_n \frac{I_0^0[w(\zeta)(1 - z\zeta^{-1})]}{I_0^0[w(\zeta)]}. \]

The system is alternatively defined by the sequence of ratios \( r_n = \frac{\phi_n(0)}{\kappa_n}, \)
known as reflection coefficients because of their role in the scattering theory formulation of OPS on the unit circle, together with a companion quantity \( \bar{r}_n = \frac{\phi_n(0)}{\kappa_n}. \)
As in the Szegö theory \[45\] \( r_n \) and \( \bar{r}_n \) are related to the above Toeplitz determinants
by

(2.19)

\[ r_n = (-1)^n \frac{I_n^1[w]}{I_n^0[w]}, \quad \bar{r}_n = (-1)^n \frac{I_n^{-1}[w]}{I_n^0[w]} \]

The Toeplitz determinants of central interest can then be recovered through the following result.

**Proposition 2.3** \([12]\). With the convention \( I_0^0 = 1 \) the sequence of \( \{I_n^0\}_{n=0}^{\infty} \)
satisfy

(2.20)

\[ \frac{I_{n+1}^0[w]I_n^0[w]}{(I_n^0[w])^2} = 1 - r_n \bar{r}_n, \quad n \geq 1. \]

subject to the condition \( r_n \bar{r}_n \neq 1 \) for all \( n \geq 1. \)

Fundamental consequences of the orthogonality condition are the following coupled linear recurrence relations.

**Proposition 2.4** \([11, 12]\).

(2.21)

\[ \kappa_n \phi_{n+1}(z) = \kappa_{n+1} z \phi_n(z) + \phi_{n+1}(0) \phi_n^*(z), \]

(2.22)

\[ \kappa_n \phi_{n+1}^*(z) = \kappa_{n+1} \phi_n^*(z) + \overline{\phi}_{n+1}(0) z \phi_n(z). \]

One finds three-term or second order recurrences for the uncoupled recurrence relations

(2.23)

\[ \kappa_n \phi_n(0) \phi_{n+1}(z) + \kappa_{n-1} \phi_{n+1}(0) z \phi_{n-1}(z) = [\kappa_n \phi_{n+1}(0) + \kappa_{n+1} \phi_n(0) z] \phi_n(z), \]

(2.24)

\[ \kappa_n \overline{\phi}_n(0) \phi_{n+1}^*(z) + \kappa_{n-1} \overline{\phi}_{n+1}(0) z \phi_{n-1}^*(z) = [\kappa_n \overline{\phi}_{n+1}(0) z + \kappa_{n+1} \overline{\phi}_n(0)] \phi_n^*(z). \]

The analogue of the Christoffel-Darboux summation formula is given by the following result.
These have integral representations analogous to (2.16,2.18)

\[ \phi_n(z) = \frac{\phi_n^*(z) - z\phi_n(z)}{1 - z}, \]

\[ = \frac{\phi_{n+1}(z) - \phi_{n+1}(z)}{1 - z}, \]

for \( z \neq 1 \). Here

Equations (2.23,2.24) being second order linear difference equations admit other linearly independent solutions \( \psi_n(z), \psi_n^*(z) \), and we define two such polynomial solutions, the polynomials of the second kind or associated polynomials

\[ \psi_n(z) := \int_{\mathbb{T}} \frac{d\zeta}{2\pi i} \frac{\zeta + z}{\zeta - z} w(\zeta)[\phi_n(\zeta) - \phi_n(z)], \quad n \geq 1, \quad \psi_0 := \kappa_0 w_0 = 1/\kappa_0, \]

and its reciprocal polynomial \( \psi_n^*(z) \). The integral formula for \( \psi_n^* \) is

\[ \psi_n^*(z) := -\int_{\mathbb{T}} \frac{d\zeta}{2\pi i} \frac{\zeta + z}{\zeta - z} w(\zeta)[z^n\phi_n(\zeta) - \phi_n^*(z)], \quad n \geq 1, \quad \psi_0^* := 1/\kappa_0. \]

A central object in the theory is the Carathéodory function, or generating function of the Toeplitz elements

\[ F(z) := \int_{\mathbb{T}} \frac{d\zeta}{2\pi i} \frac{\zeta + z}{\zeta - z} w(\zeta), \]

which has the expansions inside and outside the unit circle

\[ F(z) = \begin{cases} w_0 + 2\sum_{k=1}^{\infty} w_k z^k, & \text{if } |z| < \Delta_{\min} < 1, \\ -w_0 - 2\sum_{k=1}^{\infty} w_{-k} z^{-k}, & \text{if } |z| > \Delta_{\max} > 1. \end{cases} \]

Having these definitions one requires two non-polynomial solutions \( \epsilon_n(z), \epsilon_n^*(z) \) to the recurrences and these are constructed as linear combinations of the polynomial solutions according to

\[ \epsilon_n(z) := \psi_n(z) + F(z)\phi_n(z) = \int_{\mathbb{T}} \frac{d\zeta}{2\pi i} \frac{\zeta + z}{\zeta - z} w(\zeta)\phi_n(\zeta), \]

\[ \epsilon_n^*(z) := \psi_n^*(z) - F(z)\phi_n^*(z) = -z^n \int_{\mathbb{T}} \frac{d\zeta}{2\pi i} \frac{\zeta + z}{\zeta - z} w(\zeta)\phi_n^*(\zeta), \]

These have integral representations analogous to (2.16,2.18)

\[ \frac{\kappa_n}{2} \epsilon_n(z) = z^n \frac{I_n^1[w(\zeta)(\zeta - z)^{-1}]}{I_n^0 [w(\zeta)]}, \]

\[ \frac{\kappa_n}{2} \epsilon_n^*(z) = (z)^n \frac{I_n^{0+1}[w(\zeta)(\zeta - z)^{-1}]}{I_n^{n+1} [w(\zeta)]}. \]
for $|z| \neq 1$, which are particular cases of more general moments of the characteristic polynomial considered by Ismail and Rüdemann [32].

**Theorem 2.1** ([28], [24], [30], [35]). $\psi_n(z), \psi_n^*(z)$ satisfy the three-term recurrence relations (2.23, 2.24) and along with $\epsilon_n(z), \epsilon_n^*(z)$ satisfy a variant of (2.21) namely

\begin{align}
\kappa_n \epsilon_{n+1}(z) &= \kappa_{n+1} \epsilon_n(z) - \phi_{n+1}(0) \epsilon_n^*(z), \\
\kappa_n \epsilon_{n+1}^*(z) &= \kappa_{n+1} \epsilon_n^*(z) - \phi_{n+1}(0) \epsilon_n(z).
\end{align}

**Theorem 2.2** ([28]). The Casoratians of the polynomial solutions $\phi_n, \phi_n^*, \psi_n, \psi_n^*$ are

\begin{align}
\phi_{n+1}(z) \psi_n(z) - \psi_{n+1}(z) \phi_n(z) &= \phi_{n+1}(z) \epsilon_n(z) - \epsilon_{n+1}(z) \phi_n(z) = 2 \frac{\phi_{n+1}(0)}{\kappa_n} z^n, \\
\phi_{n+1}^*(z) \psi_n^*(z) - \psi_{n+1}^*(z) \phi_n^*(z) &= \phi_{n+1}^*(z) \epsilon_n^*(z) - \epsilon_{n+1}^*(z) \phi_n^*(z) = 2 \frac{\phi_{n+1}(0)}{\kappa_n} z^{n+1}, \\
\phi_n(z) \psi_n^*(z) + \psi_n(z) \phi_n^*(z) &= \phi_n(z) \epsilon_n^*(z) + \epsilon_n(z) \phi_n^*(z) = 2 z^n.
\end{align}

Further identities from the Szegő theory that generalise are those that relate the leading coefficients back to the reflection coefficients

\begin{align}
\kappa_n^2 &= \kappa_{n-1}^2 + \phi_n(0) \bar{\phi}_n(0), \\
\frac{l_n}{\kappa_n} &= \frac{l_{n-1}}{\kappa_{n-1}} + r_n \bar{r}_{n-1}, \\
\frac{m_n}{\kappa_n} &= \frac{m_{n-1}}{\kappa_{n-1}} + r_n \left[ \bar{r}_{n-2} + \bar{r}_{n-1} \frac{l_{n-2}}{\kappa_{n-2}} \right].
\end{align}

Some useful relations for the leading coefficients of the product of a monomial and an bi-orthogonal polynomial or its derivative are

\begin{align}
z \phi_n(z) &= \frac{\kappa_n}{\kappa_{n+1}} \phi_{n+1}(z) + \left( \frac{l_n}{\kappa_n} - \frac{l_{n+1}}{\kappa_{n+1}} \right) \phi_n(z) \\
&\quad + \left\{ \frac{l_n}{\kappa_{n-1}} \left( \frac{l_{n+1}}{\kappa_{n+1}} - \frac{l_n}{\kappa_n} \right) + \frac{m_n}{\kappa_{n-1}} - \frac{m_{n+1}}{\kappa_{n+1}} \right\} \phi_{n-1}(z) + \pi_{n-2}, \\
z^2 \phi_n(z) &= \frac{\kappa_n}{\kappa_{n+2}} \phi_{n+2}(z) + \left( \frac{l_n}{\kappa_{n+1}} - \frac{l_{n+2}}{\kappa_{n+2}} \right) \phi_{n+1}(z) \\
&\quad + \left\{ \frac{l_{n+1}}{\kappa_{n+1}} \frac{l_{n+2}}{\kappa_{n+2}} - \frac{l_n}{\kappa_n} \right\} + \frac{m_n}{\kappa_{n+1}} - \frac{m_{n+2}}{\kappa_{n+2}} \right\} \phi_n(z) + \pi_{n-1},
\end{align}
\begin{align}
\phi_n'(z) &= n \frac{\kappa_n}{\kappa_{n-1}} \phi_{n-1}(z) + \pi_{n-2}, \\
(2.45) \\
\phi_n^2(z) &= n \phi_n(z) - \frac{l_n}{\kappa_{n-1}} \phi_{n-1}(z) + \pi_{n-2}, \\
(2.46) \\
z^2 \phi_n(z) &= n \frac{\kappa_n}{\kappa_{n+1}} \phi_{n+1}(z) + \left\{ (n-1) \frac{l_n}{\kappa_n} - n \frac{l_{n+1}}{\kappa_{n+1}} \right\} \phi_n(z) + \pi_{n-1}, \\
(2.47)
\end{align}

where ' denotes the derivative with respect to \( z \) and where \( \pi_n \) denotes an arbitrary polynomial of the linear space of polynomials with degree at most \( n \). These identities can be verified directly.

We will require the leading order terms in expansions of \( \phi_n(z), \phi_n^*(z), \epsilon_n(z), \epsilon_n^*(z) \) both inside and outside the unit circle.

**Corollary 2.1.** The bi-orthogonal polynomials \( \phi_n(z), \phi_n^*(z) \) have the following expansions

\begin{align}
\phi_n(z) &= \begin{cases} 
\phi_n(0) + \frac{1}{\kappa_{n-1}}(\kappa_n \phi_{n-1}(0) + \phi_n(0)l_{n-1})z + O(z^2) & |z| < 1, \\
\kappa_n z^n + l_n z^{n-1} + O(z^{n-2}) & |z| > 1,
\end{cases} \\
(2.48) \\
\phi_n^*(z) &= \begin{cases} 
\kappa_n + l_n + O(z^2) & |z| < 1, \\
\bar{\phi}_n(0)z^n + \frac{1}{\kappa_{n-1}}(\kappa_n \bar{\phi}_{n-1}(0) + \bar{\phi}_n(0)l_{n-1})z^{n-1} + O(z^{n-2}) & |z| > 1,
\end{cases}
(2.49)
\end{align}

whilst the associated functions have the expansions

\begin{align}
\frac{\kappa_n}{2} \epsilon_n(z) &= \begin{cases} 
z^n - \frac{l_{n+1}}{\kappa_{n+1}} z^{n+1} + O(z^{n+2}) & |z| < 1, \\
\frac{\bar{\phi}_n(0)}{\kappa_{n+1}} z^{-1} + \left( \frac{\kappa_n^2}{\kappa_{n+1}^2} \phi_{n+2}(0) - \frac{\phi_n(0)}{\kappa_{n+1}} \right) z^{-2} + O(z^{-3}) & |z| > 1,
\end{cases} \\
(2.50) \\
\frac{\kappa_n}{2} \epsilon_n^*(z) &= \begin{cases} 
\bar{\phi}_n(0) z^{n+1} + \left( \frac{\kappa_n^2}{\kappa_{n+1}^2} \bar{\phi}_{n+2}(0) - \frac{\bar{\phi}_n(0)}{\kappa_{n+1}} l_{n+1} \right) z^{n+2} + O(z^{n+3}) & |z| < 1, \\
1 - \frac{l_{n+1}}{\kappa_{n+1}} z^{-1} + \left( \frac{l_n^2 + l_{n+1}^2}{\kappa_{n+1}^2 \kappa_{n+2}} - \frac{m_{n+2}}{\kappa_{n+2}} \right) z^{-2} + O(z^{-3}) & |z| > 1,
\end{cases}
(2.51)
\end{align}

**Proof.** The second line of (2.46) and the first line of (2.47) follow from the definitions. For the remaining lines of these two formulae we use (2.50)

\[ \kappa_n \phi_n'(0) = \kappa_n \phi_{n-1}(0) + \phi_n(0)l_{n-1}, \]

which results from differentiating (2.21) and setting \( z = 0 \). The first line of (2.48) and the second line of (2.49) can be derived by employing the uncoupled recurrence
A (their notation

Proof. The first, (2.52), was found in [33] where the coefficients were taken to be

with coefficient functions 

which in turn follows from combining (2.21) and (2.22).

The \( z \)-derivatives or spectral derivatives of the bi-orthogonal polynomials in general are related to two consecutive polynomials [33] and we extend this to all \( \phi_n(z), \phi'_n(z), \epsilon_n(z), \epsilon'_n(z) \) with the following parameterisation.

Proposition 2.6. The derivatives of the bi-orthogonal polynomials and associated functions are expressible as linear combinations in a related way (\( ' := d/\text{d}z \)),

(2.52) \[ W(z)\phi'_n(z) = \Theta_n(z)\phi_{n+1}(z) - (\Omega_n(z) + V(z))\phi_n(z), \]

(2.53) \[ W(z)\phi''_n(z) = -\Theta_n(z)\phi'_{n+1}(z) + (\Omega_n(z) - V(z))\phi'_n(z), \]

(2.54) \[ W(z)\epsilon'_n(z) = \Theta_n(z)\epsilon_{n+1}(z) - (\Omega_n(z) - V(z))\epsilon_n(z), \]

(2.55) \[ W(z)\epsilon''_n(z) = -\Theta_n(z)\epsilon'_{n+1}(z) + (\Omega_n(z) + V(z))\epsilon'_n(z), \]

with coefficient functions \( W(z), V(z) \) independent of \( n \).

Proof. The first, (2.52), was found in [33] where the coefficients were taken to be (their notation \( A_n, B_n \) should not be confused with our use of it subsequently)

(2.56) \[ A_n = -\frac{\kappa_{n-1}\phi_{n+1}(0)}{\kappa_n\phi_n(0)} \frac{z\Theta_n(z)}{W(z)}, \]

(2.57) \[ B_n = \frac{1}{W(z)} \left( \Omega_n(z) + V(z) - \left[ \frac{\phi_{n+1}(0)}{\phi_n(0)} + \frac{\kappa_n+1}{\kappa_n} \right] \Theta_n(z) \right). \]

The other differential relations can be found in an analogous manner. \( \square \)

The coefficient functions \( \Theta_n(z), \Omega_n(z), \Theta'_n(z), \Omega'_n(z) \) satisfy coupled linear recurrence relations themselves, one of which was reported in [33]. The full set are given in the following proposition.

Proposition 2.7. The coefficient functions satisfy the coupled linear recurrence relations

(2.58) \[ \Omega_n(z) + \Omega_{n-1}(z) - \left( \frac{\phi_{n+1}(0)}{\phi_n(0)} + \frac{\kappa_n+1}{\kappa_n} \right) \Theta_n(z) + (n-1) \frac{W(z)}{z} = 0, \]

(2.59) \[ \left( \frac{\phi_{n+1}(0)}{\phi_n(0)} + \frac{\kappa_n+1}{\kappa_n} \right) (\Omega_{n-1}(z) - \Omega_n(z)) + \frac{\kappa_n}{\kappa_n+1} \phi_{n+1}(0) \frac{z\Theta_{n+1}(z) - \kappa_{n-1}\phi_{n+1}(0)}{\phi_n(0)} z\Theta_{n-1}(z) - \frac{\phi_{n+1}(0) W(z)}{z} = 0, \]

(2.60) \[ \Omega'_n(z) + \Omega'_{n-1}(z) - \left( \frac{\kappa_n+1}{\kappa_n} + \frac{\phi_{n+1}(0)}{\phi_n(0)} \right) \Theta'_n(z) - n \frac{W(z)}{z} = 0, \]
(2.61) \( \left( \frac{\kappa_{n+1}}{\kappa_n} + \frac{\tilde{\phi}_{n+1}(0)}{\phi_n(0)} \right) \left( \Omega_{n-1}^*(z) - \Omega_n^*(z) \right) + \frac{\kappa_n \tilde{\phi}_{n+2}(0)}{\kappa_{n+1} \phi_{n+1}(0)} z \Theta_{n+1}^*(z) - \frac{\kappa_{n-1} \tilde{\phi}_{n+1}(0)}{\kappa_n \phi_n(0)} z \Theta_{n-1}^*(z) + \frac{\kappa_{n+1}}{\kappa_n} \frac{W(z)}{z} = 0, \)

(2.62) \( \Omega_{n+1}(z) + \Omega_n^*(z) - \left( \frac{\tilde{\phi}_{n+2}(0) + \kappa_{n+2}}{\phi_{n+1}(0) + \kappa_{n+1}} \right) \Theta_{n+1}(z) + \frac{\kappa_{n+1}}{\kappa_n} \left( z \Theta_n(z) - \Theta_n^*(z) \right) = 0, \)

(2.63) \( \Omega_n(z) - \Omega_{n+1}(z) + \frac{\kappa_{n+2}}{\kappa_{n+1}} \left( 1 + \frac{\tilde{\phi}_{n+2}(0) \phi_{n+1}(0)}{\kappa_{n+1} \kappa_{n+2}} \right) \Theta_{n+1}(z) + \frac{\phi_{n+1}(0) \phi_{n+1}(0)}{\kappa_{n+1} \kappa_n} z \Theta_n(z) - \frac{\kappa_{n+1}}{\kappa_n} \frac{W(z)}{z} = 0, \)

(2.64) \( \Omega_{n+1}^*(z) + \Omega_n(z) - \left( \frac{\kappa_{n+2}}{\kappa_{n+1}} \right) \left( 1 + \frac{\tilde{\phi}_{n+2}(0)}{\phi_{n+1}(0)} \right) \Theta_{n+1}^*(z) - \frac{\kappa_{n+1}}{\kappa_n} \left( z \Theta_n(z) - \Theta_n^*(z) \right) - \frac{W(z)}{z} = 0, \)

(2.65) \( \Omega_n^*(z) - \Omega_{n+1}^*(z) + \frac{\kappa_{n+2}}{\kappa_{n+1}} \left( 1 + \frac{\phi_{n+1}(0) \tilde{\phi}_{n+2}(0)}{\kappa_{n+1} \kappa_{n+2}} \right) \Theta_{n+1}^*(z) + \frac{\phi_{n+1}(0) \phi_{n+1}(0)}{\kappa_{n+1} \kappa_n} z \Theta_n(z) - \frac{\kappa_{n+1}}{\kappa_n} \Theta_n^*(z) = 0. \)

Proof. The first \( \text{(2.65)} \) was found in [33] by a direct evaluation of the left-hand side using integral definitions of the coefficient functions, however all of the relations follow from the compatibility of the differential relations and the recurrence relations. Thus \( \text{(2.62)-(2.65)} \) follow from the compatibility of \( \text{(2.52)-(2.55)} \) and \( \text{(2.22)-(2.24)} \), \( \text{(2.66)-(2.68)} \) follow from the combination of \( \text{(2.65)-(2.68)} \) and \( \text{(2.21)} \), and \( \text{(2.61)-(2.64)} \) follow from the combination of \( \text{(2.55)-(2.58)} \) and \( \text{(2.22)} \). \( \square \)

Remark 2.2. The relations given above are obviously not all independent, as for example we note that \( \text{(2.66)} \) can derived from \( \text{(2.63)} \) with the use of \( \text{(2.61)} \) below.

Corollary 2.2. Some additional identities satisfied by the coefficient functions are the following

(2.66) \( \frac{\phi_{n+1}(0)}{\phi_n(0)} \Theta_n(z) - \frac{\kappa_n}{\kappa_{n-1}} z \Theta_{n-1}(z) = \frac{\tilde{\phi}_{n+1}(0)}{\phi_n(0)} z \Theta^*_n(z) - \frac{\kappa_n}{\kappa_{n-1}} \Theta^*_n(z), \)

(2.67) \( \Omega_n^*(z) - \Omega_n(z) = -\frac{\kappa_{n+1}}{\kappa_n} (z \Theta_n(z) - \Theta^*_n(z)) + n \frac{W(z)}{z}, \)

(2.68) \( \Omega_n^*(z) + \Omega_n(z) = \frac{\kappa_n^2}{\kappa_{n+1}^2} \left[ \frac{\phi_{n+2}(0)}{\phi_{n+1}(0)} \Theta_{n+1}(z) + \frac{\kappa_{n+1}}{\kappa_n} \Theta^*_n(z) \right] + \frac{W(z)}{z}. \)
Proof. Consider (2.66) first. In (2.58) let us map \( n \mapsto n + 1 \) and add this result to (2.60). However the combination \( \Omega_{n+1}(z) + \Omega_n^*(z) \), which appears in the resulting sum in this form and also under \( n \mapsto n - 1 \), occurs in (2.62) allowing it to be eliminated. To derive (2.67) we again map \( n \mapsto n + 1 \) in (2.58) and subtract this from (2.62) eliminating \( \Omega_{n+1}(z) \). If we add (2.62) and (2.63) and simplify the result then we arrive at (2.68). \( \square \)

For a general system of bi-orthogonal polynomials on the unit circle the coupled recurrence relations and spectral differential relations can be reformulated in terms of first order \( 2 \times 2 \) matrix equations (or alternatively as second order scalar equations). Here we define our matrix variables and derive such matrix relations, and this serves as an introduction to a characterisation of the general bi-orthogonal polynomial system on the unit circle as the solution to a \( 2 \times 2 \) matrix Riemann-Hilbert problem.

Firstly we note that the recurrence relations for the associated functions \( \epsilon_n(z), \epsilon_n^*(z) \) given in (2.36, 2.37) differ from those of the polynomial systems (2.21, 2.22) by a reversal of the signs of \( \phi_n(0), \bar{\phi}_n(0) \). We can compensate for this by constructing the \( 2 \times 2 \) matrix (2.69)

\[
Y_n(z) := \begin{pmatrix} \phi_n(z) & \epsilon_n(z) \\ \phi_n^*(z) & \epsilon_n^*(z) \end{pmatrix} \frac{w'(z)}{w(z)},
\]

and note from (2.39) that \( \det Y_n = -2z^n/w(z) \).

Corollary 2.3. The recurrence relations for a general system of bi-orthogonal polynomials (2.21, 2.22) and their associated functions (2.36, 2.37) are equivalent to the matrix recurrence (2.70)

\[
Y_{n+1} := K_n Y_n = \frac{1}{\kappa_n} \begin{pmatrix} \kappa_{n+1} z & \phi_{n+1}(0) \\ \phi_{n+1}(0) z & \kappa_{n+1} \end{pmatrix} Y_n.
\]

According to (2.41) the matrix \( K_n \) has the property \( \det K_n = z \).

Corollary 2.4. The system of spectral derivatives for a general system of bi-orthogonal polynomials and associated functions (2.52-2.55) are equivalent to the matrix differential equation (2.71)

\[
Y_n' := A_n Y_n = \frac{1}{W(z)} \begin{pmatrix} \Omega_n(z) + V(z) - \frac{\kappa_{n+1}}{\kappa_n} z \Theta_n(z) & \frac{\phi_{n+1}(0)}{\kappa_n} \Theta_n(z) \\ -\frac{\phi_{n+1}(0)}{\kappa_n} z \Theta_n^*(z) & \Omega_n^*(z) - V(z) - \frac{\kappa_{n+1}}{\kappa_n} \Theta_n^*(z) \end{pmatrix} Y_n.
\]

Proof. This follows from (2.52, 2.55) and employing (2.21, 2.22, 2.36, 2.37). We note that \( \text{Tr} A_n = n/z - w'/w \) when (2.67) is employed. \( \square \)
Remark 2.3. Compatibility of the relations (2.70) and (2.71) leads to
\begin{equation}
(2.72) \quad K_n' = A_{n+1}K_n - K_nA_n,
\end{equation}
and upon examining the (1, 1)-component of this we recover the linear recurrence (2.63), the (1, 2)-component yields (2.62), whilst the (2, 1)-component gives (2.64) and the (2, 2)-component implies (2.65).

Remark 2.4. There are, in a second-order difference equation such as (2.36) or (2.37), other forms of the matrix variables and equations and these alternative forms will appear in our subsequent work. Defining
\begin{equation}
(2.73) \quad X_n(z; t) := \begin{pmatrix} \phi_{n+1}(z) & \epsilon_{n+1}(z) \\ \phi_n(z) & \epsilon_n(z) \end{pmatrix}, \quad X^*_n(z; t) := \begin{pmatrix} \phi^*_n(z) & \epsilon^*_n(z) \\ \phi^*_n(z) & \epsilon^*_n(z) \end{pmatrix},
\end{equation}
we find the spectral derivatives to be
\begin{equation}
(2.74) \quad W(z)X_n' = \begin{pmatrix} \Omega_n(z) - V(z) + n \frac{W(z)}{z} \\ \Theta_n(z) \end{pmatrix} - \frac{\kappa_n\phi_{n+2}(0)}{\kappa_{n+1}\phi_{n+1}(0)}z\Theta_{n+1}(z) X_n,
\end{equation}

\begin{equation}
(2.75) \quad W(z)X^*_n = \begin{pmatrix} -\Omega_n^*(z) - V(z) + (n + 1) \frac{W(z)}{z} \\ -\Theta_n^*(z) \end{pmatrix} - \frac{\kappa_n\phi_{n+2}(0)}{\kappa_{n+1}\phi_{n+1}(0)}z\Theta^*_{n+1}(z) X^*_n.
\end{equation}

Another system is based upon the definition
\begin{equation}
(2.76) \quad Z_n(z; t) := \begin{pmatrix} \phi_{n+1}(z) & \epsilon_{n+1}(z) \\ \phi^*_n(z) & \epsilon^*_n(z) \end{pmatrix}, \quad Z^*_n(z; t) := \begin{pmatrix} \phi^*_n(z) & -\epsilon^*_n(z) \\ \phi_n(z) & \epsilon_n(z) \end{pmatrix},
\end{equation}
and in this case the spectral derivatives are
\begin{equation}
(2.77) \quad W(z)Z_n' = \begin{pmatrix} -\Omega_n^*(z) - V(z) + \frac{\kappa_n}{\kappa_{n+1}}\Theta_n^*(z) + (n + 1) \frac{W(z)}{z} - \frac{\kappa_n\phi_{n+2}(0)}{\kappa_{n+1}^2}\Theta_{n+1}(z) \\ -\frac{\phi_{n+1}(0)}{\kappa_{n+1}}\Theta_n(z) \end{pmatrix} \times Z_n,
\end{equation}

\begin{equation}
(2.78) \quad W(z)Z^*_n = \begin{pmatrix} \Omega_n(z) - V(z) - \frac{\kappa_n}{\kappa_{n+1}}\Theta_n(z) - \frac{\kappa_n\phi_{n+2}(0)}{\kappa_{n+1}^2}\Theta^*_{n+1}(z) \\ -\frac{\phi_{n+1}(0)}{\kappa_{n+1}}\Theta_n(z) \end{pmatrix} Z^*_n.
\end{equation}
We end this section with a characterisation of a general system of bi-orthogonal polynomials on the unit circle (and their associated functions) as a solution to a particular Riemann-Hilbert problem.

**Proposition 2.8.** Consider the following Riemann-Hilbert problem for a $2 \times 2$ matrix function $Y : \mathbb{C} \to SL(2, \mathbb{C})$ defined in the following statements

1. $Y(z)$ is analytic in $\{ z : |z| > 1 \} \cup \{ z : |z| < 1 \}$,
2. on $z \in \Sigma$ where $\Sigma$ is the oriented unit circle in a counter-clockwise sense and $+(-)$ denote the left(right)-hand side or interior(exterior)

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & w(z)/z \\ 0 & 1 \end{pmatrix},$$

(2.79)

3. as $z \to \infty$

$$Y(z) = (I + O(z^{-1})) \begin{pmatrix} z^n & O(z^{-2}) \\ O(z^n) & -z^{-1} \end{pmatrix},$$

(2.80)

4. as $z \to 0$

$$Y(z) = (I + O(z)) \begin{pmatrix} O(1) & O(z^{n-1}) \\ O(1) & O(z^n) \end{pmatrix},$$

(2.81)

It is assumed that the weight function $w(z)$ satisfies the restrictions given at the beginning of this section. Then the unique solution to this Riemann-Hilbert problem is

$$Y(z) = \begin{pmatrix} \frac{\phi_n(z)}{\kappa_n} & \frac{\epsilon_n(z)}{2\kappa_n z} \\ \frac{\kappa_n \phi_n'(z)}{-\kappa_n \phi_n(z)} & -\frac{\kappa_n \epsilon_n'(z)}{2z} \end{pmatrix}, \quad n \geq 1.$$  

(2.82)

**Proof.** We firstly note from the jump condition (2.79) that $Y_{11}, Y_{21}$ are entire $z \in \mathbb{C}$. From the $(1,1)$-entry of the asymptotic condition (2.80) it is clear that $Y_{11} = \pi_n(z)$ is a polynomial of degree at most $n$. Similarly $Y_{21} = \sigma_n(z)$ from an observation of the $(2,1)$-component. From the $(1,2)$- and $(2,2)$-components of the jump condition we deduce

$$Y_{12} = \frac{w(z)}{z} Y_{11}, \quad Y_{22} = \frac{w(z)}{z} Y_{21},$$

(2.83)

and therefore

$$Y_{12} = \int_T \frac{d\zeta}{2\pi i \zeta} \frac{w(\zeta) \pi_n(\zeta)}{\zeta - z}, \quad Y_{22} = \int_T \frac{d\zeta}{2\pi i \zeta} \frac{w(\zeta) \sigma_n(\zeta)}{\zeta - z}.$$  

(2.84)

Consider the large $z$ expansion of $Y_{12}$ implied by the first of these formulae

$$Y_{12} = -z^{-1} \int_T \frac{d\zeta}{2\pi i \zeta} w(\zeta) \pi_n(\zeta) + O(z^{-2}).$$

(2.85)
According to (2.80) the integral vanishes and so $\pi_n(\zeta)$ is orthogonal to the monomial $\zeta^0$. Now take the small $z$ expansion

$$Y_{12} = \sum_{l=0}^{n-2} z^l \int_T \frac{d\zeta}{2\pi i\zeta} w(\zeta) \pi_n(\zeta) \zeta^{l+1} + O(z^{n-1}).$$

From the (1,2)-component of the condition (2.81) we observe that all terms in the sum vanish and we conclude the $\pi_n(\zeta)$ is orthogonal to the monomials $\zeta, \ldots, \zeta^{n-1}$ and the first term which survives has the monomial $\zeta^n$. Thus $\pi_n(\zeta) \propto \phi_n(\zeta)$, and from the explicit coefficient in the (1,1)-entry of (2.80) $\pi_n(\zeta)$ is the monic bi-orthogonal polynomial $\phi_n(\zeta)/\kappa_n$. We turn our attention to $Y_{22}$ and examine the small $z$ expansion

$$Y_{22} = \sum_{l=0}^{n-1} z^l \int_T \frac{d\zeta}{2\pi i\zeta} w(\zeta) \sigma_n(\zeta) \zeta^{l+1} + O(z^n).$$

The (2,2)-component of (2.81) tells us that all terms in the sum vanish and consequently $\sigma_n(\zeta)$ is orthogonal to all monomials $\zeta, \ldots, \zeta^n$. Therefore $\sigma_n(\zeta) \propto \phi_n^*(\zeta)$ and we can determine the proportionality constant from the (2,2)-component of the asymptotic formula (2.80), by comparing it with

$$Y_{22} = -z^{-1} \int_T \frac{d\zeta}{2\pi i\zeta} w(\zeta) \sigma_n(\zeta) + O(z^{-2}),$$

to conclude $\sigma_n(\zeta) = \kappa_n \phi_n^*(\zeta)$. Finally we note that

$$\int_T \frac{d\zeta}{2\pi i\zeta} \frac{w(\zeta)}{\zeta - z} \phi_n(\zeta) = \frac{1}{2z} \epsilon_n(z), \quad \int_T \frac{d\zeta}{2\pi i\zeta} \frac{w(\zeta)}{\zeta - z} \phi_n^*(\zeta) = -\frac{1}{2z} \epsilon_n^*(z),$$

when $n > 0$. We also point out $\det Y(z) = -z^{n-1}$. □

Remark 2.5. Our original matrix solution $Y_n$ specified by (2.69) is related to the solution of the above Riemann-Hilbert problem by (2.90)

$$Y_n(z) = \begin{pmatrix} \kappa_n & 0 \\ 0 & \frac{1}{\kappa_n} \end{pmatrix} Y(z) \begin{pmatrix} 1 & 0 \\ 0 & 2z \frac{w(z)}{2z} \end{pmatrix}, \quad Y(z) = \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n \end{pmatrix} Y_n(z) \begin{pmatrix} 1 & 0 \\ 0 & 2z \frac{w(z)}{2z} \end{pmatrix}. $$

Our formulation of the Riemann-Hilbert problem differs from those given in studies concerning orthogonal polynomial systems on the unit circle with more specialised weights [9], [7], [6], [5] and [8]. We have chosen this formulation as it is closest to that occurring for orthogonal polynomial systems of the line [13], the jump matrix is independent of the index $n$ which only appears in the asymptotic condition and it is simply related to our matrix formulation (2.69).

3. Regular Semi-classical Weights and Isomonodromic Deformations

All of the above results apply for a general class of weights on the unit circle but now we want to consider an additional restriction, namely the special structure of regular or generic semi-classical weights.
Definition 3.1. [37] The log-derivative of a regular or generic semi-classical weight function $w(z)$ is rational in $z$ with

$$W(z)w'(z) = 2V(z)w(z),$$

where $V(z), W(z)$ are polynomials with the following properties

1. $\deg(W) \geq 2$,
2. $\deg(V) < \deg(W)$,
3. the $m$ zeros of $W(z)$, $\{z_1, z_2, \ldots, z_m\}$ are distinct,
4. the residues $\rho_k = 2V(z_k)/W'(z_k) \notin \mathbb{Z}_{\geq 0}$.

The terminology regular refers to the connection of this definition with systems of linear second order differential equations in the complex plane which possess only isolated regular singularities, and we will see the appearance of these later.

An explicit example of such a weight is that of the form

$$w(z) = \prod_{j=1}^{m} (z - z_j)^{\rho_j}.$$  

Consistent with the requirements of Definition 3.1, we set

$$W(z) = \prod_{j=1}^{m} (z - z_j), \quad \frac{2V(z)}{W(z)} = \sum_{j=1}^{m} \frac{\rho_j}{z - z_j}.$$  

To reduce the computational labour and render the ensuing formulae simpler in appearance we are going to assume henceforth that one of the finite singular points is fixed at the origin, i.e. that $W(0) = 0$.

It follows from these definitions that the Carathéodory function satisfies an inhomogeneous form of (3.1).

Lemma 3.1 [1,37]. Let the weight $w(z)$ be such that $w(e^{\pi i}) = w(e^{-\pi i})$. The Carathéodory function satisfies the first order linear ordinary differential equation

$$W(z)F'(z) = 2V(z)F(z) + U(z),$$

where $U(z)$ is a polynomial in $z$.

Proof. Following [37] we can write

$$W(z)F(z) = \int \frac{d\zeta}{2\pi i \zeta - z} W(\zeta)w(\zeta) + \int \frac{d\zeta}{2\pi i \zeta - z} [W(z) - W(\zeta)]w(\zeta).$$

The last term is a polynomial in $z$ of bounded degree which will be denoted by $\pi_3(z)$ as we are not going to evaluate it explicitly. Differentiating this yields

$$\frac{d}{dz}W(z)F(z) = \int \frac{d\zeta}{2\pi i \zeta - (z - 2\zeta)} W(\zeta)w(\zeta) + \pi_2(z),$$
and we rewrite the first term of the right-hand side as

\[
\int \frac{d\zeta}{2\pi i \zeta (\zeta - z)^2} W(\zeta) w(\zeta) = \frac{1}{2\pi iz} \int \frac{d\zeta}{(\zeta - z)^2} W(\zeta) w(\zeta),
\]

\[
= -\frac{1}{2\pi iz} \int \frac{d\zeta}{\zeta - z} d\frac{d}{d\zeta} \frac{\zeta + z}{W(\zeta)} w(\zeta),
\]

(3.7)

Consequently we have

\[
\frac{d}{dz} W(z) F(z) = \frac{1}{z} \int \frac{d\zeta}{2\pi i \zeta} \frac{\zeta + z}{\zeta - z} \frac{d}{d\zeta} (W(\zeta) w(\zeta)) + \pi_2(z),
\]

\[
= \frac{1}{z} \int \frac{d\zeta}{2\pi i \zeta} \frac{\zeta + z}{\zeta - z} (W'(\zeta) + 2V(\zeta)) w(\zeta) + \pi_2(z),
\]

(3.8)

\[
= [W'(z) + 2V(z)] \int \frac{d\zeta}{2\pi i \zeta} \frac{\zeta + z}{\zeta - z} w(\zeta) + \pi_1(z),
\]

and (3.4) follows.

This lemma leads to the following important set of formulae.

**Proposition 3.1.** Assume that \( W(0) = 0 \). The coefficient functions \( \Theta_n(z) \), \( \Theta^*_n(z) \), \( \Omega_n(z) \), \( \Omega^*_n(z) \) are polynomials in \( z \) of degree \( m - 2, m - 2, m - 1, m - 1 \) respectively. Specifically these have leading and trailing expansions of the form

(3.9) \[ \Theta_n(z) = (n + 1 + \sum_{j=1}^{m} \rho_j) \frac{\kappa_n}{\kappa_{n+1}} z^{m-2} \]

\[ + \left\{ - (n + 1 + \sum_{j=1}^{m} \rho_j) \sum_{j=1}^{m} z_j - \sum_{j=1}^{m} \rho_j z_j \frac{\kappa_n}{\kappa_{n+1}} + (n + 2 + \sum_{j=1}^{m} \rho_j) \frac{\kappa_3}{\kappa_{n+1} \kappa_{n+2}} \frac{\phi_{n+2}(0)}{ \phi_{n+1}(0) } \right. \]

\[ - \left( n + \sum_{j=1}^{m} \rho_j \frac{\phi_{n+1}(0)}{\kappa_n} \frac{\phi_n(0)}{\kappa_{n+1}} - 2 \frac{\kappa_n}{\kappa_{n+1}^2} \right) z^{m-3} \]

\[ + O(z^{m-4}), \]
(3.10) \( \Theta_n(z) = [2V(0) - nW'(0)] \frac{\phi_n(0)}{\phi_{n+1}(0)} \)
\[\begin{align*}
+ \left\{ [2V'(0) - \frac{1}{2}nW''(0)] \frac{\phi_n(0)}{\phi_{n+1}(0)} + [2V(0) - (n - 1)W'(0)] \frac{\kappa_n \phi_{n-1}(0)}{\kappa_{n-1} \phi_{n+1}(0)} \right. \\
+ \left. \left\{ [(n + 1)W'(0) - 2V(0)] \frac{l_{n+1}}{\kappa_n} - [(n - 1)W'(0) - 2V(0)] \frac{l_{n-1}}{\kappa_{n+1}} \right\} z \right. \\
+ O(z^2),
\end{align*}\]

(3.11) \( \Theta_n^*(z) = - (n + \sum_{j=1}^m \rho_j) \frac{\tilde{\phi}_n(0)}{\phi_{n+1}(0)} z^{m-2} \)
\[\begin{align*}
+ \left\{ [(n + \sum_{j=1}^m \rho_j) \sum_{j=1}^m z_j - \sum_{j=1}^m \rho_j z_j] - \sum_{j=1}^m z_j \right\} \frac{\tilde{\phi}_n(0)}{\phi_{n+1}(0)} \right. \\
+ \left. \left[ (n + 1 + \sum_{j=1}^m \rho_j) \frac{\kappa_n \phi_{n-1}(0)}{\phi_{n+1}(0)} + \frac{\kappa_{n+1}}{\phi_{n+1}(0)} \right\} z^{m-3} \\
+ O(z^4),
\end{align*}\]

(3.12) \( \Theta_n^*(z) = [2V(0) - (n + 1)W'(0)] \frac{\phi_{n+1}(0)}{\phi_{n+1}(0)} \)
\[\begin{align*}
+ \left\{ [2V'(0) - \frac{1}{2}(n + 1)W''(0)] \frac{\phi_{n+1}(0)}{\phi_{n+1}(0)} - [2V(0) - nW'(0)] \frac{l_{n+1}}{\kappa_{n+1}} \right. \\
+ \left. \left\{ [(n + 2)W'(0) - 2V(0)] \left\{ \frac{\kappa_n}{\kappa_{n+1} + 2} \phi_{n+2}(0) \right\} + \frac{\kappa_{n+1}}{\phi_{n+1}(0)} \right\} \right\} z \right. \\
+ O(z^4),
\end{align*}\]

(3.13) \( \Omega_n(z) = \left( 1 + \frac{1}{2} \sum_{j=1}^m \rho_j \right) z^{m-1} \)
\[\begin{align*}
+ \left\{ - \frac{1}{2} \sum_{j=1}^m \rho_j (\sum_{j=1}^m z_j) + \frac{1}{2} \sum_{j=1}^m \rho_j z_j - \sum_{j=1}^m z_j \right\} \frac{\phi_{n+1}(0)}{\phi_{n+1}(0)} \right. \\
+ \left. \left[ (n + 2 + \sum_{j=1}^m \rho_j) \frac{\kappa_n}{\kappa_{n+2} \phi_{n+1}(0)} + \frac{l_{n+1}}{\kappa_{n+1}} \right\} z^{m-2} \\
+ O(z^4),
\end{align*}\]

(3.14) \( \Omega_n(z) = V(0) - nW'(0) \)
\[\begin{align*}
+ \left\{ V'(0) - \frac{1}{2} nW''(0) + \left( V(0) \frac{\kappa_n}{\kappa_{n+1}} + [V(0) - nW'(0)] \frac{\phi_{n+1}(0)}{\phi_{n+1}(0)} \right) \right. \\
+ \left. \left[ (n + 1)W'(0) \frac{l_{n+1}}{\kappa_{n+1}} - [V(0) - (n + 1)W'(0)] \frac{l_{n+1}}{\kappa_{n+1}} \right\} \right\} z \right. \\
+ O(z^4),
\end{align*}\]
(3.15) \[ \Omega_n^*(z) = -\frac{1}{2} \sum_{j=1}^{m} \rho_j z^{m-1} \]
+ \left\{ \frac{1}{2} \left( \sum_{j=1}^{m} \rho_j \right) \right\} - \frac{1}{2} \sum_{j=1}^{m} \rho_j z - (n + \sum_{j=1}^{m} \rho_j) \frac{\kappa_n}{\kappa_{n+1}} \frac{\tilde{\phi}_n(0)}{\phi_n(0)} + \frac{l_{n+1}}{\kappa_{n+1}} \right\} z^{m-2} + O(z^{m-3}),

(3.16) \[ \Omega_n(z) = (n + 1)W'(0) - V(0) \]
+ \left\{ \frac{1}{2} (n+1)W''(0) - V'(0) + [(n+2)W'(0) - 2V(0)] \right\} \frac{\kappa_n^2}{\kappa_{n+2} \kappa_{n+1}} \frac{\tilde{\phi}_{n+2}(0)}{\phi_{n+1}(0)} - W''(0) \right\} z + O(z^2).

Proof. Following the approach of Laguerre [1] we write \( F(z) \) in terms of \( \phi_n(z), \psi_n(z), \epsilon_n(z) \) and use (3.4) to deduce

\( 0 = WF' - 2VF - U, \)

\[ = W \left( \frac{\epsilon_n - \psi_n}{\phi_n} \right)' - 2V \frac{\epsilon_n - \psi_n}{\phi_n} - U, \]

\[ = \frac{W(\psi_n \phi_n' - \phi_n \psi_n') + 2V \phi_n \psi_n - U \phi_n^2}{\phi_n^2} + W \left( \frac{\epsilon_n}{\phi_n} \right)' - 2V \frac{\epsilon_n}{\phi_n}. \]

The numerator of the first term is independent of \( \epsilon_n \), and so is a polynomial in \( z \), and we denote this by

\[ 2 \frac{\phi_{n+1}(0)}{\kappa_n} z^n \Omega_n(z) = W(-\phi_n \epsilon_n + \epsilon_n \phi_n') + 2V \phi_n \epsilon_n. \]

Given that this is a polynomial we can determine its degree and minimum power of \( z \) by utilising the expansions of \( \phi_n, \epsilon_n \) both inside and outside the unit circle, namely (3.14) and (2.46). We find the degree of the right-hand side is \( n + m - 2 \) so that \( \Theta_n(z) \) is a polynomial of degree \( m - 2 \). Developing the expansions further we arrive at (3.14). An identical argument applies to the other combination

\[ 2 \frac{\phi_{n+1}(0)}{\kappa_n} z^{n+1} \Omega_n^*(z) = W(\phi_n \epsilon_n' + \epsilon_n' \phi_n') - 2V \phi_n' \epsilon_n', \]

and \( \Theta_n^*(z) \) is also a polynomial of degree \( m - 2 \) with the expansion (3.14). To establish (3.13) we utilise the other form of \( \Theta_n(z) \) and (2.48) to deduce

\[ W(\psi_n \phi_n' - \phi_n \psi_n') + 2V \phi_n \psi_n - U \phi_n^2 = 2 \frac{\phi_{n+1}(0)}{\kappa_n} z^n \Theta_n(z), \]

\[ = [\phi_{n+1} \psi_n - \psi_{n+1} \phi_n] \Theta_n(z). \]

Separating those terms with \( \phi_n \) and \( \psi_n \) as factors we have

\[ \{ \Theta_n(z) \phi_{n+1} - W \phi_n^2 - V \phi_n \} \psi_n = \{ \Theta_n(z) \psi_{n+1} - W \psi_n^2 + V \psi_n - U \phi_n \} \phi_n. \]
so that this polynomial contains both $\phi_n$ and $\psi_n$ as factors and can be written as $\Omega_n \phi_n \psi_n$ with $\Omega_n(z)$ a polynomial of bounded degree. This latter polynomial can be defined as

$$2 \frac{\phi_{n+1}(0)}{\kappa_n} z^n \Omega_n(z) = W(\psi_{n+1}' \phi_n - \phi_{n+1}' \psi_n) + V(\phi_n \psi_{n+1} + \psi_n \phi_{n+1}) - U \phi_n \phi_{n+1},$$

(3.22)

$$= W(\epsilon_{n+1}' \phi_n - \phi_{n+1} \epsilon_n') + V(\phi_n \epsilon_{n+1} + \epsilon_n \phi_{n+1}).$$

Again employing the expansions (2.46, 2.48) we determine the degree of $\Omega_n(z)$ to be $m - 1$ and the expansion (3.13) follows. Starting with the alternative definition of $\Theta^*_n(z)$ and (2.39)

$$W(\phi^*_n \psi^*_n - \phi^*_n \phi^*_n) - 2V \phi^*_n \psi^*_n - U \phi^*_n = 2 \frac{\phi_{n+1}(0)}{\kappa_n} z^{n+1} \Theta^*_n(z),$$

(3.23)

$$= [\phi^*_{n+1} \epsilon_n^* - \phi^*_n \phi^*_n] \Theta^*_n(z).$$

and using the above argument we identify for the polynomial $\Omega^*_n(z)$

$$2 \frac{\phi_{n+1}(0)}{\kappa_n} z^{n+1} \Omega^*_n(z) = W(-\psi^*_n \phi^*_{n+1} + \phi^*_n \psi^*_{n+1}) - V(\phi^*_n \psi^*_{n+1} + \psi^*_n \phi^*_{n+1})$$

$$- U \phi^*_n \phi^*_{n+1},$$

(3.24)

$$= W(-\epsilon^*_n \phi^*_{n+1} + \phi^*_n \epsilon^*_{n+1}) - V(\phi^*_n \epsilon^*_{n+1} + \epsilon^*_n \phi^*_{n+1}).$$

The degree of $\Omega^*_n(z)$ is found to be $m - 1$ and it has the expansion (3.24). $\square$

Remark 3.1. Solving for $\phi'_n$ and $\epsilon'_n$ between (3.18) and (3.22) leads to (2.52) and (2.54), whilst solving for $\phi^*_n'$ and $\epsilon^*_n'$ using (3.19, 3.24) yields (2.53) and (2.55).

Bilinear residue formulae relating products of a polynomial and an associated function evaluated at a finite singular point will arise in the theory of the deformation derivatives later. These are consequences of the workings of the proof of Proposition 3.1, so we give a complete list presently.

Corollary 3.1. Bilinear residues are related to the coefficient function residues in the following equations, valid for all $z_j$

$$\phi_n(z_j) \epsilon_n(z_j) = 2 \frac{\phi_{n+1}(0)}{\kappa_n} z^n \frac{\Theta_n(z_j)}{2V(z_j)},$$

(3.25)

$$\phi^*_n(z_j) \epsilon^*_n(z_j) = -2 \frac{\phi_{n+1}(0)}{\kappa_n} z^{n+1} \frac{\Theta^*_n(z_j)}{2V(z_j)},$$

(3.26)
Proof. These are all found by evaluating one of (3.18), (3.19), (3.22), or (3.24) at $z = z_j$ and using (2.38-2.40). □

The initial members of the sequences of coefficient functions $\{\Theta_n\}_{n=0}^{\infty}$, $\{\Theta_n^*\}_{n=0}^{\infty}$, $\{\Omega_n\}_{n=0}^{\infty}$, $\{\Omega_n^*\}_{n=0}^{\infty}$ are given by

\[
\begin{align*}
(3.35) \quad 2\frac{\phi_1(0)}{\kappa_0}\Theta_0(z) &= 2V(z) - \kappa_0^2 U(z), \\
(3.36) \quad 2\frac{\phi_2(0)}{\kappa_1}z\Theta_1(z) &= \frac{\kappa_1^2}{\kappa_0^2}z^2(2V(z) - \kappa_0^2 U(z)) - 2\kappa_1 \phi_1(0)zU(z) \\
&\quad - 2\frac{\kappa_1 \phi_1(0)}{\kappa_0^2}W(z) - \frac{\phi_1^2(0)}{\kappa_0^2}(2V(z) + \kappa_0^2 U(z)), \\
(3.37) \quad 2\frac{\tilde{\phi}_1(0)}{\kappa_0}z\Theta_0^*(z) &= -2V(z) - \kappa_0^2 U(z), \\
(3.38) \quad 2\frac{\tilde{\phi}_2(0)}{\kappa_1}z^2\Theta_1^*(z) &= \frac{\kappa_1^2}{\kappa_0^2}z^2(2V(z) - \kappa_0^2 U(z)) - 2\kappa_1 \tilde{\phi}_1(0)zU(z) \\
&\quad - 2\frac{\kappa_1 \tilde{\phi}_1(0)}{\kappa_0^2}W(z) - \frac{\tilde{\phi}_1^2(0)}{\kappa_0^2}(2V(z) + \kappa_0^2 U(z)), \\
(3.39) \quad 2\phi_1(0)\Omega_0(z) &= \kappa_1 z(2V(z) - \kappa_0^2 U(z)) - \kappa_0^2 \phi_1(0)U(z), \\
(3.40) \quad 2\tilde{\phi}_1(0)z\Omega_0^*(z) &= -\kappa_1(2V(z) + \kappa_0^2 U(z)) - \kappa_0^2 \tilde{\phi}_1(0)zU(z).
\end{align*}
\]

One can take combinations of the above functional-difference equations and construct exact differences when $z$ is evaluated at the finite singular points of the weight, i.e. $W(z) = 0$. The integration of the system is given in the following proposition.
Proposition 3.2. At all the finite singular points \( z_j, j = 1, \ldots, n \), with the exception of \( z_j = 0 \), the coefficient functions satisfy the bilinear identities

\[
\Omega_n^2(z_j) = \frac{\kappa_n \phi_n+2(0)}{\kappa_{n+1} \phi_{n+1}(0)} z_j \theta_n(z_j) \theta_{n+1}(z_j) + V^2(z_j),
\]

\[
\Omega_{n-1}^2(z_j) = \frac{\kappa_{n-1} \phi_{n-1}(0)}{\kappa_n \phi_n(0)} z_j \theta_n(z_j) \theta_{n-1}(z_j) + V^2(z_j),
\]

\[
\left[ \Omega_{n-1}(z_j) - \frac{\kappa_{n-1} \phi_{n-1}(0)}{\kappa_n \phi_n(0)} \theta_n(z_j) \right]^2 = \frac{\phi_{n+1}(0) - \phi_n(0)}{\kappa_n^2} \theta_n(z_j) \theta_{n-1}(z_j) + V^2(z_j),
\]

\[
\left[ \Omega_{n-1}^+(z_j) - \frac{\kappa_{n-1} \phi_{n-1}(0)}{\kappa_n \phi_n(0)} z_j \theta_n(z_j) \right]^2 = \frac{\phi_{n+1}(0) - \phi_n(0)}{\kappa_n^2} \theta_n(z_j) \theta_{n-1}(z_j) + V^2(z_j),
\]

\[
\frac{\phi_{n+1}(0) - \phi_n(0)}{\kappa_n^2} z_j \theta_n(z_j) \theta_{n-1}(z_j) + V^2(z_j) = \left[ \Omega_n(z_j) - \frac{\kappa_{n+1}}{\kappa_n} z_j \theta_n(z_j) \right]^2,
\]

\[
\frac{\phi_{n+1}(0) - \phi_n(0)}{\kappa_n^2} z_j \theta_n(z_j) \theta_{n-1}(z_j) + V^2(z_j) = \left[ \Omega_n^+(z_j) - \frac{\kappa_{n+1}}{\kappa_n} \theta_n(z_j) \right]^2.
\]

**First Proof.** We take the first pair of identities (3.41) and (3.42) as an example for our first proof. Multiplying the \( \Omega_n, \Omega_{n-1} \) terms of (2.60) by the corresponding terms of (2.61), evaluated at a finite singular point \( z = z_j \), one has an exact difference

\[
\Omega_n^2(z_j) - \Omega_{n-1}^2(z_j) = \frac{\kappa_n \phi_n+2(0)}{\kappa_{n+1} \phi_{n+1}(0)} z_j \theta_n(z_j) \theta_{n+1}(z_j) - \frac{\kappa_n \phi_n+2(0)}{\kappa_{n+1} \phi_{n+1}(0)} z_j \theta_n(z_j) \theta_{n-1}(z_j),
\]

assuming none of the \( z_j \) coincide with \( -r_{n+1}/r_n \) for any \( n \). Upon summing this relation the summation constant is calculated to be

\[
\Omega_0^2(z_j) - \frac{\kappa_0 \phi_2(0)}{\kappa_1 \phi_1(0)} z_j \theta_0(z_j) \theta_1(z_j) = V^2(z_j),
\]

by using the initial members of the coefficient function sequences in (3.39–3.35). The result is (3.41), whilst the second relation follows from an identical argument applied to (2.60–2.61).

**Second Proof.** The three pairs of formulae (3.41–3.42), (3.43–3.44) and (3.45–3.46) arise from the fact that at a finite singular point \( z_j \) the determinant of the matrix spectral derivative must vanish. Thus (3.41) and (3.42) express the condition that the determinant of the matrix on the right-hand sides of (2.64–2.65) and (2.66) vanish respectively. It can be shown that the same condition applied to the right-hand sides of (2.74–2.75) and (2.76–2.77) implies (3.43–3.44) respectively when one takes into account the identities (2.64–2.65) and (2.66–2.67). The last pair are a consequence of \( \det(WA_n(z_j; t)) = 0 \) along with the identity (2.67).
Third Proof. All the bilinear identities in Proposition 3.2 can be easily derived from the residue formulae (3.25-3.34) by multiplying any two of the above formulae and then factoring the resulting product in a different way. Thus (3.41) arises from multiplying (3.27) and (3.28) and then factoring the product in order to employ (3.25). Equation (3.43) comes from multiplying (3.25) and (3.26) with \( n \mapsto n - 1 \), using the recurrences (2.22), (2.37) with \( n \mapsto n - 1 \) to solve for \( \phi_{n-1}(z_j), \epsilon_{n-1}(z_j) \) and employing (3.25) along with (3.27) and (3.28) setting \( n \mapsto n - 1 \). Equation (3.45) is derived by multiplying (3.25) and (3.26) and then factoring using (3.31) and (3.33). The reciprocal versions follow from similar reasoning. □

Remark 3.2. It is clear from the first proof that the bilinear identities given in Proposition 3.2 can be straightforwardly generalised to ones that are functions of \( z \) rather than evaluated at special \( z \) values. They can be derived directly from Proposition 2.7, so apply in situations where the weights are not semi-classical, and contain additional terms with a factor of \( W(z) \) and sums of products of other coefficients ranging from \( j = 1, \ldots, n \). However because we will have no use for such relations we refrain from writing these down.

Remark 3.3. As \( z = 0 \) is a finite singular point then the limit as \( z \to 0 \) may be taken in the product of (2.58,2.59), however this does not lead to any new independent relation but simply recovers

\[
\Omega_n(0) = V(0) - nW'(0).
\]

In the case of a regular semi-classical weight function the matrix \( A_n(z; t) \) has the partial fraction decomposition

\[
A_n(z; t) := \sum_{j=1}^{m} \frac{A_{nj}}{z - z_j},
\]

under the assumptions following (3.1). Let us take the first finite singularity to be situated at the origin, \( z_1 = 0 \). Some care needs to exercised as many relations differ depending on whether \( z_j = 0 \) or not because of the additional term in (2.67). The residue matrices for the finite singularities are given by

\[
A_{nj} = \frac{\rho_j}{2V(z_j)} \begin{pmatrix}
-\Omega_n(z_j) - V(z_j) + \frac{\kappa_n+1}{\kappa_n} z_j \Theta_n(z_j) & \frac{\phi_{n+1}(0)}{\kappa_n} \Theta_n(z_j) \\
-\frac{\phi_{n+1}(0)}{\kappa_n} z_j \Theta_n^*(z_j) & -\Omega_n^*(z_j) - V(z_j) + \frac{\kappa_n+1}{\kappa_n} \Theta_n^*(z_j)
\end{pmatrix},
\]

for \( j = 2, \ldots, m \) and

\[
A_{n1} = \frac{\rho_1}{2V(0)} \begin{pmatrix}
nW'(0) - 2V(0) & [2V(0) - nW'(0)]r_n \\ 0 & 0
\end{pmatrix}.
\]
Using the identity (2.67) we note that
\[
\text{Tr} A_{nj} = -\rho_j, \quad j = 2, \ldots, m, \tag{3.52}
\]
(3.53) \[ \text{Tr} A_{n1} = n - \rho_1. \]
In either case we find that \( \det A_{nj} = 0 \) using (3.45). An alternative expression for the residue matrices in the case \( j = 2, \ldots, m \) is
\[
A_{nj} = -\frac{1}{2} \rho_j z_j^{-n} \begin{pmatrix}
\phi_n^*(z_j) \epsilon_n(z_j) & -\phi_n(z_j) \epsilon_n(z_j) \\
-\phi_n^*(z_j) \epsilon_n^*(z_j) & \phi_n(z_j) \epsilon_n^*(z_j)
\end{pmatrix}. \tag{3.54}
\]
The regular singularity at \( z = \infty \) has a residue matrix given by
\[
A_{n\infty} := \text{Res}_{z=0} \left\{ -z^{-2} A_n(z^{-1}) \right\} = -\sum_{j=1}^m A_{nj}. \tag{3.55}
\]
Using this definition and the large \( z \) terms for the coefficient functions (3.9), (3.11), (3.13), (3.15) we evaluate this matrix to be
\[
A_{n\infty} = \begin{pmatrix}
-n & 0 \\
-(n + \sum_{j=1}^m \rho_j) \bar{r}_n & \sum_{j=1}^m \rho_j
\end{pmatrix}. \tag{3.56}
\]
We read off that \( \text{Tr} A_{n\infty} = -n + \sum_{j=1}^m \rho_j \) and \( \det A_{n\infty} = -n \sum_{j=1}^m \rho_j \).
Furthermore relations (3.56) and (3.55) imply the summation identities
\[
\frac{1}{2} \sum_{j=1}^m \rho_j z_j^{-n} \phi_n(z_j) \epsilon_n(z_j) = 0, \tag{3.57}
\]
\[
\frac{1}{2} \sum_{j=1}^m \rho_j z_j^{-n} \phi_n^*(z_j) \epsilon_n(z_j) = -n, \tag{3.58}
\]
\[
\frac{1}{2} \sum_{j=1}^m \rho_j z_j^{-n} \phi_n(z_j) \epsilon_n^*(z_j) = \sum_{j=1}^m \rho_j, \tag{3.59}
\]
\[
\frac{1}{2} \sum_{j=1}^m \rho_j z_j^{-n} \phi_n^*(z_j) \epsilon_n^*(z_j) = (n + \sum_{j=1}^m \rho_j) \bar{r}_n. \tag{3.60}
\]

We wish to close this part by commenting on how one would obtain the discrete analogs of the Schlesinger equations or multi-variable extensions of the discrete Painlevé equations from the theory outlined above. If one evaluates (3.41) or (3.42) for that matter at two distinct singularities \( z_1, z_2 \), consolidates terms and then takes their ratio the result is
\[
\frac{z_1 \Theta_n(z_1) \Theta_{n+1}(z_1)}{z_2 \Theta_n(z_2) \Theta_{n+1}(z_2)} = \frac{\Omega_n(z_1) - V(z_1)}{\Omega_n(z_2) - V(z_2)} \frac{[\Omega_n(z_1) + V(z_1)]}{[\Omega_n(z_2) + V(z_2)]}. \]
This constitutes a recurrence relation for \( \Theta_n(z_1)/\Theta_n(z_2) \). To find a recurrence involving \( \Omega_n \) one adopts another method. By comparing the expansions inside and outside the unit circle, for example equations (3.10) and (3.11) for \( \Theta_n(z) \), and in particular where they overlap one can derive expressions for the sub-leading
coefficients $l_n, m_n$ in terms of the higher ones. Employing these expressions in one of the expansion forms for $\Omega_n(z)$ (3.13) or (3.14) will yield a recurrence for $\Omega_n$.

We now consider the dynamics of deforming the semi-classical weight (3.3) through a $t$-dependence of the finite singular points $z_j(t)$,

\[
\frac{d\tilde{\omega}}{\omega} = -\sum_{j=1}^{m} \rho_j \frac{\dot{z}_j}{z - z_j}, \tag{3.61}
\]

where $\dot{} := \frac{d}{dt}$. Given this motion of the finite singularities we consider the $t$-derivatives of the bi-orthogonal polynomial system.

**Proposition 3.3.** The deformation derivative of a semi-classical bi-orthogonal polynomial is

\[
\dot{\phi}_n(z) = \left\{ -\frac{\kappa_n}{\kappa_n} - \sum_{j=1}^{m} \rho_j \frac{\dot{z}_j}{z_j} + \frac{1}{2} \sum_{j=1}^{m} \rho_j \frac{\dot{z}_j}{z_j} z_j^{-n} \epsilon_n(z_j) \phi_n^*(z_j) \frac{z}{z - z_j} \right\} \phi_n(z) \tag{3.62}
\]

\[\text{whilst that of its reciprocal polynomial is}\]

\[
\dot{\phi}_n^*(z) = \left\{ -\frac{\kappa_n}{\kappa_n} + \frac{1}{2} \sum_{j=1}^{m} \rho_j \frac{\dot{z}_j}{z_j} z_j^{1-n} \epsilon_n^*(z_j) \phi_n(z_j) \frac{1}{z - z_j} \right\} \phi_n(z) \tag{3.63}
\]

The deformation derivative of an associated function is

\[
\dot{\epsilon}_n(z) = \left\{ -\frac{\kappa_n}{\kappa_n} - \frac{1}{2} \sum_{j=1}^{m} \rho_j \frac{\dot{z}_j}{z_j} z_j^{-n} \epsilon_n(z_j) \phi_n(z_j) \frac{z}{z - z_j} \right\} \epsilon_n(z) \tag{3.64}
\]

\[\text{and that of a reciprocal associated function is}\]

\[
\dot{\epsilon}_n^*(z) = \left\{ -\frac{\kappa_n}{\kappa_n} - \frac{1}{2} \sum_{j=1}^{m} \rho_j \frac{\dot{z}_j}{z_j} z_j^{1-n} \epsilon_n(z_j) \phi_n^*(z_j) \frac{1}{z - z_j} \right\} \epsilon_n(z) \tag{3.65}
\]

**Proof.** Differentiating the orthonormality condition

\[
\int \frac{d\zeta}{2\pi i\zeta} w(\zeta) \phi_n(\zeta) \overline{\phi_{n-1}(\zeta)} = \delta_{n,0},
\]
and using \(3.01\) we find

\[0 = \frac{k_n}{\kappa_n} \delta_{i,0} + \int \frac{d\zeta}{2\pi i\zeta} w(\zeta) \phi_n(z) \tilde{\phi}_{n-i} - \sum_j \rho_j \frac{\dot{z}_j}{z_j} \int \frac{d\zeta}{2\pi i\zeta} w(\zeta) \frac{1}{\zeta - z_j} \phi_n(z) \tilde{\phi}_{n-i}, \quad i = 0, \ldots, n.\]

Now

\[
\int \frac{d\zeta}{2\pi i\zeta} w(\zeta) \frac{1}{\zeta - z} \phi_n(\zeta) \tilde{\phi}_{n-i}(\zeta) = \int \frac{d\zeta}{2\pi i\zeta} w(\zeta) \phi_n(\zeta) \frac{\tilde{\phi}_{n-i}(\zeta^{-1}) - \tilde{\phi}_{n-i}(z^{-1})}{\zeta - z} \\
+ \tilde{\phi}_{n-i}(z^{-1}) \int \frac{d\zeta}{2\pi i\zeta} w(\zeta) \frac{\phi_n(\zeta)}{\zeta - z},
\]

so that

\[
0 = \left( \frac{k_n}{\kappa_n} + \sum_j \rho_j \frac{\dot{z}_j}{z_j} \right) \delta_{i,0} - 2 \sum_{j=1}^{m} \rho_j \frac{\dot{z}_j}{z_j} z_j^{-n} \phi_n^*(z_j) \epsilon_n(z_j) + \int \frac{d\zeta}{2\pi i\zeta} w(\zeta) \dot{\phi}_n \tilde{\phi}_{n-i}.
\]

In addition we can represent \(\tilde{\phi}_{n-i}(z)\) as

\[
\tilde{\phi}_{n-i}(z) = \sum_{j=0}^{n} \delta_{i,j} \tilde{\phi}_{n-j}(z),
\]

\[
= \sum_{j=0}^{n} \int \frac{d\zeta}{2\pi i\zeta} w(\zeta) \tilde{\phi}_{n-i}(\zeta) \phi_{n-j}(\zeta) \tilde{\phi}_{n-j}(z),
\]

\[
= \int \frac{d\zeta}{2\pi i\zeta} w(\zeta) \tilde{\phi}_{n-i}(\zeta) \sum_{j=0}^{n} \phi_{n-j}(\zeta) \tilde{\phi}_{n-j}(z),
\]

\[
= \int \frac{d\zeta}{2\pi i\zeta} w(\zeta) \tilde{\phi}_{n-i}(\zeta) \phi_{n-i}(\zeta) \sum_{j=0}^{n} \phi_{n-j}(\zeta) \tilde{\phi}_{n-j}(z) - \zeta \phi_{n-i}(z) \tilde{\phi}_{n-i}(z).
\]

Writing the Kronecker delta in a similar way the whole expression becomes

\[
0 = \int \frac{d\zeta}{2\pi i\zeta} w(\zeta) \tilde{\phi}_{n-i}(\zeta) \left\{ \dot{\phi}_n(\zeta) + \left( \frac{k_n}{\kappa_n} + \sum_{j=1}^{m} \rho_j \frac{\dot{z}_j}{z_j} \right) \phi_n(\zeta) \right. \\
- \left. 2 \sum_{j=1}^{m} \rho_j \frac{\dot{z}_j}{z_j} \epsilon_n(z_j) \phi_n^*(\zeta) z_j^{-n} \phi_n(z_j) - \zeta z_j^{-1} \phi_n(\zeta) z_j^{-n} \phi_n(z_j) \right\},
\]

for all \(0 \leq i \leq n\) and \(3.32\) then follows. The second relation follows by an identical argument applied to

\[
\int \frac{d\zeta}{2\pi i\zeta} w(\zeta) \phi_{n-i}(\zeta) \tilde{\phi}_{n}(\zeta) = \delta_{i,0}.
\]

The derivatives of the associated functions \(3.04, 3.05\) follow from differentiating the definitions \(2.32, 2.33\) and employing the first two results of the proposition along with the relation \(2.40\).
Corollary 3.2. The $t$-derivatives of the reflection coefficients are

\begin{align}
\frac{\dot{r}_n}{r_n} &= \frac{1}{2} \sum_{j=1}^{m} \rho_j \frac{\dot{z}_j}{z_j} \frac{\Omega_{n-1}(z_j) - V(z_j)}{V(z_j)}, \\
\frac{\dot{r}_n}{r_n} &= \frac{1}{2} \sum_{j=1}^{m} \rho_j \frac{\dot{z}_j}{z_j} \frac{\Omega_{n-1}^*(z_j) + V(z_j)}{V(z_j)}.
\end{align}

Proof. An alternative formula to (3.62) is

\begin{equation}
\phi_n(z) = - \left( \frac{\dot{\kappa}_n}{\kappa_n} + \sum_{j=1}^{m} \rho_j \frac{\dot{z}_j}{z_j} \right) \phi_n(z) + \frac{1}{2} \sum_{j=1}^{m} \rho_j \frac{\dot{z}_j}{z_j} \epsilon_n(z_j) \sum_{l=0}^{n} \bar{\phi}_{n-l}(z_j^{-1}) \phi_{n-l}(z_j),
\end{equation}

and by examining the coefficients of $z^n, z^0$ we deduce that

\begin{equation}
\frac{\dot{r}_n}{r_n} = \frac{1}{2} \kappa_{n-1} \left( \sum_{j=1}^{m} \rho_j \frac{\dot{z}_j}{z_j} \right) \phi_n(z_j) \phi_{n-1}(z_j).
\end{equation}

Noting that the derivative term of (3.62) vanishes when $z = z_j$ and employing (2.38) we arrive at (3.66). The second equation, (3.67), follows by identical reasoning. □

Sums of the bilinear residues over the finite singular points are related to deformation derivatives in the following way,

\begin{align}
2 \frac{\dot{\kappa}_n}{\kappa_n} &= - \sum_{j=1}^{m} \rho_j \frac{\dot{z}_j}{z_j} + \frac{1}{2} \sum_{j=1}^{m} \rho_j \frac{\dot{z}_j}{z_j} \frac{\dot{z}_j}{z_j} \epsilon_n(z_j) \phi_n^*(z_j), \\
\frac{\dot{\phi}_n(0)}{\phi_n(0)} + \frac{\dot{\kappa}_n}{\kappa_n} \sum_{j=1}^{m} \rho_j \frac{\dot{z}_j}{z_j} &= \frac{1}{2} \frac{\kappa_n}{\phi_n(0)} \sum_{j=1}^{m} \rho_j \frac{\dot{z}_j}{z_j} \frac{\dot{z}_j}{z_j} \epsilon_n(z_j) \phi_n(z_j), \\
(3.69) &\quad = \frac{\phi_{n+1}(0)}{\phi_n(0)} \sum_{j=1}^{m} \rho_j \frac{\dot{z}_j}{z_j} \Theta(z_j), \\
\frac{\dot{\phi}_n(0)}{\phi_n(0)} + \frac{\dot{\kappa}_n}{\kappa_n} \sum_{j=1}^{m} \rho_j \frac{\dot{z}_j}{z_j} &= - \frac{1}{2} \frac{\kappa_n}{\phi_n(0)} \sum_{j=1}^{m} \rho_j \frac{\dot{z}_j}{z_j} \frac{\dot{z}_j}{z_j} \epsilon_n^*(z_j) \phi_n^*(z_j), \\
(3.70) &\quad = \frac{\phi_{n+1}(0)}{\phi_n(0)} \sum_{j=1}^{m} \rho_j \frac{2V(z_j)}{z_j} \frac{\dot{z}_j}{z_j} \Theta^*(z_j),
\end{align}

For the regular semi-classical weights we can also formulate the system of deformation derivatives as a $2 \times 2$ matrix differential equation and demonstrate that the system preserves the monodromy data with respect to the motion of the finite singularities $z_j(t)$. 
Corollary 3.3. The deformation derivatives for a system of regular semi-classical bi-orthogonal polynomials and associated functions (3.62-3.65) are equivalent to the matrix differential equation

\[(3.72) \dot{Y}_n := B_n Y_n = \left\{ B_\infty - \sum_{j=1}^{m} \frac{\dot{z}_j}{z - \bar{z}_j} A_{nj} \right\} Y_n. \]

where

\[(3.73) B_\infty = \begin{pmatrix} \frac{\dot{k}_n}{k_n} & 0 \\ \frac{\kappa_n}{k_n} \phi_n(0) + \bar{\kappa}_n \phi_n(0) & -\frac{\kappa_n}{k_n} \end{pmatrix}. \]

Proof. This follows from a partial fraction decomposition of the system and using (3.69,3.71). □

Remark 3.4. For a special class of irregular semi-classical weights a result analogous to Corollary 3.3 has been given by Bertola, Eynard and Harnad [15].

In the case of the pair (2.70), (3.72) compatibility implies the relation

\[(3.74) \dot{K}_n = B_{n+1} K_n - K_n B_n, \]

however there are no new identities arising from this condition. Taking the (1,1)-component of both sides of this equation we see that it is identically satisfied through the use of (2.63) and (3.71). Or if we take the (1,2)-components then they are equal when use of made of (2.60) and (3.71). In a similar way we find both sides of the (2,1)-components are identical when we employ (2.58) and (3.69,3.66). Finally the (2,2)-components on both sides are the same after taking into account (2.65) and (3.69,3.71).

For the pair of linear differential relations (2.71), (3.72) compatibility leads us to the Schlesinger equations

\[(3.75) A_{nj} = [B_\infty, A_{nj}] + \sum_{k \neq j} \frac{\dot{z}_j - \dot{z}_k}{z_j - z_k} [A_{nk}, A_{nj}], \]

\[(3.76) \dot{A}_{n\infty} = [B_\infty, A_{n\infty}]. \]

Again there is not anything essentially new here, that couldn’t be derived from the system of deformation derivatives, but it is an efficient way to compute the deformation derivatives of bilinear products. Employing the explicit representations of our matrices \(A_{nj}\) we find the following independent derivatives in
component form

\begin{equation}
(3.77) \quad \frac{d}{dt} \frac{\rho_j}{2V(z_j)} \left[ \Omega_n(z_j) + V(z_j) - \frac{\kappa_{n+1}}{\kappa_n} z_j \Theta_n(z_j) \right]
= - \frac{\rho_j}{2V(z_j)} \frac{\phi_{n+1}(0)}{\kappa_n^2} \frac{d}{dt}(\kappa_n \tilde{\phi}_n(0)) \Theta_n(z_j)
- \frac{\rho_j}{2V(z_j)} \frac{\phi_{n+1}(0) \tilde{\phi}_n(0)}{\kappa_n^2} \sum_{k \neq j} \frac{\rho_k}{2V(z_k)} \frac{\tilde{z}_j - \tilde{z}_k}{z_j - z_k} \left[ z_k \Theta_n^*(z_k) \Theta_n(z_j) - z_j \Theta_n(z_k) \Theta_n^*(z_j) \right],
\end{equation}

\begin{equation}
(3.78) \quad \frac{d}{dt} \frac{\rho_j}{2V(z_j)} \frac{\phi_{n+1}(0)}{\kappa_n} \Theta_n(z_j) = \frac{\rho_j}{V(z_j)} \frac{\phi_{n+1}(0)}{\kappa_n} \left\{ \frac{\kappa_n}{\kappa_n} \Theta_n(z_j) + \sum_{k \neq j} \frac{\rho_k}{2V(z_k)} \frac{\tilde{z}_j - \tilde{z}_k}{z_j - z_k} \right\} \times \left[ \Theta_n(z_k) \left[ \Omega_n(z_j) - \frac{\kappa_{n+1}}{\kappa_n} z_j \Theta_n(z_j) \right] - \Theta_n(z_j) \left[ \Omega_n(z_k) - \frac{\kappa_{n+1}}{\kappa_n} z_k \Theta_n(z_k) \right] \right\},
\end{equation}

\begin{equation}
(3.79) \quad \frac{d}{dt} \frac{\rho_j}{2V(z_j)} \frac{\tilde{\phi}_{n+1}(0)}{\kappa_n} z_j \Theta_n^*(z_j) = \frac{\rho_j}{V(z_j)} \frac{\tilde{\phi}_{n+1}(0)}{\kappa_n} \left\{ - \frac{\kappa_n}{\kappa_n} z_j \Theta_n^*(z_j) + \frac{1}{\kappa_n \phi_{n+1}(0)} \frac{d}{dt}(\kappa_n \tilde{\phi}_n(0)) \left[ \Omega_n^*(z_j) - \frac{\kappa_{n+1}}{\kappa_n} \Theta_n^*(z_j) \right] - \sum_{k \neq j} \frac{\rho_k}{2V(z_k)} \frac{\tilde{z}_j - \tilde{z}_k}{z_j - z_k} \right\} \times \left[ z_k \Theta_n^*(z_k) \left[ \Omega_n^*(z_j) - \frac{\kappa_{n+1}}{\kappa_n} \Theta_n^*(z_j) \right] - z_j \Theta_n^*(z_j) \left[ \Omega_n^*(z_k) - \frac{\kappa_{n+1}}{\kappa_n} \Theta_n^*(z_k) \right] \right\}.
\end{equation}

The fact that the deformation equations satisfy the Schlesinger system of partial differential equations should be of no great surprise as the isomonodromic properties of the regular semi-classical weights are quite transparent.

**Proposition 3.4.** The monodromy matrix $M_j, j = 1, \ldots, m, \infty$ defined by the analytic continuation of $Y_n$ around a closed loop enclosing the singularity $z_j$

\begin{equation}
(3.80) \quad Y_n|_{z_j+\delta e^{2\pi i}} = Y_n|_{z_j+\delta} M_j,
\end{equation}

is constant with respect to the deformation variable, $M_j = 0$.

**Proof.** In the neighbourhood of any isolated finite singularity $|z - z_j| < \Delta$ the Carathéodory function is a solution of the inhomogeneous first order differential equation \[ (3.54) \] and can be decomposed as

\begin{equation}
(3.81) \quad F(z) = f_j(z) + C_j w(z).
\end{equation}

Here $f_j(z)$ is the unique, holomorphic solution of the inhomogeneous ODE in this neighbourhood whose existence is guaranteed by conditions (3) and (4) of the definition \[ (5.1) \] and which we express as

\begin{equation}
(3.82) \quad f_j(z) = \sum_{l \geq 0} a_{j,l} (z - z_j)^l.
\end{equation}
The second term of (3.81) is a solution of the homogeneous form of (3.4) and $C_j$ a coefficient depending only on $\{z_k, \rho_k\}_{k=1}^m$. We note that the deformation derivative of the Carathéodory function can be written as

$$\dot{F}(z) = \frac{w}{w_0} F(z) + \frac{1}{2} \sum_{j=1}^m \rho_j \frac{\dot{z}_j}{z_j} F(z_j) - \frac{1}{2} w_0 \sum_{j=1}^m \rho_j \frac{\dot{z}_j}{z_j},$$

under the assumptions that either $z_k \notin \mathbb{T}, k = 1, \ldots, m$ or if any $z_k \in \mathbb{T}$ then $\Re \rho < 0$, to ensure the integral (2.30) is uniformly and absolutely convergent. Differentiating (3.81) with respect to $t$ then implies the relation

$$\dot{C}_j(z - z_j)^{\rho_j} \prod_{k \neq j} (z - z_k)^{\rho_k} + \rho_j \frac{\dot{z}_j}{z_j} [f_j(z) - F(z_j)]$$

$$+ \sum_{l \geq 0} a_{j,l} (1 + l + 1)^{(l+1)} \left(\prod_{k \neq j} (z - z_k)^l \right) \sum_{l \neq j} \rho_l \frac{\dot{z}_l}{z_l} [f_j(z) - F(z_l)] + \sum_{l \neq j} \rho_l \frac{\dot{z}_l}{2z_l} [w_0 - F(z_l)] = 0.$$ 

Making the assumption $\Re \rho_j > 0$ we can use the equality $F(z_j) = f_j(z_j)$ and consequently under condition (3) of (3.1) we have

$$\dot{C}_j(z - z_j)^{\rho_j} + \text{analytic function of } z = 0,$$

for $|z - z_j| < \min \{\Delta, |z_j - z_k|, ||z_j| - 1\}$. This implies $\dot{C}_j = 0$ by condition (4).

Using this fact and that the monodromy matrix is given by

$$M_j = \begin{pmatrix} 1 & C_j (1 - e^{-2\pi i \rho_j}) \\ 0 & e^{-2\pi i \rho_j} \end{pmatrix},$$

the proof is concluded. \hfill \Box

Remark 3.5. The monodromy matrices are all upper triangular, which is consistent with the fact that we are dealing with classical solutions of the Schlesinger systems. Also, they are independent of $n$, which is to say they are preserved under the iteration $n \to n + 1$.

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