A note on exact likelihoods of the Carr-Wu models for leverage effects and volatility in financial economics.

LANCELOT F. JAMES

The Hong Kong University of Science and Technology

Recently Carr and Wu (2004, 2005) and also Huang and Wu (2004) show that most stochastic processes used in traditional option pricing models can be cast as special cases of time-changed Lévy processes. In particular these are models which can be tailored to exhibit correlated jumps in both the log price of assets and the instantaneous volatility. Naturally similar to a recent work of Barndorff-Nielsen and Shephard (2001a, b), such models may be used in a likelihood based framework. These likelihoods are based on the unobserved integrated volatility, rather than the instantaneous volatility. James (2005) establishes general results for the likelihood and estimation of a large class of such models which include possible leverage effects. In this note we show that exact expressions for likelihood models based on generalizations of Huang and Wu (2004) and Carr and Wu (2005), follow essentially from the arguments in Theorem 5.1 in James (2005) with some slight modification. We show that that an explicit likelihood for any of these types of models only requires knowledge of the characteristic functional of a suitably defined linear functional. This serves to formally verify a claim made by James (2005).

1 Introduction

Recently Carr and Wu (2004, 2005) and also Huang and Wu (2004) show that most stochastic processes used in traditional option pricing models can be cast as special cases of time-changed Lévy processes. This includes for instance the models of Duffie, Pan, and Singleton (2000) and leverage effects model of Barndorff-Nielsen and Shephard (2001a, b). In particular these are models which can be tailored to exhibit correlated jumps in both the log price of assets and the instantaneous volatility. Naturally similar to a recent work of Barndorff-Nielsen and Shephard (2001a, b), such models may be used in a likelihood based framework. James (2005) establishes general results for the likelihood and estimation of a large class of such models, based on quite general linear functionals of Poisson random measures, which include possible leverage effects. In this note we show that exact expressions for likelihood models based on generalizations of Huang and Wu (2004) and Carr and Wu (2005), follow essentially from the arguments in Theorem 5.1 in James (2005) with some slight modification. This serves to formally verify a claim made by James (2005). We shall be rather brief in our exposition and refer the reader to the above mentioned works for further references and motivation. Huang and Wu (2004) and Carr and Wu (2005) proposed a model for the log price of assets which can be written as

\[ x^*(t) = (r - q)t + J_1(\tau(t)) + J_2(\gamma(t)) + \beta \tau(t) + \alpha \gamma(t) + \sigma W_1(\tau(t)) + \sigma W_2(\gamma(t)) \]

where \((J_1, J_2)\) are independent pure jump Lévy processes, \((W_1, W_2)\) are independent standard Brownian motion independent of \((J_1, J_2)\). Furthermore \((\tau, \gamma)\) are non-negative random time changes, independent of the above processes. The independence property can be relaxed. See Carr and Wu (2004).

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Corresponding authors address. The Hong Kong University of Science and Technology, Department of Information and Systems Management, Clear Water Bay, Kowloon, Hong Kong. lancelot@ust.hk

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An example of $\tau$ and $\gamma$ are the integrated Ornstein-Uhlenbeck models of Barndorff-Nielsen and Shephard (2001a, b) which are used to model the integrated stochastic volatility. For notational convenience we shall hereafter set $\sigma = 1$. Assuming conditional independence across intervals $[i(i-1)\Delta, i\Delta]$ for $i = 1, \ldots, n$ and $\Delta > 0$, define $\tau_i := \tau((i\Delta)) - \tau(((i-1)\Delta))$ and $\gamma_i := \gamma((i\Delta)) - \gamma((i-1)\Delta))$, additionally define

$$J_{i,1} = J_1(\tau((i\Delta))) - J_1(\tau((i-1)\Delta))$$

and

$$J_{i,2} = J_2(\gamma((i\Delta))) - J_2(\gamma((i-1)\Delta)).$$

It follows from (1) that one may define a likelihood model for the aggregate returns $X_i = x^*(i\Delta) - x^*((i-1)\Delta)$ which is conditionally Normal given $(J_1, J_2, \tau, \gamma)$. Assuming that these quantities depend on some unknown Euclidean parameter $\theta$, this is in essence the type of framework addressed in James (2005) except for the appearance of the time-changed $(J_1, J_2)$. Before we proceed to derive an exact form of this present model, we will further generalize (1) by modeling the $\tau$ and $\gamma$ as linear functionals of a quite arbitrary Poisson random measure, say $N$, on a Polish space $\mathcal{V}$. In this way the $t$ in the model (1) may be replaced by a more abstract notion of time. Note that we can always choose $N$ on a big enough space so that $\tau$ and $\gamma$ are either independent or dependent.

Remark 1. Note that we take the quite general Poisson framework for explicit flexible concreteness. This includes both discrete and continuous quantities, such as shot-noise processes on abstract spaces. As we shall see, on a more abstract level we simply can specify $\tau$ and $\gamma$ so that one knows the explicit characteristic functional of linear combinations of such processes.

## 2 Exact expression for the marginal likelihood

First as in James (2005) let $N$ denote a Poisson random measure on some Polish space $\mathcal{V}$ with mean intensity,

$$\mathbb{E}[N(dx)|\nu] = \nu(dx).$$

We denote the Poisson law of $N$ with intensity $\nu$ as $\mathbb{P}(dN|\nu)$. The Laplace functional for $N$ is defined as

$$\mathbb{E}[e^{-N(f)}|\nu] = \int_{\mathcal{M}} e^{-N(f)}\mathbb{P}(dN|\nu) = e^{-\Lambda(f)}$$

where for any positive $f$, $N(f) = \int_{\mathcal{V}} f(x)N(dx)$ and $\Lambda(f) = \int_{\mathcal{V}} (1 - e^{-f(x)})\nu(dx)$. $\mathcal{M}$ denotes the space of boundedly finite measures on $\mathcal{V}$. We suppose that $\tau = N(h_i)$, for $i = 1, \ldots, n$ where $h_1, \ldots, h_n$ are positive measurable functions on $\mathcal{V}$. Similarly we suppose that for each $i$ $\gamma_i = N(g_i)$ where $g_i$ are positive measurable functions on $\mathcal{V}$. With this specification it follows from (1) that the generalized notion of aggregate returns $X_i|\tau_i, \gamma_i, J_{i,1}, J_{i,2}, \theta, \alpha, \beta, r, q$ are conditionally independent Normal random variables expressible as

$$X_i = (r - q)\Delta + J_{i,1} + J_{i,2} + \beta(\tau_i + \gamma_i) + (\alpha - \beta)\gamma_i + \sqrt{\tau_i + \gamma_i}\epsilon_i$$

where $\epsilon_i$ are independent standard Normal random variables. Now for $f_i(x) := h_i(x) + g_i(x)$ on $\mathcal{V}$, set

$$\tau_i^* := \tau_i + \gamma_i = N(f_i)$$

and set $\mu = (r - q)$ and $\alpha - \beta = \rho$. Then the conditional Normal density of each $X_i$, say $\phi(X_i|\mu\Delta + J_{i,1} + J_{i,2} + \rho\gamma_i + \beta\tau_i^*, \tau_i^*)$, can be written as

$$e^{(A_i - J_{i,1} - J_{i,2} - \rho\gamma_i)\beta} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\tau_i^*}} \left[ e^{-(A_i - J_{i,1} - J_{i,2} - \rho\gamma_i)^2/(2\tau_i^*)} \right] e^{-\tau_i^* \beta^2/2}$$
Theorem 2.1

where \( A_i = X_i - \mu \Delta \). It is not difficult to see that if \( J_{i,1} \) and \( J_{i,2} \) were removed, the subsequent likelihood model is a special case of the models handled by James (2005, Theorem 5.1). James (2005) remarks that the models such as (3) pose no additional difficulties as long as one knows the characteristic functional of say \( J_{i,1} \) and \( J_{i,2} \) and their sums. However \( J_1 \) and \( J_2 \) are pure jump Lévy processes and hence by way of the known Lévy-Khinchine formula and independence relative to \( \tau \) and \( \gamma \), this point is a trivial matter. In particular for any real or complex number, \( \omega \) write for \( j=1,2, \)

\[
E[e^{-\omega(J_j(t)-J_j(s))}] = e^{-(t-s)\psi_j(\omega)}
\]

where the explicit form of \( \psi_j(\omega) \) is given by the Lévy-Khinchine formula which can be found in Carr and Wu (2005), but is otherwise similar to the function \( \Delta \). It then follows that for each \( i \)

\[
E[e^{-\omega J_{i,1}}] = E[e^{-\tau_i \psi_1(\omega)}] = e^{-\Lambda(h_i, \psi_1(\omega))} \quad \text{and} \quad E[e^{-\omega J_{i,2}}] = e^{-\Lambda(g_i, \psi_2(\omega))}
\]

Note also conditional on \( N \), for possibly complex valued numbers \( (\omega_1, \ldots, \omega_n) \),

\[
(4) \quad E\left[ \prod_{i=1}^{n} e^{-\omega_i J_{i,1} | N} \right] = \prod_{i=1}^{n} e^{-\tau_i \psi_1(\omega_i)} \quad \text{and} \quad E\left[ \prod_{i=1}^{n} e^{-\omega_i J_{i,2} | N} \right] = \prod_{i=1}^{n} e^{-\gamma_i \psi_2(\omega_i)}.
\]

How we shall proceed is to first evaluate everything conditional on \( N \). Our results will then boil down to expectation of exponential sum of terms of the form,

\[
(5) \quad \tau_i [\psi_1(\beta + \xi y_i) + (\beta^2 + y_i^2)/2] + \gamma_i [\psi_2(\beta + \xi y_i) + \rho(\beta + \xi y_i) + (\beta^2 + y_i^2)/2]
\]

where \( \xi \) is the imaginary number. This is similar to the case of James (2005). Now for real valued numbers \( (y_1, \ldots, y_n) \), set

\[
\Omega_n(x) = \sum_{i=1}^{n} [\psi_1(\beta + \xi y_i) + (\beta^2 + y_i^2)/2] h_i(x)
\]

and

\[
\Upsilon_n(x) = \sum_{i=1}^{n} [\psi_2(\beta + \xi y_i) + \rho(\beta + \xi y_i) + (\beta^2 + y_i^2)/2] g_i(x)
\]

Note that the sum over the terms in (3) is equivalent in distribution to \( N(\Omega_n + \Upsilon_n) \). We now state the form of the likelihood.

**Theorem 2.1** Suppose that \( N \) is a Poisson random measure with intensity \( \nu \) on \( \mathcal{F} \). Furthermore suppose that \( \tau_i \) and \( \gamma_i \), defined above, are chosen such that \( \Lambda(\Omega_n + \Upsilon_n) < \infty \). Then the joint marginal density or likelihood of \( X_1, \ldots, X_n | \mu, \beta, \theta, \rho \), determined by (4) and (3), is given by,

\[
\mathcal{L}^n(X | \mu, \beta, \theta, \rho) = \frac{e^{n\tilde{A} \beta}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\Lambda(\Omega_n + \Upsilon_n)} \prod_{i=1}^{n} e^{\tilde{A} \gamma_i y_i} dy_i.
\]

Where \( \tilde{A} = \sum_{i=1}^{n} A_i / n \). The result applies for the case where \( N \) is not necessarily Poisson, by replacing \( e^{-\Lambda(\Omega_n + \Upsilon_n)} \) with \( E[e^{-N(\Omega_n + \Upsilon_n)}] \). \( \square \)

**Proof.** The proof of this result is simply a slight variation of Theorem 5.1 in James (2005). For completeness we give many of the same details. Here we use the fact that for each \( i \) one has the identity deduced from the characteristic function of a Normal distribution, with mean 0 and variance \( 1/\tau_i^* \), evaluated at \( \omega_i = A_i - J_{i,1} - J_{i,2} - \rho \gamma_i \). That is,

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\omega y_i - \tau_i y_i^2/2} dy_i = \frac{1}{\sqrt{\tau_i^*}} e^{-(\omega_i)^2/2\tau_i^*}
\]
Now the result proceeds by substituting this expression in (3) and applying Fubini’s theorem. One then integrates out the expressions involving \((J_{i,1}, J_{i,2})\) conditionally on \(N\), which results in using (4). The rest now is precisely as the proof of Theorem 5.1 in James (2005). That is after rearranging terms it remains to calculate the expectation of \(e^{-N(\Omega_n + \Upsilon_n)}\).

**Remark 2.** Note that these arguments may be used to directly evaluate the density of \(x^*(t)\) given observations \(X_1, \ldots, X_n\), which might be interesting in an option pricing context. Statistical estimation follows along the lines of James (2005).

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Lancelot F. James

The Hong Kong University of Science and Technology
Department of Information and Systems Management
Clear Water Bay, Kowloon
Hong Kong
lancelot@ust.hk