ON RESTRICTION ESTIMATES FOR DISCRETE QUADRATIC SURFACES

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Abstract. We obtain truncated restriction estimates of an unexpected form for discrete surfaces
\[ S = \{ (n_1, \ldots, n_d, R(n_1, \ldots, n_d)), n_i \in [-N, N] \cap \mathbb{Z} \}, \]
where $R$ is an indefinite quadratic form with integer matrix.

1. Introduction

We fix a non-degenerate quadratic form $R$ in $d$ variables with integer matrix. We are interested in restriction estimates for quadratic surfaces in $\mathbb{Z}^{d+1}$ of the form
\[ S = \{ (n_1, \ldots, n_d, R(n_1, \ldots, n_d)), n_i \in [-N, N] \cap \mathbb{Z} \}, \tag{1.1} \]
in the case where $R$ is indefinite. This paper should be seen as a companion to [7], which concerned the case $R(n) = n^k$ of $k$-th powers and the case $R(n) = n_1^d + \cdots + n_k^d$ of ‘$k$-paraboloids’; the methods employed here are similar but our results take a different shape.

In the case $d = 1$, $R(x) = x^2$ of the 2D parabola, Bourgain [2] resolved the natural restriction conjecture in the supercritical range, via discrete versions of the Tomas–Stein argument [13, Chapter 7] and the Hardy–Littlewood circle method. By powerful new methods of multilinear harmonic analysis, Bourgain and Demeter [4, Theorem 2.4] later established the natural restriction conjecture for arbitrary definite irrational paraboloids $R(x) = \theta_1 x_1^2 + \cdots + \theta_d x_d^2$, $\theta_i > 0$, up to $\varepsilon$ losses. In a subsequent work [3, Corollary 1.3], they also obtained the conjectured estimate for indefinite paraboloids. To state those results precisely, we set up some notation. The extension operator acting on a sequence $a : \mathbb{Z}^d \to \mathbb{C}$ supported on $[-N, N]^d$ is denoted by
\[ F_a(\alpha, \theta) = \sum_{n \in \mathbb{Z}^d} a(n)e(\alpha R(n) + \theta \cdot n) \quad (\alpha \in \mathbb{T}, \theta \in \mathbb{T}^d). \tag{1.2} \]

**Theorem 1.1** (Bourgain–Demeter [3], special case). Suppose that $R$ is a non-degenerate indefinite quadratic form in $d$ variables with integer matrix and signature $(p, q)$, and let
$$s = \min(p, q). \text{ We have}$$

$$\|F_a\|_p \lesssim \varepsilon \begin{cases} 
N^{\frac{dp}{d} - s + \varepsilon} \|a\|_2^p & \text{for } 2 \leq p \leq \frac{2(d-s+2)}{d-s}, \\
N^{\frac{dp}{d} - (d+2)} \|a\|_2^p & \text{for } p > \frac{2(d-s+2)}{d-s}.
\end{cases}$$

While this is stated only for diagonal forms in [3], a simple diagonalization argument allows one to reduce to this case. There is also an extra $N^\varepsilon$ factor in the supercritical range in that reference, which can be removed by (a minor variant of) Bourgain’s $\varepsilon$-removal estimate [2]; we refer to Appendix B for the details. The exponent of $N$ in Theorem 1.1 is sharp for even integer exponents $p > \frac{2(d-s+2)}{d-s}$, as can be seen by taking $a \equiv 1$ and using the circle method to obtain an asymptotic. As explained in [3], the lower bound $\|F_a\|_p \gtrsim N^{\frac{dp}{d} - s} \|a\|_2^p$ also holds for a sequence supported on a subspace of dimension $s$. More precisely, assume for simplicity that $R(x) = \sum_{i=1}^s x_i^2 - \sum_{i=s+1}^d x_i^2$ with $s \leq d/2$, then $a(n) = 1_{|N|2^s}(n) \prod_{i=1}^s 1_{n_i = n_{s+i}}$ is an (approximate) extremizer. Note that $|F_a| \leq N^{s/2} \|a\|_2 \leq N^{d/4} \|a\|_2$ is rather small in that case. Our main result adapts the proof of Bourgain’s $\varepsilon$-removal lemma [2] to indefinite quadratic parabolas, and we obtain an intriguing bound for the truncated integral.

**Theorem 1.2.** Let $R$ be a non-degenerate indefinite quadratic form in $d \geq 1$ variables with integer matrix and signature $(p, q)$, and let $s = \min(p, q)$. There exists $C > 0$ such that

$$\int_{|F_a| \geq CN^{d/4}} |F_a|^p dm \lesssim N^{\frac{dp}{d} - (d+2)} \|a\|_2^p$$

for $p > \frac{2(d+2)}{d}$.

Note that the upper bound above is of order less than the order $N^{\frac{dp}{d} - s + O(\varepsilon)} \|a\|_2^p$ of the complete integral, as given by Theorem 1.1. This can be seen as an inverse result saying that for sequences $a : \mathbb{Z}^d \rightarrow \mathbb{C}$ maximizing the ratio $\|F_a\|_p / \|a\|_2$, the “mass” of the integral $\int |F_a|^p dm$ is concentrated on a set where $|F_a|$ has square-root cancellation (in comparison with the trivial Cauchy-Schwarz bound $|F_a| \leq N^{d/2} \|a\|_2$). This is consistent with the above example of maximizer supported on the subvariety $n_i = n_{s+i}, 1 \leq i \leq s$. Such a behavior would be impossible in the definite case, where the tail integral over $\{|F_a| \leq N^{d/2-\varepsilon} \|a\|_2\}$ always contributes less than the main term, for any $\varepsilon > 0$.

In proving Theorem 1.2, we do not have a simple diagonalization argument at our disposal to estimate the truncated integral (1.3). Therefore we adapt the approach of Bourgain [2] for the parabola $(x_1, \ldots, x_d, x_1^2 + \cdots + x_d^2)$ to use multidimensional exponential sum estimates, whereas in the diagonal case the relevant exponential sum (4.3) splits into one-dimensional quadratic Weyl sums. This process is successful since efficient
bounds on quadratic exponential sums are known classically, and we do not encounter certain difficulties described in [7] for surfaces of high degree.

We note finally that the related problem of obtaining \( \varepsilon \)-free estimates in the full supercritical range for indefinite irrational quadratic forms \( R \) is still open, although there is partial progress in this direction by Godet and Tzvetkov [6] and Wang [12]. In the definite case, the question has been settled recently by Killip and Vișan [9].

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2. **Notation**

For functions \( f : \mathbb{T}^d \rightarrow \mathbb{C} \) and \( g : \mathbb{Z}^d \rightarrow \mathbb{C} \) we define the Fourier transforms of \( f \) and \( g \) by \( \hat{f}(k) = \int_{\mathbb{T}^d} f(\alpha) e(-\alpha \cdot k) d\alpha \) and \( \hat{g}(\alpha) = \sum_{n \in \mathbb{Z}^d} g(n) e(\alpha \cdot n) \). For a function \( h : \mathbb{R}^d \rightarrow \mathbb{C} \) we define the Fourier transform by \( \hat{h}(\xi) = \int_{\mathbb{R}^d} f(x) e(-\xi \cdot x) dx \). Given a function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) and two subsets \( A, B \) of \( \mathbb{R}^d \), we write \( A \prec f \prec B \) when \( 0 \leq f \leq 1 \) everywhere, \( f = 1 \) on \( A \) and \( f = 0 \) outside \( B \).

When \( \mathcal{P} \) is a certain property, we let \( 1_{\mathcal{P}} \) denote the boolean equal to 1 when \( \mathcal{P} \) holds and 0 otherwise, and when \( E \) is a set we define the indicator function of \( E \) by \( 1_E(x) = 1_{x \in E} \). When \( p \in [1, +\infty) \) is an exponent, we systematically denote by \( p' \in [1, +\infty] \) its dual exponent satisfying \( \frac{1}{p} + \frac{1}{p'} = 1 \). We let \( dm \) denote the Lebesgue measure on \( \mathbb{R}^d \), or on \( \mathbb{T}^d \) identified with any fundamental domain of the form \( [\theta, 1+\theta]^d \). For \( q \geq 2 \) we occasionally use \( \mathbb{Z}_q \) as a shorthand for the group \( \mathbb{Z}/q\mathbb{Z} \).

Throughout the article, we use the letter \( \varepsilon \) generically to denote a constant which can be taken arbitrarily small, and whose value may change in each occurrence.

3. **Arc mollifiers**

This section is a specialization of [7, Section 6] to the quadratic case \( k = 2 \), and we include it for completeness. Its aim is to describe a technical tool due to Bourgain [2, Section 3] and used in the proof of Theorem 1.2, which consists essentially in a partition of unity adapted to major arcs.

We fix an integer \( N \geq 1 \), to be thought of as large. We fix a smooth bump function \( \kappa \) with \( [-1, 1] \prec \kappa \prec [-2, 2] \). Let \( \tilde{N} = 2^{\lfloor \log_2 N \rfloor} \), and for every integer \( 0 \leq s \leq \lfloor \log_2 N \rfloor \) define

\[
\phi^{(s)} := \begin{cases} 
\kappa(2^s N \cdot) - \kappa(2^{s+1} N \cdot) & \text{if } 1 \leq 2^s < \tilde{N}, \\
\kappa(2^s N \cdot) & \text{if } 2^s = \tilde{N}.
\end{cases}
\]
Note that we have $\text{Supp}(\phi(s)) \subset \frac{1}{2N}I_s$, where $I_s = \pm\frac{1}{2}, 2]$ if $1 \leq 2^s < \tilde{N}$, and $I_s = [-2, 2]$ if $2^s = \tilde{N}$. Furthermore, for every dyadic integer $1 \leq Q \leq N$, we have

$$\sum_{Q \leq 2^s \leq N} \phi(s) = \kappa(QN \cdot).$$

We let $N_1 = c_1N$, for a small constant $c_1 \in (0, 1]$. It is easy to check that the intervals $\frac{a}{q} + [-\frac{2}{QN}, \frac{2}{QN}]$, $1 \leq a \leq q$, $q \sim Q$, $1 \leq Q \leq N_1$ are all disjoint. For a dyadic integer $Q$ and an integer $0 \leq s \leq \log_2 N$, we define

$$\Phi_{Q,s} = \sum_{(a,q)=1}^{q \sim Q} \tau_{-a/q}\phi(s),$$

where $\tau_{-a/q}\phi(s)(\alpha) := \phi(\alpha - a/q)$ is translation by $a/q$, so that

$$\text{Supp}(\Phi_{Q,s}) \subset \bigcup_{(a,q)=1}^{q \sim Q} \left( \frac{a}{q} + \frac{I_s}{2^sN} \right).$$

We also define the functions

$$\lambda = \sum_{Q \leq N_1} \sum_{Q \leq 2^s \leq N} \Phi_{Q,s}, \quad \rho = 1 - \lambda.$$

**Proposition 3.1.** We have $0 \leq \lambda, \rho \leq 1$ and

$$\lambda = 1, \rho = 0 \quad \text{on} \quad \bigcup_{Q \leq N_1} \bigcup_{(a,q)=1}^{q \sim Q} \left( \frac{a}{q} + \left[ -\frac{1}{QN}, \frac{1}{QN} \right] \right).$$

**Proof.** By (3.2), we can rewrite $\lambda$ as

$$\lambda = \sum_{Q \leq N_1} \sum_{(a,q)=1}^{q \sim Q} \tau_{-a/q} \left( \sum_{Q \leq 2^s \leq N} \phi(s) \right) = \sum_{Q \leq N_1} \sum_{(a,q)=1}^{q \sim Q} \tau_{-a/q} \kappa(QN \cdot).$$

The proposition follows from the localization properties of $\kappa$. 

At this stage we define the fundamental domain $U = (\frac{1}{2N_1}, 1 + \frac{1}{2N_1}]$, and we note that when $N$ is large, then for every $1 \leq a \leq q \leq Q \leq N_1$, the intervals $\frac{a}{q} + [-\frac{2}{QN}, \frac{2}{QN}]$ are contained in $U$. Therefore for $1 \leq Q \leq 2^s \leq N_1$, the functions $\phi(s)$, $\Phi_{Q,s}$ and $\lambda$ are supported on the interior of $U$, and they may be viewed as smooth functions over the torus $\mathbb{T}$, by 1-periodization from the interval $U$. We will view $\Phi_{Q,s}$ alternatively as a smooth function on the torus $\mathbb{T}$ or on the real line, but note that for an integer $n$, $\Phi_{Q,s}(n)$ has the same definition under both points of view.
For \( n \in \mathbb{Z} \) and an integer \( Q \geq 1 \) we define a truncated divisor function
\[
d(n, Q) = \sum_{1 \leq d \leq Q : d \mid n} 1.
\]
The following useful lemma is due to Bourgain [2].

**Lemma 3.2.** Let \( \delta_x \) be the Dirac function at \( x \). Then
\[
\sum_{(a,q)=1 \atop q \sim Q} \delta_{a/q}(n) \lesssim Q \cdot d(n, 2Q) \quad (n \in \mathbb{Z}).
\]

**Proposition 3.3.** We have
\[
\int \rho dm \lesssim 1,
\]
(3.6)
\[
\hat{\rho}(n) \lesssim \frac{1}{N^{1-\varepsilon}} \quad \text{for } 0 < |n| \leq AN^A.
\]
(3.9)

**Proof.** From (3.5) and (3.6), it follows that
\[
\int \rho dm = 1 - O \left( \sum_{Q \leq N_1} \sum_{Q \leq 2^s \leq N} \frac{Q^2}{2^s N} \right)
\]
\[
= 1 - O \left( \frac{1}{N} \sum_{Q < N_1} Q \right)
\]
\[
= 1 - O \left( \frac{N_1}{N} \right).
\]

**Proposition 3.4.** For every \( \varepsilon > 0 \) and \( A > 0 \), we have
\[
\int \rho dm \lesssim 1,
\]
(3.8)
\[
\hat{\rho}(n) \lesssim \frac{1}{N^{1-\varepsilon}} \quad \text{for } 0 < |n| \leq AN^A.
\]
(3.9)

**Proof.** From (3.5) and (3.6), it follows that
\[
\int \rho dm = 1 - O \left( \sum_{Q \leq N_1} \sum_{Q \leq 2^s \leq N} \frac{Q^2}{2^s N} \right)
\]
\[
= 1 - O \left( \frac{1}{N} \sum_{Q < N_1} Q \right)
\]
\[
= 1 - O \left( \frac{N_1}{N} \right).
\]
Since we have chosen \( N_1 = c_0 N \) with \( c_1 \) small enough, we have \( \int \rho \, d m \simeq 1 \) as desired. The bound on \( \hat{\rho} \) is derived from (3.7) in a similar fashion, using also the standard divisor bound \( d(n, Q) \leq d(n) \lesssim n^\varepsilon \).

4. Restriction estimates

We fix a non-degenerate integer quadratic form \( R \) in \( d \) variables. In this section, we derive Theorem 1.2 from the introduction. Note that the system of polynomials \( (R(x), x) \) has total degree \( d + 2 \), hence the critical exponent in the definite case is \( p_d = \frac{2(d+2)}{d} \). This is the exponent that arises in our argument, even in the indefinite case, due to our use of \( d \)-dimensional exponential sum estimates which do not depend on the type of quadratic form. The larger critical exponent \( p_{d,s} = \frac{2(d-s+2)}{d-s} \) of Theorem 1.1 accounts for the existence of a special linear subvariety of (5.2), but this does not influence our treatment of the truncated moment in Theorem 1.2.

We use a smooth weight function \( \omega : \mathbb{R}^d \to [0, 1] \) of the form

\[
\omega = \eta \left( \frac{\cdot}{N} \right), \quad \eta \text{ Schwarz function such that } [-1, 1]^d < \eta < [-2, 2]^d.
\]

Given a sequence \( a : \mathbb{Z}^d \to \mathbb{C} \) supported on \([-N, N]^d\) with \( \| a \|_2 = 1 \) and a weight function \( \omega : \mathbb{Z}^d \to [0, 1] \) of the form (4.1), we define

\[
F_a(\alpha, \theta) = \sum_{n \in \mathbb{Z}^d} a(n) e(\alpha R(n) + \theta \cdot n) \quad (\alpha \in \mathbb{T}, \theta \in \mathbb{T}^d),
\]

\[
F(\alpha, \theta) = \sum_{n \in \mathbb{Z}^d} \omega(n) e(\alpha R(n) + \theta \cdot n) \quad (\alpha \in \mathbb{T}, \theta \in \mathbb{T}^d),
\]

which are the extension operator of our surface \( S \) acting on the sequence \( a \) and the \( \omega \)-smoothed Fourier transform of the counting measure on \( S \), respectively.

We will quote the estimates of Section 3 extensively. Via the Tomas-Stein argument in Section 5, we will devote most of our attention to the complete exponential sum (4.3). The minor arc estimates of Appendix A yield the following in our context.

**Proposition 4.1.** **Uniformly in** \( \alpha \in \mathbb{T}, \theta \in \mathbb{T}^d \), **we have**

\[
\rho(\alpha) \neq 0 \quad \Rightarrow \quad |F(\alpha, \theta)| \lesssim N^{d/2}.
\]

**Proof.** We prove the contrapositive. If \( |F(\alpha, \theta)| \geq C_1 N^{d/2} \) for a large enough \( C_1 > 0 \), then by Proposition A.1 there exist \( a, q \in \mathbb{Z} \) such that \( |\alpha - \frac{a}{q}| \leq \frac{C_1}{qN} \), \( 1 \leq q \leq c_1 N \) and \((a, q) = 1\). Consequently there exists a dyadic integer \( Q \) such that \( q \sim Q \Rightarrow Q \leq c_1 N = N_1 \) and \( |\alpha - \frac{a}{q}| \leq \frac{1}{qQ} \), so that \( \rho(\alpha) = 0 \) by Proposition 3.1. \( \square \)
For each dyadic integer $Q \geq 1$ and each integer $s \geq 0$ such that $1 \leq Q \leq 2^s$, we define a piece of our original exponential sum by

$$
F_{Q,s}(\alpha, \theta) = F(\alpha, \theta) \left[ \Phi_{Q,s}(\alpha) - \int \frac{\Phi_{Q,s}}{\rho} \rho(\alpha) \right].
$$

We establish physical and Fourier bounds for the exponential sums $F_{Q,s}$ via the major and minor arc estimates of Appendix A. It turns out to be important to have square-root cancellation of the exponential sum $F$ on the minor arcs. We also introduce a technical device to ensure that the Fourier transforms under consideration stay inside an $N^2 \times N^2 \times \cdots \times N$ box, a fact that will prove useful later on. Specifically, we fix a trigonometric polynomial $\psi_N$ on $T^{d+1}$ such that, for a constant $C_R$ large enough with respect to $R$,

$$
[-C_R N^2, C_R N^2] \times [-2N, 2N]^d \prec \hat{\psi}_N \prec [-2C_R N^2, 2C_R N^2] \times [-4N, 4N]^d,
$$

which in particular implies that $\int_{T^{d+1}} \psi_N = 1$. When $H : T^{d+1} \to \mathbb{C}$ is a bounded measurable function, we write $\hat{H} = H * \psi_N$ for brevity; note that $\|\hat{H}\|_p \leq \|H\|_p$ for any $p \geq 1$ by Young’s inequality, and that $F = \hat{F}$ by Fourier inversion (since $\hat{F}$ is supported on the surface (5.2)).

**Proposition 4.2.** Uniformly for $(m, \ell) \in \mathbb{Z}^{d+1}$, we have

$$
\|\hat{F}_{Q,s}\|_\infty \lesssim \varepsilon \left( \frac{2^s N}{Q} \right)^{d/2} Q^\varepsilon,
$$

$$
\hat{F}_{Q,s}(m, \ell) \lesssim \varepsilon 1_{|m| \leq N^2, |\ell| \leq N} \left( \frac{Q}{2^s N} d(m - R(\ell), 2Q) + \frac{Q^2}{2^s N^{2-\varepsilon}} \right).
$$

**Proof.** When $\Phi_{Q,s}(\alpha) \neq 0$, it follows from (3.4) that there exist $a, q \in \mathbb{Z}$ such that $q \sim Q$, $(a, q) = 1$ and $|\alpha - \frac{a}{q}| \asymp \frac{1}{2^s N}$ if $2^s < \tilde{N}$, $|\alpha - \frac{a}{q}| \lesssim \frac{2}{2^s N}$ is $2^s = \tilde{N}$. By Propositions A.2 and 4.1, and by (3.6) and (3.8), it follows that, uniformly in $\theta \in \mathbb{R}^d$,

$$
|F_{Q,s}(\alpha, \theta)| \lesssim \varepsilon Q^{-\frac{d}{2}+\varepsilon} (2^s N)^{\frac{d}{2}} + \frac{Q^2}{2^s N} N^{\frac{d}{2}}
$$

$$
= \left( \frac{2^s}{Q} \right)^{\frac{d}{2}} Q^\varepsilon N^{\frac{d}{2}} + \frac{Q}{2^s} \cdot \frac{Q}{N} N^{\frac{d}{2}}
$$

$$
\leq \left( \frac{2^s N}{Q} \right)^{\frac{d}{2}} (1 + Q^\varepsilon).
$$
We let $\Psi_{Q,s} = \Phi_{Q,s} - \int \frac{\Phi_{Q,s}}{\rho} \rho$ and note that $\int \Psi_{Q,s} = 0$ for each $Q,s$. Next we observe that, for any $(m, \ell) \in \mathbb{Z}^{d+1}$,

$$
\hat{F}_{Q,s}(m, \ell) = \int_{\mathbb{T}^{d+1}} \Psi_{Q,s}(\alpha) e(-\alpha m - \theta \cdot \ell) d\alpha d\theta
$$

$$
= \sum_{n \in \mathbb{Z}^d} \omega(n) \int_{\mathbb{T}^{d+1}} \Psi_{Q,s}(\alpha) e(\alpha(R(n) - m) + \theta \cdot (n - \ell)) d\alpha d\theta
$$

$$
= \omega(\ell) \hat{\Psi}_{Q,s}(m - R(\ell)).
$$

The second bound of the proposition then follows from the identity

$$
\hat{F}_{Q,s}(m, \ell) = \psi_N(m, \ell) \omega(\ell) \hat{\Psi}_{Q,s}(m - R(\ell)) 1_{m \neq R(\ell)},
$$

and the estimates (3.6), (3.7), (3.8) and (3.9).

We now define minor and major arc pieces of our exponential sum by

$$
F_{M} = \sum_{Q \leq N_1} \sum_{Q \leq 2^s \leq N} F_{Q,s}, \quad F_{m} = F - F_{M},
$$

We can readily derive a uniform bound on the minor arc piece $F_{m}$, as an immediate consequence of the definition (3.5) and Proposition 4.1.

**Proposition 4.3.** We have $\|F_{m}\|_{\infty} \lesssim N^{d/2}$.

The previous propositions also imply simple norm estimates for the operator of convolution with a major arc piece.

**Proposition 4.4.** We have

$$
\|\hat{F}_{Q,s} * f\|_{\infty} \lesssim \left(\frac{2^s N}{Q}\right)^{\frac{d}{2}} Q^d \|f\|_1,
$$

$$
\|\hat{F}_{Q,s} * f\|_2 \lesssim \varepsilon \frac{Q}{2^s N 1 - \varepsilon} \|f\|_2.
$$

**Proof.** Note that for any bounded function $W : \mathbb{T}^{d+1} \rightarrow \mathbb{C}$, we have

$$
\|W * f\|_{\infty} \leq \|W\|_{\infty} \|f\|_1, \quad \|W * f\|_2 = \|\hat{W}\|_2 \leq \|\hat{W}\|_{\infty} \|f\|_2.
$$

It now suffices to apply these inequalities with $W = F_{Q,s}$ and insert the estimates of Proposition 4.2 (using also the bound $d(n, 2Q) \lesssim n^\varepsilon$).

By interpolation, we can obtain an estimate for all moments.

**Proposition 4.5.** Let $p_0' = \frac{2(d+2)}{d}$. For any $p' \in [2, \infty)$, we have

$$
\|\hat{F}_{Q,s} * f\|_{p'} \lesssim \left(\frac{2^s N}{Q}\right)^{(d+2)} \left(\frac{1}{p_0'} - \frac{1}{p'}\right)^{\varepsilon} N^\varepsilon \|f\|_{p'}.
$$
Proof. We interpolate between the estimates of Proposition 4.4 with \( \theta \in (0, 1) \) given by
\[
\frac{1}{p'} = \frac{1 - \theta}{\infty} + \frac{\theta}{2}, \quad \frac{1}{p} = \frac{1 - \theta}{1} + \frac{\theta}{2}.
\]
This yields
\[
\| \hat{F}_{Q,s} \ast f \|_{p'} \lesssim \left( \frac{2^s N}{Q} \right)^{(1 - \theta) \frac{d}{2}} \cdot \left( \frac{Q}{2^s N} \right)^{\theta} \cdot N^\varepsilon \cdot \| f \|_p
\]
\[
= \left( \frac{2^s N}{Q} \right)^{\frac{d}{2} - (1 + \frac{d}{2}) \theta} \cdot N^\varepsilon \cdot \| f \|_p.
\]
Since \( \theta = \frac{2}{p'} \), we may rewrite the exponent of \( \frac{2^s N}{Q} \) as \( (d + 2)(\frac{1}{p_0} - \frac{1}{p'}) \), which concludes the proof. \( \square \)

At this stage we need a preparatory lemma on truncated divisor sums from [2].

Lemma 4.6. Let \( D, Q, X \geq 1 \) and \( B \in \mathbb{N} \). When \( Q \leq 2X^{1/B} \), we have
\[
\#\{ |n| \leq X : d(n, Q) \geq D \} \lesssim_{\varepsilon, B} D^{-B} Q^\varepsilon X.
\]

Proof. We show that
\[
\sum_{|\ell| \leq X} d(\ell, Q)^B \lesssim_{\varepsilon, B} Q^\varepsilon X,
\]
from which the result follows by Markov’s inequality. In the sum above, the term \( \ell = 0 \) contributes at most \( Q^B \), and by [1, Eq. (4.31)] the other terms contribute at most \( C_{\varepsilon, B} Q^\varepsilon X \). The conclusion follows from our assumption on \( Q \). \( \square \)

We can now derive a more precise convolution bound using the previous lemma.

Proposition 4.7. Let \( B, D > 2 \). Uniformly for \( Q \leq N^{2/B} \) and \( Q \leq 2^s \leq N \), we have
\[
\| \hat{F}_{Q,s} \ast f \|_2 \lesssim_{\varepsilon, B} \frac{DQ}{2^s N} \| f \|_2 + \frac{D^{-\frac{d}{2}} Q^{1+\varepsilon}}{2^s N} N^{\frac{d}{2} + 2} \| f \|_1.
\]

Proof. By Parseval’s identity and the bounds of Proposition 4.2, we deduce that
\[
\| \hat{F}_{Q,s} \ast f \|_2 = \left[ \sum_{|m| \leq N^2} |\hat{F}_{Q,s}(m, \ell)|^2 |\hat{f}(m, \ell)|^2 \right]^{1/2}
\]
\[
\lesssim \frac{Q}{2^s N} \left[ \sum_{|m| \leq N^2} d(m - R(\ell), 2Q)^2 |\hat{f}(m, \ell)|^2 \right]^{1/2} + \frac{Q^2}{2^s N^{2-\varepsilon}} \| \hat{f} \|_2
\]
We write \( n = m - R(\ell) \), so that assuming \( Q \leq N^{2/B} \) and invoking Lemma 4.6, we obtain
\[
\|F_{Q,s} \ast f\|_2 \lesssim_{\varepsilon,B} \frac{Q}{2^s N} \left[ D^2 \|f\|_2^2 + \|\hat{f}\|_\infty N^d \times \#\{n|: d(n,2Q) > D\} \right]^{1/2} + \frac{Q^2}{2^s N^2 - \varepsilon} \|f\|_2
\]
\[
\lesssim \frac{Q}{2^s N} \left( D^2 \|f\|_2^2 + D^{-B} Q^e N^{d+2} \|f\|_1^2 \right)^{1/2} + \frac{Q}{2^s N} \cdot \frac{Q}{2^s N^{1-\varepsilon}} \|f\|_2.
\]
Since \( B > 2 \), we have that \( Q \leq N^{1-\varepsilon} \) for some \( \varepsilon > 0 \) and the last term may be absorbed into the first. Finally we obtain
\[
\|F_{Q,s} \ast f\|_2 \lesssim \frac{Q}{2^s N} \left( D \|f\|_2 + Q^e D^{-\frac{B}{2}} N^{\frac{d+2}{2}} \|f\|_1 \right).
\]
\[\square\]

This new estimate can again be interpolated with the \( L^1 \to L^\infty \) one, to obtain the following bound.

**Proposition 4.8.** Let \( B, D > 2 \). Let \( p'_0 = \frac{2(d+2)}{d} \) and \( p' \in [2, \infty) \). Uniformly for \( Q \leq N^{2/B} \) and \( Q \leq 2^s \leq N \), we have
\[
\|F_{Q,s} \ast f\|_{p'} \lesssim_{\varepsilon,B} D^\frac{2}{p'} \left( \frac{2^s N}{Q} \right)^{(d+2)(\frac{1}{p_0} - \frac{1}{p'})} Q^e \|f\|_p + D^{-\frac{B}{p'}} \left( \frac{2^s N}{Q} \right)^{(d+2)(\frac{1}{p_0} - \frac{1}{p'})} N^{\frac{d+2}{2}} Q^e \|f\|_1.
\]

**Proof.** Let \( \theta \in (0,1] \) and \( p' \geq 2 \) be such that (4.9) holds. By convexity and (4.6) and (4.10), we have
\[
\|F_{Q,s} \ast f\|_{p'} \leq \|F_{Q,s} \ast f\|_{1-\theta} \|F_{Q,s} \ast f\|_{p',2}. \]
\[
\lesssim_{\varepsilon,B} Q^e \left( \frac{2^s N}{Q} \right)^{(1-\theta)\frac{d}{2}} \cdot D^\theta \left( \frac{Q}{2^s N} \right)^\theta \cdot \|f\|_1^{1-\theta} \|f\|_2^\theta
\]
\[
+ Q^e \left( \frac{2^s N}{Q} \right)^{(1-\theta)\frac{d}{2}} \cdot D^{-\theta \frac{B}{2}} \left( \frac{Q}{2^s N} \right)^\theta \left( N^{\frac{d+2}{2}} \right)^\theta \|f\|_1.
\]
Since \( |f| \) takes values in \( \{0,1\} \), we may rewrite this as
\[
\|F_{Q,s} \ast f\|_{p'} \lesssim_{\varepsilon,B} D^\theta \left( \frac{2^s N}{Q} \right)^{\frac{d}{2} - (1+\frac{d}{2})\theta} Q^e \|f\|_p + D^{-\theta \frac{B}{2}} \left( \frac{2^s N}{Q} \right)^{\frac{d}{2} - (1+\frac{d}{2})\theta} N^{\frac{d+2}{2}} Q^e \|f\|_1.
\]
The proof is finished upon recalling that \( \theta = \frac{2}{p'} \) by (4.9). \[\square\]

We introduce a parameter \( 1 \leq Q_1 \leq N_1 \) and write \( F_{2R} = F_1 + F_2 \) with

(4.11) \[
F_1 = \sum_{Q \leq Q_1} \sum_{Q \leq 2^s \leq N} F_{Q,s}, \quad F_2 = \sum_{Q_1 < Q \leq N_1} \sum_{Q \leq 2^s \leq N} F_{Q,s}.
\]
Proposition 4.9. Suppose that $p' > p'_0 = \frac{2(d+2)}{d}$. Let $T \geq 1$ and suppose that $Q \leq N^{2/B}$. Then
\[
\| \hat{F}_1 * f \|_{p'} \lesssim T^2 N^{d-\frac{2(d+2)}{p'}} \| f \|_{p'} + T^{-B} N^{d-\frac{4(d+2)}{p'}} \| f \|_1,
\]
\[
\| \hat{F}_2 * f \|_{p'} \lesssim Q_1^{-\frac{4(d+2)}{p'}} N^{d-\frac{2(d+2)}{p'}} \| f \|_p.
\]

Proof. By the triangle inequality and Proposition 4.8 with $T = D^{1/p'}$, it follows that
\[
\| \hat{F}_1 * f \|_{p'} \lesssim \sum_{Q \leq Q_1} Q\epsilon^{-(d+2)(\frac{1}{p'_0} - \frac{1}{p'})} \sum_{2^e \leq N} (2^e)^{(d+2)(\frac{1}{p'_0} - \frac{1}{p'})} N^{(d+2)(\frac{1}{p'_0} - \frac{1}{p'})} \cdot (T^2 \| f \|_p + T^{-B} N^{\frac{4(d+2)}{p'}} \| f \|_1).
\]
\[
\lesssim T^2 N^{2(d+2)(\frac{1}{p'_0} - \frac{1}{p'})} \| f \|_p + T^{-B} N^{2(d+2)(\frac{1}{p'_0} - \frac{1}{p'}) - \frac{4(d+2)}{p'}} \| f \|_1.
\]
It is easy to rewrite the exponents of $N$ in the desired form.

Turning our attention to $F_2$, we deduce from the triangle inequality and (4.8) that
\[
\| \hat{F}_2 * f \|_{p'} \lesssim \sum_{Q > Q_1} Q^{-\frac{(d+2)}{p'_0}} \sum_{2^e \leq N} (2^e)^{(d+2)(\frac{1}{p'_0} - \frac{1}{p'})} N^{(d+2)(\frac{1}{p'_0} - \frac{1}{p'})} \cdot N^\epsilon \| f \|_p
\]
\[
\lesssim N^\epsilon Q_1^{-\frac{d+2}{p'}} N^{d-\frac{2(d+2)}{p'}} \| f \|_p.
\]

\[\square\]

5. Proof of Theorem 1.2

In this section we prove our theorem using the restriction estimates from Section 4 and Bourgain’s [1,2] discrete version of the Tomas–Stein argument [13, Chapter 7] from Euclidean harmonic analysis. We introduce a parameter $\lambda > 0$ and define
\[
E_\lambda = \{|F_a| \geq \lambda\}, \quad f = 1_{E_\lambda} \frac{F_a}{|F_a|}.
\]
Note that, by Cauchy-Schwarz in (4.2), we always have $|F_a| \leq CN^{d/2}$, and thus we assume that the parameter $\lambda$ lies in $(0, CN^{d/2})$. Our theorem will quickly follow once we establish the following sharp level set bound.

Proposition 5.1. There exists $C > 0$ such that, for $\frac{2(d+2)}{d} < q \lesssim 1$,
\[
|E_\lambda| \lesssim q N^{\frac{dq}{2} - (d+2)\lambda - q} \quad \text{for } \lambda \geq CN^{d/4}.
\]

Proof. We view $a$ and $\omega$ as functions of $(R(n), n)$ for the sake of this argument, so that $F = \omega 1_{S_{2N}}$ and $F_a = a 1_{S_{2N}}$, where
\[
S_{2N} = \{(R(n_1, \ldots, n_d), n_1, \ldots, n_d) : n_i \in [-2N, 2N] \cap \mathbb{Z}\}.
\]
By Parseval, we have
\[ \lambda |E_\lambda| \leq \langle f, F_0 \rangle_{L^2(\mathbb{T}^{d+1})} = \langle f, a1_{S_N} \rangle_{L^2(\mathbb{T}^{d+1})} = \langle \hat{f}, a \rangle_{L^2(S_N)}. \]
By Cauchy-Schwarz and under the normalization \( \|a\|_2 = 1 \), it follows that
\[ \lambda^2 |E_\lambda|^2 \leq \|f\|_{L^2(S_N)}^2 = \langle f \cdot \omega 1_{S_N}, f \rangle_{L^2(\mathbb{Z}^{d+1})}. \]
By another application of Parseval, we conclude that
\[ (5.3) \quad \lambda^2 |E_\lambda|^2 \leq \langle f * F, f \rangle_{L^2(\mathbb{T}^{d+1})} \]
We will use this inequality to obtain bounds of the expected order on the level sets \( E_\lambda \).
By our earlier observation \( F = \hat{F} \), inequality (5.3) becomes
\[ \lambda^2 |E_\lambda|^2 \leq |\langle \hat{F} * f, f \rangle|, \]
and recalling the decompositions (4.5) and (4.11), we have
\[ \lambda^2 |E_\lambda|^2 \leq |\langle \hat{F}_m * f, f \rangle| + |\langle \hat{F}_2 * f, f \rangle| + |\langle \hat{F}_1 * f, f \rangle| \]
\[ \leq \|F_m\|_{L^\infty} \|f\|_{L^1}^2 + \|\hat{F}_2 * f\|_{L^p} \|f\|_{L^p} + \|\hat{F}_1 * f\|_{L^p} \|f\|_{L^p}. \]
Let \( T \geq 1 \) be a parameter to be determined later, and assume that we have chosen \( Q_1 \) so that \( Q_1 \leq N^{2/B} \). Inserting the estimates of Propositions 4.3 and 4.9, this yields
\[ \lambda^2 |E_\lambda|^2 \lesssim N^{d/2} |E_\lambda|^2 + N^\varepsilon Q_1^{(d - d/2)/B} N^{d - 2(d + 2)/B} \|f\|_{L^p}^2 \]
\[ + T^2 N^{d - 2(d + 2)/B} \|f\|_{L^p}^2 + T^{-B} N^{d - d/2} \|f\|_{L^p} \|f\|_{L^1}. \]
Assume that \( \lambda \geq CN^{d/4} \) for \( C > 0 \) large enough and fix \( Q_1 = N^{\varepsilon_1} \), where \( \varepsilon_1 = 1/B \). For \( p' > 2(d + 2)/d \), and provided that \( \varepsilon \) is small enough, we have then
\[ \lambda^2 |E_\lambda|^2 \lesssim T^2 N^{d - 2(d + 2)/B} |E_\lambda|^{2 - \frac{2}{B'}} + T^{-B} N^{-d/2} \|E_\lambda\|^{2 - \frac{1}{B'}}. \]
Writing \( \lambda = \eta N^{d/2} \) with \( \eta \in (0, 1] \), we have therefore either
\[ |E_\lambda|^2 \lesssim T^2 N^{-\frac{2(d + 2)}{B'}} \eta^{-2} \quad \text{or} \quad |E_\lambda|^{\frac{1}{B'}} \lesssim T^{-B} N^{-\frac{d+2}{B} - \eta^{-2}}. \]
Write \( D = T^{p'} \), so that in either case
\[ |E_\lambda| \lesssim DN^{-d/2} \eta^{-p'} + D^{-B} N^{-(d+2)/2} \eta^{-2p'} \]
Choose \( D = \eta^{-\nu} \) for a parameter \( \nu > 0 \), so that
\[ |E_\lambda| \lesssim N^{-(d+2)/2} \eta^{-p' - \nu} (1 + \eta^{-p' + (B+1)\nu}). \]
Choosing \( B \geq C/\nu \) with \( C > 0 \) large enough, we deduce that \( |E_\lambda| \lesssim N^{-(d+2)/2} \eta^{-(p' + \nu)}. \)
Since \( q := p' + \nu \) can be chosen arbitrarily close to \( \frac{2(d+2)}{d} \), this finishes the proof, upon recalling that \( \eta = \lambda N^{-d/2} \). \( \square \)
Proof of Theorem 1.2. We may certainly assume that \( \|a\|_2 = 1 \) in proving this result. We apply Proposition 5.1 for a certain \( \frac{2(d+2)}{d} < q < p \) to obtain

\[
\int_{|F_a| \geq CN^{d/4}} |F_a|^p dm = p \int_{CN^{d/4}}^{N^{d/2}} \lambda^{p-1} |E_\lambda| d\lambda \\
\lesssim_p N \frac{dp}{2} - (d+2) \int_1^{N^{d/2}} \lambda^{p-q-1} d\lambda. \\
\lesssim_p N \frac{dp}{2} - (d+2).
\]

\[\square\]

Appendix A. Bounds on quadratic exponential sums

In this appendix we derive standard major and minor arc bounds on exponential sums associated to quadratic forms, which we could not locate precisely in the literature. We fix a nondegenerate quadratic form \( R \) in \( d \) variables with integer matrix, and we define \( F_R(\alpha, \theta) = \sum_n \omega(n) e(\alpha R(n) + \theta \cdot n) \quad (\alpha \in \mathbb{T}, \theta \in \mathbb{T}^d). \)

Our first minor-arc-type bound is obtained by the standard Weyl differentiation process for forms of high dimension (see [10, Section 8.3.1.1] or Davenport [5, Chapter 13]).

**Proposition A.1.** Let \( d \geq 1 \). For every \( c_0 \in (0,1] \), there exists a constant \( C > 0 \) depending at most on \( c_0, d, R \) such that the following holds. If \( |F_R(\alpha, \theta)| \geq CN^{d/2} \), there exist \( a, q \in \mathbb{Z} \) such that \( |\alpha - \frac{a}{q}| \leq \frac{c_0}{qN}, 1 \leq q \leq c_0N \) and \( (a, q) = 1 \).

**Proof.** By definition, we have \( R(x) = x^T M x \), where \( M \) is a symmetric, non-singular integer \( d \times d \) matrix. For a vector \( x \in \mathbb{R}^d \), we write \( |x| = \max(|x_1|, \ldots, |x_n|) \) and \( ||x|| = \min_{n \in \mathbb{Z}^d} |x-n| \). By squaring, we have

\[
|F_R(\alpha, \theta)|^2 = \sum_{n,m \in \mathbb{Z}^d} \omega(m)\omega(n)e(\alpha(R(m) - R(n)) + \theta \cdot (m-n)).
\]

Letting \( m = n + u \) and \( \Delta_u^\times \omega(n) = \omega(n)\omega(n + u) \), we deduce that

\[
|F_R(\alpha, \theta)|^2 = \sum_{|u| \leq 4N} e(\alpha R(u) + \theta \cdot u) \sum_{n \in \mathbb{Z}^d} \Delta_u^\times \omega(n)e(n \cdot (2\alpha M u))
\leq \sum_{|u| \leq 4N} |\Delta_u^\times \omega(2\alpha M u)|.
\]

Since \( \Delta_u^\times \omega = \eta(\frac{\cdot}{N}) \eta(\frac{\cdot+u}{N}) \) has support in \([-2N, 2N]^d\) and satisfies \( \|\partial^\alpha \Delta_u^\times \omega\|_{\infty} \lesssim N^{-|\alpha|} \) for all \( \alpha \in (\mathbb{N} \cup \{0\})^d \), one can verify through an application of Poisson’s formula that
Proposition A.2. Let \( d \geq 1 \). Suppose that \( \alpha \in \mathbb{R} \) is of the form \( \alpha = \frac{a}{q} + \beta \) with \( a, q \in \mathbb{Z} \), \( \beta \in \mathbb{R} \) such that \( \| \beta \| \lesssim \frac{1}{qN} \), \( 1 \leq q \lesssim N \) and \( (a, q) = 1 \). Then

\[
|F_R(\alpha, \theta)| \lesssim q^{-d/2 + \varepsilon} \min(N^d, |\beta|^{-d/2}).
\]

Proof. We define a Gaussian sum and an oscillatory integral by

\[
S(a, b; q) = \sum_{u \in \mathbb{Z}_q^d} e_q(aR(u) + b \cdot u), \quad I(\beta, \gamma; N) = \int_{\mathbb{R}^d} \eta(x)e(\beta N^2 R(x) + N\gamma \cdot x)dx.
\]

We write \( \alpha \equiv \frac{a}{q} + \beta \mod 1 \) and we sum over residue classes modulo \( q \) to obtain

\[
F_R(\alpha, \theta) = \sum_{u \in \mathbb{Z}_q^d} e_q(aR(u)) \sum_{n \equiv u \mod q} \omega(n)e(\beta R(n) + \theta \cdot n).
\]
Writing $1_{n \equiv u \mod q} = q^{-d} \sum_{b \in \mathbb{Z}_q^d} e_q(b \cdot (u - n))$, we arrive at

$$F_R(\alpha, \theta) = \sum_{b \in \mathbb{Z}_q^d} q^{-d} S(a, b; q) \sum_{n \in \mathbb{Z}^d} \omega(n) e(\beta R(n) + (\theta - \frac{b}{q}) \cdot n).$$

By Poisson’s formula and rescaling, it follows that

$$F_R(\alpha, \theta) = \sum_{b \in \mathbb{Z}_q^d} q^{-d} S(a, b; q) \sum_{m \in \mathbb{Z}^d} N^d \cdot I(\beta, \theta - \frac{b}{q} - m; N).$$

We write $I(\beta, \theta - \frac{b}{q} - m; N) = \int_{\mathbb{R}} \eta(x) e(N \phi_{b,m}(x)) dx$, where

$$\phi_{b,m}(x) = \beta NR(x) + (\theta - \frac{b}{q} - m) \cdot x.$$ 

On the support of $\eta$, we have $|x| \leq 2$ and therefore

$$\nabla \phi_{b,m}(x) = \theta - \frac{b}{q} - m + O(\frac{1}{q})$$

under our size condition on $\beta$. We fix a large enough constant $C > 0$.

For $|m| \geq C$, we have $|\nabla \phi_{b,m}| \simeq |m|$ on Supp $\eta$, and therefore by stationary phase [11, Chapter VII] we have $|\int_{\mathbb{R}} \eta(x) e(N \phi_{b,m})| \lesssim (N|m|)^{-(d+1)}$. For $\|\theta - \frac{b}{q}\| \geq \frac{C}{q}$, we have $|\nabla \phi_{b,m}| \asymp |\theta - \frac{b}{q} - m| \gtrsim \|\theta - \frac{b}{q}\|$ on Supp $\eta$ and $\|\frac{\phi_{b,m}}{\theta - \frac{b}{q} - m}\|_{C^2} \lesssim 1$, so that by stationary phase again we deduce that $|\int_{\mathbb{R}} \eta(x) e(N \phi_{b,m})| \lesssim (N\|\theta - \frac{b}{q}\|)^{-d}$. Finally, for $|m| \leq C$ and $\|\theta - \frac{b}{q}\| \leq \frac{C}{q}$, we note that the phase is a non-degenerate quadratic form and therefore we have an oscillatory integral estimate [13, Section 6] of the form $|\int_{\mathbb{R}} \eta(x) e(N \phi_{b,m})| \lesssim (1 + |\beta|N^2)^{-d/2}$.

For the Gaussian sum, we use the simple squaring-differencing bound $|S(a, b; q)| \lesssim q^{d/2}$ for $(a, q) = 1$ (see e.g. [8] Lemma 20.12). Inserting these various estimates into (A.1) yields

$$|F_R(\alpha, \theta)| \lesssim_q q^{-d/2} \sum_{\|\theta - \frac{b}{q}\| \leq \frac{C}{q} \atop |m| \leq C} N^d (1 + |\beta|N^2)^{-\frac{d}{2}}$$

$$+ q^{-d/2} \sum_{\|\theta - \frac{b}{q}\| \geq \frac{C}{q} \atop |m| \leq C} \|\theta - \frac{b}{q}\|^{-d} + q^{d/2} \sum_{|m| \geq C} N^{-1} |m|^{-(d+1)}$$

$$\lesssim q^{-d/2 + \varepsilon} N^d (1 + |\beta|N^2)^{-d/2} + q^{d/2 + \varepsilon}.$$ 

The second term may be absorbed into the first since $|\beta| \lesssim \frac{1}{qN}$ and $1 \leq q \lesssim N$, and this concludes the proof.

**Appendix B. A diagonalization argument**

In this section we present a simple argument, possibly well-known to experts, by which Theorem 1.1 follows from [3, Corollary 1.3].
Proof of Theorem 1.1.

Let $Q$ be a non-singular quadratic form with integer coefficients. Fix a sequence $a : \mathbb{Z}^d \to \mathbb{C}$ supported on $[-N, N]^d$; by homogeneity we may assume $\|a\|_2 = 1$. We let

$$I = \|F_a\|_p^p = \int_{[-\frac{1}{2}, \frac{1}{2}]^d} \left| \sum_{n \in \mathbb{Z}^d} a(n)e(\alpha Q(n) + \theta \cdot n) \right|^p \, d\theta.$$

We pick a linear transformation $T$ of $\mathbb{Q}^d$ such that $Q = D \circ T$, where $D$ is a diagonal form with coefficients $\pm 1$. Then by defining the lattice $\Lambda = T(\mathbb{Z}^d)$ and by a change of variables $\theta = T^*(\xi)$, we have

$$I = \int_{[-\frac{1}{2}, \frac{1}{2}]^d} \left| \sum_{m \in \Lambda} a(T^{-1}(m))e(\alpha D(m) + (T^{-1})^*(\theta) \cdot m) \right|^p \, d\theta,$$

$$= | \det T | \int_{E} \left| \sum_{m \in \Lambda} a(T^{-1}(m))e(\alpha D(m) + \xi \cdot m) \right|^p \, d\xi,$$

where $E = [-\frac{1}{2}, \frac{1}{2}] \times (T^*)^{-1}([-\frac{1}{2}, \frac{1}{2}]^d)$. We have $\Lambda \subset q^{-1}\mathbb{Z}^d$ for some $q \in \mathbb{N}$ depending on $Q$, and by a change of variables $(\alpha, \xi) \leftrightarrow (q^2\alpha, q\xi)$, we have

$$I = q^{d+2} | \det T | \int_{F} \left| \sum_{\ell \in q\Lambda} a(T^{-1}(\ell/q))e(\alpha D(\ell) + \xi \cdot \ell) \right|^p \, d\xi,$$

where $F = [-\frac{1}{2q}, \frac{1}{2q}] \times (T^*)^{-1}([-\frac{1}{2}, \frac{1}{2}]^d)$. Finally, we can cover $F$ by finitely many translated copies of $[-\frac{1}{2}, \frac{1}{2}]^{d+1}$, and since $q\Lambda \subset \mathbb{Z}^d \cap [-CN, CN]^d$, we may apply the usual restriction estimate for diagonal forms of Bourgain-Demeter [3, Corollary 1.3] to obtain the estimate

$$\|F_a\|_p^p \lesssim \begin{cases} N^{\frac{d}{d-s} + \varepsilon} \|a\|_2^p & \text{for } 2 \leq p \leq \frac{2(d-s+2)}{d-s}, \\ N^{\frac{d}{d-(d+2)\varepsilon}} \|a\|_2^p & \text{for } p > \frac{2(d-s+2)}{d-s}. \end{cases}$$

The $N^\varepsilon$ factor in the supercritical range can be removed via (a minor modification of) Bourgain’s $\varepsilon$-removal lemma for the paraboloid $(x_1, \ldots, x_d, x_1^2 + \cdots + x_d^2)$. (Alternatively, one can use Theorem 1.2 to remove this factor, via the standard $\varepsilon$-removal process [7, Lemma 3.1]).

$$\square$$

References

1. J. Bourgain, On $A(p)$-subsets of squares, Israel J. Math. 67 (1989), no. 3, 291–311.

2. , Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations, Geom. Funct. Anal. 3 (1993), no. 2, 107–156.

3. J. Bourgain and C. Demeter, Decouplings for curves and hypersurfaces with nonzero Gaussian curvature, Preprint (2015), http://arxiv.org/abs/1409.1634.

4. , The proof of the $\ell^2$ decoupling conjecture, Ann. of Math. 182 (2015), no. 1, 351–389.
5. H. Davenport, *Analytic methods for Diophantine equations and Diophantine inequalities*, second ed., Cambridge University Press, Cambridge, 2005.

6. N. Godet and N. Tzvetkov, *Strichartz estimates for the periodic non-elliptic Schrödinger equation*, C. R. Math. Acad. Sci. Paris 350 (2012), no. 21-22, 955–958.

7. K. Henriot and K. Hughes, *Restriction estimates of \( \varepsilon \)-removal type for \( k \)-th powers and paraboloids*, Preprint (2016).

8. H. Iwaniec and E. Kowalski, *Analytic number theory*, American Mathematical Society, Providence, RI, 2004.

9. R. Killip and M. Višan, *Scale invariant Strichartz estimates on tori and applications*, Preprint (2014), http://arxiv.org/abs/1409.3603.

10. A. Magyar, *On the distribution of solutions to Diophantine equations*, A panorama of discrepancy theory, Springer, Cham, 2014, pp. 487–538.

11. E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton University Press, Princeton, NJ, 1993.

12. Y. Wang, *Periodic cubic hyperbolic Schrödinger equation on \( T^2 \)*, J. Funct. Anal. 265 (2013), no. 3, 424–434.

13. T. H. Wolff, *Lectures on harmonic analysis*, vol. 29, American Mathematical Society, Providence, RI, 2003, Edited by Laba and Carol Shubin.

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