TURING INSTABILITY AND PATTERN FORMATIONS FOR REACTION-DIFFUSION SYSTEMS ON 2D BOUNDED DOMAIN

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Abstract. In this article, Turing instability and the formations of spatial patterns for a general two-component reaction-diffusion system defined on 2D bounded domain, are investigated. By analyzing characteristic equation at positive constant steady states and further selecting diffusion rate \(d\) and diffusion ratio \(\varepsilon\) as bifurcation parameters, sufficient and necessary conditions for the occurrence of Turing instability are established, which is called the first Turing bifurcation curve. Furthermore, parameter regions in which single-mode Turing patterns arise and multiple-mode (or superposition) Turing patterns coexist when bifurcations parameters are chosen, are described. Especially, the boundary of parameter region for the emergence of single-mode Turing patterns, consists of the first and the second Turing bifurcation curves which are given in explicit formulas. Finally, by taking diffusive Schnakenberg system as an example, parameter regions for the emergence of various kinds of spatially inhomogeneous patterns with different spatial frequencies and superposition Turing patterns, are estimated theoretically and shown numerically.

1. Introduction. Understanding how spatial patterns emerge is one of the major fundamental scientific challenges [17]. In 1952, Alan Turing proposed an instability mechanism for pattern formations in chemical systems [28], by setting forth that diffusion could destabilize a stable steady state and lead to nonuniform spatial patterns. Later, this kind of instability is referred as Turing instability or diffusion-driven instability in literature, and has been investigated in many chemical reaction models. For instance, a two-component model of CIMA reaction was proposed in [12], and this model has been shown to have spatially inhomogeneous patterns induced by diffusion, see [19, 37]. Thereafter, biological examples for Turing patterns have also been extensively investigated, see [18, 27, 33, 38] and references therein. Via Turing instability, diffusive systems could exhibit complex patterns, like strips, rhombic, hexagon and even mixed patterns, see [7, 8, 29]. The mechanism of diffusion-driven instability is also confirmed to relate to the formations of dissipative structures in other fields, including semiconductor physics [2], hydrodynamics [32] and astrophysics [21]. For recent review on this subject, we refer readers to [34].
In this article, we will consider the following reaction-diffusion system

\[
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= \varepsilon \Delta u(x,t) + f(u(x,t), v(x,t)), \quad x \in \Omega, t > 0, \\
\frac{\partial v(x,t)}{\partial t} &= d \Delta v(x,t) + g(u(x,t), v(x,t)), \quad x \in \Omega, t > 0, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, t \geq 0, \\
u(x,0) = \phi(x) \geq 0, v(x,0) = \varphi(x) \geq 0, \quad x \in \bar{\Omega},
\end{aligned}
\]

where \(u, v\) are densities (or concentrations) of two interacting species (or substances) at location \(x\) and time \(t\), \(f\) and \(g\) represent reaction terms, \(\Omega = (0, p) \times (0, q) \subset \mathbb{R}^2\) is bounded with \(\frac{p}{q}\) being irrational number, and \(\nu\) is unit outward normal vector on \(\partial \Omega\). Actually, Turing bifurcation of system (1) has been extensively investigated in the case of one-dimensional spatial domain. For example, it was shown in [17] that Turing bifurcation will never occur as long as \(\varepsilon > 1\). In [16], the authors presented a systematic and analytical method to study Turing instability of system (1). And in [20], the authors also briefly discussed Turing instability and pattern formations for a general two-component reaction-diffusion system subject to Neumann boundary conditions, then paid their main attention to the existence of global branch of bifurcation solutions. Moreover, in [23], necessary and sufficient conditions imposed on stability matrix which ensures that Turing instability could occur, were also obtained for a general \(n\)-component reaction-diffusion system. For two-dimensional domain, pattern selections at onset of Turing instability of system (1) were investigated in [11], by utilizing equivariant bifurcation theory. Furthermore, in [5], the authors considered Turing instability of a general two-component reaction-diffusion system with Neumann boundary conditions on two-dimensional bounded domain. However, as far as we know, these existing results on this topic don’t provide precise parameter region where Turing instability occurs.

For this purpose, we employ the method in [10] to accomplish Turing bifurcation analysis of (1) on a two-dimensional domain, by selecting \(d\) and \(\varepsilon\) as bifurcation parameters. Firstly, we determine a feasible region \(R\) in \(d-\varepsilon\) parameter plane, in which Turing bifurcation might occur when bifurcation parameters \(d\) and \(\varepsilon\) are chosen. Then, we locate all Turing bifurcation curves within this region by deriving their explicit expressions, which provide complete Turing bifurcation diagram in \(d-\varepsilon\) parameter plane for (1). Especially, these Turing bifurcation curves together determine two critical curves (which are called the first and the second Turing bifurcation curves in the context), such that stable spatially inhomogeneous steady states bifurcate from the homogeneous one as \((d, \varepsilon)\) passes through the first Turing bifurcation curve, and much more complicated patterns (such as strips, spots and hexagons) may arise as \((\varepsilon, d)\) crosses the second Turing bifurcation curve. Comparing with these existing results in literature, bifurcation diagram established in this article has the following advantages: 1) it provides sharp condition for the occurrence of Turing bifurcation; 2) one can easily tell for what values of \((d, \varepsilon)\) system (1) will exhibit some specific patterns with given wave number or exhibit superposition patterns, with the aid of these explicit expressions of Turing bifurcation curves.

Thereafter, theoretical results are applied to the following diffusive Schnakenberg system [24]
\[
\begin{align*}
    u_t(x, t) &= \varepsilon d \Delta u(x, t) + a - u(x, t) + u^2(x, t)v(x, t), \quad x \in \Omega, t > 0, \\
v_t(x, t) &= d \Delta v(x, t) + b - u^2(x, t)v(x, t), \quad x \in \Omega, t > 0,
\end{align*}
\]
(2)

under Neumann boundary conditions. Model (2) which describes spatial distribution of morphogen, has been served as a typical model for studying spatial pattern formations of reaction-diffusion equations. And, dynamical behaviors of (2) defined on one-dimensional domain have been extensively investigated from many aspects, like Turing instability [15, 36], Hopf bifurcation [14, 35], Turing-Hopf bifurcation [10, 22], asymmetric spike patterns [30]. Recently, time delay in gene expression is also proposed to be incorporated into (2), and it turns out that time delay plays a significant role in dynamics of system (2), see [6, 26, 1]. On two-dimensional domain, the formations of Turing patterns of (2) have been numerically explored in [4], and it was found that (2) can exhibit conventional Turing patterns, like hexagons and strips. Other than these, the so-called zigzag instability could also be saturated to provide unconventional stable striped patterns. By applying these theoretical results on Turing bifurcation for (1) to (2), we prove that spatially inhomogeneous steady states with different spatial frequencies will emerge, which are consistent with these numerical findings in [4]. In addition, according to bifurcation diagram of (2), one can easily figure out these questions: what kind of spatial patterns (2) may have for any \((d, \varepsilon)\) in Turing instability region; for what values of \((d, \varepsilon)\), Turing patterns with different modes coexist for (2).

The paper is organized as follows. In section 2, we investigate Turing bifurcation in \(d-\varepsilon\) bifurcation plane of system (1). And, necessary and sufficient conditions for the occurrence of Turing instability are presented, and the second Turing bifurcation curve is also derived. As an illustrative example, pattern formations and selections of diffusive Schnakenberg system (2) are explored in Section 3, by applying theoretical results. Numerical simulations are also carried out to support theoretical results. Finally, conclusions are shown in Section 4. Throughout this paper, \(\mathbb{N}\) represents the set of all positive integers, and \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\) represents the set of all non-negative integers.

2. Turing instability and spatial patterns. Suppose that reaction-diffusion system (1) admits a positive constant steady state \(E_* = (u_*, v_*)\). Let \(A\) be Jacobian matrix of (1) at \(E_*\). According to [9], \(A\) is stable if all eigenvalues of \(A\) are located at the left half plane of complex plane; and strongly stable if \(A - D\) is stable for any diagonal matrix \(D\) with nonnegative entries; and excitable if \(A\) is stable but not strongly stable. Thus, for an excitable matrix \(A\), there is always a nonnegative diagonal matrix \(D\) satisfying that \(A - D\) is unstable. Actually, there are only two possible patterns for \(A\) to be excitable, that is,

\[
A_1 = \begin{pmatrix} \alpha & -\gamma \\ \delta & -\beta \end{pmatrix} \quad \text{or} \quad A_2 = \begin{pmatrix} \alpha & \gamma \\ -\delta & -\beta \end{pmatrix}
\]
(3)

for some \(\alpha, \gamma, \delta, \beta > 0\). Especially, \(A_1\) is usually regarded to correspond to activator-inhibitor system, while \(A_2\) is regarded to correspond to positive feedback system, see [5, 25] for more details. Obviously, one necessary condition for the occurrence of Turing instability for system (1) is that \(A_1\) (or \(A_2\)) is an excitable Jacobian matrix.

The linearized system of (1) at \(E_*\) reads

\[
\frac{\partial U}{\partial t} = D\Delta U + AU,
\]
(4)

where \(U = (u, v)^T\) and \(D = \text{diag}(\varepsilon d, d)\).
Without loss of generality, for $\Omega = (0, p) \times (0, q) \subset \mathbb{R}^2$ with $\frac{p^2}{q^2}$ being irrational, suppose $p > q$. And, we further assume

$(N_0)$ $A \triangleq A_1$ (or $A_2$) is excitable, and $\text{tr}(A) = \alpha - \beta < 0$, $\det(A) = \gamma \delta - \alpha \beta > 0$.

Let $k^2 \pi^2$ be eigenvalues of $-\Delta$ on $\Omega$ subject to Neumann boundary conditions, where $k$ and $k^2$ are defined by

$$k := (n, m) \in \mathbb{N}_0 \times \mathbb{N}_0, \quad k^2 := \frac{n^2}{p^2} + \frac{m^2}{q^2}.$$

Given that $\frac{p^2}{q^2}$ is irrational number, the set of vectors $\{k \in \mathbb{N}_0 \times \mathbb{N}_0\}$ could be arranged in order, denoted by $\{k(i) \in \mathbb{N}_0 \times \mathbb{N}_0\}_{i=0}^{\infty}$, such that

$$k(i)^2 < k(i + 1)^2, \quad i \in \mathbb{N}_0.$$

Then, $k(0) = (0, 0), k(1) = (1, 0)$ because of $p > q$, and for any fixed $i \in \mathbb{N}_0$, there exist $n(i), m(i) \in \mathbb{N}_0$ such that $k(i) = (n(i), m(i))$. For simplicity of notations, denote $\mu_i \triangleq k^2(i)\pi^2 = \left(\frac{n^2(i)}{p^2} + \frac{m^2(i)}{q^2}\right)\pi^2, i \in \mathbb{N}_0$. Then, $\mu_i$ is simple and satisfies that

$$0 = \mu_0 < \mu_1 < \mu_2 < \cdots < \mu_n < \cdots \to +\infty.$$

Therefore, characteristic equation of linearized system (4) consists of the following sequences of equations

$$D_{k(i)}(\lambda, \varepsilon) := \det(A - \mu_i D - \lambda I) = \lambda^2 - TR_{k(i)}\lambda + DET_{k(i)}$$

$$= 0,$$

with

$$DET_{k(i)} = \varepsilon(d\mu_i)^2 - (\alpha - \beta \varepsilon)d\mu_i + (\gamma \delta - \alpha \beta),$$

$$TR_{k(i)} = -(\varepsilon + 1)d\mu_i + (\alpha - \beta),$$

for $k(i) \in \mathbb{N}_0 \times \mathbb{N}_0$. According to hypothesis $(N_0)$, the roots of $D_0(\lambda, \varepsilon) = 0$ have negative real parts, and $TR_{k(i)} < 0$ for all $k(i) \in \mathbb{N}_0 \times \mathbb{N}_0$. And, Turing instability indicates that there exists a $k(i) \neq 0$ such that the root $\lambda_{k(i)}$ of (5) satisfies $\lambda_{k(i)} > 0$. Actually, we have the following result.

**Lemma 2.1.** Assume that $(N_0)$ holds, and denote

$$\varepsilon_1 := \frac{1}{\beta^2} \left(\sqrt{\gamma \delta} - \sqrt{\gamma \delta - \alpha \beta}\right)^2.$$

Then, $\min_{k(i) \in \mathbb{R} \times \mathbb{R}} DET_{k(i)} < 0$ holds when $0 < \varepsilon < \varepsilon_1$. Especially, if $DET_{k(i)} < 0$ holds for some $k(i) \in \mathbb{N}_0 \times \mathbb{N}_0$, then $k(i) \neq 0$.

**Proof.** We explore the sign of $DET_{k(i)}$ by considering $k(i) \in \mathbb{R} \times \mathbb{R}$. Firstly, rewrite $DET_{k(i)}$ as

$$DET_{k(i)} = \varepsilon \left[d\mu_i - \frac{\alpha - \beta \varepsilon}{2\varepsilon}\right]^2 - \frac{1}{4\varepsilon} \left[(\alpha - \beta \varepsilon)^2 - 4\varepsilon(\gamma \delta - \alpha \beta)\right].$$

Then, according to $(N_0)$, the following inequality

$$\min_{k(i) \in \mathbb{R} \times \mathbb{R}} DET_{k(i)} = -\frac{1}{4\varepsilon} \left[(\alpha - \beta \varepsilon)^2 - 4\varepsilon(\gamma \delta - \alpha \beta)\right] < 0$$

(7)
holds if and only if
\[ 0 < \varepsilon < \frac{1}{M} \left( \sqrt{\gamma \delta - \gamma \delta - \alpha \beta} \right)^2 \quad \text{or} \quad \varepsilon > \frac{1}{M} \left( \sqrt{\gamma \delta + \gamma \delta - \alpha \beta} \right)^2. \] (8)

Therefore, if \( 0 < \varepsilon < \varepsilon_1 \), we always have \( \min_{k(i) \in \mathbb{R} \times \mathbb{R}} \text{DET}_{k(i)} < 0 \).

Furthermore, if there exists a \( k(i) \in \mathbb{N}_0 \times \mathbb{N}_0 \) satisfying that \( \text{DET}_{k(i)} < 0 \), then \( k(i) \neq 0 \) due to \( \text{DET}_0 > 0 \) by hypothesis \( (\mathbb{N}_0) \).

Actually, Lemma 2.1 indicates that if Turing instability occurs for system (1), then \((\varepsilon, d)\) must be in the following region
\[ \mathcal{R} := \{ (\varepsilon, d) : 0 < \varepsilon \leq \varepsilon_1, \ d > 0 \}. \]

In order to locate all these parameter points \((\varepsilon, d)\) in \( \mathcal{R} \), define
\[ \varepsilon_*(k(i), d) := \frac{\alpha d \mu_i - (\gamma \delta - \alpha \beta)}{d \mu_i (d \mu_i + \beta)} \] (9)
for \( d \in [d_i, \infty) \), \( k(i) \neq 0 \), and
\[ \varepsilon_*(d) := \varepsilon_*(k(i), d) \] (10)
for \( d \in [d_i, d_{i+1}, d_{i-1, i}] \), where
\[ d_i = \frac{\gamma \delta - \alpha \beta}{\alpha}, \frac{1}{\mu_i}, \]
\[ d_{i, j} = \frac{\gamma \delta - \alpha \beta}{2\alpha} \left[ \frac{1}{\mu_i} + \frac{1}{\mu_j} + \sqrt{\left( \frac{1}{\mu_i} + \frac{1}{\mu_j} \right)^2 + \frac{4\alpha \beta}{\gamma \delta - \alpha \beta} \cdot \frac{1}{\mu_i \mu_j}} \right], \] (11)

with \( i < j, \ i, j \in \mathbb{N} \) and \( d_0 := +\infty, \ d_{0, 1} := +\infty \).

Obviously,
\[ 0 < d_j < d_{j-1} \quad \text{for} \ j \in \mathbb{N}, \quad \text{and} \quad d_j \to 0 \text{ as } j \to +\infty. \]

Thus, for any \( d > 0 \), there always exists a \( j \in \mathbb{N} \) such that \( d \in [d_j, d_{j-1}) \), and for \( i \geq j \), \( \lambda(k(i)) = 0 \) is a root of (5) if and only if \( \varepsilon = \varepsilon_*(k(i), d) \).

By geometric properties of \( \varepsilon_*(d) \) and \( \varepsilon_*(k(i), d) \), we have

**Lemma 2.2.** Assume that \( (\mathbb{N}_0) \) holds. Then

1. For any \( i \in \mathbb{N} \), \( \varepsilon = \varepsilon_*(k(i), d) \) is monotonically increasing on \([d_i, d_M(i)]\) and monotonically decreasing on \((d_M(i), +\infty)\) in \( d \), and reaches its maximum \( \varepsilon_1 \) at \( d = d_M(i) \), where
\[ d_M(i) := \frac{\beta \sqrt{\gamma \delta - \alpha \beta}}{\sqrt{\gamma \delta - \gamma \delta - \alpha \beta}} \cdot \frac{1}{\mu_i}. \] (12)

And, \( d_M(i) \) is monotonically decreasing in \( i \in \mathbb{N} \).

2. For given \( i \in \mathbb{N} \), \( \{\varepsilon_*(k(j), d)\}_{j \geq i} \) is a sequence of functions of \( d \), satisfying that for any \( d > d_M(i) \),
\[ \varepsilon_*(k(i), d) > \varepsilon_*(k(i + 1), d) > \varepsilon_*(k(i + 2), d) > \cdots. \] (13)

3. For any \( i \in \mathbb{N} \) and \( j \geq i + 1, \ j \in \mathbb{N} \), the following equation
\[ \varepsilon_*(k(i), d) = \varepsilon_*(k(j), d), \ d > d_i \] (14)
has a unique root \( d_{i, j} \in (d_i, d_M(i)) \). In addition, \( d_{i, j} \) monotonically decreases in \( j \), that is,
\[ d_{i, i+1} > d_{i, i+2} > d_{i, i+3} > \cdots, \]
and for $d > d_{i,i+1}$, inequality (13) also holds.

(4) $\varepsilon_*(d) \leq \varepsilon_1$, $0 < d < +\infty$, and $\varepsilon_*(d) = \varepsilon_1$ if and only if $d = d_M(i), i \in \mathbb{N}$.

Proof. (1) By (9), we have

$$\varepsilon_*(k(i),d) = \frac{1}{\beta} \left[ \frac{\gamma_\delta}{d_{\gamma \beta} + \beta} - \frac{\gamma_\delta - \alpha \beta}{d_{\gamma \beta}} \right].$$

Then,

$$\frac{d\varepsilon_*(k(i),d)}{dd} = \left[ (d\mu_\beta + \beta)\sqrt{\gamma_\delta - \alpha \beta} \right] - (d\mu_\beta \sqrt{\gamma_\delta})^2,$$

which indicates that $\frac{d\varepsilon_*(k(i),d)}{dd} > 0$ for $d \in [d_i, d_M(i)]$, and $\frac{d\varepsilon_*(k(i),d)}{dd} < 0$ for $d \in (d_M(i), +\infty)$, and $\frac{d\varepsilon_*(k(i),d)}{dd} = 0$. Substituting $d = d_M(i)$ into (9), we have

$$\varepsilon_*(k(i),d_M(i)) = \frac{1}{\beta^2} \left( \sqrt{\gamma_\delta} - \sqrt{\gamma_\delta - \alpha \beta} \right)^2 = \varepsilon_1.$$

And, it is easy to see that $d = d_M(i)$ monotonically decreases in $i \in \mathbb{N}$, since $\mu_\beta$ monotonically increases in $i \in \mathbb{N}$.

(2) The second statement could be proved by following a similar argument to these in (1), hence the proof is omitted here.

(3) For any $i \in \mathbb{N}$, it follows from (1) that $\varepsilon = \varepsilon_*(k(i), d)$ monotonically increases from 0 to $\varepsilon_1$ in $d \in [d_i, d_M(i)]$, and monotonically decreases in $d \in (d_M(i), \infty)$ and tends to 0 as $d \to +\infty$. Then, for any $j \geq i + 1, j \in \mathbb{N}$, if $d_i > d_M(j)$, by the definition and monotonicity of $\varepsilon_*(k(i), d)$ and $\varepsilon_*(k(j), d)$, there exists a unique $d_{i,j} \in (d_i, d_M(i))$ given by (11), which satisfies (14). Similarly, if $d_i \leq d_M(j), \forall j \geq i + 1, j \in \mathbb{N}$, then for $d \in [d_M(j), d_M(i)]$, by the monotonicity of $\varepsilon_*(k(i), d)$ and $\varepsilon_*(k(j), d)$ again, there exists a unique $d_{i,j}$ given by (11), satisfying (14). According to (2), we also know that $d_{i,j}$ is decreasing in $j$, that is,

$$d_{i,i+1} > d_{i,i+2} > d_{i,i+3} > \cdots.$$

(4) The result comes directly from the proof of (1).

In Figure 1, we sketch the graphs of functions $\varepsilon = \varepsilon_1$ and $\varepsilon = \varepsilon_*(k(i), d)$ for $d \geq d_i$ and $i = 1, 2, 3 \cdots$, to help one better understand these results of Lemma 2.2.

![Figure 1](image-url)

**Figure 1.** The graphs of functions $\varepsilon = \varepsilon_1$ and $\varepsilon = \varepsilon_*(k(i), d), d \geq d_i, i = 1, 2, 3 \cdots$ in $d$-$\varepsilon$ plane.
Now, we check the transversality conditions in the following lemma.

**Lemma 2.3.** Assume that \((N_0)\) holds. For any given \(i \in \mathbb{N}\) and \(d \in (d_{i+1}, d_{i-1}, i)\), if \(\varepsilon = \varepsilon_*(d)\), characteristic equation (5) has a simple real root \(\lambda = \lambda_{k(i)}(\varepsilon)\) satisfying that \(\lambda_{k(i)}(\varepsilon_*(d)) = 0\), and

\[
\frac{d\lambda_{k(i)}(\varepsilon)}{d\varepsilon}|_{\varepsilon=\varepsilon_*(d)} < 0,
\]

and the remaining roots of (5) have negative real parts.

**Proof.** For \(d \in (d_{i+1}, d_{i-1}, i)\), if \(\varepsilon = \varepsilon_*(d)\), then \(DET_{k(i)} = 0\), therefore \(\lambda = 0\) is a root of \(D_{k(i)}(\lambda, \varepsilon_*(d)) = 0\). A direct calculation yields

\[
\frac{\partial D_{k(i)}(\lambda, \varepsilon)}{\partial \lambda}|_{\lambda=0, \varepsilon=\varepsilon_*(d)} = \frac{\partial D_{k(i)}(\lambda, \varepsilon)}{\partial \lambda}|_{\lambda=0, \varepsilon=\varepsilon_*(k(i), d)} = -TR_{k(i)} > 0,
\]

then \(\lambda = 0\) is a simple root. Let \(\lambda = \lambda_{k(i)}(\varepsilon)\) be a root of (5) satisfying \(\lambda_{k(i)}(\varepsilon_*(d)) = 0\). By

\[
\frac{dD_{k(i)}(\lambda_{k(i)}(\varepsilon), \varepsilon)}{d\varepsilon}|_{\varepsilon=\varepsilon_*(d)} = -(\alpha - \beta - (\varepsilon_*(1) + 1)d\mu_i) \frac{d\lambda_{k(i)}(\varepsilon)}{d\varepsilon}|_{\varepsilon=\varepsilon_*(d)} + d\mu_i (\beta + d\mu_i)
\]

\[
= -TR_{k(i)} \frac{d\lambda_{k(i)}(\varepsilon)}{d\varepsilon}|_{\varepsilon=\varepsilon_*(d)} + d\mu_i (\beta + d\mu_i)
\]

we have

\[
\frac{d\lambda_{k(i)}(\varepsilon)}{d\varepsilon}|_{\varepsilon=\varepsilon_*(d)} = \frac{d\mu_i (\beta + d\mu_i)}{TR_{k(i)}} < 0,
\]

which completes the proof. \(\square\)

The main results on Turing bifurcation for system (1) are given in the following theorem.

**Theorem 2.4.** Assume that \((N_0)\) holds. Then,

1. \(\varepsilon = \varepsilon_*(d), d > 0\) determines the critical curve for the occurrence of Turing instability. Specifically, if \(\varepsilon > \varepsilon_*(d)\), constant steady state \(E_*\) is asymptotically stable, and Turing instability won’t occur; if \(0 < \varepsilon < \varepsilon_*(d)\), then \(E_*\) becomes (Turing) unstable.

2. On each curve \(\varepsilon = \varepsilon_*(k(i), d), d \in (d_i, \infty), i \in \mathbb{N}\), \(k(i)-mode\) Turing bifurcation occurs for system (1) at \((u^*, v^*)\), that is, a pair of spatially inhomogeneous steady states bifurcate from \(E_*\), which can be parameterized as \((\varepsilon(\varepsilon), U(\varepsilon))\), where \(\varepsilon(\varepsilon) = \varepsilon_*(k(i), d) + \varepsilon\)

\[
U(\varepsilon) = E_* + \rho(i, \varepsilon) \cos \left(\frac{n(i)\pi x}{p}\right) \cos \left(\frac{m(i)\pi y}{q}\right), (x, y) \in \Omega
\]

(15)

for \(\varepsilon\) in one side neighborhood of 0, and \(\rho(i, \varepsilon) \to 0\) as \(\varepsilon \to 0\).

3. At the critical value \(\varepsilon = \varepsilon_*(d_{i+1}, i \in \mathbb{N}, (k(i), k(i + 1))-mode\) Turing-Turing bifurcation occurs for system (1), and the bifurcating steady states...
(d(ε₁, ε₂), d(ε₁, ε₂), U(ε₁, ε₂)) near (d_i, i+1, ε₂(d_i, i+1, E*), E*) are superpositions of two spatial patterns, and can be represented as

\[ (d(ε₁, ε₂), d(ε₁, ε₂)) = (d_i, i+1, ε₂(d_i, i+1)) + (ε₁, ε₂) \]

\[ U(ε₁, ε₂) = E* + \sum_{j=i, i+1} \pm \rho(j, ε₁, ε₂) \cos \left( \frac{n(j)πx}{p} \right) \cos \left( \frac{m(j)πy}{q} \right), \ (x, y) ∈ Ω \]

for (ε₁, ε₂) in some region of δ-neighborhood of (0, 0) for some small δ > 0, and \( ρ(j, ε₁, ε₂) \rightarrow 0 \) as (ε₁, ε₂) → 0, j = i, i + 1.

Proof. By Lemma 2.1, 2.2 and 2.3, if ε > ε*, d > 0, all roots of \( D_{k(i)}(λ, ε) = 0 \) have strictly negative real parts for any \( k(i) ∈ \mathbb{N}_0 × \mathbb{N}_0 \), hence \( E* \) is asymptotically stable for system (1); if 0 < ε < ε* for d > 0, there exists one \( i ∈ \mathbb{N} \) such that \( λ_{k(i)}(ε) > 0 \) since the transversality condition \( \frac{dλ_{k(i)}(ε)}{dε}|_{ε=ε*} < 0 \) holds by Lemma 2.3, which implies that Turing instability occurs. Furthermore, we conclude that system (1) undergoes \( k(i) \)-mode Turing bifurcation at \( (E*, ε*, d) \) for \( d ∈ (d_i, i+1, d_i-1, j) \), and \((k(i), k(i+1))\)-mode Turing-Turing bifurcation for \( d = d_i, i+1, ε = ε*, d_i, i+1) \). By a similar argument to these in [18], we could also obtain the representations for these bifurcating steady states in (15) and (16).

\[ \text{Figure 2. The first Turing bifurcation curve.} \]

Remark 1. These critical curves ε = ε*(k(i), d) and ε = ε*(d) are referred as Turing bifurcation curves and the first Turing bifurcation curve in the context, respectively. In general, the first Turing bifurcation curve is a continuous and piecewise smooth curve, and these non-smooth points are exactly Turing-Turing bifurcation points, which are denoted by \( T_{k(i), k(i+1)} \), \( i ∈ \mathbb{N} \), see Figure 2. It is observed from (9) that the expression of the (first) Turing bifurcation curve explicitly depends on wave number \( k \) and diffusion coefficient \( d \), making it more convenient for one to find stable spatial patterns with any wave number.

Remark 2. At these parameter points \( (d, ε) = (d_i, j, ε_*(k(i), (d_i, j))) \), \( i, j ∈ \mathbb{N}, i < j \), steady state bifurcation resulting from double zero eigenvalues occurs, which is called Turing-Turing bifurcation in this paper and [10]. Actually, it is never easy to discuss local structures and stabilities of these bifurcating solutions. However,
by restricting solution space to its proper subspace, one could always pick up bifurcating steady states from a double zero eigenvalue, which have the shape of \(\cos j\pi x\), see [20]. Moreover, applying the techniques of space decomposition and implicit function theorem, one could also find some superposition steady states in the form of \(s(\omega_0)\sin \omega \cos i\pi x + \cos \omega \cos j\pi x + W(\omega)\) for sufficiently small \(|\omega - \omega_0|\) which satisfies that \(s(\omega_0) = 0, W(\omega_0) = 0\), while the stability of these superposition steady states remains unknown, see [13, 31]. Recently, steady sates bifurcating from Turing-Turing bifurcation and their stabilities and spatial structures have been investigated in [3], by utilizing center manifold theory and normal form method, which theoretically tells that the superpositions of two single-mode steady states could arise from Turing-Turing bifurcation, and these superposition solutions which has the shape of \(\pm \phi_1 \cos i\pi x \pm \phi_2 \cos j\pi x\) could be stable, where \(\phi_1\) and \(\phi_2\) are some eigenvectors.

**Remark 3.** Theorem 2.4 is also applicable to system (1) defined on \(\Omega = (0, p)\) by replacing

\[
k = n, \quad \text{and} \quad k^2 = \frac{n^2}{p^2}.
\]

In the case of Dirichlet boundary conditions, one could also investigate Turing bifurcation at \(E_* = (0, 0)\) along similar lines to above discussions, and conclusions are nothing new but all the cosine functions in (15) and (16) are replaced by sine functions, and \(N_0\) is replaced by \(N\).

Recall that \(\varepsilon_*(k(i), d_i) = 0, \lim_{d\to\infty} \varepsilon_*(k(i), d) = 0\) and \(\varepsilon_*(k(i), d_M(i)) = \varepsilon_1\) by Lemma 2.2. Then, for any \(\varepsilon \in (0, \varepsilon_1)\) and \(i \in \mathbb{N}\), there exist \(d_i^- (\varepsilon) < d_M(i) < d_i^+ (\varepsilon)\) such that

\[
\varepsilon_*(k(i), d_i^- (\varepsilon)) = \varepsilon_*(k(i), d_i^+ (\varepsilon)) = \varepsilon.
\]

Theorem 2.4 implies that one has to restrict diffusion rate \(d \in (d_i^- (\varepsilon), d_i^+ (\varepsilon))\) to observe \(k(i)\)-mode Turing patterns, whenever diffusion ratio \(\varepsilon\) is fixed, that is,

**Corollary 1.** Assume that \((N_0)\) holds. For any \(\varepsilon \in (0, \varepsilon_1)\) and \(i \in \mathbb{N}\), system (1) might have \(k(i)\)-mode Turing patterns with the shape like (15) when \(d \in (d_i^- (\varepsilon), d_i^+ (\varepsilon))\).

To further determine parameter region in \(d-\varepsilon\) plane for the existence of other complex patterns of system (1), we define

\[
\varepsilon_*(d) = \begin{cases} 
\varepsilon_*(k(2), d), & d \geq d_{1,2}, \\
\varepsilon_*(k(i), d), & d \in [d_{i,i+2}, d_{i,i+1}), \ i \in \mathbb{N}, \\
\varepsilon_*(k(i + 2), d), & d \in [d_{i+1,i+2}, d_{i,i+2}), \ i \in \mathbb{N}.
\end{cases}
\] (17)

The graph of \(\varepsilon = \varepsilon_*(d)\), \(d > 0\) is called the second Turing bifurcation curve, see Figure 3. Let \(D^0, D^1\) and \(D^2\) be three subregions within \(R = \{(d, \varepsilon) : 0 < \varepsilon \leq \varepsilon_1, d > 0\}\), given by

\[
D^0 = \{(d, \varepsilon) : \varepsilon_*(d) < \varepsilon \leq \varepsilon_1, d > 0\}, \\
D^1 = \{(d, \varepsilon) : \varepsilon_*(d) < \varepsilon < \varepsilon_* (d), d > 0\}, \\
D^2 = \{(d, \varepsilon) : 0 < \varepsilon < \varepsilon_*(d), d > 0\}.
\]

Obviously, \(E_*\) is asymptotically stable for (1) when \((d, \varepsilon) \in D^0\). Define

\[
D_{k(i)} = \{(d, \varepsilon) : \varepsilon_*(d) < \varepsilon < \varepsilon_*(d), \ d \in (d_{i,i+1}, d_{i-1,i})\}
\]
Corollary 2. Assume that (N_0) holds. Then

(1) When $\varepsilon^{**}(d) < \varepsilon < \varepsilon^*(d)$ (i.e. $(d,\varepsilon) \in \mathcal{D}^1$), there exist spatially inhomogeneous patterns (single-mode Turing patterns) for system (1). More precisely, if $(d,\varepsilon) \in D_{k(i)}$ for any $i \in \mathbb{N}$, system (1) exhibits $k(i)$-mode Turing pattern with its expression given by (15).

(2) When $\varepsilon < \varepsilon^{**}(d)$ (i.e. $(d,\varepsilon) \in \mathcal{D}^2$), system (1) has spatially inhomogeneous superposition patterns (multiple-mode Turing patterns). In particular, for any $i \in \mathbb{N}, j > i$, system (1) may have $k(i), k(i+1), \ldots, k(j-1), k(j)$-mode Turing patterns and their superpositions when $(d,\varepsilon) \in D_{k(i),k(j)}$.

Figure 3. $\varepsilon = \varepsilon^{**}(d)$ is the second Turing bifurcation curve. The grey area enclosed by the first and the second Turing bifurcation curves, represents $\mathcal{D}^1$. The region below the second Turing bifurcation curve $\varepsilon = \varepsilon^{**}(d)$ is $\mathcal{D}^2$, and the blue area denoted by $D_{k(3),k(7)}$ is one component of $\mathcal{D}^2$.

3. Application to diffusive Schnakenberg system. In this section, we apply these results in previous section to the following diffusive Schnakenberg system,

$$
\begin{align*}
    &u(x,t) = \varepsilon d u(x,t) + a - u(x,t) + u^2(x,t)v(x,t), \quad x \in \Omega, \ t > 0, \\
    &v(x,t) = d \Delta v(x,t) + b - u^2(x,t)v(x,t), \quad x \in \Omega, \ t > 0, \\
    &\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, \ t \geq 0, \\
    &u(x,0) = u_0(x) \geq 0, \quad v(x,t) = v_0(x) \geq 0, \quad x \in \Omega,
\end{align*}
$$

(18)

where $u(x,t)$ and $v(x,t)$ are concentrations of activator and inhibitor at $(x,t)$ respectively; $a, b$ are positive constants; $d > 0$ is diffusion rate of inhibitor; $\varepsilon > 0$ is the ratio of diffusion rates of activator to inhibitor; and $\Omega = (0,p) \times (0,q)$ satisfies
that $2^{\frac{2}{3}}$ is irrational number. Obviously, system (18) always has a unique positive constant steady state $E_*(u_*, v_*)$, where
\[ u_* = a + b, \quad v_* = \frac{b}{(a + b)^2}. \]
The Jacobian matrix $A$ is given by
\[ A = \begin{pmatrix} 2u_*v_* - 1 & u_*^2 \\ -2u_*v_* & u_*^2 \end{pmatrix}. \]
If $b > a$, then $2u_*v_* - 1 > 0$, therefore Jacobian matrix $A$ is excitable in the form of $A_2$. Furthermore, assume that $u_*^2 > 2u_*v_* - 1$. Then, $(N_0)$ also holds. Now, $\alpha = 2u_*v_* - 1$, $\beta = \gamma = u_*^2$, $\delta = 2u_*v_*$. Substituting these values into (9) and (11), we have
\[ \varepsilon_*(k(i), d) = \frac{(2u_*v_* - 1)d\mu_i - u_*^2}{d\mu_i (d\mu_i + u_*^2)}, \quad d \in [d_i, \infty), \quad i \in \mathbb{N}, \]
and
\[ d_i = \frac{u_*^2}{2u_*v_* - 1} \cdot \frac{1}{\mu_i}, \]
\[ d_{i,j} = \frac{u_*^2}{2(2u_*v_* - 1)} \left[ \frac{1}{\mu_i} + \frac{1}{\mu_j} + \sqrt{\left( \frac{1}{\mu_i} + \frac{1}{\mu_j} \right)^2 + \frac{4(2u_*v_* - 1)}{\mu_i \mu_j}} \right], \]
with $j \geq i + 1$. According to Theorem 2.4, (19) determines the (first) Turing bifurcation curve.

Next, we numerically show how to detect various kinds of spatial patterns that system (18) can exhibit. Choose parameters
\[ p = 1, \quad q = \frac{3}{\pi}, \quad a = 0.1, \quad b = 0.9. \]
Then, $(u_*, v_*) = (1, 0.9)$ and $\varepsilon_1 = 0.1167$. Obviously, $(N_0)$ holds. By (19), (10) and (20), we have
\[ \varepsilon_*(d) = \varepsilon_*(k(i), d) := \frac{4d\mu_i - 5}{5d\mu_i (d\mu_i + 1)}, \quad d \in [d_i,i,i+1), \quad i \in \mathbb{N}, \]
and
\[ d_{i,j} = \frac{5}{8} \left[ \frac{1}{\mu_i} + \frac{1}{\mu_j} + \sqrt{\left( \frac{1}{\mu_i} + \frac{1}{\mu_j} \right)^2 + \frac{16}{5\mu_i \mu_j}} \right], \quad j \geq i + 1, \]
where $k(i) = (n(i), m(i))$ and $\mu_i = (n(i)^2 + \frac{a^2}{\pi^2}m(i)^2)\pi^2$, $i \in \mathbb{N}$. And, these values of $k(i)$, $\mu_i/\pi^2$ and $d_{i,i+1}$ for $i = 1, 2, \ldots, 18$ are listed in Table 1.

According to (1) of Corollary 2, if we select $(d, \varepsilon) \in D_{k(i)}, i \in \mathbb{N}$, then (18) will possess $k(i)$-mode spatially inhomogeneous patterns. For example, for $k(i) = (1, 1)$, a direct calculation yields $(0.13, 0.09) \in D_{k(1)}$. Therefore, for $d = 0.13, \varepsilon = 0.09$, one can observe $(1, 1)$-mode patterns for (18), see Figure 4(a). We also conduct other numerical simulations for different choices of $k(i)$, and these results are summarized in Table 2.

It follows from (3) of Theorem 2.4 that superposition patterns could be induced by consecutive Turing bifurcations with different wave numbers. As an example, when choosing $k(i) = (4, 0)$, $k(j) = (0, 4)$ and $(d, \varepsilon) = (0.025, 0.09) \in D_{k(i), k(j)}$ (see Table 3), we can detect superposition patterns of (18) which are shaped like the graph of (16) and shown in Figure 5(a) and 5(b). For the purpose of comparison,
we also plot the graph of function $z(x, y) = 0.9 + 1.8 \cos(4 \cdot \pi x) \cos(0 \cdot \frac{\pi^2}{3} y) + \cos(0 \cdot \frac{\pi^2}{3} y) \cos(4 \cdot \frac{\pi^2}{3} y)$ for $(x, y) \in (0, 1) \times (0, \frac{\pi}{2})$ in Figure 5(c). It turns out that its graph is pretty much like the pattern of $v(x, y)$ in Figure 5(b), which verifies (3) of Theorem 2.4. In Figure 6, we show another example of the superposition of two kinds of spatial patterns with spatial wave numbers $k(i) = (3, 0)$ and $k(j) = (0, 3)$. Actually, superpositions of multiple patterns are also possible. These patterns in Figure 7 are produced by the interactions of three kinds of spatial patterns with spatial wave numbers $(2, 0), (0, 2)$ and $(2, 2)$. 

Table 1. These values of $k(i)$, $\mu_i/\pi^2$ and $d_{i,i+1}$ when parameters are chosen as $(21)$, $i = 1, 2, \ldots, 28$.

| $i$ | $k(i)$ | $\mu_i/\pi^2$ | $d_{i,i+1}$ | $i$ | $k(i)$ | $\mu_i/\pi^2$ | $d_{i,i+1}$ |
|-----|--------|----------------|-------------|-----|--------|----------------|-------------|
| 1   | (1,0)  | 1              | 0.2834      | 15  | (4,0)  | 16             | 0.0179      |
| 2   | (0,1)  | 1.0966         | 0.2034      | 16  | (4,1)  | 17.0966        | 0.0171      |
| 3   | (1,1)  | 2.0966         | 0.1064      | 17  | (4,2)  | 18.5460        | 0.0164      |
| 4   | (2,0)  | 4              | 0.0709      | 18  | (4,3)  | 19.3865        | 0.0160      |
| 5   | (0,2)  | 4.3865         | 0.0629      | 19  | (4,4)  | 18.8696        | 0.0151      |
| 6   | (2,1)  | 5.0966         | 0.0566      | 20  | (4,5)  | 20.3865        | 0.0142      |
| 7   | (1,2)  | 5.3865         | 0.0449      | 21  | (4,6)  | 21.5460        | 0.0128      |
| 8   | (2,2)  | 8.3865         | 0.0342      | 22  | (5,0)  | 25             | 0.0117      |
| 9   | (3,0)  | 9              | 0.0315      | 23  | (5,1)  | 26.0966        | 0.0113      |
| 10  | (3,1)  | 10.0966        | 0.0283      | 24  | (5,2)  | 26.5460        | 0.0110      |
| 11  | (1,3)  | 10.8696        | 0.0247      | 25  | (5,3)  | 27.4156        | 0.0106      |
| 12  | (2,3)  | 13.3865        | 0.0218      | 26  | (5,4)  | 28.4156        | 0.0103      |
| 13  | (2,2)  | 13.8696        | 0.0199      | 27  | (5,5)  | 29.3865        | 0.0098      |

Table 2. Parameter values of $(d, \varepsilon)$ in $D_1$ satisfying that (18) has $k(i)$-mode Turing patterns.

| $i$ | $k(i)$ | $\varepsilon$ | $(d_i^+(\varepsilon), d_i^-(\varepsilon))$ | $d$ | $(d, \varepsilon) \in$ | Figure |
|-----|--------|----------------|----------------------------------------|-----|-----------------------|--------|
| 3   | (1,1)  | 0.09           | (0.08870, 0.2925)                       | 0.1300 | $D_{k(3),k(5)}$     | 4(a)   |
| 6   | (2,1)  | 0.09           | (0.0365, 0.12034)                      | 0.0660 | $D_{k(4),k(9)}$     | 4(b)   |
| 8   | (2,2)  | 0.09           | (0.02218, 0.0731)                      | 0.0434 | $D_{k(5),k(14)}$    | 4(c)   |
| 9   | (3,0)  | 0.09           | (0.0207, 0.0681)                       | 0.0340 | $D_{k(8),k(17)}$    | 4(d)   |
| 12  | (1,3)  | 0.0 8          | (0.0171, 0.0564)                       | 0.0270 | $D_{k(8),k(25)}$    | 4(e)   |
| 13  | (3,2)  | 0.09           | (0.0139, 0.0458)                       | 0.0260 | $D_{k(8),k(21)}$    | 4(f)   |

Table 3. Parameter values of $(d, \varepsilon)$ in $D_2$ satisfying that (18) has superposition patterns.

| $(d, \varepsilon)$     | $k(i)$ | $k(j)$ | $k(l)$ | Figure |
|------------------------|--------|--------|--------|--------|
| $(0.025, 0.09)$        | $D_{k(8),k(21)}$ | (4,0)  | (0.4)  | 5      |
| $(0.044, 0.09)$        | $D_{k(5),k(14)}$  | (3,0)  | (0.3)  | 6      |
| $(0.094, 0.09)$        | $D_{k(3),k(7)}$   | (2,0)  | (0.2)  | 7      |
Figure 4. Turing patterns for system (18) with different values of $(d, \varepsilon)$ given in Table 2.

Figure 5. (a),(b): For $k(i) = (4, 0)$ and $k(j) = (0, 4)$, there exists superposition pattern of (18) when $(d, \varepsilon) = (0.09, 0.025)$; (c): The graph of function $z(x, y) = 0.9 + 1.8 \cos(4 \cdot \pi x) \cos(0 \cdot \frac{\pi^2}{3} y) + \cos(0 \cdot \pi x) \cos(4 \cdot \frac{\pi^2}{3} y)$.

4. Conclusion. In this paper, we investigate Turing bifurcation and pattern formations for a general two-component reaction-diffusion system on a rectangle spatial domain. The explicit expressions for all Turing bifurcation curves, which explicitly depend on wave number $k$, diffusion rate $d$ and the ratio $\varepsilon$ of diffusion rates, are derived. Therefore, with the aid of these expressions, one can easily predict potential spatial patterns with arbitrary wave number that system (1) could exhibit. Furthermore, the explicit formulas for the first and the second Turing bifurcation curves, which form the boundary of Turing instability and single-mode Turing region, have also been obtained. Especially, interesting spatial phenomenon that
the stability of $E_*$ switches and spatially inhomogeneous patterns with different wave numbers successively emerge, could be observed, by choosing proper ratio $\varepsilon$ of diffusion rates and varying diffusion rate $d$. As an example, we further apply theoretical results to a diffusive Schnakenberg system, and various kinds of spatially inhomogeneous patterns with different spatial wave numbers, including $k = (1, 1), (2, 1), (2, 2), (3, 0), (1, 3), (3, 2), \ldots$, and superposition Turing patterns including superpositions of $(3, 0)$–mode and $(0, 3)$–mode patterns, $(4, 0)$–mode and $(0, 4)$–mode patterns, and $(2, 0)$–mode, $(0, 2)$–mode and $(2, 2)$–mode patterns, have been predicted theoretically and shown numerically, via choosing different diffusion rates $d$ and ratios $\varepsilon$ of diffusion rates.

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