Area-Minimizing Cones Over Grassmannian Manifolds

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Abstract
It is a well-known fact that there exists a standard minimal embedding map for the Grassmannians of $n$-planes $G(n, m; F) (F = \mathbb{R}, \mathbb{C}, \mathbb{H})$ and Cayley plane $\mathbb{O} P^2$ into Euclidean spheres, then a natural question is that if the cones over these embedded Grassmannians are area-minimizing? In this paper, detailed descriptions for this embedding map are given from the point of view of Hermitian orthogonal projectors which can be seen as a direct generalization of Gary R. Lawlor’s (23) original considerations for the case of real projective spaces, then we re-prove the area-minimization of those cones which was gradually obtained in [19], [18], and [30] from the perspectives of isolated orbits of adjoint actions or canonical embedding of symmetric $R$-spaces, all based on the method of Gary R. Lawlor’s Curvature Criterion. Additionally, area-minimizing cones over almost all common Grassmannians have been given by Takahiro Kanno, except those cones over oriented real Grassmannians $\tilde{G}(n, m; \mathbb{R})$ which are not Grassmannians of oriented 2-planes. The second part of this paper is devoted to complement this result, a natural and key observation is that the oriented real Grassmannians can be considered as unit simple vectors in the exterior vector spaces, we prove that all their cones are area-minimizing except $\tilde{G}(2, 4; \mathbb{R})$.

Keywords Area-minimizing surface · Cone · Grassmannian · Cayley plane · Plücker embedding

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1 Introduction

Backgrounds on Area-Minimizing Cones

The classical Plateau’s problem concerns on the existence of an area-minimizing surface bounded by a given Jordan curve in Euclidean space. This problem has a long and rich history and it admits many variations. Under higher dimensional situations, geometry measure theory provides effective research method for it, there the considered surfaces are often refer to currents—an type of generalized surfaces, and a compact $k$-dimensional surface with boundary is area-minimizing in $\mathbb{R}^n$ if no other integral currents with the same boundary has less surface area.

Area-minimizing cones are a class of area-minimizing surfaces whose truncated part inside the unit ball owns the least area among all integral currents with the same boundary. These cones are of particular importance for the existence of an isolated singularity at the origin which provides new examples of the solutions of noninterior regularity to Plateau’s problem.

In their fundamental paper *Calibrated geometries* in 1982 [16], Harvey and Lawson have carried out an extensive research on the geometries determined by a differential form, which was called calibration. Generally, a calibration $\varphi$ on a Riemannian manifold $X$ is a closed, smooth, differential $p$-form of comass 1 which means $\varphi$ attains maximum 1 when restricted on all the Grassmannian of oriented $p$-plane of $T_x X$ where $x$ ranges over $X$. An amazing fact is that any calibrated surface is area-minimizing in its homology class after a simple application of Stokes’ theorem.

The method of calibration can be used to determine the homologically area-minimizing circle on a given Riemannian manifold. For the Grassmannians, the research targets which we focus on this paper and also as a question posed in [16], such abundant results are presented in [11], [10], [13], [7], [9], etc.

If the ambient space of a surface $M$ is Euclidean space $\mathbb{R}^n$, its homology class just consists of those surfaces which have the same boundary with $M$, then a calibrated surface (cone) in $\mathbb{R}^n$ is (absolutely) area-minimizing. These results also hold with the smooth calibration replaced by its weak forms (like coflat calibrations, exterior derivatives of Lipschitz forms, etc, see [16], [5], [23], [27], [29]). Several calibrations like normalized powers of Kähler forms, special Lagrangian forms, Cayley calibration, and associative and coassociative calibrations are given by Harvey and Lawson, meanwhile certain area-minimizing cones are exhibited based on the calibrated theory like those homogeneous hypercones introduced in [22](be calibrated by a class of coflat calibrations), the twisted normal cones associated with the compact austere submanifolds of sphere, and the coassociative minimal cone which was first constructed in [25] as an nonparametric Lipschitz solution to the minimal surface system of high codimension, etc.

Area-minimizing cones could not always be calibrated by some smooth differential forms, with an effort for searching necessary and sufficient conditions for a cone to be area-minimizing, Gary R. Lawlor ([23]) developed a general method for proving that a cone is area-minimizing, which was called Curvature Criterion by himself.

Lawlor explained his Curvature Criterion by two equivalent concrete objects, the vanishing calibration and the area-nonincreasing retraction, both defined on certain angular neighborhood of the cone rather than in the entire Euclidean space. They are
linked by the fact that the tangent space of retraction surface is just the orthogonal complement of the dual of the vanishing calibration. They both derive an ordinary differential equation (ODE). The area-minimizing tests include if the ODE has solutions, what is the maximal existence interval of a solution, and then compare it with an important potential of the cone-normal radius. Lawlor also studied those cones which his Curvature Criterion is both necessary and sufficient like the minimal, isoparametric hypercones and the cones over principal orbits of polar group actions. Based on his method, the classification of minimal (area-minimizing, stable, unstable) cones over products of spheres is completed, he also proved some cones over unorientable real projective spaces, compact matrix groups are area-minimizing and gave new proofs of a large class of homogeneous hypercones being area-minimizing.

Other related researches or researches for the area-minimizing cones associated to Lawlor’s Curvature Criterion are a new twistor calibrated theory and applications for the Veronese cone given by Tim Murdoch [29], also see a new point of view of Veronese cone from vanishing calibration [24]; Extending the definition of coflat calibration and the illustration of special Lagrangian calibrated cones over compact matrix group which are also shown in [23] are given by Benny N. Cheng [5]; Proofs for area-minimization of Lawson-Osserman cones are given by Xiaowei Xu, Ling Yang, and Yongsheng Zhang [36]; Researches on area-minimizing cones associated with isoparametric foliations have been pioneered by Zizhou Tang and Yongsheng Zhang [34], etc.

Backgrounds on Area-Minimizing Cones Associated to Grassmannians
The Grassmannian manifolds are important symmetric spaces which can be endowed with some special geometric structures. Generally, the research objects include the Grassmannian of $n$-plane in $\mathbb{F}^m$: $G(n, m; \mathbb{F})$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$), the Grassmannian of oriented $n$-plane in $\mathbb{R}^m$: $\tilde{G}(n, m; \mathbb{R})$, and there exists an only analog for $\mathbb{F} = \mathbb{O}$, the Cayley plane: $\mathbb{O}P^2$.

There exists standard embedding maps for those Grassmannian manifolds, except those $\tilde{G}(n, m; \mathbb{R})(n, m - n \neq 2)$, into Euclidean space which can be seen as isolated orbits of some polar group actions, or standard embedding of symmetric $R$-spaces ([15], [1]). By considering $G(n, m; \mathbb{R})$ and $G(n, m; \mathbb{C})$ as isolated singular orbits of adjoint actions of special orthogonal groups and special unitary groups([19]), Michael Kerckhove proved almost all their cones are area-minimizing. From the perspective of symmetric $R$-spaces and their canonical embeddings for $G(2, 2l + 1; \mathbb{R})(l \geq 3)$, by constructing area-nonincreasing retractions directly, Daigo Hirohashi, Takahiro Kanno, and Hiroyuki Tasaki([15]) proved the area-minimization of these cones.

Later on, also from this point of view, Takahiro Kanno([18]) proved all the cones over the Grassmannian of subspaces $G(n, m; \mathbb{F})(\text{where } \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H})$ and Grassmannian of oriented 2-planes $\tilde{G}(2, m; \mathbb{R})(m \geq 5)$ are area-minimizing. Moreover, in the language of Grassmannian, he also proved the area-minimization of cones over Lagrangian Grassmannian $U(n)/O(n)$—the set of all Lagrangian $n$-planes in $\mathbb{R}^n$, complex Lagrangian Grassmannian $Sp(n)/U(n)$—the set of all complex Lagrangian $n$-planes in $\mathbb{C}^n$, $SO(2n)/U(n)$—the Grassmannian of orthogonal complex structures on $\mathbb{R}^n$, $U(2n)/Sp(n)$—the Grassmannian of quaternionic structures on $\mathbb{C}^n$, etc.
In 2015, the cone over $\mathbb{O}P^2$ was also shown as area-minimizing from the above point of view by Shinji Ohno and Takashi Sakai (30).

We summarize those results for the common Grassmannians in the following table:

| Submanifolds | Ambient spaces | Originally from | Minimizing |
|--------------|----------------|----------------|------------|
| $G(n, m; \mathbb{R})$ | $\{X \in H(m; \mathbb{R}) | trX = 0 \}$ | Kerckhove(94) | Except $\mathbb{R}P^2$ |
| $G(n, m; \mathbb{C})$ | $\{X \in H(m; \mathbb{C}) | trX = 0 \}$ | Kerckhove(94) | All |
| $G(n, m; \mathbb{H})$ | $\{X \in H(m; \mathbb{H}) | trX = 0 \}$ | Kanno(02) | All |
| $\mathbb{O}P^2$ | $\{X \in H(3; \mathbb{C}) | trX = 0 \}$ | Ohno, etc.(15) | Yes |
| $\tilde{G}(2, 2l + 1; \mathbb{R})$ | $\mathfrak{so}(2l + 1)$ | Hirohashi, etc.(00) | $l \geq 3$ |
| $\tilde{G}(2, 2l; \mathbb{R})$ | $\mathfrak{so}(2l)$ | Kanno(02) | $l \geq 4$ |

Here, the ambient space $H(m; F)(F = \mathbb{R}, \mathbb{C}, \mathbb{H})$ denotes the Euclidean space consists of all Hermitian matrices over $F$, and the Lie algebra $\mathfrak{so}(2l + 1)$, $\mathfrak{so}(2l)$ denotes the Euclidean space consists of all skew-symmetric matrices over $\mathbb{R}$. We note here Takahiro Kanno (18) actually has exhibited the area-minimization for the cones over the left $\tilde{G}(2, 5; \mathbb{R})$ and $\tilde{G}(2, 6; \mathbb{R})$, then

**Question** Similar to $\tilde{G}(2, m; \mathbb{R})$, what is the area-minimizing cones over $\tilde{G}(n, m; \mathbb{R})$?

we will give a feasible answer for this question in the final chapter.

**Statement of Results** In chapter 3, 4, 5, we re-prove the following results by regarding $G(n, m; F)(F = \mathbb{R}, \mathbb{C}, \mathbb{H}), \mathbb{O}P^2$ as Hermitian orthogonal projectors uniformly.

**Theorem 1.1** The cones over nonoriented real Grassmannian $G(n, m; \mathbb{R})$, complex Grassmannian $G(n, m; \mathbb{C})$, quaternion Grassmannian $G(n, m; \mathbb{H})$ are area-minimizing, where $m \geq 2n \geq 4$.

**Theorem 1.2** The cones over $\mathbb{C}P_{m-1}, \mathbb{H}P_{m-1}$ are area-minimizing, where $m \geq 2$.

**Theorem 1.3** The cone over $\mathbb{O}P^2$ is area-minimizing.

Additionally, for the left $\tilde{G}(n, m; \mathbb{R})$, the oriented real Grassmannian manifolds (we omit all oriented projective spaces over $\mathbb{R}$ which can be identified with the spheres), motivated by the classical Plücker embedding of Grassmannian in projective space, we also call the natural map Plücker embedding considering $\tilde{G}(n, m; \mathbb{R})$ as the collection of unit simple vectors in the exterior vector space, in chapter 6, we establish the following:

**Theorem 1.4** Except $\tilde{G}(2, 4; \mathbb{R})$, all the cones over Plücker embedding of $\tilde{G}(n, m; \mathbb{R})$ are area-minimizing.

Recently, we also give a family of area-minimizing cones over products of Grassmannians which are natural generalizations of the minimal cones over product spheres, see [17].
This paper is organized as follows:

In chapter 2, we review the knowledge of standard embeddings of Grassmannian manifolds into Euclidean spaces as Hermitian orthogonal projectors, for later citations, we add a proof for the equivalent minimality of an immersed submanifold in sphere and its cone based on the method of moving frame (6), especially the quantitative relations between their second fundamental forms which were used in [23] on many occasions.

In chapter 3, we exhibit that \( G(n, m; \mathbb{F}) (\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}) \) and \( G(m - n, m; \mathbb{F}) \) can be embedded into the same one Euclidean sphere as a pair of opposite minimal submanifolds, simultaneously, the associated cones are opposite, and these cones are proved as area-minimizing by Lawlor’s Curvature Criterion. As special cases, the proof for area-minimization of cones over projective spaces are given separately in chapter 4.

In chapter 5, based on [14], also considering the Cayley plane \( \mathbb{O}P^2 \) as the set of Hermitian orthogonal projection operators, we prove that this cone is area-minimizing.

In chapter 6, we focus on the cones over oriented real Grassmannians \( \tilde{G}(n, m; \mathbb{R}) \), the known area-minimizing cones are those cones over canonical embeddings of \( \tilde{G}(2, m; \mathbb{R}) (m \geq 5) \) into \( \text{so}(m) \) as symmetric \( \mathbb{R} \)-spaces given in [15] and [18]. Notice that the ambient spaces are the Euclidean spaces \( \text{so}(m) \) which can be identified with the exterior vector space \( \wedge^2 \mathbb{R}^m \). Similarly, we extend cones over \( \tilde{G}(2, m; \mathbb{R}) \) to cones over the general oriented Grassmannians \( \tilde{G}(n, m; \mathbb{R}) \) by embedding them in \( \wedge^n \mathbb{R}^m \) (16), (26), (10)—the so-called Plücker embedding. We note here that the second fundamental forms and some related geometries associated with this embedding map have been studied by Wei Huan Chen in [4], meanwhile, we prove that the normal radius of these cones are all equal \( \frac{\pi}{2} \). The final conclusion is that, the cone over \( \tilde{G}(2, 4; \mathbb{R}) \) is unstable, except this, all the others are area-minimizing.

2 Preliminaries

2.1 Standard Embedding of Grassmannian into Euclidean Space

The embedding of Grassmannian into Euclidean space as Hermitian orthogonal projectors is a well-known fact ([20]), for projective spaces and Cayley plane, these embeddings are just their first standard minimal immersions into unit spheres associated to the first eigenvalues of Laplace operator ([3]).

Throughout this paper, \( \mathbb{F} \) will denote the field of real numbers \( \mathbb{R} \), the field of complex numbers \( \mathbb{C} \), the normed quaternion associative algebra \( \mathbb{H} \) unless otherwise stated, this subchapter is mainly based on [3], where the author had given detailed researches for the cases of projective spaces.

We denote

\[
d = dim_{\mathbb{R}} \mathbb{F} = \begin{cases} 
1 & \text{if } \mathbb{F} = \mathbb{R}, \\
2 & \text{if } \mathbb{F} = \mathbb{C}, \\
4 & \text{if } \mathbb{F} = \mathbb{H}.
\end{cases}
\]
Let $\mathbb{F}^m$ be the right Hermitian vector space of column $m$-vectors with the inner product

$$\langle z, w \rangle_{\mathbb{F}} = z^* w = \sum_{i=1}^{m} \bar{z}^i w^i,$$

where $z = (z^i), w = (w^i) \in \mathbb{F}^m$, $\bar{z}^i$ is the standard conjugation and $z^* = \bar{z}^i$ denotes the conjugate transpose.

The Grassmannian manifold $G(n, m; \mathbb{F})$ is the set of all $n$-dimensional subspaces in $\mathbb{F}^m$, when $n = 1$, it is just the projective space. Let $\tilde{G}(n, m; \mathbb{R})$ be the set of oriented $n$-planes in $\mathbb{R}^n$, it is the double cover of $G(n, m; \mathbb{R})$.

We use the following notations: $M(m; \mathbb{F})$ denotes the space of all $m \times m$ matrices over $\mathbb{F}$, $H(m; \mathbb{F}) \subset M(m; \mathbb{F})$ denotes the space of all Hermitian matrices over $\mathbb{F}$,

$$H(m; \mathbb{F}) = \{ A \in M(m; \mathbb{F}) | A^* = A \}.$$

Let $U(m; \mathbb{F}) = \{ A \in M(m; \mathbb{F}) | AA^* = A^* A = I \}$ denote the $\mathbb{F}$-unitary group (For $\mathbb{F} = \mathbb{H}$, we will quickly see that $A$ also own two-sided inverse by considering the complex representation of $M(m; \mathbb{H})$, [37]), it preserves the Hermitian inner product of $\mathbb{F}^m$. Moreover, $U(m; \mathbb{R}) = O(m), U(m; \mathbb{C}) = U(m), U(m; \mathbb{H}) = Sp(m)$.

$H(m; \mathbb{F})$ can be identified with real Euclidean space $E^N$ with the inner product:

$$g(A, B) = \frac{1}{2} \text{Re} \, tr_{\mathbb{F}}(AB),$$

where $A, B \in H(m; \mathbb{F})$, $N = m + dm(m - 1)/2$.

$U(m; \mathbb{F})$ has an adjoint action $\rho$ on $H(m; \mathbb{F})$ given by

$$\rho : U(m; \mathbb{F}) \times H(m; \mathbb{F}) \rightarrow H(m; \mathbb{F})$$

$$(Q, P) \mapsto QPQ^*,$$

where $Q \in U(m; \mathbb{F}), P \in H(m; \mathbb{F})$.

**Proposition 2.1** $\rho^* g = g$ on $H(m; \mathbb{F})$.

**Proof** The cases $\mathbb{F} = \mathbb{R}, \mathbb{C}$ are trivial and Re is not needed in the definition of $g$. For $\mathbb{F} = \mathbb{H}$, choose $U \in Sp(m)$, then we need to verify: $\text{Re} \, tr_{\mathbb{F}}(UABU^*) = \text{Re} \, tr_{\mathbb{F}}(AB)$, the left hand is $\sum_{i,j,k} \text{Re} \, U_{ij}(AB) jk U_{ki}^+$, since $\text{Re} \, (pq) = \text{Re} \, (qp)$, then

$$\sum_{i,j,k} \text{Re} \, U_{ij}(AB) jk U_{ki}^+ = \sum_{i,j,k} \text{Re} \, U_{ki}^+ U_{ij}(AB) jk = \text{Re} \, tr_{\mathbb{F}}(AB).$$

\[ \square \]

For $L \in G(n, m; \mathbb{F})$ and $L^\perp \in G(m - n, m; \mathbb{F})$, choose an $\mathbb{F}$-unitary basis $\{z_1, \ldots, z_m\}$ of $\mathbb{F}^m$ such that $L = \text{span}_\mathbb{F} \{z_1, \ldots, z_n\}$ and $L^\perp = \text{span}_\mathbb{F} \{z_{n+1}, \ldots, z_m\}$, any other orthonormal basis of $L$ is given by $\{ (z_1, \ldots, z_n) A | A \in U(n; \mathbb{F}) \}$. We denote $P_L$ the operator of $\mathbb{F}$-Hermitian orthogonal projection onto the $n$-plane.
$L \in G(n, m; \mathbb{F})$, let $f$ be the matrix $(z_1, \ldots, z_n)$, $g$ be the matrix $(z_{n+1}, \ldots, z_m)$, then for any vector $w \in \mathbb{F}^m$,

$$P_L(w) = \sum_{i=1}^{n} z_i \langle z_i, w \rangle_\mathbb{F} = (\sum_{i=1}^{n} z_i z_i^*) w = ff^* w,$$

and

$$P_{L^\perp}(w) = \sum_{\alpha=n+1}^{m} z_\alpha \langle z_\alpha, w \rangle_\mathbb{F} = (\sum_{\alpha=n+1}^{m} z_\alpha z_\alpha^*) w = gg^* w,$$

easy to see $P_L + P_{L^\perp} = ff^* + gg^* = I$, $P_L(v) = 0$ if and only if $v \in L^\perp$.

For an $\mathbb{F}$-unitary basis $f = (z_1, \ldots, z_n)$ of $L$, we define $\tilde{\varphi}(f) = \sum_{i=1}^{n} z_i z_i^* = ff^*$, since for every $i \in \{1, \ldots, n\}$, $tr_\mathbb{F}(z_i z_i^*) = |z_i|^2 = 1$, then we see $ff^*$ is of $\mathbb{F}$-trace $n$. Let $\tilde{f} = fA$ be another $\mathbb{F}$-unitary basis for $L$, then $\tilde{\varphi}(\tilde{f}) = \tilde{f} \tilde{f}^* = fAA^* f^* = ff^*$ is still of $\mathbb{F}$-trace $n$, so $\tilde{\varphi}$ descends to an embedding map:

$$\varphi : G(n, m; \mathbb{F}) \to H(m; \mathbb{F})$$

$$L \mapsto P_L,$$

and $P_L^2 = P_L$, $L = \{ z \in \mathbb{F}^m | P_L z = z \}$, we note here though the trace of a quaternion linear endomorphism of $\mathbb{H}^m$ is not well defined([37]), but for the Hermitian orthogonal projective operator in the above, it does be.

**Proposition 2.2** Hence we get an isomorphism:

$$G(n, m; \mathbb{F}) \cong \varphi(G(n, m; \mathbb{F})) = \{ P \in H(m; \mathbb{F}) | P^2 = P, tr_\mathbb{F} P = n \}.$$  

**Proof** In the proof, we need some knowledge about quaternion linear algebra, a good reference is [32]. Follow the above discussions, we only need to prove the converse part, which is that for every element of $\{ P \in H(m; \mathbb{F}) | P^2 = P, tr_\mathbb{F} P = n \}$, there exists an $L \in G(n, m; \mathbb{F})$ associated with it, define $S = \{ z \in \mathbb{F}^m | Pz = z \}$, we want to show $dim_\mathbb{F} S = n$. Note for every $P \in H(m; \mathbb{F})$ and $P^2 = P$, there exists an unitary matrix $U \in U(m; \mathbb{F})$, such that

$$U^* P U = \Lambda = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

for an nonnegative number $r = 0, \ldots, n$, where in case of $\mathbb{F} = \mathbb{H}$, see Theorem 5.3.6 in [32] by multiplying a series of elementary matrix both in the left and right.

For $S = \{ z \in \mathbb{F}^m | Pz = z \}$, denote $z = Uw$, it is $\mathbb{F}$-linear isomorphism since $U$ is nonsingular, then $S = \{ w \in \mathbb{F}^m | \Lambda w = w \}$, hence $dim_\mathbb{F} S = r$. 
To show \( r = n \), we need to verify that the \( tr_F \) function of an Hermitian matrix (note all its diagonal elements are real numbers) is invariant under unitary adjoint action \( \rho \), this is trivial for \( F = \mathbb{R}, \mathbb{C} \).

For \( P \in H(m; \mathbb{H}) \), we consider its real matrix representation \( \chi \) which satisfy the properties: \( \chi(PT) = \chi(P)\chi(T) \) and \( \chi(P^*) = (\chi(P))^t \) for any \( P, T \in M(m; \mathbb{H}) \) ([32]). Then

\[
tr_H P = n \iff tr\chi(P) = 4n \iff tr\chi(U^*PU) = tr\chi(P) = 4n,
\]

shows that \( r = n \). \( \square \)

**Remark 2.3** An alternative form \( G(n, m; \mathbb{H}) = \{ P \in H(m; \mathbb{H}) | P^2 = P, tr\mathbb{R}P = 4n \} \) is given in [14].

From the above proposition, \( \rho \) is an isometric, transitive action when restricted on \( \varphi(G(n, m; \mathbb{H})) \), let \( P_0 = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \) be the origin of \( G(n, m; \mathbb{H}) \), \( P_0^\perp = \begin{pmatrix} 0 & 0 \\ 0 & I_{m-n} \end{pmatrix} \) be the origin of \( G(m-n, m; \mathbb{H}) \), then \( G(n, m; \mathbb{H}) \) is the \( \rho \)-orbit through \( P_0 \), \( G(m-n, m; \mathbb{H}) \) is the \( \rho \)-orbit through \( P_0^\perp \).

In the following, we will always identify \( G(n, m; \mathbb{H}) \) with \( \varphi(G(n, m; \mathbb{F})) \).

Additionally,

**Proposition 2.4** denote \( P_L^\perp = P_L^\perp \), then

\[
P_L - P_L^\perp = 2P_L - I \in U(m; \mathbb{F}) \tag{2.1}
\]

is the totally geodesic Cartan embedding of the symmetric space into associated isometry group ([2], [35]).

**Proof** Assume \( A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in U(m; \mathbb{F}) \) and \( f = \begin{pmatrix} A_1 \\ A_3 \end{pmatrix} \) (respectively, \( g = \begin{pmatrix} A_2 \\ A_4 \end{pmatrix} \)) is an \( \mathbb{F} \)-unitary basis of \( L \in G(n, m; \mathbb{F}) \) (respectively, \( L^\perp \in G(m-n, m; \mathbb{F}) \)), then \( P_L = ff^* \) and \( P_L^\perp = gg^* \), the unitary condition is

\[
ff^* + gg^* = I,
\]

the canonical involution is

\[
\sigma : U(m; \mathbb{F}) \rightarrow U(m; \mathbb{F}) \\
A \mapsto SAS,
\]

where

\[
S = \begin{pmatrix} I_n & 0 \\ 0 & -I_{m-n} \end{pmatrix}
\]
Then,

\[ A\sigma(A^{-1})S = ASA^* = ff^* - gg^* = 2P_L - I \]

So the Cartan embedding in (2.1) is equivalent to the definition in [2] up to a constant right translation. \(\square\)

**Remark 2.5** The unitary transformation \(I - 2P_L\) is actually the reflection along the subspace \(L\) through \(L^\perp([14])\).

### 2.2 Tangent and Normal Spaces of Grassmannian

We choose a curve \(\alpha(t) \in G(n, m; \mathbb{F})\), \(\alpha(0) = P, \alpha'(0) = X \in T_PG(n, m; \mathbb{F}) \subset H(m; \mathbb{F})\). Then by derivative \(\alpha(t)\alpha'(t) = \alpha(t)\), we obtain \(PX + XP = X\), i.e., \(T_PG(n, m; \mathbb{F}) \subset \{X \in H(m; \mathbb{F})|XP + PX = X\}\), they are all linear spaces over \(\mathbb{R}\). Assume \(P = QQ_0Q^*\), then set \(X = QQ_0Q^*\), we get \(X_0P_0 + P_0X_0 = X_0\). Next, we compute the dimension of the subspace \(\{X_0 \in H(m; \mathbb{F})|X_0P_0 + P_0X_0 = X_0\}\).

For any \(X_0 \in H(m; \mathbb{F})\), we put

\[ X_0 = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}, A \in H(n; \mathbb{F}), D \in H(m-n; \mathbb{F}), \]

and \(B\) is an arbitrary \(n \times (m-n)\) matrix over \(\mathbb{F}\).

Then

\[ X_0P_0 + P_0X_0 = X_0 \iff X_0 = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}, \]

and the subspace \(\{X_0 \in H(m; \mathbb{F})|X_0P_0 + P_0X_0 = X_0\}\) has the same dimension \(dn(m-n)\) with the tangent space at \(P_0\), so does for every point of \(G(n, m; \mathbb{F})\), hence:

\[ T_PG(n, m; \mathbb{F}) = T_{P^\perp}G(m-n, m; \mathbb{F}) = \{X \in H(m; \mathbb{F})|XP + PX = X\}. \]

\(G(n, m; \mathbb{F})\) is an orbit of \(\rho\) through the origin \(P_0\), for another point \(P = QQ_0Q^* = Q \cdot P_0, Q \in U(m; \mathbb{F})\), let \(X_0 \in T_PG(n, m; \mathbb{F})\), then \(X = Q \cdot X_0 = QX_0Q^* \in T_PG(n, m; \mathbb{F})\) builds isomorphism between tangent spaces, it is just isotropy representation when \(Q\) belongs to the isotropy subgroup at \(P_0\).

Let \(T_PH(m; \mathbb{F}) = T_PG(n, m; \mathbb{F}) \oplus N_PG(n, m; \mathbb{F})\). A vector \(\xi\) is in the normal space \(N_PG(n, m; \mathbb{F})\) if and only if \(g(X, \xi) = 0\) for all \(X \in T_PG(n, m; \mathbb{F})\), for \(P = QQ_0Q^*\), we set \(X = QQ_0Q^* = Q \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} Q^*, \xi = QQ_0Q^* = Q \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} Q^*\), since \(\rho\) is an isometry, then \(g(X, \xi) = 0\) is equivalent to \(g(X_0, \xi_0) = 0\), and \(\text{Re}\ tr(X_0\xi_0) = 0\) if and only if \(Y = 0\).

So,

\[ N_{P_0}G(n, m; \mathbb{F}) = \left\{ \begin{pmatrix} X & 0 \\ 0 & Z \end{pmatrix} | X \in H(n; \mathbb{F}), Z \in H(m-n; \mathbb{F}) \right\}. \quad (2.2) \]
On the other hand, $\xi P = P\xi$ iff $\xi_0 P_0 = P_0\xi_0$, and it is equivalent to $Y = 0$.

Hence,

$$N_P G(n, m; F) = N_{P\perp} G(m - n, m; F) = \{\xi \in H(m; F) | \xi P = P\xi\}.$$ 

**Lemma 2.6** Let $X, Y \in T_P G(n, m; F)$ be two tangent vectors at $P$, and $Z \in H(m; F)$ is an arbitrary vector, then (1) $P, I$ and $X Y \in N_P G(n, m; F)$; (2) $P Z + Z P - 2 P Z P \in T_P G(n, m; F)$; (3) $P X P = 0$.

**Proof** $P, I$ naturally commute with $P$; Since $P X + X P = X, P Y + Y P = Y$, then $(P X + X P) Y = X (P Y + Y P)$ shows that $XY$ commutes with $P$; easy to calculate $P (P Z + Z P - 2 P Z P) + (P Z + Z P - 2 P Z P) P = P Z + Z P - 2 P Z P$; multiplying $P$ in the right of $P X + X P = X$ shows that $P X P = 0$. □

Hence, we have the following Gauss formula for the embedding of $G(n, m; F)$ in $H(m; F)$ which is similar to the cases of projective spaces [3]:

**Proposition 2.7** Let $X, Y$ be two tangent vector fields of $G(n, m; F)$, then

$$\begin{align*}
h(X, Y) &= (XY + YX)(I - 2P); \\
\nabla_X Y &= 2(XY + YX)P + P(\tilde{\nabla}_X Y) + (\tilde{\nabla}_X Y)P,
\end{align*}$$

(2.3)

where $\tilde{\nabla}, \nabla$ are the Riemannian connections of $H(m; F), G(n, m; F)$; $h$ denotes the second fundamental form of $G(n, m; F)$ in $H(m; F)$.

**Proof** Let $P(t)$ be a curve along $G(n, m; F)$ in $H(m; F)$, consider $X = X_P \in T_P G(n, m; F)$, $P(t)$ satisfies $P(0) = P, P'(0) = X$, also let $Y = Y(t)$ be the restriction of the tangent vector field $Y(t)$ on $P(t)$, then

$$Y(t) = P(t)Y(t) + Y(t)P(t)$$

, from which we find

$$\tilde{\nabla}_X Y = P(\tilde{\nabla}_X Y) + (\tilde{\nabla}_X Y)P + XY + YX,$$

denote $f = (XY + YX)(I - 2P), g = 2(XY + YX)P + P(\tilde{\nabla}_X Y) + (\tilde{\nabla}_X Y)P$, then

$$\tilde{\nabla}_X Y = f + g.$$

From (3) of Lemma 2.6, $P(t)Y(t)P(t) = 0$, then we have $XY P + PY X + P(\tilde{\nabla}_X Y)P = 0$, since $XY P = PY X$, then

$$(XY + YX)P = - P(\tilde{\nabla}_X Y)P,$$

From (2) of Lemma 2.6, we find $g \in T_P G(n, m; F)$ and easy to check $f \in N_P G(n, m; F)$. □
2.3 Relations between minimal submanifold in sphere and its cone

Let $M$ be a $(p - 1)$-dimensional, compact submanifold which is immersed into the unit sphere of $\mathbb{R}^n$, the immersion map is denoted by $X$, then the cone $C(M)$ is described by

$$Y : M \times (0, \infty) \to \mathbb{R}^n$$

$$(m, t) \mapsto tm,$$

i.e., $Y(m, t) = tX(m)$.

The truncated cone $C(M)_\epsilon := C(M)|_{M \times [\epsilon, 1]}$ ($\epsilon > 0$).

$M$ and $C(M)$ are all immersed submanifolds in $\mathbb{R}^n$ with the canonical Euclidean metric, so we consider the following vectors being of the standard Euclidean lengths.

Locally, given the orthonormal tangent vector field $\{e_i\}$ ($i = 1, \ldots, p - 1$) and orthonormal normal vector field $\{e_\alpha\}$ ($\alpha = p, \ldots, n - 1$) of $M$ in $S^{n-1}(1)$, then $e_i(m, t) := e_i(m)$ and $e_\alpha(m, t) := e_\alpha(m)$ are orthonormal tangent vectors and orthonormal normal vectors of $C(M)$ at $(m, t)$, $m \in X(m)$.

We use the following notations for $M$: the dual frames of $M$ are $\omega^i$, the connection forms of tangent bundle are $\omega^i_j$, the connection forms of normal bundle are $\omega^\alpha_i$, the coefficients of second fundamental forms are $h^\alpha_{ij}$.

Then, restrict $\omega^\alpha$ on submanifold $M$:

$$\omega^\alpha = 0 \Rightarrow \omega^\alpha_i = h^\alpha_{ij} \omega^j_i, h^\alpha_{ij} = h^\alpha_{ji},$$

$M$ is minimal if and only if $\sum_i h^\alpha_{ii} = 0$, for every $\alpha = p, \ldots, n - 1$.

For the cone $C(M)$, set $e_0(m) = X(m)$, from

$$de_0 = dX = \omega^j_i e_i,$$

we have $\omega^i_0 = \omega^j_i, \omega^\alpha_0 = 0$.

The tangent frames of $C(M)$ in $\mathbb{R}^n$ are given by $\{e_0, e_i\}$ ($i = 1, \ldots, p - 1$), the normal frames of $C(M)$ in $\mathbb{R}^n$ are given by $\{e_\alpha\}$ ($\alpha = p, \ldots, n - 1$). Let $\{\theta^0, \theta^i\}$ be the dual frames, then from

$$dY = dtX + tdY = dte_0 + t\omega^j_i e_i,$$

we have

$$\theta^0 = dt, \theta^i = t\omega^j_i.$$

The connection forms of tangent bundle of $C(M)$ are given by

$$\theta^i_0 = \langle de_0, e_i \rangle = \omega^j_i,$$

$$\theta^j_i = \langle de_j, e_i \rangle = \omega^j_i.$$
And
\[ de_\alpha = \theta_0^0 e_0 + \theta_0^i e_i + \theta_0^\beta e_\beta \]
\[ = \omega_i^j e_i + \omega_\beta^\alpha e_\beta, \]
so \( \theta_0^0 = \theta_0^\alpha = 0, \theta_0^i = \alpha_i, \theta_0^\beta = \alpha_\beta. \)

Finally, let \( \tilde{h}_{\alpha}^\alpha(s, l = 0, 1, \ldots, p - 1) \) be the coefficients of second fundamental forms of \( C(M) \), from
\[ \theta_\alpha^i = \omega_\alpha^i = \sum_{j=1}^{p-1} h_{\alpha j}^i \omega_j^i = \sum_{l=0}^{p-1} \tilde{h}_{\alpha l}^i \theta_l^i, \]
we conclude that
\[ \tilde{h}_{i0}^\alpha = 0, \tilde{h}_{00}^\alpha = 0, \tilde{h}_{ij}^\alpha(tm) = \frac{1}{t} h_{ij}^\alpha(m), \]
(2.4)
an intuitive explanation is that at infinity, the cone behave more flat.

Let the second fundamental form of \( M \) in \( S^{n-1}(1) \) (respectively, \( C(M) \) in \( \mathbb{R}^n \)) along the normal direction \( e_\alpha \) is by \( B^\alpha \) (respectively, \( \tilde{B}^\alpha \)), then
\[ tr \tilde{B}^\alpha = \tilde{h}_{00}^\alpha + \sum_{i=1}^{p-1} \tilde{h}_{ii}^\alpha = \frac{1}{t} \sum_{i=1}^{p-1} h_{ii}^\alpha = \frac{1}{t} tr B^\alpha. \]

Hence,

**Proposition 2.8** \( M \) is minimal in \( S^{n-1}(1) \) if and only if its cone \( C(M) \) is minimal in \( \mathbb{R}^n. \)

Let the set of eigenvalues of \( B^\alpha \) (or the shape operator \( A_\alpha \)) at \( m \) be \( \{\lambda_1, \ldots, \lambda_{p-1}\} \), \( \tilde{B}^\alpha \) at \( tm \) has eigenvalues \( \{0, \frac{1}{t} \lambda_1, \ldots, \frac{1}{t} \lambda_{p-1}\} \), then

**Proposition 2.9** ([16]) \( M \) is an austere submanifold of \( S^{n-1}(1) \) if and only if its cone is an austere submanifold of \( \mathbb{R}^n. \)

### 3 Cones Over the General Grassmannians

We will adopt the following ranges of indices in this section unless otherwise stated:
\[ 1 \leq a, b, c, d \leq n; \]
\[ n + 1 \leq \alpha, \beta, \lambda, \mu \leq m; \]
\[ u, v, w, z \in \{1, i, j, k\}, \]
where \( i, j, k \) denote the imaginary units of \( F = \mathbb{H}, i \) denotes the imaginary unit of \( \mathbb{F} = \mathbb{C}. \)

\( \square \) Springer
3.1 Minimal Embedding of Grassmannian

Let \( e_a \) be the column vectors with 1 in the \( a^{th} \) slot, and \( E_{ab} = e_a e_b^T \), a simple chain rule is \( E_{ab} E_{cd} = e_a e_b^T e_c e_d^T = \delta_{bc} E_{ad} \). Now let \( F^u_{\alpha \beta} \) denotes the matrix \( u E_{\alpha \alpha} + \bar{u} E_{\beta \beta} \), where \( \bar{u} \) denote the conjugation on \( \mathbb{F} \). Then it is easy to verify that \( F^u_{\alpha \beta} \) is an orthonormal basis of \( TP_0 G(n, m; \mathbb{F}) \).

Now \( F^u_{\alpha \beta} F^v_{\beta \gamma} + F^v_{\beta \gamma} F^u_{\alpha \beta} = \delta_{\alpha \beta} (\bar{u}v E_{\alpha \beta} + \bar{v}u E_{\beta \beta}) + \delta_{\beta \gamma} (u \bar{v} E_{\alpha \beta} + v \bar{u} E_{\beta \beta}) \), then the second fundamental forms are given by

\[
 h(F^u_{\alpha \beta}, F^v_{\beta \gamma}) = -\delta_{\alpha \beta} (\bar{u}v E_{\alpha \beta} + \bar{v}u E_{\beta \beta}) + \delta_{\beta \gamma} (u \bar{v} E_{\alpha \beta} + v \bar{u} E_{\beta \beta}).
\]

So, \( h(F^u_{\alpha \beta}, F^w_{\alpha \gamma}) = 2(E_{\alpha \alpha} - E_{\alpha \gamma}) \), the mean curvature vector field at \( P_0 \) is

\[
 H_{P_0} = \frac{2}{dn(m - n)} \begin{pmatrix} -(m - n)I & 0 \\ 0 & nI \end{pmatrix} = -\frac{2m}{dn(m - n)} (P_0 - \frac{n}{m} I).
\]

Since \( G(n, m; \mathbb{F}) \) are orbits of isometry group actions and \( g(P - \frac{n}{m} I, P - \frac{n}{m} I) = \frac{n(m-n)}{2m} \). Then, after minus the center \( \frac{n}{m} I \), we see that the images of \( P - \frac{n}{m} I \) and \( P - \frac{n}{m} I = -(P - \frac{n}{m} I) \) give two minimal embeddings of \( G(n, m; \mathbb{F}) \) and \( G(m - n, m; \mathbb{F}) \) in the same one hypersphere contained in \( H(m; \mathbb{F}) \), their images and the associated cones are opposite.

We can reduce the codimension since the images of \( G(n, m; \mathbb{F}) \) and \( G(m - n, m; \mathbb{F}) \) in \( H(m; \mathbb{F}) \) are all located in the affine hyperplane \( \{P \in H(m; \mathbb{F}) | tr P = 0\}, \) hence,

**Theorem 3.1** (a). The Grassmannian \( G(n, m; \mathbb{F})(\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}) \) and \( G(m - n, m; \mathbb{F}) \) can be embedded into \( S^{N-2}(r) \) as a pair of minimal submanifolds simultaneously, where \( r = \sqrt{\frac{n(m-n)}{2m}}, N = m + dm(m-1)/2, \) moreover, their images are opposite;

(b). There exists two opposite cones associated with \( G(n, m; \mathbb{F}) \) in \( (N-1) \)-dimensional Euclidean space, one is the cone over the embedding: \( P - \frac{n}{m} I (P \in G(n, m; \mathbb{F})) \), the other one is the cone over the inverse embedding: \( P - \frac{m-n}{m} I \).

3.2 Area-Minimizing Cones Over General Grassmannians

The cone and its opposite cone share the same tangent vectors, normal vectors, second fundamental forms, and normal radius at the antipodal points, and since the second fundamental forms of projective spaces behave a little different to those of general Grassmannians, so in this subchapter, we consider cone \( C \) over the general Grassmannian manifold \( G(n, m; \mathbb{F})(m \geq 2n \geq 4) \), i.e., \( C \) is not the cone over projective spaces or its opposite cones.

Still let \( w, z \in \{1, i, j, k\} \), note the second fundamental form \( h \) is equivariant under \( \rho \), we can choose the associated equivariant normal vector fields to reduce the computation to the origin \( P_0 \).

According to (2.2), a normal basis of \( C \) at \( P_0 \) can be given by \( \xi_0 = \sqrt{\frac{2}{m}} I_m, H_l = e_1 e_1^T - e_l e_l^T (2 \leq l \leq n), H_\gamma = e_{n+1} e_{n+1}^T - e_\gamma e_\gamma^T (n + 2 \leq \gamma \leq m), \)
\[ H_{u}^{w} = w_{c}e_{d}^{T} + \bar{w}_{e}d_{e}^{T} (1 \leq c < d \leq n), \quad H_{v}^{z} = z_{e}e_{\mu}^{T} + \bar{z}_{e}e_{\lambda}^{T} (n + 1 \leq \lambda < \mu \leq m), \]
where \( T \) denotes the transpose. All normals in the above have length 1 and orthogonal to each other except any two in \( H_{I} \) or any two in \( H_{v} \). We note here \( g(H_{I}, H_{b}) = \frac{1}{2} \), and \( g(H_{v}, H_{v}) = \frac{1}{2} \), if there exits \( 2 \leq l < b \leq n \) or \( n + 2 \leq \gamma < \alpha \leq m \).

We continue to compute as follows:

\[ g(h(F_{aa}^{u}, F_{\bar{b}b}^{v}), \xi_{0}) = 0; \]
\[ g(h(F_{aa}^{u}, F_{\bar{b}b}^{v}), H_{v}) = -\delta_{ab}(\delta_{a1}\delta_{b1} - \delta_{a1}\delta_{b1})g_{uv}, \]
where in the computation, the term \( \text{Re}(\bar{u}v + uv)/2 = \text{Re}(uv) = \text{Re}(\text{Im} u, \text{Im} v) = \delta_{uv}; \)
\[ g(h(F_{aa}^{u}, F_{\bar{b}b}^{v}), H_{v}) = \delta_{ab}(\delta_{an+1}\delta_{b\bar{b}} + \delta_{a\gamma}\delta_{b\gamma})g_{uv}; \]
\[ g(h(F_{aa}^{u}, F_{\bar{b}b}^{v}), H_{cd}^{w}) = -\delta_{ab}[\text{Re}((\bar{u}v)w)g_{ad}\delta_{bc} + \text{Re}(\bar{u}v\bar{w})g_{ac}\delta_{bd}]; \]
\[ g(h(F_{aa}^{u}, F_{\bar{b}b}^{v}), H_{\lambda\mu}^{z}) = \delta_{ab}[\text{Re}(u\bar{v}z)g_{\lambda\mu} + \text{Re}(u\bar{v}\bar{z})g_{\lambda\mu}]; \]

Choose a unit normal vector at \( P_{0}, \)

\[ \xi = s_{0} + \sum_{l=2}^{n} f_{l}H_{l} + \sum_{\gamma = n+2}^{m} g_{\gamma}H_{v} + \sum_{w,c < d} p_{w}^{u}H_{cd}^{w} + \sum_{\lambda, \lambda \leq \mu} q_{\lambda\mu}^{w}H_{\lambda\mu}^{z}; \]

then

\[ s^{2} + \frac{1}{2} \sum_{l=2}^{n} f_{l}^{2} + \frac{1}{2} \left( \sum_{l=2}^{n} f_{l} \right)^{2} \]
\[ + \frac{1}{2} \sum_{\gamma = n+2}^{m} g_{\gamma}^{2} + \frac{1}{2} \left( \sum_{\gamma = n+2}^{m} g_{\gamma} \right)^{2} + \sum_{w,c < d} (p_{w}^{u})^{2} + \sum_{\lambda, \lambda \leq \mu} (q_{\lambda\mu}^{w})^{2} = 1. \quad (3.1) \]

We denote \( h_{u\alpha a,v\beta b}^{\xi} = g(h(F_{aa}^{u}, F_{\bar{b}b}^{v}), \xi), \) and \( ||h^{\xi}||^{2} = \sum_{u\alpha a,v\beta b}(h_{u\alpha a,v\beta b}^{\xi})^{2}, \) then

\[ h_{u\alpha a,v\beta b}^{\xi} = -\delta_{ab}[(\delta_{a1}\delta_{b1} - \delta_{a1}\delta_{b1})g_{uv}f_{l} + \delta_{ab}(\delta_{an+1}\delta_{b\bar{b}} + \delta_{a\gamma}\delta_{b\gamma})g_{uv}g_{\gamma} \]
\[ - \delta_{ab}[\text{Re}((\bar{u}v)w)g_{c}\delta_{bc} + \text{Re}(\bar{u}v\bar{w})g_{ac}\delta_{bd}]p_{w}^{u} \]
\[ + \delta_{ab}[\text{Re}(u\bar{v}z)g_{\lambda\mu} + \text{Re}(u\bar{v}\bar{z})g_{\lambda\mu}]q_{\lambda\mu}^{w}. \]

Note \( c < d, \lambda \leq \mu, \) all the nonzero terms are divided into eight types:
(1) \( \alpha = \beta, a > b, \) the results are \( -\text{Re}(\bar{u}vuv)p_{ba}^{u}; \)
(2) \( \alpha = \beta, a < b, \) the results are \( -\text{Re}(\bar{u}vuv)p_{ab}^{u}; \)
(3) \( \alpha < \beta, a = b, \) the results are \( \text{Re}(u\bar{v}z)q_{\beta a}^{w}; \)
(4) \( \alpha > \beta, a = b, \) the results are \( \text{Re}(u\bar{v}z)q_{\beta a}^{w}; \)
(5) \( \alpha = \beta = n + 1, a = b \geq 2, \) the results are \( \delta_{uv}(f_{a} + \sum_{\gamma} g_{\gamma}); \)
(6) \( \alpha = \beta \geq n + 2, a = b = 1, \) the results are \( \delta_{uv}(-\sum_{l} f_{l} - g_{a}); \)
(7) \( \alpha = \beta \geq n + 2, a = b \geq 2, \) the results are \( \delta_{uv}(f_{a} - g_{a}); \)
(8) \( \alpha = \beta = n + 1, a = b = 1, \) the results are \( \delta_{uv}(-\sum_{l} f_{l} + \sum_{\gamma} g_{\gamma}). \)
Proposition 3.2

\[ ||h^\xi||^2 = d \left[ (m-n) \sum_{l=2}^{n} f_l^2 + n \sum_{\gamma=n+2}^{m} g_\gamma^2 + (m-n) \left( \sum_{l=2}^{n} f_l \right)^2 + n \left( \sum_{\gamma=n+2}^{m} g_\gamma \right)^2 \right] + 2 \sum_{w, a < b} \left( p_{ab}^w \right)^2 + 2 \sum_{z, \alpha < \beta} \left( q_{\alpha\beta} \right)^2. \]

**Proof** The cases \( F = \mathbb{R} \) and \( F = \mathbb{C} \) are omitted. For \( F = \mathbb{H} \), when \( u, v \) are fixed, there are only one \( w \) or \( z \), such that \( \text{Re}^2(\bar{u}vw), \text{Re}^2(\bar{u}v\bar{w}) \), or \( \text{Re}^2(u\bar{v}z), \text{Re}^2(u\bar{v}\bar{z}) \) equals 1. The number of these nonzero terms is sixteen which is divided into four terms of \( w = 1 \), four terms of \( w = i \), four terms of \( w = j \), four terms of \( w = k \), the same for \( z \). The other nonzero terms are the sum of (5) \~ (8), then we get the conclusion. \( \square \)

The maximum of \( ||h^\xi||^2 \) can be obtained by the following discussion, from (3.1),

\[ ||h^\xi||^2 \leq d \left[ (m-n-1) \left( \sum_{l=2}^{n} f_l^2 + \left( \sum_{l=2}^{n} f_l \right)^2 \right) + (n-1) \left( \sum_{\gamma=n+2}^{m} g_\gamma^2 + \left( \sum_{\gamma=n+2}^{m} g_\gamma \right)^2 \right) 2s^2 + 2 \right] \leq 2d(m-n). \]

\( ||h^\xi||^2 = 2d(m-n) \) if and only if \( s = 0 \), and all \( g_\gamma, p_{\alpha\beta}^m, q_{\alpha\beta}^m \) are zeros.

Hence, we have

**Proposition 3.3** The upper bound of second fundamental form of cones over \( G(n, m; \mathbb{F}) \) at the points belong to unit sphere contained in \( H(m; \mathbb{F}) \) is given by \( \sup_\xi ||h^\xi||^2 = \frac{dn(m-n)^2}{m} \), where \( m \geq 2n \geq 4 \).

**Proof** Since \( G(n, m; \mathbb{F}) \) is an embedded minimal submanifold in a sphere of radius \( \sqrt{\frac{n(m-n)^2}{2m}} \), then the result follows by multiply the square of radius, see (2.4). \( \square \)

For the normal radius of \( C \), we have the following proposition, the result for \( \mathbb{F} = \mathbb{R}, \mathbb{C} \) is established in [19], more generally, the normal radius for the canonical embedding of symmetric \( R \)-spaces is computed by using the Weyl group (18).

**Proposition 3.4** The normal radius of cones over \( G(n, m; \mathbb{F}) \) and \( G(m-n, m; \mathbb{F})(\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}) \) is \( \arccos(1 - \frac{m}{n(m-n)}) \), it does not depend on the base field \( \mathbb{F} \), and this is also right for projective space and Grassmannian of hyperplanes.

**Proof** As an orbit of isometry group action, we can only compute the normal radius at the origin \( E_0 := P_0 - \frac{n}{m}I \). Choose a normal vector \( \xi \) of the cone at the origin with the same length of \( E_0 \), assume the normal geodesic \( \cos(\theta)E_0 + \sin(\theta)\xi \) intersects \( G(n, m; \mathbb{F}) \) at another point \( P \), i.e.,

\[ \cos(\theta)E_0 + \sin(\theta)\xi = P - \frac{n}{m}I. \]
Let \( \xi = \begin{pmatrix} C \\ 0 \\ 0 \\ D \end{pmatrix} \), where \( C \in H(n; \mathbb{F}), D \in H(m-n; \mathbb{F}) \), follow [32], there exist \( \mathbb{F} \)-unitary matrices \( Q, T \), such that \( QCQ^* = \Lambda_1, TDT^* = \Lambda_2 \) are real diagonal matrices. Let \( U = \begin{pmatrix} Q & 0 \\ 0 & T \end{pmatrix} \), then \( U E_0 U^* = E_0 \) and \( U \xi U^* = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \) are all diagonal matrices. Then \( U(P - \frac{n}{m}I)U^* \) is also diagonal which has diagonal elements \(-\frac{n}{m}, \frac{m-n}{m}, \ldots, -\frac{n}{m} \) of multiplicities \( m-n, n \) since \( P^2 = P \).

Now, \( E_0 = \text{diag}\{ -\frac{n}{m}, \ldots, -\frac{n}{m}, \frac{m-n}{m}, \ldots, \frac{m-n}{m} \} \), then the points which are nearest to \( E_0 \) are those interchange one pair of \( \frac{m-n}{m} \) and \( -\frac{n}{m} \) in \( P_0 - \frac{n}{m}I \), we get the conclusion. \( \square \)

Now, we recall Gary R.Lawlor’s work in [23], for the following ODE (see definition 1.1.6 in [23]):

\[
\begin{aligned}
\frac{dr}{d\theta} &= r \sqrt{r^2 k^2 \cos^2(\theta) - 2 \inf_v \det(I - \tan(\theta) h_{ij}^v)} - 1 \\
r(0) &= 1,
\end{aligned}
\]

(3.2)

where \( h_{ij}^v \) is the matrix representation of the second fundamental form of a minimal submanifold \( M \) in sphere, \( v \) is an unit normal, \( k \) is the dimension of cone \( C = C(M) \), and \( r = r(\theta) \) describes a projection curve, the ODE is built at a fixed point \( p \in M \).

Denote the real vanishing angle by \( \theta_0 \) (see Definition 1.1.7 in [23]), Lawlor uses the following estimates.

Let \( \theta_1(k, \alpha) \) be the estimated vanishing angle function replacing \( \inf_v \det(I - \tan(\theta) h_{ij}^v) \) by a smaller positive-valued function

\[
F(\alpha, \tan(\theta), k-1) = \left( 1 - \alpha \tan(\theta) \sqrt{\frac{k-2}{k-1}} \right) \left( 1 + \frac{\alpha \tan(\theta)}{\sqrt{(k-1)(k-2)}} \right)^{k-2}
\]

in (3.2), where the condition \( \alpha^2 \tan^2(\theta_1) \leq \frac{k-1}{k-2} \) should be satisfied, and \( F \) is a decreasing function of \( \alpha \) when \( \tan(\theta), k \) are fixed, it also decreases with respect to \( k \) when \( \alpha, \tan(\theta) \) are fixed.

Let \( \theta_2(k, \alpha) \) be the estimated vanishing angle function which replacing \( \inf_v \det(I - \tan(\theta) h_{ij}^v) \) by

\[
\lim_{k \to \infty} F(\alpha, \tan(\theta), k-1) = (1 - \alpha \tan(\theta)) e^{\alpha \tan(\theta)}
\]

in (3.2), where the condition \( \alpha^2 \tan^2(\theta_2) \leq 1 \) should be satisfied, and \((1 - \alpha \tan(\theta)) e^{\alpha \tan(\theta)} \) is also a decreasing function of \( \alpha \) when \( \tan(\theta) \) are fixed.

The three angles have the following relation:

\[
\theta_0 \leq \theta_1(k, \alpha) \leq \theta_2(k, \alpha),
\]

and Lawlor uses the angle function \( \theta_1 \) for \( \text{dim} \ C = \{3, \ldots, 11\} \), the angle function \( \theta_2 \) for \( \text{dim} \ C = 12 \) to gain "The table."
In the following, we compare the normal radius and vanishing angles of all the general Grassmannian manifolds \( G(n, m; \mathbb{R}) (4 \leq 2n \leq m) \).

The cones having dimension of at most 12 are \( G(2, 4; \mathbb{R}), G(2, 5; \mathbb{R}), G(2, 6; \mathbb{R}), G(2, 7; \mathbb{R}), G(3, 6; \mathbb{R}), G(2, 4; \mathbb{C}) \). We list the results in the table below:

| \( 4 \leq 2n \leq m \) | \( \dim C \) | \( \sup \| h^c \| \) | vanishing angle \( \theta_1 \) | normal radius |
|------------------------|---------|-----------------|-----------------|--------------|
| \( G(2, 4; \mathbb{R}) \) | 5       | 2               | 26.97°          | \( \pi \)    |
| \( G(2, 5; \mathbb{R}) \) | 7       | 3.6             | 16.20° \sim 16.44° | \( \arccos(\frac{1}{4}) > 80° \) |
| \( G(2, 6; \mathbb{R}) \) | 9       | 5.33            | 11.83° \sim 12.14° | \( \arccos(\frac{1}{4}) > 75° \) |
| \( G(2, 7; \mathbb{R}) \) | 11      | 7.143           | 9.41° \sim 9.54° | \( \arccos(\frac{3}{7}) > 70° \) |
| \( G(3, 6; \mathbb{R}) \) | 10      | 4.5             | 10.23°          | \( \arccos(\frac{1}{4}) > 70° \) |
| \( G(2, 4; \mathbb{C}) \) | 9       | 4               | 11.57°          | \( \pi \)    |

Two times of \( \theta_1 \) are still less than the associated normal radius, so following the Curvature Criterion (A simplified version refers to Theorem 1.3.5 in [23]), these cones are area-minimizing.

When the dimension of the cones \( C \) over Grassmannian are greater than 12, they are as follows:

1. \( G(2, m; \mathbb{R})(m \geq 8), G(3, m; \mathbb{R})(m \geq 7) \) and \( G(n, m; \mathbb{R})(m \geq 2n \geq 8) \);
2. \( G(2, m; \mathbb{C})(m \geq 5) \) and \( G(n, m; \mathbb{C})(m \geq 2n \geq 6) \);
3. \( G(n, m; \mathbb{H})(m \geq 2n \geq 4) \).

For estimating of vanishing angle, we use the following formula given by Gary R. Lawlor, if \( \dim(C) = k > 12 \), then

\[
\tan(\theta_2(k, \alpha)) < \frac{12}{k} \tan \left( \theta_2(12, \frac{12}{k} \alpha) \right),
\]

For Grassmannian \( G(n, m; \mathbb{R}) \), \( k = dn(m - n) + 1, \alpha = \sqrt{\frac{dn(m-n)^2}{m}} \), the normal radius is \( \arccos(1 - \frac{m}{n(m-n)}) \), then

\[
\tan(\theta_2(k, \alpha)) < \frac{12}{dn(m-n)+1} \tan \left( \theta_2(12, \frac{12}{dn(m-n)+1} \sqrt{\frac{dn(m-n)^2}{m}}) \right)
\]
\[
< \frac{12}{dn(m-n)+1} \tan \left( \theta_2(12, \frac{12}{\sqrt{dnm}}) \right).
\]

(1) \( d = 1, m \geq 7, \frac{12}{\sqrt{dnm}} \leq 3, \tan(\theta_2(12, 3)) = \tan(8.64°) \approx 0.152, \) then

\[
\theta_2(k, \alpha) < \arctan \left( \frac{2}{n(m-n)+1} \right) < \frac{2}{n(m-n)+1} < \frac{1}{m-2}.
\]
We will show that
\[ \frac{2}{m-2} < \arccos \left( 1 - \frac{m}{n(m-n)} \right) \] (3.3)
is always true when \( m \geq 7 \), i.e., two times of vanishing angle are still less than the normal radius which satisfy the criterion given by Gary R. Lawlor.

The right hand of (3.3) is greater than \( \arccos(1 - \frac{4}{m}) \), so it is sufficient if \( \cos(\frac{2}{m-2}) > 1 - \frac{4}{m} \), then (3.3) is true. Since \( \cos(\frac{2}{m-2}) > 1 - \frac{2}{(m-2)^2} \) and \( 2(m-2)^2 \geq m(m \geq 7) \), then we get our conclusion. Hence, the associated cones are area-minimizing.

The right hand of (3.4) is greater than \( \arccos(1 - \frac{4}{m}) \), so it is sufficient if \( \cos(\frac{1}{m-2}) > 1 - \frac{4}{m} \), then (3.4) is true. Since \( \cos(\frac{1}{m-2}) > 1 - \frac{1}{2(m-2)^2} \) and \( 8(m-2)^2 \geq m(m \geq 5) \), then we get our conclusion. Hence the associated cones are area-minimizing.

The right hand of (3.5) is greater than \( \arccos(1 - \frac{4}{m}) \), so it is sufficient if \( \cos(\frac{1}{m-2}) > 1 - \frac{4}{m} \), then (3.5) is true. Since \( \cos(\frac{1}{m-2}) > 1 - \frac{1}{32(m-2)^2} \) and \( 128(m-2)^2 \geq m(m \geq 4) \), then we get our conclusion. Hence the associated cones are area-minimizing.

**Theorem 3.5** The cones over nonoriented real Grassmannian \( G(n, m; \mathbb{R}) \), complex Grassmannian \( G(n, m; \mathbb{C}) \), quaternion Grassmannian \( G(n, m; \mathbb{H}) \) are area-minimizing, where \( m \geq 2n \geq 4 \).
4 Cones Over Projective Spaces

Now we consider projective space: \( \mathbb{F}P^{m-1} = G(1, m; \mathbb{F})(m \geq 2, \mathbb{F} = \mathbb{C}, \mathbb{H}) \), the cones over \( \mathbb{R}P^{m-1} \) have shown area-minimizing in [23] except the case \( m = 3 \). When \( m = 3 \), the cone over embedding of \( \mathbb{R}P^2 \) is the Veronese cone which owns the least area among a large class of comparison surfaces [29], [24], though, we note here the Veronese cone being area-minimizing is still an open problem. In this section, we prove that the cones over \( \mathbb{C}P^{m-1} \) and \( \mathbb{H}P^{m-1} \) are all area-minimizing for \( m \geq 2 \).

We will adopt the following ranges of indices in this section unless otherwise stated:

\[
2 \leq \alpha, \beta, \lambda, \mu \leq m;
\]

\[
u, v, z \in \{1, i, j, k\},
\]

where \( i, j, k \) denote the imaginary units of \( \mathbb{H} \).

4.1 \( m = 2 \)

These cases are trivial, the spheres \( \mathbb{F}P^1(\mathbb{F} = \mathbb{C}, \mathbb{H}) \) are embedded into spheres as minimal hypersurfaces, in fact, they are all equators by the reduction of codimension and the associated cones are plane, similar result also holds for Cayley projective line \( \mathbb{O}P^1 \) which is isometry isomorphic to \( S^8 \).

Considering the ambient space \( H(2; \mathbb{F}) \) (for \( \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \) uniformly), the origin of \( \mathbb{F}P^1 \) is \( P_0 = E_{11} = e_1e_1^T \), an orthonormal basis of tangent space of \( \mathbb{F}P^1 \) at \( P_0 \) (or minus the center) is \( E_2^\prime = ue_2e_1^T + \bar{u}e_1e_2^T(u = 1 \text{ when } \mathbb{F} = \mathbb{R}, u \in \{1, i\} \text{ when } \mathbb{F} = \mathbb{C}, u \in \{1, i, j, k\} \text{ when } \mathbb{F} = \mathbb{H}, \text{ etc}) \). The unit normal to the cone over \( \mathbb{F}P^1 \) at \( P_0 \) is \( \xi_0 = I_2 \), the identity matrix.

The second fundamental forms in \( H(2; \mathbb{F}) \) are given by

\[
h(E_2^\prime, E_2^\prime) = \delta_{uv}(-2E_{11} + 2E_{22}) = -4\delta_{uv}(P_0 - \frac{I}{2}),
\]

and when restricted on the sphere, all these second fundamental forms turn to be zero. So after minus the center, the minimal embedding \( P - \frac{I}{2} \) is totally geodesic. They are all equators by the reduction of codimension and the associated cones are 2-plane, 3-plane, 5-plane, and 9-plane which have no singularity at origin, hence are area-minimizing.

4.2 \( m \geq 3 \)

The origin of \( \mathbb{F}P^{m-1}(\mathbb{F} = \mathbb{C}, \mathbb{H}) \) is \( P_0 = E_{11} = e_1e_1^T \), an orthonormal basis of tangent space of \( \mathbb{F}P^{m-1} \) at \( P_0 \) (minus the center) is \( E_\alpha^u = ue_\alpha e_1^T + \bar{u}e_1 e_\alpha^T \) \( (2 \leq \alpha \leq m) \), a basis of normal space to the cone over \( \mathbb{F}P^{m-1} \) at \( P_0 \) is given by \( \xi_0 = \sqrt{\frac{2}{m}}I_m \), and \( H_i = e_2e_2^T - e_i e_i^T \) \( (3 \leq l \leq m) \), \( H_{\lambda, \mu} = ze_\mu e_\lambda^T + \bar{z}e_\lambda e_\mu^T \) \( (2 \leq \lambda < \mu \leq m) \), where all normals in the above have length 1 and orthogonal to each other except any two in \( H_i \), and \( g(H_i, H_k) = \frac{1}{2} \) if there exists \( 2 \leq l < b \leq m \). We note here when \( m = 3 \), the embeddings of \( \mathbb{F}P^2(\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}) \) are often called Veronese embeddings.
The second fundamental forms are given by

\[ h(E^u_\alpha, E^v_\beta) = -\delta_{\alpha\beta} (\bar{u}v + \bar{v}u)E_{11} + (u\bar{v}E_{\alpha\beta} + v\bar{u}E_{\beta\alpha}). \]

We continue to compute as follows:

\[ g(h(E^u_\alpha, E^v_\beta), \xi_0) = 0; \]
\[ g(h(E^u_\alpha, E^v_\beta), H_l) = (\delta_{\alpha\beta}2\delta_{\beta\alpha} - \delta_{\alpha l}\delta_{\beta l})\delta_{uv}f_l; \]
\[ g(h(E^u_\alpha, E^v_\beta), H^z_{\lambda\mu}) = \text{Re}(u\bar{v}\bar{z})\delta_{\alpha\mu}\delta_{\beta\lambda} + \text{Re}(u\bar{v}\bar{z})\delta_{\alpha\lambda}\delta_{\beta\mu}. \]

Choose a unit normal vector at \( P_0, \)

\[ \xi = s\xi_0 + \sum_{l=3}^m f_l H_l + \sum_{z, \lambda < \mu} q^z_{\lambda\mu} H^z_{\lambda\mu}, \]

then

\[ s^2 + \frac{1}{2} \sum_{l=3}^m f_l^2 + \frac{1}{2} \left( \sum_{l=3}^m f_l \right)^2 + \sum_{z, \lambda < \mu} (q^z_{\lambda\mu})^2 = 1. \quad (4.1) \]

We denote \( h^k_{\alpha\alpha,\nu\beta} = g(h(E^u_\alpha, E^v_\beta), \xi), \) and \( ||h^k||^2 = \sum_{\alpha\alpha,\nu\beta} (h^k_{\alpha\alpha,\nu\beta})^2, \) then

\[ h^k_{\alpha\alpha,\nu\beta} = \sum_{l=3}^m (\delta_{\alpha\beta}2\delta_{\nu\beta} - \delta_{\alpha l}\delta_{\beta l})\delta_{uv}f_l + \left[ \text{Re}(u\bar{v}\bar{z})\delta_{\alpha\mu}\delta_{\beta\lambda} + \text{Re}(u\bar{v}\bar{z})\delta_{\alpha\lambda}\delta_{\beta\mu} \right] q^z_{\lambda\mu}. \]

Note \( \lambda < \mu, \) the nonzero terms are divided into four types:

1. \( \alpha = \beta = 2, \) the result is: \( \delta_{uv} \sum f_l; \)
2. \( \alpha = \beta \geq 3, \) the results are: \(-\delta_{uv}f_4; \)
3. \( \alpha > \beta, \) the results are: \( \text{Re}(u\bar{v}\bar{z})q^z_{\beta\alpha}; \)
4. \( \alpha < \beta, \) the results are: \( \text{Re}(u\bar{v}\bar{z})q^z_{\alpha\beta}. \)

By (4.1), we have

\[ ||h^k||^2 = d \left[ 2 - 2s^2 - \left( \sum_{l=3}^m f_l \right)^2 \right] \leq 2d, \]

the equality holds if and only if \( s = 0, \sum_{l=3}^m f_l = 0. \)

By multiply the square of radius of sphere \( m^{-1}2, \) we have

**Proposition 4.1** The upper bound of second fundamental form of cones over \( \mathbb{F}P^{m-1}(m \geq 3) \) at the points belong to unit sphere contained in \( H(m; \mathbb{F}) \) is

\[ \sup_{|\xi|} ||h^k||^2 = \frac{d(m-1)}{m}. \]

The calculation of normal radius is similar to the cases of general Grassmannians, we conclude as follows:
Proposition 4.2 The normal radius of $\mathbb{F} P^{m-1}(m \geq 3, \mathbb{F} = \mathbb{C}, \mathbb{H})$ are $\arccos\left(-\frac{1}{m-1}\right)$.

Now, $k = d(m - 1) + 1$, $\alpha^2 = \frac{d(m - 1)}{m}$, we will exhibit the estimated vanishing angles for all the cones over $\mathbb{F} P^{m-1}(m \geq 3)$ in the following.

When $\mathbb{F} = \mathbb{R}$, such cases have been studied in [23].

When $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \mathbb{H}$, we list the cones having dimension no more than 12 in the table below:

| $\mathbb{F} P^{m-1}$ | dim $C$ | $\text{sup}||h^x||^2$ | vanishing angle $\theta_1$ |
|----------------------|---------|----------------------|--------------------------|
| $\mathbb{C} P^2$     | 5       | 4/3                  | 23.43° $\sim$ 23.73°    |
| $\mathbb{C} P^3$     | 7       | 3/2                  | 14.91°                   |
| $\mathbb{C} P^4$     | 9       | 8/5                  | 11.10°                   |
| $\mathbb{C} P^5$     | 11      | 5/3                  | 8.87° $\sim$ 8.88°      |
| $\mathbb{H} P^2$     | 9       | 8/3                  | 11.26° $\sim$ 11.36°    |

all the above cones have vanishing angle no more than $\frac{\pi}{4}$, so they are all area-minimizing.

When dim $C > 12$, the associated projective space are
(1) $\mathbb{C} P^{m-1}$, $m \geq 7$;
(2) $\mathbb{H} P^{m-1}$, $m \geq 4$.

For estimating of vanishing angle, we use the following formula given by Gary R. Lawlor,

$$
\tan(\theta_2(k, \alpha)) < \frac{12}{k} \tan\left(\theta_2(12, \frac{12}{k}\alpha)\right).
$$

Now, $k = d(m - 1) + 1 > 12$, $\alpha = \sqrt{\frac{d(m - 1)}{m}}$, then

$$
\tan(\theta_2(k, \alpha)) < \frac{12}{k} \tan\left(\theta_2(12, \frac{12}{d(m - 1) + 1}\sqrt{\frac{d(m - 1)}{m}})\right) < \frac{12}{k} \tan\left(\theta_2(12, \frac{12}{\sqrt{d(m - 1)m}})\right) < \tan\left(\theta_2(12, \frac{12}{\sqrt{d(m - 1)m}})\right)
$$

(1) $\mathbb{C} P^{m-1}$, $d = 2$, $m \geq 7$, then $\frac{12}{\sqrt{d(m - 1)m}} \leq \sqrt{\frac{12}{7}}$, by checking "The Table," we have

$$
\theta_2(k, \alpha) < \theta_2\left(12, \sqrt{\frac{12}{7}}\right) < 8.07° < \frac{\pi}{4}.
$$
(2) $\mathbb{H}P^{m-1}$, $d = 4$, $m \geq 4$, then $\frac{12}{\sqrt{d(m-1)m}} \leq \sqrt{3}$, by checking "The Table," we have

$$\theta_2(k, \alpha) < \theta_2(12, \sqrt{3}) = 8.15^\circ < \frac{\pi}{4}.$$ 

Hence, we have

**Theorem 4.3** The cones over $\mathbb{C}P^{m-1}, \mathbb{H}P^{m-1}$ are area-minimizing, where $m \geq 2$.

## 5 Cones Over Cayley Plane

Similar to the projective spaces, the Cayley plane $\mathbb{O}P^2$ can also be identified as the set of Hermitian orthogonal projectors, then be embedded as a minimal submanifold in a 25-dimensional sphere contained in the exceptional Jordan algebra $H(3, \mathbb{O})([33], [3])$, and it is one of the Veronese embeddings of $\mathbb{F}P^2(\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O})$. In 2015, Shinji Ohno and Takashi Sakai first confirmed its cone being area-minimizing from the point of view of canonical embedding of symmetric $\mathbb{R}$-space([30]), also see the point of view of isoparametric theory([34]). In this section, we will give a direct proof for it from the point of view of Hermitian orthogonal projectors, some basic facts associated to $\mathbb{O}P^2$ are also exhibited. This section is mainly based on [33], [16], [14].

The octonions, also called the Cayley numbers, are the last algebra in the Cayley–Dickson sequence which form a division algebra. It is $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$ with the following multiplication:

$$(a, b)(c, d) := (ac - \bar{d}b, da + b\bar{c}),$$

where $(a, b), (c, d) \in \mathbb{H} \oplus \mathbb{H}$, the conjugation operator is defined by

$$\overline{(a, b)} = (\bar{a}, -b).$$

For an octonion $u$, writing $\text{Re } u = \frac{u + \bar{u}}{2}$, $\text{Im } u = \frac{u - \bar{u}}{2}$, the octonion can be identified with the Euclidean space $\mathbb{R}^8$ with the inner product

$$\langle u, v \rangle = \text{Re } (u\bar{v}),$$

and the norm is defined as $|u| = \sqrt{\langle u, u \rangle}$.

Let $H(3, \mathbb{O})$ be the set of all $3 \times 3$ hermitian matrices whose entries are octonions, i.e., $H(3, \mathbb{O}) = \{ A \in M(3, \mathbb{O}) | A^* = A \}$, it is a Jordan algebra with the Jordan multiplication

$\mathbb{O}$ Springer
\[ A \circ B = \frac{AB + BA}{2} \text{ for } A, B \in H(3, \mathbb{O}), \]

it is also a real Euclidean space of dimension 27 with the inner product

\[ \langle A, B \rangle = \frac{tr (A \circ B)}{2} \text{ for } A, B \in H(3, \mathbb{O}), \]

and norm \(|A| := \sqrt{\langle A, A \rangle} \).

Each element \( A \in H(3, \mathbb{O}) \) has the typical form

\[
A = \begin{pmatrix}
    r_1 & x_3 & \bar{x}_2 \\
    x_3 & r_2 & x_1 \\
    \bar{x}_2 & x_1 & r_3
\end{pmatrix},
\]

where \( r_1, r_2, r_3 \in \mathbb{R}, x_1, x_2, x_3 \in \mathbb{O} \), or simply written, \( A = \{r, x\} = \{r, (x_1, x_2, x_3)\} \), if \( B = \{s, y\} \), then

\[ \langle A, B \rangle = \frac{1}{2} \langle r, s \rangle + \sum_{i=1}^{3} \langle x_i, y_i \rangle. \]

**Definition 5.1** The Cayley plane is defined by

\[ \mathbb{O}P^2 \equiv \{ A \in H(3, \mathbb{O})| A^2 = A, \ tr_{\mathbb{O}} A = 1 \}. \]

**Proposition 5.2** [14]

\[ \mathbb{O}P^2 = \left\{ a\bar{a'}| a' = (a_1, a_2, a_3) \in \mathbb{O}^3, |a_1|^2 + |a_2|^2 + |a_3|^2 = 1, [a_1, a_2, a_3] = 0 \right\}, \]

which is a 16-dimensional compact submanifold of \( H(3, \mathbb{O}) \).

**Remark 5.3** From the above theorem, for every \( A \in \mathbb{O}P^2 \), it can be written as an octonion matrix whose elements belong to one of the distinguished quaternion subalgebra \( \widehat{\mathbb{H}} \subset \mathbb{O} \), so in fact \( A \in H(3, \widehat{\mathbb{H}}) \), the set of all distinguished quaternion subalgebras, the so-called associative Grassmannian, is isomorphic to the homogeneous space \( G_2/SO(4) \) [16].

The exceptional compact lie group \( F_4 \) is defined as the automorphism group of the Jordan algebra \( H(3, \mathbb{O}) \), for \( A, B \in H(3, \mathbb{O}) \),

\[ F_4 \equiv \{ g \in GL(H(3, \mathbb{O}))| g(A \circ B) = g(A) \circ g(B) \}, \]

or equivalently,

\[ F_4 \equiv \{ g \in GL(H(3, \mathbb{O}))| g(A^2) = (g(A))^2 \}, \]

where \( A^2 \equiv A \circ A \).
Lemma 5.4 [14] If \( g \in F_4 \), then
\[
\text{tr } g(A) = \text{tr } A,
\]
for all \( A \in H(3, \mathbb{O}) \).

Since \( |A|^2 = \frac{1}{3} (A^3) \), then the action of \( F_4 \) on \( H(3, \mathbb{O}) \) is an isometric action, \( F_4 \) is a subgroup of \( O(27) \), moreover, this action is transitive.

Proposition 5.5 [14] \( F_4 \) acts transitively on the Cayley plane \( \mathbb{O}P^2 \) with isotropy subgroup equals to (an isomorphic copy of) \( \text{Spin}(9) \) at the point \( E_1 \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \), i.e.,
\[
\mathbb{O}P^2 \cong F_4/\text{Spin}(9).
\]

We choose a curve \( \alpha(t) \in \mathbb{O}P^2 \), \( \alpha(0) = P, \alpha'(0) = X \in T_P \mathbb{O}P^2 \subset H(3, \mathbb{O}). \) Then derivative \( \alpha(t)\alpha(t) = \alpha(t) \), we obtain \( PX + XP = X(\text{or } P \circ X + X \circ P = X). \)

Now Assume \( P = g(E_1) \), denote \( X = g(X_0) \), where \( g \in F_4 \), \( X_0 \in T_{E_1} \mathbb{O}P^2 \), then \( g(X_0) = g(X_0) \circ g + P \circ g = g(X_0 \circ E_1 + E_1 \circ X_0) \), tells us \( X_0 = X_0 \circ E_1 + E_1 \circ X_0 \), by a similar dimension talk like before, we see \( T_{E_1} \mathbb{O}P^2 = \{ X \in H(3, \mathbb{O})|XP + PX = X \}, T_{E_1} \mathbb{O}P^2 = \{ X \in H(3, \mathbb{O})|XE_1 + E_1 X = X \}. \)

Since \( \mathbb{O}P^2 \) is the orbit through \( E_1 \) under the isometric action of \( F_4 \), the second fundamental forms are all the same at each point, so in the following, we can only consider these at the origin \( E_1 \).

Following the discussion above, the tangent space of \( \mathbb{O}P^2 \) at \( E_1 \) is
\[
T_{E_1} \mathbb{O}P^2 = \left\{ \begin{pmatrix} 0 & u & v \\ \bar{u} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, u, v \in \mathbb{O} \right\}.
\]

The normal space at \( E_1 \) is
\[
N_{E_1} \mathbb{O}P^2 = \left\{ \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & z \\ 0 & \bar{z} & r_3 \end{pmatrix}, r_1, r_2, r_3 \in \mathbb{R}, z \in \mathbb{O} \right\}.
\]

Note that the exceptional Jordan algebra \( H(3, \mathbb{O}) \) is not associative, so it seems that there does not exist a similar Gauss formula like other Veronese embeddings of \( \mathbb{F}P^2(\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}) \), since we need a well-defined expression \( P(t)Y(t)P(t) \) in the proof in proposition 2.7.

Though, as one of the symmetric space of rank 1, \( \mathbb{O}P^2 \) still behaves like \( \mathbb{F}P^2(\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}) \) in some ways. Kunio Sakamoto [33] proved that the Veronese embedding
of $\mathbb{O}P^2$ is also a planar geodesic map just like the cases $\mathbb{F}P^2$, hence an isotropy embedding and the second fundamental form is parallel. In the next, we compute the second fundamental forms of $\mathbb{O}P^2$ and the normal radius directly.

$\mathbb{O}P^2$ is covered by three charts ([14]),

\begin{align*}
U_1 &= \left\{ \frac{a\bar{a}^t}{|a|^2} | a^t = (1, x, y) \in \mathbb{O}^3 \right\} \cong \mathbb{O}^2, \\
U_2 &= \left\{ \frac{a\bar{a}^t}{|a|^2} | a^t = (x, 1, y) \in \mathbb{O}^3 \right\} \cong \mathbb{O}^2, \\
U_3 &= \left\{ \frac{a\bar{a}^t}{|a|^2} | a^t = (x, y, 1) \in \mathbb{O}^3 \right\} \cong \mathbb{O}^2,
\end{align*}

we will do the calculation on $U_1 \cong \mathbb{O}^2$.

Set $x = \sum_{i=1}^{8} x_i u_i$ and $y = \sum_{j=1}^{8} y_j u_j$, where

$$(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8) = (1, i, j, k, e, ie, je, ke)$$

are the standard basis of $\mathbb{O}$ when identified with the Euclidean space $\mathbb{R}^8$.

Restricted on $U_1$, the embedding of $\mathbb{O}P^2$ in $H(3, \mathbb{O})$ can be given by

$$\varphi : \mathbb{O}P^2 \supset U_1 \rightarrow H(3, \mathbb{O})$$

$$(x, y) \mapsto \frac{a\bar{a}^t}{|a|^2} - \frac{1}{3} E_{11},$$

where $\varphi$ is minimal.

$$\nabla e_i e_j = \varphi_{ij}(0, 0) = 2\delta_{ij}(-E_{11} + E_{22}),$$

$$\nabla e_i e_j = \varphi_{ij}(0, 0) = 2\delta_{ij}(-E_{11} + E_{33}),$$

$$\nabla e_i e_j = \varphi_{ij}(0, 0) = \delta_{ij}(E_{23} + E_{32}),$$

where $\nabla$ is the Euclidean connection of $H(3, \mathbb{O})$, then $\sum_{i=1}^{8} \nabla e_i e_i + \sum_{j=1}^{8} \nabla e_j e_j$ is parallel to the position vector $E_1 - \frac{1}{3} E_1$ so its component in the tangent space of sphere is zero, moreover, since $\mathbb{O}P^2$ is an orbit of an isometry group action, we conclude as follows:

**Proposition 5.6** The embedding $\varphi$ is minimal.
The normal space $N_{E_1} C$ of the cone $C$ over $\odot P^2$ at $E_1$(or minus the center) has an orthonormal basis given by

$$L = \sqrt{\frac{2}{3}} I, \ M = E_{22} - E_{33}, \ N_k = u_k E_{23} + \bar{u}_k E_{32} (1 \leq k \leq 8).$$

Choose an unit normal vector $\xi \in N_{E_1} C, \ \xi = aL + bM + \sum_k c_k N_k$, i.e.,

$$a^2 + b^2 + \sum_{k=1}^{8} c_k^2 = 1.$$ 

Set $H_{AB}^\xi = \langle H(e_A, e_B), \xi \rangle$, where $H(e_A, e_B)$ is the second fundamental form of the embedding $\varphi$ into sphere, $1 \leq A, B \leq 16$, then

$$H_{AB}^\xi = \langle \varphi_{AB}, \xi \rangle = a \langle \varphi_{AB}, L \rangle + b \langle \varphi_{AB}, M \rangle + \sum_k c_k \langle \varphi_{AB}, N_k \rangle,$$

since $\xi$ is perpendicular to the position vector(or refer to [21]). The values of $H_{AB}^\xi$ are given by

$$H_{ij}^\xi = b \delta_{ij}, \ H_{ij}^\xi = c_1 \delta_{ij}, \ H_{i\bar{j}}^\xi = c_1 \delta_{i\bar{j}}, \ H_{i\bar{i}}^\xi = -b \delta_{i\bar{j}}.$$

Hence, $||H^\xi||^2 := \sum_{AB}(H_{AB}^\xi)^2 = 16(b^2 + c_1^2) \leq 16$, the equal sign holds if and only if $a = 0$ and $c_\alpha = 0$ for every $\alpha \in \{2, \ldots, 8\}$, i.e., $||H^\xi||^2$ attains its maximum at the normal direction

$$\begin{pmatrix}
0 & 0 & 0 \\
0 & b & c \\
0 & c & -b
\end{pmatrix},$$

where $b, c \in \mathbb{R}$, and $b^2 + c^2 = 1$.

**Proposition 5.7** The upper bound of second fundamental form of cone over $\odot P^2$ at the points belong to unit sphere contained in $H(3, \odot)$ is given by: $\sup_\xi ||H^\xi||^2 = \frac{16}{3}$.

Following [23], now $\dim C = 17, \ a^2 = \frac{16}{3}$, denote the estimated vanishing angle by $\theta_2$, then we have

$$\tan \left( \theta_2 \left( 17, \sqrt{\frac{16}{3}} \right) \right) < \frac{12}{17} \tan \left( \theta_2 \left( 12, \frac{12}{17} \sqrt{\frac{16}{3}} \right) \right).$$

Since $\left( \frac{12}{17} \sqrt{\frac{16}{3}} \right)^2 < 3$, we have $\theta_2 \left( 17, \sqrt{\frac{16}{3}} \right) < \theta_2(12, \sqrt{3}) = 8.15^\circ$ by Gary R.Lawlor’s table.

Now, we consider the normal radius,
Proposition 5.8 The normal radius of the cone over $\mathcal{O}P^2$ is $\frac{2\pi}{3}$.

Proof We can only do the computation at the origin $E_1 - \frac{1}{3}$, choose a normal vector of the cone $C$ of length $\frac{1}{\sqrt{3}}$: $\xi = \begin{pmatrix} r \\ 0 \\ 0 \\ s \\ z \\ 0 \\ t \end{pmatrix}$, where $r, s, t \in \mathbb{R}, z \in \mathcal{O}$, i.e.,

$$r^2 + s^2 + t^2 + 2|z|^2 = \frac{2}{3},$$

and

$$2r = s + t.$$

Set

$$\cos(\theta)(E_1 - \frac{I}{3}) + \sin(\theta)\xi = P - \frac{I}{3}, \quad (5.1)$$

where $\theta \in (0, \pi]$, $P \in \mathcal{O}P^2$.

Since trace of the right hand of (5.1) is zero, then

$$r + s + t = 0.$$

Hence

$$r = 0, t = -s, s^2 + |z|^2 = \frac{1}{3}.$$  

Now,

$$P = \begin{pmatrix} \frac{2}{3}\cos(\theta) + \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{3}\cos(\theta) + \sin(\theta)s + \frac{1}{3} & \sin(\theta)z \\ 0 & \sin(\theta)\bar{z} & -\frac{1}{3}\cos(\theta) - \sin(\theta)s + \frac{1}{3} \end{pmatrix}.$$  

From the condition $P^2 = P$, we get the solutions of (5.1):

1. $\theta = \frac{2\pi}{3}$, the set of nearest points to $E_1 - \frac{1}{3}$ is isomorphic to an immersed $S^8$ in $\mathcal{O}P^2$,

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1+\sqrt{3}s}{2} & \frac{\sqrt{3}s}{2} \\ 0 & \frac{\sqrt{3}\bar{z}}{2} & \frac{1-\sqrt{3}s}{2} \end{pmatrix},$$

where $s^2 + |z|^2 = \frac{1}{3}$.

2. $\theta = \pi$.

So, the normal radius is $\frac{2\pi}{3}$. $\square$

Finally,
Theorem 5.9 The cone over $O\mathbb{P}^2$ under embedding $\varphi$ is area-minimizing.

Remark 5.10 The four cones over Veronese embeddings of $\mathbb{P}^2(F = \mathbb{R}, \mathbb{C}, \mathbb{H}, O)$ own the same normal radius $\frac{2\pi}{3}$, and proportional length of square of second fundamental forms.

6 Cones Over Oriented Real Grassmannians

6.1 Plücker Embedding of Oriented Real Grassmannian

For searching cones over oriented real Grassmannian $\widetilde{G}(n, m; \mathbb{R})$, we should embed them in a suitable ambient Euclidean space.

In this section, we consider the Plücker embedding of all oriented real Grassmannian $\widetilde{G}(n, m; \mathbb{R})$ into unit spheres of exterior vector spaces as minimal submanifolds([16], [26]), it includes the standard embedding of complex hyperquadric $Q_l(\mathbb{C}) \cong \widetilde{G}(2, l + 2; \mathbb{R})$ into Euclidean space $\wedge^2 \mathbb{R}^{l+2}$, the calculations of second fundamental forms are based on [4] which use the method of moving frame, some good references on moving frame are [6], [12].

Let $V$ be an $m$-dimensional real inner product space(often refer to $\mathbb{R}^m$), we can define inner product spaces $\wedge^n V$. If $e_1, \ldots, e_n$ are linearly independent in $V$, then the product

$$e_\lambda = e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_n}$$

corresponding to all $\lambda \in \wedge(n, m)$ are linearly independent in $\wedge^n V$, where $\wedge(n, m)$ denote the set of all increasing maps of $\{1, \ldots, n\}$ into $\{1, \ldots, m\}$.

With respect to the induced inner product

$$\langle e_\lambda, e_\mu \rangle = det(e_{\lambda_i}, e_{\mu_j}),$$

if $\{e_1, \ldots, e_m\}$ are orthonormal basis of $V$, then $\wedge^n V$ have orthonormal basis: $\{e_\lambda\}(\lambda \in \wedge(n, m))$, and $dim \, \wedge^n V = C^n_m$, where $C^n_m$ is the combination number.

Let $\wedge^n \mathbb{R}^m$ be the vector space of all $n$-vectors of $\mathbb{R}^m$, for an oriented $n$-plane $L \in \widetilde{G}(n, m; \mathbb{R})$. Let $\{u_1, \ldots, u_n\}$ be an oriented orthonormal basis of $L$, the Plücker embedding is given by

$$i : \widetilde{G}(n, m; \mathbb{R}) \to \wedge^n \mathbb{R}^m$$

$$L \mapsto u_1 \wedge \cdots \wedge u_n,$$

the image can be seen as an orbit of the exterior power of standard representation of $SO(m)$ on $\mathbb{R}^m$, so it is equivariant.

Choose an oriented orthonormal basis of $\mathbb{R}^m$: $E = (e_1, \ldots, e_m)$, such that $E_0 = e_1 \wedge \cdots \wedge e_n$ is the origin of $\widetilde{G}(n, m; \mathbb{R})$, we give an oriented orthonormal basis for $\wedge^n \mathbb{R}^m$ as follows: set $\tilde{E}_{i_1 \ldots i_q \alpha_1 \ldots \alpha_q} = e_1 \wedge \cdots \wedge e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_q} \wedge \cdots \wedge e_n$, where $1 \leq q \leq \min(n, m - n)$, $1 \leq i_1 < \cdots < i_q \leq n$, $n + 1 \leq \alpha_1 < \cdots < \alpha_q \leq m$, and
\( e_{\alpha_1} \) is in the \( i_1 \)-position, \( \ldots, e_{\alpha_q} \) is in the \( i_q \)-position. Then, all of the \( E_0, E_{i_1 \ldots i_q \alpha_1 \ldots \alpha_q} \) give an oriented orthonormal basis of \( \wedge^n \mathbb{R}^m \) under the lexicographic arrangement.

Denote the Maurer–Cartan forms of \( SO(m) \) by \( \omega \), then \( \omega = E^{-1} dE \), where \( E \in SO(m) \), i.e., \( dE_A = E_B \omega_A^B \), it satisfies the Maurer–Cartan equation: \( d\omega = -\omega \wedge \omega \).

Now
\[
d e_i = e_j \omega_i^j + e_{\alpha} \omega_i^{\alpha},
\]
where \( 1 \leq i, j \leq n, n + 1 \leq \alpha \leq m \).

\[
d(e_1 \wedge \cdots \wedge e_n) = \sum_{i=1}^{n} e_1 \wedge \cdots \wedge de_i \wedge \cdots \wedge e_n
\]
\[
= \sum_{i, \alpha} e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{\alpha} \wedge e_{i+1} \wedge \cdots \wedge e_n \omega_i^{\alpha}
\]
\[
= E_{i\alpha} \omega_i^{\alpha},
\]

\( \tilde{G}(n, m; \mathbb{R}) \) is equipped with the induced metric: \( ds^2 = \sum_{i, \alpha} (\omega_i^{\alpha})^2 \), and \( \{E_{i\alpha}\} \) is the orthonormal tangent frame.

The orthonormal normal frame of \( i(\tilde{G}(n, m; \mathbb{R})) \hookrightarrow S^{n-1}(1) \) is \( \{E_{i_1 \ldots i_q \alpha_1 \ldots \alpha_q} \) \( q \geq 2) \}, E_0 \) is the position vector.

**6.2 Second Fundamental Forms of \( \tilde{G}(n, m; \mathbb{R}) \)**

We arrange the following indices: \( 1 \leq i_1 < \cdots < i_q \leq n, n + 1 \leq \alpha_1 < \cdots < \alpha_q \leq m \), where \( 1 \leq q \leq \min(n, m - n) \).

Now, choose an element \( \sigma \in S_q \), where \( S_q \) is the permutation group of order \( q \), let \( \tau = \sigma^{-1} \), then easy to see
\[
E_{i_{\sigma(1)} \ldots i_{\sigma(q)} \alpha_1 \ldots \alpha_q} = E_{i_1 \ldots i_q \alpha_{\tau(1)} \ldots \alpha_{\tau(q)}}
\]
where \( \alpha_{\tau(1)} \) is in the \( i_1 \)-position, \( \ldots, \alpha_{\tau(q)} \) is in the \( i_q \)-position.

Hence
\[
E_{j_1 \ldots j_q \beta_1 \ldots \beta_q} = \delta^{i_1 \ldots i_q}_{j_1 \ldots j_q} \delta_{\beta_1 \ldots \beta_q}^{\alpha_1 \ldots \alpha_q} E_{i_1 \ldots i_q \alpha_1 \ldots \alpha_q}, \tag{6.1}
\]
where \( i_1, \ldots, i_q, \alpha_1, \ldots, \alpha_q \) are not indices for summing, and
\[
\langle E_{i_1 \ldots i_q \alpha_1 \ldots \alpha_q}, E_{j_1 \ldots j_q \beta_1 \ldots \beta_q} \rangle = \delta^{i_1 \ldots i_q}_{j_1 \ldots j_q} \delta_{\beta_1 \ldots \beta_q}^{\alpha_1 \ldots \alpha_q}.
\]
Now,
\[
d E_{i\alpha} = \sum_{j \neq i} e_1 \wedge \cdots \wedge d e_j \wedge \cdots e_{\alpha} \wedge \cdots \wedge e_n + e_1 \wedge \cdots \wedge d e_{\alpha} \wedge \cdots \wedge e_n
\]
\[
= -E_0 \omega_i^\alpha + \sum_{j, \beta} E_{j\beta}(\omega_i^j \delta^\beta_\alpha + \omega_\alpha^\beta \delta^j_i) + \sum_{j \neq i, \beta \neq \alpha} E_{ji\beta\alpha} \omega_j^\beta,
\]
where \(e_\alpha, d e_\alpha\) are in the \(i\)-position.

In summary, the structure equations can be written as
\[
\begin{align*}
d E_0 &= E_{i\alpha} \omega_i^\alpha, \\
d E_{i\alpha} &= -E_0 \omega_i^\alpha + \sum_{j, \beta} E_{j\beta}(\omega_i^j \delta^\beta_\alpha + \omega_\alpha^\beta \delta^j_i) + \sum_{j \neq i, \beta \neq \alpha} E_{ji\beta\alpha} \omega_j^\beta, \\
d E_{1...i_q\alpha_1...\alpha_q} &= \text{high order terms}.
\end{align*}
\]

The dual 1-forms are \(\theta^{(i\alpha)} = (E_{i\alpha})^* = \omega_i^\alpha\), and the connection 1-forms are
\[
\theta^{(j\beta)}_{(i\alpha)} = \langle d E_{i\alpha}, E_{j\beta} \rangle = \omega_i^j \delta^\beta_\alpha + \omega_\alpha^\beta \delta^j_i.
\]

Then \(\theta^0_{(i\alpha)} = \langle d E_{i\alpha}, E_0 \rangle = -\omega_i^\alpha = -\theta^{(i\alpha)}\),

and follow (6.1), we have
\[
\theta^{(j\beta\gamma)}_{(i\alpha)} = \langle d E_{i\alpha}, E_{j\beta\gamma} \rangle = \sum_{l, \tau} \delta^j_{li} \delta^\beta_{\alpha\tau} \theta^{(l\tau)}.
\]

When \(q > 2\), \(\theta^{(i_1...i_q\alpha_1...\alpha_q)}_{(i\alpha)} = 0\).

Hence, the coefficients of second fundamental forms are given by
\[
\begin{align*}
h^0_{(i\alpha)(j\beta)} &= -\delta^j_{li} \delta^\alpha_\beta, \\
h^{(j\beta\gamma)}_{(i\alpha)(l\tau)} &= \delta^j_{li} \delta^\beta_{\alpha\tau}, \\
h_{(i_1...i_q\alpha_1...\alpha_q)} &= 0(q > 2).
\end{align*}
\]

So, the Plücker embedding is minimal, and the normals to the cone \(C\) of \(i(\tilde{G}(n, m; \mathbb{R}))\) in \(\mathbb{R}^{C_m}\) at \(E_0\) are \(E_{j\beta\gamma}\) and \(E_{i_1...i_q\alpha_1...\alpha_q}(q > 2)\), so
\[
\sup_\xi ||h^\xi||^2 = \sum_{i, a, l, \tau} ||h^{(i\alpha)(l\tau)}||^2 = \sum_{i, l, a, \tau} (\delta^i_{li})^2 (\delta^\alpha_{\alpha\tau})^2 = 4,
\]
where \(\xi\) is a unit normal of \(C\) at \(E_0\).

Hence,

**Theorem 6.1** The upper bound of second fundamental form of cones over \(\tilde{G}(n, m; \mathbb{R})\) contained in \(\wedge^n \mathbb{R}^m\) is \(\sup_\xi ||h^\xi||^2 = 4\).
6.3 Normal Radius of the Cones Over $i(\tilde{G}(n, m; \mathbb{R}))$

In this subsection, we will show that the normal radius of the cones over $\tilde{G}(n, m; \mathbb{R})$ are all at least $\frac{\pi}{2}$, in fact, they are all equal $\frac{\pi}{2}$. We note here Takahiro Kanno has confirmed it for $\tilde{G}(2, m; \mathbb{R})$ by computing with Weyl group of the symmetric pair $(SO(m)^2, SO(m))$ ([18]).

**Theorem 6.2** The normal radius of the cones over $i(\tilde{G}(n, m; \mathbb{R}))$ are all at least $\frac{\pi}{2}$.

**Proof** Let $P_0 = e_1 \wedge \cdots \wedge e_n$ be the origin, $T$ be a normal of $C$ at $P_0$, since $i$ is homogeneous, we can do the computation at the origin $P_0$.

First, we assume the normal radius is less than $\frac{\pi}{2}$, then there exists a normal $T$, such that

$$P_0 + T = \lambda P,$$

where $\lambda > 1$ is a positive real number, $P$ is another point in $i(\tilde{G}(n, m; \mathbb{R}))$, i.e., $P$ is a unit simple $n$-vector.

If $n = 2$, the proof is clear than the general cases. Now $P_0 = e_1 \wedge e_2$, the tangent vectors are $\{e_1 \wedge e_{\alpha}, e_2 \wedge e_{\alpha}\} (3 \leq \alpha \leq m)$, the normal vectors are $\{e_{\alpha} \wedge e_\beta\} (3 \leq \alpha < \beta \leq m)$.

We set $T = \sum_{\alpha < \beta} a_{\alpha\beta} e_{\alpha} \wedge e_{\beta}$, then $P$ does exist if and only if $P_0 + T$ is decomposable, i.e., it is a simple $n$-vector. By the condition about decomposable of 2-vectors ([26], [31]), it is equivalent to

$$0 = \left( e_1 \wedge e_2 + \sum_{\alpha < \beta} a_{\alpha\beta} e_{\alpha} \wedge e_{\beta} \right) \wedge \left( e_1 \wedge e_2 + \sum_{\alpha < \beta} a_{\alpha\beta} e_{\alpha} \wedge e_{\beta} \right) = 2 \sum_{\alpha < \beta} e_1 \wedge e_2 a_{\alpha\beta} e_{\alpha} \wedge e_{\beta} + \sum_{\alpha < \beta, \gamma < \tau} a_{\alpha\beta} a_{\gamma\tau} e_{\alpha} \wedge e_{\beta} \wedge e_{\gamma} \wedge e_{\tau}$$

then all $a_{\alpha\beta}$ should be zero, this is a contraction.

For general Grassmannians, we set $T = \sum_{q, l, \alpha} a_{i_1 \cdots i_q \alpha_1 \cdots \alpha_q} e_{i_1 \cdots i_q \alpha_1 \cdots \alpha_q}$, where the indices $I = \{ (i_1, \ldots, i_q) | 1 \leq i_1 < \cdots < i_q \leq n \}$, $\alpha = \{ (\alpha_1, \ldots, \alpha_q) | n + 1 \leq \alpha_1 < \cdots < \alpha_q \leq m \}$, and $2 \leq q \leq \min(n, m - n)$. Denote $P_0 = e_1 \wedge \cdots \wedge e_n$, set $w = P_0 + T$, we define a linear operator $T_w : \mathbb{R}^m \rightarrow \wedge^{n+1} \mathbb{R}^m$ by $T_w(v) = v \wedge w$, $U := \ker T_w$. Then $w$ is decomposable if and only if $\dim U = n$ ([8]).

We choose a nonzero vector $v \in U$, $v = \sum_{A=1}^{m} x_A e_A$, then

$$\sum_{A=1}^{m} x_A e_A \wedge \left( e_1 \wedge \cdots \wedge e_n + \sum_{q, l, \alpha} a_{i_1 \cdots i_q \alpha_1 \cdots \alpha_q} e_{i_1 \cdots i_q \alpha_1 \cdots \alpha_q} \right) = 0.$$

We call the axis $(n + 1)$-vectors $e_{s_1 \cdots s_p r_1 \cdots r_q} := e_{s_1} \wedge \cdots \wedge e_{s_p} \wedge e_{r_1} \wedge \cdots \wedge e_{r_q}$ is of type $(p, q)$ if $s_1, \ldots, s_p \in \{ 1, \ldots, n \}$ and $r_1, \ldots, r_p \in \{ n + 1, \ldots, m \}$, then the terms of type $(n, 1)$ are $\sum_\alpha (-1)^n x_\alpha e_1 \wedge \cdots \wedge e_n \wedge e_\alpha$. So, $x_\alpha \equiv 0$ for all $\alpha = n + 1, \ldots, m$. 

[26] [31] Springer
Then \( \text{dim } U = n \) is equivalent to \( U = \text{span}_\mathbb{R} \{ e_1, \ldots, e_n \} \), i.e.,

\[
ee_i \wedge T = 0,
\]

for all \( i \in \{1, \ldots, n\} \).

Set \( I = \{i_1, \ldots, i_q\} \), \( e_i \wedge T = 0 \) is equivalent to that the coefficients \( a_{i_1 \ldots i_q \alpha_1 \ldots \alpha_q} \) are all zeros for those indices \( i \in I \). But the value of \( i, i_1, \ldots, i_q \) could be anyone of \( 1, \ldots, n \), so all the coefficients \( a_{i_1 \ldots i_q \alpha_1 \ldots \alpha_q} \) are zeros, i.e., \( T \equiv 0 \), this is a contradiction. \( \square \)

**Remark 6.3** Assume the unit speed normal geodesic \( \cos(\theta) P_0 + \sin(\theta) \frac{T}{|T|} \) intersect the image of \( i(\tilde{G}(n, m; \mathbb{R})) \) at another point \( P \) where \( \theta \in (0, \pi] \), if \( \theta \neq \frac{\pi}{2}, \pi \), then easy to see the above discussion also derive the same contradiction, so the normal radius is just \( \frac{\pi}{2} \), the nearest points are those axis normal vectors and nonaxis normal vectors \( T \) where \( T \) is decomposable too, in special, if \( 2n \leq m \), it contains the grassmannian of oriented \( n \)-plane in normal space \( \mathbb{R}^{m-n} \).

By checking the estimated vanishing angle (we omit details here, it is similar to \( G(n, m; \mathbb{R}) \)), we conclude that

**Theorem 6.4** Except \( \tilde{G}(2, 4; \mathbb{R}) \), all the cones over Plücker embedding of \( \tilde{G}(n, m; \mathbb{R}) \) are area-minimizing.

**Remark 6.5** \( \tilde{G}(2, 4; \mathbb{R}) \) is an isometry isomorphic to \( S^2(\sqrt{2}) \times S^2(\sqrt{2}) \) ([26]), the images under Plücker embedding are the Clifford minimal hypersurfaces in \( S^5(1) \), its cone belong to the type of cones where the Curvature Criterion of Lawlor is necessary and sufficient, follow Corollary 4.4.6 in [23], it is unstable. We note here a pictured description in [28] for area-minimizing surface bounded by the product of spheres collapsed onto the cone with the increase in dimension.

**Remark 6.6** We can also talk about the cones over Plücker embedding of \( G(n, m; \mathbb{C}) \) and \( G(n, m; \mathbb{H}) \) in a similar way. Moreover, note that these submanifolds are written in \( n \)-vectors(respectively, \( 2n, 4n \)-vectors), consider their canonical forms ([16], [14]), we have the following question: could we find some calibrations which calibrate these area-minimizing cones?

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