Level sets of depth measures and central dispersion in abstract spaces

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Abstract

The lens depth of a point have been recently extended to general metric spaces, which is not the case for most depths. It is defined as the probability of being included in the intersection of two random balls centred at two random points $X_1, X_2$, with the same radius $d(X_1, X_2)$. We study the consistency in Hausdorff and measure distance, of the level sets of the empirical lens depth, based on an iid sample from a general metric space. We also prove that the boundary of the empirical level sets are consistent estimators of their population counterparts. We tackle the problem of how to order random elements in a general metric space by means of the notion of spread out and dispersive order. We present a small simulation study and analyse a real life example.

Keywords: Depth measure; Metric spaces; Levels set; Phylogenetic tree; Spread out

1 Introduction

The study of depths in statistics has gained importance in the last three decades, some significant contributions include the half space depth [29], the simplicial depth [38, 39], and the multivariate $L^1$ depth [60]. Other well-known depths are the convex hull peeling depth [6], the Oja depth [48], and the spherical depth [25], among others. They have been extended to function spaces, see [30, 41, 15, 16], to Riemannian manifolds, see [28], and also to general metric spaces, see [29]. Several applications of depths have been proposed, for instance, in classification problems, by means of the depth-depth method [61], or to functional data [46].

Most classical notions of depth, introduced initially on $\mathbb{R}^d$, can not be directly extended to general metric spaces or even to function spaces or manifolds. Some of them are computationally infeasible on high dimensional spaces, such as Liu’s or Tukey’s depth, because the computational complexity is exponential in the number of dimensions. This is not the case of the depth introduced in [29], i.e. the lens depth, whose computational complexity is of order $n^2$: it can be easily extended to general metric spaces. This make the lens depth particularly suitable for estimating its level sets, by means of the level sets of its empirical version, based on an iid sample. This is one of the main goals of this paper.

Level set estimation of depths was initially studied in [59], as a key tool for the visualization and exploration of data. As was pointed out in [40], the shape and size of these levels, as well as the direction and speed at which they expand, provides insight into the dispersion,
kurtosis, and asymmetry of the underlying distribution. It also allows extending the notion of quantile to multivariate or functional data, and the idea of outlier detection, see for instance \[26; 19\]. They are also used for supervised classification, see \[53; 35\].

A closely related problem is the estimation of the level sets of a density, a problem that has been widely studied in the literature, for instance in \[33; 50; 17; 58\]). It has also been tackled on manifolds, see for instance \[14\]. However, there has been little consideration of the estimation of the level sets of depths. An exception is \[11\], where a concentration inequality is obtained in \(\mathbb{R}^d\) for Tukey’s depth.

More formally, let \(X_1, X_2\) be two random variables defined on a (reach enough) probability space \((\Omega, \mathcal{A}, \mathbb{P})\), taking values in a complete separable metric space \((M, d)\) endowed with the Borel \(\sigma\)-algebra. Assume that they are independent. In general, the distribution of a random element \(X\) of \(M\) will be denoted by \(P_X\). Let us write \(A(x_1, x_2) = B(x_1, d(x_1, x_2)) \cap B(x_2, d(x_1, x_2))\), where \(B(p, r)\) is the closed ball centred at \(p\) with radius \(r > 0\) and \(x_1, x_2 \in M\). The lens depth of a point \(x \in M\) is defined by \(LD(x) = P(x \in A(X_1, X_2))\). Given an iid sample \(X_n = \{X_1, \ldots, X_n\}\) from a distribution \(P_X\), the empirical version of \(LD\) is given by the \(U\)-statistics of order two, \[
\hat{LD}_n(x) = \left(\frac{n}{2}\right)^{-1} \sum_{1 \leq i_1 < i_2 \leq n} 1_{A(X_{i_1}, X_{i_2})}(x). \tag{1}
\]

The general approach to obtaining consistency results (w.r.t. Hausdorff distance) for the plug-in estimator \(\{\hat{LD}_n \geq \lambda\}\) to its population counterpart \(\{LD \geq \lambda\}\) is to prove the uniform convergence of \(\hat{LD}_n\) to \(LD\). To gain some insight into the shape of \(\{\hat{LD}_n \geq \lambda\}\), see Figure 1. In general metric spaces it is necessary to restrict the convergence to compact sets. This is proved in Theorem 4, but the results also hold on \(\mathbb{R}^d\) under milder conditions, see Theorem 5.

The second goal is to tackle the problem of how to order two random elements of general metric spaces, based on the notion of spread out introduced in \[40\], and on the notion of dispersive order. This problem has been widely studied in the literature, beginning with \[36\], see \[56\] for a survey. A wide range of applications can be found in the literature, for instance in finance (see \[37; 47\]), in agriculture (see \[51\]), in economics (see \[4; 44\]), just to mention a few.

There are some extensions of the notion of dispersive order to the multivariate case, see for instance \[32; 27; 7; 63\]. They have recently been considered in general metric spaces, see \[54\]. In \[52\], a notion of multivariate dispersiveness is introduced, by means of the calculation of the volume of the level sets of Liu’s and Tukey’s depths.

This paper is organized as follows: Section 2 introduces the notation, some previous definitions, and states two important theorems that will be used to prove our main results. The almost surely (a.s.) uniform consistency of \(\hat{LD}_n\) is stated in Section 3 while its asymptotic distribution is formulated in Section 4. The a.s. consistency in Hausdorff and measure distance, for the empirical level sets \(\{\hat{LD}_n \geq \lambda\}\), as well as the a.s. consistency of its boundary, is stated in Section 5. The notions of spread out and dispersive order on general metric spaces are studied in Section 6. Lastly, in Section 7 we present first a simulation study, and then tackle the study of two real datasets, the vectocardiogram dataset (see subsection 7.2) and the influenza dataset (see subsection 7.3). All proofs are given in the Appendix.
2 Preliminaries

In this section we will introduce the notation and necessary definitions used throughout this paper. Given a metric space \((M, d)\), which will be assumed to be separable, complete and locally connected, we denote by \(B(x, r)\) the closed ball centred at \(x\) with radius \(r > 0\). The boundary of a set \(A \subset M\) is denoted by \(\partial A\). Given \(x \in A^c\), \(d(x, A) = \inf_{a \in A} d(x, a)\) and \(\text{diam}(A) = \sup_{x, y \in A} d(x, y)\). Given two closed sets \(A, B\), the Hausdorff distance between them is defined as

\[
\text{d}_H(A, B) := \max \left\{ \max_{a \in A} d(a, B), \max_{b \in B} d(b, A) \right\}
\]  

where \(d(a, B) := \inf_{b \in B} d(a, b)\). Given a function \(f : M \rightarrow \mathbb{R}\) and \(\lambda \in \mathbb{R}\), we denote by \(\{f \geq \lambda\}\) the \(\lambda\)-level set \(\{x \in M : f(x) \geq \lambda\}\).

For the estimation of level sets in general metric spaces, two main results will play a key role. The first one is Theorem 2.1 in [42]. We will make use of the following slightly restricted version.

**Theorem 1** (Molchanov, (1998)). Let \(f_n, f : M \rightarrow \mathbb{R}\) be continuous functions. Assume that for each compact set \(K_0\), \(\sup_{x \in K_0} |f_n(x) - f(x)| \rightarrow 0\). Assume that for all \(\lambda \in [c_1, c_2]\), \(\{f \geq \lambda\} \subset \{f > \lambda\}\). Then

\[
\sup_{c_1 \leq \lambda \leq c_2} \text{d}_H(\{f_n \geq \lambda\} \cap K_0, \{f \geq \lambda\} \cap K_0) \rightarrow 0.
\]

The convergence of the boundaries of the level sets is in general more involved. To prove that, we will use the following result.

**Theorem 2** (Cuevas, Gozález-Manteiga, Rodríguez-Casal, (2006)). Given a continuous function \(f : M \rightarrow \mathbb{R}\), let \((\Omega, \mathcal{A}, P)\) be a probability space and \(f_n = f_n(\omega, \cdot)\), with \(\omega \in \Omega\), a sequence of random functions, \(f_n : M \rightarrow \mathbb{R}\), \(n = 1, 2, \ldots\). Assume that for each \(n\), \(f_n\) is continuous with probability one. Assume that the following assumptions are fulfilled.

\(h_1\) \(M\) is locally connected.

\(h_2\) For all \(x \in \partial \{f \geq \lambda\}\), there exist sequences \(u_n, l_n \rightarrow x\) such that \(f(u_n) > \lambda\) and \(f(l_n) < \lambda\).
If \( \partial\{ f \geq \lambda \} \neq \emptyset \). Moreover, there exists a \( \lambda^- < \lambda \) such that the set \( \{ f \geq \lambda^- \} \) is compact. If \( \sup_{x \in M} | f(x) - f_n(x) | \to 0 \) a.s., then
\[
d_H(\partial\{ f \geq \lambda \}, \partial\{ f_n \geq \lambda \}) \to 0, \quad \text{a.s., as } n \to \infty.
\]

3 Uniform consistency of \( \hat{LD}_n \)

As mentioned above, the key points to proving the consistency in Hausdorff distance of the level sets of \( \hat{LD}_n \) are Theorems 2 and [3]. Then we have to prove that \( \hat{LD}_n \) converges uniformly to \( LD \) a.s., which is the main goal of this section.

To obtain the a.s. uniform convergence of \( \hat{LD}_n \), we will use the following version of Theorem 1 in [9].

**Theorem 3** (Billingsley and Topsoe, (1967)). Let \( \mathcal{B}(M \times M) \) be the class of all real valued, bounded, measurable functions defined on the metric space \( (M \times M, \rho) \), where \( \rho(z, y) = \max\{d(z_1, y_1), d(z_2, y_2)\} \). Suppose \( \mathcal{F} \subset \mathcal{B}(M \times M) \) is a subclass of functions. Then
\[
\sup_{f \in \mathcal{F}} \left| \int f \, dP_n - \int f \, dP \right| \to 0,
\]
for every sequence \( P_n \) that converges weakly to \( P \) if, and only if,
\[
\sup\{|f(z) - f(t)| : f \in \mathcal{F}, z = (z_1, z_2), t = (t_1, t_2) \in M \times M\} < \infty,
\]
and for all \( \epsilon > 0 \),
\[
\lim_{\delta \to 0} \sup_{f \in \mathcal{F}} P\left[ y = (y_1, y_2) : \omega_f(\mathcal{B}_\rho(y, \delta)) \geq \epsilon \right] = 0,
\]
where \( \omega_f(A) = \sup\{|f(z) - f(t)| : z, t \in A \} \) and \( \mathcal{B}_\rho(y, \delta) \) is the open ball in the metric space \( (M \times M, \rho) \) of radius \( \delta > 0 \).

To prove that (3) is fulfilled, we will use the following lemma.

**Lemma 1.** Let \( x \in M \) and \( y = (y_1, y_2) \in M \times M \) be such that \( d(x, \partial A(y_1, y_2)) > 3\delta \). Then \( \omega_{f_x}(\mathcal{B}_\rho(y, \delta)) = 0 \), where \( f_x(t_1, t_2) = I_{A(t_1, t_2)}(x) \) with \( t_1, t_2 \in M \).

**Theorem 4.** Let \( (M, d) \) be a complete separable metric space and \( P_X \) a Borel measure on \( M \). Assume that \( P_X(\partial B(x, \delta)) = 0 \) for all \( x \in M \) and \( \delta > 0 \). For any compact set \( K \subset M \),
\[
\sup_{x \in K} | \hat{LD}_n(x) - LD(x) | \to 0 \quad \text{a.s., as } n \to \infty.
\]

As a direct consequence of the previous results, if \( M \) is a compact manifold, \( \sup_{x \in M} | \hat{LD}_n(x) - LD(x) | \to 0 \). For \( \mathbb{R}^d \), uniform convergence can be obtained, as stated in the following theorem.

**Theorem 5.** Assume that \( P_X(\partial B(x, \delta)) = 0 \) for all \( x \in \mathbb{R}^d \) and \( \delta > 0 \). Then
\[
\sup_{x \in \mathbb{R}^d} | \hat{LD}_n(x) - LD(x) | \to 0 \quad \text{a.s., as } n \to \infty
\]
4 Asymptotic distribution

To obtain the asymptotic law of $\hat{LD}_n$ we will use Proposition 10 of [31], which is a very general result for empirical processes, applied to $U$ statistics. This requires proving that the family of sets $A(x,y)$ has finite Vapnik–Chervonenkis (VC) dimension (see [21]). To prove the result, we will assume that the family $\{F_p\}_{p \in M}$, where $F_p = \{(x,y) \in M \times M : d(p,x) < d(x,y)\}$, has finite VC dimension varying $p$, which is referred to in what follows as the Vapnik-ball condition.

Definition 1. A metric space $(M,d)$ fulfills the Vapnik-ball condition if the the family $\{F_p\}_{p \in M}$ where $F_p = \{(x,y) \in M \times M : d(p,x) < d(x,y)\}$ has finite VC dimension varying $p$.

Let us introduce some notation and general ideas of empirical processes, following [3]. To each $x \in M$ we associate the function $f_x : M^2 \rightarrow \mathbb{R}$ defined by $f_x(a,b) = 1_{A(a,b)}(x)$. The family of all these functions is denoted by $\mathcal{F}$. Then we can define $\hat{LD}_n$, a function of the random variable $X$ and the point $x$, by

$$\hat{LD}_n(P_X, f_x) = \left(\frac{n}{2}\right)^{-1} \sum_{1 \leq i < j \leq n} f_x(X_i, X_j),$$

where $X_1, \ldots, X_n$ is an iid sample from $X$ with distribution $P_X$. Let us denote the metric space of all real valued bounded functions defined on $\mathcal{F}$, endowed with the supremum norm, by $l^\infty(\mathcal{F})$. We can consider $\hat{LD}_n$ as a random variable with values in $l^\infty(\mathcal{F})$ given by $\omega \in \Omega \rightarrow \hat{LD}_n : \mathcal{F} \rightarrow \mathbb{R}$. We define $LD \in l^\infty(\mathcal{F})$ by $LD(f_x) = \mathbb{P}(f_x(X_1, X_2))$, with $X_1, X_2$ being independent copies of $X$. We want to study the limit law of the random element of $l^\infty(\mathcal{F})$ given by $G_n = \sqrt{n}(\hat{LD}_n - LD)$. The limit law of $G_n$ is denoted by $2GP_X$, where $GP_X$ is a Brownian bridge associated to $P_X$, which is a random element in $l^\infty(\mathcal{F})$, and is defined through its finite dimensional distributions. More precisely, for all $k$, $(GP_X(f_{x_1}), \ldots, GP_X(f_{x_k}))$ is a zero-mean Gaussian process whose $k \times k$ covariance matrix has $(i,j)$th element

$$E(GP_X(f_{x_1})GP_X(f_{x_2})) = P_2(f_{x_1}, f_{x_2}) - P_2(f_{x_1})P_2(f_{x_2}),$$

where

$$P_2(f_x) := \int_{M^2} f_x(x_1, x_2) dP_X(x_1) \times dP_X(x_2).$$

The convergence of $G_n$ to $2GP_X$ is in law, which in this case is equivalent to the convergence in distribution of the random vector $(G_n(f_{x_1}), \ldots, G_n(f_{x_k}))$ to $N(0, \Sigma)$, where the $k \times k$ matrix $\Sigma$ has $(i,j)$th element given by [4].

Theorem 6. (Limit Law) Let $M$ be a compact manifold fulfilling the Vapnik-ball condition. Following the previous notation, the stochastic processes $G_n$ converge in law to $2GP_X$, as $n \rightarrow \infty$.

5 Level set estimation

Given positive numbers $c_1 < c_2$, assume that for all $\lambda \in [c_1, c_2]$, we have $\{f \geq \lambda\} \subset \{f > \lambda\}$. From Theorem 2.1 in [42] together with Theorem 4 and the fact that $D$ is a continuous function, we get

$$\sup_{c_1 \leq \lambda \leq c_2} d_H(\{\hat{LD}_n \geq \lambda\} \cap K, \{LD \geq \lambda\} \cap K) \rightarrow 0 \ a.s., \ as \ n \rightarrow \infty$$

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for any compact set $K$. If $\nu(\{LD = \lambda\}) = 0$, it follows easily from Theorem 4 that for all compact $K$,
\[
d_\nu(\{\hat{LD}_n \geq \lambda\} \cap K, \{LD \geq \lambda\} \cap K) \to 0 \quad \text{a.s., as } n \to \infty.
\]
The following theorem states that $\partial(\{\hat{LD}_n \geq \lambda\} \cap K)$ is a consistent estimator of $\partial(\{LD \geq \lambda\} \cap K)$. The proof follows the same lines used to prove Theorem 1 in [18]. We cannot apply that theorem directly because $\hat{LD}_n(x)$ is not a continuous function, since the range of $\hat{LD}_n(x) \subset \{k(n^2)^{-1} : k = 0, \ldots, (n^2)\}$.

**Theorem 7.** Let $\lambda > 0$ be such that $\{LD \geq \lambda\} \neq \emptyset$. Under the assumptions of Theorem 4, together with hypotheses $h_1$ and $h_2$ of Theorem 2. Then
\[
\lim_{n \to \infty} d_H(\partial(\{\hat{LD}_n \geq \lambda\} \cap K), \partial(\{LD \geq \lambda\} \cap K)) = 0 \quad \text{a.s.,}
\]
for all compact sets $K$.

For the metric space $(\mathbb{R}^d, \| \cdot \|)$ endowed with the Euclidean norm we have the following trivial corollary,

**Corollary 1.** Assume that hypothesis $h_2$ of Theorem 2 is fulfilled. Assume also that $\{LD \geq \lambda^-\}$ is non-empty and compact, for some $\lambda^- < \lambda$. Then
\[
\lim_{n \to \infty} d_H(\partial(\{\hat{LD}_n \geq \lambda\} \cap K), \partial(\{LD \geq \lambda\})) = 0 \quad \text{a.s. (5)}
\]

6 Comparison of random elements of metric spaces

6.1 The notion of being spread out

The notion of the *spread out* of a random vector was introduced by [40] for $\mathbb{R}^d$. It is closely related to the notion of the dispersive order for random variables (see [56]). The idea is to compare the variation in the volume (i.e. Lebesgue measure) of the level sets of a given depth of two random vectors $X$ and $Y$. We aim to generalize that notion to random elements of general metric spaces. Let $D$ be a depth on a metric space $(M, \rho)$, and suppose given a function $\psi : \mathcal{C}_M \to \mathbb{R}$, where $\mathcal{C}_M$ is the set of all compact subsets of $M$, and two random elements $X,Y$ with probability distributions $P_X$ and $P_Y$ respectively. Then we say that $X$ is more spread out than $Y$ w.r.t. $\psi$ if for all $p_1 < p_2$ and all compact sets $K$,
\[
\psi(\{D_X \geq p_1\} \cap K) - \psi(\{D_X \geq p_2\} \cap K) \geq \psi(\{D_Y \geq p_1\} \cap K) - \psi(\{D_Y \geq p_2\} \cap K).
\]
If $X$ is more spread out than $Y$ w.r.t. $\psi$, we write $X \succeq_\psi Y$. The following are some possible choices for $\psi$.

- $\psi_1(C) = \text{diam}(C)$.
- $\psi_2(C) = \sup_{x \in C} d(x, C^c)$.
- $\psi_3(C) = \nu(C)$, if $(M, d)$ is endowed with a Borel measure $\nu$.

It is easy to see that for these functions, the relation $\succeq_\psi$ is isometric invariant (i.e. For any isometry $\Phi$, $\Phi(X) \succeq_\psi Y$ and $X \succeq_\psi \Phi(Y)$). The transitivity is clearly also fulfilled. Moreover, the point-mass at $x$, $\delta_x$, has the property that for any random element $Y$, $Y \succeq_\psi \delta_x$, for all $j = 1, 2, 3$. 

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6.2 Dispersive orders

Several dispersive orders can be defined on general metric spaces, for instance in [32] it is said that \( X \) is more disperse than \( Y \) (denoted by \( X \succeq_{\text{Disp}} Y \)) if, and only if, \( d(X, X') \succeq_{\text{st}} d(Y, Y') \), where \( X' \) and \( Y' \) are independent copies of \( X \) and \( Y \) respectively, and we write \( \succeq_{\text{st}} \) for the usual stochastic order (that is, \( X \succeq_{\text{st}} Y \), if for all \( x \in \mathbb{R} \), \( P(Y > x) \leq P(X > x) \)).

In [54] there is introduced the \textit{ball-dispersive ordering}, which is defined as follows: given two random elements \( X \) and \( Y \) of a metric space \((M,d)\), we say that \( X \) is stochastically more disperse than \( Y \) in the ball dispersive ordering, denoted by \( X \succeq_{\text{BD}} Y \), if for all \( p \in M \), there exists \( q \in M \) such that \( P(X \in B(q,r)) \geq P(Y \in B(p,r)) \). This order has some desirable properties that all orders should have, but in practice it is very difficult to verify.

Regarding the use of a depth to define orders, in [40] and [45] a dispersive order is defined using the level sets of the corresponding depth functions. More precisely, given two random vectors \( X \) and \( Y \) of \( \mathbb{R}^d \), and a depth \( D \), \( X \) is said to be more disperse than \( Y \), denoted by \( X \succeq_{\text{do}} Y \), if, and only if, for all \( \lambda \), \( \{D_Y \geq \lambda\} \subseteq \{D_X \geq \lambda\} \), where \( D_X \) and \( D_Y \) denote the depth \( \text{w.r.t.} \ X \) and \( Y \) respectively. For example, if \( D \) is the Mahalanobis depth (see [55], p. 470), and the random vectors \( X \) and \( Y \) have means \( \mu_X \) and \( \mu_Y \) and covariance matrices \( \Sigma_X \) and \( \Sigma_Y \), then \( X \succeq_{\text{do}} Y \Leftrightarrow \mu_X = \mu_Y \) and \( \Sigma_X - \Sigma_Y \) is positive semidefinite.

There are some other dispersive orders, less restrictive and more easily verified in practice. For instance, given \( \alpha \in (0,1) \), in [10] it is suggested to compare the volume of the convex hull containing the \( 100\alpha \% \) deepest sample points. Following that idea, and given a function \( \psi \) as in 6.1, we define:

1) \textbf{Strong order}: A random element \( X \) is strongly more disperse than \( Y \) if for all compact sets \( K \),

\[ \psi(\{LD_X \geq \lambda\} \cap K) \geq \psi(\{LD_Y \geq \lambda\} \cap K), \]

for all \( \lambda \geq 0 \).

2) \textbf{Weak order}: A random element \( X \) is weakly more disperse than \( Y \) if for all compact sets \( K \),

\[ \int \left( \psi(\{LD_X \geq \lambda\} \cap K) - \psi(\{LD_Y \geq \lambda\} \cap K) \right) d\lambda \geq 0, \]

for all \( \lambda \geq 0 \).

A way to quantify how disperse a random element \( X \) is with respect to another \( Y \) using a depth is to use similar ideas to the one proposed in [20; 2] for dispersive orders in \( \mathbb{R} \). For instance, if we use the lens depth, we can consider the value \( \gamma(X, Y) := \ell\{\lambda : \forall K \text{ compact } \psi(\{LD_X \geq \lambda\} \cap K) \geq \psi(\{LD_Y \geq \lambda\} \cap K)\} \),

where \( \ell \) denotes the uniform distribution on \([0, \sup_{x \in M} \max\{LD_X(x), LD_Y(x)\}]\).

7 Simulations and examples of real data

7.1 Simulations

As the first example we consider a random variable \( X \) distributed as Student’s \( t \) with \( v \) degrees of freedom and a random variable \( Y \) distributed as \( \mathcal{N}(0, \sigma^2) \). In this case it can be
proved that \( \text{LD}(x) = 2F(x)(1 - F(x)) \). If we consider the function \( \psi \) which is the Lebesgue measure of the level set, it follows easily that

\[
\gamma(X, Y) = \ell \left\{ \lambda \in (0, 1/2) : F_X^{-1} \left( \frac{1 + \sqrt{1 - 2\lambda}}{2} \right) - F_X^{-1} \left( \frac{1 - \sqrt{1 - 2\lambda}}{2} \right) \geq F_Y^{-1} \left( \frac{1 + \sqrt{1 - 2\lambda}}{2} \right) - F_Y^{-1} \left( \frac{1 - \sqrt{1 - 2\lambda}}{2} \right) \right\}
\]

Figure 2 shows \( 2\gamma(X, Y) \) for different values of \( \sigma \) and \( v \). As can be seen, the dispersive relations between them are not trivial.

![Figure 2: Plots of 5 functions depending on the value of \( v \), for \( v = 1, \ldots, 5 \). In each of them \( v \) is kept fixed and the value of the function \( 2\gamma(X, Y) \) is shown as \( \sigma \) varies over \( (0, 5] \).](image)

### 7.2 The vectocardiogram dataset

We consider a real life dataset, where the data belong to the Stiefel manifold \( SO(3, 2) \) of all orthonormal 2-frames in \( \mathbb{R}^3 \) considered as \( 3 \times 2 \) orthogonal matrices (see [34]). The dataset consists of 98 vectocardiograms from children with ages varying between 2 and 19. Among them, 56 are boys and 41 are girls. Vectocardiography is a method that produces a three dimensional curve which comprises the records of the magnitude and direction of the electrical forces generated by the heart over time. These curves are called QRS loops, see Figure 3. In [23] there is associated to each curve an element of \( SO(3, 2) \) that represents some of the information of the curve, see also [22].

This sample has been previously analysed in the literature, see for instance [13; 12; 49]. A very important problem is outlier detection, corresponding to children with (possibly) cardiological problems.

Figure 4 represents each matrix in \( SO(3, 2) \) as two points in \( S^2 \), one for each column. They are joined by an arc in \( S^2 \). The arc joining the deepest pair of observations (w.r.t. lens depth) is represented in violet, while the outliers (for a level \( \lambda = 0.10 \)) are represented as red arcs.

We also used the classification and visualization method called the depth-depth plot (see [40]) in order to identify possible differences between the vectocardiograms of boys and girls. This method assigns to each vectocardiogram \( X_i \) the two dimensional vector \((\hat{\text{LD}}_0(X_i), \hat{\text{LD}}_1(X_i))\), with \( \hat{\text{LD}}_j(X_i) \) being the empirical lens depth of \( X_i \) w.r.t. the group of boys \((j = 1)\) and girls \((j = 0)\) (see Figure 5). As can be seen, there is no clear difference between the
Figure 3: QRS curve loop. The plane in blue is the best two dimensional approximation of the curve by least squares.

Figure 4: Data visualization of the data from $SO(3, 2)$. The violet arc represents the deepest observation according to lens depth. The outliers represented by red arcs correspond to the data outside the $\lambda = 0.10$ level set of the depth.

vectocardiograms of boys and girls. However, Figure 6 shows that if we take into account some measures of the level sets (such as volume, diameter, and farthest distance to the complement of the level set), some differences between them appear. The relation between the shapes of the level sets of the depths of boys and girls is represented in the bottom right panel of Figure 6.
Figure 5: Depth-depth plot for the vectocardiogram dataset.

Figure 6: Top left panel: The volume of the level sets of the lens depth for each group ($\psi_1$). Top right panel: The diameter of the level sets of the lens depth for each group ($\psi_2$). Bottom left panel: Maximum inner radius of the lens depth for each group ($\psi_3$). Bottom right panel: A possible visual representation for the shapes of the level sets.

7.3 Influenza dataset

This dataset consists of longitudinal data of the influenza virus, belonging to the family Orthomyxoviridae. As was done in [29], it is important to model the genomic evolution of the virus, see [57]. In this paper we focuses on modelling the temporal variability of the virus by means of the lens depth, and use this to predict and anticipate a possible pandemic. The influenza virus has an RNA genomic which is very common: it produces diseases like yellow fever and hepatitis and annually costs half a million deaths worldwide. It is well
known that the viruses change their genetic pattern over time, which is vital to developing a possible vaccine. We will study the the H3N2 variant of the virus, in particular, the subtype hemagglutini (HA), which produced the SARS pandemic in 2002. This variant is known to have a variability in its genetic arrangement over time, see [14; 43].

The dataset can be found in GI-SAID EpiFlu\textsuperscript{TM} database\footnote{www.gisaid.org} providing 1089 genomic sequences of H3N1 from 1993 to 2017 in New York, aligned using MUSCULE, see [24]. We used reduced trees of 5 leaves, as in [43], to capture the structure of the data. This set of trees can be endowed with a distance, see [8]. The database used was obtained from the GitHub repository \url{https://github.com/antheamonod/FluPCA}.

For this kind of data, constructing measures of centrality and variability, as well as confidence regions, is a problem that has been previously addressed in the literature, see for instance [5; 10; 62] and references therein.

For each year we computed the empirical lens depth of the trees, considered on the manifold of phylogenetic trees. We estimated the diameter of the level sets from the sample points that belong to the level sets, see Figure 7. As can be seen, there is a larger dispersion in the years prior to the pandemic of 2002 compared to later years.

![Figure 7: Diameters of the lens depth level sets from the sample of trees for the different years, with respect to $\lambda$.](image)

**Appendix**

**Proof of Lemma [7]**

If $z = (z_1, z_2) \in B_\rho(y, \delta)$, then $d(z_1, y_1) < \delta$ and $d(z_2, y_2) < \delta$. Let us prove first that

$$\{ x \in A(y_1, y_2) : d(x, \partial A(y_1, y_2)) > 2\delta \} \subset A(z_1, z_2)$$  \hspace{1cm} (6)

for all $z = (z_1, z_2) \in B_\rho(y, \delta)$. If $x \in A(y_1, y_2)$ and $d(x, \partial A(y_1, y_2)) > 2\delta$, then $d(x, y_1) \leq d(y_1, y_2) - 2\delta \leq d(y_1, z_1) + d(z_1, z_2) + d(z_2, y_2) - 2\delta \leq d(z_1, z_2)$. In the same way, if $d(x, y_2) \leq d(z_1, z_2)$, then $x \in A(z_1, z_2)$, which implies (6). Next let us prove that

$$A(z_1, z_2) \subset \{ x : d(x, A(y_1, y_2)) \leq 3\delta \}$$  \hspace{1cm} (7)

For any $x \in A(z_1, z_2)$, $d(x, y_1) \leq d(x, z_1) + d(z_1, y_1) \leq d(z_1, z_2) + \delta \leq d(y_1, y_2) + 3\delta$. In the same way, $d(x, y_2) \leq d(x, z_2) + \delta \leq d(z_1, z_2) + \delta \leq d(y_1, y_2) + 3\delta$. Then $x \in \{ x :$
\[ d(x, A(y_1, y_2)) \leq 3\delta \}. \] To prove that \( \omega_{f_z} \{ B_{\rho}(y, \delta) \} = 0 \), it is enough to prove that for all \((z_1, z_2), (t_1, t_2) \in B_{\rho}((y_1, y_2), \delta)\)

\[ |\mathbb{I}_{A(z_1, z_2)}(x) - \mathbb{I}_{A(t_1, t_2)}(x)| = 0. \] (8)

Observe that if \( x \in A(y_1, y_2) \) and \( d(x, \partial A(y_1, y_2)) > 3\delta \), then by (6), \( x \in A(z_1, z_2) \), so \( \mathbb{I}_{A(z_1, z_2)}(x) = 0 \) and \( x \in A(t_1, t_2) \) and hence \( \mathbb{I}_{A(t_1, t_2)}(x) = 0 \) which then shows that (8) holds. Proceeding in the same way if \( x \notin A(y_1, y_2) \) and \( d(x, \partial A) > 3\delta \), by (7) \( x \notin A(z_1, z_2) \), which implies \( \mathbb{I}_{A(z_1, z_2)}(x) = 0 \). Also \( x \notin A(t_1, t_2) \), which implies that \( \mathbb{I}_{A(t_1, t_2)}(x) = 0 \), so again (8) holds.

**Proof of Theorem 4**

We will apply Billingsley’s theorem to the set of functions

\[ \mathcal{F} = \{ f_z(z, y) = \mathbb{I}_{A(z, y)}(x) : x \in K \}, \]

where the sequence \( P_n \) of probability measures on \( M \times M \) is such that \( P_n(X_i, X_j) = (1/2)^{-1} \) if \( i \neq j \) and \( P_n(X_i, X_j) = 0 \) if \( i = j \). Let \( P \) be the product measure \( P_X \times P_X \). In this case

\[ \sup_{f \in \mathcal{F}} \left| \int f dP_n - \int f dP \right| = \sup_{x \in K} |\hat{LD}_n(x) - LD(x)|. \]

Clearly, \( \sup \{ |f(z) - f(t)| : f \in \mathcal{F}, z, t \in M \times M \} = 2 \). So we have to prove that (3) holds.

By Lemma 1, to prove (3), it is enough to prove that

\[ \limsup_{\delta \to 0} \sup_{x \in K} P_X \times P_X \{ (y_1, y_2) : d(x, \partial A(y_1, y_2)) \leq 3\delta \} = 0. \]

By the dominated convergence theorem, since \( P_X(\partial B(x, \delta)) = 0 \) for all \( \delta > 0 \) and all \( x \in M \), it follows that for fixed \( \delta \), \( P_X \times P_X \{ (y_1, y_2) : d(x, \partial A(y_1, y_2)) \leq 3\delta \} \) is a continuous function of \( x \). Also, for a fixed \( x \), \( P_X \times P_X \{ (y_1, y_2) : d(x, \partial A(y_1, y_2)) \leq 3\delta \} \to 0 \), again by the dominated convergence theorem.

By Lemma 1,

\[ \sup_{x \in K} P_X \times P_X \left\{ y = (y_1, y_2) : \omega_{f_z} \{ B_{\rho}(y, \delta) \} \geq \epsilon \right\} \leq \sup_{x \in K} P_X \times P_X \{ (y_1, y_2) : d(x, \partial A(y_1, y_2)) \leq 3\delta \} \]

which converges to 0 as \( \delta \to 0 \). \( \square \)

**Proof of Theorem 5**

Let \( \epsilon > 0 \) and suppose that \( R \) is large enough so that \( P_X(B(0, R)) < \epsilon \), and put \( K = \overline{B}(0, R) \). We will apply the previously mentioned theorems of Billingsley and Topsøe to the set of functions

\[ \mathcal{F} = \{ f_z(z, y) = \mathbb{I}_{A(z, y)}(x) : x \in \mathbb{R}^d \}. \]
We start by splitting into two terms

\[
\sup_x P_X \times P_X \left\{ y = (y_1, y_2) : \omega_f \{ B_\rho(y, \delta) \} \geq \epsilon \right\} \leq \\
\sup_{x \in K} P_X \times P_X \left\{ y = (y_1, y_2) : \omega_f \{ B_\rho(y, \delta) \} \geq \epsilon \right\} + \\
\sup_{x \in K^c} P_X \times P_X \left\{ y = (y_1, y_2) : \omega_f \{ B_\rho(y, \delta) \} \geq \epsilon \right\} = I_1 + I_2.
\]

By Lemma 1,

\[ I_1 \leq \sup_{x \in K} P_X \times P_X \{ (y_1, y_2) : d(x, \partial A(y_1, y_2)) \leq 3\delta \}, \]

which converges to 0 as \( \delta \to 0 \). Regarding \( I_2 \), we bound

\[
P_X \times P_X \left\{ y = (y_1, y_2) : \omega_f \{ B_\rho(y, \delta) \} \geq \epsilon \right\} \leq \\
P_X \times P_X \left\{ y = (y_1, y_2) : y_1, y_2 \in K \text{ and } \omega_f \{ B_\rho(y, \delta) \} \geq \epsilon \right\} + \\
P_X \times P_X \left\{ y = (y_1, y_2) : y_1 \in K^c \text{ or } y_2 \in K^c \text{ and } \omega_f \{ B_\rho(y, \delta) \} \geq \epsilon \right\}
\]

The second term is bounded from above by \( 2P_X(K^c) < 2\epsilon \). Lastly, to tackle the first term, by Lemma 1, it only remains to be proved that

\[
\lim_{\delta \to 0} \sup_{x \in K^c} P_X \times P_X \left\{ y = (y_1, y_2) : y_1, y_2 \in K \text{ and } d(x, \partial A(y_1, y_2)) < 3\delta \right\} = 0
\]

To do that, let \( \delta_0 \) be small enough so that \( P_X((\partial K)^{3\delta}) < \epsilon \) for all \( \delta < \delta_0 \). If \( y_1, y_2 \in B(0, R - 3\delta) \), then \( d(x, A(y_1, y_2)) \geq 3\delta \). Then,

\[
P_X \times P_X \left\{ y = (y_1, y_2) : y_1, y_2 \in K \text{ and } d(x, \partial A(y_1, y_2)) < 3\delta \right\} \leq \\
P_X(y_1 \in (\partial K)^{3\delta} \text{ or } y_2 \in (\partial K)^{3\delta}) < 2\epsilon. \quad (9)
\]

\[ \square \]

**Proof of Theorem 4.**

By assumption, we have that \( F \) has a finite VC-dimension. This implies that the assumptions in Proposition 10 in [31] hold for an order 2 U-statistic and the asymptotic distribution is derived from Theorem 4.10 in [3].

\[ \square \]

**Proof of Theorem 7**

Let \( K \) be any compact set. Observe that \( \{ LD \geq \lambda^- \} \cap K \) is nonempty and compact for any \( 0 < \lambda^- < \lambda \). Following the same ideas as those used in the proof of Theorem 1 in [18], one can derive that for all \( \epsilon > 0 \)

\[
\partial \{ LD \geq \lambda \} \cap K \subset B(\partial \{ \widehat{LD}_n \geq \lambda \} \cap K, \epsilon).
\]

The proof of

\[
\partial \{ \widehat{LD}_n \geq \lambda \} \cap K \subset B(\partial \{ LD \geq \lambda \} \cap K, \epsilon)
\]
is slightly different from the analogous inclusion in [18]. If we proceed by contradiction, there exists an \( x_n \in \{ \text{LD}_n \geq \lambda \} \cap K \), but

\[
d(x_n, \partial \{ \text{LD} \geq \lambda \} \cap K) > \epsilon.
\]

(10)

Observe that if \( x_n \in \partial \{ \text{LD}_n \geq \lambda \} \cap K \), then \( x_n \in \partial A(X_i, X_j) \) for some \( X_i \neq X_j \). If \( x_n \) is in the boundary of two or more sets \( A(X_i, X_j) \), we can take \( y_n \) such that \( d(y_n, x_n) < \epsilon/2^n \), \( y_n \in \partial \{ \text{LD}_n \geq \lambda \} \), \( y_n \) in the boundary of only one \( \partial A(X_i, X_j) \) and \( |\text{LD}_n(y_n) - \lambda| < \binom{n}{2}^{-1} \).

If \( x_n \) is in the boundary of only one \( A(X_i, X_j) \), we choose \( x_n = y_n \). In this case we clearly have also have \( |\text{LD}_n(y_n) - \lambda| < \binom{n}{2}^{-1} \).

Since \( K \) is compact, there exists an \( x \in K \) such that \( x_n, y_n \to x \) (by considering a subsequence if necessary). Then

\[
|\text{LD}(x) - \lambda| \leq |\text{LD}(x) - \text{LD}(y_n)| + |\text{LD}(y_n) - \text{LD}(y)| + |\text{LD}(y_n) - \lambda|
\]

The first terms converge to 0 by the continuity of LD at \( x \). The second term converges to 0 a.s., because the set \( K' = \{ \cup_n \{ y_n \} \} \cup \{ x \} \) is compact and by Theorem 4, \( \sup_{z \in K'} |\text{LD}_n(z) - \text{LD}(z)| \to 0 \). The last term is bounded from above by \( \binom{n}{2}^{-1} \to 0 \) as \( n \to \infty \). Then \( \text{LD}(x) = \lambda \), which implies that \( d(x_n, \partial \{ \text{LD} \geq \lambda \} \cap K) \leq d(x_n, x) \to 0 \), which contradicts (10). \( \square \)

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