CROSS-CONNECTIONS IN CLIFFORD SEMIGROUPS

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Abstract. An inverse Clifford semigroup (often referred to as just a Clifford semigroup) is a semilattice of groups. It is an inverse semigroup and in fact, one of the earliest studied classes of semigroups [6]. In this short note, we discuss various structural aspects of a Clifford semigroup from a cross-connection perspective. In particular, given a Clifford semigroup $S$, we show that the semigroup $T_L(S)$ of normal cones is isomorphic to the original semigroup $S$, even when $S$ is not a monoid. Hence, we see that cross-connection description degenerates in Clifford semigroups. Further, we specialise the discussion to provide the description of the cross-connection structure in an arbitrary semilattice, also.

1. Introduction

Grillet introduced cross-connections as a pair of functions to describe the inter-relationship between the posets of principal left and right ideals of a regular semigroup. This construction involved building two intermediary semigroups and further identifying a fundamental image of the semigroup as a subdirect product, using the cross-connection functions. But isomorphic posets give rise to isomorphic cross-connections. So, Nambooripad replaced posets with certain small categories to overcome this limitation. Hence, using the categorical theory of cross-connections, Nambooripad constructed arbitrary semigroups from their ideal structure.

Starting from a regular semigroup $S$, Nambooripad identified two small categories: $L(S)$ and $R(S)$ which abstract the principal left and right ideal structures of the semigroup, respectively. He showed that these categories are interconnected using a pair of functors. It can be seen that the object maps of these functors coincides with Grillet’s cross-connection functions and so Nambooripad called his functors also cross-connections. Further, Nambooripad showed that this correspondence can be extended to an explicit category equivalence between the category of regular semigroups and the category of cross-connections. Hence the ideal structure of a regular semigroup can be completely captured using these ‘cross-connected’ categories.

Being a rather technical construction, it is instructive to work out the simplifications that arise in various special classes of semigroups. There has been several works in this direction [1–5,10] and in this short article, we propose to outline how the construction simplifies in a couple of very natural classes of regular semigroups: namely Clifford semigroups and semilattices. As the reader may see, this exercise also provides some useful illustrations to several cross-connection related subtleties.

In fact, Clifford semigroups are one of the first classes of inverse semigroups whose structure was studied. It was originally defined [6] as a union of groups in which idempotents commute. It may be noted that some authors refer to general union of groups also as Clifford semigroups but we shall follow [8] and refer to a
semilattice of groups as a Clifford semigroup. The following characterizations of Clifford semigroups will be useful in the sequel.

**Theorem 1.1.** [8, Theorem 4.2.1] [7, Theorem 1.3.11] Let \( S \) be a semigroup. Then the following statements are equivalent.

1. \( S \) is a Clifford semigroup;
2. \( S \) is a semilattice of groups;
3. \( S \) is a strong semilattice of groups;
4. \( S \) is regular and the idempotents of \( S \) are central;
5. Every \( \mathcal{D} \) class of \( S \) has a unique idempotent.

In Nambooripad’s cross-connection description, starting from a regular semigroup \( S \), two small categories of principal left and right ideals (denoted by \( \mathbb{L}(S) \) and \( \mathbb{R}(S) \), respectively in the sequel) are defined. Then their inter-relationship is abstracted as a pair of functors called cross-connections. Conversely, given an abstractly defined pair of cross-connected categories (with some special properties), one can construct a regular semigroup. This correspondence between regular semigroups and cross-connections is proved to be a category equivalence.

The construction of the regular semigroup from the category happens in several layers and the crucial object here is the intermediary regular semigroup \( T\mathbb{L}(S) \) of normal cones from the given category \( \mathbb{L}(S) \). We shall see that when the semigroup \( S \) is Clifford, then the semigroup \( T\mathbb{L}(S) \) is isomorphic to \( S \). It is known that this is not true in general [5] even for an inverse semigroup \( S \). This in turn, highly degenerates the cross-connection structure in Clifford semigroups. This is discussed in the next section. In the last section, we specialise our discussion to an arbitrary semilattice and describe the cross-connections therein.

As mentioned above, since the article can be seen as a part of a continuing project of studying the various classes of regular semigroups within the cross-connection framework, we refer the reader to [1,5,10] for the preliminary notions and formal definitions in Nambooripad’s cross-connection theory. We also refer the reader to [9] for the original treatise on cross-connections.

### 2. Normal categories and cross-connections in Clifford semigroups

Recall from [9] that given a regular semigroup \( S \), the normal category \( \mathbb{L}(S) \) of principal left ideals are defined as follows. The object set

\[ v\mathbb{L}(S) := \{Se : e \in E(S)\} \]

and the morphisms in \( \mathbb{L}(S) \) are partial right translations. In fact, the set of all morphisms between two objects \( Se \) and \( Sf \) may be characterised as the set

\[ \{\rho(e, u, f) : u \in eSf\} \]

where the map \( \rho(e, u, f) \) sends \( x \in Se \) to \( xu \in Sf \).

First, we proceed to discuss some special properties of the category \( \mathbb{L}(S) \) when \( S \) is a Clifford semigroup. This will lead us to the characterisation of the semigroup \( T\mathbb{L}(S) \) of all normal cones in \( S \).

**Proposition 2.1.** Let \( S \) be a Clifford semigroup. Two objects in \( \mathbb{L}(S) \) are isomorphic if and only if they are identical.

**Proof.** Clearly, identical objects are always isomorphic. Conversely, suppose \( Se \) and \( Sf \) are two isomorphic objects in \( \mathbb{L}(S) \). Then by [9] Proposition III.13(c), we have \( e \mathcal{D} f \). Recall that in a Clifford semigroup, the Green’s relations \( \mathcal{L} \), \( \mathcal{R} \) and \( \mathcal{D} \) are identical. Therefore \( e\mathcal{L} f \) and hence \( Se = Sf \). \( \square \)
Given the normal category \( \mathbb{L}(S) \) of principal left ideals of a regular semigroup \( S \), it is known that two morphisms are equal, i.e. \( \rho(e, u, f) = \rho(g, v, h) \) if and only if \( e \preceq g \), \( f \preceq h \), and \( v = gu \). Also, given two morphisms \( \rho(e, u, f) \) and \( \rho(g, v, h) \), they are composable if and only \( Sf = Sg \) so that \( \rho(e, u, f) \rho(g, v, h) = \rho(e, uv, h) \). Now, we see that the equality of morphisms simplify when \( S \) is a Clifford semigroup.

**Proposition 2.2.** Let \( S \) be a Clifford semigroup, then \( \rho(e, u, f) = \rho(g, v, h) \) in the category \( \mathbb{L}(S) \) if and only if \( e = g, u = v \) and \( f = h \).

**Proof.** Suppose that two morphisms \( \rho(e, u, f) = \rho(g, v, h) \) are equal. Recall that by [9, Lemma II.12], this implies that \( Se = Sg, Sf = Sh \) and \( v = gu \). That is, the elements \( e \) and \( g \) are two \( \mathcal{L} \) related idempotents. But since \( \mathcal{L} \) and \( \mathcal{H} \) are identical in a Clifford semigroup \( S \) and since a \( \mathcal{H} \)-class can contain at most one idempotent, we have \( e = g \). Similarly we get \( f = h \). So, the sets \( eSf = gSh \) are equal and also \( v = gu = eu = u \). \( \square \)

Now, we proceed to characterise the building blocks of the cross-connection construction, namely the normal cones in the category \( \mathbb{L}(S) \). An ‘order-respecting’ collection of morphisms in a normal category is defined as a normal cone.

**Definition 2.1.** Let \( \mathbb{L}(S) \) be the normal category of principal left ideals in a regular semigroup \( S \) and \( Sd \in v \mathbb{L}(S) \). A *normal cone with apex \( Sd \) is a function \( \gamma : v\mathbb{L}(S) \to \mathbb{L}(S) \) such that:

1. for each \( Se \in v\mathbb{L}(S) \), one has \( \gamma(Se) \in \mathbb{L}(S)(Se, Sd) \);
2. \( \iota(Sf, Sg)\gamma(Sg) = \gamma(Sf) \) whenever \( Sf \subseteq Sg \);
3. \( \gamma(Sm) \) is an isomorphism for some \( Sm \in v\mathbb{L}(S) \).

Now, for each \( a \in S \), we can define a function \( \rho^a : v\mathbb{L}(S) \to \mathbb{L}(S) \) as follows:

\[
(1) \quad \rho^a(Se) := \rho(e, ea, f) \text{ where } f \preceq a.
\]

It is easy to verify that the map \( \rho^a \) is a well-defined normal cone with apex \( Sf \in v\mathbb{L}(S) \) in the sense of Definition 2.1, see [9, Lemma III.15].

In the sequel, the normal cone \( \rho^a \) is called the *principal cone* determined by the element \( a \). In particular, observe that, for an idempotent \( e \in E(S) \), we have a principal cone \( \rho^e \) such that \( \rho^e(Se) = \rho(e, e, e) = 1_{Se} \). This leads us to the most crucial proposition of this article.

**Proposition 2.3.** In a Clifford semigroup \( S \), every normal cone in \( \mathbb{L}(S) \) is a principal cone.

**Proof.** Suppose \( \gamma \) is a normal cone in \( \mathbb{L}(S) \) with vertex \( Se \) so that \( \gamma(Se) = \rho(e, u, e) \) for some \( u \in S \). Then for any \( Sf \in v\mathbb{L}(S) \), we shall show that \( \gamma(Sf) = \rho(f, fu, e) = \rho^u \) for some \( u \in S \). To this end, first observe that since idempotents commute in \( S \), we have \( Sef = Sfe \subseteq Se \). So, by (2) of Definition 2.1, we have \( \gamma(Sef) = \rho(ef, ef, e)\gamma(Se) \). Then,

\[
\begin{align*}
\gamma(Sef) &= \rho(ef, ef, e)\gamma(Se) \\
&= \rho(ef, ef, e)\rho(e, u, e) \\
&= \rho(ef, efu, e) \\
&= \rho(ef, fu, e) \quad \text{since } u \in eSe \text{ and } ef = fe.
\end{align*}
\]
Now for any $Sf \in vL(S)$, let $\gamma(Sf) = \rho(f, v, e)$ for some $v \in fSe$. Then since $Sef \subseteq Sf$ also, we have

$$
\gamma(Sef) = \rho(ef, ef, f)\gamma(Sf)
= \rho(ef, ef, f)\rho(f, v, e)
= \rho(ef, efv, e)
= \rho(ef, ev, e)
$$

since $fv = v$.

Hence for all $Sf \in vL(S)$, we have $\gamma(Sf) = \rho(f, fu, e)$ and so $\gamma = \rho^a$.

In general, for an arbitrary regular semigroup $S$, two distinct principal cones $\rho^a$ and $\rho^b$ may be equal in $L(S)$ even when $a \neq b$. But when $S$ is Clifford, we proceed to show that it is not the case.

**Proposition 2.4.** Let $S$ be a Clifford semigroup. Given two principal cones $\rho^a$ and $\rho^b$, we have $\rho^a = \rho^b$ if and only if $a = b$.

**Proof.** Clearly when $a = b$, then $\rho^a = \rho^b$. Conversely suppose $\rho^a = \rho^b$. Then their vertices coincide and so, we have $Sa = Sb$, then since $S$ is Clifford and Green’s relations coincide, we have $aHb$. Now let $e$ be the idempotent in $H_a = H_b$, the Green’s $H$ class containing $a$ and $b$. Then $\rho^a(Se) = \rho(e, ea, e) = \rho(e, a, e)$. Similarly we get $\rho^b(Se) = \rho(e, b, e)$. Since the cones are equal, the corresponding morphism components at each vertex coincide. Hence using Proposition 2.2 we have $a = b$. □

Recall from [9, Section III.1] that the set of all normal cones in a normal category forms a regular semigroup, under a natural binary operation. So, in particular, given the normal category $L(S)$, the set $TLL(S)$ of all normal cones in the category $L(S)$ is a regular semigroup. Now, we proceed to characterise this semigroup when $S$ is Clifford.

**Theorem 2.5.** Let $S$ be a Clifford semigroup. Then the semigroup $TLL(S)$ of all normal cones in $L(S)$ is isomorphic to the semigroup $S$.

**Proof.** Recall from [9, Section III.3.2] that the map $\bar{\rho}: a \mapsto \rho^a$ from a regular semigroup $S$ to the semigroup $TL(S)$ is a homomorphism. Now, when $S$ is Clifford, by 2.3 we have seen that every normal cone in $TL(S)$ is principal and hence the map $\bar{\rho}$ is surjective. Also, by Proposition 2.4 we see that the map $\bar{\rho}$ is surjective. Hence the map $\bar{\rho}$ is an isomorphism from the Clifford semigroup $S$ to the semigroup $TLL(S)$.

The above theorem characterises the semigroup of $TLL(S)$ of all normal cones in $L(S)$; this naturally leads us to the complete description of the cross-connection structure of the semigroup $S$ as follows.

Recall from [9, Section III.4] that given a normal category, it has an associated dual category whose objects are certain set-valued functors and morphisms are natural transformations. Now we proceed to characterise the normal dual $N^*LL(S)$ of the normal category $L(S)$ of principal left ideals of a Clifford semigroup $S$. 

Theorem 2.6. Let $S$ be a Clifford semigroup. Then the normal dual $N^\ast L(S)$ of the normal category $L(S)$ of principal left ideals in $S$ is isomorphic to the normal category $R(S)$ of principal right ideals in $S$.

Proof. It is known that [9, Theorem III.25] the normal dual $N^\ast L(S)$ of the normal category $L(S)$ is isomorphic to the normal category $R(TL(S))$ of principal right ideals of the regular semigroup $TL(S)$. When $S$ is a Clifford semigroup, by Theorem 2.5, we see that the semigroup $TL(S)$ is isomorphic to $S$ and hence $N^\ast L(S)$ is isomorphic to $R(S)$ as normal categories. □

Dually, we can easily prove the following results:

Theorem 2.7. Let $S$ be a Clifford semigroup. The semigroup $TR(S)$ of all normal cones in the category $R(S)$ of all principal right ideals in $S$ is anti-isomorphic to the semigroup $S$. The normal dual $N^\ast R(S)$ of the normal category $R(S)$ is isomorphic to the normal category $L(S)$.

Recall from [9, Theorem IV.1] that the cross-connection of a regular semigroup $S$ is defined as a quadruplet $(L(S), R(S), \Gamma, \Delta)$ such that $\Gamma: R(S) \to N^\ast L(S)$ and $\Delta: L(S) \to N^\ast R(S)$ are functors satisfying certain properties. Now, using Theorems 2.6 and 2.7 we see that both the functors $\Gamma$ and $\Delta$ are in fact isomorphisms. And hence the cross-connection structure degenerates to isomorphisms of the associated normal categories in a Clifford semigroup $S$.

3. Cross-connections of a semilattice

Clearly, a semilattice is a Clifford semigroup. So, we specialise our discussion to a semilattice using the results in the previous section. The following theorem follows from Theorem 2.5.

Theorem 3.1. Let $S$ be a semilattice. Then the semigroup $TL(S)$ of all normal cones in $L(S)$ is isomorphic to the semilattice $S$.

Theorems 2.6 and 2.7 when applied to a semilattice can be unified as follows:

Theorem 3.2. Let $S$ be a semilattice. Then the normal dual $N^\ast L(S)$ of the normal category $L(S)$ of principal left ideals in $S$ is isomorphic to the normal category $R(S)$ of principal right ideals in $S$. The normal dual $N^\ast R(S)$ of the normal category $R(S)$ is isomorphic to the normal category $L(S)$.

Hence, we see that both the cross-connections functors $\Gamma$ and $\Delta$ are isomorphisms, when we have a semilattice also. So, the cross-connection structure degenerates to isomorphisms of the associated normal categories in a semilattice, too.

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