RIGIDITY OF KLEINIAN GROUPS VIA SELF-JOININGS

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Abstract. Let $\Gamma < \text{PSL}_2(\mathbb{C}) \simeq \text{Isom}^+(\mathbb{H}^3)$ be a finitely generated non-Fuchsian Kleinian group whose ordinary set $\Omega = \mathbb{S}^2 - \Lambda$ has at least two components. Let $\rho : \Gamma \to \text{PSL}_2(\mathbb{C})$ be a faithful discrete non-Fuchsian representation with boundary map $f : \Lambda \to \mathbb{S}^2$ on the limit set.

In this paper, we obtain a new rigidity theorem: if $f$ is conformal on $\Lambda$, in the sense that $f$ maps every circular slice of $\Lambda$ into a circle, then $f$ extends to a Möbius transformation $g$ on $\mathbb{S}^2$ and $\rho$ is the conjugation by $g$. Moreover, unless $\rho$ is a conjugation, the set of circles $C$ such that $f(C \cap \Lambda)$ is contained in a circle has empty interior in the space of all circles meeting $\Lambda$. This answers a question asked by McMullen on the rigidity of maps $\Lambda \to \mathbb{S}^2$ sending vertices of every tetrahedron of zero-volume to vertices of a tetrahedron of zero-volume.

The novelty of our proof is a new viewpoint of relating the rigidity of $\Gamma$ with the higher rank dynamics of the self-joining $(\text{id} \times \rho)(\Gamma) < \text{PSL}_2(\mathbb{C}) \times \text{PSL}_2(\mathbb{C})$.

1. Introduction

Let $\Gamma < \text{PSL}_2(\mathbb{C}) = \text{Isom}^+(\mathbb{H}^3)$ be a finitely generated torsion-free Kleinian group. Consider the following discreteness locus of $\Gamma$ in the space of representations of $\Gamma$ into $\text{PSL}_2(\mathbb{C})$: $\mathcal{R}_{\text{disc}}(\Gamma) = \{ \rho : \Gamma \to \text{PSL}_2(\mathbb{C}) : \text{discrete, faithful} \}$; each $\rho \in \mathcal{R}_{\text{disc}}(\Gamma)$ gives rise to a hyperbolic manifold $\rho(\Gamma) \setminus \mathbb{H}^3$ which is homotopy equivalent to $\Gamma \setminus \mathbb{H}^3$. Another commonly used notation for $\mathcal{R}_{\text{disc}}(\Gamma)$ is $\mathcal{AH}(\Gamma)$ where $\mathcal{H}$ stands for hyperbolic and $\mathcal{A}$ for the topology on this space given by the algebraic convergence (cf. [27]).

We denote by $\text{Möb}(\mathbb{S}^2)$ the group of all Möbius transformations on $\mathbb{S}^2$, by which we mean the group generated by inversions with respect to circles in $\mathbb{S}^2$. As well-known, $\text{Möb}(\mathbb{S}^2)$ is equal to the group of conformal automorphisms of $\mathbb{S}^2$. The group $\text{PSL}_2(\mathbb{C})$ can be identified with the subgroup consisting of compositions of even number of inversions with respect to circles in $\mathbb{S}^2$; in particular, it is a normal subgroup of $\text{Möb}(\mathbb{S}^2)$ of index two. This means that conjugations by elements of $\text{Möb}(\mathbb{S}^2)$ are contained in $\mathcal{R}_{\text{disc}}(\Gamma)$; we call them trivial elements of $\mathcal{R}_{\text{disc}}(\Gamma)$. Note that $\rho \in \mathcal{R}_{\text{disc}}(\Gamma)$ is trivial if and only if $\Gamma \setminus \mathbb{H}^3$ and $\rho(\Gamma) \setminus \mathbb{H}^3$ are isometric to each other.

Oh is partially supported by the NSF grant No. DMS-1900101.
The rigidity question on $\Gamma$ concerns a criterion on when a given representation $\rho \in \mathcal{R}_{\text{disc}}(\Gamma)$ is trivial. Denote by $\Lambda \subset S^2$ the limit set of $\Gamma$, that is, the set of all accumulation points of $\Gamma(o)$, $o \in \mathbb{H}^3$. A $\rho$-equivariant continuous embedding $f : \Lambda \to S^2$ is called a $\rho$-boundary map. There can be at most one $\rho$-boundary map. Two important class of representations admitting boundary maps are as follows. Firstly, if both $\Gamma$ and $\rho(\Gamma)$ are geometrically finite, and $\rho$ is type-preserving, then the $\rho$-boundary map always exists by Tukia [29]. Secondly, if $\rho$ is a quasiconformal deformation of $\Gamma$, i.e., there exists a quasiconformal homeomorphism $F : S^2 \to S^2$ such that for all $\gamma \in \Gamma$, $\rho(\gamma) = F \circ \gamma \circ F^{-1}$, then the restriction of $F$ to $\Lambda$ is the $\rho$-boundary map.

The fundamental role played by the boundary map in the study of rigidity of $\Gamma$ is well-understood, going back to the proofs of Mostow’s and Sullivan’s rigidity theorems ([19], [20], [25]). By the Ahlfors measure conjecture ([2], [3]) now confirmed by the works of Canary [7], Agol [11] and Calegari-Gabai [6], the limit set $\Lambda$ is either all of $S^2$ or of Lebesgue measure zero. Mostow rigidity theorem ([19], [20], [21]) says that if $\Gamma$ is a lattice, then any $\rho \in \mathcal{R}_{\text{disc}}(\Gamma)$ is trivial; he obtained this by showing that the $\rho$-boundary map has to be conformal on $S^2$. More generally, for any finitely generated Kleinian group $\Gamma$ with $\Lambda = S^2$, Sullivan showed that any quasiconformal deformation of $\Gamma$ is trivial [25]. In fact, Sullivan’s original theorem says that any $\rho$-equivariant quasiconformal homeomorphism of $S^2$ which is conformal on the ordinary set $\Omega = S^2 - \Lambda$ is a Möbius transformation. However Ahlfors measure conjecture implies that this is meaningful only when $\Lambda = S^2$ (cf. [14, Section 3.13]).

In this paper, we concern the case when $\Lambda \neq S^2$. For example, any geometrically finite Kleinian group which is not a lattice satisfies $\Lambda \neq S^2$ [26]. We prove that if the $\rho$-boundary map is conformal on $\Lambda$, then $\rho$ is trivial, provided the ordinary set $\Omega = S^2 - \Lambda$ has at least two connected components. By the “conformality of $f$ on $\Lambda$”, we mean that $f$ maps circles in $\Lambda$ into circles.

Circular slices. The main result of this paper is the following rigidity theorem in terms of the behavior of $f$ on circular slices of $\Lambda$: a circular slice of $\Lambda$ is a subset of the form $C \cap \Lambda$ for some circle $C \subset S^2$. We denote by $\mathcal{C}_\Lambda$ the space of all circles in $S^2$ meeting $\Lambda$.

**Theorem 1.1.** Let $\Gamma < \text{PSL}_2(\mathbb{C})$ be a finitely generated Zariski dense Kleinian group whose ordinary set $\Omega$ has at least two components. Let $\rho \in \mathcal{R}_{\text{disc}}(\Gamma)$ be a Zariski dense representation with boundary map $f : \Lambda \to S^2$.

If $f$ maps every circular slice of $\Lambda$ into a circle, then $\rho$ is a conjugation by some $g \in \text{Möb}(S^2)$ and $f = g|_\Lambda$. 


Moreover, unless \( \rho \) is a conjugation, the following subset of \( C_\Lambda \)
\[ \{ C \in C_\Lambda : f(C \cap \Lambda) \text{ is contained in a circle} \} \tag{1.1} \]
has empty interior.

We call \( \Lambda \) doubly stable if for any \( \xi \in \Lambda \), there exists a circle \( C \ni \xi \) such that for any sequence of circles \( C_i \) converging to \( C \), \( \# \limsup(C_i \cap \Lambda) \geq 2 \).

The assumption that \( \Gamma \) is finitely generated with \( \Omega \) disconnected was used only to ensure that \( \Lambda \) is doubly stable (Lemma 3.2, Theorem 4.3).

Remark 1.2. (1) This theorem holds rather trivially when \( \Lambda = S^2 \), in which case all circular slices of \( \Lambda \) are circles.

(2) If \( \Gamma < \text{PSL}_2(\mathbb{C}) \) is geometrically finite with connected limit set, then \( \Omega \) is disconnected (cf. [16, Chapter IX]); hence Theorem 1.1 applies.

Tetrahedra of zero-volume. A quadruple of points in \( S^2 \) determines an ideal tetrahedron of the hyperbolic 3-space \( \mathbb{H}^3 \). Gromov-Thurston’s proof of Mostow rigidity theorem for closed hyperbolic 3-manifolds uses the fact that a homeomorphism of \( S^2 \) mapping vertices of a maximal volume tetrahedron to vertices of a maximal volume tetrahedron is a Möbius transformation ([10] [28, Chapter 6]). In view of this, Curtis McMullen asked us whether one can consider the other extreme type of tetrahedra, namely, those of zero-volume in the study of rigidity of \( \Gamma \).

Noting that \( f : \Lambda \to S^2 \) maps every circular slice of \( \Lambda \) into a circle if and only if \( f \) maps any quadruple of points in \( \Lambda \) lying in a circle into a circle, the following is a reformulation of Theorem 1.1 which answers McMullen’s question in the affirmative:

**Theorem 1.3.** Let \( \Gamma, \rho \) be as in Theorem 1.1. If the \( \rho \)-boundary map \( f : \Lambda \to S^2 \) maps vertices of every tetrahedron of zero-volume to vertices of a tetrahedron of zero-volume, then \( f \) is the restriction of a Möbius transformation \( g \) and \( \rho \) is the conjugation by \( g \).

Cross ratios. Theorem 1.3 can also be stated in terms of cross ratios: note that for four distinct points \( \xi_1, \xi_2, \xi_3, \xi_4 \in \hat{C} \), the cross ratio \( [\xi_1 : \xi_2 : \xi_3 : \xi_4] \) is a real number if and only if all \( \xi_1, \xi_2, \xi_3, \xi_4 \) lie in a circle.

**Corollary 1.4.** Let \( \Gamma, f \) be as in Theorem 1.1. If \( [f(\xi_1) : f(\xi_2) : f(\xi_3) : f(\xi_4)] \in \mathbb{R} \) for any distinct \( \xi_1, \xi_2, \xi_3, \xi_4 \in \Lambda \) with \( [\xi_1 : \xi_2 : \xi_3 : \xi_4] \in \mathbb{R} \), then \( f \) extends to a Möbius transformation on \( \hat{C} \).

On the proof of Theorem 1.1. The novelty of our approach is to relate the rigidity question for a Kleinian group \( \Gamma < \text{PSL}_2(\mathbb{C}) \) with the dynamics of one parameter diagonal subgroups on the quotient of a higher rank semisimple real algebraic group \( G = \text{PSL}_2(\mathbb{C}) \times \text{PSL}_2(\mathbb{C}) \) by a self-joining discrete subgroup.

For a given \( \rho \in \mathcal{R}_{\text{disc}}(\Gamma) \), we consider the following self-joining of \( \Gamma \) via \( \rho \):
\[ \Gamma_\rho = (\text{id} \times \rho)(\Gamma) = \{ (\gamma, \rho(\gamma)) : \gamma \in \Gamma \} \]
which is a discrete subgroup of $G$. A basic but crucial observation is that $\rho$ is trivial if and only if $\Gamma_\rho$ is not Zariski dense in $G$ (Lemma 4.1). Our strategy is then to prove that if $f$ maps too many circular slices of $\Lambda$ into circles, then $\Gamma_\rho$ cannot be Zariski dense in $G$. We achieve this by considering the action of $\Gamma_\rho$ on the space $T_\rho$ of all tori in the Furstenberg boundary $\mathbb{S}^2 \times \mathbb{S}^2$ intersecting the limit set $\Lambda_\rho = \{(\xi, f(\xi)) \in \mathbb{S}^2 \times \mathbb{S}^2 : \xi \in \Lambda\}$. Here a torus means an ordered pair of circles in $\mathbb{S}^2$.

(1) On one hand, using the Koebe-Maskit theorem ([15], [23], see Theorem 3.4) and the hypothesis that the ordinary set $\Omega$ has at least 2 components, we show the existence of a torus $T \in T_\rho$ such that $T / \Gamma_\rho T_0$ for any torus $T_0 = (C_0, D_0)$ with $f(C_0 \cap \Lambda) \subset D_0$; in particular $\Gamma_\rho T_0 \neq T_\rho$.

(2) On the other hand, we prove in Theorem 2.1 that the Zariski density of $\Gamma_\rho$ implies the existence of a dense subset $\tilde{\Lambda}_\rho$ of $\Lambda_\rho$ such that $\Gamma_\rho T_0 = T_\rho$ for any torus $T_0$ meeting $\tilde{\Lambda}_\rho$. Denoting by $A$ the two-dimensional diagonal subgroup of $G$, the main ingredients for this step are the existence of a dense orbit of some regular one-parameter diagonal semigroup in the non-wandering set of the $A$-action on $\Gamma_\rho \setminus G$ (Theorem 2.2) as well as a theorem of Prasad-Rapinchuk [22] on the existence of $\mathbb{R}$-regular elements (Theorem 2.4). Therefore, if the subset (1.1) has non-empty interior, we can find a torus $T_0 = (C_0, D_0)$ satisfying that $f(C_0 \cap \Lambda) \subset D_0$ and $\Gamma_\rho T_0 = T_\rho$.

The incompatibility of (1) and (2) implies that either the subset (1.1) has empty interior or $\Gamma_\rho$ is not Zariski dense in $G$, as desired.

Question. There are several different proofs of Mostow rigidity theorem ([19], [20], [21]). By the viewpoint suggested in this paper, it will be interesting to find yet another proof, which directly shows the following reformulation: for any lattice $\Gamma < \text{PSL}_2(\mathbb{C})$ and $\rho \in \mathcal{R}_{\text{disc}}(\Gamma)$, the self-joining $\Gamma_\rho$ is not Zariski dense in $\text{PSL}_2(\mathbb{C}) \times \text{PSL}_2(\mathbb{C})$.

Acknowledgements. We would like to thank Curt McMullen for asking the question formulated as Theorem 1.3 as well as for useful comments on the preliminary version. We would also like to thank Yair Minsky for useful conversations.

2. Dense orbits in the space of Tori

Let $G = \text{PSL}_2(\mathbb{C}) \times \text{PSL}_2(\mathbb{C})$ and let $X = \mathbb{H}^3 \times \mathbb{H}^3$ be the Riemannian product of two hyperbolic 3-spaces. It follows from $\text{PSL}_2(\mathbb{C}) \simeq \text{Isom}^+(\mathbb{H}^3)$ that $G \simeq \text{Isom}^+(X)$. In the whole paper, we regard $G$ as a real algebraic group and the Zariski density of a subset of $G$ is to be understood accordingly. The action of $\text{PSL}_2(\mathbb{C})$ on $\mathbb{H}^3$ extends continuously to the compactification $\mathbb{H}^3 \cup \partial \mathbb{H}^3$ and its action on $\partial \mathbb{H}^3 \simeq \mathbb{S}^2$ is given by the Möbius
transformation action of $\text{PSL}_2(\mathbb{C})$ on $S^2$. We set $\mathcal{F} = S^2 \times S^2$, which coincides with the so-called Furstenberg boundary of $G$. Note that $\mathcal{F}$ is not the geometric boundary of $X$. Clearly, the action of $G$ extends continuously to the compact space $X \cup \mathcal{F}$.

For a Zariski dense subgroup $\Delta$ of $G$, its limit set $\Lambda_\Delta \subset \mathcal{F}$ is defined as all possible accumulation points of $\Delta(o)$, $o \in X$, on $\mathcal{F}$. It is a non-empty $\Delta$-minimal subset of $\mathcal{F}$ ([4, Section 3.6], [13, Lemma 2.13]).

By a torus $T$, we mean an ordered pair $T = (C_1, C_2) \subset \mathcal{F}$ of circles in $S^2$. The group $G$ acts on the space of tori by extending the action of $\text{PSL}_2(\mathbb{C})$ on the space of circles componentwise. The main goal of this section is to prove the following: denote by $\mathcal{T}_\Delta$ the space of all tori in $\mathcal{F}$ intersecting $\Lambda_\Delta$.

**Theorem 2.1.** Let $\Delta$ be a Zariski dense subgroup of $G$. There exists a dense subset $\tilde{\Lambda}_\Delta$ of $\Lambda_\Delta$ such that for any torus $T$ with $T \cap \tilde{\Lambda}_\Delta \neq \emptyset$, the orbit $\Delta T$ is dense in $\mathcal{T}_\Delta$.

This theorem may be viewed as a higher rank analogue of [13, Theorem 4.1]. The rest of this section is devoted to its proof. It is convenient to use the upper half-space model of $\mathbb{H}^3$ so that $\partial \mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$. The visual maps $G \to \mathcal{F}$, $g \mapsto g^\pm$, are defined as follows: for $g = (g_1, g_2) \in G$ with $g_i \in \text{PSL}_2(\mathbb{C})$,

$$g^+ = (g_1(\infty), g_2(\infty)) \quad \text{and} \quad g^- = (g_1(0), g_2(0)).$$

For $t \in \mathbb{C}$, we set $a_t = \text{diag}(e^{t/2}, e^{-t/2})$ and define the following subgroups of $G$:

$$A = \{(a_{t_1}, a_{t_2}) : t_1, t_2 \in \mathbb{R}\} \quad \text{and} \quad M = \{(a_{t_1}, a_{t_2}) : t_1, t_2, t_2 \in i\mathbb{R}\}.$$  

For $u = (u_1, u_2) \in \mathbb{R}^2$, we write $a_u = (a_{u_1}, a_{u_2})$ and consider the following one-parameter semigroup

$$A_u^+ = \{a_{tu} : t \geq 0\}.$$  

A loxodromic element $h \in \text{PSL}_2(\mathbb{C})$ is of the form $h = \varphi a_{t_h} m_h \varphi^{-1}$ where $t_h > 0$ and $m_h \in \text{PSO}(2)$ are uniquely determined and $\varphi \in \text{PSL}_2(\mathbb{C})$. We call $t_h > 0$ the Jordan projection of $h$ and $m_h$ the rotational component of $h$. The attracting and repelling fixed points of $h$ on $S^2$ are given by $y_h = \varphi(\infty)$ and $y_{h^{-1}} = \varphi(0)$, respectively.

For a loxodromic element $g = (g_1, g_2) \in G$, that is, each $g_i$ is loxodromic, its Jordan projection $\lambda(g)$ and the rotational component $\tau(g)$ are defined componentwise: $\lambda(g) = (t_{g_1}, t_{g_2}) \in \mathbb{R}^2_{>0}$ and $\tau(g) = (m_{g_1}, m_{g_2}) \in M$.

**Dense $A_u^+$-orbit.** For a Zariski dense subgroup $\Delta$ of $G$, we consider the following $AM$-invariant subset

$$\mathcal{R}_\Delta = \{[g] \in \Delta \backslash G : g^+, g^- \in \Lambda_\Delta\}.$$  

Let $\mathcal{L} = \mathcal{L}_\Delta \subset \mathbb{R}^2_{>0}$ denote the limit cone of $\Delta$, which is the smallest closed cone containing the Jordan projection $\lambda(\Delta) = \{\lambda(\delta) : \delta \in \Delta\}$. The Zariski density of $\Delta$ implies that $\mathcal{L}$ has non-empty interior ([4, Section 1.2].
We use the following theorem which is an immediate consequence of the result of Dang [9] (this also follows from [8] and [5]):

**Theorem 2.2.** For any Zariski dense subgroup $\Delta < G$ and any $u \in \text{int} L_\Delta$, there exists a dense $A_u^+$-orbit in $R_\Delta$.

**Proof.** As shown in [9, Theorem 7.1 and its proof], the semigroup $S^+ := \{a_n^u : n \in \mathbb{N} \cup \{0\}\}$ acts on $R_\Delta$ topologically transitively: for any non-empty open subsets $O_1, O_2$ of $R_\Delta$, $O_1 a_n^u \cap O_2 \neq \emptyset$ for some $n \in \mathbb{N}$. This implies the existence of a dense $S^+$-orbit on $R_\Delta$ (cf. [24 Proposition 1.1]). Since $S^+ \subset A_u^+$, this proves the claim. \hfill \Box

In the following, we fix $u \in \text{int} L_\Delta$ and a dense $A_u^+$-orbit, say $[g_0]A_u^+$, in $R_\Delta$, provided by Theorem 2.2. Set

$$\tilde{\Lambda}_\Delta = \Delta g_0^+ = \{ \delta g_0^+ \in \Lambda_\Delta : \delta \in \Delta \};$$

(2.1)

note that this is a dense subset of $\Lambda_\Delta$, as $\Lambda_\Delta$ is a $\Delta$-minimal subset.

Denote by $T_\Delta^\bullet$ the space of all tori $T$ with $\# T \cap \Lambda_\Delta \geq 2$.

**Corollary 2.3.** For any torus $T$ meeting $\tilde{\Lambda}_\Delta$, the closure of $\Delta T$ contains $T_\Delta^\bullet$.

**Proof.** Note that $H = \text{PGL}_2(\mathbb{R}) \times \text{PGL}_2(\mathbb{R})$ is a subgroup of $G$, as $\text{PSL}_2(\mathbb{C}) = \text{PGL}_2(\mathbb{C})$. The space $T$ of all tori in $F$ can be identified with the quotient space $G/H$. Let $T$ be a torus containing $\delta_0 g_0^+ \in \tilde{\Lambda}_\Delta$ for some $\delta_0 \in \Delta$. By the identification of $T = G/H$, we may write $T = gH$ for some $g \in G$. Then for some $h \in H$, $(gh)^+ = \delta_0 g_0^+$. If we denote by $P$ the stabilizer subgroup of $(\infty, \infty)$ in $G$, which is equal to the product of two upper triangular subgroups of $\text{PSL}_2(\mathbb{C})$, this implies that for some $p \in P$, $gh = \delta_0 g_0 p$. Write $p = nam$ where $n$ belongs to the strict upper triangular subgroup, $a \in A$ and $m \in M$. We claim that $[g]hA_u^+ \supset (R_\Delta - [g_0]A_u^+)ma$. Let $x \in R_\Delta - [g_0]A_u^+$. Since $[g_0]A_u^+ = R_\Delta$, there exists a sequence $t_i \rightarrow +\infty$ such that $x = \lim_{i \rightarrow \infty} [g_0]a_{t_i,u}$. Since $u = (u_1, u_2) \in \text{int} L_\Delta$, we have $u_1 > 0, u_2 > 0$, and hence $a_{-t_i,u}na_{t_i,u} \rightarrow e$ as $i \rightarrow \infty$.

Therefore

$$\lim_{i \rightarrow \infty} [g]ha_{t_i,u} = \lim_{i \rightarrow \infty} [g_0]nam a_{t_i,u} = \lim_{i \rightarrow \infty} [g_0]a_{t_i,u}(a_{-t_i,u}na_{t_i,u})am = xam;$$

so $xam \in [g]hA_u^+$. This proves the claim. Since $R_\Delta$ is $AM$-invariant, and $R_\Delta - [g_0]AM$ is dense in $R_\Delta$ (as $\Lambda_\Delta$ is a perfect set), it follows that

$$[g]hA_u^+ \supset R_\Delta.$$

Since $A_u^+ \subset H$, this implies that $[g]H \supset R_\Delta H$. Since $R_\Delta H = \Delta \setminus T_\Delta^\bullet$ and $T = gH$, we get $\Delta T \supset T_\Delta^\bullet$, as desired. \hfill \Box
**Loxodromic element** \( \delta \in \Delta \) with \( \tau(\delta) \) generating \( M \). We use the following special case of a theorem of Prasad and Rapinchuk [22]:

**Theorem 2.4.** [22, Theorem 1, Remark 1] Any Zariski dense subgroup \( \Delta < G \) contains a loxodromic element \( \delta \) such that \( \tau(\delta) \) generates a dense subgroup of \( M \).

**Corollary 2.5.** If \( \Delta \) is Zariski dense in \( G \), then \( \mathcal{T}_\Delta \) is dense in \( \mathcal{T}_\Delta \).

**Proof.** Let \( \delta = (\delta_1, \delta_2) \in \Delta \) be as given by Theorem 2.4. Since \( M \) has no isolated point, there exists a sequence \( m_j \), which we may assume tends to \( +\infty \), by replacing \( \delta \) by \( \delta^{-1} \) if necessary, that \( \tau(\delta)^m \) converges to \( e \). It follows that the semigroup generated by \( \tau(\delta) \) is also dense in \( M \). Let \( T = (C_1, C_2) \in \mathcal{T}_\Delta \) be any torus. It suffices to construct a sequence \( T_n = (C_{1,n}, C_{2,n}) \in \mathcal{T}_\Delta \) which converges to \( T \). We begin by fixing a point \( \xi = (\xi_1, \xi_2) \in T \cap \Lambda_\Delta \). Since \( \Delta \) acts minimally on \( \Lambda_\Delta \), there exists a sequence \( \delta_n = (\delta_{1,n}, \delta_{2,n}) \in \Delta \) such that \( \delta_n y_\delta \) converges to \( \xi \) as \( n \to \infty \); recall that \( y_\delta \in F \) denotes the attracting fixed point of \( \delta \). Fix a point \( \eta = (\eta_1, \eta_2) \in \Lambda_\Delta - \{y_\delta, y_\delta^{-1}\} \).

For each fixed \( n \in \mathbb{N} \), note that, as \( k \to \infty \), the sequence \( \delta_n \delta^k \eta \) converges to \( \delta_n y_\delta \), while rotating around \( \delta_n y_\delta \) by the amount given by \( \tau(\delta)^k \). Since \( \tau(\delta) \) generates a dense semigroup of \( M \), we can find a sequence \( k_n \to \infty \) such that for each \( i = 1, 2 \),

\[
d(\delta_{i,n} y_\delta, \delta_{i,n} \delta_i^{k_n} \eta) < \frac{1}{n} \quad \text{and} \quad \frac{n}{2} - \frac{1}{n} < \theta_{i,n} < \frac{n}{2} + \frac{1}{n}
\]

where \( \theta_{i,n} \) is the angle at \( \delta_{i,n} y_\delta \) of the triangle determined by the center of \( C_i \), \( \delta_{i,n} y_\delta \) and \( \delta_{i,n} \delta_i^{k_n} \eta_i \). For each \( i = 1, 2 \), we now choose \( p_i \in C_i - \bigcup_n \{\delta_{i,n} y_\delta, \delta_{i,n} \delta_i^{k_n} \eta_i\} \) and set \( C_{i,n} \) to be the circle passing through \( \delta_{i,n} y_\delta, \delta_{i,n} \delta_i^{k_n} \eta_i \) and \( p_i \).

From the construction, each sequence \( C_{i,n} \) converges to the circle tangent to \( C_i \) at \( \xi_i \) and passing through \( p_i \in C_i \), which must be equal to \( C_i \) itself; therefore if we set \( T_n = (C_{1,n}, C_{2,n}) \),

\[
T_n \to T \quad \text{as} \quad n \to \infty.
\]

Since \( T_n \cap \Lambda_\Delta \) contains both \( \delta_n y_\delta \) and \( \delta_n \delta^{k_n} \eta \), we have \( T_n \in \mathcal{T}_\Delta \). This completes the proof. \( \square \)

**Proof of Theorem 2.1** It suffices to consider the set \( \tilde{\Lambda}_\Delta \) defined in (2.1) by Corollary 2.3 and Corollary 2.5.

3. **Limits of circular slices and Koebe-Maskit theorem**

Let \( \Gamma < \text{PSL}_2(\mathbb{C}) \) be a non-elementary Kleinian group and \( \Omega = \mathbb{S}^2 - \Lambda \) its ordinary set, i.e., \( \Lambda \subset \mathbb{S}^2 \) denotes the limit set of \( \Gamma \). We refer to [14] and [17] for general facts on the theory of Kleinian groups.

**Definition 3.1.** (1) We call a circle \( C \) doubly stable for \( \Lambda \) if for any sequence of circles \( C_i \) converging to \( C \), \( \# \limsup(C_i \cap \Lambda) \geq 2 \).
(2) We call \( \Lambda \) doubly stable if for any \( \xi \in \Lambda \), there exists a circle \( C \ni \xi \), which is doubly stable for \( \Lambda \).

The main goal of this section is to prove the following lemma:

**Lemma 3.2.** If \( \Gamma \) is finitely generated and \( \Omega \) is not connected, then \( \Lambda \) is doubly stable.

In the rest of this section, we assume \( \Gamma \) is finitely generated. Lemma 3.2 is an immediate consequence of the following lemma, since, if \( \xi_1, \xi_2 \in \Omega \) belong to different components of \( \Omega \), then for any \( \xi \in \Lambda \), the circle \( C \) passing through \( \xi, \xi_1, \xi_2 \) is not contained in the closure of any component of \( \Omega \).

**Lemma 3.3.** Let \( C \subset S^2 \) be a circle such that \( C \not\subset \Omega_0 \) for any component \( \Omega_0 \) of \( \Omega \). If \( C_n \) is a sequence of circles converging to \( C \), then

\[
\# \limsup(C_n \cap \Lambda) \geq 2.
\]

The main ingredient is the following formulation of the Koebe-Maskit theorem ([15, Theorem 6], [23, Theorem 1]):

**Theorem 3.4.** Let \( \{ \Omega_i \} \) be the collection of all components of the ordinary set \( \Omega \). Then for any \( \alpha > 2 \), \( \sum_i \text{Diam}(\Omega_i)^\alpha < \infty \) where \( \text{Diam}(\Omega_i) \) is the diameter of \( \Omega_i \) in the spherical metric on \( S^2 \).

We will only need the following immediate corollary of Theorem 3.4:

**Corollary 3.5.** For any \( \varepsilon > 0 \), there are only finitely many components of the ordinary set of \( \Gamma \) with diameter bigger than \( \varepsilon \).

**Proof of Lemma 3.3.** Given Corollary 3.5, the proof is similar to the proof of [12, Lemma 8.1], which deals with the case when all components of \( \Omega \) are round disks.

Let \( C \) and \( C_n \to C \) be as in the statement of the lemma. It suffices to show that there exists \( \varepsilon_0 > 0 \) such that \( C_n \cap \Lambda \) contains two points of distance at least \( \varepsilon_0 \) for some infinite sequence \( n_i \to \infty \). Suppose not. Then, letting \( I_n \) be the minimal connected subset of \( C_n \) containing \( C_n \cap \Lambda \), we have \( \text{Diam}(I_n) \to 0 \) as \( n \to \infty \).

Setting \( \eta = \text{Diam}(C)/2 \), we have \( \text{Diam}(C_n) > \eta \) for all sufficiently large \( n \). Let \( 0 < \varepsilon < \eta/4 \) be arbitrary. Since \( \text{Diam}(I_n) \to 0 \), we have \( \text{Diam}(I_n) < \varepsilon \) for all large \( n \). Noting that \( C_n - I_n \) is a connected subset of \( \Omega \), let \( \Omega_n \) be the connected component of \( \Omega \) containing \( C_n - I_n \). Then \( C_n \) is contained in the \( \varepsilon \)-neighborhood of \( \Omega_n \), which implies

\[
\text{Diam}(\Omega_n) \geq \text{Diam}(C_n) - 2\varepsilon > \eta/2.
\]

By Corollary 3.5, the collection \( \{ \Omega_n : \text{Diam}(\Omega_n) > \eta/2 \} \) must be a finite set, say, \( \{ \Omega_1, \cdots, \Omega_N \} \). Therefore, for some \( 1 \leq j \leq N \), there exists an infinite sequence \( C_{n_j} \) contained in the \( \varepsilon \)-neighborhood of \( \Omega_j \). Hence \( C \) is contained in the \( 2\varepsilon \)-neighborhood of \( \Omega_j \). Since the collection \( \{ \Omega_1, \cdots, \Omega_N \} \) does not

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\( ^1 \)For a sequence of subsets \( S_n \) in a topological space, we define \( \limsup S_n = \bigcap_n \bigcup_{i \geq n} S_i \).
depend on \( \varepsilon \), we can find a sequence \( \varepsilon_k \to 0 \) and a fixed \( 1 \leq j \leq N \) such that \( C \) is contained in the \( 2\varepsilon_k \)-neighborhood of \( \Omega_j \). It follows that \( C \subset \overline{\Omega_j} \), contradicting the hypothesis on \( C \). This finishes the proof.

4. Self-Joinings of Kleinian groups and Proof of Theorem 1.1

Let \( \Gamma \subset \text{PSL}_2(\mathbb{C}) \) be a Zariski dense discrete subgroup with limit set \( \Lambda \). As before, we denote by \( \Omega = S^2 - \Lambda \) its ordinary set.

We fix a discrete faithful representation \( \rho : \Gamma \to \text{PSL}_2(\mathbb{C}) \) such that \( \rho(\Gamma) \) is Zariski dense.

We now define the self-joining of \( \Gamma \) via \( \rho \) as
\[
\Gamma_\rho := (\text{id} \times \rho)(\Gamma) = \{(\gamma, \rho(\gamma)) : \gamma \in \Gamma\},
\]
which is a discrete subgroup of \( G \).

We begin by recalling two standard facts:

**Lemma 4.1.** The subgroup \( \Gamma_\rho \) is Zariski dense in \( G \) if and only if \( \rho \) is not a conjugation by an element of \( \text{M"ob}(S^2) \).

**Proof.** It is clear that if \( \rho \) is a conjugation by an element of \( \text{M"ob}(S^2) \), then \( \Gamma_\rho \) is not Zariski dense in \( G \). To see the converse, let \( G_0 < G \) be the Zariski closure of \( \Gamma_\rho \) and suppose that \( G_0 \neq G \). Denote by \( \pi_i : G = \text{PSL}_2(\mathbb{C}) \times \text{PSL}_2(\mathbb{C}) \to \text{PSL}_2(\mathbb{C}) \) the projection onto the \( i \)-th component.

We now claim that \( \pi_1|_{G_0} \) is surjective. Since \( \Gamma \) is Zariski dense, \( \pi_1|_{G_0} \) is surjective. Hence, it suffices to show that \( \pi_1|_{G_0} \) is injective. Note that \( G_0 \cap \ker \pi_1 = G_0 \cap (\{e\} \times \text{PSL}_2(\mathbb{C})) \) is a normal subgroup of \( G_0 \). Hence, \( G_0 \cap \ker \pi_1 \) is normalized by \( \{e\} \times \text{PSL}_2(\mathbb{C}) \) since \( \rho(\Gamma) \) is Zariski dense \( \text{PSL}_2(\mathbb{C}) \).

Thus, \( G_0 \cap \ker \pi_1 \) is a normal subgroup of \( \ker \pi_1 \). As \( \ker \pi_1 \cong \text{PSL}_2(\mathbb{C}) \) is simple, \( G_0 \cap \ker \pi_1 \) is either trivial or \( \{e\} \times \text{PSL}_2(\mathbb{C}) \). In the latter case, note that \( \{e\} \times \text{PSL}_2(\mathbb{C}) < G_0 \). Since \( \pi_1|_{G_0} \) is surjective, it follows that \( G_0 = G \), yielding contradiction. Therefore \( \pi_1|_{G_0} \) is injective, and hence an isomorphism. Similarly, \( \pi_2|_{G_0} \) is an isomorphism. Hence, \( \pi_2|_{G_0} \circ \pi_1|_{G_0}^{-1} \) is a Lie group automorphism of \( \text{PSL}_2(\mathbb{C}) \). Hence it is a conjugation by a Möbius transformation (cf. [11]). Since this map restricts to \( \rho \) on \( \Gamma \), it finishes the proof. \( \square \)

Since \( \rho \) gives an isomorphism from \( \Gamma \) to \( \rho(\Gamma) \) and \( f \) is an equivariant embedding, it follows that \( \rho \) maps every loxodromic element \( \gamma \) to a loxodromic element \( \rho(\gamma) \) and \( f \) sends the attracting fixed point of \( \gamma \in \Gamma \) to the attracting fixed point of \( \rho(\gamma) \). Since the set of attracting fixed points of loxodromic elements of \( \Gamma \) is dense in \( \Lambda \), this implies the following.

**Lemma 4.2.** There can be at most one \( \rho \)-boundary map \( f : \Lambda \to S^2 \). In particular, if \( \rho \) is a conjugation by \( g \in \text{M"ob}(S^2) \), then \( f = g|_\Lambda \).

**Proof of Theorem 1.1** By Lemma 3.2 Theorem 1.1 follows from the following:
Theorem 4.3. Let $\Gamma < \text{PSL}_2(\mathbb{C})$ be a Zariski dense Kleinian group such that $\Lambda$ is doubly stable. Let $\rho \in \mathcal{R}_{\text{disc}}(\Gamma)$ be a Zariski dense representation with boundary map $f : \Lambda \to \mathbb{S}^2$. Unless $\rho$ is a conjugation, the subset

$$\Lambda_f := \bigcup \{ C \cap \Lambda : f(C \cap \Lambda) \text{ is contained in a circle} \}$$

has empty interior in $\Lambda$; hence

$$\{ C \in C_\Lambda : f(C \cap \Lambda) \text{ is contained in a circle} \}$$

has empty interior in $C_\Lambda$.

Proof. If $\Lambda = \mathbb{S}^2$, it is easy to prove this. So we assume below that $\Lambda \neq \mathbb{S}^2$. Suppose that $\rho$ is not a conjugation, so that $\Gamma_\rho$ is Zariski dense by Lemma 4.1. It follows easily from the minimality of the limit set $\Lambda_\rho$ of $\Gamma_\rho$ that

$$\Lambda_\rho = \{(\xi, f(\xi)) \in \mathbb{S}^2 \times \mathbb{S}^2 : \xi \in \Lambda \}.$$ (4.2)

Let $\tilde{\Lambda}_{\Gamma_\rho}$ be as in Theorem 2.1, which must be of the form $\{(\xi, f(\xi)) : \xi \in \tilde{\Lambda}\}$ for some dense subset $\tilde{\Lambda}$ of $\Lambda$.

Suppose on the contrary that $\Lambda_f$ has non-empty interior. Then $\Lambda_f \cap \tilde{\Lambda} \neq \emptyset$. It follows that there exists $C_0 \in C_\Lambda$ such that $C_0 \cap \tilde{\Lambda} \neq \emptyset$ and $f(C_0 \cap \Lambda)$ is contained in some circle, say, $D_0$. Set $T_0 = (C_0, D_0)$. Since $C_0 \cap \tilde{\Lambda} \neq \emptyset$, it follows from Theorem 2.1 that

$$\Gamma_\rho T_0 = T_0$$ (4.3)

where $T_\rho = T_{\Gamma_\rho}$ is the space of all tori intersecting $\Lambda_\rho$. On the other hand, we now show that the condition $f(C_0 \cap \Lambda) \subset D_0$ implies that $\Gamma_\rho T_0$ cannot be dense in $T_\rho$, using Lemma 3.3.

Step 1: There exists a circle $D$ which intersects $\Lambda_{\rho(G)}$ precisely at one point, say $f(\xi_0)$. To show this, fix any $f(\xi) \in \Lambda_{\rho(G)}$ and let $D'$ be the boundary of the minimal disk $B'$ centered at $f(\xi)$ which contains all of $\Lambda_{\rho(G)}$. By the minimality of $B'$, $D' \cap \Lambda_{\rho(G)} \neq \emptyset$. Choose $f(\xi_0) \in D' \cap \Lambda_{\rho(G)}$, and let $D$ be a circle tangent to $D'$ at $f(\xi_0)$ which does not intersect the interior of $B'$.

Step 2: By the hypothesis that $\Lambda$ is doubly stable, we can find a circle $C$ containing $\xi_0$ which is doubly stable for $\Lambda$.

Step 3: Setting $T = (C, D)$, we have $T \notin \Gamma_\rho T_1$ for any torus $T_1 = (C_1, D_1)$ with $f(C_1 \cap \Lambda) \subset D_1$. In particular, $T \notin \Gamma_\rho T_0$.

Suppose on the contrary that there exists a sequence $\gamma_n \in \Gamma$ such that $\gamma_n C_1$ converges to $C$ and $\rho(\gamma_n) D_1$ converges to $D$. Since $C$ is doubly stable for $\Lambda$, we have

$$\# \limsup(\gamma_n C_1 \cap \Lambda) \geq 2.$$ (4.4)

By the $\rho$-equivariance of $f$, we have

$$f(\gamma_n C_1 \cap \Lambda) = f(\gamma_n(C_1 \cap \Lambda)) = \rho(\gamma_n) f(C_1 \cap \Lambda) \subset \rho(\gamma_n) D_1 \cap \Lambda_{\rho(G)}.$$ Hence

$$\limsup f(\gamma_n C_1 \cap \Lambda) \subset \limsup(\rho(\gamma_n) D_1 \cap \Lambda_{\rho(G)}) \subset D \cap \Lambda_{\rho(G)}.$$
It now follows from (4.4) and the injectivity of $f$ that

$$\#D \cap \Lambda_\rho(\Gamma) \geq 2.$$ 

This contradicts the choice of $D$ made in Step (1), hence proving Step (3).

Since $(\xi_0, f(\xi_0)) \in T \cap \Lambda_\rho$, we have $T \in \mathcal{T}_\rho$. Hence we obtained a contradiction to (4.3). Therefore $\Lambda_f$ has empty interior, completing the proof. □

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