QUASICONFORMAL EXTENSIONS OF HARMONIC UNIVALENT MAPPINGS OF THE INTERIOR AND EXTERIOR UNIT DISK

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Abstract. In this note, we consider the sufficient coefficient condition for some harmonic mappings in the unit disk which can be extended to the whole complex plane. As an application of this result, we will prove that a harmonic strongly starlike mapping has a quasiconformal extension to the whole plane and will give an explicit form of its extension function. We also investigate the quasiconformal extension of harmonic mappings in the exterior unit disk.

1. Introduction

Let \( f \) be a complex-valued continuous function defined in the unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \). Let \( \mathbb{T} \) denote the unit circle \( \{ z \in \mathbb{C} : |z| = 1 \} \) and \( \overline{\mathbb{D}} \) denote the closed unit disk \( \{ z \in \mathbb{C} : |z| \leq 1 \} \). The mapping \( f = u + iv \) is said to be harmonic in \( \mathbb{D} \) if \( u \) and \( v \) are both real harmonic in \( \mathbb{D} \). It also has a canonical decomposition \( f = h + ig \), where \( h \), \( g \) are both analytic functions in \( \mathbb{D} \). Let \( \mathcal{H} \) denote the class of complex-valued harmonic functions in \( \mathbb{D} \) with the normalization \( f(0) = f_z(0) - 1 = 0 \). Each function \( f = h + ig \in \mathcal{H} \) has the power series expansions for \( h \) and \( g \) by

\[
(1.1) \quad h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad z \in \mathbb{D}.
\]

The Jacobian of \( f \) is given by

\[
J_f(z) = |h'(z)|^2 - |g'(z)|^2.
\]

A theorem due to Lewy \[14\] asserts that \( f \) is locally univalent (one-to-one) if and only if \( J_f(z) \neq 0 \) for any \( z \in \mathbb{D} \). If \( J_f(z) > 0 \), then \( f \) is said to be sense-preserving. The class of the normalized (sense-preserving) harmonic univalent mappings on the unit disk, including some subclasses, has been investigated by Clunie and Sheil-Small \[3\] (see also Duren \[4\]). A sense-preserving harmonic mapping \( f \) is said to be \( k \)-quasiconformal, if its complex dilatation \( \mu_f(z) = f_z(z)/f_{\overline{z}}(z) \), satisfies \( |\mu_f(z)| \leq k < 1 \) almost everywhere in the given domain. In the most literature, \( f \) is called a \( K \)-quasiconformal mapping with \( K = (1 + k)/(1 - k) \geq 1 \). The basic theory of quasiconformal mappings has been concluded by Lehto and Virtanen \[13\].

A well-known result given by Fait, Krzyż and Zygmunt \[5\] shows that any strongly starlike curve of order \( \alpha \) (\( 0 < \alpha < 1 \)) is a \( k \)-quasicircle with \( k \leq \sin(\alpha \pi/2) \), and any

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\]

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strongly starlike function of order $\alpha$ has an explicit quasiconformal extension on the whole plane $\mathbb{C}$. For an analytic function $h$ defined on $\mathbb{D}$ of the form $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$, with the condition $\sum_{n=2}^{\infty} n|a_n| \leq k < 1$, $h$ is a strongly starlike function and has a quasiconformal extension on the whole plane with $|\mu_h| \leq k$ almost everywhere in $\mathbb{C}$ (see [5]). It is natural to consider the question whether the same results can be obtained for a harmonic strongly starlike function of order $\alpha$. To answer this question, we introduce a class of harmonic hereditarily strongly starlike functions $f$ of order $\alpha$ ($0 < \alpha \leq 1$) defined on $\mathbb{D}$, denoted by $SS_H(\alpha)$ (see [15]). Any function $f \in SS_H(\alpha)$, satisfies

$$\left| \arg \frac{zf(z) - \bar{z}f(\bar{z})}{f(z)} \right| \leq \frac{\pi \alpha}{2}, \quad z \in \mathbb{D}\setminus\{0\}.$$ 

If $\alpha = 1$, then the inequality above can be written in the form

$$\text{Re} \left( \frac{zf(z) - \bar{z}f(\bar{z})}{f(z)} \right) > 0, \quad z \in \mathbb{D}\setminus\{0\}.$$ 

So $f$ is a harmonic fully starlike function, which inherits the hereditary property for each $|z| < r < 1$, first introduced by Chuaqui, Duren and Osgood [2].

Inspired by Ganczar [6], Hamada, Honda and Shon [7], we consider two classes of harmonic functions of $\mathbb{D}$. Let $\{\varphi_n\}_{n=2,3,...}$ and $\{\psi_n\}_{n=1,2,...}$ be two sequences of non-negative real numbers. We denote by $\mathcal{H}(\{\varphi_n\}, \{\psi_n\})$ the class of harmonic functions in $\mathcal{H}$, with the form (1.1), satisfying $0 < |b_1| < 1$ and the coefficient condition

$$\psi_1 |b_1| + \sum_{n=2}^{\infty} (\varphi_n |a_n| + \psi_n |b_n|) \leq 1.$$ 

Similarly, let $\mathcal{H}^0(\{\varphi_n\}, \{\psi_n\})$ denote the class of harmonic functions in $\mathcal{H}$, with the form (1.1), satisfying $b_1 = 0$ and

$$\sum_{n=2}^{\infty} (\varphi_n |a_n| + \psi_n |b_n|) \leq 1.$$ 

In particular, Silverman [17] proved, if $f \in \mathcal{H}^0(\{n\}, \{n\})$, then $f$ is univalent and harmonic fully starlike in $\mathbb{D}$. Jahangiri [10] proved the same result for $f$ which belongs to $\mathcal{H}(\{n\}, \{n\})$. We combine their results in Theorem [A]

**Theorem A** ([10, 17]). If $f = h + \bar{g}$ with the form (1.1), where $|b_1| < 1$, satisfies

$$\sum_{n=2}^{\infty} n |a_n| + \sum_{n=1}^{\infty} n |b_n| \leq 1,$$

then $f$ is harmonic fully starlike.

Avci and Zlotkiewicz [1] showed that a sufficient condition for a harmonic mapping $f$ to be univalent and convex in $\mathbb{D}$ is $f \in \mathcal{H}^0(\{n^2\}, \{n^2\})$. Jahangiri [9] then generalized the result to the functions $f \in \mathcal{H}(\{n^2\}, \{n^2\})$. We combine their results in Theorem [B]

**Theorem B** ([1, 9]). If $f = h + \bar{g}$ with the form (1.1), where $|b_1| < 1$, satisfies

$$\sum_{n=2}^{\infty} n^2 |a_n| + \sum_{n=1}^{\infty} n^2 |b_n| \leq 1,$$
then $f$ is harmonic fully convex.

Theorem A also showed a sufficient condition for $f \in \mathcal{H}(\{\varphi_n\}, \{\psi_n\})$ to be univalent on $D$ is that $\varphi_n \geq n \ (n \geq 2)$, and $\psi_n \geq n \ (n \geq 1)$.

Ganczar [6] gave a sufficient coefficient condition for a function $f \in \mathcal{H}^0(\{\psi_n\}, \{\psi_n\})$ to be extended to a quasiconformal homeomorphism of $\mathbb{C}$. Hamada, Honda and Shon generalized the same result to the case that $b_1$ is not necessarily 0 in [7]. We combine their results into the following theorem with a slight modification.

**Theorem C** ([6], [7]). Consider the sequence $\{\psi_n\}_{n=1,2,...}$ satisfying either of the following conditions:

(1.2) \[ \frac{\psi_n}{n} \geq \psi_1 > 1 \quad (n \geq 1); \]

(1.3) \[ \frac{\psi_n}{n} \geq \frac{\psi_2}{2} > \psi_1 = 1 \quad (n \geq 2). \]

If $f \in \mathcal{H}(\{\psi_n\}, \{\psi_n\})$, then $f$ has a homeomorphic extension to the unit circle $T$, and the image curve $f(T)$ is a quasicircle. Moreover, the mapping $F$ of the form

(1.4) \[ F(z) = \begin{cases} f(z), & |z| \leq 1, \\ z + \sum_{n=2}^{\infty} a_n z^{-n} + \sum_{n=1}^{\infty} b_n z^{-n}, & |z| \geq 1, \end{cases} \]

is a quasiconformal extension of $f$ to $\mathbb{C}$. Furthermore, the complex dilatation $\mu_F$ satisfies $|\mu_F| \leq 1/\psi_1$ if $0 < |b_1| < 1$ under the condition (1.2), or $|\mu_F| \leq 2/\psi_2$ if $b_1 = 0$ under the condition (1.3).

In Section 2, we consider a function $f \in \mathcal{H}(\{\varphi_n\}, \{\psi_n\})$, with two sequences $\{\varphi_n\}$, $\{\psi_n\}$ not exactly same. As a refinement of Theorem C, Theorem 2.1 claims that $f \in \mathcal{H}$ under the condition (2.1), has a quasiconformal extension to the whole complex plane. A result of [5] shows that any normalized bounded convex function, which is analytic and univalent in $D$, has a quasiconformal extension to $\mathbb{C}$. We cannot guarantee this property for all convex functions. But with a strong sufficient coefficient condition considered in Theorem B, a harmonic convex function can be extended to the whole plane.

In Section 3, we will give a main result as an application of Theorem 2.1 Lemma 3.1 (see also [15]) gives a sufficient condition for $f \in \mathcal{H}$ to be harmonic strongly starlike. Under this condition, Theorem 3.3 shows that the mapping $f$ admits a quasiconformal extension to $\mathbb{C}$. Compared to the analytic case discussed in [5], the condition (3.2) has no need to be restricted by an additional $k$ with $k < 1$, to obtain an explicit quasiconformal extension to $\mathbb{C}$.

In this paper, we also consider a class of harmonic sense-preserving univalent mappings defined on the exterior unit disk $\Delta = \{z \in \mathbb{C} : |z| > 1\}$ that map $\infty$ to $\infty$. This class was first introduced by Hengartner and Schober [8] in 1987. Such mappings have the representation

(1.5) \[ f(z) = \alpha z + \beta \overline{z} + \sum_{n=0}^{\infty} a_n z^{-n} + \sum_{n=1}^{\infty} b_n z^{-n} + A \log |z|, \quad z \in \Delta, \]
where $0 \leq |\beta| < |\alpha|$ and $A \in \mathbb{C}$. Let $\Sigma_H$ denote the class of mappings $f$ of the form (1.5). By applying the normalization that $\alpha = 1$, $\beta = 0$ and $a_0 = 0$, we can get the subclass of $\Sigma_H$, denoted by $\Sigma_{H}^{*}$. Then any function in $\Sigma_{H}^{*}$ has the form

$$f(z) = z + h(z) + g(z) + A \log |z|, \quad z \in \Delta,$$

where

$$h(z) = \sum_{n=1}^{\infty} a_n z^{-n}, \quad g(z) = \sum_{n=1}^{\infty} b_n z^{-n}$$

are analytic in $\Delta$.

The subclass of $\Sigma_{H}^{*}$ with $A = 0$ will be denoted by $\Sigma_{H}''$. Some properties of the harmonic meromorphic starlike functions in $\Sigma_{H}''$ have been investigated by Jahangiri [11]. In Section 4, we find a sufficient condition for $f \in \Sigma_{H}$ to be extended to a quasiconformal mapping of $\mathbb{C}$. Theorem 4.1 is given as a generalization of the case $f \in \Sigma_{H}''$, proved in [18]. Let $\hat{\Sigma}_k (0 < k < 1)$ be the class of sense-preserving homeomorphisms $h$ of the extended plane $\hat{\mathbb{C}}$ onto itself, with $h(z) = z + \sum_{n=0}^{\infty} a_n z^{-n}$ analytic univalent in $\Delta$ and $k$-quasiconformal in $\hat{\mathbb{C}}$. Krzyź [12] investigated the convolution problem of functions in $\hat{\Sigma}_k$ by the area theorem. A similar convolution problem of the harmonic functions in $\Sigma_{H}(k)$ will be considered in Section 4. The class $\Sigma_{H}(k)$ defined in this paper will be restricted by a strong coefficient condition (4.1), since the area theorem for harmonic functions (see [16]) cannot be used in the proof of Theorem 4.3.

2. A SUFFICIENT CONDITION FOR QUASICONFORMAL EXTENSIONS

For a function $f$ in $\mathcal{H}(\{\varphi_n\}, \{\psi_n\})$, we can make a refinement of Theorem C including the case that two sequences $\{\varphi_n\}, \{\psi_n\}$ are different. We obtain the following theorem.

**Theorem 2.1.** For given two real numbers $k_1$, $k_2$ with $0 < k_1 < 1$, $0 < k_2 < 1$, let $\{\varphi_n\}$ and $\{\psi_n\}$ be two sequences of positive real numbers, which satisfy

\[
\frac{\varphi_n}{n} \geq \frac{1}{k_1} \quad (n \geq 2), \\
\frac{\psi_n}{n} \geq \frac{1}{k_2} \quad (n \geq 1).
\]

Suppose that $f \in \mathcal{H}(\{\varphi_n\}, \{\psi_n\})$. Then $f$ is univalent on $\mathbb{D}$ and has a homeomorphic extension to the boundary. Moreover, the mapping $F$ of the form (1.4) is bi-Lipschitz on the plane $\mathbb{C}$. Furthermore, $F$ is a quasiconformal extension of $f$ to $\mathbb{C}$, with $|\mu_F(z)| \leq k_2$ for $z \in \mathbb{D}$, and $|\mu_F(z)| \leq k_1$ for $z \in \mathbb{C}\setminus\mathbb{D}$. Therefore, $F$ is a $k$-quasiconformal mapping of $\mathbb{C}$, where $k = \max \{k_1, k_2\}$.

**Proof.** Consider a function $f = h + \bar{g} \in \mathcal{H}(\{\varphi_n\}, \{\psi_n\})$ of the form (1.1). If $f$ satisfies condition (2.1), then we have

$$\sum_{n=1}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \leq k_1 \sum_{n=2}^{\infty} \varphi_n|a_n| + k_2 \sum_{n=1}^{\infty} \psi_n|b_n| \leq \max \{k_1, k_2\} = k < 1.$$

For any two points $z_1, z_2 \in \mathbb{D}$ with $z_1 \neq z_2$, we have

\[
0 < (1 - k)|z_1 - z_2| \leq |f(z_1) - f(z_2)| \leq (1 + k)|z_1 - z_2|.
\]
Then \( f \) is univalent on \( \mathbb{D} \) and has a homeomorphic extension to \( \overline{\mathbb{D}} \). The image \( f(\mathbb{T}) \) is a Jordan curve. For \( z_1, z_2 \in \Delta \) with \( z_1 \neq z_2 \), we compute by the form (1.4) of function \( F \) that

\[
0 < (1 - k)|z_1 - z_2| \leq |F(z_1) - F(z_2)| \leq (1 + k)|z_1 - z_2|.
\]

It means that \( F \) is bi-Lipschitz continuous on \( \Delta \), and has a homeomorphic extension to \( \overline{\Delta} \).

Next suppose \( z_1 \in \mathbb{D}, z_2 \in \Delta \). Let \( \zeta = [z_1, z_2] \cap \mathbb{T} \), and \( w_0 = [F(z_1), F(z_2)] \cap F(\mathbb{T}) = F(\zeta_0) \), where \( \zeta_0 \in \mathbb{T} \). Therefore, by (2.2) and (2.3), we obtain

\[
|F(z_1) - F(z_2)| = |f(z_1) - f(\zeta) + F(\zeta) - F(z_2)| \leq |f(z_1) - f(\zeta)| + |F(\zeta) - F(z_2)| \leq (1 + k)|z_1 - z_2|,
\]

and

\[
|F(z_1) - F(z_2)| = |F(z_1) - w_0| + |w_0 - F(z_2)| \geq (1 - k)(|z_1 - \zeta_0| + |z_2 - \zeta_0|) \geq (1 - k)|z_1 - z_2|.
\]

Hence, \( F \) is bi-Lipschitz on the whole plane \( \mathbb{C} \).

Finally, we shall compute the dilatation of the mapping \( F \). For \( z \in \mathbb{D} \), we have

\[
|\mu_F(z)| = \left| \frac{\sum_{n=1}^{\infty} n b_n z^{n-1}}{1 + \sum_{n=2}^{\infty} n a_n z^{n-1}} \right| \leq \frac{\sum_{n=1}^{\infty} n |b_n|}{1 - \sum_{n=2}^{\infty} n |a_n|} \leq \frac{k_2 \sum_{n=1}^{\infty} \psi_n |b_n|}{1 - k_1 \sum_{n=2}^{\infty} \varphi_n |a_n|} \leq k_2.
\]

Using (1.4), we compute that

\[
|\mu_F(z)| = \left| \frac{\sum_{n=1}^{\infty} n a_n z^{n-1}}{1 - \sum_{n=1}^{\infty} n b_n z^{n-1}} \right| \leq \frac{\sum_{n=1}^{\infty} n |a_n|}{1 - \sum_{n=1}^{\infty} n |b_n|} \leq \frac{k_1 \sum_{n=1}^{\infty} \varphi_n |a_n|}{1 - k_2 \sum_{n=1}^{\infty} \psi_n |b_n|} \leq k_1,
\]

for \( z \in \mathbb{C} \setminus \mathbb{D} \). Hence, for any \( z \in \mathbb{C} \), we conclude that \( |\mu_F(z)| \leq \max \{k_1, k_2\} = k \).

Therefore, \( F \) is \( k \)-quasiconformal off \( \mathbb{T} \) on \( \mathbb{C} \), since the unit circle \( \mathbb{T} \) is removable for quasiconformality. The proof is complete.

\[\square\]

**Corollary 2.2.** Consider that \( \{\varphi_n\}_{n=2,3,...} \) and \( \{\psi_n\}_{n=1,2,...} \) are two sequences of positive real numbers, satisfying \( \varphi_n \geq n, \psi_n \geq n, n = 2, 3, \cdots, \) and \( \psi_1 \geq 1 \). If \( f \in \mathcal{H}(\{\varphi_n\}, \{\psi_n\}) \), and

\[
\psi_1 |b_1| + \sum_{n=2}^{\infty} (\varphi_n |a_n| + \psi_n |b_n|) \leq k_0 < 1,
\]

then the mapping (1.4) is a quasiconformal extension of \( f \) to \( \mathbb{C} \), and \( |\mu_F(z)| \leq k_0 < 1 \) for \( z \in \mathbb{C} \).

**Proof.** We can replace the inequality (2.6) by

\[
\frac{\psi_1 |b_1|}{k_0} + \sum_{n=2}^{\infty} \left( \frac{\varphi_n |a_n|}{k_0} + \frac{\psi_n |b_n|}{k_0} \right) \leq 1.
\]

The coefficient sequences then satisfy the condition (2.1) considered in Theorem 2.1.

\[\square\]

It is shown in Theorem B that a harmonic function \( f \) in \( \mathcal{H}(\{n^2\}, \{n^2\}) \) is fully convex. Since the coefficient sequences of \( f \) satisfy the condition (2.1), \( f \) can be extended to the whole plane quasiconformally.
Corollary 2.3. Suppose \( f \in \mathcal{H}(\{n^2\}, \{n^2\}) \). Then \( f \) is univalent and convex in \( \mathbb{D} \), and has a quasiconformal extension \( F \) of the form (1.4) to \( \mathbb{C} \), with \( |\mu_F(z)| \leq 1/2 \) for \( z \in \mathbb{C} \).

3. QUASICONFORMAL EXTENSION OF HARMONIC STRONGLY STARLIKE FUNCTIONS

For \( 0 < \alpha < 1 \), we introduce the following quantities:

\[
\varphi_n(\alpha) = \frac{n - 1 + \sqrt{n^2 - 2n \cos \pi \alpha + 1}}{2 \sin (\pi \alpha / 2)},
\]
\[
\psi_n(\alpha) = \frac{n + 1 + \sqrt{n^2 + 2n \cos \pi \alpha + 1}}{2 \sin (\pi \alpha / 2)},
\]
and a result of [15] shows

\[
(3.1) \quad n < \varphi_n(\alpha) < \frac{n}{\sin (\pi \alpha / 2)} < \psi_n(\alpha)
\]
for \( n \geq 2 \).

Lemma 3.1 ([15]). Let \( f = h + \bar{g} \in \mathcal{H} \) for \( h(z) = z + a_2 z^2 + a_3 z^3 + \cdots \) and \( g(z) = b_1 z + b_2 z^2 + b_3 z^3 + \cdots \). Suppose that the inequality

\[
(3.2) \quad \sum_{n=2}^{\infty} \varphi_n(\alpha) |a_n| + \sum_{n=1}^{\infty} \psi_n(\alpha) |b_n| \leq 1
\]
holds. Then \( f \in \mathcal{SS}_H(\alpha) \).

Lemma 3.2. For \( 0 < \alpha < 1 \), we have

(i) \( \varphi_n(\alpha)/n \) is a strictly increasing function of \( n \) with \( n \geq 2 \);
(ii) \( \psi_n(\alpha)/n \) is a strictly decreasing function of \( n \) with \( n \geq 1 \).

Proof. We only give the proof of (i), and (ii) can be proved in the same way. Let \( a = 2 \sin (\pi \alpha / 2) \). To prove \( \varphi_n(\alpha)/n \) is a strictly increasing function of \( n \geq 2 \), we have to verify that

\[
\frac{\varphi_{n+1}(\alpha)}{n+1} - \frac{\varphi_n(\alpha)}{n} = \frac{1 + n \sqrt{n^2 + (n + 1)a^2} - (n + 1) \sqrt{(n - 1)^2 + na^2}}{n(n+1)a} = \frac{L(n)}{n(n+1)a}
\]
is positive. We assume first that \( L(n) \leq 0 \). A simple computation shows

\[
1 + n \sqrt{n^2 + (n + 1)a^2} \leq (n + 1) \sqrt{(n - 1)^2 + na^2}
\]

\[
\Leftrightarrow 2 \sqrt{n^2 + (n + 1)a^2} \leq (n + 1)a^2 - 2n
\]

\[
\Leftrightarrow 4 \leq a^2,
\]
which is in contradiction with the fact that \( a < 2 \). Hence, \( L(n) > 0 \), that is, \( \varphi_n(\alpha)/n \) is a strictly increasing function of \( n \).

Using Theorem 2.1, Lemmas 3.1 and 3.2, we obtain the following theorem.
Theorem 3.3. For a given $0 < \alpha < 1$, if a function $f$ belongs to $\mathcal{H}(\{\varphi_n(\alpha)\}, \{\psi_n(\alpha)\})$, then $f$ is in $\mathcal{SS}_H(\alpha)$ and has a quasiconformal extension $F$ to the whole complex plane, of the form \[ \frac{\sin (\pi \alpha/2)}{1 + \cos (\pi \alpha/2)} \] with the dilatation $|\mu_F(z)| \leq \sin (\pi \alpha/2)$ for $z \in \mathbb{D}$, and $|\mu_F(z)| \leq \sin (\pi \alpha/2)$ for $z \in \mathbb{C} \setminus \mathbb{D}$. Finally, $f$ is a $\sin (\pi \alpha/2)$-quasiconformal mapping of the whole plane.

Proof. From Lemma 3.2, $\varphi_n(\alpha)/n$ is a strictly increasing function of $n \geq 2$, and $\psi_n(\alpha)/n$ is a strictly decreasing function of $n \geq 1$. The inequality (3.1) shows that the two functions have no intersection, and both approach to $1/\sin (\pi \alpha/2)$ if $n$ tends to infinity. Following Lemma 3.1, if a harmonic function $f \in \mathcal{H}(\{\varphi_n(\alpha)\}, \{\psi_n(\alpha)\})$, $0 < \alpha < 1$, then $f \in \mathcal{SS}_H(\alpha)$. Now we apply Theorem 2.1 to the function $f$. Then, according the discussion on (2.4) and (2.5), we get

\[ |\mu_F(z)| \leq k = \max \left\{ \frac{1}{\psi_1(\alpha)}, \sin (\pi \alpha/2) \right\} \]

\[ = \max \left\{ \frac{\sin (\pi \alpha/2)}{1 + \cos (\pi \alpha/2)}, \sin (\pi \alpha/2) \right\} \]

\[ = \sin (\pi \alpha/2) < 1, \]

for $z \in \mathbb{C}$. Then the extension function $F$ is a $\sin (\pi \alpha/2)$-quasiconformal mapping of the whole plane. \hfill \Box

A simple example of harmonic strongly starlike function of order $\alpha$ constructed in [15], shows the sharp bound for the second coefficient of the co-analytic part. Here, by Theorem 3.3, an explicit form of the extension functions for more general functions will be given.

Example 3.4. For a given $\alpha$ with $0 < \alpha < 1$, we consider the mapping

\[ f_n(z) = z + b_n z^n, \quad n \geq 2, \]

where

\[ |b_n| \leq \frac{\psi_n(\alpha)}{1 + \psi_n(\alpha)} = \frac{2 \sin (\pi \alpha/2)}{(n + 1) + |n + e^{i\pi \alpha}|}. \]

The coefficients of $f_n$ satisfy the condition (3.2), so $f_n \in \mathcal{SS}_H(\alpha)$. Using Theorem 3.3, $f_n$ can be extended to $\mathbb{C}$, and the mapping of the form

\[ F_n(z) = \begin{cases} 
  z + b_n z^n, & |z| \leq 1, \\
  z + b_n z^{-n}, & |z| \geq 1,
\end{cases} \]

is a quasiconformal extension of $f_n$, with the dilatation

\[ |\mu_{F_n}(z)| \leq \frac{2n \sin (\pi \alpha/2)}{n + 1 + |n + e^{i\pi \alpha}|} \]

for $z \in \mathbb{C}$. 
4. Quasiconformal extension of harmonic mappings in the exterior disk

For a function \( f \in \Sigma_\mathbb{H} \) of the form (1.5), a simple computation shows that a sufficient condition for \( |\mu_f| \leq k < 1 \) is

\[ |\beta| + \frac{k + 1}{2} |A| + k \sum_{n=1}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \leq k|\alpha|. \]

Replacing \( k \) on the left-hand side of the inequality by 1, we obtain

\[(4.1) \quad |\beta| + |A| + \sum_{n=1}^{\infty} n(\sum_{n=1}^{\infty} n|a_n| + |b_n|) \leq k|\alpha|. \]

Let \( \Sigma_\mathbb{H}(k) \) (0 < \( k < 1 \)) be the class of functions \( f \) which belong to \( \Sigma_\mathbb{H} \) and satisfy the condition (4.1). A theorem can be stated as follows.

**Theorem 4.1.** Let \( f \in \Sigma_\mathbb{H}(k) \) be of the form (1.5) for some \( k \in (0, 1) \). Then \( f \) has a homeomorphic extension to the unit circle. Moreover, the mapping

\[ F(z) = \begin{cases} f(z), & |z| \geq 1, \\ \alpha z + |z|^\frac{\beta}{2} + \sum_{n=1}^{\infty} a_n z_n + \sum_{n=1}^{\infty} b_n z_n, & |z| \leq 1, \end{cases} \]

is a quasiconformal extension of \( f \) with the dilatation \( |\mu_F(z)| \leq k \) for \( z \in \mathbb{C} \).

**Proof.** Suppose that \( f \) belongs to the class \( \Sigma_\mathbb{H}(k) \), and \( f \) takes the form (1.5). For any different points \( z_1, z_2 \) in \( \Delta \), it is harmless to assume that \( |z_1| \geq |z_2| > 1 \). Then we compute that

\[ |f(z_1) - f(z_2)| \]

\[ = |\alpha(z_1 - z_2) + \beta(z_1^{1-n} - z_2^{1-n}) + \sum_{n=1}^{\infty} a_n (z_1^{2-n} - z_2^{2-n}) + \sum_{n=1}^{\infty} b_n (z_1^{2-n} - z_2^{2-n}) + A(\log |z_1| - \log |z_2|)| \]

\[ \geq |z_1 - z_2|(|\alpha| + |\beta| - \sum_{n=1}^{\infty} n(\sum_{n=1}^{\infty} n|a_n| + |b_n|)|z_1^{1-n} - z_2^{1-n}| - |A||z_1 - z_2|) \]

\[ = |z_1 - z_2| \left( |\alpha| + |\beta| - \sum_{n=1}^{\infty} n(\sum_{n=1}^{\infty} n|a_n| + |b_n|) \frac{|z_1^{n+1} + z_1^{n+2} z_2^1 + \cdots + z_2^{n+1}|}{|z_1 z_2|} \right) - |A| \int_{|z_2|}^{\frac{|z_1|}{|z_2|}} \frac{dt}{t} \]

\[ \geq |z_1 - z_2| \left( |\alpha| - |\beta| - \sum_{n=1}^{\infty} n(\sum_{n=1}^{\infty} n|a_n| + |b_n|) \right) - |A| \int_{|z_2|}^{\frac{|z_1|}{|z_2|}} dt \]

\[ \geq |z_1 - z_2| \left( |\alpha| - |\beta| - \sum_{n=1}^{\infty} n(\sum_{n=1}^{\infty} n|a_n| + |b_n|) - |A| \right) \]

\[ \geq |z_1 - z_2|(1 - k)|\alpha|. \]

Similarly, we obtain

\[ |f(z_1) - f(z_2)| \leq |z_1 - z_2|(1 + k)|\alpha|. \]
Therefore, we conclude that $f$ satisfies bi-Lipschitz condition

$$0 < (1 - k)|\alpha| \cdot |z_1 - z_2| \leq |f(z_1) - f(z_2)| \leq (1 + k)|\alpha| \cdot |z_1 - z_2|, \quad z \in \Delta.$$ 

It means that $f$ has a homeomorphic extension to the unit circle $T$, and $f(T)$ is a quasicircle. Next we will show the extension function $F$ of the form (4.2) is a $k$-quasiconformal mapping of the whole plane.

For $z \in \mathbb{D}$, we compute that

$$|\mu_f(z)| = \left| \frac{\beta + \sum_{n=1}^{\infty} n a_n z^{-n-1}}{\alpha + \sum_{n=1}^{\infty} n b_n z^{-n-1}} \right| \leq \frac{|\beta| + \sum_{n=1}^{\infty} n |a_n|}{|\alpha| - \sum_{n=1}^{\infty} n |b_n|} \leq k,$$

Using (4.2), we attain

$$|\mu_F(z)| = \left| \frac{\beta + \frac{4}{z^2} \sum_{n=1}^{\infty} n b_n z^{-n-1}}{\alpha + \frac{4}{z^2} \sum_{n=1}^{\infty} n a_n z^{-n-1}} \right| \leq \frac{|\beta| + \frac{4}{z^2} \sum_{n=1}^{\infty} n |b_n|}{|\alpha| - \frac{4}{z^2} \sum_{n=1}^{\infty} n |a_n|} \leq \frac{2k |\alpha| - |A|}{2 |\alpha| - |A|} \leq k,$$

for $z \in \Delta$. Hence, we conclude that $|\mu_F(z)| \leq k$ for $z \in \mathbb{C}$. The proof is complete.

**Example 4.2.** Consider a function

$$f(z) = z - \frac{i}{6} \pi + \frac{i}{4} \log |z| - \frac{i}{8} z^{-4},$$

which is a member of the class $\Sigma_H$. Since $f$ satisfies the condition (4.1), we can apply Theorem 1.1 to $f$. Then $f$ has a homeomorphic extension to the unit circle, and the mapping

$$F(z) = \begin{cases} 
  z - \frac{i}{6} \pi + \frac{i}{4} \log |z| - \frac{i}{8} z^{-4}, & |z| \geq 1, \\
  z - \frac{i}{6} \pi - \frac{i}{8} z^{-4}, & |z| \leq 1,
\end{cases}$$

is a $k$-quasiconformal extension of $f$ with $k = 7/9$. We can plot the graph of the extension function $F(z)$ by Mathematica, see Figure 1.

Krzyž [12] proved, for $f_1 \in \Sigma_{k_1}$ and $f_2 \in \Sigma_{k_2}$, the convolution function $f_1 * f_2 \in \Sigma_{k_1 k_2}$. Although we cannot expect the same result, we can establish the following theorem.

**Theorem 4.3.** If $f_1 \in \Sigma_{H}(k_1)$ and $f_2 \in \Sigma_{H}(k_2)$, then the harmonic convolution $f_1 * f_2 \in \Sigma_{H}(\sqrt{k_1 k_2})$.

**Proof.** Consider $f_1$ and $f_2$ as given by the formula (1.5). We have

$$f_1(z) = \alpha_1 z + \beta_1 \pi + \sum_{n=0}^{\infty} a_n z^{-n} + \sum_{n=1}^{\infty} b_n z^{-n} + c \log |z|,$$

and

$$f_2(z) = \alpha_2 z + \beta_2 \pi + \sum_{n=0}^{\infty} A_n z^{-n} + \sum_{n=1}^{\infty} B_n z^{-n} + C \log |z|,$$
Figure 1. The graph of $F(z)$

for $z \in \Delta$. Then the harmonic convolution of $f_1$ and $f_2$ is of the form

$$f_1 \ast f_2(z) = \alpha_1 \alpha_2 z + \beta_1 \beta_2 \overline{z} + \sum_{n=0}^{\infty} a_n A_n z^{-n} + \sum_{n=1}^{\infty} b_n B_n z^{-n} + cC \log|z|, \quad z \in \Delta.$$  \hspace{1cm} (4.3)

Since $f_1 \in \Sigma_H(k_1)$ and $f_2 \in \Sigma_H(k_2)$, it is obvious that $f_1 \ast f_2 \in \Sigma_H$. Now by the condition \((4.1)\), it suffices to show

$$M \leq \sqrt{k_1 k_2}.$$  

The quantity $M$ can be written as

$$M = \sum_{m=0}^{\infty} x_m X_m,$$

where

$$|\alpha_1| x_m = \begin{cases} \sqrt{n|a_n|^2}, & m = 2n - 1, \\ \sqrt{n|b_n|^2}, & m = 2n, \\ |\beta_1| + |c|, & m = 0, \end{cases} \quad \text{and} \quad |\alpha_2| X_m = \begin{cases} \sqrt{n|A_n|^2}, & m = 2n - 1, \\ \sqrt{n|B_n|^2}, & m = 2n, \\ |\beta_2| + |C|, & m = 0. \end{cases}$$
Hence by Cauchy-Schwarz inequality, we obtain

\[
M \leq \left( \sum_{m=0}^{\infty} x_m^2 \right)^{\frac{1}{2}} \left( \sum_{m=0}^{\infty} X_m^2 \right)^{\frac{1}{2}} = \frac{1}{|\alpha_1|^2} + \sum_{n=1}^{\infty} n(|a_n|^2 + |b_n|^2) \right)^{\frac{1}{2}} \left( \frac{1}{|\alpha_2|^2} + \sum_{n=1}^{\infty} n(|A_n|^2 + |B_n|^2) \right)^{\frac{1}{2}} \leq \sqrt{k_1 k_2}.
\]

Thus the convolution function \( f_1 \ast f_2 \) of the form (4.3) satisfies the condition (4.1), and then \( f_1 \ast f_2 \in \Sigma_\mathcal{H}(\sqrt{k_1 k_2}) \). The proof is complete.

\[\square\]

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References

1. Y. Avci and E. Zlotkiewicz, On harmonic univalent mappings, Ann. Univ. Mariae Curie-Skłodowska Sect. A 44 (1991), 1–7.
2. M. Chuaqui, P. Duren and B. Osgood, Curvature properties of planar harmonic mappings, Comput. Methods Funct. Theory 4 (2004), no. 1, 127–142.
3. J. Clunie and T. Sheil-Small, Harmonic univalent functions, Ann. Acad. Sci. Fenn. Ser. A I Math. 9 (1984), 3–25.
4. P. Duren, Harmonic Mappings in the Plane, Cambridge university press 156, 2004.
5. M. Fait, J. G. Krzyż and J. Zygmunt, Explicit quasiconformal extensions for some classes of univalent functions, Comment. Math. Helv. 51 (1976), no. 1, 279–285.
6. A. Ganczar, Explicit quasiconformal extensions of planar harmonic mappings, J. Comput. Anal. Appl. 10 (2008), no. 2, 179–186.
7. H. Hamada, T. Honda and K. H. Shon, Quasiconformal extensions of starlike harmonic mappings in the unit disc, Bull. Korean Math. Soc. 50 (2013), no. 4, 1377–1387.
8. W. Hengartner and G. Schober, Univalent harmonic functions, Trans. Amer. Math. Soc. 299 (1987), no. 1, 1–31.
9. J. M. Jahangiri, Coefficient bounds and univalence criteria for harmonic functions with negative coefficients, Ann. Univ. Mariae Curie-Skłodowska Sect. A 52 (1998), 57–66.
10. J. M. Jahangiri, Harmonic functions starlike in the unit disc, J. Math. Anal. Appl. 235 (1999), no. 2, 470–477.
11. J. M. Jahangiri, Harmonic meromorphic starlike functions, Bull. Korean Math. Soc. 37 (2000), no. 2, 291–301.
12. J. G. Krzyż, Convolution and quasiconformal extension, Comment. Math. Helv. 51 (1976), no. 1, 99–104.
13. O. Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane*, 126, Springer-Verlag, New York, 1973.

14. H. Lewy, *On the non-vanishing of the Jacobian in certain one-to-one mappings*, Bull. Amer. Math. Soc. 42 (1936), no. 10, 689–692.

15. X.-S. Ma, S. Ponnusamy and T. Sugawa, *Harmonic spirallike functions and harmonic strongly starlike functions*, preprint, available from https://arxiv.org/abs/2108.11622.

16. M. Mateljević, *Dirichlets principle, distortion and related problems for harmonic mappings*, Publ. Inst. Math. (Beograd) (N.S.) 75 (2004), no. 89, 147–171.

17. H. Silverman, *Harmonic univalent functions with negative coefficients*, J. Math. Anal. Appl. 220 (1998), no. 1, 283–289.

18. J. Widomski and M. Gregoreczyk, *Harmonic mappings in the exterior of the unit disk*, Ann. Univ. Mariae Curie-Sklodowska Sect. A 64 (2016), no. 1, 63–73.

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