BILATERAL ESTIMATES OF SOLUTIONS TO QUASILINEAR ELLIPTIC EQUATIONS WITH SUB-NATURAL GROWTH TERMS

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Abstract. We study quasilinear elliptic equations of the type 
\[-\Delta_p u = \sigma u^q + \mu \quad \text{in} \quad \mathbb{R}^n,\]
in the case \(0 < q < p - 1\), where \(\mu\) and \(\sigma\) are nonnegative measurable functions, or locally finite measures, and \(\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)\) is the \(p\)-Laplacian. Similar equations with more general local and nonlocal operators in place of \(\Delta_p\) are treated as well.

We obtain existence criteria and global bilateral pointwise estimates for all positive solutions \(u\):
\[u(x) \approx (W_p \sigma(x))^{\frac{p}{p-q-1}} + K_{p,q} \sigma(x) + W_p \mu(x), \quad x \in \mathbb{R}^n,\]
where \(W_p\) and \(K_{p,q}\) are, respectively, the Wolff potential and the intrinsic Wolff potential, with the constants of equivalence depending only on \(p, q\) and \(n\).

The contributions of \(\mu\) and \(\sigma\) in these pointwise estimates are totally separated, which is a new phenomenon even when \(p = 2\). In the homogeneous case \(\mu = 0\), such estimates were obtained earlier by a different method only for minimal positive solutions.

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1. Introduction

We present a new approach to pointwise estimates of solutions to quasilinear elliptic equations of the type

$$\begin{cases}
-\Delta_p u = \sigma u^q + \mu, & u \geq 0 \text{ in } \mathbb{R}^n, \\
\liminf_{x \to \infty} u(x) = 0,
\end{cases}$$

where $\mu, \sigma \geq 0$ are locally integrable functions, or Radon measures (locally finite) in $\mathbb{R}^n$, in the sub-natural growth case $0 < q < p - 1$.

In this paper, all solutions $u$ (possibly unbounded) are understood to be $p$-superharmonic (or equivalently locally renormalized) solutions (see [KKT]). We will assume that $u \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$, so that the right-hand side of (1.1) is a Radon measure.

We will obtain matching upper and lower estimates of solutions in terms of nonlinear potentials defined below. Our estimates hold for all $p$-superharmonic solutions $u$. In particular, they yield an existence criterion for solutions to (1.1).

In the special case $\mu = 0$, i.e.,

$$\begin{cases}
-\Delta_p u = \sigma u^q, & u \geq 0 \text{ in } \mathbb{R}^n, \\
\liminf_{x \to \infty} u(x) = 0,
\end{cases}$$

considered earlier in [CV1], the upper pointwise estimate was obtained only for the minimal solution $u$. Our proofs are new even in this case.

When $p = 2$ and $0 < q < 1$, these sublinear elliptic equations were studied by Brezis and Kamin [BK] (see also [CV2], [SV], [QV], [V], and the literature cited there).

The case $q \geq p - 1$, which comprises Schrödinger type equations with natural growth terms when $q = p - 1$, and equations of superlinear type when $q > p - 1$, is quite different in nature (see, for instance, [JMV], [JV], [PV1], [PV2]).

We observe that in general, for the existence of a nontrivial solution $u$ to (1.1), $\sigma$ must be absolutely continuous with respect to $p$-capacity, i.e., $\sigma(K) = 0$ whenever $\text{cap}_p(K) = 0$, for any compact set $K$ in $\mathbb{R}^n$.

Here the $p$-capacity of $K$ is defined by

$$\text{cap}_p(K) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla u|^p dx : u \geq 1 \text{ on } K, \quad u \in C_0^\infty(\mathbb{R}^n) \right\}.$$  

More precisely, if $u$ is a nontrivial (super)solution to (1.2) in the case $0 < q \leq p - 1$, then (see [CV1, Lemma 3.6] for a more general estimate)

$$\sigma(K) \leq \text{cap}_p(K) \frac{q}{p-1} \left( \int_K u^q d\sigma \right)^{\frac{p-1-q}{p-1}},$$

considered earlier in [CV1], the upper pointwise estimate was obtained only for the minimal solution $u$. Our proofs are new even in this case.
for all compact sets $K \subset \mathbb{R}^n$.

Among our main tools are certain nonlinear potentials associated with (1.2). We refer to the recent survey of nonlinear potentials and their applications to PDE by Kuusi and Mingione [KuMi].

Let $M^+(\mathbb{R}^n)$ denote the class of all positive (locally finite) Radon measures on $\mathbb{R}^n$. Given a measure $\sigma \in M^+(\mathbb{R}^n)$, $1 < p < \infty$ and $0 < \alpha < \frac{n}{p}$, the Havin-Maz'ya-Wolff potential, introduced in [HM] (see also [HeWo]), is defined by

\begin{equation}
W_{\alpha,p}\sigma(x) = \int_0^\infty \left[ \frac{\sigma(B(x,t))}{t^{n-\alpha p}} \right]^{\frac{1}{p-1}} \frac{dt}{t}, \quad x \in \mathbb{R}^n,
\end{equation}

where $B(x,t)$ is a ball of radius $t > 0$ centered at $x \in \mathbb{R}^n$.

Nonlinear potentials $W_{\alpha,p}\sigma$, often called Wolff potentials, occur in various problems of harmonic analysis, approximation theory, Sobolev spaces, in particular spectral synthesis problems ([AH], [HM], [HeWo], [Maz]), as well as quasilinear ([KiMa], [MZ], [PV1]) and fully nonlinear PDE ([Lab], [TW1], [TW2]).

In the linear case $p = 2$, clearly $W_{\alpha,2}\sigma = I_{2\alpha}\sigma$ (up to a constant multiple), where the Riesz potential of order $\beta \in (0, n)$ is defined by

$$I_\beta\sigma(x) = \int_{\mathbb{R}^n} \frac{d\sigma(y)}{|x-y|^{n-\beta}}, \quad x \in \mathbb{R}^n.$$

In the special case $\alpha = 1$, we will be using the notation $W_p\sigma = W_{1,p}\sigma$ $(1 < p < n)$, i.e.,

\begin{equation}
W_p\sigma(x) = \int_0^\infty \left[ \frac{\sigma(B(x,t))}{t^{n-p}} \right]^{\frac{1}{p-1}} \frac{dt}{t}, \quad x \in \mathbb{R}^n.
\end{equation}

These potentials are intimately related to the equation

\begin{equation}
\begin{cases}
-\Delta_p u = \sigma, & u \geq 0 \quad \text{in} \quad \mathbb{R}^n, \\
\liminf_{x \to \infty} u(x) = 0,
\end{cases}
\end{equation}

where $\sigma \in M^+(\mathbb{R}^n)$.

The following important global estimate, along with its local counterpart, is due to T. Kilpeläinen and J. Malý [KiMa]: Suppose $u \geq 0$ is a $p$-superharmonic solution to (1.7). Then

\begin{equation}
K^{-1}W_p\sigma(x) \leq u(x) \leq KW_p\sigma(x),
\end{equation}

where $K = K(n,p)$ is a positive constant.

It is known that a nontrivial solution $u$ to (1.7) exists if and only if

\begin{equation}
\int_1^\infty \left[ \frac{\sigma(B(0,t))}{t^{n-p}} \right]^{\frac{1}{p-1}} \frac{dt}{t} < \infty.
\end{equation}
This is equivalent to \( W_p \sigma(x) < \infty \) for some \( x \in \mathbb{R}^n \), or equivalently quasi-everywhere (q.e.) on \( \mathbb{R}^n \). In particular, (1.9) may hold only in the case \( 1 < p < n \), unless \( \sigma = 0 \).

The following bilateral pointwise estimates of nontrivial (minimal) solutions \( u \) to (1.2) in the case \( 0 < q < p - 1 \) are fundamental to our approach (see [CV1], where the upper estimate was proved only for the minimal solution):

\[
1.10 \quad c^{-1} \left[ (W_p \sigma(x))^\frac{p-1}{p-1-q} + K_{p,q} \sigma(x) \right] \leq u(x) 
\leq c \left[ (W_p \sigma(x))^\frac{p-1}{p-1-q} + K_{p,q} \sigma(x) \right], \quad x \in \mathbb{R}^n,
\]

where \( c > 0 \) is a constant which depends only on \( p, q, \) and \( n \).

Here \( K_{p,q} \sigma \) is the so-called intrinsic nonlinear potential associated with (1.2), which was introduced in [CV1]. It is defined in terms of the localized weighted norm inequalities,

\[
1.11 \quad \left( \int_B |\varphi|^q \, d\sigma \right)^\frac{1}{q} \leq \kappa(B) \| \Delta_p \varphi \|_{L^1(\mathbb{R}^n)}^{\frac{1}{p-1}},
\]

for all test functions \( \varphi \) such that \( -\Delta_p \varphi \geq 0 \), \( \liminf_{x \to \infty} \varphi(x) = 0 \). Here \( \kappa(B) \) denotes the least constant in (1.11) associated with the measure \( \sigma_B = \sigma|_B \) restricted to a ball \( B = B(x,t) \). Then the intrinsic nonlinear potential \( K_{p,q} \sigma \) is defined by

\[
1.12 \quad K_{p,q} \sigma(x) = \int_0^\infty \left[ \kappa(B(x,t))^{\frac{p(q-1)}{t^{n-p}}} \right]^{\frac{1}{p-1-q}} \frac{dt}{t}, \quad x \in \mathbb{R}^n.
\]

As was noticed in [CV1], \( K_{p,q} \sigma \not\equiv +\infty \) if and only if

\[
1.13 \quad \int_1^\infty \left[ \kappa(B(0,t))^{\frac{p(q-1)}{t^{n-p}}} \right]^{\frac{1}{p-1-q}} \frac{dt}{t} < \infty.
\]

Consequently, a nontrivial \( p \)-superharmonic solution \( u \) to (1.2) exists if and only if both \( K_{p,q} \sigma \not\equiv +\infty \) and \( W_p \sigma \not\equiv +\infty \), that is, both (1.9) and (1.13) hold.

For the existence of a nontrivial solution to equation (1.1), we need to add the condition \( W_p \mu \not\equiv +\infty \), i.e.,

\[
1.14 \quad \int_1^\infty \left[ \mu(B(0,t)) \right]^{\frac{1}{p-1-q}} \frac{dt}{t} < \infty.
\]

In this paper, we obtain the following the following criterion for existence, along with global bilateral estimates of solutions to (1.1).
Theorem 1.1. Let \( 1 < p < n, 0 < q < p - 1, \) and \( \mu, \sigma \in M^+(\mathbb{R}^n) \).
There exists a nontrivial solution \( u \) to (1.1) if and only if conditions (1.9), (1.13), and (1.14) hold. Then any nontrivial solution \( u \) satisfies the estimates
\[
\begin{align*}
C_1 \left[ (W_p \sigma(x))^{\frac{p-1}{p-1-q}} + K_{p,q} \sigma(x) + W_p \mu(x) \right] & \leq u(x) \\
\leq C_2 \left[ (W_p \sigma(x))^{\frac{p-1}{p-1-q}} + K_{p,q} \sigma(x) + W_p \mu(x) \right], \quad x \in \mathbb{R}^n.
\end{align*}
\]
where the positive constants \( C_1, C_2 \) depend only on \( p, q, \) and \( n. \)

If \( n \leq p < \infty, \) then there are no nontrivial solutions to (1.1).

The following corollary is deduced from Theorem 1.1 under the additional assumption that there exists a constant \( C = C(\sigma, p, n) \) so that
\[
(1.16) \quad \sigma(K) \leq C \text{cap}_K(K), \quad \text{for all compact sets } K \subset \mathbb{R}^n.
\]
We remark that condition (1.16) is also essential in the natural growth case \( q = p - 1 \) (see, for instance, [JMV]).

Corollary 1.2. Let \( 1 < p < n, 0 < q < p - 1, \) and \( \mu, \sigma \in M^+(\mathbb{R}^n). \)

If condition (1.16) holds, then any positive solution \( u \) to (1.1) satisfies the estimates
\[
\begin{align*}
C_1 \left[ (W_p \sigma(x))^{\frac{p-1}{p-1-q}} + W_p \mu(x) \right] & \leq u(x) \\
\leq C_2 \left[ (W_p \sigma(x))^{\frac{p-1}{p-1-q}} + W_p \sigma(x) + W_p \mu(x) \right], \quad x \in \mathbb{R}^n.
\end{align*}
\]
where \( C_1, C_2 \) are positive constants that depend only on \( p, q, n, \) and the constant \( C \) in (1.16) (in the case of \( C_2 \)).

In the special case \( \mu = 0, \) the Brezis–Kamin type pointwise estimates
\[
(1.18) \quad C_1 (W_p \sigma(x))^{\frac{p-1}{p-1-q}} \leq u(x) \leq C_2 \left[ (W_p \sigma(x))^{\frac{p-1}{p-1-q}} + W \sigma(x) \right],
\]
under the assumption (1.16) were obtained in [CV2] (the upper estimate was proved only for the minimal solution). For bounded solutions \( u, \) the term \( (W_p \sigma(x))^{\frac{p-1}{p-1-q}} \) on the right-hand side of (1.18) is redundant. This estimate in the case \( p = 2 \) was originally obtained in [BK].

Our main results are deduced via pointwise estimates of solutions to the fractional integral equation
\[
(1.19) \quad u = W_{\alpha,p}(u^q \sigma) + W_{\alpha,p} \mu, \quad u \geq 0, \quad \text{in } \mathbb{R}^n,
\]
where \( 0 < q < p - 1, \) \( 0 < \alpha < \frac{n}{p}, \) and \( \mu, \sigma \in M^+(\mathbb{R}^n). \)

Bilateral pointwise estimates of solutions to (1.19), similar to (1.10), are given in terms of nonlinear potentials \( W_{\alpha,p} \) and fractional intrinsic potentials \( W_{\alpha,p,q} \) defined in Sec. 2. In the definition of \( W_{\alpha,p,q} \), which is similar to (1.12) in the case \( \alpha = 1, \) we employ localized embedding
constants $\kappa(B)$ associated with $\sigma_B$ for balls $B = B(x,r)$, which are related to certain weighted norm inequalities for potentials $W_{\alpha,p}$.

In the special case $p = 2$, $0 < q < 1$, $0 < 2\alpha < n$, we obtain an analogue of Theorem 1.1 for the fractional Laplace problem

$$(1.20) \begin{cases} (-\Delta)^\alpha u = \sigma u^q + \mu, & u \geq 0 \text{ in } \mathbb{R}^n, \\ \liminf_{x \to \infty} u(x) = 0. \end{cases}$$

Our results on solutions to (1.19) demonstrate (see Sec. 4 below) that Theorem 1.1 remains valid for more general quasilinear operators $\text{div} A(x, \nabla u)$ in place of $\Delta_p$, under standard boundedness and monotonicity assumptions on $A(x, \xi)$ (with $\alpha = 1$, $0 < q < p - 1$), as well as for $k$-Hessian operators (with $\alpha = \frac{2k}{k+1}$, $p = k + 1$ and $0 < q < k$). The relation between equations (1.19) and the corresponding elliptic PDE is provided by the nonlinear potential theory developed in [KuMi], [Lab], [TW2].

If $q \geq p - 1$ for the quasilinear equations, or $q \geq k$ for the $k$-Hessian equations, the existence results and pointwise estimates of solutions differ greatly from Theorem 1.1. They were obtained earlier in [JV], [PV1], [PV2].

This paper is organized as follows. In Sec. 2, we recall definitions of the nonlinear potentials $W_{\alpha,p}$ and $K_{\alpha,p,q}$, and discuss some of their properties. Pointwise estimates of sub- and super-solutions of the homogeneous equation (1.2) are discussed in Sec. 3. They are extended to the non-homogeneous equation (1.1) in Sec. 4, where we prove Theorem 1.1, and its analogues for equation (1.19).

### 2. Nonlinear potentials

Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, and $0 < q < p - 1$. Let $\sigma \in M^+(\mathbb{R}^n)$. For the sake of simplicity, the nonlinear potential $W_{\alpha,p}\sigma$ defined in the Introduction will be denoted by $W\sigma$, i.e.,

$$(2.1) \quad W\sigma(x) = \int_0^\infty \left[ \frac{\sigma(B(x,t))}{t^{n-\alpha p}} \right]^{\frac{1}{p-1}} \frac{dt}{t}, \quad x \in \mathbb{R}^n.$$ 

We denote by $\kappa$ the least constant in the weighted norm inequality

$$(2.2) \quad \|W\nu\|_{L^q(\mathbb{R}^n, d\sigma)} \leq \kappa \nu(\mathbb{R}^n)^{\frac{1}{p-1}}, \quad \forall \nu \in M^+(\mathbb{R}^n).$$ 

We will also need a localized version of (2.2) for $\sigma_E = \sigma|_E$, where $E$ is a Borel subset of $\mathbb{R}^n$, and $\kappa(E)$ is the least constant in

$$(2.3) \quad \|W\nu\|_{L^q(d\sigma_E)} \leq \kappa(E) \nu(\mathbb{R}^n)^{\frac{1}{p-1}}, \quad \forall \nu \in M^+(\mathbb{R}^n).$$
In applications, it will be enough to use \( \kappa(E) \) where \( E = Q \) is a dyadic cube, or \( E = B \) is a ball in \( \mathbb{R}^n \).

It is easy to see using estimates (1.8) that embedding constants \( \kappa(B) \) in the case \( \alpha = 1 \) are equivalent to the constants \( \kappa(B) \) in (1.11).

We define the intrinsic potential of Wolff type \( K_\sigma = K_{\alpha,p,q} \sigma \) in terms of \( \kappa(B(x,t)) \), the least constant in (2.3) with \( E = B(x,t) \):

\[
K_\sigma(x) = \int_0^\infty \left[ \kappa(B(x,t)) \frac{a^{\frac{p(q-1)}{p-1-q}}}{t^{n-\alpha p}} \right]^{\frac{1}{q-1}} \frac{dt}{t}, \quad x \in \mathbb{R}^n. 
\]

Notice that \( K_{\alpha,p,q} \sigma(x) \approx K_{p,q} \sigma(x) \) in the case \( \alpha = 1 \), with the equivalence constants that depend only on \( p, q \), and \( n \) (see [CV1]). It is easy to see that \( K_\sigma(x) \not\equiv \infty \) if and only if

\[
\int_a^\infty \left[ \kappa(B(0,t)) \frac{a^{\frac{p(q-1)}{p-1-q}}}{t^{n-\alpha p}} \right]^{\frac{1}{q-1}} \frac{dt}{t} < \infty,
\]

for any (equivalently, all) \( a > 0 \). This is similar to the condition \( W_{\alpha,p} \sigma(x) \not\equiv \infty \), equivalent to (see, for instance, [CV1, Corollary 3.2])

\[
\int_a^\infty \left[ \sigma(B(0,t)) \right]^{\frac{1}{p-1}} \frac{dt}{t} < \infty.
\]

3. Homogeneous Equations

Let \( 1 < p < \infty \), \( 0 < \alpha < \frac{n}{p} \), and \( 0 < q < p - 1 \). Let us fix \( \sigma \in M^+(\mathbb{R}^n) \). We start with some estimates of solutions to the equation

\[
(3.1) \quad u(x) = W(u^q d\sigma)(x), \quad u \geq 0, \quad x \in \mathbb{R}^n,
\]

where \( u < \infty \) \( d\sigma \)-a.e. (or equivalently \( u \in L^q_{\text{loc}}(\mathbb{R}^n, \sigma) \)). Equation (3.1) can also be considered pointwise at every \( x \in \mathbb{R}^n \) where \( u(x) = W(u^q d\sigma)(x) < \infty \).

We also treat the corresponding subsolutions \( u \geq 0 \) such that

\[
(3.2) \quad u(x) \leq W(u^q d\sigma)(x) < \infty, \quad x \in \mathbb{R}^n,
\]

and supersolutions \( u \geq 0 \) such that

\[
(3.3) \quad W(u^q d\sigma)(x) \leq u(x) < \infty, \quad x \in \mathbb{R}^n,
\]

considered either \( d\sigma \)-a.e., or at every \( x \in \mathbb{R}^n \) where these inequalities hold.

For any \( \nu \in M^+(\mathbb{R}^n) (\nu \neq 0) \) such that \( W\nu \not\equiv \infty \), we set

\[
(3.4) \quad \phi_\nu(x) := W\nu(x) \left( \frac{W[(W\nu)^q d\sigma](x)}{W\nu(x)} \right)^{\frac{p-1}{p-q}}, \quad x \in \mathbb{R}^n,
\]
where we assume that $W_\nu(x) < \infty$.

Next, for $x \in \mathbb{R}^n$, we set

$$\phi(x) := \sup \{ \phi_\nu(x) : \nu \in M^+(\mathbb{R}^n), \nu \neq 0, W_\nu(x) < \infty \}.$$ 

**Theorem 3.1.** Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, and $0 < q < p - 1$. Let $\sigma \in M^+(\mathbb{R}^n)$. Then any nontrivial solution $u \geq 0$ to (3.1) satisfies the estimates

$$C \phi(x) \leq u(x) \leq \phi(x), \quad x \in \mathbb{R}^n,$$

where $C$ is a positive constant which depends only on $p$, $q$, $\alpha$ and $n$.

Moreover, the upper bound in (3.6) holds for any subsolution $u$, whereas the lower bound in (3.6) holds for any nontrivial supersolution $u$.

If $n \leq p < \infty$, then there are no nontrivial solutions to (1.1).

The proof of Theorem 3.1 is based on a series of lemmas.

**Lemma 3.2.** Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, and $0 < q < p - 1$. Let $\nu, \sigma \in M^+(\mathbb{R}^n)$. Suppose $u$ is a subsolution to (3.1). Then

$$u(x) \leq \phi(x), \quad x \in \mathbb{R}^n,$$

provided $W(u^q d\sigma)(x) < \infty$. In particular, (3.7) holds $d\sigma$-a.e.

*Proof.* Setting $d\nu = u^q d\sigma$, we see that

$$W(V)^q d\sigma)(x) = \int_0^\infty \left[ W(x,t) d\sigma(y) \right]^{p-1} dt,$$

which yields immediately (3.7). \qed

**Lemma 3.3.** Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, and $0 < q < p - 1$. Let $\nu, \sigma \in M^+(\mathbb{R}^n)$. Then there exists a positive constant $C$ which depends only on $p$, $q$, $\alpha$, and $n$ such that

$$W[(W\nu)^q d\sigma](x) \leq C (W\nu(x))^{\frac{n}{p-1}} \times \left[ W\sigma(x) + (K\sigma(x))^{\frac{n}{p-1}} \right], \quad x \in \mathbb{R}^n.$$

*Proof.* Without loss of generality we may assume that $\nu \neq 0$ and $W\nu(x) < \infty$. For $x \in \mathbb{R}^n$, we have

$$W[(W\nu)^q d\sigma](x) = \int_0^\infty \left[ \int_{B(x,t)} (W(\nu)^q d\sigma)(y) \right]^{\frac{p-1}{p}} \frac{dt}{t},$$

where

$$W_\nu(x) := \sup \{ W_\nu(x) : \nu \in M^+(\mathbb{R}^n), \nu \neq 0, W_\nu(x) < \infty \}.$$
For \( y \in B(x, t) \), we have that \( B(y, r) \subset B(x, 2t) \) if \( 0 < r \leq t \), and \( B(y, r) \subset B(x, 2r) \) if \( r > t \). Consequently, for \( y \in B(x, t) \),

\[
W_\nu(y) = \int_0^t \left[ \nu(B(y, r)) \right]^{\frac{1}{p-1}} \frac{dr}{r} + \int_t^\infty \left[ \nu(B(x, 2r)) \right]^{\frac{1}{p-1}} \frac{dr}{r}
\]

\[
\leq \int_0^t \left[ \frac{\nu(B(y, r) \cap B(x, 2t))}{r^{n-\alpha p}} \right]^{\frac{1}{p-1}} \frac{dr}{r} + \int_t^\infty \left[ \frac{\nu(B(x, 2r))}{r^{n-\alpha p}} \right]^{\frac{1}{p-1}} \frac{dr}{r}
\]

\[
\leq W_{\nu B(x, 2t)}(y) + c W_\nu(x),
\]

were \( c = 2^{\frac{n-\alpha p}{p-1}} \). Hence,

\[
\int_{B(x,t)} (W_\nu(y))^q d\sigma(y) \leq \int_{B(x,t)} (W_{\nu B(x,2t)})^q d\sigma(y) + c^q (W_\nu(x))^q \sigma(B(x,t)).
\]

Notice that by (2.3),

\[
\int_{B(x,t)} (W_{\nu B(x,2t)})^q d\sigma(y) \leq \kappa(B(x,t))^q \nu(B(x, 2t))^{\frac{q}{p-1}}.
\]

Combining the preceding estimates, we deduce

\[
\int_{B(x,t)} (W_\nu(y))^q d\sigma(y) \leq \kappa(B(x,t))^q \nu(B(x, 2t))^{\frac{q}{p-1}} + c^q (W_\nu(x))^q \sigma(B(x,t)).
\]

It follows from (3.9) and the preceding estimate,

\[
W[(W_\nu)^q d\sigma](x)
\]

\[
\leq c \int_0^\infty \left[ \frac{\kappa(B(x, t))^q \nu(B(x, 2t))^{\frac{q}{p-1}}}{t^{n-\alpha p}} \right]^{\frac{1}{p-1}} \frac{dt}{t} + c (W_\nu(x))^{\frac{q}{p-1}} \int_0^\infty \left[ \frac{\sigma(B(x, t))}{t^{n-\alpha p}} \right]^{\frac{1}{p-1}} \frac{dt}{t}
\]

\[
= c (I + II),
\]

where \( c = c(p, q, n) \).
By Hölder’s inequality with exponents \( \frac{p-1}{p-1-q} \) and \( \frac{p-1}{q} \), we estimate

\[
I = \int_0^\infty \left[ \frac{\kappa(B(x,t))q \nu(B(x,2t))^{q \frac{1}{p-1}}}{t^{n-\alpha p}} \right]^{\frac{p-1}{q}} \frac{dt}{t} \\
\leq \left( \int_0^\infty \left[ \frac{\nu(B(x,2t))}{t^{n-\alpha p}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \right)^{\frac{1}{q}} \left( \int_0^\infty \left[ \frac{\kappa(B(x,t))^{\frac{q(p-1)}{p-1-q}}}{t^{n-\alpha p}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \right)^{\frac{p-1-q}{p-1}} \\
\leq 2 \left( W\nu(x) \right)^{\frac{q}{p-1}} \left( K\sigma(x) \right)^{\frac{p-1-q}{p-1}}.
\]

Clearly,

\[
II = (W\nu(x))^{\frac{q}{p-1}} \int_0^\infty \left[ \frac{\sigma(B(x,t))}{t^{n-\alpha p}} \right]^{\frac{1}{p-1}} \frac{dt}{t} = (W\nu(x))^\frac{q}{p-1} W\sigma(x).
\]

We deduce

\[
W(W\nu)_d\sigma](x) \leq c(I + II) \\
\leq c \left( W\nu(x) \right)^{\frac{q}{p-1}} \left[ W\sigma(x) + (K\sigma(x))^{\frac{p-1-q}{p-1}} \right].
\]

This completes the proof of (3.8). \( \square \)

**Lemma 3.4.** Let \( 1 < p < \infty \), \( 0 < \alpha < \frac{n}{p} \), and \( 0 < q < p - 1 \). Let \( \sigma \in M^+(\mathbb{R}^n) \). Then there exist positive constants \( C_1, C_2 \) which depend only on \( p, q, \alpha \) and \( n \) such that

\[
(3.10) \quad C_1 \phi(x) \leq (W\sigma(x))^{\frac{p-1}{p-1-q}} + K\sigma(x) \leq C_2 \phi(x),
\]

where the lower estimate holds at all \( x \in \mathbb{R}^n \), whereas the upper estimate holds provided \( W\sigma(x) < \infty \) and \( K\sigma(x) < \infty \).

**Remark 3.5.** The assumptions \( W\sigma(x) < \infty \) and \( K\sigma(x) < \infty \) in Lemma 3.4 can be replaced with \( W\sigma \neq \infty \) and \( K\sigma \neq \infty \); then \( \phi(x) < \infty \) \( d\sigma \text{-a.e.} \), and (3.10) holds \( d\sigma \text{-a.e.} \). Moreover, the assumption \( W\sigma(x) < \infty \) in Lemma 3.4 can be dropped altogether as shown below.

**Proof of Lemma 3.4.** Let \( \nu \in M^+(\mathbb{R}^n) \), \( \nu \neq 0 \). Suppose \( W\nu(x) < \infty \). Raising both sides of (3.8) to the power \( \frac{p-1}{p-1-q} \) and multiplying by
\(W\nu(x)\), we obtain,

\[
\phi_{\nu}(x) := W\nu(x) \left( \frac{W[(W\nu)^q d\sigma](x)}{W\nu(x)} \right)^{\frac{p-1}{p-1-q}} \\
\leq C^{\frac{p-1}{p-1-q}} \left[ W\sigma(x) + (K\sigma(x))^\frac{p-1-2q}{p-1-q} \right]^{\frac{p-1}{p-1-q}}.
\]

The lower estimate in (3.10) follows immediately from the preceding inequality.

To prove the upper estimate in (3.10), notice that, since \(W\sigma(x) \neq \infty\) and \(K\sigma(x) \neq \infty\), it follows by \([CV1, \text{Theorem 4.8}]\) that there exists a (minimal) solution \(u\) to (3.1) such that

\[
c_1 \left[ (W\sigma(x))^{\frac{p-1}{p-1-q}} + K\sigma(x) \right] \leq u(x) \\
\leq c_2 \left[ (W\sigma(x))^{\frac{p-1}{p-1-q}} + K\sigma(x) \right], \quad x \in \mathbb{R}^n.
\]

Here \(c_1, c_2\) are positive constants which depend only on \(p, q, \alpha\) and \(n\), and (3.11) holds at \(x\) provided \(W\sigma(x) < \infty\) and \(K\sigma(x) < \infty\). Moreover, in this case \(u(x) = W(u^q d\sigma)(x) < \infty\). Thus, by Lemma 3.2 and the lower bound in (3.11), we have

\[
c_1 \left[ (W\sigma(x))^{\frac{p-1}{p-1-q}} + K\sigma(x) \right] \leq u(x) \leq \phi(x).
\]

The proof of Lemma 3.4 is complete. \(\square\)

**Proof of Remark 3.5.** If \(W\sigma(x) \neq \infty\) and \(K\sigma \neq \infty\), then as indicated in the above proof, there exists a solution \(u\) to (3.1) such that \(u = W(u^q d\sigma) < \infty\) \(d\sigma\)-a.e., and (3.11) holds \(d\sigma\)-a.e. In particular, \(W\sigma(x) \neq \infty\) and \(K\sigma \neq \infty\) \(d\sigma\)-a.e. Letting \(d\nu = u^q d\sigma\), we deduce \(u \leq \phi_{\nu} \leq \phi d\sigma\)-a.e., so that (3.10) holds \(d\sigma\)-a.e. as well.

Let us assume for a moment that \(W\sigma(x) < \infty\). Then, letting \(\nu = \sigma\) in the definition of \(\phi_{\nu}\), we deduce by \([CV1, \text{Lemma 3.5}]\) with \(r = q\),

\[
W[(W\sigma)^q d\sigma](x) \geq c (W\sigma(x))^{\frac{q}{p-1}+1},
\]

where \(c\) is a positive constant which depends only on \(p, q, \alpha\) and \(n\). Hence,

\[
\phi(x) \geq \phi_\sigma(x) = W\sigma(x) \left( \frac{W[(W\sigma)^q d\sigma](x)}{W\sigma(x)} \right)^{\frac{p-1}{p-1-q}} \\
\geq c (W\sigma(x))^{\frac{p-1}{p-1-q}}.
\]

Next, we observe that in the proof of the upper estimate in (3.10), we may assume without loss of generality that \(W\sigma \neq \infty\). Otherwise, we may replace \(\sigma\) with \(\sigma_{B(0,R)}\) for any \(R > 0\). Then clearly \(W\sigma_{B(0,R)} \neq \infty\),
and by the argument presented above (applied to $\sigma_{B(0,R)}$ in place of $\sigma$), we see that $\phi(x) \geq c(W\sigma_{B(0,R)}(x))^{\frac{p-1}{p+1}}$. Letting $R \to \infty$ and using the monotone convergence theorem, we see that the right-hand side tends to $\infty$ at every $x \in \mathbb{R}^n$, which forces $\phi \equiv \infty$.

Finally, if $W\sigma(x) = \infty$, but $W\sigma \not\equiv \infty$, we may consider $W\sigma_k$, where $\sigma_k$ is the $p$-measure $-\Delta_p v_k = \sigma_k$, so that $v_k \approx W\sigma_k$, with $v_k = \min(v,k)$ where $-\Delta_p v = \sigma$ and $v \approx W\sigma$. Notice that $W\sigma_k(x) = k$.

Then, clearly,

$$\phi_{\sigma_k}(x) = k^{-\frac{q}{p-1}} (W[(W\sigma_k)^q d\sigma](x))^{\frac{p-1}{p}}.$$  

For $k > 0$, we set $E_k = \{y : W\sigma(y) \geq k\}$, so that $W\sigma_{E_k}(y) = W\sigma(y)$ for $y \in E_k$. We estimate

$$W[(W\sigma_k)^q d\sigma](x) \geq k^{\frac{q}{p}} W\sigma_{E_k}(x).$$

Thus,

$$\phi(x) \geq \phi_{\sigma_k}(x) \geq (W\sigma_{E_k}(x))^{\frac{p-1}{p}}.$$  

Letting $k \to 0$, we see by the monotone convergence theorem that

$$\phi(x) \geq \phi_{\sigma_k}(x) \geq (W\sigma(x))^{\frac{p-1}{p}} = \infty.$$  

In other words, the assumption $W\sigma(x) < \infty$ in Lemma 3.4 is redundant, and actually follows from the fact that $\phi(x) < \infty$.

\[\square\]

**Proof of Theorem 3.1.** The upper bound in (3.6) for any subsolution $u$ follows from Lemma 3.2, whereas the lower bound for any nontrivial supersolution $u$ is a consequence of Lemma 3.3 and (3.11).

As a consequence of the preceding results, we obtain the following corollary.

**Corollary 3.6.** Under the assumptions of Theorem 3.1, there exist positive constants $C_1, C_2$ which depend only on $p, q, \alpha$ and $n$ such that

$$C_1 \left( (W\sigma(x))^{\frac{p-1}{p+1}} + K\sigma(x) \right) \leq u(x) \leq C_2 \left( (W\sigma(x))^{\frac{p-1}{p+1}} + K\sigma(x) \right),$$

for any solution $u$ to (3.1). Moreover, the lower estimate holds for any supersolution $u$ such that (3.3) holds at $x \in \mathbb{R}^n$, whereas the upper estimate holds for any subsolution $u$ such that (3.2) holds at $x \in \mathbb{R}^n$, and also $d\sigma$-a.e.

**Remark 3.7.** The upper estimate for $u$ in (3.12) was proved earlier in [CV1] only for the nontrivial minimal solution to (3.1), together with the lower estimate for any supersolution.
4. Non-homogeneous equations

In this section, we deduce estimates for sub- and super-solutions to the equation

\[(4.1) \quad u = W(u^q d\sigma) + W_\mu, \quad u \geq 0 \quad \text{in} \; \mathbb{R}^n,\]

in the case \(0 < q < p - 1\) which immediately yields the corresponding estimates to solutions to (1.1) via the Wolff potential estimates (1.8). The case \(\mu = 0\) was considered in Sec. 3, so we assume here that \(\mu \neq 0\).

In particular, all solutions \(u\) to (4.1) are nontrivial: \(u \geq W_\mu > 0\), and \(u < \infty \) \(d\sigma\)-a.e. (or q.e.) in \(\mathbb{R}^n\).

**Theorem 4.1.** Let \(1 < p < \infty, 0 < \alpha < \frac{n}{p}, \) and \(0 < q < p - 1\). Let \(\sigma, \mu \in M^+(\mathbb{R}^n)\). Then there exist positive constants \(C_1, C_2\) which depend only on \(p, q, \alpha\) and \(n\) such that any nonnegative solution \(u\) to (4.1) satisfies the estimates

\[(4.2) \quad C_1 \left[ (W \sigma(x))^{\frac{\alpha}{p-1}} + K \sigma(x) + W_\mu(x) \right] \leq u(x) \leq C_2 \left[ (W \sigma(x))^{\frac{\alpha}{p-1}} + K \sigma(x) + W_\mu(x) \right], \quad x \in \mathbb{R}^n,\]

where the upper estimate holds at every \(x\) where \(u(x) < \infty\), and consequently \(d\sigma\)-a.e. and q.e.

Moreover, the lower estimate in (4.2) holds for every supersolution \(u\) at every \(x \in \mathbb{R}^n\) such that

\[(4.3) \quad W(u^q d\sigma)(x) + W_\mu(x) \leq u(x) < \infty,\]

whereas the upper estimate holds for every subsolution \(u\) at every \(x \in \mathbb{R}^n\) such that

\[(4.4) \quad u(x) \leq W(u^q d\sigma)(x) + W_\mu(x) < \infty.\]

**Proof.** The case \(\mu = 0\) is considered in Sec. 3, so we may assume without loss of generality that \(\mu = 0\). Consequently, \(u(x) \geq W_\mu(x) > 0\) at every \(x \in \mathbb{R}^n\). Clearly, any supersolution to (4.1) is also a supersolution to (3.1). Hence, by Theorem 3.1, there exists a positive constant \(c = c(p, q, \alpha, n)\) such that \(u(x) \geq c \left[(W \sigma(x))^{\frac{\alpha}{p-1}} + K \sigma(x)\right]\). These two lower estimates combined yield the lower bound in (4.2) with \(C_1 = C_1(p, q, \alpha, n) > 0\).

To prove the upper bound, for any subsolution \(u\) to (4.2), we fix \(x \in \mathbb{R}^n\) such that \(u(x) \leq W(u^q d\sigma)(x) + W_\mu(x) < \infty\). Notice that if \(u\) is a solution to (4.2), then this is equivalent to \(u(x) < \infty\).

Letting \(d\omega = u^q d\sigma + d\mu\) and \(c_1 = \max \left(1, 2^{\frac{q}{p-2}}\right)\), we obviously have \(u(x) \leq c_1 W\omega(x) < \infty\) at \(x\) and \(d\sigma\)-a.e. Letting \(c_2 = \max \left(1, 2^{\frac{1}{2-p}}\right)\),
we estimate
\[ W_\omega(x) = W(u^g d\sigma + d\mu)(x) \]
\[ \leq c_2 W(u^g d\sigma)(x) + c_2 W\mu(x) \]
\[ \leq c_1 c_2 [(W_\omega)^g d\sigma](x) + c_2 W\mu(x). \]

By Lemma 3.3 with \( \omega \) in place of \( \nu \), we have
\[ W[(W_\omega)^g d\sigma](x) \leq C \left( (W_\omega(x))^\frac{q}{p-1} \left( (W_\sigma(x))^\frac{p-1}{q-1} + K\sigma(x) \right)^\frac{p-1-q}{p-1} \right), \]
where \( C = C(p, q, \alpha, n) \) is a positive constant. Combining the preceding estimates we deduce
\[ W_\omega(x) \leq c_1 c_2 \left( (W_\omega(x))^\frac{q}{p-1} \left( (W_\sigma(x))^\frac{p-1}{q-1} + K\sigma(x) \right)^\frac{p-1-q}{p-1} \right) \]
\[ + c_2 W\mu(x). \]

Using Young’s inequality with exponents \( \frac{p-1}{q} \) and \( \frac{p-1}{p-1-q} \) in the first term on the right-hand side, we estimate
\[ W_\omega(x) \leq \frac{1}{2} W_\omega(x) + C' \left( (W_\sigma(x))^\frac{p-1}{q-1} + K\sigma(x) \right) + c_2 W\mu(x), \]
where \( C' \) is a positive constant which depends only on \( p, q, \alpha \) and \( n \). Since \( W_\omega(x) < \infty \), we can move the first term on the right to the left-hand side, and obtain
\[ u(x) \leq c_1 W_\omega(x) \leq C_2 \left( (W_\sigma(x))^\frac{p-1}{q-1} + K\sigma(x) + W\mu(x) \right), \]
where \( C_2 \) is a positive constant which depends only on \( p, q, \alpha \) and \( n \). This completes the proof of the upper estimate in (4.2).

\[ \text{Proof of Theorem 1.1.} \]

Let \( d\omega = u^g d\sigma + d\mu \), where \( u \) is a solution to (1.1). Then by (1.8),
\[ (4.5) \quad K^{-1} W_p \omega(x) \leq u(x) \leq K W_p \omega(x), \]
where \( K = K(n, p) \) is a positive constant. Hence, \( u \) is a supersolution satisfying \( u \geq W_p(u^g d\sigma) + W_p \mu \), with \( \mu = c_1 \mu \) and \( \sigma = c_2 \sigma \), where \( c_1, c_2 \) depend only on \( p, q, \) and \( K \). Hence the lower estimate (1.15) of Theorem 1.1 follows from the lower estimate (4.2) of Theorem 4.1 in the special case \( \alpha = 1 \). Similarly, the upper estimate in (1.15) is deduced from the upper estimates in (4.2) and (4.5).

If a nontrivial solution \( u \) to (1.1) exists, then by the lower estimate (1.15) of Theorem 1.1 it follows that \( W_p \mu \not\equiv \infty \), \( W_p \sigma \not\equiv \infty \), and \( K_p \sigma \not\equiv \infty \), which are equivalent to conditions (1.14), (1.9), and (1.13), respectively.
Conversely, suppose that these three conditions hold. In the special case \( \mu = 0 \), a positive \( p \)-superharmonic solution \( u \in L^q_{\text{loc}}(\mathbb{R}^n) \) was constructed in [CV1, Theorem 1.1] by iterations, \( u = \lim_{j \to \infty} u_j \), where \( u_j \) is an nondecreasing sequence of \( p \)-superharmonic functions such that

\[
- \Delta_p u_{j+1} = \sigma u_j^q + \mu \quad \text{in} \; \mathbb{R}^n, \quad j = 0, 1, 2, \ldots,
\]

with an appropriate choice of \( u_0 \), namely \( u_0 = c \) \( (W_p \sigma)^{p-1-q} \), where \( c = c(p, q, n) \) is a small constant.

If \( \mu \neq 0 \), a similar iteration argument can be used with \( u_0 = 0 \) based on [PV2, Lemma 3.7 and Lemma 3.9], so that \( u_j \) satisfying (4.6) is an nondecreasing sequence of \( p \)-superharmonic functions. This part of the construction works for any \( q > 0 \) and \( p > 1 \) (see the proof of Theorem 3.10 in [PV2] for \( q > p - 1 \)). However, the way we control the growth of \( u_j \) for \( 0 < q < p - 1 \) is very different.

Since \( u_j \leq u_{j+1} \), it follows that \( u_{j+1} \) is a subsolution, so that

\[
- \Delta_p u_{j+1} \leq \sigma u_{j+1}^q + \mu \quad \text{in} \; \mathbb{R}^n, \quad j = 0, 1, 2, \ldots,
\]

By (1.8), we have

\[
u_j \leq K W_p(\sigma u_j^q + \mu)
\leq K \max(1, 2^{\frac{2}{p-q}}) \left[ W_p(\sigma u_{j+1}^q) + W_p \mu \right].
\]

After scaling by letting \( \tilde{\mu} = \sigma^{p-1} \mu \) and \( \tilde{\sigma} = \sigma^{p-1} \sigma \), where the constant \( c = K \max(1, 2^{\frac{2}{p-q}}) \), we see that \( u_{j+1} \) is a subsolution for the corresponding integral equation (4.1), i.e.,

\[
u_{j+1} \leq W_p(\tilde{\sigma} u_{j+1}^q) + W_p \tilde{\mu}, \quad j = 0, 1, 2, \ldots.
\]

It follows by induction using Lemma 3.3 with \( \nu = \tilde{\mu} \) and \( \nu' = \tilde{\sigma} u_j^q \) that the right-hand side of (4.9) is finite at every point \( x \in \mathbb{R}^n \) where \( W_p \mu(x) < \infty \), \( W_p \sigma(x) < \infty \), and \( K_p \sigma(x) < \infty \) (which is true \( \mu \)-a.e., as we demonstrate below).

By Theorem 4.1 for subsolutions, \( u_{j+1} \) has the upper bound

\[
u_{j+1}(x) \leq C \left[ (W_p \sigma(x))^{p-1-q} + K_p \sigma(x) + W_p \mu(x) \right], \quad x \in \mathbb{R}^n,
\]

with \( C \) that depends only on \( p, q, \) and \( n \), where we switched back from \( \tilde{\mu}, \tilde{\sigma} \) to \( \mu, \sigma \).

Thus, \( u = \lim_{j \to \infty} u_j \) satisfies

\[
u(x) \leq C \left[ (W_p \sigma(x))^{p-1-q} + K_p \sigma(x) + W_p \mu(x) \right], \quad x \in \mathbb{R}^n.
\]

Moreover, by [CV1, Theorem 1.1], the conditions \( W_p \sigma \neq \infty \) and \( K_p \sigma \neq \infty \) yield the existence of a positive solution \( v \in L^q_{\text{loc}}(\mathbb{R}^n, \sigma) \).
to the homogeneous equation

\[-\Delta_p v = \sigma v^q \quad \text{in } \mathbb{R}^n,\]

so that \(v\) satisfies the lower bound

\[v \geq c \left( (W_p\sigma)^{\frac{p-1}{p-1-q}} + K_p\sigma \right),\]

where \(c > 0\) is a constant that depends only on \(p, q,\) and \(n.\) Hence, 
\((W_p\sigma)^{\frac{p-1}{p-1-q}} \in L^q_{\text{loc}}(\mathbb{R}^n, \sigma)\) and \(K_p\sigma \in L^q_{\text{loc}}(\mathbb{R}^n, \sigma).\)

To verify that \(u \in L^q_{\text{loc}}(\mathbb{R}^n, \sigma),\) in view of \((4.10),\) it remains to show that \(W_p\mu \in L^q_{\text{loc}}(\mathbb{R}^n, \sigma).\) Let \(B = B(0, R)\) and let \(\mu = \mu_{2B} + \mu_{(2B)^c}.\)

Then, as in the proof of Lemma 3.3, we clearly have for all \(x \in B,
\[
W_p\mu_{(2B)^c}(x) = \int_{0}^{\infty} \left[ \frac{\mu(B(x, t) \cap (2B)^c)}{t^{n-p}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \\
\leq \int_{R}^{\infty} \left[ \frac{\mu(B(0, 2t))}{t^{n-p}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \\
= 2^{\frac{q}{p-1}} \int_{2R}^{\infty} \left[ \frac{\mu(B(0, t))}{t^{n-p}} \right]^{\frac{1}{p-1}} \frac{dt}{t}.
\]

Hence,
\[
\int_B (W_p\mu)^q d\sigma \leq c \int_B (W_p\mu_{2B})^q d\sigma + c \int_B (W_p\mu_{(2B)^c})^q d\sigma \\
\leq c \zeta(B)^q \mu(2B)^{\frac{q}{p-1}} + c \sigma(B) \left( \int_{2R}^{\infty} \left[ \frac{\mu(B(0, t))}{t^{n-\alpha_p}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \right)^q,
\]

where \(c > 0\) is a constant that depends only on \(p, q,\) and \(n.\) The right-hand side of the preceding estimate is obviously finite by (1.13) and (1.14). This proves that \(u \in L^q_{\text{loc}}(\mathbb{R}^n, \sigma).\)

Passing to the limit as \(j \to \infty\) in (4.6), we deduce as in [PV1], [PV2] for \(q > p - 1\) that \(u\) is a positive \(p\)-superharmonic solution to (1.1).

**Corollary 4.2.** The results involving pointwise estimates, as well as necessary and sufficient conditions for \(u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n), u \in L^r(\mathbb{R}^n), u \in L^p_{\text{loc}}(\mathbb{R}^n),\) etc., obtained in [CV1], [CV2], [SV], [V] for minimal solutions \(u,\) actually hold for all solutions.

**Remark 4.3.** 1. In the case \(p = 2,\) Theorem 4.1 yields bilateral pointwise estimates of solutions to the fractional Laplace equation (1.20). The case of homogeneous equations \(\mu = 0\) was considered earlier in [CV1], where the upper estimate was proved only for the minimal solution \(u.\)
2. As was mentioned in the Introduction, Theorem 1.1 is valid for general quasilinear $\mathcal{A}$-Laplace operators $\text{div}\mathcal{A}(x, \nabla u)$ in place of $\Delta_p$ under the standard structural assumptions on $\mathcal{A}$ which ensure that the Kilpeläinen–Malý estimates (1.8) hold (see [KiMa], [MZ]). The proofs in this setup are identical to those given above, with the same nonlinear potentials $W_p$ and $K_{p,q}$. The constants in our pointwise estimates (1.15) then depend on the structural constants of $\mathcal{A}$. Analogous results also hold for $k$-Hessian equations in the case $0 < q < k$ (see [CV1] for $\mu = 0$, and [PV2] for $q > k$).

3. Complete analogues of our results for (1.1) hold for the non-homogeneous problem

$$\begin{cases} -\Delta_p u = \sigma u^q + \mu, & u \geq 0 \text{ in } \mathbb{R}^n, \\ \liminf_{x \to \infty} u(x) = c, \end{cases}$$

where $c$ is a positive constant. One only needs to add $c$ to both sides of (1.15).

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