Comparing decompositions of Poincaré duality pairs

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Abstract

Analogues of JSJ decompositions were developed for Poincaré duality pairs in [19]. These decompositions depend only on the group. Our focus will be on describing the edge splittings of these decompositions more precisely. We use our results to compare these decompositions with two other closely related decompositions.
1 Introduction

In this paper, we consider algebraic analogues of previous work in the topology of 3–manifolds related to the JSJ decomposition introduced by Jaco and Shalen [8] and Johannson [9]. In [8] and [9], the authors considered a compact orientable Haken 3–manifold $M$ with incompressible boundary, and constructed the characteristic submanifold $V(M)$ as a maximal Seifert pair embedded in $M$. The frontier of $V(M)$ is a family of disjoint essential annuli and tori in $M$, which decompose $M$ into pieces either in $V(M)$ or its complement. In [12], [13], [14], [16] and [20], the emphasis turned to annuli and tori rather than Seifert pairs. In [12], the authors gave a new approach to constructing this decomposition of $M$ in which embedded essential annuli and tori were the main subject of interest. They defined an embedded essential annulus or torus in $M$ to be canonical if it can be isotoped to be disjoint from any other embedded essential annulus or torus in $M$. This led to a finer decomposition of $M$, which they called the Waldhausen decomposition (W-decomposition), from which the JSJ decomposition can be obtained in a natural way. In [16], the interest was in possibly singular essential annuli and tori in $M$. The authors defined an embedded essential annulus or torus in $M$ to be topologically canonical if it has intersection number zero with any (possibly singular) essential annulus or torus in $(M, \partial M)$. Most canonical annuli and tori in a 3–manifold are also topologically canonical, and the exceptions can be precisely described. The existence of these exceptions explains why the W-decomposition of $M$ is in general finer than the JSJ decomposition. In [16], the authors also defined an algebraic analogue in which a splitting of $\pi_1(M)$ given by an essential embedded annulus or torus in $M$ is algebraically canonical if it has intersection number zero with any almost invariant subset of $\pi_1(M)$ which is over $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$. (See [15] for a discussion of the idea of intersection numbers of almost invariant sets.) They showed that topologically canonical splittings are not quite the same as algebraically canonical ones, and gave some examples to demonstrate this. If $M$ has empty boundary, there is no difference.

In [20], (which is a revised version of [19]), an analogue of the JSJ decomposition of 3–manifolds was developed for orientable $PD(n + 2)$ pairs, with $n \geq 1$. The decomposition for a $PD(n + 2)$ pair $(G, \partial G)$ is simply the decomposition $\Gamma_{n,n+1}(G)$ of [17], which is defined for many almost finitely presented groups $G$. (If $\partial G$ is empty, so that $G$ is a $PD(n + 2)$ group, then $\Gamma_{n,n+1}(G)$ is equal to the decomposition $\Gamma_{n+1}(G)$.) See [18] for corrections.

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to [17]. Thus this decomposition depends only on the group $G$ and not on $\partial G$. In the case when $\partial G$ is empty, Kropholler [10] had earlier obtained this decomposition. In [17], the decomposition $\Gamma_{n,n+1}(G)$ was constructed as the regular neighbourhood of the family consisting of all almost invariant (a.i.) subsets of $G$ over $VPCn$ subgroups together with all a.i. subsets of $G$ over $VPC(n+1)$ subgroups which do not cross any a.i. subset of $G$ over a $VPCn$ subgroup. (A group is $VPCn$ if it is virtually polycyclic of Hirsch length $n$.) This regular neighbourhood is reduced, meaning that two adjacent vertices cannot both be isolated, except when the graph is a loop with just these two vertices. (A vertex $w$ of a $G$–tree $T$ is called isolated if it has valence 2, and these two edges have the same stabilizer as $w$. The image of $w$ in $G\backslash T$ is also called isolated.) In general, the edge groups of $\Gamma_{n,n+1}$ may not even be finitely generated, but it is shown in [20] that, in the case of $PD(n+2)$ pairs, the edge groups are all either $VPCn$ or $VPC(n+1)$.

It was shown in [20] that if $G$ is the fundamental group of a compact orientable Haken 3–manifold $M$ with incompressible boundary, then $\Gamma_{1,2}(G)$ differs from the JSJ decomposition of $M$ only in some small Seifert pieces which have no crossing annuli or tori. One can easily move from $\Gamma_{1,2}(G)$ to its completion $\Gamma_{1,2}^c(G)$, and this completion corresponds to the JSJ decomposition of $M$. The difference between $\Gamma_{1,2}(G)$ and its completion $\Gamma_{1,2}^c(G)$ is related to special properties of small Seifert fibre spaces. If the boundary of $M$ is empty, then similar comments apply to $\Gamma_2(G)$ and its completion $\Gamma_2^c(G)$. In the general case of a $PD(n+2)$ pair $(G,\partial G)$ we denote the completion of $\Gamma_{n,n+1}(G)$ by $\Gamma_{n,n+1}^c(G)$ or just $\Gamma_{n,n+1}^c$ when the group $G$ is clear from the context. The completion $\Gamma_{n,n+1}^c(G)$ is essentially obtained from $\Gamma_{n,n+1}(G)$ by re-labelling some $V_1$–vertices as $V_0$–vertices and then adding isolated $V_1$–vertices as needed to keep the graph bipartite. In particular, the edge splittings of $\Gamma_{n,n+1}$ are the same as the edge splittings of $\Gamma_{n,n+1}^c$. Again the completed decomposition depends only on the group $G$ and not on $\partial G$.

Our focus in this paper will be on the edge splittings of these decompositions of a group $G$. This is closely related to the approach in [16] in the case of 3–manifolds. Our main result, Theorem 4.1, is similar to that in [16] but is in the setting of the decomposition $\Gamma_{n,n+1}(G)$ of a $PD(n+2)$ pair $(G,\partial G)$. Our result is more detailed than that in [16], and gives a precise description of the special cases which arise. These results are new even in the setting of 3–manifolds, and they yield a substantial refinement of the results in [16].

There are other natural approaches to finding JSJ decompositions of a $PD(n+2)$ pair $(G,\partial G)$. The analogue of [12] would be to consider a max-
imal family of splittings of $G$ by annuli and tori which cross no other such splitting. (In this paper, we use the word "cross" to mean "has non-zero intersection number with".) Another approach would be simply to consider the regular neighbourhood of the family of all almost invariant subsets of $G$ which are over $VPC(n)$ or $VPC(n+1)$ subgroups. For general groups, neither of these decompositions need exist. However, in this paper, we use the results of [20] and of Theorem 4.1 to show that both decompositions exist in the setting of Poincaré duality pairs, and we compare these three different decompositions. The differences between them leads to a detailed study of various small 2-dimensional orbifolds and fibrations over them by $VPC(n)$ groups. We think that this clarifies how these various natural decompositions come about. We also discuss the special case of $PD3$ pairs where the descriptions are somewhat simpler. This seems to make an analogue of Johannson’s Deformation Theorem possible for $PD3$ pairs, and also seems relevant to some questions raised by Wall in sections 6 and 10 of [23].

In section 2 we describe the notions of annuli, tori, and their associated almost invariant sets as in [20]. We will also recall some constructions and results from [20] which we will use. In section 3 we describe examples generalizing those given in [16]. We also completely characterize all such examples. In section 4, we prove our main result, Theorem 4.1. In section 5 we compare three different versions of JSJ decompositions of $PD$–pairs. In section 6 we discuss how our results become substantially simpler when applied to the case of $PD3$ pairs, and we compare our results with those in [12]. In section 7 we discuss some related questions.

We will also use earlier definitions and results from [1], [2], [3] and [10]. There are two survey articles from around 2000, [4] and [23], which contain a number of problems related to $PD$ groups and pairs.

## 2 Preliminaries

We will consider orientable $PD(n+2)$ pairs $(G, \partial G)$ and first describe, following [20], annuli and tori in $(G, \partial G)$ and their associated almost invariant sets.

Let $H$ be a $VPC(n+1)$ subgroup of $G$. Note that as $G$ is a $PD$ group, it is torsion free. Hence $H$ is also torsion free, and so is a $PD(n+1)$ group. The double $DG$ of $G$ is an orientable $PD(n+2)$ group, and so the pair $(DG, H)$ has two ends if $H$ is orientable, and only one end otherwise. In the
first case, \(DG\) contains two complementary nontrivial \(H\)–almost invariant subsets \(X\) and \(X^*\), and any nontrivial \(H\)–almost invariant subset of \(DG\) is equivalent to one of these. Let \(Y\) denote the intersection \(X \cap G\). Thus \(Y\) and its complement \(Y^*\) in \(G\) are \(H\)–almost invariant subsets of \(G\). Further they are nontrivial unless \(H\) is peripheral in \((G, \partial G)\), i.e. \(H\) is conjugate into a group in \(\partial G\). If \(H\) is an orientable \(VPC(n+1)\) subgroup of \(G\), we call it a torus in \(G\), and \(Y\) and \(Y^*\) are the \(H\)–almost invariant subsets of \(G\) which we associate to this torus. Note that \(Y\) is automatically adapted to \(\partial G\). Conversely, suppose that \(H\) is a \(VPC(n+1)\) subgroup of \(G\), and \(Y\) is a nontrivial \(H\)–almost invariant subset of \(G\) which is adapted to \(\partial G\). Then \(Y\) extends to a nontrivial \(H\)–almost invariant subset of \(DG\). It follows that \(H\) must be orientable and hence a torus in \(G\).

The case of annuli requires more work. An annulus in a \(PD(n+2)\) pair is a certain type of orientable \(PD(n+1)\) pair. We need to consider two types of annulus. One type is \(\Lambda_H = (H, \{H, H\})\), where \(H\) is an orientable \(PDn\) group which is also \(VPCn\). We call this an untwisted annulus. The other type is \(\Lambda_H = (H, H_0)\), where \(H\) is a non-orientable \(PDn\) group which is \(VPCn\), and \(H_0\) is the orientation subgroup of \(H\). We call this a twisted annulus. Corresponding to these, we have \(K(\pi, 1)\) spaces which we denote by \((A, \partial A)\). Similarly, we denote the \(K(\pi, 1)\) pair corresponding to \((G, \partial G)\) by \((M, \partial M)\). Note that when \(n = 1\), the only \(PD1\) group is \(\mathbb{Z}\), and this is orientable. Thus twisted annuli do not appear in the theory of 3–manifolds. For simplicity, we assume that \(G\) is finitely presented so that we can identify certain cohomology groups of \(G\) with the cohomology groups with compact supports of various covers of \(M\). In the general case, when \(G\) is almost finitely presented, we have to take the appropriate ‘finitely supported’ cohomology groups of covers of \(M\).

An annulus in \((G, \partial G)\) is an injective homomorphism of group pairs \(\Theta : \Lambda_H \to (G, \partial G)\). This means that \(\Theta\) maps \(H\) to \(G\) and also maps each group in \(\partial \Lambda_H\) to a conjugate of some group in \(\partial G\). Such \(\Theta\) induces a continuous map \(\theta : (A, \partial A) \to (M, \partial M)\). Note that in the untwisted case, such a map \(\theta\) is determined up to homotopy by choosing a copy of \(H\) in two conjugates of groups in \(\partial G\), such that the two copies of \(H\) are conjugate in \(G\). And in the twisted case, \(\theta\) is determined up to homotopy by choosing a copy of \(H\) in \(G\) and a conjugate of some group in \(\partial G\) such that the intersection of \(H\) with this conjugate contains \(H_0\). Thus an annulus can be thought of purely algebraically. We call the annulus ‘essential’ if \(\theta\) cannot be homotoped relative to \(\partial A\) into \(\partial M\). It is clear that the essentiality of an annulus is also
a purely algebraic property. An untwisted annulus is essential if and only if the images of the two boundary groups are not conjugate in a group in $\partial G$. And a twisted annulus is essential if and only if $H_0$ lies in a boundary group $K$ in $\partial G$, and $H \cap K = H_0$.

We next show how to associate an almost invariant set to an essential annulus. Consider the lift $\theta_H : (A, \partial A) \to (M_H, \partial M_H)$ to the cover $M_H$ of $M$ with fundamental group $H$. Let $S$ be the component of $\partial M_H$ containing $\theta_H(\partial_0 A)$, where $\partial_0 A$ is one specified component of $\partial A$ in the untwisted case and is $\partial A$ in the twisted case.

In the untwisted case, since $\theta_H(A)$ cannot be homotoped rel $\partial A$ into $\partial M$, the other component $\partial_1 A$ must be mapped by $\theta_H$ into some other component $T$ of $\partial M_H$. Thus both $S$ and $T$ have fundamental groups isomorphic to $H$ and the images of the fundamental cycle of $H$ generate $H_n(S)$ and $H_n(T)$, both with $\mathbb{Z}$ and $\mathbb{Z}_2$ coefficients. Let $[A]$ denote the fundamental cycle of $A$ in $H_{n+1}(A, \partial A)$. Then in the boundary map

$$H_{n+1}(M_H, \partial M_H) \to H_n(\partial M_H) \simeq H_n(S) \oplus H_n(T) \oplus \cdots$$

we see that the projection of the image of $[A]$ to each of the first two direct summands of $H_n(\partial M_H)$ is a generator.

In the twisted case, $\pi_1(S)$ must be isomorphic to $H_0$ since $\theta$ cannot be homotoped into $\partial M$ relative to $\partial A$. Thus the projection of the image of $[A]$ to the summand $H_n(S)$ is a generator. So, with $\mathbb{Z}$ or $\mathbb{Z}_2$ coefficients, we have $(\theta_H)_*([A])$ is non-zero in $H_{n+1}(M_H, \partial M_H)$. Also the image with $\mathbb{Z}_2$–coefficients is the specialisation of the image with $\mathbb{Z}$–coefficients. Denote these images by $\alpha$ and $\tilde{\alpha}$, and denote the duals of these images in $H^1_c(M_H; \mathbb{Z})$ and $H^1_c(M_H; \mathbb{Z}_2)$ by $\beta$ and $\tilde{\beta}$. Again $\tilde{\beta}$ is the specialisation of $\beta$.

Next, we want to relate $\tilde{\beta}$ to the ends of the pair $(G, H)$. We have that $H$ is of infinite index in $G$, so that $H^0_\ell(M_H; \mathbb{Z}_2) = 0$ (finite cohomology is used as in section 7.4 of [3]). Thus $H^1_\ell(M_H; \mathbb{Z}_2)$ fits into the exact sequence:

$$0 \to \mathbb{Z}_2 \to H^0_\ell(M_H; \mathbb{Z}_2) \to H^1_\ell(M_H; \mathbb{Z}_2) \overset{\tau}{\to} H^1(M_H; \mathbb{Z}_2).$$

Here $H^0_\ell$ is the 0–th cohomology of the space of ends and the last map is the restriction map from finite cohomology to ordinary cohomology. Group theoretically, the above sequence identifies with the following. Let $P[H \setminus G]$ denote the power set of right cosets $Hg$ of $H$ in $G$, and let $E[H \setminus G]$ denote $P[H \setminus G]/\mathbb{Z}_2[H \setminus G]$. The above sequence can be identified with

$$0 \to \mathbb{Z}_2 \to H^0(G; E[H \setminus G]) \overset{\delta}{\to} H^1(G; \mathbb{Z}_2[H \setminus G]) \overset{\tau}{\to} H^1(H; \mathbb{Z}_2).$$
Thus $\bar{\beta}$ gives an element of $H^1(G; \mathbb{Z}_2[H\backslash G])$ which we continue to denote by $\bar{\beta}$. In order to show that this element gives a nontrivial almost invariant set, we need to know that it is non-zero in $H^1(G; \mathbb{Z}_2[H\backslash G])$, and is in the kernel of $\bar{r}$. We already know that $\bar{\beta}$ is non-zero since we started with an essential annulus. Thus it remains to show that $\bar{r}(\bar{\beta}) = 0$. Consider the following diagram:

$$
\begin{array}{c}
H^1(G; \mathbb{Z}[H\backslash G]) \xrightarrow{\tau} H^1(H; \mathbb{Z}) \\
\downarrow \rho \quad \downarrow \rho \\
H^1(G; \mathbb{Z}_2[H\backslash G]) \xrightarrow{\bar{r}} H^1(H; \mathbb{Z}_2)
\end{array}
$$

In Theorem 2 of [21], Swarup showed that $\bar{r}$ is the zero map. Since $\bar{\beta} = \rho(\beta)$ it follows that $\bar{r}(\bar{\beta}) = 0$, although in general $\bar{r}$ is not the zero map. Thus we see that $\bar{\beta} = \delta(e)$ for some element $e$ of $H^0(G; E[H\backslash G])$. Since the kernel of $\delta$ is just $\mathbb{Z}_2$, the element $e$ defines a nontrivial $H$–almost invariant subset of $G$ up to equivalence and complementation. This completes our association of an almost invariant set with an essential annulus. It turns out that given a nontrivial almost invariant subset $X$ of $G$ which is over a $VPC_n$ group $H$, there is a subgroup $H'$ of finite index in $H$ such that $X$ is a finite sum of almost invariant sets over $H'$ each associated to an annulus.

In [17], the authors considered an almost finitely presented group $G$ and an integer $n$ such that $G$ has no nontrivial almost invariant subsets over $VPC_k$ subgroups for $k < n$. Then it was shown that the family $\mathcal{F}_{n,n+1}$ of all equivalence classes of almost invariant subsets of $G$ over $VPC_n$ groups, and all $n$-canonical almost invariant subsets over $VPC(n+1)$ groups has an algebraic regular neighbourhood, denoted $\Gamma_{n,n+1}(G)$. In this setting a $H$–almost invariant subset of $G$ is $n$–canonical if it does not cross any almost invariant subset over a $VPC_n$ subgroup. If $(G, \partial G)$ is a $PD(n+2)$ pair, it was shown by Kropholler and Roller (Lemma 4.3 of [10]) that $G$ has no nontrivial almost invariant subsets over $VPC_k$ subgroups for $k < n$, so that the decomposition $\Gamma_{n,n+1}(G)$ exists. In [20], the authors showed that almost invariant subsets of $G$ over $VPC(n+1)$ subgroups which do not cross any almost invariant subset over a $VPC_n$ subgroup are automatically adapted to $\partial G$. Further, if we enlarge the family $\mathcal{F}_{n,n+1}$ to include all almost invariant sets over $VPC(n+1)$ subgroups which are adapted to $\partial G$, the new family $\mathcal{G}_{n,n+1}$ has the same regular neighbourhood $\Gamma_{n,n+1}(G)$.

If $M$ is a compact orientable Haken 3–manifold with incompressible (i.e. $\pi_1$–injective) boundary, the characteristic submanifold $V(M)$ of $M$ is a compact submanifold whose frontier consists of incompressible annuli and tori in
This decomposition of $M$ into pieces is called the JSJ decomposition. The components of $V(M)$ are Seifert fibre spaces or $I$–bundles. Cutting $M$ along the frontier of $V(M)$ yields a graph of groups structure $\Gamma(M)$ for $G = \pi_1(M)$ whose edge groups are isomorphic to $\mathbb{Z}$ or to $\mathbb{Z} \times \mathbb{Z}$. This graph is bipartite as each component of the frontier of $V(M)$ lies in the boundary of a component of $V(M)$ and a component of the complement. In [20], the authors showed that if $(G, \partial G)$ is a $PD(n+2)$ pair, then $\Gamma_{n,n+1}(G)$ has many properties in common with $\Gamma(M)$, with $V_0$–vertices of $\Gamma_{n,n+1}(G)$ corresponding to the components of $V(M)$. For the complete details the reader is referred to [20], but we will need to recall some of the definitions for use in this paper.

In [20], an important part is played by groups which are $VPCn$–by–Fuchsian. Such a group has a $VPCn$ normal subgroup whose quotient is the orbifold fundamental group of a compact 2–orbifold. Further the quotient is assumed not to be virtually cyclic. A $V_0$–vertex $v$ of $\Gamma_{n,n+1}$ such that $G(v)$ is $VPCn$–by–Fuchsian corresponds to a component $W$ of $V(M)$ which is a Seifert fibre space. Topologically there are different cases depending on how $W$ meets $\partial M$, and extra conditions are imposed on the edges of $\Gamma_{n,n+1}(G)$ which are incident to $v$ to reflect this. This is what is meant by saying that $v$ is of Seifert type. If $v$ is a $V_0$–vertex of $\Gamma_{n,n+1}$ such that $G(v)$ is $VPCn$–by–$\pi_{orb}^1(X)$, where $X$ is a 2–orbifold with virtually cyclic fundamental group, we say that $v$ is of solid torus type if $\pi_{orb}^1(X)$ is finite, and of torus type otherwise. The terminology reflects the type of the corresponding components of $V(M)$. Again conditions need to be imposed on the edges of $\Gamma_{n,n+1}(G)$ which are incident to $v$.

There are some other important special cases. In [20], the authors defined a $V_1$–vertex of $\Gamma_{n,n+1}(G)$ to be of special Seifert type if it has only one incident edge $e$ which is dual to an essential torus, and $G(e)$ is of index 2 in $G(v)$. Also a $V_1$–vertex of $\Gamma_{n,n+1}(G)$ is of special solid torus type, if $G(v)$ is $VPCn$, and either $v$ has valence 3 and each of the three edge groups is equal to $G(v)$, or $v$ has valence 1 and the single edge group has index 2 or 3 in $G(v)$. The authors also considered the completion $\Gamma_{n,n+1}^c$ of $\Gamma_{n,n+1}$. This is obtained from $\Gamma_{n,n+1}$ by re-labelling as $V_0$–vertices those $V_1$–vertices of special Seifert type or of special solid torus type, then adding isolated $V_1$–vertices to keep the graph bipartite. If the result is not reduced, we reduce it by collapsing edges.

Now the main theorem of [20] can be stated.
Theorem 2.1 (Theorem 3.14 of [20]) Let \((G, \partial G)\) be an orientable \(PD(n+2)\) pair such that \(G\) is not \(V PC\). Let \(\mathcal{F}_{n,n+1}\) denote the family of equivalence classes of all nontrivial almost invariant subsets of \(G\) which are over a \(V PCn\) subgroup, together with the equivalence classes of all \(n\)-canonical almost invariant subsets of \(G\) which are over a \(V PC(n+1)\) subgroup. Finally let \(\Gamma_{n,n+1}\) denote the reduced algebraic regular neighbourhood of \(\mathcal{F}_{n,n+1}\) in \(G\), and let \(\Gamma_{n,n+1}^c\) denote the completion of \(\Gamma_{n,n+1}\). Thus \(\Gamma_{n,n+1}\) and \(\Gamma_{n,n+1}^c\) are bipartite graphs of groups structures for \(G\), with vertices of \(V_0\)-type and of \(V_1\)-type.

Then \(\Gamma_{n,n+1}\) and \(\Gamma_{n,n+1}^c\) have the following properties:

1. Each \(V_0\)-vertex \(v\) of \(\Gamma_{n,n+1}\) satisfies one of the following conditions:
   
   (a) \(v\) is isolated, and \(G(v)\) is \(V PC\) of length \(n\) or \(n+1\), and the edge splittings associated to the two edges incident to \(v\) are dual to essential annuli or tori in \(G\).
   
   (b) \(v\) is of \(V PC(n-1)\)-by-Fuchsian type, and is of \(I\)-bundle type.
   
   (c) \(v\) is of \(V PCn\)-by-Fuchsian type, and is of interior Seifert type.
   
   (d) \(v\) is of commensuriser type. Further \(v\) is of Seifert type, or of torus type, or of solid torus type.

2. The \(V_0\)-vertices of \(\Gamma_{n,n+1}^c\) obtained by the completion process are of special Seifert type or of special solid torus type.

3. Each edge splitting of \(\Gamma_{n,n+1}\) and of \(\Gamma_{n,n+1}^c\) is dual to an essential annulus or torus in \(G\).

4. Any nontrivial almost invariant subset of \(G\) over a \(V PC(n+1)\) group and adapted to \(\partial G\) is enclosed by some \(V_0\)-vertex of \(\Gamma_{n,n+1}\), and also by some \(V_0\)-vertex of \(\Gamma_{n,n+1}^c\).

5. If \(H\) is a \(V PC(n+1)\) subgroup of \(G\) which is not conjugate into \(\partial G\), then \(H\) is conjugate into a \(V_0\)-vertex group of \(\Gamma_{n,n+1}^c\).

Remark 2.2 Recall that a vertex \(w\) of a \(G\)-tree \(T\) is called isolated if it has valence 2, and these two edges have the same stabilizer as \(w\). The image of \(w\) in \(G\backslash T\) is also called isolated.
Notice that any vertex $v$ of $\Gamma_{n,n+1}$ or $\Gamma_{c,n+1}^{n,n+1}$ has two types of "boundary" subgroups. The first type comes from the edge groups of the decomposition and the family of all these subgroups will be denoted by $\partial_1 v$. The second type comes from the decomposition of $\partial G$ by edges of the decompositions and this family will be denoted by $\partial_0 v$. The first type gives us $PD(n+1)$ pairs in $(G, \partial G)$, namely annuli or tori, and the second type gives us $PD(n+1)$ pairs which are contained in $\partial G$. In the three-dimensional topological case, $\partial_0 v$ and $\partial_1 v$ correspond to surfaces which can be amalgamated to yield the boundary of the 3–manifold $M(v)$ which corresponds to $v$. But this boundary may be compressible, and so need not yield a $PD3$–pair. In the general case we get a triple $(G(v); \partial_0 v, \partial_1 v)$ which corresponds to a Poincaré triad ([22]) but this theory in the case of groups has not been worked out.

We should also discuss the reason for excluding $VPC$ groups from consideration in Theorem 2.1. For simplicity we will consider the case when $\partial G$ is empty, so that $G$ is a $PD(n+2)$ group. Thus $\mathcal{F}_{n,n+1}$ consists of the equivalence classes of all almost invariant subsets of $G$ which are over a $VPC(n+1)$ subgroup, so that $\Gamma_{n,n+1}(G) = \Gamma_{n+1}(G)$. As $G$ is $VPC$ and $PD(n+2)$, it must be $VPC(n+2)$. As $\partial G$ is empty, cases 1b) and 1d) of Theorem 2.1 cannot arise. Also as $G$ is $VPC$, the condition of being of $VPCn$–by–Fuchsian type in case 1c) can never occur. It should be replaced by the condition of being $VPCn$–by–$VPC2$, to have a statement with some chance of holding. By the definition of $VPC$, any $VPC(n+2)$ group $G$ contains some $VPC(n+1)$ subgroup, and hence must contain a torus $T$. If $G$ admits a second torus $T'$ which crosses $T$, then $\Gamma_{n+1}(G)$ consists of a single $V_0$–vertex. But this vertex need not satisfy the modified condition 1c) in the statement of Theorem 2.1. For example, in section 7 of [7], the author gives two examples of torsion free $VPC4$ groups which are orientable $PD4$ groups, and do not contain any normal $VPC2$ subgroup. As these examples are finite extensions of $\mathbb{Z}^4$, they contain many subgroups isomorphic to $\mathbb{Z}^3$, and hence many tori, so that $\Gamma_3(G)$ consists of a single $V_0$–vertex, which cannot satisfy condition 1a) or the modified condition 1c) in the statement of Theorem 2.1. Note that torsion free $VPC3$ groups which are orientable $PD3$ groups, do satisfy the modified version of Theorem 2.1. For any such group is the fundamental group of a closed orientable 3–manifold $M$ which admits a geometric structure modeled on $E^3$, $Nil$ or $Solv$. In the first two cases, $M$ is a Seifert fibre space, and $\Gamma_2(G)$ consists of a single $V_0$–vertex of $VPC1$–by–$VPC2$ type. In the third case, either $\Gamma_2(G)$ consists of a single isolated $V_0$–vertex and a single isolated $V_1$–vertex joined by two edges, so
that $\Gamma_2(G)$ is a loop, or $\Gamma_2(G)$ consists of a single isolated $V_0$–vertex, joined
to two $V_1$–vertices of special Seifert type.

We recall Definition 5.1 and Proposition 5.3 from [20].

**Definition 2.3** An orientable $PD(n+2)$ pair $(G, \partial G)$ is atoroidal if any
orientable $VPC(n+1)$ subgroup of $G$ is conjugate into one of the groups in
$\partial G$.

**Proposition 2.4** Let $(G, \partial G)$ be an orientable atoroidal $PD(n+2)$ pair,
where $n \geq 1$. Let $A$ and $B$ be $VPC(n+1)$ groups in $\partial G$, possibly $A = B$.
Let $S$ and $T$ be $VPC(n)$ subgroups of $A$ and $B$ respectively, and let $g$ be an
element of $G$ such that $gSg^{-1} = T$. Then one of the following holds:

1. $A$ and $B$ are the same element of $\partial G$, and $g \in A$.
2. $A$ and $B$ are distinct elements of $\partial G$, are the only groups in $\partial G$, and
   $A = G = B$. Thus $(G, \partial G)$ is the trivial pair $(G, \{G, G\})$.
3. $A$ and $B$ are the same element of $\partial G$. Further $A$ is the only group in
   $\partial G$, and has index 2 in $G$.

In [20], the above proposition was applied to the $V_1$–vertices of the torus
decomposition of a $PD(n+2)$ pair $(G, \partial G)$. More precisely, let $V$ be a
$V_1$–vertex of the torus decomposition $T_{n+1}(G, \partial G)$, and let $K$ denote the
associated group $G(V)$. Let $\partial_1 K$ denote the family of subgroups of $K$ associ-
eted to the edges of $T_{n+1}(G, \partial G)$ incident to $V$, let $\partial_0 K$ denote the family of
subgroups of $K$ which lie in $\partial G$, and let $\partial K$ denote the union $\partial_1 K \cup \partial_0 K$ of
these two families. Then $(K, \partial K)$ is an orientable atoroidal $PD(n+2)$ pair.
Each group in $\partial_1 K$ is $VPC(n+1)$, so the above proposition can be applied
to any annulus in $(K, \partial K)$ with ends in $\partial_1 K$.

We now generalize this idea to apply to $V_1$–vertices of $\Gamma_{n,n+1}(G)$.

**Proposition 2.5** Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair, where $n \geq 1$.
Let $K$ be the group associated to a $V_1$–vertex $V$ of $\Gamma_{n,n+1}(G)$, and let $\partial_1 K$
denote the family of subgroups of $K$ associated to the edges of $\Gamma_{n,n+1}(G)$
incident to $V$. Let $A$ and $B$ be groups in $\partial_1 K$, possibly $A = B$. Let $S$ and $T$
be $VPC(n)$ subgroups of $A$ and $B$ respectively, and let $g$ be an element of $K$
such that $gSg^{-1} = T$. Then one of the following holds:

1. $A$ and $B$ are the same element of $\partial_1 K$, and $g \in A$. 1
2. \(A\) and \(B\) are distinct elements of \(\partial_1 K\), are the only groups in \(\partial_1 K\), and \(A = K = B\).

3. \(A\) and \(B\) are the same element of \(\partial_1 K\). Further \(A\) is the only group in \(\partial_1 K\), and has index 2 in \(K\).

**Proof.** Let \(\partial_0 K\) denote the family of subgroups of \(K\) coming from the decomposition of \(\partial G\) induced by the edge splittings of \(\Gamma_{n,n+1}(G)\). For later use, we note that any essential annulus in \((G, \partial G)\) is enclosed by a \(V_0\)-vertex of \(\Gamma_{n,n+1}(G)\). Thus if an essential annulus \(\Lambda\) in \((G, \partial G)\) is enclosed by the \(V_1\)-vertex \(V\) with associated group \(K\), it cannot be essential in \(K\). This is because there is an edge \(e\) of \(\Gamma_{n,n+1}(G)\) incident to \(V\) such that the associated edge splitting is dual to an annulus \(\Lambda'\) covered by \(\Lambda\). Note that the group associated to \(e\) lies in \(\partial_1 K\).

Now let \(\partial K\) denote the union of the two families \(\partial_0 K\) and \(\partial_1 K\). (Recall that groups in \(\partial_1 K\) are \(PD(n + 1)\) pairs in \((G, \partial G)\), and groups in \(\partial_0 K\) are \(PD(n + 1)\) pairs which are contained in \(\partial G\).) The pair \((K, \partial K)\) is again atoroidal, in the sense that any orientable \(VPC(n + 1)\) subgroup of \(K\) is conjugate into one of the groups in \(\partial K\), but \((K, \partial K)\) need not be a \(PD\)-pair. This is because the groups in \(\partial_0 K\) and \(\partial_1 K\) need not be \(PD\)-groups.

Now we let \(D K\) denote the double of \(K\) along the family \(\Sigma\) of groups in \(\partial_0 K\) which are not tori, and let \(\partial D K\) denote the family consisting of the induced double of \(\partial_1 K\) together with the double of the family of torus groups in \(\partial_0 K\). Note that as \(\partial_1 K\) consists of essential annuli and tori in \((G, \partial G)\), each group in the induced double of \(\partial_1 K\) is \(VPC(n + 1)\). For the double of an essential annulus, this is proved in section 2 of [20]. We claim that \((D K, \partial D K)\) is an orientable atoroidal \(PD(n + 2)\) pair.

Assuming this claim, we proceed as follows. The group \(A\) in \(\partial_1 K\) yields the group \(A'\) in \(\partial D K\), where \(A'\) equals \(A\) if \(A\) is a torus, and equals the double \(DA\) of \(A\) if \(A\) is an annulus. Similarly the group \(B\) in \(\partial_1 K\) yields the group \(B'\) in \(\partial D K\). The \(VPC(n)\) subgroups \(S\) and \(T\) of \(A\) and \(B\) are subgroups of \(A'\) and \(B'\) respectively, and the element \(g\) of \(K\) such that \(gSg^{-1} = T\) lies in \(D K\). Now we apply Proposition 2.4 to obtain one of the three listed cases, and the result of Proposition 2.5 follows.

It remains to prove the claim. First we need to show that \((D K, \partial D K)\) is an orientable \(PD(n + 2)\) pair. This does not follow from [1] or from [5]. Instead we start by considering the double \(D G\) of \(G\) along the family of all groups in \(\partial G\) which are not tori in \(\partial_0 K\), and let \(\partial D G\) denote the double of
the tori in $\partial_0 K$. Now it follows from \[1\] that $(DG, \partial DG)$ is an orientable $PD(n + 2)$ pair. As the pair $(DK, \partial DK)$ can be obtained from $(DG, \partial DG)$ by splitting along the $PD(n + 1)$ subgroups which form the induced double of $\partial_1 K$, it now follows from \[1\] that $(DK, \partial DK)$ is an orientable $PD(n + 2)$ pair. Finally, suppose that $(DK, \partial DK)$ admits an essential torus $T$. If $T$ is conjugate into one of the two copies of $K$, either it is an essential torus in $K$, or it is conjugate into a component of $\Sigma$. The first case cannot occur as any orientable $VPC(n + 1)$ subgroup of $K$ is conjugate into one of the groups in $\partial K$, and the second case cannot occur as no component of $\Sigma$ is a torus. Thus $T$ cannot be conjugate into either copy of $K$, and so the splittings of $DK$ over $\Sigma$ induce nontrivial splittings of $T$. As $T$ is $VPC(n + 1)$, any splitting of $T$ is over a $VPCn$ subgroup $L$, and $L$ must be normal in $T$ with quotient $\mathbb{Z}$ or $\mathbb{Z}_2 \ast \mathbb{Z}_2$. In either case, it follows that there is an essential annulus in $K$ with boundary in $\Sigma$. This would be an essential annulus in $(G, \partial G)$ enclosed by $V$, and so could not be essential in $K$, as pointed out in the first paragraph of this proof. This contradiction completes the proof of the claim that $(DK, \partial DK)$ is an orientable atoroidal $PD(n + 2)$ pair.

3 Examples of almost invariant sets

The discussion in \[20\] was mostly about almost invariant sets which are adapted to the boundary. However, in \[16\], Scott gave examples of almost invariant sets over orientable $VPC2$ subgroups of a $PD3$ pair which are not adapted to the boundary. This gave rise to the concept of special canonical torus which was used to show that the JSJ-decomposition of orientable 3–manifolds is algebraic, meaning that it depends only on the fundamental group of the manifold, not the boundary. As discussed earlier, we will say that an embedded essential annulus or torus in a 3–manifold $M$ with incompressible boundary is topologically canonical if it has intersection number zero with any (possibly singular) essential annulus or torus in $(M, \partial M)$. We will say that a splitting of $\pi_1(M)$ given by an essential annulus or torus is algebraically canonical if it has intersection number zero with any almost invariant subset of $\pi_1(M)$ which is over $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$. See \[15\] for a discussion of the idea of intersection numbers. Now we recall Scott’s example.

Example 3.1 (Scott’s example) This is Example 2.13 of \[16\]. Let $F$ be an orientable surface with at least two boundary components and let $C$ denote
one of the boundary components. Thus $\pi_1(F)$ is free, and $\pi_1(C)$ is a free factor of $\pi_1(F)$. If the rank of $\pi_1(F)$ is at least 3, then it is easy to see that there is a nontrivial splitting of $\pi_1(F)$ as an amalgamated free product over $\pi_1(C)$. Similar considerations apply to express $\pi_1(F)$ as an HNN extension if it has rank 2.

We now take two copies $F_1, F_2$ of $F$ and consider the two 3–manifolds $M_i \times S^1$, each with a boundary component $T_i$ corresponding to $C_i \times S^1$. Form a 3–manifold $M$ by gluing the $M_i$’s along $T_i$ so that the fibrations do not match. The resulting torus $T$ is a topologically canonical torus in the JSJ splitting of $M$. If each $\pi_1(F_i)$ has rank at least 3, we have $\pi_1(M_i) = A_i \ast H_i$, $i = 1, 2$, where $H_i = \pi_1(T_i)$. If $G$ denotes $\pi_1(M)$, and $H$ denotes the subgroup $H_1 = H_2$, and $A = A_1 \ast A_2$, $B = B_1 \ast B_2$, we have a splitting $G = A \ast B$ of $G$ that crosses the splitting associated to $T$. Thus although $T$ is topologically canonical, it is not algebraically canonical. Notice that embedded essential annuli in $M_1$ and $M_2$, disjoint from $T$, yield splittings of $G$ over the fibres of $M_1$ and $M_2$, so that $G$ also has splittings over incommensurable cyclic subgroups of $H$.

We construct some more examples. Consider $M_i = T_i \times I \cup N_i$, where $N_i$ is an orientable 3–manifold attached to $T_i \times \{1\}$ along at least two disjoint annuli in $\partial N_i$. Now form $M$ by identifying the $M_i$’s along $T_i \times \{0\}$, and let $T$ denote the torus $T_1 \times \{0\} = T_2 \times \{0\}$. Assume that the annuli in $T_1 \times \{1\}$ and $T_2 \times \{1\}$ used to construct $M_1$ and $M_2$ carry incommensurable subgroups of $\pi_1(T) = H$. Thus $G = \pi_1(M)$ splits over incommensurable cyclic subgroups of $H$. Again, $T$ is a topologically canonical torus in the JSJ decomposition of $M$. Now we form $H$–almost invariant subsets of $G = \pi_1(M)$ as follows. Consider the cover $M_H$ of $M$ with $\pi_1(M_H) = H$, so that each $T_i \times I$ lifts to $M_H$, and the pre-image $\widetilde{N_i}$ of each $N_i$ is disconnected. Let $C_i$, $i = 1, 2$, be one of the annuli in this lift of $T_i \times \{1\}$ used to construct $M_i$, and let $N'_i$ be the component of $\widetilde{N_i}$ attached to $C_i$. For $i = 1, 2$, let $X_i$ be the set of vertices of $M_H$ in $T_i \times [0, 1]$ together with the vertices in $N'_i$. This gives us two sets $X_1, X_2$ on different sides of $T$. To $X_1$ we add the vertices in $\widetilde{N_2} - N'_2$ and to $X_2$ we add the vertices in $\widetilde{N_1} - N'_1$ to obtain two $H$–almost invariant sets $Y_1, Y_2$. Clearly $Y_1 = Y^*_1$ and crosses the almost invariant set $X$ determined by $T$, namely $X$ consists of all vertices of $M_H$ on one side of $T$.

We can mix the above two examples to get one manifold of each type on each side of $T$. These types fall under the heading of $V_0$–vertices of
commensuriser type of Theorem 2.1. The first examples are special cases of the so called peripheral Seifert type in [20], that is, those Seifert type pieces in the decompositions $\Gamma_{n,n+1}$ and $\Gamma_{n,n+1}^c$ of a $PD(n+2)$ pair $(G, \partial G)$ which "intersect the boundary". The second are called toral type 2), in which one component of $\partial(T \times I)$ is an edge of the decomposition and the other boundary component intersects $\partial G$ in a number of parallel annuli. Note that in [20], the definition of Seifert type requires the base orbifold to have fundamental group which is not virtually cyclic. The terminology torus type is used if this base group is virtually cyclic. These examples suggest the following definition.

Definition 3.2 Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair, such that $G$ is not $VPC$. A splitting of $G$ over a $VPC(n+1)$ subgroup $H$ is called a special canonical torus if it has intersection number zero with any essential annulus in $(G, \partial G)$, and $G$ splits over incommensurable $VPCn$ subgroups of $H$.

Remark 3.3 In Proposition 3.5, we will prove that a special canonical torus in $G$ is an edge splitting of $\Gamma_{n,n+1}(G)$ or $\Gamma_{n,n+1}^c(G)$, so that this concept depends only on $G$, and not on $\partial G$. Recall that the edge splittings of $\Gamma_{n,n+1}(G)$ are the same as those of $\Gamma_{n,n+1}^c(G)$. Note that if $\partial G$ is empty, so that $G$ is a $PD(n+2)$ group, then $G$ cannot split over a $VPCn$ subgroup. Thus special canonical tori do not exist in this case.

Essentially the same definition was used in [16]. However, the above examples are not the only possibilities. Again let $M_1$ denote the 3–manifold $F_1 \times S^1$, where $F_1$ is an orientable surface with at least two boundary components. Also let $M_2$ denote the orientable 3–manifold which is a twisted $I$–bundle over the Klein bottle. Form a 3–manifold $M$ by gluing the boundary torus $T$ of $M_2$ to one of the boundary tori of $M_1$. In this case there are two distinct Seifert fibrations on $M_2$, reflecting the fact that the Klein bottle itself has two distinct Seifert fibrations. But so long as the gluing is chosen not to match the fibration of $M_1$ with either fibration of $M_2$, the torus $T$ will again be a topologically canonical torus in the JSJ splitting of $M$. It will also not be algebraically canonical but will be a special canonical torus. The easiest way to see these facts is to note that $M$ is double covered by the union of two copies of $M_1$, glued along a boundary torus so that their fibrations do not match. Thus $M$ is double covered by one of Scott’s examples.

Next we need the following technical result.
Lemma 3.4 Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair such that $G$ is not $VPC$, let $J$ and $J'$ be $VPC(n+1)$ subgroups of $G$, and let $f$ and $f'$ be edges of the universal covering $G$–tree $T$ of $\Gamma_{c,n,n+1}(G)$, with stabilizers $J$ and $J'$ respectively. If $J$ and $J'$ are commensurable, then $J = J'$ and one of the following cases holds:

1. $f = f'$.
2. $f \cap f'$ is an isolated vertex.
3. $f \cap f'$ is a $V_0$–vertex $w$ of valence 2 whose stabiliser contains an element interchanging $f$ and $f'$, and so contains $J$ with index 2.
4. There are consecutive adjacent edges $f, b, b', f'$ of $T$ such that $b \cap f$ and $b' \cap f'$ are isolated $V_1$–vertices, and $b \cap b'$ is a $V_0$–vertex $w$ of valence 2 whose stabiliser contains an element interchanging $b$ and $b'$, and so contains $J$ with index 2.

Proof. As above, we will start with a $K(\pi,1)$ pair $(M, \partial M)$ and a decomposition of $(M, \partial M)$ mimicking the decomposition $\Gamma_{c,n,n+1}(G)$. If $f = f'$, we have case 1) of the lemma, so for the rest of the proof we will assume that $f \neq f'$.

The edges $f$ and $f'$ determine splittings of $G$ over $J$ and $J'$, so that these are tori in $(G, \partial G)$. Let $L$ denote the intersection $J \cap J'$, so that $L$ is also $VPC(n+1)$, and let $\Sigma$ denote a torus with fundamental group $L$. Consider the $PD(n+2)$ pair $(K, \partial K)$ obtained by cutting $(G, \partial G)$ along these two splittings, and let $(N, \partial N)$ be obtained from $M$ in the corresponding way. (It is possible that $f$ and $f'$ yield a single splitting.) The path in $T$ between $f$ and $f'$ determines (up to homotopy) a map $F : (\Sigma \times I, \Sigma \times \partial I) \to (N, \partial N)$. We let $N_0$ denote the component of $N$ which contains the image of $F$, and consider the induced map $(\Sigma \times I, \Sigma \times \partial I) \to (N_0, \partial N_0)$ which we continue to denote by $F$. The degree of $F$ on each component of $\Sigma \times \partial I$ is non-zero, and if $F(\Sigma \times \partial I)$ is contained in a single component of $\partial N_0$, these degrees add. Thus the degree of $F$ is non-zero. It follows that $(N_0, \partial N_0)$ has a finite cover to which $F$ lifts by a map which is an isomorphism of fundamental groups. In particular, it follows that the boundary of this cover consists of two tori with fundamental groups equal to $L$. Hence either $\partial N_0$ consists of two tori with the same fundamental group as $N_0$, or $\partial N_0$ consists of a single torus with fundamental group $J$, and $J$ has index 2 in $K = \pi_1(N_0)$. In either case,
it follows that $J = J'$. In the first case, the path $\lambda$ joining $f$ and $f'$ in $T$ has stabilizer $J$, and each vertex on that path must be isolated. As $\Gamma_{n,n+1}^c(G)$ is reduced, we must have case 2) of the lemma. In the second case, the stabilizer of $\lambda$ contains $J$ with index 2, and contains a reflection. Thus there is a vertex $w$ of $\lambda$ of valence 2 whose stabilizer contains an element interchanging the two incident edges, and so contains $J$ with index 2. Further, as $\Gamma_{n,n+1}^c(G)$ is reduced, either $w$ equals $f \cap f'$ or there are consecutive adjacent edges $f, b, b', f'$ of $T$ such that $b \cap f$ and $b' \cap f'$ are isolated vertices, and $w$ equals $b \cap b'$. In either case, this implies that the image of $w$ in $\Gamma_{n,n+1}^c(G)$ is of special Seifert type, and so is a $V_0$–vertex. Thus we must have cases 3) or 4) of the lemma.

Now we can give an alternative description of special canonical tori in terms of $\Gamma_{n,n+1}^c(G)$.

**Proposition 3.5**

For an orientable $PD(n + 2)$ pair $(G, \partial G)$ such that $G$ is not $VPC$, a splitting $\alpha$ of $G$ over a $VPC(n+1)$ subgroup $H$ is a special canonical torus if and only if the following conditions hold:

1. $\alpha$ is an edge splitting of $\Gamma_{n,n+1}^c(G)$.

2. The $V_1$–vertex $w$ of $\alpha$ is isolated.

3. Each of the $V_0$–vertices adjacent to $w$ is of peripheral Seifert type, of toral type 2), or of special Seifert type.

4. At most one $V_0$–vertex can be of special Seifert type.

5. If the two edges incident to $w$ form a loop, there is only one adjacent $V_0$–vertex. In this case, that vertex must be of peripheral Seifert type.

(The concepts of peripheral Seifert type, and toral type 2) are discussed immediately preceding Definition 3.2, and “special Seifert type” was defined before Theorem 2.1. The reader is referred to [20] for full details.)

**Proof.** First suppose that $\alpha$ is a splitting of $G$ over a $VPC(n + 1)$ subgroup $H$ which satisfies conditions 1)-5) of the Proposition. As $\alpha$ is an edge splitting of $\Gamma_{n,n+1}^c(G)$, it has intersection number zero with any essential annulus in $(G, \partial G)$. It remains to show that $G$ splits over incommensurable $VPCn$ subgroups of $H$. By condition 2), the $V_1$–vertex $w$ of $\alpha$ is isolated. Suppose there is no $V_0$–vertex adjacent to $w$ of special Seifert type. Then
each of the adjacent $V_0$–vertices meets $\partial G$ in annuli (and/or tori in the case of peripheral Seifert type) and choosing an essential embedded annulus in the $V_0$–vertex and with boundary in $\partial G$ determines a splitting of $G$ over the $VPCn$ subgroup of $H$ carried by the fibres. These subgroups of $H$ must be incommensurable, as otherwise there would be an annulus in $(G, \partial G)$ which crosses $\alpha$, contradicting the fact that any edge splitting of $\Gamma_{n,n+1}^c$ crosses no annulus in $(G, \partial G)$. Thus $\alpha$ is a special canonical torus in this case. Next suppose there is a $V_0$–vertex adjacent to $w$ of special Seifert type. Denote its associated group by $\overline{H}$, and recall that $\overline{H}$ is a subgroup of index 2 in $H$. As before, choosing an essential embedded annulus in the other $V_0$–vertex with boundary in $\partial G$ determines a splitting of $G$ over the $VPCn$ subgroup $L$ of $H$ carried by the fibres. Now let $g$ denote an element of $\overline{H} - H$. We also have a splitting of $G$ over $L^g$. As $g$ normalises $H$, this is also a subgroup of $H$. Finally $L$ and $L^g$ must be incommensurable subgroups of $H$, for otherwise there would be an annulus in $(G, \partial G)$ which crosses $\alpha$, contradicting the fact that $\alpha$ is an edge splitting of $\Gamma_{n,n+1}^c$. Thus again $\alpha$ is a special canonical torus.

Now suppose that $\alpha$ is a special canonical torus. By Lemma 6.2 of [20], the fact that $\alpha$ has intersection number zero with any essential annulus in $(G, \partial G)$ implies that $\alpha$ is adapted to $\partial G$. Thus the splitting $\alpha$ is induced by a torus in $(G, \partial G)$, and so must be enclosed by some $V_0$–vertex of $\Gamma_{n,n+1}^c(G)$. As $G$ splits over a $VPCn$ subgroup $L \subset H$, the pair $(G, \partial G)$ admits an essential annulus with group $L$, and any such annulus must be enclosed by a $V_0$–vertex $v$ of $\Gamma_{n,n+1}^c(G)$, of commensurator type, so that $G(v)$ contains $H$. Thus $\alpha$ is enclosed by $v$. As $\alpha$ has intersection number zero with any essential annulus in $(G, \partial G)$, it follows that $\alpha$ must be the splitting of $G$ associated to an edge of $\Gamma_{n,n+1}^c(G)$ incident to $v$. This proves that $\alpha$ satisfies condition 1) of the proposition. Further $v$ must be of peripheral Seifert type or of toral type 2). Now we use the hypothesis that $G$ splits over two incommensurable $VPCn$ subgroups $L$ and $L'$ of $H$. So the pair $(G, \partial G)$ admits an essential annulus with group $L'$, which is enclosed by a $V_0$–vertex $v'$ of $\Gamma_{n,n+1}^c(G)$ which also has an incident edge with splitting $\alpha$. Further $v'$ must be of peripheral Seifert type or of toral type 2).

If $v$ and $v'$ are distinct, the incident edges with splitting $\alpha$ must also be distinct. As neither of $v$ and $v'$ is isolated or of special Seifert type, Lemma 3.3 implies that $v$ and $v'$ must be separated by an isolated $V_1$–vertex, so that $\alpha$ satisfies conditions 2)-5) of the Proposition, as required.

If $v$ and $v'$ coincide, there must be $g \in G$ such that $L' = L^g$. There are
now two cases, depending on whether or not there are two distinct edges incident to $v$ with associated splitting $\alpha$. Again we will apply Lemma 3.4 and use the fact that $v$ is not isolated nor of special Seifert type. If there are two distinct such edges, Lemma 3.4 implies that they meet in an isolated $V_1$–vertex. In addition, $v$ cannot be of torus type 2), as such a $V_0$–vertex can have at most one incident edge dual to a torus. It follows that $\alpha$ satisfies conditions 2)-5) of the Proposition. If there is only one such edge, Lemma 3.4 implies that there is a $V_0$–vertex $v''$ of special Seifert type which is adjacent to $v$ and separated from $v$ by an isolated $V_1$–vertex $w$. Again this implies that $\alpha$ satisfies conditions 2)-5) of the Proposition, as required.

We can now apply this Proposition to obtain the following result.

**Proposition 3.6** Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair such that $G$ is not $VP C$. Then a special canonical torus in $(G, \partial G)$ with group $H$ crosses some almost invariant subset of $G$ over $H$.

**Proof.** Let $\alpha$ be a special canonical torus in $(G, \partial G)$ with group $H$ such that $G$ splits over incommensurable subgroups $L$ and $L'$ of $H$. Let $X$ be the $H$–almost invariant subset of $G$ determined (up to equivalence and complementation) by $\alpha$. We will apply Conditions 1)-5) of Proposition 3.5. Thus $\alpha$ is an edge splitting of $\Gamma_{c,n+1}(G)$, and the $V_1$–vertex $w$ of $\alpha$ is isolated.

If there are two distinct $V_0$–vertices $v$ and $v'$ adjacent to $w$, neither of special Seifert type, we can assume that the splitting of $G$ over $L$ is enclosed by $v$, and that the splitting of $G$ over $L'$ is enclosed by $v'$. Let $Y$ denote the $L$–almost invariant subset of $G$ determined by the splitting of $G$ over $L$, and let $Y'$ denote the $L'$–almost invariant subset of $G$ determined by the splitting of $G$ over $L'$. By replacing each of $X, Y$ and $Y'$ by its complement if needed, we can arrange that $Y \leq X$ and $Y' \leq X^*$. Then $H(Y \cup Y')$ is a $H$–almost invariant subset of $G$ which crosses $X$, and hence crosses $\alpha$, as required.

If there are two distinct $V_0$–vertices $z$ and $z'$ adjacent to $w$, and if $z'$ is of special Seifert type, there is a homomorphism $G(z') \to \mathbb{Z}_2$, with kernel $H$. This extends to a homomorphism $G \to \mathbb{Z}_2$, which is trivial on all vertex groups other than $G(z')$. Let $K$ denote the kernel of this homomorphism, so that $K$ is of index 2 in $G$. This naturally has the structure of a $PD(n+2)$ pair, and there is a natural map $\Gamma_{c,n+1}(K) \to \Gamma_{c,n+1}(G)$. The pre-image of $z'$ is an isolated $V_1$–vertex of $\Gamma_{c,n+1}(K)$. The adjacent $V_0$–vertices consist of two copies of $z$. Hence the preceding paragraph yields a $H$–almost invariant subset of $K$ which crosses $\alpha$. It follows that there is also a $H$–almost invariant subset of $G$ which crosses $\alpha$, as required.
If the two edges incident to \( w \) form a loop, so there is only one adjacent \( V_0 \)-vertex \( z \), Condition 5) tells us that \( z \) must be of peripheral Seifert type. This loop determines a natural map from \( G \) to \( \mathbb{Z} \), which is trivial on all vertex groups, and hence determines a natural map from \( G \) to \( \mathbb{Z}_2 \), which is trivial on all vertex groups. The kernel is a subgroup \( K \) of \( G \) of index 2 which is still naturally a \( PD(n+2) \) pair, and again there is an edge splitting over \( H \) of \( \Gamma_{c,n,n+1}(K) \) whose \( V_1 \)-vertex is isolated. The adjacent \( V_0 \)-vertices now consist of two copies of \( z \). As before, this yields a \( H \)-almost invariant subset of \( G \) which crosses \( \alpha \), as required.

4 Proof of the main result

In this section, we prove the main theorem below and then a result about enclosings of all almost invariant sets over \( VPC(n+1) \) groups. First we need to introduce yet another version of the term ”canonical”, which generalizes the term ”algebraically canonical” discussed in the introduction. Let \( \mathcal{E}_{n,n+1}(G) \) denote the collection of all a.i. subsets of \( G \) which are over a \( VPC \) or \( VPC(n+1) \) subgroup. We will say that an element of \( \mathcal{E}_{n,n+1} \) is canonical if it has intersection number zero with every element of \( \mathcal{E}_{n,n+1} \).

**Theorem 4.1 (Main result)** Let \((G, \partial G)\) be an orientable \( PD(n+2) \) pair such that \( G \) is not \( VPC \). The edge splittings of \( \Gamma_{n,n+1}(G) \) and of \( \Gamma_{c,n,n+1}(G) \) are either canonical or are special canonical tori.

**Remark 4.2** Proposition 3.6 shows that these two conditions are mutually exclusive. Note that if \( \partial G \) is empty, the decomposition \( \Gamma_{n+1}(G) \) is an algebraic regular neighbourhood of all almost invariant subsets of \( G \) over a \( VPC(n+1) \) subgroup, so that all the edge splittings of \( \Gamma_{n+1}(G) \) and of \( \Gamma_{c,n+1}(G) \) are canonical, by definition.

We start by setting up some notation. The main step of the start of the argument is discussed in Section 6 of [20] in a different context. There the authors used it to show that \( n \)-canonical almost invariant sets over \( VPC(n+1) \) groups are automatically adapted to the boundary. Here we use it differently.

Let \((G, \partial G)\) be an orientable \( PD(n+2) \) pair, and let \( T \) denote the universal covering \( G \)-tree of the graph of groups \( \Gamma_{c,n,n+1}(G) \). Let \((M, \partial M)\) be a \( K(\pi,1) \) pair with a decomposition mimicking the decomposition \( \Gamma_{c,n,n+1} \). This induces a decomposition of the universal cover \((\widetilde{M}, \partial \widetilde{M})\) of \((M, \partial M)\).
and we have an equivariant map \( \widetilde{M} \to T \) preserving the decompositions. If \( v \) is a vertex of \( \Gamma_{n,n+1}^c \) or of \( T \), the corresponding subspaces of \( M \) or of \( \widetilde{M} \) will be denoted by \( M_v \) or \( \widetilde{M}_v \), respectively, and similarly for edges.

**Lemma 4.3** Let \( (G, \partial G) \) be an orientable PD\((n+2)\) pair such that \( G \) is not VPC, and let \( e \) be an edge of \( T \) such that the associated splitting of \( G \) is not canonical. Then there is an almost invariant set \( X \) over a VPC(\( n+1 \)) subgroup \( H \) of \( G \) which is not adapted to \( \partial G \) and crosses \( e \).

**Proof.** Recall that \( \Gamma_{n,n+1}^c(G) \) is the completion of the reduced algebraic regular neighbourhood \( \Gamma_{n,n+1}(G) \) of \( F_{n,n+1} \) in \( G \), where \( F_{n,n+1} \) denotes the family of equivalence classes of all nontrivial almost invariant subsets of \( G \) which are over a VPC\( n \) subgroup, together with the equivalence classes of all \( n \)–canonical almost invariant subsets of \( G \) which are over a VPC(\( n+1 \)) subgroup. In [20], the authors showed that \( n \)–canonical almost invariant subsets of \( G \) over VPC\( n \) subgroups are automatically adapted to \( \partial G \). Further, if we enlarge the family \( F_{n,n+1} \) to include all almost invariant sets over VPC(\( n+1 \)) subgroups which are adapted to \( \partial G \), the new family \( \mathcal{G}_{n,n+1} \) has the same regular neighbourhood \( \Gamma_{n,n+1}(G) \). In particular, no set in \( \mathcal{G}_{n,n+1} \) can cross any edge of \( \Gamma_{n,n+1}^c(G) \). Thus our assumption on \( e \) implies that there must be an almost invariant set \( X \) over a VPC(\( n+1 \)) subgroup \( H \) of \( G \) which is not adapted to \( \partial G \) and crosses \( e \), as required. \( \blacksquare \)

**Lemma 4.4** Let \( (G, \partial G) \) be an orientable PD\((n+2)\) pair such that \( G \) is not VPC, and let \( X \) be an almost invariant set over a VPC(\( n+1 \)) subgroup \( H \) of \( G \) which is not adapted to \( \partial G \). Then the following statements hold:

1. There is a group \( S \) with a conjugate in \( \partial G \) such that \( L = H \cap S \) is VPC\( n \), and \( X \cap S \) and \( X^* \cap S \) are both \( H \)–infinite.

2. There is a \( V_0 \)–vertex \( v \) of \( T \) of commensuriser type which encloses all \( L \)–almost invariant subsets of \( G \). Further, \( M_v \cap \partial \widetilde{M} \) is non-empty, \( v \) is of Seifert type or of torus type, and \( G(v) = \text{Comm}_G(L) = N_G(L) \) contains \( H \).

**Proof.** 1) Since \( X \) is not adapted to \( \partial G \), there is a group \( S \) in \( \partial G \), and \( g \in G \) such that \( X \cap gS \) and \( X^* \cap gS \) are both \( H \)–infinite. By replacing \( S \) by a conjugate if needed, we can arrange that \( X \cap S \) and \( X^* \cap S \) are both \( H \)–infinite. Consider the component \( \Sigma \) of \( \partial \widetilde{M} \) with stabilizer \( S \), and
identify $X$ and $X^*$ with subsets of the $0$–skeleton $\widetilde{M}_0$ of $\widetilde{M}$. We have that the intersections of $X$ and $X^*$ with the $0$–skeleton $\Sigma_0$ of $\Sigma$ are $H$–infinite. Hence they are $L$–infinite, where $L = H \cap S$. Thus $\varepsilon(S, L) \geq 2$. As $S$ is $PD(n + 1)$ and $H$ is $VPC$, it follows that $L$ is $VPC_n$.

2) By replacing $H$ by a subgroup of finite index if necessary, we can assume that $L$ is normal in $H$ with $L \setminus H$ infinite cyclic. Let $P_L : \widetilde{M} \to M_L$, and $P_H : \widetilde{M} \to M_H$ denote the covering projections, and let $\Sigma_L$ and $\Sigma_H$ denote the images of $\Sigma$ in $\partial M_L$ and $\partial M_H$ respectively. As $L \setminus H$ acts on $M_L$, there are infinitely many translates of $\Sigma_L$ each with fundamental group $L$, and thus infinitely many essential annuli in $M_L$. In [20], it was shown that, if the number of essential annuli in $M_L$ is at least $4$, there is a $V_0$–vertex $v$ of $T$ of commensuriser type which encloses all $L$–almost invariant subsets of $G$. Hence the stabilizer, $G(v)$, of $v$ contains $H$, and $\widetilde{M}_v \cap \partial \widetilde{M}$. It is possible that $\widetilde{M}_v \setminus \partial \widetilde{M}$ contains $\Sigma$. This happens if $S$ is $VPC(n + 1)$. In any case, $v$ is a $V_0$–vertex with $H \subset G(v)$, and $\widetilde{M}_v \cap \partial \widetilde{M}$ is non-empty, and $L$ stabilizes $\widetilde{M}_v \setminus \partial \widetilde{M}$. Thus $v$ is either of Seifert type or of toral type (see Definition 3.12 of [20]). If $v$ is of Seifert type, Lemma 5.10 of [20] tells us that $G(v) = \text{Comm}_G(L) = N_G(L)$. If $v$ is of toral type, we use the fact that $G(v)$ is $VPC(n + 1)$ and splits over $L$. Now Lemma 1.10 of [20] implies that $L$ is normal in $G(v)$ with quotient $Z$ or $Z_2 \ast Z_2$. It follows that in this case also $G(v) = \text{Comm}_G(L) = N_G(L)$. 

Note that if $X$ crosses an edge $e$ of $T$, Lemma 4.4 does not tell us that $e$ is incident to the vertex $v$ of $T$ obtained in part 2) of the above lemma. However the next lemma assures us that $X$ must cross some edge incident to $v$.

**Lemma 4.5** Let $(G, \partial G)$ be an orientable $PD(n + 2)$ pair such that $G$ is not $VPC$, let $e$ be an edge of $T$, and let $X$ be an almost invariant set over a $VPC(n + 1)$ subgroup $H$ of $G$ which is not adapted to $\partial G$ and crosses $e$.

Using the notation of Lemma 4.4, if $f$ is the first edge on the path from $v$ to $e$, then $G(f)$ is $VPC(n + 1)$, so the associated splitting of $G$ is dual to a torus, and $X$ crosses this torus.

**Remark 4.6** If $v$ is of torus type, the fact that an edge incident to $v$ determines a splitting of $G$ over a torus means that $v$ is of torus type 2).

**Proof.** Let $Z$ and $Z^*$ denote the almost invariant sets associated to $e$ chosen so that $Z$ contains $G(v)$, and let $Y$ and $Y^*$ denote the almost invariant
sets associated to \( f \) chosen so that \( Y \subset Z \) and \( Y^* \supset Z^* \). As \( X \) crosses \( Z \), we know that \( X \cap Z^* \) and \( X^* \cap Z^* \) are both \( H \)-infinite. As \( Y^* \supset Z^* \), it follows that \( X \cap Y^* \) and \( X^* \cap Y^* \) are also both \( H \)-infinite.

Now suppose that \( G(f) \) is \( VPCn \). As \( v \) is of Seifert type or of toral type, the edge group \( G(f) \) must be commensurable with \( L \). Thus \( \delta Y \) is \( L \)-finite. As \( H \subset G(v) \), and \( \delta X \) is \( H \)-finite, it follows that \( \delta X \) lies in a bounded neighbourhood of \( \tilde{M}_v \). As \( G(f) \) is commensurable with \( L \), this implies that \( \delta X \cap Y^* \) is \( L \)-finite. As \( \delta Y \) is also \( L \)-finite, it follows that \( X \cap Y^* \) and \( X^* \cap Y^* \) each have \( L \)-finite coboundary. For \( \delta(X \cap Y^*) = (X \cap \delta Y^*) \cup (\delta X \cap Y^*) \).

We conclude that each of \( X \cap Y^* \) and \( X^* \cap Y^* \) is a nontrivial \( L \)-almost invariant subset of \( G \) contained in \( Y^* \). In particular, they cannot be enclosed by \( v \), which is a contradiction. This contradiction shows that \( G(f) \) must be \( VPC(n+1) \), so that the associated splitting of \( G \) is dual to a torus, as required. It remains to show that \( X \) crosses this torus.

As \( G(f) \) determines a torus, it follows that the component \( \Sigma \) of \( \delta \tilde{M} \) with stabilizer \( S \) cannot meet this torus, and so must lie on the same side of \( \tilde{M}_f \) as does \( \tilde{M}_v \). In particular, \( S \subset Y \). As \( X \cap S \) and \( X^* \cap S \) are both \( H \)-infinite, it follows that \( X \cap Y \) and \( X^* \cap Y \) are both \( H \)-infinite. Hence all four corners of the pair \( (X, Y) \) are \( H \)-infinite, so that \( X \) crosses \( Y \), as required. \( \square \)

Next we will show that \( H \) and \( G(f) \) must be commensurable subgroups of \( G \). As the argument is quite intricate, we will break it into steps.

We split \( G \) along the torus \( G(f) \) to obtain a new \( PD(n+2) \) pair \( (G', \partial G') \) (by Theorem 8.1 of \( \Pi \)) with \( (G', \partial G') \) containing \( G(w) \), where \( v \) and \( w \) are the vertices of \( f \). Correspondingly, \( \tilde{M} \) is split along \( \tilde{M}_f \) to obtain a new space \( \tilde{N} \) containing \( \tilde{M}_w \). Thus, \( \tilde{M} \) is split along \( \tilde{M}_f \) and its translates to obtain a new space \( \tilde{N} \) containing \( \tilde{M}_w \). In particular, the boundary of \( \tilde{N} \) consists of boundary components of \( \tilde{M} \) together with translates of \( \tilde{M}_f \).

**Lemma 4.7** Using the notation of Lemma 4.5, suppose that \( H \) and \( G(f) \) are not commensurable, and let \( L' \) denote \( H \cap G(f) \). Then \( L' \) is \( VPCn \) and contains \( L \) with finite index, and there is an essential annulus in \( N \), carrying \( L' \), which lifts to an annulus \( A \) in \( P_{L'}(\tilde{N}) \) from \( P_{L'}(\tilde{M}_f) \) to a component of \( P_{L'}(\partial \tilde{N}) \).

**Proof.** Note that part 2) of Lemma 4.4 tells us that \( G(v) = Comm_G(L) = N_G(L) \) contains \( H \). As \( v \) is of Seifert type or of torus type, and \( G(f) \) is a boundary torus of \( G(v) \), it follows that \( G(f) \) also contains \( L \). In particular,
the intersection $H \cap G(f)$ contains $L$. As $H$ and $G(f)$ are not commensurable, it follows that $H \cap G(f) = L'$ is $VPCn$ and contains $L$ with finite index.

As in the proof of Lemma 4.7, we consider the intersections $X \cap Y^*$ and $X^* \cap Y^*$. Both sets are invariant under $H \cap G(f) = L'$. Again we know that $\delta X \cap Y^*$ must be $L$–finite. Suppose that $X \cap \delta Y$ is $L$–finite. Then $X \cap Y^*$ has $L$–finite coboundary and so $X \cap Y^*$ is a nontrivial $L$–almost invariant set which is not enclosed by $v$, which is again a contradiction. Thus $X \cap \delta Y$ must be $L$–infinite, and similarly $X^* \cap \delta Y$ must be $L$–infinite. Note that the intersections of $X \cap Y^*$ and $X^* \cap Y^*$ with the 0–skeleton of $\tilde{N}$ have coboundaries (in $\tilde{N}$) equal to $\delta X \cap Y^*$ and $\delta X^* \cap Y^*$ respectively, each of which is $L$–finite. As $X \cap Y^*$ and $X^* \cap Y^*$ contain $X \cap \delta Y^*$ and $X^* \cap \delta Y^*$ respectively which are both $L'$–infinite, it follows that each determines a nontrivial $L'$–almost invariant subset of $G'$. Hence there are essential annuli in $N$, carrying $L'$, one of which lifts to an annulus $A$ in $P_{L'}(\tilde{N})$ from $P_{L'}(M_f)$ to a component of $P_{L'}(\partial \tilde{N})$, as required. 

Lemma 4.8 Using the notation of Lemma 4.7, suppose that $H$ and $G(f)$ are not commensurable, and let $L'$ denote $H \cap G(f)$. Then there is an essential annulus in $N$, carrying a subgroup $L''$ of index at most 2 in $L'$, which lifts to an annulus $A$ in $P_{L'}(\tilde{N})$ from $P_{L'}(M_f)$ to a component of $P_{L'}(\partial \tilde{M})$.

Proof. By Lemma 4.7, there is an essential annulus $A$ in $P_{L'}(\tilde{N})$ from $P_{L'}(M_f)$ to a component $\Sigma_{L'}$ of $P_{L'}(\partial \tilde{M})$. If $\Sigma_{L'}$ is a component of $P_{L'}(\partial \tilde{M})$, our result is proved. So for the rest of this proof we will assume that $\Sigma_{L'}$ is not a component of $P_{L'}(\partial \tilde{M})$.

As $A$ is essential, $P_{L'}(M_f)$ and $\Sigma_{L'}$ must be distinct components of $P_{L'}(\partial \tilde{N})$. Recall that $\tilde{N}$ contains $M_w$, where $w$ is the $V_1$–vertex of the edge $f$. It follows that $A$ has a sub-annulus $A'$ which lies in $P_{L'}(M_w)$, and joins distinct boundary components. Thus the vertex $w$ has an incident edge $g$, distinct from $f$, such that $G(g)$ contains $L'$. As $G(g)$ is an edge group of $\Gamma^{c}_{n,n+1}(G)$, it must be $VPCn$ or $VPC(n + 1)$. In either case, we apply Proposition 2.5. As $f$ and $g$ are distinct, case 1) of the conclusion is not possible. It follows that $G(g)$ is $VPC(n + 1)$. Further either the pair $(G(w), \partial G(w))$ is the trivial pair, or $\partial G(w)$ consists of a single $VPC(n + 1)$ subgroup of index 2 in $G(w)$. In the second case, the vertex $w$ would be of special Seifert type and so would be a $V_0$–vertex of $\Gamma^{c}_{n,n+1}(G)$, contradicting the fact that $w$ is a $V_1$–vertex. It follows that the first case holds, so that $w$ is an isolated vertex of $\Gamma^{c}_{n,n+1}(G)$.
In particular, $G(g) = G(f)$ is $VPC(n + 1)$. Let $v'$ denote the $V_0$-vertex at the other end of the edge $g$. The part $A_1$ of $A$ in $M_{v'}$ is an essential annulus carrying $L'$ in the pair $(G(v'), \partial G(v'))$. We need to recall from Theorem 2.1 the possible types of $V_0$–vertex of $\Gamma_{n,n+1}(G)$.

If $v'$ is isolated, this would contradict the fact that $\Gamma_{n,n+1}(G)$ is reduced.

If $v'$ is of $VPC(n-1)$–by–Fuchsian type, and is of $I$–bundle type, then each edge splitting for edges incident to $v'$ would be dual to an annulus. As the edge splitting dual to $g$ is a torus, this case cannot occur.

If $v'$ is of $VPCn$–by–Fuchsian type, and is of interior Seifert type, then the essential annulus $A_1$ in $(G(v'), \partial G(v'))$ projects to an annulus in the base $2$–orbifold. As $v'$ is of $VPCn$–by–Fuchsian type, the fundamental group of this base orbifold is not virtually cyclic. It follows that it does not admit an essential annulus. We conclude that this projected annulus is inessential, which implies that $L'$ is commensurable with the $VPCn$ fibre group of $G(v')$. But this means that $G(v')$ commensurises $L'$, and hence commensurises $L$, which is again a contradiction as $G(v) = \text{Comm}_G(L)$.

Finally, if $v'$ is of commensuriser type, then $v'$ is of Seifert type, or of torus type 2), or of solid torus type. In the first case, the argument in the preceding paragraph again yields a contradiction. The last case cannot occur, as it would imply that each edge splitting for edges incident to $v'$ would be dual to an annulus, which is not possible as $g$ is dual to a torus.

We conclude that the only possibility is that $v'$ is of torus type 2). Thus $\partial_1 G(v')$ consists of the torus $G(g)$ together with annuli whose boundaries all carry the same $VPCn$ group $K$, and $\partial_0 G(v')$ consists of annuli whose boundaries all carry $K$. Recall that $A_1$ is an essential annulus carrying $L'$ in the pair $(G(v'), \partial G(v'))$. One end lies in the boundary torus $G(g)$, and the other end must lie in an annulus component of $\partial_i G(v')$, because of our assumption that $\Sigma_{L'}$ does not lie in $P_{L'}(\partial \tilde{M})$. In particular, $L'$ has a subgroup $L''$ of index at most 2 which is contained in $K$. Hence there is an essential annulus $A_2$ carrying $L''$ in the pair $(G(v'), \partial G(v'))$ which joins the boundary torus $G(g)$ to an annulus in $\partial_0 G(v')$. As $w$ is isolated, we can extend this to obtain an essential annulus $A_3$ in $P_{L'}(\tilde{N})$ which carries $L''$ and joins $P_{L'}(\tilde{M}_f)$ to a component of $P_{L'}(\partial \tilde{M})$, as required.

Lemma 4.9 Using the notation of Lemma 4.5, the subgroups $H$ and $G(f)$ of $G$ are commensurable.

Proof. Suppose that $H$ and $G(f)$ are not commensurable, and let $L' =
By Lemma 4.8, there is an essential annulus in $N$, carrying a subgroup $L''$ of index at most 2 in $L'$, which lifts to an annulus $A$ in $P_L(\tilde{N})$ from $P_L(\tilde{M}_f)$ to a component $\Sigma_{L'}$ of $P_L(\partial\tilde{M})$. Recall from Lemma 4.4 that $\tilde{M}_v$ meets a component $\Sigma$ of $\partial\tilde{M}$ with stabilizer $S$. Thus we also have an essential annulus $B$ in $M_L = P_L(\tilde{M})$ from $P_L(\Sigma) = \Sigma_L$ to $P_L(\tilde{M}_f)$. We can combine $A$ and $B$ to get an essential annulus $\theta$ in $M$ from $\Sigma_L$ to $\Sigma_L'$. The annulus $\theta$ gives us a nontrivial $L''$–almost invariant subset $X_{\theta}$ of $G$. Hence $X_{\theta}$ is enclosed by $v$. In particular, this means that $\theta$ must be properly homotopic in $M$ into $M_v$. But this contradicts the fact that $A$ is an essential annulus in $N$. This contradiction shows that $H$ and $G(f)$ must be commensurable, as required.

Combining the preceding lemmas and Remark 4.6 we have proved the following.

**Lemma 4.10** Let $(G, \partial G)$ be an orientable $PD(n + 2)$ pair such that $G$ is not $VPC$, let $T$ denote the universal covering $G$–tree of the graph of groups $\Gamma_{n,n+1}^c(G)$, and let $e$ be an edge of $T$ such that the associated splitting of $G$ is not canonical. Then the following statements hold:

1. There is an almost invariant set $X$ over a $VPC(n+1)$ subgroup $H$ of $G$ which is not adapted to $\partial G$ and crosses $e$.

2. There is a group $S$ with a conjugate in $\partial G$ such that $L = H \cap S$ is $VPC_n$, and $X \cap S$ and $X^* \cap S$ are both $H$–infinite.

3. There is a $V_0$–vertex $v$ of $T$ of commensuriser type which encloses all $L$–almost invariant subsets of $G$. Further, $\tilde{M}_v \cap \partial\tilde{M}$ is non-empty, $v$ is of Seifert type or of toral type 2), and $G(v) = \text{Comm}_G(L) = N_G(L)$ contains $H$.

4. If $f$ is the first edge on the path from $v$ to $e$, then $G(f)$ is $VPC(n+1)$ and commensurable with $H$, and $X$ crosses the torus splitting given by $f$.

Now we can complete the proof of Theorem 4.1 that the edge splittings of $\Gamma_{n,n+1}(G)$ and of $\Gamma_{n,n+1}^c(G)$ are either canonical or are special canonical tori.

**Proof.** Let $e$ be an edge of $T$ such that the associated splitting of $G$ is not canonical, and apply Lemma 4.10. Let $Y$ and $Y^*$ denote the almost invariant sets associated to $f$, chosen so that $G(v) \subset Y$. 

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As \( H \) and \( G(f) \) are commensurable, and \( X \) crosses \( Y \), the intersections \( X \cap Y^* \) and \( X^* \cap Y^* \) are nontrivial almost invariant sets over the \( VPC(n+1) \) group \( H' = H \cap G(f) \).

As \( H' \) is a torus in \( (G, \partial G) \), up to equivalence, we have only two \( H' \)-almost invariant sets which are adapted to \( \partial G \), namely \( Y \) and \( Y^* \). Thus neither of \( X \cap Y^* \) and \( X^* \cap Y^* \) is adapted to \( \partial G \). Let \( Z \) denote \( X \cap Y^* \). Then Lemma 3.4 tells us that there is a group \( S' \) such that \( Z \cap S' \) and \( Z^* \cap S' \) are both \( H' \)-infinite, and \( H' \cap S' \) is a \( VPCn \) group \( K \). Further there is a \( v_0 \)-vertex \( v' \) of \( T \), which encloses all \( K \)-almost invariant subsets of \( G \) so that \( G(v') \) contains \( H' \), and \( M_{v'} \) intersects \( \partial M \).

If \( Z \) crosses some edge \( e' \) of \( T \), Lemma 4.5 tells us that if \( f' \) is the first edge on the path from \( v' \) to \( e' \), then \( G(f') \) is \( VPC(n+1) \) and commensurable with \( H' \), and \( Z \) crosses the torus splitting given by \( f' \). In particular, \( f \) and \( f' \) have commensurable stabilizers. Thus we can apply Lemma 3.4 to deduce that \( G(f) = G(f') \). Cases 1) or 3) of that lemma would imply that \( v = v' \), so that \( Z \) is enclosed by \( v \). But this is impossible as \( Z \subset Y^* \), and \( G(v) \subset Y \). Thus we must have cases 2) or 4) of Lemma 3.4 Further, as \( v \) and \( v' \) are not isolated, in case 2), \( f \cap f' \) must be a \( V_1 \)-vertex.

If \( Z \) crosses no edge of \( T \), then \( Z \) is enclosed by some \( v_0 \)-vertex \( v' \), which again cannot be \( v \). Thus \( H' \) is a subgroup of \( G(v) \) and of \( G(v') \), and hence of the edge \( f' \) incident to \( v' \) and on the path in \( T \) from \( v \) to \( v' \). Again we must have cases 2) or 4) of Lemma 3.4, and in case 2), \( f \cap f' \) must be a \( V_1 \)-vertex.

Now Proposition 3.3 implies that in all cases, the splitting determined by \( f \) is a special canonical torus, and so is the splitting determined by \( f' \) (and the splittings determined by \( b \) and \( b' \) in case 4) of Lemma 3.4).

Finally, we will show that the original edge \( e \) that was crossed by \( X \) must equal one of \( f \) or \( f' \) (or \( b \) or \( b' \) in case 4) of Lemma 3.4). In all cases, it follows that the splitting determined by \( e \) is a special canonical torus.

Recall that \( Z = X \cap Y^* \), and that \( Z \cap S' \) and \( Z^* \cap S' \) are both \( H' \)-infinite. We claim that \( X \cap S' \) and \( X^* \cap S' \) are also both \( H' \)-infinite. As \( X \cap S' \) contains \( Z \cap S' \), the first part of the claim is clear. As \( G(f) \) determines a torus, it follows that the component of \( \partial \tilde{M} \) with stabilizer \( S' \) cannot meet this torus, and so must lie on the same side of \( \tilde{M} \) as does \( \tilde{M}_{v'} \). In particular, \( S' \subset Y^* \), so that \( Y \cap S' \) is \( H' \)-finite. Now \( Z^* = X^* \cup (X \cap Y) \), so it follows that \( X^* \cap S' \) is also \( H' \)-infinite, completing the proof of the claim.

Next we show that \( f \) is the only edge incident to \( v \) which is crossed by \( X \). For suppose that \( X \) crosses an edge \( f'' \) incident to \( v \). Lemma 4.10 shows that
$G(f'')$ is $VPC(n + 1)$ and commensurable with $H$. Now we apply Lemma 3.4 to the pair $(f, f'')$. Thus $G(f) = G(f'')$, and we must have case 1), 2), 3) or 4) of that lemma. As $f$ and $f''$ have the same $V_0$–vertex which is not isolated nor of special Seifert type, this is impossible unless $f = f''$.

Next suppose that $X$ crosses some edge $e''$ of $T$. Note that $X$ is $H'$–almost invariant, that $K = H' \cap S'$ is $VPCn$, and $X \cap S$ and $X'^* \cap S$ are both $H'$–infinite. Further $v'$ is a $V_0$–vertex of $T$ of commensuriser type which encloses all $K$–almost invariant subsets of $G$. Now we apply Lemma 4.5 with $v'$ and $e''$ in place of $v$ and $e$. This shows that if $f''$ is the first edge on the path from $v'$ to $e''$, then $G(f'')$ is $VPC(n + 1)$ and commensurable with $H'$, and $X$ crosses the torus splitting given by $f''$. Now we can argue as in the preceding paragraph to show that $f'$ is the only edge incident to $v'$ which is crossed by $X$.

As $X$ crosses the edge $e$, we conclude that $f$ is the first edge of the path joining $v$ to $e$, and that $f'$ is the first edge of the path joining $v'$ to $e$. It follows that $e$ must lie between $v$ and $v'$, so that the edge $e$ of $T$ must be equal to $f$ or $f'$ (or $b$ or $b'$), as required. It follows that each edge splitting of $\Gamma^{n,n+1}_{n}(G)$ is either canonical or is a special canonical torus, thus completing the proof of Theorem 4.1.

The following result is an easy consequence of the above arguments. Recall that a $H$–almost invariant subset $X$ of a group $G$ is enclosed by a vertex $v$ of a $G$–tree $T$, if for every edge $e$ incident to $v$, we have either $X \leq Z_e$ or $X'^* \leq Z_e$, where $Z_e$ and $Z_e^*$ are the almost invariant subsets of $G$ associated to $e$ chosen so that $v$ lies in $Z_e$. There is a natural extension of this idea as follows. If $T'$ is a subtree of $T$, we will say that $X$ is enclosed by $T'$, if for every edge $e$ incident to $T'$, but not contained in $T'$, we have either $X \leq Z_e$ or $X'^* \leq Z_e$, where $Z_e$ and $Z_e^*$ are the almost invariant subsets of $G$ associated to $e$ chosen so that $T'$ is contained in $Z_e$.

**Proposition 4.11** Let $(G, \partial G)$ be an orientable $PD(n + 2)$ pair such that $G$ is not $VPC$. Then any almost invariant subset of $G$ over a $VPC(n + 1)$ subgroup $H$ is either enclosed by a $V_0$–vertex of $T$, or is enclosed by an interval whose endpoints are the $V_0$–vertices on opposite sides of a special canonical torus edge. Further $H$ is commensurable with that splitting torus group.

**Proof.** Let $X$ be an almost invariant set over a $VPC(n + 1)$ subgroup $H$ of $G$. If $X$ is adapted to $\partial G$, then $X$ is enclosed by some $V_0$–vertex. If
X is not adapted to $\partial G$, there are two cases depending on whether or not it crosses some edge of $T$. If it crosses no edge of $T$, then $X$ is enclosed by some vertex, and hence by a $V_0$–vertex. If $X$ crosses some edge of $T$, we apply all the results above. This yields two $V_0$–vertices $v$ and $v'$ on opposite sides of a special canonical torus edge, and $X$ crosses no edges of $T$ apart from the edges $f$ and $f'$ (and $b$ and $b'$ in case 4) of Lemma 3.4 between $v$ and $v'$. Further $H$ is commensurable with $G(f)$. Thus $X$ is enclosed by the interval whose endpoints are $v$ and $v'$, the edges of this interval all have associated the same special canonical torus, and $H$ is commensurable with that splitting torus group, as required. 

5 Comparisons

As promised in the introduction, we can now give the comparisons of the JSJ decomposition of a PD($n+2$) pair $(G, \partial G)$ with two other naturally defined decompositions. The easiest to handle is the algebraic regular neighbourhood of the set $E_{n,n+1}(G)$ (the collection of all a.i. subsets of $G$ which are over a $VPCn$ or $VPC(n+1)$ subgroup). Note that it is not at all obvious that this family of almost invariant subsets of $G$ has a regular neighbourhood. Such a regular neighbourhood does not exist for general groups, as discussed in [17]. However Proposition 4.11 implies that the decomposition of $G$ obtained by collapsing to a point each interval whose endpoints are the $V_0$–vertices on opposite sides of a special canonical torus edge is the regular neighbourhood of $E_{n,n+1}(G)$. We have shown the following result.

**Theorem 5.1** Let $(G, \partial G)$ be an orientable PD($n+2$) pair such that $G$ is not VPC, and let $E_{n,n+1}(G)$ denote the collection of all a.i. subsets of $G$ which are over a $VPCn$ or $VPC(n+1)$ subgroup. Then the regular neighbourhood of $E_{n,n+1}(G)$ in $G$ exists and is obtained from $\Gamma_{n,n+1}(G)$ by collapsing to a point each interval whose endpoints are the $V_0$–vertices on opposite sides of a special canonical torus edge.

**Remark 5.2** If $\partial G$ is empty, the decomposition $\Gamma_{n+1}(G)$ is equal to the regular neighbourhood of $E_{n,n+1}(G)$ in $G$, as $E_{n,n+1}(G)$ is equal to the collection of all a.i. subsets of $G$ which are over a $VPC(n+1)$ subgroup.

Next we turn to the analogue of the topological decomposition of 3–manifolds obtained in [12]. One considers the family $S_{n,n+1}(G)$, which consists of all a.i. subsets of $G$ which are dual to splittings of $G$ over annuli.
or tori in \((G, \partial G)\). We claim that this family also has a regular neighbourhood, and that the edge splittings are those in \(S_{n,n+1}(G)\) which cross no element of \(S_{n,n+1}(G)\). As in [12], we call this the Waldhausen decomposition or W–decomposition. Again it is not at all obvious that this family of almost invariant subsets of \(G\) has a regular neighbourhood. Such a regular neighbourhood does not exist for general groups, as discussed in [17].

We will say that an element of the family \(S_{n,n+1}(G)\) which crosses no element of \(S_{n,n+1}(G)\) is isolated in \(S_{n,n+1}(G)\). We start by describing the isolated elements of \(S_{n,n+1}(G)\), and showing that there are only finitely many such elements. Trivially the edge splittings of \(\Gamma_{n,n+1}(G)\) are all isolated elements of \(S_{n,n+1}(G)\). We have the following result.

**Lemma 5.3** Let \((G, \partial G)\) be an orientable PD\((n+2)\) pair such that \(G\) is not VPC, and let \(\alpha\) be an isolated element of \(S_{n,n+1}(G)\), which is not an edge splitting of \(\Gamma_{n,n+1}(G)\). Then \(\alpha\) is enclosed by a \(V_0\)–vertex of \(\Gamma_{n,n+1}(G)\) of commensuriser type.

**Proof.** As \(\alpha\) is a splitting of \(G\) over an annulus or torus, it must be enclosed by some \(V_0\)–vertex \(v\) of \(\Gamma_{n,n+1}(G)\). Theorem 2.1 tells us that a \(V_0\)–vertex of \(\Gamma_{n,n+1}(G)\) must be isolated, of \(I\)–bundle type, of interior Seifert type, or of commensuriser type. Thus it suffices to show that the first three cases cannot occur.

If \(v\) is isolated, any splitting enclosed by \(v\) is equal to an edge splitting of \(\Gamma_{n,n+1}(G)\). Hence this case cannot occur.

If \(v\) is of \(I\)–bundle type, and so of VPC\((n-1)\)–by–Fuchsian type, then \(\alpha\) must be a splitting over an annulus. Let \(K\) denote the VPC\(n\) group carried by the splitting annulus, and let \(X_v\) denote the base 2–orbifold of \(v\). Then the image of \(K\) in the fundamental group of \(X_v\) is VPC\((\geq 1)\). As a Fuchsian group cannot have a VPC2 subgroup, it follows that the image of \(K\) must be VPC1. As \(\alpha\) is not an edge splitting of \(\Gamma_{n,n+1}(G)\), and crosses no such splitting, it determines a splitting of \(G(v)\) which is adapted to \(\partial_1 v\). This then yields a splitting over a VPC1 subgroup of the fundamental group of \(X_v\), which is adapted to the boundary \(\partial X_v\). As discussed in section 5.1.2 of [6], this splitting is dual to a ”simple closed curve” \(\gamma\) in \(X_v\), meaning that \(\gamma\) is a connected, closed 1–dimensional suborbifold of \(X_v\). Thus either \(\gamma\) is a circle or it is the quotient of a circle by a reflection involution. The assumption that \(\alpha\) is not an edge splitting of \(\Gamma_{n,n+1}(G)\), means that \(\gamma\) cannot be homotopic to a boundary curve of \(X_v\). Hence there is another ”simple closed curve” \(\delta\)
in $X_v$ whose intersection number with $\gamma$ is non-zero (Corollary 5.10 of [6]). This determines a splitting of $G$ over an annulus whose intersection number with $\alpha$ is non-zero, contradicting our assumption that $\alpha$ crosses no element of $S_{n,n+1}(G)$. We conclude that this case cannot occur.

If $v$ is of interior Seifert type, and so of $VPC_n$–by–Fuchsian type, then $\alpha$ must be a splitting over a torus. As in the preceding paragraph, this torus yields an essential "simple closed curve" in the base 2–orbifold of $v$. As in that paragraph, this implies that there is a splitting of $G$ over a torus whose intersection number with $\alpha$ is non-zero, contradicting our assumption that $\alpha$ crosses no element of $S_{n,n+1}(G)$. We conclude that this case also cannot occur, which completes the proof of the lemma.

Theorem 2.1 tells us that if $v$ is a $V_0$–vertex of $\Gamma_{n,n+1}(G)$ of commensuriser type, then $v$ is of peripheral Seifert type, or of torus type, or of solid torus type. In the penultimate section of [20], the authors discussed the structure of such vertices in great detail. They showed that in each of these cases, $G(v)$ is $VPC_n$–by–$\Gamma$, where $\Gamma$ is the fundamental group of a compact 2–dimensional orbifold $X_v$. The group $\Gamma$ is finite if $v$ is of solid torus type, is virtually infinite cyclic if $v$ is of torus type, and is not virtually cyclic if $v$ is of Seifert type. Recall that any vertex $v$ of $\Gamma_{n,n+1}$ has two types of "boundary" subgroups. The first type comes from the edge groups of the decomposition and the family of all these subgroups will be denoted by $\partial_1 v$. The second type comes from the decomposition of $\partial G$ by edges of the decompositions and this family will be denoted by $\partial_0 v$. The first type gives us $PD(n+1)$ pairs in $(G, \partial G)$, namely annuli or tori, and the second type gives us $PD(n+1)$ pairs which are contained in $\partial G$. If $v$ is of commensuriser type, these families of subgroups determine a division of the boundary $\partial X_v$ of $X_v$ into suborbifolds $\partial_0 X_v$ and $\partial_1 X_v$, where $\partial_0 X_v$ equals the closure of $\partial X_v - \partial_1 X_v$. Note that $\partial_0 X_v$ must be non-empty. It is possible that $\partial_1 X_v$ may be empty, but this happens if and only if $\Gamma_{n,n+1}(G)$ consists of the single vertex $v$, so that $G = G(v)$. Next the authors of [20] show that one can double $G(v)$ along $\partial_0 v$ which is "the intersection of $G(v)$ with $\partial G". The new object $DG(v)$ is the fundamental group of a $V_0$–vertex of $\Gamma_{n+1}(DG)$. It is $VPC_n$–by–$D\Gamma$, where $D\Gamma$ is the fundamental group of $DX_v$, the double of $X_v$ along $\partial_0 X_v$. As we are assuming that $G$ is not $VPC$, it follows that $DG$ is also not $VPC$, so that $\pi_{orb}^1(DX_v)$ cannot contain a $VPC_2$ subgroup.

In the proof of Lemma 5.3, we used the close connection between splittings of $G$ over annuli or tori enclosed by a $V_0$–vertex $v$ and "simple closed curves" in the base 2–orbifold $X_v$. Now we will extend these ideas. We will need
the idea of a "simple arc" $\lambda$ in the pair $(X_v, \partial_0 X_v)$. This means that $\lambda$ is a connected 1–dimensional suborbifold of $X_v$ with non-empty boundary, so that either $\lambda$ is an interval or it is the quotient of an interval by a reflection involution. Further $\lambda$ has boundary contained in $\partial_0 X_v$.

We will say that a "simple closed curve" $\gamma$ in the 2–orbifold $X_v$ is essential in $(X_v, \partial_0 X_v)$ if $\pi_1^{\text{orb}}(\gamma)$ injects into $\pi_1^{\text{orb}}(X_v)$ and $\gamma$ is not homotopic into $\partial_0 X_v$ or into $\partial_1 X_v$. Further $\lambda$ has boundary contained in $\partial_0 X_v$.

We will say that a "simple closed curve" $\gamma$ in the 2–orbifold $X_v$ is essential in $(X_v, \partial_0 X_v)$ if $\pi_1^{\text{orb}}(\gamma)$ injects into $\pi_1^{\text{orb}}(X_v)$ and $\gamma$ is not homotopic into $\partial_0 X_v$ or into $\partial_1 X_v$. Also we will say that a "simple arc" $\lambda$ in $(X_v, \partial_0 X_v)$ is essential in $(X_v, \partial_0 X_v)$ if $\lambda$ cannot be homotoped into $\partial_0 X_v$ nor into $\partial_1 X_v$ while keeping $\partial \lambda$ in $\partial_0 X_v$.

**Lemma 5.4** Let $(G, \partial G)$ be an orientable PD$(n + 2)$ pair such that $G$ is not VPC, and let $\alpha$ be an isolated element of $S_{n,n+1}(G)$, which is not an edge splitting of $\Gamma_{n,n+1}(G)$, and is enclosed by a $V_0$–vertex $v$ of $\Gamma_{n,n+1}(G)$ of commensuriser type, and with base 2–orbifold $X_v$. Then the following hold:

1. If $\alpha$ is a splitting of $G$ over a torus, it determines a "simple closed curve" $C$ in $X_v$ which is essential in $(X_v, \partial_0 X_v)$, and this yields a bijection between splittings of $G$ over a torus which are enclosed by $v$ and not an edge torus of $v$, and "simple closed curves" in $X_v$ which are essential in $(X_v, \partial_0 X_v)$.

2. If $\alpha$ is a splitting of $G$ over an annulus, it determines a "simple arc" $\lambda$ in $(X_v, \partial_0 X_v)$ which is essential in $(X_v, \partial_0 X_v)$, and this yields a bijection between splittings of $G$ over an annulus which are enclosed by $v$ and not an edge annulus of $v$, and "simple arcs" in $(X_v, \partial_0 X_v)$ which are essential in $(X_v, \partial_0 X_v)$.

**Proof.** 1) Note that although $\alpha$ is not an edge splitting of $\Gamma_{n,n+1}(G)$, it need not be the case that it determines a splitting of $G(v)$. This is clear if $v$ is of torus type, but the same difficulty can arise if $v$ is of Seifert type. We resolve this problem by working with the double $DG(v)$ which is the fundamental group of a $V_0$–vertex $V$ of $\Gamma_{n+1}(DG)$. Now $\alpha$ gives a splitting of $DG$ over a torus which is enclosed by $V$, and is not an edge splitting of $\Gamma_{n+1}(DG)$. As in the proof of Lemma 5.3 this splitting is dual to a "simple closed curve" $\gamma$ in $DX_v$ which cannot be homotopic to a boundary curve of $DX_v$. As $\alpha$ is enclosed by $v$, it follows that $\gamma$ can be chosen to lie in $X_v$. Further $\gamma$ must be essential in $(X_v, \partial_0 X_v)$, as $DX_v$ is the double of $X_v$ along $\partial_0 X_v$. Now reversing the arguments yields the required bijection.

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Let $A$ denote the annulus in $(G, \partial G)$ which induces the splitting $\alpha$ of $G$, let $DA$ denote the torus in $DG$ obtained by doubling $A$ along $\partial A$, and let $D\alpha$ denote the splitting of $DG$ induced by $DA$. As $\alpha$ is not an edge splitting of $\Gamma_{n,n+1}(G)$, it follows that $D\alpha$ is not an edge splitting of $\Gamma_{n+1}(DG)$. Now as in part 1), $Da$ determines a splitting of $DG(v)$ over the torus $DA$ which yields a "simple closed curve" $\gamma$ in $DX_v$ which cannot be homotopic to a boundary curve of $DX_v$. This splitting of $DG(v)$ is invariant under the involution interchanging the two copies of $G(v)$, so the curve $\gamma$ can be chosen to be invariant under the involution of $DX_v$ interchanging the two copies of $X_v$. Thus $\gamma$ is the double of a "simple arc" $\lambda$ in the base orbifold $X_v$. Further $\lambda$ has boundary contained in $\partial_0 X_v$, reflecting the fact that the boundary of the annulus lies in $\partial G$. As $\gamma$ is not homotopic to a boundary curve of $DX_v$, it follows that $\lambda$ is essential in $(X_v, \partial_0 X_v)$. One can easily reverse this argument to obtain the required bijection. 

In the preceding lemma, as $\alpha$ crosses no element of $S_{n,n+1}(G)$, it follows that $\gamma$ and $\lambda$ cross no simple closed curve in $X_v$ which is essential in $(X_v, \partial_0 X_v)$, and cross no essential simple arc in $(X_v, \partial_0 X_v)$. We will say that such $\gamma$ and $\lambda$ are isolated.

Note that in the above discussions, any compact 2–orbifold with non-empty boundary can occur as $X_v$, which is not the case in the setting of 3–manifolds. See Lemma 6.1 and the discussion at the end of section 6.

Note also that in general it is possible that an arc or closed curve in $X_v$ can be essential in $(X_v, \partial_0 X_v)$, but inessential in $(X_v, \partial X_v)$, meaning that it can be homotoped into $\partial X_v$ while keeping $\partial \lambda$ in $\partial X_v$.

**Lemma 5.5** Let $X$ be a compact 2–orbifold, and let $\partial_1 X$ denote a possibly empty compact suborbifold of $\partial X$. Let $\partial_0 X$ denote the closure of $\partial X - \partial_1 X$.

1. If the Euler characteristic $\chi(X) \leq 0$, and $\lambda$ is an isolated essential "simple arc" in $(X, \partial_0 X)$, then $\lambda$ is essential in $(X, \partial X)$.

2. If $\chi(X) \leq 0$, and $C$ is an isolated "simple closed curve" in $X$ which is essential in $(X, \partial_0 X)$, then $C$ is essential in $(X, \partial X)$.

3. If $\chi(X) > 0$, there cannot be any "simple closed curve" in $X$ which is essential in $(X, \partial_0 X)$. Also there cannot be any isolated "simple arc" in $X$ which is essential in $(X, \partial_0 X)$.

**Remark 5.6** Recall that here a "simple closed curve" is either a circle or the orbifold quotient of the circle by a reflection, and a similar comment applies.

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to the phrase "simple arc". In each case, the quotient by a reflection is a non-orientable 1–orbifold.

Proof. 1) Suppose that \( \lambda \) is orientable, so that \( \partial \lambda \) consists of two points, and is not essential in \((X, \partial X)\). Thus \( \lambda \) is parallel to an arc \( \mu \) contained in some component \( C \) of \( \partial X \). Of course, the ends of \( \mu \) lie in \( \partial_0 X \). As \( \lambda \) is essential in \((X, \partial_0 X)\), the arc \( \mu \) cannot be contained in \( \partial_0 X \). There must be at least two components of \( \partial_1 X \) in the interior of \( \mu \), so there must be at least one component \( D \) of \( \partial_0 X \) in the interior of \( \mu \). There can be no mirrors in \( X \), as otherwise we could join \( D \) to a mirror to obtain an essential "simple arc" in \((X, \partial_0 X)\) which crosses \( \lambda \). As \( X \) has no mirrors, it follows that \( C \) is a circle. But now there is an arc \( \lambda' \) with both ends in \( D \) which is parallel to an arc \( \mu' \) of \( C \), such that \( \mu \cup \mu' = C \), and \( \lambda' \) is also essential in \((X, \partial_0 X)\) and crosses \( \lambda \). This contradicts the hypothesis that \( \lambda \) is isolated, which proves the required result in the case when \( \lambda \) is orientable.

Next suppose that \( \lambda \) is not orientable, so that \( \partial \lambda \) consists of one point, and is not essential in \((X, \partial X)\). Thus \( \lambda \) is homotopic to an isomorphic 1–orbifold \( \mu \) contained in some component \( C \) of \( \partial X \). Note that the reflector point of \( \mu \) must be a reflector point of \( C \), which must be an intersection point of \( C \) with a mirror component \( m \) of \( X \). Of course, \( \partial \mu \) lies in \( \partial_0 X \). As \( \lambda \) is essential in \((X, \partial_0 X)\), the orbifold \( \mu \) cannot be contained in \( \partial_0 X \), nor in \( \partial_1 X \). It follows that there must be a component \( D \) of \( \partial_0 X \) in \( \mu \) other than the component which contains \( \partial \mu \). Note that \( D \) may contain the reflector point of \( \mu \). If \( X \) has a mirror other than \( m \), we could join \( D \) to such a mirror to obtain an essential "simple arc" in \((X, \partial_0 X)\) which crosses \( \lambda \). It follows that \( m \) is the only mirror in \( X \). In particular, \( C \) and \( m \) must together form a boundary component of the surface underlying \( X \). But now there is an arc \( \lambda' \) with both ends in \( D \) which is parallel to an arc \( \mu' \) of \( C \cup m \), such that \( \mu \cup \mu' = C \cup m \), and \( \lambda' \) is also essential in \((X, \partial_0 X)\) and crosses \( \lambda \). This again contradicts the hypothesis that \( \lambda \) is isolated, which proves the required result in the case when \( \lambda \) is not orientable.

2) Suppose that \( C \) is not essential in \((X, \partial X)\), so that \( C \) is homotopic to a boundary component \( S \) of \( X \). As \( C \) is essential in \((X, \partial_0 X)\), it follows that \( S \) is not contained in \( \partial_0 X \) or in \( \partial_1 X \). If \( X \) has negative Euler characteristic, there is a simple arc \( \mu \) in \( X \) with both ends in \( S \) which is essential in \((X, \partial X)\), and so crosses \( C \). By choosing the ends of \( \mu \) to lie in \( \partial_0 X \), we obtain a contradiction. If \( X \) has zero Euler characteristic, and is orientable, it must be an annulus or \( D^2(2,2) \), the 2–disk with two interior cone points each labeled
2, as $X$ has non-empty boundary. Note that $D^2(2,2)$ is double covered by the annulus. Thus in general, if $X$ has zero Euler characteristic, it is covered by the annulus. In particular, $\partial X$ has 1 or 2 components. In the first case, there is again an essential simple arc $\lambda$ in $(X, \partial X)$ with boundary in $\partial_0 X$ which must cross $C$. See Figures 1b), f), h), i) and j). In the second case, the two boundary components are homotopic, so that $C$ is homotopic to each. Thus neither is contained in $\partial_0 X$ or in $\partial_1 X$, and there is a simple arc in $X$ with ends in $\partial_0 X$ which joins these two boundary components, and so must be essential and cross $C$. See Figures 1a), b), c), d) and g). These contradictions complete the proof that $C$ must be essential in $(X, \partial X)$, as required.

3) As $\chi(X) > 0$, the orbifold fundamental group of $X$ must be finite, so that $X$ cannot contain any "simple closed curve" which is essential in $(X, \partial_0 X)$.

If $\chi(X) > 0$, and $\partial X$ is non-empty, the universal orbifold cover of $X$ must be the 2-disc, so that $X$ is either a cone or the quotient of a cone by a reflection. Let $D^2(p)$ denote the 2-orbifold with underlying surface the 2-disc and with a single interior cone point of order $p \geq 1$, and let $Y_p$ denote the quotient of $D^2(p)$ by a reflection. Note that $D^2(1)$ is simply the 2-disc. The underlying surface of $Y_p$ is a disk $D$, and the boundary $\partial Y_p$ consists of a single interval in $\partial D$ with refector ends. If $p = 1$, the rest of $\partial D$ is a single mirror, and if $p \geq 2$, the rest of $\partial D$ is divided into two mirrors separated by a boundary cone point, labeled $p$. Let $|\partial_0 X|$ denote the number of components of $\partial_0 X$. In all cases, there is a number $k$ depending on $X$ such that if $|\partial_0 X| < k$, then there are no essential "simple arcs" in $(X, \partial_0 X)$, and if $|\partial_0 X| \geq k$, then $X$ contains such simple arcs, but no such arc can be isolated. If $X$ is the disk $D^2(1)$, then $k = 4$. If $X$ is a cone $D^2(p)$, $p \geq 2$, then $k = 3$. If $X$ is $Y_1$, then $k = 3$, and if $X$ is $Y_p$, $p \geq 2$, then $k = 2$. Note that if $X$ is $D^2(p)$, the existence of such $k$ is clear. For if $\rho$ denotes an orientation preserving homeomorphism of $(X, \partial_0 X)$ which sends each component of $\partial_0 X$ and $\partial_1 X$ to the next such component, and if there is an essential simple arc $\lambda$ in $(X, \partial_0 X)$, then $\rho(\lambda)$ must cross $\lambda$.

Now we can prove the following result.

**Lemma 5.7** Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair such that $G$ is not $VPC$. A splitting of $G$ over an annulus or torus of $(G, \partial G)$ which is isolated in $S_{n,n+1}(G)$ is either an edge splitting of $\Gamma_{n,n+1}(G)$ or is over an annulus enclosed by a $V_0$-vertex of commensuriser type, which is not of solid torus
type. Further the family of all splittings of $G$ over an annulus or torus which are isolated in $S_{n,n+1}(G)$ is finite.

**Proof.** If $\alpha$ is a splitting over an annulus or torus and is not an edge splitting of $\Gamma_{n,n+1}(G)$, Lemma 5.3 tells us that $\alpha$ is enclosed by a $V_0$–vertex $v$ of commensuriser type.

If $\alpha$ is a splitting over a torus, it determines an isolated simple closed curve $C$ in the base orbifold $X_v$ of $v$, and $C$ is essential in $(X_v, \partial X_v)$. Thus part 2) of Lemma 5.5 implies that $C$ is essential in $(X_v, \partial X_v)$. Now Corollary 5.10 of [6] implies there is some simple closed curve in $X$ which crosses $C$, which is a contradiction. It follows that a splitting of $G$ by a torus of $(G, \partial G)$ which crosses no element of $S_{n,n+1}(G)$ must be an edge splitting of $\Gamma_{n,n+1}(G)$.

If $\alpha$ is a splitting over an annulus, it determines an isolated essential simple arc in $(X_v, \partial X_v)$, which must be essential in $(X_v, \partial X_v)$ by part 1) of Lemma 5.5. As the number of disjoint non-parallel such arcs in $X_v$ is finite, and as $\Gamma_{n,n+1}(G)$ has only finitely many vertices, the result follows. Finally part 3) of Lemma 5.5 implies that $v$ cannot be of solid torus type. 

Now we can proceed to give a complete description of the exceptional splittings of $G$. These are splittings of $G$ over an annulus of $(G, \partial G)$ which are isolated in $S_{n,n+1}(G)$ and are not edge splittings of $\Gamma_{n,n+1}(G)$. Thus each exceptional splitting is enclosed by some $V_0$–vertex $v$ of commensuriser type, which is not of solid torus type. Let $X_v$ be the base orbifold of $v$. Then our annulus determines an isolated essential simple arc $\lambda$ in $(X_v, \partial X_v)$, and so $\lambda$ is essential in $(X_v, \partial X_v)$, by Lemma 5.5.

In order to give a complete list of cases, the following lemma will be very useful.

**Lemma 5.8** Let $X$ be a compact 2–orbifold, with Euler characteristic $\chi(X) \leq 0$, and let $\partial_1 X$ denote a possibly empty compact suborbifold of $\partial X$. Let $\partial_0 X$ denote the closure of $\partial X - \partial_1 X$. Let $\lambda$ be an isolated essential simple arc in $(X, \partial_0 X)$, such that a point of $\partial \lambda$ lies in a component $C$ of $\partial X$. Then the following hold:

1. If $\chi(X) < 0$, then $C \subset \partial_0 X$.
2. If $\chi(X) = 0$, and $\partial X$ is connected, then $C \subset \partial_0 X$.

**Remark 5.9** It follows that in all cases, if $X$ admits such an arc $\lambda$, and if $\partial X$ is connected, then $\partial_1 X$ must be empty. If $\chi(X) < 0$, the same conclusion holds if $\partial X$ has two components which are joined by $\lambda$. 

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There are two orbifolds where the hypotheses of the lemma hold and \( \chi(X) = 0 \), and \( \partial X \) is not connected. In each of these cases, \( C \) need not be contained in \( \partial_0 X \). See Figures 1a), 1b) and 1c).

Proof. By Lemma 5.5 \( \lambda \) is essential in \( (X, \partial X) \).

1) As \( \chi(X) < 0 \), it is not possible to have two components of \( \partial X \) which are homotopic. Thus if \( C \) is not contained in \( \partial_0 X \), a push off \( C' \) is essential in \( (X, \partial_0 X) \) and crosses \( \lambda \). This contradiction shows that \( C \) must be contained in \( \partial_0 X \), as required.

2) As \( \partial X \) is connected, we can use the same argument as in part 1) to show that \( C \) must be contained in \( \partial_0 X \), as required.

Now we will proceed to list all cases of \( (X, \partial_0 X, \partial_1 X) \), where \( X \) is a compact 2–orbifold with \( \chi(X) \leq 0 \) and non-empty boundary, \( \partial_1 X \) is a possibly empty compact suborbifold of \( \partial X \), and \( \partial_0 X \) is the closure of \( \partial X - \partial_1 X \), and there is an isolated essential simple arc \( \lambda \) in \( (X, \partial_0 X) \). In all cases, when such an arc exists, it is unique up to isotopy. As \( \lambda \) has at least one boundary point, which must lie in \( \partial_0 X \), it follows that \( \partial_0 X \) is non-empty. To find an isolated essential simple arc \( \lambda \) we have to find all essential simple arcs and omit those that cross others. In what follows we have not shown all these arcs, only the isolated ones. (We show some examples with all possible arcs in Figure 5)

Recall from Lemma 5.4 that any isolated essential arc in the base orbifold \( X_v \) of a \( V_0 \)–vertex \( v \) of \( \Gamma_{n,n+1}(G) \) of commensuriser type gives rise to an exceptional annulus in \( S_{n,n+1}(G) \), under the assumption that \( G \) is not \( VPC \). In particular, this excludes the situation where \( \pi_1^{orb}(X_v) \) is \( VPC1 \) and \( \partial_1 X_v \) is empty. Thus in Figure 11 the isolated arcs shown in 1a), 1b) and 1c) are the only cases which are relevant to finding exceptional annuli.

However if \( G \) is \( VPC(n + 1) \) and is \( VPC n \)–by–\( \pi_1^{orb}(X) \) where \( X \) is a 2–orbifold such that \( \pi_1^{orb}(X) \) is \( VPC1 \) and \( \partial_1 X \) is empty, then an isolated arc in \( X_v \) still determines an essential annulus in \( (G, \partial G) \), but that annulus need not be isolated. For example, consider the isolated arcs shown in Figures 1b) and 1c). The orientable 3–dimensional Seifert fibre spaces over the orbifolds in these two figures are each homeomorphic to the twisted \( I \)–bundle over the Klein bottle with orientable total space, and the annuli determined by the isolated arcs cross each other. For a discussion of this example, see page 15 in [17]. Higher dimensional examples can be obtained from this example by taking the product with circles.

In drawing the orbifold \( X \), the pictured boundary consists of the orbifold
boundary $\partial X$ and mirrors. The mirrors are drawn in thick lines and $\partial X$ in thin lines. We then proceed to the division of $\partial X$ into $\partial_0 X$ and $\partial_1 X$. In the following pictures $\partial_0 X$ is still drawn in thin lines, $\partial_1 X$ in dashed lines, and the isolated arc $\lambda$ in dotted lines. Figure I shows all examples with $\chi(X) = 0$. Each of the orbifolds in Figure I is covered by the annulus, and so has orbifold fundamental group which is $VPC1$.

![Diagrams of orbifolds](image)

Figure 1: The case in which $\chi(X) = 0$

We next consider the cases with $\chi(X) < 0$. Recall that $\lambda$ is a "simple arc" in $(X, \partial_0 X)$ which is essential in $(X, \partial X)$ and crosses no essential "simple closed curve" in $X$. Corollary 5.10 of [6] tells us that $X$ admits no essential "simple closed curves" at all. Thus $X$ lies on the list of ten orbifolds given in Proposition 5.12 of [6]. However, two of these ten have no
boundary. In Figure 2, we show the remaining eight orbifolds. For each of these eight orbifolds, we use Lemmas 5.5 and 5.8 to determine the possible decompositions of $\partial X$ into $\partial_0 X$ and $\partial_1 X$ which admit an isolated essential arc, and we show all these cases in Figures 3 and 4. Figure 3 shows the cases where $\partial_1 X$ is empty, and Figure 4 shows the other cases.

![Figure 2: The eight orbifolds with $\chi < 0$, non-empty boundary, and no essential closed curves](image)

Here is a verbal description of the eight orbifolds in Figure 2 and of the possible isolated essential simple arcs.

1. $X = D^2(p, q)$, the 2-disc with two interior cone points of orders $p, q \geq 2$ with at least one strictly larger than 2. There is an isolated essential simple arc in $(X, \partial_0 X)$ iff $\partial_1 X$ is empty.

2. $X = S^1 \times I(p)$, $p \geq 2$, the annulus with one interior cone point. There is an isolated essential simple arc in $(X, \partial_0 X)$ iff $\partial_1 X$ is empty or is a component of $\partial X$.

3. $X$ is a pair of pants, with no singular points. There is an isolated essential simple arc in $(X, \partial_0 X)$ iff $\partial_1 X$ is a component of $\partial X$, or is the union of two components of $\partial X$. 

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4. The underlying surface of $X$ is a disc $D$. The boundary of $D$ contains one mirror interval, so that $\partial X$ is the closure of the complement of this mirror interval in $\partial D$, and $X$ has one interior cone point labeled $p$. There is an isolated essential arc in $(X, \partial_0 X)$ iff $\partial_1 X$ is empty.

5. The underlying surface of $X$ is an annulus $A$. The boundary of $A$ contains one mirror interval, so that $\partial X$ is the closure of the complement of this mirror interval in $\partial A$. There is an isolated essential simple arc in $(X, \partial_0 X)$ iff $\partial_1 X$ is empty or is a component of $\partial X$.

6. The underlying surface of $X$ is a disc $D$, and the boundary $\partial X$ of $X$ consists of a single interval in $\partial D$ with reflector ends, and the rest of $\partial D$ is divided into three mirrors separated by two boundary cone points, labeled $2p$ and $2q$. There is an isolated essential simple arc in $(X, \partial_0 X)$ iff $\partial_1 X$ is empty.

7. The underlying surface of $X$ is a disc $D$. The boundary $\partial X$ of $X$ consists of two disjoint intervals in $\partial D$ each with reflector ends, and the rest of $\partial D$ is divided into a single mirror and two mirrors separated by a boundary cone point, labeled $2p$. There is an isolated essential simple arc in $(X, \partial_0 X)$ iff $\partial_1 X$ is empty or is a component of $\partial X$.

8. The underlying surface of $X$ is a disc $D$. The boundary $\partial X$ of $X$ consists of three disjoint intervals in $\partial D$ each with reflector ends, and the rest of $\partial D$ consists of three mirrors. There is an isolated essential simple arc in $(X, \partial_0 X)$ iff $\partial_1 X$ is a component of $\partial X$, or is the union of two components of $\partial X$.

Thus when $\chi(X) < 0$, we have fourteen orbifolds with an isolated essential simple arc, of which the six shown in Figure 3 have $\partial_1 X$ empty. In these six cases, the group $G$ (in Theorem 5.10) is $VPC_n$–by–$\pi_1^{\text{orb}}(X)$.

Finally we can show that the family $S_{n,n+1}(G)$ has a regular neighbourhood which is a refinement $\Sigma_{n,n+1}(G)$ of $\Gamma_{n,n+1}(G)$. Every element of $S_{n,n+1}(G)$ determines a simple closed curve or simple arc in the base 2–orbifold of one of the $V_0$–vertices of $\Gamma_{n,n+1}(G)$. Now a connected compact 2–orbifold is filled by simple closed curves and simple arcs, unless it is one of the exceptional cases listed above. Thus cutting the base 2–orbifold along the exceptional arc yields 2–orbifolds which contain no isolated essential simple arc. However, in several cases the 2–orbifolds obtained by cutting along an isolated arc contain
Figure 3: Cases with $\partial_1 X = \emptyset$

Figure 4: Cases with $\partial_1 X \neq \emptyset$
non-isolated essential simple arcs. Now splitting an exceptional $V_0$–vertex $v$ along the exceptional annulus yields a vertex or vertices with base orbifold obtained by cutting $X_v$ along the isolated essential arc. These new vertices enclose elements of $S_{n,n+1}(G)$ which correspond to simple arcs in the new base orbifolds. Thus these new vertices enclose elements of $S_{n,n+1}(G)$ other than edge splittings of $\Gamma_{n,n+1}(G)$ if and only if the new base orbifold contains essential simple arcs.

Using the above notation, we can now describe the regular neighbourhood $\Sigma_{n,n+1}(G)$ of $S_{n,n+1}(G)$. It is a refinement of $\Gamma_{n,n+1}(G)$ which can be obtained essentially by splitting each exceptional $V_0$–vertex of $\Gamma_{n,n+1}(G)$ along the exceptional annulus it contains. Each non-exceptional $V_0$–vertex of $\Gamma_{n,n+1}(G)$ yields unchanged a $V_0$–vertex of $\Sigma_{n,n+1}(G)$, and each $V_1$–vertex of $\Gamma_{n,n+1}(G)$ yields unchanged a $V_1$–vertex of $\Sigma_{n,n+1}(G)$. If $v$ is an exceptional $V_0$–vertex of $\Gamma_{n,n+1}(G)$, which contains a separating exceptional annulus, then $v$ is split into two new vertices. If $v$ is an exceptional $V_0$–vertex of $\Gamma_{n,n+1}(G)$, which contains a non-separating exceptional annulus, then $v$ is split into a single new vertex. If a new vertex encloses elements of $S_{n,n+1}(G)$ other than edge splittings of $\Gamma_{n,n+1}(G)$ we label it as a $V_0$–vertex. Otherwise, we label it as a $V_1$–vertex. This yields a refinement of $\Gamma_{n,n+1}(G)$, but it may not be bipartite. By adding an isolated $V_0$–vertex between adjacent $V_1$–vertices, and an isolated $V_1$–vertex between adjacent $V_0$–vertices, and then reducing if needed, we can create a bipartite graph of groups which will be the regular neighbourhood $\Sigma_{n,n+1}(G)$ of $S_{n,n+1}(G)$. We have shown the following result.

**Theorem 5.10** Let $(G, \partial G)$ be an orientable $PD(n + 2)$ pair such that $G$ is not $VPC$, and let $S_{n,n+1}(G)$ denote the family of all a.i. subsets of $G$ which are dual to splittings of $G$ over annuli or tori in $(G, \partial G)$. Then the regular neighbourhood $\Sigma_{n,n+1}(G)$ of $S_{n,n+1}(G)$ in $G$ exists and is obtained from $\Gamma_{n,n+1}(G)$ by splitting each exceptional $V_0$–vertex along the exceptional annulus it contains, as described above.

**Remark 5.11** It follows from this theorem that if $\partial G$ is empty, so that $G$ is an orientable $PD(n + 2)$ group, then the regular neighbourhood $\Sigma_{n,n+1}(G)$ of $S_{n,n+1}(G)$ in $G$ exists and is equal to $\Gamma_{n,n+1}(G) = \Gamma_{n+1}(G)$.

In Figures 1, 3 and 4, we drew only the essential isolated arcs. In Figure 5 we draw some examples with other possible essential arcs to illustrate that they do not lead to isolated arcs. We point out the corresponding figures from the text, but omit the labels.
For Figure 3b) (b) For Figure 2c) (c) For Figure 4b) (d) For Figure 3d) (e) For Figure 2g) (f) For Figure 2h) (g) For Figure 4g)

Figure 5: Note that $\partial_1 = \emptyset$ in all cases except (c) and (g)

6 Results in dimension 3

In this section, we consider the special case of PD3 pairs and compare our results in this case with the results of Neumann and Swarup in [12]. In the previous section, a key role was played by the classification of compact 2–orbifolds with certain properties. In the case when $n = 1$, so that we are considering PD3 pairs, the following result greatly reduces the number of cases which need considering.

**Lemma 6.1** Let $(G, \partial G)$ be an orientable PD3 pair, let $v$ be a $V_0$–vertex of $\Gamma_{1,2}(G)$ which is of Seifert type or of commensuriser type, and let $X$ denote the base 2–orbifold of $v$. Then $X$ has no mirrors.

**Remark 6.2** Note that we are allowing $G$ to be VPC in the above statement. This result means that when $n = 1$, only Figures 1a), 1d)-f), 3a),b) and 4a)-c) are relevant to the results in this section. It also means that each $V_0$–vertex of $\Gamma_{1,2}(G)$ can be regarded as a Seifert fibre space or an $I$–bundle.

**Proof.** Recall that $G(v)$ is $VPC1$–by–$\pi_1^{orb}(X)$, and that $G$ is torsion free. Now a $VPC1$ group is a finite extension of $Z$, so a torsion free $VPC1$ group is $PD1$ and must also be isomorphic to $Z$. And a $VPC2$ group is a finite extension of $Z \times Z$, so a torsion free $VPC2$ group is $PD2$ and must be isomorphic to $Z \times Z$ or to $\pi_1(K)$, where $K$ denotes the Klein bottle. In particular, a torus in a PD3 pair must be isomorphic to $Z \times Z$. 

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If $X$ contains a mirror, there are three possibilities. The mirror must be a circle, or meet $\partial X$, or meet another mirror (or itself) in a corner reflector point. We will show that each of these cases is impossible, which implies that $X$ has no mirrors, as required.

First we consider a component $C$ of $\partial X$, which must be a circle or the quotient $Q$ of a circle by a reflection. We know that $\pi_1^{orb}(C)$ is the image of a torus in $\partial M_v$. As $\pi_1^{orb}(Q) \cong \mathbb{Z}_2 \ast \mathbb{Z}_2$, which is not abelian, $\pi_1(Q)$ cannot be a quotient of the abelian group $\mathbb{Z} \times \mathbb{Z}$. It follows that all components of $\partial X$ are circles. Hence no mirror of $X$ can meet $\partial X$, as required.

Next suppose that $X$ has a corner reflector. This yields a finite dihedral subgroup $D$ of $\pi_1^{orb}(X)$, where the term dihedral group includes the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. In particular $D$ is not cyclic. But the pre-image of $D$ in $G(v)$ is a torsion free $VPC1$ group and so is isomorphic to $\mathbb{Z}$, which implies that $D$ must be cyclic. This contradiction show that $X$ cannot have corner reflectors, as required.

Finally suppose that $X$ has a mirror $m$ which is a circle. Then $m$ has a neighbourhood orbifold $Y$ in $X$ with underlying space an annulus, such that $\partial Y$ is equal to one boundary component $C$ of the annulus and the other boundary component is $m$. We have $\pi_1^{orb}(C) = \pi_1(C) \cong \mathbb{Z}$, and $\pi_1^{orb}(m) = \pi_1(C) \times \mathbb{Z}_2$, and we let $R$ and $S$ denote the pre-images in $G(v)$ of $\pi_1(C)$ and $\pi_1^{orb}(m)$ respectively. Thus $R$ is a subgroup of $S$ of index 2. Each of $R$ and $S$ is a torsion free $VPC2$ group. As $C$ determines a splitting of $\pi_1^{orb}(X)$ over $\pi_1(C)$ which is adapted to $\partial X$, this yields a splitting of $G(v)$ over $R$ which is adapted to $\partial_1 v$, and hence determines a splitting of $G$ over $R$ which is adapted to $\partial G$. As the pair $(G, \partial G)$ is orientable, it follows that $R$ is orientable, and so is a torus in $G$. Also the splitting of $G$ over $R$ yields one or two $PD3$ pairs with $R$ as a boundary group. Now Lemma 2.2 of [11] implies that $R$ is maximal among torus subgroups of $G$, so that $S$ cannot be a torus. Thus $S$ is isomorphic to $\pi_1(K)$. Now consider the presentation $< a, b : bab^{-1} = a^{-1} >$ of $\pi_1(K)$. The kernel of the map $S \cong \pi_1(K) \to \pi_1^{orb}(m) \cong \mathbb{Z} \times \mathbb{Z}_2$ must be the subgroup $A$ generated by $a^2$, and this must also be the kernel of the map $R \to \pi_1(C) \cong \mathbb{Z}$. As $R$ has index 2 in $\pi_1(K)$, it must be the orientation subgroup generated by $a$ and $b^2$. But then the quotient of $R$ by $A$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$, which is a contradiction.

This completes the proof that $X$ has no mirrors.

Now we can compare our results from section 5 with those of Neumann and Swarup in [12]. Recall that in section 5 we considered an orientable $PD(n + 2)$ pair $(G, \partial G)$ such that $G$ is not $VPC$ and described the possible
exceptional annuli. These are splittings of $G$ dual to an annulus which are not edge splittings of $\Gamma_{c,n,n+1}(G)$. Each exceptional annulus is enclosed by some $V_0$–vertex $v$ of $\Gamma_{c,n,n+1}(G)$ of commensuriser type, and corresponds to an isolated arc $\lambda$ in the base 2–orbifold $X_v$ of $v$. In Figures (1a)-c), 3 and 4 we showed all possible such arcs and orbifolds. We will be interested in the special case when $n = 1$, so that $(G, \partial G)$ is an orientable $PD3$ pair. Lemma 6.1 tells us that those figures in which the orbifold $X$ has a mirror, are not relevant in this case. This substantially reduces the number of possibilities. We only need to consider the six isolated arcs shown in Figures (1a), (3a),b) and (4a)-c).

Recall from the beginning of section 2 that an annulus in a $PD(n+2)$ pair is a certain type of orientable $PD(n+1)$ pair, whose fundamental group is $VPCn$. When $n = 1$, a torsion free $VPC1$ group must be isomorphic to $\mathbb{Z}$, and this is an orientable $PD1$ group. Thus twisted annuli do not appear when considering $PD3$ pairs, and an untwisted annulus in our generalized sense is exactly the same as the ordinary annulus $S^1 \times I$. Further if our $PD3$ pair $(G, \partial G)$ comes from a compact orientable 3–manifold $M$, then there is a precise correspondence between $\Gamma_{1,2}(G)$ and the JSJ decomposition of $M$. Also any exceptional annuli in $(G, \partial G)$ correspond to embedded annuli in $M$ which cross no other embedded essential annulus in $M$ and are not splitting annuli of the JSJ decomposition of $M$. In [12], such annuli are called matched annuli, and the possibilities are listed in Lemma 3.4 of [12]. We would expect this list to be the same as our list of six possible isolated arcs in Figures (1a), (3a),b) and (4a)-c), but there are some differences. The four isolated arcs shown in Figures (1a), (3a), (4a) and (4c) yield the examples of matched annuli shown in Figure 5 of [12], but the two isolated arcs shown in (3b)) and (4b) do not correspond to matched annuli shown in Figure 5 of [12]. In (3b), $\partial_1 X$ is empty which implies that $G = G(v)$, and that $M$ is a Seifert fibre space, so this case is not of much interest. But in Figure (1b), $\partial_1 X$ is non-empty, so the isolated arc in this figure corresponds to an interesting matched annulus in $M$. This seems to be an omission in [12].

The result of Lemma 6.1 fails in higher dimensions. Mirrors of all three types discussed in the proof of Lemma 6.1 can exist in all dimensions greater than 3. We discuss some examples in dimension 4. Again higher dimensional examples can be obtained by taking the product with circles.

Our starting point is that the orientable 3–manifold $W$ which is a twisted $I$–bundle over the Klein bottle $K$ is an example of a twisted 3–dimensional annulus, and the double $DW$ of $W$ is a 3–dimensional torus. Thus the
orientable 4–manifold $DW \times I$ is the underlying space of a PD4 pair $(G, \partial G)$, and $\Gamma_{2,3}(G)$ consists of a single $V_0$–vertex $v$, so $G = G(v)$, and $v$ is of $VPC2$–by–Fuchsian type with base orbifold $Q \times I$, where $Q$ is the quotient of the circle by a reflection. This orbifold has two mirrors each meeting the orbifold boundary in reflector points. If instead one considers the manifold $DW \times S^1$, one will have two mirrors each homeomorphic to a circle.

Finally, one can also give examples with corner reflectors as follows. A useful way to think about $W$ is as the $I$–bundle over $K$ associated to the $\partial I = S^0$–bundle given by the double covering map $T \to K$. Note that this map is determined by a surjective homomorphism $\pi_1(K) \to \mathbb{Z}_2$. One way to construct $W$ is to start with the product $T \times I$ of the 2–torus with the unit interval, and consider the involution $(\tau, \sigma)$ on $T \times I$, where $\tau$ is the free involution of $T$ associated to the double covering map $T \to K$, and $\sigma$ is the reflection of $I$. As $\tau$ is free, so is $(\tau, \sigma)$, and the quotient of $T \times I$ by $(\tau, \sigma)$ is clearly $W$.

We will perform a similar construction starting with the product $T \times I \times I$, and using the natural homomorphism $\varphi : \pi_1(K) \to H_1(K; \mathbb{Z}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Only one of the three surjections $\pi_1(K) \to \mathbb{Z}_2$ yields an orientable double cover, and we will choose the basis of $H_1(K; \mathbb{Z}_2)$ so that projection onto each factor yields non-orientable double covers $K'$ and $K''$. Let $T$ denote the torus which is the 4–fold cover of $K$ corresponding to the kernel of $\varphi$. Thus $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts freely on $T$ with quotient $K$. It also acts on $I \times I$ as the group generated by reflections in each factor, and we let $X$ denote the quotient 2–orbifold of this action. The underlying space of $X$ is a disc $D$, whose boundary is divided into two mirrors and an arc of $\partial X$. The product action on $T \times I \times I$ is free and orientation preserving, so that the quotient of $T \times I \times I$ by this action is an aspherical orientable 4–manifold $Z$, and $Z$ has a natural projection to $X$. The pre-image in $Z$ of each interior point of $X$ is $T$. The pre-image of the corner reflector of $X$ is $K$, and the pre-image of all other points of one mirror is $K'$ and of the other mirror is $K''$. Finally the pre-image of all other points of $\partial X$ is $T$. Further the pre-image of $\partial X$, which is equal to $\partial Z$, consists of the union of the twisted $I$–bundle over $K'$ with boundary $T$ and the twisted $I$–bundle over $K''$ with boundary $T$, glued along $T$. The pre-image of one mirror is a twisted $I$–bundle over $K$ with boundary $K'$, and the pre-image of the other mirror is a twisted $I$–bundle over $K$ with boundary $K''$. Thus $(Z, \partial Z)$ is the underlying space of a PD4 pair $(G, \partial G)$, and $\Gamma_{2,3}(G)$ consists of a single $V_0$–vertex $v$, so $G = G(v)$, and $v$ is of $VPC2$–by–Fuchsian type with base orbifold $X$. 46
7 Some related questions

An unsatisfactory part of our work is that there is no algebraic treatment of the triple \((G(v), \partial_0 v, \partial_1 v)\) we discussed.

**Problem 7.1** Construct a theory of Poincaré triads for groups.

There is a discussion by Wall in the case of complexes \([22]\).

In Johannson’s Deformation Theorem, he considers a homotopy equivalence \(F : M \to M'\) between two Haken 3–manifolds with incompressible boundary. He shows that there is a bijection between the pieces of the JSJ decomposition of \(M\) and those of \(M'\), and that \(F\) can be homotoped to send the pieces of \(M\) to the pieces of \(M'\). In particular, the splitting annuli and tori of \(M\) are sent to splitting annuli and tori of \(M'\). For the non-characteristic pieces of \(M\), he shows one can further homotop \(F\) to arrange that the intersection with the boundary of \(M\) is mapped to the boundary of the corresponding piece of \(M'\). Finally one can arrange that the restriction of \(F\) to each non-characteristic piece is a homeomorphism to the corresponding piece of \(M'\). It is natural to ask whether there is an algebraic analogue of this. The natural analogue would be when one considers two \(PD(n + 2)\) pairs \((G, \partial G)\) and \((G', \partial G')\) with an isomorphism between \(G\) and \(G'\). There is a bijection between the underlying graphs of \(\Gamma_{n,n+1}^c(G)\) and \(\Gamma_{n,n+1}^c(G')\), and one would like to know that for a \(V_1\)–vertex \(v\) of \(\Gamma_{n,n+1}^c(G)\), the part \(\partial_0 v\) of \(\partial v\) coming from \(\partial G\) can be deformed into \(\partial_0 v'\) of the corresponding \(V_1\)–vertex of \(\Gamma_{n,n+1}^c(G')\). It seems reasonable this should hold when \(n = 1\), but this seems far from clear when \(n > 1\). The reason is that the proof of Johannson’s Deformation Theorem depends on the non-existence of certain types of essential annulus in the non-characteristic pieces of \(M\). In higher dimensions the analogous fact would be the non-existence of essential higher dimensional annuli, but that may not exclude the existence of essential maps of the 2–dimensional annulus.

**Problem 7.2** In the case of orientable \(PD3\) pairs \((G, \partial G)\) and \((G', \partial G')\), with \(G\) and \(G'\) isomorphic, for any \(V_1\)–vertex \(v\) of \(\Gamma_{n,n+1}^c(G)\), and the corresponding \(V_1\)–vertex \(v'\) of \(\Gamma_{n,n+1}^c(G')\), show that \(\partial_0 v\) can be deformed into \(\partial_0 v'\).

In Theorem 8.1 of [1], Bieri and Eckmann proved a result which we have used several times. Namely that if a \(PDn\) pair is split along a \(PD(n - 1)\) subgroup relative to the boundary, then we again get \(PDn\) pairs.
Problem 7.3 Is there an analogue of the Bieri-Eckmann Theorem when a PD$_n$ pair is split along a PD$(n-1)$ pair?

Examples in dimension 3 show that if one splits a 3–manifold with incompressible boundary along a surface with non-empty boundary, the resulting manifold may have compressible boundary. Thus if one splits a PD$_3$ pair along a PD$_2$ pair, the resulting object need not be a PD$_3$ pair. This again seems to need a theory of PD triples for groups. However, Gitik [5] has proven an analogue of the Bieri-Eckmann Theorem in the special case when splitting a PD$_n$ pair along a PD$(n-1)$ pair does yield a PD$_n$ pair.

A related natural question is:

Problem 7.4 Is there a theory of PD pairs when the maps from the boundary groups are not injective?
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