Lie systems and integrability conditions
for $t$-dependent frequency harmonic oscillators

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Abstract

Time-dependent frequency harmonic oscillators (TDFHO’s) are studied through the theory of Lie systems. We show that they are related to a certain kind of equations in the Lie group $SL(2, \mathbb{R})$. Some integrability conditions appear as conditions to be able to transform such equations into simpler ones in a very specific way. As a particular application of our results we find $t$-dependent constants of the motion for certain one-dimensional TDFHO’s. Our approach provides an unifying framework which allows us to apply our developments to all Lie systems associated with equations in $SL(2, \mathbb{R})$ and to generalise our methods to study any Lie system.

1 Introduction and motivation.

The search for solutions of autonomous integrable systems is based on the existence of a sufficient number of constants of the motion. Unfortunately, the extension of the theory to include non-autonomous systems is a difficult problem and many of the properties of $t$-dependent Hamiltonians still remain as partially understood. The situation is even worse for the corresponding quantum case for which the $t$-evolution is difficult to find, when possible.

The simplest case is that of a linear system because then there is a linear superposition principle, but the difficulty remains in the search for a
fundamental set of particular solutions.

A particular case of this linear system of a high physical interest is the $t$-dependent oscillator, i.e. variable-frequency and variable-mass oscillator, which can probably be considered as the most frequently studied of such systems [11–9] for its relevance in many different problems in Optics and Particle Physics. Our aim is to revisit the theory of TDFHO’s and some other related ones from the perspective of the theory of Lie systems [10–14] but without explicitly using the linearity properties trying to develop the theory in the framework of Lie systems and therefore following a process which can be extended to other Lie systems with the same Lie group, as the almost ubiquitous in Physics Riccati equation, the Milne-Pinney equation or the Ermakov system [15–17].

Lie systems are a generalisation of linear systems and have received much attention during the last years. Each Lie system has associated a Lie group and it can be related to another Lie system defined by right-invariant vector fields on the group, this last system being related to all Lie systems with such an associated Lie group [18].

Actually, the explicit integration of the given equations depends very much on the explicit values of the $t$-dependent functions defining the system and therefore the interest in establishing such an integrability criteria which turn out to be shared for all Lie systems with the same associated Lie algebra.

Quite often the study of such Lie systems is carried out by means of a transformation leading to a vector field and therefore to an autonomous system. As an example the standard one-dimensional harmonic oscillator described by the Hamiltonian

$$H_0(Q, P) = \frac{1}{2} (P^2 + Q^2),$$

can be transformed under a two steps transformation into the system described by the $t$-dependent Hamiltonian

$$H_F(t, q, p) = \frac{1}{2} (p^2 + F(t)q^2),$$

with $F$ being the $t$-dependent function given by

$$F = \frac{1}{\rho} \left( \frac{1}{\rho^3} - \dot{\rho} \right),$$

and where $\rho$ is an arbitrary positive function, $\rho(t) > 0$.  

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In fact, we first change the time variable
\[ T \mapsto t, \quad t = \rho^2(t)T, \]
and second, a \( t \)-dependent (but linear) canonical transformation is carried out
\[
\begin{align*}
q &= \rho Q, \\
\rho p &= P + \rho \dot{\rho} Q,
\end{align*}
\]
\[
\begin{align*}
Q &= q/\rho, \\
P &= \rho p - \dot{\rho} q.
\end{align*}
\]
Then, the Hamiltonian \( H_0 \) becomes the following \( t \)-dependent function
\[
I_\rho(t, q, p) = \frac{1}{2} \left[ (\rho p - \dot{\rho} q)^2 + \left( \frac{q}{\rho} \right)^2 \right]. \tag{2}
\]
Such a function \( I_\rho \) is a (\( t \)-dependent) constant of the motion for the \( t \)-dependent Hamiltonian \( H_\rho \). The existence of such a constant of the motion, usually called the 'Lewis invariant' \([7, 8]\), has been proved by making use of several alternative procedures (see e.g. Refs. \([1, 3, 9]\)). Notice that every particular choice for the function \( \rho(t) \) determines a \( t \)-dependent oscillator endowed with an associated \( t \)-dependent constant of the motion \( I_\rho(q, p, t) \). Conversely, suppose we consider a \( t \)-dependent oscillator \( H_\rho \) as in (1) defined by a given function \( F(t) \). Then, a solution of the auxiliary equation
\[
\ddot{\rho} + F(t)\rho = \frac{1}{\rho^3}
\]
permits constructing the constant of the motion (2). This auxiliary equation had been considered by Milne \([15]\) and its general solution was studied in 1950 by Pinney \([16]\) who proved that it belongs to the restricted subfamily of nonlinear differential equations whose exact solutions can be obtained. Nevertheless, the Pinney method makes use of two independent solutions of the associated linear problem. But this linear equation is just the equation of the \( t \)-dependent oscillator and we arrive at the annoying conclusion that (at least for a general function \( F(t) \)) we need to solve first another \( t \)-dependent equation of motion and only then we can obtain the explicit form of the constant of the motion \( I_\rho \). In any case, the above mentioned two steps procedure shows the existence of a close relationship between the two equations, the \( t \)-independent and the \( t \)-dependent one of this particular oscillator system. These relations have been explained from the point of view of Lie systems in \([17]\) where it was also studied why both differential equations (the
Milne–Pinney equation and the $t$-dependent harmonic oscillator) have the same difficulty to be solved exactly.

Integrable systems in the classical Arnold–Liouville sense are those having as many independent first-integrals in involution as degrees of freedom. Such a system is called super-integrable if, in addition, possesses more independent first-integrals than degrees of freedom. The free particle, the Kepler problem and the harmonic oscillator with rationally related frequencies are three instances of this very particular class of systems (for other super-integrable systems see Refs. [19]–[23] and references therein).

The main objective of this paper is the study of integrability conditions, in particular, those for Riccati equations or analogous systems. Such integrability conditions have been extensively studied since the investigation made by Liouville [27]. Lots of papers have treated this topic since then [28]–[31]. Recently, the theory of Lie systems has been used to analyse this subject [32]–[33]. These works show that the theory of Lie systems allows us to recover some very well-known results about the integrability of Riccati equations scattered in the literature and to collect them from an unifying perspective. In spite of their apparent simplicity, the methods already developed reduce the problem of integrability of Riccati equations to studying integrability conditions for a particular equation in the Lie group $SL(2, \mathbb{R})$. Such an equation describes the integral curves for the $t$-dependent vector fields on $SL(2, \mathbb{R})$ made up by a linear combination with $t$-dependent coefficients of right-invariant vector fields with respect to the right action of $SL(2, \mathbb{R})$ on itself. Moreover, the theory of Lie systems shows that we can apply these integrability conditions to any other Lie system related to an equation in $SL(2, \mathbb{R})$ of the same form. For instance, here we use results on integrability of Riccati equations to study TDFHO’s which are associated with the same kind of equations in $SL(2, \mathbb{R})$ as Riccati equations. In this way, we illustrate how the theory of Lie systems provides a way to extend the methods of the theory of integrability of differential equations to integrability of non-autonomous systems in Physics or other Lie systems. Furthermore, the point of view of the theory of Lie systems permits the application of the integrability conditions of common Lie systems, here Riccati equations, to the so-called SODE Lie systems, like TDFHO’s, which may be understood as Lie systems [17].

The paper is organised as follows. Section 2 is devoted to show that TDFHO’s are SODE Lie systems related to a Lie algebra of vector fields isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. In Section 3 we study some transformation properties
for these differential equations that turn out to be a straightforward generalisation of those found for Riccati equation \cite{13}. Our main result is contained in Section 4 where it is shown that the theory of Lie systems describes these transformations by means of a special kind of matrix Riccati equation. In our particular instance, this differential equation is again a Lie system related to a Lie algebra of vector fields isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$. The integrability conditions for TDFHO’s appear as conditions to be able to obtain solutions of the differential equation defining the transformation when we assume certain conditions on the solutions. This property is studied in a more detailed way in Section 5 where some previously known integrability conditions are recovered from this new perspective. Some of such an integrability conditions are applied in Sections 6 to TDFHO’s and in particular to Caldirola–Kanai TDFHO’s. In Section 7 we develop a new integrability condition and some new integrable TDFHO’s are studied. A method to obtain $t$-dependent constants of the motion is shown in Section 8.

## 2 TDFHO as a Lie system.

Any TDFHO can be considered as a Lie system related to a Lie algebra of vector fields, known as the Guldberg–Vessiot Lie algebra, isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Therefore, we can associate it with an equation in the Lie group $G = SL(2, \mathbb{R})$, see \cite{17, 33}. In fact, consider the harmonic oscillator with $t$-dependent frequency and mass described by the Hamiltonian $H \in C^\infty(\mathbb{R} \times T^*\mathbb{R})$

$$H(t, x, p) = \frac{1}{2m(t)} p^2 + \frac{1}{2} F(t) \omega^2 x^2,$$

giving rise to the Hamilton equations

$$\begin{cases}
\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m(t)}, \\
\dot{p} = -\frac{\partial H}{\partial x} = -F(t) \omega^2 x,
\end{cases} \quad (3)$$

which is a non-autonomous system of first-order ordinary differential equations in $T^*\mathbb{R}$. Its solutions are the integral curves for the $t$-dependent vector field

$$X(t) = \frac{p}{m(t)} \frac{\partial}{\partial x} - F(t) \omega^2 x \frac{\partial}{\partial p}.$$
Let $X_0, X_1$ and $X_2$ be the vector fields given by

$$X_0 = p \frac{\partial}{\partial x}, \quad X_1 = \frac{1}{2} \left( x \frac{\partial}{\partial x} - p \frac{\partial}{\partial p} \right), \quad X_2 = -x \frac{\partial}{\partial p},$$

which close on a Lie algebra with commutation relations

$$[X_0, X_2] = 2X_1, \quad [X_0, X_1] = X_0, \quad [X_1, X_2] = X_2,$$

and therefore isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Then, the $t$-dependent vector field $X$ associated with the system (3) can be written as a linear combination

$$X(t) = F(t) \omega^2 X_2 + \frac{1}{m(t)} X_0,$$

i.e. it is a linear combination with $t$-dependent coefficients

$$X(t) = \sum_{\alpha=0}^{2} b_{\alpha}(t) X_{\alpha},$$

with $b_0(t) = 1/m(t)$, $b_1(t) = 0$ and $b_2(t) = F(t) \omega^2$. Hence, $t$-dependent mass and frequency harmonic oscillators are SODE Lie systems. Moreover, for constant mass, i.e. $m(t) = 1$, we get that TDFHO’s are SODE Lie systems.

Let $V$ be the real Guldberg–Vessiot Lie algebra generated by the complete vector fields \{X_\alpha | \alpha = 0, 1, 2\} and isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. It is known that if the vector fields of such a Lie algebra are complete, there is an action $\Phi_{HO}$ of the connected Lie group $SL(2, \mathbb{R})$ on the manifold $T^* \mathbb{R}$ whose fundamental vector fields are those of $V$.

Let us choose a basis \{M_0, M_1, M_2\} of $\mathfrak{sl}(2, \mathbb{R})$ such that the fundamental vector fields corresponding to the \{M_\alpha | \alpha = 0, 1, 2\} are the respective \{X_\alpha | \alpha = 0, 1, 2\}. For example, if we consider that $\mathfrak{sl}(2, \mathbb{R})$ can be identified with the set of traceless $2 \times 2$ matrices, we can choose the basis

$$M_0 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad M_1 = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

which close the same commutation relations as the $X_\alpha$. Thus, the linear map such that the image of $M_\alpha$ is $X_\alpha$ is an isomorphism of Lie algebras and we can relate (3) to an equation in $SL(2, \mathbb{R})$ given by

$$R_{A^{-1} A} \dot{A} = -\sum_{\alpha=0}^{2} b_{\alpha}(t) M_\alpha,$$
with the initial condition $A(0) = I$ (see [33] for the reason of the minus sign).

Thus, if $A(t)$ is the solution of (7) and we denote $\xi = (x, p) \in T^*\mathbb{R}$, then the solution of (3) starting from $\xi(0)$ is $\xi(t) = \Phi_{HO}(A(t), \xi(0))$ (see e.g. [18]).

For any $t$-dependent harmonic oscillator, it can be verified that the left action $\Phi_{HO}$ of $SL(2, \mathbb{R})$ on $T^*\mathbb{R} \approx \mathbb{R}^2$ associating the elements of the basis (6) with their fundamental vector fields (4) is the linear action given by

$$\Phi_{HO} \left( A, \begin{pmatrix} x \\ p \end{pmatrix} \right) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} \quad \text{with} \quad A \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{R}).$$

In this way, the one-parameter subgroup $\exp(-t M_\alpha)$ acts on $T^*\mathbb{R}$ with infinitesimal generator $X_\alpha$.

In summary, the system (3) is a Lie system in $T^*\mathbb{R}$ related to an equation in $SL(2, \mathbb{R})$ and the solutions of the latter equation allow us to obtain the solutions of (3) in terms of the initial condition by means of the action $\Phi_{HO}$. We can treat in a similar way other Lie systems with an isomorphic Guldberg–Vessiot Lie algebra, as for instance the Milne-Pinney equation, but the action is not linear anymore.

### 3 Transformations of Lie systems in $SL(2, \mathbb{R})$.

We first recall an important property of Lie systems associated with a Lie group $G$ that in the particular case of TDFHO’s plays a very relevant rôle for establishing integrability criteria: the group $\mathcal{G} \equiv \text{Map}(\mathbb{R}, G)$ of curves in the Lie group $G$, here $SL(2, \mathbb{R})$, acts on the set of the Lie systems with such a group, see [13, 32, 33].

More explicitly, in the case we are considering each $t$-dependent frequency harmonic oscillator (3) can be considered as a curve in $\mathbb{R}^3$ through the identification with the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ corresponding to the choice of its basis $\{M_1, M_2, M_3\}$, i.e. the curve in $\mathbb{R}^3$ is defined by the coordinates of the curve $\tilde{M}(t)$ in $\mathfrak{sl}(2, \mathbb{R})$ with respect to such basis associated with the $t$-dependent vector field defining the TDFHO. Actually, we can transform every curve $\xi(t)$ in $T^*\mathbb{R}$, by an element $\tilde{A}(t)$ of $\mathcal{G}$ as follows:

$$\text{If } \tilde{A}(t) = \begin{pmatrix} \tilde{\alpha}(t) & \tilde{\beta}(t) \\ \tilde{\gamma}(t) & \tilde{\delta}(t) \end{pmatrix} \in \mathcal{G}, \quad \Theta(\tilde{A}, \xi)(t) = \begin{pmatrix} \tilde{\alpha}(t)x(t) + \tilde{\beta}(t)p(t) \\ \tilde{\gamma}(t)x(t) + \tilde{\delta}(t)p(t) \end{pmatrix}. \quad (8)$$
It can be checked \[13\] that the curve $\bar{A}(t)$ transforms the TDFHO (3) into an analogous TDFHO with new coefficients $b'_0, b'_1, b'_2$ given by

\begin{align*}
b'_2 &= \bar{\delta}^2 b_2 - \bar{\delta} \bar{\gamma} b_1 + \bar{\gamma}^2 b_0 + \bar{\gamma} \bar{\delta}^2 - \bar{\delta} \ddot{\gamma}, \\
b'_1 &= -2 \bar{\beta} \bar{\delta} b_2 + (\bar{\alpha} \bar{\delta} + \bar{\beta} \bar{\gamma}) b_1 - 2 \bar{\alpha} \bar{\gamma} b_0 + \delta \dot{\alpha} - \bar{\alpha} \bar{\delta} + \bar{\beta} \ddot{\gamma} - \bar{\gamma} \dot{\delta}, \\
b'_0 &= \bar{\beta}^2 b_2 - \bar{\alpha} \bar{\beta} b_1 + \bar{\alpha}^2 b_0 + \bar{\alpha} \dot{\beta} - \bar{\beta} \dot{\alpha}.
\end{align*}

In fact, this expression defines an affine action (see e.g. \[34\] for the general definition of this concept) of the group $G$ on the set of TDFHO's \[13\]. This means that in order to transform the coefficients of a TDFHO by means of two transformations of the above type, firstly through $A_1$ and then by means of $A_2$, it suffices to do the transformation defined by the product element $A_2 A_1$ of $G$.

The result of this action of $G$ can also be studied from the point of view of the equations in $SL(2, \mathbb{R})$. First, $G$ acts on the left on the set of curves in $SL(2, \mathbb{R})$ by left translations, i.e. a curve $\bar{A}(t)$ transforms the curve $A(t)$ into a new one $A'(t) = \bar{A}(t)A(t)$. Therefore, if $A(t)$ is a solution of (7), characterised by a curve $M(t) \in T_I SL(2, \mathbb{R})$, then the new curve satisfies a new equation like (7) but with a different right-hand side, $M'(t)$, and thus it corresponds to a new equation in $SL(2, \mathbb{R})$ associated with a new TDFHO. Of course, $A'(0) = \bar{A}(0)A(0)$, and if we want $A'(0) = \text{Id}$ we have to impose the additional condition $\bar{A}(0) = \text{Id}$. In this way $G$ acts on the set of curves in $T_I SL(2, \mathbb{R}) \simeq \mathfrak{sl}(2, \mathbb{R})$. It can be shown that the relation between both curves $M(t)$ and $M'(t)$ in $T_I SL(2, \mathbb{R})$ is given by \[18\]

\[ M'(t) = -\sum_{\alpha=0}^2 b'_\alpha(t)M_\alpha = \bar{A}(t)M(t)\bar{A}^{-1}(t) + \dot{\bar{A}}(t)\bar{A}^{-1}(t). \] (10)

Summarising, it has been shown that it is possible to associate, in a one-to-one way, TDFHO’s with equations in the Lie group $SL(2, \mathbb{R})$ and to define a group $G$ of transformations on the set of such TDFHO’s induced by the natural group action of $SL(2, \mathbb{R})$.

### 4 Lie structure of an equation of transformation of Lie systems.

Our aim in this Section is to construct, given a pair of equations in $SL(2, \mathbb{R})$ characterised by two curves $M(t), M'(t) \subset T_I SL(2, \mathbb{R})$, a system of
differential equations which is a Lie system and whose integral curves relate both systems. This system of differential equations is used in next Sections to extend the developments of [18, 32] to a broader set of cases.

Now, let us consider (10) as a system of first-order ordinary differential equations in the coefficients of the curve in $SL(2, \mathbb{R})$,

$$\bar{A}(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix},$$

relating both systems.

Rewrite (10) in a more useful way by multiplying on the right by $\bar{A}(t)$ to obtain

$$\dot{\bar{A}}(t) = M'(t)\bar{A}(t) - \bar{A}(t)M(t),$$

(11)

that is, a special kind of matrix Riccati equation, see [11]. Even if it is known that matrix Riccati equations are Lie systems, nevertheless, we consider convenient to prove this fact in detail in order to show later on how to generalise our methods.

In terms of the coefficients of $\bar{A}(t)$, this differential equation is

$$\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \\ \dot{\delta} \end{pmatrix} = \begin{pmatrix} \frac{b'_1 - b_1}{2} & b_2 & b'_0 & 0 \\ -b_0 & \frac{b'_1 + b_1}{2} & 0 & b'_0 \\ -b'_2 & 0 & -\frac{b'_1 + b_1}{2} & b_2 \\ 0 & -b'_2 & -b_0 & \frac{b'_1 - b_1}{2} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix},$$

(12)

with

$$M(t) = -\sum_{\alpha=0}^{2} b_\alpha(t)M_\alpha, \quad M'(t) = -\sum_{\alpha=0}^{2} b'_\alpha(t)M_\alpha.$$

It seems that in order to consider curves in $SL(2, \mathbb{R})$ we should also to impose that at any time $\alpha\delta - \beta\gamma = 1$. Nevertheless, it is shown later on that we can drop such a restriction for the time being because it is automatically implemented by imposing a restriction on the initial conditions. Therefore we can deal with $\delta$ in the preceding system of differential equations (12) as being independent of the other variables. This linear system of differential equations can be understood as a Lie system associated with a Lie algebra of vector fields isomorphic to $\mathfrak{gl}(4, \mathbb{R})$ but it may be seen in an alternative and more interesting way as a Lie system in one of its Lie subalgebras. In fact, let us consider the set of vector fields
\[ \begin{align*}
N_0 &= -\alpha \frac{\partial}{\partial \beta} - \gamma \frac{\partial}{\partial \delta}, \\
N_1 &= \frac{1}{2} \left( -\alpha \frac{\partial}{\partial \alpha} + \beta \frac{\partial}{\partial \beta} - \gamma \frac{\partial}{\partial \gamma} + \delta \frac{\partial}{\partial \delta} \right), \\
N_2 &= \beta \frac{\partial}{\partial \alpha} + \delta \frac{\partial}{\partial \gamma}, \\
N'_0 &= \gamma \frac{\partial}{\partial \alpha} + \delta \frac{\partial}{\partial \beta}, \\
N'_1 &= \frac{1}{2} \left( \alpha \frac{\partial}{\partial \alpha} + \beta \frac{\partial}{\partial \beta} - \gamma \frac{\partial}{\partial \gamma} - \delta \frac{\partial}{\partial \delta} \right), \\
N'_2 &= -\alpha \frac{\partial}{\partial \gamma} - \beta \frac{\partial}{\partial \delta},
\end{align*} \]

for which the non-null commutation relations among them are given by

\[ \begin{align*}
[N_0, N_1] &= N_0, & [N_0, N_2] &= 2N_1, & [N_1, N_2] &= N_2, \\
[N'_0, N'_1] &= N'_0, & [N'_0, N'_2] &= 2N'_1, & [N'_1, N'_2] &= N'_2.
\end{align*} \]

If we denote \( x \equiv (\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4 \) the column vector of the four elements of the matrix

\[ \bar{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \]

then (12) is a system of differential equations in \( \mathbb{R}^4 \) whose solutions give us the integral curves \( x(t) \) for the \( t \)-dependent vector field

\[ N(t) = \sum_{\alpha=0}^{2} \left( b_\alpha(t)N_\alpha + b'_\alpha(t)N'_\alpha \right). \] (13)

Note that \( [N_i, N'_j] = 0 \), for \( i, j = 0, 1, 2 \), and therefore the linear differential equation (12) is a Lie system in \( \mathbb{R}^4 \) associated with a Lie algebra of vector fields isomorphic to \( \mathfrak{g} \equiv \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \), i.e. this Lie algebra decomposes into a direct sum of two Lie algebras isomorphic to \( \mathfrak{sl}(2, \mathbb{R}) \) generated by \( \{N_0, N_1, N_2\} \) and by \( \{N'_0, N'_1, N'_2\} \), respectively.

The vector fields of the set \( \{N_0, N_1, N_2, N'_0, N'_1, N'_2\} \) generate an integrable distribution of rank three in almost any point of \( \mathbb{R}^4 \). Consequently, there exists, at least locally, a \( t \)-independent constant of the motion which turns out to be \( \det \bar{A} = \alpha \delta - \beta \gamma \). Such a determinant is a first-integral for all the vector fields of the module of vector fields generated by \( \{N_0, N_1, N_2, N'_0, N'_1, N'_2\} \) and, if we have a solution of (12) with an initial condition \( \det \bar{A}(0) = \alpha(0)\delta(0) - \beta(0)\gamma(0) = 1 \), then \( \det \bar{A}(t) = 1 \) at any time \( t \) and the solution can be understood as a curve in \( SL(2, \mathbb{R}) \). Therefore, we have found that the curves in \( SL(2, \mathbb{R}) \) relating the two different curves associated with two TDFHO’s can be described by the curves \( A(t) \) that are defined by the integral curves of (13) with \( \det \bar{A}(0) = 1 \), and viceversa.
Theorem 1. The curves in $SL(2, \mathbb{R})$ relating two equations in the Lie group $SL(2, \mathbb{R})$ characterised by the curves in $T_{1}SL(2, \mathbb{R})$

\[ M'(t) = -\sum_{\alpha=0}^{2} b'_{\alpha}(t)M_{\alpha}, \]
\[ M(t) = -\sum_{\alpha=0}^{2} b_{\alpha}(t)M_{\alpha}, \]

are given by the integral curves of the $t$-dependent vector field

\[ N(t) = \sum_{\alpha=0}^{2} (b_{\alpha}(t)N_{\alpha} + b'_{\alpha}(t)N'_{\alpha}), \]

such that $\det \bar{A}(0) = 1$. This system is a Lie system associated with a non-solvable Lie algebra of vector fields isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$.

Corollary 1. Given two TDFHO’s associated with the curves $M'(t)$ and $M(t)$ in $\mathfrak{sl}(2, \mathbb{R})$, there always exists a curve in $SL(2, \mathbb{R})$ relating both systems.

Proof. It is a direct consequence of the existence and unicity theorem of solutions of differential equations.

5 A review on some known integrability conditions.

Let us first remark that the Lie equations in $SL(2, \mathbb{R})$ defined by a constant curve, $M(t) = -\sum_{\alpha} c_{\alpha}M_{\alpha}$, are integrable and, consequently, the
same happens with curves of the form \( M(t) = -D(t) \left( \sum \alpha \, c_\alpha M_\alpha \right) \), where \( D \) is an arbitrary but constant sign function, because a \( t \)-reparametrisation by the function \( D(t) \) reduces the problem to the previous one. In this Section we study some cases when it is possible to find curves \( \bar{A}(t) \) in \( SL(2, \mathbb{R}) \) relating a given equation in \( SL(2, \mathbb{R}) \) characterised by a curve \( M(t) \) to an equation in \( SL(2, \mathbb{R}) \) characterised by a curve of the type \( M'(t) = -D(t)(c_0 M_0 + c_1 M_1 + c_2 M_2) \). This happens if equation (12) is easy to solve; the transfromation establishing the relation to such a TDFHO allows us to find by quadratures the solution of the given equation. We first restrict ourselves to study cases in which the curve \( \bar{A}(t) \) lies in a one-parameter subset of \( SL(2, \mathbb{R}) \). The results we show next are a direct translation to the framework of TDFHO’s of the results given in [32] for Riccati equations, for a detailed proof see [35].

**Theorem 2.** The necessary and sufficient conditions for the existence of a transformation
\[
\xi' = \Phi_{HO}(\bar{A}_0(t), \xi), \quad \xi = \left( \begin{array}{c} x \\ p \end{array} \right),
\]
with
\[
\bar{A}_0(t) = \left( \begin{array}{cc} \alpha(t) & 0 \\ 0 & \alpha^{-1}(t) \end{array} \right), \quad \alpha(t) > 0,
\]
relating the TDFHO associated with the \( t \)-dependent vector field
\[
X(t) = b_0(t)X_0 + b_1(t)X_1 + b_2(t)X_2,
\]
where \( b_0(t)b_2(t) \) has a constant sign, \( b_0(t)b_2(t) \neq 0 \), to another integrable one given by
\[
X'(t) = D(t)(c_0 X_0 + c_1 X_1 + c_2 X_2),
\]
where \( c_i \), with \( i = 0, 1, 2 \), are real numbers such that \( c_0 c_2 \neq 0 \), are
\[
D^2(t)c_0 c_2 = b_0(t)b_2(t), \quad b_1(t) + \frac{1}{2} \left( \frac{\dot{b}_2(t)}{b_2(t)} - \frac{\dot{b}_0(t)}{b_0(t)} \right) = c_1 \sqrt{\frac{b_0(t)b_2(t)}{c_0 c_2}}.
\]
Then, the transformation is uniquely defined by
\[
\bar{A}_0(t) = \left( \begin{array}{cc} \left( \frac{b_0(t)c_0}{b_0(t)c_2} \right)^{1/4} & 0 \\ 0 & \left( \frac{b_2(t)c_0}{b_0(t)c_2} \right)^{-1/4} \end{array} \right).
\]
Note that one coefficient, either \(c_0\) or \(c_2\), can be reabsorbed with a redefinition of the function \(D\). As a straightforward application of the preceding Theorem, which can be found in a similar form to that of [32], we obtain the following corollaries:

**Corollary 2.** The TDFHO (3) with \(b_0(t)b_2(t) \neq 0\) is integrable by a \(t\)-dependent change of variables

\[
\xi' = \Phi_{HO}(\bar{A}_0(t),\xi),
\]

with \(\bar{A}_0\) given by (14), if and only

\[
\sqrt{-\frac{c_0c_2}{b_0(t)b_2(t)}} \left[ b_1(t) + \frac{1}{2} \left( \frac{\dot{b}_2(t)}{b_2(t)} - \frac{\dot{b}_0(t)}{b_0(t)} \right) \right] = c_1,
\]

(17)

for certain real constants \(c_0, c_1\) and \(c_2\).

In this case

\[
D(t) = \sqrt{\frac{b_0(t)b_2(t)}{c_0c_2}},
\]

and the new system is

\[
\frac{d\xi'}{dt} = D(t) \begin{pmatrix} c_1/2 & c_0 \\ -c_2 & -c_1/2 \end{pmatrix} \xi'.
\]

(18)

**Corollary 3.** Given an integrable TDFHO characterised by a \(t\)-dependent vector field (16), the set of TDFHO’s which can be obtained through a \(t\)-dependent transformation

\[
\xi' = \Phi_{HO}(\bar{A}_0(t),\xi),
\]

with \(\bar{A}_0\) given by (14), are those of the form

\[
X(t) = b_0(t)X_0 + \left( \frac{\dot{b}_0(t)}{b_0(t)} - \frac{\dot{D}(t)}{D(t)} + c_1D(t) \right) X_1 + \frac{D^2(t)c_0c_2}{b_0(t)} X_2.
\]

(19)

Thus \(\bar{A}_0(t)\) reads

\[
\bar{A}_0(t) = \begin{pmatrix} \left( \frac{b_0(t)c_0}{b_0(b_0(t)c_2)} \right)^{1/4} & 0 \\ 0 & \left( \frac{b_2(t)c_0}{b_0(t)c_2} \right)^{-1/4} \end{pmatrix}.
\]
Therefore, starting from an integrable system we can find a family of solvable TDFHO systems whose coefficients are parametrised by $b_0(t)$. Given a TDFHO it is easy to check whether it belongs to such a family and can be easily integrated.

The integrability conditions we have described here arise as requirements on the initial $t$-dependent functions $b_\alpha$ that allow to solve exactly the initial TDFHO by a $t$-dependent transformation of the form

$$
\xi' = \Phi_{HO}(\exp(\Psi(t)v), \xi),
$$

with some $v \in \mathfrak{sl}(2,\mathbb{R})$ and $\Psi(t)$, in such a way that the initial TDFHO system (3) in the variable $\xi$ transforms into another one in the variable $\xi'$ that is a Lie system with a Guldberg–Vessiot Lie algebra of vector fields isomorphic to an appropriate Lie subalgebra of $\mathfrak{sl}(2,\mathbb{R})$ in such a way that the equation in $\xi'$ can be integrated by quadratures and the equation in $\xi$ is solvable too.

6 Applications of integrability conditions.

As a first application we show that the usual approach to the solution of the classical Caldirola–Kanai Hamiltonian [36, 37] (the solution of the quantum case can be worked in a similar way and it will be explained in a forthcoming paper) can be explained through our method. We also apply our approach to TDFHO’s. In particular, we study the case of a frequency

$$
\Omega(t) = \frac{\omega^2}{(K - c_1 t \omega)^2}.
$$

The Hamiltonian of the $t$-dependent harmonic oscillator is

$$
H(t, x, p) = \frac{1}{2} \frac{p^2}{m(t)} + \frac{1}{2} m(t) \omega^2(t) x^2.
$$

For instance, an harmonic oscillator with a damping term [36, 37] with equation of motion

$$
\frac{d}{dt}(m_0 \dot{x}) + m_0 \mu \dot{x} + k x = 0, \quad k = m_0 \omega^2,
$$

admits a Hamiltonian description with a $t$-dependent Hamiltonian

$$
H(t, x, p) = \frac{p^2}{2m_0} \exp(-\mu t) + \frac{1}{2} m_0 \exp(\mu t) \omega^2 x^2,
$$

14
i.e. $m(t)$ in (20) is $m(t) = m_0 \exp(\mu t)$. In this case the equations (3) are

$$
\begin{align*}
\dot{x} &= \frac{1}{m_0} \exp(-\mu t) p, \\
\dot{p} &= -m_0 \exp(\mu t) \omega^2 x,
\end{align*}
$$

and the $t$-dependent coefficients as a Lie system read

$$
b_0(t) = \frac{1}{m_0} \exp(-\mu t), \quad b_1(t) = 0, \quad b_2(t) = m_0 \omega^2 \exp(\mu t).
$$

Therefore, as $b_0(t)b_2(t) = \omega^2$, $b_1 = 0$ and

$$
\frac{\dot{b}_2}{b_2} - \frac{\dot{b}_0}{b_0} = 2\mu,
$$

we see that (17) holds with $c_0 = c_2 = 1, c_1 = \mu/\omega$ and the function $D$ is a constant $D = \omega$. Hence, this example reduces to the system

$$
\frac{d}{dt} \begin{pmatrix} x' \\ p' \end{pmatrix} = \begin{pmatrix} \mu/2 & \omega \\ -\omega & -\mu/2 \end{pmatrix} \begin{pmatrix} x' \\ p' \end{pmatrix}
$$

which can be easily integrated. Let $\bar{\omega}^2 = (\mu^2/4) - \omega^2$, we get

$$
\begin{pmatrix} x'(t) \\ p'(t) \end{pmatrix} = \begin{pmatrix} \cosh(\bar{\omega}t) + \frac{\mu}{2\bar{\omega}} \sinh(\bar{\omega}t) & \frac{\omega}{\bar{\omega}} \sinh(\bar{\omega}t) \\ -\frac{\omega}{\bar{\omega}} \sinh(\bar{\omega}t) & \cosh(\bar{\omega}t) - \frac{\mu}{2\bar{\omega}} \sinh(\bar{\omega}t) \end{pmatrix} \begin{pmatrix} x'(0) \\ p'(0) \end{pmatrix}
$$

and, in terms of the initial variables, obtain

$$
x(t) = \frac{e^{-\mu t/2}}{\sqrt{m_0 \bar{\omega}}} \left( \left( \cosh(\bar{\omega}t) + \frac{\mu}{2\bar{\omega}} \sinh(\bar{\omega}t) \right) \sqrt{m_0 \omega} x_0 + \frac{\omega}{\bar{\omega}} \sinh(\bar{\omega}t) \frac{p_0}{\sqrt{m_0 \omega}} \right).
$$

We can also study the TDFHO’s described by the Hamiltonian

$$
H(t, x, p) = \frac{1}{2} p^2 + \frac{1}{2} F(t) \omega^2 x^2, \quad F(t) > 0,
$$

where we assume, for simplicity, $m = 1$. The $t$-dependent vector field $X$ is

$$
X(t) = p \frac{\partial}{\partial x} - F(t) \omega^2 x \frac{\partial}{\partial p},
$$

15
which is a linear combination

\[ X(t) = F(t) \omega^2 X_2 + X_0, \]

i.e. the \( t \)-dependent coefficients in (15) are

\[ b_0(t) = 1, \quad b_1(t) = 0, \quad b_2(t) = F(t) \omega^2, \]

and the condition for \( F \) to satisfy (17) is

\[ \frac{1}{2} \frac{\dot{F}}{F} = c_1 \omega \sqrt{F}. \]

Therefore \( F \) must be of the form

\[ F(t) = \frac{1}{(L - c_1 \omega t)^2} \]

and the Hamiltonian which can be exactly integrated is

\[ H(t, x, p) = \frac{p^2}{2} + \frac{\omega^2}{2(L - c_1 \omega t)^2} x^2. \]

The corresponding Hamilton equations are

\[
\begin{cases}
\dot{x} = p, \\
\dot{p} = -\frac{\omega^2}{(L - c_1 \omega t)^2} x,
\end{cases}
\]

and the \( t \)-dependent change of variables to be performed is

\[
\begin{cases}
x' = \sqrt{\frac{\omega}{L - c_1 \omega t}} x, \\
p' = \sqrt{\frac{\omega}{L - c_1 \omega t}} p.
\end{cases}
\]

Therefore

\[
\begin{align*}
\frac{dx'}{dt} &= \frac{\omega}{L - c_1 \omega t} \left( \frac{c_1}{2} x' + p' \right), \\
\frac{dp'}{dt} &= \frac{\omega}{L - c_1 \omega t} \left( -x' - \frac{c_1}{2} p' \right).
\end{align*}
\]
Under the $t$-reparametrisation

$$\tau(t) = \int_0^t \frac{\omega dt'}{L - c_1 \omega t'} = \frac{1}{c_1} \ln \left( \frac{K'}{L - c_1 \omega t} \right),$$

the system (21) becomes

$$\begin{cases}
  \frac{dx'}{d\tau} = \frac{c_1}{2} x' + p', \\
  \frac{dp'}{d\tau} = -x' - \frac{c_1}{2} p',
\end{cases}$$

which is equivalent to the transformed Caldirola–Kanai differential equation through the change $\tau \mapsto \omega t$ and $c_1 \mapsto \mu/\omega$. In any case, the solution is

$$x'(\tau) = \left( \cosh(\bar{\omega} \tau) + \frac{c_1}{2 \omega} \sinh(\bar{\omega} \tau) \right) x'(0) + \frac{1}{\omega} \sinh(\bar{\omega} \tau) p'(0),$$

where $\bar{\omega} = \sqrt{\frac{c_1^2}{4} - 1}$ and finally

$$x(\tau(t)) = \sqrt{\frac{L - c_1 \omega t}{\omega}} \left[ \left( \cosh(\bar{\omega} \tau(t)) + \frac{c_1}{2 \omega} \sinh(\bar{\omega} \tau(t)) \right) x'(0) + \frac{1}{\omega} \sinh(\bar{\omega} \tau(t)) p'(0) \right].$$

7 Another integrability condition.

In this Section we analyse a new integrability condition that, as the preceding ones, arises as a compatibility condition for a restricted case of the system for the integral curves of (13). Nevertheless, this time the solution is restricted to one-parameter sets of matrices of $SL(2, \mathbb{R})$ that is not a group in general.

In this way, we deal with a family of transformations

$$A_0(t) = \begin{pmatrix} \frac{1}{V(t)} & 0 \\ -u_1 & V(t) \end{pmatrix}, \quad V(t) > 0,$$

where $u_1$ is a constant, i.e. we want to relate the $t$-dependent vector field

$$X(t) = X_0 + F(t) \omega^2 X_2,$$
characterised by the coefficients in (15)

\[ b_0 = 1, \quad b_1 = 0, \quad b_2 = F(t)\omega^2, \]

to an integrable one characterised by \( b'_0, b'_1 \) and \( b'_2 \), or more explicitly, to the \( t \)-dependent vector field

\[ X(t) = D(t)(c_0X_0 + c_2X_2), \]

i.e. \( b'_0 = Dc_0, b'_1 = 0, \) and \( b'_2 = Dc_2 \). Moreover, if \( c_0 \neq 0 \) we can reabsorb its value with a redefinition of \( D \) and we can assume \( c_0 = 1 \).

Under the action of (22) the original system becomes the following system

\[ \begin{aligned}
    b'_2 &= V^2b_2 + u_1Vb_1 + u_1^2b_0 - u_1\dot{V}, \\
    b'_1 &= b_1 + 2\frac{u_1}{V}b_0 - 2\frac{\dot{V}}{V}, \\
    b'_0 &= \frac{1}{V^2}b_0.
\end{aligned} \]

As \( b_1 = b'_1 = 0 \) and \( b_0 = 1 \), the second equation shows that \( \dot{V} = u_1 \), i.e. \( V(t) = u_1t + u_0 \) with \( u_0 \in \mathbb{R} \). Moreover, using this condition on the first equation together with \( b_0 = 1 \), it becomes \( b'_2 = Vb_2 \). Then, as the third equation gives us the value of \( D \) as \( D = b'_0 = 1/V^2 \), we see that \( b'_2 = Dc_2 = V^2F(t)\omega^2 \). Therefore, \( F \) must be proportional to \( (u_1t + u_0)^{-4} \),

\[ F(t) = \frac{k}{(u_1t + u_0)^4}, \quad k = \frac{c_2}{\omega^2}. \]

Let assume \( k = 1 \) and thus \( c_2 = \omega^2 \).

Then, the \( t \)-dependent transformation \( \tilde{A}_0(t) \) performing this reduction is

\[ \begin{aligned}
    x' &= \frac{x}{V(t)}, \\
    p' &= -u_1x + V(t)p.
\end{aligned} \]

Under this transformation, the initial system becomes

\[ \begin{aligned}
    \frac{dx'}{dt} &= \frac{p'}{V^2(t)}, \\
    \frac{dp'}{dt} &= \frac{\omega^2x'}{V^2(t)}. \\
\end{aligned} \]
Using the $t$-reparametrisation
\[ \tau(t) = \int_0^t \frac{dt'}{V^2(t')} = \frac{1}{u_1} \left( \frac{1}{u_0} - \frac{1}{V(t)} \right), \]
we get the next autonomous linear system
\[
\begin{cases}
\frac{dx'}{d\tau} = p', \\
\frac{dp'}{d\tau} = -\omega^2 x',
\end{cases}
\]
whose solution is
\[
\begin{pmatrix} x'(\tau) \\ p'(\tau) \end{pmatrix} = \begin{pmatrix} \cos(\omega \tau) & \sin(\omega \tau) \\ -\omega \sin(\omega \tau) & \cos(\omega \tau) \end{pmatrix} \begin{pmatrix} x'(0) \\ p'(0) \end{pmatrix}.
\]
Thus, we obtain that
\[
x(t) = V(t) \left( \cos(\omega \tau(t)) \frac{x_0}{u_0} + \frac{1}{\omega} \sin(\omega \tau(t))(-u_1 x_0 + u_0 p_0) \right).
\]

8 Some other integrable systems.

In this Section we show that the autonomisations of the transformed integrable systems obtained in latter Sections enable constructing $t$-dependent constants of the motion. Indeed, in previous cases, a TDFHO was transformed into a Lie system related to an equation in $SL(2, \mathbb{R})$
\[
R_{A^{-1} \ast A} \dot{A} = -D(t) (c_0 M_0 + c_1 M_1 + c_2 M_2),
\]
associated with a TDFHO determined by the $t$-dependent vector field
\[
X(t) = D(t)(c_0 X_0 + c_1 X_1 + c_2 X_2).
\]
Any $t$-dependent first-integral $I(t)$ of this differential equation satisfies
\[
\frac{dI}{dt} = \frac{\partial I}{\partial t} + X(t)I = 0.
\]
Thus, the function $I$ is a first-integral for the vector field on $\mathbb{R} \times T^* \mathbb{R}$
\[
\dot{X}(t) = c_0 X_0(t) + c_1 X_1(t) + c_2 X_2(t) + \frac{1}{D(t)} \frac{\partial}{\partial t}.
\]
As \( \mathbb{R} \times T^*\mathbb{R} \) is a manifold with dimension three and the differential equation we are studying are determined by a distribution of dimension one, there are two independent first-integrals.

Next, we analyse some integrable cases and their corresponding integrals.

- **Case** \( F(t) = (u_1 t + u_0)^2 \):

  In this case we obtain that by Theorem 2 the \( t \)-dependent vector field of the initial TDFHO is transformed into the following one
  \[
  X(t) = \frac{\omega}{u_1 t + u_0} \left( X_0 - \frac{u_1}{\omega} X_1 + X_2 \right)
  \]
  and thus, using the method of characteristics, we obtain the following first-integrals for this TDFHO:
  \[
  I_1 = -\frac{u_1}{\omega} p' x' + x'^2 + p'^2, \quad I_2 = \frac{(u_1 + u_0 t)^{\omega/u_1}}{\left( \left( \frac{u_1}{\omega} x' - 2p' \right) + 2\bar{\omega} x' \right)\frac{1}{\bar{\omega}}},
  \]
  with \( \bar{\omega} = \pm \sqrt{\frac{u_1^2}{4u_0^2} - 1} \).

- **Case** \( F(t) = (u_1 t + u_0)^4 \):

  In this case we see that the \( t \)-dependent vector field of the initial TDFHO is transformed into
  \[
  X(t) = \frac{1}{V^2(t)} \left( X_0 + \omega^2 X_2 \right),
  \]
  and thus, using the method of characteristics, we get the following first-integrals for the initial TDFHO
  \[
  I_1 = \left( \frac{x \omega}{V(t)} \right)^2 + (V(t)p - u_1 x)^2, \quad I_2 = \arcsin \left( \frac{x \omega}{V(t)\sqrt{I_1}} \right) + \frac{\omega}{u_1 V(t)}.
  \]

As we have two \( t \)-dependent constants of the motion in the space \( \mathbb{R} \times T^*\mathbb{R} \) and the solutions in this space are of the form \((t, x(t), p(t))\) we can obtain the solutions for our initial system.
9 Conclusions and Outlook.

The first concern of this article has been to present a discussion on integrability conditions from the viewpoint of Lie systems. Such a discussion has been used to apply some previous results of the theory of integrability of Riccati equations to investigate $t$-dependent frequency harmonic oscillators. In this way, we have illustrated the use of our theory for a well-known physical model obtaining particular integrable cases and constants of the motion.

The procedure here developed can be straightforwardly used to study any other Lie system. A detailed lecture of the paper clarify that in the particular case of Riccati equations and TDFHO’s the theory of Lie systems enable us:

- To reduce the problem of integrability for such Lie systems to the problem of integrability of equations in $SL(2, \mathbb{R})$.

- To show that integrability conditions for Lie systems in $SL(2, \mathbb{R})$ can be applied to any other Lie system associated with such equations, i.e. Milne-Pinney equations and Ermakov systems.

- To note that integrability conditions of Lie systems are closely related to some kind of matrix Riccati equations. Moreover, the study of solutions of such equations with algebraic conditions describe integrability conditions of Lie systems.

Additionally, recent results on the theory of Lie systems [38] allow us to use the procedure here developed to study at the same footing problems in Quantum Mechanics [39] and partial differential equations. It can also be shown that PDE’s consisting on matrix Riccati equations can be used to analyse integrability conditions for PDE Lie systems.

In a recent paper [40] we have proposed a generalisation of the concept of Lie system which share many of its characteristic and can be used for a more general class of systems as for instance dissipative Milne–Pinney equations [41], Emden and Abel equations.

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