HOMOLOGICAL STABILITY FOR DIFFEOMORPHISM GROUPS OF HIGH DIMENSIONAL HANDLEBODIES

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Abstract. In this paper we prove a homological stability theorem for the diffeomorphism groups of high dimensional manifolds with boundary, with respect to forming the boundary connected sum with the product $D^{p+1} \times S^q$ for $|q - p| < \min\{p, q\} - 2$. In a recent joint paper with B. Botvinnik, we prove that there is an isomorphism

$$\colim_{g \to \infty} H_*(\text{BDiff}((D^{n+1} \times S^n)^{\infty}, D^{2n}); \mathbb{Z}) \cong H_*(Q_0BO(2n + 1)(n)_+; \mathbb{Z})$$

in the case that $n \geq 4$. By combining this “stable homology” calculation with the homological stability theorem of this current paper, we obtain the isomorphism $H_k(\text{BDiff}((D^{n+1} \times S^n)^{\infty}, D^{2n}); \mathbb{Z}) \cong H_k(Q_0BO(2n + 1)(n)_+; \mathbb{Z})$ in the case that $k \leq \frac{1}{2}(g - 4)$.

1. Introduction

1.1. Main result. Let $M$ be a smooth, compact, $m$-dimensional manifold with non-empty boundary. Fix an $(m - 1)$-dimensional disk $D^{m-1} \hookrightarrow \partial M$. We denote by $\text{Diff}(M, D^{m-1})$ the group of self-diffeomorphisms of $M$ that restrict to the identity on a neighborhood of $D^{m-1}$, topologized in the $C^\infty$-topology. Let $\text{BDiff}(M, D^{m-1})$ denote the classifying space of the topological group $\text{Diff}(M, D^{m-1})$. Choose an $m$-dimensional manifold $V$ with non-empty boundary and let $M\natural V$ denote the boundary connected sum of $M$ and $V$. There is a homomorphism $\text{Diff}(M, D^{m-1}) \to \text{Diff}(M\natural V, D^{m-1})$ defined by extending a diffeomorphism identically over the boundary connect-summand $V$. This homomorphism induces a map between the classifying spaces, and iterating this map yields the direct system,

$$\text{BDiff}(M, D^{m-1}) \to \text{BDiff}(M\natural V, D^{m-1}) \to \cdots \to \text{BDiff}(M\natural V^{g\varnothing}, D^{m-1}) \to \cdots$$

In this paper we study the homological properties of this direct system in the case when $V$ is a high-dimensional handlebody $D^{p+1} \times S^q$. Our main result can be viewed as an analogue of the homological stability theorems of Harer [3], or the theorem of Galatius and Randal-Williams [4], for high-dimensional manifolds with boundary.

Below we state our main theorem. For $p, q, g \in \mathbb{Z}_{\geq 0}$, let $V_{p,q}^g$ denote the $g$-fold boundary connected sum, $(D^{p+1} \times S^q)^g$. Let $d(\pi_q(S^p))$ denote the generating set length of the homotopy group $\pi_q(S^p)$, which is the quantity, $d(\pi_q(S^p)) = \min\{k \in \mathbb{N} \mid \text{there exists an epimorphism } \mathbb{Z}^{\oplus k} \to \pi_q(S^p)\}$. We let $\kappa(\partial M)$ and $\kappa(M, \partial M)$ denote the degrees of connectivity of $\partial M$ and $(M, \partial M)$ respectively. The main result of this paper is stated below.

Theorem 1.1. Let $M$ be a compact manifold of dimension $m$ with non-empty boundary. Let $p$ and $q$ be positive integers with $p + q + 1 = m$, such that inequalities,

$$|q - p| < \min\{p, q\} - 2 \quad \text{and} \quad |q - p| < \min\{\kappa(\partial M), \kappa(M, \partial M)\} - 1$$


are satisfied. Then the homomorphism induced by the maps in (1.1),
\[ H_k(\text{BDiff}(M^nV^g_{p,q}, D^{m-1}); \mathbb{Z}) \longrightarrow H_k(\text{BDiff}(M^{n+1}V^g_{p,q}, D^{m-1}); \mathbb{Z}), \]
is an isomorphism when \( k \leq \frac{1}{2}(g-d(\pi_q(S^p))-3) \) and an epimorphism when \( k \leq \frac{1}{2}(g-d(\pi_q(S^p))-1) \).

The above theorem implies that the homology of the direct system (1.1) stabilizes in the case \( V = V^1_{p,q} \). For certain choices of \( p \) and \( q \) we can also identify the homology of the colimit. In joint work with B. Botvinnik [1] we construct a map,
\[ \colim_{g \to \infty} \text{BDiff}((D^{n+1} \times S^n)^g, D^{2n}) \longrightarrow Q_n BO(2n+1)\langle n \rangle_+, \]
which we prove induces an isomorphism in \( H_*(-; \mathbb{Z}) \) in the case that \( n \geq 4 \). Combining this homological equivalence with Theorem 1.1 we obtain the following corollary which lets us compute the homology of the classifying space \( \text{BDiff}((D^{n+1} \times S^n)^g, D^{2n}) \) in low degrees relative to \( g \).

**Corollary 1.1.1.** Let \( 2n + 1 \geq 9 \). Then there is an isomorphism,
\[ H_k(\text{BDiff}((D^{n+1} \times S^n)^g, D^{2n}); \mathbb{Z}) \cong H_k(Q_n BO(2n+1)\langle n \rangle_+; \mathbb{Z}), \]
when \( k \leq \frac{1}{2}(g - 4) \).

In addition to the classifying spaces of diffeomorphism groups of manifolds, we prove an analogous homological stability theorem for the moduli spaces of manifolds equipped with tangential structures, see Theorem 5.1. The precise statement of this theorem requires a number of preliminary definitions and so we hold off on stating this result until Section 5 where the theorem is proven.

1.2. **Ideas behind the proof.** The proof or our main theorem follows the strategy developed by Galatius and Randal-Williams in [4] used to prove homological stability for the diffeomorphism groups, \( \text{Diff}((S^n \times S^n)^g, D^{2n}) \), \( g \in \mathbb{N} \). The technique is also largely inspired by analogous the homological stability theorem of Harer [8] for the mapping class groups of surfaces. Given a compact manifold triad \( (M, \partial_0 M, \partial_1 M) \) and integers \( p + q + 1 = \dim(M) \), we construct a simplicial complex \( K^\partial(M)_{p,q} \), that admits an action of the diffeomorphism group \( \text{Diff}(M, \partial_0 M) \). Roughly, an \( l \)-simplex of \( K^\partial(M)_{p,q} \) is given by a set of embeddings, \( \phi_0, \ldots, \phi_l : (V^1_{p,q}, \partial V^1_{p,q}) \longrightarrow (M, \partial_1 M) \), for which \( \phi_i(V^1_{p,q}) \cap \phi_j(V^1_{p,q}) = \emptyset \) for all \( i \neq j \). Nearly all of the technical work of this paper is devoted to showing that the connectivity of the geometric realization \( |K^\partial(M)_{p,q}| \) increases linearly (with a factor of 1/2) in the number of boundary-connect-summands of \( V^1_{p,q} \) contained in \( M \) (see Theorem 5.2). Once high-connectivity is established, the homological stability theorem follows by the same spectral sequence argument used in [4]. Most of the simplicial and homotopy theoretic techniques that we use come from [4], but the geometric-topological the aspects of the paper require new constructions and arguments. We describe some of these new features below.

The simplicial complex \( K^\partial(M)_{p,q} \) can be viewed as a “relative version” of the semi-simplicial space constructed in our previous work [13], used to prove homological stability for the diffeomorphism groups \( \text{Diff}((S^g \times S^p)^g, \mathbb{Z}) \), \( g \in \mathbb{N} \). Our proof that the space \( |K^\partial(M)_{p,q}| \) is highly connected follows a similar strategy to what was employed in [13]. The idea is to map \( K^\partial(M)_{p,q} \) to another simplicial complex, \( L(V^g_{p,q})(M, \partial_1 M) \), that is constructed entirely out of algebraic data associated to the manifold \( M \), which we call the Wall form associated to \( M \) (see Section 5). Wall forms were initially defined in [13] to study closed manifolds and their diffeomorphisms. In this paper we have to use a relative version of the Wall form used to study manifold pairs \( (M, \partial M) \) and diffeomorphisms...
\( M \rightarrow M \) that are non-trivial on the boundary. High-connectivity of the complex \( L(\mathcal{W}_{p,q}^\partial(M, \partial_1 M)) \) follows from our work in [13] as well, and so it remains to prove that the map \( \bar{K}^\partial(M)_{p,q} \rightarrow L(\mathcal{W}_{p,q}^\partial(M, \partial_1 M)) \) is highly connected.

One of the main ingredients used in our proof that the map \( \bar{K}^\partial(M)_{p,q} \rightarrow L(\mathcal{W}_{p,q}^\partial(M, \partial_1 M)) \) is highly-connected is a new disjunction result for embeddings of high-dimensional manifolds with boundary, this is Theorem A.1 stated in the appendix. If \((P, \partial P), (Q, \partial Q) \subset (M, \partial M)\) are submanifold pairs, Theorem A.1 gives conditions for when one can find an ambient isotopy \( \Psi_t : (M, \partial M) \rightarrow (M, \partial M) \) with \( \Psi_0 = \text{Id}_M \), such that \( \Psi_1(P) \cap Q = \emptyset \). Theorem A.1 can be viewed as a generalization of the disjunction results of Wells [16], and Hatcher and Quinn [6], and thus could be of independent interest.

1.3. Plan of the paper. In Section 2 we recast the maps of the direct system (1.1) as maps arising from concatenation with a relative cobordism between two compact manifold pairs. This will enable us to restate Theorem 1.1 in a slightly more general form that will be easier for us to prove. In Section 3 we construct the main simplicial complex, \( K^\partial(M)_{p,q} \). In Section 4 we define certain algebraic invariants associated to a manifold with boundary. Section 5 is devoted to a recollection of some results from our previous work in [11] regarding Wall forms. In Section 6 we use the results developed throughout the rest of the paper to prove that \( |K^\partial(M)_{p,q}| \) is highly connected. In Section 7 we show how to obtain Theorem 2.3 (and thus Theorem 1.1) using the high connectivity of \( |K^\partial(M)_{p,q}| \). In Section 8 we show how to obtain an analogue of Theorem 2.3 for the moduli spaces of handlebodies equipped with tangential structures. In Appendix A we prove a disjunction theorem for embeddings of high-dimensional manifolds with boundary.

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2. Relative Cobordism and Stabilization

To prove Theorem 1.1 we will need to recast the direct system (1.1) as arising from concatenation with a relative cobordism between two compact manifold pairs. Doing this will make our constructions consistent with the cobordism categories considered in [2] and [4]. Below we introduce some definitions and terminology that we will use throughout the paper. Recall that a triad of topological spaces is a triple \((X; A, B)\) where \( X \) is a topological space and \( A, B \subset X \) are subspaces.

**Definition 2.1.** A manifold triad of dimension \( n \) is a triad \((W; \partial_0 W, \partial_1 W)\) where \( W \) is an \( n \)-dimensional smooth manifold, \( \partial_0 W, \partial_1 W \subset \partial W \) are submanifolds of dimension \( n - 1 \) such that \( \partial W = \partial_0 W \cup \partial_1 W \) and \( \partial(\partial_0 W) = \partial_0 W \cap \partial_1 W = \partial(\partial_1 W) \). We will denote by \( \partial_{0,1} W \) the intersection \( \partial_0 W \cap \partial_1 W \). We will refer to \( \partial_0 W \) and \( \partial_1 W \) as the faces.

Two compact \( d \)-dimensional manifold pairs \((M, \partial M)\) and \((N, \partial N)\) are said to be cobordant if there exists a \((d+1)\)-dimensional compact manifold triad \((W; \partial_0 W, \partial_1 W)\) such that \((\partial_0 W, \partial_0 1 W) = (M \sqcup N, \partial M \sqcup \partial N)\). The pair \((W, \partial W)\) is then said to be a relative cobordism between the pairs \((M, \partial M)\) and \((N, \partial N)\).

Let \((M; \partial_0 M, \partial_1 M)\) be an \( m \)-dimensional, compact manifold triad. Let \( \text{Diff}(M) \) denote the topological group of self-diffeomorphisms \( f : M \rightarrow M \) with \( f(\partial_i M) = \partial_i M \) for \( i = 0, 1 \). We are mainly interested in the subgroup \( \text{Diff}(M, \partial_0 M) \subset \text{Diff}(M) \) consisting of those self-diffeomorphisms that restrict to the identity on a neighborhood of \( \partial_0 M \). We will need the following construction.
Construction 2.1. Let \((P, \partial P)\) be an \((m - 1)\)-dimensional manifold pair and let \((K; \partial_0 K, \partial_1 K)\) be a compact manifold triad such that \(\partial_0 K = \partial_0 M \cup P\), i.e. the pair \((K, \partial_1 K)\) is a relative cobordism between \((\partial_0 M, \partial_{0.1} M)\) and \((P, \partial P)\). Let \(M \cup_{\partial_0} K\) be the manifold obtained by attaching \(K\) to \(M\) along the face \(\partial_0 M \subset \partial_0 K\). Similarly, let \(\partial_1 (M \cup_{\partial_0} K) := \partial_1 M \cup_{\partial_{0.1}} \partial_1 K\) be the manifold obtained by attaching \(\partial_1 K\) to \(\partial_1 M\) along their common boundary \(\partial_{0.1} M\). By setting \(\partial_0 (M \cup_{\partial_0} K) := P\), we obtain a new manifold triad,

\[
(M \cup_{\partial_0} K; \partial_0 (M \cup_{\partial_0} K), \partial_1 (M \cup_{\partial_0} K)).
\]

The cobordism \((K, \partial_1 K)\) induces a homomorphism, \(\text{Diff}(M, \partial_0 M) \to \text{Diff}(M \cup_{\partial_0} K, P)\), \(f \mapsto f \cup \text{Id}_K\), defined by extending diffeomorphisms identically over \(K\). This homomorphism in turn induces a map on the level of classifying spaces,

\[
(2.1) \quad \text{BDiff}(M, \partial_0 M) \to \text{BDiff}(M \cup_{\partial_0} K, P).
\]

This map should be compared to the one from \([4, \text{Equation 1.1}]\). Indeed, they are the same map in the case that \(\partial_1 M = \partial_1 K = \emptyset\).

Suppose now that \(\partial_0 M\) and \(\partial_1 M\) are non-empty. Let \(p\) and \(q\) be integers such that \(p + q + 1 = m\). Let \(K_{p,q}\) then denote the manifold obtained by forming the boundary connected sum of \(D^{p+1} \times S^q\), with \(\partial_0 M \times [0,1]\). In constructing \(K_{p,q}\) we form the boundary connected sum along a neighborhood contained in the complement of the subspace \(\partial_0 M \times \{0,1\} \subset \partial_0 M \times [0,1]\); in this way the boundary of \(K_{p,q}\) will contain \(\partial_0 M \times \{0,1\}\). We then set:

\[
\partial_0 K_{p,q} := \partial_0 M \times \{0,1\}, \quad \partial_1 K_{p,q} := \partial K_{p,q} \setminus \text{Int}(\partial_0 K_{p,q}) = (\partial_0 M \times [0,1]) \# (S^p \times S^q).
\]

The triple \((K_{p,q}; \partial_0 K_{p,q}, \partial_1 K_{p,q})\) is a manifold triad and \((K_{p,q}, \partial_1 K_{p,q})\) is a relative cobordism between the manifold pairs \((\partial_0 M \times \{0\}, \partial_{0.1} M \times \{0\})\) and \((\partial_0 M \times \{1\}, \partial_{0.1} M \times \{1\})\). We apply Construction 2.1 to this relative cobordism \((K_{p,q}, \partial_1 K_{p,q})\). We form the manifold \(M \cup_{\partial_0} K_{p,q}\) and notice that \(\partial_0 (M \cup_{\partial_0} K_{p,q}) = \partial_0 M\). We define

\[
(2.2) \quad s_{p,q}: \text{BDiff}(M, \partial_0 M) \to \text{BDiff}(M \cup_{\partial_0} K_{p,q}, \partial_0 M)
\]

to be the map on classifying spaces induced by \((K_{p,q}, \partial_1 K_{p,q})\) from (2.1). We will refer to this map as the \((p, q)\)-th stabilization map.

Remark 2.1. Note that the manifold \(M \cup_{\partial_0} K_{p,q}\) is diffeomorphic to the boundary connected sum \(M_{\natural} V^1_{p,q} = M_{\natural} (D^{p+1} \times S^q)\). By identifying \(M \cup_{\partial_0} K_{p,q}\) with \(M_{\natural} V^1_{p,q}\), (2.2) yields the map,

\[
\text{BDiff}(M, \partial_0 M) \to \text{BDiff}(M_{\natural} V^1_{p,q}, \partial_0 M).
\]

In the case that \(\partial_0 M = D^{p+q}\) we obtain the maps used in the direct system (1.1); we take this to be the definition of those maps.

We will now restate Theorem 1.1 in terms of the maps defined above. We need one more preliminary definition. Recall from the previous section the manifold \(V^g_{p,q} = (D^{p+1} \times S^q)^g\). Choose an embedded \((p+q)\)-dimensional closed disk \(D \subset \partial V^g_{p,q}\) and set \(\partial_0 V^g_{p,q} := D\) and \(\partial_1 V^g_{p,q} := \partial V^g_{p,q} \setminus \text{Int}(D)\) so as to obtain a manifold triad. With \(\dim(M) = m\) and \(p + q + 1 = m\) as above, we let \(r_{p,q}(M)\) be the integer defined by,

\[
(2.3) \quad r_{p,q}(M) = \max\{g \in \mathbb{N} \mid \text{there exists an embedding } (V^g_{p,q}, \partial_1 V^g_{p,q}) \to (M, \partial_1 M)\}.
\]
We refer to this quantity as the \((p,q)\)-rank of \(M\). This quantity \(r_{p,q}(M)\) is equivalent to the maximal number of boundary connect summands of \(D^{p+1} \times S^q\) that split off of \(M\). We emphasize that the embeddings \((V^g_{p,q}, \partial_1 V^g_{p,q}) \to (M, \partial_1 M)\) used in \([2,3]\) need not send the face \(\partial_0 V^g_{p,q}\) into \(\partial_0 M\).

**Remark 2.2.** The value \(r_{p,q}(M)\) depends on the structure of the triad \((M; \partial_0 M, \partial_1 M)\) and not just the manifold \(M\) itself. In particular, switching the roles of \(\partial_0 M\) and \(\partial_1 M\) in \([2,3]\) will change the value of the rank \(r_{p,q}(M)\).

As in the statement of Theorem 1.1 we will need to assume that the following inequalities are satisfied:

\[
q - p < \min\{p, q\} - 2, \quad |q - p| < \min\{\kappa(\partial_1 M), \kappa(M, \partial_1 M)\} - 1,
\]

where recall that \(\kappa(\partial_1 M)\) and \(\kappa(M, \partial_1 M)\) denote the degrees of connectivity of \(\partial_1 M\) and \((M, \partial_1 M)\) respectively. The main theorem that we will prove in this paper is stated below; it is a generalization of Theorem 1.1. Recall the generating set length \(\ell\).

**Theorem 2.3.** Let \((M; \partial_0 M, \partial_1 M)\) be an \(m\)-dimensional, compact, manifold triad with \(\partial_0 M\) and \(\partial_1 M\) nonempty. Let \(p\) and \(q\) be positive integers with \(p + q + 1 = m\) and suppose that the inequalities of (2.4) are satisfied. Let \(r_{p,q}(M) \geq g\). Then the homomorphism

\[
(s_{p,q})_* : H_k(\BDiff(M, \partial_0 M); \mathbb{Z}) \to H_k(\BDiff(M \cup_{\partial_0} K_{p,q}, \partial_0 M); \mathbb{Z})
\]

is an isomorphism when \(k \leq \frac{1}{2}(g - 3 - d(\pi_q(S^p)))\) and an epimorphism when \(k \leq \frac{1}{2}(g - 1 - d(\pi_q(S^p)))\).

**3. The Complex of Embedded Handles**

In this section we construct a semi-simplicial space that is analogous to [4, Definition 5.1]. The definition requires a preliminary geometric construction. For integers \(p, q,\) and \(g\), let \((V^g_{p,q}, \partial_0 V^g_{p,q}, \partial_1 V^g_{p,q})\) be the \((p + q + 1)\)-dimensional manifold triad defined in the previous section. For each \(g \in \mathbb{N}\) we let \(W^g_{p,q}\) denote the the face, \(\partial_1 V^g_{p,q} \cong (S^p \times S^q)^{\# g} \setminus \Int(D^{p+q})\). We will denote \(V_{p,q} := V^1_{p,q}\) and \(W_{p,q} := W^1_{p,q}\).

**Definition 3.1.** For an integer \(m\) let \(D^m_+\) denote the \(m\)-dimensional half-disk, i.e. the subspace given by the set \(\{x \in \mathbb{R}^m \mid |x| \leq 1, \quad x_1 \geq 0\}\). The boundary of \(D^m_+\) has the decomposition \(\partial D^m_+ = \partial_0 D^m_+ \cup \partial_1 D^m_+\) where \(\partial_0 D^m_+\) and \(\partial_1 D^m_+\) are given by \(\partial_0 D^m_+ = \{x \in D^m_+ \mid x_1 = 0\}\) and \(\partial_1 D^m_+ = \{x \in D^m_+ \mid |x| = 1\}\). In this way \((D^m_+; \partial_0 D^m_+, \partial_1 D^m_+)\) forms a manifold triad.

We will construct a slight modification of the manifold \(V_{p,q}\). Choose an embedding,

\[
\alpha : (D^{p+q}_+ \times \{1\}, \partial_0 D^{p+q}_+ \times \{1\}) \to (\partial_0 V_{p,q}, \partial_0 1 V_{p,q}).
\]

Let \(\widehat{V}_{p,q}\) denote the manifold obtained by attaching \(D^{p+q}_+ \times [0, 1]\) to \(V_{p,q}\) along the embedding \(\alpha\), i.e.

\[
\widehat{V}_{p,q} = V_{p,q} \cup_{\alpha} (D^{p+q}_+ \times [0, 1]).
\]

We then denote by \(\widehat{W}_{p,q}\) the manifold obtained by attaching \(\partial_0 D^{p+q} \times [0, 1]\) to \(\partial_1 V_{p,q}\) along the restriction of \(\alpha\) to \(\partial_0 D^{p+q} \times \{1\}\).

**Construction 3.1.** We construct a subspace of \(\widehat{V}_{p,q}\) as follows. Choose a basepoint \((a_0, b_0) \in \partial D^{p+1} \times S^q\) such that the pair \((a_0, -b_0)\) is contained in the face \(\partial_1 V_{p,q}\). Choose an embedding \(\gamma : [0, 1] \to \widehat{W}_{p,q}\) that satisfies the following conditions:
Remark 3.1. The embedding
\[ \text{We define } \phi : (a, b) \in \partial D_+^{p+q} \times \{0\}; \]
(ii) there exists \( \varepsilon \in (0,1) \) such that, \( \gamma(t) = (0, t) \in \partial D_+^{p+q} \times [0,1] \), whenever \( t \in (0, \varepsilon); \)
(iii) \( \gamma((0,1)) \cap [(D^{p+1} \times \{a_0\}) \cup \{b_0\} \times S^q]] = \emptyset. \)

We define \((B_{p,q}, C_{p,q}) \subset (\hat{V}_{p,q}, \hat{W}_{p,q})\) to be the pair of subspaces given by,
\[ (\gamma([0,1]) \cup (D^{p+1} \times \{a_0\}) \cup \{b_0\} \times S^q), \gamma([0,1]) \cup (S^p \times \{b_0\}) \cup \{a_0\} \times S^q). \]

The inclusion \((B_{p,q}, C_{p,q}) \to (\hat{V}_{p,q}, \hat{W}_{p,q})\) is clearly a homotopy equivalence of pairs. We will refer to the pair \((B_{p,q}, C_{p,q})\) as the core of \((\hat{V}_{p,q}, \hat{W}_{p,q})\).

We now use the above construction to construct a simplicial complex. Let \((M; \partial_0 M, \partial_1 M)\) be a compact manifold triad of dimension \(m\). Let \(\mathbb{R}^{m-1}\) denote \([0, \infty) \times \mathbb{R}^{m-2}\) and let \(\partial \mathbb{R}^{m-1}\) denote \(\{0\} \times \mathbb{R}^{m-2}\). Choose an embedding, \(a : [0,1] \times \mathbb{R}^{m-1} \to M\) with, \(a^{-1}(\partial_0 M) = \{0\} \times \mathbb{R}^{m-1}\) and \(a^{-1}(\partial_1 M) = [0,1) \times \partial \mathbb{R}^{m-1}\). For each pair of positive integers \(p\) and \(q\) with \(p+q+1 = m\), we define a simplicial complex \(K^\partial(M,a)_{p,q}\).

**Definition 3.2.** The simplicial complex \(K^\partial(M,a)_{p,q}\) is defined as follows:

(i) A vertex in \(K^\partial(M,a)_{p,q}\) is defined to be a pair \((t, \phi)\), where \(t \in \mathbb{R}\) and \(\phi : \hat{V}_{p,q} \to M\) is an embedding that satisfies:
   (a) \(\phi^{-1}(\partial_0 M) = D_+^{p+q} \times \{0\}\) and \(\phi^{-1}(\partial_1 M) = \hat{W}_{p,q}\);
   (b) there exists \(\varepsilon > 0\) such that for \((s,z) \in [0,\varepsilon) \times D_+^{m-1} \subset \hat{V}_{p,q}\), the equality \(\phi(s,z) = a(s, z + te_2)\) is satisfied, where \(e_2 \in \mathbb{R}^{m-1}\) denotes the second basis vector.

(ii) A set of vertices \(\{\phi_0(t_0), \ldots, \phi_j(t_j)\}\) forms an \(l\)-simplex if \(t_i \neq t_j\) and \(\phi_i(B_{p,q}) \cap \phi_j(B_{p,q}) = \emptyset\) whenever \(i \neq j\), where recall that \(B_{p,q} \subset \hat{V}_{p,q}\) is the core from Construction 3.1.

**Remark 3.1.** The embedding \(a\) was necessary in order to define the complex \(K^\partial(M,a)_{p,q}\); however, it will not play a serious role in any of our proofs latter in the paper. For this reason we will drop the embedding \(a\) from the notation and will denote \(K^\partial(M)_{p,q} := K^\partial(M,a)_{p,q}\).

The majority of the technical work of this paper is devoted to proving Theorem 3.2 stated below. The statement of this theorem will require the use of a definition from 4 which we recall below.

**Definition 3.3.** A simplicial complex \(X\) is said to be weakly Cohen-Macaulay of dimension \(n\) if it is \((n-1)\)-connected and the link of any \(p\)-simplex is \((n-p-2)\)-connected. In this case we write \(wCM(X) \geq n\). The complex \(X\) is said to be locally weakly Cohen-Macaulay of dimension \(n\) if the link of any simplex is \((n-p-2)\)-connected (but no global connectivity is required on \(X\) itself). In this case we shall write \(lCM(X) \geq n\).

**Theorem 3.2.** Let \((M; \partial_0 M, \partial_1 M)\) be an \(m\)-dimensional manifold triad with \(\partial_0 M \neq \emptyset\). Let \(p, q \in \mathbb{N}\) be such that \(p+q+1 = m\) and suppose that the inequalities,
\[ |q-p| < \min\{p,q\} - 2, \quad |q-p| < \min\{\kappa(\partial_1 M), \kappa(M, \partial_1 M)\} - 1, \]
are satisfied. Let \(d = d(\pi_q(S^p))\) be the generating set rank and suppose that \(r_{p,q}(M) \geq g\). Then the geometric realization \(|K^\partial(M)_{p,q}|\) is \(\frac{1}{2}(g-4-d)\)-connected and \(lCM(K^\partial(M)_{p,q}) \geq \frac{1}{2}(g-1-d)\).
The proof of the above theorem takes place over the course of the next three sections of the paper. In addition to the simplicial complex defined above, we will need to work with two related semi-simplicial spaces. Let \((M; \partial_0 M, \partial_1 M)\), \(p\), and \(q\) be as above and let \(a : [0, 1) \times \mathbb{R}^{m-1} \to M\) be the same embedding used in Definition 3.2. We define two semi-simplicial spaces \(K_0^\partial(M)_p,q\) and \(K_1^\partial(M,a)_p,q\) below.

**Definition 3.4.** The semi-simplicial space \(K_0^\partial(M)_{p,q}\) is defined as follows:

(i) The space of 0-simplices \(K_0^\partial(M,a)_{p,q}\) is defined to have the same underlying set as the set of vertices of the simplicial complex \(K_0^\partial(M,a)_{p,q}\). That is, \(K_0^\partial(M,a)_{p,q}\) is the space of pairs \((t, \phi)\), where \(t \in \mathbb{R}\) and \(\phi : \tilde{V}_{p,q} \to M\) is an embedding which satisfies the same condition as in part (i) of Definition 3.2.

(ii) The space of \(l\)-simplices, \(K_l^\partial(M,a) \subset (K_0^\partial(M,a))^l\) consists ordered \((l+1)\)-tuples, \(((t_0, \phi_0), \ldots, (t_l, \phi_l))\), such that \(t_0 < \cdots < t_l\) and \(\phi_i(B_{p,q}) \cap \phi_j(B_{p,q}) = \emptyset\) when \(i \neq j\).

The spaces \(K_l^\partial(M,a) \subset (\mathbb{R} \times \text{Emb}(\tilde{V}_{p,q}, M))^{l+1}\) are topologized using the \(C^\infty\)-topology on the space of embeddings. The assignments \([i] \mapsto K_l^\partial(M,a)\) define a semi-simplicial space denoted by \(K_0^\partial(M,a)_{p,q}\). The \(i\)th face map \(d_i : K_l^\partial(M,a) \to K_{l-1}^\partial(M,a)\) is given by forgetting the \(i\)th entry in the \(l\)-tuple \(((t_0, \phi_0), \ldots, (t_l, \phi_l))\).

Finally, \(K^\partial(M,a)_{p,q} \subset K_0^\partial(M,a)_{p,q}\) is defined to be the subsemi-simplicial space consisting of all simplices \(((\phi_0, t_0), \ldots, (\phi_l, t_l)) \in K_l^\partial(M,a)\) such that \(\phi_i(\tilde{V}_{p,q}) \cap \phi_j(\tilde{V}_{p,q}) = \emptyset\) whenever \(i \neq j\).

As in Remark 3.1, when working with \(K_0^\partial(M,a)_{p,q}\) and \(K_1^\partial(M,a)_{p,q}\), we will drop the embedding \(a\) from the notation and write, \(K_0^\partial(M)_{p,q} := K_0^\partial(M,a)_{p,q}\), and \(K_1^\partial(M)_{p,q} := K_1^\partial(M,a)_{p,q}\). We will ultimately need to use the fact that the geometric realizations \(|K_0^\partial(M)_{p,q}|\) and \(|K_1^\partial(M,a)_{p,q}|\) are highly connected. We prove that \(|K_0^\partial(M,a)_{p,q}|\) and \(|K_0^\partial(M,a)_{p,q}|\) are highly connected by comparing them to the simplicial complex \(K^\partial(M,a)_{p,q}\).

**Corollary 3.2.1.** Let \((M; \partial_0 M, \partial_1 M)\) be an \(m\)-dimensional manifold triad with \(\partial_0 M \neq \emptyset\). Let \(p, q \in \mathbb{N}\) be such that \(p + q + 1 = m\) and suppose that the inequalities (3.1) are satisfied. Let \(d = d(\pi_q(S^p))\) be the generating set rank and suppose that \(r_{p,q}(M) \geq g\). Then the geometric realization \(|K_0^\partial(M)_{p,q}|\) is \(\frac{1}{2}(g - 4 - d)\)-connected.

**Proof of Corollary 3.2.1.** The result is obtained from Theorem 3.2 by assembling several results from [2]. Let \(g \in \mathbb{Z}_{\geq 0}\) be given. Consider the discretization \(K_0^\partial(M)_{p,q}\). This is the semi-simplicial space obtained by defining each \(K_0^\partial(M)_{p,q}\) to have the same underlying set as \(K_1^\partial(M)_{p,q}\), but topologized in the discrete topology. The correspondence, \(((\phi_0, t_0), \ldots, (\phi_l, t_l)) \mapsto \{(\phi_0, t_0), \ldots, (\phi_l, t_l)\}\), induces a map \(|K_0^\partial(M)_{p,q}| \to |K_1^\partial(M)_{p,q}|\) which is easily seen to be a homeomorphism. Indeed, if \(\sigma = \{(\phi_0, t_0), \ldots, (\phi_l, t_l)\} \in K_0^\partial(M)_{p,q}\) is a simplex, then \(t_i \neq t_j\) if \(i \neq j\), and so there is exactly one \(l\)-simplex in \(K_0^\partial(M)_{p,q}\) with underlying set equal to \(\sigma\). It follows from this that \(|K_0^\partial(M)_{p,q}|\) is \(\frac{1}{2}(g - 4 - d)\)-connected. Next, we compare the connectivity of \(|K_0^\partial(M)_{p,q}|\) to \(|K_1^\partial(M)_{p,q}|\). By replicating the same argument used in the proof of [2] Theorem 5.5] it follows that the connectivity of \(|K_1^\partial(M)_{p,q}|\) is bounded below by the connectivity of \(|K_0^\partial(M)_{p,q}|\), and thus \(|K_0^\partial(M)_{p,q}|\) is \(\frac{1}{2}(g - 4 - d)\)-connected as well. We remark that this proof from [2] uses the fact that \(\text{ICM}(K_0^\partial(M)_{p,q}) \geq \frac{1}{2}(g - 1 - d)\) (and this is established in Theorem 3.2). Finally, to finish the proof of the lemma we observe that the inclusion \(K_0^\partial(M)_{p,q} \hookrightarrow K_1^\partial(M)_{p,q}\) is a levelwise weak homotopy equivalence. This fact is proven by
employing the same argument used in the proof of [4, Corollary 5.8]. This completes the proof of the lemma. □

High-connectivity of the space $|\tilde{K}^\bullet(M)_{p,q}|$ is the main ingredient needed for the proof of Theorem 2.3, which we carry out in Section 7. The main ingredient in proving Corollary 3.2.1 was Theorem 3.2. The next three sections then are geared toward developing all of the technical tools needed to prove Theorem 3.2.

4. The Algebraic Invariants

4.1. Invariants of manifold pairs. For what follows, let $M$ be a compact manifold of dimension $m$ with non-empty boundary. Let $A \subset \partial M$ be a submanifold of dimension $m - 1$. We will keep $M$ and $A$ fixed throughout the entire section. Let $p$ and $q$ be positive integers with $p + q + 1 = m$ and suppose further that the following inequalities are satisfied:

\begin{equation}
|q - p| < \min\{p, q\} - 2, \quad |q - p| < \min\{\kappa(A), \kappa(M, A)\} - 1,
\end{equation}

where recall $\kappa(A)$ and $\kappa(M, A)$ denote the degrees of connectivity of $A$ and $(M, A)$ respectively. We will consider the homotopy groups $\pi_q(A)$ and $\pi_{p+1}(M, A)$. We will need to be able to represent elements of these homotopy groups by smooth embeddings. The next lemma follows by assembling several results from [7] and [14]; part (i) of the lemma follows by combining the results [7, Corollary 1.1] and [7, Corollary 2.1] and part (ii) follows by combining the results [14, Lemma 1] and [14, Proposition 1].

**Lemma 4.1.** Let $M$ be a manifold of dimension $m$ with non-empty boundary and let $A \subset \partial M$ be a submanifold of dimension $m - 1$. Let $p$ and $q$ be positive integers such that $p + q + 1 = m$ and suppose that (4.1) holds. We may then draw the following conclusions about the homotopy groups $\pi_{p+1}(M, A)$ and $\pi_q(A)$:

(i) Any element of $\pi_{p+1}(M, A)$ can be represented by an embedding $(D^{p+1}, S^p) \to (M, A)$, unique up to isotopy.

(ii) Any element of $\pi_q(A)$ can be represented by an embedding $S^q \to A$, unique up to regular homotopy through immersions.

We first define a bilinear map

\begin{equation}
\tau_{p,q}^\partial : \pi_{p+1}(M, A) \otimes \pi_q(S^p) \to \pi_q(A), \quad ([f], [\phi]) \mapsto [(f|_{\partial D^{p+1}} \circ \phi)],
\end{equation}

where $f : (D^{p+1}, \partial D^{p+1}) \to (M, A)$ represents a class in $\pi_{p+1}(M, A)$ and $\phi : S^q \to S^p$ represents a class in $\pi_q(S^p)$. By the inequalities of (4.1), it follows from the Freudenthal suspension theorem that the suspension homomorphism $\Sigma : \pi_q(S^{p-1}) \to \pi_q(S^p)$ is surjective. For this reason the formula in (4.2) is indeed bilinear and thus $\tau_{p,q}^\partial$ is well defined.

We then have a bilinear intersection pairing

\begin{equation}
\lambda_{p,q}^\partial : \pi_{p+1}(M, A) \otimes \pi_q(A) \to \mathbb{Z},
\end{equation}

which is defined by sending a pair $([f], [g]) \in \pi_{p+1}(M, A) \otimes \pi_q(A)$ to the oriented algebraic intersection number associated to the maps $f|_{\partial D^{p+1}} : S^p \to A$ and $g : S^q \to A$, which we may assume are embeddings by Lemma 4.1.
We have a $(-1)^q$-symmetric bilinear pairing
\begin{equation}
\mu_q : \pi_q(A) \otimes \pi_q(A) \to \pi_q(S^p)
\end{equation}
defined in the same way as in [11 Construction 3.1]. We refer the reader there for the definition.

The next proposition shows how the maps $\tau_{p,q}^q$, $\lambda_{p,q}^q$, and $\mu_q$ are related to each other. The proof follows from [11 Proposition 3.4].

**Proposition 4.2.** Let $x \in \pi_p(M, A)$, $y \in \pi_q(A)$ and $z \in \pi_q(S^p)$. Then the equation
\[
\mu_q(\tau_{p,q}^q(y, z), x) = \lambda_{p,q}^q(y, x) \cdot z
\]
is satisfied.

**Remark 4.3.** We remark that the formula in Proposition 4.2 makes sense even in the case when $q < p$ and thus $\pi_q(S^p) = 0$.

We will also need to consider a function
\begin{equation}
\alpha_q : \pi_q(A) \to \pi_{q-1}(SO_p)
\end{equation}
defined by sending $x \in \pi_q(A)$ to the element in $\pi_{q-1}(SO_p)$ which classifies the normal bundle associated to an embedding $S^q \to A$ which represents $x$.

The map $\alpha_q$ is not in general a homomorphism. As will be seen in Proposition 4.4, the bilinear form $\mu_q$ measures the failure of $\alpha_q$ to preserve additivity. In order to describe the relationship between $\alpha_q$ and $\mu_q$, we must define some auxiliary homomorphisms. Let $\bar{\pi}_q : \pi_{q-1}(SO_p) \to \pi_q(S^p)$ be the map given by the composition $\pi_{q-1}(SO_p) \to \pi_{q-1}(S^{p-1}) \to \pi_q(S^p)$, where the first map is induced by the bundle projection $SO_p \to SO_p/SO_{p-1} \cong S^{p-1}$, and the second is the suspension homomorphism. Let $d_q : \pi_q(S^p) \to \pi_{q-1}(SO_p)$ be the boundary homomorphism associated to the fibre sequence $SO_p \to SO_{p+1} \to S^p$.

**Proposition 4.4.** The following equations are satisfied for all $x, y \in \pi_q(A)$:
\[
\alpha_q(x + y) = \alpha_q(x) + \alpha_q(y) + d_q(\mu_q(x, y)),
\]
\[
\mu_q(x, x) = \bar{\pi}_q(\alpha_q(x)).
\]

We will also need the following proposition which describes how $\alpha_q$ and $\tau_{p,q}^q$ are related.

**Proposition 4.5.** For all $(x, z) \in \pi_{p+1}(M, A) \times \pi_q(S^p)$, we have $\alpha_q(\tau_{p,q}^q(x, z)) = 0$.

**Proof.** Let $\partial_{p+1} : \pi_{p+1}(M, A) \to \pi_p(A)$ denote the boundary map. In [11 Page 9] a bilinear map $F_{p,q} : \pi_{p-1}(SO_p) \otimes \pi_q(S^p) \to \pi_{q-1}(SO_p)$ is defined. By [11 Proposition 3.8] (combined with the definition of the map $\tau_{p,q}^q$) it follows that $\alpha_q(\tau_{p,q}^q(x, z)) = F_{p,q}(\alpha_p(\partial_{p+1}(x)), z)$ for all $(x, z) \in \pi_{p+1}(M, A) \times \pi_q(S^p)$. Let $f : (D^{p+1}, \partial D^{p+1}) \to (M, A)$ be an embedding that represents the class $x \in \pi_{p+1}(M, A)$. Notice that the normal bundle of $f(D^{p+1})$ is automatically trivial since the disk is contractible. It follows that $f|_{\partial D^{p+1}} : \partial D^{p+1} \to A$ has trivial normal bundle as well, and thus it follows that $\alpha_p(\partial_{p+1}(x)) = 0$ since $f|_{\partial D^{p+1}}$ represents the class $\partial_{p+1}x \in \pi_p(A)$. Since $F_{p,q}$ is bilinear, it follows that
\[
\alpha_q(\tau_{p,q}^q(x, z)) = F_{p,q}(\alpha_p(\partial_{p+1}(x)), z) = F_{p,q}(\alpha_p(0), z) = 0
\]
for all $x$ and $z$. This concludes the proof of the proposition. \hfill \square

Using the maps defined above we will work with the algebraic structure defined by the six-tuple
\begin{equation}
(\pi_{p+1}(M, A), \pi_q(A), \tau_{p,q}^\partial, \lambda_{p,q}^\partial, \mu_q, \alpha_q).
\end{equation}
We refer to this structure as the Wall form associated to the pair $(M, A)$. We summarize the salient properties of \eqref{eq:wall-form} in the following lemma. This lemma should be compared to \cite[Lemma 3.9]{11}.

\begin{lemma}
Let $(M, A)$, $p$, and $q$ be exactly as above. The maps
- $\tau_{p,q}^\partial : \pi_{p+1}(M, A) \otimes \pi_q(S^p) \to \pi_q(A)$,
- $\lambda_{p,q}^\partial : \pi_{p+1}(M, A) \otimes \pi_q(A) \to \mathbb{Z}$,
- $\mu_q : \pi_q(A) \otimes \pi_q(A) \to \pi_q(S^p)$,
- $\alpha_q : \pi_q(A) \to \pi_{q-1}(SO_p)$,

satisfy the following conditions. For all $x, x' \in \pi_{p+1}(M, A)$, $y, y' \in \pi_q(A)$ and $z \in \pi_q(S^p)$ we have:
\begin{enumerate}[(i)]
\item $\lambda_{p,q}^\partial(x, \tau_{p,q}^\partial(x', z)) = 0$,
\item $\mu_q(\tau_{p,q}^\partial(x, z), y) = \lambda_{p,q}^\partial(x, y) \cdot z$,
\item $\alpha_q(y + y') = \alpha_q(y) + \alpha_q(y') + d_q(\mu_q(y, y'))$,
\item $\mu_q(y, y) = \pi_q(\alpha_q(y))$,
\item $\alpha_q(\tau_{p,q}^\partial(x, z)) = 0$.
\end{enumerate}
\end{lemma}

\subsection{Modifying Intersections.}

We will ultimately need to use $\lambda_{p,q}^\partial$ and $\mu_q$ to study intersections of embedded submanifolds. Let $(M, A)$, $p$ and $q$ be exactly as in the previous section. For embeddings $f : (D^{p+1}, S^p) \to (M, A)$ and $g : S^q \to A$, the integer $\lambda_{p,q}(\lceil f \rceil, \lceil g \rceil)$ is equal to the signed intersection number of $f(S^p)$ and $g(S^q)$ in $A$. Since $A$ is simply-connected, by application of the Whitney trick \cite[Theorem 6.6]{10}, one can deform $f$ through a smooth isotopy to a new embedding $f' : (D^{p+1}, S^p) \to (M, A)$ such that $f'(S^p)$ and $g(S^q)$ intersect transversally in $A$ at exactly $|\lambda_{p,q}(x, y)|$-many points, all with positive orientation. We now consider embeddings $f, g : S^q \to A$ whose images intersect transversally. The intersection $f(S^q) \cap g(S^q)$ is generically a $(q - p)$-dimensional closed manifold. We will need a higher dimensional analogue of the Whitney trick that applies to the intersection of such embeddings. The first proposition below follows from \cite{16} and \cite{6}.

\begin{thm}
(Wells, \cite{16}) Let $f, g : S^q \to A$ be embeddings. Then there exists an isotopy $\Psi_t : S^q \to A$ with $\Psi_0 = g$ and $\Psi_1(S^q) \cap f(S^q) = \emptyset$, if and only if $\mu_q([g], [f]) = 0$.
\end{thm}

We will also need a technique for manipulating the intersections of embeddings $(D^{p+1}, \partial D^{p+1}) \to (M, A)$. The following theorem is a special case of Theorem \cite{A.1}.

\begin{thm}
Let $f, g : (D^{p+1}, S^p) \to (M, A)$ be embeddings. Then there is an isotopy of embeddings $\Psi_t : (D^{p+1}, S^p) \to (M, A)$ such that $\Psi_0 = f$ and $\Psi_1(D^{p+1}) \cap g(D^{p+1}) = \emptyset$.
\end{thm}

\begin{rem}
We emphasize that in Theorem \cite{1.8} the embeddings $f$ and $g$ are completely arbitrary; the theorem holds for any two such embeddings so long as $(M, A)$, $p$, $q$ satisfy \cite{A.1}. Furthermore, we emphasize that the restriction $\Psi_t|_{S^p}$ is not in general the constant isotopy. The theorem would not be true if we insisted on keeping the restriction $\Psi_t|_{S^p}$ fixed for all $t$.
\end{rem}
We will need to apply the above theorems inductively. For the statement of the next two results, let \((M, A)\) and \(p\) and \(q\) be exactly as in the statements of the previous two theorems. The following corollary is proven in exactly the same way as \cite{1.2} Corollary 7.5 using Theorem 4.7.

**Corollary 4.9.1.** Let \(f_0, \ldots, f_m : S^q \longrightarrow A\) be a collection of embeddings such that:

(i) \(\mu_q(f_0, f_i) = 0\) for all \(i = 1, \ldots, m\);

(ii) the collection of embeddings \(f_1, \ldots, f_m\) is pairwise transverse.

Then there exists an isotopy \(\Psi_t : S^q \longrightarrow A\) with \(t \in [0, 1]\) and \(\Psi_0 = f_0\), such that \(\Psi_1(S^q) \cap f_i(S^q) = \emptyset\) for \(i = 1, \ldots, m\).

The next corollary is proven in the same way as \cite{1.1} Proposition 6.8 using Theorem 4.8.

**Corollary 4.9.2.** Let \(g_0, \ldots, g_k : (D^{p+1}, \partial D^{p+1}) \longrightarrow (M, A)\) be a collection of embeddings such that the collection of submanifolds, \(g_1(D^{p+1}), \ldots, g_k(D^{p+1}) \subset M\), is pairwise transverse. Then there exists an isotopy

\[
\Psi_t : (D^{p+1}, \partial D^{p+1}) \longrightarrow (M, A), \quad t \in [0, 1],
\]

with \(\Psi_0 = g_0\), such that \(\Psi_1(D^{p+1}) \cap g_i(D^{p+1}) = \emptyset\) for \(i = 1, \ldots, k\).

## 5. Wall Forms

We now formalize the algebraic structure studied in Section \cite{1.1}. Much of this section is a recollection of definitions and results from \cite{1.1} Section 5. We begin by introducing the category underlying our main construction.

**Definition 5.1.** Fix a finitely generated Abelian group \(H\). An object \(M\) in the category \(\text{Ab}_H^2\) is defined to be a pair of abelian groups \((M_-, M_+)\) equipped with a bilinear map, \(\tau : M_- \otimes H \longrightarrow M_+\). A morphism \(f : M \longrightarrow N\) of \(H\)-pairs is defined to be a pair of group homomorphisms \(f_- : M_- \longrightarrow N_-\) and \(f_+ : M_+ \longrightarrow N_+\) that satisfy, \(f_+ \circ \tau_M = \tau_N \circ (f_- \otimes \text{Id}_M)\). We will refer to morphisms in \(\text{Ab}_H^2\) as \(H\)-maps.

We build on the above definition as follows. Fix once and for all a finitely generated abelian group \(H\). All of our constructions will take place in the category \(\text{Ab}_H^2\). Let \(G\) be an abelian \(H\)-pair, equipped with homomorphisms \(\partial : H \longrightarrow G_+\) and \(\pi : G_+ \longrightarrow H\). Then, let \(\epsilon = \pm 1\). We call such a 4-tuple \((G, \partial, \pi, \epsilon)\) a form-parameter. Fix a form-parameter \((G, \partial, \pi, \epsilon)\) and let \(M\) be a finitely generated \(H\)-pair. Consider the following data:

- A bilinear map, \(\lambda : M_- \otimes M_+ \longrightarrow \mathbb{Z}\).
- An \(\epsilon\)-symmetric bilinear form, \(\mu : M_+ \otimes M_+ \longrightarrow H\).
- Functions, \(\alpha_\pm : M_\pm \longrightarrow G_\pm\).

Our main definition is given below.

**Definition 5.2.** The 5-tuple \((M, \lambda, \mu, \alpha)\) is said to be a Wall form with parameter \((G, \partial, \pi, \epsilon)\) if the following conditions are satisfied for all \(x, x' \in M_-\), \(y, y' \in M_+\), and \(h \in H\):

(i) \(\lambda(x, \tau_M(x', h)) = 0\),

(ii) \(\mu(\tau_M(x, h), y) = \lambda(x, y) \cdot h\),

(iii) \(\alpha_- (x + x') = \alpha_-(x) + \alpha_-(x')\),

(iv) \(\alpha_+(y + y') = \alpha_+(y) + \alpha_+(y') + \partial(\mu(y, y'))\),
The Wall form \((M, \lambda, \mu, \alpha)\) is said to be reduced if \(\alpha_-\) is identically zero. In the case of a reduced Wall form, condition (vi) then translates to \(\alpha_+(\tau_M(x, h)) = 0\) for all \(x \in M_-\) and \(h \in H\). A morphism between Wall forms (with the same form-parameter) is an \(H\)-map \(f : M \rightarrow N\) that preserves all values of \(\lambda, \mu, \) and \(\alpha\).

We will often denote a Wall form by its underlying \(H\)-pair, i.e. \(M := (M, \lambda, \mu, \alpha)\). We will need notation for orthogonal complements.

**Definition 5.3.** Let \(N \leq M\) be Wall forms. We define a new sub-Wall form \(N^\perp \leq M\) by setting:

\[
N^\perp_- := \{ x \in M_- \mid \lambda(x, w) = 0 \text{ for all } w \in N_+ \}, \\
N^\perp_+ := \{ y \in M_+ \mid \lambda(v, y) = 0 \text{ and } \mu(y, w) = 0 \text{ for all } v \in N_-, \ w \in N_+ \}.
\]

It can be easily checked that \(\tau(N^\perp_- \otimes H) \leq N^\perp_+\) and thus \(N^\perp\) actually is a sub-\(H\)-pair of \(M\). We call \(N^\perp\) the orthogonal complement to \(N\) in \(M\). Two sub-Wall forms \(N, N' \leq M\) are said to be orthogonal if \(N \cap N' = 0\), \(N \leq (N')^\perp\) and \(N' \leq N^\perp\).

We will need to use the simplicial complex from [11, Definition 4.13]. For this we must recall the definition of the standard Wall form. This requires a few steps. Fix a finitely generated Abelian group \(H\). We define an \(H\)-pair \(W \in \text{Ob}(\text{Ab}^2_H)\) by setting, \(W_- = Z\) and \(W_+ = Z \oplus H\). The map \(\tau : W_- \otimes H \rightarrow W_+\) is defined by the formula,

\[
\tau(t \otimes h) = (0, t \cdot h) \in Z \oplus H = W_+.
\]

For \(g \in \mathbb{N}\), we denote by \(W^g\) the \(g\)-fold direct-sum \(W^\otimes g\). We let \(W\) denote the \(H\)-pair \(W^1\), and \(W^0\) is understood to be the trivial \(H\)-pair. Fix elements \(a \in W_-\) and \(b \in W_+\) which correspond to \(1 \in Z\) and \((1, 0) \in Z \oplus H\) respectively. For \(g \in \mathbb{N}\), we denote by \(a_i \in W^g_-\) and \(b_i \in W^g_+\) for \(i = 1, \ldots, g\), the elements that correspond to the elements \(a\) and \(b\) coming from the \(i\)th direct-summand of \(W\) in \(W^g\). Now fix a form-parameter \((G, \partial, \pi, \epsilon)\). We endow \(W^g\) with the structure of a Wall form with parameter \((G, \partial, \pi, \epsilon)\) by setting:

\[
(5.1) \quad \lambda(a_i, b_j) = \delta_{i,j}, \quad \mu(b_i, b_j) = 0, \quad \alpha_+(b_i) = 0, \quad \alpha_- (a_i) = 0 \quad \text{for } i, j = 1, \ldots, g.
\]

These values together with the conditions imposed from Definition 5.2 determine the maps \(\lambda, \mu, \) and \(\alpha\) completely. The vanishing of \(\alpha_- (a_i)\) for all \(i\), implies that \(\alpha_-\) is identically zero, thus \((W^g, \lambda, \mu, \alpha)\) is a reduced Wall form with parameter \((G, \partial, \pi, \epsilon)\) (the fact that \(\alpha_+(b_i) = 0\) for all \(i\) does not imply that \(\alpha_+ = 0\) however). We call this the standard Wall form of rank \(g\) with parameter \((G, \pi, \partial, \epsilon)\).

We will use standard Wall form \(W\) to probe other Wall forms. The simplicial complex defined in the next Definition is an algebraic analogue of the simplicial complex defined in Section 3.

**Definition 5.4.** For a Wall form \(M\) let \(L(M)\) be the simplicial complex whose vertices are given by morphisms \(f : W \rightarrow M\). A set of vertices \(\{f_0, \ldots, f_l\}\) is an \(l\)-simplex if the sub Wall forms, \(f_0(W), \ldots, f_l(W) \leq M\) are pairwise orthogonal.

To state the main theorem regarding the simplicial complex \(L(M)\) we need to introduce a notion of rank for a Wall form. The definition below is analogous to the rank \(r_{p,q}(-)\) associated to a manifold triad \((M; \partial_0 M, \partial_1 M)\) defined back in Section 2.
**Definition 5.5.** For a Wall form $M$, the rank of $M$ is defined to be the non-negative integer,

$$r(M) := \max\{ g \in \mathbb{N} \mid \text{there exists a morphism } W^g \to M \}.$$ 

One of the key technical results proven in [11] (see [11, Theorem 5.1]) is the theorem stated below. For the statement of this theorem, let $d$ denote the generating set rank $d(H)$, which recall is the quantity, $d(H) = \min\{ k \in \mathbb{N} \mid \text{there exists an epimorphism } \mathbb{Z}^\oplus k \to H \}.$

**Theorem 5.1.** Suppose that $r(M) \geq g$. Then $\text{lCM}(L(M)) \geq \frac{1}{2}(g - 1 - d)$ and the geometric realization $|L(M)|$ is $\frac{1}{2}(g - 4 - d)$ connected.

We now show how to use the constructions from Section 4 to associate a Wall form to a pair of manifolds. Let $M$ be a compact oriented manifold of dimension $m$ with non-empty boundary. Let $A \subset \partial M$ be a submanifold of dimension $m - 1$. Let $p$ and $q$ be positive integers with $p + q + 1 = m$. Suppose that $p$ and $q$ satisfy the inequalities from (4.1). We set $H = \pi_q(S^p)$ and denote by $\mathcal{W}^{\partial}_{p,q}(M, A)$ the $H$-pair given by setting,

$$\mathcal{W}^{\partial}_{p,q}(M, A)_- := \pi_{p+1}(M, A), \quad \mathcal{W}^{\partial}_{p,q}(M, A)_+ := \pi_q(A),$$

and then by setting $\tau$ equal to the bilinear map,

$$\tau := \tau^\partial_{p,q} : \pi_{p+1}(M, A) \otimes \pi_q(S^p) \to \pi_q(A),$$

from (4.2). We need to define a suitable form-parameter. Let $G_{p,q}$ denote the abelian group $\pi_{p-1}(SO_p)$. The group $G_{p,q}$ together with the maps from Proposition 4.4, $d_q : \pi_q(S^p) \to \pi_{q-1}(SO_p)$ and $\pi_q : \pi_{q-1}(SO_p) \to \pi_q(S^p)$, make the 4-tuple $(G_{p,q}, d_q, \pi_q, (-1)^q)$ into a form parameter. It follows then directly from Lemma 4.6 that the 4-tuple,

$$(5.2) \quad (\mathcal{W}^{\partial}_{p,q}(M, A), \lambda^\partial_{p,q}, \mu_q, \alpha_q),$$

is a reduced Wall form with form-parameter $(G_{p,q}, d_q, \pi_q, (-1)^q)$. We call the Wall form of (5.2) the Wall form of degree $(p, q)$ associated to $(M, A)$. This construction should be compared to [11, Section 4.3].

We now state two basic propositions that follow directly from the definitions of $\tau^\partial_{p,q}$, $\lambda^\partial_{p,q}$, $\mu_q$, and $\alpha_q$.

**Proposition 5.2.** Let $M$ and $N$ be $m$ dimensional manifolds with non-empty boundary. Let $A \subset \partial M$ and $B \subset \partial N$ be submanifolds of dimension $m - 1$. Let $p, q \in \mathbb{Z}_{\geq 0}$ be chosen with $p + q + 1 = m$ so that the Wall forms $\mathcal{W}^{\partial}_{p,q}(N, B)$ and $\mathcal{W}^{\partial}_{p,q}(M, A)$ are defined. Then any embedding $\varphi : (N, B) \to (M, A)$ induces a unique morphism of Wall forms $\varphi_* : \mathcal{W}^{\partial}_{p,q}(N, B) \to \mathcal{W}^{\partial}_{p,q}(M, A)$.

Let $(M, A)$ and $(N, B)$ be exactly as in the previous proposition. Consider the pair $(M \natural N, A \# B)$ obtained by forming the boundary connected sum of $M$ with $N$ along embeddings, $(M, A) \hookrightarrow (D^m_+, \partial_b D^m_+) \hookrightarrow (N, B)$.

**Proposition 5.3.** There is an isomorphism of Wall forms, $\mathcal{W}^{\partial}_{p,q}(M \natural N, A \# B) \cong \mathcal{W}^{\partial}_{p,q}(M, A) \oplus \mathcal{W}^{\partial}_{p,q}(N, B)$.
6. High Connectivity of $K^\partial(M)_{p,q}$

Let $(M; \partial_0 M, \partial_1 M)$ be an $m$-dimensional manifold triad with $\partial_0 M \neq \emptyset$. Let $p$ and $q$ be positive integers with $p + q + 1 = m$ and suppose that the inequalities of (6.1) are satisfied. In this section we will prove Theorem 3.2 which asserts that $|K^\partial(M)_{p,q}|$ is $\frac{1}{2}(r_{p,q}(M) - 4 - d)$-connected and that $\text{tCM}(K^\partial(M)_{p,q}) \geq \frac{1}{2}(r_{p,q}(M) - 1 - d)$, where $d = d(\pi_q(S^p))$ is the generating set rank. Our strategy is to compare $K^\partial(M)_{p,q}$ to the complex $L(W^\partial_{p,q}(M, \partial_1 M))$ from Definition 5.4 associated to the Wall form $W^\partial_{p,q}(M, \partial_1 M)$. In view of Theorem 5.1, we will need to construct a simplicial map $K^\partial(M)_{p,q} \to L(W^\partial_{p,q}(M, \partial_1 M))$ and then prove that it is highly connected. The construction of this map and proof of its high-connectivity is carried out over the course of this section, which contains the technical core of the paper.

6.1. A simplicial map. Let $(M; \partial_0 M, \partial_1 M)$ $m$-dimensional manifold triad with $\partial_0 M \neq \emptyset$ and let $p$ and $q$ be positive integers with $p + q + 1 = m$. We will construct a simplicial map

\begin{equation}
F_{p,q} : K^\partial(M)_{p,q} \to L(W^\partial_{p,q}(M, \partial_1 M)).
\end{equation}

Recall the pair $(\hat{V}_{p,q}, \hat{W}_{p,q})$. Recall from Construction 5.1 the core, $(B_{p,q}, C_{p,q}) \cong (\hat{V}_{p,q}, \hat{W}_{p,q})$. Let $(a_0, b_0) \in S^p \times S^q$ be the basepoint used in the construction of $(B_{p,q}, C_{p,q})$ from Construction 5.1. Let $\sigma \in \pi_{p+1}(\hat{V}_{p,q}, \hat{W}_{p,q})$ be the class represented by the embedding, $(D^{p+1} \times \{b_0\}, S^p \times \{b_0\}) \hookrightarrow (B_{p,q}, C_{p,q}) \hookrightarrow (\hat{V}_{p,q}, \hat{W}_{p,q})$, and let $\rho \in \pi_q(\hat{W}_{p,q})$ be the class represented by the embedding, $\{a_0\} \times S^q \hookrightarrow C_{p,q} \hookrightarrow \hat{W}_{p,q}$. It follows directly from the construction of $\hat{V}_{p,q}$ that $\lambda_{p,q}(\sigma, \rho) = 1$ and $\alpha_q(\rho) = 0$.

Using this observation we may define a morphism of Wall forms

\begin{equation}
T_{p,q} : W \to \mathcal{W}^\partial_{p,q}(\hat{V}_{p,q}, \hat{W}_{p,q}), \quad a \mapsto \sigma, \quad b \mapsto \rho,
\end{equation}

where $a \in W_-$ and $b \in W_+$ are the standard generators used in the construction of $W$. Using the calculations,

\[
\pi_{p+1}(\hat{V}_{p,q}, \hat{W}_{p,q}) \cong \mathbb{Z}, \quad \pi_q(\hat{W}_{p,q}) \cong \pi_q(S^p) \oplus \mathbb{Z},
\]

it follows that this morphism $T_{p,q}$ is an isomorphism of Wall forms. Combining this isomorphism together with Proposition 5.3 we obtain the following result.

**Proposition 6.1.** For all $g \in \mathbb{Z}_{\geq 0}$, there is an isomorphism of Wall forms, $W^g \cong \mathcal{W}^\partial_{p,q}(V^g_{p,q}, W^g_{p,q})$.

We are now ready to define the simplicial map $F_{p,q}$ from (6.1). Let $\phi \in K^\partial(M)_{p,q}$ be a vertex. We define $F_{p,q}(\phi) \in L(\mathcal{W}^\partial_{p,q}(M, \partial_1 M))$ to be the morphism of Wall forms given by the composite,

\begin{equation}
W \xrightarrow{T_{p,q}} \mathcal{W}^\partial_{p,q}(\hat{V}_{p,q}, \hat{W}_{p,q}) \xrightarrow{\phi_*} \mathcal{W}^\partial_{p,q}(M, \partial_1 M),
\end{equation}

where the second map $\phi_*$ is the morphism of Wall forms induced by the embedding $\phi$. Using this morphism $T_{p,q}$ together with Proposition 5.3 it follows that $r(\mathcal{W}^\partial_{p,q}(M, \partial_1 M)) \geq r_{p,q}(M).

6.2. Cohen Macaulay complexes and the link lifting property. Our next step is to prove that the simplicial map $T_{p,q}$ from (6.1) is highly connected. Doing this will require us to import a simplicial technique.

**Definition 6.1.** Let $f : X \to Y$ be a simplicial map between two simplicial complexes. The map is said to have the link lifting property if the following condition holds:
Lemma 6.2. Let \( y \in Y \) be a vertex and let \( A \subset X \) be a set of vertices such that \( f(a) \in \text{lk}_Y(y) \) for all \( a \in A \). Then there exists \( x \in X \) with \( f(x) = y \) such that \( a \in \text{lk}_X(x) \) for all \( a \in A \).

In order for the map \( f \) to have the link lifting property it is necessary that the above condition be satisfied for any set of vertices \( A \subset X \). In practice, it can be difficult to verify the above lifting property for totally arbitrary subsets \( A \subset X \), and one may only be able to verify it for sets of vertices subject to some condition. We will need to work with a refined version of Definition 6.1. Recall that for any set \( K \), a symmetric relation is a subset \( R \subset K \times K \) that is invariant under the “coordinate permutation” map \( (x, y) \mapsto (y, x) \). A subset \( C \subset K \) is said to be in general position with respect to \( R \) if \( (x, y) \in R \) for any two elements \( x, y \in C \). We will need to consider symmetric relations defined on the set of vertices of a simplicial complex. Let \( X \) be a simplicial complex and let \( R \subset X \times X \) be a symmetric relation on the vertices of \( X \). The relation \( R \) is said to be edge compatible if for any 1-simplex \( \{x, y\} \subset X \), the pair \((x, y)\) is an element of \( R \).

Definition 6.2. Let \( f : X \to Y \) be a simplicial map between two simplicial complexes. Let \( R \subset X \times X \) be a symmetric relation on the set of vertices of the complex \( X \). The map \( f \) is said to have the link lifting property with respect to \( R \) if the following condition holds:

- Let \( y \in Y \) be any vertex and let \( A \subset X \) be a finite set of vertices in general position with respect to \( R \) such that \( f(a) \in \text{lk}_Y(y) \) for all \( a \in A \). Then given another finite set of vertices \( B \subset X \) (not necessarily in general position with respect to \( R \)), there exists a vertex \( x \in X \) with \( f(x) = y \) such that \( a \in \text{lk}_X(x) \) for all \( a \in A \) and \((b, x) \in R \) for all \( b \in B \).

The following lemma below is a restatement of [12, Lemma 2.3]. Its proof in [12] abstracts and formalizes the argument used in the proof of [4, Lemma 5.4]. It also uses [4, Theorem 2.4] which is a generalization of the “Coloring Lemma” of Hatcher and Wahl from [9].

Lemma 6.2. Let \( X \) and \( Y \) be simplicial complexes and let \( f : X \to Y \) be a simplicial map. Let \( R \subset X \times X \) be an edge compatible symmetric relation. Suppose that the following conditions are met:

(i) \( f \) has the link lifting property with respect to \( R \);
(ii) \( lCM(Y) \geq n \);

Then the induced map \( |f|_* : \pi_j(|X|) \to \pi_j(|Y|) \) is injective for all \( j \leq n - 1 \). Furthermore, suppose that in addition to properties (i) and (ii) the map \( f \) satisfies:

(iii) \( f(\text{lk}_X(\zeta)) \leq \text{lk}_Y(f(\zeta)) \) for all simplices \( \zeta < X \).

Then it follows that \( lCM(X) \geq n \).

6.3. Proof of Theorem 3.2. We now give the proof of Theorem 3.2. We do this by applying Lemma 6.2 to the simplicial map \( F_{p,q} : K^\partial(M)_{p,q} \to L(W_{p,q}(M, \partial_1 M)) \). In order to apply this lemma to \( F_{p,q} \), we will need to define a suitable symmetric relation on the vertices of the complex \( K^\partial(M)_{p,q} \).

Definition 6.3. We define \( T \subset K^\partial(M)_{p,q} \times K^\partial(M)_{p,q} \) to be the subset consisting of those pairs \((\phi_1, \phi_2)\) such that \( \phi_1(B_{p,q}) \) and \( \phi_2(B_{p,q}) \) are transverse in \( M \).

Clearly the subset \( T \subset K^\partial(M)_{p,q} \times K^\partial(M)_{p,q} \) is a symmetric relation on the vertices. Furthermore, this relation is edge compatible, i.e. if the set \( \{\phi_1, \phi_2\} \leq K^\partial(M)_{p,q} \) is a 1-simplex then the pair \((\phi_1, \phi_2)\) is contained in \( T \).
Proof of Theorem 6.4. Let \( r_{p,q}(M) \geq g \) and let \( d = d(\pi_q(S^p)) \) be the generating set rank of \( \pi_q(S^p) \). We will show that \( |K(M)_{p,q}| \) is \( \frac{1}{2}(g-4-d) \)-connected and \( \text{ICM}(K^\partial(M)_{p,q}) \geq \frac{1}{2}(g-1-d) \). Since \( r(W^\partial_{p,q}(M)) \geq r_{p,q}(M) \geq g \), Theorem 5.1 implies that the space \( |L(W^\partial_{p,q}(M,\partial_1 M))| \) is \( \frac{1}{2}(g-4-d) \)-connected and that \( \text{ICM}(L(W^\partial_{p,q}(M,\partial_1 M)) \geq \frac{1}{2}(g-1-d) \). The proof of the theorem will follow from Lemma 6.2 once we verify the following two properties:

(i) the map \( F_{p,q} \) has the link lifting property with respect to \( T \) (see Definition 6.2),

(ii) \( F_{p,q}(\text{lk}_{K^\partial(M)_{p,q}}(\zeta)) \leq \text{lk}_{L(W^\partial_{p,q}(M,\partial_1 M))(F_{p,q}(\zeta))} \) for any simplex \( \zeta \in K^\partial(M)_{p,q} \).

We begin by verifying property (i). Let \( f : W \rightarrow W^\partial_{p,q}(M,\partial_1 M) \) be a morphism of Wall forms, which we consider to be a vertex of \( L(W^\partial_{p,q}(M,\partial_1 M)) \). Let \( \phi_1, \ldots, \phi_k \in K^\partial(M)_{p,q} \) be a collection of vertices in general position with respect to \( T \), such that \( F_{p,q}(\phi_i) \in \text{lk}_{L(W^\partial_{p,q}(M,\partial_1 M))(f)} \) for \( i = 1, \ldots, k \). Let \( \psi_1, \ldots, \psi_m \in K^\partial(M)_{p,q} \) be another arbitrary collection of vertices. To show that \( F_{p,q} \) has the link lifting property with respect to \( T \), we will construct a vertex \( \phi \in K^\partial(M)_{p,q} \) with \( F_{p,q} = f \), such that \( \phi_i \in \text{lk}_{K^\partial(M)_{p,q}}(\phi) \) for \( i = 1, \ldots, k \) and \( \phi_j, \phi \in T \) for \( j = 1, \ldots, m \).

Let \( \zeta : (D^{p+1}, S^p) \rightarrow (M, \partial_1 M) \) and \( \xi : S^q \rightarrow \partial_1 M \) be embeddings that represent the classes

\[
-f_-(a) \in \pi_{p+1}(M, \partial_1 M) \quad \text{and} \quad f_+(b) \in \pi_q(\partial_1 M)
\]

respectively (it follows from Lemma 4.4 that these classes may be represented by embeddings). Since the cores \( \phi_1(B_{p,q}), \ldots, \phi_k(B_{p,q}) \) are transverse and \( F_{p,q}(\phi_i) \in \text{lk}_{K^\partial(M)_{p,q}}(f) \) for all \( i = 1, \ldots, k \), we may apply Corollary 4.9.1 and Corollary 4.9.2 to deform the embeddings \( \zeta \) and \( \xi \) through isotopies to new embeddings \( \zeta' \) and \( \xi' \) such that the images \( \zeta'(D^{p+1}) \) and \( \xi'(S^q) \) are disjoint from the cores \( \phi_i(B_{p,q}) \) for \( i = 1, \ldots, k \). Let \( \partial_1 M' \) denote the complement \( \partial_1 M \setminus (\bigcup_{i=1}^{k} \phi_i(C_{p,q})) \). Since \( \lambda_{p,q}(f_-(a), f_+(b)) = 1 \), we may apply the Whitney trick to deform \( \zeta' : S^q \rightarrow \partial_1 M' \) through an isotopy of embeddings into \( \partial_1 M' \), to a new embedding \( \xi'' : S^q \rightarrow \partial_1 M' \) with the property that \( \zeta''(D^{p+1}) \) and \( \xi''(S^q) \) intersect transversally at exactly one point in \( \partial_1 M' \). By Thom’s transversality theorem we may further arrange \( \zeta'(D^{p+1}) \) and \( \xi''(S^q) \) to be transverse to each of the cores \( \psi_1(B_{p,q}), \ldots, \psi_m(B_{p,q}) \) while keeping them disjoint from \( \phi_1(B_{p,q}), \ldots, \phi_k(B_{p,q}) \).

It follows by the above construction that the pair of subspaces

\[
(\zeta'(D^{p+1}) \cup \xi''(S^q), \zeta'(\partial D^{p+1}) \cup \xi''(\partial S^q))
\]

is homeomorphic to the pair \( (D^{p+1} \cup S^q, S^p \cup S^q) \). Now, both \( \zeta'(S^p) \) and \( \xi''(S^q) \) have trivial normal bundles in \( \partial_1 M \); the normal bundle of \( \xi''(S^q) \) is trivial because \( \alpha(q(f_+(b)) = 0 \) and the normal bundle of \( \zeta'(S^p) \) is trivial because it bounds the disk \( \zeta'(D^{p+1}), \zeta'(S^p) \subset (M, \partial_1 M) \) which must have trivial normal bundle since the disk is contractible. Let \( U \subset \partial_1 M \) be a regular neighborhood of \( \zeta'(S^p) \cup \xi''(S^q) \) (which is a wedge of a \( p \)-sphere with a \( q \)-sphere). The manifold \( U \) is diffeomorphic to the manifold obtained by forming the push-out of the diagram

\[
\begin{array}{c}
S^p \times D^q \xrightarrow{i_p \times \text{Id}_{D^q}} D^p \times D^q \xrightarrow{\text{Id}_{D^p} \times i_q} D^p \times S^q,
\end{array}
\]

where \( i_p : D^p \rightarrow S^p \) and \( i_q : D^q \rightarrow S^q \) are embeddings. It is easily seen that this push-out is diffeomorphic to \( W_{p,q} = S^p \times S^q \setminus \text{Int}(D^{p+q}) \) after smoothing corners, thus we have a diffeomorphism \( U \cong W_{p,q} \). By shrinking \( U \) down arbitrarily close to its “core” \( \zeta'(S^p) \cup \xi''(S^q) \cong S^p \cup S^q \), we may assume that \( U \) is disjoint from \( \phi_i(C_{p,q}) \) for all \( i = 1, \ldots, k \).

Let \( \hat{U} \subset M \) be the submanifold diffeomorphic to \( U \times [0, 1] \) obtained by adding a collar to \( U \subset \partial_1 M \) in \( M \). We extend the embedding \( \zeta' \) to an embedding \( \hat{\zeta} : (D^{p+1} \times D^q, S^p \times D^q) \rightarrow (M, \partial_1 M) \) such
that \( \zeta(S^p \times D^n) \subset U \) and \( \zeta|_{S^p \times \{0\}} = \zeta' \). We let \( V \subset M \) be the subspace obtained by forming the union of \( U \) with \( \zeta(D^{p+1} \times D^n) \). By shrinking \( \zeta(D^{p+1} \times D^n) \) down to \( \zeta(D^{p+1} \times \{0\}) \), we may assume that \( V \) is again disjoint from \( \phi_i(B_{p,q}) \) for all \( i = 1, \ldots, k \). By Proposition 6.3 (proven below) there is a diffeomorphism \( V \cong V_{p,q} = D^{p+1} \times S^q \). Furthermore, the boundary of \( V \) has the decomposition \( \partial V = \partial_0 V \cup \partial_1 V \), with \( \partial_0 V = \partial V \cap \partial_1 M \) and \( \partial_1 V = \partial V \setminus \text{Int}(\partial_1 V) \). We have diffeomorphisms \( \partial_1 V \cong W_{p,q} \) and \( \partial_0 V \cong D^{p+q} \). Using these identifications 
\[ V \cong V_{p,q}, \quad \partial_0 V \cong D^{p+q}, \quad \text{and} \quad \partial_1 V \cong W_{p,q}, \]
we obtain an embedding \( (V_{p,q}, W_{p,q}) \hookrightarrow (M, \partial_1 M) \) with image equal to \( (V, \partial_1 V) \subset (M, \partial_1 M) \).

We then choose an embedding \( \gamma : [0, 1] \hookrightarrow \partial_1 M \), disjoint from \( \phi_i(B_{p,q}) \) for all \( i = 1, \ldots, k \), and with \( \gamma(0) \in \partial_0 V \) and \( \gamma(1) \in \partial_1 M \). Taking the union of a thickening of this arc with \( V \) yields an embedding \( (\widehat{V}_{p,q}, \widehat{W}_{p,q}) \hookrightarrow (M, \partial_1 M) \) that satisfies condition i. of Definition 3.2. This in turn yields a vertex \( \phi \in K^\partial(M)_{p,q} \) with \( F_{p,q}(\phi) = f \) such that \( \phi(B_{p,q}) \cap \phi_i(B_{p,q}) = \emptyset \) for all \( i = 1, \ldots, k \). It follows that \( \phi \) is contained in the link of \( \phi \) for \( i = 1, \ldots, k \). By construction, \( \phi(B_{p,q}) \) is transverse to \( \psi_j(B_{p,q}) \) for all \( j = 1, \ldots, m \). This proves that the map \( F_{p,q} \) has the link lifting property with respect to \( T \).

By Lemma 6.2 it follows that the induced map \( \pi_i(|K^\partial(M)_{p,q}|) \hookrightarrow \pi_i(|L(W_{p,q}^\partial(M, \partial_1 M)|) \) is injective for all \( i < \frac{1}{2}(g - 1 - d) \) and thus \( |K^\partial(M)_{p,q}| \) is \( \frac{1}{2}(g - 4 - d) \)-connected. In order to conclude that \( lCM(K^\partial(M)_{p,q}) \geq \frac{1}{2}(g - 1 - d) \), we need to establish property (ii). We need to verify that,
\[ F(|K^\partial(M)_{p,q}|) \leq lK_{L(W_{p,q}^\partial(M, \partial_1 M)}(F_{p,q}(\zeta)) \]
for any simplex \( \zeta \in K^\partial(M)_{p,q} \). This property follows immediately from the fact that if \( \phi_1, \phi_2 \in K^\partial(M)_{p,q} \) are such that \( \phi_1(B_{p,q}) \cap \phi_2(B_{p,q}) = \emptyset \), then the morphisms of Wall forms \( F_{p,q}(\phi_1) \) and \( F_{p,q}(\phi_2) \) are orthogonal. This concludes the proof of the theorem.

There is one claim in the above proof that still needs verification, namely that the manifold \( V \) that we constructed is diffeomorphic to \( V_{p,q} \). We now prove that claim. Pick a base point \((a, b) \in S^p \times S^q \) such that \((S^p \times \{b\}) \cup \{(a) \times S^q \} \subset W_{p,q} = S^p \times S^q \setminus \text{Int}(D^{p+q}) \). Let \( f : S^p \hookrightarrow W_{p,q} \) be the embedding given by the chain of inclusions, \( S^p \hookrightarrow S^p \times \{b\} \hookrightarrow (S^p \times \{b\}) \cup \{(a) \times S^q \} \hookrightarrow W_{p,q} \). This embedding has a trivial normal bundle. Let \( f' : S^p \times D^q \hookrightarrow W_{p,q} \) be an embedding with \( f'|_{S^p \times \{0\}} = f \). Finally let
\[ \tilde{f} : S^p \times D^q \hookrightarrow W_{p,q} \times [0, 1] \]
denote the embedding given by, \( S^p \times D^q \xrightarrow{f'} W_{p,q} \hookrightarrow W_{p,q} \times \{1\} \hookrightarrow W_{p,q} \times [0, 1] \). Let \( V \) denote the manifold obtained by attaching the handle \( D^{p+1} \times D^q \) to \( W_{p,q} \times [0, 1] \) along the embedding \( \tilde{f} \), i.e.
\[ V = (W_{p,q} \times [0, 1]) \cup \left( D^{p+1} \times D^q \right) \].

The boundary of \( W_{p,q} \) has the decomposition, \( \partial V = (W_{p,q} \times \{0\}) \cup (\partial W_{p,q} \times [0, 1]) \cup W' \), where \( W' \) is the manifold obtained from \( W_{p,q} \) by performing surgery along the embedding \( \tilde{f} \). Consequently the manifold \( W' \) is diffeomorphic to a disk \( D^{p+q} \) and so \( \partial V \) is diffeomorphic to \( W_{p,q} \). The following proposition was used in the above proof of Theorem 3.2.

**Proposition 6.3.** Let \( p, q \in \mathbb{N} \) satisfy the inequality \( |q - p| < \min\{p, q\} - 2 \). Then the manifold \( V \) constructed above is diffeomorphic to \( D^{p+1} \times S^q = V_{p,q} \).
Proof. Let \( g : S^q \to V \) be the embedding given by the chain of inclusions,

\[
S^q \hookrightarrow \{a\} \times S^q \hookrightarrow W_{p,q} \hookrightarrow W_{p,q} \times \{\frac{1}{2}\} \hookrightarrow W_{p,q} \times [0,1] \to V.
\]

This embedding has trivial normal bundle and so extends to an embedding, \( \bar{g} : D^{p+1} \times S^q \to \text{Int}(V) \). Furthermore, \( \bar{g} \) induces an isomorphism on homology and thus is a homotopy equivalence since \( V \) is simply connected. Let \( X \) denote the complement \( V \setminus \text{Int}(\bar{g}(D^{p+1} \times S^q)) \). The boundary of \( X \) decomposes as the disjoint union \( \partial X = \partial V \sqcup \partial \bar{g}(D^{p+1} \times S^q) \). By excision we have the isomorphism

\[
0 = H_i(V, \bar{g}(D^{p+1} \times S^q)) \cong H_i(X, \partial \bar{g}(D^{p+1} \times S^q)) \quad \forall \ i \in \mathbb{Z}_{\geq 0},
\]

and for all an embedding, \( \overline{h} : D^{p+1} \times S^q \to \text{Int}(V) \), such that:

(i) \( (\partial_0 M', \partial_0 M') = (S, \partial S) \);
(ii) \( (M', \partial_1 M') \) contains \( ([0,\varepsilon) \times S, [0,\varepsilon) \times \partial S) \) for some \( \varepsilon > 0 \);
(iii) \( (M'; \partial_0 M', \partial_1 M') \) is diffeomorphic to \( (M; \partial_0 M, \partial_1 M) \).

With the above proposition established, the proof of Theorem 3.2 is now complete. By Corollary 3.2.1, it now follows that the geometric realization \( |\bar{K}^0(M)_{p,q}| \) is \( \frac{1}{2}[g - 4 - d(\pi_q(S^p))] \)-connected whenever \( r_{p,q}(M) \geq g \).

7. Homological Stability

With our main technical result Theorem 3.2 established, in this section we show how this theorem implies our homological stability theorem, Theorem 2.3. The constructions and arguments in this section are essentially the same as what was done in [3, Section 6] (or the earlier unpublished preprint [2, Section 5]) and so we merely provide an outline while referring the reader to those above mentioned papers for details.

7.1. A Model for \( \text{BDiff}(M, \partial_0 M) \). Let \( (M; \partial_0 M, \partial_1 M) \) be a compact manifold triad of dimension \( m \) with \( \partial_0 M \) and \( \partial_1 M \) non-empty. We construct a concrete model for \( \text{BDiff}(M, \partial_0 M) \). Fix once and for all an embedding, \( \theta : (\partial_0 M, \partial_0 \partial_1 M) \to (\mathbb{R}_+^\infty, \partial \mathbb{R}_+^\infty) \) and let \( (S, \partial S) \) denote the submanifold pair, \( (\theta(\partial_0 M), \theta(\partial_0 \partial_1 M)) \subset (\mathbb{R}_+^\infty, \partial \mathbb{R}_+^\infty) \).

**Definition 7.1.** We define \( \mathcal{M}(M) \) to be the set of compact \( m \)-dimensional submanifold triads

\[
(M'; \partial_0 M', \partial_1 M') \subset ([0,\infty) \times \mathbb{R}_+^\infty; [0] \times \mathbb{R}_+^\infty, [0,\infty) \times \partial \mathbb{R}_+^\infty)
\]

such that:

(i) \( (\partial_0 M', \partial_0 \partial_1 M') = (S, \partial S) \);
(ii) \( (M', \partial_1 M') \) contains \( ([0,\varepsilon) \times S, [0,\varepsilon) \times \partial S) \) for some \( \varepsilon > 0 \);
(iii) \( (M'; \partial_0 M', \partial_1 M') \) is diffeomorphic to \( (M; \partial_0 M, \partial_1 M) \).

Denote by \( \mathcal{E}(M) \) the space of embeddings, \( (M; \partial_0 M, \partial_1 M) \to ([0,\infty) \times \mathbb{R}_+^\infty; [0] \times \mathbb{R}_+^\infty, [0,\infty) \times \partial \mathbb{R}_+^\infty) \), topologized in the \( C^\infty \)-topology. The space \( \mathcal{M}(M) \) is topologized as a quotient of the space \( \mathcal{E}(M) \) where two embeddings are identified if they have the same image.
It follows from Definition [7.1] that $\mathcal{M}(M)$ is equal to the orbit space, $\mathcal{E}(M)/\text{Diff}(M, \partial_0 M)$. By [33] Lemma A.1, it follows that the quotient map $\mathcal{E}(M) \rightarrow \mathcal{E}(M)/\text{Diff}(M, \partial_0 M) = \mathcal{M}(M)$ is a locally trivial fibre-bundle. In [2] it is proven that the space $\mathcal{E}(M)$ is weakly contractible. These two facts together imply that there is a weak-homotopy equivalence, $\mathcal{M}(M) \simeq \text{BDiff}(M, \partial_0 M)$, and thus we may take the space $\mathcal{M}(M)$ to be a model for the classifying space of the diffeomorphism group $\text{Diff}(M, \partial_0 M)$.

Now let $p$ and $q$ be positive integers with $p + q + 1 = m$. Recall from Section 2 the relative cobordism, $(K_{p,q}, \partial_1 K_{p,q}) : (\partial_0 M \times \{0\}, \partial_0 1 \times \{0\}) \simeq (\partial_0 M \times \{1\}, \partial_0 1 \times \{1\})$. Choose an embedding,

$$\alpha : (K_{p,q}; \partial_0 K_{p,q}, \partial_1 K_{p,q}) \rightarrow ([0, 1] \times \mathbb{R}_+^{\infty}; \{0, 1\} \times \mathbb{R}^{\infty}, \{0, 1\} \times \partial \mathbb{R}^{\infty}),$$

that satisfies $\alpha(i, x) = (i, \theta(x))$ for all $(i, x) \in \{0, 1\} \times \partial_0 M = \partial_0 K_{p,q}$. For a submanifold $M' \subset [0, \infty) \times \mathbb{R}_+^{\infty}$, denote by $M' + e_1 \subset [1, \infty) \times \mathbb{R}_+^{\infty}$ the submanifold obtained by translating $M'$ over 1-unit in the first coordinate. For $M' \in \mathcal{M}(M)$, the submanifold $\alpha(K_{p,q}) \cup (M' \cup e_1) \subset [0, \infty) \times \mathbb{R}_+^{\infty}$ is an element of the space $\mathcal{M}(M \cup_0 K_{p,q})$, and thus we have a continuous map,

$$s_{p,q} : \mathcal{M}(M) \rightarrow \mathcal{M}(M \cup_0 K_{p,q}); \quad M' \mapsto \alpha(K_{p,q}) \cup (M' + e_1).$$

The construction of $s_{p,q}$ depends on the choice of embedding $\alpha$. Any two such embeddings are isotopic and thus it follows that the homotopy class of $s_{p,q}$ does not depend on any such choice. Using the weak homotopy equivalence $\mathcal{M}(M) \simeq \text{BDiff}(M, \partial_0 M)$, it follows that the homotopy class $s_{p,q}$ agrees with the homotopy class of the stabilization map, $\text{BDiff}(M, \partial_0 M) \rightarrow \text{BDiff}(M \cup_0 K_{p,q}, \partial_0 M)$ defined in Section 2.

### 7.2. A Semi-Simplicial Resolution

Let $(M; \partial_0 M, \partial_1 M)$ be an $m$-dimensional manifold triad as in Section 7.1. Fix $p, q \in \mathbb{N}$ such that $p + q + 1 = m$. We now construct a semi-simplicial resolution of the moduli space $\mathcal{M}(M)$. Choose once and for all a coordinate patch $c : (\mathbb{R}^{m-1}_+, \partial \mathbb{R}^{m-1}_+) \hookrightarrow (S, \partial S)$. Such a coordinate patch induces for each $M' \in \mathcal{M}(M)$ a germ of an embedding $[0, 1] \times \mathbb{R}^{m-1}_+ \rightarrow M'$ as in the definition of $K_{p,q}^\partial(M')_{p,q}$ (see Definition 3.4). For each non-negative integer $l$, we define $X^\partial_l(M)_{p,q}$ to be the set of pairs $(M', \phi)$ where $M' \in \mathcal{M}(M)$ and $\phi \in K_{l}^\partial(M')_{p,q}$. The space $X^\partial_{l+1}(M)_{p,q}$ is topologized as the quotient,

$$X^\partial_{l+1}(M)_{p,q} = ([\mathcal{E}(M) \times K_{l}^\partial(M)_{p,q}] / \text{Diff}(M, \partial_0 M)).$$

The assignments $[\ell] \mapsto X^\partial_{\ell}(M)_{p,q}$ make $X^\partial(M)_{p,q}$ into a semi-simplicial space where the face maps are induced by the face maps in $K_{p,q}^\partial(M)_{p,q}$. The forgetful maps $X^\partial_{l+1}(M)_{p,q} \rightarrow \mathcal{M}(M)$, $(M', \phi) \mapsto M'$, assemble to yield the augmented semi-simplicial space, $X^\partial(M)_{p,q} \rightarrow X^\partial_{-1}(M)_{p,q}$, where the space $X^\partial_{-1}(M)_{p,q}$ is set equal to $\mathcal{M}(M)$. We have the following proposition:

**Proposition 7.1.** The augmentation map $|X^\partial(M)_{p,q}| \rightarrow X^\partial_{-1}(M)_{p,q}$ is half-connected, where $d = d(\pi_p(S^q))$ is the generating set length.

**Proof.** For each $l \in \mathbb{Z}_{\geq 0}$ the forgetful map $X^\partial_l(M)_{p,q} \rightarrow \mathcal{M}(M)$ is a locally trivial fibre bundle with fibre given by the space $K^\partial_l(M)_{p,q}$. It follows that $|X^\partial_l(M)_{p,q}| \rightarrow X^\partial_{-1}(M)_{p,q}$ is a locally trivial fibre bundle as well with fibre given by $|K^\partial_{l+1}(M)_{p,q}|$. The proposition then follows from Corollary 3.2.1 using the long exact sequence on homotopy groups associated to a fibre-sequence.

□
7.3. Proof of Theorem [2,3] We now will show how to use the augmented semi-simplicial space $X^\partial_\bullet(M)_{p,q} \to X^\partial_{-1}(M)_{p,q}$ to complete the proof of Theorem [2,3]. First, we fix some new notation which will make the steps of the proof easier to state. For what follows let $(M; \partial_0 M, \partial_1 M)$ be a compact $m$-dimensional manifold triad with non-empty boundary. As in the previous sections, choose positive integers $p$ and $q$ with $p+q+1 = m$ that satisfy the inequalities [2,4] with respect to $(M, \partial_1 M)$. We work with the same choice of $p$ and $q$ for the rest of the section. For each $g \in \mathbb{N}$ we denote by $M_g$ the manifold obtained by forming the boundary connected-sum of $M$ with $(D^{r+1} \times S^q)^\#g$ along the face $\partial_1 M$. Clearly we have $r_{p,q}(M_g) \geq g$. We consider the spaces $\mathcal{M}(M_g)$. For each $g \in \mathbb{N}$ we have the stabilization map $s_{p,q} : \mathcal{M}(M_g) \to \mathcal{M}(M_{g+1})$. Theorem [2,3] translates to the following statement:

**Theorem 7.2.** The induced map $(s_{p,q})_* : H_k(\mathcal{M}(M_g); \mathbb{Z}) \to H_k(\mathcal{M}(M_{g+1}); \mathbb{Z})$ is an isomorphism when $k \leq \frac{1}{2}(g - 3 - d)$ and an epimorphism when $k \leq \frac{1}{2}(g - 1 - d)$.

Since $r_{p,q}(M_g) \geq g$, Proposition [2,1] implies that the map $|X^\partial_\bullet(M_{g})_{p,q}| \to X^\partial_{-1}(M_{g})_{p,q}$ is $\frac{1}{2}(g - 2 - d)$-connected. For each pair of integers $g, k \in \mathbb{Z}_{\geq 0}$ with $k \leq g$, there is a map

$$F^\partial_g : \mathcal{M}(M_{g-k-1}) \to X^\partial_k(M_{g})_{p,q}$$

defined as follows. Let $K^k_{p,q} \subset [0,k+1] \times \mathbb{R}_+^m$ denote the $(k+1)$-fold concatenation of the submanifold $\alpha(K_{p,q}) \subset [0,1] \times \mathbb{R}_+$ used in the construction of stabilization map [2,1], i.e.

$$K^k_{p,q} = \alpha(K_{p,q}) \cup [\alpha(K_{p,q})+e_1] \cup \cdots \cup [\alpha(K_{p,q})+k \cdot e_1].$$

For each $k \in \mathbb{Z}_{\geq 0}$ we fix a $k$-simplex $(\zeta_0, \ldots, \zeta_k) \in \tilde{K}^\partial(K^k_{p,q})$. The map $F^\partial_g$ from [2,2] is defined by the formula, $F^\partial_g(M') = ((\zeta_0, \ldots, \zeta_k), K^k_{p,q} \cup [(k+1) \cdot e_1 + M'])$. It follows directly from the definition of $F^\partial_g$ that for each pair $k \leq g$, the diagram

$$\begin{array}{ccc}
\mathcal{M}(M_{g-k-1}) & \xrightarrow{S_{p,q}} & \mathcal{M}(M_{g-k}) \\
\downarrow F^\partial_g & & \downarrow F^\partial_g \\
X^\partial_k(M_{g})_{p,q} & \xrightarrow{d_k} & X^\partial_{k-1}(M_{g})_{p,q}
\end{array}$$

is commutative. The following proposition is proven in the same way as [3] Propositions 5.3 and 5.5. For this reason we omit the proof and refer the reader to these analogous results from [3] and [4] for details.

**Proposition 7.3.** Let $g \geq 4 + d$. We have the following:

(i) For each $k < g$, the map $F^\partial_g : \mathcal{M}(M_{g-k-1}) \to X^\partial_k(M_{g})_{p,q}$ is a weak homotopy equivalence.

(ii) The face maps $d_i : X^\partial_k(M_{g})_{p,q} \to X^\partial_{k-1}(M_{g})_{p,q}$ are weakly homotopic.

To finish the proof of Theorem [2,2] consider the spectral sequence associated to the augmented semi-simplicial space $X^\partial_\bullet(M_{g})_{p,q} \to X^\partial_{-1}(M_{g})_{p,q}$ with $E^1$-term given by $E^1_{j,l} = H_j(X^\partial_{1}M_{g})_{p,q; \mathbb{Z}}$ for $l \geq -1$ and $j \geq 0$. The differential is given by $d^1 = \sum (-1)^i (d_i)_*$, where $(d_i)_*$ is the map on homology induced by the $i$th face map in $X^\partial_\bullet(M_{g})_{p,q}$. The group $E^\infty_{j,l}$ is a subquotient of the relative homology group

$$H_{j+l+1}(X^\partial_{-1}(M_{g})_{p,q}, |X^\partial_\bullet(M_{g})_{p,q}|; \mathbb{Z}).$$
Proposition 7.3 together with Corollary 7.1 and commutativity of diagram (7.3) imply the following facts:

(a) For \( g \geq 4 + d \), there are isomorphisms \( E^1_{j,l} \cong H_\ell(M(M_{g-j-1}); \mathbb{Z}) \).

(b) The differential \( d^1 : H_\ell(M(M_{g-j-1}); \mathbb{Z}) \cong E^1_{j,l} \to E^1_{j-1,l} \cong H_\ell(M(M_{g-j}); \mathbb{Z}) \) is equal to \( (s_{p,q})_* \) when \( j \) is even and is equal to zero when \( j \) is odd.

(c) The term \( E^\infty_{j,l} \) is equal to 0 when \( j + l \leq \frac{1}{2}(g - 2 - d) \).

To complete the proof one uses (c) to prove that the differential \( d^1 : E^1_{2j,l} \to E^1_{2j-1,l} \) is an isomorphism when \( 0 < j \leq \frac{1}{2}(g - 3 - d) \) and an epimorphism when \( 0 < j \leq \frac{1}{2}(g - 1 - d) \). This is done by carrying out the exact inductive argument given in [3, Section 5.2: Proof of Theorem 1.2]. This establishes Theorem 7.2 and the main result of this paper, Theorem 2.3.

8. Tangential Structures

In this section we prove an analogue of Theorem 1.1 for the moduli spaces of manifolds equipped with tangential structures. Recall that a tangential structure is a map \( \theta : B \to BO(d) \). A \( \theta \)-structure on a \( d \)-dimensional manifold \( M \) is a bundle map (fibrewise linear isomorphism) \( \ell : TM \to \theta^*\gamma^d \). More generally, if \( M \) is an \( l \)-dimensional manifold with \( l \leq d \), then a \( \theta \)-structure on \( M \) is a bundle map \( TM \oplus \epsilon^{d-l} \to \theta^*\gamma^d \).

Fix a tangential structure \( \theta : B \to BO(d) \). Let \( M \) be a \( d \)-dimensional manifold with boundary. Let \( P \subset \partial M \) be a codimension-0 submanifold and let \( \ell_P : TP \oplus \epsilon^1 \to \theta^*\gamma^d \) be a \( \theta \)-structure. We define

\[
\text{Bun}(TM, \theta^*\gamma^d; \ell_P) \subset \text{Bun}(TM, \theta^*\gamma^d)
\]

to be the subspace consisting of those \( \theta \)-structures on \( M \) that agree with \( \ell_P \) when restricted to \( P \).

The formula, \( \text{Bun}(TM, \theta^*\gamma^d; \ell_P) \times \text{Diff}(M, P) \to \text{Bun}(TM, \theta^*\gamma^d; \ell_P), (\ell, f) \mapsto \ell \circ Df \), defines a continuous action of the topological group \( \text{Diff}(M, P) \) on the space \( \text{Bun}(TM, \theta^*\gamma^d; \ell_P) \). We define \( \text{BDiff}_\theta(M, \ell_P) \) to be the homotopy quotient, \( \text{Bun}(TM, \theta^*\gamma^d; \ell_P)/\text{Diff}(M, P) \).

We proceed to construct stabilization maps analogous to those defined in Section 2. This will require us to make some choices. Let \( p, q \in \mathbb{Z}_{\geq 0} \) be integers such that \( p + q + 1 = d \).

**Definition 8.1.** Fix once and for all a bundle map \( \tau : \mathbb{R}^d \to \theta^*\gamma^d \). This choice determines a canonical \( \theta \)-structure on any framed \( d \)-dimensional manifold. If \( X \) is any such framed \( d \)-dimensional manifold, we denote this canonical \( \theta \)-structure on \( X \) by \( \ell_X^\tau \). Choose once and for all a framing, \( TV^1_{p,q} \cong V^1_{p,q} \times \mathbb{R}^{p+q+1} \), and consider the canonical \( \theta \)-structure, \( \ell^\tau_{V^1_{p,q}} \), induced by this chosen framing.

We call a \( \theta \)-structure \( \ell : TV^1_{p,q} \to \theta^*\gamma^d \) standard if it is homotopic to the canonical \( \theta \)-structure \( \ell^\tau_{V^1_{p,q}} \).

Let \( (M; \partial_0 M, \partial_1 M) \) be a \( d \)-dimensional manifold triad with \( \partial_0 M \) and \( \partial_1 M \) non-empty. Let \( (K_{p,q}; \partial_0 K_{p,q}) \) be the relative cobordism between \( (\partial_0 M \times \{0\}, \partial_0,1 M \times \{0\}) \) and \( (\partial_0 M \times \{1\}, \partial_1,1 M \times \{1\}) \), introduced in Section 2. Let us denote \( P := \partial_0 M \) and fix a \( \theta \)-structure \( \ell_P : TP \oplus \epsilon^1 \to \theta^*\gamma^d \). Choose a \( \theta \)-structure \( \ell_{K_{p,q}} : TK_{p,q} \to \theta^*\gamma^d \) that agrees with \( \ell_P \) on both components of \( \partial_0 K_{p,q} = \partial_0 M \times \{0,1\} \), and that is standard (in the sense of Definition 8.1) when restricted to \( V^1_{p,q} \) (where \( V^1_{p,q} \) is considered as a submanifold of \( K_{p,q} = (\partial_0 M \times [0,1])V^1_{p,q} \)). With this choice of \( \theta \)-structure
we obtain a map,

$$\text{Bun}(TM, \theta^*\gamma^d; \ell_P) \rightarrow \text{Bun}(T(M \cup P K_{p,q}), \theta^*\gamma^d; \ell_P), \quad \ell \mapsto \ell \cup \ell_{K_{p,q}}.$$  

This map is $\text{Diff}(M, P)$-equivariant, and thus it induces a map,

$$(8.1) \quad s_{p,q}^\theta : \text{BDiff}_\theta(M, \ell_P) \rightarrow \text{BDiff}_\theta(M \cup P K_{p,q}, \ell_P).$$

In addition to the inequalities imposed on $p$ and $q$ in the statement of Theorem 8.1, the following theorem will require us to also impose the further condition, $q \leq p$, and to assume that $\theta : B \rightarrow BO(d)$ is such that the space $B$ is $q$-connected.

**Theorem 8.1.** Let $p$ and $q$ be positive integers with $p + q + 1 = d = \text{dim}(M)$ and suppose that the inequalities of (8.4) are satisfied. Suppose further that $q \leq p$ and that $\theta : B \rightarrow BO(d)$ is such that $B$ is $q$-connected. Suppose that $r_{p,q}(M) \geq g$. Then the homomorphism,

$$(s_{p,q}^\theta)_* : H_k(\text{BDiff}_\theta(M, \ell_P); \mathbb{Z}) \rightarrow H_k(\text{BDiff}_\theta(M \cup P K_{p,q}, \ell_P); \mathbb{Z}),$$

is an isomorphism when $k \leq \frac{1}{2}(g - 4)$ and an epimorphism when $k \leq \frac{1}{2}(g - 2)$.

The proof of the above theorem is similar to Theorem 2.3 and requires only slight modifications. The main ingredient of the proof is to show that the tangentially structured analogue of the simplicial complex $K^\partial(M)_{p,q}$ (see Definition 8.2 below) is highly-connected relative to the rank $r_{p,q}(M)$. With this high-connectivity established, the proof of Theorem 8.1 follows in exactly the same way as the proof of Theorem 2.3 as outlined in Section 7. We will show explicitly how to prove high-connectivity of the complex (Proposition 8.3), and refer the reader to [4, Section 7] for the rest of the argument, which by this point is standard.

We proceed to construct a simplicial complex (and related semi-simplicial spaces) analogous to the one constructed in Section 3. Let $P$ and $\ell_P : TP \oplus e^1 \rightarrow \theta^* \gamma^d$ be as in the statement of Theorem 8.1. Let $(M; \partial_0 M, \partial_1 M)$ be a $d$-dimensional compact manifold triad, with $\partial_0 M = P$. Choose a $\theta$-structure $\ell_M \in \text{Bun}(TM, \theta^* \gamma^d; \ell_P)$. The definition below should be compared to [4, Definition 7.10].

**Definition 8.2.** Let $a : [0,1) \times \mathbb{R}^{d-1}_+ \rightarrow M$ be an embedding with $a^{-1}(\partial_0 M) = \{0\} \times \mathbb{R}^{d-1}_+$ and $a^{-1}(\partial_1 M) = [0,1) \times \partial \mathbb{R}^{d-1}_+$. For each pair of positive integers $p$ and $q$ with $p + q + 1 = d$, we define a simplicial complex $K^\partial(M, \ell_M, a)_{p,q}$ as follows:

(i) A vertex in $K^\partial(M, \ell_M, a)_{p,q}$ is defined to be a triple $(t, \phi, \gamma)$, where $(t, \phi)$ is an element of $K^\partial(M, a)_{p,q}$, and $\gamma$ is a path in $\text{Bun}(\hat{V}_{p,q}, \theta^* \gamma^d)$, starting at $\phi^* \ell_M$, ending at $\ell_{V_{p,q}}$, and constant on the subset $D^+_{p,q} \times \{0\} \subset \hat{V}_{p,q}$.

(ii) A set of vertices $\{(t_0, \phi_0, \gamma_0), \ldots, (t_l, \phi_l, \gamma_l)\}$ forms an $l$-simplex if $t_i \neq t_j$ and $\phi_i(B_{p,q}) \cap \phi_j(B_{p,q}) = \emptyset$ whenever $i \neq j$, just as in Definition 3.2.

As in Section 3 we will also need to work with a semi-simplicial space $K^\bullet_* (M, \ell_M, a)_{p,q}$ analogous to the simplicial complex defined above.

(i) The space of $0$-simplices $K^\partial_0(M, \ell_M, a)_{p,q}$ is defined to have the same underlying set as the set of vertices of the simplicial complex $K^\partial(M, \ell_M, a)_{p,q}$.
(ii) The space of $l$-simplices, $K^0_l(M, ℓ_M, a) \subset (K^0_0(M, ℓ_M, a))^{l+1}$ consists of the ordered $(l+1)$-tuples,
$$((t_0, φ_0, γ_0), \ldots, (t_l, φ_l, γ_l)),$$
such that $t_0 < \cdots < t_l$ and $φ_i(B_{p,q}) \cap φ_j(B_{p,q}) = \emptyset$ when $i \neq j$.

The spaces $K^0_l(M, ℓ_M, a) \subset (\mathbb{R} \times \text{Emb}(V_{p,q}, M) \times \text{Bun}(TV_{p,q}, \theta^*γ^d))^{l+1}$ are topologized using the $C^\infty$-topology on the spaces of embeddings and bundle maps. The assignments $[l] \mapsto K^0_l(M, ℓ_M, a)$ define a semi-simplicial space denoted by $K^0(M, ℓ_M, a)_{p,q}$ with face maps defined the same way as in Definition 3.4.

Finally, the sub-semi-simplicial space $K^0(M, ℓ_M, a)_{p,q} \subset K^0(M, ℓ_M, a)_{p,q}$ is defined to be the sub-semi-simplicial space consisting of all simplices $((t_0, φ_0, γ_0), \ldots, (t_l, φ_l, γ_l)) \in K^0_l(M, ℓ_M, a)$ such that $φ_i(V_{p,q}) \cap φ_j(V_{p,q}) = \emptyset$ whenever $i \neq j$.

The key technical result that we will need is the lemma stated below. This lemma is the source of the requirement that $q \leq p$ and that the space $B$ be $q$-connected. Fix a $θ$-structure $ℓ_D$ on the disk $D^{d-1}$ and fix an embedding $D^{d-1} \hookrightarrow \partial V_{p,q}$. We consider the space $\text{Bun}(TV_{p,q}, \theta^*γ^d; ℓ_D)$.

**Lemma 8.2.** Suppose that $q \leq p$ and that $θ : B \to BO(d)$ is chosen so that $B$ is $q$-connected. Then given any two elements $ℓ_1, ℓ_2 \in \text{Bun}(TV_{p,q}, \theta^*γ^d; ℓ_D)$, there exists a diffeomorphism $f \in \text{Diff}(V_{p,q}, D^{p+q})$ such that $ℓ_1$ is homotopic to $ℓ_2 \circ Df$.

**Proof.** Since the space $B$ is $q$-connected and $V_{p,q}$ is homotopy equivalent to $S^q$, it follows that the underlying map $V_{p,q} \to B$ of any $θ$-structure on $V_{p,q}$ is null-homotopic. It follows that every $θ$-structure on $V_{p,q}$ is homotopic to one that is induced by a framing of the tangent bundle. That is, every $ℓ \in \text{Bun}(TV_{p,q}, \theta^*γ^d; ℓ_D)$ is homotopic to a $θ$-structure of the form,
$$TV_{p,q} \cong V_{p,q} \times \mathbb{R}^d \xrightarrow{\text{pr}} \mathbb{R}^d \xrightarrow{\tau} \theta^*γ^d,$$
where the first arrow is a framing of the tangent bundle. Fix a framing, $φ_D : TD^{d-1} \oplus e^1 \xrightarrow{\cong} D^{d-1} \times \mathbb{R}^d$. Let $\text{Fr}(TV_{p,q}; φ_D)$ denote the space of framings of $TV_{p,q}$ that agree with $ℓ_D$ when restricted to the disk $D^{d-1} \subset \partial V_{p,q}$. From the observation made above, to prove the lemma it will suffice to prove the following statement: given any two framings $φ_1, φ_2 \in \text{Fr}(TV_{p,q}; φ_D)$, there exists $f \in \text{Diff}(V_{p,q}, D^d)$ such that $φ_1 \circ Df$ is homotopic (through framings) to $φ_2$. Let $φ_1, φ_2 \in \text{Fr}(TV_{p,q}; φ_D)$. The tangent bundle $TV_{p,q}$ has a natural splitting $E \oplus N \cong TV_{p,q}$. The bundle $E$ is the pull-back of the tangent bundle $TS^q$ over the projection $V_{p,q} \to S^q$ and $N$ is the pull-back of the normal bundle of $S^q \to V_{p,q}$. The bundle $N \to V_{p,q}$ is trivial and has fibres of dimension $p+1$. Pick once and for all a standard framing,
$$φ : TV_{p,q} \xrightarrow{\cong} V_{p,q} \times \mathbb{R}^d.$$

Since $p \geq q$ by assumption, the stabilization map $π_q(SO_{p+1}) \to π_q(SO)$ is surjective. From this is follows that $φ_i$ (for $i = 1, 2$) is homotopic to a framing of the form
$$TV_{p,q} \cong E \oplus N \xrightarrow{\text{Id}_E \oplus φ_i} E \oplus N \cong TV_{p,q} \xrightarrow{φ} V_{p,q} \times \mathbb{R}^d,$$
for some bundle isomorphism $φ_i : N \xrightarrow{\cong} N$. To prove the proposition it will suffice to show the following: given two bundle isomorphisms $α_1, α_2 : N \xrightarrow{\cong} N$ that agree when restricted to $D^{d-1} \subset \partial V_{p,q}$, there exists $f \in \text{Diff}(V_{p,q}, D^{d-1})$ such that $α_1 \circ Df = α_2$. Let $D(N)$ denote the disk
bundle associated to $N$. Let $\tilde{\alpha}_1, \tilde{\alpha}_2 : D(N) \to D(N)$ be the bundle maps induced by $\alpha_1$ and $\alpha_2$. Choose a diffeomorphism $h : V_{p,q} \to D(N)$. By setting $f$ equal to the diffeomorphism given by,

$$V_{p,q} \xrightarrow{h} D(N) \xrightarrow{\tilde{\alpha}_2 \circ \tilde{\alpha}_1^{-1}} D(N) \xrightarrow{h^{-1}} V_{p,q},$$

it follows that $\tilde{\alpha}_2 \circ D(f) = \tilde{\alpha}_1$. This concludes the proof of the lemma.

The main technical ingredient in the proof of Theorem 8.1 is the following proposition.

**Proposition 8.3.** Let $(M, \partial_0 M, \partial_1 M)$ and $p + q + 1 = d$ satisfy the inequalities (1.2). Suppose further that $q \leq p$ and that $\theta : B \to BO(d)$ is chosen so that $B$ is $q$-connected. Let $r_{p,q}(M) \geq g$. Then for any $\ell_M \in Bun(TM, \theta^* \gamma^d ; \ell_P)$, the geometric realization $\hat{\mathcal{K}}_{\bullet}^0(M, \ell_M, a)_{p,q}$ is $\frac{1}{2}(g - 4)$-connected.

**Proof.** As before, the degree of connectivity of $|\hat{\mathcal{K}}^0_{\bullet}(M, \ell_M, a)_{p,q}|$ is bounded below by the degree of connectivity of $|\mathcal{K}^0(M, \ell_M, a)_{p,q}|$. To prove the theorem it will suffice to show that $|\mathcal{K}^0(M, \ell_M, a)_{p,q}|$ is $\frac{1}{2}(g - 4)$-connected. Let $\mathcal{T} \subset \mathcal{K}^0(M, \ell_M, a)_{p,q} \times \mathcal{K}^0(M, \ell_M, a)_{p,q}$ be the symmetric relation from Definition 6.3. As in Section 6 we have a simplicial map $F_{p,q} : \mathcal{K}^0(M, \ell_M, a)_{p,q} \to L(\mathcal{W}_{p,q}^d(M, \partial_1 M))$ defined by sending $(t, \phi, \gamma)$ to the morphism of Wall forms $W_{p,q} \to W_{p,q}^d(M, \partial_1 M)$ induced by $\phi$. We will need to show that the map $F_{p,q}$ has the following properties:

(i) the map $F_{p,q}$ has the link lifting property with respect to $\mathcal{T}$ (see Definition 6.2);

(ii) $F_{p,q}(\text{lk}_{\mathcal{K}^0(M, \ell_M, a)_{p,q}}(\zeta)) \leq \text{lk}(W_{p,q}^d(M, \partial_1 M))(F_{p,q}(\zeta))$ for any simplex $\zeta \in \mathcal{K}^0(M, \ell_M, a)_{p,q}$.

Condition (ii) is proven in the same way as in the proof of Theorem 3.2. The proof of condition (i) is similar to the proof of Theorem 3.2 but requires one extra step which we describe below. This step will rely on Lemma 8.2 and thus requires the conditions that $q \leq p$ and $B$ is $q$-connected.

Let $f : \mathcal{W} \to W_{p,q}^d(M, \partial_1 M)$ be a morphism of Wall forms, i.e. a vertex of $L(\mathcal{W}_{p,q}^d(M, \partial_1 M))$. Let $(\phi_1, \beta_1), \ldots, (\phi_k, \beta_k) \in \mathcal{K}^0(M, \ell_M, a)_{p,q}$ be a collection of vertices that is in general position with respect to $\mathcal{T}$, and such that $F_{p,q}(\gamma_i, \phi_i) \in \text{lk}(f)$ for $i = 1, \ldots, k$.

(we have dropped the numbers $t_1, \ldots, t_k$ from the notation to save space). Let $(\phi'_1, \beta'_1), \ldots, (\phi'_m, \beta'_m) \in \mathcal{K}^0(M, \ell_M, a)_{p,q}$ be another arbitrary collection of vertices. To show that $F_{p,q}$ has the link lifting property with respect to $\mathcal{T}$, we need to construct a vertex $(\phi, \gamma) \in \mathcal{K}^0(M, \ell_M, a)_{p,q}$ with $F_{p,q}(\phi, \gamma) = f$, such that

$$(\phi_i, \gamma_i) \in \text{lk}(\phi, \gamma) \quad \text{and} \quad ((\psi_{j}, \gamma_j)^{i}, (\phi, \gamma)) \in \mathcal{T}$$

for all $i = 1, \ldots, k$ and $j = 1, \ldots, m$. By using the same procedure employed in the proof of Theorem 3.2 we may construct the embedding $\phi : \hat{V}_{p,q} \to M$ in the same way that was done there. However, in order to obtain the path $\gamma : [0, 1] \to \text{Bun}(T\hat{V}_{p,q}, \theta^* \gamma^d)$, we need to use Lemma 8.2. Since $q \leq p$ and the space $B$ is $q$-connected, by Lemma 8.2 we may find a diffeomorphism $\varphi : \hat{V}_{p,q} \to \hat{V}_{p,q}$, that is the identity on the half-disk $D^p_{\hat{V}_{p,q}} \subset \partial \hat{V}_{p,q}$, such that the $\theta$-structure $\varphi^* \phi^* \ell_M$ given by

$$\begin{align*}
T_{p,q} & \xrightarrow{D_{\varphi}} T_{p,q} \xrightarrow{D_{\phi}} TM \xrightarrow{\ell_M} \theta^* \gamma^d,
\end{align*}$$

(8.3)
is on the same path component of \( \text{Bun}(\hat{T}V_{p,q}, \theta^* \gamma^d) \) as the canonical \( \theta \)-structure \( \ell_{\hat{V}_{p,q}}^\gamma \). Letting \( \gamma : [0,1] \rightarrow \text{Bun}(\hat{T}V_{p,q}, \theta^* \gamma^d) \) be a path from \( \varphi^* \phi^* \ell_M \) to \( \ell_{\hat{V}_{p,q}}^\gamma \) it follows that the pair \((\phi \circ \varphi, \gamma)\) is a vertex in the complex \( K^\theta(M, \ell_M, a)_{p,q} \) that satisfies all of the desired conditions.

The construction of this vertex concludes our verification of the link lifting property. It follows from Lemma 6.2 that the degree of connectivity of \( |K^\theta(M, \ell_M, a)_{p,q}| \) is bounded below by the degree of connectivity of \( |L(W^\theta_{p,q}(M, \partial_1 M))| \). Since \( q \leq p \), \( \pi_q(S^p) \) is either isomorphic to \( \mathbb{Z} \) or is zero, thus the generating set length \( d(\pi_q(S^p)) \) is either equal to 1 or zero. It follows from Theorem 5.1 that \( |L(W^\theta_{p,q}(M, \partial_1 M))| \) is at least \( \frac{1}{2}(g - 4) \)-connected. By what was proven above it follows that \( |K^\theta(M, \ell_M, a)_{p,q}| \) is \( \frac{1}{2}(g - 4) \)-connected as well. This concludes the proof of the proposition. \( \square \)

With the above proposition established, the proof of Theorem 8.1 is obtained by implementing the same constructions from Section 7. We omit the rest of the proof and refer the reader to [4, Section 7] for details.

Appendix A. Embeddings and Disjunction

In this section we prove a disjunction result for embeddings of manifolds with boundary. This result implies Theorem 4.8 which is one of the main technical ingredients used to prove that the complex \( K(M)_{p,q} \) is highly connected.

**Theorem A.1.** Let \((M, \partial M)\) be a manifold pair of dimension \( m \). Let \((P, \partial P)\) and \((Q, \partial Q)\) be manifold pairs of dimensions \( p \) and \( q \) respectively, with \( \partial P \neq \emptyset \neq \partial Q \). Let \( f : (P, \partial P) \rightarrow (M, \partial M) \) and \( g : (Q, \partial Q) \rightarrow (M, \partial M) \) be smooth embeddings and suppose that the following conditions are met:

(i) \( m > p + q/2 + 1, \) \( m > q + p/2 + 1; \)

(ii) \( (P, \partial P), \) \( (Q, \partial Q), \) \( P, \) and \( Q \) are \( (p + q - m) \)-connected;

(iii) \( (M, \partial M) \) and \( M \) are \( (p + q - m + 1) \)-connected.

Then there exists an isotopy \( \psi_s : (P, \partial P) \rightarrow (M, \partial M), \) \( s \in [0,1], \) such that: \( \psi_0 = f \) and \( \psi_1(P) \cap g(Q) = \emptyset. \)

The proof of the above theorem is based on a technique developed by Hatcher and Quinn from [6]. We recall the results of Hatcher and Quinn in the following section, develop some new techniques in the sections that follow, and then finish the proof of Theorem A.1 in Section A.4.

A.1. The Hatcher-Quinn invariant. We now review the construction of Hatcher and Quinn from [6]. This construction involves the framed bordism groups of a space, twisted by a stable vector bundle.

**Definition A.1.** Let \( X \) be a space and let \( \zeta \) be a stable vector bundle over \( X \). For an integer \( n \), \( \Omega^X_n(X; \zeta) \) is defined to be the set of bordism classes of triples \((M, f, F)\), where \( M \) is a closed \( n \)-dimensional smooth manifold, \( f : M \rightarrow X \) is a map, and \( F : \nu_M \rightarrow f^*(\zeta) \) is an isomorphism of stable vector bundles covering the identity map on \( M \).

Let \( M, \) \( P, \) and \( Q \) be smooth manifolds of dimensions \( m, p, \) and \( q \) respectively. Let \( t \) denote the integer \( p + q - m \). Let \( f : (P, \partial P) \rightarrow (M, \partial M) \) and \( g : (Q, \partial Q) \rightarrow (M, \partial M) \) be smooth maps.
We denote by $E(f, g)$ the homotopy pull-back of the maps $f$ and $g$. Explicitly, $E(f, g)$ is the space defined by,

$$E(f, g) = \{(x, y, \gamma) \in P \times Q \times \text{Path}(M) \mid f(x) = \gamma(0), \ g(y) = \gamma(1)\}.$$

Consider the diagram

(A.1)

\[
\begin{array}{ccc}
E(f, g) & \xrightarrow{\pi_P} & P \\
\downarrow{\pi_Q} & \nearrow{\hat{s}} & \\
Q & \xrightarrow{g} & M
\end{array}
\]

where $\pi_P$ and $\pi_Q$ are the projection maps and $\hat{s}$ is the map defined by $\hat{s}(x, y, \gamma) = \gamma(1/2)$. Let $\nu_P$, $\nu_Q$ denote the stable normal bundles associated to the manifolds $P$ and $Q$ respectively. We denote by $\eta(f, g)$ the stable vector bundle over $E(f, g)$ given by the Whitney-sum $\pi_P^*\nu_P \oplus \pi_Q^*\nu_Q \oplus \hat{s}^*(TM)$.

We will need to consider the bordism group $\Omega^f_t(E(f, g); \eta(f, g))$.

Suppose now that the maps $f : (P, \partial P) \longrightarrow (M, \partial M)$ and $g : (Q, \partial Q) \longrightarrow (M, \partial M)$ satisfy $f(\partial P) \cap g(\partial Q) = \emptyset$. It follows that the pull-back, $f \cap g := (f \times g)^{-1}(\Delta_M) \subset P \times Q$, is a closed submanifold of dimension $p + q - m$. Let us denote $t := p + q - m$. Let $i : f \cap g \longrightarrow E(f, g)$ denote the canonical embedding given by the formula $(x, y) \mapsto (x, y, c(f, g))$, where $c(f, g) \in \text{Path}(M)$ is the constant path at the point $f(x) \in M$. The lemma below follows from [6, Proposition 2.1].

Lemma A.2. Let $f : (P, \partial P) \longrightarrow (M, \partial M)$ and $g : (Q, \partial Q) \longrightarrow (M, \partial M)$ be transversal smooth maps such that $f(\partial P) \cap g(\partial Q) = \emptyset$. Then there is a natural bundle isomorphism $i : \nu_{f \cap g} \cong \nu^*(\eta(f, g))$ so that the triple $(f \cap g, i, i)$ determines a well-defined element of the bordism group $\Omega^f_t(E(f, g); \eta(f, g))$.

Definition A.2. For transversal maps $f$ and $g$ with $f(\partial P) \cap g(\partial Q) = \emptyset$ as in the previous lemma, we will denote by $\alpha_t(f, g, M) \in \Omega^f_t(E(f, g); \eta(f, g))$ the element determined by the triple $(f \cap g, i, i)$ given in Lemma A.2.

The main result from [6] is the following theorem.

Theorem A.3. Let $f : (P, \partial P) \longrightarrow (M, \partial M)$ and $g : (Q, \partial Q) \longrightarrow (M, \partial M)$ be smooth embeddings such that $f(\partial P) \cap g(\partial Q) = \emptyset$. Suppose further that $m > p + q/2 + 1$ and $m > p/2 + q + 1$. If the class $\alpha_t(f, g, M)$ is equal to the zero element in $\Omega^f_t(E(f, g); \eta(f, g))$, then there exists an isotopy, $\psi_s : (P, \partial P) \longrightarrow (M, \partial M)$, such that: $\psi_0 = f$, $\psi_s|_{\partial P} = f$ for all $s \in [0, 1]$, and $\psi_1(P) \cap g(Q) = \emptyset$.

The bordism group $\Omega^f_t(E(f, g); \eta(f, g))$ in general can be quite difficult to compute. However, in the case where $P$, $Q$, and $M$ are all highly connected, the group reduces to a far simpler object.

Proposition A.4. Let $f : (P, \partial P) \longrightarrow (M, \partial M)$ and $g : (Q, \partial Q) \longrightarrow (M, \partial M)$ be smooth embeddings such that $f(\partial P) \cap g(\partial Q) = \emptyset$. Suppose that $P$ and $Q$ are $(p + q - m)$-connected and that $M$ is $(p + q - m + 1)$-connected. Then the natural map $\Omega^f_t(pt.) \longrightarrow \Omega^f_t(E(f, g), \eta(f, g))$ is an isomorphism.

Proof. Let $P$ and $Q$ be $(p + q - m)$-connected and let $M$ be $(p + q - m + 1)$-connected. There is a fibre sequence $\Omega M \longrightarrow E(f, g) \longrightarrow P \times Q$. The long exact sequence on homotopy groups implies that the space $E(f, g)$ is $(p + q - m)$-connected. The proof of the proposition follows from this. \qed
Suppose that $P$ and $Q$ are $(p + q - m)$-connected and that $M$ is $(p + q - m + 1)$-connected. If $f : (P, \partial P) \to (M, \partial M)$ and $g : (Q, \partial Q) \to (M, \partial M)$ are smooth embeddings we may consider $\alpha_t(f, g; M)$ to be an element of the framed bordism group $\Omega_t^{fr}(pt.)$, where $t = p + q - m$.

**Remark A.5.** The element $\alpha_t(f, g; M)$ is the obstruction to finding an isotopy, relative to the boundary $\partial P$, that pushes $f(P)$ off of $g(Q)$. This element $\alpha_t(f, g; M)$ very well may be non-zero for arbitrary $f$, $g$, and $M$, and thus it may appear that Theorem [A.1] is false. However, Theorem [A.1] does not assert the existence of an isotopy that fixes the boundary of $P$. Indeed, as will be seen in the following sections, $\alpha_t(f, g; M)$ is not an obstruction to the existence of an isotopy that is non-constant on the boundary of $P$.

### A.2. Relative Hatcher-Quinn Invariant.

**Definition A.3.** Let $(X, A)$ be a pair of spaces and let $\zeta$ be a stable vector bundle over $X$. For an integer $n$, $\Omega_n^{fr}((X, A), \zeta)$ is defined to be the set of bordism classes of triples $(M, f, F)$ where $(M, \partial M)$ is an $n$-dimensional manifold pair, $f : (M, \partial M) \to (X, A)$ is a map, and $F : \nu_M \to f^*(\zeta)$ is an equivalence class of bundle isomorphisms as before.

For any space pair $(X, A)$ and stable vector bundle $\zeta$ over $X$, there is a long exact sequence of bordism groups, 

\[
\ldots \to \Omega_n^{fr}(A; \zeta|_A) \to \Omega_n^{fr}(X; \zeta) \to \Omega_n^{fr}((X, A); \zeta) \to \Omega_{n-1}^{fr}(A; \zeta|_A) \to \ldots
\]

Using these relative bordism groups, we define a relative version of the Hatcher-Quinn invariant. Let $f : (P, \partial P) \to (M, \partial M)$ and $g : (Q, \partial Q) \to (M, \partial M)$ be embeddings. Unlike the case in the previous section, we now include the possibility that the intersection $f(\partial P) \cap g(\partial Q)$ be non-empty. If $f$ and $g$ are transversal (and by this we mean that both $f$ and $g$ and $f|_{\partial P}$ and $g|_{\partial Q}$ are transversal in the ordinary sense), then the pull-back $f \cap g$ is a manifold with boundary given by $\partial(f \cap g) = f|_{\partial P} \cap g|_{\partial Q} \subset \partial P \times \partial Q$. We will need to construct a relative version of the bordism invariant that was defined in the previous section.

Let $\partial E(f, g)$ denote the homotopy pull-back $E(f|_{\partial P}, g|_{\partial Q})$. The space $\partial E(f, g)$ embeds naturally as a subspace of $E(f, g)$. We have a map of pairs

\[
\iota : (f \cap g, \partial(f \cap g)) \to (E(f, g), \partial E(f, g)), \quad (x, y) \mapsto (x, y, c_f(x)).
\]

The restriction of $\eta(f, g)$ to $\partial E(f, g)$ is equal to the bundle $\eta(f|_{\partial P}, g|_{\partial Q})$. To save space we will let $\hat{E}(f, g)$ denote the pair $(E(f, g), \partial E(f, g))$. We will need to consider the relative bordism group $\Omega_t^{fr}(\hat{E}(f, g), \eta(f, g))$. Let,

\[
\hat{\partial} : \Omega_t^{fr}(\hat{E}(f, g), \eta(f, g)) \to \Omega_{t-1}^{fr}(\partial E(f, g), \eta(f, g)|_{\partial E(f, g)}),
\]

be the boundary homomorphism in the long exact sequence associated to the pair $\hat{E}(f, g)$. Using the same construction from Lemma [A.2] we obtain:

**Lemma A.6.** Let $f : (P, \partial P) \to (M, \partial M)$ and $g : (Q, \partial Q) \to (M, \partial M)$ be transversal maps. Then the pullback manifold $f \cap g$ determines a class $\alpha_t^{\partial}(f, g; M) \in \Omega_t^{fr}(\hat{E}(f, g), \eta(f, g))$. Furthermore, we have

\[
\hat{\partial}(\alpha_t^{\partial}(f, g; M)) = \alpha_{t-1}(f|_{\partial P}, g|_{\partial Q}, \partial M).
\]

Lemma [A.6] will be useful to us in order to prove the following result.
Proposition A.7. Let \( f : (P, \partial P) \rightarrow (M, \partial M) \) and \( g : (Q, \partial Q) \rightarrow (M, \partial M) \) be embeddings. Suppose that \((P, \partial P)\) and \((Q, \partial Q)\) are \((p + q - m)\)-connected and that \((M, \partial M)\) is \((p + q - m + 1)\)-connected. Then there exists an isomorphism \( \Psi_s : P \rightarrow M \) with \( s \in [0, 1] \), such that \( \Psi_0 = f \) and \( \Psi_1(\partial P) \cap g(\partial Q) = \emptyset \).

Proof. Let \( t \) denote the integer \( p + q - m \). The connectivity conditions in the statement of the proposition implies that the pair \((E(f, g), \partial E(f, g))\) is \( t \)-connected, and thus, the bordism group \( \Omega_t^{fr}(\hat{E}(f, g), \eta(f, g)) \) is trivial. It follows from this that \( \hat{\partial}(\hat{\alpha}_t(f, g, M)) = \alpha_{t - 1}(f|_{\partial P}, g|_{\partial Q}, \partial M) = 0 \).

We may then apply Theorem A.3 to obtain an isomorphism \( \psi_s : \partial P \rightarrow \partial M \) with \( \psi_0 = f|_{\partial P} \), such that \( \psi_1(\partial P) \cap g(\partial Q) = \emptyset \). The proof of the proposition then follows by application of the isomorphism extension theorem. \( \square \)

A.3. Creating intersections. In this section we develop a technique for creating intersections with prescribed Hatcher-Quinn obstructions. Let \( M \) and \( Q \) be oriented, connected manifolds of dimension \( m \) and \( q \) respectively, and let \( g : (Q, \partial Q) \rightarrow (M, \partial M) \) be an embedding. Let \( r = m - q \) and let \( f : (D^r, \partial D^r) \rightarrow (M, \partial M) \) be a smooth embedding transverse to \( g \) such that, \( f(\partial D^r) \cap g(\partial Q) = \emptyset \). Let \( j \geq 0 \) be an integer strictly less than \( r \). With \( j \)-chosen in this way it follows that \( \pi_{r+j}(S^r) \) is in the stable range and thus we have an isomorphism, \( \pi_{r+j}(D^r, \partial D^r) \cong \pi_{r+j}(S^r) \). Let \( \varphi : (D^{r+j}, \partial D^{r+j}) \rightarrow (D^r, \partial D^r) \) be a smooth map. Denote by

\[(A.3) \quad P_j : \pi_{r+j}(D^r, \partial D^r) \cong \pi_{r+j}(S^r) \xrightarrow{\cong} \Omega_{j}^{fr}(pt.) \]

the Pontryagin-Thom isomorphism. The following lemma shows how to express \( \alpha_j(f \circ \varphi, g; M) \) in terms of \( \alpha_0(f, g, M) \) and the element \( P_j([\varphi]) \in \Omega_{j}^{fr}(pt.) \).

Lemma A.8. Let \( g, f, \) and \( \varphi \) be exactly as above and suppose that \( f(\partial D^r) \cap g(\partial Q) = \emptyset \). Then

\[ \alpha_j(f \circ \varphi, g; M) = \alpha_0(f, g; M) \cdot P_j([\varphi]), \]

where the product on the right-hand side is the product in the graded bordism ring \( \Omega_{*}^{fr}(pt.) \).

Proof. Let \( \ell \in \mathbb{Z} \) denote the oriented, algebraic intersection number associated to the intersection of \( f(D^r) \) and \( g(Q) \). By application of the Whitney trick, we may deform \( f \) so that

\[(A.4) \quad f(D^r) \cap g(Q) = \{x_1, \ldots, x_\ell \}, \]

where the points \( x_i \) for \( i = 1, \ldots, \ell \) all have the same sign. It follows that, \( (f \circ \varphi)^{-1}(g(Q)) = \bigsqcup_{i=1}^\ell \varphi^{-1}(x_i) \). For each \( i \in \{1, \ldots, \ell\} \), the framing at \( x_i \) (induced by the orientations of \( f(D^r) \), \( g(Q) \) and \( M \)) induces a framing on \( \varphi^{-1}(x_i) \). We denote the element of \( \Omega_{j}^{fr}(pt.) \) given by \( \varphi^{-1}(x_i) \) with this induced framing by \([\varphi^{-1}(x_i)]\). By definition of the Pontryagin-Thom map \( P_j \), the element \([\varphi^{-1}(x_i)]\) is equal to \( P_j([\gamma]) \) for \( i = 1, \ldots, \ell \). Using the equality \( [\alpha_j] \), it follows that \( \alpha_j(f \circ \varphi, g; M) = \ell \cdot P_j([\varphi]) \). The proof then follows from the fact that \( \alpha_0(f, g, M) \) is identified with the algebraic intersection number associated to \( f(S^r) \) and \( g(Q) \). \( \square \)

We apply the above lemma to the following proposition.

Proposition A.9. Let \( Q \) and \( M \) have non-empty boundary and let \( g : (Q, \partial Q) \rightarrow (M, \partial M) \) be a smooth embedding. Let \( r \) denote the integer \( m - q \). There exists an embedding \( f : (D^r, \partial D^r) \rightarrow (M, \partial M) \) that satisfies the following conditions:

- \( f(\partial D^r) \cap g(\partial Q) = \emptyset, \)
• $f(D^r) \cap g(Q)$ consists of a single point with positive orientation,
• $f$ represents the trivial element in $\pi_r(M, \partial M)$.

Proof. We will prove the proposition by carrying out an explicit construction as follows:

(i) Choose a collar embedding $h : \partial Q \times [0, \infty) \to Q$ such that $h^{-1}(\partial Q) = \partial Q \times \{0\}$.
(ii) Choose a point $y \in \partial Q$, then define an embedding $\gamma : [0, 1] \to g(Q)$, $\gamma(t) = g(h(y,t))$. We then let $x \in g(Q)$ denote the point $\gamma(1)$.
(iii) Choose an embedding $\alpha : (D^2, \partial_0 D^2) \to (M, \partial M)$ that satisfies the following conditions:
   (a) $\alpha(D^2) \cap g(Q) = \gamma([0,1])$,
   (b) $\alpha(\partial D^2) \cap g(Q) = \{x\}$,
   (c) $\alpha(D^2)$ intersects $g(Q)$ orthogonally (with respect to some metric on $M$).
(iv) Let $r$ denote the integer $m-q$. Choose a $(r-1)$-frame of orthogonal vector fields $(v_1, \ldots, v_{r-1})$ over the embedded half-disk $\alpha(D_+^2) \subset M$ with the property that $v_i$ is orthogonal to $\alpha(D^2)$ and orthogonal to $g(Q)$ over the intersection $\alpha(D^2) \cap g(Q)$, for $i = 1, \ldots, r-1$. Since the disk is contractable, there is no obstruction to the existence of such a frame.

The orthogonal $(r-1)$-frame chosen in step (iv) induces an embedding $\bar{f} : (D^r_{+1}, \partial_0 D^r_{+1}) \to (M, \partial M)$. The orthogonality condition (condition (c)) in Step (iii) of the above construction, together with the orthogonality condition on the frame chosen in step (iv), implies that $\bar{f}(D^r_{+1})$ is transverse to $g(Q)$. Furthermore, condition (b) from step (iii) of the above construction implies that $f(\partial_1 D^r_{+1}) \cap g(\text{Int}(Q)) = \{x\}$. We then set the map $f : (D^r, \partial D^{r-1}) \to (M, \partial M)$ equal to the embedding obtained by restricting $\bar{f}$ to $(\partial_1 D^r_{+1}, \partial_0 D^r_{+1})$. This concludes the proof of the proposition. 

A.4. Proof of Theorem A.1. Let $f : (P, \partial P) \to (M, \partial M)$ and $g : (Q, \partial Q) \to (M, \partial M)$ be smooth embeddings exactly as in the statement of Theorem A.1. Suppose that conditions (i), (ii), and (iii) from the statement of Theorem A.1 are satisfied. We now have all of the necessary tools available to prove the theorem.

Proof of Theorem A.1. By Proposition A.7 we may assume that $f(\partial P) \cap g(\partial Q) = \emptyset$. Consider the element $\alpha_t(f,g,M) \in \Omega^H_t(\text{pt})$ where as before $t = p+q-m$. Let $\varphi : (D^r, \partial D^r) \to (D^{m-q}, \partial D^{m-q})$ be a map such that $P_t(\varphi) = -\alpha_t(f,g,M)$ as elements of $\Omega^H_t(\text{pt})$. By Proposition A.9 there exists a null-homotopic embedding $\phi : (D^{m-q}, \partial D^{m-q}) \to (M, \partial M)$ such that $\phi(D^{m-q})$ intersects the interior of $g(Q)$ at exactly one point. Furthermore, by general position we may assume that $\phi(\partial D^{m-q})$ is disjoint from $g(\partial Q)$. It follows from Lemma A.3 that,

\[(A.5) \quad \alpha_t(\phi \circ \varphi, g, M) = P_t(\varphi) \cdot \alpha_0(\phi, g, M) = -\alpha_t(f, g, M).\]

Now, the map $\phi \circ \varphi$ has image disjoint from $g(\partial Q) \subset \partial M$. Let $M'$ denote the complement $M \setminus g(\partial Q)$. By the connectivity and dimensional conditions from the statement of Theorem A.1 we may apply Lemma A.4 (see also [17, Theorem 1]) to obtain a homotopy

\[(A.6) \quad \tilde{\varphi}_{s} : (D^p, \partial D^p) \to (M', \partial M'), \quad s \in [0, 1],\]

with $\tilde{\varphi}_0 = \phi \circ \varphi$ such that $\tilde{\varphi}_1$ is an embedding. Let us denote, $\tilde{\varphi} := i_{M'} \circ \tilde{\varphi}_1 : (D^p, \partial D^p) \to (M, \partial M)$, where $i_{M'} : (M', \partial M') \to (M, \partial M)$ is the inclusion. Since the homotopy in (A.6) was through maps with image in $(M', \partial M')$, it follows that $\alpha_t(\tilde{\varphi}, g, M) = \alpha_t(\phi \circ \varphi, g, M)$, and then by the above
calculation (A.5) we have, \( \alpha_t(\hat{\varphi}, g, M) = -\alpha_t(f, g, M) \). Now, let \( \hat{f} : (P, \partial P) \to (M, \partial M) \) be the embedding obtained by forming the boundary-connected-sum of of the submanifolds
\[
(f(P), f(\partial P)), (\hat{\varphi}(D^p), \hat{\varphi}(\partial D^p)) \subset (M, \partial M),
\]
along an arc in \( \partial M \) that is disjoint from \( g(\partial Q) \). Clearly this embedding is homotopic (as a map) to \( f \). We emphasize that homotopy taking \( \hat{f} \) to \( f \) will not be constant on the boundary of \( P \). We then have,
\[
\alpha_t(\hat{f}, g, M) = \alpha_t(f, g, M) + \alpha_t(\hat{\varphi}, g, M) = \alpha_t(f, g, M) - \alpha_t(f, g, M) = 0.
\]
By Theorem A.3 there is a diffeotopy (relative the boundary of \( M \)) that pushes \( \hat{f}(P) \) off of \( g(Q) \). Now since \( f \) is homotopic to \( \hat{f} \), it follows that \( f \) is homotopic (through maps sending \( \partial P \) to \( \partial M \)) to an embedding with image disjoint from \( g(\partial Q) \). We then apply [6, Theorem 1.1] to conclude that the embedding \( f \) is actually isotopic (rather than just homotopic) to such an embedding with image disjoint from \( g(Q) \). Then Theorem A.1 follows by isotopy extension. □

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