A note on coupled constraint Nash games

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Abstract

In this note we are interested in a relevant generalized Nash equilibrium problem, which was proposed by Rosen in 1965. An existence result is established in the general setting of quasiconvexity, which is independent from the one given by Aussel and Dutta in 2008.

Keywords: Generalized Nash Equilibrium; Generalized Convexity; Variational Inequalities

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1 Introduction

Let \( N \) be a finite and non-empty set, representing a set of players. Let us assume that each player \( \nu \in N \) chooses a strategy \( x^\nu \) in a strategy set \( K_\nu \) of \( \mathbb{R}^{n_\nu} \). Consider the sets

\[
K = \prod_{\nu \in N} K_\nu, \quad \mathbb{R}^n = \prod_{\nu \in N} \mathbb{R}^{n_\nu}, \quad K_{-\nu} = \prod_{\mu \in N \setminus \{\nu\}} K_\mu, \quad \mathbb{R}^{-n_\nu} = \prod_{\mu \in N \setminus \{\nu\}} \mathbb{R}^{n_\mu}
\]

Also, we can write \( x = (x^\nu, x^{-\nu}) \in K \) in order to emphasize the strategy of player \( \nu \), \( x^\nu \in K_\nu \), and the strategy of the rival players \( x^{-\nu} \in K_{-\nu} \).

For each player \( \nu \), given the rivals’ strategy \( x^{-\nu} \), the player \( \nu \) chooses a strategy \( x^\nu \) such that it solves the following optimization problem

\[
\text{NEP}_\nu(x^{-\nu}) : \min_{K_\nu} \theta_\nu(\cdot, x^{-\nu}),
\]

where \( \theta_\nu : \mathbb{R}^n \to \mathbb{R} \) and \( \theta_\nu(x^\nu, x^{-\nu}) \) denotes the loss that player \( \nu \) suffers when his own strategy is \( x^\nu \) and the rivals’ strategies are \( x^{-\nu} \). Thus, a Nash equilibrium [26], associated to the loss functions \( (\theta_\nu)_\nu \), and the constraint sets \( (K_\nu)_\nu \), is a vector \( \hat{x} \in K \) such that \( \hat{x} \) solves \( \text{NEP}_\nu(\hat{x}^{-\nu}) \), for any \( \nu \).

It is well known [18, Theorem 1] in the literature that a Nash equilibrium exists when, for each \( \nu \in N \),

- the set \( K_\nu \) is compact convex and non-empty,
- the function \( \theta_\nu \) is continuous and is quasiconvex on its player’s variable.

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Since then, many authors extended this result in different ways, see for instance \[9, 18, 28, 30, 32\], where continuity assumption is relaxed, or \[22, 29\], where the quasiconvexity is weakened.

On the other hand, in a Generalized Nash Equilibrium Problem (GNEP), each player \(\nu\) possesses a constraint map \(X_\nu : K_{-\nu} \Rightarrow K_{\nu}\), so their strategy must belong to the set \(X_\nu(x^{-\nu}) \subset K_{\nu}\), where the sets \(X_\nu(x^{-\nu})\) clearly depend on the rival players’ strategies. The aim of player \(\nu\), given the rival players’ strategies \(x^{-\nu}\), is to choose a strategy \(x^\nu\) that solves the minimization problem

\[
\text{GNEP}_\nu(x^{-\nu}) : \min_{X_\nu(x^{-\nu})} \theta_\nu(\cdot, x^{-\nu}).
\]  

(2)

A Generalized Nash Equilibrium is a vector \(\hat{x} \in K\) such that \(\hat{x}^\nu\) solves GNEP\(_\nu(\hat{x}^{-\nu})\), for any \(\nu\). Arrow and Debreu [3, Lemma 2.5] proved the existence of Generalized Nash Equilibria when for all \(\nu \in N\),

- the set \(K_\nu\) is compact convex and non-empty,
- the function \(\theta_\nu\) is continuous and quasiconvex on its player’s variable,
- the map \(X_\nu\) is lower semicontinuous with closed graph and convex and non-empty values.

Clearly this existence result generalizes the first one, by considering \(X_\nu\) as the constant map \(X_\nu(x^{-\nu}) = K_{\nu}\). Since then, many authors dealt with this kind of problems, see for instance \[6, 8, 13, 17, 21, 32, 33\].

An important instance of GNEPs was presented by Rosen in [31], which is known as the jointly convex GNEP. A jointly convex GNEP is a GNEP in which the graph of each map constraint \(X_\nu\) is a certain convex, compact and non-empty subset of \(\mathbb{R}^n\), denoted by \(X\). In other words

\[
X_\nu(x^{-\nu}) = \{ x^\nu \in \mathbb{R}^n_{\nu} : (x^\nu, x^{-\nu}) \in X \},
\]

When this is the case, \(K_\nu\) is the projection of \(X\) onto \(\mathbb{R}^n_{\nu}\). This kind of GNEP have recently begun to gain more and more attention as it models problems from economics such as electricity markets, environmental games and bilateral exchange of bads, see \[11, 12, 23, 24, 34\].

On the other hand, jointly convex GNEPs can be reformulated as variational inequalities [19] when assuming pseudoconvexity and differentiability. An extension of this reformulation was presented by Aussel and Dutta in [3], using normal cones, under quasiconvexity and continuity assumptions. However, their main result concerning the existence of equilibria was established under semi strict quasiconvexity.

In this note, our main result consists to assume only quasiconvexity in order to show the existence of equilibria for the jointly convex GNEP. In Section 2 we recall some definitions of generalized convexity, continuity for functions and continuity for set-valued maps. Additionally some preliminary results are established concerning closedness and lower semicontinuity for special maps. Finally, in Section 3 we reformulate jointly convex GNEPs as variational inequality problems and show our main result.
2 Definitions, Notations and Preliminary Results

Let $U, V$ be non-empty sets. A multi-valued or set-valued map $T : U \rightrightarrows V$ is an application $T : U \rightarrow \mathcal{P}(V)$, that is, for $u \in U$, $T(u) \subseteq V$. The graph of $T$ is defined as

$$\text{gra}(T) = \{(u, v) \in U \times V : v \in T(u)\}.$$ 

Let $X, Y$ be two real Banach spaces. Recall that a set-valued map $T : X \rightrightarrows Y$ is

- **lower semicontinuous (lsc)** at $x_0 \in X$ when, for any open set $V$ such $T(x_0) \cap V \neq \emptyset$, there exists a neighborhood $U$ of $x_0$ such that $T(x) \cap V \neq \emptyset$, for every $x \in U$;
- **upper semicontinuous (usc)** at $x_0 \in X$ when, for any neighborhood $V$ of $T(x_0)$, there exists a neighborhood $U$ of $x_0$ such that $T(x) \subseteq V$, for all $x \in U$;
- **lower (respectively, upper) semicontinuous** when it is lower (resp. upper) semicontinuous at every $x_0 \in X$;
- **closed** when its graph is a closed subset of $X \times Y$.

Moreover, $T : X \rightrightarrows Y$ is lower semicontinuous at $x_0$ if, and only if, $T$ is sequentially lower semicontinuous at $x_0$, that is, for any sequence $(x_n)_{n \in \mathbb{N}}$ of $X$ converging to $x_0$ and any $y_0 \in T(x_0)$, there exists a sequence $(y_n)_{n \in \mathbb{N}}$ of $Y$ converging to $y_0$ such that $y_n \in T(x_n)$, for any $n$. The interested reader can find a proof of this fact in [20, Proposition 2.5.6].

Let $X$ be a Banach space, $X^*$ its topological dual and $\langle \cdot, \cdot \rangle$ the duality pairing. Given $A$ a subset of $X$, the closure, the interior and the convex hull of $A$ are denoted by $\overline{A}$, int$(A)$ and conv$(A)$, respectively. In addition, $-A$ denotes the set $-A = \{x \in X : -x \in A\}$. The polar cone of $A$ is the set

$$A^\circ := \{x^* \in X^* : \langle x^*, x \rangle \leq 0, \forall x \in A\},$$

which is a convex cone on $X^*$. The normal cone $N_A$ of $A$ at $x \in X$ is defined as

$$N_A(x) := \{x^* \in X^* : \langle x^*, y-x \rangle \leq 0, \forall y \in A\}.$$ 

We will consider $N_A(x) = X^*$, whenever $A = \emptyset$. It is common in this definition to impose $A$ convex and $x \in A$, however, we won’t consider such conditions in this work.

See [10] for another instance where such conditions are not considered. Note that, for all $x \in X$,

$$N_A(x) = (A-x)^\circ = N_A(x) = N_{\text{conv}(A)}(x).$$

The following lemma will be used in the proof of Proposition 2.4

**Lemma 2.1.** Let $A$ be a subset of a real Banach space $X$. If $\text{int}(A) \neq \emptyset$, then $A^\circ$ is a pointed cone, i.e. $A^\circ \cap -A^\circ = \{0\}$. Moreover, if $x \in \text{int}(A)$ and $x^* \in A^\circ$, $x^* \neq 0$, then $\langle x^*, x \rangle < 0$.

**Proof.** Note that $A^\circ \cap -A^\circ = A^\perp = \{x^* \in X^* : \langle x, x^* \rangle = 0, \forall x \in A\}$. Since $\text{int}(A) \neq \emptyset$, span$(A) = X$, so

$$A^\circ \cap -A^\circ = A^\perp = \text{span}(A)^\perp = X^\perp = \{0\}.$$
To conclude the proof, let \( x \in \text{int}(A) \), \( x^* \in A^- \), \( x^* \neq 0 \) and assume that \( \langle x, x^* \rangle \geq 0 \), hence \( \langle x, x^* \rangle = 0 \). Since \( x \in \text{int}(A) \), for some \( \varepsilon > 0 \) and all \( v \in B(0, 1) \), \( x + \varepsilon v \in A \), which implies \( \langle x + \varepsilon v, x^* \rangle \leq 0 \). Therefore \( \langle v, x^* \rangle \leq 0 \), for all \( v \in B(0, 1) \), which in turn implies that \( \langle v, x^* \rangle = 0 \), for all \( v \in B(0, 1) \). This is a contradiction, since \( x^* \neq 0 \).

Let us now recall some classical definitions of generalized convexity. Given a Banach space \( X \), a real-valued function \( h : X \to \mathbb{R} \) is

- **quasiconvex** if, for any \( x, y \in X \) and \( t \in [0, 1] \),
  \[ h(tx + (1 - t)y) \leq \max\{h(x), h(y)\}; \]

- **semistrictly quasiconvex** if it is quasiconvex and, for any \( x, y \in X \) and \( t \in ]0, 1[ \),
  \[ h(x) \neq h(y) \Rightarrow h(tx + (1 - t)y) < \max\{h(x), h(y)\}. \]

Our definition of semistrictly quasiconvex function was taken from [2] Chapter 5], and it is equivalent to the definition of pseudoconvex function given in [27] Definition 3.1]. Some authors may not include the quasiconvex condition in this definition, see for instance [2] Chapter 4].

It is well known that a function \( h \) is quasiconvex if, and only if, the **sublevel sets** \( S_{h, \lambda} = \{ y \in X : h(y) \leq \lambda \} \) are convex, for all \( \lambda \in \mathbb{R} \), and also if and only if the **strict sublevel sets** \( S_{h, \lambda}^\circ = \{ y \in X : h(y) < \lambda \} \) are convex, for all \( \lambda \in \mathbb{R} \).

Let \( X \) be a Banach space. Given \( x \in X \), \( \mathcal{V}_x \) will denote an open neighbourhood of \( x \) in \( X \). A function \( h : X \to \mathbb{R} \) is called:

- **lower semicontinuous**, if for all \( x \in X \) and \( \lambda \in \mathbb{R} \) with \( \lambda < h(x) \), there exists \( \mathcal{V}_x \) such that \( \lambda < h(x') \), for all \( x' \in \mathcal{V}_x \);

- **upper semicontinuous**, if for all \( x \in X \) and \( \lambda \in \mathbb{R} \) with \( \lambda > h(x) \), there exists \( \mathcal{V}_x \) such that \( \lambda > h(x') \), for all \( x' \in \mathcal{V}_x \);

- **lower pseudocontinuous** [25], if for all \( x, y \in X \) with \( h(y) < h(x) \), there exists \( \mathcal{V}_x \) such that \( h(y) < h(x') \), for all \( x' \in \mathcal{V}_x \);

- **upper pseudocontinuous**, if for all \( x, y \in X \) with \( h(y) > h(x) \), there exists \( \mathcal{V}_x \) such that \( h(y) > h(x') \), for all \( x' \in \mathcal{V}_x \).

It is not difficult to prove that if \( h \) is lower semicontinuous then it is lower pseudocontinuous. On the other hand, it is well known that \( h \) is lower semicontinuous if, and only if, \( S_{h, \lambda} \) is closed, for all \( \lambda \in \mathbb{R} \). In the same way, considering for \( x \in X \) the sets \( S_h(x) := S_{h, h(x)} \), \( S_h^\circ(x) := S_{h, h(x)}^\circ \), \( h \) is lower pseudocontinuous if, and only if, \( S_h(x) \) is closed, for all \( x \in X \). Also, \( h \) is upper pseudocontinuous if, and only if, \( S_h^\circ(x) \) is open, for all \( x \in X \).

Let \( X, Y \) be two real Banach spaces and \( f : X \times Y \to \mathbb{R} \) be a function. We define the following set-valued maps \( L_f^\leq, L_f : X \times Y \Rightarrow X \) as
\[
L_f^\leq(x, y) := S_{f(x, y)}^\leq(x) \quad \text{and} \quad L_f(x, y) := S_{f(x, y)}(x).
\]
and the set-valued maps \( N_f^\leq, N_f : X \times Y \to X^* \)
\[
N_f^\leq(x, y) := N_{L_f^\leq(x, y)}(x) \quad \text{and} \quad N_f(x, y) := N_{L_f(x, y)}(x).
\]
It is important to note that \( L_f^\leq (x, y) \) is not merely the projection onto \( X \) of \( S_f^\leq (x, y) \). Consider for instance the function \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, f(x, y) = x^2 + y^2 \), and take \((0, 1)\). It is not difficult to see that \( L_f^\leq (0, 1) = \emptyset \), but the projection of \( S_f^\leq (0, 1) \) onto \( \mathbb{R} \) is the interval \([-1, 1]\).

In general, we have the following relation between \( L_f^\leq (x, y) \) and \( S_f^\leq (x, y) \):

\[
L_f^\leq (x, y) \times \{y\} = S_{f(x,y)}^\leq \cap (X \times \{y\}).
\]

The following result establishes the lower semicontinuity of \( L_f^\leq \), which is inspired by [1].

**Lemma 2.2.** Let \( X, Y \) be two real Banach spaces and \( f : X \times Y \to \mathbb{R} \) be a function. If \( f \) is lower pseudoinverse on its first argument and continuous on its second one, then the map \( L_f^\leq \) is lower semicontinuous.

**Proof.** Assume, on the contrary, that \( L_f^\leq \) is not lower semicontinuous at some \((x_0, y_0) \in X \times Y\). Then, there exists an open set \( V \subset X \) with \( L_f^\leq (x_0, y_0) \cap V \neq \emptyset \) and, for every \( n, m \in \mathbb{N} \), there exist \( x_n \in B(x_0, 1/n), y_m \in B(y_0, 1/m) \), such that

\[
L_f^\leq (x_n, y_m) \cap V = \emptyset.
\]

This means that \( f(z, y_m) \geq f(x_n, y_m) \), for all \( z \in V \) and \( n, m \in \mathbb{N} \). Since \( f \) is continuous on its second argument, taking limit when \( m \to \infty \), we obtain \( f(z, y_0) \geq f(x_n, y_0) \), that is, \( x_n \in L_f(z, y_0) \), for all \( n \in \mathbb{N} \). We now take limit when \( n \to \infty \) to conclude \( x_0 \in L_f(z, y_0) \), since \( L_f(z, y_0) \) is closed, as \( f \) is lower pseudoinverse on its first argument. Therefore \( f(x_0, y_0) \leq f(z, y_0) \), for all \( z \in V \). The lemma follows by observing that, \( f(z_0, y_0) < f(x_0, y_0) \), for some \( z_0 \in V \).  

**Lemma 2.3.** Let \( X, Y \) be two real Banach spaces and \( T : X \times Y \rightrightarrows X \) be a set-valued map. If \( T \) is lower semicontinuous, then the set-valued map \( \mathcal{N}_T : X \times Y \rightrightarrows X^* \) defined as

\[
\mathcal{N}_T(x, y) := \mathcal{N}_{T(x,y)}(x)
\]

is closed.

**Proof.** Let \((x_n, y_n, x_n^*)_{n \in \mathbb{N}}\) be a sequence in the graph of \( \mathcal{N}_T \) converging to \((x_0, y_0, x_0^*)\). We aim to show that \( x_0^* \in \mathcal{N}_{T(x_0,y_0)}(y_0) \). If \( T(x_0, y_0) = \emptyset \), there is nothing to prove. Now, assume that \( T(x_0, y_0) \neq \emptyset \) and take any \( x \in T(x_0, y_0) \). By lower semicontinuity of \( T \) there exists a sequence \((z_n)_{n \in \mathbb{N}}\) converging to \( x \) such that \( z_n \in T(x_n, y_n) \), for all \( n \in \mathbb{N} \). Therefore

\[
\langle x_n^*, z_n - x_n \rangle \leq 0.
\]

The lemma follows by letting \( n \) tend to \( \infty \), to obtain \( \langle x_0^*, x - x_0 \rangle \leq 0 \).

**Proposition 2.4.** Let \( X, Y \) be two real Banach spaces and \( f : X \times Y \to \mathbb{R} \) be a function.

1. If \( f \) is lower pseudoinverse on its first argument and continuous on its second one, then the map \( \mathcal{N}_f^\leq \) is closed.

2. If \( f \) is upper pseudoinverse on its first argument then \( L_f^\geq \) is open, for all \((x, y) \in X \times Y \). In particular, if \( w^* \in \mathcal{N}_f^\geq (x, y) \), \( w^* \neq 0 \), then \( \langle w^*, z - x \rangle < 0 \), for all \( z \in L_f^\geq (x, y) \).
If $f$ is upper pseudocontinuous and quasiconvex, both on its first argument, then $\mathcal{N}_f(x, y) \neq \{0\}$, for all $(x, y) \in X \times Y$.

Proof. 1. Closedness of $\mathcal{N}_f$ follows from Lemmas 2.2 and 2.3 and the fact that $\mathcal{N}_f(x, y) = L_f(x, y)$, for any $(x, y) \in X \times Y$.

2. The set $L_f(x, y)$ is open, since $L_f(x, y) = S_f(x, y)$ and $f(\cdot, y)$ is upper pseudocontinuous. The second claim follows from the fact that $\mathcal{N}_f(x, y) = (L_f(x, y) - x)^-$ and Lemma 2.1.

3. Take $(x, y) \in X \times Y$. If $x \in \arg \min_X f(\cdot, y)$ then $L_f(x, y) = \emptyset$ and the claim trivially follows. Now assume that $x \notin \arg \min_X f(\cdot, y)$. The quasiconvexity of $f(\cdot, y)$ implies $S_f(x, y) = L_f(x, y)$ is non-empty and convex. Moreover $x \notin L_f(x, y)$ so, using the Hahn-Banach separation theorem, there exists $x^* \in X^*$, $x^* \neq 0$ such that

$$\langle x^*, z \rangle \leq \langle x^*, x \rangle, \quad \forall z \in L_f(x, y).$$

This implies that $x^* \in \mathcal{N}_f(x, y)$.

Remark 2.5. The pseudocontinuity condition on $f$ in item 3 can be dropped if $X$ is finite-dimensional.

The following example shows that continuity and convexity only in the first argument of $f$ is not enough to guarantee the closedness of $\mathcal{N}_f$, neither the lower semicontinuity of $L_f$.

Example 2.6. Define $\Theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as

$$\Theta(x, y) = \begin{cases} x, & \text{when } y \neq 1, \\ -x, & \text{when } y = 1. \end{cases}$$

It is not difficult to see that $\Theta$ is continuous and convex with respect to its first argument. Moreover, for each $(x, y) \in \mathbb{R}^2$,

$$L_{\Theta}(x, y) = \begin{cases} ]-\infty, x[ & \text{when } y \neq 1, \\ [x, +\infty[ & \text{when } y = 1. \end{cases}$$

Clearly the map $L_{\Theta}$ is not lower semicontinuous at $(0, 1)$. In addition, the strict normal operator associated to $\Theta$ is given by

$$\mathcal{N}_{\Theta}(x, y) = \begin{cases} [0, +\infty[ & \text{when } y \neq 1 \\ ]-\infty, 0[ & \text{when } y = 1. \end{cases}$$

which is not closed.

3 GNEPs and Variational Inequality Problems

Throughout this section, we will consider a jointly convex GNEP, given by a convex set $X \subset \mathbb{R}^n$. We will also consider the loss functions $\theta_\nu$ as bifunctions from $\mathbb{R}^{n_\nu} \times \mathbb{R}^{-n_\nu}$ to $\mathbb{R}$.
For each \( \nu \in N \) and \( x \in \mathbb{R}^n \), consider the set
\[
D_\nu(x) = \text{conv}\left(\mathcal{N}_{\theta_\nu}^<(x^\nu, x^{-\nu}) \cap S_\nu[0, 1]\right),
\]
where \( S_\nu[0, 1] \) is the unit sphere in \( \mathbb{R}^{n_\nu} \). This allows us to define the set-valued map
\[
T : \mathbb{R}^n \Rightarrow \mathbb{R}^n
\]
as
\[
T(x) := \prod_{\nu \in N} D_\nu(x).
\]
Clearly, for all \( x \in \mathbb{R}^n \), \( T(x) \) is compact and convex, as each \( D_\nu(x) \) is compact and convex as well. However, the operator \( T \) may not be a closed operator, as shown by the following example.

**Example 3.1.** Consider a two player NEP where each player \( \nu \in \{1, 2\} \) has a loss function defined as in Example 2.6, namely
\[
\theta_\nu(x^\nu, x^{-\nu}) = \Theta(x^\nu, x^{-\nu}) = \begin{cases}
  x^\nu, & \text{when } x^{-\nu} \neq 1, \\
  -x^\nu, & \text{when } x^{-\nu} = 1.
\end{cases}
\]
Therefore
\[
\mathcal{N}_{\theta_\nu}^<(x) = \begin{cases}
  [0, +\infty[, & \text{when } x^{-\nu} \neq 1, \\
  [-\infty, 0], & \text{when } x^{-\nu} = 1,
\end{cases}
\]
and
\[
D_\nu(x) = \begin{cases}
  \{1\}, & \text{when } x^{-\nu} \neq 1, \\
  \{-1\}, & \text{when } x^{-\nu} = 1,
\end{cases}
\]
which is not closed, so neither is \( T \).

In view of the previous example, we need additional conditions to guarantee the closedness of \( T \).

**Proposition 3.2.** Let \( T : \mathbb{R}^n \Rightarrow \mathbb{R}^n \) be defined as in (5).

1. If each loss function \( \theta_\nu \) is lower pseudocontinuous on its own player’s variable and continuous with respect to its rivals’ variables, then \( T \) is closed.

2. If each loss function \( \theta_\nu \) is quasiconvex with respect to its own player’s variable then \( T \) is non-empty valued.

**Proof.**

1. From Proposition 2.4 item 1, we have that each map \( \mathcal{N}_{\theta_\nu}^< \) is closed with convex values. This implies that the map \( D_\nu \) is closed with convex values. The result follows from the fact that the Cartesian product of closed maps is a closed map.

2. In view of Remark 2.5 we can use Proposition 2.4 item 3, to conclude that the set \( \mathcal{N}_{\theta_\nu}^<(x^\nu, x^{-\nu}) \setminus \{0\} \) is non-empty, for all \((x^\nu, x^{-\nu}) \in \mathbb{R}^n\). Thus, the map \( D_\nu \) is non-empty valued and so is \( T \). \( \square \)

Let \( S(T, \mathcal{X}) \) be the solution set of the Variational Inequality Problem associated to the map \( T \) (given as in (5)) and the set \( \mathcal{X} \), namely
\[
S(T, \mathcal{X}) = \{ x \in \mathcal{X} : \exists w_0 \in T(x), \langle w_0, y - x \rangle \geq 0, \forall y \in \mathcal{X} \}
\]
The following result establishes a link between the jointly convex GNEP and its associated variational inequality problem.
Proposition 3.3. Assume that $\mathcal{X}$ is non-empty. If every loss function $\theta_\nu$ is upper pseudocontinuous on its own player’s variable, then every point in $S(T, \mathcal{X})$ is a generalized Nash equilibrium.

Proof. Take $x \in S(T, \mathcal{X})$, so there exists $w_0 \in T(x)$ such that
\[
\langle w_0, y - x \rangle \geq 0, \quad \forall y \in \mathcal{X}.
\] (6)
Assume that $x$ is not a solution of the jointly convex GNEP. Then there exist a player $\nu$, and $z_\nu \in \mathcal{X}_\nu(x^{-\nu})$, such that
\[
\theta_\nu(z_\nu, x^{-\nu}) < \theta_\nu(x_\nu, x^{-\nu}),
\]
that is $z_\nu \in L_<^{\theta_\nu}(x_\nu, x^{-\nu})$ and $y = (z_\nu, x^{-\nu}) \in \mathcal{X}$. In view of (6),
\[
\langle w_\nu^0, z_\nu - x_\nu \rangle = \langle w_0, y - x \rangle \geq 0.
\]
This in turn implies that $w_\nu^0 = 0$, by Proposition 2.4, item 2, and the fact that $w_\nu^0 \in N_\nu^{\theta_\nu}(x_\nu, x^{-\nu})$.

On the other hand, since $w_0^\nu \in D_\nu(x)$, there exist $w_1^\nu, \ldots, w_p^\nu \in N_\nu^{\theta_\nu}(x_\nu, x^{-\nu}) \cap S_\nu[0,1], t_1, \ldots, t_p \geq 0, \sum_{i=1}^p t_i = 1$, such that $0 = w_0^\nu = \sum_{i=1}^p t_i w_i^\nu$. Take $i_0$ such that $t_{i_0} > 0$, so we have
\[
0 = \sum_{i \neq i_0} t_i w_i^\nu + t_{i_0} w_{i_0}^\nu
\]
and this implies
\[
-w_{i_0}^\nu = \sum_{i \neq i_0} \frac{t_i}{t_{i_0}} w_i^\nu.
\]
As $N_\nu^{\theta_\nu}(x)$ is a convex cone, $-w_{i_0}^\nu \in N_\nu^{\theta_\nu}(x)$, hence
\[
w_{i_0}^\nu \in N_\nu^{\theta_\nu}(x) \cap -N_\nu^{\theta_\nu}(x).
\]
However, since $L_<^{\theta_\nu}(x)$ is open, Lemma 2.1 implies that $w_{i_0} = 0$, a contradiction. The proposition follows. 

Proposition 3.3 is strongly related to [5, Theorem 3.1]. However, the authors considered continuity instead of upper pseudocontinuity.

Proposition 3.4. Assume that $\mathcal{X}$ is convex, compact and non-empty and let $T$ be defined as in (5). If for all $\nu \in N$ the following hold

1. the function $\theta_\nu$ is lower pseudocontinuous on its own player’s variable and continuous with respect to its rivals’ variables,

2. the function $\theta_\nu$ is quasiconvex with respect to its own player’s variable;

then $S(T, \mathcal{X})$ is non-empty.

Proof. Thanks to Proposition 3.2, the map $T$ is closed with convex, compact and nonempty values. Thus, $T$ is upper semicontinuous. Finally, the result follows from [4, Theorem 9.9].
Finally, we are ready for our main result, which establishes the existence of solution for jointly convex GNEPs.

**Theorem 3.5.** Assume that \( \mathcal{X} \) is convex, compact and non-empty. If for all \( \nu \in \mathcal{N} \) the following hold

1. the function \( \theta_\nu \) is pseudocontinuous on its own player’s variable and continuous on its rivals’ variables,

2. the function \( \theta_\nu \) is quasiconvex on its own player’s variable,

then there exists a generalized Nash equilibrium.

**Proof.** It is a consequence of Propositions 3.3 and 3.4.

**Remark 3.6.** Theorem 3.5 is not a consequence of [3, Theorem 2]. In fact, Theorem 3.5 improves [5, Theorem 4.2], [19, Theorem 2.1], [31, Theorem 1] and [18, Theorem 1].

The following example shows that we cannot drop the continuity of each loss function with respect to its rivals’ variables.

**Example 3.7.** Given the functions \( \theta_1, \theta_2 : \mathbb{R}^2 \rightarrow \mathbb{R} \) defined as

\[
\theta_1(x^1, x^2) := \begin{cases} 
(x^1 - \frac{x^2}{2})^2, & x^2 \in Q \\
(2x^1 - x^2)^2, & x^2 \notin Q
\end{cases}
\]

\[
\theta_2(x^1, x^2) := \begin{cases} 
(x^2 - x^1)^2, & x^1 \notin Q \\
(2x^2 - x^1)^2, & x^1 \in Q
\end{cases}
\]

Clearly, each loss function is convex and continuous with respect to its own variable. For \( X = [0, 1]^2 \), the GNEP reduces to the classic Nash equilibrium problem. Furthermore, it is no difficult to see that there is not solution to this GNEP.

We must note that the previous example directly contradicts [27, Corollary 4.3].

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