ALL POLYTOPES ARE COSET GEOMETRIES: CHARACTERIZING AUTOMORPHISM GROUPS OF k-ORBIT ABSTRACT POLYTOPES

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Abstract. Abstract polytopes generalize the classical notion of convex polytopes to more general combinatorial structures. The most studied ones are regular and chiral polytopes, as it is well-known, they can be constructed as coset geometries from their automorphism groups. This is also known to be true for 2- and 3-orbit 3-polytopes. In this paper we show that every abstract n-polytope can be constructed as a coset geometry. This construction is done by giving a characterization, in terms of generators, relations and intersection conditions, of the automorphism group of a k-orbit polytope with given symmetry type graph. Furthermore, we use these results to show that for all k ≠ 2, there exist k-orbit n-polytopes with Boolean automorphism groups , for all n ≥ 3.

1. Introduction

In the 1970s several ideas extended the geometric study of convex polytopes to generalize them from different points of view: while Tits studied incidence systems, Coxeter focused on tessellations of manifolds and Grünbaum proposed to move away from spherical tiles. In the early 1980s Danzer and Schulte put several of these ideas together to start the study of incidence polytopes, now called abstract polytopes. An abstract polytope is a ranked, partially ordered set which generalizes the face lattice of convex polytopes and tessellations. Thus, abstract polytopes generalize the classical notion of convex polytopes and tessellations to more general combinatorial structures.

The degree of symmetry of an abstract polytope is measured by counting the number of flag orbits under the action of its automorphism group, where a flag is a maximal chain of the partial order. Abstract polytopes can then be classified in terms of their so-called symmetry type graph ([2]), which encodes all the information of the local configuration of flags with respect to the automorphism group.

Polytopes with only one flag-orbit are called regular and are the most studied ones. Regular polytopes have maximal degree of symmetry and in particular their automorphism groups are generated by involutions, often called “(abstract) reflections”. The book Abstract Regular Polytopes [11] by...
McMullen and Schulte is the standard reference and it is devoted exclusively to the study of abstract regular polytopes. There are $2^n - 1$ classes of $n$-polytopes with 2 flag orbits, each of them corresponding to one symmetry type graph. Among them is the class of chiral polytopes ([14]), which have no “reflectional” symmetry, but have maximal “rotational” one.

The study of coset geometries goes back to Tits ([15]). The ideas behind this concept are to construct incidence structures using groups, and in particular were developed by Tits in connection to Coxeter groups. It is well-known that regular and chiral polytopes, as well as two-orbit polyhedra can be seen as coset geometries (see [3],[7] and [14]); this characterization of their automorphism groups constitute the most important tool to study them. In a recent paper ([8]), the authors showed that 3-orbit polyhedra, as well, can be seen as coset geometries, and this fact is used to construct 3-orbit polyhedra from symmetric groups.

The purpose of this paper is to prove the following theorem.

**Main Theorem.** Every abstract polytope can be constructed as a coset geometry.

To show this, we characterize the automorphism group of an abstract polytope with a given symmetry type graph, in terms of generators and relations, as well as some intersection conditions on some subgroups and cosets.

Throughout the paper, we work with the flag graph of a polytope, as opposed to the partial order. In fact, we shall often work with maniplexes, that is, colored graphs that generalize flag graphs of polytopes ([16]).

As pointed out before, symmetry type graphs are a great tool to classify polytopes (and manioplexes) in terms of their automorphism groups. We will rely on them heavily in our study. There are some (necessary) conditions a graph must satisfy to be the symmetry type graph of a maniplex or polytope. Graphs satisfying such conditions are called premanioplexes (or admissible graphs) and will be properly defined in Section 2.1. We believe our main theorem will shed light to the study of $k$-orbit polytopes; in particular, it opens the gate to study one of the main problems in the area (listed as Problem 12 in [1]):

**Question 1.** Given a premaniplex, does it exist a polytope (or maniplex) having such premaniplex as its symmetry type graph?

The problem of finding polytopes or manioplexes with a given symmetry type graph is in general very difficult, as one can note, for example, by looking at the history of chiral polytopes: although it was back in 1991 [14], when Schulte and Weiss studied chiral polytopes and classified their automorphism groups in terms of generators and relations, it took almost 20 years to have a construction showing that such polytopes existed for all ranks $n > 3$ (see [12]).

Very little is know about other particular instances of Question 1. It is known [2] that every premaniplex with 3 vertices with rank $n \geq 3$ is the symmetry type graph of a polytope, and in [13] Pellicer, Potočnik and Toledo proved that every premaniplex of rank $n \geq 3$ with 2 vertices is the symmetry type graph of a maniplex, but it is not known if the constructed manioplexes are polytopal (i.e. the flag graph of a polytope).

This paper is organized as follows. In Section 2 we give the basic concepts from the theory of abstract polytopes as well as manioplexes, and formally introduce the concepts of symmetry type graphs and premanioplexes. We also state a relaxed version of Question 1, that we answer later, in Section 4.1. We start Section 3 by going over the main tool used in this paper: voltage graphs. Then, by using them, in Section 3.1 we construct a maniplex $\mathcal{M}$ from a premaniplex $X$ and a group $G$ (satisfying some conditions). Then $G$ will act on $\mathcal{M}$ as a group of automorphisms and the quotient
of $M$ by the action of $G$ will be $X$. This means that $M$ will have symmetry type graph $X$ if and only if every automorphism of $M$ is represented by the action of an element of $G$.

In Section 4.1, we use the construction of Section 3, as well as the results from [4], to characterize in terms of generators and relations the groups that are automorphism groups of a polytope with a given symmetry type graph. We do so by showing that automorphism groups of polytopes must satisfy certain “intersection conditions” for some subgroups and cosets, that depend on the symmetry type graph. In other words, we give an algebraic test for the group $M$ of Section 3.1 that tells us if the constructed maniplex $M$ is polytopal or not, thus translating the problem of finding polytopes with a given symmetry type graph to a group-theoretic one. This gives an answer to Problem 1 of [1]. In Section 4.2 we give the proof to the Main Theorem by constructing a polytope as a coset geometry from a voltage group. This gives an answer to Problem 2 of [1].

We finish the paper in Section 5 by using the above result to construct $k$-orbit $n$-polytopes with Boolean automorphism groups, (that is, elementary Abelian $2$-groups), for all $k, n \geq 3$. For this, we define caterpillars as premaniplexes having exactly one generating tree, and study their coverings in order to avoid the possible extra symmetry that might happen when one uses our construction to obtain polytopes from groups.

2. Abstract polytopes and maniplexes

In this section we shall give the basic definitions and properties of abstract polytopes and maniplexes, and some relations between them. For more details, we refer the reader to [4], [11] and [16].

A partially ordered set is said to be flagged if it has a (unique) least and a (unique) greatest element and each maximal chain, called a flag, has the same finite cardinality. As all flags have the same cardinality, say $n + 2$, flagged posets naturally admit an order-preserving function, the rank function, from the poset to the set $\{-1, 0, 1, \ldots, n\}$. The rank function allows us to talk about flag-adjacencies: given two flags $\Phi$ and $\Psi$ of a flagged poset, they are said to be $i$-adjacent if they differ only in the element of rank $i$.

An (abstract) $n$-polytope (also called an (abstract) polytope of rank $n$) is a flagged poset $\mathcal{P}$ in which the following conditions hold:

- **Diamond condition:** for each flag $\Phi$ of $\mathcal{P}$ and each $i \in \{0, 1, \ldots, n-1\}$, there exists a unique $i$-adjacent flag to $\Phi$.
- **Strong flag connectedness:** for any two flags $\Phi$ and $\Psi$ of $\mathcal{P}$, there exists a sequence of adjacent flags connecting $\Phi$ to $\Psi$, such that all the flags in the sequence contain $\Phi \cap \Psi$ as a subset.

The elements of a polytope are called faces, and faces of rank $i$ are called $i$-faces. Given a flag $\Phi$, we often denote its $i$-face as $\Phi_i$, and by $\Phi^i$ its (unique!) $i$-adjacent flag. Recursively, if $w$ is a word on $\{0, \ldots, n-1\}$ we denote by $\Phi^w$ the flag $(\Phi^w)^1$. It is straightforward to see that $(\Phi^i)^j = \Phi$, and that $(\Phi^i)_j = \Phi_j$ if and only if $i \neq j$.

Given a flagged poset $\mathcal{P}$ one can define its flag graph as the graph $G(\mathcal{P})$ whose vertices are the flags of $\mathcal{P}$ and two flags $\Phi$ and $\Psi$ are connected by an edge of color $i \in \{0, 1, \ldots, n-1\}$ if and only if they are $i$-adjacent. The flag graph of an $n$-polytope is an $n$-maniplex, that is, an $n$-regular connected simple graph with a proper edge coloring with colors $\{0, \ldots, n-1\}$ such that if $i$ and $j$ are two colors satisfying that $|i - j| > 1$, then the graph induced by edges of colors $i$ and $j$ is a disjoint union of $4$-cycles. However, not every maniplex is the flag graph of a polytope.

While abstract polytopes generalize classical polytopes to combinatorial structures, maniplexes (introduced by Steve Wilson in 2012 [16]) generalize flag graphs of a polytope as well as of the flag
graphs of maps on surfaces (that is, 2-cellular embeddings of connected graphs on surfaces). In order

to unify our notation of abstract polytopes and maniplexes, when dealing with maniplexes we shall
call its vertices flags, and say that two of flags are $i$-adjacent if they are the vertices of an edge of color $i$.

It follows from the definition of a maniplex that each flag is incident to exactly one edge of each

color. Then, for each $i \in \{0, 1, \ldots, n-1\}$ one can define $r_i$ as the permutation of the set of flags that

maps each flag to its $i$-adjacent one. In other words $\Phi r_i = \Psi$ if and only if $\Phi$ and $\Psi$ are $i$-adjacent.

The permutations $r_i$, with $i \in \{0, 1, \ldots, n-1\}$, are all involutions with no fixed points and, by

connectivity, they generate a group of permutations on the flags which acts transitively on them. Furthermore, if $|i - j| > 1$ then $r_i r_j$ is also an involution with no fixed points; thus, $r_i$ and $r_j$ commute.

Since $\Phi r_i$ is the flag $i$-adjacent to $\Phi$, it is convenient to denote it by $\Phi^i$ and to follow the same recursive notation as before: $\Phi^w = (\Phi^w)^i$ where $w$ is a word on $\{0, 1, \ldots, n-1\}$.

The group $\langle r_0, r_1, \ldots, r_{n-1} \rangle$ is called the monodromy or connection group of the maniplex $M$, it shall be denoted by $Mon(M)$ and we shall call each of its elements a monodromy. If $w$ is a word on the alphabet $\{0, 1, \ldots, n-1\}$ we identify $w$ with the monodromy $x \mapsto x^w$, that is, the word $a_1 a_2 \ldots a_k$ is identified with the monodromy $r_{a_1} r_{a_2} \ldots r_{a_k}$.

A maniplex homomorphism is a graph homomorphism that preserves the color of the edges. Using

the connectedness of maniplexes one can see that every maniplex homomorphism is determined by

the image of one flag and that they are all surjective. The notions of isomorphism and automorphism

follow naturally.

As with polytopes, we denote the automorphism group of a maniplex $M$ by $\Gamma(M)$. By definition, if $\gamma \in \Gamma(M)$, then $(\Phi \gamma) (\Phi r_i) = (\Phi^i \gamma) (\Phi r_i) = (\Phi^i) r_i$ for all $i \in \{0, 1, \ldots, n-1\}$, implying that $\omega \gamma = \gamma \omega$ for all $\omega \in Mon(M)$.

Note further that, since the action of $Mon(M)$ is transitive, the action of $\Gamma(M)$ is free (or semi-regular). Of course, this is true for both polytopes and maniplexes.

Let $M$ be an $n$-maniplex. If $I \subset \{0, 1, \ldots, n-1\}$, we define $M_I$ as the subgraph of $M$ induced by

the edges of colors in $I$. If $i \in \{0, 1, \ldots, n-1\}$, we use the symbol $\overline{i}$ to denote the set $\{0, 1, \ldots, n-1\} \setminus \{i\}$, and more generally, if $K \subset \{0, 1, \ldots, n-1\}$, we denote its complement by $\overline{K}$. In particular $M_{\overline{i}}$ is the subgraph of $M$ obtained by removing the edges of color $i$. We will use this notation also for any graph with a coloring of its edges even if it is not a maniplex.

In [4] Garza-Vargas and Hubard describe how to recover a polytope $\mathcal{P}$ from its flag graph: the elements of rank i of $\mathcal{P}$ are the connected components of $M_{\overline{i}}$; the order is given as follows: given connected components $F$ and $G$ of $M_{\overline{0}}$ and $M_{\overline{j}}$, respectively, we set $F < G$ if and only if $i < j$ and $F \cap G \neq \emptyset$.

Moreover, starting with $M$ one can construct the set of $i$-faces as the connected components of

$M_{\overline{i}}$ and define the relation $F < G$ for $F \in M_{\overline{0}}$ and $G \in M_{\overline{j}}$ if and only if $i < j$ and $F \cap G \neq \emptyset$. The set of all $i$-faces, with $i \in \{0, \ldots, n-1\}$ together with $\prec$ is denoted by $\mathcal{P}(M)$. In [4] it is proved that if $M$ is any maniplex, then $\mathcal{P}(M)$ is a flagged poset.

It is not difficult to see ([4]) that $\mathcal{P}$ and $\mathcal{P}'$ are isomorphic polytopes if and only if their flag graphs $G(\mathcal{P})$ and $G(\mathcal{P}')$ are isomorphic. This fact implies the following theorem, which can be interpreted as saying that all the information of the polytope $\mathcal{P}$ is encoded in its flag graph.

**Theorem 2.1.** [4] Let $\mathcal{P}$ be a polytope and let $M = G(\mathcal{P})$ be its flag graph. Then $\mathcal{P}$ is isomorphic (as a poset) to $\mathcal{P}(M)$ and $\Gamma(\mathcal{P}) \cong \Gamma(M)$. 
Theorem 5.3 of [4] gives a characterization of polytopal maniplexes, that is, those maniplexes that are isomorphic to the flag graph of some polytope. Such characterization is given in terms of some path intersection properties of the maniplexes.

Definition 2.2. Let $\mathcal{M}$ be an $n$-maniplex. We say that $\mathcal{M}$ satisfies the strong path intersection property (or SPIP) if for every two subsets $I, J \subset \{0, 1, \ldots, n-1\}$ and for any two flags $\Phi$ and $\Psi$, whenever there is a path $W$ from $\Phi$ to $\Psi$ using only edges of colors in $I$ and also a path $W'$ from $\Phi$ to $\Psi$ using only edges of colors in $J$, there also exists a path $W''$ from $\Phi$ to $\Psi$ that uses only edges of colors in $I \cap J$.

We say that $\mathcal{M}$ satisfies the weak path intersection property (or WPIP) if for any two flags $\Phi$ and $\Psi$ and for all $k, m \in \{0, 1, \ldots, n-1\}$, whenever there is a path $W$ from $\Phi$ to $\Psi$ with only edges of colors in $[0, m] := \{0, 1, \ldots, m\}$ and a path $W'$ from $\Phi$ to $\Psi$ with only edges of colors in $[k, n-1] := \{k, k+1, \ldots, n-1\}$, then there is also a path $W''$ from $\Phi$ to $\Psi$ with only edges of colors in $[k, m] := \{k, k+1, \ldots, m\}$.

Theorem 2.3. [4, Theorem 5.3] Let $\mathcal{M}$ be a maniplex. Then the following conditions are all equivalent:

- $\mathcal{M}$ is polytopal.
- $\mathcal{M}$ satisfies the SPIP.
- $\mathcal{M}$ satisfies the WPIP.

In any of these cases $\mathcal{P}(\mathcal{M})$ is a polytope whose flag graph is isomorphic to $\mathcal{M}$.

2.1. Premaniplexes and symmetry type graphs. A $k$-orbit maniplex is one with exactly $k$ flag orbits under its automorphism group. When studying $k$-orbit maniplexes with $k > 1$ one finds that it is convenient to classify them in terms of the local structure of the flags. For this reason, in [2], Cunningham et al. introduce the concept of symmetry type graph.

Given a maniplex $\mathcal{M}$ and a subgroup $G$ of the automorphism group of $\mathcal{M}$, the symmetry type graph of $\mathcal{M}$ with respect to $G$, denoted either by $\mathcal{T}(\mathcal{M}, G)$ or by $\mathcal{M}/G$, is constructed as follows: The vertex set of $\mathcal{T}(\mathcal{M}, G)$ is the set of flag orbits of $\mathcal{M}$ under the group $G$, and if $\Phi$ and $\Psi$ are $i$-adjacent in $\mathcal{M}$ we draw an edge of color $i$ between their orbits. If $\Phi$ and $\Phi'$ are in the same orbit under $G$, we draw a semi-edge of color $i$ at the vertex corresponding to that orbit. Note that a semi-edge is different from a loop in that it consists of an edge with only one endpoint, rather than an with two equal endpoints; that is, a semi-edge is incident to its vertex once, while a loop is incident to its vertex two times.

When we speak about the symmetry type graph of $\mathcal{M}$, we mean it with respect to $\Gamma(\mathcal{M})$ and we simply write $\mathcal{T}(\mathcal{M})$ instead of $\mathcal{T}(\mathcal{M}, \Gamma(\mathcal{M}))$.

If $\mathcal{P}$ is a polytope, the symmetry type graph of $\mathcal{P}$ (with respect to $G$) is defined as the symmetry type graph (with respect to $G$) of its flag graph.

If $X$ is the symmetry type graph of an $n$-maniplex, then it is a connected graph in which every vertex is incident to exactly one edge of each color in $\{0, 1, \ldots, n-1\}$ and it satisfies that if $|i-j| > 1$, the paths of length 4 that alternate between the colors $i$ and $j$ are closed. However, it might not be a maniplex as it is not necessarily simple. We will call such a graph a $n$-premaniplex.

If $X$ is a premaniplex and $i, j \in \{0, 1, \ldots, n-1\}$ are non-consecutive, the connected components of the subgraph of $X$ induced by the edges of colors $i$ and $j$ are not necessarily 4-cycles. In fact they can be any quotient of a 4-cycle, as illustrated in Figure 1.

The notions of homomorphism, isomorphism, automorphism and monodromy group from maniplexes can all be easily extended to premaniplexes as well.

The natural projection $p : \mathcal{M} \rightarrow \mathcal{T}(\mathcal{M}, G)$ is, of course, a homomorphism (of premaniplexes).
When studying polytopes (or maniplexes) and their symmetry type graphs two natural questions occur:

- Given a premaniplex $X$, is there a polytope (or maniplex) whose symmetry type graph is $X$? (see Question 1)
- Given a premaniplex $X$, what conditions must a group $G$ satisfy so that there is a polytope (or maniplex) $P$ such that $\mathcal{T}(P, G) \cong X$?

In Section 4.1 we give a complete answer to the latter. Question 1 remains as a hard question, as even if we find a polytope $P$ and a group $G$ such that $\mathcal{T}(P, G) \cong X$, it may still happen that $G$ is a proper subgroup of $\Gamma(P)$. However, in Section 5 we give an infinite family of premaniplexes that are in fact symmetry type graphs of polytopes.

### 3. Voltage graphs

The projection $p : \mathcal{M} \to \mathcal{T}(\mathcal{M}, G)$ is an example of what is called a regular covering projection in graph theory (see [5] or [10] for more details). Given a regular covering projection $p : \overline{X} \to X$, one may recover the graph $\overline{X}$ from the graph $X$ using what is known as a voltage assignment.

We shall refer to a directed edge as a dart. The inverse of a dart $d$ is the dart $d^{-1}$ with the same edge, but different orientation. Hence, loops and links (edges between two different vertices) have two darts inverse to each other. In contrast, semi-edges have only one dart, inverse to itself. Given an edge $e$ with vertices $v$ and $u$, if a dart $d$ of $e$ has direction from $v$ to $u$, we shall say that $v$ is the initial vertex of $d$ (and denote it by $I(d)$) and $u$ is the terminal vertex of $d$ (and denote it by $T(d)$).

A path in a graph is a finite sequence of darts $W = d_1d_2\ldots,d_k$ such that the dart $d_{i+1}$ starts at the endpoint of the dart $d_i$. The start-point of $W$ is $I(d_1)$, and the endpoint of $W$ is $T(d_k)$. If $x$ and $y$ are the start-point and endpoint of $W$, respectively, we say that $W$ goes from $x$ to $y$ and we write this as $W : x \rightarrow y$. If the endpoint and start-point of a path $W$ are the same vertex $x$ we say that $W$ is a closed path based at $x$. We also consider that for every vertex $x$ there is an empty closed path based at $x$.

If a path $W$ ends at the start-point of a path $V$, we may define the product $WV$ as their concatenation.

Two paths $W$ and $W'$ with the same start-point and endpoint are said to be homotopic if one can transform $W$ into $W'$ by a finite sequence of the following operations:
• Inserting two consecutive inverse darts at any point, that is
\[ d_1d_2 \ldots d_id_{i+1} \ldots d_k \mapsto d_1 \ldots d_id_{i+1}^{-1}d_{i+1} \ldots d_k, \]
where \( I(d) = T(d_i) \);
• Deleting two consecutive inverse darts at any point, that is
\[ d_1 \ldots d_id_{i+1}^{-1}d_{i+1} \ldots d_k \mapsto d_1d_2 \ldots d_id_{i+1} \ldots d_k; \]

In this case we write \( W \sim W' \).

It is easy to see that homotopy is an equivalence relation and that if \( W \sim W' \), \( V \sim V' \), and \( W \) ends at the start-point of \( V \), then \( WV \sim W'V' \). Therefore, we can think of the product of two homotopy classes of paths. The set of all homotopy classes of paths in a graph \( X \) with this operation is called the fundamental groupoid of \( X \) and it is denoted by \( \Pi(X) \). We will often speak of a "path \( W \) in \( \Pi(X) \)", but the reader should keep in mind that we are actually referring to its homotopy class.

The subset of \( \Pi(X) \) consisting of all the (homotopy classes of) closed paths based at a vertex \( x \) forms a group known as the fundamental group of \( X \) based at \( x \) and it is denoted by \( \Pi^x(X) \).

A voltage graph is a pair \( (X, \xi) \) where \( X \) is a graph and \( \xi \) is a groupoid antimorphism from \( \Pi(X) \) to a group \( G \). In this case we say that \( \xi \) is a voltage assignment (on \( X \)) and we call \( G \) the voltage group (of \( \xi \)). The element \( \xi(W) \) is called the voltage of \( W \).

Note that a voltage assignment is completely determined by the voltages of the darts of the graph, as the voltage \( \xi(W) \) of a path \( W = d_1d_2 \ldots d_{k-1}d_k \) is simply \( \xi(d_k)\xi(d_{k-1}) \ldots \xi(d_1) \). In fact, if \( D \) is the set of darts of a graph \( X \), any function \( \xi : D \to G \) can be extended to a voltage assignment as long as \( \xi(d^{-1}) = \xi(d)^{-1} \) for every dart \( d \in D \).

Given a voltage graph \( (X, \xi) \) with voltage group \( G \), we can construct the derived graph \( X^\xi \) as follows:

• The vertex set is \( V \times G \) where \( V \) is the vertex set of \( X \).
• The dart set is \( D \times G \) where \( D \) is the dart set of \( X \).
• If the dart \( d \) goes from \( x \) to \( y \), the dart \( (d, g) \) goes from the vertex \( (x, g) \) to \( (y, \xi(d)g) \).

This is an undirected graph, as the inverse of the dart \( (d, g) \) is the dart \( (d^{-1}, \xi(d)g) \). Note that for every \( h \in G \), the mapping \( (x, g) \mapsto (x, gh) \) is an automorphism of \( X^\xi \). That is, \( G \) can be regarded as a group of automorphisms of \( X^\xi \).

Given a path \( W \) of a graph \( X \) starting at a vertex \( x \) and given an element \( g \) in the voltage group, there is a unique path \( \bar{W} \) in \( X^\xi \) that starts at \( (x, g) \) that projects to \( W \). The path \( \bar{W} \) is called a lift of \( W \) and it is easy to see that it ends at \( (y, \xi(W)g) \) where \( y \) is the endpoint of \( W \) (for details see [10]).

In our case, we will work with graphs that have a coloring of its edges (and therefore, darts), so we will define that the color of the dart \( (d, g) \) is the same as the color of \( d \).

It is known (see [5] or [10]) that if \( p : \bar{X} \to X \) is a regular covering, there is a voltage assignment \( \xi \) such that \( \bar{X} \) is isomorphic to \( X^\xi \). Moreover, given a spanning tree \( T \) of \( X \), there is a voltage assignment \( \xi \) such that \( \bar{X} \cong X^\xi \) and \( \xi(d) \equiv 1 \) for every dart \( d \) in \( T \).

3.1. Voltage graphs that give maniplexes as derived graphs. Of course, we want to use voltage graphs to obtain maniplexes (and polytopes). In this section we shall give necessary and sufficient conditions on a voltage assignment \( \xi \) on a premaniplex \( X \) so that \( X^\xi \) is a maniplex.

Let \( M \) be a maniplex, \( \Gamma \) a group of automorphisms of \( M \) and \( X = \mathcal{T}(M, \Gamma) \). Let \( T \) be a spanning tree of \( X \). As mentioned before, there is a voltage assignment \( \xi : \Pi(X) \to \Gamma \) such that \( X^\xi \)
is isomorphic to \( M \) and \( \xi(d) = 1 \) for every dart \( d \) in \( T \). The set of non-trivial voltages of darts in \( X \) then gives a generating set for the group \( \Gamma \). In fact, if we keep only one element from each pair of inverse generators, what we get is the set of \textit{distinguished generators} of \( \Gamma \), as defined in [2, Section 5].

Conversely, given a premaniplex \( X \) with fundamental groupoid \( \Pi(X) \), let \( T \) be a spanning tree of \( X \) and let \( \xi : \Pi(X) \rightarrow \Gamma \) be a voltage assignment, for some group \( \Gamma \) such that \( \xi(d) = 1 \) for every dart \( d \) in \( T \). In what follows, we find the conditions on \( \Gamma \) and \( \xi \) that ensure that \( X^\xi \) is actually a maniplex.

First we want \( X^\xi \) to be connected. It is known (see [10]) that in order for \( X^\xi \) to be connected, \( \xi(D) \) must generate \( \Gamma \), where \( D \) denotes the set of darts of \( X \).

Next, \( X^\xi \) must be a simple graph. Thus, it must not have semi-edges or multiple edges. Note that a semi-edge of \( X^\xi \) that starts at a vertex \((x, \gamma)\) ends in \((x, \xi(e)\gamma) = (x, \gamma)\), where \( e \) is a semi-edge of \( X \). This implies that if the voltage of every semi-edge of \( X \) is not trivial, we avoid semi-edges in \( X^\xi \). Since \( \xi \) is an antimorphism we should have that \( \xi(e) = \xi(e^{-1}) = \xi(e)^{-1} \), implying that the voltage of a semi-edge must have order two.

To avoid multiple edges, we need to avoid different darts with the same initial and terminal vertices; suppose \( X^\xi \) has two parallel darts \((d, \sigma) \) and \((d', \tau) \). Since both darts start at the same vertex, we have that \((I(d), \sigma) = (I(d'), \tau)\), so \( I(d) = I(d') \) and \( \sigma = \tau \). The common end-point of \((d, \sigma) \) and \((d', \tau) \) could be written as \((y, \xi(d)\sigma) \) or \((z, \xi(d')\sigma)\), where \( y \) is the end-point of \( d \) and \( z \) the end-point of \( d' \). The fact that these two are the same means that \( y = z \) and \( \xi(d) = \xi(d') \). So \((d, \sigma) \) and \((d', \sigma) \) are parallel darts in \( X^\xi \) if and only if \( d \) and \( d' \) are parallel darts in \( X \) with the same voltage. Thus, \( X^\xi \) has no parallel darts if and only if no pair of parallel darts in \( X \) has equal voltages.

Finally, we want to ensure that if \(|i - j| > 1\), the paths of length 4 in \( X^\xi \) that alternate colors between \( i \) and \( j \) are closed. Let \( \tilde{W} \) be one of these paths. Projecting \( \tilde{W} \) to \( X \) we get a path \( W \) in \( X \) of length 4 that alternates colors between \( i \) and \( j \), and since \( X \) is a premaniplex we know that \( W \) is closed. Suppose \( W \) starts at a vertex \( x \). Then \( \tilde{W} \) goes from a vertex of the form \((x, \gamma)\) to \((x, \xi(W)\gamma)\). So \( \tilde{W} \) is closed if and only if \( \xi(W)\gamma = \gamma \), or in other words, if \( W \) has trivial voltage. Summarizing this discussion, we arrive to the following lemma:

\textbf{Lemma 3.1.} Let \( X \) be a premaniplex and let \( \xi : \Pi(X) \rightarrow \Gamma \) be a voltage assignment with a spanning tree \( T \) of trivial voltage on all its darts. Then \( X^\xi \) is a maniplex if and only if

1. The set \( \xi(D) \) generates \( \Gamma \), where \( D \) is the set of darts of \( X \),
2. \( \xi(d) \) has order exactly 2 when \( d \) is a semi-edge,
3. \( \xi(d) \neq \xi(d') \) when \( d \) and \( d' \) are parallel darts, and
4. if \(|i - j| > 1\) every (closed) path \( W \) of length 4 that alternates between the colors \( i \) and \( j \) has trivial voltage.

In Section 4 we shall translate the conditions in Lemma 3.1 to relations and inequalities that the generators of a group \( \Gamma \) need to satisfy to act on a maniplex with given symmetry type graph \( X \). Before doing so, let us now take a closer look at the consequences of condition 4 of the above Lemma.

Condition 4 of Lemma 3.1 invites us to introduce a new concept of homotopy: we say that two paths \( W \) and \( W' \) are \textit{maniplex-homotopic} if we can transform one into the other by a finite sequence of inserting or deleting pairs of inverse darts, as well as switching the colors of two consecutive darts with non-consecutive colors, that is

\[ d_1 d_2 \ldots d_i d_{i+1} \ldots d_k \mapsto d_1 d_2 \ldots d'_i d'_{i+1} \ldots d_k, \]
where, if \( c(d) \) denotes the color of \( d \), we have that \(|c(d_i) - c(d_{i+1})| > 1\), \( c(d'_i) = c(d_i) \), \( I(d'_i) = I(d_i) \) and \( I(d'_{i+1}) = T[d'_i] \). Hence, a voltage assignment \( \xi \) is well defined when applied to the maniplex-homotopy class of paths if and only if it satisfies Condition 4 of Lemma 3.1. From this point on, whenever we speak about homotopy, homotopy class, fundamental groupoid, etc. we will be thinking in terms of maniplex-homotopy.

One could use the group \( \Gamma := (S \mid R) \) as the voltage group where \( S \) has a generator \( \alpha_e \) for each edge \( e \) not in the spanning tree of \( X \) and \( R \) has one element \( \alpha_e^2 \) per each semi-edge \( e \) and one element \( \alpha_{e_1}\alpha_{e_2}\alpha_{e_3}\alpha_{e_4} \), for each path of length four alternating between two non-consecutive colors. In fact every voltage group that gives a maniplex should be a quotient of this group, or in other words \( \alpha \) is the “most general” group we can use as a voltage group to get a maniplex as the derived graph. The ideas of the proof of Theorem 5.2 of [6] show that \( X = \mathcal{T}(U,G) \) where \( U \) is the universal polytope of rank \( n \) and \( G \) is some group. This means that there is some voltage assignment \( \xi \) on \( X \) with voltage group \( G \) such that \( X^\xi \) is isomorphic to the flag graph of \( U \). Because of the universality of \( U \) we get that \( G \) and \( \Gamma \) in fact are the same. In other words, if we use the most general group as our voltage group we will always get the flag graph of the universal polytope as the derived graph.

4. Intersection properties and coset geometries

In order to prove the Main Theorem, we need to characterize, in terms of generators and relations, the groups \( \Gamma \) that act by automorphisms on a polytope \( P \) in such a way that the symmetry type graph \( T(\mathcal{G}(P),\Gamma) \) is isomorphic to a given premaniplex \( X \). We shall do this in order to be able to recover the polytope \( P \) as a coset geometry using the group \( \Gamma \).

We start with the premaniplex \( X \) and provide it with a voltage assignment \( \xi \). Recall that if \((X,\xi)\) is a voltage graph with voltage group \( \Gamma \), then \( X^\xi/\Gamma \) is isomorphic to \( X \); conversely, if \( \mathcal{M}/\Gamma \) is isomorphic to \( X \) then there is a voltage assignment \( \xi \) on \( X \) such that \( X^\xi \) is isomorphic to \( \mathcal{M} \). Hence, by characterizing the voltage assignments \( \xi \) that satisfy that the derived graph \( X^\xi \) is the flag graph of a polytope, we determine the conditions that \( \Gamma \) must satisfy to be the automorphism group of a polytope with symmetry type \( X \).

4.1. Voltage graphs and the path intersection property. We have figured out how to construct a maniplex from a premaniplex via voltage assignments. It is then natural to ask: when is the obtained maniplex the flag graph of a polytope? The answer, as we shall see in this section, is closely related to Theorem 2.3. In fact, translating Theorem 2.3 to the setting of voltage assignments will give us conditions that take the form of intersection properties that certain distinguished subgroups and some left cosets must satisfy. Given two vertices \( x, y \) in \( X \) and a set of colors \( I \in \{0, 1, \ldots, n-1\} \), let us denote by \( \Pi^x,y_I(X) \) the set of (homotopy classes of) paths from \( x \) to \( y \) in \( X \) that only use darts with colors in the set \( I \). So \( \xi(\Pi^x,y_I(X)) \) denotes the set of voltages of all the paths of \( X \) from \( x \) to \( y \) whose edges have colors in \( I \).

Theorem 4.1. Let \( X \) be a premaniplex and let \( \xi : \Pi(X) \to \Gamma \) be a voltage assignment such that \( X^\xi \) is a maniplex. Then \( X^\xi \) is the flag graph of a polytope if and only if

\[
\xi(\Pi^x,y_I(X)) \cap \xi(\Pi^x,y_J(X)) = \xi(\Pi^x,y_{I \cap J}(X)),
\]

for all \( I, J \subset \{0, \ldots, n-1\} \) and all vertices \( x, y \) in \( X \).

Proof Start by assuming that \( X^\xi \) is the flag graph of a polytope. Let \( x \) and \( y \) be vertices of \( X \) and let \( I, J \subset \{0, 1, \ldots, n-1\} \). Consider two paths, \( W \in \Pi^x,y_I(X) \) and \( W' \in \Pi^x,y_J(X) \), with the same voltage, say \( \alpha \in \Gamma \). When lifting \( W \) and \( W' \) in \( X^\xi \), they lift to paths \( \tilde{W} \) and \( \tilde{W}' \), respectively, that go from \((x, 1)\) to \((y, \alpha)\) (here \( 1 \) is the identity element of \( \Gamma \)) and satisfying that \( \tilde{W} \) uses edges with colors
in $I$ while $\tilde{W}'$ uses edges with colors in $J$. In fact, one would define $\tilde{W}$ and $\tilde{W}'$ as the paths that start at $(x, 1)$ and follow the same sequence of colors as $W$ and $W'$ respectively. By Theorem 2.3 $X^\xi$ satisfies the SPIP, which implies that there is a path $\tilde{W}''$ from $(x, 1)$ to $(y, \alpha)$ that uses only colors in $I \cap J$. Then, its projection $W'' := p(\tilde{W}'')$ is a path in $X$ that goes from $x$ to $y$ that uses only colors in $I \cap J$ and has voltage $\alpha$. This proves that $\xi(\Pi_{I,J}^\alpha(X)) \cap \xi(\Pi_{I,J}^\beta(X)) \subset \xi(\Pi_{I,J}^\gamma(X))$. Since the other contention is given, equality (1) must hold.

Now let us assume that equality (1) holds for all $I, J \subset \{0, 1, \ldots, n - 1\}$ and all vertices $x$ and $y$. Let $\tilde{W}$ and $\tilde{W}'$ be paths in $X^\xi$ from a vertex $(x, \gamma)$ to a vertex $(y, \tau)$. Let $I$ and $J$ be the sets of colors of darts of $\tilde{W}$ and $\tilde{W}'$, respectively, and let $W := p(\tilde{W})$ and $W' := p(\tilde{W}')$ be the projections of the paths to $X$. Then, both $W$ and $W'$ go from $x$ to $y$, and $W$ is a path with colors in $I$, while $W'$ is a path with colors in $J$, that is, $W \in \Pi_{I,J}^\alpha(X)$ and $W' \in \Pi_{I,J}^\beta(X)$. Furthermore, since $\tilde{W}$ and $\tilde{W}'$ start at $(x, \gamma)$ and finish at $(y, \tau)$, they both have voltage $\alpha := \tau \gamma^{-1}$. By hypothesis, there exists a path $W'' \in \Pi_{I,J}^\gamma(X)$ that also has voltage $\alpha$. Then $W''$ has a unique lift $\tilde{W}''$ in $X^\xi$ starting at $(x, \gamma)$. The path $\tilde{W}''$ ends in $(y, \tau)$ and it uses darts of colors in $I \cap J$. This proves that $X^\xi$ satisfies the SPIP and therefore, by Theorem 2.3, it is the flag graph of a polytope.

Note that when $x = y$ the set $\xi(\Pi_{I,J}^\alpha(X)) = \xi(\Pi_{I,J}^\beta(X))$ is a group, since it is the image of a group under a groupoid antimorphism. Actually, we shall find a set of distinguished generators for the group $\xi(\Pi_{I,J}^\gamma(X))$ in a similar way as the distinguished generators of the automorphism group of a polytope are found in [2]. Recall that $X_f$ is the subgraph of $X$ induced by the edges with colors in $I$. Let $X_f(x)$ be the connected component of $X_f$ containing the vertex $x$. To find a set of generators of $\xi(\Pi_{I,J}^\gamma(X))$, fix a spanning tree $T_f$ for $X_f(x)$. For each dart $d$ in $X_f(x)$ but not in $T_f$ we get a cycle $C_d$ of the form $WdV$ where $W$ is the unique path contained in $T_f$ from $x$ to the initial vertex of $d$, and $V$ is the unique path contained in $T_f$ from the terminal vertex of $d$ to $x$. Then, the set $\{C_d\}$, where $d$ runs among the darts of $X_f(x)$ not in $T_f$, is a generating set for $\Pi_{I,J}^\gamma(X)$. This implies that $\{\xi(C_d)\}$, where $d$ runs among the darts of $X_f(x)$ not in $T_f$, is a set of generators for $\xi(\Pi_{I,J}^\gamma(X))$.

Since $\xi$ is a voltage assignment, we might consider only one dart $d$ for each edge. By denoting by $W_y$ the unique path contained in $T_f$ from $x$ to $y$, we can see that $\Pi_{I,J}^\gamma(X) = \Pi_{I,J}^\gamma(X) W_y$ (that is, a path from $x$ to $y$ can be written as a closed path starting and finishing at $x$, concatenated with $W_y$), which implies that $\xi(\Pi_{I,J}^\gamma(X)) = \xi(W_y) \xi(\Pi_{I,J}^\gamma(X))$. Therefore, all the intersection properties can be given in terms of left cosets of the groups $\xi(\Pi_{I,J}^\gamma)$, whose generators we already know.

Theorem 4.1 gives an intersection property for each pair of vertices $(x, y)$ and each two sets of colors $I, J \subset \{0, 1, \ldots, n - 1\}$. If we prove an intersection property for the pair $(x, y)$, by taking the inverse on both sides we get the corresponding property for the pair $(y, x)$, so we can consider only unordered pairs $(x, y)$, and this reduces the number of intersection properties to verify by a factor of 2. Still, the total number of intersection properties is quadratic on the number of vertices and exponential on the number of colors. This number gets too big too quickly; however, many of these properties are redundant, either because they are true for any group (for example, the intersection of a group and one of its subgroups is the smaller subgroup) or because they are a consequence of other intersection properties.

Fortunately we may reduce the number of intersection properties to check by following the same proof but using the weak path intersection property instead of the strong one. Doing this we get the following refinement of the previous theorem.
Theorem 4.2. Given a premaniplex \( X \) and a voltage assignment \( \xi \) such that \( X^\xi \) is a maniplex, \( X^\xi \) is the flag graph of a polytope if and only if

\[
\xi(\Pi_{[0,m]}^x \cap \Pi_{[k,n-1]}^y (X)) = \xi(\Pi_{[k,m]}^y (X)),
\]

for all \( k, m \in \{0, \ldots, n-1\} \) and all \( x, y \in \mathcal{F} \).

With Theorem 4.2 the number of intersection properties to check is now quadratic on the number of vertices and also quadratic on the rank. We can still refine this theorem a little more with the following observations:

- The cases \( k = 0 \) and \( m = n-1 \) say that the intersection of the whole voltage group with some subset is the subset itself, so they are trivially true for every voltage assignment.
- In Theorem 4.2 one has to consider the cases when \( k > m \). In such case, \( \xi(\Pi_{[k,m]}^x (X)) \) is the trivial group when \( x = y \) and the empty set when \( x \neq y \). However, if the intersection property holds for \( k = m+1 \), that is, if \( \xi(\Pi_{[0,m]}^x (X)) \cap \xi(\Pi_{[m+1,n-1]}^y (X)) \) holds for all \( k \leq m+1 \), we also need for every vertex to be in a different connected component of \( X \), which implies that its projection is a closed path \( I \). So one may only verify the intersection property for \( k \leq m+1 \).
- If \( y \) and \( y' \) are in the same connected component of \( X_{[k,m]} \) and (2) is satisfied for the pair \( (x,y) \), then it is also satisfied for the pair \( (x,y') \). To see this, let \( W \) be a path from \( y \) to \( y' \) with colors in \([k,m]\) and notice that \( \Pi_{I}^x (X) = \Pi_{I}^{x,y} (X)W \) whenever \( I \) contains \([k,m]\). By taking voltages we get that \( \xi(\Pi_{I}^x (X)) = \xi(W)\xi(\Pi_{I}^{x,y} (X)) \). This means that we can get the intersection property for the pair \( (x,y') \) by multiplying the one for pair \( (x,y) \) by \( \xi(W) \) on the left. So for each pair of numbers \((k,m)\) we only need to verify one intersection property for each pair of connected components of \( X_{[k,m]} \).

Taking the previous observations into consideration, the maximum amount of necessary intersection properties to check is \( v(v+1)/2 \sum_{m=0}^{n-2} (m+1) = v(v+1)n(n-1)/4 \). But to reach this bound we need for every vertex to be in a different connected component of \( X_{[k,m]} \) for every pair \((k,m)\) with \( 0 < k \leq m+1 \), which is only possible when every link has color 0 or \( n-1 \). This implies that every example where this bound is tight has at most 4 vertices (as for \( n \geq 3 \) the colors 0 and \( n-1 \) are not consecutive).

4.2. Constructing a polytope from the voltage group. We have seen how to recover a polytope from its flag graph (see Theorem 2.1) and when \( X^\xi \) is the flag graph of a polytope for a given premaniplex \( X \) and a voltage assignment \( \xi \) (see Theorem 4.2). By concatenating the construction of \( X^\xi \) from \( X \) and \( \xi \), and the construction of a polytope \( \mathcal{P} \) from \( X^\xi \) we get a construction of a polytope from \( X \) and \( \xi \). In this section, we translate this to give a construction only in terms of subgroups of \( \Gamma \) and their cosets. This will give the proof of the Main Theorem.

Let \( f \) be a choice function on the connected subgraphs of \( X \), that is, a function that assigns a base vertex to each such subgraph. Let \( C \) be a connected component of \( X_I \) for some \( I \subset \{0,1,\ldots,n-1\} \) and let \( x = f(C) \). Let \( \overline{C} := (X^\xi)_I(x,1) \), that is, the connected component of \( (X^\xi)_I \) containing \((x,1)\). Thus, if \((x,\gamma) \in \overline{C} \), then there is a path \( W \) from \((x,1)\) to \((x,\gamma)\) which uses only colors in \( I \), which implies that its projection is a closed path \( W \) in \( \Pi_{I}^x (X) \) with voltage \( \gamma \). This means that when considering the action of \( \Gamma \) on \( X^\xi \), the stabilizer of \( \overline{C} \) coincides with \( \xi(\Pi_{I}^x (X)) \). If we now consider a coset \( \xi(\Pi_{I}^x (X))\sigma \), this would be the set of elements of \( \Gamma \) that map \( \overline{C} \) to \((X^\xi)_I(x,\sigma) \).

We know that the \( i \)-faces of the polytope that has \( X^\xi \) as its flag graph correspond to the connected components of \( (X^\xi)_I \). This makes natural the following construction:

Given \( X \) and \( \xi \) satisfying Theorem 4.2 and a choice function \( f \) on the connected subgraphs of \( X \), we will construct a partially ordered set \( \mathcal{P}(X,\xi) \). We will start by defining the underlying set and
a rank function and define the order relation later. Let us define the elements of rank $i$ of $\mathcal{P}(X, \xi)$ as the right cosets of groups of the type $\xi(\Pi^e_i(X))$ where
\[
x \in \{ f(C) | C \text{ is a connected component of } X_i^e \}.
\]
We have to consider these groups as formal copies, that is, if $x$ and $x'$ are on different connected components we consider $\xi(\Pi^e_i(X))$ to be different than $\xi(\Pi^e_i(X))$, even if, as groups, they might be equal. Likewise, if $j \neq i$ but $\xi(\Pi^e_i(X))$ coincides with $\xi(\Pi^e_j(X))$ we consider them to be different in $\mathcal{P}(X, \xi)$. We could formalize this by saying that the elements of rank $i$ in $\mathcal{P}(X, \xi)$ are pairs $(C, \xi(\Pi^e_i(X)))$ with $C$ a connected component of $X^e_i$, $x = f(C)$ and $\gamma \in \Gamma$, but since $x$ and $i$ already appear in the notation, we may assume that $\xi(\Pi^e_i(X))\gamma$ stands for the pair $(X^e_i(x), \xi(\Pi^e_i(X))\gamma)$.

Now we shall define the order on $\mathcal{P}(X, \xi)$ as follows. First, for all $i \in \{0, 1, \ldots, n-1\}$ and every vertex $y$ in $X$ we look at the connected component $X^e_i(y)$ and fix a path going from its base vertex $x = f(X^e_i(y))$ to $y$. We call this path $W^e_i$ and we denote its voltage by $\alpha^e_i := \xi(W^e_i)$.

**Definition 4.3** (Order in $\mathcal{P}(X, \xi)$). Given $0 \leq i < j \leq n-1$, let $C$ be a connected component of $X^e_i$ and $C'$ be a connected component of $X^e_j$, and let $x$ and $x'$ be their respective base flags. Given $\gamma, \gamma' \in \Gamma$, we say that
\[
\xi(\Pi^e_i(X))\gamma < \xi(\Pi^e_j(X))\gamma'
\]
if and only if $\alpha^e_i \xi(\Pi^e_i(X))\gamma \cap \alpha^e_j \xi(\Pi^e_j(X))\gamma' \neq \emptyset$, for some $y \in C \cap C'$.

**Theorem 4.4.** Let $X$ be a premaniplex and $\xi : \Pi(X) \to \Gamma$ a voltage assignment satisfying Theorem 4.2. Let
\[
\mathcal{P}(X, \xi) := \{ \xi(\Pi^e_i(C)) : C \text{ is a connected component of } X^e_i \text{ for some } i \in \{0, n-1\}, x = f(C), \tau \in \Gamma \},
\]
together with the order given in Definition 4.3. Then $\mathcal{P}(X, \xi)$ is a polytope in which $\Gamma$ acts by right multiplication with symmetry type graph $X$.

Note that $\Gamma$ might not be the full automorphism group of $\mathcal{P}(X, \xi)$, and thus $X$ might not be the symmetry type graph of $\mathcal{P}(X, \xi)$ with respect to its full automorphism group.

**Proof** Let us denote $\mathcal{P}(X^e_i)$ as $\mathcal{P}$ and $\mathcal{P}(X, \xi)$ as $\mathcal{Q}$. Note that Theorem 4.2 implies that $\mathcal{P}$ is a polytope, and we want to show that $\mathcal{Q}$ is also a polytope (such that $\mathcal{Q}/\Gamma \cong X$).

By the discussion in Section 3.1, $\mathcal{Q}(\mathcal{P}, \Gamma) \cong X$, since, by construction of $X^e_i$, we have that $X^e_i/\Gamma \cong X$. So in order to settle the theorem it is enough to find a poset isomorphism $\varphi : \mathcal{P} \to \mathcal{Q}$ such that it commutes with the action of $\Gamma$, that is, such that $\widetilde{C}\varphi = \widetilde{C}\varphi\sigma$ for all faces $\widetilde{C}$ in $\mathcal{Q}$ and all $\sigma \in \Gamma$.

Let $\widetilde{C}$ be a face of $\mathcal{P}$. Hence, $\widetilde{C}$ is a connected component of $X^e_i$ for some color $i$. Let $C := p(\widetilde{C})$ and let $x := f(C)$. This implies that $\widetilde{C}$ has a flag $(x, \gamma)$ for some $\gamma \in \Gamma$. Let $\widetilde{K}$ be the connected component of $X^e_i$ that contains $(x, 1)$. Then, the set of elements of $\Gamma$ that map $\widetilde{K}$ to $\widetilde{C}$ is $\xi(\Pi^e_i(X))\gamma$. We want to identify $\widetilde{C}$ with this coset, so we define $\widetilde{C}\varphi := \xi(\Pi^e_i(X))\gamma$. Note that $\varphi : \mathcal{P} \to \mathcal{Q}$ is well defined, since if $(x, \gamma')$ is in $\widetilde{C}$ then $\gamma'\gamma^{-1}$ stabilizes $\widetilde{K}$, implying that $\gamma'\gamma^{-1} \in \xi(\Pi^e_i(X))$. We want to prove that $\varphi$ is a poset isomorphism and that it commutes with $\Gamma$.

Let us show first that $\varphi$ commutes with the action of $\Gamma$. Let $\widetilde{C}$ be a face in $\mathcal{P}$ and let $\sigma \in \Gamma$. By the definition of $\varphi$ we know that $\widetilde{C}\varphi = \xi(\Pi^e_i(X))\gamma$ where $i$ is the rank of $\widetilde{C}$, $x = f(p(\widetilde{C}))$ and $\gamma \in \Gamma$ is any element such that $(x, \gamma) \in \widetilde{C}$. On the other hand $(\widetilde{C}\sigma)\varphi = \xi(\Pi^e_i(X))\gamma'$ where
\(x' = f(p(\tilde{C}\sigma))\) and \((x', \gamma') \in \tilde{C}\sigma\). Also note that since \(\tilde{C}\) and \(\tilde{C}\sigma\) are in the same orbit, then \(p(\tilde{C}) = p(\tilde{C}\sigma)\), and thus \(x = x'\). Furthermore, the flag \((x, \gamma\sigma) = (x, \gamma)\sigma\) is in \(\tilde{C}\sigma\). This proves that \((\tilde{C}\sigma)\varphi = \xi(\Pi'_j(X))\gamma\sigma = (\tilde{C}\varphi)\sigma\).

Now let us prove that \(\varphi\) is an isomorphism of posets. Let \(\tilde{C}\) and \(\tilde{C}'\) be incident faces of \(P\) of ranks \(i\) and \(j\), respectively, with \(i < j\) (therefore, \(\tilde{C} < \tilde{C}'\)). Hence, there is a flag \((y, \tau)\) in \(\tilde{C} \cap \tilde{C}'\), which in turn implies that its first entry, \(y\), must be in \(C \cap C'\) where \(C = p(\tilde{C})\) and \(C' = p(\tilde{C}')\).

Let \(x = f(C)\) and \(x' = f(C')\). Note that the path \(W_i^y\) from \(x\) to \(y\) is contained in \(C\), while \(W_j^y\) from \(x'\) to \(y\) is contained in \(C'\). Then, these paths have lifts \(\tilde{W}_i^y\) and \(\tilde{W}_j^y\) respectively, that go from \((x, (\alpha_i^y)^{-1}\tau)\) and \((x', (\alpha_j^y)^{-1}\tau)\), respectively, to \((y, \tau)\). Observe that \(\tilde{W}_i^y\) is contained in \(\tilde{C}\) and \(\tilde{W}_j^y\) is contained in \(\tilde{C}'\). Thus, \((\alpha_i^y)^{-1}\tau \in \xi(\Pi'_i(X))\gamma\) and \((\alpha_j^y)^{-1}\tau \in \xi(\Pi'_j(X))\gamma'\). Therefore

\[
\tau \in \alpha_i^y \xi(\Pi'_i(X))\gamma \cap \alpha_j^y \xi(\Pi'_j(X))\gamma' = \alpha_i^y (\tilde{C}\varphi) \cap \alpha_j^y (\tilde{C}'\varphi);
\]

but this means that \(\tilde{C}\varphi < \tilde{C}'\varphi\) in \(Q\).

Conversely, suppose that \(\tilde{C}\varphi < \tilde{C}'\varphi\) in \(Q\). We want to show that \(\tilde{C} < \tilde{C}'\) in \(X^\xi\). Let us write \(\tilde{C}\varphi = \xi(\Pi'_i(X))\gamma\), where \(i\) is the rank of \(\tilde{C}\), \(x := f(p(\tilde{C}))\) and \(\gamma\) is an element of the voltage group such that \((x, \gamma) \in \tilde{C}\). Similarly, we write \(\tilde{C}'\varphi = \xi(\Pi'_j(X))\gamma'\), where \(j\) is the rank of \(\tilde{C}'\), \(x := f(p(\tilde{C}'))\) and \(\gamma'\) is an element of the voltage group such that \((x', \gamma') \in \tilde{C}'\).

By hypothesis \(\alpha_i^y \xi(\Pi'_i(X))\gamma\) and \(\alpha_j^y \xi(\Pi'_j(X))\gamma'\) have non-empty intersection for some \(y \in C \cap C'\). Let \(\tau\) be an element in such intersection. Then \((\alpha_i^y)^{-1}\tau \in \xi(\Pi'_i(X))\gamma\). This implies that \((x, (\alpha_i^y)^{-1}\tau)\) is in the same connected component of \((X^\xi)^p\) as \((x, \gamma)\), that is \((x, (\alpha_i^y)^{-1}\tau) \in \tilde{C}\). But at the same time, there is a lift of \(W_i^y\) that connects \((x, (\alpha_i^y)^{-1}\tau)\) with \((y, \tau)\), and since \(W_i^y\) does not use the color \(i\), its lift is contained in \(\tilde{C}\), which proves that \((y, \tau) \in \tilde{C}\). Similarly, the fact that \((\alpha_j^y)^{-1}\tau \in \xi(\Pi'_j(X))\gamma'\) implies that \((y, \tau) \in \tilde{C}'\). Thus, we have proved that \(\tilde{C} \cap \tilde{C}'\) is not empty, or in other words \(\tilde{C} < \tilde{C}'\) in \(P\).

Therefore, \(\varphi\) is an isomorphism and the theorem follows.

\[\square\]

5. Example: Caterpillars

If one wants to build polytopes from premaniplexes in the way we have described in this paper, it is natural to start with infinite families of premaniplexes. One could, of course, then start with premaniplexes with a fixed number \(k\) of vertices. For example, one may want to construct 2-orbit polytopes (see [9]). If, on the other hand, we do not want to limit the number of vertices of the premaniplexes in the family a first step could be to consider trees. However, since semi-edges are considered as cycles, there are no premaniplexes of rank \(n \geq 3\) whose underlying graph (without colors) is a tree. Thus, we study the closest thing to them: premaniplexes that are trees with an unlimited number of semi-edges. Hence, we define a caterpillar as a premaniplex in which every cycle is a semi-edge. In other words, a caterpillar is a premaniplex \(X\) with a unique spanning tree.

In particular, a caterpillar does not have pairs of parallel links (edges joining different vertices). If there are three links incident to one vertex, at least two of them must have colors differing by more than 1, which would imply that there is a 4-cycle. This implies that caterpillars consist in fact of a single path \(W\) (which we will call the underlying path of \(X\)) and lots of semi-edges. Of course, the colors of two consecutive edges on the path must differ by exactly one, otherwise there would be
a 4-cycle. We note here that the term *caterpillar* has been used in the graph theory literature for a very similar but slightly different concept.

Throughout this section, unless otherwise stated, $X$ will denote a finite caterpillar (that is, one with a finite number of vertices) with underlying path $W$, and its vertices will be labeled by $x_0, x_1, \ldots, x_k$, ordered as they are visited by $W$. Furthermore, we denote by $(x_i, j)$ the dart of color $j$ at $x_i$ and by $c_i$ the color of the link connecting $x_{i-1}$ and $x_i$.

### 5.1. Caterpillar coverings

We want to construct polytopes from caterpillars. As often when constructing manifolds and polytopes via voltage assignments from a premanifold $X$, the derived maniplex might not have $X$ as the symmetry type graph with respect to the full automorphism group. However, the actual symmetry type graph is a quotient of $X$. For this reason, in this section we study quotients of caterpillars.

Given a caterpillar, let us call an *endpoint* a vertex incident to just one link (that is, an endpoint of the underlying path). Note that a caterpillar is finite if and only if it has exactly two endpoints.

Every symmetry of the caterpillar must map endpoints to endpoints. If a caterpillar is finite there is at most one non-trivial symmetry and its action on the vertices $x_0, x_1, \ldots, x_k$ is given by $x_j \mapsto x_{k-j}$.

We call a finite caterpillar *symmetric* if it has a non-trivial symmetry.

A word $w$ in $\{0, 1, \ldots, n-1\}$ is simply a finite sequence $w = a_1a_2\ldots a_t$, with $a_i \in \{0, 1, \ldots, n-1\}$ for each $i = 1, 2, \ldots, t$. The *inverse* of a word $w$ is the word $w^{-1}$ that has the same colors as $w$ but written in reverse order. That is, if $w = a_1a_2\ldots a_t$ then $w^{-1} = a_ta_{t-1}\ldots a_1$. A word is said to be *reduced* if it has no occurrence of the same color twice in a row; in other words, $w = a_1a_2\ldots a_t$ is reduced if and only if $a_{i+1} \neq a_i$ for all $i = 1, 2, \ldots, t-1$. We shall work with reduced words from now on. A word $w = a_1a_2\ldots a_t$ is a *palindrome* if $a_i = a_{t+1-i}$ for all $i \in \{1, 2, \ldots, t\}$. A palindrome of even length can be written as $vw^{-1}$ for some word $v$ and cannot be reduced. A palindrome of odd length can always be written as $w = vav^{-1}$ for some color $a$ and some word $v$.

Given a segment $[x, y]$ in a caterpillar, its *underlying word* is the word $w$ consisting of the colors of the links in the path that goes from $x$ to $y$. When we speak of the underlying word of a caterpillar $X$ we are referring to the underlying word of its underlying path in a fixed orientation. Hence, we say that a segment $[x, y]$ is a *palindrome* if its underlying word $v$ is a palindrome.

**Proposition 5.1.** Let $X$ be a finite caterpillar and let $Y$ be a premanifold not isomorphic to $X$ such that there is a premanifold homomorphism $h: X \to Y$. Then $Y$ is a caterpillar. Moreover, if $Y$ has at least 2 vertices and $S = c_1c_2\ldots c_t$ is the underlying word of $X$, then there is some $r < k$ such that $w = c_1c_2\ldots c_r$ is the underlying word of $Y$ and one of the following statements is true:

1. There exist colors $a_1, a_2, \ldots, a_t \in \{c_r + 1, c_r - 1\}$ and $b_1, b_2, \ldots, b_{t-1} \in \{c_1 + 1, c_1 - 1\}$ such that $S = waww^{-1}bww^{-1}b_2w^{-1}b_{t-1}waw^{-1}$.

2. There exist colors $a_1, a_2, \ldots, a_t \in \{c_r + 1, c_r - 1\}$ and $b_1, b_2, \ldots, b_t \in \{c_1 + 1, c_1 - 1\}$ such that $S = waww^{-1}bww^{-1}b_2w^{-1}b_{t-1}waw^{-1}bw$.

In any case, if $i \equiv j \mod 2r + 2$ then $h(x_i) = h(x_j)$. Also if $i \equiv -j - 1 \mod 2r + 2$ then $h(x_i) = h(x_j)$. 

---

**Figure 2.** A finite caterpillar.
Before proving Proposition 5.1 let us remark that it simply means that the quotients of a caterpillar $X$ are those caterpillars $Y$ such that $X$ can be “folded” into $Y$. We illustrate this concept in Figure 3: the semi-edges are not drawn and the names of the vertices have been omitted, but the idea is that $X$ must be “folded” into “layers” of $r + 1$ vertices and then each vertex will be projected to the vertex on $Y$ in the same horizontal coordinate. The layer $\ell$ consists of the vertices $x_i$ where $\left\lfloor \frac{i}{r + 1} \right\rfloor = \ell$ (\(\lfloor x \rfloor\) denotes the integer part of $x$). Even layers go from left to right, while odd layers go from right to left, hence the underlying word of even layers is $w = c_1c_2\ldots c_r$ while the underlying word of odd layers is $w^{-1} = c_rc_{r-1}\ldots c_1$.

Now let us proceed with the proof.

**Proof of Proposition 5.1**

Let us consider the equivalence relation $\sim_h$ on $X$ given by $x \sim_h y$ if and only if $h(x) = h(y)$. Recall that by the definition of a premaniplex homomorphism $x \sim_h y$ implies that $x^m \sim_h y^m$ for every monodromy $m$. Therefore $Y$ is isomorphic to $X/\sim_h$.

We already know that all premaniplex homomorphisms are surjective (see Section 2). By hypothesis $h$ is not an isomorphism, which implies it cannot be injective. Let $x$ and $y$ be two different vertices on $X$ such that $x \sim_h y$. Let $x_0, x_1, \ldots, x_k$ be the sequence of vertices in the underlying path of $X$. There is a monodromy $m$ such that $x^m = x_0$, and so $x_0 \sim_h y^m$ and $y^m$ is different than $x_0$. Now let $q$ be the minimum positive number such that $x_q \sim_h x_0$. We know that $q$ exists because $x_0 \sim_h y^m$. Note that if $q = 1$ we would have that $x_0^q \sim_h x_0$ for all $i$, which would imply that the same is true for $x_1$ and in turn also for $x_2 = x_1^q$ and so on. This would mean that all the vertices of $X$ are equivalent, meaning that $Y$ has only one vertex and the proposition follows. So we may assume that $q > 1$.

We know that $x_{q-1} = x_q^q$ is equivalent $x_0^q$. If $c_q \neq c_1$, we would have that $x_0^q = x_0$. This would imply that $x_{q-1}$ is equivalent to $x_0$, contradicting the minimality of $q$. So we have proven that $c_q = c_1$. 

Figure 3. The caterpillar $X$ covers the caterpillar $Y$ if and only if it “folds” into it.
Now note that if for some $\ell$ we have that $x_{\ell} \sim_h x_1$, then $x_{\ell}^{c_1} \sim_h x_1^{c_1} = x_0$. In particular this tells us that for $\ell < q - 1$, $x_{\ell}$ cannot be equivalent to $x_1$. Now we can use the same argument we used to prove that $c_0 = c_1$ to prove that $c_{q-1} = c_2$. Analogously we can prove that $c_3 = c_0 - 2, c_4 = c_0 - 3$ and so on. In other words, $[x_0, x_q]$ is a palindrome. Since for all $i$, $c_i$ and $c_{i+1}$ are different, then $q$ is odd, say $q = 2r + 1$. Let $v$ be the underlying word of the segment $[x_0, x_q]$. Then $v$ may be written as $v = w_{r+1}^{-1}w$ for some word $w = c_1c_2\ldots c_r$. Let us call $a_1 := c_{r+1}$ and note that $a_1 \in \{c_r + 1, c_r - 1\}$.

For all $i = 0, 1, \ldots, k$, let us denote by $\hat{i}$ the residue of dividing $i$ by $2r + 2$. We will prove by induction on $i$ that $x_{\hat{i}} \sim_h x_{\hat{i}}^r$ for all $i = 0, 1, \ldots, k$ and that $c_i = c_i^r$ when $i$ is not divisible by $r + 1$, $c_i \in \{c_1 + 1, c_1 - 1\}$ if $i$ is an even multiple of $r + 1$ and $c_i = \{c_r + 1, c_r - 1\}$ when $i$ is an odd multiple of $r + 1$.

Let our induction hypothesis be that $x_{\ell} \sim_h x_{\ell}^r$ for all $\ell < i$, and that $c_\ell = c_\ell^r$ if $\ell$ is not divisible by $r + 1$.

Let us start with the case when $i = 0 \mod 2r + 2$. In this case we want to prove that $x_1 \sim_h x_1^r$ and that $c_1 \in \{c_1 + 1, c_1 - 1\}$. By our induction hypothesis we know that $x_{i-1} \sim_h x_{2r+1} \sim_h x_0$ and $c_{r-1} = c_{2r+1} = c_1$.

In particular $c_i \in \{c_1 + 1, c_1 - 1\}$. This implies that $x_i = x_{i-1}^{c_i} \sim_h x_0 = x_1^r$ (since $c_i \neq c_1$). Thus $i$ satisfies our claim.

Now let us proceed with the case when $i$ is an odd multiple of $r + 1$, that is $\hat{i} = r + 1$. In this case we want to prove that $x_{\hat{i}} \sim_h x_{\hat{i}}^r$ and that $c_{\hat{i}} \in \{c_r + 1, c_r - 1\}$. Our induction hypothesis tells us that $x_{i-1} \sim_h x_r$ and that $c_{i-1} = c_r$. Hence $c_{\hat{i}} \in \{c_r + 1, c_r - 1\}$. Note that one of the colors in $\{c_{r-1} + 1, c_{r-1} - 1\}$ is actually $c_{r+1}$, while the other is the color of a semi-edge incident to $x_r$. Since $x_{r+1} \sim_h x_r$ we have that $x_i = x_{i-1}^{c_i} \sim_h x_r \sim_h x_{r+1} = x_{\hat{i}}$. Thus $i$ satisfies our claim.

Finally let us prove our claim for the case when $i$ is not divisible by $r + 1$. Our induction hypothesis tells us that $x_{i-1} \sim_h x_{i-1}^{-1}$. Note that $\hat{i} = i - 1 + 1$. Since $i$ is not a multiple of $r + 1$ we know that $x_{\hat{i}} = x_{i-1}^{c_i^{-1}}$ is not equivalent to $x_{i-1}^{c_i^{-1}}$. This implies that $x_{\hat{i}}^{-1} = x_{i-1}^{c_i^{-1}}$ is not equivalent to $x_{i-1}^{-1}$, and since it is adjacent to $x_{i-1}$ it must be equal to either $x_i$ or $x_{i-2}$. If $i = 1 \mod 2r + 2$ then, by induction hypothesis, $x_{i-1} \sim_h x_0$, but recall that $x_0 \sim_h x_{2r+1} = x_q$ by definition, and by induction hypothesis $x_{i-2} \sim_h x_{i-2} = x_{2r+1}$. This means that $x_{i-2} \sim_h x_{i-1}$, so $x_{i-1}^r$ must be $x_i$, implying that $c_i = c_r$. Similarly, if $i \neq 1 \mod r + 1$ then our induction hypothesis tells us that $c_{i-1} = c_i^{-1} \neq c_i$, and since $x_{i-2} = x_{i-1}^{c_i^{-1}} = x_{i-1}^{-1}$ the only possibility is that $x_{i-1}^r = x_i$, implying that $c_i = c_i^r$ and that $x_{i-1} \sim_h x_{i-1}$. Thus, $i$ satisfies our claim. Note that we have also proven that if $i$ is not divisible by $r + 1$ then $x_{i-1}$ cannot be an endpoint of $X$.

We have proved that $x_i \sim_h x_{\hat{i}}$ for all $i = 0, 1, \ldots, k$. This implies automatically that if $i = j \mod 2r + 2$ then $x_i \sim_h x_j$. Moreover, if $i = -j - 1 \mod 2r + 2$, then $\hat{i} = 2r + 1 - \hat{j}$. Now since $v$ is a palindrome, we know that $x_\ell \sim_h x_{2r+1-\ell}$ for all $\ell = 0, 1, \ldots, 2r + 1$, in particular $x_{\hat{i}} = x_{2r+1-\hat{i}} \sim_h x_{\hat{j}}$. This, together with the fact that $x_i \sim_h x_{\hat{j}}$ and $x_j \sim_h x_{\hat{j}}$, implies that $x_i \sim_h x_j$.

We have already proved that $S = w_{a_1}^{-1}w_{b_1}w_{a_2}w^{-1}\ldots$ and it ends after an occurrence of $w$ or $w^{-1}$. To end the proof note that since $h$ is surjective we already know exactly what premaniplex $Y$ is: It is a caterpillar with vertex sequence $h(x_0), h(x_1), \ldots, h(x_r)$. In fact, for $i = 1, \ldots, r - 1$ and $j = 0, 1, \ldots, n - 1$ we know that $h(x_j)$ is different from $h(x_i)^j = h(x_i^j)$ if and only if $j \in \{c_{i-1}, c_i\}$; and if $i = 0$ (resp. $r$) then $h(x_i)$ is different from $h(x_i)^j$ if and only if $j = c_1$ (resp. $c_r$). □
If we look closely at the proof of Proposition 5.1 we notice that we have not actually used the fact that $X$ is finite, but only the fact that it has at least one endpoint. So the proposition may be generalized as follows:

**Proposition 5.2.** Let $X$ be a caterpillar with at least one endpoint and let $Y$ be a premaniplex such that there is a premaniplex homomorphism (or covering) $h : X \rightarrow Y$. Then $Y$ is a caterpillar. Moreover, if $Y$ has at least two vertices and $S = c_1c_2 \ldots$ is the color sequence of $X$ starting at its endpoint, then there is some $r$ such that $w = c_1c_2 \ldots c_r$ is the underlying word of $Y$ and there exist colors $a_1, a_2, \ldots \in \{c_r + 1, c_r - 1\}$ and $b_1, b_2, \ldots \in \{c_1 + 1, c_1 - 1\}$ such that $S = w a_1 w^{-1} b_1 w a_2 w^{-1} b_2 \ldots$.

If $X$ is finite it ends after an occurrence of either $w$ or $w^{-1}$. If $i \equiv j \mod 2r + 2$ then $h(x_i) = h(x_j)$. If $i \equiv j$ or $-j - 1 \mod 2r + 2$, then $h(x_i) = h(x_j)$.

Let us look now at the degenerate cases. If $Y$ is only one vertex we can think that the previous theorems still hold but with $w$ being the empty word, and since $c_1$ and $c_r$ do not exist, we get rid of the restrictions $a_i \in \{c_r + 1, c_r - 1\}$ and $b_i \in \{c_1 + 1, c_1 - 1\}$. In Figure 3 each layer of $X$ would have only one vertex and no links; essentially $X$ would be “standing up” instead of being folded. If $Y$ is isomorphic to $X$ then $S = w$ and we do not have any $a_i$ or $b_i$. In Figure 3 $X$ would have only one layer.

If one would try to prove an analogous result as Proposition 5.2 for caterpillars with no ends, there would be two additional cases to consider: One would be when $Y$ is a caterpillar with one endpoint and an underlying sequence of colors $S = c_1c_2c_3 \ldots$. In this case, the underlying infinite word of $X$ would be $S^{-1}aS$ for some color $a$, where $S^{-1} = \ldots c_3c_2c_1$. The other case would be that when $Y$ is not a caterpillar but a premaniplex consisting of a cycle with underlying word $w$ and all the edges not in the cycle are semi-edges. In this case $X$ would have the underlying infinite word $\ldots w w w \ldots$.

5.2. Caterpillars as STG of polytopes. Given a caterpillar $X$, we want to assign voltages to the semi-edges of $X$ in order to get the flag graph of a polytope as the derived maniplex. To this end, we note that in caterpillars the conditions of Theorems 4.1 and 4.2 can be simplified, as stated in the following lemma:

**Lemma 5.3.** Let $X$ be a caterpillar and let $\xi : \Pi(X) \rightarrow \Gamma$ be a voltage assignment such that all the darts in the underlying path of $X$ have trivial voltage. Then the following statements are equivalent.

1. $X^\xi$ is polytopal.
2. For every vertex $x$ in $X$ and all sets $I, J \subset \{0, 1, \ldots, n - 1\}$ the equation
   \[
   \xi(\Pi^x_I(X)) \cap \xi(\Pi^x_J(X)) = \xi(\Pi^x_{I\cup J}(X))
   \]
   holds.
3. For every vertex $x$ in $X$ and all $k, m \in \{0, 1, \ldots, n - 1\}$ the equation
   \[
   \xi(\Pi^x_{[0, m]}(X)) \cap \xi(\Pi^x_{[k, n-1]}(X)) = \xi(\Pi^x_{[k, m]}(X))
   \]
   holds.

**Proof** To prove this we have to see that for every set of colors $I \subset \{0, 1, \ldots, n - 1\}$ the set $\xi(\Pi^x_{I^y}(X))$ is either $\xi(\Pi^x_I(X))$ or empty, depending on whether or not the segment of $W$ that goes from $x$ to $y$ (which we will denote $[x, y]$) uses only colors in $I$. In fact if $[x, y]$ uses only colors in $I$ then
\[
\xi(\Pi^x_{I^y}(X)) = \xi(\Pi^x_I(X)[x, y]) = \xi([x, y]) \xi(\Pi^x_I(X)) = 1 \cdot \xi(\Pi^x_I(X)) = \xi(\Pi^x_I(X)).
\]
Note also that \( \xi(\Pi^{x,y}_I(X)) \cap \xi(\Pi^{x,y}_J(X)) \) is empty if and only if one of the factors is empty. These observations together with Theorems 4.1 prove the equivalence between conditions 1 and 2, and with Theorem 4.2 we prove the equivalence between conditions 1 and 3, thus proving the lemma.

Recall that a Boolean group (or an elementary Abelian 2-group) is a group in which every non-trivial element has order exactly 2. Thus, all Boolean groups are Abelian, and finitely generated Boolean groups are isomorphic to a direct product of cyclic groups of order 2.

**Proposition 5.4.** Every caterpillar is the quotient of the flag graph of a polytope by a Boolean group.

**Proof** Let \( X \) be a caterpillar. We shall find a Boolean group \( B \) and a voltage assignment \( \xi : \Pi(X) \to B \) such that \( X^\xi \) is polytopal in the following way.

First assign trivial voltage to all links of \( W \) and a different independent voltage to each semi-edge incident to \( x_0 \). We shall give voltage assignments to the semi-edges \( (x_i, j) \), recursively on \( i \). If \( |j - c_i| > 1 \) we assign the same voltage to \( (x_i, j) \) as we did to \( (x_{i-1}, j) \). On the other hand, if \( |j - c_i| = 1 \), we assign a new element as the voltage \( (x_i, j) \) independent from all the voltages of previous darts.

Let us call the resulting voltage group \( B \) and denote this voltage assignment by \( \xi \). By Lemma 3.1, \( X^\xi \) is a maniexip.

Note further that \( \xi \) satisfies that if \( i \in \{1, 2, \ldots, k\}; r, s \in \{0, 1, \ldots, n - 1\} \) and \( (x_{i-1}, r) \) and \( (x_i, s) \) are semi-edges, then \( \xi(x_i, s) = \xi(x_{i-1}, r) \) if and only if \( r = s \) and \( |r - c_i| \neq 1 \). It also satisfies that if \( i < \ell < j \) and \( (x_i, r) \) and \( (x_j, r) \) are semi-edges such that \( \xi(x_i, r) = \xi(x_j, r) = \gamma \), then \( (x_\ell, r) \) is also a semi-edge and \( \xi(x_\ell, r) = \gamma \).

We claim that the second statement of Lemma 5.3 is satisfied, and hence, \( X^\xi \) is the flag graph of a polytope. To show that this is the case, start by noticing that \( \xi(\Pi^**_I) \) is the group generated by the voltages of the semi-edges on the component \( X_I(x) \).

Given \( I, J \subseteq \{0, 1, \ldots, n - 1\} \), suppose that for some vertex \( x \) there is a semi-edge \( e \) in \( X_J(x) \) and a semi-edge \( e' \) in \( X_J(x) \) with \( \xi(e) = \xi(e') = \gamma \) for some \( \gamma \in \Gamma \). If \( e = e' \) then \( e \in X_{I\cap J}(x) \).

Now let \( \sigma \in \xi(\Pi^**_I(X)) \cap \xi(\Pi^**_J(X)) \) be arbitrary. Since the group \( B \) is Boolean, \( \sigma \) may be written as \( \sigma = \gamma_1 \gamma_2 \ldots \gamma_n \) where the elements \( \gamma_1, \gamma_2, \ldots, \gamma_n \) are different generators of \( B \), and this decomposition is unique up to reordering of the factors. Since \( \sigma \in \xi(\Pi^**_I(X)) \), and because the voltage of a semi-edge is always a generator, each \( \gamma_i \) is also in \( \xi(\Pi^**_I(X)) \), and since \( \sigma \in \xi(\Pi^**_J(X)) \) each \( \gamma_i \) is also in \( \xi(\Pi^**_J(X)) \). But, this implies that each \( \gamma_i \) is in \( \xi(\Pi^**_{I\cap J}(X)) \), implying that \( \sigma \in \xi(\Pi^**_{I\cap J}(X)) \). Therefore, \( \xi(\Pi^**_I) \cap \xi(\Pi^**_J) = \xi(\Pi^**_{I\cap J}) \).

The flag graphs of the polytopes constructed in the proof of Proposition 5.4 have some 4-cycles consisting of edges of consecutive colors. For this reason we say that they are degenerated.

Proposition 5.4 is still true for infinite caterpillars. Even if our algorithmic way of assigning voltages may not be doable, the voltage assignment is still well defined as a quotient of the Boolean group with one generator assigned to each semi-edge.
Note that Proposition 5.4 says that every caterpillar \( X \) is a symmetry type graph of a polytope \( P \) with respect to some group \( B \leq \Gamma(P) \). However, \( X \) is not necessarily the symmetry type graph of \( P \) with respect to the full automorphism group. That is, \( X^\xi \) might have “extra” symmetry. Thus, we would want to investigate what could be the symmetry type of \( X^\xi \) with respect to its full automorphism group.

Let \( X \) be a finite caterpillar and let \( \xi : \Pi(X) \to B \) be the voltage assignment constructed in the proof of Proposition 5.4. If \( X \) is symmetric, its non-trivial symmetry induces an automorphism of \( B \) which is just a reordering of the generators, moreover, if the generators are given the natural order, it induces the reverse order on the generators. Using [10, Theorem 7.1], we get that this symmetry induces a symmetry of the derived maniplex, so in this case the original caterpillar is not the symmetry type of the derived polytope. But we will see in Theorem 5.5 that if \( X \) is not symmetric, we can almost be certain that the caterpillar is in fact the symmetry type graph of the derived polytope by its full automorphism group. In this case, by “almost” we mean that if this is not the case, the caterpillar must have a very specific structure.

**Theorem 5.5.** Let \( X \) be a finite caterpillar of length \( k \) and rank \( n \). Let \( S = c_1c_2\ldots c_k \) be the underlying word of \( X \). Then at least one of the following statements is true:

1. \( X \) is symmetric.
2. \( X \) is the STG of a polytope with a Boolean automorphism group.
3. \( c_1 \) is in \( \{1,n-2\} \) and there exist \( r \in \{1,2,\ldots,k-1\} \), and \( a_1,a_2,\ldots,a_t \in \{0,1,\ldots,n-1\} \) where \( t = (k+1)/(2r+2) \), such that \( S = wa_1w^{-1}bwaw^{-1}b\ldots bwaw^{-1}b \) where \( w = c_1c_2\ldots c_r \) and \( b = 0 \) if \( c_1 = 1 \) and \( b = n-1 \) if \( c_1 = n-2 \).
4. There exist \( r \in \{1,2,\ldots,k-1\} \) and \( a,b \in \{0,n-1\} \) such that
   \[
   S = waw^{-1}bwaw^{-1}b\ldots bwaw^{-1}bw,
   \]
   where \( w = c_1c_2\ldots c_r \). Also \( (c_1,b), (c_r,a) \in \{(1,0), (n-2,n-1)\} \).
Proof: Suppose that $X$ is not symmetric and that it is not the STG of a polytope with a Boolean automorphism group.

Consider the voltage assignment $\xi : \Pi(X) \to B$ previously discussed. We say that two vertices $x$ and $y$ of $X$ are equivalent ($x \sim y$) if there exist (or equivalently, for all) $\sigma, \tau \in B$ such that the flags $(x, \sigma)$ and $(y, \tau)$ of $X^\xi$ are in the same orbit under the action of the automorphism group of $X^\xi$. Then $\sim$ is an equivalence relation preserved by $i$-adjacency, that is $x \sim y \Rightarrow x^i \sim y^i$. Moreover, the natural function $h : X \to X/\sim$ is a premaniplex homomorphism, so by Proposition 5.1 there exists some $r$ such that $S$ can be written as $w_1 b_1 w_2 b_2 \ldots$ ending after an occurrence of either $w$ or $w^{-1}$, where $w = c_1 c_2 \ldots c_r$, $a_i \in \{c_r + 1, c_r - 1\}$ and $b_i \in \{c_1 + 1, c_1 - 1\}$ (see Figure 3). In fact, $X/\sim$ is isomorphic to the actual symmetry type graph of $X^\xi$ (mapping the orbit of $(x, \gamma)$ to the equivalence class of $x$ is an isomorphism).

Note that since $X$ is not symmetric nor the STG of $X^\xi$ with respect to $B$, at least $b_1$ exists. Let $j \in \{0, 1, \ldots, k - 1\}$ be a number such that the segment $[x_0, x_{j+1}]$ has the underlying word $w a_1 w^{-1} b_1 w a_2 w^{-1} b_2 \ldots w^{-1} b_i$ for some $i$. We know in particular that $b_i$ differs from $c_1$ in exactly 1. We want to prove that $(c_1, b_i) \in \{(1, 0), (n - 2, n - 1)\}$. We will assume this is not the case and arrive to a contradiction.

Let $q$ be the other color that differs from $b_i$ in exactly 1 (that is $q = 2b_i - c_1$). Since $(c_1, b_i) \neq (1, 0), (n - 2, n - 1)$ we know that $q \in \{0, 1, \ldots, n - 1\}$, and thus is the color of some edges of $X$. So there are semi-edges $e, e'$ incident to $x_0$ of colors $q$ and $b_i$ respectively. Let $\alpha := \xi(e)$ and $\beta := \xi(e')$. The voltage of the closed path $e e' e'$ is $(\beta \alpha)^2 = 1$ because $B$ is Boolean. This means that its lift, the path of length 4 that starts at $(x, 1)$ in $X^\xi$ and alternates colors between $q$ and $b_i$ must be closed.

By Theorem 5.1, we know that $c_{j+1} = c_1 \neq q$, so we know that the darts $(x_{j+1}, q)$ and $(x_j, q)$ are semi-edges. Let $\kappa := \xi(x_j, q)$ and $\lambda := \xi(x_{j+1}, q)$. The path of length 4 that alternates colors between $r$ and $b_i$ starts at $x$ is closed, and its voltage is $\lambda \kappa$. Note that since $|q - b_i| = 1$ the definition of $\xi$ tells us that $\xi(x_j, q) \neq \xi(x_{j+1}, q)$, that is $\lambda \neq \kappa$, which implies $\lambda \kappa \neq 1$. This means that the path of length 4 in $X^\xi$ starting at $(x_j, 1)$ and alternating colors between $r$ and $b_i$ is not closed (it ends at $(x_j, \lambda \kappa)$).

We see that the path of length 4 in $X^\xi$ starting at $(x_j, 1)$ and alternating colors between $r$ and $c_{j+1}$ is not closed, but the one starting at $(x_0, 1)$ is. This contradicts the fact that $(x_j, 1)$ and $(x_0, 1)$ are on the same orbit. The contradiction comes from the fact that there are edges of color $q = 2b_i - c_1 \in \{0, 1, \ldots, n - 1\}$, so to avoid this we must have that $(c_1, b_i) \in \{(1, 0), (n - 2, n - 1)\}$. Since $c_1$ is fixed, every $b_i$ must be the same.
If the underlying word of $X$ ends after an occurrence of $w$ we may look at $X$ in the other direction. Then the previous result tells us that every $a_i$ is equal to some $a$ and that $(c_r,a) \in \{(1,0),(n-2,n-1)\}$.

Remark 5.6. If the third or fourth condition is the one holding in Theorem 5.5, the actual STG of $X^\xi$ might be the finite caterpillar with underlying word $w$.

By doing exactly the same proof, we obtain the following analogous result for infinite caterpillars with one end-point (this would look like a ray or half-straight line):

**Theorem 5.7.** Let $X$ be an infinite caterpillar with one end-point. Let $S$ be the sequence of colors of the underlying path of $X$ starting at its end-point. Then one of the following statements is true:

1. $X$ is the STG of a polytope with a Boolean automorphism group.
2. There exist some number $r$ and colors $b,a_1,a_2,\ldots \in \{0,1,\ldots,n-1\}$ such that $S = wa_1w^{-1}bwa_2w^{-1}bwa_3w^{-1}\ldots$ where $w = c_1c_2\ldots c_r$ and $(c_1,b) \in \{(1,0),(n-2,n-1)\}$.

Remark 5.8. If the second condition is the one holding, the actual STG of $X^\xi$ might be the finite caterpillar with underlying word $w$.

If $k \geq 3$, it is easy to construct a caterpillar of length $k-1$ (that is, with $k$ vertices and $k-1$ links) that does not satisfy properties (1), (3) and (4) (simply let $c_1 = 0$ and then avoid the color 0 for every other $c_i$). Hence, we get the following corollary:

**Corollary 5.9.** For every $n,k \geq 3$, there is an abstract $n$-polytope with Boolean automorphism group that has $k$ flag orbits.

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All polytopes are coset geometries: characterizing automorphism groups of $k$-orbit abstract polytopes

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Abstract

Abstract polytopes generalize the classical notion of convex polytopes to more general combinatorial structures. The most studied ones are regular and chiral polytopes, as it is well-know they can be constructed as coset geometries

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from their automorphism groups. This is also known to be true for 2- and 3- orbit 3-polytopes. In this paper we show that every abstract n-polytope can be constructed as a coset geometry. This construction is done by giving a characterization, in terms of generators, relations and intersection conditions, of the automorphism group of a $k$-orbit polytope with given symmetry type graph. Furthermore, we use these results to show that for all $k \neq 2$, there exist $k$-orbit $n$-polytopes with Boolean groups (elementary abelian 2-groups) as automorphism group, for all $n \geq 3$.

1 Introduction

In the 1970s several ideas extend the geometric study of convex polytopes to generalize them from different points of view: while Tits studied incidence systems, Coxeter focused on tessellations of manifolds and Grünbaum proposed to move away from spherical tiles. In the early 1980s Danzer and Schulte put several of these ideas together to start the study of incidence polytopes, now called abstract polytopes. An abstract polytope is a ranked, partially ordered set which generalizes the face lattice of convex polytopes and tessellations. Thus, abstract polytopes generalize the classical notion of convex polytopes and tessellations to more general combinatorial structures.

The degree of symmetry of an abstract polytope is measured by counting the number of flag orbits under the action of its automorphism group, where a flag is a maximal chain of the partial order. Abstract polytopes can then be classified in terms of their so-called symmetry type graph ([2]), which encodes all the information of the local configuration of flags with respect to the automorphism group.

Polytopes with only one flag-orbit are called regular and are the most studied
ones. Regular polytopes have maximal degree of symmetry and in particular
their groups are generated by involutions, often called “(abstract) reflections”.
The book [11] is the standard reference and it is devoted exclusively to the study
of abstract regular polytopes. There are $2^n - 1$ classes of $n$-polytopes with 2 flag
orbits, each of them corresponding to one symmetry type graph. Among them
is the class of chiral polytopes ([14]), which have no “reflectional” symmetry,
but have maximal “rotational” one.

The study of coset geometries goes back to Tits ([15]). The ideas behind
this concept are to construct incidence structures using groups, and in particular
were developed by Tits in connection to Coxeter groups. It is well-known that
regular and chiral polytopes, as well as two-orbit polyhedra can be seen as coset
geometries (see [3], [14] and [7]); this characterization of their automorphism
groups constitute the most important tool to study them. In a recent paper
([8]), we showed that also 3-orbit polyhedra can be seen as cosets and used this
to construct 3-orbit polyhedra from symmetric groups.

The purpose of this paper is to show the following theorem.

**Main Theorem.** *Every abstract polytope can be constructed as a coset geometry.*

To show this, we characterize the automorphism group of an abstract polytope with a given symmetry type graph, in terms of generators and relations,
as well as some intersection conditions on some subgroups and cosets.

Throughout the paper, we work with the flag graph of a polytope, as opposed
to the partial order. In fact, we shall often work with maniplexes, that is, colored
graphs that generalize flag graphs of polytopes ([16]).

As pointed out before, symmetry type graphs are a great tool to classify
polytopes (and maniplexes) in terms of their automorphism groups. We will rely
on them heavily in our study. There are some (necessary) conditions a graph
must satisfy to be the symmetry type graph of a maniplex or polytope. Graphs satisfying such conditions are called \textit{premaniplexes} (or admissible graphs) and will be properly defined in Section 2.1. We believe our main theorem will shed light to the study of $k$-orbit polytopes, at the time that opens the gate to study one of the main problems in the area (listed as Problem 12 in [1]):

**Question 1.** Given a premaniplex, does it exist a polytope (or maniplex) having such premaniplex as its symmetry type graph.

The problem of finding polytopes or manifolds with a given symmetry type graph is in general very difficult, as one can note, for example, by looking at the history of chiral polytopes: although it was back in 1991 [14], when Schulte and Weiss studied chiral polytopes and classified their automorphism groups in terms of generators and relations, it took almost 20 years to have a construction showing that such polytopes existed for all ranks $n > 3$ (see [12]).

Very little is know about other particular instances of Question 1. It is known [2] that every premaniplex with 3 vertices with rank $n \geq 3$ is the symmetry type graph of a polytope, and in [13] Pellicer, Potočnik and Toledo construct 2-orbit maniplexes, but it is not known if they are polytopal (i.e. the flag graph of a polytope).

This paper is organized as follows. In Section 2 we give the basic concepts from the theory of abstract polytopes as well as manifolds, and formally introduce the concepts of symmetry type graphs and premaniplexes. We also state a relaxed version of Question 1, that we answer later, in Section 4.1. We start Section 3 by going over the main tool used in this paper: voltage graphs. Then, by using them, in Section 3.1 we construct a maniplex $M$ from a premaniplex $X$ and a group $G$ (satisfying some conditions). Then $G$ will act on $M$ by automorphisms and the quotient of $M$ by the action of $G$ will be $X$. This means that $M$ will have symmetry type graph $X$ if and only if every automorphism
of $\mathcal{M}$ is represented by the action of an element of $G$.

In Section 4.1 we use the construction of Section 3 as well as the results from [4] to characterize in terms of generators and relations the groups that are automorphisms groups of a polytope with a given symmetry type graph. We do so by showing that automorphism groups of polytopes must satisfy certain "intersection conditions" for some subgroups and cosets, that depend on the symmetry type graph. In other words, we give an algebraic test for the group $G$ of Section 3.1 that tells us if the constructed maniplex $\mathcal{M}$ is polytopal or not, thus translating the problem of finding polytopes with a given symmetry type graph to a group-theoretic one. This gives an answer to Problem 1 of [1]. In Section 4.2 we give the proof to the Main Theorem by constructing a polytope as a coset geometry from a voltage group. This gives an answer to Problem 2 of [1].

We finish the paper in Section 5 by using the above result to construct (degenerated) $k$-orbit $n$-polytopes with Boolean groups (elementary abelian 2-groups) as automorphism group, for all $k, n \geq 3$. For this, we define caterpillars as premaniplexes having exactly one generating three and studying their coverings in order to avoid the possible extra symmetry that might happen when one uses our construction to obtain polytopes from groups.

## 2 Abstract polytopes and maniplexes

In this section we shall give the basic definitions and properties of abstract polytopes and maniplexes, and some relations between them. For more details, we refer the reader to [4], [11] and [16].

A partially ordered set is said to be flagged if it has a (unique) least and a (unique) greatest element and each maximal chain, called a flag, has the same
finite cardinality. As all flags have the same cardinality, say $n+2$, flagged posets naturally admit an order-preserving function, the rank, from the poset to the set $\{-1, 0, 1, \ldots, n\}$. The rank function allows us to talk about flag-adjacencies: given two flags $\Phi$ and $\Psi$ of a flagged poset, they are said to be $i$-adjacent if they satisfy to differ only in the element of rank $i$.

An (abstract) $n$-polytope (also called an (abstract) polytope of rank $n$) is a flagged poset $\mathcal{P}$ in which the following conditions hold:

- **Diamond condition:** for each flag $\Phi$ of $\mathcal{P}$ and each $i \in \{0, 1, \ldots, n-1\}$, there exists a unique $i$-adjacent flag to $\Phi$.

- **Strong flag connectedness:** for any two flags $\Phi$ and $\Psi$ of $\mathcal{P}$, there exists a sequence of adjacent flags connecting $\Phi$ to $\Psi$, such that all the flags in the sequence contain the faces of the intersection $\Phi \cap \Psi$.

The elements of rank $i$ in a polytope are called $i$-faces. Given a flag $\Phi$, we often denote its $i$-face as $\Phi_i$, and by $\Phi^i$ its (unique!) $i$-adjacent flag. Recursively, if $w$ is a word on $\{0, \ldots, n-1\}$ we denote by $\Phi^w$ the flag $(\Phi^w)^i$. It is straightforward to see that $(\Phi^i)^i = \Phi$, and that $\Phi^i_j = \Phi_j$ if and only if $i \neq j$.

Given a flagged poset $\mathcal{P}$ one can define its **flag graph** as the graph $G(\mathcal{P})$ whose vertices are the flags of $\mathcal{P}$ and two flags $\Phi$ and $\Psi$ are connected by an edge of color $i \in \{0, 1, \ldots, n-1\}$ if and only if they are $i$-adjacent. The flag graph of an $n$-polytope is an $n$-maniplex, that is, an $n$-regular connected simple graph with a proper edge coloring with colors $\{0, \ldots, n-1\}$ such that if $i$ and $j$ are two colors satisfying that $|i - j| > 1$, then the graph induced by edges of colors $i$ and $j$ is a disjoint union of 4-cycles. However, not every maniplex is the flag graph of a polytope.

While abstract polytopes generalize classical polytopes to combinatorial structures, maniplexes (introduced by Steve Wilson in 2012 [16]) generalize flag graphs of a polytope as well as of the flag graphs of maps on surfaces (that
is, 2-cellular embeddings of connected graph on a surface). In order to unify our notation of abstract polytopes and maniplexes, when dealing with maniplexes we shall call flags to its vertices and say that two of them are $i$-adjacent if they are the vertices of an edge of colour $i$.

It follows from the definition of a maniplex that each flag is incident to exactly one edge of each color. Then, for each $i \in \{0, 1, \ldots, n - 1\}$ one can define $r_i$ as the permutation of the set of flags that maps each flag its $i$-adjacent one. In other words $\Phi r_i = \Psi$ if and only if $\Phi$ and $\Psi$ are $i$-adjacent.

The permutations $r_i$, with $i \in \{0, 1, \ldots, n - 1\}$, are all involutions with no fixed points and, by connectivity, they generate a group of permutations on the flags which acts transitively on them. Furthermore, if $|i - j| > 1$ then $r_i r_j$ is also an involution with no fixed points; thus, $r_i$ and $r_j$ commute.

Since $\Phi r_i$ is the flag $i$-adjacent to $\Phi$, it is convenient to denote it by $\Phi^i$ and to follow the same recursive notation as before: $\Phi^{wi} = (\Phi^w)^i$ where $w$ is a word on $\{0, 1, \ldots, n - 1\}$.

The group $\langle r_0, r_1 \ldots, r_{n-1} \rangle$ is called the monodromy or connection group of the maniplex $\mathcal{M}$, it shall be denoted by $\text{Mon}(\mathcal{M})$ and we shall call each of its elements a monodromy. If $w$ is a word on the alphabet $\{0, 1, \ldots, n - 1\}$ we identify $w$ with the monodromy $x \mapsto x^w$, that is, the word $a_1 a_2 \ldots a_k$ is identified with the monodromy $r_{a_1} r_{a_2} \ldots r_{a_k}$.

A maniplex homomorphism is a graph homomorphism that preserves the color of the edges. Using the connectedness of maniplexes one can see that every maniplex homomorphism is determined by the image of one flag and that they are all surjective. The notions of isomorphism and automorphism follow naturally.

As with polytopes, we denote the automorphism group of a maniplex $\mathcal{M}$ by $\Gamma(\mathcal{M})$. By definition, if $\gamma \in \Gamma(\mathcal{M})$, then $(\Phi r_i) \gamma = (\Phi^i) \gamma = (\Phi \gamma)^i = (\Phi \gamma) r_i$, for
all \( i \in \{0, 1, \ldots, n - 1\} \), implying that \( \omega \gamma = \gamma \omega \) for all \( \omega \in \text{Mon}(M) \).

Note further that, since the action of \( \text{Mon}(M) \) is transitive, the action of \( \Gamma(M) \) is free (or semi-regular). Of course, this is true for both polytopes and maniplexes.

Let \( M \) be an \( n \)-maniplex. If \( I \subset \{0, 1, \ldots, n - 1\} \), we define \( M_I \) as the subgraph of \( M \) induced by the edges of colors in \( I \). If \( i \in \{0, 1, \ldots, n - 1\} \), we use the symbol \( i \) to denote the set \( \{0, 1, \ldots, n - 1\} \setminus \{i\} \), and more generally, if \( K \subset \{0, 1, \ldots, n - 1\} \), we denote its complement by \( \overline{K} \). In particular \( M_i \) is the subgraph of \( M \) obtained by removing the edges of color \( i \). We will use this notation also for any graph with a coloring of its edges even if its not a maniplex.

In [4] Garza-Vargas and Hubard describe how to recover a polytope \( P \) from its flag graph: the elements of rank \( i \) of \( P \) are the connected components of \( M_i \); the order is given as follows: given connected components \( F \) and \( G \) of \( M_i \) and \( M_j \), respectively, we set \( F < G \) if and only if \( i < j \) and \( F \cap G \neq \emptyset \)

In [4] it is proved that if \( M \) is any maniplex then \( P(M) \) is in fact a poset (actually, it is a flagged poset).

It is not difficult to see ([4]) that \( P \) and \( P' \) are isomorphic polytopes if and only if their flag graphs \( G(P) \) and \( G(P') \) are isomorphic. This fact implies the following theorem, which can be interpreted as saying that all the information of the polytope \( P \) is encoded in its flag graph.

**Theorem 2.1.** [4] Let \( P \) be a polytope and let \( M = G(P) \) be its flag graph. Then \( P \) is isomorphic (as a poset) to \( P(M) \) and \( \Gamma(P) = \Gamma(M) \).

Theorem 5.3 of [4] gives a characterization of *polytopal* maniplexes, that is, those maniplexes that are isomorphic to the flag graph of some polytope. Such characterization is given in terms of some path intersection properties of the maniplexes.
Definition 2.2. Let $\mathcal{M}$ be an $n$-maniplex. We say that $\mathcal{M}$ satisfies the strong path intersection property (or SPIP) if for every two subsets $I, J \subset \{0, 1, \ldots, n-1\}$ and for any two flags $\Phi$ and $\Psi$, if there is a path $W$ from $\Phi$ to $\Psi$ using only darts of colors in $I$ and also a path $W'$ from $\Phi$ to $\Psi$ using only darts of colors in $J$, then there also exists a path $W''$ from $\Phi$ to $\Psi$ that uses only darts of colors in $I \cap J$.

We say that $\mathcal{M}$ satisfies the weak path intersection property (or WPIP) if for any two flags $\Phi$ and $\Psi$ and for all $k, m \in \{0, 1, \ldots, n-1\}$, whenever there is a path $W$ from $\Phi$ to $\Psi$ with only darts of color in $[0, m] := \{0, 1, \ldots, m\}$ and a path $W'$ from $\Phi$ to $\Psi$ with only darts of colors in $[k, n-1] := \{k, k+1, \ldots, n-1\}$, then there is also a path $W''$ from $\Phi$ to $\Psi$ with only darts of colors in $[k, m] := \{k, k+1, \ldots, m\}$.

Theorem 2.3. [4] Let $\mathcal{M}$ be a maniplex. Then the following conditions are all equivalent:

- $\mathcal{M}$ is polytopal.
- $\mathcal{M}$ satisfies the SPIP.
- $\mathcal{M}$ satisfies the WPIP.

In any of these cases $\mathcal{P}(\mathcal{M})$ is a polytope whose flag graph is isomorphic to $\mathcal{M}$.

2.1 Premaniplexes and symmetry type graphs

A $k$-orbit maniplex is one with exactly $k$ flag orbits under its automorphism group. When studying $k$-orbit maniplexes with $k > 1$ one finds that it is convenient to classify them in terms of the local structure of the flags. For this reason, in [2], Cunningham et al. introduce the concept of symmetry type graph.
Given a maniplex $\mathcal{M}$ and a subgroup $G$ of the automorphism group of $\mathcal{M}$, the \textit{symmetry type graph of $\mathcal{M}$ with respect to $G$}, denoted either by $\mathcal{T}(\mathcal{M}, G)$ or by $\mathcal{M}/G$, is constructed as follows: The vertex set of $\mathcal{T}(\mathcal{M}, G)$ is the set of flag orbits of $\mathcal{M}$ under the group $G$, and if $\Phi$ and $\Psi$ are $i$-adjacent in $\mathcal{M}$ we draw an edge of color $i$ between their orbits. If $\Phi$ and $\Phi^i$ are in the same orbit under $G$, we draw a semi-edge of color $i$ at the vertex corresponding to that orbit. Recall that a semi-edge is different from a loop in that it consists of only one dart which is inverse to itself, rather than two darts starting at the same vertex; less formally, a semi-edge is incident to the vertex once, while a loop is incident to the vertex two times.

When we speak about \textit{the symmetry type graph of $\mathcal{M}$}, we mean it with respect to $\Gamma(\mathcal{M})$ and we simply write $\mathcal{T}(\mathcal{M})$ instead of $\mathcal{T}(\mathcal{M}, \Gamma(\mathcal{M}))$.

If $\mathcal{P}$ is a polytope, \textit{the symmetry type graph of $\mathcal{P}$ (with respect to $G$)} is defined as the symmetry type graph (with respect to $G$) of its flag graph.

If $X$ is the symmetry type graph of an $n$-maniplex, then it is a connected graph in which every vertex is incident to exactly one edge of each color in $\{0, 1, \ldots, n-1\}$ and it satisfies that if $|i - j| > 1$, the paths of length 4 that alternate between the colors $i$ and $j$ are closed. However, it might not be a maniplex as it is not necessarily simple. We will call such a graph a $n$-\textit{premaniplex}.

If $X$ is a premaniplex and $i, j \in \{0, 1, \ldots, n - 1\}$ are non-consecutive, the connected components of the subgraph of $X$ induced by the edges of colors $i$ and $j$ are not necessarily 4-cycles. In fact they can be any quotient of a 4-cycle, as illustrated in Figure 1.

The notions of \textit{homomorphism, isomorphism, automorphism} and \textit{monodromy group} from maniplexes can all be easily extended to premaniplexes as well.

The natural projection $p : \mathcal{M} \to \mathcal{T}(\mathcal{M}, G)$ is, of course, a homomorphism.
When studying polytopes (or maniplexes) and their symmetry type graphs two natural questions occur:

**Question 2.** Given a premaniplex $X$, is there a polytope (or maniplex) whose symmetry type graph is $X$?

**Question 3.** Given a premaniplex $X$, what conditions must a group $G$ satisfy so that there is a polytope (or maniplex) $P$ such that $T(P,G) \cong X$?

In Section 4.1 we give a complete answer to Question 3. Question 2 remains as a hard question, as even if we find a polytope $P$ and a group $G$ such that $T(P,G) \cong X$, it may still happen that $G$ is a proper subgroup of $\Gamma(P)$. However, in Section 5 we give an infinite family of premaniplexes that are in fact symmetry type graphs of polytopes.

### 3 Voltage graphs

The projection $p : \mathcal{M} \to T(\mathcal{M},G)$ is an example of what is called a *regular covering projection* in graph theory (see [5] or [10] for more details). Given a
regular covering projection $p : \tilde{X} \to X$, one may recover the graph $\tilde{X}$ from the graph $X$ using what is known as a voltage assignment.

A path in a graph is a finite sequence of darts $W = d_1d_2\ldots d_k$ such that the dart $d_{i+1}$ starts at the endpoint of the dart $d_i$. The startpoint of $W$ is the starting point of $d_1$ (denoted $I(d_1)$), and the endpoint of $W$ is the endpoint of $d_k$ (denoted $T(d_k)$). If $x$ and $y$ are the startpoint and endpoint of $W$, respectively, we say that $W$ goes from $x$ to $y$ and we write this as $W : x \to y$. If the endpoint and startpoint of a path $W$ are the same vertex $x$ we say that $W$ is a closed path based at $x$. We also consider that for every vertex $x$ there is an empty closed path based at $x$.

If a path $W$ ends at the startpoint of a path $V$, we may define the product $WV$ as their concatenation.

Two paths $W$ and $W'$ with the same startpoint and endpoint are said to be homotopic if one can transform $W$ into $W'$ by a finite sequence of the following operations:

- Inserting two consecutive inverse darts at any point, that is $d_1d_2\ldots d_id_{i+1}\ldots d_k \mapsto d_1\ldots d_idd^{-1}d_{i+1}\ldots d_k$, where $I(d) = T(d_i)$;

- Deleting two consecutive inverse darts at any point, that is $d_1\ldots d_idd^{-1}d_{i+1}\ldots d_k \mapsto d_1d_2\ldots d_id_{i+1}\ldots d_k$;

In this case we write $W \sim W'$.

It is easy to see that homotopy is an equivalence relation and that if $W \sim W'$ and $V \sim V'$, then $WV \sim W'V'$. Therefore, we can think of the product of two homotopy classes of paths. The set of all homotopy classes of paths in a graph $X$ with this operation is called the fundamental groupoid of $X$ and it is denoted.
by $\Pi(X)$. We will often speak of a "path $W$ in $\Pi(X)$", but the reader should keep in mind that we are actually referring to its homotopy class.

The subset of $\Pi(X)$ consisting of all the (homotopy classes of) closed paths based at a vertex $x$ forms a group known as the fundamental group of $X$ (based at $x$) and it is denoted by $\Pi^x(X)$.

A voltage graph is a pair $(X, \xi)$ where $X$ is a graph and $\xi$ is a groupoid antimorphism from $\Pi(X)$ to a group $G$. In this case we say that $\xi$ is a voltage assignment (on $X$) and we call $G$ the voltage group (of $\xi$). The element $\xi(W)$ is called the voltage of $W$.

Note that a voltage assignment is completely determined by the voltages of the darts of the graph, as the voltage $\xi(W)$ of a path $W = d_1d_2\ldots d_{k-1}d_k$ is simply $\xi(d_k)\xi(d_{k-1})\ldots\xi(d_1)$.

Given a voltage graph $(X, \xi)$ with voltage group $G$, we can construct the derived graph $X^\xi$ as follows:

- The vertex set is $V \times G$ where $V$ is the vertex set of $X$.
- The dart set is $D \times G$ where $D$ is the dart set of $X$.
- The dart $(d, g)$ starts at the vertex $(x, g)$ and ends at the vertex $(x, \xi(d)g)$.

This is an undirected graph, as the inverse of the dart $(d, g)$ is the dart $(d^{-1}, \xi(d)g)$.

Given a path $W$ starting at a vertex $x$ and given an element $g$ in the voltage group, there is a unique path $\tilde{W}$ in $X^\xi$ that starts at $(x, g)$ that projects to $W$. The path $\tilde{W}$ is called a lift of $W$ and it is easy to see that it ends at $(x, \xi(W)g)$ (for details see [10]).

In our case, we will work with graphs that have a coloring of its edges (and therefore, darts), so we will define that the color of the dart $(d, g)$ is the same as the color of $d$.

It is known (see [5] or [10]) that if $p : \tilde{X} \to X$ is a regular covering, there is a voltage assignment $\xi$ such that $\tilde{X}$ is isomorphic to $X^\xi$. 

13
3.1 Voltage graphs that give maniplexes as derived graphs

Of course, we want to use voltage graphs to obtain maniplexes (and polytopes). In this section we shall give necessary and sufficient conditions on a voltage assignment $\xi$ on a premaniplex $X$ so that $X^\xi$ is a maniplex.

In [2, Section 5] the authors find a set of distinguished generators for the automorphism group of a maniplex $\mathcal{M}$ with a given symmetry type graph $X$. It is easy to see that such distinguished generators can be thought of as voltages assigned to $X$ (or more precisely to $\Pi(X)$) so that the derived graph is $\mathcal{M}$. To do this, one must choose a spanning tree $T$ on $X$ and assign trivial voltage to all its darts. If we want to start with a premaniplex $X$, we need to be careful with the way we assign voltages so that the derived graph is indeed a maniplex.

Let $X$ be a premaniplex with fundamental groupoid $\Pi(X)$, and let $\xi : \Pi(X) \to \Gamma$ be a voltage assignment, for some group $\Gamma$. We want to find the conditions on $\Gamma$ and $\xi$ that ensure that $X^\xi$ is actually a maniplex, but before doing so, let us assume that there is a spanning tree $T$ of $X$ with trivial voltage in all its darts.

First we want $X^\xi$ to be connected. It is known (see [10]) that in order for $X^\xi$ to be connected, $\xi(D)$ must generate $\Gamma$, where $D$ denotes the set of darts of $D$.

Next, $X^\xi$ must be a simple graph. Thus, it must not have semi-edges nor multiple edges. Note that a semi-edge of $X^\xi$ that starts at a vertex $(x, \gamma)$ ends in $(x, \xi(e)\gamma) = (x, \gamma)$, where $e$ is a semi-edge of $X$. This implies that if the voltage of every semi-edge of $X$ is not trivial, we avoid semi-edges in $X^\xi$. Since $\xi$ is an antimorphism we should have that $\xi(e) = \xi(e^{-1}) = \xi(e)^{-1}$, implying that the voltage of a semi-edge must have order two.

To avoid multiple edges, we need to avoid different darts with the same
initial and terminal vertices; suppose $X$ has two parallel darts $(d, \sigma)$ and $(d', \tau)$. Since both darts start at the same vertex, we have that $(I(d), \sigma) = (I(d'), \tau)$, so $I(d) = I(d')$ and $\sigma = \tau$. The common end-point of $(d, \sigma)$ and $(d', \tau)$ could be written as $(y, \xi(d)\sigma)$ or $(z, \xi(d')\sigma)$, where $y$ is the end-point of $d$ and $z$ the end-point of $d'$. The fact that these two are the same means that $y = z$ and $\xi(d) = \xi(d')$. So $(d, \sigma)$ and $(d', \sigma)$ are parallel darts in $X$ if and only if $d$ and $d'$ are parallel darts in $X$ with the same voltage. Thus, $X$ has no parallel darts if and only if no pair of parallel darts in $X$ has equal voltages.

Finally, we want to ensure that if $|i - j| > 1$, the paths of length 4 in $X$ that alternate colors between $i$ and $j$ are closed. Let $W$ be one of these paths. Projecting $\tilde{W}$ to $X$ we get a path $W$ in $X$ of length 4 that alternates colors between $i$ and $j$, and since $X$ is a premaniplex we know that $W$ is closed. Suppose $W$ starts at a vertex $x$. Then $\tilde{W}$ goes from a vertex of the form $(x, \gamma)$ to $(x, \xi(W)\gamma)$. So $\tilde{W}$ is closed if and only if $\xi(W)\gamma = \gamma$, or in other words, $W$ has trivial voltage. Summarizing this discussion, we arrive to the following lemma:

**Lemma 3.1.** Let $X$ be a premaniplex and let $\xi : \Pi(X) \to \Gamma$ be a voltage assignment with a spanning tree $T$ of trivial voltage on all its darts. Then $X$ is a maniplex if and only if

1. The set $\xi(D)$ generates $\Gamma$, where $D$ is the set of darts of $X$,
2. $\xi(d)$ has order exactly 2 when $d$ is a semi-edge,
3. $\xi(d) \neq \xi(d')$ when $d$ and $d'$ are parallel darts, and
4. if $|i - j| > 1$ every (closed) path $W$ of length 4 that alternates between the colors $i$ and $j$ has trivial voltage.

In Section 4 we shall translate the conditions in Lemma 3.1 to relations and inequalities that the generators of a group $\Gamma$ need to satisfy to act on a
maniplex with given symmetry type graph $X$. Before doing so (in the next section), Let us now take a closer look at the consequences of condition 4 of the above Lemma.

Condition 4 of Lemma 3.1 invites us to introduce a new concept of homotopy: we say that two paths $W$ and $W'$ are maniplex-homotopic if we can transform one into the other by a finite sequence of inserting or deleting pairs of inverse darts, as well as switching the colors of two consecutive darts with non-consecutive colors, that is

$$d_1d_2\ldots d_id_{i+1}\ldots d_k \mapsto d_1d_2\ldots d'_i d'_{i+1}\ldots d_k,$$

where $|c(d_i) - c(d_{i+1})| > 1$, $c(d'_i) = c(d_{i+1})$, $c(d'_{i+1}) = d(d_i)$, $I(d'_i) = I(d_i)$ and $I(d'_{i+1}) = T(d'_i)$. Hence, a voltage assignment $\xi$ is well defined when applied to the maniplex-homotopy class of paths if and only if it satisfies Condition 4 of Lemma 3.1. From this point on, whenever we speak about homotopy, homotopy class, fundamental groupoid, etc. we will be thinking in terms of maniplex-homotopy.

One could use the group $\Gamma := \langle S | R \rangle$ as the voltage group where $S$ has a generator $\alpha_e$ for each edge $e$ not in the spanning tree of $X$ and $R$ has one element $\alpha_e^2$ per each semi-edge and one element $\alpha_{e_4}\alpha_{e_3}\alpha_{e_2}\alpha_{e_1}$ for each path of length four alternating between two non-consecutive colors. In fact every voltage group that gives a maniplex should be a quotient of this group, or in other words $\Gamma$ is the “most general” group we can use as a voltage group to get a maniplex as the derived graph. We know (see [6]) that $X = T(\mathcal{U}, G)$ where $\mathcal{U}$ is the universal polytope of rank $n$ and $G$ is some group. This means that there is some voltage assignment $\xi$ on $X$ with voltage group $G$ such that $X^\xi$ is isomorphic to the flag graph of $\mathcal{U}$. Because of the universality of $\mathcal{U}$ we get that $G$ and $\Gamma$ in fact are the same. In other words, if we use the most general group as our voltage group we will always get the flag graph of the universal polytope.
4 Intersection properties and coset geometries

In order to prove the Main Theorem, we need to characterize, in terms of generators and relations, the groups $\Gamma$ that act by automorphisms on a polytope $\mathcal{P}$ in such a way that the symmetry type graph $\mathcal{T}(\mathcal{G}(\mathcal{P}), \Gamma)$ is isomorphic to a given premaniplex $X$. We shall do this in order to be able to recover the polytope $\mathcal{P}$ as a coset geometry using the group $\Gamma$.

We start with the premaniplex $X$ and provide it with a voltage assignment $\xi$. Recall that if $(X, \xi)$ is a voltage graph with voltage group $\Gamma$, then $X^\xi / \Gamma$ is isomorphic to $X$; conversely, if $\mathcal{M} / \Gamma$ is isomorphic to $X$ then there is a voltage assignment $\xi$ on $X$ such that $X^\xi$ is isomorphic to $\mathcal{M}$. Hence, by characterizing the voltage assignments $\xi$ that satisfy that the derived graph $X^\xi$ is the flag graph of a polytope, we determine the conditions that $\Gamma$ must satisfy to be the automorphism group of a polytope with symmetry type $X$.

4.1 Voltage graphs and the path intersection property

We have figured out how to construct a maniplex from a premaniplex via voltage assignments. It is then natural to ask: when is the obtained maniplex the flag graph of a polytope? The answer, as we shall see in this section, is closely related to Theorem 2.3. In fact, translating Theorem 2.3 to the setting of voltage assignments will give us conditions that take the form of intersection properties that certain distinguished subgroups and some left cosets must satisfy. Given
two vertices $x, y$ in $X$ and a set of colors $I \in \{0, 1, \ldots, n - 1\}$, let us denote by
\[ \Pi^x_y(X) \] the set of (homotopy classes of) paths from $x$ to $y$ in $X$ that only use darts with colors in the set $I$. So $\xi(\Pi^x_y(X))$ denotes the set of voltages of all the paths of $X$ from $x$ to $y$ whose edges have colors in $I$.

**Theorem 4.1.** Let $X$ be a premaniplex and let $\xi : \Pi(X) \to \Gamma$ be a voltage assignment such that $X^\xi$ is a maniplex. Then $X^\xi$ is the flag graph of a polytope if and only if
\[
\xi(\Pi^x_y(I)(X)) \cap \xi(\Pi^x_y(J)(X)) = \xi(\Pi^x_y(I \cap J)(X)),
\]
for all $I, J \subset \{0, 1, \ldots, n - 1\}$ and all vertices $x, y$ in $X$.

**Proof** Start by assuming that $X^\xi$ is the flag graph of a polytope. Let $x$ and $y$ be vertices of $X$ and let $I, J \subset \{0, 1, \ldots, n - 1\}$. Consider two paths, $W \in \Pi^x_y(I)(X)$ and $W' \in \Pi^x_y(J)(X)$, with the same voltage, say $\alpha \in \Gamma$. When lifting $W$ and $W'$ in $X^\xi$, they lift to paths $\tilde{W}$ and $\tilde{W}'$, respectively, that go from $(x, 1)$ to $(y, \alpha)$ (here $1$ is the identity element of $\Gamma$) and satisfying that $\tilde{W}$ uses edges with colors in $I$ while $\tilde{W}'$ uses edges with colors in $J$. In fact, one would define $\tilde{W}$ and $\tilde{W}'$ as the paths that start at $(x, 1)$ and follow the same sequence of colors as $W$ and $W'$ respectively. By Theorem 2.3 $X^\xi$ satisfies the SPIP, which implies that there is a path $\tilde{W}''$ from $(x, 1)$ to $(y, \alpha)$ that uses only colors in $I \cap J$. Then, its projection $W'' := p(\tilde{W}'')$ is a path in $X$ that goes from $x$ to $y$ that uses only colors in $I \cap J$ and has voltage $\alpha$. This proves that $\xi(\Pi^x_y(I)(X)) \cap \xi(\Pi^x_y(J)(X)) \subset \xi(\Pi^x_y(I \cap J)(X))$. Since the other contention is given, equality (1) must hold.

Now let us assume that equality (1) holds for all $I, J \subset \{0, 1, \ldots, n - 1\}$ and all vertices $x$ and $y$. Let $\tilde{W}$ and $\tilde{W}'$ be paths in $X^\xi$ from a vertex $(x, \gamma)$ to a vertex $(y, \tau)$. Let $I$ and $J$ be the sets of colors of darts of $\tilde{W}$ and $\tilde{W}'$, respectively, and let $W := p(\tilde{W})$ and $W' := p(\tilde{W}')$ be the projections of the paths to $X$. Then, both $W$ and $W'$ go from $x$ to $y$, and $W$ is a path with
colors in $I$, while $W'$ is a path with colors in $J$, that is, $W \in \Pi^{x,y}_I(X)$ and $W' \in \Pi^{x,y}_J(X)$. Furthermore, since $\tilde{W}$ and $\tilde{W}'$ start at $(x, \gamma)$ and finish at $(y, \tau)$, they both have voltage $\alpha := \tau \gamma^{-1}$. By hypothesis, there exists a path $W'' \in \Pi^{x,y}_{I \cap J}(X)$ that also has voltage $\alpha$. Then $W''$ has a unique lift $\tilde{W}''$ which is a path in $X^\xi$ from $(x, \gamma)$ to $(y, \tau)$ and it uses darts of colors in $I \cap J$. This proves that $X^\xi$ satisfies the SPIP and therefore, by Theorem 2.3, it is the flag graph of a polytope.

Note that when $x = y$ the set $\xi(\Pi^{x,y}_I(X)) := \xi(\Pi^I_\infty(X))$ is a group, since it is the image of a group under a groupoid antimorphism. Actually, we shall find a set of distinguished generators for the group $\xi(\Pi^I_\infty(X))$ in a similar way as the distinguished generators of the automorphism group of a polytope are found in [2]. Recall that $X_I$ is the subgraph of $X$ induced by the edges with colors in $I$ and that $X_I(x)$ is the connected component of $X_I$ containing the vertex $x$. To find the distinguished generators of $\xi(\Pi^I_\infty(X))$, fix a spanning tree $T^I_x$ for $X_I(x)$. For each dart $d$ in $X_I(x)$ but not in $T^I_x$ we get a cycle $C_d$ of the form $W d V$ where $W$ is the unique path contained in $T^x_I$ from $x$ to the initial vertex of $d$, and $V$ is the unique path contained in $T^x_I$ from the terminal vertex of $d$ to $x$. Then, the set $\{C_d\}$, where $d$ runs among the darts in $X_I(x)$ not in $T^I_x$, is a generating set for $\Pi^I_x(X)$. This implies that $\{\xi(C_d)\}$, where $d$ runs among the darts of $X_I(x)$ not in $T^I_x$, is a set of generators for $\xi(\Pi^I_\infty(X))$.

Since $\xi$ is a voltage assignment, we might consider only one dart $d$ for each edge. By denoting by $W_y$ the unique path contained in $T^x_I$ from $x$ to $y$, we can see that $\Pi^{x,y}_I(X) = \Pi^I_x(X)W_y$ (that is, a path from $x$ to $y$ can be written as a closed path starting and finishing at $x$, concatenated with $W_y$), which implies that $\xi(\Pi^{x,y}_I(X)) = \xi(W_y)\xi(\Pi^I_\infty(X))$. Therefore, all the intersection properties can be given in terms of left cosets of the groups $\xi(\Pi^I_\infty)$, whose generators we already know.
Theorem 4.1 gives an intersection property for each pair of vertices \((x, y)\) and each two sets of colors \(I, J \subset \{0, 1, \ldots, n - 1\}\). If we prove an intersection property for the pair \((x, y)\), by taking the inverse on both sides we get the corresponding property for the pair \((y, x)\), so we can consider only unordered pairs \(\{x, y\}\), and this reduces the number of intersection properties to verify by a factor of 2. Still, the total number of intersection properties is quadratic on the number of vertices and exponential on the number of colors. This number gets too big too quickly; however, many of these properties are redundant, either because they are true for any group (for example, the intersection of a group and one of its subgroups is the smaller subgroup) or because they are a consequence of other intersection properties.

Fortunately we may reduce the number of intersection properties to check by following the same proof but using the weak path intersection property instead of the strong one. Doing this we get the following refinement of the previous theorem.

**Theorem 4.2.** Given a premaniplex \(X\) and a voltage assignment \(\xi\) such that \(X^\xi\) is a maniplex, \(X^\xi\) is the flag graph of a polytope if and only if

\[
\xi(\Pi^{x,y}_{[0,m]}(X)) \cap \xi(\Pi^{x,y}_{[k,n-1]}(X)) = \xi(\Pi^{x,y}_{[k,m]}(X)),
\]

for all \(k, m \in \{0, \ldots, n - 1\}\) and all \(x, y \in \mathcal{F}\).

With Theorem 4.2 the number of intersection properties to check is now quadratic on the number of vertices and also quadratic on the rank. We can still refine this theorem a little more with the following observations:

- The cases \(k = 0\) and \(m = n - 1\) say that the intersection of the whole voltage group with some subset is the subset itself, so they are trivially true for every voltage assignment.
• In Theorem 4.2 one has to consider the case when \( k > m \). In such case, 
\( \xi(\Pi_{[k,m]}^{x,y}(X)) \) is the trivial group when \( x = y \) and the empty set when \( x \neq y \). However, if the intersection property holds for \( k = m + 1 \), that is, if 
\( \xi(\Pi_{[0,m]}^{x,y}(X)) \cap \xi(\Pi_{[m+1,n-1]}^{x,y}(X)) = \xi(\Pi_{\emptyset}^{x,y}(X)) \), then it also holds for \( k > m + 1 \). So one may only verify the intersection property for \( k \leq m + 1 \).

• If \( y \) and \( y' \) are in the same connected component of \( X_{[k,m]} \) and (2) is satisfied for the pair \((x, y)\), then it is also satisfied for the pair \((x, y')\). To see this, let \( W \) be a path from \( y \) to \( y' \) with colors in \([k, m]\) and notice that 
\( \Pi_{I}^{x,y'}(X) = \Pi_{I}^{x,y}(X)W \) whenever \( I \) contains \([k, m]\). By taking voltages we get that 
\( \xi(\Pi_{I}^{x,y'}(X)) = \xi(W)\xi(\Pi_{I}^{x,y}(X)) \). This means that we can get the intersection property for the pair \((x, y')\) by multiplying the one for pair \((x, y)\) by \( \xi(W) \) on the left. So for each pair of numbers \((k, m)\) we only need to verify one intersection property for each pair of connected components of \( X_{[k,m]} \).

Taking the previous observations into consideration, the maximum amount of necessary intersection properties to check is 
\[ v(v + 1)/2 \sum_{m=0}^{n-2}(m + 1) = v(v + 1)n(n - 1)/4. \] But to reach this bound we need for every vertex to be in a different connected component of \( X_{[k,m]} \) for every pair \((k, m)\) with \( 0 < k \leq m + 1 \), which is only possible when \( v = 1 \). In other words, this bound is not tight for symmetry type graphs with more than one vertex.

4.2 Constructing a polytope from the voltage group

We have seen how to recover a polytope from its flag graph (see Theorem 2.3) and when \( X^{\xi} \) is the flag graph of a polytope for a given premaniplepx \( X \) and a voltage assignment \( \xi \) (see Theorem 4.2). By concatenating the construction of \( X^{\xi} \) from \( X \) and \( \xi \), and the construction of a polytope \( P \) from \( X^{\xi} \) we get a
construction of a polytope from $X$ and $\xi$. In this section, we translate this to give a construction only in terms of subgroups of $\Gamma$ and their cosets. This will give the proof of the Main Theorem.

Let $f$ be a choice function on the connected subgraphs of $X$, that is, a function that assigns a base vertex to each such subgraph. Let $C$ be a connected component of $X_I$ for some $I \subset \{0, 1, \ldots, n-1\}$ and let $x = f(C)$. Let $\overline{C} := (X^\xi)_I(x, 1)$, that is, the connected component of $(X^\xi)_I$ containing $(x, 1)$. Thus, if $(x, \gamma) \in \overline{C}$, then there is a path $\widetilde{W}$ from $(x, 1)$ to $(x, \gamma)$ which uses only colors in $I$, which implies that its projection is a closed path $W$ in $\Pi^\gamma_I(X)$ with voltage $\gamma$. This means that when considering the action of $\Gamma$ on $X^\xi$, the stabilizer of $\overline{C}$ coincides with $\xi(\Pi^\gamma_I(X))$. If we now consider a coset $\xi(\Pi^\gamma_I(X))\sigma$, this would be the set of elements of $\Gamma$ that map $\overline{C}$ to $(X^\xi)_I(x, \sigma)$.

We know that the $i$-faces of the polytope that has $X^\xi$ as its flag graph correspond to the connected components of $(X^\xi)_I$. This makes natural the following construction:

Given $X$ and $\xi$ satisfying Theorem 4.2 and a choice function $f$ on the connected subgraphs of $X$, we construct a partially ordered set $\mathcal{P}(X, \xi)$ with a rank function whose elements of rank $i$ are the right cosets of groups of the type $\xi(\Pi^\gamma_I(X))$ where

$$x \in \{f(C)|C \text{ is a connected component of } X^\gamma\}.$$  

We have to consider these groups as formal copies, that is, if $x$ and $x'$ are on different connected components we consider $\xi(\Pi^\gamma_I(X))$ to be different than $\xi(\Pi^\gamma_{I'}(X))$, even if, as groups, they might be equal. Likewise, if $j \neq i$ but $\xi(\Pi^\gamma_I(X))$ coincides with $\xi(\Pi^\gamma_{I'}(X))$ we consider them to be different in $\mathcal{P}(X, \xi)$.

We could formalize this by saying that the elements of rank $i$ in $\mathcal{P}(X, \xi)$ are pairs $(C, \xi(\Pi^\gamma_I(X)))$ with $C$ a connected component of $X^\gamma$, $x = f(C)$ and $\gamma \in \Gamma$, but since $x$ and $i$ already appear in the notation, we may assume that $\xi(\Pi^\gamma_I(X))\gamma$.
stands for the pair \((X_{\tau}(x), \xi(\Pi_{\tau}^x(X))\gamma)\).

Now we shall define the order on \(P(X, \xi)\) as follows. First, for all \(i \in \{0, 1, \ldots, n - 1\}\) and every vertex \(y\) in \(X\) we look at the connected component \(X_{\tau}(y)\) and fix a path going from its base vertex \(x = f(X_{\tau}(y))\) to \(y\). We call this path \(W^y_i\) and we denote its voltage by \(\alpha^y_i := \xi(W^y_i)\).

**Definition 4.3** (Order in \(P(X, \xi)\)). Given \(0 \leq i < j \leq n - 1\), let \(C\) be a connected component of \(X_{\tau}\) and \(C'\) be a connected component of \(X_{\tau'}\), and let \(x\) and \(x'\) be their respective base flags. Given \(\gamma, \gamma' \in \Gamma\), we say that \(\xi(\Pi_{\tau}^x(X))\gamma < \xi(\Pi_{\tau'}^x(X))\gamma'\) if and only if \(\alpha^y_i \xi(\Pi_{\tau}^x(X))\gamma \cap \alpha^y_j \xi(\Pi_{\tau'}^x(X))\gamma' \neq \emptyset\), for some \(y \in C \cap C'\).

**Theorem 4.4.** Let \(X\) be a premaniplex and \(\xi : \Pi(X) \to \Gamma\) a voltage assignment satisfying Theorem 4.2. Let

\[ P(X, \xi) := \{ \xi(\Pi^x_\tau(C)) : C \text{ is a connected component of } X_{\tau}, x = f(C), \tau \in \Gamma \}, \]

**Proof** Let us denote \(P(X^\xi)\) as \(P\) and \(P(X, \xi)\) as \(Q\). Note that Theorem 4.2 implies that \(Q\) is a polytope, and we want to show that \(P\) is also a polytope (such that \(P/\Gamma \cong X\)).

By the discussion in Section 3.1, \(T(P, \Gamma) \cong X\), since, by construction of \(X^\xi\), we have that \(X^\xi/\Gamma \cong X\). So in order to settle the theorem it is enough to find a poset isomorphism \(\varphi : Q \to P\) such that it commutes with the action of \(\Gamma\), that is, such that \(\tilde{C}\sigma\varphi = \tilde{C}\varphi\sigma\) for all faces \(\tilde{C}\) in \(Q\) and all \(\sigma \in \Gamma\).

Let \(\tilde{C}\) be a face of \(P\). Hence, \(\tilde{C}\) is a connected component of \(X^\xi_{\tau}\) for some color \(i\). Let \(C := p(\tilde{C})\) and let \(x := f(C)\). This implies that \(\tilde{C}\) has a flag of
the type \((x, \gamma)\) for some \(\gamma \in \Gamma\). Let \(\tilde{K}\) be the connected component of \(X_\xi^i\) that contains \((x, 1)\). Then, the set of elements of \(\Gamma\) that map \(\tilde{K}\) to \(\tilde{C}\) is \(\xi(\Pi_i^\xi(X))\gamma\).

We want to identify \(\tilde{C}\) with this coset, so we define \(\tilde{C}\varphi := \xi(\Pi_i^\xi(X))\gamma\). Note that \(\varphi : Q \to P\) is well defined, since if \((x, \gamma')\) is in \(\tilde{C}\) then \(\gamma'\gamma^{-1}\) stabilizes \(\tilde{K}\), implying that \(\gamma'\gamma^{-1} \in \xi(\Pi_i^\xi(X))\). We want to prove that \(\varphi\) is a poset isomorphism and that it commutes with \(\Gamma\).

Let us show first that \(\varphi\) commutes with the action of \(\Gamma\). Let \(\tilde{C}\) be a face in \(X^\xi\) and let \(\sigma \in \Gamma\). By the definition of \(\varphi\) we know that \(\tilde{C}\varphi = \xi(\Pi_i^\xi(X))\gamma\) where \(i\) is the rank of \(\tilde{C}\), \(x = f(p(\tilde{C}))\) and \(\gamma \in \Gamma\) is any element such that \((x, \gamma) \in \tilde{C}\). On the other hand \((\tilde{C}\sigma)\varphi = \xi(\Pi_i^\xi(X))\gamma'\) where \(x' = f(p(\tilde{C}\sigma))\) and \((x', \gamma') \in \tilde{C}\sigma\). Also note that since \(\tilde{C}\) and \(\tilde{C}\sigma\) are in the same orbit, then \(p(\tilde{C}) = p(\tilde{C}\sigma)\), and thus \(x = x'\). Furthermore, the flag \((x, \gamma\sigma) = (x, \gamma)\sigma\) is in \(\tilde{C}\sigma\). This proves that \((\tilde{C}\sigma)\varphi = \xi(\Pi_i^\xi(X))\gamma\sigma = (\tilde{C}\varphi)\sigma\).

Now let us prove that \(\varphi\) is an isomorphism of posets. Let \(\tilde{C}\) and \(\tilde{C}'\) be incident faces of \(P\) of ranks \(i\) and \(j\), respectively, with \(i < j\) (therefore, \(\tilde{C} < \tilde{C}'\)). Hence, there is a flag \((y, \tau)\) in \(\tilde{C} \cap \tilde{C}'\), which in turn implies that its first entry, \(y\), must be in \(C \cap C'\) where \(C = p(\tilde{C})\) and \(C' = p(\tilde{C}')\).

Note that the path \(W_i^y\) is contained in \(C\) while \(W_j^y\) is contained in \(C'\). Then, these paths have lifts \(\tilde{W}_i^y\) and \(\tilde{W}_j^y\) respectively, that go from \((x, (\alpha_i^y)^{-1}\tau)\) and \((x', (\alpha_j^y)^{-1}\tau)\), respectively, to \((y, \tau)\). Observe that \(\tilde{W}_i^y\) is contained in \(\tilde{C}\) and \(\tilde{W}_j^y\) is contained in \(\tilde{C}'\). Thus, \((\alpha_i^y)^{-1}\tau \in \Pi_i^\xi(X)\gamma\) and \((\alpha_j^y)^{-1}\tau \in \Pi_j^\xi(X)\gamma'\). Therefore

\[\tau \in \alpha_i^y(\xi(\Pi_i^\xi(X))\gamma) \cap \alpha_j^y(\xi(\Pi_j^\xi(X))\gamma') = \alpha_i^y(\tilde{C}\varphi) \cap \alpha_j^y(\tilde{C}'\varphi);\]

but this means that \(\tilde{C}\varphi < \tilde{C}'\varphi\) in \(Q\).

Conversely, suppose that \(\tilde{C}\varphi < \tilde{C}'\varphi\) in \(Q\). We want to show that \(\tilde{C} < \tilde{C}'\) in \(X^\xi\). Let us write \(\tilde{C}\varphi = \xi(\Pi_i^\xi(X))\gamma\), where \(i\) is the rank of \(\tilde{C}\), \(x := f(p(\tilde{C}))\) and \(\gamma\) is an element of the voltage group such that \((x, \gamma) \in \tilde{C}\). Similarly, we
write \(\tilde{C}'\varphi = \xi(\Pi^\omega_j(X))\gamma', \) where \(j\) is the rank of \(\tilde{C}'\), \(x := f(p(\tilde{C}'))\) and \(\gamma'\) is an element of the voltage group such that \((x', \gamma') \in \tilde{C}'\).

By hypothesis \(\alpha^y_i \xi(\Pi^\omega_i(X))\gamma\) and \(\alpha^y_j \xi(\Pi^\omega_j(X))\gamma'\) have non-empty intersection for some \(y \in C \cap C'\). Let \(\tau\) be an element in such intersection. Then \((\alpha^y_i)^{-1}\tau \in \xi(\Pi^\omega_i(X))\gamma\). This implies that \((x, (\alpha^y_i)^{-1}\tau)\) is in the same connected component of \((X^\xi)^\tau\) as \((x, \gamma)\), that is \((x, (\alpha^y_i)^{-1}\tau) \in \tilde{C}\). But at the same time, there is a lift of \(W^y_i\) that connects \((x, (\alpha^y_i)^{-1}\tau)\) with \((y, \tau)\), and since \(W^y_i\) does not use the color \(i\), its lift is contained in \(\tilde{C}\), which proves that \((y, \tau) \in \tilde{C}\).

Similarly, the fact that \((\alpha^y_j)^{-1}\tau \in \xi(\Pi^\omega_j(X))\gamma\) implies that \((y, \tau) \in \tilde{C}'\). Thus, we have proved that \(\tilde{C} \cap \tilde{C}'\) is not empty, or in other words \(\tilde{C} < \tilde{C}'\) in \(\mathcal{P}\).

Therefore, \(\varphi\) is an isomorphism and the theorem follows. \(\square\)

5 Example: Caterpillars

If one wants to build polytopes from premaniplexes in the way we have described in this paper, it is natural to start with infinite families of premaniplexes. One could, of course, then start with premaniplexes with a fixed number \(k\) of vertices. For example, one may want to construct 2-orbit polytopes (see [9]). If, on the other hand, we do not want to limit the number of vertices of the premaniplexes in the family a first step could be to consider trees. However, since semi-edges are considered as cycles, there are no premaniplexes whose underlying graph (without colours) is a tree. Thus, we study the closest thing to them: premaniplexes that are trees with an unlimited number of semi-edges. Hence, we define a caterpillar as a premaniplex in which every cycle is a semi-edge. In other words, a caterpillar is a premaniplex \(X\) with a unique spanning tree.

In particular, a caterpillar does not have pairs of parallel links (edges joining different vertices). If there are three links incident to one vertex, at least two of
them must have colors differing by more than 1, which would imply that there is a 4-cycle. This implies that caterpillars consist in fact of a single path $P$ (which we will call the underlying path of $X$) and lots of semi-edges. Of course, the colors of two consecutive edges on the path must differ by exactly one, otherwise there would be a 4-cycle. We note here that the term caterpillar has been used in the graph theory literature for a very similar but slightly different concept.

Throughout this section, unless otherwise stated, $X$ will denote a finite caterpillar (that is, one with a finite number of vertices) with underlying path $P$, and its vertices will be labeled by $x_0, x_1, \ldots, x_k$, ordered as they are visited by $P$. Furthermore, we denote by $(x_i, j)$ the dart of color $j$ at $x_i$ and by $c_i$ the color of the link connecting $x_{i-1}$ and $x_i$.

5.1 Caterpillar coverings

We want to construct polytopes from caterpillars. As often when constructing maniplexes and polytopes via voltage assignments from a premaniplex $X$, the derived maniplex might not have $X$ as the symmetry type graph with respect to the full automorphism group. However, the actual symmetry type graph is a quotient of $X$. For this reason, in this section we study quotients of caterpillars.

Given a caterpillar, let us call an endpoint a vertex incident to just one link (that is, an endpoint of the underlying path). Note that a caterpillar is finite if and only if it has exactly two endpoints. Every symmetry of the caterpillar must map endpoints to endpoints. If a caterpillar is finite there is at most
one non-trivial symmetry and its action on the vertices $x_0, x_1, \ldots, x_k$ is given by $x_j \mapsto x_{k-j}$. We call a finite caterpillar symmetric if it has a non-trivial symmetry.

A word $w$ in $\{0, 1, \ldots, n-1\}$ is simply a finite sequence $w = a_1 a_2 \ldots a_t$, with $a_i \in \{0, 1, \ldots, n-1\}$ for each $i = 1, 2, \ldots, t$. The inverse of a word $w$ is the word $w^{-1}$ that has the same colors as $w$ but written in reverse order. That is, if $w = a_1 a_2 \ldots a_t$ then $w^{-1} = a_t a_{t-1} \ldots a_1$. A word is said to be reduced if it has no occurrence of the same color twice in a row; in other words, $w = a_1 a_2 \ldots a_t$ is reduced if and only if $a_{i+1} \neq a_i$ for all $i = 1, 2, \ldots, t - 1$. We shall work with reduced words from now on. A word $w = a_1 a_2 \ldots a_t$ is a palindrome if $a_i = a_{t+1-i}$ for all $i \in \{1, 2, \ldots, t\}$. A palindrome word of even length can be written as $vv^{-1}$ for some word $v$ and cannot be a reduced word. A palindrome word of odd length can always be written as $w = va v^{-1}$ for some color $a$ and some word $v$.

Given a segment $[x, y]$ in a caterpillar, its underlying word is the word $w$ consisting of the colors of the links in the path that goes from $x$ to $y$. When we speak of the underlying word of a caterpillar $X$ we are referring to the underlying word of its underlying path in a fixed orientation. Hence, we say that a segment $[x, y]$ is a palindrome if its underlying word $v$ is a palindrome.

**Proposition 5.1.** Let $X$ be a finite caterpillar and let $Y$ be a premaniplex not isomorphic to $X$ such that there is a premaniplex homomorphism $h : X \to Y$. Then $Y$ is a caterpillar. Moreover, if $Y$ has at least 2 vertices and $S = c_1 c_2 \ldots c_k$ is the underlying word of $X$, then there is some $r < k$ such that $w = c_1 c_2 \ldots c_r$ is the underlying word of $Y$ and one of the following statements is true:

1. There exist colors $a_1, a_2, \ldots, a_t \in \{c_r + 1, c_r - 1\}$ and $b_1, b_2, \ldots, b_{t-1} \in \{c_1 + 1, c_1 - 1\}$ such that $S = wa_1 w^{-1} b_1 b_2 w^{-1} b_2 \ldots b_{t-1} b_t w^{-1}$.

2. There exist colors $a_1, a_2, \ldots, a_t \in \{c_r + 1, c_r - 1\}$ and $b_1, b_2, \ldots, b_t \in \{c_1 + 1, c_2 + 2, \ldots, c_{t-1} + 2, c_{t-1} - 2, \ldots, c_1 - 1\}$. 

27
In any case, if $i \equiv j \mod 2r + 2$ then $h(x_i) = h(x_j)$. Also if $i \equiv -j - 1 \mod 2r + 2$ then $h(x_i) = h(x_j)$.

Before proving Proposition 5.1 let us remark that it simply means that the quotients of a caterpillar $X$ are those caterpillars $Y$ such that $X$ can be “folded” into $Y$. We illustrate this concept in Figure 3: the semi-edges are not drawn and the names of the vertices have been omitted, but the idea is that $X$ must be “folded” into “layers” of $r + 1$ vertices and then each vertex will be projected to the vertex on $Y$ in the same horizontal coordinate. The layer $\ell$ consists of the vertices $x_i$ where $\left\lfloor \frac{i}{r+1} \right\rfloor = \ell$ ($\lfloor x \rfloor$ denotes the integer part of $x$). Even layers go from left to right, while odd layers go from right to left, hence the underlying word of even layers is $w = c_1 c_2 \ldots c_r$ while the underlying word of odd layers is $w^{-1} = c_r c_{r-1} \ldots c_1$.

Figure 3: The caterpillar $X$ covers the caterpillar $Y$ if and only if it “folds” into it.
Now let us proceed with the proof.

**Proof of Proposition 5.1** Let us consider the equivalence relation $\sim_h$ on $X$ given by $x \sim_h y$ if and only if $h(x) = h(y)$. Recall that by the definition of a premaniplex homomorphism $x \sim_h y$ implies that $x^m \sim_h y^m$ for every monodromy $m$. Therefore $Y$ is isomorphic to $X/\sim_h$.

We already know that all premaniplex homomorphisms are surjective (see Section 2). By hypothesis $h$ is not an isomorphism, which implies it cannot be injective. Let $x$ and $y$ be two different vertices on $X$ such that $x \sim_h y$. Let $x_0, x_1, \ldots, x_k$ be the sequence of vertices in the underlying path of $X$. There is a monodromy $m$ such that $x^m = x_0$, and so $x_0 \sim_h y^m$ and $y^m$ is different than $x_0$. Now let $q$ be the minimum positive number such that $x^q \sim_h x_0$. We know that $q$ exists because $x_0 \sim_h y^m$. Note that if $q = 1$ we would have that $x_i \sim_h x_0$ for all $i$, which would imply that the same is true for $x_1$ and in turn also for $x_2 = x_1^2$ and so on. This would mean that all the vertices of $X$ are equivalent, meaning that $Y$ has only one vertex and the proposition follows. So we may assume that $q > 1$.

We know that $x^q = x_0^q$ is equivalent $x_0^q$. If $c_q \neq c_1$, we would have that $x_0^q = x_0$. This would imply that $x_{q-1}$ is equivalent to $x_0$, contradicting the minimality of $q$. So we have proven that $c_q = c_1$.

Now note that if for some $\ell$ we have that $x_\ell \sim x_1$, then $x_\ell^{c_1} \sim x_1^{c_1} = x_0$. In particular this tells us that for $\ell < q - 1$, $x_\ell$ cannot be equivalent to $x_1$. Now we can use the same argument we used to prove that $c_q = c_1$ to prove that $c_{q-1} = c_2$. Analogously we can prove that $c_3 = c_{q-2}$, $c_4 = c_{q-3}$ and so on. In other words, $[x_0, x_q]$ is a palindrome. Since for all $i$, $c_i$ and $c_{i+1}$ are different, then $q$ is odd, say $q = 2r + 1$. Let $v$ be the underlying word of the segment $[x_0, x_q]$. Then $v$ may be written as $v = w^{c_{r+1}}w^{-1}$ for some word $w = c_1c_2\ldots c_r$. Let us call $a_1 := c_{r+1}$ and note that $a_1 \in \{c_r + 1, c_r - 1\}$.
For all $i = 0, 1, \ldots, k$, let us denote by $\widehat{i}$ the residue of dividing $i$ by $2r + 2$. We will prove by induction on $i$ that $x_i \sim_h x_{\widehat{i}}$ for all $i = 0, 1, \ldots, k$ and that $c_i = c_{\widehat{i}}$ when $i$ is not divisible by $r + 1$, $c_i \in \{c_1 + 1, c_1 - 1\}$ if $i$ is an odd multiple of $r + 1$ and $c_i = \{c_r + 1, c_r - 1\}$ when $i$ is an even multiple of $r + 1$.

Let our induction hypothesis be that $x_\ell \sim_h x_{\widehat{\ell}}$ for all $\ell < i$, and that $c_\ell = c_{\widehat{\ell}}$ if $\ell$ is not divisible by $r + 1$.

Let us start with the case when $i \equiv 0 \mod 2r + 2$. In this case we want to prove that $x_i \sim_h x_i$ and that $c_i \in \{c_1 + 1, c_1 - 1\}$. By our induction hypothesis we know that $x_{i-1} \sim_h x_{2r+1} \sim_h x_0$ and $c_{i-1} = c_{2r+1} = c_1$. In particular $c_i \in \{c_1 + 1, c_1 - 1\}$. This implies that $x_i = x_{i-1}^{c_i} x_0^{c_i} = x_0 = x_i$ (since $c_i \neq c_1$). Thus $i$ satisfies our claim.

Now let us proceed with the case when $i$ is an odd multiple of $r + 1$, that is $\widehat{i} = r + 1$. In this case we want to prove that $x_i \sim_h x_{\widehat{i}}$ and that $c_i \in \{c_r + 1, c_r - 1\}$. Our induction hypothesis tells us that $x_{i-1} \sim_h x_r$ and that $c_{i-1} = c_r$. Hence $c_i \in \{c_r + 1, c_r - 1\}$. Note that one of the colors in $\{c_r + 1, c_r - 1\}$ is actually $c_{r+1}$, while the other is the color of a semi-edge incident to $x_r$. Since $x_{r+1} \sim_h x_r$ we have that $x_i = x_{i-1}^{c_i} x_r^{c_i} \sim_h x_r \sim_h x_{r+1} = x_{\widehat{i}}$. Thus $i$ satisfies our claim.

Finally let us prove our claim for the case when $i$ is not divisible by $r + 1$. Our induction hypothesis tells us that $x_{i-1} \sim_h x_{\widehat{i}}$. Note that $\widehat{i} = \widehat{i-1} + 1$. Since $i$ is not a multiple of $r + 1$ we know that $x_i = x_{\widehat{i}} = x_{\widehat{i-1}}$ is not equivalent to $x_{\widehat{i}}$. This implies that $x_{i-1}^{c_i}$ is not equivalent to $x_{i-1}$, and since it is adjacent to $x_{i-1}$ it must be equal to either $x_i$ or $x_{i-2}$. If $i \equiv 1 \mod r + 1$ then by induction hypothesis $x_{i-1} \sim_h x_0$, but recall that $x_0 \sim_h x_{2r+1} = x_{\widehat{i}}$ by definition, and by induction hypothesis $x_{i-2} \sim_h x_{i-2} x_{2r+1} = x_{\widehat{i}}$. This means that $x_{i-2} \sim_h x_{i-1}$, so $x_{i-1}^{c_i}$ must be $x_i$, implying that $c_i = c_{\widehat{i}}$ if $i \neq 1 \mod r + 1$ then our induction hypothesis tells us that $c_{i-1} = c_{i-1} \neq c_{i}$, and since $x_{i-2} = x_{i-1}^{c_{i-1}} = x_{i-1}^{c_{i-1}}$, so the only possibility is that $x_{i-1}^{c_{i-1}} = x_i$, implying that $c_i = c_{\widehat{i}}$ and that $x_i \sim_h x_{\widehat{i}}$. 

30
Thus, $i$ satisfies our claim. Note that we have also proven that if $i$ is not divisible by $r + 1$ then $x_{i-1}$ cannot be an endpoint of $X$.

We have proved that $x_i \sim_h x_{\hat{i}}$ for all $i = 0, 1, \ldots, k$. This implies automatically that if $i \equiv j \mod 2r + 2$ then $x_i \sim_h x_j$. Moreover, if $i \equiv -j - 1 \mod 2r + 2$, then $\hat{i} = 2r + 1 - \hat{j}$. Now since $v$ is a palindrome, we know that $x_{\ell} \sim_h x_{2r+1-\ell}$ for all $\ell = 0, 1, \ldots, 2r + 1$, in particular $x_{\hat{i}} = x_{2r+1-\hat{j}} \sim_h x_{\hat{j}}$. This, together with the fact that $x_i \sim_h x_{\hat{i}}$ and $x_j \sim_h x_{\hat{j}}$, implies that $x_i \sim_h x_j$.

We have already proved that $S = wa_1w^{-1}b_1wa_2w^{-1}\ldots$ and it ends after an occurrence of $w$ or $w^{-1}$. To end the prove note that since $h$ is surjective we already know exactly what premaniplex $Y$ is: It is a caterpillar with vertex sequence $h(x_0), h(x_1), \ldots, h(x_r)$. In fact, for $i = 1, \ldots, r-1$ and $j = 0, 1, \ldots, n-1$ we know that $h(x_i)$ is different from $h(x_i)^j = h(x_i^j)$ if and only if $j \in \{c_{i-1}, c_i\}$; and if $i = 0$ (resp. $r$) then $h(x_i)$ is different from $h(x_i)^j$ if and only if $j = c_1$ (resp. $c_r$).

If we look closely at the proof of Proposition 5.1 we notice that we have not actually used the fact that $X$ is finite, but only the fact that it has at least one endpoint. So the proposition may be generalized as follows:

**Proposition 5.2.** Let $X$ be a caterpillar with at least one endpoint and let $Y$ be a premaniplex such that there is a premaniplex homomorphism (or covering) $h : X \rightarrow Y$. Then $Y$ is a caterpillar. Moreover, if $S = c_1c_2\ldots$ is the color sequence of $X$ starting at its endpoint, there is some $r$ such that $w = c_1c_2\ldots c_r$ is the underlying word of $Y$ and there exist colors $a_1, a_2, \ldots \in \{c_r + 1, c_r - 1\}$ and $b_1, b_2, \ldots \in \{c_1 + 1, c_1 - 1\}$ such that $S = wa_1w^{-1}b_1wa_2w^{-1}b_2\ldots$. If $X$ is finite it ends after an occurrence of either $w$ or $w^{-1}$. If $i \equiv j \mod 2r + 2$ then $h(x_i) = h(x_j)$. Also if $i \equiv -j \mod 2r + 1$ and $\left[\frac{i}{r+1}\right] \neq \left[\frac{j}{r+1}\right] \mod 2$ then $h(x_i) = h(x_j)$. 

31
Let us look now at the degenerate cases. If $Y$ is only one vertex we can think that the previous theorems still hold but with $w$ being the empty word, and since $c_1$ and $c_r$ do not exist, we get rid of the restrictions $a_i \in \{c_r + 1, c_r - 1\}$ and $b_i \in \{c_1 + 1, c_1 - 1\}$. In Figure 3 $X$ would have only one vertex and no links per layer; essentially $X$ would be “standing up” instead of being folded. If $Y$ is isomorphic to $X$ then $S = w$ and we do not have any $a_i$ or $b_i$. In Figure 3 $X$ would have only one layer.

If one would try to prove an analogous result as Proposition 5.2 for caterpillars with no ends, the only additional case to consider would be that when $Y$ is not a caterpillar but a premaniplex consisting of a cycle with underlying word $w$ and all the edges not in the cycle are semiedges. In this case $X$ would have the underlying infinite word $\cdots www \cdots$.

5.2 Caterpillars as STG of polytopes

Given a caterpillar $X$, we want to assign voltages to the semi-edges of $X$ in order to get the flag graph of a polytope as the derived maniplex. To this end, we note that in caterpillars the conditions of Theorems 4.1 and 4.2 can be simplified, as stated in the following lemma:

**Lemma 5.3.** Let $X$ be a caterpillar and let $\xi : \Pi(X) \to \Gamma$ be a voltage assignment such that all the darts in the underlying path of $X$ have trivial voltage. Then the following statements are equivalent.

1. $X^\xi$ is polytopal.

2. For every vertex $x$ in $x$ and all sets $I, J \subset \{0, 1, \ldots, n - 1\}$ the equation

   \[
   \xi(\Pi_x^I(X)) \cap \xi(\Pi_x^J(X)) = \xi(\Pi_x^{I \cap J}(X))
   \]

   holds.
3. For every vertex $x$ in $X$ and all $k,m \in \{0,1,\ldots,n-1\}$ the equation

$$\xi(\Pi^x_{[0,m]}(X)) \cap \xi(\Pi^x_{[k,n-1]}(X)) = \xi(\Pi^x_{[k,m]}(X))$$

holds.

**Proof** To prove this we have to see that for every set of colors $I \subset \{0,1,\ldots,n-1\}$ the set $\xi(\Pi^x_I(X))$ is either $\xi(\Pi^x_I(X))$ or empty, depending on whether or not the segment of $P$ that goes from $x$ to $y$ (which we will denote $[x,y]$) uses or not only colors in $I$. In fact if $[x,y]$ uses only colors in $I$ then

$$\xi(\Pi^x_{[x,y]}(X)) = \xi(\Pi^x_I(X)[x,y]) = \xi([x,y])\xi(\Pi^x_I(X)) = 1 \cdot \xi(\Pi^x_I(X)) = \xi(\Pi^x_I(X)).$$

Note also that $\xi(\Pi^x_{[x,y]}(X)) \cap \xi(\Pi^x_{[y,z]}(X))$ is empty if and only if one of the factors is empty. These observations together with Theorems 4.1 prove the equivalence between conditions 1 and 2, and with Theorem 4.2 we prove the equivalence between conditions 1 and 3, thus proving the lemma.

Recall that a **Boolean group** (or an **elementary abelian $2$-group**) is a group in which every non-trivial element has order exactly 2. Thus, all Boolean groups are abelian, and finitely generated Boolean groups are isomorphic to a direct product of cyclic groups of order 2.

**Proposition 5.4.** Every caterpillar is the quotient of the flag graph of a polytope by a Boolean group.

**Proof** Let $X$ be a caterpillar. We shall find a Boolean group $B$ and a voltage assignment $\xi : \Pi(X) \to B$ such that $X^\xi$ is polytopal in the following way.

First assign trivial voltage to all links of $P$ and a different independent voltage to each semi-edge incident to $x_0$. We shall give voltage assignments to the semi-edges $(x_i,j)$, recursively on $i$. If $|j - c_i| > 1$ we assign the same voltage
to \((x_i, j)\) as we did to \((x_{i-1}, j)\). On the other hand, if \(|j - c_i| = 1\), we assign a new element as the voltage \((x_i, j)\) independent from all the voltages of previous darts.

Let us call the resulting voltage group \(B\) and denote this voltage assignment by \(\xi\). By Lemma 3.1, \(X^\xi\) is a maniplex.

Note further that \(\xi\) satisfies that if \(i \in \{1, 2, \ldots, k\}\); \(r, s \in \{0, 1, \ldots, n - 1\}\) and \((x_{i-1}, r)\) and \((x_i, s)\) are semi-edges, then \(\xi(x_i, s) = \xi(x_{i-1}, r)\) if and only if \(r = s\) and \(|r - c_i| \neq 1\). It also satisfies that if \(i < \ell < j\) and \((x_i, r)\) and \((x_j, r)\) are semi-edges such that \(\xi(x_i, r) = \xi(x_j, r) = \gamma\), then \((x_{\ell}, r)\) is also a semi-edge and \(\xi(x_{\ell}, r) = \gamma\).

We claim that the second statement of Lemma 5.3 is satisfied, and hence, \(X^\xi\) is the flag graph of a polytope. To show that this is the case, start by noticing that \(\xi(\Pi^x_j)\) is the group generated by the voltages of the semi-edges on the component \(X_I(x)\).

Given \(I, J \subseteq \{0, 1, \ldots, n - 1\}\), suppose that for some vertex \(x\) there is a semi-edge \(e\) in \(X_I(x)\) and a semi-edge \(e'\) in \(X_J(x)\) with \(\xi(e) = \xi(e') = \gamma\) for some \(\gamma \in \Gamma\). If \(e = e'\) then \(e \in X_{I \cap J}(x)\). If \(e \neq e'\) then we have that for some \(i,j \in \{0, \ldots, k - 1\}\) and some \(r \in \{0, \ldots, n - 1\}\) occurs that \(e = (x_i, r)\) and \(e' = (x_j, r)\); in particular \(r \in I \cap J\). If \(x \in [x_i, x_j]\) then, as previously observed, \(\xi(x, r) = \gamma\) (see Figure 4) and since \((x, r) \in X_{I \cap J}(x)\) this means \(\gamma\) is a generator of \(\xi(\Pi^x_{I \cap J})\). If \(x \notin [x_i, x_j]\) consider without loss of generality that \(x_i\) is further away from \(x\) than \(x_j\), in other words, that \(x_j \in [x_i, x]\) (see Figure 5). Then \([x_i, x]\) uses only colors in \(I\) and \([x_j, x] \subset [x_i, x]\) uses only colors in \(J\). Then since \([x_j, x] \subset [x_i, x]\) we know that \([x_j, x]\) uses only colors in \(I \cap J\), meaning that \(e' \in X_{I \cap J}(x)\) and since its voltage is \(\gamma\), this shows that \(\gamma\) is always a generator of \(\xi(\Pi^x_{I \cap J})\). Therefore, we have shown that if \(\gamma\) is a generator of both \(\xi(\Pi^x_I(X))\) and \(\xi(\Pi^x_J(X))\), then it is also a generator of \(\xi(\Pi^x_{I \cap J}(X))\).
Now let $\sigma \in \xi(\Pi^x_I(X)) \cap \xi(\Pi^x_J(X))$ be arbitrary. Since the group $B$ is Boolean, $\sigma$ may be written as $\sigma = \gamma_1 \gamma_2 \ldots \gamma_s$ where the elements $\gamma_1, \gamma_2, \ldots, \gamma_s$ are different generators of $B$, and this decomposition is unique up to reordering of the factors. Since $\sigma \in \xi(\Pi^x_I(X))$, and because the voltage of a semi-edge is always a generator, each $\gamma_i$ is also in $\xi(\Pi^x_I(X))$, and since $\sigma \in \xi(\Pi^x_J(X))$ each $\gamma_i$ is also in $\xi(\Pi^x_J(X))$. But, this implies that each $\gamma_i$ is in $\xi(\Pi^x_{I \cap J}(X))$, implying that $\sigma \in \xi(\Pi^x_{I \cap J}(X))$. Therefore, $\xi(\Pi^x_I) \cap \xi(\Pi^x_J) = \xi(\Pi^x_{I \cap J})$. \hfill $\square$

Proposition 5.4 is still true for infinite caterpillars. Even if our algorithmic way of assigning voltages may not be doable, the voltage assignment is still well defined as a quotient of the Boolean group with one generator assigned to each semi-edge.

Note that Proposition 5.4 says that every caterpillar $X$ is a symmetry type graph of a polytope $P$ with respect to some group $B \leq \Gamma(P)$. However need not happen that $X$ is the symmetry type graph of $P$ with respect to the full
automorphism group. That is, $X^\xi$ might have “extra” symmetry. Thus, we would want to investigate what could be the symmetry type of $X^\xi$ with respect to its full automorphism group.

Let $X$ be a finite caterpillar and let $\xi : \Pi(X) \to B$ be the voltage assignment constructed in the proof of Proposition 5.4. If $X$ is symmetric, its non-trivial symmetry induces an automorphism of $B$ which is just a reordering of the generators, moreover, if the generators are given the natural order, it induces the reverse order on the generators. Using [10, Theorem 7.1], we get that this symmetry induces a symmetry of the derived maniplex, so in this case the original caterpillar is not the symmetry type of the derived polytope. But we will see in Theorem 5.5 that if this is not the case we can almost be certain that the caterpillar is in fact the symmetry type graph of the derived polytope by its full automorphism group. In this case, by “almost” we mean that if this is not the case, the caterpillar must have a very specific structure.

**Theorem 5.5.** Let $X$ be a finite caterpillar of length $k$ and rank $n$. Let $S = c_1 c_2 \ldots c_k$ be the underlying word of $X$. Then at least one of the following statements is true:

1. $X$ is symmetric.
2. $X$ is the STG of a polytope with a Boolean automorphism group.
3. $c_1$ is in $\{1, n-2\}$ and there exist $r \in \{1, 2, \ldots, k-1\}$, and $a_1, a_2, \ldots, a_t \in \{0, 1, \ldots, n-1\}$ where $t = (k+1)/(2r+2)$, such that $S = w a_1 w^{-1} b w a_2 w^{-1} b \ldots b w a_t w^{-1}$ where $w = c_1 c_2 \ldots c_r$ and $b = 0$ if $c_1 = 1$ and $b = n - 1$ if $c_1 = n - 2$.
4. There exist $r \in \{1, 2, \ldots, k-1\}$ and $a, b \in \{0, n-1\}$ such that

   $$S = w a w^{-1} b w a w^{-1} b \ldots b w a w^{-1} b w,$$

   where $w = c_1 c_2 \ldots c_r$. Also $(c_1, b), (c_r, a) \in \{(1, 0), (n-2, n-1)\}$.
**Proof** Suppose that $X$ is not symmetric and that it is not the STG of a polytope with a Boolean automorphism group.

Consider the voltage assignment $\xi : \Pi(X) \to B$ previously discussed. We say that two vertices $x$ and $y$ of $X$ are equivalent ($x \sim y$) if there exist (or equivalently, for all) $\sigma, \tau \in B$ such that the flags $(x, \sigma)$ and $(y, \tau)$ of $X^\xi$ are in the same orbit under the action of the automorphism group of $X^\xi$. Then $\sim$ is an equivalence relation preserved by $i$-adjacency, that is $x \sim y \Rightarrow x^i \sim y^i$. Moreover, the natural function $h : X \to X/\sim$ is a premaniplex homomorphism, so by Proposition 5.1 there exists some $r$ such that the $S$ can be written as $w_{a_1}w^{-1}_{b_1}w_{a_2}w^{-1}_{b_2} \ldots$ ending after an occurrence of either $w$ or $w^{-1}$, where $w = c_1c_2 \ldots c_r$, $a_i \in \{c_r + 1, c_r - 1\}$ and $b_i \in \{c_1 + 1, c_1 - 1\}$ (see Figure 3).

Note that since $X$ is not symmetric nor the STG of $X^\xi$ with respect to $B$, at least $b_1$ exists. Let $j \in \{0, 1, \ldots, k - 1\}$ be a number such that the segment $[x_0, x_{j+1}]$ has the underlying word $w_{a_1}w^{-1}_{b_1}w_{a_2}w^{-1}_{b_2} \ldots w^{-1}_{b_i}$ for some $i$. We know in particular that $b_i$ differs from $c_1$ in exactly 1. We want to prove that $(c_1, b_i) \not\in \{(1, 0), (n - 2, n - 1)\}$. We will assume this is not the case and arrive to a contradiction.

Let $q$ be the *other* color that differs from $b_i$ in exactly 1 (that is $q = 2b_i - c_1$). Since $(c_1, b_i) \not\in \{(1, 0), (n - 2, n - 1)\}$ we know that $q \in \{0, 1, \ldots, n - 1\}$, and thus is the color of some edges of $X$. So there are semi-edges $e, e'$ incident to $x_0$ of colors $q$ and $b_i$ respectively. Let $\alpha := \xi(e)$ and $\beta := \xi(e')$. The voltage of the closed path $ee'ee'$ is $(\beta\alpha)^2 = 1$ because $B$ is Boolean. This means that its lift, (the path of length 4 that starts at $(x, 1)$ in $X^\xi$ and alternates colors between $q$ and $b_i$) must be closed.

By Theorem 5.1, we know that $c_{j+1} = c_1 \neq q$, so we know that the darts $(x_{j+1}, q)$ and $(x_j, q)$ are semi-edges. Let $\kappa := \xi(x_j, q)$ and $\lambda = \xi(x_{j+1}, q)$. The path of length 4 that alternates colors between $r$ and $b_i$ and starts at $x_j$ is
Figure 6: The voltage of the path that alternates colors between $r$ and $b_i$ starting at $x_j$ is $\lambda \kappa \neq 1$.

closed, and its voltage is $\lambda \kappa$. Note that since $|q - b_i| = 1$ the construction of $\xi$ tells us that $\xi(x_j, q) \neq \xi(x_{j+1}, q)$, that is $\lambda \neq \kappa$, which implies $\lambda \kappa \neq 1$. This means that the path of length 4 in $X^\xi$ starting at $(x_j, 1)$ and alternating colors between $r$ and $b_i$ is not closed (it ends at $(x_j, \lambda \kappa)$).

We see that the path of length 4 in $X^\xi$ starting at $(x_j, 1)$ and alternating colors between $r$ and $c_{j+1}$ is not closed, but the one starting at $(x_0, 1)$ is. This contradicts the fact that $(x_j, 1)$ and $(x_0, 1)$ are on the same orbit. The contradiction comes from the fact that there are edges of color $q = 2b_i - c_1 \in \{0, 1, \ldots, n - 1\}$, so to avoid this we must have that $(c_1, b_i) \in \{(1, 0), (n - 2, n - 1)\}$. Since $c_1$ is fixed, every $b_i$ must be the same.

If the underlying word of $X$ ends after an occurrence of $w$ we may look at $X$ in the other direction. Then the previous result tells us that every $a_i$ is equal to some $a$ and that $(c_r, a) \in \{(1, 0), (n - 2, n - 1)\}$. \qed

Remark 5.6. If the third or fourth condition is the one holding in Theorem 5.5, the actual STG of $X^\xi$ might be the finite caterpillar with underlying word $w$.

By doing exactly the same proof, we obtain the following analogous result for infinite caterpillars with one end-point (this would look like a ray or half-straight line):

Theorem 5.7. Let $X$ be an infinite caterpillar with one end-point. Let $S$ be the sequence of colors of the underlying path of $X$ starting at its end-point. Then
one of the following statements is true:

1. X is the STG of a polytope with a Boolean automorphism group.

2. There exist some number \( r \) and colors \( b, a_1, a_2, \ldots, a_t \in \{0, 1, \ldots, n - 1\} \) such that \( S = wa_1w^{-1}bwa_2w^{-1}bwa_3w^{-1} \ldots \) where \( w = c_1c_2 \ldots c_r \) and \( (c_1, b) \in \{(1, 0), (n - 2, n - 1)\} \).

Remark 5.8. If the second condition is the one holding, the actual STG of \( X^\xi \) might be the finite caterpillar with underlying word \( w \).

If \( k \geq 3 \), it is easy to construct a caterpillar of length \( k - 1 \) (that is, with \( k \) vertices and \( k - 1 \) links) that does not satisfy properties (1), (3) and (4) (simply let \( c_1 = 0 \) and then avoid the color 0 for every other \( c_i \)). Hence, we get the following corollary:

**Corollary 5.9.** For every \( n, k \geq 3 \), there is an abstract \( n \)-polytope with Boolean automorphism group that has \( k \) flag orbits.

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