Vacuum Rabi Splitting in Nanomechanical QED System with Nonlinear Resonator

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Abstract

Considering the intrinsic nonlinearity in a nanomechanical resonator coupled to a charge qubit, vacuum Rabi splitting effect is studied in a nanomechanical QED (qubit-resonator) system. A driven nonlinear Jaynes-Cummings model describes the dynamics of this qubit-resonator system. Using quantum regression theorem and master equation approach, we have calculated the two-time correlation spectrum. Also, numerical calculations confirm these analytical results.

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I. INTRODUCTION

In quantum optics [1] and quantum information, [2] the well known Jaynes-Cummings model, [3] one of the most important models, describes the light-matter interaction between a two-level quantum system (qubit) and a boson (resonator). Generally in cavity QED system, [4] nanomechanical QED system [5] and circuit QED system, [6] vacuum Rabi splitting effect has been used to describe the qubit with frequency \( \omega_q \) and driving frequency \( \omega_p \) and driving constant \( g \) respectively. The commutation relations, \([\sigma_-, \sigma_+] = \sigma_z\) and \([a, a^\dagger] = 1\), are satisfied. The last term in Eq. (1) is a classical drive with driving constant \( \xi \) and driving frequency \( \omega_p \). The \( g \) denotes the interaction strength between the qubit and the resonator.

The paper is organized as follows. In Sec. II, using a driven nonlinear Jaynes-Cummings model, we describe the dynamics of a qubit-resonator system consisting of a superconducting qubit and a nonlinear nanomechanical resonator. In Sec. III, the two-time correlation spectrum is calculated analytically. In Sec. IV, vacuum Rabi splitting effect is studied. Also numerical simulations confirm our analytical results. Finally, our conclusions are summarized.

II. MODEL

In nanomechanical QED system, consisting a superconducting qubit and a nanomechanical resonator, we can use a driven Jaynes-Cummings model to describe the dynamics of this qubit-resonator system. [15]

\[
H_{JC}^{\text{driven}} = \omega_q \sigma_+ \sigma_- + g (a \sigma_+ + a^\dagger \sigma_-) + \omega_c a^\dagger a - \xi \sin (\omega_p t) (a + a^\dagger) .
\] (1)

Here \( H_{JC}^{\text{driven}} \) is a driven Jaynes-Cummings type Hamiltonian. The lowering (raising) operator \( \sigma_- (\sigma_+) \) and the annihilation (creation) operator \( a (a^\dagger) \) are defined to describe the qubit with frequency \( \omega_q \) and the resonator with frequency \( \omega_c \) respectively. The commutation relations, \([\sigma_-, \sigma_+] = \sigma_z\) and \([a, a^\dagger] = 1\), are satisfied. The last term in Eq. (1) is a classical drive with driving constant \( \xi \) and driving frequency \( \omega_p \). The \( g \) denotes the interaction strength between the qubit and the resonator.

Considering the nonlinearity of nanomechanical resonator, nanomechanical resonator is not assumed to be an ideal resonator again. Moving into a frame with rotating frequency \( \omega_p \), the total Hamiltonian for this qubit-resonator system writes [13]

\[
H_{t} = \Delta_a \sigma_+ \sigma_- + g (a \sigma_+ + a^\dagger \sigma_-) + \Delta_c a^\dagger a - \xi \sigma_z (a + a^\dagger) + \chi a^\dagger a + \chi (a^\dagger a)^2 ,
\] (2)

which describes the dynamics of a driven nonlinear Jaynes-Cummings model. The linear part \( \chi a^\dagger a \) and nonlinear part \( \chi (a^\dagger a)^2 \) come from a quartic potential \( x^4 \). [20]

Here the nonlinearity parameter \( \chi \) is small, \( \chi \ll g \).
The weak driving \( \xi_0 \) will induce the transitions from the vacuum state \(|00\rangle\) to the other excited states \(|01\rangle \) and \(|10\rangle\). Under the lowest order perturbation theory, we need to consider only one-phonon excitation in the qubit-resonator system,

\[
N = j + k = 0, 1. \tag{9}
\]

The Hilbert space for the reduced density matrix \( \rho \) in Eq. (8) reduces into a smaller subspace with a truncated basis

\[
\{ |j, k\rangle, \ j + k = 0, 1 \}. \tag{10}
\]

Thus, in this truncated basis, the corresponding density matrix elements satisfying the master equation in Eq. (8) are

\[
\begin{align*}
\frac{d\rho_{00,00}}{dt} &= i\xi\rho_{01,00} - i\xi\rho_{00,01} + 2\kappa\rho_{01,01} + \gamma\rho_{10,10}, \\
\frac{d\rho_{01,00}}{dt} &= [i(\Delta_{e} + 2\chi) - \kappa]\rho_{00,00} + ig\rho_{00,01} + i\xi\rho_{01,01}, \\
\frac{d\rho_{00,10}}{dt} &= [i\Delta_{e} - \gamma]\rho_{00,10} + ig\rho_{00,01} + i\xi\rho_{01,01}, \\
\frac{d\rho_{01,01}}{dt} &= -2\kappa\rho_{01,01} + ig(\rho_{01,10} - \rho_{10,01}) + i\xi\rho_{00,00}, \\
\frac{d\rho_{01,10}}{dt} &= [i(\Delta_{e} + 2\chi) - \kappa - \frac{\gamma}{2}]\rho_{01,10} + ig(\rho_{01,01} - \rho_{10,10}) + i\xi\rho_{00,10}, \\
\frac{d\rho_{00,10}}{dt} &= -\gamma\rho_{10,10} + ig(\rho_{01,10} - \rho_{10,10}). \tag{11}
\end{align*}
\]

In the long-time limit, the system stays in a steady state, we can take \( \rho_{00,00} \sim 1 \) and \( \rho_{00,00} \gg \rho_{01,01} (\rho_{10,10}) \). Here can see that \( \rho_{00,01} (\rho_{01,10}) \) scales as the order of \( \xi \) and \( \rho_{01,01} (\rho_{10,10}) \) scales as the order of \( \xi^2 \). Keeping terms of the \( \xi \) and dropping the higher terms of the \( \xi^2 \), we get

\[
\begin{align*}
\frac{d\rho_{00,01}}{dt} &= [i(\Delta_{e} + 2\chi) - \kappa]\rho_{00,01} + ig\rho_{00,10} - i\xi\rho_{00,00}, \\
\frac{d\rho_{00,10}}{dt} &= [i\Delta_{e} - \gamma]\rho_{00,10} + ig\rho_{00,01} - i\xi\rho_{00,00}. \tag{12}
\end{align*}
\]

Applying Laplace transformation, we have solved

\[
\rho_{00,01}(\tau) = \mu(\tau)\rho_{00,01}^{ss} + \nu(\tau)\rho_{00,10}^{ss} - i\xi C(\tau)\rho_{00,00}^{ss}. \tag{13}
\]

Here the coefficients \( (\mu(\tau), \nu(\tau) \text{ and } C(\tau)) \) are depen-
\[ \mu(\tau) = \frac{\Gamma_1 + i\omega_1 + (i\Delta_n - \frac{\gamma}{2})e^{-\Gamma_1\tau}e^{-i\omega_1\tau}}{2G} + \frac{\Gamma_2 + i\omega_2 + (i\Delta_n - \frac{\gamma}{2})e^{-\Gamma_2\tau}e^{-i\omega_2\tau}}{2G} \]

\[ \nu(\tau) = g \left[ e^{-\Gamma_1\tau}e^{-i\omega_1\tau} + e^{-\Gamma_2\tau}e^{-i\omega_2\tau} \right] \]

\[ C(\tau) = \frac{1}{2G} \left[ \frac{\Gamma_1 + i\omega_1 + (i\Delta_n - \frac{\gamma}{2})}{\Gamma_1 + i\omega_1} (1 - e^{-\Gamma_1\tau}e^{-i\omega_1\tau}) \right. \]

\[ \left. - \frac{1}{2G} \frac{\Gamma_2 + i\omega_2 + (i\Delta_n - \frac{\gamma}{2})}{\Gamma_2 + i\omega_2} (1 - e^{-\Gamma_2\tau}e^{-i\omega_2\tau}) \right] \]

\[ \Gamma_n = \frac{1}{2} \left( \frac{\gamma}{2} + \kappa \right) + (-1)^n \text{Im} G, \quad (14) \]

\[ \omega_n = (-1)^{n+1} \text{Re} G - \frac{1}{2} \Delta_n - \frac{1}{2} (\Delta_n + 2\chi), \quad (15) \]

And

\[ G = \sqrt{g^2 - \frac{1}{4} \left[ i(\delta + 2\chi) + \left( \frac{\gamma}{2} - \kappa \right) \right]^2} \]

for \( n = 1, 2 \).

From the above results in Eq. (12), we can calculate the single-time function

\[ \langle a(\tau) \rangle = \text{Tr} \{ a(0) \rho(\tau) \} = \rho_{01,00}(\tau) \]

where \( \rho(0) = \rho^{ss} \) and the index \( ss \) means the steady state. Through some simple calculations, we obtain the steady solution of the density matrix element \( \rho_{01,01}(\rho_{01,10}) \),

\[ \rho_{00,01}^{ss} = \frac{\xi}{(\Delta_n + 2\chi)^2 + i(\delta + 2\chi)}, \]

\[ \rho_{00,10}^{ss} = -\frac{g}{(\Delta_n + i\frac{\gamma}{2})^2} \rho_{00,01}^{ss}. \]

Using the quantum regression theorem \[15, 25, 26\], we can obtain the two-time correlation function

\[ \langle a(\tau)a^\dagger(0) \rangle = \text{Tr} \{ a(\tau)a^\dagger(0) \rho(0) \} = \mu(\tau)^*. \]

The other two-time functions can be obtained as

\[ \langle a^\dagger(\tau)a(0) \rangle = -i\xi C(\tau) \rho_{01,00}^{ss}, \]

\[ \langle a^\dagger(\tau)a^\dagger(0) \rangle = 0 \]

and

\[ \langle a(\tau)a(0) \rangle = 0. \]

Based on the above results in Eqs. (18-19-20-21), the two-time correlation function for the induced electromotive force is obtained,

\[ \langle V(\tau)V(0) \rangle \propto \mu(\tau)^*. \]

In the limit of weak driving, we can neglect the terms \( i\xi \rho_{01,00}(0), \rho_{02,00}(0), \rho_{11,00}(0), \rho_{01,01}(0) \) and \( \rho_{01,10}(0) \) which are proportional to the \( \xi^2 \) in Eq. (22).

Using the formula,

\[ \int_0^\infty dt e^{-i(\omega - \omega_0)\tau} = \frac{\Gamma - i(\omega - \omega_0)}{(\omega - \omega_0)^2 + \Gamma^2}, \]

the correlation spectrum in Eq. (6) is calculated as

\[ S_V(\omega) \propto \frac{\text{Re} \eta_2}{(\omega - \omega_1)^2 + \Gamma_1^2} + \frac{\text{Re} \eta_2}{(\omega - \omega_2)^2 + \Gamma_2^2}. \]

Where some parameters are defined,

\[ \eta_n = \frac{\Gamma_n - i\omega_n + (-i\Delta_n - \frac{\gamma}{2})}{(\omega - \omega_n)^2 + 2iG^2} [\Gamma_n - i(\omega - \omega_n)] \]

for \( n = 1, 2 \).

**IV. VACUUM RABI SPLITTING**

Generally in vacuum Rabi splitting, the splitting frequency \( \Delta \omega \) provides the information of the coupling \( g \) between the qubit and the resonator. As seen in Eq. (23), nonlinearity parameter \( \chi \) modifies the decay rate \( \Gamma_1(\Gamma_2) \) and central frequency \( \omega_1(\omega_2) \) of two peaks in the spectrum \( S_V(\omega) \). And in the limit of weak driving, the \( \xi \) does not affect the spectrum \( S_V(\omega) \), it means that we can use this driven nonlinear Jaynes-Cummings model to characterize vacuum Rabi splitting effect very well. The corresponding splitting frequency between two peaks in the spectrum \( S_V(\omega) \) is

\[ \Delta \omega = |\omega_1 - \omega_2| = 2\text{Re} G \]

which is independent of the driving strength \( \xi \) and determined by the couple strength \( g \), the detuning \( \delta \), decay rate \( \kappa \) and nonlinearity parameter \( \chi \).

Assuming nanomechanical resonator as an ideal resonator, \( \chi = 0 \), the spectrum \( S_V(\omega) \) in Eq. (23) will be same as the previous results. When the resonant condition (\( \delta = 0 \)) and the strong-coupling limit (\( \delta' > |\kappa, \gamma| \)) are adopted, we obtain the well-known splitting frequency \[4\]

\[ \Delta \omega \approx 2g. \]

To further clarify the dependence of correlation spectrum \( S_V(\omega) \) on nonlinearity parameter \( \chi \) and the driving strength \( \xi \) more clearly, the resonant condition is adopted, \( \Delta_n = \Delta_n = 1.0 \). The other parameters are \( g = 0.2, \kappa = 0.004, \gamma = 0.004 \) and 1 GHz is taken as the unit for all these parameters. \[15, 24, 27\]

Numerical calculations by QuTiP \[28\] are illustrated with some plots in the following. In Fig. [1] we choose the values of \( \xi = 0.02 \) and \( \chi = 0.01 \), the increasing of the number of total excitations \( N \) tells that the value of
$S_N = 10$ is suitable in the other plots of the spectrum. Other parameters are $\chi = 0.01$, $\xi = 0.02$. Then we take the value of $\xi = 0$. \ref{fig:corr} and \ref{fig:corr3} demonstrate the dependence of the spectrum $S_V(\omega)$ on the driving strength $\xi$ and nonlinearity parameter $\chi$. These numerical results agree with those analytical results in Eq. (23), both of them maintain that the weak driving strength $\xi$ does not change the splitting frequency $\Delta \omega$, nonlinearity parameter $\chi$ changes the heights of two peaks ($\text{Re}\eta_1$ and $\text{Re}\eta_2$) and the shifts of central frequency ($\omega_1$ and $\omega_2$) obviously.

\section{Conclusions}

In this paper, vacuum Rabi splitting effect is studied to provided the information of the coupling $g$. Considering the intrinsic nonlinearity in nanomechanical resonator, a driven nonlinear Jaynes-Cummings model is used to describe the dynamics of the qubit-resonator system. Using quantum regression theorem, the dissipative dynamics of the qubit-resonator system is solved by master equation approach. Here, the two-time correlation spectrum is analytically calculated to clarify the dependence of correlation spectrum on the driving strength $\xi$ and nonlinearity parameter $\chi$. Because of small nonlinearity in nanomechanical resonator, we find that nonlinearity parameter leads to the shifts of central frequency ($\omega_1$ and $\omega_2$) and does not change the splitting frequency $\Delta \omega$ obviously. In Fig. \ref{fig:corr} and Fig. \ref{fig:corr3} numerical results plotted by QuTiP agree with the analytical results in Eq. (23).

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