Two noncommutative parameters and regular cosmological phase transition in the semi-classical dilaton cosmology

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Abstract

We study cosmological phase transitions from modified equations of motion by introducing two noncommutative parameters in the Poisson brackets, which describes the initial- and future-singularity-free phase transition in the soluble semi-classical dilaton gravity with a non-vanishing cosmological constant. Accelerated expansion and decelerated expansion appear alternatively, where the model contains the second accelerated expansion. The final stage of the universe approaches the flat spacetime independent of the initial state of the curvature scalar as long as the product of the two noncommutative parameters is less than one. Finally, we show that the initial-singularity-free condition is related to the second accelerated expansion of the universe.

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I. INTRODUCTION

The horizon problem and flatness problem in the standard cosmology have been well appreciated by the inflation model [1] (for recent reviews, see e.g. Refs. [2, 3, 4, 5]); however, some problems still remain unsolved. One of them is the initial singularity problem, which is difficult to solve since the consistent quantum gravity has not yet been established. Another one is the cosmological coincidence problem or fine-tuning problem, where the only solution up to now may be an anthropic principle [6, 7]. Apart from these problems, we are facing to explain the late-time second accelerated expansion of the universe. So, it has been claimed that the dark energy defined by the negative equation-of-state parameter \( \omega \equiv \frac{p}{\rho} \) is responsible for this accelerated expansion [8], where \( \rho \) and \( p \) are the energy density and the pressure, respectively. It can be easily seen from the Friedmann equation and the continuity equation that the accelerated universe requires \( \omega < -\frac{1}{3} \). Note that the density of the dark energy is assumed to be positive so that the pressure should be negative. Especially, cosmologies with \( \omega < -1 \) have a defect of big rip singularity or sudden future singularity that the scale or some physical quantities become singular in a finite proper time [9, 10, 11, 12, 13, 14]. Recently, authors in Ref. [15, 16, 17, 18] showed that quantum effects can render \( \omega < -1 \) without any need of introducing ghosts or phantoms so that it is possible to have cosmologies where the equation of state parameter \( \omega < -1 \) without having big rip-like singularities.

The above mentioned cosmological problems have been studied extensively for a long time. However, they are usually hard to solve exactly and thus some simplified models may be considered in order to get some clues and insights for realistic models. Such models are exactly soluble two-dimensional dilaton gravities [19, 20, 21, 22, 23, 24, 25, 26, 27, 28], especially aiming at various cosmological problems [29, 30, 31, 32, 33, 34]. So, it would be interesting to study whether a simplified model can solve the problems and describe the late-time accelerated expansion or not. Recent works [35, 36, 37] show that noncommutative fields make it possible to obtain the transition from a decelerated universe to an accelerated universe without a cosmological constant. However, in spite of some efforts to obtain the cosmological phase transition, these models have some problems. One of them is to encounter a future singularity in a finite proper time unless an appropriate regular geometry is patched, or it does not reproduce the first accelerated expansion in the early universe. Another
interesting model in Ref. \[37\] describes an inflation in the early universe, the decelerated expansion corresponding to FRW phase, and the late-time second acceleration. However, an initial curvature singularity still exists. So, we would like to extend our previous work and obtain everywhere singularity-free cosmological solutions involving inflation, decelerated phase, and late-time second acceleration, where the whole profile is essentially similar to our universe chronologically.

For this purpose, we shall add two local counter-terms with the Polyakov action of conformal anomaly in the semi-classical action, and then impose some modified Poisson brackets with noncommutativity between relevant fields. This process naturally yields modified sets of semi-classical equations of motion involving two noncommutative parameters, and then remarkably gives desired solutions. In the next section, usual semi-classical equations of motion obeying conventional Poisson algebra will be derived in a self-contained manner, and it can be shown that the expanding universe is forever without any phase change. In Sec. III, new equations of motion are derived, which give nontrivial solutions depending on noncommutative parameters. Consequently, it can be shown that the initial- and future-singularity-free solutions along with cosmological phase transition are obtained when the product of the two noncommutative parameters is less than one. Finally, discussions will be given in Sec. IV.

II. PERMANENT ACCELERATED EXPANSION OF THE UNIVERSE

We start with the following two-dimensional dilaton gravity coupled to the conformal matter and its quantum correction,

\[ S = S_{DG} + S_{cl} + S_{qt}. \] (1)

The first term in the right-hand side is the well-known string inspired dilaton gravity action written as

\[ S_{DG} = \frac{1}{2\pi} \int d^2x \sqrt{-g} e^{-2\phi} \left[ R + 4(\nabla \phi)^2 + 4\lambda^2 \right]. \] (2)

The classical matter action \( S_{cl} \) composed of \( N \) conformal fields \( f_i \) and its one-parameter-family quantum correction \( S_{qt} \) are given by

\[ S_{cl} = \frac{1}{2\pi} \int d^2x \sqrt{-g} \left[ -\frac{1}{2} \sum_{i=1}^{N} (\nabla f_i)^2 \right]. \] (3)
\[ S_{qt} = \frac{\kappa}{2\pi} \int \sqrt{-g} \left[ -\frac{1}{4} R \frac{1}{\Box} R + (\gamma - 1)(\nabla \phi)^2 - \frac{\gamma}{2} \phi R \right], \]

respectively, where \( \kappa = (N - 24)\hbar/12 \) and \( \lambda^2 \) is a cosmological constant. The higher order of quantum corrections beyond the one-loop is negligible in the large \( N \) approximation where \( N \to \infty \) and \( \hbar \to 0 \) so that \( \kappa \) is assumed to be positive finite constant. Note that the local ambiguity terms in Eq. (4) correspond to those of the Russo-Susskind-Thorlacius (RST) model for \( \gamma = 1 \) [21], and the Bose-Parker-Peleg (BPP) model for \( \gamma = 2 \) [24]. In this work, we will assume the regularization ambiguity constant to be \( \gamma > 2 \).

In the conformal gauge, \( ds^2 = -e^{2\rho} dx^+ dx^- \), defining new fields as follows

\[
\chi = e^{-2\phi} + \kappa \left( \rho - \frac{\gamma}{2} \phi \right),
\]

\[
\Omega = e^{-2\phi} - \kappa \left( \gamma - 2 \right) \phi,
\]

the total action (1) can be written as

\[ S = \frac{1}{\pi} \int d^2 x \left[ -\frac{1}{\kappa} \partial_+ \partial_- \chi + \frac{1}{\kappa} \partial_+ \Omega \partial_- \Omega + \lambda^2 e^{2(\chi - \Omega) / \kappa} + \frac{1}{2} \sum_{i=1}^{N} \partial_+ f_i \partial_- f_i \right], \]

and constraints are given by

\[ \kappa t_{\pm} = -\frac{1}{\kappa} (\partial_{\pm} \chi)^2 + \partial_{\pm}^2 \chi + \frac{1}{\kappa} (\partial_{\pm} \Omega)^2 + \frac{1}{2} \sum_{i=1}^{N} (\partial_{\pm} f_i)^2, \]

where \( t_{\pm} \) reflects the non-locality of the anomaly term in the Polyakov action. This integration function from the non-locality should be determined by the boundary condition of the geometrical vacuum and matter state.

Assuming a homogeneous space, the Lagrangian and the constraints are reduced to

\[
L = -\frac{1}{2\kappa} \left( \frac{d\chi}{dt} \right)^2 + \frac{1}{2\kappa} \left( \frac{d\Omega}{dt} \right)^2 + 2\lambda^2 e^{2(\chi - \Omega) / \kappa} + \frac{1}{4} \sum_{i=1}^{N} \left( \frac{df_i}{dt} \right)^2, \]

\[
\kappa t_{\pm} = -\frac{1}{4\kappa} \left( \frac{d\chi}{dt} \right)^2 + \frac{1}{4\kappa} \frac{d^2 \chi}{dt^2} + \frac{1}{4\kappa} \left( \frac{d\Omega}{dt} \right)^2 + \frac{1}{8} \sum_{i=1}^{N} \left( \frac{df_i}{dt} \right)^2, \]

where the Lagrangian is defined by \( S/L_0 = \frac{1}{\pi} \int dt L \) with \( L_0 = \int d\chi \), and \( d\chi = dt \pm dx \). Then, the Hamiltonian is

\[ H = -\frac{\kappa}{2} P_{\chi}^2 + \frac{\kappa}{2} P_{\Omega}^2 - 2\lambda^2 e^{2(\chi - \Omega) / \kappa} + \sum_{i=1}^{N} P_{f_i}^2, \]

where canonical momenta are given by \( P_\chi = \frac{1}{\kappa} d\chi/dt \), \( P_\Omega = \frac{1}{\kappa} d\Omega/dt \), \( P_{f_i} = \frac{1}{2} df_i/dt \).
If we now define non-vanishing Poisson brackets as follows
\[
\{\Omega, P_\Omega\}_{PB} = \{\chi, P_\chi\}_{PB} = \{f_i, P_{f_i}\}_{PB} = 1,
\]
then Hamiltonian equations of motion [28] \(d\mathcal{O}/dt = \{\mathcal{O}, H\}_{PB}\) are
\[
\begin{align*}
\frac{d\chi}{dt} &= -\kappa P_\chi, \\
\frac{d\Omega}{dt} &= \kappa P_\Omega, \\
\frac{d f_i}{dt} &= 2 P_{f_i},
\end{align*}
\]
\[\frac{d P_\chi}{dt} = -\frac{d P_\Omega}{dt} = \frac{4\lambda^2}{\kappa} e^{2(\chi-\Omega)/\kappa}, \quad \frac{d P_{f_i}}{dt} = 0.
\] (14)

Taking \(P_{f_i} = 0\) for the sake of simplicity, these equations can be compactly written as
\[
\begin{align*}
\frac{d}{dt} (\chi + \Omega) &= -\kappa (P_\chi - P_\Omega), \\
\frac{d}{dt} (\chi - \Omega) &= -\kappa (P_\chi + P_\Omega), \\
\frac{d}{dt} (P_\chi + P_\Omega) &= 0, \\
\frac{d}{dt} (P_\chi - P_\Omega) &= \frac{8\lambda^2}{\kappa} e^{2(\chi-\Omega)/\kappa},
\end{align*}
\]
\[\] (16)

which are easily solved as
\[
\begin{align*}
\chi &= \chi_0 + \kappa P_{\chi_0} t - \frac{\lambda^2}{(P_{\chi_0} - P_{\Omega_0})^2} e^{2(\chi-\Omega)/\kappa}, \\
\Omega &= \Omega_0 + \kappa P_{\Omega_0} t - \frac{\lambda^2}{(P_{\chi_0} - P_{\Omega_0})^2} e^{2(\chi-\Omega)/\kappa},
\end{align*}
\]
\[\] (17) (18)

where \(P_{\chi_0}, P_{\Omega_0}, \chi_0,\) and \(\Omega_0\) are arbitrary constants, and we assume \(P_{\chi_0} \neq P_{\Omega_0}\). Note that these semi-classical solutions (17) and (18) from the Hamiltonian equations of motion (13) and (14) are essentially equivalent to those of Euler-Lagrangian equations of motion from the Lagrangian (9) since fields \(\Omega\) and \(\chi\) are not the quantum-mechanical operators. This is not the quantization of the semi-classical model (11). In the next section, we will modify these conventional Poisson brackets in order to obtain the modified semi-classical equations of motion.

We now turn to the issue of the expanding universe by considering the expansion rate of the scale factor,
\[
\dot{a}(\tau) = \frac{d\rho(t)}{dt} = \frac{\kappa P_{\Omega_0}(P_{\chi_0} - P_{\Omega_0}) - 2\lambda^2 e^{2(\chi-\Omega)/\kappa}}{(P_{\chi_0} - P_{\Omega_0})[-2e^{-2\phi} - \kappa(\gamma - 2)/2]} + (P_{\chi_0} - P_{\Omega_0}) \geq 0,
\] (19)

where the scale factor \(a(\tau)\) is the function of a comoving time \(\tau\), which is defined by \(ds^2 = -d\tau^2 + a^2(\tau)dx^2\) i.e. \(d\tau = e^{\rho(t)}dt\) and \(a(t) = e^{\rho(t)}\), and the overdot denotes the derivative with respect to \(\tau\). It is straightforward to show that the positive expansion follows from the condition,
\[
P_{\chi_0} - \frac{\gamma}{\gamma - 2} P_{\Omega_0} \geq 0.
\] (20)
As for the constraints, substituting the solutions (17) and (18) into the constraint equation (10) gives

$$\kappa t_{\pm} = -\frac{\kappa}{4}(P_{\chi_0}^2 - P_{\Omega_0}^2) - \frac{\lambda^2}{2}e^{2(\chi - \Omega)/\kappa},$$

where $t_{\pm}$ is determined by the matter state. The curvature scalar is calculated as

$$R = \frac{2\ddot{a}}{a} = \frac{e^{-2\phi}}{e^{-2\phi} + \kappa(\gamma - 2)/4} \left[ 4\lambda^2 + \frac{e^{-2\phi}}{[e^{-2\phi} + \kappa(\gamma - 2)/4]^2} \left( \frac{d\Omega}{dt} \right)^2 \right],$$

which cannot be negative since we have assumed that $\kappa$ is positive and $\gamma > 2$. The curvature scalar in two dimensions is directly proportional to the second derivative of the scale factor so that the universe exhibits a permanent accelerated expansion without any decelerated expansion. It means that it is nontrivial to obtain the phase transition based on the conventional Poisson brackets. In the following section, modification of the Poisson brackets (12) gives different solution showing the desired phase change without any curvature singularities.

### III. NON-SINGULAR COSMOLOGY WITH PHASE TRANSITION

We are going to extend the conventional (commutative) Poisson brackets to the modified (noncommutative) Poisson brackets characterized by the two noncommutative parameters, $\theta_1$ and $\theta_2$, which are reminiscent of the noncommutative algebra of the D-brane on the constant tensor field or a charged particle moving slowly on the constant magnetic field [38, 39, 40]. We are now trying to obtain modified semi-classical solutions from the modified semi-classical equations of motion. Now, the noncommutative Poisson algebra are given as [41, 42, 43]

$$\{\Omega, P_{\Omega}\}_{\text{MPB}} = \{\chi, P_{\chi}\}_{\text{MPB}} = 1,$$

$$\{\chi, \Omega\}_{\text{MPB}} = \theta_1, \quad \{P_{\chi}, P_{\Omega}\}_{\text{MPB}} = \theta_2,$$

where $\theta_1$ and $\theta_2$ are positive independent constants. The modified semi-classical equations of motion are given by

$$\frac{d\chi}{dt} = \{\chi, H\}_{\text{MPB}} = -\kappa P_{\chi} + \frac{4}{\kappa}\lambda^2 \theta_1 e^{2(\chi - \Omega)/\kappa},$$

$$\frac{d\Omega}{dt} = \{\Omega, H\}_{\text{MPB}} = \kappa P_{\Omega} + \frac{4}{\kappa}\lambda^2 \theta_1 e^{2(\chi - \Omega)/\kappa},$$

$$\frac{dP_{\chi}}{dt} = \{P_{\chi}, H\}_{\text{MPB}} = \frac{4}{\kappa}\lambda^2 e^{2(\chi - \Omega)/\kappa} + \theta_2 \kappa P_{\Omega},$$

$$\frac{dP_{\Omega}}{dt} = \{P_{\Omega}, H\}_{\text{MPB}} = -\frac{4}{\kappa}\lambda^2 e^{2(\chi - \Omega)/\kappa} + \theta_2 \kappa P_{\chi}.$$
Note that the original semi-classical equations of motion \[13\) and \[14\] are reproduced when \(\theta_1, \theta_2 \to 0\). In particular, if the cosmological constant vanishes, then \(\theta_1\) decouples from the equations of motion. So, we want to consider the non-vanishing cosmological constant to extend the previous works. Rewriting these equations as

\[
\frac{d}{dt}(\chi + \Omega) = -\kappa(P_x - P_{\Omega}) + \frac{8\lambda^2}{\kappa}\theta_1 e^{2(\chi-\Omega)/\kappa},
\]

(28)

\[
\frac{d}{dt}(\chi - \Omega) = -\kappa(P_x + P_{\Omega}),
\]

(29)

\[
\frac{d}{dt}(P_x + P_{\Omega}) = \theta_2\kappa(P_x + P_{\Omega}),
\]

(30)

\[
\frac{d}{dt}(P_x - P_{\Omega}) = -\theta_2\kappa(P_x - P_{\Omega}) + \frac{8\lambda^2}{\kappa}e^{2(\chi-\Omega)/\kappa},
\]

(31)

we obtain the following solutions,

\[
\chi = C_\chi - \alpha e^{\theta_2 kt} + \beta e^{-\theta_2 kt} - \frac{\lambda^2}{\theta_2^2 \kappa \alpha} e^{-\theta_2 kt} + \frac{4\lambda^2}{\theta_2^2 \kappa^2} (1 - \theta_1 \theta_2) e^{\frac{2\lambda}{\kappa}(C_\chi - C_\Omega)} \text{Ei}\left(-\frac{4\lambda}{\kappa} e^{\theta_2 kt}\right),
\]

(32)

\[
\Omega = C_\Omega + \alpha e^{\theta_2 kt} + \beta e^{-\theta_2 kt} - \frac{\lambda^2}{\theta_2^2 \kappa \alpha} e^{-\theta_2 kt} + \frac{4\lambda^2}{\theta_2^2 \kappa^2} (1 - \theta_1 \theta_2) e^{\frac{2\lambda}{\kappa}(C_\chi - C_\Omega)} \text{Ei}\left(-\frac{4\lambda}{\kappa} e^{\theta_2 kt}\right),
\]

(33)

where \(\alpha, \beta, C_\chi,\) and \(C_\Omega\) are integration constants, the exponential integral function \(\text{Ei}(z)\) is defined as \(\text{Ei}(z) = -\int_{-\infty}^{x} dx \ x^{-1} e^{-x}\), and \(\chi - \Omega = -2\alpha e^{\theta_2 kt} + C_\chi - C_\Omega\). In addition, their conjugate momenta are obtained as

\[
P_x = \theta_2 \alpha e^{\theta_2 kt} + \theta_2 \beta e^{-\theta_2 kt} - \frac{\lambda^2}{\theta_2^2 \kappa \alpha} e^{-\theta_2 kt} e^{2(\chi-\Omega)/\kappa},
\]

(34)

\[
P_{\Omega} = \theta_2 \alpha e^{\theta_2 kt} - \theta_2 \beta e^{-\theta_2 kt} + \frac{\lambda^2}{\theta_2^2 \kappa \alpha} e^{-\theta_2 kt} e^{2(\chi-\Omega)/\kappa}.
\]

(35)

From the solutions (32) and (33), we can obtain the expansion condition at asymptotic regions,

\[
\dot{a} = \frac{2}{-4e^{-2\phi} - \kappa(\gamma - 2)} \left[ \theta_2 \kappa (\alpha e^{\theta_2 kt} - \beta e^{-\theta_2 kt}) + \lambda^2 \left( \frac{4\theta_1}{\kappa t} + \frac{1}{\theta_2 \alpha} e^{-\theta_2 kt} \right) e^{[-4\alpha e^{\theta_2 kt} + 2(C_\chi - C_\Omega)]/\kappa} \right]
\]

\[
-2\theta_2 \alpha e^{\theta_2 kt} > 0.
\]

(36)

At the asymptotic future infinity and the past infinity, \(t \to \pm \infty\), the following inequalities can be derived,

\[
\alpha < 0, \quad \bar{\beta} \equiv \beta - \frac{\lambda^2}{\theta_2^2 \kappa \alpha} e^{2(C_\chi - C_\Omega)/\kappa} > 0.
\]

(37)

In the intermediate region, it is not easy to write down the condition in a simplified form. However, it will be shown in later that the positive expansion rate is possible without contraction of the universe.
FIG. 1: (a) The curvature scalar for $R_{\text{init}} > 0(C_\chi = C_\Omega = 0$, solid line), $R_{\text{init}} = 0(C_\chi = C_\Omega = 4 + (\gamma_E + \ln 5)/4$, dotted line), and $R_{\text{init}} < 0(C_\chi = C_\Omega = 10$, dashed line) are plotted with respect to the comoving time $\tau = \int_{-\infty}^{t} d\tilde{t} e^{\rho(\tilde{t})}$. (b) The future accelerated region in the box is magnified. The scalar curvatures approach zero independent of their initial behaviors. In these figures, the parameters and constants are chosen as $\kappa = 1$, $\gamma = 3$, $\lambda = 1$, $\theta_1 = 3/8$, $\theta_2 = 2$, $\alpha = -1$, $\beta = 1$.

Now, we investigate the behavior of the curvature scalar $R$, which is explicitly given by

$$R = -\frac{e^{-2\phi - 2(\chi - \Omega)/\kappa}}{e^{-2\phi} + \kappa(\gamma - 2)/4} \left\{ \theta_2^2 \kappa^2 (\alpha e^{\theta_2 \kappa \tilde{t}} + \beta e^{-\theta_2 \kappa \tilde{t}}) - 4\lambda^2 \left( 1 + \frac{4\alpha}{\kappa} \theta_1 \theta_2 e^{\theta_2 \kappa \tilde{t}} + \frac{\kappa}{4\alpha} e^{-\theta_2 \kappa \tilde{t}} \right) e^{2(\chi - \Omega)/\kappa} - \frac{e^{-2\phi}}{[e^{-2\phi} + \kappa(\gamma - 2)/4]^2} \left[ \theta_2 \kappa (\alpha e^{\theta_2 \kappa \tilde{t}} - \beta e^{-\theta_2 \kappa \tilde{t}}) + \lambda^2 \left( \frac{4\theta_1}{\kappa} + \frac{1}{\theta_2 \alpha} e^{-\theta_2 \kappa \tilde{t}} \right) e^{2(\chi - \Omega)/\kappa} \right]^2 \right\} - 4\theta_2^2 \kappa \alpha e^{\theta_2 \kappa \tilde{t}} e^{-2\phi - 2(\chi - \Omega)/\kappa}. \right)$$

(38)

It is interesting to note that the most leading term for $t \to -\infty$ is $R \simeq -\theta_2 \kappa C_{\text{cr}} t$. If the following condition is met,

$$C_{\text{cr}} \equiv -4\lambda^2 (1 - \theta_1 \theta_2) + \frac{\kappa}{4} (\gamma - 2) \theta_2^2 \kappa^2 e^{-2(C_\chi - C_\Omega)/\kappa} = 0,$$

(39)

then there does not exist any initial singularity for $\theta_1 \theta_2 < 1$.

We have assumed that the two noncommutative parameters $\theta_1$ and $\theta_2$ are positive constants for simplicity in the modified Poisson brackets [23]. The anomaly coefficient $\kappa$ is generically assumed to be an arbitrary positive constant, which is related to the large $N$ limit along with the small Plank constant $\hbar$ giving the good approximation of the one-loop correction of matter fields [19, 20, 21]. Especially, in this model, the positivity is required to obtain the forward expansion of the universe, which can be easily derived from the expansion rate $\dot{a}$ in the comoving coordinate, $\dot{a} \simeq \theta_2 \kappa / 2$ as $\tau \to 0$ and $\dot{a} \simeq \kappa / (2\theta_1)$ as
The ambiguity $\gamma$ is also an arbitrary constant satisfying $\gamma > 2$, otherwise the curvature singularity appears as seen from the curvature relations (22) and (38). Moreover, the relation of integration constants $C_\chi - C_\Omega$ is set to zero for simplicity, which is related to the origin of the conformal time $t$. Of course, the cosmological constant is still arbitrary in this formulation. Assuming $\kappa = 1$, $\gamma = 3$, $\theta_1 = 3/8$, $\theta_2 = 2$ and $C_\chi = C_\Omega$, especially, $\lambda = 1$, the behaviors of the curvature scalar and the scale factor are shown in Figs. 1 and 2 respectively. Some different choices of constants do not change the overall profile of these figures.

To compare this model with the previous work ($\lambda = 0$) [37], the crucial difference comes from the coupling of the cosmological constant and $\theta_1$, which plays an important role as seen in Eqs. (24)-(27). In Ref. [37], even though the phase changing cosmological transition has been obtained without any future singularities, the initial curvature singularity has not been avoided. At the first sight, some fine tuning of constants removes the initial singularity; however, it is not the case since the curvature singularity is a geometrically invariant quantity. For instance, if we take $\lambda^2 = 0$, then the initial singularity free condition (39) tells us that $\kappa = 0$ or $\gamma = 2$; however, it does not remove singularities since the denominator in the curvature scalar (38) may vanish [36]. Therefore, in this model the nonvanishing cosmological constant along with the noncommutativity gives the singularity free cosmological phase transition.

From now on, we will consider the singularity-free solution satisfying $C_{\text{cr}} = 0$. After some tedious calculations, the asymptotic behaviors of the curvature scalar are written as

$$R \simeq \begin{cases} \theta^2 \kappa^2 C_N + O(e^{\theta_2 \kappa t}) & \text{for } t \to -\infty, \\ \kappa^2 \lambda^2 (1 - \theta_1 \theta_2) e^{-2\theta_2 \kappa t} + O(e^{-3\theta_2 \kappa t}) & \text{for } t \to \infty, \end{cases}$$

(40)

where the unbounded constant $C_N$ is given by

$$C_N \equiv \left[ \frac{\kappa}{4} (\gamma - 2) \ln \beta - C_\Omega - \frac{4\alpha}{\kappa} \beta \right] e^{-2(C_\chi - C_\Omega)/\kappa} - \frac{4\lambda^2}{\theta_2 \kappa^2} \left[ 1 - (1 - \theta_1 \theta_2) \left( \gamma_E + \ln \left( -\frac{4\alpha}{\kappa} \right) \right) \right],$$

(41)

and the Euler’s constant $\gamma_E = \lim_{N \to \infty} \left[ \sum_{n=1}^{N} \frac{1}{n} - \ln(N) \right] \simeq 0.5772$. Since the curvature scalar is finite, eventually it is everywhere singularity-free. In addition, the curvature scalar approaches zero universally, which might be similar to the attractor mechanism in the nonlinear dynamics since the asymptotic curvature scalars are independent of $C_N$ which determines the initial state of the curvature scalars. Note that the initial state of the universe is dS-like
FIG. 2: The dotted and dashed lines show the behavior of the scale factor $a(\tau)$ and the expansion rate $\dot{a}$ with respect to the comoving time $\tau$, respectively. The solid line importantly shows the profile of the acceleration $\ddot{a}$. (a) Initially dS-like case ($C_\chi = C_\Omega = 0$) shows that the first acceleration corresponding to the first inflation starts at the comoving time $\tau = 0$. (b) The first acceleration starts after a finite time for AdS-like case ($C_\chi = C_\Omega = 10$).

for $C_N > 0$ or AdS-like for $C_N < 0$. The profile of the curvature scalar is plotted for the case of $R_{\text{init}} > 0 (C_N > 0)$, $R_{\text{init}} = 0 (C_N = 0)$, and $R_{\text{init}} < 0 (C_N < 0)$ in Fig. 1. Moreover, $\theta_1 \theta_2 < 1$ from the initial-singularity-free condition is related to the late time second accelerated expansion since the curvature scalar of the universe should approach positive zero as shown in Eq. (40).

Next, let us remind that some of dark-energy-dominant models have a defect called a big rip singularity that the scale blows up in a finite time [9, 10, 11, 12, 13]. The present model is different from the previous models, and we investigate whether this kind of singularity appears or not. The scale $a(\tau) = e^{\rho}$ is expanded as

$$\begin{align*}
a(\tau) &\simeq \begin{cases} \frac{1}{2} \theta_2 \kappa \tau \left[ 1 + \frac{1}{12} \theta_2^2 C_N \tau^2 + O(\tau^3) \right] & \text{for } \tau \to 0, \\
\frac{1}{2} \theta_2 \kappa \tau \left[ 1 + \frac{1}{\theta_1 \theta_2} \lambda^2 \tau^{-2} + O(\tau^{-4}) \right] & \text{for } \tau \to \infty,
\end{cases}
\end{align*}$$

(42)

with respect to the comoving time $\tau = \int_{-\infty}^t d\tilde{\tau} e^{\rho(\tilde{\tau})}$. Then, we see that it is definitely finite at a finite comoving time. It means that our model does not have any sudden future singularities including a big rip singularity.

Let us now study the most intriguing issue of the late-time acceleration. The acceleration is calculated as

$$\ddot{a}(\tau) \simeq \begin{cases} \frac{1}{2} \theta_2^3 \kappa^3 C_N \tau + O(\tau^3) & \text{for } \tau \to 0, \\
\frac{\kappa (1-\theta_1 \theta_2)}{\theta_1 \theta_2 \lambda^2} \tau^{-3} + O(\tau^{-5}) & \text{for } \tau \to \infty,
\end{cases}$$

(43)
where it vanishes at both ends. In the intermediate region, the profile of the acceleration is plotted in Fig. [2]. It shows that the universe starts from the inflationary era for dS-like universe ($C_N > 0$) while the inflation appears after a finite time $\tau$ for AdS-like universe ($C_N < 0$). The former case seems to be more realistic. Accelerated expansion and decelerated expansion (FRW phase) appear alternatively, and then it ends up with the second accelerated expansion. The final stage of the universe approaches the flat spacetime as long as the product of the two noncommutative parameters is less than one.

Next, in order to discuss the equation-of-state parameter, the energy-momentum tensors should be identified with the source of the constraint equation [35, 37], then

\[
T_{\pm \pm} = -\kappa t_{\pm} = \theta_2^2 \kappa \alpha \beta + \frac{\theta_2^2 \kappa^2}{4} \left( \alpha e^{\theta_2 \kappa t} - \beta e^{-\theta_2 \kappa t} \right) + \lambda^2 \frac{\kappa}{4\alpha} e^{-\theta_2 \kappa t} \epsilon^2 (\chi - \Omega) / \kappa.
\] (44)

The energy density $\varepsilon$ and the pressure $p$ in the comoving coordinates are defined by

\[
\varepsilon = T_{\tau \tau} = e^{-2\rho} \left[ T_{++} + 2T_{+-} + T_{--} \right],
\] (45)

\[
p = T_{xx} / \alpha^2(\tau) = e^{-2\rho} \left[ T_{++} - 2T_{+-} + T \right].
\] (46)

Because of $T_{+-} = 0$ from the equation of motion, the density and the pressure have the same form so that the equation-of-state parameter is simply

\[
\omega = p / \varepsilon = 1.
\] (47)

This is the same with the case of the homogeneous massless conformal fields so that the source is an ordinary matter.

Note that in spite of the plausibility of the model, it is a two dimensional toy model so that one might wonder whether such desired behaviors persist in four dimensions or not. The singularity free phase transition appears when we consider the dilaton gravity and the noncommutativity together. The action [2] is the s-wave sector of the higher-dimensional low-energy dilaton gravity [44, 45] and the quantum-mechanically induced Polyakov action [4] is the s-wave sector of the bosonized four-dimensional fermionic matter [46]. In some sense, the main body of the present model is close to the s-wave sector of the four-dimensional model. Therefore, this model is expected to be partially connected with the higher-dimensional cosmology, although the technical difficulties may arise from the nonlinearity of the higher-dimensional gravity.
On the other hand, it seems that the noncommutative parameters play some important role even when the universe is large. One should expect that they could play a role only when the universe is small. In this model, unfortunately, the curvature scalar and the acceleration definitely depend on the noncommutative parameters; however, the space time is flat and the acceleration approaches zero at the infinity, and it means that they are independent of the parameters. The most leading term of the late time scale in Eq. (42) seems to depend on $\theta_2$ because of $a(\tau) \approx \theta_2 \kappa \tau / 2$; however, it can be absorbed by redefinition $\theta_2 \tau \rightarrow \tau$. In this process, asymptotic behaviors of the curvature scalar and the acceleration have been unchanged at $\tau \rightarrow \infty$. Although the noncommutativity does not affect the geometrical behaviors at the asymptotically infinite scale, it is still hard to resolve this problem in this simplified model.

IV. DISCUSSION

We have studied the singularity-free phase transition in the semi-classically quantized dilaton gravity by assuming the noncommutativity. The basic reason for this achievement is due to the role of non-vanishing cosmological constant along with the two noncommutative parameters, which is big difference from the previous work. Especially, the parameter $\theta_1$ couples to the cosmological constant through the equations of motion. The cosmological constant with $\theta_1$ makes the initial curvature tensor finite as seen from Eq. (39), while the other noncommutative parameter $\theta_2$ plays a role of phase transition as seen in Ref. [37]. On the other hand, we have regarded the regularization ambiguity as $\gamma > 2$ since we can take the two noncommutative parameters to be small. For $\gamma = 1$ (RST model), the noncommutative parameters should be satisfied with $\theta_1 \theta_2 > 1$ for the initial-singularity-free expansion while $\theta_1 \theta_2 = 1$ for the critical case of $\gamma = 2$ (BPP model). Note that it is difficult to make the two noncommutative parameters small simultaneously unless $\gamma > 2$.

On the other hand, one might think that the noncommutative parameters play a role to the non-singular phase transition. Even if this kind of phase transition seems to be interesting, however, the origin of the noncommutativity is still unclear. In a quantum-mechanical point of view, for instance, the noncommutativity is related to the constraint problem so that the Poisson brackets for a slowly moving unit charged particle on the constant magnetic field are given by $\{x^i, x^j\} = -2 / Be^{ij}$, $\{p^i, p^j\} = -B / 2$, and $\{x^i, p^j\} = \delta_{ij}$. 

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In this work, especially for $\gamma = 2$ corresponding to the condition of $\theta_1\theta_2 = 1$, the modified Poisson brackets are the same with the point particle case as long as we identify $\theta_1 = 2/B$ and $\theta_2 = B/2$, which means that the present toy model for $\theta_1\theta_2 < 1$ suggests that some kind complicated constraint analysis should be involved.

Final comment is in order. After lots of dark energy models have been built for explanations of recent observations of the late-time accelerated expansion, there have been many attempts to constrain the dark energy equation of state by observational data sets [47, 48, 49, 50, 51, 52, 53]. As for the constant equation-of-state parameter, there has been a good agreement in the literatures at $-1.4 < \omega < -0.85$, and its value approaches $\omega = -1$ for the dark energy based on the cosmological constant. Now, one might think that this cosmological constraint excludes our model since our equation-of-state parameter (47) was fixed at $\omega = 1$. However, this is a dimensional result. Moreover, our two-dimensional toy model is not a realistic one while the cosmological constraint on the equation of state is considered in four dimensions. In fact, the equation-of-state parameter should be $\omega < -1/3$ to guarantee the positive acceleration in the four dimensional general relativity. We hope our two dimensional model can be extended to the realistic four dimensional one in the near future.

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