THE FUNDAMENTAL THEOREM FOR
WEAK BRAIDED BIMONADS

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Abstract. The theories of (Hopf) bialgebras and weak (Hopf) bialgebras have been introduced for vector space categories over fields and make heavily use of the tensor product. As first generalisations, these notions were formulated for monoidal categories, with braidings if needed. The present authors developed a theory of bimonads and Hopf monads $H$ on arbitrary categories $A$, employing distributive laws, allowing for a general form of the Fundamental Theorem for Hopf algebras. For $\tau$-bimonads $H$, properties of braided (Hopf) bialgebras were captured by requiring a Yang-Baxter operator $\tau : HH \to HH$. The purpose of this paper is to extend the features of weak (Hopf) bialgebras to this general setting including an appropriate form of the Fundamental Theorem. This subsumes the theory of braided Hopf algebras (based on weak Yang-Baxter operators) as considered by Alonso Álvarez and others.

1. Introduction

There are various generalisations of the notions of (weak) bialgebras and Hopf algebras in the literature, mainly for (braided) monoidal categories, and we refer to Böhm [7], the introductions to Alonso Álvarez e.a. [2], Böhm e.a. [10], and [14, Remarks 36.18] for more information about these.

Bimonads and Hopf monads on arbitrary categories were introduced in [24] and the purpose of the present paper is to develop a weak version of these notion, that is, the initial conditions on the behaviour of the involved distributive laws towards unit and counit are replaced by weaker conditions.

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Recall that for a bialgebra \((H, m, e, \delta, \varepsilon)\) over a commutative ring \(k\), there is a commutative diagram \((\otimes_k = \otimes)\)

\[
\begin{array}{ccc}
M & \xrightarrow{- \otimes H} & M_H^H \\
\phi_H & & U\tilde{H} \\
\\
\end{array}
\quad \quad
\begin{array}{ccc}
M & \xrightarrow{(M \otimes H, M \otimes m, M \otimes \delta)} & (M \otimes H, M \otimes m), \\
\downarrow & & \downarrow \\
\end{array}
\]

where \(M\) is the category of \(k\)-modules, \(M_H^H\) the category of right \(H\)-modules, and \(M_H^H\) denotes the category of mixed bimodules; the latter can also be considered as \((M_H^H)\tilde{H}\), that is, the category of \(\tilde{H}\)-comodules over \(M_H\) where \(\tilde{H}\) is the lifting of the comonad \(\otimes H\) to \(M_H\). \(H\) is a Hopf algebra provided the functor \(\otimes H\) is an equivalence of categories (Fundamental Theorem of Hopf modules).

Concentrating on the essential parts of this setting, we consider, for any category \(A\), the diagram

\[
(1.1)
\]

\[
\begin{array}{ccc}
A & \xrightarrow{K} & (A_T)^G \\
\phi_T & & U^G \\
\end{array}
\quad \quad
A_T,
\]

where \(T\) is some monad on \(A\), \(G\) is some comonad on the category \(A_T\) of \(T\)-modules, \(\phi_T\) and \(U^G\) denote the respective free and forgetful functors, and \(K\) is any functor making the diagram commutative.

Having such a diagram, one may ask when the functor \(K\) allows for a right adjoint \(\overline{K}\). This creates a monad \(P\) on \(A\), a monad morphism \(\iota : P \rightarrow T\), the free functor \(\phi_P : A \rightarrow A_P\), and a functor \(K_P : (A_T)^G \rightarrow A_P\), the Eilenberg-Moore comparison functor for the monad \(P\).

If \(A\) is Cauchy complete and \(P\) is a separable Frobenius monad, then the change-of-base-functor \(\iota! : A_P \rightarrow A_T\) exists (see Proposition 3.2). As a consequence, \(K_P\) has a left adjoint \(L_P\) (Proposition 4.6) leading to the commutative diagram

\[
(1.2)
\]

So asking for a fundamental theorem leads to the question for properties of \(\phi_P\) and \(K_P\). Similar constructions apply when the functor \(K\) has a left adjoint.

After assembling preliminaries in Section 2 and properties of separable Frobenius monads in Section 3, the theory just sketched is outlined in Section 4.

Section 5 deals with the application of this to endofunctors \(H\) on a category \(A\) endowed with a monad \(H\) as well as a comonad structure \(\overline{H}\), and a weak monad-comonad entwining \(\omega : HH \rightarrow HH\). Exploiting ideas from [34] and Böhm [6], these data allow for the definition of a comonad \(G\) on \(A_H\) as well as a monad \(T\) on \(A_H^H\) (Propositions 5.3, 5.4). Hereby, the category \(A_H^H(\omega)\) of mixed \(H\)-bimodules is isomorphic to the categories \((A_H^H)^G\) and \((A_H^H)^T\) (see Theorems 5.4, 5.6). If \(\omega\) is a compatible entwining (i.e. \(\delta \cdot m = Hm \cdot \omega H \cdot H\delta\), Section 6.1), there is a functor

\[
K_\omega : A \rightarrow A_H^H(\omega), \quad a \mapsto (H(a), m_a, \delta_a),
\]
leading to commutativity of the diagram corresponding to (1.1). Conditions for the existence of a right and a left adjoint functor for $K_\omega$ are investigated (Propositions 5.10, 5.11). These problems were considered in [24] for proper compatible entwinings $\omega$.

In Section 6, we define weak $\tau$-bimonads, also called weak braided bimonads (Definition 6.2), based on a weak Yang-Baxter operator $\tau: HH \to HH$ (Section 6.1). This type of operator was introduced by Alonso Álvarez e.a. in [1] for monoidal categories and is here adapted to the more general setting. The conditions on weak braided bimonads induce a weak monad-comonad entwining $\omega: HH \to HH$ as well as a weak comonad-monad entwining $\xi: HH \to HH$ and allow to refine the results in Section 5: the natural transformation $\xi := H\varepsilon \cdot \omega \cdot He: H \to H$ is idempotent and its splitting yields a separable Frobenius monad $H_\xi$ (see Proposition 6.11). Then, if $K_\omega$ has a right adjoint functor $K$, the induced monad $P$ is just $H_\xi$ and the diagram corresponding to (1.2) can be completed.

The weak bialgebras over a commutative ring $k$ as considered by Böhm e.a. in [2] are weak braided bimonads in our sense ($\tau$ the ordinary twist, $\xi = \varepsilon_s$) and for this case some of our results are shown there, including the Frobenius and separability property of $H_\xi (= \mathcal{H}_s)$ ([9, Proposition 4.4]).

Eventually, in Section 7, weak braided Hopf monads are defined as weak braided bimonads $\mathcal{H}$ with a weak antipode (Definition 7.1). In monoidal categories, these correspond to the weak braided Hopf algebras considered in [2], [3].

We show that for a braided bimonad in Cauchy complete categories the existence of an antipode is equivalent to the functor $K_\omega$ having a left adjoint and a monadic right adjoint and this leads to an equivalence between the categories of $H_\xi$-modules and $\mathcal{A}_\mathcal{H}^\mathcal{H}(\omega)$ (Fundamental Theorem 7.6).

Examples for our weak braided bimonads and weak braided Hopf monads are the weak braided Hopf algebras in strict monoidal Cauchy complete categories considered by Alonso Álvarez et al. in [1, 2, 3], the weak bimonoids and weak Hopf monoids in braided monoidal categories as defined by Pastro et. al. in [26] and also showing up in [10, Sections 3, 4]. These all subsume the weak Hopf algebras considered, e.g., by Takeuchi [32] and Schauenburg [27] and, of course, the weak Hopf algebras in vector space categories introduced by Böhm et al. in [11]. Moreover, we generalise the bimonads and $\tau$-Hopf monads defined on arbitrary categories in [24] and these include, for example, bimonoids in duoidal categories (e.g., [25]).

Opmonoidal monads $T = (T, m, e)$ on strict monoidal categories $(\mathcal{V}, \otimes, \mathbb{I})$ were also called bimomads by Bruguières et al. in [13] and these were generalised to weak bimonads in monoidal categories by Böhm et al. in [10]. As pointed out in [23, Section 5], the bimonads from [13] yield a special case of an entwining of the monad $T$ with the comonad $- \otimes T(\mathbb{I})$, where $T(\mathbb{I})$ has a coalgebra structure derived from the opmonoidality of $T$. To transfer the structures from [10] to arbitrary categories one has to consider weak entwinings between monads and comonads. It is planned to elaborate details for this in a subsequent article.

2. Preliminaries

2.1. Monads and comonads. Recall that a monad $T$ on a category $\mathcal{A}$ (or shortly an $\mathcal{A}$-monad $T$) is a triple $(T, m, e)$ where $T: \mathcal{A} \to \mathcal{A}$ is a functor with natural transformations $m: TT \to T$, $e: 1 \to T$ satisfying the usual associativity and unitality conditions. A $T$-module is an object $a \in \mathcal{A}$ with a morphism $h: T(a) \to a$ subject to associativity and unitality conditions. The (Eilenberg-Moore) category of $T$-modules is denoted by $\mathcal{A}_T$ and there is an adjunction

$$\varepsilon_T, \eta_T: \phi_T |- U_T: \mathcal{A}_T \to \mathcal{A},$$

with $\phi_T: \mathcal{A} \to \mathcal{A}_T$ and $U_T: \mathcal{A}_T \to \mathcal{A}$ given by the respective assignments

$$a \mapsto (T(a), m_a) \text{ and } (a, h) \mapsto a,$$
while \( e_T = e \) and \( (e_T)_{(a,h)} = h \) for each \( (a,h) \in \mathbb{A}_T \).

If \( T = (RF, ReF, \eta) \) is the monad generated on \( \mathbb{A} \) by an adjoint pair \( \eta, \varepsilon : F \dashv R : \mathbb{B} \to \mathbb{A} \), then there is the \textit{comparison functor} \( K_T : \mathbb{B} \to \mathbb{A}_T \) which assigns to each object \( b \in \mathbb{B} \) the \( T \)-algebra \( (R(b), R(\varepsilon_b)) \), and to each morphism \( f : b \to b' \) the morphism \( R(f) : R(b) \to R(b') \), satisfying \( U_T K_T = R \) and \( K_T F = \phi_T \). This situation is illustrated by the diagram

\[
\begin{array}{ccc}
\mathbb{B} & \xrightarrow{K_T} & \mathbb{A}_T \\
\downarrow{R} & & \downarrow{\phi_T} \\
\mathbb{A} & \xleftarrow{U_T} & \\
\end{array}
\]

The functor \( R \) is called \textit{monadic} (resp. \textit{premonadic}) if the comparison functor \( K_T \) is an equivalence of categories (resp. full and faithful).

\[2.2. \text{ Theorem.} \ (\text{Beck, [5]}) \]
Let \( \eta, \varepsilon : F \dashv R : \mathbb{B} \to \mathbb{A} \) be an adjunction, and \( T = (RF, ReF, \eta) \) the corresponding monad on \( \mathbb{A} \).

1. The comparison functor \( K_T : \mathbb{B} \to \mathbb{A}_T \) has a left adjoint \( L_T : \mathbb{A}_T \to \mathbb{B} \) if and only if for each \( (a,h) \in \mathbb{A}_T \), the pair of morphisms \( (F(h), \varepsilon_{F(a)}) \) has a coequaliser in \( \mathbb{B} \).

2. \( R \) is monadic if and only if it is conservative and for each \( (a,h) \in \mathbb{A}_T \), the pair of morphisms \((F(h), \varepsilon_{F(a)})\) has a coequaliser and this coequaliser is preserved by \( R \).

Dually, a \textit{comonad} \( G \) on \( \mathbb{A} \) (or shortly an \( \mathbb{A} \)-comonad \( G \)) is a triple \((G, \delta, \varepsilon)\) where \( G : \mathbb{A} \to \mathbb{A} \) is a functor with natural transformations \( \delta : G \to GG, \varepsilon : G \to 1 \), and \( G \)-\textit{comodules} are objects \( a \in \mathbb{A} \) with morphisms \( \theta : a \to G(a) \). Both notions are subject to coassociativity and counitality conditions. The (Eilenberg-Moore) category of \( G \)-comodules is denoted by \( \mathbb{A}^G \) and there is a cofree functor

\[ \phi^G : \mathbb{A} \to \mathbb{A}^G, \ a \mapsto (G(a), \delta_a), \]

which is right adjoint to the forgetful functor

\[ U^G : \mathbb{A}^G \to \mathbb{A}, \ (a, \theta) \mapsto a. \]

If \( \eta, \varepsilon : F \dashv R : \mathbb{A} \to \mathbb{B} \) is an adjoint pair and \( G = (FR, F\eta R, \varepsilon) \) is the comonad on \( \mathbb{A} \) associated to \((R, F)\), then one has the comparison functor

\[ K^G : \mathbb{B} \to \mathbb{A}^G, \ b \mapsto (F(b), F(\eta_b)) \]

for which \( U^G \cdot K^G = F \) and \( K^G \cdot R = \phi^G \). One says that the functor \( F \) is \textit{precomonadic} if \( K^G \) is full and faithful, and it is \textit{comonadic} if \( K^G \) is an equivalence of categories.

\[2.3. \text{ Cauchy completeness.} \]
A morphism \( e : A \to A \) in \( \mathbb{A} \) is idempotent if \( e^2 = e \) and \( \mathbb{A} \) is said to be \textit{Cauchy complete} if for every idempotent \( e : a \to a \), there exists an object \( \overline{a} \) and morphisms \( p : a \to \overline{a} \) and \( i : \overline{a} \to a \) such that \( ip = e \) and \( pi = 1_{\overline{a}} \). In this case, \((\overline{a}, i)\) is the equaliser of \( e \) and \( 1_a \) and \((p, i)\) is the coequaliser of \( e \) and \( 1_a \). Hence any category admitting either equalisers or coequalizers is Cauchy complete.

\[2.4. \text{ Proposition.} \]
Let \( G \) be a comonad on \( \mathbb{A} \). If \( \mathbb{A} \) is Cauchy complete, then so is \( \mathbb{A}^G \). Moreover, the forgetful functor \( U^G : \mathbb{A}^G \to \mathbb{A} \) preserves and creates splitting of idempotents. Explicitly, if \( e : (a, \theta) \to (a, \theta) \) is an idempotent morphism in \( \mathbb{A}^G \) and if \( a \xrightarrow{p} \overline{a} \xrightarrow{i} a \) is a splitting of \( e \) in \( \mathbb{A} \), then \((\overline{a}, G(p) \cdot \theta \cdot i)\) is a \( G \)-comodule in such a way that \( p \) and \( i \) become morphisms of \( G \)-comodules. Similarly, if \( T \) is a monad on \( \mathbb{A} \), then the forgetful functor \( U_T : \mathbb{A}_T \to \mathbb{A} \) preserves and creates equalisers.

\[ \text{ Proof.} \]
The result follows from the fact that the forgetful functor \( U^G : \mathbb{A}^G \to \mathbb{A} \) preserves and creates coequalisers, while the functor \( U_T : \mathbb{A}_T \to \mathbb{A} \) preserves and creates equalisers. \( \square \)
2.5. Split (co)equalisers. Recall (e.g. from [20]) that a diagram

\[(a \xrightarrow{\partial_0} a' \xrightarrow{p} x)\]

with \(p\partial_0 = p\partial_1\) is said to be split by a pair of morphisms \(i : x \to a'\) and \(s : a' \to a\) if \(pi = 1_x, \partial_0s = 1_{a'}\) and \(\partial_1s = ip\).

A pair of morphisms \((\partial_0, \partial_1 : a \rightrightarrows a')\) in \(\mathcal{A}\) is called split if there exists a morphism \(s : a' \to a\) with \(\partial_0s = 1\) and \(\partial_1s\partial_0 = \partial_1s\partial_1\). In this case, \(\partial_1s : a' \to a'\) is an idempotent, and if we assume \(\mathcal{A}\) to be Cauchy complete and if \(a' \xrightarrow{p} x \xrightarrow{i} a'\) is a splitting of the idempotent \(\partial_1s\), then the diagram

\[
\begin{array}{ccc}
  a & \xrightarrow{\partial_0} & a' \\
  \downarrow \partial_1 & & \downarrow p \\
  s & \xrightarrow{i} & x
\end{array}
\]

is a split coequaliser. Conversely, if the above diagram is a split coequaliser, then \(s\) makes the pair \((\partial_0, \partial_1 : a \rightrightarrows a')\) split. Thus, when \(\mathcal{A}\) is Cauchy complete, a pair \((\partial_0, \partial_1 : a \rightrightarrows a')\) is part of a split coequaliser diagram if and only if it is split. Note that split (co)equalisers are absolute, i.e., they are preserved by any functor.

If \(F : \mathcal{A} \to \mathcal{B}\) is a functor, \((\partial_0, \partial_1 : a \rightrightarrows a')\) is called \(F\)-split if the pair of morphisms \((F(\partial_0), F(\partial_1)) : F(a) \rightrightarrows F(a'))\) in \(\mathcal{B}\) is split.

The dual notions are those of cosplit pairs and \(F\)-cosplit pairs.

Given a monad \(T\) (resp. comonad \(G\)) on \(\mathcal{A}\) and a category \(\mathcal{X}\), one may consider the functor category \([\mathcal{X}, \mathcal{A}]\) and the induced monad \([\mathcal{X}, T]\) (resp. comonad \([\mathcal{X}, G]\)) thereon, whose functor part sends a functor \(F : \mathcal{X} \to \mathcal{A}\) to the composite \(TF : \mathcal{X} \to \mathcal{A}\) (resp. \(GF : \mathcal{X} \to \mathcal{A}\)). Symmetrically, one has the induced monad \([T, \mathcal{X}]\) (resp. comonad \([G, \mathcal{X}]\)) on \([\mathcal{A}, \mathcal{X}]\), whose functor-part sends \(F : \mathcal{A} \to \mathcal{X}\) to \(FT : \mathcal{A} \to \mathcal{X}\) (resp. \(FG : \mathcal{A} \to \mathcal{X}\)).

2.6. (Bi)module functors. Given a monad \(T = (T, m, e)\) on \(\mathcal{A}\) and a category \(\mathcal{X}\), a left \(T\)-module with domain \(\mathcal{X}\) is an object of the Eilenberg-Moore category \([\mathcal{X}, \mathcal{A}]_{[\mathcal{T}, \mathcal{X}]}\) of the monad \([\mathcal{T}, \mathcal{X}]\). Thus, a left \(T\)-module with domain \(\mathcal{X}\) is a functor \(M : \mathcal{X} \to \mathcal{A}\) together with a natural transformation \(q : TM \to M\), called the action (or the \(T\)-action) on \(M\), such that \(q \cdot eM = 1\) and \(q \cdot Tg = q \cdot mM\). A morphism of left \(T\)-modules with domain \(\mathcal{X}\) is a natural transformation in \([\mathcal{X}, \mathcal{A}]\) that commutes with the \(T\)-actions.

Similarly, for a category \(\mathcal{Y}\), the category of right \(T\)-modules with codomain \(\mathcal{Y}\) is defined as the Eilenberg-Moore category \([\mathcal{A}, \mathcal{Y}]_{[\mathcal{T}, \mathcal{Y}]}\) of the monad \([\mathcal{T}, \mathcal{Y}]\).

Let \(S\) be another monad on \(\mathcal{A}\). A \((T, S)\)-bimodule is a functor \(N : \mathcal{A} \to \mathcal{A}\) equipped with two natural transformations \(q_l : TN \to N\) and \(q_r : NS \to N\) such that \((N, q_l) \in [\mathcal{X}, \mathcal{A}]_{[\mathcal{T}, \mathcal{X}]}, (N, q_r) \in [\mathcal{A}, \mathcal{Y}]_{[\mathcal{T}, \mathcal{Y}]}\) and \(q_r \cdot q_l S = q_l \cdot Tq_r\). A morphism of \((T, S)\)-bimodules is a morphism of left \(T\)-modules which is simultaneously a morphism of right \(S\)-modules. We write \([\mathcal{X}, \mathcal{A}]_{[S, T]}\) for the corresponding category.

2.7. Canonical modules. Let \(T = (T, m, e)\) be an arbitrary monad on \(\mathcal{A}\). Using the associativity and unitality of the multiplication \(m\), we find that for any functor \(M : \mathcal{X} \to \mathcal{A}\), the pair \((TM, mM)\) is a left \(T\)-module. Moreover, if \(\nu : M \to M'\) is a natural transformation, then \(\nu T : TM \to TM'\) is a morphism of left \(T\)-modules.

Symmetrically, for any functor \(N : \mathcal{A} \to \mathcal{Y}\), the pair \((NT, Nm)\) is a right \(T\)-module, and if \(\nu : N \to N'\) is a natural transformation, then \(\nu T : NT \to N'T\) is a morphism of right \(T\)-modules. In particular, \((T, m)\) can be regarded as a right as well as a left \(T\)-module; again by the associativity of \(m\), \((T, m, m)\) is a \((T, T)\)-bimodule. Moreover, if \(S\) is another monad on \(\mathcal{A}\) and \(\iota : S \to T\) is a monad morphism, then
(i) \((T, ST \xrightarrow{\epsilon_T} TT \xrightarrow{m} T)\) is a left \(S\)-module;
(ii) \((T, TS \xrightarrow{T\epsilon} TT \xrightarrow{m} T)\) is a right \(S\)-module;
(iii) \((T, TT \xrightarrow{m} T, TS \xrightarrow{T\epsilon} TT \xrightarrow{m} T)\) is a \((T, S)\)-bimodule;
(iv) \((T, ST \xrightarrow{\epsilon_T} TT \xrightarrow{m} T, TT \xrightarrow{m} T)\) is an \((S, T)\)-bimodule;
(v) \((\phi_T, \varepsilon_T \phi_T \cdot \phi_T)\) is right \(S\)-module;
(vi) \((U_T, U_T \varepsilon_T \cdot \iota_T)\) is left \(S\)-module.

2.8. Tensor product of module functors. Let \(T\) be a monad on \(A\) and \(X\) and \(Y\) arbitrary categories. If \((N, \rho)\) is a left \(T\)-module and \((M, \varrho)\) a right \(T\)-module, then their tensor product (over \(T\)) is defined as the object part of the coequaliser

\[
\begin{array}{ccc}
MTN & \xrightarrow{\varrho N} & MN \\
\downarrow{M\rho} & & \downarrow{\text{can}^M_N} \\
M \otimes_T N & & \\
\end{array}
\]

in \([X, Y]\), provided that such a coequaliser exists. We often abbreviate \(\text{can}^M_N\) to \(\text{can}^{M,N}\), or even to \(\text{can}\).

2.9. Proposition. Let \(S, T\) be monads on a category \(A\). If the modules \((M, \varrho) \in [A, X]_{[S,T]}\) and \((N, \rho_T, \rho_r) \in [A, A]_{[S,T]}\) are such that the tensor product \(M \otimes_T N\) exists, then there is a natural transformation

\[
\zeta : (M \otimes_T N)_S \to M \otimes_T N
\]

making \(M \otimes_T N\) into a right \(S\)-module in such a way that the diagram

\[
\begin{array}{ccc}
(MTN, MT\rho_r) & \xrightarrow{\varrho N} & (MN, M\rho_r) \\
\downarrow{M\rho_l} & & \downarrow{\text{can}^M_N} \\
(M \otimes_T N, \zeta) & & \\
\end{array}
\]

is a coequaliser in \([A, X]_{[S,T]}\). The action \(\zeta\) is uniquely determined by the property

\[
\zeta \cdot \text{can}^{M,N}_T S = \text{can}^{M,N}_T \cdot M\rho_r.
\]

**Proof.** Note first that both \((MTN, MT\rho_r)\) and \((MN, M\rho_r)\) are objects of the category \([A, X]_{[S,T]}\), and that both \(\varrho N\) and \(M\rho_l\) can be seen as morphisms in \([A, X]_{[S,T]}\) (the former by naturality of composition, the latter because \((N, \rho_l, \rho_r) \in [A, A]_{[S,T]}\)). Now, since the functor

\[
[S, X] : [A, X] \to [A, X]
\]

preserves all colimits (hence coequalisers), the result follows from \([18\text{ Proposition 2.3}]. \)

2.10. Comodule functors. Similarly as for monads, one defines comodule functors over a comonad \(G\) on a category \(A\). A left \(G\)-comodule functor is a pair \((F, \theta)\), where \(F : X \to A\) is a functor and \(\theta : F \to GF\) is a natural transformation inducing commutativity of the diagrams

\[
\begin{array}{ccc}
F & \xrightarrow{\theta} & GF \\
\downarrow{\varepsilon_F} & & \downarrow{\delta_F} \\
F & \xrightarrow{} & GF
\end{array}
\]

Symmetrically, right \(G\)-comodule functors \(A \to Y\) are defined.

2.11. Cotensor product of comodule functors. Let \(G\) be a comonad on \(A\) and \(X, Y\) arbitrary categories. If \((C, \theta)\) is a right \(G\)-comodule functor \(A \to X\) and \((D, \vartheta)\) is a left
\( g \)-comodule functor \( Y \to A \), then their cotensor product (over \( G \)) is defined as the object part of the equaliser

\[
C \otimes^G D \xrightarrow{\text{can}_{C,D}^G} CD \xrightarrow{\theta D} CGD
\]

in \([Y, X]\), provided that such an equaliser exists. We often abbreviate \( \text{can}_{C,D}^G \) to \( \text{can}^G_{C,D} \), or even to \( \text{can} \).

Now recall the dual of [28, Lemma 21.1.5]) and Dubuc’s Adjoint Triangle Theorem [16].

2.12. Monads and adjunctions. For categories \( A, B \) and \( C \), consider the adjunctions

\( \eta, \varepsilon : F \dashv R : A \to C \) and \( \eta', \varepsilon' : F' \dashv R' : B \to C \), with corresponding monads \( T \) and \( T' \), and let

\[
\begin{array}{c}
A \\
\downarrow R \\
C
\end{array}
\xrightarrow{K}
\begin{array}{c}
B \\
\downarrow R'
\end{array}
\]

be a commutative diagram with suitable functor \( K \). Write \( \gamma_K \) for the composite

\[
F' \xrightarrow{F'\eta} F'R = F'R'KF \xrightarrow{\varepsilon'KF} KF.
\]

Then \( t_K = R'\gamma_K \) is a monad morphism \( T' \to T \) (see [16]).

2.13. Theorem. In the situation considered in 2.12, suppose that \( R \) is a premonadic functor. Then \( K \) has a left adjoint \( \underbar{K} \) if and only if the following coequaliser (used as the definition of \( \underbar{K} \)) exists in \([B, A]\),

\[
\begin{array}{c}
FR'F' \\
\downarrow F't_KR' = FR'\gamma_KR' \\
\downarrow FR'KF' = FRFR'
\end{array}
\xrightarrow{q} K
\]

When this is the case, the unit \( \eta \) of the adjunction \( \underbar{K} \dashv K \) is the unique natural transformation \( 1 \to KK \) yielding commutativity of the diagram

\[
\begin{array}{c}
F'R' \\
\gamma_KR'
\end{array}
\xrightarrow{F'R'\eta} 1
\]

while the counit \( \varepsilon \) is the unique natural transformation \( KK \to 1 \) making the diagram

\[
\begin{array}{c}
FR = FR'K \\
\downarrow qK
\end{array}
\xrightarrow{\varepsilon} 1
\]

\[
\begin{array}{c}
FR'K \\
\downarrow K\eta
\end{array}
\xrightarrow{\varepsilon} 1
\]
commute. Precomposing the image of the last square under $R'$ with $\eta R'$ and using the fact that $R' \gamma_K$ is a monad morphism $T' \to T$, we get the commutative diagram

\[
\begin{array}{c}
R' \\
\downarrow \eta R' \quad \quad \quad \quad R' F' R' \\
\downarrow R' \gamma_K R' \\
R F R' = R' K F R' \\
\downarrow R' \kappa = R q \\
R' K K
\end{array}
\]

Since $R' \varepsilon' \cdot \eta R' = 1$ by one of the triangular identities for $F' \dashv R'$, one gets

\[
R' \eta = R q \cdot \eta R'.
\]

2.14. **Comonads and adjunctions.** Again let $\eta, \varepsilon : F \dashv R : A \to B$ and $\eta', \varepsilon' : F' \dashv R' : A \to C$ be adjunctions and now consider the corresponding comonads $G$ and $G'$. Let

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{K} & \mathcal{C} \\
F & \downarrow \gamma_K & \downarrow F' \\
A & \xrightarrow{F'} & C
\end{array}
\]

be a commutative diagram and define

\[
\gamma_K : KR \xrightarrow{\eta' KR} R' F' K R = R' FR \xrightarrow{R' \varepsilon'} R'.
\]

Then $t_K = F' \gamma_K$ is a comonad morphism $G \to G'$.

2.15. **Theorem.** In the situation described in 2.14 suppose that $F'$ is precomonadic. Then $K$ has a right adjoint $K'$ if and only if the following equaliser exists in $[\mathcal{C}, \mathcal{B}]$,

\[
\begin{array}{c}
K \\
\downarrow \eta \\
RF = RF' K
\end{array}
\]

When this equaliser exists, the unit $\eta$ of the adjunction $K \dashv K'$ is the unique natural transformation $1 \to K K$ yielding commutativity of the diagram

\[
\begin{array}{c}
1 \\
\downarrow \eta \\
RF = RF' K
\end{array}
\]

while the counit $\varepsilon$ of the adjunction $K \dashv K'$ is the unique natural transformation $K K \to 1$ making the diagram

\[
\begin{array}{c}
K K \\
\downarrow \gamma_K F' \\
RF' F'
\end{array}
\]

commute. Moreover, one has

\[
(2.5) \quad F' \varepsilon = \varepsilon F' \cdot F \iota.
\]
2.16. The restriction- and change-of-base functors. Any morphism
\[ \iota : S = (S, m^S, e^S) \to T = (T, m^T, e^T) \]
of monads on \( \mathbb{A} \) (that is, a natural transformation \( \iota : S \to T \) such that \( \iota \cdot e^S = e^T \) and \( \iota \cdot m^S = m^T \cdot (\iota \iota) \)) gives rise (see [3]) to the functor
\[ \iota^* : \mathbb{A}_T \to \mathbb{A}_S, \quad (a, h) \mapsto (a, h \cdot \iota_a), \]
called the restriction-of-base functor. It is clear that \( \iota^* \) makes the diagram
\[ \begin{array}{ccc}
\mathbb{A}_T & \xrightarrow{\iota^*} & \mathbb{A}_S \\
U_T & \searrow & \downarrow U_S \\
& \mathbb{A} &
\end{array} \]
commute. Since the forgetful functor \( U_S : \mathbb{A}_S \to \mathbb{A} \) is clearly monadic, it follows from the Theorem 2.13 that \( \iota^* \) has a left adjoint \( \iota_! : \mathbb{A}_S \to \mathbb{A}_T \) if and only if the pair of natural transformations
\[
(2.6) \quad \phi_T S U_S = \phi_T U_S \phi_S U_S \xrightarrow{\phi_T U_S \varepsilon_S} \phi_T U_S \xrightarrow{q} \iota_!,
\]
where \( q : \phi_T S \to \phi_T \) is the composite \( \phi_T S \xrightarrow{\phi_T \iota^*} \phi_T T = \phi_T U_T \phi_T \xrightarrow{\varepsilon_T \phi_T} \phi_T \), has a coequaliser \( q : \phi_T U_S \xrightarrow{\iota_!} \iota_! \) in \( [\mathbb{A}_S, \mathbb{A}_T] \). This \( \iota_! : \mathbb{A}_S \to \mathbb{A}_T \), when it exists, is called the change-of-base functor. Recalling that for any \( S \)-algebra \( (a, g) \), \( U_S((e_S)_a(g)) = g \) and \( (e_T \phi_T U_S(g)) = m^T_a \), one finds that \( \iota_! \) sends an \( S \)-algebra \( (a, g) \) to the object \( \iota_! (a, g) \) in the coequaliser diagram in \( \mathbb{A}_T \),
\[
(2.7) \quad \begin{array}{ccc}
\phi_T S(a) & \xrightarrow{\phi_T(g)} & \phi_T(a) \\
& \xrightarrow{q(a, g)} & \iota_! (a, g) \\
\phi_T T(a) & \xrightarrow{m^T_a} & \phi_T(a)
\end{array}
\]

2.17. Remark. (1) Since \( (\phi_T, \varepsilon_T \phi_T \cdot \phi_T) \) is a right \( S \)-module by Section 2.7(vi), the diagram
\[
\begin{array}{ccc}
\phi_T S S & \xrightarrow{\phi_T m^S} & \phi_T S \\
& \xrightarrow{\varepsilon_T \phi_T S \cdot \phi_T S} & \phi_T T \\
& \xrightarrow{\varepsilon_T \phi_T} & \phi_T
\end{array}
\]
is a coequaliser in \( [\mathbb{A}, \mathbb{A}_T] \). Observing that the pair \( \phi_T S S \xrightarrow{\phi_T m^S} \phi_T S \) is just the pair
\[
\phi_T S U_S \phi_S \xrightarrow{\phi_T U_S \varepsilon_S \phi_S} \phi_T U_S \phi_S, \quad \text{it follows that} \quad q \phi_S = \phi_T S \xrightarrow{\iota_!} \phi_T T \xrightarrow{\varepsilon_T \phi_T} \phi_T.
\]

(2) It is easy to see that \( U_S \varepsilon_S \iota^* \phi_T \) is the composite \( ST \xrightarrow{\iota^T} TT \xrightarrow{m^T} T \) and since
\[ \begin{align*}
\bullet & \quad U_T \varepsilon_T \phi_T = U_T \phi_T = T, \\
\bullet & \quad U_T \varepsilon_T \phi_T = m^T, \\
\bullet & \quad U_T q = TS \xrightarrow{T} TT \xrightarrow{m^T} T,
\end{align*} \]
the pair
\[ (U_T \phi_T U_S \varepsilon_S \iota^* \phi_T, U_T q U_S \iota^* \phi_T) \]
is just the pair
\[ (Tm^T : TiT, m^T T : TiT). \]
Thus, if the coequaliser diagram (2.6) exists and if its image under the functor \([\mathcal{A}_S, U_T] : [\mathcal{A}_S, \mathcal{A}_T] \to [\mathcal{A}_S, \mathcal{A}_T]\) is again a coequaliser, we get a coequaliser
\[
TST \xrightarrow{Tm^T \cdot T\iota} TT \xrightarrow{U_Tq^* \phi_T} U_T\iota^* \phi_T.
\]
But since \((T, m^T \cdot T\iota)\) is a right \(S\)-module, while \((T, m^T \cdot \iota T)\) is a left \(S\)-module (see Section 2.7), one concludes that 
\[
\begin{align*}
U_T\iota^* \phi_T &= T \otimes_S T \\
U_Tq^* \phi_T &= \text{can}_{\mathcal{A}_S}.
\end{align*}
\]

2.18. Proposition. If \(\iota : S \to T\) is a morphism of \(\mathcal{A}\)-monads such that the change-of-base functor \(\iota^* : \mathcal{A}_S \to \mathcal{A}_T\) exists, then the diagram
\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\phi_S} & \mathcal{A}_S \\
\phi_T & \downarrow & \iota^* \\
& \mathcal{A}_T &
\end{array}
\]
commutes (up to isomorphism).

Proof. Since \(U_S \cdot \iota^* = U_T\) (see Section 2.16) and \(\phi_S\) (resp. \(\iota^*\)) is left adjoint to \(U_S\) (resp. \(\iota^*\)), the result follows by uniqueness of left adjoints. \(\Box\)

3. Separable Frobenius monads

The crucial role of separable Frobenius functors (e.g. [29]) in the theory of weak bimonads was pointed out by Szlachányi in [31] and such functors are used by Böhm et al. in [10] as an integral part of their definition of weak bimonads on monoidal categories. In this section we show that in our approach separable Frobenius monads \(S\) are of interest since they imply the existence of the change-of-base functor for monad morphisms \(S \to T\).

3.1. Definition. An \(\mathcal{A}\)-monad \(S = (S, m^S, e^S)\) is called Frobenius if \(S\) has an \(\mathcal{A}\)-comonad structure \(G = (S, \delta^S, \varepsilon^S)\) with commutative diagram
\[
\begin{array}{ccc}
S & \xrightarrow{\delta^S} & SS \\
\downarrow & & \downarrow \delta^S \\
S & \xrightarrow{S \delta^S} & SSS
\end{array}
\]
\[
\begin{array}{ccc}
SS & \xrightarrow{m^S} & SS \\
\downarrow & & \downarrow m^S \\
SS & \xrightarrow{S \delta^S} & SSS.
\end{array}
\]

Dually, a comonad \(G = (S, \delta^S, \varepsilon^S)\) is called Frobenius if there exists a monad structure \(S = (S, m^S, e^S)\) such that Diagram (3.1) commutes. \(S\) (resp. \(G\)) is called Frobenius separable if, moreover, \(m^S \cdot \delta^S = 1\).

3.2. Proposition. Let \(S\) be a Frobenius separable monad on a Cauchy complete category \(\mathcal{A}\). Then for any morphism \(\iota : S \to T\) of monads, the change-of-base functor \(\iota^* : \mathcal{A}_S \to \mathcal{A}_T\) exists.

Proof. We claim that, under our assumptions, (2.6) is a split pair, a splitting morphism being the composite
\[
\pi : \phi_T U_S \xrightarrow{\phi_T e^S U_S} \phi_T SU_S \xrightarrow{\phi_T \delta^S U_S} \phi_T SSU_S \xrightarrow{\phi_T SU_S} \phi_T SU_S.
\]
Indeed, that $\varrho \cdot \pi = 1$ follows from the commutativity of the diagram

Here the square and the curved region commute since $(\phi_T, \varrho)$ is a right $S$-module by Section 2.7(vii), while the triangle commutes by separability of the monad $S$.

Next, to show that $\phi_T U_S \varepsilon_S \cdot \pi \cdot \varrho U_S = \phi_T U_S \varepsilon_S \cdot \pi \cdot \phi_T U_S \varepsilon_S$, consider the diagram

in which the curved region commutes since $S$ is assumed to be Frobenius, the right-hand parallelogram commutes since $\varrho$ is a morphism of right $S$-modules, while the other regions commute by naturality of composition. Thus the whole diagram is commutative, implying – since $m^S \cdot S e^S = 1$ – that

$$\phi_T U_S \varepsilon_S \cdot \pi \cdot \varrho U_S = \phi_T U_S \varepsilon_S \cdot \varrho U_S \cdot \phi_T \delta^S U_S.$$ 

In a similar manner one proves that

$$\phi_T U_S \varepsilon_S \cdot \pi \cdot \phi_T U_S \varepsilon_S = \phi_T U_S \varepsilon_S \cdot \varrho U_S \cdot \phi_T \delta^S U_S.$$ 

So $\phi_T U_S \varepsilon_S \cdot \pi \cdot \varrho U_S = \phi_T U_S \varepsilon_S \cdot \pi \cdot \phi_T U_S \varepsilon_S$. Therefore, the pair (2.6) splits by the morphism $\pi$. Since $A$ is assumed to be Cauchy complete, $A_T$ (and hence also the functor category $[A_S, A_T]$) is also Cauchy complete (see Proposition 2.4). It then follows that the pair (2.6) admits a (split) coequaliser. Thus the extension-of-base functor $\iota : A_S \to A_T$ exists. □

Dually, we have:

3.3. Proposition. Let $F$ is a Frobenius separable comonad on a Cauchy complete category $A$. Then for any morphism $f : F \to G$ of comonads, the change-of-cobase functor $f_! : A^G \to A^F$ exists.

4. Comparison functors

Given a comonad $G$ on $A$ and a category $B$, one has the induced comonads $[B, G]$ on $[B, A]$ and $[G, B]$ on $[A, B]$.

4.1. Comodules and adjoint functors. Consider a comonad $G = (G, \delta, \varepsilon)$ on $A$ and an adjunction $\eta, \sigma : F \dashv R : B \to A$.

There exist bijective correspondences (see [17]) between:

- functors $K : B \to A^G$ with $U^G K = F$;
- left $G$-comodule structures $\theta : F \to GF$ on $F$ (i.e., $(F, \theta) \in [B, A]^{[G, B]}$);
- comonad morphisms from the comonad generated by $F \dashv R$ to the comonad $G$;
- right $G$-comodule structures $\vartheta : R \to RG$ on $R$ (i.e., $(R, \vartheta) \in [A, B]^{[G, B]}$).
These bijections are constructed as follows. If \( U^G K = F \), then \( K(b) = (F(b), \theta_b) \) for some morphism \( \alpha_b : F(b) \to GF(b) \), and the collection \( \{ \theta_b, b \in \mathcal{B} \} \) constitutes a natural transformation \( \theta : F \to GF \) making \( F \) a left \( G \)-comodule.

Conversely, if \((F, \theta) \in [\mathcal{B}, \mathcal{A}]^{[\mathcal{B}, \mathcal{G}]}\), then \( K : \mathcal{B} \to \mathcal{A}^G \) is defined by \( K(b) = (F(b), \theta_b) \).

Next, for any left \( G \)-comodule structure \( \theta : F \to GF \), the composite

\[
t_K : FR \overset{\theta_R}{\longrightarrow} GFR \overset{G\alpha}{\longrightarrow} G
\]

is a morphism from the comonad generated by \( F \) \dashv \( R \) to the comonad \( G \). Then the corresponding right \( G \)-comodule structure \( \vartheta : R \to RG \) on \( R \) is \( R \overset{\eta_R}{\longrightarrow} RFR \overset{R\delta_K}{\longrightarrow} RG \).

Conversely, for \((R, \vartheta) \in [\mathcal{A}, \mathcal{B}]^{[G, \mathcal{B}]}\), the corresponding comonad morphism \( t_K : FR \to G \) is the composite

\[
FR \overset{F\vartheta}{\longrightarrow} FRG \overset{\sigma_G}{\longrightarrow} G,
\]

while the corresponding left \( G \)-comodule structure \( \theta : F \to GF \) on \( F \) is the composite \( F \overset{F \eta}{\longrightarrow} FRF \overset{t_K}{\longrightarrow} GF \).

### 4.2. Theorem

Let \( G = (G, \delta, \varepsilon) \) be a comonad on \( \mathcal{A} \) and \( \eta, \sigma : F \dashv R : \mathcal{B} \to \mathcal{A} \) an adjunction. For a functor \( K : \mathcal{B} \to \mathcal{A}^G \) with \( U^G K = F \), the following are equivalent:

(a) \( K \) is an equivalence of categories;
(b) \( F \) is comonadic and the comonad morphism \( t_K : FR \to G \) is an isomorphism;
(c) \( F \) is comonadic and the composite

\[
\gamma_K : KR \overset{\eta^GKR}{\longrightarrow} \phi^GKR = \phi^GFR \overset{G\sigma}{\longrightarrow} \phi^G
\]

is an isomorphism.

**Proof.** (a) and (b) are equivalent by [21, Theorem 4.4]; (b) and (c) are equivalent since \( U^G \gamma_K = t_K \) by [16] and \( U^G \) reflects isomorphisms. \( \square \)

### 4.3. Right adjoint of \( K \)

Now fix a functor \( K : \mathcal{B} \to \mathcal{A}^G \) with commutative diagram

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{K} & \mathcal{A}^G \\
F & \searrow & U^G \\
& \mathcal{A}.
\end{array}
\]

Then \( \gamma_K \) is the composite \( KR \overset{\eta^GKR}{\longrightarrow} \phi^GKR = \phi^GFR \overset{G\sigma}{\longrightarrow} \phi^G \) and using the fact from Section 2.14 that \( U^G \gamma_K \) is just the comonad morphism \( t_K : FR \to G \) induced by the triangle, an easy calculation shows that

\[
\beta U^G = R U^G \gamma_K U^G \ast \eta RU^G,
\]

where \( \beta : R \to RG \) is the right \( G \)-module structure on \( R \) corresponding to the triangle \((4.1)\). Thus, when the right adjoint \( K \) of \( K \) exists, it is determined by the equaliser diagram

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{K} & \mathcal{A}^G \\
RU^G & \xrightarrow{RU^G \gamma_K} & R GU^G = RU^G \phi^G U^G.
\end{array}
\]

It is easy to see that for any \((a, \theta) \in \mathcal{A}^G\), the \((a, \theta)\)-component of \((4.2)\) is the equaliser diagram

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{t_{(a, \theta)}} & R(a) \\
\mathcal{A} & \xrightarrow{R(\theta)} & RG(a).
\end{array}
\]
It is easy to verify, using (2.5) that if \( \sigma \) is the counit of the adjunction \( K \dashv \bar{K} \), then for any \((a, \theta) \in \mathcal{A}^G\), one has

\[
U^G(\sigma_{(a, \theta)}) = \sigma_a \cdot F(\iota_{(a, \theta)}).
\]

Suppose now that \( \bar{K} \) exists, write \( P \) for the monad on \( \mathcal{B} \) generated by the adjunction \( K \dashv \bar{K} \), and consider the corresponding comparison functor \( K_P : \mathcal{A}^G \to \mathcal{B}_P \). Then \( K_P(a, \theta) = (\bar{K}(a, \theta), \bar{K}(\sigma_{(a, \theta)}) \) for any \((a, \theta) \in \mathcal{A}^G\). Moreover, \( K_P K = \phi_P \) and \( U_P K_P = \bar{K} \). The situation may be pictured as

(4.5)

In order to proceed, we need the following (see [28, Lemma 21.2.7]).

4.4. **Proposition.** Let \( \eta, \sigma : F \dashv R : \mathcal{A} \to \mathcal{C} \) and \( \eta', \sigma' : F' \dashv R' : \mathcal{B} \to \mathcal{C} \) be adjunctions with corresponding monads \( T \) and \( T' \), respectively, and let

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{K} & \mathcal{B} \\
F & \downarrow & F' \\
\mathcal{C}
\end{array}
\]

be a commutative diagram of categories and functors. Then the composite

\[
T = RF \xrightarrow{\eta'RF} R'F'R = R'KF'FR \xrightarrow{R'K\sigma'F} R'KF = R'F' = T'
\]

is a monad morphism \( T \to T' \).

Suppose again that \( \bar{K} \) exists and consider the natural transformation \( \iota : P \to RF \), where \( \iota_b = \iota_{K(b)} \) for all \( b \in \mathcal{B} \).

4.5. **Proposition.** \( \iota : P \to RF \) is a monad morphism from the monad \( P \) to the monad generated by the adjunction \( F \dashv R \).

**Proof.** Applying Proposition 4.4 to the diagram

\[
\begin{array}{ccc}
\mathcal{A}^G & \xrightarrow{U^G} & \mathcal{A} \\
K & \downarrow & \mathcal{B} \\
F & \Rightarrow & R
\end{array}
\]

in which \( U^G K = F \), gives that the natural transformation

\[
P = \bar{K}K \xrightarrow{\eta\bar{K}K} RF\bar{K}K = RU^GK\bar{K}K \xrightarrow{RU^G\sigma_K} RU^G K = RF
\]
is a monad morphism from the monad \( P \) to the monad generated by the adjunction \( F \dashv R \). Since for any \( (a, \theta) \in \mathbb{A}^G \), \( U^G(\theta(a, \theta)) = \sigma_a \cdot F(i(a, \theta)) \) by (4.3), it follows that, for each \( b \in \mathbb{B} \), the \( b \)-component of the above natural transformation is the composite

\[
P(b) \xrightarrow{\eta_P(b)} UFP(b) \xrightarrow{UF(i_b)} UFUF(b) \xrightarrow{U\sigma_F(b)} UF(b),
\]

which is easily verified to be just the morphism \( i_b : P(b) \rightarrow UF(b) \). This completes the proof. \( \Box \)

We are mainly interested in the case where the functor \( F \) is monadic. So, our standard situation of interest, and our standard notation, will henceforth be as follows. We consider a monad \( T = (T, m^T, e^T) \) on \( \mathbb{A} \), a comonad \( G \) on \( \mathbb{A_T} \) and an adjunction \( \eta_! : K \dashv \mathbb{K} : (\mathbb{A_T})^G \rightarrow \mathbb{A} \), where \( K : \mathbb{A} \rightarrow (\mathbb{A_T})^G \) is a functor with \( U^G K = \phi_T \). Write \( P = (P, m^P, e^P) \) for the monad on \( \mathbb{A} \) generated by the adjunction \( K \dashv \mathbb{K} \) and write \( i : P \rightarrow T \) for the induced morphism of monads. This is pictured in the diagram

\[
\begin{array}{ccc}
\mathbb{A} & \xrightarrow{U^P} & (\mathbb{A_T})^G \\
\bigwedge & & \\
\mathbb{A_P} & \xrightarrow{\phi_P} & (\mathbb{A_T})^G \\
\eta & \xrightarrow{K_P} & \\
\mathbb{K} & \xrightarrow{U^G} & \mathbb{A_T}
\end{array}
\]

in which \( K_P : (\mathbb{A_T})^G \rightarrow \mathbb{A_P} \) is the Eilenberg-Moore comparison functor for the monad \( P \), and thus \( K_P K = \phi_P \) and \( U^P K_P = \mathbb{K} \).

4.6. Proposition. In the situation above, the functor \( K_P : (\mathbb{A_T})^G \rightarrow \mathbb{A_P} \) admits a left adjoint \( L_P : \mathbb{A_P} \rightarrow (\mathbb{A_T})^G \) if and only if the restriction-of-base functor \( \iota^* : \mathbb{A_T} \rightarrow \mathbb{A_P} \) admits a left adjoint, i.e., the change-of-base functor \( \iota : \mathbb{A_P} \rightarrow \mathbb{A_T} \) exists. Moreover, when this is the case, \( \iota_1 = U^G L_P \).

Proof. According to Section 2.16, \( \iota_1 : \mathbb{A_P} \rightarrow \mathbb{A_T} \) exists if and only if for each \( (a, g) \in \mathbb{A_P} \), the pair of morphisms \( (\phi_T(g), m^a_T \cdot \phi_T(i_a)) \) has a coequaliser in \( \mathbb{A_T} \), while by Proposition 2.2(1), \( L_P : \mathbb{A_P} \rightarrow (\mathbb{A_T})^G \) exists if and only if the pair of morphisms \( (K(g), \sigma_{K(a)}) \) has a coequaliser in \( (\mathbb{A_T})^G \).

Since the functor \( U^G : (\mathbb{A_T})^G \rightarrow \mathbb{A_T} \) preserves and creates coequalisers, it suffices to show that the image of the pair \( (K(g), \sigma_{K(a)}) \) under \( U^G \) is just the pair \( (\phi_T(g), m^a_T \cdot \phi_T(i_a)) \). That \( U^G K(g) = \phi_T(g) \) follows from the equality \( U^G K = \phi_T \). Next, by (4.3), \( U^G(\sigma_{K(a)}) = (\varepsilon_T)_{U^G K(a)} \cdot \phi_T(i_K(a)) \). But since \( (\varepsilon_T)_{U^G K(a)} = (\varepsilon_T)(\phi_T(a)) = m^a_T \), we see that \( U^G(\sigma_{K(a)}) = m^a_T \cdot \phi_T(i_a) \).

Hence

\[
U^G(K(g), \sigma_{K(a)}) = (\phi_T(g), m^a_T \cdot \phi_T(i_a))
\]

and thus the result follows. \( \Box \)

Now assume that the change-of-base functor \( \iota_1 : \mathbb{A_P} \rightarrow \mathbb{A_T} \) exists, that is, \( K_P : (\mathbb{A_T})^G \rightarrow \mathbb{A_P} \) admits a left adjoint \( L_P : \mathbb{A_P} \rightarrow (\mathbb{A_T})^G \). Thus, for any \( (a, g) \in \mathbb{A_P} \), \( \iota_1(a, g) \) is given by be
the coequaliser

\[
\begin{array}{ccc}
T(a) & \xrightarrow{T(g)} & TT(a) \\
\downarrow{m_a} & & \downarrow{q(a,g)} \\
T(\iota a) & \xrightarrow{\iota_1} & T(a) \\
\end{array}
\]

Since \( \iota = U_G L_P \) by Proposition 4.6 and \( \iota \cdot \phi_P = \phi_T \) by Proposition 2.18, both triangles in the diagram

\[
(4.7)
\]

commute. Write \( G_1 \) (respectively \( G_2 \)) for the \( \mathbb{A}_T \)-comonad generated by the adjunction \( \phi_T \dashv U_T \) (respectively \( \iota_1 \cdot \iota^* \)), and consider the related comonad morphism \( t_{\phi_P} : G_1 \to G_2 \) (respectively \( t_{L_P} : G_2 \to G \)) corresponding to the left (respectively right) triangle in the above diagram (see Sections 2.14 and 4.1). Since \( U_P K_P = K \) and \( \phi_P \) (respectively \( L_P \)) has a right adjoint \( U_P \) (respectively \( K_P \)), it follows – by uniqueness of right adjoints – that \( L_P \cdot \phi_P = K \). Thus we may apply [23, Proposition 1.21] to obtain the equality

\[
(4.8)
\]

Recall from Section 2.16 that \( \iota_1 \) can be obtained as the coequaliser of Diagram (2.6).

4.7. Proposition. If \( K_P : (\mathbb{A}_T)^G \to \mathbb{A}_P \) admits a left adjoint, then \( t_{\phi_P} = q \iota^* \).

Proof. Applying the results of Section 2.14 to the left triangle in Diagram (4.7) gives that \( t_{\phi_P} = \iota \gamma \), where \( \gamma \) is the composite

\[
\phi_P U_P \xrightarrow{\eta_{\phi_P U_P}} \iota^* \iota_1 \phi_P U_T = \iota^* \phi_T U_T \xrightarrow{\iota^* \varepsilon_T} \iota^*.
\]

Here \( \eta \) is the unit of the adjunction \( \iota_1 \cdot \iota^* \), which (as a result of Theorem 2.15 applied to the commutative diagram \( U_P \cdot \iota^* = U_T \)) is a unique natural transformation making the diagram

\[
\begin{array}{ccc}
\phi_P U_P & \xrightarrow{\varepsilon_P} & 1 \\
\gamma' U_P & \xrightarrow{\eta} & \iota^* U_P \\
\iota^* \phi_T U_P & \xrightarrow{\iota^* \varepsilon_P} & \iota^* \iota_1
\end{array}
\]

commute. Here \( \gamma' \) is the composite

\[
\phi_P \xrightarrow{\phi_P \eta_T} \phi_P U_T \phi_T = \phi_P U_P \iota^* \phi_T \xrightarrow{\varepsilon_P \iota^* \phi_T} \iota^* \phi_T.
\]

Since \( U_P \gamma' = \iota \) and \( \iota \cdot e^P = e_T \), the diagram

\[
\begin{array}{ccc}
U_P \xrightarrow{e_U P} P U_P = U_P \phi_P U_P \xrightarrow{U_P \varepsilon_P} U_P \\
\downarrow{e^T U_P} & & \downarrow{U_P \phi_T} \\
T U_P = U_P \iota^* \phi_T U_P \xrightarrow{U_P \eta_U} U_P \iota^* \iota_1
\end{array}
\]
commute. Since $U_p e_p \cdot e^p U_p = 1$ by one of the triangular identities for the adjunction $\phi_p \dashv U_p$, it follows that $U_p \eta$ is the composite

$$U_p \xrightarrow{e^p U_p} T U_p = U_p \varepsilon^p \phi_T U_p \xrightarrow{U_p \varepsilon^p} U_p \varepsilon^p \nu_A.$$ 

In particular, $U_p \eta \phi_p$ is the composite

$$P \xrightarrow{e^P} TP = U_p \varepsilon^p \phi_T U_p \phi_p \xrightarrow{U_p \varepsilon^p \phi_p} U_p \varepsilon^p \nu_A \phi_p = U_p \varepsilon^p \phi_T = U_T \phi_T.$$ 

Since, by Remark 2.17(1), $q \phi_p$ is the composite

$$\phi_T P \xrightarrow{\phi_T \eta_T} \phi_T T \xrightarrow{e^T \phi_T} \phi_T,$$

$U_p \eta \phi_p$ is the composite

$$P \xrightarrow{e^P} TP \xrightarrow{T \iota} TT \xrightarrow{m^T} T.$$ 

Since, by functoriality of composition, $T \iota \cdot e^T P = e^T T \cdot \iota$ and since $m^T \cdot e^T = 1$, one concludes that $U_p \eta \phi_p = \iota$.

Now since for any $(a, h) \in A_T$, $(e_\iota)(a, h) = h$, one concludes that $\gamma(a, h)$ is the composite $P(a) \xrightarrow{\iota} T(a) \xrightarrow{h} a$. Now, since by Remark 2.17

$$\text{TP}(a) \xrightarrow{T(m^a_\iota)} \text{TP}(a) \xrightarrow{T(\iota_a)} TT(a) \xrightarrow{m^T_\iota} T(a)$$

is the coequaliser defining $\iota_1(P(a), m^a_\iota) = \iota_1(\phi_p(a))$, it follows that $\iota_1(\gamma(a, h)) = (t \phi_p)(a, h)$ is the unique morphism making the diagram

$$\text{TP}(a) \xrightarrow{T(\iota_a)} TT(a) \xrightarrow{m^T_\iota} T(a)$$

commute. Since $\iota_a \cdot e^a_\iota = e^a_T$ and $m^T_\iota \cdot T(e^a_\iota) = 1 = h \cdot T(e^a_T)$, it follows from the commutativity of the above diagram that $(t \phi_p)(a, h) = \iota_1(\gamma(a, h)) = q \phi(a, h)$, as claimed. $\square$

5. Weak entwinings

Let $H$ be an endofunctor on any category $\mathcal{A}$, admitting both a monad $H = (H, m, e)$ and a comonad $\overline{H} = (H, \delta, \varepsilon)$ structure, and define

$$(5.1) \quad \sigma : HH \xrightarrow{\delta H} HHH \xrightarrow{Hm} HH, \quad \overline{\sigma} : HH \xrightarrow{H \delta} HHH \xrightarrow{mH} HH.$$ 

The class $\text{Nat}(H, H)$ of all natural transformations from $H$ to itself allows for the structure of a monoid by defining the (convolution) product of any two $\phi, \phi' \in \text{Nat}(H, H)$ as the composite $\phi \ast \phi' = m \cdot \phi \phi' \cdot \delta$. The identity for this product is $e \cdot \varepsilon : H \rightarrow H$.

Recall that weak entwinings of tensor functors were defined by Caenepeel and De Groot in [15] and a more general theory was formulated by Böhm (e.g. [9] Example 5.2).
5.1. **Weak monad comonad entwinings.** For a natural transformation $\omega : HH \to HH$, define the natural transformations
\[
\begin{align*}
\xi & : H \xrightarrow{\epsilon_H} HH \xrightarrow{\omega} HH \xrightarrow{\epsilon_H} H, \\
\kappa & : HH \xrightarrow{\epsilon_{HH}} HHH \xrightarrow{\omega} HHH \xrightarrow{Hm} HH.
\end{align*}
\]

$(H, \overline{H}, \omega)$ is called a **weak entwining** (from the monad $H$ to the comonad $\overline{H}$) provided
\[
(5.2) \quad \begin{align*}
(i) & \quad \omega \cdot m_H = Hm \cdot \omega H \cdot H \delta, \\
(ii) & \quad \omega \cdot e_H = H \xi \cdot \delta, \quad \varepsilon_H \cdot \omega = H \xi \cdot \delta,
\end{align*}
\]
and is said to be **compatible** if
\[
(5.3) \quad \delta \cdot m = Hm \cdot \omega H \cdot H \delta.
\]

It is easily checked that
\[
(5.4) \quad \begin{align*}
\kappa \cdot He = \omega \cdot e_H, \quad \varepsilon_H \cdot \kappa = m \cdot \xi H, \quad \text{always hold,} \\
\xi \ast \xi = \xi, \quad \kappa \cdot \kappa = \kappa, \quad \kappa \cdot \omega = \omega, \quad \text{follow by (5.2)(i),} \\
\kappa \cdot \delta = \delta, \quad \kappa \cdot \sigma = \sigma, \quad \xi \ast 1 = 1, \quad \text{follow by (5.3).}
\end{align*}
\]

5.2. **Mixed bimodules.** We write $\mathbb{A}_{H}^{\overline{H}}(\omega)$ for the category of mixed $H$–bimodules, whose objects are triples $(a, h, \theta)$, where $(a, h) \in \mathbb{A}_{H}$, $(a, \theta) \in \mathbb{A}_{\overline{H}}$ with commutative diagram

\[
\begin{array}{ccc}
H(a) & \xrightarrow{h} & a \\
\downarrow{H(\theta)} & & \downarrow{H(h)} \\
HH(a) & \xrightarrow{\omega_a} & HH(a)
\end{array}
\]

and whose morphisms are those in $\mathbb{A}$ which are $H$–module as well as $\overline{H}$–comodule morphisms.

The following is a particular case of [6, Proposition 5.7].

5.3. **Proposition.** Let $H = (H, \overline{H}, \omega)$ be a weak entwining on $\mathbb{A}$. Then the composite
\[
\Gamma : HU_{H} \xrightarrow{e_{HU_H}} HHU_H \xrightarrow{\omega_{HU_H}} HHU_H = HU_H \phi_{H} U_H \xrightarrow{HU_{U_H} e_{H}} HU_H
\]
is an idempotent, and if
\[
\begin{array}{ccc}
HU_H & \xrightarrow{\Gamma} & HU_H \\
\downarrow{p} & & \downarrow{i} \\
G & & & \overline{G}
\end{array}
\]
is its splitting, then there is a comonad $G = (\overline{G}, \overline{\delta}, \overline{\varepsilon})$ on $\mathbb{A}_{H}$, whose functor part takes an arbitrary $(a, h) \in \mathbb{A}_{H}$ to
\[
(G(a, h), p(a,h) \cdot H(h) \cdot \omega_a \cdot H(i(a,h)) : HG(a, h) \to G(a, h)),
\]
and whose comultiplication $\overline{\delta}$ and counit $\overline{\varepsilon}$ evaluated at $(a, h)$ are the composites, respectively,
\[
G(a, h) \xrightarrow{i(a,h)} H(a) \xrightarrow{h_a} HH(a) \xrightarrow{H(p(a,h))} HG(a, h) \xrightarrow{PG(a,h)} GG(a, h),
\]

We call $G$ the **comonad induced by** $H = (H, \overline{H}, \omega)$. Obviously, $U_H \overline{G} = G$. 
5.4. Theorem. Let $H = (\overline{H}, \omega, \vartheta)$ be a weak entwining on a Cauchy complete category $\mathcal{A}$ and $G$ the induced comonad on $\mathcal{A}_H$. Then there is an isomorphism of categories

$$\Phi : \mathcal{A}_H(\omega) \to (\mathcal{A}_H)_G, \quad (a, h, \theta) \mapsto ((a, h), p(a, h) \cdot \theta),$$

with the inverse given by $\Phi^{-1}((a, h), \zeta) = (a, h, i(a, h) \cdot \zeta)$.

Proof. Since $\pi = 1$, it is clear that $\Phi \Phi^{-1} = 1$. To show that $\Phi^{-1} \Phi = 1$, consider an arbitrary object $(a, h, \theta) \in \mathcal{A}_H(\omega)$. In the diagram

```
\begin{array}{c}
a \\
\downarrow e_a \\
H(a) \\
\downarrow H(\theta) \\
\downarrow \omega \downarrow H(\theta) \\
H(a) \\
\downarrow \omega \downarrow H(\theta) \\
H(\omega) \\
\downarrow \omega \downarrow H(\theta) \\
H(a) \\
\downarrow \omega \downarrow H(\theta) \\
H(a)
\end{array}
```

the square commutes by naturality of $e$, while the trapezium commutes since $(a, h, \theta) \in \mathcal{A}_H(\omega)$. Since $h \cdot e = 1$, this means

$$\theta = \Gamma_{(a, h)} \cdot \theta = i(a, h) \cdot p(a, h) \cdot \theta.$$

Thus $\Phi^{-1} \Phi(a, h, \theta) = (a, h, \theta)$, that is, $\Phi^{-1} \Phi = 1$. \qed

Again by [5, Proposition 5.7], we get as counterpart of Proposition 5.3.

5.5. Proposition. Let $H = (\overline{H}, \omega, \vartheta)$ be a weak entwining on $\mathcal{A}$. Then

$$\Gamma' : HU^\overline{H} \longrightarrow HU^\overline{H} \varphi^\overline{H}U^\overline{H} = HHU^\overline{H} \varphi^\overline{H}U^\overline{H} \longrightarrow HU^\overline{H} \varphi^\overline{H}U^\overline{H}$$

is an idempotent, and if

```
\begin{array}{c}
HUU^\overline{H} \\
\downarrow \varphi^\overline{H} \\
\overline{T} \\
\downarrow \varphi^\overline{H} \\
HUU^\overline{H}
\end{array}
```

is its splitting, then there is a comonad $\overline{T} = (\overline{T}, \overline{m}, \overline{c})$ on $\mathcal{A}_H$, whose functor part takes an arbitrary $(a, \theta) \in \mathcal{A}_H$ to

$$(T(a, \theta), H(p'_{(a, \theta)}) \cdot \omega_a \cdot H(\theta) \cdot i'_{(a, \theta)} : T(a, \theta) \to HT(a, \theta)),$$

and whose multiplication $\overline{m}$ and unit $\overline{c}$, evaluated at an $\overline{H}$-comodule $(a, \theta)$, are the composites, respectively,

$$TT(a, \theta) \xrightarrow{i'_{(a, \theta)}} HT(a, \theta) \xrightarrow{H(i'_{(a, \theta)})} HH(a) \xrightarrow{m_a} H(a) \xrightarrow{\varphi^\overline{H}_{(a, \theta)}} T(a, \theta),$$

$$a \xrightarrow{\theta} H(a) \xrightarrow{\varphi^\overline{H}_{(a, \theta)}} T(a, \theta).$$

We call $\overline{T}$ the monad induced by $H = (\overline{H}, \overline{m}, \overline{c})$. Obviously, $U^\overline{H} \overline{T} = T$.

5.6. Theorem. Let $H = (\overline{H}, \omega, \vartheta)$ be a weak entwining on a Cauchy complete category $\mathcal{A}$ and $T$ the induced monad on $\mathcal{A}_H$. Then there is an isomorphism of categories

$$\Phi' : \mathcal{A}_H(\omega) \to (\mathcal{A}_H)_T, \quad (a, h, \theta) \mapsto ((a, \theta), (a, h) \cdot i'_{(a, \theta)}),$$

with the inverse given by $(\Phi')^{-1}((a, \theta), g) = (a, g \cdot i'_{(a, h)}, \theta)$.
5.7. Comparison functors. Let $H = (H, H, \omega)$ be a compatible weak entwining on $A$. By (5.3), there is a functor (e.g. [19, Lemma 5.1])

$$K_\omega : A \to \bar{A}_{H}(\omega), \quad a \mapsto (H(a), m_a, \delta_a).$$

Precomposing $K_\omega$ with $\Phi$ and $\Phi'$ gives functors

$$K : A \to (\bar{A}_{H})^G, \quad a \mapsto ((H(a), m_a), p_{(H(a), m_a)} \cdot \delta_a),$$

$$K' : A \to (\bar{A}_{H})^T, \quad a \mapsto ((H(a), \delta_a), m_a \cdot i'(H(a), m_a)),$$

leading to commutative diagrams

$$\begin{array}{ccc}
\bar{A}_{H} & \xrightarrow{K} & (\bar{A}_{H})^G \\
\phi_{H} \downarrow & & \downarrow \Phi \\
A_H & = & G_{\bar{A}_{H}}
\end{array}$$

$$\begin{array}{ccc}
\bar{A}_{H} & \xrightarrow{K'} & (\bar{A}_{H})^T \\
\phi'_{H} \downarrow & & \downarrow \Phi' \\
A_H & = & T_{\bar{A}_{H}}(\omega).
\end{array}$$

We will use that the splitting of $\Gamma$ (from 5.3) leads to a splitting of $\kappa$,

$$\begin{array}{ccc}
HH & \xrightarrow{\kappa} & HH \\
\overline{\pi} = p \phi_{H} \downarrow & & \overline{\pi} \leftarrow i \phi_{H} \\
\overline{G} = G_{\phi_{H}} & & \bar{G}
\end{array}$$

Symmetrically, the splitting of $\Gamma'$ (from 5.5) leads to a splitting of $\kappa'$,

$$\begin{array}{ccc}
HH & \xrightarrow{\kappa'} & HH \\
\overline{\pi}' = p' \phi_{H} \downarrow & & \overline{\pi}' \leftarrow i' \phi_{H} \\
\overline{T} = T_{\phi_{H}} & & \bar{T}
\end{array}$$

where $\kappa'$ is the composite $HH \xrightarrow{H^H} HHH \xrightarrow{H^\omega} HHH \xrightarrow{m} HH$.

5.8. Proposition. In the situation described above, consider the comonad morphism $t : \phi_{H} U_{H} \to G$ induced by the left triangle in (5.7) (see Section 4.7).

1. For any $H$-module $(a, h)$, the $(a, h)$-component for $t$ is the composite

$$t_{(a, h)} : H(a) \xrightarrow{\delta_a} HH(a) \xrightarrow{H(h)} H(a) \xrightarrow{p_{(a, h)}} G(a, h).$$

2. For any $a \in A$, the $\phi_{H}(a)$-component for $t$,

$$t_{\phi_{H}(a)} : HH(a) \xrightarrow{\delta_{H(a)}} HHH(a) \xrightarrow{H_{m_{a}}} HH(a) \xrightarrow{\overline{\pi}_{a}} \overline{G}(a),$$

is the unique morphism leading to commutativity of the diagram

$$\begin{array}{ccc}
HH(a) & \xrightarrow{t_{\phi_{H}(a)}} & \overline{G}(a) \\
\sigma_a \downarrow & & \downarrow \overline{t}_{a} \\
H(a) & & H(a).
\end{array}$$
Proof. (1) Since \( K(a) = ((H(a), m_a), p_{(H(a), m_a)} \cdot \delta_a) \), the left \( G \)-comodule structure \( \alpha : \phi_H \to G\phi_H \) on \( \phi_H \) corresponding to the left triangle in (5.5), has for its \( a \)-component \( \alpha_a = p_{(H(a), m_a)} \cdot \delta_a \). It then follows from Section 2.15 that the functor \( K : \mathcal{A} \to \mathcal{G} \) induced by the right triangle in (5.5) (see Section 4.1), has for its \( a \)-component \( t_{(a,h)} \) the composite

\[
t_{(a,h)} : H(a) \xrightarrow{\delta_a} H(a) \xrightarrow{p_{(H(a), m_a)}} G(H(a), m_a) \xrightarrow{G(h)} G(a, h),
\]

which, by naturality of composition, is the same as

\[
t_{(a,h)} : H(a) \xrightarrow{\delta_a} H(a) \xrightarrow{H(h)} H(a) \xrightarrow{p_{(a,h)}} G(a, h).
\]

Then, in particular, \( t_{\phi_H(a)} = p_{\phi_H(a)} \cdot H(m_a) \cdot \delta_{H(a)} = \overline{p}_a \cdot \sigma_a \).

(2) Since \( \overline{G} \xrightarrow{\kappa} H \xrightarrow{1} H \) is an equaliser diagram and \( \kappa \cdot \sigma = \sigma \) (see (5.4)), there is a unique morphism \( j : H \to \overline{G} \) such that \( \overline{\kappa} \cdot j = \sigma \). Then \( t_{\phi_H(a)} = \overline{p}_a \cdot \sigma_a = \overline{p}_a \cdot \overline{\kappa} \cdot \overline{\kappa} \cdot j = j_a \) and the result follows. \( \square \)

Symmetrically, we have:

5.9. Proposition. In the situation described in (5.7) the monad morphism \( t : T \to \varphi^{U_H} \) induced by the right triangle in (5.5) (see Section 4.1), has for its \( (a, \theta) \)-component

\[
t_{(a,\theta)} : T(a, \theta) \xrightarrow{\hat{t}_{(a,\theta)}} H(a) \xrightarrow{H(\theta)} H(H(a)) \xrightarrow{m_a} H(a).
\]

Our general results from Section 2 now yield:

5.10. Proposition. Let \( H = (H, \overline{H}, \omega) \) be a compatible weak entwining on \( \mathcal{A} \). Then the functor \( K : \mathcal{A} \to (\mathcal{A}_H)^\mathcal{G} \) (and hence also \( K_\omega : \mathcal{A} \to \mathcal{A}_H(\overline{H}) \)) has a right adjoint if and only if, for any \( (a, h, \theta) \in \mathcal{A}_H(\overline{H}) \), the pair of morphisms

\[
(5.8) \quad a \xleftarrow{e_a} H(a) \xrightarrow{\delta_a} H(a) \xrightarrow{H(h)} H(a)
\]

has an equaliser in \( \mathcal{A} \).

Proof. Since the functor \( U^G : (\mathcal{A}_H)^\mathcal{G} \to \mathcal{A} \) is clearly (pre)comonadic, it follows from Theorem 2.15 that the functor \( K : \mathcal{A} \to (\mathcal{A}_H)^\mathcal{G} \) admits a right adjoint if and only if for any \( ((a, h), \nu) \in (\mathcal{A}_H)^\mathcal{G} \), the pair

\[
a \xleftarrow{\beta_{(a,h)}} G(a, h),
\]

where \( \beta : U_H \to U_HG \) is the right \( G \)-comodule structure on \( U_H : \mathcal{A}_H \to \mathcal{A} \) induced by the triangle (5.5) (see Section 4.1), has an equaliser in \( \mathcal{A} \), which – since \( i : G \to HU_H \) is a (split) monomorphism – is the case if and only if the pair

\[
a \xleftarrow{i_{(a,h)} \cdot \nu} H(a)
\]

has one. According to Propositions 5.8 and 4.4 \( \beta_{(a,h)} \) is the composite

\[
a \xleftarrow{e_a} H(a) \xrightarrow{\delta_a} H(a) \xrightarrow{p_{(H(a), m_a)}} G(H(a), m_a) \xrightarrow{G(h)} G(a, h).
\]
Since \( \kappa \cdot \delta = \delta \) by (5.4) and \( \Gamma = i \cdot p \), it follows by naturality of \( i \) that the diagram

\[
\begin{array}{c}
\begin{array}{cccccccccccc}
a & \xrightarrow{\epsilon_a} & H(a) & \xrightarrow{\delta_a} & HH(a) & \xrightarrow{P(H(a),ma)} & G(H(a),ma) & \xrightarrow{G(h)} & G(a,h) \\
& & & & & & & & \\
\downarrow{\kappa_a} & & \downarrow{\Gamma(H(a),ma)} & & \downarrow{i(H(a),ma)} & & \downarrow{i(a,h)} & & \downarrow{H(h)} & & \downarrow{H(a)} \\
& & & & & & & & & & & & HH(a) \\
\end{array}
\end{array}
\]

is commutative. So we have

\[
i_{(a,h)} \cdot \beta_{(a,h)} = H(h) \cdot \delta_a \cdot e_a.
\]

Thus, the functor \( K : \mathcal{A} \to (\mathcal{A}_H)^G \) has a right adjoint if and only if for any \( ((a, h), \nu) \in (\mathcal{A}_H)^G \), the pair of morphisms

\[
a \xrightarrow{i_{(a,h)} \cdot \nu} H(a)
\]

has an equaliser. Recalling that \( \Phi : \mathcal{A}_H^H(\omega) \to (\mathcal{A}_H)^G \) is an isomorphism of categories and \( \Phi^{-1}((a, h), \nu) = (a, h, i(a,h) \cdot \nu) \) gives the desired result.

Symmetrically, we have:

5.11. **Proposition.** Let \( H = (H, \mathcal{T}, \omega) \) be a compatible weak entwining on \( \mathcal{A} \). Then the functor \( K' : \mathcal{A} \to (\mathcal{A}_H)^T \) (and hence also \( K_\omega : \mathcal{A} \to \mathcal{A}_H^H(\omega) \)) has a left adjoint if and only if for any \( (a, h, \theta) \in \mathcal{A}_H^H(\omega) \), the pair of morphisms

\[
\begin{array}{c}
\begin{array}{cccccccccccc}
H(a) & \xrightarrow{H(\theta)} & HH(a) & \xrightarrow{ma} & H(a) & \xrightarrow{\epsilon_a} & a \\
\end{array}
\end{array}
\]

has a coequaliser in \( \mathcal{A} \).

Symmetric to 5.1 one may consider

5.12. **Weak comonad monad entwinings.** For a natural transformation \( \varpi : \mathcal{T}H \to \mathcal{H} \), define the natural transformation

\[
\xi : \begin{array}{c}
\begin{array}{cccccccccccc}
H & \xrightarrow{He} & HH & \xrightarrow{\varpi H} & H \mathcal{H} & \xrightarrow{H\varepsilon} & \mathcal{H} \\
\end{array}
\end{array}
\]

\((\mathcal{T}, \mathcal{H}, \varpi)\) is called a weak entwining (from comonad \( \mathcal{T} \) to monad \( \mathcal{H} \)) provided

\[
\begin{array}{c}
\begin{array}{cccccccccccc}
(i) & \varpi \cdot Hm = mH \cdot H\varpi \cdot \varpi H, & H\delta \cdot \varpi = \varpi H \cdot H\varpi \cdot \delta H, \\
(ii) & \varpi \cdot He = \xi H \cdot \delta, & H\varepsilon \cdot \varpi = \xi H \cdot \delta, \\
\end{array}
\end{array}
\]

and is said to be compatible if

\[
\delta \cdot m = mH \cdot H\varpi \cdot \delta H.
\]

Here we get

\[
\xi \cdot \xi = \xi, \quad 1 \cdot \xi = 1.
\]
Certainly, the theory for this notion will be similar to that for monad comonad entwinings. However, the mixed bimodules (as in [5,2]) do not play the same role here but are to be replaced by liftings to Kleisli categories. Nevertheless, comonad monad entwinings will enter the picture in the next section.

6. Weak braided bimonads

In the theory of Hopf algebras $H$ over a field $k$, the twist map for $k$-vector spaces $M, N$, $\text{tw}_{M,N} : M \otimes_k N \to N \otimes_k M$, plays a crucial part. In particular it helps to commute $H \otimes_k -$ with itself by $\text{tw}_{H,H} : H \otimes_k H \to H \otimes_k H$. Generalising this to monoidal categories, often a braiding is required, that is, a condition on the whole category. It was observed (e.g. in [24]) that it can be enough to have such a twist only for the functor $H$ under consideration, that is, a natural isomorphism $\tau : HH \to HH$ satisfying the Yang-Baxter equation. For the study of weak braided Hopf algebras, Alonso Álvarez e.a. suggested in [1, Definition 1.2] to consider, for any object $D$ in a monoidal category, a weak Yang-Baxter operator $t_{D,D} : D \otimes D \to D \otimes D$, which is not necessarily invertible but only regular. Here we take up this notion and formulate it for any functor on an arbitrary category.

6.1. Weak Yang Baxter operator. Given an endofunctor $H : \mathcal{A} \to \mathcal{A}$, a pair of natural transformations $\tau, \tau' : HH \to HH$ is said to be a weak YB-pair provided the following equalities hold:

$$\tau \cdot \tau' \cdot \tau = \tau, \quad \tau' \cdot \tau \cdot \tau' = \tau', \quad \tau \cdot \tau' = \tau' \cdot \tau,$$

(6.2)

$$H\tau \cdot \tau H \cdot H\tau = \tau H \cdot H\tau \cdot \tau H,$$

$$H\tau' \cdot \tau' H \cdot H\tau' = \tau' H \cdot H\tau' \cdot \tau' H,'$$

and for $\nabla := \tau \cdot \tau'$,

(6.3)

$$\tau H \cdot H \nabla = H \nabla \cdot \tau H, \quad H\tau \cdot \nabla H = H\tau \cdot \nabla H,$$

$$\tau' H \cdot \nabla H = H \nabla \cdot \tau' H, \quad H\nabla' \cdot \nabla H = H \nabla' \cdot \nabla H.$$

The conditions in (6.2) are the usual Yang-Baxter equations for $\tau$ and $\tau'$, respectively. The equations in (6.3) are obviously satisfied if $\tau' = \tau^{-1}$ and in this case $\tau$ is known as Yang-Baxter operator.

6.2. Definition. Let $H = (H, m, e)$ a monad, $\overline{H} = (H, \delta, \varepsilon)$ a comonad on $\mathcal{A}$, and $\tau, \tau' : HH \to HH$ a weak YB-pair with $\nabla := \tau \cdot \tau'$. The triple $H = (H, \overline{H}, \tau)$ is called a weak braided bimonad (or weak $\tau$-bimonad) provided

1. $m \cdot \nabla = m, \quad \nabla \cdot \delta = \delta$;
2. $\nabla \cdot He = \tau \cdot eH, \quad H\varepsilon \cdot \nabla = \varepsilon \cdot H \nabla, \quad H \nabla \cdot eH = \tau \cdot He, \quad \varepsilon H \cdot \nabla = H \varepsilon \cdot \tau$;
3. $\delta H \cdot \tau = H\tau \cdot \tau H \cdot H\delta$, $\tau \cdot mH = Hm \cdot \tau H \cdot H\tau$;
4. $H\delta \cdot \tau = \tau H \cdot H\tau \cdot \delta H$, $\tau \cdot Hm = mH \cdot H\tau \cdot \tau H$;
5. $\delta \cdot m = mm \cdot H\tau H \cdot \delta$;
6. $\varepsilon \varepsilon \cdot mm \cdot H\delta H = \varepsilon \cdot m \cdot mH = \varepsilon \varepsilon \cdot mm \cdot H\tau' H \cdot H\delta H$;
7. $HmH \cdot \delta \cdot ee = \delta H \cdot \delta \cdot e = HmH \cdot H\tau' H \cdot \delta \delta \cdot ee$.

For vector space categories and (finite dimensional) tensor functors $H \otimes -$ with $\tau$ the twist map, these conditions were introduced in [11, Definition 1]. For monoidal categories and monoidal functors the conditions are those for a weak braided bialgebra introduced and studied by Alonso Álvarez e.a. [1] and we can - and will - freely use essential parts of their results in our situation. Note that if $\nabla$ is the identity of $H$, the conditions (1)–(4) in the definition describe the invertible double entwinings considered in [23].

The following observations provide the key to our previous results.
6.3. Entwinings. Given the data from Definition 6.2, define
\[ \omega : \text{HH} \xrightarrow{\delta H} \text{HHH} \xrightarrow{H \tau} \text{HH} \xrightarrow{mH} \text{HH}, \]
\[ \overline{\omega} : \text{HH} \xrightarrow{H \delta} \text{HHH} \xrightarrow{\tau H} \text{HH} \xrightarrow{Hm} \text{HH}. \]
From the Sections 5.1 and 5.12 we get the natural transformations
\[ \xi : H \xrightarrow{\epsilon H} \text{HH} \xrightarrow{\omega} \text{HH} \xrightarrow{\epsilon H} H, \]
\[ \overline{\xi} : H \xrightarrow{H \epsilon} \text{HH} \xrightarrow{\omega} \text{HH} \xrightarrow{H \epsilon} H, \]
and recalling \( \sigma \) and \( \overline{\sigma} \) from 5.1, we define
\[ \chi : H \xrightarrow{\epsilon H} \text{HH} \xrightarrow{\sigma} \text{HH} \xrightarrow{H \epsilon} H, \]
\[ \overline{\chi} : H \xrightarrow{H \epsilon} \text{HH} \xrightarrow{\sigma} \text{HH} \xrightarrow{H \epsilon} H. \]

With these notions we collect the basic identities proved in [2].

6.4. Proposition. Let \( H = (H, \overline{H}, \tau) \) be a weak braided bimonad on \( \mathcal{A} \). Then

1. \( \xi, \overline{\xi}, \chi \) and \( \overline{\chi} \) are idempotent (w.r.t. composition) and respect unit and counit of \( H \);
2. \( \xi \cdot m \cdot H \xi = \xi \cdot m \) and \( \overline{\xi} \cdot m \cdot \overline{\xi} H = \overline{\xi} \cdot m \);
3. \( H \xi \cdot \delta \cdot \xi = \delta \cdot \xi \) and \( \overline{\xi} H \cdot \delta \cdot \overline{\xi} = \delta \cdot \overline{\xi} \);
4. \( \sigma \cdot \epsilon H = \chi H \cdot \delta \) and \( \overline{\sigma} \cdot H \epsilon = H \overline{\chi} \cdot \delta \);
5. \( H \epsilon \cdot \sigma = m \cdot H \chi \) and \( \epsilon H \cdot \overline{\sigma} = m \cdot \overline{\chi} H \);
6. \( \xi \cdot \chi = \xi, \xi \cdot \overline{\chi} = \overline{\chi}, \chi \cdot \xi = \chi, \overline{\chi} \cdot \xi = \xi, \xi \cdot \overline{\xi} = \xi, \overline{\chi} \cdot \overline{\xi} = \overline{\xi}. \)

Proof. (1) is shown in [2, Proposition 2.9]; (2), (3) are from [2, Proposition 2.14]; (4) is shown in [2, Proposition 2.6], (5) in [2, Proposition 2.4], and (6) in [2, Proposition 2.10]. \( \square \)

The following shows the way to apply results from the preceding section.

6.5. Proposition. Let \( H = (H, \overline{H}, \tau) \) be a weak braided bimonad. Then

- \( (H, \overline{H}, \omega) \) is a compatible weak (monad comonad) entwining;
- \( (\overline{H}, H, \overline{\omega}) \) is a compatible weak (comonad monad) entwining.

Proof. As easily seen, condition (5) in 6.2 yield the equalities (5.3) and (5.11) and also implies (5.2(i)) and (5.10(i)) for \( \omega \) and \( \overline{\omega} \), respectively (e.g. [8], [23]). Now Propositions 2.3 and 2.5 in [2] show the equations in (5.2)(ii) and (5.10)(ii). \( \square \)

Direct inspection yields the technical observation:

6.6. Lemma. Suppose that \( f, g : X \to X \) are idempotents in an arbitrary category such that \( fg = g \) and \( gf = f \). If \( X \xrightarrow{f} X \xrightarrow{g} X \) (resp. \( X \xrightarrow{g} X \xrightarrow{f} X \)) is a splitting of the idempotent \( f \) (resp. \( g \)), then

\[ X_g \xrightarrow{i_g} X \xrightarrow{f} X \xrightarrow{1} X \]

is a (split) equaliser diagram, while

\[ X \xrightarrow{g} X \xrightarrow{1} X \xrightarrow{p_g} X_g \]

is a (split) coequaliser diagram.
Henceforth we work over a Cauchy complete category $\mathcal{A}$ and suppose that
\[
(6.4) \quad H \xrightarrow{\xi} H \\
\xymatrix{ & \bar{\xi} \ar[dl] \ar[d] \ar[dr] & \\
H & H & H \ar[l] \ar[l] \ar[l]}
\]
is a splitting of $\bar{\xi}$.

6.7. Proposition. In the situation of Proposition 5.10, the diagram
\[
(6.5) \quad H \bar{\xi} \xrightarrow{\xi} H \xrightarrow{\delta} H \xrightarrow{\chi H} H H \\
\xymatrix{ & H \bar{\xi} \ar[dl] \ar[d] \ar[dr] & \\
H & H H & H \ar[l] \ar[l] \ar[l]}
\]
is a (split) equaliser in $[\mathcal{A}, \mathcal{A}]$.

Proof. Since $Hm \cdot \delta H \cdot eH = \chi H \cdot \delta$ by Proposition 6.4 (4), we have to show that the diagram
\[
\xymatrix{ & H \bar{\xi} \ar[dl] \ar[d] \ar[dr] & \\
H & H H & H \ar[l] \ar[l] \ar[l]}
\]
is a split equaliser. Let us first show that the pair
\[
(6.6) \quad H \xrightarrow{\delta} H H \xrightarrow{\chi H} H H \\
\xymatrix{ & H \ar[dl] \ar[d] \ar[dr] \ar[l] & \\
H & H H & H \ar[l] \ar[l] \ar[l]}
\]
is cosplit by the morphism $H \varepsilon : H H \to H$. Indeed, since $H \varepsilon \cdot \delta = 1$ and $H \varepsilon \cdot \chi H \cdot \delta = \chi \cdot H \varepsilon \cdot \delta = \chi$, it remains to show that $\chi H \cdot \delta \cdot \chi = \delta \cdot \chi$. For this, consider the diagram
\[
\xymatrix{ & H \ar[dl] \ar[d] \ar[dr] \ar[l] & \\
H H & H H H & H \ar[l] \ar[l] \ar[l]}
\]
in which the
- regions (1), (2), (5) and (6) commute by naturality of composition;
- region (3) commutes by coassociativity of $\delta$;
- region (4) commutes by Proposition 6.4 (5);
the curved regions commute by \(6.4\) (4).

Hence the whole diagram commutes, implying

\[
\chi H \cdot \delta \cdot \chi = \delta \cdot H \varepsilon \cdot \chi H \cdot \delta = \delta \cdot \chi.
\]

So the pair \((6.6)\) is cosplit by the morphism \(H \varepsilon\) and hence one finds its equaliser by splitting the idempotent \(\chi = H \varepsilon \cdot \chi H \cdot \delta\). But since \(\bar{\xi} \cdot \chi = \chi\) and \(\chi \cdot \bar{\xi} = \bar{\xi}\) by Proposition \(6.4\) (6) and \(i\bar{\xi} \cdot q\bar{\xi}\) is the splitting of the idempotent \(\bar{\xi}\) (see \((6.4)\)), it follows from Lemma \(6.6\) that \((6.5)\) is a (split) equaliser diagram.

\[\square\]

Symmetrically, we have:

6.8. Proposition. In the situation of Proposition \(5.11\) the diagram

\[
\begin{array}{c}
   H \xrightarrow{\delta} HH \xrightarrow{m \varepsilon} HH \xrightarrow{H \varepsilon} H \\
   \downarrow H \xrightarrow{\delta} HH \xrightarrow{m \varepsilon} HH \xrightarrow{H \varepsilon} H \\
   \downarrow H \xrightarrow{\delta} HH \xrightarrow{m \varepsilon} HH \xrightarrow{H \varepsilon} H
\end{array}
\]

is a (split) coequaliser diagram in \([\mathcal{A}, \mathcal{A}]\).

Since \(Hm \cdot \delta H \cdot eH = \chi H \cdot \delta\) and \(\varepsilon H \cdot mH \cdot H \delta = m \cdot \chi H\) by Proposition \(6.4\) (4) and (5), Propositions \(6.7\) and \(6.8\) immediately yield:

6.9. Corollary. Let \(H = (\mathcal{H}, \mathcal{H}, \tau)\) be a weak braided bimonad \(\mathcal{A}\). Then

\[
\begin{array}{c}
   H \xrightarrow{i\bar{\xi}} H \delta \xrightarrow{\chi H} HH \\
   \downarrow \bar{\xi} \xrightarrow{\chi H} \downarrow \bar{\xi} \xrightarrow{\chi H} \downarrow \bar{\xi}
\end{array}
\]

is a (split) equaliser yielding the monad

\[
H\bar{\xi} = (H\bar{\xi}, m\bar{\xi}, e\bar{\xi}) \quad \text{with} \quad m\bar{\xi} = q\bar{\xi} \cdot m \cdot (i\bar{\xi} \bar{\xi}) \quad \text{and} \quad e\bar{\xi} = q\bar{\xi} \cdot e,
\]

and

\[
\begin{array}{c}
   H \xrightarrow{m \varepsilon} HH \xrightarrow{H \varepsilon} H \bar{\xi} \\
   \downarrow H \xrightarrow{m \varepsilon} HH \xrightarrow{H \varepsilon} H \bar{\xi} \\
   \downarrow H \xrightarrow{m \varepsilon} HH \xrightarrow{H \varepsilon} H \bar{\xi}
\end{array}
\]

is a (split) coequaliser yielding the comonad

\[
\overline{H\xi} = (H\bar{\xi}, \delta \bar{\xi}, \varepsilon \bar{\xi}) \quad \text{with} \quad \overline{H\xi} = H\bar{\xi}, \delta \bar{\xi} = (q\bar{\xi} \bar{\xi}) \cdot \delta \cdot i\bar{\xi} \quad \text{and} \quad \varepsilon \bar{\xi} = \varepsilon \cdot i\bar{\xi}.
\]

The next result provides the technical data to show Frobenius separability.

6.10. Lemma. Let \(H = (\mathcal{H}, \mathcal{H}, \tau)\) be a weak braided bimonad and consider the composite

\[
v : 1 \xrightarrow{\varepsilon} H \xrightarrow{\delta} HH \xrightarrow{q\bar{\xi}} H\bar{\xi} H\bar{\xi}.
\]

For this, \(m\bar{\xi} \cdot v = e\bar{\xi}\) and one has commutativity of the diagrams

\[
\begin{array}{c}
   H\bar{\xi} \xrightarrow{\varepsilon} H \xrightarrow{\delta} HH \xrightarrow{q\bar{\xi}} H\bar{\xi} \xrightarrow{m \varepsilon} H\bar{\xi} H\bar{\xi} \\
   \downarrow \bar{\xi} \xrightarrow{\delta} \downarrow \bar{\xi} \xrightarrow{\delta} \downarrow \bar{\xi} \xrightarrow{\delta}
\end{array}
\]

\[
\begin{array}{c}
   H\bar{\xi} \xrightarrow{i\bar{\xi}} H \xrightarrow{\delta} HH \xrightarrow{q\bar{\xi}} H\bar{\xi} \xrightarrow{m \varepsilon} H\bar{\xi} H\bar{\xi} \\
   \downarrow \bar{\xi} \xrightarrow{\delta} \downarrow \bar{\xi} \xrightarrow{\delta} \downarrow \bar{\xi} \xrightarrow{\delta}
\end{array}
\]

\[
\begin{array}{c}
   H\bar{\xi} \xrightarrow{i\bar{\xi}} H \xrightarrow{\delta} HH \xrightarrow{q\bar{\xi}} H\bar{\xi} \xrightarrow{m \varepsilon} H\bar{\xi} H\bar{\xi} \\
   \downarrow \bar{\xi} \xrightarrow{\delta} \downarrow \bar{\xi} \xrightarrow{\delta} \downarrow \bar{\xi} \xrightarrow{\delta}
\end{array}
\]

\[
\begin{array}{c}
   H\bar{\xi} \xrightarrow{i\bar{\xi}} H \xrightarrow{\delta} HH \xrightarrow{q\bar{\xi}} H\bar{\xi} \xrightarrow{m \varepsilon} H\bar{\xi} H\bar{\xi} \\
   \downarrow \bar{\xi} \xrightarrow{\delta} \downarrow \bar{\xi} \xrightarrow{\delta} \downarrow \bar{\xi} \xrightarrow{\delta}
\end{array}
\]
**Proof.** The diagram

\[
\begin{array}{ccccccccc}
1 & e & H & \delta & HH & HHe & HHH & HHH & HHH \\
\downarrow H & \downarrow He & \downarrow \delta H & \downarrow mH & \downarrow He & \downarrow mH & \downarrow HHe & \downarrow HHH \\
H & HH & HH & HHH & HH & HH & HH & HH & HH \\
\end{array}
\]

commutes by naturality and since \(\varpi\) is an entwining (Proposition 6.5). Now the equation \(\xi \cdot m \cdot \xi H = \xi \cdot m\) (Proposition 6.4(2)) yields the commutative diagram

\[
\begin{array}{ccccccccc}
1 & e & H & \delta & HH & \xi & H & HH & HH \\
\downarrow \xi & \downarrow H \xi & \downarrow \delta H & \downarrow mH & \downarrow H H & \downarrow H H & \downarrow H H & \downarrow H H \\
H & H & H & H & H & H & H & H & H \\
\end{array}
\]

and the splitting \(\xi = \iota \cdot q\xi\) implies \(m\xi \cdot \nu = e\xi\).

In the diagram

\[
\begin{array}{ccccccccc}
H\xi & eH\xi & HH\xi & \delta H\xi & HHH\xi & q\xi HH\xi & HH\xi H\xi & H\xi H\xi & H\xi H\xi \\
\downarrow H & \downarrow H H & \downarrow \delta H & \downarrow HH \xi & \downarrow \xi H H & \downarrow H H H & \downarrow H H H & \downarrow H H H \\
H H & HH & HH & HHH & HH & HHH & HH & HH & HH \\
\end{array}
\]

- the triangle commutes by the splitting of \(\xi\);
- region (a) commutes by Proposition 6.4(4);
- region (b) commutes because Proposition 6.4(2) induces \(q\xi \cdot m \cdot \xi H = q\xi \cdot m\);
- the other regions commute by naturality.

Since \(\chi H \cdot \delta \cdot \iota \xi = \delta \cdot \iota \xi\) by Proposition 6.7 we obtain from this commutativity of the left hand diagram.
Next, in the diagram

![Diagram](image-url)

- the top triangle commutes by the splitting of $\xi$;
- region (a) commutes since from Proposition 6.4(1) and (3) we get
  $$\xi H \cdot \delta \cdot e = \xi H \cdot \delta \cdot e = \delta \cdot e = \delta \cdot e;$$
- region (b) commutes by Proposition 6.4(4);
- the other regions commute by naturality.

Thus the whole diagram is commutative, and in the light of the equality $q\xi \cdot \chi = q\xi$, derived from $\xi \cdot \chi = \xi$ in Proposition 6.4(6), commutativity of the right hand diagram follows. □

The following generalises [27, Proposition 4.2] and [30, Proposition 1.6] (see also [9, Proposition 4.4]).

6.11. **Proposition.** Let $H = (H, H, \tau)$ be a weak braided bimonad on $\mathcal{A}$. Then the quintuple $(H^\xi, m^\xi, e^\xi; \delta^\xi, \epsilon^\xi)$ is a Frobenius separable (co)monad.

**Proof.** Using the results of Lemma 6.10 it is easy to verify that the diagram corresponding to (3.1) is commutative and that $m^\xi \cdot \delta^\xi = 1_H^\xi$.

The statement about the comonad $H^\xi$ holds by symmetry. □

6.12. **Proposition.** Let $H = (H, H, \tau)$ be a weak braided bimonad on $\mathcal{A}$ and suppose the functor $K_\omega : \mathcal{A} \to K_H^\omega(\omega)$ admits a right adjoint $K : K_H^\omega(\omega) \to \mathcal{A}$. Then $H^\xi = (H^\xi, m^\xi, e^\xi)$ is the monad generated by the adjunction $K_\omega \dashv K$ and $\iota^\xi$ is the corresponding morphism of monads $H^\xi \to H$ (as in Proposition 4.5).

**Proof.** Suppose there is an adjunction

$$\eta, \tau : K_\omega \dashv K : K_H^\omega(\omega) \to \mathcal{A};$$

consider the monad $(K K_\omega, K \tau \epsilon K_\omega, \eta)$ it generates on $\mathcal{A}$. Write $\iota : K K_\omega \to H$ for the corresponding morphism of monads (as in Proposition 4.5). Since, by Proposition 5.10 the functor $K_\omega : \mathcal{A} \to K_H^\omega(\omega)$ admits a right adjoint if and only if for every $(a, h, \theta) \in K_H^\omega(\omega)$, the pair of morphisms $5.8$ has an equaliser in $\mathcal{A}$, and since $K_\omega(a) = (H(a), m_a, \delta_a)$ for all $a \in \mathcal{A}$, it follows from Proposition 6.7 that $K K_\omega = H^\xi$ and that $\iota^\xi = \iota$. 

□
According to Theorem 2.15, the unit of the adjunction $K_\omega \dashv K$ is the (unique) natural transformation $\eta: 1 \to \overline{KK}_\omega = \overline{H}$ such that the diagram
\[
\begin{array}{c}
H\overline{H}(a) \\
\downarrow \overline{\eta}_a \\
\overline{H}(a)
\end{array}
\begin{array}{c}
\overline{\varepsilon}_a \\
\uparrow e_a \\
a
\end{array}
\begin{array}{c}
\overline{H}(a)
\end{array}
\]
commutes, for all $a \in \mathcal{A}$. It then follows that $\overline{\eta}_a = q_{\overline{a}} \cdot i_{\overline{a}} \cdot \overline{\eta}_a = q_{\overline{a}} \cdot e_a$.

Next, since for any $a \in \mathcal{A}$, $\overline{\varepsilon}_{K_\omega(a)} = m_a \cdot H(i_{\overline{a}})$ by (4.4), it follows that
\[
\overline{K}(\overline{\varepsilon}_{K_\omega(a)}) : \overline{KK}_\omega K_\omega(a) = H\overline{H}(\overline{a}) \to \overline{KK}_\omega(a) = H\overline{a}
\]
is the unique morphism making the diagram
\[
\begin{array}{c}
H\overline{H}(\overline{a}) \\
\downarrow \overline{\eta}_a \\
\overline{H}(a)
\end{array}
\begin{array}{c}
\overline{\varepsilon}_a \\
\uparrow e_a \\
a
\end{array}
\begin{array}{c}
\overline{H}(a)
\end{array}
\]
commute and we calculate (recall (6.4))
\[
\overline{K}(\overline{\varepsilon}_{K_\omega(a)}) = q_{\overline{a}} \cdot i_{\overline{a}} \cdot \overline{K}(\overline{\varepsilon}_{K_\omega(a)}) = q_{\overline{a}} \cdot m_a \cdot H(i_{\overline{a}}) \cdot i_{\overline{a}} H\overline{K}_\omega(a) = q_{\overline{a}} \cdot m_a \cdot (i_{\overline{a}} i_{\overline{a}}).
\]
This completes the proof. 

By symmetry, we also have:

6.13. Proposition. Let $H = (H, \overline{H}, \tau)$ be a weak braided bimonad on $\mathcal{H}$ and suppose the functor $K_\omega : \mathcal{H} \to \overline{\mathcal{H}}(\omega)$ admits a left adjoint $K : \overline{\mathcal{H}}(\omega) \to \mathcal{H}$. Then $\overline{K}(\overline{\varepsilon}_H) = (H\overline{H}, \delta_H, \varepsilon_H)$ is the comonad generated by the adjunction $K \dashv K_\omega$ and $q_\overline{H}$ is a comonad morphism $H \to \overline{H}$.

According to Section 2.7, the monad morphism $\overline{i} : H\overline{H} \to H$ equips $H$ with an $H\overline{H}$-bimodule structure, where the left action is the composite
\[
\rho_l : H\overline{H} \overline{i} \overline{H} \to HH \overset{m}{\to} H,
\]
and the right action is the composite
\[
\rho_r : HH\overline{H} \overset{H\overline{i}}{\to} HH \overset{m}{\to} H.
\]

6.14. Proposition. For a weak braided bimonad $H = (H, \overline{H}, \tau)$ on $\mathcal{H}$, $\sigma = Hm \cdot \delta H$ coequalises the pair $(\rho_r H, H \rho_l)$, i.e.,
\[
\sigma \cdot \rho_r H = \sigma \cdot H \rho_l.
\]

Proof. Since (6.5) is an equaliser diagram and since in Figure 1
- Diagram (1) commutes since $e$ is the unit of the monad $T$;
- Diagrams (2) and (10) commute by Proposition 6.5;
- Diagrams (3), (7) and (9) commute by associativity of $m$;
- Diagrams (4), (5), (6) and (8) commute by functoriality of composition,
one sees that $\sigma \cdot \rho_r H = H_m \cdot \delta H \cdot H m \cdot H \xi H = \sigma \cdot H \rho_l$.

Suppose now that the tensor product $H \otimes_{H^\xi} H$ exists, i.e., there is a coequaliser diagram

\begin{equation}
H H \xrightarrow{\rho_r H} H H \xrightarrow{l} H \otimes_{H^\xi} H,
\end{equation}

where $l = \text{can}_{H, H}^{H^\xi}$. Note that, by Proposition 2.9, $H \otimes_{H^\xi} H$ has a right $H$-module structure such that $l$ is a morphism of right $H$-modules. Moreover, since $\sigma$ coequalises $\rho_r H$ and $H \rho_l$, the composite $\overline{\rho} \cdot \sigma : HH \to \overline{G}$ (see (5.6)) also coequalises them, and since Diagram (6.7) is a coequaliser, there exists a unique natural transformation $\gamma : H \otimes_{H^\xi} H \to \overline{G}$ making the diagram

\begin{equation}
H H \xrightarrow{\rho_r H} H H \xrightarrow{l} H \otimes_{H^\xi} H
\end{equation}

commute. It follows – since $U_{H^\xi H} \phi U_{H^\xi H} = \overline{\rho} \cdot \sigma$ by Proposition 5.8 – that the diagram

\begin{equation}
H H \xrightarrow{\sigma} H H \xrightarrow{l} H \otimes_{H^\xi} H \xrightarrow{\overline{\rho} \cdot \sigma} \overline{G}
\end{equation}

commutes. Precomposing this square with $He$ and using $\sigma \cdot He = \delta$ (e.g. [24, 5.2]), we get

\begin{equation}
\overline{\rho} \cdot \delta = \gamma \cdot l \cdot He.
\end{equation}
6.15. *$H$ as $H^\xi$-bicomodule.* The comonad morphism $q^\xi : \mathbb{H} \to \mathbb{H}^\xi$ equips $H$ with an $H^\xi \mathbb{H}$-bicomodule structure, where the left coaction is the composite

$$\theta_l : H \xrightarrow{\delta} HH \xrightarrow{\varphi H} H^\xi H,$$

and the right coaction is the composite

$$\theta_r : H \xrightarrow{\delta} HH \xrightarrow{Hq^\xi} HH^\xi.$$

Moreover, there is a unique natural transformation $\gamma' : T \to H \otimes H^\xi H$ making the triangle in the diagram (6.11) commute. Here $T'$ and $T$ are as defined in the diagram (5.7).

6.16. **Proposition.** Let $H = (H, \varphi, \tau)$ be a weak braided bimonad on $\mathbb{A}$. Viewing $H$ as a right $H^\xi \mathbb{H}$-module by the structure map $HH^\xi \xrightarrow{\varphi^\xi} HH \xrightarrow{m} H$, then $q^\xi : H \to H^\xi$ is a morphism of right $H^\xi \mathbb{H}$-modules.

**Proof.** For this, we have to show commutativity of the diagram

$$HH^\xi \xrightarrow{q^\xi H} H^\xi HH \xrightarrow{m^\xi} H^\xi.$$

Since $m^\xi = q^\xi \cdot m \cdot (\iota^\xi \iota^\xi)$, this diagram can be rewritten as

and in this diagram the triangle commutes by naturality of composition, and the trapezoid commutes, since $\iota^\xi$ is a (split) monomorphism and $\iota^\xi \cdot m \cdot \iota^\xi H = \iota^\xi \cdot m$ by Proposition 6.4(2). This completes the proof. $\square$
Consider now the diagram

\[
\begin{array}{ccc}
H \otimes \xi H & \xrightarrow{\rho_1 H} & HH \\
\downarrow q^e H & & \uparrow l \\
H^e \xi \xi H & \xrightarrow{m^e H} & HH
\end{array}
\]

in which \(q^e H \cdot \rho_1 = m^e H \cdot q^e \xi H\), since \(q^e\) is a morphism of right \(\xi\)-modules by Proposition 6.16 \(q^e H \cdot H \rho_l = H^e \rho_l \cdot H^e \xi H\), because of naturality of composition, and the bottom row is a split coequaliser, since the pair \((H, \rho_l)\) is a left \(\xi\)-module (see Remark 2.17 (1)). It then follows

6.17. **Proposition.** Let \(H = (\underline{H}, \tau)\) be a weak braided bimonad on \(\mathbb{A}\). In the situation described above, there is a unique morphism \(\tilde{q} : H \otimes \xi H \to H\) making the trapezoid in the diagram (6.12) commutative and this is a morphism of right \(H\)-modules.

**Proof.** According to Section 2.7, the morphisms \(q^e H, \xi H\) and \(m\) are morphisms of right \(H\)-modules. Then the composite \(m \cdot \tilde{\tau} \cdot q^e = \rho_l \cdot q^e\) (and hence also \(\tilde{q} \cdot l\)) is a morphism of right \(H\)-modules, implying – since \(l\) and \(H\) are both epimorphisms of right \(H\)-modules – that \(\tilde{q}\) is also a morphism of right \(H\)-modules. \(\Box\)

6.18. **Proposition.** Suppose \(\gamma : H \otimes \xi H \to \xi \) in (6.8) to be an epimorphism. If the morphisms \(f, g : H \to H\) are such that \(f \cdot 1 = g \cdot 1\), then \(f \cdot \xi = g \cdot \xi\).

**Proof.** If \(f, g : H \to H\) are morphisms such that \(f \cdot 1 = g \cdot 1\), then

\[
m \cdot f H \cdot \sigma = m \cdot g H \cdot \sigma
\]

and since \(\sigma \cdot He = \delta\) (e.g. [24 (5.2)]), we have

\[
m \cdot f H \cdot \sigma \cdot He = m \cdot g H \cdot \sigma \cdot He.
\]

According to Section 2.7, \(fH\) and \(gH\) can be seen as morphisms of the right \(H\)-module \((HH, Hm)\) to itself, while \(m\) is a morphism from the right \(H\)-module \((HH, Hm)\) to the \(H\)-module \((H, m)\). Moreover, \(\sigma\) is also a morphism of right \(H\)-modules (e.g. [24 Section 5.1]). Thus the composites \(m \cdot f H \cdot \sigma\) and \(m \cdot g H \cdot \sigma\) both are morphisms of right \(H\)-modules. It then follows from the right version of [24 Lemma 3.2] that

\[
m \cdot f H \cdot \sigma = m \cdot g H \cdot \sigma.
\]

Next, since \(\sigma = \kappa \cdot \sigma = \tilde{\tau} \cdot \overline{\sigma}\) and \(\overline{\sigma} \cdot \sigma = \gamma \cdot l\) (by (6.9)), we have

\[
m \cdot f H \cdot \tilde{\tau} \cdot \gamma \cdot l = m \cdot g H \cdot \tilde{\tau} \cdot \gamma \cdot l,
\]

and \(l\) and \(\gamma\) being epimorphisms we get \(m \cdot f H \cdot \tilde{\tau} = m \cdot g H \cdot \tilde{\tau}\), thus

\[
m \cdot f H \cdot \kappa = m \cdot g H \cdot \kappa.
\]

Recalling that \(\kappa \cdot He = \omega \cdot e H\) and \(\omega \cdot e H = H \xi \cdot \delta\) (see (5.4), (5.2)(ii)), we get

\[
m \cdot f H \cdot \kappa \cdot He = m \cdot f H \cdot \omega \cdot e H = m \cdot f H \cdot H \xi \cdot \delta = f \cdot \xi,
\]

and similarly, one derives \(m \cdot g H \cdot \kappa \cdot He = g \cdot \xi\). Thus, \(f \cdot \xi = g \cdot \xi\). \(\Box\)

Since \(\kappa\) is (clearly) a morphism of right \(H\)-modules, Proposition 2.4 yields
6.19. **Proposition.** The composite $\overline{G}H \xrightarrow{\overline{p}} HHH \xrightarrow{Hm} HH \xrightarrow{i} G$ makes $\overline{G}$ into a right $H$-module such that $\overline{p} : HH \to \overline{G}$ and $i : G \to HH$ both are morphisms of right $H$-modules.

6.20. **Corollary.** $\gamma : H \otimes_H \tilde{H} \to \overline{G}$ is a morphism of right $H$-modules.

**Proof.** Since, in Diagram (6.9), the morphisms $\sigma, p$ and $l$ are all morphisms of right $H$-modules (see [24, 5.1] and Proposition 6.19), and since $l$ and $lH$ are both epimorphisms, it follows that $\gamma$ is also a morphism of right $H$-modules. $\square$

7. **Weak braided Hopf monads**

In this section, we define a weak antipode for weak braided bimonads $H = (H, \overline{H}, \tau)$ and formulate various forms of a Fundamental Theorem. The definition corresponds to that in [11], [2] and in other papers on (generalisations of) weak Hopf algebras. For the notations we refer to the preceding section.

7.1. **Definition.** Given a weak braided bimonad $H$, a natural transformation $S : H \to H$ is called a weak antipode, provided

\[ 1 \ast S = \xi, \quad S \ast 1 = \overline{\xi}, \quad S \ast 1 \ast S = S. \]

Since $\xi \ast 1 = 1 = 1 \ast \xi$ (see (5.4), (5.12)), we also get $1 \ast S \ast 1 = 1$.

A weak braided bimonad $H$ with a weak antipode is called a weak braided Hopf monad or a weak $\tau$-Hopf monad.

7.2. **Proposition.** Let $H = (H, \overline{H}, \tau)$ be a weak braided Hopf monad on $\mathbb{A}$. Then the functor $K_\omega : \mathbb{A} \to \mathbb{A}_{H}(\omega)$ admits a right adjoint $K : \mathbb{A}_{H}(\omega) \to \mathbb{A}$.

**Proof.** Suppose that $H$ has an antipode $S$. By Proposition 5.10 we have to show that for any $(a, h, \theta) \in \mathbb{A}_H(\omega)$, the pair (5.8) has an equaliser. We claim that the pair is cosplit by the composite $d : H(a) \xrightarrow{S_a} H(a) \xrightarrow{h} a$. Indeed, in the proof of [2 Proposition 3.5 (ii)] it is shown that

\[ H(h) \cdot \delta_a \cdot e_a \cdot d \cdot \theta = \theta \cdot d \cdot \theta. \]

It remains to prove that

\[ d \cdot H(h) \cdot \delta_a \cdot e_a = 1. \]

For this, consider the diagram

\[
\begin{array}{ccccccccc}
 a & \xrightarrow{e_a} & H(a) & \xrightarrow{\delta_a} & HH(a) & \xrightarrow{H(h)} & H(a) \\
 & & \Uparrow{\overline{S_a}} & & & & \Uparrow{S_a}
\end{array}
\]

\[
\begin{array}{ccccccccc}
 H(a) & \xrightarrow{H(h)} & H(a) \\
 & \xrightarrow{H(h)} & a, \\
 & & \Uparrow{h}
\end{array}
\]

\[
\begin{array}{ccccccccc}
 HH(a) & \xrightarrow{H(h)} & H(a) \\
 & \xrightarrow{H(h)} & H(a) \\
 & \xrightarrow{H(h)} & a, \\
 & & \Uparrow{h}
\end{array}
\]

in which

- the left triangle commutes by Proposition 6.4 (1);
- the right triangle commutes because $S$ is an antipode;
- the top square commutes by naturality of $S$;
- the bottom square commutes since $(a, h) \in \mathbb{A}_H$. 

7.3. Proposition. Let $H = (H, \overline{H}, \tau)$ be a weak braided Hopf monad on $\mathcal{A}$. Then the functor $K_\omega : \mathcal{A} \to \mathcal{A}_H(\omega)$ admits a left adjoint $\overline{K} : \mathcal{A}_H(\omega) \to \mathcal{A}$ that takes $(a, h, \theta) \in \mathcal{A}_H(\omega)$ to the object $\overline{a}$ which splits the idempotent $a \overset{a}{\to} H(a) \overset{S_a}{\to} H(a) \overset{h}{\to} a$. Moreover, the diagram

$$
\begin{align*}
H(a) & \xrightarrow{h} HH(a) & \overset{m_a}{\xrightarrow{}} & H(a) & \xrightarrow{\epsilon_a} & a & \overset{q_a}{\to} & \overline{a}
\end{align*}
$$

is a (split) coequaliser.

7.4. Proposition. Let $H = (H, \overline{H}, \tau)$ be a weak braided Hopf monad on a Cauchy complete category $\mathcal{A}$. Then, for any $(a, h, \theta) \in \mathcal{A}_H(\omega)$, the diagram

$$
\begin{align*}
a & \xrightarrow{\theta} H(a) \\
\downarrow & \quad \quad \quad \downarrow H(\beta_a) \\
a & \xrightarrow{h} H(a)
\end{align*}
$$

with the idempotent $\beta_a : a \overset{\theta}{\to} H(a) \overset{S_a}{\to} H(a) \overset{h}{\to} a$ (see proof of Proposition 7.2) is commutative.

Proof. We compute

\[
\begin{align*}
h \cdot H(\beta_a) \cdot \theta &= h \cdot H(h) \cdot H(S_a) \cdot H(\theta) \cdot \theta \\
&= h \cdot m_a \cdot H(S_a) \cdot \delta \cdot \theta \\
&= h \cdot e_{H(a)} \cdot \omega_a \cdot e_{H(a)} \cdot \theta \\
&= \epsilon_a \cdot H(h) \cdot \omega_a \cdot H(\theta) \cdot e_a \\
&= \epsilon_a \cdot \theta \cdot h \cdot e_a = 1.
\end{align*}
\]

The second and sixth equations hold since $(a, h) \in \mathcal{A}_H$ and $(a, \theta) \in \mathcal{A}_H$, the third one holds by the definition of an antipode, the forth one by naturality of $e$ and $\epsilon$, and the fifth one by the fact that $(a, h, \theta) \in \mathcal{A}_H(\omega)$. 

7.5. Proposition. Let $H = (H, \overline{H}, \tau)$ be a weak braided Hopf monad on a Cauchy complete category $\mathcal{A}$. Then

1. $K_\omega : \mathcal{A} \to \mathcal{A}_H(\omega)$ admits both a left and a right adjoint $\overline{K}, K : \mathcal{A}_H(\omega) \to \mathcal{A}$;
2. the unit $\eta : 1 \to KK$ of $K \dashv K$ is a split monomorphism, while the counit $\varepsilon : K\overline{K} \to 1$ of $K \dashv \overline{K}$ is a split epimorphism.
Proof. According to Propositions 7.2 and 7.3, \( K_\omega : \mathcal{A} \to \mathcal{A}_H(H(\omega)) \) admits both left and right adjoints \( K \rightleftarrows K \) and they can be constructed as follows. For any object \((a, h, \theta) \in \mathcal{A}_H(H(\omega))\), \( \beta_a : a \xrightarrow{\theta} H(a) \xrightarrow{\delta_{\theta}} H_h(a) \xrightarrow{\epsilon_{\theta}} a \) is idempotent (see proof of Proposition 7.2) with a splitting \( a \xrightarrow{\eta_a} a \xrightarrow{\iota_a} a \) and \( K(a, h, \theta) = K_\omega(a, h, \theta) = a \).

Moreover,

\[
\begin{array}{cccccc}
H(a) & \xrightarrow{H(\theta)} & H H(a) & \xrightarrow{m_a} & H(a) & \xrightarrow{\epsilon_a} & a \xrightarrow{q_a} \bar{a}
\end{array}
\]

is the defining coequaliser diagram for \( K(a, h, \theta) \), while

\[
\begin{array}{ccccccc}
a \xrightarrow{\iota_a} & H(a) & \xrightarrow{H(\theta)} & H H(a) & \xrightarrow{\delta_a} & H(h) H(a) & \xrightarrow{H(h)} & H(a)
\end{array}
\]

is the defining equaliser diagram for \( K(a, h, \theta) \).

It is easy to verify directly, using (2.4) and (2.5), that \( \eta(a, h, \theta) = H(q_a) \cdot \theta \) and \( \varepsilon(a, h, \theta) = h \cdot H(\iota_a) \). Now, in view of Proposition 7.3, we compute

\[
\varepsilon(a, h, \theta) \cdot \eta(a, h, \theta) = h \cdot H(\iota_a) \cdot H(q_a) \cdot \theta = h \cdot H(\beta_a) \cdot \theta = 1.
\]

Thus \( \varepsilon(a, h, \theta) \) is a split epimorphism, while \( \eta(a, h, \theta) \) is a split monomorphism. \( \square \)

We are now ready to state and prove our main result. It subsumes the original version of the Fundamental Theorem proved for weak Hopf algebras over fields in \( \text{[11, Theorem 3.9]} \) as well as various generalisations, for example, for algebras over commutative rings in \( \text{[33, Theorem 5.12]}, \text{[14, Theorem 36.16]} \), for Hopf algebroid in \( \text{[7, Theorem 4.14]} \), and for weak braided Hopf algebras on monoidal categories \( \text{[2, Proposition 3.6]} \).

7.6. Fundamental Theorem. Let \( H = (H, \mathcal{H}, \tau) \) be a weak braided bimonad on a Cauchy complete category \( \mathcal{A} \), and \( \mathcal{T} \) and \( \mathcal{G} \) the monad and comonad induced on \( \mathcal{A}_H(H(\omega)) \) and on \( \mathcal{A}_H \), respectively. Then the following are equivalent:

(a) \( H \) is a weak braided Hopf monad;

(b) the functor \( K_\omega : \mathcal{A} \to \mathcal{A}_H(H(\omega)) \) admits both left and right adjoints, and the right adjoint is monadic;

(c) the functor \( K_\omega : \mathcal{A} \to \mathcal{A}_H(H(\omega)) \) admits both left and right adjoints, and the left adjoint is comonadic;

(d) the induced natural transformation \( \gamma : H \otimes_\mathcal{H} \mathcal{H} \to \mathcal{G} \) is an isomorphism;

(e) the induced natural transformation \( \gamma' : \mathcal{T} \to H \otimes_\mathcal{H} \mathcal{T} \mathcal{H} \) is an isomorphism.

Moreover, if the (equivalent) conditions above hold, then there is an equivalence of categories \( \mathcal{A}_H \simeq \mathcal{A}_H(H(\omega)) \).

Proof. (a) \( \Rightarrow \) (b). If \( H \) is a weak braided Hopf monad, then, by Propositions 7.2 and 7.3, the functor \( K_\omega : \mathcal{A} \to \mathcal{A}_H(H(\omega)) \) admits a right adjoint \( K \rightleftarrows K \) and a left adjoint \( K \). So it remains to prove that \( K \) is monadic. By Proposition 7.3, the counit of the adjunction \( K \rightleftarrows K \) is a
split epimorphism. Moreover, since \( A \) is assumed to be Cauchy complete, so is \( \mathcal{H}(\omega) \) by Proposition 2.23. Applying now the dual of [22] Proposition 3.16 gives that \( \overline{K} \) is monadic.

(b) \( \Rightarrow \) (d). \( \mathcal{H} \) is separable Frobenius by Proposition 6.11 and hence, by Proposition 3.2 the change-of-base functor \((i\overline{\xi})_! : \mathcal{H} \rightarrow \mathcal{H} \) exists. It then follows from Proposition 4.6 that the comparison functor \( K : \mathcal{H} \rightarrow (\mathcal{H}^G) \) admits a left adjoint \( L : \mathcal{H} \rightarrow (\mathcal{H}^G) \) such that \( U^GL = (i\overline{\xi})_! \). The situation is illustrated by the diagram

Next, from the proof of Proposition 3.2 we know that the defining coequaliser of \((i\overline{\xi})_!\),

\[
\phi_H^tU = \phi_H^tU\phi_H^tU\phi_H^t \xrightarrow{\phi_H^tU\phi_H^t} \phi_H^tU\phi_H^t \xrightarrow{\phi_H^t} \phi_H^tU\phi_H^t \xrightarrow{q} (i\overline{\xi})_!,
\]

where \( \rho \) is the composite \( \phi_H^tU\phi_H^t \xrightarrow{\phi_H^t} \phi_H^tH = \phi_H^tU\phi_H^t \xrightarrow{\phi_H^t} \phi_H^t \), is absolute (i.e., is preserved by any functor). It then follows from Remark 2.17 (2) that the tensor product \( H \otimes \mathcal{H} \) is just the functor \( UH(i\overline{\xi})_!(i\overline{\xi})^* \phi_H^t \). Moreover, \( l = UHq(i\overline{\xi})^* \phi_H^t \).

Since \( UH(i\overline{\xi})^* = UH \), it follows from Proposition 4.7 and Equation 4.8 that the diagram

\[
\phi_H^tU\phi_H^t(i\overline{\xi})^* = \phi_H^tU\phi_H^t \xrightarrow{q(i\overline{\xi})^*} (i\overline{\xi})_!(i\overline{\xi})^* \xrightarrow{t_K} G \rightarrow G
\]

commutes, and since \( UH \mathcal{G} = G \), \( G\phi_H = \mathcal{G} \), and (7.2) is absolute, the diagram

\[
HH \xrightarrow{l} H \otimes \mathcal{H} \xrightarrow{UHt_H\phi_H} \mathcal{H} \xrightarrow{\phi_H^t} \mathcal{H}
\]

commutes. Since \( l \) is an epimorphism and (6.9) is also commutative, it follows that \( \gamma = UHt_H\phi_H \).

Now, since \( \overline{K} \) is assumed to be monadic, the comparison functor \( K : (\mathcal{H}^G) \rightarrow (\mathcal{H}^G) \) is an equivalence, and hence its left adjoint \( L : \mathcal{H} \rightarrow (\mathcal{H}^G) \) is also an equivalence.
Applying Theorem 4.2 to the right commutative triangle in Diagram (7.1) gives that $t_{L,\bar{\eta}}$ is an isomorphism. Quite obviously, $\gamma = U_{H,\bar{\eta}}\phi_{H}$ is then also an isomorphism.

(d) $\Rightarrow$ (a). Suppose that the natural transformation $\gamma : H \otimes_{\bar{\eta}} H \to \bar{G}$ is an isomorphism. Then we claim that the composite

$$S : H \xrightarrow{H \epsilon} HH \xrightarrow{\bar{p}} \bar{G} \xrightarrow{\gamma^{-1}} H \otimes_{\bar{\eta}} H \xrightarrow{\bar{q}} H$$

is an antipode for $H$. Note first that by Propositions 6.17, 6.19 and Corollary 6.20, the composite

$$\bar{q}_1 : HH \xrightarrow{\bar{p}} \bar{G} \xrightarrow{\gamma^{-1}} H \otimes_{\bar{\eta}} H \xrightarrow{\bar{q}} H$$

is a morphism of right $H$-modules. To show that $S \ast 1 = \xi$, consider the diagram

\[
\begin{array}{ccc}
H & \xrightarrow{\delta} & HH \\
H \xrightarrow{He} & & HH \\
\end{array}
\]

\[
\begin{array}{ccc}
HH & \xrightarrow{H \epsilon H} & HHH \\
\xrightarrow{HeH} & & \xrightarrow{\bar{p}} \bar{G} \\
\xrightarrow{Hm} & & \xrightarrow{\gamma^{-1}} H \otimes_{\bar{\eta}} H \\
\xrightarrow{H \epsilon} & & \xrightarrow{\bar{q}} H \\
\end{array}
\]

in which

- the rectangle commutes since the $\bar{q}_1$ is a morphism of right $H$-modules;
- the triangle commutes since $e$ is the unit of the monad $H$;
- part (1) commutes since $\bar{p} \cdot \delta = \gamma \cdot l \cdot He$ by (6.10), and therefore $\gamma^{-1} \cdot \bar{p} \cdot \delta = l \cdot He$.

So the whole diagram commutes and we have $S \ast 1 = \bar{q} \cdot l \cdot He$.

Commutativity of the trapezoid in (6.12) shows $\bar{q} \cdot l = m \cdot \bar{\xi} H \cdot He = m \cdot He \cdot \bar{\xi} = \xi$ by naturality of composition, we get

$$S \ast 1 = \bar{q} \cdot l \cdot He = m \cdot \bar{\xi} H \cdot He = m \cdot He \cdot \bar{\xi} = \xi,$$

and from (6.12) we conclude $1 \ast S \ast 1 = 1 \ast \xi = 1$.

In the diagram

\[
\begin{array}{ccc}
H & \xrightarrow{\delta} & HH \\
H \xrightarrow{He} & & HH \\
\xrightarrow{H \epsilon} & & HH \\
\end{array}
\]

\[
\begin{array}{ccc}
HH & \xrightarrow{\epsilon} & HHH \\
\xrightarrow{H \epsilon H} & & \xrightarrow{\bar{p}} \bar{G} \\
\xrightarrow{Hm} & & \xrightarrow{\gamma^{-1}} H \otimes_{\bar{\eta}} H \\
\xrightarrow{\epsilon} & & \xrightarrow{\bar{q}} H \\
\end{array}
\]

- part (1) commutes (see (5.3) and (5.2)(i),
- the top squares commute by naturality of composition,
- the bottom square commutes since $\bar{q}_1$ is a morphism of right $H$-modules, and
- the triangle commutes since $e$ is the unit of the monad $H$.

Now, commutativity of the diagram and $\bar{p} \cdot \kappa = \bar{p} \cdot \bar{\eta} \cdot \bar{p} = \bar{p}$ give

$$S \ast \xi = m \cdot He \cdot \bar{q}_1 H \cdot He H \cdot \delta = \bar{q}_1 \cdot \kappa \cdot He$$

$$= \bar{q} \cdot \gamma^{-1} \cdot \bar{p} \cdot \kappa \cdot He = \bar{q} \cdot \gamma^{-1} \cdot \bar{p} \cdot He = S.$$
Recalling $1 \xi = 1 = \xi 1$ (from (5.12), (5.4)) yields

$$1 \ast S \ast 1 = 1 \ast \xi 1 = \xi \ast 1$$

and applying Lemma 6.18 shows $1 \ast S = 1 \ast \xi 1 = \xi \ast 1 = S \ast 1 \ast S = S$.

Thus, $S$ is an antipode and hence $H$ is a weak braided Hopf monad, as asserted.

The implications (a)$\Rightarrow$(c)$\Rightarrow$(e)$\Rightarrow$(a) hold by symmetry and the last assertion follows from the proof of the implication (b)$\Rightarrow$(d).

7.7. Remark. When a weak braided bimonad $H$ has an antipode $S$, it can be shown that the composite

$$G \xrightarrow{\xi} H \xrightarrow{\delta H} HH \xrightarrow{HSH} HHH \xrightarrow{Hm} HHH \xrightarrow{1} H \otimes H \xi H$$

is the two-sided inverse of $\gamma : H \otimes H \xi H \rightarrow G$, while the composite

$$H \otimes H \xi H \xrightarrow{can} HH \xrightarrow{H \delta} HHH \xrightarrow{HSH} HHH \xrightarrow{Hm} HHH \xrightarrow{\varphi'} H \xrightarrow{\mathcal{T}}$$

is the two-sided inverse of $\gamma' : \mathcal{T} \rightarrow H \otimes H \xi H$.

7.8. Theorem. Let $H = (H, \mathcal{H}, \tau)$ be a weak braided bimonad on a Cauchy complete category $\mathcal{A}$, and $\mathcal{T}$ and $G$ the monad and comonad it induces on $\mathcal{A}$ and $\mathcal{A} \mathcal{H}$, respectively.

(1) If $H$ preserves the existing coequalisers, the following are equivalent:

(a) $H$ is a weak braided Hopf monad;

(b) the functor $K_\omega : \mathcal{A} \rightarrow \mathcal{A} \mathcal{H}(\omega)$ admits a monadic right adjoint;

(c) the induced natural transformation $\gamma : H \otimes H \xi H \rightarrow G$ is an isomorphism.

(2) If $H$ preserves the existing equalisers, the following are equivalent:

(a) $H$ is a weak braided Hopf monad;

(b) the functor $K_\omega : \mathcal{A} \rightarrow \mathcal{A} \mathcal{H}(\omega)$ admits a comonadic left adjoint;

(c) the induced natural transformation $\gamma' : \mathcal{T} \rightarrow H \otimes H \xi H$ is an isomorphism.

Moreover, if the (equivalent) conditions above hold, then there is an equivalence of categories $\mathcal{A} \mathcal{H} \tau \simeq \mathcal{A} \mathcal{H}(\omega)$.

Proof. By symmetry, it suffices to prove (1). Then, in the light of Proposition 7.6 we need only to show that (b) implies (c). So suppose the functor $K_\omega : \mathcal{A} \rightarrow \mathcal{A} \mathcal{H}(\omega)$ to admit a monadic right adjoint $\mathcal{K}$. Then the comparison functor $K_{\mathcal{H} \tau} : (\mathcal{A} \mathcal{H})^G \rightarrow \mathcal{A} \mathcal{H} \tau$ is an equivalence and hence has a left adjoint left inverse $L_{\mathcal{H} \tau} : \mathcal{A} \mathcal{H} \tau \rightarrow (\mathcal{A} \mathcal{H})^G$. It then follows from Proposition 4.6 that $U^G L_{\mathcal{H} \tau} = (\xi \tau)$.

Since $H$ is assumed to preserve the existing coequalisers, the forgetful functor $U_T : \mathcal{A}_T \rightarrow \mathcal{A}$ also preserves the existing coequalisers (e.g. [12 Proposition 4.3.2]). Since colimits in functor categories are calculated componentwise, this implies that the image of the coequaliser (7.2) under the functor $[A, A \mathcal{H}] : [A, \mathcal{A} \mathcal{H}, \mathcal{A}] \rightarrow [A, \mathcal{H} \tau, \mathcal{A}]$ is again a coequaliser. Then $\gamma = U_{\mathcal{A} \mathcal{H} \tau} L_{\mathcal{H} \tau} \phi$ by exactly the same argument used in the proof of the implication (b)$\Rightarrow$(d) of Proposition 7.6.
Now, since $L_{HR}$ is an equivalence of categories and $U^G L_{HR} = (\xi^{-1})$, it follows by Theorem 4.2 that $t_{L_{HR}}$ (and hence also $\gamma$) is an isomorphism. Thus (b) implies (c), as required. \qed

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