On various integrable discretizations of a general two-component Volterra system

Corina N Babalic\textsuperscript{1,2} and A S Carstea\textsuperscript{2,3}

\textsuperscript{1} Department of Physics, University of Craiova, Craiova, Romania
\textsuperscript{2} Department of Theoretical Physics, National Institute of Physics and Nuclear Engineering, Atomistilor 407, 077125, Magurele, Bucharest, Romania
\textsuperscript{3} Author to whom any correspondence should be addressed.

E-mail: carstea@gmail.com

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Abstract

We present two integrable discretizations of a general differential–difference bicomponent Volterra system. The results are obtained by discretizing directly the corresponding Hirota bilinear equations in two different ways. Multisoliton solutions are presented together with a new discrete form of Lotka–Volterra equation obtained by an alternative bilinearization.

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(Some figures may appear in colour only in the online journal)

1. Introduction

One of the main difficulties in the topic of integrable systems is obtaining an integrable discretization of a given partial differential or differential–difference integrable system. Applying integrability criteria like complexity growth [1] or singularity confinement [2] is not always easy because the discrete lattice equations have, in general, complicated forms. However, it is much more convenient to start with some general lattice equation and impose some simpler (but more restrictive) integrability requirements (like, for instance, cube consistency [3] which leads immediately to the discrete variant of ‘zero curvature representation’).

One of the powerful methods among the methods of finding integrable discretizations is the Hirota bilinear method. The idea is quite simple. First, the integrable differential or differential–difference integrable system has to be correctly bilinearized (in the sense of allowing the construction of a general multisoliton solution). After that, in the first step, one has to replace differential Hirota operators with discrete ones preserving the gauge invariance. Of course, the resulting bilinear fully discrete system is not necessarily integrable, so in the second step, the multisoliton solution must be found [4]. If this exists, then the discrete bilinear
system is integrable and, in the final step which is rather complicated, the nonlinear form has to be recovered. Here, it is possible to introduce an auxiliary function, as Hirota has shown in [5]. We have to clarify here the concept of integrability. Usually, the integrability for a given partial differential or a partial discrete equation relies on the existence of an infinite number of independent integrals in involution that usually can be computed from the Lax pairs. Accordingly, if the equation can be written as a compatibility condition of two non-trivial linear operators (i.e. Lax pairs), then it is automatically completely integrable. However, the alternative formulation given by the Hirota bilinear form is also used in proving integrability by requiring the existence of a general multisoliton solution. Here, the word general means the solution describing multiple collisions of an arbitrary number of solitons having arbitrary parameters and phases and also for all branches of dispersion relations. Any constraint in the form of a multisoliton solution breaks complete integrability. The integrals of motion are related to the soliton parameters which, because of elastic collisions, remain unchanged (and they are also related to the spectrum of the Lax operator invariant as well since the equation itself is an isospectral deformation). The construction of the multisoliton solution is quite difficult, but it has been observed that once a general three-soliton solution is constructed, then it can be proved by induction that the general N-soliton can be constructed as well. The existence of the three-soliton solution has been used in the classification of completely integrable bilinear equations [11, 12] as an integrability criterion.

In this paper, we are going to give two integrable discretizations of a general two-component bidirectional Volterra system (in the sense that the dispersion relation has two branches). This system is rather old. It was formulated for the first time by Hirota and Satsuma [7] but with constraints on parameters. Then, various solutions (rational, white and dark solitonic) have been obtained [8–10]. Curiously enough, the full discretization has not been studied so far, although the initial system can be formulated as a coupled differential–difference focusing or defocusing mKdV system with non-zero boundary conditions. In this paper, we present two fully discrete bilinear forms, show the structure of an N-soliton solution and then recover the nonlinear form. For the second discretization, we extend the approach of Hirota presented in [5] and use two auxiliary functions. An important fact is that the multisoliton solution in the case of second discretization has practically the same phase factors and interaction terms as in the differential–difference one. The same fact has been observed by Hirota and Tsujimoto in [5, 14, 6] for other cases. They have shown that for many examples including lattice mKdV, lattice NLS, lattice coupled mKdV, the structure of a soliton solution remains the same as in the case of differential–difference analogues (although, it is true that all their examples support only unidirectional solitons). We think that this happens in many other cases and as another example we construct a new completely integrable discretization of the Lotka–Volterra equation.

2. Discretization of the general Volterra system

The differential–difference system under consideration is

\[
\begin{align*}
\dot{Q}_n &= (c_0 + c_1 Q_n + c_2 Q_n^2) (R_{n+1} - R_{n-1}) \\
\dot{R}_n &= (c_0 + c_1 R_n + c_2 R_n^2) (Q_{n+1} - Q_{n-1}),
\end{align*}
\] (1)

where \(c_0\), \(c_1\) and \(c_2\) are arbitrary constants and the dot means a derivative with respect to time. In [8], we have shown that following the simple scalings and translations,

\[
\begin{align*}
\bar{u}_n &= \frac{2c_2}{\sqrt{4c_0c_2 - c_1^2}} \left( Q_n + \frac{c_1}{2c_2} \right), \\
\bar{v}_n &= \frac{2c_2}{\sqrt{4c_0c_2 - c_1^2}} \left( R_n + \frac{c_1}{2c_2} \right),
\end{align*}
\]
we can cast the system in the following form:
\[
\dot{u}_n = \left(1 + u_n^2\right)(v_{n+1} - v_{n-1}) \\
\dot{v}_n = \left(1 + v_n^2\right)(u_{n+1} - u_{n-1}),
\]
where \(u_n, v_n \to x\) as \(n \to \pm\infty\) and \(\alpha = c_1\left(4c_0c_2 - c_1^2\right)^{-1/2}\). These transformations are valid only in the case of non-zero \(c_2\). Of course, one must be careful about the square root appearing in the definition of \(\alpha\). In the case of a negative argument, the system is changed to a defocusing form:
\[
\dot{u}_n = \left(1 - u_n^2\right)(v_{n+1} - v_{n-1}) \\
\dot{v}_n = \left(1 - v_n^2\right)(u_{n+1} - u_{n-1}).
\]
From the point of view of integrability, this is not crucial since an overall imaginary unit factor for \(u_n, v_n\) will change one system into the other. But for the soliton dynamics in real space (which we are not discussing here), the solutions of the above systems behave in a completely different manner (only the uniconponent case has been analyzed quite recently in [10]).

For the system (2), the Hirota bilinear form is given by
\[
f_n g_n - f_n \dot{g}_n = (1 + \alpha^2)(F_{n+1} G_{n-1} - G_{n+1} F_{n-1}) \\
\dot{F}_n G_n - F_n \dot{G}_n = (1 + \alpha^2)(f_{n+1} g_{n-1} - g_{n+1} f_{n-1}) \\
(1 + i\alpha) F_{n+1} G_{n-1} + (1 - i\alpha) F_{n-1} G_{n+1} = 2f_{n} g_{n} \\
(1 + i\alpha) f_{n+1} g_{n-1} + (1 - i\alpha) f_{n-1} g_{n+1} = 2F_{n} G_{n},
\]
where the nonlinear substitutions are
\[
u_n = \alpha - i \frac{\partial}{\partial t} \ln \frac{f_n(t)}{g_n(t)}, \quad v_n = \alpha - i \frac{\partial}{\partial t} \ln \frac{F_n(t)}{G_n(t)}.
\]
The bilinear system (4)–(7) is completely integrable [8] and has the following \(N\)-soliton solution:
\[
f_n = \sum_{\mu_1:...:\mu_N \in \{0,1\}} \left( \prod_{i=1}^{N} (\beta_i p_i^{\mu_i} e^{\omega_i})^{\mu_i} \prod_{i<j} A_{ij}^{\mu_i \mu_j} \right) \\
F_n = \sum_{\mu_1:...:\mu_N \in \{0,1\}} \left( \prod_{i=1}^{N} (\beta'_i p_i^{\mu_i} e^{\omega_i})^{\mu_i} \prod_{i<j} A_{ij}^{\mu_i \mu_j} \right) \\
g_n = \sum_{\mu_1:...:\mu_N \in \{0,1\}} \left( \prod_{i=1}^{N} (\gamma_i p_i^{\mu_i} e^{\omega_i})^{\mu_i} \prod_{i<j} A_{ij}^{\mu_i \mu_j} \right) \\
G_n = \sum_{\mu_1:...:\mu_N \in \{0,1\}} \left( \prod_{i=1}^{N} (\gamma'_i p_i^{\mu_i} e^{\omega_i})^{\mu_i} \prod_{i<j} A_{ij}^{\mu_i \mu_j} \right),
\]
where the dispersion relation and phase factors are
\[
\omega_i = \epsilon_i(1 + \alpha^2)\frac{1 - p_i^2}{p_i}, \\
\beta_i = \frac{i\alpha}{p_i} \left(1 + \frac{\epsilon_i}{2} \left( p_i + p_i^{-1} \right) \right), \quad \gamma_i = \frac{i\alpha}{p_i} \left(1 + \frac{\epsilon_i}{2} \left( p_i + p_i^{-1} \right) \right) + \frac{\epsilon_i}{2},
\]
\[ p'_i = \frac{i\alpha \epsilon_i (1 - \frac{\epsilon_i}{2} (p_i + p_i^{-1}))}{(p_i - p_i^{-1})} + \frac{1}{2} \quad y'_i = \frac{i\alpha \epsilon_i (1 - \frac{\epsilon_i}{2} (p_i + p_i^{-1}))}{(p_i - p_i^{-1})} - \frac{1}{2} \]

\[ A_{ij} = \left( \frac{\epsilon_i p_i - \epsilon_j p_j}{1 - \epsilon_i \epsilon_j p_i p_j} \right)^2. \]

In order to construct an integrable discretization, we replace time derivatives in (4) and (5) with finite differences \((t \to m)\),
\[ f_n \to \frac{1}{\delta} (f(n, m + \delta) - f(n, m)), \]
and impose the invariance of the resulting bilinear equation with respect to multiplication with \(\exp(\mu n + \nu m)\) for any \(\mu, \nu\) (bilinear gauge invariance). In this first discretization, we discretize also equations (6) and (7) by assuming a gauge-invariant shift in the time variable.

The fully discrete gauge-invariant bilinear equations are given by
\[ \tilde{f}_n g_n - f_n \tilde{g}_n = \delta (1 + \alpha^2) (\tilde{F}_{n+1} G_{n-1} - \tilde{G}_{n+1} F_{n-1}) \]
\[ \tilde{F}_n G_n - F_n \tilde{G}_n = \delta (1 + \alpha^2) (\tilde{F}_{n+1} \tilde{G}_{n-1} - \tilde{G}_{n+1} f_{n-1}) \]
\[ (1 + i\alpha) \tilde{F}_{n+1} G_{n-1} + (1 - i\alpha) F_{n+1} \tilde{G}_{n-1} = \tilde{f}_n g_n + f_n \tilde{g}_n \]
\[ (1 + i\alpha) \tilde{f}_{n+1} g_{n-1} + (1 - i\alpha) \tilde{f}_{n-1} \tilde{g}_{n+1} = \tilde{F}_n G_n + F_n \tilde{G}_n, \]
where \(\tilde{f}_n = f(n, m + \delta)\) etc. In order to check the integrability, one has to compute the general \(N\)-soliton solution of the above system. It can be extended to the \(N\)-soliton solution in the form (the proof is given in the appendix):
\[ f_n = \sum_{\mu_1, \ldots, \mu_N \in \{0,1\}} \left( \prod_{i=1}^{N} (a_i p_i^\mu q_i^\mu)^{\mu_i} \prod_{i<j}^{N} M_{ij}^{\mu \mu_j} \right) \]
\[ F_n = \sum_{\mu_1, \ldots, \mu_N \in \{0,1\}} \left( \prod_{i=1}^{N} (A_i p_i^\mu q_i^\mu)^{\mu_i} \prod_{i<j}^{N} M_{ij}^{\mu \mu_j} \right) \]
\[ g_n = \sum_{\mu_1, \ldots, \mu_N \in \{0,1\}} \left( \prod_{i=1}^{N} (b_i p_i^\mu q_i^\mu)^{\mu_i} \prod_{i<j}^{N} M_{ij}^{\mu \mu_j} \right) \]
\[ G_n = \sum_{\mu_1, \ldots, \mu_N \in \{0,1\}} \left( \prod_{i=1}^{N} (B_i p_i^\mu q_i^\mu)^{\mu_i} \prod_{i<j}^{N} M_{ij}^{\mu \mu_j} \right). \]

where the dispersion relation and phase factors are given by
\[ q_i = \left( \frac{p_i + \delta \epsilon_i (1 + \alpha^2)}{p_i + p_i^{-1} \delta \epsilon_i (1 + \alpha^2)} \right)^{1/\delta} \]
\[ a_i = \frac{i\alpha (-1 + \frac{\epsilon_i}{2} (p_i + p_i^{-1})) - \frac{\epsilon_i}{2}}{(p_i - p_i^{-1})(1 + \delta + \delta \alpha^2)} \quad b_i = \frac{i\alpha (-1 + \frac{\epsilon_i}{2} (p_i + p_i^{-1})) + \frac{\epsilon_i}{2}}{(p_i - p_i^{-1})(1 + \delta + \delta \alpha^2)} \]
\[ A_i = \frac{i\alpha \epsilon_i (1 - \frac{\epsilon_i}{2} (p_i + p_i^{-1}))}{(p_i - p_i^{-1})(1 + \delta + \delta \alpha^2)} + \frac{1}{2} \quad B_i = \frac{i\alpha \epsilon_i (1 - \frac{\epsilon_i}{2} (p_i + p_i^{-1}))}{(p_i - p_i^{-1})(1 + \delta + \delta \alpha^2)} - \frac{1}{2} \]
Accordingly, our bilinear system is an integrable one. Now, we can proceed to recover the nonlinear form. Dividing (8) by (10) and (9) by (11) and taking into account that

\[
    \tan(a_n - a_{n-1}) = \frac{\delta(1 + \alpha^2) \tan(R_{n+1} - R_{n-1})}{1 + \alpha \tan(R_{n+1} - R_{n-1})},
\]

we obtain the following system:

\[
    \tan(\tilde{a}_n - a_n) = \frac{\delta(1 + \alpha^2) \tan(\tilde{R}_{n+1} - \tilde{R}_{n-1})}{1 + \alpha \tan(\tilde{R}_{n+1} - \tilde{R}_{n-1})},
\]

where \( a_n = \frac{1}{2} \log(f_n/g_n) \), \( R_n = \frac{1}{2} \log(F_n/G_n) \). To our knowledge, this system is a new one. We do not know how it is related to other integrable discretizations of Volterra systems. In the case of \( \alpha \to 0 \), the classical lattice self-dual network of Hirota is obtained.

Since the phase factors are defined up to the multiplication with the same constant factor (which we take to be the imaginary unit), the \( f_n, g_n \) (and \( F_n, G_n \) as well) will be complex conjugated, so the physical fields \( a_n \) and \( R_n \) are the real functions. Here we give the one- and two-soliton solutions (see also figure 1):

\[
    a_n = \frac{1}{2} \log \left( \frac{1 + i a_1 p_n^1 q_n^{1\text{d}}} {1 + i b_1 p_n^1 q_n^{1\text{d}}} \right), \quad R_n = \frac{i}{2} \log \left( \frac{1 + i A_1 p_n^1 q_n^{1\text{d}}}{1 + i B_1 p_n^1 q_n^{1\text{d}}} \right).
\]

\[
    q_n = \frac{1}{2} \log \left( \frac{1 + ia_1 p_n^1 q_n^{1\text{d}} + ia_2 p_n^2 q_n^{2\text{d}}}{1 + ib_1 p_n^1 q_n^{1\text{d}} + ib_2 p_n^2 q_n^{2\text{d}}} - a_1 A_2 M_{12}(p_1 p_2)^\mu(q_1 q_2)^{\mu\text{d}} \right),
\]

\[
    R_n = \frac{1}{2} \log \left( \frac{1 + ia_1 p_n^1 q_n^{1\text{d}} + ia_2 p_n^2 q_n^{2\text{d}}}{1 + ib_1 p_n^1 q_n^{1\text{d}} + ib_2 p_n^2 q_n^{2\text{d}}} - A_1 A_2 M_{12}(p_1 p_2)^\mu(q_1 q_2)^{\mu\text{d}} \right).
\]

However, one can obtain a different nonlinear form by taking \( q_n = f_n/g_n \) and \( r_n = F_n/G_n \), namely

\[
    \tilde{a}_n = \frac{z_1 \tilde{r}_{n+1} + z_2 r_{n-1}}{z_1^2 \tilde{r}_{n+1} + z_1 r_{n-1}}, \quad \tilde{r}_n = \frac{z_1 \tilde{q}_{n+1} + z_2 q_{n-1}}{z_1^2 \tilde{q}_{n+1} + z_1 q_{n-1}},
\]

where \( z_1 = 1 + i \alpha + \delta(1 + \alpha^2) \), \( z_2 = 1 - i \alpha - \delta(1 + \alpha^2) \). In the case of \( q_n = r_n \), the system becomes the Nijhoff–Capel lattice mKdV equation [13], but with complex coefficients. Of course, a complete description imposes finding the Lax pairs, Hamiltonian structure using \( r \)-matrices and so on [18].
3. Second discretization

We may discretize the bilinear equations (4)–(7) in a simpler way. Namely we only discretize the dispersion equations (4) and (5) which depend explicitly on time and leave (6) and (7) unmodified. We obtain

\[
\tilde{f}_n g_n - f_{n+1} \bar{g}_{n+1} = \delta(1 + \alpha^2) (F_{n+1} G_{n-1} - \bar{G}_{n+1} F_{n-1}) 
\]

(20)

\[
\tilde{F}_n G_n - F_{n+1} \bar{G}_{n+1} = \delta(1 + \alpha^2) (\tilde{f}_{n+1} g_{n-1} - \bar{g}_{n+1} f_{n-1}) 
\]

(21)

\[
(1 + i\alpha) F_{n+1} G_{n-1} + (1 - i\alpha) F_{n-1} G_{n+1} = 2 f_n g_n
\]

(22)

\[
(1 + i\alpha) f_{n+1} g_{n-1} + (1 - i\alpha) f_{n-1} g_{n+1} = 2 F_n G_n. 
\]

(23)

This bilinear system is again completely integrable. It has the N-soliton solution of the same form:

\[
f_n = \sum_{\mu_1, \ldots, \mu_N \in \{0, 1\}} \left( \prod_{i=1}^{N} (\beta_i p_i^{(n)} q_i^{(m)})^{\mu_i} \prod_{i<j} A_{ij}^{\mu_i \mu_j} \right) 
\]

(24)

\[
F_n = \sum_{\mu_1, \ldots, \mu_N \in \{0, 1\}} \left( \prod_{i=1}^{N} (\beta'_i p_i^{(n)} q_i^{(m)})^{\mu_i} \prod_{i<j} A_{ij}^{\mu_i \mu_j} \right) 
\]

(25)

\[
g_n = \sum_{\mu_1, \ldots, \mu_N \in \{0, 1\}} \left( \prod_{i=1}^{N} (\gamma_i p_i^{(n)} q_i^{(m)})^{\mu_i} \prod_{i<j} A_{ij}^{\mu_i \mu_j} \right) 
\]

(26)

\[
G_n = \sum_{\mu_1, \ldots, \mu_N \in \{0, 1\}} \left( \prod_{i=1}^{N} (\gamma'_i p_i^{(n)} q_i^{(m)})^{\mu_i} \prod_{i<j} A_{ij}^{\mu_i \mu_j} \right). 
\]

(27)

The dispersion relation and the phase factors are

\[
q_i = \left( \frac{p_i + \delta \epsilon_i (1 + \alpha^2)}{p_i + \alpha^2 \epsilon_i} \right)^{1/\beta}
\]

\[
\beta_i = \frac{i \alpha (1 + \frac{\alpha}{2} (p_i + p_j^{-1})) - \epsilon_i}{2}, \quad \gamma_i = \frac{i \alpha (1 + \frac{\alpha}{2} (p_i + p_j^{-1})) + \epsilon_i}{2}
\]

\[
\beta'_i = \frac{i \alpha \epsilon_i (1 + \frac{\alpha}{2} (p_i + p_j^{-1})) + \frac{1}{2}}{2}, \quad \gamma'_i = \frac{i \alpha \epsilon_i (1 + \frac{\alpha}{2} (p_i + p_j^{-1})) - \frac{1}{2}}{2}
\]

\[
A_{ij} = \left( \frac{\epsilon_i p_i - \epsilon_j p_j}{1 - \epsilon_i \epsilon_j p_i p_j} \right)^2.
\]

The nonlinear form can be easily recovered, although the system now is more complicated and it will involve two auxiliary functions \(w_n\) and \(v_n\). We divide (20) by \(g_n \bar{g}_n\) and (21) by \(G_n \bar{G}_n\). Calling \(x_n = f_n / g_n, y_n = F_n / G_n, w_n = G_{n-1} G_{n+1} / g_n \bar{g}_n, v_n = g_{n-1} \bar{g}_{n+1} / G_n \bar{G}_n\), we obtain

\[
\tilde{x}_n - x_n = \delta(1 + \alpha^2) (\tilde{y}_{n+1} - y_{n-1}) w_n
\]

\[
\tilde{y}_n - y_n = \delta(1 + \alpha^2) (\tilde{x}_{n+1} - x_{n-1}) v_n.
\]
But one can see immediately that
\[ u_{n+1}/v_n = \frac{G_n \tilde{G}_{n+2} G_n \tilde{G}_n}{G_{n+1} \tilde{G}_{n+1} G_{n-1} \tilde{G}_{n+1}} = \left( \frac{G_n^2}{G_{n+1}^2} \right) \left( \frac{\tilde{G}_{n+2} \tilde{G}_n}{\tilde{G}_{n+1} \tilde{G}_{n+1}} \right). \]

The factors in parentheses can be computed easily from (22) and (23) by dividing them into \( G_{n-1} G_{n+1} \) and \( g_{n-1} g_{n+1} \). Finally, the nonlinear form of our system is
\[ \tilde{x}_n - x_n = \delta (1 + \alpha^2) (\tilde{y}_{n+1} - y_{n-1}) u_n \]
\[ \tilde{y}_n - y_n = \delta (1 + \alpha^3) (\tilde{x}_{n+1} - x_{n-1}) v_n \]
\[ u_{n+1} = v_n \frac{x_{n+1} \tilde{x}_{n+1} (1 + i\alpha) + x_{n-1} \tilde{x}_{n+1} (1 - i\alpha)}{y_n \tilde{y}_{n+2} (1 + i\alpha) + y_{n-1} \tilde{y}_{n+2} (1 - i\alpha)} \]
\[ v_{n+1} = u_n \frac{y_{n+1} \tilde{y}_{n+1} (1 + i\alpha) + y_{n-1} \tilde{y}_{n+1} (1 - i\alpha)}{x_n \tilde{x}_{n+2} (1 + i\alpha) + x_{n-1} \tilde{x}_{n+2} (1 - i\alpha)}. \]

We can eliminate the auxiliary functions \( u_n \) and \( v_n \) and obtain the following higher order system:
\[ \tilde{x}_{n+1} - x_{n+1} = \frac{\tilde{y}_n - y_n}{\tilde{x}_{n+1} - x_{n+1}} x_{n+1} \tilde{x}_{n+1} (1 + i\alpha) + x_{n-1} \tilde{x}_{n+1} (1 - i\alpha) \]
\[ \tilde{y}_{n+1} - y_{n+1} = \frac{\tilde{x}_n - x_n}{\tilde{y}_{n+1} - y_{n+1}} y_{n+1} \tilde{y}_{n+1} (1 + i\alpha) + y_{n-1} \tilde{y}_{n+1} (1 - i\alpha) \]
\[ \tilde{x}_{n+2} - x_n = \frac{\tilde{y}_n - y_n}{\tilde{x}_{n+2} - x_n} x_{n+1} \tilde{x}_{n+2} (1 + i\alpha) + x_{n-1} \tilde{x}_{n+2} (1 - i\alpha). \]

An important remark is that we have the same phase factors and interaction term as in the differential–difference case, namely the system (4)–(7) (and that is why we kept the same notation). Also, in both equations (28) and (29), the step of time discretization \( \delta \) disappeared. So there is no trace of discretization in the solutions, except the dispersion relation. This means that the structure of the soliton solution is the same at the level of tau functions. However, the nonlinear form is different. For instance, the one- and two-soliton solutions
\[ x_n = \frac{1 + \beta_1 p_n^{1m}}{1 + \gamma_1 p_n^{1m}}, \quad \gamma_n = \frac{1 + \beta_1' p_n^{1m}}{1 + \gamma_1' p_n^{1m}}, \]
\[ x_n = \frac{1 + \beta_1 p_n^{1m} + \beta_2 p_n^{2m} + \beta_1 \beta_2 A_{12} (p_1 p_2)^{m} (q_1 q_2)^{m}}{1 + \gamma_1 p_n^{1m} + \gamma_2 p_n^{2m} + \gamma_1 \gamma_2 A_{12} (p_1 p_2)^{m} (q_1 q_2)^{m}}, \]
\[ y_n = \frac{1 + \beta_1 p_n^{1m} + \beta_2 p_n^{2m} + \beta_1' \beta_2' A_{12} (p_1 p_2)^{m} (q_1 q_2)^{m}}{1 + \gamma_1' p_n^{1m} + \gamma_2' p_n^{2m} + \gamma_1' \gamma_2' A_{12} (p_1 p_2)^{m} (q_1 q_2)^{m}}, \]

are complex functions (since the phase factors are complex). But if we define new fields \( \phi_n \) and \( \psi_n \) by \( x_n = \exp(i\phi_n) \) and \( y_n = \exp(i\psi_n) \), then the soliton solutions expressed by \( \phi_n \) and \( \psi_n \) are real functions and have the same shape as the ones in the first discretization.

**Remark.** This type of discretization involving auxiliary functions has also been done by Hirota and Tsujimoto in [5, 14, 6]. In their examples also the form and phase factors of soliton solutions remain the same. We believe that this happens in more general cases, although it must be proved rigorously.
We consider the substitution un and J. Phys. A: Math. Theor. 46 Of course, this equation can be seen as a particular case of the system (1) for Qn/Gamma1.

This bilinear system is integrable because it is equivalent (if we take the right-hand side) and the second one will remain the same. We shall obtain (4)–(7).

Discretizing the first two equations and after making the same steps as in the previous case, we would have obtained almost a similar form:

\[
\begin{align*}
\hat{Q}_n - Q_n &= \delta(R_{n+1} - R_{n-1})V_n \\
\hat{R}_n - R_n &= \delta(Q_{n+1} - Q_{n-1})W_n
\end{align*}
\]

\[
W_{n+1} = V_n \frac{c_0 + c_1 \hat{Q}_{n+1} + c_2 \hat{Q}^2_{n+1}}{c_0 + c_1 \hat{R}_n + c_2 \hat{R}^2_n}
\]

\[
V_{n+1} = W_n \frac{c_0 + c_1 \hat{R}_{n+1} + c_2 \hat{R}^2_{n+1}}{c_0 + c_1 \hat{Q}_n + c_2 \hat{Q}^2_n}.
\]

The main drawback of this system is that the soliton solution has a very complicated form.

4. A new form of the Lotka–Volterra equation

Let us give a nice and simple example related to the above construction, namely the differential–difference Lotka–Volterra equation:

\[
\frac{du_n}{dt} = u_n(u_{n+1} - u_{n-1}).
\]

Of course, this equation can be seen as a particular case of the system (1) for Q_n = R_n ≡ un and c_0 = c_2 = 0, c_1 = 1. However, because the simplified form (2) has been obtained only for c_2 ≠ 0, this equation cannot be studied as a particular case of the bilinear system (4)–(7). We consider the substitution \( u_n = G_n/F_n \) and obtain the following:

\[
\begin{align*}
DG_n, F_n &= (G_{n+1}F_{n-1} - G_{n-1}F_{n+1}) \\
G_nF_n &= F_{n+1}F_{n-1}.
\end{align*}
\]

Now, we are going to discretize only the first bilinear equation (imposing gauge invariance on the right-hand side) and the second one will remain the same. We shall obtain (t → mδ),

\[
\begin{align*}
\hat{G}_nF_n - G_n\hat{F}_n &= \delta(\hat{G}_{n+1}F_{n-1} - G_{n-1}\hat{F}_{n+1}) \quad (30) \\
G_nF_n &= F_{n+1}F_{n-1}. \quad (31)
\end{align*}
\]

This bilinear system is integrable because it is equivalent (if we take \( G_n = f_{n-1}f_{n+2}, F_n = f_nf_{n+1} \)) with a quadrilinear equation reducible to

\[
\begin{align*}
\tilde{f}_{n+1} &+ \delta f_{n-1}f_{n+2} - (1 + \delta)f_nf_{n+1} = 0, \quad (32)
\end{align*}
\]

which is the integrable bilinear form of the discrete Lotka–Volterra (with \( u_n = f_{n-1}f_{n+2}/f_nf_{n+1} \) [4],

\[
\tilde{u}_n = u_n \frac{1 - \delta + \delta u_{n-1}}{1 - \delta + \delta u_{n+1}}
\]

\[\]
However, the nonlinear form recovered from (30) and (31) is different, and we proceed as in the case of the system (28)–(29). Calling $x_n = G_n/F_n$ we obtain

$$\tilde{x}_n - x_n = \delta(\tilde{x}_{n+1} - x_{n-1}) w_n$$

(33)

$$w_{n+1} = w_n \frac{\tilde{x}_{n+1}}{x_n}. \quad (34)$$

This is a different variant of the lattice Lotka–Volterra equation. We can solve the second linear equation in $w_n$ and obtain

$$\tilde{x}_n - x_n = \delta(\tilde{x}_{n+1} - x_{n-1}) \prod_{k=-\infty}^{n} \frac{x_k}{x_{k-1}},$$

or we can eliminate $w_n$ from the first equation and find the following nice form:

$$\frac{\tilde{x}_{n+1} - x_{n+1}}{x_{n+1}} \frac{\tilde{x}_{n+1} - x_{n-1}}{x_{n-1}} \frac{x_n}{x_{n+1}} = 1.$$  

The first two factors on the left-hand side look like a discrete Schwartzian derivative [13], although the expression is not a cross ratio of four points but a cross ratio of diagonals of the two adjacent parallelograms formed by the six points. We think that this is a new equation, although its symmetric form may be related in a way (unknown to us) to some well-known one.

5. Conclusions

In this paper, we have presented two integrable discretizations of a general bicomponent differential–difference Volterra system. The main procedure was discretizing the differential Hirota bilinear operator and then recovering the nonlinear form with the aid of some auxiliary functions. This approach may lead to higher order nonlinear equations. We applied this procedure also to the well-known Lotka–Volterra equation and found a new discrete form; however, we started from a different bilinearization involving two tau functions. Relying on the fact that the structure of soliton solutions remains the same, we believe that this procedure will be effective in discretizing even non-integrable equations of the reaction–diffusion type (because it keeps the same structure of traveling waves).

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Appendix A. Proof of the $N$-soliton solution

As we said in the introduction, checking the existence of the three-soliton solution for a bilinear system is enough to prove integrability. However, because the structure of the soliton solution is rather complicated, we are going to sketch the proof for the $N$-soliton solution in the case of the first discretization (for the second one the phase factors and bilinear equations are simpler and everything goes in the same way). We follow the same procedure as in the old papers of Hirota [15, 16, 7, 17]. For simplicity, let us call $\Delta = \delta(1 + \sigma^2)$ and redefine $q_i \to q_i^\prime$. Plugging the $N$-soliton solution (12)–(15) into the first bilinear equation (8) we obtain
\begin{align}
\sum_{\mu=0,1} \sum_{\mu'=0,1} \left[ \prod_{i=1}^{N} a_{i}^{\mu} b_{i}^{\mu'} \left( \prod_{i=1}^{N} q_{i}^{\mu} - \prod_{i=1}^{N} q_{i}^{\mu'} \right) - \Delta \prod_{i=1}^{N} A_{i}^{\mu} B_{i}^{\mu'} \left( \prod_{i=1}^{N} q_{i}^{\mu} P_{i}^{\mu'-\mu} - \prod_{i=1}^{N} q_{i}^{\mu'} P_{i}^{\mu-\mu'} \right) \right] \\
\times \prod_{i<j} M_{ij}^{(\mu,\mu')+(\mu',\mu)} N^{p_{i}^{(\mu,\mu')+(\mu',\mu)m} q_{i}^{(\mu,\mu')+(\mu',\mu)m}} = 0. \tag{A.1}
\end{align}

Let the coefficient of the factor $\prod_{i=1}^{v_0} p_{i}^{\mu} \prod_{i=v_{0}+1}^{v_{1}} p_{i}^{2\mu} q_{i}^{\mu}$ be $F$. We have

\[ F = \sum_{\mu,\mu'=0,1} c_{\mu,\mu'} \left[ \prod_{i=1}^{v_0} a_{i}^{\mu} b_{i}^{\mu'} \left( \prod_{i=1}^{v_0} q_{i}^{\mu} - \prod_{i=1}^{v_0} q_{i}^{\mu'} \right) \right. \]
\[ \left. - \Delta \prod_{i=1}^{v_0} A_{i}^{\mu} B_{i}^{\mu'} \left( \prod_{i=1}^{v_0} q_{i}^{\mu} P_{i}^{\mu'-\mu} - \prod_{i=1}^{v_0} q_{i}^{\mu'} P_{i}^{\mu-\mu'} \right) \right] \prod_{i<j} M_{ij}^{(\mu,\mu')+(\mu',\mu)} , \]

where $c_{\mu,\mu'}$ implies that the summation over families of indices $\mu$ and $\mu'$ should be done under the requirements

\[ \mu_{i} + \mu'_{j} = 1, \quad i = 1, \ldots, v_0 \]
\[ \mu_{i} = \mu'_{j} = 1, \quad i = v_{0} + 1, \ldots, v_1 \]
\[ \mu_{i} = \mu'_{j} = 0, \quad i = v_{1} + 1, \ldots, N. \]

Substituting the expressions of the dispersion relation and phase factors (16)–(18) and introducing the multi-index $\sigma = \mu - \mu'$, we find that $F = \text{const.}\hat{F}$, where

\[ \hat{F} = \sum_{\sigma = \pm 1} \left\{ \prod_{i=1}^{v_0} \left( 1 + \epsilon_{i} \Delta p_{i}^{\sigma} \right) - \prod_{i=1}^{v_0} \left( 1 + \epsilon_{i} \Delta p_{i}^{-\sigma} \right) \right\} \prod_{i=1}^{v_0} \left( i\alpha - \cosh(\sigma \epsilon_{i}) \right) \]
\[ - \epsilon_{i} \sinh(\sigma \epsilon_{i})(1 + \Delta) - \Delta \left( \prod_{i=1}^{v_0} \left( 1 + \epsilon_{i} \Delta p_{i}^{\sigma} \right) p_{i}^{\sigma} - \prod_{i=1}^{v_0} \left( 1 + \epsilon_{i} \Delta p_{i}^{-\sigma} \right) p_{i}^{-\sigma} \right) \]
\[ \times \prod_{i=1}^{v_0} (i\epsilon_{i} - \cosh(\sigma \epsilon_{i}) + \sinh(\sigma \epsilon_{i})(1 + \Delta)) \right\} \]
\[ \times \prod_{i<j} (\epsilon_{i} \epsilon_{j} - \cosh(\sigma \epsilon_{i} - \sigma \epsilon_{j}))^{2}. \]

Making the notation $\chi_{i} = (\epsilon_{i} p_{i})^{1/2}$, $z = (1 + \Delta + i\alpha)/2$, $z^{*} = (1 + \Delta - i\alpha)/2$, the above relation becomes proportional to the following expression:

\[ \hat{F} = \sum_{\sigma = \pm 1} \left\{ \prod_{j=1}^{v_0} \left( 1 + \Delta x_{j}^{\sigma} \right) - \prod_{j=1}^{v_0} \left( 1 + \Delta x_{j}^{-\sigma} \right) \right\} \prod_{j=1}^{v_0} (-i\alpha - z^{*} x_{j}^{\sigma} + z x_{j}^{-\sigma}) \]
\[ - \Delta \left( \prod_{j=1}^{v_0} \left( 1 + \Delta x_{j}^{\sigma} \right) x_{j}^{\sigma} - \prod_{j=1}^{v_0} \left( 1 + \Delta x_{j}^{-\sigma} \right) x_{j}^{-\sigma} \right) \]
\[ \times \prod_{j=1}^{v_0} (i\alpha + z^{*} x_{j}^{\sigma} - z x_{j}^{-\sigma}) \right\} \prod_{i<j} \left( x_{i}^{\sigma} x_{j}^{-\sigma} - x_{i}^{-\sigma} x_{j}^{\sigma} \right)^{2}. \]

Now in order to show that (35) holds, we have to prove that $\hat{F} = 0$ for any $n = 1, \ldots, N$. We shall prove this by induction. First, it is easily seen that this expression has the following properties.
(i) $\hat{F}(x_1, \ldots, x_n)$ is a symmetric and even function of $x_1, \ldots, x_n$ and invariant under the transformation $x_j \to 1/x_j$ for any $j$.

(ii) $\hat{F}(x_1, \ldots, x_n)|_{x_1=1} = 0$.

(iii) $\hat{F}(x_1, \ldots, x_n)|_{x_1=x_2} = 2(1 + \Delta x_1^2)(1 + \Delta x_1^{-2})(-i\alpha - z^* x_1^2 + z x_1^{-2})(-i\alpha - z^* x_1^{-2} + z x_1^2)\hat{F}(x_3, \ldots, x_n)$.

Now, we assume that $\hat{F} = 0$ for $n-1$ and $n-2$. From (ii) and (iii), $\hat{F}(x_1, \ldots, x_n)|_{x_i=1} = 0$ and $\hat{F}(x_1, \ldots, x_n)|_{x_i=x_j} = 0$ for any $i, j$. Then, based on the property (i), Hirota proved [15–17] that the expression of $\hat{F}$ can be factored by a function

$$
\frac{\prod_{i=1}^{n} (x_i^2 - 1)^2 \prod_{i<j} (x_i^2 - x_j^2) (x_i^2 x_j^2 - 1)^2}{\prod_{i=1}^{n} x_i^2 \prod_{i<j} x_i^4 x_j^4}
$$

which has $2n + 2n(n-1)$ zeros of order 2. Accordingly, $\hat{F}$ would have at least $4n^2$ polynomials in the numerator. On the other hand, the functions

$$
\hat{F}(x_1, \ldots, x_n) \prod_{i=1}^{n} x_i^2 \prod_{i<j} x_i^4 x_j^4
$$

are polynomials of degree $2n^2 + 6n$ at most. Hence, $\hat{F} = 0$ for any $n$. In the same way, one can prove that the $N$-soliton solution obeys the other bilinear equations.

For instance in the case of the third bilinear equation (10) introducing the expressing of a $n$-soliton solution (12–15), we obtain

$$
\sum_{\mu, \mu'=0}^{N} \left[ \prod_{i=1}^{n} q_j^{(\mu_i \mu_j)} \left( \prod_{i=1}^{n} q_j^{(\mu_i)'} + \prod_{i=1}^{n} q_j^{(\mu_i)''} \right) \right. \\
- \prod_{j=1}^{n} A_j^{(\mu_j \mu_j')} \left( \prod_{i=1}^{n} (1 + i\alpha) q_j^{(\mu_j)'} p_j^{(\mu_j-i\mu_j)} + \prod_{j=1}^{n} (1 - i\alpha) q_j^{(\mu_j)'} p_j^{(\mu_j+i\mu_j)} \right) \\
\times \prod_{i<j} M_j^{(\mu_j \mu_j)} q_i^{(\mu_i)'} p_i^{(\mu_i+m)} q_i^{(\mu_i)'} p_i^{(\mu_i+m)} = 0. \quad (A.2)
$$

From this expression, we obtain by the same steps as above that the coefficient of the factor $\prod_{j=1}^{m} p_j^{(\mu_j \mu_j)} \prod_{j=1}^{m} p_j^{(\mu_j \mu_j)} q_j^{(\mu_j)'} p_j^{(\mu_j)'} q_j^{(\mu_j)'} p_j^{(\mu_j)'}$ is proportional to the following expression:

$$
\hat{K} = \sum_{\sigma = \pm}^{\pm} \left\{ \prod_{j=1}^{m} \left( 1 + \Delta x_j^{2+i\sigma_j} \right) + \prod_{j=1}^{m} \left( 1 + \Delta x_j^{2-i\sigma_j} \right) \right\} \prod_{j=1}^{m} (-i\alpha - x_j^{2+i\sigma_j} + x_j^{-2-i\sigma_j}) \\
- \left( \prod_{j=1}^{m} (1+i\alpha) (1+\Delta x_j^{2+i\sigma_j}) x_j^{i\sigma_j} + \prod_{j=1}^{m} (1-i\alpha) (1+\Delta x_j^{-2+i\sigma_j}) x_j^{-i\sigma_j} \right) \\
\times \prod_{j=1}^{m} \left( \alpha + x_j^{2+i\sigma_j} - x_j^{-2-i\sigma_j} \right) \prod_{i<j} \left( x_i^{i\sigma_j} - x_j^{-i\sigma_j} - x_i^{i\sigma_j} - x_j^{-i\sigma_j} \right)^2.
$$

Immediately, one can see that $\hat{K}$ is an invariant, symmetric and even function satisfying the same properties $K(x_1, \ldots, x_n)|_{x_1=1} = 0$, $K(x_1, \ldots, x_n)|_{x_i=x_j} = 2(1 + \Delta x_1^2)(1 + \Delta x_1^{-2})(-i\alpha - z^* x_1^2 + z x_1^{-2})(-i\alpha - z^* x_1^{-2} + z x_1^2)K(x_3, \ldots, x_n)$, so the induction method can be applied in the same way.
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