BRAUER GROUP OF THE MODULI SPACES OF STABLE VECTOR BUNDLES OF FIXED DETERMINANT OVER A SMOOTH CURVE

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Abstract. Let $X$ be an irreducible smooth projective curve, defined over an algebraically closed field $k$, of genus at least three and $L$ a line bundle on $X$. Let $\mathcal{M}_X(r, L)$ be the moduli space of stable vector bundles on $X$ of rank $r$ and determinant $L$ with $r \geq 2$. We prove that the Brauer group $\text{Br}(\mathcal{M}_X(r, L))$ is cyclic of order $g\cdot\text{c.d.}(r, \text{degree}(L))$. We also prove that $\text{Br}(\mathcal{M}_X(r, L))$ is generated by the class of the projective bundle obtained by restricting the universal projective bundle. These results were proved earlier in [BBGN] under the assumption that $k = \mathbb{C}$.

1. Introduction

Let $X$ be a compact connected Riemann surface of genus $g$, with $g \geq 3$. Fix a holomorphic line bundle $L$ over $X$ and also fix an integer $r \geq 2$. Let $\mathcal{M}_X(r, L)$ denote the moduli space of stable vector bundles on $X$ of rank $r$ and determinant $L$, which is a smooth quasiprojective complex variety of dimension $(r^2 - 1)(g - 1)$. There is a Poincaré vector bundle over $X \times \mathcal{M}_X(r, L)$ if and only if $r$ and $\text{degree}(L)$ are coprime [Ra]. When $r$ and $\text{degree}(L)$ are coprime, any two Poincaré vector bundle over $X \times \mathcal{M}_X(r, L)$ differ by tensoring with a line bundle pulled back from $\mathcal{M}_X(r, L)$. Hence the projectivized Poincaré bundle is unique. Even when $r$ and $\text{degree}(L)$ are not coprime, there is a unique projective Poincaré bundle over $X \times \mathcal{M}_X(r, L)$, although it is not a projectivization of a vector bundle.

In [BBGN] it was proved that the Brauer group of $\mathcal{M}_X(r, L)$ is cyclic of order $g\cdot\text{c.d.}(r, \text{degree}(L))$. As mentioned above, there is a universal projective bundle $\mathcal{P}$ on $X \times \mathcal{M}_X(r, L)$. Fixing a point $x \in X$, let $\mathcal{P}_x$ be the projective bundle on $\mathcal{M}_X(r, L)$ obtained by restricting $\mathcal{P}$ to $\{x\} \times \mathcal{M}_X(r, L)$. In [BBGN] it was also shown that the Brauer group $\text{Br}(\mathcal{M}_X(r, L))$ is generated by the class of $\mathcal{P}_x$.

Our aim here is to prove these results for all algebraically closed fields; see Theorem 2.3.

The computation in [BBGN] crucially uses the calculation of the Picard group of $\mathcal{M}_X(r, L)$. It may be mentioned that the assumption in [DN] that the characteristic of the base field is zero is used in the computation of the Picard group of the moduli space $\mathcal{M}_X(r, L)$. In particular, the Reynolds’ operators, which play a crucial role in the computation, are valid only in characteristic zero. A recent theorem of Hoffmann shows that the Picard group of the moduli space does not depend on the base field [Hof]. The proof of Theorem 2.3 follows the strategy of [BBGN]; some details not given in [BBGN] are given here.
2. Universal projective bundle and Brauer group

Let \( k \) be an algebraically closed field. Let \( X \) be an irreducible smooth projective curve, defined over \( k \), of genus \( g \), with \( g \geq 3 \). Fix an integer \( r \geq 2 \) and also fix a line bundle \( L \) over \( X \). The degree of \( L \) will be denoted by \( d \). Let \( \mathcal{M}_X(r, d) \) be the moduli space of stable vector bundles on \( X \) of rank \( r \) and degree \( d \). Consider the morphism

\[
\phi : \mathcal{M}_X(r, d) \to \text{Pic}^d(X), \quad E \mapsto \bigwedge^r E.
\]

Let \( \mathcal{M}_X = \mathcal{M}_X(r, L) := \phi^{-1}(L) \) be the fiber of \( \phi \) over the point \( L \in \text{Pic}^d(X) \). This moduli space \( \mathcal{M}_X \) is canonically identified with the following two moduli spaces:

1. moduli space of pairs of the form \((E, \xi)\), where \( E \) is a stable vector bundle over \( X \) of rank \( r \), and \( \xi : \bigwedge^r E \to L \) is an isomorphism, and
2. the moduli space of stable vector bundles \( E \) on \( X \) of rank \( r \) such that \( \bigwedge^r E \) is isomorphic to \( L \)

(see \cite{Ho}, p. 1308, Proposition 2.1).

It is known that there is a universal projective bundle

\[
\mathcal{P} \to X \times \mathcal{M}_X
\]

(\cite{Ra}, \cite{Ne}). This follows from the construction of the moduli space and the fact that the global automorphisms of a stable vector bundle are nonzero constant scalar multiplications; this projective bundle \( \mathcal{P} \) is described in the proof of Proposition 2.1. Fix a closed point \( x \in X \). Let

\[
\mathcal{P}_x := \mathcal{P}|_{\{x\} \times \mathcal{M}_X} \to \mathcal{M}_X
\]

be the restriction of \( \mathcal{P} \) to \( \{x\} \times \mathcal{M}_X \).

For any quasiprojective variety \( Y \) defined over the field \( k \), the Brauer group \( \text{Br}(Y) \) of \( Y \) is defined to be the Morita equivalence classes of Azumaya algebras over the variety \( Y \). It is known that this Brauer group \( \text{Br}(Y) \) coincides with the equivalence classes of all principal \( \text{PGL}_k \)-bundles over \( Y \), where two principal \( \text{PGL}_k \)-bundles \( P \) and \( Q \) are identified if there are two vector bundles \( V_1 \) and \( V_2 \) over \( Y \) satisfying the condition that the two principal \( \text{PGL}_k \)-bundles \( P \otimes \mathbb{P}(V_1) \) and \( Q \otimes \mathbb{P}(V_2) \) are isomorphic. The addition of two projective bundles \( P \) and \( Q \) in the Brauer group \( \text{Br}(Y) \) is defined to be the equivalence class of the projective bundle \( P \otimes Q \). The inverse of a projective bundle \( P \) in \( \text{Br}(Y) \) is the equivalence class of the dual projective bundle \( P^* \). (See \cite{Gr1}, \cite{Gr2}, \cite{Gr3}, \cite{Mi}, \cite{Ga} for properties of Brauer groups.) The cohomological Brauer group \( \text{Br}'(Y) \) of the variety \( Y \) is the torsion part of the étale cohomology group \( H^2_{\text{et}}(Y, \mathbb{G}_m) \). There is a natural injective homomorphism \( \text{Br}(Y) \to \text{Br}'(Y) \) which is in fact an isomorphism by a theorem of Gabber \cite{dJ}, \cite{Ho}.

**Proposition 2.1.** The Brauer group \( \text{Br}(\mathcal{M}_X) \) is generated by the class \( \text{cl}(\mathcal{P}_x) \in \text{Br}(\mathcal{M}_X) \) of the projective bundle \( \mathcal{P}_x \) defined in (2.2).
**Proof.** Given any line bundle $L_0$ on $X$, the morphism

$$\mathcal{M}_X = \mathcal{M}_X(r, L) \longrightarrow \mathcal{M}_X(r, \mathcal{L} \otimes L_0^r), \quad E \mapsto E \otimes L_0$$

is an isomorphism. The natural isomorphism of $\mathbb{P}(E \otimes L_0)$ with $\mathbb{P}(E)$ produces an isomorphism between the universal projective bundles over $X \times \mathcal{M}_X(r, L)$ and $X \times \mathcal{M}_X(r, \mathcal{L} \otimes L_0^r)$. Therefore, after tensoring with a line bundle $L_0$ of sufficiently large degree, we may assume that

$$\frac{d}{r} > 2g - 1.$$  

Let $\overline{M}_X$ denote the moduli space of semistable vector bundles $E$ on $X$ of rank $r$ with $\bigwedge^r E = L$.

The cotangent bundle of $X$ will be denoted by $K_X$. For any vector bundle $E \in \overline{M}_X$ and any point $y \in X$,

$$H^1(Y, E \otimes \mathcal{O}_X(-y)) = H^0(Y, E^* \otimes K_X \otimes \mathcal{O}_X(y))^* = 0$$

because $\deg(E^* \otimes K_X \otimes \mathcal{O}_X(y)) < 0$ and $E^* \otimes K_X \otimes \mathcal{O}_X(y)$ is semistable. So from the long exact sequence of cohomologies associated to the short exact sequence

$$0 \longrightarrow E \otimes \mathcal{O}_X(-y) \longrightarrow E \longrightarrow E_y \longrightarrow 0$$

it follows that the evaluation homomorphism $H^0(X, E) \longrightarrow E_y$ is surjective; hence $E$ is generated by its global sections.

Take any $E \in \overline{M}_X$. Since the vector bundle $E$ is generated by its global sections, there is a short exact sequence

$$0 \longrightarrow \mathcal{O}_X^{\oplus(r-1)} \longrightarrow E \longrightarrow \bigwedge^r E = L \longrightarrow 0. \quad (2.3)$$

This short exact sequence does not split because $E$ is semistable and $\deg(L) > 0$. All such nontrivial extensions are parameterized by

$$\mathbb{P}(H^1(X, \text{Hom}(L, \mathcal{O}_X^{\oplus(r-1)}))^*) = \mathbb{P}((H^1(X, L^*)^{\oplus(r-1)})^*) = \mathbb{P}((H^1(X, L^*)^{\oplus(r-1)})) \cdot \mathbb{P}(GL(r - 1, k) \cdot \mathbb{P}(GL(r - 1, k) \cdot \mathbb{P}(GL(r - 1, k) = \overline{M}_X$$

(see [Né], [DN]).

The tautological line bundle $\mathcal{O}_{\mathbb{P}((H^1(X, L^*)^{\oplus(r-1)})^*)}$ on $\mathbb{P}((H^1(X, L^*)^{\oplus(r-1)})^*)$ will be denoted by $\mathcal{L}_0$. Let

$$p_1 : X \times \mathbb{P}((H^1(X, L^*)^{\oplus(r-1)})^*) \longrightarrow X, \quad p_2 : X \times \mathbb{P}((H^1(X, L^*)^{\oplus(r-1)})^*) \longrightarrow \mathbb{P}((H^1(X, L^*)^{\oplus(r-1)})^*)$$

be the natural projections. There is a universal extension over $X \times \mathbb{P}((H^1(X, L^*)^{\oplus(r-1)})^*)$

$$0 \longrightarrow (p_1^* \mathcal{O}_X^{\oplus(r-1)}) \otimes p_2^* \mathcal{L}_0 \longrightarrow \mathcal{E} \longrightarrow p_1^* \mathcal{L} \longrightarrow 0. \quad (2.4)$$
Let \( U \subset \mathbb{P}((H^1(X, L^*)^{r-1})^*) \) be the subset defined by all points \( t \in \mathbb{P}((H^1(X, L^*)^{r-1})^*) \) such that the vector bundle \( \mathcal{E}|_{X \times \{ t \}} \) is stable. This subset \( U \) is nonempty Zariski open. Let
\[
\theta : U \longrightarrow U / \text{PGL}(r - 1, k) = \mathcal{M}_X \tag{2.5}
\]
be the quotient map. Consider the action of \( \text{PGL}(r - 1, k) \) on \( X \times \mathbb{P}((H^1(X, L^*)^{r-1})^*) \) given by the trivial action on \( X \) and the above action of \( \mathbb{P}((H^1(X, L^*)^{r-1})^*) \). This action lifts to an action of \( \text{PGL}(r - 1, k) \) on \( \mathbb{P}(\mathcal{E}) \). The corresponding geometric invariant theoretic quotient
\[
\mathcal{P} := (\mathbb{P}(\mathcal{E})|_{X \times U}) / \text{PGL}(r - 1, k)
\]
is the universal projective bundle on \( X \times \mathcal{M}_X \) (see (2.1)).

Consider the map
\[
F := \text{Id}_X \times f : X \times \mathcal{P}_x \longrightarrow X \times \mathcal{M}_X, \tag{2.6}
\]
where \( f \) is the projection in (2.2). We will construct a vector bundle
\[
\mathcal{V} \longrightarrow X \times \mathcal{P}_x
\]
with the property that \( \mathbb{P}(\mathcal{V}) = F^* \mathcal{P} \).

Let \( \mathcal{E}_x := \mathcal{E}|_{\{ x \} \times U} \longrightarrow U \) be the vector bundle obtained by restricting \( \mathcal{E} \) in (2.4) to \( \{ x \} \times U \). Let
\[
\mathcal{Q} := \mathbb{P}(\mathcal{E}_x) \xrightarrow{\beta'} U
\]
be the corresponding projective bundle. Define
\[
\beta := \text{Id}_X \times \beta' : X \times \mathcal{Q} \longrightarrow X \times U,
\]
and consider the pulled back vector bundle
\[
\tilde{\mathcal{E}} := (\beta^* \mathcal{E}) \otimes (q_2^* \mathcal{O}_\mathcal{Q}(-1)) \longrightarrow X \times \mathcal{Q},
\]
where \( q_2 : X \times \mathcal{Q} \longrightarrow \mathcal{Q} \) is the natural projection, and
\[
\mathcal{O}_\mathcal{Q}(1) \longrightarrow \mathbb{P}(\mathcal{E}_x) = \mathcal{Q}
\]
is the tautological line bundle. For the natural action of \( \text{GL}(r - 1, k) \) on \( \tilde{\mathcal{E}} \), the center \( \mathbb{G}_m \) of \( \text{GL}(r - 1, k) \) acts trivially on \( \tilde{\mathcal{E}} \). Consequently, the geometric invariant theoretic quotient
\[
\mathcal{V} := \tilde{\mathcal{E}} / \text{GL}(r - 1, k) \longrightarrow X \times (\mathcal{Q} / \text{GL}(r - 1, k)) = X \times \mathcal{P}_x
\]
is a vector bundle. It is straight-forward to check that
\[
\bullet \mathbb{P}(\mathcal{V}) = F^* \mathcal{P}, \text{ where } F \text{ is the map in (2.6), and}
\bullet \text{for each point } y \in \mathcal{P}_x, \text{ the vector bundle } \mathcal{V}|_{X \times \{ y \}} \text{ on } X \text{ lies in the isomorphism class of vector bundles associated to the point } f(y) \in \mathcal{M}_X, \text{ where } f \text{ is defined in (2.2)}.
\]

Let \( B : X \times \mathcal{P}_x \longrightarrow \mathcal{P}_x \) be the natural projection. Consider the direct image \( B_* \mathcal{V} \longrightarrow \mathcal{P}_x \).

Let \( Z \subset \mathcal{P}_x \) be a nonempty Zariski open subset such that the restriction \( (B_* \mathcal{V})|_Z \) is a trivial vector bundle. Fix a trivialization of \( (B_* \mathcal{V})|_Z \). Take a point \( y_0 \in Z \) and choose \( r - 1 \) linearly independent sections
\[
s_1, \ldots, s_{r-1} \in H^0(X \times \{ y_0 \}, \mathcal{V}|_{X \times \{ y_0 \}})
\]
such that the coherent subsheaf of \( \mathcal{V}|_{X \times \{ y_0 \}} \) generated by \( s_1, \ldots, s_{r-1} \) is a subbundle of \( \mathcal{V}|_{X \times \{ y_0 \}} \) of rank \( r - 1 \); we note that from (2.3) it follows immediately that such \( r - 1 \)}
linearly independent sections exist. Extend each $s_i$ to a section $\tilde{s}_i$ of $V|_{X \times Z}$ using the above trivialization of $(B_\ast V)|_Z$. There is a Zariski open subset $Z' \subset Z$ containing $y_0$ such that the coherent subsheaf of $V|_{X \times Z}$ generated by $\tilde{s}_1, \ldots, \tilde{s}_{r-1}$ is a subbundle of $V|_{X \times Z'}$ of rank $r-1$. Note that this subbundle over $X \times Z'$ is trivial and a trivialization is given by the images of $\tilde{s}_1, \ldots, \tilde{s}_{r-1}$. Therefore, on $X \times Z'$, we have a short exact sequence of vector bundles

$$0 \longrightarrow \mathcal{O}^{\oplus(r-1)}_{X \times Z'} \longrightarrow V|_{X \times Z'} \longrightarrow L' \longrightarrow 0,$$

where $L'$ is a line bundle on $X \times Z'$. Considering the top exterior products it follows that for each point $y \in Z'$, the restriction $L'|_{X \times \{y\}}$ is isomorphic to the line bundle $L$. Now from the seesaw theorem (see [Mu, p. 51, Corollary 6]) it follows that there is a line bundle $L''$ on $Z'$ such that the line bundle $L' \otimes B^*L''$ on $X \times Z'$ is isomorphic to the pullback of $L$ to $X \times Z'$. We may trivialize $L''$ over suitable nonempty Zariski open subsets of $Z'$. Therefore, it follows that there is a nonempty Zariski open subset

$$\iota : \mathcal{W} \hookrightarrow Z' \subset \mathcal{P}_x \quad (2.7)$$

such that the restriction $L'|_{X \times \mathcal{W}}$ is isomorphic to the pullback of $L$ to $X \times \mathcal{W}$.

Consequently, there is a morphism

$$\varphi : \mathcal{W} \longrightarrow \mathcal{U} \subset \mathbb{P}((H^1(X, L^*)^{r-1})^*)$$

such that the following diagram is commutative

$$\begin{array}{ccc}
\mathcal{W} & \xrightarrow{\varphi} & \mathcal{U} \\
\downarrow{\iota} & & \downarrow{\theta} \\
\mathcal{P}_x & \xrightarrow{f} & \mathcal{M}_X
\end{array} \quad (2.8)$$

where $\theta$, $\iota$ and $f$ are the morphisms in (2.5), (2.7) and (2.2) respectively.

The codimension of the complement

$$\mathbb{P}((H^1(X, L^*)^{r-1})^*) \setminus \mathcal{U} \subset \mathbb{P}((H^1(X, L^*)^{r-1})^*)$$

is at least two. To prove this, note that $\text{Pic}(\mathcal{U}) = \mathbb{Z}$ [DN, p. 89, Proposition 7.13] (here we need the assumption that $g \geq 3$); this immediately implies that the codimension of the complement $\mathcal{U}^c$ is at least two. Since the Brauer group of a projective space is zero, in view of this codimension estimate, it follows from the “Cohomological purity” [Mi, p. 241, Theorem VI.5.1] (it also follows from [Gr1, p. 292–293]) that

$$\text{Br}(\mathcal{U}) = 0. \quad (2.9)$$

From the commutativity of (2.8) we conclude that the pullback homomorphism

$$\iota^* \circ f^* = (f \circ \iota)^* : \text{Br}(\mathcal{M}_X) \longrightarrow \text{Br}(\mathcal{W})$$

coincides with the homomorphism $\varphi^* \circ \theta^* : \text{Br}(\mathcal{M}_X) \longrightarrow \text{Br}(\mathcal{W})$. On the other hand, from (2.9) we know that $\varphi^* \circ \theta^* = 0$. Hence

$$\iota^* \circ f^* = 0.$$
On the other hand, the homomorphism

\[ \iota^* : \text{Br}(\mathcal{P}_x) \longrightarrow \text{Br}(\mathcal{W}) \]

is injective because \( \mathcal{W} \) is a Zariski open dense subset of \( \mathcal{P}_x \) (see [Mi, p. 142, Theorem 2.5]). Consequently,

\[ f^* : \text{Br}(\mathcal{M}_X) \longrightarrow \text{Br}(\mathcal{P}_x) \]

is the zero homomorphism. On the other hand, the kernel of the above homomorphism \( f^* \) is generated by the class of \( \mathcal{P}_x \) [Ga, p. 193, Theorem 2]. Therefore, we conclude that \( \text{Br}(\mathcal{M}_X) \) is generated by the class of \( \mathcal{P}_x \). This completes the proof. \( \square \)

We will denote the integer g.c.d.(\( r, d \)) by \( \delta \).

**Lemma 2.2.** The order of the Brauer class \( cl(\mathcal{P}_x) \in \text{Br}(\mathcal{M}_X) \) is \( \delta \).

**Proof.** Let \( \widetilde{\mathcal{M}}_X = \widetilde{\mathcal{M}}_X(r, L) \) be the moduli stack of pairs of the form \((E, \xi)\), where \( E \) is a stable vector bundle over \( X \) of rank \( r \) and \( \xi : \bigwedge^r E \longrightarrow L \) is an isomorphism. Let \( \mu_r \) denote the kernel of the homomorphism

\[ \mathbb{G}_m \longrightarrow \mathbb{G}_m, \ z \mapsto z^r. \]

The natural morphism

\[ \gamma : \widetilde{\mathcal{M}}_X \longrightarrow \mathcal{M}_X \]

makes \( \widetilde{\mathcal{M}}_X \) a \( \mu_r \)-gerbe over \( \mathcal{M}_X \). Let

\[ c_0 \in H^2(\mathcal{M}_X, \mu_r) \]

be the class of this \( \mu_r \)-gerbe. The Brauer class

\[ cl(\mathcal{P}_x) \in \text{Br}(\mathcal{M}_X) = H^2(\mathcal{M}_X, \mathbb{G}_m) \]

coincides with the image of \( c_0 \) under the homomorphism

\[ \eta : H^2(\mathcal{M}_X, \mu_r) \longrightarrow H^2(\mathcal{M}_X, \mathbb{G}_m) \]

given by the inclusion of \( \mu_r \) in \( \mathbb{G}_m \).

There is a short exact sequence

\[ 0 \longrightarrow \text{Pic}(\mathcal{M}_X) \stackrel{\gamma}{\longrightarrow} \text{Pic}(\widetilde{\mathcal{M}}_X) \stackrel{\nu}{\longrightarrow} \text{Hom}(\mu_r, \mathbb{G}_m) = \mathbb{Z}/r\mathbb{Z} \longrightarrow \text{Br}(\mathcal{M}_X) \]

[BH, p. 232, Lemma 4.4], where \( \nu \) sends a line bundle on the \( \mu_r \)-gerbe \( \widetilde{\mathcal{M}}_X \) to the weight associated to the action of \( \mu_r \) on it. The image

\[ \alpha(1) \in \text{Br}(\mathcal{M}_X) = H^2(\mathcal{M}_X, \mathbb{G}_m) \]

coincides with the image of the class of the \( \mu_r \)-gerbe \( \widetilde{\mathcal{M}}_X \) under the above homomorphism \( \eta \). From this it follows that

\[ \alpha(1) = cl(\mathcal{P}_x) \]

because \( cl(\mathcal{P}_x) \) also coincides with the image of the class of the \( \mu_r \)-gerbe \( \widetilde{\mathcal{M}}_X \) under the above homomorphism \( \eta \).
From [Hof, p. 1311, Lemma 3.6], [Hof, p. 1311, Lemma 3.3] and [Hof, p. 1310, Theorem 3.1] it follows that

\[ \mathbb{Z}/\text{image}(\nu) = \mathbb{Z}/\delta \mathbb{Z}, \]

where \( \nu \) is the homomorphism in (2.10). Therefore, from (2.10) we conclude that the order of \( \alpha(1) \in \text{Br}(\mathcal{M}_X) \) is \( \delta \). Now the lemma follows from (2.11).

\[ \square \]

Combining Proposition 2.1 and Lemma 2.2 we have:

**Theorem 2.3.** The Brauer group \([\text{Br}(\mathcal{M}_X)]\) is cyclic of order \( \delta = \gcd(r,d) \). The group \([\text{Br}(\mathcal{M}_X)]\) is generated by the class \( \text{cl}(\mathcal{P}_x) \in \text{Br}(\mathcal{M}_X) \) of the projective bundle \( \mathcal{P}_x \) defined in (2.2).

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