GRADIENT ESTIMATES ON WARPED PRODUCT GRADIENT ALMOST RICCI SOLITONS

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Abstract. In this paper, by slightly modifying Li-Yau’s technique so that we can handle drifting Laplacians, we were able to find three different gradient estimates for the warping function, one for each sign of the Einstein constant of the fiber manifold. As an application, we exhibit a nonexistence theorem for gradient almost Ricci solitons possessing certain metric properties on the base of the warped product.

1. Introduction and main results

A gradient almost Ricci soliton [11] is a complete Riemannian manifold \((M^n, g)\) together with functions \(h, \rho : M \to \mathbb{R}\) satisfying the equation

\[ \text{Ric}_g + \text{Hess}_g h = \rho g, \]

where \(\text{Ric}_g\) and \(\text{Hess}_g h\) stand, respectively, for the Ricci tensor and the Hessian of \(h\). The function \(h : M \to \mathbb{R}\) is usually referred to as the potential function of the soliton in the literature. The quadruple \((M^n, g, h, \rho)\) is classified into three types according to the sign of \(\rho\): expanding if \(\rho < 0\), steady if \(\rho = 0\) and shrinking if \(\rho > 0\). If \(\rho\) occurs as a constant, the soliton is usually referred to as a gradient Ricci soliton.

From the above it is readily seen that Einstein manifolds, that is, those for which

\[ \text{Ric}_g = \rho g, \]

trivially fulfill the requirement for a gradient almost Ricci soliton with a constant \(\rho\) and Killing vector field \(\nabla h\), not to exclude the possibility of an also constant \(h\).

Recent works like [5, 6, 7, 8, 12] exhibit the fact that warped product manifolds constitute a highly profitable ground for the construction of gradient almost Ricci solitons. It’s worth noticing that [12] contain infinitely many solutions to (1) in closed form.

Definition 1.1. Let \((B^n, g_B)\) and \((F^k, g_F)\) be two Riemannian manifolds and \(f > 0\) on \(B\). The product manifold \(B \times F\) furnished with metric tensor

\[ g = \pi^* g_B + (f \circ \pi)^2 \sigma^* g_F, \]

where \(\pi : B \times F \to B\), \(\sigma : B \times F \to F\) are the projections on the first and second factor, respectively, is called warped product manifold and will be denote by \(B \times_f F\). The function \(f\) is called warping function, \(B\) is called the base and \(F\) the fiber.

In [5] the authors investigated warped product gradient Ricci solitons and proved that either the warping function \(f\) is constant or the potential function satisfies

\[ h = h_B \circ \pi, \quad h_B \in C^\infty(B). \]
In this very spirit the authors of [6] studied warped product gradient almost Ricci solitons with the assumption that

$$
\rho = \rho_B \circ \pi, \quad h = h_B \circ \pi, \quad \rho_B, h_B \in C^\infty(B),
$$

and prove the following result:

**Proposition 1.2.** Let \((B^n \times_f F^k, g, h, \rho)\) be a gradient almost Ricci soliton, then the functions \(f, h_B\) defined on \((B^n, g_B)\) satisfy

$$
Ric_{g_B} + Hess_{g_B} h_B = \rho_B g_B + \frac{k}{f} Hess_{g_B} f,
$$

and the fiber \((F^k, g_F)\) is Einstein with \(Ric_f = \theta g_F\), where

$$
\theta = \rho_B f^2 + f \Delta f + (k - 1)|\nabla f|^2 - f \nabla h_B(f).
$$

The authors make use of the equation (3) in combination with the maximum principle to provide triviality results for warping functions reaching a maximum. As a consequence one sees that it is impossible to achieve the structure of a gradient almost Ricci soliton on warped product manifolds possessing a compact base and \(\rho_B f^2 \geq \theta\).

Proposition 1.2 is known to be true for Einstein manifolds \((B^n \times_f F^k, g)\) since [8]. Besides, with the change \(f = \exp(k^{-1}u)\) in (4), we get that

$$
Ric_{g_B} + Hess_{g_B} u - \frac{1}{k} du \otimes du = \rho_B g_B.
$$

Objects satisfying equation (6) are called quasi-Einstein manifold [8] and play an important role in the analysis of solutions for (3). For instance, in [2], by starting off with (6), the author developed a gradient estimate for \(u\) and concluded that there are no nonconstant solutions for (3) if \(\rho_B = 0\) and \(\theta \leq 0\). Therefore, there are no nontrivial Ricci flat warped products such that its fiber manifold has a nonpositive Einstein constant. In the same way, in [10] it’s shown some triviality results for the warping function \(f\) in terms of gradient estimates, however, in the case \(\rho_B \leq 0\).

By the above exposed it is interesting to investigate whether or not we have local gradient estimates for warping solutions of (6) in the general case. By the change \(f = v^2\) in equation (5) we obtain that

$$
\Delta h_B v + \rho_B k^2 v - \theta k v^{1-\frac{1}{k}} = 0.
$$

where \(\Delta_w = e^w \text{div}(e^{-w} \nabla w)\) is the so called drifting Laplacian on the Bakry-Émery geometry. In order to accomplish such a task, we focus our attention on gradient estimates for the positive solutions of the nonlinear equation (7) that, in compliment to [8], are defined on a noncompact base. We mainly follow the lines of P. Li and S.T. Yau’s proof in [8].

**Theorem 1.3.** Let \((B^n \times_f F^k, g, h, \rho)\) be a complete gradient almost Ricci soliton with noncompact base satisfying

$$
Ric_{g_B} + Hess_{g_B} h_B - \frac{1}{m} dh_B \otimes dh_B \geq -K, \quad \Delta h_B \rho_B \geq 0, \quad |\nabla \rho_B| \leq \gamma(2R),
$$

for \(K \geq 0\) in the metric ball \(B(p, 2R)\) of the base. Then, for any \(\beta \in (0, 1)\) with \(\beta > 1 - \frac{2}{m}\), the warping function \(f\) satisfies the following gradient estimates:

- If \(\theta < 0\), we have

$$
\beta \frac{|\nabla f|^2}{f^2} + \frac{\rho_B}{k} - \frac{\theta}{k f^2} \leq \frac{(n + m)}{k^2 \beta} P + \frac{n + m}{2 k^4 \beta} Q^2, \quad \text{in} \quad B(p, R),
$$

If $\theta \geq 0$, assume that $f$ is bounded in $B(p, 2R)$, then we have
\[
\beta \frac{\|\nabla f\|^2}{f^2} + \frac{\rho_B}{k} - \frac{\theta}{kf^2} \leq \frac{(n + m)c_1^2}{(n + m)c_1^2} \left[ P + 2\theta M \right] + \sqrt{\frac{n + m}{2\beta k^4}} (Q + S)^{\frac{1}{2}}, \quad \text{in} \quad B(p, R),
\]
where
\[
M = \sup_{B(p, 2R)} f^{-2}, \quad P = \frac{c_1}{4R^2\beta(1 - \beta)} + \frac{c_1}{R^2} c_2 + 2c_2^2, \quad Q = \frac{3\beta}{2} \left[ \frac{n + m}{4} (k\gamma)^4 (1 - \beta^2) \right]^{\frac{1}{2}} + \left( \frac{n + m}{2} \beta(1 - \epsilon)^{-1}(1 - \beta)^{-2} K^2, \right.
\]
\[
S = \frac{\beta(n + m)}{2(1 - \epsilon)(1 - \beta)} \left\{ \left[ \frac{M\theta}{\beta} \left( \beta - 1 + \frac{2}{k} \right) \right]^2 + \frac{2KM\theta}{\beta} \left( \beta - 1 + \frac{2}{k} \right) \right\},
\]
c_1, c_2 are positive constants and $\epsilon \in (0, 1)$.

By letting $R \to \infty$ we get the following global gradient estimates.

**Corollary 1.4.** Let $(B^n \times F^k, g, h, \rho)$ be a complete gradient almost Ricci soliton with no compact base satisfying
\[
Ric_{gh} + Hess_{gh} h_B - \frac{1}{m} dh_B \otimes dh_B \geq -K, \quad \Delta h_B \rho_B \geq 0, \quad \|\nabla \rho_B\| \leq \gamma,
\]
for $K \geq 0$ on $B$. Then, for any $\beta \in (0, 1)$ with $\beta > 1 - \frac{2}{k}$, the warping function $f$ satisfies the following gradient estimates:

- **If $\theta < 0$, we have**
  \[
  \beta \frac{\|\nabla f\|^2}{f^2} + \frac{\rho_B}{k} - \frac{\theta}{kf^2} \leq \sqrt{\frac{n + m}{2\beta k^4}} Q^{\frac{1}{2}}, \quad \text{in} \quad B^n,
  \]

- **If $\theta = 0$, we have**
  \[
  \beta \frac{\|\nabla f\|^2}{f^2} + \frac{\rho_B}{k} \leq \sqrt{\frac{n + m}{2\beta k^4}} Q^{\frac{1}{2}}, \quad \text{in} \quad B^n,
  \]

- **If $\theta > 0$, assume that $f$ is bounded, then we have**
  \[
  \beta \frac{\|\nabla f\|^2}{f^2} + \frac{\rho_B}{k} - \frac{\theta}{kf^2} \leq \frac{2(n + m)\theta M'}{k^2\beta} + \sqrt{\frac{n + m}{2\beta k^4}} (Q + S)^{\frac{1}{2}}, \quad \text{in} \quad B^n,
  \]
where
\[
M' = \sup_B f^{-2}, \quad Q = \frac{3\beta}{2} \left[ \frac{n + m}{4} (k\gamma)^4 (1 - \beta^2) \right]^{\frac{1}{2}} + \left( \frac{n + m}{2} \beta(1 - \epsilon)^{-1}(1 - \beta)^{-2} K^2, \right.
\]
\[
S = \frac{\beta(n + m)}{2(1 - \epsilon)(1 - \beta)} \left\{ \left[ \frac{M\theta}{\beta} \left( \beta - 1 + \frac{2}{k} \right) \right]^2 + \frac{2KM\theta}{\beta} \left( \beta - 1 + \frac{2}{k} \right) \right\},
\]
and $\epsilon \in (0, 1)$.

As an application we obtain the following two results.
Corollary 1.5. There is no complete gradient Ricci soliton \((B^n \times_f F^k, g, h, \rho)\) satisfying
\[
Ric_g + Hess_g h_B - \frac{1}{m} dh_B \otimes dh_B \geq 0,
\]
if either \(\rho = 0, \theta < 0\) or \(\rho = cte > 0, \theta = 0\).

Example 1.6. Let \((H^n, g_{-1}), (\mathbb{R}^n, g_0)\) and \((S^n, g_1)\) be, in this order, the hyperbolic and Euclidean spaces as well as the round sphere, all with standard metrics. Consider the manifolds \((B, g_B) = (S^n \times \mathbb{R}^n, g_1 + g_0), (F, g_F) = (H^k, g_{-1})\) and define
\[
h_v : S^n \to \mathbb{R}, \quad p \mapsto g_1(p, v),
\]
for any given \(v \in S^n\). Then, it’s readily seen that
\[
h_B : S^n \times \mathbb{R}^n \to \mathbb{R}, \quad (x, y) \mapsto - \log \left(\frac{3 - h_v(x)}{n}\right),
\]
satisfies
\[
Ric_g + Hess_g h_B - \frac{1}{m} dh_B \otimes dh_B \geq 0,
\]
and, therefore, by Corollary 1.5, there cannot exist an \(f\) such manifold as the base of a warped product gradient Ricci soliton \((B^n \times F^k, g)\) with either a negative scalar curvature of the fiber manifold \(\leq 0\) and \(\rho = 0\) or null scalar curvature of the fiber manifold \(\neq 0\) and \(\rho > 0\).

In particular, for Einstein metrics, we foresee that

Corollary 1.7. There is no Einstein warped product \((B^n \times F^k, g)\) such that \(\rho = 0\) and \(Ric_g \geq 0\) with either \(\theta < 0\) or \(\theta = 0\) and \(f \neq constant\).

Example 1.8. By taking \((B, g_B) = (\mathbb{R}^n, g_0)\) and \((F, g_F) = (H^k, g_{-1})\) we see, now from Corollary 1.7, that none of the warped product manifolds \((\mathbb{R}^n \times H^k, g)\) are Ricci flat.

2. Proofs

Proof of Theorem 1.3: Applying the change \(f = v^\frac{1}{k}\) in equation (5) of Proposition 1.2 we obtain that
\[
(8) \quad \Delta h_B v + \rho_B k v - \theta k v^{1 - \frac{1}{k}} = 0.
\]
Let \(v\) a positive solution to (8). Then, \(u = \log v\) satisfies
\[
\Delta h_B u = (\beta - 1)|\nabla u|^2 - L, \quad \text{where} \quad L := \beta |\nabla u|^2 + \rho_B k - \theta k e^{-\frac{2u}{k}}\quad \beta \in (0, 1), \quad \beta > 1 - \frac{2}{k}.
\]
Now, consider a cut-off function \(\xi\) satisfying
\[
\xi(r) = \begin{cases} 
1 & \text{if } r \in [0, 1] \\
0 & \text{if } r \in [2, \infty) 
\end{cases}, \quad -c_1 \leq \frac{\xi'(r)}{\xi^2(r)} \leq 0, \quad -c_2 \leq \xi''(r), \quad c_1, c_2 \in (0, \infty)
\]
and define
\[
\psi(x) = \xi \left(\frac{r(x)}{R}\right),
\]
where \( r(x) \) is the distance function from \( p \) to \( x \). Using an argument of Calabi [1] (see also Cheng and Yau [4]), we can assume without loss of generality that the function \( \psi \) is smooth in \( B(p, 2R) \). Then, the function defined by \( G = \psi L \) is smooth in \( B(p, 2R) \).

Let \( x_0 \in B(p, 2R) \) be a point at which \( G \) attains its maximum value \( G_{\text{max}} \), and suppose that \( G_{\text{max}} > 0 \) (otherwise the proof is trivial). At the point \( x_0 \), we have

\[
\nabla(G) = \psi \nabla L + L \nabla \psi = 0.
\]

Moreover,

\[
0 \geq \Delta_h G, \\
= \psi \Delta_h L + L \Delta_h \psi + 2 \langle \nabla \psi, \nabla L \rangle, \\
= \psi \Delta_h L + L \Delta_h \psi - 2L \frac{|\nabla \psi|^2}{\psi}.
\]

In order to estimate the right-hand side of (9) we prove the following lemma:

**Lemma 2.1.** Let \((B^n, g_B)\) be a complete noncompact Riemannian manifold satisfying,

\[
\text{Ric}_{g_B} + \text{Hess}_{g_B} h_B - \frac{1}{m} dh_B \otimes dh_B \geq -K,
\]

for \( K \geq 0 \) in the metric ball \( B(p, 2R) \subset B \), and let \( L \) and \( \psi \) as above. Then, we have that

\[
\Delta_h \psi \geq -\frac{(n-1 + R\sqrt{nk})c_1 + c_2}{R^2},
\]

\[
\Delta_h L \geq 2\beta \frac{(\Delta_h u)^2}{n+m} + 2(1-\beta)k(\nabla u, \nabla \rho_B) - 2(\nabla u, \nabla L) - 2\beta K|\nabla u|^2 +
\]

\[
+ k\Delta_h \rho_B - 2\theta e^{-\frac{2}{k}} \left( (\beta - 1 + \frac{2}{k})|\nabla u|^2 + L \right).
\]

**Proof of Lemma 2.1:** Equation (11) follows from the calculation

\[
\frac{|\nabla \psi|^2}{\psi} = \frac{1}{\xi} \left\langle \xi' \frac{\nabla r}{R}, \xi' \frac{\nabla r}{R} \right\rangle = \frac{(\xi')^2}{\xi} \frac{1}{R^2} (\nabla r, \nabla r) \leq \frac{c_1^2}{R^2}.
\]

It has been shown by Qian [13], the following estimate

\[
\Delta_h r^2 \leq n \left( 1 + \sqrt{1 + \frac{4Kr^2}{n}} \right)
\]

which implies

\[
\Delta_h r = \frac{1}{2r} (\Delta_h r^2 - 2|\nabla r|^2)
\]

\[
\leq \frac{n-2}{2r} + \frac{n}{2r} \left( 1 + \sqrt{1 + \frac{4Kr^2}{n}} \right)
\]

\[
= \frac{n-1}{r} + \sqrt{nK}.
\]
Then, we obtain
\[ \Delta_{h_B} \psi = \frac{\xi''(r)|\nabla r|^2}{R^2} + \frac{\xi'(r)\Delta_{h_B} r}{R} \geq -\frac{(n - 1 + R\sqrt{nK})c_1 + c_2}{R^2}. \]
which proves (12).

From the Bochner formula for the \( m \)-Bakry-Émery Ricci tensor and the lower bound hypothesis (10), we obtain
\[ \frac{1}{2} \Delta_{h_B} |u|^2 \geq \frac{(\Delta_{h_B} u)^2}{n + m} + \langle \nabla u, \nabla \Delta_{h_B} u \rangle - K|\nabla u|^2. \]

Therefore,
\[ \Delta_{h_B} L = \beta \Delta_{h_B} |\nabla u|^2 + k\Delta_{h_B} \rho_B - k\theta \Delta_{h_B} e^{-\frac{2u}{k}} \]
\[ \geq 2\beta \frac{(\Delta_{h_B} u)^2}{n + m} + 2\beta \langle \nabla u, \nabla \Delta_{h_B} u \rangle - 2\beta K|\nabla u|^2 + k\Delta_{h_B} \rho_B - k\theta \Delta_{h_B} e^{-\frac{2u}{k}}. \]

Notice that
\[ 2\beta \langle \nabla u, \nabla \Delta_{h_B} u \rangle = \langle \nabla u, \nabla \left[ \left( 1 - \frac{1}{\beta} \right) \left( -k\rho_B + k\theta e^{-\frac{2u}{k}} \right) - \frac{L}{\beta} \right] \rangle \]
\[ = 2(1 - \beta)k\langle \nabla u, \nabla \rho_B \rangle + 2(\beta - 1)k\theta \langle \nabla u, \nabla e^{-\frac{2u}{k}} \rangle - 2\langle \nabla u, \nabla L \rangle \]
\[ = 2(1 - \beta)k\langle \nabla u, \nabla \rho_B \rangle - 4(\beta - 1)\theta e^{-\frac{2u}{k}}|\nabla u|^2 - 2\langle \nabla u, \nabla F \rangle, \]
and
\[ \Delta_{h_B} e^{-\frac{2u}{k}} = \frac{2}{k} e^{-\frac{2u}{k}} \left[ \frac{2}{k} |\nabla u|^2 - \Delta_{h_B} u \right] \]
\[ = \frac{2}{k} e^{-\frac{2u}{k}} \left[ \frac{2}{k} |\nabla u|^2 - (\beta - 1)|\nabla u|^2 + L \right] \]
\[ = \frac{2}{k} e^{-\frac{2u}{k}} \left( \frac{2}{k} - \beta + 1 \right) |\nabla u|^2 + L \right]. \]

It follows that
\[ \Delta_{h_B} L \geq 2\beta \frac{(\Delta_{h_B} u)^2}{n + m} + 2(1 - \beta)k\langle \nabla u, \nabla \rho_B \rangle - 2\langle \nabla u, \nabla L \rangle - 2\beta K|\nabla u|^2 + \\
+ k\Delta_{h_B} \rho_B - 2\theta e^{-\frac{2u}{k}} \left( (\beta - 1 + \frac{2}{k})|\nabla u|^2 + L \right), \]
which completes the proof of lemma. \( \square \)

Continuing, by Lemma 2.1 and (9), we obtain at the point \( x_0 \),
\[ \psi \left\{ 2\beta \frac{(\Delta_{h_B} u)^2}{n + m} + 2(1 - \beta)k\langle \nabla u, \nabla \rho_B \rangle - 2\langle \nabla u, \nabla L \rangle - 2\beta K|\nabla u|^2 + k\Delta_{h_B} \rho_B + \\
- 2\theta e^{-\frac{2u}{k}} \left( (\beta - 1 + \frac{2}{k})|\nabla u|^2 + L \right) \right\} \leq LH, \]
where
\[ H = \left(\frac{(n-1 + R \sqrt{nK})c_1 + c_2 + 2c_2}{R^2}\right). \]

From the fact that \(0 \leq \psi \leq 1\), we have
\[-2\psi \langle \nabla u, \nabla L \rangle = 2L \langle \nabla u, \nabla \psi \rangle \geq -2L |\nabla u| |\nabla \psi| \geq -\frac{2c_1}{R} \psi \frac{1}{2} L |\nabla u|.\]

Then
\[
2\beta \psi \frac{(\Delta h_B u)^2}{n + m} + 2(1 - \beta)k \psi \langle \nabla u, \nabla \rho_B \rangle - \frac{2c_1}{R} \psi \frac{1}{2} L |\nabla u| - 2\beta \psi K |\nabla u|^2 + \]
\[
+k \psi \Delta h_B \rho_B - 2\theta e^{-\frac{2K}{K}} \psi \left[ (\beta - 1 + \frac{2}{\beta}) |\nabla u|^2 + L \right] \leq LH. \tag{14}
\]

In the sequel we distinguish between two cases: (a) \(\theta < 0\) and (b) \(\theta \geq 0\).

Case (a): \(\theta < 0\). Since \(\Delta h_B \rho_B \geq 0\), \(|\nabla \rho_B| \leq \gamma (2R)\), then (14) yields
\[
2\beta \psi \frac{(\Delta h_B u)^2}{n + m} - 2(1 - \beta)k \psi \gamma |\nabla u| - \frac{2c_1}{R} \psi \frac{1}{2} L |\nabla u| - 2\beta \psi K |\nabla u|^2 \leq LH.
\]

Multiplying both sides of the above equation by \(\psi\) and using the fact that \(0 \leq \psi \leq 1\), we obtain
\[
2\beta \frac{(\psi \Delta h_B u)^2}{n + m} + 2(\beta - 1)k \psi \frac{\gamma}{\beta} |\nabla u| - \frac{2c_1}{R} \psi \frac{1}{2} L |\nabla u| - 2\beta \psi K |\nabla u|^2 \leq \psi LH.
\]

Let
\[
y = \psi |\nabla u|^2, \quad z = \psi (-k \rho_B + k \rho e^{-\frac{2K}{K}}).
\]

Then we have
\[
\frac{2\beta}{n + m} \left\{ (y - z)^2 + \frac{(\beta - 1)k \gamma (n + m) y^\frac{\gamma}{\beta}}{\beta} - \frac{(n + m)c_1}{R} y \frac{1}{\beta} \left( y - \frac{z}{\beta} \right) - (n + m) K y \right\} \leq \psi LH.
\]

From Li-Yau’s arguments ([6], pg.161-162), we know that
\[
(y - z)^2 - (n + m)c_1 R^{-1} y \frac{1}{\beta} \left( y - \frac{z}{\beta} \right) - (n + m) K y - (n + m) \left( \frac{1}{\beta} - 1 \right) \gamma y \frac{1}{\beta} \geq \]
\[
\geq \left( \frac{1}{\beta} \right)^{-2} \left( y - \frac{z}{\beta} \right)^2 - \frac{(n + m)^2 c_1}{8} \left( \frac{1}{\beta} \right)^2 \left( \frac{1}{\beta} - 1 \right)^{-1} R^{-2} \left( y - \frac{z}{\beta} \right) + \]
\[- \frac{3}{4} 4^{-\frac{1}{2}} (n + m) \frac{\gamma}{\beta} \left[ \gamma^4 \left( \frac{1}{\beta} - 1 \right)^2 \left( \frac{1}{\beta} \right)^2 \epsilon^{-1} \right] - \frac{(n + m)^2}{4} (1 - \epsilon)^{-1} \left( \frac{1}{\beta} - 1 \right)^{-2} \left( \frac{1}{\beta} \right)^2 K^2.
\]

for any \(0 < \epsilon < 1\).
Therefore in our context, we obtain
\[
\frac{2\beta}{n+m} \left\{ (\beta y - z)^2 - \frac{(n+m)^2 c_1^2}{8R^2 \beta (1-\beta)} (\beta y - z) - \frac{3}{4} \left( \beta + \frac{1}{2} \right) (n+m) \frac{2}{\beta} \left( 1 - \beta \right) \right\} +
\frac{3}{4} \frac{(n+m)^2}{(1-\varepsilon)^{-1}} (1-\beta)^{-2} K^2 \right\} \leq (\beta y - z) H.
\]
Hence,
\[
\frac{2\beta}{n+m} (\psi L)^2 - P(\psi L) - Q \leq 0,
\]
where,
\[
P = \frac{(n+m)c_1}{4R^2 \beta (1-\beta)} + H,
\]
\[
Q = \frac{3\beta}{2} \left\{ n+m \frac{2}{\beta} (k\gamma)^4 \left( 1 - \beta \right) \frac{1}{\beta} \left( 1 - \beta \right) \right\} + \frac{\beta(n+m)}{2} (1-\varepsilon)^{-1} (1-\beta)^{-2} K^2.
\]
Using the inequality \( az^2 - bz \leq c \), one obtains
\[
z \leq \frac{2b}{a} + \sqrt{\frac{c}{a}}.
\]
Then
\[
\sup_{x \in B(p,R)} L(x) \leq (\psi L)(x_0) \leq \frac{n+m}{\beta} P + \sqrt{\frac{n+m}{2\beta} Q^2},
\]
and hence,
\[
\beta |\nabla u|^2 + k\rho B - k\theta e^{-\frac{n+m}{2p}} \leq \frac{n+m}{\beta} P + \sqrt{\frac{n+m}{2\beta} Q^2}.
\]
Replacing the function \( u = \log \frac{f^{2^+}}{1} \) back into the above equation we obtain the desired estimate when \( \theta < 0 \).

Case (b): \( \theta \geq 0 \). Since
\[
\Delta_{\rho B} \rho B \geq 0, \quad |\nabla \rho B| \leq \gamma(2R),
\]
then (14) yields
\[
2\beta \psi \frac{(\Delta_{\rho B} u)^2}{n+m} - 2(1-\beta)k\psi |\nabla u| \gamma - \frac{2c_1}{R} \psi \frac{2}{R} L |\nabla u| - 2\beta\psi K |\nabla u|^2 +
-2\theta e^{-\frac{n+m}{2p}} \psi \left( (\beta - 1 + \frac{2}{k}) |\nabla u|^2 + L \right) \leq LH.
\]
Multiplying both sides of the above equation by \( \psi \), and using the fact that \( 0 \leq \psi \leq 1 \), we obtain
\[
2\beta \psi \frac{(\Delta_{\rho B} u)^2}{n+m} + 2(\beta - 1)k\psi |\nabla u| \gamma - \frac{2c_1}{R} \psi \frac{2}{R} L |\nabla u| - 2\beta\psi K |\nabla u|^2 +
-2\theta \psi \left( (\beta - 1 + \frac{2}{k}) |\nabla u|^2 + L \right) \leq \psi LH,
\]
where \( M = \sup_{B(p,2R)} e^{-\frac{n+m}{2p}} \).
Let
\[
y = \psi |\nabla u|^2, \quad z = \psi (-k\rho B + k\theta e^{-\frac{n+m}{2p}}).
\]
Then we have
\[
\frac{2\beta}{n+m} \left\{ (y-z)^2 - \frac{c_1(n+m)}{R} y^\frac{2}{\beta} + (n+m) \frac{(\beta-1)}{\beta} k\gamma y^\frac{2}{\beta} + \right.
\]
\[
- (n+m) \left\{ \frac{\theta M}{\beta} (\beta - 1 + \frac{2}{k}) + K \right\} \right \} \leq \psi (H + 2\theta M).
\]
Hence, again by the Li-Yau arguments, we get that
\[
\frac{2\beta}{n+m} \left\{ (\beta y-z)^2 - \frac{(n+m)^2 c_1^2}{8R^2 \beta^2 (1-\beta)} (\beta y-z) - \frac{3}{4} (1-\frac{1}{k})(n+m) \frac{4}{\beta^4} (1-\beta)^2 \left( \frac{\theta M}{\beta} (\beta - 1 + \frac{2}{k}) + K \right) \right \} \leq (\beta y-z)H,
\]
which means that
\[
\frac{2\beta}{n+m} (\psi L)^2 - \left\{ P + 2\theta M \right\} (\psi L) - (Q + S) \leq 0,
\]
where
\[
S = \frac{\beta (n+m)}{2(1-\varepsilon)(1-\beta)^2} \left\{ \left\{ \frac{M\theta}{\beta} (\beta - 1 + \frac{2}{k}) \right\}^2 + \frac{2KM\theta}{\beta} (\beta - 1 + \frac{2}{k}) \right\}.
\]
Therefore
\[
\beta |\nabla u|^2 + k\rho_B - k\theta e^{-\frac{2u}{\beta}} \leq \frac{n+m}{\beta} (P + 2\theta M) + \sqrt{\frac{n+m}{2\beta} (Q + S)^2}.
\]
Replacing the function \( u = \log f^{\frac{m}{n+m}} \) back into the above equation we obtain the desired estimate when \( \theta \geq 0 \).

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