KERR-SCHILD METRICS REVISITED I. THE GROUND STATE†

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ABSTRACT

The Kerr-Schild pencil of metrics \( g_{ab} + \Lambda l^al^b \) is investigated in the generic case when it maps an arbitrary vacuum space-time with metric \( g_{ab} \) to a vacuum space-time. The theorem is proved that this generic case, with the field \( l \) shearing, does not contain the shear-free subclass as a smooth limit. It is shown that one of the Kóta-Perjés metrics is a solution in the shearing class.

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1. INTRODUCTION

The challenge of Kerr-Schild metrics for researchers in general relativity appears unabated for many years now. The particular way Kerr-Schild metrics incorporate a congruence of null curves in space-time geometry is a sure source of the fascination. And then, an eminent member of this class is the Kerr black hole. A brief critical review of the literature below will help us through the ups and downs of the subject.

The original Kerr-Schild Ansatz maps\(^1\) Minkowski space-time with a null field \( l \), to an empty curved space-time with a metric quadratic in \( l \). Boyer and Lindquist\(^2\) show that the image of the map is an algebraically special space-time in which \( l \) is a multiple principal null direction of the Weyl tensor. Hence \( l \) is tangent to a congruence of null geodesics. The problem has been investigated in detail by Grses and Grsey\(^3\), who derive many of the properties of the resulting empty space-time. Their treatment is unduly cumbersome though, as they introduce a large number of various new fields.

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Another important contribution to the subject by Debever\textsuperscript{4} employs a Newman-Penrose (NP) null tetrad adapted to the vector field $l$. He normalises the tangent vector $l$ of the null geodesics such that the parameter is not affine. Unfortunately, however, his definition of the shear, a key notion in the paper, is the one used elsewhere in the affine gauge. McIntosh\textsuperscript{5} and McIntosh and Hickman\textsuperscript{6} have considered the conditions that a Kerr-Schild vacuum have twistfree principal null directions and to be of Petrov type D. For a review of the original Kerr-Schild Ansatz, cf. Chandrasekhar\textsuperscript{7}.

Remarkably, even the general solution of the original Kerr-Schild problem is known\textsuperscript{8}. The solution is provided by the Kerr theorem, stating that there is a one-to-one correspondence between geodesic and shear-free null congruences in Minkowski space-time and zeroes of analytic functions in three complex variables.

The status of the original Kerr-Schild Ansatz and its generalisation where both the base space-time (to be named \textit{parent} space-time) and the total space-time are curved is reviewed in the Exact Solution Book\textsuperscript{9}. However, the subject appears to be plagued by false statements. It is claimed in connection with the works of Thompson\textsuperscript{10} and Dozmorov\textsuperscript{11}, that the vector $l$ must be a multiple null principal vector of the curvature in both space-times\textsuperscript{12}. (There is, moreover, a list of corrections issued by the authors of the book which advises to cancel the paragraph on double Kerr-Schild metrics in Sec. 28.5.) Thompson\textsuperscript{10} has obtained various results on the generic pencil. Thus he has proved that the null vector field $l$ of the pencil is geodesic provided it is a geodesic of the parent space-time. For the vacuum-vacuum Kerr-Schild maps, Xanthopoulos\textsuperscript{13} has proven that the full field equations are implied by the equations linear in the pencil parameter $\Lambda$. Urbantke\textsuperscript{14} has derived Kerr-Schild solutions in alternative theories of gravitation.

Recently, Nahmad-Achar\textsuperscript{15} has investigated the types of energy-momentum tensors which can be generated from an arbitrary space-time by the generalised Kerr-Schild map. By using the NP approach, he proved that the ensuing space-time is algebraically special whenever the parent space-time is an algebraically special vacuum.

Despite these developments, there remain some tantalising questions about the
Kerr-Schild metrics. The original motivation for our work was question (Q1): Is it possible to construct a ‘Fock space’ for vacuum gravitational fields by creating a multitude of Kerr-Schild congruences on some fixed vacuum base space-time? We propose that the answer is no. This conclusion is based on our research on question (Q2): what is the condition on a vacuum space-time that it can contain a Kerr-Schild congruence mapping to a vacuum space-time. In this paper we find a restriction on the parent space-time of the vacuum-vacuum Kerr-Schild map. This may be concisely put such that the generic Kerr-Schild map does not arise as a smooth extension of the shear-free class. Our result casts a serious doubt on the chances that attempts relying on the Kerr theorem to carry over to general relativity a complex-manifold description of Minkowski space time can be successful.

In Sec. 2, we obtain the set of three field equations of the vacuum-vacuum Kerr-Schild map. One of these is the condition that $l$ is a geodesic vector field, a fact already known to Thompson\textsuperscript{10}. In Sec. 3, we present the spin-coefficient form of the pencil equations in the affinely parametrized gauge. Hence we get the affine-parameter dependence of the field quantities $\rho, \sigma$ and $\Psi_0$. Sec. 4 contains our main theorem, the proof of which employs the set of coupled equations for the spin coefficient fields $\alpha, \beta, \pi, \tau$ and $\Psi_1$. It follows from our theorem that in the generic case, this set of equations is homogeneous and linear. In Sec. 5, we get the ‘ground solution’ of the general Kerr-Schild map by taking the trivial solution of this set and of the coupled homogeneous linear equations for $\lambda, \mu$ and $\Psi_2$. The solution is then shown to be one of the Kóta-Perj’és metrics, by use of an Eddington-type coordinate transformation.

\section{2. VACUUM CONDITIONS}

Let $g_{ab}$ be the metric of a vacuum space-time and $\tilde{g}_{ab}$ a pencil of vacuum metrics of the Kerr-Schild form

$$\tilde{g}_{ab} = g_{ab} + \Lambda l_a l_b \quad l_a l^a = 0 \quad (2.1)$$

where the real constant $\Lambda$ is the pencil parameter and $l^a$ is a null congruence. In consequence of (2.1) we have

$$\tilde{g}^{ab} = g^{ab} - \Lambda l^a l^b \quad l^a = \tilde{g}^{ab} l_b = g^{ab} l_b \quad \tilde{g} = g \quad . \quad (2.2)$$
The covariant derivatives $\tilde{\nabla}_a$ and $\nabla_a$ annihilating $\tilde{g}_{ab}$ and $g_{ab}$ respectively, have the difference tensor $C^c_{ab}$. The Ricci tensor $\tilde{R}_{ac}$ may be computed by use of the relations\[16\]

\[\tilde{\nabla}_a \omega_b - \nabla_a \omega_b = -C^c_{ab} \omega_c\]

\[C^c_{ab} = \frac{1}{2} \tilde{g}^{cd} (\nabla_a \tilde{g}_{bd} + \nabla_b \tilde{g}_{ad} - \nabla_d \tilde{g}_{ab})\] \hspace{1cm} (2.3)

\[\tilde{R}_{ac} = R_{ac} - 2\nabla[a C^b_{c|c} + 2C^e_{c[a} C^b_{b]e}\]

where $R_{ac}$ is the Ricci tensor of the metric $g_{ab}$, defined in terms of the Riemann tensor, $R_{ac} = R_{babc}$. The Ricci tensor $\tilde{R}_{ac}$ is a polynomial of degree 3 in $\Lambda$. As the vacuum Einstein equations $\tilde{R}_{ac} = 0$ hold for all real values of the pencil parameter $\Lambda$, we can equate the coefficients of each power of $\Lambda$ with zero. We get four tensor relations, but one of them is satisfied identically. Thus we are left with the equations:

\[\nabla_b [\nabla_a (l_c l^b) + \nabla_c (l_a l^b) - \nabla^b (l_a l_c)] = 0\] \hspace{1cm} (2.4)

\[(\nabla b^b)l_{(a} Dl_{c)} + \frac{1}{2} (Dl_a)(Dl_c) + l_{(a} D Dl_{c)} + l_{a} l_c (\nabla b l_{d})(\nabla^{[b} l^{d]}) - (Dl^b)(\nabla b l_{(a})l_{c)} = 0\] \hspace{1cm} (2.5)

\[l_{a} l_{c} (Dl_b)(Dl^b) = 0\] \hspace{1cm} (2.6)

where the tensor notation refers to the parent space-time and $D = l^a \nabla_a$. From (2.6), the vector $l^a$ satisfies the geodesic condition:

\[Dl^a = f l^a\] \hspace{1cm} (2.7)

Using (2.7) and the Ricci identity

\[(\nabla_a \nabla_b - \nabla_b \nabla_a) l^c = R^c_{dab} l^d\] \hspace{1cm} (2.8)

we get from (2.5) and (2.4):

\[f \nabla_b l^b + \frac{1}{2} f^2 + D f + (\nabla b l_{d})(\nabla^{[b} l^{d]}) = 0\] \hspace{1cm} (2.9)

\[\nabla^b l_{b} (l_{a} l_{c}) + 2R_{abcd} l^b l^d - 2\nabla_{(a}[l_{c)} (\nabla b l^b + f)] = 0.\] \hspace{1cm} (2.10)

Equations (2.7), (2.9) and (2.10) ensure that a Kerr-Schild pencil will generate a vacuum space-time from a vacuum space-time. For future reference, we now rewrite these
equations in a Newman-Penrose form\textsuperscript{17}, choosing $l$ a vector of the null tetrad ($l, n, m, \bar{m}$). We choose the phase of the complex vector $m$ such that the spin coefficient $\epsilon$ is real,

$$\epsilon = \bar{\epsilon} .$$ \hspace{1cm} (2.11)

We can still perform spatial rotations

$$\tilde{m} = e^{i\Phi} m,$$ \hspace{1cm} (2.12)

with $D\Phi = 0$, as well as arbitrary null rotations about $l$,

$$\tilde{l} = l$$

$$\tilde{m} = m + El$$

$$\tilde{n} = n + \bar{E}m + E\bar{m} + E\bar{E}l .$$ \hspace{1cm} (2.13)

The geodesic condition (2.7) becomes

$$\kappa = 0, \quad f = 2\epsilon .$$ \hspace{1cm} (2.14)

Eq.(2.9) takes the form

$$2D\epsilon = \frac{1}{2}(\rho - \bar{\rho})^2 + 2\epsilon(\rho + \bar{\rho}) - 4\epsilon^2 .$$ \hspace{1cm} (2.15)

The tetrad components of Eq.(2.10) are

$$\Psi_0 = -\sigma(4\epsilon + \rho - \bar{\rho})$$

$$4\epsilon(\rho + \bar{\rho}) = (\rho + \bar{\rho})^2 - 2(\rho\bar{\rho} + \sigma\bar{\sigma})$$

$$D(4\epsilon - \rho - \bar{\rho}) = -8\epsilon^2 + 2\epsilon(\rho + \bar{\rho}) - 2(\rho\bar{\rho} + \sigma\bar{\sigma})$$

$$D\tau - \delta\sigma - \delta\bar{\rho} - 4\delta\epsilon - 2\Psi_1 = -4\epsilon(\tau - \bar{\alpha} - \beta)$$

$$- \sigma\bar{\tau} + \sigma(\alpha + 5\bar{\beta}) - \rho\tau + (2\rho - \bar{\rho})(\bar{\alpha} + \beta) + \pi\sigma + \bar{\pi}\rho$$

$$D(\gamma + \bar{\gamma}) - 2\Delta\epsilon + \Delta(\rho + \bar{\rho}) - \delta(\alpha + \bar{\beta}) - \bar{\delta}(\bar{\alpha} + \beta) - 2Re\Psi_2 =$$

$$\pi(\tau + \bar{\alpha} + \beta) + \bar{\pi}(\bar{\tau} + \alpha + \bar{\beta}) - 4\epsilon(\gamma + \bar{\gamma}) - 2\epsilon(\mu + \bar{\mu})$$

$$- \sigma\lambda - \bar{\sigma}\bar{\lambda} - \rho\mu - \bar{\rho}\bar{\mu} - \tau(\alpha + \bar{\beta}) - \bar{\tau}(\bar{\alpha} + \beta)$$

$$+ 4(\alpha + \bar{\beta})(\bar{\alpha} + \beta) - (\alpha + \bar{\beta})(\bar{\alpha} - \beta) - (\alpha - \bar{\beta})(\bar{\alpha} + \beta) .$$ \hspace{1cm} (2.16.e)

These relations are complemented by the vacuum NP equations (NP 4.2), the Bianchi identities (NP 4.5) and the commutators (NP 4.4) of the four derivative operators $D, \Delta, \delta$ and $\bar{\delta}$. Equations (2.15) and (2.16.c) turn out to be consequences of (2.16.a),
(2.16.b), (NP 4.2.a) and (NP 4.2.b). Hence, Eq. (2.5) follows from Eq. (2.4). Moreover, Xanthopoulos¹³ proves that also the geodesic condition (2.6) follows from (2.4).

The task of solving the field equations is, however, better served by a different gauge in which an affinely parametrized tangent vector \( l' \) is selected for the Kerr-Schild congruence. This gauge will be used in the subsequent sections.

### 3. THE GAUGE WITH AFFINE PARAMETRIZATION

The Kerr-Schild map (2.1) may be postulated in the slightly different form

\[
\tilde{g}_{ab} = g_{ab} + V l'_a l'_b
\]

where \( V \) is a scalar function. Reparametrizing by

\[
l = \phi l',
\]

where \( \phi \) is a real function, the pencil (2.1) becomes (3.1) with

\[
V = \Lambda \phi^2.
\]

The scale of \( l' \) is arbitrary in (3.1), and we fix it by adopting an affine parametrization

\[
D' l'_a = 0,
\]

where \( D' = l'^a \nabla_a \). The rest (2.4) and (2.9) of the Kerr-Schild pencil equations take then the form

\[
\nabla_b \left[ \nabla_a \left( V l'_c l'^b \right) + \nabla_c \left( V l'_a l'^b \right) - \nabla^b \left( V l'_a l'_c \right) \right] = 0
\]

(2.4′)

\[
D' D' V + (\nabla^a l'_a) D' V + 2 V (\nabla b l'_a) \nabla [b l'^d] = 0.
\]

(2.9′)

Some useful spin coefficients transform under (3.2) as follows: \( 2 \epsilon = D' \phi, \quad \rho = \phi \rho', \quad \sigma = \phi \sigma' \) and \( \Psi_0 = \phi^2 \Psi_0' \).

Dropping the primes, we have in terms of the affine parameter \( r \):

\[
D = \partial / \partial r,
\]
\[ \kappa = \epsilon = 0 . \] 

We find that a step-by-step process of integrating the field equations can be launched in this gauge. The key observation is that Eqs. (2.16a) and (2.16b), together with (NP 4.2.a) and (NP 4.2.b), \textit{i.e.},

\[ \Psi_0 = -\sigma (2D\ln \phi + \rho - \bar{\rho}) \]

\[ 2(\rho + \bar{\rho})D\ln \phi = (\rho + \bar{\rho})^2 - 2(\rho \bar{\rho} + \sigma \bar{\sigma}) \]

\[ D\rho = \rho^2 + \sigma \bar{\sigma} \]

\[ D\sigma = (\rho + \bar{\rho})\sigma + \Psi_0, \] 

form a closed set of equations. Introducing the real functions \( x, y \) and \( z \) by

\[ x = \rho + \bar{\rho} \quad y = \rho \bar{\rho} \quad z = \sigma \bar{\sigma}, \] 

we obtain the autonomous system

\[ Dx = x^2 + 2(z - y) \]

\[ Dy = x(y + z) \] 

\[ xDz = 4z(y + z) . \] 

Solution of Eqs. (3.8), by taking the \( D \) derivatives and decoupling, yields the spin coefficients

\[ \rho = -\frac{1}{2r} \left[ 1 + \cos \eta \frac{r \cos \eta}{r \cos \eta + iB} \right] \] 

\[ \sigma = -\frac{\sin \eta}{2r} \frac{r \cos \eta + iB}{r \cos \eta - iB} . \] 

Here one of the constants has been eliminated by the appropriate choice of the origin of the affine parameter \( r \). Also we used the remaining spatial rotations (2.12) to eliminate the \( r \)-independent part of the phase factor of \( \sigma \). The integration functions \( \eta \) and \( B \) may depend on the coordinates \((u, x^2, x^3)\). While \( B \) can take any real value, \( \eta \) ranges in the interval \([0, 360^\circ)\).
For the value $B = 0$, the null congruence with tangent $l$ is twist-free, and for $\eta = 0$ or $\eta = 180^\circ$, it is shear-free. When both $B = 0$ and $\eta = 180^\circ$ holds, there is no expansion.

From (3.6), (3.9) and (3.10) we get the curvature quantity

$$\Psi_0 = -\frac{\sin 2\eta}{4r^2}$$

and for $r \geq 0$, the scaling function can be written

$$\phi = A^2 \left( \frac{r \cos \eta}{r^2 \cos \eta + B^2} \right)^{1/2}$$

where $A$ is another integration function of $u, x^2$ and $x^3$.

We have yet the freedom of performing the null rotations (2.13). We use these to set

$$\pi = \alpha + \bar{\beta}, \quad (3.13)$$

removing thereby the term with $D$ from the commutator $[D, \delta]$. There remain further null rotations (2.13) compatible with (3.13) where $E$ satisfies

$$DE = \bar{\rho}E + \sigma \bar{E}. \quad (3.14)$$

Let us denote the components of the complex vector $m$ by

$$m = \Omega \frac{\partial}{\partial r} + m^i \frac{\partial}{\partial x^i}, \quad x^i = u, x^2, x^3. \quad (3.15)$$

The second commutator in (NP 4.4), when applied to the coordinates, yields the equations:

$$D\Omega = \bar{\rho}\Omega + \sigma \Omega \quad (3.16)$$

$$D m^i = \bar{\rho} m^i + \sigma \bar{m}^i. \quad (3.17)$$

Noticing that equations (3.14) and (3.16) have identical forms, we use the remaining null rotations to arrange

$$\Omega = 0. \quad (3.18)$$
We next integrate Eq. (3.17) to obtain the rest of the components of \( m \):

\[
m^j = \frac{1}{\sqrt{r^2 - iB}} \left[ Q^j_1 r^\cos \eta + iQ^j_2 r^\sin \eta \right] \quad j = 1, 2, 3 \tag{3.19}
\]

where \( Q^j_1 \) and \( Q^j_2 \) are again real functions of \( u, x^2, x^3 \).

### 4. THE MAIN THEOREM

We now rewrite the remaining Kerr-Schild equations (2.16.d) and (2.16e) in the affine gauge. Eq. (2.16.d) takes the form\(^{18} \), using (NP 4.2.c), (NP 4.2.k), (2.14), (3.2) and (3.4):

\[
\delta \left( \frac{\Psi_0}{2\sigma} \right) + \frac{\Psi_0}{\sigma} \delta \ln \phi - 2\sigma \delta \ln \phi - \Psi_1 = \frac{\Psi_0}{2\sigma} (\tau - \bar{\alpha} - \bar{\beta}) - 2\sigma (\bar{\tau} - \alpha - \bar{\beta}) + \bar{\tau} \sigma - \tau \rho \tag{4.1}
\]

Note that the \( \sigma \to 0 \) limit is well-behaved as follows from the relation \( \frac{\Psi_0}{2\sigma} = -\frac{1}{2\tau} - \rho \).

From (2.16e), with (NP 4.2.f), (NP 4.2.q), (2.14),(3.2) and (3.4):

\[
\delta(\bar{\tau} - \alpha - \bar{\beta}) + \delta(\tau - \bar{\alpha} - \beta) - (\delta \bar{\tau} + \delta \ln \phi + (\rho + \bar{\rho}) \Delta \ln \phi
\]

\[
+ (2\bar{\tau} - 3\alpha - 5\bar{\beta}) \delta \ln \phi + (2\tau - 3\bar{\alpha} - 5\beta) \delta \ln \phi - 4(\delta \ln \phi)(\bar{\delta} \ln \phi) =
\]

\[
2 \text{Re} \Psi_2 - (\gamma + \bar{\gamma})(\rho + \bar{\rho}) + \frac{1}{2}(\mu + \bar{\mu})(\frac{\Psi_0}{\sigma} + \rho - \bar{\rho}) - (\mu - \bar{\mu})(\rho - \bar{\rho})
\]

\[
+ \tau(\bar{\tau} - \alpha - 3\bar{\beta}) + \bar{\tau}(\tau - \bar{\alpha} - 3\beta)
\]

\[
+ 4(\alpha + \bar{\beta})(\bar{\alpha} + \beta) - (\alpha + \bar{\beta})(\bar{\alpha} - \beta) - (\alpha - \bar{\beta})(\bar{\alpha} + \beta) \tag{4.2}
\]

Using (3.13) in (NP 4.2.c), (NP 4.2.d), (NP 4.2.e), and the first Bianchi equation (NP 4.5), we have

\[
D\tau = \rho(\tau + \bar{\pi}) + \sigma(\bar{\tau} + \pi) + \Psi_1 \tag{4.3a}
\]

\[
D\pi = 2\rho \pi + 2\bar{\sigma} \bar{\pi} + \bar{\Psi}_1 \tag{4.3b}
\]

\[
D\alpha = \rho(\pi + \alpha) + \bar{\sigma}(\bar{\pi} - \bar{\alpha}) \tag{4.3c}
\]

\[
D\beta = \bar{\rho} \beta + \sigma(2\pi - \bar{\beta}) + \Psi_1 \tag{4.3d}
\]

\[
D\Psi_1 = 4\rho \Psi_1 + (\bar{\delta} + \pi - 4\alpha)\Psi_0. \tag{4.3e}
\]

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This is a set of coupled linear, inhomogeneous equations for the functions \( \tau, \pi, \alpha, \beta \) and \( \Psi_1 \) and their complex conjugates, with the constraints (3.13) and (4.1), algebraic in \( r \). Furthermore, we need Eq. (NP 4.2k),

\[
\delta \rho - \bar{\delta} \sigma = \rho \bar{\pi} + \sigma ( \pi - 4 \alpha ) + ( \rho - \bar{\rho} ) \tau - \Psi_1 \quad (4.4)
\]

for the proof of our

**Theorem:** For a generalized vacuum-vacuum Kerr-Schild pencil, either of the following conditions hold:

(i) The parameter \( \eta \) assumes one of the special values given by

\[
sin \eta = 0, \pm 1, \pm 2^{\frac{1}{4}} \quad (4.5)
\]

(ii) The \( \delta \) derivatives are restricted by

\[
\delta \rho = \delta \bar{\rho} = \delta \sigma = \delta \bar{\sigma} = \delta \Psi_0 = \delta \phi = 0 \quad (4.6)
\]

We prove the theorem by taking the \( D \) derivative of both sides of (4.1) and using Eqs. (4.3), (3.9), (3.10), (3.11) and the second commutator (NP 4.4) for eliminating the \( D \) derivatives. We remove \( \pi - 4 \alpha \) from (4.3e) by Eq. (4.4). All the unknown spin coefficients cancel and we arrive at an equation of the form \( a \delta \phi + b \bar{\delta} \phi = 0 \). Eliminating next \( \bar{\delta} \phi \) with the help of the complex conjugate equation, we get

\[
\frac{\cos \eta ( r^2 \cos \eta + B^2 ) + r^2 \cos \eta - B^2 }{r^{10} ( r^2 \cos \eta + B^2 )} P \delta \phi = 0 \quad (4.7)
\]

where \( P \) is the polynomial in \( \sin \eta \) and \( \cos \eta \)

\[
P = (2 \sin^2 \eta - 1) \sin^3 \eta \cos \eta \quad (4.8)
\]

with roots given in (i). For other values of \( \eta \), equation (4.7) together with (3.12) yields \( \delta A = \delta B = \delta \eta = 0 \). Using this information, part (ii) of the theorem is proved.
Condition (4.7) is trivially satisfied in the absence of shear, $\sin \eta = 0$. Thus the Kerr solution will not emerge as a limiting case of the shearing solutions.

Setting aside case (i), the parameter $\eta$ may take real values. At the next step of the integration process, we are confronted with the set of coupled homogeneous linear equations (4.3) for the field quantities $\alpha, \beta, \pi, \tau, \Psi_1$ and their complex conjugates. Two linear algebraic relations among these quantities follow from the adopted gauge, $\pi - \alpha - \bar{\beta} = 0$, and from the Kerr-Schild condition (4.1):

$$\Psi_1 = \frac{\Psi_0}{2\sigma} (\bar{\pi} - \tau) + \sigma (\bar{\tau} - 2\pi) + \rho \tau. \quad (4.9)$$

Equation (4.4) yields a further algebraic constraint:

$$4\sigma \alpha = \sigma (3\pi - \bar{\pi}) + \left(\rho - \frac{\Psi_0}{2\sigma}\right) \bar{\pi} + \left(\frac{\Psi_0}{2\sigma} - \bar{\rho}\right) \tau. \quad (4.10)$$

Eqs. (4.3c), (4.3d) and (4.3e) are a consequence of the algebraic relations and Eqs. (4.11). Therefore, the knowledge of the solution of Eqs. (4.11) suffices for determining the solution of the complete system (4.1), (4.3) and (4.4).

One can make use of the algebraic constraints to obtain equations for various closed subsets of the unknown functions. For example, using the Kerr-Schild constraint (4.9) for eliminating $\Psi_1$ in (4.3a) and (4.3b), we get a set of four coupled equations for $\pi, \tau$:

$$D\tau = (2\rho - \frac{\Psi_0}{2\sigma})\tau + \sigma (2\bar{\pi} - \pi) + (\rho + \frac{\Psi_0}{2\sigma}) \bar{\pi}$$

$$D\pi = (2\rho + \frac{\Psi_0}{2\sigma})\pi + \bar{\sigma} \tau + (\bar{\rho} - \frac{\Psi_0}{2\sigma}) \bar{\pi} \quad (4.11)$$

and their complex conjugates. Eqs. (4.3c), (4.3d) and (4.3e) are a consequence of the algebraic relations and Eqs. (4.11). Therefore, the knowledge of the solution of Eqs. (4.11) suffices for determining the solution of the complete system (4.1), (4.3) and (4.4).

Alternatively, Eqs. (4.3b), (4.3c) and (4.3e) form a closed set for $\alpha, \pi, \Psi_1$ and their complex conjugates.

Systems of homogeneous linear differential equations always have nontrivial solutions in the domain of continuity of the coefficients. The general solution of $n$
coupled equations is given by $n$ linearly independent solution vectors. We will not attempt to tackle this problem here, but in the next section we shall seek for space-times characterised by the trivial solution.

5. A PARTICULAR SOLUTION

The purpose of this section is to demonstrate that the class of vacuum space-times (2.1) with nonvanishing shear $\sigma$ is not empty. It will suffice for us to take the trivial solution of Eqs. (4.3):

$$\tau = \pi = \alpha = \beta = \Psi_1 = 0.$$  \hfill (5.1)

Equations (4.9) and (4.10) are identically satisfied. Eqs. (NP 4.2.g), (NP 4.2.h) and the second relation of (NP 4.5) form a closed system of linear homogeneous equations for the field quantities $\lambda, \mu$ and $\Psi_2$:

$$D\lambda = \rho \lambda + \bar{\sigma} \mu \quad \text{(5.2a)}$$
$$D\mu = \bar{\rho} \mu + \sigma \lambda + \Psi_2 \quad \text{(5.2b)}$$
$$D\Psi_2 = 3\rho \Psi_2 - \lambda \Psi_0 \quad \text{(5.2c)}$$

Equations (NP 4.2.f) and (NP 4.2.l) involve the spin coefficient $\gamma$,

$$D\gamma = \Psi_2 \quad \text{(5.3)}$$
$$\Psi_2 = \mu \rho - \lambda \sigma + \gamma (\rho - \bar{\rho}).$$

When $B \neq 0$, we may use the second equation (5.3) to obtain $\gamma$. The first equation is compatible with this. The curl-free fields with $B = 0$ will be considered elsewhere.

In the case when one of the quantities $\lambda, \mu, \Psi_2, \gamma$ vanishes, Eqs. (5.2) and (5.3) imply that the remainder must also vanish. (The general solution of the system will be given in Paper II.) From (NP 4.2.o) $\nu = 0$, from (NP 4.2.i) $\Psi_3 = 0$ and from (NP 4.2.j) we get $\Psi_4 = 0$. Let us choose again this simplest case:

$$\lambda = \mu = \gamma = \nu = \Psi_2 = \Psi_3 = \Psi_4 = 0.$$  \hfill (5.4)
From the commutators (NP 4.4) we obtain that $\Delta$ commutes both with $\delta$ and $D$. Thus we can adopt such coordinates that $\Delta = \frac{\partial}{\partial u}$, $Q_1^j = Q_1^j(x^2, x^3)$ and $Q_2^j = Q_2^j(x^2, x^3)$. The functions $A, B$ and $\eta$ are, in fact, constants in this case because we already have $\delta A = \delta B = \delta \eta = 0$, the fifth relation of (NP 4.5) implies $\Delta \Psi_0 = 0$ or $\Delta \eta = 0$ and from (4.2) we have $\Delta ln \phi = 0$ or $\Delta A = \Delta B = 0$. Furthermore $Q_1 = Q_1^j \frac{\partial}{\partial x^j}$ and $Q_2 = Q_2^j \frac{\partial}{\partial x^j}$ are noncommutative but they commute with $\Delta$ and $D$. This enables us to choose the coordinates such that

$$Q_1^j = (\cos \eta Bx^2, 0, 1)$$
$$Q_2^j = (0, 1, 0) .$$

(5.5)

By use of the completeness relation

$$g^{ab} = l^a n^b + n^a l^b - m^a \bar{m}^b - \bar{m}^a m^b ,$$

we obtain the inverse metric

$$g^{ab} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & -hQ_1^2 & 0 & -hQ_1 \\
0 & -hQ_1 & -hr^2 \sin \eta & 0 \\
0 & -hQ_1 & 0 & -h \\
\end{pmatrix} \quad (5.7)$$

where

$$h(r) = 2 \frac{r \cos \eta - \sin \eta - 1}{r^2 \cos \eta + B^2} \quad Q_1 = \cos \eta Bx^2 .$$

Hence we get the metric

$$ds^2 = ds^2 - V_0 \frac{r \cos \eta}{r^2 \cos \eta + B^2} (du - \cos \eta Bx^2 dx^3)^2$$

(5.8)

where $V_0 = \lambda A^4$ is a constant and $ds^2$ is the parent metric of the Kerr-Schild pencil:

$$ds^2 = 2dr du - 2cBx^2 dr dx^3 - \frac{r \cos \eta + B^2}{2r \cos \eta} \left[ r^{1 - \sin \eta} (dx^2)^2 + r^{1 + \sin \eta} (dx^3)^2 \right]$$

(5.9)

By performing the Eddington-type coordinate transformation

$$du = dt + \frac{r \cos \eta + B^2}{V_0 r \cos \eta} dr ,$$

(5.10)
the metric assumes the form
\[
    ds^2 = -V_0 r^{2\cos\eta} + B^2 (dt - \cos\eta B x^2 dx^3)^2 \\
    - \frac{r^{2\cos\eta} + B^2}{2r^{\cos\eta}} \left[ \frac{-2}{V_0} dr^2 + r^{1-\sin\eta}(dx^2)^2 + r^{1+\sin\eta}(dx^3)^2 \right].
\]  

(5.11)

This is one of the Kóta-Perjés\textsuperscript{20} vacuum solutions, and thus (5.11) admits a Killing vector \(\partial/\partial t\) whose eigenrays are geodesic.

6. CONCLUDING REMARKS

The picture we glean in this work is as follows. The vacuum space-times in which vacuum Kerr-Schild metrics can be generated are characterized either (i) by special values of the parameter \(\eta\), or by condition (ii) of our theorem. The metrics in class (ii) are a dynamical system governed by two sets of homogeneous linear equations (4.3), with \(\bar{\Psi}_0 = 0\), and (5.2). The spectrum of the system is constrained by the nonradial Newman-Penrose and Kerr-Schild equations. The ground state is a Kóta-Perjés space-time. At the moment of writing it appears to us that the excited states can all be generated in terms of elementary functions. This issue will be discussed in Paper II.

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