Gravity with Perturbative Constraints: Dark Energy Without New Degrees of Freedom

Alan Cooney  
Department of Physics, University of Arizona, Tucson, AZ 85721, USA

Simon DeDeo  
Kavli Institute for Cosmological Physics, University of Chicago, Chicago, IL 60637, USA

&  
Institute for the Physics and Mathematics of the Universe, University of Tokyo, Kashiwano-ha 5-1-5, Kashiwa-shi, Chiba 277-8582, Japan

Dimitrios Psaltis  
Departments of Astronomy and Physics, University of Arizona, Tucson, AZ 85721, USA

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Major observational efforts in the coming decade are designed to probe the equation of state of dark energy. Measuring a deviation of the equation-of-state parameter $w$ from -1 would indicate a dark energy that cannot be represented solely by a cosmological constant. While it is commonly assumed that any implied modification to the $\Lambda$CDM model amounts to the addition of new dynamical fields, we propose here a framework for investigating whether or not such new fields are required when cosmological observations are combined with a set of minimal assumptions about the nature of gravitational physics. In our approach, we treat the additional degrees of freedom as perturbatively constrained and calculate a number of observable quantities, such as the Hubble expansion rate and the cosmic acceleration, for a homogeneous Universe. We show that current observations place our Universe within the perturbative validity of our framework and allow for the presence of non-dynamical gravitational degrees of freedom at cosmological scales.

I. INTRODUCTION

Cosmological observations suggest the existence of dark matter. There is, also, by now ample evidence that the Universe at late times is accelerating. Do either of these facts require the introduction of new, dynamical degrees of freedom in the gravitational and matter sectors?

In the case of dark matter, the analysis of “bullet cluster” objects such as 1E 0657-56 [1] and MACSJ0025.4-1222 [2] comes as conclusive evidence for the existence of new degrees of freedom independent of the visible baryonic matter. While the exact nature of these new degrees of freedom are still under debate, the simplest choice – a single, non-relativistic, dark matter fluid – is the center of the cosmological standard model.

Even if we choose to account for the galactic rotation curves with a modification to gravity, it is difficult to do it without the addition of extra degrees of freedom. A consequence of the Lovelock-Grigore theorem [3, 4] is that most local, classical, and covariant modifications to the gravitational action lead to new fields [3]. Alternatives to dark matter that have these properties – such as TeVeS [5] and STVG [6] – themselves have new fields as a consequence of the Lovelock-Grigore theorem.

Today, cold dark matter with a cosmological constant ($\Lambda$CDM) satisfies all current observational constraints [8]. However, a number of theoretical issues [9] have prompted observational campaigns to search for deviations from the predicted $\Lambda$CDM behavior [10]. These observations will measure the equation-of-state parameter $w$ of the dark energy and search for deviations from the value $w = -1$ that corresponds to a cosmological constant. Should their results force us to abandon $\Lambda$CDM, we will face a problem for dark energy similar to that we faced for dark matter: do such observations require new dynamical degrees of freedom? So far, the two major approaches to dark energy modeling that predict values of $w \neq -1$, quintessence [11] and $f(R)$ gravity [12], incorporate new dynamical degrees of freedom, which can be cast, in both cases, in the form of a single scalar field.

A number of attempts have been made for modifying General Relativity in a way that generates cosmologies with $w \neq -1$ but without introducing new dynamical fields. Modified source gravity [13], based on insights from the Palatini formulation of the Einstein equations [14] forces the (Einstein-frame) scalar field to be non-dynamical by erasing the kinetic term. Cuscuton cosmology [15, 16] also modifies the kinetic term; it appears to have new dynamical degrees of freedom but these are frozen out at the perturbative level. Non-local [17, 18] or holographic [19] theories may also avoid the introduction of new degrees of freedom. Albeit ambitious, these attempts can face serious problems [20].

In the opposite extreme, because of the Lovelock-Grigore theorem, the theoretical pressures on purely phenomenological models that deviate from $\Lambda$CDM but that do not introduce new dynamical degrees of freedom are...
strong. The fact that observations favoring dark energy are made on relativistic, Hubble scales makes it even harder to avoid the consequence of the Lovelock-Grigore theorem, while remaining internally consistent. Requiring general covariance from the start, for example, prevents one from even phrasing many minimal theories the way MOND did on galactic scales \cite{21}. In general, while many theories similar or equivalent to Brans-Dicke \cite{33} may, in the limit of small post-Newtonian corrections, appear only to modify the General Relativistic equations of motion, the new scalars such theories introduce appear as truly independent degrees of freedom in the relativistic regime – for example, on scales approaching the horizon size.

Approaches, however, that tie together two particular sets of observations – homogeneous expansion and linear structure formation – in a phenomenological but self-consistent fashion have also been recently explored. For example, the parametrized post-Friedmann framework \cite{22} makes minimal assumptions about the underlying physics, requiring only causality, metric structure, and energy-momentum conservation. The Cardassian model \cite{28} introduces a power-law term in the Friedmann equation, which leads to accelerated expansion even in a matter dominated universe. The approach of Ref. \cite{3} is similar in spirit, and explicitly shows how a subset of modifications will introduce new degrees of freedom. Although these approaches are general and simple to use in connection to observational data, they can not, by their very nature, generate predictions for other astrophysical settings – the solar system, for example, or nearby (non-cosmological) event horizons. As a result, they can not guarantee that dynamical degrees of freedom are not required for consistency in these other situations.

We follow here a different approach intermediate to the two outlined above, which we call Gravity with Perturbative Constraints (see also Ref. \cite{24} for an initial discussion of cosmology with perturbative constraints). In contrast to the fundamental approach, we do not propose a basic theory but rather remain largely neutral on the details of how a particular set of observations are generated. In contrast to the phenomenological approach, our models have a wide range of applicability and one can, for example, ask whether the cosmological predictions that do not add extra degrees of freedom are consistent with observations in the solar system or of compact objects.

Our aim is to cover as large a class as possible of modifications to gravity that do not require new degrees of freedom. Within our framework, it is possible in principle to incorporate all such modifications that are covariant, metric (in the sense of obeying the Einstein equivalence principle), and conserve energy-momentum. The price we will pay for such generality is that we must remain perturbatively close to General Relativity. While we discuss in detail the exact conditions for perturbative validity below, we caution here that this validity does not require weak fields (as defined with reference to either the Minkowski or homogenous FRW metrics as the background).

Our method of generating such models is known in the literature as that of perturbative constraints \cite{25}; it is an alternative way of associating, in a physically unambiguous fashion, a general action of the gravitational field with a field equation that is of second order. Developed to deal with approximate local actions derived from fundamentally non-local theories, it has a wide range of applicability and does not rely on the existence of an underlying non-locality.

The method of perturbative constraints we use here allows us to retain many of the important features of the action of the gravitational field. In particular, it allows us to retain the translational symmetries that, under Noether's principle, lead to energy-momentum conservation, as well as enforcing diffeomorphism invariance – the invariance of physical law under changes of coordinates. It does so without introducing new degrees of freedom, while at the same time producing behavior that differs from the General Relativistic case, and allowing for a consistent set of predictions across a wide variety of spacetimes.

II. GRAVITY WITH PERTURBATIVE CONSTRAINTS

In this paper, we specialize to the case of a flat, homogenous FRW metric,

$$ds^2 = -dt^2 + a^2(t) [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)] . \tag{1}$$

It is important to emphasize here, however, that because we will be modifying gravity in ways that affect cosmological expansion, standard relationships between, e.g., the energy density of various components and the rate of expansion that hold in a flat Universe within General Relativity will not necessarily hold here.

We start with an action for the gravitational field that is a general function $f(R)$ of the Ricci scalar curvature. The field equation for this general theory is

$$f' R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f' + g_{\mu\nu} \Box f' - \nabla_\mu \nabla_\nu f' = 8\pi G T_{\mu\nu} , \tag{2}$$

where $f' = \frac{\partial f}{\partial R}$. For a flat FRW spacetime, the Ricci scalar curvature is

$$R = 6 \left[ \frac{\dot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 \right] . \tag{3}$$

The above field equation obeys the Bianchi identity and hence $\nabla^\mu T_{\mu\nu} = 0$, which results in the usual conservation law

$$\dot{\rho}_i + 3 \frac{\dot{a}}{a} (\rho_i + P_i) = 0 . \tag{4}$$

Here $\rho_i$ and $P_i$ are the energy density and pressure, respectively, of the $i^{th}$ constituent of the Universe. In this
paper, we will consider Lagrangian densities of the form

$$f(R) = R - 2\Lambda + \frac{\mu^{2(n+1)}}{R^n}, \quad (5)$$

where \(n\) and \(\mu\) are free parameters and the last term in the action is considered only at the perturbative level. The case \(\Lambda = 0, n = 1\) was first discussed in Ref. [24].

Henceforth we shall adopt geometerized units where \(8\pi G = 1\). Following standard procedure, we evaluate the \((\mu, \nu) = (0, 0)\) component of the field equation as well as its trace and obtain

$$-3f'\dot{a} + \frac{1}{2}f + 3f''\ddot{R} = \rho \quad (6)$$

$$Rf' - 2f - 3\left[f''\ddot{R} + \dddot{a} f' + 3\dot{a}\ddot{R} + Rf''\right] = (-\rho + 3P). \quad (7)$$

In this last equation, \(\rho\) and \(P\) denote the total density and pressure of all constituents of the Universe.

When the function \(f(R)\) is non-linear in the Ricci scalar curvature, Eq. (5) is not a first-order ordinary differential equation in time, as is the standard Friedmann equation. Indeed, as discussed earlier, a non-linear Lagrangian introduces additional degrees of freedom to the gravitational field. The key feature of gravity with perturbative action shows that we can expand the final solution for the scale factor to be unity in the present epoch.

In practice, we combine Eq. (6) with the conservation equation in time, as is the standard Friedmann equation. Consistency requires that terms of order \(\mu^4\) and \(\mu^8\) and higher are negligible and can, therefore, be dropped.

When discussing our results, we will occasionally express them in terms of the contributions of matter and of the cosmological constant to the energy density of the Universe, which are given by

$$\Omega_m = \frac{\rho_m}{3H^2} \quad (12)$$

and

$$\Omega\Lambda = \frac{\Lambda}{3H^2}, \quad (13)$$

respectively, where \(H \equiv (\dot{a}/a)\). Finally, in order to compare our results to the usual parametrization of the dark energy equation of state, we will also write the Friedmann equation in the XCDM form [27] as

$$3\left(\frac{\dot{a}}{a}\right)^2 = \rho_m + \rho_X, \quad (14)$$

and the equation for acceleration as

$$6\ddot{a} = -\rho_m - (1 + 3w_X)\rho_X, \quad (15)$$

where \(\rho_X\) is the energy density of an equivalent “dark energy” component with an equation of state \(P_X = w_X\rho_X\). Note that this term includes contributions from both the cosmological constant and by the perturbative term that modifies the Einstein equations.

A. Case 1: \(\Lambda = 0, n = 1\)

This situation corresponds to a Universe with zero cosmological constant and \(f(R) = R + \mu^4/R\). Since we are interested in deviations in the matter dominated regime, we will only consider matter with \(P = 0\). In this case, the \((\mu, \nu) = (0, 0)\) component of the field equation and its trace become

$$3\left(\frac{\dot{a}}{a}\right)^2 + 3\mu^4 \left[\frac{R}{a} + \frac{\dot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)\frac{\ddot{R}}{R}\right] = \rho_m \quad (16)$$

and

$$R + 3\mu^4 \left[\frac{R}{a} + \frac{\dot{a}}{a} + 6\left(\frac{\dot{a}}{a}\right)^2 + 6\frac{\dot{a}}{a}\frac{\ddot{R}}{R}\right] = \rho_m, \quad (17)$$

respectively.

When \(\mu = 0\), the equation for the scale factor describes the familiar matter-dominated Einstein cosmology, for which

$$a(t) = \left(\frac{\sqrt{3\rho_m}}{\mu}\right)^{\frac{2}{n}}. \quad (18)$$
As a result

\[
\left( H^{(0)} \right)^2 \equiv \left( \frac{\dot{a}^{(0)}}{a^{(0)}} \right)^2 = \frac{\rho_{m,0}}{3 \left( a^{(0)} \right)^3} \tag{19}
\]

and

\[
R^{(0)} = 6 \left[ \frac{\dot{a}^{(0)}}{a^{(0)}} + (H^{(0)})^2 \right] = \frac{\rho_{m,0}}{a^{(0)}}. \tag{20}
\]

Now to order \( \mu^4 \) Eq. \( \ref{eq:16} \) gives

\[
2H^{(0)}\ddot{a}^{(4)} + \frac{\mu^4}{(R^{(0)})^2} \left( \frac{R^{(0)}}{6} + \frac{\dot{a}^{(0)}}{a^{(0)}} + 2 \frac{\dot{R}^{(0)}}{R^{(0)}} H^{(0)} \right) = -a^{(4)} \frac{\rho_{m,0}}{(a^{(0)})^3}. \tag{21}
\]

Solving for \( a^{(4)}(t) \) using Eq. \( \ref{eq:19} \) and \( \ref{eq:20} \) the contribution to order \( \mu^8 \) is

\[
a^{(4)}(t) = \frac{9 \rho_{m,0}^2 \mu^4}{40}. \tag{22}
\]

We then perform the same expansion up to order \( \mu^8 \). The contribution is

\[
a^{(8)}(t) = -\frac{18819 \rho_{m,0}^4 \mu^8}{1600}. \tag{23}
\]

As a boundary condition, we require that at the present epoch, which we denote by \( t_0 \), the scale factor is unity. Thus, as we evaluate \( a(t) \) to higher order in \( \mu \), we get corrections to the value of \( t_0 \) from General Relativity, i.e.,

\[
t_0 = \frac{2}{\sqrt{3 \rho_{m,0}}} \left( 1 - \frac{3 \mu^4}{\rho_{m,0}^2} + \frac{115}{2} \frac{\mu^8}{\rho_{m,0}^4} + \ldots \right) \tag{24}
\]

As discussed earlier, it is practically impossible to check the convergence of our method without knowing the form of the Lagrangian action to higher order in the parameter \( \mu \). We can provide, however, a necessary condition for convergence in calculating an observable quantity, by requiring that the correction to the value of this observable calculated to order \( \mu^8 \) is not larger than the correction calculated to order \( \mu^4 \).

As an example, we calculate the value of the Hubble constant at the present epoch, \( H_0 \equiv H(t = t_0) \), as

\[
H_0 = \sqrt{\frac{\rho_{m,0}}{3}} \left( 1 + \frac{3 \mu^4}{\rho_{m,0}^2} - \frac{1017}{2} \frac{\mu^8}{\rho_{m,0}^4} + \ldots \right). \tag{25}
\]

Similarly we evaluate the deceleration parameter \( q_0 \equiv -\ddot{a}/a^2 \) at the present epoch as

\[
q_0 = \frac{1}{2} \left( 1 - 36 \frac{\mu^4}{\rho_{m,0}^2} + 12312 \frac{\mu^8}{\rho_{m,0}^4} + \ldots \right). \tag{26}
\]

Requiring the correction to order \( \mu^8 \) in the expression for the Hubble constant to be smaller than the correction to order \( \mu^4 \) results in the bound \( \mu^4/\rho_{m,0}^2 < 6/1017 \), whereas the same requirement for the deceleration parameter leads to \( \mu^4/\rho_{m,0}^2 < 36/12312 \). Both of these constraints at the present epoch are very similar and require \( \mu^4 \) to have very small values, which cannot lead to an accelerated expansion within the limits of perturbative validity.

An additional problem of using a cosmology with \( n = 1 \) and \( \Lambda = 0 \) to account for the observations can be seen by writing the Friedmann equation to order \( \mu^4 \) (as in \cite{24}), i.e.,

\[
3 \left( \frac{\dot{a}}{a} \right)^2 = \rho_m + \frac{6 \mu^4}{\rho_m}. \tag{27}
\]

Transforming this equation in the XCDM form requires that

\[
\rho_X = \frac{6 \mu^4}{\rho_m}. \tag{28}
\]

Clearly the matter-dominated perturbative gravity theory to order \( \mu^4 \) behaves as a dark energy fluid with \( w_X = \frac{\mu^4}{\rho_m} \), which is inconsistent with the combined WMAP and BAO data \cite{3}.

**B. Case 2: \( \Lambda \neq 0, n = 1 \)**

We have found that, while the case with \( n = 1 \) and \( \Lambda = 0 \) may generate accelerating solutions, these solutions lie outside the range where the perturbative expansion to the action is valid. Intuitively, this comes from the fact that by the time acceleration has begun, the matter density is low enough that the zeroth-order relation, \( R_0 = \rho_{m,0} \), implies that the contribution of the additional term, \( \mu^4/R \), is too large to be treated as a small perturbation to the standard gravitational action. This is not the case, however, in a Universe with a non-zero cosmological constant, as we will discuss below.

In the case of a Universe with a non-zero cosmological constant, the \( (\mu, \nu) = (0, 0) \) component of the field equation and the trace are

\[
3 \left( \frac{\dot{a}}{a} \right)^2 + \frac{3 \mu^4}{R^2} \left[ \frac{R}{6} + \frac{\dot{a}}{a} + 2 \left( \frac{\dot{a}}{a} \right) \frac{\dot{R}}{R} \right] = \rho_m + \Lambda, \tag{29}
\]

and

\[
R + \frac{3 \mu^4}{R^2} \left[ \frac{R + 2 \frac{\dot{R}}{R}}{6} \left( \frac{\dot{R}}{R} \right)^2 + 6 \left( \frac{\dot{a}}{a} \right) \frac{\dot{R}}{R} \right] = \rho_m + 4 \Lambda, \tag{30}
\]

respectively.

We will again seek a perturbative solution to this equation of the form \( \ref{eq:11} \).
tion is the same as for \( \Lambda \text{CDM} \), i.e.,

\[
a^{(0)}(t) = \left[ \frac{\rho_{m,0}}{\Lambda} \sinh \left( \frac{\sqrt{3\Lambda}}{2} t \right) \right]^{\frac{2}{3}}. \tag{31}
\]

The time at the present epoch, i.e., the one for which the \( \Lambda \text{CDM} \) solution gives a scale factor of unity, is

\[
t_0 = \frac{2}{\sqrt{3\Lambda}} \sinh^{-1} \left( \sqrt{\frac{\Lambda}{\rho_{m,0}}} \right). \tag{32}
\]

At orders \( \mu^4 \) and \( \mu^8 \), the solutions of the equation for the scale factor, \( a^{(4)}(t) \) and \( a^{(8)}(t) \), respectively, as well as the time \( t_0 \) that corresponds to the present epoch can be found in closed form but are too long to be displayed here (we present the solution to order \( \mu^4 \) in Appendix A). Instead, we write the Friedmann equation to order \( \mu^4 \)

\[
3 \left( \frac{\dot{a}}{a} \right)^2 = \rho_m + \Lambda - \frac{\mu^4}{(\rho_m + 4\Lambda)^2} \left[ 3\Lambda - 6(\rho_m + \Lambda) \left( \frac{\rho_m}{\rho_m + 4\Lambda} \right) \right] \tag{33}
\]

and the acceleration equation to the same order

\[
6 \ddot{a} = -\rho_m + 2\Lambda - \frac{\mu^4}{(\rho_m + 4\Lambda)^2} \left[ 6\Lambda + 24(\rho_m + \Lambda) \left( \frac{\rho_m}{\rho_m + 4\Lambda} \right) \right.
\]

\[
\left. -54(\rho_m + \Lambda) \left( \frac{\rho_m}{\rho_m + 4\Lambda} \right)^2 \right]. \tag{34}
\]

These two equations allow us to calculate to order \( \mu^4 \) two observable quantities at the present epoch, as a function of the parameters \( \mu^4 \) and \( \Lambda \): the Hubble constant, or equivalently the parameter \( \Omega_m \), and the deceleration parameter \( q_0 \). We show the result of this calculation in Fig. 1 and 2 respectively, together with the regions where the calculation of each observable is not perturbatively valid.

In both figures, the horizontal axis, i.e., \( \mu^4/(\rho_{m,0} + 4\Lambda^2) = 0 \), corresponds to a General Relativistic Universe with a cosmological constant. The WMAP results, interpreted within General Relativity, correspond to a Universe with \( \Lambda/\rho_{m,0} \approx 2.65 \). Keeping this value constant and moving towards positive values of \( \mu^4/(\rho_{m,0} + 4\Lambda^2) \), the predicted value of \( \Omega_m \) increases, because the perturbative term in the modified Friedman equation contributes negatively to the rate of expansion of the universe, and the deceleration parameter decreases for the same reason. The opposite is true when the parameter \( \mu^4/(\rho_{m,0} + 4\Lambda^2) \) becomes increasingly negative.

The hatched areas in both figures show the regions of the parameter space where the corrections to each observable calculated to order \( \mu^8 \) are not negligible and hence the solution is no longer of the same order as the field equation. This is the criterion we have discussed earlier in determining the perturbative validity of our calculations. The particular shapes of the hatched regions are determined by the dependence of the perturbative terms in Eqs. 33 and 34 both of which can be negative, positive, or even zero for different values of the ratio \( \Lambda/\rho_{m,0} \). In both figures, however, the excluded regions lie far from the parameters of the General Relativistic Universe that are consistent with the WMAP data, suggesting that small potential deviations from the General Relativistic predictions can be modeled successfully within our framework.

Finally, we investigate the parameter \( w_X \) at the present
epoch for an equivalent XCDM cosmology that would result in the same cosmological expansion rate and acceleration as that predicted by the last two equations. We calculate the equivalent equation-of-state parameter by comparing the last two equations to those of the XCDM framework. The result is, to order $\mu^4$,

$$w_X \sim -1 + \frac{\mu^4}{(\rho_m + 4\Lambda)^2} \left( \frac{\rho_m + \Lambda}{\Lambda} \right) \left[ 12 \left( \frac{\rho_m}{\rho_m + 4\Lambda} \right) - 18 \left( \frac{\rho_m}{\rho_m + 4\Lambda} \right)^2 \right].$$

(35)

In Fig. 3, we show contours of constant values of the parameter $w_X$. The current observed limits $-0.14 < w_X < 0.12$ on the equation-of-state parameter $w_X$ suggest that deviations from the $\Lambda$CDM model that do not introduce new degrees of freedom fall within the perturbative limits of our approach and cannot be ruled out a priori.

**C. Case 3: $\Lambda = 0$, $n \neq 1$**

Finally, we examine a more general class of theories with $n \neq 1$ that are nevertheless matter dominated. We focus here on the case $\Lambda = 0$ in order to isolate the expected dependence of our previous results on the value of the parameter $n$.

In this case, the $(\mu, \nu) = (0, 0)$ field equation becomes

$$3 \left( \frac{\ddot{a}}{a} \right)^2 + 3 \mu^2 \rho_m \left[ \frac{R}{6} + n \frac{\ddot{a}}{a} + n(n + 1) \left( \frac{\dot{a}}{a} \right) \frac{R}{a} \right] = \rho_m$$

(36)

and its solution can be expanded again in terms of the perturbative parameter $\mu^2/R_0$. The constraint on this parameter for perturbative validity becomes increasingly less restrictive as the exponent $n$ decreases.

As with Case 1, we can write the Friedmann equation to order $\mu^{2(n+1)}$ and obtain

$$3 \left( \frac{\ddot{a}}{a} \right)^2 = \rho_m + \frac{\mu^{2(n+1)}}{2 \rho_m^2} [6n(n + 1) + n - 1].$$

(37)

At the same order the acceleration equation is given by

$$6 \frac{\ddot{a}}{a} = -\rho_m + \frac{(2 + 3n)\mu^{2(n+1)}}{2 \rho_m^2} [6n(n + 1) + n - 1].$$

(38)

If we write the Friedmann equations in the flat XCDM form, Eq. [14], in the present epoch, we obtain for the equivalent equation-of-state parameter,

$$w_X = -(n + 1).$$

(39)

As a result, increasing the value of the exponent $n$ makes the equation-of-state parameter $w_X$ more negative.

**III. CONCLUSIONS**

We have presented a formalism – Gravity with Perturbative Constraints – that describes deviations from General Relativistic predictions at cosmological scales without introducing new degrees of freedom. Our formalism preserves diffeomorphism invariance, the Einstein equivalence principle, and energy-momentum conservation. Because it is based on modifications of the Lagrangian of the gravitational field, it allows us to link a wide variety of observations – from compact objects and solar system phenomena, to non-linear clustering and linear structure formation – in order to test for the consistency of assuming no new gravitational degrees of freedom.

Our method is quite different from the Palatini formulation of $1/R$ gravity discussed by, for example, Ref. [32], and expanded on as “modified source gravity” in Ref. [13]. Since both our methods and those related to Palatini formulations, however, eliminate the extra degree of freedom associated with modifying General Relativity it is worth comparing some of their features. In particular, because we are able to treat our system perturbatively – and, explicitly, because our predictions for densities $\rho \gtrsim \rho_{\text{crit}}$ do not demand a particular behavior when $\rho \ll \rho_{\text{crit}}$ – we do not run into the issues found by Ref. [14], where the microscopic discreteness expected for ordinary baryons and dark matter particles no longer “averages out” correctly to return the homogenous Friedmann equations.

Put explicitly, Ref. [14] noted first that the microscopic discreteness of matter did not strongly affect the averaged metric in General Relativity: i.e., when taking the matter field to be composed of discrete particles (small but still larger than their Schwarzschild radius) the gravitational fields relative to the background are still small and $g_{\mu\nu}$ can be consistently be replaced by $(g_{\mu\nu})$ plus a small, linear perturbation. This behavior was then contrasted with the Palatini $1/R$ formulation, where the $\Phi$ field, a
function of the matter density that appears as a modification to the Einstein equation, has strong variations from point to point since the behavior of $\Phi$ is strongly nonlinear in the range $\rho \in [0, \rho_{\text{crit}}]$ regardless of the length scale of the perturbation.

In the case of gravity with perturbative constraints, however, describing the behavior of the equations of motion when $\rho \ll \rho_{\text{crit}}$ may involve increasingly higher order terms in the approximate equations of motion, and – likely for many, though perhaps not all, models where the first terms in the equations of motion are generated by $1/R^n$ cases discussed in this paper – eventually goes outside the radius of convergence of the series. That our theory can not in all cases make predictions for atomic-scale physics is a feature of the perturbative approach; in contrast to Palatini $1/R$, the behavior at $\rho \gtrsim \rho_{\text{crit}}$ decouples from these more difficult questions and no particular atomic-scale behavior is required by us for consistency with larger scales.

In this first approach to the problem, we chose to look for modifications of the Einstein-Hilbert action that involve the addition of terms proportional to an inverse power of the Ricci scalar curvature. We were lead to this choice by our goal to describe potential deviations at cosmological scales, without affecting significantly the behavior of gravity in the solar system. One might object that our particular choice of parametrization – a two dimensional space, ($\Lambda, n$) – too drastically narrows down the possible models considered. In a sense, this is a problem with any attempt to cover a space of functions with a finite number of parameters.

Our approach, however, follows the general direction of testing for modifications of gravity in the solar system and in other astrophysical systems by using the Parametrized Post-Newtonian (PPN) framework. In a similar fashion to our work, the PPN framework provided a way to link together different observations in various astrophysical settings in a consistent and physically rigorous way. While the PPN model itself required a certain degree of arbitrariness – the functional forms of the potentials – it turned out that the original choices of these functional forms were sufficient to cover nearly every theory proposed in the following thirty-seven years. Only in a few cases have failures of PPN been due to an overly restrictive choice of parameters (see, e.g. [30]). The usefulness of the framework we propose here will depend, of course, on its ability to capture the behavior of plausible gravitational theories that go beyond General Relativity.

The particular set of terms that we included in the Lagrangian of the gravitational field allow us also to study in a more systematic way other phenomenological cosmologies that produce accelerating expansion. For example, the set of parameters discussed in Sec. II C, i.e., $\Lambda = 0$ and $n \neq 1$, provide a homogenous cosmology identical to the Cardassian model of Ref. [23]. Within our formalism, however, we can also make predictions for the formation of structure in this cosmology, by taking explicitly into account the modifications to the Poisson equation.

Our formalism is also capable of describing the cosmological expansion histories that are generated by the phenomenological XCDM model [31]. This is demonstrated in Fig. 3 for a flat universe, where the values of the equivalent equation-of-state parameter $w$ are shown as a function of the perturbative parameter $\mu^4$, the matter density, and the value of the cosmological constant. In a similar manner, our framework can also describe the cosmological expansion generated by the model suggested by the Dark Energy Task Force [10], in which the equation-of-state parameter is allowed to change in time as $w_0 + (1 - a)w_a$, since the values of the equivalent parameter $w$ calculated here depend indeed on cosmic time (cf. Eq. 35). In contrast to the phenomenological model, however, our framework offers additional insight as to how the resulting gravity modifications agree with structure formation observations, or with “fifth-force” constraints from other arenas.

We will study the behavior of the cosmological expansion predicted by gravity with perturbative constraints for a Universe with a finite spatial curvature as well as compare these predictions directly to observations in forthcoming papers.

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**IV. APPENDIX: PERTURBATIVE SOLUTION TO 4TH ORDER FOR $\Lambda \neq 0$**

As discussed in the main text, perturbations to $\Lambda$CDM at order $\mu^4$ has closed-form solution. In particular:

$$a^{(4)}(t) = \left( \frac{\rho_{m,0}}{\rho_{\Lambda}} + 4 \right)^2 \left\{ \frac{2 \tan^{-1} \left[ \sqrt{3} \tanh \left( \frac{\sqrt{3} \Lambda t}{2} \right) \right]}{96 \sqrt{3} \tanh \left( \frac{\sqrt{3} \Lambda t}{2} \right)} - 9 \sqrt{3} t \right\} + \frac{17 - 23 \cosh(\sqrt{3} M) + 8 \cosh(2\sqrt{3} M)}{48 \left[ 1 - 2 \cosh(\sqrt{3} M) \right]^2}.$$
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