Nonabelian Duality and Solvable Large $N$ Lattice Systems.

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Abstract

We introduce the basics of the nonabelian duality transformation of $SU(N)$ or $U(N)$ vector-field models defined on a lattice. The dual degrees of freedom are certain species of the integer-valued fields complemented by the symmetric groups’ $\otimes_n S(n)$ variables. While the former parametrize relevant irreducible representations, the latter play the role of the Lagrange multipliers facilitating the fusion rules involved. As an application, I construct a novel solvable family of $SU(N)$ $D$-matrix systems graded by the rank $1 \leq k \leq (D-1)$ of the manifest $[U(N)]^\otimes k$ conjugation-symmetry. Their large $N$ solvability is due to a hidden invariance (explicit in the dual formulation) which allows for a mapping onto the recently proposed eigenvalue-models [7] with the largest $k = D$ symmetry. Extending [7], we reconstruct a $D$-dimensional gauge theory with the large $N$ free energy given (modulo the volume factor) by the free energy of a given proposed $1 \leq k \leq (D-1)$ $D$-matrix system. It is emphasized that the developed formalism provides with the basis for higher-dimensional generalizations of the Gross-Taylor stringy representation of strongly coupled $2d$ gauge theories.

Keywords: Lattice, Yang-Mills, Duality, Solvability

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1 Introduction

A Duality of the $D = 4$ continuum Yang-Mills gauge system to a kind of string theory remains to be one of a few intuitive guiding principles to attack nonperturbative dynamics of the strong interactions. Among the circumstantial evidences, the central role is played by the Wilson’s $D \geq 2$ string-like representation \[1\] of the strong-coupling (SC) series but in a lattice cousin of the continuum YM theory. As it is well known, this particular expansion (running in terms of the inverse powers of the bare coupling constant) can not be directly extended into the weak-coupling phase relevant for the continuum limit. Nevertheless, we believe that (properly chosen) lattice $YM$ systems hide a stringy pattern relevant for the $D \geq 2$ continuum gauge theories at least in the regime being the continuum counterpart of the lattice SC phase. To this aim, one is to consider a continuum $D \geq 2$ $YM_D$ model with a finite ultra-violate cut off $\tilde{\Lambda}_{UV}$ and sufficiently large coupling constant(s).

To support this idea, we refer to the well-studied $D = 2$ case, where Gross and Taylor proposed an elegant stringy representation \[2\] of the large $N$ SC series in the continuum $SU(N)$ gauge system on an arbitrary $2d$ surface. Recall that a continuum $YM_2$ can be directly reproduced through the corresponding lattice gauge model with the action defined via the associated self-reproducing plaquette-factor \[3, 4\]. As a result, the pattern of the proposed $D = 2$ representation is the same both in the SC regime of a given continuum $YM_2$ theory and in the SC phase of the corresponding self-reproducing lattice model. The only considerable distinction is that in the latter case one would deal with the discretized surfaces rather than with the $2d$ manifolds. The crucial point is that, as far as the ‘built in’ topological data is concerned, this difference does not matter. As a result, in both instances the appropriate SC series can be reinterpreted in terms of statistics of all admissible branched coverings (associated to the base-surface) described canonically in terms of the symmetric groups’ elements.

The key-ingredient of the above $D = 2$ construction is the so-called Schur-Weyl complementarity (see e.g. \[5\] for a review) between the Lie and symmetric groups. Altogether, for $YM_2$ it fulfils the role of a bridge between the symmetry and topology that makes it particularly suitable for construction of the gauge string representation. Unfortunately, the proposed in \[2\] technology can not be directly extended to $D \geq 3$. The purpose of the
The present paper is to develop the basics of an approach which, among other things, renders accessible higher-dimensional generalizations of the Gross-Taylor pattern.

One of the central elements of our approach is the nonabelian duality transformation. On the one hand, the latter can be considered as a natural extension of the Schur-Weyl duality. On the other hand, it is to be viewed as a realization of the long sought nonabelian version of the abelian transformation well known in the context of the pure $U(1)$ lattice system (see [11] for a review). This generalization can be compared, in particular, with the recent conjecture of Polyakov [6]: He advocated that the lattice abelian transformation encodes the $N = 1$ string-like pattern which might be generalized (in a yet unknown way) for the $U(N)$ continuum gauge theory with an arbitrary $N$.

The Gauge String construction, we keep in mind, is facilitated by the formalism which is to synthesize both the nonabelian duality (i.e. symmetry) and the topology of the branched coverings. We find it appropriate to introduce the former ingredient in a simpler setting that avoids entanglement with the topology. For this purpose, we find a simpler application of the duality transformation constructing a family of solvable large $N$ $SU(N)$ matrix systems which are not tractable by other methods. The $D > 2$ generalization of the Gross-Taylor 2d stringy pattern on a lattice will be given in a forthcoming paper [16].

In what follows, employing the nonabelian duality we design a mapping between the recently proposed solvable $D$-matrix eigenvalue-theories [7] and a novel class of the $D$-matrix models which apparently are not of the eigenvalue-type. There are a few reasons why the latter models are worth studying by themselves. First, it provides with a rare example of solvable multimeatrix models nontrivially depending on the nondiagonal components of the $SU(N)$ or $U(N)$ matrices involved. As we will see, the mechanism behind their solvability is different from that in the popular systems computable owing to the 'built in' Itzykson-Zuber integral [8]: the Kazakov-Migdal model [9] together with the conventional and conformal multimeatrix systems (see e.g. [10] for a review). Second, generalizing the prescription of [7], the new models can be viewed as the large $N$ reduction of the associated $D$-dimensional lattice gauge theories. In other words, the large $N$ partition function (PF) $\tilde{X}_{LD}$ of the latter theory (defined on a cubic lattice
of $D$-volume $L^D$) can be reproduced
\[ \lim_{N \to \infty} \tilde{X}_{L^D} = \lim_{N \to \infty} (\tilde{X}_r)^{L^D}, \tag{1.1} \]
through the PF $\tilde{X}_r$ of the corresponding $D$-matrix model with the reduced space-time dependence. The latter models are computable owing to the claimed duality to the basic $SU(N)$ eigenvalue-family \[7\]

\[ e^{-S^{(2)}(\{U_\mu\})} = \sum_{\{R_\phi\}} e^{-S^{(2)}(\{R_\phi\})} \prod_{\mu \nu} \chi_{R_{\mu\nu}}(U_\mu) \chi_{R_{\mu\nu}}(U_\nu)^2, \tag{1.2} \]
formulated in terms of the eigenvalues of the $D$-matrices $U_\rho \in SU(N)$, or to its minor modification
\[ e^{-S^{(1)}(\{U_\mu\})} = \sum_{\{R_\phi\}} e^{-S^{(1)}(\{R_\phi\})} \prod_{\mu \nu} \chi_{R_{\mu\nu}}(U_\mu^+) \chi_{R_{\mu\nu}}(U_\nu^+) \prod_{\rho=1}^D \chi_{R_{\rho}}(U_\rho), \tag{1.3} \]
both being defined on a single $D$-cube with periodic boundary conditions: $\{\mu \nu\} = \{1, ..., D(D-1)/2\}$. The relevant sums run over all $SU(N)$ irreducible representations (irreps) $R_\phi \in Y_{n(\phi)}(N)$, $\phi \in \{\mu \nu\}, \{\rho\}$, and in the case of (1.3) it is postulated that the numbers of boxes $n(\rho)$ in the associated Young tableau are constrained by $n(\rho) = \sum_{\nu \neq \rho} n(\rho \nu)$.

The key-advantage of the eigenvalue-systems (1.2),(1.3) is that their large $N$ PF $\tilde{X}_r^{(m)}$ is explicitly computed \[7\] employing the saddle-point (SP) method applied to the irreps $\{R_\phi\}$. To construct the purported solvable noneigenvalue deformations of (1.2),(1.3), we first rewrite the PF $\tilde{X}_r^{(m)}$ of the latter systems in terms \[7\] of the $D$-products
\[ \lim_{N \to \infty} \tilde{X}_r^{(m)} = \lim_{N \to \infty} \left[ \sum_{\{R_\phi\}} e^{-S(\{R_\phi\})} m \prod_{\rho=1}^{D(D-1)/2} L_{R_\rho(\{R_{\rho \mu}\})}^{(D-1)} \right]^m \tag{1.4} \]
of the generalized Littlewood-Richardson (GLR) coefficients of $(D-1)th$ order
\[ L_{R_\rho(\{R_{\rho \mu}\})}^{(D-1)} = \int d\tilde{U}_\rho^{SU(N)} \chi_{R_\rho}(\tilde{U}_\rho^+) \left[ \otimes_{\mu \neq \rho} \chi_{R_{\mu \rho}}(\tilde{U}_\rho) \right] \in \mathbb{Z}_{\geq 0}. \tag{1.5} \]
which encode the fusion rules of the $SU(N)$ characters. I assert that the reduction of the PF to the GLR generating functional (1.4) takes place in a
larger, apparently noneigenvalue variety of $D$-matrix models like

$$e^{-S_r(U_{\rho})} = \sum_{\tilde{n}(+) \in \mathbb{Z}_{\geq 0}} \sum_{\{w_q^{(s)}\}_{+\tilde{n}(+)}} e^{-A\{w_q^{(s)}\}} \Re[\otimes_{q=1}^p \text{tr}(U_{\mu_q}U_{\nu_q}...U_{\lambda_q})], \quad (1.6)$$

where the $\{w_q^{(s)}\}_{+\tilde{n}(+)}$-sum runs over all cyclic-symmetrized words $w_q^{(s)} \equiv [\mu_q\nu_q...\lambda_q]^{(s)}$ (made of the $2 \otimes D$ different $\rho$-, $\rho^{-1}$-'letters') of lengths $n_k$ which can be composed from the total number $\tilde{n}(+)$ of the $\{U_{\rho^\pm}\}$ factors: $\sum_{q=1}^p n_q = \tilde{n}(+)$. The representation (1.6) is not particularly helpful to distinguish the GLR computable variety, we are interested in, from the generic $D$-matrix system. The only explicit general structure is that the family (1.6) forms a natural hierarchy graded by the rank $k = D,...,1$, of the $[U(N)]^\otimes k$ conjugation-invariance $[U(N)]^\otimes k : U_{\rho^{(\beta)}} \to g_\beta^+ U_{\rho^{(\beta)}} g_\beta$, $\beta = 1,...,k$, $g_\beta \in U(N)$, \quad (1.7)

where $U_{\rho}$ factors are recollected into a set of $\beta$-families $\{\rho(\beta)\}$. The deep reason for the GLR solvability of the $k < D$ noneigenvalue-systems (i.e. for their duality to the $k = D$ models (1.2),(1.3)) is a specific hidden symmetry. The latter becomes manifest only in the dual representation for the PF of a $k \leq D$ subvariety of (1.6) (including (1.2), (1.3)). Extending the abelian transformation \[14\], we propose the following dual variables: the $\phi$-species of the integer-valued $\{\lambda(\phi)\}$-sets (parametrizing associated $SU(N)$ irreps $R_\phi$) combined with the elements of the symmetric groups $\otimes_n S(n)$. The $S(n)$-valued degrees of freedom (being composed into the characters of the corresponding tensor representations) act as the Lagrange multipliers. They facilitate the nonabelian fusion-rule constraints for the complementary integer-valued fields $\lambda_i(\phi)$, $i = 1,...,N - 1$, entering the construction within the canonical $S(n_\phi)$-valued Young projectors $P_{R_\phi}$, $R_\phi \in Y_{n_\phi}^{(N)}$.

In the dual representation, the GLR coefficients in the PF (1.4) can be referred (owing to the Schur-Weyl complementarity \[12\]) to the simplest available fusion-pattern of the $S(n_\phi)$-valued Young idempotents (YI) $C_{R_\phi} \sim P_{R_\phi}$. It is the nonabelian duality which, as we will see, allows to reveal that among (1.6) there are $k < D$ systems with the same (as in the $k = D$ case (1.2), (1.3)) GLR pattern (1.4) of the underlying YI fusion-rules. Complementary, the pivotal role of the $\otimes_n S(n)$-variables in fact foreshadows a
tight relation to the $D \geq 2$ Gauge String construction generalizing the 2d pattern [4].

Finally, the organization of the paper is as following. The details of the exact duality transformation, applied first to the eigenvalue-models (1.2), (1.3), are discussed at length in in Section 2. In particular, for the latter models we rederive the GLR form (1.4) of the PF $\tilde{X}_r(\mu)$ directly in the framework of the dual representation. Building on this formalism, in Section 3 we construct the GLR computable $k \leq D - 1$ subvariety of the $D$-matrix systems (1.6). Among the deformations, we select a $k = 1$ noneigenvalue-family specifically suitable for the discussion of the continuum limit (CL) in the corresponding $D$-dimensional induced gauge theories. The algorithm to reconstruct the latter theories is formulated in Section 4. To address the issue of the CL, in Section 5 we first transform the associated large $N$ GLR functional (1.4) into the 1-matrix representation [7]. Then we prove that the simple criterion imposed on the latter 1-matrix system (formulated in [7] for the $k = D$ family (1.2), (1.3)) is valid in the case of the selected $k = 1$ variety as well. For this purpose, following [7] we demonstrate that the link-variables in the corresponding induced lattice gauge theory are localized $\{U_\rho(z) \to 1\}$ (modulo the relevant symmetries) which is tantamount to the regime of the CL.

Our conclusion emphasizes the major novel possibilities open by the proposed approach. Among other things, we assert one of the expected features of the $D > 2$ Gauge String representation (of the strongly coupled $YM_D$ theories) novel compared to the $D = 2$ construction of Gross and Taylor [2].

A few Appendices contain relevant technical details of the derivations used in the main text.

2 The Dual form of $D$-matrix models.

Let us introduce the concept of the nonabelian Duality which is built on the complementarity of the Lie and symmetric groups (reviewed in Appendix A). In our opinion, the dual representation provides with the appropriate mathematical framework to operate with the results of the generic multiple $U(N)$ or $SU(N)$ integrations like those defining the PF $\tilde{X}_r$ of an arbitrary $D$-matrix model (1.6).
It is instructive to view the nonabelian transformation as the extension of the abelian construction [11] dating back to the classical paper due to Kramers and Wannier who discovered the selfduality of the 2d Ising model. For a brief review, take the most relevant for our analysis option of the pure lattice gauge theory based on the $U(1)$ group, i.e. compact QED in $D \geq 2$. Its partition function is defined as the multiple link-integral of the product composed of the standard $U(1)$ plaquette-factors $Z(U) = \sum_{n\in\mathbb{Z}} \chi_n(U) Q(n)$

\[
\tilde{X}^{U(1)}_{N_p,N_l} = \prod_{p=1}^{N_p} \prod_{l=1}^{N_l} \int dU^{U(1)}_l Z(U(p)) , \quad \int dU^{U(1)}_l \equiv \int_{-\pi}^{\pi} \frac{d\theta_l}{2\pi} , \quad (2.1)
\]

where $U^{U(1)}_l = e^{i\theta_l}$, while $U(p) = e^{i[\nabla \theta](p)}$ is the holonomy around the $p$th elementary plaquette of the base-lattice $\{p;l\}$ consisting of $N_l$ links and $N_p$ plaquettes. In the abelian case, the crucial simplification arises due to the fact that all $U(1)$ irreps (labelled by $n \in \mathbb{Z}$) are one-dimensional, while the $U(1)$ characters

\[
U(1) : \quad \chi_n(U) = U^\oplus n = e^{i\theta_n} , \quad \chi_{n_1}(U) \chi_{n_2}(U) = \chi_{n_1+n_2}(U) , \quad (2.2)
\]

form the so-called character-group [11] isomorphic to $\mathbb{Z}$. Combining eq. (2.2) with the simple structure the generic $U(1)$ 1-link integral

\[
\int dU^{U(1)} U^\oplus m (U^+)^\oplus m = \delta[n,m] , \quad n, m \in \mathbb{Z}_{\geq 0} \quad (2.3)
\]

one easily derives the dual form of the partition function (2.1) as the constrained multiple sum over the integer-valued variables (assigned to the plaquettes)

\[
\tilde{X}^{U(1)}_{N_p,N_l} = \prod_{p=1}^{N_p} \sum_{n(p)\in\mathbb{Z}} Q(n(p)) \prod_{l=1}^{N_l} \delta[(\sum_{\tilde{p}_l=1}^{2(D-1)} n(\tilde{p}_l)), 0] \quad (2.4)
\]

Here the sum in the argument of the Kronecker delta-function, running over the $2(D-1)$ plaquettes $\tilde{p}_l$ which share a given link $l$ in common, represents the relevant $U(1)$ fusion-rule algebra.

Finally, one observes that the relevant $U(1)$ (and more generally $U(N)$ or $SU(N)$) $D$-matrix models [1,3] are defined on the following reduced base-lattice. Topologically, this lattice is made (as it is clear e.g. from the
pattern of eq. (1.2)) from the set of the $D(D - 1)/2$ distinct $\mu\nu$-plaquettes through the identification of all $D - 1$ their $\rho$-links to match with the topology of a $D$-cube with periodic boundary conditions. Altogether this conglomerate can be visualized as $D(D - 1)/2$ mutually intertwined 2-tori.

2.1 The Dual form of the $U(N)$ measure.

Let us proceed introducing the dual representation of the functional measure in the nonabelian lattice vector-field theories. Recall that the measure, considered as a distribution, can be defined specifying all the 'moments' of this distribution. On a given base-lattice, these 'moments' are specified defining at each link the set of generic 1-link integrals

$$M_{G}^{p_{1}...q_{m}}(n,m) \equiv \int dU \left( U^{p_{1}}...U^{q_{m}}(U^{+})^{l_{1}}...U^{l_{m}} \right)$$

composed from the $N \times N$ matrices $(U)_{jk}^{p_{k}}$, $(U^{+})_{jk}^{q_{k}}$ in the (anti)fundamental representation of the Lie group $G$ in question. The crucial observation is that $M_{G}^{p_{1}...q_{m}}(n,m)$ can be dually reformulated in terms of the $S(n)$-valued variables. In particular, in the $G = U(N)$ case $M_{U(N)}^{p_{1}...q_{m}}(n,m)$ reads

$$M_{U(N)}^{U(N)}(n,m)_{j_{1}...l_{m}}^{p_{1}...q_{m}} = \delta[n,m] \sum_{\delta \in S(n)} D(\delta^{-1} \Lambda^{(-1)}_{n})_{\{i_{\ominus n}\}}^{\{j_{\ominus n}\}} D(\delta)_{\{l_{\ominus n}\}}^{\{p_{\ominus n}\}}$$

generalizing the $U(1)$ pattern (2.3). The derivation of the identity (2.6) is given in Appendix B, and here we simply explain the meaning of its building blocks. The factor $D(\Psi)$ stands for the canonical tensor representation of a given $S(n)$-algebra element $\Psi$ deduced (by linearity) from the representation [12] of a $S(n)$-group element $\sigma$

$$D(\sigma)_{\{i_{\ominus n}\}}^{\{j_{\ominus n}\}} = \delta^{i_{\sigma(1)}}_{j_{1}} \delta^{i_{\sigma(2)}}_{j_{2}} ... \delta^{i_{\sigma(n)}}_{j_{n}} ; \quad \hat{\sigma} : k \to \sigma(k) , \quad k = 1,...,n ,$$

while the introduced in (2.6) operator $\Lambda^{(-1)}_{n} \in S(n)$ can be viewed as belonging to the family

$$\Lambda_{n}^{(m)} = \sum_{R \in Y_{n}(N)} d_{R} (n! \dim R/d_{R})^{m} C_{R} , \quad m \in \mathbb{Z} ,$$

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which is expressed in terms of the dimensions \( d_R \) and \( \text{dim}R \) of the \( S(n) \)- and (chiral) \( U(N) \)-irreps \( R \in Y^{(N)}_n \) respectively. As for the operator \( C_R \in S(n) \) in eq. (2.8), it denotes the canonical Young idempotent proportional \( P_R = d_R C_R \) to the Young projector \( P_R \) (see Appendix A for more details)

\[
P_R = \frac{d_R}{n!} \sum_{\sigma \in S(n)} \chi_R(\sigma) \, \sigma , \quad R \in Y_n ,
\]

where \( \chi_R(\sigma) \) is the corresponding character. Summarizing, one observes that the \( S(n) \)-representation (2.6) of the 1-link integral (2.5) establishes a remarkable duality extending the Schur-Weyl complementarity (see Appendix A for a review of the latter). Considered as the operators acting on \(|\rho >^{\otimes n} \otimes |q >^{\otimes n}\)-space, the left and right hand sides of eq. (2.6) belong to the complementary structures: the Lie group ring and the symmetric group algebra (being augmented by the integer-valued fields parametrizing irreps \( R \)) respectively.

Next, in contradistinction to the \( U(N) \) case, the \( SU(N) \) 1-link integral (2.5) doesn’t vanish provided that \( n = m \mod N \) which makes it generically more complex. In the context of the \( D \)-matrix systems (1.6), the important simplification arises because (without loss of generality) the total amounts \( n_{\pm}(\rho) \) of the \( U_{\rho_{\pm}} \) factors in each trace-product of (1.6) can be constrained to be equal for each \( \rho \). In other words, given the numbers \( n^{(q)}_{\pm}(\rho) \) of the \( U_{\rho_{\pm}} \) factors entering corresponding traces \( tr(U_{\mu_q} U_{\nu_q} ... U_{\lambda_q}) \), we postulate that

\[
n_+ (\rho) = n_-(\rho) \equiv n(\rho) , \quad \forall \rho \quad ; \quad n_{\pm}(\rho) = \sum_{q=1}^{p} n^{(q)}_{\pm}(\rho).
\]

In turn, the constraint (2.10) ensures that the action in the associated \( D \)-matrix \( SU(N) \) subfamily of (1.6) is invariant under the \( D \) copies of the transformations

\[
[U(1)]^{\otimes D} : U_\rho \to t_\rho U_\rho , \quad t_\rho \in U(1) ,
\]

taking values in \( U(1) \) rather than in the center-subgroup \( T = \mathbb{Z}_N \) of \( SU(N) \). As a result, the nondiagonal moments \( M^{SU(N)}(n, m) \), \( n \neq m \), do not contribute into the PF \( \tilde{X}_r \), while in the remaining diagonal integrals \( M^{SU(N)}(n, n) \) the \( SU(N) \) link-variables can be substituted by the
\[ U(N) = [SU(N) \otimes U(1)]/\mathbb{Z}_N \text{ ones} \]

\[ M^{SU(N)}(n,n) = M^{U(N)}(n,n) , \quad \forall n \in \mathbb{Z}_{\geq 0} , \quad (2.12) \]
as it is proven in Appendix B.

### 2.2 The Dual form of the \( D \)-matrix actions.

Complementary to the reformulation (2.9) of the measure, a generic \( D \)-matrix action (1.6) can be rewritten in a more concise synthetic form combining both the 'normal' \( U_j \)-variables and the dual degrees of freedom. As a result, integrating out \( \{ U_\rho \} \) with the help of the \( S(n) \)-formula (2.9), the PF \( \tilde{X}_p \) of (1.6) can be expressed in terms of the dual variables only.

It is appropriate to recall first the synthetic representation of the \( SU(N) \) group characters (see e.g. [5])

\[ \chi_R(U) = 1 \frac{1}{n!} \sum_{\sigma \in S(n)} \chi_R(\sigma) Tr_n[D(\sigma)U^{\otimes n}] = Tr_n[D(C_R)U^{\otimes n}] , \quad \quad (2.13) \]

associated to the set of irreps \( R \in Y^{(N)}_n \) with the Young tableaus containing a given number \( n \) of boxes. Eq. (2.13) simply rewrites the Frobenius formula \([14]\) in terms the conventional algebraic notations \([14]\) in terms the conventional algebraic notations

\[ Tr_n[D(\sigma)U^{\otimes n}] = \sum_{i_1i_2...i_n=1}^{N} U^{i_{\sigma(1)}}_{i_1} U^{i_{\sigma(2)}}_{i_2} ... U^{i_{\sigma(n)}}_{i_n} = \prod_{k=1}^{n} (tr(U^{(k)}))^p_k , \quad (2.14) \]

where \( \sigma \in [1^{p_1}2^{p_2}...n^{p_n}] \), i.e. \( \sigma \) belongs to the \( S(n) \) conjugacy class \([\sigma]\) defined by the associated partition of \( n : \sum_{k=1}^{n} k p_k = n \). Note also that the complete set of \( \chi_R(U) \) is expressed (see e.g. \([14]\)) with the help of \( (U^+)^j \)-factors, while \( (U^+)^j \) is not engaged.

Topologically, each individual trace \( tr(U^k) \) in eq. (2.14) can be visualized by the \( k \)-fold winding of a path around a single base-cycle associated to the \( U \)-factor. To generalize the 1-cycle construction (2.14) for the case of the \( D \)-matrix action (1.6), consider first a single 'closed' \( q \)-loop \( tr(U_{\mu_1}U_{\nu_2}...U_{\lambda_q}) \).

One observes that the latter trace can be visualized now by a path wrapped,
according to the structure of the associated word $w^{(s)}_q$, around the $D$ independent $\rho$-cycles of the base-lattice. In turn, any $(C(2m))$-cyclic symmetrized word $\mu \nu ... \lambda$ (constrained for simplicity by (2.10)) of a length $2m = 2 \bigoplus_{\rho=1}^D m_\rho$ can be reproduced

$$tr(U_\mu U_\nu ... U_\lambda) = Tr_{2m}[D(\alpha_{\{m\}}) \bigotimes_{\rho=1}^D \left( (U_\rho)^{\oplus m_\rho} \otimes (U_\rho^+)^{\oplus m_\rho} \right)], \quad (2.15)$$

with the help of the equivalence class $[\alpha_{\{m\}}]$ of the $2m$-cycle permutations $\alpha_{\{m\}} \in C(2m)$ defined modulo certain conjugations (immaterial for our present discussion). More explicitly, the structure of the r.h.s. of eq. (2.15) adopts the pattern (2.14) to the presence of the $2D$ different $U_\rho$, $U_\rho^+$ basis-factors

$$\left[(U_1)^{i_{\alpha(1)}} \cdots (U_1)^{i_{\alpha(m_\rho)}}(U_1^+)^{k_{\alpha(m_\rho+1)}} \cdots (U_1^+)^{k_{\alpha(2m_\rho)}}\right] \otimes \cdots \left[(U_2)^{k_{\alpha(2m_\rho)}}\right], \quad (2.16)$$

where the mapping $n \to \alpha(n)$, $n = 1, \ldots, 2m$, defines the $S(2m)$ permutation $\alpha_{\{m\}}$.

Next, eq. (2.15) by the same token represents a generic $\{w^{(s)}_k\}_{\tilde{n}(\pm)}$ product of traces (with $2m = \tilde{n}(\pm)$ ) entering the $D$-matrix action (1.6) constrained by (2.10). For this purpose, one is to choose such $\sigma(\{w^{(s)}_k\}) \in S(2m)$ that can be decomposed into the ordered product $\sigma = P \otimes_{k=1}^{4n_+} c_{n_k}$ of the $n_k$-cycle permutations $c_{n_k}$ reproducing the trace-product in question. Summarizing, the exponent of the $D$-matrix action (1.6) can be reformulated in the following synthetic form

$$e^{-S_r([U_\rho])} = \sum_{\{n_\rho\}} Re \left[ \sum_{\sigma} \psi_{\{n_\rho\}}(\sigma) Tr_{4n_+}[D(\sigma) \bigotimes_{\rho=1}^D (U_\rho)^{\oplus n_\rho} \otimes (U_\rho^+)^{\oplus n_\rho}]\right], \quad (2.17)$$

where $n_\rho \in \mathbb{Z}_{\geq 0}$, $2n_+ = \sum_{\rho=1}^D n_\rho$, and $\sigma \in S(4n_+)$. We remark also that in fact the associated to (2.17) PF $\tilde{X}_r$ remains invariant under the substitution $Re[..] \to [..]$, i.e. one could omit the selection of the real part of the combination in the rectangular brackets of (2.17).

As our attention is restricted to the solvable deformations of (1.2),(1.3), we impose extra condition that $\psi_{\{n_\rho\}}(\sigma)$ is functionally parametrized by a
set \( \{ R_\phi \in Y_n^{(N)} \} \) of the relevant \( SU(N) \) irreps

\[
\psi_{\{ n_\rho \}}(\sigma) = \sum_{\{ R_\phi \}} e^{-S(\{ R_\phi \})} \psi_{\{ n_\rho \}}(\sigma|\{ R_\phi \}) , \quad n_\rho = \sum_{\nu \neq \rho} n_{\rho\nu} . \tag{2.18}
\]

As a result, the \( D \)-matrix action \((2.17)/(2.18)\), being defined in terms of the \( \{ \Xi_{\{ n(\rho) \}} \} \)-set of the \( S(4n_+) \)-algebra elements

\[
\Xi_{\{ n(\rho) \}} = \sum_{\sigma \in S(4n_+)} \psi_{\{ n(\rho) \}}(\sigma) \sigma = \sum_{\{ R_\phi \}} e^{-S(\{ R_\phi \})} \Xi_{\{ n(\rho) \}}(\{ R_\phi \}) , \tag{2.19}
\]

can be resummed in terms of the alternative set of the (properly normalized) operators \( \Xi_{\{ n(\rho) \}}(\{ R_\phi \}) \in S(4n_+) \), \( \phi \in \{ \mu\nu \}, \{ \rho \} \), weighted by a numerical factor \( e^{-S(\{ R_\phi \})} \). Note also the advantage of choosing the \( SU(N) \) option of the \( D \)-matrix models \((1.6), (2.10)\), where the \( SU(N) \) link-variables can be extended (according to \((2.12)\)) to the \( U(N) \) ones. In this way, we combine the simpler structure of the \( U(N) \) 1-link integral \((2.6)\) with the more compact pattern \((2.13)\) of the \( SU(N) \) characters (implicitly entering the action through the operators \((2.19)\)).

Summarizing, in the nonabelian case the dual representation introduces the extended (compared to \((2.4)\)) set of the dual variables: the integer-valued \( \{ \lambda(\phi) \} \)-fields parametrizing the relevant irreps \( R_\phi \) are complemented by the elements of the \( \otimes_n S(n) \)-algebra. In the particular case of the \( SU(N) \) GLR generating functionals \((1.2), (1.3)\), the pertinent dual degrees of freedom fit the pattern

\[
\{ \otimes_n S(n) ; \{ \lambda \} \in [Z^N/S(N)]^{\oplus D(D-1)/2} \otimes [Z^N/S(N)]^{\oplus D} \}, \tag{2.20}
\]

where, for a given \( \phi \in \{ \mu\nu \}, \{ \rho \} \), each \( \{ \lambda(\phi) \} \)-sector is composed of the \( SU(N) \) sets of \( N-1 \) nonnegative integers \( \{ \lambda^{SU(N)} \} = \{ \lambda_1 > \lambda_2 > . . . > \lambda_{N-1} > 0 \} \). The latter enter the scene through the relevant Young idempotents \( C_{R_\phi} \).

### 2.3 D-matrix amplitudes v.s. \( Tr_{4n(+)} \)-characters.

Let us now put together the dual pattern \((2.6)\) of the measure and the synthetic representation \((2.17)\) of the \( D \)-matrix action. To take advantage of the
large $N$ limit, we change the relative order between the integration $\prod_{\{\rho\}} dU_\rho$ and the summation (2.18) over $\{R_\phi\}$: for a finite $N$, one is to integrate out the $\{U_\rho\}$-variables (containing the $O(N^2)$ degrees of freedom) in the first place. To justify this interchange, the weight-function $\psi_{\{n_\rho\}}(\sigma)$ in eq. (2.17) should provide, for any finite $N$, with the absolute and uniform in $U_\rho \in U(N)$ convergence of the $\{n_\rho\}$-series. In the dual reformulation (2.20), the integral over $\{U_\rho\}$ is traded for the sums (2.6). Therefore, the summation over the $\mathcal{O}_\rho S(n)$-elements (for each particular $\{n_\rho\}$) is to be performed prior to the remaining sum over the $O(N)$ degrees of freedom parametrizing irreps $R_\phi$ involved into (2.18) and (2.6). The resulting effective $\{R_\phi\}$-theory (generalizing the functional (1.4)) can be approached, at least in principle, by the subsequent large $N$ saddle-point analysis.

Following the proposed strategy, one is to express the $D$-matrix $\text{PF} \tilde{X}_r$ as the weighted sum of the master-integrals

$$Tr_{4n(+)}[D(A_{\{n_\rho\}})] = \int Tr_{4n(+)}[D(\Xi_{\{n_\rho\}})D(\{U_\rho \otimes U_\rho^+\})] \prod_{\tilde{\rho}=1}^{D} dU_{\tilde{\rho}}, \quad (2.21)$$

equal to the character of the corresponding master-elements $A_{\{n_\rho\}}$ (belonging to the the $S(4n(+))$-algebra) in the tensor representation. In eq. (2.21) we have defined (following eq. (2.17))

$$D(\{U_\rho \otimes U_\rho^+\}) \equiv \bigotimes_{\rho=1}^{D} \left( (U_\rho)^{\oplus n_\rho} \otimes (U_\rho^+)^{\oplus n_\rho} \right), \quad (2.22)$$

where $\Xi_{\{n_\rho\}} \in S(4n(+))$ is introduced in eq. (2.19) so that the block (2.22) is associated in eq. (2.17) to the subset of the terms summed up for the particular partition $\{n_\rho \equiv n(\rho)\}$ of a given $2n_+ \equiv 2n(+)$.

To derive the master-element $A_{\{n_\rho\}}$, in eq. (2.21) the result of the $D$ different $U_\rho$-integrations (2.6) is to be represented as an operator embedded to act in the same enveloping $S(4n(+))$-space where both $\Xi_{\{n_\rho\}}$ and the complementary block (2.22) (being considered as the operator) act.

For this purpose, let us first specify a $S(4n(+))$-basis suitable to accomplish our program. As it is reviewed in Appendix A, each individual $\hat{U}_\rho$ or $\hat{U}_\rho^+$ acts on the associated elementary subspace $\hat{U}_\rho |i_- (\rho) > = (U_\rho)^{i}_j |j_- (\rho) >$, $\hat{U}_\rho^+ |i_+ (\rho) > = (U_\rho^+)^{i}_j |j_+ (\rho) >$; $i,j = 1, ..., N$. Thus, a given realization of
the $S(4n(\pm))$-basis is to be constructed as a properly ordered outer product of the elementary building blocks $|i_\pm(\rho)>$. In particular, the ordering of the $\{U_\rho\}$-factors in eqs. (2.21),(2.16) is associated to the following basis

$$|\tilde{I}_{4n(\pm)}> = \bigotimes_{\rho=1}^D |I_{2n(\rho)}> ; \quad |I_{2n(\rho)}> = |I_{n(\rho)}^{(+)}> \otimes |I_{n(\rho)}^{(-)}>,$$  \hspace{1cm} (2.23)

$$|I_{n(\rho)}^{(\pm)}> = \bigotimes_{\nu \neq \rho}^{D-1} |I_{n(\rho)}^{(\pm)}> ; \quad |I_{n(\rho)}^{(\pm)}> = |i_\pm(\rho)> \oplus |i_\pm(\nu)> \oplus |i_\pm(\rho)> \oplus |i_\pm(\nu)> ;$$ \hspace{1cm} (2.24)

where $2n(\pm) = \sum_{\rho=1}^D n(\rho)$.

Returning to the $S(4n(\pm))$ representation of the $D$ 1-link integrations, it is more effective to employ the alternative, $S(2n)$-reformulation of the $S(n) \otimes S(n)$ formula (2.6) (see Appendix B) which in the $|I_{2n}> = |I_{n}^{(+)}> \otimes |I_{n}^{(-)}>$ basis reads

$$\int dU \, D(U)^{j_1\cdots j_n}_{i_1\cdots i_n} \, D(U^+)^{j_{n+1}\cdots j_{2n}}_{i_{n+1}\cdots i_{2n}} = D(\Phi_{2n}) \, (\Lambda_n^{(-1)} \otimes \tilde{1}_{[n]}) \{j_{\oplus 2n}^{\oplus 2n}\};$$ \hspace{1cm} (2.25)

where $|I_n^{(\pm)}> = |i_\pm> \oplus |n>$ matches with $|I_{n(\rho)}^{(\pm)}>$ of eq. (2.24), and $\tilde{1}_{[n]}$ denotes the 'unity'-permutation of the $S(n)$ group. As for the operator $\Lambda_n^{(-1)} \in S(n)$, it is defined by eq. (2.8), while

$$D(\Gamma(2n)) \{j_{\oplus 2n}^{\oplus 2n}\} = \sum_{\delta \in S(n)} D(\delta)^{j_1\cdots j_n}_{i_1\cdots i_n} \otimes D(\delta)^{j_{n+1}\cdots j_{2n}}_{i_{n+1}\cdots i_{2n}} \in S(2n).$$ \hspace{1cm} (2.26)

Let us restore the $\rho$-labels, i.e. $n \rightarrow n(\rho)$. For a given link $\rho$, the left and the right $S(n(\rho))$-subblocks of $\Gamma(2n(\rho))$ in eq. (2.23) act respectively on the 'chiral', $|I_{n(\rho)}^{(+)}>$, and the 'antichiral', $|I_{n(\rho)}^{(-)}>$, $S(n(\rho))$-subspaces of $|I_{2n(\rho)}>$ entering eq. (2.23). The same convention is used for the $S(n(\rho))$-subblocks in the direct product $\Lambda_n^{(-1)} \otimes \tilde{1}_{[n]}$ entering eq. (2.25). The remaining $S(2n)$-operator $\Phi_{2n}$, being considered in the alternative ordered basis for each $|I_{2n(\rho)}>$ -subsector

$$|I_{2n(\rho)}> \rightarrow |\tilde{I}_{2n(\rho)}> = (|i_+(\rho)> \otimes |i_-(\rho)>)^{\oplus n(\rho)},$$ \hspace{1cm} (2.27)

(with $|i_\pm(\rho)>^{\oplus n(\rho)} = \otimes_{\nu \neq \rho}^{D-1} |i_\pm(\rho)>^{\oplus n(\rho)}$), takes the simple form of the outer product of the 2-cycle permutations $c_2 \in C(2)$

$$\Phi_{2n(\rho)} = (c_2)^{\oplus n(\rho)} \in S(2n(\rho)) ; \quad c_2 : \{12\} \rightarrow \{21\},$$ \hspace{1cm} (2.28)
with each \( c_2 \in S(2) \) acting on the 'elementary' sector \( |i_+(\rho) > \otimes |i_- (\rho) > \). Combining all the pieces together, the master-element \( A_{\{n(\rho)\}} \) is supposed to be constructed as the composition of the involved (into eqs. (2.19) and (2.23)) \( S(n(\phi)) \)-valued operators embedded to act in the common 'enveloping' \( S(4n(\pm)) \)-space.

Let us apply this algorithm in order to rederive the GLR pattern (1.4) directly evaluating the master-integrals (2.21) associated to the eigenvalue-models (1.2) or (1.3). Given the basis (2.23), the corresponding to (1.3) operator \( \Xi_{\{n(\rho)\}} \) assumes (after identification \( S^{(1)}(\{R_\phi\}) = S(\{R_\phi\}) \) in eq. (2.19)) the following form

\[
\Xi_{\{n(\rho)\}}^{(1)}(\{R_\phi\}) = \bigotimes_{\rho = 1}^D K_{2n(\rho)}^{(1)} ; \ K_{2n(\rho)}^{(1)} = (C_{R_\rho} \otimes [\otimes_{\nu \neq \rho} D^{-1} C_{R_\nu}]); \tag{2.29}
\]

while the substitution \( C_{R_\rho} \rightarrow [\otimes_{\nu \neq \rho} D^{-1} C_{R_\nu}] \) reproduces \( K_{2n(\rho)}^{(2)} \) of the option (1.2). By definition, each \( C_{R_\rho} \)-factor in eq. (2.29) is postulated to act on the corresponding \( |I_{n(\rho)}^{(\pm)} > \) subspace of (2.24), while each \( C_{R_\rho} \)-factor acts on the associated \( |I_{n(\rho)}^{(\pm)} > \) subspace.

Actually, a preliminary variant of the GLR pattern (1.4) can be deduced from (2.23) already at this step. For this purpose, one is to combine the peculiar structure of the dual \( \Xi_{\{n(\rho)\}}^{(m)}(\{R_\phi\}) \)-operator \( (2.29) \) with the invariance of any \( D \)-matrix action (2.17) with respect to the substitution of \( \Xi_{\{n(\rho)\}} \) by its 'twisted' partner

\[
\Xi_{\{n(\rho)\}} \rightarrow \sum_{\{\sigma_{\pm}(\rho)\}} [\sigma_+ \otimes \sigma_-]^{-1} \Xi_{\{n(\rho)\}} [\sigma_+ \otimes \sigma_-] ; \ \sigma_\pm = \otimes_{\rho = 1}^D \sigma_\pm(\rho); \tag{2.30}
\]

where \( \sigma_\pm(\rho) \in S(n(\rho)) \) is postulated to act on the corresponding \( |I_{n(\rho)}^{(\pm)} > \) subspace of the alternative \( S(4n(\pm)) \) basis

\[
|I_{4n(\pm)}^{(\pm)} > = |I_{2n(\pm)}^{(\pm)} > \otimes |I_{2n(\pm)}^{(\pm)} > ; \ |I_{2n(\pm)}^{(\pm)} > = \otimes_{\rho = 1}^D |I_{n(\rho)}^{(\pm)} > , \tag{2.31}
\]

with \( |I_{n(\rho)}^{(\pm)} > \) being defined by eq. (2.24). Performing the substitution (2.30) for \( \Xi_{\{n(\rho)\}}^{(m)}(\{R_\phi\}) \) defined by (2.29), we finally employ the fusion rules of the Young idempotents (see Appendix D)

\[
\sum_{\delta \in S(n_+)} \frac{[\delta (\otimes_{k=1}^p C_{R_k}) \delta^{-1}]}{(n_+) !} = \bigoplus_{R_+ \in Y_{n_+}} \left[ L_{R_+}^{(p)}(R_k) C_{R_+} \right] ; \ n_+ = \sum_{k=1}^p n_k; \tag{2.32}
\]
to arrive at the structure which foreshadows the $D$-products (1.4) of the GLR coefficients.

As for the invariance (2.30), it is a consequence of the basic commutativity $[D(\sigma), U^{\otimes n}] = 0, \forall \sigma \in S(n)$ (see Appendix A). The latter ensures that the ordered product (2.22) (associated to the $|I_{4n(+)}>$-basis (2.23)) is equal to its 'twisted' counterpart

$$D \bigotimes_{\rho=1}^{D} \sum_{\{\sigma_{\pm}(\rho)\}} (\sigma_{+}(\rho) \otimes \sigma_{-}(\rho))^{-1}[(U_{\rho})^{\otimes n_{\rho}} \otimes (U_{\rho}^{+})^{\otimes n_{\rho}}](\sigma_{+}(\rho) \otimes \sigma_{-}(\rho)), \quad (2.33)$$

where $\sigma_{\pm}(\rho) \in S(n(\rho))$. Being rewritten in the $|I'_{4n(+)}>$ basis (2.31), it matches with (2.30). Remark also that, compared to eq. (1.5), the decomposition (2.32) involves a larger set of the GLR coefficients of $p$th order: the involved irreps $R_{\psi}$ are parametrized by the $S(n(\psi))$, rather than $SU(N)$, Young tableaus $Y_{n(\psi)}$ (see Appendices A and D for the relevant details).

Returning to the derivation of the master-element $A^{(m)}_{\{n(\rho)\}}$ associated to the eigenvalue-models (1.2),(1.3), one is to put together eqs. (2.21), (2.25) and (2.29). Prior to the twisting (2.30), it results in

$$A^{(1)}_{\{n(\rho)\}} = \bigotimes_{\rho=1}^{D} \Phi_{2n(\rho)} \frac{\Gamma(2n(\rho))}{n!} \frac{d_{R_{\rho}}}{dim R_{\rho}} K^{(1)}_{2n(\rho)}, \quad (2.34)$$

while the substitution $K^{(1)}_{2n(\rho)} \rightarrow [(\bigotimes_{\nu \neq \rho}^{D-1} C_{R_{\nu}})] P_{R_{\rho}} \otimes [\bigotimes_{\mu \neq \rho}^{D-1} C_{R_{\mu}}]$ reproduces $A^{(2)}_{\{n(\rho)\}}$ corresponding to the option (1.2). Let us show how, inside the $Tr_{4n(+)}$ character (2.21), the expression (2.34) can be further simplified to end up with the GLR functional (1.4). First, according to the invariance (2.30), the operator $\Xi^{(m)}_{\{n(\rho)\}}(\{R_{\phi}\})$ can be substituted by its $(\sigma_{+} \otimes \sigma_{-})$-twisted partner. Combining it the identity which in the $|I_{2n}>$-basis reads

$$\Gamma(2n) [\sigma_{+} \otimes \sigma_{-}] = [\sigma_{-} \otimes \hat{I}_{[n]}] \Gamma(2n) [\sigma_{+} \otimes \hat{I}_{[n]}], \quad \forall \sigma_{\pm} \in S(n), \quad (2.35)$$

inside the character one can substitute $A^{(1)}_{\{n(\rho)\}} \rightarrow \tilde{A}^{(1)}_{\{n(\rho)\}}$ where

$$\tilde{A}^{(1)}_{\{n(\rho)\}} = \bigotimes_{\rho=1}^{D} \Phi_{2n(\rho)} \frac{\Gamma(2n(\rho))}{n!} \sum_{R_{\rho} \in Y_{n(\rho)}} \left( \frac{L^{(D-1)}_{R_{\rho}}(R_{\rho})}{dim R_{\rho}} C_{R_{\rho}} \otimes \hat{I}_{[n(\rho)]} \right). \quad (2.36)$$

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Let us note that the invariance with respect to the latter substitution independently follows from \([P_R, \sigma] = 0, \forall \sigma\), and the identity \[3.7\] of Appendix B. The latter formulas ensure that in eq. (2.34) the product of the operators, complementary to \(\Xi^{(1)}_{n(\rho)}\{\{R_\phi\}\}\), belongs to the center of 
\[^nD_{\rho=1}S(n(\rho))] \otimes[^nD_{\rho=1}S(n(\rho))]\) (with each of the \(S(n(\rho))\)-factors acting on the corresponding \([I_n^{(\pm)} >]-\)subspaces of (2.34)), i.e. the subgroup corresponding to the conjugations (2.30). In turn, together with (2.35) it justifies the validity of the twisting (2.36) of \(A^{(1)}_{n(\rho)}\). Similarly, for the option \(A^{(2)}_{n(\rho)}\) one reproduces the product of the GLR coefficients which matches (after some auxiliary trick discussed in Section 5.1) with the \(m = 2\) case of (1.4).

To ensure that the remaining factors in (2.36) conspire to reproduce exactly the correct GLR functional (1.4), one is to use first the defining property of \(\Phi_{2n}\) which in the basis \(|I_{2n}^{(\pm)} >|= |I_n^{(\pm)} > \otimes[I_n^{(-)} >| assumes the form

\[
Tr_{2n}[D((\sigma_+ \otimes \sigma_-) \Phi_{2n})] = Tr_n[D(\sigma_+ \sigma_-)]\) , \(\forall \sigma_\pm \in S(n)\) , (2.37)

where \(\sigma_\pm\) acts on the corresponding \(|I_n^{(\pm)} >\)-subspace. Combining (2.37) with the completeness condition

\[
\frac{1}{n!} \sum_{\sigma \in S(n)} \chi_{R_1}(\sigma \rho) \chi_{R_2}(\sigma^{-1} \alpha) = \delta_{R_1, R_2} \frac{\chi_{R_1}(\rho \alpha)}{d_{R_1}}\) (2.38)

and the projection-formula, which tells that \(Tr_n[D(P_R \sigma)]\) (where \(P_R = d_R C_R\) and \(R \in Y_{n(N)}\)) is nonzero only for \(R \in Y_n\) when

\[
Tr_n[D(P_R \sigma)] = \dim R \chi_R(\sigma)\) , \(\forall R \in Y_{n(N)}\) , (2.39)

we finally rederive (1.4).

In conclusion, let us compare how the nonabelian fusion-rules’ constraints are realized in the original and dual representations. In the former, ‘normal’ formulation (1.4), the conditions on the admissible irreps \(R_\phi\) are imposed by the products of the GLR coefficients. On the side of the dual representation, the fusion rules are realized in terms of the \(\otimes_n S(n)\) degrees of freedom (2.20) combined into the \(Tr_{4n(+)}\)-characters (2.21). Owing to the inherent combinations of the Young idempotents \(C_{R_\phi}\), algebraically the symmetric groups’ variables play the role of the \(S(n)\)-valued Lagrange multipliers (absent in the abelian case (2.2)).
3 GLR-computable $D$-matrix PFs.

Now we are in a position to formulate the concept of the symmetry, which underlies the resolution of the fusion rule algebra in terms of the GLR coefficients \(1.3\) for a subvariety of the $D$-matrix systems \(1.6, 2.10\) graded by the rank \(k = 1, \ldots, D\) of the conjugation-invariance \(1.7\). Since this symmetry is \textit{not} manifest in the original formulation \(1.6\) (based on the word-parametrization of the traces involved), one can view it as the \textit{hidden} symmetry of the system.

The idea is to induce the $k < D$ GLR solvable systems generalizing the $k = D$ operator $\Xi_\{n_\rho\}(\{R_\phi\})$ (defined for $m = 1$ in eq. \(2.29\)) in such a way so that the following basic property of the latter operator is preserved. Namely, the invariance of a generic $D$-matrix action under the twisting \(2.30\) of $\Xi_\{n_\rho\}(\{R_\phi\})$ must be transformed into that under the complementary twisting (akin to \(2.32\)) of the $\{C_{R_\phi}\}$-factors entering $\Xi_\{n_\rho\}(\{R_\phi\})$ in the master-integral \(2.21\). One easily observes that the required \textit{symmetry} (with respect to the 'switching' of the $(\sigma_+ \otimes \sigma_-)$-twist \(2.33\) from the $\{U_{\rho^\pm}\}$ - to the $\{C_{R_\phi}\}$ -block) holds true provided in eq. \(2.21\) $\Xi_\{n_\rho\}$ is chosen in the following form

$$
\Xi_\{n_\rho\} = \sum_{\{R_\phi\}} e^{-E(\{R_\phi\})} \Xi_\{n_\rho\}(\{R_\phi\}) \Psi_\{n_\rho\}(\{R_\phi\}),
$$

(3.1)

where $\Xi_\{n_\rho\}(\{R_\phi\})$ is determined by eq. \(5.4\), $E(\{R_\phi\})$ is some numerical weight-factor, while $\phi \in \{\mu\nu\}, \{\rho\}$ and $m = 1, 2$. As for $\Psi_\{n_\rho\}$, it can be an \textit{arbitrary} element belonging to the \textit{center} of any subalgebra $\tilde{S} \otimes \tilde{S}$ of $S(2n_\pm) \otimes S(2n_\pm)$ (in what follows we employ the $|I_{n_\pm}^\prime>$-basis \(2.31\)) which contains the subsubalgebra

$$
[\otimes_{\rho=1}^D S(n_\rho)] \otimes [\otimes_{\rho=1}^D S(n_\rho)]
$$

(3.2)

inherent in the twisting \(2.33\). More generally, $\Psi_\{n_\rho\}$ should commute with \textit{any} element of the group-product \(3.2\) that can be concisely formalized by the pattern

$$
\Psi_\{n_\rho\} = \frac{1}{[n(S)]^2} \sum_{\{\sigma_+ \otimes \sigma_-\} \in \tilde{S}} [\sigma_+ \otimes \sigma_-]^{-1} M_{\{n_\rho\}}(\{R_\phi\}) [\sigma_+ \otimes \sigma_-],
$$

(3.3)
where \( n(\tilde{S}) \) denotes the number of the elements in \( \tilde{S} \), while \( M_{\{n_\rho\}} \) a pri-
ori may be a generic element of the \( S(4n_+) \) algebra (consistent with the convergence of the final summation \((2.17)\) over \( \{n_\rho\} \in [Z_{\geq 0}]^D \).

Depending on the choice of \( M_{\{n_\rho\}} \) and the admissible subgroup \( \tilde{S} \in S(2n_+) \), the resulting (via \((3.1)\) and \((2.24)\)) \( D \)-matrix action \((1.7)\) is endowed with the conjugation-symmetry \((1.7)\) of a different rank \( k \). In what follows, we concentrate on the simplest case when \( \tilde{S} = S(2n_+) \), \( n(\tilde{S}) = (2n_+)! \), resulting in the \( k = 1 \) subvariety of the models \((1.7)\) which can be viewed as the noneigenvalue deformations of the two basic \( k = D \) systems \((1.2),(1.3)\).

To develop an intuition in what is going on, let us obtain the explicit form

\[
\sum_{\sigma_+ \in S(2n_+)} (\sigma_+ \otimes \sigma_-)^{-1}[(U)_{\otimes 2n_+} \otimes (U^+)_{\otimes 2n_+}] (\sigma_+ \otimes \sigma_-), \tag{3.4}
\]

onto the complementary \( \{C_{R_\rho}\} \)-block of \((3.1)\). To be more specific, consider the \( m = 2 \) option of \((3.1)\) and choose the simplest separable form of \( M_{\{n_\rho\}} \):

\[
M_{\{n_\rho\}} = M_{2n_+}^{(1)}(\{R_\rho\}) \otimes M_{2n_+}^{(2)}(\{R_\rho\}) , \quad M_{2n_+}^{(1,2)} \in S(2n_+), \tag{3.5}
\]

where \( \phi \in \{\mu \nu\}, \{\rho\} \). Taking into account the identity

\[
\sum_\sigma \sigma(\otimes_{\rho=1}^D P_{R_\rho}[\otimes_{\nu \neq \rho}^{D-1} C_{R_{\nu \rho}}])^{-1}/(2n_+)! = \prod_{R_+} L_{R_+}^{(D-1)} L_{R_+}^{(D)} C_{R_+}, \tag{3.6}
\]

(where \( \sigma \in S(2n_+) \), \( R_+ \in Y_{2n_+} \)) and summing up \( \sum_{R_\rho \in Y_{n_\rho}} P_{R_\rho} = 1 \), one obtains

\[
e^{-\tilde{S}_{\{n_\rho\}}^2(U)} = \sum_{\{R_\rho^{(q)}_\phi\}} e^{-E(\{R_{\mu \rho}\})} H(\{R_\rho^{(q)}_\phi\}) 2^{\sum_{p=1}^D R_\rho^{(p)}(M_{2n_+}^{(p)}) \chi_\rho^{(p)}(U^{(p)}) d_{R_\rho^{(p)}}}, \tag{3.7}
\]

\[
H(\{R_\rho^{(q)}_\phi\}) = \prod_{p=1}^2 [ L_{R_+^{(D-1)}}^{(D)} \otimes \chi_\rho^{(D)} L_{R_+^{(D-1)}}^{(D-1)} \otimes \chi_\rho^{(D-1)}], \tag{3.8}
\]

where \( U^{(2)} \equiv U^+ \), \( U^{(1)} \equiv U \) and \( \phi \in +, \{\rho\}, \{\mu \nu\} \).
Next, the analysis of the continuum limit (in the associated induced gauge theory) will require the knowledge of the explicit relation between the two weights: \( E(\{ R_\phi \} ) \) (entering \( e^{-\tilde{S}_t(\{ U \} )} \equiv e^{-\tilde{S}_t(\{ U_\rho \} )}_{|\{ U_\rho = U \} } \) and \( S(\{ R_\phi \} ) \) involved into the \( GLR \) computable \( D \)-matrix partition function (1.4). To derive a transparent example of such a relation, we concentrate on the deformations of (1.3) and specify the operator \( M_{\{ n_\rho \}} \) further in the form generalizing (3.3)

\[
M_{\{ n_\rho \}}(\{ R_\phi \} ) = \left[ (\tilde{M}_{2n_+}^{(1)} F_{2n_+}) \otimes \tilde{M}_{2n_+}^{(2)} \right] \frac{\Gamma(4n_+)}{(2n_+)!}, \tag{3.9}
\]

where \( \Gamma(4n_+) \in S(4n_+) \) is defined by eq. (2.26) (with each \( S(2n_+) \) -subblock acting on the corresponding \( \{ I_{2n_+}^\pm \} \) -subspace of (2.31)). The auxiliary factor

\[
F_{2n_+} = \sum_{R_+ \in Y_{2n_+}^{(N)}} e^{-E(\{ R_{\mu \nu} \}, R_+) + E(\{ R_{\mu \nu} \})} P_{R_+}. \tag{3.10}
\]

is introduced to trade \( E(\{ R_{\mu \nu} \} ) \) in the final amplitudes for its \( R_+ \) dependent counterpart (and for simplicity we consider \( \{ R_\rho \} \) -independent weights).

To begin with, employing (3.6) and the identities listed in the end of the previous section, one easily obtains for the associated to (3.9) deformation of the eigenvalue-action (1.3) (considered for the coinciding arguments)

\[
e^{-\tilde{S}_t^{(1)}(\{ U \} )} = \sum_{\{ R_\phi^{(q)} \}} e^{-E} Q(\{ R_\phi^{(q)} \} ) \frac{\chi_{R_+}(\tilde{M}_{2n_+}^{(1)} \tilde{M}_{2n_+}^{(2)})}{d_{R_+}^3}|\chi_{R_+}(U)|^2, \tag{3.11}
\]

\[
Q(\{ R_\phi^{(q)} \} ) = 2 \prod_{p=1}^{2} L_{R_+|\{ R_\phi^{(p)} \} }^{(D)} \left( \otimes_{\mu=1}^{D} L_{R_\phi^{(p)}|\{ R_{\mu \nu} \} }^{(D-1)} \right). \tag{3.12}
\]

Let us now turn to the evaluation of the PF \( \tilde{X}_r \) corresponding to (3.11). Similarly to eq. (2.39), the master-integral (2.21) associated to (3.9) is expressed in terms of the following \( S(4n_+) \) -algebra element \( \tilde{A}_{2n_+} \)

\[
\sum_{\{ R_\phi \}} e^{-E[\tilde{M}_{2n_+}^{(1)} \otimes \tilde{M}_{2n_+}^{(2)}]} \Phi_{4n_+} \left( \sum_{\{ \sigma_\pm \} } [\sigma_+ \otimes \sigma_-] \frac{W_{4n_+}}{(2n_+)!} [\sigma_+ \otimes \sigma_-]^{-1} \right), \tag{3.13}
\]
\[ W_{4n^+} = \left[ P_{R+} \otimes_{\rho=1}^{D} C_{R_{\rho}} \frac{L_{\rho}^{(D-1)}}{d_{R_{\rho}}(R_{\rho} + 1)} \right] \otimes \mathbb{I}_{[2n^+]!} \Gamma(4n^+) \frac{[\otimes_{\rho=1}^{D} \Gamma(2n_{\rho})]}{[2n_{\rho}]!}, \] (3.14)

where the second sum runs over \( \sigma_\pm \in S(2n^+) \) and \( \Phi_{4n^+} = \otimes_{\rho=1}^{D} \Phi_{2n_{\rho}} \). To simplify eq. (3.13) further, we first note that the factor \( \otimes_{\rho=1}^{D} \Gamma(2n_{\rho})/[2n_{\rho}]! \) in the \((\sigma_+ \otimes \sigma_-)\)-twisted operator \( W_{4n^+} \) can be substituted by the \( \mathbb{I}_{[2n^+]!} \)-unity employing the proper change of the variables. Indeed, let \( \Gamma(4n^+) \) and \( \Gamma(2n_{\rho}) \) be defined by eq. (2.26) in terms of \( \delta \in S(2n^+) \) and \( \delta_{\rho} \in S(n_{\rho}) \) elements respectively. Making (for a given \( \delta \)) the two shifts \( \delta \to \delta[\otimes_{\rho=1}^{D} \delta_{\rho}^{-1}] \), \( \sigma_- \to \sigma_-[\otimes_{\rho=1}^{D} \delta_{\rho}^{-1}] \), one eliminates the dependence of (3.13) on \( \{\delta_{\rho}\} \).

Second, the resulting form of eq. (3.13) can be shown to be invariant under the substitution of \( [P_{R+} \otimes_{\rho=1}^{D} C_{R_{\rho}}] \) by

\[ \sum_{\tilde{\sigma} \in S(2n^+)} \tilde{\sigma} [P_{R+} \otimes_{\rho=1}^{D} C_{R_{\rho}}] \tilde{\sigma}^{-1} = L_{R+\{R_{\rho}\}}^{(D)} C_{R+}, \] (3.15)

made in the ‘chiral’ \( S(2n^+) \)-block of the combination in the first rectangular brackets of (3.14). To this aim, one is to perform (for a given \( \tilde{\sigma} \)) the following composition of the shifts and the conjugation: \( \sigma_+ \to \sigma_+\tilde{\sigma}^{-1}, \delta \to \tilde{\sigma}\delta\tilde{\sigma}^{-1}, \sigma_- \to \sigma_-\tilde{\sigma}^{-1} \). Altogether, the tensor-dependent part of eq. (3.13) can be rewritten as

\[ (C_{R^+} \tilde{M}_{2n^+}^{(1)} \otimes \tilde{M}_{2n^+}^{(2)}) \Phi_{4n^+} \sum_{\{\sigma_\pm\}} (\sigma_+ \otimes \sigma_-) \frac{\Gamma(4n^+)}{([2n^+]!)^3} (\sigma_+ \otimes \sigma_-)^{-1}, \] (3.16)

where \( \sigma_\pm \in S(2n^+) \). Employing (2.39) together with (2.37) and introducing \( \tilde{\sigma}_+ = [\sigma_-^{-1} \tilde{M}_{2n^+}^{(2)}] \sigma_+ \) instead of \( \sigma_+ \), one transforms eq. (3.16) into

\[ \frac{\text{dim}R_{+}}{d_{R^+}} \sum_{\delta,\tilde{\sigma}_+,\sigma_-} \chi_{R^+} (\delta^{-1}\tilde{\sigma}_+\delta\tilde{\sigma}_- \{\sigma_-^{-1} \tilde{M}_{2n^+}^{(2)} \tilde{M}_{2n^+}^{(1)} \sigma_-\}) \frac{\Gamma(4n^+)}{([2n^+]!)^3}, \] (3.17)

where the sums over \( \delta,\tilde{\sigma}_+,\sigma_- \) run over the \( S(2n^+) \)-group elements. Finally, combining \( \sum_{\{\delta,\sigma \in S(n)\}} \chi_{R}(\delta\sigma\delta^{-1}\sigma^{-1})/(n!)^2 = 1/d_{R} \) (see [2]) together with

\[ \sum_{\sigma \in S(n)} \frac{\sigma B_n \sigma^{-1}}{n!} = \sum_{R \in Y_n} \frac{\chi_{R}(B_n)}{d_{R}} P_{R}, \quad \forall B_n \in S(n), \] (3.18)
(see Appendix D), and with \( \chi_{R_1}(P_2\sigma) = \delta_{R_1,R_2'R_1}(\sigma) \), we finally obtain

\[
\tilde{X}_r = \sum_{\{R_\phi\}} e^{-E(\{R_{\mu\nu}\},R_+)} \tilde{Q}(\{R_\phi\}) \, \frac{\chi_{R_+}(\tilde{M}_{2n_1}'(1) \tilde{M}_{2n_1}'(2)) \dim R_+}{d_{R_+}^{\beta} \otimes_{\rho=1}^{D} \dim R_{\rho}},
\]

(3.19)

\[
\tilde{Q}(\{R_\phi\}) = L_{R_+|\{R_\rho\}}^D \left( \otimes_{\mu=1}^{D} L_{R_\mu|\{R_{\mu\nu}\}}^{D-1} \right),
\]

(3.20)

where the sum runs over the SU\((N)\) irreps \(\{R_\phi\}, \phi \in +, \{\rho\}, \{\mu\nu\}\).

In conclusion, one observes that both in eq. (3.19) and in eq. (3.11) there appears the same factor \( K(R_+) = \chi_{R_+}(\tilde{M}_{2n_1}'(1) \tilde{M}_{2n_1}'(2))/d_{R_+}^{\beta} \) violating the invariance under the \(\mathbb{Z}_2\)-conjugation: \( \otimes_\rho R_\phi \leftrightarrow \otimes_\rho \bar{R}_\phi \), where \( \phi \in \{\mu\nu\}, \{\rho\}, + \). To retain this auxiliary symmetry (and make contact with \( S(\{R_\phi\}) \) in (5.7) of Section 5), we redefine

\[
e^{-E(\{R_{\mu\nu}\},R_+)} K(R_+) = e^{-E(\{R_{\mu\nu}\},R_+)} = e^{-S(\{R_\phi\})} \otimes_{\rho=1}^{D} \dim R_{\rho}/\dim R_+,
\]

(3.21)

postulating that \( \tilde{E}(\{R_{\mu\nu}\},R_+) \) is \(\mathbb{Z}_2\)-invariant.

4 Mapping onto the induced gauge theory.

In [7] we have developed the algorithm that associates to the \( k = D \) eigenvalue models like (1.2),(1.3) the \( D \)-dimensional induced gauge theory in such a way that the correspondence (1.1) (between the the PFs) holds true. Our present purpose is to induce, preserving (1.1), gauge theories from the generic \( k \leq D \) \( D \)-matrix systems (1.6) (invariant under (1.7) and (2.11)) including those belonging to the GLR computable variety defined via eqs. (3.1),(3.3).

To begin with, taking (1.2) as an example, let us briefly review the algorithm designed in [7] for the \( k = D \) \( D \)-matrix models. It consists of the two steps. First, one employs the large \( N \) saddle-point method to prove that the \( SU(N) \) system (1.2) is reduced (eliminating the space-time dependence) from the following \( D \)-dimensional eigenvalue-system. The latter is defined associating to each site \( x \) (of \( L^D \) lattice) the factor

\[
\sum_{\{R_{\mu\nu}\}} e^{-S} \prod_{\{\mu\nu\}} \chi_{R_{\mu\nu}}(U_\mu(x)) \chi_{R_{\mu\nu}}(U_{\nu}(x + \mu)) \chi_{R_{\mu\nu}}(U_{-\mu}(x + \nu)) \chi_{R_{\mu\nu}}(U_{-\nu}(x)),
\]

(4.1)
where $S \equiv S(\{R_{\mu\nu}\})$ and $R_{\mu\nu} \equiv R_{\mu\nu}(x)$. Observe that the correspondence between (1.2) and (4.1) implies the particular choice of the mapping \(\{U_{\rho}\} \to \{U_{\rho}(z)\}\) between the link-variables entering the PFs $\bar{X}_r$ and $\bar{X}_{LD}$ respectively.

The constructed in this way intermediate system (4.1) is invariant under the local $[U(N)]^{oD}$ conjugation-symmetry

\[ U_{\rho}(z) \to h^+_{\rho}(z) U_{\rho}(z) h_{\rho}(z), \quad h_{\rho} \in U(N), \quad \rho = 1, ..., D, \quad (4.2) \]

combined with the reduced gauge symmetry with respect to the center $T$ of the Lie group

\[ U_{\rho}(z) \to H^+(z) U_{\rho}(z) H(z + \rho), \quad H(z) \in T, \quad T = Z_N, \quad (4.3) \]

complemented by the global $[Z_N]^{\oplus D}$-invariance

\[ \{T\}^{\oplus D} : U_{\rho}(z) \to t_{\rho} U_{\rho}(z), \quad t_{\rho} \in T = Z_N. \quad (4.4) \]

The latter two symmetries substantiate consistency of the second step: the gauge theory is induced from the system (1.1) through the ‘gauge transformation’

\[ U_{\rho}(z) \to \tilde{\mathcal{G}}^+(z) U_{\rho}(z) \tilde{\mathcal{G}}(z + \rho), \quad \rho = 1, ..., D, \quad (4.5) \]

introducing the auxiliary $SU(N)$ scalar field $\tilde{\mathcal{G}}(z)$ assigned to the lattice sites. Integration over $\tilde{\mathcal{G}}(z)$ with the Haar measure (normalized by $\int d\tilde{\mathcal{G}}(z) = 1$) results [7] in the associated effective theory with the manifestly gauge-invariant action $\tilde{S}_{\text{eff}}(\{U_{\rho}(z)\})$.

The pairing between the local conjugation-symmetry (1.2) and its global $k = D$ counterpart (1.7) is crucial for maintaining the large $N$ correspondence (1.1) between (1.1) and its reduced partner (1.2). Being intended to induce a gauge theory from the $k < D$ non-eigenvalue $D$-matrix models (1.6)/(3.1), we propose to map preliminary these models onto the associated effective $D$-matrix eigenvalue-theories. Then, to maintain the correspondence (1.1) with the latter effective theory, an appropriate modification of the associated pattern (4.1) of the intermediate $D$-dimensional eigenvalue-theory will be found. Finally, the mapping (4.5) will produce, as previously, an induced gauge theory.
To fulfil this program, let us start with the construction of the effective eigenvalue-theory associated to a given \( k < D \) model (1.6) which for simplicity is restricted to satisfy (2.10) with \( n_\rho \) being additionally constrained by

\[
n_\rho = \sum_{\nu \neq \rho} n_{\rho \nu}, \quad n_{\mu \nu} \in \mathbb{Z}_{\geq 0}.
\]  

(4.6)

For this purpose, in (1.6) one is to rewrite \( U_\rho = \Omega_\rho \text{diag}[e^{i\omega(\rho)}] \Omega_\rho^+ \) and then integrate over \( \Omega_\rho, \rho = 1, \ldots, D \), employing (2.12) and the decomposition of the \( U(N) \) measure

\[
\int_{U(N)} dU = \int d\Omega \prod_{k=1}^{N} \int_{-\pi}^{+\pi} \frac{d\omega_k}{2\pi} \prod_{i<j} |2\sin(\frac{\omega_i - \omega_j}{2})|^2,
\]  

(4.7)

The point is that the integrations over the right-cosets \( \Omega_\rho \in U(N)/[U(1)]^\otimes N \) can be extended, introducing the auxiliary matrix \( \tilde{T}_\rho \in [U(1)]^\otimes N \), to those over \( W_\alpha \) spanning the full \( U(N) \) group-manifold (so that the dual representation (2.6) of the 1-link integral is applicable to this preliminary mapping).

Second, the remaining integrals over \( T_\rho \equiv \text{diag}[e^{i\omega(\rho)}] \in [U(1)]^\otimes N \) by the same token can be promoted (introducing \( \Upsilon_\rho \in U(N)/[U(1)]^\otimes N \)) to those over \( \tilde{U}_\rho \in U(N) \) which altogether reads \( d\Omega_\rho \to d(\Omega_\rho T_\rho) \equiv dW_\rho, \ dT_\rho \to d(\Upsilon_\rho T_\rho \Upsilon_\rho^+) \equiv d\tilde{U}_\rho \). In terms of this extended set of the variables, the PF of a generic \( D \)-matrix model (1.6) can be rewritten

\[
\tilde{X}_r = \int_{U(N)} e^{-S_{\text{eff}}(\{\tilde{U}_\rho\})} \prod_{\tilde{\rho}=1}^{D} d\tilde{U}_{\tilde{\rho}} = \int_{SU(N)} e^{-S_{\text{eff}}(\{\tilde{U}_\rho\})} \prod_{\tilde{\rho}=1}^{D} d\tilde{U}_{\tilde{\rho}}
\]  

(4.8)

as the PF of the associated effective eigenvalue-theory manifestly invariant under the \( k = D \) conjugation-symmetry (1.7)

\[
S_{\text{eff}}(\{\tilde{U}_\rho\}) = -\ln\left[ \int_{U(N)} e^{-S_{\text{eff}}(W_\rho \tilde{U}_\rho W_\rho^+)} \prod_{\tilde{\rho}=1}^{D} dW_{\tilde{\rho}} \right],
\]  

(4.9)

where the particular normalization \( \int_{U(N)} dU = 1 \) of the Haar measure is used. In the derivation of the second, \( SU(N) \) form of \( \tilde{X}_r \) in (4.8), we employ the invariance of \( S_{\text{eff}}(\{\tilde{U}_\rho\}) \) under (2.11) (i.e. (2.10)) that allows to apply the identity (2.12). To return from (4.8) to the original representation,
one absorbs $\Upsilon_\rho$ by the opposite 'shift' $W_\rho \to W_\rho \Upsilon^+_{\rho}$ of the $W_\rho$-variables and then employs the commutativity $\hat{T}_\rho \text{diag}[e^{i\omega(\rho)}] \hat{T}_\rho^+ = \text{diag}[e^{i\omega(\rho)}]$ . As for the maximal $k = D$ symmetry (1.7) of (4.9), it follows from the possibility to reabsorb the $\Psi_\rho$-rotations in the same way as we have done for $\Upsilon_\rho$.

The action of the resulting effective $SU(N)$ eigenvalue-theory can be defined by in the following form

$$e^{-\hat{S}_\nu((U_\rho))} = \sum_{\{n_\rho\} \{R^{(q)}_\rho\} \in Y_{n_\rho}(N)} e^{-A(\{R^{(q)}_\rho\})} \text{Re}\left[ \prod_{\rho=1}^D \chi_{R^{(-)}_\rho}(U_\rho_\rho) \chi_{R^{(+)}_\rho}((U_\rho^+)_\rho) \right],$$

(4.10)

where $A \equiv A(\{R^{(q)}_\rho\})$ and the associated to $\{R^{(q)}_\rho\}$ numbers of boxes $\{n_\rho(q)\}$ satisfy (owing to (2.10)) $n_\rho(1) = n_\rho(2) \equiv n_\rho, \forall \rho$.

To reconstruct an associated intermediate $D$-dimensional eigenvalue-system, let us first compare the pattern (4.10) with the one (where $U^{(+)}_\rho \equiv U^+_\rho, U^{(-)}_\rho \equiv U^-_\rho$)

$$\sum_{\{n_{\mu\nu}\} \{R^{(q)}_{\mu\nu}\} \in Y_{n_{\mu\nu}}(N)} e^{-A(\{R^{(q)}_{\mu\nu}\})} \text{Re}\left[ \prod_{\mu\nu=1}^{D(D-1)/2} \prod_{q=\pm} \chi_{R^{(q)}_{\mu\nu}}(U^{(q)}_{\mu\nu}) \chi_{R^{(q)}_{\mu\nu}}((U^{(q)}_{\nu\mu}) \right],$$

(4.11)

that generalizes (1.2) remaining compatible with the algorithm employed in [7] to induce a gauge theory. The $D$-dimensional eigenvalue-system corresponding to (4.11) can be deduced from (4.1) substituting each $\mu\nu$-block of the characters by the more general block (where $R^{(q)}_{\mu\nu} = R^{(q)}_{\mu\nu}(x)$)

$$\chi_{R^{(q)}_{\mu\nu}}(U^{(q)}_{\mu\nu}(x)) \chi_{R^{(q)}_{\mu\nu}}((U^{(q)}_{\nu\mu}(x + \mu)) \chi_{R^{(q)}_{\mu\nu}}((U^{(q)}_{\nu\mu}(x + \nu)) \chi_{R^{(q)}_{\mu\nu}}((U^{(q)}_{\nu\mu}(x + \nu))),$$

(4.12)

with the overall weight being given by $S \equiv S(\{R^{(q)}_{\mu\nu}(x)\})$.

One observes that for $D \geq 3$ the pattern (4.10) of the effective eigenvalue-theory, according to Frobenius formula (2.13),(2.14), generically can not be reproduced in terms of (4.11) (resulting from (4.12) after the large $N$ SP reduction of the space-time dependence of $U_\mu(z), R^{(q)}_{\mu\nu}(x)$). To adjust the algorithm of [7] to the more general $D \geq 3$ family (4.10), we note first that the latter systems can be reduced (eliminating the space-time dependence) from the following $D$-dimensional eigenvalue-systems. The latter are defined
associating to each site \( x \) (instead of (4.1)) the factor

\[
\sum_{\{n_\rho\}} \sum_{\{R_\rho\} \in \mathcal{Y}_{n_\rho}(N)} e^{-A(\{R_\rho^{(q)}\})} \text{Re}\left[ \prod_{\rho=1}^{D} \prod_{q=\pm} X_{R_\rho^{(q)}(U_\rho(x))} \right], \tag{4.13}
\]

which provides with the mapping \( \{U_\rho\} \to \{U_\rho(z)\} \) alternative to the one encoded in the pairing between (4.11) and (4.12). By the same token as in [7], the PF \( \tilde{X}_{L,D} \) of (4.13) is related to that \( X_r \) of (4.10) through the large \( N \) correspondence (1.1).

Next, application of the second mapping (4.5) converts the \( D \)-dimensional eigenvalue-system (4.13) into an induced gauge theory. Altogether, this prescription provides with the algorithm which induces a gauge model from a generic \( D \)-matrix system (4.3)/(2.10) including the \( k \leq D \) family (3.1),(3.3) with the GLR computable PF. The subtlety is that the intermediate eigenvalue-system (4.12) in addition is invariant under the (finite \( N \)) local \([Z_N]^{\otimes D}\) symmetry

\[
[Z_N]^{\otimes D} : U_\rho(z) \to t_\rho(z) U_\rho(z) , \quad t_\rho(z) \in Z_N , \tag{4.14}
\]

'much larger' than the \( Z_N \) gauge invariance (4.3). In turn, symmetry (4.14) is present in the induced via (4.5) gauge theory that is known to set zero the average of any Wilson loop \( W_C(U) = \text{tr}(U_\mu(x)U_\nu(x+\mu)\ldots U_\rho(x-\rho)) \) provided the corresponding to the contour \( C \) (minimal) area does not vanish. Therefore, it calls for a modification of the prescription to get rid of the unwanted invariance (4.14) keeping (4.3) intact.

To circumvent this problem, we propose the following synthetic algorithm defining the mapping \( \{U_\rho\} \to \{U_\rho(z)\} \) for the link-variables of (4.10). First, one is to use the Frobenius formula (2.13),(2.14) expanding the characters in (4.10) in terms of the trace products. In the resulting sum, consider a particular term (substituting \( U_\rho \to U_\rho \))

\[
\sim \prod_{\rho=1}^{D} \prod_{k} \text{tr}((U_\rho^{(k)})^{p_k(\rho)}) \prod_{k} \text{tr}((U_\rho^{+(k)})^{p_k^+(\rho)}) , \quad \sum_{k} k p_k^+(\rho) = n_\rho , \tag{4.15}
\]

containing total amount \( n_\rho \) of the \( U_\rho \) (or, equally, \( U_\rho^+ \)) factors. Second, let us separate (in a way specified below) the product (4.13) into two blocks
splitting the partitions \([p_1^\pm(\rho) 2p_2^\pm(\rho) \ldots]\) according to 
\[p_k^\pm(\rho) = t_k^\pm(\rho) + f_k^\pm(\rho)\]
so that
\[
\sum_k k t_k^\pm(\rho) = m_\rho^{(1)} , \quad \sum_k k f_k^\pm(\rho) = m_\rho^{(2)} , \quad m_\rho^{(1)} + m_\rho^{(2)} = n_\rho.
\] (4.16)

Given (4.16), perform (with the help of the second Frobenius formula (A.6)) multiple Fourier expansion of the first, \(\{t_k^\pm(\rho)\}\)-block
\[
\prod_{\{\rho,k,q\}} \text{tr}((U_{\rho}^{(q)})^k) = \sum_{\{R_{\rho}^{(q)}\}} e^{-B_1(\{R_{\rho}^{(q)}\})} \prod_{\{\rho,q\}} \chi_{R_{\rho}^{(q)}}(U_{\rho}^{(q)}).
\] (4.17)

Let the link-variables \(U_{\rho}\) in (4.17) be mapped onto \(U_{\rho}(z)\) of the intermediate
\(D\)-dimensional eigenvalue-system in compliance with the pattern (4.13). We postulate that the set \(\{m_\rho^{(1)}\}\), minimizing the function \(\sum_{\rho=1}^D [m_\rho^{(1)}]^2\), is constrained by the following condition. There should exist a set \(\{m_{\mu\nu} \in \mathbb{Z}_{\geq 0}\}\) of \(D(D-1)/2\) integers so that the second \(\{f_k^\pm(\rho)\}\) block can be represented in the form
\[
\prod_{\{\rho,k,q\}} \text{tr}((U_{\rho}^{(q)})^k) = \sum_{\{R_{\mu\nu}^{(q)}\}} e^{-B_2(\{R_{\mu\nu}^{(q)}\})} \prod_{\{\mu\nu,q\}} \chi_{R_{\mu\nu}^{(q)}}(U_{\mu}^{(q)}) \chi_{R_{\mu\nu}^{(q)}}(U_{\nu}^{(q)}),
\] (4.18)

(where \(B_2 \equiv B_2(\{R_{\mu\nu}^{(q)}\})\)) that matches with the pattern (4.11). In eq. (4.18), it is supposed that \(R_{\mu\nu}^{(q)} \in Y(N)\) with \(m_\rho^{(1)} = \sum_{\rho \neq \nu} m_{\rho\nu}\). As a result, the latter condition evidently allows to map \(\{U_{\rho}\}\) in (4.18) onto \(\{U_{\rho}(z)\}\) according to the pattern (4.12).

Upon a reflection, the above prescription remains essentially ambiguous. To fix the freedom of choosing the \(\{m_{\mu\nu} \in \mathbb{Z}_{\geq 0}\}\) set, we impose extra constraint that the latter integers minimize \(\sum_{(\mu\nu)} [m_{\mu\nu} - n_{\mu\nu}]^2\) (where \(n_{\mu\nu}\) enters (4.6)). In case if there remains (accidental) residual ambiguity, one is to symmetrize over all the admissible options. Summarizing, we have formulated the algorithm to map a given \(D\)-matrix model (4.4) (constrained by (2.10),(4.6)) onto the intermediate \(D\)-dimensional eigenvalue-system which is invariant under (4.3),(4.2). Applying to the latter the final mapping (4.5), we induce the theory invariant under the conventional \(SU(N)\) gauge invariance.
5 Continuum limit of the induced theories.

The analysis of the continuum limit in the lattice gauge theories, induced from the GLR computable $k < D$ models (1.6)/(4.6), essentially follows the route employed in [7] for gauge theories induced directly from the $k = D$ models like (1.2). In what follows we briefly sketch the major steps with the emphasis on a few novel details.

The idea is to take advantage of the fact that the infinite correlation length in a gauge theory is supposed to entail the following effect. Namely, the link-variables $U_\rho(z)$ are supposed to be localized (modulo (4.4) and the gauge transformations) in the infinitesimal vicinity, scaling as $O(N^0)$, of the group-unity element $\hat{1}$. The gauge-invariant representation of this condition implies the existence of some $O(N^0)$ functional $\tilde{g}^2N \equiv \tilde{g}^2(\{g_k\})N \to 0$ (of the relevant coupling constants $\{g_k\}$) so that

$$\lim_{N \to \infty} \lim_{\tilde{g}^2N \to 0} \frac{1}{N} < tr[U(pl)] > -1 \sim O(\tilde{g}^2N), \quad (5.1)$$

where $U(pl) = U_\mu(x)U_\nu(x + \mu)U_\nu^+(x + \nu)U_\mu^+(x)$ stands for the holonomy around an elementary plaquette in an arbitrary $\mu\nu$-plane. Let us fix the 'maximal tree' gauge [14] putting $U_\rho(z) = \hat{1}$ on a largest possible tree (made of the links) which by definition does not contain nontrivial 1-cycles. Then, introducing the quantum fluctuations $A^{ab}_\rho(z) = -i\ln[U^{ab}_\rho(z)]$, the required localization can be formulated in the large $N$ limit in the form

$$\lim_{N \to \infty} \lim_{\tilde{g}^2N \to 0} < [A^{ab}_\rho(z)]^2 > \sim O(\tilde{g}^2) \quad \text{mod (4.4)}, \quad \forall a, b = 1, ..., N. \quad (5.2)$$

Following [7], we intend to prove that in the induced theory the constraint (5.2) is fulfilled if in the associated $D$-matrix model (defined for definiteness by eqs. (3.9),(3.21)) the condition

$$\lim_{N \to \infty} \lim_{\tilde{g}^2N \to 0} < A^{ab}_\rho^2 > \sim O(\tilde{g}^2) \quad \text{mod (2.11)}, \quad (5.3)$$

(where $A^{ab}_\rho = -i\ln[U^{ab}_\rho]$), is valid for any given $a, b = 1, ..., N$. In turn, to ensure scaling (5.3) we first represent the $D$-matrix PF $\lim_{N \to \infty} \tilde{X}^{(m)}_\tau$ as the $(mD)$th power of the effective 1-matrix $SU(N)$ theory formulated in terms of irreps $R$

$$\lim_{N \to \infty} \tilde{X}^{(m)}_\tau = \lim_{N \to \infty} \left[ \sum_{R} e^{-S^{(m)}(R[D])} \right]_{mD}. \quad (5.4)$$
which is valid \([7]\) provided \(-\ln[\tilde{X}^{(m)}_\phi] \sim O(N^2)\) and the weight \(S^{(m)}(\{R_\phi\})\) in eq. (1.4) (defining \(X^{(m)}_\phi\)) is invariant under the group-product

\[
S(D) \otimes S(D(D-1)/2) \otimes \mathbb{Z}_2; \quad \mathbb{Z}_2 : \otimes_\phi R_\phi \leftrightarrow \otimes_\phi \bar{R}_\phi, \tag{5.5}
\]

combining the separate permutations within the two sets \((\{\mu \nu\} \text{ and } \{\rho\})\) of the irrep-indices \(\phi \in \{\rho\}, \{\mu \nu\}\) together with the simultaneous conjugation of all the involved into eq. (1.4) irreps \(R_\phi\).

As we will demonstrate, the condition (5.3) can be reformulated as the following constraint on the saddle-point (SP) values of the \(\lambda_j \in \mathbb{Z}\) fields canonically parametrizing the irreps \(R\) in the effective 1-matrix system (5.4).

Namely, the SP values \(\lambda_i^{(0)} = N\tilde{\lambda}_i^{(0)}\) should approach 'infinity' according to the complementary scaling-condition

\[
\lim_{N \to \infty} \lim_{\bar{g}^2 \to 0} |\lambda_i^{(0)}| \sim O(N/[\bar{g}N^{1/2}]) \iff |\bar{\lambda}_i^{(0)}| \sim O([\bar{g}N^{1/2}]^{-1}), \quad (5.6)
\]

with the functional \(\bar{g}(\{g_k\})\) (which enters (5.1),(5.3)) tending to zero.

5.1 The effective \(N \to \infty\) 1-matrix theory.

To prove that (5.3) yields (5.1), we first derive the 1-matrix representation (5.4) of the large \(N\) PF (1.4) associated to the \(D\)-matrix model (1.6) specified by (3.9),(3.21). As we will see, the sum \(\sum_{R''}\) in (5.4) is in fact constrained \([7]\) by the condition that both \(n(R)\) and \(n(\bar{R})\) must be nonnegative multiples of \((D-1)\) (where \(n(R)\) is the number of boxes in the Young tableau \(Y^{(N)}_{n(R)}\) associated to \(R\)).

Upon a reflection, the pattern (3.20) suggests to start with a little bit more specific \(m = 1\) form of (1.4)

\[
\tilde{X}_r = \sum_{\{R_\phi\}} e^{-S(\{R_\phi\})} L^{(D)}_{R_+ \{R_\phi\}} \left( \otimes_{\mu=1}^{D} L^{(D-1)}_{R_{\nu} \{R_{\mu \nu}\}} \right), \tag{5.7}
\]

where each sum over \(R_\phi \in Y^{(N)}_{n_\phi}, \phi \in \{\mu \nu\}, \{\rho\}, +, \) runs over the \(\phi\)-species of the \(SU(N)\) irreps. As for the weight-factor \(S(\{R_\phi\})\), it is defined in (3.21) being invariant under the group-product (5.3), where \(\mathbb{Z}_2\) symmetry is extended for \(\phi = +, \{\rho\}, \{\mu \nu\}\). Next, 'integrating out' in eq. (5.7) the
auxiliary $R_+$ variable, one brings it into the required $m = 1$ form (1.4) with the identification

$$e^{-S(R_{\rho})} = \sum_{R_+} e^{-S(R_{\rho}, R_{\mu\nu})} \mathcal{L}_{R_+}(D) \cdot$$

Returning to the reduction of (1.4) to (5.4), it is built on the localization of the large $N$ summations over $\{R_{\rho}\} \otimes \{R_{\mu\nu}\}$ on the solution $\{R^{(0)}_{\rho}\} \otimes \{R^{(0)}_{\mu\nu}\}$ of the corresponding saddle-point equations. We refer to [7] for the discussion of these equations, and now simply assert the properties of the solution in the case when the constraints (5.5) are additionally imposed. To be more specific, we select the option when the effective 1-matrix system in eq. (5.4) is reduced to the simplest solvable class of the $SU(N)$ or $U(N)$ models with $S(R|D)$ being defined as

$$e^{-S(\{\lambda\})} = |\text{dim} R(\{\lambda\})|^q e^{-\sum_{n=1}^{M_0} \sum_{\tilde{r}(\{p\}) \in Y_{2n}} g_{\tilde{r}(\{p\})} \prod_{k=1}^{N} \sum_{i=1}^{N} (\lambda_i - \frac{N-1}{2})^{pk}}$$

where $q > 0$, and in the $SU(N)$ case the set $\{g_{\tilde{r}(\{p\})}\}$ is supposed to maintain invariance of $S(\{\lambda\})$ under the translations $\lambda_i \to \lambda_i + m$. The latter is to match with the fact that $U(N)$ irreps are labelled by a set of $N$ integers $\lambda^{U(N)}_i$ (constrained by $\sum_{i=1}^{N} \lambda_i = (N - 1)/2$)

$$\{\lambda^{U(N)}\} = \{\lambda_1 > \lambda_2 > \ldots > \lambda_{N-1} > \lambda_N\} \in \mathbb{Z}^{2N}/S(N)$$

generated from the $SU(N)$-set of $N - 1$ nonnegative integers $\{\lambda^{SU(N)}\} = \{\lambda_1 > \lambda_2 > \ldots > \lambda_{N-1} > 0\}$ by the extra integer number $\lambda_N \geq 0$ or $\lambda_N < 0$. As for the sum $\sum_{p}$ in the exponent of (5.9), it runs over the irreps $\tilde{r} \equiv \tilde{r}(\{p\}) \in Y_{2n}$ of the even symmetric group $S(2n)$ (labelled by the partitions $\{p\} = [1^{p_1}2^{p_2}\ldots n^{p_{2n}}]$ of $2n$: $\sum_{k=1}^{n} kp_k = 2n$) with $n \leq M_0 \in \mathbb{Z}_{\geq 1}$. Under these conditions, the saddle-point $SU(N)$-set $\{R^{(0)}_{\rho}\} \otimes \{R^{(0)}_{\mu\nu}\}$

is supposed to be unique, $\{\rho\} \otimes \{\mu\nu\}$ independent respectively, and selfdual. Consequently, the generating functional (1.4) is equivalent in the large $N$ limit to the reduced system resulting after the identification

$$R_{\mu\nu} \equiv R_2 \in Y^{(N)}_{n_2}, \forall \mu\nu \ ; \ R_{\rho} \equiv R \in Y^{(N)}_{n}, \forall \rho ,$$

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with the remaining summations over \( R, R_2 \) being localized on the same saddle-point values (7.11).

The effective action for \( R, R_2 \), resulting from the reduction (5.12), contains (according to (1.4)) the \( D \)th power of \( L^{(D-1)}_{R|R_2} \). To simplify this expression further, one can employ the \( \gamma = D \) case of the identity [7]

\[
\lim_{N \to \infty} \int \prod_{\alpha=1}^{p} \sum_{R_{\alpha}} e^{-S_p(R_{\alpha})} = \lim_{N \to \infty} \left[ \int \prod_{\alpha=1}^{p} \sum_{R_{\alpha}} e^{-S_p(R_{\alpha})/\gamma} \right]^\gamma ,
\]

valid provided that \( \gamma > 0 \), the saddle-point values of both \( e^{-S_p(R_{\alpha}^{(0)})} \) and \( e^{-S_p(R_{\alpha}^{(0)})/\gamma} \) are unique and positive, while the corresponding free energy is \( \sim O(N^2) \). Summarizing, it defines the following one-matrix representation of the large \( N \) family (1.4)

\[
\lim_{N \to \infty} \tilde{X}_r = \lim_{N \to \infty} \left[ \int dU \sum_{R, R_2} e^{-S^{(D)}(R, R_2)/mD} \chi_R(U^+) \chi_{R_2}(U)^{D-1} \right]^{mD} ,
\]

where the weight \( S^{(D)}(R, R_2) \) is deduced from \( S\{R_\phi, \{R_{\mu\nu}\}\} \) of (1.4) through ‘dimensional reduction’ (5.12). Next, owing to the invariance of (1.5) (in (1.4)) under (2.11), the SU(\( N \)) partition function \( \tilde{X}_r \) is invariant under the substitution \( \beta \) of the \( SU(\mathbb{N}) \) link-variables by the \( U(\mathbb{N}) = [SU(\mathbb{N}) \otimes U(1)]/\mathbb{Z}_N \) ones. Therefore, the sum in (5.14) over \( SU(\mathbb{N}) \) irreps \( R \) is effectively constrained by the \( \mathbb{Z}_2 \)-invariant pair of the \( U(\mathbb{N}) \) conditions (nontrivial in \( D > 2 \))

\[
L^{(D-1)}_{R|R_2^2} \neq 0 \Rightarrow n(R) = n(R_2)(D-1) , \ n(\bar{R}) = n(\bar{R}_2)(D-1) .
\]

Here the integers \( n(R_\phi), n(\bar{R}_\phi) \in \mathbb{Z}_{\geq 0} \) denote the number of boxes in the \( \mathbb{Z}_2 \)-invariant pair of the Young tableaus corresponding to (5.12):

\[
n(R(\{\lambda\})) = \sum_{i=1}^{N} n_i = \sum_{i=1}^{N} (\lambda_i - N + i) ,
\]

while \( \mathbb{Z}_2 \) conjugation \( R \leftrightarrow \bar{R} \) reads: \( \{\lambda_i\} \leftrightarrow \{-\lambda_{N-i+1} + \beta\} \), where \( \beta^{U(\mathbb{N})} = (N - 1) \) and \( \beta^{SU(\mathbb{N})} = \lambda_1 \) in the \( U(\mathbb{N}) \) and \( SU(\mathbb{N}) \) cases respectively.

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Finally, in order to recast eq. (5.14) into the form of eq. (5.4), let us first introduce
\[ e^{-H(R|D)} = \sum_{R_2} e^{-S^{(D)}(R,R_2)/mD} L^{(D-1)}_{R[R_2]} . \]  
(5.17)

By the same token as in [7], the \( Z_2 \)-invariant pair of the \( D > 2 \) \( SU(N) \) conditions (5.15) is the only constraint defining the whole \( e^{-H(R|D)} \)-family induced from the \( e^{-S^{(D)}(R,R_2)/mD} \)-variety via (5.17). It suggests to factorize the latter constraints out
\[ e^{-H(R|D)} = e^{-S(R|D)} \sum_{k,\bar{k}\in Z} \delta_{n(R),[D-1]k} \delta_{n(\bar{R}),[D-1]\bar{k}} ; \]  
(5.18)

so that, for a fixed \( D \), in (5.4) any residual \( R \)-valued function \( e^{-S(R|D)} \) (consistent with the scaling \( -\ln[\tilde{X}_r] \sim O(N^2) \) and with \( Z_2 \)-invariance \( S(R|D) = S(\tilde{R}|D) \)) can be induced through (5.17) provided the judicious adjustment of \( e^{-S^{(D)}(R,R_2)/mD} \). The two periodic Kronecker delta-functions are supposed to be defined [7] via certain '\( \varepsilon \)-regularization' of the Poisson re-summation formula (with explicit form of the latter being immaterial for the present discussion). We note also that similar analysis of the \( U(N) \) GLR functionals (1.2),(1.3) (with the sum running over the \( U(N) \) irreps) results in the \( U(N) \) counterpart of eq. (5.4).

In conclusion, let us remark that in the selected model (5.9) the appropriate scaling \( -\ln[\tilde{X}_r] \sim O(N^2) \), \( |\lambda_j| \sim O(N) \), is adjusted [7] provided for each \( r(\{p\}) \in Y_{2n} \) the following pattern is valid
\[ \bar{\lambda}_j = \lambda_j/N \ , \ g_{r(\{p\})} = b_{r(\{p\})} N^\gamma_r \ , \ \gamma_{r(\{p\})} = 2 - 2n - \sum_{k=1}^{2n} p_k , \]  
(5.19)

where it is postulated that \( b_{r(\{p\})} \sim O(N^0) \).

5.2 The \( \{(<tr[U(pl)] > /N) \to \hat{1}\} \) localization.

Let us prove that, in the gauge theories induced via (4.5) from the \( k = 1 \) system (1.3) (specified by (3.4),(3.21)), the required localization (5.1) of \( U(\rho(z)) \) is predetermined by the scaling-condition (5.7) for the \( \{\lambda\} \) fields entering (5.4),(5.5).
To begin with, we recall that the constraint (5.6) is dynamically fulfilled provided in (5.9) all coupling constants (rescaled according to (5.19)) tend to zero: \( \{ b_{r(p)} \} \to 0 \). In the simplest case of the \( U(N) \) action (5.9) with \( M_0 = 1 \), the functional \( \tilde{g}^2 \) is to be introduced by

\[
\sum_{R(\{\lambda\})} |dimR(\{\lambda\})|^q \exp[-g_{\tilde{r}_0} \sum_{i=1}^N \left( \lambda_i - \frac{N-1}{2} \right)^2] \quad \tilde{g}^2/2 = g_{\tilde{r}_0} ,
\]

where \( g_{\tilde{r}_0} \sim O(1/N) \) and \( \tilde{r}_0 = [2^1] \). In a general case (5.9),(5.19), one is to choose \( b_{2k} = \lim \sup |b_r| \) as the largest \( |b_r(\{p\})| \) in each \( k \)-subset of \( r \in Y_{2k} \). Then, \( N\tilde{g}^2 \) is equated with the \( \lim \sup \left[ \left( b_{2k} \right)^\frac{1}{k} \right] \) found among all \( k \leq M_0 \) (provided the associated leading terms, by themselves, ensure the convergence of the \( \{\lambda\} \)-series in (5.9)).

Next, building on the results of [7], one might expect that the scaling (5.6) results in the complementary localization (5.3) of the link-variables \( \{U_\rho\} \) in the \( D \)-matrix systems (1.6) specified by (3.9),(3.21). This assertion, in particular, employs that the action (of the WC perturbation theory) in the latter system evidently contains (owing to (3.11)) the quadratic in \( A^\rho_{ab} \) term. In turn, the patterns of the involved mappings (4.8) and (4.5) suggest that (5.3) indeed entails the required localization (5.1) in the gauge theory induced from (1.6)/(3.9),(3.21).

To substantiate these statements by an explicit computation, we consider the large \( N \) WC asymptotics \( \tilde{g}^2 N \to 0 \) of the properly normalized partition function (PF)

\[
X_{L^D}^{(in)} = \int \prod_{\{\rho, z\}} dU_\rho(z) \exp[-S(\{U_\rho(z)\}) - S(\{\hat{1}\})] \quad (5.21)
\]

associated to a (induced) gauge theory on a cubic lattice with \( L^D \) sites. In [7] it is shown that the localization (5.4) generically results in the power-like large \( N \) WC asymptotics

\[
\lim_{N \to \infty} \lim_{\tilde{g}^2 N \to 0} \frac{X_{L^D}^{(in)}(\tilde{g})}{C \tilde{g} N^{\frac{D}{2}(D-1)N^2L^D}} = C > 0 , \quad (5.22)
\]

where the \( \int dU = 1 \) normalization of the Haar measure is used, and \( C \) is a model dependent constant. Thus, our aim is to prove that (5.22) is valid.
in the specific case of the gauge theories induced from the $D$-matrix models (1.4)/(3.9),(3.21) constrained by (3.6).

For this purpose one first observes that, according to the mapping (4.5), the factor $e^{-S(\{\hat{U}\})}$ can be rewritten as the partition function (PF) $\tilde{X}_{L_D}^{(a)}$ of the auxiliary $D$-dimensional model. The latter is deduced from the intermediate $D$-dimensional eigenvalue-system (induced on $L_D$ lattice via the decomposition (4.17),(4.18)) in the following way. Namely, in the plaquette-factor defining the latter eigenvalue-system, one is to substitute the link-variables $U_\rho(z)$ by the 'composite' field

$$U_\rho(z) \rightarrow \tilde{G}^+(z)\tilde{G}(z+\rho)$$

as it is predetermined by the pattern (4.5). Consequently, the properly normalized PF of the induced gauge theory can be represented as the ratio

$$X_{L_D}^{(in)} = \frac{X_{L_D}}{X_{L_D}^{(a)}} ,$$

where $X_{L_D}$ and $X_{L_D}^{(a)}$ are the PFs (both normalized akin to (5.21)) associated to the intermediate $D$ dimensional eigenvalue-system and the auxiliary model defined through (5.23) respectively.

Next, the correspondence (1.1) allows to express the large $N$ limit of $X_{L_D}$ as the $L_D$th power of the PF $X_r$ of the $k = 1$ $D$-matrix model (1.4)/(3.9),(3.21). As we will prove in the end of this subsection for the particular case of the latter model, the $\{\lambda\}$-localization (5.6) results in the power-like asymptotics (provided $\int dU = 1$)

$$\lim_{N \rightarrow \infty} \lim_{\tilde{g}^2 N \rightarrow 0} X_r(\{g_r\}) = [B \tilde{g}N^2]^{D_N^2} , \ B > 0 .$$

Being combined with the pattern (3.11)/(3.21) of $e^{-\tilde{S}_a(U)}$, it ensures the complementary $\{U_\rho \rightarrow 1\}$ localization (5.3) of the fluctuations $A_{\rho}^{ab}$. Similarly to [7], the pattern (5.23) is tantamount to the following large $N$ scaling-law (for each particular $j$)

$$\lim_{N \rightarrow \infty} \lim_{\tilde{g}^2 N \rightarrow 0} \omega(j)^2(\rho) \sim O(\tilde{g}^2 N) \ mod (2.11) ,$$

where $\omega(j), \ j = 1,\ldots,N,$ are the eigenvalues of $U_\rho = \Omega_\rho \ diag[e^{i\omega(\rho)}] \Omega_\rho^+$ entering the effective eigenvalue-theory (1.8). In turn, by the same token as
In [7], it ensures that in the auxiliary model (5.23) the $SU(N)$ field $\tilde{G}(z)$ is localized (modulo (4.4)) in the vicinity of $\hat{1}$ so that

$$\lim_{N \to \infty} \lim_{N \to 0} X_{L^D}^{(a)}(\{g_r\}) = [\tilde{B} \, \tilde{g} N^{\frac{1}{2}}]^{L^D N^2}, \quad \tilde{B} > 0.$$  \hfill (5.27)

Summarizing, we reproduce the purported asymptotics (5.22) of the PF of the gauge theory induced from the $k = 1$ GLR computable $D$-matrix model (1.6)/(3.9),(3.21).

In conclusion, let us demonstrate that in the $k = 1$ $D$-matrix model (1.6)/(3.9),(3.21) the large $N$ WC asymptotics (5.25) is indeed valid. To begin with, one readily obtains (from (3.11),(3.21)) for the action of the latter model

$$e^{-\tilde{S}^{(1)}(1)} = \sum_{R^{(q)}_\phi} e^{-\tilde{E}} \dim R_+ \prod_{p=1}^{2} L^{(D)}_{R_+ | \{R^{(p)}_\rho\}} \left( \otimes_{\mu=1}^{D} L^{(D-1)}_{R^{(2)}_\mu | \{R_{\mu\nu}\}} \right), \hfill (5.28)$$

where $\tilde{E}$ is defined by eq. (5.8). Thus, the asymptotics of the ratio (5.24) is predetermined by the $\tilde{g}^2 N$-scaling of the factor

$$\left[ L^{(D)}_{R_+ | \{R^{(1)}_\phi\}} \dim R_+ \otimes_{\mu=1}^{D} \dim R^{(2)}_\mu \right]^{-1} \hfill (5.29)$$

responsible for the ‘mismatch’ between (3.19) and (3.11) (where we have used that $R^{(2)}_\mu$ in (3.12) can be identified with $R_\mu$ of eq. (3.20)). Next, recall that according to eq. (5.12), the irrep $R$ in eq. (5.4) represents the irreps $\{R^{(p)}_\rho\}$ entering the GLR fusion-rules (1.4). Owing to the pattern (3.11),(3.13),(3.21) of the involved GLR fusion rules, the scaling-condition (5.6) is valid for the characteristic values of all species $\{\lambda^{(0)}(\phi)\}$, $\phi \in \{\mu\nu\}, \{\rho\}, +$, parametrizing the SP irreps $\{R^{(0)}_\phi\}$ (on which the relevant large $N$ sums are localized).

Next, the dimension-formula $\chi_R(\lambda)(U) = det_{k,j}(e^{i\lambda_k \omega_j})/det_{k,j}(e^{i(N-k) \omega_j})$ together with (5.6) predetermines that each $\dim R_\phi$ contributes in the limit $N \to \infty$ with the scaling-factor $[\tilde{g} N^{\frac{1}{2}}]^{-N^2}$. Complementary, given (5.6), the eigenvalues $\tilde{\omega}_j(\phi)$ of $\tilde{U}^{U(N)}_{\phi}$ (entering the definition (1.3) of the relevant GLR coefficients (5.24) modified by the extension (2.12)) satisfy (5.26) as in [7]. Combining it with pattern (4.4) of the $U(N)$ Haar measure ($\int dU = 1$), one concludes that the GLR coefficient of $K$th order scales in the large $N$ WC
limit as $[\text{dim}R]^{K-1} \sim O([\tilde{g}^2N]^{-\frac{N^2(K-1)}{4}})$. Putting together all the scaling-factors inherent in (5.25), one reproduces (5.25).

As a side remark, had we retained in $\tilde{X}_r$ of eq. (3.19) the $S(2n_+)$-factor $K(R_+) = \chi_{R_+}(\tilde{M}_{2n+}^{(1)} \tilde{M}_{2n+}^{(2)})/d_{R_+}^4$, while keeping $E(\{R(\phi)\}) \mathbb{Z}_2$-invariant, there would be no way to adjust the parameters of the latter weight to set up the $\{\lambda\}$-localization (5.6). Finally, we note also that the pattern of the large $N$ phase transitions in the gauge theories induced from (1.6)/(3.9),(3.21) can be analysed in essentially the same way as it is done for the theories [7] induced from the eigenvalue-models (1.2),(1.3).

6 Conclusions.

In this paper we have introduced the basic concepts of the nonabelian duality transformation and applied them constructing a novel family of solvable $D$-matrix models (defined via (3.3)) graded by the rank $1 \leq k \leq D - 1$ of the manifest $[U(N)]^{\oplus k}$ conjugation-symmetry (1.7). The key-ingredients of the transformation are the dual representation (2.6) of the $U(N)$ 1-link integral and the synthetic form (2.17) of a generic $D$-matrix $SU(N)$ or $U(N)$ system (1.6)/(2.10). Combining these ingredients together, the partition function of any matrix theory (1.6) can be rewritten in terms of the $Tr_{4n_+}$ characters. The latter are the traces of the different $S(n_\phi)$ tensors (represented via (2.7)) which, being composed of the dual variables, are embedded into the enveloping $S(4n_+)$ space. The dual set consists of the integer-valued $\{\lambda\}$ fields (parametrizing via (2.18) relevant irreps $\{R_\phi\}$ that are complemented by the $\otimes n S(n)$-valued degrees of freedom (facilitating the fusion-rule algebra of the Young idempotents $C_{R_\phi}$ involved).

So far, the available solvable $D$-matrix models of the $1 \leq k \leq D - 1$ type are mainly associated to the situations [13, 9] where an application of the Itzykson-Zuber formula [8] transforms the model into some $k = D$ eigenvalue-theory of $q$ (hermitean or unitary) matrices. The proposed nonabelian duality suggests the alternative ’mechanism’ of the solvability realized for the subclass (3.3) of (1.6). Here the underlying reason is the hidden symmetry of the action which becomes manifest after reformulation in terms of the dual variables. It is this somewhat unconventional symmetry which pre-
determines that the involved Young idempotents satisfy the simplest pattern (2.32) of the fusion-rules encoded by the GLR coefficients (1.3). In turn, the GLR pattern allows to map the associated $1 \leq k \leq D - 1$ systems (1.6),(3.3) onto the $D$-matrix eigenvalue-models (1.2),(1.3) endowed with the largest possible $k = D$ conjugation-symmetry (1.7).

The latter $k = D$ eigenvalue-models, being solvable in the limit $N \to \infty$, has been recently proposed [7] to reproduce the large $N$ free energy (FE) of the associated lattice gauge theory in $D$ dimensions. Generalizing the algorithm of [7], the prescription is developed to reconstruct the gauge theory with the FE $-\ln[\hat{X}_{L,0}]$ equal (modulo the $L^D$-volume factor (1.1)) to the FE $-\ln[\hat{X}_r]$ of a given $1 \leq k \leq D - 1$ system (1.6),(2.10). The new algorithm is applicable to any hypothetical solvable $D$-matrix model (i.e. is not necessarily restricted to the subvariety (3.3)) consistent with the $[Z_N]^D$-invariance (2.11).

As well as in [7], we address the question of the continuum limit in the gauge systems induced from (1.6),(3.3). To clarify this issue, we choose the judiciously constructed $k = 1$ family (3.9) constrained by (3.21). To be even more specific, the 1-matrix model of the representation (5.4) is selected in the simplest form (5.9). Given this choice, we prove that the proposed in [7] scaling-condition (5.6) (imposed on the effective 1-matrix system (5.4)) does ensure the required localization $\{U_\rho(z) \to \hat{1}\}$ of the link-variables in the associated induced gauge theories.

The major motivation for this project is to develop the formalism which makes accessible the Gauge String representation (of strongly coupled gauge theories) yielding the $D > 2$ extension of the $D = 2$ construction due to Gross and Taylor [2]. One observes that the proposed approach deals with the structures which are already very similar in nature to those of [2]. The precise reformulation of the amplitudes (in a lattice $YM$ system), in compliance with the pattern defining the data of the properly associated (generalized) branched covering spaces, will be given in the forthcoming paper [16]. Here we announce only one important novel feature of the $D > 2$ stringy pattern that is not present in the $D = 2$ construction. Compared to the latter case, in $D > 2$ one is forced to introduce certain generalizations of the canonical branched covering spaces. This is foreshadowed algebraically by the necessity to embed various $S(n_\phi)$ operators (acting in different subspaces) into the common enveloping space in a manner similar to what we have done in
Sections 2 and 3.

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A: $SU(N)/\otimes_n S(n)$ complementarity.

To facilitate the nonabelian Duality transformation on the lattice, one needs a piece of the formalism which we now focus on. Let us start with the basic facts about the action of the $GL(N)$ (which can be further restricted to $U(N)$ or $SU(N)$) and the symmetric groups on the tensor spaces associated to the structures introduced in the Section 2.

Recall that $GL(N)$ is the group of the automorphisms of nondegenerate ($\det V \neq 0$) complex $N \times N$ matrices $V_i^j$. Given any basis $\{|i>, \ i = 1, 2, ..., N\}$ on a $N$-dimensional vector space $X_N$, the fundamental matrix representation of $GL(N)$ is defined canonically as $\hat{V}|i> = V_i^j |j>$. Given $X_N$, one introduces the basis $|i\oplus_n > = |i_1> \otimes |i_2> \otimes ... |i_n>$ for the direct product space $X_N\oplus_n$. The elements of $GL(N)$ act on $X_N\oplus_n$ according to the standard rule $\hat{V}\otimes_{p=1}^n |i_p> = D(V)^{\{j\oplus_n\}}_{\{i\oplus_n\}} |j\oplus_n>\equiv V^{\oplus n}$, where conventionally $D(V)^{\{j\oplus_n\}}_{\{i\oplus_n\}} = (V^j_{i_1})(V^j_{i_2})...(V^j_{i_n})\equiv V^{\oplus n}$.

Next, the representation theory proves (see e.g. [12]) that the symmetric group $S(n)$ is the most general group of transformations commuting with the elements $V^{\oplus n}$ of the $GL(N)$ on the $X_N\oplus_n$. The elements of the $S(n)$-group are represented by the linear transformation

$$\sigma|i\oplus_n> = |\sigma^{-1}\sigma(1)c> \otimes |\sigma^{-1}\sigma(2)c> \otimes ... |\sigma^{-1}\sigma(nc)\equiv D(\sigma)^{\{j\oplus_n\}}_{\{i\oplus_n\}} |j\oplus_n> \quad (A.1)$$

where $D(\sigma)^{\{j\oplus_n\}}_{\{i\oplus_n\}}$ is given by eq. (2.4), with the basic property being the commutativity

$$[D(\sigma), D(V)] = 0 \quad \forall \sigma \in S(n), \forall V \in G. \quad (A.2)$$

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The representation of the $S(n)$-algebra elements is deduced from the group-representation (2.7) by linearity. Remark that the unity element $\hat{1}_{[n]}$ of the $S(n)$ group is represented by the ’trivial’ permutation (A.1) $\hat{1}_{[n]}$: $\sigma(k) = k$, $k = 1, \ldots, n$.

The central operation, we will employ, is the decomposition of the direct product space $V^\otimes n$ (or, equally, $(V^+)^{\otimes n}$) into the irreps of the Lie group or, dually, into the irreps of $S(n)$. Recall that in the $S(n)$ group the irreps are labelled [12] by a set of $k$ nonnegative, nonincreasing integers $\{n_i; n_1 \geq n_2 \geq \ldots \geq n_k \geq 0\}$ constrained by the single condition $\bigoplus_{i=1}^{k} n_i = n$ and visualized as the Young tableau $Y_n$ with $n$ boxes. The $U(N)$ or $GL(N)$ irreps can be parametrized in a similar fashion by a set of $N$ integers $\{n_i; n_1 \geq n_2 \geq \ldots \geq n_N\}$ that is related to the alternative classification (5.10) identifying $n_i = \lambda_i - N + i$. When all $n_i$, $i = 1, \ldots, N$, are nonnegative (nonpositive), the associated $U(N)$-characters are expressed in terms of the $V - (V^+)$ tensors only. The corresponding ’(anti)chiral’ sector of $U(N)$-irreps can be visualized [12] by the $U(N)$ Young tableaus $Y_n^{(N)}$ containing $n = \sum_{i=1}^{N} n_i$ boxes distributed in not more than $N$ rows. In the $SU(N)$ case, the complete set of irreps is labelled by the $SU(N)$ Young tableaus $Y_n^{(N)}$ containing not more than $N - 1$ rows, i.e. $n_N = 0$.

The required decomposition of $V^\otimes n$ (or $(V^+)^{\otimes n}$) can be canonically generated by the $Y_n^{(N)}$-subset of the Young projectors $\{P_R\}$ (defined by eq. (2.3)) that belong to the center of the $S(n)$ algebra: $[P_R, \sigma] = 0$, $\forall \sigma \in S(n)$, $\forall R \in Y_n$. The tensor product $V^\otimes n$ is mapped [12] by an admissible projector $P_R$, $R \in Y_n^{(N)}$ onto the $\dim R$ copies of the $S(n)$ irrep $\bar{T}_R$ or equivalently onto the $d_R$ copies of the Lie group irrep $T_R$

$$D(P_R)V^\otimes n = (\bar{T}_R)^{\otimes \dim R} = (T_R)^{\otimes d_R} = \bar{T}_R \otimes T_R \ , \ R \in Y_n^{(N)} \ , \quad (A.3)$$

where $d_R, \dim R$ stand respectively for the dimensions of $S(n)$- and the chiral $GL(N)$- (or, equivalently, $U(N)$-) irreps $R$ respectively. In the case of $G = SU(N)$, one obtains in this way all the irreducible representations. For $G = U(N)$, the space $V^\otimes n$ contains irreps included into the single chiral sector of $U(N)$-irreps parametrized by the Young tableaus $Y_n^{(N)}$.

Combining equation (A.3) with the defining properties of $\{P_R\}$

$$P_{R_1}P_{R_2} = \delta_{R_1,R_2}P_{R_1} \ , \quad \sum_{R \in Y_n} P_R = 1 \quad (A.4)$$

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we deduce one of the central results of the representation theory, the so-called Schur-Weyl duality (see e.g. [4])

\[ V^\oplus n = \sum_{R \in Y_n(N)} \tilde{T}_R \otimes T_R \]  

(A.5)

which formalizes the complementary roles of the Lie and symmetric groups.

Employing eqs. (A.3),(A.5), the formulas relevant for the Duality transformation can be represented in the concise algebraic form. First, taking trace of the Schur-Weyl decomposition (A.5) and using the completeness condition (A.4) for \( \{ P_R \} \), one deduces (see e.g. [5]) the second Frobenius formula

\[ \Upsilon_{[\sigma]}(V) = Tr_n[D(\sigma)V^\oplus n] = \sum_{R \in Y_n(N)} \chi_R(\sigma)\chi_R(V) \]  

(A.6)

or its modification \( Tr_n[D(P_R)D(\sigma)V^\oplus n] = \chi_R(\sigma)\chi_R(V) \) following from eq. (A.3). Similarly, the first Frobenius formula (2.13) for \( \chi_R(V) \) can be obtained multiplying eq. (A.5) by \( D(\sigma) \) and taking the trace as previously.

---

**B: The Dual form of the 1-link integral.**

In this Appendix, we derive the dual representation (2.6) of the 1-link integral (2.5). To compute \( M^G(n,m)_{j_1,...,j_m}^{p_1,...,q_m} \) of eq. (2.5) for a particular Lie group \( G \), the starting idea [13, 14] is to differentiate the simple generating function

\[ F_{n,m}^G(A,B) = \int dV \ (Tr[AV])^n \ (Tr[BV^+])^m \]  

(B.1)

with respect to \( A, B \in G \).

**B.1 The \( S(n) \)-form of the \( U(N) \) formula.**

In the \( U(N) \)-case the invariance (2.11) of the integral (B.1) under the \( U(1) \)-subgroup of \( U(N) = (SU(N) \otimes U(1))/Z_N \) ensures that it doesn’t vanish only for \( n = m \). For \( F_{n,n}(A,B), A,B \in U(N) \), one derives [13, 14]

\[ F_{n,n}^{U(N)}(A,B) = \sum_{R \in Y_n(N)} \frac{d_R^2}{dim_R} \chi_R(AB) \]  

(B.2)
where the second Frobenius formula (A.6) and the standard orthogonality condition of the characters have been employed. Applying to $F_{n,n}(A,B)$ the operator $(n!)^{-2} \prod_{k=1}^{n} \partial^2 / \partial A_{pk} \partial B_{qk}$ (where $A, B$ in the r.h.s. of eq. (B.2) can be extended to $GL(N)$), one obtains for the $U(N)$ 1-link integral (2.5)

\[
\frac{1}{(n!)^2} \sum_{R \in Y_n(N)} \frac{d_R}{\text{dim} R} \sum_{\rho, \sigma \in S(n)} D(C_R)_{q_{\sigma(1)}q_{\sigma(2)}...q_{\sigma(n)}}^{q_{\rho(1)}q_{\rho(2)}...q_{\rho(n)}} \delta_{i_{\sigma(1)}i_{\sigma(2)}...i_{\sigma(n)}}^{p_{\rho(1)}p_{\rho(2)}...p_{\rho(n)}}.
\]

Introducing $\delta = \rho \sigma^{-1} \in S(n)$, we arrive at the concise representation given by the eq. (2.6), with the $S(n)$-tensors $D(\delta)$ and $\Lambda_n^{(-1)}$ being defined by eqs. (2.7) and (2.8) respectively. Note that, in $\Lambda_n^{(-1)}$ the sum over the $U(N)$ G-irreps $R \in Y_n(N)$ is restricted to the single chiral sector parametrized by the $U(N)$ Young tableaus $Y_n(N)$ (containing not more than $N$ rows). Let us remark also that eq. (2.6) is complementary to already existing representations [13, 14], [2] of (2.5) which are not suitable for our present purposes.

**B.2 The $S(2n)$-form of the $U(N)$ formula.**

The integration formula (2.6) can be represented in terms of the elements of the $S(2n)$-algebra 'enveloping' $S(n) \otimes S(n)$. Employing the ordered link-basis $|I_{2n} > := |I_{n}^{(+)} > |I_{n}^{(-)} >$ (with $|I_{n}^{(\pm)} > := |i_{\pm} >^{\oplus n}$ akin to (2.24)), one rewrites eq. (2.6) in the $S(2n)$-form of eqs. (2.25),(2.26). By construction, the 'chiral', $|I_{n}^{(+)} >$, and the 'antichiral', $|I_{n}^{(-)} >$, sectors are associated respectively to the first and to the second $S(n)$-subblocks of (2.25),(2.26). As for the operator $\Phi_{2n} \in S(2n)$, comparing (2.6) and (2.25) one deduces for its explicit form

\[
D(\Phi_{2n})_{\{j_{\oplus 2n}\}}^{\{i_{\oplus 2n}\}} = \delta_{i_{1}}^{j_{1}} \delta_{i_{2}}^{j_{2}} \ldots \delta_{i_{n}}^{j_{n}} \delta_{i_{n+1}}^{j_{n+1}} \delta_{i_{n+2}}^{j_{n+2}} \ldots \delta_{i_{2n}}^{j_{2n}}
\]

which in turn can be concisely represented in the alternative ordered $S(2n)$-basis $|\tilde{I}_{2n} > := (|i_{+} > |i_{-} >)^{\oplus n}$ as the outer product

\[
D(\Phi_{2n}) = D((c_2)^{\oplus n}) \in S(2n)
\]

of the 2-cycle permutations $c_2 \in C_2$, $c_2 : \{12\} \to \{21\}$, where each individual $c_2 \in S(2)$ acts on the $(|i_{+} > |i_{-}>)$-subspace of $|\tilde{I}_{2n} >$. 

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Finally, we mention the commutation-rules

\[
[ \Phi_{2n}, \Gamma(2n) ] = [ \Gamma(2n), (A_n^{(m)} \otimes \hat{1}_{[n]} ) ] = 0 , \tag{B.6}
\]

where in the second case the above \(|I_{2n}\rangle\) -basis is employed. In particular, it makes the relative order of the operators in the product (2.25) immaterial. Also, let us include the following useful identities

\[
\Gamma(2n) \cdot (E_n^{(+)} \otimes E_n^{(-)}) = (E_n^{(-)} \otimes E_n^{(+)}) \cdot \Gamma(2n) ;
\]
\[
\Phi_{2n} \cdot (E_n^{(+)} \otimes E_n^{(-)}) = (E_n^{(-)} \otimes E_n^{(+)}) \cdot \Phi_{2n} , \tag{B.7}
\]
valid for \(\forall E_n^{(\pm)} \in S(n)\) where again the \(|I_{2n}\rangle\) -basis is implied.

**B.3 \(V^{SU(N)} \rightarrow V^{U(N)}\) extension for \(F_{n,n}^{SU(N)}(A, B)\).**

Let us prove that the diagonal \(SU(N)\) moments \(F_{n,n}^{SU(N)}(A, B)\), defined by eq. (B.4), are invariant under the substitution of the \(SU(N)\) link-variables by the \(U(N) = [SU(N) \otimes U(1)]/Z_N\) ones

\[
V^{SU(N)} \rightarrow (V^{SU(N)} \otimes V^{U(1)})/Z_N = V^{U(N)} , \quad dV^{SU(N)} \rightarrow dV^{U(N)} , \tag{B.8}
\]
that is tantamount to eq. (2.12). Actually, the identity (2.12) for \(n < N\) directly follows from equivalence [14] of any polynomial \(U(N)\) representation with \(n < N\) to the corresponding \(SU(N)\) one. But for \(n \geq N\) one needs a more refined consideration (valid for any \(n\)) we now focus on.

First, we note that the extra \(U(1)\) -factor in eq. (B.8) matches with the auxiliary \(U(1)\) -invariance (2.11) which is implicit in the diagonal ’moments’ \(F_{n,n}^{SU(N)}(A, B)\) (while nondiagonal \(n \neq m\) \(SU(N)\) integrals (2.5) are only \(Z_N\) -invariant). To promote (within \(F_{n,n}^{SU(N)}(A, B)\)) the \(Z_N\) center-subgroup (2.11) into \(U(1)\), one is to multiply each \(V^{SU(N)}\) by the auxiliary factor \(V^{U(1)}\) and then integrate \(dV^{U(1)}\) with the \(U(1)\) Haar measure normalized to unity. The resulting pattern is to be confronted with the factorized representation of the \(U(N)\) measure

\[
\int_{U(N)} dV^{U(N)} \ldots \rightarrow \int_{U(1)} dV^{U(1)} \int_{SU(N)} dV^{SU(N)} \ldots \tag{B.9}
\]
that will be derived below. Identifying the $U(1)$ sector in eq. (B.9) with the averaging over the auxiliary $U(1)$-transformation (that leaves diagonal integrals $F_{n,n}^{SU(N)}(A,B)$ invariant), we justify the required formula (2.12).

To obtain eq. (B.9), recall first that the explicit expressions [14] for the $U(N)$ and $SU(N)$ measures (after decomposition $V = \Omega diag[e^{i\omega}] \Omega^+$) are given respectively by eq. (4.7) and

$$
\int dV^{SU(N)} \ldots = \int_{-\pi}^{+\pi} \prod_{k=1}^{N} \frac{d\omega_k}{2\pi} \delta^{(2\pi)}(\omega_+) |\Delta(\{\omega_p\})|^2 \ d\Omega \ldots , \quad (B.10)
$$

where $\omega_+ = \sum_{k=1}^{N} \omega_k$, and $\delta^{(2\pi)}(\phi) = 2\pi \sum_{n \in \mathbb{Z}} \delta(\phi - 2\pi n)$ is the periodic $\delta$-function.

Next, the factorized form (B.9) of the $U(N)$ measure (4.7) is predetermined by the decomposition $V^{U(N)} = (V^{U(N)}/\det[V^{U(N)}]^{1/N}) \otimes \det[V^{U(N)}]^{1/N}$ that allows to identify

$$
(V^{U(N)}/\det[V^{U(N)}]^{1/N}) \cong V^{SU(N)}(\{\omega^{SU(N)}_k\}, \Omega) , \quad (B.11)
$$

$$
V^{U(1)} \cong \det[V^{U(N)}]^{1/N} = e^{i\omega^U(N)/N} , \quad (B.12)
$$

where $\omega^U(N)/N \in [-\pi, +\pi] \mod 2\pi$. As a result, in eq. (B.9) $V^{U(1)} \equiv V^{U(1)}(\omega_+/N)$, $\int dV^{U(1)} = \int_{-\pi}^{+\pi} d(\omega_+/N)/2\pi$, while

$$
\omega_k^{SU(N)} = \omega_k^{U(N)} - \frac{\omega_+^{U(N)}}{N} ; \quad \omega_k^{SU(N)} = \sum_{k=1}^{N} \omega_k^{SU(N)} = 0 \mod 2\pi . \quad (B.13)
$$

In order to convert the decomposition (B.11),(B.12) into (B.9), one is to change variables in eq. (4.7) going over from $\{\omega_k^{U(N)}\}$ to the overcomplete set $\{\omega_k^{SU(N)}; \ k = 1, \ldots, N\} \otimes (\omega_+^{U(N)}/N)$ entering eq. (B.13). First, $\Delta(\{\omega_k^{U(N)}\}) = \Delta(\{\omega_k^{SU(N)}\})$. Second, the constraint $\omega_+^{SU(N)} = 0 \mod 2\pi$ for the $\{\omega_k^{SU(N)}\}$-set can be imposed by the $\delta^{(2\pi)}$-function (reminiscent of the $SU(N)$ measure (B.10)), and no additional Jacobian is needed. Summarizing, we arrive at the identity

$$
dV^{U(N)}(\{\omega_k^{U(N)}\}, \Omega) = dV^{U(1)}(\omega_+^{U(N)}/N)dV^{SU(N)}(\{\omega_k^{SU(N)}\}, \Omega) , \quad (B.14)
$$

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\[ \omega_k^{SU(N)} \in [-\pi + \frac{\omega^U(N)}{N}, \pi + \frac{\omega^U(N)}{N}] \mod 2\pi, \quad k = 1, ..., N, \quad (B.15) \]

where \(\int dV^U(N)\) and \(\int dV^{SU(N)}\) are defined by eqs. (4.7) and (B.10) respectively. Evaluating the diagonal \(SU(N)\) 1-link integral \(F_{n,n}^{SU(N)}(A, B)\), one observes that the overall \((\omega^U(N)/N)\)-shift (B.15) of the \(\{\omega^V_k\}\)-variables doesn’ affecting the result. Altogether, it transforms the equality (B.14) into the announced decomposition (B.9) of the \(U(N)\) measure.

Finally, the invariance (2.11) of the diagonal \(SU(N)\) 1-link integral (B.1) under the extension of the measure \(V^{SU(N)}(\{\omega^{SU(N)}_k\}, \Omega) \rightarrow V^U(1)(\omega^{U(N)/N}) V^{SU(N)}(\{\omega^{SU(N)}_k\}, \Omega) \quad (B.16)\)

allows to identify \(F_{n,n}^{SU(N)}(A, B)\) with the \(F_{n,n}^U(A, B)\). For this purpose, one is to integrate \(\int^{+\pi}_{-\pi} d(\omega^+/N)/2\pi\) that trades (B.10) for the factorized pattern (B.9) of the \(U(N)\) measure. This completes the proof of the identity (2.12).

**C: Fusion rules of Young idempotents.**

In this Appendix we derive the identities relating Young idempotent \(C_{R_+} \in S(n_+), \quad n_+ = \sum_{k=1}^p n_k\), with the corresponding direct product \(\otimes_{k=1}^p C_{R_k}\). Let us start with the following observation. The outer product of two idempotents \(C_{R_1} \otimes C_{R_2} \in S(n_1) \otimes S(n_2)\), being embedded into \(S(n_1 + n_2)\), ceases to be a composition \(\phi_\alpha\) of the \(S(n_1 + n_2)\) Young idempotents defined as

\[ \phi_\alpha = \sum_{\sigma \in S(n_+)} \alpha([\sigma]) \sigma = \sum_{R_+ \in Y_{n_+}^{(N)}} \alpha_{R_+} C_{R_+}, \quad (C.1) \]

where \(\alpha([t\sigma t^{-1}]) = \alpha([\sigma]), \quad \forall t \in S(n_1 + n_2).\) Indeed, (for \(R_k \in Y_{n_k}^{(N)}\)) the product \((C_{R_2} \otimes C_{R_1}) V^{\oplus(n_1+n_2)}\), being a Lie group representation

\[ Tr_{n_+}[(\otimes_{k=1}^p C_{R_k}) V^{\oplus n_+}] = \otimes_{k=1}^p \chi_{R_k}(V) ; \quad R_k \in Y_{n_k}^{(N)}, \quad n_+ = \sum_{k=1}^p n_k, \quad (C.2) \]
does not generate a $S(n_1 + n_2)$-representation. To preserve the Schur-Weyl duality (A.3), we introduce the ’twisted’ deformation

$$C_{R_1} \otimes C_{R_2} \to \frac{1}{(n_1 + n_2)!} \sum_{\delta \in S(n_1 + n_2)} \left[ \delta (C_{R_1} \otimes C_{R_2}) \delta^{-1} \right]. \quad \text{(C.3)}$$

By construction, the ’twisted’ product (C.3) commutes with $\forall \in S(n_1 + n_2)$ and thus belongs to the center (C.1) of the $S(n_1 + n_2)$-algebra. More generally, given $V \in U(N)$ we arrive at the pattern (2.32) of the fusion rules of the $S(n_\psi)$-valued Young idempotents. By the same token, one arrives at the inverse of the identity (2.32)

$$C_{R_+} = \bigoplus_{\{R_k\} \in Y_{n_k}} L^{(p)}_{\{R_k\}, R_+} \sum_{\delta \in S(n+) \delta} \left[ \delta \left( \otimes_{k=1}^{p} C_{R_k} \right) \delta^{-1} \right], \quad R_+ \in Y_{n+}, \quad \text{(C.4)}$$

expressed, strictly speaking, in terms of some other set of the GLR coefficients $L^{(p)}_{\{R_k\}, R_+, R_+}$ of $p$th order.

Next, in the framework of the duality transformation the fusion rules (2.32) and (C.4) are realized in the tensor representation (2.7) and enter only inside the associated traces $Tr_{n_+}$. Consequently, according to the second Frobenius formula (A.6), it effectively eliminates the contribution of those $S(n_\psi)$-irreps $R_\psi \in Y_{n_\psi}$ which do not correspond to a $U(N)$ irrep (i.e. do not belong to $Y^{(N)}_{n_\psi}$). Therefore, we confine our attention to the GLR coefficients $L^{(p)}_{\{R_k\}, R_+}$ with all $R_\psi \in Y^{(N)}_{n_\psi}$. Upon a reflection, this subset coincides with the coefficients in the associated via (C.2) decomposition of the $U(N)$ characters (with the irreps restricted to the chiral sector), i.e. assume the integral form

$$L^{(p)}_{\{R_k\}, R_+} = \int_{U(N)} dV \chi_{R_+}(V) \otimes_{k=1}^{p} \chi_{R_k}(V), \quad \text{(C.5)}$$

where $n_+ = \sum_{k=1}^{p} n_k$. To prove this statement, one first applies both sides of (2.32) (or (C.4)) to $V^{\otimes n_+} \in U(N)$ and then takes the overall trace $Tr_{n_+}$. Employing the commutativity (A.2), we get rid of the sum over $\delta \in S(n_+)$ and use the identity (C.2) which altogether results in (C.3). Finally, let us remark that a generic coefficient $L^{(p)}_{\{R_k\}, \forall R_k \in Y^{(N)}_{n_k}}$, can be represented as a combination of the ’elementary’, $p = 2$ Littlewood-Richardson (LR) coefficients $L^{(2)}_{\{R_k\}, R_1, R_2}$ [12] (entering the two-character fusion rules).
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