The power of quantum channels for creating quantum correlations

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Abstract

Local noise can produce quantum correlations on an initially classically correlated state, provided that it is not represented by a unital or semi-classical channel [1]. We find the power of any given local channel for producing quantum correlations on an initially classically correlated state. We introduce a computable measure for quantifying the quantum correlations in quantum-classical states, which is based on the non-commutativity of ensemble states in one party of the composite system. Using this measure we show that the amount of quantum correlations produced, is proportional to the classical correlations in the initial state. The power of an arbitrary channel for producing quantum correlations is found by averaging over all possible initial states. Finally we compare our measure with the geometrical measure of quantumness for a subclass of quantum-classical states, for which we have been able to find a closed analytical expression.

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1 Introduction

One of the essential features of quantum mechanics is entanglement, which is a well known resource for quantum computation and communication tasks [2, 3]. However, it is increasingly become clear that certain kinds of separable states, with vanishing entanglement, exhibit some type of quantum correlation which turns out to be useful in information processing tasks. For example it has been shown [4] that it can be helpful in mixed state quantum computation [5], local broadcasting [6], quantum state merging [7], quantum communication [8, 9, 10] and quantum state discrimination [11]. Different measures have been introduced to quantify this kind of correlation [12, 13, 14, 15, 16, 17, 18, 19, 20]. An interesting features of this kind of correlation is that it can be produced by local actions on classically correlated states [1, 21, 22, 23]. The capability of creating this kind of correlation by unitary transformation [24] and its behavior under dissipation is studied [25, 26].

As an example which has no kind of quantum correlation, consider the following separable state:

$$\rho_{cc} = \sum_{i,j} p_{ij} \left| u_i \right\rangle \langle u_i \left| \otimes \left| v_j \right\rangle \langle v_j \right| \quad (1)$$

in which, \{\left| u_i \right\rangle\} and \{\left| v_j \right\rangle\} are orthogonal bases for each part, \(p_{ij}\)s are non-negative and \(\sum_{i,j} p_{ij} = 1\). The states \left| u_i \right\rangles are completely distinguishable in party A and the same holds for the states \left| v_j \right\rangles in party B. Such states are known as classical-classical (CC), or classically correlated states [13, 27].

Another class of separable states are of the form

$$\rho_{qc} = \sum_{i,j} p_{ij} \rho_i \otimes \left| v_j \right\rangle \langle v_j \right| \quad (2)$$

where \(\rho_i\)s are arbitrary pure or mixed but non-orthogonal density matrices. In these states, called Quantum-Classical, the states in party A are not necessarily distinguishable and this quantumness feature shows itself in the correlations between parts of this composite system.

States with quantum correlations of the form (2) can be obtained from classically correlated states in (1) by local noisy channels which are described by CPT (completely positive trace preserving) maps. It has been shown [1] that a local channel can produce quantum correlations on a CC state provided that the channel is neither unital nor semi-classical. Furthermore, it has been shown that the necessary and sufficient conditions for local creation of quantum correlations is that it is not a commutativity-preserving channel [22]. In [23] the maximum amount of quantum correlations that can be created by the channel from a classically correlated state has been found, using discord as a measure of quantum correlations.

As it is shown in [1] local channels which are not unital or semi-classical can produce quantum correlations, on suitable initial states. It is then natural to ask how much quantum correlations a general local channel can produce, when it acts on a classically correlated state. Clearly this question has operational and experimental significance. The amount of quantum correlations produced, depends not only on the local noise, but also on the initial classically correlated state. Here we find the amount of quantum correlations that a given channel can produce on an arbitrary classically correlated state. Furthermore, we find the average performance of a channel by averaging the amount of correlation which it creates on all classically correlated input states. To this end, we introduce a computable measure for quantum correlations and justify it in several ways. In particular for a subset of QC states (2) in which \(\rho_1\) and \(\rho_2\) are arbitrary pure states, we do perform an analytical optimization to obtain a closed form for the geometric measure of correlations introduced in [1] and show that our measure is monotonic with the geometric measure. Although this measure can only be used to quantify the amount of correlations in the states of form (2), the advantage of it is that no optimization is required for calculating it.
The structure of the paper, is as follows. In section (2) we introduce a simple measure for classical correlations of bi-partite qubit systems and remind the readers of the conditions under which \[1\], a local channel can or cannot create quantum correlations in such states. In section (3), we recapitulate what is known about qubit channels and add some new results on characterization of semi-classical qubit channels and their relation with unital channels. In section (4), a computable measure for quantifying the correlations in quantum-classical states, is introduced, where its properties are studied in detail. In section (5) the performance of a general qubit channel for producing quantum correlations is discussed and its correlating power is calculated. Section (6) is devoted to some explicit examples including amplitude damping channel. Finally in section (7), we derive a closed expression for geometrical measure of quantumness for a subclass of QC states \((\mathcal{E})\) in which \(\rho_1\) and \(\rho_2\) are arbitrary pure states, and compare it with our measure. The paper ends with a conclusion.

2 Classical and quantum correlations of qubit states

When the states in possession of the two parties, belong to two-level systems or qubits, many of the considerations, i.e. quantifying the classical and quantum correlations and also the characterization of quantum channels greatly simplify and pave the way for analytical treatments. In particular, as we will show, one can introduce computable measures for quantum correlations.

Let us start with a CC state as in \((\mathcal{E})\). For the two-level case this state is written explicitly as
\[
\sigma = p_{00}|u_0⟩⟨v_0| + p_{01}|u_0⟩⟨v_1| + p_{10}|v_0⟩⟨u_1| + p_{11}|v_1⟩⟨u_1|.
\]
we define its measure of classical correlations by the following quantity:
\[
C(\sigma) := 4|p_{00}p_{11} - p_{01}p_{10}|.
\]
In this way an uncorrelated state (i.e. a product state) for which \(p_{ij} = p_ip_j\) has zero measure of correlations and a state with maximal classical correlations, has \(C = 1\). An example of such a state is given by
\[
\sigma^{\text{max}} = \frac{1}{2}(|0⟩⟨0| + |1⟩⟨1|).
\]
The states in possession of party A, are not identical or orthogonal, the shared state between the two parties, although being separable and having no entanglement, is known to exhibit some degree of non-classical correlations. An example of this kind of state is
\[
\rho = \frac{1}{2}(|0⟩⟨0| + |+⟩⟨+|).
\]
where \(|+⟩ = \frac{1}{\sqrt{2}}(|0⟩ + |1⟩)\). Indeed various measures for determining the amount of quantum correlations in these states have been proposed in the literature \([1, 21, 22, 23]\).

An interesting question which has recently been investigated \([1, 21, 22, 23]\) is whether one of the parties, say Alice, can generate quantum correlations by performing a general quantum channel \(\mathcal{E}\) on her qubit. In such a case, the resulting state is
\[
\rho = (\mathcal{E} \otimes I)(\sigma) = \sum_{i,j} p_{ij}\mathcal{E}(|u_i⟩⟨u_i|) \otimes |v_j⟩⟨v_j|.
\]
This question was first posed in \([1]\) where it was proved that for qubits, this is not possible if the channel \(\mathcal{E}\) is unital or semi-classical (see the next section for their definition).
In view of these results, a natural question is how much a general non-unital channel is effective in creating quantum correlations starting from a classically correlated state. Clearly this question has operational and experimental significance. By local operation and classical communication Alice and Bob can prepare a classically correlated state of the form (1). Then Alice can perform a quantum channel on her qubit to turn the classically correlated state into a state with some degree of quantum correlations. One can then ask that given a fixed input state, what kind of channel Alice should use to create the highest amount of correlations. Or one can ask: given a fixed quantum channel, what kind of classically correlated input state, create the largest amount of quantum correlations.

3 Preliminaries on qubit channels

In this section we remind the reader of a few basic facts about qubit channels. We also discuss the characterization of unital and semi-classical channels and their relations with each other. It is well known that a qubit channel $\mathcal{E}$ induces an affine transformation on the Bloch sphere, $\mathcal{E} : \mathbf{r} \rightarrow \Lambda \mathbf{r} + \mathbf{t}$. A qubit channel can always be decomposed as $\mathcal{E} = \mathcal{U} \circ \mathcal{E}_c \circ \mathcal{V}$, where $\mathcal{U}$ and $\mathcal{V}$ are unitary channels and $\mathcal{E}_c$ is the canonical form of the channel. In other words, for every channel $\mathcal{E}$, one can write

$$\mathcal{E}(\rho) = U\mathcal{E}_c(V\rho V^{-1})U^{-1},$$

where $\mathcal{E}_c(\rho)$ is a canonical channel whose action on the Bloch vectors is given by $\mathcal{E}_c : \mathbf{r} \rightarrow \Lambda_D \mathbf{r} + \mathbf{t}$, in which $\Lambda_D = SAT$ is a diagonal matrix, where $S$ and $T$ are rotations in Bloch sphere induced by the unitary operators $U$ and $V$ [28]. A unital channel is one for which $\mathcal{E}(I) = I$ and a semi-classical channel is such its action on any input state can be written as

$$\mathcal{E}_{sc}(\rho) = \sum_k f_k(\rho) |k\rangle \langle k|,$$

where $\{|k\rangle\}$ is a fixed orthogonal set independent of the state $\rho$. Unital and semi-classical channels can be characterized in a simple way and for qubits, at least for qubit channels. For unital channels $\mathbf{t} = 0$. To characterize semi-classical qubit channels, let the fixed bases of the channel be as $\{|b\rangle, |-b\rangle\}$. Then we have

$$\mathcal{E}_{sc}(\rho) = f_+(\mathbf{r}) |b\rangle \langle b| + f_- (\mathbf{r}) |-b\rangle \langle -b|$$

$$= f_+(\mathbf{r}) \frac{1}{2} (I + \mathbf{b} \cdot \sigma) + f_- (\mathbf{r}) \frac{1}{2} (I - \mathbf{b} \cdot \sigma),$$

where $\mathbf{b}$ is a fixed basis vector.
Figure 2: (Color Online) The action of a unital channel (left) and a semi-classical channel (right) on the Bloch sphere. A unital channel always maps co-linear vectors to co-linear vectors. A semi-classical channel maps every Bloch vector to a fixed direction. None of them can create quantum correlations, since the resulting states of Alice, can be diagonalized in the same basis.

Figure 3: (Color Online) Semi-classical and unital channels. The channel $\mathcal{E}_1$ which is unital but not semi-classical, has $t = 0$, its $|\Lambda\rangle$ is not a product state. The channel $\mathcal{E}_2$ has $t = 0$ and its $|\Lambda\rangle$ is a product state. The channel channel $\mathcal{E}_3$, which is semi-classical but not unital, has its $|\Lambda\rangle$ a product, but its $t$ is non-zero. Finally the channel $\mathcal{E}_4$ has a non-zero $t$ and its $|\Lambda\rangle$ is not a product state. Only these types of channels which are neither unital nor semi-classical can create quantum correlations.

where in the second line we have written $f_\pm(r)$ to stress the dependence of $f_\pm$ on the Bloch vector $r$ of the state $\rho$. Using the fact that $\mathcal{E}_{sc}$ should be convex-linear, we find that $f_\pm$ should be affine transformations on $r$ and hence, without loss of generality, they can be parameterized as $f_\pm(r) = \frac{1}{2}(1 \pm t \pm a \cdot r)$ where we have used the fact that $f_+ + f_- = 1$. Here $a$ and $t$ are a real vector and a real number respectively. Putting all this together, we find

$$\mathcal{E}_{sc}(\rho) = \frac{1}{2}(I + (a \cdot r + t)b \cdot \sigma),$$

which means that the affine transformation induced by a semi-classical channel is given by

$$t_{sc} = tb, \quad \Lambda_{sc}r = (a \cdot r)b.$$  \hspace{1cm} (12)

This means that semi-classical channels are parameterized by 7 parameters, pertaining to the real number $t$ and the two vectors $a$ and $b$.

These considerations teach us how to characterize semi-classical and non-semi-classical qubit channels. A channel is non-semi-classical if $t \not\parallel \Lambda r$, i.e. either when $t = tb$ and $\Lambda r = (a \cdot r)c$ with
when \( \Lambda_{ij} \) as a tensor cannot be decomposed into the product of two vectors. In fact we note from (12) that \( (\Lambda_{sc})_{ij} = (b)_i(a)_j \propto (t_{sc})_i(a)_j \). That is, \( \Lambda \) as a tensor is a product of \( t_{sc} \) and another vector. In other words, if we vectorize the matrix \( \Lambda \) as \( |\Lambda\rangle := \sum_{i,j} \Lambda_{ij} |i,j\rangle \), then a qubit channel is semi-classical when its \( |\Lambda\rangle \) is as follows
\[
|\Lambda_{sc}\rangle \propto |t_{sc}\rangle|a\rangle,
\]
that is, if \( |\Lambda_{sc}\rangle \) is a product state. Otherwise it is a non-semi-classical channel. Figure (2) shows, the actions of unital and semi-classical channels on the Bloch vector and figure (3) shows the relation between these two classes of channels.

### 4 A computable measure for quantum correlations

One can quantify the quantum correlations in a given bi-partite state \( \rho \) in various ways, for example by using a measure based on distance. One such measure is [1]
\[
Q_G(\rho) := 1 - \max_{\sigma \in cc} F(\rho, \sigma),
\]
where \( cc \) denotes the set of all classically correlated states. However such measures are not easy to calculate, specially when we note that the set of classically correlated is not convex. For our purpose, i.e. for calculating the power of an arbitrary quantum channel for generating quantum correlations, we need a simple measure which has a simple analytic expression and yet, it retains many of the properties that other measures of quantum correlations have. We introduce this measure and then will explain its properties later on. As for the measure (14), in section (7), we derive a closed analytical expression for an important subclass and show that it qualitatively agrees with our measure for this subclass.

Consider a state of the form
\[
\rho = X_0 \otimes |0\rangle\langle 0| + X_1 \otimes |1\rangle\langle 1|,
\]
where \( X_1 \) and \( X_2 \) are two positive operators on the qubit space. We define the degree of quantum correlations of this state to be given by
\[
Q(\rho) := 4\| [X_0, X_1] \|_1
\]
where \( [A, B] = AB - BA \) and \( \| A \|_1 \) is the trace-norm given by \( \| A \|_1 = \text{tr}(\sqrt{A^\dagger A}) \). It is based on the fact that when \( [X_1, X_2] = 0 \), then they can be diagonalized in the same basis and hence the state is obviously classically correlated. Indeed if \( \rho_0 = \frac{1}{2}(I + r_0 \cdot \sigma) \) and \( \rho_1 = \frac{1}{2}(I + r_1 \cdot \sigma) \) are two density matrices, and
\[
\rho = p_0 \rho_0 \otimes |0\rangle\langle 0| + p_1 \rho_1 \otimes |1\rangle\langle 1|,
\]
then our measure gives, after a simple calculation
\[
Q(\rho) := 4p_0 p_1 |r_0 \times r_1|.
\]
This gives a value 0 for states of the form (5) and a value 1 for the state (6).

Besides its simplicity, this measure has many interesting properties which we now explain. Obviously it vanishes for classically correlated states and gives the maximum value of unity for states of the form (6). Consider now a general classically correlated state as
\[
\sigma = p_{00} |n\rangle\langle n| \otimes |0\rangle\langle 0| + p_{01} |n\rangle\langle n| \otimes |1\rangle\langle 1| + p_{10} |n\rangle\langle n| \otimes |0\rangle\langle 1| + p_{11} |n\rangle\langle n| \otimes |1\rangle\langle 1|.
\]
Let Alice acts on her qubit by a general quantum channel. From
\[ \mathcal{E}(|n\rangle\langle n|) = \frac{1}{2}(I + (\Lambda n + t) \cdot \sigma), \quad \mathcal{E}(|-n\rangle\langle -n|) = \frac{1}{2}(I + (-\Lambda n + t) \cdot \sigma). \] (20)
the resulting bi-partite state will then be given by
\[ (\mathcal{E} \otimes I)(\sigma) = X_0 \otimes |0\rangle\langle 0| + X_1 \otimes |1\rangle\langle 1|, \] (21)
where
\[ X_0 = \frac{1}{2} [(p_{00} + p_{10})(I + t \cdot \sigma) + (p_{00} - p_{10})\Lambda n \cdot \sigma] \] (22)
\[ X_1 = \frac{1}{2} [(p_{01} + p_{11})(I + t \cdot \sigma) + (p_{01} - p_{11})\Lambda n \cdot \sigma] \] (23)
Therefore using (21) we find that
\[ Q((\mathcal{E} \otimes I)\sigma) = 4||X_0, X_1||_1 = 4|p_{00}p_{11} - p_{01}p_{10}| \times 2|t \times \Lambda n| = 2|t \times \Lambda n|C(\sigma). \] (24)
This result says that the amount of quantum correlations produced is directly proportional to the amount of classical correlations already present, in the form \(4|p_{00}p_{11} - p_{01}p_{10}|\). Therefore no quantum correlations is created when the initial state has no classical correlations and the maximum quantum correlations is created only when the initial state has maximum classical correlations. Moreover for a unital channel (for which \(t = 0\)) or a semiclassical channel (for which \(t||\Lambda n\), see (12)) no quantum correlations is created.
Henceforth we take all input states to be of the form (5), for which we have
\[ Q((\mathcal{E} \otimes I)\sigma_{max}) = 2|t \times \Lambda n|. \] (25)
It is also proportional to degree of non-unitality of the quantum channel, measured by the magnitude of the vector \(t\). As a second merit of our measure, we show that unital channels, not only cannot create quantum correlations, they cannot increase the amount of quantum correlations for an arbitrary input state, which may happen to have some degree of quantum correlations. That is we show that for any quantum-classical (QC) input state \(\rho\), and any unitary channel \(\mathcal{E}_u\)
\[ Q((\mathcal{E}_u \otimes I)(\rho)) \leq Q(\rho). \] (26)
To prove this consider a QC state of the form
\[ \rho = p_0 \rho_0 \otimes |0\rangle\langle 0| + p_1 \rho_1 \otimes |1\rangle\langle 1|, \] (27)
where \(\rho_0\) and \(\rho_1\) do not necessary commute. In other words, \(\rho_i = \frac{1}{2}(I + r_i \cdot \sigma)\) where \(r_0\) and \(r_1\) are not necessarily co-linear or unit vectors. A unital channel acting on this state will produce
\[ (\mathcal{E}_u \otimes I)\rho = p_0 \mathcal{E}_u(\rho_0) \otimes |0\rangle\langle 0| + p_1 \mathcal{E}_u(\rho_1) \otimes |1\rangle\langle 1|, \] (28)
where \(\mathcal{E}_u(\rho_i) = \frac{1}{2}(I + \Lambda r_i \cdot \sigma), \quad i = 0, 1\). The quantum correlations of the new state, measured by (16) is given by
\[ Q((\mathcal{E}_u \otimes I)\rho) = 4p_0p_1 ||\mathcal{E}_u(\rho_0), \mathcal{E}_u(\rho_1)|| = 4p_0p_1|\Lambda r_0 \times \Lambda r_1|. \] (29)
However we know from the classification of qubit channels (28), that \(\Lambda = S \Lambda_D T\), where \(S\) and \(T\) are two orthogonal matrices which do the singular value decomposition of \(\Lambda\) and \(\Lambda_D = diag(\lambda_1, \lambda_2, \lambda_3)\) is a diagonal matrix with \(|\lambda_i| \leq 1\). Using the orthogonality of \(S\), we find that
\[ Q((\mathcal{E}_u \otimes I)\rho) = 4p_0p_1|SA_D T r_0 \times SA_D T r_1| = 4p_0p_1|\Lambda_D T r_0 \times \Lambda_D T r_1|. \] (30)
Using the conditions on the values of \(\lambda_i\), i.e. \(|\lambda_i| \leq 1\), we find that \(|\Lambda_D T r_0 \times \Lambda_D T r_1| \leq |T r_0 \times T r_1|\) and again using the orthogonality of \(T\), we find that this is less than or equal to \(|r_0 \times r_1|\). Therefore we have proved that
\[ Q((\mathcal{E}_u \otimes I)\rho) \leq 4p_0p_1|r_0 \times r_1| = Q(\rho). \] (31)
5 The power of quantum channels for generating quantum correlations

Let $\mathcal{E}$ be a completely positive trace-preserving map acting on a qubit. We ask how much power this quantum channel has for creating quantum correlations, when it acts on maximally classically correlated states. As explained in the introduction, the amount of correlations produced, depends on the initial state. Moreover as shown in (24), it is directly proportional to the amount of classical correlations already present in the state. Therefore to compute the power of a quantum channel, the best to do is to average over all the possible input states of the form (5) which have maximal classical correlations and then see how much quantum correlations on the average, a given channel $\mathcal{E}$ can produce when Alice enacts it on her state.

Therefore we define the power of the channel as follows:

$$ P(\mathcal{E}) := \int Q ((E \otimes I)(\sigma^{\max})) \, dn, $$

(32)

where $dn$ is an invariant measure over the Bloch sphere.

Using (25) and (32) we find that

$$ P(\mathcal{E}) = \int d\Omega 2|t \times \Lambda|.$$

(33)

We now show that with the measure defined in (16), the power of the channel $\mathcal{E}$ is the same as that of its canonical form $\mathcal{E}_c$. This is again an interesting property of the measure (16) which is not clear to hold for other kinds of measures, which are based on optimization, like the geometric measure in [1]. To do this we note that inserting (8) inside the commutator, the unitary operators $U$ and $U^{-1}$ will be eliminated, due to the property $[UaU^{-1}, UbU^{-1}] = U[a, b]U^{-1}$ and we are left with

$$ P(\mathcal{E}) = \int d\Omega \| E_c(\Omega |n\rangle\langle n|V\rangle(V\Omega)^\dagger, E_c(\Omega |n\rangle\langle n|V\rangle(V\Omega)^\dagger) \|_1. $$

(34)

If we now note that the states $|n\rangle$ and $|n\rangle$ can be obtained by the action of a unitary operator $\Omega$ on the states $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$, i.e. $|n\rangle = \Omega |0\rangle$, $|n\rangle = \Omega |1\rangle$, then we find that

$$ P(\mathcal{E}) = \int d\Omega \| E_c(\Omega |0\rangle\langle 0|V\rangle(V\Omega)^\dagger, E_c(\Omega |1\rangle\langle 1|V\rangle(V\Omega)^\dagger) \|_1. $$

(35)

Using the invariance of the measure $d\Omega = d(V\Omega)$, we finally find

$$ P(\mathcal{E}) = \int d\Omega \| E_c(\Omega |0\rangle\langle 0|\Omega^\dagger, E_c(\Omega |1\rangle\langle 1|\Omega^\dagger) \|_1 = P(\mathcal{E}_c). $$

(36)

In view of this result, hereafter we can calculate the power of quantum channels when they are in the canonical form.

6 Examples

In this section, we study the correlations created by a few non-unital and non-semi-classical channels.
Figure 4: (Color Online) Correlating power of amplitude damping channel, based on two different measures for correlations, the quantum discord (dashed red line) and our measure (solid blue line). The power based on our measure is closed to the one based on quantum deficit (not shown). See figure (1) of [23].

6.1 The amplitude damping channel

This channel describes the leakage of a photon in a cavity to an environment which has no photon, and is described by the Kraus operators

\[ E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}. \]

(37)

The parameters of the corresponding affine transformation are given by

\[ \Lambda = \begin{pmatrix} \sqrt{1-\gamma} & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 \\ 0 & 0 & 1 - \gamma \end{pmatrix}, \quad t = \begin{pmatrix} 0 \\ 0 \\ \gamma \end{pmatrix}. \]

(38)

From (33) we find after a simple integration

\[ \mathcal{P}(E_{AD}) = \frac{\pi \gamma \sqrt{1-\gamma}}{2}. \]

(39)

Obviously when \( \gamma = 0 \), \( E_{AD} = \text{id} \), no correlations can be produced. In the other extreme, when \( \gamma = 1 \), the channel maps every state to \( |0\rangle \langle 0| \) and again no correlations can be produced. Channels for which \( \gamma = \frac{2}{3} \) have the highest correlating power equal to \( \frac{\pi}{3\sqrt{3}} \). Figure (4) shows the correlating power of amplitude damping channel as a function of \( \gamma \) based on two different measures. It is intriguing that a channel which is dissipative in nature can create correlations, although its correlating power is small, see the other examples.

6.2 Measurements followed by preparations

Another interesting class of non-unital channels is adaptive preparation of states, depending on the outcome of a projective measurement. Let Alice measures her qubit in the basis \( \{ |a\rangle, |-a\rangle \} \). In case her outcome is \( |a\rangle \), she replaces her qubit with \( |m_0\rangle \) and in case her outcome is \( |-a\rangle \), she replaces her qubit with \( |m_1\rangle \), where \( |m_0\rangle \) and \( |m_1\rangle \) are two non-orthogonal pure states. Such a channel can be described as

\[ \mathcal{E}(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger \]

(40)
with $A_0 = |m_0\rangle\langle a|$ and $A_1 = |m_1\rangle\langle -a|$. To find the correlating power of this channel, according to (33), we have to find the affine transformation corresponding to this channel. To do this we note that

$$\mathcal{E}(\rho) = |m_0\rangle\langle m_0|\rho|a\rangle\langle a| + |m_1\rangle\langle m_1|\langle -a\rangle\langle -a|$$  \hspace{1cm} (41)

Writing the pure states in terms of their Bloch representation, namely i.e. $\rho = \frac{1}{2} (1 + n \cdot \sigma)$ and

$$|m_i\rangle\langle m_i| = \frac{1}{2} (1 + m_i \cdot \sigma), \quad |\pm a\rangle\langle \pm a| = \frac{1}{2} (1 \pm a \cdot \sigma),$$  \hspace{1cm} (42)

we find that

$$\mathcal{E} : n \rightarrow \frac{1}{2} (m_0 - m_1) a \cdot n + \frac{1}{2} (m_0 + m_1).$$  \hspace{1cm} (43)

Therefore we find the quantum correlations in the resulting state to be

$$Q ((\mathcal{E} \otimes I)\sigma) = |(m_0 \times m_1)(a \cdot n)|$$  \hspace{1cm} (44)

Thus the quantum correlations is maximized when $a \parallel n$ and when $m_0 \perp m_1$. This means that to create maximum correlations, Alice should measure her qubit in the same basis as present in the initial state and she should also prepare the corresponding states to have orthogonal vectors on the Bloch sphere, i.e. $|0\rangle$ and $|+\rangle$. From (33), the correlating power of this channel turns out to be $P(\mathcal{E}) = \frac{1}{2} |m_0 \times m_1|$.

### 6.3 A non-semi-classical channel

Consider a non-semi-classical channel $\mathcal{E}$ with $\Lambda r = (a \cdot r)c$ and $t = tb$, with $c \not\parallel b$. When acting on a maximally classically correlated state, the resulting state has a quantum correlations given by $2|t \times \Lambda n| = 2|a \cdot n| |b \times c|$ and the power of the channel will be given by $P(\mathcal{E}) = t|a||b \times c|$.

### 6.4 General qubit channels

In this subsection we discuss the correlating power of channels in general. In view of equation (36), we need only consider the canonical form of channels for which $\Lambda_D$ is diagonal. Every such channel is described by 6 parameters, $\{\lambda_i, t_i\}$, $i = 1, 2, 3$. First we note from (25), that for a fixed channel the best initial state is when $t \cdot \Lambda_D n = 0$. In this case, $|t \times \Lambda_D n|$ reduces to $|t||\Lambda_D n|$ and from this we find the best choice of $n$ is that $n$ be parallel to $\mathcal{A} := (\lambda_1, \lambda_2, \lambda_3)$. Putting all this together we find from (25) that the maximum correlations this channel can produce is given by

$$Q_{max}(\mathcal{E}) = 2|t| \sqrt{\frac{A_1^2 + A_2^2 + A_3^2}{A_1^2 + A_2^2 + A_3^2}}.$$  \hspace{1cm} (45)

The correlating power of a general abstract channel is given by the integral (32).

### 7 A note on geometric measure of correlations

We have based our discussion on a simple and easily computable measure of quantum correlations, introduced in (10). It is desirable to compare this measure with the geometric measure (14), introduced in (11). The point is that the later measure needs an optimization which does not necessarily lead to a closed analytical form. However in this section we derive a closed expression for the geometric measure of quantum correlations, for a class of states in the form

$$\rho = p_0 |n_0\rangle\langle n_0| \otimes |0\rangle\langle 0| + p_1 |n_1\rangle\langle n_1| \otimes |1\rangle\langle 1|,$$  \hspace{1cm} (46)
where \(|n_0\rangle\) and \(|n_1\rangle\) are two arbitrary pure states. The results in this section, which are a byproduct of our investigations on this problem, can indeed be read independently from the rest of the paper. In fact these results have their own interest, since both the question we pose and also the method of analysis which is based on an analytic optimization problem, are interesting in their own right. Finally we show that our measure agrees qualitatively with this measure based on fidelity and distance. The question we ask is this:

**Question:** Let \(\rho\) be a state as in (46). What is the nearest classically correlated state to this state? By nearest, we mean the state with the highest fidelity. Therefore we want to find the classically correlated state \(\sigma_{cc}\) of the form (1) which has the highest fidelity \(F(\rho, \sigma_{cc})\). Following (14) we then regard \(1 - F(\rho, \sigma_{cc})\) as the quantum correlations of the state \(\rho\).

Given the huge space of classically correlated states, (see (1)) which is parameterized by the classical probability distribution \(p_{ij}\) and the orthonormal qubit bases states \(\{|u_A\rangle\} \) and \(\{|v_B\rangle\}\), it is clear that this optimization problem is quite non-trivial. Nevertheless we find an exact answer for this question in an analytic way. We first present our answer to this question in the form of a theorem and then detail our proof.

**Theorem:** Given the bi-partite state

\[
\rho = p_0 |n_0\rangle \langle n_0| \otimes |0\rangle \langle 0| + p_1 |n_1\rangle \langle n_1| \otimes |1\rangle \langle 1|,
\]

a classically correlated state which is nearest to this state is of the following form:

\[
\sigma_{cc} = \frac{1 + \xi}{2} |s_0\rangle \langle s_0| \otimes |0\rangle \langle 0| + \frac{1 - \xi}{2} |s_1\rangle \langle s_1| \otimes |1\rangle \langle 1|
\]

where depending on the angle between the vectors \(n_0\) and \(n_1\), (Fig. (5)) we have

i: If \(n_0 \cdot n_1 \geq 0\), then

\[
s_0 = s_1 = \frac{p_0 n_0 + p_1 n_1}{\sqrt{1 - 2p_0 p_1 (1 - n_0 \cdot n_1)}}, \quad \xi = \frac{p_0 - p_1}{\sqrt{1 - 2p_0 p_1 (1 - n_0 \cdot n_1)}},
\]

ii: If \(n_0 \cdot n_1 \leq 0\), then

\[
s_0 = -s_1 = \frac{p_0 n_0 - p_1 n_1}{\sqrt{1 - 2p_0 p_1 (1 + n_0 \cdot n_1)}}, \quad \xi = \frac{p_0 - p_1}{\sqrt{1 - 2p_0 p_1 (1 + n_0 \cdot n_1)}}.
\]

In the following discussion and proofs, we designate the corresponding CC states in the above two cases by \(\sigma^{cc}_{cc}\) (case i) and \(\sigma^{cc}_{cc}\) (case ii) respectively.

Note that in case i), the classically correlated state is indeed a product state while in the other case, it has some classical correlations. In both cases, the fidelity between the state \(\rho\) and this nearest state \(\sigma_{cc}\) is given by

\[
F_{max} = F(\rho, \sigma_{cc}) = \frac{1}{2} \left[ \sqrt{p_0 (1 + \xi)} (1 + n_0 \cdot s_0) + \sqrt{p_1 (1 - \xi)} (1 + n_1 \cdot s_1) \right].
\]

In particular it directly follows that when \(p_0 = p_1 = \frac{1}{2}\), the fidelity is given by

\[
F_{max} = \sqrt{\frac{1}{2} \left[ 1 + \sqrt{\frac{1 + |n_0 \cdot n_1|}{2}} \right]},
\]

implying that the largest quantum correlations belongs to states of the form

\[
\rho = \frac{1}{2} |0\rangle \langle 0| \otimes |0\rangle \langle 0| + |\phi\rangle \langle \phi| \otimes |1\rangle \langle 1|,
\]
where $|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle)$ is an equatorial states on the Bloch sphere.

**Lemma 1:** The nearest classically correlated state $\sigma_{cc}$ is of the form

$$\sigma_{cc} = x_0 \otimes |0\rangle\langle 0| + x_1 \otimes |1\rangle\langle 1|.$$  

(53)

Note that by this lemma, we are excluding the possibility of the second bases to be any bases other than $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$.

**Proof:** We first rewrite $\rho$ in the matrix notation as

$$\rho = \left( \begin{array}{cc} p_0 \rho_0 & 0 \\ 0 & p_1 \rho_1 \end{array} \right),$$

(54)

where $\rho_0 = |n_0\rangle\langle n_0|$ and $\rho_1 = |n_1\rangle\langle n_1|$. This state clearly has the invariance property

$$(I \otimes Z)\rho(I \otimes Z) = \rho,$$  

(55)

where $Z = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$. Now assume that $\sigma_{cc}$ be a state which has the maximum fidelity with this state. Then from the above invariance and from the invariance property of the fidelity, under local unitary transformations, we find

$$F(\rho, \sigma_{cc}) = F((I \otimes Z)\rho(I \otimes Z), \sigma_{cc}) = F(\rho, (I \otimes Z)\sigma_{cc}(I \otimes Z))$$  

(56)

Therefore either $\sigma_{cc}$ and $(I \otimes Z)\sigma_{cc}(I \otimes Z)$ are the same state, or else we can form an invariant state in the form

$$\sigma_{cc}^{new} = \frac{1}{2}(\sigma_{cc} + (I \otimes Z)\sigma_{cc}(I \otimes Z))$$  

(57)

which has higher fidelity with $\rho$ in view of the convex property of the fidelity $F(\rho, \lambda \sigma_1 + (1 - \lambda)\sigma_2) \geq \lambda F(\rho, \sigma_1) + (1 - \lambda)F(\rho, \sigma_2)$. Note that since we have to make this maximization over the set of classically correlated states, it is important to note if $\sigma_{cc}$ is classically correlated, then $\sigma_{cc}^{new}$ is also classically correlated. This shows that the closest CC state to $\rho$ has the same invariance property 53 and so is of the same form as $\rho$ itself, hence the lemma is proved.

**Lemma 2:** The classically correlated state $\sigma_{cc}$ nearest to (47) is of the form

$$\sigma_{cc} = q_0 |s_0\rangle \langle s_0| \otimes |0\rangle\langle 0| + q_1 |s_1\rangle \langle s_1| \otimes |1\rangle\langle 1|.$$  

(58)
where $s_0$ and $s_1$ are two unit vectors on the Bloch sphere which are co-linear, (i.e. they are either the same $s_1 = s_0$ or opposite to each other $s_1 = -s_0$). By this lemma we are excluding the possibility of $x_0$ and $x_1$ to be mixed states.

**Proof:** The proof of this lemma is actually by calculation. From lemma 1, and noting that $[x_0, x_1] = 0$, we let $\{\mid s \rangle, \mid -s \rangle \}$ be the basis which diagonalize $x_0$ and $x_1$. Then the state $\sigma_{cc}$ will be of the form

$$\sigma_{cc} = (q_{00} \mid s \rangle \langle s \mid + q_{10} \mid -s \rangle \langle -s \mid) \otimes \mid 0 \rangle \langle 0 \mid + (q_{01} \mid s \rangle \langle s \mid + q_{11} \mid -s \rangle \langle -s \mid) \otimes \mid 1 \rangle \langle 1 \mid. \quad (59)$$

We want to maximize the fidelity of the state (47) with this state, given the general definition $F(\rho, \sigma) = \text{tr} \sqrt{\rho^2 \sigma \rho \sigma^2}$. From the form of (47) and (59) and the purity of all the states $\mid n_i \rangle$ and $\mid s_i \rangle$, and using the fact that $|\langle m | n \rangle| = \sqrt{1 + \cos^2 \theta}^2$, we find

$$F(\rho, \sigma_{cc}) = \sqrt{p_0 \frac{1}{2} (q_{00} + q_{10} + q_{01} - q_{11}) s \cdot n_0} + \sqrt{p_1 \frac{1}{2} (q_{01} + q_{11} + q_{00} - q_{10}) s \cdot n_1}. \quad (60)$$

We now have to maximize this expression with respect to the variables $q_{ij}$, subject to the constraint $\sum_{ij} q_{ij} = 1$ and the direction of the vector $s$. The first thing to note is that setting the variations of $F$ with respect to $s$ (using Lagrange multipliers to account for its normalization), one finds that $s$ should lie in the same plane as $n_0$ and $n_1$. One then set the variations of $F$ with respect to $q_{ij}$ equal to zero, again taking into account the constraint $\sum_{ij} q_{ij} = 1$ with a Lagrange multiplier. The result is that from all the $q_{ij}$ only two should be non-vanishing. This proves the lemma.

Having this, and taking for concreteness $q_{11} = q_{10} = 0$ we now re-write (60) as

$$F(\rho, \sigma) = \sqrt{\frac{p_0 q_0 (1 + n_0 \cdot s_0)}{2}} + \sqrt{\frac{p_1 q_1 (1 + n_1 \cdot s_1)}{2}}. \quad (61)$$

We now take either $s_1 = s_0$ or $s_1 = -s_0$ in order not to loose generality. To proceed with the optimization, we now parametrize the vectors $n_0$ and $n_1$ and $s_0$ as in figure (6), that is:

$$n_0 = (-\sin \alpha, \cos \alpha), \quad n_1 = (\sin \alpha, \cos \alpha), \quad s_0 = (\sin \theta, \cos \theta), \quad s_1 = \pm s_0. \quad (62)$$
Also we set
\[ q_0 := \frac{1 + \xi}{2}, \quad q_1 := \frac{1 - \xi}{2}, \]  
where \( 1 \leq \xi \leq 1 \). Using (62) and (63), equation (61) is rewritten as
\[ F(\rho, \sigma_{cc}^+) = \sqrt{p_0(1 + \xi)} \cos\left(\frac{\theta + \alpha}{2}\right) + \sqrt{p_1(1 - \xi)} \cos\left(\frac{\theta - \alpha}{2}\right). \]  

and
\[ F(\rho, \sigma_{cc}^-) = \sqrt{p_0(1 + \xi)} \cos\left(\frac{\theta + \alpha}{2}\right) + \sqrt{p_1(1 - \xi)} \sin\left(\frac{\theta - \alpha}{2}\right). \]  

We now have to maximize each of these expressions with respect to the two parameters \( \theta \) and \( \xi \). It is convenient to do this for the two states \( \sigma_{cc}^\pm \) separately.

**Case i:** Consider the expression (64). Setting \( \frac{\partial F}{\partial \xi} = 0 \), we find
\[ \sqrt{p_0(1 + \xi)} \cos\left(\frac{\theta + \alpha}{2}\right) = \sqrt{p_1(1 - \xi)} \cos\left(\frac{\theta - \alpha}{2}\right). \]  

Setting \( \frac{\partial F}{\partial \theta} = 0 \), we obtain
\[ \sqrt{p_0(1 + \xi)} \sin\left(\frac{\theta + \alpha}{2}\right) = -\sqrt{p_1(1 - \xi)} \sin\left(\frac{\theta - \alpha}{2}\right). \]  

From these two expressions, one can obtain the optimal values of \( \xi \) and \( \theta \), which ultimately determine the vector \( s_0 \) and \( q_0 \) and hence the classically correlated state \( \sigma_{cc}^+ \) which has the maximal fidelity with the given state \( \rho \). To proceed further, we note that by multiplying equations (66) and (67) we obtain
\[ p_0 \sin(\theta + \alpha) = p_1 \sin(\alpha - \theta). \]  

This equation determines the angle \( \theta \). Dividing (69) by (66) we obtain
\[ (1 + \xi) \tan\left(\frac{\alpha + \theta}{2}\right) = (1 - \xi) \tan\left(\frac{\alpha - \theta}{2}\right). \]  

From this second equation we can then determine \( \xi \). In order to express everything in terms of the original data of the problem, that is \( p_0, n_0 \) and \( n_1 \), we note equation (68) is equivalent to
\[ p_0 n_0 \times s_0 = p_1 n_1 \times s_0, \]  

from which we find that \( s \propto p_0 n_0 + p_1 n_1 \) or after normalization,
\[ s_0 = \frac{p_0 n_0 + p_1 n_1}{\sqrt{1 - 2p_0p_1(1 - n_0 \cdot n_1)}}. \]  

To express \( \xi \) directly in terms of the initial data of the problem we note that after some algebra, equation (69) gives
\[ \xi = -\frac{\sin \theta}{\sin \alpha}. \]  

It is now straightforward to start from (72) and verify the following relations:
\[ \sin \alpha = \sqrt{\frac{1 - n_0 \cdot n_1}{2}}, \quad x = \frac{n_1 - n_0}{\sqrt{2(1 - n_0 \cdot n_1)}}. \]
From the fact that \( \sin \theta = \mathbf{x} \cdot \mathbf{s}_0 \) and the above expressions, we can put everything together using (71) and (73) and obtain \( \xi \) as

\[
\xi = \frac{p_0 - p_1}{\sqrt{1 - 2p_0 p_1 (1 - \mathbf{n}_0 \cdot \mathbf{n}_1)}}. 
\]  
(74)

In this way we obtain the one of the nearest classically correlated state to our quantum-classical state \( \rho \).

**Case ii:** In this case where \( s_1 = -s_0 \), we will have

\[
F(\rho, \sigma_{cc}^-) = \sqrt{\frac{p_0}{1 + \xi}} \cos\left(\frac{\theta + \alpha}{2}\right) + \sqrt{\frac{p_1}{1 - \xi}} \sin\left(\frac{\theta - \alpha}{2}\right). 
\]  
(75)

Setting \( \frac{\partial F}{\partial \xi} = 0 \), we now find

\[
\sqrt{\frac{p_0}{1 + \xi}} \cos\left(\frac{\theta + \alpha}{2}\right) = \sqrt{\frac{p_1}{1 - \xi}} \sin\left(\frac{\theta - \alpha}{2}\right). 
\]  
(76)

Setting \( \frac{\partial F}{\partial \theta} = 0 \), we obtain

\[
\sqrt{p_0 (1 + \xi)} \sin\left(\frac{\theta + \alpha}{2}\right) = \sqrt{p_1 (1 - \xi)} \cos\left(\frac{\theta - \alpha}{2}\right). 
\]  
(77)

Now instead of (76) and (77) we will have

\[
p_0 \sin(\theta + \alpha) = p_1 \sin(\theta - \alpha). 
\]  
(78)

Instead of (70) we now have

\[
p_0 \mathbf{n}_0 \times \mathbf{s}_0 = -p_1 \mathbf{n}_1 \times \mathbf{s}_0, 
\]  
(79)

leading to

\[
\mathbf{s}_0 = \frac{p_0 \mathbf{n}_0 - p_1 \mathbf{n}_1}{\sqrt{1 - 2p_0 p_1 (1 + \mathbf{n}_0 \cdot \mathbf{n}_1)}}. 
\]  
(80)

Instead of (69) we will have

\[
(1 + \xi) \tan\left(\frac{\theta + \alpha}{2}\right) = (1 - \xi) \cot\left(\frac{\theta - \alpha}{2}\right). 
\]  
(81)

which leads to

\[
\xi = \frac{\cos \theta}{\cos \alpha}. 
\]  
(82)

To obtain an explicit expression for \( \xi \), we now use

\[
\cos \alpha = \sqrt{\frac{1 + \mathbf{n}_0 \cdot \mathbf{n}_1}{2}}, \quad z = \frac{\mathbf{n}_1 + \mathbf{n}_0}{\sqrt{2(1 + \mathbf{n}_0 \cdot \mathbf{n}_1)}}, 
\]  
(83)

and the facts that \( \cos \theta = z \cdot \mathbf{s}_0 \) to obtain

\[
\xi = \frac{p_0 - p_1}{\sqrt{1 - 2p_0 p_1 (1 + \mathbf{n}_0 \cdot \mathbf{n}_1)}}. 
\]  
(84)

It is interesting to compare the value of quantum correlations in the state (46) as given by our measure (16) and as given by the geometric measure in (14). Figure (7) shows the amount of correlations as a function of \( \mathbf{n}_0 \cdot \mathbf{n}_1 \) for the case \( p_0 = p_1 = \frac{1}{2} \). It is seen that while both measures agree, our measure is such that it is normalized to 1 for the state \( \frac{1}{2}(|0\rangle \langle 0| + |+\rangle \langle +| + |1\rangle \langle 1|) \). This is yet another good feature of our measure in addition to its simplicity and computability.
Figure 7: (Color Online) The quantum correlations in the state (47) (for $p_0 = p_1 = \frac{1}{2}$), as measured by our measure (solid blue line) and by the measure (14) (dashed red line). The horizontal axis is $n_0 \cdot n_1$.

8 Conclusion

In conclusion, we have studied the power of local channels in producing quantum correlations when they act on classically correlated states. To quantify the performance of an arbitrary channel in producing quantum correlations production, we first introduce a computable measure for which calculation no optimization is required. This measure is 0 for product states and is one for states of the form which are maximal quantum correlations. This maximality is related to the fact the maximal indistinguishability of the states in possession of Alice. Furthermore, this measure in invariant under local unitary evolution. We have also calculated in closed form, the geometric measure of quantum correlations introduced in [1], for a subclass of states (which allowed analytical calculations) and have shown that it is monotonic with our measure.

Using this measure, the amount of quantum correlations produced by an arbitrary unital or semi-classical channel is shown to be zero, as it is expected [1]. Furthermore, we have shown that the amount of correlations produced, is proportional to the classical correlations in the initial state. We also show that the power of all qubit quantum channels are equal to the power of their canonical form $E_c$.

We expect that a modification of this measure can also quantify quantum correlations in higher than two dimensions. In these dimensions, even unital channels may produce quantum correlations. Using this measure, which is easy to compute, one can analyze the performance of unital and non-unital channels in higher dimension. The only property that is not valid in higher dimensions is the equality of the average power of channels related through unitary evolutions before and after their action, which is due to the fact that $\mathcal{E} = \mathcal{U} \circ E_c \circ \mathcal{V}$ holds just in dimension two.

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