Partial transpose is an important operation for quantifying entanglement. In this study, the (partial) transpose of any single (two-mode) operator is investigated. Using the Fock basis expansion, it is found that the transposed operator of an arbitrary operator can be obtained by replacing $a^\dagger a$ with $aa^\dagger$, rather than the c-number within the normal ordering form. The transpose of the displacement and Wigner operators is also investigated, from which the relation of the Wigner function, characteristics function, and average values such as covariance matrix is constructed between the density operator and transposed density operator. These observations can be further extended to multi-mode cases. As for the application, partial transpose of the two-mode squeezed operator and the entanglement of the two-mode squeezed vacuum through a laser channel is considered.

1. Introduction

Quantum entanglement is an important resource for achieving different quantum tasks, including quantum teleportation, key distribution, dense code, and metrology.[1] Specifically for quantum systems with continuous variables, justifying the existence of quantum entanglement has been a challenge. To address this issue, numerous entanglement criteria and measure methods have been proposed[2-13]; these include the Peres-Horodecki, Simon, Shchukin and Vogel, Hillery and Zubairy, Agarwal and Biswas, and Nha and Kim criteria, as well as the entropy of entanglement, entanglement of formation, and logarithmic negativity. Among the aforementioned methods, the logarithmic negativity is considered a computable measure of entanglement for arbitrary bipartite systems, including both Gaussian and non-Gaussian states.[11] In these criteria or measures, the key starting point is that the partial transpose on the density operator of the two-mode entangled states, where the resulting operator is, may be unqualified quantum states whose eigenvalues are non-negative real numbers. The logarithmic negativity is proposed based on the negativity of partial transposition, that is, the negative eigenvalues. To handle the partial transpose or calculate the logarithmic negativity, on the one hand, it is a common method to appeal to a special basis (say Fock basis) or corresponding matrix on a selected basis. Conversely, for quantum systems with continuous variables, the method in the phase space is also used to discuss the entanglement through the partial transposition. For instance, Simon extended the Peres-Horodecki criterion to bipartite continuous-variable states from the viewpoint of a phase space.[2] It is shown that the partial transpose corresponds to a mirror reflection, which can be observed using the definition of the Wigner function in coordinate representation. The aforementioned two methods can be considered indirect for discussions about entanglement or negativity.

Interestingly, treating the issue of quantum decoherence in an open system, which is described by the master equation, two different ways have been developed and used widely. One is converting the master equation to the equation of the distribution function in the phase space, such as the P-function, and the Wigner function and then deriving their input-output relations.[14-16] The other is to directly solve the master equation to derive the sum representation of the Kraus operator from the viewpoint of an operator.[17,18] Recently, the Kraus representation has been used to handle the precision of quantum metrology in the presence of noise.[19-21] It is necessary to further reveal the nature and eigen spectrum of the operator. Therefore, an interesting question arises naturally: how do we directly achieve the transpose or partial transpose of any operator, including the density operator from the viewpoint of an operator?

In this study, starting from the definition of an operator transpose in a Fock space, we examine the transpose or partial transpose of any operator within a normal ordering form. It is shown that the (partially) transposed operator can be obtained by replacing creation and annihilation operators with annihilated and created ones within normal ordering form, respectively, while classical numbers remain unchanged. Note that the replacement must be made within normal ordering form especially for a partial transpose case. Therefore, we further examine the relations...
between operators, distribution functions in phase space and average values of operators before and after the transpose operation. These discussions can be directly extended to multi-mode cases. In particular, as an effective and direct application, we analytically calculate the entanglement degree of the two-mode squeezed vacuum (TMSV) when passing through the laser channel using the logarithmic negativity.

The remainder of this paper is organized as follows. In Section 2, we define an operator transpose in a Fock space and then derive a lemma of the operator transpose for a normally ordering operator. The transpose of some operators is discussed, including the displacement, Wigner, and single-mode squeezing operators. In addition, the relations about distribution functions and average values are derived before and after the transpose. This case is extended to a two-mode case by considering partial transpose in Section 3. Section 4 presents the derivation of the degree of entanglement of TMSV through a laser channel. Finally, our conclusions are presented in Section 5.

2. Transpose of the Operator

Herein, we introduce the definition of an operator transpose. First, we consider a simple case, that is, single-mode operator transpose, including the displacement, Wigner, and density operators. For any single-mode operator \( \hat{O} \), it can be expanded in a Fock space as

\[
\hat{O} = \sum_{j,k} O_{jk} \hat{j} \langle k | = \langle j | \hat{O} | k \rangle
\]  

its transpose, denoted as \( \hat{O}^T \), can be defined as

\[
\hat{O}^T = \sum_{j,k} O_{jk} \hat{j} \rho \langle k | = \sum_{j,k} O_{jk} \langle j | \hat{k} \rho
\]  

or

\[
\hat{O}^T = \sum_{j,k} O_{jk} \hat{j} \langle k |\)

Note that the defined transpose for an operator is basis dependent. For a different basis, such as coherent state basis, the corresponding results will be mutually different.

2.1. Transpose of Any Normally Ordering Operator

Here, we consider a lemma: for any normally ordering operator, its transpose can be obtained by replacing \( a a^\dagger \) with \( a^\dagger a \) within its normal ordering form; that is,

\[
\left[ F(a, a^\dagger) \right]^T = F(a^\dagger, a)
\]

where the symbol \( 
\) denotes the normal ordering. The form of the operator within normal ordering remains unchanged.

Proof of Equation (4): Expanding operator \( F(a, a^\dagger) \): in the Fock space, we have

\[
F(a, a^\dagger) = \sum_{j,k} \langle j | \hat{j} : F(a, a^\dagger) : \langle k | \hat{k}
\]

Using the coherent state representation of the Fock state, that is, \( |k\rangle = \frac{1}{\sqrt{k!}} e^{a^\dagger \alpha} |0\rangle \) in which \( |\alpha\rangle = e^{a^\dagger} |0\rangle \) is an un-normalized coherent state, one can observe that

\[
|j\rangle : F(a, a^\dagger) : |k\rangle = \frac{1}{\sqrt{j! k!}} \frac{\partial^{j+k}}{\partial^j \alpha^k} \left( \langle \tau | : F(a, a^\dagger) : | \tau \rangle \right)_{\tau = 0}
\]

\[
= \frac{1}{\sqrt{j! k!}} \frac{\partial^{j+k}}{\partial^j \alpha^k} \left( \langle \tau | : F(a', a) : | \tau \rangle \right)_{\tau = 0}
\]

\[
= \langle k | : F(a, a^\dagger) : |j\rangle
\]

Thus, substituting Equation (6) into Equation (5) and using the definition in Equation (2), we have

\[
\left[ F(a, a^\dagger) \right]^T = \sum_{j,k} \langle k | \langle j | : F(a, a^\dagger) : |j | \rangle
\]

\[
= : F(a^\dagger, a):\]

indicating that under the normal ordering form, the transpose operator can be obtained by making a replacement with \( a \leftrightarrow a^\dagger \), instead of the c-number. For example, for \( (a^2 a)^T = (a^2 a)^T = a^2 a^\dagger \leftrightarrow a^\dagger a^\dagger \).

Using this property in Equation (4), we can obtain the transpose of the displacement operator

\[
[D(a)]^T = e^{a a^\dagger - a^\dagger a} = D(-a^\dagger)
\]

and the transpose of the Wigner operator \( (a^\dagger a)^T \)

\[
[D(a^\dagger a)]^T = \frac{1}{\pi} : \exp \left\{ -2(a - a^\dagger)(a^\dagger - a) \right\} :\]

\[
= \Delta(a^\dagger, a)
\]

Equation (8) indicates that the transpose of the displacement operator can be obtained by replacing \( a \) with \( -a^\dagger \), or

\[
D(q, p) \rightarrow D(-q, p) = [D(q, p)]^T
\]

where \( q = (q + ip)/\sqrt{2} \) and \( p = (p - iq)/\sqrt{2} \). Meanwhile, the transpose of the Wigner operator can be obtained by replacing \( a \leftrightarrow a^\dagger \) or \( p \leftrightarrow -p \), which is different from the case of displacement. In addition, the transposed Wigner operator \( [\Delta(a^\dagger, a)]^T \) is still an Hermite operator owing to

\[
[\Delta(a^\dagger, a)]^T = \Delta(a^\dagger, a)
\]

which can be clearly observed in Equation (9). Actually, Equation (10) can also be derived from Equations (9) and (2), respectively. Interestingly, a single-mode squeezing operator \( \hat{S}_x(\xi) = \hat{a} \hat{a}^\dagger e^{-i\xi} \)
The transpose of $S_1(\xi)$ is given by
\[ S_1(\xi) = \text{sech}^{1/2} r : \exp \left\{ \frac{1}{2} (e^{i\theta} a^\dagger a - e^{-i\theta} a a^\dagger) \tanh r \right\} \times \exp \left\{ (\text{sech} r - 1) a^\dagger a \right\} : \] (13)

Thus, according to Equation (4), the transpose of $S_1(\xi)$ is
\[ [S_1(\xi)]^T = \exp \left\{ \frac{1}{2} (\xi a^\dagger - \xi^* a^\dagger) \right\} = S_1(-\xi^*) \] (14)

It is evident that, for complex parameter $\xi$, $[S_1(\xi)]^T = S_1(-\xi^*) \neq S_1(-\xi) = [S_1(\xi)]^T$. In particular, for a real parameter $\xi$, $[S_1(\xi)]^T = [S_1(\xi)]^\dagger$. This case is not true for the two-mode correlated operation (see Equation (51)).

2.2. Characteristics Function and Wigner Function of Transposed Density Operator

2.2.1. Characteristics Function of the Transposed Density Operator

For any single-mode density operator $\rho$, which can be expanded according to the displacement operator, that is, the Weyl representation of the density operator\(^{[26]}\)
\[ \rho = \int_{-\infty}^{\infty} d^2 \alpha \chi(\alpha, \alpha^*) D(-\alpha) \] (15)

or
\[ \rho = \int_{-\infty}^{\infty} dq dp \chi(q, p) D(-q, -p) \] (16)

Thus, using Equation (8), the transpose of the density operator $\rho$ is obtained by
\[ \rho^T = \int_{-\infty}^{\infty} d^2 \alpha \chi(\alpha, \alpha^*) D(\alpha^*) \]
\[ = \int_{-\infty}^{\infty} d^2 \alpha \chi(-\alpha^*, -\alpha) D(-\alpha) \] (17)

Equation (17) indicates that the characteristic function of a transposed density operator is obtained by replacing $\alpha$ with $-\alpha^*$, which is similar to the case of the displacement operator. That is, after the transpose for a density operator, we have
\[ \chi(\alpha, \alpha^*) \rightarrow \chi(-\alpha^*, -\alpha) \] (18)
\[ \chi(q, p) \rightarrow \chi(-q, -p) \] (19)

2.2.2. Wigner Function of the Transposed Density Operator

Here, we examine the corresponding Wigner function after the transpose of the density operator. For this purpose, we can expand the single-mode density operator $\rho$ using the Wigner operator $\Delta(q, p)\,[23,24]$; that is,
\[ \rho = 2\pi \int_{-\infty}^{\infty} W(q, p) \Delta(q, p) dq dp \] (20)

where $W(q, p) = W(\alpha, \alpha^*)$ is the Wigner function of the density operator $\rho$ and the Wigner operator $\Delta(q, p)$ is defined by
\[ \Delta(q, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy e^{iyp} \left| q + \frac{y}{2} \right| \left| q - \frac{y}{2} \right| \]
\[ = \frac{1}{\pi} \exp \left\{ -2(a^\dagger - a)(\alpha - \alpha^*) \right\} : \]
\[ \equiv \Delta(\alpha, \alpha^*) \] (23)

where the symbol $::$ denotes the normal ordering.

Observing that the transposed Wigner operator is given by
\[ [\Delta(q, p)]^T = \Delta(q, -p) \] (24)
\[ [\Delta(\alpha, \alpha^*)]^T = \Delta(\alpha^*, \alpha) \] (25)

the transposed density operator is obtained as
\[ \rho^T = 2\pi \int_{-\infty}^{\infty} W(q, p) \Delta(q, -p) dq dp \]
\[ = 2\pi \int_{-\infty}^{\infty} W(q, -p) \Delta(q, p) dq dp \] (26)

Thus, the Wigner function of the transposed density operator $\rho^T$ can be obtained by replacing $p$ with $-p$ that is,
\[ W(q, p) \rightarrow W(q, -p) \]
\[ W(\alpha, \alpha^*) \rightarrow W(\alpha^*, \alpha) \] (27)

Note that the Wigner function of the transposed density operator $\rho^T$ remains a real function that can be observed from Equation (12), and $W(q, -p)$ remains a quasi-probability function because the transpose cannot change the trace of $\rho$; that is,
\[ \text{tr} \rho^T = \int_{-\infty}^{\infty} W(q, -p) dq dp = 1 \] (28)

Interestingly, after the transpose, the Wigner and characteristic functions can be obtained by mirror reflection along $q$ and $p$ axes, respectively.\(^{[22]}\) In addition, from Equation (26), we obtain
\[ (\rho^T) = 2\pi \int_{-\infty}^{\infty} [W(q, -p)]^T \Delta(q, p) dq dp \]
\[ = 2\pi \int_{-\infty}^{\infty} W(q, -p) \Delta(q, p) dq dp \] (29)
Equation (29) implies that \( \rho^T \) is still a Hermite operator. Note that the transpose cannot change the eigenvalues. In other words, both \( \rho^T \) and \( \rho \) share the same non-negative real eigenvalues. Thus, for a single-mode system, the transposed density operator is still used a density operator, which can represent a quantum state (eigenvalues of the density operator are non-negative real numbers).

### 2.3. Average Value of the Operator in the Transposed Density Operator

In this subsection, we consider the average value of an operator under the transposed density operator and construct the relation between these average values after and before the transpose operation.

#### 2.3.1. Average Value in Normally Ordering Form

For any operator \( \hat{O} \) whose normal ordering form is defined as : \( F(a, a^\dagger) \) : , its average value under the transposed density operator \( \rho^T \) can be calculated as

\[
\langle : F(a, a^\dagger) : \rangle_{\rho^T} = \langle : F(a^\dagger, a) : \rangle_{\rho} \tag{30}
\]

where the right-hand average value is under the density operator \( \rho \). Equation (30) can be proved by extending \( \rho \) in the Fock space and using a method similar to that used to derive Equation (6). In particular, when there is only a real parameter within : \( F(a^\dagger, a) \) :, then one can have \( \langle : F(a, a^\dagger) : \rangle_{\rho^T} = \langle : [F(a, a^\dagger)]^T : \rangle_{\rho} \). The above results can be extended directly to the multimode transpose case.

#### 2.3.2. Covariance Matrix of the Transposed Single-Mode Gaussian State

In the quantum system with a continuous variable, the covariance matrix, which comprises some average values, is often calculated. Forexample, the covariancematrix is used to derivethe accuracy matrix, which comprises some average values, is often calculated. For a single-mode Gaussian state, its characteristic function \( \chi(q, p) \) can be given by \( \chi^{[37]} \)

\[
\chi(q, p) = \exp \left\{ -\frac{1}{2} \xi \Sigma \Omega \Sigma^T \xi^T + iX^T \xi \right\} \tag{31}
\]

where \( X = \text{tr}(\hat{X}_{\rho}) \), \( \xi = (q \quad p) \) and \( V = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \) is only the covariance matrix, as well as \( \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). After transposition, according to Equation (19), the characteristic function of the transposed density operator is

\[
\chi^T(q, p) = \chi(-q, p) = \exp \left\{ -\frac{1}{2} \xi \Sigma \Omega \Sigma^T \xi^T \right\} \tag{32}
\]

where \( \Sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \) and we have assumed \( X = 0 \) for simplicity. Noticing that \( \Omega \Omega^T = \Omega^T \Omega = 1 \), the covariance matrix of the transposed density operator is given by

\[
\Omega^T \Sigma \Omega \Sigma^T \Omega = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} V \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^T = \begin{pmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{pmatrix} \tag{33}
\]

Equation (33) is the relation between the two covariance matrices before and after the transpose.

To check the relationship in Equation (33), we further consider the special covariance matrix, which is defined by

\[
V = \begin{pmatrix} 14 & 12 \\ 12 & 22 \end{pmatrix} \tag{34}
\]

where \( Q = a + a^\dagger \) and \( P = (a - a^\dagger) / i \). Then, we have

\[
Q^2 = a^2 + a^\dagger a + a^\dagger a^\dagger = a^2 + a^\dagger a + 2a^\dagger a + 1 : \tag{35}
\]

According to Equation (30), we have

\[
\langle Q^2 \rangle_{\rho} = \langle : a^2 + a^\dagger a + 2a^\dagger a + 1 : \rangle_{\rho} = \langle : Q^2 \rangle_{\rho} \tag{36}
\]

Similarly, we have \( \langle QP + PQ \rangle_{\rho} = -(\langle QP + PQ \rangle)_{\rho} \), as expected.

### 3. Two-Mode Operator’s Partial Transpose

In this section, we extend the single-mode case to the two-mode case. If we create the transpose for two parties of the two-mode quantum system, then the case is similar to the aforementioned one, which will be a trivial extension. Here, we only consider partial transpose with respect to one part of the two mode, that will be different from the above case to a certain extent and can also be easily extended to a multi-mode case.

#### 3.1. Partial Transpose for Any Two-Mode Operator

In the Fock space, any two-mode operator \( \hat{O}^{AB} \) can expanded as

\[
\hat{O} = \sum_{j,k,l} O_{jk,l}(k, l) \langle j, k | \hat{O} | l, k \rangle \tag{37}
\]

its partial transpose with respect to mode \( B \), denoted by \( (\hat{O}^{AB})^T_B \), can be defined as

\[
(\hat{O}^{AB})^T_B = \sum_{j,k,l} O_{jk,l}(k, l) \langle i, j | \hat{O} | i, j \rangle \tag{38}
\]
Similarly, to derive Equation (4) for a two-mode normally ordered operator: $F(a, a^\dagger; b, b^\dagger)$, its partial transpose for mode $B$ is given by

\[
\left[ F(a, a^\dagger; b, b^\dagger) \right]^{T_B} = F(a, a^\dagger; b, b^\dagger).
\]

This indicates that the partial transpose for the operator, such as: $F(a, a^\dagger; b, b^\dagger)$, can be achieved by replacing $b(b^\dagger)$ with $b'(b)$, respectively, while the other mode $A$ remains unchanged. Note that the replacement must be taken under the normal ordering form. This will be evident in the following section.

### 3.2. Partially Transposed Density Operator in Phase Space

Next, considering any two-mode Bose density operator $\rho_{AB}$, it can always be expanded as

\[
\rho_{AB} = \int_{-\infty}^{\infty} dq_1 dp_1 dq_2 dp_2 \exp \left( \chi(q_1, p_1; q_2, p_2) \right) \\
\times D_a(-q_1, -p_1) D_b(-q_2, -p_2)
\]

or

\[
\rho_{AB} = \int_{-\infty}^{\infty} dq_1 dp_1 dq_2 dp_2 W(q_1, p_1; q_2, p_2) \\
\times (2\pi)^2 \Delta_a(q_1, p_1) \Delta_b(q_2, p_2)
\]

Here, we consider the partial transpose with respect to mode $B$. Using Equations (11) and (24), we have

\[
\chi(q_1, p_1; q_2, p_2) \rightarrow \chi(q_1, p_1; -q_2, p_2)
\]

and

\[
W(q_1, p_1; q_2, p_2) \rightarrow W(q_1, p_1; q_2, -p_2)
\]

and then we have

\[
\left( \rho_{AB} \right)^{T_B} = \int_{-\infty}^{\infty} dq_1 dp_1 dq_2 dp_2 W(q_1, p_1; q_2, -p_2) \\
\times (2\pi)^2 \Delta_a(q_1, p_1) \Delta_b(q_2, p_2)
\]

and

\[
\left( \rho_{AB} \right)^{T_B} = \int_{-\infty}^{\infty} dq_1 dp_1 dq_2 dp_2 \exp \left( \chi(q_1, p_1; -q_2, p_2) \right) \\
\times D_a(-q_1, -p_1) D_b(-q_2, -p_2)
\]

Similarly, one can observe that

\[
\left[ \left( \rho_{AB} \right)^{T_B} \right]^{T_B} = \left( \left( \rho_{AB} \right)^{T_B} \right)^T
\]

and

\[
1 = \text{tr} \left( \left( \rho_{AB} \right)^{T_B} \right) = \int_{-\infty}^{\infty} dq_1 dp_1 dq_2 dp_2 \exp \left( \chi(q_1, p_1; q_2, -p_2) \right)
\]

Note that, although the partially transposed density operator $(\rho_{AB})^{T_B}$ presents the properties of both quasi-probability and Hermite, $(\rho_{AB})^{T_B}$ and $\rho_{AB}$ will not share the same non-negative real eigenvalues. This implies that, for a two-mode system, the partially transposed density operator may not be qualified as a density operator.

### 3.3. Partial Transpose of the Two-Mode Squeezing Operator

Here, we examine the partial transpose of a two-mode squeezing operator as an example. Theoretically, the two-mode squeezing operator is expressed as\(^{[28]}\)

\[
S(\xi) = \exp \left\{ (\xi a^\dagger b^\dagger - \xi^* ab) \right\}, \xi = re^{i\phi}
\]

whose normal ordering form is given by\(^{[25]}\)

\[
S(\xi) = \text{sech} r : \exp \left\{ a^\dagger b e^{i\phi} \tanh r \right\} \\
\times \exp \left\{ (a^\dagger a + b^\dagger b) \left( \text{sech} r - 1 \right) \right\} \\
\times \exp \left\{ -abe^{-i\phi} \tanh r \right\}
\]

Similarly, to derive Equation (7), we obtain

\[
[S(\xi)]^{T_B} = \text{sech} r : \exp \left\{ a^\dagger b e^{i\phi} \tanh r \right\} \\
\times \exp \left\{ (a^\dagger a + b^\dagger b) \left( \text{sech} r - 1 \right) \right\} \\
\times \exp \left\{ -abe^{-i\phi} \tanh r \right\}
\]

Accordingly, for a beam splitter operator, we have\(^{[29,30]}\)

\[
B(\theta) = \exp \left\{ \theta \left( a^\dagger b e^{i\phi} - ab^\dagger e^{-i\phi} \right) \right\} \\
= : \exp \left\{ (a^\dagger a + b^\dagger b) \left( \cos \theta - 1 \right) \right\} \\
\times \exp \left\{ (a^\dagger b^\dagger + ab^\dagger e^{-i\phi}) \sin \theta \right\}
\]

Comparing Equations (49) and (50), we note that if we take $\cos \theta = \text{sech} r, \sin \theta = \tanh r, \theta = \arg \sin[\tanh r]$, then Equation (49) can be rewritten as

\[
[S(\xi)]^{T_B} = \cos \theta \exp \left\{ \theta \left( a^\dagger b e^{i\phi} - ab^\dagger e^{-i\phi} \right) \right\} \\
= \cos \theta \times B(\theta)
\]

It is shown that the partial transpose of the two-mode squeezing operator can be equivalent to a beam splitter operator except for a factor $\cos \theta$. Note that the partial transpose of the two-mode squeezing operator cannot be obtained by replacing $b(b^\dagger)$ with $b'(b)$ from Equation (47). This operation is valid only within the normal ordering form.

### 3.4. Covariance Matrix of Partially Transposed Two-Mode Gaussian States

For two-mode Gaussian states, the characteristic function of the density operator $\rho_{AB}$ can be expressed as\(^{[27]}\)

\[
\chi(\xi) = \exp \left\{ -\frac{1}{2} \xi \Omega \Omega^T \xi^* \right\}
\]
where we have assumed $\vec{X} = 0$ again for simplicity, and $\vec{X} = \text{tr}(\hat{X} \rho)$, $\xi = (q_1, p_1, q_2, p_2)$ and the covariance matrix $V$ and $\Omega$, respectively, are given by

$$
V = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{pmatrix} = V^T
$$

\hspace{1cm} (53)

$$
\Omega = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix} \Omega V^T = 1
$$

\hspace{1cm} (54)

According to Equation (42), the characteristic function of partially transposed density operator $(\rho_{AB})^T_x$ is given by $\chi_{(\rho_{AB})^T_x}(\xi) = \exp[-1/2 \Sigma \Omega V \Omega^T \Sigma^T \xi]$, with $\Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$. Thus the covariance matrix of partially transposed density operator is given by

$$
V_{(\rho_{AB})^T_x} = \Omega^T \Sigma \Omega \Sigma^T \Omega = \Sigma V \Sigma^T
$$

\hspace{1cm} (55)

with $\tilde{\Sigma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}$. Equation (55) is just the relation between the two covariance matrices before and after the partial transpose. This case can be easily extended to multi-mode case.

To check the relationship in Equation (55), we consider the average value of $a_{14} = \langle Q_1 P_2 \rangle$ where $Q_1 = (a_1 + a_1^\dagger)$ and $P_2 = (a_2 - a_2^\dagger)/i$. We can observe that $Q_1 P_2 = \langle (a_1 + a_1^\dagger)(a_2 - a_2^\dagger)/i \rangle$. Using Equation (30), we have

$$
\langle Q_1 P_2 \rangle_{(\rho_{AB})^T_x} = \langle (a_1 + a_1^\dagger)(a_2 - a_2^\dagger)/i \rangle_{\rho_{AB}} = -\langle Q_1 P_2 \rangle_{\rho_{AB}}
$$

\hspace{1cm} (56)

as expected.

4. Entanglement of Two-Mode Squeezed States in Laser Channels

In this section, we further consider another important application of Equation (39). The partial transpose is used to examine whether an entanglement exists for a multi-mode quantum system after one or more modes are transposed while others remain unchanged. Here, we consider the two-mode case as an example and derive analytically the entanglement degree in laser channels. For this purpose, we first briefly review the logarithmic negativity and then derive analytical expressions of both the normal ordering form of the density operator and its logarithmic negativity in the laser channels.

4.1. Negativity and Logarithmic Negativity

If a two-mode quantum state is separable, then the density operator can be expressed as

$$
\rho_{AB} = \sum_i p_i \rho_{A_i} \otimes \rho_{B_i}
$$

\hspace{1cm} (57)

Thus, the corresponding partial transpose with respect to $B$ mode is given by

$$
(\rho_{AB})^T_B = \sum_i p_i \rho_{A_i} \otimes (\rho_{B_i})^T
$$

\hspace{1cm} (58)

As mentioned previously, $(\rho_B)^T$ remains in a quantum state. Thus, for a separable state, the partially transposed operator in Equation (58) can be qualified as a quantum state. The non-negative eigenvalues are shared by the two states after and before the partial transpose. However, for an entangled state $\rho_{AB}$, cannot be expressed in the form of Equation (57). Thus, its partial transpose may not be a quantum state although the partially transposed operator satisfies both Equations (45) and (46). This implies that there may be negative eigenvalues for $(\rho_{AB})^T$. Based on this point, we can quantify the degree of entanglement using negative eigenvalues [11] by defining

$$
N(\rho_{AB}) = \frac{1}{2} \text{Tr} \left[ \sqrt{(\rho_{AB})^T_x^2 - (\rho_{AB})^T_x} \right]
$$

\hspace{1cm} (59)

and

$$
E(\rho_{AB}) = \log_2 \left[ 1 + 2N(\rho_{AB}) \right] = \log_2 \left\| (\rho_{AB})^T_x \right\| + 1
$$

\hspace{1cm} (60)

where $\left\| \hat{O} \right\|$ is the trace norm, $\left\| \hat{O} \right\| = \sqrt{\text{Tr} \hat{O}^\dagger \hat{O}}$. Equations (59) and (60) are negativity and logarithmic negativity, respectively. They are both effective for Gaussian and non-Gaussian states and are much more amenable to evaluation than distillable entanglement, relative entropy entanglement, and entanglement cost.

4.2. Normal Ordering Form of Density Operator in the Laser Channel

Next, we consider the evolution of the density operator in a laser channel, which includes photon loss and the thermal channel as its special cases. For a single-mode system, the master equation describing the dynamical process is given by [14]

$$
\frac{d}{dt} \rho(t) = g \left[ 2a^\dagger \rho a - aa^\dagger \rho - \rho aa^\dagger \right] + \kappa \left[ 2a^\dagger \rho a - a^\dagger a \rho - \rho a^\dagger a \right]
$$

\hspace{1cm} (61)

where $g$ and $\kappa$ are the gain and loss factors, respectively.
Using the entangled state representation from Equation (61), we can obtain the evolution of the Wigner function as:

\[
W(\alpha, t) = \sum_{\beta} \frac{1}{A} \int \frac{d^2 \beta}{\pi} e^{\frac{1}{2} \left[ - (\beta^2 - 4) \right]} W(\beta, 0)
\]

where \( W(\beta, 0) \) is the initial Wigner function and

\[
A = \frac{\kappa + \overline{a}}{\kappa - \bar{a}} \left( 1 - e^{-2(\kappa - \bar{a})} \right)
\]

For a continuous variable system, two-mode entangled Gaussian states are the main resource of quantum information and processing and are easily generated experimentally. Thus, many researchers have focused on them.\(^{[31-35]}\) Here, we consider the two modes of the TMSV passing through the channels, independently. The initial Wigner function of the TMSV is given by

\[
W(\beta_1, \beta_2, 0) = \frac{1}{\pi^3} \exp \left\{ -2 \left[ |\beta_1|^2 + |\beta_2|^2 \right] \cosh 2r \right\} \\
\times \exp \left\{ 2 (\beta_1 \beta_2 + \beta_1^* \beta_2^*) \sinh 2r \right\}
\]

Substituting Equation (64) into Equation (62), we can obtain the Wigner function after the laser channels; that is,

\[
W(\alpha_1, \alpha_2, t) = \frac{2}{A_1 A_2} \int \frac{d^2 \alpha_1 d^2 \alpha_2}{\pi^2} W(\beta_1, \beta_2, 0) \\
\times \exp \left\{ -2 A_1 \left[ |\alpha_1 - \beta_1 e^{-\left( (\kappa-\bar{a}) \right)}|^2 \right] \right\} \\
\times \exp \left\{ -2 A_2 \left[ |\alpha_2 - \beta_2 e^{-\left( (\kappa-\bar{a}) \right)}|^2 \right] \right\}
\]

where \( g_i \) and \( \kappa_i (i = 1, 2) \) are the gain and loss factors, respectively, and

\[
A_j = \frac{\kappa_j + \overline{a}}{\kappa_j - \bar{a}} \left( 1 - e^{-2(\kappa - \bar{a})} \right), j = 1, 2
\]

From Equation (65), we obtain

\[
W(\alpha_1, \alpha_2, t) = \frac{1}{\pi^2} \exp \left\{ -2 \lambda_1 |\alpha_1|^2 - 2 \lambda_2 |\alpha_2|^2 \right\} \\
\times \exp \left\{ 2 \lambda_{12} (\alpha_1^* \alpha_2^* + \alpha_1 \alpha_2) \right\}
\]

where we have set

\[
\lambda_1 = \frac{1}{2} \left( \lambda_{12} + e^{-2(\kappa-\bar{a})} \right) \cosh 2r
\]

\[
\lambda_2 = \frac{1}{2} \left( \lambda_{12} - e^{-2(\kappa-\bar{a})} \right) \cosh 2r
\]

\[
\lambda_{12} = \frac{1}{2} e^{-2(\kappa-\bar{a})} e^{-2(\kappa-\bar{a})} \sinh 2r
\]

and

\[
\Gamma = \left( A_1 e^{-2(\kappa-\bar{a})} + A_2 e^{-2(\kappa-\bar{a})} \right) \cosh 2r \\
+ A_1 A_2 + e^{-2(\kappa-\bar{a})} e^{-2(\kappa-\bar{a})}
\]

Subsequently, using Equation (21) or Equation (41), and the integration within an ordered product of operators (IWOP) technique,\(^{[36,37]}\) one can obtain the final density operator by performing the integration; that is,

\[
\rho_{ab} = (4\pi)^2 \int W(\alpha_1, \alpha_2, t) \Delta_\alpha (\alpha_1) \Delta_\gamma (\alpha_2) d^2 \alpha_1 d^2 \alpha_2
\]

\[
= 4 \Omega_0 : e^{2\Omega_{0} \left( a^* a + 2 \Omega_{0} b^* b + 2 \Omega_{0} (ab + a^* b^*) \right)} : \]

where

\[
\Omega_0 = \frac{1}{\Gamma} \left( \frac{\lambda_{12} - \lambda_1 - \lambda_2 \lambda_{12}}{(\lambda_{12} + 1)(\lambda_{12} + 1) - \lambda_{12}^2} \right) > 0
\]

\[
\Omega_1 = \frac{\lambda_{12}^2 - \lambda_1 - \lambda_{12} \lambda_2}{(\lambda_{12} + 1)(\lambda_{12} + 1) - \lambda_{12}^2}
\]

\[
\Omega_2 = \frac{\lambda_{12}^2 - \lambda_2 - \lambda_{12} \lambda_1}{(\lambda_{12} + 1)(\lambda_{12} + 1) - \lambda_{12}^2}
\]

\[
\Omega_3 = \frac{\lambda_{12}^2 - \lambda_{12} - \lambda_{12}^2}{(\lambda_{12} + 1)(\lambda_{12} + 1) - \lambda_{12}^2}
\]

Equation (70) is the normal ordering form of the density operator after the laser channels.

4.3. Trace Norm of decohered TMSV in the Laser Channel

According to Equation (4), we obtain the partial transpose for any \( b \) of \( \rho_{ab} \) (i.e., replacing \( b^* \) with \( b \)). Then,

\[
\left[ \rho_{ab} \right]^{T_b} = 4 \Omega_0 : e^{2\Omega_{0} \left( a^* a + 2 \Omega_{0} b^* b + 2 \Omega_{0} (ab + a^* b^*) \right)} : \\
= 4 \Omega_0 : e^{2\left( a^* b^* \right) \left( \Omega_{12} \Omega_{2} | a \rangle \langle b | \right)} :
\]

Using the completeness of the two-mode coherent states or the following formula,\(^{[38,39]}\)

\[
\{ : \exp[\alpha^* (u - l) a] : \} \{ : \exp[\alpha^* (u' - l) a] : \} \\
= : \exp[\alpha^* (uu' - l) a] : \]

where \( a = (a \ b \ \cdots \ d)^T \) and \( \alpha' = (a^* \ b^* \ \cdots \ d^* \) , the squared of \( \left[ [\rho_{ab}]^{T_b} \right]^2 \) can be calculated as

\[
\left[ [\rho_{ab}]^{T_b} \right]^2 = (4 \Omega_0)^2 : \exp \left\{ (a^* b^*) [M - I] (\alpha^* b) \right\} :
\]

where we have set

\[
M = \begin{pmatrix}
w & \nu \\
\nu & u
\end{pmatrix}
\]

\[
w = (1 + 2 \Omega_{0})^2 + (2 \Omega_{0})^2
\]

\[
u = (1 + 2 \Omega_{0})^2 + (2 \Omega_{0})^2
\]

\[
u = 4 \Omega_{0} (1 + \Omega_{1} + \Omega_{2})
\]
Clearly, \( w > 0, u > 0 \) and \( wu - v^2 \geq 0 \). Using the formula converting normal ordering form of an operator to its close exponential form\(^{[38,39]}\)

\[
\hat{\Theta} \equiv \exp \{ \hat{a}^\dagger (N - I) a \} := \exp \{ \hat{a}^\dagger (\ln N) a \}
\]

leading to

\[
\sqrt{\Theta} =: \exp \left\{ \left( a^\dagger \ b^\dagger \right) \left( \sqrt{N} - I \right) \left( a \ b \right) \right\} : 
\]

then we have

\[
\sqrt{\left[ \rho_{ab} \right]}^2 = 4 \Omega_0 : \exp \left\{ \left( a^\dagger \ b^\dagger \right) \left( \sqrt{M} - I \right) \left( a \ b \right) \right\} : 
\]

where we may define \( \sqrt{M} = \begin{pmatrix} p & m \\ m & q \end{pmatrix} \) owing to the symmetry of \( M \).

Further, using the completeness of the coherent state representation again, one can obtain

\[
\left\| \left[ \rho_{ab} \right] \right\| = \frac{4 \Omega_0}{\det(I - \sqrt{M})} = \frac{4 \Omega_0}{\Delta} 
\]

where \( \Delta = (1 - p)(1 - q) - m^2 = 1 + \sqrt{wu - v^2} - \Lambda_+ - \Lambda_- \), and \( \Lambda_{\pm} = \pm \sqrt{wu - v^2} \pm \sqrt{wu - v^2} \). In particular, the condition \( \det M = wu - v^2 \geq 0 \) must be satisfied, leading to \( \Lambda_{\pm} \) being positive real numbers. Thereafter, the logarithmic negativity of the decohered TMSV in the laser channel is given by

\[
E_N = \log_2 \frac{4 \Omega_0}{\Delta} 
\]

It is evident that a non-zero degree of entanglement exists only when \( 4 \Omega_0 > \Delta \). This condition will determine the critical time or the minimum entanglement time.

### 4.4. Some Special Cases

Next, we focus on special cases, including the ideal TMSV: the TMSV is decohered by bi-symmetrical photon loss as well as gain and loss.

**Entanglement of the TMSV:** For this case, we should take \( t = 0 \), which leads to \( 2 \Omega_0 = 2 \Omega = -1, 4 \Omega = \tanh^2 r \), and \( w = u = \tanh^2 r, v = 0 \), and \( \Lambda_+ = \tanh r \). Thus, the logarithmic negativity of the TMSV is given by \( E_N = \log_2 e^{2r} \), as expected.

**Entanglement of the TMSV with Bi-Symmetrical Photon-Loss:** For this case, we can obtain \( g_i = 0 \) and \( \kappa_1 = \kappa_2 = \kappa \). Thereafter, we have

\[
w = u = \frac{T^2 (1 + R^2 \tanh^2 r)}{(1 - R^2 \tanh^2 r)^2} \tanh^2 r
\]

\[
v = \frac{2RT^2 \tanh^3 r}{(1 - R^2 \tanh^2 r)^2}
\]

Thus,

\[
E_N = \log_2 \frac{1}{1 - T(1 - e^{-2r})}
\]

where \( R = 1 - T \) and \( T = e^{-2x^2} \), respectively. In particular, when \( T = 1 \), Equation (83) is reduced to the negativity \( E_N = \log_2 e^{2x^2} \) of the TMSV. Equation (83) is in agreement with the results in Ref. \([34, 40]\), as expected. One can observed that \( E_N \) increases and decreases as \( r \) and \( 1 - T \) increase. In the limit \( T \to 0 \), we have \( E_N \to 0 \). This case is true for a single-side photon-loss case, in which \( w = T \tanh^2 r, v = \sqrt{TR} \tanh^3 r \), and \( u = (R^2 \tanh^2 r + T) \tanh^3 r \), and \( E_N \) can be obtained by substituting these into Equations (79)-(80), which are not shown here for simplicity. This indicates that the entanglement always exists only when the loss is present.

**Bi-Symmetrical Photon Loss and Gain:** For this case, we can consider \( g_i = g, \kappa_1 = \kappa_2 = \kappa \). From Equation (80), we have that

\[
t_c = \frac{1}{2(\kappa - g)} \ln \frac{g + \kappa \tanh r}{g(1 + \tanh r)}
\]

for any squeezing parameter \( r > 0 \). If \( t \geq t_c \), then \( E_N = 0 \). Noticeably, a threshold value of time \( t_c \) exists when the system passes a channel with gain and loss. This threshold time is different from the case only with loss.

### 5. Conclusion

In this study, we examined both the transpose of a single-mode operator and the partial transpose of a two-mode operator. Based on the definition of the operator transpose in the Fock space and the normal ordering form of the operator, we derived a lemma; that is, the transposed operator for both the single-mode and the two-mode cases can be obtained by replacing the transposed mode with respect to \( b(b^\dagger) \) with \( b^\dagger(b) \) within normal ordering form. The classical numbers involved remain unchanged. Using this property, we further considered some transposed operators, such as displaced, Wigner, and single and/or two-mode squeezing operators. Thereafter, we constructed the relation between distribution functions in the phase space before and after (partial) transpose, including the Wigner and characteristic functions. Further, the relation of average values is bridged, including the covariance matrix. The above discussions can be extended to a multimode case. As an effective and direct application of the lemma, we analytically calculated the entanglement degree of the TMSV when passing through the laser channel using the logarithmic negativity. In addition, it will be effective for the calculation of the entanglement of non-Gaussian states using the normal ordering form of the partially transposed density operator. Our method could be effective for entangled Fermi systems, which will be further considered in the future.

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Conflict of Interest
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