FAST GROWTH IN FØLNER SETS FOR THOMPSON’S GROUP \( F \)

JUSTIN TATCH MOORE

Abstract. The purpose of this note is to prove a lower bound on the Følner function for Thompson’s groups \( F \). In particular I will prove that there is a constant \( C \) such that \( \text{Føl}_F(C^n) \geq \exp_n(0) \) where \( \exp_n(k) \) is the \( n \)-fold iteration of the function \( k \mapsto 2^k \).

1. Introduction

In this paper we will study the Følner function for Thompson’s group \( F \). Recall that a finite subset \( A \) of a finitely generated group \( G \) is \( \varepsilon \)-Følner if

\[
\sum_g |(A \cdot g) \setminus A| < \varepsilon |A|
\]

where the sum is taken over some fixed finite generating set \( S \) and \( \setminus \) denotes symmetric difference. The Følner function of \( G \) is defined by

\[
\text{Føl}_G(n) = \min\{|A| : A \subseteq G \text{ is } \frac{1}{n}\text{-Følner}\}
\]

with \( \text{Føl}_G(n) = \infty \) if there is no \( 1/n \)-Følner set. Here \( S \) is implicit and, while it does influence the exact value of \( \text{Føl}_G(n) \), it does not affect its asymptotics. Notice that \( G \) is amenable if and only if its Følner function is finite valued.

Thompson’s group \( F \) has many equivalent definitions; we will use the formulation in terms of tree diagrams defined below. The standard presentation of \( F \) is infinite, with generators \( x_i \) \((i \in \mathbb{N})\) satisfying \( x_i^{-1}x_nx_i = x_{n+1} \) for all \( i < n \). It is well known, however, that \( F \) admits the finite presentation

\[
\langle A, B \mid [AB^{-1}, A^{-1}BA] = [AB^{-1}, A^{-2}BA^2] = \text{id} \rangle
\]

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Geoghegan conjectured that \( F \) is not amenable \([1]\) and at present this problem remains open. The goal of this paper is to establish the following lower bound on the Følner function for \( F \).

**Theorem 1.1.** There is a constant \( C > 1 \) such that if \( A \subseteq F \) is a \( C^{-n} \)-Følner set, then \( A \) contains at least \( \exp_n(0) \) elements. In particular \( F_\infty \) is not eventually dominated by any finite iterate of the exponential function.

Here \( \exp_n(k) \) is the \( n \)-fold composition of the exponential function defined by \( \exp_0(k) = k \) and \( \exp_{n+1}(k) = 2^{\exp_n(k)} \). If it turns out that \( F \) is amenable, then Theorem 1.1 would be a step toward answering (negatively) the following question of Gromov \([7]\).

**Question 1.2.** Is there a primitive recursive function which eventually dominates every Følner function of an amenable finitely presented group?

While it is known that the Følner functions of solvable finitely generated groups can grow arbitrarily fast \([6]\), this is not the case for finitely presented groups (since there are only countably many such groups). See \([5]\) for what can be accomplished via wreath products of \( \mathbb{Z} \).

This note is organized as follows. In Section 2, I will review some of the basic definitions associated with \( F \) and fix some notational conventions. Those familiar with \( F \) will only need to scan the choice of generators and the multiplication convention (the conventions follow \([2]\), not \([4]\)). In Section 3, I will introduce the notion of a marginal set.

These are sets which must have small intersections with Følner sets. Section 4 recasts the amenability of \( F \) in terms of its right action on finite binary trees. Section 5 defines an operation on binary trees which exponentially decreases their size. It is shown that the trees which are trivialized by this operation are marginal and it is this that allows the proof of Theorem 1.1.

2. **Notation and background**

I will follow the notational conventions of Belk’s thesis \([2]\) (which often conflict with \([4]\)).

The collection of finite binary sequences is equipped with the operation of concatenation (denoted \( u \cdot v \)), the partial order \( \subseteq \) of extension (defined by \( u \subseteq u \cdot v \)), and the lexicographic order (denoted \( u <_{\text{lex}} v \)). Note that \( u \) extends \( v \) includes the possibility that \( u = v \). Let \( \mathcal{T} \) denote the collection of all finite rooted binary trees. For concreteness, we will view elements \( T \) of \( \mathcal{T} \) as finite sets of binary sequences which have the following properties:
• if \( t \) is in \( T \), then so are all initial parts of \( t \).
• if \( t \) is in \( T \), then either \( t \) is maximal in \( T \) or else \( t^0 \) and \( t^1 \) are both in \( T \).

In particular \( \langle \rangle \) denotes the root, and the maximal elements are the leaves of \( T \). The set of leaves of \( T \) will be denoted \([T]\) and has a canonical enumeration which is \(<_{\text{lex}}\)-increasing. Terminology such as “maximal leaf” will always refer to this order.

The trivial tree is the one which consists only of its root. If \( T \) is in \( \mathcal{T} \) and \( u \) is in \( T \), define \( T/u \) to be the set of all \( s \) such that \( u^s \) is in \( T \) (i.e., \( T/u \) is the tree of descendants of \( u \)).

A tree diagram is a pair \((L, R)\) of elements of \( \mathcal{T} \) such that \(|L| = |R|\). We view a tree diagram as describing a map of sequences defined by

\[
s_i^\ast x \mapsto t_i^\ast x
\]

where \( s_i \) and \( t_i \) are the \( i \)th leaves of \( L \) and \( R \) respectively and \( x \) is any sequence. The value of the associated map \( f \) at a sequence \( t \) will be denoted \( t \cdot f \). If two tree diagrams define the same map on infinite sequences, then I will say they are equivalent. Every tree diagram is equivalent to a unique minimal tree diagram (see [4]); such a tree diagram is said to be reduced.

Thompson’s group \( F \) is the collection of reduced tree diagrams with the operation of composition. In this note \( f \cdot g \) is defined to mean “\( f \) followed by \( g \)” (i.e. \( g \circ f \)). \( F \) is generated by \( \{x_0, x_1\} \) where \( x_0 \) and \( x_1 \) are specified by:

\[
x_0 = \begin{cases} 00 \mapsto 0 \\ 01 \mapsto 10 \\ 1 \mapsto 11 \end{cases} \quad x_1 = \begin{cases} 0 \mapsto 0 \\ 100 \mapsto 10 \\ 101 \mapsto 110 \\ 11 \mapsto 111 \end{cases}
\]

In our discussion of \( F \) “generator” will mean either \( x_0 \) or \( x_1 \) (this is really only relevant in Lemma 5.7). I will also identify elements of \( F \) with the corresponding maps on sequences. If \( T \) is in \( \mathcal{T} \) and \( f \) is in \( F \) with \( f \) defined on the leaves of \( T \), then \( T \cdot f \) is the element of \( \mathcal{T} \) which has \( \{t \cdot f : t \in [T]\} \) as its set of leaves. This defines a partial right action of \( F \) on \( \mathcal{T} \). If \( f \) is in \( F \), I will write \((L_f, R_f)\) to denote the reduced tree diagram for \( f \).

3. Marginal sets

Throughout this section “right” in an implicit adjective whenever applicable, although all statements have their corresponding “left” analogs. Suppose that \( G \) is a group admitting a partial action on a
set $S$. (By partial action I mean a partial map $\cdot : S \times G \to S$ such that $x \cdot (gh) = (x \cdot g) \cdot h$ whenever all instances of evaluating the action are defined.) If $E \subseteq S$ and $g$ is in $G$, define $E \cdot g = \{ x \cdot g : x \in E \}$.

**Definition 3.1.** A weighted $\varepsilon$-Følner set is a function $\mu$ from a finite subset of $S$ into $(0, \infty)$ which satisfies

$$\sum_g \sum_s |\mu(s \cdot g) - \mu(s)| < \varepsilon \sum_s \mu(s),$$

where $g$ ranges over the generators with the conventions that $\mu(s) = 0$ if $s$ is not in the domain of $\mu$ and $\mu(s \cdot g) = 0$ if $s \cdot g$ is undefined. The set $\{ s \in S : \mu(s) > 0 \}$ will be referred to as the support of $\mu$. It will be convenient to let $\mu(A)$ denote $\sum_s \mu(s)$.

Hence an $\varepsilon$-Følner set is a set $A \subseteq S$ such that $\chi_A$ is a weighted $\varepsilon$-Følner set. We will need the following property of weighted Følner sets.

**Lemma 3.2.** Suppose $G$ acts partially on sets $S$ and $T$ and that $\mu$ is a weighted $\varepsilon$-Følner set with respect to the action on $S$. If $h : S \to T$ satisfies that $h(s \cdot g) = h(s) \cdot g$ (with both quantities defined) whenever $\mu(s) + \mu(s \cdot g) > 0$ and $g$ is a generator or the inverse of a generator, then $\nu(t) = \sum_{h(s) = t} \mu(s)$ defines a weighted $\varepsilon$-Følner set (with respect to the action on $T$).

We will be interested in the following notion which ensures that a set has small intersection with any Følner set. The definition is motivated by the following simple observation. If $\mu$ is a finitely additive invariant probability measure on a group $G$ and $E \subseteq G$ satisfies that, for some $g$ in $G$, $\{ E \cdot g^i : i \in \mathbb{N} \}$ is a pairwise disjoint family, then $\mu(E) = 0$.

**Definition 3.3.** If $g \in G$, $I \subseteq S$, and $E \subseteq S$, then I will say that $g$ marginalizes $E$ off of $I$ if for every $x$ in $E$ if $x \cdot g^k \in E$ and $k > 0$, then there is an $i < k$ such that $x \cdot g^i$ is in $I$ or is undefined. If $I$ is the empty-set, then I will simply write $g$ marginalizes $E$.

**Remark 3.4.** Since the exponentiation is defined in $G$, it is entirely possible in general for $x \cdot g^i$ to be undefined while $x \cdot g^j$ is defined for some $i < j$.

**Definition 3.5.** The $k$-marginal sets for the partial action of $G$ on $S$ are defined recursively as follows. The empty-set is 0-marginal. If $E = \bigcup_{i=0}^n E_i \subseteq S$ and for each $i$, there is a $g$ and a $k$-marginal set $I$ such that $g$ marginalizes $E_i$ off $I$, then $E$ is $(k + 1)$-marginal. $E \subseteq S$ is marginal if it is $k$-marginal for some $k < \infty$. 
We will need the following lemma which shows that marginal sets have small intersections with Følner sets.

**Lemma 3.6.** Suppose that $G$ is a group partially acting on $S$, and $\mu$ is a weighted $\varepsilon$-Følner set with support $A$. If $E \subseteq S$ and $g$ marginalizes $E$ off $S \setminus A$, then $\mu(E) < d_g \varepsilon \mu(S)$, where $d_g$ is the distance from $g$ to the identity.

**Proof.** For each $x$ in $E \cap A$, let $F(x)$ denote the set of all $x \cdot g^i$ such that

1. $x \cdot g^j$ is defined and in the support of $\mu$ for all $j \leq i$ and
2. $\mu(x \cdot g^{i+1}) < \mu(x \cdot j)$ for all $j \leq i$.

Observe that each $F(x)$ is finite and non empty. Because $E$ is marginalized off $A$ by $g$, $\{F(x) : x \in E\}$ is a pairwise disjoint family. Observe that for a fixed $x$ in $E$

$$\mu(x) \leq \sum_{y \in F(x)} \mu(y) - \mu(y \cdot g) = \sum_{y \in F(x)} |\mu(y \cdot g) - \mu(y)|.$$

Summing over $x$ in $E$ and utilizing the assumption that $\mu$ is $\varepsilon$-Følner, now gives the desired estimate. $\square$

The following lemmas are straightforward computations.

**Lemma 3.7.** If $\mu$ is a weighted $\varepsilon$-Følner set and $\nu$ is a function from a finite subset of $S$ into $[0, \infty)$ with $\nu \leq \mu$ pointwise and $\nu(S) \geq (1 - \delta) \mu(S)$, then $\nu$ is a weighted $(2\delta + \varepsilon)/(1 - \delta)$-Følner set.

**Lemma 3.8.** Suppose that $E \subseteq S$ is $k$-marginal. Then there is a $C$ such that if $\mu$ is a weighted $\varepsilon$-Følner set, then $\mu \upharpoonright (S \setminus E)$ is a weighted $C\varepsilon$-Følner set.

4. Følner sets of trees

Rather than studying Følner sets in $F$ directly, it will be easier to deal with weighted Følner sets in the partial right action of $F$ on $\mathcal{T}$ which I will refer to as *weighted Følner sets of trees*. These are essentially weighted right Følner sets consisting of positive elements of $F$. The reformulation of the amenability of $F$ in terms of the existence of Følner sets of positive elements is well known, but we will need the more precise analytic content of the following lemmas.

**Lemma 4.1.** For every finite binary sequence $u$, the set of all $f$ in $F$ such that $u$ is not in $R_f$ is marginal.
Proof. Notice that since every element of $\mathcal{T}$ contains the empty string, it suffices to consider $u$ of positive length. Let $u$ be given and let $v$ be the immediate predecessor of $u$ in the order of extension. Let $y$ be the element of $F$ which is the identity on sequences which do not extend $v$ and which satisfies $(v^* a) \cdot y = v^*(a \cdot x_0)$ for all sequences $a$. Observe that if $u$ is not in $R_f$, then $f \cdot y^n$ has a reduced tree diagram of the form

$$(L, R_f \cup R_{y^n})$$

for some $L$ in $\mathcal{T}$. If $n > 0$, then $u$ is in $R_f \cdot y^n$ and hence $y$ marginalizes the set of all $f$ in $F$ such that $u \not\in R_f$. □

Lemma 4.2. There is a constant $C$ such that if $A \subseteq F$ is a right $\varepsilon$-Følner set, then there is a weighted $C\varepsilon$-Følner set of trees supported on a subset of $\{R_f : f \in A\}$.

Proof. Fix a $U$ in $\mathcal{T}$ such that if $x$ is a generator, then $L_x \cup R_x \subseteq U \setminus [U]$. By Lemma 4.1, the set of $f$ in $F$ such that $U \not\subseteq R_f$ is 1-marginal. Let $A_0 = \{f \in A : U \subseteq R_f\}$. By Lemma 3.3, $A_0$ is $C\varepsilon$-Følner for some $C$ which does not depend on $A$. Observe that if $x$ is a generator and $f$ is in $A_0$, then $L_{f \cdot x} = L_f$ and $R_{f \cdot x} = R_f \cdot x$. We are now finished by Lemma 3.2 applied to $h(f) = R_f$ and $\mu(s) = \chi_{A_0}(s)$.

5. An Operation on Elements of $\mathcal{T}$

In this section I will define an operation $\partial$ on elements of $\mathcal{T}$ which reduces their size logarithmically.

If $T$ is in $\mathcal{T}$, then the end leaves of $T$ are the maximum and minimum leaves of $T$. All other leaves of $T$ are interior leaves.

Definition 5.1. Suppose that $T$ is in $\mathcal{T}$. $\partial T$ is the maximum $U \in \mathcal{T}$ contained in $T$ which satisfies the following defining conditions:

1. $U$ contains both 01 and 10;
2. one of the following holds:
   - if $u <_{\text{lex}} v$ are interior leaves of $U$, then $2|T/u| \leq |T/v|/2$;
   - if $u <_{\text{lex}} v$ are interior leaves of $U$, then $2|T/v| \leq |T/u|/2$;
3. the minimum (maximum) interior leaves of $U$ terminate with a 1 (respectively 0).

If no such $U$ exists, then $\partial T$ is defined to be the trivial tree.

Lemma 5.2. If there is a $U$ satisfying the defining conditions for $\partial T$, then there is a maximum such $U$.

Proof. Suppose for contradiction that $U$ and $V$ are distinct maximal elements of $\mathcal{T}$, each contained in $T$, which satisfy the defining conditions of $\partial T$. First I claim that condition [2] is satisfied in the same way for
extends a leaf of $V$ adding to $U$ the roles of its predecessors. If $w$ be the greatest leaf of $u$ is non-empty, there is a leaf of $W$ extending $01$ and $w'$ be the leftmost leaf of $W$ extending $10$. Observe that condition 2 implies that $|[T/01]| < 2|[T/w]|$. It follows that $|[T/01]| < 2|[T/w]| \leq |[T/w']| \leq |[T/10]|$.

For simplicity we will assume that the quantity $|[T/u]|$ is increasing as $u$ increases in $[U]$ (or equivalently as $u$ increases in $[V]$). Since $U \cup V$ is non-empty, there is a $<_\text{lex}$-minimal leaf of $U$ which either properly extends a leaf of $V$ or is properly extended by a leaf of $V$; by exchanging the roles of $U$ with $V$ if necessary, we may assume the former occurs. First suppose that the minimum leaves of $U$ and $V$ are the same. Let $u$ be the greatest leaf of $U$ such that it and all of its $<_\text{lex}$-predecessors extend (not necessarily properly) a leaf of $V$. Let $W$ be obtained by adding to $V$ those elements of $U$ which are extended by $u$ or one of its predecessors. If $w$ is a leaf of $W$ which occurs at or before $u$, then $|[T/w]| \leq |[T/v]|$, where $v$ is the leaf of $V$ which $w$ extends. It follows that condition 2 is satisfied. Also notice that $U$ and $V$ have the same minimum interior leaf. Also, since $V \setminus U$ is non empty, $u$ is not the maximum interior leaf of $W$. Therefore $W$ satisfies condition 3. But now $W$ satisfies the defining conditions for $\partial T$, contradicting the maximality of $V$.

Now suppose that $u$ is the minimum leaf of $U$. Let $v$ be such that $v^*0$ is the minimum leaf of $V$, noting that $v^*1$ is the minimum interior leaf of $V$. Since $v^*0$ is in $U$, it must be that $v^*1$ is also in $U$. Let $W$ be the union of $V$ together with all elements of $U$ which extend $v^*0$. The minimum (maximum) interior leaf of $W$ is the same as $U$ (respectively $V$) and therefore $W$ satisfies condition 3. Also, for the same reasons as above, $W$ satisfies condition 3. Again, $W$ satisfies the defining conditions for $\partial T$, contradicting the maximality of $V$.

**Lemma 5.3.** If $\partial T$ has $n$ leaves, then $T$ has more than $2^{n-2}$ leaves.

**Proof.** There are $n - 2$ interior leaves of $\partial T$ and by our assumption, the total number of leaves of $T$ which extend an interior leaf of $\partial T$ is at least $\sum_{i=1}^{n-2} 2^{i-1} = 2^{n-2} - 1$. Since $\partial T$ is a subset of $T$, there are at least two leaves of $T$ remaining to be counted, putting the total greater than $2^{n-2}$. □

**Lemma 5.4.** If $L_g \subseteq \partial T$ and does not contain the end leaves of $\partial T$, then $\partial(T \cdot g) = (\partial T) \cdot g$. 
Proof. First observe that if \( g \) is in \( F \), \( u \) is finite binary sequence and \( u \) properly extends an element of \( [L_g] \), then \( u \) and \( u \cdot g \) have the same final digit. Consequently, \( (\partial T) \cdot g \) satisfies condition 3. Since \( g \) preserves lexicographic order and extension of sequences, \( (\partial T) \cdot g \) satisfies condition 3. Finally, the hypothesis of the lemma implies that \( \partial T \cdot g \) satisfies condition 3. It follows that \( (\partial T) \cdot g \subseteq \partial(T \cdot g) \).

For the reverse implication, the above argument establishes that \( L_g^{-1} = R_g \) is contained in \( \partial(T \cdot g) \). A similar argument shows that \( \partial(T \cdot g) \cdot g^{-1} \) is contained in \( \partial(T \cdot g \cdot g^{-1}) = \partial T \). Since \( g \) and \( g^{-1} \) are injections, it follows that \( \partial(T \cdot g) \) and \( (\partial T) \cdot g \) have the same elements and hence are equal. \( \square \)

Definition 5.5. Let \( \mathcal{E} \) be the set of all \( T \) in \( \mathcal{T} \) such that neither of the following inequalities hold:

\[
\begin{align*}
(+) && |T/001| < |T/01| < |T/10| \\
(-) && |T/001| > |T/01| > |T/10|
\end{align*}
\]

Lemma 5.6. \( \mathcal{E} \) is 2-marginal.

Proof. Define

\[
a = x_0^2 x_1 x_2^{-2} x_0^{-2} \\
b = x_0^2 x_1 x_3^{-1} x_0^{-2}
\]

and note that \( a \) and \( b \) define the following maps:

\[
a = \begin{cases}
000 &\mapsto 000 \\
0010 &\mapsto 001 \\
0011 &\mapsto 0100 \\
01 &\mapsto 0101 \\
100 &\mapsto 011 \\
101 &\mapsto 10 \\
11 &\mapsto 11
\end{cases}
\]

\[
b = \begin{cases}
000 &\mapsto 000 \\
0010 &\mapsto 001 \\
0011 &\mapsto 01 \\
01 &\mapsto 100 \\
10 &\mapsto 101 \\
11 &\mapsto 11
\end{cases}
\]

Define

\[
\mathcal{E}_a = \{ T \in \mathcal{T} : \max(|T/001|, |T/10|) = |T/01| \} \\
\mathcal{E}_b = \{ T \in \mathcal{T} : \max(|T/001|, |T/10|) < |T/01| \}
\]

Observe that for all \( T \) in \( \mathcal{T} \) such that \( T \cdot a \) is defined, we have:

\[
|((T \cdot a)/001)| = |T/0010| < |T/001| \\
|((T \cdot a)/01)| = |T/0011| + |T/01| + |T/100| > |T/01| \\
|((T \cdot a)/10)| = |T/101| < |T/10|
\]
Therefore if $k > 0$ and $T$ is in $\mathcal{E}_a \cdot a^k$, then
\[
\max(||T/001||, ||T/10||) < ||T/01||
\]
and, in particular, $T$ is not in $\mathcal{E}_a$. This shows that $a$ marginalizes $\mathcal{E}_a$.

Observe that for all $T$ in $\mathcal{F}$ such that $T \cdot b$ is defined, we have:
\[
||T \cdot b/001|| = ||T/001|| < ||T/001||
\]
\[
||T \cdot b/01|| = ||T/001|| < ||T/001||
\]
\[
||T \cdot b/10|| = ||T/01|| + ||T/10|| > ||T/10||
\]
Therefore if $k > 0$ and $T$ is in $\mathcal{E}_b \cdot b^k$, then
\[
\max(||T/001||, ||T/01||) < ||T/10||
\]
and, in particular, $T$ is not in $\mathcal{E}_b$. This shows that $b$ marginalizes $\mathcal{E}_b$.

Now notice that $T$ is in $\mathcal{E}_a \cup \mathcal{E}_b$ iff
\[
||T/01|| = \max(||T/001||, ||T/01||, ||T/10||)
\]
Clearly the elements of $\mathcal{E} \setminus (\mathcal{E}_a \cup \mathcal{E}_b)$ lie in one of the following sets:
\[
\mathcal{E}_1 = \{ T \in \mathcal{F} : ||T/01|| = ||T/10|| < ||T/001|| \}
\]
\[
\mathcal{E}_2 = \{ T \in \mathcal{F} : ||T/01|| = ||T/001|| < ||T/10|| \}
\]
\[
\mathcal{E}_3 = \{ T \in \mathcal{F} : ||T/01|| < \min(||T/10||, ||T/001||) \}
\]
It is therefore sufficient to show that each of these sets is marginalized by $x_0$ off $\mathcal{E}_a \cup \mathcal{E}_b$. As the argument is similar for each of these sets, I will focus on $\mathcal{E}_1$. Let $T$ be in $\mathcal{E}_1$ and suppose that $k > 0$ is such that for all $i < k$, $T \cdot x_0^i$ is defined and is not in $\mathcal{E}_a \cup \mathcal{E}_b$. Consider the sequence $n_i$ for $0 \leq i \leq k + 1$ defined by
\[
\begin{align*}
n_i &= ||T/0^i + 1|| = ||(T \cdot x_0^i)/01||
\end{align*}
\]
Since $T$ is in $\mathcal{E}_1$, $n_0 < n_1$. Also, if $0 < i < k$ and $n_{i-1} < n_i$, then $n_i < n_{i+1}$ since
\[
\begin{align*}
n_{i-1} &= ||(T \cdot x_0^i)/10||
\end{align*}
\]
\[
\begin{align*}
n_i &= ||(T \cdot x_0^i)/01||
\end{align*}
\]
\[
\begin{align*}
n_{i+1} &= ||(T \cdot x_0^i)/001||
\end{align*}
\]
and $T \cdot x_0^i$ is not in $\mathcal{E}_a \cup \mathcal{E}_b$. It follows inductively that $n_{i+1} > n_i$ for all $i < k$. Consequently, $T \cdot x_0^k$ is not in $\mathcal{E}_1$. This shows that $x_0$ marginalizes $\mathcal{E}_1$ off $\mathcal{E}_a \cup \mathcal{E}_b$.

\begin{lemma}
There is an $N$ such that if $T$ is in $\mathcal{F}$ and $T \cdot g$ is not in $\mathcal{E}$ for any $g$ with $d_g \leq N$, then one of the following inequalities holds:
\[
2||T/01|| \leq ||T/10||
\]
\[
2||T/10|| \leq ||T/01||
\]
\end{lemma}
Proof. Let \( c = x_0^2 x_2^{-1} x_0^{-1} \) and \( d = x_0^2 x_1^{-1} x_0^{-1} \), noting that \( c \) and \( d \) define the following maps:

\[
c = \begin{cases} 
000 & \mapsto 00 \\
001 & \mapsto 01 \\
01 & \mapsto 100 \\
10 & \mapsto 101 \\
11 & \mapsto 11 
\end{cases} \quad d = \begin{cases} 
000 & \mapsto 00 \\
001 & \mapsto 01 \\
01 & \mapsto 011 \\
10 & \mapsto 10 \\
11 & \mapsto 11 
\end{cases}
\]

If \( T \) satisfies \((+)\), then we have that
\[
2[|(T \cdot c)/01]| = 2[|T/001|] < |[T/01]| + |[T/10]| = |[(T \cdot c)/10]|.
\]
Similarly if \( T \) satisfies \((-)\), then we have that
\[
2[|(T \cdot d)/10]| = 2[|T/10|] < |[T/001]| + |[T/01]| = |[(T \cdot d)/01]|.
\]

Next observe that if \( g \) is a generator or its inverse (here it is important that the generators have the form specified in the introduction) and \( T \) satisfies \((+)\), then \( T \cdot g \) can not satisfy \((-)\) (and similarly with \((+)\) and \((-)\) interchanged). Let \( N \) be the maximum of the distances of \( c^{-1} \) and \( d^{-1} \) to the identity. If \( T \) is such that whenever \( d_g \leq N \), then \( T \cdot g \) is not in \( \mathcal{E} \), then either \( T \cdot c^{-1} \) and \( T \cdot d^{-1} \) both satisfy \((+)\) or both satisfy \((-)\). In the former case \(2|T/01| \leq |T/10| \) and in the latter case \(2|T/10| \leq |T/01| \). Hence \( N \) satisfies the conclusion of the lemma.

Lemma 5.8. There is a constant \( C \) such that if \( \mu \) is a weighted \( \varepsilon \)-Følner set of trees, then there is a weighted \( C\varepsilon \)-Følner set of trees which is supported on a subset of \( \partial T : \mu(T) > 0 \).

Proof. Let \( \mu \) be a given weighted \( \varepsilon \)-Følner set of trees. If \( N \) satisfies the conclusion of Lemma 5.7, let \( \mathcal{A}' \) be the set of all \( T \) in the support of \( \mu \) such that if \( g \) is in \( F \) and \( d_g \leq N + 2 \), then \( T \cdot g \) is not in \( \mathcal{E} \). By Lemmas 3.8 and 5.6 \( \mu \upharpoonright (\mathcal{T} \setminus \mathcal{E}) \) is \( C_0\varepsilon \)-Følner for some \( C_0 \). By applying Lemma 3.7 and the definition of \( \mu \upharpoonright (\mathcal{T} \setminus \mathcal{E}) \) being \( C_0\varepsilon \)-Følner, it follows that \( C \) can be chosen so that \( \mu \upharpoonright \mathcal{A}' \) is \( C\varepsilon \)-Følner. If \( T \) is in \( \mathcal{A}' \), then for each \(-1 \leq i \leq 2\), one of the following inequalities is satisfied:

\[
2[|(T \cdot x_0^{-i})/01]| \leq |[(T \cdot x_0^{-i})/10]| \\
2[|(T \cdot x_0^{-i})/10]| \leq |[(T \cdot x_0^{-i})/01]|.
\]

Moreover, since \( T \cdot x_0^{-i} \) is not in \( \mathcal{E} \) for \(-1 \leq i \leq 2\), it must be that each \( T \cdot x_0^{-i} \) satisfies the same inequality for all \(-1 \leq i \leq 2\). Hence if \( T \) is in \( \mathcal{A}' \) and \( i \in \{0, 1\} \), then

\[
\{000, 001, 01, 10, 110, 1110, 1111\} \subseteq \partial T
\]
and hence $L_{x_i} \subseteq \partial T$ and $L_{x_i}$ does not contain the end leaves of $\partial T$. By Lemma 5.4

$$(\partial T) \cdot x_i = \partial(T \cdot x_i)$$

for $i \in \{0, 1\}$. Applying Lemma 3.2 to $\mu \upharpoonright \mathcal{A}'$ and $h = \partial$ gives the desired conclusion.

Now we are ready to complete the proof of Theorem 1.1. This is proved by exhibiting

(*) If $A \subseteq F$ is a $C^{-n}$-Følner set, then $A$ contains an element with a tree diagram containing at least $\exp_n(0)$ leaves.

To see why this is sufficient, first recall that there is a linear relationship between the distance of an element $f$ of $F$ to the identity and the size of the trees occurring in a tree diagram for $f$ [3]. Also recall that every $\varepsilon$-Følner set contains a connected component which is an $\varepsilon$-Følner set. Furthermore, a (right) Følner set for $F$ can be assumed to contain the identity by replacing it by an appropriate left translate. Therefore (*) implies that, for some $C$ not depending on $\varepsilon$, both the diameter and the cardinality of a $C^{-n}$-Følner set are at least $\exp_n(0)$.

Now we will prove (*). By Lemmas 4.2 and 5.8 there is a constant $C$ such that:
- if $A \subseteq F$ is an $\varepsilon$-Følner set, then there is a weighted $C\varepsilon$-Følner set of trees $\mu$ with support contained in $\{R_f : f \in A\}$;
- if $\mu$ is a weighted $\varepsilon$-Følner set of trees, then there is a $C\varepsilon$-Følner set of trees $\nu$ which is supported on $\{\partial T : (\mu(T) > 0) \land (\partial T \text{ is non trivial})\}$

Notice that if $A \subseteq F$ is $C^{-n-1}$-Følner, then there is an $A' \subseteq A$ such that $\{R_f : f \in A'\}$ is $C^{-n}$-Følner and hence there is a $f$ in $A$ such that $\partial^n R_f$ is non trivial. Let $k_i = ||[\partial^{n-i} R_f]||$ and observe that by Lemma 5.3 $k_0 = 4$ and $k_{i+1} \geq 2k_i^{-2}$. It follows by induction that $\exp_i(0) + 2 < k_i$ and in particular that that $R_f$ contains at least $\exp_n(0)$ leaves. Finally, by adjusting the constant $C$ if necessary, we may arrange that if $A$ is $C^{-n}$-Følner, then there is an $f$ in $A$ such that $R_f$ has at least $\exp_n(0)$ leaves. This finished the proof of Theorem 1.1.

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Department of Mathematics, Cornell University, Ithaca, NY 14853–4201, USA

E-mail address: justin@math.cornell.edu