Characteristic Polynomials and Eigenvalues for Certain Classes of Pentadiagonal Matrices

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Received: 13 May 2020; Accepted: 28 June 2020; Published: 1 July 2020

Abstract: There exist pentadiagonal matrices which are diagonally similar to symmetric matrices. In this work we describe explicitly the diagonal matrix that gives this transformation for certain pentadiagonal matrices. We also consider particular classes of pentadiagonal matrices and obtain recursive formulas for the characteristic polynomial and explicit formulas for their eigenvalues.

Keywords: pentadiagonal matrix; symmetric pentadiagonal matrix; eigenvalue; characteristic polynomial

MSC: 15B99; 15A18

1. Introduction

Tridiagonal and pentadiagonal matrices appear in several areas of mathematics and engineering, specially involving linear systems of differential equations. In [1], the authors give necessary and sufficient conditions for a matrix to be diagonally similar to a symmetric matrix. For tridiagonal matrices, the explicit construction of the diagonal matrix involved in this similar transformation is given in [2]. There are several results concerning different types of tridiagonal matrices and the obtaining of their eigenvalues and eigenvectors, see for instance [3] and [4] and the references therein. In the case of pentadiagonal matrices, there are many articles concerning algorithms for solving systems of equations associated with them. Among these works, we mention [5–12]. Furthermore, there are some results for particular cases of pentadiagonal matrices. In [13], the author gives a recurrence formula for the determinant of pentadiagonal matrices \( A = (A_{ij}) \), such that \( A_{ij} \neq 0 \) for \( |i - j| = 1 \). In [14], an algorithm is given to find the determinant of pentadiagonal matrices satisfying \( A_{ij+2} \neq 0 \). In [15], the author shows that the characteristic polynomial for such matrices is the product of two polynomials given in terms of Chebyshev polynomials. In [16], the authors study pentadiagonal Toeplitz matrices and give determinantal identities for the symmetric and skew-symmetric cases. In [17] and [18], banded Toeplitz matrices are studied and, in particular, results on particular banded pentadiagonal Toeplitz matrices are obtained.

Since every \( 3 \times 3 \) matrix is a pentadiagonal one, it is clear that not every pentadiagonal matrix is similar to a symmetric matrix. In this work, we consider two classes of pentadiagonal matrices and obtain recursive formulas for the characteristic polynomials and explicit formulas for the eigenvalues of these classes of pentadiagonal matrices. The paper is organized as follows: in Section 2, we present two classes of pentadiagonal matrices and show explicitly that these matrices are similar to symmetric pentadiagonal ones. In Section 3 we obtain recursive formulas for the characteristic polynomials of this
type of matrices and a result regarding their eigenvalues. In Section 4, we consider special subclasses of pentadiagonal matrices and, using the results of Section 3, we show their eigenvalues and provide some results on the nullity and spectral radius of these matrices.

Let $\mathbb{M}_n(\mathbb{R})$ be the set of square matrices of order $n$ with real entries and let $S \subset \mathbb{M}_n(\mathbb{R})$ be the set of symmetric matrices. We denote the $(i,j)$ entry of a matrix $A$ by $A_{i,j}$, and in some cases, to avoid confusion, we will use the notation $[A]_{i,j}$. The highest integer lower than or equal to $x$ will be denoted by $\lfloor x \rfloor$. A matrix $A \in \mathbb{M}_n(\mathbb{R})$ is called a pentadiagonal matrix if $A_{i,j} = 0$ whenever $|i - j| > 2$. The class of pentadiagonal matrices, denoted by $P_n$, consists of matrices of the form

$$A = \begin{pmatrix} a_1 & b_1 & d_1 & 0 & \ldots & 0 \\ c_1 & a_2 & b_2 & d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & \ldots & c_{n-1} & b_{n-1} \\ 0 & \ldots & 0 & c_{n-2} & a_n \end{pmatrix}.$$  

We will distinguish two subclasses of pentadiagonal matrices, namely:

$$C_1 = \left\{ A \in P_n : b_{2i} = c_{2i} = d_{2i-1} = e_{2i-1} = 0, \text{ for } i = 1, 2, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor \right\}$$

and

$$C_2 = \left\{ A \in P_n : b_{2i-1} = c_{2i-1} = d_{2i} = e_{2i} = 0, \text{ for } i = 1, 2, \ldots, \frac{n}{2} \right\}.$$ 

For a real symmetric matrix $A = (A_{i,j})$ of order $n$, the non-directed graph $G(A)$ associated to $A$ consists of vertices $\{1, 2, \ldots, n\}$ and edges $\{i,j\}$ for which $i \neq j$ and $A_{i,j}A_{j,i} \neq 0$. The matrix $A$ is acyclic if $G(A)$ has no cycles.

2. Pentadiagonal Matrices Similar to Symmetric Matrices

In [1], the authors give necessary and sufficient conditions for a matrix $A = (A_{i,j})$ to be diagonally similar to a symmetric matrix, namely

$$A_{i,j} \neq 0 \text{ implies } A_{j,i} \neq 0,$$

and for any sequence of integers $i_1, \ldots, i_r$ such that $1 \leq i_k \leq n$, $k = 1, \ldots, r$ the following holds

$$A_{i_1,i_2}A_{i_2,i_3}\ldots A_{i_{r-1},i_r}A_{i_r,i_1} = A_{i_1,i_2}A_{i_2,i_3}\ldots A_{i_{r-1},i_r}A_{i_r,i_1}.$$

The construction of the diagonal matrix for the case of tridiagonal matrices is done in [2]. For acyclic matrices is given in [19] and [20]. In the following theorem we impose conditions to matrices $A \in C_1 \cup C_2$ and give the explicit construction of a diagonal matrix $D$ such that $DAD^{-1}$ is a symmetric pentadiagonal matrix.

**Theorem 1.** Let $A \in C_1 \cup C_2$. If any of the following conditions holds:

(i) $A \in C_1$ satisfies $b_ic_i > 0$ or $b_i = c_i = 0$, for $i = 1, \ldots, n - 1$;

(ii) $A \in C_2$ satisfies $d_ie_i > 0$ or $d_i = e_i = 0$, for $i = 1, \ldots, n - 2$,

then $A$ is similar to a symmetric pentadiagonal matrix.

**Proof.** Under assumptions (i) or (ii), $A$ satisfies conditions (1) and (2), therefore the existence of $D$ is guaranteed. If $b_ic_i > 0$ (respectively $d_ie_i > 0$) then the sign of $b_i$ and $c_i$ (respectively $d_i$ and $e_i$) are
the same. Furthermore, if \( b_i = 0 \) (respectively if \( d_i = 0 \)) then \( c_i = 0 \) (respectively \( e_i = 0 \)). We define \( \sigma_i \) (respectively \( \tau_i \)) in the following manner:

\[
\sigma_i = \begin{cases} 
1, & \text{if } b_i > 0; \\
0, & \text{if } b_i = 0; \\
-1, & \text{if } b_i < 0,
\end{cases} \quad \text{and} \quad \tau_i = \begin{cases} 
1, & \text{if } d_i > 0; \\
0, & \text{if } d_i = 0; \\
-1, & \text{if } d_i < 0.
\end{cases}
\]

We will consider each condition separately. Assume \( A \) satisfies condition (i). We will prove that \( A \) is similar to a symmetric pentadiagonal matrix \( R_n \) of order \( n \) of the form:

- If \( n \) is even:

\[
R_n = \begin{pmatrix}
a_1 & \sqrt{e_1} & 0 & 0 & \cdots & 0 \\
\sqrt{e_1} & a_2 & \sqrt{e_2} & 0 & \cdots & 0 \\
0 & \sqrt{e_2} & a_3 & \sqrt{e_3} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \sqrt{e_{n-1}} & a_{n-1} & \sqrt{e_n} \\
0 & \cdots & 0 & \sqrt{e_{n-1}} & a_{n-1} & a_n
\end{pmatrix}
\]

(3)

- If \( n \) is odd:

\[
R_n = \begin{pmatrix}
R_{n-1} & 0 \\
0 & a_n
\end{pmatrix},
\]

where \( R_{n-1} \) is of the form (3).

Let \( D = \text{diag}\{a_1, a_2, \ldots, a_n\} \) be the diagonal matrix whose diagonal entries are given, recursively, by:

\[
\begin{align*}
a_1 &= 1; \\
n_k &= \sqrt{\frac{b_{2k-1}}{e_{2k-1}}} a_{2k-1}, & & \text{for } k = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor; \\
n_k &= \sqrt{\frac{d_{2k}}{e_{2k}}} a_{2k}, & & \text{for } k = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1; \\
a_n &= 1, & & \text{if } n \text{ is odd.}
\end{align*}
\]

In the previous formula:

- If \( b_{2j-1} = c_{2j-1} = 0 \) for some \( j \), then we replace \( a_{2j} = a_{2j-1} = \sqrt{\frac{d_{2j-2}}{e_{2j-2}}} a_{2j-2} \).

- If \( d_{2j} = c_{2j} = 0 \) for some \( j \), then we replace \( a_{2j+1} = \sqrt{\frac{c_{2j+1}}{b_{2j+1}}} a_{2j} \).

Consider now \( X = DAD^{-1} \). Then \( X = (X_{ij}) = (a_i a_j^{-1}) \), where

- If \( i \) is odd, then

\[
A_{ij} = \begin{cases} 
\alpha_i, & \text{if } j = i; \\
b_i, & \text{if } j = i + 1; \\
0, & \text{elsewhere,}
\end{cases} \quad \text{and} \quad [R_n]_{ij} = \begin{cases} 
\alpha_i, & \text{if } j = i; \\
\sqrt{e_i} a_i, & \text{if } j = i + 1; \\
0, & \text{elsewhere.}
\end{cases}
\]

- If \( i \) is even, then

\[
A_{ij} = \begin{cases} 
e_{i-2}, & \text{if } j = i - 2; \\
\tau_{i-1}, & \text{if } j = i - 1; \\
\alpha_i, & \text{if } j = i; \\
\sqrt{d_i} a_i, & \text{if } j = i + 2; \\
0, & \text{elsewhere,}
\end{cases} \quad \text{and} \quad [R_n]_{ij} = \begin{cases} 
\tau_{i-2} \sqrt{d_i e_{i-2}}, & \text{if } j = i - 2; \\
\tau_{i-1} \sqrt{b_i e_{i-1}}, & \text{if } j = i - 1; \\
\alpha_i, & \text{if } j = i; \\
\tau_i \sqrt{d_i e_i}, & \text{if } j = i + 2; \\
0, & \text{elsewhere.}
\end{cases}
\]
Then we obtain:

(a) If $i$ is odd, then

$$X_{i,i} = a_i = [R_n]_{i,i}, \quad \text{and} \quad X_{i,i+1} = a_i b_i a_{i+1}^{-1} = a_i b_i \sqrt{\frac{c_i}{b_i}} a_{i+1}^{-1} = c_i \sqrt{b_i c_i} = [R_n]_{i,j+1}.$$ 

(b) If $i$ is even, then

$$X_{i,i-2} = a_i c_{i-2} a_{i-1}^{-1} = \sqrt{\frac{b_{i-1}}{c_{i-1}}} \sqrt{\frac{d_{i-2}}{e_{i-2}}} \sqrt{\frac{c_{i-1}}{b_{i-1}}} a_{i-2} a_{i-1}^{-1} = \tau_{i-2} \sqrt{d_{i-1} e_{i-2}} = [R_n]_{i,i-2},$$

$$X_{i,i-1} = a_i c_{i-1} a_{i-1}^{-1} = \sqrt{\frac{b_{i-1}}{c_{i-1}}} a_{i-1} c_{i-1} a_{i-1}^{-1} = \tau_{i-1} \sqrt{b_{i-1} c_{i-1}} = [R_n]_{i,i-1},$$

$$X_{i,i} = a_i a_i a_i^{-1} = [R_n]_{i,i},$$

$$X_{i,i+2} = a_i d_i a_{i+2}^{-1} = a_i \sqrt{\frac{c_{i+1}}{b_{i+1}}} \sqrt{\frac{d_{i+1}}{e_{i+1}}} \sqrt{\frac{b_{i+1}}{c_{i+1}}} a_{i+1}^{-1} = \tau_i \sqrt{d_i e_i} = [R_n]_{i,i+2}.$$ 

Therefore $X = R_n$ and $A$ is similar to a symmetric pentagonal matrix.

Assume now that $A$ satisfies condition (ii). We will prove that $A$ is similar to a symmetric matrix $T_n$ of the form:

- If $n$ is odd:

$$T_n = \begin{pmatrix}
    a_1 & 0 & \frac{\tau_1 \sqrt{d_1}}{c_1} & 0 & \ldots & 0 \\
    0 & a_2 & 0 & \tau_2 \sqrt{d_2} c_2 & 0 & \ldots \\
    \frac{\tau_1 \sqrt{d_1}}{c_1} & 0 & a_3 & 0 & \tau_3 \sqrt{d_3} c_3 & 0 & \ldots \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\
    0 & \ldots & 0 & \tau_{n-2} \sqrt{d_{n-2}} c_{n-2} & a_{n-1} & 0 & \tau_{n-1} \sqrt{d_{n-1}} c_{n-1} & 0 \\
    0 & \ldots & 0 & 0 & a_n & \tau_{n-1} \sqrt{d_{n-1}} c_{n-1} & \ldots & \ldots & a_n
\end{pmatrix}$$

(4)

- If $n$ is even:

$$T_n = \begin{pmatrix}
    T_{n-1} & 0 \\
    0 & a_n
\end{pmatrix}$$

where $T_{n-1}$ is of the form (4).

Let $D = \text{diag}\{\beta_1, \beta_2, \ldots, \beta_n\}$ be the diagonal matrix defined by:

$$\beta_1 = 1;$$

$$\beta_{2k} = \sqrt{\frac{d_{2k-1}}{c_{2k-1}}} \sqrt{\frac{c_{2k}}{b_{2k}}} \beta_{2k-1}, \quad \text{for} \quad k = 1, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor;$$

$$\beta_{2k+1} = \sqrt{\frac{b_{2k}}{c_{2k}}} \beta_{2k}, \quad \text{for} \quad k = 1, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor;$$

$$\beta_n = 1, \quad \text{if} \quad n \text{ is even.}$$

In the previous formula:

- If $b_{2j} = c_{2j} = 0$ for some $j$, then we replace $\beta_{2j+1} = \beta_{2j} = \sqrt{\frac{d_{2j-1}}{c_{2j-1}}} \beta_{2j-1}.$

- If $d_{2j-1} = c_{2j-1} = 0$ for some $j$, then we replace $\beta_{2j} = \sqrt{\frac{c_{2j}}{b_{2j}}} \beta_{2j-1}.$

Analogously to case (i), we obtain $DAD^{-1} = T_n$ and the proof is complete.
Corollary 1. If $A$ is a pentadiagonal matrix under the hypothesis of Theorem 1, then all the eigenvalues of $A$ are real and $A$ is diagonalizable.

Example 1. Consider the pentadiagonal matrix,

$$A = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
3 & 2 & -2 & 0 & 0 \\
0 & 0 & -1 & -2 & 0 \\
0 & -1 & -5 & 3 & 0 \\
0 & 0 & 0 & 0 & 27 \\
0 & 0 & 0 & 5 & 7
\end{pmatrix}.$$  

Since $A$ satisfies the conditions of Theorem 1, it is similar to the symmetric pentadiagonal matrix

$$B = \begin{pmatrix}
1 & \sqrt{3} & 0 & 0 & 0 \\
\sqrt{3} & 2 & 0 & -\sqrt{2} & 0 \\
0 & 0 & -1 & -\sqrt{10} & 0 \\
0 & -\sqrt{2} & -\sqrt{10} & 3 & \sqrt{20} \\
0 & 0 & 0 & 0 & 27 \\
0 & 0 & 0 & \sqrt{20} & 7
\end{pmatrix},$$

by the diagonal matrix $D = \text{diag} \{1, \sqrt{3}, \sqrt{5}, \sqrt{2}, \sqrt{8}, \sqrt{15}, \sqrt{15}\}$.

Remark 1. The pentadiagonal matrices of Theorem 1 are, in general, not similar by permutation to tridiagonal matrices. In fact, the graph associated to a pentadiagonal matrix considered in Theorem 1 is a caterpillar, while the graph associated to a tridiagonal matrix is a path. Since a caterpillar is, in general, not a path, the problem of finding a symmetric matrix similar to a given pentadiagonal matrix one, is not equivalent to the related problem regarding tridiagonal matrices.

3. On Characteristic Polynomials

In this section we give recursive formulas for the characteristic polynomials of the considered matrices.

Let $A_n \in \mathcal{C}_1 \cap \mathcal{S}$ and $B_n \in \mathcal{C}_2 \cap \mathcal{S}$, i.e.,

$$A_n = \begin{pmatrix}
a_1 & b_1 & 0 & 0 & 0 & \cdots & 0 \\
b_1 & a_2 & d_2 & 0 & 0 & \cdots & 0 \\
0 & a_3 & b_3 & d_3 & 0 & \cdots & 0 \\
0 & d_3 & b_3 & a_5 & d_5 & \cdots & 0 \\
0 & 0 & 0 & a_5 & b_5 & \cdots & 0 \\
0 & 0 & 0 & d_5 & b_5 & a_6 & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & a_{n-2} & 0 & d_{n-2} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & a_{n-1} & b_{n-1} \\
0 & 0 & 0 & 0 & 0 & \cdots & d_{n-2} & b_{n-1} & a_n
\end{pmatrix} \quad \text{and} \quad B_n = \begin{pmatrix}
a_1 & 0 & d_1 & 0 & 0 & \cdots & 0 \\
0 & a_2 & b_2 & 0 & 0 & \cdots & 0 \\
0 & d_2 & a_3 & b_3 & 0 & \cdots & 0 \\
0 & 0 & d_3 & a_5 & b_5 & \cdots & 0 \\
0 & 0 & 0 & a_5 & b_5 & \cdots & 0 \\
0 & 0 & 0 & 0 & a_6 & \cdots & 0 \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & a_{n-2} & 0 & d_{n-2} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & a_{n-1} & b_{n-1} \\
0 & 0 & 0 & 0 & 0 & \cdots & d_{n-2} & b_{n-1} & a_n
\end{pmatrix},$$

where the entries $b_{n-1}$ and $d_{n-2}$ are equal to zero according to the parity of the order of each matrix.
Proposition 1. Let $P_0(\lambda) = 1$. The characteristic polynomial $P_n(\lambda)$ of $A_n$ is given by the recursion formula:

\[ P_1(\lambda) = (a_1 - \lambda), \]
\[ P_2(\lambda) = (a_2 - \lambda)(a_1 - \lambda) - b_1^2, \]
\[ P_n(\lambda) = \left\{ \begin{array}{ll}
(a_n - \lambda)P_{n-1}(\lambda) & \text{if } n \text{ is odd, } n \geq 3 \\
[(a_n - \lambda)(a_{n-1} - \lambda) - b_{n-1}^2]P_{n-2}(\lambda) - d_{n-2}^2(a_n - \lambda)(a_{n-3} - \lambda)P_{n-4}(\lambda) & \text{if } n \text{ is even, } n \geq 4.
\end{array} \right. \]

Proof. By computing the determinant along the last row, we obtain

\[ P_n(\lambda) = (a_n - \lambda)P_{n-1}(\lambda) - b_{n-1}^2P_{n-2}(\lambda) - d_{n-2}^2(a_n - \lambda)P_{n-3}(\lambda). \]

Clearly, if $k$ is odd, we obtain that $P_k(\lambda) = (a_k - \lambda)P_{k-1}(\lambda)$. By replacing in the previous formula, we obtain the result. \(\square\)

Analogously, we obtain:

Proposition 2. The characteristic polynomial $Q_n(\lambda)$ of $B_n$ is given by the recursion formula:

\[ Q_1(\lambda) = a_1 - \lambda, \]
\[ Q_2(\lambda) = (a_2 - \lambda)(a_1 - \lambda), \]
\[ Q_3(\lambda) = (a_1 - \lambda)(a_2 - \lambda)(a_3 - \lambda) - b_2^2(a_1 - \lambda) - d_1^2(a_2 - \lambda), \]
\[ Q_n(\lambda) = \left\{ \begin{array}{ll}
(a_n - \lambda)Q_{n-1}(\lambda) & \text{if } n \text{ is even, } n \geq 4 \\
[(a_n - \lambda)(a_{n-1} - \lambda) - b_{n-1}^2]Q_{n-2}(\lambda) - d_{n-2}^2(a_n - \lambda)(a_{n-3} - \lambda)Q_{n-4}(\lambda) & \text{if } n \text{ is odd, } n \geq 5.
\end{array} \right. \]

The following two corollaries show that the symmetric pentadiagonal matrices in $C_1$ or $C_2$ have exactly $n$ distinct eigenvalues.

Corollary 2. Let $A_n \in C_1 \cap S$ such that $a_i = a$ for $i = 1 \ldots n$. If $b_{2j-1}d_{2k} \neq 0$ for $j = 1 \ldots \lfloor \frac{n}{2} \rfloor$ and $k = 1 \ldots \lfloor \frac{n-2}{2} \rfloor$, then the eigenvalues of $A_n$ are all simple.

Proof. First of all, notice that $P_{2k}(a) \neq 0$ for $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$. If $P_{2k}(a) = 0$ for some $k$, this would imply that $P_{2j}(a) = 0$ for $1 \leq j < k$, but $P_2(a) = -b_1^2 \neq 0$, contradicting the assumption.

Since for a real symmetric matrix, the eigenvalues of a submatrix of lower order interlace the eigenvalues of the matrix, then the zeros of $P_{2k-1}(\lambda)$ interlace the zeros of $P_{2k}(\lambda)$. Assume now that $P_{2k}(\lambda)$ and $P_{2k-1}(\lambda)$ have a common zero $\mu$. If $0 = P_{2k-1}(\mu) = (a - \mu)P_{2k-2}(\mu)$ we must have $\mu = a$ or $P_{2k-2}(\mu) = 0$. By the previous, $\mu = a$ is not a zero of $P_{2k}(\lambda)$, therefore $\mu \neq a$ and $P_{2k-2}(\mu) = 0$. In this case, $0 = P_{2k}(\mu) = -d_{2k-2}(a - \mu)^2P_{2k-4}(\mu)$ and then $P_{2k-4}(\mu) = 0$. This implies that $P_{2j}(\mu) = 0$ for $1 \leq j < k$, but $P_{2j}(\mu) = -d_{2j}^2(a - \mu)^2P_{2j}(\mu) \neq 0$ contradicting the assumption, therefore $P_{2k}(\lambda)$ and $P_{2k-1}(\lambda)$ do not have a common zero and then all the eigenvalues of $A_n$ are simple. \(\square\)

Corollary 3. Let $B_n \in C_2 \cap S$ such that $a_i = a$ for $i = 1 \ldots n$. If $b_{2j}d_{2k-1} \neq 0$ for $j = 1 \ldots \lfloor \frac{n-2}{2} \rfloor$ and $k = 1 \ldots \lfloor \frac{n}{2} \rfloor$, then the eigenvalues of $B_n$ are all simple.

Proof. Follows analogously to the proof of Corollary 2. \(\square\)

Remark 2. As we mentioned before, there are some works concerning the obtaining of formulas for the characteristic polynomial, eigenvalues and eigenvectors of a pentadiagonal matrix $A = (A_{ij})$. In [13], the author consider matrices satisfying $A_{ij} \neq 0$ for $|i-j| = 1$. In [14], the matrices are constrain to the condition $A_{i,i+2} \neq 0$. As one can easily see, the classes $C_1$ and $C_2$ are not contained in the previous cases.
4. Spectrum of Special Classes of Pentadiagonal Matrices

Here we apply the previous results to special classes of pentadiagonal matrices to obtain explicit formulas of the eigenvalues and some results concerning them.

4.1. The Class $C^a_1$

Consider the families of subclasses of $C_1$, given by:

$$C^a_1 = \begin{cases} 
A_n \in C_1 : 
& a_i = a, \quad \text{for } i = 1, \ldots, n; \\
& b_{2i-1} = c_{2i-1} = b \geq 0, \quad \text{for } i = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor; \\
& d_{2i} = e_{2i} = d \geq 0, \quad \text{for } i = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1.
\end{cases}$$

So, the elements of $C^a_1$ are of the form:

$$A_{2k} = \begin{pmatrix} 
& a & b & 0 & 0 & \ldots & 0 \\
& b & a & 0 & d & \ddots & \\
& 0 & 0 & \ddots & \ddots & \ddots & 0 \\
& \vdots & \ddots & \ddots & \ddots & \ddots & d \\
& 0 & \ddots & \ddots & \ddots & \ddots & b \\
& 0 & \ldots & 0 & d & b & a
\end{pmatrix} \quad \text{and} \quad A_{2k+1} = \begin{pmatrix} 
A_{2k} \\
0 \\
a
\end{pmatrix}.$$

**Remark 3.** Every matrix $A \in C^a_1$ can be decomposed as $A = aI + B$ where $B \in C^0_1$. Then every eigenvalue of $A$ is of the form

$$\lambda_i(A) = a + \lambda_i(B) \quad \text{for } i = 1, \ldots, n.$$

Therefore, in order to obtain the spectra of $C^a_1$, it is enough to study the class $C^0_1$.

In view of this, we present in the next theorem the characteristic polynomial for matrices in this class.

**Theorem 2.** Let $P_n(\lambda)$ be the characteristic polynomial of $A_n \in C^0_1$.

(i) If $n = 2k$, then

$$P_{2k}(\lambda) = \sum_{l=0}^{k} (-1)^l \binom{k-l}{l} (\lambda^2 - b^2)^{k-l} (d\lambda)^{2l}.$$  

(ii) If $n = 2k + 1$, then $P_{2k+1}(\lambda) = -\lambda P_{2k}(\lambda)$.

**Proof.** By Proposition 1 we obtain that

$$P_{2k}(\lambda) = (\lambda^2 - b^2)P_{2k-2}(\lambda) - d^2\lambda^2 P_{2k-4}(\lambda).$$

Let $r = \lambda^2 - b^2$ and $s = d^2\lambda^2$. Then $P_{2k}(\lambda)$ can be obtained as

$$P_{2k}(\lambda) = ax_1^k + bx_2^k,$$

where $x_1, x_2$ are the roots of $x^2 - rx + s = 0$, this is

$$x_{1,2} = \frac{r \pm \sqrt{r^2 - 4s}}{2}.$$ 

Set $r = 2\sqrt{s} \cos \theta$. Then

$$x_{1,2} = \frac{2\sqrt{s} \cos \theta \pm \sqrt{4s \cos^2 \theta - 4s}}{2} = \frac{2\sqrt{s} \cos \theta \pm i2\sqrt{s} \sin \theta}{2} = \sqrt{s} e^{\pm i\theta} = d\lambda e^{\pm i\theta}.$$
Theorem 3. The eigenvalues of $A$ are obtained by replacing the formula with the initial conditions

$$P_0(\lambda) = 1, \quad P_2(\lambda) = \lambda^2 - b^2.$$ 

Then

$$1 = \alpha + \beta, \quad 2\sqrt{s} \cos \theta = \alpha \sqrt{s} [\cos \theta + i \sin \theta] + \beta \sqrt{s} [\cos \theta - i \sin \theta],$$

and we obtain that

$$\alpha = \frac{1}{2} \left[ 1 - i \frac{\cos \theta}{\sin \theta} \right] \quad \text{and} \quad \beta = \frac{1}{2} \left[ 1 + i \frac{\cos \theta}{\sin \theta} \right].$$

Therefore

$$P_{2k}(\lambda) = \frac{1}{2} \left[ 1 - i \frac{\cos \theta}{\sin \theta} \right] (d\lambda)^k e^{i\theta} + \frac{1}{2} \left[ 1 + i \frac{\cos \theta}{\sin \theta} \right] (d\lambda)^k e^{-i\theta},$$

and by reducing this expression, we obtain

$$P_{2k}(\lambda) = (d\lambda)^k \frac{\sin((k+1)\theta)}{\sin \theta}. \quad (6)$$

Now, by considering the recursion formula

$$\sin(m\theta) = \sin \theta \sum_{l=0}^{m-1} (-1)^l \binom{m-1}{l} 2^{m-2l-1} (\cos \theta)^{m-2l-1}, \quad (7)$$

and since $\cos \theta = \frac{\lambda^2 - b^2}{2d\lambda}$, we obtain that

$$P_{2k}(\lambda) = \sum_{l=0}^{k-1} (-1)^l \binom{k-1}{l} (\lambda^2 - b^2)^{k-2l} (d\lambda)^{2l}. \quad (8)$$

As we mentioned in the proof of Proposition 1, $P_{2k+1}(\lambda) = -\lambda P_{2k}(\lambda)$ and the result follows. \(\square\)

**Corollary 4.** Consider $A_n \in C_1^0$. Then

$$\det(A_n) = \left\{ \begin{array}{ll} (-1)^k b^{2k} & \text{if } n = 2k, \\ 0 & \text{if } n = 2k + 1. \end{array} \right.$$ 

Now we are ready to stabilise the main result of this section:

**Theorem 3.** The eigenvalues of $A_n \in C_1^0$ of order $n = 2k$, are given by the following formulas:

$$\mu_j = d \cos \left( \frac{\pi j}{k+1} \right) + \sqrt{d^2 \cos^2 \left( \frac{\pi j}{k+1} \right) + b^2} \quad \text{and} \quad \mu_{k+j} = d \cos \left( \frac{\pi j}{k+1} \right) - \sqrt{d^2 \cos^2 \left( \frac{\pi j}{k+1} \right) + b^2},$$

for $j = 1, \ldots, k$.

**Proof.** Consider equation (6). Then $P_{2k}(\lambda) = 0$ if and only if $\sin((k+1)\theta) = 0$ or $\lambda = 0$.

Set now $t^\frac{1}{2} = e^{i\theta}$. Then

$$\sin((k+1)\theta) = \frac{t^{k+1} - t^{-k+1}}{2i} = \frac{\mu^{k+1} - 1}{2it^{k+1}} \quad \text{and} \quad \sin \theta = \frac{t^{1/2} - t^{-1/2}}{2i} = \frac{t - 1}{2it^{1/2}}.$$ 

Therefore

$$0 = \frac{\sin((k+1)\theta)}{\sin \theta} = \left( \frac{1}{\mu^{k+1}} \right) \frac{\mu^{k+1} - 1}{t - 1}.$$
and all the roots of this equation are the \((k + 1)\)-th roots of unit distinct of 1. Let \(\xi_j = e^{i(\frac{2\pi j}{k+1})}\) for \(j = 1, \ldots, k\) be these roots. Then \(e^{\theta_j} = \xi_j^{\frac{1}{k}} = \left(e^{i(\frac{2\pi j}{k+1})}\right)^{\frac{1}{2}}\) and \(\theta_j = \frac{\pi j}{k+1}\) for \(j = 1, \ldots, k\).

Since, by the proof of Theorem 2, \(\cos \theta = \frac{\lambda^2 - b^2}{2d\lambda}\), we obtain that \(\lambda^2 - (2d\cos \theta)\lambda - b^2 = 0\), and then

\[
\lambda = \frac{2d\cos \theta \pm \sqrt{4d^2 \cos^2 \theta + 4b^2}}{2}.
\]

By replacing,

\[
\lambda = \frac{2d\cos \left(\frac{\pi j}{k+1}\right) \pm \sqrt{4d^2 \cos^2 \left(\frac{\pi j}{k+1}\right) + 4b^2}}{2} = d \cos \left(\frac{\pi j}{k+1}\right) \pm \sqrt{d^2 \cos^2 \left(\frac{\pi j}{k+1}\right) + b^2}.
\]

Therefore, all the eigenvalues of \(P_{2k}(\lambda)\) are given by:

\[
\mu_j = d \cos \left(\frac{\pi j}{k+1}\right) + \sqrt{d^2 \cos^2 \left(\frac{\pi j}{k+1}\right) + b^2} \quad \text{and} \quad \mu_{k+j} = d \cos \left(\frac{\pi j}{k+1}\right) - \sqrt{d^2 \cos^2 \left(\frac{\pi j}{k+1}\right) + b^2} \quad (9)
\]

for \(j = 1, \ldots, k\).

Since we have obtained \(2k\) roots of \(P_{2k}(\lambda)\), we conclude that these are all the eigenvalues of \(A_n\). \(\square\)

**Remark 4.** \(\lambda = 0\) is a root of \(P_{2k}(\lambda)\) if and only if \(b = 0\). In fact, if \(\lambda = 0\) is a root of \(P_{2k}(\lambda)\) then \(0 = P_{2m}(0)\) for every \(m < k\). Since \(P_2(0) = -b^2\) then we must have \(b = 0\). Equation (9) says that the condition \(b = 0\) is necessary in order to obtain \(\lambda = 0\) as an eigenvalue of an even order matrix \(A_n\).

**Corollary 5.** If \(b = 0\) in \(A_n\), then the multiplicity of \(\lambda = 0\) is

\[
\text{mult}(0) = \begin{cases} 
2m, & \text{if } n = 4m; \\
2m + 1, & \text{if } n = 4m + 1; \\
2m + 2, & \text{if } n = 4m + 2; \\
2m + 3, & \text{if } n = 4m + 3. 
\end{cases}
\]

**Proof.** If \(b = 0\) and \(n = 2k\), we obtain by equation (9), that \(\mu_j = 2 \cos \left(\frac{\pi j}{k+1}\right)\) and \(\mu_{k+j} = 0\) for \(j = 1, \ldots, k\). Since \(\cos \left(\frac{\pi j}{k+1}\right) = 0\) if and only if \((k + 1)\) is even and \(j = \frac{k + 1}{2}\), we obtain multiplicity \(k\) for \(n = 2(2m)\) and multiplicity \(k + 1\) for \(n = 2(2m + 1)\). If \(n\) is odd, the eigenvalues of \(A_n\) are \(\lambda = 0\) and the \((n - 1)\) eigenvalues of \(A_{n-1}\). Then the result follows. \(\square\)

**Corollary 6.** The spectral radius of \(A_n \in C^0_{\lambda}\) is \(\rho = d \cos \left(\frac{\pi}{k+1}\right) + \sqrt{d^2 \cos^2 \left(\frac{\pi}{k+1}\right) + b^2}\).

**Proof.** Consider \(f(j) = d \cos \left(\frac{\pi j}{k+1}\right) + \sqrt{d^2 \cos^2 \left(\frac{\pi j}{k+1}\right) + b^2}\). Then

\[
f'(j) = -f(j) \frac{d \sin \left(\frac{\pi j}{k+1}\right) \frac{\pi}{k+1}}{\sqrt{d^2 \cos^2 \left(\frac{\pi j}{k+1}\right) + b^2}}
\]
By Remark 4, \( f(j) = 0 \) if and only if \( b = 0 \) and in this case \( f'(j) = -2d \sin \left( \frac{\pi}{k+1} \right) \frac{\pi}{k+1} \neq 0 \) for \( j = 1, \ldots, k \). Therefore the critical points of \( f(j) \) are in \( j = 1 \) and \( j = k \). If \( b \neq 0 \), we conclude by the previous analysis, that \( j = 1 \) and \( j = k \) are the critical points. Moreover, since \( f(1) > 0 \) and \( f(k) < 0 \), we obtain that \( f(j) \) has a maximum value at \( j = 1 \), and the proof is complete.

**Example 2.** If \( d = 0 \) in \( \mathcal{A}_n \) of even order \( n = 2k \), we obtain a tridiagonal matrix and the eigenvalues are \( b \) and \(-b\), both with multiplicity \( k \).

**Example 3.** Consider \( k = 5 \). Then \( \theta_j = \frac{\pi j}{6} \) for \( j = 1, \ldots, 5 \). Therefore the eigenvalues of \( A_{10} \), are:

\[
\begin{align*}
\mu_1 &= d \left( \frac{\sqrt{3}}{2} \right) + \sqrt{d^2 \left( \frac{3}{4} \right) + b^2} = \frac{d\sqrt{3} + \sqrt{3d^2 + 4b^2}}{2}, \\
\mu_2 &= d \left( \frac{1}{2} \right) + \sqrt{d^2 \left( \frac{1}{4} \right) + b^2} = \frac{d + \sqrt{d^2 + 4b^2}}{2}, \\
\mu_3 &= \sqrt{b^2} = b, \\
\mu_4 &= d \left( -\frac{1}{2} \right) + \sqrt{d^2 \left( \frac{1}{4} \right) + b^2} = \frac{-d + \sqrt{d^2 + 4b^2}}{2}, \\
\mu_5 &= d \left( -\frac{\sqrt{3}}{2} \right) + \sqrt{d^2 \left( \frac{3}{4} \right) + b^2} = \frac{-d\sqrt{3} + \sqrt{3d^2 + 4b^2}}{2}, \\
\mu_6 &= d \sqrt{3} - \sqrt{3d^2 + 4b^2}, \\
\mu_7 &= \frac{d - \sqrt{d^2 + 4b^2}}{2}, \\
\mu_8 &= -b, \\
\mu_9 &= \frac{-d - \sqrt{d^2 + 4b^2}}{2}, \\
\mu_{10} &= \frac{-d\sqrt{3} - \sqrt{3d^2 + 4b^2}}{2}.
\end{align*}
\]

4.2. On the Class \( C^0_2 \)

Analogously to the class \( C^1_2 \), one can define the families of subclasses of \( C_2 \), given by:

\[
C^0_2 = \left\{ B_n \in C_2 : \begin{array}{ll}
    a_i = a & \text{for } i = 1, \ldots, n; \\
    b_{2i} = c_{2i} = b & \text{for } i = 1, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor; \\
    d_{2i-1} = e_{2i-1} = d & \text{for } i = 1, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor.
\end{array} \right\}
\]

Then elements of \( C^0_2 \) are of the form:

\[
B_{2k-1} = \begin{pmatrix}
a & 0 & d & 0 & \cdots & 0 \\
0 & a & b & 0 & \cdots & \vdots \\
d & b & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & d \\
0 & \cdots & 0 & d & b & a
\end{pmatrix}
\]

and

\[
B_{2k} = \begin{pmatrix}
B_{2k-1} & 0 \\
0 & a
\end{pmatrix}.
\]

We will only consider the class \( C^0_2 \), since the general case \( C^q_2 \) follows by Remark 3. Using Propositions 1 and 2 and replacing the coefficients by \( b \) and \( d \) properly, we obtain the following recursion formula:

\[
Q_{2k+1}(\lambda) = -\lambda P_{2k}(\lambda) + \lambda d^2 P_{2k-2}(\lambda), \\
Q_{2k+2}(\lambda) = -\lambda Q_{2k+1}(\lambda).
\]

(10)
By equations (10) and (6), we obtain

\[ Q_{2k+1}(\lambda) = \lambda \left[ d^k \lambda^{k-1} \left( -\lambda \frac{\sin((k+1)\theta)}{\sin \theta} + d \frac{\sin(k\theta)}{\sin \theta} \right) \right]. \]

Therefore, \( Q_{2k+1}(\lambda) \) has one root \( \lambda = 0 \) and the remaining are given by \( \lambda = \frac{d \sin(k\theta)}{\sin((k+1)\theta)} \).

Furthermore, by the proof of Theorem 2, \( \lambda^2 - (2d \cos \theta)\lambda - b^2 = 0 \). These together imply

\[ d^2 \sin^2(k\theta) - 2d^2 \cos \theta \sin(k\theta) \sin((k+1)\theta) - b^2 \sin^2((k+1)\theta) = 0. \]

A straightforward computation shows that

\[ d^2 \sin(k\theta) \sin((k+2)\theta) + b^2 \sin^2((k+1)\theta) = 0. \]

Again, if \( t^\frac{1}{2} = e^{i \theta} \), then we obtain

\[ \frac{(t^k - 1)(t^{k+2} - 1)}{(t^{k+1} - 1)^2} = -\frac{b^2}{d^2}, \]

and

\[ \sum_{i=0}^{k-1} t^{i} \frac{t^{k+1} \sum_{j=0}^{i}}{t^{i}} = -\frac{b^2}{d^2}. \]

Therefore

\[ \sum_{s=0}^{k-1} (s+1)(b^2 + d^2)t^s + (kd^2 + (k+1)b^2)t^k + \sum_{s=0}^{k-1} (s+1)(b^2 + d^2)t^{2k-s} = 0. \quad (11) \]

If \( t_i, \ i = 1, \ldots, 2k, \) are the roots of equation (11), then

\[ \lambda_i = \frac{dt^\frac{1}{2}(t_i^k - 1)}{(t_i^{k+1} - 1)}, \quad i = 1, \ldots 2k \]

are the remaining roots of \( Q_{2k+1}(\lambda) \).

**Remark 5.** If \( b \neq 0 \) or \( d \neq 0 \) in \( B_n \), the multiplicity of \( \lambda = 0 \) is

\[ \text{mult}(0) = \begin{cases} 1, & \text{if } n \text{ is odd;} \\ 2, & \text{if } n \text{ is even.} \end{cases} \]

5. Conclusions

In this work, we have considered certain pentadiagonal matrices that are diagonally similar to symmetric pentadiagonal matrices. We provided recursive formulas for the characteristic polynomial of symmetric pentadiagonal matrices, and for particular subclasses we have given explicit formulas for the eigenvalues.

Author Contributions: All authors have contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Fondo Puente de Investigación de Excelencia—FPI-18-02 and Coloquio de Matemática from Universidad de Antofagasta.
Acknowledgments: The authors thank the referees for their comments and suggestions which improved the presentation of this paper. The researchers A.E. Brondani and F.A.M. França thank the hospitality of Departamento Matemática of Universidad de Antofagasta where this paper was started. M.A. Alvarez and L. Medina thank the support of Vicerrectoría de Investigación, Innovación y Postgrado from Universidad de Antofagasta.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Parter, S.V.; Youngs, J.W.T. The symmetrization of matrices by diagonal matrices. *J. Math. Anal. Appl.* **1962**, *4*, 102–110.
2. Rozsa, R. On periodic continuants. *Linear Algebra Appl.* **1969**, *2*, 267–274.
3. Losonczi, L. Eigenvalues and eigenvectors of some tridiagonal matrices. *Acta Math. Hung.* **1992**, *60*, 309–322.
4. da Fonseca, C.M.; Kowalenko, V. Eigenpairs of a family of tridiagonal matrices: three decades later. *Acta Math. Hung.* **2020**, *160*, 376–389.
5. Askar, S.S.; Karawia, A.A. On Solving Pentadiagonal Linear Systems via Transformations. *Math. Probl. Eng.* **2015**, *2015*, 232456. doi:10.1155/2015/232456.
6. Batista, M. A method for solving cyclic block penta-diagonal systems of linear equations. *arXiv* **2008**, arXiv:0806.3639V5.
7. Jia, J.T.; Sogabe, T. A novel algorithm for solving quasi penta-diagonal linear systems. *J. Math. Chem.* **2013**, *51*, 881–889.
8. Karawia, A.A. A computational algorithm for solving periodic pentadiagonal linear systems. *Appl. Math. Comput.* **2006**, *174*, 613–618.
9. Lv, X.-G.; Le, J. A note on solving nearly pentadiagonal linear systems. *Appl. Math. Comput.* **2008**, *204*, 707–712.
10. El-Mikkawy, M.; El-Desouky Rahmo. Symbolic algorithm for inverting cyclic pentadiagonal matrices recursively—Derivation and implementation. *Comput. Math. Appl.* **2010**, *59*, 1386–1396.
11. Navon, I.M. A periodic pentadiagonal systems solver. *Commun. Appl. Numer. Methods* **1987**, *3*, 63–69.
12. Nguetchue, S.N.N.; Abelman, S. A computational algorithm for solving nearly penta-diagonal linear systems. *Appl. Math. Comput.* **2008**, *203*, 629–634.
13. Sweet, R.A. A recursive relation for the determinant of a pentadiagonal matrix. *Commun. ACM* **1969**, *12*, 330–332.
14. Hadj, A.D.A.; Elouafi, M. On the characteristic polynomial, eigenvectors and determinant of a pentadiagonal matrix. *Appl. Math. Comput.* **2008**, *198*, 634–642.
15. Elouafi, M. On formulae for the determinant of symmetric pentadiagonal Toeplitz matrices. *Arab. J. Math.* **2018**, *7*, 91–99.
16. Andelic, M.; da Fonseca, C.M. Some determinantal considerations for pentadiagonal matrices. *Linear Multilinear Algebra* **2019**, doi:10.1080/03081087.2019.1708845.
17. Trench, W.F. On the Eigenvalue Problem for Toeplitz Band Matrices. *Linear Algebra Appl.* **1985**, *64*, 199–214.
18. Ekström, S.-E.; Serra-Capizzano, S. Eigenvalues and eigenvectors of banded Toeplitz matrices and the related symbols. *Numer. Linear Algebra Appl.* **2018**, *25*, e2137.
19. Berman, A.; Hershkowitz, D. Matrix diagonal stability and its applications. *SIAM J. Algebr. Discrete. Methods* **1983**, *4*, 377–382.
20. Berman, A.; Hershkowitz, D. Characterization of acyclic d-stable matrices. *Linear Algebra Appl.* **1984**, *58*, 17–31.

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