ON THE STEINNESS OF A CLASS OF KÄHLER MANIFOLDS

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1. Introduction

Let \((M^n, g_0)\) be a complete non-compact Kähler manifold with complex dimension \(n\) and with bounded nonnegative holomorphic bisectional curvature. Let \(R\) be the scalar curvature and define

\[ k(x, r) := \frac{1}{V_x(r)} \int_{B_x(r)} R dV. \]

In [8], it was proved by the authors that if \(M\) has maximum volume growth, then \(M\) is biholomorphic to \(\mathbb{C}^n\). There, the authors used a result of Ni in [22] (see also [11, 13]) which states that the condition of maximum volume growth on \(M\) implies that

\[ k(x, r) \leq C \frac{1}{1 + r^2} \]

for some \(C\) for all \(x\) and \(r\). In [9], the authors proved that condition (1.1) implies that \(M\) is holomorphically covered by \(\mathbb{C}^n\), without assuming the maximum volume growth condition. The proof is obtained by studying the Kähler-Ricci flow:

\[ \frac{dg_{i\bar{j}}}{dt} = -R_{i\bar{j}} \]

with initial data \(g_0\). It is well-known by [30] that if the scalar curvature decays linearly in the average sense:

\[ k(x, r) \leq C/(1 + r) \]

for some constant \(C\) for all \(x\) and \(r\) then (1.2) has long time solution with uniformly bounded curvature. By the results in [13, 26], the linear decay condition (1.3) is true in most case, at least for constant \(C\) which may depend on \(x\).

In this paper, we will prove the following:

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Theorem 1.1. Let \((M^n, g_0)\) be a complete non-compact Kähler manifold with bounded non-negative holomorphic bisectional curvature. Suppose the scalar curvature of \(g_0\) satisfies the linear decay condition (1.3). Then \(M\) is holomorphically covered by a pseudoconvex domain in \(\mathbb{C}^n\) which is homeomorphic to \(\mathbb{R}^{2n}\). Moreover, if \(M\) has positive bisectional curvature and is simply connected at infinity, then \(M\) is biholomorphic to a pseudoconvex domain in \(\mathbb{C}^n\) which is homeomorphic to \(\mathbb{R}^{2n}\), and in particular, \(M\) is Stein.

Remark 1.1. By a result of Yau [32], the pseudoconvex domain in Theorem 1.1 has infinite Euclidean volume. The authors would like to thank Shing Tung Yau for providing this information.

If we assume that \(k(r) = \frac{C}{1 + r^2}, \) for \(\epsilon > 0\), the result that \(M\) is biholomorphic to a pseudoconvex domain was proved by Shi [30] under the additional assumption that \((M, g)\) has positive sectional curvature. Note that if \(M\) has positive sectional curvature, then it is well-known that \(M\) is diffeomorphic to \(\mathbb{R}^{2n}\) by [16], and is Stein by [15]. Under the same decay condition and assuming maximum volume growth, similar results were obtained by Chen-Zhu [10]. All these works are before [8, 9].

As in the above mentioned works, our proof of Theorem 1.1 is based on the Kähler-Ricci flow (1.2). In fact, Theorem 1.1 will be proved as a consequence of the following more general:

Theorem 1.2. Let \(M^n\) be a complex noncompact manifold. Suppose there exist a sequence of complete Kähler metrics \(g_i\), for \(i \geq 1\), on \(M\) such that

(a1) \(c g_i \leq g_{i+1} \leq g_i\) for some \(1 > c > 0\) for all \(i\).
(a2) \(|Rm(g_i)| + |\nabla Rm(g_i)| \leq c'\) for some \(c'\) on \(B_i(p, r_0)\) for some \(p \in M\) and \(r_0 > 0\) for all \(i\) where \(B_i(p, r_0)\) is the geodesic ball with respect to \(g_i\).
(a3) \(g_i\) is contracting in the following sense: For any \(\epsilon,\) for any \(i,\) there exists \(i' > i\) with
\[
g_{i'} \leq \epsilon g_i
\]
in \(B_i(p, r_0)\).

Then \(M\) is covered by a pseudoconvex domain in \(\mathbb{C}^n\) which is homeomorphic to \(\mathbb{R}^{2n}\).

To prove Theorem 1.1, the solution \(g(t)\) to the Kähler-Ricci flow on \(M\) will be used to produce a sequence of Kähler metrics \(g_i\) satisfying the hypothesis of the Theorem 1.2. The main steps in proving
Theorem 1.2 can be sketched as follows. The idea is to consider the sequence of holomorphic normal “coordinate charts” around some \( p \in M \) corresponding to the sequence \( g_i \). We then use this sequence of charts, together with a gluing technique as in [30], to build a map from an open set in \( \mathbb{C}^n \) onto \( M \). In general however, these charts will only be locally biholomorphic, and to build such a map one generally needs to control the sets around \( p \) on which these charts are injective\(^1\). We will not assume any control these sets. Instead, we will control the sets where the corresponding coordinate transition functions are injective using a method developed by the authors in [9]. Once the transition functions are established, we then following similar techniques in [30] and [9], to build a covering map from an open set \( \Omega \) in \( \mathbb{C}^n \) onto \( M \). By its construction, \( \Omega \) will be shown to be pseudoconvex and homeomorphic to \( \mathbb{R}^{2n} \).

2. HOLOMORPHIC COORDINATE “COVERING” CHARTS

Let \( M, g_i, p, r_0 \) be as in Theorem 1.2.

**Lemma 2.1.** There exists \( r > 0 \), and a family of holomorphic maps

\[
\Phi_i : D(r) \to M
\]

for all \( i \geq 1 \) with the following properties:

(i) \( \Phi_i \) is a local biholomorphism from \( D(r) \subset \mathbb{C}^n \) onto its image,
(ii) \( \Phi_i(0) = p \),
(iii) \( \Phi_i^*(g_i)(0) = g_e \),
(iv) \( \frac{1}{C}g_e \leq \Phi_i^*(g_i) \leq Cg_e \) in \( D(r) \),

where \( g_e \) is the standard metric on \( \mathbb{C}^n \), \( C \) is a constant independent of \( t \) and \( p \).

**Proof.** Using condition (a2) and by considering the pullback metric under the exponential map within the conjugate locus, one can apply Proposition 2.1 in [8] to obtain the results. \( \square \)

**Corollary 2.1.** \( B_i(C^{-1}\rho) \subset \Phi_i(D(\rho)) \subset B_i(C\rho) \) for some \( C > 0 \) for all \( 0 < \rho < r \) and \( i \geq 1 \).

The following two lemmas are from [9, Lemmas 3.2 and 3.3].

**Lemma 2.2.** For any \( 0 < \rho \leq r \), where \( r \) is as in Lemma 2.1, there exists \( 0 < \rho_1 < r_0 \), independent of \( i \), satisfying the following:

\[^1\text{In [30], positive sectional curvature was used to produce a sequence of strictly convex domains around } p \text{ exhausting } M, \text{ which were then used to control the injectivity of the charts. In [10], maximal volume growth was used to control the injectivity radius under the Kähler-Ricci flow.}\]
(i) For any $q \in B_i(p, \rho_1)$, there is $z \in D(\frac{\rho}{5})$ such that $\Phi_i(z) = q$.
(ii) For any $q \in B_i(p, \rho_1)$, $z \in D(\frac{\rho}{5})$ with $\Phi_i(z) = q$, and any smooth curve $\gamma$ in $M$ with $\gamma(0) = q$ such that $L_i(\gamma) < \rho_1$, there is a unique lift $\tilde{\gamma}$ of $\gamma$ by $\Phi_i$ so that $\tilde{\gamma}(0) = z$ and $\tilde{\gamma} \subset D(\frac{\rho}{5})$.

**Lemma 2.3.** Fix $i \geq 1$. Let $0 < \rho \leq r$ be given and let $\rho_1$ be as in Lemma 2.2. Given any $\epsilon > 0$, there exists $\delta > 0$, which may depend on $i$, satisfying the following properties:

Let $\gamma(\tau), \beta(\tau), \tau \in [0,1]$ be smooth curves from $q \in B_i(p, \rho_1)$ with length less than $\rho_1$ with respect to $g_i$ and let $z_0 \in D(\frac{1}{5}\rho)$ with $\Phi_i(z_0) = q$.

Let $\tilde{\gamma}, \tilde{\beta}$ be the liftings from $z_0$ of $\gamma$ and $\beta$ as described in Lemma 2.2. Suppose $d_i(\gamma(\tau), \beta(\tau)) < \delta$ for all $\tau \in [0,1]$, then $d_e(\tilde{\gamma}(1), \tilde{\beta}(1)) < \epsilon$.

Here $d_i$ is the distance in $\gamma(\cdot, \tau)$ as in Lemma 2.2 from $z_0$. Then $\tilde{\gamma}_r(1) = \tilde{\gamma}_0(1)$ for all $\tau$.

**Proof.** By Lemma 2.2 $\tilde{\gamma}_r(1) \in D(\frac{1}{5}\rho)$ for all $\tau$. Let $\epsilon > 0$ be such that $\Phi_i$ is injective on $D(w, \epsilon)$ for all $w \in D(\frac{1}{5}\rho)$. Let $\delta > 0$ be as in Lemma 2.3. Let $m$ be large enough, such that $d_i(\gamma(s, j/m), \gamma(s, (j + 1)/m)) < \delta$ for all $s$ for $0 \leq j < m - 1$. By Lemma 2.3 we have $|\tilde{\gamma}_{j/m}(1) - \tilde{\gamma}_{(j+1)/m}(1)| < \epsilon$ for $0 \leq j \leq m - 1$. Since $\Phi_i \circ \tilde{\gamma}_r(1) = q_2$, and $\Phi_i$ is injective in $D(\tilde{\gamma}_r(1), \epsilon)$, we have

$$\tilde{\gamma}_{j/m}(1) = \tilde{\gamma}_{(j+1)/m}(1).$$

From this the Corollary follows.

**3. HoloMorphic Transition Functions**

Let $M, g_i, p, r_0$ be as in Theorem 1.2.

**Lemma 3.1.** Let $r$ be as in Lemma 2.2. There exists $r > \rho > 0$ such that for every $i \geq 1$, where there is a map $F_{i+1}$ from $D(\rho)$ to $D(r)$ such that $\Phi_i = \Phi_{i+1} \circ F_{i+1}$ on $D(\rho)$. Moreover, $\Phi_i(D(\rho)) \subset B_i(p, r_0)$ where $r_0$ is the constant in (a2).

**Proof.** In Lemma 2.2, let $\rho = r$ and let $\rho_1$ be as in the conclusion of the Lemma. Note that $\rho_1$ is independent of $i$. Now let $0 < K < 1$ be a constant to be determined. For any $z \in D(K\rho_1)$, let $\gamma^*(\tau), 0 \leq \tau \leq 1$, be the line segment from 0 to $z$, and let $\gamma = \Phi_i \circ \gamma^*$. By (a1) and
Lemma 2.1, we see that there is a constant \( C_1 > 0 \) independent of \( i \) such that
\[
(3.1) \quad L_{i+1}(\gamma) \leq L_i(\gamma) < C_1 K \rho_1.
\]
Now choose \( K \) so that \( C_1 K < 1 \) and let \( \rho = K \rho_1 \). Since \( \gamma(0) = p \), by Lemma 2.2 there is a unique lift \( \tilde{\gamma} \) of \( \gamma \) by \( \Phi_i+1 \) so that \( \tilde{\gamma}(0) = 0 \) and \( \tilde{\gamma} \subset D(\frac{1}{2}r) \). We define \( F_{i+1}(z) = \tilde{\gamma}(1) \). \( F_{i+1} \) is then a well-defined map from \( D(\rho) \) to \( D(r) \) by the uniqueness of the lifting. Also, by construction we have \( \Phi_i = \Phi_{i+1} \circ F_{i+1} \) on \( D(\rho) \). By choosing a smaller \( \rho \), we also have \( \Phi_i(D(\rho)) \subset B_i(p, r_0) \). This completes the proof of the Lemma. □

Lemma 3.2. Let \( \rho \) be as in Lemma 3.1. Then for any \( i \geq 1 \), the map \( F_{i+1} \) satisfies the following:

(a) \( F_{i+1}(0) = 0 \)

(b) \( F_{i+1} \) is a local biholomorphism.

(c) \( b_1|v| \leq |F_{i+1}'(0)v| \leq b_2|v| \)
for some \( 0 < b_1 \leq b_2 \leq 1 \) independent of \( i \), and for all vectors \( v \in \mathbb{C}^n \), where \( F' \) is the Jacobian of \( F \).

(d) There exist \( \rho_1 \) and \( \rho_2 \) independent of \( i \), each in \( (0, \rho) \), such that \( F_{i+1}(D(\rho_1)) \subset D(\rho_2) \), and \( F_{i+1}^{-1} \) exists on \( D(\rho_2) \).

Proof. (a) follows from the definition of \( F_{i+1} \).

(b) can be proved as in the proof of Lemma 3.4 part (b) in [9], using Lemmas 2.3 and 3.1.

(c) follows from the (a1) and Lemma 2.1.

(d) follows from the proof of part (d) of Lemma 3.4 in [9]. □

Corollary 3.1. Let \( \rho_1 \) be as in Lemma 3.2. Then for any \( i \geq 1 \), \( F_{i+1}(D(\rho_1)) \) is Runge in \( \mathbb{C}^n \).

Proof. Let \( i \geq 1 \) be given. Then given any holomorphic function \( f \) on \( F_{i+1}(D(\rho_1)) \subset \mathbb{C}^n \), we must show that \( f \) can be approximated by entire functions on \( \mathbb{C}^n \) uniformly on compact subsets of \( F_{i+1}(D(\rho_1)) \).
Consider the holomorphic function \( f \circ F_{i+1} \) defined on \( D(\rho_1) \). Since \( D(\rho_1) \) is just a ball in \( \mathbb{C}^n \), \( f \circ F_{i+1} \) can be approximated uniformly on compact subsets of \( D(\rho_1) \) by entire functions \( h \) on \( \mathbb{C}^n \). By part (d) of Lemma 3.2 \( h \circ F_{i+1}^{-1} \) are defined on \( D(\rho_2) \) and holomorphic. We see that these approximate \( f \) uniformly on compact subsets of \( F_{i+1}(D(\rho_1)) \) \( \subset D(\rho_2) \). Finally, as \( D(\rho_2) \) is just a ball in \( \mathbb{C}^n \), we see that the functions \( h \circ F_{i+1}^{-1} \) can themselves be approximated uniformly
on compact subsets of $D(\rho_2)$ by entire functions. Thus by part (d) of Lemma 3.2, $f$ can be uniformly approximated on compact subsets of $F_{i+1}(D(\rho_1))$ by entire functions. This completes the proof of the Corollary.

**Corollary 3.2.** Let $\rho_1$ be as in Lemma 3.2. Then for any $i \geq 1$, $F_{i+1}$ can be approximated uniformly on compact subsets of $F_i(D(\rho_1))$ by entire functions. This completes the proof of the Corollary.

**Proof.** This follows from Corollary 3.1 and Theorem 2.1 in [1].

4. **Construction of a Map onto $M$**

Let $M, g_i, p, r_0$ be as in Theorem 1.2. We begin with the following lemma on the transition functions $F_i$ which basically says that the maps are contracting.

**Lemma 4.1.** Let $\rho_1$ be as in Lemma 3.2. Then there exists positive constants $\rho_3$ and $C$ with $C > 1$, $C\rho_3 < \rho_1$, such that for every $i$ and $k \geq 1$

$$F_{i+k} \circ \cdots \circ F_{i+1}(D(\rho_3)) \subset D(C\rho_3).$$

**Proof.** Let $C_1 > 1$ be the constant in property (iv) of $\Phi_i$ in Lemma 2.1 and $\rho_1$ as in Lemma 3.2. Let $C = C_1^2$ and $\rho_3 = \rho_1/(2C)$. For any $i$, we want to prove that

$$F_{i+1}(D(\rho_3)) \subset D(C\rho_3).$$

Let

$$A = \{\eta \in (0, \rho_3] \mid F_{i+1}(D(\eta)) \subset D(C\rho_3)\}$$

Since $F_{i+1}(0) = 0$ and $F_{i+1}$ is a local biholomorphism, it is easy to see that $A$ is nonempty in $(0, \rho_3]$. Let $r \in A$ and $z \in D(\eta)$. Since $C\rho_3 < \rho_1$, $\Phi_i(tz)$ and $\Phi_{i+1} \circ F_{i+1}(tz)$, $0 \leq t \leq 1$, are defined and are equal by Lemmas 3.1, 3.2. By Lemma 2.1, we have

$$|\frac{d}{dt}F_{i+1}(tz)|_{g_e} \leq C_1|\frac{d}{dt}F_{i+1}(tz)||\Phi^*_{i+1}(g_{i+1})$$

$$= C_1|\frac{d}{dt}\Phi_{i+1} \circ F_{i+1}(tz)||g_{i+1})$$

$$\leq C_1|\frac{d}{dt}\Phi_i(tz)||g_{i}(i)$$

$$= C_1|\frac{d}{dt}(tz)||\Phi^*_i(g_{i})$$

$$\leq C_1^2|z|$$

$$< C_1^2\rho_3.$$
Steinness of a class of Kähler manifolds

where we have used (a1). From this, it is easy to see that \( A \) is both open and closed in \((0, \rho_3]\) and \( F_{i+1}(D(\rho_3)) \subset D(\rho_3) \).

Suppose \( k > 1 \) such that
\[
F_{i+1} \circ \cdots \circ F_{i+1}(D(\rho_3)) \subset D(\rho_3)
\]
for all \( 1 \leq l < k \). As before, let
\[
B = \{ \eta \in (0, \rho_3] | F_{i+k} \circ \cdots \circ F_{i+1}(D(\eta)) \subset D(\rho_3) \}.
\]
Again, \( B \) is nonempty in \((0, \rho_3]\). Suppose \( r \in B \) and \( z \in D(r) \). Then \( \Phi_i(tz) \) and \( \Phi_i \circ F_{i+k} \circ \cdots \circ F_{i+1}(tz), 0 \leq t \leq 1 \), are well-defined and equal. As before, we can prove that \( B \) is open and closed in \((0, \rho_3]\) and
\[
F_{i+k} \circ \cdots \circ F_{i+1}(D(\rho_3)) \subset D(\rho_3).
\]
This completes the proof of the Lemma.

\[\square\]

Remark 4.1. For later use, we will assume that \( \rho_3 < \frac{1}{8} r \), where \( r \) is as in Lemma 2.1.

Lemma 4.2. Let \( \rho_3 \) as in Lemma 4.1. There exists a positive increasing sequence \( n_i \) for \( i \geq 1 \) such that \( n_1 = 1 \) and
\[
F_{n_{i+1}} \circ \cdots \circ F_{n_{i+2}} \circ F_{n_{i+1}}(D(\rho_3)) \subset D\left(\frac{\rho_3}{2}\right)
\]
for every \( i \).

Proof. Let \( C \) be the constant in Lemma 4.1. Let \( n_1 = 1 \). By Lemma 4.1 for all \( k \), \( F_{n_{i+k}} \circ \cdots \circ F_{n_{i+1}} \) is defined in \( D(\rho_3) \) for all \( k \geq 1 \). As in the proof of Lemma 4.1 for all \( z \in D(\rho_3) \), and \( 0 \leq t \leq 1 \):
\[
\frac{d}{dt} F_{n_{i+k}} \circ \cdots \circ F_{n_{i+1}}(tz) |_{g_e} \leq C_1 \frac{d}{dt} F_{n_{i+k}} \circ \cdots \circ F_{n_{i+1}}(tz) |_{\Phi_{k+n_1}(g_{n_1+1})}
\]
\[
= C_1 \frac{d}{dt} \Phi_{k+n_1} \circ F_{n_{i+k}} \circ \cdots \circ F_{n_{i+1}}(tz) |_{g_{k+n_1}}
\]
\[
= C_1 \frac{d}{dt} \Phi_{n_3}(tz) |_{g_{k+n_1}}
\]
where \( C_1 \) is as in the proof Lemma 4.1. Since \( \Phi_{n_1}(D(\rho_3)) \subset B_{n_1}(p, r_0) \) by the choice of \( \rho \) in Lemma 3.1 by (a3) and (iv) in Lemma 2.1 we can find \( n_2 > n_1 \) such that
\[
F_{n_2} \circ \cdots \circ F_{n_{i+1}}(D(\rho_3)) \subset D\left(\frac{1}{2} \rho_3\right).
\]
Similarly, one can choose \( n_3, n_4, \ldots \) inductively which satisfy the conclusion of the lemma.

\[\square\]
We now want to construct an appropriate sequence \( \tilde{F}_j \in \text{Aut}(\mathbb{C}^n) \) which will approximate the sequence \( F_j \) for \( j \geq 2 \).

Let \( n_i \) be as in Lemma 4.2. By Lemmas 4.1, 4.2 and Corollary 3.2, we can find \( \tilde{F}_2, \ldots, \tilde{F}_{n_2} \) in \( \text{Aut}(\mathbb{C}^n) \) such that

\[
\tilde{F}_{k+1} \circ \cdots \circ \tilde{F}_2(D(\rho_3)) \subset D(\rho_1)
\]

for \( 2 \leq k \leq n_2 \) and

\[
\tilde{F}_{n_2} \circ \cdots \circ \tilde{F}_2(D(\rho_3)) \subset D(\rho_3).
\]

Since \( \Phi_{n_2} \) is a local biholomorphism, we have

\[
|D\Phi_{n_2} \circ \tilde{F}_{n_2} \circ \cdots \circ \tilde{F}_2(z)(v)|_{g_1} \geq b_2 > 0
\]

for all \( z \in D(\rho_3) \) and unit vectors \( v \in \mathbb{C}^n \).

Let \( S_2 = (\tilde{F}_{n_2} \circ \cdots \circ \tilde{F}_2)^{-1}(D(\rho_3)) \). Use Lemmas 4.1, 4.2 and Corollary 3.2 again, we can find \( n_3 > n_2 \) and \( \tilde{F}_{n_2+1}, \ldots, \tilde{F}_{n_3} \) in \( \text{Aut}(\mathbb{C}^n) \) such that

\[
\tilde{F}_{n_3} \circ \cdots \circ \tilde{F}_{n_2+1}(D(\rho_3)) \subset D(\rho_3).
\]

Since

\[
\Phi_{n_2} = \Phi_{n_3} \circ \tilde{F}_{n_3} \circ \cdots \circ \tilde{F}_{n_2+1}
\]
on \( D(\rho_3) \), we may choose \( \tilde{F}_{n_2+1}, \ldots, \tilde{F}_{n_3} \) such that they also satisfy:

\[
d_{g_1}(\Phi_{n_3} \circ \tilde{F}_{n_3} \circ \cdots \circ \tilde{F}_{n_2+1} \circ \tilde{F}_{n_2} \circ \cdots \circ \tilde{F}_2(z), \Phi_{n_2} \circ \tilde{F}_{n_2} \circ \cdots \circ v \circ \tilde{F}_2(z)) \leq \frac{1}{2^2}
\]

for \( z \in S_2 \) and

\[
|D(\Phi_{n_3} \circ \tilde{F}_{n_3} \circ \cdots \circ \tilde{F}_{n_2+1} \circ \tilde{F}_{n_2} \circ \cdots \circ \tilde{F}_2)(z)(v)|_{g_1} - |D(\Phi_{n_2} \circ \tilde{F}_{n_2} \circ \cdots \circ \tilde{F}_2)(z)(v)|_{g_1} \leq \frac{b_2}{2^2}
\]

for all \( z \in S_2 \), and for all unit vector in \( \mathbb{C}^n \). Let \( 0 < b_3 < b_2 \) be such that

\[
|D\Phi_{n_3} \circ \tilde{F}_{n_3} \circ \cdots \circ \tilde{F}_2(z)(v)|_{g_1} \geq b_3 > 0
\]

for all \( z \in S_2 \) and unit vector \( v \in \mathbb{C}^n \). Let \( S_3 = (\tilde{F}_{n_3} \circ \cdots \circ \tilde{F}_2)^{-1}(D(\rho_3)) \). Inductively in this way, we have the following:

**Lemma 4.3.** There exist \( \tilde{F}_2, \ldots, \tilde{F}_j, \ldots \) in \( \text{Aut}(\mathbb{C}^n) \), such that the following conditions are satisfied for all \( i \geq 2 \):

\[
\tilde{F}_{n_i+1} \circ \cdots \circ \tilde{F}_{n_i+1}(D(\rho_3)) \subset D(\rho_3).
\]

\[
d_{g_1}(\Phi_{n_i+1} \circ \tilde{F}_{n_i+1} \circ \cdots \circ \tilde{F}_{n_i+1} \circ \tilde{F}_{n_i} \circ \cdots \circ \tilde{F}_{n_1+1}(z), \Phi_{n_i} \circ \tilde{F}_{n_i} \circ \cdots \circ \tilde{F}_{n_1+1}(z)) \leq \frac{1}{2^{i+1}}
\]
for all \( z \in S_i \).

\[
\| D(\Phi_{n+1} \circ \bar{F}_{n+1} \circ \cdots \circ \bar{F}_n \circ \cdots \circ \bar{F}_{n+1})(z)(v) \|_{g_1} - \| D(\Phi_n \circ \bar{F}_n \circ \cdots \circ \bar{F}_{n+1})(z)(v) \|_{g_1} \leq \frac{b_i}{2^{i+1}}
\]

(4.16)

for all \( z \in S_i \) and Euclidean unit vectors \( v \), where the sequence \( b_i \) is positive, decreases, and satisfies

\[
\| D(\Phi_n \circ \bar{F}_n \circ \cdots \circ \bar{F}_{n+1})(z)(v) \|_{g_1} \geq b_i
\]

(4.17)

for all \( z \in S_i \) and Euclidean unit vectors \( v \). Here

\[
S_i = (\bar{F}_n \circ \cdots \circ \bar{F}_2)^{-1}(D(p_3)).
\]

Corollary 4.1. Let \( S_i \) be as above. Then \( S_i \) is an increasing sequence of open sets in \( \mathbb{C}^n \).

Proof. From (4.14) we have

\[
\bar{F}_{n+1} \circ \cdots \circ \bar{F}_n \circ \cdots \circ \bar{F}_2(S_i) = \bar{F}_{n+1} \circ \cdots \circ \bar{F}_n(D(p_3)) \subset D(p_3)
\]

and thus

\[
S_i \subset \bar{F}_2^{-1} \circ \cdots \circ \bar{F}_{n+1}^{-1}(D(p_3)) = S_{i+1}
\]

Definition 4.1. Let the sequences \( S_i \) and \( n_i \) be as above. Let

\[
\Omega = \bigcup_{i=2}^{\infty} S_i
\]

Corollary 4.2. \( \Omega \) is pseudoconvex and is homeomorphic to \( \mathbb{R}^{2n} \).

Proof. Since each \( S_i \) is pseduoconvex in \( \mathbb{C}^n \), \( \Omega \) is pseudoconvex. Since each \( S_i \) is homeomorphic to the unit ball in \( \mathbb{R}^{2n} \), \( \Omega \) is also homeomorphic to \( \mathbb{R}^{2n} \) by [2]. This completes the proof of the corollary.

We now begin to use the maps \( \bar{F}_i \) to construct a map from \( \Omega \) onto \( M \). We need the following lemma.

Lemma 4.4. Let \( M, g, p \) be as in Theorem 1.2. Then for all \( \epsilon > 0 \), \( \bigcup_i B_i(\epsilon) = M \), where \( B_i(\epsilon) = B_i(p, \epsilon) \).

Proof. Let \( 0 < 3\epsilon < r_0 \), where \( r_0 \) is as in (a3). Obviously, \( B_1(\epsilon) \subset \bigcup_i B_i(\epsilon) \). We claim that if \( B_1(k\epsilon) \subset \bigcup_i B_i(\epsilon), k \geq 1 \), then \( B_1((k+1)\epsilon) \subset \bigcup_i B_i(\epsilon) \).

Suppose \( B_1(k\epsilon) \subset \bigcup_i B_i(\epsilon) \), then \( \bar{B}_1(k\epsilon - \frac{1}{2}\epsilon) \subset B_i(\epsilon) \) provided \( i \) is large enough. Hence \( B_1((k+1)\epsilon) \subset B_i(\epsilon + \frac{3}{2}\epsilon) \subset B_i(3\epsilon) \) for \( i \) large.
enough by (a1). By (a3), we can find \(i\) such that \(B_1((k+1)\epsilon) \subset B_i(\epsilon)\). This completes the proof of the lemma.

**Lemma 4.5.** Let \(\Gamma_i := \Phi_{n_i} \circ \tilde{F}_{n_i} \circ \cdots \circ \tilde{F}_2\). Then the following map \(\Psi : \Omega \to M\) is well defined.

\[
(4.20) \quad \Psi(z) = \lim_{i \to \infty} \Gamma_i(z).
\]

**Proof.** This follows from (4.15) in Lemma 4.3, Corollary 4.1 and the definition of the maps \(\Gamma_i\).

**Lemma 4.6.** \(\Psi\) is a local biholomorphism and onto.

**Proof.** By property (iv) of the maps \(\Phi_i\), and the fact that \(\bigcup_i B_i(p, \epsilon) = M\) for all \(\epsilon\), given any \(R > 0\) we can find \(n_i\) such that

\[
(4.21) \quad B_1(p, R) \subset B_{n_i}(p, \epsilon) \subset \Phi_{n_i}(D(C_1\epsilon)),
\]

for some constant \(C_1^2\) is the constant in Lemma 2.1(iv). Here we have used Corollary 2.1 provided \(C_1\epsilon < \rho_3\). Choose such an \(\epsilon\). Then

\[
\Gamma_i(S_i) = \Phi_{n_i} \circ \tilde{F}_{n_i} \circ \cdots \circ \tilde{F}_2(S_i) = \Phi_{n_i}(D(\rho_3) \supset B_1(p, R)).
\]

Thus by (4.15) and the fact that the \(S_i\)'s are increasing it follows that

\[
(4.22) \quad B_1(p, R - 1) \subset \Gamma_j(S_i)
\]

for all \(j \geq i\). From the definition of the map \(\Psi\), we see that

\[
(4.23) \quad B_1(p, R - 1) \subset \Psi(\Omega).
\]

Hence \(\Psi(\Omega) = M\).

We now show that \(\Psi\) is a local biholomorphism. Observe that \(\Omega\) is open and \(\Psi\) is a holomorphic map. Now to show \(\Psi\) is a local biholomorphism on \(\Omega\), it will be sufficient to show it is a local biholomorphism on the sets \(S_i\) for each \(i\). Fix some \(i\). Then by (4.16) and the fact that the \(b_i\)'s are decreasing,

\[
(4.24) \quad \|D(\Gamma_j)(z)(v)\|_{g_1} \geq b_i - \frac{b_i}{2}
\]

for all \(j \geq i\), \(z \in S_i\) and all unit vectors \(v\) at \(z\). Thus by the definition of \(\Psi\), (4.24) implies \(\Psi\) is a local biholomorphism on \(S_i\). Noting that \(i\) is arbitrary, this completes the proof of the Lemma.

\[\square\]
5. Proof of Theorem 1.2

Let $M$ and $g_i$ satisfy (a1)–(a3). And let $\Psi$ be the map constructed in the previous section. If we take $\pi : \hat{M} \to M$ to be a universal holomorphic covering of $M$ and let $\hat{g}_i = \pi^*(g_i)$, then $(\hat{M}, \hat{g}_i)$ will still satisfy (a1)–(a3). Thus to prove Theorem 1.2 it will be sufficient to prove that $\Psi$ is injective assuming that $M$ is simply connected. Before we prove this, let us first prove the following:

**Lemma 5.1.** Let $\alpha(s), 0 \leq s \leq 1$ be a smooth curve in $M$. Then there exists $\epsilon > 0$ such that if $\beta(s)$ is another smooth curve $M$ with same end points as $\alpha(s)$ such that $d_1(\alpha(s), \gamma(s)) < \epsilon$ for all $s$, then there is a smooth homotopy $\gamma(s, \tau)$ with end points fixed such that $\gamma(s, 0) = \alpha(s)$ and $\gamma(s, 1) = \beta(s)$. Moreover, there is a constant $L$ depending only on $(M, g_1), \max_{0 \leq s \leq 1} \{\alpha'(s)|_{g_1} + |\beta'(s)|_{g_1}\}$, such that the length of $\gamma(\cdot, \tau)$ with respect to $g_1$ is bounded above by $L$.

**Proof.** In the following, all lengths on $M$ will be computed with respect to the metric $g_0$. Let $\alpha(s)$ be given. Then there is $R > 0$ such that $\alpha \subset B_1(p, R/2)$. First let $\epsilon > 0$ be the lower bound for the injectivity radius of $B_1(p, R)$. Suppose $\beta(s)$ is another smooth curve on $M$ with same end points as $\alpha(s)$. Then there is a smooth homotopy $\gamma(s, \tau)$ such that $\gamma(s, \tau)$, for $0 \leq \tau \leq 1$ is the minimal geodesic from $\alpha(s)$ to $\beta(s)$. Then for each $s$, $J = \gamma_s$ is a Jacobi field along the geodesic $\gamma(s, \tau)$ for $0 \leq \tau \leq 1$, with boundary value $J(0) = \alpha'(s)$ and $J(1) = \beta'(s)$. With respect to an orthonormal frame $\{e_i\}$ parallel along $\gamma(s, \tau)$, $0 \leq \tau \leq 1$, the components $y_i$ of $J$ satisfies

$$\begin{bmatrix}
y''_1 \\
\vdots \\
y''_{2n}
\end{bmatrix} = A \begin{bmatrix}
y_1 \\
\vdots \\
y_{2n}
\end{bmatrix}$$

where $A_{ij} = \langle R(\gamma_{\tau}, e_i)\gamma_{\tau}, e_j \rangle$. Here $'$ means derivatives with respect to $\tau$. Note that $|\gamma_{\tau}| \leq \epsilon$ and the curvature is bounded from below, we have

$$\left(\sum_i y_i^2\right)'' \geq -C_1\epsilon^2 \sum_i y_i^2$$

for some constant $C_1 > 0$ depending only on the lower bound of the curvature and $n$. Hence if $\epsilon > 0$ is small enough depending only on the curvature, we can compare $\sum_i y_i^2$ with the solution $f$ of $f'' = -C_1\epsilon^2 f$ with the same boundary value as $\sum_i y_i^2$. Hence

$$|\gamma_s|^2 = |J|^2 = \sum_i y_i^2 \leq C_2$$
for some $C_2$ depending only on $g_0$ and $\max_{0 \leq t \leq 1} \{|\alpha'(t)| + |\beta'(t)|\}$. \hfill \Box

We now complete the proof of Theorem 1.2 by proving the following:

**Lemma 5.2.** If $M$ is simply connected, then $\Psi$ is injective.

**Proof.** Suppose the lemma is false. Then there are distinct points $z_1, z_2 \in \Omega$ such that $\Psi(z_1) = \Psi(z_2) = q$. Let $\tilde{\gamma}(s)$ be a smooth curve in $\Omega$ for $s \in [0, 1]$, joining $z_1$ to $z_2$ parametrized proportional to arc length with respect to the Euclidean metric, and let $\gamma(s) = \Psi \circ \tilde{\gamma}(s)$. Then $\gamma(0) = \gamma(1) = q$. Let $\gamma(s, \tau)$ be a smooth homotopy of $\gamma$ for $(s, \tau) \in [0, 1] \times [0, 1]$ such that $\gamma(s, 0) = \gamma(s), \gamma(s, 1) = q$ for all $s \in [0, 1]$, and $\gamma(0, \tau) = \gamma(1, \tau) = q$ for all $\tau \in [0, 1]$. Let $L_1 = \max\{l(\gamma(\cdot, \tau)) \mid \tau \in [0, 1]\}$, where $l(\gamma(\cdot, \tau))$ is the length of $\gamma(\cdot, \tau)$ with respect to $g_1$.

Let $R > 0$ be fixed, such that $\gamma(s, \tau) \in B_1(p, R)$ for all $0 \leq s, \tau \leq 1$.

By (1.2) and the fact that $S_i \subset S_{i+1}$ for all $i$, there exists $i_0$ such that

$$B_1(p, R) \subset \Gamma_j(S_i)$$

for all $j \geq i \geq i_0$, and that $\tilde{\gamma} \subset S_{i_0}$.

Since $\Psi$ is a local biholomorphism, it is easy to see that for any $a > 0$ there is $b > 0$ such that for all $i$ large enough, $\Gamma_i(D(z_k, a)) \supset B_1(\Gamma_i(z_k), b), k = 1, 2$. Since $\Gamma_i(z_k) \rightarrow \Psi(z_k) = q$, by choosing an even larger $i_0$, for all $i \geq i_0$ there exist $\zeta_{1,i} \neq \zeta_{2,i} \in S_{i_0}$ such that $\Gamma_i(\zeta_{1,i}) = \Gamma_i(\zeta_{2,i}) = q$ and that $\zeta_{1,i} \rightarrow z_1$ and $\zeta_{2,i} \rightarrow z_2$. Now for $i$ large enough, we can join $\zeta_{1,i}$ to $\zeta_{2,i}$ by first joining $\zeta_{1,i}$ to $z_1$, then $z_1$ to $z_2$ along $\tilde{\gamma}$, then $z_2$ to $\zeta_{2,i}$. Let us denote this curve by $\tilde{\gamma}_i(s), s \in [0, 1]$ parametrized proportional to arc length. We may assume $\tilde{\gamma}_i(s)$ is smooth, $\tilde{\gamma}_i(s) \subset K \subset S_{i_0}$ for some compact set $K$, and $|\tilde{\gamma}'_i| \leq C_1$ for some constant independent of $i$ for all $i \geq i_0$. Moreover, we have $|\tilde{\gamma}(s) - \tilde{\gamma}_i(s)| \rightarrow 0$ uniformly over $s$ as $i \rightarrow \infty$. Since $\Psi$ is a local biholomorphism, there is a constant $C_2$ independent of $i$ such that if $\gamma_i = \Psi \circ \tilde{\gamma}_i$, then

$$|\gamma_i'(s)|_{g_1} \leq C_2.$$

For the curve $\gamma(s)$, let $\epsilon$ be as in Lemma 5.1. Since $\Gamma_i$ converge to $\Psi$ uniformly on compact sets together with first derivatives, if $i_0$ is chosen large enough, then the following are true:

(i) $d_1(\gamma_i(s), \gamma_i(s)) < \frac{\epsilon}{2}$

(ii) $d_1(\Gamma_i \circ \tilde{\gamma}(s), \gamma_i(s)) = d_1(\Gamma_i \circ \tilde{\gamma}(s), \Psi \circ \tilde{\gamma}(s)) < \frac{\epsilon}{2}$

(iii) $|(\Gamma_i \circ \tilde{\gamma})'(s)|_{g_1} \leq |(\Psi \circ \tilde{\gamma})'(s)|_{g_1} + C_2 = |\gamma_i'(s)|_{g_1} + C_2 \leq 2C_2$

for $i, j \geq i_0$. 

By (i) and (ii), we have:

\[ d_1(\gamma(s), \Gamma_i \circ \tilde{\gamma}_i(s)) < \epsilon \]

for all \( i \geq i_0 \). Thus by Lemma 5.1 and (5.2), for each \( i \geq i_0 \) we can find a homotopy which deforms \( \gamma(s) \) to \( \Gamma_i \circ \tilde{\gamma}_i(s) \), with end points fixed, so that each curve in the homotopy has length (with respect to \( g_1 \)) bounded by some constant \( L \) independent of \( i \).

Now let \( \rho_1 \) be as in Lemma 2.2 corresponding to \( \rho = r \). Then we can choose \( i \geq i_0 \) large enough but fixed, such that \( B_1(p, L + L_1 + R + 1) \subset \Phi_{n_i}(D(\rho_3)) \), and any curve \( \beta \) in the above homotopies is in \( B_1(p, L + L_1 + R + 1) \) and satisfies \( L_i(\beta) \leq 1/(L + L_1 + R + 1) \rho_3 \). Here we have used (a3).

Let \( w_k = \tilde{F}_{n_i} \circ \cdots \circ \tilde{F}_2(\zeta_{k,i}) \), \( k = 1, 2 \). Then \( w_1 \neq w_2 \). Note that \( \tilde{F}_{n_i} \circ \cdots \circ \tilde{F}_2(S_{i_0}) \subset \tilde{F}_{n_i} \circ \cdots \circ \tilde{F}_2(S_i) \subset D(\rho_3) \).

By Corollary 2.2, since the lift of \( \Gamma_i \circ \tilde{\gamma}_i(s) \) in the Lemma 2.2 from \( w_1 \) by \( \Phi_{n_i} \) is \( \tilde{F}_{n_i} \circ \cdots \circ \tilde{F}_2 \circ \tilde{\gamma}_i(s) \), the lift \( \tilde{\sigma} \) of \( \gamma(s, 1) \) satisfies \( \tilde{\sigma}(1) = \tilde{F}_{n_i} \circ \cdots \circ \tilde{F}_2 \circ \tilde{\gamma}_i(1) = w_2 \). This is impossible because \( \Phi_{n_i} \circ \tilde{\sigma}(s) = \gamma(s, 1) \) is a constant map, \( \tilde{\sigma}(0) = w_1 \neq w_2 = \tilde{\sigma}(1) \) and \( \Phi_{n_i} \) is a local biholomorphism.

\[ \square \]

6. Proof of Theorem 1.1

In this section we prove Theorem 1.1. We begin proving a general theorem on complete solutions to the Kähler-Ricci flow

(6.1)

\[ \frac{\partial g_{ij}}{\partial t} = -R_{ij} \]

**Theorem 6.1.** Let \( g(t) \) be a complete solution to (6.1) with non-negative holomorphic bisectional curvature such that \( g(0) \) has bounded curvature. Fix some \( p \in M \) and let \( \lambda_i(t) \) be the eigenvalues of \( Rc(p, t) \) arranged in increasing order. Then

\[ t \lambda_k(t) \]

is nondecreasing in \( t \) for all \( 1 \leq k \leq n \).

**Proof.** To prove the theorem we may assume again that \( M \) is simply connected and by the result of [5], we may further assume that the Ricci curvature is positive for all \( x \in M \) and for all \( t > 0 \).

Now let \( k \geq 1 \), and let \( h(t) \) be any positive function with \( h'(t) > 0 \) for all \( t \). We claim that for any \( t_0 \) there is \( \epsilon > 0 \) such that \( th(t)\lambda_k(t) < \)
$t_0 h(t_0) \lambda_k(t_0)$ for all $t \in (t_0 - \epsilon, t_0)$. By taking $h(t) = 1 + \delta t$ with $\delta > 0$ and then let $\delta \to 0$, we see that the theorem will follow from this claim which we now prove.

For any $t$, let $0 < \lambda_1(t) \leq \cdots \leq \lambda_n(t)$ be the eigenvalues of $R_{ij}(p, t)$. For any $\sigma > 0$ let $E_\sigma(t)$ be the direct sum of the corresponding eigenspaces with eigenvalues $\lambda < \sigma$. Now let $t_0$ be fixed and let $m \geq k$ be the largest integer such that $\lambda_j(t_0) = \lambda_k(t_0)$ for $m \geq j \geq k$. Let $\sigma > 0$ be such that $\lambda_k(t_0) < \sigma < \lambda_{m+1}(t_0)$ if $m < n$ and $\sigma > \lambda_k(t_0)$ if $m = n$. Then there exists $\epsilon > 0$ such that for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$, $\lambda_m(t) < \sigma < \lambda_{m+1}(t)$ if $m < n$ and $\lambda_n(t) > \sigma$ if $m = n$. In any case, the orthogonal projection $P_\sigma(t)$ onto $E_\sigma(t)$ is smooth in $(t_0 - \epsilon, t_0 + \epsilon)$, see [8, p.501] for example. For any $t_1 \in (t_0 - \epsilon, t_0 + \epsilon)$, let $v_1$ be an eigenvector of $R_{ij}(t)$ corresponding to $\lambda_m(t_1)$ with length 1. Let

$$v(t) = \frac{P_\sigma(t)v_1}{|P_\sigma(t)v_1|}.$$ 

Note that for $t$ close to $t_1$, $P_\sigma(t)v_1 \neq 0$. In local holomorphic coordinates $z^i$, let $a(t) = R_{ij} v^i \bar{v}^j$ where $v(t) = v^i(t) \frac{\partial}{\partial z^i}$. Note that $P_\sigma(t_1)(v_1) = v_1$ and thus $a(t_1) = \lambda_m(t_1)$. Also note that we have $a(t) \leq \lambda_m(t)$ for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$ where $a(t)$ is defined. Now note that

$$0 = \frac{d}{dt} \langle v(t), v(t) \rangle_t = -R_{ij}(p, t)v^i \bar{v}^j + 2\text{Re}(g_{ij} \frac{dv^i}{dt} \bar{v}^j).$$

Hence at $t_1$ we have

$$\text{Re}(g_{ij} \frac{dv^i}{dt} \bar{v}^j) = \frac{a(t_1)}{2} = \frac{\lambda_m(t_1)}{2}.$$ 

By the Harnack Inequality in [4] we have

$$\frac{\partial R_{ij}}{\partial t} + g^{kl} R_{ik} R_{kj} + \frac{R_{ij}}{t} \geq 0.$$ 

for all $t$. Thus at $t_1$ we have
\[ 0 \leq \frac{\partial R_{i\bar{j}}}{\partial t} v^i v^\bar{j} + g^{k\bar{l}} R_{k\bar{j}} v^i v^\bar{j} + \frac{R_{i\bar{j}}}{t} v^i v^\bar{j} \]
\[ = \frac{d}{dt} (R_{i\bar{j}} v^i v^\bar{j}) - 2 \Re \left( R_{i\bar{j}} \left( \frac{d}{dt} v^i \right) v^\bar{j} \right) + g^{k\bar{l}} R_{k\bar{j}} v^i v^\bar{j} + \frac{R_{i\bar{j}}}{t} v^i v^\bar{j} \]
\[ = \frac{d}{dt} (R_{i\bar{j}} v^i v^\bar{j}) - \lambda^2_m(t_1) + \lambda^2_m(t_1) + \frac{R_{i\bar{j}}}{t} v^i v^\bar{j} \]
\[ = \frac{d}{dt} a + a \]
\[ (6.5) \]

where the third equality follows from writing the expressions in a holomorphic coordinate \( z^i \) such that \( \frac{\partial}{\partial z^i} \) form a basis of eigenvectors of \( Rc(p, T) \) with \( v_1 = \frac{\partial}{\partial z^1} \) at \( p \), and (6.3). Since \( a(t_1) > 0 \), \( h(t) > 0 \) and \( h'(t) > 0 \), we conclude that
\[ \frac{d}{dt} (th(t)a(t)) > 0 \]
at \( t_1 \) and hence \( th(t)a(t) \) is increasing in \( t \) for \( t \in (t_1 - \epsilon_1, t_1 + \epsilon_1) \) for some \( \epsilon_1 > 0 \). Hence
\[ t_1 h(t_1) \lambda_m(t_1) = t_1 h(t_1) a(t_1) \]
\[ < th(t)a(t) \]
\[ \leq th(t)\lambda_m(t) \]
\[ (6.6) \]
for all \( t_1 < t < t_1 + \epsilon_1 \). As \( t_1 \) was chosen arbitrarily in \( (t_0 - \epsilon, t_0 + \epsilon) \), we conclude that \( th(t)\lambda_m(t) \) is increasing in \( (t_0 - \epsilon, t_0 + \epsilon) \). In particular,
\[ t_0 h(t_0) \lambda_k(t_0) = t_0 h(t_0) \lambda_m(t_0) \]
\[ > th(t)\lambda_m(t) \]
\[ \geq th(t)\lambda_k(t) \]
\[ (6.7) \]
for all \( t \in (t_0 - \epsilon, t_0) \). This proves the claim and the theorem. \( \square \)

**Proof of Theorem 1.1.** We begin by observing that if \( M \) has positive holomorphic bisectional curvature and is simply connected near infinity, then it is actually simply connected. Indeed, if \( M \) were not simply connected there would exist a nontrivial minimizer of a free homotopy class. This however is impossible by the fact that the bisectional curvature is positive, and an argument as in the proof of Sygne theorem. By the remarks at the beginning of §5, we may assume that \( M \) is simply connected in Theorem [7,7] and by [7] we may also assume that the Ricci curvature is positive in spacetime.
Now by the long time existence results in [30] we know that under the hypothesis of Theorem 1.1, (6.1) has a long time solution $g(t)$ with uniformly bounded non-negative holomorphic bisectional curvature together with the covariant derivatives of the curvature tensor. Since $Rc > 0$, Theorem 6.1 implies that given any compact set $\Omega$ we can find $C > 0$ such that $Rc(t) \geq \frac{C}{t}g(t)$ on $\Omega$ for all $t$. From this, (6.1) and recalling that the curvature of $g(t)$ is uniformly bounded on $[0, \infty) \times M$, it is not hard to see show that the sequence of metrics $g_i = g(i)$ on $M$ satisfies the hypothesis of Theorem 1.2 and thus Theorem 1.1 follows. \hfill $\Box$

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