New Pricing Framework:
Options and Bonds

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Abstract

A unified analytical pricing framework with involvement of the shot noise random process has been introduced and elaborated. Two exactly solvable new models have been developed.

The first model has been designed to value options. It is assumed that stock price stochastic dynamics follows a Geometric Shot Noise motion. A new arbitrage-free integro-differential option pricing equation has been found and solved. The put-call parity has been proved and the Greeks have been calculated. Three new Greeks associated with model market parameters have been introduced and evaluated. It has been shown that in diffusion approximation the developed option pricing model incorporates the well-known Black-Scholes equation and its solution. The stochastic dynamic origin of the Black-Scholes volatility has been discussed.

The new option pricing model has been generalized based on a stock price dynamics modeled by the superposition of Geometric Brownian motion and Geometric Shot Noise.

To model stochastic dynamics of a short term interest rate, the second model has been introduced and developed based on Langevin type equation with shot noise. It has been found that the model provides affine term structure. A new bond pricing formula has been obtained. It has been shown that in diffusion approximation the developed bond pricing formula goes into the well-known Vasiček solution.

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The stochastic dynamic origin of the long-term mean and instantaneous volatility of the Vasiček model have been explained.

A generalized bond pricing model has been introduced and developed based on short term interest rate stochastic dynamics modeled by superposition of Wiener process and shot noise.

Despite the lack of normality of probability distributions involved, all newly elaborated models have the same degree of analytical tractability as the Black–Scholes model and the Vasiček model. It allows to obtain new exact simple formulas to value options and bonds.

Key words: Financial derivatives fundamentals, Shot noise, Option pricing equation, Green function, Put-call parity, Greeks, Black–Scholes equation, Short term interest rate, Vasiček model, Affine term structure.

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1 Introduction

The aim of this paper is to introduce and elaborate a new unified analytical framework to value options and bonds.

The options pricing approach is based on a stock price dynamics that has been modelled by the stochastic differential equation with involvement of shot noise. It results in Geometric Shot Noise motion of stock price. A new arbitrage-free integro-differential option pricing equation has been developed and solved. New exact formulas to value European call and put options have been obtained. The put-call parity has been proved. Based on the solution of the option pricing equation the Greeks have been calculated. Three new Greeks associated with the model market parameters have been introduced and evaluated. It has been shown that the developed option pricing framework incorporates the well-known Black-Scholes equation [1]. The Black-Scholes equation and its solutions emerge from our integro-differential option pricing equation in the special case which we call “diffusion approximation”. The Geometric Shot Noise model in diffusion approximation gives natural explanation of the stochastic dynamic origin of volatility in the Black-Scholes model.

The bonds pricing analytical approach is based on the Langevin type stochastic differential equation with additive shot noise to model a short term interest rate dynamics. It results in non-Gaussian random motion of short term interest rate. A bond pricing formula has been obtained and it has been shown that the model provides affine term structure. The new bond pricing formula incorporates the well-known Vasicek solution [2]. The Vasicek solution comes out from our bond pricing formula in diffusion approximation. The stochastic dynamic origin of the Vasicek long-term mean and instantaneous volatility have been discussed.

It has to be emphasized that despite the lack of normality of probability distributions involved, two developed quantitative models to value options and bonds have the same degree of analytical tractability as the Black–Scholes model [1] and the Vasicek model [2]. Analytical tractability allows to obtain new exact simple formulas to value options and bonds.

The paper’s main results are presented by Eqs. (1), (2), (16), (22), (39), (43), (113)-(116), (124), (131), (136), (142), (144), (162) and (175).

New formulae to evaluate the common Greeks for call and put options have been obtained based on the integro-differential option pricing equation. The formulas are listed in Table 1, which summarizes the common Greeks.
for European call options, and in Table 2, which summarizes the common Greeks for European put options.

The paper is organized as follows.

In Sec.2 Geometric Shot Noise motion has been introduced and applied to model a stock price dynamics.

A new arbitrage-free integro-differential equation to value options is presented in Sec.3. It has been shown that Green’s function method is an effective mathematical tool to solve this equation. The exact analytical solutions to this equation have been found for European call and put options.

The Put-Call parity has been proved in Sec.4.

The Greeks, including three newly introduced Greeks, have been calculated in Sec.5. The new Greeks are option sensitivities associated with the market parameters involved into the definition of the shot noise process. The Gaussian jumps model has been considered to find the formulae for three new Greeks.

The diffusion approximation to the integro-differential option pricing equation has been defined and elaborated in Sec.6. It has been shown that the well-known Black-Scholes equation and its volatility come out from the integro-differential pricing equation in diffusion approximation. It has been shown as well that the solution to the Black-Scholes equation straightforwardly follows from the exact solution to integro-differential pricing equation. The well-known Black-Scholes Greeks for European call options have been replicated from the common Greeks in diffusion approximation. The Black-Scholes Greeks for European call options are summarized by Table 3.

The generalized option pricing framework has been presented in Sec.7. The generalization comes from the idea to accommodate the superposition of Geometric Brownian motion and Geometric Shot Noise in the equation for a stock price dynamics. The outcome of implementing this idea is a generalized pricing equation. New formulas to value European call and put options have been obtained as solutions to the generalized pricing equation. The special limit cases of those formulas have been presented and discussed.

In Sec.8 short term interest rate has been modeled by the Langevin stochastic differential equation with additive shot noise. It is interesting that despite the lack of normality, the model possesses exactly the same mean and variance as the Vasicek model, which results in the Normal distribution of a short term interest rate. A new bond pricing formula has been found, and it has been shown that the new model provides affine term structure. It has been shown that the bond pricing formula is the solution to the term
structure equation. The well-known Vasiček model for short term interest rate with its long-term mean and instantaneous volatility comes out from our model in diffusion approximation. It has been shown as well that the Vasiček bond pricing formula comes out from the new bond pricing formula in diffusion approximation.

A generalized bond pricing model has been introduced and developed in Sec.9 based on short term interest rate stochastic dynamics modeled by superposition of Wiener process and shot noise. A new bond pricing formula has been found and it has been shown that the generalized model provides affine term structure. The term structure equation has been presented and solved.

The paper’s main results and findings have been summarized and discussed in the Conclusion.

The Appendix presents Green’s function approach to solve the Black-Scholes equation for European call options. The well-known solution has been obtained straightforwardly without preliminary transformations to convert the Black-Scholes equation to the heat equation. To our best knowledge, the implemented Green’s function approach has not yet been presented anywhere.

2 Stock price stochastic dynamics

2.1 Geometric Shot Noise motion

It is supposed that a stock price $S(t)$ follows the stochastic differential equation

$$dS = -qSdt + SF(t)dt,$$

(1)

with $q$ being a continuously compounding dividend yield and random force $F(t)$ modeled by the shot noise process (see, Eq.(6) in [3])

$$F(t) = \sum_{k=1}^{n} \eta_k \varphi(t - t_k),$$

(2)

where random jumps $\eta_k$ of stock price are statistically independent and distributed with probability density function $p(\eta)$, random time points $t_k$, which are arrival times of stock price jumps, are uniformly distributed on
time interval \([t, T]\), so that their total number \(n\) obeys the Poisson law with parameter \(\lambda\), and deterministic function \(\varphi(t)\) is the response function.

It is supposed that defined by Eq. (2) shot noise process \(F(t)\) describes the influence of different fluctuating factors on stock price dynamics. A single shot noise pulse \(\eta_k \varphi(t - t_k)\) describes the influence of a piece of information which has become available at random moment \(t_k\) on the stock price at a later time \(t\). Amplitude \(\eta_k\) responds to the magnitude of the stock price pulse \(\varphi(t - t_k)\). Amplitudes \(\eta_k\) are random statistically independent variables, subject to the market information available. We assume as well that each pulse has the same functional form or, in other words, one general response function \(\varphi(t)\) can be used to describe the stock price dynamics.

We call the stochastic dynamics introduced by Eq. (1) Geometric Shot Noise motion.

### 2.2 Characteristic functional of shot noise

By definition the characteristic functional \(\Phi[\alpha(\tau)]\) of random force \(F(t)\) is

\[
\Phi[\alpha(\tau)] = \left\langle \exp \left\{ i \int_t^T d\tau \alpha(\tau) F(\tau) \right\} \right\rangle,
\]

where \(\alpha(\tau)\) is an arbitrary sufficiently smooth function and \(\langle \ldots \rangle\) stands for the average over all randomness involved into the random force \(F(t)\).

The characteristic functional contains all information about statistical moments of random force \(F(t)\). For example, the mean value of \(F(t)\) can be calculated as functional derivative

\[
\langle F(t) \rangle = \frac{1}{i} \frac{\delta}{\delta \alpha(t)} \Phi[\alpha(\tau)]|_{\alpha(\tau) = 0}, \quad (4)
\]

while the correlation function \(\langle F(t_1)F(t_2) \rangle\) is given by the second order functional derivative

\[
\langle F(t_1)F(t_2) \rangle = \frac{1}{i^2} \frac{\delta^2}{\delta \alpha(t_1) \delta \alpha(t_2)} \Phi[\alpha(\tau)]|_{\alpha(\tau) = 0}. \quad (5)
\]

To evaluate the characteristic functional \(\Phi[\alpha(\tau)]\) we need to define probabilistic characteristics of each of three sources of randomness involved in
Eq. (2). Therefore, assuming that these three sources of randomness are independent of each other, we have three statistically independent averaging procedures [3]:

1. Averaging over uniformly distributed time points \( t_k \) on the interval \([t, T]\),

\[
\langle \ldots \rangle_{T-t} = \prod_{k=1}^{n} \frac{1}{T-t} \int_{t}^{T} dt_k \ldots.
\]  

(6)

2. Averaging over random stock price jumps, which are statistically independent and distributed with probability density function \( p(\eta) \),

\[
\langle \ldots \rangle_{\eta} = \prod_{k=1}^{n} \int_{-\infty}^{\infty} d\eta_k p(\eta_k) \ldots.
\]  

(7)

3. Averaging over random number \( n \) of price jumps,

\[
\langle \ldots \rangle_{n} = e^{-\lambda(T-t)} \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^{n}}{n!} \ldots,
\]  

(8)

which is in fact the averaging with the Poisson probability density function and \( \lambda \) is the rate of arrival of price jumps, i.e. the number of jumps per unit of time.

Now we are in position to calculate \( \Phi[\alpha(\tau)] \). First, by performing steps \#1 and \#2 we obtain

\[
\Phi[\alpha(\tau)] = \left\langle \exp \left\{ i \int_{t}^{T} d\tau \alpha(\tau) \sum_{k=1}^{n} \eta_k \varphi(\tau - t_k) \right\} \right\rangle_{t_k, \eta} 
\]

(9)

\[
= \left[ \frac{1}{T-t} \int_{t}^{T} dt' \int_{-\infty}^{\infty} d\eta p(\eta) \exp \left\{ i \eta \int_{t'}^{T} d\tau \alpha(\tau) \varphi(\tau - t') \right\} \right]^{n}.
\]

Then, let us do step \#3,

\[
\Phi[\alpha(\tau)] = \left\langle \left[ \frac{1}{T-t} \int_{t}^{T} dt' \int_{-\infty}^{\infty} d\eta p(\eta) \exp \left\{ i \eta \int_{t'}^{T} d\tau \alpha(\tau) \varphi(\tau - t') \right\} \right]^{n} \right\rangle_{n}
\]
\[ e^{-\lambda(T-t)} \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n}{n!} \left[ \frac{1}{T-t} \int_{t}^{T} dt' \right] \]

\[ \times \int_{-\infty}^{\infty} d\eta \eta \exp \left\{ i\eta \int_{t}^{T} d\tau \alpha(\tau) \chi(\tau-t') \right\} \]

\[ = \exp \left\{ \lambda \int_{t}^{T} dt' \int_{-\infty}^{\infty} d\eta \eta \exp \left\{ i\eta \int_{t}^{T} d\tau \alpha(\tau) \chi(\tau-t') \right\} - 1 \right\} . \]

Hence, we found the equation for the characteristic functional \( \Phi[\alpha(\tau)] \) defined by Eq.(3)

\[ \Phi[\alpha(\tau)] = \exp \left\{ \lambda \int_{t}^{T} dt' \int_{-\infty}^{\infty} d\eta \eta \left( e^{i\eta \alpha(\tau)} - 1 \right) \right\} . \]  

Further, we chose, as an example, the response function

\[ \chi(t) = \delta(t), \]  

where \( \delta(t) \) is delta-function. In this case we have for the characteristic functional \( \Phi[\alpha(\tau)] \)

\[ \Phi[\alpha(\tau)] = \exp \left\{ \lambda \int_{t}^{T} d\tau \int_{-\infty}^{\infty} d\eta \eta (e^{i\eta \alpha(\tau)} - 1) \right\} . \]  

Now, having Eq.(13) and definitions (4) and (5) one can easily obtain the mean value \( <F(t)> \)

\[ <F(t)> = \lambda \int_{-\infty}^{\infty} d\eta \eta, \]  

and the correlation function \( <F(t_1)F(t_2)> \)
\[ < F(t_1)F(t_2) > = \lambda \left( \int_{-\infty}^{\infty} d\eta p(\eta)\eta^2 \right) \delta(t_1-t_2) + < F(t_1) < F(t_2) > . \] (15)

3 Option pricing equation and its solutions

3.1 A new arbitrage free integro-differential pricing equation

Having the characteristic functional \( \Phi[\alpha(\tau)] \) given by Eq.(13) and assuming frictionless and no-arbitrage market, a constant risk-free interest rate \( r \), and a stock price dynamics governed by the Geometric Shot Noise motion given by Eq.(1), we introduce a new arbitrage free integro-differential option pricing equation

\[
\frac{\partial C(x,t)}{\partial t} + (r-q)\frac{\partial C(x,t)}{\partial x} + \lambda \int_{-\infty}^{\infty} d\eta p(\eta) \left\{ C(x+\eta,t) - C(x,t) - (e^{\eta}-1)\frac{\partial C(x,t)}{\partial x} \right\}
\]

\[-rC(x,t) = 0,
\]

where

\[ x = \ln \frac{S}{K}, \] (17)

and \( C(x,t) \) is the value of European call option\(^1\). \( S \) is a stock price governed by Eq.(1), \( K \) is the strike price, \( r \) is the risk-free interest rate, \( q \) is continuously compounding dividend yield, which is a constant, \( p(\eta) \) is probability density function involved into Eq.(2). Equation (16) is the integro-differential equation.

The terminal condition (or payoff function) for European call option is

\(^1\)An option which gives the owner the right, but not the obligation, to buy a stock, at a specified price (strike price \( K \)), by a predetermined date (maturity date \( T \)).
$$C(S, T) = \max(S - K, 0).$$ \hspace{1cm} (18)

If we take into account Eq. (17) then we can write Eq. (18) as

$$C(x, T) = K \max(e^x - 1, 0).$$ \hspace{1cm} (19)

Thus, the new generalized option pricing framework has been introduced by Eqs. (16) and (19).

To value European put option\footnote{An option which gives the owner the right, but not the obligation, to sell an asset, at a strike price \( K \), on the maturity date \( T \).} \( P(x, t) \) we have the same equation as Eq. (16) while the terminal condition is

$$P(S, T) = \max(K - S, 0),$$ \hspace{1cm} (20)

With help of Eq. (17) the terminal condition (20) for European put option becomes

$$P(x, T) = K \max(1 - e^x, 0).$$ \hspace{1cm} (21)

If we go from \( C(x, t) \) to \( C(S, t) \) where \( x \) and \( S \) are related each other by Eq. (17), then we can write an option pricing equation (16) in the form

$$\frac{\partial C(S, t)}{\partial t} + (r - q)S\frac{\partial C(S, t)}{\partial S}$$

$$+ \lambda \int_{-\infty}^{\infty} d\eta p(\eta) \left\{ C(Se^\eta, t) - C(S, t) - (e^\eta - 1)S\frac{\partial C(S, t)}{\partial S} \right\} - rC(S, t) = 0,$$

with the terminal condition given either by Eq. (18) or by Eq. (20).

### 3.2 Solution to the integro-differential equation

#### 3.2.1 Call option

To solve Eq. (16) subject to terminal condition given by (19) we will use the Green’s function method.
By definition Green’s function $G(x-x', T-t)$ satisfies the integro-differential equation

$$
\frac{\partial G(x-x', T-t)}{\partial (T-t)} + (r-q) \frac{\partial G(x-x', T-t)}{\partial x} + \lambda \int_{-\infty}^{\infty} d\eta \eta p(\eta) \left\{ G(x+\eta-x', T-t) - G(x-x', T-t) \right\} - rG(x-x', T-t) = 0,
$$

and the terminal condition

$$G(x-x', T-t)|_{t=T} = G(x-x', 0) = \delta(x-x'),$$

where $T$ is an option maturity time, consequently, $T-t$ is the time to maturity.

Having Green’s function $G(x-x', T-t)$, the solution to Eq. (16) with condition (19) becomes

$$C(x,T-t) = K \int_{-\infty}^{\infty} dx' G(x-x', T-t) \max(e^{x'} - 1, 0),$$

and the solution to Eq. (16) with condition (21) becomes

$$P(x,T-t) = K \int_{-\infty}^{\infty} dx' G(x-x', T-t) \max(1 - e^{x'}, 0).$$

Green’s function introduced by Eqs. (23) and (24) can be found by the Fourier transform method. Green’s function $G(x-x', T-t)$ reads

$$G(x-x', T-t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ik(x-x')} G(k, T-t),$$

where $G(k, T-t)$ is the Fourier transform of the Green’s function defined by
\[ G(k, T - t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{-ikx} G(x, T - t). \] (28)

In terms of \( G(k, T - t) \) Eq. (23) takes the form

\[ \frac{\partial G(k, T - t)}{\partial (T - t)} = [-r + ik(r - q)] \]

\[ + \lambda \int_{-\infty}^{\infty} d\eta p(\eta) \{ e^{ik\eta} - 1 - ik(e^{\eta} - 1) \} G(k, T - t), \]

while the terminal condition (24) becomes

\[ G(k, T - t)|_{t=T} = G(k, 0) = 1. \] (30)

The solution of the problem defined by Eqs. (29) and (30) is

\[ G(k, T - t) = e^{-r(T-t)} \]

\[ \times \exp \left\{ [ik(r - q) + \lambda \int_{-\infty}^{\infty} d\eta p(\eta) \{ e^{ik\eta} - 1 - ik(e^{\eta} - 1) \}] (T-t) \right\}, \]

Then Eq. (27) gives us Green’s function \( G(x - x', T - t) \)

\[ G(x - x', T - t) = \frac{e^{-r(T-t)}}{2\pi} \int_{-\infty}^{\infty} dk e^{ik(x-x')} \]

\[ \times \exp \left\{ [ik(r - q) + \lambda \int_{-\infty}^{\infty} d\eta p(\eta) \{ e^{ik\eta} - 1 - ik(e^{\eta} - 1) \}] (T-t) \right\}. \] (32)

Substituting Eq. (32) into Eq. (25) yields for the value of European call option \( C(x, T - t) \)
\[ C(x, T - t) = \frac{Ke^{-r(T-t)}}{2\pi} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dk e^{ik(x-x')} \]
\[ \times \exp \left\{ ik(r - q) + \lambda \int_{-\infty}^{\infty} d\eta p(\eta) \{ e^{ik\eta} - 1 - ik(e^{\eta} - 1) \} (T - t) \right\} \] (33)
\[ \times \max(e^{x'} - 1, 0). \]

Further, changing integration variable \( x' \) to \( z \)
\[ z = x - x' + (r - q - \lambda \varsigma)(T - t), \] (34)
and introducing parameter \( l \) defined by
\[ l = x + (r - q - \lambda \varsigma)(T - t), \] (35)
yield
\[ C(x, T - t) = Ke^{-r(T-t)} \frac{e^l}{2\pi} \int_{-\infty}^{l} dz \int_{-\infty}^{\infty} dk e^{ikz} \exp \{ \lambda(T - t)\xi(k) \} \] (36)
\[ -\frac{Ke^{-r(T-t)}}{2\pi} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dk e^{ikz} \exp \{ \lambda(T - t)\xi(k) \}, \]
where \( \varsigma \) and \( \xi(k) \) have been introduced by
\[ \varsigma = \int_{-\infty}^{\infty} d\eta p(\eta)(e^{\eta} - 1), \] (37)
and
\[ \xi(k) = \int_{-\infty}^{\infty} d\eta p(\eta)(e^{ik\eta} - 1), \] (38)
with \( p(\eta) \) being the probability density function of the magnitude of jumps.

It is easy to see that Eq. (36) can be written in the form

\[
C(S, T - t) = S e^{-q(T-t)} L_1(l) - K e^{-r(T-t)} L_2(l),
\]

where functions \( L_1(l) \) and \( L_2(l) \) are defined as

\[
L_1(l) = \frac{e^{-\lambda x(T-t)}}{2\pi} \int_{-\infty}^{l} dz \int_{-\infty}^{\infty} dk e^{ikz} \exp \{ \lambda (T-t) \xi(k) \},
\]

\[
L_2(l) = \frac{1}{2\pi} \int_{-\infty}^{l} dz \int_{-\infty}^{\infty} dk e^{ikz} \exp \{ \lambda (T-t) \xi(k) \},
\]

with \( \varsigma \) and \( \xi(k) \) given by Eqs. (37) and (38).

Thus, we found the new equation (39) to value European call option when the stochastic dynamics of a stock price is governed by Eq. (1).

### 3.2.2 Put option

Substitution of Eq. (32) into Eq. (26) yields for the value of European put option

\[
P(x, T - t) = \frac{K e^{-r(T-t)}}{2\pi} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dk e^{i k (x - x')}
\]

\[
\times \exp \left\{ ik(r - q) + \lambda \int d\eta p(\eta) \left\{ e^{i k \eta} - 1 - i k (e^{\eta} - 1) \right\} (T-t) \right\}
\]

\[
\times \max(1 - e^{x'}, 0).
\]

This equation can be presented as

\[
P(S, T - t) = K e^{-r(T-t)} L_2(l) - S e^{-q(T-t)} L_1(l),
\]

where \( l \) is given by Eq. (35) while functions \( L_1(l) \) and \( L_2(l) \) are introduced by
\( L_1(l) = \frac{e^{-\lambda(T-t)}}{2\pi} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dk e^{ikz} \exp \{ \lambda(T-t)\xi(k) \}, \quad (44) \)

\( L_2(l) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dk e^{ikz} \exp \{ \lambda(T-t)\xi(k) \}, \quad (45) \)

with \( \varsigma \) and \( \xi(k) \) defined by Eqs. (37) and (38).

Thus, we obtained new equation (43) to value European put option when the stochastic dynamics of stock price is governed by Eq. (1).

4 Put-Call parity

The put-call parity is a fundamental relationship between the values of European call and put options, both with the same strike price \( K \) and time to expiry \( T - t \). This relationship is a manifestation of the no-arbitrage principle. The put-call parity equation is model independent and has the form

\( C(S,T-t) - P(S,T-t) = Se^{-q(T-t)} - Ke^{-r(T-t)}. \quad (46) \)

To prove this relationship we use Eqs. (39) and (43) to obtain

\( C(S,T-t) - P(S,T-t) = \)

\( = Se^{-q(T-t)}(L_1(l) + L_1(l)) - Ke^{-r(T-t)}(L_2(l) + L_2(l)), \)

From the definitions of functions \( L \) given by Eqs. (40) and (41), and functions \( L \) given by Eqs. (44) and (45) we have

\( L_1(l) + L_1(l) = 1, \quad (48) \)

and

\( L_2(l) + L_2(l) = 1. \quad (49) \)

Indeed, with help of Eqs. (40) and (44) we write for Eq. (48)
\[ L_1(l) + \mathcal{L}_1(l) = e^{-\lambda c(T-t)} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dke^{ikz-z} \exp\{\lambda(T-t)\xi(k)\} \]
\[ + \int_{l}^{\infty} dke^{ikz-z} \exp\{\lambda(T-t)\xi(k)\} \]
\[ = e^{-\lambda c(T-t)} \int_{-\infty}^{\infty} dk\delta(k - 1/i) \exp\{\lambda(T-t)\xi(k)\} = 1, \]
we used \(\xi(1/i) = \varsigma\), which follows straightforwardly from Eqs.(37) and (38) for \(\varsigma\) and \(\xi(k)\).

The similar consideration can be provided to prove Eq.(49).

By substituting Eqs.(48) and (49) into Eq.(47) we complete the proof of Put-Call parity equation (46).

5 Greeks

The purpose of this Section is to calculate the Greeks based on the integro-differential pricing equation (16).
In option pricing fundamentals and option trading the Greek letters are used to defined sensitivities (Greeks) of the option value in respect to a change in either underlying price (i.e., stock price) or parameters (i.e., risk-free rate, time to maturity, etc.). The Greeks can be considered as effective tools to measure and manage the risk in an option position. The most common Greeks are the first order derivatives: delta, rho, psi and theta as well as gamma, a second-order derivative of the option value over underlying price.
To simplify the Greeks calculations let’s note that the following equations hold
\[ Se^{-q(T-t)} \frac{\partial L_1(l)}{\partial l} = Ke^{-r(T-t)} \frac{\partial L_2(l)}{\partial l}, \]  
\[ Se^{-q(T-t)} \frac{\partial \mathcal{L}_1(l)}{\partial l} = Ke^{-r(T-t)} \frac{\partial \mathcal{L}_2(l)}{\partial l}. \]
where functions \(L_1(l), L_2(l), \mathcal{L}_1(l), \mathcal{L}_2(l)\) are given by Eqs.(40), (41), (44) and (45).
5.1 Common Greeks

The Greek delta for a call option $\Delta_C$ is given by

$$\Delta_C = \frac{\partial C(S, T - t)}{\partial S} = e^{-q(T-t)}L_1(l). \quad (53)$$

The Greek rho $\rho_C$ for a call option is

$$\rho_C = \frac{\partial C(S, T - t)}{\partial r} = (T - t)Ke^{-r(T-t)}L_2(l). \quad (54)$$

The Greek psi $\psi_C$ for a call option is the first partial derivative of option value $C(S, T - t)$ with respect to the dividend rate $q$,

$$\psi_C = \frac{\partial C(S, T - t)}{\partial q} = -(T - t)Se^{-q(T-t)}L_1(l). \quad (55)$$

The Greek theta $\Theta_C$ for a call option is defined by

$$\Theta_C = -\frac{\partial C(S, T - t)}{\partial (T - t)} = \frac{\partial C(S, T - t)}{\partial t}. \quad (56)$$

Thus, we have

$$\Theta_C = qSe^{-q(T-t)}L_1(l) - rKe^{-r(T-t)}L_2(l) + Se^{-q(T-t)}\lambda \xi L_1(l)$$

$$- \lambda Se^{-q(T-t)}\frac{e^{-\lambda \xi (T-t)}}{2\pi} \int_{-\infty}^{l} dze^{-z} \int_{-\infty}^{\infty} dk e^{ikz} \xi (k) \exp \left\{ \lambda(T - t)\xi (k) \right\}.$$  

$$+ \frac{\lambda Ke^{-r(T-t)}}{2\pi} \int_{-\infty}^{l} dz \int_{-\infty}^{\infty} dk e^{ikz} \xi (k) \exp \left\{ \lambda(T - t)\xi (k) \right\}.$$  

Finally, the second order sensitivity gamma $\Gamma_C$ for a call option is

$$\Gamma_C = \frac{\partial^2 C(S, T - t)}{\partial S^2} = e^{-q(T-t)}\frac{\partial L_1(l)}{\partial l} \frac{\partial l}{\partial S} = \frac{e^{-q(T-t)}}{S} \frac{\partial L_1(l)}{\partial l}. \quad (57)$$

Functions $L_1(l)$ and $L_2(l)$ in the formulas above are given by Eqs.(40) and (41).

Table 1 summarizes the common Greeks for a call option.
\[ \Delta_C = \frac{\partial C(S,T-t)}{\partial S} = e^{-q(T-t)} L_1(l) \]

\[ \Gamma_C = \frac{\partial^2 C(S,T-t)}{\partial S^2} = \frac{e^{-q(T-t)} \partial L_1(l)}{S} = \frac{K e^{-r(T-t)} \partial L_2(l)}{\partial l} \]

\[ \rho_C = \frac{\partial C(S,T-t)}{\partial r} = (T-t) K e^{-r(T-t)} L_2(l) \]

\[ \psi_C = \frac{\partial C(S,T-t)}{\partial q} = -(T-t) S e^{-q(T-t)} L_1(l) \]

\[ \Theta_C = \frac{\partial^2 C(S,T-t)}{\partial t \partial q} = \frac{q Se^{-q(T-t)} L_1(l) - r K e^{-r(T-t)} L_2(l)}{2\pi} \int_{-\infty}^{t} \int_{-\infty}^{\infty} dze^{-z} \int_{-\infty}^{t} dke^{ikz} \xi(k) e^{\lambda(T-t)\xi(k)} \]

\[ + \frac{\lambda K e^{-r(T-t)}}{2\pi} \int_{-\infty}^{t} \int_{-\infty}^{\infty} dz \int_{-\infty}^{t} dke^{ikz} \xi(k) e^{\lambda(T-t)\xi(k)} \]

Table 1. Common Greeks (Call option)

The common Greeks for a put option can be easily found by using the Put-Call parity equation (46).

The Greek delta for a put option \( \Delta_P \) is

\[ \Delta_P = \frac{\partial P(S,T-t)}{\partial S} = -e^{-q(T-t)} L_1(l) = \Delta_C - e^{-q(T-t)}, \quad (58) \]

where \( \Delta_C \) is the Greek delta for a call option given by Eq.(53).

The Greek rho \( \rho_P \) for a put option is

\[ \rho_P = \frac{\partial P(S,T-t)}{\partial r} = -(T-t) K e^{-r(T-t)} L_2(l) = \rho_C - (T-t) K e^{-r(T-t)}, \quad (59) \]

where \( \rho_C \) is the Greek rho for a call option given by Eq.(54).

The Greek psi \( \psi_P \) for a put option is the first partial derivative of option value \( P(S,T-t) \) with respect to the dividend rate \( q \),

\[ \psi_P = \frac{\partial P(S,T-t)}{\partial q} = (T-t) S e^{-q(T-t)} L_1(l) = \psi_C + (T-t) S e^{-q(T-t)}, \quad (60) \]

where \( \psi_C \) is the Greek psi for a call option given by Eq.(55).
The Greek theta $\Theta_P$ for a put option is defined by

\[
\Theta_P = -\frac{\partial P(S, T - t)}{\partial (T - t)} = \frac{\partial P(S, T - t)}{\partial t}.
\]

Thus, from the Put-Call parity equation (46) we have

\[
\Theta_P = \Theta_C - qSe^{-q(T-t)} + rKe^{-r(T-t)},
\]

where $\Theta_C$ is the Greek theta for a call option given by Eq.(56).

Finally, the second order sensitivity gamma $\Gamma_P$ for a put option is

\[
\Gamma_P = \frac{\partial^2 P(S, T - t)}{\partial^2 t} e^{-q(T-t)} = \frac{\partial L_1(l)}{\partial l} = \Gamma_C,
\]

where $\Gamma_C$ is the Greek gamma for a call option given by Eq.(55).

Hence, we conclude that

\[
\Gamma_P = \Gamma_C.
\]

Table 2 summarizes the common Greeks for a put option.

| Function | Put |
|----------|-----|
| Delta, $\Delta_P$ | $-e^{-q(T-t)} \mathcal{L}_1(l) = \Delta_C - e^{-q(T-t)}$ |
| Gamma, $\Gamma_P$ | $e^{-q(T-t)} \frac{\partial \mathcal{L}_1(l)}{\partial l} = \frac{\partial L_1(l)}{\partial l}$ |
| Rho, $\rho_P$ | $-(T - t)Ke^{-r(T-t)} \mathcal{L}_2(l) = \rho_C - (T - t)Ke^{-r(T-t)}$ |
| Psi, $\psi_P$ | $(T - t)Se^{-q(T-t)} \mathcal{L}_1(l) = \psi_C + (T - t)Se^{-q(T-t)}$ |
| Theta, $\Theta_P$ | $\Theta_C - qSe^{-q(T-t)} + rKe^{-r(T-t)}$ |

Table 2. Common Greeks (Put option)

### 5.2 New Greeks

#### 5.2.1 Gaussian jumps model

The new Greeks are an option sensitivities associated with the market parameters involved in the definition of shot noise process $F(t)$ (see, Eq.(1)). Besides the parameter $\lambda$, which is the rate of a stock price jump arrivals, there can be other market parameters associated with probability density function $p(\eta)$ involved into the definition of shot noise $F(t)$ (see, Eq.(1)). As
soon as we specify the probability density function \( p(\eta) \), we will get the market parameters associated with it. Having these parameters we can introduce a few new Greeks.

At this point we assume, as an example, that probability density function of stock price jumps is a normal distribution

\[
p(\eta) = \frac{1}{\sqrt{2\pi}\delta} \exp\left\{ -\frac{(\eta - \nu)^2}{2\delta^2} \right\},
\]

where the market parameters \( \nu \) and \( \delta^2 \) are the mean and variance of a stock price jump magnitude.

The first order derivatives of call option with respect to the parameters \( \lambda \), \( \nu \) and \( \delta^2 \) will bring three new Greeks, which do not exist in the Black-Scholes option pricing framework.

It follows immediately from Put-Call parity equation \([46]\) that all three new Greeks are the same for call and put options. Thus, we need to calculate these Greeks for call option only.

### 5.2.2 Greek \( \kappa \)

We introduce the notation \( \kappa_C \) for the derivative of call option with respect to \( \lambda \), which is the rate of jump arrivals, see Eq.\([5]\). Hence, the definition of the Greek ”kappa” \( \kappa_C \) is

\[
\kappa_C = \frac{\partial C(S,T-t)}{\partial \lambda}.
\]

The Greek \( \kappa_C \) is a new Greek that does not exist in the Black-Scholes framework, because of absence of the parameter \( \lambda \).

To find the Greek ”kappa” we differentiate Eq.\([39]\) with respect to \( \lambda \). The result is

\[
\frac{\kappa_C}{T-t} = -\varsigma Se^{-q(T-t)} L_1(l)
\]

\[
+ Se^{-q(T-t)} \frac{e^{-\lambda\xi(T-t)}}{2\pi} \int_{-\infty}^{l} dz \int_{-\infty}^{\infty} dk e^{ikz-z\xi(k)} \exp \{ \lambda(T-t)\xi(k) \} \]

\[(66)\]
\[- \frac{Ke^{-r(T-t)}}{2\pi} \int_{-\infty}^{l} \int_{-\infty}^{\infty} dze^{ikz} \xi(k) \exp \{\lambda(T-t)\xi(k)\}.\]

Comparing Eqs. (61) and (66) let’s obtain the relationship between \(\Theta_C\) and \(\kappa_C\).

\[\kappa_C = \frac{T-t}{\lambda} (qSe^{-q(T-t)}L_1(l) - rKe^{-r(T-t)}L_2(l) - \Theta_C),\]  
\[(67)\]

or

\[\Theta_C = qSe^{-q(T-t)}L_1(l) - rKe^{-r(T-t)}L_2(l) - \frac{\lambda\kappa_C}{T-t}.\]  
\[(68)\]

Thus, we discovered a new fundamental relationship between common Greek \(\Theta_C\) and newly introduced Greek \(\kappa_C\).

It has already been mentioned that the ”kappa” for a put option \(\kappa_P\) is the same as ”kappa” for a call option \(\kappa_C\),

\[\kappa_P = \kappa_C,\]  
\[(69)\]

because of the Put-Call parity equation (46).

The fundamental relationship between the common Greeks \(\Theta_P\) and newly introduced \(\kappa_P\) has a form

\[\Theta_P = -qSe^{-q(T-t)}L_1(l) + rKe^{-r(T-t)}L_2(l) - \frac{\lambda\kappa_P}{T-t},\]  
\[(70)\]

which can be easily verified with help of Eqs. (61), (68) and (69).

Straightforward substitution of Eq. (39) into the definition (65) and calculation of the derivative of \(C(S,T-t)\) with respect to \(\lambda\) yield the identities for \(\kappa_C\) with involvement of some common Greeks. For example, it is easy to see that the identity holds

\[\kappa_C = S \frac{\partial \Delta_C}{\partial \lambda} - \frac{1}{T-t} \frac{\partial \rho_C}{\partial \lambda},\]  
\[(71)\]

where \(\Delta_C\), and \(\rho_C\) are defined by Eqs. (53) and (54).
5.2.3 Greek $\mu$

Let us introduce the Greek "mu" $\mu_C$ as the first order derivative of a call option with respect to the parameter $\nu$, which is the mean of a stock price jump magnitude,

$$\mu_C = \frac{\partial C(S, T - t)}{\partial \nu}. \quad (72)$$

The Greek $\mu_C$ is a new Greek that does not exist in the Black-Scholes framework, because of absence of the market parameter $\nu$.

Calculating the derivative of $C(S, T - t)$ with respect to $\nu$ yields

$$\frac{\mu_C}{\lambda(T - t)} = -\frac{\partial \varsigma}{\partial \nu} Se^{-q(T-t)} L_1(l)$$

$$+ S e^{-q(T-t)} \frac{e^{-\lambda \varsigma(T-t)}}{2\pi} \int_{-\infty}^{l} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dk e^{ikz} \frac{\partial \xi(k)}{\partial \nu} \exp \{ \lambda(T - t) \xi(k) \} \quad (73)$$

$$- Ke^{-r(T-t)} \frac{e^{-\lambda \varsigma(T-t)}}{2\pi} \int_{-\infty}^{l} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dk e^{ikz} \frac{\partial \xi(k)}{\partial \nu} \exp \{ \lambda(T - t) \xi(k) \}.$$

It is easy to see from Eqs. (37) and (38) that for $p(\eta)$ given by Eq. (64) we have

$$\frac{\partial \varsigma}{\partial \nu} = e^{\nu \xi + \frac{\xi^2}{2}} = (\varsigma + 1), \quad (74)$$

and

$$\frac{\partial \xi(k)}{\partial \nu} = e^{ik\nu - \frac{k^2\varsigma}{2}} = ik(\xi(k) + 1). \quad (75)$$

Therefore, Eq. (73) can be rewritten as

$$\frac{\mu_C}{\lambda(T - t)} = -Se^{-q(T-t)}(\varsigma + 1) L_1(l)$$

$$+ S e^{-q(T-t)} \frac{e^{-\lambda \varsigma(T-t)}}{2\pi} \int_{-\infty}^{l} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dk e^{ikz} \frac{1}{ik(\xi(k) + 1)} \exp \{ \lambda(T - t) \xi(k) \} \quad (76)$$
Further, taking into account that
\[ ike^{ikz} = \frac{\partial}{\partial z}e^{ikz}, \]
and performing integration over \( dz \) we obtain
\[
\frac{\mu_C}{\lambda(T - t)} = -\varsigma Se^{-q(T-t)}L_1(l) + \frac{\mu_C}{\lambda(T - t)} e^{-\varsigma(T-t)} \int_{-\infty}^{\infty} dze^{-z} \int_{-\infty}^{\infty} dke^{ikz} \xi(k) \exp \{\lambda(T - t)\xi(k)\},
\]
which can be considered as the equation to calculate the Greek \( \mu_C \).

Let us show that Eq.(77) can be used to find new identities with involvement of the Greek \( \mu_C \) and some common Greeks. For instance, if we rewrite Eq.(77) in the form
\[
\mu_C \lambda = Se^{-q(T-t)} L_1(l) \left( \frac{\partial L_1(l)}{\partial \lambda} - \frac{\partial L_1(l)}{\partial l} \frac{\partial l}{\partial \lambda} \right),
\]
then it is easy to see that the identity holds
\[
\frac{\mu_C}{\lambda} = Se^{-q(T-t)} \int_{-\infty}^{\infty} dze^{-z} \int_{-\infty}^{\infty} dke^{ikz} \xi(k) \exp \{\lambda(T - t)\xi(k)\},
\]
which establishes the relationship between newly introduced Greeks \( \kappa_C \) and \( \mu_C \).
5.2.4 Greek $\epsilon$

A new Greek "epsilon" $\epsilon_C$ is introduced as the derivative of a call option with respect to the parameter $\delta$, which is the standard deviation of a stock price jump magnitude,

$$\epsilon_C = \frac{\partial C(S, T - t)}{\partial \delta}. \quad (81)$$

The Greek $\epsilon_C$ is a new Greek that does not exist in the Black-Scholes framework, because of absence of the market parameter $\delta$.

The derivative of $C(S, T - t)$ with respect to $\delta$ can be expressed as

$$\frac{\epsilon_C}{\lambda(T - t)} = -\frac{\partial \zeta}{\partial \delta} S e^{-q(T-t)} L_1(l)$$

$$+ S e^{-q(T-t)} \frac{e^{-\lambda \zeta(T-t)}}{2\pi} \int_{-\infty}^{l} dz e^{-z} \int_{-\infty}^{\infty} dk e^{ikz} \frac{\partial \xi(k)}{\partial \delta} \exp \{\lambda(T - t) \xi(k)\} \quad (82)$$

$$- K e^{-r(T-t)} \int_{-\infty}^{l} dz \int_{-\infty}^{\infty} dk e^{ikz} \frac{\partial \xi(k)}{\partial \delta} \exp \{\lambda(T - t) \xi(k)\}.$$  

It follows from Eqs. (37) and (38) that

$$\frac{\partial \zeta}{\partial \delta} = e^{\nu + \frac{\varepsilon^2}{2}} = \delta(\zeta + 1), \quad (83)$$

and

$$\frac{\partial \xi(k)}{\partial \delta} = e^{ik\nu - \frac{\varepsilon^2 k^2}{2}} = -k^2 \delta(\xi(k) + 1). \quad (84)$$

Further, taking into account that

$$-k^2 e^{ikz} = \frac{\partial^2}{\partial z^2} e^{ikz},$$

we have

$$\frac{\epsilon_C}{\delta \lambda(T - t)} = -(\zeta + 1) S e^{-q(T-t)} L_1(l).$$
\[ + S e^{-q(T-t)} \frac{e^{-\lambda_c(T-t)}}{2\pi} \int_{-\infty}^{l} dz e^{-z} \int_{-\infty}^{\infty} dk \frac{\partial^2}{\partial z^2} e^{ikz} (\xi(k) + 1) \exp \{ \lambda(T-t)\xi(k) \} \]

\[ - Ke^{-r(T-t)} \frac{1}{2\pi} \int_{-\infty}^{l} dz \int_{-\infty}^{\infty} dk \frac{\partial^2}{\partial z^2} e^{ikz} (\xi(k) + 1) \exp \{ \lambda(T-t)\xi(k) \} \]

Integration by parts over \( dz \) yields

\[ \frac{\epsilon_C}{\lambda(T-t)\delta} = -S e^{-q(T-t)} L_1(l) \]

\[ + S e^{-q(T-t)} \frac{e^{-\lambda_c(T-t)}}{2\pi} \int_{-\infty}^{l} dz e^{-z} \int_{-\infty}^{\infty} dk e^{ikz} \xi(k) \exp \{ \lambda(T-t)\xi(k) \} \]

\[ + K e^{r(T-t)} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikl} (\xi(k) + 1) \exp \{ \lambda(T-t)\xi(k) \} , \]

which can be considered as the equation to calculate the Greek \( \epsilon_C \).

Comparing the equation above with Eq.(79) let’s conclude that the following relationship holds

\[ \frac{\epsilon_C}{\lambda(T-t)\delta} = \frac{\mu_C}{\lambda(T-t)} + S^2 \Gamma_C \] (86)

where \( \Gamma_C \) is given by Eq.(57).

Straightforward substitution of Eq.(89) into definition (81) and calculation of the derivative of \( C(S, T - t) \) with respect to \( \delta \) yield the identities for \( \epsilon_C \) with involvement of common Greeks \( \Delta_C \) and \( \rho_C \). For example, it is easy to see that the identity holds

\[ \epsilon_C = S \frac{\partial \Delta_C}{\partial \delta} - \frac{1}{T-t} \frac{\partial \rho_C}{\partial \delta} , \]

where \( \Delta_C \) and \( \rho_C \) are defined by Eqs.(53) and (54).
6 Black Scholes equation

6.1 Diffusion approximation

We are aiming to show that the well-known Black-Scholes equation can be obtained from the integro-differential option pricing equation Eq. (16). To get the Black-Scholes equation let’s consider the market situation when variance of a stock price jump magnitude $\delta^2 \to 0$, while the arrival rate of price jumps $\lambda \to \infty$ in such way that the product $\lambda \int_{-\infty}^{\infty} d\eta p(\eta)\eta^2$ remains finite. We call this case ”diffusion approximation”.

Due to the condition $\delta^2 \to 0$ the expression under the integral sign in Eq.(16) can be expanded in $\eta$ up to the second-order

$$\frac{\partial C(x,t)}{\partial t} + (r - q)\frac{\partial C(x,t)}{\partial x}$$

$$+ \lambda \int_{-\infty}^{\infty} d\eta p(\eta)\eta^2 \left\{ \frac{\partial^2 C(x,t)}{\partial x^2} - \frac{\partial C(x,t)}{\partial x} \right\} - rC(x,t) = 0.$$ (87)

It is convenient to introduce the notation

$$\sigma^2 = \lambda \int_{-\infty}^{\infty} d\eta p(\eta)\eta^2.$$ (88)

to simplify the equations in downstream consideration. In general, the ”diffusion approximation” is the case when the mean of stock price jumps $\nu \to 0$, $\delta^2 \to 0$ and $\lambda \to \infty$ while the products $\lambda \int_{-\infty}^{\infty} d\eta p(\eta)\eta$ and $\lambda \int_{-\infty}^{\infty} d\eta p(\eta)\eta^2$ remain finite.

6.2 Option pricing equation

With help of notation given by Eq.(88) equation (87) can be rewritten as
\[
\frac{\partial C(x,t)}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 C(x,t)}{\partial x^2} + (r - q - \frac{\sigma^2}{2}) \frac{\partial C(x,t)}{\partial x} - rC(x,t) = 0. \tag{89}
\]

Then the substitution

\[
\frac{S}{K} = e^x, \tag{90}
\]

yields

\[
\frac{\partial C(S,t)}{\partial t} + \frac{\sigma^2}{2} \left( S \frac{\partial}{\partial S} \right)^2 C(S,t) + (r - q - \frac{\sigma^2}{2}) S \frac{\partial C(S,t)}{\partial S} - rC(S,t) = 0, \tag{91}
\]

or

\[
\frac{\partial C(S,t)}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 C(S,t)}{\partial S^2} + (r - q) S \frac{\partial C(S,t)}{\partial S} - rC(S,t) = 0. \tag{92}
\]

This is the famous Black-Scholes equation \[1\], which has to be supplemented with the terminal condition given by Eq.\(18\) for call option and by Eq.\(20\) for put option.

Thus, it has been shown that new pricing equation \(16\) goes into the Black-Scholes equation \(92\) in diffusion approximation. The parameter \(\sigma\) introduced by Eq.\(88\) is called "volatility" of a stock price \(S\).

### 6.3 The solution to Black Scholes equation

The straightforward solution of Eq.\(92\) with the terminal condition \(18\) has been presented in the Appendix.

Here we show that in diffusion approximation solution \(39\) to the integro-differential pricing equation \(16\) goes into the Black-Scholes solution Eq.\(92\).

In the diffusion approximation we have

\[
\lambda \varsigma = \lambda \int_{-\infty}^{\infty} \! d\eta p(\eta)(\eta + \frac{\eta^2}{2}) = \lambda \nu + \frac{\sigma^2}{2}, \tag{93}
\]

and
\[ \lambda \xi(k) = \lambda \int_{-\infty}^{\infty} d\eta p(\eta)(ik\eta - \frac{k^2\eta^2}{2}) = ik\lambda \nu - \frac{k^2\sigma^2}{2}, \quad (94) \]

where \( \sigma^2 \) is introduced by Eq. (88) and \( \varsigma \) and \( \xi(k) \) are defined by Eqs. (37) and (38).

### 6.3.1 Function \( L_1(l) \) in diffusion approximation

To find the equation for \( L_1(l) \) in diffusion approximation we substitute Eqs. (93) and (94) into Eq. (40)

\[ L_1(l) \rightarrow_{\text{diff}} L_1^{(\text{diff})} = \frac{e^{-\frac{\lambda \nu + \frac{\sigma^2}{2}}{(T-t)}}}{2\pi} \int_{-\infty}^{\infty} dze^{-z} \quad (95) \]

Calculation of the integral over \( dk \) results in

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} dke^{ikz} \exp \left\{ (ik\lambda \nu - \frac{k^2\sigma^2}{2})(T-t) \right\} \quad (96) \]

Substituting Eq. (96) into Eq. (95) and changing integration variable \( z \rightarrow y \)

\[ z \rightarrow y = z + \lambda \nu(T-t), \]

yield

\[ L_1^{(\text{diff})} = \frac{1}{\sigma \sqrt{2\pi(T-t)}} \]

\[ x+(r-q-\frac{\sigma^2}{2})(T-t) \]

\[ \times \int_{-\infty}^{\infty} dy \exp \left\{ -\frac{1}{2} \left( \frac{y}{\sigma \sqrt{T-t}} + \sigma \sqrt{T-t} \right)^2 \right\}. \]
With help of new integration variable $w$

$$y \rightarrow w = \frac{y}{\sigma \sqrt{T-t}} + \sigma \sqrt{T-t},$$

we rewrite Eq.(97) in the following way

$$L_1^{(\text{diff})} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x + (r - q - \frac{\sigma^2}{2})(T-t) + \sigma \sqrt{T-t}} dw e^{-\frac{w^2}{2}}. \quad (98)$$

Then function $L_1^{(\text{diff})}$ takes the form

$$L_1^{(\text{diff})} = N(d_1), \quad (99)$$

where function $N(d)$ is the cumulative distribution function of the standard normal distribution \[1\]

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} dw e^{-\frac{w^2}{2}}, \quad (100)$$

and parameter $d_1$ has been introduced by

$$d_1 = x + (r - q - \frac{\sigma^2}{2})(T-t) + \frac{\sigma \sqrt{T-t}}{\sigma} + \sigma \sqrt{T-t} = x + (r - q + \frac{\sigma^2}{2})(T-t). \quad (101)$$

Thus, it has been shown that in diffusion approximation function $L_1(l)$ goes into function $N(d_1)$,

$$L_1(l) \rightarrow L_1^{(\text{diff})} = N(d_1). \quad (102)$$

The similar consideration brings the equation for function $L_1(l)$ in diffusion approximation

$$L_1(l) \rightarrow L_1^{(\text{diff})} = N(-d_1), \quad (103)$$

with $N(d)$ and $d_1$ defined by Eqs.(100) and (101).
6.3.2 Function $L_2(l)$ in diffusion approximation

To find the equation for $L_2(l)$ in diffusion approximation we substitute Eqs. (93) and (94) into Eq. (41)

$$L_2(l) \rightarrow L_2^{(\text{diff})} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{x+(r-q-\lambda \nu- \frac{\sigma^2}{2})(T-t)}{dz}$$ (104)

$$\times \int_{-\infty}^{\infty} dke^{ikz} \exp \left\{ (ik\lambda \nu - \frac{k^2\sigma^2}{2})(T-t) \right\} ,$$

Following the consideration provided while we calculated $L_1^{(\text{diff})}$ we find

$$L_2^{(\text{diff})} = N(d_2),$$ (105)

where function $N(d)$ is defined by Eq. (100) and $d_2$ has been introduced by

$$d_2 = \frac{x+(r-q-\frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t}$$ (106)

Thus, it has been shown that in diffusion approximation function $L_2(l)$ goes into function $N(d_2)$.

In the similar way the equation for $L_2(l)$ can be obtained

$$L_2(l) \rightarrow L_2^{(\text{diff})} = N(-d_2),$$ (107)

with $N(d)$ and $d_2$ defined by Eqs. (100) and (106).

6.3.3 Solution to option pricing equation in diffusion approximation

Using Eqs. (99) and (105) we can write Eq. (39) in the form

$$C_{BS}(S, T-t) = Se^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2),$$ (108)

where $C_{BS}(S, T-t)$ stands for the value of European call option in the Black-Scholes model.

Using Eqs. (103) and (107) we can write Eq. (43) in the form
\[ P_{BS}(S, T - t) = Ke^{-r(T-t)}N(-d_2) - Se^{-q(T-t)}N(-d_1), \quad (109) \]

where \( P_{BS}(S, T - t) \) stands for the value of European put option in the Black-Scholes model.

Hence, we see that in diffusion approximation, solutions (39) and (43) go into the Black-Scholes solutions for call and put options. The key feature of diffusion approximation is the Black-Scholes volatility defined by Eq.(88). In other words, Eq.(88) sheds light into the stochastic dynamic origin of the Black-Scholes volatility, which emerges naturally in the diffusion approximation.

6.4 Greeks in diffusion approximation

6.4.1 Greek vega

In the Black-Scholes world there is a volatility \( \sigma \). Hence, we can consider the derivative of call and put options with respect to \( \sigma \). This is the Greek “vega”.

For instance, for European call option we have vega \( \nu_C \)

\[ \nu_C = \frac{\partial C(S, T - t)}{\partial \sigma} = Se^{-q(T-t)}\sqrt{T-t}N'(d_1) = Ke^{-r(T-t)}\sqrt{T-t}N'(d_2), \quad (110) \]

where \( N'(d) \) stands for derivative of \( N(d) \) with respect to \( d \),

\[ N'(d) = \frac{1}{\sqrt{2\pi}} \exp(-d^2/2), \quad (111) \]

and \( d_1 \) and \( d_2 \) are defined by Eqs.(101) and (106).

It follows from the Put-call parity law (46), which holds for the Black-Scholes solutions, that

\[ \nu_P = \frac{\partial P(S, T - t)}{\partial \sigma} = \frac{\partial C(S, T - t)}{\partial \sigma} = \nu_C, \quad (112) \]

which means that the Greeks vega for call and put options are the same.
6.4.2 Black-Scholes Greeks

It follows from Eqs. (102), (103), (105), and (107) that in the diffusion approximation all common Greeks presented by Tables 1 and 2 go into the well-known Black-Scholes Greeks.

Table 3 summarizes the Black-Scholes Greeks for a call option

| Greek | Formula |
|-------|---------|
| Delta, $\Delta_C$ | $\frac{\partial C(S,T-t)}{\partial S} = e^{-q(T-t)} N(d_1)$ |
| Gamma, $\Gamma_C$ | $\frac{\partial^2 C(S,T-t)}{\partial S^2} = \frac{e^{-q(T-t)}}{S\sigma \sqrt{T-t}} N'(d_1) = \frac{K e^{-r(T-t)}}{S^2 \sigma \sqrt{T-t}} N''(d_2)$ |
| Rho, $\rho_C$ | $\frac{\partial C(S,T-t)}{\partial \rho} = (T-t)Ke^{-r(T-t)} N(d_2)$ |
| Psi, $\psi_C$ | $\frac{\partial C(S,T-t)}{\partial q} = -\frac{\sigma S e^{-q(T-t)} N'(d_1)}{2\sqrt{T-t}} + q Se^{-q(T-t)} N(d_1) - r Ke^{-r(T-t)} N(d_2)$ |
| Theta, $\Theta_C$ | $\frac{\partial C(S,T-t)}{\partial \tau} = \frac{Se^{-q(T-t)} \sqrt{T-t} N'(d_1)}{\sqrt{T-t}} = K e^{-r(T-t)} \sqrt{T-t} N'(d_2)$ |

Table 3. Black-Scholes Greeks (Call option)

7 Generalized option pricing framework

7.1 Geometric Brownian motion and Geometric Shot Noise

The new option pricing framework presented in Sec.3 can be easily generalized to accommodate the superposition of Geometric Brownian motion and a Geometric Shot Noise motion. Indeed, if besides the Geometric Shot Noise motion we have the Geometric Brownian motion, then Eq.(11) becomes

$$dS = (\mu - q)Sdt + \sigma SdW + SF(t)dt,$$

where $\mu$ and $\sigma$ are constants belonging to the Brownian motion process, and $W$ is a Wiener process, while all other notations are the same as in Eq.(1).

7.2 Generalized arbitrage-free integro-differential option pricing equation

If a stock price follows Eq.(113) then the generalized arbitrage-free integro-differential option pricing equation has a form
\[
\frac{\partial C(x,t)}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 C(x,t)}{\partial x^2} + \left( r - q - \frac{\sigma^2}{2} \right) \frac{\partial C(x,t)}{\partial x} + \lambda \int d\eta p(\eta) \left\{ C(x + \eta, t) - C(x, t) - (e^{\eta} - 1) \frac{\partial C(x,t)}{\partial x} \right\} 
\]

\[+ rC(x,t) = 0, \]

where \( x \) is given by Eq.(17).

If we go from \( C(x,t) \) to \( C(S,t) \) where \( x \) and \( S \) are related to each other by Eq.(17), then we can write the generalized option pricing equation in the form

\[
\frac{\partial C(S,t)}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 C(S,t)}{\partial S^2} + (r - q) S \frac{\partial C(S,t)}{\partial S} + \lambda \int d\eta p(\eta) \left\{ C(S e^{\eta}, t) - C(S, t) - (e^{\eta} - 1) S \frac{\partial C(S,t)}{\partial S} \right\} - rC(S,t) = 0, \]

with the terminal condition given either by Eq.(18) or by Eq.(20).

In the case when \( \lambda = 0 \) Eq.(115) becomes the Black-Scholes equation (see, Eq.(92)). In the case when \( \sigma = 0 \) Eq.(115) becomes the integro-differential option pricing equation (see, Eq.(22)).

The solution to Eq.(115) with the terminal condition (18) is

\[
C(S, T - t) = S e^{-q(T-t)} L_1(l_\sigma) - K e^{-r(T-t)} L_2(l_\sigma), \]

where new functions \( L_1(l_\sigma) \) and \( L_2(l_\sigma) \) are introduced by

\[
L_1(l_\sigma) = \frac{e^{-\frac{\sigma^2}{2} - l_\sigma(T-t)}}{2\pi} \int_{-\infty}^{l_\sigma} \int_{-\infty}^{\infty} dk e^{ikz - z} \exp \left\{ -\frac{\sigma^2 k^2}{2} + \lambda(T-t) \xi(k) \right\}, \]

and

34
\[
\mathbf{L}_2(l_\sigma) = \frac{1}{2\pi} \int_{-\infty}^{l_\sigma} dz \int_{-\infty}^{\infty} dke^{ikz} \exp\left\{-\frac{\sigma^2 k^2}{2} + \lambda(T-t)\xi(k)\right\},
\]

(118)

with \(\varsigma\) and \(\xi(k)\) given by Eqs.(37) and (38), and the parameter \(l_\sigma\) is

\[
l_\sigma = x + (r - q - \frac{\sigma^2}{2} - \lambda\varsigma)(T-t),
\]

(119)

Generalized option pricing formula (116) for European call option follows from a stock price stochastic dynamics (113) modeled by the superposition of Geometric Brownian motion and a Geometric Shot Noise.

Let us consider two limit cases of Eq.(116).

1. In the limit case when \(\sigma = 0\) we have

\[
\mathbf{L}_1(l_\sigma) \to L_1(l),
\]

(120)

\[
\mathbf{L}_2(l_\sigma) \to L_2(l),
\]

(121)

and Eq.(116) goes into the option pricing equation (39) with \(l\) defined by Eq.(35).

2. In the limit case when \(\lambda = 0\) we have

\[
\mathbf{L}_1(l_\sigma) \to N(d_1),
\]

(122)

\[
\mathbf{L}_2(l_\sigma) \to N(d_2),
\]

(123)

and Eq.(116) goes into the Black-Scholes option pricing formula (108) with \(d_1\) and \(d_2\) defined by Eqs.(103) and (106).

Hence, Eq.(116) describes the impact on the value of European call option as the interplay between Gaussian Geometric Brownian motion and non-Gaussian Geometric Shot Noise involved into a stock price stochastic dynamics.

On a final note, let us present the generalized option pricing formula for European put option. The solution to Eq.(115) with the terminal condition (20) is
\[ P(S, T - t) = Ke^{-r(T-t)} \mathcal{L}_2(l_\sigma) - Se^{-q(T-t)} \mathcal{L}_1(l_\sigma), \]  

(124)

where new functions \( \mathcal{L}_1(l) \) and \( \mathcal{L}_2(l) \) are introduced by

\[
\mathcal{L}_1(l_\sigma) = \frac{e^{-\frac{\sigma^2}{2} - \lambda(T-t)}}{2\pi} \int_{l_\sigma}^{\infty} dz \int_{-\infty}^{\infty} dk e^{ikz-z} \exp \left\{ -\frac{\sigma^2 k^2}{2} + \lambda(T-t)\xi(k) \right\},
\]

(125)

and

\[
\mathcal{L}_2(l_\sigma) = \frac{1}{2\pi} \int_{l_\sigma}^{\infty} dz \int_{-\infty}^{\infty} dk e^{ikz} \exp \left\{ -\frac{\sigma^2 k^2}{2} + \lambda(T-t)\xi(k) \right\},
\]

(126)

and the parameter \( l_\sigma \) is defined by Eq.(119).

Let us consider two limit cases of Eq.(125).

1. In the limit case when \( \sigma = 0 \) we have

\[
\mathcal{L}_1(l_\sigma) \xrightarrow{\sigma=0} \mathcal{L}_1(l),
\]

(127)

\[
\mathcal{L}_2(l_\sigma) \xrightarrow{\sigma=0} \mathcal{L}_2(l),
\]

(128)

and Eq.(125) goes into the option pricing equation (43) with \( l \) defined by Eq.(35).

2. In the limit case when \( \lambda = 0 \) we have

\[
\mathcal{L}_1(l_\sigma) \xrightarrow{\lambda=0} N(-d_1),
\]

(129)

\[
\mathcal{L}_2(l_\sigma) \xrightarrow{\lambda=0} N(-d_2),
\]

(130)

and Eq.(124) goes into the Balck-Scholes option pricing formula (109) with \( d_1 \) and \( d_2 \) defined by Eqs.(101) and (106).

Hence, Eqs.(124) describes the impact on the value of European put option as the interplay between Gaussian Geometric Brownian motion and non-Gaussian Geometric Shot Noise involved into a stock price stochastic dynamics modeled by Eq.(113).
8 Short term interest rate dynamics

8.1 Langevin equation with shot noise

We introduce a new stochastic model to describe the short term interest rate dynamics,

\[ dr = -ardt + F(t)dt, \quad (131) \]

which is stochastic differential equation of the Langevin type with the shot noise \( F(t) \) given by Eq.(2) and the parameter \( a \) \((a > 0)\) being the speed at which \( r(t) \) goes to its equilibrium level at \( t \to \infty \).

The short term interest rate \( r(t) \) governed by Eq.(131) is non-Gaussian random process.

It is easy to see from Eq.(131) that if at the time moment \( t \) the short term interest rate was \( r(t) \) then later on, at the time moment \( s, s > t \), it will be

\[ r(s) = e^{-a(s-t)}r(t) + \int_t^s d\tau e^{-a(s-\tau)}F(\tau). \quad (132) \]

or

\[ r(s) = e^{-as}r_0 + \int_0^s d\tau e^{-a(s-\tau)}F(\tau), \quad (133) \]

with \( r_0 = r(t)|_{t=0} \) if we choose \( t = 0 \).

8.2 Bond Price

8.2.1 Affine term structure

The price at time \( t \) of a zero-coupon bond \( P(t, T, r) \) which pays one currency unit at maturity \( T, (0 \leq t \leq T) \), is

\[ P(t, T, r) = \exp \left\{ - \int_t^T dsr(s) \right\} >_F, \quad (134) \]

subject to terminal condition
Here \( r(s) \) is given by Eq. (132), \( r = r(t) \) is the rate at the time \( t \), and \(<...>_F \) stands for the average with respect to all randomness involved into the random force \( F(t) \).

Further, substitution of \( r(s) \) from Eq. (132) into Eq. (134) and straightforward integration over \( ds \) yields

\[
P(t, T, r) = \exp \{ A(t, T) - B(t, T)r(t) \},
\]

where the following notations have been introduced

\[
B(t, T) = \int_t^T ds e^{-a(s-t)} = \frac{1 - e^{-a(T-t)}}{a},
\]

and

\[
\exp \{ A(t, T) \} = < \exp \left\{ - \int_t^T ds B(s, T)F(s) \right\} >_F.
\]

Therefore, we conclude that the new model introduced by Eq. (131) provides affine term structure.

To calculate average in Eq. (138) we use Eqs. (3) and (13) to obtain

\[
\exp \{ A(t, T) \} = \exp \left\{ \lambda \int_t^T ds \int_{-\infty}^{\infty} d\eta p(\eta)(e^{-\eta B(s, T)} - 1) \right\},
\]

from which it follows directly that

\[
A(t, T) = \lambda \int_t^T ds \int_{-\infty}^{\infty} d\eta p(\eta)(e^{-\eta B(s, T)} - 1).
\]

where \( B(s, T) \) is defined by Eq. (137) and \( p(\eta) \) given by Eq. (64) is the probability density function of shot noise amplitude \( \eta \).

The calculation of the integral over \( \eta \) in Eq. (140) gives the result
\[ \int_{-\infty}^{\infty} d\eta p(\eta) (e^{-\eta B(s,T)} - 1) = \exp\{-\nu B(s,T) + \frac{\delta^2}{2} B^2(s,T)\} - 1, \quad (141) \]

therefore, we have for \( A(t,T) \)

\[ A(t,T) = \lambda \int_{t}^{T} ds \left\{ \exp\{-\nu B(s,T) + \frac{\delta^2}{2} B^2(s,T)\} - 1 \right\}. \quad (142) \]

By introducing a new integration variable \( s \to y = B(s,T) \), we can express \( A(t,T) \) in the form

\[ A(t,T) = \lambda \int_{0}^{B(t,T)} dy \frac{d}{dy} \left\{ e^{-\nu y + \frac{\delta^2}{2} y^2} - 1 \right\}. \quad (143) \]

Thus, based on short term interest rate dynamics introduced by Eq. (131), we obtained formula (136) for price of a zero coupon bond, where \( B(t,T) \) is given by Eq. (137) and \( A(t,T) \) is given by Eqs. (142) or (143).

### 8.2.2 The term structure equation

Defined by Eq. (134) the price of zero-coupon bond \( P(t,T,r) \) solves the following integro-differential equation

\[ \frac{\partial P(t,T,r)}{\partial t} - ar \frac{\partial P(t,T,r)}{\partial r} \]

\[ + \lambda \int_{-\infty}^{\infty} d\eta p(\eta) \{P(t,T,r+\eta) - P(t,T,r)\} = rP(t,T,r), \quad (144) \]

subject to terminal condition (135), with \( \lambda \) being the number of short term interest rate jumps per unit of time, \( p(\eta) \) being the probability density function of jump magnitude, and the parameter \( a \) being the speed at which the short term interest rate goes to its equilibrium level at \( t \to \infty \).
Following Vasiček \cite{2} terminology, we call Eq.\textit{(144)} ”the term structure equation”.

To find the solution to Eq.\textit{(144)} subject to terminal condition \textit{(135)} let’s note that the affine term structure means that the bond price admits solution of the form given by Eq.\textit{(136)}. Substitution of Eq.\textit{(136)} into Eq.\textit{(144)} gives the two ordinary differential equations to obtain $A(t,T)$ and $B(t,T)$

$$\frac{dA(t,T)}{dt} + \lambda \int_{-\infty}^{\infty} d\eta p(\eta) \{e^{-\eta B(s,T)} - 1\} = 0,$$ \hspace{1cm} (145)

and

$$\frac{dB(t,T)}{dt} - a B(t,T) + 1 = 0,$$ \hspace{1cm} (146)

subject to terminal conditions at $t = T$

$$A(T,T) = 0,$$ \hspace{1cm} (147)

and

$$B(T,T) = 0.$$ \hspace{1cm} (148)

It is easy to see, that the system of equations (145)-(147) brings the solutions for $A(t,T)$ and $B(t,T)$ given exactly by Eqs.\textit{(140)} and \textit{(137)}.

Thus, we proved that the price of zero-coupon bond $P(t,T,r)$ defined by Eq.\textit{(134)} solves the new term structure equation \textit{(144)}.

8.3 Vasiček model
8.3.1 Diffusion approximation

To obtain the Vasiček bond pricing formula from Eqs.\textit{(136)}, \textit{(137)} and \textit{(142)}, we consider the diffusion approximation when $\nu \rightarrow 0$, $\delta^2 \rightarrow 0$ and $\lambda \rightarrow \infty$ while the products $\lambda \int_{-\infty}^{\infty} d\eta p(\eta) \eta$ and $\lambda \int_{-\infty}^{\infty} d\eta p(\eta) \eta^2$ remain finite.

It follows from Eq.\textit{(140)} that $A_{\text{diff}}(t,T)$ in this approximation is

$$A(t,T) \rightarrow A_{\text{diff}}(t,T)$$ \hspace{1cm} (149)
\[
\lambda \int_{-\infty}^{\infty} d\eta \rho(\eta) \{-\eta B(s, T) + \frac{\eta^2}{2} B^2(s, T)\} = -\lambda \nu B(s, T) + \frac{\sigma^2}{2} B^2(s, T),
\]

where \(B(s, T)\) is given by Eq.(137), the parameter \(\nu\) is

\[
\nu = \int_{-\infty}^{\infty} d\eta \rho(\eta) \eta,
\]

and \(\sigma^2\) has been introduced by Eq.(88).

### 8.3.2 Long-term mean and instantaneous volatility

Substituting \(B(s, T)\) into Eq.(149) and evaluating the integral over \(ds\) yield

\[
A_{\text{diff}}(t, T) = -\lambda \nu \left( T - t - B(t, T) \right) + \frac{\sigma^2}{2a} \left( T - t - 2B(t, T) + 1 - e^{-2a(T-t)} \right) \left( T - t - B(t, T) + \frac{1}{2a} \right).
\]

Let us introduce the notation

\[
b = \frac{\lambda \nu}{a}.
\]

Then we have

\[
A_{\text{diff}}(t, T) \equiv A_{\text{Vasiček}}(t, T) = (b - \frac{\sigma^2}{2a^2}) [B(t, T) - (T - t)] - \frac{\sigma^2}{4a} B^2(t, T).
\]

A bond pricing equation (136) in diffusion approximation has a form

\[
P_{\text{diff}}(t, T) = \exp \{ A_{\text{diff}}(t, T) - B(t, T)r(t) \}.
\]

We recognize the Vasiček bond pricing formula [2] in Eq.(154) with \(A_{\text{Vasiček}}(t, T)\) given by Eq.(153) and \(B(t, T)\) given by Eq.(137). Hence, we come to conclusion that \(b\) defined by Eq.(152) is long term mean level and \(\sigma\) introduced by Eq.(88) is instantaneous volatility of the Vasiček short term interest rate.
model. The parameter $\sigma^2/2a^2$ appearing in Eq.(153) is sometimes called long term variance.

Let us remind that the Vasiček short term interest rate model has been introduced by means of the following stochastic differential equation [2]

$$dr = a(b - r)dt + \sigma dW(t),$$

where $a$ is the parameter $a$ ($a > 0$) being the speed at which $r(t)$ goes at $t \to \infty$ to its equilibrium level $b$, the parameter $\sigma$ is instantaneous volatility and $W(t)$ is a standard Wiener process.

The parameters $a$, $b$, and $\sigma$ are positive constants. The parameter $a$ is the speed at which $r(t)$ goes to its equilibrium level $b$ at $t \to \infty$. Hence, the parameter $a$ in Vasiček mode has exactly the same meaning as in our model introduced by Eq.(131). The existence and meaning of the phenomenological long term mean level $b$ in Vasiček model (155) is explained in our model by Eq. (152) which emerges naturally in the diffusion approximation. In other words, our model provides quantitative background to introduce the parameter $b$ and lets us to conclude that the long term mean level has stochastic dynamic origin. The origin of the instantaneous volatility $\sigma$ in Vasiček model is explained in our model by Eqs.(149) and (88), that is our model provides quantitative background to introduce the Vasiček parameter $\sigma$.

Thus, we see that the short term interest rate model defined by Eq.(131) in diffusion approximation goes exactly into the well-known Vasiček model [2]. The long term mean level given by Eq.(152) and instantaneous volatility given by Eq.(88) are presented in terms of the parameters involved into probability density function of interest rate jumps magnitude $p(\eta)$. In other words, the new stochastic model (131) gives insight into stochastic dynamic origin of the Vasiček long term mean level and a short term interest rate instantaneous volatility.

Let us emphasize that the stochastic dynamics (131) initiates non-Normal probability distribution of short term interest rate in contrast with the Vasiček model, which gives the Normal distribution. However, despite the lack of normality the stochastic model (131) has the same degree of analytical tractability as the Vasiček model.

8.3.3 Mean and variance

It is interesting that despite the lack of normality the model introduced by Eq.(131) possesses exactly the same mean and variance as the Vasiček model,
which results in the Normal distribution of short term interest rate.
Indeed, it follows from Eq. (133) that the mean $<r(s)>$ is

$$<r(s)> = e^{-as}r_0 + \int_0^s d\tau e^{-a(s-\tau)} <F(\tau) >,$$  \hspace{1cm} (156)

and the variance is

$$\text{Var}(r(s)) = <(r(s) - <r(s)>)^2 > \hspace{1cm} (157)$$

$$= <\left(\int_0^s d\tau e^{-a(s-\tau)} (F(\tau) - <F(\tau)>)\right)^2 >,$$

where $<...>$ stands for average defined by Eqs. (6)-(8).

Taking into account Eq. (14) and using Eq. (64) we obtain for the mean

$$<r(s)> = e^{-as}r_0 + b(1 - e^{-as}),$$ \hspace{1cm} (158)

where $b$ is given by Eq. (152).

To calculate the variance $\text{Var}(r(s))$ let’s use Eqs. (15) and (64) to find

$$\text{Var}(r(s)) = <(r(s) - <r(s)>)^2 > = \frac{\sigma^2}{2a}(1 - e^{-2as}),$$ \hspace{1cm} (159)

where $\sigma$ is given by Eq. (88).

The conditional mean and variance are

$$<r(s)|r(t)> = e^{-as}r(t) + b(1 - e^{-a(s-t)}),$$ \hspace{1cm} (160)

and

$$\text{Var}(r(s)|r(t)) = \frac{\sigma^2}{2a}(1 - e^{-2a(s-t)}).$$ \hspace{1cm} (161)

The equations for $<r(s)|r(t)>$ and $\text{Var}(r(s)|r(t))$ coincide exactly with the Vasiček’s equations (see, Eqs. (25) and (26) in [2]) for conditional mean and variance of short term interest rate.
9 Generalized short term interest rate model

9.1 Vasiček model with shot noise

Now we introduce a generalized short term interest rate model as superposition of the new model defined by Eq.(131) and the Vasiček short term interest rate model given by Eq.(155)

\[ dr = a(b - r)dt + \sigma dW(t) + F(t)dt, \]

where all notations are the same as they were defined for Eqs.(131) and (155).

It is easy to see from Eq.(162) that if at the time moment \( t \) the short term interest rate was \( r(t) \) then later on, at the time moment \( s, s > t \), it will be

\[ r(s) = e^{-a(s-t)}r(t) + b(1 - e^{-a(s-t)}) \]

\[ + \sigma \int_t^s dW(\tau)e^{-a(s-\tau)} + \int_t^s d\tau e^{-a(s-\tau)}F(\tau), \]

or

\[ r(s) = e^{-as}r_0 + b(1 - e^{-as}) + \sigma \int_t^s dW(\tau)e^{-a(s-\tau)} + \int_0^s d\tau e^{-a(s-\tau)}F(\tau), \]  

with \( r_0 = r(t)|_{t=0} \) if we choose \( t = 0 \).

9.2 Bond Price

The price at time \( t \) of a zero-coupon bond \( P(t, T, r) \) which pays one currency unit at maturity \( T, (0 \leq t \leq T) \), is

\[ P(t, T, r) = \exp \left\{ -\int_t^T dsr(s) \right\} > W, \ F, \quad P(T, T, r) = 1. \]
where \( r(s) \) is given by Eq.(163) and \( < ... >_W, F \) stands for average over Wiener process \( W(t) \) and all randomness involved into shot noise \( F(t) \).

Further, substitution of \( r(s) \) from Eq.(163) into Eq.(165) and straightforward integration over \( ds \) yields

\[
P(t,T,r) = \exp \{ A(t,T) - B(t,T)r(t) \},
\]

where \( B(t,T) \) has been introduced by Eq.(137), and

\[
\exp \{ A(t,T) \} = \exp \{ -b(T - t - B(t,T)) \}
\]

(167)

\[
\times < \exp \left\{ -\sigma \int_t^T dW(s)B(s,T) \right\} >_W < \exp \left\{ -\int_t^T dsB(s,T)F(s) \right\} >_F,
\]

because Wiener process and shot noise \( F(t) \) are independent of each other. Therefore, we conclude that the generalized model introduced by Eq.(162) provides affine term structure.

Further, from Eq.(139) we obtain

\[
< \exp \left\{ -\int_t^T dsB(s,T)F(s) \right\} >_F
\]

(168)

\[
= \lambda \int_t^T ds \left\{ \exp \left\{ -\nu B(s,T) + \frac{\delta^2}{2} B^2(s,T) \right\} - 1 \right\}.
\]

While for average with respect to the Wiener process\(^3\) we have

\[
\Phi[\alpha(\tau)] = \left\langle \exp \left\{ i\sigma \int_t^T d\tau \alpha(\tau)W(\tau) \right\} \right\rangle_W = \exp \left\{ -\frac{\sigma^2}{2} \int_t^T d\tau \alpha^2(\tau) \right\},
\]

(169)

where \( \alpha(\tau) \) is arbitrary sufficiently smooth function and \( \langle ... \rangle_W \) stands for average over the Wiener process.

\[\text{45}\]
\[ < \exp \left\{ -\sigma \int_t^T dW(s)B(s, T) \right\} >_W = \exp \left\{ \frac{\sigma^2}{2} \int_t^T ds B^2(s, T) \right\} \] (170)

\[ = \exp \left\{ \frac{\sigma^2}{2a^2}(T - t) - \frac{\sigma^2}{2a^2}B(t, T) - \frac{\sigma^2}{4a}B^2(t, T) \right\}. \]

It follows from Eqs. (167)-(170) that

\[ \mathcal{A}(t, T) = (b - \frac{\sigma^2}{2a^2})[B(t, T) - (T - t)] - \frac{\sigma^2}{4a}B^2(t, T) \] (171)

\[ + \lambda \int_t^T ds \int_{-\infty}^{\infty} d\eta p(\eta)(e^{-\eta B(s, T)} - 1), \]

which can be written as

\[ \mathcal{A}(t, T) = A_{\text{Vasiček}}(t, T) + A(t, T), \] (172)

with \( A_{\text{Vasiček}}(t, T) \) and \( A(t, T) \) defined by Eqs. (171) and (142).

Thus, based on generalized short term interest rate dynamics introduced by Eq. (162), we obtained the formula (166) for price of a zero coupon bond, where \( B(t, T) \) is given by Eq. (137) and \( A(t, T) \) is given by Eq. (171).

There are the following limit cases of Eq. (172)

\[ \mathcal{A}(t, T) \xrightarrow{\lambda = 0} A_{\text{Vasiček}}(t, T), \] (173)

which results in the Vasiček formula for a bond price, and

\[ \mathcal{A}(t, T) \xrightarrow{b = 0, \sigma = 0} A(t, T), \] (174)

which results in bond pricing formula (136) with \( B(t, T) \) given by Eq. (137) and \( A(t, T) \) given by Eqs. (142).

Hence, the generalized short term interest rate dynamics (162) describes the impact on a bond price the interplay between the Vasiček stochastic dynamics (155) and jump dynamics (131) modeled by the Geometric Shot Noise motion.
9.2.1 Generalized term structure equation

Defined by Eq.(165) the price of zero-coupon bond \( P(t, T, r) \) solves the following integro-differential equation

\[
\frac{\partial P(t, T, r)}{\partial t} + ab \frac{\partial P(t, T, r)}{\partial r} + \sigma^2 \frac{\partial^2 P(t, T, r)}{\partial r^2} + \lambda \int_{-\infty}^{\infty} d\eta p(\eta) \left\{ P(t, T, r + \eta) - P(t, T, r) \right\} = rP(t, T, r),
\]

subject to terminal condition (135), with the parameters \( a, b, \) and \( \sigma \) being Vasiček’s parameters in Eq.(155), \( \lambda \) being the number of short term interest rate jumps per unit of time, and \( p(\eta) \) being the probability density function of jump magnitude.

This is a generalized term structure equation in the case when the short term interest rate solves the stochastic differential equation (162).

To find the solution to Eq.(175) subject to terminal condition (135) let’s note that the affine term structure means that the bond price admits solution of the form given by Eq.(166). Substitution of Eq.(166) into Eq.(175) gives the two ordinary differential equations to obtain \( A(t, T) \) and \( B(t, T) \) The ordinary differential equation to find \( A(t, T) \) has a form

\[
\frac{dA(t, T)}{dt} + abB(t, T) + \frac{\sigma^2}{2} B^2(t, T) + \lambda \int_{-\infty}^{\infty} d\eta p(\eta) \left\{ e^{-\eta B(s, T)} - 1 \right\} = 0,
\]

subject to terminal condition at \( t = T \)

\[
A(T, T) = 0,
\]

while for \( B(t, T) \) we have Eq.(146) subject to terminal condition given by Eq.(148).

Therefore, \( B(t, T) \) is given by Eq.(137).

The solution to Eq.(176) in terms of \( B(t, T) \) is

\[
A(t, T) = \int_{t}^{T} ds \left\{ abB(t, T) + \frac{\sigma^2}{2} B^2(s, T) + \lambda \int_{-\infty}^{\infty} d\eta p(\eta)(e^{-\eta B(s, T)} - 1) \right\},
\]
which can straightforwardly be transformed into Eq. (171).

Thus, it has been proved that the price of zero-coupon bond \( P(t, T, r) \) defined by Eq. (165) solves the generalized term structure equation (175).

10 Conclusion

The unified framework consisting of two analytical approaches to value options and bonds has been introduced and elaborated.

The options pricing approach is based on a stock price dynamics that has been modeled by the stochastic differential equation with involvement of shot noise. It results in Geometric Shot Noise motion of stock price. A new arbitrage-free integro-differential option pricing equation has been developed and solved. New exact formulas to value European call and put options have been obtained. The put-call parity has been proved. Based on the solution of the option pricing equation the Greeks have been calculated. Three new Greeks associated with the model market parameters have been introduced and evaluated. It has been shown that the developed option pricing framework incorporates the well-known Black-Scholes equation \([1]\). The Black-Scholes equation and its solutions emerge from our integro-differential option pricing equation in the special case which we call ”diffusion approximation”. The Geometric Shot Noise model in diffusion approximation gives natural explanation of the stochastic dynamic origin of volatility in the Black-Scholes model.

The generalized option pricing framework has been introduced and developed based on stock price stochastic dynamics modeled by superposition of Geometric Brownian motion and Geometric Shot Noise in the equation for a stock price dynamics. New formulas to value European call and put options have been obtained as the solutions to generalized pricing equation. The special limit cases of those formulas have been presented and discussed.

The bonds pricing analytical approach has been developed based on the Langevin type stochastic differential equation with additive shot noise to model a short term interest rate dynamics. It results in non-Gaussian random motion of short term interest rate. It is interesting that despite the lack of normality the model possesses exactly the same mean and variance as the Vasiček model, which results in the Normal distribution of short term interest rate. A bond pricing formula has been obtained and it has been shown that the model provides affine term structure. The new bond pricing
formula incorporates the well-known Vasiček solution \cite{2}. The term structure equation has been obtained and it has been shown that the bond pricing formula solves the term structure equation. The well-known Vasiček model for short term interest rate with its long-term mean and instantaneous volatility comes out from our model in diffusion approximation. The stochastic dynamic origin of the Vasiček long-term mean and instantaneous volatility have been discussed.

A generalized bond pricing model has been introduced and developed based on short term interest rate stochastic dynamics modeled by superposition of Wiener process and shot noise. A new bond pricing formula has been found and it has been shown that the generalized model provides affine term structure. The generalized term structure equation has been presented and solved.

It has to be emphasized that despite the lack of normality of probability distributions involved, all newly developed quantitative models to value options and bonds, have the same degree of analytical tractability as the Black–Scholes model \cite{1} and the Vasiček model \cite{2}. Analytical tractability allows to obtain new exact formulas to value options and bonds.

The unified framework can be easily extended to cover valuation of other types of options and a variety of financial products with option features involved, and to elaborate enhanced short term interest rate models to value interest rate contingent claims.

References

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11 Appendix

Let us show how to obtain the Black-Scholes formula to value European call option without converting the Black-Scholes equation into the heat equation. In other words, we are aiming to get a straightforward solution to the Black-Scholes problem given by Eqs. (92) and (18). Using notation (17), the Black-Scholes problem (92) with the terminal condition given by Eq. (18) can be rewritten in a mathematically equivalent form as Eq. (89) with the terminal condition given by Eq. (19). The Green’s function method is a convenient way to solve the problem given by Eqs. (89) and (19).

The Green’s function $G(x - x', T - t)$ satisfies the partial differential equation

$$\frac{\partial G(x - x', T - t)}{\partial (T - t)} = \frac{\sigma^2}{2} \frac{\partial^2 G(x - x', T - t)}{\partial x^2}$$

$$+ (r - q - \frac{\sigma^2}{2}) \frac{\partial G(x - x', T - t)}{\partial x} - rG(x - x', T - t) = 0,$$

and the terminal condition

$$G(x - x', T - t)|_{t=T} = G(x - x', 0) = \delta(x - x').$$

Having Green’s function $G(x - x', T - t)$, we can write the solution of Eq. (89) with condition Eq. (19) in the form

$$C(x, T - t) = K \int_{-\infty}^{\infty} dx' G(x - x', T - t) \max(e^{x'} - 1, 0).$$

Green’s function introduced by Eq. (179) with the terminal condition (180) can be found by the Fourier transform method. With help of definitions (27) and (28), equation (179) reads

$$\frac{\partial G(k, T - t)}{\partial (T - t)} = \left\{-\frac{\sigma^2 k^2}{2} + ik(r - q - \frac{\sigma^2}{2}) - r\right\} G(k, T - t),$$

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and the terminal condition \((180)\) for \(G(k, T - t)\) is

\[
G(k, T - t)|_{t=T} = G(k, 0) = 1. \tag{183}
\]

The solution of the problem defined by Eqs.\((182)\) and \((183)\) is

\[
G(k, T - t) = e^{-r(T-t)} \exp \left\{ \left[ -\frac{\sigma^2 k^2}{2} + ik(r-q - \frac{\sigma^2}{2}) \right](T-t) \right\}. \tag{184}
\]

Substitution of Eq.\((184)\) into Eq.\((27)\) gives us the Green’s function \(G(x - x', T - t)\) of the Black-Scholes equation \((89)\)

\[
G(x - x', T - t) = \frac{e^{-r(T-t)}}{2\pi} \int dke^{ik(x-x')} \times \exp \left\{ \left[ -\frac{\sigma^2 k^2}{2} + ik(r-q - \frac{\sigma^2}{2}) \right](T-t) \right\} \\
= \frac{e^{-r(T-t)}}{\sigma \sqrt{2\pi(T-t)}} \exp \left\{ -\frac{[x - x' + (r-q - \frac{\sigma^2}{2})(T-t)]^2}{2\sigma^2(T-t)} \right\}. \tag{185}
\]

Then Eq.\((181)\) yields for the value of European call option \(C(x, T - t)\)

\[
C(x, T - t) = \frac{Ke^{-r(T-t)}}{\sigma \sqrt{2\pi(T-t)}} \int dx' \exp \left\{ -\frac{[x - x' + (r-q - \frac{\sigma^2}{2})(T-t)]^2}{2\sigma^2(T-t)} \right\} \\
\times \max(e^{x'} - 1, 0). \tag{186}
\]

By introducing a new integration variable \(z\), \(x' \rightarrow z\),

\[
z = \frac{x - x' + (r-q - \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}},
\]

and using Eq.\((90)\) we find for the European call option \(C(S, T - t)\)

\[
C(S, T - t) = \frac{Se^{-q(T-t)}}{\sqrt{2\pi}} e^{-\frac{S^2}{2}(T-t)} \int dz e^{-\frac{z^2}{2} e^{-z\sigma \sqrt{T-t}}} \quad \tag{187}
\]
\[- Ke^{-r(T-t)} \frac{d_2}{\sqrt{2\pi}} \int_{-\infty}^{d_2} dz \exp \left\{ -\frac{z^2}{2} \right\},\]

where the parameter \(d_2\) is given by

\[d_2 = \frac{\ln \frac{S}{K} + (r - q - \frac{\sigma^2}{2})(T - t)}{\sigma \sqrt{T - t}}.\] (188)

By changing integration variable \(z \rightarrow y, y = z + \sigma \sqrt{T - t}\) in the first term of Eq. (187) we obtain

\[C(S, T - t) = \frac{Se^{-q(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{d_2 + \sigma \sqrt{T - t}} dy e^{-\frac{y^2}{2}},\]

\[- Ke^{-r(T-t)} \frac{d_2}{\sqrt{2\pi}} \int_{-\infty}^{d_2} dz \exp \left\{ -\frac{z^2}{2} \right\}.\]

It is easy to see from the equation above that we came to the well-known solution to the Black-Scholes equation (92)

\[C(S, T - t) = Se^{-q(T-t)} N(d_1) - Ke^{-r(T-t)} N(d_2),\]

where \(N(d)\) is the cumulative distribution function of the standard normal distribution given by Eq. (100)

\[N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} dz \exp \left\{ -\frac{z^2}{2} \right\},\]

and the parameter \(d_1\) is

\[d_1 = d_2 + \sigma \sqrt{T - t} = \frac{\ln \frac{S}{K} + (r - q + \frac{\sigma^2}{2})(T - t)}{\sigma \sqrt{T - t}},\] (189)

with \(d_2\) defined by Eq. (188).