Loop Quantum Gravity:  
An Inside View

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Abstract

This is a (relatively) non–technical summary of the status of the quantum dynamics in Loop Quantum Gravity (LQG). We explain in detail the historical evolution of the subject and why the results obtained so far are non–trivial. The present text can be viewed in part as a response to an article by Nicolai, Peeters and Zamaklar. We also explain why certain no go conclusions drawn from a mathematically correct calculation in a recent paper by Helling et al are physically incorrect.
1 Introduction

Loop Quantum Gravity (LQG) [1, 2, 3] has become a serious competitor to string theory as a candidate theory of quantum gravity. Its popularity is steadily growing because it has transpired that the major obstacle to be solved in combing the principles of General Relativity and Quantum Mechanics is to preserve a key feature of Einstein's theory, namely background independence. LQG, in contrast to the present formulation of string theory, has background independence built in by construction.

Loosely speaking, background independence means that the spacetime metric is not an external structure on which matter fields and gravitational perturbations propagate. Rather, the metric is a dynamical entity which becomes a fluctuating quantum operator. These fluctuations will be huge in extreme astrophysical (centre of black holes) and cosmological (big bang singularity) situations and the notion of a (smooth) background metric disappears, the framework of quantum field theory on (curved) background metrics [4, 5] becomes meaningless. Since quantum gravity is supposed to take over as a more complete theory precisely in those situations when there is no meaningful concept of a (smooth) metric at all available, background independence is a necessary feature of a successful quantum gravity theory.

Indeed, the modern formulation of ordinary QFT on background spacetimes uses the algebraic approach [6] and fundamental for this framework is the locality axiom: Two (scalar) field operators $\hat{\phi}(f), \hat{\phi}(f')$ which are smeared with test functions $f, f'$ whose supports are spacelike separated are axiomatically required to commute. In other words, the causality structure of the external background metric defines the algebra of field operators $A$. One then looks for Hilbert space representations of $A$. It follows, that without a background metric one cannot even define quantum fields in the usual setting.

Notice that background independence implies that a non-perturbative formulation must be found. For, if we split the metric as $g = g_0 + h$ where $g_0$ is a given background metric and treat the deviation $h$ (graviton) as a quantum field propagating on $g_0$, then we break background independence because we single out $g_0$. We also break the symmetry group of Einstein’s theory which is the group of diffeomorphisms of the given spacetime manifold $M$. Moreover, it is well-known that this perturbative formulation leads to a non-renormalisable theory, with [7] or without [8] supersymmetry. String theory [9] in its current formulation is also background dependent because one has to fix a target space background metric (mostly Minkowski space or maybe AdS) and let strings propagate on it. Some excitations of the string are interpreted as gravitons and one often hears that string theory is perturbatively finite to all orders, in contrast to perturbative quantum gravity. However, this has been established only to second order and only for the heterotic string [10] which is better but still far from a perturbatively finite theory. In fact, even perturbative finiteness is not the real issue because one can formulate perturbation theory in such a way that UV divergences never arise [11]. The issue is whether 1. only a finite number of free renormalisation constants need to be fixed (predictability) and 2. whether the perturbation series converges. Namely, in a fundamental theory as string theory claims to be, there is no room for singularities such as a divergent perturbation series. This is different from QED which is believed to be only an effective theory. Hence, before one does not prove convergence of the perturbation series, string theory has not been shown to be a fundamental theory. All these issues are obviously avoided in a manifestly non-perturbative formulation.

One of the reasons why LQG is gaining in its degree of popularity as compared to string theory is that LQG has “put its cards on the table”. LQG has a clear conceptual setup which follows from physical considerations and is based on a rigorous mathematical formulation. The “rules of the game” have been written and are not tinkered with. This makes it possible even for outsiders of the field [12, 13] to get a relatively good understanding of the physical and mathematical details. There is no room for extra, unobserved structures, the approach is intendedly minimalistic. In LQG one just tries to make quantum gravity and general relativity work together harmonically. However, in order to do so one must be ready to go beyond some of the mathematical structures that we got used to from ordinary QFT as we have explained. Much of the criticism against LQG of which some can be found in [12] has to do with the fact that physicists equipped with a particle physics background feel uneasy when one explains to them that in LQG we cannot use perturbation theory, Fock spaces, background metrics etc. This is not the fault of
LQG. It will be a common feature of all quantum gravity theories which preserve background independence. In such theories, the task is to construct a new type of QFT, namely a QFT on a differential manifold $M$ rather than a QFT on a background spacetime $(M, g_0)$. Since such a theory “quantises all backgrounds at once” in a coherent fashion, the additional task is then to show that for any background metric $g_0$ the theory contains a semiclassical sector which looks like ordinary QFT on $(M, g_0)$. This is what LQG is designed to do, not more and not less.

Another criticism which is raised against LQG is that it is not a unified theory of all interactions in the sense that string theory claims to be. Indeed, in LQG one quantises geometry together with the fields of the (supersymmetric extension of the) standard model. At present there seem to be no constraints on the particle content or the dimensionality of the world. In fact, this is not quite true because the size of the physical Hilbert space of the theory may very well depend on the particle content and moreover, the concrete algebra $\mathfrak{g}$ which one quantises in LQG is available only in $3+1$ dimensions. But apart from that it is certainly true that LQG cannot give a prediction of the matter content. The fact that all matter can be treated may however be an advantage because, given the fact that in the past 100 years we continuously discovered substructures of particles up to the subnuclear scale makes it likely that we find even more structure until the Planck scale which is some sixteen orders of magnitude smaller than what the LHC can resolve. Hence, LQG is supposed to deliver a universal framework for combining geometry and matter, however, it does not uniquely predict which matter and does not want to. Notice that while theorists would find a “unique” theory aesthetical, there is no logical reason why a fundamental theory should be mathematically unique.

In this context we would like to point out the following: One often hears that string theory is mathematically unique, predicting supersymmetry, the dimensionality (ten) of the world and the particle content. What one means by that is that a consistent quantum string theory based on the Polyakov action on the Minkowski target spacetime exists only if one is in ten dimensions and only if the theory is supersymmetric and there are only five such theories. However, this is not enough in order to have a unique theory because string theory must be decompactified from ten to four dimensions with supersymmetry being broken at sufficiently high energies in order to be phenomenologically acceptable. Recent findings [14] show that for Minkowski space there are an at least countably infinite number of physically different, phenomenologically acceptable ways to compactify string theory from ten to four dimensions. These possibilities are labelled by flux vacua and the resulting collection of quantum string theories is called the landscape. In this sense, string theory is far from being mathematically unique. The presence of an infinite number of possibilities could be interpreted as the loss of predictability of string theory and the use of the anthropic principle was proposed [15]. The question, whether a physical theory that needs the anthropic principle still can be called a fundamental theory, was discussed in [16].

Our interpretation of the landscape is the following which is in agreement with [17]: The anthropic principle should be avoided by all means in a fundamental theory, hence a new idea is needed and it is here where background independence could help. We notice that each landscape vacuum is based on a different background structure (flux charges, moduli). In addition, a landscape will exist for each of the uncountably infinite number of target space background spacetimes\(^1\) Thus, the full string theory landscape is presumably labelled not only by flux vacua on a given spacetime but also by the background spacetimes. Suppose one could rigorously quantise string theory on all of these background structures. Then, if one knew how to combine all of these distinct quantum theories into a single one, thus achieving background independence, then the landscape would have disappeared. The understanding of the present author is that this is what M – theory is supposed to achieve but currently, to the best knowledge of the author, there seems to be no convincing idea for how to do that.

In the present article we summarise the status of the quantum dynamics of LQG which has been the

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\(^1\)Currently string theory can be quantised only on a hand full of target space background spacetimes, mostly only on those on which the theory becomes a free field theory. For instance, on AdS$_5 \times $S$^5$, which is much discussed in the context of the famous AdS/CFT correspondence [18], string theory is interacting and to the best knowledge of the author currently no quantum string theory on this spacetime was constructed.
focal point of criticism in [12]. Our intention is to give a self-contained inside point of view of the subject which is complementary to [12] in the sense that we explain in some detail why the constructions used in LQG are physically well motivated and sometimes even mathematically unique, hence less ambiguous than described in [12]. We exactly define what is meant by canonical quantisation of general relativity, indicating explicitly the freedom that one has at various stages of the quantisation programme. We will see that the theory has much less freedom than [12] makes it look like. In particular, we evaluate “what has been gained in LQG as compared to the old geometrodynamics approach” and we will see that the amount of progress is non-trivial. We also include more recent results such as the Master Constraint Programme [19] which tightens the implementations of the quantum dynamics and enables to systematically construct the physical inner product, which was not possible as of three years ago. Furthermore, in order to show that there is not only mathematical progress in LQG, we also fill the gaps that [12] did not report on such as the semi-classical sector of the theory, matter coupling, quantum black hole physics, quantum cosmology and LQG phenomenology. Finally we also show that the key technique that was used to make the Wheeler–DeWitt operator well-defined [20] is also the underlying reason for the success of Loop Quantum Cosmology (LQC) which is the usual cosmological minisuperspace toy model quantised with the methods of LQG.

Here we only sketch these results since we want to reach a rather general audience. All the technical details can be found in the comprehensive and self-contained monograph [2].

Remark
Since no theoretical Ansatz concerning Quantum Gravity has been yet brought to completion each of these Ansätze is understandably subject to criticisms. This is clearly the case also for LQG, and in particular such criticisms can be found e.g., in [12] [13]. We want to stress that the purpose of this article, while in part a response to [12], is not to criticise [12]. In fact, the considerable effort of the non expert authors of [12] to present LQG in as much technical detail as they did from a particle physicist’s perspective is highly appreciated. Rather, what we have in mind is to draw a more optimistic picture than [12] did, to hopefully resolve confusions that may have arisen from gaps in [12] and to give a more complete picture of all the research being done in LQG than [12] did. The discussion will be kept objectively, problems with the present formulation of LQG will not be swept under the rug but rather discussed in great detail together with their possible solutions.
2 Classical preliminaries

The starting point is a Lagrangean formulation of the classical field theory, say the Einstein Hilbert Lagrangean for General Relativity. Hence one has an action

\[ S = \int_M d^{n+1}x\ L(\Phi, \partial \Phi) \quad (2.1) \]

where \( L \) is the Lagrangean density, that is, a scalar density of weight one constructed in a covariant fashion from the fields \( \Phi \) and their first partial derivatives\(^2\) which is sufficient for gravity and all known matter. Here \( \Phi \) stands for a collection of fields including the metric and all known matter. \( M \) is an \((n+1)\) – dimensional, smooth differential manifold.

If one wants to have a well – posed initial value formulation, then the metric fields \( g \) that live on \( M \) are such that \((M, g)\) is globally hyperbolic which implies \(^3\) that \( M \) is diffeomorphic to the direct product \( \mathbb{R} \times \sigma \) where \( \sigma \) is an \( n \)-dimensional smooth manifold. Since the action \((2.1)\) is invariant under \( \text{Diff}(M) \), the diffeomorphisms \( Y : \mathbb{R} \times \sigma \to M ; (t, x) \to Y_t(x) \) are a symmetry of the action. For each \( Y \) we obtain a foliation of \( M \) into one parameter family of spacelike hypersurfaces \( \Sigma_t = Y_t(\sigma) \). One now pulls all fields back by \( Y \) and obtains an action on \( \mathbb{R} \times \sigma \). Due to the arbitrariness of \( Y \), this action contains \( n + 1 \) fields, usually called lapse and shift fields, which appear without time derivatives, they are Lagrange multipliers.

The Legendre transformation is therefore singular and leads to constraints on the resulting phase space \(^4\). They can be obtained by extremisation of the action with respect to the Lagrange multipliers.

Hence, after the Legendre transformation we obtain a phase space \( \mathcal{M} \) of canonical fields \( \phi \) which are the pull – backs to \( \sigma \) of the spacetime fields \( \Phi \) together with their canonically conjugate momenta \( \pi \). The symplectic manifold \( \mathcal{M} \) with coordinates \( (\phi(x), \pi(x))_{x \in \sigma} \) equipped with the corresponding canonical bracket \(^4\)

\[ \{\phi(x), \pi(x')\} = \delta^{(0)}(x, x') \]

is a time \( t \) independent object.

As one can show by purely geometric arguments \(^5\), these constraints are automatically of first class in the terminology of Dirac, that is, they close under their mutual Poisson brackets, irrespective of the matter content of the theory. As we will need them in some detail later on, let us display this so called Dirac algebra \( \mathfrak{D} \) in more detail

\[
\begin{align*}
\{D(\tilde{N}), D(\tilde{N}')\} &= 8\pi G_{\text{Newton}} \ D(L_{\tilde{N}}\tilde{N}') \\
\{D(\tilde{N}), H(N')\} &= 8\pi G_{\text{Newton}} \ H(L_{\tilde{N}}N') \\
\{H(N), H(N')\} &= 8\pi G_{\text{Newton}} \ D(q^{-1}N dN' - N' dN) 
\end{align*}
\]

The notation is as follows: We distinguish between the so – called spatial diffeomorphism constraints \( D(\tilde{N}) \), \( x \in \sigma \) and the Hamiltonian constraints \( H(x) \), \( x \in \sigma \). Notice that these are infinitely many constraints, \( n + 1 \) per \( x \in \sigma \). We smear them with test functions \( N^a, N \), specifically

\[ D(\tilde{N}) = \int_{\sigma} d^3x \ N^a D_a \quad \text{and} \quad H(N) = \int_{\sigma} d^3x \ N H. \]

Finally, \( q_{ab} \) is the pull – back to \( \sigma \) of the spacetime metric with inverse \( q^{ab} \) and \( L \) denotes the Lie derivative.

The algebra \( \mathfrak{D} \) is universal and underlies the canonical formulation of any field theory based on an action which is \( \text{Diff}(M) \) invariant and contains General Relativity in \( n + 1 \) dimensions as for instance the closed bosonic string\(^5\).

As we can read off from \((2.2)\), it has the following structure: The first line in \((2.2)\) says that elements of the form \( D(\tilde{N}) \) generate a subalgebra which can be identified with the Lie algebra \( \text{diff}(\sigma) \) of the spatial

\(^1\)Higher derivative theories can also be treated canonically \(^2\), however, they are generically pathological, that is, unstable \(^2\).

\(^2\)Unless otherwise stated we take \( \sigma \) to be compact without boundary in order to avoid a lengthy discussion of boundary terms.

\(^3\)We suppress spatial tensor and internal Lie algebra indices.

\(^4\)A closed bosonic string is an embedding of a circle \( \sigma := S^1 \) into a \( D + 1 \) dimensional target space background manifold, mostly \( D + 1 \) dimensional Minkowski space. The spacetime manifold of the string is therefore \( M = \mathbb{R} \times S^1 \), i.e. a 2D cylinder. In 2D gravity is topological, hence gravity is also trivially contained in string theory.
diffeomorphism group $\text{Diff}(\sigma)$ of $\sigma$. This is why the $D(\vec{N})$, where the $\vec{N}$ are arbitrary smooth vector fields on $\sigma$ of rapid decrease, are called spatial diffeomorphism constraints. The second line in (2.2) says that $\text{diff}(\sigma)$ is not an ideal of $\mathfrak{D}$ because the Hamiltonian constraints $H(N)$, where the $N$ are arbitrary smooth functions on $\sigma$ of rapid decrease, are not $\text{diff}(\sigma)$ invariant. The name Hamiltonian constraint stems from the fact that the Hamiltonian flow of this constraint on the phase space generates gauge motions which, when the equations of motion hold, can be identified with spacetime diffeomorphisms generated by vector fields orthogonal to the hypersurfaces $\Sigma_t$. Finally the third line in (2.2) says that (2.2) is not a Lie algebra in the strict sense of the word because, while the right hand side of the Poisson bracket between two Hamiltonian constraints is a linear combination of spatial diffeomorphism constraints, the coefficients in that linear combination have non--trivial phase space dependence through the tensor $q^{ab}(x)$.

A peculiarity happens in the case $n = 1$, such as the closed bosonic string: In $n = 1$ dimensions, $p$--times contravariant and $q$--times covariant tensors are the same thing as scalar densities of weight $q - p$. For this reason, in contrast to $n > 1$ dimensions, in $n = 1$ dimensions the constraints come with a natural density weight of two rather than one while the smearing functions acquire density weight $-1$ rather than $0$. One can think of this as if the actual constraints had been multiplied by a factor of $\sqrt{q}$ while the smearing functions had been multiplied by a factor of $\sqrt{q}^{-1}$ which however does not change the Poisson bracket because gravity is not dynamical in 2D. For this reason the factor $q^{-1}$ must be absent in the third relation in (2.2) in order to match the density weights on both sides of the equations. This is why only in 2D the Dirac algebra $\mathfrak{D}$ is a true Lie algebra. In fact, using the combinations $V_\pm := H \pm D$ one realises that the Dirac algebra acquires the structure a direct sum of loop algebras (Lie algebra of the diffeomorphism group of the circle) $\mathfrak{D} \cong \text{diff}(S^1) \oplus \text{diff}(S^1)$, see [28] for all the details. Thus, in 2D the Dirac algebra trivialises. It does not even faintly display the complications that come with the non--Lie algebra structure of $\mathfrak{D}$ in realistic field theories, i.e. $D = 4$. Therefore, any comparisons made between structures in 2D and 4D which hide this important difference are void of any lesson.

Proceeding with the general classical theory, what we are given is a phase space $\mathcal{M}$ subjected to a collection of constraints $C_I$, $I \in \mathcal{I}$ where in our case the labelling set comprises the $N, \vec{N}$. These constraints force us to consider the constraint hypersurface $\mathcal{M} := \{ m \in \mathcal{M}; C_I(m) = 0 \ \forall \ I \in \mathcal{I} \}$. The closure of $\mathfrak{D}$ means that the Hamiltonian flow of the $C_I$ preserves $\mathcal{M}$. Since the $C_I$ generate gauge transformations (namely spacetime diffeomorphisms), all the points contained in the gauge orbit $\{m\}$ through $m \in \mathcal{M}$ must be identified as physically equivalent. As one can show in general [29], the set of orbits $\mathcal{M} := \{ \{m\}; m \in \mathcal{M} \}$ is again a symplectic manifold and known as the reduced phase space.

It is mathematically more convenient to consider functions on all of $\mathcal{M}$ which are invariant under gauge transformations, called Dirac observables. Their restrictions to $m \in \mathcal{M}$ are completely determined by $[m]$. The physical idea to construct such functions is due to Rovelli [30] and its mathematical implementation has been much improved recently in [31] (see also [32]). For a particularly simple realisation of this so--called relational Ansatz in terms of suitable matter see [33]. We consider functions $T_I$ on $\mathcal{M}$ which have the property that the matrix with entries $A_{IJ} := \{C_I, T_J\}$ is invertible (at least locally). Let $X_I$ be the Hamiltonian vector field of the constraint $C_I := \sum_\lambda (A^{-1})_{IJ} C_J$. The set of constraints $C'_I := \sum_\lambda (A^{-1})_{IJ} C_J$. The set of constraints $C'_I$ is equivalent to the set of the $C_I$ but the $C'_I$ have the advantage that the vector fields $X_I$ are weakly (that is, on $\mathcal{M}$) mutually commuting. Now given any smooth function $f$ on $\mathcal{M}$ and any real numbers $\tau_I$ in the range of the $T_I$ consider

$$O_f(\tau) := [\alpha_I(f)]_{\tau - T}, \quad \alpha_I(f) := [\exp(\sum_I t_I X_I) \cdot f]$$

(2.3)

Notice that one is supposed to first evaluate $\alpha_I(f)$ with $t_I$ considered as real numbers and then evaluates the result at the phase space dependent point $t_I = \tau_I - T_I$. It is not difficult to show that (2.3) is a weak Dirac observable, that is $\{C_I, O_f(\tau)\}_{\mathcal{M}} = 0$. It has the physical interpretation of displaying the value of $f$ in the gauge $T_I = \tau_I$. Equivalently, it is the gauge invariant extension of $f$ off the gauge cut $T = \tau$ and in fact can be expanded in a power series in $\tau - T$ by expanding the exponential function in (2.3).

The relational Ansatz solves the problem of time of canonical quantum gravity: By this one means that in generally covariant systems there is no Hamiltonian, there are only Hamiltonian constraints. Since the
observables of the theory are the gauge invariant functions on phase space, that is, the Dirac observables, “nothing moves in canonical quantum gravity” because the Poisson brackets between the Hamiltonian constraints and the Observables vanishes (weakly) by construction. The missing evolution of the Dirac observables \(O_f(\tau)\) is now supplied as follows: Using the fact that the map \(\alpha_t\) in eq. (2.3) is actually a Poisson autormorphism (i.e. a canonical transformation) one can show that if 1. the phase space coordinates can be subdivided into canonical pairs \((T_I, \pi_I)\) and \((q^a, p_a)\) and 2. if \(f\) is a function of only \(q^a, p_a\) then the evolution in \(\tau\) has a Hamiltonian generator \[H(\tau)\]. That is, there exist Dirac observables \(H_I(\tau)\) such that \(\partial O_f(\tau)/\partial \tau_I = \{O_f(\tau), H_I(\tau)\}\).

The task left is then to single out a one parameter family \(s \mapsto \tau_I(s)\) such that the corresponding Hamiltonian

\[
H(s) = \sum_I \tau_I(s) H_I(\tau(s))
\] (2.4)

is positive, \(s\)-independent and reduces to the Hamiltonian of the standard model on flat space. This has been achieved recently in [33] using suitable matter which supplies the clocks \(T_I\) with the required properties. It follows that the gauge invariant functions \(O_f(s)\) then evolve according to the physical Hamiltonian \(H\). Moreover, they satisfy the algebra \(\{O_f(s), O_{f'}(s)\} = O_{\{f, f'\}}(s)\) because the \(s\) evolution has the canonical generator \(H\).

To summarise:

Classical canonical gravity has a clear conceptual and technical formulation with no mysteries or unsolved conceptual problems. Certainly classical General Relativity is not an integrable system and thus not everything is technically solvable (for instance not all solutions to the field equations are known) but one exactly knows what to do in order to try to solve a given problem. The canonical formulation that we have used here for a generally covariant field theory is widely used in numerical General Relativity with great success. General covariance is manifestly built into the framework and is faithfully represented in terms of the Dirac algebra \(\mathcal{D}\) which is the key object to construct the invariants \(\mathcal{I}\) of the theory and the physical Hamiltonian \(H\) according to which they evolve. At no point in those constructions did one use a background metric or did one violate spacetime diffeomorphism invariance. This is because, while one did use a split of spacetime into space and time, one did consider all splits simultaneously which is reflected in the constraints that in turn enforce spacetime diffeomorphism invariance.

For clarity we mention that diffeomorphism invariance should not be confused with Poincaré invariance. Poincaré invariance is an invariance of a special solution to Einstein’s vacuum equations. It is not a symmetry or a gauge invariance of the theory. The gauge group is \(\text{Diff}(M)\) which is a background metric independent object because it only refers to the differential manifold \(M\) but to no metric. In fact, if \(\sigma\) is compact as appropriate for certain cosmological models, then the Poincaré group \(\mathcal{P}\) has no place in the theory. If \(M\) is equipped with asymptotically flat boundary conditions then in fact one can in addition define Poincaré generators of \(\mathcal{P}\) as functions on phase space, called ADM charges [24]. These are particular Dirac observables. Notice that \(\mathcal{P}\) is not contained in \(\text{Diff}(M)\) because diffeomorphisms are of rapid decrease at spatial infinity (at least they vanish there). This must be because \(\mathcal{P}\) is a symmetry and not a local gauge invariance like \(\text{Diff}(M)\).

### 3 Canonical quantisation programme

The programme of canonical quantisation is a mathematical formalism which seeks to provide a quantum field theory from a given classical field theory. There are several choices to be made within the formalism and the outcome depends on it. This applies to ordinary field theories such as free scalar fields on Minkowski space as well as to more complicated situations. In the presence of constraints such as in General Relativity one would ideally solve the constraints classically before quantising the theory. That is, one studies the representation

\footnote{Nothing is lost by this assumption because \(T_I\) is pure gauge and the constraints can be solved for \(\pi_I\) in terms of \(q^a, p_a, T_I\).}
theory of the algebra of invariants such as \( \mathcal{E} \). Unfortunately, this is generically to difficult because the algebra of invariants is complicated and thus usually prevents one from using standard representations for simple algebras such as a Fock representation for usual CCR (canonical commutation relation) or CAR (canonical anticommutation relation) algebras.

Thus, in order to start the quantisation process one follows Dirac \[25\] and starts with a redundant set of functions on phase space which generate a sufficiently simple Poisson algebra so that suitable representations thereof can be found. These functions are not gauge invariant but provide a system of coordinates for \( \mathcal{M} \). Then, in a second step, provided that the constraints themselves can be represented on the chosen, so called kinematical, Hilbert space as closable\(^7\) and densely defined operators, one looks for the generalised joint kernel of the constraint operators. Here generalised refers to the fact that the joint kernel typically has trivial intersection with the Hilbert space, that is, the non zero solutions of the constraints are not normalisable. Rather, they are elements of the physical Hilbert space which is not a subspace of the kinematical Hilbert space. The physical Hilbert space is induced from the kinematical Hilbert space by applying standard spectral theory to the constraint operators. Once the physical Hilbert space is known, at least in principle, it automatically carries a self – adjoint representation of the algebra of strong observables, that is, those operators that commute with all quantum constraints and for which (2.3) and (2.4) are the classical counterparts.

All of this is of course difficult, if not impossible, to carry out exactly and in full completeness for General Relativity because, after all, one is dealing with a rather non – linear and highly interacting QFT. Hence, in praxis one will have to develop and rely on approximation schemes. However, these are only technical difficulties coming from the complexity of the theory. There are no in principle obstacles, the programme of canonical quantisation follows a clear sequence of steps at each of which one knows exactly what one has to do and sometimes one has a certain freedom which one will exploit using physical intuition.

After the above sketch of the programme, we will now become somewhat more detailed and pin down explicitly the freedom that one has and the choices that one has to make.

The starting point is a then a symplectic manifold \( \mathcal{M} \) subject to real valued, first class constraints \( C_I, \ I \in \mathcal{I} \). That is, we have \( \{ C_I, C_J \} = f_{I,J}^K C_K \) for some, possibly phase space dependent functions \( f_{I,J}^K \), called structure functions. We will assume for simplicity, as it is the case in General Relativity, that we are dealing with a completely parametrised system, that is, there is no apriori gauge invariant Hamiltonian. In order to simplify the discussion for the purposes of this short review, we display here for concreteness a recently proposed strategy \[19, 34\] to deal with those constraints: Consider instead of the individual constraints \( C_I \) the single Master Constraint \( M := \sum_I C_I K_{IJ} C_J \). Here \( K = (K_{IJ}) \) is a positive definite matrix valued function on phase space. The Master Constraint contains the same information about the gauge redundancy of the system as the individual \( C_I \) since \( M = 0 \) is equivalent with \( C_I = 0 \) for all \( I \) and the equation \( \{ O, \{ O, M \} \}_{M=0} \) is equivalent with \( \{ O, C_I \}_{M=0} \) for all \( I \). Hence the Master Constraint selects the the same reduced phase space as the original set of constraints. The reason for the using the matrix \( K \) is that we can and often must use the associated freedom to regularise the square of the constraints: namely, typically the \( C_I \) become operator valued distributions and their square is therefore ill – defined. By a judicious choice of \( K \) (which also becomes an operator) one can remove the corresponding UV singularity. See e.g. \[35\] for examples.

Given this set up, the programme of canonical quantisation consists of the following\(^8\):

**I. Algebra of elementary functions \( \mathcal{E} \)

Select a Poisson sub\(^*\)– algebra \( \mathcal{E} \) of \( C^\infty(\mathcal{M}) \), called elementary functions, which separates the points of \( \mathcal{M} \). That is, \( \mathcal{E} \) should be closed under taking Poisson brackets and complex conjugation and for any \( m \neq m' \) there exists \( e \in \mathcal{E} \) such that \( e(m) \neq e(m') \). The latter property implies that any \( f \in C^\infty(\mathcal{M}) \) can be thought of as a function of the elements of \( \mathcal{E} \) so that \( \mathcal{E} \) is a sytstem of coordinates for \( \mathcal{M} \) (which

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\(^7\)That is, the adjoint is also densely defined.

\(^8\)What follows is still a simplified version. See \[2\] for a complete discussion.
III. Kinematical Hilbert space

Next we study the representation theory of the kinematical Hilbert space $H$. Regularisation and renormalisation. For the purpose of this discussion it will be sufficient to insist that defined operators, possibly subject to some choice of factor ordering and maybe after some sort of problem is provided for instance by asking that (parts of) the gauge group be represented unitarily on the corresponding Hilbert space or that the constraints be represented at all as closable and densely defined operators, possibly subject to some choice of factor ordering and maybe after some sort of regularisation and renormalisation. For the purpose of this discussion it will be sufficient to insist that the kinematical Hilbert space $H$ carries the Master Constraint Operator $\hat{M}$ as a positive and self–adjoint operator.

IV. Physical Hilbert space

The idea to solve the Master Constraint is to apply spectral theory to it \[34\]. Suppose that the Hilbert $H$ decomposes into separable $\hat{M}$–invariant subspaces $H_\theta$ where $\theta$ labels the corresponding sectors. Then it is well known that $H_\theta$ is unitarily equivalent to a direct integral Hilbert space

$$H_\theta \cong H_\theta^\oplus := \int_{\text{spec}(\hat{M})} d\mu(\lambda) H_\lambda^\theta$$

Here the measure class $\mu$ on the spectrum $\text{spec}(\hat{M})$ of the Master Constraint is unique and the multiplicities $\dim(H_\lambda^\theta)$ are unique up to $\mu$–measure zero sets. The unitary map $U : H_\theta \to H_{\theta}^\oplus; \psi \mapsto (\hat{\psi}(\lambda))_\lambda$ is a generalisation of the Fourier transform and is such that $U \hat{M} \psi = (\lambda \hat{\psi}(\lambda))_\lambda$, that is $U \hat{MU}^{-1}$ is represented as multiplication by $\lambda$ on $H_\lambda^\theta$. We have

$$<\psi, \psi^\prime>_{H_\theta} = <U \psi, U \psi^\prime>_{H_\theta^\oplus} = \int_{\text{spec}(\hat{M})} d\mu(\lambda) <\hat{\psi}(\lambda), \hat{\psi^\prime}(\lambda)>_{H_\lambda^\theta}$$

The physical Hilbert space is now defined as

$$H_{\text{phys}} := \oplus_\theta H_{\lambda=0}^\theta$$

There are several remarks in order about $\text{33}$:

1. In order that this works one must Lebesgue decompose every space $H_\theta$ into the $\hat{M}$–invariant pure
point, absolutely continuous and continuous singular pieces and then decompose them as a direct integral.

2. The spaces $\mathcal{H}_\lambda^0$ are uniquely determined in the pure point case but in the absolutely continuous case (and continuous singular case, which usually is absent in practice) further input is required because here the set $\{\lambda = 0\}$ is of $\mu$-measure zero. Roughly speaking, one requires that the space $\mathcal{H}_\lambda^0$ carries a non-trivial, irreducible representation of the algebra of (strong) observables. See [44] for details.

3. Due to a bad choice of factor ordering involved in the construction of $\hat{M}$ it may happen that $0 \not\in \text{spec}(\hat{M})$. This typically happens when the quantum constraints $\hat{C}_I$ that enter the definition of $\hat{M}$ are anomalous, that is, if they do not close as a quantum algebra. This can easily happen especially in the case that the classical constraint algebra involves non-trivial structure functions rather than structure constants. Hence, although the Master Constraint always trivially forms a non anomalous algebra, possible anomalies in the original algebra are detected by it, so nothing is swept under the rug. In this case, following [36], we propose to replace $\hat{M}$ by $\hat{M} = \hat{M} - \lambda_0$ where $\lambda_0 = \text{min} (\text{spec}(\hat{M}))$. Here $\lambda_0$ should be finite and $\lim_{\eta \to 0} \lambda_0 = 0$ in order that both $\hat{M}$, $\hat{M}'$ have the same classical limit. This has worked so far in all studied cases [35] where $\lambda_0$ is related to a reordering or normal ordering of the constraints into a non anomalous form.

4. In case that the constraints can be exponentiated to a Lie group $\mathfrak{G}$ one can avoid the construction of the Master Constraint and apply a more heuristic technique called group averaging [37]. This is at most possible if the constraints form an honest Lie algebra with structure constants rather than structure functions. Since we may assume without loss of generality that the constraints and the structure constants are real valued, we assume that we are given a unitary representation $U$ of $\mathfrak{G}$ on $\mathcal{H}$. Assume also that there is a Haar measure $\nu$ on $\mathfrak{G}$, that is, a not necessarily normalised but bi-invariant (with respect to group translations) positive measure on $\mathfrak{G}$. Fix a dense domain $\mathcal{D}$ and let $\mathcal{D}^*$ be the algebraic dual of $\mathcal{D}$, i.e. linear functionals on $\mathcal{D}$ with the topology of pointwise convergence of nets. We define the rigging map

$$\eta : \mathcal{D} \to \mathcal{D}^*; \ f \mapsto \int_{\mathfrak{G}} d\nu(g) \ <U(g)f, . >$$

(3.4)

The reason for restricting the domain of $\eta$ to a dense subset $\mathcal{D}$ of $\mathcal{H}$ is that in general only then defines an element of $\mathcal{D}^*$. The image of $\eta$ are solutions to the constraints in the sense that

$$[\eta(f)](U(g)f') = [\eta(f)](f')$$

(3.5)

for all $g \in \mathfrak{G}$ and all $f' \in \mathcal{D}$. Notice that if we would identify the distribution $\eta(f)$ with the formal vector

$$\eta'(f) := \int_{\mathfrak{G}} d\nu(g) \ U(g)f$$

(3.6)

then its norm diverges unless $\nu$ is normalisable, that is, unless $\mathfrak{G}$ is compact so that $\eta'(f)$ is not an element of $\mathcal{H}$ in general. However, formally we have $<\eta'(f), f' >= [\eta(f)](f')$ and thus

$$<U(g)\eta'(f), f'> = <\eta'(f), U(g^{-1})f'> = \eta(f)[U(g^{-1})f'] = <\eta'(f), f'>$$

(3.7)

for all $g, f'$. Thus formally $U(g)\eta'(f) = \eta'(f)$ which shows that $\eta'(f)$ is a (generalised, since not normalisable) eigenvector of all the $U(g)$ with eigenvalue equal to one as appropriate for a solution to the constraints. Hence [33] is the rigorous statement of the formal computation [37].

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9In general, given an operator $O$ which together with its adjoint $O^\dagger$ is densely defined on $\mathcal{D} \subset \mathcal{H}$ and preserves $\mathcal{D}$ we define the dual $O'$ on the algebraic dual $\mathcal{D}^*$ by $[O'](f) := l(O^\dagger f)$ for all $f \in \mathcal{D}$. 

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We define the physical inner product on the image of $\eta$ by

$$<\eta(f),\eta(f')>_{\text{phys}} := \eta(f')[f]$$  (3.8)

and the physical Hilbert space is the completion of $\eta(\mathcal{D})$ in the corresponding norm$^{10}$.  

The spectral decomposition solution of the constraint can be seen as a special case of group averaging in the sense that in case of a single self–adjoint constraint $\hat{M}$ we can indeed exponentiate it to obtain a one parameter unitary, Abelian group $U(t) = \exp(it\hat{M})$. The Haar measure in this case would seem to be the Lebesgue measure $d\nu(t) = dt/(2\pi)$. We then formally have (we drop the label $\theta$)

$$<\eta(f),\eta(f')>_{\text{phys}} = \int_\mathbb{R} d\nu(t) <U(t)f',f>$$

$$= \int_\mathbb{R} d\nu(t) \int_{\text{spec}(\hat{M})} d\mu(\lambda) e^{-it\lambda} <\tilde{f}'(\lambda),\tilde{f}(\lambda)>_{\mathcal{H}_\lambda}$$

$$= \int_{\text{spec}(\hat{M})} d\mu(\lambda) <\tilde{f}'(\lambda),\tilde{f}(\lambda)>_{\mathcal{H}_\lambda} \int_\mathbb{R} d\nu(t) e^{-it\lambda}$$

$$= c <\tilde{f}'(0),\tilde{f}(0)>_{\mathcal{H}_0}$$  (3.9)

where $c = [\int_{\text{spec}(\hat{M})} d\mu(\lambda) \delta(\lambda)]$. This calculation is formal in the sense that we have interchanged the sequence of the integrations. Also the constant $c$ can be vanishing or divergent which is one of the reasons why group averaging is only formal. For instance in the case of a pure point spectrum the appropriate measure is not the Lebesgue measure but rather the Haar measure on the Bohr compactification of the real line. See$^{31}$ for the details. However, at least heuristically one sees how these methods are related.

This ends the outline of the quantisation programme. We now apply it to General Relativity.

### 4 Status of the quantisation programme for Loop Quantum Gravity (LQG)

In this section we describe to what extent the canonical quantisation programme has been implemented for General Relativity, that is, we give the status of Loop Quantum Gravity. As an aside we sketch the historical development of the subject. We will mostly consider pure gravity, matter coupling works completely similar$^{38}$. Also, in order to avoid technicalities about boundary terms (which can be dealt with$^{38}$) consider compact $\sigma$ without boundary unless stated otherwise.

#### 4.1 Canonical quantum gravity before LQG

The canonical quantisation of General Relativity in terms of the ADM variables$^4$ culminated in the seminal work by DeWitt$^{39}$ which formally carried out many of the steps outlined in the previous section. These crucial papers laid the foundations for a substantial amount of work on the canonical quantisation of General Relativity that followed. However, the stress is here on the word formally. We mention just some problems with these pioneering papers.

**1. Kinematics:**

The Hilbert space representation used there was given in terms of a formal path integral which must be called ill – defined by the standards of a mathematical physicist. For instance, the “measure” was defined to be an infinite Lebesgue measure $[Dq]$ over a space of three metrics, an object that does

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$^{10}$Provided that$^{35}$ is positive semidefinite and with removal of zero norm vectors understood.
not exist mathematically; the integration space, which should be given the appropriate structure of a measurable space was not specified etc.

Nonetheless, if one defines the three metric operator as a multiplication operator and the conjugate momentum operator as a functional differential operator times $i\ell^2$, then one arrives at a formal representation of the canonical commutation relations such that the canonical coordinates are represented as formally symmetric operators. Moreover, the formal Lebesgue measure is formally invariant under infinitesimal spatial diffeomorphisms.

2. Dynamics

2a. Spatial Diffeomorphism Invariance:

In order to solve the spatial diffeomorphism constraint one can assume that wave functions are normalisable functionals of spatially diffeomorphism invariant functions of the tree metric such as integrals over $\sigma$ of scalar densities of weight one constructed from the metric, the curvature tensor and all its covariant derivatives. In order that those derivatives make sense one must assume that the functional integral is over smooth three metrics. However, even if the wave function is of the form $\exp\left(-\int_\sigma d^3x \sqrt{\det(q)}\right)$ which is damped for large $q$ then the functional integral is ill defined: Due to spatial diffeomorphism invariance of the wave function and measure, the infinite volume of $\text{Diff}(\sigma)$ must be factored out. But even after that, the space of smooth metrics is typically of measure zero with respect to the Gaussian type measure $[Dq] \exp(-2 \int_\sigma d^3x \sqrt{\det(q)})$. Finally the function $\det(q)$ can stay small while components of $q_{ab}$ can become large, hence the exponent has flat directions so that the integral also has divergent modes. Hence the norm of these type of states are dangerously close to being either plain infinite or plain zero.

2b. Hamiltonian Constraints

The infinite number of Hamiltonian constraints were formally given as a functional differential equation of second order, which goes by the famous name Wheeler – DeWitt equations. However, these “operators”, which are really products of operator valued distributions multiplied at the same point in $\sigma$, are hopelessly divergent on the space of wave functions just specified where the divergence really originates from the product of operator valued distributions. There was no “normal ordering” or renormalisation possible because no exact vaccum state could be found with respect to which one should normal order.

It is therefore fair to say that canonical Quantum Gravity got stuck at the level of \[39\] in the mid sixties of the past century.

4.2 The new phase space

In a sense, in terms of the ADM variables one could never even find a proper, background independent representation of the canonical commutation relations. Thus, even leaving the dynamics aside, one could never even finish the kinematical part of the programme.

With the advent of the new variables \[40\] there was new hope. Initially the new variables consisted, instead of a three metric and (essentially) the extrinsic curvature as a canonical pair, of an $\text{SL}(2,\mathbb{C})$ connection $A^\mathbb{C}$ and an imaginary $\text{sl}(2,\mathbb{C})$ valued, pseudo two – form \[11\] $E^\mathbb{C}$. This was attractive because the Hamiltonian constraint, after multiplying it with the non – polynomial factor $\sqrt{\det(q)}$, becomes a fourth order polynomial $\tilde{H} = \sqrt{\det(q)}H$ in terms of $A^\mathbb{C}$, $E^\mathbb{C}$ which is no worse than in Yang – Mills theory. Hence the dynamics seemed to be drastically simplified as compared to the ADM formulation with its non – polynomial Hamiltonian constraint.

\[11\] A pseudo – two form is dual, via the totally antisymmetric, metric independent symbol, to a vector density.

\[12\] The determinant of the three metric is required to be everywhere non vanishing classically, hence the modified constraint captures the same information about the reduced phase space as the original one.
The catch, however, was in the reality conditions: Namely, in order to deal with real rather than complex General Relativity one had to impose the reality conditions

\[ A^\mathbf{T} + A^\Psi = 2\Gamma, \quad E^\mathbf{T} + E^\Psi = 0 \]

where \( \Gamma \) is spin connection of the triad \( e \) determined by the three metric. Since essentially \( E^\Psi = -i\sqrt{\det(q)}e \) it follows that \( \Gamma \) and thus \( (4.1) \) take a highly non polynomial form. In fact, \( \Gamma \) is a fraction whose numerator and denominator are homogeneous polynomials of degree three in \( E^\mathbf{T} \) and its first partial derivatives. It is clear that to find a representation of the formal \( \ast \)-algebra \( \mathfrak{A} \) with \( (4.1) \) as \( \ast \)-relations is hopeless and to date nobody was successful.

Despite of this, in \([41, 42]\) an honest representation for a canonical theory based on an \( SU(2) \) connection \( A \) and a real \( su(2) \) valued pseudo two form \( E \) was constructed\(^{13}\). More precisely, \([41]\) constructs a measurable space of generalised (distributional) connections \( \mathfrak{A} \) which turns out to be the Gel’fand spectrum of an Abelian \( C^* \)-subalgebra of the corresponding kinematical algebra \( \mathfrak{A} \). In \([42]\) a (regular, Borel, probability) measure \( \mu_0 \) on \( \mathfrak{A} \) was constructed. Thus, the corresponding Hilbert space \( \mathcal{H} := L_2(\mathfrak{A},d\mu_0) \) is a space of square integrable functions on \( \mathfrak{A} \). As expected \([43]\), the space of classical (smooth) connections \( \mathcal{A} \) is contained in a measurable subset of \( \mathfrak{A} \) of measure zero. Hence, any (formal) state which requires to be restricted to smooth connections in order that, say the Hamiltonian constraint be defined on it, has zero norm and thus can be discarded from \( \mathcal{H} \). For the first time, these and related questions could be answered with absolute precision.

However, what does this Hilbert space have to do with General Relativity if the true phase space is in terms of \( SL(2,\mathbb{C}) \) plus complicated reality conditions rather than \( SU(2) \) with simple reality conditions? In \([45]\) it was pointed out that the Hilbert space \( \mathcal{H} \) can still be considered as a representation space for the quantum kinematics of General Relativity. In fact, the connection \( A \) and pseudo two form \( E \) are related to triad \( e \) and extrinsic curvature \( K \) by\(^{14}\)

\[ A^a_j = \Gamma^j_a + \beta K_{ab}e_j^b, \quad E^a_j = \sqrt{\det(q)}e^a_j/\beta \]

where \( a,b,c,.. = 1,2,3 \) are spatial tensor indices, where \( j,k,l,.. = 1,2,3 \) are \( su(2) \) Lie algebra indices and the real number \( \beta \) is called the Immirzi parameter \([46]\). For any (nonvanishing) value of \( \beta \), the variables \((A,E)\) are canonically conjugate and thus can be used as a kinematical starting point for the quantisation programme.

The price to pay is that, in order to keep it polynomial, one has to multiply the Hamiltonian constraint (which of course depends explicitly on \( \beta \)) by a sufficiently large power of \( \det(q) = |\det(E)| \). This was considered to be rather unattractive because these very high degree polynomials would intuitively drastically worsen the UV singularity structure of the Hamiltonian constraint as compared to the ADM formulation. In fact, this UV problem was already noticed at a rather formal level with the quantum version of \( \tilde{H} \) in terms of the complex variables \([47]\): All the formal solutions to the Hamiltonian constraint were soltions at the regularised level only (in some ordering). When taking the (point splitting) regulator away, the result would be of the type zero times infinity. These problems were expected to even worsen when increasing the polynomial degree of the Hamiltonian constraint. Hence, the initial excitement that formally Wilson loop functions of smooth and non intersecting loops were formally annihilated by the Hamiltonian constraint dropped significantly.

Hence, a critic could have said at this point:

\textit{You have made the theory more complicated and you have not gained anything: You may have a rigorous kinematical framework but that framework does not support the quantum dynamics of the theory.}

In \([48]\) it was demonstrated how these obstacles can be overcome:

\(^{13}\)That in this representation the pseudo two form is indeed an essentially selfadjoint operator valued distribution was only shown later in \([43]\).

\(^{14}\)If the \( SU(2) \) Gauss law holds.
One can show that General Relativity or any other background independent quantum field theory is UV self-regulating provided one equips the Hamiltonian constraint with its natural density weight equal to one as it automatically appears in the classical analysis. Notice that the Hamiltonian of the standard model on Minkowski space has density weight two rather than one. This is the reason why in background dependent quantum field theories UV singularities appear. One can intuitively understand this as follows: In background dependent theories, Hamiltonians are spatial integrals over sums of products of operator valued distributions evaluated at the same point. Such products are therefore divergent. In background independent theories such polynomials $P$ also appear, however, they appear as numerators in a fraction $P/Q$. The denominator $Q$ of that fraction is an appropriate power of $\sqrt{\det(q)}$ such that $P/Q$ is a scalar density of weight one. As one can show, if the numerator has the singularity structure of the $(n+1) -$th power of the $\delta -$distribution (and its spatial derivatives) then the denominator has the singularity structure of the $n -$th power. This must be the case in order that the operator valued distribution has the correct density weight. Hence, in a proper (point splitting) regularisation of $P/Q$ one can, intuitively speaking, “factor out” $n$ of those $\delta -$distributions and one is left with a well-defined integral after removing the regulator.

In other words, it was wrong to assume that the Hamiltonian constraint should be polynomial. Rather, it must be non-polynomial in order that it is well defined. Of course, the details are not as simple as that and we will explain the open issues in the next section. However, even at this stage one can say:

What has been gained is that not only a rigorous kinematical framework has been erected, that framework also supports the quantum dynamics. In particular, the original problem of the reality conditions is completely resolved.

One of the most important issues is whether that dynamics defined by the final, regulator free, Hamiltonian constraint operator, which underwent rather non trivial regularisation steps until one removed the regulator, reduces to the classical one in an appropriate classical limit. We will have much to say about this point in subsequent sections.

Before we close this section, let us comment on some criticism that one might have encountered [49, 50]: The complex connection is actually the pull back to $\sigma$ of the (anti) self – dual part of the 4D spin connection. Hence it has a covariant interpretation. The real valued connection is not related to a covariant action as simply as that. The relation is as follows: Additional to the Palatini action one considers a term which is topological on shell which amounts to the total action

$$ S = \int_M F_{IJ} \wedge *(e^I \wedge e^J) + \frac{1}{\beta} \int_M F_{IJ} \wedge (e^I \wedge e^J) $$

(4.3)

Here $I, J, K, .. = 0, 1, 2, 3$ are Lorentz indices, $e^I$ is the cotetrad one form and $F_{IJ}$ is the curvature for a Lorentz connection $A_{IJ}$. The first term in (4.3) is the Palatini action. The second term is a total derivative when substituting the equation of motion for the connection $A^{IJ}$. Now when performing the Legendre transform of (4.3) one encounters second class constraints [26]. These must be eliminated by using the Dirac bracket or by partially fixing the Lorentz gauge symmetry $SO(1, 3)$ (or its universal cover $SL(2, \mathbb{C})$) to $SO(3)$ (or $SU(2)$) respectively, called the time gauge.

Using the Dirac bracket leads to a Poisson structure with respect to which connections are not Poisson commuting. Hence, while manifestly originating from a covariant action, the Lorentz connections cannot be used as a configuration space in the quantisation programme, that is, they cannot be represented as (commuting) multiplication operators [50]. In fact, to date there is no honest representation based on Lorentz connections available. On the other hand, the time gauge immediately leads to the phase space description sketched above with Immirzi parameter $\beta$. The manifest covariant origin of the phase space is lost due to the gauge fixing of the Lorentz group [49], however, one can show easily [2, 13] that symplectic

15Notice that the $\delta -$distribution $\delta(x, y)$ transforms as a density of weight one in, say $x$ and as a scalar in, say $y$. 

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reduction with respect to the $SU(2)$ Gauss constraint results in the manifestly covariant ADM phase space. Hence, both criticisms are of purely aethetical nature and do not give rise to either an obstacle or an insight concerning the quantisation.

4.3 Quantum kinematics

4.3.1 Elementary functions

Having convinced ourselves that the cotangent bundle $\mathcal{M} := T^*(\mathcal{A})$ over the space of smooth $SU(2)$ connections is an appropriate kinematical phase space of General Relativity we are supposed to choose an appropriate Poisson $*$--subalgebra $\mathfrak{E}$ of elementary functions. Experience from lattice gauge theory shows that it is convenient to work with $SU(2)$ valued magnetic holonomies

$$A(e) := \mathcal{P} \exp(\int_e A)$$

and real valued electric fluxes

$$E_f(S) := \int_S \text{Tr}(n \ast E)$$

Here $e$ is a path in $\sigma$, $S$ is a two surface in $\sigma$ and $n$ is a Lie algebra valued scalar. These functions separate the points of $\mathcal{M}$ since $G = SU(2)$ is compact. Moreover, they satisfy the reality conditions

$$\overline{A(e)} = [A(e^{-1})]^T, \quad \overline{E_n(S)} = E_n(S)$$

as well as the Poisson brackets

$$\{A(e), A(e')\} = 0, \quad \{E_f(S), A(e)\} = 8\pi G_{\text{Newton}} \beta A(e_1) f(S \cap e) A(e_2)$$

Here we have assumed that $e$ and $S$ intersect transversally in an interior point of both $S, e$ thus splitting the path $e$ at $S \cap e$ as $e = e_1 \circ e_2$ and $G$ is Newton’s constant. Similar formulae can be derived if $S, e$ intersect in a more complicated way.

The algebra $\mathfrak{E}$ can now be described as follows: Consider the algebra $\text{Cyl}$ of cylindrical functions, that is, those which depend on a finite number of holonomies only. Hence, a cylindrical function is of the form

$$\text{Cyl}$$

This way we do not need a background

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16For simplicity we assume that the $SU(2)$ principal bundle is trivial which is always possible. The final quantum theory turns out not to be affected by this assumption. The paths and surfaces are piecewise analytic for technical reasons.
metric in order raise or lower indices. Moreover our holonomies and fluxes transform in a simple way under
the kinematical part of the gauge group, that is, $SU(2)$ gauge transformations and spatial diffeomorphisms.
In fact, consider the smeared Gauss constraint and spatial diffeomorphism constraint respectively given by

$$G(\Lambda) := \int \sigma \, d^3x \, \Lambda^j \, G_j, \quad D(v) := \int \sigma \, d^3x \, v^a \, C_a$$

(4.9)

(where $\Lambda$, $v$ are test functions) where\(^{17}\)

$$G_j = \text{Tr}(\tau_j D_a E^a), \quad D_a = \text{Tr}(F_{ab} E^b)$$

(4.10)

and the one parameter families of canonical transformations generated by them. These are explicitly given
by, say on $f \in \text{Cyl}$

$$\alpha_{\exp(t\Lambda)}(f) := \sum_{n=0}^{\infty} \frac{t^n}{n!} \{C(\Lambda), f\}_{(n)}$$

$$\alpha_{\varphi_t^v}(f) := \sum_{n=0}^{\infty} \frac{t^n}{n!} \{C(v), f\}_{(n)}$$

(4.11)

where $b(e)$, $f(e)$ denote respectively the beginning and final point of $e$. Here $t \mapsto \varphi_t^v$ is the one parameter
family of diffeomorphisms generated by $v$. It is not difficult to see that $\alpha_{\exp(t\Lambda)}$ is the restriction to local
gauge transformations of the form $g = \exp(t\Lambda)$ and spatial diffeomorphisms of the form $\varphi = \varphi_t^v$ of the following
action of the semidirect product $\mathfrak{G} = \mathfrak{G} \rtimes \text{Diff}(\mathcal{S})$ on $\text{Cyl}$ given by

$$[\alpha_g(f)](A) = f_\gamma(\{g(b(e))A(e)g(f(e))^{-1}\}_{e \in E(\gamma)})$$

$$[\alpha_\varphi(f)](A) = f_\gamma(\{A(\varphi(e))\}_{e \in E(\gamma)})$$

(4.12)

There is a similar action on the vector fields $u_{n,S}$. As the notation suggests, the maps $\alpha_g$, $\alpha_\varphi$ are automor-
phisms of $\mathfrak{E}$, that is $\alpha_\cdot(\{a,b\}) = \{\alpha(a), \alpha(b)\}$ for any $a, b \in \mathfrak{E}$ as one can easily verify. Hence we have a
representation of $\mathfrak{G}$ as automorphisms on $\mathfrak{E}$.

4.3.2 Quantum $\ast$–algebra

We follow the standard construction of section $\S$.

Consider the free $\ast$–algebra $\mathfrak{A}$ generated by $\mathfrak{E}$. That is, we consider finite linear combinations of “words”
w constructed from $\mathfrak{E}$. A word is simply a formal finite sequence $w = (a_1..a_N)$ of elements $a_k$ of $\mathfrak{E}$. Multiplication of words is defined as $w \cdot w' = (a_1..a_Na'_1..a'_{N'})$ where $w = (a_1..a_N)$, $w' = (a'_1..a'_{N'})$. The $\ast$
operation on $\mathfrak{A}$ is $w^\ast = (a_N..a_1)$.

Consider the two sided ideal $\mathcal{J}$ in $\mathfrak{A}$ generated by elements of the form

$$(a) \cdot (b) - (b) \cdot (a) - i\hbar(\{a, b\})$$

(4.13)

for all $a, b \in \mathfrak{E}$. Then the quantum $\ast$–algebra is defined as the quotient

$$\mathfrak{A} := \mathfrak{A} / \mathcal{J}$$

(4.14)

We can now simply lift the automorphisms labelled by $\mathfrak{G}$ from $\mathfrak{E}$ to $\mathfrak{A}$ by $\alpha_\cdot(w) = (\alpha(a_1)..\alpha(a_N)).$\(^{17}\) and $F$ are respectively the covariant differential and curvature determined by $A$ and $\tau_j$, $j = 1, 2, 3$ denotes a basis of $su(2)$.
4.3.3 Representations of $\mathfrak{A}$

In quantum field theory representations of $\mathfrak{A}$ are never unique in contrast to the situation in quantum mechanics where the Stone–von Neumann theorem guarantees that irreducible and weakly continuous representations of the Weyl algebra generated by the unitary operators $U(x) = \exp(ixq)$, $V(y) = \exp(iyp)$, $x, y \in \mathbb{R}$ are automatically unitarily equivalent to the Schrödinger representation. For example, an appeal to Haag’s theorem reveals that Fock representations for free massive scalar fields with different masses are unitarily inequivalent representations of the corresponding Weyl algebra. Hence already in this simplest case we have an uncountably infinite number of unitarily inequivalent representations of the canonical commutation relations and all of them satisfy the Wightman axioms, e.g., Poincaré invariance. In order to single out preferred representations one must use additional criteria from physics. In the case of the scalar field, the representation is fixed if we insist on the Wightman axioms plus specifying the mass of the scalar field. Hence we need dynamical input as pointed out in [56].

In the case of our algebra $\mathfrak{A}$ the idea is to use dynamical input from the kinematical gauge algebra $\mathfrak{G}$. Namely, we want a unitary representation of $\mathfrak{G}$ on the Hilbert space. To do this, recall that for any $\sigma$−algebra such as our $\mathfrak{A}$ it is true that any representation is a (possibly uncountably infinite) direct sum of cyclic representations. Hence it is sufficient to consider cyclic representations. Next, any cyclic representation is in one to one correspondence with a state $\omega$ on $\mathfrak{A}$ via the GNS construction [50]. Here a state is defined as a positive linear functional on $\mathfrak{A}$, that is, $\omega(w^*w) \geq 0$ for all $w \in \mathfrak{A}$. It is not to be confused with vectors, that is, elements of some Hilbert space. Hence, it suffices to consider states on $\mathfrak{A}$.

The physical input to have a unitary representation of $\mathfrak{A}$ on the GNS Hilbert space $H_\omega$ determined by $\omega$ now amounts to asking that the state $\omega$ be $\mathfrak{G}$−invariant. To see this we have to recall some elements of the GNS construction:

The GNS construction means that there is a one to one correspondence between states $\omega$ on a (unital) $\sigma$−algebra $\mathfrak{A}$ and GNS data $(H_\omega, \pi(\omega), \Omega_\omega)$. Here $H_\omega$ is a Hilbert space, $\pi(\omega)$ is a representation of $\mathfrak{A}$ by densely defined and closable operators on $H_\omega$ and $\Omega_\omega$ is a cyclic vector in $H_\omega$. Cyclic means that $\pi_\omega(\mathfrak{A})\Omega_\omega$ is dense in $H_\omega$. This is done as follows: Consider the subspace of $\mathfrak{A}$ (considered as vector space) defined by $\mathfrak{J} := \{ w \in \mathfrak{A}; \omega(w^*w) = 0\}$. It is not difficult to show that this is a left ideal. Consider the equivalence classes $\{w\} := \{w + w' ; w' \in \mathfrak{J}\}$. Then $H_\omega$ is the closure of the vector space $\mathfrak{A}/\mathfrak{J}$ of equivalence classes, $\Omega_\omega := [1]$ and $\pi_\omega(w)[w^\prime] := [ww^\prime]$. The scalar product is defined as $\langle w, w^\prime \rangle_{H_\omega} := \omega(w^*w^\prime)$. Now if $\omega$ is in addition $\mathfrak{G}$ invariant then by using the automorphism property it is easy to see that $U_\omega(\mathfrak{g})[w] := [\alpha_\mathfrak{g}(w)]$ is a unitary representation of $\mathfrak{G}$ with $\mathfrak{G}$ − invariant cyclic vector $\Omega_\omega$.

The surprising result is now the following structural theorem [57].

Theorem 4.1.

The only $\mathfrak{G} −$ invariant state on the holonomy – flux algebra $\mathfrak{A}$ is the Ashtekar – Isham – Lewandowski state $\omega_{\text{AIL}}$ whose GNS data coincide with the Ashtekar – Isham - Lewandowski representation.

The surprising aspect to this theorem is that not the full gauge symmetry of the theory associated with the Hamiltonian constraint had to be used. In fact it is actually sufficient to just use the spatial diffeomorphism invariance in order to prove the theorem\textsuperscript{18}.

The assumptions of the theorem are fairly weak as one can see. A possible generalisation is as follows: We have implicitly assumed that the flux operators themselves exist as self − adjoint operators on the Hilbert space. This is equivalent to asking that the state is regular, i.e. weakly continuous with respect to the one parameter unitary groups they generate. This needs not to be the case. In [58] it was shown that including non − regular states into the analysis does not change the uniqueness result modulo a slight additional assumption that one has to make. This is to say that the uniqueness result is fairly robust. It is rather

\textsuperscript{18}The careful statement of the theorem uses semianalytic rather than smooth structures on $\sigma$. For every smooth structure there is always a semianalytic structure and semianalytic charts are equivalent up to smooth diffeomorphisms. Semianalyticity is the rigorous formulation of the more intuitive notion of piecewise analyticity. See [57] for details.
There are several complementary characterisations of the kinematical Hilbert space $\mathcal{H} := \mathcal{H}_\omega$ which are useful in different contexts. This section is for the mathematically inclined reader and can be skipped by readers interested only in the conceptual framework.

1. **Positive linear functional characterisation**
   We notice first of all that every word $w$ can be written, using the commutation relations (4.7), (4.13) as a finite linear combination of reduced words. A reduced word is of the form $f_{u_1}, \ldots, f_{u_N}$ with $f \in \text{Cyl}$ and arbitrary $n_k, S_k$ and $N = 0, 1, \ldots$. Due to linearity it suffices to specify $\omega$ on reduced words. The definition is
   \[ \omega(w) = \begin{cases} 0 & \text{if } N > 0 \\ \omega_0(f) & \text{if } N = 0 \end{cases} \]  
   (4.15)
   Here $\omega_0$ is the so-called Ashtekar – Lewandowski positive linear functional on the $C^*$-algebra completion $\text{Cyl}$ of Cyl with respect to the sup norm. It can be explicitly written as
   \[ \omega_0(f) = \int_{SU(2)^n} d\mu_H(h_1) \ldots d\mu_H(h_n) f_\gamma(h_1, \ldots, h_n) \]  
   (4.16)
   for $f(A) = f_\gamma(A(e_1)), \ldots, A(e_n)$, i.e. $f$ cylindrical over a graph with $n$ edges. Here $\mu_H$ is the Haar measure on $SU(2)$. The Hilbert space $\mathcal{H}$ is the GNS Hilbert space derived from (4.15).

2. **$C^*$-algebraic characterisation**
   The completion $\overline{\text{Cyl}}$ of Cyl with respect to the sup norm $||f|| := \sup_{A \in \mathcal{A}} |f(A)|$ defines an Abelean $C^*$-algebra $\overline{\mathcal{A}}$ with its Gel’fand spectrum $\Delta(\overline{\text{Cyl}})$, also called the Ashtekar – Isham space of generalised connections. By the Gel’fand isomorphism we can think of $\overline{\text{Cyl}}$ as the space $C(\overline{\mathcal{A}})$ of continuous functions on the spectrum. The spectrum of an Abelean $C^*$-algebra is a compact Haussorff space if equipped with the Gel’fand topology of pointwise convergence of nets. Hence, by the Riesz – Markov theorem (60) the positive linear functional $\omega_0$ in (4.16) is in one to one correspondence with a (regular, Borel) measure $\mu_0$ on $\overline{\mathcal{A}}$ also called the Ashtekar – Lewandowski measure. The Hilbert space $\mathcal{H} := L_2(\mathcal{A}, d\mu_0)$ is the space of square integrable functions on $\overline{\mathcal{A}}$ with respect to that measure.

3. **Projective limit characterisation**
   The spectrum of $\overline{\text{Cyl}}$ abstractly defined above can be given a concrete geometric interpretation. It can be identified set theoretically and topologically as the set of homomorphisms from the groupoid $\mathcal{P}$ of paths into $SU(2)$, that is, there is a homomorphism $\mathcal{A} \to \text{Hom}(\mathcal{P}, SU(2))$ (61). Here the groupoid of paths is defined, roughly speaking, as the set of (piecewise analytic) paths modulo retractions and reparametrisations together with the operations of 1. connecting paths with common beginning or end point and 2. inversion of orientation. Now recall that an element $A \in \mathcal{A}$ is a homomorphism from Cyl into the complex numbers. Consider a function $f \in \text{Cyl}$ cylindrical over some graph $\gamma$. Since $A$ is a homomorphism we have $A(f) = f_{\gamma}(\{A(h_e)\}_{e \in E(\gamma)})$ where for $A \in \mathcal{A}, h_e(A) = A(e)$ is the holonomy map. Hence it suffices to consider the action of $A \in \mathcal{A}$ on holonomy maps. Now since $h_e h_{e'} = h_{e e'}$, $h^{-1}_e = (h_e)^{-1}$ and $A$ is a homomorphism it follows that every point in the spectrum defines an element of $\text{Hom}(\mathcal{P}, SU(2))$. That also the converse is true is shown e.g. in (62), hence there is a bijection.

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19 That is, the set of all homomorphisms from the algebra into the complex numbers.
To see that this bijection is a homeomorphism we must specify a topology on \( \text{Hom}(\mathcal{P}, SU(2)) \). To do this, we describe the space \( \text{Hom}(\mathcal{P}, SU(2)) \) as a projective limit: For every graph \( \gamma \) we consider the space \( \text{Hom}(\gamma, SU(2)) \) of homomorphisms from the subgroupoid of paths within \( \gamma \) (also denoted \( \gamma \)) into the gauge group. Since such homomorphisms are completely specified by their action on the edges of the graph, the sets \( \text{Hom}(\gamma, SU(2)) \) are identified topologically with \( SU(2)^n \) where \( n \) is the number of edges of the graph. As such, \( \text{Hom}(\gamma, SU(2)) \) is a compact Hausdorff space. The set of subgroupoids is partially ordered and directed with respect to the inclusion relation, that is, for any two \( \gamma, \gamma' \) there is \( \bar{\gamma} \) (e.g. \( \gamma \cup \gamma' \)) such that \( \gamma, \gamma' \subset \bar{\gamma} \). Given \( \gamma \subset \gamma' \) we say that \( A_{\gamma'} \in \text{Hom}(\gamma', SU(2)) \) is compatible with \( A_{\gamma} \in \text{Hom}(\gamma, SU(2)) \) provided that the restriction of \( A_{\gamma'} \) to \( \gamma \) coincides with \( A_{\gamma} \), that is, \( A_{\gamma'}|_{\gamma} = A_{\gamma} \). The projective limit \( \text{Hom}(\mathcal{P}, SU(2)) \) (of the spaces \( \text{Hom}(\gamma, SU(2)) \)) is the (automatically closed) subset of the infinite direct product \( X \) of the Hom\((\gamma, SU(2)) \) restricted to the compatible points. The space \( X \) carries the natural Tychonov topology \(^{63}\) with respect to which it is compact and Hausdorff. This property is inherited by the closed subset \( \text{Hom}(\mathcal{P}, SU(2)) \) in the subspace topology. As one can show, the compact Hausdorff topologies on \( X \) and \( \text{Hom}(\mathcal{P}, SU(2)) \) are identified by the above mentioned bijection \( A \mapsto (A|_{\gamma})_{\gamma} \) where \( A|_{\gamma} \) is the restriction of \( A \) to \( \gamma \).

Also the measure \( \mu_0 \) abstractly defined via the Riesz – Markov theorem can be given a nice projective description: On each subgroupoid \( \gamma \) we consider the product Haar measure \( \mu_{0,\gamma} \) as in \(^{64,65}\). Let \( p_\gamma : \text{Hom}(\mathcal{P}, SU(2)) \to \text{Hom}(\gamma, SU(2)) \); \( A \mapsto A|_{\gamma} \) be the restriction map. The system of measures \( \mu_{0,\gamma} \) satisfies the following compatibility condition: For every \( \gamma \subset \gamma' \) we have \( \int d\mu_{0,\gamma'} f_{\gamma} = \int d\mu_{0,\gamma} f_{\gamma} \) for every \( f = f_{\gamma} \circ p_\gamma \) cylindrical over \( \gamma \). This property qualifies the \( \mu_{0,\gamma} \) as the cylindrical projections \(^{64}\) \( \mu_{0,\gamma} = \mu_0 \circ p_\gamma^{-1} \) of a measure on the projective limit. Here the translation invariance and normalisation of the Haar measure are absolutely crucial to establish this property.

4. Inductive limit characterisation

We consider the Hilbert spaces \( \mathcal{H}_{\gamma}(\text{Hom}(\gamma, SU(2)), d\mu_{0,\gamma}) \). For every \( \gamma \subset \gamma' \) there is an isometric embedding \( U_{\gamma'}: \mathcal{H}_{\gamma} \to \mathcal{H}_{\gamma'} \). These isometries satisfy \( U_{\gamma'} = U_{\gamma''} U_{\gamma'} \) for all \( \gamma \subset \gamma' \subset \gamma'' \). This qualifies the \( \mathcal{H}_{\gamma} \) as an inductive system of Hilbert spaces. The Hilbert space \( \mathcal{H} \) is the corresponding inductive limit.

It is not difficult to show that this representation of \( \mathfrak{A} \) is irreducible \(^{65}\). Moreover, it turns out that the Hilbert space \( \mathcal{H} \) has an orthonormal basis over which one has complete control, the spin network basis \(^{66}\). These provide an indispensible tool in all analytical calculations in LQG. A spin network (SNW) is a quadruple \(^{20}\) \( s = (\gamma, j, m, n) \) consisting of a graph \( \gamma \), a collection of spin quantum numbers \( j = \{j_e\}_{e \in E(\gamma)} \) and two collections of magnetic quantum numbers \( m = \{m_e\}_{e \in E(\gamma)} \), \( n = \{n_e\}_{e \in E(\gamma)} \) subject to the conditions \( j_e = 1/2, 1, 3/2, \ldots \) and \( m_e, n_e \in \{-j_e, -j_e+1, \ldots, j_e\} \). The analytical expression for a spin network function (SNWF) is given by (we write \( A(e) := A(h_e) \) for \( A \in \mathfrak{A} \))

\[
T_s(A) := \prod_{e \in E(\gamma)} \sqrt{2j_e + 1} \left[ \pi_{j_e}(A(e)) \right]_{m_e n_e}
\]

(4.17)

Here \( \pi_j \) is the spin \( j \) irreducible representation of \( SU(2) \). Its dimension is \( 2j + 1 \) and we label the entries of the corresponding matrices by \( [\pi(h)]_{mn} \).

Three important properties of \( \mathcal{H} \) follow from the existence of the SNW basis:

1. Since the set of finite graphs is an uncountably infinite set, the kinematical Hilbert space is therefore non separable since it does not have a countable basis.

2. \(^{20}\)It is understood that at bivalent vertices such that the incident edges are at least \( C^{(1)} \) continuations of each other, then in the intertwiner decomposition of the state (see below) no trivial representation occurs. Otherwise this leads to an overcounting problem. Hence, if the intertwiner is trivial then such points are not counted as vertices.
Consider a vector field \( v \) on \( \sigma \) and let \( t \rightarrow \varphi^v_t \) be the one parameter family of spatial diffeomorphisms generated by it\(^{21}\). Then the one parameter unitary group \( t \rightarrow U(\varphi^v_t) \) is not weakly continuous, that is, it does not hold that \( \lim_{t \to 0} \langle \psi, U(\varphi^v_t)\psi' \rangle = \langle \psi, \psi' \rangle \) for all \( \psi, \psi' \in \mathcal{H} \). To see this choose \( \psi = \psi' = T_s \) such that the graph \( \gamma \) underlying \( s \) has support in the support of \( v \). Then \( \langle T_s, U(\varphi^v_t)T_s \rangle = 0 \) for all \( \epsilon > |t| > 0 \) for some \( \epsilon \) because \( U(\varphi)T_s = T_{\varphi s} \) if \( \varphi = (\gamma, j, m, n) \) if \( s = (\gamma, j, m, n) \). By Stone's theorem\(^{65}\) this means that the infinitesimal generators of spatial diffeomorphisms do not exist as (self-adjoint) operators on \( \mathcal{H} \).

3.

On SNWF’s the operators \( A(e) \) act by multiplication while \( E_{n,S} := u_{n,S} \) becomes a linear combination of the right invariant vector fields \( X^i_d = \text{Tr}([\tau_j A(e)]^T \partial/\partial A(e)) \) on a copy of \( SU(2) \) coordinatised by \( A(e) \).

### 4.4 Quantum dynamics

The quantum dynamics consists in two steps: 1. Reduction of the system with respect to the gauge transformations generated by the constraints and 2. Introduction of a notion of time with respect to which observables (gauge invariant operators) evolve. It is convenient to subdivide the discussion of the reduction step into the gauge transformations corresponding to \( \mathbb{G} \) and those generated by the Hamiltonian constraint. We will also mention spin foam models which are the path integral formulation of LQG. Spin foam models were completely neglected in \( \{12\} \) although half of the current activity in LQG is devoted to them. This was partly corrected in \( \{13\} \). The presentation will be brief since our main focus is on the criticisms of \( \{12\} \) towards the canonical formulation.

#### 4.4.1 Reduction of Gauss – and spatial diffeomorphism constraint

##### 4.4.1.1 Gauss constraint

The SNWF are not invariant under \( \mathbb{G} \). It is easy to make them gauge invariant as follows: Pick a vertex \( v \in V(\gamma) \) in the vertex set of \( \gamma \) and consider the edges \( e_1, \ldots, e_N \) incident at it. Let us assume for simplicity that the edges are all outgoing from \( v \), the general case is similar but requires more book keeping. It is easy to see that at \( v \) the state transforms in the tensor product representation \( j_1 \otimes \ldots \otimes j_N \) where \( j_k := j_{e_k} \). Hence in order to make the state gauge invariant, all we need to do is to couple the \( N \) spins \( j_1, \ldots, j_N \) to resulting spin zero. This is familiar from the quantum mechanics of the angular momentum: We begin with

\[
|j_1m_1 \otimes j_2m_2 \rangle = \sum_{j_{12}} <j_{12}m_1 + m_2|j_1m_1;j_2m_2 > |j_{12}m_1 + m_2 >
\]

The recoupling quantum numbers take range in \( j_{12} \in \{ |j_1 - j_2|, \ldots, j_1 + j_2 \} \) and \( <j_{12}m_1 + m_2|j_1m_1;j_2m_2 > \) is the familiar Clebsch – Gordan coefficient (CGC). Next we repeat \( \{11\} \) with the substitutions \( (j_1, m_1; j_2, m_2) \rightarrow (j_{12}, m_1 + m_2; j_3, m_3) \).

The procedure is now iterated until all spins have been recoupled to total angular momentum \( J = 0 \) and total magnetic quantum number \( M = m_1 + \ldots + m_N = 0 \). Consider the corresponding coefficients \( < j_{11}m_1; \ldots, j_Nm_N |JM > \) in the decomposition of \( |j_1m_1 \otimes \ldots \otimes j_Nm_N > \) into the \( |JM > \). As we just showed, these can be written explicitly as polynomials of CGC’s. We are interested only in those coefficients with \( J = 0 \), called intertwiners \( I_{J} \). This imposes some restriction on the range of the \( j_k \) in order that this is possible at all. The number of those intertwiners does not depend on the sequence in which we couple those spins which is called a recoupling scheme. Different recoupling schemes are related by a unitary transformation. We now take one of those intertwiners and sum the SNWF times the intertwiner over all \( m_k \in \{-j_k, \ldots, j_k\} \). The result is a state which is gauge invariant at \( v \). Now repeat this for all vertices.

\(^{21}\) These are obtained by computing the integral curves \( c^v_\epsilon(t) \) defined by \( c^v_\epsilon(t) = v(c^v_\epsilon(t)) \), \( c^v_\epsilon(0) = x \) and setting \( \varphi^v_\epsilon(x) := c^v_\epsilon(t) \).
The resulting states are gauge invariant and orthonormal with respect to the kinematical inner product by the properties of the CGC’s and they define an orthonormal basis of the \( G \) invariant Hilbert space. We will also denote them by \( T_s \) where now \( s = (\gamma, j, I) \) and \( I = \{ I_v \}_{v \in V(\gamma)} \).

### 4.4.1.2 Spatial diffeomorphism constraint

While the solutions to the Gauss constraint were normalisable with respect to the kinematical inner product, this turns out to be no longer the case with respect to the spatial diffeomorphism constraint. Let \( D \) be the finite linear span of SNWF’s which by construction is dense in \( H \). We will look for solutions to the spatial diffeomorphism constraint in the algebraic dual \( D^* \) of \( D \). The algebraic dual of \( D \) are simply linear functionals on \( D \) equipped with the topology of pointwise convergence of nets (weak *-topology). It is clear that an element \( l \in D^* \) is completely specified by the numbers \( l_s := l(T_s) \). Hence we can write any element of \( D^* \) formally as the uncountable direct sum

\[
l = \sum_s l_s < T_s, >
\]  

where the sum is over all gauge invariant spin network labels.

Following the group averaging technique described earlier, we say that an element \( l \in D^* \) is spatially diffeomorphism invariant provided that

\[
l(U(\varphi)f) = l(f)
\]  

for all \( \varphi \in \text{Diff}(\sigma) \) and all \( f \in D \). As we have seen, this definition is a direct generalisation from vectors \( \psi \in H \) to distributions of the equation \( U(\varphi)\psi = \psi \) for all \( \varphi \in \text{Diff}(\sigma) \). The latter equation has only one solution (up to a constant) in \( H \), namely \( \psi = \Omega_\omega = 1 \), the trivial spin network state.

In order to see what this requirement amounts to we notice that it is sufficient to restrict attention to the \( f = T_s \). Let \( [s] = \{ \varphi \cdot s; \varphi \in \text{Diff}(\sigma) \} \) be the orbit of \( s \). Then it is not difficult to see that it amounts to asking that \( l_s = l_{s'} \) whenever \( [s] = [s'] \). Thus \( l_s = l_{[s]} \) just depends on the orbit and not on the representative. It is therefore clear that no non zero solution except for the vector 1 is normalisable with respect to the spatial diffeomorphism constraint. Let \( \psi \in H \) to distributions of the equation \( U(\varphi)\psi = \psi \) for all \( \varphi \in \text{Diff}(\sigma) \). The latter equation has only one solution (up to a constant) in \( H \), namely \( \psi = \Omega_\omega = 1 \), the trivial spin network state.

We therefore have to define a new inner product on the solution space \( D_{Diff}^* \). This can be systematically done using the group averaging technique described in section 3. The only known Haar measure on \( \text{Diff}(\sigma) \) is the counting measure. Indeed, it is almost true that \( T_s \) coincides with the image of the rigging map

\[
\eta(T_s) := \sum_{\varphi \in \text{Diff}(\sigma)} < U(\varphi)T_s, >
\]  

if it was not for fact that \( \text{Diff}(\sigma) \) contains an uncountably infinite number of elements which have trivial action on any given \( s \). These trivial action diffeomorphisms form a subgroup (but not an invariant one) but that subgroup evidently depends on \( s \). Hence one cannot take a universal factor group (rather: coset) for the averaging in order to get rid of the associated infinity. However, we notice that formally \( \eta(T_s)|T'_s = 0 \) whenever \( [s] \neq [s'] \). Hence it is justified to decompose the kinematical Hilbert space into the direct sum of \( \text{Diff}(\sigma) \) invariant subspaces \( H_\gamma \) consisting of the finite linear span of SNWF’s over the graphs \( \gamma' \) in the orbit \([\gamma]\) of \( \gamma \). The group averaging can now be done on these subspaces separately because in any case their images under \( T_s \) would be orthogonal. This is done by identifying a subset \( \text{Diff}_{[\gamma]}(\sigma) \) which is in one to one correspondence\(^{22}\) with the points in \([\gamma]\). When restricting \( (4.20) \) only to those diffeomorphisms and a discrete set of additional graph symmetries\(^{23}\) then we indeed get the \( \eta(T_s) = k_1 s T_s \) where \( k_1 s \) is a positive

\(^{22}\)That is, fix a representative \( \gamma_0 \) in every orbit and select diffeomorphisms which map \( \gamma_0 \) to every point in the orbit.

\(^{23}\)These are diffeomorphisms which leave the range of the representative \( \gamma_0 \) invariant but permute the edges among each other.
constant which is of the form of a positive number \(k_{[\gamma(s)]}\) times an integer which is the ratio of the orbit sizes of the least symmetric \([s']\) with \([\gamma(s')] = [\gamma(s)]\) and the orbit size of \([s]\). See [43] for the details. It follows that the spatially diffeomorphism invariant inner product is determined by

\[
<T_{[s]}, T_{[s']} >_{\text{Diff}} = \frac{1}{k_{[s]}k_{[s']}} \eta(T_s), \eta(T_{s'}) >_{\text{Diff}} = \frac{1}{k_{[s]}k_{[s']}} \eta(T_{s'})[T_s] = \frac{\delta_{[s],[s']}}{k_{[s]}} \tag{4.22}
\]

Notice, however, that the relative normalisation of the \(T_{[s]}\) is only fixed for those \(s\) which have diffeomorphic underlying graphs because we applied the averaging to all those “sectors” separately. In order to fix the normalisations between the sectors one needs to consider diffeomorphism invariant operators which are classically real valued and map between these sectors and require that they be self – adjoint (or at least symmetric).

Finally we mention that \(\mathcal{H}_{\text{Diff}}\) just like \(\mathcal{H}\) is still not separable because the set of singular knot classes \([\gamma]\) has uncountably infinite cardinality [68]. This is easy to understand from the fact that the group of semianalytic diffeomorphism reduces to \(GL(3, \mathbb{R})\) at each vertex. Hence, for vertices of valence higher than nine one cannot arbitrarily change, in a coordinate chart, all the angles between the tangents of the adjacent edges. It turns out that valence five is already sufficient, that is, there are diffeomorphism invariant “angles”, called moduli \(\theta\) in all vertices of valence five or higher. There are several proposals for an enlargement of the group of diffeomorphisms [69] [70] [71], however, these groups do not interact well with certain crucial operators in the the theory such as the volume operator which depend on at least \(C^{(1)}(\sigma)\) structures while those extensions basically replace diffeomorphisms by homeomorphisms or even mor general bijective maps on \(\sigma\). We will see, however, that the non separability of \(\mathcal{H}_{\text{Diff}}\) is immaterial when we pass to the physical Hilbert space \(\mathcal{H}_{\text{phys}}\).

### 4.4.2 Reduction of the Hamiltonian constraint

The informed reader knows that the implementation of the Hamiltonian constraint is the most important technical problem in canonical quantum gravity ever since. The source of these technical problems within LQG can be appreciated when recalling the Dirac algebra \(\mathcal{D}\) [22]:

1. The first relation in [22] means that \(\text{diff}(\sigma)\) is a subalgebra. However, the second relation says that this subalgebra is not an ideal. In other words, the Hamiltonian constraints are not spatially diffeomorphism invariant. In particular, if there is a quantum operator \(\hat{H}(N)\) associated with \(H(N)\) then it cannot be defined on \(\mathcal{H}_{\text{Diff}}\). We stress this simple observation here because one often hears statements saying the contrary in the literature. Spatially diffeomorphism invariant states do play a role but a quite different one as we will see shortly. The constraint operators \(\hat{H}(N)\) must be defined on the kinematical Hilbert space \(\mathcal{H}\) and nowhere else. One could try, as suggested in [22] [72] to define the dual \(\hat{H}'(N)\) of the constraint operator on some subspace \(\mathcal{D}_s^*\) of \(\mathcal{D}^*\) invariant under the \(\hat{H}'(N)\) via

\[
[\hat{H}'(N)f] := l(\hat{H}(N)f) \tag{4.23}
\]

for all \(f \in \mathcal{D}\). However, in order to solve all constraints, eventually one wants to restrict \(\mathcal{D}_s^*\) to the space of spatially diffeomorphism invariant distributions on which \(\hat{H}'(N)\) is ill defined. Hence the definition on \(\mathcal{H}\) is the only option.

2. We have seen that the kinematical Hilbert Hilbert space is, under rather mild assumptions, uniquely selected. In other words, there is no other choice. Unfortunately, as we have seen, in this representation the diffeomorphisms are not represented weakly continuously and there is no way out of this fact. This poses a problem in representing [22] on \(\mathcal{H}\) because evidently [22] involves the infinitesimal generators \(D(N)\) of spatial diffeomorphisms which are obstructed to exist as quantum operators as we just have said. As far as the first two relations in [22] are concerned, there is a substitute involving finite
(exponentiated) diffeomorphisms only. It is given by

\[
U(\varphi)U(\varphi')U(\varphi)^{-1} = U(\varphi \circ \varphi' \circ \varphi^{-1}) \\
U(\varphi)\hat{H}(N)U(\varphi)^{-1} = \hat{H}(N \circ \varphi)
\]  

(4.24)

Indeed, if \(\hat{\mathcal{D}}(\vec{N})\) would exist then one parameter subgroups of spatial diffeomorphisms would be given by \(U(\varphi t) = \exp(it\hat{\mathcal{D}}(\vec{N})/(\hbar 8\pi G_{\text{Newton}}))\) and then (4.24) would be equivalent to

\[
\begin{align*}
[\hat{\mathcal{D}}(\vec{N}), \hat{\mathcal{D}}(\vec{N}')] &= i8\pi G_{\text{Newton}}\hbar \hat{\mathcal{D}}(\mathcal{L}_{\vec{N},N}') \\
[\hat{\mathcal{D}}(\vec{N}), \hat{H}(N')] &= i8\pi G_{\text{Newton}}\hbar \hat{H}(\mathcal{L}_{\vec{N},N}')
\end{align*}
\]  

(4.25)

upon taking the derivative at \(t = 0\).

3. In other words there is a finite diffeomorphism reformulation of the first two relations in (2.2). However, this is no longer possible for the third relation in (2.2). The problem is the structure function involved on the right hand side of this relation which prevents us from exponentiating the Hamiltonian constraints. The commutator algebra of the Hamiltonian constraints is simply so complicated that the Dirac algebra \(\mathcal{D}\) is no longer a Lie group. Therefore we cannot exponentiate the third relation in (2.2) and there seems to be no chance to find a substitute involving finite diffeomorphisms only.

4. Even if the problem just mentioned could be solved, we would still have no idea for how to find the physical inner product because group averaging only works for Lie algebra valued constraints.

These remarks sound like an obstruction to implement the operator version of the Hamiltonian constraint in LQG at all. In what follows we describe the progress that has been made over the past ten years with regard to this task. There are two constructions: The first, surprisingly, indeed proposes a quantisation of the Hamiltonian constraints as operators on the kinematical Hilbert space. The algebra of these operators is non – Abelean and closes in a precise sense as we will see. We again stress this because one sometimes reads that the Hamiltonian constraint algebra is Abelean \([72, 73]\) which is simply wrong. However, no physical scalar product using these operators has so far been constructed due to the non – Lie algebra structure mentioned above. Also, so far the correctness of the semiclassical limit of these constraint operators has not been established, in particular it is unsettled in which sense the third relation in (2.2) is implemented in the quantum theory.

To make progress on these two open problems, the construction of the physical scalar product and the establishment of the correct classical limit which are interlinked in a complicated way as we will see, the Master Constraint Programme was launched \([19, 75, 34, 35]\). We have outlined it already in section 3 for a general theory and will apply it to the Hamiltonian constraints below.

This section is organised as follows:

The Master Constraint Programme overcomes many of the shortcomings of the Hamiltonian constraint and is the modern version of the implementation of the Hamiltonian constraint in LQG. We will still describe first the old Hamiltonian constraint programme \([20, 38, 75]\) in order to address the criticisms spelled out in [12] and because the quantisation technique in [19, 75] is still heavily based on the key techniques developed in [20]. Indeed, without the techniques developed in [20] the recent intriguing results of Loop Quantum Cosmology (LQC)\textsuperscript{24} \([76]\) such as avoidance of the big bang singularity could never have been achieved. A large amount of the success of LQC is a direct consequence of [20].

Then we describe the Master Constraint Programme which was not mentioned at all in [12] although it removes many of the criticisms stated there. In particular we describe recent progress made in a particular

\textsuperscript{24}LQC is the usual homogeneous (and isotropic) cosmological model quantised by LQG methods. It is not the cosmological sector of LQG because LQG is a quantum field theory (infinite number of degrees of freedom) while LQG is a quantum mechanical toy model (finite number of degrees of freedom) in which the inhomogeneous excitations are switched off by hand.
version of the Master Constraint Programme called Algebraic Quantum Gravity (AQG)\textsuperscript{77} which establishes that the Master Constraint Operator has the correct semiclassical limit. The work\textsuperscript{77} also removes the criticism of\textsuperscript{12} that no calculations involving the volume operator can be carried out in LQG. This, together with the general direct integral construction of the physical inner product already described make the Master Constraint Programme a promising step forward in LQG.

Unfortunately the subsequent discussion is rather complicated because the problems of anomaly freeness, semiclassical limit, dense definition, representation of the Dirac algebra etc. for the Hamiltonian constraints are interlinked in a complex way. In order to appreciate these interdependencies we have to go into some detail about the actual constructions. We will try our best at keeping the discussion as non-technical as possible.

4.4.2.1 Hamiltonian constraint

The task is to quantise the Hamiltonian constraints which on the new phase space are given by\textsuperscript{25}

$$H(N) = \int_{\sigma} d^3x \ N(x) \frac{\text{Tr}(F_{ab}E^a E^b)\left(x\right)}{\sqrt{|\det(E)|}}$$

(4.26)

where $N$ is a test function. The non polynomial character of (4.26) is evident and it is hard to imagine that there is any way to tame (4.26) when replacing $A,E$ by their operator equivalents.

We now sketch the key tools developed in\textsuperscript{20}. Let $R_x$ be any region in $\sigma$ containing $x$ as an interior point then

$$e^j_\alpha(x) = -\{A^j_\alpha(x), V(R_x)\}/\kappa$$

(4.27)

where $\kappa = 8\pi G_{\text{Newton}}$ and

$$V(R_x) := \int_{R_x} d^3y \sqrt{|\det(E)|}(y)$$

(4.28)

is the volume of the region $R_x$. Using the relation $e^j_\alpha = E^j_\alpha/\sqrt{|\det(E)|}$ we can rewrite (4.26) as

$$H(N) = -\frac{1}{\kappa} \int_{\sigma} N(x) \text{Tr}(F(x) \wedge \{A(x), V(R_x)\})$$

(4.29)

where now all the dependence on $E$ resides in the volume function $V(R_x)$. The point of doing this that $V(R_x)$ admits a well defined quantisation as a positive essentially self – adjoint operator\textsuperscript{26} $\hat{V}(R_x)$ on $\mathcal{H}$. Following the rules of canonical quantisation one would then like to replace the Poisson bracket between the functions appearing in (4.29) by the commutator between the corresponding operators divided by $i\hbar$.

The problem is that the connection operator $\hat{A}(x)$ does not exist on the Hilbert space $\mathcal{H}$. To see this, note the classical identity $A_\alpha(x)\dot{p}^{\alpha}(0) = (d/dt)_{t=0} A(p_t)$ where $p : [0,1] \rightarrow \sigma$; $s \mapsto p(s)$ is a path, $p_t(s) = p(ts)$ for $t \in [0,1]$, $p(0) = x$ and $A(p_t)$ is the holonomy along $p_t$. By varying the path we can recover the connection from the holonomy. Hence we would like to define the connection operator by this formula for the holonomy operator. However, this does not work since the family of operators $t \mapsto A(p_t)$ is not weakly continuous on $\mathcal{H}$. Hence the derivative at $t = 0$ is ill defined. It follows that the UV singularity structure of the Hamiltonian constraints is not at all determined by the $E$ dependence but rather by the $A$ dependence. In particular, the ambiguities discussed below coming from the loop attachment purely stem from the $A$ dependence.

It is at this point where we must regularise (4.29). We consider a triangulation $\tau$ of $\sigma$ by tetrahedra $\Delta$. For each $\Delta$, let us single out a corner $p(\Delta)$ and denote the edges of $\Delta$ outgoing from $p(\Delta)$ by $s_I(\Delta)$, $I = 1,2,3$.\textsuperscript{26}This is only the simplest piece of the geometry part of the constraint, the remaining piece as well as matter contributions can be treated analogously\textsuperscript{35} and will be neglected here for pedagogical reasons.\textsuperscript{26}Actually there are two inequivalent volume operators\textsuperscript{78}\textsuperscript{79} which result from using two different background independent regularisation techniques. However, in a recent mathematical self – consistency analysis\textsuperscript{80} the operator\textsuperscript{78} could be ruled out.

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Likewise, denote by $s_{IJ} (\Delta)$ the edges of $\Delta$ connecting the end points of $s_I (\Delta)$, $s_J (\Delta)$ such that the loop $\beta_{IJ} (\Delta) = s_I (\Delta) \circ s_J (\Delta) s_J (\Delta)^{-1}$ is the boundary of a face of $\Delta$. In particular $s_{IJ} (\Delta) = s_{IJ} (\Delta)^{-1}$. It is then not difficult to see that

$$H_\tau (N) := \frac{1}{\kappa} \sum_{\Delta \in \tau} N_p(\Delta) \sum_{IJK} \epsilon_{IJK} \text{Tr} (A(\beta_{IJ}(\Delta))A(s_K(\Delta))\{A(s_K(\Delta))^{-1}, V(R_p(\Delta))\})$$

is a Riemann sum approximation to (4.29), that is, it converges to (4.29) as we refine the triangulation to the continuum. We will denote the refinement limit by $\tau \to \sigma$.

Since (4.30) is now written in terms of quantities of which the quantisation is known we immediately get a regularised Hamiltonian constraint operator on $\mathcal{H}$ given by

$$\hat{H}_\tau(N) := \frac{1}{i \ell_P^2} \sum_{\Delta \in \tau} N_p(\Delta) \sum_{IJK} \epsilon_{IJK} \text{Tr} (A(\beta_{IJ}(\Delta))A(s_K(\Delta))\{A(s_K(\Delta))^{-1}, \hat{V}(R_p(\Delta))\})$$

The reason for the adjoint operation in (4.31) is due to the definition of the dual action in the footnote before (3.5) on elements in $\mathcal{D}^*$ which in turn would coincide with the action of $\hat{H}(N)$ if elements of $\mathcal{D}^*$ would be normalisable. Notice that (4.30) is real valued so that classically $H_\tau(N) = \overline{H_\tau(N)}$ so we may denote its operator equivalent with or without adjoint operation. It is not difficult to see that in this ordering the operator is densely defined\(^{27}\) on $\mathcal{D}$ and closable (its adjoint is also densely defined on $\mathcal{D}$). However, it is not even symmetric in this ordering. This may seem strange at first, however it is not logically required because we are only interested in the zero point of its spectrum. It is not even possible to have a symmetric ordering as pointed out in [81] where it is shown that for reasons of anomaly freeness in constraint algebras with structure functions symmetric orderings are ruled out.

What we are interested in is in which operator topology (if any) the limit $\tau \to \sigma$ exists. Since $\hat{H}_\tau(N)$ is not bounded, convergence in the uniform topology is ruled out. For the same reason that connection operators are not defined, convergence in the weak (and thus also strong) operator topology is ruled out. Hence we are looking for a weaker topology. There is only one natural candidate available: The weak* topology with respect to the algebraic dual $\mathcal{D}^*$ or a suitable subspace thereof. The only natural subspace is the space of spatially diffeomorphism invariant distributions $\mathcal{D}^*_\text{Diff}$ (finite linear combinations of the $T[s]$ defined in section 4.3.1).\(^{28}\)

Before we do this, we must tame the limit $\tau \to \sigma$ somewhat: Notice that classically $\lim_{\tau \to \sigma} H_\tau(N) = H(N)$ no matter how we refine the triangulation. This observation suggests the following strategy: Given a graph $\gamma$ we consider a family $\epsilon \mapsto \tau_\epsilon^\gamma$ of triangulations adapted to $\gamma$ where $\epsilon$ denotes the fineness of the triangulation and $\epsilon \to 0$ corresponds to $\tau \to \sigma$. This family is equipped with the following properties: For each vertex $v \in V(\gamma)$ and each triple of edges $e_1, e_2, e_3 \in E(\gamma)$ incident at $v$ consider a tetrahedron $\Delta_v^\epsilon(e_1, e_2, e_3)$ such that $p(\Delta_v^\epsilon(e_1, e_2, e_3)) = v$, such that $s_I(\Delta_v^\epsilon(e_1, e_2, e_3))$ is a proper segment of $e_I$, such that the $s_{IJ}(\Delta_v^\epsilon(e_1, e_2, e_3))$ do not intersect $\gamma$ except in their end points and such that the $\Delta_v^\epsilon(e_1, e_2, e_3)$ are diffeomorphic for different values of $\epsilon$. That such tetrahedra always exist is proved in [20].

Consider seven additional tetrahedra $\Delta_v^\epsilon(e_1, e_2, e_3), ..., \Delta_v^\epsilon(e_1, e_2, e_3)$ which are obtained by analytically continuing\(^{28}\) the segments $s_I(\Delta_v^\epsilon(e_1, e_2, e_3))$ through the vertex so that we obtain altogether eight tetrahedra of equal coordinate volume which are like the eight octants of a Cartesian coordinate system. Denote by $W^\epsilon_v(e_1, e_2, e_3)$ the neighbourhood of $v$ they fill. Let $W_v^\epsilon$ be the region occupied by the union of the $W^\epsilon_v(e_1, e_2, e_3)$ as we vary the unordered triples of edges incident at $v$. For sufficiently fine triangulation, the $W^\epsilon_v$ are mutually disjoint. Finally let $W^\epsilon_{v, 1}$ be the union of the $W^\epsilon_v$. We have the following identity for any classical integral

$$\int_{\sigma} = \int_{\sigma} - \sum_{v \in V(\gamma)} \frac{1}{n(\gamma)} \sum_{e_1 \cap e_2 \cap e_3 = v} \{ \int_{W^\epsilon_v(e_1, e_2, e_3)} + \int_{W^\epsilon_v(e_1, e_2, e_3)} \}$$

\(^{27}\)This is basically due to the properties of the volume operator $\hat{V}(\mathcal{R})$: If the graph of a spin network state does not contain a vertex inside the region $\mathcal{R}$ then it is annihilated.

\(^{28}\)For sufficiently fine triangulation the segments can be taken to be analytic.
where \( n_v \) is the valence of \( v \). We now triangulate the regions \( \sigma - W^\epsilon_1, W^\epsilon_2 \) arbitrarily and use the classical approximation \( \int W^\epsilon_2(e_1, e_2, e_3) \approx 8 \int \Delta^\epsilon_2(e_1, e_2, e_3) \). Then, the tetrahedra within \( \sigma - W^\epsilon_1, W^\epsilon_2 - W^\epsilon_3(e_1, e_2, e_3) \) can be shown not to contribute to the action of the operator \( \hat{H}_{\gamma_2}^\dagger \) on any SNWF \( T_s \) over \( \gamma \) so that we obtain

\[
\hat{H}_{\gamma_2}^\dagger(N)T_s = \frac{1}{v_T} \sum_{v \in V(\gamma)} N(v) \frac{8}{(\frac{n_v}{4})^3} \sum_{e_1 \in \epsilon_2} \sum_{e_3} \epsilon_{IJK} \times \\
\times \text{Tr}(A(\beta_{IJ}(\Delta^\epsilon_2(e_1, e_2, e_3)))A(s_K(\Delta^\epsilon_2(e_1, e_2, e_3)))A(s_K(\Delta^\epsilon_3(e_1, e_2, e_3))))^{-1} \hat{V}(R(p(\Delta))) T_s
\]

(4.33)

For each \( \gamma \) choose some \( \epsilon_\gamma \) once and for all such that \( \gamma_\gamma := \tau_\gamma^\epsilon \) satisfies the required properties and define \( \hat{H}^\dagger(N)T_s := \hat{H}_{\gamma_\gamma}^\dagger T_s \) and \( \hat{H}_\gamma^\dagger(N)T_s := \hat{H}_{\gamma_\gamma}^\dagger T_s \) whenever \( s = (\gamma, j, I) \). Then, due to spatial diffeomorphism invariance we have the following notion of convergence

\[
\lim_{\epsilon \to 0} ||l(\hat{H}_\gamma^\dagger(N)f) - l(\hat{H}_\epsilon^\dagger(N)f)|| = 0
\]

(4.34)

for all \( l \in \mathcal{D}_\text{Diff}^* \) and all \( f \in \mathcal{D} \). It is quite remarkable that precisely the space of diffeomorphism invariant distributions which is selected by one of the gauge symmetries of the theory naturally allows us to define an appropriate operator topology with respect to which it is possible to remove the regulator of the Hamiltonian constraints. Notice that despite the fact that we have worked with triangulations adapted to a graph, the operator is a linear operator on \( \mathcal{H} \) where it is, together with its adjoint, densely defined on \( \mathcal{D} \).

One of the most striking features is that the Hamiltonian constraint operators do not suffer from UV singularities as we anticipated in a background independent theory.

Several remarks are in order:

1. **Quantum Spin Dynamics (QSD)**

   Intuitively, the action of the Hamiltonian constraint operator on spin network functions over a graph \( \gamma \) is by creating the new edges \( s_{I,J}(\Delta^\epsilon_2(e_1, e_2, e_3)) \) coloured with the spin \( 1/2 \) representation and by changing the spin on the segments \( s_{I}(\Delta^\epsilon_2(e_1, e_2, e_3)) \) from \( j \) to \( j \pm 1/2 \). Hence in analogy to QCD one could call QCD Hamiltonian constraint operators do not suffer from UV singularities as we anticipated in a background independent theory.

   **Quantum Spin Dynamics (QSD)**

2. **Locality**

   The action of the Hamiltonian constraint operator has been criticised to be too local in the following sense: The modifications that the Hamiltonian constraint operator performs at a given vertex do not propagate over the whole graph but are confined to a neighbourhood of the vertex. In fact, repeated action of the Hamiltonian generates more and more new edges ever closer to the vertex never intersecting each other thus producing a fractal structure. In particular there is no action at the new vertices created. This is not what happens in lattice gauge theory where no new edges are created.

   Notice, however, that there is a large conceptual difference between lattice gauge theory which is a background dependent and regulator dependent (discretised) theory while LQG is a background independent and regulator independent (continuum) theory. Even the role of the single QCD Hamiltonian (generator of physical time evolution) and the infinite number of Hamiltonian constraints (generator of unphysical time reparametrisations) is totally different. Hence there is no logical reason why one should compare the lattice QCD Hamiltonian with the QSD (or LQG) Hamiltonian constraints. In particular, by inspection the infinite number of constraints \( H(x) = 0 \) have a more local structure than a Hamiltonian \( H = \int d^3 x H(x) \).

   Next, it is actually technically incorrect that the actions of the Hamiltonian constraints \( \hat{H}_v, \hat{H}_v \) at different vertices \( v, v' \) do not influence each other: In fact, these two operators do not commute,

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\(^{29}\text{Use the axiom of choice for each diffeomorphism equivalence class of loop assignments.}\)
instance if $v, v'$ are next neighbour, because for any choice function $\gamma \mapsto \epsilon_\gamma$ what is required is that the loop attachments at $v, v'$ do not intersect which requires that the action at $v'$ after the action at $v$ attaches the loop at $v'$ closer to $v'$ than it would before the action at $v$ and vice versa.

Finally, the action of the Hamiltonian constraints on spin network states did not fall from the sky but was derived from a proper regularisation. In particular it is not difficult to see that the operator would become anomalous (see below) if it would act at the vertices that it creates. This would indeed happen if one used the volume operator \textcolor{red}{78} rather than \textcolor{red}{79}. Fortunately, the volume operator \textcolor{red}{78} was shown to be inconsistent \textcolor{red}{80} for totally independent reasons.

In summary, there is no conclusive reason for why this locality property of the constraints is a bad feature. In fact, in 3D \textcolor{red}{38} the solution space of those constraints selects precisely the physical Hilbert space of \textcolor{red}{83}.

3. Ambiguities

The final Hamiltonian constraint operators seem to be highly ambiguous. There are several qualitatively different sources of ambiguities:

3.a. Factor ordering ambiguities

We decided to order the $E$ dependent terms in \textcolor{red}{(4.30)} to the right of the $A$ dependent terms. Could we have reversed the order? The answer is negative \textcolor{red}{20}: Any other ordering results in an expression which is no longer densely defined because the operator would map any spin network state to a state which is a linear combination of SNWF’s whose underlying graphs are all tetrahedra of the triangulation. The resulting “state” is not normalisable in the infinite refinement limit. Thus the factor ordering chosen is in fact unique.

3.b. Representation ambiguities

When replacing connections by holonomies we have used the holonomy in the defining representation of $\text{SU}(2)$. However, as pointed out in \textcolor{red}{84} we could also work with higher spin representations without affecting the limit of the Riemann sum approximation. Recently it was shown \textcolor{red}{85} that higher spin leads to spurious solutions to the Hamiltonian constraints in 3D (where all solutions are known) and therefore very likey also in 4D. Hence this representation ambiguity is very likely to be absent.

Notice also that such kind of ambiguities are also present in ordinary QFT: Consider a $\lambda \phi^4$ QFT. Classically we could replace $\pi(x)$ by $\pi f(x) := e^{i\phi(f)} \pi(x) e^{-i\phi(f)}$ in the Hamiltonian where $\phi(f) = \int_{\mathbb{R}^3} d^3x f(x) \phi(x)$ with some suitable test function $f$ and $\phi, \pi$ are canonically conjugate. One could even replace $\exp(i\phi(f))$ by some other invertible functional $F$ of $\phi$ and consider $[F \pi F^{-1} + F^{-1}\pi F]/2$. In quantum theory the Hamiltonian does change when performing this substitution leading to a different spectrum. Of course in QFT one would never do that because this factor ordering ambiguity generically spoils polynomiality of the Hamiltonian, so one is guided by some simplicity or naturalness principle. In General Relativity the Hamiltonian constraint is non polynomial from the outset, however, still $j = 1/2$ is the simplest choice.

3.c. Loop assignment ambiguities

The largest source of ambiguities is in the choice of the family of triangulations $\epsilon \mapsto \tau_\epsilon$ adapted to a graph. In particular, while it is natural to align the edges of the tetrahedra of the triangulations with the beginning segments of the edges of the graph\textsuperscript{30} because there are no other natural tetrahedra available in the problem, it is not the only logically possible choice. For instance, one could slightly detach the loops $\beta_{ij}(\Delta_\epsilon'(e_1, e_2, e_3))$ from the beginning segments of $e_1, e_2, e_3$ as mentioned in the review by Ashtekar and Lewandowski in \textcolor{red}{31} which found its way into \textcolor{red}{12}. Our statement here is as follows: First of all there is an additional, heuristic argument in favour of the

\textsuperscript{30}There is no ambiguity in the fact that the only contributions of the operator result from the vertices of the graph. This is a direct consequence of the properties of the volume operator \textcolor{red}{79} and the unique factor ordering mentioned above.
alignment. Secondly, even if one does not accept that argument, all of these uncountably infinite number of ambiguities at the level of $\mathcal{H}$ are reduced to a countable number at the level of $\mathcal{H}_{\text{phys}}$ of which all but a few are rather pathological in the sense that one could also use them in lattice gauge theory but does not due to reasons of naturalness.

Concerning the first claim, we want to point out that one of the reasons for why we have decided to work with the expression \{\(A^i_\beta(x), V(R_x)\)\} rather than \(\{(e_{ab}E^b_j E^j_\epsilon) e_{\epsilon jkl}/\sqrt{\det(E)}\}(x)\) is that direct quantisation of the latter would formally result on a spin network state over a graph $\gamma$ in an expression of the form (before introducing the point splitting regulator)

$$\int_{\sigma} d^3x \frac{1}{\sum_{v'\in V(\gamma)} \delta(x,v') \hat{V}_v} \sum_{v\in V(\gamma)} \sum_{e_1\cap e_2=v} \int_0^1 dt \dot{e}_1^a(t) \delta(x,e_1(t)) \int_0^1 ds \dot{e}_2^b(s) \delta(x,e_2(s)) \times$$

$$\times F^j_{ab}(e_1(t) + e_2(s)) e_{\epsilon jkl} X^k_{e_1} X^l_{e_2}$$

(4.35)

where $X^k_e$ is a right invariant vector field on the copy of $SU(2)$ corresponding to $A(e)$. Likewise $\hat{V}_v$ is a well defined operator (not an operator valued distribution) built from those vector fields. Clearly (4.35) involves the holonomy of an infinitesimal loop whose tangents at the $v$ are pairs of edges incident at $v$. This motivates the alignment mentioned above\(^{31}\). The only reason why (4.35) is not used in place of (4.30) is that the the operator $\hat{V}_v$ has zero modes so that its inverse is not even densely defined.

Concerning the second claim we notice that solutions to all constraints will be elements $l$ of $D^*_{\text{Diff}}$ which satisfy $l(H(N)^j f) = 0$ for all $N$ and all $f \in D$. Now since $l$ is spatially diffeomorphism invariant, the space of solutions to all constraints only depends on the spatially, piecewise analytic diffeomorphism invariant characteristics of the loops $\beta_{IJ}(\Delta_v^{e_1}(e_1,e_2,e_3))$. Hence it matters whether or not the tetrahedra $\Delta$ are just continuous at their corners or of higher differentiability class, how the additional edges are routed or braided through the edges of the graph and whether or not the tetrahedra $\Delta$ are just continuous at their corners. Concerning the braiding, a natural choice is the one displayed in [20] which makes use of Puiseaux’ theorem\(^{32}\). It follows that the seemingly uncountably infinite set of possible loop assignments is reduced to a discrete number of choices, of which all but a finite number is unnatural\(^{33}\), once we construct solutions of all constraints.

In summary, the most natural proposal is such that the edges $s_{IJ}(\Delta_v^{e_1}(e_1,e_2,e_3))$ intersect the graph $\gamma$ transversally with the braiding described in [20]. This defines a concrete and non ambiguous operator in the sense that it uniquely selects a subspace of $D^*_{\text{Diff}}$ as the space of solutions to all constraints.

3.d. Habitat ambiguities

In [12] we find an extensive discussion about “habitats” $D^*_{\text{dif}}$. A habitat is a subspace of $D^*$ containing $D^*_{\text{Diff}}$ with the minimal requirement that it is preserved by the dual action of the

\(^{31}\)A careful point splitting regularisation removes the $\delta$–distribution in both numerator and the denominator as well as the the integral over $\sigma$ leaving only an integral over $s,t$ with support in infinitesimal neighbourhoods of the vertices of the graph in question.

\(^{32}\)Basically one wants that the arcs intersect the graph only in their end points which for sufficiently fine triangulations can only happen for edges $e$ incident at the vertex in question. One first shows that there always exists an adapted frame, that is, a frame such that $s_1, s_2$ lie in the $x, y$ plane for sufficiently short $s_1, s_2$. Now one shows that for any other edge $e$ of the graph whose beginning segment is not aligned with either $s_1$ or $s_2$ there are only two possibilities: A. Either for all adapted frames the beginning segment of $e$ lies above or below the $x, y$ plane and whether it is above or below is independent of the adapted frame. B. Or there exists an adapted frame such that $e$ lies above the $x, y$ plane. This can be achieved simultaneously for all edges incident at the vertex in question. The natural prescription is then to let the edge $s_{1,2} \ell$ be the straight line in the selected frame connecting the end points of $s_1, s_2$ at which it intersects transversally.

\(^{33}\)Like winding the segments $s_1$ of the tetrahedra of the triangulation an arbitrary number of times around the edges $e_1$ of the graph. Such a ridiculous choice could also be made in lattice gauge theory but is not considered there due to reasons of naturalness.
Hamiltonian constraints. Habitats were introduced in [12]. The idea was to take the limit \( \epsilon \to 0 \) for the duals of the Hamiltonian constraints on such a habitat in the sense of pointwise convergence. The habitat ambiguity is that there may be billions of habitats on which a limit of this kind can be performed. As was shown in those papers, there exists at least one such habitat and it has the property that the limit dual operators are Abelian.

We now show that this habitat ambiguity is actually absent: Namely, the habitat spaces must be genuine extensions of \( \mathcal{D}_{\text{Diff}}^* \). Hence these spaces are not in the kernel of the spatial diffeomorphism constraint and are therefore unphysical. Hence the only domain where to define the Hamiltonian constraints (rather than their duals) is on \( \mathcal{D} \), i.e. on a dense subspace of the kinematical Hilbert space \( \mathcal{H} \). This is the same domain as for the spatial diffeomorphism constraints which thus treats both types of constraints democratically. This fact is widely appreciated in the LQG community and not a matter of debate, the habitat construction presented in [12] is outdated. Habitats are unphysical and completely irrelevant in LQG.

On the kinematical Hilbert space the Hamiltonian constraints are non commuting, see below. The apparent contradiction with the Abelian nature of the limits of the duals on the above mentioned habitat is resolved by the fact that effectively the commutator of the limiting duals on the habitat is the dual of the commutator on \( \mathcal{D} \). While the commutator on \( \mathcal{D} \) is non vanishing, its dual annihilates \( \mathcal{D}_{\text{Diff}}^* \) and also the habitat \( \mathcal{D}_{\text{Diff}}^* \) chosen which is a sufficiently small extension of \( \mathcal{D}_{\text{Diff}}^* \).

Hence we see that the amount of ambiguity is far less severe than [12] perhaps make it sound once we pass to \( \mathcal{H}_{\text{phys}} \). In fact, there are only a handful full of natural proposals available.

4. **Anomalies**

As already mentioned, the constraint algebra can only be checked in the form that only involves finite diffeomorphisms. Indeed it is not difficult to see that the first two relations of the Dirac algebra (2.2) really hold in the form (4.24) on \( \mathcal{H} \) up to a spatial diffeomorphism [2]. Likewise one can check that

\[
I((\hat{H}^\dagger(N), \hat{H}^\dagger(N'))|f) = 0
\]

for all test functions \( N, N' \) all \( f \in \mathcal{D} \) and all \( l \in \mathcal{D}_{\text{Diff}}^* \). This can be read as an implementation of the third relation in (2.2) because that relation involves an infinitesimal spatial diffeomorphism constraint whose dual action should annihilate \( \mathcal{D}_{\text{Diff}}^* \). Of course, the commutator does not involve an infinitesimal diffeomorphism which does not exist in our theory. Rather what happens is the following: The commutator \( [\hat{H}^\dagger(N), \hat{H}^\dagger(N')] \) is non vanishing on \( \mathcal{H} \). However, it can be shown that on SNWF’s it is a finite linear combination of terms of the form \( [U(\varphi) - U(\varphi')]|\hat{O} \) where \( \hat{O} \) is some operator on \( \mathcal{H} \).

5. **On shell closure versus off shell closure**

As we just saw, the quantum constraint algebra is consistent, i.e. non – anomalous. More precisely, the first relation in (2.2) holds, in exponentiated form, on \( \mathcal{H} \) exactly, it is non anomalous in every sense. The second relation in (2.2) also holds in exponentiated form on \( \mathcal{H} \) but only modulo a spatial diffeomorphism. How about the third relation in (2.2)? In [38] it is shown that an independent quantisation of the classical function \( D(q^{-1}(N'd\hat{N} - N\hat{d}N')) \) appearing on the right hand side of (2.2) can be given. There is no contradiction to the non existence of the operator corresponding to \( D(N) \) because \( D(q^{-1}(N'd\hat{N} - N\hat{d}N')) \) is not of the form \( D(N) \) due to the structure function \( q^{-1} \) which is responsible for the existence of the composite operator corresponding to \( D(q^{-1}(N'd\hat{N} - N\hat{d}N')) \). That operator is constructed in a way analogous to the Hamiltonian constraint operator and is formulated in terms of the operators \( U(\varphi) \). The duals of both operators \( [\hat{H}^\dagger(N), \hat{H}^\dagger(N')] \) and \( D(q^{-1}(N'd\hat{N} - N\hat{d}N')) \) annihilate \( \mathcal{D}_{\text{Diff}}^* \).

Hence what one can say is the following: We define two operators on \( \mathcal{H} \) as equivalent \( \hat{O}_1 \sim \hat{O}_2 \) provided that the dual of \( \hat{O}_1 - \hat{O}_2 \) annihilates \( \mathcal{D}_{\text{Diff}}^* \). Then the classical identities (2.2) holds on \( \mathcal{H} \) in the sense of equivalence classes (the first relation even identically).
One could call this partly on-shell closure (partly because we did not use the full \( \mathcal{H}_{\text{phys}} \) but only \( \mathcal{H}_{\text{Diff}} \) in the equivalence relation). While it would be more satisfactory to have full off-shell closure, it is not logically required: At the end we are only interested in physical states and these are in particular spatially diffeomorphism invariant. Those states cannot distinguish between different representatives of the equivalence classes.

6. **Semiclassical limit**

The problem with demonstrating off-shell closure is that, in contrast to the first two, the third relation in (2.2) does not hold by inspection, not even modulo a diffeomorphism. This is not surprising because even classically one needs a full page of calculation in order to bring the Poisson bracket between two Hamiltonian constraints into the form of the right hand side of the third relation in (2.2). This calculation involves reordering of terms, differential geometric identities and integrations by parts etc. which are difficult to perform at the operator level. In order to make progress on this issue one would therefore like to probe the Dirac algebra with semiclassical states, the idea being that in expectation values with respect to semiclassical states the operators can be replaced by their corresponding classical functions and commutators by Poisson brackets, up to \( \hbar \) corrections.

There are two immediate obstacles with this idea:

The first is that the volume operator involved is not analytically diagonalisable. Recently, however, it was shown that analytical calculations involving the volume operator can be performed precisely using coherent states on \( \mathcal{H} \), so this problem has been removed. The second is that the existing semiclassical tools are only appropriate for graph non-changing operators such as the volume operator. Namely, as we will see, in order to be normalisable, coherent states are (superpositions of) states defined on specific graphs. The Hamiltonian constraint operator, however, is graph changing. This means that it creates new modes on which the coherent state does not depend and whose fluctuations are therefore not suppressed. Therefore the existing semiclassical tools are insufficient for graph changing operators such as the Hamiltonian constraint. The development of improved tools is extremely difficult and currently out of reach.

7. **Solutions and physical inner product**

Solutions to all constraints can be constructed algorithmically. These are the full LQG analogs of the LQC solutions of the difference equation that results from the single Hamiltonian constraint of LQC. They are the first rigorous solutions ever constructed in canonical quantum gravity, have non-zero volume and are labelled by fractal knot classes because the iterated action of the Hamiltonian constraint creates a self-similar structure (spiderweb) around each vertex. However, as in LQC these solutions are not systematically derived from a rigging map which is why a physical inner product is currently missing for those solutions.

This finishes the discussion of the properties of the Hamiltonian constraint operators. We want to stress that while evidently several issues need to be resolved, this is the first time in history that canonical quantum gravity was brought to a level such that

1. these and related questions could meaningfully be asked and analysed with mathematical precision.
2. a concrete, natural proposal for the Hamiltonian constraint operators can be derived which is consistent (anomaly free), namely the one where the segments \( s_I \) and \( s_{IJ} \) respectively are aligned and transversal to the graph respectively and where the resulting loops \( \beta_{IJ} \) are in the \( j = 1/2 \) representation.

Nobody in the LQG community believes that this concrete model is the “right” or “final” one, but it provides a concrete proposal which can be studied and further improved.

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34 Notice that it is not possible to probe \( \mathcal{D} \) with semiclassical spatially diffeomorphism invariant states because none of the operators involved preserves \( \mathcal{H}_{\text{Diff}} \).
As discussed, the most important open issues are the semiclassical limit and the physical inner product. These issues are overcome to a large extent by the Master Constraint Programme.

4.4.2.2 Master Constraint Programme

The idea of the Master Constraint is to sidestep the complications of the Hamiltonian constraints that have their origin in the non Lie algebra structure of the Dirac algebra \( \mathcal{D} \). Consider the Master constraint

\[
M := \int_\sigma d^3x \frac{H(x)^2}{\sqrt{\det(q)(x)}}
\]

(4.37)

It is not difficult to see that (4.37) has the following properties:

1. \( M = 0 \) is equivalent with \( H(N) = 0 \) for all test functions \( N \).
2. \( \{ F, \{ F, M \} \} \equiv 0 \) is equivalent with \( \{ F, H(N) \} = 0 \) when \( H(N') = 0 \) for all test functions \( N, N' \).
3. \( M \) is spatially diffeomorphism invariant.

The first property says that the single constraint \( M = 0 \) encodes the same constraint surface as the infinite number of Hamiltonian constraints while the second property says that the single double Poisson bracket with \( M \) selects the same weak Dirac observables as the infinite number of single Poisson brackets with Hamiltonian constraints. In other words the Master constraints defines the same reduced phase space as the infinite number of Hamiltonian constraints.

The third property means that the complicated Dirac algebra \( \mathcal{D} \) can be replaced by the comparatively trivial Master Algebra \( \mathfrak{M} \)

\[
\{ D(\widetilde{N}), D(\widetilde{N}') \} = \kappa D(\mathcal{L}_{\widetilde{N}} D(\widetilde{N}'))
\]

\[
\{ D(\widetilde{N}), M \} = 0
\]

\[
\{ M, M \} = 0
\]

(4.38)

which now is a true Lie algebra. This removes almost all obstacles that we encountered with the Hamiltonian constraints:

1. **Role of \( \mathcal{H}_{\text{Diff}} \)**

   Since \( M \) is spatially diffeomorphism invariant, its operator version \( \hat{M} \) can be defined directly on the spatially diffeomorphism invariant Hilbert space \( \mathcal{H}_{\text{Diff}} \). In fact, if \( \hat{M} \) is a knot class changing spatially diffeomorphism invariant operator, then it *must* be defined on \( \mathcal{H}_{\text{Diff}} \), it cannot be defined on \( \mathcal{H} \) \[11\].

   In retrospect, this justifies the construction of \( \mathcal{H}_{\text{Diff}} \) because for the solution of the Hamiltonian constraints the Hilbert space \( \mathcal{H}_{\text{Diff}} \) is unsuitable as an intermediate step towards the physical Hilbert space as it is not left invariant by the Hamiltonian constraints.

2. **Regulator removal**

   Remember the awkward role of spatially diffeomorphism invariant states in the removal of the regulator of the Hamiltonian constraint operators using a special type of weak* operator topology. It turns out \[24\] that a knot class changing operator can indeed be constructed on \( \mathcal{H}_{\text{Diff}} \) by directly implementing the techniques of \[20\] sketched above. In the construction of this operator, the removal of the regulator is now in the standard weak operator topology of \( \mathcal{H}_{\text{Diff}} \).

3. **Physical Hilbert space**

   Since the infinite number of Hamiltonian constraints was replaced by a single constraint, provided \( \hat{M} \) can be defined as a positive self - adjoint operator on \( \mathcal{H}_{\text{Diff}} \) and provided that \( \mathcal{H}_{\text{Diff}} \) decomposes into a direct sum of \( \hat{M} \)-invariant, separable Hilbert spaces, we know that direct integral decomposition guarantees the existence of a physical Hilbert space with a positive definite inner product induced from \( \mathcal{H}_{\text{Diff}} \). In order to construct it explicitly one needs to know the projection valued measure associated with \( \hat{M} \), that is, only standard spectral theory is required. While this is a difficult task to carry out explicitly due to the complexity of the operator \( \hat{M} \), we therefore have an existence proof for \( \mathcal{H}_{\text{phys}} \).
4. Anomalies: Ambiguities, locality and the semiclassical limit

Since there is only one Master constraint operator, it is trivially anomaly free. This enables one to consider a wider class of loop attachments, in particular those that would lead to an anomaly in the algebra of the Hamiltonian constraints. As examples show [35], such quantisations of the Master constraint based on anomalous individual constraints lead to spectrum which does not include zero. However, the prescription to subtract the spectrum gap from the Master constraint as mentioned in section 3 works in all examples studied. One might worry that this spectrum gap, which in free field theories is related to a normal ordering constant, is infinite. However, this is not the case: The master constraint is not just a plain sum of squares of the individual constraints, it is a \textit{weighted} sum. The weight in the case of gravity is the factor \( 1/\sqrt{\det(q)} \) in (4.37) which is the natural object to consider in order to make (4.37) spatially diffeomorphism invariant. For the case of the Maxwell field Gauss constraint studied in [35] the associated weight had to be a certain trace class operator on the one particle subspace of the Fock space. The weight function (operator) thus makes the normal ordering constant finite. Hence the master constraint programme can handle anomalous constraints.

For gravity of particular interest are constraints which are not graph changing, although the corresponding Hamiltonian constraints would be anomalous, for three reasons:

4.a. \textit{Ambiguities}

Since the loop to be attached is already part of the graph, the situation becomes closer to the situation in lattice gauge theory. This tremendously reduces the number of choices for the loop attachment and makes it no worse than the choice of a fundamental Hamiltonian in Wilson’s approach to the renormalisation group.

4.b. \textit{Locality}

With this option we are free to consider for instance “next neighbour” loop attachments. This leads to a spreading of the influence of the action of the Hamiltonian constraint from one vertex to all others thus removing the criticism of [82].

4.c. \textit{Semiclassical limit}

A non graph changing Master constraint can be defined on the kinematical Hilbert space. This has the advantage that the semiclassical states which so far in LQG are elements of \( \mathcal{H} \) can be directly used to analyse the semiclassical properties of \( \hat{\mathcal{M}} \). This has been done recently in [77] with the expected result that the infinitesimal generators (the Hamiltonian constraints) do have the correct semiclassical limit. Since these constraints determine the physical Hilbert space, this is an important step towards showing that gauge invariant operators commuting with \( \hat{\mathcal{M}} \) have the correct semiclassical limit on the physical Hilbert space. The semiclassical limit is of course only reached on graphs which are sufficiently fine. Graphs with huge holes would correspond to spacetimes with degenerate metrics in macroscopic regions which is not allowed in classical General Relativity.

Notice that semiclassical states have so far not been constructed on \( \mathcal{H}_{\text{Diff}} \). This means that the semiclassical limit of the graph changing Master constraint is currently out of reach, thus favouring the graph non changing version.

In what follows we will sketch both the graph changing Master constraint operator on \( \mathcal{H}_{\text{Diff}} \) and the graph non changing operator on \( \mathcal{H} \).

4.4.2.2.1 Graph changing Master constraint

We follow closely [74]. We notice that classically (\( \tau \) is again a triangulation)

\[
M = \lim_{\tau \to \sigma} \sum_{\Delta \in \tau} [\hat{\mathcal{C}}(\Delta)]^2
\]

\[(4.39)\]
where $\bar{C}(\Delta)$ coincides with (4.30) for the smearing function $N = \chi_\Delta$ (the characteristic function for a tetrahedron) and with $V(\Delta)$ replaced by $\sqrt{V(\Delta)}$. Thus, the heuristic idea is to define the quadratic form on $D^*_\text{Diff}$ by

$$Q_M(l, l') := \lim_{\tau \to \sigma} \sum_{\Delta \in \tau} < l, [\hat{C}'(\Delta)]^* [\bar{C}(\Delta)] l' >_{\text{Diff}}$$

(4.40)

where the prime denotes the operator dual as usual and * denotes the adjoint on $\mathcal{H}_{\text{Diff}}$. Unfortunately (4.40) is ill defined as it stands because the operators $\hat{C}'(\Delta)$ do not preserve $D^*_\text{Diff}$. The cure is to extend $< .. >_{\text{Diff}}$ to an inner product $< .. >_{\text{Diff}}$ on all of $D^*$. The final result turns out to be insensitive to the details of the extension because in the limit $\tau \to \sigma$ the Riemann sum becomes well defined on $D^*_\text{Diff}$.

Rather than going through the rigorous argument which can be found in [74] we will present here the shortcut already sketched in [19]: Pretending that (4.40) is well defined we can insert a resolution of unity to an inner product $< .. >$

$V(\Delta)$.

Expression (4.43) defines a positive quadratic form. However, it is not obvious that it presents the matrix elements of a positive operator. In [74] it is shown that (4.43) is closable thus presenting the matrix elements of a positive, self adjoint operator $\hat{M}$ on $\mathcal{H}_{\text{Diff}}$. Moreover, the non separable Hilbert space $\mathcal{H}_{\text{Diff}}$ decomposes into an uncountable direct sum $\mathcal{H}_{\text{Diff}} = \oplus_\sigma \mathcal{H}^\sigma_{\text{Diff}}$. Here the sectors $\mathcal{H}^\sigma_{\text{Diff}}$ are separable and are labelled by the angle moduli mentioned earlier. They are left invariant by $\hat{M}$ basically because it only creates three valent vertices which do not have moduli. It follows that the direct integral method is applicable to $\hat{M}$ thus resulting in the physical Hilbert space $\mathcal{H}_{\text{phys}}$ induced from $\mathcal{H}_{\text{Diff}}$.

It is easy to show that $\hat{M}$ allows for an infinite number of zero eigenvectors (elements of $\mathcal{H}_{\text{Diff}}$). This follows immediately from the properties of the Hamiltonian constraints. One just has to choose $\gamma(s)$ to be out of the range of the graphs underlying the SNW’s generated by the Hamiltonian constraints. Hence zero is contained in the point spectrum of this operator which constructed using non anomalous constraints. However, due to the present lack of graph changing and even spatially diffeomorphism invariant coherent states, a verification of the correct semiclassical limit of the graph changing $\hat{M}$ is currently out of reach.

$^55$Modulo some subtleties which can be found in [74] and that can be dealt with.
4.4.2.2 Non graph changing (extended) Master constraint

In order to have control on the semiclassical limit one must currently use a non graph changing operator and an operator which can be defined on $\mathcal{H}$. This can only be done by using underlying Hamiltonian constraints which are anomalous in the naive discretisation displayed below. However, there are techniques known from lattice gauge theory \[77\] which make use of the renormalisation group flow and which might enable one to work with non anomalous constraints. This amounts to considering more sophisticated discretisations. We see here that the issues of the semiclassical limit and the anomaly freeness are interlinked in a complicated way. Fortunately, anomalies do not pose any obstacles to the Master constraint programme.

In order to define such an operator we need the notion of a minimal loop: Given a vertex $v$ of a graph $\gamma$ and two edges $e, e'$ outgoing from $v$, a loop $\beta(\gamma, v, e, e')$ within $\gamma$ based at $v$, outgoing along $e$ and incoming along $e'$ is said to be minimal if there is no other loop within $\gamma$ with the same properties and fewer edges traversed. Let $L(\gamma, v, e, e')$ be the set of minimal loops with the data indicated. Notice that this set is always non empty but may consist of more than one element. We now define $\tilde{MT}_s := \tilde{M}_sT_s$ on spin network states $T_s$ over $\gamma$ where

$$
\tilde{M}_\gamma := \sum_{v \in V(\gamma)} \hat{C}_v \hat{C}_v
$$

$$
\hat{C}_v := \frac{1}{|T(\gamma, v)|} \sum_{e_1, e_2, e_3 \in T(\gamma, v)} \epsilon_v(e_1, e_2, e_3) \frac{\epsilon_v(e_1, e_2, e_3)}{|L(\gamma, v, e_1, e_2)|} \sum_{\beta \in L(\gamma, v, e_1, e_2)} \text{Tr}([A(\beta) - A(\beta)^{-1}]A(e_3)[A(e_3)^{-1}, \sqrt{\hat{V}_v}])
$$

Here $T(\gamma, v)$ is the number of ordered triples of edges incident at $v$ (taken with outgoing orientation) whose tangents are linearly independent\[^{36}\] and $\epsilon_v(e_1, e_2, e_3) = \text{sgn}(\det(\dot{e}_1(0), \dot{e}_2(0), \dot{e}_3(0)))$. The volume operator is given explicitly by

$$
\hat{V}_v = \frac{\sqrt{i}}{48} \sum_{e_1, e_2, e_3 \in T(\gamma, v)} \epsilon_v(e_1, e_2, e_3) \epsilon_{jkl}X_{e_1}^jX_{e_2}^kX_{e_3}^l
$$

where $X^jT_a = \text{Tr}([\tau_j A(e)]^T \partial / \partial A(e))$ is the right invariant vector field on the copy of $SU(2)$ determined by the holonomy $A(e)$ as introduced earlier.

It is easy to see that the definition (4.44) is spatially diffeomorphism invariant. Moreover, the results of \[77\] imply that expectation values with respect to the coherent states constructed in \[86\] which were barely mentioned in [12], defined on graphs which are sufficiently fine, the zeroth order in $\hbar$ of $\tilde{M}_\gamma$ coincides with the classical expression. In other words, the correctness of the classical limit of $\tilde{M}$ has been established recently. The results of \[77\] also imply that the commutator between the $\sum_v N_v \hat{C}_v$ reproduces the third relation in (2.2) in the sense of expectation values with respect to coherent states where $\hat{C}_v$ is the same as $\hat{C}_v$ in (4.41) just that $\sqrt{\hat{V}_v}$ is replaced by $\hat{V}_v$. This removes a further criticism spelled out in [12], namely we have off shell closure of the Hamiltonian constraints to zeroth order in $\hbar$. Possible higher order corrections (anomalies) are no obstacle for the Master constraint programme as already said.

4.4.2.3 Brief note on the volume operator

In order to show this one has to calculate the matrix elements of (4.45) which is non trivial because the spectrum of that operator is not accessible exactly\[^{37}\]. However, one can perform an error controlled $\hbar$ expansion within coherent state matrix elements and compute the matrix elements of every term in that expansion analytically \[77\]. The idea is extremely simple and it will surprise nobody that this works: In applications we are interested in expressions of the form $Q^r$ where $Q$ is a positive operator, $0 < r \leq 1/4$ is a

\[^{36}\]We set $\hat{C}_v = 0$ if $T(\gamma, v) = \emptyset$.

\[^{37}\]The matrix elements of the argument of the square root are known in closed form \[88\].
rational number and its relation to the volume operator is \( V = \sqrt[4]{Q} \). The matrix elements of \( Q \) in coherent states can be computed in closed form. Now use the Taylor expansion of the function \( f(x) = (1 + x)^r \) up to some order \( N \) including the remainder with \( x = Q/\langle Q \rangle - 1 \) where \( \langle Q \rangle \) is the expectation value of \( Q \) with respect to the coherent state of interest. The operators \( x^n \) in that expansion can be explicitly evaluated in the coherent state basis while the remainder can be estimated from above and provides a higher \( \hbar \) correction than any of the \( x^n \), \( 0 \leq n \leq N \). This completely removes the criticism of [12] that “nothing can be computed”.

In [12] we also find a lengthy discussion about the regularisation of the volume operator in terms of flux operators. Actually the discussion in [12] follows closely [79], however, the additional averaging step performed in [79] is left out in [12] for reasons unclear to the present author. However, even if one considers that averaging procedure unconvincing or unmotivated, there are completely independent abstract reasons for why (4.45) is the only possibility to define the volume operator which were spelled out in [79]: Namely, the argument of the volume operator, which classically is given as the integral of \( \sqrt{|\det(E)|} \) must be a completely skew expression in the right invariant vector fields because the only way to regularise it is in terms of flux operators. Now the relative coefficients between the terms for each triple are fixed, up to an overall constant, by spatial diffeomorphism covariance and cylindrical consistency\(^{38}\). The task to do was to show that a regularisation indeed exists which produces (4.45) which was done in [79] and to fix the constant which was done in [80]. In addition, there is an alternative point splitting regularisation [80] which does not use the averaging which also results in (4.45). Hence there can be absolutely no debate [12] about the correctness of (4.45) in particular that now we know from [77] that its classical limit is correct.

Finally, [12] stresses that the final operator that one gets should be independent of the regularisation scheme and it is criticised that the regularisation scheme that one uses for the volume operator seems to depend on ad hoc choices so that different choices could give a different operator. Again we state that [79, 80] fix the volume operator uniquely. Apart from that we would like to stress that in ordinary QFT there are only a hand full of regularisation schemes that one tests: Pauli – Villars, minimal subtraction, dimensional regularisation, point splitting. Here two different schemes were used [79, 89] which resulted in the same operator and thus the test is of the same order of “generality”.

4.4.2.4 Algebraic Quantum Gravity (AQG)

Notice that the framework of Algebraic Quantum Gravity (AQG) proposed in [77] in many ways supersedes LQG: In contrast to LQG, AQG is a purely combinatorial theory, that is, topology and differential structure of \( \sigma \) are semiclassical notions and not elements of the combinatorial formulation. Next, there are not an uncountably infinite number of finite, embedded graphs, there is only one countably infinite algebraic (or abstract) graph [90]. In particular, the theory loses its graph dependence, only in the semiclassical sectors (corresponding to different \( \sigma \)) do embedded graphs play a role. Hence AQG can possibly deal with topology change.

The Hilbert space of AQG is still not separable, but for an entirely different reason than in LQG: Since the graph is infinite we have to deal with an infinite tensor product of Hilbert spaces [86]. However, as von Neumann showed, these Hilbert spaces naturally decompose into separable Hilbert spaces which in our case turn out to be invariant under the algebraic version of \( \hat{M} \) so that on each sector the physical inner product exists by direct integral decomposition. Hence, non separability poses absolutely no obstacle. Some of these sectors can presumably be identified as approximations to Quantum Field Theories on curved backgrounds (namely when the geometry fluctuations around that background are small). In some sense, all QFT’s on curved spacetimes are included which must be the case in order to have a background independent theory. The Hilbert space therefore has to be non separable for we do not expect QFT’s on different backgrounds to be unitarily equivalent and there are certainly uncountably many non diffeomorphic backgrounds.

\(^{38}\)That is, the expression (4.45) for a given graph reduces to the one on any smaller graph when applied to spin network functions over the smaller graphs.
Finally, since the natural representation $U(\varphi)$ of $\text{Diff}(\sigma)$ is not available in the combinatorial theory (there is no $\sigma$), spatial diffeomorphism invariance has to be dealt with in an algebraic way. This possible by using the extended Master constraint whose classical expression for given $\sigma$ is given by

$$M_E = M + \int_{\sigma} d^3x \frac{q^{ab}D_aD_b}{\sqrt{\det(q)}} \tag{4.46}$$

It turns out that the additional piece in (4.46) just like $M$ itself can be lifted to the algebraic level thus abstracting from the given $\sigma$. Actually, the additional piece could also be defined in LQG\footnote{Despite the fact that $D_a$ does not exist as an operator valued distribution in LQG. The too singular $D_a$ are tamed by the additional operator $q^{ab}/\sqrt{\det(q)}$.} and the results of \cite{77} also imply that $M_E$ has the correct classical limit in both LQG and AQG. However, within LQG $M_E$ is somewhat unmotivated because one already has the representation $U(\varphi)$ of spatial diffeomorphisms. In AQG on the other hand there is no choice and the advantage of $M_E$ is that it treats the Hamiltonian constraint and the spatial diffeomorphism constraint on equal footing (rather than defining the infinitesimal generator for the Hamiltonian constraints but only exponentiated diffeomorphisms).

We refrain from displaying more details about AQG here as this is a rather recent proposal and because this is a review about LQG. The interested reader is referred to \cite{77}.

### 4.4.3 Dirac observables and physical Hamiltonian

As mentioned in section\cite{2} General Relativity is an already parametrised system and in order to extract gauge invariant information and a notion of physical time evolution among observables one must deparameterise it, e.g. using the relational framework sketched in section\cite{2}. There are many ways to do this but a minimal requirement is that the physical Hamiltonian is close to the Hamiltonian of the standard model at least when spacetime is close to being flat. In \cite{33} a particularly simple way of deparametrisation which fulfills this requirement has been recently proposed using scalar phantom matter. In fact one can write the Hamiltonian constraints in the equivalent form $H(x) = \pi(x) + C(x)$ where $\tau$ is the momentum conjugate to the phantom field $\phi$ and $C$ is a positive function on phase space which depends on all remaining matter and geometry only. Let now for any real number $\tau$

$$h_\tau := \int_{\sigma} d^3x (\tau - \phi)(x) C(x), \quad h := \int_{\sigma} d^3x C(x) \tag{4.47}$$

Given a spatially diffeomorphism invariant function $F$ we set

$$F(\tau) := \sum_{n=0}^{\infty} \frac{1}{n!} \{h_\tau, F\}_{(n)} \tag{4.48}$$

Then $F(\tau)$ is a one parameter family of Dirac observables and $dF(\tau)/d\tau = \{h, F(\tau)\}$. In particular, $h$ is itself a Dirac observable, namely the physical Hamiltonian that drives the physical time evolution of the Dirac observables.

This holds for the classical theory. In quantum theory (4.48) should be replaced by

$$\hat{F}(\tau) := \exp(\hat{h}_\tau/(ih)) \hat{F} \exp(-\hat{h}_\tau/(ih)) \tag{4.49}$$

provided we can make sense out of $\hat{h}_\tau$ as a self – adjoint operator. This is work in progress.

### 4.4.4 Brief note on spinfoam models

Spin foam models\cite{91} are an attempt at a path integral definition of LQG. They were heuristically defined in the seminal work\cite{92} which attempted at the construction of the physical inner product via the formal
exponentiation of the Hamiltonian constraints of [20]. The reason that this approach was formal is that the Hamiltonian constraints do not form a Lie algebra and they are not even self–adjoint. Thus, there are mathematical (exponentiation of non normal operators) and physical (non Lie group structure of the constraints prohibiting the possibility that functional integration over $N$ of $\exp(i\hat{H}(N))$ leads to a (generalised) projector) issues with this proposal.

This is why spin foam models nowadays take a different starting point. Namely, one starts from the Palatini action and writes it as a topological BF theory $S_{BF} = \int_M \text{Tr}(B \wedge F)$ together with additional simplicity constraint action $S(\Lambda, B) = \int_M \text{Tr}(\Lambda \otimes B \wedge B)$ where $\Lambda$ is a Lagrange multiplier tensor field with certain symmetry properties. Extremisation with respect to $\Lambda$ imposes the condition that the $B$ field two form comes from the wedge product of two tetrads. The advantage of this formulation is that a lot is known about the topological BF theory and one can regard the additional simplicity constraint as a kind of “interaction” term in addition to the “free” BF term. In order to define the spinfoam model one has to regularise it as in the canonical theory by introducing a finite 4D triangulation $\tau$ and a corresponding discretisation of the action like Wilson’s action for Yang Mills theory. The connection $A$ underlying the curvature $F$ is located as a holonomy on the edges of the dual triangulation $\tau^*$ while the $B$ field is located on the faces of $\tau$. One integrates $\exp(iS_{BF} + iS(\Lambda, B))$ over $A$ with respect to the Haar measure and over $B$ and the Lagrange multiplier $\Lambda$ with respect to Lebesgue measure. The integral over $\Lambda$ results in a $\delta$–distribution in $B$. This can be heuristically replaced by a $\delta$ distribution in the right invariant vector fields corresponding to the holonomies of the connection. One can then perform the $B$ integral resulting in an additional $\delta$ distribution in the holonomies which then are written as a sum over representations using the Peter&Weyl theorem. The simplicity constraints in terms of the right invariant vector fields then impose restrictions on the occurring representations on the edges and intertwiners at the vertices.

These steps are simplest illustrated by modelling the situation by a one dimensional system $S_{BF} = BF, S(\Lambda, B) = \Lambda B^2$. Then formally

$$\int dF \int dB \int d\Lambda \exp(i[BF + \Lambda B^2]) = \int dF \int dB \int \delta(B^2) \exp(iBF)$$

$$= \int dF \int dB \int \delta(-(d/dF)^2) \exp(iBF) = \int dF \int \delta(-(d/dF)^2) \delta(F) \quad (4.50)$$

This brief paragraph does not reflect at all the huge body of research performed on spin foam models, we have barely touched only those aspects directly connected with the canonical formulation. Please refer to [91] and references therein for a more complete picture describing the beautiful connection with state sum models, TQFT’s, categorification, 4D manifold invariants (Donaldson theory), non–commutative geometry, emergence of Feynman graph language and renormalisation groups etc.

From the canonical perspective, spin foam models are very important as they provide a manifestly spacetime diffeomorphism covariant formulation of LQG. In order to reach this goal, the following issues have to be overcome:

1. The relation with the canonical formulation is somehow lost. In fact, it is well known how to obtain a path integral formulation of a given canonical constrained theory [20]. The integration measure cannot be the naive one as used above if there are second class constraints. That this is indeed the case has been shown in the important work [93] which is, in the mind of the author, not sufficiently appreciated.

2. While the simplicity constraints expressed in terms of $B$ are mutually commuting as operators, their replacement in terms of right invariant vector fields do not and in fact they do not form a closed algebra. Hence, considered as quantum constraints they are anomalous and it is remarkable that there exists a unique non trivial solution to the simplicity constraints [94] at all. For the corresponding model one can show that the path integral is dominated by degenerate metrics [95], hence it seems
not to have the correct semiclassical behaviour which is then maybe not too surprising. There should be a way to implement the simplicity constraints in their non anomalous form.

3. In contrast to pure BF theory these constrained BF theories are no longer topological and thus not independent of the triangulation. Thus, in order to get rid of the triangulation dependence one could sum over triangulations and the weights with which this should be done are motivated by group field theory \[94\]. The result is supposed to give a formula defining a rigging map. While there are attractive features such as an emergent Feynman graph language, it is presently unclear whether the sum converges (as it should in a fundamental theory) or whether it is maybe not more appropriate to perform a refinement limit as in the theory of dynamical triangulations \[97\].

5 Physical applications

We have so far mostly reported about the status of the quantisation programme. Since LQG is a non perturbative approach, preferrably one would complete the quantisation programme before one studies physical applications. Since the programme reached its current degree of maturity only relatively recently, physical applications could so far not attract much attention. Certainly what is needed in the future is an approximation scheme with respect to which physical states, the physical inner product, Dirac observables and the physical Hamiltonian can be computed with sufficient detail. The semiclassical states \[86\] provide a possible avenue especially with respect to applications for which the quantum geometry can be regarded as almost classical. Namely we can consider kinematical semiclassical states which are peaked on the constraint surface and on the gauge cut defined by the clock variables. These states are then approximately annihilated by the Master constraint and the power series defining the Dirac Observables can be terminated after a few terms just like in perturbation theory. This procedure could be called quantum gauge fixing because we do not fix a gauge classically but rather suppress the fluctuations off the constraint surface and off the gauge cut.

Despite the fact that such an approximation scheme has so far not been worked out in sufficient detail\[40\] there are already some physical applications of LQG which are insensitive to the details of such an approximation scheme. These are 1. matter coupling, 2. kinematical geometrical operators, 3. Quantum black hole physics, 4. semiclassical states, 5. Loop quantum cosmology and 6. LQG phenomenology. We will say only very little about these topics here because our main focus is the mathematical structure of LQG. Hence we will restrict ourselves to the salient results and ideas.

5.1 Matter coupling

We have so far hardly mentioned matter. However, in LQG all (supersymmetric) standard matter can be straightforwardly coupled as well \[38, 75\]. As far as the kinematics is concerned, the background independent representation for the gauge fields of the standard matter is the same as for the gravitational sector because all the constructions work for an arbitrary compact gauge group. For Higgs fields, which are located at the vertices of the graph and other scalar matter one has a similar construction just that now states are labelled by points rather than edges. Finally for fermionic matter one uses a standard Berezin integral kind of construction. As far as the dynamics is concerned, the key technique of section 4.4.2.1 applies. All negative powers of \(\det(q)\) which appear in the matter terms and which are potentially singular can be replaces by commutators between fractional powers of the volume operator and gravitational holonomy operators.

The corresponding contributions to the Hamiltonian constraint have to be added up and are then squared in the Master constraint again without picking up an UV divergence. It is often criticised that LQG therefore does not impose any restriction on the allowed matter coupling. While that may turn out to be

\[40\]In particular one would like to know how close the approximate kinematical calculations are to the actual calculations on the physical Hilbert space.
phenomenologically attractive for the reasons mentioned in the introduction, it may actually be technically incorrect: For it could be that the answer to the question, whether the spectrum of the Master constraint contains zero, critically depends on which type of matter we couple. This is due to the fact that the shift of the minimum of the spectrum of the Master constraint away from zero is typically due to a kind of normal ordering correction. Now intuition from ordinary QFT suggests that there must be a critical balance between bosonic and fermionic matter in order that positive bosonic corrections cancel negative fermionic ones. Hence, maybe after all the spectrum only contains zero if we allow for supersymmetric matter. In order to decide this a more detailed knowledge of the spectrum of the Master constraint is required.

5.2 Kinematical geometric operators

One of the most cited results of LQG is the discreteness of the spectrum of kinematical geometric operators such as the volume operator, the area operator [78, 98] or the length operator [99]. The origin of this pure point spectrum is that these operators are functions of right invariant vector fields on various copies of SU(2) and thus they are diagonalised by linear combinations spin network states with fixed graph and edge spin but varying intertwiners. Since for fixed edge spin the space of intertwiners is finite dimensional, it follows that these operators reduce to finite dimensional Hermitian matrices on these fixed graph and spin subspaces.

However one should stress that the discreteness of the spectrum is a kinematical feature: None of these operators commutes with the spatial diffeomorphism or the Master constraint. Whether or not these operators retain this property after having them made true Dirac observables via the relational machinery depends on the clock matter that is used to deparametrise the theory. See [100] for a discussion.

However, if the discreteness of the spectrum is retained then this could be interpreted as saying that in LQG the geometry is discontinous or distributive at Planck scale. At macroscopic scales there is a correspondence principle at work, that is, the difference between subsequent eigenvalues rapidly decays for large eigenvalues.

5.3 Semiclassical states

As already mentioned, the development of semiclassical tools represent an important area of research in the development of LQG because they allow to test whether LQG is really a quantum theory of General Theory and not of some pathological phase thereof. These developments were hardly mentioned in [12]. Semiclassical states for LQG [86, 101, 102] have so far been constructed only for the kinematical Hilbert space because the primary goal was so far to test the semiclassical limit of the constraint operators which by definition annihilate physical semiclassical states and thus cannot be tested by them. However, we will present some ideas of how spatially diffeomorphism invariant or even physical semiclassical states might be constructed.

The kinematical semiclassical states are actually coherent states and are all based on the complexifier technique [86] which we will briefly sketch now: Suppose that we are given a phase space of cotangent bundle structure $\mathcal{M} = T^*\mathcal{A}$ where $\mathcal{A}$ is the configuration space. A complexifier is, roughly speaking, a positive function $C$ on $\mathcal{M}$ with the dimension of an action which grows stronger than linearly as $E^I \to \infty$ where $E^I$ denotes the momentum coordinates on $\mathcal{M}$ and $I \in \mathcal{I}$ is a labelling set. Denoting the points in $\mathcal{A}$ by $A_I$ we define complex configuration coordinates

$$Z_I = \sum_{n=0}^{\infty} \frac{i^n}{n!}\{A_I, C\}$$

explaining the name complexifier. Suppose that the theory can be canonically quantised such that $\hat{C}$ becomes a positive, self – adjoint operator on a Hilbert space $\mathcal{H} = L_2(\mathcal{A}, d\mu)$ of square integrable functions on some
distributional extension $\overline{A}$ of $A$ with respect to some measure $\mu$. The quantum analog of (5.1) becomes, upon replacing Poisson brackets by commutators divided by $i\hbar$, the *annihilation operator*

$$\hat{Z}_I = e^{-\hat{C}/\hbar} \hat{A}_I e^{\hat{C}/\hbar}$$

(5.2)

which explains the dimensionality of $C$. The operators $\hat{Z}_I$ are mutually commuting. The exponentials are defined via the spectral theorem. Let $\delta_A$ be the $\delta$--distribution with respect to $\mu$ with support at $A$ and consider the distribution

$$\Psi_A := e^{-\hat{C}/\hbar} \delta_A$$

(5.3)

Due to the positivity of $\hat{C}$ the operator $\exp(-\hat{C}/\hbar)$ is a smoothening operator, explaining the required positivity of $C$. In fact, if $\mathcal{H}$ is separable, then $\Psi_A$ will be an element of $\mathcal{H}$ (normalisable) if $C$ is suitably chosen. Now the growth condition in the definition of $C$ typically ensures that $\Psi_A$ is *analytic* in $A$ and hence can be analytically continued, as an $L_2$ function, to $Z$. Denoting the analytically continued object by $\Psi_Z$ we obtain immediately the defining property of a *coherent state* to be a simultaneous eigenstate of the annihilation operators

$$\hat{Z}_I \Psi_Z = Z_I \Psi_Z$$

(5.4)

Notice, however, that if $\mathcal{H}$ is not separable then $\Psi_Z$ is only a coherent distribution even if $C$ has all the required properties.

This construction in fact covers all coherent states that have been considered for finite or infinite systems of uncoupled harmonic oscillators, in particular the “classical” coherent states for the Maxwell field (QED). For the Maxwell field the complexifier turns out to be

$$C = \frac{1}{2e^2} \int_{\mathbb{R}^3} d^3x \, \delta_{ab} \, E^a (-\Delta)^{-1/2} E^b$$

(5.5)

where $e$ is the electric charge, $E^a$ the electric field and $\Delta$ the flat space Laplacian.

In fact, quadratic expressions in the momentum operators always are good choices for $C$. However, for LQG we may not use background dependent objects such as $\Delta$. In [80] quadratic expressions in the area operator (see below) were used and semiclassical properties such as peakedness in phase space, infinitesimal Ehrenfest property, overcompleteness, semiclassical limit and small fluctuations were established. Of course, since the kinematical LQG Hilbert space $\mathcal{H}$ is not separable, one must restrict the complexifier construction to separable subspaces. Natural candidates are the Hilbert spaces $\mathcal{H}_\gamma$ (closure of the span of SNWF’s over $\gamma$) and $\mathcal{H}_\gamma'$ (closure of the span of SNWF’s over all subgraphs of $\gamma$). The resulting states $\Psi_{Z,\gamma} = \sum_{\gamma(s)=\gamma} \Psi_{Z,s} T_s$ and $\Psi_{Z,\gamma'} = \sum_{\gamma(s)\subset\gamma} \Psi_{Z,s} T_s$ are respectively the spin network or cylindrical projections of the distributions $\Psi_Z = \sum_s \Psi_{Z,s} T_s$ (the sum is over all SNW’s) and are called shadows [102] or cut – offs [80] of $\Psi_Z$ respectively.\(^{41}\)

This graph dependence of the present semiclassical framework of LQG is an unpleasant feature which so far has prevented one from establishing the semiclassical limit of graph changing operators such as the Hamiltonian constraint. This is because the Hamiltonian constraint creates new edges whose fluctuations are not controlled by these graph dependent states. Hence the above mentioned semiclassical properties only hold for graph non changing operators and this is why the graph non changing Master constraint is under much better control than the Hamiltonian constraints. In AQG [77] even the graph dependence is lost because there is only one fundamental graph.

Finally, let us address the question of spatially diffeomorphism invariant or physical states. These Hilbert spaces do not obviously have a representation as $L_2$ spaces and moreover it is not easy to find complexifiers with the required properties which are either spatially diffeomorphism invariant or Dirac observables. Hence the complexifier idea is not immediately applicable. However, we have shown that there

\(^{41}\)In order to avoid confusion which may arise form corresponding remarks in [12]: These states are functions of distributional connections $A \in \overline{A}$ labelled by smooth fields $Z$. This is even the case for Maxwell coherent states. Hence one can surely get back the smooth fields of the classical theory in the classical limit.
are (anti) linear rigging maps \( \eta_{\text{Diff}} : D \to \mathcal{H}_{\text{Diff}} \) and \( \eta_{\text{phys}} : D^{*}_{\text{Diff}} \to \mathcal{H}_{\text{phys}} \) respectively. Now, given a, say cutoff state \( \Psi_{Z,\gamma} \), we obtain spatially diffeomorphism invariant states \( \Psi_{\text{Diff}}^{Z,\gamma} := \eta_{\text{Diff}}(\Psi_{Z,\gamma}) \) and physical states \( \Psi_{\text{phys}}^{Z,\gamma} := \eta_{\text{phys}} \circ \eta_{\text{Diff}}(\Psi_{Z,\gamma}) \) which can serve as Ansätze for semiclassical states in the corresponding Hilbert spaces. Whether they continue to have the desired semiclassical properties with respect to spatially diffeomorphism invariant or Dirac observables respectively is the subject of current research.

5.4 Quantum black hole physics

The main achievement of LQG in this application is to provide a microscopic explanation of the Bekenstein Hawking entropy, see [103] and references therein. The classical starting point is the theory of isolated and dynamical horizons [104] which is somehow a local\(^{42}\) definition of an event horizon and captures the intuitive idea of a black hole in equilibrium. The notion of an isolated horizon uses, among other things, the classical field equations and therefore is a classical concept which is imported into the quantum theory by hand. In other words, the presence of the black hole is put in classically leading to an inner boundary of spacetime. It would be more desirable to have entirely quantum criteria at one’s disposal, see e.g. [105] for first steps, however the following partly semiclassical framework is completely consistent and satisfactory.

The presence of the inner boundary leads to boundary conditions which, intuitively speaking, reduce the gauge freedom at the boundary and thus give rise to boundary degrees of freedom. Remarkably, their dynamics is described by a \( U(1) \) quantum Chern Simons theory. On the other hand, the bulk is described by LQG. In order to compute the entropy of the black hole one counts the number of eigenstates of the area operator of the \( S^2 \) cross sections \( S \) of the horizon\(^{43}\) whose eigenvalues fit into the interval \([A_{R_0} - \ell_P^2, A_{R_0} + \ell_P^2]\) where \( A_{R_0} \) is some macroscopic area.

This number would be infinite if \( S \) would be an arbitrary surface. Namely a bulk state is described, near the horizon, by the ordered sets of punctures of the bulk graph with the surface \( S \) and at each such puncture \( p \) by the total spin \( j_p \) to which the edges running into \( p \) couple. The area eigenvalue for such a configuration is given by \( \lambda = \hbar \kappa \beta \sum_p \sqrt{j_p(j_p + 1)} \) \hspace{1cm} (5.6)

For fixed \( j_p \) there are an infinite number of spin network states which couple to total \( j_p \) (for instance let two edges run into \( p \) with spins \( j_1, j_2 = j_1 + j_p \) where \( j_1 \) is arbitrary). Hence if we would count bulk states, the entropy would diverge.

However, the physical reasoning is that what we must count are horizon states of the Chern Simons theory because the horizon degrees of freedom are the intrinsic description of the black hole and not the bulk degrees of freedom. Due to the quantum boundary conditions, the surface and bulk degrees of freedom are connected in the following way in the quantum theory: Around each puncture, the holonomy along the loop of the \( U(1) \) Chern Simons connection must be, roughly speaking, equal to the signed area (flux) of the surface bounded by the loop. Hence what matters to the surface theory is the number \( j_p \) and not the detailed recoupling that created it. In other words, one ignores the multiplicities of the \( j_p \).

With this in mind one can count now the number of eigenvalues. This would again be infinite if there would not be an area gap, i.e. a smallest non vanishing area eigenvalue which one can read off from (5.6). The result is \( \ln(N) = \frac{\beta A_{R_0}}{\beta_0 4\ell_P^2} + O(\ln(A_{R_0}/\ell_P^2)) \) \hspace{1cm} (5.7)

where \( \beta_0 \) is a numerical constant. This is the Bekenstein Hawking formula if we set \( \beta = \beta_0 \) which has been suggested to be one way to fix the Immirzi parameter. This would be inconsistent if \( \beta_0 \) would depend on

\(^{42}\)The usual definition of a black hole region as the complement of the past of future null infinity obviously requires the knowledge of the entire spacetime and is inappropriate to do local physics.

\(^{43}\)Due to the boundary conditions this turns out to be a Dirac observable. In particular, different cross sections have the same area.
the hair of the black hole. However, the constant $\beta_0$ is universal, all black holes of the Schwarzschild – Reissner – Nordström – Newman – Kerr family are allowed as well as Yang Mills and dilatonic hair. Notice that these black holes are of astrophysical interest, they are non supersymmetric and far from extremal, in contrast to the similar calculations in string theory which heavily depend on extremality.

In summary, there is an unexpected, consistent interplay between classical black hole physics, quantum Chern Simons theory and LQG. Future improvements should include the development of quantum horizons and Hawking radiation.

5.5 Loop quantum cosmology (LQC)

Loop quantum cosmology (LQC) is not the cosmological sector of LQG\textsuperscript{44}. Rather it is the usual homogeneous minisuperspace quantised by the methods of LQG. This has a kinematical and a dynamical side [109]. As far as the kinematics is concerned, although LQC has only a finite number of degrees of freedom one can circumvent the Stone – von Neumann uniqueness theorem for the representations of the canonical commutation relations by dropping the assumption of weak continuity of the Weyl operators. This is in complete analogy to LQG where holonomies but no connections are well defined as operators for precisely the same reason. The corresponding Hilbert space is then not of the Schrödinger type $L_2(\mathbb{R}, dx)$ but rather of the Bohr type $L_2(\mathbb{R}, d\mu_0)$. Here $\mathbb{R}$ is the Bohr compactification of the real line. It is the counterpart of $\mathcal{A}$ and can be defined as the Gel’fand spectrum of the Abelian $C^*$ algebra generated by the functions $q \rightarrow \exp(i\mu q)$. This algebra is called the algebra of quasiperiodic functions and is the counterpart of $\mathcal{Cyl}$. Finally $\mu_0$ is the Haar measure on $\mathbb{R}$ which is in complete analogy to the Ashtekar – Lewandowski measure of LQG.

On the dynamical side the situation in LQG is matched in the sense that in the Hamiltonian constraint one cannot work with connections but only with holonomies. Hence one has to modify the classical constraint by working with say $\sin(\mu_0 q)/\mu_0$ rather than $q$ where $\mu_0$ is an arbitrarily small but finite constant\textsuperscript{45}. Since also inverse powers of the momentum $p$ conjugate to $q$ appear in the Hamiltonian constraint one uses the same key kind of key identities as in LQG such as

$$ir\mu_0 \frac{\text{sgn}(p)}{|p|^{1-r}} = e^{-i\mu_0 q}\{[p]^r, e^{i\mu_0 q}\} \quad (5.8)$$

where $0 < r < 1$ is a rational number, in order to avoid negative powers of the $p$ operator in the quantum theory.

The main advantage of this model is that one can carry out almost all steps of the quantisation programme and compare it with the conventional Schrödinger quantisation (Wheeler DeWitt theory). The predictions of the model are in fact quite intriguing: Avoidance of curvature singularities, deterministic quantum gauge flow through the would be singularity, inflation from quantum geometry, avoidance of chaos in Bianchi IX cosmologies, recovery of conventional cosmology at large scale factor etc. See [76] for a review. The most mathematically precise treatment can be found in [110].

Of course one is never sure whether the simplifications that are made in toy models spoil its predictive power, that is, whether the results of the toy model continue to hold in the full theory. Partial confirmation of LQC singularity avoidance results within full LQG can be found in [112], although via a completely different mechanism. However, at least the model tests important aspects of the full theory, in particular the key identities of the type (4.27) without which these spectacular results of LQC would not have been possible.

Notice that LQG and in particular LQC can easily deal with de Sitter space kind of situations while this seems to be harder in superstring theory whose effective low energy limit should be supergravity on de Sitter

\textsuperscript{44} So far there is no satisfactory derivation of LQC from LQG, LQC is constructed “by analogy”.

\textsuperscript{45} See [109] for arguments to fix the value of $\mu_0$. In LQG the analog of $\mu_0$ would be the regulator $\epsilon$ in the loop attachment, however, in LQG all values of $\epsilon$ are equivalent due to spatial diffeomorphism invariance. This does not happen in LQC because in LQC the spatial diffeomorphism group is gauge fixed so that the appearance of $\mu_0$ could be considered as an artefact of the simplicity of the model.
space. However, the de Sitter algebra does not admit a positive Hamiltonian indicating that supergravity on de Sitter space is unstable. This is potentially alarming because recent observations indicate that the universe currently undergoes a de Sitter phase.

5.6 LQG phenomenology

The field of LQG phenomenology has just started to develop, mostly because in the majority of cases there is no clear cut derivation of the phenomenological assumptions made from full LQG. See [113, 114] for a review. One of the hottest topics in this fields are signatures of Lorentz invariance violation. A phenomenological description of this could be doubly special relativity (DSR) [115], a theory in which not only the speed of light but also the Planck energy is (inertial) frame independent. In 3D it turns out to be possible to connect DSR [116], non commutative geometry [117] and LQG in the spin foam formulation but in 4D this is still elusive.

The intuitive idea behind Lorentz invariance violation in quantum gravity is the apparently Planck scale discreteness of LQG: If true, then quantum geometry looks more like a crystal than vacuum even if the gravitational vacuum state looks like Minkowski space on large scales. Hence there could be non trivial dispersion relations for light propagation leading to energy dependent time of arrival delays of photons of high energies that have travelled a long distance. One possible source of such signals are γ-ray bursts at cosmological distances [118, 119] which would be detectable by the GLAST detector provided that the effect is linear in $E/E_P$ where $E$ is the photon energy and $E_P$ the Planck energy. For first steps towards a systematic derivation from LQG see [120, 121]. Notice that for the five perturbative string theories on the Minkowski target space Lorentz invariance is built in axiomatically, hence Lorentz invariance violation could discriminate between LQG and string theory.

Another hot topic concerns cosmology. To this realm belong the prediction of deviations from the scale invariance of the power spectrum of the cosmic microwave background radiation (CMBR) [122, 123] using LQC (related) methods which might be detectable by the WMAP or PLANCK detectors.

Suffice it to say that this field is largely unexplored and that it needs more input both from experiments and theory.

6 Conclusions and outlook

We hope to have given a brief but self critical account of the status of LQG with special focus on the most important issue, the implementation of the quantum dynamics. In particular we hope to have addressed most if not all issues that have been brought up in [12]. We presented them from an “inside” point of view and showed why the mostly technically correct description in [12], in our mind, is often unnecessarily sceptic, inconclusive or incomplete. Notice also that we only reported results well known in the LQG literature. We emphasise this because the unfamiliar reader may have the impression that only [12] unveiled the issues discussed there.

We have indicated why non separable Hilbert spaces are no obstacle in LQG, they may even be welcome! There has been important progress recently on the frontiers of the semiclassical limit, the physical Hilbert space, physical (Dirac) observables, the physical Hamiltonian, the constraint algebra, the avoidance of anomalies and quantisation ambiguities, the covariant formulation [13] as well as physical applications which were insufficiently appreciated in [12]. The report given in [12] in many ways displays the field of LQG as it was a decade ago and thus ignores the progress made since then during which the field quadrupled in size. We hope to have clarified in this report that important developments were left out in [12] thus hopefully reducing the negative flavour conveyed there.

Let us discuss further issues touched upon in [12] which were not yet mentioned in this article:

1. A folklore statement that seems to have entered several physics blogs is that weakly discontinuous representations of the kind used in LQG do not work for the harmonic oscillator so why should they
work for more complicated theories? This is the conclusion reached in [124]. As we will now show, while [124] is technically correct, its physical conclusion is completely wrong. In [124] one used a representation discussed first for QED [125] in order to avoid the negative norm states of the Gupta – Bleuler formulation. In this representation neither position $q$ nor momentum $p$ operators are well defined, only the Weyl operators $U(a) = \exp(iaq)$, $V(b) = \exp(ibp)$ exist. Hence the usual harmonic oscillator Hamiltonian $H = q^2 + p^2$ does not exist in this representation. Consider the substitute $H_\epsilon = [\sin^2(\epsilon q) + \sin^2(\epsilon p)]/\epsilon^2$. What is shown in [124] is that this operator is ill defined as $\epsilon \to 0$. Is this a surprise? Of course not, we knew this without calculation because the representation is not weakly continuous. The divergence of $H_\epsilon$ as $\epsilon \to 0$ in discontinuous representations is therefore a trivial observation. However, what is physically much more interesting is the following. Fix an energy level $E_0$ above which the harmonic oscillator becomes relativistic and thus becomes inappropriate to model the correct physics. Let\(^{46}\) $a_\epsilon^\dagger := [\sin(\epsilon q) + i \sin(\epsilon p)]/\epsilon$. Consider the finite number of observables

$$b_{\epsilon,n} := \frac{1}{n!}(a_\epsilon)^n (a_\epsilon^\dagger)^n, \quad n = 0, \ldots, N = E_0/\hbar$$ (6.1)

Let $\Omega_0$ be the Fock vacuum in the Schrödinger representation and $\omega$ the state underlying the discontinuous representation. Fix a finite measurement precision $\delta$. Since the Fock representation is weakly continuous we find $c_0(N, \delta)$ such that $|<\Omega_0, b_{\epsilon,n}\Omega_0> - nh| < \delta/2$ for all $\epsilon \leq \epsilon_0$. On the other hand, by Fell’s theorem\(^{47}\) we find a trace class operator $\rho_{N, \delta}$ in the GNS representation determined by $\omega$ such that $|\text{Tr}(\rho_{N, \delta} b_{\epsilon,n}) - <\Omega_0, b_{\epsilon,n}\Omega_0> | < \delta/2$ for all $n = 0, 1, \ldots, N$. It follows that with arbitrary, finite precision $\delta > 0$ we find states in the Fock and discontinuous representations respectively whose energy expectation values are given with precision $\delta$ by the usual value $nh$. This implies that the two states cannot be physically distinguished.

In [127] [128] even more was shown\(^{48}\): There the spectrum of the operator $H_\epsilon$ was studied and the eigenvectors and eigenvectors were determined explicitly. One could show that by tuning $\epsilon$ according to $N, \delta$ even the first $N$ eigenvalues do not differ more than $\delta$ from $(n + 1)\hbar$. Moreover, having fixed such an $\epsilon$, the non separable Hilbert space is a direct sum of separable $H_\epsilon$ invariant subspaces and if we just consider the algebra generated by $a_\epsilon$ each of them is superselected. Hence we may restrict to any one of these irreducible subspaces and conclude that the physics of the discontinuous representation is indistinguishable from the physics of the Schrödinger representation within the error $\delta$. This should be compared with the statement found in [124] that in discontinuous representations the physics of the harmonic oscillator is not correctly reproduced.

2. Actually the paper [124] was triggered by [129] where the following was shown: Using discontinuous representations one can quantise the closed bosonic string in any spacetime dimension without encountering anomalies, ghosts (negative norm states) or a tachyon state (instabilities). The representation independent and purely algebraic no go theorem of [130] that the Virasoro anomaly is unavoidable is circumvented by quantising the Witt group $\text{Diff}(S^1) \rtimes \text{Diff}(S^1)$ rather than its algebra $\text{diff}(S^1) \otimes \text{diff}(S^1)$. Since the representation of the Witt group is discontinuous, the infinitesimal generators do not exist and there is no Virasoro algebra in this discontinuous representation, exactly like in LQG. However, as in LQG, a unitary representation of the Witt group is sufficient in order to obtain the Hilbert space of physical states via group averaging techniques and even a representation of the invariant charges [131] of the closed bosonic string.

\(^{46}\)Notice that classically $H_\epsilon = |a_\epsilon|^2$.

\(^{47}\)The abstract statement is [6]: The folium of a faithful state on a $C^*$-algebra is weakly dense in the set of all states. Here the folium of a states are all trace class operators on the corresponding GNS Hilbert space. The theorem applies to the unique $C^*$ algebra generated by the Weyl operators $U(a), V(b)$ which we are considering here. The representations considered in [124] [125] are faithful.

\(^{48}\)In a representation which was continuous in one of $p$ or $q$ but discontinuous in the other. But similar results hold in this completely discontinuous representation considered here.
Does this mean that the magical dimension $D = 26$ cannot be seen in this representation? Of course it can: One way to detect it in the usual Fock representation of the string is by considering the Poincaré algebra (in the lightcone gauge) and ask that it closes. For the LQG string (129) again the Poincaré group is represented unitarily but weakly discontinuously. However, we can approximate the generators as above in terms of the corresponding Weyl operators using some tiny but finite parameter $\epsilon$. Since these are a finite number of operators in the corresponding $C^*$ algebra, an appeal to Fell’s theorem and using continuity of the Weyl operators in the Fock representation guarantees that we find a state in the folium of the LQG string with respect to which the expectation values of the approximate Poincaré generators coincides with their vacuum (or higher excited state) expectation values in the Fock representation to arbitrary precision $\delta$.

Thus $D = 26$ is also hidden in this discontinuous representation, it is just that for no $D$ there is a quantisation obstruction. Of course, much still has to be studied for the LQG string, e.g. a formulation of scattering theory, however, the purpose of (129) was not to propose a phenomenologically interesting model but rather to indicate that $D = 26$ is not necessarily sacred but rather a feature of the specific Fock quantisation used.

3. In (6) we find the statement that the instantaneous fields (smearred only in 3D rather than in 4D) are too singular in interacting quantum field theories. Indeed, in Wightman field theories one can read Haag’s theorem as saying that the representation of the interacting field algebra (which contains dynamical information) is never unitarily equivalent to the representation of the canonical commutation relations of the free field algebra (which lacks the information about the interaction). This seems to imply an obstruction to canonical quantisation where one precisely starts from a purely kinematical representation of the Poincaré algebra of the instantaneous fields. The catch is in the assumptions of Haag’s theorem. In LQG this no go theorem is circumvented because the quantum field theory that one constructs is not a Wightman field theory: It is a QFT of a new kind, namely a background independent QFT to which Haag’s theorem as stated above does not apply because the Wightman axioms do not hold. In fact, in LQG the interaction is encoded in the self-adjoint Master constraint which is well (densely) defined.

4. In (12) we find the question where in LQG does one find the counter terms (7) of perturbative quantum gravity? More generally, how does one make contact with perturbative QFT and what role does the renormalisation group play in LQG, if any? These perturbative questions are hard to answer in a theory which is formulated non-perturbatively, however, let us make a guess:

Once a physical Hamiltonian such as the one of section 4.4.4 has been successfully quantised one can in principle define scattering theory in the textbook way, that is, one would compute transition amplitudes between initial and final physical states. This may be hard to do technically but there is no obstruction in principle. In order to recover perturbation theory around Minkowski space one will consider a physical state (vacuum) which is a minimal energy state with respect to that Hamiltonian and peaked on Minkowski space. The physical excitations of that state can be considered as the analogs of the graviton excitations of the perturbative formulation. Now by construction the transition amplitudes (n point functions) are finite, however, there will be of course screening effects, i.e. effectively a running of couplings where the energy scale at which one measures is fed in by the physical state by which one probes a given operator. This is the way we expect to recover renormalisation effects.

As far as counterterms are concerned, as we have frequently stated, there are correction terms in all semiclassical computations done so far which depend on the Planck mass, see e.g. (121) where a finite but large quantum gravity correction to the cosmological constant (50) is computed which results

49 Of course one could ask whether the question is meaningful if quantum gravity, which is believed to be non renormalisable, simply does not admit a perturbative formulation. However, it is believed that perturbative quantum gravity does make sense as an effective theory.

50 The cosmological “constant” therefore becomes dynamical.
from photon field propagation on fluctuating spacetimes. Similar results are expected with respect to graviton propagation. These counter term operators are formulated in terms of the canonical fields but using the field equations one can presumably recast their classical limits into covariant counter term Lagrangeans.

Of course it is on the burden of LQG to show that this is really what happens, but it is not that there are no ideas for possible mechanisms.

We will now answer a number of frequently asked questions which one can find in \[12\]:

1. **Is there only mathematical progress in LQG?**

   A continuously updated and fairly complete list of all LQG publications to date can be found in \[132\]. A brief look at this list will show that there are papers of all levels of rigour and that mathematically more sophisticated papers were motivated and driven by less rigorous papers which started from a physical idea. It is true that in LQG one puts stress on mathematical rigour. The reason is that developing background independent QFT’s means walking on terra incognita. Hence, one does not have the luxury to be cavalier about mathematical details as in background dependent QFT’s where well established theorems ensure that there are rigorous versions of formal calculations.

   Section 5 should have indicated that current research is focussed on hot research topics such as semiclassical quantum gravity (contact with QFT on curved spacetimes), quantum cosmology and quantum black hole physics. These results together with the huge body of work on spin foam models was hardly mentioned in \[12\], see however \[13\]. Ignoring this research performed over the past ten years means giving an out of date presentation of LQG which would be similar to writing a review on string theory without mentioning D – branes, M – theory and the landscape.

   Notice also that being a much smaller and younger field than string theory or high energy physics\footnote{LQG is the focal point of only an order of $10^2$ researchers worldwide.} which in addition cannot just use the techniques from ordinary particle physics but in fact must first develop its own mathematical framework from scratch, the amount of results obtained so far is naturally smaller due to lack of man power.

2. **Has there anything been gained as compared to the Wheeler – DeWitt framework?**

   Any serious theoretical physicist will confirm that it is almost a miracle that one can tame mathematical monsters such as the area, volume, Hamiltonian constraint or Master constraint operator at all. These operators are integrals over delicate, nonpolynomial functions of operator valued distributions evaluated at the same point which are hopelessly singular in usual background dependent Fock representations. Moreover, not only can one give mathematical sense to them, they are even free of UV divergences. This is the beauty of background independence and provides a precise implementation of the old physical idea that quantum gravity should provide the *natural regulator of ordinary QFT UV divergences*.

   For the first time one can write down a concrete, mathematically well defined proposal for the Hamiltonian or Master constraint and study its physical properties. For the first time one can actually construct rigorous solutions thereof. For the first time one can precisely define a kinematical, spatially diffeomorphism invariant or physical Hilbert space. For the first time one could show that the semiclassical limit of at least the graph non changing Master constraint is the correct one with respect to rigorously defined, kinematical coherent states\footnote{Notice that it is meaningless the semiclassical limit of a constraint operator with physical coherent states which by definition are in its (generalised) kernel.}.

   It is true that not all questions have been answered in connection with the quantum dynamics and research on it will continue to occupy many researchers during many years to come. However, what is asked for in \[12\] is too much: Nobody expects that one can completely solve the theory. We cannot even solve classical General Relativity completely and we will probably never be able to.
and even more so LQG are not integrable systems such as string theory on Minkowski background target space which is mathematically relatively trivial as a field theory. Even today people working in classical General Relativity struggle to get analytical results for the gravitational waves radiated by, say, a black hole merger. The problem was posed almost half a century ago but recent progress is mostly due to increasing computing power. Gravitational waves is just a tiny sector of classical General Relativity and in LQG we also must hope that we can at least analyse the theory in sufficient detail in those sectors.

In [12] the authors ask for (approximately) physical semiclassical states that enable one to investigate the QFT on curved spacetimes limit of the theory. We claim that these exist: Kinematical coherent states have been introduced in [86]. We can choose to have them peaked on the constraint surface and then those states solve the Master constraint approximately [77], that is, $\|\hat{M}\psi\| \approx 0$. These states will enable us to perform semiclassical perturbation theory as described in [77] and the non-diagonisability of the volume operator poses absolutely no problem here.

Finally, again a glance at [132] reveals that in LQG there is linear progress on the quantisation programme outlined in section 3 over the past twenty years. One never changed the rules of that programme which means that the velocity of progress is naturally decreasing as one faces the ever tougher steps of that programme. We just mention that because from [12] one could sometimes get the impression that what is criticised is that researchers in LQG did not identify the open problems mentioned in [12]. They did as one can see from the publications, but some of the problems simply have not yet been solved and are topics of current research. That does not mean that they cannot be solved at all and so far every hurdle in LQG was taken.

3. Is general covariance broken in LQG?

When reading [12] one may get the impression that spacetime diffeomorphism invariance is broken right from the beginning just because one performs a 3 + 1 split of the action. This impression is wrong. As we have tried to explain in section 2 the constraints require that physical observables are independent of the foliation that one introduces in the canonical formulation. This is the same in string theory where the Witt constraints require worldsheet diffeomorphism invariance of physical observables. These constraints are implemented in the LQG Hilbert space and their kernel defines spacetime diffeomorphism invariant states. The question of off – shell versus on – shell closure is still open for the Hamiltonian constraint. This is why the Master constraint programme was introduced as a possible alternative which seems to work in the sense that for the Master constraint one could show that those constraints and their algebra have the correct classical limit [77]. They are maybe implemented anomalously in the Master constraint but by subtracting from the Master constraint the minimum of the spectrum the anomaly can be cancelled which corresponds to some kind of normal ordering. Notice that this is an off shell closure of the constraints as asked for in [12].

In [12] the authors illustrate the importance of on – shell versus off – shell closure by multiplying a given set of non anomalous quantum constraints with structure constants with a central operator. These modified constraints still close on shell while the off shell algebra would now close only with structure functions. However, there are now possibly extra solutions, namely those in the kernel of the central operator, hence the physical Hilbert space would suffer from an infinite number of ambiguities. We find this example inconclusive for the following reason: In LQG one did not randomly multiply the Hamiltonian constraint by something else but rather just used the classical expression and directly quantised it by reasonable regularisation techniques. Furthermore, when modifying the classical constraints by multiplying it with a Dirac observable, the modified constraints define the same constraint surface as the original ones only where the Dirac observable does not vanish. Hence the extra solutions in the kernel of the central operator are physically not allowed and therefore, in this example at least, the modified quantum constraints in fact define the same physical Hilbert space as the original constraints.
4. Does non-separability of the Hilbert space prevent the emergence of the continuum in the semiclassical limit?

In [12] the authors point out the non-separability of the kinematical Hilbert space which originates from the weak discontinuity of the holonomy operators. They call this the *pulverisation of the continuum* in the sense that all, even infinitesimally different, edges lead to orthogonal spin network states. The only topology on the set graphs with respect to which the scalar product is continuous is the *discrete topology* (every subset is open). They then ask whether the continuum can be recovered in the semiclassical limit. The answer is in the affirmative: The approximately physical states \[86\] (kinematical coherent states which are peaked on the constraint surface of the phase space) are labelled by smooth classical fields and the corresponding expectation values of physical operators such as the Master constraint depend even smoothly on those fields, not only continuously, see e.g. \[77\].

5. Is the ambiguity in the Hamiltonian constraint comparable to non renormalisability of perturbative quantum gravity?

Definitely not:
First of all there is a crucial qualitative difference: Perturbative quantum gravity cannot make sense as a fundamental theory because the perturbation series diverges for all possible choices of the renormalisation constants. LQG is a finite theory for any choice of the ambiguity parameters. Next, the countably infinite number of renormalisation constants in perturbative quantum gravity take continuous values so that the number of ambiguities is uncountably infinite while the physical Hilbert space of LQG depends only on a discrete number of ambiguities. Finally, in perturbative quantum gravity all values of the infinite number of renormalisation constants are, a priori, all equally natural while in LQG all of the discrete choices are pathological except for a finite number. As we have explained, without some notion of naturalness, even ordinary QFT suffers from an infinite number of ambiguities (such as all possible discretisations of Yang Mills theories on all possible lattices). Hence applying naturalness, the amount of ambiguity in LQG reduces to a finite number of ambiguities which is comparable to the degree of ambiguity of a renormalisable ordinary quantum field theory.

An interesting speculation is that the ambiguities in the definition of the Hamiltonian constraints are maybe not unlike the discretisation ambiguities in Wilson’s approach to (Euclidean) quantum Yang–Mills theories. A priori, there are infinitely many possible definitions of the microscopic Lagrangean. However, their flow under (some background independent version of) the renormalisation group may end up in a common UV fix point. All those Lagrangeans are then in the same universality class and something similar might happen in LQG to the Hamiltonian constraints.

In the appendix the interested reader can find an example where the necessity of mathematical machinery is illustrated in a concrete physical question, namely whether the so called Kodama state is a physical state of LQG. This would not be possible without it and therefore exemplifies “what has been gained”.

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A The Kodama state

We end this article by displaying a concrete example which illustrates the necessity of all the mathematical machinery in order to reach a conclusive answer for precise questions. The example is the so called Kodama state [133] which is frequently claimed to be an exact solution to all constraints of LQG [134]. In fact the Kodama state has attracted much attention in the early days of LQG (see [135] and references therein) because of its formal connection with the Jones polynomial [136] which would therefore seem to be an exact solution (in the loop representation) of all the constraints of LQG.

We will now show that this does not hold for various reasons. To see what is going on, consider pure gravity with a cosmological constant. After multiplying the Hamiltonian constraint by the factor \( \sqrt{\det(q)} \) it is given by

\[
\tilde{H} = \epsilon^{jkl} \epsilon_{abc} E^a_j E^b_k [B^c_j + \Lambda E^c_j]
\]

where \( \Lambda \) is the cosmological constant and \( B^c_j = e_{abc} E_{aj} / 2 \) the magnetic field of the complex connection \( A^\mathbb{C} \) which is the pull back to \( \sigma \) of the self dual part of the spin connection (annihilating the tetrad).

The idea underlying the Kodama state is that the \( SL(2,\mathbb{C}) \) Chern – Simons action

\[
S_{CS}[A^\mathbb{C}] := \int_\sigma \text{Tr}(F^\mathbb{C} \wedge A^\mathbb{C} - \frac{1}{3} A^\mathbb{C} \wedge A^\mathbb{C} \wedge A^\mathbb{C})
\]

is the generating functional of the magnetic field, that is, \( \delta S_{CS} / \delta A^{\mathbb{C} ij}(x) = B^{\mathbb{C}a}(x) \) where the functional derivative is in the sense of the space of smooth \( SL(2,\mathbb{C}) \) connections \( A^{\mathbb{C}} \). Now the canonical brackets \( \{ E^a_j(x), A^{\mathbb{C}jk}(y) \} = i\kappa \delta(x,y) \) suggest to formally define a Hilbert space \( \mathcal{H}_{\mathbb{C}} = L_2(A^\mathbb{C}, [dA^\mathbb{C} \, d\bar{A}^\mathbb{C}]) \) of square integrable, holomorphic functions on \( A^\mathbb{C} \) with respect to formal Lebesgue measure and to represent \( A^{\mathbb{C}ij}(x) \) as a multiplication operator and \( E^a_j(x) \) as \(-\ell^2 \delta / \delta A^{\mathbb{C}ij}(x) \). This formally satisfies the canonical commutation relations.

As is well known, the Chern – Simons action is invariant under infinitesimal gauge transformations and as an integral of a three form over all of \( \sigma \) it is also spatially diffeomorphism invariant. Moreover, in the ordering (A.1) the Kodama state

\[
\Psi_{\text{Kodama}} = e^{\frac{i}{2\kappa^2} S_{CS}}
\]

is annihilated by the Hamiltonian constraint. This is exciting because the nine conditions \( B = -\Lambda E \) satisfied by this state is easily seen to correspond to de Sitter space for the appropriate sign of \( \Lambda [134] \). Hence, the Kodama state could be argued to correspond to the de Sitter vacuum of LQG.

There are several flaws with this formal calculation:

A. Adjointness relations

The formal representation of the canonical commutation relations just outlined is not a representation of the *algebra generated by \( E, A^\mathbb{C} \), that is, the adjointness relations are not satisfied. These demand that \( E \) is self adjoint and that \( A^\mathbb{C} + \overline{A^\mathbb{C}} = 2\Gamma(E) \) where \( \Gamma \) is the spin connection of \( E \). It is clear that the “measure” \([dA^\mathbb{C} \, d\overline{A^\mathbb{C}}]\) cannot implement these adjointness relations, hence we have to incorporate a formal kernel \( K(A^\mathbb{C}, \overline{A^\mathbb{C}}) \). A formal Fourier transform calculation [2] reveals that

\[
K(A^\mathbb{C}, \overline{A^\mathbb{C}}) = \int [dE] \exp(i \int d^3x \left[ A^{\mathbb{C}ij}_a + \overline{A^{\mathbb{C}ij}_a} - \Gamma^{ij}_a \right] E^a_j)
\] (A.4)

Even without specifying the details of this functional integral, with this kernel the inner product is no longer positive (semi) definite.
B. \textit{Euclidean gravity}

Suppose we replace $A$ by a real connection. This formally corresponds to Euclidean gravity and now the formal Hilbert space would be $H = L_2(A[dA])$ which does give rise to a formal representation of the algebra underlying $A, E$ if $E^a_j(x) = i\ell P_\delta \delta A^a_j(x)$. Now the Kodama state becomes

$$\Psi_{\text{Kodama}} = e^{i\Lambda \sqrt{E}} S_{CS}$$  \hspace{1cm} (A.5)

Being a pure phase, it is not normalisable in that formal inner product.

C. \textit{Measurability}

Now consider instead the rigorous Ashtekar – Isham – Lewandowski representation $L_2(A, d\mu_0)$. Certainly the operator corresponding to (A.1) does not exist in that representation but let us forget about the origin of $\Psi_{\text{Kodama}}$ and just ask whether it defines an element of that Hilbert space. Being a pure phase it is formally normalisable because the measure $\mu_0$ is normalised. However, the question is whether the Kodama state is a measurable function in order that we can compute inner products between the Kodama state and, say, spin network functions. It is easy to see that this is not the case. For instance this follows from the fact that if we would triangulate the integral over the the Chern – Simons action in order to replace the integral by a Riemann sum over certain holonomies (these are measurable functions) and consider the infinite refinement limit, then in this limit the Kodama state has non vanishing inner product with an uncountably infinite number of spin network functions. Thus, it is not normalisable when viewed as a proper $L_2$ function. This can also be interpreted differently: Recall that $L_2$ functions are only defined up to sets of measure zero. The Chern Simons action is a priori defined only on the measure zero subset of smooth connections. The extension to $\mathcal{A}$ that we just tried by representing it as a linear combination of spin network functions is no longer a phase.

D. \textit{Distributional solution}

One could interpret the last item as saying that the Chern Simons state defines a rigorous element of $\mathcal{D}^*$ and now the question is whether it is annihilated by the rigorously defined dual of the Hamiltonian constraint constructed in section 4. It is easy to see that this is not the case because the Hamiltonian constraint with a cosmological constant term, although its dual formally acquires the ordering as in (A.1), is not proportional to $B + \Lambda E$ because the volume operator that enters the cosmological constant term is not quantised in the form $E^a_j c^c_a$ but rather as $\sqrt{\det(|E|)}$. One could of course write the smeared constraint in the form

$$H(N) = \int_{\sigma} N \text{Tr}([F + \Lambda \star E] \wedge \{A, V\})$$  \hspace{1cm} (A.6)

where $V$ is the volume functional and proceed as in section 4 although it would be somewhat awkward to define the volume operator in this way. However, even if this would work, this would still only define a solution to the Euclidean constraint.

This discussion hopefully illustrates the physical importance of the mathematical notions introduced and shows that LQG has been brought to a level of mathematical rigour that allows to actually answer physical questions. Without it we could not have decided if and in which sense the Kodama state is a physical state.

References

[1] C. Rovelli. \textit{Quantum Gravity}, (Cambridge University Press, Cambridge, 2004).

[2] T. Thiemann. \textit{Modern Canonical Quantum General Relativity}, (Cambridge University Press, Cambridge, 2006) (at press). \texttt{gr-qc/0110034}

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\footnote{A function is said to be measurable if the preimages of open subsets of $\mathbb{C}$ are measurable subsets. In our case, the measurable sets are generated by the Borel sets of $\mathcal{A}$.}
[3] C. Rovelli. Loop quantum gravity. *Living Rev. Rel.* **1** (1998), 1. [gr-qc/9710008]

C. Rovelli. Strings, loops and others: a critical survey of the present approaches to quantum gravity. plenary lecture given at 15th Intl. Conf. on Gen. Rel. and Gravitation (GR15), Pune, India, Dec 16-21, 1997. [gr-qc/9803024]

M. Gaul and C. Rovelli, Loop quantum gravity and the meaning of diffeomorphism invariance. *Lect. Notes Phys.* **541** (2000), 277-324. [gr-qc/9910079]

T. Thiemann. Lectures on loop quantum gravity. *Lect. Notes Phys.* **631** (2003), 41-135. [gr-qc/0210094]

A. Ashtekar and J. Lewandowski. Background independent quantum gravity: a status report. *Class. Quant. Grav.* **21** (2004), R53. [gr-qc/0404018]

L. Smolin. An invitation to loop quantum gravity. [hep-th/0408048]

[4] R. M. Wald. *Quantum field theory in curved space-time and black hole thermodynamics*, (Chicago University Press, Chicago, 1995).

[5] R. Brunetti, K. Fredenhagen and R. Verch. The generally covariant locality principle: a new paradigm for local quantum field theory. *Commun. Math. Phys.* **237** (2003), 31-68. [math-ph/0112041]

[6] R. Haag. *Local Quantum Physics*, 2nd ed., (Springer Verlag, Berlin, 1996).

[7] N. Marcus and A. Sagnotti. The ultraviolet behavior of N=4 Yang-Mills and the power counting of extended superspace. *Nucl. Phys.* **B256** (1985), 77.

M. H. Goroff and A. Sagnotti. The ultraviolet behavior of Einstein gravity. *Nucl. Phys.* **B266** (1986), 709.

Non – renormalizability of (last hope) $D = 11$ supergravity with a terse survey of divergences in quantum gravities. [hep-th/9905017]

[8] M. H. Goroff and A. Sagnotti. Quantum gravity at two loops. *Phys. Lett.* **B160** (1985), 81.

[9] J. Polchinski. *String Theory*, vol.1: An introduction to the bosonic string, vol. 2: Superstring theory and beyond, (Cambridge University Press, Cambridge, 1998).

[10] E. D’Hoker and D.H. Phong. Lectures on Two Loop Superstrings. [hep-th/0211111]

[11] G. Scharf. *Finite Quantum Electrodynamics: The Causal Approach*, (Springer Verlag, Berlin, 1995).

[12] H. Nicolai, K. Peeters and M. Zamaklar. Loop quantum gravity: an outside view. *Class. Quant. Grav.* **22** (2005), R193. [hep-th/0501114]

[13] H. Nicolai and K. Peeters. Loop and spin foam quantum gravity: a brief guide for beginners. [gr-qc/0601129]

[14] F. Denef and M. Douglas. Distributions of flux vacua. *JHEP* **0405** (2004), 072. [hep-th/0404116]

J. Shelton, W. Taylor, B. Wecht. Generalized flux vacua. [hep-th/0607015]

[15] L. Susskind. The anthropic landscape of string theory. [hep-th/0302219]

[16] L. Smolin. Scientific alternatives to the anthropic principle. [hep-th/0407213]

[17] L. Smolin. The case for background independence. [hep-th/0507235]

[18] J. Maldacena. The large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.* **2** (1998), 231-252. [hep-th/9711200]

[19] T. Thiemann. The Phoenix project: master constraint programme for loop quantum gravity. *Class. Quant. Grav.* **23** (2006), 2211-2248. [gr-qc/0305080]

[20] T. Thiemann. Quantum Spin Dynamics (QSD). *Class. Quantum Grav.* **15** (1998), 839-873. [gr-qc/9606089]

[21] D.M. Gitman and I. V. Tyutin. *Quantization of Fields with Constraints*, (Springer-Verlag, Berlin, 1990).

[22] R. P. Woodard. Avoiding dark energy with 1/r modifications of gravity. [astro-ph/0601672]

[23] R. Geroch. The domain of dependence. *Journ. Math. Phys.*, **11** (1970), 437 - 509.

[24] R. Beig and O. Murchadha. The Poincaré group as the symmetry group of canonical general relativity. *Ann. Phys.* **174** (1987), 463.

[25] P. A. M. Dirac. *Lectures on Quantum Mechanics*, (Belfer Graduate School of Science, Yeshiva University Press, New York, 1964).

[26] M. Henneaux and C. Teitelboim. *Quantization of Gauge Systems*, (Princeton University Press, Princeton, 1992).
[27] S. A. Hojman, K. Kuchar and C. Teitelboim. Geometrodynamics regained. *Annals Phys.* **96** (1976), 88-135.

[28] T. Thiemann. The LQG string: loop quantum gravity quantization of string theory I: Flat target space. *Class. Quant. Grav.* **23** (2006), 1923-1970. [hep-th/0401172]

[29] N. M. J. Woodhouse. *Geometric Quantization*, 2nd ed., (Clarendon Press, Oxford, 1991).

[30] C. Rovelli. What is observable in classical and quantum gravity? *Class. Quantum Grav.* **8** (1991), 297-316.

C. Rovelli. Quantum reference systems. *Class. Quantum Grav.* **8** (1991), 317-332.

C. Rovelli. Time in quantum gravity: physics beyond the Schrodinger regime. *Phys. Rev.* **D43** (1991), 442-456.

C. Rovelli. Quantum mechanics without time: a model. *Phys. Rev.* **D42** (1990), 2638-2646.

[31] B. Dittrich. Partial and complete observables for Hamiltonian constrained systems. [gr-qc/0411013]

B. Dittrich. Partial and complete observables for canonical general relativity. [gr-qc/0507106]

[32] T. Thiemann. Reduced phase space quantization and Dirac observables. *Class. Quant. Grav.* **23** (2006), 1163-1180. [gr-qc/0411031]

[33] T. Thiemann. Solving the problem of time in general relativity and cosmology with phantoms and k-essence. [astro-ph/0607380]

[34] B. Dittrich and T. Thiemann. Testing the master constraint programme for loop quantum gravity: I. General framework. *Class. Quant. Grav.* **23** (2006), 1025-1066. [gr-qc/0411138]

B. Dittrich and T. Thiemann. Testing the master constraint programme for loop quantum gravity: II. Finite – dimensional systems. *Class. Quant. Grav.* **23** (2006), 1067-1088. [gr-qc/0411139]

B. Dittrich and T. Thiemann. Testing the master constraint programme for loop quantum gravity: III. SL(2R) models. *Class. Quant. Grav.* **23** (2006), 1089-1120. [gr-qc/0411140]

B. Dittrich and T. Thiemann. Testing the master constraint programme for loop quantum gravity: IV. Free field theories. *Class. Quant. Grav.* **23** (2006), 1121-1142. [gr-qc/0411141]

B. Dittrich and T. Thiemann. Testing the master constraint programme for loop quantum gravity: V. Interacting field theories. *Class. Quant. Grav.* **23** (2006), 1143-1162. [gr-qc/0411142]

[35] J. Klauder. Universal procedure for enforcing quantum constraints. *Nucl. Phys.* **B547** (1999), 397-412. [hep-th/9901010]

A. Kempf and J. R. Klauder, On the implementation of constraints through projection operators, *J. Phys.* **A34** (2001), 1019-1036. [quant-ph/0009072]

D. Giulini and D. Marolf. On the generality of refined algebraic quantization. *Class. Quant. Grav.* **16** (1999), 2479-2488. [gr-qc/9812024]

[36] T. Thiemann. Quantum Spin Dynamics (QSD): II. The kernel of the Wheeler-DeWitt constraint operator. *Class. Quantum Grav.* **15** (1998), 875-905. [gr-qc/9606090]

T. Thiemann. Quantum Spin Dynamics (QSD): III. Quantum constraint algebra and physical scalar product in quantum general relativity. *Class. Quantum Grav.* **15** (1998), 1207-1247. [gr-qc/9705017]

T. Thiemann. Quantum Spin Dynamics (QSD): IV. 2+1 Euclidean quantum gravity as a model to test 3+1 Lorentzian quantum gravity. *Class. Quantum Grav.* **15** (1998), 1249-1280. [gr-qc/9705018]

T. Thiemann. Quantum Spin Dynamics (QSD): V. Quantum gravity as the natural regulator of the Hamiltonian constraint of matter quantum field theories. *Class. Quantum Grav.* **15** (1998), 1281-1314. [gr-qc/9705019]

T. Thiemann. Quantum Spin Dynamics (QSD): VI. Quantum Poincaré algebra and a quantum positivity of energy theorem for canonical quantum gravity. *Class. Quantum Grav.* **15** (1998), 1463-1485. [gr-qc/9705020]

[37] B. S. DeWitt. Quantum theory of gravity. I. The canonical theory. *Phys. Rev.* **160** (1967), 1113-1148.

B. S. DeWitt. Quantum theory of gravity. II. The manifestly covariant theory. *Phys. Rev.* **162** (1967), 1195-1238.

B. S. DeWitt. Quantum theory of gravity. III. Applications of the covariant theory. *Phys. Rev.* **162** (1967), 1239-1256.
[40] A. Ashtekar. New variables for classical and quantum gravity. *Phys. Rev. Lett.* **57** (1986), 2244-2247.
A. Ashtekar. New Hamiltonian formulation of general relativity. *Phys. Rev.* **D36** (1987), 1587-1602.

[41] A. Ashtekar and C.J. Isham. Representations of the holonomy algebras of gravity and non-Abelean gauge theories. *Class. Quantum Grav.* **9** (1992), 1433. [hep-th/9202053]

[42] A. Ashtekar and J. Lewandowski. Representation theory of analytic holonomy $C^*$ algebras. In *Knots and Quantum Gravity*, J. Baez (ed.), (Oxford University Press, Oxford 1994). [gr-qc/9311010]

[43] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão and T. Thiemann. Quantization of diffeomorphism invariant theories of connections with local degrees of freedom. *Journ. Math. Phys.* **36** (1995), 6456-6493. [gr-qc/9504018]

[44] J.M. Mourão, T. Thiemann and J.M. Velhinho. Physical properties of quantum field theory measures. *J. Math. Phys.* **40** (1999), 2337-2353. [hep-th/9711139]

[45] F. Barbero. Real Ashtekar variables for Lorentzian signature space times. *Phys. Rev.* **D51** (1995), 5507-5510.
F. Barbero. Reality conditions and Ashtekar variables: a different perspective. *Phys. Rev.* **D51** (1995), 5498-5506.

[46] G. Immirzi. Quantum gravity and Regge calculus. *Nucl. Phys. Proc. Suppl.* **57** (1997), 65. [gr-qc/9701052]
C. Rovelli and T. Thiemann. The Immirzi parameter in quantum general relativity. *Phys. Rev.* **D57** (1998), 1009-1014. [gr-qc/9705059]

[47] B. Brügmann and J. Pullin. Intersecting N loop solutions of the Hamiltonian constraint of quantum gravity. *Nucl. Phys. B363* (1991), 221-246.
B. Brügmann, J. Pullin and R. Gambini. Knot invariants as nondegenerate quantum geometries. *Phys. Rev. Lett.* **68** (1992), 431-434. B. Brügmann, J. Pullin and R. Gambini. Jones polynomials for intersecting knots as physical states of quantum gravity. *Nucl. Phys. B385* (1992), 587-603.

[48] T. Thiemann. Anomaly-free formulation of non-perturbative, four-dimensional Lorentzian quantum gravity. *Physics Letters B380* (1996), 257-264. [gr-qc/9606088]

[49] J. Samuel. Canonical gravity, diffeomorphisms and objective histories. *Class. Quant. Grav.* **17** (2000), 4645-4654. [gr-qc/0005094]
J. Samuel. Is Barbero’s Hamiltonian formulation a gauge theory of Lorentzian gravity? *Class. Quantum Grav.* **17** (2000), L141. [gr-qc/0005095]

[50] S. Alexandrov. SO(4,C) covariant Ashtekar-Barbero gravity and the Immirzi parameter. *Class. Quant. Grav.* **17** (2000), 4255-4268. [gr-qc/0005085]

[51] R. Gambini and A. Trias. Second quantization of the free electromagnetic field as quantum mechanics in the loop space. *Phys. Rev.* **D22** (1980), 1380.
C. Di Bartolo, F. Nori, R. Gambini and A. Trias. Loop space quantum formulation of free electromagneticism. *Lett. Nuov. Cim.* **38** (1983), 497.
R. Gambini and A. Trias. Gauge dynamics in the C representation. *Nucl. Phys.* **B278** (1986), 436.
R. Giles. The reconstruction of gauge potentials from Wilson loops. *Phys. Rev.* **D8** (1981), 2160.
T. Jacobson and L. Smolin. Nonperturbative quantum geometries. *Nucl. Phys.* **B299** (1988), 295.
C. Rovelli and L. Smolin. Loop space representation of quantum general relativity. *Nucl. Phys.* **B331** (1990), 80.

[52] A. Ashtekar, A. Corichi and J.A. Zapata. Quantum theory of geometry III: Non-commutativity of Riemannian structures. *Class. Quant. Grav.* **15** (1998), 2955-2972 [gr-qc/9806041]

[53] H. Araki. Hamiltonian formalism and the canonical commutation relations in quantum field theory. *J. Math. Phys.* **1** (1960), 492.

[54] J. Lewandowski, A. Okolow, H. Sahlmann and T. Thiemann. Uniqueness of diffeomorphism invariant states on holonomy – flux algebras. *Comm. Math. Phys.* **267** (2006), 703-733. [gr-qc/0504147]

[55] C. Fieschhacker. *Representations of the Weyl algebra in quantum geometry*. math-ph/0407006

[56] O. Brattelli and D. W. Robinson. *Operator algebras and quantum statistical mechanics*, vol. 1,2, (Springer Verlag, Berlin, 1997).

[57] W. Rudin. *Real and complex analysis*, (McGraw-Hill, New York, 1987).
P. Hajíček and K. Kuchař. Constraint quantization of parametrized relativistic gauge systems in curved space-times. *Phys. Rev.* **D41** (1990), 1091-1104.

P. Hajíček and K. Kuchař. Transversal affine connection and quantization of constrained systems. *Journ. Math. Phys.* **31** (1990), 1723-1732.

L. Smolin. The classical limit and the form of the Hamiltonian constraint in non-perturbative quantum general relativity. [gr-qc/9609034]

E. Witten. (2+1)-dimensional gravity as an exactly solvable system. *Nucl. Phys.* **B311** (1988), 46.

M. Gaul and C. Rovelli. A generalized Hamiltonian constraint operator in loop quantum gravity and its simplest Euclidean matrix elements. *Class. Quant. Grav.* **18** (2001) 1593-1624. [gr-qc/0011106]

A. Perez. On the regularization ambiguities in loop quantum gravity. *Phys. Rev.* **D73** (2006), 044007. [gr-qc/0509118]

T. Thiemann. Quantum Spin Dynamics (QSD): VII. Symplectic structures and continuum lattice formulations of gauge field theories. *Class. Quant. Grav.* **18** (2001) 3293-3338. [hep-th/0005232]

T. Thiemann. Complexifier coherent states for canonical quantum general relativity. *Class. Quant. Grav.* **23** (2006), 2063-2118. [gr-qc/0206037]

T. Thiemann. Gauge Field Theory Coherent States (GCS): I. General properties. *Class. Quant. Grav.* **18** (2001), 2025-2064. [hep-th/0005233]

T. Thiemann and O. Winkler. Gauge Field Theory Coherent States (GCS): II. Peakness properties. *Class. Quant. Grav.* **18** (2001) 2561-2636. [hep-th/0005237]

T. Thiemann and O. Winkler. Gauge Field Theory Coherent States (GCS): III. Ehrenfest theorems. *Class. Quantum Grav.* **18** (2001), 4629-4681. [hep-th/0005234]

T. Thiemann and O. Winkler. Gauge field theory coherent states (GCS): IV. Infinite tensor product and thermodynamic limit. *Class. Quantum Grav.* **18** (2001), 4997-5033. [hep-th/0005235]

H. Sahlmann, T. Thiemann and O. Winkler. Coherent states for canonical quantum general relativity and the infinite tensor product extension. *Nucl. Phys.* **B606** (2001) 401-440. [gr-qc/0102038]

P. Hasenfratz. The Theoretical Background and Properties of Perfect Actions. [hep-lat/9803027]

S. Hauswith. Perfect Discretizations of Differential Operators. [hep-lat/0003007]; The Perfect Laplace Operator for Non-Trivial Boundaries. [hep-lat/0003007]; The Perfect Laplace Operator for Non-Trivial Boundaries. [hep-lat/0010033]

T. Thiemann and J. Brunnemann. Simplification of the spectral analysis of the volume operator in loop quantum gravity. *Class. Quant. Grav.* **20** (2003), 1289-1346. [gr-qc/0305060]

T. Thiemann. Closed formula for the matrix elements of the volume operator in canonical quantum gravity. *Journ. Math. Phys.* **39** (1998), 3347-3371. [gr-qc/9606091]

D. Stauffer and A. Aharony. *Introduction to Percolation Theory*, 2nd ed., (Taylor and Francis, London, 1994).

D. M. Cvetovic, M. Doob and H. Sachs. *Spectra of Graphs*, (Academic Press, New York, 1979).

A. Perez. Spin foam models for quantum gravity. *Class. Quant. Grav.* **20** (2003), R43. [gr-qc/0301113]

M. Reisenberger and C. Rovelli. Sum over surfaces form of loop quantum gravity. *Phys. Rev.* **D56** (1997), 3490-3508. [gr-qc/9612035]

E. Buffenoir, M. Henneaux, K. Noui and Ph. Roche. Hamiltonian analysis of Plebanski theory. *Class. Quant. Grav.* **21** (2004), 5203-5220. [gr-qc/0404041]

J. W. Barrett and L. Crane. Relativistic spin networks and quantum gravity. *J. Math. Phys.* **39** (1998), 3296-3302. [gr-qc/9709025]

J. W. Barrett and L. Crane. A Lorentzian signature model for quantum general relativity. *Class. Quant. Grav.* **17** (2000) 3101-3118. [gr-qc/9904025]

J. C. Baez, J. D. Christensen, T. R. Halford and D. C. Tsang. Spin foam models of Riemannian quantum gravity. *Class. Quant. Grav.* **19** (2002), 4627-4648. [gr-qc/0202017]

J. C. Baez and J. D. Christensen. Positivity of spin foam amplitudes. *Class. Quant. Grav.* **19** (2002), 2291-2306. [gr-qc/0110044]

L. Freidel. Group field theory: an overview. *Int. J. Theor. Phys.* **44** (2005), 1769-1783. [hep-th/0505016]

J. Ambjorn, M. Carfora and A. Marzuoli. *The geometry of dynamical triangulations*, (Springer-Verlag, Berlin, 1998).
[98] A. Ashtekar and J. Lewandowski. Quantum theory of geometry I: Area Operators. *Class. Quantum Grav.* **14** (1997), A55-A82. gr-qc/9602046

[99] T. Thiemann. A length operator for canonical quantum gravity. *Journ. Math. Phys.* **39** (1998), 3372-3392. gr-qc/9606092

[100] B. Dittrich and T. Thiemann. Facts and fiction about Dirac observables. (to appear)

[101] M. Varadarajan. Fock representations from U(1) holonomy algebras. *Phys. Rev.* **D61** (2000), 104001. gr-qc/0001050

M. Varadarajan. Photons from quantized electric flux representations. *Phys. Rev.* **D64** (2001), 104003. gr-qc/0104051

M. Varadarajan. Gravitons from a loop representation of linearized gravity. *Phys. Rev.* **D66** (2002), 024017. gr-qc/0204067

M. Varadarajan. The Graviton vacuum as a distributional state in kinematic loop quantum gravity. *Class. Quant. Grav.* **22** (2005), 1207-1238. gr-qc/0411012

[102] A. Ashtekar and J. Lewandowski. Relation between polymer and Fock excitations. *Class. Quant. Grav.* **18** (2001), L117-L128. gr-qc/0107043

[103] A. Ashtekar. Classical and quantum physics of isolated horizons: a brief overview. *Lect. Notes Phys.* **541** (2000) 50-70.

[104] S. Hayward. Marginal surfaces and apparent horizons. gr-qc/9303006

S. Hayward. On the definition of averagely trapped surfaces. *Class. Quant. Grav.* **10** (1993), L137-L140. gr-qc/9304042

S. Hayward. General laws of black hole dynamics. *Phys. Rev.* **D49** (1994), 6467-6474.

S. Hayward, S. Mukohyama and M.C. Ashworth. Dynamic black hole entropy. *Phys. Lett.* **A256** (1999), 347-350. gr-qc/9810006

A. Ashtekar and B. Krishnan. Dynamical horizons and their properties. *Phys. Rev.* **D68** (2003), 104030. gr-qc/0308033

[105] V. Husain and O. Winkler. Quantum black holes. *Class. Quant. Grav.* **22** (2005), L135-L142. gr-qc/0412039

[106] A. Ashtekar, J. C. Baez and K. Krasnov. Quantum geometry of isolated horizons and black hole entropy. *Adv. Theor. Math. Phys.* **4** (2001), 1-94. gr-qc/0005126

[107] M. Domagala and J. Lewandowski. Black hole entropy from quantum geometry. *Class. Quant. Grav.* **21** (2004), 5233-5244. gr-qc/0407051

[108] K. Meissner. Black hole entropy in loop quantum gravity. *Class. Quant. Grav.* **21** (2004), 5245-5252. gr-qc/0407052

[109] A. Ashtekar, M. Bojowald and J. Lewandowski. Mathematical structure of loop quantum cosmology. *Adv. Theor. Math. Phys.* **7** (2003), 233. gr-qc/0304074

[110] A. Ashtekar, T. Pawlowski and P. Singh. Quantum nature of the big bang. *Phys. Rev. Lett.* **96** (2006), 141301. gr-qc/0602086

A. Ashtekar, T. Pawlowski and P. Singh. Quantum nature of the big bang: an analytical and numerical investigation. I. *Phys. Rev.* **D73** (2006), 124038. gr-qc/0604013

A. Ashtekar, T. Pawlowski and P. Singh. Quantum nature of the big bang: improved dynamics. gr-qc/0607039

[112] J. Brunnemann and T. Thiemann. On (cosmological) singularity avoidance in loop quantum gravity. *Class. Quant. Grav.* **23** (2006), 1395-1428. gr-qc/0505032

J. Brunnemann and T. Thiemann. Unboundedness of triad – like operators in loop quantum gravity. *Class. Quant. Grav.* **23** (2006), 1429-1484. gr-qc/0505033

[113] T. Jacobson, S. Liberati and D. Mattingly. Lorentz violation at high energy: concepts, phenomena and astrophysical constraints. *Annals Phys.* **321** (2006), 150-196. astro-ph/0505267

[114] S. Hossenfelder. Interpretation of quantum field theories with a minimal length scale. *Phys. Rev.* **D73** (2006), 105013. hep-th/0603032
[115] J. Kowalski-Glikman. Introduction to doubly special relativity. Lect. Notes Phys. 669 (2005), 131-159. [hep-th/0405273]

[116] L. Freidel, J. Kowalski-Glikman and L. Smolin. 2+1 gravity and doubly special relativity. Phys. Rev. D69 (2004), 044001. [hep-th/0307085]

[117] L. Freidel and S. Majid. Noncommutative harmonic analysis, sampling theory and the Duflo map in 2+1 quantum gravity. [hep-th/0601004]

[118] G. Amelino-Camelia, John R. Ellis, N.E. Mavromatos, D.V. Nanopoulos and Subir Sarkar. Potential sensitivity of gamma ray burster observations to wave dispersion in vacuo. Nature. 393 (1998) 763-765. [astro-ph/9712103]

[119] S. D. Biller et al. Limits to quantum gravity effects from observations of TeV flares in active galaxies. Phys. Rev. Lett. 83 (1999), 2108-2111. [gr-qc/9810044]

[120] R. Gambini and J. Pullin, Nonstandard optics from quantum spacetime. Phys. Rev. D59 (1999), 124021. [gr-qc/9809038]

[121] H. Sahlmann and T. Thiemann. Towards the QFT on curved spacetime limit of QGR. 1. A general scheme. Class. Quant. Grav. 23 (2006), 867-908. [gr-qc/0207030]

[122] H. Sahlmann and T. Thiemann. Towards the QFT on curved spacetime limit of QGR. 2. A concrete implementation. Class. Quant. Grav. 23 (2006), 909-954. [gr-qc/0207031]

[123] S. Hofmann and O. Winkler. The spectrum of fluctuations in inflationary cosmology. astro-ph/0411124

[124] Robert C. Helling, G. Policastro. String quantization: Fock vs. LQG representations. hep-th/0409182

[125] H. Narnhofer and W. Thirring. Covariant QED without indefinite metric. Rev. Math. Phys. SI1 (1992), 197 – 211.

[126] J. Slawny. On factor representations and the C*-algebra of canonical commutation relations. Comm. Math. Phys. 24 (1972), 151 – 170.

[127] A. Ashtekar, S. Fairhurst and J. L. Willis. Quantum gravity, shadow states and quantum mechanics. Class. Quant. Grav. 20 (2003), 1031. [gr-qc/0207106]

[128] K. Fredenhagen, F. Reszewski. Polymer state approximations of Schrodinger wave functions. gr-qc/0606090

[129] T. Thiemann. The LQG string: loop quantum gravity quantization of string theory I: Flat target space. Class. Quant. Grav. 23 (2006), 1923-1970. [hep-th/0401172]

[130] G. Mack, “Introduction to Conformal Invariant Quantum Field Theory in two and more Dimensions”, in: Cargese 1987, “Nonperturbative Quantum Field Theory”, 1987; Preprint DESY 88-120

[131] K. Pohlmeyer. A group theoretical approach to the quantization of the free relativistic closed string. Phys. Lett. B119 (1982), 100.

[132] D. Bahns. The invariant charges of the Nambu – Goto string and canonical quantisation. J. Math. Phys. 45 (2004), 4640-4660. [hep-th/0403108]

[133] A. Hauser and A. Corichi. Bibliography of publications related to classical self-dual variables and loop quantum gravity, gr-qc/0509039

[134] H. Kodama. Holomorphic wave function of the universe. Phys. Rev. D42 (1990), 2548-2565.

[135] L. Freidel and L. Smolin. Linearization of the Kodama state. Class. Quant. Grav. 21 (2004), 3831-3844. [hep-th/0310224]

[136] R. Gambini and J. Pullin. Loops, Knots, Gauge Theories and Quantum Gravity, (Cambridge University Press, Cambridge, 1996).

[137] E. Witten. Quantum field theory and the Jones polynomial. Comm. Math. Phys. 121 (1989), 351-399.