Logical coherence in Bayesian simultaneous three-way hypothesis tests

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Abstract: This paper studies whether Bayesian simultaneous three-way hypothesis tests can be logically coherent. Two types of results are obtained. First, under the standard error-wise constant loss, only for a limited set of models can a Bayes simultaneous test be logically coherent. Second, if more general loss functions are used, then it is possible to obtain Bayes simultaneous tests that are always logically coherent. An explicit example of such a loss function is provided.

1. Introduction

In a three-way decision problem [Yao, 2012, Liu and Liang, 2014, Yao, 2015] one must classify objects into three categories. While a two-way decision necessarily leads to an affirmation or a negation, a three-way decision also allows non-commitment or pause to gather more evidence. Such a flexible approach has led to advances in areas such as clustering [Yu, 2017], classification [Zhou, 2014, Zhang et al., 2019], multi-agent decisions [Yang and Yao, 2012] game theory [Herbert and Yao, 2011, Azam and Yao, 2014, Bashir et al., 2021], and recommender systems [Zhang et al., 2017].
In particular, three-way decisions can also be applied to statistical hypothesis testing [Wald, 1945, Kaiser, 1960, Tukey, 1960, Harris, 2016, Berg, 2004, Goudey, 2007, Esteves et al., 2016, Stern et al., 2017]. In this context, one gathers data, \( x \in \mathcal{X} \), to decide whether an unobserved quantity, \( \theta \in \Theta \), satisfies \( \theta \in H \), for \( H \subseteq \Theta \). While standard hypothesis tests allow only the rejection or non-rejection of \( H \), three-way (agnostic) tests allow \( H \) to be accepted, rejected or remain undecided. In the statistical literature, such a decision is usually represented by a function, \( \varphi_H : \mathcal{X} \to \{0, \frac{1}{2}, 1\} \). In this context, \( \varphi_H(x) = 0, \varphi_H(x) = 1, \) and \( \varphi_H(x) = \frac{1}{2} \) mean that one decides to, respectively, accept, reject and remain undecided about \( H \) after observing \( x \).

This definition can be identified with the standard three-decision regions:

\[
\begin{align*}
POS(H_0) &:= \{ x \in \mathcal{X} : \varphi_H(x) = 0 \} \\
NEG(H_0) &:= \{ x \in \mathcal{X} : \varphi_H(x) = 1 \} \\
BND(H_0) &:= \left\{ x \in \mathcal{X} : \varphi_H(x) = \frac{1}{2} \right\}
\end{align*}
\]

In order to determine the optimal decision regions, one can use Bayesian decision theory [Yao, 2010]. In this context, one possible approach is to use an error-constant (EC) loss function (Definition 1.1), as presented in Example 1.2.

**Definition 1.1** (Error-wise constant loss function). Let \( H \) be an hypothesis. The error-wise constant (EC) loss function, \( L_H \), is given by table 1, where \( 0 < \lambda_{BP}^H < \lambda_{NP}^H \), \( 0 < \lambda_{BN}^H < \lambda_{PN}^H \), and \( (\lambda_{PN}^H - \lambda_{BN}^H)\lambda_{NP}^H > \lambda_{BP}^H\lambda_{PN}^H \). These restrictions are made so that the loss for each type of error corresponds to its intuitive meaning. For instance, when \( H \), accepting \( H \) is better than not deciding, which in turn is better than rejecting \( H \). Also, not deciding is always better than deciding randomly between accepting or rejecting \( H \).

**Example 1.2** (Posterior probability three-way tests). Under the EC loss (Definition 1.1), Yao
|                | $\theta \in H$ | $\theta \notin H$ |
|----------------|--------------|------------------|
| **accept**     | $0$          | $\lambda_H^{HN}$|
| **undecided**  | $\lambda_H^{BP}$ | $\lambda_H^{BN}$|
| **reject**     | $\lambda_H^{NP}$ | $0$              |

Table 1: Error-constant loss function

[2007] determines the optimal three-way decision regions for hypothesis tests:

\[
POS(H) = \{ x \in \mathcal{X} : \mathbb{P}(\theta \in H | x) > \beta^H \}, \\
NEG(H) = \{ x \in \mathcal{X} : \mathbb{P}(\theta \in H | x) < \alpha^H \}, \text{ and} \\
BND(H) = \{ x \in \mathcal{X} : \alpha^H \leq \mathbb{P}(\theta \in H | x) \leq \beta^H \},
\]

where $\beta^H = \frac{\lambda_H^{HN} - \lambda_H^{BN}}{\lambda_H^{HN} - \lambda_H^{BP} + \lambda_H^{NP}} < 1$, and $\alpha^H = \frac{\lambda_H^{BN}}{\lambda_H^{HN} - \lambda_H^{BP} + \lambda_H^{NP}} > 0$. (1)

A more general setting occurs in simultaneous hypothesis testing, in which one wishes to test a collection of hypotheses, $\sigma(\Theta)$, at the same time [Shaffer, 1995, Lehmann et al., 2005]. Definition 1.3 describes Bayesian optimality in this context:

**Definition 1.3** (Bayesian optimality for simultaneous hypothesis tests). For each hypothesis, $H \in \sigma(\Theta)$, let $L_H$ be a loss function. A simultaneous hypothesis test, $\varphi$, is Bayes with respect to $L$ if, for every hypothesis, $H$, $\varphi_H$ is a Bayes test for testing $H$ against $L_H$.

The following example shows that posterior-probability based simultaneous tests are obtained from the EC loss in a similar fashion as in Example 1.2:

**Example 1.4** (Simultaneous test based for error-wise constant (EC) losses). Let $L$ be a loss function such that, for each hypothesis, $H$, $L_H$ is the loss function presented in table 1. In this case, the simultaneous test that satisfies eq. (1) for each $H$ is Bayes with respect to $L$.

**Definition 1.5** (Simultaneous test based for trivial error-wise constant (TEC) losses). If for each $H \in \sigma(\Theta)$, $L_H$ is such that the constants in table 1 do not depend on $H$, then $L$ is said to be a trivial error-wise constant loss (TEC). In this case, the Bayes simultaneous test given by eq. (1) is such that $\alpha^H$ and $\beta^H$ do not depend on $H$. 

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In the context of simultaneous tests, one is often interested in an overall interpretation of all the tests. One condition that is required for the interpretability of the tests is their logical coherence. For instance, if \( x \in \text{POS}(\theta > 1) \) and also \( x \in \text{NEG}(\theta > 0) \), then, after observing \( x \), one would believe both that “\( \theta > 1 \)′′ is true and that “\( \theta > 0 \)′′ is false, a logical contradiction. Such contradictory conclusions are hard to interpret and should be avoided.

Based on this challenge and on previous proposals for logical requirements [Gabriel, 1969, Schervish, 1996, Lavine and Schervish, 1999, Hommel and Bretz, 2008, Romano et al., 2011, Izbicki and Esteves, 2015, Hansen and Rice, 2022], the concept of logical coherence in simultaneous hypothesis testing is proposed [Esteves et al., 2016]:

**Definition 1.6 (Logical coherence).** A simultaneous hypothesis test is logically coherent if:

1. (Propriety) \( \text{POS}(\Theta) = \mathcal{X} \),

2. (Monotonicity) If \( H_1 \subseteq H_2 \), then \( x \in \text{POS}(H_1) \) implies that \( x \in \text{POS}(H_2) \) and \( x \in \text{BND}(H_1) \) implies that \( x \in \text{BND}(H_2) \cup \text{POS}(H_2) \),

3. (Intersection consonance) If \( x \in \text{POS}(H_1) \) and \( x \in \text{POS}(H_2) \), then \( x \in \text{POS}(H_1 \cap H_2) \),

4. (Invertibility) If \( x \in \text{POS}(H) \), then \( x \in \text{NEG}(H^c) \).

This paper studies under what conditions it is possible to obtain a Bayes simultaneous test that is logically coherent. Section 2 reviews a useful characterization of logical coherence in terms of region estimators. Using this characterization, Section 3 explores the relation between the EC loss and logical coherence. This section shows that it is impossible to fully reconcile Bayesian decision theory with logical coherence while using the EC loss. Given this impossibility, Section 4 explores more general loss functions. This section defines the GFBST loss and shows that, under this loss, the Bayes test is always logically coherent.
2. Characterization of logical coherence

Logically coherent tests can be characterized in terms of region estimators [Esteves et al., 2016]. A region estimator, \( R(x) \), is usually interpreted as a set of likely values for \( \theta \). Region estimators are formalized below:

**Definition 2.1** (Region estimator). A region estimator is a function \( R : \mathcal{X} \rightarrow \mathcal{P}(\Theta) \), where \( \mathcal{P}(\Theta) \) is the collection of all subsets of \( \Theta \).

A particular type of region estimator is the highest posterior density (HPD) set. The HPD contains the parameter values with posterior density above a given threshold. If \( \Theta \) is finite, then the posterior density is often taken as the posterior probability, that is, the HPD contains the most probable values for \( \theta \).

**Example 2.2** (Highest posterior density set). A region estimator, \( R(x) \), is a highest posterior density set with respect to a posterior density, \( f(\theta|x) \), if there exists \( k \) such that

\[
R(x) = \{ \theta \in \Theta : f(\theta|x) \geq k \}.
\]

Using region estimators, one can construct a simultaneous test, as illustrated in fig. 1. A test based on a region estimator, \( R(x) \), accepts \( H \) if \( R(x) \subseteq H \), that is, all likely values for \( \theta \) reside in \( H \). Similarly, it reject \( H \) if \( H \cap R(x) = \emptyset \), that is no likely value of \( \theta \) resides in \( H \). Otherwise, the test remains agnostic about \( H \).
**Definition 2.3** (Region-based test). \( \varphi \) is a region-based test if there exists a region estimator, \( R \), such that \( x \in \text{POS}(H) \) if \( R(x) \subseteq H \), \( x \in \text{NEG}(H) \) if \( R(x) \cap H = \emptyset \), and \( x \in \text{BND}(H) \), otherwise, that is,

\[
\varphi_H(x) = \begin{cases} 
0 & \text{if } R(x) \subseteq H \\
1 & \text{if } R(x) \cap H = \emptyset \\
\frac{1}{2} & \text{otherwise}
\end{cases}
\]

The (non-invariant) Generalized Full Bayesian Significance Test (GFBST; Stern et al. 2017) is a particular type of test based on a region estimator. It uses an HPD as region estimator.

**Example 2.4** (GFBST). The GFBST is a region-based test in which \( R(x) \) is an HPD.

Example 2.5 describes a GFBST.

**Example 2.5.** Consider that \( n \) balls are removed without replacement from a box with \( \theta \) blue balls and \( N - \theta \) yellow balls, where \( n < N \). The total number of sampled blue balls, \( X \), follows \( X|\theta \sim \text{Hypergeometric}(\theta, N - \theta, n) \). Also, consider that, a priori, \( \theta \sim \text{Binomial}(N, 0.5) \).

It can be shown that \( \theta - X|X \sim \text{Binomial}(N - n, 0.5) \). Hence, for every \( k > 0 \) and \( 1 \leq x \leq n \),

\[
R_k(x) = \left\{ 0 \leq i \leq N : \frac{(N - n)}{2} + x - i \right\} \text{ is a HPD.}
\]

In this case, a GFBST accepts \( H \) if it contains all points close to \( \frac{N - n}{2} + x \), rejects \( H \) if it contains none of these points, and otherwise remains agnostic.

Under special circumstances all logically coherent simultaneous tests are based on region estimators [Esteves et al., 2016]. In particular, this relation is valid when \( \Theta \) is a finite set:

**Theorem 2.6.** If \( \Theta \) is finite and \( \varphi \) is a logically coherent simultaneous test, then \( \varphi \) is based on a region estimator.

The next section studies under what circumstances a Bayes test against an EC loss can be logically coherent.
3. The relation between Bayesian optimality and logical coherence under error-wise constant loss

A logically coherent test, \( \varphi \), that is Bayes against an EC loss admits further characterization. In such a case, not only is \( \varphi \) a region-based test, but also based on an HPD. That is, a logically coherent test that is Bayes against an EC loss is a GFBST, as presented in Theorem 3.1.\(^1\)

**Theorem 3.1.** Let \( \Theta \) be a finite set. If there exists a probability, \( P \), and a TEC loss, \( L \), such that a logically coherent simultaneous test, \( \varphi \), is Bayes against \( L \) according to \( P \), then \( \varphi \) is a GFBST.

However, do there exist actual cases in which a test is both Bayes with respect to a TEC loss and also logically coherent? Section 2 shows that a logically coherent test must be based on a region estimator. Also, Yao [2007] shows that a Bayes test against a TEC loss must be a probability-based test. Despite these strong restrictions, Theorem 3.2 shows that every logically coherent test is Bayes against a TEC loss for some probability measure.

**Theorem 3.2.** Let \( \Theta \) and \( \mathcal{X} \) be finite sets. If \( \varphi \) is a logically coherent simultaneous test, then there exists a probability, \( P \), and a TEC loss function, \( L \), such that \( \varphi \) is Bayes against \( L \).

Theorem 3.2 shows that, for each logically coherent test, there exists a choice of \( P \) and \( L \) such that the test is also Bayes with respect to a TEC loss.\(^2\) Example 3.3 shows a choice of \( L \) and \( P \) so that a logically coherent test is Bayes and, therefore, also is a GFBST.

**Example 3.3.** Let \( \Theta = \{1, 2, 3, 4\} \), \( \mathcal{X} \in \{0, 1\} \), \( R(0) = \{1, 2\} \), \( R(1) = \{3, 4\} \), and \( \varphi \) be a test based on \( R \). Let \( L \) be a TEC loss so that \( \beta = \frac{7}{10} \) and \( \alpha = \frac{3}{10} \). Also, let \( P(1|x) = P(2|x) = \frac{4}{10} - x \) and \( P(3|x) = P(4|x) = \frac{4}{10} x \). Let \( \varphi^* \) be the Bayes test according to \( L \). Note that the two least probable outcomes sum up a probability of \( \frac{2}{10} \). Hence, every hypothesis that contains none of the most probable outcomes is rejected by \( \varphi^* \). Next, if an hypothesis contains both of the most

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\(^1\)Lemma A.3, in the Appendix, is used to prove Theorem 3.1. Recall that if a test is logically coherent and \( \Theta \) is a finite set, then the test is based on a region estimator. Lemma A.3 shows that, if a Bayes test is based on a region estimator, then there exists a loss such that the region estimator is Bayes. That is, a Bayes logically coherent test is necessarily based on a region estimator which is also Bayes.

\(^2\)Under mild assumptions, Theorems 3.1 and 3.2 also hold when \( \Theta \) is a countable set.
probable outcomes, than its probability is at least \( \frac{8}{10} \), so it is accepted by \( \varphi^* \). Finally, if an hypothesis contains only one of the most probable outcomes, than its probability is between \( \frac{4}{10} \) and \( \frac{6}{10} \), so \( \varphi^* \) remains agnostic about \( H \). From the previous conclusions, obtain that \( \varphi = \varphi^* \), that is, \( \varphi \) is a logically coherent test that is Bayes against \( L \) according to \( \mathbb{P} \). Finally, note that when using \( \mathbb{P} \), \( R \) is an HPD, that is, \( \varphi \) is a GFBST, as also known from Theorem 3.1.

Example 3.3 shows that, for a given region-based test, a specific choice of TEC loss and \( \mathbb{P} \) are required so that the test is Bayes. However, in most settings \( \mathbb{P} \) is given and one wishes to choose \( L \) so that the Bayes test is logically coherent. Theorem 3.4 shows that there is no choice of an EC loss such that the Bayes test is logically coherent for every \( \mathbb{P} \).

**Theorem 3.4.** Let \( |\Theta| \geq 3 \). For each \( \mathbb{P} \) and \( L \), let \( \varphi_{\mathbb{P},L} \) be a Bayes simultaneous test against \( L \) according to \( \mathbb{P} \). If \( L \) is an EC loss, then there exists \( \mathbb{P} \) such that \( \varphi_{\mathbb{P},L} \) is not logically coherent.

Theorem 3.4 shows that, if \( L \) is an EC loss, then there exists a probability, \( \mathbb{P} \), such that the resulting Bayes test is not logically coherent. Hence, a procedure that yields Bayes tests that are logically coherent must be based on more general loss functions. The next section explores these losses.

4. A logically coherent Bayesian procedure

This section develops a loss function such that, for every probability, \( \mathbb{P} \), the resulting Bayes test is logically coherent. This loss is presented in Definition 4.1:

**Definition 4.1** (GFBST loss). Let \( \mu \) be a measure over \( \Theta \) such that \( \mathbb{P}(\theta|\mathbf{x}) \) is absolutely continuous with respect to \( \mu \) for every \( \mathbf{x} \in \mathcal{X} \) and \( f(\theta|\mathbf{x}) := \frac{d\mathbb{P}(\theta|\mathbf{x})}{d\mu} \). The tangent set to hypothesis \( H \) according to \( \mu \), \( T^H_x \), is defined as \( T^H_x := \{ \theta \in \Theta : f(\theta|\mathbf{x}) > \sup_{\theta' \in H} f(\theta'|\mathbf{x}) \} \). The GFBST loss according to \( \mu \) for testing \( H \) is given by Table 2.

The GFBST loss, which generalizes the two-way counterpart in Madruga et al. [2001], admits an intuitive interpretation [Stern, 2003]. Observe that \( T^H_x \subseteq H^c \) is the collection of values
in $\Theta$ that are more likely than every point in $H$. Hence, $T^H_x$ and $T^{H^c}_x$ can be interpreted as the set of points that are strong contenders for, respectively, $H$ and $H^c$. The GFBST loss is lowest, 0, when either $H$ is rejected and $\theta$ is a strong contender for $H$ or $H$ is accepted and $\theta$ is a strong contender for $H^c$. Also the GFBST is largest, $b + c$, when either $H$ is rejected and $\theta$ is a strong contender for $H^c$ or $H$ is accepted and $\theta$ is a strong contender for $H$. Finally, the GFBST loss assumes intermediate values, when either $\theta$ is not a strong contender for $H$ or $H^c$ or when the agnostic decision is chosen.

**Theorem 4.2.** For every probability $P$, if $\phi$ is a Bayes simultaneous test against the GFBST loss, then $\phi$ is a GFBST.

Theorem 4.2 shows that, if the GFBST loss is used, then the Bayes test is a GFBST. Therefore, for every probability measure, the Bayes test against the GFBST loss is logically coherent. Hence, using loss functions that are more general than the EC loss, it is possible to always reconcile Bayesian decision theory with logical coherence.

### 5. Final remarks

Simultaneous three-way decisions may require more constraints than are typically used in individual decisions. In particular, when performing simultaneous hypothesis test, one might expect logical coherence between conclusions. This paper presents results on whether it possible to obtain logical coherence together with Bayesian optimality.

Two types of results are obtained. If an error-wise constant loss is used, then only for a limited set of models can a Bayes simultaneous test be logically coherent. This result motivated
the investigation of other types of loss functions which might provide a better reconciliation between Bayesian optimality and logical coherence. We propose the GFBST loss and show that every Bayes test against this loss is a GFBST. Since every GFBST is logically coherent, the GFBST loss yields Bayes tests that are always logically coherent.

The above results show that the GFBST loss can lead to simultaneous tests that yield conclusions which are more interpretable than the ones obtained from the EC loss. The results also show that simultaneous three-way decisions can yield a layer of complexity that is not present in individual decision problems. Further investigation might determine whether this layer of complexity is also present in other applications of three-way decisions, such as classification or clustering.

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A. Proofs

Proof of Theorem 2.6. Let \( \mathcal{F} = \{ H \in \sigma(\Theta) : \forall H^* \in \sigma(\Theta), H \cap H^* \in \{ \emptyset, H \} \} \). Since \( \Theta \) is finite and \( \sigma(\Theta) \) is a \( \sigma \)-field, \( \mathcal{F} \) partitions \( \Theta \). Define the equivalence relation \( \sim \) such that \( \theta_1 \sim \theta_2 \) if there exists \( F \in \mathcal{F} \) such that \( \theta_1 \in F \) and \( \theta_2 \in F \). Define \( \Theta^* \) as the quotient space \( \Theta \setminus \sim \). Also, let \( \sigma(\Theta^*) \) and \( \varphi^* \) be the quotient \( \sigma \)-field of \( \sigma(\Theta) \) and the quotient test of \( \varphi \) over \( \sim \). It follows from construction that \( \sigma(\Theta^*) \) includes the singleton. Hence, Esteves et al. [2016] obtains that \( \varphi^* \) is based on a region estimator, \( R^* \). Conclude that \( \varphi \) is based on a region estimator, \( R \).  

Lemma A.2. If \( L \) is a proper loss, then
\[
\min \left( \mathbb{E} \left[ L_{\varphi'} \left( \frac{1}{2}, \theta \right) \right] \left| x \right. \right), \left( \mathbb{E} \left[ L_{\varphi'} \left( 0, \theta \right) \right] \left| x \right. \right) \leq \mathbb{E} \left[ L_{\varphi'} \left( 1, \theta \right) \right] \left| x \right.
\]
implies that
\[
\mathbb{E} \left[ L_{\varphi'} \left( \frac{1}{2}, \theta \right) \right] \left| x \right. \leq \mathbb{E} \left[ L_{\varphi'} \left( 1, \theta \right) \right] \left| x \right.
\]

Proof. It is sufficient to prove that, if \( \mathbb{E} \left[ L_{\varphi'} \left( 0, \theta \right) \right] \left| x \right. \leq \mathbb{E} \left[ L_{\varphi'} \left( 1, \theta \right) \right] \left| x \right. \), then \( \mathbb{E} \left[ L_{\varphi'} \left( \frac{1}{2}, \theta \right) \right] \left| x \right. \leq \mathbb{E} \left[ L_{\varphi'} \left( 1, \theta \right) \right] \left| x \right. \). Let \( \mathbb{E} \left[ L_{\varphi'} \left( 0, \theta \right) \right] \left| x \right. \leq \mathbb{E} \left[ L_{\varphi'} \left( 1, \theta \right) \right] \left| x \right. \). Since \( L \) is proper,
\[
\mathbb{E} \left[ L_{\varphi'} \left( \frac{1}{2}, \theta \right) \right] \left| x \right. \leq \frac{\mathbb{E} \left[ L_{\varphi'} \left( 0, \theta \right) \right] \left| x \right.}{2} + \frac{\mathbb{E} \left[ L_{\varphi'} \left( 1, \theta \right) \right] \left| x \right.}{2}
\]
\[
\leq \mathbb{E} \left[ L_{\varphi'} \left( 1, \theta \right) \right] \left| x \right.
\]

Lemma A.3. Let \( \Theta \) be finite, \( \sigma(\Theta) \) include the unitary sets, and \( \varphi \) be generated by the region estimator, \( R \). If there exists a probability, \( \mathbb{P} \), and a proper loss, \( L \), such that \( \varphi \) is Bayes against \( L \)
according to \( P \), then \( R \) is a Bayes region estimator against \( \bar{L} \) according to \( P \), where

\[
\bar{L}(A, \theta) = \sum_{\theta' \in A} \left[ L_{\theta'}\left(\frac{1}{2}, \theta\right) - L_{\theta'}(1, \theta) \right].
\]

**Proof.** The Bayes region estimator against \( \bar{L} \), \( R^* \), satisfies:

\[
R^*(x) := \left\{ \theta' \in \Theta : E \left[ L_{\theta'}\left(\frac{1}{2}, \theta\right) \bigg| x \right] \leq E \left[ L_{\theta'}(1, \theta) \bigg| x \right] \right\}.
\]

Hence, it is sufficient to prove that \( R \equiv R^* \). Since \( \varphi \) is Bayes against \( L \), \( \varphi_{\theta'}(x) < 1 \) if and only if

\[
\min \left\{ E \left[ L_{\theta'}\left(\frac{1}{2}, \theta\right) \bigg| x \right], E \left[ L_{\theta'}(0, \theta) \bigg| x \right] \right\} < E \left[ L_{\theta'}(1, \theta) \bigg| x \right].
\]

Using Lemma A.2, conclude that \( \varphi_{\theta'}(x) < 1 \) if and only if

\[
E \left[ L_{\theta'}\left(\frac{1}{2}, \theta\right) \bigg| x \right] \leq E \left[ L_{\theta'}(1, \theta) \bigg| x \right].
\]

Since \( \varphi \) is generated by \( R \), it follows that

\[
R(x) = \left\{ \theta' : \varphi_{\theta'}(x) < 1 \right\} = R^*(x).
\]

**Proof of Theorem 3.1.** Since \( \varphi \) is logically coherent, it follows from Theorem 2.6 that \( \varphi \) is based on a region estimator, \( R \). It follows from Lemma A.3 that \( R \) is a Bayes region estimator against \( \bar{L} \). Since \( L \) is a TEC loss, which is proper, \( \bar{L}(A, \theta) = \lambda_{BN}|A| - (\lambda_{BN} - \lambda_{BP})I_A(\theta) \). That is,

\[
R(x) = \left\{ \theta \in \Theta : P(\theta\mid x) \geq \frac{\lambda_{BN}}{\lambda_{BN} - \lambda_{BP}} \right\}.
\]

Conclude that \( R(x) \) is a HPD. \( \square \)

**Lemma A.4** (Union consonance). Let \( \varphi \) be logically coherent. If \( H_1 \) and \( H_2 \) are such that \( \varphi_{H_1}(x) = 1 \) and \( \varphi_{H_2}(x) = 1 \), then \( \varphi_{H_1 \cup H_2}(x) = 1 \).

**Proof.** It follows from invertibility that \( \varphi_{H_1}(x) = 0 \) and \( \varphi_{H_2}(x) = 0 \). Hence, from intersection
Table 3.: Loss function used in the proof of Theorem 3.2.

| Decision       | state of the nature | $\theta \in A$ | $\theta \notin A$ |
|----------------|---------------------|----------------|------------------|
| 0 (accept $A$) | 0                   | $k$            |                  |
| $\frac{1}{2}$  | 1                   | 1              |                  |
| 1 (reject $A$) | $k$                 | 0              |                  |

consonance, $\varphi_{H_1 \cap H_2}(x) = 0$. Finally, conclude from invertibility that $\varphi_{H_1 \cup H_2}(x) = 1$. 

**Lemma A.5.** Let $\Theta$ be a finite set. If $\varphi$ is a logically coherent simultaneous test, then:

(a) For every $x \in \mathcal{X}$, there exists $\theta_0 \in \Theta$ such that $\varphi_{\{\theta_0\}}(x) < 1$.

(b) For every $x \in \mathcal{X}$, if $\varphi_{\{\theta_0\}}(x) = 0$, then $\varphi_{\{\theta\}}(x) = 1$, $\forall \theta \neq \theta_0$.

**Proof.** (a) Assume that there exists $x \in \mathcal{X}$ such that $\varphi_{\{\theta\}}(x) = 1$, for every $\theta \in \Theta$. It follows from Lemma A.4 that $\varphi_{\Theta}(x) = 1$, which contradicts the propriety of $\varphi$. (b) Let $\theta_0$ be such that $\varphi_{\{\theta_0\}}(x) = 0$. It follows from invertibility that $\varphi_{\{\theta_0\}^c}(x) = 1$. Conclude from monotonicity that, for every $\theta \neq \theta_0$, $\varphi_{\{\theta\}}(x) = 1$.

**Proof of Theorem 3.2.** Since $\varphi$ is logically coherent, it follows from Esteves et al. [2016] that there exists $R(x)$ such that, $\varphi_{H}(x) = 1 \iff H \cap R(x) = \emptyset$, $\varphi_{H}(x) = 0 \iff R(x) \subseteq H$ and $\varphi_{H}(x) = \frac{1}{2}$, otherwise. Using Lemma A.5, conclude that $R(x) \neq \emptyset$. In the following, we determine a loss, $L$, and a joint probability, $\mathbb{P}(\theta, x)$, such that $\varphi$ is Bayes.

Let $|\Theta| = k$. Also, let $L$ be the TEC given by table 3. It follows from Yao [2007] that $\varphi$ is Bayes with respect to $L$ when:

$$\varphi_H(x) = \begin{cases} 1 & \text{if } \mathbb{P}(\theta \in H|x) < \frac{1}{k} \\ 0 & \text{if } \mathbb{P}(\theta \in H|x) > \frac{k-1}{k} \\ \frac{1}{2} & \text{otherwise.} \end{cases} \tag{2}$$

Next, we determine $\mathbb{P}(\theta, x)$ such that these conditions hold.
In order to determine $P(\theta, x)$ it is sufficient to choose $P(x)$ and $P(\theta|x)$. For each $A \in \mathcal{X}$, let $P(x \in A) = \frac{|A|}{|\mathcal{X}|}$, that is, the uniform distribution over $\mathcal{X}$. Also, for $H \subset \Theta$,

$$P(\theta \in H|x) = \frac{1}{2k} \cdot \frac{|H|}{|\mathcal{X}|} + \frac{2k-1}{2k} \cdot \frac{|H \cap R(x)|}{|R(x)|}. \quad (3)$$

It remains to show that $\varphi$ is Bayes with respect to $L$ and $P$. We study three cases: (i) If $\varphi_H(x) = 1$, then $H \cap R(x) = \emptyset$. Using eq. (3), conclude that $P(\theta \in H|x) \leq \frac{1}{2k} \cdot 1 + \frac{2k-1}{2k} \cdot 0 < \frac{1}{k}$. (ii) If $\varphi_H(x) = 0$, then $R(x) \subseteq H$. Using eq. (3), conclude that $P(\theta \in H|x) \geq \frac{1}{2k} \cdot 0 + \frac{2k-1}{2k} \cdot 1 > \frac{k-1}{k}$, (iii) If $\varphi_H(x) = \frac{1}{2}$, then $R(x) \cap H^c \neq \emptyset$ and $R(x) \cap H \neq \emptyset$, that is, $1 \leq |H \cap R(x)| < |R(x)| \leq k$. Using eq. (3), conclude that $P(\theta \in H|x) \geq \frac{1}{2k} \cdot \frac{1}{2} + \frac{2k-1}{2k} \cdot \frac{1}{2} = \frac{1}{k}$. Also, $P(\theta \in H|x) \leq \frac{1}{2k} \cdot \frac{k-1}{k} + \frac{2k-1}{2k} \cdot \frac{k-1}{k} = \frac{k-1}{k}$. That is, $\frac{1}{k} \leq P(\theta \in H|x) \leq \frac{k-1}{k}$. It follows from eq. (2) that $\varphi$ is Bayes with respect to $L$ using $P$. \hfill \square

**Lemma A.6.** Let $L$ be an EC loss Definition 1.1 and, for each $P$, let $\varphi_{P,L}$ be a Bayes simultaneous test for $P$ against $L$. If, for every $P$, $\varphi_{P,L}$ is logically coherent, then $\varphi_{P,L}$ is a simultaneous test such as in Example 1.2 and:

1. for every $A, B \in \sigma(\Theta)$ such that $\emptyset \neq A \subseteq B \neq \emptyset$, $\alpha^A \geq \alpha^B$.
2. for every $A, B \in \sigma(\Theta)$ such that $A - B \neq \emptyset$, $B - A \neq \emptyset$, and $A \cup B \neq \emptyset$: $\alpha^A + \alpha^B \leq \alpha^{A \cup B}$.

**Proof.** Let $x \in \mathcal{X}$ be arbitrary.

If $\alpha^A < \alpha^B$, then for $P$ such that $P(\theta \in A|x) = P(\theta \in B|x) = 0.5(\alpha^A + \alpha^B)$, $\varphi_{P,L}(A) < 1$ and $\varphi_{P,L}(B) = 1$, that is, $\varphi_{P,L}$ does not satisfy monotonicity. Conclude that, if $\varphi_{P,L}$ is logically coherent for every $P$, then $\alpha^A \geq \alpha^B$ for every $\emptyset \neq A \subseteq B \neq \emptyset$.

If $\alpha^A + \alpha^B > \alpha^{A \cup B}$, then let $\delta := (\alpha^A + \alpha^B) - \alpha^{A \cup B} > 0$. By taking $P$ such that

$$P(\theta \in A|x) = \max(0, \alpha^A - 0.4\delta),$$

$$P(\theta \in B|x) = \max(0, \alpha^B - 0.4\delta),$$

$$P(\theta \in A \cup B|x) = \min(1, \alpha^A + \alpha^B - 0.8\delta),$$


obtain $\varphi_{P,L}(A) = 1$, $\varphi_{P,L}(B) = 1$, and $\varphi_{P,L}(A \cup B) < 1$, that is, it follows from Lemma A.4 that $\varphi_{P,L}$ is not logically coherent. Conclude that, if $\varphi_{P,L}$ is logically coherent for every $P$, then $\alpha^A + \alpha^B \leq \alpha^{A \cup B}$.

Proof of Theorem 3.4. Assume that, for every $P$, $\varphi_{L,P}$ is logically coherent. Let $\theta_1, \theta_2 \in \Theta$ and $A = \{\theta_1\}$, $B = \{\theta_2\}$. Since $|\Theta| \geq 3$, $A - B \neq \emptyset$, $B - A \neq \emptyset$ and $A \cup B \neq \Omega$. Hence, it follows from Lemma A.6 that

$$\alpha^A \geq \alpha^{A \cup B},$$
$$\alpha^B \geq \alpha^{A \cup B},$$
$$\alpha^{A \cup B} \geq \alpha^A + \alpha^B.$$ 

That is, $\alpha^A = \alpha^B = \alpha^{A \cup B} = 0$, a contradiction with Example 1.2. Conclude that there exists $P$ such that $\varphi_{L,P}$ is not logically coherent.

Proof of Theorem 4.2. The posterior expected losses for each decision are given by:

$$E[L_A(0, (\theta, x)|x)] = bP(\theta \notin T^A_x \cup T^{A'}_x | x) + (b + c)P(\theta \in T^A_x | x),$$
$$E \left[ L_A \left( \frac{1}{2}, (\theta, x) \middle| x \right) \right] = v + cP(\theta \in T^A_x \cup T^{A'}_x | x),$$
$$E[L_A(1, (\theta, x)|x)] = bP(\theta \notin T^A_x \cup T^{A'}_x | x) + (b + c)P(\theta \in T^{A'}_x | x).$$

Next, it follows from definition that $T^A_x \subseteq A^{c}$ and $T^{A'}_x \subseteq A$. Hence, $T^A_x \cap T^{A'}_x = \emptyset$. Hence,

$$E[L_A(0, (\theta, x)|x)] - E \left[ L_A \left( \frac{1}{2}, (\theta, x) \middle| x \right) \right] = (b + c)P(\theta \notin T^{A'}_x | x) - (v + c)$$
$$E[L_A(0, (\theta, x)|x)] - E[L_A(1, (\theta, x)|x)] = (b + c) \left[ P(\theta \in T^A_x | x) - P(\theta \in T^{A'}_x | x) \right]$$
$$E[L_A(1, (\theta, x)|x)] - E \left[ L_A \left( \frac{1}{2}, (\theta, x) \middle| x \right) \right] = (b + c)P(\theta \notin T^A_x | x) - (v + c)$$

Also, recall from definition that either $T^A_x = \emptyset$ or $T^{A'}_x = \emptyset$. Hence, since $0 < v < b$ and $c > 0$,
if $\varphi$ is Bayes, then $\varphi_H(x) = 0$ if and only if $\mathbb{P}(\theta \notin T^A_x | x) < \frac{\nu + c}{b + c}$ and $\varphi_H(x) = 1$ if and only if $\mathbb{P}(\theta \notin T^A_x | x) < \frac{\nu + c}{b + c}$. It follows from Esteves et al. [2016] that $\varphi$ is the GFBST. □