Fermionic superfluidity: from cold atoms to neutron stars

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Abstract

From flow without dissipation of energy to the formation of vortices when placed within a rotating container, the superfluid state of matter has proven to be a very interesting physical phenomenon. Here we present the key mechanisms behind superfluidity in fermionic systems and apply our understanding to one of the most exotic systems in the universe: the superfluid interior of a neutron star. The extreme conditions of neutron stars prevent us from directly probing the internal superfluid properties, however, we can experimentally realize conditions resembling the interior through the use of cold atoms prepared in a laboratory and simulated on a computer. Key insights can be gained by simulating the neutron star superfluid using another system with analogous properties: a cold atomic Fermi gas. Computational physicists are leveraging the power of supercomputers to simulate interacting atomic systems with unprecedented accuracy. In this paper we provide a pedagogical introduction to the physics, guiding the reader through the major conceptual steps to understand the relation between cold atoms, superfluids, and neutron stars. We stress the surprising similarity between these systems, which stems from universality, a fundamental notion in many-body physics. These topics are available in advanced textbooks, but introductory materials are harder to come by; this paper is intended to fill the gap for curious undergraduate and graduate students. We will show how cold atoms can help us make significant strides towards understanding the exotic physics found deep within the universe.

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Supplementary material for this article is available online

(Some figures may appear in colour only in the online journal)

1. Introduction

Moving from the ‘comfort zone’ of classical mechanics to the more mysterious realm of quantum physics opens the possibility to explore the behavior of nature at a deeper and very fascinating level. The transition from classical to quantum mechanics prompts the investigation of microscopic systems: we know that the motion of matter at the length scale of nuclei and electrons is governed by the Schrödinger equation. Nevertheless, in some important cases, quantum effects do not remain hidden in the realm of microscopic systems but manifest in macroscopic systems, giving rise to plenty of exciting and puzzling physical behaviors that challenge our intuition and understanding. In fact, the subtle interplay between quantum mechanics, quantum statistics, and interatomic forces gives rise to a number of fascinating phenomena.

A crucial example of such a phenomenon is superfluidity, closely related to the perhaps more famous superconductivity. Superfluidity is a unique state of matter characterized by the ability of a system to flow without friction. The superfluid state of matter displays other unique properties like, for example, the formation of quantum vortices when the system is enclosed in a rotating container. Although historically superfluidity was first observed in bosonic systems, namely in $^4$He samples at temperatures below the ‘lambda-temperature’, $T = 2.2$ K, (please see, e.g. [1, 2] and references therein) we now know that the phenomenon can also be observed in fermionic systems. To our knowledge, superfluidity, and in particular fermionic superfluidity, is described only in very advanced books and papers (e.g. [3, 4]), which makes it hard for a student to have access to this beautiful chapter in physics. Although the phenomenon appears, or is expected to appear, in some of the most complicated systems in nature, like quantum liquids, the interiors of neutron stars, and in superconductors, the underlying physics can be captured by a few simple and fundamental ‘ingredients’.

The purpose of this work is to present the key physical mechanisms underlying superfluidity in a simple but precise way, so that students or teachers can comfortably make the connection between basic quantum mechanical problems, like free particles, harmonic oscillators, and simple atoms, to more mysterious and fascinating systems such as the neutron superfluid deep inside a neutron star. We will shed light into the connection between two apparently disconnected fields of physics, namely atomic physics and nuclear astrophysics. The term ‘universality’ frequently emerges in physics, and in this context it may be the key to ‘reproduce’ the conditions that exist deep within a neutron star in a laboratory on earth. In a few words, if we put together a collection of fermions, such as $^6$Li or $^{40}$K atoms cooled down to temperatures on the order of nano-kelvins in a laboratory or neutrons in a star, and we have an attractive force acting among them, then we have all the necessary conditions to observe a Fermi superfluid. The behavior of this superfluid will be universal—largely independent from the properties of the microscopic constituents. We will discuss the foundations and the evidence that strongly suggests that such a system indeed exists inside a neutron star.

This paper is organized as follows: we start with an introduction to Fermi gases (sections 2.1 and 2.2), beginning with a description of non-interacting systems. We then introduce superfluidity by moving to attractive systems and we discuss the crucial notion of ‘universality’, which allows us to draw the connection between an ultracold interacting Fermi gas made of
Lithium or Potassium atoms (section 3) and the proposed superfluid deep within a neutron star (sections 4 and 5). We then present the crucial concept of the existence of a ‘macroscopic’ wave function of a superfluid (section 6) together with its spectacular consequence of the formation of vortices (section 7) and the role of such vortices in the behavior of a neutron star (section 7). Further discussion about the implications of superfluidity in measurable properties of the stars is presented in section 8, before we draw our conclusions in section 9.

This work is complemented by a set of exercises that can be found in the supplementary material (https://stacks.iop.org/EJP/43/065801/mmedia) and that will be referred to throughout the paper. We recommend the reader think about these problems while following the discussion. We invite a reader who is interested in diving more deeply into some of the topics we discuss in this paper to read the excellent book by Tilley and Tilley on superconductivity and superfluidity [5].

2. Fermi gases

Three ‘ingredients’ play a crucial role in fermionic superfluidity: quantum mechanics, Fermi statistics and an attractive force among the fermions. We will now examine these in some detail and we will discuss the emergence of superfluidity from the interplay among the three.

From fundamental quantum mechanics, we know how to begin building the mathematical description of the motion of \(N\) spin-1/2 fermions (for example electrons or neutrons) of mass \(m\). We ultimately want to find the wave function of the system, \(\Psi (r_1\sigma_1, \ldots, r_N\sigma_N, t)\), where \(r_i \in \mathbb{R}^3\) is the position variable, \(\sigma_i = \uparrow, \downarrow\) is the spin variable and \(t\) is time. The wave function satisfies Schrödinger’s equation:

\[
i\hbar \frac{\partial}{\partial t} \Psi = \hat{H} \Psi,\tag{1}\]

where \(\hat{H}\) is the Hamiltonian operator of the system:

\[
\hat{H} = \hat{K} + \hat{V},
\]

\[
\hat{K} = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \nabla_i^2,
\]

\[
\hat{V} = \sum_{i<j=1}^{N} v(\hat{r}_i - \hat{r}_j),\tag{2}\]

where \(v(\hat{r}_i - \hat{r}_j)\) is the interaction potential energy of particles \(i\) and \(j\). It is possible to add an external field in which our fermions move, like the electric field created by the nuclei in a solid, but we will not need these extensions for the purpose of this paper. For a more detailed discussion of this topic, please see [6]. Since we are dealing with fermions, the wave function \(\Psi\) is antisymmetric with respect to an exchange of particle labels, that is:

\[
\Psi(\ldots r_i\sigma_i, \ldots, r_j\sigma_j, \ldots, t) = -\Psi(\ldots r_j\sigma_j, \ldots, r_i\sigma_i, \ldots, t).\tag{3}\]

We invite the reader to note that, from (3), it follows that we can never have two fermions with the same spin orientation occupying the same position.

To understand this point better, imagine we have a source preparing two quantum particles, say two neutrons, with the same spin orientation \(\sigma\), for example \(\sigma = \uparrow\). For simplicity, we assume the particles move in one dimension. The wave function of the system, \(\Psi(x_1, x_2)\), where
we omit the spin variables to keep the notation simple, yields the probability density for the positions of the two particles,

$$|\Psi(x_1, x_2)|^2 \, dx_1 \, dx_2. \tag{4}$$

In other words, it is the probability to observe one neutron between $x_1$ and $x_1 + dx_1$ and one neutron between $x_2$ and $x_2 + dx_2$. A fundamental principle in quantum statistics is that identical particles are indistinguishable: it is impossible to design an experiment which allows us to label the particles. This implies that the probability density must remain invariant if we swap the variables,

$$|\Psi(x_1, x_2)|^2 = |\Psi(x_2, x_1)|^2. \tag{5}$$

When the particles are fermions, in particular, the above condition is realized by antisymmetric wave functions:

$$\Psi(x_1, x_2) = -\Psi(x_2, x_1). \tag{6}$$

An example of an antisymmetric wave function is illustrated in figure 1. An immediate consequence is that the wave function identically vanishes whenever $x_1 = x_2$, the dotted line in the figure, which implies that two fermions with the same spin orientation cannot be in the same position. More generally, two fermions cannot be in the same quantum state. We invite the reader to note that in the framework of old quantum theory, this is known as the Pauli principle.

2.1. The Fermi sea

Whenever the interactions are negligible and thus the Hamiltonian has only the kinetic term $\hat{H} = \hat{K}$, it is relatively simple to solve Schrödinger’s equation. In particular, we can explicitly find the lowest energy stationary state of the system, called the ground state of the system. Mathematically, the ground state is the eigenvector of the Hamiltonian corresponding to the minimum eigenvalue. Physically, the ground state wave function is the equilibrium state of the system when the temperature is $T = 0$ K and it is frequently called the Fermi sea. In this section we will describe the Fermi sea, focusing on the physical interpretation.

As is traditionally done in condensed matter textbooks, we would like to take a moment to address the wave nature of particles. In the quantum regime, each fermion has a wave function to describe its motion. In the Fermi sea, the free fermions’ wave functions take the form of plane waves. A plane wave has the mathematical expression $\varphi_k(r) \propto \exp(ik \cdot r)$ and it describes a fermion with a well defined momentum $p = \hbar k$ ($k$ is typically called the wave vector) and a well defined kinetic energy $\varepsilon(k) = \frac{\hbar^2 k^2}{2m} = \frac{p^2}{2m}$. Incidentally, we invite the reader to think of Heisenberg’s principle: as the momentum $p$ can be measured with arbitrary precision when a fermion is in a plane wave state, the particle is completely delocalized: in fact, the probability density for the position of the fermion $|\varphi_k(r)|^2$ is independent from its position. Due to the antisymmetry of the wave function, each plane wave can accommodate up to two fermions, one with spin up ($\uparrow$) and one with spin down ($\downarrow$) [7]. We now set $\frac{\hbar^2}{2m} = 1$ for the rest of this discussion to focus on the concepts at hand.

In simple words, each fermion has a well defined value of the momentum, or equivalently of the velocity. In a classical context, at $T = 0$ K, all the particles would be ‘frozen’, and all the velocities would be equal to zero. On the other hand, because of the antisymmetry, only two particles, one with spin up and one with spin down, are allowed to have zero velocity. Therefore, all of the other fermions must ‘move’. From the mathematical point of view, it is
Figure 1. Illustration of fermion wave function antisymmetry.

convenient to enclose our fermions in a cubic box of volume $V$ and to fix the number of particles to $N$. We choose simple periodic boundary conditions on the walls of the box: the idea is that the box is not a physical container, but it is a region inside a huge Fermi gas, where we are floating, and periodic boundary conditions allow us to mimic an infinite system made of an infinite number of replicas of our box. The volume $V$ induces a discretization in the momenta, as only plane waves whose wavelengths fit the box are allowed. This is analogous to a vibrating string with the two ends fixed: not all vibration modes are allowed.

In the minimum energy state, the plane wave states will be filled starting from the lowest kinetic energy at $k = 0$ and continuing until all $N$ fermions have been placed. All the plane waves corresponding to higher kinetic energies remain unoccupied. The Fermi sea in one and two dimensions can be illustrated as in figures 2 and 3, where we show the fermions ‘sitting’ on the available momenta. For each momentum, we will have one fermion with spin up (↑) and one with spin down (↓). It is useful to show such available momenta together with the related kinetic energies, often denoted as $\omega$: these plots, shown in figures 2 and 3 for one and two dimensional Fermi gases respectively, are known as dispersion curves. We observe that the dispersion relations sketched are very similar to the dispersion relations of a particle in a potential box; the two become identical in the limit of infinite volume. Geometrically, the set of occupied momenta fills a sphere in three dimensions and is visualized in figure 4, where the fermions are represented schematically as small arrows. This sphere is called the Fermi sphere and it fully characterizes the state of the fermions in equilibrium at $T = 0$ K. The radius of the Fermi sphere is given by
Figure 2. Illustration of a one-dimensional dispersion relation, showing also the fermions ‘sitting’ on the available momenta. The green dashed lines are occupied energy levels and the red circles highlight the Fermi surface (in this case, two momenta).

\[ k_F = \left( \frac{3\pi^2 N}{V} \right)^{\frac{1}{3}}, \]  

and is called the Fermi momentum of the system. The surface of this sphere, which separates the occupied momenta from the non-occupied momenta, is called the Fermi surface. Associated with the Fermi momentum we have the Fermi energy, which is the maximum kinetic energy occupied by a fermion at \( T = 0 \, \text{K} \), defined as

\[ E_F = k_F^2. \]  

Thus, fermions are considered to be within the Fermi sphere if

\[ k^2 = k_x^2 + k_y^2 + k_z^2 < E_F \]  

is satisfied. Intuitively, when we observe the system in momentum space, the distribution of fermions in the Fermi sea resembles a parking lot, which is completely filled up to the Fermi surface, and empty outside the Fermi sphere. In this language, the antisymmetry of the fermionic wave function simply states that two cars cannot share the same parking spot. One spot corresponds to a single pair of numbers: the particular value of \( k \) along with one of the two possible spin orientation values.

A consequence of this organization is the concept of degenerate Fermi pressure. The amount of ‘space’ is limited in the parking lot, and once full, we can no longer accommodate new arriving cars. The system thus exerts a ‘pressure’ on any newcomers trying to squeeze themselves in, barring any entrance into the Fermi sphere. Similarly, the system will resist any external forces trying to compact the Fermi surface: the parking lot size is fixed and unchanging. This
Figure 3. In two dimensions, the dispersion relation takes the form of a paraboloid. The red circle represents the Fermi surface, the green represents occupied energy levels, and the black represents unoccupied energy levels.

feature of Fermi gases can be found in many different types of systems. For example, in the realm of solid state physics [8], it is well known that the simple model of a non-interacting Fermi gas allows us to capture, at least at the qualitative level, the physical mechanisms underlying several important phenomena in metals, for example electrical conductivity. In this work we aim to focus on ultracold atoms (see e.g. [9–11]) and on the gas of neutrons within the core of a neutron star (see e.g. [12–14]).

2.2. How cold is a Fermi sea?

It is very common in physics to ‘transform’ energies into temperatures. In our context, the Fermi temperature of a system is defined as $T_F = \frac{E_F}{k_B}$, where $k_B$ is the usual Boltzmann’s constant. We described above the minimum energy state of the system, that is, the state that the system chooses when thermodynamic equilibrium is established at $T = 0$ K. However, it can be shown that as long as the physical temperature remains much lower than the Fermi temperature of the system, $T \ll T_F$, the temperature dependence of the macroscopic properties of the system, for example pressure and energy, is negligible and the system can be described as if it were at $T = 0$ K. More precisely, the temperature-dependence of all the thermodynamic quantities is (at least) $O(T/T_F)^2$, and thus negligible for $T \ll T_F$. The reader may refer to problem (1) in our supplemental material for some insight on thermal effects in Fermi gases. So, while classical systems immediately start to move as we heat them up starting from $T = 0$ K, fermionic systems will just ‘ignore’ the change in temperature until the temperature approaches the order of the Fermi temperature. Further discussion can be found in any book about statistical mechanics, for example [15]. Applying this concept can be very counter-intuitive when looking at ‘natural’ Fermi gases, such as neutron stars, since we typically consider them to be ‘very hot’. 
Figure 4. Illustration of a three-dimensional Fermi sphere. The Fermi surface is the surface of the sphere.

When describing the interior temperatures of neutron stars [16], which are on the order of $10^8$ K, we observe them to be well below their Fermi temperatures, $T_F \approx 10^{12}$ K. This huge value is due to the density of the star: in fact, from (7), we see that the Fermi energy, and thus the Fermi temperature, depends on the density of the particles. The reader may also refer to problem (2) for an order of magnitude estimation of Fermi temperatures for three different systems. Therefore, we can consider the interiors of these stars as ‘very cold’ even if their actual temperature is hundreds of millions of Kelvins!

In the laboratory, it is now possible to artificially generate ultracold Fermi gases [9–11], also referred to as fermionic cold atoms, typically consisting of $^6$Li or $^{40}$K atoms. These gases can reach temperatures as low as of $T \approx 10^{-9}$ K, which can be achieved via laser and magnetic trapping and cooling techniques (see e.g. the very nice reviews [17, 18]). Experimental methods now allow us to generate clouds of atoms which behave as mixtures of two ‘spin species’, typically referred to as ‘up’ and ‘down’, that are indeed Fermi gases. More precisely, the two spin species are two lowest energy hyperfine states [11] of $^6$Li or $^{40}$K atoms embedded in an external magnetic field that can be controlled by the experimentalist. These gases are far less dense than a neutron star (and indeed far less dense than ordinary matter) and have a Fermi temperature around $T_F \approx 10^{-8}$ K [9]. Since $T \ll T_F$, these systems can also be considered as ‘very cold’.
3. Fermi superfluids

We stress that the Fermi sea as we describe it above cannot be a superfluid: there is no way our collection of fermions, 'piled-up' like cars in a full parking lot, can flow without friction, which is the defining property of a superfluid. In the case of the Fermi sea, energy exchange is readily available: injecting energy into the Fermi sea simply promotes one or more fermions to some unoccupied momenta above the Fermi surface. In order to have a flow without friction, we need to have a mechanism that prevents the energy exchange between the system and the walls of the container in which the system is flowing. We need to dig deeper by introducing interactions.

The model of a non-interacting Fermi gas is extremely useful, and it helps us shed light into the microscopic and macroscopic behavior of plenty of physical systems. But the picture becomes even more intriguing if we allow the fermions to interact at low temperatures, in particular when the interaction is attractive. The physical origin of the attraction depends on the details of the physical system. For example, we know that the gas of electrons in metals experiences an attraction as a consequence of their interaction with the vibrating lattice of the positive charges (the nuclei of the metal atoms) [19]. While this attraction is normally negligible with respect to the Coulomb repulsion among the electrons, at very low temperature it becomes important, and it results in the phenomenon of superconductivity: electric current is able to flow without resistance. Superconductivity is the direct analogue to superfluidity, they are two sides of the same coin. In cold atoms, the attraction is due to the effective interaction between two $^6\text{Li}$ and $^{40}\text{K}$ atoms in the lowest hyperfine states [20]. Such states arise from the coupling between the total angular momentum—orbital and spin—of the valence electron of the alkali elements and the nuclear spin, and are sensitive to the presence of an external magnetic field [11]. As a result, the force between two atoms can be tuned by changing the external magnetic field, which is a unique feature of cold atoms. In fact, while there is no way to change the electron–electron interaction in a material or the nuclear interaction, cold atoms allow us to explore different interaction strengths, making them a fascinating and unprecedented many-body laboratory [21]. The details of the tuning are a bit beyond the scope of this paper, and it involves the subtle physics of the Feshbach resonances, which is crucial for the formation of molecules in the gas, as we will discuss below. We refer the interested reader to [20]. Finally, it is key to note that in a system of interacting neutrons, the nuclear force is known to be attractive in a given distance range (see e.g. [22, 23]).

The detailed form of the interatomic potential from (2), $v(r_i - r_j)$, will naturally be specific to the physical system that we are investigating. In general, though, it will always have a range $r_0$ such that $v(r_i - r_j) \simeq 0$ if $|r_i - r_j| > r_0$: two particles need to be close enough to feel the attractive force. In most situations involving Fermi gases, it happens that the average distance between two fermions, which can be defined as $d = \left( \frac{V}{\pi} \right)^{1/3}$, is much larger than the range $r_0$. This implies that the ‘fine’ details of the interatomic force can be safely neglected and so we say that the systems are ‘dilute’. This means that the very complicated, and partly unknown, details of the force between two neutrons can be neglected, and the effect of the interaction can be taken into account by focusing on the relative motion of two neutrons when they are very far from each other. The possibility to ignore most of the details of the interatomic forces allows us to expect that dilute gases are ‘universal’. This allows us to look for similarities between cold atoms and the fluid of neutrons within a neutron star. In fact, both systems can be considered to be cold, as discussed previously, and also to be dilute.

In the inner crust of a neutron star, for example, the average distance among neutrons [16] is much bigger than the range of nuclear forces, which is comparable to the size of nuclei, of the order of $10^{-15}$ m [22]. This is also the typical situation in (most) experiments in cold atoms,
which can be engineered to be dilute ‘by design’. For a more in-depth discussion of the range of interactions in cold atoms, we refer the reader to [20, 21]. Scattering theory [24] dramatically simplifies the description of this ‘dilute’ regime: at low energy, in fact, only very few physical quantities are enough to capture the effect of the interatomic forces. In the simplest case, these parameters are the density of the gas itself, or, equivalently, the Fermi momentum $k_F$, and the scattering length $a$ [9, 24, 25], which characterizes the strength of the interactions. As a result, the properties of ultracold, dilute Fermi gases made of lithium or potassium atoms and the fluid of neutrons inside neutron stars, are ‘universal’ in the sense that they depend only on the product $k_F a$ [25].

The interesting question is: what happens to a Fermi gas when we switch on an attractive interatomic interactions? This is where the crucial notion of fermionic pairing enters the game: the attractive forces may induce the fermions to organize in pairs, typically called Cooper pairs [19]. In an ordinary superconductor, two electrons will form a singlet pair at very low temperature [19]. In an ultracold gas, two atoms in different hyperfine states will form a pair (the formation of pairs in the same hyperfine state, say a ‘triplet pair’, is normally strongly suppressed by the antisymmetry of the many-body wave function) [20]. Similarly, two neutrons in the outer crust of a neutron star are expected to form a singlet pair, as we will discuss below [12]. In this context, using the language of the excellent review by Ketterle and Zwierlein [11], ultracold atoms are a ‘real gift from nature’ as they allow us to deeply explore this pairing phenomenon through our control of the strength of the interaction [20]. This leads to the fascinating physics of the Bose–Einstein condensation (BEC)-Bardeen–Cooper–Schrieffer (BCS) crossover, that we will briefly introduce here. A reader interested in a more detailed presentation can see, for example, the two excellent reviews [11, 21]. As we discussed above, if no interaction is present, the particles are organized in momentum space, with all plane waves states occupied up to the Fermi energy. If the attractive interaction is weak, one fermion, say spin-up, with momentum $k$ will find a ‘partner’ with spin-down with momentum $-k$, where the fact that the pair has a vanishing center-of-mass momentum derives simply from the energy minimization in the ground state. This mechanism is the backbone of the celebrated BCS [19] theory which explained ordinary superconductivity for the first time in 1956. It is worth observing that, in some sense, the phenomenon ‘takes place’ in momentum space, while the real space picture still closely resembles the behavior of the non-interacting Fermi gas. In fact, in this regime it turns out that the average distance between the paired fermions is much larger than the typical interatomic distance [11, 19]: one may imagine a many-body picture where several atoms (or electrons) are moving between the two paired ones. Despite this, the formation of pairs has a deep effect on the mechanism through which the system can exchange energy with the environment, which is the root of superfluidity, as we will discuss in more detail later on. Before dwelling on superfluidity, we find it useful to discuss what happens when the interactions become strong, which can be readily realized in ultracold atoms [20]: as we increase the strength of the interaction, the Cooper pairs become tightly bound, gradually assuming the shape of molecules [11]. The average distance becomes much smaller than the interatomic distance and the picture in real space is now dramatically different: a gas of fermionic atoms ‘becomes’ a gas of bosonic molecules. In fact, composite states of two spin 1/2 fermions are bosons. In addition, the spectacular phenomenology related to Bose statistics enters the game. In particular, BEC takes place, leading a macroscopic number of pairs to populate a single quantum state [6]. This is usually called the BEC regime [11], and the smooth change from the many-body BCS regime and this molecular BEC regime is called the BEC-BCS crossover [11]. Incidentally, we comment that, in three-dimensional gases, as we move across the crossover, we ‘meet’ a particular situation when the scattering length $a$ does diverge: this is called unitarity [9]. With a divergent scattering length, the details of the interatomic interaction are entirely
irrelevant and the system has only one significant length scale related to the Fermi momentum: this is the regime where cold gases are most universal. Interested readers can explore [9].

In both the BCS and BEC regimes, the system behaves in a coherent way [6]: in very simple words, a huge number of pairs are doing exactly the same thing! In more precise terms, we have an overpopulation of a quantum state that manifests at the macroscopic level. Particles, or more precisely pairs, lose their individual identities as their wave functions overlap and begin to act in unison. This paves the way for superfluid behavior. The key point is that, in this regime, there is a gap in the energies available for the particles to populate. In other words, there is a ‘forbidden region’ which imposes severe constraints on the energy exchange the system can have with its environment. The ground state of the system is now ‘protected’ and can flow without dissipation. This has spectacular consequences like the flow of persistent currents in superconductors [5], the non-viscous flow in cold atoms, as well as the formation of superfluid vortices [26]. When the interaction is strong and the particles are organized in tightly bound molecules, this gap possesses the simple interpretation of the energy needed to break a pair of fermions and it is usually called the superfluid gap.

When the superfluid is on earth, the gap can be measured experimentally with spectroscopy experiments (please see the very good book [27] for an in-depth explanation of such experiments and the theory of them). Such experiments measure the so called spectral function $A(k, \omega)$, yielding a map of the quantum states that are available for the particles in the system [27, 28]. The same function can be also estimated using sophisticated theoretical and computational techniques which can include mean field theories, perturbative expansions, field-theoretical methods, or quantum Monte Carlo methods [29]. An example is shown in figure 5, where we present results for a three dimensional Fermi gas with attractive interactions at unitarity in order to help the reader grasp the intuitive meaning of the spectral function. The color plot has been obtained using quantum Monte Carlo methods by ourselves, using the approach extensively discussed in [29, 30]. The Fermi surface is by construction at energy $\omega = 0$. More precisely, the energy scale is shifted by the chemical potential corresponding to the given particle density. For $\omega < 0$, the spectral function tells us the probability to find a particle in a plane-wave state with a certain momentum $k$ and a corresponding energy $\omega$ (here, in units of the Fermi momentum and Fermi energy). The broadening which is evident in the color plot arises from the fact that the momentum of one particle is not a good quantum number for an interacting system. These states can be probed experimentally by resolving the energy and momentum of a particle that is ejected from the system due to an interaction with an external probe [27]. We invite the reader to reflect on the fact that this is the same mechanism that takes place in a solar panel. In the region $\omega > 0$, the spectral function shows us the available plane-wave states for a new particle that we may inject into the system via a spectroscopy experiment. The darkest regions correspond to areas of the energy–momentum plane where there are many available states. Figure 5 clearly shows the gap as the region around $\omega = 0$ where no available states are present. Injected particles can populate the states above the gap, though such particles must ‘pay the entrance fee’ of at least $2\Delta$. The observation of a finite gap is a pointer to the existence of the superfluid state of the system: as we already explained, the gap acts as a ‘protector’, in the sense that it regulates the possible energy exchanges.

We find it useful to comment that, in scientific papers, it is very common to see figures of dispersion curves, which correspond to the maxima of the spectral functions in the momentum–energy plane. Whenever the spectral function is peaked around such lines, which happens mostly in the BCS regime, these curves are a realistic representation of the spectral functions itself, and thus provide the theoretical foundation of the notion of a quasi-particle with a given momentum and energy, which is a key concept in many-body theory. In simple words, a point
on such curves, say in the positive energy region, tell us that a quasi-particle state is available in
the system and such state can be excited in a spectroscopic experiment. It is useful to mention
that, in the realm of Fermi superfluids, BCS theory [19] allows us to provide a precise definition
of a quasi-particle, which turns out to be a quantum ‘mixture’ of a particle and a hole. A reader
interested in deepening the understanding of quasi-particles in the broader field of many-body
physics may see, e.g., the classic book [31]. In figure 6 we plot the dispersion relation for a
weakly attractive Fermi gas in the BCS regime, together with the dispersion curve for a non-
interacting Fermi gas. In both cases the zero energy is set as the Fermi energy (or chemical
potential for the interacting system), such that, as in figure 5, negative $\omega$ corresponds to occu-
pied states and positive $\omega$ corresponds to unoccupied states. In the non-interacting case, no
energy is expended for a particle to move into a higher energy state. However, if we juxtapose
the non-interacting dispersion curve with the interacting dispersion curve, we can clearly see
the presence of a gap (highlighted) around zero energy. In the interacting case, the dispersion
curves follow the approximate relationship [19]

$$\omega = \pm \sqrt{(k^2 - E_F)^2 + \Delta^2},$$  \hspace{1cm} (10)

where $\Delta$ is the gap. An interacting system has two branches, separated by the energy gap
required to break apart a pair of fermions. As the interaction becomes stronger, the gap
increases, while simultaneously, the minimum of the available states and the maximum of
the populated states ‘shift’ toward $k = 0$ [29], where we can clearly see the superfluid gap of
the system. Further discussion can be found in Rinott et al [32].

Before concluding this section, we reiterate the crucial physical insight: the gas can flow
without dissipation so long as the container is unable to supply an amount of energy at least
equal to the gap. At this point, the reader might wonder if there is a way to access the gap
of the neutron superfluid inside a neutron star. Considering it is no easy task to travel to a
neutron star to setup a spectroscopy experiment, and that the neutron–neutron interaction is
very complicated and partially unknown, is there a way for us to measure such a gap? For this
to be possible, we would need the existence of a physical property of the star that is dependent
on the gap value and that can be measured, thus allowing us to have ‘indirect’ access to the
gap. Very interestingly, as we will discuss below, it is widely believed that such a property
does exist—the cooling curve of a neutron star. This paves the way for an exciting research
question: if we can infer the value of a gap from measurements of a star’s cooling curve, can
we take advantage of the universality that we discussed above and of the tunability of cold
Figure 6. (Left) Dispersion curve for the non-interacting system. (Right) Dispersion curve of an interacting system. The interacting system has a prohibited region where no fermions may occupy energy states known as the superfluid pairing gap, $\Delta$ (highlighted in orange is $2\Delta$ symmetric about 0 energy).

gases to ‘reproduce’ the superfluid within a neutron star in a laboratory on earth? For a deeper discussion, also involving aspects of theoretical approaches, please see [25].

4. The formation of a neutron star

Now that we have presented the key ingredients that allow us to understand the physical mechanisms underlying the spectacular phenomenology of superfluids, we find it a beneficial example for deepening our understanding to discuss one of the most mysterious superfluids that exists in nature: the fluid of neutrons in a neutron star. We will certainly need some context, so we will briefly provide a general description of the birth of neutron stars. Later, we will discuss how the superfluid influences, or is speculated to influence, the behavior of the star through a description of glitches and the cooling process.

Interstellar gas clouds are diffuse and primarily composed of hydrogen molecules and dust (particles made of carbon, silicon, and oxygen atoms). These gaseous conglomerates can span distances from less than a light-year to several hundred light-years. We see the temperature vary inside the interstellar clouds: dust absorbs visible and ultraviolet light, cooling the interior regions. In denser regions, a lack of thermal pressure creates an instability, and a perturbation to the cloud, such as a shock wave from a supernova or nearby colliding galaxies, can induce collapse. Under the influence of gravity, gas will fall to the center of mass. Gravitational potential energy is converted into heat energy under this compression, creating a thermal pressure gradient. If the core temperature reaches $1.5 \times 10^7$ K, nuclear fusion jump-starts and the cloud becomes a ‘protostar’, beginning its journey on the main sequence of stars. At this point, the star’s initial mass determines its future form. If the star is less than eight solar masses, we can expect a compact object about the size of Earth—a white dwarf. If the star is more than 30 solar masses, we can expect a black hole to form in due time, while a star above eight solar
masses has the potential to form a neutron star [16]. We assume this last scenario and proceed with our description of the natural processes that take place to create a neutron star.

Thermonuclear fusion sustains a star as thermal and radiation pressure balance gravity for millions to billions of years. If the mass of the star is big enough, the core will burn through a chain of elements: hydrogen, helium, carbon, neon, oxygen, magnesium, silicon, and iron. As each element is exhausted, the core contracts until the ignition temperature for the next step of the elemental chain is reached [16]. When iron, the most stable nucleus, is burnt, a critical (and quite dramatic) stage is reached: there will not be a new fusion reaction. The thermonuclear energy is not able to compete against gravitational collapse, and the star caves in on itself. The situation inside the core during this critical stage is really unique: the energy is such that electrons will combine with the protons to form neutrons and neutrinos, in a process called inverse $\beta$-decay [33]. The neutrons will organize in a Fermi sea, which, as we discussed earlier, resists the compression of the core with degenerate Fermi pressure, making it ‘stiff’. Infalling material will thus rebound, sending out a shock wave. This wave will stall a few hundred kilometers from the center as its energy is dissipated via neutrino losses and photodisintegration of nuclei. Free-falling material once supported by the core will collide with this stalled shock front to produce another shock wave that ejects all but the stiffened core in a supernova explosion. As a residual of the explosion, a protoneutron star is born [16].

5. The structure of a neutron star and the neutron superfluid

Glancing at a neutron star, one will immediately notice a hydrogen atmosphere. Directly beneath will lie a thermal insulator, known as the envelope, which acts as a barrier between the hot interior and the surface atmosphere. Diving deeper, we expect to see four internal regions of the neutron star that are distinguishable by nucleon densities: the outer crust, the inner crust, the outer core, and the inner core (depicted in figure 7). The outer crust is a few hundred meters below the envelope and is expected to be a nuclear lattice composed of heavy nuclei. The inner crust contains a lattice of neutron rich nuclei in equilibrium with a superfluid of neutrons which are paired in the $^1S_0$ quantum singlet state and a degenerate electron gas [34]. Here the spectroscopic notation $^1S_0$ simply means that the neutrons in the inner crust form pairs made of one spin-up and one spin-down, as discussed in previous sections [7].

As we venture into the outer core, the system becomes more and more exotic: we do not have nuclei any more; instead, we have a fluid of free neutrons, protons, and electrons. It is assumed that both the neutrons and the protons are superfluids. The proton fluid is charged and would be considered a type II superconductor. The neutrons would pair with the same spin orientation (a triplet state), while the protons would form pairs in the more ‘conventional’ $^1S_0$ singlet state. A ‘soup’ of two Fermi superfluids of protons and neutrons is thus expected to form the deepest region of the neutron stars. The interaction between the two components and the extreme density can make the physics very challenging. The inner core is mysterious because its composition and equation of state are unknown to the scientific community, however, there are several propositions for the composition of the core matter: nucleons; pion or kaon condensates; hyperons; and quarks [14, 34, 35]. For the purpose of this paper, we focus on the superfluid of neutrons in the inner crust, which is ‘dilute’ and universal, in the sense discussed in section 3; if we knew the interaction strength, or the superfluid gap, we would be able to ‘observe’ it in a laboratory in the form of a cold atomic Fermi gas. Cold atoms would ‘impersonate’ the neutrons and we would be able to tune the interaction strength to reproduce the physics inside the mysterious star.
Figure 7. Structure of a neutron star. The layers and their respective thicknesses are: the atmosphere 1 cm (a), and the outer crust (0.3–0.5) km (b), the inner crust (1–2) km (c), where the superfluid transition occurs, the outer core (2–3) km (d), and the inner core (2–3) km (e). The densities are $4 \times 10^{11}$ g cm$^{-3}$ between (b) and (c), $0.5 \rho_0$ between (c) and (d), $2\rho_0$ between (d) and (e), and $(3–9)\rho_0$ at (e). $\rho_0 = 2.8 \times 10^{14}$ g cm$^{-3}$ is the normal nuclear density, the mass density of heavy atomic nuclei [36].

6. The wavefunction of the coherent state

The superfluid state is a condensate and thus it is enough to solve for the wave function of a single boson. A percentage of the particles will be in a higher energy state when the system has a finite temperature, but for the sake of simplicity we ignore these particles if the temperature is sufficiently low. In order to keep the math as simple as possible, we will disregard the internal structure of a pair of fermions and will treat the pair just like a structureless boson. Therefore, the wave function governing the macroscopic motion of the superfluid will have the general form

$$\Psi(r) = R(r)e^{i\Theta(r)}, \quad (11)$$

where $\rho(r) = R(r)^2$ is the probability density for the position of the bosons, while the phase $\Theta(r)$ controls the current density, that is, the local velocity of the particles. Note that (11) has no dependence on time. In general, the wave function is a function of time, however, here we have assumed the system has already relaxed to equilibrium. This equilibrium function is also known as a steady state. Once the steady state is reached, the state will not change, unless the system is somehow perturbed and driven out of equilibrium. The wave function (11) entirely governs the motion of a superfluid, and in particular, it allows us to understand the formation of quantum vortices. This is one of the crucial features of superfluids and is the main reason why it is widely believed that neutrons deep in a neutron star form a superfluid.
7. Vortices and glitches

We now provide a short mathematical discussion of the superfluid vortices inside the rotating crust of a neutron star. A general expression for the velocity field of the condensate in equilibrium, given the wave function (11), is

\[ \mathbf{v}(r) = \frac{\hbar}{m} \nabla \Theta(r) \]  

(12)

(see problems (3) and (4)). A special case is for a steady superfluid (when (11) is a constant): as long as the rotation is slow, the superfluid will remain steady, since the container is not able to drag the nonviscous fluid. The situation changes when the rotation becomes fast. The wave function that describes the motion of the system will correspond to the minimum energy when studied from a reference frame that is rotating with the container. Imagine, for simplicity, that the container is a cylinder that is rotating around its axis, which we chose to be the \( z \) axis of our reference frame. It is natural, simply by transforming the energy into a rotating reference frame, to expect that the wave function \( \psi(r) \) will be an eigenvector of the component of the angular momentum along \( z, \hat{L}_z \), which is a conserved quantity, as we are assuming the interaction only depends on the inter-particle distance and that the rotation is about the \( z \) axis. This implies that \( \Theta(r) = m \varphi \), where \( m \) is an integer number, while \( \varphi \) is the cylindrical angular coordinate of the position \( r = (x, y, z) \): \( \varphi = \arcsin(y/r) \), with \( r = \sqrt{x^2 + y^2} \) being the distance from the axis. The relation (12) leads us to the interesting result

\[ \mathbf{v}(r) \propto \frac{1}{r} \hat{e}_\varphi, \]  

(13)

where \( \hat{e}_\varphi \) is the unit vector along \( \varphi \) (see problem (5)). This velocity field describes a vortex, as seen in figure 8, where we show a top-view stream-plot of the vector field (13). So, the fact that a superfluid is a coherent state of a macroscopic number of pairs described by a single-pair quantum wave function naturally allows us to conclude that, within a rotating container (provided that the rotation is fast enough), the superfluid will form vortices. From thermodynamic arguments it may be seen that the higher the speed or rotation, the greater is the number of vortices [6, 37–40]. Please see problems (6) and (7) for a derivation of this concept.

We comment that such vortices in rotating superfluids have been indeed observed in cold atomic systems, both bosonic and fermionic. In such experiments, the superfluid is embedded in a rotating perturbation designed by revolving laser beams around the gas [41], mimicking a rotating container: the vortices can then be probed using resonant absorption imaging [42], which consists of observing the shadow imprinted by the superfluid on a low-intensity probe beam of laser light. In the paper [41] spectacular images of a very regular lattice of up to 130 vortices in a system of ‘rotating’ Na (bosonic) atoms can be found. A conceptually very similar experiment on ‘rotating’ Fermi superfluids made of the lowest hyperfine states of \(^{6}\)Li atoms is described in [26]: regular arrays of long-lived vortices were observed both in the BEC and in the BCS regime. Also, in [43], the fascinating case of fermionic superfluids with imbalanced populations in the two hyperfine states, mimicking magnetic superconductors, was explored and vortices were observed. Such spectacular experiments confirm that the formation of vortices in rotating containers is a unique manifestation of superfluids, and can be used as a probe to detect superfluidity in a system. In addition, these experiments, through the observation of the crystal structure of the vortex lattices, together with irregularities such as dislocations or grain boundaries [41], shed light into the possibility of studying vortex matter in complex systems like in a neutron star.
Let us now discuss the evidence for the existence of a neutron superfluid inside a neutron star: the phenomenon of glitches. A detailed description of the origin of neutron star glitches would certainly require a highly sophisticated presentation, but we can capture the essence of the phenomenon with a few simple ingredients: a fluid of fermions (the neutrons) which attract each other, form pairs, and as the star is rotating, form vortices. If the star is rotating (rotating neutron stars are typically called pulsars), the neutron superfluid inside a neutron star is expected to host an array of vortices. The fact that the superfluid coexists with a lattice of nuclei makes it possible for the vortices to be pinned to the lattice sites, similar to vortices in rivers that are pinned to rocks on the riverbed. The rotation of the star gradually slows down as the star emits energy. As this happens, the vortices will feel a force, known as a Magnus force, that pulls them towards the surface of the star. This force can break the pinning, and the vortex may hit the surface. As this happens, a glitch takes place—the star spins faster for a short time—and observations of these glitches give indirect evidence of a superfluid core [34, 44–47]. It is quite amazing to see how the simple physical mechanisms discussed in this paper allow us to give a possible explanation to a phenomenon that happens in the deep universe. We comment that other models have been proposed to explain the glitches, for example star-quakes, but to our knowledge the role of the superfluid vortices and their unpinning is widely accepted in the astrophysics community as the leading mechanism.

8. Cooling of neutron stars

While the glitches are certainly the most spectacular manifestation of the superfluid existing deep inside a neutron stars, the superfluid neutrons are expected to also affect other properties
of the star, in particular, the cooling process. A detailed description can be found in [12], here we give a brief and non-exhaustive description in order to allow the reader to understand the role played by the superfluid.

Neutron stars undergo three main cooling stages. In stage one, the crust is thermally decoupled from the interior such that the surface temperature of the star reflects the thermal state of the crust. In the early life of a neutron star, 10–100 years, cooling primarily happens via neutrino emission. During stage two, the surface temperature of the crust will adjust to the internal temperature, such that the core and crust reach thermal equilibrium. Within the period of 10–100 years \(< t < 10^5–10^6\) years, neutrino emission continues to dominate. Stage three refers to a mature neutron star experiencing photon emission via the transport of heat from the core to its surface. From \(10^5–10^6\) years and onward, the evolution of the core temperature is thus governed by this radiation [48].

Cooling curves depict the dependence of the surface temperature on the age of the star. They depend strongly on the properties of matter, such as the local density within the core of the neutron star. In particular, they are sensitive to the existence of a superfluid. For a non-superfluid core, the transition from stage one to stage two consists of a sharp drop off in temperature due to the direct Urca process [49–51]. A direct Urca process consists of a pair of reactions:

\[ n \rightarrow p + e^- + \nu_e, \]
\[ p + e^- \rightarrow n + \nu_e, \]

where \(n\) is a neutron, \(p\) is a proton, \(e^-\) is an electron, and \(\nu_e\) is an electron neutrino. The emitted neutrinos are responsible for the cooling of the star because they transport energy away from the star. These reactions are possible given the right condition of density, \(\rho \geq 4.62\rho_{\text{nuclear}}\), where \(\rho_{\text{nuclear}} = 2.3 \times 10^{17} \text{ kg m}^{-3}\), which is possible for stars whose mass is \(M_{\text{NS}} \geq 1.35M_\odot\), where \(M_\odot\) is the mass of the Sun. The resource of proton concentration must be sufficient (a ratio of the number of protons to the number of nucleons needs to be approximately 0.1) [51].

In contrast, superfluidity suppresses the neutrino luminosity, slowing down the cooling in stage two [52]. This is due to the fact that nucleons must be excited above the pairing gap (discussed in section 3) inherent in a superfluid to participate in the direct Urca process. Interestingly, an accurate analysis of the cooling curve may yield an estimate of the superfluid gap, which, in turn, may give us information about the Fermi superfluid, namely the interaction strength. This information, thanks to the universality of Fermi superfluids, could allow us to mimic the neutron superfluid using cold atoms.

9. Summary and conclusions

Using quantum mechanics we can describe the nature of Fermi gases, which are a useful resource we can utilize to study unique states of matter. Fermions are governed by quantum statistics which determines the organization of the Fermi sea: fermions will occupy all energy states up to the density dependent Fermi energy. Looking at a non-interacting Fermi gas in momentum space, we see that if any amount of energy is introduced to the system, fermions can easily use it to occupy states above the Fermi energy. Fermi gases have internal temperature scales dependent upon the Fermi temperature. So long as the Fermi gas’s temperature is well below the Fermi temperature, the system’s behavior can be approximated as if it were at absolute zero and is considered ‘cold’. Introducing an attractive interaction between the fermions will result in pairs of fermions forming a condensate. These integer spin pairs no longer obey Fermi statistics and ‘collapse’ into the single-pair ground state, leading to a macroscopic manifestation of a quantum phenomenon known as superfluidity, in which the pairs of fermions...
behave coherently. In contrast to the non-interacting cold Fermi gas, an energy barrier, known as the superfluid pairing gap, is formed to protect the superfluid state, resulting in nonviscous flow. The energy states above the Fermi energy become prohibited unless energy greater than or equal to twice the pairing gap is injected into the system.

If we allow the fermions to interact and if the average distance between the fermions is much larger than the range of the force resulting from the interparticle potential, we say a Fermi gas is dilute. We can compare and connect two systems literally light years apart from each other due to the universality of dilute and cold Fermi gases: cold atoms made in a laboratory and the theorized superfluid cores of neutron stars. Observations of glitches in neutron stars provide indirect evidence for the existence of a superfluid in the deep universe: a rapid acceleration in the rotation rate of a neutron star. These are predicted to be a result of the quantized vortices that can form within a rotating superfluid. The cooling curves of neutron stars are thought to be dependent on the superfluid pairing gap. Simulating superfluids with different, measurable pairing gaps to generate cooling curves that can then be compared to the cooling curves of actual neutron stars is one way to shed light on these fascinating stellar objects.

Science is ultimately about observations and experiments on real physical systems. Experiments tend to be expensive and difficult, and thus the number of experiments and observations a researcher would like to do is much smaller than the number of experiments they actually can do. In addition, for practical reasons many systems in nature are impossible to study directly, such as most astronomical phenomena. Obviously the era of space exploration has allowed for us to have direct contact with the greater universe, but for the foreseeable future we are going to be confined within the Solar System. The nearest neutron star is approximately 400 light years away, roughly 25 million times the distance between the Earth and the Sun. The technical challenge of recreating certain systems in the laboratory is also a major impediment to experimental science. Rapid progress is being made in realizing superfluid and supersolid phases in cold atomic gas systems. Because of the universality described above, these may serve as useful real models of the superfluid interiors of neutron stars.

We want to stress the importance of computational physics for investigating systems such as quantum fluids. Purely theoretical investigations are extremely difficult for complex systems. Computational physics can be seen as an implementation of theoretical physics at a much larger scale than can be done with pencil and paper work alone. Furthermore many of the methods implemented by computational physicists were invented before modern computers existed or were powerful enough to tackle physically meaningful systems. Computational physics will not be a replacement for experimental or theoretical work. However, it combines the inexpensiveness of theory with the ability to account for the greater complexity of real physical systems. Computational results also serve as useful benchmarks that provide a roadmap for more ambitious experiments within the next decade.

Appendix A. Derivation and motivation for the probability current

Here we would like to provide a motivation for (12). The explanation is straight forward but without it, this equation can seem to have appeared from nowhere and at first sight, the exclusive dependence on the phase can be quite perplexing. To understand this equation will start by deriving an expression for the probability current, and from there find (12). Suppose we have a region of space \( \mathcal{V} \) and a particle with wave function \( \Psi(\mathbf{r}, t) \). We would like to know how the probability of finding the particle in \( \mathcal{V} \) changes with time. We write the probability as

\[
P(\mathcal{V}) = \int_{\mathcal{V}} \rho \, d^3 \mathbf{r}.
\]  

(A1)
Where the probability density is \( \rho = R^2 = |\Psi|^2 = \Psi^*\Psi \). Taking the time derivative of the density we have
\[
\frac{\partial \rho}{\partial t} = \frac{\partial \Psi^*}{\partial t} \Psi + \Psi^* \frac{\partial \Psi}{\partial t}
\] (A2)

Substituting the Schrödinger equation, (where we assume we are working in units where \( \hbar = 1 \))
\[
i \frac{\partial \Psi}{\partial t} = -\frac{1}{2m} \nabla^2 \Psi + V \Psi
\] (A3)

(and its complex conjugate), into (A2) we obtain
\[
\frac{\partial \rho}{\partial t} = -\frac{i}{2m} (\Psi \nabla^2 \Psi^* - \Psi^* \nabla^2 \Psi) = -\nabla \cdot J.
\] (A4)

Here we have defined a new quantity, the probability current,
\[
J = \frac{i}{2m} (\Psi \nabla \Psi^* - \Psi^* \nabla \Psi).
\] (A5)

Using (A1) and (A4) and applying the divergence theorem, we can write
\[
\int_{\mathcal{V}} \nabla \cdot J \, d^3r = \int_{\partial \mathcal{V}} J \cdot n \, d^2r,
\] (A6)

where \( \partial \mathcal{V} \) is the surface surrounding the region \( \mathcal{V} \).

Intuitively, this means we can treat the probability density like an inhomogeneous fluid, that is, a fluid with a variable density, like a gas. As time passes in some regions of space probability can become more rarefied and in other regions probability can accumulate. The probability is a conserved quantity (the total amount is always equal to one) and obeys the continuity equation,
\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot J,
\] (A7)

which intuitively means that if a fluid moves from one region to another, it must move through all the regions in-between. This means any concentration of probability in one region will be at the expense of probability flowing out of adjacent regions.

We now want to obtain (12). We start by writing the wave function in polar form using (11) and substituting this into the probability current given in (A5). This gives
\[
J = \frac{i}{2m} [\text{Re} e^{i\Theta} \nabla \text{Re} e^{-i\Theta} - \text{Re} e^{-i\Theta} \nabla \text{Re}^{i\Theta}],
\] (A8)

where we have suppressed the \( \vec{q} \) and \( t \) dependence. Evaluating the gradients and simplifying we obtain
\[
J = \frac{\rho}{m} \nabla \Theta (\mathbf{r}, t).
\] (A9)

In classical hydrodynamics the current is given as \( J = \rho \mathbf{v} \). Here \( \rho \) is the density of a fluid parcel moving with velocity \( \mathbf{v} \). From comparison, we can conclude
\[ \mathbf{v}(\mathbf{r}, t) = \frac{1}{m} \nabla \Theta(r, t), \quad (A10) \]

which gives us (12): the velocity field of a particle in the superfluid.

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