ON EXTENSION OF CLOSED COMPLEX (BASIC) DIFFERENTIAL FORMS: (BASIC) HODGE NUMBERS AND (TRANSVERSELY) \( p \)-KÄHLER STRUCTURES

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In Memory of Jean-Pierre Demailly (1957-2022)

Abstract. Inspired by a recent work of D. Wei–S. Zhu on the extension of closed complex differential forms and Voisin’s usage of the \( \partial \overline{\partial} \)-lemma, we obtain several new theorems of deformation invariance of Hodge numbers and reprove the local stabilities of \( p \)-Kähler structures with the \( \partial \overline{\partial} \)-property. Our approach is more concerned with the \( d \)-closed extension by means of the exponential operator \( e^{\iota \phi} \). Furthermore, we prove the local stabilities of transversely \( p \)-Kähler structures with mild \( \partial \overline{\partial} \)-property by adapting the power series method to the foliated case, which strengthens the works of A. El Kaïmi Alaoui–B. Gmira and P. Raźny on that of the transversely Kähler foliations with homologically orientability. We observe that a transversely Kähler foliation, even without homologically orientability, also satisfies the \( \partial \overline{\partial} \)-property. So even when \( p = 1 \) (transversely Kähler), our results are new as we can drop the assumption in question on the initial foliation. Several theorems on the deformation invariance of basic Hodge/Bott–Chern numbers with mild \( \partial \overline{\partial} \)-properties are also presented.

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1. Introduction

The classical deformation theory of compact complex manifolds, developed by Kodaira–Nirenberg–Spencer and Kuranishi [KNS58, KS60, Ku64, Ko86], intensively studies the complex structures ‘close to’ a given one. Subsequently, the theory had been extended to the case of complex pseudogroup structures such as [Ko60, KS61]. In particular, many excellent works concerning the deformations of transversely holomorphic foliations and holomorphic foliations appear, see [DK79, DK80, GM81, GHS83, EKA88, EKAN89, GN89, Gl92, EKAG97, BHTF21, Rz21], etc. Most of these works essentially dealt with the extension of closed complex (basic) differential forms.

As an important and direct application, one considers the deformation invariance of Hodge numbers. It becomes a useful tool in the study of the deformation limit problems. For instance, let \(\pi : X \to \Delta\) be a holomorphic family over an open disk in \(\mathbb{C}\). D. Popovici (resp. I-H. Tsai—the first author) showed that if the fiber \(X_t := \pi^{-1}(t)\) is projective (resp. Moishezon) for every \(t \in \Delta^\ast\) (resp. for each nonzero \(t\) in an uncountable subset of \(\Delta\)), and the reference fiber \(X_0\) satisfies the local deformation invariance for Hodge number of type \((0,1)\) or admits a strongly Gauduchon metric, then \(X_0\) is still Moishezon, cf. [Po13, RT21]. It is well known that each Hodge number takes a constant value along the small differentiable deformation \(X_t\) of \(X_0\) when the central fiber \(X_0\) satisfies the (standard) \(\bar{\partial}\bar{\partial}\)-property or more generally, the Frölicher spectral sequence of \(X_0\) degenerates at the \(E_1\)-level, cf. [Gi65, Section 5.1] or [Vo02, Proposition 9.20]. Recall that the \((\text{standard})\ \bar{\partial}\bar{\partial}\)-property refers to: for every pure-type \(d\)-closed form on a compact complex manifold, the properties of \(d\)-exactness, \(\bar{\partial}\)-exactness, \(\bar{\partial}\bar{\partial}\)-exactness and \(\partial\bar{\partial}\)-exactness are equivalent. So it is natural to study this topic with more general conditions on \(X_0\), such as some ‘weak’ \(\bar{\partial}\bar{\partial}\)-properties as stated in Subsection 2.1. Such results were first given in [ZR15, Rz18].

Recently, W. Xia studied this further in terms of canonical deformations [Xi21b, Theorem 1.3]. Drawing on Xia’s work and a recent work of Wei–Zhu [WZ20], we obtain several new theorems on the deformation invariance of Hodge numbers:

**Theorem 1.1** (\(-\text{Theorem 4.1}\). Let \(\pi : X \to B\) be the differentiable family of compact complex \(n\)-dimensional manifolds over a sufficiently small domain in \(\mathbb{R}^d\) as in Definition 3.1 with the central fiber \(X_0 := \pi^{-1}(0)\) and the general fibers \(X_t := \pi^{-1}(t)\). Consider the function

\[
B \ni t \mapsto h^{p,q}_{\bar{\partial}t}(X_t) := \dim_{\mathbb{C}} H^{p,q}_{\bar{\partial}t}(X_t), \quad \text{for any non-negative integers } p, q \leq n.
\]

If the injectivity of the mapping \(i^{p,q+1}_{BC,\bar{\partial}}\), the surjectivity of the mapping \(i^{p,q}_{BC,\bar{\partial}}\) on the central fiber \(X_0\) and the deformation invariance of the \((p,q-1)\)-Hodge number \(h^{p,q-1}_{\bar{\partial}t}(X_t)\) hold, then \(h^{p,q}_{\bar{\partial}t}(X_t)\) are independent of \(t\). See the notations \(i^{p,q+1}_{BC,\bar{\partial}}, i^{p,q}_{BC,\bar{\partial}}\) in Subsection 2.1.

Examples 4.2, 4.3, and 4.4 show that the deformation invariance of the \((p,q)\)-Hodge number fails when one of the three conditions in Theorem 1.1 is not satisfied, while
the other two hold, thanks to the Kuranishi family of the Iwasawa manifold (see [Ag13, Appendix] and [Sc07, Section 1.c]).

Following [RZ18, Notation 3.5], we say that a compact complex manifold $X$ satisfies $S^{p,q}$ (resp. $B^{p,q}$), if for any $\dbar$-closed $\partial g \in A^{p,q}(X)$, the equation

$$\dbar x = \partial g$$

has a solution (resp. a $\partial$-exact solution) of pure-type complex differential form. Similarly, a compact complex manifold $X$ is said to satisfy $S^{p,q}$ (resp. $B^{p,q}$), if for any $\dbar$-closed $g \in A^{p-1,q}(X)$, the equation (1.2) has a solution (resp. a $\partial$-exact solution) of pure-type complex differential form.

The speciality of the types may lead to the weakening of the conditions in Theorem 1.1, such as $(p,0)$ and $(0,q)$:

**Theorem 1.2 (=Theorems 4.6+4.7).** With the setting of Theorem 1.1,

1. if $X_0$ satisfies $B^{p,1}$ (i.e., the mapping $\iota_{BC,0}^{p,1}$ is injective) and $S^{p+1,0}$, then $h_{\partial \bar{\partial}}^{p,0}(X_t)$ are deformation invariant;
2. if $X_0$ satisfies $B^{1,q}$ (i.e., the mapping $\iota_{BC,0}^{0,q}$ is surjective) and $h_{\partial \bar{\partial}}^{0,q-1}(X_t)$ satisfies the deformation invariance, then $h_{\partial \bar{\partial}}^{0,q}(X_t)$ are independent of $t$.

Note that Theorem 1.2(2) first appeared as [RZ18, Theorem 3.7]. So we can also obtain [RZ18, Remark 3.8, Corollary 3.9] as a consequence. Furthermore, an example applicable to Theorem 1.2(1) in Remark 4.8 shows that the function of Theorem 1.2(1) for $(p,0)$-Hodge numbers goes beyond Kodaira–Spencer’s squeeze [KS60, Theorem 13] sometimes.

Before setting out the strategy to prove Theorems 1.1 and 1.2, we first briefly state some knowledge of analytic deformation theory of complex structures to be introduced in detail in Subsection 3.1. Let $\pi : X \to B$ be a differentiable family as aforementioned inducing a canonical differentiable family of integrable Beltrami differentials on $X_0$, denoted by $\varphi(z,t), \varphi(t)$, and $\varphi$ interchangeably. Now consider the exponential operator of contraction operators

$$e^{\varphi} := \sum_{k=0}^{\infty} \frac{t^k \varphi(t)}{k!},$$

and one can show that

$$e^{\varphi} : A^{p,0}(X_0) \to A^{p,0}(X_t).$$

However, $e^{\varphi}$ can’t preserve $(p,q)$-forms to $X_t$ when $1 \leq q \leq n$. A useful fact due to [FM09, Theorem 5.1] or [WZ20, Section 3.1] shows that it actually maps from a filtration on $X_0$ to one on $X_t$, that is

$$e^{\varphi} : F^p A^k(X_0) = \bigoplus_{p\leq s \leq k} A^{s,k-s}(X_0) \to F^p A^k(X_t).$$

According to [WZ20, Xi21b], one can define the projection operator

$$\mathcal{P}_\varphi : A^{0,1}(X_0) \to A^{0,1}(X_t).$$

We then use the exponential operator $e^{\varphi}$ and the projection operator $\mathcal{P}_\varphi$ to define the extension operator

$$\rho_{\varphi} : A^{p,q}(X_0) \to A^{p,q}(X_t)$$

as in (3.4). To prove the deformation invariance of Hodge numbers, Q. Zhao—the first author [ZR15, RZ18] introduced an extension map

$$e^{\varphi \rho_{\varphi}} : A^{p,q}(X_0) \to A^{p,q}(X_t),$$


which can preserve all \((p,q)\)-forms and played an important role in their subsequent papers [RWZ19, RWZ21]. By comparing the explicit local expressions, one can deduce the relationship between these two extension maps:

\[
\rho_{\varphi} = e^{\varphi} |\nabla| \circ (1 - \overline{\varphi})^{-1} \partial, 
\]

where the notation \(\partial\) denotes the simultaneous contraction on each component of a complex differential form.

An application of Kuranishi’s completeness theorem [Ku64] can reduce our Theorems 1.1 and 1.2 to the Kuranishi family, which contains all sufficiently small differentiable deformations \(X_0\) in some sense (see Subsection 3.2). So to prove them, we just need to consider a Kuranishi family of deformations of \(X_0\) over \(\Delta_\varepsilon := \{ t \in \mathbb{C} \mid |t| \leq \varepsilon \}\) with small \(\varepsilon\), and there exists a family of integrable Beltrami differentials \(\{\varphi(t)\}_{|t| \leq \varepsilon}\) depending on \(t\) holomorphically and describing the variations of complex structures on \(X_0\).

Recently, Wei–Zhu [WZ20] applied the \(\partial \overline{\partial}\)-Hodge theory to extend a \(d\)-closed \((p,q)\)-form on \(X_0\) to a \(d\)-closed filtrated \((p+q)\)-form on \(X_t\), whose \((p, q)\)-part on \(X_t\) is \(\overline{\partial}\)-closed via the extension operator \(\rho_{\varphi}\) in (1.3). This is surely an interesting and important result. We will reprove their theorem in Subsection 3.3, from the perspective of Bott–Chern theory.

**Theorem 1.3** ([WZ20, Theorem 1.1]). Let \(X_0\) be a compact complex manifold that satisfies \(\mathbb{B}^{p,q+1}\). Given a \(d\)-closed \(\mu_0 \in A^{p,q}(X_0)\), one can construct \(\mu(z, t) \in A^{p,q}(X_0)\) satisfying \(d(e^{\varphi}(\mu(z, t))) = 0\) on \(X_0\) (or \(X_t\) and \(\mu(z,0) = \mu_0(z)\), which is holomorphic in small \(t\). Furthermore, the extension \(\rho_{\varphi}(\mu(z, t)) \in A^{p,q}(X_t)\) is \(\overline{\partial}\_t\)-closed.

Now we are ready to describe our strategy to consider the deformation invariance of Hodge numbers briefly. The Kodaira–Spencer’s upper semi-continuity theorem ([KS60, Theorem 4]) tells us that the function (1.1) is always upper semi-continuous for \(t \in \Delta_\varepsilon\) and thus, to approach the deformation invariance of \(h^{p,q}_{\overline{\partial}}(X_t)\), we only need to obtain the lower semi-continuity. And, our main strategy is to look for an injective extension map from \(H^{p,q}_{\overline{\partial}}(X_0)\) to \(H^{p,q}_{\overline{\partial}}(X_t)\) with the help of Theorem 1.3. More precisely, we first need to find a nice uniquely-chosen \(d\)-closed representative \(\mu_0\) of the given initial Dolbeault cohomology class in \(H^{p,q}_{\overline{\partial}}(X_0)\) (see Lemma 4.10), and then apply power series method and Bott–Chern theory to construct \(\mu(z, t) \in A^{p,q}(X_0)\), such that it is smooth in \((z, t)\) and holomorphic in small \(t\) and \(\rho_{\varphi}(\mu(z, t))\) is \(\overline{\partial}\_t\)-closed in \(A^{p,q}(X_t)\) by Theorem 1.3. Finally, we try to verify that the extension map

\[
H^{p,q}_{\overline{\partial}}(X_0) \rightarrow H^{p,q}_{\overline{\partial}}(X_t) : [\mu_0]_{\overline{\partial}} \mapsto [\rho_{\varphi}(\mu(z, t))]_{\overline{\partial}_t}
\]

is injective.

Our Theorem 1.1 and the first assertion of Theorem 1.2 on the deformation invariance of Hodge numbers are different from the results in [RZ18], cf. Remarks 4.5, 4.13, and 4.14. Our approach focuses more on the \(d\)-closed extension, while they concentrated on the specific \(\overline{\partial}\)-extension from \(A^{p,q}(X_0)\) to \(A^{p,q}(X_t)\) (see [RZ18, Proposition 1.2]) by use of their extension map (1.4).

As another application of the extension of closed complex differential forms in Theorem 1.3, we study the local stabilities of several special complex structures. Inspired by the proof of [Vo02, Theorem 9.23], we can take advantage of Theorem 1.3 and also the deformation openness of the \(\partial \overline{\partial}\)-property to prove the local stabilities of \(p\)-Kähler structures (for this concept, one can refer to Appendix A for more details) with the \(\partial \overline{\partial}\)-property.
Theorem 1.4 ([RWZ21, Theorem 4.9]). For any positive integer \( p \leq n - 1 \), any small differentiable deformation \( X_t \) of a \( p \)-Kähler manifold \( X_0 \) satisfying the \( \partial\overline{\partial} \)-property is still \( p \)-Kählerian.

Corollary 1.5. Let \( \pi : \mathcal{X} \to B \) be a differentiable family of compact complex manifolds.

(i) ([KS60, Theorem 15]) If a fiber \( X_0 := \pi^{-1}(t_0) \) admits a Kähler metric, then for a sufficiently small neighborhood \( U \) of \( t_0 \) on \( B \), the fiber \( X_t := \pi^{-1}(t) \) over any point \( t \in U \) still admits a Kähler metric, which coincides for \( t = t_0 \) with the given Kähler metric on \( X_0 \).

(ii) ([Wu06, Theorem 5.13]) Let the fiber \( X_0 \) be a balanced manifold and satisfy the \( \partial\overline{\partial} \)-property. Then \( X_t \) also admits a balanced metric for small \( t \).

Notice that [RWZ21] presented a power series proof for Kodaira–Spencer’s local stabilities theorem of Kähler structures via their extension map (1.4), which is a problem that dates back to [MK71, Remark 1 on p. 180]: ‘A good problem would be to find an elementary proof (for example, using power series methods). Our proof uses nontrivial results from partial differential equations’. In our proof, our primary focus is on the d-closed extension, by means of the exponential operator \( e^{\varphi} \), which is more natural and succinct in some sense. However, we still need to use [KS60, Theorem 7] to guarantee the positivity of the constructed explicit \( p \)-Kähler form, see Subsection 5.2 for more details.

A challenge problem proposed by [WZ20] is how to prove the first assertion of Corollary 1.5 without using [KS60, Theorem 7].

It is worth mentioning that in [RWZ19], X. Wan–Q. Zhao—the first author looked deeper into the local stabilities of \( p \)-Kähler structure when the central fiber \( X_0 \) satisfies some ‘weak’ \( \partial\overline{\partial} \)-properties. More concretely, for any positive integer \( p \leq n - 1 \), any small differentiable deformation \( X_t \) of an \( n \)-dimensional \( p \)-Kähler manifold \( X_0 \) satisfying the \( (p, p + 1) \)-th mild \( \partial\overline{\partial} \)-property is still \( p \)-Kählerian ([RWZ19, Theorem 1.1]). However, in our approach, it seems that the standard \( \partial\overline{\partial} \)-property condition on \( X_0 \) in Theorem 1.4 can’t be weakened, see Remark 5.5.

As is well known, the Bott–Chern numbers are always upper semi-continuous with respect to the ordinary topology in a small differentiable family. Since the ordinary topology is much finer than the analytic Zariski topology, it’s natural to ask

Question 1.6 (=Question 5.8). For a holomorphic family \( \{X_t\} \), is the function

\[
B \ni t \mapsto h^{p,q}_{BC}(X_t) := \dim_{\mathbb{C}} H^{p,q}_{BC}(X_t)
\]

upper semi-continuous with respect to the analytic Zariski topology?

To solve this question, we wish to use Grauert’s upper semi-continuity theorem 5.7. In other words, one needs to find some holomorphic vector bundle \( V \) on \( \mathcal{X} \), such that \( H^{p,q}_{BC}(X_t) \simeq H^{q}(X_t, V_{|X_t}) \). This seems hard to achieve due to the results in [Bg69, AN71], see Subsection 5.3 for more details. Recently, Xia [Xi21a, Theorem 1.1 and Remark 3.6] confirmed Question 1.6 when the type is \( (p, 0) \) or \( (0, q) \). One motivation to affirm Question 1.6 is to obtain Theorem 5.10, as long as one notices that the \( \partial\overline{\partial} \)-property in Theorem 1.4 actually can be replaced by the deformation invariance of \( (p, p) \)-Bott–Chern numbers as shown in [RWZ21, Remark 4.13].

As is widely recognized, the transversely Kähler structures hold a central position within the field of foliation theory and are closely linked to a wealth of geometric structures. For instance, Vaisman manifolds, LVM manifolds (a generalized version of Calabi–Eckmann manifolds), and Sasakian manifolds all possess transversely Kähler structures despite not being Kähler themselves, see [Me00, BG08, MSY08, No08, FOW09, OV22], etc. Recently, the transversely balanced structures (or, more generally, the transversely
Gauduchon structures, see Remark 6.11), are also actively studied, as evidenced by [FZ19, BH22], etc. In light of this, it naturally becomes a logical progression to extend the concept of compact $p$-Kähler manifolds, as originally introduced by L. Alessandrini–M. Andreatta [AA87, Definition 1.11], to the transverse context. This extension lends us to the introduction of \textit{transversely $p$-Kähler foliations} (Definition 6.9). Remarkably, this overarching notion unifies the two aforementioned structures, specifically when $p$ takes values of $1$ and $r-1$ (in the context of a transversely holomorphic foliation of codimension $r$), respectively (Remark 6.11). When delving into the intermediate cases ($1 < p < r-1$), a thought-provoking question naturally arises: Does there exist a non-trivial (specifically, an analytic) deformation of $(M,\mathcal{F})$ for every $t$ sufficiently close to $0$, depending differentiably on $t$ with $\sigma_0 = \sigma$. See also [Rż21, Theorem 5.2]. One can refer to Subsections 6.1–6.5 for relevant definitions. Notice that if the deformation $\mathcal{F}_t$ is arbitrary, then the above assertion doesn’t hold, due to the example built in [EKAN90], namely an analytic family $\{\mathcal{F}_t\}_{t \in \mathbb{C}}$ of holomorphic foliations on a compact complex nilmanifold such that $\mathcal{F}_0$ is transversely Kähler, but for any $t \neq 0$, $\mathcal{F}_t$ has no transversely Kähler structure. See also the discussion in [Rż21, Section 6] based on the examples from [No08, GNT16].

Inspired by [RWZ19], we extend the ‘weak’ $\partial \bar{\partial}$-properties to the foliated version (Definition 6.22), and then prove the following local stabilities theorem by adapting the power series method to our setting. The main ingredients in this proof are the transversely elliptic operator theory initiated from [EKAN90] and the Kuranishi family constructed by [EKAN89], see Subsection 6.6 for more details.

\textbf{Theorem 1.7.} Let $\{\mathcal{F}_t\}_{t \in U}$ be a smooth family of transversely Hermitian structures on a compact foliated manifold $(M,\mathcal{F})$ with fixed differentiable type ($\mathcal{F}$ is of complex codimension $r$), parametrized by an open neighborhood $U$ of $0$ in $\mathbb{R}^d$. If $\mathcal{F} = \mathcal{F}_0$ is transversely $p$-Kähler and satisfies the $(p, p+1)$-th mild $\partial \bar{\partial}$-property for $1 \leq p \leq r-1$, then $\mathcal{F}_t$ is also transversely $p$-Kähler for every $t$ sufficiently close to $0$.

This result strengthens the works of [EKAG97, Rż21] on that of the transversely Kähler foliations with homologically orientability, and can be viewed as a generalization of [RWZ19, Theorem 1.1] to the foliated case. In a certain sense, the approach we have adopted has resulted in greater efficiency compared to the methods used in [EKAG97, Rż21], cf. Remark 6.35 (a).

Notice that even when $p = 1$ (transversely Kähler) Theorem 1.7 is new.

\textbf{Corollary 1.8.} Let $\{\mathcal{F}_t\}_{t \in U}$ be a smooth family of transversely Hermitian structures on a compact foliated manifold $(M,\mathcal{F})$ with fixed differentiable type, parametrized by an open neighborhood $U$ of $0$ in $\mathbb{R}^d$. If $\mathcal{F} = \mathcal{F}_0$ is transversely Kähler, then $\mathcal{F}_t$ is also transversely Kähler for every $t$ sufficiently close to $0$. 6
Compared with the previous works [EKAG97, Rż21], Corollary 1.8 can drop the homologically orientability assumption. To this end, we observe that a transversely Kähler foliation, even without homologically orientability, also satisfies the $\partial\overline{\partial}$-property and, therefore, satisfies $(1,2)$-th mild $\partial\overline{\partial}$-property, cf. Subsection 6.4 and Remark 6.35 (b).

Finally, several theorems concerning the deformation invariance of basic Hodge/Bott–Chern numbers with ‘weak’ $\bar{\partial}\partial$-properties are displayed, see Theorem 6.37. In these theorems, the homologically orientability assumption is necessary, as indicated in Subsection 6.7. We also require the foliation to remain unchanged in this context. For further insights, one can refer to Question 6.38 posed by Raźny (and also the preceding discussions) regarding the invariance of basic Hodge numbers under arbitrary deformations of transversely Kähler foliations.

Convention 1.9. All compact complex (or smooth) manifolds in this paper are assumed to be connected unless mentioned otherwise and $\pi : X \to B$ will always denote a differentiable or holomorphic family of $n$-dimensional compact complex manifolds, whose central fiber is $(X_0, z)$ with local holomorphic coordinates $z := (z^i)_{i=1,...,n}$ and general fiber is $X_t := \pi^{-1}(t)$.

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2. Variations of $\partial\overline{\partial}$-property and $\partial\overline{\partial}$-equation

In this section, we collect some basics to be used later. Throughout this section, we will always denote by $X$ a compact complex manifold of complex dimension $n$.

2.1. Cohomology groups and variations of $\partial\overline{\partial}$-property. We will often use the commutative diagram:

Recall that Dolbeault cohomology groups $H^{\bullet,\bullet}_\partial(X)$ of $X$ are defined by:

$$H^{\bullet,\bullet}_\partial(X) := \frac{\ker \partial}{\text{im } \partial},$$

with $H^{\bullet,\bullet}_\partial(X)$ likewise defined, while Bott–Chern and Aeppli cohomology groups are defined as

$$H^{\bullet,\bullet}_{BC}(X) := \frac{\ker \partial \cap \ker \overline{\partial}}{\text{im } \partial + \text{im } \overline{\partial}}$$

and

$$H^{\bullet,\bullet}_A(X) := \frac{\ker \partial \cap \ker \overline{\partial}}{\text{im } \partial + \text{im } \overline{\partial}}$$
respectively. The dimensions of $H^{p+q}_{\partial\bar{\partial}}(X)$, $H^{p,q}_{\partial\bar{\partial}}(X)$, $H^{p,q}_{\partial}(X)$, $H^{p,q}_{\partial}(X)$ and $H^{p,q}_{\bar{\partial}}(X)$ over $\mathbb{C}$ are denoted by $b_{p+q}(X)$, $\overline{b}_{p,q}(X)$, $b_{p,q}(X)$, $b_{A}(X)$ and $\overline{b}_{\bar{\partial}}(X)$, respectively, and the first four of them are usually called the $(p+q)$-th Betti numbers, $(p, q)$-th Hodge numbers, Bott–Chern numbers and Aeppli numbers of $X$, respectively. From the very definition of these cohomology groups, the following equalities clearly hold

$$h_{BC}^{p,q} = h_{BC}^{q,p} = h_{A}^{n-q,n-p} = h_{A}^{n-p,n-q} = h_{\bar{\partial}}^{p,q} = h_{\partial}^{q,p} = h_{\partial}^{n-q,n-p}.$$  

So the (standard) $\partial\bar{\partial}$-property, which means for every pure-type $d$-closed form on a compact complex manifold, the properties of $d$-exactness, $\partial$-exactness and $\partial\bar{\partial}$-exactness are equivalent, is equivalent to the following mappings

$$i_{BC,dR}^{p,q}: H^{p,q}_{\partial}(X) \rightarrow H^{p+q}_{\partial\bar{\partial}}(X)$$

are injective for all $p, q$, or to the isomorphisms of all the maps in Diagram (2.1) by [DGMS75, Remark 5.16].

Recall that a compact complex manifold $X$ satisfies $\mathbb{S}^{p,q}$ (resp. $\mathbb{B}^{p,q}$) if for any $\partial$-closed $\partial g \in A^{p,q}(X)$, the equation (1.2) has a solution (resp. a $\partial$-exact solution) of pure-type differential form. Similarly, a compact complex manifold $X$ is said to satisfy $\mathbb{S}^{p,q}$ (resp. $\mathbb{B}^{p,q}$), if for any $\partial$-closed $g \in A^{p,q}(X)$, the equation (1.2) has a solution (resp. a $\partial$-exact solution) of pure-type complex differential form. It is easy to verify the following implications:

$$\begin{align*}
\mathbb{B}^{p,q} &\Rightarrow \mathbb{S}^{p,q} \\
\downarrow & \\
\mathbb{B}^{p,q} &\Rightarrow \mathbb{S}^{p,q}.
\end{align*}$$

And it is apparent that a compact complex manifold $X$, where the $\partial\bar{\partial}$-property holds, satisfies $\mathbb{B}^{p,q}$ for any $(p, q)$.

It is easy to check that the following statements are equivalent:

1. the injectivity of $i_{BC,dR}^{p,q}$ holds on $X \iff X$ satisfies $\mathbb{B}^{p,q}$;
2. the injectivity of $i_{A,dR}^{p,q}$ holds on $X \iff X$ satisfies $\mathbb{S}^{p,q}$;
3. the surjectivity of $i_{BC,dR}^{p-1,q}$ holds on $X \iff X$ satisfies $\mathbb{B}^{p,q}$.

2.2. Hodge theory on compact complex manifolds: Bott–Chern. Let $X$ be a compact complex manifold. The Bott–Chern cohomology group has been introduced in (2.2). Also, the Bott–Chern Laplacian is given by

$$\Box_{BC} := \partial\partial\bar{\partial} \partial^* + \bar{\partial} \bar{\partial}^* \partial \bar{\partial} + \partial \partial^* \partial \bar{\partial} + \partial^* \partial \partial \partial^* \partial + \partial^* \partial \bar{\partial} \partial^*,$$

and $G_{BC}$ is the associated Green’s operator of this fourth order Kodaira–Spencer operator. Then we have the Hodge decomposition of $\Box_{BC}$ on $X$:

$$A^{p,q}(X) = \ker \Box_{BC} \oplus \ker (\partial \bar{\partial}) \oplus (\partial^* \partial^* + \partial \bar{\partial})$$

whose three parts are orthogonal to each other with respect to the $L^2$-scalar product defined by a Hermitian metric on $X$, combined with the equality

$$1 = \mathbb{H}_{BC} + \Box_{BC} G_{BC} = \mathbb{H}_{BC} + G_{BC} \Box_{BC},$$

where $\mathbb{H}_{BC}$ is the harmonic projection operator. And it should be noted that

$$\ker \Box_{BC} = \ker \partial \cap \ker \bar{\partial} \cap \ker (\partial \bar{\partial})^*.$$

We get the following two observations:

1. $\Box_{BC} (\partial \bar{\partial})^* = \partial \bar{\partial} (\partial \bar{\partial})^* \Box_{BC}$;
2. $G_{BC} (\partial \bar{\partial})^* = \partial \bar{\partial} (\partial \bar{\partial})^* G_{BC}$.

For the resolution of $\partial \bar{\partial}$-equations, we need a crucial lemma.
Lemma 2.1 ([Po15, Theorem 4.1]). Let \((X, \omega)\) be a compact Hermitian complex manifold with a pure-type complex differential form \(\alpha\). Assume that the \(\partial \bar{\partial}\)-equation
\[
\partial \bar{\partial} x = \alpha
\]
admits a solution of pure-type complex differential form. Then an explicit solution of the \(\partial \bar{\partial}\)-equation (2.4) can be chosen as
\[
(\partial \bar{\partial})^* G_{BC} \alpha,
\]
which uniquely minimizes the \(L^2\)-norms of all the solutions with respect to \(\omega\). Besides, the equalities hold
\[
G_{BC}(\partial \bar{\partial}) = (\partial \bar{\partial}) G_A \quad \text{and} \quad (\partial \bar{\partial})^* G_{BC} = G_A (\partial \bar{\partial})^*,
\]
where \(G_{BC}\) and \(G_A\) are the associated Green's operators of \(\Box_{BC}\) and \(\Box_A\), respectively. Here \(\Box_{BC}\) is defined in (2.3) and \(\Box_A\) is the second Kodaira–Spencer operator (often also called Aeppli Laplacian)
\[
\Box_A = \partial^* \partial \partial \partial + \bar{\partial} \partial \partial^* \partial^* + \partial^* \bar{\partial} \partial^* + \partial \partial^* + \partial \partial^*.
\]

3. Extension of closed forms: Bott–Chern approach

Wei–Zhu [WZ20] applied the \(\partial \bar{\partial}\)-Hodge theory to extend a \(d\)-closed \((p, q)\)-form on \(X_0\) to a \(d\)-closed filtrated \((p + q)\)-form on \(X_i\), whose \((p, q)\)-part on \(X_i\) is \(\partial \bar{\partial}\)-closed by type projection. We will reprove their theorem in this section by applying Bott–Chern theory.

3.1. Beltrami differentials and extension map. By a holomorphic family \(\pi : X \to B\) of compact complex manifolds from a complex manifold to a connected complex manifold, we mean that \(\pi\) is a proper and surjective holomorphic submersion, as in [Ko86, Definition 2.8]; while for differentiable one, we adopt:

Definition 3.1 ([Ko86, Definition 4.1]). Let \(X\) be a differentiable manifold, \(B\) a domain of \(\mathbb{R}^d\) and \(\pi\) a smooth map of \(X\) onto \(B\). By a differentiable family of \(n\)-dimensional compact complex manifolds we mean the triple \(\pi : X \to B\) satisfying the following conditions:

(a) The rank of the Jacobian matrix of \(\pi\) is equal to \(d\) at every point of \(X\);
(b) For each point \(t \in B\), \(\pi^{-1}(t)\) is a compact connected subset of \(X\);
(c) \(\pi^{-1}(t)\) is the underlying differentiable manifold of the \(n\)-dimensional compact complex manifold \(X_t\) associated to each \(t \in B\);
(d) There is a locally finite open covering \(\{U_j \mid j = 1, 2, \ldots\}\) of \(X\) and complex-valued smooth functions \(\zeta_j^1(p), \ldots, \zeta_j^n(p)\), defined on \(U_j\) such that for each \(t\),
\[
\{p \mapsto (\zeta_j^1(p), \ldots, \zeta_j^n(p)) \mid U_j \cap \pi^{-1}(t) \neq \emptyset\}
\]
form a system of local holomorphic coordinates of \(X_t\).

Beltrami differential plays an important role in deformation theory. For a compact complex manifold \(X\), we call an element in \(A^{0,1}(X, T_X^{1,0})\) a Beltrami differential, where \(T_X^{1,0}\) denotes the holomorphic tangent bundle of \(X\). Then \(\iota_{\varphi}\) or \(\varphi_{\perp}\) denotes the contraction operator with \(\varphi \in A^{0,1}(X, T_X^{1,0})\) alternatively if there is no confusion. One similarly follows the notation
\[
e^{-\Box} = \sum_{k=0}^{\infty} \frac{1}{k!} \Box^k,
\]
where \(\Box^k\) denotes \(k\)-time action of the operator \(\Box\). The summation in the above formulation is often finite since the dimension of \(X\) is finite.

We will always consider the differentiable family \(\pi : X \to B\) of compact complex \(n\)-dimensional manifolds over a sufficiently small domain in \(\mathbb{R}^d\) with the reference fiber
$X_0 := \pi^{-1}(0)$ and the general fibers $X_t := \pi^{-1}(t)$. For simplicity we set $d = 1$. Denote by  $\zeta := (\zeta^0_j(z, t))$ the holomorphic coordinates of $X_t$ induced by the family with the holomorphic coordinates $z := (z^i)$ of $X_0$, under a coordinate covering $\{U_i\}$ of $X$, when $t$ is assumed to be fixed, as the standard notions in deformation theory described at the beginning of [MK71, Chapter 4]. This family induces a canonical differentiable family of integrable Beltrami differentials on $X_0$, denoted by $\varphi(z, t), \varphi(t)$ and $\varphi$ interchangeably.

In the sequel we will state some explicit computations as presented in [MK71, Chapter 4.1] or [RZ18, Section 2.1]. A Beltrami differential can be written as

\[
\varphi(t) = \left( \frac{\partial}{\partial z} \right)^T \left( \frac{\partial \zeta}{\partial z} \right)^{-1} \partial \zeta,
\]

where $\partial = \left( \begin{array}{c} \frac{\partial}{\partial z^1} \\ \vdots \\ \frac{\partial}{\partial z^n} \end{array} \right)$, $\partial \zeta = \left( \begin{array}{c} \partial \zeta^1 \\ \vdots \\ \partial \zeta^n \end{array} \right)$, $\frac{\partial}{\partial z}$ stands for the matrix $\left( \frac{\partial \zeta^\alpha}{\partial z^j} \right)_{1 \leq \alpha \leq n}$ and $\alpha, j$ are the row and column indices. Here $(\frac{\partial}{\partial z})^T$ is the transpose of $\frac{\partial}{\partial z}$ and $\partial$ denotes the Cauchy–Riemann operator with respect to the holomorphic structure on $X_0$. Locally, $\varphi(t)$ is expressed as $\varphi \cdot dz^i \otimes \frac{\partial}{\partial z^i} \in \mathcal{A}^{0,1}(X_0, T^0_{X_0})$, so it can be considered as a matrix $\left( \varphi^i_j \right)_{1 \leq i \leq n}$. By (3.1), this matrix can be explicitly written as

\[
\varphi = \left( \varphi^i_j \right)_{1 \leq i \leq n} = \varphi(t) \left( \frac{\partial}{\partial z^i}, dz^i \right) = \left( \left( \frac{\partial \zeta}{\partial z} \right)^{-1} \left( \frac{\partial \zeta}{\partial z} \right) \right)^i_j.
\]

A fundamental fact is that the Beltrami differential $\varphi(t)$ defined as above satisfies the integrability:

\[
\partial \varphi(t) = \frac{1}{2} [\varphi(t), \varphi(t)],
\]

and we then call $\varphi(t)$ an \textit{integrable Beltrami differential}. Now consider the exponential operator of contraction operators

\[
ed^\varphi := \sum_{k=0}^{\infty} \frac{\varphi(t)}{k!}.
\]

A calculation shows that

\[
ed^\varphi \left( dz^i \wedge \cdots \wedge dz^p \right) = (dz^i + \varphi(t)_i dz^i) \wedge \cdots \wedge (dz^p + \varphi(t)_p dz^p).
\]

Then one obtains

\[
ed^\varphi : \mathcal{A}^{p,0}(X_0) \longrightarrow \mathcal{A}^{p,0}(X_t)
\]

by

\[
d \zeta^\beta = \frac{\partial \zeta^\beta}{\partial z^i} dz^i + \frac{\partial \zeta^\beta}{\partial \zeta^j} dz^j = \frac{\partial \zeta^\beta}{\partial z^i} e^\varphi (dz^i)
\]

according to (3.2). However, $e^\varphi$ can’t preserve $(p, q)$-forms for $1 \leq q \leq n$. An interesting observation in [FM09, Theorem 5.1] or [WZ20, Section 3.1] tells that it in fact maps from a filtration on $X_0$ to one on $X_t$, i.e.,

\[
ed^\varphi : \mathcal{F}^p A^k(X_0) = \bigoplus_{p \leq q \leq k} A^{s,k-s}, X_0) \longrightarrow \mathcal{F}^p A^k(X_t).
\]

Actually, for $\sigma \in A^{s,k-s}(X_0)$ ($p \leq s \leq k$) with the local expression

\[
\sigma = \sigma_{i_1 j_1 \cdots i_s j_s} dz^{i_1} \wedge \cdots \wedge dz^{i_s} \wedge dz^{j_1} \wedge \cdots \wedge dz^{j_s},
\]

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one has
\[
e^{\phi}(\sigma) = \sigma_{i_1 \cdots i_p j_1 \cdots j_q} e^{\phi} \left( dz^{i_1} \wedge \cdots \wedge dz^{i_p} \right) \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q} \\
= \sigma_{i_1 \cdots i_p j_1 \cdots j_q} \left( dz^{i_1} + \varphi(t) \wedge dz^{i_1} \right) \wedge \cdots \wedge \left( dz^{i_p} + \varphi(t) \wedge dz^{i_p} \right) \\
\wedge \left( \frac{\partial \bar{z}^{j_1}}{\partial \zeta^\beta} d\zeta^\beta + \frac{\partial \bar{z}^{j_q}}{\partial \bar{\zeta}^\beta} d\bar{\zeta}^\beta \right),
\]
which clearly belongs to $F^pA^q(X_t)$. Next, we will state the definitions of the projection operator and the extension operator, originating from [WZ20, Xi21b]. For $\sigma \in A^{0,1}(X_0)$ with the local expression $\sigma = \sigma_j d\bar{z}^j$, one defines the projection operator
\[
\mathcal{P}_\varphi : A^{0,1}(X_0) \rightarrow A^{0,1}(X_t)
\]
by
\[
(3.3) \quad \mathcal{P}_\varphi (\sigma) = \mathcal{P}_\varphi \left( \sigma_j \left( \frac{\partial \bar{z}^j}{\partial \zeta^\beta} d\zeta^\beta + \frac{\partial \bar{z}^j}{\partial \bar{\zeta}^\beta} d\bar{\zeta}^\beta \right) \right) := \sigma_j \left( (1 - \varphi \bar{\varphi})^{-1} \right)_k \left( d\bar{z}^k + \varphi(t) \wedge d\bar{z}^k \right),
\]
where $\varphi \bar{\varphi} = \bar{\varphi} \varphi, \varphi \varphi = \varphi \bar{\varphi}$ and $I$ is the identity matrix. Since $\varphi(t)$ is a well-defined, global $(1,0)$ vector valued $(0,1)$-form on $X_0$, $\mathcal{P}_\varphi$ is globally well-defined thanks to the last equality in (3.3) from [RZ18, (2.18)]. We then use the exponential operator $e^{\phi}$ and the projection operator $\mathcal{P}_\varphi$ to define the extension operator
\[
\rho_\varphi : A^{p,q}(X_0) \rightarrow A^{p,q}(X_t).
\]
Concretely, for $\sigma \in A^{p,q}(X_0)$ with the local expression
\[
\sigma = \sigma_{i_1 \cdots i_p j_1 \cdots j_q} \left( dz^{i_1} \wedge \cdots \wedge dz^{i_p} \right) \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q},
\]
one has
\[
(3.4) \quad \rho_\varphi (\sigma) := \sigma_{i_1 \cdots i_p j_1 \cdots j_q} e^{\phi} \left( dz^{i_1} \wedge \cdots \wedge dz^{i_p} \right) \wedge \mathcal{P}_\varphi (d\bar{z}^{j_1}) \wedge \cdots \wedge \mathcal{P}_\varphi (d\bar{z}^{j_q}) \\
= \sigma_{i_1 \cdots i_p j_1 \cdots j_q} \left( dz^{i_1} + \varphi(t) \wedge dz^{i_1} \right) \wedge \cdots \wedge \left( dz^{i_p} + \varphi(t) \wedge dz^{i_p} \right) \\
\wedge \left( (1 - \varphi \bar{\varphi})^{-1} \right)_k \left( d\bar{z}^k + \varphi(t) \wedge d\bar{z}^k \right) \wedge \cdots \wedge \left( (1 - \varphi \bar{\varphi})^{-1} \right)_k \left( d\bar{z}^k + \varphi(t) \wedge d\bar{z}^k \right).
\]
Likewise, $\rho_\varphi$ is also globally well-defined.

In [ZR15, RZ18], Zhao–the first author introduced an extension map
\[
e^{\phi|\tau} : A^{p,q}(X_0) \rightarrow A^{p,q}(X_t),
\]
which can preserve all $(p,q)$-forms and played an important role in their subsequent papers [RWZ19, RWZ21]. More precisely, with the above notations, for $\sigma \in A^{p,q}(X_0)$, one defines [RZ18, Definition 2.8]:
\[
(3.5) \quad e^{\phi|\tau}(\sigma) = e_{i_1 \cdots i_p j_1 \cdots j_q} e^{\phi} \left( dz^{i_1} \wedge \cdots \wedge dz^{i_p} \right) \wedge e^{\tau} \left( d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q} \right) \\
= \sigma_{i_1 \cdots i_p j_1 \cdots j_q} \left( dz^{i_1} + \varphi(t) \wedge dz^{i_1} \right) \wedge \cdots \wedge \left( dz^{i_p} + \varphi(t) \wedge dz^{i_p} \right) \\
\wedge \left( d\bar{z}^{j_1} + \varphi(t) \wedge d\bar{z}^{j_1} \right) \wedge \cdots \wedge \left( d\bar{z}^{j_q} + \varphi(t) \wedge d\bar{z}^{j_q} \right).
\]
For example, for \( \sigma \in A^{p,q}(X_0) \) with the above notation, \((1 - \bar{\phi} \varphi + \bar{\varphi}) \downarrow \sigma \) means that the operator \((1 - \bar{\phi} \varphi + \bar{\varphi}) \) acts on \( \sigma \) simultaneously as:

\[
(1 - \bar{\phi} \varphi + \bar{\varphi}) \downarrow (\sigma_{i_1 \ldots i_p j_1 \ldots j_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q})
\]

\[
= \sigma_{i_1 \ldots i_p j_1 \ldots j_q} (1 - \bar{\phi} \varphi + \bar{\varphi}) \downarrow dz^{i_1} \wedge \cdots \wedge (1 - \bar{\phi} \varphi + \bar{\varphi}) \downarrow d\bar{z}^{j_q}.
\]

By comparing (3.4) and (3.5), we get the relationship between the two extension maps:

\[
(3.6) \quad \rho_{\varphi} = e^{i \varphi} \tau \circ (1 - \bar{\varphi} \varphi)^{-1} \downarrow.
\]

Zhao—the first author obtained the \( \bar{\partial} \)-extension obstruction for \((p, q)\)-forms of the smooth family via the extension map \( e^{i \varphi} \tau \) (cf. [RZ18, Proposition 2.13]), namely

\[
(3.7) \quad \bar{\partial}_t (e^{i \varphi} \tau (\sigma)) = e^{i \varphi} \tau (1 - \bar{\varphi} \varphi)^{-1} \downarrow ([\partial_t, \varphi] + \bar{\partial}) (1 - \bar{\varphi} \varphi) \downarrow \sigma),
\]

where \( \sigma \in A^{p,q}(X_0) \). One concludes that (3.7) is equivalent to the following assertion due to (3.6):

\[
\rho_{\varphi}^{-1} \bar{\partial}_t \rho_{\varphi} = \bar{\partial} - \mathcal{L}^{1,0}_{\varphi(t)} \quad \text{on } A^{p,q}(X_0),
\]

which is exactly [Xi21b, Theorem 2.9, the case of \( E = \Omega^p \)].

**Remark 3.2.** In [Tu21], J. Tu used the pair deformation \( \{(X_t, E_t)\} \) to give a correspondence between \( E_0 \)-valued \((p, q)\)-forms on \( X_0 \) and \( E_t \)-valued \((p, q)\)-forms on \( X_t \) by

\[
P_t : A^{p,q}(X_0, E_0) \longrightarrow A^{p,q}(X_t, E_t).
\]

Recall that the pair deformation \( \{(X_t, E_t)\} \) is a holomorphic family of pairs \( \{(X_t, E_t)\} \) where each \( E_t \) is a holomorphic vector bundle over a compact complex manifold \( X_t \). The holomorphic structures of \( E_t \) and complex structures of \( X_t \) vary simultaneously. In the case when each \( E_t \) is the trivial line bundle, \( P_t \) coincides with (3.6) and, therefore [Tu21, Theorem 2] is equivalent to [RZ18, Proposition 2.13].

The following proposition is vital in this paper:

**Proposition 3.3** ([LRY15, Theorem 3.4], [RZ18, Proposition 2.2]). Let \( \phi \in A^{0,1}(X, T_X^{1,0}) \) on a complex manifold \( X \). Then on the space \( A^{*,*}(X) \),

\[
(3.8) \quad e^{-i \phi} \circ d \circ e^{i \phi} = d - \mathcal{L}_{\phi}^{1,0} + i \bar{\partial}_{\phi} \frac{1}{2} [\phi, \varphi],
\]

where \( \mathcal{L}_{\phi}^{1,0} := i \bar{\partial}_{\phi} \partial - \partial_{\phi} \) is the Lie derivative.

From the proof of Proposition 3.3, we see that (3.8) is a natural generalization of Tian–Todorov Lemma [Ti87, To89], whose variants appeared in [Fr91, BK98, LSY09, Cm05] and also [LR12, LRY15] for vector bundle valued forms.

**Lemma 3.4.** For \( \phi, \psi \in A^{0,1}(X, T_X^{1,0}) \) and \( \alpha \in A^{*,*}(X) \) on an \( n \)-dimensional complex manifold \( X \),

\[
[\phi, \psi] \downarrow \alpha = -\partial (\psi \downarrow (\phi \downarrow \alpha)) - \psi \downarrow (\phi \downarrow \partial \alpha) + \phi \downarrow \partial (\psi \downarrow \alpha) + \psi \downarrow \partial (\phi \downarrow \alpha),
\]

where

\[
[\phi, \psi] := \sum_{i,j=1}^n (\phi^i \wedge \partial_i \psi^j + \psi^i \wedge \partial_i \phi^j) \otimes \partial_j
\]

for \( \varphi = \sum_i \varphi^i \otimes \partial_i \) and \( \psi = \sum_i \psi^i \otimes \partial_i \).
3.2. **Kuranishi family.** We introduce some basics on Kuranishi family of complex structures in this subsection originally from [Ku64].

By (the proof of) Kuranishi’s completeness theorem [Ku64], for any compact complex manifold $X_0$, there exists a complete holomorphic family $\varpi : \mathcal{K} \to T$ of complex manifolds at the reference point $0 \in T$ in the sense that for any differentiable family $\pi : X \to B$ with $\pi^{-1}(s_0) = \varpi^{-1}(0) = X_0$, there exist a sufficiently small neighborhood $E \subseteq B$ of $s_0$, and smooth maps $\Phi : X_E \to \mathcal{K}$, $\tau : E \to T$ with $\tau(s_0) = 0$ such that the diagram commutes

$$
\begin{array}{ccc}
X_E & \xrightarrow{\Phi} & \mathcal{K} \\
\pi \downarrow & & \downarrow \varpi \\
(E, s_0) & \xrightarrow{\tau} & (T, 0),
\end{array}
$$

$\Phi$ maps $\pi^{-1}(s)$ biholomorphically onto $\varpi^{-1}(\tau(s))$ for each $s \in E$, and

$$
\Phi : \pi^{-1}(s_0) = X_0 \longrightarrow \varpi^{-1}(0) = X_0
$$
is the identity map. This family is called the *Kuranishi family* and constructed as follows. Let $\{\eta_{\nu}\}_{\nu=1}^m$ be a base for the harmonic space $H^{0,1}(X_0, T_{X_0}^1)$, where some suitable Hermitian metric is fixed on $X_0$ and $m \geq 1$; Otherwise the complex manifold $X_0$ would be *rigid*, i.e., for any differentiable family $\phi : M \to P$ with $s_0 \in P$ and $\phi^{-1}(s_0) = X_0$, there is a neighborhood $V \subseteq P$ of $s_0$ such that $\phi : \phi^{-1}(V) \to V$ is trivial. Then one can construct a holomorphic family

$$
\varphi(t) = \sum_{|I|=1}^{\infty} \varphi_I t^I := \sum_{j=1}^{\infty} \varphi_j(t), \ I = (i_1, \cdots, i_m), \ t = (t_1, \cdots, t_m) \in \mathbb{C}^m,
$$

for small $t$, of Beltrami differentials as follows:

$$
\varphi_1(t) = \sum_{\nu=1}^{m} t_\nu \eta_\nu
$$

and for $|I| \geq 2$,

$$
\varphi_I = \frac{1}{2} \bar{\partial} \mathcal{G} \sum_{J+L=I} [\varphi_J, \varphi_L].
$$

Clearly, $\varphi(t)$ satisfies the equation

$$
\varphi(t) = \varphi_1 + \frac{1}{2} \bar{\partial} \mathcal{G} [\varphi(t), \varphi(t)].
$$

Let

$$
T = \{ t \mid \mathbb{H}[\varphi(t), \varphi(t)] = 0 \}
$$

where $\mathbb{H}$ is the harmonic projection. So for each $t \in T$, $\varphi(t)$ satisfies

$$
(3.9) \quad \bar{\partial} \varphi(t) = \frac{1}{2} [\varphi(t), \varphi(t)],
$$

and determines a complex structure $X_t$ on the underlying differentiable manifold of $X_0$. More importantly, $\varphi(t)$ represents the complete holomorphic family $\varpi : \mathcal{K} \to T$ of complex manifolds. Roughly speaking, the Kuranishi family $\varpi : \mathcal{K} \to T$ contains all small differentiable deformations of $X_0$. 
3.3. $d$-closed extension of $(\mu,q)$-form: Bott–Chern approach. Assume that a compact complex manifold $X_0$ satisfies $\mathbb{B}^{p,q+1}$. Now, considering a Kuranishi family of deformations of $X_0$ over $\Delta_\epsilon$, for small $\epsilon$, one can find a family of integrable Beltrami differentials $\{\varphi(t)\}_{|t|\leq \epsilon}$ depending on $t$ holomorphically and describing the variations of complex structure on $X_0$.

In this subsection, we will reprove the following theorem due to [WZ20], from the perspective of Bott–Chern Hodge theory.

**Theorem 3.5.** Suppose that the compact complex manifold $X_0$ satisfies $\mathbb{B}^{p,q+1}$. Given a $d$-closed $\mu_0 \in A^{p,q}(X_0)$, one can construct $\mu(z,t) \in A^{p,q}(X_0)$ satisfying $d(e^{\varphi}(\mu(z,t))) = 0$ on $X_0$ (or $X_t$) and $\mu(z,0) = \mu_0(z)$, which is smooth in $(z,t)$ and holomorphic in small $t$. Furthermore, the extension $\rho_{\varphi}(\mu(z,t)) \in A^{p,q}(X_t)$ is $\bar{\partial}_{\mu}$-closed.

**Proof.** The proof will be divided into four steps.

**Step (I).** Transfer to the certain differential equations.

For the integrable Beltrami differential $\varphi(t)$ on $X_0$, we get the following formula by virtue of Proposition 3.3:

$$e^{-i\varphi} \circ d \circ e^{i\varphi} = d - L_\varphi^1 = d + \partial \varphi - \iota \varphi \partial.$$

Hence, for $\mu \in A^{p,q}(X_0)$, $e^{\varphi}(\mu)$ is $d$-closed on $X_0$ (or $X_t$) if and only if

$$(d + \partial \varphi - \iota \varphi \partial) \mu = 0.$$

By comparing the types, one knows that the above equation amounts to

$$\begin{cases}
\partial \mu = 0, \\
\bar{\partial} \mu = -\partial(\varphi, \mu).
\end{cases}$$

(3.10)

**Step (II).** Study the integral equation.

Given any $d$-closed $(\mu,q)$-form $\mu_0 \in A^{p,q}(X_0)$, one studies the integral equation as follows:

$$(3.11) \quad \mu + \partial(\overline{\partial})^* \mathbb{G}_{BC} \partial(\varphi(t), \mu) = \mu_0.$$

In the sequel we will prove that the equation (3.11) possesses a unique solution $\mu(z,t) \in A^{p,q}(X_0)$ which is smooth in $(z,t)$ and holomorphic in $t \in \Delta_\epsilon$.

Denote the completion of the norm space $(A^{p,q}(X_0), \| \cdot \|_{k,\alpha})$ by $\mathcal{E}$, and consider the linear operator

$$Q_{\varphi(t)} := -\partial(\overline{\partial})^* \mathbb{G}_{BC} \partial \varphi(t)$$

on $\mathcal{E}$. Then (3.11) is equivalent to the following equation

$$(3.12) \quad (I - Q_{\varphi(t)}) \mu = \mu_0.$$

One can easily see that $Q_{\varphi(t)}$ satisfies $\|Q_{\varphi(t)}\| < 1$ on $\mathcal{E}$ due to the standard estimates for Green’s operator $\mathbb{G}_{BC}$ as $t \in \Delta_\epsilon$. We then apply [Yo80, Chapter II.1, Theorem 2] to $T = I - Q_{\varphi(t)}$ to obtain the unique solution

$$\mu(z,t) = (I - Q_{\varphi(t)})^{-1} \mu_0 = \mu_0 + \sum_{k=1}^{\infty} Q_{\varphi(t)}^k(\mu_0)$$

of (3.12) for any $|t| \leq \epsilon$.

The integrable Beltrami differential $\{\varphi(t)\}_{|t|\leq \epsilon}$ can be written as a convergent power series in $t$ since it depends on $t$ holomorphically. Thus, $\mu(z,t)$ is also a power series in $t$ by (3.13). Moreover, it is easy to verify by the standard elliptic estimates and (3.12) that the $\| \cdot \|_{k,\alpha}$-norm of $\mu(z,t)$ is finite which implies that $\mu(z,t)$ is convergent for $|t| \leq \epsilon$. 

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So $\mu(z,t)$ is holomorphic in $t$ for $|t| \leq \epsilon$. Since we have got the $C^k$-continuity of $\mu(z,t)$ from above, one can use the standard regularity theory for elliptic differential operator such as in [MK71, Proposition 2.6 of Chapter 4] to obtain that the unique solution $\mu(z,t)$ satisfying (3.11) is a smooth $(p,q)$-form in $(z, t)$ for small $t$.

**Step (III). Show the solution $\mu(z, t)$ obtained in Step (II) satisfies (3.10).**

The unique solution $\mu(z, t)$ obtained in Step (II) clearly satisfies the first equality of (3.10) and, therefore we just need to check the second equality.

We can locally express $\varphi(t)$ and $\mu(z, t)$ as $\sum_{i \geq 1} \varphi_i t^i$ and $\sum_{i \geq 0} \mu_i t^i$, respectively, since they are both holomorphic in small $t$. Then the equation (3.9) is equivalent to

$$
\left\{ \begin{array}{l}
\overline{\partial} \varphi_1 = 0, \\
\overline{\partial} \varphi_k = \sum_{i+j=k \atop i,j \geq 1} \frac{1}{2} [\varphi_i, \varphi_j], \quad k \geq 2.
\end{array} \right.
$$

Rewrite equation (3.11) as

$$
\mu_k = -\partial(\overline{\partial})^* G_{BC} \partial \left( \sum_{i+j=k \atop i \geq 1, j \geq 0} \varphi_{i,j} \mu_j \right), \quad k \geq 1.
$$

We aim to show that

$$
\overline{\partial} \mu_k = -\partial \left( \sum_{i+j=k \atop i \geq 1, j \geq 0} \varphi_{i,j} \mu_j \right), \quad k \geq 1.
$$

When $k = 1$, one has by (3.14)

$$
\overline{\partial} \partial(\varphi_{1,0}) = -\overline{\partial} \partial(\varphi_{1,0}) = -\partial \left( \overline{\partial} \varphi_{1,0} + \varphi_{1,0} \overline{\partial} \mu_0 \right) = 0.
$$

Since $X$ satisfies $\mathbb{H}^{p,q+1}$, there exists some $\beta_1$ such that $\partial(\varphi_{1,0}) = \partial \overline{\partial} \beta_1$, implying $\mathbb{H}_{BC} \partial(\varphi_{1,0}) = 0$ because of (2.2). We then obtain

$$
\overline{\partial} \mu_1 = (\overline{\partial})^* (\overline{\partial})^* G_{BC} \partial(\varphi_{1,0}) = G_{BC} (\overline{\partial})^* \partial(\varphi_{1,0}) = G_{BC} \Box_{BC} \partial(\varphi_{1,0}) = \partial(\varphi_{1,0}) - \mathbb{H}_{BC} \partial(\varphi_{1,0}).
$$

Here in (3.16), the second, third and fourth equalities follow from the observation (2), (2.3) together with (3.14) and (2.2), respectively. Now, by induction hypothesis, assume that (3.15) holds for $k \leq l - 1$, and then similarly for $k = l$, one has

$$
\overline{\partial} \mu_k = \partial \overline{\partial} (\overline{\partial})^* G_{BC} \partial \left( \sum_{i+j=k \atop i \geq 1, j \geq 0} \varphi_{i,j} \mu_j \right) = \partial \left( \sum_{i+j=k \atop i \geq 1, j \geq 0} \varphi_{i,j} \mu_j \right) - \mathbb{H}_{BC} \partial \left( \sum_{i+j=k \atop i \geq 1, j \geq 0} \varphi_{i,j} \mu_j \right).
$$

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Hence, it suffices to show

\[
(3.17) \quad H_{BC} \partial \left( \sum_{i+j=k \atop i \geq 1, j \geq 0} \varphi_i \mu_j \right) = 0.
\]

By explicit computation, we have

\[
\bar{\partial} \partial \left( \sum_{i+j=k \atop i \geq 1, j \geq 0} \varphi_i \mu_j \right) = -\partial \bar{\partial} \left( \sum_{i+j=k \atop i \geq 1, j \geq 0} \varphi_i \mu_j \right)
\]

\[
= -\partial \sum_{i+j=k \atop i \geq 1, j \geq 0} (\bar{\partial} \varphi_i \mu_j + \varphi_i \bar{\partial} \mu_j)
\]

\[
= -\partial \sum_{i+j=k \atop i \geq 1, j \geq 0} \left( \frac{1}{2} \sum_{a+b=i} [\varphi_a, \varphi_b] \mu_j - \sum_{a+b=j} \varphi_i \bar{\partial} (\varphi_{a+b} \mu_b) \right)
\]

\[
= -\partial \sum_{a+b+c=l} (-\partial (\varphi_{a+b} \varphi_c \mu_c)) = 0,
\]

where the last but one equality comes from Lemma 3.4 together with the \(\partial\)-exactness of \(\mu_j\). By applying the same method as before, one obtains (3.17). Thus, the solution \(\mu(z, t)\) obtained in Step (II) satisfies (3.10), i.e., it satisfies \(d(e^{\varphi}(\mu(z, t))) = 0\) on \(X_0\) (or \(X_t\)).

**Step (IV).** Extend \(\mu(z, t)\) by \(\rho_{\varphi}\).

Now, as \(e^{\varphi}(\mu(z, t))\) is \(d\)-closed in \(F^{p}A^{p+q}(X_t)\), one writes

\[
e^{\varphi}(\mu(z, t)) = \alpha_{t}^{p,q} + \alpha_{t}^{p+1,q-1} + \cdots + \alpha_{t}^{p+q,0}
\]

\[
\in A^{p,q}(X_t) \oplus A^{p+1,q-1}(X_t) \oplus \cdots \oplus A^{p+q,0}(X_t),
\]

and gets \(\alpha_{t}^{p,q} = \rho_{\varphi}(\mu(z, t))\). Since \(d(e^{\varphi}(\mu(z, t))) = 0\) and \(d = \partial_t + \bar{\partial}_t\), we have \(\bar{\partial}_t (\alpha_{t}^{p,q}) = 0\) by comparing the types.

The proof of Theorem 3.5 is completed. \(\square\)

**Remark 3.6.** Notice that in [WZ20], Wei–Zhu first obtained Theorem 3.5 by use of \(\bar{\partial} \partial\)-Lapalacian \(\Box_{\bar{\partial} \partial}\). The integral equation what they study therein is

\[
(3.18) \quad \mu = \mu_0 - \partial (\bar{\partial} \bar{\partial}^* G_{\bar{\partial} \partial} \partial (\varphi(\mu)),
\]

where \(\mu_0 \in A^{p,q}(X_0)\) is the given \(d\)-closed \((p, q)\)-form and \(G_{\bar{\partial} \partial}\) is the associated Green’s operator of the 4th order real elliptic differential operator \(\Box_{\bar{\partial} \partial}\) defined by

\[
\Box_{\bar{\partial} \partial} = \bar{\partial} \partial (\bar{\partial} \partial)^* + (\partial \bar{\partial})^* (\partial \bar{\partial}).
\]

In addition, they used Banach fixed point theorem to get the unique solution of (3.18) motivated by [LZ18, LZ20]. Our bounded linear operator \(Q_{\varphi(t)}\) in Step (II) coincides with their contraction mapping.

4. *Deformation invariance of Hodge numbers*

On deformation invariance of Hodge numbers, one only needs to consider a Kuranishi family of deformations \(\pi : X \rightarrow \Delta_\epsilon\) of \(n\)-dimensional compact complex manifolds over a small complex disk with the general fibers \(X_t := \pi^{-1}(t)\) throughout this section, and always fixes a Hermitian metric \(g\) on the central fiber \(X_0\).
4.1. **Main results and examples.** We state the main theorem of this section, whose proof will be postponed to Subsection 4.2.

**Theorem 4.1.** If the injectivity of the mapping $\iota_{BC,\vartheta}^{p,q+1}$, the surjectivity of the mapping $\iota_{BC,\vartheta}^{p,q}$ on the central fiber $X_0$ and the deformation invariance of the $(p, q-1)$-Hodge number $h_{\vartheta}^{p,q-1}(X_t)$ hold, then $h_{\vartheta}^{p,q}(X_t)$ are independent of $t$.

Next, we will state the following three examples that the deformation invariance of the $(p, q)$-Hodge number fails when one of the three conditions in Theorem 4.1 is not satisfied, while the other two hold, with the help of Hodge, Bott–Chern and Aeppli numbers of manifolds in the Kuranishi family of the Iwasawa manifold (see [Ag13, Appendix] and [Sc07, Section 1.c]). It implies that the three conditions in Theorem 4.1 cannot be dropped for the sake of the validity of a theorem concerning the deformation invariance of all the $(p, q)$-Hodge numbers.

Let $\mathbb{I}_3$ be the Iwasawa manifold of complex dimension 3 with $\varphi^1, \varphi^2, \varphi^3$ denoted by the basis of the holomorphic one form $H^0(\mathbb{I}_3, \Omega^1(1))$ of $\mathbb{I}_3$, satisfying the relation

$$d\varphi^1 = 0, \ d\varphi^2 = 0, \ d\varphi^3 = -\varphi^1 \wedge \varphi^2.$$  

And the convention $\varphi^{123} := \varphi^1 \wedge \varphi^2 \wedge \overline{\varphi^1} \wedge \overline{\varphi^2} \wedge \overline{\varphi^3}$ will be used for simplicity. In order to reduce the number of cases under consideration, notice that, on a compact complex Hermitian manifold $X$ of complex dimension $n$, for any integer $0 \leq p, q \leq n$, the Hodge-$*$-operator induces an isomorphism between

$$H^p_{\vartheta}(X) \xrightarrow{\sim} H^{n-q,n-p}_{\vartheta}(X) \cong H^{n-p,n-q}_{\vartheta}(X).$$

**Example 4.2** ($(p, q) = (1, 0)$). The injectivity of $\iota_{BC,\vartheta}^{1,0}$ holds on $\mathbb{I}_3$ with the deformation invariance of $h_{\vartheta}^{1,0-1}(X_t)$ trivially established but $\iota_{BC,\vartheta}^{1,0}$ is not surjective. In this case, $h_{\vartheta}^{1,0}(X_t)$ are deformation variant.

*Proof.* [Ag13, Appendix] presents that $h_{BC}^{1,1} = 4, h_{\vartheta}^{1,1} = 6$, and $h_{BC}^{1,0} = 2, h_{\vartheta}^{1,0} = 3$. More precisely, one has by [Sc07, p. 6]:

\[ H_{BC}^{1,1}(X) = \langle [\varphi^{11}]_{BC}, [\varphi^{12}]_{BC}, [\varphi^{21}]_{BC}, [\varphi^{22}]_{BC} \rangle, \]

\[ H_{\vartheta}^{1,1}(X) \cong H_{\vartheta}^{1,1}(X) = \langle [\varphi^{11}]_{\vartheta}, [\varphi^{12}]_{\vartheta}, [\varphi^{21}]_{\vartheta}, [\varphi^{22}]_{\vartheta}, [\varphi^{13}]_{\vartheta}, [\varphi^{31}]_{\vartheta}, [\varphi^{23}]_{\vartheta} \rangle, \]

which gives the injectivity of $\iota_{BC,\vartheta}^{1,1}$. However, according to [Sc07, Last para. in Section 1.c], $\iota_{BC,\vartheta}^{1,0}$ is not surjective. The deformation variance of $h_{\vartheta}^{1,0}(X_t)$ follows from [Ag13, Appendix].

**Example 4.3** ($(p, q) = (2, 0)$). The surjectivity of $\iota_{BC,\vartheta}^{2,0}$ holds on $\mathbb{I}_3$ with the deformation invariance of $h_{\vartheta}^{2,0-1}(X_t)$ trivially established but $\iota_{BC,\vartheta}^{2,1}$ is not injective. In this case, $h_{\vartheta}^{2,0}(X_t)$ are deformation variant.

*Proof.* By [Sc07, p. 6], we get

\[ H_{BC}^{2,0}(X) = \langle [\varphi^{12}]_{BC}, [\varphi^{13}]_{BC}, [\varphi^{23}]_{BC} \rangle, \]

\[ H_{\vartheta}^{2,0}(X) = \langle [\varphi^{12}]_{\vartheta}, [\varphi^{13}]_{\vartheta}, [\varphi^{23}]_{\vartheta} \rangle, \]

which implies the surjectivity of $\iota_{BC,\vartheta}^{2,0}$. However, the following fact shows the non-injectivity of $\iota_{BC,\vartheta}^{2,1}$:

\[ H_{BC}^{2,1}(X) = \langle [\varphi^{121}]_{BC}, [\varphi^{122}]_{BC}, [\varphi^{131}]_{BC}, [\varphi^{132}]_{BC}, [\varphi^{231}]_{BC}, [\varphi^{232}]_{BC} \rangle, \]

\[ H_{\vartheta}^{2,1}(X) \cong H_{\vartheta}^{2,1}(X) = \langle [\varphi^{131}]_{\vartheta}, [\varphi^{132}]_{\vartheta}, [\varphi^{133}]_{\vartheta}, [\varphi^{231}]_{\vartheta}, [\varphi^{232}]_{\vartheta}, [\varphi^{233}]_{\vartheta} \rangle. \]
Moreover, [Ag13, Appendix] gives the fact of the deformation variance of $h_{\mathcal{I}_i}^{2,0}(X_i)$. □

**Example 4.4** ($(p, q) = (2, 3)$). The mapping $\iota_{BC, T}^{2,3}$ on $\mathbb{I}_3$ is surjective and the injectivity of $\iota_{BC, T}^{2,4}$ trivially holds, but $h_{\mathcal{I}_i}^{2,2}(X_i)$ are deformation variant. In this case, $h_{\mathcal{I}_i}^{2,3}(X_i)$ are deformation variant.

**Proof.** It is clear that $\iota_{BC, T}^{2,3}$ is surjective since by [Sc07, p. 6],

$$H_{BC}^{2,3}(X) = \langle [\varphi^{123}]_{BC}, [\varphi^{132}]_{BC}, [\varphi^{231}]_{BC} \rangle,$$

$$H_{\mathcal{I}}^{2,3}(X) = \langle [\varphi^{123}]_{\mathcal{I}}, [\varphi^{132}]_{\mathcal{I}}, [\varphi^{231}]_{\mathcal{I}} \rangle.$$ 

The deformation variance of $h_{\mathcal{I}_i}^{2,2}(X_i)$ and $h_{\mathcal{I}_i}^{2,3}(X_i)$ can also be got from [Ag13, Appendix]. □

**Remark 4.5.** The three examples above partially coincide with those listed in [RZ18, Examples 3.2, 3.3, and 3.4], where Zhao—the first author verified the indispensability of the three conditions therein to state a theorem for the deformation invariance of all the $(p, q)$-Hodge numbers. More precisely, their three conditions are the injectivity of the mappings $\iota_{BC, T}^{p+1,q}$, $\iota_{\mathcal{I}, A}^{p+1}$ and the deformation invariance of the $(p, q - 1)$-Hodge number (see also our Remarks 4.13, 4.14).

The speciality of the types may lead to the weakening of the conditions in Theorem 4.1, such as $(p, 0)$ and $(0, q)$. Hence, another two theorems follow, whose proofs will be given in Subsection 4.3. Note that Theorem 4.7 first appeared as [RZ18, Theorem 3.7]. So we can as well obtain [RZ18, Remark 3.8, Corollary 3.9] as a consequence.

**Theorem 4.6.** If $X_0$ satisfies $\mathbb{B}^{p,1}$ (i.e., the mapping $\iota_{BC, T}^{p,1}$ is injective) and $\mathbb{B}^{p+1,0}$, then $h_{\mathcal{I}_i}^{2,0}(X_i)$ are deformation invariant.

**Theorem 4.7.** If $X_0$ satisfies $\mathbb{B}^{1,q}$ (i.e., the mapping $\iota_{BC, T}^{0,q}$ is surjective) and $h_{\mathcal{I}_i}^{1,q-1}(X_i)$ satisfies the deformation invariance, then $h_{\mathcal{I}_i}^{0,q}(X_i)$ are independent of $t$.

**Remark 4.8.** Drawing on the work of S. Console–A. Fino–Y.-S. Poon [CFP16], Zhao—the first author constructed an example, namely a holomorphic family of nilmanifolds of complex dimension 5, whose central fiber is endowed with an abelian complex structure. This family admits the deformation invariance of the $(p, 0)$-Hodge numbers for $1 \leq p \leq 5$, but not the $(1, 1)$-Hodge number or $(1, 1)$-Bott–Chern number. By verification, this example is also applicable to our Theorem 4.6, which shows the function of Theorem 4.6 possibly beyond Kodaira–Spencer’s squeeze [KS60, Theorem 13] in this case. One can see Example 3.11 in the update version on the first author’s researchgate [RZ] for more details.

4.2. **Proofs of the invariance of Hodge numbers** $h_{\mathcal{I}_i}^{p,q}(X_i)$. We aim to prove Theorem 4.1 in this subsection, which can be restated as the following theorem thanks to the equivalent statements (1) and (3) displayed in Subsection 2.1.

**Theorem 4.9.** If the central fiber $X_0$ satisfies both $\mathbb{B}^{p,q+1}$ and $\mathbb{B}^{p+1,q}$ with the deformation invariance of $h_{\mathcal{I}_i}^{p,q-1}(X_i)$ established, then $h_{\mathcal{I}_i}^{p,q}(X_i)$ are independent of $t$.

We have described our strategy in the Introduction 1, so the next lemma, motivated by [Po19, Lemma 3.1] is crucial, which can help us to achieve the first step. For the convenience of the readers, we recall the proof.
Lemma 4.10 ([RZ18, Lemma 3.13]). Assume that the compact complex manifold $X$ satisfies $\mathcal{B}^{p+1,q}$. Then each Dolbeault class $[\sigma]_{\bar{\nabla}}$ of the type $(p, q)$ can be canonically represented by a uniquely-chosen $d$-closed $(p, q)$-form $\gamma_{\sigma}$.

Proof. We first choose the unique harmonic representative of $[\sigma]_{\bar{\nabla}}$, still denoted by $\sigma$. It is clear that the $d$-closed representative $\gamma_{\sigma} \in A^{p,q}(X)$ satisfies

$$\sigma + \bar{\partial} \beta_{\sigma} = \gamma_{\sigma}$$

for some $\beta_{\sigma} \in A^{p,q-1}(X)$. This is equivalent that some $\beta_{\sigma} \in A^{p,q-1}(X)$ solves the following equation

$$\bar{\partial} \partial \beta_{\sigma} = - \partial \sigma.$$ 

The existence of $\beta_{\sigma}$ is assured by our assumption on $X$ and uniqueness with $L^2$-norm minimum by [RZ18, Lemma 3.12], that is, one can choose $\beta_{\sigma}$ as $-(\partial \bar{\partial})^* G_{BC} \partial \sigma$. □

By Theorem 3.5, one has

Proposition 4.11. Assume that the compact complex manifold $X_0$ satisfies $\mathcal{B}^{p,q+1}$ and $\mathcal{B}^{p+1,q}$. Then for each Dolbeault class in $H^{p,q}_{\bar{\nabla}}(X_0)$ with the unique canonical $d$-closed representative $\mu_0$ given as Lemma 4.10, we can get $\mu(z, t) \in A^{p,q}(X_0)$ with $\mu(z, 0) = \mu_0(z)$, which is smooth in $(z, t)$ and holomorphic in $t$ for small $t$, such that $\rho_\varphi(\mu(z, t))$ is $\partial_t$-closed in $A^{p,q}(X_t)$.

According to the strategy as we represented after the statement of Theorem 1.3, Theorem 4.1 follows immediately from the following result:

Proposition 4.12. If the $\bar{\partial}$-extension of $H^{p,q}_{\bar{\nabla}}(X_0)$ as in Proposition 4.11 holds for a complex manifold $X_0$, then the deformation invariance of $h^{p,q-1}_{\partial_t}(X_t)$ assures that the extension map

$$H^{p,q}_{\bar{\nabla}}(X_0) \to H^{p,q}_{\partial_t}(X_t) : [\mu_0]_{\bar{\nabla}} \mapsto [\rho_\varphi(\mu(z, t))]_{\partial_t}$$

is injective.

Proof. Fix a smooth family of Hermitian metrics $\{h_t\}_{t \in \Delta_t}$ for the infinitesimal deformation $\pi : X \to \Delta_t$ of $X_0$. So, if the Hodge numbers $h^{p,q-1}_{\partial_t}(X_t)$ are deformation invariant, the Green’s operator $G_t$, acting on the $A^{p,q-1}(X_t)$, depends differentiably with respect to $t$ from [KS60, Theorem 7]. Applying this, one guarantees that this extension map can not send a non-zero class in $H^{p,q}_{\bar{\nabla}}(X_0)$ to a zero class in $H^{p,q}_{\partial_t}(X_t)$.

If one supposes that

$$\rho_\varphi(\mu(z, t)) = \bar{\partial}_t \eta_t$$

for some $\eta_t \in A^{p,q-1}(X_t)$ when $0 \neq t \in \Delta_t$, the Hodge decomposition of $\bar{\partial}_t$ and the commutativity of $G_t$ with $\bar{\partial}_t$ and $\partial_t$ imply that

$$\rho_\varphi(\mu(z, t)) = \partial_t \bar{\eta}_t = \partial_t (\bar{\partial}_t \partial_t \eta_t + \square_t G_t \eta_t)$$

$$= \partial_t (\partial_t \partial_t \eta_t)$$

$$= \partial_t G_t (\partial_t \partial_t \eta_t)$$

$$= \partial_t G_t (\partial_t \rho_\varphi(\mu(z, t))),$$

where $\bar{\partial}_t$ and $\square_t$ are the harmonic projectors and the Laplace operators with respect to $(X_t, \omega_t)$, respectively. Let $t$ converge to 0 on both sides of the equality

$$\rho_\varphi(\mu(z, t)) = \partial_t G_t (\partial_t \rho_\varphi(\mu(z, t))),$$

which implies that $\mu_0$ is $\bar{\partial}_t$-exact on the central fiber $X_0$. Here the Green’s operator $G_t$ depends differentiably with respect to $t$. □
Remark 4.13. In [RZ18, Theorem 1.3 or the First para. in Section 3.2], Zhao–the first author also got a theorem about the invariance of Hodge numbers $h_{p,q}^{p,q}(X_t)$. By comparison, one can see that both need the deformation invariance of $h_{p,q}^{p,q+1}(X_t)$, and the condition $'X_0$ satisfies $\mathbb{B}^{p,q}$ (resp. $\mathbb{B}^{p+1,q}$)” used by us is stronger (resp. weaker) than their condition $'X_0$ satisfies $\mathbb{S}^{p,q+1}$ (resp. $\mathbb{S}^{p+1,q}$).

4.3. Proofs of the invariance of Hodge numbers $h_{p,q}^{p,q}(X_t)$, $h_{0,q}^{p,q}(X_t)$: special cases.

Proof of Theorem 4.6. Since any holomorphic $(p,0)$-form on $X_0$ satisfying $S^{p+1,0}$ is actually $d$-closed and Proposition 4.12 holds automatically in the $(p,0)$-case, one can complete the proof only by Proposition 4.11.

Proof of Theorem 4.7. Compared with Theorem 4.9, this one drops the condition $\mathbb{B}^{0,q+1}$ since it automatically holds.

Remark 4.14. In [RZ18, Theorem 3.18], Zhao–the first author considered the invariance of Hodge numbers $h_{p,q}^{p,q}(X_t)$, where their condition $'\mathbb{S}^{p+1,0}$ (resp. $\mathbb{S}^{p+1,0}$) on $X_0$ ‘is stronger (resp. weaker) than ours.

5. Local stabilities of $p$-Kähler structures

Local stabilities of complex structures are significant in deformation theory of complex structures. Alessandri–Bassanelli [AB90] proved that the $(n-1)$-Kählerian property is not preserved under the small deformations for balanced manifolds, nor for more general $p$-Kähler manifolds (the basics of this concept are introduced in Appendix A), where $p > 1$, while, motivated by the proof of [Vo02, Theorem 9.23], we plan to reprove the local stabilities of $p$-Kähler structures with the $\overline{\partial}\partial$-property in this section. This will be based on the extension of closed complex differential forms discussed in Subsection 3.3, as well as the deformation openness of the $\overline{\partial}\partial$-property.

Theorem 5.1 ([RZ21, Theorem 4.9]). For any positive integer $p \leq n-1$, any small differentiable deformation $X_t$ of a $p$-Kähler manifold $X_0$ satisfying the $\overline{\partial}\partial$-property is still $p$-Kählerian.

5.1. Introduction: useful results and classical proofs. The $\overline{\partial}\partial$-property is a powerful tool to study Kähler manifolds, such as the functional of solving the $\overline{\partial}$-equation $\partial x = \partial\alpha$ with closed $\partial\alpha$ on a Kähler manifold and the degeneration of Frölicher spectral sequence. As we know, there are many interesting alternative characterizations of the $\overline{\partial}\partial$-property on compact complex manifolds, among which are P. Deligne–P. Griffiths–J. Morgan–D. Sullivan’s one [DGMS75] by the decomposition type of Dolbeault complex and D. Angella–A. Tomassini’s one [AT13] by Frölicher-type equality in terms of Betti, Bott–Chern and Aeppli numbers. Recently, Y. Zou–the first author [RZ20] characterized various (weak) $\overline{\partial}\partial$-properties by de Rham, Dolbeault, Bott–Chern, Aepli and Varouchas’ cohomology groups in a uniform way, from the perspective of double complex.

Recall that a complex manifold is $p$-Kählerian if it admits a $p$-Kähler form, i.e., a $d$-closed transverse $(p,p)$-form as in Definition A.1. One notices that each $n$-dimensional complex manifold is $n$-Kählerian and there are two basic properties of $p$-Kählerian structures:

Lemma 5.2 ([AA87, Proposition 1.15] or [RZ21, Corollary 4.6]). A complex manifold $M$ is $1$-Kähler if and only if $M$ is Kähler; an $n$-dimensional complex manifold $M$ is $(n-1)$-Kähler if and only if $M$ is balanced, i.e., it admits a real positive $(1,1)$-form $\omega$, satisfying

$$d(\omega^{n-1}) = 0.$$
Thus, as a direct corollary of Theorem 5.1, one has:

**Corollary 5.3.** Let \( \pi : X \to B \) be a differentiable family of compact complex manifolds.

(i) ([KS60, Theorem 15]) If the fiber \( X_0 := \pi^{-1}(t_0) \) admits a Kähler metric, then for a sufficiently small neighborhood \( U \) of \( t_0 \) on \( B \), the fiber \( X_t := \pi^{-1}(t) \) over any point \( t \in U \) still admits a Kähler metric, which coincides for \( t = t_0 \) with the given Kähler metric on \( X_0 \).

(ii) ([Wu06, Theorem 5.13]) Let the fiber \( X_0 \) be a balanced manifold and satisfy the \( \ddbar \)-property. Then \( X_t \) also admits a balanced metric for small \( t \).

The first assertion of Corollary 5.3 is the fundamental Kodaira–Spencer’s local stability theorem of Kähler structures, and motivates the second one of Corollary 5.3 and many other related works on local stabilities of complex structures in [FY11, Vo02, Wu06, AU16, AU17, RWZ19].

Let us sketch Kodaira–Spencer’s proof of local stability theorem [KS60]. Let \( \mathbb{H}_{BC,t} \) be the orthogonal projection to the kernel \( F_{BC,t} \) of the Bott–Chern Laplacian
\[
\Box_{BC,t} = \partial \bar{\partial} \tilde{\alpha}_t + \bar{\partial} \partial \tilde{\alpha}_t + \partial \bar{\partial} \tilde{\alpha}_t + \bar{\partial} \partial \tilde{\alpha}_t + \partial \partial \tilde{\alpha}_t + \bar{\partial} \partial \tilde{\alpha}_t
\]
and \( G_{BC,t} \) the corresponding Green’s operator with respect to \( \tilde{\alpha}_t \) on \( X_t \). Here \( \alpha_t = \sqrt{-1} g_\gamma(\zeta, t) d\zeta^i \wedge d\overline{\zeta}^j \) is a Hermitian metric on \( X_t \) depending differentiably on \( t \) with \( \alpha_0 \) being a Kähler metric on \( X_0 \), and \( \overline{\partial}_t \) (resp. \( \partial_t \)) is the dual of \( \overline{\partial}_t \) (resp. \( \partial_t \)) with respect to \( \alpha_t \). By a cohomologically argument with the upper semi-continuity theorem, they prove that \( \mathbb{H}_{BC,t} \) and \( G_{BC,t} \) depend differentiably on \( t \). Then they can construct the desired Kähler metric on \( X_t \) as
\[
\tilde{\alpha}_t = \frac{1}{2} (\mathbb{H}_{BC,t} \alpha_t + \overline{\mathbb{H}_{BC,t}} \alpha_t).
\]

We next roughly state Voisin’s proof of local stability theorem [Vo02, Theorem 9.23]. Let \( \pi : X \to B \) be a family of complex manifolds with Kählerian \( X = X_0 \). One puts a Hermitian metric on \( X \) to get an induced Hermitian metric on each \( X_t := \pi^{-1}(t) \). We can identify \( H^1(X_b, \Omega^1_{X,b}) \) with the forms of type \((0, 1)\) with values in \( \Omega^1_{X,b} \) which are harmonic for the Laplacian associated to the operator \( \overline{\partial} \). By virtue of the deformation invariance of Hodge numbers [Vo02, Proposition 9.20] together with the deformation theory from [MK71], she proved the existence of a \( C^\infty \) section \( (\omega_b)_{b \in B} \), \( \omega_0 = \omega \) (\( d \)-closed Kähler form) of the bundle \( \Omega^1_{X/B} \otimes \Omega^{0,1}_{X/B} \) such that \( \omega_b \) is \( \overline{\partial}_b \)-closed for every \( b \) sufficiently near 0, where
\[
\Omega^\varnothing_{X/B} := \Omega^1_X / \pi^* \Omega^1_B,
\]
and
\[
\Omega^{0,1}_{X/B} := \bigwedge^q \Omega^{0,1}_{X/B}, \quad \Omega^{0,1}_{X/B} := \Omega^{0,1}_X / \pi^* \Omega^{0,1}_B.
\]
Finally, she got the desired Kähler form \( \Re \omega' \) from a \( d \)-closed \((1, 1)\)-form \( \omega' \) constructed on \( X_b \) due to her (separate) usage of deformation openness of \( \overline{\partial} \)-property [Vo02, Propositions 9.20 and 9.21] and some uniform convergence arguments. One should notice that the former property was first showed by Voisin, while one realizes that the \( \overline{\partial} \)-property is equivalent to the Frölicher (or Hodge to de Rham) spectral sequence degenerates at \( E_1 \) plus the formal Hodge decomposition. See [DGMS75, (5.21)], [Wu06, Theorem 5.12] and [AT13, Corollary 3.7] for alternative proofs.

It’s worth noting that when using Voisin’s approach to deal with the local stabilities of \( p \)-Kähler structures \((p > 1)\), one might encounter some obstacles. Unlike the case of 1-Kähler forms, a \( p \)-Kähler form \( \omega_0 \) on \( X_0 \) is not necessarily \( \overline{\partial} \)-harmonic. Hence, given an
initial $d$-closed $\omega_0 \in A^{p,p}(X_0)$, one first needs to find a $\overline{\partial}$-harmonic representative which is much possibly different form $\omega_0$, and then imitates the subsequent actions. However, it seems that the transversality probably gets damaged in this process.

Motivated by Voisin’s proof, we now briefly describe our method to prove local stabilities of $p$-Kähler structures, which is quite different from Kodaira–Spencer’s. By means of the Kuranishi’s completeness theorem introduced in Subsection 3.2, we can reduce Theorem 5.1 to the Kuranishi family $\varpi : \mathcal{X} \to T$ by shrinking $E$ if necessary, that is, we will explicitly construct a $p$-Kähler extension $\hat{\omega}_t$ of the $p$-Kähler form $\omega_0$ on $X_0$, such that $\hat{\omega}_t$ is a $p$-Kähler form on the general fiber $\varpi^{-1}(t) = X_t$.

In this method, the following observation plays a prominent role.

**Proposition 5.4** ([RWZ21, Proposition 4.12]). Let $\pi : \mathcal{X} \to B$ be a differentiable family of compact complex $n$-dimensional manifolds and $\Omega_\tau$ a family of real $(p,p)$-forms with $p < n$, depending smoothly on $t$. Assume that $\Omega_0$ is a transverse $(p,p)$-form on $X_0$. Then $\Omega_t$ is also transverse on $X_t$ for small $t$.

This proposition demonstrates that any smooth real extension of a transverse $(p,p)$-form remains transverse. Therefore, the only obstruction to extending a $d$-closed transverse $(p,p)$-form on a compact complex manifold lies in its $d$-closed property. We will provide the proof of Theorem 5.1 in the next subsection.

### 5.2. Proof of Theorem 5.1

Consider a Kuranishi family of deformations $\pi : \mathcal{X} \to \Delta$, (with small $\epsilon$) of $n$-dimensional complex manifolds over a small complex disk with the general fibers $X_t := \pi^{-1}(t)$, and fix a smooth family of Hermitian metrics $\{h_t\}_{t \in \Delta}$. Now given the $p$-Kähler form $\omega_0$ on $X_0$, namely a $d$-closed transverse $(p,p)$-form, based on Theorem 3.5, one can find a unique solution $\omega(t) \in A^{p,p}(X_0)$ satisfying the equation (3.11) which is smooth in $(z,t)$ and holomorphic in small $t$. Thus, we apply (3.3) to our situation to see

$$e^{\varphi}(\omega(t)) = \omega_t^{p,p} + \omega_t^{p+1,p-1} + \cdots + \omega_t^{2p,0}$$

$$\in A^{p,p}(X_t) \oplus A^{p+1,p-1}(X_t) \oplus \cdots \oplus A^{2p,0}(X_t)$$

is $d$-closed in $F^pA^{2p}(X_t)$. Then one has the observations due to type reason:

$$\begin{aligned}
\partial_t \omega_t^{p,p} &= 0, \\
\partial_t \omega_t^{p,p} + \overline{\partial}_t \omega_t^{p+1,p-1} &= 0.
\end{aligned}$$

Since $X_0$ satisfies the $\partial\overline{\partial}$-lemma, so does the general fiber $X_t$. Thus, there exists $\beta_t \in A^{p,p-1}(X_t)$ satisfying

$$\partial_t \omega_t^{p,p} = \partial_t \overline{\partial}_t \beta_t$$

by virtue of the second equality of (5.1). Set

$$\tilde{\omega}_t := \omega_t^{p,p} - \overline{\partial}_t \beta_t.$$  

Hence,

$$\tilde{\omega}_t := \frac{1}{2} \left( \tilde{\omega}_t + \overline{\omega}_t \right)$$

is real and $d$-closed. Thanks to Proposition 5.4, to prove that $\tilde{\omega}_t$ is the $p$-Kähler extension of the $p$-Kähler form $\omega_0$ on $X_0$, it suffices to show that $\overline{\partial}_t \beta_t$ converges uniformly to zero as $t$ tends to 0.

Following the notations introduced in Section 5.1, one knows that $\mathbb{H}_{BC,t}$ and $\mathbb{G}_{BC,t}$ are $C^\infty$ differentiable in $t$ since $\dim F_{BC,t}$ is deformation invariant, because of [KS60, Theorem 7] and [Wu06, Theorem 5.12]. Choose the explicit solution

$$\beta_t = (\partial_t \overline{\partial}_t)^* \mathbb{G}_{BC,t} \partial_t \omega_t^{p,p},$$

...
as shown in Lemma 2.1. One thus obtains that \( \overline{\partial}_t \beta_t \) converges uniformly to zero with \( t \), since
\[
\beta_t|_{t=0} = (\partial_t \overline{\partial}_t)^* G_{BC,0} \partial_t \omega_0 = 0.
\]

Theorem 5.1 is, thus, proved.

**Remark 5.5.** It is natural to ask if we can weaken the standard \( \partial \bar{\partial} \)-property that appeared in Theorem 5.1 in our approach as in [RWZ19, Theorem 1.1].

(a) By [RWZ19, Definition 3.1], we say that a complex manifold \( X \) satisfies the \((p, q)\)-th mild \( \partial \bar{\partial} \)-property if for any complex differential \((p - 1, q)\)-form \( \xi \) with \( \partial \bar{\partial} \xi = 0 \) on \( X \), there exists a \((p - 1, q - 1)\)-form \( \theta \) such that \( \partial \bar{\partial} \theta = \partial \bar{\partial} \xi \). Then obviously, this condition is equivalent to the condition \( \mathbb{H}^{p,q} \) as introduced in [RZ18, Notation 3.5].

In our proof of Theorem 5.1, the existence of \( \beta_t \) relies on the deformation openness of \( \partial \bar{\partial} \)-property. However, the mild one doesn’t possess this property (see [UV15, Example 3.7]).

(b) There is another weak version of the \( \partial \bar{\partial} \)-property, namely the \((p, q)\)-th strong \( \partial \bar{\partial} \)-property on \( X \), first proposed by Angella–Ugarte [AU17] in the case \((p, q) = (n - 1, n)\), stated that the induced mapping \( p_{BC,A}^{p,q} : H_{BC}^{p,q}(X) \to H_A^{p,q}(X) \) by the identity map is injective (see Diagram (2.1)), which is equivalent to that for any \( d \)-closed \((p, q)\)-form \( \Gamma \) of the type \( \Gamma = \partial \xi + \bar{\partial} \psi \), there exists a \((p - 1, q - 1)\)-form \( \theta \) such that
\[
\partial \bar{\partial} \theta = \Gamma.
\]

It is a well-known fact that the \((p, q)\)-th strong \( \partial \bar{\partial} \)-property can imply the \((p, q)\)-th mild \( \partial \bar{\partial} \)-property.

The most noteworthy is that Angella–Ugarte [AU17, Proposition 4.8] showed the deformation openness of the \((n - 1, n)\)-th strong \( \partial \bar{\partial} \)-property. So, it’s spontaneous to ask if one can weaken our second part of Corollary 5.3 to \((n - 1, n)\)-th strong \( \partial \bar{\partial} \)-property version by using our method. One can see that in the proof of Theorem 5.1, it is crucial to ensure the \( C^\infty \) differentiability of \( G_{BC,1} \) in \( t \) when \( t \) converges to 0, which is benefited from the deformation invariance of Bott–Chern number. However, [AU17, Example 4.10] shows that a small deformation of a completely-solvable Nakamura threefold, which is balanced and satisfies the \((2, 3)\)-th strong \( \partial \bar{\partial} \)-property, is also balanced. But the \((2, 2)\)-th Bott–Chern number varies along this deformation.

Hence, among all the (weak) versions of the \( \partial \bar{\partial} \)-properties as far as we know, it seems that the standard \( \partial \bar{\partial} \)-property on \( X_0 \) is the most general condition to prove the local stabilities of \( p \)-Kähler structures in this approach.

### 5.3. Upper semi-continuity of Bott–Chern numbers.

In several complex variables, one has the remarkable Grauert’s semi-continuity theorem:

**Theorem 5.6** ([BS76, Theorem 4.12.(i) of Chapter III] or [Gr60]). Let \( f : X \to Y \) be a proper morphism of complex spaces and \( S \) a coherent analytic sheaf on \( X \) which is flat with respect to \( Y \) (or \( f \)), which means that the \( \mathcal{O}_{f(x)} \)-modules \( S_x \) are flat for all \( x \in X \). Set \( S(y) \) as the analytic inverse image with respect to the embedding \( X_y \) in \( X \). Then for any integers \( i, d \geq 0 \), the set
\[
\{ y \in Y \mid \dim \mathbb{C} H^i (X_y, S(y)) \geq d \}
\]
is an analytic subset of \( Y \).

The topology on \( Y \) defined by closed sets as analytic sets is referred to as the analytic Zariski topology. The statement of Theorem 5.6 implies the upper semi-continuity of \( h^i (X_y, S(y)) \) concerning this analytic Zariski topology.

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The following result, a straightforward application of Theorem 5.6, plays a crucial role in deformation theory.

**Theorem 5.7** ([GR84, §10.5.4]). Let $f : X \to Y$ be a holomorphic family of compact complex manifolds with connected complex manifolds $X, Y$ and $V$ a holomorphic vector bundle on $X$. Then for any integers $i, d \geq 0$, the set

$$\{ y \in Y \mid \dim \mathbb{C} H^i(X_y, V|_{X_y}) \geq d \}$$

is an analytic subset of $Y$.

It may be noteworthy to mention that semi-continuity theorem of Grauert is much stronger than the corresponding Kodaira–Spencer’s result, since the ordinary topology is obviously much finer. For example, consider a Kuranishi family of deformations $\pi : \Delta \times X \to X$ of $n$-dimensional compact complex manifolds with small $\epsilon$. Kodaira–Spencer’s result tells us that the function $\Delta \ni t \mapsto h_{BC}^{p,q}(X_t)$ is always upper semi-continuous (with respect to the original topology), but Grauert’s theorem gives the additional information that this function is actually constant outside a complex analytic set of $\Delta$. In other words, in this Kuranishi family if jumping of Hodge numbers occurs it must do so only on a complex analytic set. This additional information could not be got just from the original Kodaira–Spencer’s theorem. Here we apply Theorem 5.7 to the holomorphic vector bundle of relative differential forms

$$\Omega_{p,X/\Delta}^q := \bigwedge^p \Omega_{X/\Delta}^q,$$

noting that $\Omega_{p,X/\Delta}^q|_{X_t} \cong \Omega_{X_t}^p$. One can also refer to [Xi21b, Proposition 5.8] for a new proof concerning some cases of Theorem 5.7.

Now, we turn to study the Bott–Chern numbers under small deformation. With the setting as stated in the last paragraph, the function is always upper semi-continuous with respect to the classical topology for $t \in \Delta$, thanks to [KS60, Theorem 4]. So it’s natural to ask if the function $t \mapsto h_{BC}^{p,q}(X_t)$ is also upper semi-continuous with respect to the analytic Zariski topology. One possible approach is to use Grauert’s upper semi-continuity Theorem 5.7. In other words, one needs to find some holomorphic vector bundle $V$ on $X$, such that $H_{BC}^{p,q}(X_t) \simeq H^q(X_t, V|_{X_t})$. Note that Aeppli cohomology is the dual of Bott–Chern cohomology (see (2.1)). Hence, in order to associate it with the cohomology group of some sheaf, we next will list a few results concerning the resolutions of the sheaf $\mathcal{H}$ of germs of pluriharmonic functions originating from [Bg69, AN71].

Denote by $\mathcal{A}^{p,q}$ the sheaf of germs of $C^\infty$ complex differential forms of type $(p, q)$ on an $n$-dimensional complex manifold $X$ and set $A^{p,q}(X) = H^q(X, \mathcal{A}^{p,q})$. One also calls $\Omega^p$ (resp. $\overline{\Omega}^p$) the sheaf of germs of holomorphic (resp. antiholomorphic) $p$-forms on $X$.

First of all, we have the exact sequence

$$0 \to \mathbb{C} \xrightarrow{\alpha} \mathcal{O} \oplus \overline{\mathcal{O}} \xrightarrow{\beta} \mathcal{H} \to 0$$

where $\alpha(c) = c \oplus (-c)$ and $\beta(f \oplus g) = f + g$.

Secondly we have the following resolutions of types $(1, 1)$ and $(2, 3)$:
and in general a resolution of type \((p, q)\):  

\[
0 \rightarrow \mathcal{A}^0 \rightarrow h_0 \mathcal{A}^1 \rightarrow h_1 \mathcal{A}^2 \rightarrow \ldots \rightarrow h_{q-1} \mathcal{A}^q \rightarrow H^q \rightarrow 0
\]

where

\[\mathcal{A}^p \rightarrow \mathcal{A}^{p+1} \rightarrow \ldots \rightarrow \mathcal{A}^{p+q-1} \rightarrow h_p \mathcal{A}^{p+q} \rightarrow \mathcal{A}^{p+q+1},\]

\footnote{Here by (2.1) we may assume \(1 \leq p \leq q \leq n\). One should aware that the resolutions are invalid for types \((p, 0)\) and \((0, q)\) when trying to relate the Aeppli cohomology groups to some sheaves as shown later.}
is: 

\[ \mathcal{B}^i = \begin{cases} \bigoplus_{s=0}^{i} \mathcal{A}^{s,i-s}, & \text{for } 0 \leq i \leq p-1; \\ \bigoplus_{s=0}^{p} \mathcal{A}^{s,i-s}, & \text{for } p \leq i \leq q-1; \\ \bigoplus_{s=i-q}^{p} \mathcal{A}^{s,i-s}, & \text{for } q \leq i \leq q+p-1, \end{cases} \]

and

\[ \mathcal{A}^i = \begin{cases} \Omega^{i+1} \oplus \mathcal{B}^i \oplus \Omega^{i+1}, & \text{for } 0 \leq i \leq p-1; \\ \Omega^{i+1} \oplus \mathcal{B}^i, & \text{for } p \leq i \leq q-1, \end{cases} \]

and where the maps are induced by \( \partial, \overline{\partial} \) and injection due to the bidegree.

To use the above resolution, we set

\[ \mathcal{L}^j = \ker h_j, \]

and then get the exact sequence of sheaves:

\[ 0 \longrightarrow \mathcal{L}^q \longrightarrow \mathcal{B}^q \xrightarrow{h_q} \mathcal{B}^{q+1} \longrightarrow \cdots \longrightarrow \mathcal{B}^{p+q-1} \xrightarrow{\partial \overline{\partial}} \mathcal{A}^{p+1,q+1}, \]

and

\[ 0 \longrightarrow \mathcal{L}^{q-1} \longrightarrow \mathcal{A}^{q-1} \longrightarrow \mathcal{L}^q \longrightarrow 0, \]

\[ 0 \longrightarrow \mathcal{L}^{q-2} \longrightarrow \mathcal{A}^{q-2} \longrightarrow \mathcal{L}^{q-1} \longrightarrow 0, \]

\[ \vdots \]

\[ 0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{A}^0 \longrightarrow \mathcal{L}^1 \longrightarrow 0. \]

Since the sheaves \( \mathcal{B}^s \) \((q \leq s \leq p + q - 1)\) are fine sheaves, one obtains

\[ (5.2) \quad H_{BC}^{n-p,n-q}(X) \cong H_{BC}^{n-q,n-p}(X) \cong H_A^{p,q}(X) \cong H_{A}^{p,q}(X) \cong H^p(X, \mathcal{L}^q). \]

However, it is easy to show that the sheaf \( \mathcal{L}^q \) is not coherent analytic, not to mention \( \pi \)-flat. Thus, it seems difficult to apply Theorem 5.6 directly to get the desired consequence. The readers probably observe that here we get (5.2) in a roundabout way. In fact, one can also gain the similar result straightforwardly, i.e., find some sheaves \( \mathcal{V}^q \), such that \( H_{BC}^{p,q}(X) \cong H^p(X, \mathcal{V}^q) \) by virtue of the corresponding resolutions. These two ways are actually equivalent.

Based on the above discussions, here appears a basic but possibly difficult question:

**Question 5.8.** Let \( \pi : (X, X) \to (B, 0) \) be a small holomorphic deformation of a compact complex manifold \( X \). Is the function \( B \ni t \mapsto h_{BC}^{p,q}(X_t) \) upper semi-continuous with respect to the analytic Zariski topology?

**Remark 5.9.** One can give an affirmative answer to Question 5.8 in a special case, that is any compact complex manifold \( X \) satisfying \( b_2 = 0 \) and \((p, q) = (1, 1)\), where \( b_k(X) = \dim_C H^k(X, \mathbb{C}) \). Indeed, under these assumptions, we have (cf. [Mc19, Theorem 6.2] for example)

\[ h_{BC}^{1,1} = h_{A}^{n-1,n-1} = 2h_{BC}^{0,1} - b_1. \]

Recently, Xia [Xi21a, Theorem 1.1 and Remark 3.6] confirmed the above question when the type is \((p, 0)\) or \((0, q)\). One motivation to consider Question 5.8 is:

**Theorem 5.10.** Suppose that the answer to Question 5.8 with \( \dim B = 1 \) is positive, and the fibers \( X_t \) in an uncountable subset of \( B \) are \( p \)-Kähler. Then any very general fiber is still \( p \)-Kähler. Here a very general fiber \( X_b \) is a fiber over \( b \in B \) that belongs to \( V \) which is a complement of countably many Zariski closed proper subsets \( Z_i \) of \( B \).
Proof. The theorem follows as long as one notices that the \( \partial \overline{\partial} \)-property in Theorem 5.1 actually can be replaced by the deformation invariance of \((p, p)\)-Bott–Chern numbers as shown in [RWZ21, Remark 4.13]. □

Noteworthy is the following question about the analytic openness property of the \( \partial \overline{\partial} \)-property.

**Question 5.11** ([Xi21a, p. 2]). With the setting of Question 5.8, is the set

\[ T := \{ t \in B \mid X_t \text{ satisfies the } \partial \overline{\partial}_t \text{-property} \} \]

an analytic open set (i.e., complement of an analytic subset) of \( B \)?

If one can confirm Question 5.11 with \( T \neq \emptyset \), then \( T = B \) or \( T = B \setminus \{0\} \) possibly after shrinking the open base \( B \subset \mathbb{C} \).

**Proposition 5.12.** If the answer to Question 5.8 is positive, then so is the answer to Question 5.11.

**Proof.** [AT13, Theorem] tells us that for every \( k \in \mathbb{N} \) and \( t \in B \),

\[ h^k_{BC}(X_t) + h^k_A(X_t) \geq 2b_k(X_t) = 2b_k, \]

where \( h^k_{BC}(X_t) := \sum_{p+q=k} h^{p,q}_{BC}(X_t) \) and \( h^k_A(X_t) := \sum_{p+q=k} h^{p,q}_A(X_t) \). Moreover, the equality in (5.3) holds if and only if \( X_t \) satisfies the \( \partial \overline{\partial}_t \)-property.

As the sum of finitely many upper semi-continuous functions (with respect to any topology) is still upper semi-continuous, one can combine it with (2.1) to see that

\[ T = B \setminus \bigcup_{k \in \mathbb{N}} \{ t \in B \mid h^k_{BC}(X_t) + h^k_A(X_t) \geq 2b_k + 1 \} \]

is an analytic open subset of \( B \). □

6. Local stabilities of transversely \( p \)-Kähler foliations

We first introduce the concept of transversely \( p \)-Kähler foliation, then extend the ‘weak’ \( \partial \overline{\partial} \)-properties to the foliated case. We strengthen the local stabilities theorems in [EKAG97, Rź21] to the ‘higher positivity’ cases, which can also be viewed as the foliated version of the main results in [RWZ19]. In a sense, our use of the power series approach has led to increased efficiency when contrasted with the techniques employed in [EKAG97, Rź21], as illustrated in Remark 6.35.

6.1. Transversely \( p \)-Kähler structures. In this subsection, we first provide a rapid review of transverse structures on foliations (see [Mo88, Ni11] for example), and then give the definition of transversely \( p \)-Kähler foliation. Furthermore, a class of examples of homologically orientable transversely Hermitian foliations on compact nilmanifolds that are transversely \( p \)-Kähler (with \( p > 1 \)), but not transversely Kähler are displayed.

**Definition 6.1.** A foliation \( \mathcal{F} \) of codimension \( s \) on an \( m \)-dimensional smooth manifold \( M \) is a cocycle \( \mathcal{U} = (\{U_i\}, \{\pi_i\}, \{\gamma_{ij}\}) \) modelled on an \( s \)-dimensional smooth manifold \( N_0 \) consisting of

(i) an open covering \( \{U_i\} \) of \( M \),
(ii) submersions \( \pi_i : U_i \rightarrow N_0 \) with connected fibers defining \( \mathcal{F} \),
(iii) diffeomorphism transition functions \( \gamma_{ij} : \pi_j(U_i \cap U_j) \rightarrow \pi_i(U_i \cap U_j) \) such that \( \pi_i = \gamma_{ij} \circ \pi_j \).
We call the disjoint union \( N = \coprod_{U_i \in \mathcal{U}} \pi_i(U_i) \) the transverse manifold of \( \mathcal{F} \) associated to the cocycle \( \mathcal{U} \). The local diffeomorphisms \( \gamma_{ij} \) generate a pseudogroup \( \mathcal{H} \) of transformations on \( N \). One can equate the space of leaves \( M/\mathcal{F} \) of the foliation \( \mathcal{F} \) with \( N/\mathcal{H} \).

A transverse structure to \( \mathcal{F} \) is an \( \mathcal{H} \)-invariant geometric structure on \( N \). For instance:

**Definition 6.2.** The foliation \( \mathcal{F} \) on \( M \) is:

(i) Riemannian if there exists an \( \mathcal{H} \)-invariant Riemannian metric on \( N \), which is equivalent to the existence of a Riemannian metric \( g \) with \( L_X g = 0 \) for any \( X \) in the normal bundle \( \mathcal{N} \mathcal{F} := TF/\mathcal{T}\mathcal{F} \), where \( \mathcal{T}\mathcal{F} \) is the bundle tangent to the leaves;

(ii) transversely holomorphic if on \( N \) there exists a complex structure of which \( \mathcal{H} \) is a pseudogroup of local holomorphic transformations, which is equivalent to the existence of an almost complex structure \( J \) on \( \mathcal{N}\mathcal{F} \) such that:

(a) \( L_X J = 0 \) for any vector field \( X \) tangent to the leaves,

(b) for any two sections \( Z_i \) \( (i = 1, 2) \) of the normal bundle, we have

\[
N_j[Z_1, Z_2] := [JZ_1, JZ_2] - J[Z_1, Z_2] - J[JZ_1, Z_2] + J^2[Z_1, Z_2] = 0;
\]

(iii) transversely Hermitian if it is transversely holomorphic and there exists an \( \mathcal{H} \)-invariant Hermitian metric on \( N \);

(iv) transversely Kähler if on \( N \) there exists an \( \mathcal{H} \)-invariant Kähler structure.

**Remark 6.3.** By Definition 6.2.(ii), one knows that a transversely holomorphic foliation of complex codimension \( s \) with \( m = 2s \) is nothing but a complex structure on \( M \), likewise for the last two cases.

**Definition 6.4.** A foliation \( \mathcal{F} \) is said to be transversely orientable if the normal bundle \( \mathcal{N}\mathcal{F} \) is orientable. This is equivalent to the orientability of \( N \) plus the orientation preservation of all \( \gamma_{ij} \).

**Definition 6.5.** A foliation is said to be transversely parallelizable (abbreviated as TP) if there exist \( s \) linearly independent \( \mathcal{H} \)-invariant vector fields.

**Definition 6.6.** A smooth form \( \alpha \) on \( M \) is called basic if for any vector field \( X \) tangent to the leaves of \( \mathcal{F} \) the equalities hold

\[
\iota_X \alpha = \iota_X d\alpha = 0.
\]

Basic 0-forms all also called basic functions.

Basic forms on the foliated manifold \( (M, \mathcal{F}) \) are in one-to-one correspondence with \( \mathcal{H} \)-invariant forms on \( N \). For any basic form \( \alpha \), it is easy to see that \( \iota_X \alpha \) is also basic. Hence, the set \( A^\bullet(M/\mathcal{F}) \) of basic forms is a subcomplex of \( (A^\bullet(M), d) \). The cohomology of \( (A^\bullet(M/\mathcal{F}), d) \) is called the basic cohomology of \( (M, \mathcal{F}) \) and denoted by \( H^\bullet(M/\mathcal{F}) \). For a Riemannian foliation of codimension \( s \) on a connected compact manifold \( M \), \( H^s(M/\mathcal{F}) = \mathbb{R} \) or \( H^s(M/\mathcal{F}) = 0 \) by a result of [EKASH85]. One then has

**Definition 6.7.** The foliated manifold \( (M, \mathcal{F}) \) (or just \( \mathcal{F} \)) is called homologically orientable if \( H^s(M/\mathcal{F}) = \mathbb{R} \).

**Remark 6.8.** The above condition is equivalent to the existence of a (real) volume form on the leaves \( \chi \) which is relatively closed, that is, \( d\chi(X_1, \ldots, X_{m-s}, Y) = 0 \) for \( X_1, \ldots, X_{m-s} \) tangent to \( \mathcal{F} \), cf. [Ru79]. In that case, one can complete the transverse metric by a Riemannian metric on the whole manifold for which the leaves are minimal and \( \chi \) is associated to this metric and we then can also say that \( \mathcal{F} \) is taut.
Furthermore, suppose that the foliation $\mathcal{F}$ is transversely holomorphic of complex codimension $r$ with $s = 2r$. We say that a system of local coordinates

$$(x, z) = (x^1, \ldots, x^{m-s}, z^1, \ldots, z^r)$$

on $U_i$ is adapted to $\mathcal{F}$ (or more precisely to the submersion $\pi_i$) if for each point $w \in U_i$, one has $\pi_i(w) = (z^1(w), \ldots, z^r(w))$. On $U_i$, and in terms of adapted coordinates, the leaves of $\mathcal{F}$ are defined by $z^j = \text{constant}$. This notion surely can be defined similarly for general foliations. So in adapted coordinates $(x, z)$ the tangent bundle $T\mathcal{F}$ is generated by the vector fields $\partial_{x^1}, \ldots, \partial_{x^{m-s}}, \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^r}$. That is, $T\mathcal{F}$ is the kernel of the differential $d\pi_i$. As in the case of a complex structure, the complexification $N\mathcal{F}^C$ of the normal bundle $N\mathcal{F}$ of a transversely holomorphic foliation $\mathcal{F}$ decomposes as

$$N\mathcal{F}^C = N\mathcal{F}^{1,0} \oplus N\mathcal{F}^{0,1}$$

and the corresponding dual bundles fulfill the same relation $N^*\mathcal{F}^C = N^*\mathcal{F}^{1,0} \oplus N^*\mathcal{F}^{0,1}$. Notice that, locally,

$$N^*\mathcal{F}^{1,0} = \langle dz^1, \ldots, dz^r \rangle_C \quad \text{and} \quad N^*\mathcal{F}^{0,1} = \langle d\bar{z}^1, \ldots, d\bar{z}^r \rangle_C.$$

Let $\tau : E \to M$ be a complex vector bundle of rank $N$ defined by a cocycle $\{V_i, g_{ij}\}$, where $\{V_i\}$ is an open cover of $M$ and $g_{ij}$ are the transition functions $g_{ij} : V_i \cap V_j \to \text{GL}(N, \mathbb{C})$ satisfying the cocycle condition:

$$g_{ij}(w) = g_{ik}(w) \cdot g_{kj}(w) \quad \text{for} \quad w \in V_i \cap V_j \cap V_k.$$ 

We say that $E$ is an $\mathcal{F}$-bundle if the functions $g_{ij}$ are basic on $V_i \cap V_j$. Then for a $\mathcal{F}$-bundle $E$ on $M$, a section $\alpha \in C^\infty(M, E)$ is said to be basic if the local representative functions of $\alpha$ with respect to an adapted system of local coordinates which can trivialize $E$ are basic. It can be proved easily that $\bigwedge^k(N^*\mathcal{F}^C)$, the $k$-th exterior product of the complexification of the conormal bundle of $\mathcal{F}$, is an $\mathcal{F}$-bundle. Therefore, we can view basic $k$-forms as basic sections of $\bigwedge^k(N^*\mathcal{F}^C)$. And a basic $k$-form $\alpha$ is of pure type $(p, q)$ if for any point of $M$ there exists an adapted system of local coordinates described as above such that

$$\alpha(x, z) = \sum_{1 \leq i_1 < \cdots < i_p \leq r, 1 \leq j_1 < \cdots < j_q \leq r} f_{i_1 \cdots i_p j_1 \cdots j_q}(z, \bar{z}) dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}.$$

Let us denote by

$$A^k_c(M/\mathcal{F}) \quad \text{(resp.} \quad A^p,q_c(M/\mathcal{F})\text{)}$$

the space of complex valued basic $k$-forms (resp. the space of complex valued basic forms of pure type $(p, q)$) on the transversely holomorphic foliated manifold $(M, \mathcal{F})$. Then,

$$A^k_c(M/\mathcal{F}) = \bigoplus_{p+q=k} A^p,q_c(M/\mathcal{F}).$$

Also note that the exterior derivative $d$ can be decomposed into the sum of operators $\partial$ and $\bar{\partial}$ of order $(1, 0)$ and $(0, 1)$, respectively. In other words,

$$\partial : A^{p,q}(M/\mathcal{F}) \to A^{p+1,q}(M/\mathcal{F}) \quad \text{and} \quad \bar{\partial} : A^{p,q}(M/\mathcal{F}) \to A^{p,q+1}(M/\mathcal{F}).$$

The differential complex $\{(A^{p,q}(M/\mathcal{F}), \partial, \bar{\partial})\}$ is called the $p$-th basic Dolbeault complex of $(M, \mathcal{F})$. Its cohomology is called the $p$-th basic Dolbeault cohomology of $(M, \mathcal{F})$ which is denoted by $H^{p,q}(M/\mathcal{F})$. We denote by $h^{p,q}_c(M/\mathcal{F})$ the dimension of $H^{p,q}(M/\mathcal{F})$ over $\mathbb{C}$ and call it the $(p, q)$-basic Hodge number.

Next, we introduce the following new notion on the transverse structures of foliations, which can be seen as the transverse version of Definition A.1.
**Definition 6.9.** A transversely holomorphic foliation $\mathcal{F}$ of complex codimension $r$ is said to be *transversely $p$-Kähler* for the integer $p$ with $1 \leq p \leq r$ if there exists an $\mathcal{H}$-invariant $p$-Kähler structure on $N$, or equivalently, if $\mathcal{F}$ admits a *transversely $p$-Kähler form*, that is a $d$-closed transverse positive $(p,p)$-basic form. Here we call a real $(p,p)$-basic form $\Omega$ *transverse positive* if at any given $w \in M$, for any independent $\tau_j \in (\mathcal{N}\mathcal{F}^{1,0})_w$, $1 \leq j \leq r-p,$

$$\Omega(w) \wedge \sqrt{-1} \tau_1 \wedge \overline{\tau_1} \wedge \cdots \wedge \sqrt{-1} \tau_{r-p} \wedge \overline{\tau_{r-p}}$$

is strictly positive, or equivalently, if for any point of $M$ there exists an adapted system of local coordinates $(U, x, z)$ containing it such that for any independent $\tau_j \in \mathcal{N}\mathcal{F}^{1,0}|_U$, $1 \leq j \leq r-p,$

$$\Omega|_U \wedge \sqrt{-1} \tau_1 \wedge \overline{\tau_1} \wedge \cdots \wedge \sqrt{-1} \tau_{r-p} \wedge \overline{\tau_{r-p}}$$

is strictly positive.

**Remark 6.10.** (1) Notice that in Definition 6.9, $\tau_j$ is not required to come from a global basic form. Indeed, different from the classical case, we can’t always extend a nonzero local basic form to be a global one due to the lack of ‘transverse version’ of unity of partition. See for example [BH22, Definition of the degree of a transversely Hermitian vector bundle (1.3)] for the same reason.

(2) Our ‘transverse positive’ definition for a basic form is equivalent to the one in [ZZ21, page 8] (although they are more concerned with homologically orientable transversely holomorphic foliation).

**Remark 6.11.** Analogously to Lemma 5.2, one can show that a foliation $\mathcal{F}$ is transversely 1-Kähler (resp. transversely $(r-1)$-Kähler) if and only if $\mathcal{F}$ is transversely Kähler (resp. transversely balanced). Here, we say that $\mathcal{F}$ is *transversely balanced* if it admits a real transverse positive $(1,1)$-basic form $\omega$ such that $\omega^{-1}$ is $d$-closed, or equivalently, if it admits a $d$-closed strictly positive $(r-1,r-1)$-basic form, cf. [Mi82, (4.8)]. Similarly, we say that $\mathcal{F}$ is *transversely Gauduchon* if it admits a real transverse positive $(1,1)$-basic form $\omega$ such that $\omega^{-1}$ is $\partial\overline{\partial}$-closed mimicking the definition as in the complex manifold case, cf. [Ga77]. Clearly, every transversely balanced foliation is also transversely Gauduchon.

Next, we shall present a class of examples of homologically orientable transversely Hermitian foliations on compact nilmanifolds which are transversely $p$-Kähler (with $p > 1$), but not transversely Kähler, motivated by [CW90] and [AB91].

**Example 6.12.** Let $N_{2r+1}$ be the following Heisenberg type group (which is a simply connected nilpotent Lie group):

$$N_{2r+1} := \left\{ A \in \text{GL}(r+2, \mathbb{C}) \mid A = \begin{bmatrix} 1 & X & z \\
0 & I_r & Y \\
0 & 0 & 1 \end{bmatrix} \right\},$$

and let $G_{2r+1}$ be the discrete subgroup of $N_{2r+1}$ all of whose entries are Gaussian integers $\alpha_1 + \sqrt{-1}\alpha_2$ with $\alpha_i \in \mathbb{Z}$. Then the homogenous manifold $N_{2r+1}/G_{2r+1}$ becomes a holomorphically parallelizable compact connected complex manifold, denoted by $\eta\beta_{2r+1}$. For $r = 1$, $\eta\beta_3$ is the *Iwasawa manifold*, the classical example of compact non-Kähler threefold, cf. [AB91, page 205].

Now following the ideas presented in the construction of examples for foliations with specific structures (as discussed in [CW90, Section 2]), we intend to create a class of examples in question by use of the manifold $\eta\beta_{2r+1}$. Let $\Gamma_{2r+1}$ be the following finitely
generated subgroup of \( N_{2r+1} \) of matrices of the form (which contains the uniform group \( G_{2r+1} \) of \( N_{2r+1} \)): 

\[
\begin{bmatrix}
1 & X + sX' & z + sz' \\
0 & I_r & Y \\
0 & 0 & 1
\end{bmatrix}
\]

where \( X, X', Y \) are matrices with Gaussian integer entries, \( z, z' \) also are Gaussian integers and \( s \) is a fixed irrational number.

Then \( \Gamma_{2r+1} \) can be identified with the Gaussian integer lattice of the group \( U_{3r+2} \) of matrices of the form:

\[
\begin{bmatrix}
1 & x_1 & \cdots & x_r & x'_1 & \cdots & x'_r & z & z' \\
0 & 1 & \cdots & 0 & 0 & \cdots & 0 & y_1 & 0 \\
0 & 0 & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 1 & 0 & \cdots & 0 & y_r & 0 \\
0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & y_1 \\
0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & y_r \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

where \( x_i, x'_i, y_i, y'_i, z, z' \in \mathbb{C} \) for any \( i \in \{1, \ldots, r\} \). Furthermore, there exists a surjective submersion \( u_r: U_{3r+2} \to N_{2r+1} \) (also with connected fibers) given by the correspondence

\[(x_i, x'_i, z, z', y_i, y'_i) \mapsto (x_i + s x'_i, z + s z', y_i).
\]

The foliation defined by the submersion \( u_r \) is \( \Gamma_{2r+1} \)-invariant and therefore it projects to a complex \((2r+1)\)-codimensional foliation, denoted by \( F_{2r+1} \) (which depends on \( U_{3r+2}, \Gamma_{2r+1} \) and \( u_r \)), on a complex \((3r+2)\)-dimensional compact manifold \( M_{3r+2} := U_{3r+2}/\Gamma_{2r+1} \). Noteworthy to mention that any foliated geometric structures on the foliated manifold \( (M_{3r+2}, F_{2r+1}) \) correspond bijectively to \( \Gamma_{2r+1} \)-invariant ones on \( N_{2r+1} \). In particular, the basic forms on \( (M_{3r+2}, F_{2r+1}) \) are in one-to-one correspond with the \( \Gamma_{2r+1} \)-invariant forms on \( N_{2r+1} \). So the same reasoning as in [AB91, Section 4] will enable one to show that for arbitrary \( r \), \( F_{2r+1} \) is not transversely \( p \)-Kähler for \( 1 \leq p \leq r \) and it is transversely \( p \)-Kähler for \( r+1 \leq p \leq 2r+1 \). The homologically orientability property of \( F_{2r+1} \) can be proved utilizing the method provided in [CW91, page 182] similarly.

### 6.2. Transversely elliptic operators

In this subsection, we recall the notion of a transversely elliptic basic differential operator and the construction of the scalar product both focusing on the space of basic forms, cf. [EKA90]. For the rest of this paper, unless otherwise stated, we always assume that the foliation \( F \) is transversely Hermitian of complex codimension \( r \) with \( s = 2r \) and that the manifold \( M \) is compact.

**Definition 6.13.** A basic differential operator of order \( l \) on the space of complex valued basic forms is a linear map \( D: A^\bullet_c(M/\mathcal{F}) \to A^\bullet_c(M/\mathcal{F}) \) such that in adapted coordinates \( (x^1, \ldots, x^{n-1}, y^1, \ldots, y^s) \) it has the expression:

\[
D = \sum_{|\mu| \leq l} a_\mu(y) \frac{\partial^{|\mu|}}{\partial^{\mu_1} y_1 \ldots \partial^{\mu_s} y^s}
\]

where \( \mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}^s, |\mu| = \mu_1 + \cdots + \mu_s \) and \( a_\mu \) are matrices of appropriate size with basic functions as coefficients. The principal symbol of \( D \) at the point \( w = (x, y) \)
and the basic covector $\xi \in (N^*\mathcal{F})_w$, $\xi = (\xi_1, \ldots, \xi_s)$ is the linear map
\[
\sigma(D)(w, \xi) : \bigwedge(N^*\mathcal{F}^C)_w \rightarrow \bigwedge(N^*\mathcal{F}^C)_w
\]
defined by
\[
\sigma(D)(w, \xi)(\eta) = \sum_{|\alpha| = l} \xi_1^{\alpha_1} \cdots \xi_s^{\alpha_s} a_\alpha(y)(\eta).
\]
We say that $D$ is transversely elliptic if $\sigma(D)(z, \xi)$ is an isomorphism for every $w \in M$ and every basic covector $\xi$ different from 0.

**Construction 6.14.** Now we come to briefly introduce the construction of the scalar product on the space $A^{p,q}(M/\mathcal{F})$ of $(p, q)$-basic forms, viewed as basic sections of
\[
N_{p,q} := \bigwedge^p (N^*\mathcal{F}^C).
\]
In this paper, we will always follow this construction. One starts with the principal $SO(s)$-bundle
\[
p : M^\# \rightarrow M
\]
of orthonormal frames transversal to $\mathcal{F}$, lifting the foliation $\mathcal{F}$ to a TP transversely Hermitian foliation $\mathcal{F}^\#$ on $M^\#$ with $\dim M^\# = \dim \mathcal{F}$. In addition, the foliation is $SO(s)$-invariant, i.e., for any element $a \in SO(s)$ and any leaf of $\mathcal{F}^\#$, $a(L)$ is also a leaf of $\mathcal{F}^\#$. We can further choose some transverse metric such that it is invariant with respect to the $SO(s)$-action and the fibers of $p : M^\# \rightarrow M$ are of measure 1.

Let $N_{p,q}^\# = \pi^*N_{p,q}$. Then $N_{p,q}^\#$ is an $SO(s)$-bundle and in fact a Hermitian $\mathcal{F}^\#$-bundle (cf. [EKA90, Definition 2.5.2]). Also, one has the following canonical isomorphism:
\[
A^{p,q}(M/\mathcal{F}) \rightarrow C^\infty_{SO(s)}(N_{p,q}^\# / \mathcal{F}^\#),
\]
where $C^\infty_{SO(s)}(N_{p,q}^\# / \mathcal{F}^\#)$ is the space of basic sections of $N_{p,q}^\#$ which are invariant under the action of $SO(s)$. By invoking the results in [Mo82], one can get a compact manifold $W$ and a fiber bundle
\[
\pi : M^\# \rightarrow W
\]
such that the fibers amount to the closures of leaves of $\mathcal{F}^\#$, and then extend the above transverse metric to a Hermitian metric on $M^\#$ such that the fibers of $\pi : M^\# \rightarrow W$ also possess measure 1. We then call the manifold $W$ the basic manifold of $\mathcal{F}$. The $SO(s)$-action on $M^\#$ descends to an $SO(s)$-action on $W$. The bundle $N_{p,q}^\#$ on $M^\#$ is associated with a Hermitian bundle denoted by $\widetilde{N}_{p,q}$ on $W$, which will be called the useful bundle corresponding to $N_{p,q}$. We will denote the Hermitian scalar product on $\widetilde{N}_{p,q}$ by $\widetilde{h}_{p,q}$, which is indeed induced by the transversely Hermitian metric on $(M, \mathcal{F})$. One can show that $\widetilde{N}_{p,q}$ is also an $SO(s)$-bundle and there exists a canonical isomorphism of $A(W)$-module (here $A(W)$, as the ring of smooth functions on $W$, is identified with the ring of basic functions on $M$) by utilizing (6.1):
\[
\mathcal{B} : A^{p,q}(M/\mathcal{F}) \rightarrow C^\infty_{SO(s)}(\widetilde{N}_{p,q}).
\]
where $C^\infty_{SO(s)}(\widetilde{N}_{p,q})$ denotes the space of smooth sections of $\widetilde{N}_{p,q}$ which are invariant under the action of $SO(s)$.

Now, for arbitrary $\alpha, \beta \in A^{p,q}(M/\mathcal{F})$, we define their scalar product by:
\[
\langle \alpha, \beta \rangle := \int_W \widetilde{h}_{p,q}(\mathcal{B}(\alpha), \mathcal{B}(\beta)) (w) \, d\mu(w),
\]
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where $\mu$ is the volume form associated with the Hermitian metric induced by that of $M^\#$. The transverse $\star$-operator can be defined fiberwise on the orthogonal complements of the spaces tangent to the leaves standardly. Furthermore, one can define the Hölder norms or Sobolev norms in the space $A^{p,q}(M/\mathcal{F})$ thanks to the isomorphism (6.2). For example, for $\varphi \in A^{p,q}(M/\mathcal{F})$ and $k \geq 0$ ($k \in \mathbb{Z}$), $0 < \alpha < 1$ ($\alpha \in \mathbb{R}$), we set

$$||\varphi||_{k+\alpha} := ||\mathcal{D}(\varphi)||_{k+\alpha}.$$ 

The readers can refer to [EKA90, Section 3.5.6] for more details.

**Remark 6.15.** The scalar product constructed as above can be used to define $\delta$ (resp. $\partial^*$, resp. $\overline{\partial}'$) as the operators adjoint to $d$ (resp. $\partial$, resp. $\overline{\partial}$). We then can define the basic Dolbeault Laplacian operator (likewise for $\overline{\partial}$ and its joint transverse dual Lefschetz operator)

$$\square_b^p := \overline{\partial} \partial^* + \partial^* \overline{\partial}.$$ 

Noteworthy is the following formula showed in [ER96, Proposition 2.2], cf. also [AL92]:

$$\delta \beta = (-1)^{(p+1)+1} \star (d + (P \zeta) \wedge) \star \beta,$$

where $\beta$ is a complex valued $p$-basic form and $P \zeta$ is a 1-basic form dependent on $\mathcal{F}$ (a marginally modified mean curvature). One can also refer to [ER96, line 10 in page 1262] for the explicit definition for the operator

$$\star : A^p(M/\mathcal{F}) \to A^{n-p}(M/\mathcal{F}).$$

And we have

$$\star^2 = (-1)^{p(s-p)} \text{id}.$$ 

We split $P \zeta$ into forms $\zeta_1$ and $\zeta_2$ of types $(1,0)$ and $(0,1)$, respectively, to get

$$\partial^* \beta = (-1)^{s(p+1)+1} \star (\partial + \zeta_1 \wedge) \star \beta,$$

and $\overline{\partial}' \beta = (-1)^{s(p+1)+1} \star (\overline{\partial} + \zeta_2 \wedge) \star \beta$

with respect to the induced operator

$$\star : A^{p_1,p_2}(M/\mathcal{F}) \to A^{r-p_1,r-p_2}(M/\mathcal{F})$$

with $p_1 + p_2 = p$.

If $\mathcal{F}$ is further required to be transversely Kähler with a transverse Kähler form $\omega$. We then can define the transverse Lefschetz operator:

$$L : A^{p_1,p_2}(M/\mathcal{F}) \to A^{p_1+1,p_2+1}(M/\mathcal{F}), \; L \alpha := \alpha \wedge \omega$$

and its joint transverse dual Lefschetz operator

$$\Lambda : A^{p_1,p_2}(M/\mathcal{F}) \to A^{p_1-1,p_2-1}(M/\mathcal{F}), \; \Lambda := \star^{-1} L \star$$

with respect to the operator $\star$ (6.4). One may observe that the equalities (6.15) and (6.3) are not so concise as those in the Hermitian manifold case. Generally, the forms $\zeta_i$ ($i = 1, 2$) are exactly nonzero, see [Ca84] and [Mo88, Example 2.3 in Appendix B] for example. Sometimes to make things simpler, one may need the homologically orientability condition, see Remark 6.16 for more details.

**Remark 6.16.** The homologically orientability or taut (cf. Definition 6.7 and Remark 6.8) hypothesis will enable one to define an inner product on $A^p(M/\mathcal{F})$ without using the basic manifold $W$ introduced in Construction 6.14. As in the classical case, one can define the Hodge star operator

$$\star_b : A^r(M/\mathcal{F}) \to A^{s-r}(M/\mathcal{F})$$

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in the following way. Let $U$ be an open set on which the foliation is trivial. Let $\theta_1, \ldots, \theta_s$ be real 1-forms such that $(\theta_1, \ldots, \theta_s)$ is an orthonormal basis of the free module $\Omega^1(U/\mathcal{F})$. Then define $\ast_b$ by
\[
\ast_b(\theta_{i_1} \wedge \cdots \wedge \theta_{i_p}) = \epsilon \theta_{j_1} \wedge \cdots \wedge \theta_{j_{s-p}},
\]
where $\{j_1, \ldots, j_{s-p}\}$ is the increasing complementary sequence of $\{i_1, \ldots, i_p\}$ in the set $\{1, \ldots, s\}$ and $\epsilon$ is the signature of the permutation $\{i_1, \ldots, i_p, j_1, \ldots, j_{s-p}\}$. A straightforward calculation shows that $\ast_b$ satisfies the identity $\ast_b \ast_b = (-1)^{p(s-p)} \text{id}$. For $\alpha, \beta \in A^p(M/\mathcal{F})$, we define a Hermitian product by
\[
\langle \alpha, \beta \rangle = \int_M \alpha \wedge \ast_b \beta \wedge \chi.
\]
Then it is easy to see that the adjoint operator $\delta$ of $d$ with respect to (6.5) satisfies
\[
\delta = (-1)^{s(p-1) - 1} \ast_b d \ast_b.
\]
The construction can be repeated for complex valued basic forms on transversely Hermitian foliations, and similarly, one can obtain the analogous clean inequalities for $\partial, \bar{\partial}$ and their adjoint operators.

Remark 6.17. Definition 6.13 and Construction 6.14 are the special cases of the results in [EKA90]. Actually, for a compact manifold endowed with a Riemannian foliation $\mathcal{F}$, El Kacimi Alaoui considered a differential operator $D$ of order $l$ on $C_{\infty}(E/\mathcal{F})$ and can define the scalar product on it, where $E$ is a Hermitian $\mathcal{F}$-bundle and $C_{\infty}(E/\mathcal{F})$ denotes the space of basic sections of $E$ (see these concepts in Subsection 6.1).

El Kacimi Alaoui showed that if a basic differential operator $D$ is (strongly) transversely elliptic, then so is its adjoint operator. Furthermore, a useful fact says that the study of a (strongly) transversely elliptic basic differential operator is equivalent to that of a (strongly) elliptic differential operator in the usual sense of the same order on the basic manifold $W$ acting on the useful bundle of the Hermitian $\mathcal{F}$-bundle $E$ under the action of a compact Lie group, cf. [EKA90, Propositions 2.7.7–2.7.8].

6.3. Basic Bott–Chern cohomology and mild $\partial\bar{\partial}$-property for foliations. We start this subsection by stating some results mainly from [EKA90, EKAG97, RZ17] to be used later, and then extend several notions, lemmata as aforementioned to the foliated case.

One can define the basic Bott–Chern cohomology of $\mathcal{F}$:
\[
H_{BC}^\ast \bullet(M/\mathcal{F}) := \ker \partial \cap \ker \bar{\partial} / \text{im } \partial \bar{\partial}
\]
with the help of the basic Dolbeault double complex, see (6.1). Also, by virtue of Construction 6.14 and Remark 6.15 we can define the basic Bott–Chern Laplacian
\[
\Box_{BC}^b := \partial \bar{\partial} + \bar{\partial} \partial + \partial^* \bar{\partial} + \bar{\partial}^* \partial
\]
and $G_{BC}^b$, is the associated basic Green’s operator. We then have the basic Hodge decomposition of $\Box_{BC}^b$ on $(M, \mathcal{F})$:
\[
A^{p,q}(M/\mathcal{F}) = \ker \Box_{BC}^b \oplus \text{im } (\partial \bar{\partial}) \oplus (\text{im } \partial^* + \text{im } \bar{\partial}^*),
\]
whose three parts are orthogonal to each other with respect to the $L^2$-scalar product (existing on the completion of $A^{p,q}(M/\mathcal{F})$) defined by transversely Hermitian metric, combined with the equality
\[
1 = \mathbb{H}_B^b + \Box_{BC}^b G_{BC}^b = \mathbb{H}_B^b + G_{BC}^b \Box_{BC}^b,
\]
where $\mathbb{H}_B^b$ is the basic harmonic projection operator.
Remark 6.18. In the case when \( F \) is transversely Hermitian, El Kacimi Alaoui established a Hodge–Kodaira decomposition for the basic Dolbeault complex, so our (6.7) is an application of [EKA90, Theorem 3.3.3].

With these preparations, one may carry over the approach verbatim in Lemma 2.1 to obtain

**Lemma 6.19.** Let \( \beta \) be a pure-type basic form on a transversely Hermitian foliation \((M, F)\). Assume that the \( \partial \bar{\partial} \)-equation

\[
\partial \bar{\partial} x = \beta
\]

admits a solution of pure-type basic form. Then an explicit solution of the \( \partial \bar{\partial} \)-equation (6.8) can be chosen as

\[
(\partial \bar{\partial})*G^\beta_{BC, A} \beta,
\]

which uniquely minimizes the \( L^2 \)-norms of all the solutions with respect to the transversely Hermitian metric. Besides, the equalities hold

\[
G^\beta_{BC} (\partial \bar{\partial}) = (\partial \bar{\partial})*G^\beta_{A} \quad \text{and} \quad (\partial \bar{\partial})*G^\beta_{BC} = G^\beta_{A} (\partial \bar{\partial})*,
\]

where \( G^\beta_{BC} \) and \( G^\beta_{A} \) are the associated basic Green’s operators of \( \square^\beta_{BC} \) and \( \square^\beta_{A} \), respectively. Here \( \square^\beta_{BC} \) is defined in (6.6) and \( \square^\beta_{A} \) is the basic Aeppli Laplacian

\[
\square^\beta_{A} = \partial^* \partial \partial^* + \partial^* \partial \partial + \partial \partial^* \partial^* + \partial \partial \partial^* + \partial \partial^* \partial + \partial \partial \partial.
\]

We have the following important properties.

**Proposition 6.20** ([EKAG97, Proposition 6.2], [Rź17, Proposition 3.1]). \( \square^\beta_{BC} \) and \( \square^\beta_{A} \) are all self-adjoint and transversely elliptic.

**Example 6.21.** According to (6.2) and Remark 6.17, we have the following commutative diagram:

\[
\begin{array}{c}
A^{p,q}(M/F) \xrightarrow{\square^\beta_{BC}} A^{p,q}(M/F) \\
\bigg\downarrow \big\Downarrow \\
C^\infty(S\mathcal{O}(s)}(\tilde{N}_{p,q}) \xrightarrow{\square^\beta_{BC}} C^\infty(S\mathcal{O}(s})(\tilde{N}_{p,q}),
\end{array}
\]

where \( \square^\beta_{BC} \) is an ordinary elliptic operator induced by the basic Bott–Chern Laplacian.

Recall that a foliation satisfies the \( \partial \bar{\partial} \)-property if for pure-type basic forms,

(6.9)

\[
\ker \partial \cap \text{im} \bar{\partial} = \ker \bar{\partial} \cap \text{im} \partial = \text{im} \partial \bar{\partial}.
\]

This property is thoroughly studied in the foliated case in [Rź17, Rź21]. Hence, we can extend the ‘weak’ \( \partial \bar{\partial} \)-properties as mentioned before to the foliation version without any difficulties. For example:

**Definition 6.22.** The foliation satisfies the \( (p,q) \)-th mild \( \partial \bar{\partial} \)-property or \( \mathbb{B}^{p,q} \) if for any \((p−1, q)\)-basic form \( \xi \) with \( \partial \bar{\partial} \xi = 0 \) on \((M, F)\), there exists a \((p−1, q−1)\)-basic form \( \theta \) such that \( \partial \bar{\partial} \theta = \partial \xi \).

Inspired by [RWZ21, Observation 2.11], we have

**Lemma 6.23.** Assume that \((M, F)\) satisfies the \( (p, q + 1) \)- and \((q, p + 1)\)-th mild \( \partial \bar{\partial} \)-properties. The system of equations

(6.10)

\[
\begin{cases}
\partial x = \bar{\partial} \zeta, \\
\bar{\partial} x = \partial \xi,
\end{cases}
\]

\(35\)
has a solution of \((p, q)-\text{basic form}\), where \(\zeta, \xi\) are \((p + 1, q - 1)\)- and \((q + 1, p - 1)\)-basic forms, respectively, if and only if
\[
\begin{aligned}
\bar{\partial} \partial \zeta &= 0, \\
\bar{\partial} \partial \xi &= 0.
\end{aligned}
\]

6.4. \( \bar{\partial} \partial \)-property for foliations without homologically orientability. L. A. Cordero–R. A. Wolak [CW91, Lemma 1 in Section 4] showed that a homologically orientable transversely Kähler foliation on a compact manifold satisfies the \( \bar{\partial} \partial \)-property (6.9), cf. also [RŻ17, Theorem 7.1]. In this subsection, we aim to show that the homologically orientability assumption is indeed unnecessary. This result may be well-known to experts; nevertheless, we provide a proof here for the convenience of the readers. This observation will help us remove the \((1, 2)\)-th mild \( \bar{\partial} \partial \)-property assumption in Theorem 1.7 when \( p = 1 \) (which is Corollary 1.8), see also Remark 6.35 (b).

We first establish the following transverse Kähler identities, which eliminate the need for the homologically orientability assumption in [EKA90, Lemma 3.4.4 and Proposition 3.4.5].

**Lemma 6.24.** Let \( \mathcal{F} \) be a transversely Kähler foliation (which is not necessarily homologically orientable) with a transverse Kähler form \( \omega \) on a compact manifold. Then following the notations in Remark 6.15, the identities below hold true:

1. \( [\bar{\partial}, L] = [\partial, L] = 0 \) and \( [\bar{\partial}, \Lambda] = [\partial^*, \Lambda] = 0 \).
2. \( [\bar{\partial}, \Lambda] = -\sqrt{-1} \Lambda \) and \( [\Lambda, \bar{\partial}] = -\sqrt{-1} \partial^* \) and \( [\Lambda, \partial] = \sqrt{-1} \bar{\partial} \).
3. \( \square_b^* = \square_b^* = \frac{1}{2} \square_a^* \) and \( \square_a^* \) commutes with \( \ast, \partial, \bar{\partial}, \partial^*, \bar{\partial}^* \), \( L \), and \( \Lambda \).

**Proof.** Keeping with the same spirit as in the classical setting (cf. e.g. [GH78, page 111] or [Hb05, Proposition 3.1.12]), we will only give a sketch here.

Let us first prove (1). The first assertion in (1) holds thanks to the \( d \)-closedness of the transverse Kähler form \( \omega \). The second assertion in (1) follows from the first one and (6.15): for a complex valued \( p \)-basic form \( \beta \) we have
\[
[\bar{\partial}, \Lambda](\beta) = (-1)^s(p+1)^1 \ast (\bar{\partial} \ast \zeta_2 \wedge) \ast \ast \ast^{-1} L \ast \beta - \ast^{-1} L \ast (-1)^s(p+1)^1 \ast (\bar{\partial} \ast \zeta_2 \wedge) \ast \beta
= (-1)^s(p+1)^1 \ast ([\bar{\partial}, L] + [\zeta_2 \wedge, L]) \ast \beta
= 0.
\]

Similarly for \( [\partial^*, \Lambda] = 0 \).

(2) comes from (1) plus the transverse Lefschetz decomposition theorem for basic forms (the proof is almost the same as the one on an Hermitian manifold). That is, let \( \mathcal{F} \) be a transversely Hermitian foliation, then the following decomposition holds:
\[
A^k_c(M/\mathcal{F}) = \bigoplus_{i \geq 0} L^i(P^{k-2i}(M/\mathcal{F}))
\]
where \( P^{k-2i}(M/\mathcal{F}) = \ker (\Lambda : A^{k-2i}_c(M/\mathcal{F}) \to A^{k-2i-2}_c(M/\mathcal{F})) \) and an element \( \alpha \in P^{k-2i}(M/\mathcal{F}) \) is called **primitive**.

One can derive (3) and (4) via (1) and (2) and the proofs are trivial. For instance,
\[
\sqrt{-1}(\bar{\partial} \partial + \partial \bar{\partial}) = \partial [\Lambda, \partial] + [\Lambda, \partial] \partial
= \partial \Lambda \partial - \partial \Lambda \partial = 0.
\]

The proof is thus completed. \( \square \)
With Lemma 6.24 and the decomposition of the basic Dolbeault cohomology, which also doesn’t require the homologically orientability assumption (see [EKA90, Theorem 3.3.3] and Remark 6.18), in hand, we now can achieve our goal in this subsection. The idea of the proof is the same as that of [Hb05, Corollary 3.2.10] for example so we will omit the proof.

**Theorem 6.25.** With the same settings as in Lemma 6.24, \( \mathcal{F} \) satisfies the \( \partial \bar{\partial} \)-property.

**Remark 6.26.** Similarly, one can show that transverse Kähler foliations on compact manifolds, even without the homologically orientability assumption, can possess some properties as Kähler structures on compact manifolds, including the Hodge decomposition, Hard Lefschetz decomposition, etc. However, it’s important to note that the assumption in question is still required for certain duality-type theorems due to the existence of the forms \( \zeta_i, (i = 1, 2) \) (6.3). Examples of such theorems include the transverse Serre duality and the duality theorem for basic Bott–Chern and Aeppli cohomology (cf. [Rź17, Corollary 3.1]).

**Remark 6.27.** Noteworthy to mention that the converse of Theorem 6.25 isn’t always true. Cordero–Wolak [CW90, Section 2.3] constructed a transversely symplectic but not transversely Kähler foliation on a compact non-complex nilmanifold. Recently, Raźny [Rź17, 6.3] has shown that this foliation satisfies the \( \partial \bar{\partial} \)-property.

### 6.5. Deformation theory of foliations.

In this subsection, we will give some preliminaries on the deformation theory of foliations following [EKAG97, Section 4].

We are going to recall the definition of a deformation of a foliation in a general way. Let \( M \) be a smooth manifold of dimension \( m \). For each \( x \in M \), let \( G_x(M, s) \) be the Grassmannian manifold of \( s \)-planes in \( T_xM \). Then \( G(M, s) := \bigcup_{x \in M} G_x(M, s) \) can be given a structure of a differentiable manifold such that the canonical projection \((x, \tau) \in G(M, s) \rightarrow x \in M \) is a locally trivial fibration, whose fibre is the Grassmannian \( G(m, s) \) of \( s \)-planes in the space \( \mathbb{R}^m \). Then a subbundle of rank \( s \) of \( TM \) is just a section of the bundle \( G(M, s) \rightarrow M \). Denote by \( C^\infty(G(M, s)) \) the space of sections of this bundle. Let \( \tau \in C^\infty(G(M, s)) \). By Frobenius theorem, \( \tau \) is tangent to the foliation if and only if, for any pair \((U, V)\) of (global) sections of \( \tau \), the Lie bracket \([U, V]\) is also a section of \( \tau \). Let \((X_1, \ldots, X_s)\) be a local basis of \( \tau \). Then

\[
U = \sum_{i=1}^{s} a^i X_i \quad \text{and} \quad V = \sum_{j=1}^{s} b^j X_j.
\]

So the bracket \([U, V]\) can be expressed as

\[
[U, V] = \sum_{i,j=1}^{s} \left\{ a^i b^j [X_i, X_j] + (a^i X_i(b^j)) X_j - b^j X_j(a^i X_i) \right\}.
\]

So the value of \([U, V]\) in \( \nu \tau = TM/\tau \) at a point \( x \in M \) depends only on the value of \( U \) and \( V \) at \( x \). Hence, \( Q_\tau(U, V) = \pi ([U, V]) \) is a skew-symmetric bilinear map \( Q_\tau: \tau \times \tau \rightarrow \nu \tau \) where \( \pi: TM \rightarrow \nu \tau \) is the canonical projection. In other words, \( Q_\tau \) is a global section of the vector bundle \( \Lambda^2(\tau, \nu \tau) \) of skew-symmetric bilinear forms on the bundle \( \tau \). The integrability condition of \( \tau \) is equivalent to \( Q_\tau \) identically equals to 0. So we get a map

\[
Q: C^\infty(G(M, s)) \rightarrow \mathcal{Z}^r,
\]
where \( \mathcal{Z} \) is a fibre bundle over \( \mathcal{G}(M, s) \) whose fibre over a point \( \sigma \in \mathcal{G}(M, s) \) is the infinite-dimensional space \( \Omega^2(\sigma, v\sigma) \) of global sections of the bundle \( \Lambda^2(\tau, v\tau) \). The space \( \text{Fol}(M, s) \) of codimension \( s \) foliations on \( M \) is exactly the set \( \{Q = 0\} \). It will be equipped with the \( \mathcal{C}^\infty \)-topology induced by the topology of the Fréchet manifold \( \mathcal{C}^\infty(\mathcal{G}(M, s)) \). Let \( \mathcal{D}(M) \) be the diffeomorphism group of \( M \). Then \( \mathcal{D}(M) \) acts on \( \mathcal{C}^\infty(\mathcal{G}(M, s)) \) and the action preserves \( \text{Fol}(M, s) \). We denote by \( \mathcal{O}_\mathcal{F} \) the orbit of \( \mathcal{F} \in \text{Fol}(M, s) \).

**Definition 6.28.** A deformation of \( \mathcal{F} \) parametrized by an open neighborhood \( U \) of 0 in \( \mathbb{R}^d \) is a continuous map \( \rho : U \to \text{Fol}(M, s), t \mapsto \mathcal{F}_t \) such that \( \rho(0) = \mathcal{F}_0 = \mathcal{F} \).

**Definition 6.29.** A deformation \( \rho : U \to \text{Fol}(M, s), t \mapsto \mathcal{F}_t \) of \( \mathcal{F} \) parametrized by an open neighborhood \( U \) of 0 in \( \mathbb{R}^d \) is called with fixed differentiable type if for all \( t \in U \), \( \rho(t) \in \mathcal{O}_\mathcal{F}, \) i.e., there exists a differentiable family \( \{h_t\}_{t \in U} \) of diffeomorphisms on \( M \) such that \( h_t^* \mathcal{F}_t = \mathcal{F} \).

One has the following nice properties when the initial foliation is transversely holomorphic.

**Proposition 6.30 ([EKAG97, Proposition 4.3]).** Let \( \mathcal{F}_t \) be the deformation of an \( r \)-complex codimensional \( (s = 2r) \) transversely holomorphic foliation \( \mathcal{F} \) with fixed differentiable type.

(i) Suppose that \( \mathcal{F} \) is Riemannian. Then for any \( t \in U \), \( \mathcal{F}_t \) is also Riemannian.

(ii) The family \( \{\mathcal{F}_t\}_{t \in U} \) is lifted to \( M^\# \) in a new family of TP foliations \( \{\mathcal{F}_t^\#\}_{t \in U} \) invariant under \( \text{SO}(s) \) with fixed differentiable type.

(iii) Assume that the foliations \( \mathcal{F}_t \) are transversely holomorphic and \( \mathcal{F}_0 = \mathcal{F} \) possesses a transversely Hermitian metric \( \Omega_0 \). Then \( \{\mathcal{F}_t\}_{t \in U} \) are provided with transversely Hermitian metrics \( \Omega_t \) varying differentiably depending on \( t \).

6.6. **Proof of Theorem 1.7.** We split the proof of Theorem 1.7 in four steps.

**Step (I). Integrable Beltrami differentials, Kuranishi family: foliated version.**

We first briefly introduce the main results obtained in [EKAN89], which is of great significance in our proof.

With the same setting as in Theorem 1.7, El Kacimi Alaoui–M. Nicolau proved the existence of versal space (also called Kuranishi space) for deformations mimicking the construction of the Kuranishi space presented in [Ku64], and the universal property of the versal space was strengthened in [Gb92]. As a result, one can get a family, denoted by

\[ \phi(t) \in A^{0,1}(M/\mathcal{F}, N\mathcal{F}^{1,0}) \]

for the Kuranishi family, satisfying the equation

\[ \overline{\partial} \phi(t) = \frac{1}{2} [\phi(t), \phi(t)] \]

on some analytic set and determining the transversely holomorphic structure on \( (M, \mathcal{F}_t) \). Hence, this construction and \( \phi(t) \) will play the same roles as the Kuranishi’s completeness theorem (introduced in Section 3.2) and the integrable Beltrami differentials \( \varphi(t) \), respectively. Let \( z^1, \ldots, z^r \) and \( \zeta^1, \ldots, \zeta^r \) be the transverse holomorphic coordinates on \( \mathcal{F}_0 \) and \( \mathcal{F}_t \), respectively. Then \( \phi(t) \) can be defined by
\[ (6.11) \quad \phi(t) := \left( \frac{\partial \zeta}{\partial z} \right)^{-1} \left( \frac{\partial \zeta}{\partial \bar{z}} \right)^i d\bar{z}^j \otimes \frac{\partial}{\partial \bar{z}^i}. \]

It will be often written as \( \phi \) briefly. Also, we can define the extension map
\[ e^{\phi|\tau} : A^{p,q}(M/F_0) \rightarrow A^{p,q}(M/F_t), \]
alogous to (3.5) which can preserve all \((p, q)\)-basic forms.

**Step (II). Transverse positive \((p, p)\)-basic forms: deformation openness.**

To prove Theorem 1.7, we need the following key observation, which will be the deformation openness property of the transverse positivity for \((p, p)\)-basic forms. Here we follow the idea as that of \[\text{[RWZ21, the second proof of Theorem 4.9]}\]. Note that to make the proposition holds, the new extension map \( e^{\phi|\tau} \) is a key player. We will use its two properties in the following proof: one is that it depends smoothly on \( t \); the other one is that it can send each kind of (strictly) positive \((p, p)\)-basic forms on \((M, F_t)\) bijectively onto the corresponding one on \((M, F_0)\), which can be showed by an argument similar to the one in \[\text{[RWZ19, Proposition 4.11]}\].

**Proposition 6.31.** Let \( \{F_t\}_{t \in U} \) be a smooth family of transversely Hermitian structures on a compact foliated manifold \((M, \mathcal{F})\) with fixed differentiable type, and \( \Omega_t \) a family of real \((p, p)\)-basic forms, depending smoothly on \( t \). Assume that \( \Omega_0 \) is a transverse positive \((p, p)\)-basic form on \((M, F_0)\). Then \( \Omega_t \) is also transverse positive on \((M, F_t)\) for sufficiently small \( t \).

**Proof.** Let \( \Omega_0 \) be a transverse positive \((p, p)\)-basic form on \((M, F_0)\) and \( \Omega_t \) its real extension on \((M, F_t)\). It is sufficient to show that: at any given point \( x \in M \), there exists a uniform small constant \( \epsilon \) such that for any \( t \in U_\epsilon \), where \( U_\epsilon = \{ t \in \mathbb{R}^d \mid |t| < \epsilon \} \), and any nonzero decomposable \( \tau \in \wedge^{q,0}(N^*F_0)_x \) with \( p + q = r \), we have
\[ (6.12) \quad \Omega_t(x) \cap \sigma_q e^\phi(\tau) \cap e^\phi(\bar{\tau}) > 0. \]

Indeed, let \( \omega_0 \) be a transversely Hermitian metric on \((M, F_0)\). For any fixed point \( x \in M \), we define a continuous function \( f_x(t, [\tau]) \) on \( U_\epsilon \times Y_x \) by
\[ (6.13) \quad f_x(t, [\tau]) := \frac{\Omega_t(x) \cap \sigma_q e^\phi(\tau) \cap e^\phi(\bar{\tau})}{|\tau|_\omega_0(x) \cdot \omega_0(x)^r}, \]
where \( Y_x = \rho(G(q, r))|_x \subset \mathbb{P}(\wedge^{q,0}(N^*F_0)_x) \) is compact. By assumption,
\[ f_x(0, [\tau]) = \frac{\Omega_0(x) \cap \sigma_q \tau \cap \bar{\tau}}{|\tau|_\omega_0(x) \cdot \omega_0(x)^r} > 0. \]
Hence, by the continuity of \( f_x(0, [\tau]) \) on \( t \) and \( [\tau] \) plus the compactness of \( Y_x \), there exists a constant \( \epsilon_x > 0 \) depending only on \( x \), such that
\[ f(x, \overline{U}_{\epsilon_x/2}, Y_x) := f_x(\overline{U}_{\epsilon_x/2}, Y_x) > 0. \]

Let \( \{T_j\}_{j \in J} \) be trivializing adapted covering of \( M \), and choose any \( x \in T_j \) for some \( j \). So we can identify \( Y_x \) and \( Y_y \) for any point \( y \subset T_j \), and \( f_y \) is defined on \( Y_x \). Similarly, by the continuity of \( f \) on \( \overline{U}_{\epsilon_x/2} \times Y_x \), there exists an open neighbourhood \( R_x \subset T_j \) of \( x \) such that
\[ f(R_x, \overline{U}_{\epsilon_x/2}, Y_x) > 0. \]

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Since $M$ is compact, one gets a finite open covering $R_{x_i}, i = 1, \ldots, m, M = \bigcup_{i=1}^{m} R_{x_i}$. Set
\[
\varepsilon := \min_{1 \leq i \leq m} \epsilon_{x_i}/2 > 0.
\]
Then
\[
f(x, \overline{\mathcal{U}_\varepsilon}, Y_{x}) = f_x(\overline{\mathcal{U}_\varepsilon}, Y_{x}) > 0
\]
for any $x \in M$.

Therefore, by virtue of the definition (6.13), for any $|t| \leq \varepsilon$, we have
\[
\Omega_t(x) \wedge \sigma_{\varepsilon} e^{\phi}(\tau) \wedge e^{\varepsilon}(\overline{\tau}) = f_x(t, [\tau])|\tau|_{\omega_0(x)}^2 \cdot \omega_0(x)^t > 0
\]
for any nonzero decomposable $\tau \in \Lambda^{p,0}(N^*F_0)$. This is the desired inequality (6.12).

With these preparations, to prove Theorem 1.7, it suffices to prove the special case $p = q$ of the following theorem.

**Theorem 6.32.** Let $\{F_t\}_{t \in \Delta}$ be a smooth family of transversely Hermitian structures on a compact foliated manifold $(M, \mathcal{F})$ with fixed differentiable type, parametrized by a small complex disc. If $\mathcal{F}_0$ satisfies the $(p, q + 1)$-th and $(q, p + 1)$-th mild $\partial\bar{\partial}$-properties, then there is a d-closed $(p, q)$-basic form $\Omega(t)$ on $(M, \mathcal{F}_t)$ depending smoothly on $t$ with $\Omega(0) = \Omega_0$ for any d-closed $\Omega_0 \in A^{p,q}(M/\mathcal{F})$.

Next we will sketch the proof of Theorem 6.32 by adopting the power series idea used in [RWZ19, Section 4]. Here we just indicate the main procedures emphasizing how we adapt the method to our new setting.

**Step (III). Obstruction equation and construction of power series.**

As both $e^{(1-\delta\phi)^{-1}\tilde{\phi}}$ and $e^{\phi}$ are invertible operators when $t$ is sufficiently small by (6.11), it follows that for any $\Omega \in A^{p,q}(M/\mathcal{F})$,
\[
e^{\phi\bar{\Phi}}(\Omega) = e^{\phi} \circ e^{(1-\delta\phi)^{-1}\tilde{\phi}} \circ e^{-1}(1-\delta\phi)^{-1}\tilde{\phi} \circ e^{-\phi} \circ e^{\phi\bar{\Phi}}(\Omega).
\]

Here we follow the notations: $\overline{\phi\phi} = \phi \cdot \overline{\phi}, \mathbb{1}$ is the identity operator defined as:
\[
\mathbb{1} = \frac{1}{p+q} \left( \sum_{i=1}^{r} dz_i \otimes \frac{\partial}{\partial z_i} + \sum_{i=1}^{r} d\bar{z}_i \otimes \frac{\partial}{\partial \bar{z}_i} \right)
\]
when it acts on $(p, q)$-basic forms, and likewise for others. Set
\[
\bar{\Omega} = e^{-t(1-\delta\phi)^{-1}\tilde{\phi}} \circ e^{-\phi} \circ e^{\phi\bar{\Phi}}(\Omega) = (\mathbb{1} - \overline{\phi\phi}) \cdot \Omega,
\]
where $\Omega$ and $\bar{\Omega}$ are apparently one-to-one correspondence. Here the notation $\cdot \Omega$ denotes the simultaneous contraction. By observing the local expression for any element in $A^{0,1}(M/\mathcal{F}, \mathcal{N}\mathcal{F}^{1,0})$ (see (6.1)), we have

**Proposition 6.33.** Let $\phi \in A^{0,1}(M/\mathcal{F}, \mathcal{N}\mathcal{F}^{1,0})$ on $(M, \mathcal{F})$. Then on the space $A^{\bullet\bullet}(M/\mathcal{F})$,
\[
e^{-\phi} \circ d \circ e^{\phi} = d - \mathcal{L}_\phi^{1,0} + i\overline{\phi\phi}^{-\frac{1}{2}}[\phi, \phi]^t
\]
where $\mathcal{L}_\phi^{1,0} := i\phi \partial - \partial i\phi$ is the Lie derivative.
Together with \((6.14)\) and \((6.15)\), Proposition 6.33 implies that

\[
d(e^{t\phi} l \tilde{\phi}(\Omega)) = d \circ e^{t\phi} \circ e^{t(1-\phi)-1\tilde{\phi}}(\Omega)
\]

\[
= e^{t\phi} \circ (\tilde{\partial} + [\partial, t\phi] + \partial) \circ e^{t(1-\phi)-1\tilde{\phi}}(\tilde{\Omega})
\]

\[
= e^{t\phi}(\tilde{\partial}_\phi + \partial) \sum_{k=0}^{+\infty} A_k
\]

\[
= e^{t\phi}\left(\tilde{\partial}_\phi A_0 + \sum_{k=0}^{+\infty}(\partial A_k + \tilde{\partial}_\phi A_{k+1})\right),
\]

where

\[
A_k := \frac{t^k}{k!}(1-\phi)(1-\tilde{\phi})(\tilde{\Omega})
\]

is a \((p + k, q - k)\)-basic form and

\[
\tilde{\partial}_\phi := \tilde{\partial} + [\partial, t\phi].
\]

Thus, \(d(e^{t\phi} l \tilde{\phi}(\Omega)) = 0\) amounts to

\[
(6.16) \quad \tilde{\partial}_\phi A_0 = 0, \quad \partial A_k + \tilde{\partial}_\phi A_{k+1} = 0, \quad k = 0, 1, 2, \ldots
\]

Following the proof given in [RWZ21, Propositions 4.2 and 4.5], one sees that \((6.16)\) is equivalent to

\[
(6.17) \quad \sum_{k=0}^{+\infty}\left(\tilde{\partial} \circ \frac{k^{b-1}}{(k-1)!} + \partial \circ \frac{k^b}{k!}\right) A_{r-(t-k)} = 0,
\]

where \(\max\{1, r - p - q\} \leq l \leq \min\{2r - p - q, r + 1\}\), \(t^k\phi = 0\) for \(k < 0\) and \(0! = 1\). And the system \((6.17)\) of obstruction equations can be reduced to the following one with only two equations:

\[
(6.18) \quad \begin{cases}
\sum_{k=0}^{\infty}\left(\tilde{\partial} \circ \frac{k^{b-1}}{(k-1)!} + \partial \circ \frac{k^b}{k!}\right) \circ \frac{t^k}{k!}(1-\phi)(1-\tilde{\phi})(\tilde{\Omega}) = 0, \\
\sum_{k=0}^{\infty}\left(\tilde{\partial} \circ \frac{k^{b-1}}{(k-1)!} + \partial \circ \frac{k^b}{k!}\right) \circ \frac{t^{k-1}}{(k-1)!}(1-\phi)(1-\tilde{\phi})(\tilde{\Omega}) = 0.
\end{cases}
\]

Rewrite \((6.18)\) as

\[
\begin{align*}
\tilde{\Omega}' &= -\tilde{\partial}\sum_{k=1}^{\infty}\frac{k^{b-1}}{(k-1)!} \circ \frac{t^k}{k!}(1-\phi)(1-\tilde{\phi})(\tilde{\Omega}), \\
\tilde{\Omega}' &= -\partial\sum_{k=0}^{\infty}\frac{k^{b+1}}{(k+1)!} \circ \frac{t^{k+1}}{k!}(1-\phi)(1-\tilde{\phi})(\tilde{\Omega}),
\end{align*}
\]

where

\[
\tilde{\Omega}' := \tilde{\Omega} + \sum_{k=1}^{\infty}\frac{k^b \phi}{k!} \circ \frac{t^k}{k!}(1-\phi)(1-\tilde{\phi})(\tilde{\Omega}).
\]

By Lemma 6.23, we need to prove

\[
(6.19) \quad \begin{cases}
\tilde{\partial} \sum_{k=1}^{\infty}\frac{k^{b-1}}{(k-1)!} \circ \frac{t^k}{k!}(1-\phi)(1-\tilde{\phi})(\tilde{\Omega}) = 0, \\
\partial \sum_{k=0}^{\infty}\frac{k^{b+1}}{(k+1)!} \circ \frac{t^{k+1}}{k!}(1-\phi)(1-\tilde{\phi})(\tilde{\Omega}) = 0,
\end{cases}
\]

at the \((N + 1)\)-th order if it has not been exceeding for the orders \(N\). Analogous to [RWZ19, p. 471], we write the power series \(\alpha(t)\)

\[
\alpha(t) = \sum_{k=0}^{\infty} \sum_{i+j=k} \alpha_{i,j} t^i \hat{\Omega}^j,
\]

...
of (bundle-valued) \((p, q)\)-basic forms as

\[
\begin{align*}
\alpha(t) &= \sum_{k=0}^{\infty} \alpha_k, \\
\alpha_k &= \sum_{i+j=k} \alpha_{ij} t^i \bar{t}^j,
\end{align*}
\]

where \(\alpha_k = \sum_{i+j=k} \alpha_{ij} t^i \bar{t}^j\) is the \(k\)-order homogeneous part in the expansion of \(\alpha(t)\) and all \(\alpha_{ij}\) are smooth (bundle-valued) \((p, q)\)-basic forms on \(\mathcal{S}_0\) with \(\alpha(0) = \alpha_{0,0}\).

Indeed, one can prove (6.19) by repeating the procedures presented in [RWZ19, p. 476]. Then by Lemmata 6.19 and 6.23, one has:

**Proposition 6.34.** With the notations of Lemmata 6.19 and 6.23, the system of equations (6.10) has a canonical solution

\[ x = \overline{\partial} (\overline{\partial})^* G_{BC} \bar{\partial} \zeta - \partial (\overline{\partial})^* G_{BC} \bar{\partial} \bar{\zeta}. \]

Using Proposition 6.34, one obtains a formal solution of (6.18) by induction, (6.20)

\[
\begin{align*}
\tilde{\Omega}_l &= - \left( \sum_{k=1}^{\infty} \frac{t^k}{k!} \circ \frac{\ell_{(1-\phi)}^{-1}\phi}{k!} (\overline{\partial}) (\overline{\partial})^* G_{BC} \bar{\partial} \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} \circ \frac{\ell_{(1-\phi)}^{-1}\phi}{k!} (\overline{\partial}) \right)_l \\
&+ \partial (\overline{\partial})^* G_{BC} \bar{\partial} \left( \sum_{k=0}^{\infty} \frac{t^{k+1}}{k!} \circ \frac{\ell_{(1-\phi)}^{-1}\phi}{k!} (\overline{\partial}) \right)_l \\
&- \left( \sum_{k=1}^{\min\{q,r,p\}} \frac{t^k}{k!} \circ \frac{\ell_{(1-\phi)}^{-1}\phi}{k!} (\overline{\partial}) \right)_l \\
&+ \partial (\overline{\partial})^* G_{BC} \bar{\partial} \left( \sum_{k=0}^{\min\{q,r,p\}} \frac{t^{k+1}}{k!} \circ \frac{\ell_{(1-\phi)}^{-1}\phi}{k!} (\overline{\partial}) \right)_l.
\end{align*}
\]

**Step (IV). Regularity argument.**

From the induction expression (6.20), one obtains the formal expression of \(\tilde{\Omega}\) :

\[
\tilde{\Omega} = - \left( \sum_{k=1}^{\min\{q,r,p\}} \frac{t^k}{k!} \circ \frac{\ell_{(1-\phi)}^{-1}\phi}{k!} (\overline{\partial}) \right)_l \\
+ \partial (\overline{\partial})^* G_{BC} \bar{\partial} \left( \sum_{k=0}^{\min\{q,r,p\}} \frac{t^{k+1}}{k!} \circ \frac{\ell_{(1-\phi)}^{-1}\phi}{k!} (\overline{\partial}) \right)_l + \Omega_0.
\]

Set

\[
F = \partial (\overline{\partial})^* G_{BC} \bar{\partial} \sum_{k=0}^{\min\{q,r,p\}} \frac{t^{k+1}}{(i+1)!} \circ \frac{\ell_{(1-\phi)}^{-1}\phi}{k!}
\]

and write

\[
(6.22) \quad \Omega_0 = (1 - F)\tilde{\Omega}.
\]

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We claim that $\widetilde{\Omega}(t)$ converges in Hölder norm (6.14) as $t \to 0$. Here, we resort to Construction 6.14 and Remark 6.17 to get two transversely elliptic estimates for a basic form $\psi$,

$$||\bar{\partial}^*\psi||_{k-1,\alpha} \leq C_1 ||\psi||_{k,\alpha}$$

and

$$||\mathcal{G}^b_{BC}\psi||_{k,\alpha} \leq C_{k,\alpha} ||\psi||_{k-4,\alpha},$$

where $k > 3$ and $C_{k,\alpha}$ depends on only on $k$ and $\alpha$, not on $\psi$. As $\phi(t)$ converges smoothly to zero as $t \to 0$, one estimates by (6.22),

$$||\Omega_0||_{k,\alpha} \geq (1 - \epsilon_{k,\alpha}) ||\tilde{\Omega}||_{k,\alpha},$$

where $0 < \epsilon_{k,\alpha} \ll 1$ is some constant depending on $k, \alpha$.

Finally, we proceed to the regularity of $\Omega(t)$ since there is possibly no uniform lower bound for the convergence radius obtained as above in the $C^{k,\alpha}$-norm when $k$ converges to $+\infty$. Similarly, we can transfer the world of transversely elliptic theory to the one of elliptic theory in the ordinary sense as aforementioned. So our argument also lies heavily in the elliptic estimates [Ko86, Appendix 8], [DN55] and also [RWZ21, Subsection 3.2]. Without loss of generality, we just consider the equation:

$$\square^b_{\Omega} = -\square^b_{\Omega} - \sum_{k=1}^{\min(q,r-p)} \frac{k}{k!} \frac{t^k (1-\phi)^{-1}\phi}{k!} (\tilde{\Omega} - \bar{\partial}\bar{\partial}^*(\partial\bar{\partial})^*G^b_{BC}\partial) \sum_{k=1}^{\min(q,r-p)} \frac{t_\phi^{k-1}}{(k-1)!} \frac{t^k (1-\phi)^{-1}\phi}{k!} (\tilde{\Omega})$$

$$+ \square^b_{\partial}(\partial\bar{\partial})^*G^b_{BC}\partial \sum_{k=0}^{\min(q,r-p)} \frac{t_\phi^{k+1}}{(k+1)!} \frac{t^k (1-\phi)^{-1}\phi}{k!} (\tilde{\Omega})$$

by applying the basic Dolbeault Laplacian $\square^b_{\Omega} = \bar{\partial}\partial + \bar{\partial}\bar{\partial}$ to the expression formula (6.21) and omitting the lower-order term $\square^b_{\Omega} \Omega_0$ in this expression.

For each $l = 1, 2, \cdots$, choose a smooth function $\eta^l(t)$ with values in $[0, 1]$:

$$\eta^l(t) = \begin{cases} 1, & \text{for } |t| \leq \left(\frac{1}{2} + \frac{1}{2l+1}\right)\gamma; \\ 0, & \text{for } |t| \geq \left(\frac{1}{2} + \frac{1}{2l+1}\right)\gamma, \end{cases}$$

where $\gamma$ is a positive constant to be determined. Inductively, by Douglis–Nirenberg’s interior estimates [Ko86, Appendix Theorem 2.3], [DN55], for any $l = 1, 2, \cdots$, $\eta^{2l+1}\Omega(t)$ is $C^{k+1,\alpha}$, where $\gamma$ can be chosen independent of $l$. Since $\eta^{2l+1}(t)$ is identically equal to 1 on $|t| < \frac{3}{2}$ which is independent of $l$, $\Omega(t)$ is $C^{\infty}$ on $(M, \mathcal{F}_0)$ with $|t| < \frac{3}{2}$. Then $\Omega(t)$ can be considered as a real analytic family of $(p, q)$-basic forms in $t$ and thus is smooth on $t$.

The proof of Theorem 6.32 (and thus Theorem 1.7) is completed.

**Remark 6.35.** On the assumptions of Theorem 1.7, we should notice:

(a) One can conclude from our proof that the homologically orientability assumption used in [EKAG97, R21] is not necessary in our approach. In the previous works, they follow the original cohomological argument used by Kodaira–Spencer, and it seems that the assumption in question on $\mathcal{F}$ is necessary for the upper semi-continuity theorem for transversely elliptic operators to work (see [R21, Remark 4.6] for example). In our proof, we directly construct the desired transversely $p$-Kähler form. We more faithfully follow the scalar product or Hölder norm on the space of basic forms introduced in Construction 6.14 and allow the existence of a correction term involving the mean curvature of the foliation when defining the adjoint operators, see Remarks 6.15, 6.16.
(b) As shown in Subsection 6.4, any transversely Kähler foliation, even if not necessarily homologically orientable, satisfies the $\overline{\partial}\partial$-property (6.9) and, therefore, obviously fulfills the mild $(1,2)$-th $\overline{\partial}\partial$ property. Thus, compared to the works of [EKAG97, R221], Theorem 1.7 requires weaker assumptions when $p = 1$, specifically, it drops the homologically orientability assumption.

6.7. Deformation invariance of basic Hodge/Bott–Chern numbers. Recently, Raźny showed the rigid property of basic Hodge numbers under deformations of homologically orientable transversely Hermitian foliations on a compact manifold when $\mathcal{F}_0$ satisfies $\partial\overline{\partial}$-property and the foliations are fixed [R221, Corollary 5.3]. Motivated by this, we study the deformation invariance of basic Hodge/Bott–Chern numbers with ‘weak’ $\partial\overline{\partial}$-properties in this subsection.

The following theorem can be seen as the foliated version of Kodaira–Spencer’s results. It seems that we cannot drop the homologically orientability assumption as already mentioned in Remark 6.35 (a).

**Theorem 6.36 ([EKAG97, Theorem 5.4], [R221, Theorem 4.5]).** Let $\{\mathcal{F}_t\}_{t \in U}$ be a smooth family of homologically orientable transversely Hermitian structures on a compact manifold $M$ with fixed differentiable type, parametrized by an open neighborhood $U$ of 0 in $\mathbb{R}^d$. Fix any two non-negative integers $p, q$. Let $D_t : A^{p,q}(M/\mathcal{F}_t) \rightarrow A^{p,q}(M/\mathcal{F}_t)$ be a family of transversely elliptic operators of even order. Denote $h(t) := \dim(\ker(D_t))$. Then $h(t)$ is upper semi-continuous. Furthermore, if $h(t)$ is independent of $t \in U$, then the operators $G_t$ and $H_t$ depend differentiably on $t \in U$. Here, $G_t$ and $H_t$ are the associated Green’s operator and harmonic projection operator, respectively.

Consequently, one can repeat the same procedures in Section 4, [RZ18, Section 3] and [RWZ19, Section 5] to get

**Theorem 6.37.** Let $\{\mathcal{F}_t\}_{t \in U}, M, U$ be as in Theorem 6.36 and $\mathbb{S}^{p,q}, \mathbb{B}^{p,q}$ be defined for $\mathcal{F}_0$ similarly to Definition 6.22.

(a) If $\mathcal{F}_0$ satisfies both $\mathbb{S}^{p,q+1}$ and $\mathbb{B}^{p+1,q}$ with the deformation invariance of $h_{\mathcal{F}_0}^{p,q-1}(M/\mathcal{F}_0)$ established, then $h_{\mathcal{F}_t}^{p,q}(M/\mathcal{F}_t)$ are independent of $t$.

(b) If $\mathcal{F}_0$ satisfies both $\mathbb{S}^{p,q+1}$ and $\mathbb{B}^{p+1,q}$ with the deformation invariance of $h_{\mathcal{F}_0}^{p,q-1}(M/\mathcal{F}_0)$ established, then $h_{\mathcal{F}_t}^{p,q}(M/\mathcal{F}_t)$ are independent of $t$.

(c) If $\mathcal{F}_0$ satisfies both $\mathbb{S}^{p,q+1}$ and $\mathbb{B}^{p+1,q}$ with the deformation invariance of the $(p-1, q-1)$-basic Aeppli number $h_{\Delta}^{p-1,q-1}(M/\mathcal{F}_0)$ established, then the $(p,q)$-basic Bott–Chern number are independent of $t$.

Notice that in this subsection we always assume that the foliations are fixed. We should then mention an interesting question proposed by Raźny if the foliations are allowed to vary. For the case of a family of Sasakian manifolds which is in particular a family of homologically orientable transversely Kähler foliations, Raźny [R221, Theorem 1] affirmed this question (see also an alternative proof based on Vaisman geometry, [OV22, Theorem 31.25]) which completes earlier results of C. P. Boyer–K. Galicki [BG08] and of O. Goertsches–H. Nozawa–D. Töben [GNT16] who proved the invariance for special types of deformations. Note that Raźny’s result boosts the study of basic Hodge numbers by showing they can help distinguish between different deformation classes of Sasakian structures, without relying on contact topology, see [KP22] and the references therein for some recent progress. Very recently, Raźny also gave a positive answer to this question for a new class of transversely Kähler foliations, cf. [R223].
Question 6.38 ([Rz21, Question 1.2]). Are the basic Hodge numbers rigid under deformations of (homologically orientable) transversely Kähler foliations on compact manifolds?

APPENDIX A. THE $p$-KÄHLER STRUCTURES

In this appendix, we state some basic knowledge about the $p$-Kähler structures. Let $V$ be a complex $n$-dimensional vector space and $V^*$ its dual space, i.e., the space of complex linear functionals over $V$. Denote the complexified space of the exterior $m$-vectors of $V^*$ by $\Lambda^m_{\mathbb{C}} V^*$, which admits a natural direct sum decomposition

$$\bigwedge^m_{\mathbb{C}} V^* = \bigoplus_{r+s=m} \bigwedge^{r,s} V^*,$$

where $\bigwedge^{r,s} V^*$ is the complex vector space of $(r, s)$-forms on $V^*$. The case $m = 1$ exactly reads

$$\bigwedge^1_{\mathbb{C}} V^* = V^* \bigoplus \overline{V^*},$$

where the natural isomorphism $V^* \cong \bigwedge^{1,0} V^*$ is used. Let $q \in \{1, \ldots, n\}$ and $p = n - q$. Clearly, the complex dimension $N$ of $\bigwedge^{q,0} V^*$ equals to the combination number $C^q_n$. After fixing a basis $\{\beta_i\}_{i=1}^N$ of the complex vector space $\bigwedge^{q,0} V^*$, we can give the canonical Plücker embedding as in [GH78, p. 209] by

$$\rho: \quad G(q, n) \twoheadrightarrow \mathbb{P}(\bigwedge^{q,0} V^*)$$

Here $G(q, n)$ denotes the Grassmannian of $q$-planes in the vector space $V^*$ and $\mathbb{P}(\bigwedge^{q,0} V^*)$ is the projectivization of $\bigwedge^{q,0} V^*$. A $q$-plane in $V^*$ can be represented by a decomposable $(q, 0)$-form $s \in \bigwedge^{q,0} V^*$ up to a nonzero complex number, and $\{s_i\}_{i=1}^N$ are exactly the coordinates of $s$ under the fixed basis $\{\beta_i\}_{i=1}^N$. Decomposable $(q, 0)$-forms are those forms in $\bigwedge^{q,0} V^*$ that can be expressed as $\gamma_1 \wedge \cdots \wedge \gamma_q$ with $\gamma_i \in V^* \cong \bigwedge^{1,0} V^*$ for $1 \leq i \leq q$. The $pq$-dimensional locus $\rho(G(q, n))$ in $\mathbb{P}(\bigwedge^{q,0} V^*)$ characterizes the decomposable $(q, 0)$-forms in $\mathbb{P}(\bigwedge^{q,0} V^*)$.

Now we list several positivity notions (cf. [HK74, Hv77, Dm12] for more details). A $(q, q)$-form $\Theta$ in $\bigwedge^{q,q} V^*$ is defined to be strictly positive (resp., positive) if

$$\Theta = \sigma_q \sum_{i,j=1}^N \Theta_{ij} \beta_i \wedge \overline{\beta_j},$$

where $\Theta_{ij}$ is a positive (resp. semi-positive) Hermitian matrix of the size $N \times N$ with $N = C^q_n$ under the basis $\{\beta_i\}_{i=1}^N$ of the complex vector space $\bigwedge^{q,0} V^*$ and $\sigma_q$ is defined to be the constant $2^{-q}(\sqrt{-1})^q$. According to this definition, the fundamental form of a Hermitian metric on a complex manifold is actually a strictly positive $(1, 1)$-form everywhere. A $(p, p)$-form $\Gamma \in \bigwedge^{p,p} V^*$ is said to be strictly weakly positive (resp. weakly positive) if the volume form

$$\Gamma \wedge \sigma_q \tau \wedge \overline{\tau}$$

is strictly positive (resp. positive) for every nonzero decomposable $(q, 0)$-form $\tau$ of $V^*$, while a $(q, q)$-form $\Upsilon \in \bigwedge^{q,q} V^*$ is called strongly positive if $\Upsilon$ is a convex combination

$$\Upsilon = \sum_s \gamma_s \sqrt{-1} \alpha_{s,1} \wedge \overline{\alpha}_{s,1} \wedge \cdots \wedge \sqrt{-1} \alpha_{s,q} \wedge \overline{\alpha}_{s,q},$$

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where \( \alpha_{s,i} \in V^* \) and \( \gamma_s \geq 0 \). As shown in [Dm12, Chapter III.§ 1.A], the sets of weakly positive and strongly positive forms are closed convex cones, and by definition, the weakly positive cone is dual to the strongly positive cone via the pairing
\[
\bigwedge^{p,p} V^* \times \bigwedge^q V^* \to \mathbb{C}.
\]
Then all weakly positive forms are real. An element \( \Xi \) in \( \bigwedge^{p,p} V^* \) is called transverse, if it is strictly weakly positive. There exist many various names for this terminology and we refer to [AB91, Appendix] for a list.

These positivity notions on complex vector spaces can be extended pointwise to complex differential forms on a complex manifold.

**Definition A.1** ([AA87, Definition 1.11], for example). For an \( n \)-dimensional complex manifold \( M \) and a positive integer \( p \leq n \), \( M \) is called a \( p \)-Kähler manifold if there exists a \( p \)-Kähler form, that is a \( d \)-closed transverse \( (p,p) \)-form on \( M \).

The readers can refer to [Su76] for more related concepts (such as differential form transversal to the cone structure on a real differentiable manifold) to \( p \)-Kähler structures.

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