Combining an optimized expansion scheme in the spirit of the background field method with the Coleman’s normal-ordering renormalization prescription, we calculate the effective potential of sine-Gordon field theory beyond the Gaussian approximation. The first-order result is just the sine-Gordon Gaussian effective potential (GEP). For the range of the coupling $\beta^2 \leq 3.4\pi$ (an approximate value), a calculation with Mathematica indicates that the result up to the second order is finite without any further renormalization procedure and tends to improve the GEP more substantially while $\beta^2$ increases from zero.

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I. INTRODUCTION

The effective potential (EP) of sine-Gordon (sG) field theory is useful and important for studying sG field theory itself and its equivalent models in quantum field theory and condensed matter physics. It has been calculated with the one-loop (1L) and Gaussian approximations, yielding the 1L and Gaussian EPs, respectively. In 1980s, the sG 1L EP or plus its thermal correction was used for calculating quantum or thermal correction to the masses of the sG soliton or breathers. Analyzing the sG 1L EP with thermal or finite-size effect gave the information on the vacuum structure of the sG system under the finite-temperature or -size condition in 1990s. Very recently, the sG 1L EP was used to investigate the phase structure of a compact U(1) gauge theory. The sG Gaussian EP without or with thermal effect has been analyzed to discuss vacuum stability and structure of the sG system. Moreover, in 1990s, the sG Gaussian EP was employed to discuss the sG field theory with a finite cutoff and obtain phase diagrams of a two-dimensional neutral classical Coulomb gas. Ref. also mentioned that the sG Gaussian EP can produce interesting results on the phase structure for the above-mentioned compact U(1) gauge theory. Besides, based on the GEP method, the two-particle excited states can be constructed to study bound states and scattering states in sG field theory.

In the weak coupling limit, the sG Gaussian EP can be reduced to the sG 1L EP, and the sG Gaussian EP provides more information than the sG 1L EP. However, no work estimated the approximation accuracy of the sG Gaussian EP. Furthermore, the sG 1L EP is ill-defined for the domain where the secondary derivative of the classical potential $\frac{\partial^2 V(\phi)}{\partial \phi^2}$ is negative and the curves of the sG Gaussian EP is not smooth but sharp when $\frac{\partial^2 V(\phi)}{\partial \phi^2}$ vanishes. Although there has been a long history in studying the sG system and some interesting as well as important quantities of it have been found or established rigorously, the exact EP of sG field theory has not been obtained yet. Under such a circumstance, a better approximated EP of sG field theory beyond the Gaussian approximation will be worth considering and may give a substantial correction to the sG Gaussian EP. Recently, the calculation of the EP beyond the Gaussian approximation has attracted much interest and the $\lambda\phi^4$ model as well as some simple fermion models were considered. Very recently, we also considered a class of scalar field theories with the optimized Rayleigh-Schrödinger expansion. Therefore, it is interesting to add an exponential interaction model, for example the sG field theory, to this list. This paper will report such a work. Using an optimized expansion scheme in the spirit of the background field method, which was developed by Stancu and Stevenson, together with the Coleman’s normal-ordering renormalization prescription, we calculate the EP of sG field theory up to the second order. No explicit divergences exist in the resultant expression and the first-order result is just the sG Gaussian EP. A further numerical calculation with Mathematica indicates that the result up to the second order is...
finite for the case of the coupling $\beta^2 \leq 3.4\pi$ and its correction to the Gaussian EP becomes more substantial while $\beta^2$ increases from zero.

Next section, the optimized expansion scheme in Ref. [13] and the Coleman’s normal-ordering prescription in the functional integral formalism [16] will be introduced into sG field theory to calculate its EP. We will calculate the approximated sG EP up to the second order in Sect. III and have a simple discussion on its features in Sect. IV. A brief conclusion will be made in Sect. V.

II. OPTIMIZED EXPANSION OF SG EP

For the sG field theory, we use the following Lagrangian

$$L_x = \frac{1}{2} \partial_\mu \phi_x \partial^\mu \phi_x - \frac{m^2}{\beta^2} [1 - \cos(\beta \phi_x)] ,$$

(1)

where the subscript $x = (x, t)$ represents the coordinates in a $(1 + 1)$-dimensional Minkowski space, $\partial_\mu$ and $\partial^\mu$ are the corresponding covariant derivatives, $\phi_x$ the scalar field at $x$, $m$ a mass parameter and the dimensionless $\beta$ the coupling parameter. It is always viable to have $\beta \geq 0$ without loss of generality. Obviously, the classical potential $V(\phi_x) = \frac{m^2}{\beta^2} [1 - \cos(\beta \phi_x)]$ is invariant under the transform $\phi \to \phi + \frac{2m}{\beta}$ with any integer $n$, and so the classical vacua are infinitely degenerate.

There exist several ways of defining the effective potential, and we will adopt the functional integral formalism and start from the generating functional for Green’s functions [17]

$$Z_M[J] = \int D\phi \exp\{ i \int dxdt [L_x + J_x \phi_x] \} ,$$

(2)

where $J_x$ is an external source at $x$, $D\phi$ the functional measure and the subscript $M$ implies Minkowski space. In the functional integral of Eq.(2), the integrand is oscillatory. One way of calculating $Z_M[J]$ is to transform it into a generating functional in Euclidean space

$$Z[J] = \int D\phi \exp\{ - \int d^2r [\frac{1}{2} \nabla_r \phi_r \nabla_r \phi_r + V(\phi_r) - J_r \phi_r] \}$$

(3)

with a time continuation $t \to -i \tau$ [17]. Here, $r = (x, \tau)$ and $\nabla_r$ is the gradient with respect to $r$ in two-dimensional Euclidean space. Taking $W[J] = \ln(Z[J])$, which is the generating functional for the connected Green’s functions, one can get the vacuum expectation value of the field $\phi_r$ in the presence of $J_r$

$$\varphi_r = \frac{\delta W[J]}{\delta J_r} .$$

(4)

Thus, the effective potential in Euclidean space can be defined as

$$V(\varphi) = \left. \frac{W[J] - \int d^2r J_r \varphi_r}{\int d^2r} \right|_{\varphi_r = \varphi} ,$$

(5)

where $\varphi$ is independent of the coordinate $r$. In principle, returning to Minkowski space from Eq.(5), one can get the effective potential of sG field theory. Nevertheless, it is not necessary to continue Eq.(5) back to Minkowski space. In fact, the proper functions in Minkowski space coincide with those in Euclidean space [18], and discussions on the $\lambda \phi^4$ model in Refs. [13, 14, 16] have suggested and supported this point.

For any $(1 + 1)$-dimensional scalar field theory without derivative interactions, the Coleman’s normal-ordering prescription [15] is useful for renormalizing the theory. Hence, we introduce the prescription according to Ref.[16] and rewrite $Z[J]$ as

$$Z[J] = \exp\{ \int d^2r [\frac{1}{2} I_{(0)}(M^2) - \frac{1}{2} \lambda^2 I_{(4)}(M^2)] \}$$

$$\times \int D\phi \exp\{ - \int d^2r [\frac{1}{2} \phi_r (-\nabla_r)^2 \phi_r - J_r \phi_r + N_M[V(\phi_r)]] \}$$

(6)

where,

$$I_{(n)}(Q^2) = \begin{cases} \int \frac{d^2p}{(2\pi)^2} (p^2 + Q^2)^{n}, & \text{for } n \neq 0 \\ \int \frac{d^2p}{(2\pi)^2} \ln(p^2 + Q^2), & \text{for } n = 0 \end{cases}$$
Now the optimized expansion scheme in Ref.[13] can be applied to sG field theory. One can start it from a modification of Eq.(6) with the following steps. First, a parameter \(\mu\) is introduced by adding a vanishing term \(\int d^2rΦ_ν(\mu^2 - \mu^2)ϕ_r\) into the exponent of the functional integral in Eq.(6). Then, make a shift \(ϕ_r → ϕ_r + \Phi\) with \(Φ\) a constant background field. Finally, inserting an expansion factor \(δ\), one can modify \(Z[J]\) as

\[
Z[J; \Phi, δ] = \exp\{-\int d^2r\left[\frac{1}{2}I_0(\mu^2) - \frac{1}{2}M^2I_1(\mu^2) + \frac{m^2}{β^2} + J, \Phi\} \right] \\
× ∫ Dϕ \exp(-\int d^2r\frac{1}{2}ϕ_r(-∇^2_r + μ^2)ϕ_r - J, ϕ_r]) \exp{-\int d^2rH_I(ϕ, ϕ, μ)} \exp\{\frac{β^2}{2}I_1(I_{\mu}(\mu^2))\}.
\]

by finishing the Gaussian integral

\[
∫ Dϕ \exp\{-∫ d^2r\frac{1}{2}ϕ_r(-∇^2_r + μ^2)ϕ_r - J, ϕ_r]) \exp\{-\frac{1}{2}∫ d^2rI_0(\mu^2) \exp\{\frac{1}{2}Jf^{-1}J\}.
\]

By substitution, \(Jf^{-1}J = ∫ d^2r_1d^2r_2J_{r_1}f_{r_1r_2}J_{r_2}\) with

\[
f_{r_1r_2}^{-1} = ∫ \frac{d^2p}{(2π)^2} \frac{1}{p^2 + μ^2} \exp^p{(r_2 - r_1)} = \frac{1}{2π}K_0(μ|r_2 - r_1|).
\]

In Eq.(11), \(K_0(μ|r_2 - r_1|)\) is the modified Bessel function of the second kind. Thus, correspondingly to the above modifications, \(W[J]\) becomes \(W[J; \Phi, δ] = \ln(Z[J; \Phi, δ])\). Cumulatively expanding \(\exp(-\int d^2rH_I(ϕ, ϕ, μ))\) and \(\ln(\exp(-\int d^2rH_I(ϕ, ϕ, μ))\exp(\frac{1}{2}Jf^{-1}J))\) with Taylor series of the exponential and logarithmic functions, one has

\[
W[J; \Phi, δ] = -∫ d^2r\frac{1}{2}I_0(\mu^2) - \frac{1}{2}M^2I_1(\mu^2) + \frac{m^2}{β^2} - J, \Phi\} + \frac{1}{2}Jf^{-1}J + \sum_{l=1}^{∞} \frac{(-1)^l+1}{l!} \left[ \exp\{-\frac{1}{2}Jf^{-1}J\} \right] \times \sum_{n=1}^{∞} \frac{(-1)^n}{n!} δ^n \int d^2r_kH_I(ϕ, ϕ, μ) \exp\{\frac{1}{2}Jf^{-1}J\} \right]^l.
\]

Substituting Eq.(12) into Eqs.(4) and (5) will give rise to an expansion of the sG EP which is independent of the parameter \(μ\) for the extrapolating case of \(δ = 1\). If the series is truncated at any order of \(δ\), then the truncated result will depend on arbitrary parameters \(μ\) and \(Φ\). The background field \(Φ\) does not affect the EP as long as the wave function renormalization is not involved and so \(Φ\) can conveniently and directly be rendered as \(ϕ_r\), the vacuum expectation value of the field operator \(ϕ\). As for \(μ\), it should be determined according to the principle of minimal sensitivity. That is, under the principle of minimal sensitivity, \(μ\) will be chosen from roots which make the first (or second) derivative of the truncated result with respect to \(μ\) vanish. Thus, \(μ\) will depend on the truncated order. It is this dependence that makes the truncated result approach the exact EP order by order. Consequently, the above procedure provides an approximate method of calculating the sG EP. It should be noted that in the approximation, taking \(Φ = ϕ_r\) in Eq.(4) will yield \(J\) which should be approximated up to the same order of \(δ\) as the truncated \(W[J; \Phi, δ]\) is.
III. APPROXIMATING SG EP UP TO THE SECOND ORDER

Now, we concretely calculate the sG EP up to the second order. Using the formulae \((x + y)^n = \sum_{k=0}^{n} C^n_k x^{n-k} y^k\), one can have the following expressions

\[
\int \prod_{k=1}^{n} d^2 r_k \mathcal{H}_1(\frac{\delta}{\delta J_{r_k}}, \Phi, \mu) \exp\{\frac{1}{2} Jf^{-1}J\} = (-1)^n \int \prod_{k=1}^{n} d^2 r_k \sum_{k=0}^{n} C^n_k (\frac{\mu^2}{2})^{n-k} (\frac{m^2}{\beta^2})^k \exp\{\frac{k\beta^2}{2} I(1)(M^2)\} \\
\times \prod_{j_1=1}^{n-k} \frac{\delta^2}{\delta J_{\beta r_{j_1}}} \prod_{j_2=n-k+1}^{n} \cos(\beta(\frac{\delta}{\delta J_{\beta r_{j_2}}} + \Phi)) \exp\{\frac{1}{2} Jf^{-1}J\} \tag{13}
\]

and

\[
\prod_{j=1}^{k} \cos(\beta(\frac{\delta}{\delta J_{\beta r_{j}}} + \Phi)) \exp\{\frac{1}{2} Jf^{-1}J\} = 2^{-k} \sum_{j=0}^{k} C^k_j \exp\{i(2-k)j\beta\Phi\} \exp\{\frac{1}{2} \int d^2r' d^2r'' [J_{\beta r'} + i\beta(\sum_{j_1=1}^{k-j} \delta(r' - r_{j_1}) \]
\]

\[- \sum_{j_2=k-j+1}^{k} \delta(r'' - r_{j_2})]\] \] \[\frac{\delta}{\delta J_{\beta r_{j_1}}} f_{r_{j_1}r_{j_2}} - \frac{\delta}{\delta J_{\beta r_{j_2}}} f_{r_{j_1}r_{j_2}}\} \cos(\beta(\sum_{j_1=1}^{k-j} \delta(r'' - r_{j_1}) - \sum_{j_2=k-j+1}^{k} \delta(r'' - r_{j_2}))\} \). \tag{14}

To obtain Eq.(14), we brought its left back to its original functional integral expression and used the exponential definition of the cosine function as well as the result of the Gaussian integral, Eq.(10). From Eq.(14), one can write down

\[
\cos(\beta(\frac{\delta}{\delta J_{\beta r_1}} + \Phi)) \exp\{\frac{1}{2} Jf^{-1}J\} = \exp\{-\frac{\beta^2}{2} f_{r_1}^{-1}\cos(\beta(\int d^2r' f_{r_1}^{-1} J_{\beta r'} + \Phi)) \exp\{\frac{1}{2} Jf^{-1}J\} \tag{15}
\]

and

\[
\cos(\beta(\frac{\delta}{\delta J_{\beta r_1}} + \Phi)) \cos(\beta(\frac{\delta}{\delta J_{\beta r_2}} + \Phi)) \exp\{\frac{1}{2} Jf^{-1}J\} = \exp\{-\beta^2(f_{r_1}^{-1} + f_{r_2}^{-1})\} \cos(\beta(\int d^2r'(f_{r_1}^{-1} + f_{r_2}^{-1}) J_{\beta r'} + 2\Phi)) \]
\[
+ \exp\{-\beta^2(f_{r_1}^{-1} - f_{r_2}^{-1})\} \exp\{-i\beta(\int d^2r'(f_{r_1}^{-1} - f_{r_2}^{-1}) J_{\beta r'} \exp\{\frac{1}{2} Jf^{-1}J\} \). \tag{16}
\]

Employing Eqs.(13), (15) and (16), one can truncate Eq.(12) at the zeroth, first and second orders of \(\delta\) to get \(W^0[\Phi, J, \delta], W^1[\Phi, J, \delta] \) and \(W^2[\Phi, J, \delta]\), respectively and then obtain the approximated EP up to the corresponding orders. (Here, the Greek number means up to the corresponding order of \(\delta\).)

At the zeroth order, \(W^0[\Phi, J, \delta]\) is just the first two terms in Eq.(12) and taking the zeroth order expression of Eq.(4) as \(\Phi \) leads to \(J^0_{\beta r} = 0\). Thus, the sG EP at the zeroth order is

\[
W^0[\Phi, J, \delta] = W^0[\Phi, J, \delta] + \delta \int d^2r' \left\{ \frac{\mu^2}{2} [I_{r_1}^{-1} + (\int d^2r'' f_{r_1}^{-1} J_{\beta r''})^2] \right\} ^\frac{m^2}{\beta^2} e^{-\frac{\beta^2}{2} (J_{r_1}^{-1} - I(1)(M^2))} \cos(\int d^2r'' f_{r_1}^{-1} J_{\beta r''} + \beta\Phi) \right\}, \tag{17}
\]

Up to the first order, one has

\[
W^1[\Phi, J, \delta] = W^0[\Phi, J, \delta] + \delta \int d^2r' \left\{ \frac{\mu^2}{2} [I_{r_1}^{-1} + (\int d^2r'' f_{r_1}^{-1} J_{\beta r''})^2] \right\} ^\frac{m^2}{\beta^2} e^{-\frac{\beta^2}{2} (J_{r_1}^{-1} - I(1)(M^2))} \cos(\int d^2r'' f_{r_1}^{-1} J_{\beta r''} + \beta\Phi) \right\}, \tag{18}
\]
and \(-\frac{\delta W^{I}[J;\Phi,\delta]}{\delta J_r} = \varphi_r' = \Phi\) yields
\[
\int d^2 r' f^{-1}_{r'r} J_{r'} + \delta \int d^2 r' f^{-1}_{r'r} \{ \mu^2 \int d^2 r'' f^{-1}_{r'r''} J_{r''} + m^2 \frac{\beta}{\beta^2} e^{-\frac{\beta^2}{2}(\frac{1}{2} J^2_{r'r} - \frac{1}{2} J^2_{r''}) \sin(\int d^2 r'' f^{-1}_{r'r''} J_{r''} + \beta \Phi)} \} = 0 .
\] (19)

From Eq.(19), \(J_r\) up to the first order is
\[
J^I = \delta \frac{m^2}{\beta^2} e^{-\frac{\beta^2}{2}(J^2_{r'r} - \frac{1}{2} J^2_{r''}) \sin(\beta \Phi)} .
\] (20)

Because \(J^I_r = 0\) and there exists no linear term of \(J_r\) in \(W^{I}[J;\Phi,\delta] - \int d^2 r J_r \Phi\) but the quadratic one, only \(J^I_r\) is needed to extract the sG EP up to the first order. In fact, to obtain the EP up to the \(n\)th order, one need the approximated \(J\) only up to the \((n-1)\)th order. Now one can write down the sG EP up to the first order.

\[
W^{I}[J;\Phi,\delta] = W^{I}[J;\Phi,\delta] + \delta^2 \frac{1}{2} \int d^2 r' d^2 r'' \left\{ \frac{1}{2} \mu^2 (f^{-1}_{r'r})^2 - \left(\frac{m^2}{\beta^2}\right)^2 \exp\{-\beta^2 f^{-1}_{r'} - I_{(1)}(M^2)\} \cos(\beta \Phi) \right. \\
- \beta^2 \mu^2 (\frac{m^2}{\beta^2})^2 \exp\{-\beta^2 f^{-1}_{r'} - I_{(1)}(M^2)\} (f^{-1}_{r'r''})^2 \cos(\beta \Phi) \\
- \left. \frac{1}{2} (\frac{m^2}{\beta^2})^2 [\exp\{-\beta^2 (f^{-1}_{r'} - I_{(1)}(M^2) + f^{-1}_{r''})\} \cos(2\beta \Phi) + \exp\{-\beta^2 (f^{-1}_{r'} - I_{(1)}(M^2) - f^{-1}_{r''})\} \cos(\beta \Phi)] \right\} .
\] (22)

\(\varphi_{x}^{II} = -\frac{\delta E_{x}^{II}[J;\Phi,\delta]}{\delta J_x} = 0\) can be solved for \(J^{II}\). In the present case, however, it is enough to use \(J^I\). Substituting Eq.(20) into Eq.(22), keeping terms up to the second order of \(\delta\), one finds that those terms with the factor of squared volume are cancelled out and the sG EP up to the second order is as follows.

\[
\varphi_{x}^{II} = \varphi_{x}^{I}(\Phi, \delta) - \delta^2 \frac{1}{2} \left\{ \frac{1}{2} \mu^2 I^{(2)}(\mu^2) - \frac{m^2}{\beta^2} \beta^2 \mu^2 I^{(2)}(\mu^2) e^{-\frac{\beta^2}{2}(I_1(\mu^2) - I_{(1)}(M^2))} \cos(\beta \Phi) \right. \\
+ \left. \left(\frac{m^2}{\beta^2}\right)^2 \sum_{k=1}^{\infty} \frac{\beta^2 (2k+1)!}{(2k+1)!} I^{(2k+1)}(\mu^2) e^{-\beta^2 (I_1(\mu^2) - I_{(1)}(M^2))} \sin^2(\beta \Phi) \right) \\
+ \left. \left(\frac{m^2}{\beta^2}\right)^2 \sum_{k=1}^{\infty} \frac{\beta^2 (2k)!}{(2k)!} I^{(2k)}(\mu^2) e^{-\beta^2 (I_1(\mu^2) - I_{(1)}(M^2))} \cos^2(\beta \Phi) \right\} .
\] (23)

with

\[
I^{(k)}(\mu^2) = \int \frac{d^2 r' d^2 r'' (f^{-1}_{r'r})^k}{d^2 r'} = \frac{1}{(2\pi)^k-1} \mu^2 \int_0^\infty dRR(K_R(R))^k \\
= \int \frac{d^2 p_1 \cdots d^2 p_{k-1}}{(2\pi)^2(2k-1)} \frac{1}{(p_1^2 + \mu^2) \cdots (p_{k-1}^2 + \mu^2)(p_1 + \cdots + p_{k-1})^2 + \mu^2} .
\] (24)

where we use Eq.(11) to obtain Eq.(24). In Eqs.(21) and (23), the arbitrary \(\mu\) should be determined according to the principle of minimal sensitivity [13, 19].
In the same way, employing Eqs.(13) and (14), one can obtain the sG EP up to higher orders, albeit it may be difficult to solve the renormalization problem and perform numerical calculations. Next section, we will have a simple discussion on the sG EP up to the second order, Eq.(23).

For schemes beyond Gaussian approximation, the calculation of \( W[J; \Phi, \delta] \) becomes lengthy and tedious even only up to the second order. Nevertheless, one can simplify it by using the replacement trick in Ref. [12]. Furthermore, if the trick is combined with a similar procedure to that in our former work Ref. [13], the calculation would become a little more simplified.

In passing, we point out that one can also obtain the above results by borrowing Feynman diagrammatic technique in Ref. [21] with the propagator of Eq.(11). To make it clear, we rewrite the interaction Eq.(8) as \( \mathcal{H}_I(\phi_r, \Phi, \mu) = V_\phi \phi^2 + \sum_{n=0}^{\infty} V_n \phi^{2n} + \sum_{n=0}^{\infty} V_{in} \phi^{2n+1} \). Here, \( V_\phi = -\mu^2 \) is the coefficient attached to the vertex with two legs, \( V_n = -\frac{\mu^2}{2\pi} \exp(\frac{\pi}{2} I_{(1)}(\mathcal{M}^2)) \cos(\beta \Phi)(-1)^n \frac{\mu^{2n+1}}{(2n+1)!} \) the coefficient attached to vertices with \( 2n \) legs and \( V_n = \frac{\mu^2}{2\pi} \exp(\frac{\pi}{2} I_{(1)}(\mathcal{M}^2)) \sin(\beta \Phi)(-1)^n \frac{\mu^{2n+1}}{(2n+1)!} \) those attached to vertices with \( (2n+1) \) legs \( (n = 0, 1, 2, \ldots) \). Obviously, the sG EP is just the sum of connected one-particle-irreducible vacuum diagrams consisting of the propagator and \( V_\phi, V_n, V_{in} \)-vertices.

For example, in the great bracket of Eq.(23), the first term comes from the diagram consisting of two \( V_\phi \)-vertices, the second term is the sum of diagrams consisting of one \( V\phi \)-and one \( V_{in} \)-vertices, the third the sum of diagrams consisting of two \( V_{in} \)-vertices and the fourth the sum of diagrams consisting of two \( V_{in} \)-vertices. When the diagrammatic technique is used, one has to be careful to count the symmetric factor for every topologically equivalent diagram correctly.

Additionally, the sG EP beyond the sG Gaussian EP can also be calculated in the way of Ref. [14] and the result in Ref. [14] is easily used to produce the sG EP up to the second order which should be identical to Eq.(23).

IV. DISCUSSIONS ON SG POST-GAUSSIAN EP

Noting Eq.(25) and the results \( \frac{1}{2} I_{(0)}[\mu^2] - I_{(0)}[\mathcal{M}^2] = \frac{1}{2} \mathcal{M}^2 I_{(1)}[\mathcal{M}^2] - \frac{1}{2} \mu^2 I_{(1)}[\mu^2] = \frac{\mu^2 - \mathcal{M}^2}{2\pi} \) and \( I_{(1)}[\mu^2] - I_{(1)}[\mathcal{M}^2] = -\frac{1}{\mu^2} \ln(\frac{\mu^2}{\mathcal{M}^2}) \), one can find that there exist no explicit divergences in Eqs.(21) and (23) for the extrapolating case of \( \delta = 1 \). This is because we used the Coleman’s normal-ordering prescription at the beginning. Thus, no further renormalization procedure is needed for the sG EP up to the first order. As for Eq.(23), we should not draw the same conclusion without further investigation because there exist series in it. Employing Eq.(24), one can easily rewrite Eq.(23) as

\[
\mathcal{V}^{II}(\Phi) = \mathcal{V}^{I}(\Phi, \delta) - \delta^2 \frac{1}{2} \left( \frac{1}{2} \mu^2 I^{(2)}(\mu^2) - \frac{m^2}{\beta^2} \frac{\mu^2 I^{(2)}(\mu^2)}{\beta^2} e^{\frac{\beta^2}{2}(I_{(1)}(\mu^2) - I_{(1)}(\mathcal{M}^2))} \cos(\beta \Phi) \right)
+ \left( \frac{\mu^2}{\beta^2} \right)^2 \frac{2\pi}{\mu^2} \left[ A_\phi(\beta) \sin^2(\beta \Phi) + A_c(\beta) \cos^2(\beta \Phi) \right] e^{-\beta^2(I_{(1)}(\mu^2) - I_{(1)}(\mathcal{M}^2))}
\]

(26)

with the coefficients \( A_\phi(\beta) = \int_0^{\infty} dRR[\sinh(\frac{\beta^2}{2}\mathcal{M}^2 K_0(R))] - \int_0^{\infty} dRR[\cos(\frac{\beta^2}{2}\mathcal{M}^2 K_0(R)) - 1] \) and \( A_c(\beta) = \int_0^{\infty} dRR[\cosh(\frac{\beta^2}{2}\mathcal{M}^2 K_0(R)) - 1] \). Mathematica program can readily give values of \( A_\phi(\beta) \) and \( A_c(\beta) \) for the case of \( \beta^2 \leq 3.431556 \pi \) and \( \beta^2 \leq 3.415399 \pi \), respectively, and one has \( A_\phi(\beta) > A_c(\beta) > 0 \). For larger values of \( \beta^2 \), Mathematica program shows that \( A_\phi(\beta) \) and \( A_c(\beta) \) are not convergent. In fact, for the range of \( 4\pi \leq \beta^2 < 8\pi \), divergences which can not be removed with the help of Coleman’s normal-ordering prescription were observed early in 1977 [21], and it has been shown that they can be eliminated by the introduction of constant counterterms or in some alternative ways [22]. Here, we will discuss Eq.(26) only for the range of \( \beta^2 \leq 3.4\pi \).

First we give a brief discussion on the sG EP up to the first order, Eq.(21). Taking \( \delta = 1 \), Eq.(21) is easily rewritten as

\[
\mathcal{V}^{I}(\Phi) = \frac{m^2}{\beta^2} + \frac{\mu^2 - \mathcal{M}^2}{8\pi} - \frac{m^2}{\beta^2} \frac{\mu^2}{\mathcal{M}^2} \cos(\beta \Phi) \frac{\phi^2}{\mu^2}.
\]

(27)

To determine \( \mu \), one can require \( \frac{d\mathcal{V}^{I}(\Phi)}{d\mu} = 0 \), which gives rise to \( \mu = \mathcal{M}(\frac{m^2 \cos(\beta \Phi)}{\mathcal{M}^2} + \frac{4\pi}{\beta^2}) \). Defining the renormalized mass \( m_R \) in the same way as Ref. [1], i.e., \( m_R^2 = \frac{d\mathcal{V}^{I}(\Phi)}{d\mu}|_{\Phi=0} \), one has \( m_R^2 = \mathcal{M}^2 \left( \frac{m^2}{\mathcal{M}^2} + \frac{4\pi}{\beta^2} \right) \). If the normal-ordering mass \( \mathcal{M} \) is taken as \( m_R \), then Eq.(27) is just Eq.(2.13) in Ref. [1] except for a divergent constant. That is, the sG EP up to the first order is nothing but the sG Gaussian EP. Thus, the scheme in the present paper provides a systematic tool of improving the sG Gaussian EP. Note that \( \mu \) has no real value when \( \cos(\beta \Phi) < 0 \). In this case, because the sG Gaussian EP was originally obtained variationally with the variational parameter \( \mu \), one usually consider \( \mathcal{V}^{I}(\Phi) \) at the end points of the range \( 0 \leq \mu < \infty \) to choose the minimal one as the Gaussian EP [1]. Thus, the sG Gaussian
EP is a constant for the case of \( \cos(\beta \Phi) \leq 0 \) because \( \mu \) should be chosen as zero, and accordingly, the sG Gaussian EP is not smooth when \( \cos(\beta \Phi) = 0 \).

Now we analyze the sG EP up to the second order, Eq.(26). It is the next order result to the sG Gaussian EP and usually called post-Gaussian EP \[13\]. Finishing integrals in Eq.(26) and taking \( \delta = 1 \), one can reach

\[
\psi^{II}(\Phi) = \frac{m^2}{\beta^2} - \frac{M^2}{8\pi} + \frac{\mu^2}{16\pi} + \left( \frac{m^2}{8\pi} - \frac{m^2}{\beta^2} \right) \frac{\mu^2}{M^2} \cos(\beta \Phi) \\
- \frac{\pi}{M^2} \left( \frac{m^2}{\beta^2} \right)^2 \left[ A_s(\beta) \sin^2(\beta \Phi) + A_c(\beta) \cos^2(\beta \Phi) \right] \left( \frac{\mu^2}{M^2} \right)^{\beta^2} - 1. \tag{28}
\]

Note that the stationary condition \( \frac{\partial \psi^{II}(\Phi)}{\partial \mu} = 0 \) has no real root for \( \mu \), and thus, one has to appeal to \( \frac{\partial^2 \psi^{II}(\Phi)}{\partial \mu^2} = 0 \).

It can be explicitly solved with Mathematica and yields

\[
\mu^{\beta^2 - 1} = \left\{ \beta^2 (4\pi - \beta^2)^{\frac{1}{2}} (8\pi - \beta^2) \cos(\beta \Phi) + \beta^4 (4\pi - \beta^2)(8\pi - \beta^2)^2 \cos^2(\beta \Phi) + 2048 \pi^4 (6\pi - \beta^2)(A_s(\beta) \sin^2(\beta \Phi) + A_c(\beta) \cos^2(\beta \Phi)) \right\}^{\frac{1}{2}} \\
\times \frac{2^{-5}\pi^{-2}M^{-2}m^{-2}\beta^2}{(4\pi - \beta^2)^{\frac{1}{2}} (6\pi - \beta^2)[A_s(\beta) + A_c(\beta) + (A_c(\beta) - A_s(\beta)) \cos(2\beta \Phi)]}. \tag{29}
\]

Obviously, Eq.(29) gives a real \( \mu \) for the case of \( \beta^2 < 4\pi \) (in the case of finite \( A_s(\beta) \) and \( A_c(\beta) \)), and so the sG post-Gaussian EP can have an explicit expression by substituting Eq.(29) into Eq.(28). Taking \( M = m = 1 \), one can turn Eqs.(28) and (29) dimensionless for a numerical calculation. Comparing with the classical potential and Gaussian EP, the sG post-Gaussian EP possesses the following features for the case of \( \beta^2 \leq 3.4\pi \). Firstly, like the sG Gaussian EP \[3, 4\], the sG post-Gaussian EP has the same periodicity as the sG classical potential. This can be seen from Fig.1 and Fig.2. In Figs.1 and 2, the sG post-Gaussian EPs (solid curves) are compared with the classical potentials (dotted curves) at \( \beta^2 = 0.5\pi \) and \( 3.4\pi \), respectively. In Figs.1 and 2 as well as the latter Figs.3 and 4, the longitudinal axes represent potentials \( V(\phi) \), \( \psi^{II}(\Phi) \) or \( \psi^{II}(\Phi) \) and the horizontal axes represent \( \phi \) or \( \Phi \). Secondly, the sG post-Gaussian EP is well defined and smooth for the whole domain and Figs.1 and 2 illustrate this point. Thirdly, one observes that in Figs.1 and 2, whereas for smaller \( \beta^2 \) peaks of the sG post-Gaussian EP are higher than those of the classical potential, for larger \( \beta^2 \) (still \( \leq 3.4\pi \)) peaks of the sG post-Gaussian EP are lower than those of the classical potentials. Finally, Figs.1 and 2 suggest wells of the sG post-Gaussian EP are wider than those of the classical potentials. Figs.3 and 4 give more detailed comparison on the last point among wells of the classical potentials (dotted), the sG Gaussian (dashed) and the post-Gaussian EPs (solid) for \( \beta^2 = 2\pi \) and \( 3.4\pi \), respectively. Noting that in Fig.1, the wells of the sG post-Gaussian EP are almost identical to those of the classical potentials, one can conclude that the correction of the sG post-Gaussian EP from the sG Gaussian EP becomes more substantial with the increase of \( \beta^2 \).
FIG. 2: Similar to Fig.1 but at $\beta^2 = 3.4\pi$.

FIG. 3: Comparison between the classical potential (dotted curve), the sG Gaussian (dashed curve) and post-Gaussian EP (solid curve) at $\beta^2 = 2\pi$. The longitudinal axis represents potentials $V(\phi), V'(\Phi)$ or $V''(\Phi)$ and the horizontal axis represents the classical field $\phi$ or $\Phi$.

V. CONCLUSION

In this paper, we have performed an optimized expansion in the spirit of the background field method plus the Coleman's normal-ordering renormalization prescription to calculate the EP of sG field theory beyond the Gaussian approximation. We obtained the sG EP up to the second order, Eq.(28) together with Eq.(29), which is finite for the range of $\beta \leq 3.4\pi$ (an approximate value from Mathematica) without the need of any further renormalization procedure. It is well-defined for the whole domain of $\Phi$, and give more substantial correction to the sG Gaussian EP with the increase of $\beta^2$. In view of the existence of many equivalents to the sG model and the importance of going beyond the Gaussian approximation, we believe that our investigation in this paper is meaningful and interesting. Still, some further investigations are needed. Without any doubt, it will be important to renormalize Eq.(23) beyond the range of $\beta^2$ which we have treated in the present paper and apply the result to many sister systems of the sG model. Additionally, a generalization of the present work to a finite-temperature case will be also interesting because the finite-temperature effect at the 1L and Gaussian approximation may change the periodicity of the classical potential.
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