Using a combination of analytical and numerical methods, we obtain a two-dimensional spacetime describing a black hole with tachyon hair. The physical ADM mass of the black hole is finite. The presence of tachyon hair increases the Hawking temperature.
The fundamental role of black holes in the physics of the four-dimensional real world is currently not well understood. Two basic problems must be solved in order to elucidate matters. First, we need to construct a satisfactory quantum theory of gravitation. Second, we need to be able to resolve the Hawking paradox, namely, the apparent conflict between the laws of quantum mechanics and the way in which black holes evaporate.

One of the ingredients in these puzzles is the observation that black holes have a large entropy. For non-rotating, uncharged black holes in four dimensions, the entropy $S$ is proportional to the area of the event horizon and is given by $\frac{4\pi GM^2}{\bar{h}}$, where the mass of the black hole is $M$. In contrast, as $M$ gets large, the entropy of most isolated systems, e.g., stars or relativistic gas, grows at most like $M$. Thus, a black hole has much more entropy than a corresponding ‘normal’ system. According to the Boltzmann interpretation of entropy, this reflects a rapid growth in the corresponding density of states $\rho(M)$. In fact, since $\rho(M) = e^S$, it follows that $\rho(M)$ fails to be bounded by the Boltzmann factor $e^{-M/T}$ for large $M$ for any value of the temperature $T$. The negative specific heat of the Schwarzschild black hole is associated with this fact.

In the classical limit $\bar{h} \to 0$ the entropy becomes infinite, which is reflected in the black-hole uniqueness theorems (the ‘no-hair’ theorems). This powerful collection of results in classical general relativity shows that the only degrees of freedom of black holes observable from outside the event horizon are those related to longitudinal degrees of freedom of gauge fields that are also physical. Thus, a black hole is characterized by its mass, momentum, and angular momentum (from the gravitational field), and electric and magnetic charge (from a $U(1)$ gauge field) $[3, 4, 5, 6]$. Similar considerations apply to non-abelian Yang-Mills fields and systems with Higgs fields $[7, 8, 9, 10, 11]$. 

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One testing ground for exploration of these puzzles is the case of two spacetime dimensions, where a satisfactory nonperturbative version of string theory exists, thereby providing a quantum theory of gravity. Within this framework, a black-hole solution has recently been found \[12, 13, 14, 15\]. The entropy of this black hole, at least in the sigma-model approximation, is given by

\[ S = \frac{M}{T_c}. \]

Here, \( M \) is the ADM mass \[16\] and \( T_c \) is the temperature, which is a fixed parameter derived from the central charge of the theory.

This two-dimensional result is dramatically different from that found in four dimensions. The entropy only grows linearly with the mass, like normal systems. Consequently, the density of states is bounded by the Boltzmann factor provided the physical temperature \( T \) obeys \( T < T_c \). Presumably there is a phase transition at \( T = T_c \), but the physical significance of this is not currently understood. In any event, the slower growth of the density of states in two dimensions suggests that the black hole uniqueness theorems might not have as broad an applicability in \( d = 2 \) as in \( d = 4 \). This would appear to be especially important for the phase \( T < T_c \).

Let us examine the uniqueness theorems at the semiclassical level for the case of two-dimensional bosonic string theory. The physical states of the theory are divided into two classes: the tachyon field \( T \), which is a massless scalar field, and an infinite set of discrete states, of which the graviton and dilaton are examples. In the \( \sigma \)-model approximation to string theory, target-space physics can be described by an effective spacetime action \[17\]

\[ I = -\frac{1}{2\kappa^2} \int d^2 x \sqrt{g} e^\phi \left( \hat{c} - R - (\nabla \phi)^2 + (\nabla T)^2 + 2V(T) \right), \]

where \( R \) is the Ricci scalar of the metric \( g_{ab} \), \( \phi \) is the dilaton, and \( T \) is the tachyon with potential \( V(T) = -2T^2/\alpha' + \ldots \). The constant \( \hat{c} \) is related to the central charge of the string theory and is given by

\[ \hat{c} = \frac{2(d-26)}{3\alpha'}, \]

while \( \alpha' \) is the inverse string tension and \( \kappa \) corresponds to the Newton constant in two dimensions.
Varying the action (4) with respect to the metric and tachyon yields the corresponding beta-function equations at one loop:

\[ R_{\mu\nu} = \nabla_\mu \nabla_\nu \phi + \nabla_\mu T \nabla_\nu T \]  

(6)

for the metric, and

\[ \Box T + \nabla_\mu \phi \nabla^\mu T = \frac{\partial V}{\partial T} \]  

(7)

for the tachyon. Variation of the action with respect to the dilaton gives an equation consistent with the Bianchi identities, representing a first integral of Eq. (4) with \( \hat{c} \) as the constant of integration. It can be written

\[ \hat{c} = R - (\nabla \phi)^2 - 2 \Box \phi - (\nabla T)^2 - 2V(T) \]  

(8)

One expects black-hole equilibrium states to be static and asymptotically flat, so we assume the spacetime possesses a timelike Killing vector \( k^\mu \), at least in the asymptotic region. Since the black hole is an isolated object, at large distances from it the target space can be taken as the linear dilaton vacuum. A general form of the metric for a static \( d = 2 \) spacetime is

\[ ds^2 = -v^2 dt^2 + \frac{dr^2}{w^2} \]

(9)

where \( v \) and \( w \) are taken as functions of \( r \) only. With this parametrization, the beta-function equation (4) for the metric becomes the two equations

\[ \frac{v''}{v} + \frac{v' w'}{v w} + \frac{v'}{v} \phi' = 0 \]

(10)

and

\[ \frac{v''}{v} + \frac{v' w'}{v w} + \frac{w'}{w} \phi' + T'^2 = 0 \]

(11)

where a prime denotes \( d/dr \). The tachyon equation (7) becomes

\[ T'' + \left( \frac{v' w'}{v w} + \phi' \right) T' - \frac{1}{w^2} \frac{\partial V}{\partial T} = 0 \]

(12)

and the dilaton equation (5) becomes

\[ \frac{v''}{v} + \frac{v' w'}{v w} + \phi'^2 + \frac{1}{2} \phi'^2 + \left( \frac{v' w'}{v w} \right) \phi' + \frac{1}{2} T'^2 + \frac{1}{w^2} \left( V + \frac{1}{2} \hat{c} \right) = 0 \]

(13)
To simplify formulae in what follows, we adopt units such that $\alpha' = 2$ and we keep only the quadratic terms in the tachyon potential. This means, for example, that $V(T) = -T^2$ and $\dot{c} = -8$. Integrating Eq. (14) and rearranging the other equations yields the convenient forms

$$v' = \frac{k}{w} e^{-\phi},$$

(14)

where $k$ is an integration constant,

$$\phi'' + \left(\frac{v'}{v} + \frac{w'}{w} + \phi'\right) \phi' + \frac{2}{w^2} (4 + T^2) = 0,$$

(15)

$$T'' + \left(\frac{v'}{v} + \frac{w'}{w} + \phi'\right) T' + \frac{2}{w^2} T = 0,$$

(16)

and the first integral,

$$T'^2 = \phi'^2 + 2 \frac{v'}{v} \phi' - \frac{2}{w^2} (4 + T^2).$$

(17)

These equations contain the essential information about the metric-dilaton-tachyon system.

If the tachyon background vanishes, $T = 0$, then there are only two relevant solutions to these equations. Their expression is simplified by working with the convenient local-gauge choice $w = 1$. The first solution is the linear-dilaton vacuum, in which the spacetime is flat and the dilaton is given by

$$\phi = \phi_0 + \sqrt{8} r.$$

(18)

The second is the black-hole solution for which

$$v(r) = \tanh \sqrt{2} r$$

(19)

and

$$\phi(r) = \phi_0 + 2 \ln \cosh \sqrt{2} r.$$

(20)

As $r \to \infty$, the black-hole spacetime tends to the linear-dilaton vacuum. The mass of the black hole is given by $M = \sqrt{8} e^{\phi_0}$.

(21)
The precise definition of $M$ is discussed below. The black hole has an event horizon at $r = 0$, where $v(r) = 0$. Assuming that $v \to 1$ at spatial infinity, the Hawking temperature $T_H$ of the horizon is given generally by

$$T_H = \frac{1}{2\pi} v'|_{\text{horizon}} \quad (22)$$

and so, evaluating it in this case, we find

$$T_H = \frac{1}{\sqrt{2} \pi} = T_c \quad . \quad (23)$$

An immediate question about tachyonic hair is whether one can prove a no-hair theorem analogous to those for general relativity. A no-hair theorem for scalar fields in two dimensions, valid under certain conditions, has been presented in Ref. [19]. However, this result does not apply here because the tachyon has derivative couplings to the dilaton.

Given the absence of a definitive result excluding tachyon hair, the first step is to examine Eqs. (14) – (17) to see if we can find tachyon hair for the case of linearized perturbations about the black-hole metric. Thus, taking Eq. (16) and using Eqs. (19) and (20) for the background, we find

$$T'' + \sqrt{2} \left( \frac{1 + 2 \sinh^2 \sqrt{2} r}{\sinh \sqrt{2} r \cosh \sqrt{2} r} \right) T' + 2 \sqrt{2} T = 0 \quad . \quad (24)$$

This is essentially a hypergeometric equation [20]. Define the new radial coordinate

$$\xi = \cosh^2 \sqrt{2} r \quad . \quad (25)$$

This maps the event horizon $r = 0$ into $\xi = 1$ and $r$-spatial infinity into $\xi$-spatial infinity. The spacetime singularity, which is not covered by the coordinate system defined by Eq. (19), now appears at $\xi = 0$. Equation (24) becomes

$$8\xi(\xi - 1) \frac{d^2T}{d\xi^2} + 8(2\xi - 1) \frac{dT}{d\xi} + 2T = 0 \quad . \quad (26)$$

This hypergeometric equation is represented by

$$P \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ 0 & 0 & 1/2 \\ 0 & 0 & 1/2 \end{array} ; \xi \right\} \quad \text{(27)}$$
in the Riemann-Papperitz scheme. Near $\xi = 1$, the two linearly independent solutions are of the form

$$ T \sim \text{constant} \quad , \quad T \sim \ln |\xi - 1| \quad , $$

(28)

while as $\xi \to \infty$ they are of the form

$$ T \sim \frac{1}{\sqrt{\xi}} \quad , \quad T \sim \frac{\ln |\xi|}{\sqrt{\xi}} \quad . $$

(29)

The solution that is regular at the horizon is given by

$$ T = \frac{2T_0}{\pi} \sech \sqrt{2 \, r} \, K(\tanh \sqrt{2 \, r}) = T_0 \, P_{-1/2}(2\xi - 1) = T_0 \, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \xi\right) \quad , $$

(30)

where $K$ is the complete elliptic integral of the first kind, $T_0$ is the value of the tachyon field at the horizon, $P_{-1/2}$ is the Legendre function of order $-\frac{1}{2}$, sometimes called a conical function, and ${}_2F_1$ is the hypergeometric function. A plethora of details about the latter can be found in Ref. [21]. The first representation is convenient for examining the behavior of $T$ at spatial infinity, whereas the latter two are convenient for the behavior near the horizon. The other solution to Eq. (27) diverges at the horizon and therefore is not physically acceptable.

Whilst regularity of the fields at infinity and at the horizon are necessary conditions for a physically acceptable solution, we must also check that the energy in the field is finite. The energy-momentum tensor $\Theta_{\mu\nu}$ of the tachyon field is given by

$$ \Theta_{\mu\nu} = \partial_{\mu} T \partial_{\nu} T - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \partial_{\rho} T \partial_{\sigma} T \quad . $$

(31)

Therefore, the energy in the field in the domain of outer communication is

$$ E_T = \int_{\mathcal{H}}^{\infty} k^\mu k^\nu \Theta_{\mu\nu} \, dr \quad , $$

(32)

where $\mathcal{H}$ is the event horizon. Substituting for the specific form of the metric (19) and the tachyon perturbation (30) leads to the expression

$$ E_T = \frac{1}{2} \int_{0}^{\infty} \tanh^2 \sqrt{2 \, r} \, T'^2 \, dr \quad . $$

(33)

Amusingly, Eq. (33) can be written in terms of a generalized hypergeometric function. Since

$$ T = T_0 \, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; -\sinh^2 \sqrt{2 \, r}\right) \quad , $$

(34)
it follows that
\[ T' = \frac{-T_0}{\sqrt{2}} \sinh \sqrt{2} r \cosh \sqrt{2} r \ \mathcal{F}_1\left(\frac{3}{2}, \frac{3}{2}; 2; -\sinh^2 \sqrt{2} r \right). \] (35)

By using the quadratic transformations of Gauss and Kummer, we can rewrite this expression as
\[ T' = \frac{-T_0}{\sqrt{8}} \sinh \sqrt{8} r \ \mathcal{F}_1\left(\frac{3}{4}, \frac{3}{4}; 2; -\sinh^2 \sqrt{8} r \right). \] (36)

Clausen has developed an expression for the square of this function [22], which gives
\[ T'^2 = \frac{T_0^2}{8} \sinh^2 \sqrt{8} r \ \mathcal{F}_2\left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}; 2, 3; -\sinh^2 \sqrt{8} r \right). \] (37)

Thus,
\[ E_T = \frac{1}{4} T_0^2 \int_0^\infty dr \ \sinh^4 \sqrt{2} r \ \mathcal{F}_2\left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}; 2, 3; -\sinh^2 \sqrt{8} r \right). \] (38)

Although an analytical form for this integral might exist, we do not pursue this approach further here. Instead, we restrict ourselves to proving that this expression is finite. Both functions \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) have singularities only when their argument takes the values 0, 1, or \( \infty \). This follows because they obey second- and third-order differential equations, respectively, that have ordinary points everywhere except for regular singular points at these three locations. Expanding \( T(r) \) for small \( r \) gives
\[ T(r) = T_0 \left( 1 - \frac{1}{8} r^2 + \frac{1}{48} r^4 + \ldots \right). \] (39)

Thus, for small \( r \) the integrand in Eq. (38) is of order \( r^6 \), and so the contribution to the integral is finite. As \( r \) increases to infinity, the argument of the hypergeometric function decreases, and so the only remaining potential singularity in the integral appears as \( r \to \infty \). However, in this limit
\[ T(r) \to \frac{4T_0}{\pi} e^{-\sqrt{2} r} \left( \sqrt{2} r + \ln 2 + O\left(\frac{1}{r}\right) \right). \] (40)

The integrand in Eq. (38) therefore varies as \( e^{-\sqrt{8} r} \) in the limit as \( r \to \infty \), so finiteness is guaranteed. The finiteness of the perturbative contribution to the energy strongly suggests that there is enough energy in the tachyon field to further curve the spacetime, but insufficient energy to destroy asymptotic flatness.
Our analysis so far makes plausible the existence of black-hole spacetimes with tachyon hair. To prove that this indeed occurs, the complete nonlinear equations (14) – (17) must be solved, and the solutions must be shown to have the required finiteness properties. This is sometimes termed the ‘back-reaction’ problem. It has been examined by several authors in the context of various approximation schemes. Ref. [23] considers lowest-order perturbation theory and argues that the event horizon remains regular. In contrast, Ref. [24] sets $V(T) = 0$ and claims that the event horizon becomes singular. Ref. [25] uses perturbation theory to argue that the energy of the spacetime is infinite, a conclusion also reached in Ref. [26]. Finally, Ref. [27] argues that no black-hole spacetime is possible for nonzero tachyon background. The situation is evidently rather confused.

One practical approach to settling this issue is the numerical integration of Eqs. (14) – (17). It suffices to integrate the three equations (14), (15), and (17). For the numerical analysis, it is convenient to work in the the gauge $w = 1$ and to convert these three equations into four first-order equations. We have performed the numerical integration using a fourth-order generalized Runge-Kutta method implemented on an HP Apollo Series 700 workstation. The program is initialized as follows. Values of the tachyon charge $T_0$ and the dilaton strength $\phi_0$ at the horizon $r = 0$ are selected. As described below, perturbation theory in $r$ near the horizon, applied for the full nonlinear equations, is then used to fix the initial values of $v$, $\phi$, $\phi'$ and $T$ at some specified small $r$. The integration is continued until the asymptotic regimes of the fields are attained.

Figures 1-4 show the results for a tachyon charge $T_0 = 0.1$ with $\phi_0 = 0$. The associated metric is displayed in Figure 1. The invariance of Eqs. (14)–(17) under scaling of the metric $v$ has been used to set the asymptotic value to the canonical choice of one. As can be seen, the metric is a smooth function of $r$ that rises to close to its asymptotic value within about two radial units. Figure 2 displays the dilaton, while Figure 3 shows its slope. The dilaton begins at zero and within about two radial units converts to a linearly rising trajectory, with asymptotic slope $\sqrt{8} \approx 2.8$. In contrast, as can be seen in Figure 4, the tachyon begins at $T_0 = 0.1$ and falls smoothly, approaching zero after about six radial units. The results indicate that
solutions to the nonlinear equations do indeed exist and that $v$, $\phi$ and $T$ are smooth functions everywhere outside the event horizon.

It is useful to examine the solutions analytically near the horizon and asymptotically at spatial infinity to gain insight into the physics. There are two natural methods for extracting results for the event horizon: using perturbation theory either in $r$ or in $T_0$. Both yield insights, so we present them in turn.

First, consider the expansion in positive powers of $r$. The symmetries of Eqs. (14) – (17) in the gauge $w = 1$ imply that the tachyon and dilaton can be expanded in even powers of $r$, while the metric can be expanded in odd powers. Substitution into Eqs. (14) – (17) and collection of coefficients generates three series for the metric, the dilaton, and the tachyon. The coefficients can be written as functions of the square of the tachyon charge, $T_0^2$. We use the lower-order terms in these three series to initialize our numerical analysis.

For the tachyon, we find

$$T = T_0 \left( 1 - \frac{1}{2} r^2 + \frac{11 + 2T_0^2}{48} r^4 - \ldots \right) .$$

(41)

This provides the nonlinear correction at order $T_0^2$ to Eq. (39), which was obtained in the linearized theory. For the dilaton, we get the equation

$$\phi = \phi_0 + \frac{1}{2} (4 + T_0^2) r^2 - \ldots .$$

(42)

The constant $\phi_0$ can be set to zero without loss of generality, as it merely represents a shift in the definition of the origin of the dilaton field. Finally, for the metric we obtain

$$v = v_1 \left( r - \frac{4 + T_0^2}{6} r^3 + \ldots \right) .$$

(43)

Here, $v_1$ corresponds to the arbitrary scale choice that can be made for the metric. We have adopted the convention that $v_1$ is fixed so that $v \to 1$ as $r \to \infty$. Using Eq. (22), it then follows that

$$v_1 = 2\pi T_H .$$

(44)

Further results for the event horizon can be extracted via an expansion in powers of $T_0$ about the analytical black-hole solution (19) and (20). Taking Eq. (30) for the
tachyon field, we write
\[ v = \tanh \sqrt{2} r + T_0^2 g(r) + O(T_0^4) \]  \hspace{1cm} (45)

The dilaton can be eliminated from Eq. (17) using Eq. (14). In the gauge \( w = 1 \), Eq. (17) then reduces to a second-order differential equation for the metric perturbation \( g \), given by
\[ -2T^2 - T'^2 = 2 \left( \coth \sqrt{2} r + 2 \sinh \sqrt{2} r \cosh \sqrt{2} r \right) g'' + 8\sqrt{2} \sinh^2 \sqrt{2} r \left( g' + \frac{8}{\sinh \sqrt{2} r \cosh \sqrt{2} r} g \right). \]  \hspace{1cm} (46)

The relevant solution of this equation satisfies the boundary conditions \( g \rightarrow v_1 r \) as \( r \rightarrow 0 \) and \( g \rightarrow 0 \) as \( r \rightarrow \infty \). By treating the left-hand side of this equation as a source and obtaining the Green function for the second-order differential operator on the right-hand side, an integral expression for \( g \) satisfying the boundary conditions can be found.

The explicit form is not needed here. Instead, we seek the value of \( g'(0) \), since this information gives the Hawking temperature as a function of \( T_0^2 \) via Eqs. (44) and (47),
\[ T_H(T_0) = \frac{1}{2\pi} \left( \sqrt{2} + g'(0)T_0^2 + O(T_0^4) \right). \]  \hspace{1cm} (47)

The Green-function method yields the integral expression
\[ g'(0) = \frac{1}{4} \int_0^\infty dr \frac{\sinh \sqrt{8} r}{\cosh^2 \sqrt{8} r} \left( 2T^2 + T'^2 \right). \]  \hspace{1cm} (48)

The integrand is positive. This implies that the Hawking temperature is increased above \( T_c \) when a tachyon charge is added to the black hole. Qualitatively, this effect is different from the situation for the Reissner-Nordstrom solutions in four dimensions, where the addition of electric charge decreases the Hawking temperature instead.

Although the integral might be performed analytically using the explicit expressions for the tachyon field given in Eq. (30), we do not pursue this here. In practice our numerical methods suffice to determine the Hawking temperature for any given situation. For example, we find numerically that \( T_H \simeq 0.2253 \) for the solution displayed in Figures 1 to 4. In contrast, the uncharged black hole has \( T_H \simeq 0.2251 \).
In fact, we have determined that the Hawking temperature of the black hole with tachyon hair is given approximately as

$$T_H = \frac{1}{\sqrt{2} \pi} \left( 1 + \frac{1}{10} T_0^2 + O(T_0^4) \right). \quad (49)$$

In principle, an analytical expression for $T_H$ might be found that is exact to all orders in $T_0$. However, in practice this is difficult because Eqs. (22) and (44) only hold if $v \to 1$ at spatial infinity, so knowledge of the asymptotic behavior of $v, \phi, \text{and } T$ associated with a particular behavior near the horizon is required.

Next, we turn to an investigation of the analytical behavior of our solutions at infinity. The gauge choice $w = 1$ is less convenient for this issue, so instead we define a new radial coordinate $\zeta$ such that

$$\phi = \tilde{\phi}_0 + \ln \zeta, \quad (50)$$

where $\tilde{\phi}_0$ is a constant. In the absence of a tachyon background, $T = 0, \phi_0 = \tilde{\phi}_0$, and the new coordinate $\zeta$ reduces to the coordinate $\xi$ previously introduced. In this gauge, Eq. (13) becomes

$$\dot{T}^2 = \frac{1}{\zeta} \left( \frac{\ddot{v}}{\dot{v}} + \frac{\dot{v}}{v} \right) + \frac{2}{\zeta^2}, \quad (51)$$

where the dot signifies a $\zeta$ derivative.

Given a functional form for $T$, this equation can be integrated directly to give $v$. As $r \to \infty$, the asymptotic form of $T$ is given by Eq. (10). Note that this equation is only valid for small $T_0$ because there are corrections to the tachyon equation of order $T_0^2$ coming from the metric and dilaton. Since $\zeta = \cosh^2 \sqrt{2} r$, the tachyon can be written as

$$T = \frac{T_0}{\pi \sqrt{\zeta}} \left( \ln \zeta + 4 \ln 2 \right) + O \left( \zeta^{-\frac{3}{2}} \right). \quad (52)$$

Integrating Eq. (51) yields

$$\zeta^2 v \dot{v} = C \exp \left( \int \zeta \dot{T}^2 \, d\zeta \right), \quad (53)$$

where $C$ is a integration constant. Substituting for $\dot{T}$ using Eq. (52) gives

$$v \dot{v} = \frac{C}{\zeta^2} \exp \left[ \frac{1}{4 \pi^2 \zeta} T_0^2 \left( -(\ln \zeta)^2 + (2 - 8 \ln 2) \ln \zeta - 2 + 8 \ln 2 - 16(\ln 2)^2 \right) + O(\zeta^{-2}) \right]. \quad (54)$$
As $\zeta \to \infty$ we can approximate this expression by taking only the first terms in the expansion of the exponential. Integrating the resulting expression gives

$$\frac{1}{2}v^2 = D - \frac{C}{\zeta} + \frac{C}{8\pi^2 \zeta^2} T_0^2 \left( (\ln \zeta)^2 - \ln \zeta (1 - 8 \ln 2) + \frac{3}{2} - 4 \ln 2 + 16(\ln 2)^2 \right) + O(\zeta^{-3}),$$

(55)

where $D$ is another integration constant that sets the scale of the time coordinate. We choose the conventional time coordinate with $v \to 1$ as $\zeta \to \infty$, which gives $D = \frac{1}{2}$. Thus,

$$v = 1 - \frac{C}{\zeta} + \frac{C}{8\pi^2 \zeta^2} T_0^2 \left( (\ln \zeta)^2 - \ln \zeta (1 - 8 \ln 2) + \frac{3}{2} - 4 \ln 2 + 16(\ln 2)^2 - \frac{2\pi^2 C}{T_0^2} \right) + O(\zeta^{-3}).$$

(56)

This analytical expression correctly reproduces the behavior of our numerical solutions.

With these results, the metric coefficient $w$ in the new gauge can be found from Eq. (14). Asymptotic flatness together with the conventional normalization requires that $w \to \sqrt{8} \zeta$ as $\zeta \to \infty$. This fixes the value of $k$ in terms of $C$ and $\tilde{\phi}_0$:

$$k = \sqrt{8} \ C \ e^{\tilde{\phi}_0}.$$

(57)

Explicit evaluation of $w$ leads to

$$w = \sqrt{8} \ \zeta - \frac{1}{\sqrt{2} \ \pi^2} T_0^2 \left( -(\ln \zeta)^2 + \ln \zeta (2 - 8 \ln 2) - 2 + 8 \ln 2 - 16(\ln 2)^2 + \frac{2\pi^2 C}{T_0^2} \right) + O(\zeta^{-1}).$$

(58)

The remaining issue is the physical meaning of the constant of integration $C$. Rescaling the coordinate $\zeta$ allows the elimination of $C$ from the system of equations. Therefore, $C = 1$ is a legitimate choice. However, in the process the gauge condition (50) must be preserved. The effect of this rescaling is therefore to renormalize the value of $\tilde{\phi}_0$. Henceforth, we choose $C = 1$.

Next, we calculate the ADM mass of this spacetime. The canonical treatment proceeds as follows. Starting from the action (4), one first needs to obtain
the hamiltonian. This requires a separation of the coordinates into space and time, denoted by \( x \) and \( t \) here. In the remainder of this paper, we redefine a prime to indicate an \( x \) derivative and a dot to indicate a \( t \) derivative.

The metric decomposes in the usual manner as

\[
g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta^2/\gamma & \beta \\ \beta & \gamma \end{pmatrix} .
\] (59)

The variable \( \gamma \) can be regarded as a one-dimensional spatial metric. The next step is substitution of this metric form into the action and the determination of the canonical momenta conjugate to the canonical coordinates \( \alpha, \beta, \gamma, \phi, \) and \( T \). These are

\[
\pi_\alpha = 0 , \quad \pi_\beta = 0 , \quad \pi_\gamma = \frac{1}{\alpha \sqrt{\gamma}} \dot{\phi} e^\phi
\] (60)

and

\[
\pi_\phi = \frac{2 \sqrt{\gamma}}{\alpha} \dot{T} e^\phi , \quad \pi_T = -\frac{2 \sqrt{\gamma}}{\alpha} \dot{\phi} e^\phi - \frac{1}{\alpha \sqrt{\gamma}} \dot{\gamma} e^\phi .
\] (61)

Both \( \pi_\alpha \) and \( \pi_\beta \) vanish by virtue of the constraints of the system. For simplicity, we have worked in the gauge \( \beta = 0 \).

For time-independent field configurations, the canonical hamiltonian then becomes

\[
H = \int dx \ e^\phi \left( -\alpha \sqrt{\gamma} \dot{c} - \frac{2 \alpha}{\sqrt{\gamma}} \dot{\phi} - \frac{\alpha}{\sqrt{\gamma}} \phi'^2 + \frac{\alpha}{\sqrt{\gamma}} T'^2 - \alpha \sqrt{\gamma} V(T) \right) .
\] (62)

However, in a reparametrization-invariant system, \( H \) must vanish modulo a boundary term. This boundary term is the ADM energy of the system. To convert \( H \) into a boundary term, it suffices to substitute the beta-function equation for the dilaton, Eq. (13), into Eq. (62). This gives

\[
H = \int dx \ \left[ -\frac{2}{\sqrt{\gamma}} e^\phi (\alpha' + \alpha \phi') \right]' .
\] (63)

Thus, the ADM energy is

\[
E = \frac{2}{\sqrt{\gamma}} e^\phi (\alpha' + \alpha \phi') \bigg|_{x \to \infty} .
\] (64)

In two dimensions, \( E \) is infinite even for flat space. The physically important quantity for a curved spacetime \( \mathcal{M} \) with given metric and dilaton on the boundary
at infinity is the difference between the ADM energy $E_M$ and the corresponding quantity $E_F$ for flat spacetime $F$ with the same values of the metric and dilaton on the boundary.

For our black hole with tachyon charge, the first step in obtaining the ADM mass is to construct a flat metric such that it coincides at $\zeta = \hat{\zeta}$, say, with that of the tachyonic black hole. The canonically normalized linear-dilaton vacuum, which in the present variables is given by Eq. (50) and

$$v = \alpha = 1, \quad w = \frac{1}{\sqrt{\gamma}} = \sqrt{8} \zeta,$$  

(65)

does not satisfy this requirement. Instead, suppose that at $\zeta = \hat{\zeta}$ the variables $v, w, \phi$ take the values $\hat{v}, \hat{w}, \hat{\phi}$, respectively. Then, flat space has the form

$$\phi = \hat{\phi}_0 + \ln \frac{\zeta}{\hat{\zeta}}, \quad v = \alpha = \hat{v}, \quad w = \frac{1}{\sqrt{\gamma}} = \sqrt{8} \zeta.$$  

(66)

This can be compared with the metric of the tachyonic black hole, Eqs. (56) and (58), at the point $\hat{\zeta}$. The fiducial flat-space metric has the same values of $v, w, \phi$ at that point.

Subtracting the ADM energy of the flat spacetime $F$ from that of the curved spacetime $M$ and cancelling terms where possible gives

$$\Delta E = E_F - E_M = 2e^{\hat{\phi}_0} \hat{w} \frac{\partial v}{\partial \zeta}.$$  

(67)

Explicitly,

$$\Delta E = 2e^{\hat{\phi}_0} \hat{\zeta} \left( \sqrt{8} \zeta + O \left( (\ln \zeta)^2 \right) \right) \left( \frac{1}{\zeta^2} + O \left( \frac{1}{\zeta^3} \right) \right).$$  

(68)

For large $\zeta$, this expression has a well-defined and finite limit, which we interpret as the physical mass $M$ of the tachyonic black hole. We find

$$M = \sqrt{32} e^{\hat{\phi}_0}.$$  

(69)

Note that this general expression agrees with the results for the uncharged black hole $[13, 14, 15]$, up to an overall numerical factor that arises because the constant $\tilde{\phi}_0$ has not been canonically fixed.
We thus conclude that in two spacetime dimensions it is possible to find black holes with arbitrary finite physical mass and tachyon charge, with Hawking temperature above that of the corresponding uncharged black holes.

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**Figure Captions**

Figure 1. The metric $v$ as a function of $r$ in dimensionless units.
Figure 2. The dilaton $\phi$ as a function of $r$ in dimensionless units.
Figure 3. The dilaton slope $\phi'$ as a function of $r$ in dimensionless units.
Figure 4. The tachyon $T$ as a function of $r$ in dimensionless units.
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