Geometry of Associated Quantum Vector Bundles and the Quantum Gauge Group

Gustavo Amilcar Saldaña Moncada

Instituto de Matemáticas, Universidad Nacional Autonoma de México, UNAM, Area de la Investigación Científica, Circuito Exterior, Ciudad Universitaria, Ciudad de México, CP 04510, México.
email: gamilcar@ciencias.unam.mx

Abstract
In Differential Geometry, it is well–known that given a principal $G$–bundle with a principal connection, for every unitary finite–dimensional linear representation of $G$, one can induce a linear connection and a hermitian structure on associated vector bundles, which are compatible. Furthermore, the gauge group acts on the space of principal connections and on the space of linear connections defined on associated vector bundles. This paper aims to present the non–commutative counterpart of all these classical facts in the theory of quantum bundles and quantum connections.

Keywords: Quantum Connections, Hermitian Structures, Quantum Gauge Group.

2000 MSC: 46L87, 16T99.

1. Introduction

Non–Commutative Geometry, also known as Quantum Geometry, arises as a kind of algebraic and physical generalization of geometrical concepts [1, 2, 3]. There are a variety of reasons to believe that this branch of mathematics could solve some of the Standard Model’s fundamental problems [1].

In total agreement with this philosophy, M. Durdevich developed on [4, 5, 6] a formulation of the theory of principal bundles and principal connections in Non–Commutative Geometry’s framework. This theory used the concept of quantum group presented by S. L. Woronovicz on [7, 8] playing the role of the structure group on the bundle. However, it uses a more general differential calculus on the quantum group that allows to extend the
complete structure of $\ast$–Hopf algebra; reflecting the classical fact that for every Lie group its tangent bundle is a Lie group as well. Furthermore, Durdevich’s formulation can embrace other classical concepts like characteristic classes and classifying spaces [9, 10].

On [11] one can appreciate a categorical equivalence between principal bundles with principal connections over a fixed base space $M$ and the category of associated functors called gauge theory sectors; Durdevich’s theory allows to recreate this result for quantum principal bundles and regular quantum principal connections [12]. All of these are clear reasons to keep developing this theory.

The purpose of this work is to extend the theory of associated quantum vector bundles following the line of research of M. Durdevich [10, 12] and also in agreement with [13, 14, 15], to add canonical hermitian structures as well as studying the relationship between this new structure and induced quantum linear connections for any real quantum principal connection. Moreover, we will introduce and analyze an ad hoc definition of quantum gauge group and its natural action on the space of quantum principal connections and the space of quantum linear connections.

We believe that the approach presented is important not only because of the results as such that we will show, which reflect the analogy with the classical case and extends the theory (the reader should put particular attention on Theorems 3, 4, Definition 3, and Proposition 4 among other results), but also because it opens the door to many other research lines, for example, the moduli space of quantum connections and Yang–Mills models and Field Theory, in accordance with the work presented in [14, 15]. The study of Yang–Mills models and Field Theory is the next step of our research and it will be explored in detail in forthcoming publications [16, 17, 18].

The paper is organized into five sections. In the second one we are going to present preparatory material, and it is broken down into two subsections. The first one is about compact matrix quantum group, its representations, and the universal differential envelope $\ast$–calculus of a $\ast$–FODC. In the second subsection, we will present all the basic notions of quantum principal bundles and quantum principal connections; however, we will change the standard definition of quantum principal connections in order to embrace a more general theory. The third section consists of two subsections. In the first subsection, we will show the general theory of associated quantum vector bundles and induced quantum linear connections. In contrast, in the second one, we will introduce a canonical hermitian structure that results compatible
with induced quantum linear connections. The fourth section is about the quantum gauge group and its action on quantum connections. Finally, for this paper not to be too long, in the fifth section, we are going to present three illustrative examples by using trivial quantum principal bundles (in the sense of [5]), the quantum Hopf fibration and a special classical/quantum hybrid principal bundle. The final section is about some concluding comments.

We shall use the notation used in [12], for example all quantum spaces will be formally represented by associative unital \(*\)-algebras over \(\mathbb{C}\), \((X, \cdot, 1, \ast)\) interpreted like the \(*\)-algebra of smooth \(\mathbb{C}\)-valued functions on the quantum space. We will identify the quantum space with its algebra, so in general, we are going to omit the words associative and unital. Also, all our \(*\)-algebra morphisms will be unital and in the whole work, we are going to use Sweedler notation.

In literature there are other viewpoints on quantum principal bundles, for example [19, 20, 21]. All these formulations are intrinsically related by the theory of Hopf–Galois extensions [22]. We have decided to use Durdevich's formulation of quantum principal bundles because of its purely geometrical–algebraic framework, when differential calculus (which link geometry, analysis and algebra), connections, their curvature and their covariant derivatives are the most relevant objects.

It is worth mentioning that in general, we will not assume that the connections are strong either regular [23, 5]. Even though this text is based on Durdevich's theory, our definition of the quantum gauge group will be the one presented in [24], but at the level of differential calculus.

2. Preparatory Material

Like in the classical case, the idea of group will play a fundamental role in the concept of principal bundle. So it is necessary to start checking the highlights of this theory developed by S. L. Woronovicz on [7, 8], and the highlights of the universal differential envelope \(*\)-calculus presented on [4, 25].

2.1. About Quantum Groups

A compact matrix quantum group (cmqg) will be denoted by \(\mathcal{G}\); while its dense \(*\)-Hopf (sub)algebra will be denoted by

\[\mathcal{G}^\infty := (G, \cdot, 1, \phi, \epsilon, \kappa, \ast),\]
where $\phi$ is the coproduct, $\epsilon$ is the counity and $\kappa$ is the coinverse. In accordance with [8], it shall treat as the algebra of all smooth $C^\infty$–valued functions defined on $G$. In the same way a (smooth right) $G$–representation on a $\mathbb{C}$–vector space $V$ is a linear map
\[ \alpha : V \longrightarrow V \otimes G \]
such that
\[ (\text{id}_V \otimes \epsilon) \circ \alpha \cong \text{id}_V \] (1)
and
\[ (\text{id}_V \otimes \phi) \circ \alpha = (\alpha \otimes \text{id}_G) \circ \alpha. \] (2)
We say that the representation is finite–dimensional if $\dim_{\mathbb{C}}(V) < |N|$. $\alpha$ usually receives the name of (right) coaction or (right) corepresentation of $G$ on $V$. It is worth mentioning that in the general theory ([7]), Equation (1) is not necessary.

Given two $G$–representations $\alpha$, $\beta$ acting on $V$, $W$, respectively, a representation morphism is a linear map
\[ T : V \longrightarrow W \]
such that the following equation holds
\[ (T \otimes \text{id}_G) \circ \alpha = \beta \circ T \] (3)
The set of all representation morphisms between two representations $\alpha$, $\beta$ will be denoted as
\[ \text{Mor}(\alpha, \beta) \] (4)
and the set of all finite–dimensional $G$–representations will be denoted by
\[ \text{Obj}(\text{Rep}_G) \] (5)
A $G$–representation is reducible if there exists a subspace $L$ such that $\alpha(L) = L \otimes G$ and it is unitary if viewed has an element of $B(V) \otimes G$ with $B(V) := \{ f : V \longrightarrow V \mid f \text{ is linear} \}$ it is unitary. In [7], Woronovicz proved that for every finite–dimensional $G$–representation acting on $V$ there is an inner product $\langle -|- \rangle$ (not necessarily unique) such that it becomes into a unitary one. This inner product is $\alpha$–invariant in the sense of
\[ \langle v_1|v_2 \rangle \otimes 1 = \sum_{k,l} \langle v_{1k}|v_{2l} \rangle \otimes g_{1k}^* g_{2l} \] (6)
for all $v_1, v_2 \in V$ with $\alpha(v_i) = \sum_k v_{ik} \otimes g_{ik}$. Even more on [7] one can find a prove of the following theorem

**Theorem 1.** Let $T$ be a complete set of mutually non-equivalent irreducible unitary (necessarily finite-dimensional) $G$–representations with $\alpha^{\text{triv}} \in T$ (the trivial representation on $\mathbb{C}$). For any $\alpha \in T$ that acts on $(V^\alpha, \langle \cdot | \cdot \rangle)$,

$$\alpha(e_i) = \sum_{j=1}^{n_\alpha} e_j \otimes g_{ij}^\alpha, \quad (7)$$

where $\{e_i\}_{i=1}^{n_\alpha}$ is an orthonormal basis of $V^\alpha$ and $\{g_{ij}^\alpha\}_{ij=1}^{n_\alpha} \subseteq G$. Then $\{g_{ij}^\alpha\}_{\alpha,i,j}$ is a linear basis of $G$, where the index $\alpha$ runs on $T$ and $i, j$ run from 1 to $n_\alpha$.

The basis $\{g_{ij}^\alpha\}_{\alpha,i,j}$ satisfies

$$\sum_{k=1}^{n_\alpha} g_{ik}^\alpha \kappa(g_{kj}^\alpha) = \delta_{ij} 1, \quad \sum_{k=1}^{n_\alpha} g_{ik}^\alpha \kappa g_{jk}^\alpha = \delta_{ij} 1 \quad (8)$$

with $\delta_{ij}$ being the Kronecker delta, among other properties [7].

Given $(\Gamma, d)$ a bicovariant first-order differential $\ast$–calculus (\ast–FODC [25]) on $G$, the universal differential envelope $\ast$–calculus

$$(\Gamma^\wedge, d, \ast) \quad (9)$$

is introducing as the graded differential $\ast$–algebra defined by

$$\Gamma^\wedge := \otimes_G^\ast \Gamma / \mathcal{Q}, \quad \otimes_G^\ast \Gamma := \bigoplus_k (\otimes_G^k \Gamma) \quad \text{with} \quad \otimes_G^k \Gamma := \underbrace{\Gamma \otimes_G \cdots \otimes_G \Gamma}_{k \text{ times}}$$

with $\mathcal{Q}$ the bilateral ideal of $\otimes_G^\ast \Gamma$ generated by

$$\sum_i dg_i \otimes_G dh_i \quad \text{such that} \quad \sum_i g_i dh_i = 0, \quad g_i, h_i \in G.$$
allow us to extend ∗-algebra morphisms of \( G \) and linear ∗-antimultiplicative morphisms of \( G \), always that they commute with \( d \). By using these properties one can define a graded differential ∗-Hopf algebra structure on \( \Gamma^\wedge \) which extends \( G^\infty \) [4]

\[
\Gamma^\wedge^\infty := (\Gamma^\wedge, \cdot, 1, \phi, \epsilon, \kappa, d, *)
\]

and \( \epsilon \) has the particularity that \( \epsilon(\vartheta) = 0 \) if \( \vartheta \notin G \). Now it is possible to define the right adjoint action of \( \Gamma^\wedge \)

\[
\text{Ad} : \Gamma^\wedge \rightarrow \Gamma^\wedge \otimes \Gamma^\wedge
\]

such that \( \text{Ad}(\vartheta) = (-1)^{\partial x(1)\partial x(2)} \vartheta(2) \otimes \kappa(\vartheta(1))\vartheta(3) \), where \( \partial x \) denotes the grade of \( x \) and (using Sweedler’s notation)

\[
(id_{\Gamma^\wedge} \otimes \phi)\phi(\vartheta) = (\phi \otimes id_{\Gamma^\wedge})\phi(\vartheta) = \vartheta(1) \otimes \vartheta(2) \otimes \vartheta(3).
\]

Clearly \( \text{Ad} \) extends the right adjoint action \( \text{Ad} \) of \( G \) [25]. Define

\[
i_{\text{inv}}\Gamma^\wedge = \{ \theta \in \Gamma \mid \Phi_{\Gamma^\wedge}(\theta) = 1 \otimes \theta \},
\]

with \( \Phi_{\Gamma^\wedge} \) the extension of the canonical left representation of \( G \) on \( \Gamma \) [25]. This space is a graded \( \mathbb{C} \)-vector space, and it is well–known that

\[
i_{\text{inv}}\Gamma := i_{\text{inv}}\Gamma^\wedge^1 \cong \text{Ker}(\epsilon)/\mathcal{R},
\]

where \( \mathcal{R} \subseteq \text{Ker}(\epsilon) \) is the canonical right \( G \)-ideal of \( G \) associated to \( (\Gamma, d) \) [8, 25]. This space plays the role of the quantum (dual) Lie algebra and it allows us to consider the quantum germs map [4, 25]

\[
\pi : G \rightarrow i_{\text{inv}}\Gamma
\]

\[
g \mapsto \kappa(g(1))dg(2)
\]

where \( \phi(g) = g^{(1)} \otimes g^{(2)} \). This map has several useful properties, for example the restriction map \( \pi|_{\text{Ker}(\epsilon)} \) is surjective and

\[
\text{ker}(\pi) = \mathcal{R} \oplus \mathbb{C}1,
\]

\[
\pi(g) = -(d\kappa(g^{(1)}))g^{(2)}, \quad dg = g^{(1)}\pi(g^{(2)}),
\]

\[
d\kappa(g) = -\pi(g^{(1)})\kappa(g^{(2)}), \quad \pi(g)^* = -\pi(\kappa(g)^*), \quad d\pi(g) = -\pi(g^{(1)})\pi(g^{(2)}).
\]
for all \( g \in G \) \([4, 25]\). The canonical right representation of \( G \) on \( \Gamma \) leaves \( \text{inv} \Gamma \) invariant and denoting it by

\[
\text{ad} : \text{inv} \Gamma \longrightarrow \text{inv} \Gamma \otimes G
\]  

it satisfies \([4, 25]\)

\[
\text{ad} \circ \pi = (\pi \otimes \text{id}_G) \circ \text{Ad}.
\]

There is a right \( G \)-module structure in \( \text{inv} \Gamma \) given by

\[
\theta \circ g = \kappa(g^{(1)})\theta g^{(2)} = \pi(hg - \epsilon(h)g)
\] (15)

if \( \theta = \pi(h) \). It is worth mentioning that

\[
(\theta \circ g)^* = \theta^* \circ \kappa(g)^*
\]

and also we have

\[
\text{inv} \Gamma^\wedge = \otimes \text{inv} \Gamma / S^\wedge, \quad \otimes \text{inv} \Gamma := \bigoplus_k (\otimes^k \text{inv} \Gamma) \text{ with } \otimes^k \text{inv} \Gamma := \underbrace{\text{inv} \Gamma \otimes \cdots \otimes \text{inv} \Gamma}_{k \text{ times}}
\]

where \( S^\wedge \) is the bilateral ideal of \( \otimes \text{inv} \Gamma \) generated by

\[
\pi(g^{(1)}) \otimes \pi(g^{(2)})
\]

with \( g \in \mathcal{R} \). In this way it is possible to extend the module structure to \( \text{inv} \Gamma^\wedge \) by means of

\[
1 \circ g = \epsilon(g), \quad (\theta_1 \theta_2) \circ g = (\theta_1 \circ g^{(1)}) (\theta_2 \circ g^{(2)}).
\]

Finally the following identification holds

\[
\Gamma^\wedge \cong G \otimes \text{inv} \Gamma^\wedge.
\]

2.2. Basics of Quantum Principal Bundles

Now we can introduce the notions of quantum principal bundles and quantum principal connections. As we mentioned before, we are going to base on the theory developed by M. Durdevich but changin the standard definition of quantum principal connections and we will use the notation showed on \([12]\). If the reader prefers, it is possible to check this theory in the original work \([4, 5, 6]\).
Let \((M, \cdot, 1, \ast)\) be a quantum space and let \(\mathcal{G}\) be a cmqg. A quantum principal \(\mathcal{G}\)-bundle over \(M\) (qpb) is a quantum structure formally represented by the triplet
\[
\zeta = (GM, M, GM\Phi),
\]
(16)
where \((GM, \cdot, 1, \ast)\) is a quantum space called the quantum total space with \((M, \cdot, 1, \ast)\) as quantum subspace which receives the name of quantum base space, and
\[
GM\Phi : GM \rightarrow GM \otimes G
\]
is a \(*\)-algebra morphism that satisfies
1. \(GM\Phi\) is a \(G\)-representation.
2. \(GM\Phi(x) = x \otimes 1\) if and only if \(x \in M\).
3. The linear map \(\beta : GM \otimes GM \rightarrow GM \otimes G\) given by
\[
\beta(x \otimes y) := x \cdot GM\Phi(y) = (x \otimes 1) \cdot GM\Phi(y)
\]
is surjective.

Given \(\zeta\) a qpb over \(M\), a differential calculus on it is:
1. A graded differential \(*\)-algebra \((\Omega^\bullet(GM), d, \ast)\) generated by \(\Omega^0(GM) = GM\) (quantum differential forms on \(GM\)).
2. A bicovariant \(*\)-FODC over \(G\) \((\Gamma, d)\).
3. The map \(GM\Phi\) is extendible to a graded differential \(*\)-algebra morphism
\[
\Omega\Psi : \Omega^\bullet(GM) \rightarrow \Omega^\bullet(GM) \otimes \Gamma^\wedge.
\]
It is worth mentioning that \(\Omega\Psi\) is a representation of \(\Gamma^\wedge\) acting on \(\Omega^\bullet(GM)\).

In this way, the space of horizontal forms is defined as
\[
\text{Hor}^\bullet GM := \{\varphi \in \Omega^\bullet(GM) \mid \Omega\Psi(\varphi) \in \Omega^\bullet(GM) \otimes G\}.
\]
(17)
and it is a graded \(*\)-subalgebra of \(\Omega^\bullet(GM)\). Since
\[
\Omega\Psi(\text{Hor}^\bullet GM) \subseteq \text{Hor}^\bullet GM \otimes G,
\]
the map
\[
H\Phi := \Omega\Psi|_{\text{Hor}^\bullet GM}
\]
(18)
turns into a $\mathcal{G}$–representation [25]. Also, one can define the space of base forms (quantum differential forms on $M$) as

$$\Omega^\bullet(M) := \{\mu \in \Omega^\bullet(GM) \mid \Omega\Psi(\mu) = \mu \otimes 1\}. \quad (19)$$

The space of base forms is a graded differential $\ast$–subalgebra of $\Omega^\bullet(GM)$ and in general, it is not generated by $M$. As a little example, let $\mathcal{G}$ be the quantum group associated to $U(1)$ and $(\mathcal{G} = \mathbb{C}[z, z^*], \cdot, 1, \phi, \epsilon, \kappa, \ast)$ be its dense $\ast$–Hopf algebra. Then by considering $M := \{\lambda 1 \mid \lambda \in \mathbb{C}\}$, the triplet $\zeta = (GM := G, M, GM\Phi := \phi)$ is a quantum principal $\mathcal{G}$–bundle over $M$. For differential forms on $GM = G$, let us take any universal differential envelope $\ast$–calculus $(\Gamma^\wedge, d, \ast)$ over $GM = G$ without elements of degree $n \geq 2$ and $\dim(\text{inv} \Gamma) = 1$; and for differential forms on $G$ (as the structure group) let us take the trivial differential calculus, i.e., $d = 0$. By defining

$$\Omega\Psi : \Gamma^\wedge \longrightarrow \Gamma^\wedge \otimes G$$

such that for degree 0 is $\phi$ and for degree 1 is given by $\Omega\Psi(g\pi(z)) = \phi(g)(\pi(z) \otimes 1)$ where $g \in G$, we get a differential calculus on $\zeta$. It is easy to see that

$$\Omega^1(M) = \{\lambda \pi(z) \mid \lambda \in \mathbb{C}\},$$

but this space cannot be generated by $M$.

In accordance with [25], by using $\text{inv} \Gamma^\wedge$, it can be defined the space of vertical forms.

Now we introduce the concept of quantum principal connection (qpc) as a linear map

$$\omega : \text{inv} \Gamma \longrightarrow \Omega^1(GM) \quad (20)$$

that satisfies

$$\Omega\Psi(\omega(\theta)) = (\omega \otimes \text{id}_G) \circ \text{ad}(\theta) + 1 \otimes \theta.$$\hspace{1cm} A qpb with a qpc will be denoted by $(\zeta, \omega)$. In analogy with the classical case, it can be proved that the set

$$\text{qpc}(\zeta) := \{\omega : \text{inv} \Gamma \longrightarrow \Omega^1(GM) \mid \omega \text{ is a qpc on } \zeta\} \quad (21)$$

is not empty for any qpb $\zeta$ ([5]), and it is an affine space modeled by the vector space of connection displacements $\overrightarrow{\text{qpc}(\zeta)} := \text{Mor}^1(\text{ad}, H\Phi)$, where

$$\text{Mor}^1(\text{ad}, H\Phi) = \{\lambda : \text{inv} \Gamma \longrightarrow \Omega^1(GM) \mid (\lambda \otimes_G \text{id}_G) \circ \text{ad} = H\Phi \circ \lambda\}. \quad (22)$$
There is a canonical involution on $qpc(\zeta)$ given by
\[ \hat{\omega} := \ast \circ \omega \circ \ast \] (23)
and we define the dual qpc of $\omega$ as $\hat{\omega}$. A qpc $\omega$ is real if $\hat{\omega} = \omega$ and we say that it is imaginary if $\hat{\omega} = -\omega$. It is worth mentioning that in order to embrace a more general theory, our definition of qpcs is a little different that the standard on literature, for example on [4, 5, 6, 25] because in these papers qpcs are always real.

A qpc is called regular if for all $\phi \in \text{Hor}^k GM$ and $\theta \in \text{inv} \Gamma$, we have
\[ \omega(\theta) \phi = (-1)^k \phi^{(0)}(0) \omega(\theta \circ \varphi^{(1)}), \] (24)
where $H\Phi(\varphi) = \varphi^{(0)} \otimes \varphi^{(1)}$. A qpc $\omega$ is called multiplicative if
\[ \omega(\pi(g^{(1)})) \omega(\pi(g^{(2)})) = 0 \] (25)
for all $g \in \mathcal{R}$.

The curvature of a qpc is defined as the linear map
\[ R^\omega := d \circ \omega - \langle \omega, \omega \rangle : \text{inv} \Gamma \longrightarrow \Omega^2(GM) \] (26)
with
\[ \langle \omega, \omega \rangle := m_\Omega \circ (\omega \otimes \omega) \circ \delta : \text{inv} \Gamma \longrightarrow \Omega^2(GM), \]
where $\delta : \text{inv} \Gamma \longrightarrow \text{inv} \Gamma \otimes \text{inv} \Gamma$ is an embedded differential ([5, 25]) and $m_\Omega$ is the multiplication map of $\Omega^\bullet(GM)$. A really important property of multiplicative qpcs is the fact that for these connections the curvature does not depend on the map $\delta$ [5, 25]. The curvature of any qpc fulfills
\[ R^\omega \in \text{MOR}(\text{ad}, H\Phi). \] (27)
and also we can define
\[ \hat{R}^\omega := \ast \circ R^\omega \circ \ast = R^{\hat{\omega}}. \] (28)

To conclude this section we are going to define the covariant derivative of a qpc as the first–order linear map
\[ D^\omega : \text{Hor}^\bullet GM \longrightarrow \text{Hor}^\bullet GM \] (29)
such that for every \( \varphi \in \text{Hor}^k GM \)
\[
D^\omega(\varphi) = d\varphi - (-1)^k \varphi^{(0)}(\omega(\varphi^{(1)})).
\]

On the other hand, the first-order linear map
\[
\hat{D}^\omega := \ast \circ D^\omega \circ \ast
\]
is called the dual covariant derivative of \( \omega \). It can be proved that
\[
D^\omega \in \text{Mor}(H\Phi, H\Phi), \quad D^\omega_{|\Omega^* (M)} = d_{|\Omega^* (M)},
\]
and
\[
\hat{D}^\omega(\varphi) = D^\omega(\varphi) + \ell^\omega(\pi(\kappa^{-1}(\varphi^{(1)})), \varphi^{(0)}) + (-1)^k \varphi^{(0)}(\omega - \hat{\omega})(\pi(\varphi^{(1)})),
\]
where
\[
\ell^\omega : \text{inv} \Gamma \times \text{Hor}^* GM \longrightarrow \text{Hor}^* GM
\]
\[
(\theta, \varphi) \mapsto \omega(\theta) \varphi - (-1)^k \varphi^{(0)}(\omega - \hat{\omega})(\pi(\varphi^{(1)})).
\]
The map \( \ell^\omega \) measures the lack of regularity of \( \omega \) in the sense of \( \ell^\omega = 0 \) if and only if \( \omega \) is regular. Moreover, if \( \omega \) is real and regular, \( D^\omega = \hat{D}^\omega = \hat{D}^\omega \) and if \( \omega \) is regular, then \( D^\omega \) and \( \hat{D}^\omega \) satisfy the graded Leibniz rule. In general
\[
D^\omega(\varphi \psi) = D^\omega(\varphi) \psi + (-1)^k \varphi D^\omega(\psi) + (-1)^k \varphi^{(0)} \ell^\omega(\pi(\varphi^{(1)}), \psi)
\]
\[
D^\omega(\psi^*) = D^\omega(\psi^*) + \ell^\omega(\pi(\kappa(\psi^{(1)})^*), \psi^{(0)*})
\]
for all \( \varphi \in \text{Hor}^k GM, \psi \in \text{Hor}^* GM \).

**Remark 1.** Let \( \mathcal{T} \) a complete set of mutually non-equivalent irreducible \( G \)-representations with \( \alpha^\text{triv}_\zeta \in \mathcal{T} \). From this point until the end of this paper we are going to consider for a given \( \zeta = (GM, M, G_M \Phi) \) and each \( \alpha \in \mathcal{T} \) that there exists
\[
\{T_k^L\}_{k=1}^{d_\alpha} \subseteq \text{Mor}(\alpha, G_M \Phi)
\]
for some \( d_\alpha \in \mathbb{N} \) such that
\[
\sum_{k=1}^{d_\alpha} x^{\alpha}_k x^{\alpha}_k = \delta_{ij} \mathbb{1},
\]
with $x_{ki}^\alpha := T^L_k(e_i)$, where $\{e_i\}_{i=1}^{n_\alpha}$ is some fixed orthonormal basis of the vector space $V^\alpha$ on which $\alpha$ acts. Also we are going to assume that the following relation holds

$$W^\alpha X^\alpha = \text{Id}_{n_\alpha}, \quad \text{where} \quad W^\alpha = (w^\alpha_{ij}) = Z^\alpha X^\alpha C^\alpha - 1$$

(35)

for each $\alpha$. Here $X^\alpha = (x^\alpha_{ij}) \in M_{d_\alpha \times n_\alpha}(GM)$, $X^\alpha^* = (x^\alpha_{ij}^*)$, while $\text{Id}_{n_\alpha}$ is the identity element of $M_{n_\alpha}(GM)$ and $Z^\alpha = (z^\alpha_{ij}) \in M_{d_\alpha}(\mathbb{C})$ is a strictly positive element. Finally $C^\alpha \in M_{n_\alpha}(\mathbb{C})$ is the matrix written in terms of the basis $\{e_i\}_{i=1}^{n_\alpha}$ of the canonical representation isomorphism between $\alpha$ and $\alpha^{cc} := (\text{id}_{V^\alpha} \otimes \kappa^2)\alpha$, and $W^\alpha X^\alpha$ is the transpose matrix of $W^\alpha [5]$.

It is worth mentioning that in terms of the theory of Hopf–Galois extension ([22]), the first condition of Remark 1 guarantees that $GM$ is principal [23]. Furthermore, the second condition implies the existence of a right $M$–linear right $G$–colinear splitting of the multiplication $GM \otimes M \rightarrow GM$.

We added these conditions because in general, in Durdevich’s theory it is not necessary to work with Hopf–Galois extensions and/or principal Hopf–Galois extensions (for example, for all the theory presented before Remark 1 it is not necessary a Hopf–Galois extension [25]); however, it is a well-established and common framework along many papers ([19, 20, 21]) that allows us to keep developing the theory. Moreover, we decided to use Equation 34 and 35 instead of just imposing that $GM$ is left/right principal because in this way, it is possible to do explicit calculations as the reader will verify in the next section; not to mention that these equations are essential in other papers of Durdevich’s theory, for example [5, 6].

3. Associated Quantum Vector Bundles, Induced Quantum Linear Connections and Hermitian Structures

The first purpose of this paper is to present some of the essential aspects of associated quantum vector bundles, induced quantum linear connections and the definition of the canonical hermitian structure, as well as the relationship between these structures, showing an analogy with the classical case. In this section we shall deal with all of this. Since in this paper we are not interested in the categorical point of view, we can weaken some conditions imposed in [12].
3.1. Associated Quantum Vector Bundles and Induced Quantum Linear Connections

Let us start taking a quantum $G$–bundle $\zeta = (G, M, \Phi)$ and a $G$–representation $\alpha \in \mathcal{T}$ acting on $V^\alpha$. The $C^*$–vector space $\text{Mor}(\alpha, G, \Phi)$ has a natural $M$–bimodule structure given by multiplication with elements of $M$. Define

$$
\varrho^\alpha_{kl} : M \rightarrow M
$$

(36)

$$
p \mapsto \sum_{i=1}^{n_\alpha} x_{ki}^\alpha p x_{li}^{\alpha*}
$$

with $k, l \in \{1, ..., d_\alpha\}$. For all $p, q \in M$ we have

$$
\varrho^\alpha_{kl}(p)^* = \varrho^\alpha_{lk}(p^*) , \quad \sum_{i=1}^{n_\alpha} \varrho^\alpha_{ki}(p) \varrho^\alpha_{il}(q) = \varrho^\alpha_{kl}(pq);
$$

so there is a linear multiplicative $*$–preserving (in general not–unital) map

$$
\varrho^\alpha : M \rightarrow M_{d_\alpha} (M)
$$

(37)

$$
p \mapsto (\varrho^\alpha_{kl}(p)) ,
$$

where $M_{d_\alpha} (M)$ denotes $d_\alpha \times d_\alpha$ matrices with coefficients in $M$ [5]. Let us take the free left $M$–module $M^{d_\alpha}$, its canonical basis $\{ \overline{e}_1, ..., \overline{e}_{d_\alpha} \}$ and the left $M$–submodule

$$
M^{d_\alpha} \cdot \varrho^\alpha (1) \subseteq M^{d_\alpha}.
$$

The operation

$$
\cdot : M^{d_\alpha} \cdot \varrho^\alpha (1) \otimes M \rightarrow M^{d_\alpha} \cdot \varrho^\alpha (1)
$$

(38)

given by

$$
\varepsilon \otimes p \mapsto \varepsilon \cdot \varrho^\alpha (p)
$$

induces a $M$–bimodule structure on $M^{d_\alpha} \cdot \varrho^\alpha (1)$. Define the left $M$–module morphism

$$
H : M^{d_\alpha} \rightarrow \text{MOR}(\alpha, G, \Phi)
$$

(39)

such that $H(\overline{e}_k) = T^L_k$. Thus according to [5]

$$
H|_{M^{d_\alpha} \cdot \varrho^\alpha (1)} : M^{d_\alpha} \cdot \varrho^\alpha (1) \rightarrow \text{MOR}(\alpha, G, \Phi)
$$

is a $M$–bimodule isomorphism. This shows that

$$
\zeta^L_\alpha := (\Gamma^L, V^\alpha M) := \text{MOR}(\alpha, G, \Phi), +, \cdot)
$$

(40)
is a finitely generated projective left $M$–module. In particular for all $T \in \Gamma(M, V^\alpha M)$,

$$T = \sum_{k=1}^{d_\alpha} p_k^T T_k^L, \quad p_k^T = \sum_{i=1}^{n_\alpha} T(e_i) x_{ki}^* \in M.$$  \hspace{1cm} (41)

We can repeat this process to conclude that

$$\zeta_\alpha^R := (\Gamma^R(M, V^\alpha M) := \text{Mor}(\alpha, GM \Phi), +, \cdot)$$  \hspace{1cm} (42)

is a finitely generated projective $M$–right module too [5]. In this case for every $T \in \text{Mor}(\alpha, GM \Phi)$

$$T = \sum_{i=1}^{d_\alpha} T_k^R \tilde{p}_k^R \quad \text{with} \quad T_k^R = \sum_{i=1}^{d_\alpha} z_{ki} T_i^L, \quad \tilde{p}_k^R = \sum_{i,j=1}^{d_\alpha, n_\alpha} g_{ij}^\alpha w_{ij}^* T(e_j) \in M. \hspace{1cm} (43)$$

where $Y^\alpha = (y_{ij}^\alpha) \in M_{d_\alpha}(\mathbb{C})$ is the inverse of $Z^\alpha$.

In Differential Geometry, given a principal $G$–bundle

$$\pi : GM \rightarrow M$$

($GM$ is the total space, $M$ is the base space, $G$ is the structure group and $\pi$ is the bundle projection), sections of the associated vector bundle $\Gamma(M, V^\alpha M)$ given $\alpha$ a linear representation of $G$ acting on the space $V^\alpha$, can be viewed as maps $T : GM \rightarrow V$. By dualizing this result and in the light of [1, 26] and the Serre–Swan theorem, $\zeta_\alpha^L$ has to be interpreted as the associated left quantum vector bundle (associated left qvb) and $\zeta_\alpha^R$ as the associated right quantum vector bundle (associated right qvb). Our notation is just an apology of the classical case, but considering the left/right structures.

Let us fix a qpc $\omega$. Then the map

$$\Upsilon^{-1}_\alpha : \Omega^\bullet(M) \otimes_M \Gamma^L(M, V^\alpha M) \rightarrow \text{Mor}(\alpha, H \Phi)$$

such that $\Upsilon^{-1}_\alpha(\mu \otimes_M T) = \mu T$ is a graded $M$–bimodule isomorphism, where $\text{Mor}(\alpha, H \Phi)$ has the $M$–bimodule structure similar to the one of $\text{Mor}(\alpha, GM \Phi)$; and its inverse function is given by

$$\Upsilon_\alpha(\tau) = \sum_{k=1}^{d_\alpha} \mu_k^T \otimes_M T_k^L, \quad \mu_k^T = \sum_{i=1}^{n_\alpha} \tau(e_i) x_{ki}^*.$$  \hspace{1cm} (44)
Elements of this tensor product can be interpreted as left qvb–valued differential forms. Thus the linear map

\[ \nabla^\omega_\alpha : \Gamma^L(M, V^\alpha M) \longrightarrow \Omega^1(M) \otimes_M \Gamma^L(M, V^\alpha M) \]

\[ T \longmapsto \Upsilon_\alpha \circ D^\omega \circ T, \]  

(45)
is a quantum linear connection on \( \zeta^L_\alpha \) in the sense of [26]. Furthermore, the linear map

\[ \tilde{\Upsilon}^{-1}_\alpha : \Gamma^R(M, V^\alpha M) \otimes_M \Omega^\bullet(M) \longrightarrow \text{Mor}(\alpha, H \Phi) \]
such that \( \tilde{\Upsilon}^{-1}_\alpha (T \otimes_M \mu) = T \mu \) is also a graded \( M \)--bimodule isomorphism as well with its inverse function given by

\[ \tilde{\Upsilon}_\alpha (\tau) = \sum_{k=1}^{d_\alpha} T_k \otimes_M \tilde{\mu}_k^\tau, \quad \tilde{\mu}_k^\tau = \sum_{i,j=1}^{d_\alpha n_\alpha} t_i^\tau w_j^\alpha \tau(e_j). \]  

(46)

Elements of this tensor product can be interpreted as right qvb–valued differential forms. Hence the linear map

\[ \hat{\nabla}^\omega_\alpha : \Gamma^R(M, V^\alpha M) \longrightarrow \Gamma^R(M, V^\alpha M) \otimes_M \Omega^1(M) \]

\[ T \longmapsto \tilde{\Upsilon}_\alpha \circ \hat{D}^\omega \circ T, \]  

(47)
is a quantum linear connection in \( \zeta^R_\alpha \). \( \nabla^\omega_\alpha \) and \( \hat{\nabla}^\omega_\alpha \) receive the name of induced quantum linear connections (induced qlc) of \( \omega \). In addition, by defining

\[ \sigma_\alpha := \tilde{\Upsilon}_\alpha \circ \Upsilon^{-1}_\alpha \]  

(48)
we obtain that when \( \omega \) is real and regular

\[ \sigma_\alpha \circ \nabla^\omega_\alpha = \hat{\nabla}^\omega_\alpha, \]

which is the case of the theory presented on [12].

Extending \( \nabla^\omega_\alpha \) to the exterior covariant derivative

\[ d^{\nabla^\omega_\alpha} : \Omega^\bullet(M) \otimes_M \Gamma^L(M, V^\alpha M) \longrightarrow \Omega^\bullet(M) \otimes_M \Gamma^L(M, V^\alpha M) \]
such that for all \( \mu \in \Omega^k(M) \)

\[ d^{\nabla^\omega_\alpha} (\mu \otimes_M T) = d\mu \otimes_M T + (-1)^k \mu \nabla^\omega_\alpha (T), \]  

(49)
the curvature of $\nabla_\alpha$ is defined as
\[
R^{\nabla_\alpha} := d^{\nabla_\alpha} \circ \nabla_\alpha : \Gamma^L(M, V^\alpha) \to \Omega^2(M) \otimes_M \Gamma^L(M, V^\alpha M)
\] (50)
and the following formula holds
\[
d^{\nabla_\alpha} = \Upsilon_\alpha \circ D^\omega \circ \Upsilon_\alpha^{-1}.
\] (51)
In the same way, by using the exterior covariant derivative of $\hat{\nabla}_\alpha$
\[
d^{\hat{\nabla}_\alpha} : \Gamma^R(M, V^\alpha M) \otimes_M \Omega^\bullet(M) \to \Gamma^R(M, V^\alpha M) \otimes_M \Omega^\bullet(M)
\]
which is given by
\[
d^{\hat{\nabla}_\alpha}(T \otimes_M \mu) = \hat{\nabla}_\alpha^\omega(T) \mu + T \otimes_M d\mu,
\] (52)
the curvature is defined as
\[
R^{\hat{\nabla}_\alpha} := d^{\hat{\nabla}_\alpha} \circ \hat{\nabla}_\alpha : \Gamma^R(M, V^\alpha M) \to \Gamma^R(M, V^\alpha M) \otimes_M \Omega^2(M)
\] (53)
and the following formula holds
\[
d^{\hat{\nabla}_\alpha} = \hat{\Upsilon}_\alpha \circ \hat{D}^\omega \circ \hat{\Upsilon}_\alpha^{-1}.
\] (54)
The theory of connections on finitely generated projective (left/right) modules has been studied along too many years, for example on [29] and we will follow this line of research. In particular, on [29] it is showed a kind of Bianchi identity that all connections satisfy, but just when $M$ is commutative.

All our constructions can be extended in a very natural way by using the direct sum operator for every $\alpha \in \text{Obj}(\text{Rep}_G)$, not just for elements of $\mathcal{T}$. It is worth mentioning that definitions of $\zeta_\alpha^L$, $\zeta_\alpha^R$, the fact that $\Upsilon_\alpha$ and $\tilde{\Upsilon}_\alpha$ are $M$–bimodule isomorphisms, as well as Equations 51, 54 are clearly the non–commutative counter part of the Gauge Principle [27, 11].

In accordance with [28], $GM \Box^G V^\alpha \ast \cong \Gamma^L(M, V^\alpha M)$ (for the natural left coaction on $V^\alpha \ast$, the dual space of $V^\alpha$), which is the commonly accepted construction of the associated qvb. Nevertheless, we have decided to use $\Gamma^L(M, V^\alpha M)$ and $\Gamma^R(M, V^\alpha M)$ because in this way, the definitions of $\nabla_\alpha^\omega$, $\hat{\nabla}_\alpha^\omega$ are completely analogous to its classical counterparts (in Differential Geometry, both connections are the same); not to mention that it is easier to work with, because it will allow us to do explicit calculations, like the reader will verify in the rest of this paper. In addition, by using intertwining maps the definition of the canonical hermitian structure will look more natural.
3.2. The Canonical Hermitian Structure

One of the purposes of this paper is to introduce a hermitian structure on associated qvbs (as well as showing some of its properties) compatible with induced qlcs. This result will be showed until Theorem 3.

For \( \alpha \in T \), the left \( M \)-module morphism given by \( H \) followed by \( H|_{M^{d}\alpha \cdot \varrho^{\alpha}(\mathbb{I})}^{-1} \), which is

\[
\overline{e}_i \mapsto \sum_{k=1}^{d_\alpha} p_{ik} \overline{e}_k,
\]

with \( p_{ij} = \sum_{k=1}^{d_\alpha} x_{ik}^{\alpha} x_{jk}^{\alpha \ast} \in M \), induces a canonical hermitian structure on \( \zeta^L_{\alpha} \), i.e., a \( M \)-valued sesquilinear map (antilinear in the second coordinate)

\[
\langle -, - \rangle_L : \Gamma^L(M, V^{\alpha}M) \times \Gamma^L(M, V^{\alpha}M) \longrightarrow M
\]
such that for all \( p \in M, T_1, T_2 \in \Gamma^k(M, V^{\alpha}M) \)

1. \( \langle T_1, p T_2 \rangle_L = \langle T_1, T_2 \rangle_L p^\ast \).
2. \( \langle T_1, T_2 \rangle_L^* = \langle T_2, T_1 \rangle_L \).
3. \( \langle T_1, T_1 \rangle_L \in M^+ \), where \( M^+ \) is the the pointed convex cone generated by elements of the form \( \{pp^\ast\} \).

Explicitly

\[
\langle T_1, T_2 \rangle_L = \sum_{k=1}^{n_\alpha} T_1(e_k)T_2(e_k)^\ast.
\] (55)

Even more

\[
\langle T_1 p, T_2 \rangle_L = \langle T_1, T_2 p^\ast \rangle_L.
\] (56)

It is worth mentioning that \( \langle -, - \rangle_L \) does not depend on the orthonormal basis \( \{e_k\}_{k=1}^{n_\alpha} \) used.

**Remark 2.** The associated matrix of \( H \) followed by \( H|_{M^{d}\alpha \cdot \varrho^{\alpha}(\mathbb{I})}^{-1} \) written in terms of the basis \( \{\overline{e}_1, ..., \overline{e}_{d_\alpha}\} \) is exactly \( \varrho^{\alpha}(\mathbb{I}) \) (see Equation 37) and a direct calculation shows that

\[
\varrho^{\alpha}(\mathbb{I}) = \varrho^{\alpha}(\mathbb{I})^\dagger,
\]

where \( \dagger \) denotes the composition of the \( \ast \) operation on \( M \) and the usual matrix transposition. General theory ([29]) tells us that in this situation \( \langle -, - \rangle_L \) is non–singular, i.e., there is a Riesz representation theorem in terms of left \( M \)-modules.
Let \( \alpha \in \text{Obj}(\text{Rep}_G) \) acting on \( V^\alpha \). Then there exist \( \alpha_i \in \mathcal{T} \) acting on \( V^{\alpha_i} \) such that \( \alpha \cong \bigoplus_{i=1}^n \alpha_i \). [7]. Assume that \( f \) is a representation isomorphism between them. Thus
\[
A_f : \bigoplus_{i=1}^n \Gamma^L(M, V^{\alpha_i} M) \longrightarrow \Gamma^L(M, V^\alpha M)
\]
\[
T \longmapsto T \circ f
\]
is a \( M \)-bimodule isomorphism and its inverse is \( A_{f^{-1}} \) [12]. We can define an hermitian structure on \( \Gamma^L(M, V^\alpha M) \) given by
\[
\langle - , - \rangle_L : \Gamma^L(M, V^\alpha M) \times \Gamma^L(M, V^\alpha M) \longrightarrow M
\]
\[
( T_1 , T_2 ) \longmapsto \sum_{k=1}^{n_\alpha} (T_1 \circ f^{-1})(v_k) (T_2 \circ f^{-1})(v_k)^*,
\]
with \( \{v_k\} \) an orthonormal basis of \( \bigoplus_{i=1}^n V^{\alpha_i} \). For any unitary representation morphism \( f \), the previous equation agrees with the canonical hermitian structure induced by the direct sum of the canonical hermitian structures of \( \zeta_{\alpha_i}^L \), so we can take Equation 58 as our definition for a general finite-dimensional \( G \)-representation, especially because there always exists unitary representation morphisms. In fact, according to [7] \( V^\alpha \) decomposes into an orthogonal direct sum of subspaces \( V^{\alpha_i} \) such that \( \alpha \mid V^{\alpha_i} \cong \alpha_i \) and these restrictions are unitary. This tells us that it is enough to find a unitary representation morphism between \( \alpha \mid V^{\alpha_i} \) and \( \alpha_i \). In accordance with [7]
\[
\text{Mor}(\alpha \mid V^{\alpha_i}, \alpha_i) = \{ \hat{f} \mid \hat{\lambda} \in \mathbb{C} \},
\]
where \( \hat{f} \) is a representation isomorphism. By considering \( f = 1 / (\det(\hat{f}))^{1/n_\alpha} \hat{f} \) it can be showed that \( f^* = \lambda f^{-1} \) since \( f^* \circ f = \lambda \text{id}_{V^{\alpha_i}} \) and \( \lambda \in \mathbb{C} \) has to be a \( n_\alpha \)-root of unity. Due to the fact that \( f^* \circ f \) is a positive element and \( \lambda \) is also an eigenvalue, we conclude that \( \lambda = 1 \).

**Definition 1 (Canonical hermitian structure).** For every \( \alpha \in \text{Obj}(\text{Rep}_G) \) we define the canonical hermitian structure on the associated left qvb \( \zeta_{\alpha}^L \) as the sesquilinear map given by
\[
\langle - , - \rangle_L : \Gamma^L(M, V^\alpha M) \times \Gamma^L(M, V^\alpha M) \longrightarrow M
\]
\[
( T_1 , T_2 ) \longmapsto \sum_{k=1}^{n_\alpha} T_1(e_k) T_2(e_k)^*,
\]
where \( \{e_i\}_{i=1}^{n_\alpha} \) is any orthonormal basis of \( V^\alpha \).
It is worth mentioning that despite that we have used the word *canonical* on its name, \( \langle -, - \rangle_L \) depends on the inner product \( \langle - | - \rangle \) of \( V^\alpha \) for which \( \alpha \) is unitary.

On [12] it was defined a canonical \( M \)-bimodule isomorphism

\[
A_{\alpha_1, \alpha_2} : \Gamma^L(M, (V_1 \otimes V_2)M) \longrightarrow \Gamma^L(M, V_1M) \otimes_M \Gamma^L(M, V_2M)
\]

for all \( \alpha_1, \alpha_2 \in T \). The proof of the following proposition is a straightforward calculation

**Proposition 1.** By considering the tensor product hermitian structure on the space \( \Gamma^L(M, V_1M) \otimes_M \Gamma^L(M, V_2M) \) given by

\[
\langle T_1 \otimes_M T_2, U_1 \otimes_M U_2 \rangle_L^\otimes = \langle T_1(T_2)_L, U_1 \rangle_L,
\]

all maps \( A_{\alpha_1, \alpha_2} \) are isometries.

Since \( \langle -, - \rangle_L \) does not depend on the orthonormal basis used to calculate it, one gets

**Proposition 2.** If \( f : V \longrightarrow W \) is a unitary corepresentation morphism between \( \alpha \) and \( \beta \), then the \( M \)-bimodule morphism

\[
A_f : \Gamma^L(M, WM) \longrightarrow \Gamma^L(M, VM)
\]

\[
T \mapsto T \circ f
\]

is an isometry.

Due to the fact that the canonical hermitian structure is induced by

\[
(p, q) \longmapsto \sum_{i=1}^{d_\alpha} p_i q_i^*,
\]

where \( p = (p_1, ..., p_{d_\alpha}), q = (q_1, ..., q_{d_\alpha}) \in M^{d_\alpha} \), it follows that [29]

**Theorem 2.** Let \( \zeta \) be a qpq and let us assume that \( (M, \cdot, 1, *) \) has structure of \( C^* \)-algebra (or assume that it can be completed to a \( C^* \)-algebra). Then for all \( \alpha \in \text{Obj}(\text{Rep}_G) \),

\[
(\zeta^L_\alpha, \langle -, - \rangle_L)
\]

is a Hilbert \( C^* \)-module (or it can be completed to a Hilbert \( C^* \)-module).
The previous theorem is important since, in its context, one can apply all the theory about Hilbert $C^*$–modules ([30]) to associated left qvbs.

**Definition 2 (Canonical hermitian structure).** For every $\alpha \in \text{Obj}(\text{Rep}_G)$ we define the canonical hermitian structure on the associated right qvb $\zeta^R_\alpha$ as the sesquilinear map (now antilinear in the first coordinate) given by

$$\langle -,- \rangle_R : \Gamma^R(M,V^\alpha M) \times \Gamma^R(M,V^\alpha M) \longrightarrow M$$

$$(T_1, T_2) \longmapsto \sum_{k=1}^{n_\alpha} T_1(e_k)^* T_2(e_k),$$

where $\{e_i\}_{i=1}^{n_\alpha}$ is any orthonormal basis of $V^\alpha$.

It is worth mentioning that $\langle -,- \rangle_R$ does not come from the generators $\{T^R\}$; however, it shares with $\langle -,- \rangle_L$ similar properties; for example, it is non–singular.

The introduction of hermitian structures on associated left/right qvbs opens the door to study adjointable operators $\text{End}(\zeta^L_\alpha),\text{End}(\zeta^R_\alpha)$, and unitary operators $U(\zeta^L_\alpha), U(\zeta^R_\alpha)$ [29].

By taking a differential calculus on $\zeta$, the canonical hermitian structures can be extended to $\Omega^\bullet(M)$–valued sesquilinear maps

$$\langle -,- \rangle_L : (\Omega^\bullet(M) \otimes_M \Gamma^L(M,V^\alpha M)) \times (\Omega^\bullet(M) \otimes_M \Gamma^L(M,V^\alpha M)) \longrightarrow \Omega^\bullet(M)$$

$$\langle -,- \rangle_R : (\Gamma^R(M,V^\alpha M) \otimes_M \Omega^\bullet(M)) \times (\Gamma^R(M,V^\alpha M) \otimes_M \Omega^\bullet(M)) \longrightarrow \Omega^\bullet(M)$$

by means of [29]

$$\langle \mu_1 \otimes_M T_1, \mu_2 \otimes_M T_2 \rangle_L = \mu_1 \langle T_1, T_2 \rangle_L \mu_2^*$$

and

$$\langle T_1 \otimes_M \mu_1, T_2 \otimes_M \mu_2 \rangle_R = \mu_1^* \langle T_1, T_2 \rangle_R \mu_2.$$

In the previous context, it is common to say that a qlc $\nabla$ is compatible with a hermitian structure $\langle -,- \rangle$ or that it is a hermitian qlc if

$$\langle \nabla(x_1), x_2 \rangle + \langle x_1, \nabla(x_2) \rangle = d\langle x_1, x_2 \rangle$$

for all elements $x_1, x_2$ of the qvb.

**Theorem 3.** Let $(\zeta, \omega)$ be a qpb with a real qpc and $\alpha \in \text{Obj}(\text{Rep}_G)$. Then the induced qlc is hermitan.
Proof. It is enough to prove the statement for \( \alpha \in \mathcal{T} \). In this way

\[
\langle \nabla_\alpha^\omega(T_1), T_2 \rangle_L + \langle T_1, \nabla_\alpha^\omega(T_2) \rangle_L = \sum_{k,i=1}^{d_\alpha, n_\alpha} \mu_k D\omega \circ T_1^L(k, T_2) + T_1(e_i) T_2^L(e_i)^*
\]

\[
+ T_1(e_i) T_2^L(e_i)^*(\mu_k D\omega \circ T_2)^*
\]

\[
= \sum_{k,i,j=1}^{d_\alpha, n_\alpha} D\omega(T_1(e_j)) x_{k,j}^A x_{k,i}^A T_2(e_i)^*
\]

\[
+ T_1(e_i) x_{k,i}^A x_{k,j}^A D\omega(T_2(e_j))^*
\]

\[
= \sum_{i=1}^{n_\alpha} D\omega(T_1(e_i)) T_2(e_i)^* + T_1(e_i) D\omega(T_2(e_i))^*
\]

\[
= \sum_{i=1}^{n_\alpha} D\omega(T_1(e_i)) T_2(e_i)^*
\]

\[
- \sum_{i,j=1}^{n_\alpha} T_1(e_j) \ell^\omega(\pi(g_{ij}^A), T_2(e_i)^*)
\]

\[
+ \sum_{i,j=1}^{n_\alpha} T_1(e_i) \ell^\omega(\pi(g_{ij}^A)^*, T_2(e_j)^*)
\]

\[
= \sum_{i=1}^{n_\alpha} D\omega(T_1(e_i)) T_2(e_i)^*
\]

\[
- \sum_{i,j=1}^{n_\alpha} T_1(e_j) \ell^\omega(\pi(g_{ij}^A), T_2(e_i)^*)
\]

\[
+ \sum_{i,j=1}^{n_\alpha} T_1(e_i) \ell^\omega(\pi(g_{ij}^A), T_2(e_j)^*)
\]

\[
= D\omega \langle T_1, T_2 \rangle_L = d(T_1, T_2)_L.
\]

for all \( T_1, T_2 \in \Gamma(M, V^\alpha M) \). Another explicit calculation like the previous one shows

\[
\langle \nabla_\alpha^\omega(T_1), T_2 \rangle_R + \langle T_1, \nabla_\alpha^\omega(T_2) \rangle_R = d(T_1, T_2)_R.
\]

In the context of [12], Propositions 1, 2 are telling us that the functor qAss can be defined by adding the inner product that turns \( \alpha \) into a unitary representation and the canonical hermitian structure, at least for degree 0 morphisms.
Before concluding this section, it is important to mention that the definition of the canonical hermitian structure on associated left/right qvbs is based on the fact that in Differential Geometry, one can define a metric on associated vector bundles by using the inner product that turns the linear representation into a unitary one (invariant inner products). Even more, by taking a principal connection, the induced linear connection is compatible with this metric, and one can appreciate the non–commutative version of this result in Theorem 3.

The last result is the core of this paper since it will allow to prove the existence of formally adjoint operators for the covariant derivatives \( d\nabla_\omega \), \( d\hat{\nabla}_\omega \) for any quantum principal connection \( \omega \), without any restriction on it: it is not necessary to assume that \( \omega \) is real, or regular or even strong, which is a big difference with almost all literature of the subject. With the formally adjoint operators, it is possible to develop Field theory by using Laplacians just like in the classical case [16, 17, 18].

4. The Quantum Gauge Group and its Action on Quantum Connections

The other purpose of this paper is to present an ad hoc definition of the quantum gauge group for a given qpb and its action on the space of qpcs and the spaces of induced qlc. We will deal with all of these in this section. We are quite interested in the action of the quantum gauge group on qpcs because as the reader can check on [17], one of our final goals is to develop Yang–Mills theory in Durdevich’s framework and for that it is necessary classifying qpcs via gauge transformations.

Our definition of the quantum gauge group will be inspired by the one presented in [24] and for that we need to start defining the quantum translation map [24, 31, 32].

4.1. The Quantum Translation Map and The Quantum Gauge Group

Given a qpb, the surjective map \( \beta \) can be used to define the linear isomorphism

\[
\tilde{\beta} : GM \otimes_M GM \rightarrow GM \otimes G
\]

such that \( \tilde{\beta}(x \otimes_M y) = \beta(x \otimes y) = (x \otimes 1) \cdot_{GM} \Phi(y) \). The degree zero quantum translation map is defined as

\[
qtrs : G \rightarrow GM \otimes_M GM
\]
such that \( qtrs(g) = \tilde{\beta}^{-1}(1 \otimes g) \). Explicitly, by taking the linear basis \( \{g_{ij}^\alpha\}_{\alpha, i, j} \) (see Theorem 1 and Remark 1) we have

\[
qtrs(g_{ij}^\alpha) = \sum_{k=1}^{da} x_{ki}^\alpha \otimes_M x_{kj}^\alpha.
\]

Sometimes for comfort, we are going to use the Sweedler’s notation \( qtrs(g) = [g]_1 \otimes_M [g]_2 \). The map \( \tilde{\beta} \) has a natural extension to a graded linear isomorphism [31, 32]

\[
\tilde{\beta} : \Omega^\bullet(GM) \otimes_{\Omega^\bullet(M)} \Omega^\bullet(GM) \longrightarrow \Omega^\bullet(GM) \otimes \Gamma^\wedge.
\]  

(60)

given by

\[
\tilde{\beta}(w \otimes z) = (w \otimes 1) \cdot \Omega \Psi(z).
\]

On the other hand, by taking a qpc \( \omega \) (which always exists [5]), the degree zero quantum translation map can be extended to

\[
qtrs : \Gamma^\wedge \longrightarrow \Omega^\bullet(GM) \otimes_{\Omega^\bullet(M)} \Omega^\bullet(GM)
\]

(61)

and it can be proved that [31, 32]

\[
qtrs(\vartheta) = \tilde{\beta}^{-1}(1 \otimes \vartheta).
\]

The quantum translation map satisfies

1. \( qtrs \circ d = d_{\otimes^\bullet} \circ qtrs \), where \( d_{\otimes^\bullet} \) is the differential map of \( \Omega^\bullet(GM) \otimes_{\Omega^\bullet(M)} \Omega^\bullet(GM) \).
2. \( qtrs(\theta) = 1 \otimes_{\Omega^\bullet(M)} \omega(\theta) - (m_{\Omega} \otimes_{\Omega^\bullet(M)} \text{id}_{GM})(\omega \otimes qtrs)\text{ad}(\theta) \) for all \( \theta \in \text{inv} \Gamma \), where \( m_{\Omega} \) is the multiplication map of \( \Omega^\bullet(GM) \).
3. \( [\vartheta]_1[\vartheta]_2 = \epsilon(\vartheta) \) for all \( \vartheta \in \Gamma^\wedge \).
4. \( (\text{id}_{\Omega^\bullet(GM)} \otimes_{\Omega^\bullet(M)} \Omega \Psi) \circ qtrs = (qtrs \otimes \text{id}_{\Gamma^\wedge}) \circ \phi \).
5. \( (\Omega \Psi \otimes_{\Omega^\bullet(M)} \text{id}_{\Omega^\bullet(GM)}) \circ qtrs = (\sigma \otimes_{\Omega^\bullet(M)} \text{id}_{\Omega^\bullet(GM)}) \circ (\kappa \otimes qtrs) \circ \phi \), where

\[
\sigma : \Gamma^\wedge \otimes \Omega^\bullet(GM) \longrightarrow \Omega^\bullet(GM) \otimes \Gamma^\wedge
\]

is the canonical graded twist map, i.e., \( \sigma(\vartheta \otimes w) = (-1)^{kl} w \otimes \vartheta \) if \( w \in \Omega^k(M) \) and \( \vartheta \in \Gamma^\wedge \).
6. \( \mu qtrs(\vartheta) = (-1)^{k\mu} qtrs(\vartheta) \mu \) for all \( \mu \in \Omega^k(M), \vartheta \in \Gamma^\wedge \).
among other properties [31, 32]. It is worth mentioning that since qtrs agrees
with $\tilde{\beta}^{-1}(\mathbb{1} \otimes -)$, it does not depend on the choice of $\omega$.

Let

$$f_1, f_2 : \Gamma^\wedge \longrightarrow \Omega^\bullet(GM)$$

be two graded linear maps. The convolution product of $f_1$ with $f_2$ is defined by

$$f_1 \ast f_2 = m_\Omega \circ (f_1 \otimes f_2) \circ \phi : \Gamma^\wedge \longrightarrow \Omega^\bullet(GM),$$

where $m_\Omega$ is the product map. Now we will just consider graded maps $\hat{f}$ such that

$$\hat{f}(\mathbb{1}) = \mathbb{1}$$

and

$$(\hat{f} \otimes \text{id}_{\Gamma^\wedge}) \circ \text{Ad} = \Omega \Psi \circ \hat{f} \quad (62)$$

where $\text{Ad} : \Gamma^\wedge \longrightarrow \Gamma^\wedge \otimes \Gamma^\wedge$ is the extension of $\text{Ad} : G \longrightarrow G \otimes G$ (see Equation 10). We say that $\hat{f}$ is a convolution invertible map if there exists a graded linear map $\hat{f}^{-1} : \Gamma^\wedge \longrightarrow \Omega^\bullet(GM)$ such that

$$\hat{f} \ast \hat{f}^{-1} = \hat{f}^{-1} \ast \hat{f} = \mathbb{1}.$$ 

A direct calculation shows that the set of all convolution invertible maps is a group with respect to the convolution product.

**Proposition 3.** There exists a group isomorphism between the group of all graded left $\Omega^\bullet(M)$–module isomorphisms $\mathfrak{F} : \Omega^\bullet(GM) \longrightarrow \Omega^\bullet(GM)$ that satisfy $\mathfrak{F}(\mathbb{1}) = \mathbb{1}$ and such that the following equation holds

$$(\mathfrak{F} \otimes \text{id}_{\Gamma^\wedge}) \circ \Omega \Psi = \Omega \Psi \circ \mathfrak{F} \quad (63)$$

and the group of all convolution invertible maps. Here we have considered $(\mathfrak{F}_1 \circ \mathfrak{F}_2)(w) = \mathfrak{F}_2(\mathfrak{F}_1(w))$.

**Proof.** For a map $\mathfrak{F}$, consider

$$\hat{f}_\mathfrak{F} := m_{\Omega^\bullet} \circ (\text{id}_{\Omega^\bullet(GM)} \otimes \mathfrak{F}) \circ \text{qtrs} : \Gamma^\wedge \longrightarrow \Omega^\bullet(GM),$$

where $m_{\Omega^\bullet} : \Omega^\bullet(GM) \otimes \Omega^\bullet(M) \longrightarrow \Omega^\bullet(GM)$ is the multiplication map; and for $f$ define

$$\mathfrak{F}_f := m_{\Omega} \circ (\text{id}_{\Omega^\bullet(GM)} \otimes \hat{f}) \circ \Omega \Psi : \Omega^\bullet(GM) \longrightarrow \Omega^\bullet(GM).$$

Now the proof is completely analogous to the one presented in [24] but by considering $\Gamma^{\wedge \infty}$ the graded–differential $*$–Hopf algebra structure of $\Gamma^\wedge$ instead of $G^\infty$. 

24
Despite the following definition does not recreate the classical case, \textit{a priori} motivation to consider it is the fact that in Non–Commutative Geometry, we identify the set of sections of the associated bundle with equivariant maps. On \cite{11, 17, 18} one can find \textit{a posteriori} motivation of this definition based on the orbits of Yang–Mills qpcs.

**Definition 3.** (The quantum gauge group) Let $\zeta = (\mathcal{G}M, M, GM \Phi)$ be a quantum $\mathcal{G}$–bundle over $(M, \cdot, 1, *)$ with a differential calculus. We define the quantum gauge group $q\mathcal{G}\Phi$ as the group of all convolution invertible maps. Elements of $q\mathcal{G}\Phi$ are usually called \textit{quantum gauge transformations} (qgts).

Let $\zeta$ be a quantum principal $\mathcal{G}$–bundle with a differential calculus. The set of all characters of $\mathcal{G}$

$$G_{cl} := \{ \chi : \mathcal{G} \to \mathbb{C} \mid \chi \text{ is a character} \}$$

has a group structure with the multiplication $\chi_1 \ast \chi_2 := (\chi_1 \otimes \chi_2) \circ \phi$, the unity is $\epsilon$ and the inverse of a character $\chi$ is defined by $\chi^{-1} := \chi \circ \kappa$ \cite{7, 4}. In agreement with the Gelfand–Naimark theorem, this group can be interpreted as the group of all \textit{classical points} of $\mathcal{G}$ and it is isomorphic to a compact subgroup of $U(n)$ for some $n \in \mathbb{N}$ \cite{4, 7}. Every character $\chi$ can be extended to

$$\chi : \Gamma^\wedge \to \mathbb{C}$$

by means of $\chi|_G := \chi, \chi|_{\Gamma^\wedge k} := 0$ for $k \geq 1$. Consider

$$\mathfrak{F}_\chi := (\text{id}_{\Omega^*(GM)} \otimes \chi) \circ \Omega \Psi : \Omega^*(GM) \to \Omega^*(GM).$$

This map is a graded differential $\ast$–algebra isomorphism with inverse

$$\mathfrak{F}_\chi^{-1} := \mathfrak{F}_\chi^{-1}$$

because of ($\Omega \Psi$ is an action of $\Gamma^\wedge$)

$$\mathfrak{F}_\chi^{-1} \circ \mathfrak{F}_\chi = (\text{id}_{\Omega^*(GM)} \otimes \chi^{-1}) \circ \Omega \Psi \circ (\text{id}_{\Omega^*(GM)} \otimes \chi) \circ \Omega \Psi$$

$$= (\text{id}_{\Omega^*(GM)} \otimes \chi^{-1} \otimes \chi) \circ (\Omega \Psi \otimes \text{id}_{\Gamma^\wedge}) \circ \Omega \Psi$$

$$= (\text{id}_{\Omega^*(GM)} \otimes \chi^{-1} \otimes \chi) \circ (\text{id}_{\Omega^*(GM)} \otimes \phi) \circ \Omega \Psi$$

$$= (\text{id}_{\Omega^*(GM)} \otimes ((\chi^{-1} \otimes \chi) \circ \phi)) \circ \Omega \Psi = (\text{id}_{\Omega^*(GM)} \otimes \epsilon) \circ \Omega \Psi$$

$$= \text{id}_{\Omega^*(GM)}$$
and a similar calculation works to prove that \( \mathcal{F}_\chi \circ \mathcal{F}_\chi^{-1} = \text{id}_{\Omega^\bullet(GM)} \). Due to the fact that

\[
\mu \in \Omega^\bullet(M) \iff \omega\Psi(\mu) = \mu \otimes 1
\]

it is clear that \( \mathcal{F}_\chi|_{\Omega^\bullet(M)} = \text{id}_{\Omega^\bullet(M)} \). Finally a direct calculation as before proves that \( \mathcal{F}_\chi \) induces a qgt \( f_\chi \) by means of Proposition 3 if and only if

\[
(id_{\Gamma^\wedge} \otimes \chi) \circ \phi = (\chi \otimes id_{\Gamma^\wedge}) \circ \phi.
\]

In this way, if one considers the submonoid \( \hat{G}_{cl} \) of \( G_{cl} \) such that Equation 68 holds, then it is possible to define the monoid morphism.

\[
\Delta : \hat{G}_{cl} \longrightarrow \mathfrak{qG\mathfrak{S}}
\]

\[
\chi \mapsto f_\chi.
\]

If \( G \) is abelian in the sense of [7] (for example, for the cmqg associated to the Lie group \( U(1) \)), then \( \hat{G}_{cl} = G_{cl} \) and \( \Delta \) is a group morphism. This is the quantum counterpart of the classical fact that for a given principal \( G \)-bundle \( \pi : GM \longrightarrow M \) with \( G \) abelian, the diffeomorphism

\[
r_g : GM \longrightarrow GM
\]

\[
x \mapsto xg
\]

is a gauge transformation for all \( g \in G \).

4.2. Action on Quantum Connections

The main idea of gauge theory is to study classes of objects transformable one to another by gauge transformations and in particular, to study gauge-invariant objects. A condition is called gauge-invariant if it is satisfied by the whole gauge class. Probably one of the most important examples of gauge-equivalent objects arises when the gauge group acts on the set of principal connections by means of the pull-back, since this reverberates in an action on associated connections. The purpose of this subsection is to get the non-commutative geometrical counterpart of the gauge group’s action on connections.

The proof of the following theorem is straightforward and hence we will omit it.

**Theorem 4.** Let \( \zeta = (GM, M, GM\Phi) \) be a qpb over \((M, \cdot, 1, *)\) with a qpc \( \omega \) and a qgt \( f \). Then
1. The linear map $f \wr \omega$ defined by

$$f \wr \omega(\theta) := \mathfrak{F}_f(\omega(\theta))$$

is again a qpc, and this defines a right group action of the qgg in the set of all qpcs $\text{qpc}(\zeta)$ of $\zeta$.

2. If $\mathfrak{F}_f$ preserves the $\star$ operation, $f \wr \omega$ is real when $\omega$ is real.

3. If $\mathfrak{F}_f$ is a graded $\star$–algebra morphism, $f \wr \omega$ is regular when $\omega$ is regular.

4. If $\mathfrak{F}_f$ is a graded $\star$–algebra morphism, $f \wr \omega$ is multiplicative when $\omega$ is multiplicative.

It is worth mentioning that to define $\mathfrak{F}_f$ as a graded left $\Omega^\bullet(M)$–module isomorphism such that it satisfies Equation 63 and $\mathfrak{F}_f(1) = 1$, it is not necessary the quantum translation map. The last theorem provides us gauge–equivalence classes of quantum principal connections and the moduli space of quantum connections.

Like in the classical case, it is possible to find an explicit formula of the gauge action on connections and their curvatures.

**Proposition 4.** Given a qpc $\omega$, we get

$$f^\omega(\theta) = m_\Omega(\omega \otimes f)\text{ad}(\theta) + f(\theta)$$

for all $\theta \in \text{inv} \Gamma$. Even more, if $\mathfrak{F}_f$ is a graded differential $\star$–algebra morphism, then the curvature satisfies

$$\mathfrak{F}_f(R^\omega(\theta)) = R^{f^\omega}(\theta) = m_\Omega(R^\omega \otimes f)\text{ad}(\theta).$$

**Proof.** For all $\theta \in \text{inv} \Gamma$

$$f(\theta) = m_{\Omega^\bullet}(\text{id}_{\Omega^\bullet(GM)} \otimes_{\Omega^\bullet(M)} \mathfrak{F}_f) \text{qtrs}(\theta)$$

$$= \mathfrak{F}_f(\omega(\theta)) - \omega(\theta^{(0)}) [\theta^{(1)}]_1 \mathfrak{F}_f([\theta^{(1)}]_2) = \mathfrak{F}_f(\omega(\theta)) - \omega(\theta^{(0)}) f(\theta^{(1)}),$$

where $\text{ad}(\theta) = \theta^{(0)} \otimes \theta^{(1)}$. The last equality implies that

$$f^\omega(\theta) = \omega(\theta^{(0)}) f(\theta^{(1)}) + f(\theta) = m_\Omega(\omega \otimes f)\text{ad}(\theta) + f(\theta).$$

On the other hand assume that $\mathfrak{F}_f$ is a graded differential $\star$–algebra. If $\theta = \pi(g)$ such that $\delta(\theta) - \pi(g^{(1)}) \otimes \pi(g^{(2)})$ with $\phi(g) = g^{(1)} \otimes g^{(2)}$ ([25]), we have

$$\mathfrak{F}_f(R^\omega(\theta)) = \mathfrak{F}_f(d\omega(\theta)) - \mathfrak{F}_f(\langle \omega, \omega \rangle(\theta)) = df^\omega(\theta) - \langle f^\omega, f^\omega \rangle(\theta)$$

$$= R^{f^\omega}(\theta);$$
\[ \mathcal{F}_f \circ R^\omega = m_\Omega \circ (\text{id}_{\Omega^\ast(GM)} \otimes f) \circ \Omega \circ R^\omega = m_\Omega \circ (R^\omega \otimes f) \circ \text{ad}, \]

which completes the proof.

It is worth mentioning that the covariant derivative fulfills

\[ D^\phi \omega(\varphi) = d\varphi - (-1)^k \varphi^{(0)} \mathcal{F}_f(\omega(\varphi^{(1)})) \]  

(73)

where \( \varphi \in \text{Hor}^k GM \).

From this point and in the rest of this section, we shall assume that \( \mathcal{F}_f \) is a graded differential \(*\)-algebra morphism. This happens, for example, for qgtts induced by elements of \( \hat{G}_{cl} \).

### 4.3. On Induced Quantum Linear Connections

In Differential Geometry the gauge group acts on associated vector bundles via vector bundle isomorphisms. Specifically, given a principal \( G \)-bundle \( \pi : GM \longrightarrow M \) and an associated vector bundle \( \pi_\alpha : V^\alpha M \longrightarrow M \) with \( V^\alpha M := (GM \times V^\alpha)/G \) for the linear representation \( \alpha : G \longrightarrow GL(V^\alpha) \), a gauge transformation \( F \) induces a vector bundle isomorphism defined by \[ A_F : V^\alpha M \longrightarrow V^\alpha M \]

\[ [x,v] \mapsto [F(x),v]. \]

If \( \omega \) is a principal connection of \( \zeta \), then \( A_F \) is a parallel vector bundle isomorphism between \( (\zeta_\alpha, \nabla_\alpha^\omega) \) and \( (\zeta_\alpha, \nabla_\alpha^{F^\ast \omega}) \) and we have

\[ \nabla^{F^\ast \omega}_\alpha = (\text{id}_{\Omega^1(M)} \otimes \tilde{A}_F) \circ \nabla^\omega_\alpha \circ \tilde{A}^{-1}_F \]

(the tensor product is over \( C^\infty(M) \)), where here \( \tilde{A}_F \) denotes the isomorphism of sections induced by \( A_F \) [12]. The curvature follows a similar formula. These facts inspire the following theorem.

**Theorem 5.** Let \( (\zeta, \omega) \) be a qpb with a qpc. Then a quantum gauge transformation \( f \) defines a left \( M \)-module automorphism \( A_f \) of \( \zeta^L_\alpha \) such that

\[ (\text{id}_{\Omega^1(M)} \otimes_M A_f) \circ \nabla^\omega_\alpha = \nabla^{f^\ast \omega}_\alpha \circ A_f \]

for a fixed \( G \)-representation \( \alpha \). Furthermore we have \( (A_f \otimes_M \text{id}_{\Omega^1(M)}) \circ \sigma_\alpha = \sigma_\alpha \circ (\text{id}_{\Omega^1(M)} \otimes_M A_f) \) (see Equation 48).
Proof. It is enough to prove the theorem for $\alpha \in T$. Let us start noticing that by Equation 63, $D_{f}^{\varphi \omega} \circ \tilde{\mathcal{S}}_{f} = \tilde{\mathcal{S}}_{f} \circ D^{\omega}$ and also the map

$$A_{f} : \Gamma^{L}(M, V^{\alpha}M) \longrightarrow \Gamma^{L}(M, V^{\alpha}M) \quad T \longmapsto \tilde{\mathcal{S}}_{f} \circ T$$

(74)

is well-defined. Even more, it is a $M$–bimodule morphism, and its inverse is given by the composition with $\tilde{\mathcal{S}}_{f}^{-1}$. In this way, for all $T \in \Gamma^{L}(M, V^{\alpha}M)$

$$(\nabla_{\alpha}^{f \omega} \circ A_{f})(T) = \nabla_{\alpha}^{f \omega}(\tilde{\mathcal{S}}_{f} \circ T) = \sum_{k=1}^{d_{\alpha}} \mu_{D^{\phi \omega} \circ \tilde{\mathcal{S}}_{f} \circ T} \otimes_{M} T_{k}$$

$$= \sum_{k=1}^{d_{\alpha}} \mu_{\tilde{\mathcal{S}}_{f} \circ D^{\omega} \circ T} \otimes_{M} T_{k};$$

so $(\Upsilon_{\alpha}^{-1} \circ \nabla_{\alpha}^{f \omega} \circ A_{f})(T) = \tilde{\mathcal{S}}_{f} \circ D^{\omega} \circ T$. On the other hand

$$(\text{id}_{\Omega^{*}(M)} \otimes_{M} A_{f}) \circ \nabla_{\alpha}^{\omega}(T) = \sum_{k=1}^{d_{\alpha}} (\text{id}_{\Omega^{*}(M)} \otimes_{M} A_{f})(\mu_{D^{\omega} \circ T} \otimes_{M} T_{k})$$

$$= \sum_{k=1}^{d_{\alpha}} \mu_{D^{\omega} \circ T} \otimes_{M} A_{f}(T_{k}) = \sum_{k=1}^{d_{\alpha}} \mu_{D^{\omega} \circ T} \otimes_{M} \tilde{\mathcal{S}}_{f} \circ T_{k};$$

thus

$$(\Upsilon_{\alpha}^{-1} \circ (\text{id}_{\Omega^{*}(M)} \otimes_{M} A_{f}) \circ \nabla_{\alpha}^{\omega})(T) = \sum_{k=1}^{d_{\alpha}} \mu_{\tilde{\mathcal{S}}_{f} \circ D^{\omega} \circ T} \otimes_{M} T_{k} = \tilde{\mathcal{S}}_{f} \circ D^{\omega} \circ T.$$

By using the fact that $\Upsilon_{\alpha}^{-1}$ is bijective, we conclude that $A_{f}$ satisfies the first part of the statement.

Let us take $\psi \in \Omega^{*}(M) \otimes_{M} \Gamma(M, V^{\alpha}M)$. Then if $\Upsilon_{\alpha}^{-1}(\psi) = \sum_{k} T^{R}_{k} \tilde{\mu}_{k}$ we get

$$(A_{f} \otimes_{M} \text{id}_{\Omega^{*}(M)})\sigma_{\alpha}(\psi) = \sum_{k} A_{f}(T^{R}_{k}) \otimes_{M} \tilde{\mu}_{k} = \sum_{k} \tilde{\mathcal{S}}_{f} \circ T^{R}_{k} \otimes_{M} \tilde{\mu}_{k}$$

so $\tilde{\Upsilon}_{\alpha}^{-1}(A_{f} \otimes_{M} \text{id}_{\Omega^{*}(M)})\sigma_{\alpha}(\psi) = (\tilde{\mathcal{S}}_{f} \circ \Upsilon_{\alpha}^{-1})(\psi)$; while

$$\sigma_{\alpha}(\text{id}_{\Omega^{*}(M)} \otimes_{M} A_{f})(\psi) = \sum_{k} T^{R}_{k} \otimes_{M} \mu'_{k},$$

29
if \((\mathfrak{f}_f \circ \Upsilon^{-1})(\psi) = \sum_k T_k^R \mu'_k\) because of \((\Upsilon^{-1} \circ (\text{id}_{\Omega^*} \otimes_M A_f))(\psi) = (\mathfrak{f}_f \circ \Upsilon^{-1})(\psi)\). Hence \(\tilde{\Upsilon}^{-1} \sigma_{\alpha} (\text{id}_{\Omega^*} \otimes_M A_f)(\psi) = (\mathfrak{f}_f \circ \Upsilon^{-1})(\psi)\) and the theorem follows because \(\tilde{\Upsilon}^{-1}\) is bijective.

**Corollary 1.** The following formula holds:

\[
\nabla^{f \omega}_\alpha = (\text{id}_{\Omega^*} \otimes_M A_f) \circ \nabla^{\omega}_\alpha \circ A_f^{-1} = (\text{id}_{\Omega^*} \otimes_M A_f) \circ \nabla^{\omega}_\alpha \circ A_f^{-1}.
\]

**Proposition 5.** We have \(A_f \in U(\zeta^L_{\alpha})\).

**Proof.** As before, it is enough to prove the proposition for \(\alpha \in \mathcal{T}\). In this way by taking \(T_1, T_2 \in \Gamma^{L}(M, V^\alpha M)\)

\[
\langle A_f(T_1), T_2\rangle_L = \sum_{k=1}^{n_\alpha} \mathfrak{f}_f(T_1(e_k))T_2(e_k)^* = \sum_{k=1}^{n_\alpha} \mathfrak{f}_f(T_1(e_k)\mathfrak{f}_f^{-1}(T_2(e_k))^*)
\]

\[
= \sum_{k=1}^{n_\alpha} T_1(e_k)\mathfrak{f}_f^{-1}(T_2(e_k))^*
\]

\[
= \langle T_1, \mathfrak{f}_f^{-1} \circ T_2 \rangle_L
\]

\[
= \langle T_1, A_f^{-1}(T_2) \rangle_L
\]

where we have used that \(\sum_{k=1}^{n_\alpha} T_1(e_k)\mathfrak{f}_f^{-1}(T_2(e_k))^* \in M\). This allows us to conclude that \(A_f\) is adjointable with respect to \(\langle -, - \rangle_L\) and \(A_f^\dagger = A_f^{-1}\).

**Corollary 2.** Last corollary turns into \(\nabla^{f \omega}_\alpha = (\text{id}_{\Omega^*} \otimes_M A_f) \circ \nabla^{\omega}_\alpha \circ A_f^\dagger\).

A direct calculation shows

\[
R^{\nabla^{f \omega}_\alpha} = (\text{id}_{\Omega^*} \otimes_M A_f) \circ R^{\nabla^\alpha} \circ A_f^\dagger.
\] (75)

Of course, there are similar results for \((\zeta^R_{\alpha}, \nabla^\omega_{\alpha}), (\zeta^R_{\alpha}, \nabla^{f \omega}_\alpha)\) and

\[
\tilde{A}_f: \Gamma^R(M, V^\alpha M) \longrightarrow \Gamma^R(M, V^\alpha M)
\]

\[T \longmapsto \tilde{\mathfrak{f}}_f \circ T\] (76)

where \(\tilde{\mathfrak{f}}_f := * \circ \mathfrak{f}_f \circ *\).

30
Remark 3. Notice that in order to define $A_f$ and $\hat{A}_f$, it is not necessary that $\mathfrak{g}_f$ be a graded differential $\ast$–algebra morphism, our definition works for any qgt $\mathfrak{g}$; so this induces a natural right action of $q\mathfrak{G} \mathfrak{S}$ on $\Gamma^L(M, V^\alpha M)$ and $\Gamma^R(M, V^\alpha M)$.

5. Examples

This section is dedicated to present three examples in order to illustrate the theory developed in this paper.

5.1. Trivial Quantum Principal Bundles

Mirroring the classical case, trivial quantum principal bundles are perhaps the first examples that one has in mind. We will consider the theory of trivial qpbs developed in [5] to show our example. It is worth mentioning that this kind of bundles are also trivial in the sense of [19].

In [5] there is a characterization of all real qpcs that we can extend to any qpc (not necessary real) by linear maps

$$A^\omega : \text{inv}\Gamma \rightarrow \Omega^1(M),$$

which can be interpreted as the non–commutative gauge potentials. In this way, every qpc $\omega$ is of the form

$$\omega = (A^\omega \otimes \text{id}_G) \circ \text{ad} + \omega^{\text{triv}},$$

where

$$\omega^{\text{triv}} : \text{inv}\Gamma \rightarrow \Omega^1(M \otimes G)$$

$$\theta \rightarrow 1 \otimes \theta$$

receives the name of trivial qpc. Of course, this characterization extends to the curvature by using the non–commutative field strength $F^\omega$ [5]

$$R^\omega = (F^\omega \otimes \text{id}_G) \circ \text{ad}, \quad F^\omega = dA^\omega - \langle A^\omega, A^\omega \rangle : \text{inv}\Gamma \rightarrow \Omega^2(M).$$

Proposition 6. Let $\zeta^{\text{triv}}$ be a trivial quantum principal $\mathcal{G}$–bundle with the trivial differential calculus. Given $\mathcal{T}$ a complete set of mutually non–equivalent irreducible unitary $\mathcal{G}$–representations, we have

1. If $\alpha \in \mathcal{T}$ acts on a $\mathbb{C}$–vector space of dimension $n_\alpha$, then there exists a left–right $M$–basis $\{T^\alpha_k\}_{k=1}^{n_\alpha} \subseteq \text{Mor}(\alpha, GM \Phi)$ such that Equation 34 holds. In particular, associated left/right qvbs always exists for any $\alpha \in \text{Obj}(\text{Rep}_\mathcal{G})$ and they are trivial qvbs in the sense of [12].

31
2. For the trivial qpc (which is real, regular, multiplicative and flat [5])
\[ D^{\omega_{\text{triv}}} \circ T_k^\alpha = 0, \quad \hat{D}^{\omega_{\text{triv}}} \circ T_k^\alpha = 0 \]
for all \( k \).

3. The induced qlc for \( \omega_{\text{triv}} \) and \( \alpha \in \mathcal{T} \) can be expressed by
\[
\nabla_{\omega_{\text{triv}}}^\alpha (T) = \sum_{k=1}^{n_\alpha} d_p T_k^\alpha \otimes_M T_k^\alpha, \quad \hat{\nabla}_{\omega_{\text{triv}}}^\alpha (T) = \sum_{k=1}^{n_\alpha} T_k^\alpha \otimes_M d_p T_k^\alpha
\]
(see Equation 41). In particular, it is a trivial qlc in the sense of [12].

4. The exterior covariant derivatives are given by
\[
d^{\nabla_{\omega_{\text{triv}}}^\alpha} (\mu \otimes_M T) = \sum_{k=1}^{n_\alpha} d(\mu p_k^T) \otimes_M T_k^\alpha,
\]
and
\[
d^{\hat{\nabla}_{\omega_{\text{triv}}}^\alpha} (T \otimes_M \mu) = \sum_{k=1}^{n_\alpha} T_k^\alpha \otimes_M d(\mu p_k^T)
\]
for all \( \mu \in \Omega^* (M) \). In particular \( \nabla_{\omega_{\text{triv}}}^\alpha \) and \( \hat{\nabla}_{\omega_{\text{triv}}}^\alpha \) are flat.

5. On the context of Remark 2 and the point 1 we have \( g^\alpha (1) = 1 \otimes \text{Id}_{n_\alpha} \).

**Proof.** 1. Consider \( G^\alpha = (g_{ij}^\alpha) \in M_{n_\alpha} (G) \). Then the linear maps
\[ T_k^\alpha : V^\alpha \longrightarrow M \otimes G \]
defined by \( T_k^\alpha (e_i) = 1 \otimes g_{ki}^\alpha \), form a left–right \( M \)-basis of \( \text{Mor}(\alpha, GM \Phi) \) that satisfy Equation 34 since according to [7] \( G^\alpha G^\alpha \dagger = G^\alpha \dagger G^\alpha = \text{Id}_{n_\alpha} \).

2. Because of \( dg = g^{(1)} \pi (g^{(2)}) \) for all \( g \in G \) we get
\[
D^{\omega_{\text{triv}}} (T_k^\alpha (e_i)) = D^{\omega_{\text{triv}}} (1 \otimes g_{ki}^\alpha) = 1 \otimes dg_{ki}^\alpha - (1 \otimes g_{ki}^{\alpha (1)}) \omega_{\text{triv}} (\pi (g_{ki}^{\alpha (2)}))
= 1 \otimes dg_{ki}^\alpha - 1 \otimes g_{ki}^{\alpha (1)} \pi (g_{ki}^{\alpha (2)}) = 0
\]
This shows that \( D^{\omega_{\text{triv}}} \circ T_k^\alpha = 0 \) for \( k = 1, \ldots, n_\alpha \). A similar calculation proves \( \hat{D}^{\omega_{\text{triv}}} \circ T_k^\alpha = 0 \).

3. In this case for all \( T \in \text{Mor}(\alpha, GM \Phi) \) we get \( T = \sum_{k=1}^{n_\alpha} p_k^T T_k^\alpha = \sum_{k=1}^{n_\alpha} T_k^\alpha p_k^T \).

The previous point together with Equation 32 show that \( D^{\omega_{\text{triv}}} \circ T = \sum_{k=1}^{n_\alpha} dp_k^T T_k^\alpha, \hat{D}^{\omega_{\text{triv}}} \circ T = \sum_{k=1}^{n_\alpha} T_k^\alpha dp_k^T \) and hence Equation 79 holds.
4. It follows from the previous point.
5. It follows from the definition of $g^\alpha(1)$.

Point 3 of the last proposition characterizes the left/right quasibundle (qlc) associated to the trivial qpc and $\alpha \in T$; while point 5 shows explicitly that the canonical hermitian structure (induced by the basis of point 1) is always non-singular, in accordance with the theory showed on [29].

Let us fix an example using quantum line bundles. In this case we will consider that the *-FODC on (the quantum group associated to) $U(1)$ is given by the right ideal $\text{Ker}^2(\epsilon)$ and hence the universal differential envelope *-calculus $(\Gamma^\wedge, d, *)$ is the classical algebra of differential forms of $U(1)$. Moreover, let us consider $(\Omega^*(M), d, *)$ as the graded differential *-algebra used on [16] (the Chevalley–Eilenberg complex for square matrices of order 2 with complex coefficients). In an abuse of notation, we shall identify $U(1)$ with the Laurent polynomial algebra.

For $U(1)$ it is well-known that a complete set of mutually inequivalent irreducible unitary (co)representation $\tau$ is in bijection with $\mathbb{Z}$. Let $n \in \mathbb{Z}$. In this way the left–right $M$–basis of the last proposition is given by

$$T^n : \mathbb{C} \longrightarrow M \otimes U(1)$$

$$w \longmapsto w \text{Id}_2 \otimes z^n$$

and hence every $T \in \text{Mor}(n, GM\Phi)$ is of the form $T = p^T T^n = T^n p^T$, where $p^T = T(1)(\text{Id}_2 \otimes z^{*n})$ (see Equations 41, 43). For any qpc $\omega$ (see Equation 77) we have

$$\nabla^n_\alpha(T) = (dp - np\mu) \otimes_M T^n, \quad  \widehat{\nabla}^n_\alpha(T) = T^n \otimes_M (dp + n\mu^*p),$$

where $T(1) = p \otimes z^n$ and $A^\omega(\pi(z)) = \mu \in \Omega^1(M)$.

On the other hand, the left and right canonical hermitian structures are given by

$$\langle T_1, T_2 \rangle_L = T_1(1)T_2(1)^* = p_1 p_2^*, \quad \langle T_1, T_2 \rangle_R = T_1(1)^* T_2(1) = p_1^* p_2,$$

where $T_i(1) = p_i \otimes z^n$ for $i = 1, 2$; which we know is non-singular.

For an explicit example of Theorem 3 let us take a real qpc $\omega$. This
implies that $\mu^* = -\mu$ and then
\[
\langle \nabla_\alpha^\omega(T_1), T_2 \rangle_L + \langle T_1, \nabla_\alpha^\omega(T_2) \rangle_L = \langle (dp_1 - np_1\mu) \otimes_M T^n, T_2 \rangle_L 
+ \langle T_1, (dp_2 - np_2\mu) \otimes_M T^n \rangle_L 
= dp_1p_2^* - np_1\mu p_2^* + p_1dp_2^* - np_1^*p_2^* 
= dp_1p_2^* + p_1dp_2^* = d\langle T_1, T_2 \rangle_L;
\]
while
\[
\langle \tilde{\nabla}_\alpha^\omega(T_1), T_2 \rangle_R + \langle T_1, \tilde{\nabla}_\alpha^\omega(T_2) \rangle_R 
= \langle T^n \otimes_M (dp_1 + np_1^*p_1), T_2 \rangle_R 
+ \langle T_1, T^n \otimes_M (dp_2 + np_2^*p_2) \rangle_R 
= dp_1^*p_2 + np_1^*p_2 + p_1^*dp_2 + np_1^*p_2 
= dp_1^*p_2 + p_1^*dp_2 = d\langle T_1, T_2 \rangle_R,
\]
where $T_1$ and $T_2$ are like in the previous paragraph.

Let us return to the general case. For degree zero the quantum translation
map is given by
\[
\text{qtrs}(g_{ij}) = \sum_{k=1}^n (1 \otimes g_{ki}^\alpha) \otimes_M (1 \otimes g_{kj}^\alpha),
\]
and by considering the trivial qpc we get that for all $\theta \in \text{inv} \Gamma$
\[
\text{qtrs}(\theta) = 1 \otimes_{\Omega^\bullet(M)} (1 \otimes \theta) - (1 \otimes \theta^{(0)}) \text{qtrs}(\theta^{(1)}),
\]
where $\text{ad}(\theta) = \theta^{(0)} \otimes \theta^{(1)}$.

Now let us return to quantum line bundles with the same $\ast$–FODC on $U(1)$. If $\mathfrak{f}$ is a graded differential $\ast$–algebra (like in the classical case), then $\text{Im}(\mathfrak{f})$ is graded commutative. Since $z$ is a unitary element and $\Gamma^{\wedge k} = \{0\}$ for $k \geq 2$, it follows that $\mathfrak{f}$ has to satisfy
\[
\mathfrak{f}(z) = p \quad \text{where} \quad pp^* = p^*p = 1 \quad \text{and} \quad p^*dp \in \text{Ker}(d).
\]
The action of this kind of qgts on qpcs is determined by (see Equation 77)
\[
\mathfrak{f}^\pi \omega^{\text{triv}}(\zeta) = A^\omega(\zeta) \otimes 1 + 1 \otimes \zeta \quad \text{with} \quad A^\omega(\zeta) = p^*dp,
\]
where $\pi(z) = \zeta$. At least, we can guarantee that qgts given by $\tilde{G}_{cl}$, which
in this case are of the form $f_x = e^{ix}1$ with $x \in \mathbb{R}$, are graded differential
$\ast$–algebra morphisms. The existence of more graded differential $\ast$–algebra morphisms that also are qgts depends on the explicit form of $(\Omega^\ast(M), d, \ast)$. For example if $(\Omega^\ast(M), d, \ast)$ is the graded differential $\ast$–algebra used on [16], then there are no more qgts of this type.

When $(\Omega^\ast(M), d, \ast)$ comes from a smooth (compact) manifold, then we get all the classical gauge transformations; however, there are qgts that do not give rise to smooth maps and hence they are useless in the framework of Differential Geometry.

It is worth mentioning that on [18] we used the trivial quantum principal bundle over the two–point space with $S_2$ as structure group, as an example of this paper and [16].

5.2. The Quantum Hopf Fibration

In Differential Geometry, the Hopf fibration is one of the most basic and well–established examples of principal bundles. We will use the quantum version of this bundle to illustrate our theory.

Let us consider the quantum bundle and its differential calculus showed on [17]. Like the reader can verify on [17], this quantum bundle satisfy the assumptions of the Remark 1, where the elements $\{x^\alpha_{k+1}\}_{k=0}^n$ correspond to the first column of the representation matrix for spin $l = n/2$ and since this matrix is hermitian we explicitly have that $\rho^\alpha(1) = \rho^\alpha(1)^\dagger$ (see Remark 2) and hence the canonical hermitian structure is non–singular [29].

Since the quantum Hopf fibration has $U(1)$ as structure group, we can consider $Z$ as $T$ again. Now let us take the canonical qpc $\omega^c$ (the quantum version of the principal connection on the Hopf fibration associated to the Levi–Civittâ connection) [17] and we can compute its induced qlc. For example, taking $n \in \mathbb{N}$, for $T \in \text{Mor}(n, G_M \Phi)$ such that $T(1) = \alpha^n$, we have

$$\nabla^c_n(T) = -\frac{(1 - q^{2n})q^{3-2n}}{1 - q^2} \sum_{k=0}^{n} \alpha^{n-1} \gamma^* \eta_+ x_{k+11}^{n+1} \otimes_{S_2} T^n_{k+11},$$

$$\hat{\nabla}^c_n(T) = -\frac{(1 - q^{2n})q^{3-2n}}{1 - q^2} \sum_{k=0}^{n} q^{2k} T^n_{k+11} \otimes_{S_2} x_{k+11}^{n+1} \alpha^{n-1} \gamma^* \eta_+.$$

The left and right canonical hermitian structure are given by

$$\langle T_1, T_2 \rangle_L = T_1(1)T_2(1)^*, \quad \langle T_1, T_2 \rangle_R = T_1(1)^* T_2(1)$$
and for example, $\langle T, T \rangle_L = \alpha^n \alpha^* n$ and $\langle T, T \rangle_R = \alpha^* n \alpha^n$ for $T(1) = \alpha^n$.

Moreover, by using Equation 34 and some basic relations of the bundle and its calculus we have

$$
\langle \nabla^\omega^c (T), T \rangle_L = \frac{(1 - q^{2n}) q^{3 - 2n}}{1 - q^2} \sum_{k=0}^{n} \alpha^{n-1} \gamma^* \eta_+ x_{k+1}^{n+1} \langle T_{k+11}^n, T \rangle_L
$$

$$
= \frac{(1 - q^{2n}) q^{3 - 2n}}{1 - q^2} \sum_{k=0}^{n} \alpha^{n-1} \gamma^* \eta_+ x_{k+1}^{n+1} \alpha^n
$$

$$
= \frac{(1 - q^{2n}) q^{3 - 2n}}{1 - q^2} \alpha^{n-1} \gamma^* \gamma^* \eta_+
$$

and

$$
\langle T, \nabla^\omega^c (T) \rangle_L = \frac{(1 - q^{2n}) q^{3 - 2n}}{1 - q^2} \sum_{k=0}^{n} \langle T, T_{k+11}^n \rangle_L x_{k+1}^{n+1} (\alpha^{n-1} \gamma^* \eta_+)^*
$$

$$
= \frac{(1 - q^{2n}) q^{3 - 2n}}{1 - q^2} \sum_{k=0}^{n} \alpha^n x_{k+1}^{n+1} x_{k+1}^{n+1} (\alpha^{n-1} \gamma^* \eta_+)^*
$$

$$
= \frac{(1 - q^{2n}) q}{1 - q^2} \alpha^n \alpha^{n-1} \gamma^* \gamma^* \eta_+;
$$

which shows that

$$
\langle \nabla^\omega^c (T), T \rangle_L + \langle T, \nabla^\omega^c (T) \rangle_L = d(\alpha^n \alpha^* n) = d(T, T)_L.
$$

In the same way, for an explicitly calculation of the other part of Theorem 3 we get

$$
\langle \nabla^\omega^c (T), T \rangle_R = \frac{(1 - q^{2n}) q^{3 - 2n}}{1 - q^2} \sum_{k=0}^{n} q^{2k} (\alpha^{n-1} \gamma^* \eta_+)^* x_{k+1}^{n+1} \langle T_{k+11}^n, T \rangle_R
$$

$$
= \frac{(1 - q^{2n}) q^{3 - 2n}}{1 - q^2} \sum_{k=0}^{n} q^{2k} (\alpha^{n-1} \gamma^* \eta_+)^* x_{k+1}^{n+1} x_{k+1}^{n+1} \alpha^n
$$

$$
= \frac{(1 - q^{2n}) q^{1 - 2n}}{1 - q^2} \alpha^{n-1} \alpha^n \gamma^* \gamma^* \eta_-;
$$
and
\[
\langle T, \hat{\nabla}_\alpha^\omega (T) \rangle_R = \frac{1 - q^{2n}}{1 - q^2} q^{2n-2n} \sum_{k=0}^{n} q^k \langle T, T_{k+1}^n \rangle_R \alpha^{n-1} \gamma^{*} \eta_+ \\
= \frac{1 - q^{2n}}{1 - q^2} q^{2n-2n} \sum_{k=0}^{n} q^k \alpha^{*} \gamma \otimes \alpha^{n-k} \gamma^k \\
= \frac{1 - q^{2n}}{1 - q^2} \alpha^{*} \gamma \otimes \alpha^{n-1} \gamma^{*} \eta_+ ,
\]
where we have used the Equation 35 for this bundle [17] as well as some basic relations of the bundle and its calculus; hence
\[
\langle \hat{\nabla}_\alpha^\omega (T), T \rangle_R + \langle T, \hat{\nabla}_\alpha^\omega (T) \rangle_R = d(\alpha^{*} \gamma) = d\langle T, T \rangle_R.
\]

For degree zero elements we get
\[
\text{qtrs}(z^n) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] q^{-2k} a^{*} \gamma^{k} \gamma^{n-k} \otimes \gamma^k 
\]
and for \( \theta \in \text{inv} \Gamma \), \text{qtrs} is like on Equation 81.

On the other hand, let \( f \) be a qgt. Then Equation 62 implies \( \text{Im}(f) \subseteq \Omega^*(S^2_q) \) since \( \text{Ad}(\vartheta) = \vartheta \otimes 1 \) for all \( \vartheta \in \Gamma^\wedge \). In the way that we have defined the qgg, in general, it is very large and it could be quite challenging to calculate its explicit form. For example, for any \( \mu \in \Omega^1(S^2_q) \) there exists a qgt \( f \) such that \( f(\varsigma) = \mu \). Indeed, taking \( f : \Gamma^\wedge \rightarrow \Omega^*(S^2_q) \) such that \( f|_{U(1)} = \epsilon \) and \( f(g\varsigma) = \epsilon(g)\mu \) we get that \( f \in \text{qGG} \).

Let \( \omega \) be another qpc. Then according to [5, 25]
\[
\omega = \omega^c + \lambda,
\]
where \( \lambda : \text{inv} \Gamma \rightarrow \Omega^1(SU_q(2)) \) is a connection displacement (see Equation 22) and \( \lambda \) has to satisfy \( \lambda(\varsigma) \in \Omega^1(S^2_q) \) since \( \text{ad}(\theta) = \theta \otimes 1 \) for all \( \theta \in \text{inv} \Gamma \). Thus by the previous result we can conclude that the action of the qgg on the spaces of qpcs is transitive (see Equation 71).

Due to \( \text{qGG} \) is in general quite large, in [16, 17, 18] we work with some interesting subgroups: subgroups that let invariant the respective Lagrangian. For example, if we denote by \( \text{qGG}_{\text{YM}} \) the subgroup of all qgts that let invariant the Noncommutative geometrical Yang–Mills Lagrangian and since qgg acts transitively on qpcs, we concluded that for the quantum Hopf fibration
\[
\text{qGG}_{\text{YM}} := \{ f \in \text{qGG} | f^\theta \omega = \omega^c + \lambda \text{ with } d\lambda(\varsigma) = 0 \}
\]
and up to qgt, $\omega^c$ is the only one Yang–Mills qpc, just like in the classical case [17].

5.3. Quantum Principal Bundles and Dunkl Operators

For our final example we will use a no–common quantum bundle: the quantum principal $G$–bundle showed on [25, 36]. We shall denoted it by

$$\zeta = (GM, M_{GM}\Phi).$$

This qpb is defined by the dualization of a classical principal bundle with a (finite) Coxeter group $W(G)$ as the structure group, which we shall denoted it by

$$\rho : P \to P/W(G),$$

where $P$ is the classical total space, $P/W(G)$ is the classical base space and the map $\rho : P \to P/W(G)$ is the canonical projection (which is the bundle projection).

On the other hand, horizontal forms on $\zeta$ are given by the complexification of the de–Rham graded differential $\star$–algebra on $P$, quantum differential forms on $M$ are given by the complexification of the de–Rham graded differential $\star$–algebra on $P/W(G)$ and $(\Gamma, d)$ is given by the theory of $\star$–FODCs on finite groups [36]. It is worth mentioning that in this quantum space exists a canonical qpc $\omega^c$, which has the form of the one presented on Equation 78 and its covariant derivative is exactly the de-Rham differential.

The attractive point about this quantum/classical hybrid bundle $\zeta$ is the existence of qpcs $\omega$ called Dunkl connections, whose covariant derivatives are Dunkl Operators!. It is worth emphasizing that Dunkl operators were introducing in 1989 in the framework of Harmonic Analysis [37], in principle, there is not a relation between these operators and Non–Commutative Geometry and of course, they cannot be covariant derivatives in the classical sense [25].

**Proposition 7.** The quantum bundle $\zeta$ satisfies all conditions written on Remark 1.

**Proof.** Let us take a complete set of mutually inequivalent irreducible unitary (co)representation $\tau$. An element of $\tau$ shall treated indistinctly as a (co)representation of $G$ and as a representation of $W(G)$, depending on the context. Let $\alpha \in \tau$. If $\{e_i\}_{i=1}^{n_\alpha}$ is the corresponding orthonormal basis of $V^\alpha$, then

$$\alpha(e_j) = \sum_{k,h} e_k \otimes \lambda_{kj}^h \Delta_h,$$
where \( k \) runs from 1 to \( n_\alpha \), \( h \) runs on \( W(G) \), \( \Delta_h \) is the characteristic function of \( h \) and we are considering \( h \cdot e_j = \sum_{k=1}^{n_\alpha} \lambda^h_{kj} e_k \). For each fiber of \( P \), fix an element \( p \). Thus let us consider
\[
 f_{ij}^\alpha : P \longrightarrow \mathbb{C} \\
 x \longrightarrow \lambda^h_{ij},
\]
where \( h \in W(G) \) is such that \( ph = x \). Then a direct calculation shows that the maps
\[
 T^\alpha_i : V^\alpha \longrightarrow GM
\]
such that \( T^\alpha_i(e_j) = f_{ij}^\alpha \) are elements of \( \text{Mor}(\alpha, GM\Phi) \) that satisfy the conditions written on Remark 1, considering \( C^\alpha = Z^\alpha = \text{Id}_{n_\alpha} \).

It follows from the last result that \( \varrho^\alpha(\mathbb{1}) = \left( \sum_{i=1}^{n_\alpha} f_{ki}^\alpha f_{li}^{\alpha*} \right)_{kl} \) (see Remark 2) is an hermitian matrix because the matrix \( (\lambda^h_{ij})_{ij} \) is always hermitian for all \( h \in W(G) \) and hence, we explicitly have that the canonical hermitian structure is non–singular [29].

In order to show a concrete example let us focus our study on standard Dunkl connections using the development showed on [36]. It is worth mentioning that since we have change the standard definition of qpcs in order to embrace a more general theory (see Equation 20), to us, Dunkl displacements \( \lambda : \text{inv} \Gamma \longrightarrow \Omega^1(GM) \) do not need the \( i = \sqrt{-1} \) factor. These qpcs are given by (see Equation 22)
\[
 \omega = \omega^c + \lambda,
\]
where \( \lambda \) satisfies \( \lambda(\pi(\Delta_\sigma)) = \varrho_\sigma r^\# \),
\[
 \varrho_\sigma : P \longrightarrow \mathbb{R} \\
 x \longrightarrow \frac{\kappa(r)}{\langle r|x \rangle},
\]
\( \kappa : R \longrightarrow \mathbb{R} \) is a multiplicative function (it is \( W(G) \)-invariant), \( r \in R \) with \( R \) the root system and \( r^\# \) is the element of the dual space associated to \( r \). Covariant derivatives of these kind of qpcs are given by
\[
 (D^\omega f)(x) = df(x) + \sum_{r \in R^+} \kappa(r) \frac{f(x) - f(x\sigma_r)}{\langle r|x \rangle} r
\]
39
for every \( f \in GM \); which is a Dunkl operator in vector form [25].

Let \( \alpha \in T \). Then

\[
\nabla^\omega_{\alpha} T = \sum_{i=1}^{n_\alpha} \mu_i^{D^\omega} \otimes_M T_\alpha^i, \quad \nabla^\omega_{\alpha} T = \sum_{i=1}^{n_\alpha} T_\alpha^i \otimes_M \mu_i^{D^\omega}
\]

where \( \mu_i^{D^\omega} = \sum_{i=1}^{n_\alpha} D^\omega T(e_i) f_{ki}^\alpha \) (see Equations 44, 46) and \( T(e_k) \in GM \).

Notice that Dunkl connections are multiplicative, but not regular [25]. Furthermore, it is worth mentioning that in the classical sense, there are not principal connections on this bundle.

On the other hand, the left and right canonical hermitian structures are given by

\[
\langle T_1, T_2 \rangle_L = \sum_{k=1}^{n_\alpha} T_1(e_k) T_2(e_k)^* , \quad \langle T_1, T_2 \rangle_R = \sum_{k=1}^{n_\alpha} T_1(e_k)^* T_2(e_k) .
\]

For an explicit example of Theorem 3 let us take a real Dunkl connection. These connections are characterized by \( \omega = \omega^c + \tilde{\lambda} \), with \( \tilde{\lambda} = i\lambda \). Therefore by using Equation 34

\[
\langle \nabla^\omega_{\alpha}(T_1), T_2 \rangle_L = \sum_{i=1}^{n_\alpha} \mu_i^{D^\omega} T_1^\alpha \langle T_1^\alpha, T_2 \rangle_L = \sum_{k=1}^{n_\alpha} \langle D^\omega T_1(e_k) \rangle T_2(e_k)^*
\]

and evaluating on any \( x \in P \) we have

\[
\sum_{k=1}^{n_\alpha} \left( dT_1(e_k)(x) T_2(e_k)^*(x) + i \sum_{r \in R^+} \kappa(r) T_1(e_k)(x) - T_1(e_k)(x \sigma_r) \right) \langle r|x \rangle r T_2(e_k)^*(x) \right) ;
\]

and

\[
\langle T_1, \nabla^\omega_{\alpha}(T_2) \rangle_L = \sum_{i=1}^{n_\alpha} \langle T_1, T_1^\alpha \rangle_L \mu_i^{D^\omega} T_2^\alpha \langle T_2^\alpha, T_2 \rangle_L = \sum_{k=1}^{n_\alpha} T_1(e_k) (D^\omega T_2(e_k))^*
\]

and evaluating on any \( x \in P \) we get

\[
\sum_{k=1}^{n_\alpha} \left( T_1(e_k)(x) dT_2(e_k)^*(x) - i T_1(e_k)(x) \sum_{r \in R^+} \kappa(r) T_2(e_k)^*(x) - T_2(e_k)^*(x \sigma_r) \right) \langle r|x \rangle r .
\]
This implies for all \( x \in P \) that \( \langle \nabla_\alpha(T_1), T_2 \rangle_L + \langle T_1, \nabla_\alpha(T_2) \rangle_L \) is equal to
\[
d\langle T_1, T_2 \rangle_L(x) + i \sum_{r \in R^+} \kappa(r) \frac{q(x) - q(x\sigma_r)}{\langle r|x \rangle} r,
\]
where \( q = \sum_{k=1}^{n_\alpha} T_1(e_k)(\sigma_r T_2(e_k))^* \in GM \). However since \( T_1, T_2 \in \text{MOR}(\alpha, GM\Phi) \), we have \( \sigma_r T_i(e_j) = \sum_{k=1}^{n_\alpha} \lambda_{kj}^\sigma T_i(e_k) \); thus \( q(x) = q(x\sigma_r) \) for all \( x \in P \). In summary we have proven explicitly that
\[
\langle \nabla_\alpha(T_1), T_2 \rangle_L + \langle T_1, \nabla_\alpha(T_2) \rangle_L = d\langle T_1, T_2 \rangle_L.
\]
A very similar argumentation proves that
\[
\langle \hat{\nabla}_\alpha(T_1), T_2 \rangle_R + \langle T_1, \hat{\nabla}_\alpha(T_2) \rangle_R = d\langle T_1, T_2 \rangle_R.
\]
Defining \( g^\alpha_{ij}: W(G) \longrightarrow \mathbb{C} \) by \( g^\alpha_{ij}(h) = \lambda^h_{ij} \) we get
\[
\text{qtrs}(g^\alpha_{ij}) = \sum_{k=1}^{n_\alpha} f^\alpha_{ki} \otimes_M f^\alpha_{kj},
\]
and by considering the canonical qpc we have that for all \( \theta \in \text{inv}_\Gamma \)
\[
\text{qtrs}(\theta) = 1 \otimes \Omega^*(M) (1 \otimes \theta) - (1 \otimes \theta^{(0)}) \text{qtrs}(\theta^{(1)}), \tag{81}
\]
where \( \text{ad}(\theta) = \theta^{(0)} \otimes \theta^{(1)} \).

Now let us consider a graded left \( \Omega^*(M) \)-module isomorphism
\[
\mathfrak{F}: \Omega^*(M) \longrightarrow \Omega^*(M)
\]
such that for irreducible degree 1 elements is given by
\[
\mathfrak{F}(\mu + x \otimes \theta) = \mu + x\lambda(\theta) + x \otimes \theta,
\]
where \( \lambda \) is a Dunkl displacement, \( \mu \in \text{Hor}^1 GM, x \in GM \) and \( \theta \in \text{inv}_\Gamma \). This map satisfies Equation 63 and by Proposition 3 it induces a qgt \( \mathfrak{f} \). Finally, a direct calculation shows that \( \hat{\mathfrak{f}}^\alpha \omega^c \) is a standard Dunkl connections. By using the form of the last map \( \mathfrak{F} \) every single classical gauge transformation can be extended into a quantum one.

Just like in our previous example, the qgg is quite large and acts transitively on the space of qpcs. However, for the purpose of this line of research, it is not necessary to work with the whole qgg, just with special subgroups, like \( \text{qG} \mathfrak{S}_\text{YM} \) [16].
6. Concluding Comments

This paper extends the work presented in [12] in the sense that we use general qpcs, not just the real and regular ones, and we have defined other geometrical structures.

First of all, we would like to highlight the importance of the universal differential envelope $\ast$–calculus $(\Gamma^\wedge, d, \ast)$ as quantum differential forms on $\mathcal{G}$. This space is one of the principal differences with the theory of [19], and it does not only allows to extend the $\ast$–Hopf algebra structure of $\mathcal{G}_\infty$ to $\Gamma^\wedge\infty$, but it is maximal with this property [4]. Moreover, this space allowed us to define the quantum translation map at the level of differential calculus and with that, the qgg.

Now let us talk a little more about Remark 1. These conditions are not so restrictive as it seems at first sight, especially the first one since according to [6], the set of left generators $\{T^L_k \}_{k=1}^d$ always exists if $M$ is (a subalgebra of a $C^*$–algebra) stable under holomorphic functional calculus.

The fact that $GM$ is a left/right principal Hopf–Galois extension (under the assumptions of Remark 1) guarantees that $\text{Mor}(\alpha, GM\Phi)$ is a finitely generated projective left/right $M$–module. However, Equations 34, 35 not only allow to prove explicitly this, for example, by using the map $H$ (see Equation 39), but also allow to define the canonical hermitian structures and showing that they are non–singular (see Remark 2); not to mention that thank to these equations we can realize other explicit calculations, for example on Theorem 3. In the last example of the previous section we showed how this generators look like in a classical principal bundle.

Moreover, assumptions of Remark 1 are essential along Durdevich’s theory, for example on [9, 10]. On [9] these assumptions as well as the explicit form of the map $H$ are used to create classifying space and classifying maps; while on [10] these assumptions are used for a non–commutative version of Weyl theory about characteristic classes.

Since $\text{Mor}(\alpha, GM\Phi)$ is a $M$–bimodule in a natural way, we deiced to deal with the left and right structures, and Durdevich’s theory allows to develop the theory for the left/right associated qvbs. In [16, 17, 18] one can appreciate more explicitly the importance of taking into account both associated qvbs and their induced qlcs. For smooth compact manifolds, both associated qvbs are the same and since (classical) principal connections are always regular and real, both induced qlcs are the same.

It is worth mentioning that there are some other papers dealing with
hermitian structures on quantum spaces, for example \cite{33} in which the author presented a notion of Spin Geometry on Quantum Groups. On \cite{33} quantum differential forms on $G$ are given by the braided exterior calculus (\cite{8}) instead of universal differential envelope $\ast$–calculus that we used. Nevertheless, there is a morphism between these two spaces and with that one could try to integrate both ideas in order to develop a theory for spinor quantum bundles.

Now let us talk about the qgg. As we have mentioned before, Definition 3 is the one presented on \cite{24} but at the level of differential calculus, and as we have just seen, it does not recreate the classical case.

In Durdevich’s theory there are some attempts to get a definition of the qgg, for example on \cite{32,34}. To accomplish one of the purposes of this paper, the definition of the qgg presented on \cite{32} is useless because it does not create an action on the space of qpcs. On the other hand, the formulation showed on \cite{34} is just for the special case $M := C^*_\mathbb{C}(X) := \{ \text{f is smooth} \}$, where $X$ a compact smooth manifold; and for an special graded differential $\ast$–algebra on $G$: the minimal admissible calculus. This is why we decided to use Definition 3, despite of the fact that it does not recreate the classical case.

One possible option to recreate the classical case is by considering convolution invertible maps that also are graded differential $\ast$–algebras and defining the qgg as the group generated by these elements. Another option is defining the qgg as the group of all graded differential $\ast$–algebra isomorphisms $\tilde{\mathcal{G}} : \Omega^\ast(GM) \to \Omega^\ast(GM)$ that satisfy Equation 63. With these options, depending on the qpb, the qgg could not have enough elements.

This is a problem, for example, when we talk about Yang–Mills theory in Non–Commutative Geometry since the orbit of Yang–Mills qpcs could be trivial \cite{16,17,18}; from a physical point of view, this implies that there could be too many non–gauge–equivalent boson fields. To prevent this fact, we have decided to define the qgg in the most general way and use Equation 70 for the action on qpcs.

In the literature, for example \cite{35}, there is a common accepted action of the qgg as

$$ f \ast \omega \ast f^{-1} + f \ast (d \circ f^{-1}); $$

nevertheless, this definition is not well–defined in Durdevich’s framework since qpcs are defined on the quantum Lie algebra $\text{inv}\Gamma$. This is another reason to use our definition of the qgg and Equation 70 as the action of $qG\mathcal{G}$ on the space of qpcs. In Differential Geometry, the action of the gauge
group on principal connections is via the pull–back; Equation 70 is just the dualization of that.

On the other hand, although this work has been developed in the framework of Non–Commutative Geometry, the quantum gauge group is a classical group. Therefore, an exciting research would be to explore a way to define the qgg as a quantum group; although there would be coactions instead of actions in this situation.

The reader is invited to notice the incredible geometric–dual similarity of this theory with Differential Geometry, particularly our construction of the canonical hermitian structure (see Definitions 1, 2), Theorems 3, 4 and Propositions 4.

References

[1] CONNES, A. : Noncommutative Geometry, https://www.alainconnes.org/docs/book94bigpdf.pdf, 2019

[2] PRUGOVECKI, E. : Quantum Geometry: A framework for Quantum General Relativity–Fundamental Theories of Physics, Springer, 1992.

[3] WORONOWICZ, S, L. : Pseudospaces, Pseudogroups and Pontryagin Duality, in: Proceedings of International Conference on Mathematics and Physics, Lausanne 1979, Lectures Notes in Physics, 115, 407–412 (1980).

[4] DURDEVICH, M. : Geometry of Quantum Principal Bundles I, Commun Math Phys 175 (3), 457-521 (1996).

[5] DURDEVICH, M. : Geometry of Quantum Principal Bundles II, Rev. Math. Phys. 9 (5), 531—607 (1997).

[6] DURDEVICH, M. : Geometry of Quantum Principal Bundles III, Algebras, Groups & Geometries. 27, 247–336 (2010).

[7] WORONOWICZ, S, L. : Compact Matrix Pseudogroups, Commun. Math. Phys. 111, 613-665 (1987).

[8] WORONOWICZ, S, L. : Differential Calculus on Compact Matrix Pseudogroups (Quantum Groups), Commun. Math. Phys. 122, 125-170 (1989).
[9] **Durdevich, M.** : *Quantum Classifying Spaces and Universal Quantum Characteristic Classes*, Banach Center Publications **40**, 315-327 (1997).

[10] **Durdevich, M.** : *Characteristic Classes of Quantum Principal Bundles*, Algebras, Groups and Geometries **26**, 241-341 (2009).

[11] **Saldaña, M, G, A. & Weingart, G.** : *Functoriality of Principal Bundles and Connections*, arXiv:1907.10231v2, 18 Apr 2020. *To be published in Cahiers de Topologie et Géométrie Différentielle Catégoriques, Volume LXIV, 2023.*

[12] **Saldaña, M, G.** : *Functoriality of Quantum Principal Bundles and Quantum Connections*, arXiv:2002.04015, 10 Feb 2020.

[13] Hajac, P & Majid, S. : *Projective Module Description of the q–monopole*, Commun. Math. Phys., **206**, 247–264 (1999).

[14] **Landi, G.; Reina, C. & Zampini, A.** : *Gauge Laplacians on Quantum Hopf Bundles*, Commun. Math. Phys., **287**, 179–209 (2009).

[15] **Zampini, A.** : *Warped Products and Yang-Mills Equations on Noncommutative Spaces*, Lett. Math. Phys., **105** (2), 221–243 (2015).

[16] **Saldaña, M, G, A.** : *Quantum Principal Bundles and Yang–Mills–Scalar–Matter Fields*. arXiv:2109.01554v3, 30 Apr 2022.

[17] **Saldaña, M, G, A.** : *Yang–Mills–Matter–Fields in the Quantum Hopf Fibration*. arXiv:2112.01973v2, 30 Apr 2022.

[18] **Saldaña, M, G, A.** : *Yang–Mills–Matter–Fields in the Two–Point Space*. arXiv:2112.00647v1, 1 Dec 2021.

[19] **Brzeziński, T & Majid, S.** : *Quantum Group Gauge Theory on Quantum spaces*, Commun. Math. Phys. **157**, 591–638 (1993). *Erratum: Commun. Math. Phys. 167–235 (1995).*

[20] **Budzyński, R., J. & Kondracki, W.** : *Quantum Principal Bundles: Topological Aspects*, Rep. Math. Phys., **37**(3), 365-385 (1996).
[21] PFLAUM, M., J. : *Quantum Groups on Fiber Bundles*, Commun. Math. Phys., 116 (2), 279–315 (1994).

[22] KREIMER, H., F. & TAKEUCHI, M. : *Hopf Algebras and Galois Extensions of an Algebra*, Indiana Univ., Math. J, 30 (5), 675-692 (1981).

[23] BAUM, P., F., DE COMMER, K. & HAJAC, P. : *Free Actions of Compact Quantum Groups on Unital C*-algebras*, Doc. Math., 22, 825-849 (2017).

[24] BRZEZIŃSKI, T : *Translation Map in Quantum Principal Bundles*, J. Geom. Phys. 20, 349 (1996).

[25] SONTZ, S, B. : *Principal Bundles: The Quantum Case*, Universitext, Springer, 2015.

[26] DUBOIS-VIOLETTE, M. : *Lectures on Graded Differential Algebras and NonCommutative Geometry*, arXiv:math/9912017v3, 21 Jun 2000.

[27] KOLÁR, I., MICHIR, P. W. & SLOVÁK, J. : *Natural Operations in Differential Geometry*, internet book http://www.mat.univie.ac.at/ michor/kmsbookh.pdf

[28] BRZEZIŃSKI, T. : *On Modules Associated to Coalgebra Galois Extension*, J. Algebra 215, 290-317 (1999).

[29] LANDI, G. : *An Introduction to Noncommutative Spaces and Their Geometries*, Springer, 1997.

[30] LANCE, E. C. : *Hilbert C*-modules*, Cambridge University Press, 1995.

[31] DURDEVICH, M. : *Quantum Principal Bundles & Hopf-Galois Extensions*, https://www.matem.unam.mx/ micho/papers.html

[32] DURDEVICH, M. : *Quantum Gauge Transformations & Braided Structure on Quantum Principal Bundles*, https://www.matem.unam.mx/ micho/papers.html
[33] **Heckenberger, I.** : *Spin Geometry on Quantum Groups via Covariant Differential Calculi*, Adv. Math. **175** 192–242 (2003).

[34] **Durdevich, M.** : *Quantum Principal Bundles and Corresponding Gauge Theories*, J. Phys. A: Math. Gen. **30** 2027 (1997).

[35] **Hajac, P.** : *Strong Connections on Quantum Principal Bundles*, Commun. Math. Phys., **182** (3), 579–617 (1996).

[36] **Durdevich, M. & Sontz, S. B.** : *Dunkl Operators as Covariant Derivatives in a Quantum Principal Bundle*, SIGMA **9**, 040 (2013).

[37] **Dunkl, C. F.** : *Differential Difference Operators Associated to Reflection Groups*, Trans. Am. Math. Soc. **3**, 457–521 (1996).