Ground state uniqueness of the twelve site RVB spin-liquid parent Hamiltonian on the kagome lattice

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Anderson’s idea of a (short-ranged) resonating valence bond (RVB) spin liquid has been the first ever proposal of what we now call a topologically ordered phase. Since then, a wealth of exactly solvable lattice models have been constructed that have topologically ordered ground states. For a long time, however, it has been difficult to realize Anderson’s original vision in such solvable models, according to which the ground state has an unbroken SU(2) spin rotational symmetry and is dominated by fluctuation of singlet bonds. The kagome lattice is the simplest lattice geometry for which parent Hamiltonians stabilizing a prototypical spin-1/2 short-ranged RVB wave function has been constructed and strong evidence has been given that this state belongs to a topological phase. The uniqueness of the desired RVB-type ground states has, however, not been rigorously proven for the simplest possible such Hamiltonian, which acts on 12 spins at a time. Rather, this uniqueness has been demonstrated for a longer ranged (19-site) variant of this Hamiltonian by Schuch et al., via making contact with powerful results for projected entangled-pair states. In this paper, we extend this result to the 12-site Hamiltonian. Our result is based on numerical studies on finite clusters, for which we demonstrate a “ground state intersection property” with implications for arbitrary system size. We also review the relations between various constructions schemes for RVB parent-Hamiltonians found in the literature.

I. INTRODUCTION

The study of topological phases of matter has become a dominant theme of contemporary research into condensed matter physics. The notion of a “topological phase” has become attached to a wide range of systems, from non-interacting ones such as Chern insulators to interacting systems displaying non-trivial topological orders. The latter have found a multitude of realizations in the physics of the fractional quantum Hall effect and are considered possible hardware for topological quantum computation. However, the earliest incarnation of topological order and, more generally, systems with fractionalized excitations that surfaced in the condensed matter physics literature was proposed by Anderson in the context of quantum magnetism. The key question asked in this seminal work was whether a quantum antiferromagnet could retain all symmetries of the system in its ground state. These symmetries were understood to include both the space group of the underlying lattice structure and SU(2) rotational invariance in spin space, as well as time-reversal symmetry. For this Anderson developed the scenario of the so-called “resonating-valence-bond” (RVB) spin liquid. While the term is used broadly for physically distinct situations, namely both long-ranged or critical RVB spin liquids and short-ranged ones, it became clear through a number of subsequent developments that the latter are characterized by a number of interesting topological properties. These include, in particular, semionic fractional statistics and non-trivial topological degeneracy which are nowadays easily recognized as the hallmarks of topological order. The topological quantum numbers that the short-ranged RVB spin liquid was argued to have are those of the \( \mathbb{Z}_2 \) topological phase.

In the original RVB scenario, the spin liquid wave function is a superposition of many different valence bond states, where each valence bond corresponds to a singlet pairing between two spin 1/2 degrees of freedoms localized on different lattice sites. It proved difficult, however, to realize such a scenario in both realistic and toy-model Hamiltonians for frustrated quantum anti-ferromagnets. Instead, initial successful demonstrations of the \( \mathbb{Z}_2 \) topological phase was given in models that completely abandoned the RVB-constraint of SU(2) invariance. The first such demonstration was Kitaev’s the toric code model. More true to the original RVB idea was the construction of quantum dimer models (QDMs) where the first demonstration of the \( \mathbb{Z}_2 \) phase was given for the triangular lattice QDM by Moessner and Sondhi. In QDMs, the valence bonds of RVB physics are mimicked by hard-core bosonic degrees of freedom (“dimers”). However, there is no non-trivial way in which a global SU(2) symmetry is realized. The recipe of Ref. was successfully generalized by several other works but at first without reinstating SU(2) invariance. It remains non-trivial to address the general question where, if anywhere, within the phase diagram of SU(2)-invariant local spin 1/2 Hamiltonians a topological spin liquid can be stabilized. Intuitively, one might argue that the breaking of symmetries becomes the harder to avoid the more sym-
metries there are that could possibly be broken. Positive results, however, were obtained early on for large-N generalizations of the SU(2) symmetry.\textsuperscript{19} Moreover, for SU(2) spins on a highly decorated lattice, a controlled procedure was given in Ref. \textsuperscript{17} to show that wave functions of the nearest-neighbor RVB form can be stabilized by a local parent Hamiltonian. By this we mean wave functions of the following kind:

$$|RVB\rangle = \sum_D |D\rangle,$$

(1)

where the sum goes over all dimerizations of the lattice into nearest neighbor pairs, and $|D\rangle$ denotes a state where each pair of the dimerization carries a singlet, following some sign convention. This settles the existence of general lattice structures that support topological spin liquids, for some choice of Hamiltonian. However, one may still ask the same question with a given and fairly simple lattice in mind, such as the square, triangular, or kagome. It is then natural to seek parent Hamiltonians for states of the form \textsuperscript{1} defined for the specific lattice structure in question. On general grounds, however, it is expected that Eq. \textsuperscript{1}, with only nearest neighbor valence bonds appearing, describes a stable phase only on a non-bipartite lattice.

On bi-partite lattices, any parent Hamiltonian for Eq. \textsuperscript{1} will generally inherit the extensive ground state degeneracy of QDMs defined on bipartite lattices.\textsuperscript{19} This extensive ground state degeneracy is intimately related to the critical behavior of correlation functions. This is demonstrated by the cases considered by Fujimoto\textsuperscript{20} where parent Hamiltonians for Eq. \textsuperscript{1} for both the square lattice and the honeycomb lattice were constructed (although the square lattice Hamiltonian of Ref. \textsuperscript{19} was deficient, in that it admitted exponentially more ground states than intended, as pointed out by Cano and Fendley\textsuperscript{20} who gave a valid construction for the square lattice and a simplified one for the honeycomb). Indeed, the local parent Hamiltonians known to stabilize Eq. \textsuperscript{1} for these lattices have (at least) the full extensive (multi-critical) ground state degeneracy of the associated QDMs. Moreover, recent Monte-Carlo studies\textsuperscript{21,22} of correlation functions in the singlet sector of Eq. \textsuperscript{1} on the square lattice demonstrate the critical behavior that is predicted qualitatively by the QDMs on the same lattice. In contrast, for non-bipartite lattices, QDMs have exponentially decaying correlations, and, over some range of parameters, no broken symmetry combined with the topological degeneracy expected of the $Z_2$ phase.\textsuperscript{13,14}

A. RVB parent Hamiltonian(s) on the kagome lattice

The above considerations motivate the search for parent Hamiltonians for the $|RVB\rangle$ state on simple non-bipartite lattices. To the best of our knowledge, the first such has been given by one of us in Ref. \textsuperscript{23}. Subsequent studies have strongly supported the exponentially decaying nature of correlation functions\textsuperscript{24,25} and the liquid, symmetry unbroken character of the ground state.\textsuperscript{26}

In addition to showing these properties for the “special” ground states at the solvable point, one desires evidence that said properties are stable to perturbation. Ideally, one would prove that the parent Hamiltonian has an energy gap, as befits a topological phase. This, however, is usually difficult to achieve in two and higher dimensions, unless the Hamiltonian is so tuned that it is the sum of commuting operators. On the other hand, the exponential decay of correlation functions, together with the fact that the Hamiltonian has the correct (finite) ground state degeneracy for any finite lattice size, is widely regarded strong circumstantial evidence for the presence of an energy gap. The uniqueness of the desired ground states, modulo topological degeneracy, is also an essential ingredient in making the case that the parent Hamiltonian is not “sick” in the sense that it admits many more ground states than intended, perhaps extensively so or worse. In Ref. \textsuperscript{23}, it was argued that the most likely subspace to harbor additional ground states is the space spanned by the nearest neighbor valence bond state $|D\rangle$. It was shown there that this subspace does not contain any ground states besides those of the form \textsuperscript{1}, where, in the presence of periodic boundary conditions, the sum may be restricted to valence bond states within one of four topological sectors.\textsuperscript{11} This result made use of the linear independence of the set $|D\rangle$, the general question of which is a long-standing problem in the field of short-range RVB physics. In Ref. \textsuperscript{23}, this question was answered positively for the kagome, and for many other lattices in Ref. \textsuperscript{27} expanding earlier results.\textsuperscript{28}

The result for the ground state uniqueness as given in Ref. \textsuperscript{23} is partial, in that it applies strictly only to the subspace spanned by nearest neighbor valence bond states. It does, however, apply to a large class of parent Hamiltonians defined in the same paper, which can be labeled by the basic local cell $C$ on which local terms in the respective parent Hamiltonian act. The smallest possible cell $C$ for which the construction of Ref. \textsuperscript{23} yields a non-trivial parent Hamiltonian is the twelve-site cell shown in FIG. 1. The 12-site “star” cell, a), and the 19-site “double star” cell, b). The kagome lattices we consider can be covered by either type of cell, and various parent Hamiltonians for the RVB state Eq. \textsuperscript{1} are discussed that are given by sums of local terms acting simultaneously on spins within cells of either type a) or b).
We conclude in Section IV.

Their ground state spaces that has been of much interest results should also be of interest in the general context that for kagome lattices of any size, the ground state using properties of frustration free Hamiltonians, imply RVB configurations and in Ref. 24 using properties of kagome lattice RVB state as done in Ref. 23 using local review the constructions of parent Hamiltonians for the following. This question only depends on the ground state uniqueness of general PEPS parent Hamiltonians.

For \( H_{19} \) (though not for some larger basic cells also considered), the result of Ref. 24 also makes use of the linear independence property proven in Ref. 23. Here, in turn, we will show that the ground state uniqueness, modulo topological degeneracy, of \( H_{19} \) also implies that of \( H_{12} \). The latter had been conjectured earlier by one of us24 This then completes the demonstration of the catalog of desirable properties discussed above for the “simpler” kagome lattice RVB parent Hamiltonian \( H_{12} \). Our results should also be of interest in the general context of the study of frustration free lattice Hamiltonians and their ground state spaces that has been of much interest recently.25-34

The paper is organized as follows. In Section II we review the constructions of parent Hamiltonians for the kagome lattice RVB state as done in Ref. 23 using local RVB configurations and in Ref. 24 using properties of PEPS states, and show that they are identical. In section III we discuss numerical studies on small clusters, which, using properties of frustration free Hamiltonians, imply that for kagome lattices of any size, the ground state spaces of the two Hamiltonians \( H_{12} \) an \( H_{19} \) are the same. We conclude in Section IV.

II. VARIOUS CONSTRUCTIONS SCHEMES FOR RVB PARENT HAMILTONIANS

A. Defining local RVB states and parent Hamiltonians

A standard procedure to construct a pair (\( |\psi\rangle, H \)) consisting of a ground state \( |\psi\rangle \) describing, by assumption, a certain phase and a parent Hamiltonian \( H \) is to attempt to make this pair “frustration free”. This means that \( H = \sum_i h_i \) is the sum of not necessarily commuting local terms \( h_i \), such that \( |\psi\rangle \) is a common ground of each \( h_i \). It is then necessarily also a ground state of \( H \). To determine the degeneracy of this ground state is usually a non-trivial problem that will be of interest in the following. This question only depends on the ground state spaces of the individual \( h_i \), which therefore may be taken to be the negatives of the projection operators onto their respective ground state spaces, or \( h_i = -P_i \).

One generally desires the ground state degeneracy (of \( H \)) to be small, especially when describing a stable phase. For this the rank of the operator \( P_i \), viewed as acting on the local Hilbert space of a small cell, should not be too large. In contrast, when choosing all \( P_i \) to be the identity operator, every state \( |\psi\rangle \) would be a ground state, and such a Hamiltonian would of course be entirely trivial. Somewhat in between these two extremes are the so-called Klein models55 whose ground states are exponentially degenerate in the system size, and include the subspace of nearest neighbor valence bond states \( |D\rangle \). Here one is interested on conditions such that the ground state space is exactly spanned by the exponentially many states \( |D\rangle \). This is the case for some lattices,28 but not for others. Generally, it is thus desirable to have the rank of \( P_i \) as small as possible. The condition that the state \( |\psi\rangle \) is a ground state of \( h_i \) is equivalent to the statement that the support of the local density matrix \( \rho_i \) is contained in the local ground state subspace of the operator \( h_i \) (whose dimension is the rank of \( P_i \)). Here, by \( \rho_i \) we mean the density matrix obtained by tracing \( |\psi\rangle\langle\psi| \) over the complement of the local cell on which \( h_i \) acts, and by the support of \( \rho_i \), we mean the orthogonal complement of its kernel, that is, the direct sum of its non-zero eigenvalue eigenspaces. It follows that the smaller the rank of \( P_i \), the more restrictive is the condition for a state to be the simultaneous eigenstate of all the operators \( h_i \). For given state \( |\psi\rangle \), the smallest possible rank that \( P_i \) can have is obviously equal to the dimension of the support of \( \rho_i \), corresponding to the case where \( P_i \) is simply the projection operator onto said support. Hence, to construct a parent
Hamiltonian for given $|\psi\rangle$ in this manner, it is necessary to identify local cells for which the local density matrix $\rho_i$ derived from $|\psi\rangle$ does not have full rank. The smaller the rank of $\rho_i$ (the dimension of its support), the more likely it is that the state $|\psi\rangle$ is the unique ground state of the parent Hamiltonian thus constructed, or at least is one of a fixed number of degenerate ground states, independent of system size.

In Ref. 23 a general recipe was given how to construct a local subspace $RL(C)$ of “resonant loop” states that necessarily contains the support of the local density matrix $\rho_C$ for the RVB state, $|RVB\rangle$, for a given lattice with local cell $C$. Here we briefly review this recipe. Consider a local cell $C$, as depicted in Fig. 2(b)), which is part of some larger lattice. For such a cell, we define a local dimerization as a pairing of sites of the cell into nearest neighbors, such that each inner site participates in a pair (dimer), but boundary sites may or may not. Here, by inner sites we mean sites for which all nearest neighbors in the lattice topology also belong to the cell, whereas boundary sites of the cell do not satisfy this property. An example for a local dimerization is also shown in Fig. 2(b).

We will refer to the boundary sites not participating in a local dimerization as the “free sites” of that dimerization.

We may regard two local dimerizations $D, D'$ as “dynamically equivalent” if one can be transformed into the other by certain local dimer moves. These dimer moves are the same as those appearing in the Hamiltonian of a suitable quantum dimer model on the same lattice. For the kagome lattice, they consist of moves shifting dimers long closed paths that contain exactly one hexagon.13 These dimer moves have the benefit of being ergodic within topological sectors. Note that for local cells and local dimerizations as defined here, dimer moves never change the set of free sites. We may therefore define local “resonant loop” state as a state of the form

$$|D\rangle\langle f| = \sum_{D' \in [D]} |D'\rangle \otimes |f\rangle$$

(2)

Here, by $[D]$ we mean a dynamical equivalence class of local dimer coverings of the cell $C$, and $|f\rangle$ denotes a fixed spin configuration of the free boundary sites of $[D]$. The ket $|D'\rangle$ denotes a state where each pair in $D'$ carries a singlet, and the overall sign of the resulting product of singlets is determined by the link orientation also shown in Fig. 2(b). The desired subspace $RL(C)$ is then the linear hull of all possible states $\psi_{[D],f}$,

$$RL(C) = \{|\psi_{[D],f}\rangle\},$$

(3)

for all possible dynamical equivalence classes $[D]$ and, for given $[D]$, all possible free site configurations $f$ (a complete set of $|f\rangle$s). The local subspace $RL(C)$ contains the support of the local density matrix of the state $|RVB\rangle$ for the cell $C$.13 This is also true in the presence of periodic boundary conditions, where one has four degenerate ground states, one for each of four topological sectors $i = 1 \ldots 4$:

$$|RVB, i\rangle = \sum_{D \in [i]} |D\rangle,$$

(4)

where presently, $D$ again denotes a global dimerization of the lattice. We introduce the orthogonal projection operator $P_C$ onto the space $RL(C)$. We are interested in conditions where the states $|RVB, i\rangle$ are unique ground states of the parent Hamiltonian

$$H = -\sum_C P_C,$$

(5)

where the sum goes over all cells of a certain “type” of geometry. The smallest type of cell for which one may hope that this is the case is the 12-site star shaped cell of Fig. 1(b). We will call the corresponding parent Hamiltonian $H_{12}$. The purpose of this paper is to establish that the unique ground states of $H_{12}$ are the four-fold degenerate states $1$. We note that in this case, as the cell contains only a single hexagon, every local dimerization $D$ is dynamically equivalent to a single partner $D^*$. We should therefore think of the space $RL(C_{12})$ as

$$\{(|D\rangle + |D^*\rangle) \otimes |f\rangle\}.$$  

(6)

The other case of special importance is that of $H_{19}$, where the geometry of the underlying cell is that of the 19-site “double star” cell of Fig. 1(b). In this case, a local dynamical equivalence class similarly consists of four distinct local dimerizations. This follows from the fact that elementary resonance moves can be carried out around each of the two hexagons contained in the cell, and moves around different hexagons commute.13

### B. The PEPS parent Hamiltonian

In Ref. 24 parent Hamiltonians have been constructed utilizing the notion that the state $1$ can be written as a PEPS state. We focus here on the case where the Hamiltonian is the sum over local projection operators acting on the 19-cite cell discussed above. Here we first show that the PEPS Hamiltonian constructed for this cell is identical to the Hamiltonian $H_{19}$ introduced earlier. This is necessary because we have only shown thus far that the subspace $RL(C_{19})$ projected on by the local terms in $H_{19}$ contains the support of the local density matrix, but we have not shown identity. Hence one might suspect that PEPS-Hamiltonian could be different from $H_{19}$, in that its local operators project onto a smaller, more optimized subspace, but we will show here that this is not the case. We do this by first describing the ground state subspace of local terms in the PEPS parent Hamiltonian, and show it to be identical to $RL(C_{19})$.

In the PEPS formalism, the parent Hamiltonian is constructed from local projections onto a subspace that is the
range of a map from a virtual space into the local Hilbert space of “real” degrees of freedom. This map is described by a tensor associated to the local cell. The details are given in Refs. 24 and 29. Here we restrict ourselves to the necessary definitions that determine the subspace in question. We thus assign to each site of the kagome lattice two 3-qutrit states, one associated to each adjacent triangle. For each triangle, we may define the state

$$|\epsilon\rangle = \sum_{i,j,k=0}^{2} \epsilon_{ijk}|ijk\rangle + |222\rangle$$

(7)

Consider now a 19-site double star cell, where for each boundary site, both qutrits are included. For this cell we now consider virtual states of the following form:

$$|\phi_{i1...i8}\rangle = |\epsilon\rangle^{10} \otimes |i_1...i_8\rangle,$$

(8)
i.e., a state where each of the ten triangles of the cell is in the state $|\epsilon\rangle$, and each of the eight boundary qutrits that are not part of a triangle of the cell is assigned a qutrit value $i_j \in \{0,1,2\}$. For each site we now introduce an operator $P$ that maps the two adjacent qutrits onto a real spin 1/2 degree of freedom associated with this site:

$$P = | \downarrow \rangle \langle 02| + | 20\rangle \langle 01| + | \uparrow \rangle \langle 12| + | 21\rangle \langle 1|.$$

(9)

In essence, therefore, a virtual $|0\rangle$ signifies a down-spin at the same site, a virtual $|1\rangle$ signifies an up-spin, whereas a virtual $|2\rangle$ signifies the absence of a valence bond belonging to the triangle associated to the qutrit and touching the site at which the qutrit resides. Now consider the operator $P^{19}$, the tensor product of 19 copies of the operator $P$ applied to the 19 sites of the cell. We will only be interested in the restriction of this operator to the virtual subspace spanned by the states $|\phi_{i1...i8}\rangle$, which is parameterized by the eight virtual boundary qutrits, and is thus isomorphic to $(\mathbb{C}^3)^{\otimes 8}$. The subspace onto which local terms in the PEPS parent Hamiltonian project is just the range of the operator $P^{19}$, i.e., the local subspace spanned by the state $P^{19}|\phi_{i1...i8}\rangle$. Therefore, to show that this Hamiltonian is identical to $H_{19}$ as defined above through local RVB states, we all need to demonstrate is that there exists, essentially, a one-to-one correspondence between the states $P^{19}|\phi_{i1...i8}\rangle$ and the local RVB states $\psi_{|D\rangle, f}$, for the 19-site cell defined earlier. We will do so by observing that the 0’s and 1’s among the entries $i_1...i_8$ define a free spin configuration $f$, and that the 2’s define a class $|D\rangle$ of local dimer patterns connected by resonance moves. The states $\psi_{|D\rangle, f}$ and $P^{19}|\phi_{i1...i8}\rangle$ thus identified are indeed the same.

Before we give some details, we must make this one-to-one correspondence more precise. We observe that for $P^{19}|\phi_{i1...i8}\rangle$ to be non-zero, the indices $i_1...i_8$ must contain an odd number of 2’s. We see this by noting that any qutrit configuration surviving the action of $P^{19}$ must have exactly one qutrit valued 2 at each site, i.e., nineteen 2’s altogether. The state $|\phi_{i1...i8}\rangle$ has the property that on the inside of each of the ten triangles, there is always an odd number of 2’s, which guarantees that the total number of 2’s inside those triangles is even.

We must thus have an odd number of 2’s sitting outside of those triangles, which is by definition the number of 2’s among the indices $i_1...i_8$. We may thus further restrict the operator $P^{19}$ to the linear span of the states $|\phi_{i1...i8}\rangle$ with $Z|\phi_{i1...i8}\rangle = -|\phi_{i1...i8}\rangle$, where $Z$ is the generator of $Z_2$ represented as $(-1)^n$, and $n = 2$ is the operator that counts the number of 2’s in the virtual state. The one-to-one correspondence, via the map $P^{19}$, which we seek to establish is thus one between the states $\psi_{|D\rangle, f}$ and those states $|\phi_{i1...i8}\rangle$ whose quantum number $Z$ equals $-1$.

To complete the argument, it is now clear that any configuration of virtual boundary qutrits $i_1...i_8$ with $Z = -1$ specifies a possible configuration of free boundary spins $f$ in a state $\psi_{|D\rangle, f}$, via the identification $0 \equiv \downarrow$, $1 \equiv \uparrow$. The virtual 2’s among the qutrits $i_1...i_8$, on the other hand, specify those boundary sites that participate in resonating valence bonds. It is easy to see that specifying the boundary sites that participate in valence bonds precisely identifies a class $|D\rangle$ of local valence bond configurations. With $|D\rangle$ and $f$ thus given by the virtual labels $i_1...i_8$, we indeed have (with appropriate normalization conventions)

$$P^{19}|\phi_{i1...i8}\rangle = \psi_{|D\rangle, f}$$

(10)

This establishes the desired identity between the PEPS parent-Hamiltonian for the 19-cite cell and $H_{19}$.

We note in passing that, for the theory of Ref. 24, it was of some importance that the map $P^{19}$ has a left inverse when restricted to the subspace spanned by the states $|\phi_{i1...i8}\rangle$ with $Z = -1$. This has been coined “$Z_2$-injectivity”24,29. We see here that this is equivalent to the linear independence of the states $\psi_{|D\rangle, f}$, which in turn is a consequence of the linear independence of the states $|D\rangle \otimes |f\rangle$ in Eq. (2) observed in Ref. 23. There, this latter

![FIG. 3.](https://example.com/image3.png)
property had been used to prove the linear independence of global nearest neighbor valence bond states on general kagome-type lattices.

III. GROUND STATE UNIQUENESS FOR THE PARENT HAMILTONIAN $H_{12}$

For definiteness, in the following we will always use the definition of parent Hamiltonians given in Ref. 23 as reviewed in Sec. II A. For the 19-site cell, this is equivalent to the PEPS definition, as shown above. In Ref. 24 it has been shown that for $H_{19}$ and a kagome lattice with periodic boundary conditions, the ground state space is spanned by the four topologically degenerate RVB wave functions $|\psi_1\rangle$, i.e., the four RVB states are the unique ground states of $H_{19}$, up to linear combinations. The frustration free character of the Hamiltonians $H_{19}$ and $H_{12}$ now allows for a simple criterion that is sufficient in order for $H_{12}$ to ‘inherit’ this ground state uniqueness property from $H_{19}$. In PEPS-terminology, we demonstrate an “intersection property”29 for the ground state spaces of $H_{12}$ and $H_{19}$.

Consider a lattice consisting just of one 19-site cell $C_{19}$. On this lattice, $H_{19}$ is equal to (the negative of) just a single projection operator onto the space $RL(C_{19})$, $H_{19} = -P_{C_{19}}$, whereas $H_{12} = H_L + H_R$, with $H_L$, $H_R$ each being minus the projection operator onto the space $RL(C_{12}) = -P_{C_{12}}$, for the left/right 12-site star of the cell, respectively. The claim is now that if for this 19-site cell, every ground state of $H_{12}$ is also one of $H_{19}$, then this is also true for any larger kagome lattice that can be covered by 19-site cells. For, if $|\psi\rangle$ is the ground state of $H_{12} = - \sum_{C_{12}} P_{C_{12}}$, for some such lattice, then $|\psi\rangle$ is a ground state of each individual operator $-P_{C_{12}}$ in the sum, and therefore also of each operators $H_L + H_R$ defined as above for any particular 19-site cell $C_{19}$ of the lattice. Then, assuming that we can show that any ground state of $H_L + H_R$ is also a ground state of $-P_{C_{19}}$, the state $|\psi\rangle$ must also be a ground state of $H_{19}$, since the last argument can be made for any 19-site cell of the lattice. Hence $H_{12}$ cannot have more ground states than $H_{19}$, and, by construction, has the same four RVB ground states $|\psi_j\rangle$. This only relies on the statement that any ground state of $H_L + H_R$ is also a ground state of $-P_{C_{19}}$, which is apparently a local statement, i.e., it can be checked on a 19-site lattice. For the 19-site cell, the set of states $|\psi_{|D|,j}\rangle$ spanning the space $RL(C_{19})$ consists of 3280 states. They are linearly independent. This is therefore the ground state degeneracy of $-P_{C_{19}}$. As is elementary to see, each of the states $|\psi_{|D|,j}\rangle$ (for the 19-site cell) is also a ground state of $H_L + H_R$. The ground state degeneracy of $H_L + H_R$ can therefore be only greater than or equal to that of $-P_{C_{19}}$, with the equality implying identical ground state spaces. We have shown that this is indeed that case, using two different numerical methods. The first is by straightforward diagonalization, using both $S_z$-conservation and the two mirror symmetries of the 19-site cell. The second is specific to finding the ground state subspace of frustration free Hamiltonians. Since any ground state of $H_L + H_R$ must be a ground state of both $H_L$ and $H_R$, we may first obtain a complete set of ground states of $H_L$, working only on the 12-site cell. These are just the states $|\psi_{|D|,j}\rangle$ for the 12-site cell. (These are likewise linearly independent which leads to a $Z_2$-injectivity property for the 12-site cell noted in Ref. 24 which is analogous to that for the 19-site cell already mentioned.) Let us denote the latter by $|\tilde{\psi}_{|D|,j}\rangle$ to emphasize that these are states of 12-spins. On the 19-site cell, the ground states of $H_L$ are thus of the form $|\tilde{\psi}_{|D|,j}\rangle \otimes |r\rangle$, where $|r\rangle$ is an arbitrary configuration of the remaining spins of the 19-site double star cell that do not belong to the left star. The number of these states is now rather small compared to the full dimension $2^{19}$ of the Hilbert space of the 19-site cell. The ground states of $H_L + H_R$, being ground states of both $H_L$ and $H_R$, must now be a linear combination of the states $|\tilde{\psi}_{|D|,j}\rangle \otimes |r\rangle$. We may thus diagonalize $H_R$ within this subspace. By the variational principle, eigenstates within this subspace that have the lowest possible eigenenergy of $-1$ are true eigenstates also of the unrestricted $H_R$, and correspond to ground states of $H_L + H_R$. Conversely, every ground state of $H_L + H_R$ can be obtained in this two step procedure, which allows one to avoid ever working with the full Hilbert space of dimension $2^{19}$. ($S_z$-conservation may further be used in this case as well.) We have used both of the above methods to confirm that the ground state degeneracy of $H_L + H_R$ is identical to that of $-P_{C_{19}}$, thus the ground state spaces of these two operators are identical.

The above observations complete the demonstration that on any finite kagome lattice that can be covered by 19-site cells, the ground state spaces of $H_{12}$ and $H_{19}$ are identical. Hence for any such lattice to which the proof of Ref. 24 applies (in particular for the periodic boundary condition chosen there), the ground state space of $H_{12}$ is spanned by the fourfold topologically degenerate RVB-states $|\psi_j\rangle$. This property of $H_{12}$ had originally been conjectured in Ref. 23 where it was proven to hold only within the subspace of nearest neighbor valence bond coverings. We may ask if the method described here can be applied to different ground states and their parent Hamiltonians. A natural modification of the RVB states is to utilize a different sign convention. Hence, replace Eq. (1) with

$$\langle \text{RVB}, i \rangle = \sum_{D \in [i]} (-1)^{\tilde{N}} |D\rangle.$$  

(11)

Here, $\tilde{N}$ is an operator that counts the number of resonance moves required to transform the dimer covering $D$ into a given reference dimer covering in the topological sector $[i]$. That this is well defined (modulo 2) can be seen from the following, alternative definition. Pair up all hexagons on the lattice, say, into nearest neighbor pairs. Then connect members of a pair through paths that start
and end at the midpoints of the respective hexagons, and intersect links of the kagome lattice, avoiding sites. The operator \( \hat{N} \) may then be defined as the number of dimers crossed by these paths in the dimer covering \( D \). (By the linear independence of the states \( |D\rangle \) \(^{23} \) this is indeed a well defined operator within the subspace of nearest neighbor valence bond states, although this is perhaps not essential in defining the state \( |11\rangle \), where it is sufficient that a phase \((-1)^{N_D}\) can be associated to each dimer covering \( D \).) It is easy to see that \((-1)^{N_D}\) changes sign upon a dimer resonance move around any hexagon. This construction can be thought of as a variational excited state for the original parent Hamiltonian (having the states \( |1\rangle \) as ground states), where a “vison” type excitation is placed in each hexagon.

One may instead want to construct a new parent Hamiltonian for this new variational wave function. It is easy to generalize the construction of parent Hamiltonians given in Sec. IIA to the present situation. For the 12-site version of the parent Hamiltonian, all one needs to do is to replace the generating set of \( RL(C_{12}) \), Eq. [6], with

\[
\{(|D\rangle - |D^\ast\rangle) \otimes |f\rangle\}. \tag{12}
\]

Similarly, an additional sign could be introduced as described above in the generating set of the subspace \( RL(C_{19}) \). We denote the resulting parent Hamiltonians by \( \tilde{H}_{12} \) and \( \tilde{H}_{19} \), respectively. We have shown using the same methods described above in this section that the ground state spaces of \( \tilde{H}_{12} \) and \( \tilde{H}_{19} \) are identical. We do not know at present if the methods developed in Ref. \(^{24} \) to show uniqueness of the fourfold degenerate RVB ground states carry over to \( \tilde{H}_{19} \). We note that one essential ingredient of these methods, the \( \mathbb{Z}_2 \)-injectivity of the tensor associated with the 19-cite cell, still holds in this case. This is so for reasons identical to those given above for the original RVB-states, namely, the linear independence of the states \( |D\rangle \otimes |f\rangle \) for the 19-site cell. For the 12-site cell, however, the corresponding \( \mathbb{Z}_2 \)-injectivity no longer holds. This is so since the generating states of the subspace \( RL(C_{12}) \) listed in Eq. \([12]\) are no longer linearly independent, as already noted in Ref. \(^{23} \). We leave further investigation of the ground state uniqueness of \( \tilde{H}_{12} \) and \( \tilde{H}_{19} \) for future work.

\section{IV. Conclusion}

In this work, we have clarified connections between parent Hamiltonians for nearest neighbor resonating valence bond states arising from the PEPS construction and from earlier considerations. The case that these Hamiltonians demonstrate the existence of \( SU(2) \) invariant topological spin liquids rests primarily on the nature of correlations in their ground state, as well as the uniqueness of these ground states, i.e., in particular, the fact that they display the correct degeneracy on any finite lattice. This ground state uniqueness was first proven for a Hamiltonian that acts on 19 spins a time.\(^{23} \) Here we have combined the latter result with numerical work on finite clusters to establish this ground state uniqueness for a simpler 12-site Hamiltonian, completing the proof of an earlier conjecture. Technically, this was done by demonstrating a ground state intersection property for the finite clusters studied. We believe that this 12-site Hamiltonian is the smallest parent Hamiltonian for the prototypical nearest neighbor resonating valence bond state \( |1\rangle \) on the kagome lattice that has these desired features. Our results should also be of interest in the broader context of frustration free two-dimensional Hamiltonians.

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However, in keeping with the definition of Ref. 23, we have defined \([D]\) through local “resonance moves”. It is easy to see that the two definitions coincide. For, consider any two local valence bond configurations on the 19-site cell that touch upon the same sites. Then their overlap graph must satisfy one of the following three conditions: i) there are only short (length 2) loops, ii) there is one long loop encircling exactly one of the two hexagons, iii) there is one long loop encircling both hexagons. These three cases correspond, respectively, to situations where the dimer configurations are i) identical, ii) related by a single resonance move, iii) related by two (commuting) resonance moves. I.e., they are in the same class \([D]\) as defined by resonance moves. Easy extension of this argument shows that resonance moves on the kagome are ergodic within topological sectors, as first stated in Ref. 14.

This number is easily obtained as \(\binom{8}{1} \times 2^1 + \binom{8}{3} \times 2^3 + \binom{8}{5} \times 2^5 + \binom{8}{7} \times 2^7\), e.g., by counting the number of allowed virtual indices in the states \(|\phi_{i_1...i_8}\rangle\).