Non-Markovian Persistence and Nonequilibrium Critical Dynamics

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The persistence exponent $\theta$ for the global order parameter, $M(t)$, of a system quenched from the disordered phase to its critical point describes the probability, $p(t) \sim t^{-\theta}$, that $M(t)$ does not change sign in the time interval $t$ following the quench. We calculate $\theta$ to $O(\epsilon^2)$ for model A of critical dynamics (and to order $\epsilon$ for model C) and show that at this order $M(t)$ is a non-Markov process. Consequently, $\theta$ is a new exponent. The calculation is performed by expanding around a Markov process, using a simplified version of the perturbation theory recently introduced by Majumdar and Sire [Phys. Rev. Lett. \textbf{77}, 1420 (1996)].

The ‘persistence exponent’, $\theta$, which characterizes the decay of the probability that a stochastic variable exceeds a threshold value (typically its mean value) throughout a time interval, has attracted a great deal of recent interest \cite{MS}. One of the most surprising properties of this exponent is that its value is highly non-trivial even in simple systems. For example, $\theta$ is irrational for the $q > 2$ Potts model in one dimension \cite{MS} (where the fraction of spins that have not changed their state in the time $t$ after a quench to $T = 0$ decays as $t^{-\theta}$) and is apparently not a simple fraction for the diffusion equation \cite{MS} (where the fraction of space where the diffusion field has always exceeded its mean decays as $t^{-\theta}$).

A recent study of non-equilibrium model A critical dynamics, where a system coarsens at its critical point starting from a disordered initial condition, looked at the probability $P(t_1, t_2)$ that the global magnetization does not change sign during the interval $t_1 < t < t_2$ \cite{MS}. The persistence exponent for this system is defined by $P(t_1, t_2) \sim (t_1/t_2)^\theta$ in the limit $t_2/t_1 \to \infty$. Explicit results were obtained for the 1D Ising model, the $n \to \infty$ limit of the $O(n)$ model, and to order $\epsilon = 4 - d$ near dimension $d = 4$. For these systems it was found that the value of $\theta$ was related to the dynamic critical exponent $z$, the static critical exponent $\eta$, and ‘nonequilibrium’ exponent $\lambda$ (which describes the decay of the autocorrelation with the initial condition, $\langle \phi(x,t)\phi(x,0) \rangle \sim t^{-\lambda/z}$) by the scaling relation $\theta \equiv \lambda - d + 1 - \eta/2$. This relation may be derived from the assumption that the dynamics is Markovian, which is indeed the case for all of the cases considered in that paper.

From a consideration of the structure of the diagrams which appear at order $\epsilon^2$ (and higher order), however, it was argued that the dynamics of the global order parameter should not be Markovian to all orders, implying that the exponent $\theta$ does not obey exactly that ‘Markovian scaling relation’ \cite{MS}. This means that $\theta$ is a new exponent. Monte-Carlo simulations in 2 dimensions indeed suggest weak violation of the Markov scaling relation \cite{MS}.

In this Rapid Communication we present an explicit calculation of the non-Markovian properties of the global order parameter. The nonequilibrium magnetization-magnetization correlation function is calculated to order $\epsilon^2$, and this is then used to calculate $\theta$ to the same order, using a perturbative method proposed by Majumdar and Sire (MS) \cite{MS}, valid in the vicinity of a Markov process. The Markov scaling relation is shown explicitly to be violated at order $\epsilon^2$, so $\theta$ is indeed a new independent exponent.

Before discussing the calculation of $\theta$, however, we provide first a simpler, and more transparent, formulation of the perturbation theory than that given in MS. In particular the final result, Eq. (40), does not appear explicitly in MS [12].

Let $y(t)$ be a Gaussian stochastic process with zero mean, whose correlation function obeys dynamical scaling, i.e. $\langle y(t_1)y(t_2) \rangle = t_1^{\alpha} \Phi(t_1/t_2)$. Let $T = \ln t$ and $x(T) = y(t)/\langle y^2(t) \rangle^{1/2}$. Then $x(t)$ is a Gaussian stationary process with zero mean, i.e. its correlation function is translationally invariant, $\langle x(T_1)x(T_2) \rangle = A(T_2 - T_1)$. Notice that $A(0) = 1$ by construction, which convention we shall adopt throughout this Communication (in contrast to that of ref. \cite{MS}). If the persistence probability of $y$ decays algebraically in $t$, then the persistence probability of $x(T)$ decays as $\sim \exp(-\theta T)$ for $T \to \infty$.

The persistence probability may be expressed as the ratio of two path integrals, as follows \cite{MS}:

$$P(x(T') > 0; 0 < T' < T) = \int_{x(0)}^{x(T')} Dx(T) \exp(-S) \int Dx(T) \exp(-S),$$

where

$$S = \frac{1}{2} \int_0^T dT_1 \int_0^T dT_2 x(T_1)G(T_1, T_2)x(T_2).$$

Here $G(T_1, T_2)$ is the matrix inverse of the correlation matrix $\langle x(T_1)x(T_2) \rangle = A(T_2 - T_1)$. Notice that $G$ is not
simply a function of $T_2 - T_1$ (unless we impose periodic boundary conditions).

In MS this path-integral formalism was used to map the Markov process onto a quantum harmonic oscillator in imaginary time, developing the perturbation theory in the formalism of quantum mechanics. We shall merely use path integrals as a convenient notation, performing all our calculations within the natural framework of stochastic processes.

Let $x^0(T)$ be a stationary Gaussian Markov process, i.e. one defined by

$$\frac{dx^0}{dT} = -\mu x^0 + \xi(T),$$

where $\xi$ is a Gaussian white noise, with $\langle \xi(T)\xi(T') \rangle = 2\mu \delta(T - T')$. The noise strength has been chosen so that the autocorrelation function is $A^0(T) = \exp(-\mu T)$.

Suppose the process $x(T)$ is perturbatively close to a Markov process, in the sense that $G = G^0 + \epsilon g$. Then we can expand the exponentials in the path integrals in (4) and re-expONENTiate, so that to $O(\epsilon)$ the numerator becomes

$$\int_C Dx(T)e^{-S} = \int_C Dx(T) \exp \left( -S^0 - \frac{\epsilon}{2} \gamma(T) + O(\epsilon^2) \right),$$

where the subscript $C$ represents the constraint $x(T') > 0$ (0 < $T' < T$) on the paths in the integral in the numerator of (4), and

$$\gamma(T) = \int_0^T dT_1 \int_0^T dT_2 g(T_1, T_2) A^0_\epsilon(T_1, T_2),$$

where

$$A^0_\epsilon(T_1, T_2) \equiv \frac{\int_C Dx(T) x(T_1) x(T_2) e^{-S^0}}{\int_C Dx(T) e^{-S^0}}$$

is the correlation function for the Markov process, averaged (and normalized) only over the paths consistent with the constraint $C$. The denominator in (4) is given by an identical expression, except that $A^0_\epsilon$ is replaced by $A^0$, the unconstrained correlation function.

By virtue of the constraint, $A^0_\epsilon$ will not be strictly translationally invariant for finite $T$. In the limit $T \to \infty$, however, the double time-integral in (4) reduces to $T$ times an infinite integral over the relative time $T_2 - T_1$, with $A^0_\epsilon(T_1, T_2)$ replaced by its stationary limit $A^0_\epsilon(T_2 - T_1)$. Similarly, $g$ will be translationally invariant in this regime, giving

$$\gamma(T) \to T \int_{-\infty}^{\infty} (d\omega/2\pi) \tilde{g}(\omega) \tilde{A}^0_\epsilon(\omega),$$

where we have used the translational invariance to write the final result in Fourier space $\tilde{A}^0_\epsilon$. Note that the zeroth-order result

$$\int_{x>0} Dx(T) \exp(-S^0) / \int Dx(T) \exp(-S^0)$$

is just the persistence probability of the stationary Gaussian Markov process $x^0(T)$, which decays as $\exp(-\mu T)$ as $T \to \infty$.

Using (4), (5) and (7), we find that the persistence exponent may be written in the form

$$\theta \equiv \lim_{T \to \infty} \frac{1}{T} \log \left[ \frac{P(x(T') > 0; 0 < T' < T)}{1} \right] = \mu + \epsilon \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{g}(\omega) \left[ \tilde{A}^0_\epsilon(\omega) - \tilde{A}^0(\omega) \right] + O(\epsilon^2).$$

Similarly, $A^0_\epsilon$ is the correlation function for the Markov process, averaged only over paths where $x(T)$ is always positive, we consider a path starting at $(x_1, T_1)$, passing through $(x_1, T_1)$ and $(x_2, T_2)$ without ever crossing the origin. Then the required stationary limit is

$$P^+(x_1, T_1; x_2, T_2) = \lim_{T_1 \to \infty, T_1 \to \infty} \frac{Q^+(f; 2; 1|i)}{Q^+(f;i)},$$

where we have adopted an obvious shorthand notation for the arguments of $Q^+$.

To calculate the joint probability $P^+(x_1, T_1; x_2, T_2)$ that the process goes through $(x_1, T_1)$ and $x_2$ at $T_2$, averaged only over paths where $x(T)$ is always positive, we consider a path starting at $(x_1, T_1)$, passing through $(x_1, T_1)$ and $(x_2, T_2)$ without ever crossing the origin. Then the required stationary limit is

$$P^+(x_1, T_1; x_2, T_2) = \frac{2}{\pi} \left[ 1 - e^{-2\mu T} \right]^{-1/2} x_1 x_2 e^{\mu T} \sinh \left( \frac{x_1 x_2}{2 \sinh \mu T} \right).$$

It is now straightforward to evaluate the autocorrelation function:

$$A^0_\epsilon(T) = \int_0^\infty dx_1 \int_0^\infty dx_2 x_1 x_2 P^+(x_1, 0; x_2, T)$$

$$= \frac{2}{\pi} \left[ 3 \left( 1 - e^{-2\mu T} \right)^{1/2} + (e^{\mu T} + 2e^{-\mu T}) \sin^{-1} e^{-\mu T} \right].$$
Eq. (8) for \( \theta \) can now be expressed as a real-time integral as follows. We first write \( A(T) = A^0(T) + \alpha(T) \), and we note that in Fourier space \( \hat{A}(\omega)^{-1} = \hat{G}(\omega) = \hat{G}^0(\omega) + \hat{g}(\omega) \). Using \( A^0 = \exp(-\mu T) \) gives \( \hat{g}(\omega) = -\hat{a}(\omega)(\omega^2 + \mu^2)/4\mu^2 \). Inserting this in (8), and transforming to real time, gives

\[
\theta = \mu - \frac{\epsilon}{4\mu^2} \int_0^\infty dT a(T) \left( \mu^2 - \partial_T^2 \right) \left[ A^0(T) - A^0(0) \right]
\]

\[
= \mu \left\{ 1 - 2\mu \int_0^\infty \frac{a(T)}{1 - \exp(-2\mu T)^{3/2}} dT \right\}.
\]

(16)

The final result is remarkably compact. Since \( a(T) \) is just the perturbation to the Markov correlator \( A^0(T) = e^{-\mu T} \), the normalization \( A(T) = 1 \) forces \( a(0) = 0 \). This is sufficient to converge the integral in (16). Provided \( a(T) \) vanishes more rapidly than \( T^{1/2} \). Eq. (16) has recently been used to calculate persistence exponents for interface growth in a class of generalized Edwards-Wilkinson models [13].

As was remarked earlier, the problem of non-equilibrium critical dynamics is Markovian to first order in \( \epsilon = 4 - d \). In the thermodynamic limit the global order parameter is Gaussian because, at time \( t \), it is the sum of \( [L/\xi(t)]^d \) (essentially) statistically independent contributions, where \( L \) is the system size and \( \xi \sim t^{1/2} \) is the length scale over which critical correlations have been established. Corrections to the Gaussian distribution can be expressed in terms of higher cumulants of the normalized total magnetization \( M(t)/\langle M^2(t) \rangle^{1/2} \). Using the translational invariance of the system with respect to space it is easy to show that for large \( L \) the 2N-point cumulant is smaller by a factor \( (1^{1/2}/L)^{(N-1)d} \) compared to the Gaussian part of the 2N-point correlation function. The perturbative approach discussed in the first part of this Communication can therefore be applied. To calculate the lowest non-Markovian term in \( \theta \), we need to calculate the autocorrelation function of the total magnetization \( M(t) \) to order \( \epsilon^2 \), i.e. we need to calculate the autocorrelation function \( A(t_1, t_2) = \langle M(t_1) M(t_2) \rangle/\langle M^2(t_1) \rangle^{1/2} \langle M^2(t_2) \rangle^{1/2} \), which in the scaling regime depends only on the ratio \( t_2/t_1 \). The necessary techniques of dynamical field-theory, incorporating the extra renormalization associated with the random initial condition (and responsible for the nonequilibrium exponent \( \lambda \)), have been developed by Janssen et al [13,14]. We first consider ‘model A’ dynamics [14] for a nonconserved \( O(n) \) model (where ‘per-
than the measured value $\theta_z = 0.505 \pm 0.020$ (the finite-size scaling method used in [11] naturally determines the combination $\theta = 2\mu$). The non-Markov correction factor in $A(T)$ is, for $n = 1$, $(1 + 0.007543\ldots\epsilon^2) \approx 1.030$ for $\epsilon = 2$. The ‘improved’ estimate for $\theta_z$ becomes $0.474 \pm 0.006$, closer to, but still somewhat smaller than, the numerical estimate.

For $d = 3$, one has $z = 2.032 \pm 0.004$, $\lambda = 2.789 \pm 0.006$ [21] and $\eta = 0.032 \pm 0.003$, giving $\mu = 0.380 \pm 0.003$. Multiplying by the non-Markov correction factor for $\epsilon = 1$, i.e. $1.0075$, gives $\theta = 0.383 \pm 0.003$, compared to the numerical result $\theta \approx 0.41$ [15]. A direct expansion to order $\epsilon^2$ using the known expansions for $z$, $\lambda$, and $\eta$, gives (specializing to $n = 1$) $\theta = 1/2 - \epsilon/12 + \alpha - 2\ln 3)\epsilon^2/72 - 2\epsilon^2/81 + O(\epsilon^3)$, i.e. $\theta \approx 0.365$ for $d = 3$, slightly lower than that obtained using the best numerical estimates of $z$, $\lambda$ and $\eta$ and only using the $\epsilon$-expansion for the non-Markov correction.

A similar approach can be applied to ‘model C’ critical dynamics [3], in which a nonconserved order parameter field is coupled to a conserved density. In this case, one obtains non-Markovian corrections already at order $\epsilon$.

The autocorrelation function is given by (for $n = 1$)

$$A(T) = \exp(-\mu T) \left[ 1 - \frac{\epsilon}{6} F_C(e^T) + O(\epsilon^2) \right]$$

(19)

$$F_C(x) = \ln 2 - \frac{x - 1}{2x} + x \ln x - \frac{x - 1}{2} \ln(x - 1) - \frac{x + 1}{2} \ln(x + 1).$$

(20)

Again, $F_C(e^T)$ vanishes like $T \ln T$ for $T \to 0$, while $F_C(\infty) = \ln 2 - 1/2$. Inserting $\alpha = A(T) = A(T) - \exp(-\mu T)$ from (19) into (19) gives

$$\theta = \mu \left[ 1 + \frac{7 - 4\sqrt{2}}{12} \epsilon + O(\epsilon^2) \right],$$

(21)

where $\mu = (\lambda - d + 1 - \eta/2)/z$ as before, but now the dynamical exponents $z$ and $\lambda$ take their model-C values [17,18].

In summary, we have computed to order $\epsilon^2$ the persistence exponent $\theta$ for the global order parameter $M(t)$ of models A and C. At this order, the dynamics of $M(t)$ are non-Markovian, and $\theta$ is a new exponent, not related to the usual static and dynamic exponents. The calculation was performed by expanding around a Markov process, using a simplified form of the perturbation theory introduced by Majumdar and Sire.

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