A general transformation formula for conformal fields

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Abstract

An explicit transformation formula for chiral conformal fields under arbitrary holomorphic coordinate transformations is established. As an application I calculate the transformation law of the general quasiprimary field at level 4.

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1. Introduction

An important feature of two-dimensional conformal field theory is the fact that the algebra of conformal transformations can be understood as the direct product of two infinite dimensional algebras. The theory factorises therefore into a holomorphic and an anti-holomorphic subtheory, which have the group of analytic substitutions of the (anti-)analytic variable \( z(\bar{z}) \) as reparametrisation symmetries \([1]\). The central extension of this algebra is the Virasoro algebra \( L_n \), which is thus the remains of the conformal symmetry for each of the chiral theories.

The Virasoro algebra contains the algebra of Möbius transformations \( L_j, j = -1, 0, 1 \), namely the algebra of automorphisms of the Riemann sphere, as a subalgebra. For the Möbius subgroup the transformation formula of (chiral) conformal fields is known \([2]\). Namely, for \( \psi \) a \( su(1, 1) \) highest weight vector of conformal weight \( h \), i. e. \( L_1 \psi = 0 \) and \( L_0 \psi = h \psi \), the corresponding vertex operator transforms as

\[
D_\gamma V(\psi, z)D_\gamma^{-1} = \left[ \frac{d\gamma(z)}{dz} \right]^h V(\psi, \gamma(z)),
\]

where

\[
D_\gamma = \exp \left\{ \frac{b}{d} L_{-1} \right\} \left( \frac{\sqrt{ad - bc}}{d} \right)^{2L_0} \exp \left\{ -\frac{c}{d} L_1 \right\}
\]

and the Möbius transformation is given by

\[
\gamma(z) = \frac{az + b}{cz + d}
\]

In this letter I want to show how arbitrary holomorphic transformations can be implemented in the operator framework of (chiral) conformal field theory.

2. The main result

Let \( \psi \) be a primary field of weight \( h \), i. e. \( L_0 \psi = h \psi \) and \( L_n \psi = 0 \) for \( n > 0 \). Let \( f \) be an analytic function defined in some open neighbourhood of the origin \( f : D \rightarrow \mathbb{C} \) which is locally invertible, i. e. \( f'(0) \neq 0 \). Let furthermore the action of the exponentials of the Virasoro generators be local with respect to the given system of vertex operators
in the sense of [2]. Then the local transformation of the primary field $\psi$ is given as
\[
e^{f(0)L_{-1}} f'(0) L_0 \prod_{n=1}^{\infty} e^{S_n L_n} V(\psi, z)  \prod_{n=\infty}^{1} e^{-S_n L_n} f'(0) - L_0 e^{-f(0)L_{-1}} = f'(z)^h V(\psi, f(z)).
\]
(4)
The $S_n$ are defined to be $S_n = T_n(z)|_{z=0}$ and the $T_n(z)$ are recursively defined
\[
T_0(z) = \log f'(z),
\]
(5)
\[
T_n(z) = \frac{1}{n+1} \left( T_{n-1}^n(z) - A_n(z) \right) \quad n \geq 1.
\]

$A_n(z)$ is of the form
\[
A_n(z) = \sum_{\sum_{i=1}^m k_i l_i = n} C^m(k_1, l_1; \ldots; k_m, l_m) T_{k_1}^{l_1}(z) \cdots T_{k_m}^{l_m}(z),
\]
(6)
where the sum extends over those $2m$-tupels, where $1 \leq k_1 < k_2 < \cdots < k_m$ and $l_i$ is positive. The coefficients $C^m(k_1, l_1; \ldots; k_m, l_m)$ are zero unless $K_p > k_{p+1}$ for $p = 1, \ldots, m-1$, where
\[
K_p := \left( \sum_{i=1}^p k_i l_i \right) - 1.
\]
(7)
If this condition is met they are given as
\[
C^1(k, l) = \begin{cases} 
\frac{k+1}{l(l-2)!} \prod_{p=1}^{l-2} (pk-1) & l \geq 3 \\
\frac{k+1}{2} & l = 2 \\
0 & l = 1,
\end{cases}
\]
(8)
and for $m \geq 2$ as
\[
C^m(k_1, l_1; \ldots; k_m, l_m) = \left( \prod_{j=2}^m \frac{1}{l_j} \prod_{q_j=1}^{l_j} (K_{j-1} + (q_j - 2)k_j) \right) C^1(k_1, l_1).
\]
(9)
Before proving the formula I would like to remark that $3! T_2(z)$ is the Schwarzian derivative of the function $f$
\[
(D f)(z) = \frac{f''(z)}{f'(z)} - \frac{3}{2} \left( \frac{f''(z)}{f'(z)} \right)^2.
\]
(10)
Because of (8), $C^l(1, l) = 0$ for $l \neq 2$ and $C^m(1, 2; \ldots ) = 0$ for $m \geq 2$, as $K_1 = 1 \leq k_2$. Thus all $S_n, n \geq 2$ are sums of products of derivatives of the Schwarzian derivative. This implies that in the case of the Möbius transformations the formula (11) reduces to (9), as the Schwarzian derivative vanishes if and only if $f$ is a Möbius transformation.

To prove the formula it is enough to consider the case where $f(0) = 0$ and $f'(0) = 1$, as I can use (1, 2). By the uniqueness theorem of [2] it is furthermore sufficient to check the identity when applying both sides to the vacuum.

The strategy of the proof is to show (by induction) that all coefficients of the power series expansions in $z$ agree. The induction start is trivial, since both sides are just $\psi$.

To prove the induction step suppose that the coefficients for the power series expansion agree up to order $n - 1$. Taking the $n$-th derivative on the left-hand-side, evaluated at zero, I get

$$e^{S_0 L_0} \ldots e^{S_n L_n} (L_{-1})^n \psi,$$

which equals

$$L_{-1} \{ e^{S_0 L_0} \ldots e^{S_{n-1} L_{n-1}} (L_{-1})^{n-1} \psi \}$$

$$+ \sum_{k=0}^{n} e^{S_0 L_0} \ldots e^{S_{k-1} L_{k-1}} [e^{S_k L_k}, L_{-1}] e^{S_{k+1} L_{k+1}} \ldots e^{S_{n-1} L_{n-1}} (L_{-1})^{n-1} \psi. \quad (13)$$

Using the formula

$$[e^A, B] = \int_0^1 dt \ e^{(1-t)A} [A, B] \ e^{tA}$$

I can rewrite this expression as

$$L_{-1} \{ e^{S_0 L_0} \ldots e^{S_{n-1} L_{n-1}} (L_{-1})^{n-1} \psi \}$$

$$+ \sum_{k=1}^{n} (k+1) S_k e^{S_0 L_0} \ldots e^{S_{k-1} L_{k-1}} L_{k-1} e^{S_k L_k} \ldots e^{S_{n-1} L_{n-1}} (L_{-1})^{n-1} \psi \quad (15)$$

$$+ \sum_{k=1}^{n} (k+1) S_k \int_0^1 dt \ e^{S_0 L_0} \ldots e^{S_{k-1} L_{k-1}} [e^{(1-t)S_k L_k}, L_{k-1}]$$

$$e^{tS_k L_k} e^{S_{k+1} L_{k+1}} \ldots e^{S_{n-1} L_{n-1}} (L_{-1})^{n-1} \psi, \quad (16)$$

since the $k = 0$-term does not contribute because of $S_0 = 0$. Using the definition of $S_k$ this expression now becomes
\[ L_{-1} \left\{ e^{S_0 L_0} \ldots e^{S_{n-1} L_{n-1}} (L_{-1})^{n-1} \psi \right\} \]  
\[ + \sum_{k=1}^{n} \left. \frac{d}{d S_k} \right|_{z=0} \left\{ e^{S_0 L_0} \ldots e^{S_{n-1} L_{n-1}} (L_{-1})^{n-1} \psi \right\} \]  
\[ + \sum_{k=1}^{n} (k+1) S_k \int_0^1 dt \ e^{S_0 L_0} \ldots e^{S_{k-1} L_{k-1}} \left( e^{(1-t) S_k L_k}, L_{k-1} \right) \]  
\[ e^{t S_k L_k} e^{S_{k+1} L_{k+1}} \ldots e^{S_{n-1} L_{n-1}} (L_{-1})^{n-1} \psi \]  
\[ - \sum_{k=1}^{n} A_k(0) e^{S_0 L_0} \ldots e^{S_{k-1} L_{k-1}} L_{k-1} e^{S_k L_k} \ldots e^{S_{n-1} L_{n-1}} (L_{-1})^{n-1} \psi. \]  

Equations (17) and (18) are just the \( n \)-th derivative of the right-hand-side at \( z = 0 \), since the term in brackets is (by the induction hypothesis) the \((n-1)\)st derivative of the right-hand-side at \( z = 0 \). Thus it remains to prove that the remaining terms cancel. As the term \( k = n \) in (19) does not contribute, \( A_n \) has to solve

\[ A_n L_{n-1} (L_{-1})^{n-1} \psi = \sum_{k=1}^{n} (k+1) T_k^2 \int_0^1 dt^{(1)} \ (1 - t^{(1)}) \int_0^{t^{(1)}} dt^{(2)} \ e^{T_0 L_0} \ldots e^{T_{k-1} L_{k-1}} \]  
\[ e^{(1-t^{(1)})(1-t^{(2)}) T_k L_k} L_{2k-1} e^{(1-(1-t^{(1)})(1-t^{(2)}) T_k L_k} e^{T_{k+1} L_{k+1}} \ldots e^{T_{n-1} L_{n-1}} (L_{-1})^{n-1} \psi \]  
\[ - \sum_{k=1}^{n} A_k(0) e^{T_0 L_0} \ldots e^{T_{k-1} L_{k-1}} L_{k-1} e^{T_k L_k} \ldots e^{T_{n-1} L_{n-1}} (L_{-1})^{n-1} \psi, \]  

where I have used (14) to rewrite the commutator and I have replaced \( S_l \) by \( T_l \), since \( A_n \) has to satisfy the above equation for a neighbourhood of \( z = 0 \).

To check (21,22) I commute the Virasoro generator \( L_{2k-1} \) so that it stands next to the exponential containing \( L_{2k-1} \) to get a term of the form (23). I thereby obtain a sum of commutators of the form (19), which I can rewrite using (14) to get terms of the form (21,22). I then repeat the above procedure for each of these terms. For a given \( n \) only those terms contribute, whose Virasoro generator \( L_p \) satisfies \( p \leq n - 1 \). I therefore have to do the above algorithm only finitely many times. Furthermore, in (21,23) all terms with \( p < n - 1 \) cancel, since by induction \( A_p \) with \( p < n - 1 \) satisfies an equation corresponding to (21,23). Hence there is a solution for \( A_n \) and it has to be of the form (3). To calculate the numerical coefficients \( C^m(k_1, l_1; \ldots; k_m, l_m) \) one finally observes that the term containing \( T_{k_1}^{i_1} \ldots T_{k_m}^{i_m} \) can be obtained in one way only, namely by successively commuting \( L_{2k_1-1} \) with \( e^{it_{k_1} L_{k_1}} \) \( (l_1 - 2 \) times), then commuting the resulting Virasoro
generator with \(e^{T_{k_2}L_{k_2}}\) \((l_2\) times) and so forth. Calculating the corresponding multiple integrals (coming from the successive use of (14)) I arrive at (8) and (9).

The proof of the formula holds formally only for meromorphic fields, i.e. fields with integral conformal weight. However, as the action of the Virasoro algebra is local with respect to arbitrary fields, the result should extend to the more general case as well.

3. Consequences

It was a priori to be expected that (4) holds for some coefficients \(S_n\). The interesting new piece of information is therefore the explicit formula for these coefficients (8 - 9). Furthermore, the above derivation suggests how the result generalises to arbitrary conformal fields:

Let \(f : D \rightarrow \mathbb{C}\) be a function as above and let the \(S_n\)'s be defined as in (5 - 9). Let \(\phi\) be a secondary field of weight \(h + m\) in the family of \(\psi\). Let \(H(p)(f^{(i)}(z))\) be given as

\[
f'(z)^{L_0} \prod_{n=1}^{m} e^{T_n(z)L_n} \phi = \sum_{(p)} H(p)(f^{(i)}(z)) L_{p_1} \cdots L_{p_k} \phi, \tag{24}
\]

where \(p\) is a \(k\)-tuple of integers \(p_1, \ldots, p_k\) and the sum extends over all \(p\) for which \(L_{p_1} \cdots L_{p_k} \phi \neq 0\). Then the transformation law for \(\phi\) is given as

\[
e^{f(0)L_{-1}} f'(0)^{L_0} \prod_{n=1}^{\infty} e^{S_nL_n} V(\phi, z) \prod_{n=\infty}^{1} e^{-S_nL_n} f'(0)^{-L_0} e^{-f(0)L_{-1}}
\]

\[
= \sum_{(p)} H(p)(f^{(i)}(z)) V(L_{p_1} \cdots L_{p_k} \phi, f(z)). \tag{25}
\]

The proof of (25) is a replica of the above proof — the only difference being the fact that to prove the induction step from \(n-1\) to \(n\) the sum in (13) extends now to \(k = n + m\) and similarly in (21-23). However, since the \(A_p\) are defined recursively, so as to cancel (identically) all terms of the corresponding form, all remaining arguments go through.

The above formulae show that the product of exponentials in (4), resp. (25), implements arbitrary holomorphic coordinate transformations in the operator framework of chiral conformal field theory. They provide thus a link between two different concepts of primary/secondary/etc. fields: the one defined in terms of transformation properties of
fields and the one defined in terms of annihilation properties of the corresponding vectors under the action of the Virasoro algebra.

Furthermore, (24, 25) is a powerful tool for the determination of the transformation law for arbitrary conformal fields. As an application I have calculated the transformation formula for the general quasiprimary field at level 4 in the family of the primary field $\psi$. This fields is given as

$$\psi_4 = \left( b_4 L_{-1}^4 + a_4 L_{-2}^2 + m_1 L_2 L_{-1}^2 + m_2 L_{-3} L_{-1} + m_3 L_{-4} \right) \psi,$$

where

$$m_1 = -4b_4 \frac{2h + 3}{3}$$

$$m_2 = \frac{4b_4(4h^2 + 8h + 3) - 9a_4}{6}$$

$$m_3 = \frac{-4b_4(4h^2 + 8h + 3) + 9a_4(h - 1)}{15}$$

Using (24, 25) I find that $\psi_4$ transforms as

$$\psi_4(z) \rightarrow f'(z)^{h+4} \psi_4(f(z)) + \frac{a_4 \beta_1 + b_4 \beta_2}{a_2} (Df)(z) f'(z)^{h+2} \psi_2(f(z))$$

$$+ (a_4 \gamma_1 + b_4 \gamma_2) \frac{2}{2h + 1} (Df)(z) f'(z)^{h+2} \psi''(f(z))$$

$$+ (a_4 \gamma_1 + b_4 \gamma_2) \left( +2(Df)(z) f''(z) f'(z)^h - (Df)'(z) f'(z)^{h+1} \right) \psi'(f(z))$$

$$+ \left[ (a_4 \gamma_1 + b_4 \gamma_2) h \left( (Df)(z) f'''(z)^2 f'(z)^{h-2} - (Df)'(z) f''(z) f'(z)^{h-1} \right. \right.$$  

$$\left. + \frac{1}{5} (Df)'(z) f'(z)^{h} \right) + (Df)^2(z) \left( a_4 \delta_1 + b_4 h \left( \frac{4}{5} \gamma_2 + \delta_2 \right) \right) \psi(f(z)),$$

where $\psi_2$ is the general quasiprimary field at level 2

$$\psi_2 = a_2 \left( L_{-1}^2 - \frac{2(2h + 1)}{3} L_{-2} \right) \psi$$

and

$$\beta_1 = \frac{-5c + 58h + 22}{20(2h + 1)}$$
\[
\beta_2 = \frac{8 h^3 + 36 h^2 + 36 h}{5 (2 h + 1)} \quad (33)
\]
\[
\gamma_1 = \frac{1}{40} (5 c + 8 h - 3) \quad (34)
\]
\[
\gamma_2 = -\frac{1}{90} \left( 32 h^3 - 76 h^2 + 104 h - 15 + 5 c \left( 4 h^2 + 8 h + 3 \right) \right) \quad (35)
\]
\[
\delta_1 = \frac{c^2 + c (40 h + 11)}{144} + \frac{h (544 h - 329)}{360} + \frac{900}{900} \quad (36)
\]
\[
\delta_2 = \frac{(8 c h^2 + 6 c h - 9 c - 100 h^2 + 75 h)}{45} \quad (37)
\]

In the case of the vacuum representation, i.e. \( \psi = \Omega \) and \( h = 0 \), \( b_4 \), \( m_1 \) and \( m_2 \) can be chosen arbitrarily and \( m_3 = -3/5 a_4 \). The transformation formula then simplifies to

\[
\psi_4^0(z) \rightarrow f'(z)^4 \psi_4^0(f(z)) + a_4 \frac{5 c + 22}{30} (Df)(z) f'(z)^2 T(f(z))
\]
\[
+ \frac{c}{12} \frac{1}{2} a_4 \frac{5 c + 22}{30} (Df)^2(z) \Omega,
\]

where \( T \) is the holomorphic component of the stress-energy tensor, which is the quasiprimary field at level 2 with \( a_2 = -\frac{3}{2} \) in the family of the identity.

I have also checked that \((24, 25)\) reproduces the transformation law of \([3]\) for the quasiprimary field at level 6 in the 1-family.

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**References**

[1] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. 241 (1984) 333

[2] P. Goddard, Meromorphic conformal field theory, in: Proc. CIRM Conference on Infinite Dimensional Lie Algebras (Luminy, July 1988), ed. V.G. Kac (World Scientific, Singapore, 1989) p. 556.

[3] K. Rehren and B. Schroer, Nucl. Phys. B 295 (1988) 229