THE HIERARCHY OF POISSON BRACKETS FOR
THE OPEN TODA LATTICE AND ITS’ SPECTRAL
CURVES

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Abstract. We establish a new representation of the infinite hierarchy of Poisson
brackets (PB) for the open Toda lattice in terms of its spectral curve. For the classical
Poisson bracket (PB) we give a representation in the form of a contour integral of some
special Abelian differential (meromorphic one-form) on the spectral curve. All higher
brackets of the infinite hierarchy are obtained by multiplication of the one-form by a power
of the spectral parameter.

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1. Introduction.

All known integrable hierarchies of equations like Toda lattice, Camassa-Holm equation, Korteweg de Vriez equation, Nonlinear Schrödinger equation, sine-Gordon equation, Landau-Lifshitz equations are Hamiltonian systems. In fact on their phase space $\mathcal{M}$ there exists a finite or infinite set of commutative vector fields $X_1, X_2, \ldots$ that are compatibility conditions for the Lax’s equations. This vector fields can be written with the classical Poisson bracket $\{ , \}_{\pi_0}$ and different Hamiltonians $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \ldots$; as

$$X_k = \{ , \mathcal{H}_k \}_{\pi_0}, \quad k = 1, 2, \ldots$$

In all these examples there exists a second bracket $\{ , \}_{\pi_1}$ compatible with the first $\{ , \}_{\pi_0}$. For the KdV the bracket $\{ , \}_{\pi_1}$ is called the Lenard-Magri bracket. All vector fields can be written with respect to this second bracket.

In this paper we study a Hamiltonian theory of the finite open Toda lattice. The open finite Toda lattice is a mechanical system of $N$–particles connected by elastic strings. The Hamiltonian of the system is

$$H = \sum_{k=0}^{N-1} \frac{p_k^2}{2} + \sum_{k=0}^{N-2} e^{q_{k+1} - q_k} - e^{q_{k} - q_{k+1}}.$$  

Introducing the classical Poisson bracket

$$\{f, g\}_{\pi_0} = \sum_{k=0}^{N-1} \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k},$$ (1.1)

we write the equations of motion as

$X_1: \qquad q_k^* = \{q_k, H\} = p_k,$

$$p_k^* = \{p_k, H\} = -e^{q_{k+1} - q_k} + e^{q_{k-1} - q_k}, \quad k = 1, \ldots, N - 1.$$ We put $q_{-1} = -\infty$, $q_N = \infty$ in all formulas. These equations define the vector field $X_1$.

The next, the quadratic bracket $\{ , \}_{\pi_1}$ for the Toda was found by M. Adler, [11]. The third, the cubic bracket $\{ , \}_{\pi_2}$ was discovered by B. Kupershmidt, [12]. Contrary to the KdV case and the like the recurrence operator for the open Toda lattice is not known.

P. Damianou proved, [3], that on the phase space of the open Toda lattice there exists an infinite sequence of Poisson brackets

$$\{ , \}_{\pi_0} \quad \{ , \}_{\pi_1} \quad \{ , \}_{\pi_2} \quad \ldots$$ (1.2)
Figure 1. The reducible Riemann surface $\Gamma$ which consist of two components $\Gamma_-$ and $\Gamma_+$. These components are copies of the Riemann sphere attached to each other at the points $z_0, z_1, \ldots, z_{N-1}$.

Any vector field of the open Toda hierarchy can be written using these brackets and different Hamiltonians $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \ldots$

$$X_k = \{ , \mathcal{H}_{k-p}\}_{\pi_p}, \quad k = 1, 2, \ldots; \quad 0 \leq p \leq k.$$ 

P. Damianou proved this result using implicit inductive procedure employing sequence of vector fields which are called master symmetries. L. Fayusovich and M. Gekhhtman, [5] associated this family of Poisson brackets with multiple Poisson structures on the Riemann sphere.

For the open Toda lattice the associated spectral curve, the reducible Riemann surface $\Gamma$, was introduced in [11] and presented on Figure 1. In [17], in the context of Nonlinear Schrödinger equation, the author introduced a new parametrization of the phase space in terms of the
associated spectral curve $\Gamma$ and the Weyl function $\chi = \chi(q), q \in \Gamma$;

$$\mathcal{M} \longrightarrow (\Gamma, \chi).$$

This parametrisation also can be constructed for the open Toda lattice. For the Toda lattice we used it to study Poisson bracket $\{ \cdot , \}_{\pi_0}$ for the first time in [15].

The goal of the present paper is to write Poisson brackets for the open Toda in the explicit form

$$\{\chi(p), \chi(q)\}^f = \sum_{k=0}^{N-1} \int_{O_k} \omega_{pq}^f,$$

where the evaluation map is defined as

$$q : (\Gamma, \chi) \rightarrow \chi(q), \quad q \in \Gamma.$$

The meromorphic one-form $\omega_{pq}^f$ depends on the functional parameter $f$. When $f$ is $n$-th power of the spectral parameter the formula defines the Poisson bracket $\{ \cdot , \}^{\pi_n}$. The one-form has poles at the points $z_0, \ldots, z_{N-1}$; at the points $P, Q$ and at infinity. The small circles $O_k$ surround points $z_k$.

An explicit formula for $\omega_{pq}^f$ will be given in the text of the paper. We also demonstrate how to get from this formula various Darboux systems for the bracket. We construct two such systems. One is associated with poles of the Weyl function and another is associated with its zeros.

It was conjectured in [18] that this form is universal. Namely, the formula of this type represents a hierarchy of Poisson brackets of any integrable hierarchy which can be integrated by methods of algebraic geometry. This paper is a confirmation of this conjecture for the finite open Toda lattice.

The Sections 2 through 6 are devoted to the direct spectral transform. We describe the image and the range of the phase space under the direct spectral transform. The image is the space where we construct the Poisson brackets. In Section 7 we present the universal formula for the Poisson brackets on the space of rational functions on the Riemann surface. These PB produce Poisson brackets of the Camassa–Holm equation, [16], and magnetic monopoles, [2]. We present a direct proof of the Jacobi’s identity for these Poisson structures. Due to importance of this result we give two independent proofs. In Section 8 we construct Dirac’s restriction of these Poisson structures on the image of the direct spectral transform. In Section 9 we present action–angle coordinates. In Section 10 we describe another system of Darboux’s coordinates for this family of brackets.
2. THE COMMUTATOR FORMALISM.

Following [9, 13], introduce the new Flaschka–Manakov variables
\[ c_k = e^{q_k - q_{k+1}/2}, \quad k = 0, 1, \ldots, N - 2; \]
\[ v_k = -p_k, \quad k = 0, 1, \ldots, N - 1. \]

The equations of motion take the form
\[ X_1 : \quad v_k^* = c_k^2 - c_{k-1}^2, \quad c_k^* = c_k(v_{k+1} - v_k)/2. \]

These equations are compatibility conditions for the Lax equation \( L^* = [A, L] \), where
\[
L = \begin{bmatrix}
  v_0 & c_0 & 0 & \cdots & 0 \\
  c_0 & v_1 & c_1 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & \cdots & c_{N-3} & v_{N-2} & c_{N-2} \\
  0 & \cdots & 0 & c_{N-2} & v_{N-1}
\end{bmatrix}
\]
and
\[
2A = \begin{bmatrix}
  0 & c_0 & 0 & \cdots & 0 \\
  -c_0 & 0 & c_1 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & \cdots & -c_{N-3} & 0 & c_{N-2} \\
  0 & \cdots & 0 & -c_{N-2} & 0
\end{bmatrix}.
\]

The Lax formula implies that the spectrum \( z_0 < \ldots < z_{N-1} \) is fixed.

3. THE DIRECT SPECTRAL PROBLEM.

We associate with \( L \) the eigenvalue problem
\[ v_0y_0 + c_0y_1 = zy_0, \quad (3.1) \]
\[ c_0y_0 + v_1y_1 + c_1y_2 = zy_1, \quad (3.2) \]
\[ c_{n-1}y_{n-1} + v_ny_n + c_ny_{n+1} = zy_n, \quad n = 2, \ldots, N - 2; \]
\[ c_{N-2}y_{N-2} + v_{N-1}y_{N-1} + c_{N-1}y_N = zy_{N-1}. \quad (3.3) \]

The coefficient \( c_{N-1} \) is defined by the formula \( c_{N-1} = \prod_{k=0}^{N-2} c_k^{-1} \). For the system 3.1 3.3 we introduce the solution
\[ P(z) : \quad P_{-1}(z) = 0, \quad P_0(z) = 1, \ldots, P_N(z); \]
and for the system 3.2 3.3 another solution
\[ Q(z) : \quad Q_0(z) = 0, \quad Q_1(z) = \frac{1}{c_0}, \ldots, Q_N(z). \]
The Weyl function $\chi(z)$ is defined as

$$\chi(z) = -\frac{Q_N(z)}{P_N(z)};$$

To state properties of the function $\chi(z)$ we establish a determinant representation of the polynomials $P(z)$ and $Q(z)$.

**Lemma 3.1.** Let $L_{[k,p]}$ be the truncated matrix

$$L_{[k,p]} = \begin{bmatrix} v_k & c_k & \cdots & 0 \\ c_k & v_{k+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & c_{p-1} \\ \cdots & \cdots & \cdots & v_p \end{bmatrix}. \quad (3.4)$$

Then for $n = 1, 2, \ldots, N$; we have

$$P_n(z) = (-1)^n \frac{\det(L_{[0,n-1]} - zI)}{\prod_{k=0}^{n-1} c_k}, \quad (3.5)$$

and

$$Q_n(z) = (-1)^{n+1} \frac{\det(L_{[1,n-1]} - zI)}{\prod_{k=0}^{n-1} c_k}. \quad (3.6)$$

These formulas can be proved by expanding determinants over the lower row and showing that the polynomials satisfy the three term recurrent relation and the initial conditions.

Formulas (3.5)-(3.6) imply

$$\chi(z) = \frac{(-1)^{N+1} \det(L_{[1,N-1]} - zI)}{(-1)^N \det(L - zI)} = \prod_{s=1}^{N-1} (\gamma_s - z) \left/ \prod_{n=0}^{N-1} (z - \gamma_n) \right. ;$$

$$\chi(z) = -\frac{(-1)^N \det(L_{[1,N-1]} - zI)}{(-1)^{N+1} \det(L - zI)} = \prod_{s=1}^{N-1} (z - \gamma_s) \left/ \prod_{n=0}^{N-1} (z - z_n) \right. ;$$

where the roots $z$ and $\gamma$ interlace

$$z_0 < \gamma_1 < z_1 < \cdots < z_{N-2} < \gamma_{N-1} < z_{N-1}; \quad (3.8)$$

due to the Sturm theorem, [6]. Any such function $\chi(z)$ has the form

$$\chi(z) = \sum_{k=0}^{N-1} \frac{\rho_k}{z_k - z}; \quad (3.9)$$

with $\rho_k > 0$ and $\sum_{k=0}^{N-1} \rho_k = 1$. For $z$ on the real line and below/above spectrum the function $\chi(z)$ is positive/negative.
The function \( \chi(z) \) can be expanded at infinity as

\[
\chi(z) = -\frac{s_0}{z} - \frac{s_1}{z^2} - \frac{s_2}{z^3} - \ldots,
\]

where \( s_p = \sum_{k=0}^{N-1} z_k^p \rho_k, \quad p = 0, 1, 2, \ldots \); and \( s_0 = 1 \).

Following [10], consider functions \( \chi(l) \) with the properties

i. analytic in the half–planes \( \Im z > 0 \) and \( \Im z < 0 \).

ii. \( \chi(\bar{z}) = \chi(z) \), if \( \Im z \neq 0 \).

iii. \( \Im \chi(z) > 0 \), if \( \Im z > 0 \).

All such function are called \( R \)-functions. They play central role in the spectral theory of selfadjoint operators. The Weyl function of a Jacobi matrix is an \( R \)-function.

4. The spaces of rational functions \( \text{Rat}_N \) and \( \text{Rat}'_N \).

Consider a connected set of functions \( \chi(z) \) on \( \mathbb{CP}^1 \) with the property

\( \chi(\infty) = 0 \) and \( N \) simple poles at \( z_0, z_1, \ldots, z_{N-1} \). We denote all such functions as \( \text{Rat}_N \). Apparently any function from \( \text{Rat}_N \) can be uniquely written as

\[
\chi(z) = -\frac{q(z)}{p(z)}, \quad \text{where} \quad p(z) = \prod_{k=0}^{N-1} (z-z_k), \quad q(z) = q_0 \prod_{k=1}^{N-1} (z-%20\gamma_k).
\]

The space \( \text{Rat}_N \) has complex dimension \( 2N \) and \( z - q(z) \) complex coordinates

\( z_0, \ldots, z_{N-1}; q(z_0), \ldots, q(z_{N-1}) \).

We will need more detailed representation

\[
\chi(z) = -\frac{q_0 \prod_{s=1}^{N-1} (z-%20\gamma_s)}{\prod_{n=0}^{N-1} (z-z_n)} = -\frac{q_0 z^{N-1} + q_1 z^{N-2} + \ldots + q_{N-1}}{z^N + p_0 z^{N-1} + \ldots + p_{N-1}}. \tag{4.1}
\]

Any such function can be represented as

\[
\chi(z) = \sum_{k=1}^{N} \frac{\rho_k}{z_k - z}, \quad \rho_k = -\text{res}_{z_k} \chi(z). \tag{4.2}
\]

We have another set of \( z - \rho \) coordinates

\( z_0, \ldots, z_{N-1}; \rho_0, \ldots, \rho_{N-1} \).

The function \( \chi(z) \) can be expanded at infinity as

\[
\chi(z) = -\frac{s_0}{z} - \frac{s_1}{z^2} - \frac{s_2}{z^3} - \ldots,
\]

where \( s_p = \sum_{k=0}^{N-1} z_k^p \rho_k, \quad p = 0, 1, 2, \ldots \) The polynomials \( p(z) \) and \( q(z) \) can be reconstructed from this asymptotic expansion if one takes sufficiently many terms of the expansion at infinity.
We denote by $\text{Rat'}_N$ the subset of all functions from $\text{Rat}_N$ which satisfy the condition

$$q_0 = \sum_{n=1}^{N} \rho_n = 1.$$ 

The Weyl functions of a Jacobi matrix are exactly those $R$–functions that belong to $\text{Rat'}_N$. This implies that all $z_k$ are real and $\rho_k > 0$ in the representation.

5. The Toda lattice hierarchy.

We denote by $\mathcal{M} = \mathbb{R}^N \times \mathbb{R}_+^{N-1}$ the space of all possible $2N - 1$ dimensional vectors

$$(v_0, v_1, \ldots, v_{N-1}; c_0, c_1, \ldots, c_{N-2}).$$

Apparently, $\mathcal{M} = \mathbb{R}^N \times \mathbb{R}_+^{N-1}$.

5.1. The hierarchy of Toda flows. As was proved by J. Moser, $[14]$, that the vector field $X_1$ on the space $\mathcal{M}$ is the first in the hierarchy of commutative vector fields. In fact, there exist $N$ matrices

$$A_1 = A, A_2, A_3, \ldots, A_N;$$

which produce a family of commutative vector fields

$$X_k : \quad \frac{\partial L}{\partial t_k} = [A_k, L], \quad k = 1, 2, \ldots, N; \quad (5.1)$$

and

$$\frac{\partial}{\partial t_p} \frac{\partial L}{\partial t_k} = \frac{\partial}{\partial t_k} \frac{\partial L}{\partial t_p}.$$ 

Theorem 5.1. $[14]$. The Lax equation $\[5.7\]$ implies that under the action of the vector field $X_k$, $k = 1, \ldots, N$; the parameters $z - \rho$ change according to the rule

$$\frac{\partial}{\partial t_k} z_n = 0;$$

$$\frac{\partial}{\partial t_k} \rho_n = \left(z_n^k - \sum_{s=0}^{N-1} z_s^k \rho_s\right) \rho_n.$$

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5.2. **Three Poisson structures on** $\mathcal{M}$. Any PB bracket on $\mathcal{M}$ which is $2N - 1$ dimensional manifold will be degenerate. Now we give explicit formulas for the first three PB together with their Casimirs.

The first linear bracket $\{ , \}_{\pi_0}$ is the classical Poisson bracket $\Pi_{\text{I}}$ in $c - v$ coordinates

$$
\{c_k, v_k\}_{\pi_0} = -c_k/2, \quad \{c_k, v_{k+1}\}_{\pi_0} = c_k/2. \quad (5.2)
$$

All other brackets vanish. The functional $\Phi = \text{trace } L$ is the Casimir of the bracket.

The second quadratic bracket $\{ , \}_{\pi_1}$ appeared in the work Adler, [1], and it is defined by the relations

$$
\{c_k, c_{k+1}\}_{\pi_1} = c_k c_{k+1}/2, \quad \{c_k, v_k\}_{\pi_1} = -c_k v_k, \\
\{c_k, v_{k+1}\}_{\pi_1} = c_{k} v_{k+1}, \quad \{v_k, v_{k+1}\}_{\pi_1} = 2v_k^2.
$$

All other brackets vanish. The functional $\Phi = \text{det } L$ is the Casimir of this bracket.

The third cubic bracket $\{ , \}_{\pi_2}$ appeared in the work of Kupershmidt, [2], and it is defined by the relations

$$
\{c_k, c_{k+1}\}_{\pi_2} = c_k c_{k+1} v_{k+1}, \quad \{c_k, v_k\}_{\pi_2} = -c_k v_k^2 - c_k^3, \\
\{c_k, v_{k+1}\}_{\pi_2} = c_k v_{k+1}^2 + c_k^3, \quad \{c_k, v_{k+2}\}_{\pi_2} = c_k c_{k+1}^2, \\
\{c_{k+1}, v_k\}_{\pi_2} = -c_k^2 c_{k+1}, \quad \{v_k, v_{k+1}\}_{\pi_2} = 2c_k^2 (v_k + v_{k+1}).
$$

All other brackets vanish. The functional $\Phi = \text{trace } L^{-1}$ is the Casimir of this bracket.

5.3. **The Hamiltonian formulation of the hierarchy of Toda flows.** It is known, [3], that the Poisson brackets introduced above are the first three of the infinite sequence of compatible Poisson structures $\pi_k$, $k = 0, 1, \ldots$. These higher PB brackets are constructed in [3] using master symmetries. The construction is based on the inductive procedure. The general formula for these PB in the $c - v$ coordinates is not known.

The vector fields $X_p$ defined by Lax equations $\Pi_{\text{I}}$ are Hamiltonian. Introduce the functions

$$
\mathcal{H}_n = \frac{1}{n+1} \text{trace } L^{n+1}, \quad n = 0, 1, \ldots \quad (5.3)
$$

Then the vector field $X_k$ is produced by the Hamiltonian $\mathcal{H}_k$ with the Poisson bracket $\{ , \}_{\pi_0}$. Moreover, the vector field $X_k$ is produced by the Hamiltonian $\mathcal{H}_{k-p}$ with the Poisson bracket $\{ , \}_{\pi_p}$, where integer $p$ is such that $k \geq p \geq 0$.  

---

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6. **The spectral curve and the new parametrization** $(\Gamma, \chi)$.

Now we describe the direct spectrum transform from the space of Jacobi matrices $\mathcal{M}$ to the space of pairs $(\Gamma, \chi)$

$$\mathcal{M} \longrightarrow (\Gamma, \chi).$$

The spectral curve $\Gamma = \Gamma_+ \sqcup \Gamma_-$ consists of two copies of $\mathbb{CP}^1$ glued together at the points of the spectrum $z_0, z_1, \ldots, z_{N-1}$; see Figure 1. The points of $\Gamma$ we denote by $q = (z, \pm)$. We define $\chi(q) = \chi(z)$, $q \in \Gamma_-$. In fact we deal not with the whole $\Gamma$ but only with its' $\Gamma_-$ component.

The direct spectral transform is injective. We described its' range when we described all functions which can appear as Weyl functions of a Jacobi matrix. There are two classical effective ways from the 19th century to invert this map. One due to Stieltjes, [6], is to expand $\chi$ into a continuous fraction and another due to Jacobi, [8], is to construct the orthogonal polynomials $P(z)$ and $Q(z)$ using the moments $s_p$. The new approach based on the notion of the BA function for the reducible curve is given in [11].

Now we can write a representation of the flows of the Toda hierarchy and PB in terms of the new parametrization.

7. **The hierarchy of Poisson brackets on** $(\Gamma, \chi)$, where $\chi \in \text{Rat}_N$.

A simplectic structure on such space of pairs $(\Gamma, \chi)$ was introduced by Atiyah and Hitchin in [2] as

$$\sum_{k=0}^{N-1} \frac{d q(z_k)}{q(z_k)} \wedge d p(z_k).$$

The corresponding Poisson structure is given by the formula

$$\{\chi(p), \chi(q)\} = \frac{(\chi(p) - \chi(q))^2}{p - q}. \quad (7.1)$$

This form was found in the paper of Faybusovich and Gekhtman, [5]. For the Atiyah-Hitchin bracket [7] it was shown in [16] that it corresponds to the main Poisson bracket for the Camassa-Holm equation written in terms of the Weyl function, [19], of the associated Krein's string spectral problem. Faybusovich and Gekhtman also found higher brackets of the infinite hierarchy of Toda flows with (7.1) being the first bracket. In our paper [7] we found an algebraic-geometrical representation of all these brackets. It was explained in [7] that these brackets produce hierarchy of Poisson brackets of the Camassa-Holm equation.
To introduce a formula for the hierarchy we consider a meromorphic differential \( \omega^f_{pq} \) on \( \Gamma \) which depends on the entire function \( f(z) \) and two points \( p \) and \( q \):

\[
\omega^f_{pq} = \frac{\epsilon_{pq}(z)}{p - q} \times f(z) \chi(z) (\chi(p) - \chi(q)),
\]

where

\[
\epsilon_{pq}(z) = \frac{1}{2\pi i} \left[ \frac{1}{z - p} - \frac{1}{z - q} \right] dz;
\]

is the standard differential Abelian differential of the third kind with residues \( \pm 1 \) at the points \( p \) and \( q \). It also can be written as

\[
\omega^f_{pq} = \epsilon_{pq}(z) \times f(z) \chi(z) (\chi(p) - \chi(q)),
\]

where

\[
\epsilon_{pq}(z) = \frac{1}{2\pi i} \left[ \frac{1}{(z - p)(z - q)} \right] dz.
\]

The analytic Poisson brackets are defined on \( (\Gamma, \chi) \), where \( \chi \in \text{Rat}_N \), by the formula, see [7]:

\[
\{\chi(p), \chi(q)\}_f = \sum_{k=0}^{N-1} \int_{O_k} \omega^f_{pq},
\]

where the circles \( O_k \) are traversed clockwise and surround points \( z_k \).

**Theorem 7.1.** The Poisson bracket \( \{ \cdot, \cdot \}_f \) satisfies the Jacobi identity

\[
\{\{\chi(p), \chi(q)\}, \chi(r)\} + cp(p, q, r) = 0.
\]

We gave an indirect proof in [7] by constructing such coordinates that \( \{ \cdot, \cdot \}_f \) has the standard constant form

\[
J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.
\]

(7.3)

Now we found a direct proof of Theorem 7.1 which is rather difficult.

7.1. **The first proof of Jacobi identity.** First we give a proof of the Jacobi identity that does not use explicit form of differentials \( \epsilon_{pq}(z) \). Every we below we omit the superscript \( f \) in the formula \( \{ \cdot, \cdot \} = \{ \cdot, \cdot \} \).

**Proof.** From the definition

\[
\{\chi(p), \chi(q)\} = \frac{1}{2\pi i} \int_{\bigcup O_k} \frac{dz f(z) \chi(z)}{(z - p)(z - q)} \times (\chi(p) - \chi(q)).
\]
Therefore,

\[
\{\{\chi(p), \chi(q)\}, \chi(r)\} = \frac{1}{2\pi i} \int_{\mathcal{O}_k} \frac{dz f(z)}{(z-p)(z-q)} \times \{\chi(z) (\chi(p) - \chi(q)), \chi(r)\} \\
= \frac{1}{2\pi i} \int_{\mathcal{O}_k} \frac{dz f(z)\chi(z)}{(z-p)(z-q)} \times \{\chi(p) - \chi(q)\}, \chi(r)\} + \\
+ \frac{1}{2\pi i} \int_{\mathcal{O}_k} \frac{dz f(z)}{(z-p)(z-q)} \times \{\chi(z), \chi(r)\} (\chi(p) - \chi(q))
\]

= \ I + II.

For the first term we have

\[
I = \frac{1}{2\pi i} \int_{\mathcal{O}_k} \frac{dz f(z)\chi(z)}{(z-p)(z-q)} \times \frac{1}{2\pi i} \int_{\mathcal{O}_{k'}} \frac{d\eta f(\eta)\chi(\eta)}{(\eta-p)(\eta-r)} (\chi(p) - \chi(r))
\]

\[
- \frac{1}{2\pi i} \int_{\mathcal{O}_k} \frac{dz f(z)\chi(z)}{(z-p)(z-q)} \times \frac{1}{2\pi i} \int_{\mathcal{O}_{k'}} \frac{d\eta f(\eta)\chi(\eta)}{(\eta-q)(\eta-r)} (\chi(q) - \chi(r))
\]

\[
= \frac{1}{(2\pi i)^2} \int_{\mathcal{O}_k \cup \mathcal{O}_{k'}} \int_{\mathcal{O}_{k'} \cup \mathcal{O}_k} \frac{dzd\eta f(z)f(\eta)\chi(z)\chi(\eta)(\chi(p) - \chi(r))(z-r)(\eta-q)}{(z-p)(z-q)(z-r)(\eta-p)(\eta-r)(\eta-q)}
\]

\[
- \frac{1}{(2\pi i)^2} \int_{\mathcal{O}_k \cup \mathcal{O}_{k'}} \int_{\mathcal{O}_{k'} \cup \mathcal{O}_k} \frac{dzd\eta f(z)f(\eta)\chi(z)\chi(\eta)(\chi(q) - \chi(r))(z-r)(\eta-p)}{(z-p)(z-q)(z-r)(\eta-p)(\eta-r)(\eta-q)}.
\]

Denoting \(\mathcal{P}(z) = (z-p)(z-q)(z-r)\),

\[
I = \frac{1}{(2\pi i)^2} \int_{\mathcal{O}_k \cup \mathcal{O}_{k'}} \int_{\mathcal{O}_{k'} \cup \mathcal{O}_k} \frac{dzd\eta f(z)f(\eta)\chi(z)\chi(\eta)(\chi(p) - \chi(r))(z-r)(\eta-q)}{\mathcal{P}(z)\mathcal{P}(\eta)}
\]

\[
- \frac{1}{(2\pi i)^2} \int_{\mathcal{O}_k \cup \mathcal{O}_{k'}} \int_{\mathcal{O}_{k'} \cup \mathcal{O}_k} \frac{dzd\eta f(z)f(\eta)\chi(z)\chi(\eta)(\chi(q) - \chi(r))(z-r)(\eta-p)}{\mathcal{P}(z)\mathcal{P}(\eta)}
\]

After simple algebra

\[
I + c.p. = \chi(p)(q-r) \frac{1}{(2\pi i)^2} \int_{\mathcal{O}_k \cup \mathcal{O}_{k'}} \int_{\mathcal{O}_{k'} \cup \mathcal{O}_k} \frac{dzd\eta f(z)f(\eta)\chi(z)\chi(\eta)(\eta + p - 2z)}{\mathcal{P}(z)\mathcal{P}(\eta)}
\]

\[
+ \chi(q)(r-p)...
\]

\[
+ \chi(r)(p-q)....
\]
The last integral vanishes due to skew symmetry.

Similar for the second term we have

\[
II = \frac{1}{2\pi i} \int_{O_k^r} \frac{dzf(z)}{(z-p)(z-q)} (\chi(p) - \chi(q)) \times \frac{1}{2\pi i} \int_{O_k^l} \frac{d\eta f(\eta)\chi(\eta)(\chi(z) - \chi(r))}{(\eta - z)(\eta - r)}
\]

\[
= \frac{1}{(2\pi i)^2} \int_{O_k^r} \int_{O_k^l} d\eta dz f(z)f(\eta)\chi(\eta)(\chi(z) - \chi(q))(\eta - p)(\eta - q)(z - r)
\]

\[
+ \frac{1}{(2\pi i)^2} \int_{O_k^r} \int_{O_k^l} d\eta dz f(z)f(\eta)\chi(\eta)(\chi(z) - \chi(r))(\eta - p)(\eta - q)(z - r)
\]

\[
= A + B.
\]

From simple algebra

\[
B + c.p. = \chi(q)\chi(p)(p - q) \frac{1}{(2\pi i)^2} \int_{O_k^r} \int_{O_k^l} d\eta dz f(z)f(\eta)\chi(\eta)(\eta - r)
\]

\[
+ \chi(p)\chi(r)(r - p)...
\]

\[
+ \chi(r)\chi(q)(q - r)...
\]

Changing the order of integration

\[
\int_{O_k^r} \int_{O_k^l} d\eta dz f(z)f(\eta)\chi(\eta)(\eta - r) = \int_{O_k^l} \int_{O_k^r} d\eta f(\eta)\chi(\eta)(\eta - r) \frac{1}{\mathcal{P}(z)}
\]

The differential \(dzf(z)/\mathcal{P}(z)\) is analytic inside the circles \(O_k\) and the integral vanishes due to the Cauchy theorem. Therefore,

\[
B + c.p. = 0.
\]

This implies

\[
II + c.p. = A + c.p. = \chi(p)(q - r) \frac{1}{(2\pi i)^2} \int_{O_k^r} \int_{O_k^l} dz d\eta f(z)f(\eta)\chi(z)\chi(\eta)(\eta - p)
\]

\[
+ \chi(q)(r - p)...
\]

\[
+ \chi(r)(p - q)....
\]

Finally,

\[
I + II + c.p. = [\chi(p)(q - r) + \chi(q)(r - p) + \chi(r)(p - q)] \times \frac{1}{(2\pi i)^2} \int_{O_k^r} \int_{O_k^l} d\eta dz f(z)f(\eta)\chi(z)\chi(\eta)(\eta - 2) - 2z)
\]

The last integral vanishes due to skew symmetry.
7.2. The second proof of Jacobi identity. Here we give a second proof of the Jacobi identity that does not use explicit form of differentials $\epsilon_{pq}(z)$.

Proof 2. From the definition

$$\{\chi(p), \chi(q)\} = \int \sum_{O_k} [\epsilon_{pq}^o f](z) \times \chi(z)(\chi(p) - \chi(q)).$$

Therefore,

$$\{\{\chi(p), \chi(q)\}, \chi(r)\} = \int \sum_{O_k} [\epsilon_{pq}^o f](z) \times \chi(z)\{\chi(p) - \chi(q), \chi(r)\}$$

$$= \int \sum_{O_k} [\epsilon_{pq}^o f](z) \times \chi(z)\{\chi(p) - \chi(q)\} + \int \sum_{O_k} [\epsilon_{pq}^o f](z) \times (\chi(p) - \chi(q)) \{\chi(z), \chi(r)\}$$

$$= I + II.$$
For the first term we have

\[ I = \int \sum_{O_k} \left[ \epsilon^{\circ}_{pq} f \chi \right](z) \times \{ \chi(p), \chi(r) \} - \int \sum_{O_k} \left[ \epsilon^{\circ}_{pq} f \chi \right](z) \times \{ \chi(q), \chi(r) \} \]

\[ = \int \sum_{O_k} \left[ \epsilon^{\circ}_{pq} f \chi \right](z) \times \int \sum_{O_{k'}} \left[ \epsilon^{\circ}_{pr} f \chi \right](\eta) \times (\chi(p) - \chi(r)) \]

\[ - \int \sum_{O_k} \left[ \epsilon^{\circ}_{pq} f \chi \right](z) \times \int \sum_{O_{k'}} \left[ \epsilon^{\circ}_{qr} f \chi \right](\eta) \times (\chi(q) - \chi(r)) \]

\[ = + \chi(p) \int \sum_{O_k} \int \sum_{O_{k'}} \left[ \epsilon^{\circ}_{pq} f \chi \right](z) \left[ \epsilon^{\circ}_{pr} f \chi \right](\eta) \]

\[ - \chi(r) \int \sum_{O_k} \int \sum_{O_{k'}} \left[ \epsilon^{\circ}_{pq} f \chi \right](z) \left[ \epsilon^{\circ}_{pr} f \chi \right](\eta) \]

\[ + \chi(r) \int \sum_{O_k} \int \sum_{O_{k'}} \left[ \epsilon^{\circ}_{pq} f \chi \right](z) \left[ \epsilon^{\circ}_{qr} f \chi \right](\eta) \]

\[ - \chi(q) \int \sum_{O_k} \int \sum_{O_{k'}} \left[ \epsilon^{\circ}_{pq} f \chi \right](z) \left[ \epsilon^{\circ}_{qr} f \chi \right](\eta). \]

After simple algebra

\[ I + c.p. = \chi(p) \int \sum_{O_k} \int \sum_{O_{k'}} \left[ \epsilon^{\circ}_{pq} f \chi \right](z) \left[ \epsilon^{\circ}_{pr} f \chi \right](\eta) - \left[ \epsilon^{\circ}_{rp} f \chi \right](z) \left[ \epsilon^{\circ}_{pq} f \chi \right](\eta) \]

\[ - \left[ \epsilon^{\circ}_{qr} f \chi \right](z) \left[ \epsilon^{\circ}_{qp} f \chi \right](\eta) + \left[ \epsilon^{\circ}_{qr} f \chi \right](z) \left[ \epsilon^{\circ}_{rp} f \chi \right](\eta) \]

\[ + \chi(q)...

\[ + \chi(r).... \]

Using the first identity

\[ \frac{\epsilon_{ab}(z)}{z-c} = \frac{\epsilon_{a'b'}(z)}{z-c'}, \quad (7.4) \]

where \((a', b', c')\) is an arbitrary permutation of the points \((a, b, c)\), and the second identity

\[ (z-r)(\eta-q)-(z-q)(\eta-r)=(z-p)(\eta-r)+(z-p)(\eta-q) = (\eta+p-2z)(q-r), \]

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we transform the expression under integral sign to the form

\[ I + c.p. = \chi(p)(q - r) \int \int \frac{[\epsilon_{pq}^0 f\chi](z) [\epsilon_{pr}^0 f\chi](\eta)}{(z - r)(\eta - q)}(\eta + p - 2z) \]

\[ + \chi(q) \]
\[ + \chi(r) \]

Similar for the second term we have

\[ II = (\chi(p) - \chi(q)) \times \sum_{O_k} [\epsilon_{pq}^0 f\chi](z) \int \sum_{O_{k'}} [\epsilon_{zr}^0 f\chi](\eta) \times (\chi(z) - \chi(r)) \]

\[ = (\chi(p) - \chi(q)) \times \sum_{O_k} [\epsilon_{pq}^0 f\chi](z) \int \sum_{O_{k'}} [\epsilon_{zr}^0 f\chi](\eta) \]

\[ - (\chi(p) - \chi(q)) \chi(r) \times \sum_{O_k} [\epsilon_{pq}^0 f\chi](z) \int \sum_{O_{k'}} [\epsilon_{zr}^0 f\chi](\eta) \]

\[ = A - B. \]

It is easy to see

\[ A + c.p. = \chi(p) \left[ \int \sum_{O_k} [\epsilon_{pq}^0 f\chi](z) \int \sum_{O_{k'}} [\epsilon_{zr}^0 f\chi](\eta) - \int \sum_{O_k} [\epsilon_{rp}^0 f\chi](z) \int \sum_{O_{k'}} [\epsilon_{zq}^0 f\chi](\eta) \right] \]

\[ + \chi(q) \]
\[ + \chi(r) \]

Using the identity

\[(z - r)(\eta - p)(\eta - q) - (z - q)(\eta - p)(\eta - r) = (\eta - p)(r - q)(z - \eta)\]

we obtain

\[ A + c.p. = \chi(p)(r - q) \int \sum_{O_k} \sum_{O_{k'}} \frac{[\epsilon_{pq}^0 f\chi](z) [\epsilon_{zr}^0 f\chi](\eta)}{(z - r)(\eta - q)}(z - \eta) \]

\[ + \chi(q) \]
\[ + \chi(r) \]

Using the identity

\[ \frac{\epsilon_{zr}^0(\eta)}{\eta - q}(z - \eta) = -\epsilon_{rq}^0, \]

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we have

\[ A + c.p. = \chi(p)(q-r) \int \int_{O_k O_k'} \frac{[\epsilon_{pq}f\chi](z) [\epsilon_{rq}f\chi](\eta)}{(z-r)(\eta-p)} (\eta-p) \]

\[ + \chi(q) \ldots \]

\[ + \chi(r) \ldots \]

From simple algebra

\[ B + c.p. = \chi(q)w(p) \left[ \int \int_{O_k O_k'} \frac{[\epsilon_{zq}f](z) [\epsilon_{zr}f\chi](\eta) - \int \int_{O_k O_k'} \frac{[\epsilon_{zp}f](z) [\epsilon_{zq}f\chi](\eta)}{(z-p)(\eta-q)(\eta-r)} \right] \]

\[ + \chi(p)\chi(r) \ldots \]

\[ + \chi(r)\chi(q) \ldots \]

We are going to transform the expression in the square bracket using the first identity [7.4] and the second identity

\[(z-p)(\eta-q)(\eta-r) - (z-q)(\eta-p)(\eta-r) = (\eta-r)(z-\eta)(p-q).\]

Therefore,

\[ B + c.p. = \chi(q)\chi(p) \times \left[ \int \int_{O_k O_k'} \frac{[\epsilon_{zq}f](z) [\epsilon_{zr}f\chi](\eta) (\eta-r)(z-\eta)(p-q)}{(z-p)(\eta-q)(\eta-r)} \right] \]

\[ + \chi(p)\chi(r) \ldots \]

\[ + \chi(r)\chi(q) \ldots \]

Note,

\[ \epsilon_{zq}(\eta)(z-\eta) = -\frac{d\eta}{\eta-p} = \epsilon_{p\infty}(\eta).\]

Changing the order of integration

\[ \int \int_{O_k O_k'} \frac{[\epsilon_{zq}f](z) [\epsilon_{zq}f\chi](\eta)(\eta-r)(z-\eta)(p-q)}{(z-p)(\eta-q)(\eta-r)} \]

\[ = \int \int_{O_k O_k'} \frac{[\epsilon_{zq}f\chi](\eta)(\eta-r)(p-q)}{(\eta-q)(\eta-r)} \int \int_{O_k} \frac{[\epsilon_{zq}f](z)}{z-p} \]

The differential is analytic inside the circles \(O_k\) and the integral vanishes due to the Cauchy theorem. Therefore,

\[ B + c.p. = 0.\]
This implies

\[ II + c.p. = A + c.p. = \chi(p)(q-r) \int \sum_{O_k} \sum_{O_{k'}} \int \frac{[\epsilon_{pq}f\chi](z) [\epsilon_{rq}f\chi](\eta)}{(z-r)(\eta-p)} (\eta-p) \]

\[ + \chi(q)(r-p) \ldots \]

\[ + \chi(r)(p-q) \ldots \]

Finally,

\[ I + II + c.p. = [\chi(p)(q-r) + \chi(q)(r-p) + \chi(r)(p-q)] \times \]

\[ \int \sum_{O_k} \sum_{O_{k'}} \int \frac{[\epsilon_{pq}f\chi](z) [\epsilon_{pq}^2f\chi](\eta)}{(z-r)(\eta-p)} (2\eta - 2z). \]

The last integral vanishes due to skew symmetry.

7.3. **Two quadratic algebras.** By the Cauchy formula from 7.2 we have for any entire \( f(z) \)

\[ \{\chi(p), \chi(q)\}^f = \text{res}_p \omega_{pq}^f + \text{res}_q \omega_{pq}^f + \text{res}_\infty \omega_{pq}^f \]

\[ = \frac{f(p)\chi(p) - f(q)\chi(q)}{p-q} (\chi(p) - \chi(q)) + \text{res}_\infty \omega_{pq}^f. \]

If \( f(z) = z^n, n = 0, 1, \ldots; \) then the residue at infinity vanishes identically only for \( n = 0 \) or 1. When \( f(z) = 1 \) we obtain quadratic Poisson algebra corresponding to the *rational* solution of CYBE

\[ \{\chi(p), \chi(q)\}^1 = (\chi(p) - \chi(q)) \frac{\chi(p) - \chi(q)}{p-q}. \] (7.5)

Another quadratic Poisson algebra is obtained for \( f(z) = z \) and it corresponds to the *trigonometric* solution of CYBE

\[ \{\chi(p), \chi(q)\}^z = (p\chi(p) - q\chi(q)) \frac{\chi(p) - \chi(q)}{p-q}. \] (7.6)

It can be verified directly that 7.5 and 7.6 satisfy Jacobi identity.

7.4. **The Darboux coordinates.** The following result is proved using residues:

**Theorem 7.2.** ([14], Theorem 2.1) The Poisson bracket 7.2 in \( z - \rho \) coordinates has the form

\[ \{\rho_k, \rho_n\} = \frac{(f(z_k) + f(z_n))\rho_k \rho_n}{z_n - z_k} (1 - \delta_n^k), \] (7.7)

\[ \{\rho_k, z_n\} = \rho_k f(z_n) \delta_n^k, \] (7.8)

\[ \{z_k, z_n\} = 0. \] (7.9)
This result together with the formula \( q(z_k) = p'(z_k) \rho_k \) implies that in \( z - q(z) \) coordinates takes the form
\[
\{q(z_k), z_n\} = f(z_n)q(z_k) \delta^n_k;
\]
and all other brackets vanish
\[
\{q(z_k), q(z_n)\} = \{z_k, z_n\} = 0.
\]
In the coordinates
\[
I_k = \int_\infty^{z_k} \frac{d\zeta}{f(\zeta)}, \quad k = 0, 1, \ldots, N - 1; \quad (7.10)
\]
and
\[
y_k = \ln q(z_k), \quad k = 0, 1, \ldots, N - 1.
\]
We define \( \infty \) some fixed point on the Riemann sphere. We need a convergence of the integral. The bracket \( 7.2 \) takes the standard constant form
\[
\{y_k, I_n\} = \delta^n_k; \quad \{y_k, y_n\} = \{I_k, I_n\} = 0.
\]

8. The hierarchy of Poisson brackets on \((\Gamma, \chi)\), where \( \chi \in \text{Rat}_N' \).

**Lemma 8.1.** For two functionals
\[
\Phi_1 = I_0 + I_1 + \ldots + I_{N-1}, \quad \Phi_2 = \log q_0;
\]
where \( I_k \) are defined by \( 7.10 \), the bracket \( 7.2 \) is
\[
\{\Phi_1, \Phi_2\}' = 1.
\]

**Proof.** Defined function \( F(z) \) by the formula
\[
F(z) = \int_\infty^z \frac{ds}{f(s)}.
\]
We assume that zeros of \( f(z) \) are distinct from the poles \( z_k \). Therefore, \( F(z) \) is well defined in the vicinity of the polls. Integrating by parts
\[
\Phi_1 = I_0 + I_1 + \ldots + I_{N-1} = F(z_1) + F(z_2) + \ldots + F(z_{N-1})
\]
\[
= \sum_k \frac{1}{2\pi i} \int_{O_k} F(\zeta) d\ln \chi(\zeta)
\]
\[
= \sum_k \frac{1}{2\pi i} \int_{O_k} \frac{1}{f(\zeta)} \ln \chi(\zeta) d\zeta.
\]
Using $q_0 = - \lim_{y \to \infty} y \chi(y)$, we have

$$\{ \Phi_1, q_0 \} = - \lim_{y \to \infty} y \{ \Phi_1, \chi(y) \} = - \sum_k \frac{1}{2\pi i} \int_{O_k} \frac{1}{f(\zeta) \chi(\zeta)} \lim_{y \to \infty} y \{ \chi(\zeta), \chi(y) \} d\zeta.$$

Using 7.2, the Cauchy’s formula for sufficiently large $R$

$$\lim_{y \to \infty} y \{ \chi(\zeta), \chi(y) \} = \lim_{y \to \infty} \sum_{k'} \frac{1}{2\pi i} \int_{O_{k'}} \frac{y dz}{(z - \zeta)(z - y)} f(z) \chi(z) (\chi(\zeta) - \chi(y))$$

$$= - \chi(\zeta) \sum_{k'} \frac{1}{2\pi i} \int_{O_{k'}} \frac{dz}{z - \zeta} f(z) \chi(z)$$

$$= - f(\zeta) \chi^2(\zeta) + \chi(\zeta) \frac{1}{2\pi i} \int_{O_R} \frac{dz}{z - \zeta} f(z) \chi(z) = I + II.$$

Changing the order of integration for any $k$ we see that the integral vanishes due to the Cauchy theorem

$$- \int_{O_k} \frac{d\zeta}{f(\zeta)} \int_{O_R} \frac{dz}{z - \zeta} f(z) \chi(z) = \int_{O_R} \frac{d\zeta}{f(\zeta)} \chi(z) \int_{O_k} \frac{dz}{z - \zeta} f(\zeta) = 0.$$

Therefore, contribution of the term $II$ is zero. For the term $I$ we have

$$\{ \Phi_1, q_0 \} = - \lim_{y \to \infty} y \{ \Phi_1, \chi(y) \} = \sum_k \frac{1}{2\pi i} \int_{O_k} \frac{d\zeta}{f(\zeta) \chi(\zeta)} f(\zeta) \chi^2(\zeta)$$

$$= \sum_k \frac{1}{2\pi i} \int_{O_k} \chi(\zeta) d\zeta = \frac{1}{2\pi i} \int_{O_R} \chi(\zeta) d\zeta = q_0,$$

where $R$ is sufficiently large number. □

**Theorem 8.2.** For any choice of an entire function $f(z)$ a Dirac restriction of the Poisson bracket 7.2 on the sub-manifold

$$\Phi_1 = c_1, \quad \Phi_2 = c_2;$$

where $x_k$ are defined by 7.10, is given by the formula

$$\{ \chi(p), \chi(q) \}^f = \sum_{k=0}^{N-1} \int_{O_k} \omega^f_{p,q},$$

where $\omega^f_{p,q}$ is the Poisson bracket 7.2, is given by the formula

$$\{ \chi(p), \chi(q) \}^f = \sum_{k=0}^{N-1} \int_{O_k} \omega^f_{p,q}.$$
where the circles $O_k$ are traversed clockwise and surround points $z_k$. The new modified differential $\tilde{\omega}_{pq}^f$ is

$$\tilde{\omega}_{pq}^f = \frac{\epsilon_{pq}(z)}{p-q} \times f(z)\chi(z)(\chi(p) - \chi(q)) - \epsilon_{pq}(z) \times f(z)\chi(z)\chi(p)\chi(q)e^{-c2}.$$  

**Proof.** According to the Dirac’s recipe, 4, we modify the original bracket \(\{ , \}\) by adding two extra terms

\[ F_1^* = \{F_1, F_2\}' = \{F_1, F_2\} + \sigma_1 \{F_1, \Phi_1\} + \sigma_2 \{F_1, \Phi_2\}. \]

The constants $\sigma_1$ and $\sigma_2$ are specified by conditions

\[ \Phi_1^* = \{\Phi_1, F_2\}' = \{\Phi_1, F_2\} + \sigma_1 \{\Phi_1, \Phi_1\} + \sigma_2 \{\Phi_1, \Phi_2\} = 0, \]

\[ \Phi_2^* = \{\Phi_2, F_2\}' = \{\Phi_2, F_2\} + \sigma_1 \{\Phi_2, \Phi_1\} + \sigma_2 \{\Phi_2, \Phi_2\} = 0. \]

Using Lemma 8.1 we have

$$\sigma_1 = \{\Phi_2, F_2\} \quad \quad \sigma_2 = -\{\Phi_1, F_2\},$$

and

$$\{F_1, F_2\}' = \{F_1, F_2\} + \{\Phi_2, F_2\} \{\Phi_1, F_1\} - \{\Phi_1, F_2\} \{\Phi_1, F_2\}. $$

Therefore,

$$\{\chi(x), \chi(y)\}' = \{\chi(x), \chi(y)\} + \{\Phi_2, \chi(y)\} \{\Phi_1, \Phi_1\} - \{\Phi_1, \chi(y)\} \{\chi(x), \Phi_2\}. $$

Using,

$$\{\Phi_1, \chi(y)\} = \chi(y); \quad (8.3)$$

and

$$\{\Phi_2, \chi(y)\} = -\chi(y)e^{-\Phi_2} \sum_k \frac{1}{2\pi i} \int_{O_k} \frac{d\zeta}{\zeta-y} f(\zeta)\chi(\zeta). \quad (8.4)$$

we obtain the result.

To prove 8.3 we use formula for $\Phi_1$ obtained in the proof of Lemma 8.1

$$\Phi_1 = \sum_k \frac{1}{2\pi i} \int_{O_k} \frac{1}{f(\zeta)} \ln \chi(\zeta) d\zeta.$$  

Therefore,

$$\{\Phi_1, \chi(y)\} = \sum_k \frac{1}{2\pi i} \int_{O_k} \frac{1}{f(\zeta)} \chi(\zeta) \{\chi(\zeta), \chi(y)\} d\zeta$$

$$= \sum_k \frac{1}{2\pi i} \int_{O_k} \frac{d\zeta}{f(\zeta)\chi(\zeta)} \sum_{k'} \frac{1}{2\pi i} \int_{O_{k'}} \frac{d\eta}{(\eta - \zeta)(\eta - y)} f(\eta)\chi(\eta)(\chi(\zeta) - \chi(y)).$$
When $k \neq k'$ we have
\[
\int_{O_k} \frac{d\zeta}{f(\zeta)\chi(\zeta)} \int_{O_{k'}} \frac{d\eta}{(\eta - \zeta)(\eta - y)} f(\eta)\chi(\eta) = 0
\]
due to the Cauchy theorem. When $k = k'$ we have changing the order of the integration
\[
\therefore = \sum_k \frac{1}{2\pi i} \int_{O_k} \frac{d\zeta}{f(\zeta)\chi(\zeta)} \int_{O_k} \frac{d\eta}{(\eta - \zeta)(\eta - y)} f(\eta)\chi(\eta)
\]
\[
= \sum_k \frac{1}{2\pi i} \int_{O_k} \frac{d\eta}{(\eta - y)} f(\eta)\chi(\eta) \int_{O_k} \frac{d\zeta}{f(\zeta)\chi(\zeta)(\eta - \zeta)}
\]
\[
= \sum_k \frac{1}{2\pi i} \int_{O_k} \frac{\chi(\eta)d\eta}{\eta - y} = \frac{1}{2\pi i} \int_{O_R} \frac{\chi(\eta)d\eta}{\eta - y} = \chi(y).
\]
To prove 8.4 we have
\[
\{ q_0, \chi(y) \} = -\lim_{x \to \infty} x\chi(x), \chi(y) \}
\]
\[
= -\lim_{x \to \infty} \sum_k \frac{1}{2\pi i} \int_{O_k} \frac{xd\zeta}{(\zeta - x)(\zeta - y)} f(\zeta)\chi(\zeta)(\chi(x) - \chi(y))
\]
\[
= -\chi(y) \sum_k \frac{1}{2\pi i} \int_{O_k} \frac{d\zeta}{\zeta - y} f(\zeta)\chi(\zeta).
\]
This implies the result. \(\square\)

As a byproduct we have that the bracket defined by 8.2 satisfies the Jacobi identity. This statement also can be proved directly, similar to Theorem 7.1.

The bracket 8.2 has two Casimirs $\Phi_1$ and $\Phi_2$ and, therefore, degenerates on $\text{Rat}_N$ with the rank $2N - 2$. The bracket 8.2 produces vector fields in $\text{Rat}_N$ that are tangent to $\text{Rat}'_N$ and therefore it can be restricted to this subspace. We call it a bracket on $(\Gamma, \chi)$, where $\chi \in \text{Rat}'_N$.

For $f(z) = 1$ we have
\[
\Phi_1 = z_0 + z_1 + \ldots + z_{N-1} = p_0.
\]
For $f(z) = z$ we have
\[
\Phi_1 = \ln z_0 + \ln z_1 + \ldots + \ln z_{N-1} = \ln p_{N-1}.
\]
For \( f(z) = z^{n+1}, n = 1, 2, \ldots \); we have
\[
\Phi_1 = \frac{1}{z_0^n} + \frac{1}{z_1^n} + \ldots + \frac{1}{z_{N-1}^n}.
\]
These are the Casimirs of three Poisson structures considered in Section 5.2.

### 8.1. The Toda flows in terms of \((\Gamma, \chi)\).

Now we compute brackets 8.2 in terms of the coordinates \(z - \rho\) similar to Theorem 2.1 in [7].

**Theorem 8.3.** The Poisson bracket 8.2 in \(z - \rho\) coordinates has the form
\[
\{\rho_k, \rho_n\}^f = (f(z_k) + f(z_n) - f(z_k) \delta_n^k) \times \left( \frac{\rho_k \rho_n}{z_k - z_n} - \frac{\rho_m}{\sum_{m \neq k} z_m - z_k} - \frac{\rho_m}{\sum_{m \neq n} z_m - z_n} \right),
\]
\[
\{\rho_k, z_n\}^f = \rho_k f(z_n) \delta_n^k - f(z_n) \rho_k \rho_n, \quad (8.6)
\]
\[
\{z_k, z_n\}^f = 0. \quad (8.7)
\]

Now consider Hamiltonians defined by 5.3
\[
H_n = \frac{1}{n+1} \text{trace } L^{n+1} = \frac{1}{n+1} \sum_{k=0}^{N-1} z_k^{n+1}, \quad n = 0, 1, \ldots.
\]

Due to Theorem 8.3, one has for the brackets \(\{ , \}_z^p\) and integer \(k, p: k \geq p \geq 0\);
\[
z_n^* = \{z_n, H_{k-p}\}_z^{p} = 0
\]
and
\[
\rho_n^* = \{\rho_n, H_{k-p}\}_z^{p} = \left( z_n^{k} - \sum_{s=0}^{N-1} z_s^k \rho_s \right) \rho_n.
\]

These equations coincide with the equations for the \(X_k\) vector field in Theorem 5.1.

### 8.2. Two quadratic algebras.

As before, from 8.2 for any entire \(f(z)\) we have
\[
\{\chi(p), \chi(q)\}_z^f = \text{res}_p \tilde{\omega}_{pq}^f + \text{res}_q \tilde{\omega}_{pq}^f + \text{res}_\infty \tilde{\omega}_{pq}^f
\]
\[
= (f(p)\chi(p) - f(q)\chi(q)) \left( \frac{\chi(p) - \chi(q)}{p - q} \right) + \text{res}_\infty \tilde{\omega}_{pq}^f.
\]
When \(f(z) = 1\) we obtain quadratic Poisson algebra
\[
\{\chi(p), \chi(q)\}_z^1 = (\chi(p) - \chi(q)) \left( \frac{\chi(p) - \chi(q)}{23} \right).
\]

(8.8)
For $f(z) = z$ we have

$$\{\chi(p), \chi(q)\}^z = (p\chi(p) - q\chi(q)) \left( \frac{\chi(p) - \chi(q)}{p - q} - \chi(p)\chi(q)e^{-cz} \right).$$

(8.9)

When $f(z) = z^n$, $n = 0, 1, \ldots$; then the residue at infinity vanishes identically only for $n = 0$ or 1. This implies that higher Toda brackets with $n \geq 2$ do not form the quadratic Poisson algebra and therefore do not admit $R$–matrix representation.

9. **The action-angle coordinates**

In the cooordinates

$$I_k = \int_\infty^{z_k} \frac{d\zeta}{f(\zeta)}; \quad \theta_k = \ln \left( \frac{(-1)^k q(z_k)}{q(z_0)} \right), \quad k = 1, \ldots, N - 1.$$

where $\infty$ is a fixed point on the Riemann sphere, the bracket $8.2$ takes the standard constant form. As it is explained in [15] the variables $\theta_k$ are angles on the isospectral manifold parametrized by the non-standard Abel map constructed from the differentials of the third type on the spectral curve $\Gamma$. The ispectral manifold is not compact and the angles are not a cyclic variables; the vector $(\theta_1, \ldots, \theta_{N-1})$ takes value in $\mathbb{R}^{N-1}$.

The following theorem was proved in [15] for $f = 1$. The methods of this paper extend easily to the case of general $f$.

**Theorem 9.1.** The Poisson bracket $8.2$ in $I - \theta$ coordinates has the form

$$\{\theta_k, \theta_n\}^f = 0, \quad (9.1)$$

$$\{I_k, I_n\}^f = 0, \quad (9.2)$$

$$\{\theta_k, I_n\}^f = 0. \quad (9.3)$$

10. **Another system of Darboux’s coordinates**

In this section we construct system of Darboux coordinates for $8.2$ on $\text{Rat}_N$ associated with zeroes of these functions. Let us define

$$\pi_k = \log((-1)^{N+k}p(\gamma_k)), \quad k = 1, \ldots, N - 1.$$ 

In additions to $\Phi_1$ and $\Phi_2$ we use

$$\gamma_1, \ldots, \gamma_{N-1}; \quad \pi_1, \ldots, \pi_{N-1}.$$

The following result was proved (Theorem 3 in [15]) for $f = 1$. Now we can formulate it for general $f$. 

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Theorem 10.1. The Poisson bracket in $\gamma - \pi$ coordinates has the form

$$\{\gamma_k, \pi_n\}^f = \delta_k^n, \quad (10.1)$$

$$\{\Phi_1, \Phi_2\}^f = 1. \quad (10.2)$$

All other brackets vanish.

From this the relations follow easily for the restricted bracket 8.2.

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