Notes on area operator, geometric 2-rough paths and Young integral when \( p^{-1} + q^{-1} = 1 \)

Danyu Yang∗

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Abstract

When the norm on continuous bounded variation paths weakened to 2-variation, the area operator is not continuous nor bounded, but is closable in 2-rough norm, and paths in the closure (i.e. paths which admits an enhancement into a geometric 2-rough path) is not linear.

For path \( \gamma \) with vanishing 2-variation, the Riemann-Stieltjes integral \( \int_{s<u_1<u_2<t} d\gamma(u_1) \cdot d\gamma(u_2) \) is the only possible candidate to enhance \( \gamma \) into a geometric 2-rough path, but the integral may not exist, so not every path with vanishing 2-variation admits an enhancement.

Young integral is extended to the case \( p^{-1} + q^{-1} = 1 \) by assuming a finer scale continuity. As a consequence, when \( p = q = 2 \), by adding a log term (and log log term, etc.) in the modulus of continuity, there exists a sequence of nested spaces of enhancible paths.

1 Definitions and notations

Firstly, we define \( p \)-variation seminorm on the space of continuous paths, which is important in rough path theory (see [3], [4] and [1]).

**Definition 1** A finite set of points \( D = \{ t_j \}_{j=0}^n \) is said to be a finite partition of interval \([0, T]\), if \( 0 = t_0 < t_1 < \cdots < t_n = T \).

**Notation 2** Suppose \( D = \{ t_j \}_{j=0}^n \) is a finite partition of \([0, T]\). Denote \( |D| := \max_{0 \leq j \leq n-1} \{|t_{j+1} - t_j|\} \) as the mesh of \( D \).

**Notation 3** Denote \( V \) as a Banach space with norm \( \|\cdot\| \).

**Notation 4** For \( T > 0 \), denote \( C ([0, T], V) := \{ \gamma | \gamma : [0, T] \to V \text{ is continuous} \} \); denote \( \Delta_{[0, T]} := \{(s, t) | 0 \leq s \leq t \leq T \} \) and

\[ C \left( \Delta_{[0, T]}, V \right) := \{ \alpha | \alpha : \Delta_{[0, T]} \to V \text{ is continuous}, \alpha(t, t) = 0, \forall t \in [0, T] \} \. \]

∗University of Oxford and Oxford-Man Institute, Email: yangd@maths.ox.ac.uk
Definition 5 Suppose \( \alpha \in C (\Delta_{[0,T]}, \mathcal{V}) \). For \( p > 0 \), define the \( p \)-variation of \( \alpha \) on \([0,T]\) as

\[
\|\alpha\|_{p\text{-var},[0,T]} := \left( \sup_{D \subseteq [0,T], \, |D| \leq \delta} \sum_{j, t_j \in D} \|\alpha(t_j, t_{j+1})\|^p \right)^{\frac{1}{p}},
\]

where the supremum is over all finite partitions of \([0,T]\).

When \( p = \infty \), define \( \|\alpha\|_{\infty\text{-var},[0,T]} := \sup_{0 \leq s < t \leq T} \|\alpha(s, t)\| \).

For any fixed \( \alpha \in C (\Delta_{[0,T]}, \mathcal{V}) \), the function \( p \mapsto \|\alpha\|_{p\text{-var},[0,T]} \) on \( p \in (0, \infty] \) is non-increasing and continuous where it is finite.

Definition 6 Suppose \( \alpha \in C (\Delta_{[0,T]}, \mathcal{V}) \). Then \( \alpha \) is said to be of vanishing \( p \)-variation for some \( p > 0 \), if

\[
\lim_{\delta \to 0} \omega_p (\alpha, \delta) := \lim_{\delta \to 0} \left( \sup_{D \subseteq [0,T], \, |D| \leq \delta} \sum_{j, t_j \in D} \|\alpha(t_j, t_{j+1})\|^p \right)^{\frac{1}{p}} = 0. \tag{2}
\]

Definition 7 Suppose \( \gamma \in C ([0,T], \mathcal{V}) \). Define \( \tilde{\gamma} \in C (\Delta_{[0,T]}, \mathcal{V}) \) by setting

\[
\tilde{\gamma}(s, t) := \gamma(t) - \gamma(s), \forall 0 \leq s \leq t \leq T. \tag{3}
\]

Then define \( \|\gamma\|_{p\text{-var},[0,T]} := \|\tilde{\gamma}\|_{p\text{-var},[0,T]} \), \( \omega_p (\gamma, \delta) := \omega_p (\tilde{\gamma}, \delta) \) and that \( \gamma \) is said to be of vanishing \( p \)-variation if \( \lim_{\delta \to 0} \omega_p (\gamma, \delta) = 0 \).

Both \( p \)-variation norm and being of vanishing \( p \)-variation are invariant under reparametrisation (i.e. continuous non-decreasing \( \varphi : [0,T] \to \mathbb{R}^+ \), continuity preserves compactness and being non-decreasing preserves the order).

Notation 8 For \( p > 0 \), denote \( C^p\text{-var} (\Delta_{[0,T]}, \mathcal{V}) \) and \( C^{0,p}\text{-var} (\Delta_{[0,T]}, \mathcal{V}) \) as subspaces of \( C (\Delta_{[0,T]}, \mathcal{V}) \):

\[
C^p\text{-var} (\Delta_{[0,T]}, \mathcal{V}) := \left\{ \alpha \in C (\Delta_{[0,T]}, \mathcal{V}) \mid \|\alpha\|_{p\text{-var},[0,T]} < \infty \right\},
\]

\[
C^{0,p}\text{-var} (\Delta_{[0,T]}, \mathcal{V}) := \left\{ \alpha \in C (\Delta_{[0,T]}, \mathcal{V}) \mid \lim_{\delta \to 0} \omega_p (\alpha, \delta) = 0 \right\}.
\]

Similarly, for \( p \geq 1 \), denote \( C^p\text{-var} ([0,T], \mathcal{V}) \) and \( C^{0,p}\text{-var} ([0,T], \mathcal{V}) \) as subspaces of continuous paths \( C ([0,T], \mathcal{V}) \):

\[
C^p\text{-var} ([0,T], \mathcal{V}) := \left\{ \gamma \in C ([0,T], \mathcal{V}) \mid \|\gamma\|_{p\text{-var},[0,T]} < \infty \right\},
\]

\[
C^{0,p}\text{-var} ([0,T], \mathcal{V}) := \left\{ \gamma \in C ([0,T], \mathcal{V}) \mid \lim_{\delta \to 0} \omega_p (\gamma, \delta) = 0 \right\}.
\]
Then (based on Proposition 5.6 [1]), \( C^{0,p-\text{var}} ([\Delta_0,T], \mathcal{V}) \subset C_{p-\text{var}} ([\Delta_0,T], \mathcal{V}) \) and \( C^{0,p-\text{var}} ([0,T], \mathcal{V}) \subset C_{p-\text{var}} ([0,T], \mathcal{V}) \). Moreover, \( C_{p-\text{var}} ([0,T], \mathcal{V}) \) can be treated as a subspace of \( C_{p-\text{var}} ([\Delta_0,T], \mathcal{V}) \) in which functions on \( \Delta_0[T] \) are generated from paths (by \( \triangle \)). Similarly, \( C^{0,p-\text{var}} ([0,T], \mathcal{V}) \) can be treated as a subspace of \( C^{0,p-\text{var}} ([\Delta_0,T], \mathcal{V}) \). Therefore, we have the inclusions of spaces:

\[
\bigcup C^{0,p-\text{var}} ([\Delta_0,T], \mathcal{V}) \subset C_{p-\text{var}} ([\Delta_0,T], \mathcal{V}) \setminus \bigcup C^{0,p-\text{var}} ([0,T], \mathcal{V}) \subset C_{p-\text{var}} ([0,T], \mathcal{V})
\]

For paths in \( C^{0,p-\text{var}} ([0,T], \mathcal{V}) \) an explicit characterization is available.

**Notation 9** Suppose \( \gamma : [0,T] \to \mathcal{V} \) is a continuous path, and \( D = \{ t_j \} \) is a finite partition of \([0,T]\). Denote \( \gamma^D \) as the piecewise linear path which coincides with \( \gamma \) on points in \( D \), i.e.

\[
\gamma^D (t) = \begin{cases} \frac{t - t_j}{t_{j+1} - t_j} \gamma (t_{j+1}) + \frac{t_{j+1} - t}{t_{j+1} - t_j} \gamma (t_j), & t \in [t_j, t_{j+1}] \end{cases}.
\]

(4)

Then when \( 1 < p < \infty \), for \( \gamma \in C_{p-\text{var}} ([0,T], \mathcal{V}) \), the following three statements are equivalent (Wiener's characterization, Thm 5.31 [1]):

\[
\gamma \in C^{0,p-\text{var}} ([0,T], \mathcal{V}) \Leftrightarrow \exists \{ \gamma_n \}_{n=0}^{\infty} \subset C^{1-\text{var}} ([0,T], \mathcal{V}) \text{ s.t. } \lim_{n \to \infty} \| \gamma_n - \gamma \|_{p-\text{var},[0,T]} = 0
\]

\[
\Leftrightarrow \lim_{|D| \to 0} \| \gamma^D - \gamma \|_{p-\text{var},[0,T]} = 0.
\]

(In Thm 5.31 [1], the equivalency is identified for paths taking value in \( \mathbb{R}^d \), but can be extended to paths taking value in Banach space \( \mathcal{V} \).) When \( p = 1 \), the latter two are equivalent to the absolutely continuity of \( \gamma \) (Proposition 1.32 [1]), while \( \gamma \) is of vanishing 1-variation if and only if it is a constant.

**Notation 10** Denote \( \otimes \) as tensor product. Suppose \( (\mathcal{V}, \| \cdot \|_\mathcal{V}) \) and \( (\mathcal{U}, \| \cdot \|_\mathcal{U}) \) are two Banach spaces. Denote \( (\mathcal{V} \otimes \mathcal{U}, \| \cdot \|_{\mathcal{V} \otimes \mathcal{U}}) \) is the Banach space defined as the completion of \( \{ \sum_{i=1}^{n} v_i \otimes u_i, v_i \in \mathcal{V}, u_i \in \mathcal{U}, n \geq 1 \} \) w.r.t. \( \| \cdot \|_{\mathcal{V} \otimes \mathcal{U}} \).

**Notation 11** For Banach space \( (\mathcal{V}, \| \cdot \|_\mathcal{V}) \) and \( v_1, v_2 \in \mathcal{V} \), denote \([v_1, v_2] := v_1 \otimes v_2 - v_2 \otimes v_1 \). Denote \( (\mathcal{V}, \mathcal{W}, \| \cdot \|_{\mathcal{V} \otimes \mathcal{W}}) \) as the Banach space defined as the completion of \( \{ \sum_{i=1}^{n} [v_i, v_2], v_i, v_2 \in \mathcal{V}, n \geq 1 \} \) w.r.t. \( \| \cdot \|_{\mathcal{V} \otimes \mathcal{W}} \).

In this manuscript, we assume \( \| v \otimes u \|_{\mathcal{V} \otimes \mathcal{U}} \leq \| v \|_\mathcal{V} \| u \|_\mathcal{U}, \forall v \in \mathcal{V}, \forall u \in \mathcal{U} \).

**Definition 12** Suppose \( \mathcal{V} \) and \( \mathcal{U} \) are two Banach spaces, and \( \gamma_1 \in C^{1-\text{var}} ([0,T], \mathcal{V}) \), \( \gamma_2 \in C^{1-\text{var}} ([0,T], \mathcal{U}) \).

Define the iterated integral of \( \gamma_1 \) and \( \gamma_2 \), \( I \) \( (\gamma_1, \gamma_2) \in C ([\Delta_0,T], \mathcal{V} \otimes \mathcal{U}) \), as

\[
I (\gamma_1, \gamma_2) (s, t) = \int_{s < u_1 < u_2 < t} d\gamma_1 (u_1) \otimes d\gamma_2 (u_2), \forall 0 \leq s < t \leq T.
\]
When \( \mathcal{U} = \mathcal{V} \) (so \( \gamma_i \in C^{1-\text{var}} ([0, T], \mathcal{V}), i = 1, 2 \)), define \( A (\gamma_1, \gamma_2) \in C (\Delta_{[0, T]}, [\mathcal{V}, \mathcal{V}]) \) as

\[
A (\gamma_1, \gamma_2) (s, t) = \frac{1}{2} \int \int_{s < u_1 < u_2 < t} [d\gamma_1 (u_1), d\gamma_2 (u_2)] , \forall 0 \leq s \leq t \leq T.
\]

The notation \( I (\gamma_1, \gamma_2) \) is used in the proof of extension of Young integral, \( A (\gamma_1, \gamma_2) \) is used to estimate \( A (\gamma) \) when \( \gamma = \gamma_1 + \gamma_2 \).

**Definition 13 (area)** Suppose \( \gamma \in C^{1-\text{var}} ([0, T], \mathcal{V}) \). Define the area of \( \gamma \), \( A (\gamma) \in C (\Delta_{[0, T]}, [\mathcal{V}, \mathcal{V}]) := A (\gamma, \gamma) \).

**Definition 14 (area operator)** The area operator is the operator defined on the set of continuous bounded variation paths which sends \( \gamma \) to \( A (\gamma) \).

The area operator can be extended where the Riemann-Stieltjes integral \( A (\gamma) \) is well-defined (e.g. \( \mathcal{G}_2 (\mathcal{V}) \) defined below).

When \( \gamma \in C^{1-\text{var}} ([0, T], \mathcal{V}) \), based on Young integral (i.e. \( \mathcal{G}_2 (\mathcal{V}) \) below),

\[
A (\gamma) \in C^{1-\text{var}} (\Delta_{[0, T]}, [\mathcal{V}, \mathcal{V}]) \subseteq C^{0,1-\text{var}} (\Delta_{[0, T]}, [\mathcal{V}, \mathcal{V}]).
\]

On the other hand, because \( \omega_1 (\alpha, \delta) \leq \| \alpha - \alpha_n \|_{1-\text{var}} + \omega_1 (\alpha_n, \delta) \) (\( \omega_1 \) defined at \( \mathcal{G}_2 (\mathcal{V}) \)), \( C^{0,1-\text{var}} (\Delta_{[0, T]}, [\mathcal{V}, \mathcal{V}]) \) is closed under \( 1 \)-variation. Thus,

\[
\{ A (\gamma) | \gamma \in C^{1-\text{var}} ([0, T], \mathcal{V}) \}_{1-\text{var}} \subseteq C^{0,1-\text{var}} (\Delta_{[0, T]}, [\mathcal{V}, \mathcal{V}]). \tag{6}
\]

**Definition 15 (weak geometric 2-rough path)** Suppose \( \gamma \in C ([0, T], \mathcal{V}), \alpha \in C (\Delta_{[0, T]}, [\mathcal{V}, \mathcal{V}]) \). Then \( \Gamma := (\gamma, \alpha) \in C (\Delta_{[0, T]}, \mathcal{V} \oplus [\mathcal{V}, \mathcal{V}]) \) is called a weak geometric 2-rough path, if for any \( 0 \leq s \leq u \leq t \leq T \),

\[
\alpha (s, t) = \alpha (s, u) + \alpha (u, t) + \frac{1}{2} [\gamma (u) - \gamma (s), \gamma (t) - \gamma (u)] , \tag{7}
\]

and \( \| \Gamma \|_{G^{(2)}} := \left( \| \gamma \|^2_{2-\text{var}} + \| \alpha \|_{1-\text{var}} \right)^{1/2} < \infty \).

Property at \( \mathcal{G}_2 (\mathcal{V}) \) is called multiplicativity. \( \| \cdot \|_{G^{(2)}} \) is called 2-rough norm.

**Definition 16 (geometric 2-rough path)** \( \Gamma := (\gamma, \alpha) \in C (\Delta_{[0, T]}, \mathcal{V} \oplus [\mathcal{V}, \mathcal{V}]) \) is a geometric 2-rough path, if there exist \( \{ \gamma_n \}_n \subset C^{1-\text{var}} ([0, T], \mathcal{V}) \) such that

\[
\lim_{n \to \infty} \| \Gamma - (\gamma_n, A (\gamma_n)) \|_{G^{(2)}} = 0.
\]

One can verify that a geometric 2-rough path is a weak geometric 2-rough path.

Thus, if \( (\gamma, \alpha) \) is a geometric 2-rough path, then \( \gamma \) is of vanishing 2-variation (because of \( \mathcal{G}_2 (\mathcal{V}) \)) and \( \alpha \) is of vanishing 1-variation (because of \( \mathcal{G}_2 (\mathcal{V}) \)). Suppose \( \gamma \in C^{0,2-\text{var}} ([0, T], \mathcal{V}), \) then we say \( \gamma \) can be enhanced into a geometric 2-rough path (or enhancible), if there exists \( \alpha \in C^{0,1-\text{var}} (\Delta_{[0, T]}, [\mathcal{V}, \mathcal{V}]) \) such that \( (\gamma, \alpha) \) is a geometric 2-rough path.

**Notation 17** Denote \( \mathcal{G}_2 (\mathcal{V}) \subseteq C^{0,2-\text{var}} ([0, T], \mathcal{V}) \) as the set of paths which admits an enhancement into a geometric 2-rough path.

\( \mathcal{G}_2 (\mathcal{V}) \) is invariant under reparametrisation and contains, e.g. \( C^{1-\text{var}} ([0, T], \mathcal{V}) \).
2 Questions, answers and results

Suppose \( \gamma_1 \) and \( \gamma_2 \) are continuous paths on \([0, T]\), consider the Riemann-Stieltjes integrals (whenever they exist):

\[
\begin{align*}
\alpha(s, t) &= \int_{s < u_1 < u_2 < t} d\gamma_1(u_1) \otimes d\gamma_2(u_2), (s, t) \in \Delta_{[0, T]} \quad (8) \\
i(t) &= \int_0^t \gamma_1(u) \otimes d\gamma_2(u), t \in [0, T].
\end{align*}
\]

If \( \gamma_1 \) is continuous and \( \gamma_2 \) of bounded variation, then \( \alpha \) and \( i \) are of bounded variation, and

\[
\|\alpha\|_{1-\text{var}, [0, T]} \vee \|i\|_{1-\text{var}, [0, T]} \leq \|\gamma_1\|_{\infty-\text{var}, [0, T]} \|\gamma_2\|_{1-\text{var}, [0, T]}.
\]

Young [6] demonstrated that, if \( \gamma_1 \) is of finite \( p \)-variation, \( \gamma_2 \) of finite \( q \)-variation, and \( p > 1, q > 1, p^{-1} + q^{-1} > 1 \), then \( \alpha \) and \( i \) are still well-defined, and (based on Thm 1.16 in [3])

\[
\begin{align*}
\|\alpha\|_{(p^{-1} + q^{-1})^{-1}-\text{var}, [0, T]} &\leq C_{p, q} \|\gamma_1\|_{p-\text{var}, [0, T]} \|\gamma_2\|_{q-\text{var}, [0, T]}, \\
\|i\|_{p-\text{var}, [0, T]} &\leq (C_{p, q} + 1) \|\gamma_1\|_{p-\text{var}, [0, T]} \|\gamma_2\|_{q-\text{var}, [0, T]}.
\end{align*}
\]

(\( \alpha \) is of finite \( (p^{-1} + q^{-1})^{-1} \)-variation, \( (p^{-1} + q^{-1})^{-1} < 1 \); \( i \) is of finite \( q \) variation, \( q > 1 \), the same as \( \gamma_2 \).) However, the existence of integral is problematic when \( p^{-1} + q^{-1} = 1 \). In the special case \( \gamma_1 = \gamma_2 := \gamma \), the definition of \( \int \gamma \otimes d\gamma \) is problematic when \( \gamma \) is of (vanishing) 2-variation.

While according to rough path theory, if a vanishing 2-variation path \( \gamma \) can be enhanced into a geometric 2-rough path, then one can give meaning to differential equation driven by enhanced \( \gamma \), and the solution exists and is unique under certain regularity assumptions on the vector field (see [3], [4], [1]).

In this manuscript, we study the properties of the area operator and of \( G_2(V) \), through several questions. (This manuscript is intended to be some notes about area and geometric 2-rough paths, and main results are as listed in the abstract.)

**Problem 18** Suppose \( V \) is a Banach spaces, and \( \gamma \in C^{0, 2-\text{var}}([0, T], V) \). Does the Riemann-Stieltjes integration \( \int_0^T \gamma \otimes d\gamma \) exist; if it exists, what is the regularity of path \( t \mapsto \int_0^t \gamma \otimes d\gamma \)?

In 2009, P. L. Lions [2] sketched a proof of the statement that: if \( \gamma_1 \) and \( \gamma_2 \) are of vanishing 2-variation, then \( \int_0^T \gamma_1 \otimes d\gamma_2 \) can be defined through Riemann sums and is of vanishing 2-variation. His statement, however, is incorrect: first of all, the Riemann-Stieltjes integral may not exist (Example [38]); secondly, (when restricted to continuous bounded variation paths equipped with 2-variation) the path-\( \rightarrow \)area operator is not bounded (even when area equipped with uniform norm).
In Example 1(p194), the authors give an example of possible divergence of Riemann sums (w.r.t. finite partition \(D\)) as \(|D| \to 0\). Here we modify the example and get non-existence.

For Riemann-Stieltjes integral \(\int \gamma \otimes d\gamma\), selecting different representative points only produces a negligible error when \(\gamma \in C^{0,2-\varphi}(0, T), V\). Actually, suppose \(\gamma\) is a path defined on \([0, T]\) of vanishing 2-variation, and \(D = \{t_j\}\) is a finite partition satisfying \(|D| \leq \delta\). Then for any \(\{\eta_j, \xi_j\}_j\) satisfying \(t_j \leq \eta_j, \xi_j \leq t_{j+1}\), we have (assume \(|u \otimes v| \leq |u||v|\)):

\[
\left\| \sum_j \left(\gamma (\eta_j) - \gamma (\xi_j)\right) \otimes \left(\gamma (t_{j+1}) - \gamma (t_j)\right) \right\| \\
\leq \left( \sum_j \| \gamma (\eta_j) - \gamma (\xi_j) \|^2 \right)^{\frac{1}{2}} \left( \sum_j \| \gamma (t_{j+1}) - \gamma (t_j) \|^2 \right)^{\frac{1}{2}}.
\]

Since \(\{\eta_j, \xi_j\}_j\) can be treated as points in another finite partition whose mesh is less or equal 2\(\delta\), so

\[
\lim_{\delta \to 0} \sup_{D, |D| \leq \delta} \left\| \sum_{j, t_j \in D} \left(\gamma (\eta_j) - \gamma (\xi_j)\right) \otimes \left(\gamma (t_{j+1}) - \gamma (t_j)\right) \right\| \\
\leq \lim_{\delta \to 0} \sup_{D, |D| \leq 2\delta} \sum_{t_j \in D} \| \gamma (t_{j+1}) - \gamma (t_j) \|^2 = 0.
\]

However, problem may occur when one keeps on inserting partition points—the area generated by the added points could be infinite. In Example 38 we give a path \(f \in C^{0,2-\varphi}(0, 1), \mathbb{C}\):

\[
f (t) = \sum_{n=1}^{\infty} \sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k! 2^k} \exp \left(2\pi i (-1)^n 2^k t\right), \quad t \in [0, 1],
\]

where \(c > \pi, \quad c^n \leq \sum_{k=l_n}^{l_{n+1}-1} k^{-1} \leq c^n + 1, \forall n \geq 1,
\]

which satisfies that, for any \(a \in [-\infty, \infty]\), there exists a sequence of finite partitions \(\{D_n^a\}\) of \([0, 1]\) \((x := \text{Re } f, y := \text{Im } f)\),

\[
\lim_{n \to \infty} |D_n^a| = 0 \quad \text{but} \quad \lim_{n \to \infty} \sum_{k, t_k \in D_n^a} x(t_k) y(t_{k+1}) - x(t_{k+1}) y(t_k) = a.
\]

As a result, since the Riemann sum w.r.t. finite partition \(D\) is

\[
\sum_{k, t_k \in D} \frac{1}{2} \left( f(t_k) + f(t_{k+1}) \right) \otimes \left( f(t_{k+1}) - f(t_k) \right)
\]

\[
= \frac{1}{2} \sum_{k, t_k \in D} \left[ f(t_k), f(t_{k+1}) \right] + \frac{1}{2} f(T)^{\otimes 2} - \frac{1}{2} f(0)^{\otimes 2},
\]

6.
which does not have a limit as $|D| \to 0$ because of (11), so the Riemann-Stieltjes integral $\int_0^1 f \otimes df$ does not exist.

$f$ at (10) is in $C^{0,2-\var}([0, T], C)$. Similar argument can be applied to $C^{0,2-\var}([0, T], V)$ when $\dim(V) \geq 2$. Select $e_1, e_2 \in V$, s.t. $\langle e_1, e_2 \rangle \neq 0$. With $f$ at (19), define $\tilde{f} = (\text{Re } f) e_1 + (\text{Im } f) e_2$. Then following similar reasoning, the Riemann-Stieltjes integral $\int_0^1 \tilde{f} \otimes d\tilde{f}$ does not exist, and for any $a \in [-\infty, \infty]$, there exists a sequence of finite partitions $\{D_n^a\}_n$ of $[0, 1]$, s.t.

$$\lim_{n \to \infty} |D_n^a| = 0 \quad \text{but} \quad \lim_{n \to \infty} \left\| \sum_{k, t_k \in D_n^a} \| \tilde{f} (t_k), \tilde{f} (t_{k+1}) \| \right\| = a. \quad (12)$$

When $\dim(V) = 1$, the Riemann-Stieltjes integral $\int_0^T \gamma d\gamma$ does exist for any $\gamma \in C^{0,2-\var}([0, T], V)$ and equals to $2^{-1} (\gamma^2 (T) - \gamma^2 (0))$, because the vector field is commutative in one-dimensional case, so the Lie bracket vanishes. Thus, any one-dimensional vanishing 2-variation path is in $G_2(V)$, and

$$G_2(V) = C^{0,2-\var}([0, T], V) \quad \text{when} \quad \dim(V) = 1. \quad (13)$$

**Problem 19** When equipping $C^{1-\var}([0, T], V)$ with 2-variation norm, is the area operator continuous, or bounded?

When $\dim(V) = 1$, area vanishes, so the area operator is trivial. In that case it is continuous and bounded. When $\dim(V) \geq 2$, as a consequence of possible non-existence of the Riemann-Stieltjes integral (12), the area operator is not continuous nor bounded.

Actually, suppose $\dim(V) \geq 2, \gamma \in C^{0,2-\var}([0, T], V)$, and $\gamma^D$ the piecewise linear paths coincides with $\gamma$ on points in $D$ (as defined at (13)). Then after direct computation, the Riemann sum of $\int \gamma \otimes d\gamma$ w.r.t. $D$ equals to $A(\gamma^D)(0, T)$ plus a constant:

$$\sum_{k, t_k \in D} \frac{1}{2} \left( \gamma (t_k) + \gamma (t_{k+1}) \right) \otimes \left( \gamma (t_{k+1}) - \gamma (t_k) \right) \quad (14)$$

$$= \frac{1}{2} \sum_{k, t_k \in D} \left[ \gamma (t_k), \gamma (t_{k+1}) \right] + \frac{1}{2} \gamma^2 (T) - \frac{1}{2} \gamma^2 (0)$$

$$= A(\gamma^D)(0, T) + \frac{1}{2} (\gamma (T) + \gamma (0)) \otimes (\gamma (T) - \gamma (0)).$$

Thus, based on (12), there exists a path $f : [0, 1] \to V$ of vanishing 2-variation, such that for any $a \in [-\infty, \infty]$, there exists a sequence of finite partitions $\{D_n^a\}_n$ of $[0, 1]$, satisfying $\lim_{n \to \infty} |D_n^a| = 0$ but $\lim_{n \to \infty} \| A(\gamma^D_n)(0, 1) \| = a$. While $f^{D_n^a}$ converges to $f$ in 2-variation when $n$ tends to infinity (based on (5)). Thus, the area operator is not continuous and not bounded, at least when area is equipped with uniform norm. Thus, there is No universal constant $C$, s.t. $\| A(\gamma) \|_{\infty-\var} \leq C \| \gamma \|_{2-\var}^2$ for all $\gamma \in C^{1-\var}([0, T], V)$. Compare with Young
integral: for any \( p \in [1, 2) \), there exists \( C_p \), s.t. for any \( \gamma \in C^{p-\text{var}}([0, T], V) \),
\[
\|A(\gamma)\|_{\frac{p}{\text{var}}} \leq C_p \|\gamma\|^2_{p-\text{var}} \text{ (i.e. (2))}.
\]

Moreover, by modifying our example, we get a sequence of continuous bounded variation paths (Example 41 at p26) converging to zero in 2-variation, but their area diverge at any non-trivial point: \( (s, t) \in \Delta_{[0,T]}, s < t \). Therefore, when equipping bounded variation paths with 2-variation, the area operator is not continuous nor bounded, even in the sense of at some single point. (The paths in Example 41 are in \( C^{0,2-\text{var}}([0,1], \mathbb{C}) \), but can be generalized to \( C^{0,2-\text{var}}([0,1], \mathbb{V}) \) whenever \( \dim(\mathbb{V}) \geq 2 \).)

**Problem 20** When \( C^{1-\text{var}}([0, T], \mathbb{V}) \) is equipped with 2-variation norm, is the path→area operator closable in p-variation? In other words, if \( \{\gamma_n\}_n \) and \( \{\gamma_m\}_m \) are two sequence of paths in \( C^{1-\text{var}}([0, T], \mathbb{V}) \) converging in 2-variation to the same limit, and \( \{A(\gamma_n)\}_n \) and \( \{A(\gamma_m)\}_m \) converge in p-variation respectively. Then is that true that \( \{A(\gamma_n)\}_n \) and \( \{A(\gamma_m)\}_m \) converge to the same limit?

When \( p > 1 \), not true. When \( p = 1 \), is true. (We assume \( \dim(\mathbb{V}) \geq 2 \), because area vanishes for one-dimensional paths.)

For \( p > 1 \), an illustrative example is \( r_n(t) = \left(\frac{\cos nt}{\sqrt{n}}, \frac{\sin nt}{\sqrt{n}}\right) \), \( t \in [0, 2\pi] \), \( n \geq 1 \). \( r_n \) converges to 0 in q-variation for any \( q > 2 \), but their area converge to \( t - s \) in p-variation for any \( p > 1 \):
\[
\int_s^t \left(\frac{\cos nu}{\sqrt{n}} - \frac{\cos ns}{\sqrt{n}}\right) d\frac{\sin nu}{\sqrt{n}} - \left(\frac{\sin nu}{\sqrt{n}} - \frac{\sin ns}{\sqrt{n}}\right) d\frac{\cos nu}{\sqrt{n}} = t - s - \frac{\sin n(t-s)}{n}.
\]
and
\[
\left\|\frac{1}{\sqrt{n}} \exp(i nt)\right\|_{q-\text{var}} \lesssim \left\|\frac{\sin n(t-s)}{n}\right\|_{p-\text{var}} \lesssim \frac{1}{n^{1-\frac{1}{p}}}.\]

Thus, \((0, 0)\) and \((0, t - s)\) are two geometric q-rough paths with the same first level path for any \( q \in (2, 3) \). (Geometric q-rough paths \( q \in (2, 3) \) are elements in the closure of \( \{(\gamma, A(\gamma)) | \gamma \in C^{1-\text{var}}([0, T], \mathbb{V})\} \) under the metric
\[
d((\gamma_1, A(\gamma_1)), (\gamma_2, A(\gamma_2))) := \left(\|\gamma_1 - \gamma_2\|^q_{q-\text{var}} + \|A(\gamma_1) - A(\gamma_2)\|^\frac{q}{2-\text{var}}\right)^\frac{1}{q}.
\]

However, \( r_n \) are uniformly bounded in 2-variation, but do not converge in 2-variation (\(||n^{-\frac{1}{q}} \cos (nt) - (2n)^{-\frac{1}{q}} \cos (2nt)||_{2-\text{var}} \geq 2, \forall n \)). To construct our example, we add in a decay factor, sum finitely of them together to compensate the decaying effect on \( t - s \), and end up with functions \( \{g_n\}_n \subset C^{0,2-\text{var}}([0,1], \mathbb{C}) \) (Example 42 at p26)
\[
g_n(t) = \left(\pi \sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k}\right)^{-\frac{1}{2}} \sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k^{\frac{1}{2}} 2^k} \exp(2\pi i 2^kt), t \in [0, 1], \quad (15)
\]
where \( \sum_{k=l_n}^{l_{n+1}-1} k^{-1} \geq 1, \forall n \geq 1 \).
We prove that $g_n$ converge in 2-variation to zero as $n$ tends to infinity, but their area converge to $t - s$ in $p$-variation, for any $p > 1$.

For Banach space $\mathcal{V}$, $\dim (\mathcal{V}) \geq 2$, select $e_1, e_2 \in \mathcal{V}$, s.t. $[e_1, e_2] \neq 0$. With $g_n$ defined at (13), define $\tilde{g}_n := (\text{Re } g_n) e_1 + (\text{Im } g_n) e_2$. Then $\{\tilde{g}_n\} \subset C^{1-\text{var}} ([0, 1], \mathcal{V})$, $\lim_{n \to \infty} \|\tilde{g}_n\|_{2-\text{var}} = 0$ and

$$
\lim_{n \to \infty} \|A (\tilde{g}_n) (s, t) - (t - s) [e_1, e_2]\|_{p-\text{var}} = 0 \text{ for any } p > 1.
$$

When $p = 1$, if $(\gamma, \alpha_1)$ and $(\gamma, \alpha_2)$ are two geometric 2-rough paths, then $\alpha_1 - \alpha_2 := \varphi$ is additive thus a path. Moreover, based on (14), $\alpha_1$ and $\alpha_2$ are in $C^{0,1-\text{var}} ([\triangle_{0, T}], \mathcal{V})$ of vanishing 1-variation, then $\varphi$ is of vanishing 1-variation. While a path of vanishing 1-variation is constant, so $\alpha_1 = \alpha_2$.

For the same reason we have: the projection of a geometric $n$-rough path to its first $n - 1$ level elements is injective for any $n \in \mathbb{N}$, $n \geq 2$. While in Remark 9.13 (case ii b2) in [1], the authors commented that the projection is not a injection without providing a proof.

**Problem 21** Is that true that every path in $C^{0,2-\text{var}} ([0, T], \mathcal{V})$ admits an enhancement into a (weak) geometric 2-rough path? (i.e. is the inclusion $\mathcal{G}_2 (\mathcal{V}) \subseteq C^{0,2-\text{var}} ([0, T], \mathcal{V})$ strict?)

When $\dim (\mathcal{V}) = 1$, $\mathcal{G}_2 (\mathcal{V}) = C^{0,2-\text{var}} ([0, T], \mathcal{V})$ (see (13)).

When $\dim (\mathcal{V}) \geq 2$, $\mathcal{G}_2 (\mathcal{V}) \nsubseteq C^{0,2-\text{var}} ([0, T], \mathcal{V})$, and an example is given in Thm 9.12 [1]. Actually, following the same reasoning as in Thm 9.12 [1], we use $f$ defined at (10) to prove that $\mathcal{G}_2 (\mathcal{V}) \nsubseteq C^{0,2-\text{var}} ([0, T], \mathcal{V})$ when $\dim (\mathcal{V}) \geq 2$. Select $e_1, e_2 \in \mathcal{V}$, s.t. $[e_1, e_2] \neq 0$. With $f$ at (10), denote $\tilde{f} := (\text{Re } f) e_1 + (\text{Im } f) e_2$, so $\tilde{f} \in C^{0,2-\text{var}} ([0, T], \mathcal{V})$. Assume that $(\tilde{f}, \alpha)$ is a weak geometric 2-rough path. Then using multiplicativity of $(\tilde{f}, \alpha)$ (i.e. (7)), for any finite partitions $D$, we have

\[
\left| \sum_{j,t_j \in D} \left[ \tilde{f} (t_j), \tilde{f} (t_{j+1}) \right] - \left[ \tilde{f} (0), \tilde{f} (1) \right] \right| = 2 \left\| \alpha (0, 1) - \sum_{j,t_j \in D} \alpha (t_j, t_{j+1}) \right\| \\
\leq 4 \| \alpha \|_{1-\text{var}} < \infty.
\]

Then contradiction is established, if $\sum_{j,t_j \in D} \left[ \tilde{f} (t_j), \tilde{f} (t_{j+1}) \right]$ are not uniformly bounded for all finite partitions, which is true because of (12).

Then a natural question arises:

**Problem 22** What is the condition for vanishing 2-variation paths to be enhancible (i.e. in $\mathcal{G}_2 (\mathcal{V})$)?

We prove that:

**Theorem 23** Suppose $\gamma \in C^{0,2-\text{var}} ([0, T], \mathcal{V})$. Then $\gamma \in \mathcal{G}_2 (\mathcal{V})$ if and only if $A (\gamma^D)$ converges in 1-variation as $|D| \to 0$. 

9
Enhanced into a geometric 2-rough path.

It is necessary to enhance a path in $\mathbb{C}_q$, i.e. $(\gamma, \alpha)$ is a geometric 2-rough path, then there exists a sequence of continuous bounded variation paths $\{\gamma_n\}_n$, s.t. $(\gamma_n, A(\gamma_n))$ converge to $(\gamma, \alpha)$ in 2-rough norm $\|\cdot\|_{C(2)}$. However, their construction of $\{\gamma_n\}$ depends on $\alpha$ (i.e. Chow-Rashevskii connectivity theorem), while not $\gamma^D$ in general.

For any $0 \leq s \leq t \leq T$ and any finite partition $D$ of $[s, t]$, the Riemann sums of $2^{-1} \int_s^t [\gamma(u) - \gamma(s), d\gamma(u)]$ w.r.t. $D \subset [s, t]$ is

$$2^{-1} \sum_{k,t_k \in D} \frac{1}{2} [\gamma(t_k) + \gamma(t_{k+1}), \gamma(t_{k+1}) - \gamma(t_k)] - 2^{-1} [\gamma(s), \gamma(t)]$$

$$= 2^{-1} \sum_{k,t_k \in D} [\gamma(t_k), \gamma(t_{k+1})] - 2^{-1} [\gamma(s), \gamma(t)].$$

On the other hand, direct computation gives us

$$A(\gamma^D)(s,t) = 2^{-1} \sum_{k,t_k \in D \subset [s.t]} [\gamma(t_k), \gamma(t_{k+1})] - 2^{-1} [\gamma(s), \gamma(t)].$$

Thus, the Riemann-Stieltjes integral $2^{-1} \int_s^t [\gamma(u) - \gamma(s), d\gamma(u)]$ is the pointwise limit of $A(\gamma^D)$ as $|D| \to 0$. Hence, if $\gamma$ is in $\mathcal{G}_2(\mathcal{V})$, then $A(\gamma^D)$ converge in 1-variation (Theorem 23), so converge pointwisely, to $2^{-1} \int_s^t [\gamma(u) - \gamma(s), d\gamma(u)]$.

Therefore, the Riemann-Stieltjes integral $2^{-1} \int_s^t [\gamma(u) - \gamma(s), d\gamma(u)]$ is the only possible candidate to enhance $\gamma$: If the integral does not exist, or $(\gamma, 2^{-1} \int_s^t [\gamma(u) - \gamma(s), d\gamma(u)])$ is not a geometric 2-rough path, then $\gamma$ cannot be enhanced into a geometric 2-rough path.

While when $p > 2$, the convergence of $A(f^D)$ as $|D| \to 0$ is not necessary to enhance a path in $C^{0/p-\text{var}}([0,T], \mathcal{V})$. Our path $f$ at (10) is in $C^{0,2-\text{var}}([0,T], \mathcal{V}) \subset C^{p-\text{var}}([0,T], \mathcal{V}) \subset \cap_{p > 2} C^{0,p-\text{var}}([0,T], \mathcal{V})$. Based on [5], finite $p$-variation paths can be enhanced into a geometric $q$-rough path for any $q > p$, so $f$ can be enhanced into a geometric $p$-rough path for any $p > 2$. While $\sup_{D \subset [0,1]} A(f^D)(0,1)$ is not bounded, so $A(f^D)$ do not converge in $p$-variation, for any $p \in [1, \infty]$.

Similar to Theorem 23 we proved that:

**Theorem 24** Suppose $\gamma \in C^{2-\text{var}}([0,T], \mathcal{V})$. Then $\gamma$ can be enhanced into a weak geometric 2-rough path if and only if

$$\sup_{D} \|A(\gamma^D)\|_{1-\text{var}, [0,T]} < \infty \text{ and } \{A(\gamma^D)\}_D \text{ are equicontinuous}.$$

The proof is given at page 37.

**Problem 25** Is $\mathcal{G}_2(\mathcal{V})$ a linear space?
$\mathcal{G}_2 (V)$ is linear when dim $(V) = 1$; is not linear when dim $(V) \geq 2$.

Based on 13 we got at the end of Problem 18 when dim $(V) = 1$, $\mathcal{G}_2 (V) = C^{0,2-var} ([0, T], V)$ thus a space. When dim $(V) \geq 2$, based on the reasoning in Problem 21 $\mathcal{G}_2 (V)$ is not a space.

The non-linearity of $\mathcal{G}_2 (V)$ is inherited from the non-linearity of the area operator.

**Proposition 26** When dim $(V) \geq 2$, both $\mathcal{G}_2 (V)$ and $C^{0,2-var} ([0, T], V) / \mathcal{G}_2 (V)$ are dense in $C^{0,2-var} ([0, T], V)$ under 2-variation norm.

**Proof.** $\mathcal{G}_2 (V)$ is dense in $C^{0,2-var} ([0, T], V)$, because (based on 13)

$$C^{1-var} ([0, T], V) \subseteq \mathcal{G}_2 (V) \subseteq C^{0,2-var} ([0, T], V) =: \overline{C^{1-var} ([0, T], V)}^{2-var}.$$

On the other hand, when dim $(V) \geq 2$, suppose $\gamma \in C^{0,2-var} ([0, T], V)$. We want to find a non-enhancible path $\tilde{\gamma}$ in the 2-variation neighborhood of $\gamma$. Based on the definition of $f$ at (10), define

$$f_N (t) := \sum_{n=N}^{\infty} \sum_{k=0}^{l_n-1} \frac{1}{k+1} \exp (2\pi i (-1)^n 2^k t), t \in [0, 1].$$

Then (based on Lemma 37 below, which is used in the proof of the non-enhancibility of $f$), $\sup_N \| f_N \|_{2-var} := C < \infty$. On the other hand, for each fixed $N$, $\sup_D \| A((f_N)^D) (0, 1) \| = C < \infty$ (because $\sup_D \| A (f^D) (0, 1) \| = \infty$ and $f - f_N$ is smooth). Select $e_1, e_2 \in V$, s.t. $[e_1, e_2] \neq 0$. For any $\epsilon > 0$, choose integer $K$, s.t. $2^{-2K} < T$, $\| \gamma \|_{2-var,[0,2^{-2K}]} < \epsilon$ and $\| e_1 \| + \| e_2 \|$ $(C|\gamma|_{K+1}) < \epsilon$.

Define $g \in C^{0,2-var} ([0,1], V)$ by

$$g (t) = (\Re (f_{K+1} (t) - f_{K+1} (1))) e_1 + (\Im (f_{K+1} (t) - f_{K+1} (1))) e_2, t \in [0, 1].$$

Then $g (1) = 0$ and

$$\| g \|_{2-var,[0,1]} \leq (\| e_1 \| + \| e_2 \|) \| f_{K+1} \|_{2-var,[0,1]}$$

$$\leq (\| e_1 \| + \| e_2 \|) (C|\gamma|_{K+1}) < \epsilon,$$

$$\sup_D \| A(g^D) (0, 1) \| = \sup_D \| A((f_{K+1})^D) (0, 1) \| \| e_1, e_2 \| = \infty.$$

Define

$$\tilde{\gamma} (t) = \begin{cases} g (2^{2(K+1)} t) + (\gamma (\frac{1}{2^{2(K+1)} t}), & t \in [0, \frac{1}{2^{2(K+1)} T}] \\ \text{linear}, & t \in [\frac{1}{2^{2(K+1)} T}, \frac{1}{2^{2K} T}] \\ \gamma (t), & t \in [\frac{1}{2^{2K} T}, T] \end{cases}.$$ 

Then $\tilde{\gamma}$ is continuous and

$$\| \gamma - \tilde{\gamma} \|_{2-var} \leq 2 \| \gamma \|_{2-var,[0,2^{-2K}]} + \| g \|_{2-var,[0,1]} < 3\epsilon.$$
On the other hand,

\[
\sup_{D \subseteq [0, 1]} \left\| A(\bar{\gamma}^D) \right\|_{1\text{-var}} \geq \sup_{D \subseteq [0, 1]} \left\| A(\bar{\gamma}^D) \right\|_{\infty\text{-var}} \\
\geq \sup_{D \subseteq [0, 1]} \left\| A(\bar{\gamma}^D)(0, 1) \right\| = \sup_{D \subseteq [0, 1]} \left\| A(g^D)(0, 1) \right\| = \infty.
\]

Thus \(A(\bar{\gamma}^D)\) do not converge in 1-variation as \(|D| \to 0\), and based on Theorem 23 \(\bar{\gamma}\) is not enhancible. ■

When \(\gamma\) is a path of finite \(p\)-variation, \(p \in [1, 2]\), based on Young integral and Theorem 23, the enhancement of \(\gamma\) to geometric 2-rough path exists uniquely in the form of Riemann-Stieltjes integral. Thus \(\cup_{1 \leq p < 2} C^{p\text{-var}}([0, T], V) \subseteq \mathcal{G}_2(V)\).

**Problem 27** Is the inclusion \(\cup_{1 \leq p < 2} C^{p\text{-var}}([0, T], V) \subseteq \mathcal{G}_2(V)\) strict?

Yes, it is. When \(\dim(V) = 1\), \(\mathcal{G}_2(V) = C^{0,2\text{-var}}([0, 1], V)\) (based on (13)). Select \(e \in V, e \neq 0\), and define \(h(t) = \left(t^2 \cos^2 \left( \frac{t}{\lambda} \right) / \ln t \right) e, t \in [0, 1]\). Then

\[h \in C^{0,2\text{-var}}([0, T], V) \setminus \cup_{1 \leq p < 2} C^{p\text{-var}}([0, T], V)\] (Exer5.35[1]).

When \(\dim(V) \geq 2\), the inclusion is strict because \(\cup_{1 \leq p < 2} C^{p\text{-var}}([0, T], V)\) is a space, but \(\mathcal{G}_2(V)\) is not (Problem 23).

Although \(\mathcal{G}_2\) is not a space, it can be shifted in any of the "Young" direction.

**Proposition 28** \(\mathcal{G}_2(V) + \cup_{1 \leq p < 2} C^{p\text{-var}}([0, T], V) = \mathcal{G}_2(V)\).

Suppose \(\gamma_1 \in \mathcal{G}_2(V)\), then \(\gamma_1\) is of finite 2-variation. For any \(\gamma_2\) of finite \(p\)-variation, \(p \in [1, 2]\), according to Young integral (i.e.(11)), \(A(\gamma_1^D, \gamma_2^D)\) converge in \((2^{-1} + p^{-1})^{-1}\)-variation as \(|D| \to 0\) \((p < 2\), so converge in 1-variation). Similarly, \(A(\gamma_2^D, \gamma_1^D)\) and \(A(\gamma_2^D, \gamma_2^D)\) converge in 1-variation as \(|D| \to 0\). On the other hand, \(\gamma_1 \in \mathcal{G}_2(V)\), so apply Theorem 23 \(A(\gamma_1^D) := A(\gamma_1^D, \gamma_1^D)\) converge in 1-variation. Therefore \(A((\gamma_1 + \gamma_2)^D) = \sum_{i,j=1}^{2} A(\gamma_i^D, \gamma_j^D)\) converge in 1-variation as \(|D| \to 0\) and \(\gamma_1 + \gamma_2\) is enhancible (Theorem 23).

In the way of exploring paths in \(\mathcal{G}_2(V)\), we get an extension to Young [6].

**Theorem 29** Let \(V_i, i = 1, 2\), be two Banach spaces and \(\gamma_i : [0, 1] \to V_i\) be two continuous paths. If there exist \(p > 1, q > 1\), \(p^{-1} + q^{-1} = 1\), and two non-decreasing functions \(m_i : [0, 1] \to \mathbb{R}^+, i = 1, 2\), satisfying

\[
\lim_{t \to 0} m_i(t) = 0, m_i(1) \leq 1, \text{ and } \int_0^1 \frac{m_1(t) m_2(t)}{t} dt < \infty,
\]

such that

\[
\sup_{0 \leq s \leq t \leq 1} \frac{\|\gamma_1(t) - \gamma_1(s)\|}{|t-s|^q m_1(t-s)} := C_1 < \infty, \quad \sup_{0 \leq s \leq t \leq 1} \frac{\|\gamma_2(t) - \gamma_2(s)\|}{|t-s|^q m_2(t-s)} := C_2 < \infty.
\]

(16)
Then the Riemann-Stieltjes integral \( \int_0^1 \gamma_1(t) \otimes d\gamma_2(t) \), \( t \in [0, 1] \) exists, and
\[
\left\| \int_0^1 \gamma_1(t) \otimes d\gamma_2(t) \right\|_{q \text{-var}} \leq 8C_1C_2 \left( 2 + \int_0^1 \frac{m_1(t) m_2(t)}{t} dt \right).
\]

Theorem 29 is proved in page 30.

**Remark 30** When \( m_1(x) = x^a, m_2(x) = x^b \), \( a > 0, b > 0 \), we get Young integral [10].

**Remark 31** In the proof of Theorem 29, we get an estimation of the iterated integral of \( \gamma_1 \) and \( \gamma_2 \) (Definition 12):
\[
\|I(\gamma_1, \gamma_2)\|_{1 \text{-var}} \leq C_1C_2 \left( 15 + 8 \int_0^1 \frac{m_1(t) m_2(t)}{t} dt \right).
\]

On the other hand, \( \int_0^1 \frac{m_1(t)m_2(t)}{t} dt < \infty \) is necessary in the sense of the following example.

**Example 32** Suppose \( m_i : [0, 1] \to \mathbb{R^+} \) are two non-decreasing functions, satisfying \( \lim_{t \to 0} m_i(t) = 0, \|m_i\| \leq 1, i = 1, 2, \) and \( \int_0^1 \frac{m_1(t)m_2(t)}{t} dt = \infty \). Then for any \( p > 1, q > 1, p^{-1} + q^{-1} = 1 \), there exist two continuous real-valued paths \( \gamma_i : [0, 1] \to \mathbb{R}, i = 1, 2, \) s.t.
\[
\sup_{0 \leq s < t \leq 1} \left| \frac{\gamma_1(t) - \gamma_1(s)}{t - s} \right| < \infty, \quad \sup_{0 \leq s < t \leq 1} \left| \frac{\gamma_2(t) - \gamma_2(s)}{t - s} \right| < \infty,
\]
but the Riemann-Stieltjes integral \( \int_0^1 \gamma_2(t) \otimes d\gamma_2(t) \) does not exist.

Proof of Example 32 is given in page 33.

As a consequence of refined Young integral, we have a sufficient condition for path to be in \( \mathcal{G}_2(\mathcal{V}) \).

**Theorem 33** Let \( \gamma : [0, 1] \to \mathcal{V} \) be a continuous paths. If there exists a non-decreasing function \( m : [0, 1] \to \mathbb{R^+} \) satisfying
\[
\lim_{t \to 0} m(t) = 0, m(1) \leq 1, \text{ and } \int_0^1 \frac{m^2(t)}{t} dt < \infty,
\]
such that
\[
\sup_{0 \leq s < t \leq 1} \frac{\|\gamma(t) - \gamma(s)\|}{t - s m(t - s)} < \infty. \tag{17}
\]
Then \( \gamma \in \mathcal{G}_2(\mathcal{V}) \).

Theorem 33 is proved in page 33.

**Remark 34** In Theorem 33 by adding a log term and log-log term so on and so forth, one can get a sequence of nested spaces in \( \mathcal{G}_2(\mathcal{V}) \). Because of inclusion, their union is still a space in \( \mathcal{G}_2(\mathcal{V}) \).

**Remark 35** As a consequence of Example 32, for any non-decreasing function \( m : [0, 1] \to \mathbb{R^+} \), \( \lim_{t \to 0} m(t) = 0, m(1) \leq 1 \) and \( \int_0^1 \frac{m^2(t)}{t} dt = \infty \), there exists \( \gamma : [0, 1] \to \mathbb{C} \) satisfying (17) but not in \( \mathcal{G}_2(\mathcal{C}) \).
3 Proofs

Recall $\triangle_{[0,1]} = \{(s,t) | 0 \leq s \leq t \leq 1\}$.

**Lemma 36** For any $p > 1$ and any $a > 0$, there exists constant $C_{a,p} > 0$, such that for any integer $m \geq 1$,

$$\sum_{k=1}^{m} \frac{2^{(1-\frac{1}{p})k}}{k^a} \leq C_{a,p} \frac{2^{(1-\frac{1}{p})m}}{m^a}.$$

**Proof.** Fix $p > 1$. Denote $b := 2^{(1-\frac{1}{p})}$. Firstly, suppose $c > 0$ is a constant, and $\sum_{k=1}^{m} k^{-a}b^k \leq c(m_1)^{-a}b^{m_1}$. Then $\sum_{k=1}^{m_1+1} k^{-a}b^k \leq c(m_1 + 1)^{-a}b^{m_1+1}$ would hold provided:

$$\frac{b^{m_1}}{m_1^a} + \frac{b^{m_1+1}}{(m_1 + 1)^a} \leq c \left( \frac{b^{m_1+1}}{(m_1 + 1)^a} \right)^{\frac{1}{a}},$$

i.e. $(c - 1)^\frac{1}{a} b^\frac{1}{a} - c^\frac{1}{a} = m_1 \geq \frac{1}{a}.$

Then we choose $C$ in this way: Fix constant $C_1 > \frac{b}{a}$, and let

$$C_{a,p} := C_1 \land \max \left\{ m_1^a \sum_{k=1}^{m} \frac{b^k}{k^a}, 1 \leq m \leq \left[ \frac{C_1^{\frac{1}{a}}}{(C_1 - 1)^\frac{1}{a} b^\frac{1}{a} - C_1^{\frac{1}{a}}} \right] + 1 \right\}.$$

The following lemma is in the form of Exercise 9.14 in [1], only that we give an uniform estimates.

**Lemma 37** Suppose $\mathcal{V}$ is a Banach space, $\varphi_n : \triangle_{[0,1]} \to \mathcal{V}$, $n \geq 1$, and there exists constant $M > 0$ s.t.

$$\| \varphi_n (s,t) \| \leq M (1 \land |t-s|), \forall (s,t) \in \triangle_{[0,T]}, \forall n \geq 1.$$

For $p \in (1, \infty)$, $a \in (0, \infty)$ and integers $1 \leq N_1 \leq N_2 \leq \infty$, define

$$g^{a,p}_{N_1,N_2} (s,t) = \sum_{k=N_1}^{N_2} \frac{1}{k^a 2^{2k}} \varphi_k (2^{2k}s, 2^{2k}t), t \in [0,1].$$

Then

(i) $\sup_{1 \leq N_1 \leq N_2 \leq \infty} \sup_{0 \leq \tau \leq 1} \left\| g^{a,p}_{N_1,N_2} (s,t) \right\|_{-a} \leq C_{a,p,M} < \infty$; (18)

for any $\delta \in (0, 1)$ (recall $\omega_p(\gamma, \delta)$ defined at (2)), we have

(ii) $\sup_{1 \leq N_1 \leq N_2 \leq \infty} \omega_p \left( g^{a,p}_{N_1,N_2}, \delta \right) \leq C_{a,p,M} \left( \ln \left( \frac{2}{\delta} \right) \right)^{-a}$; (19)
and for any fixed \( N_1 \geq 1 \),

\[
(iii) \quad \sup_{N_1 \leq N_2 \leq N_1^a} \| f^p_{N_1, N_2} \|_{p \text{-var}, [0, 1]} \leq \frac{C_{\text{a,p}, M}}{N_1^a}, \tag{20}
\]

where \( C_{\text{a,p}, M} = (\ln 4)^a 2^{-\frac{a}{2}} M \left( 8C_{\text{a,p}} + \left( 2^\frac{a}{2} - 1 \right)^{-1} \right) \) with \( C_{\text{a,p}} \) from Lemma 36, and \( C_{\text{a,p}, M} = \left( (\ln 4)^{-a} C_{\text{a,p}, M} + 2M^p \left( 1 - 2^{-\frac{a}{2}} \right)^{-p} \right)^{\frac{1}{p}} \).

**Proof.** For (18). Fix 0 \( \leq s < t \leq 1 \). Denote \( n := \left[ \log_4 \frac{8}{t-s} \right] \), then use \( \| \varphi_k (s, t) \| \leq M (1 \wedge |t-s|) \), we get

\[
\left\| g^{a,p}_{N_1, N_2} (s, t) \right\| \leq \sum_{k=1}^{n} \frac{1}{k a 2^\frac{a}{2} \pi} \left\| \varphi_k (2^{2k} s, 2^{2k} t) \right\| + \sum_{k=n+1}^{\infty} \frac{1}{k a 2^\frac{a}{2} \pi} \left\| \varphi_k (2^{2k} s, 2^{2k} t) \right\|
\]

\[
\leq M \sum_{k=1}^{n} \frac{2^{2(\frac{1}{2})^k k}}{a} |t-s| + \sum_{k=n+1}^{\infty} \frac{M}{k a 2^\frac{a}{2} \pi}.
\]

Based on Lemma 36 there exists \( C_{\text{a,p}}, s.t. \) for any \( m \geq 1 \), \( \sum_{k=1}^{m} k^{-a} 2^{2(\frac{1}{2})^k k} \leq C_{\text{a,p}} m^{-a} 2^{2(\frac{1}{2})^m} \). Thus (\( n > \log_4 \frac{8}{t-s} \) and \( \frac{2}{t-s} < 2^{2n} \leq \frac{8}{t-s} \)),

\[
\left\| g^{a,p}_{N_1, N_2} (s, t) \right\| \leq M C_{\text{a,p}} \frac{2^{2(\frac{1}{2})^n n}}{n^a} |t-s| + M \frac{1}{2^\frac{a}{2} - 1 n^a 2^\frac{a}{2} \pi}.
\]

\[
\leq M \left( 8C_{\text{a,p}} + \frac{1}{2^\frac{a}{2} - 1} \right) \frac{1}{n^a 2^\frac{a}{2} \pi} |t-s|^\frac{a}{2} \left( \ln \frac{2}{t-s} \right)^{-a}.
\]

Since our estimates holds for any 0 \( \leq s < t \leq 1 \) and any integers \( 1 \leq N_1 \leq N_2 \leq \infty \), (18) is done.

Based on (18), for any \( \delta \in (0, 1) \), and any finite partition \( D = \{ t_j \} \), \( |D| \leq \delta \), we have

\[
\sum_{j, t_j \in D} \left\| g^{a,p}_{N_1, N_2} (t_j, t_{j+1}) \right\|^p \leq C_{\text{a,p}, M} \left( \ln \frac{2}{\delta} \right)^{-ap} \sum_{j, t_j \in D} |t_{j+1} - t_j| = C_{\text{a,p}, M} \left( \ln \frac{2}{\delta} \right)^{-ap}.
\]

It holds for any \( D, |D| \leq \delta \), and any integers \( 1 \leq N_1 \leq N_2 \leq \infty \), so (19) holds.

Then we prove (20). Fix \( N_1 \). Finite partitions whose mesh less then \( 2^{-2N_1} \), is done in (19):

\[
\sup_{N_1 \leq N_2 \leq \infty} \sup_{|D| \leq 2^{-2N_1}} \sum_{j, t_j \in D} \left\| g^{a,p}_{N_1, N_2} (t_j, t_{j+1}) \right\|^p \leq \frac{C_{\text{a,p}, M}^p}{(\ln 4)^{ap} N_1^p}. \tag{21}
\]
For finite partitions $D = \{t_j\}$ satisfying $|D| > 2^{-2N_1}$, we denote $J_{N_1} := \{|j| |t_{j+1} - t_j| > 2^{-2N_1}\}$. Since there can not be more than $2 \times 2^{2N_1}$ many subintervals in $J_{N_1}$, (and using $|\varphi_n(s,t)| \leq M$)

$$\sum_{t_j \in D, j \in J_{N_1}} \left\| g_{N_1,N_2}^{a,p} (t_j,t_{j+1}) \right\|^p \leq 2^{2N_1+1} \left( \sum_{k=N_1}^{M} \frac{2^k}{2^p} \right)^p \leq 2 \left( \frac{2^p M}{2^p - 1} \right)^p \frac{1}{N_1^{np}}.$$  

The intervals in $D$ which are not in $J_{N_1}$ can be treated as subintervals in another finite partition $D'$, $|D'| \leq 2^{-2N_1}$, so using (21) to bound them, we get

$$\sum_{t_j \in D} \left\| g_{N_1,N_2}^{a,p} (t_j,t_{j+1}) \right\|^p \leq \sum_{j \notin J_{N_1}} + \sum_{j \in J_{N_1}} \leq \left( \frac{C_p^{a,p} M}{(\ln 4)^{2p}} + 2 \left( \frac{2^p M}{2^p - 1} \right)^p \right) \frac{1}{N_1^{np}}.$$  

Our estimates hold for any finite partition $D$, and for any integer $N_2 \geq N_1$, so (20) holds. ■

Example 38 Suppose $c > \pi$ is a constant, and $\{l_n\}$ is a sequence of increasing integers, satisfying

$$c^n \leq \sum_{k=1}^{l_n+1} \frac{1}{k} \leq c^n + 1, \quad \forall n \geq 1. \quad (22)$$

If define $f : [0,1] \to \mathbb{C}$ as

$$f(t) = \sum_{n=1}^{\infty} \sum_{k=l_n}^{l_n+1} \frac{1}{k+2^k} \exp \left( 2\pi i (-1)^n 2^k t \right), \quad t \in [0,1].$$

Then $f$ is of vanishing 2-variation, and for any $a \in [-\infty,\infty]$, there exists a sequence of finite partition $\{D_n^a\}$ of $[0,1]$ satisfying (with $x := \text{Re} f$, $y := \text{Im} f$)

$$\lim_{n \to \infty} |D_n^a| = 0 \quad \text{and} \quad \lim_{n \to \infty} \sum_{l,t_i \in D_n^a} (x(t_l) y(t_{l+1}) - y(t_l) x(t_{l+1})) = a. \quad (23)$$

The $(-1)^n$ ensure that the limit oscillates. If without $(-1)^n$ we only get divergence, while not non-existence.

**Proof.** $f$ of vanishing 2-variation follows from (19) in Lemma 37 (with $a = \frac{1}{2}$, $p = 2$, $M = 1$, $N_1 = 1$, $N_2 = \infty$). Suppose $N \geq 1$ is an integer, denote

$$D_N := \{l 2^{-2N} \}_{l=0}^{2^{2N}-1}, \quad t_{l,N}^N := l 2^{-2N}, \quad l = 0, 1, \ldots, 2^{2N}, \quad (24)$$

and $(f,D_N) := \sum_{l=0}^{2^{2N}-1} (x(t_{l,N}^N) y(t_{l+1}^N) - y(t_{l,N}^N) x(t_{l+1}^N))^2$. \quad (25)

We want to prove that for each $a \in [-\infty,\infty]$, there exists a sequence of finite partitions $\{D_n^a\}_n \subset \{D_N\}_N$, satisfying $\lim_{n \to \infty} (f,D_n^a) = a.$
Denote

\[ e_k = (-1)^N, \quad k = l_n, \ldots, l_{n+1} - 1, \quad c_k^N = 2\pi^{2k-2N}e_k, \quad k = l_1, \ldots, N - 1. \]

Then \( 2\pi\epsilon_k2^{2k}t_l = \epsilon_k^N, \) and

\[
x (t_l^N) y (t_{l+1}^N) - y (t_l^N) x (t_{l+1}^N)
\]

\[
= \sum_{j=l_1}^{N-1} \frac{1}{j+2l} \cos \left( 2\pi\epsilon_j2^{2j}t_l^N \right) \sum_{k=l_1}^{N-1} \frac{1}{k+2k} \sin \left( 2\pi\epsilon_k2^{2k}t_{l+1}^N \right)
\]

\[
- \sum_{j=l_1}^{N-1} \frac{1}{j+2l} \sin \left( 2\pi\epsilon_j2^{2j}t_l^N \right) \sum_{k=l_1}^{N-1} \frac{1}{k+2k} \cos \left( 2\pi\epsilon_k2^{2k}t_{l+1}^N \right)
\]

\[
= \sum_{k,j=l_1}^{N-1} \frac{1}{k+2k+2l+j} \sin \left( (l+1)c_j^N - lc_j^N \right)
\]

\[
= \sum_{k=l_1}^{N-1} \frac{1}{k+2k} \sin \left( 2\pi\epsilon_k2^{2k-2N} \right)
\]

\[
+ \sum_{l_1 \leq k < j \leq N-1} \frac{1}{k+2k+j} \sum_{l=0}^{2^{2N-1}} \sin \left( l \left( c_k^N - c_j^N \right) + c_k^N \right) + \sin \left( l \left( c_j^N - c_k^N \right) + c_j^N \right)
\]

Sum \( l \) from 0 to \( 2^{2N} - 1 \),

\[
\langle f, D_N \rangle = \sum_{l=0}^{2^{2N-1}} x (t_l^N) y (t_{l+1}^N) - y (t_l^N) x (t_{l+1}^N)
\]

\[
= \sum_{k=l_1}^{N-1} \frac{1}{k+2k-2N} \sin \left( 2\pi\epsilon_k2^{2k-2N} \right)
\]

\[
+ \sum_{l_1 \leq k < j \leq N-1} \frac{1}{k+2k+j} \sum_{l=0}^{2^{2N-1}} \sin \left( l \left( c_k^N - c_j^N \right) + c_k^N \right) + \sin \left( l \left( c_j^N - c_k^N \right) + c_j^N \right)
\]

Since

\[
\sum_{l=0}^{2^{2N-1}} \sin \left( l \left( c_k^N - c_j^N \right) + c_k^N \right) = \sum_{l=0}^{2^{2N-1}} \sin \left( l \left( c_j^N - c_k^N \right) + c_j^N \right) = 0,
\]

so

\[
\langle f, D_N \rangle = \sum_{k=l_1}^{N-1} \frac{1}{k+2k-2N} \sin \left( 2\pi\epsilon_k2^{2k-2N} \right)
\]

\[
= \sum_{j=1}^{j-1} (-1)^j s_j^N + \sum_{k=l_1}^{N-1} \frac{1}{k+2k-2N} \sin \left( 2\pi2^{2k-2N} \right).
\]
Thus using since \(a\) is odd and \(\sum_{j=1}^{m} (-1)^{j} s_{j}^{N}\), when \(m\) is even and \(m \geq 4\), for any \(N \geq l_{m}\),
\[
\sum_{j=1}^{m-1} (-1)^{j} s_{j}^{N} = -(s_{m-1}^{N} - s_{m-2}^{N}) - \cdots - s_{1}^{N} \quad (26)
\]
\[
\leq -\frac{4c - 2\pi}{c^{2} - 1} (e^{m} - c^{2}) + \pi (m - 2) - 4c.
\]
Similarly, when \(m\) is odd and \(m \geq 5\), for any \(N \geq l_{m}\),
\[
\sum_{j=1}^{m-1} (-1)^{j} s_{j}^{N} = (s_{m-1}^{N} - s_{m-2}^{N}) + \cdots + (s_{2}^{N} - s_{1}^{N}) \quad (27)
\]
\[
\geq \frac{4c - 2\pi}{c^{2} - 1} (e^{m} - c) - \pi (m - 1);
\]
and when \(m\) is odd and \(m \geq 5\), for any \(N \geq l_{m}\), the upper bound:
\[
\sum_{j=1}^{m-1} (-1)^{j} s_{j}^{N} = s_{m-1}^{N} - (s_{m-2}^{N} - s_{m-3}^{N}) - \cdots - s_{1}^{N} \quad (28)
\]
\[
\leq 2\pi \times (e^{m-1} + 1) - \frac{4c - 2\pi}{c^{2} - 1} (e^{m-1} - c^{2}) + \pi (m - 3) - 4c
\]
\[
= \left(\frac{2\pi}{c} - \frac{4c - 2\pi}{c(c^{2} - 1)}\right) e^{m} + \pi (m - 1) + \frac{4c - 2\pi}{c^{2} - 1} c^{2} - 4c.
\]
Since we assumed \(c > \pi\), so in (26) and (27), \(\frac{4c - 2\pi}{c^{2} - 1} > 0\). On the other hand, since \(\langle f, D_{l_{m}} \rangle = \sum_{j=1}^{m-1} (-1)^{j} s_{j}^{m}\), so based on (26) and (27), we have
\[
\lim_{n \to \infty} \langle f, D_{l_{2n}} \rangle = -\infty \quad \text{and} \quad \lim_{n \to \infty} \langle f, D_{l_{2n+1}} \rangle = +\infty.
\]
Thus, if when \(a = +\infty\) let \(D_{l_{a}} := D_{l_{2n+1}}\), when \(a = -\infty\) let \(D_{l_{a}} := D_{l_{2n}}\), then when \(a = +\infty\) or \(-\infty\), we have \(\lim_{n \to \infty} |D_{l_{a}}^{n}| = 0\), and \(\lim_{n \to \infty} \langle f, D_{l_{a}}^{n} \rangle = a\).

Fix \(a \in (-\infty, \infty)\). Firstly, we assumed \(c > \pi\), so
\[
0 < \frac{2\pi}{c} - \frac{4c - 2\pi}{c(c^{2} - 1)} < \frac{2\pi}{c} < 2.
\]
For our fixed $c > \pi$, choose integer $M_c \geq 1$, s.t. for any $m \geq M_c$,

$$
\left(\frac{2\pi}{c} - \frac{4c - 2\pi}{c(c^2 - 1)}\right) c^m + \pi (m - 1) + \frac{4c - 2\pi}{c^2 - 1} c^2 - 4c \leq 2c^m.
$$

Thus, combined with (28), when $m$ is odd and $m \geq 5 \vee M_c$, for any $N \geq l_m$, we have

$$
\sum_{j=1}^{m-1} (-1)^j s_j^N \leq 2c^m. \quad (29)
$$

Then for our fixed $a \in (-\infty, \infty)$, choose odd integer $M(a) \geq 5 \vee M_c$ such that, for any odd integer $m \geq M(a)$, and any $N \geq l_m$, we have

$$
\sum_{j=1}^{m-1} (-1)^j s_j^N > |a| + 10\pi, \quad (30)
$$

which is possible because of (27).

We prove that for any odd integer $m \geq M(a)$, there exists $N_m(a)$, $l_m < N_m(a) < l_{m+1}$, s.t.

$$
|\langle f, D_{N_m(a)} \rangle - a| \leq \frac{\pi}{l_m}.
$$

Fix odd integer $m \geq M(a)$. For any $N \geq l_m$ (use $c^m \leq \sum_{k=l_m}^{l_{m+1}-1} k^{-1}$, i.e. (29)),

$$
|a| + 10\pi < \sum_{j=1}^{m-1} (-1)^j s_j^N \leq 2c^m \leq 2 \sum_{k=l_m}^{l_{m+1}-1} k^{-1}. \quad (31)
$$

Thus, when $N = l_{m+1}$ in (31), we have

$$
\langle f, D_{l_{m+1}} \rangle = \sum_{j=1}^{m-1} (-1)^j s_j^{l_{m+1}} - \sum_{k=l_m}^{l_{m+1}-1} \frac{\sin \left(2\pi 2^{2k-2l_{m+1}} \right)}{k 2^{2k-2l_{m+1}}} \leq \sum_{j=1}^{m-1} (-1)^j s_j^{l_{m+1}} - 4 \sum_{k=l_m}^{l_{m+1}-1} k^{-1} \leq -2 \sum_{k=l_m}^{l_{m+1}-1} k^{-1} < -|a| - 10\pi. \quad (32)
$$

While in (31) let $N = l_m$, we have

$$
\langle f, D_{l_m} \rangle = \sum_{j=1}^{m-1} (-1)^j s_j^m > |a| + 10\pi. \quad (33)
$$

Combine (32) with (33), if $|\langle f, D_N \rangle - \langle f, D_{N+1} \rangle|$ is uniformly small when $l_m \leq N \leq l_{m+1} - 1$, then $\exists N_m(a)$, $l_m \leq N_1(a) \leq l_{m+1}$, s.t. $\langle f, D_{N_m(a)} \rangle$ is in the neighborhood of $a$. 

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Actually, for any \( N \geq l_1 + 1 \),

\[
|\langle f, D_{N+1} \rangle - \langle f, D_N \rangle| \leq \sum_{k=1}^{N-1} \frac{1}{k^{2k-2(N+1)}} \left| \frac{\sin \left( \frac{2\pi 2^{2k-2N}}{4} \right) - \sin \left( \frac{2\pi 2^{2k-2(N+1)}}{4} \right)}{l_1} \right| + \frac{4}{N}.
\]

For any \( \theta \in [0, \frac{\pi}{2}] \), using \( \sin (2\theta) = 2\sin \theta \cos \theta \), we have

\[
\frac{1}{\theta} \left| \frac{\sin (4\theta)}{4} - \sin \theta \right| = \frac{\sin \theta}{\theta} |\cos \theta \cos 2\theta - 1| \\
\leq \left| \left( 1 - 2\sin^2 \frac{\theta}{2} \right) (1 - 2\sin^2 \theta) - 1 \right| \leq 14\sin^2 \frac{\theta}{2} \leq \frac{7}{2} \theta^2.
\]

Thus let \( \theta = 2\pi 2^{2k-2(N+1)} \), we have

\[
\frac{1}{2^{2k-2(N+1)}} \left| \frac{\sin \left( \frac{2\pi 2^{2k-2N}}{4} \right) - \sin \left( \frac{2\pi 2^{2k-2(N+1)}}{4} \right)}{l_1} \right| \leq 28\pi^3 \left( \frac{1}{2^{2(N+1)-2k}} \right)^2.
\]

Thus, when \( l_m \leq N \leq l_{m+1} - 1 \),

\[
|\langle f, D_{N+1} \rangle - \langle f, D_N \rangle| \leq 28\pi^3 \sum_{k=1}^{N-1} \frac{1}{k} \left( \frac{1}{2^{2(N+1)-2k}} \right)^2 + \frac{4}{N}. \tag{34}
\]

While one can prove that for any \( m \geq 2 \), \( \sum_{k=1}^{m-1} \frac{2^k}{k} \leq \frac{2^{m+1}}{m} \) by using mathematical induction, so for any \( N \geq l_1 + 1 \),

\[
\sum_{k=1}^{N-1} \frac{1}{k} \left( \frac{1}{2^{2(N+1)-2k}} \right)^2 \leq \frac{1}{2^{4N+4}} \sum_{k=1}^{N-1} \frac{2^{4k}}{k} \leq \frac{1}{16N}. \tag{35}
\]

Then, combined (34) with (35), we get when \( l_m \leq N \leq l_{m+1} - 1 \),

\[
|\langle f, D_{N+1} \rangle - \langle f, D_N \rangle| \leq \left( \frac{7}{4} \pi^3 + 4 \right) \frac{1}{N} < \frac{20\pi}{l_m}.
\]

Thus, combined with (32) and (33), there exists integer \( N_m (a), l_m \leq N_m (a) \leq l_{m+1} \), s.t.

\[
|\langle f, D_{N_m (a)} \rangle - a| < \frac{10\pi}{l_m}.
\]

Moreover, since \( \langle f, D_{l_m} \rangle > |a| + 10\pi \geq |a| + \frac{10\pi}{l_m} \), \( \langle f, D_{l_m+1} \rangle < -|a| - 10\pi \leq -|a| - \frac{10\pi}{l_m} \), so \( l_m < N_m (a) < l_{m+1} \).

Therefore, if let \( D_m := D_{N_m (a)} \), \( m \geq 1 \), then \( \{D_m\}_{m} \) is a sequence of finite partitions, whose mesh tends to zero, but the limit of the corresponding Riemann sum is \( a \).

Next, we demonstrate that when the space of smooth paths is equipped with 2-variation, the area operator is unbounded, and non-closable when the area is equipped with \( p \)-variation, \( p > 1 \).
Lemma 39 Suppose \( \{l_n\}_n \) is a sequence of strictly increasing integers. Then

\[
\lim_{n \to \infty} \left\| \sum_{k=l_n}^{l_n+1} \frac{1}{k2^{2k}} \sin \left(2\pi 2^{2k} (t-s) \right) \right\|_{p-var,[0,1]} = 0 \text{ for any } p > 1.
\]

Proof. We do estimation for fixed \( p > 1 \) and fixed sufficiently large \( n \).

For integer \( m \geq l_n \), denote \( I_m := (2^{-2p(m+1)}, 2^{-2pm}) \), and denote \( I_{n+} := (2^{-2pn}, 1] \). Suppose \( D = \{t_j\} \) is a finite partition satisfying that \( \{|t_{j+1} - t_j|\}_j \subset \cup_{i=1}^{s} I_{m_i} \cup I_{n+} \) with \( \min_{1 \leq i \leq s} m_i \geq l_n \). Denote \( J_{m_i} := \{j| t_{j+1} - t_j \in I_{m_i}\} \) and \( J_{n+} := \{j| t_{j+1} - t_j \in I_{n+}\} \). We assume that \( J_{m_i} \) is not empty for each \( i \). For \( J_{n+} \), since we can not have more than \( 2^{2pl_n+1} \sum_{j \in J_{n+}} (t_{j+1} - t_j) \) intervals in \( J_{n+} \), so

\[
\sum_{j,j \in J_{n+}} \left( \sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k2^{2k}} \sin \left(2\pi 2^{2k} (t_{j+1} - t_j) \right) \right)^p \leq 2^{2pl_n+1} \left( \sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k2^{2k}} \right)^p \sum_{j,j \in J_{n+}} (t_{j+1} - t_j) \leq \frac{2^{2p+1}}{3p^{m_n}} \sum_{j,j \in J_{n+}} (t_{j+1} - t_j).
\]

Then we do estimation for fixed \( i, \ i = 1, 2, \ldots, s \). Suppose \( t_{j+1} - t_j \in I_{m_i} \), then

\[
\left( \sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k2^{2k}} \sin \left(2\pi 2^{2k} (t_{j+1} - t_j) \right) \right)^p \leq 2^{p-1} \left( 2\pi \sum_{k=l_n}^{m_i} \frac{1}{k} \right)^p |t_{j+1} - t_j|^p + \left( \sum_{k=m_i+1}^{\infty} \frac{1}{k2^{2k}} \right)^p \leq 2^{p-1} \left( 2\pi (1 + \ln m_i) \right)^p \left( \frac{1}{2^{2p(m_i)} \ln m_i} + \frac{1}{3p^{m_i} 2^{2pm_i}} \right).
\]

Since there can not be more than \( 2 \times 2^{2p(m+1)} \sum_{j,j \in J_{m_i}} (t_{j+1} - t_j) \) many intervals whose length fall into the category \( I_{m_i} \), so

\[
\sum_{j,j \in J_{m_i}} \left( \sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k2^{2k}} \sin \left(2\pi 2^{2k} (t_{j+1} - t_j) \right) \right)^p \leq 2^{p-1} \left( 2\pi \right)^p \left( 1 + \ln m_i \right)^p \left( \frac{1}{2^{2p(m_i)} \ln m_i} + \frac{1}{3p^{m_i} 2^{2pm_i}} \right) \times 2^{2p(m_i+1)+1} \sum_{j,j \in J_{m_i}} (t_{j+1} - t_j) \leq 2^{3p} \left( 2\pi \right)^p \left( 1 + \ln m_i \right)^p \left( \frac{1}{2^{2p(p-1)m_i} + \frac{1}{3p^{m_i}} \right) \sum_{j,j \in J_{m_i}} (t_{j+1} - t_j).
\]

Since \( \{l_n\} \) are strictly increasing integers, so \( \lim_{n \to \infty} l_n = +\infty \). Thus, for our fixed \( p > 1 \), there exists \( N(p) \geq 1 \), s.t. for any \( n \geq N(p) \) and any \( m_i \geq l_n \), we
have
\[
\left(1 + \ln m_i\right)^p \leq \frac{1}{m_i^p}.
\]

Therefore, for any fixed finite partition \(D = \{t_j\}\) of \([0,1]\), when \(n \geq N(p)\), we have (using (36), (37) and \(\sum_{i=1}^{n} \sum_{j \in J_{m_i}} (t_{j+1} - t_j) + \sum_{j \in J_{n+}} (t_{j+1} - t_j) = 1\), \(\min_{1 \leq i \leq s} m_i \geq l_n\)

\[
\sum_{j,j \in D} \left(\sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k^{2k}} \sin \left(2\pi 2^{2k} (t_{j+1} - t_j)\right)\right)^p
\]

\[
\leq 2^{3p} \left(2\pi^p + \frac{1}{3p}\right) \left(\sum_{i=1}^{s} \frac{1}{m_i} \sum_{j \in J_{m_i}} (t_{j+1} - t_j)\right) + \frac{2^{2p+1}}{3p^p} \sum_{j,j \in J_{n+}} (t_{j+1} - t_j)
\]

\[
\leq 2^{3p} \left(2\pi^p + \frac{1}{3p}\right) \frac{1}{l_n^p}.
\]

Hence, for any fixed \(p > 1\), there exists integer \(N(p)\), s.t. for any \(n \geq N(p)\),

\[
\left\|\sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k^{2k}} \sin \left(2\pi 2^{2k} (t - s)\right)\right\|^p \leq 2^{3p} \left(2\pi^p + \frac{1}{3p}\right) \frac{1}{l_n^p}.
\]

Proof finishes. ■

**Lemma 40** Suppose \(\{l_n\}_n\) is a sequence of strictly increasing integers. Define

\[
g_n(t) = \sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k^{2k}} \exp \left(2\pi i 2^{2k} t\right) , \quad t \in [0,1].
\]

Then \(\lim_{n \to \infty} \|g_n\|_{2-var} = 0\), and for any \(p > 1\),

\[
\lim_{n \to \infty} \left\| A(g_n)(s,t) - \left(\pi \sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k} \right) (t-s)\right\|_{p-var,[0,1]} = 0.
\]

**Proof.** Since trigonometric functions are Lipschitz and bounded, so according to (20) in Lemma 37 with \(p = 2\), \(\lim_{n \to \infty} \|g_n\|_{2-var,[0,1]} = 0\).

According to the definition of area, if denote \(x_n := \text{Re} g_n\), \(y_n := \text{Im} g_n\), and

\[
p_n(s,t) = \int_s^t x_n(u) dy_n(u) - y_n(u) dx_n(u),
\]

\[
q_n(s,t) = y_n(s) x_n(t) - x_n(s) y_n(t),
\]

we have

\[
A(g_n)(s,t) = \frac{1}{2} \left(p_n(s,t) + q_n(s,t)\right).
\]

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Firstly, for $p_n (s,t)$,

$$p_n (s,t) = 2\pi \int_s^t \left( \sum_{i=l_n}^{l_n+1-1} \frac{1}{i^2 + 2^i} \cos (2\pi 2^{2i}u) \right) \left( \sum_{j=l_n}^{l_n+1-1} \frac{2j}{j^2 + 2^j} \cos (2\pi 2^{2j}u) \right) du$$

$$+ \left( \sum_{j=l_n}^{l_n+1-1} \frac{1}{j^2 + 2^j} \sin (2\pi 2^{2j}u) \right) \left( \sum_{i=l_n}^{l_n+1-1} \frac{2i}{i^2 + 2i} \sin (2\pi 2^{2i}u) \right) du$$

$$= 2\pi \int_s^t \left( \frac{1}{k} \sum_{k=l_n}^{l_n+1-1} \frac{1}{i^2 + 2^j} \cos (2\pi 2^{2i}u) \cos (2\pi 2^{2j}u) + \frac{2i-j}{i^2 + 2i} \sin (2\pi 2^{2i}u) \sin (2\pi 2^{2j}u) \right) du$$

$$= 2\pi \int_s^t \left( \frac{1}{k} \sum_{k=l_n}^{l_n+1-1} \frac{2j-i}{i^2 + 2i} \sin (2\pi 2^{2j}u) \sin (2\pi 2^{2j}u) \right) du$$

$$= \left( \frac{1}{k} \sum_{k=l_n}^{l_n+1-1} \frac{2j-i}{i^2 + 2i} \right) (t-s) + \mu \sum_{l_n \leq i < j \leq l_n+1-1} \frac{1}{i^2 + 2i} p_{i,j} (s,t) ,$$

where

$$p_{i,j} (s,t) := \left( \frac{2^{2j} + 2^{2i}}{2^{2j} - 2^{2i}} \right) \left( \sin \left( 2\pi \left( 2^{2j} - 2^{2i} \right) \right) - \sin \left( 2\pi \left( 2^{2j} - 2^{2i} \right) \right) \right) .$$

While, for $q_n (s,t)$,

$$q_n (s,t) = y_n (s) x_n (t) - x_n (s) y_n (t)$$

$$= \left( \sum_{i=l_n}^{l_n+1-1} \frac{1}{i^2 + 2^i} \sin (2\pi 2^{2i}u) \right) \left( \sum_{j=l_n}^{l_n+1-1} \frac{1}{j^2 + 2j} \cos (2\pi 2^{2j}u) \right)$$

$$- \left( \sum_{j=l_n}^{l_n+1-1} \frac{1}{j^2 + 2j} \cos (2\pi 2^{2j}u) \right) \left( \sum_{i=l_n}^{l_n+1-1} \frac{1}{i^2 + 2i} \sin (2\pi 2^{2i}u) \right)$$

$$= \sum_{i,j=l_n}^{l_n+1-1} \frac{1}{i^2 + 2i + 2j} \sin (2\pi 2^{2i}u - 2^{2j}t)$$

$$= \sum_{k=l_n}^{l_n+1-1} \frac{1}{2^{2k}} \sin (2\pi 2^{2k} (t-s)) + \mu \sum_{l_n \leq i < j \leq l_n+1-1} \frac{1}{i^2 + 2i + 2j} q_{i,j} (s,t) ,$$

where

$$q_{i,j} (s,t) = \sin \left( 2\pi \left( 2^{2i} s - 2^{2j} t \right) \right) + \sin \left( 2\pi \left( 2^{2i} s - 2^{2j} t \right) \right) .$$
Thus

\[ A(g_n)(s,t) - \left( \pi \sum_{k=1}^{l_{n+1}-1} \frac{1}{k} \right) (t-s) \]

\[ = \frac{1}{2} \left( - \sum_{k=1}^{l_{n+1}-1} \frac{1}{k2^{2k}} \sin \left( 2\pi 2^{2k} (t-s) \right) + \sum_{l_n \leq i \leq l_{n+1}-1} \frac{1}{i^2 j^2 2^{i+j}} (p_{i,j} (s,t) + q_{i,j} (s,t)) \right). \]

Based on Lemma 39, \[ \sum_{k=1}^{l_{n+1}-1} k^{-1}2^{-2k} \sin \left( 2\pi 2^{2k} (t-s) \right) \] converge to 0 as \( n \) tends to infinity in \( p \)-variation for any \( p > 1 \), so we are left with

\[ \sum_{l_n \leq i < j \leq l_{n+1}-1} \frac{1}{i^2 j^2 2^{i+j}} (p_{i,j} (s,t) + q_{i,j} (s,t)). \]

While

\[ p_{i,j} (s,t) + q_{i,j} (s,t) \]

\[ = \left( \frac{2^{2j} + 2^{2i}}{2^{2j} - 2^{2i}} \right) \sin \left( 2\pi 2^{2j} (2^{2j} - 2^{2i}) (t-s) \right) \]

\[ + \sin \left( 2\pi 2^{2j} (2^{2j} - 2^{2i}) t \right) + \sin \left( 2\pi 2^{2j} (2^{2j} - 2^{2i}) s \right) \]

\[ = \left( \frac{2 \times 2^{2i}}{2^{2j} - 2^{2i}} \right) \sin \left( 2\pi 2^{2j} (2^{2j} - 2^{2i}) t \right) - \sin \left( 2\pi 2^{2j} (2^{2j} - 2^{2i}) s \right) \]

\[ + \sin \left( 2\pi 2^{2j} (2^{2j} - 2^{2i}) t \right) + \sin \left( 2\pi 2^{2j} (2^{2j} - 2^{2i}) s \right) \]

\[ = \left( \frac{2 \times 2^{2i}}{2^{2j} - 2^{2i}} \right) \sin \left( 2\pi 2^{2j} (2^{2j} - 2^{2i}) t \right) - \sin \left( 2\pi 2^{2j} (2^{2j} - 2^{2i}) s \right) \]

\[ - 2 \cos \left( 2\pi 2^{2j} \left( \frac{t-s}{2} \right) \right) \sin \left( 2\pi 2^{2j} \left( \frac{t-s}{2} \right) \right) - 2 \cos \left( 2\pi 2^{2j} (2^{2j} - 2^{2i}) \left( \frac{t-s}{2} \right) \right) \]

\[ = \left( -4 \times 2^{2i} \right) \cos \left( 2\pi 2^{2j} (2^{2j} - 2^{2i}) \left( \frac{t-s}{2} \right) \right) \sin \left( 2\pi 2^{2j} (2^{2j} - 2^{2i}) \left( \frac{t-s}{2} \right) \right) \]

\[ - 4 \cos \left( 2\pi 2^{2i} \frac{t-s}{2} \right) \sin \left( 2\pi 2^{2i} \frac{t-s}{2} \right) \]

\[ = 4 \cos \left( 2\pi 2^{2i} \frac{t-s}{2} \right) \sin \left( 2\pi 2^{2i} \frac{t-s}{2} \right) \]

Therefore,

\[ |p_{i,j} (s,t) + q_{i,j} (s,t)| \]

\[ \leq 4 \left( \frac{2^{2i+2j}}{2^{2j} - 2^{2i}} \right) \left| \frac{\sin \left( 2\pi 2^{2j} \frac{t-s}{2} \right)}{2^{2j} - 2^{2i}} \right| \cdot \left| \cos \left( 2\pi 2^{2i} \frac{t-s}{2} \right) \right| \]

\[ \leq 4 \left( \frac{2^{2i+2j}}{2^{2j} - 2^{2i}} \right) \left| \frac{\sin \left( 2\pi 2^{2j} \frac{t-s}{2} \right)}{2^{2j} - 2^{2i}} \right| \cdot \left| \cos \left( 2\pi 2^{2i} \frac{t-s}{2} \right) \right| \]
While, since for any \( \theta \), and any integer \( n \geq 1 \),
\[
\sin \theta \prod_{k=0}^{n-1} \cos (2^k \theta) = \frac{\sin (2^n \theta)}{2^n},
\]
so, when \( j > i \),
\[
\sin \left( \frac{2^{2j} \theta}{2^{2j}} \right) = \sin \left( \frac{2^{2i} \theta}{2^{2i}} \right) \prod_{k=2i}^{2j-1} \cos (2^k \theta).
\]
Thus when \( \theta = \pi (t - s) \), continue with \( \text{(38)} \), we have
\[
| p_{i,j} (s,t) + q_{i,j} (s,t) | \leq 4 \left( \frac{2^{2i} + 2j}{2^{2i} - 2^{2j}} \right) \sin \left( \frac{2\pi 2^{2i} t - s}{2^{2i}} \right) \cos \left( \frac{2\pi 2^{2j} t - s}{2^{2j}} \right) - \sin \left( \frac{2\pi 2^{2i} t - s}{2^{2i}} \right) \cos \left( \frac{2\pi 2^{2j} t - s}{2^{2j}} \right)
\]
\[
= 4 \left( \frac{2^{2i} + 2j}{2^{2i} - 2^{2j}} \right) \sin \left( \frac{2\pi 2^{2i} t - s}{2^{2i}} \right) \left| \cos \left( \frac{2\pi 2^{2i} t - s}{2^{2i}} \right) \prod_{k=2i}^{2j-1} \cos (2^k \theta) - \cos \left( \frac{2\pi 2^{2j} t - s}{2^{2j}} \right) \right|
\]
\[
\leq 4 \left( \frac{2^{2i} + 2j}{2^{2i} - 2^{2j}} \right) \sin \left( \frac{2\pi 2^{2i} t - s}{2^{2i}} \right) \times 2 \leq \frac{32}{3} \left| \sin \left( \frac{2\pi 2^{2i} t - s}{2^{2i}} \right) \right| \text{ when } j > i
\]
Therefore, for any \( p \in (1, 2) \),
\[
\sum_{l_n \leq i \leq j, j \leq l_{n+1}-1} \frac{1}{i^{\frac{2}{2} + 2j}} | p_{i,j} (s,t) + q_{i,j} (s,t) | \leq \frac{32}{3} \sum_{j=l_{n+1}}^{l_{n+1}-1} \frac{1}{j^{2(2-\frac{1}{p})}} \left( \sum_{i=l_n}^{j-1} \frac{1}{i^{\frac{2}{2} + 2i}} \right) \left| \sin \left( \frac{2\pi 2^{2i} t - s}{2^{2i}} \right) \right|
\]
\[
\leq \frac{32}{3} \sum_{j=l_{n+1}}^{l_{n+1}-1} \frac{1}{j^{2(2-\frac{1}{p})}} \left( \sum_{i=l_n}^{j-1} \frac{1}{i^{\frac{2}{2} + 2i}} \right) \left| \sin \left( \frac{2\pi 2^{2i} t - s}{2^{2i}} \right) \right| \leq \frac{C_{\perp, p, 1}}{l_n^2}.
\]
While since \( | \sin (t - s) | \leq 1 \land | t - s | \), based on \( \text{(20)} \) in Lemma \( \text{(37)} \) for any \( p > 1 \), there exists a constant \( C_{\perp, p, 1} \), s.t. for any \( l_n \) and any \( j > l_n \), we have,
\[
\left\| \sum_{i=l_n}^{j-1} \frac{1}{i^{\frac{2}{2} + 2i}} \left| \sin \left( \frac{2\pi 2^{2i} t - s}{2^{2i}} \right) \right| \right\|_{p-var} \leq \frac{C_{\perp, p, 1}}{l_n^2}.
\]
Therefore, for any \( p \in (1, 2) \), since \( \cdot \|_{p-var} \) is a norm, combined with \( \text{(39)} \)
\[
\left\| \sum_{l_n \leq i \leq j, j \leq l_{n+1}-1} \frac{1}{i^{\frac{2}{2} + 2j}} (p_{i,j} (s,t) + q_{i,j} (s,t)) \right\|_{p-var} \leq \frac{32}{3} \sum_{j=l_{n+1}}^{l_{n+1}-1} \frac{1}{j^{2(2-\frac{1}{p})}} \left( \sum_{i=l_n}^{j-1} \frac{1}{i^{\frac{2}{2} + 2i}} \right) \left| \sin \left( \frac{2\pi 2^{2i} t - s}{2^{2i}} \right) \right|_{p-var}
\]
\[
\leq \frac{32C_{\perp, p, 1}}{3l_n^2} \sum_{j=l_{n+1}}^{l_{n+1}-1} \frac{1}{j^{2(2-\frac{1}{p})}} \leq \frac{32C_{\perp, p, 1}}{3(2^{2(1-\frac{1}{p})} - 1) l_n 2^{2(1-\frac{1}{p})} l_n} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]
Thus, for any $p > 1$ (since $p$-variation is non-increasing, so if converge in $p$-variation, $p \in (1, 2)$, then converge in $p$-variation, $p > 1$)

$$\lim_{n \to \infty} \left\| A(g_n)(s, t) - \left( \pi \sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k} \right) (t - s) \right\|_{p-\text{var}} = 0.$$  

Example 41 Suppose $\{l_n\}$ is a sequence of increasing integers, satisfying that for any $n \geq 1$, $\sum_{k=l_n}^{l_{n+1}-1} k^{-1} \geq n$. Define 

$$f_n(t) = \frac{1}{k+2^k} \exp \left( 2\pi i 2^k t \right), \quad t \in [0, 1].$$  

(40) Then $\lim_{n \to \infty} \|f_n\|_{2-\text{var}, [0, 1]} = 0$, but for any $0 \leq s < t \leq 1$, $\lim_{n \to \infty} A(f_n)(s, t) = +\infty$.

**Proof.** Follows from Lemma 40

$$\lim_{n \to \infty} \left\| A(f_n)(s, t) - \left( \pi \sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k} \right) (t - s) \right\|_{p-\text{var}} = 0, \quad \text{for any } p > 1.$$  

As a clear consequence of this example, when the space of smooth paths is equipped with 2-variation, the area operator is not continuous, nor bounded.

Example 42 Suppose $\{l_n\}$ is a sequence of increasing integers, satisfying that for any $n \geq 1$, $\sum_{k=l_n}^{l_{n+1}-1} k^{-1} \geq 1$. Define

$$g_n(t) = \left( \pi \sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k} \right)^{-\frac{1}{2}} \sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k+2^k} \exp \left( 2\pi i 2^k t \right), \quad t \in [0, 1].$$

Then $\lim_{n \to \infty} \|g_n\|_{2-\text{var}, [0, 1]} = 0$, and for any $p > 1$,

$$\lim_{n \to \infty} \|A(g_n)(s, t) - (t - s)\|_{p-\text{var}, [0, 1]} = 0.$$  

**Proof.** Follows from Lemma 40

The convergence of $A(g_n)$ to $t - s$ can not hold in 1-variation, because $h_n$ is a sequence of smooth paths, so the limit of $A(g_n)$ in 1-variation is of vanishing 1-variation, while $t - s$ is not. Actually, since $g_n$ converge to zero in 2-variation, so if $A(g_n)$ converge in 1-variation then should converge to 0 (closable when area equipped with 1-variation).

Example 42 demonstrates that when the space of smooth paths is equipped with 2-variation and their area with $p$-variation, $p > 1$, the area operator is not closable.
Next, we extend Young integral \[ \text{[3]} \] to the case \( p^{-1} + q^{-1} = 1 \) by assigning a finer scale continuity (e.g. logarithmic). Before that, we prove a lemma. Recall definition of \( \omega_p (\gamma, \delta) \) at \[ \text{[2]} \].

**Lemma 43** Suppose \( \gamma_1 \in C^{p\text{-var}} ([0,1], \mathcal{V}) \), \( \gamma_2 \in C^{q\text{-var}} ([0,1], \mathcal{V}) \), \( p^{-1} + q^{-1} = 1 \). \( D_1 = \{ t_k \} \) and \( D_2 = \{ s_j \} \) are two finite partitions of \([0,1]\), and \( D_2 \) is a refinement of \( D_1 \), i.e. for any \( k \), there exist integers \( n_k < n_{k+1} \), s.t. \( t_k = s_{n_k} < s_{n_k+1} < \ldots < s_{n_{k+1}} = t_{k+1} \). Then if denote \( I^D := I (\gamma^D_1, \gamma^D_2) \) (see definition at \[ \text{[12]} \]) and suppose \(|D_1| \leq \delta \), we have

\[
\| I^{D_1} - I^{D_2} \|_{1\text{-var},[0,1]} \leq \sum_k \| I^{D_1} \|_{1\text{-var},[t_k,t_{k+1}]} + \sum_k \| I^{D_2} \|_{1\text{-var},[t_k,t_{k+1}]} + 2\omega_p (\gamma_1, \delta) \| \gamma_2 \|_{q\text{-var},[0,1]} + 2\omega_q (\gamma_2, \delta) \| \gamma_1 \|_{p\text{-var},[0,1]} .
\]

**Proof.** Denote \( \Delta \gamma_i := \gamma^D_i - \gamma^D_2 \), \( i = 1,2 \), denote \( \Delta I := I^{D_1} - I^{D_2} \). For any \( (s,t) \in \Delta [0,\tau] \),

\[
\Delta I (s,t) = \int_s^t (\gamma^D_1 (u) - \gamma^D_1 (s)) \otimes d\gamma^D_1 (u) - \int_s^t (\gamma^D_2 (u) - \gamma^D_2 (s)) \otimes d\gamma^D_2 (u)
\]

\[
= \int_s^t (\Delta \gamma_1 (u) - \Delta \gamma_1 (s)) \otimes d\gamma^D_1 (u) + \int_s^t (\gamma^D_2 (u) - \gamma^D_2 (s)) \otimes d\Delta \gamma_2 (u)
\]

\[
= : I (\Delta \gamma_1, \gamma^D_1) (s,t) + I (\gamma^D_2, \Delta \gamma_2) (s,t) .
\]

Suppose \( t_{k_i} - 1 < s \leq t_{k_1} \leq t_{k_2} \leq t < t_{k_2+1} \), then

\[
I (\Delta \gamma_1, \gamma^D_2) (s,t) = I (\Delta \gamma_1, \gamma^D_1) (s,t_{k_1}) + I (\Delta \gamma_1, \gamma^D_1) (t_{k_1}, t_{k_2}) + I (\Delta \gamma_1, \gamma^D_1) (t_{k_2}, t)
\]

\[
+ (\Delta \gamma_1 (t_{k_1}) - \Delta \gamma_1 (s)) \otimes \gamma^D_1 (t_{k_1}) - \gamma^D_1 (t_{k_1})
\]

\[
+ (\Delta \gamma_1 (t_{k_2}) - \Delta \gamma_1 (t_{k_1})) \otimes \gamma^D_1 (t_{k_2}) - \gamma^D_1 (t_{k_2}) .
\]

where the last term vanishes, because \( \Delta \gamma_1 (t_{k_1}) = \Delta \gamma_1 (t_{k_2}) \). Similar result holds for \( I (\gamma^D_2, \Delta \gamma) (s,t) \):

\[
I (\gamma^D_1, \Delta \gamma_2) (s,t) = I (\gamma^D_1, \Delta \gamma_2) (s,t_{k_1}) + I (\gamma^D_1, \Delta \gamma_2) (t_{k_1}, t_{k_2}) + I (\gamma^D_1, \Delta \gamma_2) (t_{k_2}, t)
\]

\[
+ \left( \gamma^D_1 (t_{k_1}) - \gamma^D_1 (s) \right) \otimes \Delta \gamma_2 (t_{k_1}) - \Delta \gamma_2 (t_{k_1})
\]

Thus (since \( \Delta I = I (\Delta \gamma, \gamma_1) + I (\gamma_2, \Delta \gamma) \), \( \| u \otimes v \| \leq \| u \| \| v \| \))

\[
\| \Delta I (s,t) \| \leq \| \Delta I (s,t_{k_1}) \| + \| \Delta I (t_{k_1}, t_{k_2}) \| + \| \Delta I (t_{k_2}, t) \|
\]

\[
+ \| \Delta \gamma_1 (t_{k_1}) - \Delta \gamma_1 (s) \| \left( \| \gamma^D_1 (t) - \gamma^D_1 (t_{k_1}) \| \right)
\]

\[
+ \| \gamma^D_1 (t_{k_2}) - \gamma^D_1 (s) \| \| \Delta \gamma_2 (t) - \Delta \gamma_2 (t_{k_2}) \| .
\]

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For $\Delta I(t_k, t_{k+1})$, by using multiplicativity and $\triangle \gamma_i(t_k) = 0, \forall k, i = 1, 2$, we get
\[
\Delta I(t_k, t_{k+1}) = \sum_{j=k_1}^{k_2-1} \Delta I(t_j, t_{j+1}).
\]
(42)

Thus, combine (41) with (42), we decompose $[s, t]$ into the union of three kinds of subintervals: $[s, t_k]$, $[t_j, t_{j+1}]$ and $[t_k, t]$, and each of them is a subinterval of some $[t_k, t_{k+1}]$. Thus, for any finite partition, applying our estimates to each subinterval, summing them together, and taking supremum over all finite partitions. By using Hölder inequality, we get
\[
\|\Delta I\|_{1-var} \leq \sum_k \|\Delta I\|_{1-var, [t_k, t_{k+1}]} + \left( \sum_k \|\triangle \gamma_1\|_{p-var, [t_k, t_{k+1}]} \right)^\frac{1}{p} \|\gamma_1\|_{q-var, [0, 1]}
\]
(43)
\[
+ \left( \sum_k \|\triangle \gamma_2\|_{q-var, [t_k, t_{k+1}]} \right)^\frac{1}{q} \|\gamma_1\|_{p-var, [0, 1]}
\]
On the other hand, when $i = 1, 2$,
\[
\sup_{D} \|\gamma_i^{D}\|_{p-var, [0, 1]} \leq \|\gamma_i\|_{p-var, [0, 1]},
\]
and since $\triangle \gamma_i := \gamma_i^{D_1} - \gamma_i^{D_2}$,
\[
\|\Delta \gamma_i\|_{p-var, [t_k, t_{k+1}]} \leq \|\gamma_i^{D_1}\|_{p-var, [t_k, t_{k+1}]} + \|\gamma_i^{D_2}\|_{p-var, [t_k, t_{k+1}]}
\]
(45)
\[
\leq 2 \|\gamma_i\|_{p-var, [t_k, t_{k+1}]}.
\]
Therefore, combine (43), (44) with (45),
\[
\|\Delta I\|_{1-var, [0, 1]} \leq \sum_k \|\Delta I\|_{1-var, [t_k, t_{k+1}]} + 2 \left( \sum_k \|\gamma_1\|_{p-var, [t_k, t_{k+1}]} \right)^\frac{1}{p} \|\gamma_2\|_{q-var, [0, 1]}
\]
\[
+ 2 \left( \sum_k \|\gamma_2\|_{q-var, [t_k, t_{k+1}]} \right)^\frac{1}{q} \|\gamma_1\|_{p-var, [0, 1]}
\]
Since $\|\Delta I\|_{1-var, [t_k, t_{k+1}]} \leq \|I^{D_1}\|_{1-var, [t_k, t_{k+1}]} + \|I^{D_2}\|_{1-var, [t_k, t_{k+1}]}$ and $|D_1| \leq \delta$, recall definition of $\omega_p(\gamma, \delta)$ at (2), proof finishes. ■

The following lemma will be used in the proof of Theorem 29.

Lemma 44 Suppose $\gamma_i : [0, T] \to \mathcal{V}$, $i = 1, 2$, are two continuous piecewise linear paths obtained by interpolating on the same finite partition of $[0, T]$. Then
for any $p > 1$, $q > 1$, $p^{-1} + q^{-1} = 1$, there exists finite partition $D = \{t_k\}$ of $[0, T]$, $|D| \leq 2^{-1}T$, s.t.

$$
\|I(\gamma_1, \gamma_2)\|_{1-var,[0, T]} \leq \sum_{k, t_k \in D} \|I(\gamma_1, \gamma_2)\|_{1-var,[t_k, t_{k+1}]} + 2 \|\gamma_1\|_{p-var,[0, T]} \|\gamma_2\|_{q-var,[0, T]}.
$$

If $\gamma_i$ are linear on $[0, T]$ then

$$
\|I(\gamma_1, \gamma_2)\|_{1-var,[0, T]} \leq \|\gamma_1\|_{p-var,[0, T]} \|\gamma_2\|_{q-var,[0, T]}.
$$

Proof. Denote $I := I(\gamma_1, \gamma_2)$ and denote $D' = \{t_j\}_{j=0}^n$ as the finite partition on which $\gamma_i$, $i = 1, 2$, are interpolated.

When $n = 1$, $\{t_j\}_{j=0}^n = \{0, T\}$, then $\gamma_i$ are linear on $[0, T]$, $i = 1, 2$. After computation, one gets (assume $\|u \otimes v\| \leq \|u\| \|v\|)$

$$
\|I\|_{1-var,[0, T]} = \|I_1(0) - I_2(0)\|_{1-var,[0, T]} \leq \|\gamma_1(T) - \gamma_2(0)\|_{1-var,[0, T]} \leq \|\gamma_1\|_{p-var,[0, T]} \|\gamma_2\|_{q-var,[0, T]}.
$$

When $n \geq 2$, denote $t_{j_i} := \min \{t_j | t_j \leq 2^{-1}T\}$.

If $t_{j_i} = 0$, then $j_1 = 0$, and $t_{j_i+1} = t_i > 2^{-1}T$. Thus

$$
\|I\|_{1-var,[0, T]} \leq \|I\|_{1-var,[0, t_1]} + \|I\|_{1-var,[t_1, T]} + \|\gamma_1\|_{p-var,[0, t_1]} \|\gamma_2\|_{q-var,[t_1, T]}.
$$

Use (46) for $\|I\|_{1-var,[0, t_1]}$

$$
\|I\|_{1-var,[0, T]} \leq \|I\|_{1-var,[t_1, T]} + \|\gamma_1\|_{p-var,[0, t_1]} \|\gamma_2\|_{q-var,[0, t_1]} + \|\gamma_1\|_{p-var,[0, t_1]} \|\gamma_2\|_{q-var,[t_1, T]} \leq \|I\|_{1-var,[t_1, T]} + 2^{1+} \|\gamma_1\|_{p-var,[0, T]} \|\gamma_2\|_{q-var,[0, T]}.
$$

Since $T - t_1 < 2^{-1}T$, lemma holds.

If $t_{j_i} > 0$, then

$$
\|I\|_{1-var,[0, T]} \leq \|I\|_{1-var,[0, t_{j_i}]} + \|I\|_{1-var,[t_{j_i}, T]} + \|\gamma_1\|_{p-var,[0, t_{j_i}]} \|\gamma_2\|_{q-var,[t_{j_i}, T]}.
$$

Then if $t_{j_i+1} = T$, $\gamma_i$ are linear on $[t_{j_i}, T]$, $i = 1, 2$, so similar as above,

$$
\|I\|_{1-var,[0, T]} \leq \|I\|_{1-var,[0, t_{j_i}]} + 2^{1+} \|\gamma_1\|_{p-var,[0, T]} \|\gamma_2\|_{q-var,[0, T]}.
$$

Since $t_{j_i} \leq 2^{-1}T$, lemma holds.

If $t_{j_i+1} < T$, then $0 < t_{j_i} < 2^{-1}T < t_{j_i+1} < T$, continue with (47),

$$
\|I\|_{1-var,[0, t_{j_i}]} \leq \|I\|_{1-var,[0, t_{j_i}]} + \|I\|_{1-var,[t_{j_i}, t_{j_i+1}]} + \|\gamma_1\|_{p-var,[0, t_{j_i}]} \|\gamma_2\|_{q-var,[t_{j_i}, t_{j_i+1}]}.
$$
Since $\gamma_i$ are linear on $[t_{j_i}, t_{j_i+1}]$, so
\begin{equation}
\|I\|_{1-\text{var},[t_{j_i}, t_{j_i+1}]} \leq \|\gamma_1\|_{p-\text{var},[t_{j_i}, t_{j_i+1}]} \|\gamma_2\|_{q-\text{var},[t_{j_i}, t_{j_i+1}]}.
\end{equation}

Thus, combine (47), (48) with (49), using Hölder inequality, we get
\begin{align*}
\|I\|_{1-\text{var},[0,T]} & \leq \|I\|_{1-\text{var},[0,t_{j_1}]} + \|I\|_{1-\text{var},[t_{j_1}, t_{j_1+1}]} + 2 \|\gamma_1\|_{p-\text{var},[0,T]} \|\gamma_2\|_{q-\text{var},[0,T]}.
\end{align*}

Since $t_{j_1} \leq 2^{-1}T$ and $t_{j_1+1} > 2^{-1}T$, so lemma holds. Proof finishes. $\blacksquare$

**Theorem 29** Let $\gamma_i : [0, 1] \rightarrow V_i$, $i = 1, 2$, be two continuous paths. If there exist $p > 1$, $q > 1$, $p^{-1} + q^{-1} = 1$, and two non-decreasing functions $m_i : [0, 1] \rightarrow \mathbb{R}^+$, $i = 1, 2$, satisfying
\begin{equation}
\lim_{t \to 0} m_i(t) = 0, \quad m_i (1) \leq 1, \quad i = 1, 2, \quad \text{and} \quad \int_0^1 \frac{m_1(t) m_2(t)}{t} \, dt < \infty.
\end{equation}

such that
\begin{align*}
C_1 := \sup_{0 \leq s < t \leq 1} \frac{\|\gamma_1(t) - \gamma_1(s)\|}{|t-s|^p} m_1(t-s) < \infty,
C_2 := \sup_{0 \leq s < t \leq 1} \frac{\|\gamma_2(t) - \gamma_2(s)\|}{|t-s|^q} m_2(t-s) < \infty.
\end{align*}

Then the Riemann-Stieltjes integral $\int_0^t \gamma_1(t) \otimes d\gamma_2(t)$, $t \in [0, 1]$, exists, and
\begin{equation}
\left\| \int_0^t \gamma_1(t) \otimes d\gamma_2(t) \right\|_{q-\text{var}} \leq 8C_1C_2 \left( 2 + \int_0^1 \frac{m_1(t) m_2(t)}{t} \, dt \right).
\end{equation}

**Proof.** Recall the definition of $I \left( \gamma_1^D, \gamma_2^D \right)$ at (12):
\begin{equation}
I \left( \gamma_1^D, \gamma_2^D \right)(s, t) = \int_s^t \left( \gamma_1^D(u) - \gamma_1^D(s) \right) \otimes d\gamma_2^D(u), \quad 0 \leq s < t \leq 1.
\end{equation}

Denote $I^D_i := I \left( \gamma_1^D, \gamma_2^{D_i} \right)$, $i = 1, 2$. Firstly, we prove that $I^D$ converge in 1-variation as $|D| \to 0$.

Since $m_i$ are non-decreasing, so ($\omega_p$ defined at (2))
\begin{equation}
\omega_p(\gamma_1, \delta) \leq C_1 m_1(\delta), \quad \omega_q(\gamma_2, \delta) \leq C_2 m_2(\delta);
\end{equation}

since $|m_i| \leq 1$ so $\|\gamma_1\|_{p-\text{var},[0,T]} \leq C_1, \|\gamma_2\|_{q-\text{var},[0,T]} \leq C_2$.

Based on Lemma 13, for any finite partition $D_1 \subset D_2 \subset [0, 1]$, if $|D_1| \leq \delta$ then
\begin{equation}
\|I^D_1 - I^D_2\|_{1-\text{var}} \leq \sum_k \|I^D_1\|_{1-\text{var},[t_k, t_{k+1}]} + \sum_k \|I^D_2\|_{1-\text{var},[t_k, t_{k+1}]} + 2\omega_p(\gamma_1, \delta) \|\gamma_2\|_{q-\text{var},[0,1]} + 2\omega_q(\gamma_2, \delta) \|\gamma_1\|_{p-\text{var},[0,1]}.
\end{equation}
Therefore, using Hölder inequality,
\[ 2\omega_p (\gamma_1, \delta) \|\gamma_2\|_{q-\text{var},[0,1]} + 2\omega_q (\gamma_2, \delta) \|\gamma_1\|_{p-\text{var},[0,1]} \leq 2C_1C_2 (m_1 (\delta) + m_2 (\delta)) \] (52)

For \( \sum_k \| D^1_{D_k} \|_{1-\text{var},[t_k,t_{k+1}]} \), since \( D_1 \) is linear on \([t_k, t_{k+1}]\), so
\[ \| D^1_{D_k} \|_{1-\text{var},[t_k,t_{k+1}]} \leq \| \gamma_1 (t_{k+1}) - \gamma_1 (t_k) \|_{p-\text{var},[t_k,t_{k+1}]} - \| \gamma_2 (t_{k+1}) - \gamma_2 (t_k) \|_{q-\text{var},[t_k,t_{k+1}]} \]
Therefore, using Hölder inequality,
\[ \sum_k \| D^1_{D_k} \|_{1-\text{var},[t_k,t_{k+1}]} \leq \left( \sum_k \| \gamma_1 \|_{p-\text{var},[t_k,t_{k+1}]} \right)^{\frac{p}{p+q}} \left( \sum_k \| \gamma_2 \|_{q-\text{var},[t_k,t_{k+1}]} \right)^{\frac{q}{p+q}} \]
\[ \leq m_p (\gamma_1, \delta) m_q (\gamma_2, \delta) \leq C_1C_2m_1 (\delta) m_2 (\delta) \] (53)

For \( \sum_k \| D^2_{D_k} \|_{1-\text{var},[t_k,t_{k+1}]} \), applying Lemma 14 to \( \| D^2_{D_k} \|_{1-\text{var},[t_k,t_{k+1}]} \), \( \forall k \), then there exists a finite partition \( D^{(1)} = \{ u^1_j \}_{j} \), \( |D^{(1)}| \leq 2^{-1} \delta \), s.t.
\[ \sum_k \| D^2_{D_k} \|_{1-\text{var},[t_k,t_{k+1}]} \leq \sum_{j,u^1_j \in D^{(1)}} \| D^2_{D_j} \|_{1-\text{var},[u^1_j,u^1_{j+1}]} + 2 \sum_k \| \gamma_1 \|_{p-\text{var},[t_k,t_{k+1}]} \| \gamma_2 \|_{q-\text{var},[t_k,t_{k+1}]} \]
\[ \leq \sum_{j,u^1_j \in D^{(1)}} \| D^2_{D_j} \|_{1-\text{var},[u^1_j,u^1_{j+1}]} + 2 \left( \sum_k \| \gamma_1 \|_{p-\text{var},[t_k,t_{k+1}]} \right)^{\frac{p}{p+q}} \left( \sum_k \| \gamma_2 \|_{q-\text{var},[t_k,t_{k+1}]} \right)^{\frac{q}{p+q}} \]
\[ \leq \sum_{j,u^1_j \in D^{(1)}} \| D^2_{D_j} \|_{1-\text{var},[u^1_j,u^1_{j+1}]} + 2C_1C_2m_1 (\delta) m_2 (\delta) \]

Continue the process: applying Lemma 14 to \( \| D^2_{D_k} \|_{1-\text{var},[u^1_j,u^1_{j+1}]} \), \( \forall j \), then there exists a finite partition \( D^{(2)} = \{ u^2_j \}_{j} \), \( |D^{(2)}| \leq 2^{-2} \delta \), s.t.
\[ \sum_{j,u^1_j \in D^{(1)}} \| D^2_{D_j} \|_{1-\text{var},[u^1_j,u^1_{j+1}]} \leq \sum_{j,u^2_j \in D^{(2)}} \| D^2_{D_j} \|_{1-\text{var},[u^2_j,u^2_{j+1}]} + 2C_1C_2m_1 \left( \frac{\delta}{2} \right) m_2 \left( \frac{\delta}{2} \right) \]

So on and so forth, and we get (for fixed \( D_2 \), \( I^{D_2} \) is of vanishing 1-variation)
\[ \sum_k \| I^{D_2} \|_{1-\text{var},[t_k,t_{k+1}]} \leq 2C_1C_2 \sum_{n=0}^{\infty} m_1 \left( \frac{\delta}{2^n} \right) m_2 \left( \frac{\delta}{2^n} \right) \] (54)
Since \( m_1 \) and \( m_2 \) are non-decreasing, so when \( n \geq 1 \),

\[
m_1 \left( \frac{\delta}{2^n} \right) m_2 \left( \frac{\delta}{2^n} \right) \leq \left( \frac{\delta}{2^n} \right)^{-1} \int_{\frac{t}{2^n}}^{\frac{t}{2^{n-1}}} m_1 (t) m_2 (t) \, dt \leq 2 \int_{\frac{t}{2^n}}^{\frac{t}{2^{n-1}}} \frac{m_1 (t) m_2 (t)}{t} \, dt.
\]

Thus

\[
\sum_{n=0}^{\infty} m_1 \left( \frac{\delta}{2^n} \right) m_2 \left( \frac{\delta}{2^n} \right) \leq m_1 (\delta) m_2 (\delta) + 2 \int_0^{\delta} \frac{m_1 (t) m_2 (t)}{t} \, dt.
\]

Combined with (54),

\[
\sum_k \left\| I^{D_k} \right\|_{1 \text{-var}, [t_k, t_{k+1}]} \leq 2C_1 C_2 \left( m_1 (\delta) m_2 (\delta) + 2 \int_0^{\delta} \frac{m_1 (t) m_2 (t)}{t} \, dt \right). 
\] 

Therefore, combine (51), (52), (53) with (55), we get

\[
\| I^{D_1} - I^{D_2} \|_{1 \text{-var}} \leq C_1 C_2 \left( 2 (m_1 (\delta) + m_2 (\delta)) + 3m_1 (\delta) m_2 (\delta) + 4 \int_0^{\delta} \frac{m_1 (t) m_2 (t)}{t} \, dt \right).
\]

In the above we assume \( D_2 \subset D_1 \). For two general finite partitions \( D \) and \( D' \), \( |D| \lor |D'| \leq \delta \), denote \( D'' := D \cup D' \), apply our estimates to \( D, D'' \) and \( D', D'' \), we get

\[
\| I^{D} - I^{D'} \|_{1 \text{-var}} \leq 2C_1 C_2 \left( 2 (m_1 (\delta) + m_2 (\delta)) + 3m_1 (\delta) m_2 (\delta) + 4 \int_0^{\delta} \frac{m_1 (t) m_2 (t)}{t} \, dt \right).
\]

Because we assumed that \( \lim_{t \to 0} m_i (t) = 0 \) and \( \int_0^1 \frac{m_i (t) m_j (t)}{t} \, dt < \infty \), so the Riemann-Stieltjes integral \( I (\gamma_1, \gamma_2) \) exists, \( I (\gamma_1^D, \gamma_2^D) \) converge in 1-variation to \( I (\gamma_1, \gamma_2) \) as \( |D| \to 0 \), and \( (|m_i| \leq 1, i = 1, 2) \)

\[
\sup_D \| I (\gamma_1, \gamma_2) - I (\gamma_1^D, \gamma_2^D) \|_{1 \text{-var}} \leq 2C_1 C_2 \left( 7 + 4 \int_0^1 \frac{m_1 (t) m_2 (t)}{t} \, dt \right).
\]

Moreover, if denote finite partition \( D_0 := \{0, 1\} \) then

\[
\| I^{D_0} \|_{1 \text{-var}} \leq \| (\gamma_1 (1) - \gamma_1 (0)) \lor (\gamma_2 (1) - \gamma_2 (0)) \| \leq C_1 C_2.
\]

Thus

\[
\| I (\gamma_1, \gamma_2) \|_{1 \text{-var}} \leq C_1 C_2 \left( 15 + 8 \int_0^1 \frac{m_1 (t) m_2 (t)}{t} \, dt \right).
\]

Then we work out \( \left\| I (\gamma_1, \gamma_2) \circ d\gamma_2 (u) \right\|_{q \text{-var}} \) from \( \| I (\gamma_1, \gamma_2) \|_{1 \text{-var}} \). Since

\[
I (\gamma_1, \gamma_2) (s, t) = \int_s^t (\gamma_1 (u) - \gamma_1 (s)) \circ d\gamma_2 (u)
= \int_s^t \gamma_1 (u) \circ d\gamma_2 (u) - \gamma_1 (s) \circ (\gamma_2 (t) - \gamma_2 (s))
\]

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Thus, combined with (56), we get

Then

Proof.

Let

but the Riemann-Stieltjes integral

Therefore, if define function \( \beta : \Delta_{[0,1]} \to \mathcal{V}_1 \otimes \mathcal{V}_2 \) by setting

\[
\beta (s, t) := \gamma_1 (s) \otimes (\gamma_2 (t) - \gamma_2 (s)), \quad \forall (s, t) \in \Delta_{[0,1]}.
\]

Then

\[
\| \beta \|_{q \text{-var}} \leq \| \gamma_1 \|_{\infty \text{-var}} \| \gamma_2 \|_{q \text{-var}} \leq C_1 C_2.
\]

Thus, combined with (56), we get

\[
\left\| \int_0^1 \gamma_1 (u) \otimes d\gamma_2 (u) \right\|_{q \text{-var}} \leq \| I (\gamma_1, \gamma_2) \|_{1 \text{-var}} + \| \beta \|_{q \text{-var}} \leq 8C_1 C_2 \left( 2 + \int_0^1 \frac{m_1 (t) m_2 (t)}{t} \, dt \right).
\]

Proof finishes. \( \blacksquare \)

When \( m_1 (t) = t^a, m_2 (t) = t^b, a > 0, b > 0 \), we get Young integral.

The condition \( \int_0^1 \frac{m_1 (t) m_2 (t)}{t} \, dt < \infty \) is necessary in the sense of following example.

**Example 32** Suppose \( m_i : [0, 1] \to \mathbb{R}^+ \) are two non-decreasing functions, satisfying \( \lim \epsilon \to 0 m_i (t) = 0, |m_i| \leq 1, i = 1, 2, \) and \( \int_0^1 \frac{m_1 (t) m_2 (t)}{t} \, dt = \infty \). Then for any \( p > 1, q > 1, p^{-1} + q^{-1} = 1 \), there exist two continuous real-valued paths \( \gamma_i : [0, 1] \to \mathbb{R}, i = 1, 2, \) s.t.

\[
C_1 := \sup_{0 \leq s < t \leq 1} \frac{|\gamma_1 (t) - \gamma_1 (s)|}{|t - s|^{\frac{1}{p}} m_1 (t - s)} < \infty, \quad C_2 := \sup_{0 \leq s < t \leq 1} \frac{|\gamma_2 (t) - \gamma_2 (s)|}{|t - s|^{\frac{1}{q}} m_2 (t - s)} < \infty,
\]

but the Riemann-Stieltjes integral \( \int_0^1 \gamma_1 (t) \, d\gamma_2 (t) \) does not exist.

Proof. Let \( \epsilon_k = 1 \) or \( -1, \) \( \forall k, \) and define

\[
\gamma_1 (t) = \sum_{k=1}^{\infty} \frac{m_1 (2^{-2k})}{2^{\frac{4k}{p}}} \cos (2\pi 2^{2k} t), \quad t \in [0, 1],
\]

\[
\gamma_2 (t) = \sum_{k=1}^{\infty} \epsilon_k \frac{m_2 (2^{-2k})}{2^{\frac{4k}{p}}} \sin (2\pi 2^{2k} t), \quad t \in [0, 1].
\]

Then \( \gamma_i \) satisfy (57). Take \( \gamma_1 \) as an example. For \( 0 \leq s < t \leq 1 \), let \( n = \left\lfloor \log_4 \frac{4}{|t-s|} \right\rfloor \), we have

\[
|\gamma_1 (t) - \gamma_1 (s)| \leq 2\pi \left( \sum_{k=1}^{n} m_1 (2^{-2k}) 2^{2(1-\frac{1}{p})k} \right)|t-s| + 2 \sum_{k=n+1}^{\infty} \frac{m_1 (2^{-2k})}{2^{\frac{4k}{p}}}.
\]

(58)
Since \( \lim_{m \to 0} \frac{m(t)}{m(t)^2} = 1 \) (\( \int_0^1 \frac{m(t)}{t} \, dt \geq \int_0^1 \frac{m(t)m(t)}{t} \, dt = \infty \)) so \( \lim_{m \to 0} \frac{m(t)}{m(t)^2} = \infty \), so using L'Hospital's rule,

\[
\begin{align*}
\lim_{n \to \infty} & \frac{m_1(2-2n) \, 2^{2(\frac{1}{2})n}}{\sum_{k=1}^{n} m_1(2-2k) \, 2^{2(\frac{1}{2})k}} \\
= & \lim_{n \to \infty} \frac{m_1(2-2n) \, 2^{2(\frac{1}{2})n} - m_1(2-2(n-1)) \, 2^{2(\frac{1}{2})(n-1)}}{m_1(2-2n) \, 2^{2(\frac{1}{2})n}} \\
= & \lim_{t \to 0} \frac{m_1(t) \, 2^{2(\frac{1}{2})} - m_1(4t)}{m_1(t) \, 2^{2(\frac{1}{2})}} = \frac{2^{2(\frac{1}{2})} - 1}{2^{2(\frac{1}{2})}}.
\end{align*}
\]

Therefore, there exists constant \( C_1 \), s.t. for any \( n \geq 1 \),

\[
\sum_{k=1}^{n} m_1(2-2k) \, 2^{2(\frac{1}{2})k} \leq C_1 m_1(2-2n) \, 2^{2(\frac{1}{2})n},
\]

Continue with (58), since \( m_1 \) is non-decreasing (\( n = \left\lfloor \log_2 \frac{t}{r} \right\rfloor \)) so \( 2^{-2n} < |t-s| \leq 2^{-2(n-1)} \)

\[
|\gamma_1(t) - \gamma_1(s)| \leq 2\pi C_1 m_1(2-2n) \, 2^{2(\frac{1}{2})n} |t-s| + \frac{2}{2^p - 1} m_1(2-2n) \, 2^{-\frac{p}{2}n} \\
\leq \left( 8\pi C_1 + \frac{2}{2^p - 1} \right) |t-s|^{\frac{1}{2}} m_1(t-s).
\]

Then we prove the Riemann-Stieltjes integral \( \int_0^1 \gamma_1(t) \, d\gamma_2(t) \) does not exist. First, the limit of Riemann sum as \( |D| \to 0 \) does not depend on the selection of representative points, because \( \gamma_1 \in C^{0,p-var} \), \( \gamma_2 \in C^{0,q-var} \). Actually, since \( \gamma_i \) satisfy (17) and \( m_i \) are non-decreasing, so \( \omega_p(\gamma_1, \delta) \leq C_1 m_1(\delta) \) and \( \omega_q(\gamma_2, \delta) \leq C_2 m_2(\delta) \). Suppose \( D = \{t_k\} \) is a finite partition of \([0,1]\), then the error occurred to the Riemann sum of \( \int_0^1 \gamma_1(t) \, d\gamma_2(t) \) w.r.t. \( D \) from selecting different representative points is bounded by

\[
\begin{align*}
\sum_k |\gamma_1(t_{k+1}) - \gamma_1(t_k)| |\gamma_2(t_{k+1}) - \gamma_2(t_k)| \\
\leq \left( \sum_k |\gamma_1(t_{k+1}) - \gamma_1(t_k)|^p \right)^{\frac{1}{p}} \left( \sum_k |\gamma_2(t_{k+1}) - \gamma_2(t_k)|^q \right)^{\frac{1}{q}} \\
\leq C_1 C_2 m_1(|D|) m_2(|D|),
\end{align*}
\]

which tends to zero as \( |D| \to 0 \). On the other hand, since

\[
\begin{align*}
\sum_k \frac{1}{2} (\gamma_1(t_{k+1}) + \gamma_1(t_k))(\gamma_2(t_{k+1}) - \gamma_2(t_k)) \\
= \frac{1}{2} \sum_k (\gamma_1(t_k) \gamma_2(t_{k+1}) - \gamma_2(t_k) \gamma_1(t_{k+1})) + \frac{1}{2} \gamma_1(1) \gamma_2(1) - \frac{1}{2} \gamma_1(0) \gamma_2(0),
\end{align*}
\]

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so the existence of Riemann-Stieltjes integral \( \int_0^1 \gamma_1(t) \, d\gamma_2(t) \) is equivalent to the existence of
\[
\lim_{|D| \to 0} \sum_{k, i \in D} (\gamma_1(t_k) \gamma_2(t_{k+1}) - \gamma_2(t_k) \gamma_1(t_{k+1})).
\]

Similar as the estimates in Example 38, if denote finite partition \( D_{2N} = \{ t_i^N \} \) where \( t_i^N := l2^{-2N}, l = 0, 1, \ldots, 2^{2N} \), we get
\[
\left\langle \int \gamma_1 d\gamma_2, D_{2N} \right\rangle := \sum_{i=0}^{2^N-1} (\gamma_1(t_i^N) \gamma_2(t_{i+1}^N) - \gamma_2(t_i^N) \gamma_1(t_{i+1}^N)) \tag{59}
\]
\[
= \sum_{k=1}^{N-1} \epsilon_k m_1(2^{-2k}) m_2(2^{-2k}) 2^{2k-2N} \sin(2\pi 2^{2k-2N}).
\]

While since \( m_i \) are non-decreasing, so for any \( k \geq 1 \),
\[
m_1(2^{-2k}) m_2(2^{-2k}) \geq \frac{1}{3} \int_{2^{-2(k+1)}}^{2^{-2k}} m_1(t) m_2(t) \frac{dt}{t},
\]
so based on our assumption \( \int_0^1 \frac{m_1(t) m_2(t)}{t} dt = \infty \), we have
\[
\sum_{k=1}^{\infty} m_1(2^{-2k}) m_2(2^{-2k}) = \infty.
\]

Thus, since \( m_i \) are non-decreasing and \( \lim_{t \to 0} m_i(t) = 0 \), so using exactly the same estimates as in Example 38, for any sequence of strictly increasing integers \( \{l_n\} \) satisfying for some \( c > \pi \)
\[
c^n \leq \sum_{k=l_n}^{l_{n+1}-1} m_1(2^{-2k}) m_2(2^{-2k}) \leq c^n + 1, \forall n \geq 1,
\]
we let \( \epsilon_k = (-1)^n \), when \( l_n \leq k \leq l_{n+1} - 1 \).

Then, for any \( a \in [-\infty, \infty] \), there exists a finite partition \( \{D_n^a\}_n \subset \{D_{2N}\}_N \),
\[
\lim_{n \to \infty} |D_n^a| = 0,
\]
but \( \lim_{n \to \infty} \langle \int \gamma_1 d\gamma_2, D_n^a \rangle = a. \]

Next, we want to prove that a vanishing 2-variation path \( \gamma \) can be enhanced into a geometric 2-rough path, if and only if \( A(\gamma^D) \) (the areas of piecewisely linear approximation) converge in 1-variation as \( |D| \to 0 \).

\textbf{Lemma 45} Suppose \( \gamma \in C^{0,2-var}([0, T], \mathcal{V}) \). \( D_1 = \{ t_k \}_k \) and \( D_2 = \{ s_j \}_j \) are two finite partitions of \([0, T]\), and \( D_2 \) is a refinement of \( D_1 \), i.e. for any \( k \), there exist integers \( n_k < n_{k+1} \), s.t. \( t_k = s_{n_k} < s_{n_k+1} < \cdots < s_{n_{k+1}} = t_{k+1} \). Then if \( |D_1| \leq \delta \), we have
\[
\| A(\gamma^{D_1}) - A(\gamma^{D_2}) \|_{1-var} \leq \sum_k \| A(\gamma^{D_2}) \|_{1-var, [t_k, t_{k+1}]} + 4 \| \gamma \|_{2-var, [0, T]} \omega_2(\gamma, \delta).
\]

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\textbf{Proof.} Almost the same as that of Lemma \ref{lemma13} when \( p = q = 2 \), by using \( \| [u, v] \| \leq 2 \| u \| \| v \| \). \( \sum_k \| A^{D_k} \|_{1-\text{var}, [t_k, t_{k+1}]} = 0 \) because \( A^{D_k} \) is linear on \([t_k, t_{k+1}], \forall k \). \( \square \)

\textbf{Lemma 46} Suppose \((\gamma, \alpha)\) is a weak geometric 2-rough path, and \( D = \{t_k\}_{k=0}^n \) is a finite partition of \([0, T]\). Then

\[
A^{(\gamma^D)}(0, T) = \frac{1}{2} \sum_{k=0}^{n-1} [\gamma(t_k), \gamma(t_{k+1})] - \frac{1}{2} [\gamma(0), \gamma(T)],
\]

\[
\alpha(0, T) = \sum_k \alpha(t_k, t_{k+1}) + A^{(\gamma^D)}(0, T).
\]

\textbf{Proof.} The first is obtained from directly computation, the second is got by using multiplicativity of \((\gamma, \alpha)\) (i.e. \( \ref{lemma14} \)). \( \square \)

\textbf{Theorem 23} Suppose \( \gamma \in C^{0,2-\text{var}}([0, T], V) \). Then \( \gamma \in G_2(V) \) if and only if \( A^{(\gamma^D)} \) converges in 1-variation as \( |D| \to 0 \).

\textbf{Proof.} \( \Leftarrow \) is clear; we prove \( \Rightarrow \). Suppose \((\gamma, \alpha)\) is a geometric 2-rough path, so \( \gamma \) is of vanishing 2-variation, \( \alpha \) is of vanishing 1-variation. Thus, for any \( \epsilon > 0 \), there exists \( \delta > 0 \), s.t. for any finite partition \( D \) of \([0, T]\) satisfying \( |D| \leq \delta \), \( \sum_{k,t_k \in D} \| \gamma \|_{2-\text{var}, [t_k, t_{k+1}]}^2 < \epsilon \) and \( \sum_{k,t_k \in D} \| \alpha(t_k, t_{k+1}) \| < \epsilon \).

Suppose \( D_1 = \{t_k\}_k \) and \( D_2 = \{s_j\}_j \) are two finite partitions of \([0, T]\) satisfying \( |D_1| \leq \delta, |D_2| \leq \delta \). \( D_2 \) is a refinement of \( D_1 \). Based on Lemma \ref{lemma15}

\[
\| A^{(\gamma^{D_1})} - A^{(\gamma^{D_2})} \|_{1-\text{var}} \leq \sum_k \| A^{(\gamma^{D_2})} \|_{1-\text{var}, [t_k, t_{k+1}]} + 4 \| \gamma \|_{2-\text{var}, [0, T]} \epsilon^2.
\]

For \( \sum_k \| A^{(\gamma^{D_2})} \|_{1-\text{var}, [t_k, t_{k+1}]} \). Since \( \gamma^{D_2} \) is a piecewisely linear path on each \([t_k, t_{k+1}], \) so we only consider finite partitions, whose points are all "corner" points. Suppose \( D_3 \) is a finite partition satisfying \( D_1 \subset D_3 = \{u_i\} \subset D_2 \). Suppose \( u_i = s_{m_i} < s_{m_i+1} < \cdots < s_{m_i+1} = u_{i+1} \), then based on Lemma \ref{lemma16} for each \( i \),

\[
\| A^{(\gamma^{D_2})}(u_i, u_{i+1}) \| \leq \| \alpha(u_i, u_{i+1}) \| + \sum_{j=m_i+1}^{m_i+1-1} \| \alpha(s_j, s_{j+1}) \|.
\]

Sum over \( i \), then

\[
\sum_{i, u_i \in D_3} \| A^{(\gamma^{D_2})}(u_i, u_{i+1}) \| \leq \sum_{i, u_i \in D_3} \| \alpha(u_i, u_{i+1}) \| + \sum_i \sum_{j=m_i+1}^{m_i+1-1} \| \alpha(s_j, s_{j+1}) \|.
\]

Since \( |D_2| \leq |D_3| \leq |D_1| \leq \delta \), so as we assumed, \( \sum_{i, u_i \in D_3} \| \alpha(u_i, u_{i+1}) \| < \epsilon, \)

\[
\sum_i \sum_{j=m_i+1}^{m_i+1-1} \| \alpha(s_j, s_{j+1}) \| = \sum_{j,s_j \in D_2} \| \alpha(s_j, s_{j+1}) \| < \epsilon.
\]

Thus

\[
\sum_{i, u_i \in D_3} \| A^{(\gamma^{D_2})}(u_i, u_{i+1}) \| < 2\epsilon.
\]

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Therefore, taking supremum over all possible $D_3$, we get

$$
\sum_k \| A(\gamma^{D_k}) \|_{1-\text{var},[t_k, t_{k+1}]} \leq 2\epsilon.
$$

Thus

$$
\| A(\gamma^{D_1}) - A(\gamma^{D_2}) \|_{1-\text{var}} \leq 2\epsilon + 4\| \gamma \|_{2-\text{var},[0,T]} \epsilon^2.
$$

For any finite partition $D$ and $D'$, denote $D'' = D \cup D'$, and use the above estimates for $D$, $D''$ and $D'$, $D''$. Proof finishes. □

Therefore, if a vanishing 2-variation path $\gamma$ can be enhanced into a geometric weak geometric 2-rough path, then $A(\gamma^{D_1})$ converge in 1-variation as $|D| \to 0$, so converge pointwisely to the Riemann-Stieltjes integral $2^{-1} \int_s^t [\gamma(u) - \gamma(s), d\gamma(u)]$.

**Theorem 24** Suppose $\gamma \in C^{2-\text{var}}([0,T], \nu)$. Then $\gamma$ can be enhanced into a weak geometric 2-rough path if and only if

$$
\sup_D \| A(\gamma^D) \|_{1-\text{var},[0,T]} < \infty \text{ and } \{ A(\gamma^D) \}_D \text{ are equicontinuous.}
$$

**Proof.** $\Leftarrow$ Suppose $\{D_n\}_n$ is a sequence of finite partitions of $[0,T]$ satisfying $\lim_{n \to \infty} |D_n| = 0$. Since $\{ A(\gamma^{D_n}) \}_n$ are uniformly bounded and equicontinuous, so based on Arzelà-Ascoli theorem, there exists a subsequence $\{ A(\gamma^{D_{n_k}}) \}_k$ which converge in uniform norm. Denote the limit as $\alpha$.

$\gamma$ is continuous, so $\gamma^{D_{n_k}}$ converge to $\gamma$ in uniform norm as $k$ tends to infinity. Since multiplicativity is preserved under pointwise convergence, $(\gamma, \alpha)$ is multiplicaticve. On the other hand, use the lower semi-continuity of $p$-variation,

$$
\| \alpha \|_{1-\text{var},[0,T]} \leq \lim_{n \to \infty} \| A(\gamma^{D_n}) \|_{1-\text{var},[0,T]} \leq \sup_D \| A(\gamma^D) \|_{1-\text{var},[0,T]} < \infty.
$$

Thus, $(\gamma, \alpha)$ is a weak geometric 2-rough path.

$\Rightarrow$ Suppose $(\gamma, \alpha)$ is a weak geometric 2-rough path. Fix finite partition $D = \{t_k\}$ and $(s, t) \in \triangledown[0,T]$. Suppose $t_{k-1} < s \leq t_k \leq t_k \leq t < t_{k+1}$, then based on Lemma [46]

$$
\| A(\gamma^D)(s, t) \| \leq \| \alpha(s, t) \| + \| \alpha(s, t_{k_1}) \| + \sum_{k = k_1}^{k_2-1} |\alpha(t_k, t_{k+1})| + \| \alpha(t_{k_2}, t) \| \leq 2\| \alpha \|_{1-\text{var},[s,t]}.
$$

Thus $\{ A(\gamma^D) \}_D$ are equicontinuous, and (based on 60),

$$
\sup_D \| A(\gamma^D) \|_{1-\text{var},[0,T]} \leq 2\| \alpha \|_{1-\text{var},[0,T]} < \infty.
$$
Lemma 47 Suppose $\gamma : [0,T] \to V$, is a continuous finitely piecewise linear path. Then for any $p > 1$, $q > 1$, $p^{-1} + q^{-1} = 1$, there exists finite partition $D = \{t_k\}$ s.t. $|D| \leq 2^{-1}T$ and

$$\|A(\gamma)\|_{1-\text{var},[0,T]} \leq \sum_{k, t_k \in D} \|A(\gamma)\|_{1-\text{var},[t_k, t_{k+1}]} + 2\|\gamma\|^2_{2-\text{var},[0,T]}.$$  

Proof. Almost the same as that of Lemma 44 when $p = q = 2$, by using $\|u, v\| \leq 2\|u\|\|v\|$. ■

Theorem 33 Let $\gamma : [0,1] \to V$ be a continuous paths. Then if there exists an non-decreasing function $m : [0,1] \to \mathbb{R}^+$ satisfying

$$\lim_{t \to 0} m(t) = 0, \ m(1) \leq 1, \ \text{and} \ \int_0^1 \frac{m^2(t)}{t} \ dt < \infty,$$

such that

$$\sup_{0 \leq s < t \leq 1} \frac{\|\gamma(t) - \gamma(s)\|}{|t - s|^{\frac{1}{2}} m(t - s)} < \infty.$$

Then $\gamma \in G_2(V)$.

Proof. Denote

$$C := \sup_{0 \leq s < t \leq 1} \frac{\|\gamma(t) - \gamma(s)\|}{|t - s|^{\frac{1}{2}} m(t - s)} < \infty.$$

Then $\|\gamma\|_{2-\text{var},[0,T]} \leq C$; $\omega_2(\gamma, \delta) \leq Cm(\delta)$. Using Lemma 45

$$\|A(\gamma^{D_1}) - A(\gamma^{D_2})\|_{1-\text{var}} \leq \sum_k \|A(\gamma^{D_2})\|_{1-\text{var},[t_k, t_{k+1}]} + 4\|\gamma\|_{2-\text{var},[0,T]} \omega_2(\gamma, \delta).$$

While, apply Lemma 47 to bisect intervals, and use similar reasoning as that lead to (44) in proof of Theorem 29 (starting from page 33), we get

$$\sum_k \|A^{D_2}\|_{1-\text{var},[t_k, t_{k+1}]} \leq 2C^2 \sum_{n=0}^{\infty} m^2\left(\frac{\delta}{2^n}\right) \leq 2C^2 \left( m^2(\delta) + 2\int_0^\delta \frac{m^2(t)}{t} \ dt \right).$$

Thus, $\|A(\gamma^{D_1}) - A(\gamma^{D_2})\|_{1-\text{var}} \leq 2C^2 \left( 2m(\delta) + m^2(\delta) + 2\int_0^\delta \frac{m^2(t)}{t} \ dt \right).$

Since $\lim_{\delta \to 0} m(\delta) = 0$ and $\int_0^1 \frac{m^2(t)}{t} \ dt < \infty$, so $A(\gamma^D)$ converge in 1-variation as $|D| \to 0$. Based on Theorem 29 $\gamma$ is in $G_2(V)$. ■

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