Small $x$ behavior of the slope $d \ln F_2/d \ln (1/x)$ in the framework of perturbative QCD

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Abstract

Using an analytical parameterization for the behavior of the $x$ slope of the structure function $F_2$ at small $x$ in perturbative QCD, at the leading twist approximation of the Wilson operator product expansion, and applying a flat initial condition in the DGLAP evolution equations, we found very good agreement with new precise deep inelastic scattering experimental data from HERA.

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1 Introduction

The measurement of the deep-inelastic scattering (DIS) structure function $F_2$ \(^1\), and the derivatives $dF_2/d\ln(Q^2)$ \(^1\) and $d\ln F_2/d\ln(1/x)$ \(^2\) in HERA have permitted the access to a very interesting kinematical range for testing the theoretical ideas on the behavior of quarks and gluons \(^3\) carrying a very low fraction of momentum of the proton, the so-called small $x$ region. In this limit one expects that nonperturbative effects may give essential contributions. However, the reasonable agreement between HERA data and the next-to-leading (NLO) approximation of perturbative QCD has been observed for $Q^2 \geq 2$ GeV\(^2\) (see review Ref. \(^7\) and references therein) and, thus, perturbative QCD could describe the evolution of $F_2$ and its derivatives up to very low $Q^2$ values, traditionally explained by soft processes. It is of fundamental importance to find out the kinematical region where the well-established perturbative QCD formalism can be safely applied at small $x$.

The standard program to study the $x$ behavior of quarks and gluons is carried out by comparison of data with the numerical solution of the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP)\(^3\) equations \(^8\) by fitting the parameters of the $x$ profile of partons at some initial $Q_0^2$ and the QCD energy scale $\Lambda$ \(^10\). However, for analyzing exclusively the small $x$ region, there is the alternative of doing a simpler analysis by using some of the existing analytical solutions of DGLAP in the small $x$ limit \(^12\)-\(^15\). This was done so in Ref. \(^12\) where it was pointed out that the HERA small $x$ data can be interpreted in terms of the so-called doubled asymptotic scaling (DAS) phenomenon related to the asymptotic behavior of the DGLAP evolution discovered many years ago \(^16\).

The study of Ref. \(^12\) was extended in Ref. \(^13, 14, 15\) to include the finite parts of anomalous dimensions of Wilson operators and Wilson coefficients\(^2\), which predicts \(^14, 15\) the small $x$ asymptotic form of parton distributions (PD) in the framework of the DGLAP equation starting at some $Q_0^2$ with the flat function:

$$f_a(Q_0^2) = A_a \quad (\text{hereafter } a = q, g),$$  

where $f_a$ are the parton distributions multiplied by $x$ and $A_a$ are unknown parameters to be determined from data.

¿From now on, we refer to the approach of Ref. \(^13, 14, 15\) as generalized DAS approximation. In generalized DAS the flat initial conditions written in Eq. (1) determine, as in the standard case \(^12\), the basic role of the anomalous dimensions singular parts while the contribution from finite parts of anomalous dimensions and also from Wilson coefficients can be considered as corrections which however are important to have better agreement with experimental data \(^14\). In the present work, similarly to Refs. \(^12\)-\(^15\), the contribution from the non-singlet quark component was neglected.

The usage of the flat initial condition given in Eq. (1) is supported by the actual experimental situation: low-$Q^2$ data \(^17, 1\) \(^18\) \(^19\) are well described for $Q^2 \leq 0.4$ GeV\(^2\)

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\(^1\) At small $x$ there is a different approach based on the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation \(^9\), whose application is out of the scope of this work. However, sometimes we use below the BFKL-based predictions in the discussions and for the comparison with our results in generalized DAS.

\(^2\) In the standard DAS approximation \(^16\) only the singular parts of the anomalous dimensions are used.
by Regge theory with Pomeron intercept $\alpha_P(0) \equiv \lambda_P + 1 = 1.08$, closed to the standard ($\alpha_P(0) = 1$) one. The small rise observed in HERA data \[1, 2, 18, 19\] at low $Q^2$ can be naturally explained by including higher twist terms (see \[13, 20\]). Moreover, HERA data \[1, 2, 18, 19\] with $Q^2 > 1$ GeV$^2$ are in good agreement with the predictions from the GRV parton densities \[11\] which supports our aim to develop an analytical form for the parton densities at small $x$ because, at least conceptually, our method is very closed to the GRV approach.

The purpose of this work is to extend the study of Ref. \[14\] to compare the predictions from the generalized DAS approach with the new precise H1 data \[4\] for the $F_2$ $x$ slope.

The article is organized as follows. In Section 2 we address the present situation with experimental data for the slope $d \ln F_2 / d \ln (1/x)$ and we shortly review some approaches to describe them. Sections 3 and 4 contain, for completeness, a compilation of the basic formulae in the generalized DAS approximation from Ref. \[14\] needed for the present study. In Section 5 we compare our predictions for the derivative $d \ln F_2 / d \ln (1/x)$ with experimental data and discuss the obtained results.

2 The slope $d \ln F_2 / d \ln (1/x)$: experimental data and QCD phenomenology

Various groups have been able to fit the available data (mostly separating the low and high $Q^2$ regions) using a hard input at small $x$, $x^{-\lambda}$, $\lambda > 0$. This is clearly different from the flat input in the DAS approach of Refs. \[12\]-\[15\], also describing reasonably well the experimental results. In some sense it is not very surprising because modern HERA data (at large $Q^2$) cannot distinguish between the behavior from a steep input parton parameterization at quite large $Q^2$, and the steep form acquired after dynamical evolution from a flat initial condition at quite low $Q^2$ values.

Moreover, for the $Q^2$ evolution based on the full set of anomalous dimensions obtained at $x \to 0$ in Ref. \[21\] within the BFKL formalism \[9\], the results weakly depend on the form of the initial condition \[22\], preserving the steep ones and changing the flat ones. In the case of working with anomalous dimensions at fixed order in $\alpha_s$, the initial conditions are important when the data are considered in a wide $Q^2$ range and it is necessary to choose adequately the form of the PD asymptotics at some $Q^2_0$.

As we have already discussed in the introduction, the usage of a flat initial condition leads to the (generalized) DAS approximation \[13\]-\[15\] (see the compilation of the most important formulas in the following two sections). An alternative to this is the choice of a steep initial condition at some quite large $Q^2$: $f_a(x, Q^2) \sim x^{-\lambda_c}$ (subindex c stays for constant), that leads to the following $Q^2$-dependence for $f_a(x, Q^2)$ \[23\]-\[27\] (if $x^{-\lambda_c} >> Const$):

$$\frac{f_a(x, Q^2)}{f_a(x, Q^2_c)} \sim \frac{M_a^+(1 + \lambda_c, Q^2)}{M_a^+(1 + \lambda_c, Q^2_c)},$$

where $M_a^+(1 + \lambda_c, Q^2)$ is the analytical continuation (from integer $n$ to real $1 + \lambda_c$ values)
of the ‘+’ component of the Mellin moment of \( f_a(x, Q^2) \):

\[
M_a(n, Q^2) = \int_0^1 dx x^{n-2} f_a(x, Q^2)
\]  

(3)

When \( x^{-\lambda_c} \gg Const \), the slope \( \lambda_c \) should be \( Q^2 \)-independent and the whole \( Q^2 \)-dependence for \( f_a(x, Q^2) \) comes from the factor \( \sim M^+_a(1 + \lambda_c, Q^2) \) in front of \( x^{-\lambda_c} \) in Eq. (2). Approximations similar to Eq. (2) have been successfully applied in studying the \( Q^2 \)-dependence of HERA data at large \( Q^2 \) values (see Ref. [28] and references therein).

Considering separately the low \( Q^2 \) region, it is also possible to have a good agreement between \( F_2 \) data and its Regge-like behavior [4]. Indeed, \( F_2 \) at \( Q^2 \to 0 \) can be extracted from the relation:

\[
F_2 = \frac{Q^2}{4\pi\alpha_{em}} \sigma_{\gamma p}^\gamma
\]  

(4)

where \( \alpha_{em} \) is electromagnetic coupling constant and \( \sigma_{\gamma p}^\gamma \) is the total (virtual) photoproduction cross-section.

A large amount of experimental data on hadronic cross-sections for many different processes shows an universal rise at large energies that gives the possibility to parameterize all these cross-sections as the sum of two different components

\[
\sigma_{tot}^\gamma = A_P s^{\alpha_P(0)-1} + A_R s^{\alpha_R(0)-1},
\]  

(5)

being \( s \) the center of mass energy squared. The constants \( A_P \) and \( A_R \) are process-dependent magnitudes and the intercepts \( \alpha_P(0) \approx 1.08 \) and \( \alpha_R(0) \approx 0.5 \) (see [29]) are universal process-independent constants. The first and second terms in Eq. (5) correspond to (soft) Pomeron and Reggeon exchange, respectively.

From Eqs. (4) and (5) one immediately obtains that for \( Q^2 \to 0 \)

\[
F_2(x, Q^2) \sim x^{-\varepsilon} \quad \text{and, hence,} \quad f_a(x, Q^2) \sim x^{-\varepsilon} \quad (\varepsilon = \alpha_P(0) - 1 \approx 0.08),
\]

because \( s = Q^2/x \) at small \( x \).

There were many attempts to study the whole \( Q^2 \) region in the framework of Regge-asymptotics (see, for example, the reviews in Ref. [7]). As the reports in Ref. [7] contain quite numerous sets of models, we will restrict ourselves to discuss below only two of them.

In Ref. [30] the fit to \( F_2 \) experimental data has been done under the approach

\[
f_a(x, Q^2) \sim x^{-\lambda(Q^2)}
\]  

(6)

and a fast changing of the \( \lambda(Q^2) \) has been found in the transition range \( Q^2 \sim 5\div10 \) GeV\(^2\). Unfortunately, it is rather difficult to reconcile for the whole \( Q^2 \) range the Regge-like behavior given by Eq. (6) with DGLAP evolution. Some progress along this line has been done in Ref. [27] that was also based on the flat initial conditions as given by Eq. (1). However, in Ref. [27] the PD structure is limited by the Regge-like form of Eq. (6), permitting to reconcile it with DGLAP evolution only separately at low \( Q^2 \), where \( \lambda(Q^2) \)
is close to 0 (or to $\varepsilon$), and at large $Q^2$, where $\lambda(Q^2) \sim \lambda_c$. The structure function $F_2$ and parton distributions have been obtained in Ref. [27] for the whole $Q^2$ range only as a combination of these two representations.

For other types of models (see [32, 33]) the phenomenological $Q^2$-dependence of $\lambda(Q^2)$ has been introduced in the form:

$$\lambda(Q^2) = \varepsilon \left(1 + \frac{Q^2}{Q^2 + c}\right)$$

with a fitted constant $c$, that produces the soft values of the slope $\lambda(Q^2)$ close to $\varepsilon$ at low $Q^2$ and the hard ones $\lambda(Q^2) \sim \lambda_c \sim 0.2 \div 0.3$ at $Q^2 \geq 20$ GeV$^2$.

Very recently new precise experimental data on $\lambda(Q^2)$ has become available [3]. The H1 data points are shown in Fig. 1 where one can observe that for fixed $Q^2$, $\lambda$ is independent on $x$ within the experimental uncertainties, in the range $x < 0.01$. Indeed, H1 data are well described by the power behavior [5]:

$$F_2(x, Q^2) = Cx^{-\lambda(Q^2)},$$

where $\lambda(Q^2) = \hat{a} \ln(Q^2/\Lambda^2)$ with $C \approx 0.18, \hat{a} \approx 0.048$ and $\Lambda = 292$ MeV. The linear rise of $\lambda$ with $\ln Q^2$ given by Eq. (7) is plotted in Fig 2.

A similar analysis for extracting $\lambda(Q^2)$ as a function of $x$ has been carried out by the ZEUS Collaboration. As it is possible to see in Fig. 8 of Ref. [4], the ZEUS data for $\lambda(Q^2)$ are compatible with a constant value $\sim 0.1$ at $Q^2 < 0.6$ GeV$^2$, as it is expected under the assumption of single soft Pomeron exchange within the framework of Regge phenomenology. For the case of H1, this behavior can also be inferred from the new preliminary H1 data [3] at quite low values of $Q^2$.

Let us point out that, even though our results in the framework of the generalized DAS approximation (Eqs. (8)-(11) below) do not have an explicit power-like behavior, they really mimic a power law shape over a limited region of $x$ and $Q^2$ (see Section 4). In addition we have observed earlier [14] that the $x$ dependence of the effective slopes in generalized DAS is not strong and the $F_2$ effective slope was in good agreement with old (less precise) H1 data [1]. We repeat in Section 5 the analysis performed in Ref. [14] but now using the new precise H1 data for the slope [3].

### 3 $Q^2$ dependence of $F_2$ and parton distributions in generalized DAS approximation

To begin with, we shortly write down the results of the generalized DAS firstly presented in Ref. [14]. The small $x$ behavior of the parton densities and $F_2$ at NLO has the form:

$$f_a(z, Q^2) = f_a^+(z, Q^2) + f_a^-(z, Q^2)$$

and

Hereafter $z = x/x_0$, where $x_0$ is a free parameter which limits the applicability range of formulae (8)-(11) and can be fitted from experimental data together with the magnitudes of gluon and sea quark distributions at $Q_0^2$. As it has been shown in Ref. [14], the fits to $F_2$ HERA data depend very slightly on the concrete $x_0$ value.
\[ f_a^-(z, Q^2) \sim \exp(-d_-(1)s - D_-(1)p) + O(z) \]
\[ f_g^+(z, Q^2) \sim I_0(\sigma) \exp(-\overline{d}_+(1)s - \overline{D}_+(1)p) + O(\rho) \]
\[ f_q^+(z, Q^2) \sim f_g^+(z, Q^2) \cdot \left(1 - \overline{d}_+^+(1)a_s(Q^2)\right) \frac{\beta_1(\sigma)}{I_0(\sigma)} + 20a_s(Q^2) + O(\rho) \]
\[ F_2(z, Q^2) = e \cdot \left(f_q(z, Q^2) + \frac{2}{3}f_a(Q^2)f_g(z, Q^2)\right), \]

where \( e = \sum_i e_i^2/f \) is the average charge square of \( f \) effective quarks, \( a_s = \alpha_s/(4\pi) \) and

\[ s = \ln \left( \frac{a_s(Q_0^2)}{a_s(Q^2)} \right), \quad p = a_s(Q_0^2) - a_s(Q^2), \quad D_\pm = d_\pm - \frac{\beta_1}{\beta_0}d_\pm, \]
\[ \sigma = 2\sqrt{(d_+s + D_+p)\ln z}, \quad \rho = \sqrt{\frac{(d_+s + D_+p)\ln z}{\ln z}} = \frac{\sigma}{2\ln(1/z)}, \]

and \( \beta_0 \) and \( \beta_1 \) are first two terms of QCD \( \beta \)-function.

The components of the leading order (LO) AD \( d_-(n) \) and singular parts \( \hat{d}_+ \) and regular ones \( \overline{d}_+(n) \) of the LO AD \( d_+(n) = \hat{d}_+(n-1) + \overline{d}_+(n) \) have the following form (at \( n \to 1 \)):

\[ \hat{d}_+ = -\frac{12}{\beta_0}, \quad \overline{d}_+(1) = 1 + \frac{20f}{27\beta_0}, \quad d_-(1) = \frac{16f}{27\beta_0} \]

The corresponding NLO components can be represented as follows:

\[ \hat{d}^{++} = \frac{412}{27\beta_0}f, \quad \hat{d}^{+-} = -20, \quad \hat{d}^{--} = 0, \]
\[ \overline{d}^{++}(1) = \frac{8}{\beta_0} \left(36\zeta_3 + 33\zeta_2 - \frac{1643}{12} + \frac{2}{9}f\left[\frac{68}{9} - 4\zeta_2 - \frac{13}{243}f\right] \right), \]
\[ \overline{d}^{+-}(1) = \frac{134}{3} - 12\zeta_2 - \frac{13}{81}f, \quad \overline{d}^{--}(1) = \frac{80}{81}f, \]
\[ d^{--}(1) = \frac{16}{9\beta_0} \left(2\zeta_3 - 3\zeta_2 + \frac{13}{4} + f\left[\frac{23}{18} - \frac{13}{243}f\right] \right), \]
\[ d^{+-}(1) = 0, \quad d^{++}(1) = -3\left(1 + \frac{f}{81}\right). \]

Some interesting features of the results given in Eqs. (8)-(12) are summarized below:

- Both, the gluon and quark singlet densities given above are presented in terms of two components ('+' and '−') which are obtained from the analytical \( Q^2 \)-dependent expressions of the corresponding ('+' and '−') components of PD moments.

- The '−' component is constant at small \( x \), whereas the '+' component grows at \( Q^2 \geq Q_0^2 \) as \( \sim \exp(\sigma) \), where \( \sigma \) contains the positive LO term \( |\hat{d}_+|s\ln(1/z) \) and the negative NLO one \( |\overline{D}_+|p\ln(1/z) \) (see Eq. (12)). So, the most important part from the NLO corrections (i.e. the singular part at \( x \to 0 \)) is taken in a proper way: it comes directly into the argument of the Bessel functions and does not spoil the applicability of perturbation theory at low \( x \) values.
4 \( Q^2 \) dependence of the slope \( d \ln F_2 / d \ln (1/x) \) in generalized DAS approximation

The behavior of the parton densities and the structure function \( F_2 \) within generalized DAS approach, given by Eqs. (8)-(11), can be represented by a power law shape over a limited region of \( x \) and \( Q^2 \) [4, 11]:

\[
f_a(x, Q^2) \sim x^{-\lambda_a^{\text{eff}}(x, Q^2)} \quad \text{and} \quad F_2(x, Q^2) \sim x^{-\lambda_{F_2}^{\text{eff}}(x, Q^2)}
\]

Because \( d/d \ln x = d/d \ln z \), the effective slopes can be obtained directly from Eqs. (8)-(11). They have the form:

\[
\begin{align*}
\lambda_g^{\text{eff}}(z, Q^2) &= \frac{f^+_g(z, Q^2)}{f_g(z, Q^2)} \cdot \rho \cdot \frac{I_1(\sigma)}{I_0(\sigma)} \\
\lambda_q^{\text{eff}}(z, Q^2) &= \frac{f^+_q(z, Q^2)}{f_q(z, Q^2)} \cdot \rho \cdot \frac{I_2(\sigma)(1 - \frac{\rho}{\lambda^+}(1)a_s(Q^2)) + 20a_s(Q^2)I_1(\sigma)/\rho}{I_1(\sigma)(1 - \frac{\rho}{\lambda^+}(1)a_s(Q^2)) + 20a_s(Q^2)I_0(\sigma)/\rho} \\
\lambda_{F_2}^{\text{eff}}(z, Q^2) &= \frac{\lambda_q^{\text{eff}}(z, Q^2) \cdot f^+_q(z, Q^2) + (2f)/3a_s(Q^2) \cdot \lambda_g^{\text{eff}}(z, Q^2) \cdot f^+_g(z, Q^2)}{f_q(z, Q^2) + (2f)/3a_s(Q^2) \cdot f_g(z, Q^2)}
\end{align*}
\]

It is interesting to emphasize that from Eq. (14) one obtains that the gluon effective slope \( \lambda_g^{\text{eff}} \) is larger than the quark slope \( \lambda_q^{\text{eff}} \), which is in excellent agreement with MRS [35] and GRV [11] analyses (see also Ref. [10]).

On the other hand the effective slopes \( \lambda_g^{\text{eff}} \) and \( \lambda_{F_2}^{\text{eff}} \) in Eq. (15) depend on the magnitudes of the initial PD, \( A_n \), and also on the chosen input values \( Q_0^2 \) and \( \Lambda \). However, at quite large \( Q^2 \), where the “-” component is negligible, the dependence on the initial PD disappears, having in this case, for the asymptotic behavior, the following expressions:

\[
\begin{align*}
\lambda_g^{\text{eff,as}}(z, Q^2) &= \rho \cdot \frac{I_1(\sigma)}{I_0(\sigma)} \approx \rho - \frac{1}{4 \ln (1/z)} \\
\lambda_q^{\text{eff,as}}(z, Q^2) &= \rho \cdot \frac{I_2(\sigma)(1 - \frac{\rho}{\lambda^+}(1)a_s(Q^2)) + 20a_s(Q^2)I_1(\sigma)/\rho}{I_1(\sigma)(1 - \frac{\rho}{\lambda^+}(1)a_s(Q^2)) + 20a_s(Q^2)I_0(\sigma)/\rho} \\
&\approx \left( \rho - \frac{3}{4 \ln (1/z)} \right) \left( 1 - \frac{10a_s(Q^2)}{(d_s + D_s)} \right) \\
\lambda_{F_2}^{\text{eff,as}}(z, Q^2) &= \lambda_q^{\text{eff,as}}(z, Q^2) \frac{1 + 6a_s(Q^2)/\lambda_g^{\text{eff,as}}(z, Q^2)}{1 + 6a_s(Q^2)/\lambda_g^{\text{eff,as}}(z, Q^2)} + O(a_s^2(Q^2)) \\
&\approx \lambda_q^{\text{eff,as}}(z, Q^2) + \frac{3a_s(Q^2)}{\ln (1/z)},
\end{align*}
\]

where the symbol \( \approx \) denotes that an approximation was done in the expansion of the modified Bessel functions \( I_n(\sigma) \) \( n = 0, 1, 2 \). These approximations are accurate only at large \( \sigma \) values (i.e. at large \( Q^2 \) and/or small \( x \)).

\textsuperscript{4} The asymptotic formulae given in Eq. (16) work quite well at any \( Q^2 \geq Q_0^2 \) values, because at \( Q^2 = Q_0^2 \) the values of \( \lambda_g^{\text{eff}} \) and \( \lambda_{F_2}^{\text{eff}} \) are equal zero. The use of approximations in Eq. (15) instead of the exact results given in Eq. (16) underestimates (overestimates) only slightly the gluon (quark) slope at \( Q^2 \geq Q_0^2 \). For the \( F_2 \) case, the similarity of \( \lambda_{F_2}^{\text{eff}} \) and \( \lambda_{F_2}^{\text{eff,as}} \) values is shown in Fig 1.
Finally, we note that at LO the $F_2$ slope $\lambda_{F_2}^{eff,as}$ is equal to the quark slope $\lambda_q^{eff,as}$ and it coincides, for very large values of $\sigma$, with the result from Ref. [36] in the case of a flat input (see also the first article in Ref. [7]). At NLO, $\lambda_{F_2}^{eff,as}$ lies between the quark and gluon slopes but closer to the former (see Fig. 3 in Ref. [14]).

5 Comparison with experimental data

Using the results of previous section we have analyzed HERA data for the slope $d \ln F_2/d \ln(1/x)$ at small $x$ from the H1 Collaboration [3].

Initially our results for $\lambda_{F_2}^{eff}$ depends on five parameters, i.e. $Q_s^0$, $x_0$, $A_q$, $A_g$ and $\Lambda_{\overline{MS}}(f=4)$. In our previous article [14] we have fixed $\Lambda_{\overline{MS}}(f=4)=250$ MeV which was a reasonable value extracted from the traditional (higher $x$) experiments. All other parameters have been fitted and very good agreement with $F_2$ HERA data has been found for $Q_s^0 \sim 1$ GeV$^2$ (all results depend very slightly on $x_0$).

In this work we take $\Lambda_{\overline{MS}}(f=4)=292$ MeV in agreement with the more recent H1 results [5] and other analyses (see Ref. [37] and references therein) and we fit directly the slope $d \ln F_2/d \ln(1/x)$ data using Eq. (15). The result is shown in Fig. 1. For comparison we also plot the curves from a fit to $F_2$ data in Ref. [14] where the value of $250$ MeV was used. The results are very similar and demonstrate the very important feature of an approximate $x$-independence of $\lambda_{F_2}^{eff}$ as given by Eq. (15), which is in agreement with H1 data [5].

We also present in Fig. 1 the asymptotic values for the slope $\lambda_{F_2}^{eff,as}$ as obtained from Eq. (16). The agreement with data and with the other curves is also rather good if one takes in consideration that, in this case, there is no fit involved because the only parameters entering Eq. (16) are the fixed values $\Lambda_{\overline{MS}}(f=4)=292$ MeV and $x_0 = 1$.

Thus, the tiny $x$ dependence of the slope $\lambda_{F_2}^{eff}(x,Q^2)$ in the considered region of $x$ and $Q^2$ support the possibility to successfully use our generalized DAS approximation in the type of $x$ independent analysis for the $F_2$ slope.

Fig. 2 shows the experimental data for $\lambda_{F_2}^{eff}$ and the corresponding H1 parameterization [5] written above in Eq. (7). We have also plotted the result from Eq. (16) and one from Eq. (13) using the parameters from our previous article [14] as it was presented before in Fig. 1. For both cases, we give it for two representative $x$ values.

From the visual inspection of Fig. 1, one can notice that the boundaries and mean values of the experimental $x$ ranges [3] increase proportionally with $Q^2$, what is related to the kinematical restrictions in the HERA experiments: $x \sim 10^{-4} \cdot Q^2$ (see Refs. [3, 4, 5, 13] and, for example, Fig. 1 of [4]).

Fig. 3 shows H1 experimental data [5] for $\lambda_{F_2}^{eff}$ and the H1 parameterization (Eq. (7)), as given in Fig. 2, but this time in comparison with the asymptotic values $\lambda_{F_2}^{eff,as}$ calculated from Eq. (16) using $x = a \cdot 10^{-4} \cdot Q^2$ with $a = 0.1, 1$ and 10. One has a reasonable agreement with H1 data for $Q^2 > 2$ GeV$^2$ using $a$ between 0.1 and 1 (the two
lower dashed curves in Fig. 3), which approximately corresponds to the middle points of the measured $x$ range.

6 Conclusions

We have studied the $Q^2$ dependence of the slope $\lambda_{F_2}^{\text{eff}} = d\ln F_2/d\ln(1/x)$ at small $x$ in the framework of perturbative QCD. Our results are in good agreement with new precise experimental H1 data \[3\] at $Q^2 \geq 2 \text{ GeV}^2$, where perturbative theory can be applicable.

Despite the fact that our approach, which can be called generalized doubled asymptotic scaling approximation, is based on pure perturbative grounds: a flat initial conditions at $Q^2_0 \approx 1 \text{ GeV}^2$ and dynamical evolution to $Q^2 \geq Q^2_0$ (it is very close conceptually to GRV approach but contains exact analytical $Q^2$ evolution), it can be reasonable applied for the new precise data of the slope $\lambda_{F_2}^{\text{eff}}$.

The agreement between $\lambda_{F_2}^{\text{eff}}$ data and perturbative QCD has been already observed by H1\[4\] and ZEUS \[1\] collaborations. The obtained linear rise of $\lambda(Q^2)$ with $\ln Q^2$ (see for example Figs. 2 and 3), parametrized by H1 as in Eq. (7), can be naively interpreted in the strong nonperturbative way, i.e. $\lambda(Q^2) \sim 1/\alpha_s(Q^2)$. Our analysis, however, demonstrates that the rise can be explained as $\sim \ln \ln Q^2$, that is natural in perturbative QCD at low $x$ (see \[1\], \[2\]-\[15\] and references therein): when the coupling constant is running, the renormalization group leads to the small $x$ behavior for the PD $\sim \ln(\alpha_s(Q^2))$ at the LO of perturbation theory and $\sim \alpha_s(Q^2)$ at NLO (see Eqs. (8)-(12) and discussions after Eq. (14)).

The good agreement between perturbative QCD and the experiment obtained here and in Ref. \[14, 15\] demonstrate the fact that for $Q^2 > 2 \text{ GeV}^2$ non-perturbative contributions as shadowing effects \[38\], higher twist effects \[39\] and others seem to be quite small (see also Ref. \[40\] and references therein) or they seem to be canceled between them and/or with $\ln(1/x)$ terms contained in the higher orders of perturbation theory. Note, however, that higher twist corrections are important at $Q^2 \leq 1 \text{ GeV}^2$, as it has been demonstrated in Ref. \[15, 20, 37\]. To clear up the correct contributions from nonperturbative dynamics and higher orders containing strong $\ln(1/x)$ terms, it is necessary further efforts in the development of theoretical approaches.

Moreover, the good agreement between perturbative QCD and experimental data at low $Q^2$ can be explained with a larger effective scale for the QCD coupling constant \[14, 15\]. The similar behavior has already been observed in the framework of perturbative QCD \[41\] and in BFKL-motivated approaches \[12\]-\[14\] (see the recent review in Ref. \[43\] and discussions therein).

Note that large NLO corrections calculated recently in the framework of BFKL \[47\] (see also \[17\]) lead to a strong suppression of the LO BFKL results for the high energy asymptotic behavior of the cross-section (see, for example, \[12\] an \[13\]). The careful inclusion of NLO corrections leads to results which are quite close to those obtained in pure perturbative QCD \[43\]. This fact can give an additional support for the good applicability of perturbation theory in the small $x$ range, where, as was expected before, nonperturbative effects should give an essential contribution.

As the next step, it could be very useful to apply the generalized doubled asymptotic scaling approach to make a combined analysis of HERA data for $F_2$, $dF_2/d\ln(Q^2)$,
\(d \ln F_2/d \ln(1/x)\) and \(F_L\). We hope to consider this work in a forthcoming article, including higher twist corrections in the \(Q^2\)-evolution approach given by Eqs. (8)-(11). We also plan to consider additional terms in the initial condition given by Eq. (4) proportional to \(\ln(1/x)\) and \(\ln^2(1/x)\).

We hope that the analysis will be relevant to find out the kinematical region where the well-established perturbative QCD formalism can be safely applied at small \(x\). Moreover, the study should clear up the reason of the good agreement between the small \(x\) relation of \(F_L, F_2\) and \(dF_2/d \ln(Q^2)\) obtained in pure perturbative QCD in Ref. [48] (and based on previous works [26, 49]), the experimental data for these structure functions [50, 2] and the predictions of Ref. [51] in the framework of \(k_t\)-factorization [21, 22, 52].

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Figure captions

Figure 1. The derivative \( \frac{d \ln F_2}{d \ln (1/x)} \) (effective slope \( \lambda \)) as a function of \( x \) for different \( Q^2 \) values. Data points are from H1 [5]. Only statistical uncertainties are shown. The solid line is the result from a fit using \( \lambda_{F_2}^{\text{eff}} \) in Eq. (15) with fixed \( Q_0^2 = 1 \) GeV\(^2\) and \( x_0 = 1 \). The dotted line is the same as the solid but with the parameters from a fit to \( F_2 \) data in Ref. [14]. The dashed line corresponds to the asymptotic expression \( \lambda_{F_2}^{\text{eff,as}} \) in Eq. (16).

Figure 2. The derivative \( \frac{d \ln F_2}{d \ln (1/x)} \) (effective slope \( \lambda \)) as a function of \( Q^2 \). Data points are from H1 [5]. Outer error bars include statistic and systematic added in quadrature. Inner bars correspond to statistic errors. The solid line corresponds to the H1 parameterization [5] given in Eq. (7). Dotted and dashed curves are produced as in Fig. 1. For the lower (upper) curves, the value \( x = 10^{-4} \) (\( x = 10^{-2} \)) was used.

Figure 3. The derivative \( \frac{d \ln F_2}{d \ln (1/x)} \) (effective slope \( \lambda \)) as a function of \( Q^2 \). Data points are from H1 [5]. Error bars and solid line are as indicated in Fig. 2. The dashed lines were calculated with Eq. (16) using \( x = a \cdot 10^{-4} \cdot Q^2 \) with \( a = 0.1, 1 \) and 10. Upper curves correspond to larger \( x \).
Fig. 1

\[ \lambda(x, Q^2) \]

- \( Q^2 = 2 \text{ GeV}^2 \)
- \( Q^2 = 2.5 \text{ GeV}^2 \)
- \( Q^2 = 3.5 \text{ GeV}^2 \)
- \( Q^2 = 5 \text{ GeV}^2 \)
- \( Q^2 = 6.5 \text{ GeV}^2 \)
- \( Q^2 = 8.5 \text{ GeV}^2 \)
- \( Q^2 = 12 \text{ GeV}^2 \)
- \( Q^2 = 15 \text{ GeV}^2 \)
- \( Q^2 = 20 \text{ GeV}^2 \)
- \( Q^2 = 25 \text{ GeV}^2 \)
- \( Q^2 = 35 \text{ GeV}^2 \)
- \( Q^2 = 45 \text{ GeV}^2 \)
- \( Q^2 = 60 \text{ GeV}^2 \)
- \( Q^2 = 90 \text{ GeV}^2 \)
- \( Q^2 = 120 \text{ GeV}^2 \)
- \( Q^2 = 150 \text{ GeV}^2 \)
