BACKGROUND GEOMETRY
IN GAUGE GRAVITATION THEORY

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Abstract

Dirac fermion fields are responsible for spontaneous symmetry breaking in gauge gravitation theory because the spin structure associated with a tetrad field is not preserved under general covariant transformations. Two solutions of this problem can be suggested.

(i) There exists the universal spin structure $S \to X$ such that any spin structure $S^h \to X$ associated with a tetrad field $h$ is a subbundle of the bundle $S \to X$. In this model, gravitational fields correspond to different tetrad (or metric) fields. (ii) A background tetrad field $h$ and the associated spin structure $S^h$ are fixed, while gravitational fields are identified with additional tensor fields $q^{\lambda \mu}$ describing deviations $\tilde{h}_a^\lambda = q^{\lambda \mu} h_\mu^a$ of $h$. One can think of $\tilde{h}$ as being effective tetrad fields. We show that there exist gauge transformations which keep the background tetrad field $h$ and act on the effective fields by the general covariant transformation law. We come to Logunov’s Relativistic Theory of Gravity generalized to dynamic connections and fermion fields.

1 Introduction

Existence of Dirac fermion fields implies that, if a world manifold $X$ is non-compact in order to satisfy causality conditions, it is parallelizable, that is, the tangent bundle $TX$ is trivial and the principal bundle $LX$ of oriented frames in $TX$ admits a global section $[4]$.

Dirac spinors are defined as follows $[2, 3]$. Let $M$ be the Minkowski space with the metric

$$\eta = \text{diag}(1, -1, -1, -1),$$

written with respect to a basis $\{e^a\}$. By $C_{1,3}$ is meant the complex Clifford algebra generated by $M$. This is isomorphic to the real Clifford algebra $\mathbb{R}_{2,3}$ over $\mathbb{R}^5$. Its subalgebra generated
by \( M \subset \mathbb{R}^5 \) is the real Clifford algebra \( \mathbb{R}_{1,3} \). A Dirac spinor space \( V_s \) is defined as a minimal left ideal of \( \mathbb{C}_{1,3} \) on which this algebra acts on the left. We have the representation

\[
\gamma : M \otimes V_s \rightarrow V_s, \quad \bar{e}^a = \gamma(e^a) = \gamma^a,
\]

of elements of the Minkowski space \( M \subset \mathbb{C}_{1,3} \) by the Dirac matrices on \( V_s \). The explicit form of this representation depends on the choice of the ideal \( V_s \). Different ideals lead to equivalent representations (1). The spinor space \( V_s \) is provided with the spinor metric

\[
a(v, v') = \frac{1}{2}(v^+ \gamma^0 v' + v'^+ \gamma^0 v).
\]

Let us consider morphisms preserving the representation (1). By definition, the Clifford group \( G_{1,3} \) consists of the invertible elements \( l_s \) of the real Clifford algebra \( \mathbb{R}_{1,3} \) such that the inner automorphisms

\[
l_s e l_s^{-1} = l(e), \quad e \in M,
\]

defined by these elements preserve the Minkowski space \( M \subset \mathbb{R}_{1,3} \). Since the action (3) of the group \( G_{1,3} \) on \( M \) is not effective, one usually considers its spin subgroup

\[
L_s = \text{Spin}^0(1, 3) \simeq \text{SL}(2, \mathbb{C}).
\]

This is the two-fold universal covering group \( z_L : L_s \rightarrow L \) of the proper Lorentz group \( L = SO^0(1, 3) \). The group \( L \) acts on \( M \) by the generators

\[
L_{ab}^c_d = \eta_{ad} \delta^c_b - \eta_{bd} \delta^c_a.
\]

The spin group \( L_s \) acts on the spinor space \( V_s \) by the generators

\[
L_{ab} = \frac{1}{4}[\gamma_a, \gamma_b].
\]

Since, \( L_{ab}^+ \gamma^0 = -\gamma^0 L_{ab} \), the group \( L_s \) preserves the spinor metric (2). The transformations (4) and (3) preserve the representation (1), that is,

\[
\gamma(lM \otimes l_s V_s) = l_s \gamma(M \otimes V_s).
\]

A Dirac spin structure on a world manifold \( X \) is said to be a pair \((P_s, z_s)\) of an \( L_s \)-principal bundle \( P_s \rightarrow X \) and a principal bundle morphism of \( P_s \) to the frame bundle \( LX \) with the structure group \( GL_4 = GL^+(4, \mathbb{R}) \) [2, 4, 5]. Owing to the group epimorphism \( z_L \), every bundle morphism \( z_s \) factorizes through a bundle epimorphism of \( P_s \) onto a subbundle \( L^h X \subset LX \) with
the structure group $L$. Such a subbundle $L^hX$ is called a Lorentz structure [3, 7]. It exists since $X$ is parallelizable.

By virtue of the well-known theorem [10], there is one-to-one correspondence between the Lorentz subbundles of the frame bundle $LX$ and the global sections $h$ of the quotient bundle

$$\Sigma = LX/L \to X.$$  (6)

This is the two-fold covering of the bundle of pseudo-Riemannian metrics in $TX$, and sections of $\Sigma$ are tetrad fields. Let $P^h$ be the $L_s$-principal bundle covering $L^hX$ and

$$S^h = (P^h \times V_s)/L_s$$  (7)

the associated spinor bundle. Sections $s^h$ of $S^h$ describe Dirac fermion fields in the presence of the tetrad field $h$.

Indeed, every tetrad field $h$ yields the structure

$$T^*X = (L^hX \times M)/L$$

of the $L^hX$-associated bundle of Minkowski spaces on $T^*X$, and defines the representation

$$\gamma_h : T^*X \otimes S^h = (P^h \times (M \otimes V_s))/L_s \to (P^h \times (M \otimes V_s))/L_s = S^h,$$

$$\gamma_h : T^*X \ni t^* = \hat{x}_\lambda dx^\lambda \mapsto \hat{x}_\lambda \tilde{dx}^\lambda = \hat{x}_\lambda h^a_\lambda(x)\gamma^a,$$  (8)

of covectors on $X$ by the Dirac matrices on elements of the spinor bundle $S^h$. The crucial point is that different tetrad fields $h$ define non-equivalent representations (8) [8, 9]. It follows that every Dirac fermion field must be considered in a pair with a certain tetrad field.

Thus, we come to the well-known problem of describing fermion fields in the presence of different gravitational fields and under general covariant transformations. Recall that general covariant transformations are automorphisms of the frame bundle $LX$ which are the canonical lift of diffeomorphisms of a world manifold $X$. They do not preserve the Lorentz subbundles of $LX$. The following two solutions of this problem can be suggested.

(i) One can describe the total system of the pairs of spinor and tetrad fields if any spinor bundle $S^h$ is represented as a subbundle of some fibre bundle $S \to X$ [11-14]. To construct $S$, let us consider the two-fold universal covering group $\widetilde{GL}_4$ of the group $GL_4$ and the corresponding principal bundle $\widetilde{LX}$ covering the frame bundle $LX$ [2,15-17]. Note that the group $\widetilde{GL}_4$ admits only infinite-dimensional spinor representations [18]. At the same time, $\widetilde{LX}$ has the structure of the $L_s$-principal bundle $\widetilde{LX} \to \Sigma$ [13, 14]. Then let us consider the associated spinor bundle

$$S = (\widetilde{LX} \times V_s)/L_s \to \Sigma$$
which is the composite bundle $S \to \Sigma \to X$. We have the representation
\begin{align*}
\gamma_\Sigma : & (\Sigma \times T^*X) \otimes \Sigma \to S, \\
\gamma_\Sigma : & dx^\lambda \mapsto \sigma^\lambda_a \gamma_a.
\end{align*}
(9)
of covectors on $X$ by the Dirac matrices. One can show that, for any tetrad field $h$, the restriction of $S \to \Sigma$ to $h(X) \subset \Sigma$ is a subbundle of $S \to X$ which is isomorphic to the spinor bundle $S^h$, while the representation $\gamma_\Sigma$ (9) restricted to $h(X)$ is exactly the representation $\gamma_h$ (8). The bundle $\tilde{L}X$ inherits general covariant transformations of $LX$ [15]. They, in turn, induce general covariant transformations of $S$, which transform the subbundles $S^h \subset S$ to each other and preserve the representation (9) [13, 14].

(ii) This work is devoted to a different model. A background tetrad field $h$ and the associated background spin structure $S^h$ are fixed, while gravitational fields are identified with the sections of the group bundle $Q \to X$ associated with $LX$ (in the spirit of Logunov’s Relativistic Gravitation Theory (RGT) [19]). We will show that there exists an automorphism $\tilde{f}_h$ of $LX$ over any diffeomorphism $f$ of $X$ which preserves the Lorentz subbundle $L^hX \subset LX$.

2 Gauge transformations

With respect to the tangent holonomic frames $\{\partial_\mu\}$, the frame bundle $LX$ is equipped with the coordinates $(x^\lambda, p^\lambda_a)$ such that general covariant transformations of $LX$ over diffeomorphisms $f$ of $X$ take the form
\[
\tilde{f} : (x^\lambda, p^\lambda_a) \mapsto (f^\lambda(x), \partial_\mu f^\lambda(x)p^\mu_a).
\]
They induce general covariant transformations
\[
\tilde{f} : (p, v) \cdot GL_4 \mapsto (\tilde{f}(p), v) \cdot GL_4
\]
of any $LX$-associated bundle
\[
Y = (LX \times V)/GL_4,
\]
where the quotient is defined by identification of elements $(p, v)$ and $(pg, g^{-1}v)$ for all $g \in GL_4$.

Given a tetrad field $h$, any general covariant transformation of the frame bundle $LX$ can be written as the composition $\tilde{f} = \Phi \circ \tilde{f}_h$ of its automorphism $\tilde{f}_h$ over $f$ which preserves $L^hX$ and some vertical automorphism
\begin{equation}
\Phi : p \mapsto p\phi(p), \quad p \in LX,
\end{equation}
(10)
where $\phi$ is a $GL_4$-valued equivariant function on $LX$, i.e.,

$$\phi(pg) = g^{-1}\phi(p)g, \quad g \in GL_4.$$

Since $X$ is parallelizable, the automorphism $\tilde{f}_h$ exists. Indeed, let $z^h$ be a global section of $L^hX$. Then, we put

$$\tilde{f}_h : L_xX \ni p = z^h(x)g \mapsto z^h(f(x))g \in L_{f(x)}X.$$

The automorphism $\tilde{f}_h$ restricted to $L^hX$ induces an automorphism of the principal bundle $P^h$ and the corresponding automorphism $\tilde{f}_s$ of the spinor bundle $S^h$, which preserve the representation (8).

Turn now to the vertical automorphism $\Phi$. Let us consider the group bundle $Q \to X$ associated with $LX$. Its typical fibre is the group $GL_4$ which acts on itself by the adjoint representation. Let $(x^\lambda, q^\lambda_\mu)$ be coordinates on $Q$. There exist the left and right canonical actions of $Q$ on any $LX$-associated bundle $Y$:

$$\rho_l : Q \times X \to Y,$$

$$\rho_l : ((p, g) \cdot GL_4, (p, v) \cdot GL_4) \mapsto (p, gv) \cdot GL_4,$$

$$\rho_r : ((p, g) \cdot GL_4, (p, v) \cdot GL_4) \mapsto (p, g^{-1}v) \cdot GL_4.$$

Let $\Phi$ be the vertical automorphism (10) of $LX$. The corresponding vertical automorphisms of an associated bundle $Y$ and the group bundle $Q$ read

$$\Phi : (p, v) \cdot GL_4 \mapsto (p, \phi(p)v) \cdot GL_4,$$

$$\Phi : (p, g) \cdot GL_4 \mapsto (p, \phi(p)g\phi^{-1}(p)) \cdot GL_4.$$

For any $\Phi$, there exists the fibre-to-fibre morphism

$$\overline{\Phi} : (p, q) \cdot GL_4 \mapsto (p, \phi(p)q) \cdot GL_4$$

of the group bundle $Q$ such that

$$\rho_l(\overline{\Phi}(Q) \times Y) = \Phi(\rho_l(Q \times Y)), \quad (11)$$

$$\rho_r(\overline{\Phi}(Q) \times \Phi(Y)) = \rho_r(Q \times Y). \quad (12)$$

For instance, if $Y = T^*X$, the expression (11) takes the coordinate form

$$\rho_l : (x^\lambda, q^\lambda_\mu, \dot{x}_\mu) \mapsto (x^\lambda, \dot{x}_\lambda q^\lambda_\mu),$$

$$\overline{\Phi} : (x^\lambda, q^\lambda_\mu) \mapsto (x^\lambda, S^\lambda_\nu q^\nu_\mu),$$

$$\rho_r(x^\lambda, S^\lambda_\nu q^\nu_\mu, \dot{x}_\alpha(S^{-1})^\alpha_\lambda) = (x^\lambda, \dot{x}_\lambda q^\lambda_\mu).$$
Hence, we obtain the representation

\[ \gamma_Q : (Q \times T^* X) \otimes (Q \times S^h) \to (Q \times S^h), \]

\[ \gamma_Q = \gamma_h \circ \rho_r : (q, t^*) \mapsto \dot{x}_Q^\lambda \mu \hat{d}x^\mu = \dot{x}_h^\lambda \mu h_{a}^\mu \gamma^a, \] (13)

on elements of the spinor bundle \( S^h \). Let \( q_0 \) be the canonical global section of the group bundle \( Q \to X \) whose values are the unit elements of the fibres of \( Q \). Then, the representation \( \gamma_Q \) (13) restricted to \( q_0(X) \) comes to the representation \( \gamma_h \) (8).

Sections \( q(x) \) of the group bundle \( Q \) are dynamic variables of the model under consideration. One can think of them as being tensor gravitational fields of Logunov’s RTG. There is the canonical morphism

\[ \rho_l : Q \times \Sigma \to \Sigma, \]

\[ \rho_l : ((p, g) \cdot GL_4, (p, \sigma) \cdot GL_4) \mapsto (p, g\sigma) \cdot GL_4, \quad p \in LX, \]

\[ \rho_l : (x^\lambda, q^\lambda_\mu, \sigma^\mu_a) \mapsto (x^\lambda, q^\lambda_\mu h^\mu_a). \]

This morphism restricted to \( h(X) \subset \Sigma \) takes the form

\[ \rho_h : Q \to \Sigma, \]

\[ \rho_h : ((p, g) \cdot L, (p, \sigma_0) \cdot L) \to (p, g\sigma_0) \cdot L, \quad p \in L^h X, \quad \]

\[ \rho_h : (x^\lambda, q^\lambda_\mu) \mapsto (x^\lambda, q^\lambda_\mu h^\mu_a), \] (14)

where \( \sigma_0 \) is the center of the quotient \( GL_4/L \).

Let \( \Sigma_h \), coordinatized by \( \tilde{\sigma}^\mu_a \), be the quotient of the bundle \( Q \) by the kernel \( \text{Ker}_h \rho_h \) of the morphism (14) with respect to the section \( h \). This is isomorphic to the bundle \( \Sigma \) provided with the Lorentz structure of an \( L^h X \)-associated bundle. Then the representation (13), which is constant on \( \text{Ker}_h \rho_h \), reduces to the representation

\[ (\Sigma_h \times T^* X) \otimes (\Sigma_h \times S^h) \to (\Sigma_h \times S^h), \]

\[ (\tilde{\sigma}, t^*) \mapsto \dot{x}_h^\lambda \mu \tilde{\sigma}^\mu_a \gamma^a. \] (15)

Thence, one can think of a section \( \tilde{h} \neq h \) of the bundle \( \Sigma_h \) as being an effective tetrad field, and can treat \( \tilde{g}^{\mu\nu} = \tilde{h}^\mu_a \tilde{h}^\nu_b \eta_{ab} \) as an effective metric. A section \( \tilde{h} \) is not a true tetrad field, while \( \tilde{g} \) is not a true metric. Covectors \( \tilde{h}^a = \tilde{h}^a_\mu \hat{d}x^\mu \) have the same representation by \( \gamma \)-matrices as the covectors \( h^a = h^a_\mu \hat{d}x^\mu \), while Greek indices go down and go up by means of the background metric \( \hat{g}^{\mu\nu} = h^a_\mu h^b_\nu \eta_{ab} \).
Given a general covariant transformation \( \tilde{f} = \Phi \circ \tilde{f}_h \) of the frame bundle \( LX \), let us consider the morphism

\[
\tilde{f}_Q : Q \to \Phi \circ \tilde{f}_h(Q), \quad S^h \to \tilde{f}_s(S^h), \quad T^*X \to \tilde{f}(T^*X).
\]

This preserves the representation (13), i.e., \( \gamma_Q \circ \tilde{f}_Q = \tilde{f}_s \circ \gamma_Q \), and yields the general covariant transformation \( \tilde{\sigma}_a^\lambda \mapsto \partial_\mu f^\lambda \tilde{\sigma}_a^\mu \) of the bundle \( \Sigma_h \).

Thus, we recover RTG [19] in the case of a background tetrad field \( h \) and dynamic gravitational fields \( q \).

### 3 Gauge theory of RTG

We follow the geometric formulation of field theory where a configuration space of fields, represented by sections of a bundle \( Y \to X \), is the finite dimensional jet manifold \( J^1Y \) of \( Y \), coordinatized by \((x^\lambda, y^i, y^{i\lambda})\) [14, 21]. Recall that \( J^1Y \) comprises the equivalence classes of sections \( s \) of \( Y \to X \) which are identified by their values and values of their first derivatives at points \( x \in X \), i.e.,

\[
y^i \circ s = s^i(x), \quad y^{i\lambda} \circ s = \partial_\lambda s^i(x).
\]

A Lagrangian on \( J^1Y \) is defined to be a horizontal density

\[
L = \mathcal{L}(x^\lambda, y^i, y^{i\lambda})\omega, \quad \omega = dx^1 \cdots dx^n, \quad n = \dim X.
\]

The notation \( \pi^\lambda_i = \partial_\lambda y^i \mathcal{L} \) will be used.

A connection \( \Gamma \) on the bundle \( Y \to X \) is defined as a section of the jet bundle \( J^1Y \to Y \), and is given by the tangent-valued form

\[
\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_i).
\]

For instance, a linear connection \( K \) on the tangent bundle \( TX \) reads

\[
K = dx^\lambda \otimes (\partial_\lambda + K^\mu_{\nu, \alpha} \partial_\nu \partial_\alpha).
\]

Every world connection \( K \) yields the spinor connection

\[
K_h = dx^\lambda \otimes [\partial_\lambda + \frac{1}{4}(\eta^{kb} h^a_\mu - \eta^{ka} h^b_\mu)(\partial_\lambda h^a_\mu - h^a_\mu K^\mu_{\nu, \nu}) L_{ab}^A B y^B \partial_A].
\]
on the spinor bundle $S^h$ with coordinates $(x^\lambda, y^A)$, where $L_{ab}$ are generators \cite{3, 12, 14, 22}.

Using the connection $K_h$ and the representation $\gamma_Q$ \cite{13}, one can construct the following Dirac operator on the product $Q \times S^h$:

$$D_Q = q^\lambda \mu h^\mu_\gamma A D_\lambda,$$

where $D_\lambda = y^A_\lambda - K^A_\lambda$ are the covariant derivatives relative to the connection \cite{17}. The operator \cite{18} restricted to $q_0(X)$ recovers the familiar Dirac operator on $S^h$ for fermion fields in the presence of the background tetrad field $h$ and the world connection $K$.

Thus, we obtain the metric-affine generalization of RTG where dynamic variables are tensor gravitational fields $q$, general linear connections $K$ and Dirac fermion fields in the presence of a background tetrad field $h$ \cite{20}. The configuration space of this model is the jet manifold $J^1Y$ of the product

$$Y = Q \times C_K \times S^h,$$

where $C_K = J^1LX/GL_4$ is the bundle whose sections are world connections $K$. The bundle \cite{19} is coordinatized by $(x^\mu, q^\mu_\nu, k^\alpha_\mu_\nu, y^A)$. A total Lagrangian on this configuration space is the sum

$$L = L_{MA} + L_q(q, g) + L_D$$

where $L_{MA}$ is a metric-affine Lagrangian, expressed into the curvature

$$R^\alpha_\mu_\beta = k^\alpha_\mu_\beta - k^\alpha_\mu_\lambda k_\lambda^\beta + k^\alpha_\mu_\epsilon k^\epsilon_\lambda^\beta - k^\alpha_\epsilon k^\epsilon_\mu^\beta$$

and the effective metric $\tilde{\sigma}^{\mu\nu} = \tilde{\sigma}^\alpha_\mu_\beta \tilde{\sigma}^\beta_\nu \eta^{ab}$, the Lagrangian $L_q$ depends on tensor gravitational fields $q$ and the background metric $g$, and

$$L_D = \{ \frac{i}{2} \tilde{\sigma}^\lambda_\mu [y^A_\mu (\gamma^0 \gamma^q)^A_B (y^B_\lambda - \frac{1}{4} (\eta^{kh} \sigma^{-1a}_\mu - \eta^{ka} \sigma^{-1b}_\mu) (\tilde{\sigma}^\mu_\lambda - \tilde{\sigma}^\mu_\epsilon k^\epsilon_\lambda^\lambda_\nu) L_{ab}^C c y^C - (y^A_\lambda - \frac{1}{4} (\eta^{kh} \sigma^{-1a}_\mu - \eta^{ka} \sigma^{-1b}_\mu) (\tilde{\sigma}^\mu_\lambda - \tilde{\sigma}^\mu_\epsilon k^\epsilon_\lambda^\lambda_\nu) y^A C L_{ab}^C A (\gamma^0 \gamma^q)^A_B c y^C ] - m y^A_\lambda (\gamma^0)^A_B c y^B \} | \tilde{\sigma} |^{-1/2}, \quad \tilde{\sigma} = \det(\tilde{\sigma}^{\mu\nu})$$

is the Lagrangian of fermion fields in metric-affine gravitation theory \cite{12-14}, where tetrad fields are replaced with the effective tetrad fields $\tilde{\sigma}$. If

$$L_{AM} = (-\lambda_1 R + \lambda_2) | \tilde{\sigma} |^{-1/2}, \quad L_q = \lambda_3 g_{\mu\nu} \tilde{\sigma}^{\mu\nu} | \sigma |^{-1/2}, \quad L_D = 0,$$

where $R = \tilde{\sigma}^{\mu\nu} R^\alpha_\mu_\alpha \rho$, the familiar Lagrangian of RTG is recovered.
4 Energy-momentum conservation law

We follow the standard procedure of constructing Lagrangian conservation laws which is based on the first variational formula \[14, 23\]. This formula provides the canonical splitting of the Lie derivative of a Lagrangian \(L\) along a vector field \(u\) on \(Y \to X\). We have

\[
\partial_\lambda u^\lambda \mathcal{L} + \left[ u^\lambda \partial_\lambda + u^i \partial_i + (d_\lambda u^i - y_\mu^i \partial_\lambda u^\mu) \partial_i^\lambda \right] \mathcal{L} = (22)
\]

\[(u^i - y_\mu^i u^\mu)(\partial_i - d_\lambda \partial_i^\lambda)\mathcal{L} - d_\lambda [\pi^\lambda_\iota(u^\mu y_\mu^i - u^i) - u^\lambda \mathcal{L}].\]

This identity restricted to the shell

\[(\partial_i - d_\lambda \partial_i^\lambda)\mathcal{L} = 0, \quad d_\lambda = \partial_\lambda + y_\mu^i \partial_i + y_\mu^\lambda \partial_i^\mu,\]

comes to the weak equality

\[
\partial_\lambda u^\lambda \mathcal{L} + \left[ u^\lambda \partial_\lambda + u^i \partial_i + (d_\lambda u^i - y_\mu^i \partial_\lambda u^\mu) \partial_i^\lambda \right] \mathcal{L} \approx (23)
\]

\[-d_\lambda [\pi^\lambda_\iota(u^\mu y_\mu^i - u^i) - u^\lambda \mathcal{L}].\]

If the Lie derivative of \(L\) along \(u\) vanishes (i.e., \(L\) is invariant under the local 1-parameter group of gauge transformations generated by \(u\)), we obtain the weak conservation law

\[0 \approx -d_\lambda [\pi^\lambda_\iota(u^\mu y_\mu^i - u^i) - u^\lambda \mathcal{L}].\]

For the sake of simplicity, let us replace the bundle \(Q\) in the product \((19)\) with the bundle \(\Sigma_h\), and denote

\[Y' = \Sigma_h \times C_K \times S^h.\]

There exists the canonical lift on \(Y'\) of every vector field \(\tau\) on \(X\):

\[
\tilde{\tau} = \tau^\mu \partial_\mu + (\partial_\nu \tau^\alpha k_{\mu}^{\alpha \beta} - \partial_\beta \tau^\nu k_{\mu}^{\alpha \beta} - \partial_\mu \tau^\nu k_{\nu}^{\alpha \beta} + \partial_{\mu \beta} \tau^\alpha) \frac{\partial}{\partial k_{\mu}^{\alpha \beta}} + \partial_\nu \tau^\mu \tilde{\sigma}^\nu \frac{\partial}{\partial \tilde{\sigma}^\nu} + (24)
\]

\[
\frac{1}{4}(\eta_{k_{\mu}^{a \beta}} - \eta_{k_{\mu}^{b \beta}})(\tau^\lambda \partial_\lambda k_{\mu}^{a \beta} - k_{\mu}^{a \beta} \partial_\nu \tau^\nu)(L_{ab}^e B^e \partial_A - L_{ab}^e c^d \tilde{\sigma}^\nu \frac{\partial}{\partial \tilde{\sigma}^\nu}),
\]

where \(L_{ab}^c d\) are generators \((4)\). This lift is the generator of gauge transformations of the bundle \(Y'\) induced by morphisms \(\tilde{f}_Q\) \((16)\). Its part acting on Greek indices is the familiar generator of general covariant transformations, whereas that acting on the Latin ones is a local generator of vertical Lorentz gauge transformations.
Let us examine the weak equality (23) in the case of the Lagrangian (20) and the vector field (24). In contrast with $L_q$, the Lagrangians $L_{MA}$ and $L_D$ are invariant under the above-mentioned gauge transformations. Then, using the results of [12, 14], we bring (23) into the form
\[
\partial_\lambda (\tau^\lambda L_q) + (\partial_\alpha \tau^\mu \tilde{g}^{\alpha\nu} + \partial_\alpha \tau^\nu \tilde{g}^{\alpha\mu}) \frac{\partial L_q}{\partial \tilde{g}^{\mu\nu}} \approx 0
\] (25)
d\lambda (2\tau^\mu \tilde{g}^{\lambda\alpha} \frac{\partial L_q}{\partial \tilde{g}^{\alpha\mu}} + \tau^\lambda L_q - d_\mu U^{\mu\lambda}),
\]
where
\[
U = 2 \frac{\partial L_{AM}}{\partial R_{\mu\lambda\alpha\nu}} (\partial_\nu \tau^\alpha - k_{\sigma}^{\alpha} \tau^\sigma)
\]
is the generalized Komar superpotential of the energy-momentum of metric-affine gravity [12, 14, 24].

A glance at the expression (25) shows that, if the Lagrangian $L$ (20) contains the Higgs term $L_q$, the energy-momentum flow is not reduced to a superpotential, and the familiar covariant conservation law
\[
\tilde{\nabla}_\alpha \hat{t}_\lambda \approx 0, \quad \hat{t}_\lambda = 2\tilde{g}^{\alpha\mu} \frac{\partial L_q}{\partial \tilde{g}^{\mu\nu}} \] (26)
takes place. Here, $\tilde{\nabla}_\alpha$ are covariant derivatives relative to the Levi-Civita connection of the effective metric $\tilde{g}$. In the case of the standard Lagrangian $L_q$ (21) of RTG, the equality (26) comes to the well-known condition
\[
\nabla_\alpha (\tilde{g}^{\alpha\mu} \sqrt{|\tilde{g}|}) \approx 0,
\]
where $\nabla_\alpha$ are covariant derivatives relative to the Levi-Civita connection of the background metric $g$. On solutions satisfying this condition, the energy-momentum flow in RTG reduces to the generalized Komar superpotential just as it takes place in General Relativity [25], Palatini formalism [26, 27], metric-affine and gauge gravitation theories [12, 14, 24].

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