Pseudo-Distance-Regularised Graphs Are Distance-Regular or Distance-Biregular

M.A. Fiol
Universitat Politècnica de Catalunya
Departament de Matemàtica Aplicada IV
Barcelona, Catalonia
e-mail: fiol@ma4.upc.edu

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Abstract
The concept of pseudo-distance-regularity around a vertex of a graph is a natural generalization, for non-regular graphs, of the standard distance-regularity around a vertex. In this note, we prove that a pseudo-distance-regular graph around each of its vertices is either distance-regular or distance-biregular. By using a combinatorial approach, the same conclusion was reached by Godsil and Shawe-Taylor for a distance-regular graph around each of its vertices. Thus, our proof, which is of an algebraic nature, can also be seen as an alternative demonstration of Godsil and Shawe-Taylor’s theorem.

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1 Introduction

Distance-regularity around a vertex of a (regular) graph is the local analogue of distance-regularity. More precisely, a graph $\Gamma$ with vertex set $V$ is distance-regular around a vertex $u$ if the distance partition of $V$ induced by $u$ is regular (see, for instance, Brouwer, Cohen and Neumaier [3]). In [14], Godsil and Shawe-Taylor defined a distance-regularised graph as that being distance-regular around each of its vertices. The interest of these graphs relies on the fact that they are a common generalization of distance-regular graphs and generalized polygons. The authors of [14] used a combinatorial approach to prove that distance-regularised graphs are either distance-regular or distance-biregular. For some properties of the latter, see Delorme [7]. More recently, Fiol, Garriga and Yebra
introduced the concept of pseudo-distance-regularity around a vertex, as a natural generalization for not necessarily regular graphs, of distance regularity around a vertex. In this note, we prove that the same conclusion obtained in \cite{4} can be reached when $\Gamma$ is pseudo-distance-regularized; that is, pseudo-distance-regular graph around each of its vertices. In fact, this was already obtained in \cite{12}, but making strong use of the result in \cite{4}. Here we provide an independent direct proof, which is simple and of algebraic nature. Thus, our contribution can be seen as an alternative demonstration of Godsil and Shawe-Taylor’s theorem. Moreover, it turns out that the same conclusion of the theorem is obtained from the seemingly weaker condition of pseudo-distance-regularity.

## 2 Preliminaries

Let us first give some basic notation and results on which our proof is based. For more background on graph spectra, distance-regular and pseudo-distance-regular graphs see, for instance, \cite{4 2 3 4 5 6 8 9 11}.

Throughout this note, $\Gamma$ is a connected graph with vertex set $V = V(\Gamma)$, $n = |V|$ vertices, adjacency matrix $A$ and spectrum $\text{sp} \Gamma = \text{sp} A = \{\lambda_0^m, \lambda_1^m, \ldots, \lambda_d^m\}$, where the different eigenvalues of $\Gamma$ are in decreasing order, $\lambda_0 > \lambda_1 > \cdots > \lambda_d$, and the superscripts stand for their multiplicities $m_i = m(\lambda_i)$, $i = 0, 1, \ldots, d$. Then, as it is well known, $\lambda_0$, with multiplicity $m_0 = 1$, coincides with the spectral radius of $A$, and has a positive (column) eigenvector (the Perron vector) $\alpha$, which we normalize in such a way that $\|\alpha\|_2 = n$. Let $\text{dist}(u, v)$ represent the distance between the vertices $u, v \in V$. Then, $\Gamma_i(u) = \{v \mid \text{dist}(u, v) = i\}$, the eccentricity of a vertex $u$ is $\text{ecc}(u) = \max\{\text{dist}(u, v) \mid v \in V\}$, and the diameter of $\Gamma$ is $D = \max\{\text{ecc}(u) \mid u \in V\}$. For every $0 \leq i \leq D$, the distance-$i$ matrix $A_i$ has entries $(A_i)_{uv} = 1$ if $\text{dist}(u, v) = i$, and $(A_i)_{uv} = 0$, otherwise. We also use the weighted distance-$i$ matrix, defined as $A^\ast_i = A_i \circ J^\ast$, where $J^\ast = \alpha\alpha^\ast$ and ‘$\circ$’ stands for the Hadamard product; that is, $(A^\ast_i)_{uv} = \alpha_u \alpha_v$ if $\text{dist}(u, v) = i$, and $(A^\ast_i)_{uv} = 0$, otherwise.

### 2.1 Local spectrum and predistance polynomials

Given a graph $\Gamma$ with adjacency matrix $A$ and spectrum as above, its idempotents $E_i$ correspond to the orthogonal projections onto the eigenspaces $\ker(A - \lambda_i I)$, $i = 0, 1, \ldots, d$. Their entries $m_{uv}(\lambda_i) = (E_i)_{uv}$ are called the (crossed) uv-local multiplicities of $\lambda_i$. In particular, the diagonal elements $m_u(\lambda_i) = m_{uu}(\lambda_i)$ are the so-called u-local multiplicities of $\lambda_i$, because they satisfy properties similar to those of the (global) multiplicities $m(\lambda_i)$, but when $\Gamma$ is “seen” from the base vertex $u$. Indeed,

$$a^{(i)}_{uu} = (A^\ast_i)_{uu} = \sum_{i=0}^d m_u(\lambda_i)\lambda_i^i, \quad \text{and} \quad \sum_{u\in V} m_u(\lambda_i) = m(\lambda_i). \quad (1)$$

The u-local spectrum of $\Gamma$ is constituted by the eigenvalues of $A$, say $\mu_0, \mu_1, \ldots, \mu_{d_u}$, with nonzero u-local multiplicity. Then, it is known that the vector space spanned by the
vectors \((A^\ell)_{u, \ell \geq 0}\), \(\ell \geq 0\), (that is, the \(u\)-th columns of the matrices \(A^\ell\), \(\ell \geq 0\)) has dimension \(d_u\), and the eccentricity of \(u\) satisfies \(\text{ecc}(u) \leq d_u\). When \(\text{ecc}(u) = d_u\) we say that \(u\) is extremal (for more details, see [11]).

An orthogonal base for such a vector space is the following. The \(u\)-local predistance polynomials \(p_0^u, p_1^u, \ldots, p_d^u\), \(\deg p_i = i\), associated to a vertex \(u\) of \(\Gamma\) with nonzero local multiplicities \(m_u(\mu_i), i = 0, 1, \ldots, d_u\), are a sequence of orthogonal polynomials with respect to the scalar product

\[
\langle f, g \rangle_u = \langle f(A)g(A) \rangle_{uu} = \sum_{i=0}^{d_u} m_u(\mu_i) f(\mu_i) g(\mu_i) = \sum_{i=0}^{d} m_u(\lambda_i) f(\lambda_i) g(\lambda_i),
\]

normalized in such a way that \(\|p_i^u\|^2_u = \alpha_u^2 p_i^u(\lambda_0)\). We notice that in [11] the normalization condition was \(\|p_i^u\|^2_u = p_i^u(\lambda_0)\) but, although the theory remains unchanged, it seems more convenient to use the above. Then, in particular, \(p_0^u = \alpha_u^2\) and \(p_1^u = \frac{\alpha_u^2 \lambda_0}{\alpha_u} x\). Indeed, they are orthogonal since \(\langle 1, x \rangle_u = \sum_{i=0}^{d_u} m_u(\lambda_i) \lambda_i = 0\), and the normalization condition is fulfilled:

\[
\|\alpha_u^2\|^2_u = \alpha_u^4 \sum_{i=0}^{d} m_u(\lambda_i) = \alpha_u^4 \alpha_u^2 p_0^u(\lambda_0).
\]

\[
\|\frac{\alpha_u^2 \lambda_0}{\alpha_u} x\|^2_u = \frac{\alpha_u^4 \lambda_0^2}{\alpha_u^2} \sum_{i=0}^{d} m_u(\lambda_i) \lambda_i^2 = \frac{\alpha_u^4 \lambda_0^2}{\alpha_u} = \alpha_u^2 p_1^u(\lambda_0).
\]

If \(\Gamma\) is \(\delta\)-regular, then \(\alpha_u = 1\), \(\lambda_0 = \delta\), and we have \(p_0 = 1\) and \(p_1 = x\), which are the first distance polynomials for every distance-regular graph. More generally, and as expected, if \(\Gamma\) is distance-regular, the \(u\)-local predistance polynomials are independent of \(u\) and become the distance polynomials \(p_i, i = 0, 1, \ldots, D\), satisfying

\[
p_i(A) = A_i, \quad i = 0, 1, \ldots, D.
\]

As every sequence of orthogonal polynomials, the \(u\)-local predistance polynomials satisfy a three-term recurrence of the form

\[
xp_i^u = b_{i-1}^u p_{i-1}^u + a_i^u p_i^u + c_{i+1}^u p_{i+1}^u, \quad i = 0, 1, \ldots, d_u,
\]

where \(b_{i-1}^u = c_{d_u+1}^u = 0\), and the other numbers \(b_i^u, a_i^u,\) and \(c_{i+1}^u\) are the Fourier coefficients of \(xp_i^u\) in terms of \(p_{i-1}^u, p_i^u,\) and \(p_{i+1}^u\), respectively.

### 2.2 Pseudo-distance-regularity around a vertex

Given a graph \(\Gamma\) as above, consider the mapping \(\rho : V \rightarrow \mathbb{R}^n\) defined by \(\rho(u) = \alpha_u e_u\), where \(e_u\) is the coordinate vector. Note that, since \(\|\rho(u)\| = \alpha_u\), we can see \(\rho\) as a function which assigns weights to the vertices of \(\Gamma\). In doing so we “regularize” the graph, in the sense that the average weighted degree of each vertex \(u \in V\) becomes a constant:

\[
\delta_u^* = \frac{1}{\alpha_u} \sum_{v \in \Gamma(u)} \alpha_v = \lambda_0,
\]
where $\Gamma(u) = \Gamma_1(u)$. Using these weights, we consider the following concept. A partition $\mathcal{P}$ of the vertex set $V = V_1 \cup \cdots \cup V_m$ is called pseudo-regular (or pseudo-equitable) whenever the pseudo-intersection numbers

$$b_{ij}^*(u) = \frac{1}{\alpha_u} \sum_{v \in \Gamma(u) \cap V_j} \alpha_v, \quad i,j = 0,1,\ldots,m$$

do not depend on the chosen vertex $u \in V_i$, but only on the subsets $V_i$ and $V_j$. In this case, such numbers are simply written as $b_{ij}^*$, and the $m \times m$ matrix $B^* = (b_{ij}^*)$ is referred to as the pseudo-quotient matrix of $A$ with respect to the (pseudo-regular) partition $\mathcal{P}$. Pseudo-regular partitions were introduced by Fiol and Garriga [10], as a generalization of the so-called regular partitions, where the above numbers are defined by $b_{ij}^*(u) = |\Gamma(u) \cap V_j|$ for $u \in V_i$. A detailed study of regular partitions can be found in Godsil [12] and Godsil and McKay [13].

Let $u$ be a vertex of $\Gamma$ with eccentricity $\text{ecc}(u) = \varepsilon_u$. Then $\Gamma$ is said to be pseudo-distance-regular around $u$ (or $u$-local pseudo-distance-regular) if the distance-partition around $u$, that is $V = C_0 \cup C_1 \cup \cdots \cup C_{\varepsilon_u}$, where $C_i = \Gamma_i(u)$ for $i = 0,1,\ldots,\varepsilon_u$, is pseudo-regular. From the characteristics of the distance-partition, it is clear that its pseudo-quotient matrix is tridiagonal $B^* = (b_{ij}^*)$ with nonzero entries $c_i^* = b_{i-1,i}^*$, $a_i^* = b_{i+1,i}^*$ and $b_i^* = b_{i+1,i}^*$, $0 \leq i \leq \varepsilon_u$, with the convention $c_0^* = b_{\varepsilon_u}^* = 0$. By [11], notice that $a_i^* + b_i^* + c_i^* = \lambda_0$ for $i = 0,1,\ldots,\varepsilon_u$. These parameters are called the $u$-local (pseudo-)intersection numbers. In [11], it was shown that local pseudo-distance regularity is a generalization of distance-regularity around a vertex. Indeed, if $\Gamma$ is distance-regular around $u \in V$, with intersection numbers $a_i,b_i,c_i$, then the entries of the Perron vector $\alpha$ have a constant value, say $\alpha_i$, on each of the sets $\Gamma_i(u)$, $i = 0,1,\ldots,\varepsilon_u$, and $\Gamma$ turns out to be pseudo-distance-regular around $u$, with $u$-local intersection numbers

$$a_i^* = a_i, \quad b_i^* = \frac{\alpha_i+1}{\alpha_i} b_i, \quad c_i^* = \frac{\alpha_i-1}{\alpha_i} c_i, \quad i = 0,1,\ldots,\varepsilon_u.$$  

Conversely, when the eigenvector $\alpha$ of a pseudo-distance-regular graph $\Gamma$ exhibits such a regularity (which is the case for all $u$ when $\Gamma$ is regular or bipartite biregular), we have that $\Gamma$ is also distance-regular around $u$ with intersection parameters given again by (6).

As happens for distance-regular graphs, the existence of the so-called $u$-local distance polynomials, satisfying the “local version” of (2), guarantees that $\Gamma$ is pseudo-distance regular around $u$.

**Theorem 2.1** ([11]). Let $\Gamma$ be a graph having a vertex $u$ with eccentricity $\varepsilon_u$. Then, $\Gamma$ is pseudo-distance-regular around $u$ if and only if the $u$-local predistance polynomials satisfy

$$(p_i^n(A))_u = (A_i^*)_u, \quad i = 0,1,\ldots,\varepsilon_u.$$  

Moreover, if this is the case, $u$ is extremal, $\varepsilon_u = d_u$, and the $u$-local intersection numbers $a_i^*,b_i^*$ and $c_i^*$ coincide with the Fourier coefficients of the recurrence (3).

By this result, note that the $u$-local intersection numbers are univocally determined, through the $u$-local predistance polynomials, by the $u$-local spectrum.
3 The proof

Now we are ready to give the algebraic proof that a graph which is pseudo-distance-regular around each of its vertices is either distance-regular or distance-biregular.

Theorem 3.1. Every pseudo-distance-regularized graph $\Gamma$ is either distance-regular or distance-biregular.

Proof. Let $v, w$ be two vertices adjacent to a vertex $u$. Then, by Theorem 2.1,

$$\alpha_u \alpha_v = (p^v_1(A))_{uv} = \frac{\alpha_u^2 \lambda_0}{\delta_u} (A)_{uv} = \frac{\alpha_u^2 \lambda_0}{\delta_u} (A)_{uw} = (p^w_1(A))_{uw} = \alpha_u \alpha_w,$$

and we infer that $\alpha_u = \alpha_w$ since $\alpha_u > 0$. Hence, since $\Gamma$ is connected, all vertices at even (respectively odd) distance from $u$ have component $\alpha_u$ (respectively $\alpha_v$), and $\Gamma$ is either bipartite biregular if $\alpha_u \neq \alpha_v$, or regular otherwise.

Moreover, by using the orthogonal decomposition of $x^\ell$ in terms of the base $\{p^u_i\}_{0 \leq i \leq d_u}$, the number of $\ell$-walks between two adjacent vertices $u, v$ is

$$(A^\ell)_{uv} = \sum_{i=0}^{d_u} \frac{\langle x^\ell, p^u_i \rangle}{\|p^u_i\|_2} (p^v_i(A))_{uv} = \frac{\langle x^\ell, p^u_i \rangle}{\|p^u_i\|_2} \alpha_u \alpha_v = \frac{\alpha_v}{\alpha_u} \frac{1}{\lambda_0} \sum_{i=0}^{d_u} m_u(\lambda_i) \lambda_i^{\ell+1} \quad (7)$$

and, similarly,

$$(A^\ell)_{uv} = \frac{\alpha_u}{\alpha_v} \frac{1}{\lambda_0} \sum_{i=0}^{d_v} m_v(\lambda_i) \lambda_i^{\ell+1} \quad (8)$$

Hence, from (1), (7), and (8) we have that, given the local multiplicities of $u$, those of $v$ are uniquely determined from the system

$$\sum_{i=0}^{d_u} m_u(\lambda_i) = 1$$

$$\sum_{i=0}^{d_v} m_v(\lambda_i) \lambda_i^{r} = \frac{\alpha_u^2}{\alpha_v^2} \sum_{i=0}^{d_u} m_u(\lambda_i) \lambda_i^{r}, \quad r = 1, 2, \ldots, d,$$

with $d$ equations, $d$ unknowns $m_v(\lambda_i)$ ($i = 1, 2, \ldots, d$), and Vandermonde determinant. Then, we have only two possible cases:

1. If $\Gamma$ is regular, then $\alpha_u = \alpha_v = 1$ for every $u, v \in V$ and, hence, every vertex has the same local spectrum (this is known to be equivalent to say that $\Gamma$ is walk-regular [14]), and the same $u$-local distance-polynomials. Thus, $\Gamma$ is distance-regular.

2. If $\Gamma$ is bipartite $(\delta_1, \delta_2)$-biregular, then $\alpha_u = \sqrt{(\delta_1 + \delta_2)/2\delta_2}$ for every $u \in V_1$ and $\alpha_v = \sqrt{(\delta_1 + \delta_2)/2\delta_1}$ for every $v \in V_2$. In this case, every vertex of $V_1$ has the same local spectrum and the same holds for every vertex in $V_2$ (in this case, we could say that $\Gamma$ is walk-biregular). Thus, the sequence of local $u$-predistance polynomials only depend on the partite set where $u$ belongs to and, consequently, $\Gamma$ is distance-biregular.
This completes the proof. □

As mentioned above, notice that, since every regular or biregular graph which is pseudo-distance-regular around a vertex is also distance-regular around such a vertex, Theorem 3.1 implies the result of Godsil and Shawe-Taylor in [14].

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