REGULARITY FOR HARMONIC MAPS INTO CERTAIN PSEUDO-RIEMANNIAN MANIFOLDS

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Abstract. In this article, we investigate the regularity for certain elliptic systems without a $L^2$-antisymmetric structure. As applications, we prove some $\epsilon$-regularity theorems for weakly harmonic maps from the unit ball $B = B(m) \subset \mathbb{R}^m$ ($m \geq 2$) into certain pseudo-Riemannian manifolds: standard stationary Lorentzian manifolds, pseudospheres $S^{n}_v \subset \mathbb{R}^{n+1}$ ($1 \leq v \leq n$) and pseudohyperbolic spaces $H^{n}_v \subset \mathbb{R}^{n+1+1}$ ($0 \leq v \leq n - 1$). Consequently, such maps are shown to be Hölder continuous (and as smooth as the regularity of the targets permits) in dimension $m = 2$. In particular, we prove that any weakly harmonic map from a disc into the De-Sitter space $S^n_1$ or the Anti-de-Sitter space $H^n_1$ is smooth. Also, we give an alternative proof of the Hölder continuity of any weakly harmonic map from a disc into the Hyperbolic space $H^n_0$ without using the fact that the target is nonpositively curved. Moreover, we extend the notion of generalized (weakly) harmonic maps from a disc into the standard sphere $S^n$ to the case that the target is $S^{n}_v$ ($1 \leq v \leq n$) or $H^n_v$ ($0 \leq v \leq n - 1$), and obtain some $\epsilon$-regularity results for such generalized (weakly) harmonic maps.

1. Introduction

In the recent papers by Rivièvre [36] and Rivièvre-Struwe [39], the following regularity results for elliptic systems with a $L^2$-antisymmetric structure are established:

**Theorem 1.1** (Rivièvre [36] for $m = 2$, Rivièvre-Struwe [39] for $m \geq 3$). Let $B = B(m) \subset \mathbb{R}^m$ ($m \geq 2$) be the unit ball. There exists $\epsilon_m > 0$ such that for every $\Omega \in L^2(B, so(n) \otimes \wedge^1 \mathbb{R}^m)$ and for every weak solution $u \in W^{1,2}(B, \mathbb{R}^n)$ of the following elliptic system:

$$-\text{div}\, \nabla u = \Omega \cdot \nabla u$$

satisfying

$$\sup_{B_{\mu(x)}(x)} \left( R^{2-m} \int_{B_{\mu(x)}(x)} |\nabla u|^2 + |\Omega|^2 \right)^{\frac{1}{2}} < \epsilon_m,$$

we have that $u$ is Hölder continuous in $B$.

One of the main applications of the above results is the regularity theory for harmonic map systems into closed Riemannian manifolds, where the $L^2$-antisymmetric property of the potential $\Omega$ in (1.1) relies on the fact that the target manifolds are compact and Riemannian. For classical regularity results of weakly harmonic maps, see e.g. the books by Hélein [22] and Lin-Wang [30] and references therein.

In this paper, we shall study the regularity for weakly harmonic maps from the unit ball $B = B(m) \subset \mathbb{R}^m$ ($m \geq 2$) into certain pseudo-Riemannian manifolds from different points of view. Analytically, it is interesting to know how the structure of the harmonic map system is affected when the target manifolds

\[\text{Date: March 22, 2012.}\]

\[2000\ Mathematics\ Subject\ Classification.\ 58E20; 53C50; 35J60; 35B65.\]

\[\text{Key words and phrases.} \quad \text{Harmonic map; Regularity; Lorentzian manifold; Pseudo-Riemannian manifold.}\]

\[\text{This work was supported by the Forschungsinstitut für Mathematik at ETH Zürich and partially by The Leverhulme Trust.}\]
become pseudo-Riemannian. As we will see later, in general, the $L^2$-antisymmetric structure for harmonic map systems into closed Riemannian manifolds may not be preserved any more when the target manifolds become non-compact or non-Riemannian. Therefore, we would like to explore the extent to which the results developed by Rivièrè [36] and Rivièrè-Struwe [39] can be generalized to elliptic systems without a $L^2$-antisymmetric structure. Geometrically, considering the link between harmonic maps into $S^4 \subset \mathbb{R}^5$ and the conformal gauss maps of Willmore surfaces in $S^3$ (see Bryant [6]), and the regularity results for weak Willmore immersions established by Rivièrè [37], we are strongly encouraged to find a method to study the regularity for weakly harmonic maps into $S^4_1$ and then extend it to the cases of more general targets. Physically, it is known that harmonic maps play an important role in string theory (see e.g. [11, 27]). One of the most significant results in string theory is the AdS/CFT correspondence (Anti-de-Sitter space/Conformal Field Theory correspondence) proposed in 1997 by Maldacena [31]. In view of the recent work on minimal surfaces in Anti-de-Sitter space and its applications in theoretical physics (see e.g. Alday-Maldacena [1]), we are interested in extending the regularity theory for harmonic maps into closed Riemannian manifolds to the cases that the targets are some model spacetimes (which are non-compact and Lorentzian) considered in General Relativity (see e.g. [28, 34]), for instance, standard stationary Lorentzian manifolds, De-Sitter space $\mathbb{D}^n_2$ (also denoted by $d_{n}\mathbb{D}$) and Anti-de-Sitter space $\mathbb{A}^n_2$ (also denoted by $AdS_n$).

In the present work, we solve these problems by using the theory of integrability by compensation developed in [47, 53, 10, 14, 15] and some conservation laws, due to the symmetries of the target manifolds considered. We point out that our results partially realize the perspectives (proposed by Rivièrè [37], p.3-4) developed in [47, 33, 10, 14, 15] and some conservation laws, due to the symmetries of the target manifolds.

First, we observe that, by slightly adapting the techniques used by Rivièrè-Struwe [39], similar regularity results as in Theorem 1.1 extend to certain elliptic systems with a potential $a$ priori in $L^\infty(B)$ but not necessary antisymmetric. To see this, recall that for $1 \leq s < \infty$, the Morrey norm $\| \cdot \|_{M^s(B)}$ of a function $f \in L^s_{\text{loc}}(B)$ is

$$\|f\|_{M^s(B)} = \sup_{B_\varepsilon(x) \subset B} \left( R_{s-m} \int_{B_\varepsilon(x)} |f|^s \right)^{\frac{1}{s}},$$

then we have the following

**Theorem 1.2.** For $m \geq 2$ and for any $\Lambda > 0$, there exists $\varepsilon_{m,\Lambda} > 0$ such that for every $\Theta \in L^2(B, so(n) \otimes \Lambda^1\mathbb{R}^m)$, $\zeta \in W^{1,2}(B, M(n) \otimes \Lambda^2\mathbb{R}^m)$, $F \in W^{1,2}(B, M(n))$, $G \in W^{1,2}(B, \mathbb{M}(n))$ and $Q \in W^{1,2}(B, GL(n))$ and for every weak solution $u \in W^{1,2}(B, \mathbb{R}^m)$ of the following elliptic system:

$$- \text{div} (Q \nabla u) = \Theta \cdot \nabla u + F \text{curl} \zeta \cdot G \nabla u$$

satisfying

$$\|\nabla u\|_{M^s_1(B)} + \|\Theta\|_{M^s_2(B)} + \|\text{curl} \zeta\|_{M^s_3(B)} + \|\nabla Q\|_{M^s_4(B)} + \|\nabla F\|_{M^s_5(B)} + \|\nabla G\|_{M^s_6(B)} < \varepsilon_{m,\Lambda}$$

and

$$|Q| + |Q^{-1}| + |F| + |G| \leq \Lambda, \quad \text{a.e. in } B,$$

we have that $u$ is Hölder continuous in $B$.

The result in Theorem 1.2 was partially obtained by Hajlasz-Strzelecki-Zhong [18], Theorem 1.2 for the case $m = 2, \Theta = 0, Q = I_n$, and by Schikorra [44], Remark 3.4 for the case $m \geq 2, \zeta = 0$.

Note that the elliptic system (1.3) can be written as

$$- \text{div} (Q \nabla u) = \left( \Theta + F \text{curl} \zeta (GQ^{-1}) \right) \cdot (Q \nabla u)$$

(1.6)
or equivalently as
\[- \text{div } \nabla u = \left\{ Q^{-1} \nabla Q + Q^{-1} \left( \Theta + F \text{ curl } \zeta (GQ^{-1}) \right) \right\} \cdot \nabla u. \tag{1.7} \]
Considering \(Q\) as a kind of gauge transformation, we interpret the elliptic system (1.7) as follows: its potential
\[ Q^{-1} \nabla Q + Q^{-1} \left( \Theta + F \text{ curl } \zeta (GQ^{-1}) \right) Q \]
is gauge equivalent to a new one
\[ \Theta + F \text{ curl } \zeta (GQ^{-1}) \]
which can be decomposed into an antisymmetric part \(\Theta\) and an almost divergence free part \(F \text{ curl } \zeta (GQ^{-1})\).

As an application of Theorem 1.2, we shall study the regularity for weakly harmonic maps into standard stationary Lorentzian manifolds. A standard stationary Lorentzian manifold (see e.g. \([28, 34]\)) is a product manifold \(\mathbb{R} \times M\) equipped with a metric
\[ g = - (C \circ \pi_M)(\pi^\alpha \rho_t + \pi^\alpha_M \omega) \otimes (\pi^\beta dt + \pi^\beta_M \omega) + \pi^\alpha_M g_M, \tag{1.8} \]
where \((\mathbb{R}, dt^2)\) is the 1-dimensional Euclidean space, \((M, g_M)\) is a closed Riemannian manifold of class \(C^3\), \(C\) is a positive \(C^2\) function on \(M, \omega\) is a \(C^2\) 1-form on \(M, \pi_R\) and \(\pi_M\) are the natural projections on \(\mathbb{R}\) and \(M\), respectively. For simplicity of notations, we shall write the metric (1.8) as
\[ g = -\beta(dt + \omega)^2 + g_M. \tag{1.9} \]
By Nash’s embedding theorem, we embed \((M, g_M)\) isometrically into some Euclidean space \(\mathbb{R}^n\). Then, there exists a tubular neighborhood \(V_\delta M\) of radius \(\delta > 0\) of \(M\) in \(\mathbb{R}^n\) and a \(C^2\) projection map \(\Pi\) from \(V_\delta M\) to \(M\) (see Hélein’s book \([22]\), Chapter 1). Moreover, we pull back \(\beta\) and \(\omega\) via the projection \(\Pi\) and obtain \(\Pi^* \beta \in C^2(V_\delta M, (0, \infty))\) and \(\Pi^* \omega \in C^2(Q_1(V_\delta M))\), respectively. For simplicity, we shall still denote \(\Pi^* \beta\) and \(\Pi^* \omega\) by \(\beta\) and \(\omega\), respectively. Write \(\omega = \sum_{i=1}^n \omega_i dy^i, y = (y^1, ..., y^n) \in V_\delta M \subset \mathbb{R}^n\), where \(\omega_i \in C^2(V_\delta M)\).

To study the regularity for weakly harmonic maps into \((\mathbb{R} \times M, g)\), we consider the space
\[ W^{1,2}(B, \mathbb{R} \times M) := \left\{ (t, u) \in W^{1,2}(B, \mathbb{R}) \times W^{1,2}(B, \mathbb{R}^n) \mid u(x) \in M \text{ a.e. } x \in B \right\} \tag{1.10} \]
For a map \((t, u) \in W^{1,2}(B, \mathbb{R} \times M)\), we define the following Lagrangian:
\[ E(t, u) = - \frac{1}{2} \int_B \beta(u)|\nabla t + \omega_i(u) \nabla u^i|^2 + \frac{1}{2} \int_B |\nabla u|^2 \tag{1.11} \]

**Definition 1.1.** A map \((t, u) \in W^{1,2}(B, \mathbb{R} \times M)\) is called a weakly harmonic map from \(B\) into \((\mathbb{R} \times M, g)\), if it is a critical point of the Lagrangian functional (1.11).

The Euler-Lagrange equation (see Section 3) for a weakly harmonic map \((t, u) \in W^{1,2}(B, \mathbb{R} \times M)\) from \(B\) into \((\mathbb{R} \times M, g)\) is an elliptic system of the form (1.3), which can be geometrically interpreted as follows: the antisymmetric term \(\Theta\) corresponds to the Riemannian structure of the closed spacelike hypersurfaces \(\{t\} \times M\) and the divergence free term \(\text{curl } \zeta\) corresponds to the following conservation law
\[ \text{div } \left\{ \beta(u)(\nabla t + \omega_i(u) \nabla u^i) \right\} = 0, \text{ in } \mathcal{D}'(B) \tag{1.12} \]
due to the symmetry of the target generated by the timelike Killing vector field \(\partial_t\). Applying Theorem 1.2, we have the following \(\epsilon\)-regularity result:

**Theorem 1.3.** For \(m \geq 2\), there exists \(\epsilon_m > 0\) depending on \((\mathbb{R} \times M, g)\) such that any weakly harmonic map \((t, u) \in W^{1,2}(B, \mathbb{R} \times M)\) from \(B\) into \((\mathbb{R} \times M, g)\) satisfying
\[ ||\nabla t||_{L^2(B)} + ||\nabla u||_{L^2(B)} < \epsilon_m, \tag{1.13} \]
is Hölder continuous (and as smooth as the regularity of the target permits) in \(B\).
In dimension $m = 2$, we notice that the Morrey norm $\| \cdot \|_{M^p_0}$ reduces to the norm $\| \cdot \|_{L^2}$. Therefore, by conformal invariance and rescaling in the domain, we obtain the following regularity result:

**Theorem 1.4.** For $m = 2$, any weakly harmonic map $(t, u) \in W^{1,2}(B, \mathbb{R} \times M)$ from $B$ into $(\mathbb{R} \times M, g)$ is Hölder continuous (and as smooth as the regularity of the target permits) in $B$.

In Theorem 1.4, if the target $(\mathbb{R} \times M, g)$ is a standard static Lorentzian manifold (see e.g. [28], [34]), namely, the 1-form $\omega$ in the metric $g$ (see (1.9)) vanishes identically, then the corresponding regularity result was proved by Isobe [24] (using Helein’s method of moving frame [22]).

Next, we shall consider, in a certain sense, elliptic systems of the form (1.1) with the potential $\Omega$ a priori only in $L^p$ for some $1 < p < 2$. Note that, if $\Omega$ is not in $L^2$, then the right hand side of (1.1) is not in $L^1$ and thus the equation makes no sense any more (not even in the distribution sense!). However, we observe that, if in addition, $\Omega$ is divergence free, namely,

$$\text{div } \Omega = 0, \quad \text{in } \mathcal{D}'(B),$$

(1.14) then the equation (1.1) can be written in the following form:

$$- \text{div } (\nabla u + \Omega u) = 0, \quad \text{in } \mathcal{D}'(B),$$

(1.15) This new form (1.15) has the advantage that it is still meaningful in the distribution sense if $\Omega$ is a priori only in $L^p$ for some $1 < p < 2$. Moreover, under the further assumption that the Morrey norms $\|\nabla u\|_{M^p_0(B)}$ and $\|\Omega\|_{M^p_0(B)}$ are sufficiently small, the Hölder continuity of the weak solution $u$ holds.

**Theorem 1.5.** For $m \geq 2$ and for any $1 < p < \frac{2m}{m-1}$, there exists $\epsilon_{m,p} > 0$ such that for any $\Omega \in L^p(B, (M(n) \otimes \Lambda^1 \mathbb{R}^m))$ satisfying (1.14) and for any weak solution $u \in W^{1,2}(B, \mathbb{R}^n)$ of the elliptic system (1.15) satisfying

$$\|\nabla u\|_{M^p_0(B)} + \|\Omega\|_{M^p_0(B)} < \epsilon_{m,p},$$

(1.16) we have that $u$ is Hölder continuous in $B$.

As applications of Theorem 1.5, we shall study the regularity for weakly harmonic maps into pseudo-spheres and pseudo-hyperbolic spaces. For this purpose, we recall some facts about these target spaces and refer to O’Neill’s book [34] for more details.

Let $n \in \mathbb{N}$ and let $\nu \in \mathbb{N}$ satisfy $0 \leq \nu \leq n$. Denote

$$\mathcal{E} = (e_{ij}) := \begin{pmatrix} -I_n & 0 \\ 0 & I_{n+1-\nu} \end{pmatrix}.$$  

(1.17) The pseudo-Euclidean space $\mathbb{R}_{\nu}^{n+1}$ of signature $(\nu, n + 1 - \nu)$ is the space $\mathbb{R}^{n+1}$ equipped with a metric

$$\langle v, w \rangle_{\nu}^{n+1} := v^T \mathcal{E} w = - \left( v^1 w^1 + \cdots + v^\nu w^\nu \right) + \left( v^{\nu+1} w^{\nu+1} + \cdots + v^{n+1} w^{n+1} \right),$$

for all $v = (v^1, ..., v^{\nu+1})^T \in \mathbb{R}^{\nu+1}$ and $w = (w^1, ..., w^{n+1})^T \in \mathbb{R}^{n+1}$. The pseudosphere $\mathbb{S}_{\nu}^n$ in $\mathbb{R}_{\nu}^{n+1}$ is defined as

$$\mathbb{S}_{\nu}^n := \left\{ y \in \mathbb{R}_{\nu}^{n+1} \mid \langle y, y \rangle_{\nu}^{n+1} = y^T \mathcal{E} y = 1 \right\}$$

(1.18) with the induced metric. In particular, $\mathbb{S}_0^0 \subset \mathbb{R}_{0}^{1+1}$ is the standard sphere $\mathbb{S} \subset \mathbb{R}^{n+1}$ and $\mathbb{S}_1^n \subset \mathbb{R}_1^{n+1}$ is the De-Sitter space $dS_n$ in General Relativity. The linear isometries of $\mathbb{R}_{\nu}^{n+1}$ form the group

$$O(\nu, n + 1 - \nu) = \left\{ P \in \text{GL}(n + 1) \mid P^T = \mathcal{E} P^{-1} \mathcal{E} \right\}.$$  

(1.19) Denote by $\text{SO}^+(\nu, n + 1 - \nu)$ the identity component of $O(\nu, n + 1 - \nu)$. The lie algebra of $\text{SO}^+(\nu, n + 1 - \nu)$ is

$$\text{so}(\nu, n + 1 - \nu) = \left\{ A \in \text{GL}(n + 1) \mid A^T = -\mathcal{E} A \mathcal{E} \right\}.$$  

(1.20)
Using the isometric embedding $\mathbb{S}^n_v \subset \mathbb{R}^{n+1}_v$, we set
\[
W^{1,2}(B, \mathbb{S}^n_v) := \left\{ u = (u^1, u^2, ..., u^{n+1})^T \in W^{1,2}(B, \mathbb{R}^{n+1}_v) \mid u^T E u = 1 \text{ a.e. in } B \right\}.
\] (1.21)

For a map $u \in W^{1,2}(B, \mathbb{S}^n_v)$, we define the following Lagrangian:
\[
E(u) := \frac{1}{2} \int_B (\nabla u)^T E \nabla u = -\frac{1}{2} \int_B (|\nabla u|^2 + |\nabla u^-|^2) + \frac{1}{2} \int_B (|\nabla u^{n+1}|^2 + |\nabla u^{n+1}|^2)
\] (1.22)

**Definition 1.2.** A map $u \in W^{1,2}(B, \mathbb{S}^n_v)$ is called a weakly harmonic map from $B$ into $\mathbb{S}^n_v$, if it is a critical point of the Lagrangian functional (1.22).

Denote
\[
\mathcal{F} = (\sigma_{ij}) := \begin{pmatrix}
-I_{n+1} & 0 \\
0 & I_{n-1}
\end{pmatrix}
\] (1.23)

The pseudohyperbolic space $\mathbb{H}^n_v$ in $\mathbb{R}^{n+1}_v$ is defined as
\[
\mathbb{H}^n_v := \left\{ y \in \mathbb{R}^{n+1}_v \mid \langle y, y \rangle_{\mathbb{R}^{n+1}_v} = y^T \mathcal{F} y = 1 \right\}
\] (1.24)

with the induced metric. In particular, $\mathbb{H}^n_v \subset \mathbb{R}^{n+1}_v$ is a hyperboloid containing two copies of the Hyperbolic space $\mathbb{H}^n$ and $\mathbb{H}^1_v \subset \mathbb{R}^{n+1}_v$ is the Anti-de-Sitter space $AdS_n$ in General Relativity.

Using the isometric embedding $\mathbb{S}^n_v \subset \mathbb{R}^{n+1}_v$, we set
\[
W^{1,2}(B, \mathbb{H}^n_v) := \left\{ u = (u^1, u^2, ..., u^{n+1})^T \in W^{1,2}(B, \mathbb{R}^{n+1}_v) \mid u^T \mathcal{F} u = -1 \text{ a.e. in } B \right\}.
\] (1.25)

For a map $u \in W^{1,2}(B, \mathbb{H}^n_v)$, we define the following Lagrangian:
\[
E(u) := \frac{1}{2} \int_B (\nabla u)^T E \nabla u = -\frac{1}{2} \int_B (|\nabla u|^2 + |\nabla u^-|^2) + \frac{1}{2} \int_B (|\nabla u^{n+1}|^2 + |\nabla u^{n+1}|^2)
\] (1.26)

**Definition 1.3.** A map $u \in W^{1,2}(B, \mathbb{H}^n_v)$ is called a weakly harmonic map from $B$ into $\mathbb{H}^n_v$, if it is a critical point of the Lagrangian functional (1.26).

Notice that the following anti-isometry (see O’Neill’s book [34]):
\[
\sigma : \mathbb{R}^{n+1}_v \rightarrow \mathbb{R}^{n+1}_v, \quad (y_1, ..., y_{n+1}) \mapsto (y_1, ..., y_{n+1}, y_{n+1}, ..., y_1)
\]
induces an anti-isometry from $\mathbb{S}^n_v$ to $\mathbb{H}^n_{-v}$. In the sequel, we shall only consider the cases of $\mathbb{S}^n_v$ ($0 \leq v \leq n$).

To proceed, we recall that a weakly harmonic map $u \in W^{1,2}(B, \mathbb{S}^n_v)$ satisfies the following conservation laws (due to Shatah [42] and Chen [9]. See also Rubinstein-Sternberg-Keller [40] and Hélein’s book [22]):
\[
\text{div} \left( u^i \nabla u^i - u^j \nabla u^j \right) = 0, \quad \forall i, j = 1, 2, ..., n+1,
\] (1.27)

which can be interpreted by Noether theorem, using the symmetries of $\mathbb{S}^n_v$. Note that the pseudospheres $\mathbb{S}^n_v$ ($1 \leq v \leq n$) have isometry groups $O(v, n+1-n)$ and hence they are all maximally symmetric. With the help of the symmetric properties, we are able to extend the conservation laws (1.27) to weakly harmonic maps into these more general targets.

**Proposition 1.1.** For $m \geq 2$. Let $u \in W^{1,2}(B, \mathbb{S}^n_v) (1 \leq v \leq n)$ be a weakly harmonic map. Then the conservation laws (1.27) hold.

For a weakly harmonic map $u \in W^{1,2}(B, \mathbb{S}^n_v) (0 \leq v \leq n)$, we define the following matrix valued vector field
\[
\Theta = (\Theta^i) := (u^i \nabla u^i - u^j \nabla u^j), \quad i, j = 1, 2, ..., n+1.
\] (1.28)
In the case of a compact target $S^n$, $\Theta$ is in $L^2(B, M(n) \otimes \Lambda^1\mathbb{R}^m)$ and $u$ weakly solves the following elliptic system (see Hélein [19, 22])

$$- \text{div} \nabla u = \Theta \cdot \nabla u.$$

Since $\Theta$ is divergence free (due to the conservation laws (1.27)), the continuity of $u$ in dimension $m = 2$ follows immediately from Wente’s lemma [57].

However, in the case of a non-compact target $S^n$ ($1 \leq n \leq n$), $\Theta$ is only in $L^p(B, M(n) \otimes \Lambda^1\mathbb{R}^m)$ for any $1 < p < 2$. Proposition 1.1 indicates that $\Theta$ is still divergence free. In what follows, we show that $u$ is a weak solution of an elliptic system of the form (1.15) with its potential satisfying (1.14). Moreover, by making use of the conservation laws (1.27), we are able to estimate $\|\Theta\|_{M^p(B, \mathbb{R}^n)}$ by $\|\nabla u\|_{M^p(B)}$, where $B_{1/2} = B_{1/2}(m) \subset \mathbb{R}^m (m \geq 2)$ is the ball centered at 0 and of radius $1/2$.

**Proposition 1.2.** For $m \geq 2$. Let $u \in W^{1,2}(B, S^n)$ ($1 \leq n \leq n$). Then

$$\nabla u + \Theta \mathcal{E} u = 0, \quad \text{a.e. in } B. \quad (1.29)$$

where $\Theta$ is defined as in (1.28). Consequently, we have

$$- \text{div} (\nabla u + \Theta \mathcal{E} u) = 0 \quad \text{in } \mathcal{D}'(B). \quad (1.30)$$

Furthermore, suppose that $u$ is weakly harmonic and for any fixed $1 < p < \frac{m}{m-1}$ there holds $\|\nabla u\|_{M^p(B)} < \infty$, then we have the following estimate:

$$\|\Theta\|_{M^p(B, \mathbb{R}^n)} \leq C \|\nabla u\|_{M^p(B)}^2. \quad (1.31)$$

Since $\mathcal{E}$ is a constant matrix, applying Theorem 1.5 with $\Omega = \Theta \mathcal{E}$ and using a rescaling of the domain gives the following $\epsilon$-regularity result:

**Theorem 1.6.** For $m \geq 2$ and for any $1 < p < \frac{m}{m-1}$, there exists $\epsilon_{m,p} > 0$ such that any weakly harmonic map $u \in W^{1,2}(B, S^n)$ ($1 \leq n \leq n$) satisfying

$$\|\nabla u\|_{M^p(B)}^2 < \epsilon_{m,p} \quad (1.32)$$

is Hölder continuous (and hence smooth) in $B$.

In dimension $m = 2$, a straightforward calculation gives that $\|\nabla u\|_{M^p(B)} \leq \|\nabla u\|_{L^p(B)}$ for any $1 < p < 2$. Therefore, by conformal invariance, we have

**Theorem 1.7.** For $m = 2$, any weakly harmonic map $u \in W^{1,2}(B, S^n)$ ($1 \leq n \leq n$) is Hölder continuous (and hence smooth) in $B$.

In particular, we prove that any weakly harmonic map from a disc into the De-Sitter space $S^n$ or the Anti-de-Sitter space $\mathbb{H}_1^n \equiv S^n_{n-1}$ is smooth. Also, we give an alternative proof of the Hölder continuity of weakly harmonic maps from a disc into the Hyperbolic space $\mathbb{H}^n$ (one component of $\mathbb{H}^n_1 \equiv S^n_{n-1}$) without using the fact that the target has non-positive sectional curvature (for a proof using the curvature property, we refer to Jost’s book [26]). We expect that the results in Theorem 1.6 and Theorem 1.7 can be extended in the same spirit of Hélein’s setting in [20] to certain homogeneous pseudo-Riemannian manifolds.

Furthermore, we observe that the methods used in the proofs of Proposition 1.2 and Theorem 1.5 can be applied to study the $\epsilon$-regularity of maps in the spaces of distributions of lower regularity. This motivates us to extend the notion of generalized (weakly) harmonic maps from a disc into the standard sphere $S^n$ (introduced by Almeida [2]) to the cases that the targets are pseudospheres $S^n$ ($1 \leq n \leq n$) (see Section 5). To see this, we recall the notion of generalized (weakly) harmonic maps into $S^n$.

**Definition 1.4** (Almeida [2]). For $m = 2$, a map $u \in W^{1,1}(B, S^n)$ is called a generalized (weakly) harmonic map if (1.27) hold.
Generalized (weakly) harmonic maps into $\mathbb{S}^n$ might be not continuous. However, there are some $\epsilon$-regularity results for such maps. Almeida [2] showed that any generalized harmonic map $u \in W^{1,1}(B, \mathbb{S}^n)$ with $||\nabla u||_{L^2(B)}$ small is smooth (an alternative proof was given by Ge [16]). Moser [32] proved that any generalized harmonic map $u \in W^{1,p}_{\text{loc}}(B, \mathbb{S}^n)$ with $p \in (1,2)$ is smooth if $p$ is sufficiently close to 2 and $||u||_{BMO}$ is small. Strzelecki [40] showed that any generalized harmonic map $u \in W^{1,p}_{\text{loc}}(B, \mathbb{S}^n)$ with $p \in (1,2)$ is smooth provided that $||u||_{BMO}$ is small.

To extend the notion of generalized (weakly) harmonic maps into the pseudospheres $\mathbb{S}^n_\nu$ $(1 \leq \nu \leq n)$, we observe that a $W^{1,1}$ map from a disc into any of these non-compact targets is not a priori in $L^\infty$ and hence the conservation laws (1.27) make no sense for such a map. Therefore, we need to require that the map $u$ belongs to the sobolev space $W^{1,2}$ so that

$$u'\nabla u' - u'\nabla u' \in L^1_{\text{loc}}(B), \quad \forall i, j = 1, 2, ..., n+1.$$  

and hence the conservation laws (1.27) become meaningful.

**Definition 1.5.** For $m = 2$, a map $u \in W^{1,\frac{4}{3}}(B, \mathbb{S}^n_\nu) \ (1 \leq \nu \leq n)$ is called a generalized (weakly) harmonic map if (1.27) hold.

Analogously to Theorem 1.6, we have the following $\epsilon$-regularity result.

**Theorem 1.8.** For $m = 2$ and for any $\frac{4}{3} < p < 2$, there exists $\epsilon_p > 0$ such that any generalized (weakly)
harmonic map $u \in W^{1,\frac{4}{3}}(B, \mathbb{S}^n_\nu) \ (1 \leq \nu \leq n)$ satisfying

$$||\nabla u||^2_{M^p(B)} < \epsilon_p$$  

is Hölder continuous (and hence smooth) in $B$.

Finally, we study the regularity for an elliptic system of the form (1.1) with $\Omega \in L^2(B, so(1,1) \otimes \Lambda^1 \mathbb{R}^2)$ in dimension $m = 2$ and show by constructing an example that weak solutions in $W^{1,2}$ to such an elliptic system might be not in $L^\infty$.

The paper is organized as follows. In Section 2, we prove Theorem 1.2 and Theorem 1.5. In Section 3, we apply Theorem 1.2 to prove the $\epsilon$-regularity (Theorem 1.3) of weakly harmonic maps into standard stationary Lorentzian manifolds. In Section 4, we first show Proposition 1.1 and Proposition 1.2. Then we prove the regularity results (Theorem 1.6 and Theorem 1.7) for weakly harmonic maps into pseudospheres. In Section 5, the $\epsilon$-regularity result (Theorem 1.8) for generalized (weakly) harmonic maps from a disc into pseudospheres is proved. In Section 6, we study an elliptic system with a $L^2-so(1,1)$ structure in dimension $m = 2$.

**Notation:** For a 2-vector field $\xi = \xi_i \partial_i \wedge \partial_x$, curl $\xi$ denotes the vector field $\left( \sum \partial_i \xi_{ij} \right) \partial_j$, and $d\xi$ denotes the 3-vector field $\left( \partial_i \xi_{ij} \right) \partial_i \wedge \partial_x \wedge \partial_y$. A constant $C$ may depend on $m, n, p$.

**Acknowledgements** The author would like to thank Professor Tristan Rivière for many useful discussions, consistent support and encouragement. Thanks also to Professor Jürgen Jost and Professor Yuxin Ge for helpful conversations.

2. Proofs of Theorem 1.2 and Theorem 1.5

In this section, we will prove Theorem 1.2 and Theorem 1.5.
First, combining the div-curl inequality by Coifman-Lions-Meyer-Semmes \[10\] (see Müller \[33\] for an earlier contribution), the Hardy-BMO duality by Fefferman \[14\] (see also Fefferman-Stein \[15\] and Stein \[43\]) and the observation that the Morrey spaces $M_s^p(\mathbb{R}^n)$ ($1 \leq s < \infty$) are contained in the space BMO($\mathbb{R}^m$) (due to Evans \[13\]), we give the following lemma (see Proposition III.2 in Bethuel \[5\], Lemma 3.1 in Schikorra \[41\] and Strzelecki \[45\] p.234-235. See also Chanillo \[7\] and Chanillo-Li \[8\]).

**Lemma 2.1.** For $m \geq 2$, $1 \leq s < \infty$ and $1 < p < \infty$. Let $1 < q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. For any ball $B_{R}(x) \subset \mathbb{R}^m$, $f \in W^{1,p}(B_{R}(x))$, $g \in W^{1,q}(B_{R}(x), \mathbb{L}^2 \mathbb{R}^m)$ satisfying

$$f|_{\partial B_{R}(x)} = 0 \quad \text{or} \quad g|_{\partial B_{R}(x)} = 0$$

(2.1)

and $h \in W^{1,s}(B_{2R}(x))$ satisfying

$$\|\nabla h\|_{M_s^p(B_{2R}(x))} < \infty,$$

(2.2)

there holds:

$$\int_{B_{R}(x)} (\nabla f \cdot \text{curl} \, g) \, h \leq C \|\nabla f\|_{L^p(B_{R}(x))} \|\text{curl} \, g\|_{L^q(B_{R}(x))} \|\nabla h\|_{M_s^p(B_{2R}(x))},$$

(2.3)

where $C = C_{m,s,p} > 0$ is a uniform constant independent of $R > 0$.

Next, with the help of the above lemma, we follow the approach used by Rivi`ere-Struwe \[39\] to prove Theorem 1.2 and Theorem 1.5.

**Proof of Theorem 1.2:** Fix $m \geq 2$ and $\Lambda > 0$. Choose $\epsilon_{m,\Lambda} > 0$ sufficiently small, then by assumption \[1.4\] and the existence of Coulomb gauge (due to Rivi`ere \[36\], for $m = 2$ and Rivi`ere-Struwe \[39\] for $m \geq 3$), we conclude that there are $P \in W^{1,2}(B, SO(n))$ and $\xi \in W^{1,2}_0(B, so(n) \otimes \mathbb{L}^2 \mathbb{R}^m)$ with $d\xi = 0$ such that

$$P^{-1} \nabla P + P^{-1} \Theta P = \text{curl} \, \xi \quad \text{in} \, B.$$  

(2.4)

and the following estimate holds

$$\|\nabla P\|_{M^2(B)} + \|\nabla \xi\|_{M^2(B)} \leq C \|\Theta\|_{M^2(B)} \leq C \epsilon_{m,\Lambda}.$$  

(2.5)

Using \[4.4\], we rewrite the system \[1.3\] as

$$- \text{div} \left( P^{-1} Q \nabla u \right) = \left( P^{-1} \nabla P + P^{-1} \Theta P \right) \cdot P^{-1} Q \nabla u + P^{-1} F \text{ curl} \, \zeta \cdot G \cdot \nabla u$$

(2.6)

Write $P^{-1} = (P_{ij}), \Theta = (\Theta^{ij}), Q = (Q^{ij}), G = (G^{ij})$ and $Q = (Q^{ij}).$ Then the above equation can be written as

$$- \text{div} \left( \sum_{j,k} P_{ij} Q^{jk} \nabla u^k \right) = \sum_{j,k,l} \text{curl} \, \xi^{ij} \cdot P_{jk} Q_{kl} \nabla u^l + \sum_{j,k,l} P_{ij} F_{jk} \text{ curl} \, \xi^{kl} \cdot G^{lr} \nabla u^r$$

(2.7)

$$= \sum_{j,k,l} P_{jk} Q^{kl} \text{ curl} \, \xi^{ij} \cdot \nabla u^l + \sum_{j,k,l} P_{ij} F_{jk} G^{lr} \text{ curl} \, \xi^{kl} \cdot \nabla u^r$$

Since $P \in W^{1,2}(B, SO(n)), F \in W^{1,2} \cap L^\infty(B, M(n)), G \in W^{1,2} \cap L^\infty(B, M(n))$ and $Q \in W^{1,2} \cap L^\infty(B, GL(n))$, we have $P^{-1} Q \in W^{1,2} \cap L^\infty(B, GL(n)), P_{ij} F_{jk} G^{lr} \in W^{1,2} \cap L^\infty(B)$. Using the assumption \[1.5\], one can verify that

$$\|\nabla (P^{-1} Q)\|_{M^2(B)} + \sum_{i,j,l,r} \|\nabla (P_{ij} F_{jk} G^{lr})\|_{M^2(B)} \leq C(\Lambda) \left( \|\nabla P\|_{M^2(B)} + \|\nabla Q\|_{M^2(B)} + \|\nabla F\|_{M^2(B)} + \|\nabla G\|_{M^2(B)} \right).$$

(2.8)

Here and in the sequel, $C(\Lambda) > 0$ is a constant also depending on $\Lambda.$
Combining (2.5), (2.8) and assumption (1.4), we get
\[ \|\nabla u\|_{M^q_{\closedone}(B)} + \|\nabla (P^{-1}Q)\|_{M^q_{\closedone}(B)} + \sum_{i,j,k,l\neq i,j,k,l} \|\nabla (P_{ij}^{kl}G^{ij})\|_{M^q_{\closedone}(B)} + \|\nabla \xi\|_{M^q_{\closedone}(B)} + \|\text{curl} \ \xi\|_{M^q_{\closedone}(B)} \leq C(\Lambda)\varepsilon_{m,\Lambda}. \] (2.9)

On the other hand, since \( P^{-1} \) takes values in \( \text{SO}(n) \), it follows from assumption (1.5) that
\[ C(\Lambda)^{-1} \|\nabla u\| \leq |P^{-1}Q| \|\nabla u\| = \|Q \|\nabla u\| \leq C(\Lambda) \|\nabla u\| \quad \text{a.e. in } B. \] (2.10)

Similarly to the approach by Rivièrë-Strauwe (39), Section 3, Proof of Theorem 1.1, p.459-460. See also Schikorra (41), p.510-511, we apply Hodge decomposition (see [25]) to \( P^{-1}Q \nabla u \), use (2.7), (2.9), (2.10), Lemma 2.1, and take \( \varepsilon_{m,\Lambda} > 0 \) sufficiently small to get the Morrey type estimates for \( \nabla u \). Finally, we apply an iteration argument as in [17] to obtain the Hölder continuity of \( u \) in \( B \).

**Proof of Theorem 1.5:** Fix any \( 1 < p < \frac{m}{m-1} \). Since \( \text{div} \ \Omega = 0 \), by Hodge decomposition, there exists \( \xi \in W^{1,p}(B; M(n) \otimes \wedge^2 \mathbb{R}^m) \) such that
\[ \Omega = \text{curl} \ \xi. \] (2.11)

Let \( B_{2R}(x_0) \subset B \) and let \( w \in W^{1,2}(B_{2R}(x_0), \mathbb{R}^n) \) be solving
\[ \begin{cases} -\text{div} \ \nabla w &= 0, &\text{in } B_{R}(x_0) \\ w &= u, &\text{on } \partial B_{R}(x_0) \end{cases} \] (2.12)

Then \( v := u - w \in W^{1,2}(B_{2R}(x_0), \mathbb{R}^n) \) solves
\[ \begin{cases} -\text{div} \ (\nabla v + \Omega u) &= 0, &\text{in } B_{R}(x_0) \\ v &= 0, &\text{on } \partial B_{R}(x_0) \end{cases} \] (2.13)

Let \( q = \frac{p}{p-1} > m \) be the conjugate exponent of \( p \). For any \( \varphi \in W^{1,q}_0(B_{R}(x_0)) \) with \( \|\varphi\|_{W^{1,q}(B_{2R}(x_0))} \leq 1 \). Using assumption (1.6), Lemma 2.1, (2.11) and (2.13), we estimate for each \( i \),
\[ \int_{B_{R}(x_0)} \nabla v^i \cdot \nabla \varphi = - \int_{B_{R}(x_0)} (\Omega^{ij} u^j) \cdot \nabla \varphi \]
\[ = - \int_{B_{R}(x_0)} (\text{curl} \ \xi^{ij} \cdot \nabla \varphi) u^j \]
\[ \leq C \|\text{curl} \ \xi^{ij}\|_{L^p(B_{2R}(x_0))} \|\nabla \varphi\|_{L^q(B_{2R}(x_0))} \|\nabla u\|_{M^q_{\closedone}(B_{2R}(x_0))} \]
\[ \leq C \|\Omega^{ij}\|_{L^p(B_{2R}(x_0))} \|\nabla u\|_{M^q_{\closedone}(B_{2R}(x_0))} \]
\[ \leq CR^{-1} \|\Omega^{ij}\|_{M^q_{\closedone}(B_{2R}(x_0))} \|\nabla u\|_{M^q_{\closedone}(B_{2R}(x_0))} \]
\[ \leq CR^{-1} \varepsilon_{m,p} \|\nabla u\|_{M^q_{\closedone}(B_{2R}(x_0))}. \] (2.14)

Since \( v\big|_{\partial B_{R}(x_0)} = 0 \), by duality (similarly to Rivièrë-Strauwe (39)) there holds:
\[ \|\nabla v\|_{L^p(B_{R}(x_0))} \leq C \sup_{\varphi \in W^{1,q}_0(B_{R}(x_0)), \|\varphi\|_{L^q(B_{R}(x_0))} \leq 1} \int_{B_{R}(x_0)} \nabla v \cdot \nabla \varphi. \] (2.15)

Combining (2.14) and (2.15) gives
\[ \|\nabla v\|_{L^p(B_{R}(x_0))} \leq CR^{-1} \varepsilon_{m,p} \|\nabla u\|_{M^q_{\closedone}(B_{2R}(x_0))}. \] (2.16)

Next, we see from (2.12) that \( w \) is harmonic in \( B_{2R}(x_0) \) and hence \( \nabla w \) is also harmonic in \( B_{R}(x_0) \). By Campanato estimates for harmonic functions (see (17)), we have that for any \( r < R \) the following holds:
\[ \int_{B_{r}(x_0)} |\nabla w|^p \leq C \left( \frac{r}{R} \right)^m \int_{B_{R}(x_0)} |\nabla w|^p. \] (2.17)
Using that fact that \( u = v + w \) and combining (2.16), (2.17), we estimate
\[
\int_{B_{\epsilon}(x_0)} |\nabla u|^p \leq C \int_{B_{\epsilon}(x_0)} |\nabla w|^p + C \int_{B_{\epsilon}(x_0)} |\nabla v|^p
\leq C \left( \frac{R}{\epsilon} \right)^m \int_{B_{\epsilon}(x_0)} |\nabla w|^p + C \int_{B_{\epsilon}(x_0)} |\nabla v|^p
\leq C \left( \frac{R}{\epsilon} \right)^m \int_{B_{\epsilon}(x_0)} |\nabla u|^p + C \int_{B_{\epsilon}(x_0)} |\nabla v|^p
\leq C \left( \frac{R}{\epsilon} \right)^m \int_{B_{\epsilon}(x_0)} |\nabla u|^p + C R^{m-p} (\epsilon_{\text{reg},p}) \int_{M_\epsilon^0(B_{2\epsilon}(x_0))} |\nabla u|^p.
\] (2.18)

For the rest of the proof, we can apply the same arguments as in Schikorra ([41], p.510-511), similarly to Rivièere-Struwe ([39], Proof of Theorem 1.1, p.459-460), to obtain the Morrey type estimates for \( \nabla u \). The Hölder continuity of \( u \) in \( B \) follows immediately from an iteration argument as in [17]. \( \square \)

**Remark 2.1.** By slightly modifying the proof, we will see that the regularity result in Theorem 1.5 still hold if the elliptic system (1.15) is replaced by the following:
\[- \text{div} (Q \nabla u + \Omega u) = 0, \quad \text{in } \mathcal{D}'(B)\] (2.19)
with \( Q \in W^{1,2} \cap L^\infty(B, GL(n)) \) satisfying \(|Q| + |Q^{-1}| \leq \Lambda \), a.e. in \( B \), for some uniform constant \( \Lambda > 0 \). The proof relies on applying Hodge decomposition to \( Q \nabla u \) to get the Morrey type estimates for \( \nabla u \) as is done by Rivièere-Struwe ([39]. See also Schikorra [41]).

3. **Harmonic maps into standard stationary Lorentzian manifolds**

In this section, we shall first show that the Euler-Lagrangian equations for weakly harmonic maps into standard stationary Lorentzian manifolds are elliptic systems of the form (1.3) and then apply Theorem 1.2 to prove the \( \epsilon \)-regularity (Theorem 1.3) for such maps.

**Proof of Theorem 1.3:** Let \((t, u) \in W^{1,2}(B, \mathbb{R} \times M)\) be a weakly harmonic map from \( B \) into \((\mathbb{R} \times M, g)\), where the metric \( g \) is defined as in (1.9). For any \( s \in W^{1,2}_0 \cap L^\infty(B, \mathbb{R}) \) and for any \( v \in W^{1,2}_0 \cap L^\infty(B, \mathbb{R}^n) \), we have that
\[ t_\epsilon = t + \epsilon s \quad \text{and} \quad u_\epsilon = \Pi(u + \epsilon v) \] (3.1)
are well defined for sufficiently small \( \epsilon > 0 \). Hence \((t_\epsilon, u_\epsilon) \in W^{1,2}(B, \mathbb{R} \times M)\) gives an admissible variation for \((t, u)\). By Definition 1.1, there holds
\[
\frac{d}{d\epsilon} E(t_\epsilon, u_\epsilon) \bigg|_{\epsilon=0} = 0, \quad \forall s \in W^{1,2}_0 \cap L^\infty(B, \mathbb{R}), \forall v \in W^{1,2}_0 \cap L^\infty(B, \mathbb{R}^n) \] (3.2)
A straightforward calculation gives
\[
\int_B \left\{ \frac{1}{2} (\nabla \beta \cdot w) |\nabla t + \omega_\beta \nabla u|^2 - \beta (\nabla t + \omega_\beta \nabla u) \cdot (\nabla s + \omega_\beta \nabla w) + (\nabla \omega_\beta \cdot w) \nabla t + \nabla u \cdot \nabla w \right\} = 0, \] (3.3)
where \( w = d\Pi(u) v, v \in W^{1,2}_0 \cap L^\infty(B, \mathbb{R}^n) \).

To deduce the Euler-Lagrangian equations, we shall choose appropriate admissible variations in (3.3).
First, taking \( s \in W^{1,2}_0 \cap L^\infty(B) \) and \( v \equiv 0 \) in (3.3), we obtain
\[
0 = \int_B -\beta(u)(\nabla t + \omega_i(u)\nabla v^i) \cdot \nabla s.
\]
Since \( s \in W^{1,2}_0 \cap L^\infty(B) \) is arbitrarily chosen, we get the following conservation law
\[
- \text{div} \left\{ \beta(u) \left( \nabla t + \omega_i(u)\nabla v^i \right) \right\} = 0. \tag{3.4}
\]
Next, taking \( w = d\Pi(u)v, v \in W^{1,2}_0 \cap L^\infty(B, \mathbb{R}^n) \) and \( s \equiv 0 \) in (3.3) gives
\[
0 = \int_B \left\{ -\frac{1}{2}(\nabla \beta \cdot w)\nabla t + \omega_i\nabla u^i |^2 - \beta(\nabla t + \omega_i\nabla u^i) \cdot \left( \omega_j\nabla u^j + (\nabla \omega_k \cdot w)\nabla u^k \right) + \nabla u \cdot \nabla w \right\}
\]
\[
= \int_B \left\{ -\frac{1}{2} \left( \frac{\partial \beta}{\partial y^j} \cdot w^j \right)\nabla t + \omega_i\nabla u^i |^2 - \beta(\nabla t + \omega_i\nabla u^i) \cdot \left( \omega_j\nabla u^j + \nabla u^k \frac{\partial \omega_k}{\partial y^j} \cdot w^j \right) + \nabla u \cdot \nabla w \right\}
\]
\[
= \int_B \left\{ -\text{div} (\nabla u^i) \cdot w^j + \beta(\nabla t + \omega_i\nabla u^i) \cdot \nabla u^l \left( \frac{\partial \omega_l}{\partial y^j} - \frac{\partial \omega_k}{\partial y^j} \right) \cdot w^j - \frac{1}{2} |\nabla t + \omega_i\nabla u^i|^2 \sqrt{\beta} \cdot w^j \right\} \tag{3.5}
\]
where in the last step we have used (3.4) and integration by part. Denote \( H := (H^1, ..., H^n) \) with
\[
H^j := \beta(\nabla t + \omega_i\nabla u^i) \cdot \nabla u^l \left( \frac{\partial \omega_l}{\partial y^j} - \frac{\partial \omega_k}{\partial y^j} \right). \tag{3.6}
\]
Then (3.5) becomes
\[
0 = \int_B (- \text{div} \nabla u + H) \cdot d\Pi(u)v, \quad \forall v \in W^{1,2}_0 \cap L^\infty(B, \mathbb{R}^n)
\]
Since \( v \in W^{1,2}_0 \cap L^\infty(B, \mathbb{R}^n) \) is arbitrarily chosen, we have (similarly to the calculations in [20], Chapter 1.)
\[
- \text{div} \nabla u - A(u)(\nabla u, \nabla u) + d\Pi(u)H = 0, \tag{3.7}
\]
where \( A \) is the second fundamental form of \( M \) in \( \mathbb{R}^n \). Let \( v_l, l = d+1, d+2, ..., n \) be an orthonormal frame for the normal bundle \( T^\perp \mathcal{M} \) (and still denote by \( v_l \) the corresponding normal frame along the map \( u \)), then we can rewrite (3.7) as follows:
\[
- \text{div} \nabla u = v_l \nabla v_l \cdot \nabla u - H + \langle H, v_l \rangle v_l, \tag{3.8}
\]
where \( \langle \cdot, \cdot \rangle \) denotes the Euclidean metric on \( \mathbb{R}^n \). We have thus obtained the Euler-Lagrangian equations:
\[
- \text{div} \left\{ \beta(u) \left( \nabla t + \omega_i(u)\nabla v^i \right) \right\} = 0, \tag{3.9}
\]
\[
- \text{div} \nabla u = v_l \nabla v_l \cdot \nabla u - H + \langle H, v_l \rangle v_l. \tag{3.10}
\]
To proceed, we write the system of equations (3.9) and (3.10) in the form of (1.3). By Hodge decomposition, we conclude from the conservation law (3.4) that there exists \( \eta \in W^{1,2}(B, \wedge^2 \mathbb{R}^n) \) such that
\[
\beta(u) \left( \nabla t + \omega_i(u)\nabla v^i \right) = \text{curl} \eta. \tag{3.11}
\]
Then, by (3.6), we can rewrite the equation (3.10) as:
\[
- \text{div} \nabla u^l = \Theta_{jk} \cdot \nabla u^k + a_{jk} \cdot \text{curl} \eta \cdot \nabla u^k + b_{jk} \cdot \text{curl} \eta \cdot \beta(u)(\nabla t + \omega_i(v^i)), \tag{3.12}
\]
where

\begin{align}
\Theta^k_j &:= y^k_j \nabla y^k_j - y^k_i \nabla y^i_j, \\
a^k_j &:= -\left( \frac{\partial \omega_j}{\partial y^k} - \frac{\partial \omega_k}{\partial y^j} \right) + \left( \frac{\partial \omega_i}{\partial y^k} - \frac{\partial \omega_k}{\partial y^i} \right)y^k_j, \\
b_j &:= \frac{1}{2\beta(u)} \left( \frac{\partial \beta}{\partial y^j} y^j_i \right) y^k_j.
\end{align}

Now we can write the Euler-Lagragian equations (3.9) and (3.10) as the following elliptic system:

\begin{align}
- \text{div} \left( \bar{Q} \cdot \left( \begin{array}{c}
\nabla t \\
\nabla u
\end{array} \right) \right) &= \Theta \cdot \left( \begin{array}{c}
\nabla t \\
\nabla u
\end{array} \right) + F \text{ curl } \zeta \cdot \left( \begin{array}{c}
\nabla t \\
\nabla u
\end{array} \right),
\end{align}

where

\begin{align}
\bar{Q} &= \tilde{Q} \circ u, \\
\bar{Q} &= \left( \begin{array}{c}
\beta \\
0
\end{array} \right), \\
\Theta &= \left( \begin{array}{cc}
0 & \beta \omega \\
0 & I_n
\end{array} \right), \\
\omega &= (\omega_1, \omega_2, ..., \omega_n), \\
\zeta &= \text{diag}(\eta, \eta, ..., \eta).
\end{align}

Since \( M \) is compact, \( \beta \in C^2(M, (0, \infty)) \) and \( \omega \in C^2(\Omega^1(M)) \), there exists \( \lambda > 0 \) depending only on the target \( (\mathbb{R} \times M, g) \) such that for any \( y \in M \) there hold

\begin{align}
0 < \lambda^{-1} \leq \beta(y), \\
|\beta(y)| + |\nabla \beta(y)| + |\nabla^2 \beta(y)| \leq \lambda < \infty, \\
|\omega(y)| + |\nabla \omega(y)| + |\nabla^2 \omega(y)| \leq \lambda < \infty .
\end{align}

Using the notations (3.13)-(3.15), (3.17)-(3.20) and the above estimates (3.21), we can easily verify that \( \Theta \in L^2(B, so(n+1) \otimes \wedge^1 \mathbb{R}^m), \ F \in W^{1,2} \cap L^{\infty}(B, M(n+1)), \ z \in W^{1,2}(B, M(n+1) \otimes \wedge^2 \mathbb{R}^m), \ Q \in W^{1,2} \cap L^{\infty}(B, GL(n+1)) \) and the following estimates hold:

\begin{align}
||Q||_{M^2(B)} + ||F||_{M^2(B)} + ||\nabla \bar{Q}||_{M^2(B)} &\leq C_2(\lambda) ||\nabla u||_{M^2(B)} ,
\end{align}

where \( C_1(\lambda) > 0 \) and \( C_2(\lambda) > 0 \) are constants also depending on \( \lambda \).

To estimate \( |\bar{Q}|^{-1} \), we note that

\begin{align}
\bar{Q}^{-1} &= \left( \begin{array}{cc}
\beta^{-1} & -\omega \\
0 & I_n
\end{array} \right).
\end{align}

Hence, by (3.21), there exists some constant \( C_3(\lambda) > 0 \) such that

\begin{align}
|\bar{Q}^{-1}| &= |\bar{Q}^{-1} \circ u| \leq C_3(\lambda), \ \text{a.e. in } B.
\end{align}

On the other hand, it follows from (3.11) and (3.21) that

\begin{align}
|\text{curl } \eta| \leq C_4(\lambda) (||\nabla t|| + ||\nabla u||), \ \text{a.e. in } B.
\end{align}

By (3.20) and the above inequality, we verify that

\begin{align}
||\text{curl } \zeta||_{M^2(B)} \leq C_5(\lambda) \left( ||\nabla t||_{M^2(B)} + ||\nabla u||_{M^2(B)} \right).
\end{align}
Combining (3.22) and (3.24) gives
\[ |Q| + |Q^{-1}| + |F| \leq C_1(\lambda) + C_3(\lambda), \quad \text{a.e. in } B. \]  
(3.26)

Combining (3.23) and (3.25) gives
\[ \|\nabla f\|_{M^2(B)} + \|\nabla u\|_{M^2(B)} + \|\text{curl} \, \xi\|_{M^2(B)} + \|\Theta\|_{M^2(B)} + \|\nabla F\|_{M^2(B)} + \|\nabla Q\|_{M^2(B)} \]
\[ \leq (1 + C_2(\lambda) + C_3(\lambda)) \left( \|\nabla f\|_{M^2(B)} + \|\nabla u\|_{M^2(B)} \right). \]  
(3.27)

Take \( \Lambda := C_1(\lambda) + C_3(\lambda) > 0 \), then \( \Lambda \) depends only on \( (\mathbb{R}^m, g) \). Let \( \epsilon_m > 0 \) be the small constant (depending on \( m \) and \( \Lambda \)) as in Theorem 1.2. Take
\[ \epsilon_m := \frac{\epsilon_{m,\Lambda}}{1 + C_2(\lambda) + C_3(\lambda)}, \]
then \( \epsilon_m > 0 \) depends only on \( (\mathbb{R}^m, g) \). Applying Theorem 1.2 to the elliptic system (3.16), we conclude from (3.26) and (3.27) that \( (t, u) \) is Hölder continuous in \( B \) if \( \|\nabla f\|_{M^2(B)} + \|\nabla u\|_{M^2(B)} < \epsilon_m \). By standard elliptic regularity theory, \( (t, u) \) is as smooth as the regularity of the target \( (\mathbb{R}^m, g) \) permits. \( \square \)

4. Harmonic maps into pseudospheres \( S_v^n (1 \leq v \leq n) \)

In this section, we shall first prove Proposition 1.1 and Proposition 1.2. Then, with the help of these two propositions, we apply Theorem 1.5 to prove the regularity results (Theorem 1.6 and Theorem 1.7) for weakly harmonic maps into pseudospheres \( S_v^n (1 \leq v \leq n) \).

**Proof of Proposition 1.1:** Fix \( i \neq j \in \{1, 2, ..., n+1\} \). Let \( E_{ij} \in so(n+1) \) be the matrix whose \((i, j)\)-component is 1, \((j, i)\)-component is -1 and all the other components are 0. Let \( E \) be the matrix defined as in (1.17). Then one verifies that \( E_{ij} E \in so(n+1 - v) \) and \( e^{E_{ij} E} \in O(n+1 - v) \) (see e.g. [34]). For any \( \varphi \in C_0^\infty(B) \), define
\[ R_i := e^{\varphi E_{ij} E} \in C_0^\infty(B, O(n+1-v)). \]  
(4.1)

Using the property of an element in the group \( O(n+1-v) \) (see (1.19)), we have
\[ (R_i u, R_i u)_{B_0^{n+1}} = (R_i u)^T E R_i u = u^T R_i^T E R_i u = u^T E u = 1, \quad \text{a.e. in } B. \]  
(4.2)

It follows that \( R_i u \in W^{1,2}(B, S_v^n) \). Since \( u \) is weakly harmonic, by Definition 1.2, we calculate
\[ 0 = \frac{d}{dt} \bigg|_{t=0} E(R_i u) = \int_B \left( \nabla (R_i u) \right)^T E \frac{d}{dt} \bigg|_{t=0} (\nabla (R_i u)) \]
\[ = \int_B \left( \nabla u \right)^T E \left( \nabla (\varphi E_{ij} E u) \right) \]
\[ = \int_B \left( \nabla u \right)^T E E_{ij} E u \nabla \varphi + (\nabla u)^T E E_{ij} E \nabla u \varphi \]
\[ = \int_B \left( \nabla u \right)^T E E_{ij} E u \nabla \varphi \]
\[ = (e_{ij} E_{ij}) \int_B \left( u^T \nabla u^T - u^T \nabla u^T \right) \nabla \varphi, \]  
(4.3)

where we have used the fact that \( E E_{ij} E \in so(n+1) \) and hence
\( (\nabla u)^T E E_{ij} E \nabla u = 0 \quad \text{a.e. in } B. \)
Since $\varphi \in C_0^\infty(B)$ is arbitrary and $\epsilon_j \epsilon_{jj}$ is either 1 or $-1$ (see (1.17)), we conclude from (4.3) that the conservation laws (1.27) hold for $i \neq j$.

The case of $i = j$ is trivial. This completes the proof. 

**Proof of Proposition 1.2:** First, by definition of the space $W^{1,2}(B, \mathbb{S}^n)$ (see (1.21)), we have

$$u^j \epsilon_{jk} u^k = 1 \text{ a.e. in } B.$$  

(4.4)

Taking $\nabla$ on both sides of (4.4) gives

$$\nabla u^j \epsilon_{jk} u^k = 0 \text{ a.e. in } B.$$  

(4.5)

Recall that (see (1.28)) $\Theta = (\Theta^j) = (u^j \nabla u^i - u^i \nabla u^j)$. Combining (4.4) and (4.5), we calculate

$$\nabla u^i + \Theta^i \epsilon_{jk} u^k = \nabla u^i + (u^j \nabla u^i - u^i \nabla u^j) \epsilon_{jk} u^k$$

$$= \nabla u^i (1 - u^i \epsilon_{jk} u^k) + u^i (\nabla u^i \epsilon_{jk} u^k)$$

$$= 0 \text{ a.e. in } B.$$  

(4.6)

This proves (1.29).

Since $u \in W^{1,2}(B, \mathbb{R}^{n+1})$, one verifies that $\nabla u^i + \Theta^i \epsilon_{jk} u^k \in L^1(B)$ for each $i$. Taking $-\text{div}$ on both sides of (4.6) gives

$$- \text{div}(\nabla u + \Theta \; \mathcal{E} \; u) = 0, \quad \text{in } \mathcal{D}'(B).$$

Next, we assume that $u$ is weakly harmonic and for any fixed $1 < p < \frac{m}{m-1}$ there holds $\|\nabla u\|_{L^p(B)} < \infty$.

We shall derive the estimate (1.31).

Let $q = \frac{m}{p-1} > m$ be the conjugate exponent of $p$. Let $B_R(x_0) \subset B_{1/2}$. For any $\Phi \in L^q(B_R(x_0), \wedge L^m)$ with $\|\Phi\|_{L^q(B_R(x_0))} \leq 1$ and for any $0 < \rho < R$, let $\tau = \tau(\rho) \in C_0^\infty(B_R(x_0), [0, 1])$ be a cut-off function satisfying

$$\tau \equiv 1, \quad \text{on } B_\rho(x_0),$$

then $\tau \Phi$ is supported in $B_R(x_0)$ and vanishes on $\partial B_R(x_0)$. By Hodge decomposition, there exist $\alpha \in W^{1,q}_0(B_R(x_0)), $ $\beta \in W^{1,q}_0(B_R(x_0), \wedge L^m)$ and a harmonic $h \in C_0^\infty(B_R(x_0), \wedge L^m)$ such that

$$\tau \Phi = \nabla \alpha + \text{curl} \beta + h.$$  

(4.7)

Moreover, we have

$$\|\nabla \alpha\|_{L^q(B_R(x_0))} + \|\nabla \beta\|_{L^q(B_R(x_0))} \leq C\|\tau \Phi\|_{L^q(B_R(x_0))} \leq C\|\Phi\|_{L^q(B_R(x_0))} \leq C,$$  

(4.8)

where $C > 0$ is a constant independent of $\rho$ and $R$. Recall that $\tau \in C_0^\infty(B_R(x_0))$, we get $h \mid_{\partial B_R(x_0)} = (\tau \Phi) \mid_{\partial B_R(x_0)} = 0$. Since $h$ is harmonic, it follows that $h \equiv 0$ in $B_R(x_0)$.

Since $u$ is weakly harmonic, by Proposition 1.1, $\Theta = (\Theta^j) = (u^j \nabla u^i - u^i \nabla u^j)$ is divergence free. Then, using (4.7), (4.8) and the fact that $h \equiv 0$ in $B_R(x_0)$, and applying Lemma 2.1, we estimate for fixed $i, j \in$
Furthermore, using the definition of the Morrey norm
\[ \int_{B_R(x_0)} (\tau \Theta^{i}) \cdot \Phi = \int_{B_R(x_0)} \Theta^{i} \cdot (\tau \Phi) \]
\[ = \int_{B_R(x_0)} \Theta^{i} \cdot (\nabla \alpha + \text{curl} \beta) \]
\[ = \int_{B_R(x_0)} \Theta^{i} \cdot \text{curl} \beta \]
\[ = \int_{B_R(x_0)} (\nabla u^i - u^i \nabla u^i) \cdot \text{curl} \beta \]
\[ = \int_{B_R(x_0)} (\nabla u^i \cdot \text{curl} \beta) u^i - (\nabla u^i \cdot \text{curl} \beta) u^i \]
\[ \leq C \| \nabla u \|_{L^p(B_{\rho}(x_0))} \| \text{curl} \beta \|_{L^q(B_{\rho}(x_0))} \| \nabla u \|_{M^p(B_{\rho}(x_0))} \]
\[ \leq C \| \nabla u \|_{L^p(B_{\rho}(x_0))} \| \text{curl} \beta \|_{L^p(B_{\rho}(x_0))} \| \nabla u \|_{M^p(B_{\rho}(x_0))} \]
\[ \leq C \| \nabla u \|_{L^p(B_{\rho}(x_0))} \| \nabla u \|_{M^p(B_{\rho}(x_0))} \]  
\[ (4.9) \]
By duality characterization of $L^p$ functions, we have
\[ \| (\tau \Theta^{i}) \|_{L^p(B_{\rho}(x_0))} \leq C \| \nabla u \|_{L^p(B_{\rho}(x_0))} \| \nabla u \|_{M^p(B_{\rho}(x_0))} \]  
\[ (4.10) \]
It follows that
\[ \| \Theta^{i} \|_{L^p(B_{\rho}(x_0))} \leq \| (\tau \Theta^{i}) \|_{L^p(B_{\rho}(x_0))} \leq C \| \nabla u \|_{L^p(B_{\rho}(x_0))} \| \nabla u \|_{M^p(B_{\rho}(x_0))} \]  
\[ (4.11) \]
Since $\rho \in (0, R)$ is arbitrary, let $\rho \nearrow R$, then we get
\[ \| \Theta^{i} \|_{L^p(B_{\rho}(x_0))} \leq C \| \nabla u \|_{L^p(B_{\rho}(x_0))} \| \nabla u \|_{M^p(B_{\rho}(x_0))} \]  
\[ (4.12) \]
Furthermore, using the definition of the Morrey norm $\| \nabla u \|_{M^p(B_{\rho}(x_0))}$ and the fact that $B_{2R}(x_0) \subset B$, we estimate
\[ \| \Theta \|_{L^p(B_{\rho}(x_0))} = \sum_{i,j} \| \Theta^{i} \|_{L^p(B_{\rho}(x_0))} \leq C \| \nabla u \|_{L^p(B_{\rho}(x_0))} \| \nabla u \|_{M^p(B_{\rho}(x_0))} \]
\[ \leq C R^{\frac{m-1}{2}} \| \nabla u \|_{M^p(B_{\rho})} \| \nabla u \|_{M^p(B_{\rho})} \]
\[ = C R^{\frac{m-1}{2}} \| \nabla u \|_{M^p(B)}^2 \]
Since the ball $B_{\rho}(x_0) \subset B_{1/2}$ is arbitrary, it follows that
\[ \| \Theta \|_{M^p(B_{1/2})} = \sup_{B_{\rho}(x_0) \subset B_{1/2}} \left( \int_{B_{\rho}(x_0)} |\Theta|^p \right)^{\frac{1}{p}} \leq C \| \nabla u \|_{M^p(B)}^2 \]
Thus, we have completed the proof.

**Proof of Theorem 1.6:** Note that $E$ is a constant matrix. Combining Proposition 1.1, Proposition 1.2, Theorem 1.5 and using a rescaling of the domain gives that $u$ is Hölder continuous in $B$. Moreover, since $\text{div} \Theta = 0$, we can rewrite the equation in (1.30) as
\[ - \text{div} \nabla u = \Theta \cdot E \cdot \nabla u. \]
By standard elliptic regularity theory, $u$ is smooth in $B$. 

**Proof of Theorem 1.7:** Fix some $1 < p \leq \frac{m}{m-1} = 2$. By conformal invariance in dimension $m = 2$ and rescaling in the domain, we assume W.L.O.G that
\[ \| \nabla u \|_{L^p(B_{1/2})} < \varepsilon_{2,p}, \]  
\[ (4.13) \]
where $\epsilon_{z,p}$ is given in Theorem 1.6 with $m = 2$. By a straightforward calculation, it follows that

$$\|\nabla u\|_{M^p(B)}^2 \leq \|\nabla u\|_{L^2(B)}^2 < \epsilon_{2,p}. \quad (4.14)$$

Applying Theorem 1.6 with $m = 2$ gives that $u$ is Hölder continuous (and hence smooth) in $B$. \hfill \Box

5. Generalized (weakly) harmonic maps into $\mathbb{S}^n_\nu$ ($1 \leq \nu \leq n$)

In this section, we shall prove the $\epsilon$-regularity result (Theorem 1.8) for generalized (weakly) harmonic maps into $\mathbb{S}^n_\nu$ ($1 \leq \nu \leq n$). Throughout this section, $B$ will denote the unit disc in $\mathbb{R}^2$.

**Proof of Theorem 1.8:** Slightly modifying some arguments in the proofs of Proposition 1.2 and Theorem 1.5 will be sufficient to prove this theorem.

Fix any $\frac{1}{2} < p < 2$ and let $u \in W^{1,p}(B, \mathbb{S}^n_\nu)$ ($1 \leq \nu \leq n$) be a generalized (weakly) harmonic map satisfying

$$\|\nabla u\|_{M^p(B)}^2 < \epsilon_p \quad (5.1)$$

with $\epsilon_p > 0$ being determined later. Then $u \in W^{1,p}(B)$ and hence $\Theta = (\Theta^{ij}) := (u^i\nabla u^j - u^j\nabla u^i) \in L^{p'}(B)$, where $p' = \frac{2p}{p-1} \in (1, p)$. By Definition 1.5, there holds

$$\text{div } \Theta = 0, \quad \text{in } D'(B). \quad (5.2)$$

Applying similar arguments as in the proof of Proposition 1.2 with $m = 2$ gives that

$$\nabla u + \Theta E u = 0 \quad \text{a.e. in } B \quad (5.3)$$

and

$$\|\Theta\|_{M^{p'}(B_{1/2})} \leq C_{p'} \|\nabla u\|_{M^p(B)}^2 \leq C_{p'} \|\nabla u\|_{M^p(B)}^2 \leq C_{p'} \epsilon_p. \quad (5.4)$$

Let $B_{2R}(x_0) \subset B_{1/2}$ and let $w \in W^{1,p}(B_{R}(x_0), \mathbb{R}^{n+1})$ be solving

$$\begin{cases}
- \text{div } \nabla w &= 0, \quad \text{in } B_R(x_0) \\
w &= u, \quad \text{on } \partial B_R(x_0)
\end{cases} \quad (5.5)$$

and define $v := u - w \in W^{1,p}_{0}(B_{R}(x_0), \mathbb{R}^{n+1})$.

Let $q' = \frac{p'}{p' - 1}$ be the conjugate exponent of $p'$. Then for any $\varphi \in W^{1,q'}_{0}(B_{R}(x_0))$ with $\|\varphi\|_{W^{1,q'}(B_{R}(x_0))} \leq 1$. Using (5.3) and (5.5), we get

$$\int_{B_{R}(x_0)} \nabla v^i \cdot \nabla \varphi = \int_{B_{R}(x_0)} \nabla u^i \cdot \nabla \varphi - \int_{B_{R}(x_0)} \nabla w^i \cdot \nabla \varphi = - \epsilon_{ij} \int_{B_{R}(x_0)} \Theta^{ij} u^i \cdot \nabla \varphi$$

Then using (5.1), (5.2), (5.4), Lemma 2.1 and taking $\epsilon_p > 0$ sufficiently small, we can apply the same arguments as in the proof of Theorem 1.5 with $m = 2$ and use a rescaling of the domain to conclude that $u$ is Hölder continuous and hence smooth (by standard elliptic regularity) in $B$. \hfill \Box

Furthermore, we observe that the $\epsilon$-regularity result in Theorem 1.8 still hold if the Morrey norm $\|\nabla u\|_{M^p(B)}$ is replaced with the Lorentz norm $\|\nabla u\|_{L^{1,\infty}(B)}$ (which was used in Almeida [2]). To see this, we recall the following:
Lemma 5.1 (Almeida [2], Lemma 9). Suppose $D$ has finite measure. Let $1 < p < p_1 < \infty$. Then, there is a constant $C$ such that, for all $q, q_1 \in [1, \infty]$ and for any $f \in L^{p_1} (D)$,

$$
\| f \|_{L^p(D)} \leq C \left( \mu(D) \right)^{\frac{p-1}{p_1}} \| f \|_{L^{p_1}(D)}
$$

(5.6)

Recall that $L^{p,p} = L^p$. Consequently, we have

Lemma 5.2. Let $1 < p < 2$. Then, there is a constant $C$ such that, for any $f \in L^{2,\infty}(B)$,

$$
\| f \|_{M^p(B)} \leq C \| f \|_{L^{2,\infty}(B)}
$$

(5.7)

Proof. Take $p = q, p_1 = 2, q_1 = \infty$ in Lemma 5.1 and let $D$ run over all discs $B(x_0) \subset B$.

Combining Theorem 1.8 and Lemma 5.2 gives the following $\varepsilon$-regularity result (using the Lorentz norm).

Theorem 5.1. There exists $\varepsilon > 0$ such that any generalized (weakly) harmonic map $u \in W^{1,\frac{2}{\varepsilon}} (B, \mathbb{R}^n)$ $(1 \leq \nu \leq n)$ satisfying

$$
\| \nabla u \|_{L^{2,\infty}(B)} < \varepsilon
$$

is smooth in $B$.

6. Regularity for an elliptic system with a potential in $so(1, 1)$

Throughout this section, $B$ will denote the unit disc in $\mathbb{R}^2$. We consider the elliptic system (1.1) with a potential $\Omega \in L^2(B, so(1, 1) \otimes \Lambda^1 \mathbb{R}^2)$. By Hodge decomposition, there exist $\Omega_1 \in W^{1,2}(B, so(1, 1))$ and $\Omega_2 \in W^{1,2}(B, so(1, 1) \otimes \Lambda^2 \mathbb{R}^2)$ such that

$$
\Omega = \nabla \Omega_1 + \text{curl } \Omega_2.
$$

(6.1)

Theorem 6.1. Let $u \in W^{1,2}(B, \mathbb{R}^2)$ be a weak solution of the elliptic system (1.1) with a potential $\Omega \in L^2(B, so(1, 1) \otimes \Lambda^1 \mathbb{R}^2)$. Decompose $\Omega$ as in (6.1). If $\Omega_1 \in L^\infty(B, so(1, 1))$, then $u$ is Hölder continuous in $B$.

Proof. Since $\Omega_1$ takes values in $so(1, 1)$, we can write (see O’Neill’s book [34])

$$
\Omega_1 = \left( \begin{array}{cc} 0 & s \\ s & 0 \end{array} \right), \quad \text{for some } s \in W^{1,2}(B).
$$

(6.2)

Consequently, we have $\nabla \Omega_1 = \Omega_1 \nabla \Omega_1$ and hence $\nabla (e^{\Omega_1}) = e^{\Omega_1} \nabla \Omega_1$. Then we calculate

$$
- \text{div} (e^{\Omega_1} \nabla u) = - e^{\Omega_1} \nabla \Omega_1 \cdot \nabla u + e^{\Omega_1} \Omega_1 \cdot \nabla u = e^{\Omega_1} \text{curl } \Omega_2 \cdot \nabla u.
$$

(6.3)

Using (6.2), we get

$$
e^{\Omega_1} = (e^{\Omega_1})^T = \left( \begin{array}{cc} \cosh s & \sinh s \\ \sinh s & \cosh s \end{array} \right), \quad e^{-\Omega_1} = (e^{\Omega_1})^{-1} = \left( \begin{array}{cc} \cosh s & -\sinh s \\ -\sinh s & \cosh s \end{array} \right).
$$

Since $\Omega_1 \in L^\infty(B)$, there exists a constant $\lambda \in (0, \infty)$, such that $|s| \leq \lambda$, a.e. in $B$. Therefore, we have

$$
|e^{\Omega_1}| + |(e^{\Omega_1})^{-1}| \leq C(\lambda), \quad \text{a.e. in } B
$$

for some constant $C(\lambda) > 0$ depending on $\lambda$.

On the other hand, one verifies that $e^{\Omega_1} \in W^{1,2} \cap L^\infty(B, M(2))$. Recall that $\Omega_2 \in W^{1,2}(B, so(1, 1) \otimes \Lambda^2 \mathbb{R}^2)$. Applying Theorem 1.2 (with $m = 2$ and $\Lambda = C(\lambda)$) to the elliptic system (6.3), using the conformal invariance in dimension $m = 2$ and rescaling in the domain, we get the Hölder continuity of $u$ in $B$. □
Theorem 6.1 is optimal. To see this, we set
\[ s(x) = \log \log \frac{2}{|x|}, \quad u_1(x) = \log \log \frac{2}{|x|}, \quad u_2(x) = \log \log \frac{2}{|x|}, \quad x \in B. \]

Then the map \( u = (u_1, u_2)^T \in W^{1,2}(B, \mathbb{R}^2) \) is a weak solution to the elliptic system (1.1) with a potential \( \Omega \) satisfying
\[
\Omega = \begin{pmatrix} 0 & \nabla s \\ \nabla s & 0 \end{pmatrix} \in L^2(B, so(1, 1) \otimes \Lambda^1 \mathbb{R}^2) \text{ and } s \text{ is not in } L^\infty(B).
\]

However, \( u \) is not in \( L^\infty(B) \).

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