Abstract

The purpose of this article is to give a result of localization in space of the ground state photons, in some sense, of a Hamiltonian modelling nuclear magnetic resonance in quantum electrodynamics. The asymptotic at infinity obtained is $|x|^{-5}$ where $x$ is considered here as the position of the photons. Moreover, the number of photons at large distance is the smallest in the ground state total spin direction.

Keywords: Ground state, photon localization, photon asymptotic, nuclear magnetic resonance, quantum electrodynamics

1 Introduction.

This article is concerned with the ground states of the Hamiltonian modelling nuclear magnetic resonance (NMR) in the framework of quantum electrodynamics (QED), in particular with the distribution in $\mathbb{R}^3$ of photons in these states.

NMR is the interaction of a finite number of fixed particles of $\mathbb{R}^3$ with a constant magnetic field and with the quantized field. This phenomenon is represented by a Hamiltonian $H(g)$ depending on a positive parameter $g$ (the coupling constant) and acting on a Hilbert space $\mathcal{H}$. This Hamiltonian, introduced by Reuse [29], may also be seen as a simplification of the Pauli-Fierz Hamiltonian [6, 11]. It is recalled in Section 2.

The Hilbert space $\mathcal{H}$ is a completed tensor product $\mathcal{H} = \mathcal{H}_{ph} \otimes \mathcal{H}_{sp}$, where $\mathcal{H}_{ph}$ is the space of the photons and $\mathcal{H}_{sp}$, the space of the particles with spin (cf. Section 2). The space $\mathcal{H}_{ph}$ is a Fock space, in which the usual operators, as the number operator $N$ or the annihilation operators $a(k), k \in \mathbb{R}^3$, are defined.

It was shown in [6, 16, 31, 12, 15]… that the infimum of the spectrum is an eigenvalue. This result is recalled in Theorem 2.1. The eigenvectors are called ground states. Their properties are studied in, e.g., [20, 5, 21, 9, 32, 3]. The main result, Theorem 1.1, is concerned with the localization in space of the photons of the ground state, for a non zero external field and a positive, but sufficiently small, coupling constant $g$.

The problem of the localization of photons is mentioned in Reuse [29] (page 296-297) and [10]. See [23] and [4] as well.

We first define a function on $\mathbb{R}^3$, which could be thought of as the density in space of the numbers of photons. It is well known (see [9, 18], see also [16] and [6, 12]) that every ground state $U$ of the Hamiltonian $H(g)$ is in the domain $D(N^m \otimes I)$ for an arbitrary $m \geq 0$, where $N$ is the number operator.
For any $U$ in $D(N_{1/2} \otimes I)$, one recalls in the Appendix A the definition of the function $k \to (a(k) \otimes I)U$ an an element of $L^2(\mathbb{R}^3, \mathcal{H}^3)$.

Therefore one may define, for any ground state $U$, the function $k \to (a(k) \otimes I)U$ in $L^2(\mathbb{R}^3, \mathcal{H}^3)$, and hence its Fourier transform, which is in $L^2(\mathbb{R}^3, \mathcal{H}^3)$ too. We denote this transform by $x \to (\hat{a}(x) \otimes I)U$.

By abuse of notation, one writes:

$$
(\hat{a}(x) \otimes I)U = \int_{\mathbb{R}^3} e^{-ix \cdot k} (a(k) \otimes I)U dk.
$$

(1.1)

One has:

$$
(2\pi)^{-3} \int_{\mathbb{R}^3} ||(\hat{a}(x) \otimes I)U||^2 dx = < (N \otimes I)U, U >.
$$

Since $< (N \otimes I)U, U >$ is the average number of photons, one could imagine that the average number of photons in a Borel set $E$ of $\mathbb{R}^3$ is given by:

$$
<N_E f, f > = (2\pi)^{-3} \int_E ||\hat{a}(x) \otimes I)U||^2 dx.
$$

In this way, the function $x \to ||(\hat{a}(x) \otimes I)U||^2$ can be seen as the density, in space, of the number of photons.

We aim at studying this function when $U = U_g$ is a ground state of $H(g)$, under the assumption that the constant field $B^{ext}$ is not zero and that $g$ is positive but sufficiently small. In this case, according to [31] and [19], the space of the ground states has dimension 1. This result is recalled in Theorem 2.1.

The operator $\sigma^{[\lambda]}_m$ is defined in (2.3). The main result is the following.

**Theorem 1.1.** Suppose that $B^{ext} \neq 0$. Let $U_g$ be a normalized ground state of Theorem 2.1, where the coupling constant $g$ is small enough to ensure that the space of ground states has dimension 1. Then:

1. The function $x \to |x|^{5/2}(\hat{a}(x) \otimes I)U_g$ is bounded and continuous on $\mathbb{R}^3$, with values in $\mathcal{H}^3$.

2. For every unitary vector $v$ of $\mathbb{R}^3$, one has:

$$
\lim_{|x| \to \infty} |x|^{5/2}(\hat{a}(|x|v) \otimes I)U_g = -\frac{3}{\sqrt{2}} \chi(0)(v \times S^{tot})U_g
$$

where $S^{tot}$ is the vector of $\mathbb{R}^3$ with coordinates:

$$
S^{tot}_j = \sum_{\lambda=1}^P < (I \otimes \sigma^{[\lambda]}_j)U_g, U_g >.
$$

In particular, for the local density of the photons of the ground state, there exists $C > 0$ such that

$$
||(\hat{a}(x) \otimes I)U_g||^2 \leq C(1 + |x|)^{-5}.
$$

Since $U_g$ is normalized, one has, for all unitary vectors $v$ in $\mathbb{R}^3$:

$$
\lim_{|x| \to \infty} |x|^5 ||(\hat{a}(|x|v) \otimes I)U_g||^2 = (9/2)|\chi(0)|^2((v \times S^{tot}))^2
$$

One sees that, for large distances, the photons are fewer in the direction of the total spin.
The Hamiltonian and its ground state.

The Hilbert space of the states of our system is a completed tensor product $\mathcal{H}_{ph} \otimes \mathcal{H}_{sp}$, where $\mathcal{H}_{ph}$ is the Hilbert space of the free photons and $\mathcal{H}_{sp}$, the space of the particles with spin.

Photons. The one photon configuration Hilbert space $\mathcal{H}$ is the set of mappings $f \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ satisfying $k \cdot f(k) = 0$ almost everywhere in $k \in \mathbb{R}^3$ (see [26]) where $|f|^2 = \int_{\mathbb{R}^3} |f(k)|^2 dk$. One denotes by $\langle f, g \rangle$ the scalar product of two elements $f$ and $g$ of $\mathcal{H}$, where the mapping $g \rightarrow \langle f, g \rangle$ is antilinear. The Hilbert space $\mathcal{H}_{ph}$ of photon quantum states is the symmetrized Fock space $\mathcal{F}_s(\mathcal{H}_{C})$ over the complexified space of $\mathcal{H}$. We follow [27] for Fock spaces considerations and notations, in particular, for the usual operators in these spaces: the Segal field $\Phi_S(V)$ associated with an element $V$ in $\mathcal{H}^2$, the $\Gamma(T)$ and $d\Gamma(T)$ operators associated with some operator $T$ acting in $\mathcal{H}^2$. Note that, throughout this paper, the space $\mathcal{H}^2$ is sometimes identified to the complexified space $\mathcal{H}_{C}$ but this identification is not everywhere systematically effectuated in order to avoid possible confusions.

Let $M_\omega$ be the operator with domain $D(M_\omega) \subset \mathcal{H}$ such that $M_\omega q(k) = |k|q(k)$ almost everywhere in $k \in \mathbb{R}^3$. We denote in the same way the analogous operators defined on $\mathcal{H}^2$ or on the complexified space $\mathcal{H}_{C}$. In the Fock space framework, the photon free energy Hamiltonian operator $H_{ph}$ is usually defined as $H_{ph} = d\Gamma(M_\omega)$.

The photon number operator denoted by $N$ is $N = d\Gamma(I)$.

The three components of the magnetic field at each point $x$ in $\mathbb{R}^3$ are defined using the elements $B_{jx}$ belonging to $\mathcal{H}^2$ and written as follows, when one identifies $\mathcal{H}^2$ with the complexified space $\mathcal{H}_{C}$:

$$B_{jx}(k) = \frac{i\chi(|k|)|k|^\frac{1}{2}}{(2\pi)^{\frac{3}{2}}}e^{-i(k \cdot x)}\frac{k \times e_j}{|k|}, \quad k \in \mathbb{R}^3 \setminus \{0\} \quad (2.1)$$

where the function $\chi$ (ultraviolet cutoff) belongs to $S(\mathbb{R})$.

One then defines the magnetic fields components operators at each point $x$ of $\mathbb{R}^3$ by:

$$B_m(x) = \Phi_S(B_{mx}),$$

for $m = 1, 2, 3$.

Spins. The configuration space of the system of $P$ spins is then the space $\mathcal{H}_{sp} = (\mathbb{C}^2)^{\otimes P}$. The fermion property for the spin-$\frac{1}{2}$ fixed particles is omitted here. In the space $\mathcal{H}_{sp}$, we use the operators related to the spins of the different particles. Let $\sigma_j$ ($1 \leq j \leq 3$) be the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.2)$$

For all $\lambda \leq P$ and all $m \leq 3$, we denote by $\sigma_\lambda^m$ the operator in $\mathcal{H}_{sp}$ defined by:

$$\sigma_\lambda^m = I \otimes \cdots \otimes I \otimes \sigma_m \otimes I \otimes \cdots \otimes I, \quad (2.3)$$

where $\sigma_m$ is located at the $\lambda^{th}$ position.
The Hamiltonian. This Hamiltonian is often used for modeling NMR in quantum field theory (see \cite{28,24} and Section 4.11 of \cite{29}). It is a selfadjoint extension of the following operator, initially defined in a dense subspace of $\mathcal{H}_{ph} \otimes \mathcal{H}_{sp}$:

$$H(g) = H_0 + g H_{int},$$

(2.4)

where $g$ is a positive constant and:

$$H_0 = H_{ph} \otimes I + \sum_{\lambda=1}^{P} \sum_{m=1}^{3} B_{m}^{\text{ext}} \otimes \sigma_{m}^{[\lambda]}$$

(2.5)

where $H_{ph} = d\Gamma(M_{\omega})$ is the photon free energy operator, acting in a domain $D(H_{ph}) \subset \mathcal{H}_{ph}$ and $B_{\text{ext}}^{\text{ext}} = (B_{1}^{\text{ext}}, B_{2}^{\text{ext}}, B_{3}^{\text{ext}}) \neq 0$ is the constant magnetic field. Moreover:

$$H_{int} = \sum_{\lambda=1}^{P} \sum_{m=1}^{3} B_{m}(x_{\lambda}) \otimes \sigma_{m}^{[\lambda]}$$

(2.6)

and the $x_{\lambda}$ ($1 \leq \lambda \leq N$) are the points of $\mathbb{R}^3$ where the fixed particles are located.

If an element $U$ of $\mathcal{H}^2$ lies in the domain $D(M_{\omega}^{-1/2})$ then the Segal field $\Phi_{S}(U)$ is bounded from $D(H_{ph})$ into $\mathcal{H}_{ph}$, see point ii) of Proposition 3.4 in \cite{1} or see \cite{8}. This is therefore the case for the operators $B_{j}(x)$ and $E_{j}(x)$ according to the assumptions on the ultraviolet cutoff function $\chi$ in \cite{21}. Thus, according to the Kato-Rellich Theorem, $H(h)$ has a selfadjoint extension with the same domain as the free operator $H_0 = H_{ph} \otimes I$ domain.

The ground state. Let us now recall the results of \cite{16}, \cite{31} and \cite{12} about the ground state of this Hamiltonian, or more precisely of Hamiltonians very close to this one.

**Theorem 2.1.** The operator $H(g)$ defined by (2.4), (2.5) and (2.6), admits a selfadjoint extension with the same domain as the free operator $H_0$. There exists a unitary element $U_g$ of $D(H_0)$, such that $H(g)U_g = E(g)U_g$, where $E(g)$ is the infimum of the spectrum of $H(g)$. This element $U_g$ is in the domain of $N_{m} \otimes I$, for all $m \geq 0$. If $g$ is small enough and if $B_{\text{ext}}^{\text{ext}} \neq 0$, this element $U_g$ is unique up to a multiplicative factor.

This theorem follows from \cite{12} (Theorem 1 page 447), and, for the uniqueness, from \cite{31} and \cite{19}.

3 Asymptotic behavior at infinity of the photon number.

Theorem \cite{11} is proved in this Section. We first note that, according to equality (2.1), the mapping $k \rightarrow (a(k) \otimes I)U$ taking values in $\mathcal{H}^3$, belongs to $L^1(\mathbb{R}^3)$, and consequently, its Fourier transform denoted by $x \rightarrow (\hat{a}(x) \otimes I)U$, also taking values in $\mathcal{H}^3$, is continuous on $\mathbb{R}^3$.

The following result is useful in order to prove the other points in Theorem \cite{11}.

**Proposition 3.1.** Let $(H, D(H))$ be a self-adjoint operator in a Hilbert space $\mathcal{H}$ and suppose that the spectrum of the operator $H$ is the half line $[E, \infty)$ (with $E \in \mathbb{R}$). Then,

$$F(z, H, E) = z(H - E + z)^{-1}$$
Proof. For any $x$ belonging to the spectrum of $H$, set:

$$\varphi_n(x) = \frac{z_n}{z_n + x - E}.$$  

If Re\(z_n\) $\geq 0$ then $|\varphi_n(x)| \leq 1$ for all $x$ in the spectrum of $H$. In addition, $\varphi_n(x)$ tends to $\varphi(x)$ for each $x$ in the spectrum of $H$. According to a standard result (for example, see [7] Lemma 3, Chapter 5, Section 4), $\varphi_n(H)f$ tends to $\varphi(H)f$ for each $f$. Observe that $\varphi_n(H)f = F(z_n, H, E)f$. It is also well known (see [7], Chapter 6, Section 1 and see also [25]) that $\varphi(H)$ is the orthogonal projection on ker$(H - E)$. The proof of the Proposition then follows.

The mapping $(\hat{a}(x) \otimes I)U_g$ is given by (11.1) and $(a(k) \otimes I)U_g$ by (B.1) and (2.1) where $U_g$ is the ground state. That is:

$$(\hat{a}(x) \otimes I)U_g = -\frac{ig}{4\pi^2} \sum_{\lambda=1}^{P} \sum_{m=1}^{3} \int_{\mathbb{R}^3} e^{-ix \cdot k} \chi(|k|) |k|^{1/2} \frac{k \times e_m}{|k|} (H - E + |k|)^{-1} f_m^{[\lambda]} dk$$

with $f_m^{[\lambda]} = (I \otimes \sigma_m^{[\lambda]})U_g$. In the sequel, $\chi(|k|)$ is approximated by $\chi(0)e^{-|k|}$. Therefore, the following function is under consideration:

$$b(x)U_g = -\frac{ig}{4\pi^2} \sum_{\lambda=1}^{P} \sum_{m=1}^{3} \int_{\mathbb{R}^3} e^{-ix \cdot k} \chi(0)e^{-|k|} |k|^{1/2} \frac{k \times e_m}{|k|} (H - E + |k|)^{-1} f_m^{[\lambda]} dk.$$  

(3.1)

Lemma 3.2. There exists $C > 0$ such that:

$$\| (\hat{a}(x) \otimes I)U - b(x)U \| \leq \frac{Cg}{|x|^3}.$$  

Proof of the Lemma. The standard measure on the unit sphere $S^2$ is denoted by $\mu$. For any $v \in S^2$, $m \leq 3$ and $\lambda > 0$, one has:

$$\int_{S^2} e^{-i\lambda v \cdot \omega} (\omega \times e_m) d\mu(\omega) = 4i\pi (v \times e_m) \left[ \frac{\cos \lambda}{\lambda} - \frac{\sin \lambda}{\lambda^2} \right].$$  

(3.2)

One then deduces the existence of $C > 0$ satisfying:

$$\| (\hat{a}(x) \otimes I)U - b(x)U \| \leq C \sum_{m=1}^{3} |I_m(x)|$$

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with, if $x = |x|v, |v| = 1$:

$$I_m(x) = \int_0^\infty \left[ \cos \frac{|x|\rho}{|x|\rho} - \sin \frac{|x|\rho}{(|x|\rho)^2} \right] \Phi_m(\rho) d\rho$$

where:

$$\Phi_m(\rho) = \rho^{5/2}\left(\chi(\rho) - \chi(0)e^{-\rho}\right)(H - E + \rho)^{-1}f_m.$$

One checks that:

$$\int_0^\infty \frac{\sin(|x|\rho)}{|x|^2\rho^2} (\Phi_m(\rho)) d\rho = \frac{1}{|x|^3} \int_0^\infty \cos(|x|\rho) \frac{d}{d\rho} \left[ \frac{\Phi_m(\rho)}{\rho^2} \right] d\rho$$

and:

$$\int_0^\infty \frac{\cos(|x|\rho)}{|x|\rho} (\Phi_m(\rho)) d\rho = \frac{1}{|x|^3} \int_0^\infty \cos(|x|\rho) \frac{d^2}{d\rho^2} \left[ \frac{\Phi_m(\rho)}{\rho} \right] d\rho.$$

These integrations by parts holds true since as $\rho \to 0$:

$$\left| \frac{d}{d\rho} \left[ \frac{\Phi_m(\rho)}{\rho^2} \right] \right| + \left| \frac{d^2}{d\rho^2} \left[ \frac{\Phi_m(\rho)}{\rho} \right] \right| \leq \frac{C}{\rho^{1/2}}.$$

Moreover, the left hand side is rapidly decreasing as $\rho \to \infty$. The proof is thus completed.

□

**Proof of Theorem 1.1**  It is sufficient to consider $b(x)U_g$. For each $v \in S^2$, one sees using (3.1) and (3.2):

$$b(|x|v)U_g = \chi(0)\pi^{-1/2} \sum_{\lambda=1}^{\infty} \sum_{m=1}^{3} v \times e_m I_m^{[\lambda]}(|x|)$$

with:

$$I_m^{[\lambda]}(|x|) = \int_0^\infty \left[ \cos \frac{|x|\rho}{|x|\rho} - \sin \frac{|x|\rho}{(|x|\rho)^2} \right] \Psi_m(\rho) d\rho$$

where:

$$\Psi_m(\rho) = \rho^{5/2}e^{-\rho}(H - E + \rho)^{-1}f_m^{[\lambda]}.$$

One has:

$$I_m^{[\lambda]}(|x|) = \frac{2}{|x|^5/2} \int_0^\infty (t^2 \cos(t^2) - \sin(t^2))e^{-t^2/|x|}F\left(\frac{t^2}{|x|}, H, E\right)f_m^{[\lambda]} dt \quad (3.3)$$

where $F(s, H, E) = s(H - E + s)^{-1}$.

We can write $I_m^{[\lambda]} = I_m^{[\lambda]+} + I_m^{[\lambda]-}$ setting:

$$I_m^{[\lambda]+} = \frac{1}{|x|^{5/2}} \int_0^\infty e^{\varepsilon t^2} (t^2 + \varepsilon) e^{-t^2/|x|} F\left(\frac{t^2}{|x|}, H, E\right)f_m^{[\lambda]} dt \quad (3.4)$$

with $\varepsilon = \pm$. If $|x| \geq 1$, a change of contour of integration shows that:

$$I_m^{[\lambda]} = \frac{1}{|x|^{5/2}} (\varepsilon e^{\varepsilon x^2/4}) \int_0^\infty (t^2 + 1) e^{-t^2/|x|}e^{-\varepsilon t^2/|x|} F\left(\frac{\varepsilon t^2}{|x|}, H, E\right)f_m^{[\lambda]} dt. \quad (3.5)$$
It is therefore deduced that:

\[ |x|^{5/2} f_m^{(\lambda, \text{aux})}(|x|) \leq 2 |f_m^{(\lambda)}| \int_0^\infty (r^2 + 1)e^{-r^2} \, dr. \]  

(3.6)

Consequently, the function \( x \rightarrow |x|^{5/2} b(x) \) is bounded on \( \{|x| \geq 1\} \) proving point i) of the Theorem when using Lemma 3.2.

Using Lebesgue’s dominated convergence Theorem, point ii) comes from:

\[
\lim_{|x| \to \infty} |x|^{5/2} f_m^{(\lambda, \text{aux})}(|x|) = \left( \varepsilon i e^{\varepsilon x/4} \right) \lim_{z \to 0, \text{Re} z \geq 0} F(z, H, E) f_m^{(\lambda)} \int_0^\infty (r^2 + 1)e^{-r^2} \, dr.
\]

The existence of the above limit comes from Proposition 3.1 which also shows:

\[
\lim_{z \to 0, \text{Re} z \geq 0} F(z, H, E) f_m^{(\lambda)} = P f_m = < f_m^{(\lambda)} , U_g > U_g
\]

where \( P \) is the projection on the eigenspace of the infimum of the spectrum of \( H(g) \). Therefore:

\[
\lim_{|x| \to \infty} |x|^{5/2} b(|x|v) U_g = -\sqrt{2/\pi} \chi(0) \sum_{\lambda=1}^P \sum_{m=1}^3 v \times e_m < (I \otimes s_m^{(\lambda)})U_g, U_g > U_g \int_0^\infty (r^2 + 1)e^{-r^2} \, dr
\]

\[
= -\sqrt{2/\pi} \chi(0)(v \times S^{(\text{tot})}) U_g \int_0^\infty (r^2 + 1)e^{-r^2} \, dr
\]

\[
= -\frac{3}{\sqrt{2}} \chi(0)(v \times S^{(\text{tot})}) U
\]

proving point ii) with the help of Lemma 3.2.

\[ \square \]

### Appendices

#### A Standard facts on annihilation operators.

The following results are classical but adapted to the Hilbert space \( H \) in Section 3. We denote by \( F_s^{(\text{reg})}(H_C) \) the subspace of \( F_s(H_C) \) constituted with the finite linear combinations of the symmetrized products of \( g_1 \otimes \cdots \otimes g_m \) where the \( g_j \in S(R^3, R^3) \) satisfy \( \langle k, g_j(k) \rangle = 0 \) for all \( k \in R^3 \).

For all \( k \in R^3 \) and \( f \in F_s^{(\text{reg})}(H_C) \), \( a(k)f \) is classically defined as following. For any (non symmetrized) product \( g = g_1 \otimes \cdots \otimes g_m \), with the \( g_j \) satisfying the above conditions, one set:

\[
a(k)g = \sqrt{m} g_1(k) g_2 \otimes \cdots \otimes g_m.
\]

This definition is next adapted to the symmetrized space. Since the function \( g_1 \) takes values in \( R^3 \), one sees that \( k \rightarrow a(k)f \) takes values in \( (F_s(H_C))^3 \) and belongs to \( S(R^3) \). One also notes:

\[
< Nf, f > = \int_{R^3} ||a(k)f||^2 \, dk.
\]
The next two propositions are useful for the purpose of an extension by density in $D(N^{1/2})$.

**Proposition A.1.** $S(\mathbb{R}^3, \mathbb{R}^3) \cap \mathcal{H}$ is dense in $\mathcal{H}$.

**Proof.** Set $f \in \mathcal{H}$ and $\rho$ a $C^\infty$ smooth cut-off function defined on $\mathbb{R}^+$, vanishing on $[0,1]$ and equal to one on $[2, +\infty)$. One has:

$$\lim_{n \to \infty} \int_{\mathbb{R}^3} |f(k) - f(k)\rho(n|k|^2)|^2 \, dk = 0$$

using Lebesgue Theorem. The above norm $\| \|$ is the one in $\mathbb{R}^3$. Thus, the set $\mathcal{H}_0$ of functions in $\mathcal{H}$ vanishing on a ball centered at the origin is dense in $\mathcal{H}$. Take $f \in \mathcal{H}_0$ vanishing on a ball centered at 0 with radius $\eta > 0$. There exist sequences $f_i^{(n)}$ of $S(\mathbb{R}^3, \mathbb{R})$ converging to $f_i$, for $i = 1, 2, 3$ ($n$ is the coordinate index for these sequences). It is naturally not clear that:

$$\sum_{i=1}^{3} k_i f_i^{(n)}(k) = 0$$

and therefore that $f^{(n)}$ (whose components are the $f_i^{(n)}$) belongs to $\mathcal{H}$. We then set, again with a cut-off function:

$$g^{(n)} = (f^{(n)} - \frac{k}{|k|^2} f^{(n)} \cdot k) \rho(\frac{4}{\eta^2}|k|^2). \quad (A.1)$$

One checks that $g^{(n)} \in \mathcal{H} \cap S(\mathbb{R}^3, \mathbb{R}^3)$ and tends to $f$ in $\mathcal{H}$, i.e., with the $L^2(\mathbb{R}^3, \mathbb{R}^3)$ norm.

\[\square\]

**Proposition A.2.** $\mathcal{F}_s^{reg}(\mathcal{H}_C)$ is dense in $D(N^{1/2})$.

**Proof.** One chooses a basis $(u_i)$ of $\mathcal{H}$ in $\mathcal{H} \cap S(\mathbb{R}^3, \mathbb{R}^3)$ and set:

$$\zeta_\alpha = \sqrt{\frac{\alpha!}{\alpha!}} S_n \left( \otimes_{i} u_i^{\otimes \alpha_i} \right), \quad n = |\alpha|. \quad (A.2)$$

The set of $\zeta_\alpha$ (with $|\alpha| = n$) is an othonormal basis of $\otimes^n \mathcal{H}$ (see Janson [22]). Thus, the set of finite linear combinations of $\zeta_\alpha$ (with arbitrary $|\alpha|$) is dense in the set of finite number particles $\mathcal{F}_s^{fin}(\mathcal{H})$ which is dense in $D(N^{1/2})$. The proof of the Proposition is then complete since the set of finite linear combinations of $\zeta_\alpha$ is included in $\mathcal{F}_s^{reg}(\mathcal{H}_C)$.

\[\square\]

We can now extend the definition of $a(k)$ to $D(N^{1/2})$.

**Proposition A.3.** Set $f \in D(N^{1/2})$ and a sequence $(f_n)$ in $\mathcal{F}_s^{reg}(\mathcal{H}_C)$ converging to $f$ in $D(N^{1/2})$. Then, the $k \to a(k)f_n$ has a limit in $L^2(\mathbb{R}^3, (\mathcal{F}_s(\mathcal{H}_C))^3)$. This limit is denoted (definition) by $a(k)f$.

Indeed, according to the previous points, one has if $m < n$:

$$\int_{\mathbb{R}^3} ||a(k)(f_m - f_n)||^2 dk \leq (||N^{1/2}f_m|| + ||N^{1/2}f_n||)||N^{1/2}(f_m - f_n)||$$

and one also has ($\varphi \in D(N^{1/2})$):

$$||N^{1/2}\varphi||^2 = \int_{\mathbb{R}^3} ||a(k)\varphi||^2 dk. \quad (A.3)$$
B Pull Through Formula

Let $U_g$ be a normalized ground state given by Theorem 2.1. It is recalled in Theorem 2.1 that $U_g$ belongs to the domain $N^m \otimes I$ for all integers $m$. According to Proposition A.3, the function $k \mapsto (a(k) \otimes I)U_g$ is well defined as an element of $L^2(\mathbb{R}^3, \mathcal{H})$. In particular, this function is defined almost everywhere. We give in the next result an explicit expression of this function, recalling the Pull through formula and its proof (see [12] and see also [30, 13, 11, 16, 14, 6, . . .]). The uniqueness of the ground state is not useful for that purpose.

**Theorem B.1.** Let $U_g$ be a (normalized) ground state of Theorem 2.1. Then, for almost every $k \in \mathbb{R}^3 \setminus \{0\}$:

$$ (a(k) \otimes I)U_g = -\frac{g}{\sqrt{2}} \sum_{\lambda=1}^{P} \sum_{m=1}^{3} B_{m,x^\lambda}(k) (H(g) - E(g))^{-1} (I \otimes \sigma_m^{[\lambda]}) U_g $$

where $B_{m,x^\lambda}(k)$ is defined in (2.1). Consequently:

$$ \|(a(k) \otimes I)U_g\| \leq \frac{g}{\sqrt{2}} \sum_{\lambda=1}^{P} \sum_{m=1}^{3} \left|B_{m,x^\lambda}(k)\right| \|(I \otimes \sigma_m^{[\lambda]}) U_g\|. $$

**Proof.** In view of Lemma 2.5 in [3]:

$$ [d\Gamma(M_{\omega}), a(k)] = -|k|a(k). $$

(B.3)

Thus, from (2.4) and (2.5), for almost every $k$ in $\mathbb{R}^3$:

$$ (H(g) - E(g) + |k|)(a(k) \otimes I)U_g = (a(k) \otimes I)(H(g) - E(g))U_g + g[H_{\text{int}}, (a(k) \otimes I)]U_g. $$

One has:

$$ [H_{\text{int}}, (a(k) \otimes I)] = \sum_{\lambda=1}^{P} \sum_{m=1}^{3} [B_{m,x^\lambda}(x), a(k)] \otimes \sigma_m^{[\lambda]}.$$

Besides:

$$ \sqrt{2}[B_{m,x^\lambda}(x), a(k)] = [a^*(B_{m,x^\lambda}), a(k)] = -B_{m,x^\lambda}(k). $$

(B.4)

Consequently:

$$ [H_{\text{int}}, (a(k) \otimes I)] = -(1/\sqrt{2}) \sum_{\lambda=1}^{P} \sum_{m=1}^{3} B_{m,x^\lambda}(k) (I \otimes \sigma_m^{[\lambda]}). $$

For all $k \neq 0$, the operator $(H(g) - E(g) + |k|)$ is invertible. One then deduces equality (B.1). □

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