INEQUALITY OF NOETHER TYPE FOR SMOOTH MINIMAL 3-FOLDS OF GENERAL TYPE

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Abstract. Let $X$ be a smooth minimal projective 3-fold of general type. We give a sharp inequality between canonical volume $K_X^3$ and $\chi(\omega_X)$:

$$K_X^3 \geq \frac{4}{3} \chi(\omega_X) - 2.$$ 

1. Introduction

Throughout this paper, we work over the complex number field $\mathbb{C}$. Let $C$ be a nonsingular projective curve, we have the equality

$$\deg(K_C) = 2g(C) - 2,$$

where $g(C)$ is the geometric genus of $C$.

On a smooth minimal projective surface of general type $S$, we have the Noether inequality (see [12]):

$$K_S^2 \geq 2\chi(O_S) - 6,$$

where $K_S$ is the canonical divisor of $S$ and $\chi(O_S)$ is the Euler characteristic of $S$. Together with the Bogomolov-Miyaoka-Yau inequality (cf. [11], [15]): $K_S^2 \leq 9\chi(O_S)$, they have ever played very important roles in surface theory.

Let $X$ be a nonsingular projective 3-fold of general type, there have been many papers which give effective Noether type of inequalities (see [1], [2], [5], [6], [7], [10], [13] and [14] etc.).

Now let’s restrict our interest to smooth minimal projective 3-fold of general type. The known result is as follows: D. K. Shin (1997, [14]) gave a effective Noether type inequality for smooth minimal 3-fold of general type. Meng Chen and Christopher D. Hacon (2008, [7]) improved the main theorem of [14] and gave an effective Noether type inequality.

The main result in this paper is the following:

Main Theorem. Let $X$ be a nonsingular projective minimal 3-fold of general type. Then the following inequality holds:

$$K_X^3 \geq \frac{4}{3} \chi(\omega_X) - 2.$$
Remark 1.1. M. Kobayashi (1992, [10]) found an infinite number of examples (Proposition 3.2 in [10]) satisfying the equality:

\[ K^3_X = \frac{4}{3} \chi(\omega_X) - 2. \]

According to M. Kobayashi’s interesting examples, the inequality in Main Theorem is sharp.

Remark 1.2. According to Jungkai A. Chen and Meng Chen’s recent results ([see [3]]), the method in this paper can be easily generalized to Gorenstein minimal 3-fold of general type. The result is the following:

Theorem 1.3. Let \( X \) be a Gorenstein minimal 3-fold of general type, we have the following inequality:

\[ K^3_X \geq \frac{4}{3} \chi(\omega_X) - 2. \]

2. Preliminaries

2.1. Conventions. Let \( X \) be a nonsingular projective variety of dimension \( d \). A \( \mathbb{Q} \)-divisor \( D \) is called nef (or numerically effective) if \( D \cdot C \geq 0 \) for any effective curve \( C \subset X \). A nef divisor \( D \) is called big if \( D^d > 0 \). Denote by \( K_X \) the canonical divisor of \( X \).

The symbols \( \sim \), \( \equiv \) and \( =_{\mathbb{Q}} \) respectively stands for linear, numerical and \( \mathbb{Q} \)-linear equivalences.

Let \( S \) be a smooth projective surface of general type, assume its minimal model is \( S_0 \), \( S \) is called a surface of type \((a,b)\) if \( K^2_{S_0} = a \), \( p_g(S_0) = b \).

2.2. Set up for canonical maps. Let \( X \) be a smooth minimal projective 3-fold of general type with \( p_g(X) \geq 2 \). Denote by \( \phi_1 \) the canonical map which is usually a rational map. Take the birational modification \( \pi : X' \to X \), following Hironaka, such that

1. \( X' \) is smooth;
2. the movable part of \( |K_{X'}| \) is base point free;
3. there exists a canonical divisor \( K_X \) such that \( \pi^*(K_X) \) has support with only simple normal crossings.

Denote by \( g \) the composition \( \phi_1 \circ \pi \). So \( g : X' \to W' \subseteq \mathbb{P}^{p_g(X)-1} \) is a morphism. Let \( g : X' \to B \to W' \) be the Stein factorization of \( g \). We can write

\[ K_{X'} = \pi^*(K_X) + E = M + Z \]

where \( M \) is the movable part of \( |K_{X'}| \), \( Z \) is the fixed part and \( E \) is an effective divisor which is a sum of distinct exceptional divisors.

If \( \dim \phi_1(X) < 3 \), \( f \) is called the induced fibration of \( \phi_1 \).

If \( \dim \phi_1(X) = 2 \), a general fibre of \( f \) is a smooth projective curve \( C \) of genus \( g \geq 2 \).
If \( \dim \phi_1(X) = 1 \), a general fibre \( S \) of \( f \) is a smooth projective surface of general type. Denote by \( S_0 \) the minimal model of \( S \) and by \( \sigma : F \to F_0 \) the contraction map. Denote by \( b \) the genus of the base curve \( B \).

### 3. Several Lemmas

**Lemma 3.1.** Let \( S \) be a smooth projective surface of general type and \( L \) a nef divisor on \( S \). The following holds.

1. Suppose that \( |L| \) gives a non-birational, generically finite map onto its image. Then \( L^2 \geq 2h^0(S, \mathcal{O}_S(L)) - 4 \).
2. Suppose that there exists a linear subsystem \( \Lambda \subset |L| \) such that \( \Lambda \) defines a generically finite map of degree \( d \) onto its image. Then \( L^2 \geq d[\dim \mathbb{C}\Lambda - 1] \) where \( \dim \mathbb{C}\Lambda \) denotes the projective dimension of \( \Lambda \).

**Proof.** See [Lemma 2.2, [5]].

**Lemma 3.2.** Let \( S \) be a smooth minimal projective surface of general type. The following holds:

1. \( |mK_S| \) is base point free for all \( m \geq 4 \);
2. \( |3K_S| \) is base point free provided \( K_S^2 \geq 2 \);
3. \( |3K_S| \) is base point free provided \( p_g(S) > 0 \) and \( p_g(S) \neq 2 \);
4. \( |2K_S| \) is base point free provided \( p_g(S) > 0 \) or \( K_S^2 \geq 5 \).

**Proof.** See [Lemma 2.4, [5]].

**Lemma 3.3.** Let \( S \) be a smooth projective surface of general type. Let \( \sigma : S \to S_0 \) be the contraction onto the minimal model. Suppose that there is an effective irreducible curve \( C \) on \( S \) such that \( h^0(S, C) = 2 \).

If \( K_{S_0}^2 = p_g(S) = 1 \), then \( C \cdot \sigma^*(K_{S_0}) \geq 2 \).

**Proof.** See [Lemma 2.5, [5]].

**Lemma 3.4.** Let \( f : X \to C \) be a minimal fibration of surfaces of general type over \( C \), a smooth projective curve of genus \( b \). Let \( F \) be a general fibre of \( f \).

1. If \( p_g(F) \geq 3 \) and \( |K_F| \) is not composed of a pencil, then

\[
K_X^3 \geq \frac{4(p_g(F) - 2)}{p_g(F)} \left\{ \frac{(3K_F^2 - 2\chi(\mathcal{O}_F)p_g(F) + 4\chi(\mathcal{O}_F))}{2(p_g(F) - 2)} \right\} (b - 1) - \chi(\mathcal{O}_X)
\]

2. If \( |K_F| \) is composed of a pencil and \( F \) is not a surface with \( K_F^2 = 1, p_g(F) = 2 \), then

\[
K_X^3 \geq \frac{4(p_g(F) - 1)}{p_g(F)} \left\{ \frac{(3K_F^2 - 2\chi(\mathcal{O}_F)p_g(F) + 2\chi(\mathcal{O}_F))}{2(p_g(F) - 1)} \right\} (b - 1) - \chi(\mathcal{O}_X)
\]

3. If \( K_F^2 = 1 \), then

\[
K_X^3 \geq 3(b - 1) - \chi(\mathcal{O}_X)
\]
If \( p_g(F) = 1 \), then
\[
K_X^3 \geq K_F^2(6 - \chi(O_F))(b - 1) - \chi(O_X)
\]
(4) If \( p_g(F) = 0 \), then
\[
K_X^3 \geq \begin{cases} 
6K_F^2(b - 1) + \frac{2l(2)}{3} & \text{when } K_F^2 \geq 2 \\
6(b - 1) + \frac{6l(2)}{3} & \text{when } K_F^2 = 1
\end{cases}
\]
Proof. See [Main Theorem 1, [13]].

Lemma 3.5. Let \( X \) be a nonsingular projective minimal 3-fold of general type. We have a sharp inequality
\[
K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}.
\]
Proof. See [Theorem 1.2, [2]].

Lemma 3.6. Let \( V \) be a smooth projective 3-fold of general type with \( p_g(V) > 0 \). Then \( \chi(\omega_V) \leq p_g(V) \) unless a generic irreducible component in the general fiber of the Albanese morphism is a surface \( V_y \) with \( q(V_y) = 0 \), in which case one has the inequality
\[
\chi(\omega_V) \leq (1 + \frac{1}{p_g(V_y)})p_g(V).
\]
Proof. See [Proposition 2.1, [7]].

Lemma 3.7. Let \( X \) be a smooth projective 3-fold of general type. Suppose \( p_g(X) \geq 3 \), \( \dim \phi_1(X) = 1 \). Keep the same notations as in the Set up for canonical maps. If \( F \) is a surface with invariants \((1,2)\), then one of the following holds:
1. \( b = 1 \), \( q(X) = 1 \) and \( h^2(O_X) = 0 \);
2. \( b = 0 \), \( q(X) = 0 \) and \( h^2(O_X) \leq 1 \).
Proof. See [Lemma 4.5, [5]].

Lemma 3.8. Suppose that \( X \) is a smooth projective 3-fold. Let \( M \) be a divisor on \( X \) such that \( h^0(X, M) \geq 2 \) and that \( |M| \) has base points but no fixed part. By Hironaka’s theorem ([9]), we may take successive blow-ups
\[
\pi : X' = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X
\]
such that
1. \( \pi_i \) is a single blow-up along smooth center \( W_i \) on \( X_i - 1 \) for all \( i \);
2. \( W_i \) is contained in the base locus of the movable part of
\[
|((\pi_1 \circ \pi_2 \circ \cdots \circ \pi_i - 1)^*M)|
\]
and thus \( W_i \) is a reduced closed point or a smooth projective curve on \( X_i - 1 \);
3. the movable part of \( \pi^*(M) \) has no base points.
It is clear that the resulting 3-fold $X'$ is still smooth. Let $E_i$ be the exceptional divisor on $X'$ corresponding to $W_i$. Then we may write

$$K_{X'} = \pi^*(K_X) + \sum_{i=1}^{n} b_i E_i, \pi^*(M) = M + \sum_{i=1}^{n} e_i E_i,$$

where $b_i, e_i \in \mathbb{Z}, b_i \geq 0$ and $M$ is the movable part of $|\pi^*(M)|$. From the definition of $\pi$, we see $e_i > 0$ for all $i$. For all $i$, we have $b_i \leq 2e_i$.

Proof. See [Lemma 4.2, [5]] □

4. PROOF OF THE MAIN THEOREM

Proposition 4.1. Let $X$ be a nonsingular projective minimal 3-fold of general type. Suppose the general fiber of the Albanese morphism is neither a surface of type $(1, 1)$ nor a surface of type $(1, 2)$. Then

$$K_X^3 \geq \frac{4}{3} \chi(\omega_X) - 2$$

Proof. Note that since $K_X^3 > 0$ is an even integer, the inequality holds for $\chi(\omega_X) \leq 3$. Therefore we may assume $\chi(\omega_X) \geq 4$.

Case 1. $q(X) = 0$.

In this case, we have

$$\chi(\omega_X) = p_g(X) + q(X) - h^2(O_X) - 1 \leq p_g(X) - 1.$$

According to Lemma 3.5, the required inequality holds.

Case 2. $q(X) \geq 1$ and the dimension of the image of the Albanese map is 2 or 3.

According to [Proposition 2.9, [4]], $|4K_X|$ is birational. According to [Theorem 1.5, [8]], we have

$$K_X^3 \geq 2p_g(X) - 6.$$

Since $K_X^3 > 0$ is an even integer, we have

$$K_X^3 \geq 2p_g(X) - 4$$

or

$$K_X^3 = 2p_g(X) - 6.$$

According to Lemma 3.6, we have $\chi(\omega_X) \leq p_g(X)$. If $K_X^3 \geq 2p_g(X) - 4$, we have

$$K_X^3 \geq 2\chi(\omega_X) - 4,$$

which is stronger than the required inequality.

Now suppose $K_X^3 = 2p_g(X) - 6$, we have $p_g(X) \geq 4$.

According to [Theorem 1.3, [8]], [Theorem 1.4, [8]] and [Theorem 4.1, [8]], we see that the canonical map of $X$ is generically finite. According to [Proposition 2.2, [10]] and [Proposition 2.5, [10]], we see that $|K_X|$ is base point free and the canonical map is a generically finite morphism of degree 2. Let $S \in |K_X|$ be a general member, then $S$ is a smooth
minimal projective surface of general type. We have a short exact sequence

\[ 0 \to \mathcal{O}_X \to \mathcal{O}_X(K_X) \to \mathcal{O}_S(S) \to 0. \]

According to Lemma 3.1 and our assumption, we can see that the natural map

\[ H^0(X, K_X) \to H^0(S, S|_S) \]

is surjective. So the natural map

\[ H^1(X, \mathcal{O}_X) \to H^1(X, K_X) \]

is injective. Thus \( q(X) \leq h^2(\mathcal{O}_X) \), we have

\[ K_X^3 = 2pg(X) - 6 \geq 2\chi(\omega_X) - 4, \]

which is stronger than the required inequality.

**Case 3.** The dimension of the image of the Albanese morphism is 1 and the general fiber of the Albanese map is neither a surface of type \((1,1)\) nor a surface of type \((1,2)\).

Let \( a : X \to B \) be the Albanese morphism. Denote by \( F \) a general fiber of the Albanese morphism, then \( F \) is a smooth minimal projective surface. By the assumption and the surface theory, we have \( K_F^2 \geq 2 \).

According to Lemma 3.4, we have \( K_X^3 \geq \frac{3}{2}\chi(\omega_X) \), which is stronger than the required inequality.

**Proposition 4.2.** Let \( X \) be a smooth minimal projective 3-fold of general type. Suppose the general fiber of the Albanese morphism is a surface of type \((1,1)\), then

\[ K_X^3 \geq 2\chi(\omega_X) - 2. \]

**Proof.** Since \( K_X^3 > 0 \) is an even integer, the inequality holds if \( \chi(\omega_X) \leq 2 \). We may assume \( \chi(\omega_X) \geq 3 \). According to [4] and our assumption, we may assume \( pg(X) \geq 2 \).

Denote by \( a : X \to B \) the Albanese map. Let \( F \) be a general fiber of the Albanese map, then \( F \) is a smooth minimal projective surface of general type. Since \( F \) is a surface of type \((1,1)\), the canonical map of \( X \) maps \( F \) to a point. Therefore \(|K_X|\) is composed of a pencil.

**Case 1.** \( pg(X) = 2 \).

In this case we have

\[ \chi(\omega_X) \leq q(X) + 1. \]

Since \( X \) is a smooth minimal projective 3-fold of general type, the divisor \( K_X - 2(q(X) - 1)F \) is a nef divisor. Then

\[ K_X^3 \geq 6q(X) - 6 \geq 2\chi(\omega_X) - 2. \]

**Case 2.** \( pg(X) \geq 3 \).

We will deduce our inequality by studying the bicanonical map of \( X \). The canonical map of \( X \) induces a fibration \( f : X' \to B \) and we can assume that the bicanonical map of \( X' \) is a morphism. Denote by
$M'$ the movable part of $|2K_{X'}|$. Let $S$ be a general fiber of $f$. Consider the natural map

$$H^0(X', M'_2) \rightarrow V_2 \subset H^0(S, 2K_S),$$

where $V_2$ is the image of $r$.

Denote by $\Lambda_2$ the linear system corresponding to $V_2$. According to the surface theory, we have $h^0(S, 2K_S) = 3$. Since $V_2$ is a non-zero subspace of $H^0(S, 2K_S)$, we have $1 \leq \dim V_2 \leq 3$.

**Subcase 2.1.** $\dim V_2 = 1$.

The linear system $|M'_2|$ is composed of a free pencil in this case. Since $p_g(X) > 0$, the canonical map and the bicanonical map induce the same fibration $f$. We have

$$M_2 \sim \sum_{i=1}^{a_2} S_i \equiv a_2 S,$$

where $S_i's$ are distinct fibers of $f$ and $S$ is a general fiber of $f$.

Since $B$ is a nonsingular projective curve of genus $q(X)$, one has $a_2 \geq P_2(X)$ and $\pi^*(K_X)|_S \sim \sigma^*(S_0)$ ($\sigma : S \rightarrow S_0$ is the contraction of $S$ onto its minimal model $S_0$). Therefore, we have

$$K_X^3 \geq \frac{1}{2} a_2 \geq \frac{1}{2} P_2(X).$$

Since $P_2(X) = \frac{1}{2} K_X^3 + 3 \chi(\omega_X)$, we have

$$K_X^3 \geq 2 \chi(\omega_X),$$

which is stronger than the required inequality.

**Subcase 2.2.** $\dim V_2 = 2$.

In this case, the Stein factorization of the morphism given by the linear system $|M'_2|$ gives a fibration $f_2 : X' \rightarrow B_2$, where $B_2$ is a normal projective surface. Let $C$ be a general fiber of $f_2$, then $S$ is natural fibred by curves with the same numerical type as $C$. $\Lambda_2$ is a subsystem of $|2K_S|$ and is composed of a free pencil. Since $S$ is a regular surface, $\Lambda_2$ is a free pencil over the smooth rational curve. Denote by $C'$ a general member of $\Lambda_2$, then $h^0(S, C') = 2$. We see that a general member of $|C'|$ is a nonsingular projective curve. According to Lemma 3.3, we have $(\sigma^*(K_{S_0}) \cdot C') \geq 2$. Then $\pi^*(K_X) \cdot C \geq 2$.

Let $S_2$ be a general member of $|M'_2|$, then $S_2$ is a smooth projective surface of general type. On the surface $S_2$, we have

$$S_2|_{S_2} \sim \sum_{i=1}^{a_2} C_i \equiv a_2 C,$$

where $a_2 \geq P_2(X) - 2$ and $C_i's$ are distinct fibers of $f_2$. Since $2 \pi^*(K_X) \geq M'_2$, we get

$$4K_X^3 \geq \pi^*(K_X) \cdot M_2 \cdot S_2 \geq (\pi^*(K_X) \cdot C)a_2 \geq 2P_2(X) - 4.$$
So we obtain
\[ K_X^3 \geq 2\chi(\omega_X) - \frac{4}{3}, \]
which is stronger than the required inequality.

**Subcase 2.3.** \( \dim V_2 = 3 \).

In this case, the linear system \( \Lambda_2 \) is just the movable part of \( |2K_S| \). According to surface theory, \( \Lambda_2 \) gives a generically finite morphism of degree 4. So the bicanonical map of \( X' \) is a generically finite morphism of degree 4. Let \( S_2 \in |M'_2| \) be a general member, then \( S_2 \) is a smooth projective surface of general type. On the surface \( S_2 \), denote \( L_2 := S_2|S_2 \). The linear system \( |S_2||S_2 \) gives a generically finite morphism of degree 4. Since \( 2\pi^*(K_X) \geq S_2 \) and according to Lemma 3.1, we have
\[ K_X^3 \geq \frac{1}{8}L_2^2 \geq \frac{1}{2}(P_2(X) - 3). \]

Therefore
\[ K_X^3 \geq 2\chi(\omega_X) - 2. \]
We get the required inequality. \( \Box \)

**Proposition 4.3.** Let \( X \) be a nonsingular projective minimal 3-fold of general type. Suppose the general fiber of the Albanese morphism is a surface of type \((1,2)\). Then
\[ K_X^3 \geq 4\chi(\omega_X) - 2 \]

*Proof.* Note that since \( K_X^3 > 0 \) is an even integer, the inequality holds for \( \chi(\omega_X) \leq 3 \). Therefore we may assume \( \chi(\omega_X) \geq 4 \). According to [7] and our assumption, one has \( p_g(X) \geq 3 \). Since \( |K_X||F| \) is a sub-system of \( |K_F| \), the canonical map cannot be a generically finite map.

**Case 1.** The dimension of the image of the canonical map of \( X \) is 1.

According to Lemma 3.7, we can see that \( \chi(\omega_X) = p_g(X) \).

According to [Theorem 4.1, [2]], we have
\[ K_X^3 \geq \frac{7}{3}\chi(\omega_X) - 2, \]
which is stronger than the required inequality.

**Case 2.** The dimension of the image of the canonical map of \( X \) is 2.

Let \( a : X \to Y \) be the Albanese map. The canonical map induces a fibration \( f : X' \to B \) where \( B \) is a normal projective surface. Let \( C \) be a general fiber of \( f \). Denote by \( |M| \) the movable part of \( |K_{X'}| \). Let \( S \) be a general member of \( |M| \), then \( S \) is a smooth projective surface of general type. Denote by \( F_1 \) the general fiber of \( a \circ \pi : X' \to Y \). \( F_1 \) is a smooth projective surface of type \((1,2)\) and is naturally fibred by curves with the same numerical type as \( C \). Since \( \dim \phi_1(X) = 2 \),
$|M||F_i|$ is the movable part of $|K_{F_i}|$. According to the surface theory, $C$ is a smooth curve of genus 2.

On the surface $S$, $|S|_s$ and $S \subset X' \to Y$ gives the same fibration (This follows from the fact that $S^2 \cdot F_1 = 0$). So the complete linear system $|S|_s$ induces a fibration over a curve of genus $q(X)$. We may write

$$M|_S \sim \sum_{i=1}^{a_1} C_i \equiv a_1 C,$$

where the $C_i$s are distinct fibers of the fibration. By Riemann-Roch formula and Clifford’s inequality, we have $a_1 \geq 2p_g(X) - 4$ or $a_1 \geq p_g(X) + q(X) - 2$.

In order to get our inequality, we will take the modification $\pi : X' \to X$ as in Lemma 3.8. Set $|K_X| = |M| + \overline{Z}$, where $|M|$ is the movable part of $|K_X|$ and $\overline{Z}$ is the fixed part. We may take the same successive blow-ups

$$\pi : X' = X_n \to X_{n-1} \to \cdots \to X_i \to X_{i-1} \to \cdots \to X_1 \to X_0 = X$$

as in the set up for Lemma 3.8.

We have

$$K_{X'} = \pi^*(K_X) + E = \pi^*(K_X) + \sum_{i=0}^{n-1} b_i E_i$$

and $\pi^*(M) \sim M + \sum_{i=0}^{n-1} e_i E_i$. We know that $b_i \geq 0$, $e_i > 0$ and both $b_i$ and $e_i$ are integers for all $i$. We also have

$$\pi^*(K_X) = \pi^*(M) + \pi^*(\overline{Z}) = M + \sum_{i=0}^{n-1} e_i' E_i + \sum_{j=1}^{q} d_j D_j = M + E',$$

where $e_i' \geq e_i$, $d_j > 0$, $E_i \neq D_j$ and $D_{j_1} \neq D_{j_2}$ provided $j_1 \neq j_2$.

We have

$$K_X^3 = (\pi^*(K_X))^3 \geq (\pi^*(K_X) \cdot C)a_1 + \pi^*(K_X)|_S \cdot E'|_S.$$

If $(\pi^*(K_X) \cdot C) \geq 2$, then we have

$$K_X^3 \geq 4p_g(X) - 8$$

or

$$K_X^3 \geq 2p_g(X) + 2q(X) - 2.$$

According to Lemma 3.6, we have $\chi(\omega_X) \leq \frac{3}{2} p_g(X)$. So we get

$$K_X^3 \geq \frac{4}{3} \chi(\omega_X) - 2$$

or

$$K_X^3 \geq 2\chi(\omega_X).$$

Thus we can get the required inequality.
From now on, we suppose \((\pi^*(K_X) \cdot C) = 1\). Note that, in this situation, \(|\mathcal{M}|\) definitely has base points. (Otherwise, we can take \(\pi =\) identity, then we have
\[
(\pi^*(K_X) \cdot C) = K_X \cdot C = (K_X + S)|_S \cdot C = K_S \cdot C = 2,
\]
which contradicts to the assumption.)

Denote \(E'|_S = E'_V + E'_H\), where \(E'_V\) is the vertical part, i.e., \(f|_S\) maps \(E'_V\) to finite points, and \(E'_H\) is the horizontal part, i.e., \(E'_H \cdot C > 0\). Since \(E'|_S \cdot C = \pi^*(K_X) \cdot C = 1\), we have \(E'_H \cdot C = 1\). So \(E'_H\) is a section of the fibration \(f|_S\) and we can easily see that the section \(E'_H\) is a nonsingular projective curve. Denote \(E|_S = E_V + E_H\), where \(E_V\) is the vertical part and \(E_H\) is the horizontal part. Since \(E|_S \cdot C = K_X \cdot C - \pi^*(K_X) \cdot C = 1\), \(E_H\) is an irreducible curve and \(E_H\) comes from some exceptional divisor \(E_i\) with \(b_i = 1\). We may suppose that \(E_H\) comes from \(E_0\). Then \(b_0 = 1\). Since \(E_0|_S\) has only one horizontal part, \(E_H\) and \(E'_H\) coincide and we denote this curve by \(\Gamma\). We have
\[
E_V = \sum_{i=1}^{n-1} b_i (E_i|_S) + (E_0|_S - \Gamma)
\]
\[
E'_V = \sum_{i=1}^{q} e'_i(E_i|_S) + \sum_{j=1}^{r} d_j(D_j|_S) + (E_0|_S - \Gamma)
\]

According to Lemma 3.8, we have the following inequality:
\[
E_V \leq 2E'_V.
\]

On the surface \(S\), we have \(E_V \cdot \Gamma \leq 2E'_V \cdot \Gamma\). Since \(\Gamma \cdot C = 1\), \(\Gamma\) is a nonsingular projective curve with geometric genus \(q(X)\).

By the adjunction formula, we have
\[
2q(X) - 2 \leq 2g(\Gamma) - 2 = (K_S + \Gamma) \cdot \Gamma
\]

On the other hand, we have
\[
(K_S + \Gamma) \cdot \Gamma = (K_X')|_S + S|_S + \Gamma) \cdot \Gamma = (\pi^*(K_X)|_S + E_V + S|_S + 2\Gamma) \cdot \Gamma.
\]

Since \(E_V \leq 2E'_V\), we can get
\[
(K_S + \Gamma) \cdot \Gamma \leq (3\pi^*(K_X) - S|_S) \cdot \Gamma.
\]

Therefore
\[
\pi^*(K_X)|_S \cdot E'|_S \geq \frac{1}{3}(a_1 + 2q(X) - 2).
\]

So we have
\[
K_X^3 \geq \frac{4}{3}a_1 + \frac{2}{3}q(X) - \frac{2}{3}.
\]

**Subcase 2.1.** \(a_1 \geq p_g(X) + q(X) - 2\).

In this case, we have \(a_1 \geq \chi(\omega_X) - 1\) and \(q(X) \geq 1\). So we obtain
\[
K_X^3 \geq \frac{4}{3}\chi(\omega_X) - \frac{4}{3}.
\]
which is stronger than the required inequality.

**Subcase 2.2.** \(a_1 \geq 2p_g(X) - 4\).

In this case, we have

\[
K_X^3 \geq \frac{8}{3} p_g(X) + \frac{2}{3} q(X) - 6.
\]

According to Lemma 3.6, we have

\[
K_X^3 \geq 2p_g(X) + \frac{2}{3} p_g(X) + \frac{2}{3} q(X) - 6 \geq \frac{4}{3} \chi(\omega_X) + \frac{2}{3} p_g(X) + \frac{2}{3} q(X) - 6
\]

Since \(p_g(X) \geq 3\) and \(q(X) \geq 1\), we will get the required inequality unless \(p_g(X) + q(X) \leq 5\).

From now on, we may assume \(p_g(X) + q(X) \leq 5\). In this case, we have \(q(X) \leq 2\).

Since \(K_X^3 \geq \frac{8}{3} p_g(X) + \frac{2}{3} q(X) - 6\), we get

\[
K_X^3 \geq \frac{4}{3} \chi(\omega_X) + \frac{4}{3} p_g(X) - \frac{2}{3} q(X) - \frac{14}{3} \geq \frac{4}{3} \chi(\omega_X) - 2.
\]

So the required inequality holds. \(\square\)

**Main Theorem.** Let \(X\) be a nonsingular projective minimal 3-fold of general type. Then the following inequality holds:

\[
K_X^3 \geq \frac{4}{3} \chi(\omega_X) - 2.
\]

**Proof.** This is a direct result of Proposition 4.1, Proposition 4.2 and Proposition 4.3. \(\square\)

5. **Acknowledgment**

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