Coupling constant dependence of the shear viscosity in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory

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Abstract: Gauge theory - gravity duality predicts that the shear viscosity of $\mathcal{N} = 4$ supersymmetric $SU(N_c)$ Yang-Mills plasma at temperature $T$ in the limit of large $N_c$ and large 't Hooft coupling $g_{YM}^2 N_c$ is independent of the coupling and equals to $\pi N_c^2 T^3 / 8$. In this paper, we compute the leading correction to the shear viscosity in inverse powers of 't Hooft coupling using the $\alpha'$-corrected low-energy effective action of type IIB string theory. We also find the correction to the ratio of shear viscosity to the volume entropy density (equal to $1 / 4\pi$ in the limit of infinite coupling). The correction to $1 / 4\pi$ scales as $(g_{YM}^2 N_c)^{-3/2}$ with a positive coefficient.

Keywords: AdS/CFT correspondence, thermal field theory
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### 1. Introduction

Transport coefficients such as viscosity, diffusion constants, thermal and electric conductivities are the key ingredients in describing the hydrodynamic regime of any medium [1]. These coefficients are usually obtained from experiment rather than computed from first principles of an underlying microscopic theory, because the study of realistic strongly interacting systems often remains beyond the reach of currently available theoretical methods.

For finite-temperature quantum field theories, and thermal gauge theories in particular, computations based on the Boltzmann equation in the regime of weak coupling $g \ll 1$ show [2, 3, 4, 5] that the shear viscosity behaves as

$$\eta \sim \frac{C N_c^2 T^3}{g^4 \log 1/g^2},$$

(1.1)

where $N_c^2$ is the number of colors and $C$ is a large numerical coefficient.
Since the entropy scales as \( S \sim N_c^2 T^3 V_3 \), the ratio of shear viscosity to the volume entropy density \( \frac{\eta}{s} = \frac{S}{V_3} \) behaves as

\[
\eta \sim \frac{1}{g^4 \log 1/g^2} \gg 1
\]  

(1.2)
in the regime of weak coupling. On the other hand, hydrodynamic models used to describe elliptic flows observed in recent heavy ion collision experiments at RHIC seem to favor small values of the ratio \( \eta/s \). This is not a contradiction, however, since the experimental data were obtained for the range of energies where the coupling remains relatively large, whereas the result (1.2) is valid for small coupling.

It is therefore desirable to obtain results for the viscosity-entropy ratio in the regime of intermediate and strong coupling. Lattice simulations cannot address the issue of real-time dynamics directly facing (among other things) a formidable problem of analytic continuation. (For indirect approaches, see [9], [10], [11], [12].)

In the absence of more conventional methods, the AdS/CFT (or gauge theory/gravity) duality conjecture in string theory [13, 14, 15, 16] emerged as a source of insights into the non-perturbative regime of thermal gauge theories. The best studied example of the duality relates a four-dimensional finite-temperature \( \mathcal{N} = 4 \) \( SU(N_c) \) supersymmetric Yang-Mills (SYM) theory in the limit of large \( N_c \) and large 't Hooft coupling \( g_{YM}^2 N_c \) to the supergravity background corresponding to a stack of \( N_c \) near-extremal black three-branes.

Using the AdS/CFT conjecture, one is able to predict the behavior of the entropy of \( \mathcal{N} = 4 \) SYM in the regime of strong coupling [17], [18]. In the large \( N_c \) limit, the entropy is given by

\[
S = \frac{2\pi^2}{3} N_c^2 V_3 T^3 f(g_{YM}^2 N_c),
\]  

(1.3)
where the function \( f(g_{YM}^2 N_c) \) interpolates (presumably smoothly) between 1 at zero coupling and \( 3/4 \) at infinite coupling. The strong coupling expansion for \( f \) was obtained by Gubser, Klebanov and Tseytlin

\[
f(g_{YM}^2 N_c) = \frac{3}{4} + \frac{45}{32} \zeta(3) (g_{YM}^2 N_c)^{-3/2} + \cdots.
\]  

(1.4)
At weak coupling, \( f \) can be found by using finite-temperature perturbation theory [19]

\[
f(g_{YM}^2 N_c) = 1 - \frac{3}{2\pi^2} g_{YM}^2 N_c + \frac{3 + \sqrt{2}}{\pi^3} (g_{YM}^2 N_c)^{3/2} + \cdots
\]  

(1.5)

\footnote{Normalization convention for \( g_{YM} \) adopted in the current version of this paper differs from the one used in [13]. See Eq. (2.6) and footnote 5.}
Our goal in the present paper is to obtain the analogue of the strong coupling expansion (1.4) for the shear viscosity and for the ratio \( \eta/s \). In the limit of infinite \( N_c \) and infinite \( \text{'t Hooft coupling}, the shear viscosity of \( \mathcal{N} = 4 \) SYM plasma was found to be 

\[
\eta = \frac{\pi}{8} N_c^2 T^3.
\] (1.6)

Taking into account the leading term in (1.4), one may notice that \[\nonumber\]

\[
\frac{\eta}{s} = \frac{1}{4\pi} + O\left(1/N_c^2\right) + O\left((g_{YM}^2 N_c)^{-3/2}\right).
\] (1.7)

The shear viscosity (1.6) was obtained by a number of interconnected methods \[\nonumber\], all of which relied on the validity of the conjectured gauge/gravity correspondence. In that sense, the result (1.6) does not enjoy the status of a firmly established field-theoretical calculation and should be viewed as a prediction of a gauge/gravity duality for a particular supersymmetric theory\(^2\).

It was further found in \[\nonumber\] that the ratio \( \eta/s \) remains equal to \( 1/4\pi \) for other gauge/gravity duals where \( \eta \) and \( s \) alone behave very differently compared to the basic example of \( \mathcal{N} = 4 \) SYM. The same feature holds for theories dual to \( M \)-branes \[\nonumber\]. The calculations in \[\nonumber\] were done independently by using the traditional AdS/CFT strategy, and also by deriving a generic formula for \( \eta/s \) by a method reminiscent of the old “membrane paradigm” approach\(^3\). It was later shown in \[\nonumber\] that the type II supergravity equations of motion guarantee that the generic value for \( \eta/s \) is universally equal to \( 1/4\pi \). Whenever the near-horizon supergravity geometry dual to a finite temperature gauge theory allows for an extension to asymptotically flat space-time\(^4\), the universality of \( \eta/s \) can be related \[\nonumber\] to the universality of low energy absorption cross sections for black holes observed in \[\nonumber\].

If the conjectured universality is indeed true, it is most probably associated with the regime of strong coupling in various theories. For weakly coupled theories, the ratio \( \eta/s \) is typically very large, even at large \( N_c \). It was conjectured in \[\nonumber\] that \( 1/4\pi \) is the lowest possible value for \( \eta/s \) for all substances in nature. In view of this conjecture, finding the first nontrivial term in the strong coupling expansion for \( \eta/s \) is especially interesting.

Our paper is organized as follows. In Section 2 we outline the method of computing the strong coupling expansion for shear viscosity of \( \mathcal{N} = 4 \) SYM from supergravity.

\(^2\) Over the years, AdS/CFT has survived numerous checks at zero temperature \[\nonumber\]. For a check at finite temperature, see \[\nonumber\].

\(^3\) From the AdS/CFT point of view, this method presumably computes the lowest (hydrodynamic) quasinormal frequency of a given gravitational background. The corresponding link is not firmly established at the moment, see \[\nonumber\], \[\nonumber\].

\(^4\) Such an extension is not known for explicit examples discussed in \[\nonumber\].
Section 3 we consider gravitational perturbations of the $\alpha'$-corrected black brane metric and compute the boundary term of the corresponding on-shell action. In Section 4 we use the Minkowski AdS/CFT prescription to find the retarded thermal correlator of a stress-energy tensor in the hydrodynamic approximation and compute the correction to the shear viscosity. Finally, in appendix A we collect the coefficients of the effective action which are too cumbersome to appear in the main text.

2. Shear viscosity of $\mathcal{N} = 4$ SYM from supergravity

Shear viscosity is the transport coefficient that appears in the hydrodynamic constitutive relation for the spatial components of the stress-energy tensor (see [21], [31] for more hydrodynamics preliminaries).

Among the methods developed in [20], [21], [23] for computing the shear viscosity from supergravity, the most straightforward and the least technically complicated one [20] is based on the Kubo relation

$$\eta = \lim_{\omega \to 0} \frac{1}{2\omega} \int dt \, d^3 x \, e^{i\omega t} \langle [T_{xy}(x)T_{xy}(0)] \rangle. \quad (2.1)$$

The Kubo formula is a particular case of the celebrated fluctuation-dissipation theorem which relates fluctuations of a medium to the response of the medium to an external action. The formula expresses transport coefficients of a slightly non-equilibrium system in terms of the real-time correlation functions computed in an equilibrium thermal ensemble. Eq. (2.1) can be written in the form

$$\eta = \lim_{\omega \to 0} \frac{1}{2\omega i} \left[ G^A_{xy,xy}(\omega, 0) - G^R_{xy,xy}(\omega, 0) \right], \quad (2.2)$$

where the retarded Green’s function is defined as

$$G^R_{\mu\nu,\lambda\rho}(\omega, q) = -i \int d^4 x \, e^{-iq\cdot x} \theta(t) \langle [T_{\mu\nu}(x), T_{\lambda\rho}(0)] \rangle, \quad (2.3)$$

and $G^A(\omega, q) = (G^R(\omega, q))^*$. 

To compute the hydrodynamic limit of the retarded correlation function (2.3) from supergravity, one can use either the Minkowski AdS/CFT prescription [26], [32], or the method based on calculating the absorption cross-section of gravitons by non-extremal three-branes [33, [20]. In the absence of $\alpha'$ corrections, to the lowest order in $\omega/T \ll 1$, $q/T \ll 1$ the retarded correlation function is given by [21]

$$G^R_{xy,xy}(\omega, q) = -\frac{N_c^2 T^2}{16} (i2\pi T \omega + q^2 + \cdots). \quad (2.4)$$
From Eq. (2.2) one finds the shear viscosity
\[
\eta = \frac{\pi N_c^2 T^3}{8} + O \left( \frac{1}{N_c^2} \right) + O \left( \left( g_{YM} N_c \right)^{-3/2} \right). \tag{2.5}
\]

Corrections to the shear viscosity in powers of the inverse 't Hooft coupling correspond on the string theory side of AdS/CFT duality to \( \alpha' \) corrections to classical general relativity. The precise relation between the expansion parameters in supergravity and Yang-Mills theory is given by\(^5\) [16]
\[
\frac{\alpha'}{L^2} = \frac{1}{\sqrt{g_{YM} N_c}}. \tag{2.6}
\]

Since the leading order stringy correction to type IIB supergravity is proportional to \( \alpha'^3 \), on the gauge theory side we expect the coupling constant correction to shear viscosity to scale as \( (g_{YM} N_c)^{-3/2} \). At the same time, finding \( 1/N_c^2 \) corrections requires quantum gravity calculations in the black three-brane background.

In this paper, we use the prescription of [26] to compute the retarded correlator \( G_{xy,xy}^R \) from the \( \alpha' \)-corrected black three-brane metric found in [18]. The coupling constant correction to viscosity then follows from the Kubo formula (2.2).

3. Perturbations of the \( \alpha' \)-corrected near-extremal three-brane background

We start with the tree level type IIB low-energy effective action in ten dimensions taking into account the leading order string corrections [35, 36]
\[
S = \frac{1}{2 \kappa_{10}^2} \int d^{10}x \sqrt{g} \left[ R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{4 \cdot 5!} (F_5)^2 + \cdots + \gamma e^{-\frac{3}{2} \phi} W + \cdots \right], \tag{3.1}
\]

\[
\gamma = \frac{1}{8} \zeta(3) (\alpha')^3, \tag{3.2}
\]

where \( W \) may be chosen in an appropriate scheme to be proportional to the fourth power of the Weyl tensor [18]
\[
W = C^{hmnk} C_{pmnq} C_h^{rs} C^q_{rsk} + \frac{1}{2} C^{hkmn} C_{pqmn} C_h^{rs} C^q_{rsk}. \tag{3.3}
\]

---

\(^5\) The normalization convention \( L^4/\alpha'^2 = 4\pi g_s N_c = g_{YM}^2 N_c \) [14] used in the current version of this paper differs from the normalization \( L^4/\alpha'^2 = 2g_{YM}^2 N_c \) used in the earlier versions and in some other works by a factor of 2. In general, this normalization depends on the normalization of the generators of the gauge group: \( g_{YM}^2 = 2\pi g_s/c \) for \( \text{tr} (T^a T^b) = c \delta^{ab} \). See e.g. Section 13.4 of [34].
It is helpful to rewrite $W$ in the form [37]

$$W = B_{ijkl} \left( 2B^{iklj} - B^{lijk} \right), \quad B_{ijkl} = C_{ijmn} \cdot C_{lkkm}. \quad (3.4)$$

In (3.1), ellipses stand for other fields not essential for the present analysis. As usual, it is assumed that the self-duality condition on $F_5$ is imposed after the equations of motion are derived. The form of the action (3.1) and subtleties associated with the self-duality condition on the five-form are discussed in [18], [37]. We emphasize that the possibility of additional corrections of order $(\alpha')^3$ associated with the five-form is not excluded. However, indirect arguments presented in [18], [37] suggest that these corrections, if present, will not affect the near-horizon geometry of the black three-brane solution. We proceed with our calculation while ignoring this potential problem.

3.1 The $\alpha'$-corrected black three-brane background and its $S^5$ reduction

In the absence of $\alpha'$ corrections, the metric describing the non-extremal black three-brane solution trivially factorizes into the five-sphere part and the “AdS” part. However, this is no longer the case for the $\alpha'$-corrected solutions, as was first emphasized by Pawelczyk and Theisen [40]. Thus in considering both the $\alpha'$-corrected black brane solution and its fluctuations, the correct procedure is to work in ten dimensions and then perform the $S^5$ reduction to obtain the five-dimensional asymptotically AdS geometry. An additional subtlety comes from dealing with the self-dual five-form $F_5$, which identically squares to zero by virtue of ten-dimensional self-duality.

The ten-dimensional metric can be taken in the form of a standard ansatz used for the $S^5$ reduction of type IIB supergravity (see e.g. [38])

$$ds_{10}^2 = e^{-\frac{\nu}{3} \nu(x)} g_{5mn}(x) dx^m dx^n + e^{2\nu(x)} d\Omega_5^2,$$

where the five-dimensional asymptotically AdS metric $g_{5mn}(x)$ has a general form consistent with the symmetries of the problem

$$ds_5^2 = \frac{r_0^2}{u} e^{c(u)} \left( -f e^{a(u)} dt^2 + dx^2 + dy^2 + dz^2 \right) + \frac{du^2}{4u^2 f} e^{b(u)}.$$

Here $f(u) = 1 - u^2$, $r_0$ is the parameter of non-extremality of the black brane geometry, and we set the “AdS radius” $L$ to one. When $\gamma = 0$ in (3.1), the metric (3.3) with $\nu = 0, a = 0, b = 0, c = 0$ is the standard solution of type IIB low-energy equations of motion describing the near-horizon limit of non-extremal three-branes. Corrections

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6In a dual thermal field theory, Poincare invariance is broken, but the system is assumed to be isotropic.
to that solution can be found by solving equations of motion perturbatively in \( \gamma \). To
leading order in \( \gamma \), functions \( a, b, c, \nu \) were found in [18], [40]

\[
a(u) = -15 \gamma (5u^2 + 5u^4 - 3u^6), \quad (3.7a)
\]
\[
b(u) = 15 \gamma (5u^2 + 5u^4 - 19u^6), \quad (3.7b)
\]
\[
c(u) = 0, \quad (3.7c)
\]
\[
\nu(u) = \frac{15\gamma}{32} u^4 (1 + u^2). \quad (3.7d)
\]

The Hawking temperature corresponding to the metric (3.3) is

\[
T = \frac{r_0}{\pi} (1 + 15\gamma). \quad (3.8)
\]

The dilaton \( \phi \) also receives \( \alpha' \) corrections [18]. Since to leading order in \( \gamma \) they do not
mix with the gravitational perturbations which we are considering in this paper, we do
not write them explicitly.

Substituting the ansatz (3.5) into the action (3.1) we find

\[
S_5 = \frac{\pi^3}{2\kappa_{10}^2} \int d^5x \sqrt{g_5} \left[ R_5 + 20e^{-\frac{16}{3}\nu} - \frac{1}{2} (\partial \phi)^2 - \frac{40}{3} (\partial \nu)^2 - 8e^{-\frac{40}{3}\nu} + \gamma e^{-\frac{3}{2}\phi} e^{-\frac{40}{3}\nu} W(u, \nu, \gamma) \right],
\]
 \( (3.9) \)

where the ten-dimensional gravitational constant \( \kappa_{10} = \sqrt{8\pi G} \) is related to the number
\( N_c \) of coincident branes and the AdS radius \( L \) by \( \kappa_{10} = 2\pi^2 \sqrt{\pi} L^4 / N_c \).

For small \( \nu \sim \gamma \) we have

\[
e^{-\frac{40}{3}\nu} W(u, \nu, \gamma) = 180u^8 + 1800u^8 \nu(u) + O(\nu^2). \quad (3.10)
\]

(Note that the expression for \( W(u, \nu, \gamma) \) is not merely the five-dimensional \( W \) computed
with the metric (3.6).)

As mentioned above, there is a subtlety in reducing the term involving the five-form
field strength. The correct procedure is to assume that \( F_5 \) has components only along
the five-sphere, and double that contribution in the effective ten-dimensional action.
Another way of saying this is that the term \( -8e^{-\frac{40}{3}\nu} \) was added to (3.9) by hand to
ensure that, upon variation, the correct reduced equations of motion in five dimensions
are reproduced.

3.2 Metric perturbations

The coupling between the boundary value of the graviton and the stress-energy tensor of
a gauge theory is given by \( h^x_y(x)T^y_x/2 \). According to the AdS/CFT prescription, in order
to compute the retarded thermal two-point function of the components of the stress-energy tensor entering the Kubo formula (2.2), we should add a small bulk perturbation $h_{xy}(u, x)$ to the metric (3.5), and compute the on-shell action as a functional of its boundary value $h_{xy}(x)$.

Simple symmetry arguments [21] show that for a perturbation of this type and a metric of the form (3.5) all other components of a generic perturbation $h_{\mu\nu}$ can be consistently set to zero. (We have checked explicitly that to linear order in the perturbation, the equation for $h_{xy}$ decouples from all other equations describing a generic perturbation of the background (3.3).)

It will be convenient\(^7\) to introduce a field $\phi(u, x)$

$$\phi(u, x) = \frac{u}{r^2_0} h_{xy}(u, x)$$

(note that $h_{\gamma} = g^{x\gamma}h_{xy} = e^{i\omega\tau} \phi(u, x) = \phi(u, x) + O(u^4)$) and use the Fourier decomposition

$$\phi(u, x) = \int \frac{d^4k}{(2\pi)^4} e^{-i\omega t + i k \cdot x} \phi_k(u).$$

Since in (2.2) we need the correlator at vanishing spatial momentum, it will be sufficient to consider perturbations which depend on the radial coordinate and time only.

### 3.3 The effective action

Substituting the perturbed $\alpha'$-corrected black brane metric into the ten-dimensional action (3.1) and performing the five-sphere reduction keeping in mind the subtleties mentioned in Section 3.1, we obtain the effective action for the field $\phi_k(u)$. To quadratic order in $\phi_k(u)$ and linear order in $\gamma$ this action is given by

$$S_5 = S^{(0)} + S^{(2)},$$

where $S^{(0)}$ is independent of $\phi_k(u)$, and

$$S^{(2)} = \frac{N_c^2}{8\pi^2} \int \frac{d^4k}{(2\pi)^4} \int_0^1 du \left[ A \phi_k^{''} \phi_{-k} + B \phi_k^{'} \phi_{-k} + C \phi_k^{'} \phi_{-k} \\
+ D \phi_k \phi_{-k} + E \phi_k^{'''} \phi_{-k} + F \phi_k^{'} \phi_{-k} \right].$$

The coefficients $A, B, C, D, E, F$ are even functions of the momentum. They are given explicitly in appendix A.

\(^7\)In the absence of $\alpha'$-corrections the effective action of $\phi(u, x)$ is that of a minimally coupled scalar in the AdS black hole background.
Variation of $S^{(2)}$ leads to

$$
\delta S^{(2)} = \frac{N_c^2}{8\pi^2} \int \frac{d^4k}{(2\pi)^4} \left[ \int_0^1 du \left[ EOM \right] \delta \varphi_{-k} + \left( B_1 \delta \varphi_{-k} + B_2 \delta \varphi'_{-k} \right) \right],
$$

where the coefficients of the boundary term are given by

$$
B_1 = -(A\varphi_k)' + 2B\varphi'_k + C\varphi_k - 2(E\varphi''_k)' + F\varphi''_k - (F\varphi'_k)',
$$

$$
B_2 = A\varphi_k + F\varphi'_k + 2E\varphi''_k,
$$

and EOM denotes the left hand side of the Euler-Lagrange equation

$$
A\varphi''_k + C\varphi'_k + 2D\varphi_k - \frac{d}{du} \left( 2B\varphi'_k + C\varphi_k + F\varphi''_k \right) + \frac{d^2}{du^2} \left( A\varphi_k + 2E\varphi''_k + F\varphi'_k \right) = 0.
$$

In order to have a well-defined variational principle, one has to add a generalized Gibbons-Hawking boundary term to the action (3.14). Variation of this additional term is supposed to cancel the contribution $B_2 \delta \varphi'_{-k}$ in (3.15) without leading to any new boundary terms involving first and higher derivatives of $\varphi_k$. In the absence of higher-derivative terms in the action, this is achieved by adding the standard Gibbons-Hawking term proportional to the trace of the extrinsic curvature of the boundary, i.e. by adding the boundary term $-A\varphi_k \varphi'_k$ to the action (3.14). Similarly, one can cancel the second term in $B_2$ by adding the boundary term $-F\varphi''_k \varphi'_{-k}/2$. However, the last term in (3.17) poses a problem: adding a term $-2E\varphi''_k \varphi'_{-k}$ to cancel it would upon variation produce second derivatives of $\delta \varphi_{-k}$. This problem is not unexpected because we have a theory whose equation of motion is a fourth order differential equation. In our case, however, the difficulty can be avoided since we intend to treat the higher-derivative term as a perturbation, and we only solve the equation of motion (3.18) perturbatively to the first non-trivial order in $\gamma$. This means that effectively we are still solving a second order equation. Let us write Eq. (3.18) in the form

$$
\varphi''_k + p_1 \varphi'_k + p_0 \varphi_k = O(\gamma),
$$

where all $\gamma$-dependent terms are exiled to the right. Now, if we add to the action (3.14) the generalized Gibbons-Hawking term

$$
K = -A\varphi_k \varphi'_{-k} - \frac{F}{2} \varphi_k \varphi'_{-k} + E \left( p_1 \varphi'_k + 2p_0 \varphi_k \right) \varphi'_{-k},
$$

and note that the coefficient $E$ is proportional to $\gamma$, then upon variation we find that the boundary term involving $\delta \varphi'_{-k}$ vanishes on shell up to terms of order $\gamma^2$. Thus, to linear order in $\gamma$, the variational problem is now well-posed.
The bulk action (3.14) can be rewritten in the form

\[ S^{(2)} = \frac{N_c^2}{8\pi^2} \int \frac{d^4k}{(2\pi)^4} \int_0^1 du \left( \partial_u \mathcal{B} + \frac{1}{2} [EOM] \varphi_{-k} \right), \]  

where

\[ \mathcal{B} = -\frac{A'}{2} \varphi_k \varphi_{-k} + B \varphi_k' \varphi_{-k} + C \varphi_k \varphi_{-k} - E' \varphi_k'' \varphi_{-k} + E \varphi_k' \varphi_{-k} - E \varphi_k'' \varphi_{-k} \]

\[ + \frac{F'}{2} \varphi_k' \varphi_{-k} - \frac{F'}{2} \varphi_k \varphi_{-k} \]  

(3.22)

and EOM denotes the left hand side of (3.18). Adding to this the generalized Gibbons-Hawking term (3.20), we observe that the complete on shell action reduces to a boundary term

\[ \int \frac{d^4k}{(2\pi)^4} F_k \bigg|_0^1, \]  

(3.23)

where

\[ F_k = \frac{N_c^2 r_0^4}{8\pi^2} \left[ (B - A) \varphi_k' \varphi_{-k} + \frac{1}{2} (C - A') \varphi_k \varphi_{-k} - E' \varphi_k'' \varphi_{-k} + E \varphi_k' \varphi_{-k} \right. \]

\[ - E \varphi_k'' \varphi_{-k} - \frac{F'}{2} \varphi_k' \varphi_{-k} - E \frac{1}{u f} \varphi_k' \varphi_{-k} + 2E \frac{u w^2}{u f^2} \varphi_k' \varphi_{-k} \bigg], \]  

(3.24)

and \( w \equiv \omega/2r_0 \).

### 3.4 The solution for \( h_{xy} \)

We now turn to finding the solution to the equation of motion (3.18). Explicitly, Eq. (3.18) reads

\[ u f^2 \phi_k'' - f (1 + u^2) \phi_k' + w^2 \phi_k = -\gamma \frac{37}{3} u^7 f^3 \phi_k^{IV}(r) + \gamma \frac{74}{3} u^6 f^2 (9u^2 - 5) \phi_k''(r) \]

\[ - \gamma \frac{u^3 f}{8} \left( -600 + 6193 u^2 - 25552 u^4 + 22327 u^6 + 592 u^3 w^2 \right) \phi_k''(r) \]

\[ - \gamma \frac{u^2 f}{8} \left( -600 + 819 u^2 - 3440 u^4 + 7105 u^6 + 2368 u^3 w^2 \right) \phi_k'(r) \]

\[ - \gamma \frac{u^2}{8} \left( 100 u - 550 u^5 + 1000 u^7 - 450 u^9 + 600 w^2 - 719 u^2 w^2 + 3968 u^4 w^2 \right. \]

\[ - 2665 u^6 w^2 + 4 u^3 (74 w^4 - 25) \right) \phi_k(r). \]  

(3.25)
where we put all \( \gamma \)-dependent terms to the right. Eq. (3.25) can be solved perturbatively in \( \gamma \) by writing
\[
\varphi_k(u) = \varphi_k^{(0)}(u) + \gamma \varphi_k^{(1)}(u),
\]  
where \( \varphi_k^{(0)}(u) \) is the solution to the minimally coupled massless scalar equation (i.e. Eq. (3.25) with \( \gamma = 0 \)) found in [21],
\[
\varphi_k^{(0)}(u) = (1 - u)^{-i\frac{w}{2}} \left( 1 - \frac{iu}{2} \ln \frac{1 + u}{2} + O(w^2) \right).
\]  
The full solution obeying the incoming wave boundary condition at the horizon \( u = 1 \) [21] and normalized to one at the boundary \( u = 0 \) is
\[
\varphi_k(u) = (1 - u)^{-i\frac{w}{2}} G(u),
\]  
where the function \( G(u) \) is regular at \( u = 1 \), and is given explicitly by
\[
G(u) = 1 - \gamma \frac{25}{16} u^4 (1 + u^2) - \frac{iw}{2} \left[ 1 - \frac{25}{16} \gamma u^4 (1 + u^2) \right] \log (1 + u)
+ \gamma \frac{iw u^2}{2} \left( 43 u^4 + 135 u^2 + 195 \right) + O(\gamma^2, w^2).
\]  

4. Coupling constant correction to shear viscosity

Having found the solution for a gravitational perturbation, we can compute the correlation function \( G_{xy,xy}(\omega, q) \) by applying the Minkowski AdS/CFT prescription [26]
\[
G_{xy,xy}^{R}(\omega, q) = \lim_{u \to 0} 2\mathcal{F}_k,
\]  
where \( \mathcal{F}_k \) is the boundary term (3.24). (It is assumed that all contact and momentum-independent terms in (4.1) should be discarded.) Computing the limit, to the lowest order in \( w \) we find
\[
\lim_{u \to 0} 2\mathcal{F}_k = \frac{N_c r_0^4}{4\pi^2} \left( 1 + 75\gamma - \frac{1}{u^2} - iw(1 + 195\gamma) + O(w^2) \right).
\]  

On a technical side, we note that the asymptotic behavior of the coefficients (A.1)–(A.6) and the regularity of the solution (3.28) at \( u \to 0 \) imply that only the first two terms in (3.24) contribute to the limit (4.2), with the nontrivial contribution coming only from the term \( (B - A) \varphi_k \varphi_{-k} \). Moreover, contributions to the coefficients \( A \) and \( B \) coming from the Weyl term \( W \) vanish in the limit. The \( W \) term therefore plays the role of a “grey cardinal” influencing the result indirectly through the correction to the black brane metric and the equation of motion.
Thus the retarded correlator at zero spatial momentum and to the leading order in $\omega$ is
\[ G_{xy,xy}(\omega,0) = -i \frac{N_c^2 T^3 \omega}{8\pi^2} (1 + 195\gamma) = -i \frac{N_c^2 T^3 \omega}{8} (1 + 150\gamma) \quad (4.3) \]
where we used the expression (3.8) for the Hawking temperature of the $\alpha'$-corrected metric. Applying the Kubo formula (2.2), we immediately find
\[ \eta = \frac{\pi}{8} N_c^2 T^3 (1 + 150\gamma) \quad (4.4) \]
Combining (2.6) and (3.2), we obtain
\[ \eta = \frac{\pi}{8} N_c^2 T^3 \left( 1 + \frac{75}{4} \zeta(3) (g_{YM}^2 N_c)^{-3/2} + \cdots \right) \quad (4.5) \]
Using the result of Gubser, Klebanov and Tseytlin (1.4) for the entropy density we find the strong coupling expansion for the ratio $\eta/s$
\[ \frac{\eta}{s} = \frac{1}{4\pi} (1 + 135\gamma) \quad (4.6) \]
Thus for large 't Hooft coupling $g_{YM}^2 N_c \gg 1$ the correction to the ratio of shear viscosity to the entropy density in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory is positive,
\[ \frac{\eta}{s} = \frac{1}{4\pi} \left( 1 + \frac{135}{8} \zeta(3) (g_{YM}^2 N_c)^{-3/2} + \cdots \right) \quad (4.7) \]
Formulas (1.3) and (1.7) are the main result of this paper. Thus we have shown that the conjecture of (23) that $\eta/s \geq 1/4\pi$ remains valid to the next order in the 't Hooft coupling expansion.

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\[8\] See footnote 5 regarding the gauge coupling constant normalization.
A. Coefficients of the effective action

The coefficients of the five-dimensional effective action (3.14) are given by

\[ A = \frac{\pi^4 T^4}{2u} \left[ 8f(u) + \gamma u^2 \left( -600 + 25u^2 + 1760u^4 - 1185u^6 - 44u^3w^2 \right) \right], \quad (A.1) \]

\[ B = \frac{\pi^4 T^4}{8u} \left[ 24f(u) + \gamma u^2 \left( -1800 + 179u^2 + 5424u^4 - 3387u^6 - 768u^3w^2 \right) \right], \quad (A.2) \]

\[ C = -\frac{\pi^4 T^4}{4u^2 f} \left[ 8f(u)(3 + u^2) + \gamma u^2 \left( 600 - 825u^2 - 16945u^4 + 34005u^6 - 16835u^8 \right. \right. \]
\[ \left. + 104u^2w^2 + 872u^5w^2 \right] \right], \quad (A.3) \]

\[ D = \frac{\pi^4 T^4}{8u^3 f^2} \left[ 16f(u)^2 + 8uf(u)w^2 + \gamma u^3 \left( 250u + 1570u^5 + 2220u^7 - 1920u^9 \right. \right. \]
\[ \left. + 600w^2 + 25u^2w^2 - 2400u^4w^2 + 1007u^6w^2 + 8u^3(37w^4 - 265) \right] \right], \quad (A.4) \]

\[ E = \gamma 37 \frac{\pi^4 T^4 u^5 f(u)^2}{}, \quad (A.5) \]

\[ F = \gamma 2\pi^4 T^4 u^4 f(u) (11 - 37u^2). \quad (A.6) \]

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