THE RATIONAL HOMOLOGY OF THE OUTER AUTOMORPHISM GROUP OF $F_7$

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Abstract. We compute the homology groups $H_*(\text{Out}(F_7); \mathbb{Q})$ of the outer automorphism group of the free group of rank 7.

We produce in this manner the first rational homology classes of $\text{Out}(F_n)$ that are neither constant ($\ast = 0$) nor Morita classes ($\ast = 2n - 4$).

1. Introduction

The homology groups $H_k(\text{Out}(F_n); \mathbb{Q})$ are intriguing objects. On the one hand, they are known to “stably vanish”, i.e. for all $n \in \mathbb{N}$ we have $H_k(\text{Out}(F_n); \mathbb{Q}) = 0$ as soon as $k$ is large enough [3]. Hatcher and Vogtmann prove that the natural maps $H_k(\text{Out}(F_n); \mathbb{Q}) \to H_k(\text{Aut}(F_n); \mathbb{Q})$ and $H_k(\text{Aut}(F_n); \mathbb{Q}) \to H_k(\text{Aut}(F_{n+1}); \mathbb{Q})$ are isomorphisms for $n \geq 2k + 2$ respectively $n \geq 2k + 4$, see [4, 5]. On the other hand, $H_k(\text{Out}(F_n); \mathbb{Q}) = 0$ for $k > 2n - 3$, since $\text{Out}(F_n)$ acts geometrically on a contractible space (the “spine of outer space”, see [2]) of dimension $2n - 3$. Combining these results, the only $k \geq 1$ for which $H_k(\text{Out}(F_n); \mathbb{Q})$ could possibly be non-zero are in the range $\frac{n}{2} - 2 < k \leq 2n - 3$. Morita conjectures in [9, page 390] that $H_{2n-3}(\text{Out}(F_n); \mathbb{Q})$ always vanishes; this would improve the upper bound to $k = 2n - 4$, and $H_{2n-4}(\text{Out}(F_n); \mathbb{Q})$ is also conjectured to be non-trivial.

We shall see that the first conjecture does not hold. Indeed, the first few values of $H_k(\text{Out}(F_n); \mathbb{Q})$ may be computed by a combination of human and computer work, and yield

| $n$ | $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|-----|-----|---|---|---|---|---|---|---|---|---|---|---|----|
| 2   | 1   | 0 |
| 3   | 1   | 0 | 0 | 0 |
| 4   | 1   | 0 | 0 | 0 | 1 | 0 |
| 5   | 1   | 0 | 0 | 0 | 0 | 0 | 0 |
| 6   | 1   | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 7   | 1   | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |

The values for $n \leq 6$ were computed by Ohashi in [12]. They reveal that, for $n \leq 6$, only the constant class ($k = 0$) and the Morita classes $k = 2n - 4$ yield non-trivial homology. The values for $n = 7$ are the object of this Note, and reveal that the picture changes radically:

**Theorem.** The non-trivial homology groups $H_k(\text{Out}(F_7); \mathbb{Q})$ occur for $k \in \{0, 8, 11\}$ and are all 1-dimensional.

Previously, only the rational Euler characteristic $\chi_\mathbb{Q}(\text{Out}(F_7)) = \sum (-1)^k \dim H_k(\text{Out}(F_7); \mathbb{Q})$ was known [10], and shown to be 1. These authors computed in fact the rational Euler characteristics up to $n = 11$ in that paper and the sequel [11].

2. Methods

We make fundamental use of a construction of Kontsevich [10], explained in [1]. We follow the simplified description from [12].

Let $F_n$ denote the free group of rank $n$. This parameter $n$ is fixed once and for all, and will in fact be omitted from the notation as often as possible. An admissible graph of rank $n$ is a
graph $G$ that is 2-connected ($G$ remains connected even after an arbitrary edge is removed), without loops, with fundamental group isomorphic to $F_n$, and without vertex of valency $\leq 2$. Its degree is $\deg(G) := \sum_{v \in V(G)} (\deg(v) - 3)$. In particular, $G$ has $2n - 2 - \deg(G)$ vertices and $3n - 3 - \deg(G)$ edges, and is trivalent if and only if $\deg(G) = 0$. If $\Phi$ is a collection of edges in a graph $G$, we denote by $G/\Phi$ the graph quotient, obtained by contracting all edges in $\Phi$ to points.

A forested graph is a pair $(G, \Phi)$ with $\Phi$ an oriented forest in $G$, namely an ordered collection of edges that do not form any cycle. We note that the symmetric group $\text{Sym}(k)$ acts on the set of forested graphs whose forest contains $k$ edges, by permuting the forest’s edges.

For $k \in \mathbb{N}$, let $C_k$ denote the $\mathbb{Q}$-vector space spanned by isomorphism classes of forested graphs of rank $n$ with a forest of size $k$, subject to the relation

$$(G, \pi \Phi) = (-1)^\pi(G, \Phi) \quad \text{for all } \pi \in \text{Sym}(k).$$

Note, in particular, that if $(G, \Phi) \sim (G, \pi \Phi)$ for an odd permutation $\pi$ then $(G, \Phi) = 0$ in $C_k$. These spaces $(C_*)$ form a chain complex for the differential $\partial = \partial_C - \partial_R$, defined respectively on $(G, \Phi) = (G, \{e_1, \ldots, e_p\})$ by

$$\partial_C(G, \Phi) = \sum_{i=1}^p (-1)^i (G/e_i, \Phi \setminus \{e_i\}),$$

$$\partial_R(G, \Phi) = \sum_{i=1}^p (-1)^i (G, \Phi \setminus \{e_i\}),$$

and the homology of $(C_*, \partial)$ is $H_*(\text{Out}(F_n); \mathbb{Q})$.

The spaces $C_k$ may be filtered by degree: let $F_p C_k$ denote the subspace spanned by forested graphs $(G, \Phi)$ with $\deg(G/\Phi) \leq p$. The differentials satisfy respectively

$$\partial_C(F_p C_k) \subseteq F_p C_{k-1}, \quad \partial_R(F_p C_k) \subseteq F_{p-1} C_{k-1}.$$ 

A spectral sequence argument gives

$$H_p(\text{Out}(F_n); \mathbb{Q}) = E^2_{p,0} = \frac{\ker(\partial_C|_{F_p C_p}) \cap \ker(\partial_R|_{F_p C_p})}{\partial_R(\ker(\partial_C|_{F_{p+1} C_{p+1}}))}. \quad (1)$$

Note that if $(G, \Phi) \in F_p C_p$ then $G$ is trivalent. We compute explicitly bases for the vector spaces $F_p C_p$, and matrices for the differentials $\partial_C, \partial_R$, to prove the theorem.

3. Implementation

We follow for $n = 7$ the procedure sketched in [12]. Using the software program nauty [8], we enumerate all trivalent graphs of rank $n$ and vertex valencies $\geq 3$. The libraries in nauty produce a canonical ordering of a graph, and compute generators for its automorphism group. We then weed out the non-2-connected ones.

For given $p \in \mathbb{N}$, we then enumerate all $p$-element oriented forests in these graphs, and weed out those that admit an odd symmetry. These are stored as a basis for $F_p C_p$. Let $a_p$ denote the dimension of $F_p C_p$.

For $(G, \Phi)$ a basis vector in $F_p C_p$, the forested graphs that appear as summands in $\partial_C(G, \Phi)$ and $\partial_R(G, \Phi)$ are numbered and stored in a hash table as they occur, and the matrices $\partial_C$ and $\partial_R$ are computed as sparse matrices with $a_p$ columns.

The nullspace $\ker(\partial_C|_{F_p C_p})$ is then computed: let $b_p$ denote its dimension; then the nullspace is stored as a sparse $(a_p \times b_p)$-matrix $N_p$. The computation is greatly aided by the fact that $\partial_C$ is a block matrix, whose row and column blocks are spanned by $\{(G, \Phi) : G/\Phi = G_0\}$ for all choices of the fully contracted graph $G_0$. The matrices $N_p$ are computed using the linear algebra library linbox [7], which provides exact linear algebra over $\mathbb{Q}$ and finite fields.

Finally, the rank $c_p$ of $\partial_R \cap N_p$ is computed, again using linbox. By (1), we have

$$\dim H_p(\text{Out}(F_n); \mathbb{Q}) = b_p - c_p - c_{p+1}.$$
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For memory reasons (the computational requirements reached 200GB of RAM at its peak), some of these ranks were computed modulo a large prime ($65521$ and $65519$ were used in two independent runs).

Computing modulo a prime can only reduce the rank; so that the values $c_p$ we obtained are underestimates of the actual ranks of $\partial_N \circ N_p$. However, we also know a priori that $b_p - c_p - c_{p+1} \geq 0$ since it is the dimension of a vector space; and none of the $c_p$ we computed can be increased without at the same time causing a homology dimension to become negative, so our reduction modulo a prime is legal.

For information, the parameters $a_p, b_p, c_p$ for $n = 7$ are as follows:

| $p$ | $a_p$ | $b_p$ | $c_p$ |
|-----|------|------|------|
| 0   | 365  | 364  | 364  |
| 1   | 3712 | 1784 | 5642 |
| 2   | 23227| 14766| 5642 |
| 3   | 854k | 30326| 4222 |
| 4   | 1.6m | 38113| 1054 |
| 5   | 2.3m | 28588| 1054 |
| 6   | 2.6m | 16741| 1054 |
| 7   | 1.2m | 6931 | 1054 |
| 8   | 376k | 1682 | 1054 |
| 9   | 179  | 179  | 1054 |
| 10  | 179  | 179  | 1054 |
| 11  | 179  | 179  | 1054 |

The largest single matrix operations that had to be performed were computing the nullspace of a $2038511 \times 536647$ matrix (16 CPU hours) and the rank modulo $65519$ of a (less sparse) $1355531 \times 16741$ matrix (10 CPU hours).

The source files used for the computations are available as supplemental material. Compilation requires \texttt{g++} version 4.7 or later, a functional \texttt{linbox} library, available from the site \url{http://www.linalg.org}, as well as the \texttt{nauty} program suite, available from the site \url{http://pallini.di.uniroma1.it}. It may also be directly downloaded and installed by typing \texttt{make nauty25r9} in the directory in which the sources were downloaded. Beware that the calculations required for $n = 7$ are prohibitive for most desktop computers.

### Conclusion

Computing the dimensions of the homology groups is only the first step in understanding them; much more interesting would be to know visually, or graph-theoretically, where these non-trivial classes come from.

It seems almost hopeless to describe, via computer experiments, the non-trivial class in degree 8. It may be possible, however, to arrive at a reasonable understanding of the non-trivial class in degree 11.

This class may be interpreted as a linear combination $w$ of trivalent graphs on 12 vertices, each marked with an oriented spanning forest. There are $376365$ such forested graphs that do not admit an odd symmetry. The class $w \in \mathbb{Q}^{376365}$ is an $\mathbb{Z}$-linear combination of $70398$ different forested graphs, with coefficients in \{±1, ... , ±16\}. For example, eleven graphs occur with coefficient ±13; four of them have indices 25273, 53069, 53239, 53610 respectively, and are, with the spanning tree in bold,
The coefficients of $w$, and corresponding graphs, are distributed as ancillary material in the file $w_{\text{cycle}}$, in format ‘coefficient [edge1 edge2 ...]’, where each edge is ‘x–y’ or ‘x+y’ to indicate whether the edge is absent or present in the forest. Edges always satisfy $x \leq y$, and the forest is oriented so that its edges are lexicographically ordered. Edges are numbered from 0 while graphs are numbered from 1. There are no multiple edges.

ACKNOWLEDGMENTS

I am grateful to Alexander Berglund and Nathalie Wahl for having organized a wonderful and stimulating workshop on automorphisms of free groups in Copenhagen in October 2015, when this work began; to Masaaki Suzuki, Andy Putman and Karen Vogtmann for very helpful conversations that took place during this workshop; and to Jim Conant for having checked the cycle $w$ (after finding a mistake in its original signs) with an independent program.

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