APPROXIMATE DYNAMICS OF A CLASS OF STOCHASTIC WAVE EQUATIONS WITH WHITE NOISE

GUANGGAN CHEN* AND QIN LI
School of Mathematical Science, and V.C. & V.R. Key Lab
Sichuan Normal University
Chengdu, 610068, China

YUNYUN WEI
College of Management Science
Chengdu University of Technology
Chengdu, 610059, China

(Communicated by Björn Schmalfuss)

Abstract. This work is concerned with a stochastic wave equation driven by a white noise. Borrowing from the invariant random cone and employing the backward solvability argument, this wave system is approximated by a finite dimensional wave equation with a white noise. Especially, the finite dimension is explicit, accurate and determined by the coefficient of this wave system; and further originating from an Ornstein-Uhlenbek process and applying Banach space norm estimation, this wave system is approximated by a finite dimensional wave equation with a smooth colored noise.

1. Introduction. Stochastic wave equations play a very important role, when random fluctuations are taken into account, to describe the propagation of waves in certain systems or media, such as the atmosphere, oceans, sonic booms, traffic flows, optic devices, and quantum fields (see [6, 7, 11, 15, 27, 29, 34] and the references therein). There are much more researches on them recently [4, 5, 21, 25].

In this paper, we investigate a stochastic wave equation driven by a white noise (also called SWE for short)

\[ u_{tt} + au_t = \nu \Delta u + bu + f(u) \dot{W}(t), \quad t \geq 0, \quad x \in D, \]
\[ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in D, \]
\[ u(t, 0) = u(t, \pi) = 0, \quad t \geq 0, \]

where \( u \) is a real wave function from \( \mathbb{R}^+ \times D \) to \( \mathbb{R} \), and the domain \( D \) is \([0, \pi] \). The coefficients \( a, \nu \) and \( b \) are positive real constants satisfying \( \frac{\nu}{2} > \left[ \sqrt{\frac{b}{2}} \right] + 1 \), where \( \left[ \sqrt{\frac{b}{2}} \right] \) denotes the maximum integer less than \( \sqrt{\frac{b}{2}} \). The nonlinear term \( f \) is globally Lipschitz continuous on \( L^2(D) \) with the Lipschitz constant \( L_f \). The white

\[ 2020 \text{ Mathematics Subject Classification. Primary: 37L55, 35L05, 37D10; Secondary: 58J65.} \]

\[ \text{Key words and phrases. Stochastic wave equations, finite dimensional approximation, invariant random cone, colored noise.} \]

The first author is supported by the National Science Foundation of China (Grants No. 11571245).

* Corresponding author: Guanggan Chen.
noise $\dot{W}(t)$ is the general derivative of the Wiener process $W(t)$ specified in the
next section: when $l(u) = \sigma$ (a real parameter), it is additive; when $l(u) = u$, it is
multiplicative.

For stochastic wave equations, the infinite dimension makes the system too com-
plex to geometrically visualize, too difficult to theoretically analyze; meanwhile, the
driven noise makes the system too uncertain to accurately predict, too expensive
to numerically simulate.

To overcome the trouble from the infinite dimension, Fan and Wang [10] derived
the existence of random attractor of the stochastic wave equation, which has a finite
fractal Hausdorff dimension. Lu and Schmalfuß [21] established the existence of
random invariant manifolds of the stochastic wave equation, which provides a finite
dimensionally geometrical characterization. Lv, Wang and Roberts [23] further
showed the approximation of the random inertial manifold of the stochastic wave
equation.

The first purpose of this paper, therefore, is to derive approximate dynamics of
Eq.(1) with a finite dimension. Borrowing from the invariant random cone theory
and employing the backward solvability argument, we finally reduce Eq.(1) to a
finite dimensional stochastic wave equation with a white noise on a random invariant
manifold. Furthermore, the finite dimension is $[\sqrt{\frac{b}{\nu}}]$ with the condition $\frac{a}{2} > [\sqrt{\frac{b}{\nu}}]+1$.

To deal with the trouble from the white noise, Hale and Raugel [14], Mora [24],
Cerrai and Freidlin [3] and Lv and Wang [22] investigated the approximation of
the stochastic wave equation as the noise intensity goes to zero. Chen, Duan and
Zhang [4] derived the effective behaviour for a class of the stochastic wave equation
for the small enough noise. On the other hand, since the Wiener process $W(t)$ is
continuous everywhere but no differentiable everywhere, Acquistapace and Terreni
[1] proposed a smooth $\Phi^\varepsilon(t)$ to approximate $W(t)$. Wong and Zakai [35, 36] applied
a piecewise linear difference scheme to approximate $W(t)$. More approximations
works of noises please see [12, 13, 16, 17, 18, 19, 20, 26, 28, 30, 31, 32, 37] and the
references therein. However, there is few on this issue of stochastic wave equations.

Hence, the second purpose of this paper is to explore approximate dynamics of
Eq.(1) driven by a smooth colored noise. More precisely, we will investigate the
approximate system

$$X^\varepsilon_{tt} + aX^\varepsilon_t = \nu \Delta X^\varepsilon + bX^\varepsilon + f(X^\varepsilon) + l(X^\varepsilon)\dot{\Phi}^\varepsilon(t), \hspace{1cm} (4)$$

$$X^\varepsilon(0, x) = X^\varepsilon_0(x), \hspace{0.5cm} X^\varepsilon(t, 0) = X^\varepsilon(t, \pi) = 0, \hspace{0.5cm} t \geq 0, \hspace{1cm} (5)$$

where $\varepsilon$ is a small enough parameter ($0 < \varepsilon \ll 1$), and the definition of color noise
$\dot{\Phi}^\varepsilon(t)$ will be given in detail in the next section. It is deserved to note that the
approximation $\dot{\Phi}^\varepsilon(t)$ of $W(t)$ is smooth, which contributes the higher regularity of
Eq.(4) than that of Eq.(1) driven by the white noise. We finally show that Eq.(1)
is reduced to a finite dimensional wave equation with a smooth colored noise.

This paper is organized as follows. In the next section, we state some preliminar-
ies including random dynamical system, invariant random cone and the Langevin
equation. In the third section, we analyze the spectrum of the wave operator and
the characteristics of the linear wave equation. In the fourth section, we address
the stationary solution of stochastic wave equations with additive noise and multi-
plicative noise, and establish the existence of random invariant manifolds. In the
last two sections, we express our main results: to verify the original system (1)-(3)
2. Preliminaries.

2.1. Random dynamical systems. We first state some basic concepts of random
dynamic systems (also see [2, 9]).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. A flow $\theta$ of mappings $\{\theta_t\}_{t \in \mathbb{R}}$ is
defined on the sample space $\Omega$ such that

$$\theta : \mathbb{R} \times \Omega \to \Omega, \quad \theta_0 = \text{id}_\Omega, \quad \theta_t \circ \theta_r = \theta_{t+r},$$

for $t, \tau \in \mathbb{R}$. Further, the flow $\{\theta_t\}_{t \in \mathbb{R}}$ is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$-measurable, where $\mathcal{B}(\mathbb{R})$ is the Borel $\sigma$-algebra of $\mathbb{R}$. In addition, the measure $\mathbb{P}$ is ergodic with respect to this flow $\{\theta_t\}_{t \in \mathbb{R}}$. Then $\Theta = (\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is called a metric dynamical system.

For our application, we will consider a special but very important metric dynamical system induced by the Wiener process. Let $\{W(t)\}_{t \in \mathbb{R}}$ be a two-sided Wiener process taking values in a Hilbert space $\mathcal{H}$. Its sample paths are in the space $C_0(\mathbb{R}, \mathcal{H})$ of real continuous functions defined on $\mathbb{R}$ with the compact open topology, taking zero value at zero time. On this set we consider the measurable flow $\theta = \{\theta_t\}_{t \in \mathbb{R}}$, defined by

$$\theta_t u = \omega(\cdot + t) - \omega(t), \quad u \in \Omega, \quad t \in \mathbb{R}.$$ 

The distribution of this process induces a probability measure on $\mathcal{B}(C_0(\mathbb{R}, \mathcal{H}))$, which is ergodic with respect to $\theta_t$ (see Arnold [2]).

A cocycle $\phi$ is a mapping from $\mathbb{R} \times \Omega \times \mathcal{H}$ to $\mathcal{H}$, which is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H}))$-measurable such that

$$\phi(0, \omega) u = u, \quad \phi(t + \tau, \omega) u = \phi(t, \theta_r \omega) \phi(\tau, \omega) u$$

for all $t, \tau \in \mathbb{R}^+, \omega \in \Omega$ and $u \in \mathcal{H}$. Then $\phi$ together with the metric dynamical system $\theta$ forms a random dynamical system.

2.2. Invariant random cone. We recall some definitions about random manifold
and random cone (also see [8, 33]).

A random set $M(\omega)$ is called an invariant set for a random dynamical system $\phi(t, \omega)$ if

$$\phi(t, \omega) M(\omega) \subset M(\theta_t \omega) \quad \text{for } t \geq 0.$$ 

Furthermore, if there exists a Lipschitz mapping $h(\cdot, \omega)$ from $\mathcal{H}^+$ to $\mathcal{H}^-$ with $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ (the definitions and properties of $\mathcal{H}^+$ and $\mathcal{H}^-$ are given in Section 3) such that

$$M(\omega) = \{\xi + h(\xi, \omega) | \xi \in \mathcal{H}^+\},$$

then $M(\omega)$ is called a Lipschitz invariant manifold.

An invariant manifold $M(\omega)$ of a random dynamical system $\phi(t, \omega)$ is called
almost surely asymptotically complete if for every $u \in \mathcal{H}$, there exists $\tilde{u} \in M(\omega)$ such that

$$||\phi(t, \omega) u - \phi(t, \omega) \tilde{u}||_\mathcal{H} \leq D(\omega)||u - \tilde{u}||_\mathcal{H} e^{-kt}, \quad t \geq 0,$$
for almost all $\omega \in \Omega$, where $k$ is a positive constant and $D(\omega)$ is a positive random variable.

For a positive random variable $\delta(\omega)$, the random set

$$C_{\delta(\omega)}(\omega) := \{u \mid (u, \omega) \in H \times \Omega \text{ and } ||\Pi_- u||_H \leq \delta(\omega)||\Pi_+ u||_H\}$$

is called a random cone, where $\Pi_+$ and $\Pi_-$ are the projections specified in Section 3.

Let $C_{\delta(\omega)}(\omega)$ be a random cone. For arbitrary $u$ and $\tilde{u}$ in $H$ with $u - \tilde{u} \in C_{\delta(\omega)}(\omega)$, there exists a random variable $\overline{\delta}(\omega) \in (0, \delta(\omega))$ almost surely, such that

$$\phi(t, \omega)u - \phi(t, \omega)\tilde{u} \in C_{\overline{\delta}(\theta_t\omega)}(\theta_t\omega)$$

for almost all $\omega \in \Omega$.

Then we call the random dynamical system $\phi(t, \omega)$ has invariant random cone property for the random cone $C_{\delta(\omega)}(\omega)$.

2.3. Langevin equation. For the Langevin equation

$$\begin{cases}
dz^\varepsilon = -\frac{1}{\varepsilon}z^\varepsilon dt + \frac{1}{\varepsilon}dW(t), \\
z^\varepsilon(0) = \frac{1}{\varepsilon} \int_{-\infty}^{0} e^{\frac{s}{\varepsilon}}dW(s),
\end{cases}$$

(8)

it has the unique solution $z^\varepsilon(\theta_t\omega)$ (see [8]), which is also called an Ornstein-Uhlenbeck process. Take $\varepsilon = \frac{1}{n}$ ($n = 1, 2, \ldots$) to be discrete. Then that $\varepsilon \to 0$ actually means that $n \to \infty$. Furthermore, it possesses the following properties (also see [8, 9]).

(i) There exists a $\{\theta_t \mid t \in \mathbb{R}\}$ invariant set $\Omega$ of full measure such that the sample paths $\omega(t)$ satisfy

$$\lim_{t \to \pm \infty} \frac{\omega(t)}{t} = 0, \ \omega \in \Omega.$$

(ii) The random variable

$$z^\varepsilon(\omega) = \int_{-\infty}^{0} \frac{1}{\varepsilon} e^{\frac{s}{\varepsilon}}dW(s) = -\int_{-\infty}^{0} \frac{1}{\varepsilon^2} e^{\frac{s}{\varepsilon}} \omega(s) ds, \ \omega \in \Omega,$$

is well-defined and the unique stationary solution of Eq.(8) admits

$$z^\varepsilon(\theta_t\omega) = -\int_{-\infty}^{0} \frac{1}{\varepsilon^2} e^{\frac{s}{\varepsilon}} \omega(t + s) ds = \frac{1}{\varepsilon} \omega(t) - \int_{-\infty}^{0} \frac{1}{\varepsilon^2} e^{\frac{s}{\varepsilon}} \omega(t + s) ds.$$

Moreover, the mapping $t \to z^\varepsilon(\theta_t\omega)$ is continuous.

(iii) 

$$\lim_{t \to \pm \infty} \frac{|z^\varepsilon(\theta_t\omega)|}{|t|} = 0 \text{ and } \lim_{t \to \pm \infty} \frac{1}{t} \int_{0}^{t} z^\varepsilon(\theta_s\omega) ds = 0.$$

Now, we define

$$\Phi^\varepsilon(t) := \int_{0}^{t} z^\varepsilon(\theta_s\omega) ds.$$

Lemma 2.1. [1] Let $W(t)$ be a scalar two-sided Brownian motion. For every fixed $T > 0$, then $\Phi^\varepsilon(t)$ almost surely converges $W(t)$ uniformly in $[0, T]$ as $\varepsilon \to 0$. 
3. **Spectrum of SWE.** Let $L^2(D)$ be the usual Hilbert space on $D = [0, \pi]$ with the inner product $(\cdot, \cdot)$ and the norm $\| \cdot \|_{L^2(D)}$. And let $H^1_0(D)$ be the usual Sobolev space with a compact support, whose norm is denoted as $\| \cdot \|_{H^1}$ for simplicity, we denote $\mathcal{H} := H^1_0(D) \times L^2(D)$.

Eq. (1) is equivalent to
\[
\begin{pmatrix}
u \Delta + bI & I \\ -aI & 0
\end{pmatrix}
\begin{pmatrix}
    u \\ v
\end{pmatrix}
= \begin{pmatrix}
    0 \\ f(u)
\end{pmatrix}
+ \int_0^t \begin{pmatrix}
    0 \\ l(u)W(t)
\end{pmatrix} dt ,
\] (9)
where $(u, v)^T \in \mathcal{H}$ and the superscript "T" denotes the matrix transposition.

Define operator
\[ A := \begin{pmatrix} 
u \Delta + bI & I \\ -aI & 0 \end{pmatrix}, \]
where $I$ is the identity operator. Since that the eigenvalues of the Laplacian operator $\Delta$ on $[0, \pi]$ are $\{ -k^2 \}_{k \in \mathbb{N}}$ with the corresponding eigenvectors $\{ \sin kx \}_{k \in \mathbb{N}}$, the eigenvalues of the operator $A$ are $\{ \lambda^+_k, \lambda^-_k \}_{k \in \mathbb{N}}$ with
\[
\lambda^+_k = -\frac{a}{2} + \sqrt{\frac{a^2}{4} + b - \nu k^2}, \quad \lambda^-_k = -\frac{a}{2} - \sqrt{\frac{a^2}{4} + b - \nu k^2},
\]
whose corresponding eigenvectors are $\{ e^+_k, e^-_k \}_{k \in \mathbb{N}}$ with
\[
e^+_k = \begin{pmatrix} \sin kx \\ \lambda^+_k \sin kx \end{pmatrix}, \quad e^-_k = \begin{pmatrix} \sin kx \\ \lambda^-_k \sin kx \end{pmatrix}.
\]

When $0 < k \leq \sqrt{b/\nu}$, it is known that $b - \nu k^2 \geq 0$; when $k > \sqrt{b/\nu}$, it is known that $b - \nu k^2 < 0$. Then the eigenvalue $\lambda^+_k$ is a positive real number, the eigenvalue $\lambda^-_{k_{\sqrt{b/\nu}}}$ is a negative real number, and the eigenvalues $\lambda^+_k \in \mathbb{R}$ and $\lambda^\pm_{k_{\sqrt{b/\nu} + 1}}$ are the complex numbers with negative real part $\text{Re} \lambda^+_k \in \mathbb{R}$ are composed of $\mathbb{R}$.

For simplicity, here and hereafter, we denote $N := \lfloor \sqrt{b/\nu} \rfloor$, and
\[
\mathcal{H}_+ := \text{span}\{ e^-_k \mid 1 \leq k \leq N \},
\mathcal{H}_N^- := \text{span}\{ e^-_k \mid 1 \leq k \leq N \},
\mathcal{H}_N^+ := \text{span}\{ e^+_k \mid k \geq N + 1 \},
\]
and
\[
\mathcal{H}_- := \mathcal{H}_N^- + \mathcal{H}_N^+, \quad \mathcal{H}_N := \mathcal{H}_+ + \mathcal{H}_N^-.
\]

For $U = (u_1, u_2)^T$ and $V = (v_1, v_2)^T$ in $\mathcal{H}$, the usual inner product on $\mathcal{H}$ is defined as
\[
\langle U, V \rangle = \langle u_1, v_1 \rangle + \langle \Delta^{1/2} u_1, \Delta^{1/2} v_1 \rangle + \langle u_2, v_2 \rangle.
\]
In the sense of the usual inner product on $\mathcal{H}$ (resp. $\mathcal{H}_+$) is not orthogonal with $\mathcal{H}_N^-$. But $\mathcal{H}_+ \perp \mathcal{H}_N^+$ and $\mathcal{H}_N^- \perp \mathcal{H}_N^+$. Then $\mathcal{H}_N \perp \mathcal{H}_N^+$. Further $\mathcal{H}_N \oplus \mathcal{H}_N^+ = \mathcal{H}$.

To eliminate the trouble that $\mathcal{H}_+$ is not orthogonal with $\mathcal{H}_N^-$, originating from [24], we now begin to define a new inner product on $\mathcal{H}$. 


Firstly for arbitrary \( U_1 = (u_{11}, u_{12})^T \) and \( V_1 = (v_{11}, v_{12})^T \) in \( \mathcal{H}_N \), and \( U_2 = (u_{21}, u_{22})^T \) and \( V_2 = (v_{21}, v_{22})^T \) in \( \mathcal{H}_{N^+} \), the inner products on \( \mathcal{H}_N \) and \( \mathcal{H}_{N^+} \) are defined, respectively, as
\[
\langle U_1, V_1 \rangle_{\mathcal{H}_N} := \frac{a^2}{4} (u_{11}, v_{11}) + \langle \Delta u_{11}, v_{11} \rangle + \langle \frac{a}{2} u_{11} + u_{12}, \frac{a}{2} v_{11} + v_{12} \rangle,
\]
\[
\langle U_2, V_2 \rangle_{\mathcal{H}_{N^+}} := (\Delta u_{21}, v_{21}) + \left( \frac{a^2}{4} - 2(N + 1)^2 \right) \langle u_{21}, v_{21} \rangle + \left( \frac{a}{2} u_{21} + u_{22}, \frac{a}{2} v_{21} + v_{22} \right).
\]
Then for arbitrary \( U \) and \( V \) in \( \mathcal{H}_N \), since \( \mathcal{H}_N \oplus \mathcal{H}_{N^+} = \mathcal{H} \), there exist \( U_1 \) and \( V_1 \) in \( \mathcal{H}_N \), and \( U_2 \) and \( V_2 \) in \( \mathcal{H}_{N^+} \) such that \( U = U_1 + U_2 \) and \( V = V_1 + V_2 \). Then the new inner product on \( \mathcal{H} \) is defined as
\[
\langle U, V \rangle_{\mathcal{H}} = \langle U_1, V_1 \rangle_{\mathcal{H}_N} + \langle U_2, V_2 \rangle_{\mathcal{H}_{N^+}},
\]
which, with the assumption \( \frac{a}{2} > \left[ \sqrt{\frac{1}{2}} \right] + 1 \), induces a new orthogonality as follows
\[
\mathcal{H}_+ \perp \mathcal{H}_N^-, \quad \mathcal{H}_- \perp \mathcal{H}_{N^+}^-, \quad \mathcal{H}_{N^-} \perp \mathcal{H}_{N^+}.
\]
Then \( \mathcal{H}_N \perp \mathcal{H}_N^+, \) and \( \mathcal{H}_- \perp \mathcal{H}_N^+ \). Therefore, \( \mathcal{H}_+ \oplus \mathcal{H}_- = \mathcal{H} \).

Since \( \mathcal{H}_+ \oplus \mathcal{H}_- \) equals \( \mathcal{H} \), and \( \mathcal{H}_+ \) is the finite dimensional state space with the dimension \( N \), we denote \( \Pi_+ \) as the projection operator from \( \mathcal{H} \) to \( \mathcal{H}_+ \), and denote \( \Pi_- := I - \Pi_+ \) as the operator from \( \mathcal{H} \) to \( \mathcal{H}_- \), where \( I \) is the identity operator. For any \((u, v)^T \in \mathcal{H}_N \), we sometimes use the notations \( u_+, v_+, u_- \) and \( v_- \) to be as \( \Pi_+ u, \Pi_+ v, \Pi_- u, \) and \( \Pi_- v \), respectively. We also use the operators \( A_+, A_-, A_{N^-}, A_{N^+} \) to denote the restricting operator \( A \) onto the spaces \( \mathcal{H}_+, \mathcal{H}_-, \mathcal{H}_{N^-}, \mathcal{H}_{N^+} \), respectively.

Furthermore,
\[
\| e^{A_+ t} \|_{\mathcal{H}} \leq e^{\lambda_+ t}, \quad \text{for } t \leq 0,
\]
\[
\| e^{A_- t} \|_{\mathcal{H}} \leq e^{\lambda_- t}, \quad \text{for } t \geq 0,
\]
\[
\| e^{A_{N^-} t} \|_{\mathcal{H}} \leq e^{\alpha t}, \quad \text{for } t \geq 0,
\]
\[
\| e^{A_{N^+} t} \|_{\mathcal{H}} \leq e^{\beta t}, \quad \text{for } t \geq 0,
\]
which imply that there exist positive constants \( \alpha \) and \( \beta \) with \( 0 \leq \alpha < \beta \) such that
\[
\| e^{A_+ t} x \|_{\mathcal{H}} \leq e^{\alpha t} \| x \|_{\mathcal{H}}, \quad \text{for } t \leq 0,
\]
\[
\| e^{A_- t} x \|_{\mathcal{H}} \leq e^{-\beta t} \| x \|_{\mathcal{H}}, \quad \text{for } t \geq 0.
\]

**Hypothesis (H).** (Lipschitz condition) The nonlinear term \( f : H_0^1 \rightarrow L^2(D) \) is assumed to be Lipschitz continuous, i.e.,
\[
\| f(u_1) - f(u_2) \|_{L^2(D)} \leq L_f \| u_1 - u_2 \|_{H_0^1},
\]
where the Lipschitz constant \( L_f \) is positive.

Next, we state the well-posedness of the stochastic system (1)-(3).

**Lemma 3.1.** [21] Assume that the Hypothesis (H) holds, and the initial value \((u_0, v_1)^T \in \mathcal{F}_0\)-measurable in \( L^2(\Omega, \mathcal{H}) \). Then the system (1)-(3) has a unique global solution \((u(t), v(t))^T \in C([0, +\infty), \mathcal{H}) \). Furthermore, the solution mapping
\[
(t, \omega, (u_0, u_1)^T) \mapsto \phi(t, \omega)(u_0, u_1)^T := (u(t), v(t))^T
\]
generates a continuous random dynamical system. In fact the mapping \( \phi \) is \((\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathcal{F}) \otimes \mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H}))\)-measurable.
In this paper, without confusion, we would denote \( u, v, X^\varepsilon, Y^\varepsilon \) as the stochastic solutions, \( \pi, \bar{\pi}, \bar{X}, \bar{Y} \) as the random solutions, \( u^*, v^*, X^{*\varepsilon}, Y^{*\varepsilon} \) as the stationary solutions, \( \bar{u}, \bar{v}, \bar{X}, \bar{Y} \) as the solutions on the invariant manifold. In addition, like \( u_+, u_-, v_+ \) and \( v_- \) defined as before, we use the subscript “+” and “-” to denote the solutions restricting onto the spaces \( \mathcal{H}_+ \) and \( \mathcal{H}_- \), respectively, such as \( X^+_\varepsilon, X^-_\varepsilon, Y^+_\varepsilon \) and \( Y^-_\varepsilon \), etc. Here we do not list and explain them one by one.

4. Random invariant manifold. In this section, using the Lyapunov-Perron method (also see [9]), we briefly exhibit that Eq.(1) and Eq.(4) admit the random invariant manifolds.

4.1. Additive noise case. When \( \ell(u) = \sigma \), Eq.(1) and Eq.(4) are equivalent to the following equations, respectively,

\[
\begin{pmatrix}
  u \\
  v 
\end{pmatrix}_t = \begin{pmatrix}
  0 & I \\
  \nu \Delta + bI & -aI 
\end{pmatrix} \begin{pmatrix}
  u \\
  v 
\end{pmatrix} + \begin{pmatrix}
  0 \\
  f(u) 
\end{pmatrix} + \begin{pmatrix}
  0 \\
  \sigma \dot{W}(t) 
\end{pmatrix},
\]

and

\[
\begin{pmatrix}
  X^\varepsilon \\
  Y^\varepsilon 
\end{pmatrix}_t = \begin{pmatrix}
  0 & I \\
  \nu \Delta + bI & -aI 
\end{pmatrix} \begin{pmatrix}
  X^\varepsilon \\
  Y^\varepsilon 
\end{pmatrix} + \begin{pmatrix}
  0 \\
  f(X^\varepsilon) 
\end{pmatrix} + \begin{pmatrix}
  0 \\
  \sigma \dot{\Phi}(t) 
\end{pmatrix}.
\]

Now we consider the stationary solutions of the following two linear equations

\[
\begin{cases}
  \begin{pmatrix}
    u \\
    v 
  \end{pmatrix} = A \begin{pmatrix}
    u \\
    v 
  \end{pmatrix} + \begin{pmatrix}
    0 \\
    \sigma \dot{W}(t) 
  \end{pmatrix}, \\
  \begin{pmatrix}
    u(0) \\
    v(0) 
  \end{pmatrix} = \int_{-\infty}^{0} e^{-A_{-}\tau} \begin{pmatrix}
    0 \\
    \sigma 
  \end{pmatrix} dW(\tau) + \int_{0}^{\infty} e^{-A_{+}\tau} \begin{pmatrix}
    0 \\
    \sigma 
  \end{pmatrix} dW(\tau),
\end{cases}
\]

and

\[
\begin{cases}
  \begin{pmatrix}
    X^\varepsilon \\
    Y^\varepsilon 
  \end{pmatrix} = A \begin{pmatrix}
    X^\varepsilon \\
    Y^\varepsilon 
  \end{pmatrix} + \begin{pmatrix}
    0 \\
    \sigma \dot{\Phi}(t) 
  \end{pmatrix}, \\
  \begin{pmatrix}
    X^\varepsilon(0) \\
    Y^\varepsilon(0) 
  \end{pmatrix} = \int_{-\infty}^{0} e^{-A_{+}\tau} \begin{pmatrix}
    0 \\
    \sigma 
  \end{pmatrix} d\Phi^\varepsilon(\tau) + \int_{0}^{\infty} e^{-A_{+}\tau} \begin{pmatrix}
    0 \\
    \sigma 
  \end{pmatrix} d\Phi^\varepsilon(\tau).
\end{cases}
\]

Lemma 4.1. There exist the stationary solutions \((u^*(\theta_\omega), v^*(\theta_\omega))^\top\) and \((X^{*\varepsilon}(\theta_\omega), Y^{*\varepsilon}(\theta_\omega))^\top\) of Eq.(18) and Eq.(19), respectively. Furthermore,

\[
\begin{pmatrix}
  u^*(\theta_\omega) \\
  v^*(\theta_\omega) 
\end{pmatrix} = \int_{-\infty}^{t} e^{-A_{-}(\tau-t)} \begin{pmatrix}
    0 \\
    \sigma 
  \end{pmatrix} dW(\tau)
\]

\[
+ \int_{0}^{t} e^{-A_{+}(\tau-t)} \begin{pmatrix}
    0 \\
    \sigma 
  \end{pmatrix} dW(\tau),
\]

\[
\begin{pmatrix}
  X^{*\varepsilon}(\theta_\omega) \\
  Y^{*\varepsilon}(\theta_\omega) 
\end{pmatrix} = \int_{-\infty}^{t} e^{-A_{-}(\tau-t)} \begin{pmatrix}
    0 \\
    \sigma 
  \end{pmatrix} d\Phi^\varepsilon(\tau)
\]

\[
+ \int_{\infty}^{t} e^{-A_{+}(\tau-t)} \begin{pmatrix}
    0 \\
    \sigma 
  \end{pmatrix} d\Phi^\varepsilon(\tau).
\]

Proof. Put \( U(t, \omega; (u_0, v_0)^\top) \) to be the solution of Eq.(18) with the initial value \((u_0, v_0)^\top\). Obviously, \( U(0, \omega; (u_0, v_0)^\top) = (u_0, v_0)^\top \). For Eq.(18), combining the
conditions (14) and (15), it implies that
\[ U(t, \omega; (u^*(\omega), v^*(\omega))^\top) \]
\[ = e^{At} \begin{pmatrix} u^*(\omega) \\ v^*(\omega) \end{pmatrix} + \int_0^t e^{-A(\tau-t)} \begin{pmatrix} 0 \\ 0 \end{pmatrix} dW(\tau) \]
\[ = e^{At} \int_{-\infty}^0 e^{-A(\tau-t)} \begin{pmatrix} 0 \\ 0 \end{pmatrix} dW(\tau) + e^{At} \int_0^0 e^{-A+\tau} \begin{pmatrix} 0 \\ 0 \end{pmatrix} dW(\tau) \]
\[ + \int_0^t e^{-A+\tau} \begin{pmatrix} 0 \\ 0 \end{pmatrix} dW(\tau) \]
\[ = \int_{-\infty}^0 e^{-A-\tau} \begin{pmatrix} 0 \\ 0 \end{pmatrix} dW(\tau) + \int_0^0 e^{-A+\tau} \begin{pmatrix} 0 \\ 0 \end{pmatrix} dW(\tau) \]
\[ + \int_t^0 e^{-A+\tau} \begin{pmatrix} 0 \\ 0 \end{pmatrix} dW(\tau) \]
\[ = \int_{-\infty}^0 e^{-A-\tau} \begin{pmatrix} 0 \\ 0 \end{pmatrix} dW(\tau) + \int_0^t e^{-A+\tau} \begin{pmatrix} 0 \\ 0 \end{pmatrix} dW(\tau) \]
\[ = \begin{pmatrix} u^*(\theta_1\omega) \\ v^*(\theta_1\omega) \end{pmatrix}, \]
which further satisfies
\[ U(0, \omega; (u^*(\omega), v^*(\omega))^\top) \]
\[ = \int_{-\infty}^0 e^{-A-\tau} \begin{pmatrix} 0 \\ 0 \end{pmatrix} dW(\tau) + \int_0^0 e^{-A+\tau} \begin{pmatrix} 0 \\ 0 \end{pmatrix} dW(\tau) \]
\[ = (u^*(\theta_1\omega), v^*(\theta_1\omega))^\top, \]
and \( E(u^*(\theta_1\omega), v^*(\theta_1\omega))^\top = E(u^*(\omega), v^*(\omega))^\top. \)

Therefore, the process \((u^*(\theta_1\omega), v^*(\theta_1\omega))^\top\) is the stationary solution of Eq.(18).

Using the same argument, it is shown that \((X^*\varepsilon(\theta_1\omega), Y^*\varepsilon(\theta_1\omega))^\top\) is also the stationary solution of Eq.(19). \(\square\)

Put
\[ \begin{pmatrix} \pi \\ \nu \end{pmatrix} := \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} u^*(\theta_1\omega) \\ v^*(\theta_1\omega) \end{pmatrix}, \]
(22)
and
\[ \begin{pmatrix} X^\varepsilon \\ Y^\varepsilon \end{pmatrix} := \begin{pmatrix} X^\varepsilon \\ Y^\varepsilon \end{pmatrix} - \begin{pmatrix} X^*\varepsilon(\theta_1\omega) \\ Y^*\varepsilon(\theta_1\omega) \end{pmatrix}. \]
(23)

Then Eq.(16) and Eq.(17) can be written, respectively, as
\[ \begin{pmatrix} \pi \\ \nu \end{pmatrix}_t = A \begin{pmatrix} \pi \\ \nu \end{pmatrix} + \begin{pmatrix} 0 \\ g_1(\theta_1\omega, \pi) \end{pmatrix}, \]
(24)
and
\[ \begin{pmatrix} X^\varepsilon \\ Y^\varepsilon \end{pmatrix}_t = A \begin{pmatrix} X^\varepsilon \\ Y^\varepsilon \end{pmatrix} + \begin{pmatrix} 0 \\ g_2(\theta_1\omega, X^\varepsilon) \end{pmatrix}, \]
(25)
where \(g_1(\theta_1\omega, \pi) := f(\pi + u^*(\theta_1\omega))\) and \(g_2(\theta_1\omega, X^\varepsilon) := f(X^\varepsilon + X^*\varepsilon(\theta_1\omega))\) with the same Lipschitz constant \(L_f\) as the function \(f\).

We introduce the random transformation
\[ T_a(\omega, (u, v)^\top) := (u, v)^\top - (u^*(\omega), v^*(\omega))^\top, \]
with \( T_a^{-1}(\omega, (u, v)^\top) = (u, v)^\top + (u^*(\omega), v^*(\omega))^\top. \)
(26)
and
\[
T_\alpha^\varepsilon(\omega, (X^\varepsilon, Y^\varepsilon)^\top) := (X^\varepsilon, Y^\varepsilon)^\top - (X^{*\varepsilon}(\omega), Y^{*\varepsilon}(\omega))^\top,
\]
with 
\[
T_\alpha^\varepsilon^{-1}(\omega, (X^\varepsilon, Y^\varepsilon)^\top) = (X^\varepsilon, Y^\varepsilon)^\top + (X^{*\varepsilon}(\omega), Y^{*\varepsilon}(\omega))^\top.
\] (27)

Then using a similar method in [8], we have the following conclusion.

**Lemma 4.2.** Suppose that \((\bar{\pi}, \overline{\nu})^\top\) and \((\bar{X}^\varepsilon, \overline{Y}^\varepsilon)^\top\) are the solutions of Eq.(24) and Eq.(25), respectively. Then \(T_\alpha^\varepsilon^{-1}(\theta_\varepsilon \omega, (\bar{\pi}, \overline{\nu})^\top)\) and \(T_\alpha^\varepsilon^{-1}(\theta_\varepsilon \omega, (\bar{X}^\varepsilon, \overline{Y}^\varepsilon)^\top)\) are solutions of Eq.(16) and Eq.(17), respectively.

Now, we briefly state the skeleton of deriving the random invariant manifolds for Eq.(24) and Eq.(25) by the Lyapunov-Perron method (also see [9]).

Project Eq.(24) onto \(H_+\) and \(H_-\), respectively, as
\[
\begin{pmatrix}
\bar{\pi}_+

\bar{\nu}_+
\end{pmatrix}_t = A_+ \begin{pmatrix}
\bar{\pi}_+

\bar{\nu}_+
\end{pmatrix} + \begin{pmatrix}
0

g_1+(\theta_\varepsilon \omega, (\bar{\pi}_+ + \bar{\nu}_-))
\end{pmatrix},
\] (28)
and
\[
\begin{pmatrix}
\bar{\pi}_-

\bar{\nu}_-
\end{pmatrix}_t = A_- \begin{pmatrix}
\bar{\pi}_-

\bar{\nu}_-
\end{pmatrix} + \begin{pmatrix}
0

g_1-(\theta_\varepsilon \omega, (\bar{\pi}_+ + \bar{\nu}_-))
\end{pmatrix},
\] (29)
where 
\[
g_1+(\theta_\varepsilon \omega, (\bar{\pi}_+ + \bar{\nu}_-)) := \Pi_+ g_1(\theta_\varepsilon \omega, (\bar{\pi}_+ + \bar{\nu}_-)) \text{ and } g_1-(\theta_\varepsilon \omega, (\bar{\pi}_+ + \bar{\nu}_-)) := \Pi_- g_1(\theta_\varepsilon \omega, (\bar{\pi}_+ + \bar{\nu}_-)).
\]

For arbitrary \(\eta \in (-\beta, -\alpha)\), we define the Banach space
\[
C^-_\eta = \{ \varphi | \varphi \text{ maps } (-\infty, 0] \text{ to } H: \varphi \text{ is continuous and } \sup_{t \in (-\infty, 0]} e^{-\eta t}\|\varphi(t)\|_H < \infty \}
\]
with the norm
\[
\|\varphi(\cdot)\|_{C^-_\eta} = \sup_{t \in (-\infty, 0]} e^{-\eta t}\|\varphi(t)\|_H.
\]

Define the nonlinear operators \(\mathcal{I}\) and \(\mathcal{J}\) on \(C^-_\eta\) as
\[
\mathcal{I}((\bar{\pi}, \overline{\nu})^\top, \xi)(t, \omega) = \varepsilon A_+^* \xi + \int_0^t e^{A_+(t-\tau)} \begin{pmatrix}
0

g_1+(\theta_\varepsilon \omega, (\bar{\pi}_+ + \bar{\nu}_-))
\end{pmatrix} d\tau + \int_{-\infty}^t e^{A_-^*(t-\tau)} \begin{pmatrix}
0

g_1-(\theta_\varepsilon \omega, (\bar{\pi}_+ + \bar{\nu}_-))
\end{pmatrix} d\tau,
\] (30)
and
\[
\mathcal{J}((\bar{X}^\varepsilon, \overline{Y}^\varepsilon)^\top, \xi)(t, \omega) = \varepsilon A_+^t \xi + \int_0^t e^{A_+(t-\tau)} \begin{pmatrix}
0

g_2+(\theta_\varepsilon \omega, (\bar{X}^\varepsilon_+ + \bar{X}^\varepsilon_-))
\end{pmatrix} d\tau + \int_{-\infty}^t e^{A_-^t(t-\tau)} \begin{pmatrix}
0

g_2-(\theta_\varepsilon \omega, (\bar{X}^\varepsilon_+ + \bar{X}^\varepsilon_-))
\end{pmatrix} d\tau,
\] (31)
where \(\xi = \Pi_+(\bar{\pi}_0, \overline{\nu}_0)^\top \in H_+,\ g_2+(\theta_\varepsilon \omega, (\bar{X}^\varepsilon_+ + \bar{X}^\varepsilon_-)) := \Pi_+ g_2(\theta_\varepsilon \omega, (\bar{X}^\varepsilon_+ + \bar{X}^\varepsilon_-)),\ g_2-(\theta_\varepsilon \omega, (\bar{X}^\varepsilon_+ + \bar{X}^\varepsilon_-)) := \Pi_- g_2(\theta_\varepsilon \omega, (\bar{X}^\varepsilon_+ + \bar{X}^\varepsilon_-)).\) Then for any \(\xi \in H_+\) and
Lemma 4.3. (Invariant manifold of random additive system) Suppose the assumptions about the linear operator \( A \) and the nonlinearity \( f \) in subsection 3 hold, and assume further the following spectrum gap condition holds, that is,

\[
\sup_{\tau \leq 0} e^{-\eta \tau} \left\| \int_{0}^{\tau} e^{A_{+}(-\tau)} \left( g_{1+}(\theta_{\tau}, \pi_{1}) - g_{1+}(\theta_{\tau}, \pi_{2}) \right) d\tau \right\|_{H} < 1
\]

Then there exist the Lipschitz continuous random invariant manifolds \( M(\omega) \) and \( M^{\varepsilon}(\omega) \) for the random systems (24) and (25), respectively. Moreover,

\[
M(\omega) = \{ (\xi, h(\xi, \omega)) | \xi \in \mathcal{H}_{+} \} \quad \text{and} \quad M^{\varepsilon}(\omega) = \{ (\xi, h^{\varepsilon}(\xi, \omega)) | \xi \in \mathcal{H}_{+} \},
\]

where the Lipschitz functions \( h(\xi, \omega) \) and \( h^{\varepsilon}(\xi, \omega) \) are defined as (42) and (43).

Using the transformation \( T_{a} \) and \( T_{a}^{\varepsilon} \) defined as (26) and (27), we obtain the invariant manifold of the stochastic additive system.

Lemma 4.4. (Invariant manifold of stochastic additive system) Suppose the assumptions about the linear operator \( A \) and the nonlinearity \( f \) in subsection 3 hold, and assume further the following spectrum gap condition holds, that is,

\[
L_{f} \left( \frac{1}{\eta + \beta} + \frac{1}{\alpha - \eta} \right) < 1.
\]
Then Eq.(16) and Eq.(17) have the Lipschitz invariant manifolds $M^u(\omega) = T^{-1}_a(\omega, M(\omega))$ and $M^{X^\varepsilon}(\omega) = T^{-1}_a(\omega, M^{X^\varepsilon}(\omega))$, respectively. Moreover,

$$M^u(\omega) = \{ (\xi, h^u(\xi, \omega)) | \xi \in \mathcal{H}_+ \} \quad \text{and} \quad M^{X^\varepsilon}(\omega) = \{ (\xi, h^{X^\varepsilon}(\xi, \omega)) | \xi \in \mathcal{H}_+ \}$$

with

$$h^u(\xi, \omega) = \begin{pmatrix} h_1^u(\xi, \omega) \\ h_2^u(\xi, \omega) \end{pmatrix} := h(\xi - \int_0^\infty e^{-A+\tau} \begin{pmatrix} 0 \\ \sigma \end{pmatrix} dW(\tau), \omega)
+ \int_{-\infty}^0 e^{-A-\tau} \begin{pmatrix} 0 \\ \sigma \end{pmatrix} dW(\tau),$$

and

$$h^{X^\varepsilon}(\xi, \omega) = \begin{pmatrix} h_1^{X^\varepsilon}(\xi, \omega) \\ h_2^{X^\varepsilon}(\xi, \omega) \end{pmatrix} := h^{\varepsilon}(\xi - \int_0^\infty e^{-A+\tau} \begin{pmatrix} 0 \\ \sigma \end{pmatrix} d\Phi^{\varepsilon}(\tau), \omega)
+ \int_{-\infty}^0 e^{-A-\tau} \begin{pmatrix} 0 \\ \sigma \end{pmatrix} d\Phi^{\varepsilon}(\tau),$$

where the Lipschitz functions $h(\xi, \omega)$ and $h^{\varepsilon}(\xi, \omega)$ are defined as (42) and (43).

4.2. Multiplicative noise case. When $l(u) = u$, Eq.(1) and Eq.(4) is equivalent to the following equations

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & I \\ \nu \Delta + bI & -aI \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ f(u) \end{pmatrix} + \begin{pmatrix} 0 \\ uW(t) \end{pmatrix},$$

and

$$\begin{pmatrix} X^\varepsilon \\ Y^\varepsilon \end{pmatrix}_t = \begin{pmatrix} 0 & I \\ \nu \Delta + bI & -aI \end{pmatrix} \begin{pmatrix} X^\varepsilon \\ Y^\varepsilon \end{pmatrix} + \begin{pmatrix} 0 \\ f(X^\varepsilon) \end{pmatrix} + \begin{pmatrix} 0 \\ X^\varepsilon \dot{\Phi}^\varepsilon(t) \end{pmatrix},$$

respectively.

Let $\Xi = u$ and $\Psi = u_t - \varepsilon uz^\varepsilon$, where $z^\varepsilon(\theta_t, \omega)$ be the stationary solution of the Langevin equation (8). Then, we can write Eq.(35) as

$$\begin{pmatrix} \Xi \\ \Psi \end{pmatrix}_t = A \begin{pmatrix} \Xi \\ \Psi \end{pmatrix} + \begin{pmatrix} 0 \\ f(\Xi) \end{pmatrix} + \begin{pmatrix} \varepsilon z^\varepsilon(\theta_t, \omega) \\ \varepsilon z^\varepsilon(\theta_t, \omega) - \varepsilon a z^\varepsilon(\theta_t, \omega) - (\varepsilon z^\varepsilon(\theta_t, \omega))^2 - \varepsilon z^\varepsilon(\theta_t, \omega) \end{pmatrix} \begin{pmatrix} \Xi \\ \Psi \end{pmatrix}.$$  

(37)

We introduce the random transformation

$$T_m(\omega, (u, v)^\top) := (u, v - \varepsilon uz^\varepsilon(\omega), \quad (u, v)^\top \in \mathcal{H}.$$  

(38)

The following equation will play an important role

$$\begin{pmatrix} \hat{X}^\varepsilon \\ \hat{Y}^\varepsilon \end{pmatrix}_t = A \begin{pmatrix} \hat{X}^\varepsilon \\ \hat{Y}^\varepsilon \end{pmatrix} + \begin{pmatrix} 0 \\ f(\hat{X}^\varepsilon) \end{pmatrix} + \begin{pmatrix} 0 \\ X^\varepsilon \dot{\Phi}^\varepsilon(t) - \varepsilon ax^\varepsilon \dot{\Phi}^\varepsilon(t) - \varepsilon a\hat{X}^\varepsilon(\Phi^\varepsilon(t))^2 - \varepsilon \hat{Y}^\varepsilon \dot{\Phi}^\varepsilon(t) \end{pmatrix}.$$  

(39)

Let $\hat{X}^\varepsilon = X^\varepsilon$ and $\hat{Y}^\varepsilon = X_t^\varepsilon - \varepsilon \hat{X}^\varepsilon \varepsilon z^\varepsilon$, we obtain

$$\begin{pmatrix} \hat{X}^\varepsilon \\ \hat{Y}^\varepsilon \end{pmatrix}_t = A \begin{pmatrix} \hat{X}^\varepsilon \\ \hat{Y}^\varepsilon \end{pmatrix} + \begin{pmatrix} 0 \\ f(\hat{X}^\varepsilon) \end{pmatrix} + \begin{pmatrix} 0 \\ Z^\varepsilon(\theta_t, \omega) \end{pmatrix} \begin{pmatrix} \hat{X}^\varepsilon \\ \hat{Y}^\varepsilon \end{pmatrix},$$  

(40)
where \( Z^\varepsilon(\theta_1\omega) := \begin{pmatrix} \varepsilon z^\varepsilon(\theta_1\omega) & 0 \\ z^\varepsilon(\theta_1\omega) - \varepsilon a z^\varepsilon(\theta_1\omega) - (\varepsilon z^\varepsilon(\theta_1\omega))^2 & -\varepsilon z^\varepsilon(\theta_1\omega) \end{pmatrix} \).

Project Eq.(37) onto \( \mathcal{H}_+ \) and \( \mathcal{H}_- \), respectively, as

\[
\begin{pmatrix} \overline{\pi}_+ \\ \overline{\nu}_+ \end{pmatrix}_t = A_+ \begin{pmatrix} \overline{\pi}_+ \\ \overline{\nu}_+ \end{pmatrix} + \begin{pmatrix} 0 \\ f_+(\overline{\pi}_+ + \overline{\nu}_-) \end{pmatrix} + Z^\varepsilon(\theta_1\omega) \begin{pmatrix} \overline{\pi}_+ \\ \overline{\nu}_+ \end{pmatrix},
\]

and

\[
\begin{pmatrix} \overline{\pi}_- \\ \overline{\nu}_- \end{pmatrix}_t = A_- \begin{pmatrix} \overline{\pi}_- \\ \overline{\nu}_- \end{pmatrix} + \begin{pmatrix} 0 \\ f_-(\overline{\pi}_+ + \overline{\nu}_-) \end{pmatrix} + Z^\varepsilon(\theta_1\omega) \begin{pmatrix} \overline{\pi}_- \\ \overline{\nu}_- \end{pmatrix},
\]

where \( f_+(\overline{\pi}_+ + \overline{\nu}_-) := \Pi_+ f(\overline{\pi}_+ + \overline{\nu}_-) \) and \( f_-(\overline{\pi}_+ + \overline{\nu}_-) := \Pi_- f(\overline{\pi}_+ + \overline{\nu}_-) \).

Denote \((\overline{\pi}, \overline{\nu})^T\) and \((X^\varepsilon, Y^\varepsilon)^T\) as the solution of Eq.(37) and Eq.(36) with the initial data \((\overline{\pi}_0, \overline{\nu}_0)^T = (\Pi_+ (\overline{\pi}_0, \overline{\nu}_0)^T, \Pi_- (\overline{\pi}_0, \overline{\nu}_0)^T)^T\) and \((X^\varepsilon_0, Y^\varepsilon_0)^T = (\Pi_+ (X^\varepsilon_0, Y^\varepsilon_0)^T, \Pi_- (X^\varepsilon_0, Y^\varepsilon_0)^T)^T\), respectively.

Similar to the definition of Banach space in [33]. For arbitrary \( \eta \in (-\beta, -\alpha) \), we define two Banach spaces

\[ C_{\eta, \varepsilon} = \{ \varphi \mid \varphi \text{ maps } (-\infty, 0) \text{ to } \mathcal{H} : \varphi \text{ is continuous and } \sup_{t \in (-\infty, 0]} e^{-\eta t - \int_0^t Z^\varepsilon(\theta_1\omega) ds} \| \varphi(t) \| \mathcal{H} < \infty \} \]

with the norm

\[ \| \varphi(\cdot) \|_{C_{\eta, \varepsilon}} = \sup_{t \in (-\infty, 0]} e^{-\eta t - \int_0^t Z^\varepsilon(\theta_1\omega) ds} \| \varphi(t) \| \mathcal{H}, \]

and

\[ C_{\eta, B^\varepsilon} = \{ \varphi \mid \varphi \text{ maps } (-\infty, 0] \text{ to } \mathcal{H} : \varphi \text{ is continuous and } \sup_{t \in (-\infty, 0]} e^{-\eta t - \int_0^t B^\varepsilon(\theta_1\omega) ds} \| \varphi(t) \| \mathcal{H} < \infty \} \]

with the norm

\[ \| \varphi(\cdot) \|_{C_{\eta, B^\varepsilon}} = \sup_{t \in (-\infty, 0]} e^{-\eta t - \int_0^t B^\varepsilon(\theta_1\omega) ds} \| \varphi(t) \| \mathcal{H}, \]

where \( B^\varepsilon(\theta_1\omega) := \begin{pmatrix} 0 & 0 \\ z^\varepsilon(\theta_1\omega) & 0 \end{pmatrix} \).

Define the nonlinear operators \( \mathcal{N} \) on \( C_{-\eta, \varepsilon} \) as

\[
\mathcal{N}((\overline{\pi}, \overline{\nu})^T, \xi)(t, \omega) = e^{A_+ t + \int_0^t f_+^r Z^\varepsilon(\theta_r\omega) dr} \xi \\
+ \int_0^t e^{A_+ (t-r) + \int_r^t f_+^r Z^\varepsilon(\theta_r\omega) dr} \begin{pmatrix} 0 \\ f_+(\theta_r\omega, (\overline{\pi}_+ + \overline{\nu}_-)) \end{pmatrix} dr \\
+ \int_{-\infty}^t e^{A_- (t-r) + \int_r^t f_-^r Z^\varepsilon(\theta_r\omega) dr} \begin{pmatrix} 0 \\ f_-(\theta_r\omega, (\overline{\pi}_+ + \overline{\nu}_-)) \end{pmatrix} dr,
\]

where \( \xi = \Pi_+(\overline{\pi}_0, \overline{\nu}_0)^T \in \mathcal{H}_+ \).
Then for any $\xi \in H_+$ and $(\overline{u}_1, \overline{v}_1) \top$, $(\overline{u}_2, \overline{v}_2) \top$ in $C_{0, \varepsilon}^-$, it has
\[
\|N((\overline{u}_1, \overline{v}_1) \top, \xi) - N((\overline{u}_2, \overline{v}_2) \top, \xi)\|_{C_{0, \varepsilon}^-} \\
\leq \sup_{t \leq 0} e^{-\eta t} - f_0^t Z^*(\theta, \omega)dr \int_{-\infty}^t e^{\alpha(t - \tau) + f_1^\tau Z^*(\theta, \omega)dr} \left( f_+ (\overline{u}_1) - f_+ (\overline{u}_2) \right) d\tau\|_{H} \\
+ \sup_{t \leq 0} e^{-\eta t} - f_0^t Z^*(\theta, \omega)dr \int_{-\infty}^t e^{-\beta(t - \tau) + f_1^\tau Z^*(\theta, \omega)dr} \left( f_+ (\overline{u}_1) - f_+ (\overline{u}_2) \right) d\tau\|_{H} \\
\leq \sup_{t \leq 0} e^{-\eta t} - f_0^t Z^*(\theta, \omega)dr \int_{-\infty}^t e^{-\alpha(t - \tau) + f_1^\tau Z^*(\theta, \omega)dr} \left( \overline{u}_1(\tau) - \overline{u}_2(\tau) \right) \|_{H} \|_{H} d\tau \\
+ \sup_{t \leq 0} e^{-\eta t} - f_0^t Z^*(\theta, \omega)dr \int_{-\infty}^t e^{-\beta(t - \tau) + f_1^\tau Z^*(\theta, \omega)dr} \left( \overline{v}_1(\tau) - \overline{v}_2(\tau) \right) \|_{H} \|_{H} d\tau \\
\leq L_f \left( \frac{1}{\eta + \beta} + \frac{1}{\alpha - \eta} \right) \left\| \left( \begin{array}{c} \overline{u}_1(\tau) - \overline{u}_2(\tau) \\ \overline{v}_1(\tau) - \overline{v}_2(\tau) \end{array} \right) \right\|_{C_{0, \varepsilon}^-}.
\]

If
\[
L_f \left( \frac{1}{\eta + \beta} + \frac{1}{\alpha - \eta} \right) < 1,
\]
then by the fixed point argument, $N((\overline{u}, \overline{v}) \top, \xi)$ has a unique fixed point $(\overline{u}(t, \omega; \xi), \overline{v}(t, \omega; \xi)) \top \in C_{0, \varepsilon}^-$, which is the solution of Eq.(37). Define a map
\[
h(\xi, \omega) = (h_1(\xi, \omega), h_2(\xi, \omega)) \top := \Pi_-(\overline{u}(0, \omega; \xi), \overline{v}(0, \omega; \xi)) \top
\]
from $H_+ \times \Omega$ to $H_-$ such that
\[
h(\xi, \omega) = \left( \begin{array}{c} h_1(\xi, \omega) \\ h_2(\xi, \omega) \end{array} \right) \\
= \int_{-\infty}^0 e^{-A_- \tau - f_0^\tau Z^*(\theta, \omega)dr} \left( \begin{array}{c} 0 \\ f_-(\theta, \omega; \overline{X}(\tau, \omega; \xi)) \end{array} \right) d\tau.
\]

Similarly, for Eq.(40), it has that
\[
\hat{h}(\xi, \omega) = \left( \begin{array}{c} \hat{h}_1(\xi, \omega) \\ \hat{h}_2(\xi, \omega) \end{array} \right) \\
:= \int_{-\infty}^0 e^{-A_- \tau - f_0^\tau Z^*(\theta, \omega)dr} \left( \begin{array}{c} 0 \\ f_-(\theta, \omega; \overline{X}(\tau, \omega; \xi)) \end{array} \right) d\tau.
\]

Using the same arguments as above, on $C_{0, \varepsilon}^-$, the mapping of Eq.(36) can be expressed analogously as
\[
h^\varepsilon(\xi, \omega) = \left( \begin{array}{c} h_1(\xi, \omega) \\ h_2(\xi, \omega) \end{array} \right) := \int_{-\infty}^0 e^{-A_- \tau - f_0^\tau B^*(\theta, \omega)dr} \left( \begin{array}{c} 0 \\ f_-(X^\varepsilon(\tau, \omega; \xi)) \end{array} \right) d\tau.
\]

**Lemma 4.5.** (Invariant manifold of random multiplicative system) Suppose the assumptions about the linear operator $A$ and the nonlinearity $f$ in subsection 3 hold, and assume further the following spectrum gap condition holds, that is,
\[
L_f \left( \frac{1}{\eta + \beta} + \frac{1}{\alpha - \eta} \right) < 1.
\]

Then there exist the Lipschitz invariant random manifolds for Eq.(37), Eq.(40) and Eq.(36)
\[
M(\omega) = \{ \xi + h(\xi, \omega) | \xi \in H_+ \} ,
\]
\[
\hat{M}(\omega) = \{ \xi + \hat{h}(\xi, \omega) | \xi \in H_+ \} .
\]
and
\[ M^c(\omega) = \{ \xi + h^c(\xi, \omega) | \xi \in \mathcal{H}_+ \}, \]  
(48)
where \( h = (h_1(\xi, \omega), h_2(\xi, \omega))^\top \), \( \hat{h} = (\hat{h}_1(\xi, \omega), \hat{h}_2(\xi, \omega))^\top \) and \( h^c = (h^c_1(\xi, \omega), h^c_2(\xi, \omega))^\top \) are Lipschitz continuous mappings from \( \mathcal{H}_+ \times \Omega \) to \( \mathcal{H}_- \) defined as Eq.(42), Eq.(43) and Eq.(44).

Using the transformation \( T_m \) and \( T_m^{-1} \) defined as (38), we obtain the invariant manifold of the stochastic multiplicative system.

**Lemma 4.6.** (Invariant manifold of stochastic multiplicative system) Suppose the assumptions about the linear operator \( A \) and the nonlinearity \( f \) in subsection 3 hold, and assume further the following spectrum gap condition holds, that is,
\[ L_f \left( \frac{1}{\eta + \beta} + \frac{1}{\alpha - \eta} \right) < 1. \]

Then Eq.(35) and Eq.(39) have the Lipschitz invariant manifolds \( M^w(\omega) = T_m^{-1}(\omega, M(\omega)) \) and \( M^{x^c}(\omega) = T_m^{-1}(\omega, M^x(\omega)) \), respectively.

5. Approximate dynamics of a finite dimensional SWE driven by a white noise. In this section, we will employ the invariant random cone to verify the almost surely asymptotical completion of the invariant random manifold. Then we approximate Eq.(1) to the finite dimension wave equation with a white noise. Furthermore, the finite dimension is \( \lfloor \sqrt{\frac{\beta}{\alpha}} \rfloor \) with the condition \( \frac{\alpha}{2} > \lfloor \sqrt{\frac{\beta}{\alpha}} \rfloor + 1. \)

5.1. Additive noise case.

**Lemma 5.1.** Suppose the assumptions about the linear operator \( A \) and the nonlinearity \( f \) in subsection 3 hold, and assume further the Lipschitz constant \( L_f \) is small enough. Then the random dynamical system \( \phi(t, \omega) \) generated by Eq.(24) has the invariant random cone property for the the random cone \( C_3(\omega) \) with a deterministic positive parameter \( \delta \). In addition, if there exists \( t_0 > 0 \) such that \( (\hat{u}_1, \hat{v}_1)^\top, (\hat{u}_2, \hat{v}_2)^\top \in \mathcal{H} \) and \( \phi(t_0, \omega)(\hat{u}_1, \hat{v}_1)^\top - \phi(t_0, \omega)(\hat{u}_2, \hat{v}_2)^\top \notin C_3(\theta_0, \omega) \), then \( \| \phi(t, \omega)(\hat{u}_1, \hat{v}_1)^\top - \phi(t, \omega)(\hat{u}_2, \hat{v}_2)^\top \|_{\mathcal{H}} \leq D(\omega)\| (\hat{u}_1, \hat{v}_1)^\top - (\hat{u}_2, \hat{v}_2)^\top \|_{\mathcal{H}^e} \) for any \( t \in [0, t_0] \), where \( D(\omega) \) is a positive random variable and \( k = \beta - L_f - \delta^{-1}L_f > 0. \)

Proof. Let \( (\pi_1, \pi_1^\top) \) and \( (\pi_2, \pi_2^\top) \) be two solutions of Eq.(24) with the initial value \( (\hat{u}_1, \hat{v}_1)^\top \) and \( (\hat{u}_2, \hat{v}_2)^\top \), respectively, and put \( (p_1, p_2)^\top := (\pi_1 + \pi_2, \pi_1 + \pi_2)^\top \), \( (q_1, q_2)^\top := (\pi_1 - \pi_2, \pi_1 - \pi_2)^\top \). Then
\[
\begin{pmatrix}
  p_1 \\
  p_2
\end{pmatrix}_t = A_+(p_1, p_2) + \begin{pmatrix}
  0 \\
  g_{1_+}(\theta_t \omega, \pi_1^+ + \pi_1^-)
\end{pmatrix}
- \begin{pmatrix}
  0 \\
  g_{1_+}(\theta_t \omega, \pi_2^+ + \pi_2^-)
\end{pmatrix},
\]  
(49)
and
\[
\begin{pmatrix}
  q_1 \\
  q_2
\end{pmatrix}_t = A_-(q_1, q_2) + \begin{pmatrix}
  0 \\
  g_{1_-}(\theta_t \omega, \pi_1^+ + \pi_1^-)
\end{pmatrix}
- \begin{pmatrix}
  0 \\
  g_{1_-}(\theta_t \omega, \pi_2^+ + \pi_2^-)
\end{pmatrix},
\]  
(50)
which imply that
\[
\frac{1}{2} \frac{d}{dt} \left\| \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \right\|^2_{\mathcal{H}} \geq -\alpha \left\| \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \right\|^2_{\mathcal{H}} - L_f \left\| \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \right\|^2_{\mathcal{H}} - L_f \left\| q_1 \right\|_{\mathcal{H}} \cdot \left\| q_2 \right\|_{\mathcal{H}},
\]
and
\[
\frac{1}{2} \frac{d}{dt} \left\| \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\|^2_{\mathcal{H}} \leq -\beta \left\| \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\|^2_{\mathcal{H}} - L_f \left\| q_1 \right\|_{\mathcal{H}} \cdot \left\| q_2 \right\|_{\mathcal{H}} - L_f \left\| \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \right\|^2_{\mathcal{H}} \left\| \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\|_{\mathcal{H}}.
\]

Then it follows from (51) and (52) that
\[
\frac{1}{2} \frac{d}{dt} \left\| \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\|^2_{\mathcal{H}} - \delta^2 \left\| \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \right\|^2_{\mathcal{H}} \leq -\beta \left\| \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\|^2_{\mathcal{H}} + L_f \left\| \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \right\|^2_{\mathcal{H}} + \delta^2 \left\| \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\|^2_{\mathcal{H}} + \delta L_f \left\| \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\|^2_{\mathcal{H}} + \delta L_f \left\| \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \right\|^2_{\mathcal{H}} + \delta L_f \left\| \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\|^2_{\mathcal{H}}.
\]

Also notice that if \((p_1, p_2)^\top, (q_1, q_2)^\top\) is in the boundary of the random cone \(C_\delta(\omega)\), then \(||(q_1, q_2)^\top||_{\mathcal{H}} = \delta ||(p_1, p_2)^\top||_{\mathcal{H}}\). Therefore
\[
\frac{1}{2} \frac{d}{dt} \left\| \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\|^2_{\mathcal{H}} - \delta^2 \left\| \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \right\|^2_{\mathcal{H}} \leq (\alpha - \beta + 2L_f + \delta L_f + \delta^{-1}L_f) \left\| \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\|^2_{\mathcal{H}}.
\]

By the assumption \(L_f\) is small enough such that
\[
\alpha - \beta + 2L_f + \delta L_f + \delta^{-1}L_f < 0,
\]
then \(||(q_1, q_2)^\top||_{\mathcal{H}}^2 - \delta^2 ||(p_1, p_2)^\top||_{\mathcal{H}}^2\) is decreasing on the boundary of the random cone \(C_\delta(\omega)\). Consequently, it is concluded that \(\phi(t, \omega)(\tilde{u}_1, \tilde{v}_1)^\top - \phi(t, \omega)(\tilde{u}_2, \tilde{v}_2)^\top \in C_\delta(\theta_t\omega)\) whenever \((\tilde{u}_1, \tilde{v}_1)^\top - (\tilde{u}_2, \tilde{v}_2)^\top \in C_\delta(\omega)\). The first result holds in Lemma 5.1.

In the next, we prove the second result in Lemma 5.1.

If there exists a positive time \(t_0\) such that \(\phi(t_0, \omega)(\tilde{u}_1, \tilde{v}_1)^\top - \phi(t_0, \omega)(\tilde{u}_2, \tilde{v}_2)^\top \notin C_\delta(\theta_t\omega), \quad 0 \leq t \leq t_0\),

that is,
\[
\left\| \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} \right\|_{\mathcal{H}} > \delta \left\| \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} \right\|_{\mathcal{H}}, \quad 0 \leq t \leq t_0,
\]
which implies from (52) that
\[
\frac{1}{2} \frac{d}{dt} \left\| \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\|^2_{\mathcal{H}} \leq -\beta \left\| \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\|^2_{\mathcal{H}} - L_f \left\| \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\|^2_{\mathcal{H}} - \frac{1}{\delta^2} \left\| \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} \right\|_{\mathcal{H}}^2, \quad 0 \leq t \leq t_0.
\]
Then
\[
\left\| \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} \right\|_{\mathcal{H}}^2 < \frac{1}{\delta^2} \left\| \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} \right\|_{\mathcal{H}}^2 \leq \frac{1}{\delta^2} e^{-2kt} \left\| \begin{pmatrix} q_1(0) \\ q_2(0) \end{pmatrix} \right\|_{\mathcal{H}}^2, \quad 0 \leq t \leq t_0.
\]
Therefore there exits a positive random variable $D(\omega)$ such that
\[
\|\phi(t, \omega)(\tilde{u}_1, \tilde{v}_1)^T - \phi(t, \omega)(\tilde{u}_2, \tilde{v}_2)^T\|_{\mathcal{H}} \\
\leq D(\omega)\|((\tilde{u}_1, \tilde{v}_1)^T - (\tilde{u}_2, \tilde{v}_2)^T\|_{\mathcal{H}}e^{-kt}, 0 \leq t \leq t_0,
\]
which completes the proof.

\textbf{Remark 1.} Notice that the smallness of Lipschitz constant $L_f$ in Lemma 5.1 is to ensure that the condition (53) holds.

For the purpose of the finite dimensional approximation, we will use the backward solvability argument to study Eq. (24) on the invariant manifold $M(\omega)$.

For arbitrary given end time $T_f > 0$ and $t \in [0, T_f]$, we consider Eq. (24) in the following form
\[
\left( \begin{array}{c} \pi_+ \\ \pi_- \end{array} \right)_t = A_+ \left( \begin{array}{c} \pi_+ \\ \pi_- \end{array} \right) + \left( \begin{array}{c} 0 \\ g_{1+}(\theta, \omega, (\pi_+ + \pi_-)) \end{array} \right),
\]
with the initial value $(\pi_+(T_f), \pi_+(T_f))^T = \xi \in \mathcal{H}_+$, and
\[
\left( \begin{array}{c} \pi_- \\ \pi_- \end{array} \right)_t = A_- \left( \begin{array}{c} \pi_- \\ \pi_- \end{array} \right) + \left( \begin{array}{c} 0 \\ g_{1-}(\theta, \omega, (\pi_+ + \pi_-)) \end{array} \right),
\]
with the initial value $(\pi_-(0), \pi_-(0))^T = h((\pi_+(0), \pi_+(0))^T, \omega)$, where $h$ is defined as (42).

Also, for $t \in [0, T_f]$, Eq. (24) is equivalent to the integral equations
\[
\left( \begin{array}{c} \pi_+(t) \\ \pi_+(t) \end{array} \right) = e^{A_+(t-T_f)}\xi + \int_{T_f}^te^{A_+(t-\tau)} \left( \begin{array}{c} 0 \\ g_{1+}(\theta, \omega, \pi(\tau)) \end{array} \right) d\tau,
\]
(56) and
\[
\left( \begin{array}{c} \pi_-(t) \\ \pi_-(t) \end{array} \right) = e^{A_-(t-\tau)}h((\pi_+(0), \pi_+(0))^T, \omega) \\
+ \int_0^te^{A_-(-\tau)} \left( \begin{array}{c} 0 \\ g_{1-}(\theta, \omega, \pi(\tau)) \end{array} \right) d\tau.
\]
(57)

\textbf{Remark 2.} Similarly, for $t \in [0, T_f]$, Eq. (25) can be written as
\[
\left( \begin{array}{c} X_+^+(t) \\ Y_+^+(t) \end{array} \right) = e^{A_+(t-T_f)}\xi + \int_{T_f}^te^{A_+(t-\tau)} \left( \begin{array}{c} 0 \\ g_{2+}(\theta, \omega, X(\tau)) \end{array} \right) d\tau
\]
(58) with the initial value $(X_+(T_f), Y_+(T_f))^T = \xi \in \mathcal{H}_+$, and
\[
\left( \begin{array}{c} X_-^+(t) \\ Y_-^+(t) \end{array} \right) = e^{A_-t} \left( \begin{array}{c} X_0^- \\ Y_0^- \end{array} \right) + \int_0^te^{A_-(-\tau)} \left( \begin{array}{c} 0 \\ g_{2-}(\theta, \omega, X(\tau)) \end{array} \right) d\tau
\]
(59) with the initial value $(X_-(0), Y_-(0))^T = h^\varepsilon((X_0^+, Y_0^+))^T, \omega)$, where $h^\varepsilon$ is defined as (43).

\textbf{Lemma 5.2.} Suppose the assumptions about the linear operator $A$ and the nonlinearity $f$ in subsection 3 and the spectrum gap condition (41) hold. Then for any end time $T_f > 0$, there exists a unique solution $((\pi_+(\cdot), \pi_+(\cdot))\top, (\pi_-(\cdot), \pi_-(\cdot))\top) \in C([0, T_f]; \mathcal{H}_+ \times \mathcal{H}_-)$ for Eq. (56) and Eq. (57). Moreover, for any $t \geq 0$, $((\pi_+(t, \theta, \omega), \pi_+(t, \theta, \omega))^T, (\pi_-(t, \theta, \omega), \pi_-(t, \theta, \omega))^T) \in M(\omega)$ for almost all $\omega \in \Omega$.

\textbf{Proof.} Under the assumption of $A$ and $f$, it is easy to obtain the existence and uniqueness of the small time interval by using the contraction argument, and then generalize it to any time interval (also see [8]).
Theorem 5.3. (Approximation of a finite dimensional SWE driven by an additive white noise) Suppose the assumptions about the linear operator $A$ and the nonlinearity $f$ in subsection 3 hold. Assume further the Lipschitz constant $L_f$ is small enough. Then Eq. (24) has a finite dimensional invariant manifold $M(\omega)$. Furthermore, for any solution $(\Pi(t,\theta_{-r}\omega), \nu(t,\theta_{-r}\omega))^\top$ of Eq. (24), there exists a positive random variable $D(\omega)$, a positive constant $k$, and an orbit $(\tilde{u}(t,\theta_{-r}\omega), \tilde{v}(t,\theta_{-r}\omega))^\top$ on the invariant manifold $M(\omega)$ such that for any $t \geq 0$

$$||\left(\begin{array}{c} \Pi(t,\theta_{-r}\omega) \\ \nu(t,\theta_{-r}\omega) \end{array}\right) - \left(\begin{array}{c} \tilde{u}(t,\theta_{-r}\omega) \\ \tilde{v}(t,\theta_{-r}\omega) \end{array}\right)||_H \leq D(\omega)||\left(\begin{array}{c} \Pi(0) \\ \nu(0) \end{array}\right) - \left(\begin{array}{c} \tilde{u}(0) \\ \tilde{v}(0) \end{array}\right)||_H e^{-kt}.$$  

(60)

Meanwhile, Eq. (24) is approximated to the finite dimensional random equation on $\mathcal{H}_+$

$$\left(\begin{array}{c} \Pi_+ \\ \nu_+ \end{array}\right)_t = A_+ \left(\begin{array}{c} \Pi_+ \\ \nu_+ \end{array}\right) + \left(\begin{array}{c} 0 \\ g_{1+}(\theta_{-r}\omega, \Pi_+ + h_1((\Pi_+, \nu_+)^\top, \omega)) \end{array}\right),$$

(61)

where $h_1$ is defined as (42). Moreover, Eq. (16) is approximated to the finite dimensional SWE on $\mathcal{H}_+$

$$\left(\begin{array}{c} u_+ \\ v_+ \end{array}\right)_t = A_+ \left(\begin{array}{c} u_+ \\ v_+ \end{array}\right) + \left(\begin{array}{c} f_+(u_+ + h_1^\omega((u_+, v_+)^\top, \omega)) \\ 0 \end{array}\right) + \left(\begin{array}{c} 0 \\ \sigma W(t) \end{array}\right),$$

(62)

where $h_1^\omega$ is given in Lemma 4.4.

Proof. By Lemma 5.1, it only needs to prove (60).

For any solution $(\Pi_1(t,\theta_{-r}\omega), \nu_1(t,\theta_{-r}\omega))^\top = ((\Pi_{1+}(t,\theta_{-r}\omega), \Pi_{1+}(t,\theta_{-r}\omega))^\top, (\Pi_{1-}(t,\theta_{-r}\omega), \nu_{1-}(t,\theta_{-r}\omega))^\top)$ of Eq. (24), it follows from Lemma 5.2 that for any end time $T_f > 0$, there exists a solution $(\Pi_2(t,\theta_{-r}\omega), \nu_2(t,\theta_{-r}\omega))^\top \in C([0, T_f]; \mathcal{H}_+ \times \mathcal{H}_-)$ of Eq. (24) on the invariant random manifold $M(\omega)$ such that $(\Pi_2(T_f, \theta_{-T_f}\omega), \nu_2(T_f, \theta_{-T_f}\omega))^\top = (\Pi_1(T_f, \theta_{-T_f}\omega), \nu_1(T_f, \theta_{-T_f}\omega))^\top$.

Obviously, $(\Pi_2(t,\theta_{-r}\omega), \nu_2(t,\theta_{-r}\omega))^\top$ depends on the end time $T_f$. Hence we use the notation $(\Pi_2(t,\theta_{-r}\omega; T_f), \nu_2(t,\theta_{-r}\omega; T_f))^\top = ((\Pi_{2+}(t,\theta_{-r}\omega; T_f), \Pi_{2+}(t,\theta_{-r}\omega; T_f))^\top, (\Pi_{2-}(t,\theta_{-r}\omega; T_f), \nu_{2-}(t,\theta_{-r}\omega; T_f))^\top)$ to denote $(\Pi_2(t,\theta_{-r}\omega), \nu_2(t,\theta_{-r}\omega))^\top = ((\Pi_{2+}(t,\theta_{-r}\omega), \Pi_{2+}(t,\theta_{-r}\omega))^\top, (\Pi_{2-}(t,\theta_{-r}\omega), \nu_{2-}(t,\theta_{-r}\omega))^\top)$. Taking $t = 0$, obviously, $(\Pi_{2+}(0,\omega; T_f), \nu_{2+}(0,\omega; T_f))^\top = (\Pi_{2+}(0,\omega), \nu_{2+}(0,\omega))^\top$ and $(\Pi_{2-}(0,\omega; T_f), \nu_{2-}(0,\omega; T_f))^\top = (\Pi_{2-}(0,\omega), \nu_{2-}(0,\omega))^\top$.

Since the solution $((\Pi_{2+}(t,\theta_{-r}\omega; T_f), \nu_{2+}(t,\theta_{-r}\omega; T_f))^\top, (\Pi_{2-}(t,\theta_{-r}\omega; T_f), \nu_{2-}(t,\theta_{-r}\omega; T_f))^\top$ is on the invariant random manifold $M(\omega)$, then it infers

$$||\left(\begin{array}{c} \Pi_{2-}(0,\omega; T_f) \\ \nu_{2-}(0,\omega; T_f) \end{array}\right)||_H \leq \int_0^\infty e^{-\beta r} \left(g_{1-}(\theta_{-r}\omega, \Pi_{2-}(r))\right) ||_H dr \leq L_f \int_0^\infty e^{-(\beta + \eta)r + \eta r} \left(\frac{\Pi_{2-}(r)}{\nu_{2-}(r)} + u^* (\theta_{-r}\omega) + v^* (\theta_{-r}\omega)\right) ||_H dr \leq L_f \left(||u^* (\theta_{-r}\omega)||_{C_{\eta}^-} + ||v^* (\theta_{-r}\omega)||_{C_{\eta}^-}\right) \int_0^\infty e^{-(\beta + \eta)r} dr,$$  

(63)
where $A_{L_f}(\omega)$ is a finite tempered random variable due to $\eta \in (-\beta, -\alpha)$.

Noticing that $(\overline{\pi}_2+(T_f, \theta_{-T_f} \omega), \overline{\pi}_2+(T_f, \theta_{-T_f} \omega))^T = (\overline{\pi}_1+(T_f, \theta_{-T_f} \omega), \overline{\pi}_1+(T_f, \theta_{-T_f} \omega))^T$, it then deduces that $(\overline{\pi}_2+(T_f, \theta_{-T_f} \omega), \overline{\pi}_2+(T_f, \theta_{-T_f} \omega))^T - (\overline{\pi}_1+(T_f, \theta_{-T_f} \omega), \overline{\pi}_1+(T_f, \theta_{-T_f} \omega))^T \notin C_5(\omega)$, which implies from the invariant random cone property of Eq. (24) (Lemma 5.1) that

$$
\left( \begin{array}{c}
\overline{\pi}_2+(t, \theta_t \omega) \\
\overline{\pi}_2+(t, \theta_t \omega)
\end{array} \right) - \left( \begin{array}{c}
\overline{\pi}_1+(t, \theta_t \omega) \\
\overline{\pi}_1+(t, \theta_t \omega)
\end{array} \right) \notin C_5(\omega), \quad 0 \leq t \leq T_f.
$$

Particularly, $(\overline{\pi}_2+(0, \omega), \overline{\pi}_2+(0, \omega))^T - (\overline{\pi}_1+(0, \omega), \overline{\pi}_1+(0, \omega))^T \notin C_5(\omega)$, which immediately implies from (63) that

$$
\| \left( \begin{array}{c}
\overline{\pi}_2+(0, \omega; T_f) \\
\overline{\pi}_2+(0, \omega; T_f)
\end{array} \right) \|_{\mathcal{H}} \leq \| \left( \begin{array}{c}
\overline{\pi}_1+(0, \omega) \\
\overline{\pi}_1+(0, \omega)
\end{array} \right) \|_{\mathcal{H}} + \| \left( \begin{array}{c}
\overline{\pi}_1+(0, \omega) \\
\overline{\pi}_1+(0, \omega)
\end{array} \right) \|_{\mathcal{H}}.
$$

Then the random set $\Gamma(\omega) := \{(\overline{\pi}_2+(0, \omega; T_f), \overline{\pi}_2+(0, \omega; T_f))^T \mid T_f \geq 0\}$ is bounded in $\mathcal{H}$. For almost all $\omega \in \Omega$, choose a sequence $\{T_{f_n}\}_{n \in \mathbb{N}}$, which tends to infinity as $n$ goes infinity, such that $\lim_{n \to \infty} (\overline{\pi}_2+(0, \omega; T_{f_n}), \overline{\pi}_2+(0, \omega; T_{f_n}))^T = (\overline{\pi}_2+(\omega), \overline{\pi}_2+(\omega))^T$, which is $\mathcal{F}$-measurable.

Let $(\overline{u}(t, \theta_{-t} \omega), \overline{v}(t, \theta_{-t} \omega))^T = ((\overline{u}_+(t, \theta_{-t} \omega), \overline{v}_+(t, \theta_{-t} \omega), \overline{v}_-(t, \theta_{-t} \omega))^T, (\overline{u}_-(t, \theta_{-t} \omega), \overline{v}_-(t, \theta_{-t} \omega))^T)$ be a solution of Eq. (24) with the initial value $(\overline{u}(0, \omega), \overline{v}(0, \omega))^T = ((\overline{u}_+(\omega), \overline{v}_+(\omega))^T, h((\overline{u}_+(\omega), \overline{v}_+(\omega))^T, \omega))^T$, where $h$ is defined as (42). Then it follows from Lemma 5.1 and Lemma 5.2 that

$$
\left( \begin{array}{c}
\overline{\pi}_1(t, \theta_{-t} \omega) \\
\overline{\pi}_1(t, \theta_{-t} \omega)
\end{array} \right) - \left( \begin{array}{c}
\overline{\pi}_2(t, \theta_{-t} \omega) \\
\overline{\pi}_2(t, \theta_{-t} \omega)
\end{array} \right) \notin C_5(\omega), \quad 0 \leq t < \infty,
$$

which implies again from Lemma 5.1 that (60) holds.

Furthermore, it is easy to deduce that Eq. (24) is approximated to Eq. (61). Then by Lemma 4.2, Eq. (16) is approximated to Eq. (62). The proof is completed. \hfill \Box

5.2. Multiplicative noise case. In this subsection, we investigate the finite dimensional approximation of Eq. (37).

First, similarly as the additive noise case, Eq. (37) is equivalent to the following equations

$$
\left( \begin{array}{c}
\overline{\pi}_+(t) \\
\overline{\pi}_+(t)
\end{array} \right) = e^{A_+(t-T_f)+\int_{T_f}^t f_+(\theta_r \omega) \, dr} \xi + \int_{T_f}^t e^{A_+(t-\tau)+\int_{\tau}^t f_+(\theta_r \omega) \, dr} \left( \begin{array}{c}
0 \\
0
\end{array} \right) d\tau
$$

with the initial value $(\overline{\pi}_+(T_f), \overline{\pi}_+(T_f))^T = \xi \in \mathcal{H}_+$, and

$$
\left( \begin{array}{c}
\overline{\pi}_-(t) \\
\overline{\pi}_-(t)
\end{array} \right) = e^{A_-(t-T_f)+\int_0^t f_-\xi \, d\tau} \left( \begin{array}{c}
0 \\
0
\end{array} \right) + \int_0^t e^{A_-(t-\tau)+\int_{\tau}^t f_-\xi \, d\tau} \left( \begin{array}{c}
0 \\
0
\end{array} \right) d\tau
$$

(64)

(65)
with the initial value \((\overline{\pi}_-(0), \overline{v}_-(0))^\top = h((\overline{\pi}_+(0), \overline{v}_+(0))^\top, \omega)\), where \(h\) is defined as (42).

And Eq. (40) is equivalent to the following equations

\[
\begin{pmatrix}
\dot{X}^\varepsilon(t)
\dot{Y}^\varepsilon(t)
\end{pmatrix} = e^{A_+(t-T_f) + \int_{T_f}^t Z^\varepsilon(\theta, \omega) d\theta} \begin{pmatrix} 0 \\ f_+(\dot{X}^\varepsilon(\tau)) \end{pmatrix} d\tau
\]

with the initial value \((\dot{X}^\varepsilon(T_f), \dot{Y}^\varepsilon(T_f))^\top = \xi \in \mathcal{H}^+\), and

\[
\begin{pmatrix}
\dot{X}^\varepsilon(t)
\dot{Y}^\varepsilon(t)
\end{pmatrix} = e^{A_-(t-T_f) + \int_0^t Z^\varepsilon(\theta, \omega) d\theta} \begin{pmatrix} X^\varepsilon(0) \\ Y^\varepsilon(0) \end{pmatrix}
\]

\[
+ \int_0^t e^{A_-(t-\tau) + \int_\tau^t Z^\varepsilon(\theta, \omega) d\theta} \begin{pmatrix} 0 \\ f_-(\dot{X}^\varepsilon(\tau)) \end{pmatrix} d\tau
\]

with the initial value \((\dot{X}^\varepsilon(0), \dot{Y}^\varepsilon(0))^\top = \hat{h}(X^\varepsilon(0), Y^\varepsilon(0))^\top, \omega)\), where \(\hat{h}\) is defined as (43).

From Wang and Duan [33], we have the finite dimensional approximation of the multiplicative noise case as follows.

**Theorem 5.4.** (Approximation of a finite dimensional SWE driven by multiplicative white noise) Suppose the assumptions about the linear operator \(A\) and the nonlinearity \(f\) in subsection 3 hold, and assume further the Lipschitz constant \(L_f\) is small enough. Then Eq. (37) has a finite dimensional invariant manifold \(M(\omega)\). Furthermore, for any solution \((\overline{\pi}(t, \theta_{-t}\omega), \overline{v}(t, \theta_{-t}\omega))^\top\) of Eq. (37), there exists a positive random variable \(D(\omega)\), a positive constant \(k\), and an orbit \((\tilde{\pi}(t, \theta_{-t}\omega), \tilde{v}(t, \theta_{-t}\omega))^\top\) on the invariant manifold \(M(\omega)\) such that for any \(t \geq 0\)

\[
|| \begin{pmatrix}
\overline{\pi}(t, \theta_{-t}\omega)
\overline{v}(t, \theta_{-t}\omega)
\end{pmatrix} - \begin{pmatrix}
\tilde{\pi}(t, \theta_{-t}\omega)
\tilde{v}(t, \theta_{-t}\omega)
\end{pmatrix} ||_{\mathcal{H}}
\leq D(\omega)|| \begin{pmatrix}
\overline{\pi}(0)
\overline{v}(0)
\end{pmatrix} - \begin{pmatrix}
\tilde{\pi}(0)
\tilde{v}(0)
\end{pmatrix} ||_{\mathcal{H}} e^{-kt}, t > 0.
\]

Meanwhile, Eq. (37) is approximated to the following random equation on \(\mathcal{H}^+\)

\[
\begin{pmatrix}
\overline{\pi}_+
\overline{v}_+
\end{pmatrix} = A_+ \begin{pmatrix}
\overline{\pi}_+
\overline{v}_+
\end{pmatrix} + \begin{pmatrix} 0 \\ f_+(\overline{\pi}_+ + h_1((\overline{\pi}_+ + \overline{v}_+)^\top, \omega)) \end{pmatrix} + \begin{pmatrix}
\varepsilon z^\varepsilon(\theta_{t}\omega)
\varepsilon z^\varepsilon(\theta_{t}\omega) - \varepsilon a z^\varepsilon(\theta_{t}\omega) - (\varepsilon z^\varepsilon(\theta_{t}\omega))^2 - \varepsilon z^\varepsilon(\theta_{t}\omega)
\end{pmatrix} \begin{pmatrix}
\overline{\pi}_+
\overline{v}_+
\end{pmatrix},
\]

where \(h_1\) is as defined in (42). Moreover, by the transformation (38), Eq. (35) can be approximated to a finite dimensional equation on \(\mathcal{H}^+\) as follows

\[
\begin{pmatrix}
u_+
\nu_+
\end{pmatrix}
\]
6. Approximate dynamics of a finite dimensional SWE driven by a colored noise.

6.1. Additive noise case. In this section, we prove the second main result, that is, Eq.(16) is approximated by a finite dimensional wave equation with a smooth colored noise. First, we state the result.

**Theorem 6.1.** (Approximation of a finite dimensional SWE driven by a additive smooth colored noise) Assume that assumptions about the linear operator $A$, the nonlinearity $f$ in subsection 3 and the spectrum gap condition (41) hold. Then the solution of Eq.(16) almost surely converges the solution of the finite dimensional SWE driven by a additive smooth colored noise (71) in $\mathcal{H}$ as $\varepsilon \to 0$, that is, for any $t \geq 0$,

$$
\| (u(t, \theta_{-\omega}), v(t, \theta_{-\omega})) - (X^{\varepsilon}_{+}(t, \theta_{-\omega}), Y^{\varepsilon}_{+}(t, \theta_{-\omega})) \|_{\mathcal{H}} \to 0, \quad \text{as} \quad \varepsilon \to 0, \quad \text{a.s.}
$$

Here $(X^{\varepsilon}_{+}, Y^{\varepsilon}_{+})$ satisfies

$$
\begin{pmatrix}
X^{\varepsilon}_{+} \\
Y^{\varepsilon}_{+}
\end{pmatrix}_t = A_+ \begin{pmatrix}
X^{\varepsilon}_{+} \\
Y^{\varepsilon}_{+}
\end{pmatrix} + \begin{pmatrix}
0 \\
0
\end{pmatrix}
+ \begin{pmatrix}
f_+(X^{\varepsilon}_{+} + h^{1\varepsilon}_1((X^{\varepsilon}_{+}, Y^{\varepsilon}_{+})^\top, \omega)) \\
\sigma \tilde{\Phi}^0(t)
\end{pmatrix},
$$

(71)

where $h^{1\varepsilon}_1$ is given in Lemma 4.4.

To prove the Theorem 6.1, basing on Theorem 5.3 and using (27), we only need to establish that a similar result (Proposition 1) as Theorem 5.3 for $(X^{\varepsilon}_{+}, Y^{\varepsilon}_{+})^\top$ and the following result (Lemma 6.5)

$$(u_+(t, \theta_{-\omega}), v_+(t, \theta_{-\omega}))^\top \to (X^{\varepsilon}_{+}(t, \theta_{-\omega}), Y^{\varepsilon}_{+}(t, \theta_{-\omega}))^\top, \quad \text{as} \quad \varepsilon \to 0, \quad \text{a.s. in} \quad \mathcal{H},$$

where $(u_+(t, \theta_{-\omega}), v_+(t, \theta_{-\omega}))^\top$ satisfies (62). To this end, we give some preliminaries.

First similar to Theorem 5.3, we have the following results for Eq.(17).

**Proposition 1.** Suppose that assumptions about linear operator $A$ and nonlinearity $f$ in subsection 3 hold. Assume further the Lipschitz constant $L_f$ is small enough. Then Eq.(25) has a finite dimensional invariant manifold $M^\varepsilon(\omega)$. Furthermore, for any solution $(X^{\varepsilon}(t, \theta_{-\omega}), Y^{\varepsilon}(t, \theta_{-\omega}))^\top$ of Eq.(25), there exists a positive random variable $D(\omega)$, a positive constant $k$, and an orbit $(\tilde{X}^{\varepsilon}(t, \theta_{-\omega}), \tilde{Y}^{\varepsilon}(t, \theta_{-\omega}))^\top$ on the invariant manifold $M^X(\omega)$ for any $t \geq 0$ such that

$$
\| (\tilde{X}^{\varepsilon}(t, \theta_{-\omega}), \tilde{Y}^{\varepsilon}(t, \theta_{-\omega})) - \begin{pmatrix}
\tilde{X}^{\varepsilon}(0, \theta_{-\omega}) \\
\tilde{Y}^{\varepsilon}(0, \theta_{-\omega})
\end{pmatrix} \|_{\mathcal{H}} \leq D(\omega) \| (X^{\varepsilon}(0), Y^{\varepsilon}(0)) - \begin{pmatrix}
\tilde{X}^{\varepsilon}(0) \\
\tilde{Y}^{\varepsilon}(0)
\end{pmatrix} \|_{\mathcal{H}} e^{-kt}.
$$

(72)

Meanwhile, Eq.(25) is approximated to the following random equation on $\mathcal{H}_+$

$$
\begin{pmatrix}
\tilde{X}^{\varepsilon}_{+} \\
\tilde{Y}^{\varepsilon}_{+}
\end{pmatrix}_t = A_+ \begin{pmatrix}
\tilde{X}^{\varepsilon}_{+} \\
\tilde{Y}^{\varepsilon}_{+}
\end{pmatrix} + \begin{pmatrix}
0 \\
0
\end{pmatrix}
+ g_{2+}(\theta_{\omega}, \tilde{X}^{\varepsilon}_{+} + h^{1\varepsilon}_1((\tilde{X}^{\varepsilon}_{+}, \tilde{Y}^{\varepsilon}_{+})^\top, \omega))
$$

(73)

where $h^{1\varepsilon}_1$ is defined as (43). Moreover, Eq.(17) is approximated to Eq.(71) on $\mathcal{H}_+$. 

Lemma 6.2. [17] Let $W(t)$ be a scalar two-sided Brownian motion, then for any $\epsilon > 0, \sigma > 0$, there exist $T > 0$ and $\varepsilon_1 > 0$ such that $e^{-\sigma t}|W(t) - \Phi^\varepsilon(t)| \leq \epsilon$, uniformly for both $t \geq T$ and $\varepsilon \leq \varepsilon_1$.

Lemma 6.3. Let $(u^*(\theta \omega), v^*(\theta \omega))^\top$ and $(X^\varepsilon(\theta \omega), Y^\varepsilon(\theta \omega))^\top$ be the stationary solutions of (18) and (19), respectively. Then for any $\eta \in (-\beta, -\alpha)$, it has $\|(u^*(\theta \omega), v^*(\theta \omega))^\top - (X^\varepsilon(\theta \omega), Y^\varepsilon(\theta \omega))^\top\|_{C^\varepsilon_0} \to 0$ as $\varepsilon \to 0$, a.s.

Proof. It follows from Eq.(20) and Eq.(21) that

$$
\| \left( \begin{array}{c} u^*(\theta \omega) - X^\varepsilon(\theta \omega) \\ v^*(\theta \omega) - Y^\varepsilon(\theta \omega) \end{array} \right) \|_{C^\varepsilon_0}
= \sup_{t \leq 0} e^{-\eta t} \left\| \int_0^t e^{-A_+ (\tau - t)} \sigma d \left( \begin{array}{c} 0 \\ W(\tau) - \Phi^\varepsilon(\tau) \end{array} \right) + \int_{-\infty}^0 e^{-A_+ (\tau - t)} \sigma d \left( \begin{array}{c} 0 \\ W(\tau) - \Phi^\varepsilon(\tau) \end{array} \right) \right\|_\mathcal{H}
\leq \sup_{t \leq 0} e^{-\eta t} \left\| \int_0^t e^{-A_+ (\tau - t)} \sigma d \left( \begin{array}{c} 0 \\ W(\tau) - \Phi^\varepsilon(\tau) \end{array} \right) \right\|_\mathcal{H}
+ \sup_{t \leq 0} e^{-\eta t} \left\| \int_{-\infty}^t e^{-A_+ (\tau - t)} \sigma d \left( \begin{array}{c} 0 \\ W(\tau) - \Phi^\varepsilon(\tau) \end{array} \right) \right\|_\mathcal{H}
:= I_1 + I_2.
$$

For $I_1$, since that

$$
\int_{-\infty}^t e^{-A_+ (\tau - t)} \sigma d \left( \begin{array}{c} 0 \\ W(\tau) - \Phi^\varepsilon(\tau) \end{array} \right)
= \sigma \left( \begin{array}{c} 0 \\ W(\tau) - \Phi^\varepsilon(\tau) \end{array} \right) + \int_{-\infty}^t \left( \begin{array}{c} 0 \\ W(\tau) - \Phi^\varepsilon(\tau) \end{array} \right) e^{-A_+ (\tau - t)} A_+ \sigma d\tau,
$$

then

$$
I_1 \leq \sup_{t \leq 0} e^{-\eta t} ||\sigma|| \left\| \left( \begin{array}{c} 0 \\ W(\tau) - \Phi^\varepsilon(\tau) \end{array} \right) \right\|_\mathcal{H}
+ \sup_{t \leq 0} e^{-\eta t} \int_t^\infty ||\sigma|| e^{-A_+ (\tau - t)} ||A_+||_\mathcal{H} || \left( \begin{array}{c} 0 \\ W(\tau) - \Phi^\varepsilon(\tau) \end{array} \right) \right\|_\mathcal{H} d\tau
\leq \sup_{t \leq 0} ||\sigma|| e^{-\eta t} \left\| \left( \begin{array}{c} 0 \\ W(\tau) - \Phi^\varepsilon(\tau) \end{array} \right) \right\|_\mathcal{H}
+ \sup_{t \leq 0} ||\sigma|| e^{-\eta t} \int_t^\infty e^{\beta (\tau - t)} ||A_+||_\mathcal{H} \left( \begin{array}{c} 0 \\ W(\tau) - \Phi^\varepsilon(\tau) \end{array} \right) \right\|_\mathcal{H} d\tau
:= I_{11} + I_{12},
$$

where $I_{11}$ obviously converges to 0 as $\varepsilon$ tends to 0.

Now we turn to investigate $I_{12}$. By Lemma 6.2, for any $\epsilon > 0$, there exist $T > 0$ and $\sigma \in (\eta, 0)$ such that $e^{-\delta t}|W(t) - \Phi^\varepsilon(t)| \leq \epsilon$ uniformly for both $t \geq T$ and $\varepsilon \leq \varepsilon_1$. For fixed $T$, by Lemma 2.1, there exists $\varepsilon_0 > 0$ such that $|W(t) - \Phi^\varepsilon(t)| \leq \epsilon$ for $\varepsilon \leq \varepsilon_0$ and $t \in [0, T]$. 
If $t \geq T$, then
\[
e^{-\eta t} \int_t^\infty |\sigma| e^{-\alpha(\tau-t)} ||A_+||_H \left( \begin{array}{c} 0 \\ W(\tau) - \Phi^\tau(\tau) \end{array} \right) \|H \| d\tau \\
\leq |\sigma| e^{-\eta t} ||A_+||_H \int_t^\infty e^{-\alpha \delta} \|H \| d\tau \\
< \frac{\epsilon |\sigma| e^{-\eta t} ||A_+||_H}{\alpha - \delta}.
\]

If $t \leq T$, then
\[
e^{-\eta t} \int_t^T |\sigma| e^{-\alpha(\tau-t)} ||A_+||_H \left( \begin{array}{c} 0 \\ W(\tau) - \Phi^\tau(\tau) \end{array} \right) \|H \| d\tau \\
\leq e^{-\eta t} \int_t^T |\sigma| e^{-\alpha(\tau-t)} ||A_+||_H \left( \begin{array}{c} 0 \\ W(\tau) - \Phi^\tau(\tau) \end{array} \right) \|H \| d\tau \\
+ e^{-\eta t} \int_t^T |\sigma| e^{-\alpha(\tau-t)} ||A_+||_H \left( \begin{array}{c} 0 \\ W(\tau) - \Phi^\tau(\tau) \end{array} \right) \|H \| d\tau \\
\leq |\sigma| e^{-\eta t} ||A_+||_H \int_t^T e^{-\alpha \delta} d\tau + |\sigma| e^{-\eta t} ||A_+||_H \int_t^\infty e^{-\alpha \delta} \|H \| d\tau.
\]

Consequently, it has
\[
I_{12} < \frac{|\sigma| ||A_+||_H}{\alpha} + \frac{|\sigma| ||A_+||_H}{\alpha - \delta}.
\]

Then it immediately implies that $I_{12} \to 0$ as $\epsilon \to 0$, which, combining (76) with the convergence of $I_{11}$, deduces that $I_1 \to 0$ as $\epsilon \to 0$.

Similarly, it is shown that $I_2 \to 0$ as $\epsilon \to 0$. This completes the proof of Lemma 6.3.

Lemma 6.4. Suppose that assumptions about the linear operator $A$, the nonlinearity $f$ in subsection 3 and the spectrum gap condition (41) hold. Then for any $\eta \in (\beta, -\alpha)$, the solution $(u(t), v(t))^T$ of Eq.(16) almost surely converges the solution $(X^\tau(t), Y^\tau(t))^T$ of Eq.(17) in $C_\eta$ as $\epsilon \to 0$, that is, $||(u(\cdot), v(\cdot))^T - (X^\tau(\cdot), Y^\tau(\cdot))^T||_{C_\eta} \to 0$ as $\epsilon \to 0$, a.s.

Proof. It follows from (30), (31), (22) and (23) that
\[
\| \begin{pmatrix} \bar{u}(\cdot) \\ \bar{v}(\cdot) \end{pmatrix} - \begin{pmatrix} \bar{X}(\cdot) \\ \bar{Y}(\cdot) \end{pmatrix} \|_{C_\eta} \\
\leq \sup_{t \leq 0} e^{-\eta t} ||Lf \begin{pmatrix} \bar{u}(t) \\ \bar{v}(t) \end{pmatrix} \|H \\
+ \sup_{t \leq 0} e^{-\eta t} ||Lf \int_t^t e^{\alpha(t-\tau)} \left( \begin{array}{c} \bar{u}(\tau) \\ \bar{v}(\tau) \end{array} \right) - \left( \begin{array}{c} X^\tau(\tau) \\ Y^\tau(\tau) \end{array} \right) \|H d\tau \\
+ \sup_{t \leq 0} e^{-\eta t} ||Lf \int_t^\infty e^{-\beta(t-\tau)} \left( \begin{array}{c} \bar{u}(\tau) \\ \bar{v}(\tau) \end{array} \right) - \left( \begin{array}{c} X^\tau(\tau) \\ Y^\tau(\tau) \end{array} \right) \|H d\tau \\
+ \sup_{t \leq 0} e^{-\eta t} ||Lf \int_{-\infty}^t e^{-\beta(t-\tau)} \left( \begin{array}{c} \bar{u}(\tau) \\ \bar{v}(\tau) \end{array} \right) - \left( \begin{array}{c} X^\tau(\tau) \\ Y^\tau(\tau) \end{array} \right) \|H d\tau \\
\leq Lf \begin{pmatrix} \frac{1}{\alpha - \eta} \\ \frac{1}{\alpha - \eta} \end{pmatrix} \| \begin{pmatrix} \bar{u}(\cdot) \\ \bar{v}(\cdot) \end{pmatrix} - \begin{pmatrix} \bar{X}(\cdot) \\ \bar{Y}(\cdot) \end{pmatrix} \|_{C_\eta} \\
+ Lf \begin{pmatrix} \frac{1}{\beta + \eta} \\ \frac{1}{\beta + \eta} \end{pmatrix} \| \begin{pmatrix} u^*(\theta, \omega) \\ v^*(\theta, \omega) \end{pmatrix} - \begin{pmatrix} X^*(\theta, \omega) \\ Y^*(\theta, \omega) \end{pmatrix} \|_{C_\eta}.
\]
Then
\[
\left[ 1 - L_f \left( \frac{1}{\eta + \beta} + \frac{1}{\alpha - \eta} \right) \right] \| u_+^* (\omega, t, \theta - t) - X_+^* (t, \theta - t) \|_{C_t^0} \leq L_f \left( \frac{1}{\eta + \beta} + \frac{1}{\alpha - \eta} \right) \| u_+ (\omega, t, \theta) - X_+ (t, \theta) \|_{C_t^0}.
\]
So
\[
\| \left( \begin{array}{c} u_+ (\omega, t, \theta) - X_+ (t, \theta) \\ v_+ (\omega, t, \theta) - Y_+ (t, \theta) \end{array} \right) \|_{C_t^0} \leq \frac{L_f \left( \frac{1}{\eta + \beta} + \frac{1}{\alpha - \eta} \right)}{1 - L_f \left( \frac{1}{\eta + \beta} + \frac{1}{\alpha - \eta} \right)} \left( \begin{array}{c} u_+^* (\omega, t, \theta - t) - X_+^* (t, \theta - t) \\ v_+^* (\omega, t, \theta - t) - Y_+^* (t, \theta - t) \end{array} \right) \|_{C_t^0},
\]
which implies from Lemma 6.3 that \( \| (\pi_+ (\omega), \pi_+ (\omega))^T - (X_+^* (\omega), Y_+^* (\omega))^T \|_{C_t^0} \to 0 \) as \( \varepsilon \to 0 \). Furthermore, using (26) and (27), Lemma 6.4 holds.

\textbf{Lemma 6.5.} Suppose that assumptions about linear operator \( A \) and nonlinearity \( f \) in subsection 3 and the spectrum gap condition (41) hold. Then the finite dimension approximating of Eq.(16) almost surely converges that of Eq.(17) in \( \mathcal{H} \) as \( \varepsilon \to 0 \), that is, for any \( t \geq 0 \),
\[
\| \left( \begin{array}{c} u_+ (\omega, t, \theta - t) \\ v_+ (\omega, t, \theta - t) \end{array} \right) - \left( \begin{array}{c} X_+^* (t, \theta - t) \\ Y_+^* (t, \theta - t) \end{array} \right) \|_{\mathcal{H}} \to 0 \quad \text{as} \quad \varepsilon \to 0, \quad \text{a.s.}
\]

\textbf{Proof.} For any end time \( T_f > 0 \), let that \( (\pi_+ (T_f), \pi_+ (T_f))^T = (X_+ (T_f), Y_+ (T_f))^T \). Then it follows from Eq.(56) and Eq.(58) that

\[
\left\| \begin{array}{c} \pi_+ (t, \theta - t) \\ \pi_+ (t, \theta - t) \end{array} - \left( \begin{array}{c} X_+ (t, \theta - t) \\ Y_+ (t, \theta - t) \end{array} \right) \right\|_{\mathcal{H}} \leq L_f \int_t^{T_f} e^{\alpha (t - r)} \left( \begin{array}{c} g_1 (\theta - r, \omega, \pi (r)) - g_2 (\theta - r, \omega, X^* (r)) \\ u^* (\theta - r, \omega) - X^* (\theta - r, \omega) \end{array} \right) d\tau \right\|_{\mathcal{H}}
\]

Notice that for any \( \eta \in (-\beta, -\alpha) \) and \( t \geq 0 \), the integral \( \int_{T_f}^{T_f} e^{\alpha (t - r)} r \) is finite whenever \( T_f \in [0, \infty) \). It follows from Lemma 6.3, Lemma 6.4 and (77) that \( I_3 \to 0 \) and \( I_4 \to 0 \) as \( \varepsilon \to 0 \).

Consequently,
\[
\left\| \begin{array}{c} u_+ (\omega, t, \theta - t) \\ v_+ (\omega, t, \theta - t) \\ \pi_+ (t, \theta - t) \\ \pi_+ (t, \theta - t) \\
\end{array} - \left( \begin{array}{c} X_+^* (t, \theta - t) \\ Y_+^* (t, \theta - t) \\ X_+ (t, \theta - t) \\ Y_+ (t, \theta - t) \end{array} \right) \right\|_{\mathcal{H}} \to 0, \quad \text{as} \quad \varepsilon \to 0, \quad \text{a.s.,} \quad \text{(78)}
\]
which implies from (26) and (27) that Lemma 6.5 holds. This completes the proof.

6.2. Multiplicative noise case. In this subsection, we demonstrate the multiplicative case, Eq. (35), is approximated by a finite dimensional wave equation with a smooth colored noise. First, the main result is as follows.

**Theorem 6.6.** *(Approximation of a finite dimensional SWE driven by a multiplicative smooth colored noise) Suppose that assumptions about linear operator $A$ and nonlinearity $f$ in subsection 3 and the spectrum gap condition (45) hold. Then the Eq. (35) almost surely converges the finite dimensional SWE of Eq. (36) in $\mathcal{H}$ as $\varepsilon \to 0$, that is, for any $t \geq 0$,

$$
\| \begin{pmatrix} u(t, \theta_{-\omega}) \\ v(t, \theta_{-\omega}) \end{pmatrix} - \begin{pmatrix} X_\varepsilon^+(t, \theta_{-\omega}) \\ Y_\varepsilon^+(t, \theta_{-\omega}) \end{pmatrix} \|_{\mathcal{H}} \to 0, \quad \text{as } \varepsilon \to 0, \quad a.s.,
$$

where $(X_\varepsilon^+(t, \theta_{-\omega}), Y_\varepsilon^+(t, \theta_{-\omega}))^T$ satisfies

$$
\begin{pmatrix} X_\varepsilon^+ \\ Y_\varepsilon^+ \end{pmatrix}_t = A_+ \begin{pmatrix} X_\varepsilon^+ \\ Y_\varepsilon^+ \end{pmatrix} + \begin{pmatrix} 0 \\ f_+ (X_\varepsilon^+ + h_\varepsilon^+ ((X_\varepsilon^+, Y_\varepsilon^+)^T, \omega)) \end{pmatrix},
$$

where $h_\varepsilon^+$ is as defined in (44).

To prove the Theorem 6.6, based on Theorem 5.4, we only need to derive that a similar result (Proposition 2) as Theorem 5.4 for $(X_\varepsilon^+, Y_\varepsilon^+)^T$ and the following result (Lemma 6.12)

$$(u_+ (t, \theta_{-\omega}), v_+ (t, \theta_{-\omega}))^T \to (X_\varepsilon^+(t, \theta_{-\omega}), Y_\varepsilon^+(t, \theta_{-\omega}))^T, \quad \text{as } \varepsilon \to 0, \quad a.s. \text{ in } \mathcal{H},$$

where $(u_+ (t, \theta_{-\omega}), v_+ (t, \theta_{-\omega}))^T$ satisfies (70).

**Lemma 6.7.** [33] Assume the spectrum gap condition (45) holds. Then for any $T_f > 0$, Eq. (37) has a unique solution $((\pi_+ (\cdot), \pi_+ (\cdot)), (\pi_- (\cdot), \pi_- (\cdot)))^T \in C(0, T_f; \mathcal{H}_+ \times \mathcal{H}_-)$, and Eq. (40) has a unique solution $((\hat{X}_\varepsilon^+, \hat{Y}_\varepsilon^+), (\hat{X}_\varepsilon^-, \hat{Y}_\varepsilon^-))^T \in C(0, T_f; \mathcal{H}_+ \times \mathcal{H}_-)$.

Further, for any $t > 0$, it holds

$$
((\pi_+ (t, \theta_{-\omega}), \pi_+ (t, \theta_{-\omega})), (\pi_- (t, \theta_{-\omega}), \pi_- (t, \theta_{-\omega})))^T \in M(\omega),
$$

$$
((\hat{X}_\varepsilon^+ (t, \theta_{-\omega}), \hat{Y}_\varepsilon^+ (t, \theta_{-\omega})), (\hat{X}_\varepsilon^- (t, \theta_{-\omega}), \hat{Y}_\varepsilon^- (t, \theta_{-\omega})))^T \in \hat{M}(\omega),
$$

where the invariant manifolds $M(\omega)$ and $\hat{M}(\omega)$ are defined as (46) and (47), respectively.

**Lemma 6.8.** [33] Assume that the Lipschitz constant $L_f$ is small enough. Then the stochastic dynamic system of Eq. (36) and Eq. (40) has a cone invariance.

**Lemma 6.9.** [33] Assume that the Lipschitz constant $L_f$ is small enough such that $\beta - L_f - \delta^{-1} L_f > 0$, denoted as $k$ for simplicity. If there further exist a time $t_0 > 0$ and two points $(\hat{u}_1, \hat{v}_1)^T, (\hat{u}_2, \hat{v}_2)^T \in \mathcal{H}$ such that $\varphi^\varepsilon (t_0, \omega)(\hat{u}_1, \hat{v}_1)^T - \varphi^\varepsilon (t_0, \omega)(\hat{u}_2, \hat{v}_2)^T \notin C_k(\hat{\theta}_0, \omega)$, then

$$
\|\varphi^\varepsilon (t_0, \omega)(\hat{u}_1, \hat{v}_1)^T - \varphi^\varepsilon (t_0, \omega)(\hat{u}_2, \hat{v}_2)^T\|_{\mathcal{H}} \leq D(\omega) \| (\hat{u}_1, \hat{v}_1)^T - (\hat{u}_2, \hat{v}_2)^T\|_{\mathcal{H}} e^{-kt}, \quad 0 \leq t \leq t_0,
$$

where $D(\omega)$ is a slowly increasing random variable.
Proposition 2. Suppose that assumptions about linear operator $A$ and nonlinearity $f$ in subsection 3 hold. Assume further the Lipschitz constant $L_f$ is small enough. Then Eq. (36) has a finite dimensional invariant manifold $M^\varepsilon(\omega)$. Furthermore, for any solution $(X^\varepsilon(t, \theta-\omega), Y^\varepsilon(t, \theta-\omega))^\top$ of Eq. (36), there exists a positive random variable $D(\omega)$, a positive constant $k$, and an orbit $(\hat{X}^\varepsilon(t, \theta-\omega), \hat{Y}^\varepsilon(t, \theta-\omega))^\top$ on the invariant manifold $M^\varepsilon(\omega)$ for any $t \geq 0$ such that

\[
\| (X^\varepsilon(t, \theta-\omega), Y^\varepsilon(t, \theta-\omega)) - (\hat{X}^\varepsilon(t, \theta-\omega), \hat{Y}^\varepsilon(t, \theta-\omega)) \|_{\mathcal{H}} \leq D(\omega) \| (X^\varepsilon(0), Y^\varepsilon(0)) - (\hat{X}^\varepsilon(0), \hat{Y}^\varepsilon(0)) \|_{\mathcal{H}} e^{-kt}, \quad t > 0.
\]

Moreover, the orbit $(X^\varepsilon, Y^\varepsilon)^\top = ((X^\varepsilon_+, Y^\varepsilon_+), (X^\varepsilon_-, Y^\varepsilon_-))^\top$ of Eq. (36) can be approximated to finite space $\mathcal{H}_+$ and satisfied Eq. (79).

Using the same arguments as the proof of Theorem 5.3, we easily get Proposition 2.

Lemma 6.10. Suppose that assumptions about linear operator $A$ and nonlinearity $f$ in subsection 3 hold. Assume further the Lipschitz constant $L_f$ is small enough. And let $(X^\varepsilon_+(t, \theta-\omega), Y^\varepsilon_+(t, \theta-\omega))^\top$ and $(X^\varepsilon_-(t, \theta-\omega), Y^\varepsilon_-(t, \theta-\omega))^\top$ be the finite dimensional approximation solutions of Eq. (39) and Eq. (36), respectively. Then for almost all $\omega \in \Omega$,

\[
\| (X^\varepsilon_+(t, \theta-\omega), Y^\varepsilon_+(t, \theta-\omega)) - (X^\varepsilon_+(t, \theta-\omega), Y^\varepsilon_+(t, \theta-\omega)) \|_{\mathcal{H}} \to 0, \quad \text{as} \quad \varepsilon \to 0.
\]

Proof. It follows from (39) and (36) that

\[
\begin{bmatrix}
X^\varepsilon_+ \\
Y^\varepsilon_+
\end{bmatrix}_t = A_+ \begin{bmatrix}
X^\varepsilon_+ \\
Y^\varepsilon_+
\end{bmatrix} + \begin{bmatrix}
f(X^\varepsilon_+) \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
\varepsilon aX^\varepsilon_+ \Phi^\varepsilon(t) - \varepsilon aX^\varepsilon_+ \Phi^\varepsilon(t) - \varepsilon aX^\varepsilon_+ (\Phi^\varepsilon(t))^2 - \varepsilon Y^\varepsilon_+ \Phi^\varepsilon(t)
\end{bmatrix}.
\]

According to the definition of the $\Phi^\varepsilon(t)$ and the properties of the $z^\varepsilon(\theta_\omega)$ in subsection 2.3, the result Eq. (81) holds. The proof is completed.

Lemma 6.11. Suppose that assumptions about linear operator $A$ and nonlinearity $f$ in subsection 3 hold. Assume further the Lipschitz constant $L_f$ is small enough. And let $(\pi^\varepsilon_+(t, \theta-\omega), \pi^\varepsilon_+(t, \theta-\omega))^\top$ and $(\hat{X}^\varepsilon_+(t, \theta-\omega), \hat{Y}^\varepsilon_+(t, \theta-\omega))^\top$ be the finite dimensional approximation solutions of Eq. (37) and Eq. (40), respectively. Then for almost all $\omega \in \Omega$,

\[
\| (\pi^\varepsilon_+(t, \theta-\omega), \pi^\varepsilon_+(t, \theta-\omega)) - (\hat{X}^\varepsilon_+(t, \theta-\omega), \hat{Y}^\varepsilon_+(t, \theta-\omega)) \|_{\mathcal{H}} \to 0, \quad \text{as} \quad \varepsilon \to 0.
\]
Proof. By the definition of (64), (66) and $\Phi^\varepsilon$, for any $t \in [0, T_f]$, there is

\[
\left\| \begin{pmatrix} \overline{u}_+^\varepsilon(t) \\ \overline{v}_+^\varepsilon(t) \end{pmatrix} - \begin{pmatrix} \overline{X}_+^\varepsilon(t) \\ \overline{Y}_+^\varepsilon(t) \end{pmatrix} \right\|_{\mathcal{H}} 
\]

\[
= \left\| e^{A_+(t-T_f)+f_+^\varepsilon} Z^\varepsilon(\theta, \omega) - \begin{pmatrix} \overline{u}_+^\varepsilon(T_f) \\ \overline{v}_+^\varepsilon(T_f) \end{pmatrix} \right\|_{\mathcal{H}} 
\]

\[
+ \int_{T_f}^t e^{A_+(t-\tau)+f_+^\varepsilon} Z^\varepsilon(\theta, \omega) d\tau \left( \begin{pmatrix} \overline{u}_+^\varepsilon(T_f) \\ \overline{v}_+^\varepsilon(T_f) \end{pmatrix} - \begin{pmatrix} \overline{X}_+^\varepsilon(T_f) \\ \overline{Y}_+^\varepsilon(T_f) \end{pmatrix} \right) d\tau 
\]

\[
\leq \left\| e^{A_+(t-T_f)+f_+^\varepsilon} Z^\varepsilon(\theta, \omega) - \begin{pmatrix} \overline{u}_+^\varepsilon(T_f) \\ \overline{v}_+^\varepsilon(T_f) \end{pmatrix} \right\|_{\mathcal{H}} 
\]

\[
+ \int_{T_f}^t e^{A_+(t-\tau)+f_+^\varepsilon} Z^\varepsilon(\theta, \omega) d\tau \left( \begin{pmatrix} \overline{u}_+^\varepsilon(T_f) \\ \overline{v}_+^\varepsilon(T_f) \end{pmatrix} - \begin{pmatrix} \overline{X}_+^\varepsilon(T_f) \\ \overline{Y}_+^\varepsilon(T_f) \end{pmatrix} \right) d\tau 
\]

\[
:= I_1 + I_2. 
\]

For $I_1$, we assume that the initial values of the original system (35) and the approximation system (36) are equal, which implies $u_+(T_f) = X_+^\varepsilon(T_f)$ and $v_+(T_f) = Y_+^\varepsilon(T_f)$. By the Lemma 6.10, one gets $X_+^\varepsilon(T_f) \rightarrow X_+^\varepsilon(T_f)$ and $Y_+^\varepsilon(T_f) \rightarrow Y_+^\varepsilon(T_f)$ as $\varepsilon \rightarrow 0$ in $\mathcal{H}$. Furthermore, in the light of the transformation $T_m$ defined as (38), we can obtain that $\overline{u}_+(T_f) \rightarrow \overline{X}_+^\varepsilon(T_f)$ and $\overline{v}_+(T_f) \rightarrow \overline{Y}_+^\varepsilon(T_f)$ as $\varepsilon \rightarrow 0$. Thus, $I_1 \rightarrow 0$ as $\varepsilon \rightarrow 0$ in $\mathcal{H}$.

As for the term $I_2$, combined with Hypothesis(H), we deduce

\[
I_2 = \left\| \int_{T_f}^t e^{A_+(t-\tau)+f_+^\varepsilon} Z^\varepsilon(\theta, \omega) d\tau \left( \begin{pmatrix} \overline{u}_+(\tau) \\ \overline{v}_+(\tau) \end{pmatrix} - \begin{pmatrix} \overline{X}_+^\varepsilon(\tau) \\ \overline{Y}_+^\varepsilon(\tau) \end{pmatrix} \right) \right\|_{\mathcal{H}} 
\]

\[
\leq \int_{T_f}^t e^{a(t-\tau)+f_+^\varepsilon} Z^\varepsilon(\theta, \omega) d\tau \left( \begin{pmatrix} \overline{u}_+(\tau) \\ \overline{v}_+(\tau) \end{pmatrix} - \begin{pmatrix} \overline{X}_+^\varepsilon(\tau) \\ \overline{Y}_+^\varepsilon(\tau) \end{pmatrix} \right) \right\|_{\mathcal{H}} d\tau 
\]

\[
\leq L_f \left\| \begin{pmatrix} \overline{u}(-T_f) - \overline{X}_+^\varepsilon(-T_f) \\ \overline{v}(-T_f) - \overline{Y}_+^\varepsilon(-T_f) \end{pmatrix} \right\|_{C_{\overline{u}, \overline{v}}} \int_{-T_f}^{-t} e^{a(t+\tau)-\eta(t+\tau)} d\tau. 
\]
Assume further that the initial value \((\overline{u}_0, \overline{v}_0)^\top = (\hat{X}_0^\varepsilon, \hat{Y}_0^\varepsilon)^\top\). Then
\[
\| \begin{pmatrix} \overline{u}(\cdot) - \hat{X}^\varepsilon(\cdot) \\ \overline{v}(\cdot) - \hat{Y}^\varepsilon(\cdot) \end{pmatrix} \|_{C_{0,1}^-} \\
\leq \sup_{t \leq T_1} e^{-\eta t - J_0^t \mathbf{Z}^\varepsilon(\theta_t)dt} L_f \| \int_0^t e^{\alpha(t-\tau)+J_0^\tau \mathbf{Z}^\varepsilon(\theta_t)dt} \left( \begin{pmatrix} \overline{u} - \hat{X}^\varepsilon \\ \overline{v} - \hat{Y}^\varepsilon \end{pmatrix} \right) dt \|_{\mathcal{H}} \\
+ \sup_{t \leq T_1} e^{-\eta t - J_0^t \mathbf{Z}^\varepsilon(\theta_t)dt} L_f \| \int_{-\infty}^t e^{-\beta(t-\tau)+J_0^\tau \mathbf{Z}^\varepsilon(\theta_t)dt} \left( \begin{pmatrix} \overline{u} - \hat{X}^\varepsilon \\ \overline{v} - \hat{Y}^\varepsilon \end{pmatrix} \right) dt \|_{\mathcal{H}} \\
\leq L_f \| \begin{pmatrix} \overline{u}(\cdot) - \hat{X}^\varepsilon(\cdot) \\ \overline{v}(\cdot) - \hat{Y}^\varepsilon(\cdot) \end{pmatrix} \|_{C_{0,1}^-} \int_0^t e^{\alpha(t-\tau)-\eta(t-\tau)} dt \\
+ L_f \| \begin{pmatrix} \overline{u}(\cdot) - \hat{X}^\varepsilon(\cdot) \\ \overline{v}(\cdot) - \hat{Y}^\varepsilon(\cdot) \end{pmatrix} \|_{C_{0,1}^-} \int_{-\infty}^t e^{-\beta(t-\tau)-\eta(t-\tau)} dt \\
\leq \| \begin{pmatrix} \overline{u}(\cdot) - \hat{X}^\varepsilon(\cdot) \\ \overline{v}(\cdot) - \hat{Y}^\varepsilon(\cdot) \end{pmatrix} \|_{C_{0,1}^-} L_f \frac{1}{\eta + \beta} + \frac{1}{\alpha - \eta},
\]
which implies that
\[
\| \begin{pmatrix} \overline{u}(t) - \hat{X}^\varepsilon(t) \\ \overline{v}(t) - \hat{Y}^\varepsilon(t) \end{pmatrix} \|_{C_{0,1}^-} \to 0.
\]
Since \(\int_{-T_1}^{-t} e^{\alpha(t+\tau)-\eta(\tau+t)} d\tau\) is finite, then \(I_2 \to 0\). Thus,
\[
\| \begin{pmatrix} \overline{u}(t) \\ \overline{v}(t) \end{pmatrix} - \begin{pmatrix} \hat{X}^\varepsilon(t) \\ \hat{Y}^\varepsilon(t) \end{pmatrix} \|_{\mathcal{H}} \to 0 \text{ as } \varepsilon \to 0.
\]
The proof is completed. \(\square\)

**Lemma 6.12.** Suppose that assumptions about linear operator \(A\) and nonlinearity \(f\) in subsection 3 hold. Assume further the Lipschitz constant \(L_f\) is small enough. And let \((X^\varepsilon_+(t,\theta_t,\omega), Y^\varepsilon_+(t,\theta_t,\omega))^\top\) and \((u_+(t,\theta_t,\omega), v_+(t,\theta_t,\omega))^\top\) be the finite dimensional approximation solutions of Eq.(36) and Eq.(35), respectively. Then for almost all \(\omega \in \Omega\),
\[
\| \begin{pmatrix} X^\varepsilon_+(t,\theta_t,\omega) \\ Y^\varepsilon_+(t,\theta_t,\omega) \end{pmatrix} - \begin{pmatrix} u_+(t,\theta_t,\omega) \\ v_+(t,\theta_t,\omega) \end{pmatrix} \|_{\mathcal{H}} \to 0, \text{ as } \varepsilon \to 0.
\]

**Proof.** It is obviously that Lemma 6.12 holds by Lemma 6.10 and Lemma 6.11. \(\square\)

**REFERENCES**

[1] P. Acquistapace and B. Terreni, An approach to Itô linear equations in Hilbert spaces by approximation of white noise with coloured noise, *Stoch. Anal. Appl.*, 2 (1984), 131–186.

[2] L. Arnold, *Random Dynamical Systems*, Springer-Verlag, Heidelberg, 1998.

[3] S. Cerrai and M. I. Freidlin, On the Smoluchowski-Kramers approximation for a system with an infinite number of degrees of freedom, *Probab. Theory Rel.*, 135 (2006), 363–394.

[4] G. Chen, J. Duan and J. Zhang, Approximating dynamics of a singularly perturbed stochastic wave equation with a random dynamical boundary condition, *SIAM. J. Math. Anal.*, 45 (2013), 2790–2814.

[5] P. L. Chow, Asymptotics of solutions to semilinear stochastic wave equations, *Ann. Appl. Probab.*, 16 (2006), 757–780.

[6] P. L. Chow, Stochastic wave equations with polynomial nonlinearity, *Ann. Appl. Probab.*, 12 (2002), 361–381.
[7] G. da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, UK, 1992.

[8] J. Duan, K. Lu and B. Schmalfuß, Invariant manifolds for stochastic partial differential equations, *Ann. Probab.*, 31 (2003), 2109–2135.

[9] J. Duan, K. Lu and B. Schmalfuß, Smooth stable and unstable manifolds for stochastic evolutionary equations, *J. Dyn. Differ. Equ.*, 16 (2004), 949–972.

[10] X. Fan and Y. Wang, Fractal dimension of attractors for a stochastic wave equation with nonlinear damping and white noise, *Stoch. Anal. Appl.*, 25 (2007), 381–396.

[11] J. Garcia-Ojalvo and J. M. Sancho, *Noise in Spatially Extended Systems*, Springer-Verlag, Berlin, 1999.

[12] Z. Guo, X. Yan, W. Wang and X. Liu, Approximate the dynamical behavior for stochastic systems by Wong-Zakai approaching, *J. Math. Anal. Appl.*, 457 (2018), 214–232.

[13] M. Hairer and E. Pardoux, A Wong-Zakai theorem for stochastic PDEs, *J. Math. Soc. Jpn.*, 67 (2015), 1551–1604.

[14] J. Hale and G. Raugel, Upper semicontinuity of the attractor for a singularly perturbed hyperbolic equation, *J. Differ. Equations*, 73 (1988), 197–214.

[15] W. Horsthemke and R. Lefever, *Noise-induced Transitions: Theory and Applications in Physics, Chemistry, and Biology*, Springer Series in Synergetics, Berlin, Springer, 1984.

[16] N. Ikeda, S. Nakao and Y. Yamato, A class of approximations of Brownian motion, *Publ. Res. Inst. Math. Sci.*, 13 (1977), 285–300.

[17] T. Jiang, X. Liu and J. Duan, A Wong-Zakai approximation for random invariant manifolds, *J. Math. Phys.*, 58 (2017), 122701.

[18] D. Kelley and I. Melbourne, Smooth approximation of stochastic differential equations, *Ann. Probab.*, 44 (2016), 479–520.

[19] F. Konetny, On Wong-Zakai approximation of stochastic differential equations, *J. Multivariate Anal.*, 13 (1983), 605–611.

[20] T. Kurz and P. Protter, Weak limit theorems for stochastic integrals and stochastic differential equations, *Ann. Probab.*, 19 (1991), 1035–1070.

[21] K. Lu and B. Schmalfuß, Invariant manifolds for stochastic wave equations, *J. Differ. Equations*, 236 (2007), 460–492.

[22] Y. Lv and W. Wang, Limiting dynamics for stochastic wave equations, *J. Differ. Equations*, 244 (2008), 1–23.

[23] Y. Lv, W. Wang and A. J. Roberts, Approximation of the random inertial manifold of singularly perturbed stochastic wave equations, *Stoch. Dyn.*, 14 (2014), 1350018.

[24] X. Mora, Finite-dimensional attracting invariant manifold of damped semilinear wave equations, *Contributions to Nonlinear Partial Differential Equations*, 2 (1985), 172–183.

[25] C. Mueller, Long time existence for the wave equation with a noise term, *Ann. Probab.*, 25 (1997), 133–151.

[26] S. Nakao, On weak convergence of sequences of continuous local martingale, *Ann. I. H. Poincare B*, 22 (1986), 371–380.

[27] E. Pardoux and A. Piatnitski, Homogenization of a singular random one-dimensional PDE with time-varying coefficients, *Ann. Probab.*, 40 (2012), 1316–1356.

[28] P. Protter, Approximations of solutions of stochastic differential equations driven by semimartingales, *Ann. Probab.*, 13 (1985), 716–743.

[29] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II*, Academic Press, New York, 1975.

[30] J. Shen and K. Lu, Wong-Zakai approximations and center manifolds of stochastic differential equations, *J. Differ. Equations*, 263 (2017), 4929–4977.

[31] G. Tessitore and J. Zabczyk, Wong-Zakai approximation of stochastic evolution equations, *J. Evol. Equ.*, 6 (2006), 621–655.

[32] X. Wang, K. Lu and B. Wang, Wong-Zakai approximations and attractors for stochastic reaction-diffusion equations on unbounded domains, *J. Differ. Equations*, 264 (2018), 378–424.

[33] W. Wang and J. Duan, A dynamical approximation for stochastic partial differential equations, *J. Math. Phys.*, 48 (2007), 102701.

[34] G. Whitham, *Linear and Nonlinear Waves*, Wiley, New York, 1974.

[35] E. Wong and M. Zakai, On the relation between ordinary and stochastic differential equations, *Int. J. Eng. Sci.*, 3 (1965), 213–229.
[36] E. Wong and M. Zakai, On the convergence of ordinary integrals to stochastic integrals, *Ann. Math. Stat.*, 36 (1965), 1560–1564.

[37] X. Yan, X. Liu and M. Yang, Random attractors of stochastic partial differential equations: A smooth approximation approach, *Stoch. Anal. Appl.*, 35 (2017), 1007–1029.

Received August 2020; revised November 2020.

E-mail address: chenguanggan@hotmail.com
E-mail address: liqin.liqin@foxmail.com
E-mail address: yunyunwei@sina.com