ALMOST-LINEAR-TIME WEIGHTED $\ell_p$-NORM SOLVERS IN SLIGHTLY DENSE GRAPHS VIA SPARSIFICATION

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Abstract. We give almost-linear-time algorithms for constructing sparsifiers with $n \text{poly}(\log n)$ edges that approximately preserve weighted ($\ell_2^2 + \ell_p^p$) flow or voltage objectives on graphs. For flow objectives, this is the first sparsifier construction for such mixed objectives beyond unit $\ell_p$ weights, and is based on expander decompositions. For voltage objectives, we give the first sparsifier construction for these objectives, which we build using graph spanners and leverage score sampling. Together with the iterative refinement framework of [Adil et al, SODA 2019], and a new multiplicative-weights based constant-approximation algorithm for mixed-objective flows or voltages, we show how to find $(1 + 2^{-\text{poly}(\log n)})$ approximations for weighted $\ell_p$-norm minimizing flows or voltages in $p(m^{1+o(1)} + n^{4/3+o(1)})$ time for $p = \omega(1)$, which is almost-linear for graphs that are slightly dense ($m \geq n^{4/3+o(1)}$).

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1. Introduction

Network flow problems are some of the most extensively studied problems in optimization (e.g. see [AMO93; Sch02; GT14]). A general network flow problem on a graph \(G(V, E)\) with \(n\) vertices and \(m\) edges can be formulated as

\[
\min_{B^\top f = d} \text{cost}(f),
\]

where \(f \in \mathbb{R}^E\) is a flow vector on edges satisfying net vertex demands \(d \in \mathbb{R}^V\), \(B \in \mathbb{R}^{E \times V}\) is the signed edge-vertex incidence matrix of the graph, and \(\text{cost}(f)\) is a cost measure on flows. The weighted \(\ell_\infty\)-minimizing flow problem, i.e., \(\text{cost}(f) = \|S^{-1}f\|_\infty\), captures the celebrated maximum-flow problem with capacities \(S\); the weighted \(\ell_1\)-minimizing flow problem, \(\text{cost}(f) = \|Sf\|_1\), captures the transshipment problem generalizing shortest paths with lengths \(S\); and \(\text{cost}(f) = f^\top Rf = \|R^{1/2}f\|_2^2\) captures the electrical flow problem [ST04].

Dual to flow problems are voltage problems, which can be formulated as

\[
\min_{d^\top v = 1} \text{cost}'(Bv),
\]

Analogous to the flow problems, picking \(\text{cost}'(Bv) = \|SBv\|_1\) captures the capacitated min-cut problem, \(\text{cost}'(Bv) = \|S^{-1}Bv\|_\infty\) captures vertex-labeling [Kyn+15], and \(\text{cost}'(Bv) = (Bv)^\top R^{-1}Bv = \|R^{-1/2}Bv\|_2^2\) captures the electrical voltages problem.

The seminal work of Spielman and Teng [ST04] gave the first nearly-linear-time algorithm for computing \((1+1/poly(n))\)-approximate solutions to electrical (weighted \(\ell_2\)-minimizing) flow/voltage problems. This work spurred the “Laplacian Paradigm” for designing faster algorithms for several classic graph optimization problems including maximum flow [Chr+11; She13; Kel+14], multi-commodity flow [Kel+14], bipartite matching [Mad13], transshipment [She17], and graph partitioning [OSV12], culminating in almost-linear-time or nearly-linear-time low-accuracy algorithms (i.e. \(1 + \varepsilon\) approximations with \(poly(\varepsilon)\) running time dependence) for many of these problems.

Progress on high-accuracy algorithms (i.e. algorithms that return \((1 + 1/poly(n))\)-approximate solutions with only a \(poly(\log n)\) factor overhead in time) for solving these problems has been harder to come by, and for many flow problems has been based on interior point methods [DS08]. E.g. the best running time for maximum flow stands at \(\tilde{O}(\min(m\sqrt{n}, n^{\omega} + n^{2+1/6}))\) [LS14; CLS19] and \(\tilde{O}(m^{4/3})\) for unit-capacity graphs [Mad13; LS20a; LS20a]. Other results making progress in this direction include works on shortest paths with small range negative weights [Coh+17b], and matrix-scaling [Coh+17a; All+17]. Recently, there has been progress on the dense case. In [van+20], the authors developed an algorithm for weighted bipartite matching and transshipment running in \(\tilde{O}(m + n^{3/2})\) time. This is a nearly-linear-time algorithm in moderate dense graphs.

Bubeck et al. [Bub+18] restarted the study of faster high-accuracy algorithms for the weighted \(\ell_p\)-norm objective, \(\text{cost}(f) = \|Sf\|_p\), a natural intermediate objective between \(\ell_2\) and \(\ell_\infty\). This result improved the running time significantly over classical interior point methods [NN94] for \(p\) close to 2. Adil et al. [Adi+19] gave a high-accuracy algorithm for computing \(\ell_p\)-norm minimizing flows in time \(\min\{m^{3/4+o(1)}, n^{\omega}\}\) for \(p \in (2, \sqrt{\log n})\). Building on their work, Kyng et al. [Kyn+19] gave an almost-linear-time high-accuracy algorithm for unit-weight \(\ell_p\)-norm minimizing flows \(\text{cost}(f) = \|f\|_p\) for large \(p \in (\omega(1), \sqrt{\log n})\). More generally, they give an almost-linear-time high-accuracy algorithm for mixed \(\ell_2^2 + \ell_p\) objectives as long as the \(\ell_p\)-norm is unit-weight, i.e.,

\[
\text{cost}(f) = \|R^{1/2}f\|_2^2 + \|f\|_p^p.
\]

Their algorithm for \((\ell_2^2 + \ell_p^p)\)-minimizing flows was subsequently used as a key ingredient in recent results improving the running time for high-accuracy/exact maximum flow on unit-capacity graphs to \(m^{4/3+o(1)}\) [LS20a; LS20a].

1
In this paper, we obtain a nearly-linear running time for weighted \( \ell_2^2 + \ell_p^p \) flow/voltage problems on graphs. Our algorithm requires \( p(m^{1+o(1)} + n^{4/3+o(1)}) \) time for \( p = \omega(1) \) which is almost-linear-time for \( p \leq m^{o(1)} \) in slightly dense graphs, \( (m \geq n^{4/3+o(1)}) \).

Our running time \( m^{1+o(1)} + n^{4/3+o(1)} \) is even better than the \( \tilde{O}(m + n^{3/2}) \) time obtained for bipartite matching in \cite{Van+20}. Our result beats the \( \Omega(n^{3/2}) \) barrier that arises in \cite{Van+20} from the use of interior point methods that maintain a vertex dual solution using dense updates across \( \sqrt{n} \) iterations. The progress on bipartite matching relies on highly technical graph-based inverse maintenance techniques that are tightly interwoven with interior point method analysis. In contrast, our sparsification methods provide a clean interface to iterative refinement, which makes our analysis much more simple and compact.

**Graph Sparsification.** Various notions of graph sparsification – replacing a dense graph with a sparse one, while approximately preserving some key properties of the dense graph – have been key ingredients in faster low-complexity algorithms. Benčík and Karger \cite{BK96} defined cut-sparse-ifiers that approximately preserve all cuts, and used them to give faster low-complexity approximation algorithms for maximum-flow. Since then, several notions of sparsification have been studied extensively and utilized for designing faster algorithms \cite{Alt+93, PS89, BS07, Kel+14, RST14, Chu+18}.

Sparsification has had a smaller direct impact on the design of faster high-complexity algorithms for graph problems, limited mostly to the design of linear system solvers \cite{ST04, KMP11, PS14, Kel+14, RST14, CP15, Kyn+16, Dur+17, Chu+18}. Kyng et al. \cite{Kyn+19} constructed sparsifiers for weighted \( \ell_2^2 \) + unweighted \( \ell_p^p \)-norm objectives for flows. In this paper, we develop almost-linear-time algorithms for building sparsifiers for weighted \( \ell_2^2 + \ell_p^p \) norm objectives for flows and voltages,

\[
\text{cost}(f) = \| R\frac{1}{2} f \|_2^2 + \| Sf \|_p^p \quad \text{and cost}'(Bv) = \| W\frac{1}{2} Bv \|_2^2 + \| UBv \|_p^p,
\]

and utilize them as key ingredients in our faster high-complexity algorithms for optimizing such objectives on graphs. Our construction of sparsifiers for flow objectives builds on the machinery from \cite{Kyn+19}, and our construction of sparsifiers for voltage objectives builds on graph spanners \cite{Alt+93, PS89, BS07}.

2. **Our Results**

Our main results concern flow and voltage problems for mixed \((\ell_2^2 + \ell_p^p)\)-objectives for \( p \geq 2 \). Since our algorithms work best for large \( p \), we restrict our attention to \( p = \omega(1) \) in this overview. Section 3 provides detailed running times for all \( p \geq 2 \). We emphasize that by setting the quadratic term to zero in our mixed \((\ell_2^2 + \ell_p^p)\)-objectives, we get new state of the art algorithms for \( \ell_p \)-norm minimizing flows and voltages.

**Mixed \( \ell_2, \ell_p \)-norm minimizing flow.** Consider a graph \( G = (V, E) \) along with non-negative diagonal matrices \( R, S \in \mathbb{R}^{E \times E} \), and a gradient vector \( g \in \mathbb{R}^E \), as well as demands \( d \in \mathbb{R}^V \). We refer to the diagonal entries of \( R \) and \( S \) as \( \ell_2 \)-weights and \( \ell_p \)-weights respectively. Let \( B \) denote the signed edge-vertex incidence of \( G \) (see Appendix A.4). We wish to solve the following minimization problem with the objective \( \mathcal{E}(f) = g^T f + \| R^{1/2} f \|_2^2 + \| S f \|_p^p \)

\[
\min_{B^T f = d} \mathcal{E}(f)
\]

We require \( g \perp \{ \ker(R) \cap \ker(S) \cap \ker(B) \} \) so that the problem has bounded minimum value, and \( d \perp 1 \) so a feasible solution exists. These conditions can be checked in linear time and have a simple combinatorial interpretation. Note that the choice of graph edge directions in \( B \) matters for the value of \( g^T f \). The flow on an edge is allowed to be both positive or negative.
Mixed $\ell_2$-$\ell_p$-norm minimizing voltages. Consider a graph $G = (V, E)$ along with non-negative diagonal matrices $W \in \mathbb{R}^{E \times E}$ and $U \in \mathbb{R}^{E \times E}$, and demands $d \in \mathbb{R}^V$. We refer to the diagonal entries of $W$ and $U$ as $\ell_2$-conductances and $\ell_p$-conductances respectively. In this case, we want to minimize the objective $\mathcal{E}(v) = d^T v + \|W^{1/2}Bv\|_2^2 + \|Uv\|_p^p$ in minimization problem

$$\min_v \mathcal{E}(v)$$

In the voltage setting, we only require $d \perp 1$ so the problem has bounded minimum value.

Obtaining good solutions. For both these problems, we study high accuracy approximation algorithms that provide feasible solutions $x$ (a flow or a voltage respectively), that approximately minimize the objective function from some starting point $x^{(0)}$, i.e., for some small $\varepsilon > 0$, we have

$$\mathcal{E}(x) - \mathcal{E}(x^*) \leq \varepsilon(\mathcal{E}(x^{(0)}) - \mathcal{E}(x^*))$$

where $x^*$ denotes an optimal feasible solution. Our algorithms apply to problems with quasipolynomially bounded parameters, including quasipolynomial bounds on non-zero singular values of matrices we work with. Below we state our main algorithmic results.

**Theorem 2.1** (Flow Algorithmic Result). Consider a graph $G$ with $n$ vertices and $m$ edges, equipped with non-negative $\ell_2$ and $\ell_p$-weights, as well as a gradient and demands, all with quasipolynomially bounded entries. For $p = \omega(1)$, in $p(m^{1+o(1)} + n^{4/3+o(1)}) \log^2 1/\varepsilon$ time we can compute an $\varepsilon$-approximately optimal flow solution to Problem (1) with high probability.

This improves upon [Adi+19; APS19; AS20] which culminated in a $pm^{4/3+o(1)} \log^2 1/\varepsilon$ time algorithm.

**Theorem 2.2** (Voltage Algorithmic Result). Consider a graph $G$ with $n$ vertices and $m$ edges, equipped with non-negative $\ell_2$ and $\ell_p$-conductances, as well as demands, all with quasipolynomially bounded entries. For $p = \omega(1)$, in $p(m^{1+o(1)} + n^{4/3+o(1)}) \log^2 1/\varepsilon$ time we can compute an $\varepsilon$-approximately optimal voltage solution to Problem (2) with high probability.

**Background: Iterative Refinement for Mixed $\ell_2$-$\ell_p$-norm Flow Objectives.** Adil et al. [Adi+19] developed a notion of iterative refinement for mixed $(\ell_2^2 + \ell_p^p)$-objectives which in the flow setting, i.e. Problem (1), corresponds to approximating $\mathcal{E}'(\delta) = \mathcal{E}(f + \delta)$ using another $(\ell_2^2 + \ell_p^p)$-objective which roughly speaking corresponds to the 2nd degree Taylor series approximation of $\mathcal{E}'(\delta)$ combined with an $\ell_2$-norm term $\|S\delta\|_p^p$, while ensuring feasibility of $f + \delta$ through a constraint $B\delta = 0$. We call the resulting problem a residual problem. Adil et al. [Adi+19] showed that obtaining a constant-factor approximate solution to the residual problem in $\delta$ is sufficient to ensure that $\mathcal{E}(f + \delta)$ is closer to the optimal solution by a multiplicative factor depending only on $p$. In [APS19], this result was sharpened to show that such an approximate solution for the residual problem can be used to make $(1 - \Omega(1/p))$ multiplicative progress to the optimum, so that $O(p \log(m/\varepsilon))$ iterations suffice to produce an $\varepsilon$-accurate solution.

In order to solve the residual problem to a constant approximation, Adil et al. [Adi+19] developed an accelerated multiplicative weights method for $(\ell_2^2 + \ell_p^p)$-flow objectives, or more generally, for mixed $(\ell_2^2 + \ell_p^p)$-regression in an underconstrained setting.

**Sparsification results.** Our central technical results in this paper concern sparsification of residual flow and voltage problems, in the sense outlined in the previous paragraph. Concretely, in nearly-linear time, we can take a residual problem on a dense graph and produce a residual problem on a sparse graph with $O(n)$ edges, with the property that constant factor solutions to the sparse residual problem still make $(1 - \Omega(m^{-2/p}p))$ multiplicative progress on the original problem. This leads to an iterative refinement that converges in $O(pm^{4/3} \log(m/\varepsilon))$ steps. However, the accelerated
multiplicative weights algorithm that we use for each residual problem now only requires $\tilde{O}(n^{4/3})$ time to compute a crude solution.

**Flow residual problem sparsification.** In the flow setting, we show the following:

**Theorem 2.3** (Informal Flow Sparsification Result). Consider a graph $G$ with $n$ vertices and $m$ edges, equipped with non-negative $\ell_2$ and $\ell_p$-weights, as well as a gradient. In $\tilde{O}(m)$ time, we can compute a graph $H$ with $n$ vertices and $\tilde{O}(n)$ edges, equipped with non-negative $\ell_2$ and $\ell_p$-weights, as well as a gradient, such that a constant factor approximation to the flow residual problem on $H$, when scaled by $mv^{\frac{1}{p-1}}$ results in an $\tilde{O}(mv^{\frac{2}{p-1}})$ approximate solution to the flow residual problem on $G$. The algorithm works for all $p \geq 2$ and succeeds with high probability.

Our sparsification techniques build on [Kyn+19], require a new bucketing scheme to deal with non-uniform $\ell_p$-weights, as well as a pre-processing step to handle cycles with zero $\ell_2$-weight and $\ell_p$-weight. This pre-processing scheme in turn necessitates a more careful analysis of additive errors introduced by gradient rounding, and we provide a more powerful framework for this than [Kyn+19].

**Voltage residual problem sparsification.** In the voltage setting, we show the following.

**Theorem 2.4** (Voltage Sparsification Result (Informal)). Consider a graph $G$ with $n$ vertices and $m$ edges, equipped with non-negative $\ell_2$ and $\ell_p$-conductances. In $\tilde{O}(m)$ time, we can compute a graph $H$ with $n$ vertices and $\tilde{O}(n)$ edges, equipped with non-negative $\ell_2$ and $\ell_p$-conductances, such that constant factor approximation to the voltage residual problem on $H$, when scaled by $mv^{\frac{1}{p-1}}$ results in an $\tilde{O}(m^{\frac{1}{p}})$ approximate solution to the voltage residual problem on $G$. The algorithm works for all $p \geq 2$ and succeeds with high probability.

Note that our voltage sparsification is slightly stronger than our flow sparsification, as the former loses only a factor $\tilde{O}(m^{\frac{1}{p}})$ in the approximation while the latter loses a factor $\tilde{O}(m^{\frac{1}{p-1}})$. Our voltage sparsification uses a few key observations: In voltage space, surprisingly, we can treat the $\ell_2$ and $\ell_p$ costs separately. This behavior is very different than the flow case, and arises because in voltage space, every edge provides an “obstacle”, i.e. adding an edge increases cost, whereas in flow space, every edge provides an “opportunity”, i.e. adding an edge decreases cost. This means that in voltage space, we can separately account for the energy costs created by our $\ell_2$ and $\ell_p$ terms, whereas in flow space, the $\ell_2$ and $\ell_p$ weights must be highly correlated in a sparsifier. Armed with this decoupling observation, we preserve $\ell_2$ cost using standard tools for spectral graph sparsification, and we preserve $\ell_p$ cost approximately by a reduction to graph distance preservation, which we in turn achieve using weighted undirected graph spanners.

**Voltage space accelerated multiplicative weights solver.** The algorithm from [Adi+19] for constant approximate solutions to the residual problem works in the flow setting. Using iterative refinement, the algorithm could be used to compute high-accuracy solutions. Because we can use high-accuracy flow solutions to extract high-accuracy solutions to the dual voltage problem, [Adi+19] were also able to produce solutions to $\ell_q$-norm minimizing voltage problems (where $\ell_q$ for $q = p/(p-1)$ is the dual norm to $\ell_p$). Hence, by solving $\ell_p$-flow problems for all $p \in (2, \infty)$, [Adi+19] were able to solve $\ell_q$-norm minimizing voltage problems for all $q \in (1, 2)$.

Our sparsification of flow and voltage problems works only for $p \geq 2$. Thus, in order to solve for $q$-norm minimizing voltages for $q > 2$, we require a solver that works directly in voltage space for mixed ($\ell_2^2 + \ell_p^p$)-objectives.

We develop an accelerated multiplicative weights algorithm along the lines of [Chr+11; Chi+13; Adi+19] that works directly in voltage space for mixed ($\ell_2^2 + \ell_p^p$)-objectives, or more generally for overconstrained mixed ($\ell_2^2 + \ell_p^p$)-objective regression. Concretely, this directly gives an algorithm
for computing crude solutions to the residual problems that arise from applying [Adi+19] iterative refinement to Problem (2). Our solver produces an improved $O(1)$-approximation to the residual problem rather than a $p^{O(p)}$-approximation from [Adi+19]. This gives an $\hat{O}(m^{4/3})$ high-accuracy algorithm for mixed ($\ell^2 + \ell^p$)-objective voltage problems for $p > 2$, unlike [Adi+19], which could only solve pure $p > 2$ voltage problems. We then speed this up to a $p(m^{1-o(1)} + n^{4/3 + o(1)})$ time algorithm for $p = \omega(1)$ by developing a sparsification procedure that applies directly to mixed ($\ell^2 + \ell^p$)-voltage problems for $p > 2$.

**Mixed $\ell^2$-$\ell^p$-norm regression.** Our framework can also be applied outside of a graph setting, where our new accelerated multiplicative weights algorithm for overconstrained mixed ($\ell^2 + \ell^p$)-regression gives new state-of-the-art results in some regimes when combined with new sparsification results. In this setting we develop sparsification techniques based on the Lewis weights sampling from the work of Cohen and Peng [CP15]. We focus on the case $2 < p < 4$, where [CP15] provided fast algorithms for Lewis weight sampling.

**Theorem 2.5 (General Matrices Sparsification Result).** Let $p \in [2, 4)$, let $M \in \mathbb{R}^{m_1 \times n}, N \in \mathbb{R}^{m_2 \times n}$ be matrices, $m_1, m_2 \geq n$, and let $\text{LSS}(B)$ denote the time to solve a linear system in $B^T B$. Then, we may compute $\tilde{M}, \tilde{N} \in \mathbb{R}^{O(n^{p/2} \log(n)) \times n}$ such that with probability at least $1 - \frac{1}{n^{20}}$, for all $\Delta \in \mathbb{R}^n$,

$$\|\tilde{M} \Delta\|_2^2 + \|\tilde{N} \Delta\|_p^p \approx_{O(1)} \|M \Delta\|_2^2 + \|N \Delta\|_p^p,$$

in time $\hat{O}(\text{nnz}(M) + \text{nnz}(N) + \text{LSS}(\tilde{M}) + \text{LSS}(\tilde{N}))$, for some $\tilde{M}$ and $\tilde{N}$ each containing $O(n \log(n))$ rescaled rows of $M$ and $N$, respectively.

**Theorem 2.6 (General Matrices Algorithmic Result).** For $p \in [2, 4)$, with high probability we can find an $\varepsilon$-approximate solution to (3) in time

$$\hat{O}\left((\text{nnz}(M) + \text{nnz}(N) + (\text{LSS}(\tilde{M}) + \text{LSS}(\tilde{N})) n \frac{n^{p-2}}{6p-4}) \log^2(1/\varepsilon)\right),$$

for some $\tilde{M}$ and $\tilde{N}$ each containing $O(n^{p/2} \log(n))$ rescaled rows of $M$ and $N$, respectively, where $\text{LSS}(A)$ is the time required to solve a linear equation in $A^T A$ to quasipolynomial accuracy.

Note that for all $p \in (2, 4)$, we have that the exponent $\frac{n^{p-2}}{6p-4} \leq 0.4$.

**Remark 2.7.** By [Coh+12], a linear equation in $A^T A$, where $A \in \mathbb{R}^{m \times n}$ can be solved to quasipolynomial accuracy in time $\hat{O}(\text{nnz}(A) + n^\omega)$.

Using the above result for solving the required linear systems, we get a running time of $\hat{O}(\text{nnz}(M) + \text{nnz}(N) + (n^{p/2} + n^\omega)n \frac{n^{p-2}}{6p-4})$, matching an earlier input sparsity result by Bubeck et al. [Bub+18] that achieves $\hat{O}((\text{nnz}(M) + \text{nnz}(N))(1 + n^{\frac{5}{2}}m^{-\frac{3}{2}}) + m^{\frac{1}{2}} + n^{2 + n^\omega})$, where $M \in \mathbb{R}^{m_1 \times n}, N \in \mathbb{R}^{m_2 \times n}$ and $m = \max\{m_1, m_2\}$.

3. Main Algorithm

In this section, we prove Theorems 2.1, 2.2 and 2.6. We first design an algorithm to solve the following general problem:

**Definition 3.1.** For matrices $M \in \mathbb{R}^{m_1 \times n}, N \in \mathbb{R}^{m_2 \times n}$ and $A \in \mathbb{R}^{d \times n}$, $m_1, m_2 \geq n, d \leq n$, and vectors $b \perp \{\ker(M) \cap \ker(N) \cap \ker(A)\}$ and $c \in \text{im}(A)$, we want to solve

$$\begin{align*}
\min_x & \quad b^T x + \|M x\|_2^2 + \|N x\|_p^p \\
\text{s.t.} & \quad Ax = c.
\end{align*}$$

(3)
In order to solve the above problem, we use the iterative refinement framework from [APS19] to obtain a residual problem which is defined as follows.

**Definition 3.2.** For any \( p \geq 2 \), we define the residual problem \( res(\Delta) \), for (3) at a feasible \( x \) as,

\[
\max_{A\Delta = 0} \quad res(\Delta) \overset{\text{def}}{=} g^\top \Delta - \Delta^\top R\Delta - \|N\Delta\|^p_p, \quad \text{where,}
\]

\[
g = \frac{1}{p} b + \frac{2}{p} M^\top M x + |N x|^{p-2} N x \quad \text{and} \quad R = \frac{2}{p^2} M^\top M + 2N^\top \text{Diag}(|N x|^{p-2}) N.
\]

This residual problem can further be reduced by moving the term linear in \( x \) to the constraints via a binary search. This leaves us with a problem of the form,

\[
\min_{\Delta} \quad \Delta^\top R\Delta + \|N\Delta\|^p_p
\]

\[
\text{s.t.} \quad g^\top \Delta = a, A\Delta = 0,
\]

for some constant \( a \).

In order to solve the above problem with \( \ell_2^p + \ell_p^0 \) objective, we reduce the instance size via a sparsification routine, and then solve the smaller problem by a multiplicative weights algorithm. We adapt the multiplicative-weights algorithm from [Adi+19] to work in the voltage space while improving the dependence of the runtime from \( pO(p) \) to \( p \), and the approximation quality from \( pO(p) \) to \( O(1) \). The precise sparsification routines are described in later sections.

For large \( p \), i.e., \( p > \log m \), in order to get a linear dependence on the running time on \( p \), we need to reduce the residual problem in \( \ell_p \)-norm to a residual problem in \( \log m \)-norm by using the framework from [AS21].

The entire meta-algorithm is described formally in Algorithm 1 and its guarantees are described by the next theorem. Most proof details are deferred to Appendix D.

**Theorem 3.3.** For an instance of Problem (3), suppose we are given a starting solution \( x^{(0)} \) that satisfies \( Ax^{(0)} = c \) and is a \( \kappa \) approximate solution to the optimum. Consider an iteration of the while loop, line 8 of Algorithm 1 for the \( \ell_p \)-norm residual problem at \( x^{(t)} \). We can define \( \mu_1 \) and \( \kappa_1 \) such that if \( \Delta \) is a \( \beta \) approximate solution to a corresponding \( p' \)-norm residual problem, then \( \mu_1 \Delta \) is a \( \kappa_1 \)-approximate solution to the \( p \)-residual problem. Further, suppose we have the following procedures,

1. **Sparsify:** Runs in time \( K \), takes as input any matrices \( R, N \) and vector \( g \) and returns \( \vec{R}, \vec{N}, \vec{g} \) having sizes at most \( \tilde{n} \times n \) for the matrices , such that if \( \Delta \) is a \( \beta \) approximate solution to,

\[
\max_{A\Delta = 0} \quad \vec{g}^\top \Delta - \|\vec{R}\Delta\|^2_2 - \|\vec{N}\Delta\|^p_{p'},
\]

for any \( p' \geq 2 \), then \( \mu_2 \tilde{\Delta} \), for a computable \( \mu_2 \) is a \( \kappa_2 \)-approximate solution for,

\[
\max_{A\Delta = 0} \quad res(\Delta) \overset{\text{def}}{=} g^\top \Delta - \|R^{1/2}\Delta\|^2_2 - \|N\Delta\|^p_{p'}.
\]

2. **Solver:** Approximately solves (4) to return \( \tilde{\Delta} \) such that \( \|\tilde{R}\Delta\|^2_2 \leq \kappa_3 \nu \) and \( \|\tilde{N}\Delta\|^p_{p'} \leq \kappa_4 \nu \) in time \( \tilde{K}(\tilde{n}) \) for instances of size at most \( \tilde{n} \).

Algorithm 1 finds an \( \varepsilon \)-approximate solution for Problem (3) in time

\[
\tilde{O}\left(\frac{p \kappa_4^{1/(p-1)} \kappa_3 \kappa_2 \kappa_1 (K + \tilde{K}(\tilde{n}))}{\varepsilon} \log \left(\frac{n p}{\varepsilon}\right)^2\right).
\]
Algorithm 1 Meta-Algorithm for $\ell_p$ Flows and Voltages

1: procedure SPARSIFIED-$p$-PROBLEMS($A, M, N, c, b, p$)
2:  $x \leftarrow x^{(0)}$, such that $f(x^{(0)}) \leq \kappa \text{OPT}$
3:  $T \leftarrow \tilde{O}(p^{k_1}k_2\kappa_3 \log(\frac{\kappa}{\epsilon}))$
4:  for $t = 0$ to $T$ do
5:     At $x^{(t)}$ define $g, R, N$ and $\text{res}(\Delta)$, the residual problem (Definition 3.2)
6:     $a \leftarrow \frac{1}{2}, b \leftarrow 1, \mu_1 \leftarrow 1, \kappa_1 \leftarrow 1$
7:     $\nu \leftarrow f(x^{(0)})$
8:     while $\nu \geq \frac{\epsilon f(x^{(0)})}{\kappa p}$ do
9:         if $p > \log m$ then $\triangleright$ Convert $\ell_p$-norm residual to $\log m$-norm residual
10:            $p' \leftarrow \log m$
11:            $N' \leftarrow \frac{1}{2^{p'}}(\frac{\nu}{m})^{-\frac{1}{p'}} N$
12:            $a \leftarrow \frac{1}{2^{p}}, b \leftarrow O(1)m^{o(1)}$
13:            $\mu_1 \leftarrow m^{-o(1)}, \kappa_1 \leftarrow m^{o(1)}$ $\triangleright$ Lose $\kappa_1$ in approx. when scaled by $\mu_1$
14:            $(\tilde{g}, \tilde{R}, \tilde{N}) \leftarrow \text{SPARSIFY}(g, R, N')$ $\triangleright$ Lose $\kappa_2$ in approx. when scaled by $\mu_2$
15:        else
16:            $(\tilde{g}, \tilde{R}, \tilde{N}) \leftarrow \text{SPARSIFY}(g, R, N)$ $\triangleright$ Lose $\kappa_2$ in approx. when scaled by $\mu_2$
17:            $p' \leftarrow p$
18:        Use Solver to compute $\kappa_3, \kappa_4$ approximate solution to
19:            \begin{equation}
20:                \tilde{\Delta}(\nu) \leftarrow \arg\min_{\Delta} \frac{\|\tilde{R}\|_2^2}{\|\tilde{O}\|_{p'}} + \|\tilde{N}\|_{p'}^p
21:                \text{s.t. } \tilde{g}^\top \Delta = a\nu, \quad A\Delta = 0.
22:            \end{equation}
23:            $\nu \leftarrow \nu/2$
24:            $\Delta \leftarrow \arg\min_{\Delta} f(\Delta)
25:            x \leftarrow x - \frac{\Delta}{p}$
26:        return $x$

3.1. Algorithms for $\ell_p$-norm Problems. The problems discussed in Section 2 are special cases of Problem (3), which means we can use Algorithm 1. To prove our results, we will utilize Theorem 3.3 with the respective sparsification procedures and the following multiplicative-weights based algorithm for solving problems of the form,

\begin{equation}
\min_{\Delta} \Delta^\top M^\top M\Delta + \|N\Delta\|_p^p
\text{s.t. } A\Delta = c.
\end{equation}

We describe our solver formally and prove the following theorem about its guarantees in Appendix C.

Theorem 3.4. Let $p \geq 2$. Consider an instance of Problem (5) described by matrices $A \in \mathbb{R}^{d \times n}, N \in \mathbb{R}^{m_1 \times n}, M \in \mathbb{R}^{m_2 \times n}, d \leq n \leq m_1, m_2$, and vector $c \in \mathbb{R}^d$. If the optimum of this problem is at most $\nu$, Procedure RESIDUAL-SOLVER (Algorithm 3) returns an $x$ such that $Ax = c$,
and $\mathbf{x}^\top \mathbf{M}^\top \mathbf{M} \mathbf{x} \leq O(1)\nu$ and $\|\mathbf{N} \mathbf{x}\|^p \leq O(3^p)\nu$. The algorithm makes $O\left(pm\frac{p-2}{p-1}\right)$ calls to a linear system solver.

We utilize Procedure Residual-Solver as the Procedure Solver in Algorithm Sparsified-p-Problems. The algorithm uses the procedure only for solving problems instances with $p \leq \log m$. Thus, its running time is

$$K(\tilde{n}) = \tilde{O}\left(\frac{n^{\frac{2}{p-1}}}{m^{\frac{1}{p-1}}} \cdot LSS(\tilde{n})\right) \leq \tilde{O}\left(\frac{n^{1/3}}{m} \cdot LSS(\tilde{n})\right),$$

where LSS(\tilde{n}) denotes the time required to solve a linear system in matrices of size \(\tilde{n}\). We also have, $\kappa_3 = O(1), \kappa_4^{1/(p-1)} = O(1)$.

We next estimate the values of $\kappa_1$ and $\mu_1$. If $p \leq \log m$, we have $\mu_1 = 1$ and $\kappa_1 = 1$. Otherwise, $\mu_1 = \tilde{O}(1)$ and $\kappa_1 = O(m^{o(1)})$ (Refer to Lemma D.4 in Appendix D).

In order to obtain an initial solution, we usually solve an $\ell_2$-norm problem. This gives an $m^{p/2}$ approximate initial solution which results in a factor of $p^2$ in the running time. To avoid this, we can do a homotopy on $p$ similar to [AS20], i.e., start with an $\ell_2$ solution and solve the $\ell_{p2}$ problem to a constant approximation, followed by $\ell_{21}, \ldots, \ell_p$. We note that a constant approximate solution to the $\ell_{p2}$-norm problem gives an $O(m)$ approximation to the $\ell_p$ problem and thus, we can solve $\log p$ problems where we can assume $\kappa = O(m)$.

We now complete the proof of our various algorithmic results by utilizing sparsification procedures specific to each problem.

\textbf{$\ell_p$ Flows.} We will prove Theorem 2.1 (Flow Algorithmic Result), with explicit $p$ dependencies.

\textbf{Proof.} From Theorem 2.3, we obtain a sparse graph in $K = \tilde{O}(m)$ time with $\tilde{n} = \tilde{O}(n)$ edges. A constant factor approximation to the flow residual problem on this sparse graph when scaled by $\mu_2 = m^{-\frac{1}{p-1}}$ gives a $\kappa_2 = \tilde{O}\left(m^{\frac{2}{p-1}}\right)$-approximate solution to the flow residual problem on the original graph. We can solve linear systems on the sparse graph in $\tilde{O}(\tilde{n}) = \tilde{O}(n)$ time using fast Laplacian solvers. Using all these values in Theorem 3.3 we get the final runtime to be $pm\frac{2}{p-1} + o(1)\left(m + n^{1+\frac{p-2}{p-1}}\right)\log^2(\frac{pm}{\varepsilon})$ as claimed. We prove Theorem 2.3 in Section A.

\textbf{$\ell_p$ Voltages.} We will prove Theorem 2.2 (Voltage Algorithmic Result), with explicit $p$ dependencies.

\textbf{Proof.} From Theorem 2.4, we obtain a sparse graph in $K = \tilde{O}(m)$ time with $\tilde{n} = \tilde{O}(n)$ edges. A constant factor approximation to the voltage residual problem on this sparse graph when scaled by $\mu_2 = m^{-\frac{1}{p-1}}$ gives a $\kappa_2 = \tilde{O}\left(m^{\frac{2}{p-1}}\right)$-approximate solution to the voltage residual problem on the original graph. We can solve linear systems on the sparse graph in $\tilde{O}(\tilde{n}) = \tilde{O}(n)$ time using fast Laplacian solvers. Using all these values in Theorem 3.3 we get the final runtime to be $pm\frac{1}{p-1} + o(1)\left(m + n^{1+\frac{p-2}{p-1}}\right)\log^2(\frac{pm}{\varepsilon})$ as claimed. We prove Theorem 2.4 in Section A.

\textbf{General Matrices.} We will now prove Theorem 2.6.

\textbf{Proof.} We assume Theorem 2.5 which we prove in Appendix B. From the theorem, we have $\kappa_2 = O(1)$ and $\mu_2 = O(1)$. Note that $K = \text{LSS}(	ilde{M}) + \text{LSS}(	ilde{N})$ for some $\tilde{M}, \tilde{N} \in \mathbb{R}^{O(n \log(n)) \times n}$, which is the time required to solve linear systems in $\tilde{M}^\top \tilde{M}$ and $\tilde{N}^\top \tilde{N}$, respectively. Since, by Theorem 2.5 the size of $\tilde{M}$ and $\tilde{N}$ is $\tilde{n} = O(n^{p/2} \log(n))$, the cost from the solver in Theorem 3.3 is $\tilde{O}_p\left((\text{LSS}(	ilde{M}) + \text{LSS}(	ilde{N}))n^{\frac{p(p-2)}{6p-4}}\right)$.
4. Construction of Sparsifiers for $\ell_2^2 + \ell_p^p$ Voltages

In this section, we prove a formal version of the voltage sparsification result (Theorem 2.4):

**Theorem 4.1.** Consider a graph $G = (V, E)$ with non-negative 2-weights $w \in \mathbb{R}^E$ and non-negative $p$-weights $s \in \mathbb{R}^E$, with $m$ and $n$ vertices. We can produce a graph $H = (V, F)$ with edges $F \subseteq E$, $\ell_2$-weights $u \in \mathbb{R}^F$, and $\ell_p$-weights $t \in \mathbb{R}^F$, such that with probability at least $1 - \delta$ the graph $H$ has $O(n \log(n/\varepsilon))$ edges and

$$\frac{1}{1.5} \| WB_G x \|_2 \leq \| UB_H x \|_2 \leq 1.5 \| WB_G x \|_2$$

and for any $p \in [1, \infty]$

$$\frac{1}{m^{1/p} \log(n)} \| SB_G x \|_p \leq \| TB_H x \|_p \leq \| SB_G x \|_p$$

where $W = \text{Diag}(w)$, $U = \text{Diag}(u)$, $S = \text{Diag}(s)$, $T = \text{Diag}(t)$. We denote the routine computing $H$ and $u, t$ by $\text{SpannerSparsify}$, so that $(H, u, t) = \text{SpannerSparsify}(G, w, s)$. This algorithm runs in $O(m \log(1/\delta))$ time.

We will first define some terms required for our result. Given a undirected graph $G = (V, E)$, with edge lengths $l \in \mathbb{R}^E$, and $u, v \in V$, we let $d_G(u, v)$ denote the shortest path distance in $G$ w.r.t $l$, so that if $P$ is the shortest path w.r.t $l$ then

$$d_{G,l}(u, v) = \sum_{e \in P} l(e)$$

**Definition 4.2.** Given a undirected graph $G = (V, E)$ with edge lengths $l \in \mathbb{R}^E$, a $K$-spanner is a subgraph $H$ of $G$ with the same edge lengths s.t. $d_H(u, v) \leq Kd_G(u, v)$.

Baswana and Sen showed the following result on spanners [BS07].

**Theorem 4.3.** Given an undirected graph $G = (V, E, l)$ with $m$ edges and $n$ vertices, and an integer $k > 1$, we can compute a $(2k - 1)$-spanner $H$ of $G$ with $O(n^{1+1/k})$ edges in expected time $O(km)$.

**Lemma 4.4.** Given an undirected graph $G = (V, E)$ with positive edge lengths $l \in \mathbb{R}^E$, and a $K$-spanner $H = (V, F)$ of $G$, for all $x \in \mathbb{R}^V$ we have

$$\max_{(u, v) \in F} \frac{1}{l(u, v)} | x(u) - x(v) | \leq \max_{(u, v) \in E} \frac{1}{l(u, v)} | x(u) - x(v) | \leq K \max_{(u, v) \in F} \frac{1}{l(u, v)} | x(u) - x(v) |$$

Proof. The inequality $\max_{(u, v) \in F} \frac{1}{l(u, v)} | x(u) - x(v) | \leq \max_{(u, v) \in E} \frac{1}{l(u, v)} | x(u) - x(v) |$ is immediate from $F \subseteq E$.

To prove the second inequality, we note that if $(u, v) \in E$ has shortest path $P$ in $H$ then

$$\frac{1}{l(u, v)} | x(u) - x(v) | \leq \sum_{(z, y) \in P} \frac{K}{l(z, y)} | x(z) - x(y) | \leq \max_{(z, y) \in P} \frac{K}{l(z, y)} | x(z) - x(y) |.$$

**Definition 4.5.** Given a undirected graph $G = (V, E)$ with $m$ edges and $n$ vertices with positive edge $\ell_2$-weights $w \in \mathbb{R}^E$, a spectral $\varepsilon$-approximation of $G$ is a graph $H = (V, F)$ with $F \subseteq E$ with positive edge $\ell_2$-weights $u \in \mathbb{R}^F$ s.t.

$$\frac{1}{1 + \varepsilon} \| WB_G x \|_2 \leq \| UB_H x \|_2 \leq (1 + \varepsilon) \| WB_G x \|_2$$

where $W = \text{Diag}(w)$ and $U = \text{Diag}(u)$. 

9
The following result on spectral sparsifiers was shown by Spielman and Srivastava \(SS11\) (see also \(Spi15\)).

**Theorem 4.6.** Given a graph \(G = (V, E)\) with positive \(\ell_2\)-weights \(w \in \mathbb{R}^E\) with \(m\) edges and \(n\) vertices, for any \(\varepsilon \in (0, 1/2]\), we can produce a graph \(H = (V, F)\) with edges \(F \subseteq E\) and \(\ell_2\)-weights \(u \in \mathbb{R}^F\) such that \(H\) has \(O(ne^{-2}\log(n/\delta))\) edges and with probability at least \(1 - \delta\) we have that \((H, u)\) is a spectral \(\varepsilon\)-approximation of \((G, w)\). We denote the routine computing \(H\) and \(u\) by \textsc{SpectralSparsify}, so that \((H, u) = \textsc{SpectralSparsify}(G, s, \varepsilon, \delta)\). This algorithm runs in \(O(m)\) time. Furthermore, if the weights \(w\) are quasipolynomially bounded, then so are the weights of \(u\).

We can now prove our main result.

**Proof of Theorem 4.1.** We consider a graph \(G = (V, E)\) with \(m\) edges and \(n\) vertices, and with non-negative \(\ell_p\)-weights \(r \in \mathbb{R}^E\), non-negative \(\ell_2\)-weights \(s \in \mathbb{R}^E\). We define \(\hat{E} \subseteq E\) to be the edges s.t. \(s(e) > 0\), and then let \(l \in \mathbb{R}^{\hat{E}}\) by \(l(e) = 1/s(e)\), and \(\hat{G} = (V, \hat{E})\). We then apply Theorem 4.3 to \(\hat{G}\) with \(l\) as edge lengths, and with \(k = \log(n)\). We turn the algorithm of Theorem 4.3 into running time \(\tilde{O}(m \log(1/\delta))\), instead of expected time \(\tilde{O}(m)\), by applying the standard Las Vegas to Monte-Carlo reduction. With probability \(1 - \delta/2\), this gives us a \(\log n\)-spanner \(H_1\) of \(\hat{G}\), and we define \(t\) by restricting \(s\) to the edges of \(H_1\). By Lemma 4.3, we then have

\[
\|TB_{H_1}x\|_\infty \leq \|SB_{G}x\|_\infty \leq \log(n)\|TB_{H_1}x\|_\infty
\]

Because \(TB_{H_1}x\) is a restriction of \(SB_{G}x\) to a subset of the coordinates, we always have for any \(p \geq 1\) that \(\|TB_{H_1}x\|_p \leq \|SB_{G}x\|_p\).

At the same time, we also have

\[
\|SB_{G}x\|_p \leq m^{1/p}\|SB_{G}x\|_\infty \leq m^{1/p}\log(n)\|TB_{H_1}x\|_\infty \leq m^{1/p}\log(n)\|TB_{H_1}x\|_p
\]

We define \(\hat{E} \subseteq E\) to be the edges s.t. \(r(e) > 0\), and the let \(\hat{G} = (V, \hat{E})\). Now, appealing to Theorem 4.6 we let \((H_2, u) = \textsc{SpectralSparsify}(\hat{G}, r, 1/2, \varepsilon/2)\).

Finally, we form \(H\) by taking the union of the edge sets of \(H_1\) and \(H_2\) and extending \(u\) and \(t\) to the new edge set by adding zero entries as needed. By a union bound, the approximation guarantees of Equations 6 and 7 simultaneously hold with probability at least \(1 - \delta\).

The edge set remains bounded in size by \(\tilde{O}(n \log n)\).

To see Theorem 2.4, note that from Theorem 4.1 we get,

\[
m^{-\frac{1}{p-1}} \left(m^{-\frac{1}{p-1}} \|WB_{G}x\|_2^2 + m^{-1}\|SB_{G}x\|_p^p\right) \leq m^{-\frac{1}{p-1}} \left(\|UB_{H}x\|_2^2 + \|TB_{H}x\|_p^p\right)
\]

The other direction is easy to see.

5. Extensions of Our Results and Open Problems

**Solving dual problems:** \(q\)-norm minimizing flows and voltages for \(q < 2\). When the mixed \((\ell_2^2 + \ell_p^2)\)-objective flow problem (Problem 11) is restricted to the case \(g = 0\) and \(R = 0\), it becomes a pure \(\ell_p\)-norm minimizing flow problem, and its dual problem can be slightly rearranged to give

\[
\min_v d^T v + \left\|S^{-1}Bv\right\|_q^q
\]

where \(q = p/(p-1) = 1 + 1/(p-1)\). We refer to the diagonal entries of \(S^{-1}\) as \(\ell_q\)-conductances. Because we can solve Problem 11 to high-accuracy in near-linear time for \(p = \omega(1)\), this allows us to solve Problem 8, the dual voltage \(\ell_q\)-norm minimization, in time \(p(m^{1+o(1)} + n^{4/3+o(1)})\log^2 1/\varepsilon\) (see \(Adi+19\), Section 7 for the reduction). We summarize this in the theorem below.
**Theorem 5.1** (Voltage Algorithmic Result, $q < 2$ (Informal)). Consider a graph $G$ with $n$ vertices and $m$ edges, equipped with positive $\ell_q$-conductances, as well as a demand vector. For $1 < q < 2$, when $q = 1 + o(1)$, in poly\(\left(\frac{1}{\epsilon}\right)\left(m^{1+o(1)} + n^{4/3+o(1)}\right)\log^2 1/\epsilon\) time, we can compute an $\epsilon$-approximately optimal voltage solution to Problem \(\text{(3)}\) with high probability.

Similarly, we can solve $\ell_q$-norm minimizing flows for $q < 2$ as dual to the $\ell_p$-voltage problem, a special case of the mixed \((\ell_q^2 + \ell_q^2)\)-voltage problem. Picking $W = 0$ in Problem \(\text{(2)}\), we obtain a pure $\ell_p$-norm minimizing voltage problem, and its dual problem can be slightly rearranged to give

\[
\min_{B^Tf = d} \left\| U^{-1}f \right\|_q^q
\]

where $q = p/(p - 1) = 1 + 1/(p - 1)$. We refer to the diagonal entries of $U^{-1}$ as $q$-weights. Again, because we can solve Problem \(\text{(2)}\) to high-accuracy in near-linear time for $p = \omega(1)$, this allows us to solve Problem \(\text{(4)}\), the dual flow $\ell_q$-norm minimization, in time $p(m^{1+o(1)} + n^{4/3+o(1)})\log^2 1/\epsilon$.

**Theorem 5.2** (Flow Algorithmic Result, $q < 2$ (Informal)). Consider a graph $G$ with $n$ vertices and $m$ edges, equipped with positive $q$-weights, as well as a demand vector. For $1 < q < 2$, when $q = 1 + o(1)$, in poly\(\left(\frac{1}{\epsilon}\right)\left(m^{1+o(1)} + n^{4/3+o(1)}\right)\log^2 1/\epsilon\) time, we can compute an $\epsilon$-approximately optimal flow solution to Problem \(\text{(4)}\) with high probability.

**Open Questions.** **Mixed $\ell_2, \ell_q$ problems for small $q < 2$.** In this work, we provided new state-of-the-art algorithms for weighted mixed $\ell_2, \ell_p$-norm minimizing flow and voltage problems for $p >> 2$, and for pure $\ell_q$-norm minimizing flow and voltage problems for $q$ near 1.

A reasonable definition of mixed $\ell_2, \ell_p$-norm problems for $q < 2$ is based on gamma-functions as introduced in \cite{Bub+18} and used in \cite{Adi+19}. We believe that with minor adjustments to our multiplicative weights solver, these objectives could be handled too, by solving their dual $\ell_2, \ell_p$-gamma function problem for $p > 2$.

**Directly sparsifying mixed $\ell_2, \ell_q$ problems for $q < 2$.** A second approach to developing a fast $\ell_2, \ell_q$-gamma function solver for $q < 2$ would be to directly develop sparsification in this setting. We believe this might be possible, and in the general matrix setting might provide better algorithms than alternative approaches.

**Removing the $m^{\frac{O(1)}{p-1}}$ loss in sparsification.** Our current approaches to graph mixed $\ell_2, \ell_p$-sparsification lose a factor $m^{\frac{O(1)}{p-1}}$ in their quality of approximation, which leads to a $m^{\frac{O(1)}{p-1}}$ factor slowdown in running time, and makes our algorithms less useful for small $p$. We believe a more sophisticated graph sparsification routine could remove this loss and result in significantly faster algorithms for $p$ close to 2.

**Using mixed $\ell_2, \ell_p$-objectives as oracles for $\ell_\infty$ regression.** The current state-of-the-art algorithm for computing maximum flow in unit capacity graphs runs in $\tilde{O}(m^{1/3})$ time \cite{LS20a}, and uses the almost-linear-time algorithm from \cite{Kyn+19} for solving unweighted $\ell_2^2 + \ell_p^2$ instances as a key ingredient.

**References**

\cite{Adi+19} D. Adil, R. Kyng, R. Peng, and S. Sachdeva. “Iterative refinement for $\ell_p$-norm regression”. In: Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms. SIAM. 2019, pp. 1405–1424 (cit. on pp. 13 4 5 6 10 11 30 32).

\cite{All+17} Z. Allen-Zhu, Y. Li, R. Oliveira, and A. Wigderson. “Much faster algorithms for matrix scaling”. In: 2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS). IEEE. 2017, pp. 890–901 (cit. on p. 11).
[Coh+17b] M. B. Cohen, A. Madry, P. Sankowski, and A. Vladu. “Negative-Weight Shortest Paths and Unit Capacity Minimum Cost Flow in $\tilde{O}(m^{10/7} \log W)$ Time (Extended Abstract)”. In: Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2017, Barcelona, Spain, Hotel Porta Fira, January 16-19, 2017, pp. 752–771 (cit. on p. 1).

[CP15] M. B. Cohen and R. Peng. “$\ell_p$ Row Sampling by Lewis Weights”. In: Proceedings of the Forty-Seventh Annual ACM Symposium on Theory of Computing. STOC ’15. Portland, Oregon, USA: Association for Computing Machinery, 2015, pp. 183–192. isbn: 9781450335362 (cit. on pp. 2, 5, 27).

[DS08] S. I. Daitch and D. A. Spielman. “Faster approximate lossy generalized flow via interior point algorithms”. In: Proceedings of the 40th annual ACM symposium on Theory of computing. STOC ’08. Available at http://arxiv.org/abs/0803.0988. Victoria, British Columbia, Canada: ACM, 2008, pp. 451–460. isbn: 978-1-60558-047-0 (cit. on p. 1).

[Dur+17] D. Durfee, J. Peebles, R. Peng, and A. B. Rao. “Determinant-Preserving Sparsification of SDD Matrices with Applications to Counting and Sampling Spanning Trees”. In: FOCS. IEEE Computer Society, 2017, pp. 926–937 (cit. on p. 2).

[Fos+53] F. G. Foster et al. “On the stochastic matrices associated with certain queuing processes”. In: The Annals of Mathematical Statistics 24.3 (1953), pp. 355–360 (cit. on p. 26).

[GT14] A. V. Goldberg and R. E. Tarjan. “Efficient maximum flow algorithms”. In: Commun. ACM 57.8 (2014), pp. 82–89 (cit. on p. 1).

[Kel+14] J. A. Kelner, Y. T. Lee, L. Orecchia, and A. Sidford. “An Almost-Linear-Time Algorithm for Approximate Max Flow in Undirected Graphs, and its Multicommodity Generalizations”. In: Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2014, Portland, Oregon, USA, January 5-7, 2014, 2014, pp. 217–226 (cit. on pp. 1, 2).

[KMP11] I. Koutis, G. L. Miller, and R. Peng. “A Nearly-$m \log n$ Time Solver for SDD Linear Systems”. In: Proceedings of the 2011 IEEE 52nd Annual Symposium on Foundations of Computer Science. FOCS ’11. Washington, DC, USA: IEEE Computer Society, 2011, pp. 590–598. isbn: 978-0-7695-4571-4 (cit. on p. 2).

[Kyn+15] R. Kyng, A. Rao, S. Sachdeva, and D. A. Spielman. “Algorithms for Lipschitz Learning on Graphs”. In: Proceedings of The 28th Conference on Learning Theory. Ed. by P. Grünwald, E. Hazan, and S. Kale. Vol. 40. Proceedings of Machine Learning Research. Paris, France: PMLR, 2015, pp. 1190–1223 (cit. on p. 1).

[Kyn+16] R. Kyng, Y. T. Lee, R. Peng, S. Sachdeva, and D. A. Spielman. “Sparsified Cholesky and multigrid solvers for connection laplacians”. In: Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing. ACM. 2016, pp. 842–850 (cit. on p. 2).

[Kyn+19] R. Kyng, R. Peng, S. Sachdeva, and D. Wang. “Flows in Almost Linear Time via Adaptive Preconditioning”. In: Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing. STOC 2019. Phoenix, AZ, USA: ACM, 2019, pp. 902–913. isbn: 978-1-4503-6705-9 (cit. on pp. 1, 2, 4, 11, 18, 19, 22).

[Lew78] D. Lewis. “Finite dimensional subspaces of $L(p)$”. eng. In: Studia Mathematica 63.2 (1978), pp. 207–212 (cit. on p. 26).

[LS14] Y. T. Lee and A. Sidford. “Path Finding Methods for Linear Programming: Solving Linear Programs in $\tilde{O}(\text{vrank})$ Iterations and Faster Algorithms for Maximum Flow”. In: FOCS. 2014 (cit. on p. 1).
Graphs”. In: *2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS)*. IEEE, 2020, pp. 919–930 (cit. on pp. [1] [2]).
APPENDIX A. CONSTRUCTION OF SPARSIFIERS FOR $\ell_2^2 + \ell_p^p$ FLOWS

In this section we will prove the following formal version of Theorem 2.3.

**Theorem A.1.** Consider an instance $G = (V^G, E^G, r^G, s^G, g^G)$ with $n$ vertices and $m$ edges. Suppose we want to solve, $\min_{B,G} f = \mathcal{E}(f)$. We can compute in time $\tilde{O}(m)$ an instance $\mathcal{H} = (V^H, E^H, r^H, s^H, g^H)$ such that with probability $1 - \varepsilon$, $\mathcal{H}$ has $n$ vertices and $m_{\mathcal{H}} = n \text{polylog}(n/(\varepsilon\delta))$ edges, and for all $f^\mathcal{H}$ we can compute a corresponding $f^G$ in $\tilde{O}(m)$ time such that,

$$\mathcal{H} \preceq_{\kappa, \delta} G \quad \text{and} \quad G \preceq_{\kappa, \delta} \mathcal{H},$$

where $\kappa = m^{1/(p-1)} \text{polylog}(n/(\varepsilon\delta))$.

**A.1. Preliminaries.**

**Smoothed $\ell_p$ norm functions.** We consider $p$-norms smoothed by the addition of a quadratic term. First we define such a smoothed $p^{th}$-power on $\mathbb{R}$.

**Definition A.2** (Smoothed $p^{th}$-power). Given $r, x \in \mathbb{R}, r \geq 0$ define the $r$-smoothed $s$-weighted $p^{th}$-power of $x$ to be

$$h_p(r, s, x) = rx^2 + s|x|^p.$$

This definition can be naturally extended to vectors to obtained smoothed $\ell_p$-norms.

**Definition A.3** (Smoothed $\ell_p$-norm). Given vectors $x \in \mathbb{R}^m, r, s \in \mathbb{R}_{\geq 0}^m$, define the $r$-smooth $s$-weighted $p$-norm of $x$ to be

$$h_p(r, s, x) = \sum_{i=1}^m h_p(r_i, s_i, x_i) = \sum_{i=1}^m (r_i x_i^2 + s_i |x_i|^p).$$

**Flow Problems and Approximation.** We will consider problems where we seek to find flows minimizing smoothed $p$-norms. We first define these problem instances.

**Definition A.4** (Smoothed $p$-norm instance). A **smoothed $p$-norm instance** is a tuple $G$,

$$G = (V^G, E^G, g^G, r^G, s^G),$$

where $V^G$ is a set of vertices, $E^G$ is a set of undirected edges on $V^G$, the edges are accompanied by a gradient, specified by $g^G \in \mathbb{R}^{E^G}$, the edges have $\ell_2^2$-resistances given by $r^G \in \mathbb{R}_{\geq 0}^{E^G}$, and $s \in \mathbb{R}^{E^G}$ gives the $p$-norm scaling.

**Definition A.5** (Flows, residues, and circulations). Given a smoothed $p$-norm instance $G$, a vector $f \in \mathbb{R}^{E^G}$ is said to be a flow on $G$. A flow vector $f$ satisfies residues $b \in \mathbb{R}^{V^G}$ if $(B^G)^\top f = b$, where $B^G \in \mathbb{R}^{E^G \times V^G}$ is the edge-vertex incidence matrix of the graph $(V^G, E^G)$, i.e., $(B^G)_{(u,v)} = 1_u - 1_v$.

A flow $f$ with residue $0$ is called a circulation on $G$.

Note that our underlying instance and the edges are undirected. However, for every undirected edge $e = (u, v) \in E$, we assign an arbitrary fixed direction to the edge, say $u \rightarrow v$, and interpret $f_e \geq 0$ as flow in the direction of the edge from $u$ to $v$, and $f_e < 0$ as flow in the reverse direction. For convenience, we assume that for any edge $(u, v) \in E$, we have $f_{(u,v)} = -f_{(v,u)}$.

**Definition A.6** (Objective, $\mathcal{E}(G)$). Given a smoothed $p$-norm instance $G$, and a flow $f$ on $G$, the associated objective function, or the energy, of $f$ is given by

$$\mathcal{E}(G) = (g^G)^\top f - h_p(r, s, f).$$
Definition A.7 (Smoothed $p$-norm flow / circulation problem). Given a smoothed $p$-norm instance $G$ and a residue vector $b \in \mathbb{R}^{E_G}$, the smoothed $p$-norm flow problem $(G, b)$, finds a flow $f \in \mathbb{R}^{E_G}$ with residues $b$ that maximizes $\mathcal{E}_G(f)$, i.e.,

$$\max_{f : (B^G)^\top f = b} \mathcal{E}_G(f).$$

If $b = 0$, we call it a smoothed $p$-norm circulation problem.

Note that the optimal objective of a smoothed $p$-norm circulation problem is always non-negative, whereas for a smoothed $p$-norm flow problem, it could be negative.

Approximating Smoothed $p$-norm Instances. Since we work with objective functions that are non-standard (and not even homogeneous), we need to carefully define a new notion of approximation for these instances.

Definition A.8 ($\mathcal{H} \preceq_{\kappa, \delta} G$). For two smoothed $p$-norm instances, $G, \mathcal{H}$, we write $\mathcal{H} \preceq_{\kappa, \delta} G$ if there is a linear map $M_{\mathcal{H} \to G} : \mathbb{R}^{E_{\mathcal{H}}} \to \mathbb{R}^{E_G}$ such that for every flow $f^\mathcal{H}$ on $\mathcal{H}$, we have that $f^G = M_{\mathcal{H} \to G}(f^\mathcal{H})$ is a flow on $G$ such that

1. $f^G$ has the same residues as $f^\mathcal{H}$ i.e., $(B^G)^\top f^G = (B^\mathcal{H})^\top f^\mathcal{H}$, and
2. has energy bounded by:

$$\frac{1}{\kappa} \left( \mathcal{E}_{\mathcal{H}}(f^\mathcal{H}) - \delta \|f^\mathcal{H}\|_1 \right) \leq \mathcal{E}_G\left(\frac{1}{\kappa} f^G\right).$$

For some of our transformations on graphs, we will be able to prove approximation guarantees only for circulations. Thus, we define the following notion restricted to circulations.

Definition A.9 ($\mathcal{H} \preceq_{\text{cycle}, \kappa, \delta} G$). For two smoothed $p$-norm instances, $G, \mathcal{H}$, we write $\mathcal{H} \preceq_{\text{cycle}, \kappa, \delta} G$ if there is a linear map $M_{\mathcal{H} \to G} : \mathbb{R}^{E_{\mathcal{H}}} \to \mathbb{R}^{E_G}$ such that for any circulation $f^\mathcal{H}$ on $\mathcal{H}$, i.e., $(B^\mathcal{H})^\top f^\mathcal{H} = 0$, the flow $f^G = M_{\mathcal{H} \to G}(f^\mathcal{H})$ is a circulation, i.e., $(B^G)^\top f^G = 0$, and satisfies

$$\frac{1}{\kappa} \left( \mathcal{E}_{\mathcal{H}}(f^\mathcal{H}) - \delta \|f^\mathcal{H}\|_1 \right) \leq \mathcal{E}_G\left(\frac{1}{\kappa} f^G\right).$$

Observe that $\mathcal{H} \preceq_{\kappa, \delta} G$ implies $\mathcal{H} \preceq_{\text{cycle}, \kappa, \delta} G$.

We define the usual induced matrix 1-to-1 norm as

$$\|M\|_{1 \to 1} = \max_{f \in \mathbb{R}^E} \frac{\|Mf\|_1}{\|f\|_1}.$$

We define a special matrix 1-to-1 norm over circulations by

$$\|M\|_{\text{cycle} 1 \to 1} = \max_{Bf = 0} \frac{\|Mf\|_1}{\|f\|_1}.$$

Definition A.10 ($\mathcal{H} \preceq_{\kappa} G$ and $\mathcal{H} \preceq_{\text{cycle}, \kappa} G$). We abbreviate $\mathcal{H} \preceq_{\kappa, 0} G$ as $\mathcal{H} \preceq_{\kappa} G$, and $\mathcal{H} \preceq_{\text{cycle, \kappa}, 0} G$ as $\mathcal{H} \preceq_{\text{cycle, \kappa}} G$.

Definition A.11. In the context of a problem with $m$ vertices and $n$ edges, we say a real number $x$ is quasi-polynomially bounded

$$2^{-\text{polylog}(n)} \leq |x| \leq 2^{\text{polylog}(n)}.$$
Definition A.12. Consider a smoothed $p$-norm flow problem $(\mathcal{G}, b)$ where $\mathcal{G} = (V, E, r, s, g)$ with $n$ vertices and $m$ edges. We say that the instance is quasipolynomially-bounded if the entries of $r$ and $s$ are quasipolynomially bounded and
\[
\max_{f : (B^G) \setminus f = b} \mathcal{E}^G(f) \leq 2^{\text{polylog}(n)}.
\]

Definition A.13 (Touched and untouched cycles). We say a cycle of edges in $E$ is touched if it contains an edge $e$ s.t. $r(e) \neq 0$ or $s(e) \neq 0$. Otherwise, we say the cycle is untouched.

Definition A.14 (Cycle-touching instance). We say an instance $\mathcal{G} = (V, E, r, s, g)$ is cycle-touching if every cycle of edges in $e$ is touched.

A.2. Additional Properties of Flow Problems and Approximation. The definitions in Section A.1 satisfy most properties that we want from comparisons. The following lemma, slightly extends a similar statement in [Kvn+19].

Lemma A.15 (Reflexivity). For every smoothed $p$-norm instance $\mathcal{G}$, and every $\kappa \geq 1$, $\delta \geq 0$, we have $\mathcal{G} \preceq_{\kappa, \delta} \mathcal{G}$ with the identity map.

Proof. Consider the map $M_{\mathcal{G} \rightarrow \mathcal{G}}$ such that for every flow $f^G$ on $\mathcal{G}$, we have $M_{\mathcal{G} \rightarrow \mathcal{G}}(f^G) = f^G$. Thus,
\[
\mathcal{E}^G\left(\kappa^{-1} M_{\mathcal{G} \rightarrow \mathcal{G}}(f^G)\right) = \mathcal{E}^G\left(\kappa^{-1} f^G\right)
= \left(g^G\right)^\top \left(\kappa^{-1} f^G\right) - h_p(r, \kappa^{-1} f^G)
\geq \kappa^{-1} \left(g^G\right)^\top f^G - \kappa^{-2} h_p(r, f^G)
\geq \kappa^{-1} \left(g^G\right)^\top f^G - \kappa^{-1} h_p(r, f^G)
= \kappa^{-1} \mathcal{E}^G(f^G)
\geq \kappa^{-1} \mathcal{E}^G(f^G) - \delta \left\| f^G \right\|_1
\]
Moreover $(B^G)^\top M_{\mathcal{G} \rightarrow \mathcal{G}}(f^G) = B^G f^G$. Thus, the claims follow. \qed

It behaves well under composition.

Lemma A.16 (Composition). Given two smoothed $p$-norm instances, $\mathcal{G}_1, \mathcal{G}_2$, such that $\mathcal{G}_1 \preceq_{\kappa_1, \delta_1} \mathcal{G}_2$ with the map $M_{\mathcal{G}_1 \rightarrow \mathcal{G}_2}$ and $\mathcal{G}_2 \preceq_{\kappa_2, \delta_2} \mathcal{G}_3$ with the map $M_{\mathcal{G}_2 \rightarrow \mathcal{G}_3}$, then $\mathcal{G}_1 \preceq_{\kappa, \delta} \mathcal{G}_3$ with the map $M_{\mathcal{G}_1 \rightarrow \mathcal{G}_3} = M_{\mathcal{G}_2 \rightarrow \mathcal{G}_3} \circ M_{\mathcal{G}_1 \rightarrow \mathcal{G}_2}$ and $\kappa = \kappa_1 \kappa_2$ and $\delta = \delta_1 + \delta_2 \left\| M_{\mathcal{G}_1 \rightarrow \mathcal{G}_2} \right\|_{1 \rightarrow 1}$.

Similarly, given two smoothed $p$-norm instances, $\mathcal{G}_1, \mathcal{G}_2$, such that $\mathcal{G}_1 \preceq_{\kappa_1, \delta_1} \mathcal{G}_2$ with the map $M_{\mathcal{G}_1 \rightarrow \mathcal{G}_2}$ and $\mathcal{G}_2 \preceq_{\kappa_2, \delta_2} \mathcal{G}_3$ with the map $M_{\mathcal{G}_2 \rightarrow \mathcal{G}_3}$, then $\mathcal{G}_1 \preceq_{\kappa, \delta} \mathcal{G}_3$ with the map $M_{\mathcal{G}_1 \rightarrow \mathcal{G}_3} = M_{\mathcal{G}_2 \rightarrow \mathcal{G}_3} \circ M_{\mathcal{G}_1 \rightarrow \mathcal{G}_2}$ and $\kappa = \kappa_1 \kappa_2$ and $\delta = \delta_1 + \delta_2 \left\| M_{\mathcal{G}_1 \rightarrow \mathcal{G}_2} \right\|_{1 \rightarrow 1}$.

Proof. We can simply chain together the guarantees to see that:
\[
\mathcal{E}^{\mathcal{G}_1}\left( f^{\mathcal{G}_1} \right) \leq \kappa_1 \mathcal{E}^{\mathcal{G}_2}\left( \frac{1}{\kappa_1} f^{\mathcal{G}_2} \right) + \delta_1 \left\| f^{\mathcal{G}_1} \right\|_1
\leq \kappa_1 \kappa_2 \left( \mathcal{E}^{\mathcal{G}_3}\left( \frac{1}{\kappa_1 \kappa_2} f^{\mathcal{G}_3} \right) + \delta_2 \left\| \frac{1}{\kappa_1} f^{\mathcal{G}_2} \right\|_1 \right) + \delta_1 \left\| f^{\mathcal{G}_1} \right\|_1
= \kappa_1 \kappa_2 \mathcal{E}^{\mathcal{G}_3}\left( \frac{1}{\kappa_1 \kappa_2} f^{\mathcal{G}_3} \right) + \delta_2 \left\| f^{\mathcal{G}_2} \right\|_1 + \delta_1 \left\| f^{\mathcal{G}_1} \right\|_1
\]
\[
\leq \kappa_1 \kappa_2 \mathcal{E}_3^{G_3} \left( \frac{1}{\kappa_1 \kappa_2} f_{G_3} \right) + \left( \delta_2 \| \mathcal{M}_{G_1 \rightarrow G_2} \|_{1 \rightarrow 1} + \delta_1 \right) \| f_{G_1} \|_1
\]

A similar calculation gives the cycle composition guarantee, but now allows us to bound norms using the \(\| \mathcal{M}_{G_1 \rightarrow G_2} \|_{1 \rightarrow 1}^{\text{cycle}}\) norm. In all cases, our maps preserve that flows route the correct demands. \(\square\)

The most important property of this is that this notion of approximation is also additive, \(i.e.,\) it works well with graph decompositions.

**Definition A.17** (Union of two instances). Consider smoothed \(p\)-norm instances, \(G_1, G_2\), with the same set of vertices, \(i.e., V^{G_1} = V^{G_2}\). Define \(G = G_1 \cup G_2\) as the instance on the same set of vertices obtained by taking a disjoint union of the edges (potentially resulting in multi-edges). Formally,

\[G = (V^{G_1}, E^{G_1} \cup E^{G_2}, (g^{G_1}, g^{G_2}), (r^{G_1}, r^{G_2}), (s^{G_1}, s^{G_2})).\]

We prove the following lemma, which closely follows an analogous statement in [Kyn+19].

**Lemma A.18** (Union of instances). Consider four smoothed \(p\)-norm instances, \(G_1, G_2, H_1, H_2\), on the same set of vertices, \(i.e., V^{G_1} = V^{G_2} = V^{H_1} = V^{H_2}\), such that for \(i = 1, 2\), \(H_i \preceq_{\kappa, \delta} G_i\) with the map \(\mathcal{M}_{H_i \rightarrow G_i}\). Let \(G \overset{\text{def}}{=} G_1 \cup G_2\), and \(H \overset{\text{def}}{=} H_1 \cup H_2\). Then, \(H \preceq_{\kappa, \delta} G\) with the map

\[\mathcal{M}_{H \rightarrow G} (f^H) = (f^{H_1}, f^{H_2}) \overset{\text{def}}{=} (\mathcal{M}_{H_1 \rightarrow G_1} (f^{H_1}), \mathcal{M}_{H_2 \rightarrow G_2} (f^{H_2})),\]

where \((f^{H_1}, f^{H_2})\) is the decomposition of \(f^H\) onto the supports of \(H_1\) and \(H_2\).

**Proof.** Let \(f^H\) be a flow on \(H\). We write \(f^H = (f^{H_1}, f^{H_2})\). Let \(f^G \overset{\text{def}}{=} \mathcal{M}_{H \rightarrow G} (f^H)\). If \(f^{G_i}\) denotes \(\mathcal{M}_{H_i \rightarrow G_i} (f^{H_i})\) for \(i = 1, 2\), then we know that \(f^G = (f^{G_1}, f^{G_2})\). Thus, the objectives satisfy

\[\mathcal{E}^G (\kappa^{-1} f^G) = \mathcal{E}^{G_1} (\kappa^{-1} f^{G_1}) + \mathcal{E}^{G_2} (\kappa^{-1} f^{G_2}) \geq \kappa^{-1} \left( \mathcal{E}^{H_1} (f^{H_1}) - \delta \| f^{H_1} \|_1 \right) + \kappa^{-1} \left( \mathcal{E}^{H_2} (f^{H_2}) - \delta \| f^{H_2} \|_1 \right) = \kappa^{-1} \left( \mathcal{E}^H (f^H) - \delta \| f^H \|_1 \right)\]

For the residues, we have,

\[(B^G)^\top (f^G) = (B^{G_1})^\top (f^{G_1}) + (B^{G_2})^\top (f^{G_2}) = (B^{H_1})^\top (f^{H_1}) + (B^{H_2})^\top (f^{H_2}) = (B^H)^\top (f^H)\]

Thus, \(H \preceq_{\kappa, \delta} G\). \(\square\)

This notion of approximation also behaves nicely with scaling of \(\ell_2\) and \(\ell_p\) resistances.

**Lemma A.19.** For all \(\kappa > 1\), and for all pairs of smoothed \(p\)-norm instances, \(G, H\), on the same underlying graphs, \(i.e., (V^G, E^G) = (V^H, E^H)\), such that,

1. the gradients are identical, \(g^G = g^H\),
2. the \(\ell_2^2\) resistances are off by at most \(\kappa\), \(i.e., r^G_e \leq \kappa r^H_e\) for all edges \(e\), and
3. the \(p\)-norm scaling is off by at most \(\kappa^{p-1}\), \(i.e., s^G \leq \kappa^{p-1} s^H\),

then \(H \preceq_{\kappa} G\) with the identity map.

**Proof.** Follows from Lemma 2.13 in [Kyn+19]. \(\square\)
A.3. **Orthogonal Decompositions of Flows.** At the core of our graph decomposition and sparsification procedures is a decomposition of the gradient $g$ of $G$ into its cycle space and potential flow space. We denote such a splitting using

$$g^G = \hat{g}^G + B^G \psi^G, \text{ s.t. } B^G \hat{g}^G = 0.$$  

Here $\hat{g}$ is a circulation, while $B\psi$ gives a potential induced edge value. We will omit the superscripts when the context is clear.

The following minimization based formulation of this splitting of $g$ is critical to our method of bounding the overall progress of our algorithm.

**Fact A.20.** The projection of $g$ onto the cycle space is obtained by minimizing the Euclidean norm of $g$ plus a potential flow. Specifically,

$$\|\hat{g}\|_2^2 = \min_x \|g + Bx\|_2^2.$$  

**Lemma A.21.** Given a graph/gradient instance $G$, consider $H$ formed from a subset of its edges. The projections of $g^G$ and $g^H$ onto their respective cycle spaces, $\hat{g}^G$ and $\hat{g}^H$ satisfy:

$$\|\hat{g}^H\|_2^2 \leq \|\hat{g}^G\|_2^2 \leq \|g^G\|_2^2.$$  

A.4. **Numerics, Conditioning, Inexact Laplacian Solvers.** Because we allow instances $G = (V, E, r, s, g)$ with $r(e) = 0$ and $s(e) = 0$ for some edges $e \in E$ (and this is important for our sparsification procedures), we need to be somewhat careful about disallowing instances with a cycle that has zero $r$ and $s$ values and non-zero gradient: In that case, our “energy” can diverge to $+\infty$.

**Definition A.22** (Unbounded and constant cycles). An untouched cycle $C \subseteq E$ is unbounded if the sum of terms around the cycle is non-zero, i.e. $\sum_{e \in C} g(e) \neq 0$. If the sum is zero, we call the unbounded cycle a constant cycle.

**Lemma A.23.** Consider a smoothed $p$-norm flow problem $(G, b)$, i.e.

$$\max_{f : (B^G)^\top f = b} \mathcal{E}^G(f).$$

The problem is unbounded, i.e. has objective value $\mathcal{E}^G \to \infty$ if and only if $G$ contains an unbounded cycle.

**Proof.** The problem is unbounded exactly when the gradient has non-zero inner product with some element of the subspace $\ker(B^\top) \cap \ker(R) \cap \ker(S)$. The first condition means it $\ker(B^\top)$ tells us the element must be in the cycle space, while the latter two tells us it must be supported edges with non-zero $R$ and $S$. Writing the element as a linear combination of particular cycles, the gradient must have a non-zero inner product with one untouched cycle. □

**Lemma A.24.** The algorithm `detectUnbounded` takes as input a smoothed $p$-norm problem $(G, b)$ and if $G$ contains an unbounded cycle, the algorithm detects this and outputs an unbounded cycle (an arbitrary one if there are multiple).

**Proof.** We run a DFS only using edges outside the union of the support of $r$ and $s$ (just delete the others before), and assign voltages. On an edge to already visited vertex, check if new voltage agrees, if output cycle on stack: it is an unbounded cycle. At the end, consistent voltages certify that gradient is an electrical flow and hence objective is bounded. Overall time is linear, as each edge is processed only once. □

**Lemma A.25.** The algorithm `constantCycleContraction` takes as input a smoothed $p$-norm problem $(G, b)$ which has bounded objective value and
Furthermore, if and for zero-demand flows the maps 
\[ M = 0 \]
Once we know the problem is bounded, and hence contains no unbounded cycles, we can 
Suppose the entries of \( G \) are quasi-polynomially bounded, then so are the entries of \( (G', b') \).

Furthermore, if \( b = 0 \) then \( b' = 0 \) and the solution maps preserve the objective value exactly, i.e. \( c = 0 \) and hence
\[ G \preceq_G G' \text{ and } G' \preceq_G G, \]
and for zero-demand flows the maps \( M_{G \to G'} \) and \( M_{G' \to G} \) satisfy
\[ \|M_{G \to G'}\|_{1 \to 1}^{\text{cycle}} \leq 1 \text{ and } \|M_{G' \to G}\|_{1 \to 1}^{\text{cycle}} \leq m. \]

Proof. Once we know the problem is bounded, and hence contains no unbounded cycles, we can look at the connected components consisting of untouched edges, and we can repeatedly contract cycles of these parts. In the case where demands are zero initially, we produce a smaller instance where again demands are zero. To lift to the larger instance, we can simply put no flow on these edges. Because having a cycle flow on these edges does not increase the objective, our mapping guarantees hold. Overall time is linear, as each edge is processed only once.

\[ \square \]

Lemma A.26. Consider a smoothed \( p \)-norm circulation problem \( (G, 0) \), where \( G = (V, E, r, s, g) \). Suppose the entries of \( r \) and \( s \) are quasipolynomially bounded and \( G \) is cycle-touching, and suppose \( \|g\|_{\infty} \leq 2^{\text{polylog}(n)} \). Then \( (G, 0) \) is quasi-polynomially bounded.

Proof. If every cycle that every non-zero \( g \) edges appears in contains a non-zero entry of \( r \) or \( s \), then increasing flow along that cycle will eventually lead to a decrease in objective.

\[ \square \]

Remark A.27. We are analyzing our algorithm in the Real RAM model, but by applying the tools from this section, it can also be analyzed in fixed precision arithmetic with polylogarithmic bit complexity per number: In this model, our DETECTUNBOUNDED and CONSTANTCYCLECONTRACTION procedures still work and returns a cycle-touching instance. Once an instance is cycle-touching and the non-zero vectors it returns are not too small or big, it is possible to manage errors from fixed point arithmetic. This can also allow us to work with inexact Laplacian solvers using quasipolynomial errors, which we can compute in this model in nearly-linear time.

A.5. Main Sparsification Theorem for Flows.

Theorem A.28 (Instance Sparsification). Consider an instance \( G = (V^G, E^G, r^G, s^G, g^G) \) with \( n \) vertices and \( m \) edges, with \( r^G \) and \( s^G \) quasipolynomially bounded, and \( \|g^G\|_F \leq 2^{\text{polylog}(n)} \), and suppose the instance is cycle-touching. We can compute an instance \( H = (V^H, E^H, r^H, s^H, g^H) \) with \( n \) vertices and \( m_H = n\text{polylog}(n/(\varepsilon \delta)) \) edges, again with \( r^H \) and \( s^H \) quasipolynomially bounded, and \( \|g^H\|_\infty \leq 2^{\text{polylog}(n)} \), in time \( O(m) \) such that with probability \( 1 - \varepsilon \) the maps \( M_{G \to H} \) and \( M_{H \to G} \)
certify
\[ H \preceq_{\kappa, \delta} G \text{ and } G \preceq_{\kappa, \delta} H, \]
where \( \kappa = m^{1/(p-1)} \text{polylog}(n/(\varepsilon \delta)) \). Furthermore, these maps can be applied in time \( \tilde{O}(m) \).

21
Definition A.29. A graph $G$ is a $\alpha$-uniform $\phi$-expander (or uniform expander when parameters not spelled out explicitly) if

1. $r$ on all edges are the same.
2. $s$ on all edges are the same.
3. $G$ has conductance\footnote{We use an instance and its underlying graph interchangeably in our discussion.} at least $\phi$.
4. The projection of $\mathbf{g}$ onto the cycle space of $G$, $\hat{\mathbf{g}}_G = (I - BLB^\top)\mathbf{g}$, is $\alpha$-uniform (see next definition), where $B$ is the edge-vertex incidence matrix of $G$, and $L = B^\top B$ is the Laplacian.

Definition A.30. A vector $\mathbf{y} \in \mathbb{R}^m$ is said to be $\alpha$-uniform if
\[
\|\mathbf{y}\|_2^2 \leq \frac{\alpha}{m}\|\mathbf{y}\|_\infty^2.
\]
We abuse the notation to also let the all zero vector $0$ be 1-uniform.

The next 2 theorems are from [Kyn+19]

Theorem A.31 (Decomposition into Uniform Expanders). Given any graph/gradient/resistance instance $\mathcal{G}$ with $n$ vertices, $m$ edges, all equal to $r$, $p$-weights all equal to $s$, and gradient $\mathbf{g}^\mathcal{G}$, along with a parameter $\delta$, DECOMPOSE($\mathcal{G}, \delta$) returns disjoint vertex subsets $V_1, V_2, \ldots$ in $O(m \log n \log(n/\delta))$ time such that if we let $\mathcal{G}_1, \mathcal{G}_2, \ldots$ be the instances obtained by restricting $\mathcal{G}$ to the induced graphs on the $V_i$ sets, then at least $m/2$ edges are contained in these subgraphs, and each $\mathcal{G}_i$ satisfies (for some absolute constant $c_{\text{partition}}$):

1. The graph $(V^\mathcal{G}_i, E^\mathcal{G}_i)$ has conductance at least
\[
\phi = \frac{1}{c_{\text{partition}} \cdot \log^3 n \cdot \log(n/\delta)},
\]
and degrees at least $\phi \cdot \frac{m}{2n}$, where $c_{\text{partition}}$ is an absolute constant.
2. The projection of its gradient $\hat{\mathbf{g}}^\mathcal{G}_i$ into the cycle space of $\mathcal{G}_i$, $\hat{\mathbf{g}}^\mathcal{G}_i$ satisfies one of:
   (a) $\hat{\mathbf{g}}^\mathcal{G}_i$ is $O(\log^8 n \log^3(n/\delta))$-uniform,
   \[
   \left(\hat{\mathbf{g}}^\mathcal{G}_i\right)_e^2 \leq \frac{O\left(\log^{14} n \log^5(n/\delta)\right)}{m_i} \|\mathbf{g}^\mathcal{G}_i\|_2^2 \quad \forall e \in E(\mathcal{G}_i).
   \]
   Here $m_i$ is the number of edges in $\mathcal{G}_i$.
   (b) The $l^2_2$ norm of $\hat{\mathbf{g}}^\mathcal{G}_i$ is smaller by a factor of $\delta$ than the unprojected gradient:
   \[
   \|\hat{\mathbf{g}}^\mathcal{G}_i\|_2^2 \leq \delta \cdot \|\mathbf{g}^\mathcal{G}_i\|_2^2.
   \]

Theorem A.32 (Sampling Uniform Expanders). Given an $\alpha$-uniform $\phi$-expander $\mathcal{G} = (V^\mathcal{G}, E^\mathcal{G}, r^\mathcal{G}, s^\mathcal{G}, \mathbf{g}^\mathcal{G})$ with $m$ edges and vertex degrees at least $d_{\text{min}}$, for any sampling probability $\tau$ satisfying
\[
\tau \geq c_{\text{sample}} \log(n/\varepsilon) \cdot \left(\frac{\alpha}{m} + \frac{1}{\phi^2 d_{\text{min}}}\right),
\]

\footnote{$2\tau$ are uniform, so conductance is defined as in unweighted graphs. We use the standard definition of conductance. For graph $G = (V, E)$, the conductance of any $\emptyset \neq S \subseteq V$ is $\phi(S) = \frac{\delta(S)}{\min(\text{vol}(S), \text{vol}(V \setminus S))}$ where $\delta(S)$ is the number of edges on the cut $(S, V \setminus S)$ and $\text{vol}(S)$ is the sum of the degree of nodes in $S$. The conductance of a graph is $\phi_G = \min_{S \neq \emptyset, V} \phi(S)$.}
Remark A.34. in logarithmic time in the Real RAM with comparisons. apply this procedure to quasipolynomially bounded numbers, which ensures it can be implemented

Lemma A.35 (2-rounding). We are given an instance \( \mathcal{G} = (V, E, r, s, g) \). The graph \( H \) has the same vertex set as \( G \), and \( H \) has at most \( 2\tau m \) edges. Furthermore, \( r^H = \tau \cdot r^G \) and \( s^H = r^G \). The maps \( \mathcal{M}_{\mathcal{G} \rightarrow \mathcal{H}} \) and \( \mathcal{M}_{\mathcal{H} \rightarrow \mathcal{G}} \) certify

\[
\mathcal{H} \preceq_{\kappa} \mathcal{G} \text{ and } \mathcal{G} \succeq_{\kappa} \mathcal{H},
\]

where \( \kappa = m^{1/(p-1)} \phi^{-9} \log^3 n \). This map can be applied in time \( \tilde{O}(m) \).

Definition A.33 (2-Rounded instance). We call an instance \( \mathcal{G} = (V^G, E^G, r^G, s^G, g^G) \), 2-rounded, if every non-zero entry of \( s \) and \( r \) has absolute value equal to a power of two (which can be negative).

Given an instance \( \mathcal{G} = (V^G, E^G, r^G, s^G, g^G) \), we compute a 2-rounded instance \( \mathcal{G}' = (V^G, E^G, r^G, s^G, g^G) \). We round each \( r^G \) of edges \( e \in E^G \) down to the nearest power of 2 (can be less than 1). Similarly, we round each \( s^G \) of edges \( e \in E^G \) down to the nearest power of 2 (can be less than 1). We denote this rounding procedure by \( \text{instanceRound} \), s.t. \( \mathcal{G}' = \text{instanceRound}(\mathcal{G}) \). We will only need to apply this procedure to quasipolynomially bounded numbers, which ensures it can be implemented in logarithmic time in the Real RAM with comparisons.

Remark A.34. Because it is applied to quasipolynomially bounded entries, the rounding can be implemented using a polylogarithmic number of bit operations in fixed point arithmetic.

Lemma A.35 (2-rounding). Consider an instance \( \mathcal{G} = (V, E, r, s, g) \). Let \( \mathcal{G}' = \text{instanceRound}(\mathcal{G}) \). Then the identity map between the instances certifies

\[
\mathcal{G} \preceq_1 \mathcal{G}' \text{ and } \mathcal{G}' \preceq_2 \mathcal{G}.
\]

and \( \mathcal{G}' \) is 2-rounded.

Proof of Theorem A.28. We are given an instance \( \mathcal{G} = (V^G, E^G, r^G, s^G, g^G) \) with \( n \) vertices and \( m \) edges, with \( r^G \) and \( s^G \) quasipolynomially bounded, and \( \|g^G\|_{\infty} \leq 2^{\text{polylog}(n)} \), and cycle-touching.

We then compute \( \mathcal{G}' = \text{instanceRound}(\mathcal{G}) \), to get the 2-rounded, cycle-touching instance \( \mathcal{G}' = (V^{G'}, E^{G'}, r^{G'}, s^{G'}, g^{G'}) \) with \( r^{G'} \) and \( s^{G'} \) quasipolynomially bounded, and \( \|g^{G'}\|_{\infty} \leq 2^{\text{polylog}(n)} \) with using the identity map between the instances

\[
\mathcal{G} \preceq_1 \mathcal{G}' \text{ and } \mathcal{G}' \preceq_2 \mathcal{G}.
\]

Because \( \mathcal{G}' \) is 2-rounded and \( r^{G'} \) and \( s^{G'} \) quasipolynomially bounded, the entries of these two vectors only take on polylog\((n)\) different values. Thus we can divide the edges \( E^{G'} \) into polylog\((n)\) buckets such that in every bucket \( i \), all edges have the same 2-resistance value \( r_e = r^{(i)} \) and the same \( p \)-weight value \( s(e) = s^{(i)} \).

Let \( \mathcal{G}^{(i)} = (V^{G^{(i)}}, E^{G^{(i)}}, r^{G^{(i)}}, g^{G^{(i)}}) \) be the instance arising from restricting \( \mathcal{G}' \) to the edges of bucket \( i \), while letting \( V^{G^{(i)}} \) to be the set of vertices incident on edges of \( E^{G^{(i)}} \). There will be exactly one bucket containing all the edges \( e \) with \( r(e) = s(e) = 0 \). We let this bucket have index \( i = 0 \). This bucket cannot contain a cycle of edges, because if it did then \( \mathcal{G}'' \) would contain an untouched cycle, contradiction that it is cycle-touching. Hence \( \mathcal{G}^{(0)} \) contains at most \( n - 1 \) edges.

We now sparsify the edges in every bucket separately, except bucket \( i = 0 \), which we do not sparsify. For \( i > 0 \), we define \( \mathcal{G}^{(i,0)} = \mathcal{G}^{(i)} \), and let \( j \leftarrow 0 \). As long as \( \mathcal{G}^{(i, j)} \) contains more than \( n \log^{10} n \) edges we then repeat the following: Appealing to Theorem A.31, we now call \( \text{DECOMPOSE}(\mathcal{G}^{(i, j)}, \delta) \) with \( \tilde{\delta} = 2^{-\log^c n} \delta \) for some universal constant \( c \) large enough that \( \|g^{G^{(i, j)}}\|_{\infty} \tilde{\delta} \leq \delta \). This produces a partition of \( V^{(i, j)} \) into disjoint \( V^{(i, j)}_1, \ldots, V^{(i, j)}_k \). The \( \text{DECOMPOSE} \) algorithm defines \( \mathcal{G}^{(i, j)} \) to be
the instance given by restricting $G^{(i,j)}$ to the induced graph on $V^{(i,j)}_1$ and let $G^{(i,j+1)}$ be the instance arising from restricting $G^{(i,j)}$ to the graph consisting of edges crossing between the vertex partitions, and the vertices incident on these edges. We then let $j = j + 1$ and repeat the decomposition if necessary. By Theorem A.31 $G^{(i,j+1)}$ contains at most half of the edges of $G^{(i,j+1)}$, we call DECOMPOSE at most $\log n$ times as $\log_2(m/n) \leq \log(n)$. For each bucket $i$, we let $j_i$ denote the last instance produced, i.e. $G^{(i,j_i)}$ is the this final instance, which is not included in any expander.

For every instance $G^{(i,j)}$, we let $m^{(i,j)} = |E^{G^{(i,j)}}|$ denote the number of edges of the graph, and $n^{(i,j)} = |V^{G^{(i,j)}}|$ the number of vertices. Note that the edges of $G^{(i)}$ are partitioned between the $G^{(i,j_i)}$ instances and the $G^{(i,j)}$ instances, i.e. each edges of $G^{(i)}$ is contained in exactly instance, either a $G^{(i,j_i)}$ or a $G^{(i,j)}$. Thus the union of all these in the sense of Definition A.17 is exactly $G^{(i)}$, and the union $G^{(i)}$ is $G'$.

We will not sparsify the final $G^{(i,j_i)}$, which for each bucket $i$ have at most $n \log^{10} n$ edges. Again, for every instance $G^{(i,j)}$, we let $m^{(i,j)} = |E^{G^{(i,j)}}|$ denote the number of edges of the graph.

By Theorem A.31

- The graph associated with each $G^{(i,j)}$ has conductance at least $\phi \geq \frac{1}{C_1 \log^3 n - \log(n/\delta)}$, for some universal constant $C_1$.
- For each $G^{(i,j)}$ either
  - ("Uniform case") the instance is $\alpha$-uniform with $\alpha \leq C_2 \log^8 n \log^3 (n/\delta)$ for some universal constant $C_2$.
  - ("Small case") The $\ell^2_3$ norm of $\hat{g}^{G^{(i,j)}}$, the gradient projected to the cycle space, is smaller by a factor of $\tilde{\delta}$ than the unprojected gradient of the original graph

$$\left\| \hat{g}^{G^{(i,j)}} \right\|_2^2 \leq \tilde{\delta} \cdot \left\| g^{G^{(i,j)}} \right\|_2^2 \leq \tilde{\delta} \cdot \left\| g \right\|_2^2.$$

- The minimum degree of each $G^{(i,j)}$ graph is at least $\phi \cdot \frac{m^{(i,j)}}{3n(i,j)}$.

We let $\bar{\varepsilon} = \varepsilon/m$. For instances $G^{(i,j)}_l$ in the “Uniform case”, we let

$$\tau^{(i,j)}_l = c_{\text{sample}} \log(n/\bar{\varepsilon}) \left( \frac{\alpha}{m^{(i,j)}} + \frac{1}{\phi^2 d_{\min}} \right) \leq C_3 \log(n/\bar{\varepsilon}) \log^{12} (n/\delta) \left( \frac{1}{m^{(i,j)}} + \frac{n^{(i,j)}}{m^{(i,j)}} \right)$$

for some universal constant $C_3$, and we call $\text{SampleAndFixGradient}(G^{(i,j)}_l, \tau^{(i,j)}_l)$, which returns instance $H^{(i,j)}_l$ and maps $M_{H^{(i,j)}_l} \rightarrow H^{(i,j)}_l$ and $M_{G^{(i,j)}_l} \rightarrow G^{(i,j)}_l$.

For instances $G^{(i,j)}_l$ in the “small case”, we define $\hat{G}^{(i,j)}_l$ to be the instance $G^{(i,j)}_l$ with all gradient entries set to zero. We let

$$\tau^{(i,j)}_l = c_{\text{sample}} \log(n/\bar{\varepsilon}) \frac{1}{\phi^2 d_{\min}} \leq C_3 \log(n/\bar{\varepsilon}) \log^{12} (n/\delta) \frac{n^{(i,j)}}{m^{(i,j)}}$$

And call $\text{SampleAndFixGradient}(\hat{G}^{(i,j)}_l, \tau^{(i,j)}_l)$ which again returns an instance $H^{(i,j)}_l$ and maps $M_{\hat{G}^{(i,j)}_l} \rightarrow H^{(i,j)}_l$ and $M_{H^{(i,j)}_l} \rightarrow \hat{G}^{(i,j)}_l$.

The instance $H^{(i,j)}_l$ has the same vertex set as $G^{(i,j)}_l$ and at most $\tau^{(i,j)}_l m^{(i,j)}$ edges.

In the “uniform case”, with probability at least $1 - \bar{\varepsilon}$, the maps certify

$$H^{(i,j)}_l \preceq_n G^{(i,j)}_l$$

and $G^{(i,j)}_l \preceq_n H^{(i,j)}_l$,
where $\kappa \leq m^{1/(p-1)} \log^{12}(n/\hat{\delta})$, directly by Theorem [A.32]. In the “small case”, with probability at least $1 - \hat{\varepsilon}$, the maps certify with the same $\kappa$ that

$$\mathcal{H}^{(i,j)}_l \succeq_{\kappa, \delta} \mathcal{G}^{(i,j)}_l \text{ and } \mathcal{G}^{(i,j)}_l \succeq_{\kappa, \delta} \mathcal{H}^{(i,j)}_l,$$

because by Theorem [A.32]

$$\mathcal{H}^{(i,j)}_l \succeq_{\kappa} \mathcal{G}^{(i,j)}_l \text{ and } \mathcal{G}^{(i,j)}_l \succeq_{\kappa} \mathcal{H}^{(i,j)}_l,$$

and our rounding of the gradients implies (certified by the identity map) that

$$\mathcal{G}^{(i,j)}_l \succeq_{1, \hat{\delta}} \|g\|_{\infty} \mathcal{G}^{(i,j)}_l \text{ and } \mathcal{G}^{(i,j)}_l \succeq_{1, \hat{\delta}} \|g\|_{\infty} \mathcal{G}^{(i,j)}_l,$$

with finally our choice of $\hat{\delta} = 2^{\log^c(n)} \delta$ for some large enough universal constant $c$ ensures $\hat{\delta} \|g\|_{\infty} \leq \delta$, and by Lemma [A.16] the guarantees compose to give Equation (12). By Lemma [A.16], the guarantees compose and we get that the total edge count in the sampled instances $\mathcal{H}^{(i,j)}_l$ associated with $\mathcal{G}^{(i,j)}$ is bounded by

$$\sum_l 2\tau^{(i,j)}_l m^{(i,j)}_l \leq \sum_l \frac{3}{2} C_3 \log(n/\hat{\varepsilon}) \log^{12}(n/\hat{\delta}) \left( 1 + \frac{n^{(i,j)} m^{(i,j)}_l}{m^{(i,j)}_l} \right)$$

$$\leq 3C_3 \log(n/\hat{\varepsilon}) \log^{12}(n/\hat{\delta}) n^{(i,j)}$$

$$\leq 3C_3 \log(n/\hat{\varepsilon}) \log^{12}(n/\hat{\delta}) n.$$

Since the number of buckets (indexed by $i$) is bounded by polylog($n$) and the number DECOMPOSE calls for each bucket (indexed by $j$) is bounded by log($n$), we get that the total number of edges summed across all the $\mathcal{H}^{(i,j)}_l$ for all $i, j, l$ is bounded by

$$n \log(1/\hat{\varepsilon}) \log^{12}(1/\hat{\delta}) \text{ polylog}(n).$$

Summed over all $i$, the total number of edges in the $\mathcal{G}^{(i,j)}$ instances is

$$n \text{ polylog}(n).$$

We return a sparsifier instance $\mathcal{H}$ consisting of the union over all $i$ and over all $j$ of the $\mathcal{H}^{(i,j)}_l$ and the $\mathcal{G}^{(i,j)}_l$, and $\mathcal{G}^{(0)}$ (the bucket with edges where 2 and $p$ weights are both zero), with a total number of edges bounded by

$$n \log(1/\hat{\varepsilon}) \log^{12}(1/\hat{\delta}) \text{ polylog}(n) \leq n \text{ polylog}(n/(\varepsilon \delta)).$$

Because the union of the original instances $\mathcal{G}^{(i,j)}$, and $\mathcal{G}^{(0)}$ gives us $\mathcal{G}'$, by Lemma [A.18]

$$\mathcal{G}' \succeq_{\kappa, \delta} \mathcal{H} \text{ and } \mathcal{H} \succeq_{\kappa, \delta} \mathcal{G}'$$

where again $\kappa \leq m^{1/(p-1)} \log^{42}(n/\hat{\delta}) \leq m^{1/(p-1)} \text{ polylog}(n/\hat{\delta})$. Then, because Equation (11) holds using the identity map between $\mathcal{G}$ and $\mathcal{G}'$, we have

$$\mathcal{G} \succeq_{2\kappa, \delta} \mathcal{H} \text{ and } \mathcal{H} \succeq_{2\kappa, \delta} \mathcal{G}.$$

We sparsified at most $m$ different expanders, since each contains a distinct edges of $\mathcal{G}$, and each sparsification fails with probability at most $\hat{\varepsilon} = \varepsilon / m$, so by a union bound, the probability none of the sparsifications fail is at least $1 - \varepsilon$. That weights of $\mathcal{H}$ are quasipolynomially bounded follows from the explicit weights given in Theorem [A.32]. The overall time bound to apply the union map also follows immediately from Theorem [A.32].\[\square\]
Proof of Theorem A.1

Proof. We will use Theorem A.28. This requires that the instance is cycle-touching. So we first convert out instance \( \mathcal{G} \) to \( \mathcal{G}' \) using Lemma A.25. We thus have maps \( \mathcal{M}_{\mathcal{G} \to \mathcal{G}'} \) and \( \mathcal{M}_{\mathcal{G}' \to \mathcal{G}} \) that can be applied in \( O(m) \) time, where \( \| \mathcal{M}_{\mathcal{G} \to \mathcal{G}'} \|_{2 \to 1} \leq 1 \) and \( \mathcal{G}' \preceq_{1,0} \mathcal{G} \), \( \mathcal{G} \preceq_{1,0} \mathcal{G}' \) (since we are solving the residual problem, our demand vector is 0). We can apply Theorem A.28 on \( \mathcal{G}' \) to get an instance \( \mathcal{H} \) with at most \( m_{\mathcal{H}} = n \polylog(n/(\epsilon \delta)) \) edges, maps \( \mathcal{M}' \) that can be computed in \( O(m) \) time, and

\[
\mathcal{H} \preceq_{\kappa,\delta} \mathcal{G}' \quad \text{and} \quad \mathcal{G}' \preceq_{\kappa,\delta} \mathcal{H}
\]

where \( \kappa = m^{-1/(p-1)} \). Now to go from \( \mathcal{G} \) to \( \mathcal{H} \), we will compose these two approximations and we thus have from Lemma A.16

\[
\mathcal{G} \preceq_{\kappa,\delta} \mathcal{G} \quad \text{and} \quad \mathcal{G} \preceq_{\kappa,\delta} \mathcal{H}
\]

Finally, as \( \| \mathcal{M}_{\mathcal{G} \to \mathcal{G}'} \|_{2 \to 1} \leq 1 \), this completes our proof. We remark that any quasipolynomial blow-up in this error would also be acceptable. \( \Box \)

Appendix B. Sparsification for General \( \ell_2^2 + \ell_p^p \) Objectives Using Lewis Weights

We will prove Theorem 2.5

B.1. Leverage Scores and Lewis Weights. For \( \alpha \geq 1 \), and \( x, y > 0 \), we say \( x \approx_\alpha y \) if \( \frac{1}{\alpha} x \leq y \leq \alpha x \). The statistical leverage score of a row \( a_i \) of a matrix \( A \) is defined as

\[
\tau_{2,i}(A) \stackrel{\text{def}}{=} a_i^\top (A^\top A)^{-1} a_i = \| (A^\top A)^{-1/2} a_i \|_2^2.
\]

The generalization of statistical leverage scores to \( \ell_p \)-norms is given by \( \ell_p \) Lewis weights [Lew78], which are defined as follows:

Definition B.1. For a matrix \( A \) and for \( p \geq 1 \), we define the \( \ell_p \) Lewis weights \( \{ \tau_{p,i} \}_i \) to be the unique weights such that

\[
\tau_{p,i} = \tau_{2,i}(\text{DIAG}(\tau_{p,i})^{1/2-1/p} A).
\]

Equivalently,

\[
a_i^\top \left( A^\top \text{DIAG}(\tau_{p,i})^{1-2/p} A \right)^{-1} a_i = \tau_{p,i}^{2/p}.
\]

When the matrix \( A \) is not obvious from the context, we will denote the Lewis weights by \( \tau_{p,i}(A) \).

We use \( \tilde{\tau}_{p,i} \) to denote \( \beta \)-approximate Lewis weights, i.e., \( \tilde{\tau}_{p,i} \approx_\beta \tau_{p,i} \).

Lemma B.2 (Foster’s Theorem [Fos+53]). For any matrix \( A \in \mathbb{R}^{m \times n} \), \( m \geq n \), we have \( \sum_i \tau_{2,i}(A) = \text{rank}(A) \leq n \).

As a simple corollary, we get that the \( \ell_p \) Lewis weights also sum to \( n \).

Corollary B.3. For any matrix \( A \in \mathbb{R}^{m \times n} \), \( m \geq n \), and any \( p \), we have \( \sum_i \tau_{p,i}(A) \leq n \).

Proof. By definition and existence of the Lewis weights,

\[
\sum_i \tau_{p,i}(A) = \sum_i \tau_{2,i}(\text{DIAG}(\tau_{p,i})^{1/2-1/p} A) = \text{rank}(\text{DIAG}(\tau_{p,i})^{1/2-1/p} A) \leq n,
\]

where the second equality follows from Lemma B.2. \( \Box \)
As we will see, having access to $\tau_{2,i}$ would allow us to determine a spectral approximation to $A$, though with many fewer rows. Unfortunately, the naive approach to calculating them requires computing $(A^T A)^{+}$, which would defeat the purpose of finding a smaller spectral approximation in the first place. Thus, the key insight of work by [Coh+15] is that a certain uniform sampling-based approach is sufficient to determine approximate leverage scores, as established by the following important lemma.

**Lemma B.4** (Lemma 7, [Coh+15]). Given matrix $A$, $p \in [2, 4)$, $\theta < 1$, and a matrix $B$ containing $O(n \log(n))$ rescaled rows of $A$, there is an algorithm that, w.h.p. in $n$, computes $n^\theta$-approximate $\tau_{2,i}$ for $A$ in time $O((\text{LSS}(B) + \text{nnz}(A))\theta^{-1})$, where $\text{LSS}(B)$ is the time required to solve a linear equation in $B^T B$ to quasipolynomial accuracy.

While the previous lemma provides approximations to $\tau_{2,i}$, it was later shown by [CP15] that in fact we may use such a routine as a black-box for determining approximations $\tau_{p,i}(A)$, for $p \in [2, 4)$.

**Lemma B.5** (Lemma 2.4, [CP15]). For any fixed $p < 4$, given a routine $\text{APPROXLEVERAGESCORES}$ for computing, with high probability in $n$, $\beta$-approximate statistical leverage scores of rows of matrices of the form $W A$ for $\beta = n^{\frac{1-\log p}{p}}$, there is an algorithm $\text{APPROXLEWISWEIGHTS}$ $n^\theta$-approximate $\ell_p$ Lewis weights for $A$ with $O\left(\frac{\log(\theta^{-1})}{1-\frac{p}{2}}\right)$ calls to $\text{APPROXLEVERAGESCORES}$.

Combining these two lemmas, we arrive at the following overall computational cost for finding $\tilde{\tau}_{p,i}$.

**Theorem B.6.** Given matrix $A$, $p \in [2, 4)$, $\theta < 1$, there is an algorithm that computes $n^\theta$-approximate $\tau_{p,i}$ for $A$ in time

$$O\left( \frac{p}{(1-|p/2-1|)^2} (\text{LSS}(B) + \text{nnz}(A))\theta^{-1} \log(\theta^{-1}) \right),$$

where $\text{LSS}(B)$ is the time required to solve a linear equation in $B^T B$ to quasipolynomial accuracy for some matrix $B$ containing $O(n \log(n))$ rescaled rows of $A$.

**Proof.** The theorem follows immediately from Lemmas B.4 and B.5.

**Lemma B.7** ($\ell_2$ Matrix Concentration Bound (Lemma 4, [Coh+15])). There exists an absolute constant $C_2$ such that for any matrix $A \in \mathbb{R}^{m \times n}$, and any set of sampling values $\nu_i$ satisfying

$$\nu_i \geq \tau_{2,i}(A) \cdot C_2 \varepsilon^{-2} \log n,$$

if we generate a matrix $S$ with $N = \sum_i \nu_i$ rows, each chosen independently as the $i$th standard basis vector times $\frac{1}{\sqrt{\nu_i}}$, with probability $\frac{1}{N}$, then with probability at least $1 - \frac{1}{n m \log n}$, we have

$$\forall x \in \mathbb{R}^n, \|SAx\|_2 \approx_{1+\varepsilon} \|Ax\|_2.$$

**Lemma B.8** ([BLM89], [CP15] (Lemma 7.1)). For $p \geq 2$, there exists an absolute constant $C_p$ such that for any matrix $A \in \mathbb{R}^{m \times n}$, and any set of sampling values $\nu_i$ satisfying

$$\nu_i \geq \tau_{p,i}(A) \cdot C_p n^{-2} \varepsilon^{-5} \log n \log^{1/\varepsilon},$$

if we generate a matrix $S$ with $N = \sum_i \nu_i$ rows, each chosen independently as the $i$th standard basis vector times $\frac{1}{\sqrt{\nu_i}}$, with probability $\frac{1}{N}$, then with probability at least $1 - \frac{1}{n m \log n}$, we have

$$\forall x \in \mathbb{R}^n, \|SAx\|_p \approx_{1+\varepsilon} \|Ax\|_p.$$

**Lemma B.9.** For $p \in [2, 4)$, given matrices $C, D \in \mathbb{R}^{m \times n}$, there exist $\nu_i > 0, i \in [m]$ with $N = \sum_i \nu_i \leq O(1)n^{p/2} \log n$ such that, if we generate a matrix $S$ with $N$ rows, each chosen independently
Proof. Let $\tilde{\tau}_{2,i}(C)$ be 2-approximate leverage scores of $C$ and $\tilde{\tau}_{p,i}(D)$ be 2-approximate $\ell_p$ Lewis weights for $D$. Define

$$\nu_i = C_{2,p} \max\{\tilde{\tau}_{2,i}(C) \cdot \log n, \tilde{\tau}_{p,i}(D) \cdot n^{2-1} \log n\},$$

where $C_{2,p}$ is a large enough absolute constant we specify later. Since $\sum_i \tilde{\tau}_{2,i}(C) \leq 2 \sum_i \tau_{2,i}(C) \leq 2n$ and $\sum_i \tilde{\tau}_{p,i}(D) \leq 2 \sum_i \tau_{p,i}(D) \leq 2n$ from Corollary \[\text{B.3}\] we get $N = \sum_i \nu_i \leq O(C_{2,p}) n^{2-1} \log n$. Let $S$ be defined as in the lemma statement, i.e.

$$S_{ab} = \begin{cases} \frac{1}{\nu_i^{1/p}} & \text{if } \text{th basis vector is chosen for row } a, \\ 0 & \text{otherwise.} \end{cases}$$

Let us assume for row $a$, we have chosen the $b$th basis vector. Now define the diagonal matrix $R$ as

$$R_{aa} = \frac{1}{\nu_i^{1/p}}.$$

Note that $\tilde{S} = RS$ is a matrix with $N$ rows, each chosen independently as the $i$th standard basis vector times $\frac{1}{\nu_i^{1/p}}$ with probability $\frac{1}{\nu_i^{1/p}}$. We can pick $C_{2,p}$ large enough so that $\nu_i \geq \tau_{2,i}(C) \cdot C_{2} \log n$, and we can apply Lemma \[\text{B.7}\] for some constant $\varepsilon < 1$ to obtain that with probability at least $1 - \frac{1}{n^{1/\varepsilon}}$, we have

$$\forall x \in \mathbb{R}^n, \|RSCx\|_2 = \|\tilde{S}Cx\|_2 \approx_2 \|Cx\|_2.$$  

Similarly, we can pick $C_{2,p}$ large enough so that we have $\nu_i \geq \tau_{p,i}(D) \cdot n^{2-1} \log n$. Thus, using Lemma \[\text{B.8}\] we get that with probability at least $1 - \frac{1}{n^{1/\varepsilon}}$, we have

$$\forall x \in \mathbb{R}^n, \|SDx\|_p \approx_2 \|Dx\|_p.$$  

Combining the above two claims, and applying a union bound, we obtain our lemma. \[\square\]

**Lemma B.10.** Let $p \in [2, 4)$, let $M, N, A$ be matrices such that $M \in \mathbb{R}^{m_1 \times n}$, $N \in \mathbb{R}^{m_2 \times n}$, $m_1, m_2 \geq n$, and $A \in \mathbb{R}^{d \times n}$, $d \leq n$, and consider the problem

$$\min_{\Delta} \Delta^\top M^\top M \Delta + \|N \Delta\|_p^p$$  

subject to $A \Delta = c$, with optimum at $\Delta^\ast$. Then, with high probability we may compute $\tilde{M}, \tilde{N} \in \mathbb{R}^{O(n^{p/2} \log(n)) \times n}$ such that, for a $\kappa$-approximate solution $\tilde{\Delta}$ to the problem

$$\min_{\Delta} \tilde{\Delta}^\top \tilde{M}^\top \tilde{M} \tilde{\Delta} + \|\tilde{N} \tilde{\Delta}\|_p^p$$  

subject to $A \tilde{\Delta} = c$, with optimum at $\tilde{\Delta}^\ast$, $\tilde{\Delta}$ is a $O(\kappa)$-approximate solution to \[\text{(15)}\].

**Proof.** Let $\tilde{M} = RMS$ and $\tilde{N} = SN$ be as provided by Lemma \[\text{B.9}\]. It follows that

$$\tilde{\Delta}^\top \tilde{M}^\top \tilde{M} \tilde{\Delta} + \|\tilde{N} \tilde{\Delta}\|_p^p \leq 2^p \left( \tilde{\Delta}^\top \tilde{M}^\top \tilde{M} \tilde{\Delta} + \|\tilde{N} \tilde{\Delta}\|_p^p \right) \leq 2^p \kappa \left( \tilde{\Delta}^\ast^\top \tilde{M}^\top \tilde{M} \tilde{\Delta}^\ast + \|\tilde{N} \tilde{\Delta}^\ast\|_p^p \right) \leq 2^p \kappa \left( \Delta^\ast^\top \Delta^\ast + \|\Delta^\ast\|_p^p \right).$$  

28
\[ \leq 2^{2p} \kappa \left( \Delta^\ast M^\top M \Delta^\ast + \| N \Delta^\ast \|_p^p \right) \]
\[ \leq 256 \kappa \left( \Delta^\ast M^\top M \Delta^\ast + \| N \Delta^\ast \|_p^p \right), \]
where the last inequality follows from our bound on \( p \).

We now recall the main Lewis weights-based sparsification result, Theorem 2.6, which was proven in Section 3.1.

This result gives us the following corollaries which distinguish the general problem from the more structured graph problem, whereby the latter may take advantage of fast Laplacian solvers. Note that, for Theorem 2.6 and its corollaries, we use \( \tilde{O}(p) \) to suppress a \((1 - |p/2 - 1|)^{-2}\) term, which will become large as \( p \) approaches 4.

**Proof of Theorem 2.5.** Follows from Lemmas B.6 and B.10. \( \square \)

**Corollary B.11** (General matrix setting). Consider (3) for arbitrary \( M \in \mathbb{R}^{m_1 \times n}, N \in \mathbb{R}^{m_2 \times n} \). Then, for \( p \in [2, 4) \), with high probability, we can find an \( \varepsilon \)-approximate solution in time

\[ \tilde{O}( \left( \text{nnz}(M) + \text{nnz}(N) + \text{nnz}(\tilde{M}) + \text{nnz}(\tilde{N}) + n^\omega \right) n \frac{p}{p-2} \log^2(1/\varepsilon) ) \]

for some \( \tilde{M} \) and \( \tilde{N} \) each containing \( O(n^p/\log(n)) \) rescaled rows of \( M \) and \( N \), respectively.

**Corollary B.12** (Graph setting). Consider (3) for \( M \in \mathbb{R}^{m_1 \times n}, N \in \mathbb{R}^{m_2 \times n} \), \( m_1, m_2 \geq n \), given as the edge-vertex incidence matrices for some graphs. Then, for \( p \in [2, 4) \), with high probability, we can find an \( \varepsilon \)-approximate solution to (3) in time

\[ \tilde{O}( \left( m_1 + m_2 + n^\frac{p}{2} \left( 1 + \frac{p-2}{p-3} \right) \right) \log^2(1/\varepsilon) ) \]

**Appendix C. Width-Reduced Approximate Solver for \( \ell_2^2 + \ell_p^p \) Problems**

We will solve problems of the form,

\[ \min_{\Delta} \quad \Delta^\top M^\top M \Delta + \| N \Delta \|_p^p \]
\[ \text{s.t.} \quad A \Delta = c, \]

which have an optimum at most \( \nu \). We first scale the problem down to a new problem with optimum at most 1. Note that there exists \( \Delta^\ast \) such that \( A \Delta^\ast = c \) and \( \Delta^\ast^\top M^\top M \Delta^\ast + \| N \Delta^\ast \|_p^p \leq \nu \). Let \( \tilde{M} = \nu^{-\frac{p}{2}} M \) and \( \tilde{\Delta} = \nu^{-1/p} \Delta^\ast \). The following problem has optimum at most 1 since \( \tilde{\Delta} \) is a feasible solution.

\[ \min_{\Delta} \quad \tilde{\Delta}^\top \tilde{M}^\top \tilde{M} \tilde{\Delta} + \| N \Delta \|_p^p \]
\[ \text{s.t.} \quad A \Delta = \nu^{-1/p} c, \]

Now, let \( \bar{\Delta} \) denote a feasible solution such that \( \bar{\Delta}^\top \tilde{M}^\top \tilde{M} \bar{\Delta} \leq \alpha \) and \( \| N \Delta \|_p^p \leq \beta \). Note that \( \Delta = \nu^{1/p} \bar{\Delta} \) satisfies the constraints of (17) and,

\[ \Delta^\top M^\top M \Delta \leq \alpha \nu, \quad \text{and}, \quad \| N \Delta \|_p^p \leq \beta \nu. \]

It is thus sufficient to solve Problem (18) to an \( \alpha, \beta \) approximation. We thus have the following result which follows from Theorem C.1.
Theorem 3.4. Let $p \geq 2$. Consider an instance of Problem (5) described by matrices $A \in \mathbb{R}^{d \times n}$, $N \in \mathbb{R}^{m_1 \times n}$, $M \in \mathbb{R}^{m_2 \times n}$, $d \leq n \leq m_1, m_2$, and vector $c \in \mathbb{R}^d$. If the optimum of this problem is at most $\nu$, Procedure Residual-Solver (Algorithm 3) returns a $x$ such that $Ax = c$, and $x^\top M^\top M x \leq O(1)\nu$ and $\|N x\|_p^p \leq O(3^p)\nu$. The algorithm makes $O\left(\frac{p^2}{(p-2)}\right)$ calls to a linear system solver.

C.1. Solving Scaled Problem. We will show that we can solve problems of the form
\begin{equation}
\begin{aligned}
\min_{\Delta} & \quad \Delta^\top M^\top M \Delta + \|N \Delta\|_p^p \\
\text{s.t.} & \quad A \Delta = c,
\end{aligned}
\end{equation}
which have an optimum value at most 1. We will use the following oracle in our algorithm.

Algorithm 2 Oracle
\begin{algorithmic}[1]
\Procedure{Oracle}{$A, M, N, c, w$}
\State $r_e \leftarrow w_e^{p-2}$
\State Compute,
\State \hspace{1em} $\Delta = \arg\min_{\Delta : A \Delta = c} m_1^p \Delta^\top M^\top M \Delta' + \frac{1}{3^{p-2}} \sum_e r_e (N \Delta_e)^2$
\State \Return $\Delta$
\EndProcedure
\end{algorithmic}

We can use now use Algorithm 4 from [Adi+19],

Notation. We will use $\Delta^*$ to denote the optimum of (19) and $\tilde{\Delta}$ to denote the solution returned by the oracle (Algorithm 2). We thus have,
\begin{itemize}
\item $\Delta^*^\top M^\top M \Delta^* \leq 1$,
\item $\|N \Delta^*\|_p \leq 1$
\item $r_e \geq 1, \forall e$.
\end{itemize}

We will prove the following main theorem:

Theorem C.1. Let $p \geq 2$. Given matrices $A \in \mathbb{R}^{d \times n}$, $N \in \mathbb{R}^{m_1 \times n}$, $M \in \mathbb{R}^{m_2 \times n}$, $m_1, m_2 \geq n$, $d \leq n$, and vector $c$, Algorithm 3 uses $O\left(\frac{p^2}{(p-2)}\right)$ calls to the oracle (Algorithm 2) and returns a vector $x$ such that $Ax = c$, $x^\top M^\top M x \leq O(1)$ and $\|N x\|_p^p = O(3^p)$.

Analysis of Algorithm 3. Similar to [Adi+19] we will track two potentials $\Phi$ and $\Psi$ which we define as,
\begin{align*}
\Phi(w(i)) & \overset{\text{def}}{=} \|w\|_p^p \\
\Psi(r) & \overset{\text{def}}{=} \min_{\Delta : A \Delta = c} m_1^p \Delta^\top M^\top M \Delta + \frac{1}{3^{p-2}} \sum_e r_e (N \Delta_e)^2.
\end{align*}

Note that these potentials have a similar idea as [Adi+19] but are defined differently. Our proof will follow the following structure,
\begin{itemize}
\item[(1)] Provided the total number of width reduction steps, $K$, is not too big, $\Phi(\cdot)$ is small. This in turn helps upper bound the value of the solution returned by the algorithm.
\item[(2)] Showing that $K$ cannot be too big, because each width reduction step cause large growth in $\Psi(\cdot)$, while we can bound the total growth in $\Psi(\cdot)$ by relating it to $\Phi(\cdot)$.
\end{itemize}
Algorithm 3 Algorithm for the Scaled down Problem

1: procedure Residual-Solver($A, M, N, c$)
2: \( w^{(0,0)} \leftarrow 1 \)
3: \( x \leftarrow 0 \)
4: \( \rho \leftarrow \Theta \left( \left( \frac{p^2 - 4p + 2}{m(3p - 2)} \right) \right) \) \hspace{1cm} \triangleright \text{width parameter} \)
5: \( \beta \leftarrow \Theta \left( \left( \frac{p - 2}{m(3p - 2)} \right) \right) \) \hspace{1cm} \triangleright \text{resistance threshold} \)
6: \( \alpha \leftarrow \Theta \left( \left( \frac{p - 1}{m} - \frac{p^2}{m(3p - 2)} \right) \right) \) \hspace{1cm} \triangleright \text{step size} \)
7: \( \tau \leftarrow \Theta \left( \left( \frac{m(p - 1) - 2}{m(3p - 2)} \right) \right) \) \hspace{1cm} \triangleright \ell_p \text{ energy threshold} \)
8: \( T \leftarrow \alpha^{-1} m^1/p = \Theta \left( \frac{pm^{p - 2}}{p(3p - 2)} \right) \)
9: \( i \leftarrow 0, k \leftarrow 0 \)
10: while \( i < T \) do
11: \( \Delta = \text{Oracle} \left( A, M, N, c, w \right) \)
12: if \( \|N\Delta\|_p^{p} \leq \tau \) then \hspace{1cm} \triangleright \text{flow step} \\
13: \( w^{(i+1,k)} \leftarrow w^{(i,k)} + \alpha|N\Delta| \)
14: \( x \leftarrow x + \alpha \Delta \)
15: \( i \leftarrow i + 1 \)
16: else \hspace{1cm} \triangleright \text{width reduction step} \\
17: \quad \text{For all edges } e \text{ with } |N\Delta|_{e} \geq \rho \text{ and } r_{e} \leq \beta \\
18: \quad w_{e}^{(i,k+1)} \leftarrow 2^{\frac{1}{p - 2}} w_{e} \)
19: \( k \leftarrow k + 1 \)
20: return \( m^{-\frac{2}{p}} x \)

We start by proving some results that we need to prove our final result, Theorem [C.1]. The proofs of all lemmas are in Section [C.2].

Lemma C.2. Let \( p \geq 2 \). For any \( w \), let \( \tilde{\Delta} \) be the solution returned by Algorithm 2. Then,
\[
\sum_{e} (N\tilde{\Delta})_{e}^{2} \leq \sum_{e} r_{e} (N\tilde{\Delta})_{e}^{2} \leq \|w\|^{p - 2}
\]

We next show through the following lemma that the \( \Phi \) potential does not increase too rapidly. The proof is through induction and can be found in the Appendix.

Lemma C.3. After \( i \) flow steps, and \( k \) width-reduction steps, provided
\[
(1) \ p^{p} \alpha^{p} \tau \leq p \alpha_{1} m_{1}^{\frac{1}{p}}, \ (\text{controls } \Phi \text{ growth in flow-steps}) \\
(2) \ k \leq 2^{\frac{p}{p - 2}} r_{p} m_{1}^{2/p} \beta^{\frac{2}{p - 2}}, \ (\text{acceptable number of width-reduction steps})
\]
the potential \( \Phi \) is bounded as follows:
\[
\Phi \left( w^{(i,k)} \right) \leq \left( 2\alpha i + m_{1}^{1/p} \right)^{p} \exp \left( \frac{2^{\frac{p}{p - 2}} \left( \frac{k}{p^{2} m_{1}^{2/p} \beta^{\frac{2}{p - 2}}} \right) }{\rho^{p} m_{1}^{2/p} \beta^{\frac{2}{p - 2}}} \right).
\]

We next show how the potential \( \Psi \) changes with a change in resistances. The proof is in the Appendix.
Lemma C.4. Let \( \tilde{\Delta} = \arg\min_{A} \Delta^\top M \Delta + \frac{1}{p^{2-p}} \sum_e r_e (N \Delta)_e^2 \). Then one has for any \( r' \) and \( r \) such that \( r' \geq r \),
\[
\Psi(r') \geq \Psi(r) + \sum_e \left( 1 - \frac{r_e}{r'_e} \right) r_e (N \tilde{\Delta})_e^2.
\]

The next lemma gives a lower bound on the energy in the beginning and an upper bound on the energy at each step.

Lemma C.5. Initially, we have,
\[
\Psi(r^{(0,0)}) \geq \frac{\|M + N\|_{\min}^2\|c\|_2^2}{\|A\|^2} \overset{\text{def}}{=} L,
\]
where \( \|M + N\|_{\min} = \min_{Ax = c} \|Mx\|_2^2 + \|Nx\|_2^2 \) and \( \|A\| \) is the operator norm of \( A \). Moreover, at any step \((i, k)\) we have,
\[
\Psi(r^{(i,k)}) \leq m_p - 2 + \frac{1}{3p-2} \Phi(i, k)^{\frac{p-2}{p}}.
\]

We next bound the change in energy in a flow step and a width reduction step. This lemma is directly from \([\text{Adi}+19]\) and the proof is also very similar. We include it here for completeness.

Lemma C.6. Suppose at step \((i, k)\) we have \( \frac{\|N \Delta\|_p}{p} > \tau \) so that we perform a width reduction step (line \([7]\)). If
\begin{enumerate}
\item \( \tau^{2/p} > \Omega(1) \Psi(r)^{\frac{1}{p}} \), and
\item \( \tau > \Omega(1) \Psi(r)^{\frac{1}{p}} \),
\end{enumerate}
then
\[
\Psi(r^{(i,k+1)}) \geq \Psi(r^{(i,k)}) + \Omega(1) \tau^{2/p}.
\]
Furthermore, if at \((i, k)\) we have \( \frac{\|N \Delta\|_p}{p} \leq \tau \) so that we perform a flow step, then
\[
\Psi(r^{(i,k+1)}) \geq \Psi(r^{(i,k)}).
\]

Proof of Theorem [C.1]

Proof. We begin by setting all our parameter values.

- \( \alpha \leftarrow \Theta\left(m_1^2 \frac{p^2 - 5p + 2}{p^{2p - 2}}\right) \)
- \( \tau \leftarrow \Theta\left(m_1^{\frac{(p-1)(p-2)}{(3p-2)}}\right) \)
- \( \beta = \Theta\left(m_1^{\frac{p-2}{p}}\right) \)
- \( \rho = \Theta\left(m_1^{\frac{p^2 - 4p + 2}{p(3p-2)}}\right) \)

Note that the above values satisfy the relations \( p^\alpha \rho^\tau = p \alpha m_1^{\frac{p-2}{p}} \).

Let \( m_1^{-1/p} x \) be the solution returned by Algorithm [3]. Note that this satisfies the linear constraint required. We will now bound the values of \( m_1^{-2/p} x^\top M^\top M x \) and \( m_1^{-1} \|N x\|_p^p \). If the algorithm
terminates in \( T = \alpha^{-1} m_1^{1/p} \) flow steps and \( K \leq 2^{-\frac{p}{p-2}} \rho^2 m_1^{2/p} \beta^{-\frac{2}{p-2}} \) width reduction steps, then from Lemma [C.3]

\[
\Phi(w^{(T,K)}) \leq O(3^p)m_1 e^1 = O(3^p m_1)
\]

Note that throughout the algorithm \( w \geq |N x| \). This means that the algorithm returns \( m_1^{-\frac{1}{p}} x \) with

\[
m_1^{-1}\|N x\|_p^p \leq \frac{1}{m_1} \|w^{(T,K)}\|_p^p = \frac{1}{m_1} \Phi(w^{(T,K)}) \leq O(3^p).
\]

To bound the other term, let \( \tilde{\Delta}^{(t)} \) denote the solution returned by the oracle in iteration \( t \). Note that, since \( \Phi \leq O(3^p)m_1 \) for all iterations, we always have \( \Psi(r) \leq O(1)m_1^{-\frac{p}{p-2}} \). We claim that

\[
\left( \tilde{\Delta}^{(t)} \right)^\top M^\top M \tilde{\Delta}^{(t)} \leq \Psi(r) \leq O(1)\frac{\tilde{\Delta}^{(t)}^\top M \tilde{\Delta}^{(t)} \leq O(1)}{m_1^{-\frac{p}{p-2}}}
\]

This concludes the first part of the proof that if the number of width reduction steps are bounded, then we return a solution with the required values. We will now show that we cannot have more width reduction steps.

Suppose to the contrary, the algorithm takes a width reduction step starting from step \((i, k)\) where \( i < T \) and \( k = 2^{-\frac{p}{p-2}} \rho^2 m_1^{2/p} \beta^{-\frac{2}{p-2}} \). Since the conditions for Lemma [C.3] hold for all preceding steps, we must have \( \Phi(w^{(i,k)}) \leq O(3^p)m_1 \). We note that our parameter values satisfy \( \tau^{2/p} \geq \Omega(1)\frac{\bar{\Psi}}{m_1^{2/p}} \) and \( \tau \geq \Omega(1)\rho^{-2}\Psi \) since \( \Psi \leq O(1)m_1^{-\frac{p}{p-2}} \).

This means that at every step \((j, l)\) preceding the current step, the conditions of Lemma [C.6] are satisfied, so we can prove by a simple induction that

\[
\Psi(r^{(i,k+1)}) \geq \Psi(r^{(0,0)}) + \Omega(1)\tau^{2/p} k.
\]

Since our parameter choices ensure \( \tau^{2/p} k > \Theta(m_1) \),

\[
\Psi(r^{(i,k+1)}) - \Psi(r^{(0,0)}) > \Omega(m_1).
\]

Since \( \Phi(w^{(i,k)}) \leq O(3^p)m_1 \),

\[
\Psi(r^{(i,k+1)}) - \Psi(r^{(0,0)}) \leq O\left(\frac{\bar{\Psi}}{m_1^{2/p}}\right),
\]

which is a contradiction. We can thus conclude that we can never have more than \( K = 2^{-\frac{p}{p-2}} \rho^2 m_1^{2/p} \beta^{-\frac{2}{p-2}} \) width reduction steps, thus concluding the correctness of the returned solution. We next bound the number of oracle calls required. The total number of iterations is at most,

\[
T + K \leq \alpha^{-1} m_1^{1/p} + 2^{-p/(p-2)} \rho^2 m_1^{2/p} \beta^{-\frac{2}{p-2}} \leq O\left(\frac{\bar{\Psi}}{pm_1^{2/p-2}}\right).
\]

\[\square\]
C.2. Missing Proofs.

**Lemma C.2.** Let $p \geq 2$. For any $w$, let $\Delta$ be the solution returned by Algorithm 2. Then,
\[
\sum_e (N\Delta_e)^2 \leq \sum_e r_e (N\Delta_e)^2 \leq \|w\|^{p-2}
\]

**Proof.** Since $\Delta$ is the solution returned by Algorithm 2 and $\Delta^*$ satisfies the constraints of the oracle, we have,
\[
\sum_e r_e (N\Delta_e)^2 \leq \sum_e r_e (N\Delta^*_e)^2 = \sum_e w_e^{p-2} (N\Delta^*)_e^2 \leq \|w\|^{p-2}.
\]

In the last inequality we use,
\[
\sum_e w_e (N\Delta^*)_e^2 \leq \left( \sum_e (N\Delta^*)_e^2 \right)^{2/p} \left( \sum_e |w_e|^{(p-2)/p} \right)^{(p-2)/p} = \|N\Delta^*\|_p^2 \|w\|_p^{(p-2)/p} \leq \|w\|^{(p-2)/p}, \text{ since } \|N\Delta^*\|_p \leq 1.
\]

Finally, using $r_e \geq 1$, we have $\sum_e (N\Delta_e)^2 \leq \sum_e r_e (N\Delta_e)^2$, concluding the proof. □

**Lemma C.3.** After $i$ flow steps, and $k$ width-reduction steps, provided

1. $p^6 \alpha^p \tau \leq \text{pom}_1^i$, (controls $\Phi$ growth in flow-steps)
2. $k \leq 2^{-\frac{p}{p-2}} p^2 m_1^{2/p} \beta^{-\frac{2}{p-2}}$, (acceptable number of width-reduction steps)

the potential $\Phi$ is bounded as follows:
\[
\Phi(w^{(i,k)}) \leq \left( 2 \alpha i + m_1^i \right)^p \exp \left( 2^{-\frac{p}{p-2}} \frac{k}{p^2 m_1^{2/p} \beta^{-\frac{2}{p-2}}} \right).
\]

**Proof.** We prove this claim by induction. Initially, $i = k = 0$, and $\Phi(w^{(0,0)}) = m_1$, and thus, the claim holds trivially. Assume that the claim holds for some $i, k \geq 0$. We will use $\Phi$ as an abbreviated notation for $\Phi(w^{(i,k)})$ below.

Flow Step. For brevity, we use $w$ to denote $w^{(i,k)}$. If the next step is a flow step,
\[
\Phi(w^{(i+1,k)}) = \|w^{(i,k)} + \alpha N\Delta\|^p_p \leq \|w\|^p_p + \alpha \left( N\Delta \right)^T \left| \nabla \|w\|_p^p \right| + 2p \alpha^2 \sum_e |w_e|^{p-2} \left| (N\Delta)_e \right|^2 + \alpha^p p^p \|N\Delta\|_p^p
\]

by Lemma B.1 of [APS19]

We next bound $\left| (N\Delta) \right|^T \left| \nabla \|w\|_p^p \right|$ as,
\[
\sum_e \left| (N\Delta)_e \right| \left| \nabla_e \|w\|_p^p \right| \leq p \|w\|_p^{p-1}.
\]

Using Cauchy Schwarz’s inequality,
\[
\left( \sum_e \left| (N\Delta)_e \right| \left| \nabla_e \|w\|_p^p \right| \right)^2 \leq p^2 \left( \sum_e \left| (N\Delta)_e \right| |w_e|^{p-2} \right)^2.
\]
\[ \leq p^2 \left( \sum_e |w_e|^{p-2} w_e^2 \right) \left( \sum_e |w_e|^{p-2} (N \Delta)^2_e \right) \]
\[ = p^2 \|w\|_p^p \sum_e r_e (N \Delta)^2_e \]
\[ \leq p^2 \|w\|_p^{2p-2} \]

We thus have,
\[ \sum_e |N \Delta|_e \|w\|_p^p \leq p \|w\|_p^{p-1}. \]

Using the above bound, we now have,
\[ \Phi \left( w^{i+1, k} \right) \leq \|w\|_p^p + p\alpha \|w\|_p^{p-1} + 2p^2 \alpha^2 \|w\|_p^{p-2} + p^p \alpha P \|N \Delta\|_p^p \]
\[ \leq \|w\|_p^p + p\alpha \|w\|_p^{p-1} + 2p^2 \alpha^2 \|w\|_p^{p-2} + p\alpha \Phi \left( N \Delta \right)_e^{\frac{p-1}{p}}, \]

by Assumption 1 of this Lemma.

Recall \( \|w\|_p^p = \Phi(w) \). Since \( \Phi \geq m_1 \), we have,
\[ \leq \Phi(w) + p\alpha \Phi(w) \frac{p-1}{p} + 2p^2 \alpha^2 \Phi(w) \frac{p-2}{p} + p\alpha \Phi \left( N \Delta \right)_e^{\frac{p-1}{p}} \]
\[ \leq (\Phi(w)^{1/p} + 2\alpha)^p. \]

From the inductive assumption, we have
\[ \Phi(w) \leq \left( 2\alpha i + m_1^{1/p} \right)^p \exp \left( O_p(1) \frac{k}{\rho^2 m_1^{2/p} \beta^{-\frac{1}{p-2}}} \right). \]

Thus,
\[ \Phi(i, k) \leq (\Phi(w)^{1/p} + 2\alpha)^p \leq \left( 2\alpha (i + 1) + m_1^{1/p} \right)^p \exp \left( O_p(1) \frac{k}{\rho^2 m_1^{2/p} \beta^{-\frac{1}{p-2}}} \right) \]

proving the inductive claim.

Width Reduction Step. We have the following:
\[ \sum_{e \in H} r_e \leq \rho^{-2} \sum_{e \in H} r_e (N \Delta)^2_e \leq \rho^{-2} \sum_{e} r_e (N \Delta)^2_e \leq \rho^{-2} \|w\|_p^{p-2} \leq \rho^{-2} \Phi \frac{p-2}{p}, \]

and
\[ \Phi(i, k + 1) \leq \Phi + \sum_{e \in H} |w_e|^{k+1} \|
\[ \leq \Phi + 2^\frac{p}{p-2} \sum_{e \in H} |w_e|^{p} \]
\[ \leq \Phi + 2^\frac{p}{p-2} \sum_{e} r_e^{\frac{p}{p-2}} \]
\[ \leq \Phi + 2^\frac{p}{p-2} \left( \sum_{e \in H} r_e \right) \left( \max_{e \in H} r_e \right)^{\frac{p}{p-2} - 1} \]
\begin{align*}
\leq \Phi + 2\rho^{p-2}\Phi^{\frac{p-2}{p}}\beta^{\frac{p-2}{p}}.
\end{align*}

Again, since \( \Phi(w) \geq m_1 \),

\[
\Phi(i, k + 1) \leq \Phi \left( 1 + 2\rho^{p-2}m_1^{-\frac{2}{p}}\beta^{\frac{p-2}{p}} \right) \leq \left( 2\alpha_i + m_1^{1/p} \right)^p \exp \left( 2\rho^{\frac{p}{2}} \frac{k + 1}{\rho^2 m_1^{2/p} \beta^{p-2}} \right)
\]
proving the inductive claim.

Lemma C.4. Let \( \bar{\Delta} = \arg\min_{\Delta = c m_1^p} \Delta^\top M^\top M \Delta + \frac{1}{2\rho} \sum_e r_e (N \Delta)_e^2 \). Then one has for any \( r' \) and \( r \) such that \( r' \geq r \),

\[
\Psi(r') \geq \Psi(r) + \sum_e \left( 1 - \frac{r_e}{r'_e} \right) r_e (N \bar{\Delta})_e^2.
\]

Proof.

\[
\Psi(r) = \min_{Ax = c} x^\top M^\top M x + x^\top N^\top R N x.
\]

Constructing the Lagrangian and noting that strong duality holds,

\[
\Psi(r) = \min_x \max_y x^\top M^\top M x + x^\top N^\top R N x + 2y^\top (c - Ax)
\]

\[
= \max_y \min_x x^\top M^\top M x + x^\top N^\top R N x + 2y^\top (c - Ax).
\]

Optimality conditions with respect to \( x \) give us,

\[
2M^\top M x^* + 2N^\top R N x^* = 2A^\top y.
\]

Substituting this in \( \Psi \) gives us,

\[
\Psi(r) = \max_y 2y^\top c - y^\top A \left( M^\top M + N^\top R N \right)^{-1} A^\top y.
\]

Optimality conditions with respect to \( y \) now give us,

\[
2c = 2A \left( M^\top M + N^\top R N \right)^{-1} A^\top y^*,
\]

which upon re-substitution gives,

\[
\Psi(r) = c^\top \left( A \left( M^\top M + N^\top R N \right)^{-1} A^\top \right)^{-1} c.
\]

We also note that

\[
x^* = \left( M^\top M + N^\top R N \right)^{-1} A^\top \left( A \left( M^\top M + N^\top R N \right)^{-1} A^\top \right)^{-1} c.
\]

We now want to see what happens when we change \( r \). Let \( R' = R + S \), where \( S \) is the diagonal matrix with the changes in the resistances. We will use the following version of the Sherman-Morrison-Woodbury formula multiple times,

\[
(X + U CV)^{-1} = X^{-1} - X^{-1} U (C^{-1} + V X^{-1} U)^{-1} V X^{-1}.
\]

We begin by applying the above formula for \( X = M^\top M + N^\top R N, \ C = I, \ U = N^\top S^{1/2} \) and \( V = S^{1/2} N \). We thus get,

\[
\left( M^\top M + N^\top R' N \right)^{-1} = \left( M^\top M + N^\top R N \right)^{-1} - \left( M^\top M + N^\top R N \right)^{-1} N^\top S^{1/2}
\]

\[
\left( I + S^{1/2} N \left( M^\top M + N^\top R N \right)^{-1} N^\top S^{1/2} \right)^{-1} S^{1/2} N \left( M^\top M + N^\top R N \right)^{-1}.
\]

36
We next claim that,

\[
I + S^{1/2}N \left( M^\top M + N^\top RN \right)^{-1} N^\top S^{1/2} \preceq I + S^{1/2} R^{-1} S^{1/2},
\]

which gives us,

\[
\begin{align*}
(M^\top M + N^\top R' N)^{-1} & \preceq (M^\top M + N^\top R N)^{-1} - \\
(M^\top M + N^\top R N)^{-1} N^\top S^{1/2} (I + S^{1/2} R^{-1} S^{1/2})^{-1} S^{1/2} N (M^\top M + N^\top R N)^{-1}.
\end{align*}
\]

This further implies,

\[
\begin{align*}
A \left( M^\top M + N^\top R' N \right)^{-1} A^\top & \preceq A \left( M^\top M + N^\top R N \right)^{-1} A^\top - \\
A \left( M^\top M + N^\top R N \right)^{-1} N^\top S^{1/2} (I + S^{1/2} R^{-1} S^{1/2})^{-1} S^{1/2} N \left( M^\top M + N^\top R N \right)^{-1} A^\top.
\end{align*}
\]

We apply the Sherman-Morrison formula again for, \(X = A \left( M^\top M + N^\top R N \right)^{-1} A^\top, C = -(I + S^{1/2} R^{-1} S^{1/2})^{-1}, U = A \left( M^\top M + N^\top R N \right)^{-1} N^\top S^{1/2} \) and \(V = S^{1/2} N \left( M^\top M + N^\top R N \right)^{-1} A^\top \).

Let us look at the term \(C^{-1} + VX^{-1} U\).

\[
\begin{align*}
- \left( C^{-1} + VX^{-1} U \right)^{-1} & = \left( I + S^{1/2} R^{-1} S^{1/2} - VX^{-1} U \right)^{-1} \preceq (I + S^{1/2} R^{-1} S^{1/2})^{-1}.
\end{align*}
\]

Using this, we get,

\[
\begin{align*}
\left( A \left( M^\top M + N^\top R' N \right)^{-1} A^\top \right)^{-1} & \succeq X^{-1} + X^{-1} U (I + S^{1/2} R^{-1} S^{1/2})^{-1} VX^{-1},
\end{align*}
\]

which on multiplying by \(c^\top\) and \(c\) gives,

\[
\Psi(r') \geq \Psi(r) + c^\top X^{-1} U (I + S^{1/2} R^{-1} S^{1/2})^{-1} VX^{-1} c.
\]

We note from Equation (20) that \(x^* = \left( M^\top M + N^\top R N \right)^{-1} A^\top X^{-1} c\). We thus have,

\[
\begin{align*}
\Psi(r') & \geq \Psi(r) + (x^*)^\top N^\top S^{1/2} (I + S^{1/2} R^{-1} S^{1/2})^{-1} S^{1/2} N x^* \\
& = \Psi(r) + \sum_e \left( \frac{r'_e - r_e}{r'_e} \right) r_e (N x^*)_e.
\end{align*}
\]

\[
\square
\]

**Lemma C.5.** Initially, we have,

\[
\Psi(r^{(0,0)}) \geq \frac{\|M + N\|^2_{\min} \|c\|^2_2}{\|A\|^2} \overset{\text{def}}{=} L,
\]

where \(\|M + N\|_{\min} = \min_{Ax = c} \|M x\|^2_2 + \|N x\|^2_2\) and \(\|A\|\) is the operator norm of \(A\). Moreover, at any step \((i, k)\) we have,

\[
\Psi(r^{(i,k)}) \leq m_1 \frac{\rho - 2}{\rho} + \frac{1}{3^{\rho - 2}} \Phi(i, k) \frac{\rho - 2}{\rho}.
\]
Proof. For the lower bound in the initial state, since \( r_e \geq 1 \), for any solution \( \Delta \), we have
\[
\Psi(r^{(0,0)}) \geq m_1^{\frac{p-2}{p}} \bar{\Delta}^\top \sum e \, r^{(0,0)}(N\bar{\Delta})_e^2 = \|M\bar{\Delta}\|^2 + \frac{1}{3^{p-2}} \sum e \, r_e(N\bar{\Delta})_e^2 \geq \|M + N\|^2 \|ar{\Delta}\|^2,
\]
where \( \|M + N\|^2 = \min_{Ax = \epsilon} \|Mx\|^2 + \|Nx\|^2 \). Note that if \( \|M + N\|^2 = 0 \) then the oracle has returned the optimum in the first iteration. On the other hand, because
\[
\|c\|_2 = \|A\bar{\Delta}\|_2 \leq \|A\|\|ar{\Delta}\|_2,
\]
where \( \|A\| \) is the operator norm of \( A \). We get
\[
\|ar{\Delta}\|_2 \geq \frac{|c|_2}{\|A\|},
\]
upon which squaring gives the lower bound on \( \Psi(r^{(0,0)}) \).
For the upper bound, Lemma C.2 implies that,
\[
\Psi\left( r^{(i,k)} \right) = m_1^{\frac{p-2}{p}} \bar{\Delta}^\top M\bar{\Delta} + \frac{1}{3^{p-2}} \sum e \, r_e(N\bar{\Delta})_e^2 \leq m_1 \bar{\Delta}^\top M\bar{\Delta} + \frac{1}{3^{p-2}} \sum e \, r_e(N\bar{\Delta})_e^2 \leq m_1 + \frac{1}{3^{p-2}} \|w\|_p^2 \leq m_1 + \frac{1}{3^{p-2}} \Phi(i,k) \frac{p-2}{p}.
\]

Lemma C.6. Suppose at step \((i,k)\) we have \( \|N\bar{\Delta}\|^p \geq \tau \) so that we perform a width reduction step (line 17). If
\begin{enumerate}
\item \( \tau^{2/p} \geq \Omega(1) \Psi(r) \), and
\item \( \tau \geq \Omega(1) \Psi(r) \|p-2\| \).
\end{enumerate}
Then
\[
\Psi\left( r^{(i,k+1)} \right) \geq \Psi\left( r^{(i,k)} \right) + \Omega(1) \tau^{2/p}.
\]
Furthermore, if at \((i,k)\) we have \( \|N\bar{\Delta}\|^p \leq \tau \) so that we perform a flow step, then
\[
\Psi\left( r^{(i,k+1)} \right) \geq \Psi\left( r^{(i,k)} \right).
\]

Proof. It will be helpful for our analysis to split the index set into three disjoint parts:
\begin{itemize}
\item \( S = \{ e : |N\Delta_e| \leq \rho \} \)
\item \( H = \{ e : |N\Delta_e| > \rho \text{ and } r_e \leq \beta \} \)
\item \( B = \{ e : |N\Delta_e| > \rho \text{ and } r_e > \beta \} \).
\end{itemize}
Firstly, we note
\[
\sum_{e \in S} |N\Delta|^p_e \leq \rho^{p-2} \sum_{e \in S} |N\Delta|^2_e \leq \rho^{p-2} \sum_{e \in S} r_e |N\Delta|^2_e \leq \rho^{p-2} \Psi(r).
\]

hence, using Assumption [2]
\[
\sum_{e \in H \cup B} |N\Delta|^p_e \geq \sum_e |N\Delta|^p_e - \sum_{e \in S} |N\Delta|^p_e \geq \tau - \rho^{p-2} \Psi(r) \geq \Omega(1) \tau.
\]

38
This means,

\[ \sum_{e \in B} (N \Delta)_e^2 \geq \left( \sum_{e \in B} |N \Delta|^p_e \right)^{2/p} \geq \Omega(1) \tau^{2/p}. \]

Secondly we note that,

\[ \sum_{e \in B} (N \Delta)_e^2 \leq \beta^{-1} \sum_{e \in B} r_e (N \Delta)_e^2 \leq \beta^{-1} \Psi(r). \]

So then, using Assumption \[1\]

\[ \sum_{e \in H} (N \Delta)_e^2 = \sum_{e \in H} (N \Delta)_e^2 - \sum_{e \in B} (N \Delta)_e^2 \geq \Omega(1) \tau^{2/p} - \beta^{-1} \Psi(r) \geq \Omega(1) \tau^{2/p}. \]

As \( r_e \geq 1 \), this implies \( \sum_{e \in H} r_e (N \Delta)_e^2 \geq \Omega(1) \tau^{2/p} \). We note that in a width reduction step, the resistances change by a factor of 2. Thus, combining our last two observations, and applying Lemma \[C.4\] we get

\[ \Psi \left( r^{(i,k+1)} \right) \geq \Psi \left( r^{(i,k)} \right) + \Omega(1) \tau^{2/p}. \]

Finally, for the “flow step” case, we use the trivial bound from Lemma \[C.4\] ignoring the second term,

\[ \Psi \left( r^{(i,k+1)} \right) \geq \Psi \left( r^{(i,k)} \right). \]

\[ \Box \]

**Appendix D. \( \ell_p \)-Regression**

**Definition D.1 (\( \kappa \)-approximate solution).** Let \( \kappa \geq 1 \). A \( \kappa \)-approximate solution for the residual problem is \( \Delta \) such that \( A \Delta = 0 \) and \( \text{res}(\Delta) \geq \frac{1}{\kappa} \text{res}(\Delta^*) \), where \( \Delta^* = \text{argmax}_{A \Delta = 0} \text{res}(\Delta) \).

**Lemma D.2. (Iterative Refinement \[APS19\])**. Let \( p \geq 2 \), and \( \kappa \geq 1 \). Starting from an initial feasible solution \( x^{(0)} \), and iterating as \( x^{(t+1)} = x^{(t)} - \frac{\Delta}{p} \), where \( \Delta \) is a \( \kappa \)-approximate solution to the residual problem (Definition \[D.2\]), we get an \( \varepsilon \)-approximate solution to \([3]\) in at most \( O \left( pk \log \left( \frac{f(x^{(0)}) - \text{OPT}}{\varepsilon \text{OPT}} \right) \right) \) calls to a \( \kappa \)-approximate solver for the residual problem.

**Proof.** Let \( f(x) = b^\top x + \|Mx\|_2^2 + \|Nx\|_p^p \) and \( \text{res}(\Delta) = g^\top \Delta - \Delta^\top R\Delta - \|N\Delta\|_p^p \). Observe that,

\[ b^\top (x + \Delta) = b^\top x + b^\top \Delta, \]

and

\[ \|M(x + \Delta)\|_2^2 = \|Mx\|_2^2 + 2 \Delta^\top M^\top Mx + \|M\Delta\|_2^2. \]

Using Lemma B.1 from \[APS19\] we have,

\[ \|N(x + \Delta)\|_p^p \leq \|Nx\|_p^p + p(N\Delta)^\top |Nx|_{p-2}^p N x + 2p^2 \Delta^\top N^\top \text{Diag}(|Nx|_{p-2}^p) N \Delta + p^p \|N\Delta\|_p^p, \]

and,

\[ \|N(x + \Delta)\|_p^p \geq \|Nx\|_p^p + p(N\Delta)^\top |Nx|_{p-2}^p N x + \frac{p}{8} \Delta^\top N^\top \text{Diag}(|Nx|_{p-2}^p) N \Delta + \frac{1}{2p+1} \|N\Delta\|_p^p. \]

Using these relations, we have,

\[ f(x + \Delta) \leq f(x) + pg^\top \Delta + p^2 \Delta^\top R\Delta + p^p \|N\Delta\|_p^p, \]

or

\[ f \left( x - \frac{\Delta}{p} \right) \leq f(x) - \text{res}(\Delta). \]
A lower bound looks like,
\[ f(x + \Delta) \geq f(x) + pg^\top \Delta + \frac{p}{16} \Delta^\top R \Delta + \frac{1}{2^{p+1}} \|N \Delta\|_p^p. \]
For \( \lambda = 16p \),
\[
f(x) - f\left(x - \frac{\lambda \Delta}{p}\right) \leq \lambda g^\top \Delta - \frac{\lambda^2}{16p} \Delta^\top R \Delta - \frac{\lambda p}{p^p 2^{p+1}} \|N \Delta\|_p^p
\]
\[
\leq \lambda \left( g^\top \Delta - \frac{\lambda}{16p} \Delta^\top R \Delta - \frac{\lambda p}{p^p 2^{p+1}} \|N \Delta\|_p^p \right)
\]
\[
\leq \lambda \left( g^\top \Delta - \Delta^\top R \Delta - \|N \Delta\|_p^p \right) = \lambda \text{res}(\Delta).
\]
These relations are the same as Lemma B.2 of [APS19]. We can follow the proof further from [APS19] to get our result.

\[\square\]

Solving the Residual Problem.

**Lemma D.3.** Let \( \Delta^* \) denote the optimum of the residual problem at \( x^{(t)} \) and \( \text{OPT} \) denote the optimum of Problem (24). We have that \( \text{res}(\Delta^*) \in (\nu/2, \nu] \) for some \( \nu \in \left[ \varepsilon \frac{\text{OPT}}{\lambda}, f\left(x^{(0)}\right) - \text{OPT} \right] \).

**Proof.** From the above proof, we note that for any \( x \), let \( \Delta^* \) be the optimum of the residual problem.
\[
\text{res}(\Delta^*) \leq f(x) - f\left(x - \frac{\Delta^*}{p}\right) \leq f\left(x^{(0)}\right) - \text{OPT}.
\]
Let \( \Delta \) be the step we take to reach the optimum from \( x \).
\[
\text{res}(\Delta^*) \geq \text{res}(\Delta) \geq \frac{f(x) - \text{OPT}}{\lambda} \geq \varepsilon \frac{\text{OPT}}{\lambda},
\]
where the last inequality follows since otherwise \( x \) is an \( \varepsilon \) approximate solution. The lemma thus follows. \( \square \)

**Lemma D.4.** Let \( p \geq p' \) and \( \nu \) be such that \( \text{res}_p(\Delta^*) \in (\nu/2, \nu] \), where \( \Delta^* \) is the optimum of the residual problem for \( p \)-norm (Definition 3.2). The following problem has optimum between \( \left[ \nu/32, O(1)m^{\frac{1}{p-1}}\nu \right] \text{def} = (a\nu, b\nu) \).

\[
(24) \quad \max_{\Delta = 0} g^\top \Delta - \Delta^\top R \Delta - \frac{1}{2} \left( \frac{\nu}{m} \right)^{1 - \frac{p'}{p}} \|N \Delta\|_p^{p'}
\]
For \( \beta \geq 1 \), if \( \tilde{\Delta} \) is a \( \beta \)-approximate solution to the above problem, then \( \alpha \tilde{\Delta} \) gives an \( 8 \frac{\lambda^2}{m^{p-1}} m^{p-1} \left( \frac{1}{p} - \frac{1}{p'} \right)^{1 - \frac{1}{p'}} \) approximate solution to the residual problem, where \( \alpha = \frac{g}{4 \beta^2} m^{\frac{1}{p-1}} \left( \frac{1}{p} - \frac{1}{p'} \right)^{-\frac{1}{p'}} \).

**Proof.** We will first show that the optimum of Problem (24) is at most \( O(m^{\frac{1}{p-1}}\nu) \). Suppose the optimum \( \Delta^* \) is such that it gives an objective value of \( \beta \nu \). Using a scaling argument as in the above proof, we can conclude that,
\[
g^\top \Delta^* - \Delta^*^\top R \Delta^* - \frac{1}{2} \left( \frac{\nu}{m} \right)^{1 - \frac{p'}{p}} \|N \Delta^*\|_p^{p'} = \Delta^*^\top R \Delta^* + (p' - 1) \frac{1}{2} \left( \frac{\nu}{m} \right)^{1 - \frac{p'}{p}} \|N \Delta^*\|_p^{p'} = \beta \nu.
\]
This implies that $g^\top \Delta^* \geq \beta \nu$, $\Delta^*^\top R \Delta^* \leq \beta \nu$ and $\|N \Delta^*\|_{p'}^p \leq 2\beta \nu^p_m 1^{-p'/p}$. We thus have,

$$\|N \Delta^*\|_p^p \leq (2\beta)^{p/p'} m^{p/p'-1} \nu.$$ 

Let $\alpha = \frac{1}{16} m^{-\nu/p}(\frac{1}{p'} - \frac{1}{p})$ be some scaling factor. Now,

$$\alpha^2 \Delta^*^\top R \Delta^* \leq \alpha \frac{\beta \nu}{8},$$

and

$$\alpha^p \|N \Delta^*\|_p^p \leq \alpha \frac{1}{16 \nu} m^{-\frac{p}{p'-1}} \frac{\beta \nu}{8} \leq \alpha \frac{\beta \nu}{8}.$$ 

Consider,

$$\text{res}_p \left( \beta^{-\frac{p}{p'-1}} \alpha \Delta^* \right) \geq \left( \beta^{-\frac{p}{p'-1}} \alpha \right) \left( g^\top \Delta^* - \frac{\beta \nu}{8} - \frac{\beta \nu}{8} \right) \geq \beta \nu \alpha.$$ 

Since $\text{res}_p(\cdot)$ has optimum at most $\nu$, we must have

$$\beta \nu \alpha \leq 1,$$

which gives,

$$\beta \leq O(1)m^{\nu/(p'-1)}.$$

Thus Problem (24) has an optimum at most $O(1)m^{\nu/(p'-1)} \nu = b \nu$. To obtain a lower bound, consider $\Delta$ obtained in Lemma 3.1 of [AS20]. We will evaluate the objective at $\frac{1}{8} \Delta$,

$$g^\top \frac{1}{8} \Delta - \frac{1}{8^2} \Delta^\top R \Delta - \frac{1}{8^2} \left( \frac{\nu}{m} \right)^{1-\frac{p'}{p}} \|N \Delta\|_{p'}^p \geq \frac{\nu}{16} \frac{\nu}{32} = \frac{\nu}{32} = \frac{\nu}{32} = a \nu.$$

Therefore, the optimum of Problem (24) must be at least $\frac{\nu}{32}$. We next look at how a $\beta$-approximate solution of (24) translates to an approximate solution of the residual problem for $p$. Let $\tilde{\Delta}$ be a $\beta$-approximate solution to (24) and $\Delta^*$ denote its optimum. Denote the objective at $\Delta$ for (24) as $\text{res}_p(\Delta)$. We know that $\text{res}_p(\Delta^*) \geq \text{res}_p(\tilde{\Delta}) \geq \frac{\beta}{2} \text{res}_p(\Delta^*)$. Note that, $g^\top \tilde{\Delta}$ has to be between $a \nu / \beta$ and $b \nu$. This ensures that,

$$\tilde{\Delta}^\top R \tilde{\Delta} + \frac{1}{2} \left( \frac{\nu}{m} \right)^{1-\frac{p'}{p}} \|N \tilde{\Delta}\|_{p'}^p \leq g^\top \tilde{\Delta} \leq b \nu.$$

Let $\alpha = \frac{a}{4b \nu m^{-\frac{p}{p'}(\frac{1}{p} - \frac{1}{p'})}}$. Following the calculations above, we have,

$$\alpha^2 \tilde{\Delta}^\top R \tilde{\Delta} \leq \alpha \frac{a \nu}{4 \beta},$$

and

$$\alpha^p \|N \tilde{\Delta}\|_p^p \leq \alpha \frac{a^{p-1}}{4 p-1 b^p \beta^{p-1}} m^{-\frac{p}{p'}(\frac{1}{p'} - \frac{1}{p})} \|N \tilde{\Delta}\|_p^p \leq \alpha \frac{a \nu}{4 \beta}.$$ 

Therefore,

$$\text{res}_p(\alpha \tilde{\Delta}) \geq \frac{a \alpha}{2 \beta}.$$ 

□
Lemma D.5. At \( x^{(t)} \), let \( \nu \) be such that \( \text{res}(\Delta^*) \in (a\nu, b\nu) \) for some values \( a \) and \( b \). The following problem has optimum at most \( b\nu \).

\[
\begin{align*}
\min_{\Delta} & \quad \Delta^\top R\Delta + \|N\Delta\|_p^p \\
\text{s.t.} & \quad g^\top \Delta = a\nu, A\Delta = 0.
\end{align*}
\]

(26)

Further, if \( \bar{\Delta} \) is a feasible solution to Problem (26) such that \( \bar{\Delta}^\top R\bar{\Delta} \leq a\nu \) and \( \|N\bar{\Delta}\|_p \leq b\nu \), then we can pick a scalar \( \mu = \frac{\nu}{4\alpha\beta^{1/(p-1)}b} \) such that \( \mu\bar{\Delta} \) is a \( \frac{2\alpha^2\beta^{1/(p-1)}}{a^2} \)-approximate solution to the residual problem.

Adapted from the proof of Lemma B.4 of [APS19]

Proof. The assumption on the residual is

\[
\text{res}(\Delta^*) = g^\top \Delta^* - \Delta^*^\top R\Delta^* - \|N\Delta^*\|_p^p \in (a\nu, b\nu).
\]

Since the last 2 terms are strictly non-positive, we must have, \( g^\top \Delta^* \geq a\nu \). Since \( \Delta^* \) is the optimum and satisfies \( A\Delta^* = 0 \),

\[
\frac{d}{d\lambda}(g^\top \lambda\Delta^* - \lambda^2 \Delta^*^\top R\Delta^* - \lambda^p \|N\Delta^*\|_p^p)_{\lambda=1} = 0.
\]

Thus,

\[
g^\top \Delta^* - \Delta^*^\top R\Delta^* - \|A\Delta^*\|_p^p = \Delta^*^\top R\Delta^* + (p-1)\|N\Delta^*\|_p^p.
\]

Since \( p \geq 2 \), we get the following

\[
\Delta^*^\top R\Delta^* + \|N\Delta^*\|_p^p \leq g^\top \Delta^* - \Delta^*^\top R\Delta^* - \|A\Delta^*\|_p^p \leq b\nu.
\]

For notational convenience, let function \( h_p(R, \Delta) = \Delta^\top R\Delta + \|N\Delta\|_p^p \). Now, we know that, \( g^\top \Delta^* \geq a\nu \) and \( g^\top \Delta^* - h_p(R, \Delta^*) \leq b\nu \). This gives,

\[
a\nu \leq g^\top \Delta^* \leq h_p(R, \Delta^*) + b\nu \leq 2b\nu.
\]

Let \( \Delta = \delta \Delta^* \), where \( \delta = \frac{a\nu}{g^\top \Delta^*} \). Note that \( \delta \in [a/2b, 1] \). Now, \( g^\top \Delta = a\nu \) and,

\[
h_p(R, \Delta) \leq \max\{\delta^2, \delta^p\} h_p(R, \Delta^*) \leq b\nu.
\]

Note that this \( \Delta \) satisfies the constraints of program (26) and has an optimum at most \( b\nu \). So the optimum of the program must have an objective at most \( b\nu \). Now let \( \bar{\Delta} \) be a \((\alpha, \beta)\) approximate solution to (26), i.e.,

\[
\bar{\Delta}^\top R\bar{\Delta} \leq a\nu \quad \text{and} \quad \|N\bar{\Delta}\|_p^p \leq b\nu.
\]

Let \( \mu = \frac{a}{4\alpha\beta^{1/(p-1)}b} \). We have,

\[
g^\top (\mu \bar{\Delta}) - h_p(R, \mu \bar{\Delta}) \geq \mu \left( a\nu - \mu \bar{\Delta}^\top R\bar{\Delta} - \mu^{p-1} \|N\bar{\Delta}\|_p^p \right)
\]

\[
\geq \mu \left( a\nu - \frac{a\nu}{4\beta^{1/(p-1)}} - \frac{a\nu}{4\alpha} \right)
\]

\[
\geq \mu \left( a\nu - \frac{a\nu}{4} - \frac{a\nu}{4} \right)
\]

\[
\geq \frac{a\nu}{4b} b\nu \geq \frac{a^2}{16\alpha^2\beta^{1/(p-1)}} \text{OPT}.
\]

\[
\square
\]
Theorem 3.3. For an instance of Problem (3), suppose we are given a starting solution \( \mathbf{x}^{(0)} \) that satisfies \( \mathbf{A} \mathbf{x}^{(0)} = \mathbf{c} \) and is a \( \kappa \) approximate solution to the optimum. Consider an iteration of the while loop, line 4 of Algorithm 4 for the \( \ell_p \)-norm residual problem at \( \mathbf{x}^{(t)} \). We can define \( \mu_1 \) and \( \kappa_1 \) such that if \( \Delta \) is a \( \beta \) approximate solution to a corresponding \( p' \)-norm residual problem, then \( \mu_1 \Delta \) is a \( \kappa_1 \)-approximate solution to the \( p \)-residual problem. Further, suppose we have the following procedures,

1. **Sparseify:** Runs in time \( K \), takes as input any matrices \( \mathbf{R}, \mathbf{N} \) and vector \( \mathbf{g} \) and returns \( \mathbf{R}, \mathbf{N}, \mathbf{g} \) having sizes at most \( \tilde{n} \times n \) for the matrices, such that if \( \Delta \) is a \( \beta \) approximate solution to,

   \[
   \max_{A \Delta = 0} \mathbf{g}^\top \Delta - \| \mathbf{R} \Delta \|_2^2 - \| \mathbf{N} \Delta \|_{p'}^p,
   \]

   for any \( p' \geq 2 \), then \( \mu_2 \Delta \), for a computable \( \mu_2 \) is a \( \kappa_2 \beta \)-approximate solution for,

   \[
   \max_{A \Delta = 0} \text{res}(\Delta) \overset{\text{def}}{=} \mathbf{g}^\top \Delta - \| \mathbf{R}^{1/2} \Delta \|_2^2 - \| \mathbf{N} \Delta \|_{p'}^p.
   \]

2. **Solver:** Approximately solves (4) to return \( \Delta \) such that \( \| \mathbf{R} \Delta \|_2^2 \leq \kappa_3 \nu \) and \( \| \mathbf{N} \Delta \|_{p'}^p \leq \kappa_4 \nu \) in time \( \tilde{K}(\tilde{n}) \) for instances of size at most \( \tilde{n} \).

Algorithm 4 finds an \( \varepsilon \)-approximate solution to Problem (3) in time

\[
\tilde{O} \left( p \kappa_4^{1/(p-1)} \kappa_3 \kappa_2 \kappa_1 (K + \tilde{K}(\tilde{n})) \log \left( \frac{\kappa_{4}}{\varepsilon} \right)^2 \right).
\]

Proof. From Lemma D.2 we know that given an instance of Problem (3), at every \( \mathbf{x}^{(t)} \) we can define a residual problem and if \( \Delta \) is a \( \beta \) approximate solution of the residual problem, updating \( \mathbf{x}^{(t)} \) to \( \mathbf{x}^{(t)} - \frac{\Delta}{\beta} \), we can find the required \( \varepsilon \) approximate solution in

\[
O \left( p \beta \log \left( \frac{f(\mathbf{x}^{(0)}) - \text{OPT}}{\varepsilon \text{OPT}} \right) \right) \leq O \left( p \beta \log \left( \frac{\kappa_{4}}{\varepsilon} \right) \right)
\]

iterations. The last inequality follows since \( \frac{f(\mathbf{x}^{(0)}) - \text{OPT}}{\varepsilon \text{OPT}} \leq \frac{\kappa_{\text{OPT}}}{\varepsilon} = \kappa \). It is thus sufficient to solve a residual problem at \( \mathbf{x} \) that looks like,

\[
\max_{A \Delta = 0} \mathbf{g}^\top \Delta - \Delta^\top \mathbf{R} \Delta - \| \mathbf{N} \Delta \|_{p'}^p.
\]

Here \( \mathbf{g} \) and \( \mathbf{R} \) depend on \( \mathbf{x}^{(t)}, \mathbf{M}, \mathbf{N} \). Now, suppose we have \( \text{res}(\Delta^*) \in (\nu/2, \nu] \). We will consider the following cases:

1. \( p \leq \log m \):
   We apply Sparseify to \( \mathbf{g}, \mathbf{R}, \mathbf{N} \) to get \( \tilde{\mathbf{g}}, \tilde{\mathbf{R}}, \tilde{\mathbf{N}} \). Now is \( \Delta \) a \( \beta \) approximate solution to

   \[
   \max_{A \Delta = 0} \tilde{\mathbf{g}}^\top \Delta - \Delta^\top \tilde{\mathbf{R}} \Delta - \| \tilde{\mathbf{N}} \Delta \|_{p'}^p,
   \]
   then \( \mu_2 \Delta \) is a \( \kappa_2 \beta \)-approximate solution to the residual problem. We will now solve the above problem. Note that the size of this problem is \( m \). Let \( \Delta \) be a \( \kappa_3, \kappa_4 \) approximate solution to Problem (4). From D.5 \( \frac{a}{2b \kappa_3 \kappa_4^{1/(p-1)}} \Delta \) is a \( 4\kappa_3 \kappa_4^{1/(p-1)} \nu^2 \) approximation to (27).

Now, going back, \( \frac{a}{2b \kappa_3 \kappa_4^{1/(p-1)}} \mu_2 \Delta \) is a \( 4\kappa_3 \kappa_4^{1/(p-1)} \nu^2 \)-approximate solution to the residual problem. Now, \( \mu_1 = \kappa_1 = 1 \) and thus we have that, in this case if \( \text{res}(\Delta^*) \in (\nu/2, \nu] \), then

\[
\frac{a}{2b \kappa_3 \kappa_4^{1/(p-1)}} \mu_2 \Delta \text{ is a } 4\kappa_3 \kappa_4^{1/(p-1)} \nu^2 \text{-approximate solution to the residual problem.}
\]
(2) $p > \log m$:

In this case, there is an additional step that converts the residual problem to an instance of the previous case for $p = \log m$. We apply \textsc{Sparsify} to $g, R, N'$ and set $p = \log m$. If $\tilde{\Delta}$ is the $\alpha$ solution returned by the previous case on $p = \log m$ and $N = N'$, then $\mu_1 \tilde{\Delta}$ is a $\kappa_1 \alpha$-approximation for the residual problem. Thus $\frac{a}{2b\kappa_3 \kappa_4^{1/(p-1)} \mu_2 \mu_1 \tilde{\Delta}}$ is a $\frac{4\kappa_3 \kappa_4^{1/(p-1)} b^2}{a^2} \kappa_2 \kappa_1$-approximate solution to the residual problem.

From the above discussion we conclude that, if $\text{res}(\Delta^*) \in (\nu/2, \nu]$, then we get a solution $\tilde{\Delta}$ such that $\text{res}(\tilde{\Delta}) \geq \frac{a}{4 \kappa_4^{1/(p-1)} \kappa_3 \kappa_2 \kappa_1 b^2} \text{res}(\Delta^*)$. From Lemma D.3 we know that $\text{res}(\Delta^*) \in (\nu/2, \nu]$ for some $\nu \in \left[ \frac{\text{Opt}}{p}, f\left(\mathbf{x}^{(0)}\right) - \text{OPT} \right]$. Since $\text{OPT} \geq \frac{f\left(\mathbf{x}^{(0)}\right)}{\kappa}$ and $\text{OPT} \geq 0$, it is sufficient to change $\nu$ in the range $\left[ \frac{\varepsilon \text{Opt}}{p}, f\left(\mathbf{x}^{(0)}\right) \right]$.

We finally look at the running time. We start with a residual problem. We require time $K$ to apply procedure \textsc{Sparsify}. We next require time $\tilde{K}(m)$ to solve [1]. Now this either gives us an $\frac{4\kappa_4^{1/(p-1)} \kappa_3 \kappa_2 \kappa_1 b^2}{a^2}$-approximate solution to the residual problem. For every residual problem we repeat the above process at most $\log \frac{\kappa \epsilon}{\varepsilon}$ times (corresponding to the number of values of $\nu$). We use the fact that $a, b \leq m^{o(1)}$. Thus the total running time is,

$$O\left( \frac{4p \kappa_4^{1/(p-1)} \kappa_3 \kappa_2 \kappa_1 b^2}{a^2} \left( K + \tilde{K}(m) \right) \log \left( \frac{\kappa}{\varepsilon} \right) \log \left( \frac{\kappa \epsilon}{\varepsilon} \right) \right)$$

$$\leq \tilde{O}\left( \frac{\kappa_4^{1/(p-1)} \kappa_3 \kappa_2 \kappa_1}{\epsilon} \left( K + \tilde{K}(m) \right) \log \left( \frac{\kappa \epsilon}{\varepsilon} \right)^2 \right)$$

$\Box$