Cosmological Correlations in Power Law Inflation models

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Scalar field with non-minimal coupling to curvature scalar is studied in Robertson-Walker background. The infrared limit of two point function, and, in turn, of the energy-momentum tensor of scalar field have been considered in the power law inflation model. In this limit, within the perfect fluid model, consistent evolution of scale factor following power law inflation gives rise to growing mode solution for negative value of coupling constant. A simplified calculation for density perturbation in power law inflationary models is presented with these mode functions. Salient features of the perturbation spectra has been commented upon.

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I. INTRODUCTION

Apart from solving some of the major problems of standard cosmology, inflationary models provide the most elegant solution to the formation of large scale structure in the Universe. The small inhomogeneity in density, as observed in the present day Universe, is considered to be originated from the quantum fluctuation in energy density of scalar field within a small causal region at the early epoch which subsequently grew into super horizon size, leaving horizon during rapidly expanding inflationary phase of the Universe. These super horizon size perturbations on re-entering the horizon at the post inflationary phase leaves imprint of primordial fluctuation on the cosmic microwave background radiation (CMBR). The density perturbation spectrum thus generated is nearly scale-invariant and the observable deviation from this scale invariant nature can discriminate among inflationary models.

The evaluation of the density perturbation depends on the time evolution of scalar field in the time-dependent cosmological background described by Robertson-Walker metric. The observed perturbation being very small, closely Gaussian, one may feel that free scalar field theory can adequately describe this fluctuation given two point correlation, thereby determining the necessary power spectrum, and all higher even correlations are factorized in terms of them. Moreover, the scalar field are usually considered to be ultra-light \((m << H)\) as the long wavelength limit is truly important in determining perturbation spectrum.

Recent times, the scope of precision observation of CMBR has generated considerable interest to search for its non-Gaussian nature, albeit very small, following the seminal work of Maldacena. Within the single field inflationary model, an estimate of the non-Gaussian nature was given for three point correlations involving different numbers of scalar and gravitational fields to first order in their interactions. Later on, Weinberg has refined the formulation using closed time path (‘in-in formalism’) and has shown a general method of calculation of cosmological correlations to higher orders in perturbation theory. Subsequently, Weinberg also pointed out some of the pertinent issues related to infrared divergence and the scheme of loop calculations, in general. Some of the works employing the ‘in-in formalism’ can be mentioned in this context.

The non-Gaussian distributions can be obtained in many ways such as going beyond models of single-field inflation, using extended kinetic terms, or by breaking the slow-roll conditions in single-field inflation. This has been comprehensively reviewed.

It is worth mentioning that the scalar field dynamics in time-dependent Robertson -Walker background has been studied earlier using real time formalism starting both from initial condition in equilibrium or in non-equilibrium with techniques similar to flat space-time. Typically this involves evolution of field in complex time domain, resulting in multiple Green functions which may be used to develop a consistent scheme of perturbation theory. In particular, considerable interest has been shown towards de Sitter space-time which has cosmological relevance.

Usually, most inflationary models employs de Sitter phase of expansion with horizon staying fixed but physical wavelength growing and the nature of potential driving such inflation should obey slow roll conditions to generate enough inflation to cover present horizon. The power law evolution of scale factor \(a(t) \sim t^p\), with \(p\) large, has been shown to be adequate in solving almost all the problems that are addressed by the usual model of inflation. The power law inflation has been studied in the context of extended inflation. Further works in the context can be seen in . The power law inflation are mostly associated with canonical scalar field with exponential potential. It may also happen otherwise, such as in K-inflation model. In view of recent available data, such as Planck, the feasibility of power law inflation can be explored in several fronts.

In this work we have considered scalar field coupled to gravity with arbitrary coupling to curvature scalar. For ultra-light scalars, the mode functions are obtained in the Robertson-Walker background with scale factor grows according to power law. The expectation value of energy-momentum tensor operator in the infrared limit is considered perfect fluid model in the spirit of . As the infrared modes are relevant for structure formation at large scale afterwards, we
address the issue of calculation of density perturbation in power inflation with real time formulation a la Semenoff-Wise. The time evolution is followed using density operator at the onset of inflation which mimics equilibrium form with parametrization. The correlation of density fluctuation is obtained by restricting up to quadratic order term to see small effect of nonlinearity. Finally, the dependence of spectral index on nonminimal coupling and power of scale factor has been pointed.

In section 2 basic features of scalar field theory with non-minimal coupling to curvature scalar is stated. The mode function in the background of Robertson-Walker is given and its relevant asymptotic form is stated. Section 3 contains the form of energy-momentum tensor in the infrared limit. Particularly, the consistent picture of fluid model in this context has been discussed and the relevance of power law inflation has been identified. Section 3, contains a sample calculation of density perturbation in power inflation model employing real time path formalism. Some of the features of the obtained perturbation spectrum has been discussed.

II. SCALAR FIELD IN ROBERTSON-WALKER BACKGROUND

The action for a real scalar field \( \phi \) in a general space-time background with non-minimal coupling to scalar curvature \( \mathcal{R}(t) \) is given by

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi(x) \partial_\nu \Phi(x) - \frac{1}{2} m^2 + \xi \mathcal{R}(t) \right] \Phi^2,
\]

where metric \( g^{\mu\nu} \) is the space-time metric and \( \xi \) denotes the coupling constant, taken arbitrary. The coupling constant \( \xi = 0 \) corresponds to minimally coupled scalar field while \( \xi = \frac{1}{6} \) corresponds to conformally coupled scalar field. The field equation obtained by varying the action is given by

\[
\left[ \frac{1}{\sqrt{-g}} \partial_\mu (g^{\mu\nu} \sqrt{-g} \partial_\nu \phi) + m^2 + \xi \mathcal{R}(t) \right] \phi(x) = 0.
\]

Considering Robertson-Walker metric background for spatially flat section

\[
ds^2 = dt^2 - a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2)],
\]

the above equation leads to

\[
\ddot{\phi} + 3H \dot{\phi} - \frac{1}{a^2} \nabla^2 \phi + (m^2 + \xi \mathcal{R}(t)) \phi = 0.
\]

The field can be Fourier decomposed into modes

\[
\phi(x) = \int d^3k [a_k e^{ik \cdot x} f_k(t) + a_k^* e^{-ik \cdot x} f_k^*(t)].
\]

The time dependent mode function, \( f_k(t) \) obeys

\[
\left[ \frac{d^2}{dt^2} + 3 \frac{\dot{a}}{a} \frac{d}{dt} + \frac{k^2}{a^2(t)} + m^2 + \xi \mathcal{R}(t) \right] f_k(t) = 0.
\]

Writing \( f_k(t) = a^{-3/2} g_k(t) \), the above equation reduces to that of harmonic oscillator with time dependent frequency

\[
\ddot{g}_k(t) + \omega_k^2(t) g_k(t) = 0,
\]

where

\[
\omega_k^2(t) = \left( \frac{k^2}{a^2(t)} + m^2 \right) + (\xi - 1/6) \mathcal{R}(t) - \frac{1}{2} \left[ \dot{H}(t) + \frac{1}{2} H^2(t) \right].
\]

Note that, the first two terms within parenthesis do not involve any time derivative of scale factor while the remaining terms are comprised of such time derivatives, at most to second order.

The Wronskian of complex mode functions obey

\[
\dot{f}_k f_k^* - f_k \dot{f}_k^* = -i/a^3(t),
\]

\[ (t_a) = \int d^4x \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi(x) \partial_\nu \Phi(x) - \frac{1}{2} m^2 + \xi \mathcal{R}(t) \right] \Phi^2,
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\]

Note that, the first two terms within parenthesis do not involve any time derivative of scale factor while the remaining terms are comprised of such time derivatives, at most to second order.

The Wronskian of complex mode functions obey

\[
\dot{f}_k f_k^* - f_k \dot{f}_k^* = -i/a^3(t),
\]
which may be chosen so as to satisfy at the initial time, say $t_0$. Written in terms of $g_k$, the Wronskian condition reads

$$\dot{g}_k g_k^* - g_k \dot{g}_k^* = -i.$$  

(9)

which is time dependent and therefore it is satisfied for all subsequent times. It is obvious that the detailed form of mode function $g_k(t)$ depends on the time dependence of the scale factor. We are primarily interested in cases where the scale factor either grows exponentially with time, as in de-Sitter phase, or it follows a simple power law dependence on time.

For power law behaviour of the scale factor, $a(t) \sim t^p$, the mode equation (6) reduces to

$$\ddot{g}_k(t) + \left[ \frac{k^2}{a^2(t)} + H^2(t) \left\{ \frac{m^2}{H^2(t)} + 12\xi - \frac{9}{4} \left( 1 - \frac{2}{3p} \right) \right\} \right] g_k(t) = 0.$$  

(10)

The mode equations are solved in general using WKB approximation. However, in the limit where mass term $m^2/H^2(t)$ is either vanishingly small as for ultra light scalar $m << H$ or time independent, such as in effective mass description, the mode functions obeying Wronskian (9) can be obtained exactly

$$g_k(t) = \frac{1}{2} \sqrt{\frac{p}{p-1}} \frac{\pi}{H} H_{\nu}^{(1,2)} \left( \frac{k}{p-1} a(t) H(t) \right),$$  

(11)

where $H_{\nu}^{(1,2)}$ are the Hankel functions of first and second kind respectively with order $\nu$ with

$$\nu^2 = \frac{(3p-1)^2}{4(p-1)^2} - 6\xi\left( \frac{2p-1}{p-1} \right) - \left( \frac{p}{p-1} \right) \left( \frac{m^2(t)}{H^2(t)} \right).$$  

(12)

Incidentally, in the de-Sitter phase $a(t) \sim e^{\eta H_0 t}$, $H_0$ being a constant, the corresponding equation is

$$\ddot{g}_k(t) + \left[ \left( \frac{m^2}{H_0^2} + 12\xi - \frac{9}{4} \right) H_0^2 + \frac{k^2}{a^2(0)} e^{-2\eta H_0 t} \right] g_k(t) = 0.$$  

(13)

and the solution is given by

$$g_k(t) = \frac{1}{2} \sqrt{\frac{\pi}{H_0}} H_{\nu}^{(1,2)} \left( \frac{k}{a(t) H_0} \right),$$  

(14)

with $\nu^2 = 9/4 - 12\xi - m^2/H_0^2$.

Often it is convenient to use conformal time $\eta = \int dt/a(t) (-\infty < \eta < 0)$. In terms of conformal time field decomposed in modes

$$\phi(x) = \int d^3k [a_k e^{i \mathbf{k} \cdot \mathbf{x}} f_k(\eta) + a_k^* e^{-i \mathbf{k} \cdot \mathbf{x}} f_k^*(\eta')].$$  

(15)

The mode functions written in general,

$$f_k(\eta) = \alpha \left( \frac{\pi \eta}{4} \right)^{1/2} \eta H \left[ C_1 H_{\nu}^{(1)}(k \eta) + C_2 H_{\nu}^{(2)}(k \eta) \right],$$  

(16)

where $\alpha = 1$ or $\frac{e^{-1}}{|k|}$ for de Sitter or power law behaviour of scale factor, respectively. Here the coefficients $C_1, C_2$ are in general $k$ dependent function. The Wronskian condition (9) imposes relation between coefficients; $| C_2 |^2 - | C_1 |^2 = 1$.

The vacuum state is defined by

$$a_k \ | 0 \rangle = 0.$$  

(17)

where, $a_k$ is the annihilation operator of the field quanta.

In contrast to Minkowski space field theory where unique choice of vacuum is guided by Poincaré invariance, such choice of vacuum is absent in curved space-time. Different choices of constants $C_1$ and $C_2$ obeying above condition leads to plethora of inequivalent vacua, all related by Bogoliubov transformation. This may also exhibit infrared divergence even in the absence of interaction, except gravity, as shown in Robertson-Walker background. Importantly, based on the Hadamard structure of two point correlation, in general background, a state exhibiting infrared divergence cannot arise from dynamical evolution from regular initial condition.

The two point correlation is given by

$$\langle \phi(x) \phi(x') \rangle = \frac{d^3k}{(2\pi)^3} e^{i \mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} f_k(\eta) f_k^*(\eta').$$  

(18)
Here we are interested in the problems of density fluctuation involving modes of long wavelength, typically of the order of Hubble radius. The ultraviolet divergence associated with high energy modes and its renormalization has been investigated thoroughly, which is hardly a practicable issue from the perspective of calculating density perturbation.

The infrared contribution of the above integral can be obtained by employing expansion of Hankel function for small argument \((\nu > 0)\) and up to first non leading term:

\[
H_{\nu}^{(1,2)}(z) = \mp \frac{i}{\pi} \Gamma(\nu) \left( \frac{z}{2} \right)^{-\nu} \left[ 1 + \frac{1}{\nu - 1} \left( \frac{z}{2} \right)^{2} \right] \tag{19}
\]

giving mode function

\[
f_k(\eta) \sim (C_2 - C_1) A^\nu \eta^{-\nu} k^{-\nu} [1 + \gamma k^2 \eta^2], \tag{20}
\]

where \(\mu = \frac{3p - 1}{2(2p - 1)}, \quad A = \pm \frac{i}{\pi} \Gamma(\nu) 2^{\nu - 1} (1 - p)^{p/(p - 1)}\) and \(\gamma = 1/(\nu - 1)\).

Clearly, the source of divergence is the integral in the infrared limit

\[
|A|^2 \Gamma^2(\nu) \left( \frac{\eta \eta'}{4} \right)^{\mu - \nu} \int d^3k k^{-2\nu} |C_2 - C_1|^2. \tag{21}
\]

Assuming the coefficients are regular as \(k \to 0\), the integral diverges for \(\nu \geq 3/2\).

One may note that similar marginal divergence happens for a minimally coupled massless scalar field in de Sitter phase.

A plot of such condition is shown in \((\xi, p)\) plane, exactly same as is shown in Fig. 1, indicating the shaded region where the two point function diverges, in general. However, the coincident limit indicated there is hardly of practical interest as pointed later.

For power law expansion, with regular coefficients the infrared divergence in two point function occurs over a wide range of values of \(p\), governed by the inequality

\[
\xi \leq \frac{3p - 2}{6p(2p - 1)}. \tag{22}
\]

This also shows that for minimally coupled scalar field \((\xi = 0)\), the divergence occurs for \(2/3 \leq p < 1\) and \(p > 1\). In particular, \(\xi < 0\), is interesting in several aspect such as particle production. Also, as we shall show here that power law inflation \(p > 1\) with growing modes is supported in this region.

Further, the infrared divergence can be ameliorated with suitable choice of mode function such that \(|C_2 - C_1|\) increases at a rate faster than \(k^{2\nu - 2}\). The \(k\) dependent coefficients can be determined such as inflation dominated pre-inflationary phase.

\[\text{FIG. 1: IR divergent regions (shaded) of two point function.}\]

The classical energy momentum tensor of a non minimally coupled scalar field can be obtained by varying action (1) with respect to the metric \(g^{\mu\nu}\):

\[
T_{\mu\nu} = (1 - 2\xi) g_{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (1 - 4\xi) g_{\mu\nu} \partial_\mu \phi \partial^2 \phi + 2\xi \phi \partial^2 \phi + 2g_{\mu\nu} \phi \nabla^2 \phi - \xi (\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R}) \phi^2 + \frac{1}{2} m^2 g_{\mu\nu} \phi^2. \tag{22}
\]

The trace of energy momentum tensor is given by

\[
T \equiv g^{\mu\nu} T_{\mu\nu} = (6\xi - 1) \nabla_\mu (\phi \nabla^\mu \phi) + m^2 \phi^2 + \phi (\Box + m^2 + \mathcal{R}) \phi, \tag{23}
\]
where the last term vanishes by equation of motion (2).
Treating \( \phi \) as the full quantum field with \( \langle \phi \rangle = 0 \), the energy density and the homogeneous pressure are given by
\[
\rho = \langle T_{00} \rangle, \quad p = -\frac{1}{a^2} \langle T_{ii} \rangle \quad \text{(no sum over } i) \tag{24}
\]
Considering fluid model, one obtains
\[
w = \frac{1}{3} \left( 1 - \frac{\langle T \rangle}{\langle T_{00} \rangle} \right). \tag{25}
\]
In spatially flat Robertson-Walker metric, the classical energy density (22) becomes,
\[
T_{00} = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \left( \frac{\nabla \phi}{a} \right)^2 + 6\xi H \phi \dot{\phi} - \frac{2\xi}{a^2} \left( (\nabla \phi)^2 - \phi \nabla^2 \phi \right) + \frac{1}{2} \dot{\phi}^2 + 3\xi H^2 \phi^2, \tag{26}
\]
and the trace of energy-momentum tensor (23) is given by
\[
T = (6\xi - 1) \left[ \dot{\phi}^2 - \left( \frac{\nabla \phi}{a} \right)^2 + \phi \ddot{\phi} + 3H \phi \dot{\phi} - \phi \left( \frac{\nabla^2 \phi}{a^2} \right) \right] + m^2 \phi^2. \tag{27}
\]
Using mode expansion (17), and \( \phi \) as quantum field, the vacuum expectation value of (symmetrized) \( T_{00} \) is obtained;
\[
\rho = \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{2a^2} | f'_k |^2 + \frac{1}{2} \left( m^2 + \frac{k^2}{a^2} + 3H^2 \xi \right) | f_k |^2 + 3\xi H (f_k f'_{k'} + f_k f'_{k'}) \right]. \tag{28}
\]
Similarly the trace of energy momentum tensor (27) obtained;
\[
\langle T \rangle = \int \frac{d^3k}{(2\pi)^3} \left[ (6\xi - 1) \left( \frac{1}{a^2} | f'_{k'} |^2 - \omega_k^2 | f_k |^2 \right) + m^2 | f_k |^2 \right] \tag{29}
\]
where \( \omega_k^2 = k^2/a^2(t) + m^2 + \xi R \) and prime denotes derivative with respect to conformal time \( \eta \). In the massless limit, the long wavelength behaviour of the energy density and the trace of energy-momentum tensor can be evaluated using approximation (19).
For \( \nu \neq \mu \),
\[
\rho \overset{k \to 0}{\sim} | C_2 - C_1 |^2 | A |^2 \eta^{-2(\mu - \nu)} \frac{1}{2\pi^2 a^2 \eta^2} \left\{ \frac{1}{2} (\nu - \mu)^2 + \frac{3}{4} \xi (2\mu - 1)^2 + \frac{3}{2} (2\mu - 1)(\nu - \mu) \right\} \int dkk^{2-2\nu} \tag{30}
\]
\[
\langle T \rangle \overset{k \to 0}{\sim} | C_2 - C_1 |^2 | A |^2 \eta^{-2(\nu - \mu)} \frac{(6\xi - 1)}{2\pi^2 a^2 \eta^2} \left\{ (\nu - \mu)^2 - \frac{3}{2} (4\mu^2 - 1) \xi \right\} \int dkk^{2-2\nu}. \tag{31}
\]
For \( \nu = \mu \) (with \( \mu > 0 \)),
\[
\rho \overset{k \to 0}{\sim} | C_2 - C_1 |^2 | A |^2 \left\{ \frac{3}{2} \frac{\xi}{\pi^2 a^2 \eta^2} \int dkk^{2-2\mu} + \frac{1}{4\pi^2 a^2} \left( 1 - 6\xi \frac{2\mu - 1}{\mu - 1} \right) \int dkk^{4-2\mu} \right\}, \tag{32}
\]
\[
\langle T \rangle \overset{k \to 0}{\sim} | C_2 - C_1 |^2 | A |^2 \left\{ \frac{3}{4} \frac{\xi (6\xi - 1)}{\pi^2 a^2 \eta^2} \int dkk^{2-2\mu} - \frac{6\xi - 1}{2\pi^2 a^2} \int dkk^{4-2\mu} \right\}. \tag{33}
\]
Now, for the consistency of Friedmann equations, with vanishing cosmological constant, the power law behaviour of scale factor requires \( w = 2/3p - 1 \) where \( w \) contains the ratio which is well behaved in the long wavelength limit.
Using eqn(30),(31) we can have from eqn(25) the explicit expression of \( W \) as,
\[
w = \frac{1}{3} \left[ 1 - \frac{\left\{ (\mu - \nu)^2 - \frac{3}{2} (4\mu^2 - 1) \right\} (6\xi - 1)}{\left\{ (\mu - \nu)^2 + \frac{3}{2} (2\mu - 1)^2 + \frac{3}{2} (2\mu - 1)(\mu - \nu) \right\}} \right] \tag{34}
\]
Also the power law behaviour needs
\[
\frac{\langle T \rangle}{\langle T_{00} \rangle} = 4 - 2/p. \tag{35}
\]
Combining together, for \( \nu \neq \mu \), one arrives at the condition,

\[
\frac{8(6\xi - 1)(\nu - \mu)\nu}{2(\nu - \mu)^2 + 6\xi(\nu - \mu)(2\mu - 1) + 3\xi(2\mu - 1)^2} = \frac{2(2\mu + 1)}{2\mu - 1},
\]

(36)

Next we checked for a consistent solution obeying the constraint, \( \nu^2 = \mu^2 - \frac{3}{2}(4\mu^2 - 1) \geq 9/4 \) for growing modes. This is indeed possible as shown by solid contour falling in shaded region of fig-1. Thus it is possible to have modes leading to growing structure in the power law inflation models and it happens for \( \xi < 0 \). Also a mode convergent initially always remains so in course of time evolution.

Thus the energy density of non minimally coupled scalar field shows infrared divergence in the same range \( \nu \geq 3/2 \) as the two point function. However, for \( \xi = 0 \), the energy density of massless scalar is divergent for \( \nu \geq 5/2 \) \( \text{i.e.} \ 3/4 \leq p \leq 2, \ (p \neq 1) \), which is smaller than the corresponding range for two point function\(^\text{22}\). It may be noted that in Fig. 2, in this region \( \rho \) and \( p \) differs in sign, as expected in inflation.

For \( \nu = \mu \), which essentially corresponds to \( \xi = 0 \) (as \( \nu > 0 \)), one obtains \( w = -1/3 \). The solution of Friedmann equation and also fig. 3 gives \( p = 1 \), which is in contradiction with divergence range stated earlier\(^\text{22}\).

FIG. 2: Long wavelength behaviour of energy-density and pressure

Fig.3 shows the distribution of \( w \). The gray part represents the right hand side of equation (34) and the coloured
part is \( w = 2/3p - 1 \) which intersect along a curve. Apart from several divergent spikes at follows the same trend as shown by solid line in Fig. 1.

Summarily, for non minimally coupled scalar field, the infrared behaviour of states with \( C_1 \) and \( C_2 \) regular is such that metric with power law expansion may have self consistent solutions of Friedmann’s equation but it admits infrared divergence, as shown for a minimally coupled scalar\(^{22}\). In other words, inflation needs choice of special kind states with \( C_1 \) and \( C_2 \) irregular and \( xi \) negative.

### III. DENSITY FLUCTUATION IN INFLATIONARY EPOCH

Considering a large volume \( V \) at cosmological scale, the density inhomogeneity, at time \( t \) is measured by the mean square fluctuation in the density function

\[
\left( \frac{\delta \rho}{\bar{\rho}} \right)^2 = \left( \frac{\langle \rho(\vec{x}, t) - \bar{\rho}(t) \rangle^2}{\bar{\rho}(t)} \right)_{x},
\]

where \( \langle \cdots \rangle_x \) denotes average over all space and \( \bar{\rho} \) is the homogeneous background energy density, \( \bar{\rho} = \langle \rho(\vec{x}, t) \rangle_x \). The density contrast \( \delta(\vec{x}, t) \), excess over the averaged energy density \( \bar{\rho}(t) \), can be Fourier decomposed as

\[
\delta(\vec{x}, t) = \frac{1}{\sqrt{V}} \sum_k \delta_k(t) e^{i\vec{k} \cdot \vec{x}}.
\]

The mean square fluctuation (39) can be written in terms of Fourier modes

\[
\left( \frac{\delta \rho}{\bar{\rho}} \right)^2 = \frac{1}{V} \sum_k |\delta_k(t)|^2 \longrightarrow \int d(ln k) \frac{k^3}{2\pi^2} |\delta_k(t)|^2 .
\]

A convenient measure of fluctuation is the power per logarithmic interval in wave number or variance

\[
\left( \frac{\delta \rho}{\bar{\rho}} \right)^2_k = \frac{k^3}{2\pi^2} |\delta_k(t)|^2 \equiv \Delta^2(k).
\]

As the density fluctuation arises from quantum fluctuation of field in the inflationary phase, one may obtain similar expression for density inhomogeneity, with \( \rho(\vec{x}, t) \) replaced by the corresponding energy density operator \( \hat{\rho}(\vec{x}, t) \) with expectation value in an appropriate state. However, the energy density operators involves product of fields at the distance behaviour, as usual, not important in this context as measurements involve smearing of operator.

The energy density operator \( \hat{\rho}(\vec{x}, t) \) is the time-time component of the energy-momentum tensor of inflaton field:

\[
T_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - g_{\mu\nu} \left( \frac{1}{2} g^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi - V(\Phi) \right),
\]

where the potential function \( V(\Phi) \) depends on the model chosen. With a shift of the field \( \Phi(x) \) by homogeneous classical field \( \phi(t) \),

\[
\Phi(\vec{x}, t) = \phi(t) + \phi(\vec{x}, t),
\]

such that for the quantum field \( \langle \phi(x) \rangle = 0 \), the energy density operator reads \( \hat{\rho}(x) \) obtained from (43) is

\[
\hat{\rho}(x) \equiv \bar{\rho}(t) + \hat{U}(x, t),
\]

where \( \bar{\rho}(t) = \frac{1}{2} \dot{\phi}^2 + V(\phi) \) and \( \hat{U}(x, t) \) involves the quantum field \( \phi \). The formal structure of \( \hat{U}(x, t) \), up to quadratic terms is

\[
\hat{U}(x) = r(t) \phi(x) + s(t) \dot{\phi}(x) + u(t) \phi^2(x) + v(t) \dot{\phi}^2(x) + w(t) \partial_i \phi \partial_j \phi,
\]

The coefficients \( r(t), s(t), u(t), v(t), w(t) \) depend on the classical field and the parameters appearing in the potential \( V(\phi) \). In the potential driven inflation model e.g. \( v(t) = \frac{1}{4} \) and \( w(t) = 1/2a^2(t) \).

The inclusion of terms higher order in \( \phi \) and its derivatives allows nonlinearity of small amount, usually improving previous calculation to density fluctuation\(^{33}\). The issue of renormalization of composite operators, determining short distance behaviour, as usual, not important in this context as measurements involve smearing of operator.

As the fluctuation size increases, a relativistic treatment is mandatory for superhorizon sized perturbation. In contrast to density fluctuation for modes well within the horizon, the density fluctuation corresponding to superhorizon
modes are afflicted with the gauge ambiguity problem. The evolution of perturbation in this regime is kinematic in nature involving the evolution of curvature perturbation in space-time. A crucial result of such analysis is the constancy of a gauge invariant quantity $\xi$ for superhorizon sized modes. This is particularly useful in following the density perturbation of a given mode from its exit from horizon during inflationary phase until its reentry in radiation or matter dominated epoch at a later stage. In the uniform Hubble constant gauge, $\xi$ assumes a particularly simple form at the horizon crossings (i.e. $k/aH \sim 1$):

$$\xi = \frac{\delta \rho}{\rho + \rho^0}.$$ 

It may be noted that $\delta \rho$ is gauge non-invariant, thus

$$\xi = (1 + p/\rho)^{-1} \left( \frac{\delta \rho}{\rho} \right)_H$$

is not manifestly gauge invariant.

**IV. DENSITY PERTURBATION IN POWER LAW INFLATION MODELS**

In this section we consider fluctuation in energy density of scalar field in the background admitting power law inflation. Typically, we are interested in a scheme of calculating density perturbation spectrum that may be applicable to models such as in extended inflation. Here we assume that, without much ado, the classical inflationary solution for the scalar field joins smoothly to the solutions before and after the inflation, which indeed is a subtle issue.

On separating the field in classical and quantum part, the classical part of energy density (41) becomes

$$\bar{\rho}(t) = \frac{1}{2} \dot{\psi}^2 + V(\psi),$$

and the quantum part given by $U(x, t)$ given in (43).

Note that the correlation of a composite operator $\mathcal{O}(x)$ involving product of field and its derivatives is given by

$$\langle \mathcal{O}(x) \mathcal{O}(x') \rangle - \langle \mathcal{O}(x) \rangle \langle \mathcal{O}(x') \rangle.$$ 

The correlation in energy density operator given in terms of expectation values of the field and its derivatives (43) leads to

$$\langle U(x, t)U(x', t') \rangle = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} \tilde{P}(k^2, t, t')$$

where

$$\tilde{P}(k^2; t, t') = [r(t)r(t') + s(t)s(t')\partial_t + \{r(t)s(t')\partial_t + (t \leftrightarrow t')\}]\tilde{P}(k^2, t, t')$$

$$+ \frac{1}{2} \int \frac{d^3k'}{(2\pi)^3} \left[ 4u(t)u(t') + \partial_t^{(1)} \partial_t^{(2)} + \partial_t^{(2)} \partial_t^{(2)} + \left\{ \frac{2k^2(k - k')}{a(t)a(t')} \right\}^2 + 2\left\{ u(t)\partial_t^{(1)} \partial_t^{(2)} + (t \leftrightarrow t') \right\} - \tilde{k} \cdot (\tilde{k} - \tilde{k}') \left\{ \frac{4u(t^2)}{a(t^2)^2} + \frac{2}{a(t)^2} \partial_t^{(1)} \partial_t^{(2)} + (t \leftrightarrow t') \right\} \right] \tilde{P}^{(1)}(k^2, t, t') \tilde{P}^{(2)}((\tilde{k} - \tilde{k}')^2, t, t')$$

(44)

Here $P(k^2, t, t')$ is related to expectation value of fields as

$$\langle \phi(x, t)\phi(x', t') \rangle = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} \tilde{P}(k^2, t, t')$$

(45)

It may be mentioned that the dropped terms involves highly divergent contribution from composite operators, but measurement of a quantity at sharply defined instant or at a precise point are hardly considered in quantum theory, rather it is considered in a smeared sense. This is also sensible as the long wavelength limit is really of interest in present context.

The two point function given by real-time formulation is given by

$$\tilde{P}(k^2, t, t') = \frac{1}{[a(t)a(t')]^{3/2}} [g^+(t)g^-(t') + n(\omega_0)(g^+(t)g^-(t') + g^-(t)g^+(t'))]$$

(46)

with $n(\omega_0) = (e^{\delta_0\omega_0} - 1)^{-1}$. 

Here density matrix at the onset of inflation is parametrized as $e^{-\beta_0 H(t_0)}$ and we are temporarily leaving aside the question whether truly thermal equilibrium prevailed till before inflation and the reason for its departure thereof.

The other quantity $\omega_0$ is related to the wavelength of the modes concerned as

$$\omega_0^2 = \left(\frac{k}{a(t_0)}\right)^2 + H^2(t_0) \left(12\xi - \frac{9}{4}\right)$$

(47)

and mode functions $g^\pm$ are given in (11).

Note that for the purpose of calculation of density perturbation the time at which mode function leaves the horizon (say, $t_H$) is important. So the concerned mode also expands by a factor $a(t_H)/a(t_0)$ during the time interval of onset of inflation ($t_0$) and that of horizon exit ($t_H$). Also the horizon size potentially sets the maximal limit on scale of fluctuation to be considered, and the oscillatory nature of mode functions are clearly defined.

Using expression (46), the spectrum is given by

$$a^3(t) \hat{p}_i(p, t^2, t) = \left[r(t) + (\nu - 3/2)H(t)s(t)\right]^2 \hat{p}(p, t) + \frac{1}{2} \int \frac{d^3p'}{(2\pi)^3} \left\{2u(t) + (\nu - 3/2)^2H^2(t) - \hat{p}', \left(\frac{a(t')}{a(t)} \hat{p} - \hat{p}'\right)\right\}^2 \hat{p}(p', t) \hat{p} \left(\frac{a(t')}{a(t)} \hat{p} - \hat{p}'\right)^2 , t, t'$$

(48)

where, $\hat{p} = \vec{k}/a(t)$ and

$$\nu = \sqrt{\frac{(3p - 1)^2}{4(p - 1)^2} - 6\xi \frac{p(p - 1)}{(p - 1)^2}}.$$ 

(49)

The first term in eqn. (48) can be evaluated using asymptotic form (11) for small argument $k/a << H$, modes will within horizon at the onset of inflation $t = t_0$

$$\frac{k^3}{2\pi^2} \hat{p}(k, t_0) \simeq \left(\frac{k}{2\pi aH}\right)^{3-2\nu} H^2(t_0) \left(\frac{p}{p - 1}\right)^{1/2 - \nu} \pi^{-2\nu} \Gamma^2(\nu).$$

(50)

Note that in the limit $p >> 1, \nu = 3/2$ and it reduces to the usual scale invariant result in de Sitter case. For large $p$ the contribution from first term is given by

$$\left[r(t_0) + \left(\nu - \frac{3}{2}\right)H(t_0)s(t_0)\right]^2 \left(\frac{p}{p - 1}\right) \frac{H(t_0)^2}{2k^3}$$

(51)

However, the usual approach of considering that at the horizon crossing, the argument of Hankel function being large enough to justify the asymptotic expansion $H_{\nu}^{(1,2)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{\pm iz}$ leads to

$$\frac{k^3}{2\pi^2} \hat{p}(k, t_H) \simeq \left(\frac{k}{aH}\right)^2 \left(\frac{H(t_H)}{2\pi}\right) = H^2(t_H),$$

(52)

which is independent of $\nu$.

Contribution from the second term in the leading order can be estimated by evaluating the integral at $t = t_0$, with terms proportional to $a^{-2}$ and $a^{-4}$ neglected, as

$$\int \frac{d^3p_0}{(2\pi)^3} \left(\frac{1}{\omega_0}\right)^{2\nu} \left(1 + \frac{2}{e^{\beta_0 \omega_0} - 1}\right) \left(\frac{1}{\omega_0}\right)^{2\nu} \left(1 + \frac{2}{e^{\beta_0 \omega_0} - 1}\right).$$

(53)

Now in the limit $\beta_0 \omega_0 \rightarrow 0$, a legitimate approximation for the modes of concerned at very early stage, equation(54) approximates to,

$$\int \frac{d^3p_0}{(2\pi)^3} \left(\frac{1}{\omega_0}\right)^{2\nu} \left(1 + \frac{2}{\beta_0 \omega_0}\right) \left(\frac{1}{\omega_0}\right)^{2\nu} \left(1 + \frac{2}{\beta_0 \omega_0}\right).$$

(54)

Using Feynman parametric integral representation

$$\frac{1}{a^\alpha b^\beta} = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dz \frac{z^{\alpha - 1}(1 - z)^{\beta - 1}}{|az + (1 - z)b|^{|\alpha + \beta|}}$$


the factors in equation(52) may be combined to write as\textsuperscript{27}

\[
\frac{1}{8\pi^{3/2}k^{4\nu-3}\Gamma(\nu)} \int_0^1 d\theta \left( \frac{[\theta(1-\theta)]^{\nu-1}}{[\theta(1-\theta)+C]^{2\nu-3/2}} + \frac{2\Gamma(2\nu-1)}{\Gamma(\nu+1/2)} B_0 \int_0^1 d\theta \frac{\theta^{\nu-1/2}(1-\theta)^{\nu-1}}{[\theta(1-\theta)+C]^{2\nu-1}} \right)
\]

(55)

where \( \theta = \frac{a(t)}{a(t_0)}(\frac{k}{p}) = k' / k \) and \( C = (12\xi - 9/4)H^2(t_0) \).

Now, for \( p \gg 1 \) eqn. (53) gives to the leading order contribution as,

\[
\frac{1}{\pi^2 k^{2(4\nu-3)}} \{ A + \ln k \}
\]

(56)

where \( A \) is a constant depending on \( \xi \) and \( p \).

So, the complete expression for the density perturbation can be written as,

\[
\left( \frac{\delta \rho}{\rho} \right)_0^2 = k^3 \tilde{P}_U = \left\{ r(t_0) + \left( \nu - \frac{3}{2} \right) H(t_0)s(t_0) \right\}^2 \left\{ \frac{p}{2(p-1)} \right\} H(t_0)^2 \]

\[ + k^{6-4\nu} \left\{ 2\nu(t_0) + \left( \nu - \frac{3}{2} \right)^2 H^2(t_0) \right\}^2 \left\{ \frac{p}{2(p-1)} \right\} \frac{H^4(t_0)}{2\pi^2} \{ A + \ln k \} \]

(57)

The perturbation relevant at horizon crossing \( t_H \) can be conveniently converted from that at the onset of inflation \( t_0 \) by

\[
\frac{k}{a(t_0)} = \frac{a(t_H)}{a(t_0)} \cdot \frac{k}{a(t_H)}.
\]

(58)

Note that the perturbation spectrum depends both on \( p \) and \( \xi \) intricately. So one may constrain the parameter space in this quantities to fit with the observational limits. Also the inclusion of quadratic terms gives rise to logarithmic dependence on wave number \( k \) which is in resemblance with the usual results obtained as in\textsuperscript{25}.

From eqn (57), the spectral index defined as \( n_s - 1 = d(\ln P_t) / d\ln k \) gives approximately by

\[
n_s - 1 \simeq \frac{a(6-4\nu)}{2\hat{b}(6-4\nu)} \ln(k)
\]

(59)

where, \( a = \frac{A}{8\pi^2} \left( \frac{p}{p-1} \right)^2 H^4 \left[ \frac{2}{p}(3 - \frac{1}{p}) + (\nu - 3/2)^2 \right] \), and \( \hat{b} = \frac{1}{8\pi^2} \left( \frac{p}{p-1} \right)^2 H^4 \left[ \frac{2}{p}(3 - \frac{1}{p}) + (\nu - 3/2)^2 \right] \).

Fig. 4 shows the variation of \( n_s \) with \( p \) and coupling parameter \( \xi \), It is within the range \( 0.94 \leq n_s \leq 0.99 \) and in agreement with the observational data\textsuperscript{38,39}.

![FIG. 4: Distribution of spectral index, \( n_s \)](image)

The running spectral index defined \( n' = dn_s / d(\ln k) \) gives , \( n' = 2\hat{b}'(6 - 4\nu) \) and the corresponding range is \(-0.001 \leq n' \leq 0.004 \) which is in agreement with the observational data\textsuperscript{38,39}. 
V. CONCLUSION

Study of scalar field in time dependent cosmological background is important as its quantum fluctuation in a small causal region at an early epoch, in principle, can account the observed density fluctuation in Universe. We have addressed the issue of density perturbation within the scope of power law inflation where scalar field is coupled to curvature scalar. In this context, the infrared modes of the ultra-light scalar field are considered. The average of energy-momentum tensor operator is fitted with perfect fluid model and growing mode solutions are shown to occur at ξ < 0 region. As the density perturbation spectrum is related to the correlation of energy-momentum tensor, the two point function determines the power spectra in its leading part. The real time formalism of Semenoff-Wise has been used for two point function. However, this precisely depends on the time of onset of inflation which differs from the time of leaving the horizon depending upon modes. Inclusion of quadratic term in fields has been shown to lead to logarithmic correction in the formula, usually found in the literature. The spectral index found depends on ξ and p, therefore relevant parameter space can be explored so that power law inflation may become consistent with observational data.

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