Compression of sources of probability distributions and density operators

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Abstract—We study the problem of efficient compression of a stochastic source of probability distributions. It can be viewed as a generalization of Shannon’s source coding problem. It has relation to the theory of common randomness, as well as to channel coding and rate–distortion theory: in the first two subjects “inverses” to established coding theorems can be derived, yielding a new approach to proving converse theorems, in the third we find a new proof of Shannon’s rate–distortion theorem.

After reviewing the known lower bound for the optimal compression rate, we present a number of approaches to achieve it by code constructions. Our main results are: a better understanding of the known lower bounds on the compression rate by means of a strong version of this statement, a review of a construction achieving the lower bound by using common randomness which we complement by showing the optimal use of the latter within a class of protocols. Then we review another approach, not dependent on common randomness, to minimizing the compression rate, providing some insight into its combinatorial structure, and suggesting an algorithm to optimize it.

The second part of the paper is concerned with the generalization of the problem to quantum information theory: the compression of mixed quantum states. Here, after reviewing the known lower bound we contribute a strong version of it, and discuss the relation of the problem to other issues in quantum information theory.

I. SOURCES OF DISTRIBUTIONS

A theorem of Shannon [10] basic to all information theory describes the optimum compression of a discrete memoryless source, showing that the minimum achievable rate is the entropy of the source distribution. The situation is the following:

Let $P$ be a probability distribution on the finite set $\mathcal{X}$. We call $(E, D)$ an $(n, \lambda)$–code for the discrete memoryless source $P$, if

$$
E : \mathcal{X}^n \to \mathcal{C}, \\
D : \mathcal{C} \to \mathcal{Y}^n
$$

are stochastic maps, with a finite set $\mathcal{C}$, such that

$$
\sum_{x^n \in \mathcal{X}^n} P^n(x^n) \Pr\{x^n = D(E(x^n))\} \geq 1 - \lambda,
$$

where

$$
\Pr\{x^n = D(E(x^n))\} = D(E(x^n))\{x^n\}.
$$

Denoting the minimal $|\mathcal{C}|$ such that an $(n, \lambda)$ code exists, by $M(n, \lambda)$, Shannon [10] shows that for $\lambda \in (0, 1)$

$$
\lim_{n \to \infty} \frac{1}{n} \log M(n, \lambda) = H(P),
$$

where $H(P) = -\sum_x P(x) \log P(x)$ is the entropy of the distribution.

Motivated by the work [1], and by a construction in [3] (in footnote 4), we study here the following modification of this problem:

To each $x \in \mathcal{X}$ is associated a probability distribution $W_x$ on the finite set $\mathcal{Y}$ (thus $W$ is a stochastic map, or channel, form $\mathcal{X}$ to $\mathcal{Y}$). An $(n, \lambda)$–code is now a pair $(E, D)$ of stochastic maps

$$
E : \mathcal{X}^n \to \mathcal{C}, \\
D : \mathcal{C} \to \mathcal{Y}^n
$$

(compare with eq. (1)), and instead of condition (2) we impose

$$
\sum_{x^n \in \mathcal{X}^n} P^n(x^n) \frac{1}{2} \|W^n_x - D(E(x^n))\|_1 \leq \lambda,
$$

where $\| \cdot \|_1$ is the $\ell^1$–norm on function on $\mathcal{Y}^n$: $\|f\|_1 = \sum_{y^n} |f(y^n)|$. Note that for two probability distributions $P$ and $Q$, $\frac{1}{2} \|P - Q\|_1$ equals their total variational distance $d_{TV}(P, Q) = \sup_{A \subset \mathcal{Y}^n} |P(A) - Q(A)|$ of the two. We define $M(n, \lambda)$ to be the minimal $|\mathcal{C}|$ of an $(n, \lambda)$–code.

Note that for $\mathcal{Y} = \mathcal{X}$, and $W_x$ the point–mass $\delta_x$ in $x$, the new notion of $(n, \lambda)$–code coincides with the previous one. Notice further, that we allow probabilistic choices in the encoding and decoding. While it is easy to see that this freedom does not help in Shannon’s problem, it is crucial for the more general form, that we will study in this paper.

The basic problem of course is to find the optimum rate of compression (if the limit exists; otherwise lim sup is to be considered), and especially the behaviour of this function at $\lambda \to 0$.

For the case $\lambda = 0$, i.e. perfect restitution of the distributions $W_x$, these definitions in principle make sense, but we don’t expect a neat theory to emerge. Instead we define

$$
S(n) = \min H(E(P^{\otimes n})),
$$

the minimal entropy of the distribution on $\mathcal{C}$ induced by the encoder $E$ (with the idea that blocks of these $n$–blocks we may data compress to this rate). Obviously $S(n_1 + n_2) \leq S(n_1) + S(n_2)$, so the limit

$$
\Gamma(P, W) = \lim_{n \to \infty} \frac{1}{n} S(n)
$$

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exists, and is equal to the infimum of the sequence. To evaluate this quantity is another problem we would like to solve.

The structure of this paper is as follows: first we find lower bounds (section II), then discuss upper bounds, preferably by constructing codes: in section III we show how the lower bound is approached by using the additional resource of common randomness, in section IV we prove achievability of it under a letterwise fidelity criterion as a consequence of this result, section V presents a construction to upper bound $\Gamma$ and $\Gamma_\lambda$. In section VI applications of the results and conjectures are presented: first, we make it plausible that the distillation procedure of [2] is asymptotically reversible, second we show that Shannon’s coding theorem allows an “inverse” (at least in situations where unlimited common randomness is around), third we give a simple proof that feedback does not increase the rate of a discrete memoryless channel, and fourth demonstrate, how Shannon’s rate–distortion theorem follows as a corollary. The compression result (with or without common randomness) thus reveals a great unifying power in classical information theory. Finally, in section VII we discuss extensions of our results to the case of a source of mixed quantum states: the present discussion fits into this modality.

Let us mention here the previous work on the problem: the major initiating works are [11] and [13]. The latter introduced the distinction between blind and visible coding, and between the block– and letterwise fidelity criterion. In contrast to the pure state case the four possible combinations of these conditions seem to lead to rather different answers. The case of blind coding with either the letter– or blockwise fidelity criterion was solved recently by Koashi and Imoto [22]. Otherwise in this paper, we will only address the visible case. An attempt on the letterwise fidelity case with either blind or visible encoding was made in [13]. However, an examination of the approach of this work shows that it does not fit into any of the the classes of fidelity criteria proposed by [13]: for a code $(E, D)$ one could either apply the global criterion, which is essentially our eq. (4), that is definitely not what is considered in [13], there being employed rate distortion theory.

Or one could impose that the output $E(D(x^n))$ is good on the average letterwise (the local criterion of [13]):

$$\sum_{x^n} P^n(x^n) \left[ \frac{1}{n} \sum_{k=1}^n d(W_{xk}, D(E(x^n))_k) \right] \leq \lambda, \quad (5)$$

where $D(E(x^n))_k$ denotes the marginal distribution of $D(E(x^n))$ on the $k^{\text{th}}$ factor in $Y^n$, and $d$ is any distance measure on probability distributions (that we require only to be convex in the second variable). For $d(P, Q) = \frac{1}{2} \| P - Q \|_1$ this is implied by eq. (4). This, too, is not met in [13], as there $E$ and $D$ are constructed as deterministic maps, while to satisfy eq. (5) one needs at least a small amount of randomness.

To achieve this one could base the fidelity condition on looking at individual letter positions of source and output simultaneously:

$$\sum_x P(x) \left[ \frac{1}{n} \sum_{k=1}^n d \left( W_{x_k}, \sum_{x_k : x_k = x} P^n(x^n) D(E(x^n))_k \right) \right] \leq \lambda. \quad (6)$$

Condition (5) being weaker than (4), this one is still weaker. However, this, too, does not coincide with the criterion of [13]: denoting by $G$ the joint distribution of $x$ and $y$ according to $P$ and $W$, i.e. $G(xy) = P(x)W_x(y)$, one considers

$$\sum_{x^n} P^n(x^n) d \left( G, \frac{1}{n} \sum_{k=1}^n \delta_{x_k} \otimes D(E(x^n))_k \right) \leq \lambda \quad (7)$$

(This is implied by eq. (1) of [13] for $\epsilon = \delta = \lambda/2$, which in turn is implied by eq. (6) for $\epsilon = \delta = \sqrt{\lambda}$). It is not at all clear how to connect this with any of the above: eq. (6) is about the empirical joint distribution of letters in $x^n$ and $D(E(x^n))$ (assume for simplicity, as indeed the authors of [13] do, that $E$ and $D$ are deterministic), that is about a distribution created by selecting a position $k$ randomly, while eqs. (5) to (6) are about distributions created either by the coding process alone or in conjunction with the source. Our view is confirmed in an independent recent analysis of [13] by Soljanin [18], to the same effect.

An interesting new twist was added when in [14] (and later in a more extended way in [16] and the recent [18]) the use of unlimited common randomness between the sender and receiver was allowed in the visible coding model with blockwise fidelity criterion. As already mentioned, we reproduce this result here in detail, with special attention to the resource of common randomness: we present a protocol for which we prove that it has minimum common randomness consumption in the class of protocols which even simulate full passive feedback of the received signal to the sender.

II. LOWER BOUND AND CONJECTURES

Let the random variable $X^n = (X_1, \ldots, X_n)$ be distributed according to $P^n$. Then we can define $Y^n$ by

$$\Pr\{Y^n = y^n | X^n = x^n\} = W^n_{zy}(y^n).$$

By (4) we have the Markov chain

$$X^n \rightarrow \rightarrow E(X^n) \rightarrow \rightarrow D(E(X^n)) \approx Y^n.$$  

Using data processing inequality as follows:

$$\log |C| \geq H(E(X^n))$$

$$\geq I(X^n \land E(X^n))$$

$$\geq I(X^n \land D(E(X^n))$$

$$\geq I(X^n \land Y^n) - nf(\lambda),$$

with $f(\lambda) \rightarrow 0$ for $\lambda \rightarrow 0$. To be precise, one may choose (for $\lambda \leq 1/2$)

$$f(\lambda) = \lambda(\log |X| + 2 \log |Y|) + 2h(\lambda),$$
employing the following well known result with eq. (3).

**Lemma 1:** Let $P$ and $Q$ be probability distributions on a set with finite cardinality $a$, such that $\|P - Q\|_1 \leq 1/2$. Then

$$|H(P) - H(Q)| \leq a h\left(\frac{1}{a}\right) := -\lambda \log \frac{1}{a}.$$  

**Proof.** See e.g. [8].

Thus we arrive at

**Theorem 2:** For any $n$ and $0 < \lambda < 1$:

$$\frac{1}{n} \log M(n, \lambda) \geq I(P; W) - f(\lambda),$$

where

$$I(P; W) = H(PW) - \sum_x P(x)H(W_x)$$

is the mutual information of the channel $W$ between the input distribution $P$ and the output distribution $PW = \sum_x P(x)W_x$.

By using slightly stronger estimates, we even get

**Theorem 3:** For every $\lambda \in (0, 1)$

$$\liminf_{n \to \infty} \frac{1}{n} \log M(n, \lambda) \geq I(P; W).$$

**Proof.** Let $(E, D)$ be an optimal $(n, \lambda)$-code. From eq. (4) we find (by a Markov inequality argument) that

$$P^n\left\{ x^n : \frac{1}{2}\|W^n_{x^n} - D(E(x^n))\|_1 \leq \sqrt{\lambda}\right\} \geq 1 - \sqrt{\lambda}.$$  

Denote the intersection of this set with the typical sequences $T^n_{\lambda, \delta}$ (see eq. (8) below) by $A$, with $\delta = \sqrt{\frac{2\lambda}{1-\sqrt{\lambda}}}$. Then

$$P^n(A) \geq \frac{1 - \sqrt{\lambda}}{2} =: \lambda',$$

and there exists an $(n, \lambda')$-transmission code $U \subset A$ for the channel $W^n$ with $|U| \geq \exp(nI(P; W) - O(\sqrt{n}))$, see [8] (the case of a classical–quantum channel $W$ was done in [21]). By construction this is a $(n, 1 - \lambda')$-code for the channel $D \circ E$.

We want now view $E$ as belonging to the message encoder, and $D$ as belonging to the message decoder, the resulting code being one for the identical channel on $C$. Let us denote the concatenation of the map $D$ with the channel decoder by $\delta$. On the other hand, we may replace $E$ by a deterministic map $\epsilon$, because randomization at the encoder never decreases error probabilities: $(\epsilon, \delta)$ still is an $(n, 1 - \lambda')$-code. It is now obvious that $|\epsilon^{-1}(c)| \leq \lambda'^{-1}$ for every $c \in C$, hence

$$M(n, \lambda) = |C| \geq \lambda'|U| = \exp(nI(P; W) - O(\sqrt{n})),$$

and we are done.

It might be a bit daring to formulate conjectures at this point, so we content ourselves with posing the following questions:

**Question 4:** Is it true that for all $\lambda \in (0, 1)$

$$\lim_{n \to \infty} \frac{1}{n} \log M(n, \lambda) = I(P; W) ?$$

In fact, we would like to go present a slightly stronger statement:

**Question 4':** For every $\lambda \in (0, 1)$, $\epsilon > 0$, $\delta > 0$, and large enough $n$ does there exists a $(n, \lambda)$-code with

$$\frac{1}{n} \log |C| \leq I(P; W) + \epsilon$$

and with the additional property that

$$\forall x^n \in T^n_{\lambda, \delta} \frac{1}{2}\|W^n_{x^n} - D(E(x^n))\|_1 \leq \lambda ?$$

Here $T^n_{\lambda, \delta}$ is the set of typical sequences:

$$T^n_{\lambda, \delta} = \{ x^n : \forall x : |N(x|x^n) - nP(x)| \leq \delta \sqrt{n}\sigma_x \},$$

where $N(x|x^n)$ counts the number of occurrences of $x$ in $x^n$, and $\sigma_x := \sqrt{nP(x)(1 - P(x))}$. Observe that by Chebyshev’s inequality

$$P^n(T^n_{\lambda, \delta}) \geq 1 - \frac{\lambda}{\delta^2}.$$  

In fact, by employing the Chernoff bound we even obtain

$$P^n(T^n_{\lambda, \delta}) \geq 1 - |\lambda| \exp(-\delta^2).$$

With these bounds it is easily seen that a positive answer to the latter question implies the same to the former. But also conversely, it is not difficult to show that a “yes” to question 4 implies a “yes” to question 4'.

III. … AND HOW TO ACHIEVE IT (CHEATING SLIGHTLY)

The following construction is a generalization and refinement of the one by Bennett et al. [3] (footnote 4), found independently by Dür, Vidal, and Cirac [8]. The idea there is to use common randomness between the sender and the receiver of the encoded messages. Formally this means that $E$ and $D$ also depend on a common random variable $\nu$, uniformly distributed and independent of all others. Note that this has a nice expression when viewing $E$ and $D$ as map valued random variables: here we allow dependence (via $\nu$) between $E$ and $D$, while in the initial definition, eq. (3), $E$ and $D$ are independent (as random variables). It seems that the power of allowing the use of common randomness can be understood from this point of view: it is a “convexification” of the theory with deterministic or independent encoders and decoders.

It is easy to see that the lower bound of theorem 2 still applies here. We only have to modify the derivation a little bit:

$$\log |C| \geq H(E(X^n) | \nu) \geq I(X^n \land Y^n | \nu) \geq I(X^n \land D(E(X^n)) | \nu).$$

We obtain

$$I(X^n \land Y^n | \nu) = I(X^n \land D(E(X^n)) | \nu) \geq I(X^n \land Y^n | \nu).$$
with a slight variant \( \tilde{f} \) of \( f \).

We shall apply an explicit large deviation estimate for sampling probability distributions from \( \mathcal{I} \) (extended to density operators in \( \mathbb{I} \)), which we state separately without proof:

**Lemma 5:** Let \( X_1, \ldots, X_M \) be independent identically distributed (i.i.d.) random variables with values in the function algebra on the finite set \( \mathcal{K} \), which are bounded between 0 and 1, the constant function with value 1. Assume that the average \( \mathbb{E} X_\mu = \sigma \geq s \mathbb{1} \). Then for \( 0 < \eta < 1/2 \)

\[
\Pr \left\{ \frac{1}{M} \sum_{\mu=1}^{M} X_\mu \not\in [(1 \pm \eta)\sigma] \right\} \leq 2|\mathcal{K}| \exp \left( -M \frac{\eta^2 s^2}{2 \ln 2} \right),
\]

where \( [(1 \pm \eta)\sigma] = [(1 - \eta)\sigma; (1 + \eta)\sigma] \) is an interval in the value–wise order of functions: \( [A; B] = \{ X : \forall k A(k) \leq X(k) \leq B(k) \} \).

Before we prove our main theorem, we need three lemmas on exact types and conditional types. The first is a simple yet crucial observation:

**Lemma 6:** Let \( W \) be a channel from \( \mathcal{X} \) to \( \mathcal{Y} \), \( P \) a p.d. on \( \mathcal{X} \), \( Q = PW \) the induced distribution on \( \mathcal{Y} \) and \( V \) the transpose channel from \( \mathcal{Y} \) to \( \mathcal{X} \).

Let \( R, S \) be exact \( n \)-types of \( \mathcal{X}, \mathcal{Y} \), respectively that are marginals of a joint exact \( n \)-type \( T \) of \( \mathcal{X} \times \mathcal{Y} \). Consider the uniform distribution \( P^n_R \) on \( T^n_R \) on \( T^n_R \), which has the property

\[
P^n_R(x^n) = \frac{1}{|T^n_R|} = \frac{P^n(x^n)}{|P^n(T^n_R)|} \quad \text{for } x^n \in T^n_R,
\]

and the channel from \( T^n_R \) to \( T^n_S \),

\[
W^n_T(y^n|x^n) = \frac{1}{|T^n_T(x^n)|} T^n(x^n y^n)
= \frac{|T^n_T|}{|T^n_T|} = \frac{|T^n_T(x^n)|}{|T^n_T(x^n)|} \quad \text{for } x^n y^n \in T^n_T,
\]

where \( T^n_T(x^n) := T^n_T(\{ x^n \} \times T^n_S) \) is the set of conditional exact typical sequences of \( x^n \).

Then the induced distribution \( Q^n_S = P^n_R W^n_T \) on \( T^n_S \) is the uniform distribution, i.e.

\[
Q^n_S(y^n) = \frac{1}{|T^n_S|} = \frac{Q^n(y^n)}{|Q^n(T^n_S)|} \quad \text{for } y^n \in T^n_S,
\]

and the transpose channel to \( W^n_T \) is indeed \( V^n_T \), defined by

\[
V^n_T(x^n y^n) = \frac{1}{|T^n_T(y^n)|} T^n(x^n y^n)
= \frac{|T^n_T(y^n)|}{|T^n_T(y^n)|} = \frac{|T^n_T(x^n y^n)|}{|T^n_T(y^n)|} \quad \text{for } x^n y^n \in T^n_T,
\]

with \( T^n_T(y^n) := T^n_T \cap (T^n_R \times \{ y^n \}) \).

**Proof.** Straightforward.

**Lemma 7:** There is an absolute constant \( K \) such that for all distributions \( P \) on \( \mathcal{X} \), \( x^n \in T^n_R \), channels \( W: \mathcal{X} \to \mathcal{Y} \) and \( \delta > 0 \)

\[
|T^n_P \delta | \leq \exp(nH(P) + K \delta |\mathcal{X}| \sqrt{n}),
|T^n_W \delta | \geq \exp(nH(P) - K \delta |\mathcal{X}| \sqrt{n}),
|T^n_W \delta (x^n) | \leq \exp(nH(W|R) + K \delta |\mathcal{X} \times \mathcal{Y}| \sqrt{n}),
|T^n_W \delta (x^n) | \geq \exp(nH(W|R) - K \delta |\mathcal{X} \times \mathcal{Y}| \sqrt{n}).
\]

For \( \delta = 0 \), consider a joint \( n \)-type \( T \) on \( \mathcal{X} \times \mathcal{Y} \) with marginals \( R \) on \( \mathcal{X} \) and \( S \) of \( \mathcal{Y} \). Then, introducing the channel \( Z \) with \( T(xy) = R(x)Z(y|x) \):

\[
|T^n_R | \leq \exp(nH(R)),
|T^n_R | \geq (n + 1)^{-|\mathcal{X}|} \exp(nH(R)),
|T^n_T(x^n) | \leq \exp(nH(Z|R)),
|T^n_T(x^n) | \geq (n + 1)^{-|\mathcal{X} \times \mathcal{Y}|} \exp(nH(Z|R)).
\]

**Proof.** See [24].

The third contains the central insight for our construction:

**Lemma 8:** With the hypotheses and notation of lemma 7 there exist families \( (Y^{(\nu)}_\mu)_{\mu=1, \ldots, M}, \nu = 1, \ldots, N \), from \( T^n_S \) such that for all \( \nu \)

\[
\frac{1}{M} \sum_{\mu} V^n_T(\cdot |Y^{(\nu)}_\mu) \in [(1 - \epsilon)P^n_R, (1 + \epsilon)P^n_R],
\]

and

\[
\frac{1}{NM} \sum_{\mu, \nu} \delta_{Y^{(\nu)}_\mu} \in [(1 - \epsilon)Q^n_S, (1 + \epsilon)Q^n_S],
\]

for all \( M \) and \( N \) that satisfy

\[
M > \frac{2 \ln 2 |T^n_R| |T^n_S|}{c^2} \log (4N|T^n_R|),
\]

\[
NM > \frac{2 \ln 2 |T^n_S| |T^n_R|}{c^2} \log (4|T^n_S|).
\]

**Proof.** Introduce i.i.d. random variables, distributed on \( T^n_S \) according to \( Q^n_S \) (i.e. uniformly). Then for all \( \nu, \mu \):

\[
\mathbb{E} \delta_{Y^{(\nu)}_\mu} = Q^n_S, \quad \mathbb{E} V^n_T(\cdot |Y^{(\nu)}_\mu) = P^n_R.
\]

Hence lemma 7 applies and we find

\[
\forall \nu \quad \Pr \{-N_\nu \leq 2|T^n_R| \exp \left( -M \frac{c^2}{2 \ln 2 |T^n_R|} \right) \},
\]

and

\[
\Pr \{-N_\mu \leq 2|T^n_S| \exp \left( -NM \frac{c^2}{2 \ln 2 |T^n_S|} \right) \}.
\]

By choosing \( N \) and \( M \) according to the lemma we enforce that the sum of these probabilities is less than 1, hence there are actual values of the \( Y^{(\nu)}_\mu \) such that all (I) and (II) are satisfied.

With this we are ready to prove:

**Theorem 9:** There exists an \((n, \lambda)\)-code \((E_\nu, D_\nu)_{\nu=1, \ldots, N}\) with

\[
|C| \leq \exp(nI(P; W) + O(\sqrt{n})
\]
and common randomness consumption

\[ N \leq \exp(nH(W|P) + O(\sqrt{n})). \]

In fact, not only the condition \( [2] \) is satisfied but the even stronger

\[ \forall x^n \in T^n, \frac{1}{2}||W_{x^n} - D(E(x^n))||_1 \leq \lambda. \quad (11) \]

**Proof.** Suppose \( x^n \) is seen at the source, and that its type is \( R \). For each joint \( n \)-type \( T \) of \( X \times Y \) we assume that families \( (Y^{\nu}_n) \) as described in lemma \( [3] \) are fixed throughout.

Then the protocol the sender follows is:

1. Choose a joint type \( T \) on \( X \times Y \) with probability \( W^n(T^n(x^n)|x^n) \) and send it. Note that \( T \) can be written \( T(xy) = R(x)Z(y|x) \), with the marginal \( R \) on \( X \) and a channel \( Z : X \rightarrow Y \).
2. If \( R \) is not typical or \( T \) is not jointly typical then terminate.
3. Use the common randomness to choose \( \nu \) uniformly.
4. Choose \( \mu \) according to

\[
\Pr\{\mu|x^n\} = \frac{W^n(Y^{\nu}_n|x^n)}{\sum_{\nu'} W^n(Y^{\nu'}_n|x^n)},
\]

and send it.

The receiver chooses \( y^n = Y^{\nu}_n \), using the common randomness sample \( \nu \). Let us first check that this procedure works correctly:

For typical \( x^n \) we can calculate the distribution of \( y^n \) conditional on the event that their joint type is \( T \): this is then a distribution on \( T^n(x^n) \), and we assume \( T \) to be typical.

\[
\Pr\{x^n, T\} = \frac{1}{N} \sum_{\nu, \mu = 1}^{N,M} \frac{W^n(Y^{\nu}_n|x^n)}{W^n(Y^{\nu'}_n|x^n)} \delta_{Y^{\nu}_n(x^n)}
= \frac{1}{NM} \sum_{\nu, \mu = 1}^{N,M} P^n_R(x^n) W^n(y^n|x^n)
= \frac{1}{NM} \sum_{\nu, \mu = 1}^{N,M} \frac{1}{1 + B(\epsilon)} Q^n_{\nu}(y^n) \delta_{Y^{\nu}_n(x^n)}
= \frac{1}{1 + B(\epsilon)} W^n(y^n|x^n),
\]

with the “big-B” notation: \( B(\epsilon) \) signifies any function whose modulus is bounded by \( \epsilon \). Here we have used the definition of the protocol, then lemma \( [3] \) (for the definition of \( V^n_T \) and the fact that \( W^n(y^n|x^n) \) does not depend on \( y^n \in T^n(x^n) \)), then lemma \( [3] \). So, the induced distribution is, up to a factor between \( \frac{1}{1 + B(\epsilon)} \) and \( \frac{1}{1 + \epsilon} \), equal to the correct output distribution \( W^n_T(x^n) \). Now averaging over the typical \( T \) gives eq. \( [11] \).

What is the communication cost? Sending \( T \) is asymptotically for free, as the number of joint types is bounded by the polynomial \( (n + 1)^{|X| \times |Y|} \). Sending \( \mu \) costs \( \log M \) bits, with \( M \) bounded according to lemma \( [3] \). That is,

\[
\log M \leq n \left( \max_{T \text{ typical}} I(R; Z) \right) + O(\log n)
\leq nI(P; W) + O(\sqrt{n}),
\]

On the other hand

\[
\log N \leq n \left( \max_{T \text{ typical}} (H(PZ) - I(R; Z)) \right)
\leq nH(W|P) + O(\sqrt{n}),
\]

and we are done. \( \blacksquare \)

**Remark 10:** In the above statement of theorem \( [2] \) we assumed \( \lambda \) to be a constant, absorbed into the “\( O(\sqrt{n}) \)” in the code length estimate. Using the Chernoff estimate \( [11] \) on the probabilities of typical sets in the above proof in fact shows the existence of an \( (n, \lambda) \)-code satisfying \( [11] \)

\[
|C| \leq \exp(nI(P; W) + O(-\log \lambda)\sqrt{n})
\]

In the line of \( [2] \), the interpretation of this result is that investing common randomness at rate \( H(W|P) \), one can simulate the noisy channel \( W \) by a noiseless one of rate \( I(P; W) \), when sending only \( P \)-typical words.

Considering the construction again, we observe that in fact not only it provides a simulation of the channel \( W \), but additionally of the **noiseless passive feedback**. Simply because the sender can read off from his random choices the \( y^n \) obtained by the receiver, too. This observation is the key to show that our above construction is optimal under the hypothesis that the channel with noiseless passive feedback is simulated: in fact, both sender and receiver can observe the very output sequence \( y^n \) of the channel, which has entropy \( H(PW) \), they are able to generate common randomness at this rate. Since communication was only at rate \( I(P; W) \), the difference must by invested in prepared common randomness: otherwise we would get more of it out of the system than we could have possibly invested. Formally this insight is captured by the following result:

**Theorem 11:** If the decoder of a \( (n, \lambda) \)-code \( (E, D) \) with common randomness consumption \( \nu \in [N] \) (with distribution \( \xi \)) depends deterministically on \( \nu \) and \( \epsilon \in C \) (which is precisely the condition that the encoder can recover the receiver’s output) then

\[
|C| \geq \exp\left(nI(P; W) - O(\sqrt{n})\right),
|N| \geq \exp\left(nH(PW) - O(\sqrt{n})\right).
\]

**Proof.** For the first inequality introduce the channels \( A^{\nu}_n = D_{\nu}(E_{\nu}(x^n)) \), and their induced distributions \( R^{\nu} \) on \( Y^n \) and transposes channels \( B^{\nu}_y \) with respect to \( P^n \), i.e.

\[
P^n(x^n)A^{\nu}_n(y^n) = R^{\nu}(y^n)B^{\nu}_y(x^n).
\]

Then we can rewrite eq. \( [3] \) as

\[
\sum_{y^n} \frac{1}{2}||Q^n(y^n)W^{\nu}_n - \sum_{\nu} \xi_{\nu} R^{\nu}(y^n)B^{\nu}_y||_1 \leq \lambda.
\]
This inequality obviously remains valid if we restrict the sum to \( y^n \in T^n_{Q,\delta} \) and replace \( V_{y^n}^n \) and \( B_{y^n}^{(\nu)} \) by their restrictions to \( T^n_{Q,\delta}(y^n) \): \( V_{y^n}^n \) and \( B_{y^n}^{(\nu)} \), respectively.

On the other hand, choosing \( \delta = \sqrt{\frac{2|X|\log|Y|}{|X|}} \), we have
\[
\sum_{y^n \in T^n_{Q,\delta}} Q^n(y^n) V_{y^n}^n(\lambda^n) \geq \frac{1 + \lambda}{2} =: \lambda',
\]
which yields
\[
\sum_{\nu} \xi_n u \sum_{y^n \in T^n_{Q,\delta}} R^{(\nu)}(y^n) B_{y^n}^{(\nu)}(\lambda^n) \geq 1 - \lambda'.
\]

Hence there exists at least one \( \nu \) such that
\[
\sum_{y^n \in T^n_{Q,\delta}} R^{(\nu)}(y^n) B_{y^n}^{(\nu)}(\lambda^n) \geq 1 - \lambda'.
\]

Note that, as functions on \( \lambda^n \),
\[
\sum_{y^n \in T^n_{Q,\delta}} R^{(\nu)}(y^n) B_{y^n}^{(\nu)}(\lambda^n) \leq P^n,
\]
so, when we introduce the support \( S \) of the left hand side, we arrive at
\[
P^n(S) \geq 1 - \lambda',
\]
from which our claim follows by a standard trick \([24]\): let
\[
S' = S \cap T^n_{P,\delta'}, \text{ with } \delta' = \sqrt{\frac{2|X|\log|Y|}{|X|}}. \text{ Then}
\]
\[
P^n(S') \geq 1 - \lambda',
\]
and using the fact that
\[
\forall x^n \in T^n_{P,\delta'}, P^n(x^n) \leq \exp(-nH(P) + K|\lambda'|\delta'\sqrt{n}),
\]
this implies
\[
|S| \geq |S'| \geq \frac{1 - \lambda'}{2} \exp(nH(P) - K|\lambda'|\delta'\sqrt{n}).
\]

Now only note that (since \( D_\nu \) is deterministic)
\[
|S| \leq |C| \max_{y^n \in T^n_{Q,\delta}} |T^n_{Q,\delta}(y^n)| \leq |C| \exp(nH(V|Q) + O(\sqrt{n})�,\]
and by \( I(P;W) = I(Q;V) = H(P) - H(V|Q) \) we are done.

Now for the second inequality: from the definition we get, by summing over \( x^n \),
\[
\frac{1}{2} \left\| (PW)^n - \sum_{\nu} \xi_n \sum_{x^n} P^n_{x^n} D_\nu(E_\nu(x^n)) \right\|_1 \leq \lambda.
\]

Because the \( D_\nu \) are all deterministic, the distributions \( D_\nu(E_\nu(x^n)) \) are all supported on sets of cardinality \( |C| \). Hence the support \( S \) of \( \sum_{\nu} x^n \nu \sum_{x^n} P^n_{x^n} D_\nu(E_\nu(x^n)) \) can be estimated \( |S| \leq N|C| \).

On the other hand, we deduce
\[
(PW)^n(S) \geq 1 - \lambda,
\]
which, by the same standard trick \([24]\) as before, yields our estimate: with \( \delta = \sqrt{\frac{2|X|\log|Y|}{|X|}} \), the set \( S' = S \cap T^n_{PW,\delta} \) satisfies
\[
(PW)^n(S') \geq \frac{1 - \lambda}{2},
\]
but since for all \( y^n \in T^n_{PW,\delta} \)
\[
(PW)^n(y^n) \leq \exp(-nH(PW) + K|Y|\delta\sqrt{n}),
\]
we can conclude
\[
N|C| \geq |S| \geq |S'| \geq \frac{1 - \lambda}{2} \exp(-nH(PW) + K|Y|\delta\sqrt{n}).
\]

Collecting these results we can state

Corollary 12: For any simulation of the channel \( W \) together with its noiseless passive feedback with error \( \lambda < 1 \), at rate \( R \) and common randomness consumption rate \( C \):
\[
R \geq C(W) = \max_P I(P;W), \quad R + C \geq \max_P H(PW).
\]

Conversely, these rates are also achievable.

Proof. A simulation of the channel must be in the error bound for every input \( x^n \), hence eq. \((\text{[22]}\)) will be satisfied for every distribution \( P \). The lower bounds follow now from theorem \((\text{[12]}\)) by choosing \( P \) to maximize \( I(P;W) \) and \( H(PW) \), respectively.

To achieve this, the encoder, on seeing \( x^n \) reports its type to the receiver (asymptotically free) and then they use the protocol of theorem \((\text{[3]}\)) for \( P = P_{x^n} \), the empirical distribution of \( x^n \). Possibly they have to use the channel at rate \( C(W) - I(P;W) \) to set up additional common randomness beyond the given \( \max_P H(PW) - C(W) \).

At this point we would like to point out a remarkable parallel of methods and results to the work \([22]\): our use of lemma \((\text{[8]}\)) is the classical case of of the use of its quantum version from \([3]\), and the main result of the cited paper is the quantum analog of the present theorem \((\text{[12]}\)) and \((\text{[3]}\)) and \((\text{[13]}\)), and even the construction of the following section has its counterpart there.

The use of common randomness turned out to be remarkably powerful, and it is known in various occasions to make problems more tractable: a major example is the arbitrarily varying channel (see for example the review \([14]\)). While for discrete memoryless channels it does not lead to improved rates or error bounds, it there allows for a “reverse” of Shannon’s coding theorem \([3]\) in the sense of simulating efficiently a noisy channel by a noiseless one. This viewpoint seems to extend to quantum channels as well, assisted by entanglement rather than common randomness: see \([3]\). We shall expand on the power of the “randomness assisted” viewpoint in section \((\text{[4]}\)).
IV. Solution under a letterwise criterion

Here we show that from the theorem of the previous section a solution to the compression problem under a slightly relaxed distance criterion follows: whereas previously we had to employ common randomness to achieve the lower bound $I(P;W)$, this will turn out to be unnecessary now.

Specifically, our condition will be eq. (3):

\[ |C| \leq \exp \left( nI(P;W) + O(\sqrt{n}) \right), \]

which is less than 1 for

\[ Q > \frac{4 \ln 2}{e^2 u} (n \log |\mathcal{X}| + \log(2|\mathcal{Y}|)) . \]

Hence there exist actual values $T_1, \ldots, T_Q$ such that

\[ \forall x^n \in T^n P, x_k - D_{T_q}(X^n)_k \leq 3 \epsilon, \]

which is what we wanted to prove: observe that $Q$ grows only polynomially.

As we remarked already in the introduction, [3] proposed to prove this result (and indeed more, being interested in the tradeoff between rate and error), but eventually turned to the much softer condition (3), which originates from the traditional model of rate distortion theory.

V. A general construction

Nice though the idea of the previous section is, the lower bound results show that on this road we cannot hope to approach the conjectured bound, because without common randomness at hand we have to spend communication at the same rate to establish it (compare [15], appendix, for this rather obvious-looking fact).

In this section we want to study the perfect restitution of the probability distributions $W_x$ (i.e. $\lambda = 0$):

Recall that here we want to minimize $H(E(X^n))$, and this minimum we call $S(n)$. Obviously $S(n_1 + n_2) \leq S(n_1) + S(n_2)$, so the limit

\[ \Gamma(P, W) = \lim_{n \to \infty} \frac{1}{n} S(n) \]

exists, and is equal to the infinum of the sequence.

Then we have

**Theorem 14:** For all $\lambda \in (0, 1)$

\[ \limsup_{n \to \infty} \frac{1}{n} \log M(n, \lambda) \leq \Gamma(P, W). \]

\[ \Pr \left\{ \frac{1}{Q} \sum_{q=1}^{Q} X^{T_q}_{x^n}[k] \not\in [(1 \pm \epsilon)X_{x^n}[k] \text{ on supp } W_x] \right\} \leq 2|\mathcal{Y}| \exp \left( -Q \frac{e^2 u}{4 \ln 2} \right). \]

Hence the sum of these probabilities is upper bounded by

\[ 2|\mathcal{Y}| |\mathcal{X}| \exp \left( -Q \frac{e^2 u}{4 \ln 2} \right), \]

By lemma [3] we obtain

\[ \frac{1}{Q} \sum_{q=1}^{Q} X^{T_q}_{x^n}[k] = (1 \pm \epsilon)X_{x^n}[k] \text{ on supp } W_x \]

which is equal to $\lambda = 0$.

\[ \frac{1}{Q} \sum_{q=1}^{Q} D_{T_q}(X^n)_k \leq 3 \epsilon, \]

Thus we have

\[ \Pr \left\{ \frac{1}{Q} \sum_{q=1}^{Q} D_{T_q}(X^n)_k \leq 3 \epsilon \right\} \leq 2|\mathcal{Y}| \exp \left( -Q \frac{e^2 u}{4 \ln 2} \right). \]

\[ \forall x \in \mathcal{X} \forall y \in \mathcal{Y} \quad W_x(y) = \sum_{c \in \mathcal{C}} D_{cy} E_{xc}. \]
Examples of this constructions are discussed in [3] (where it was in fact invented), and here we want to add some general remarks on optimizing it, as well thoughts on a possible algorithm to do that.

We begin with a general observation on the number of intermediate nodes:

**Theorem 15** ("c dice with d sides") An optimal zero error code for $W$ requires at most $CD - 1$ intermediate nodes, with $C = |\mathcal{X}|$, $D = |\mathcal{Y}|$.

**Proof.** For a fixed set $\mathcal{C}$ the problem is the following: Under the constraints

\begin{align}
\forall xc \quad & E_{xc} \geq 0, \quad \forall x \sum_c E_{xc} = 1, \quad (12) \\
\forall cy \quad & D_{cy} \geq 0, \quad \forall c \sum_y D_{cy} = 1, \quad (13) \\
\forall xy \quad & \sum_c E_{xc}D_{cy} = W_x(y), \quad (14)
\end{align}

minimize the entropy $H(\mu)$, where $\mu_c = \sum_x P_x E_{xc}$.

Observe that for each fixed set of $D_{cy}$ the constraints define a convex admissible region for the $E_{xc}$, of which a concave function is to be minimized. Hence, the minimum will be achieved at an extreme point of the region, that we rewrite as follows:

$$
\left\{ E_{xc} \geq 0 : \forall xy \sum_c E_{xc}D_{cy} = W_x(y) \right\}
$$

$$
= \bigoplus_x \left\{ E_{xc} \geq 0 : \forall y \sum_c E_{xc}D_{cy} = W_x(y) \right\}.
$$

An extreme point must be extremal in every of the summand convex bodies $B_x$. On the other hand, an extreme point of $B_x$ must meet $\dim B_x$ many of the inequalities ($E_{xc} \geq 0$) with equality. Since $\dim B_x \geq |\mathcal{C}| - D$ there remain only at most $D$ nonzero $E_{xc}$, for every $x$. In particular, only at most $CD$ many $c \in \mathcal{C}$ are accessed at all. In fact, to minimize $H(\mu)$, at most $CD - 1$, otherwise $c$ would contain full information about $x$. \hfill \blacksquare

**Remark 16:** The last argument can be improved: for $C, D \geq 2$ we can even assume $|\mathcal{C}| \leq CD - C + 1$.

The argument of the proof gives us the idea that maybe by an alternating minimization we can find the optimal code:

Indeed, conditions (12) and (14) for fixed $D$ are linear in $E$, and the target function is concave (entropy of a linear function of $E$), so we can find it’s minimum at an extreme point of the admissible region. This part is solved by standard convex optimization methods. On the other hand, for fixed $E$, eqs. (13) and (14) are linear in $D$. However, variation does not change the aim function. Still we have freedom to choose, and this might be a good rule: let $D$ maximize the conditional entropy $H(D|\mu)$. The rationale is that this entropy signifies the ignorance of the sender about the actual output. If it does not approach $H(W|P)$ in the limit this means that the protocol simulates partial feedback of the channel $W$, which could be used to extract common randomness. This amount is a lower bound to what the protocol has to communicate in excess of $I(P;W)$. We have, however, no proof that this rule converges to an optimum.

**VI. Applications**

In this section we point out three important connections to other questions, some of which depend on positive answers to the questions [3] and [4].

**A. Common randomness**

It is known that if two parties (say, Alice and Bob) have access to many independent copies of the pair of random variables $(X,Y)$ (which are supposed to be correlated), then they can, by public discussion (which is overheard by an eavesdropper), create common randomness at rate $I(X \land Y)$, almost independent of the eavesdropper’s information. For details see [4], where this is proved, and also the optimality of the rate. One might turn around the question and ask, how much common randomness is required to create the pair $(X^n,Y^n)$ approximately. This question, in the vein of that of the previous subsection, is really about reversibility of transformations between different appearances of correlation. Note that this was confirmed in [3] for the case of deterministic correlation between $X$ and $Y$, i.e. $H(Y|X) = H(X|Y) = 0$, which there was paralleled to entanglement concentration and dilution for pure states.

An affirmative answer to question [3], surprisingly implies that a rate of $I(X \land Y)$ of common randomness is sufficient, with no further public discussion to create pairs $X,Y$. This is done by first creating the distribution $Q$ of $E(X^n)$ on $C$ from the common randomness (this Alice and Bob do each on their own!): this may be not altogether obvious as the common randomness is assumed in pure form (i.e. a uniform distribution on $N$ alternatives), while the distribution
$Q$ may have no regularity. To overcome this difficulty fix an $\epsilon > 0$ and let
\[
k = \left\lfloor \frac{\log |C| - \log \epsilon}{\epsilon} \right\rfloor.
\]
Now we partition the unit interval into the subintervals
\[
I_a = \left( (1 + \epsilon)^{-(a-1)}, (1 + \epsilon)^{-a} \right), \quad a = 1, \ldots, k,
\]
\[
I_\infty = \left[ (1 + \epsilon)^{-k}, 0 \right],
\]
and define $C_a = \{ c \in C : Q(c) \in I_a \}$, $q_a = Q(C_a)$. Notice that for $a < \infty$ the probabilities for $c$’s belonging to the same set $C_a$ differ from each other only by a factor between $1-\epsilon$ and $1+\epsilon$, and that $q_\infty \leq \epsilon$, because of $(1+\epsilon)^{-k} \leq \epsilon/|C|$, by definition of $k$. Hence, defining uniform distributions $U_a$ on $C_a$ for $a < \infty$, it is immediate that
\[
\frac{1}{2} \left\| Q - \sum_{a=1}^{k} \frac{q_a}{1-q_\infty} U_a \right\|_1 \leq 2\epsilon.
\]
Now the distribution on the $a = 1, \ldots, k$ in this formula can be approximated to within $1/k$ by a $k^2$ distribution, which in turn can be obtained directly from a uniform distribution on $k^2$ alternatives. In this way we reduced everything to a number of uniform distributions, maybe on differently sized sets, all bounded by $|C|$ and a helper uniform distribution on a set of size $k^2$. However, it is well known that these can be obtained from a uniform distribution on $O(k^2|C|)$ items within arbitrarily small error.

Given this distribution on $C$, Bob applies $D$, whereas Alice applies the transpose channel $E^\prime$ to $E$. One readily checks that this produces the joint distribution of $X^n, Y^n$, up to arbitrarily small disturbance in the total variational norm.

Note that this result would imply a new proof of the optimality of of the rate $I(X \land Y)$ of common randomness distillation from $X^n, Y^n$: because we can simulate the latter pair of random variables with this rate of common randomness, we would obtain a net increase of common randomness after application of the distillation, which clearly cannot be.

**B. Channel coding**

It was already pointed out that this study has the paper\footnote{[1]} as one motivation, with its idea to prove the optimality of Shannon’s coding theorem by showing that every noisy channel $W$ can be simulated by a binary noiseless one operating at rate $C(W)$. Shannon’s theorem is understood as saying that the noisy channel can simulate a binary noiseless one of rate $C(W)$. Both simulations are allowed to perform with small error. Note that an affirmation of question\footnote{[2]} implies that this can be done, without the common randomness consumption like in section\footnote{[3]}. As indicated, this provides a proof of the converse to Shannon’s coding theorem:

The idea is that otherwise we could, given a rate of $C(W)$ noiseless bits simulate the channel, which in turn could be used to transmit at a rate $R > C(W)$. The combination of simulation and coding yields a coding method for transmitting $R$ bits over a channel providing $C(W)$ noiseless bits, which is absurd (in [3] this reasoning is called “causality argument”). Theorem\footnote{[4]} allows us to prove even more:

**Theorem 17 (Shannon [4])** For the channel $W$ with noiseless feedback (i.e. after each symbol $x$ transmitted the sender gets a copy of the symbol $y$ read by the receiver, and may react in her encoding) the capacity is given by $C(W)$. In fact, for the maximum size $M_f(n, \lambda)$ of an $(n, \lambda)$–feedback code
\[
M_f(n, \lambda) \leq \exp(nC(W) + O(\sqrt{n})).
\]

**Proof.** Let an optimal $(n, \lambda)$–feedback code for the channel $W^n$ with noiseless feedback be given. We will construct an $(n^2, \lambda')$–code with shared randomness, as follows:

Choose a simulation of the channel $W$ on $n$–blocks sending $nC(W) + O(\sqrt{n} \log n)$ bits, and using shared randomness, and with error bounded by $\epsilon = \frac{1-\lambda}{2\lambda'}$ (this is possible by the construction of theorem\footnote{[5]} — see remark\footnote{[6]}). We shall use $N$ independent copies of the feedback code in parallel: in each round $n$ inputs symbols are prepared, sent through the channel, yielding $n$ respective feedback symbols. Obviously, each round can be simulated with an error in the output distribution bounded by $\epsilon$, using our simulation of the channel $W$ (which, as we remarked earlier, simulates even the feedback). In each of the parallel executions of the feedback code thus accumulates an error of at least $\frac{Nn}{2}$, increasing the error probability of the code to $\frac{1-N}{2}$. Hence on the block of all the $n$ feedback codes we can bound the error probability by $\lambda' = 1 - \frac{1-N}{2}$. But this is subexponentially (in $N = n^2$) close to 1, so a standard argument applies:

First, by considering average error probability we can get rid of the shared randomness: there exists one value of the shared random variable for which the average error probability is bounded by $\lambda'$. Then we can argue that there is a subset $U$ of the constructed code’s message set $M^n$ which has maximal error probability bounded by $\lambda'' = \frac{1}{2\lambda'}$ and
\[
|U| \geq (1 - \lambda'')|M^n|.
\]

What we achieved so far hence is this: a code of $|U|$ messages with error probability $\lambda''$ and using $N(C(W) + o(N))$ noiseless bits. Clearly, we may assume the encoder to be deterministic without losing in error probability. But then at most $(1 - \lambda'')^{-1}$ messages can be mapped to the same codeword without violating the error condition.

Collecting everything we conclude
\[
|M^n| \leq (1 - \lambda'')^{-1}|U| \\
\leq (1 - \lambda'')^{-2} \exp(n^2C(W) + o(n^2)) \\
= \left[ \exp(nC(W) + o(n)) \right]^n,
\]

implying the theorem. \hfill \blacksquare

**Remark 18:** The weak converse (i.e. the statement that the rate for codes with error probability approaching 0 is bounded by $C(W)$) is much easier to obtain, by simply
keeping track of the mutual information between the message and the channel output through the course of operating a feedback code, using some well-known information identities, and finally estimating the code rate employing Fano’s inequality.

C. Rate–distortion theorem

Let \( d : X \times Y \to \mathbb{R}_{\geq 0} \) be any distortion measure, i.e. a non-negative real function. This function is extended to words \( X^n \times Y^n \) by letting

\[
d^n(x^n, y^n) = \sum_{k=1}^{n} d(x_k, y_k).
\]

Shannon’s rate distortion theorem is about the following problem: construct an \( n \)-block code \((E, D)\) (which may be chosen to be deterministic) such that for a given \( d \geq 0 \)

\[
d(E, D) := \sum_{x^n} P^n(x^n) d^n(x^n, D(E(x^n))) \leq nd,
\]

i.e., the average distortion between source and output word is bounded by \( nd \).

A pair \((R, d)\) of non-negative real numbers is said to be achievable if there exist \( n \)-block codes with code rate tending to \( R \) and distortion rate asymptotically bounded by \( d \). Define the rate–distortion function \( R(d) \) as the minimum \( R \) such that \((R, d)\) is achievable.

**Theorem 19** (Shannon [17]) The rate distortion function is given by the following formula:

\[
R(d) = \min \{ I(P; W) : W \text{ channel s.t. } \text{Ed}(X, Y) \leq d \},
\]

where \( \text{Ed}(X, Y) = \sum_{x,y} P(x)W_x(y)d(x,y) \) is the expected (single–letter) distortion when using the channel \( W \).

The proof of “\( \geq \)” here is a simple exercise using convexity of mutual information in the channel and standard entropy inequalities. We can give a simple proof of the “\( \leq \)”–part of this result, using theorem [1].

Choose some channel \( W \) satisfying the distortion constraint. Then mapping \( x^n \) to \( W^n_x \) obviously satisfies the distortion constraint on the code in the sense that the expected distortion between input and output, over source and channel, is bounded by \( nd \). Of course, sampling \( W^n_x \) at the encoder and sending some \( y^n \) will not meet the bound \( I(P; W) \). However, we can apply theorem [1] to approximately simulate the joint distribution of \( x^n \) and \( y^n \) by using some common randomness \( \nu \) and a deterministic code \((E_\nu, D_\nu)\) sending \( nI(P; W) + O(\sqrt{n}) \) bits. Hence, invoking linearity of the definition of \( d(E, D) \),

\[
\sum_{\nu} x_\nu d(E_\nu, D_\nu) \leq nd + O(\epsilon),
\]

so there must be one \( \nu \) such that \( d(E_\nu, D_\nu) \leq nd + O(\epsilon) \), which ends our proof.

At this point we would like to advertise our point of view that theorem [13] and even more so theorem [1] is what rate–distortion is actually about: the former theorem shows how to simulate a given channel on all individual positions of a transmission, and this is what we need in rate–distortion. In fact, rate–distortion theory is unchanged when instead of the one convex condition (“distortion bound”) on the code we have several, effectively restricting the admissible approximate joint types of input and output to any prescribed convex set — in particular a single point.

The strength of theorem [13] in comparison to such a development of rate–distortion theory lies in the fact that with its help we satisfy the convex conditions in every letter, not just in the block average. And theorem [1] gives the analogue of this even with the condition imposed on the whole block, yielding results that are not obtainable by simply applying rate–distortion tools (see e.g. [11]).

VII. Compression of sources of quantum states

The problem studied in this paper has a natural extension to quantum information theory: now the source emits (generally mixed) quantum states \( W_x \) on the Hilbert space \( \mathcal{Y} (x \in \mathcal{X}) \), with probabilities \( P(x) \), and an \((n, \lambda)\)-code is a pair \((E, D)\) of maps

\[
E : \mathcal{X}^n \to \mathcal{S}(\mathcal{C}),
D : \mathcal{S}(\mathcal{C}) \to \mathcal{S}(\mathcal{Y}^\otimes n),
\]

where \( \mathcal{S}(\mathcal{C}) \) is the set of states on the code Hilbert space \( \mathcal{C} \) and \( D \) is completely positive, trace preserving, and linear. The condition to satisfy is

\[
\sum_{x^n \in \mathcal{X}^n} P^n(x^n) \frac{1}{2} \| W^n_x - D(E(x^n)) \|_1 \leq \lambda,
\]

with the trace norm \( \| \cdot \|_1 \) on density operators. Define, like before, \( M(n, \lambda) \) as the minimum \( \text{dim} \mathcal{C} \) of an \((n, \lambda)\)-code.

Sometimes, the stronger condition

\[
\forall x^n \in \mathcal{T}_{P, \delta} \frac{1}{2} \| W^n_x - D(E(x^n)) \|_1 \leq \lambda
\]

will be applied.

Notice that this contains our original problem as the special case of a quasiclassical ensemble, when all the \( \rho_x \) commute (which means they can be interpreted as probability distributions on a set of common eigenstates).

This problem (with a number of variations, which we explained in the introductory section [1] for the classical case) is studied in [1]. There (and previously in [11]) it is shown that the lower bound theorem [2] holds in the quantum case, too, with understanding \( H \) as von Neumann entropy.

**Theorem 20:** For all \( n, \lambda \)

\[
\frac{1}{n} M(n, \lambda) \geq I(P; W) - f(\lambda),
\]

with a function \( f(\lambda) \to 0 \) for \( \lambda \to 0 \).

Let us improve this slightly by proving the strong version of this result:

**Theorem 21:** For all \( \lambda \in (0, 1) \)

\[
\lim inf_{n \to \infty} \frac{1}{n} M(n, \lambda) \geq I(P; W).
\]
Proof. By much the same method as the proof of theorem [3], the changes are that we need the more general code selection result of [2], thm. II.4, instead of the classical theorem [8], and which we state separately below: if \( (E, D) \) is an optimal \((n, \lambda)\)-code, define

\[
\mathcal{A} = \left\{ x^n : \frac{1}{2}||W^n_x - D(E(x^n))||_1 \leq \sqrt{\lambda} \right\}.
\]

Obviously \( P^n(A) \geq 1 - \sqrt{\lambda} > 0 \), so we can apply lemma [2] and find an \((n, \epsilon)\)-transmission code \( U \subset A \) for \( W^n \) such that

\[ |U| \geq \exp(nI(P;W) - O(\sqrt{n})). \]

This is an \((n, \lambda')\)-code for the channel \( D \circ E \), with \( \lambda' = \sqrt{\lambda + \epsilon} < 1 \), if we choose \( \epsilon \) small enough. Combining \( E \) with the transmission encoder, and \( D \) with the transmission decoder, we obtain an \((n, \lambda)\)-transmission code for \( |U| \) many messages over a noiseless system with Hilbert space \( \mathcal{C} \) of dimension \( M(n, \lambda) \).

To each message \( u \in U \) there belongs a decoding operator \( \Delta_u \geq 0 \) on the coding space \( \mathcal{C} \), forming together a POVM:

\[ \sum_u \Delta_u = \mathbb{1}. \]

Now to decode correctly with probability \( 1 - \lambda' \), for each \( u \) we must have

\[ \text{Tr} \Delta_u \geq 1 - \lambda'. \]

On the other hand, by \( \sum_u \text{Tr} \Delta_u = M(n, \lambda) \), we conclude

\[
M(n, \lambda) \geq \dim \mathcal{C} \geq \frac{1}{1 - \lambda'} |U| \geq \frac{1}{1 - \lambda'} \exp(nI(P;W) - O(\sqrt{n})),
\]

and we are done.

**Lemma 22:** For \( 0 < \tau, \lambda < 1 \) there is a constant \( K' \) and \( \delta > 0 \) such that for every discrete memoryless quantum channel \( W \) and distributions \( P \) on \( X \) the following holds: if \( A \subset 2^W \) is such that \( P^n(A) \geq \tau \) then there exists an \((n, \lambda)\)-transmission code \( (E, D) \) with the properties

\[
\forall m \in \mathcal{M} \quad E(m) \in A \quad \text{and} \quad \text{Tr} D_m \leq \text{Tr} \Pi^f_{H,m}, \delta ,
\]

\[
|\mathcal{M}| \geq \exp(nI(P;W) - K'\sqrt{n}).
\]

*Proof.* See [21], thm. II.4.

Progress on the problem of achievability of this bound is not known to us. It is remarkable that Koashi and Imoto [12] could obtain the exact optimal bound in the case of blind coding. It is indirectly defined via a canonical joint decomposition of the source states, but it can be derived from their result that generically the optimum rate is \( H(PW) \), which is achieved by simply Schumacher encoding the ensemble \( \{ P(x), W_x \} \).

Nevertheless, the results obtained in the classical case are very encouraging, so we state two conjectures:

**Conjecture 23:** For \( 0 < \lambda < 1 \) there exist \((n, \lambda)\)-codes with common randomness, asymptotically achieving transmission rate \( I(P;W) \) and common randomness consumption \( H(W|P) \).

If it turns out true, and also question [4] has a positive answer, we might even hope that also

**Question 24:** For \( 0 < \lambda < 1 \), is

\[
\lim_{n \to \infty} \frac{1}{n} \log M(n, \lambda) = I(P;W) ?
\]

[Note that, as in the case of question [4], codes achieving the optimal bound may also be constructed to satisfy eq. [3].] The implications of these statements, if they are true, would be of great significance to quantum information theory: not only would we get a new proof of the capacity of a classical–quantum channel being bounded by the maximum of the Holevo information and for the optimality of common randomness extraction from a class of bipartite quantum sources [11], but also the achievability of \( I(P;W) \) in the quantum rate distortion problem [23] with visible coding would follow, that until now has escaped all attempts.

**VIII. CONCLUDING REMARKS**

We demonstrated the current state of knowledge in the problem of visible compression of sources of probability distributions and its extension to mixed state sources in quantum information theory. Apart from reviewing the currently known constructions we contributed a better understanding of the resources involved: in particular the use of common randomness in some of them, and providing strong converses. Also we showed the numerous applications the result (and sometimes the conjectures) have throughout information theory, making the matter an eminent unifying building block within the theory.

We would like to draw the attention of the reader once more to our questions [3] and [24] and especially the conjecture [23] offering them as a challenge to continue this work.

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