On time-dependent symmetries and formal symmetries of evolution equations

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Abstract

We present the explicit formulae, describing the structure of symmetries and formal symmetries of any scalar (1+1)-dimensional evolution equation. Using these results, the formulae for the leading terms of commutators of two symmetries and two formal symmetries are found. The generalization of these results to the case of system of evolution equations is also discussed.

1 Introduction

It is well known that provided scalar (1+1)-dimensional evolution equation possesses the infinite-dimensional commutative Lie algebra of time-independent generalized (Lie – Bäcklund) symmetries, it is either linearizable or integrable via inverse scattering transform (see e.g. [1], [2], [3] for the survey of known results and [4] for the generalization to (2+1) dimensions). The existence of such algebra is usually proved by exhibiting the recursion operator [3] or mastersymmetry [5]. But in order to possess the latter, the equation in question must have some (possibly nonlocal) time-dependent symmetries. This fact is one of the main reasons of growing interest to the study of whole algebra of time-dependent symmetries of evolution equations [6], [7].

However, to the best of author’s knowledge, there exist almost no results, describing the structure of this algebra for the generic scalar evolution equation (even in 1+1 dimensions) without any initial conjectures about the specific properties of this equation, like the existence of formal symmetry, Lax pair, mastersymmetry, etc. [1] Although Vinogradov et al. [10] had outlined the general scheme of study of local and nonlocal symmetries of evolution equations, it remained unrealized in its general form until now.

In this paper we put the part of this scheme into life and go even further. Namely, in Theorems 1 and 2 we describe the general structure of local symmetries and formal symmetries of scalar evolution equation, and in Theorems 3 and 5 we present the formulae for the leading terms of the commutators of two symmetries (of sufficiently high order) and of two formal symmetries. We present also the generalization of these results to the case of systems of evolution equations.

1 Nevertheless, let us mention the papers of Flach [8] and Magadeev [9].
2 Basic definitions and known facts

We consider the scalar 1+1-dimensional evolution equation
\[
\frac{\partial u}{\partial t} = F(x, t, u, u_1, \ldots, u_n), \quad n \geq 2, \quad \partial F/\partial u_n \neq 0, \tag{1}
\]
where \(u_l = \partial^l u/\partial x^l, l = 0, 1, 2, \ldots, u_0 \equiv u\), and the symmetries of this equation, i.e. the right hand sides \(G\) of evolution equations
\[
\frac{\partial u}{\partial \tau} = G(x, t, u, u_1, \ldots, u_k), \tag{2}
\]
compatible with equation (1). The greatest number \(k\) such that \(\partial G/\partial u_k \neq 0\) is called the order of symmetry and is denoted as \(k = \text{ord} G\). If \(G\) is independent of \(u, u_1, \ldots\), we assume that \(\text{ord} G = 0\). Let \(S^{(k)}\) be the space of symmetries of order not higher than \(k\) of (1) and \(S = \bigcup_{j=0}^{\infty} S^{(k)}\). \(S\) is Lie algebra with respect to the so-called Lie bracket \([1, 3]\)
\[
\{h, r\}\quad \text{where for any sufficiently smooth function} \quad f \quad \text{of} \quad x, t, u, u_1, \ldots, u_s \quad \text{we have introduced the notation}
\]
\[
f_* = \sum_{i=0}^{s} \frac{\partial h}{\partial u_i} D^i, \quad \nabla f = \sum_{i=0}^{\infty} D^i(f) \partial/\partial u_i,
\]
where \(D = \partial/\partial x + \sum_{i=0}^{\infty} u_{i+1} \partial/\partial u_i\).

\(G\) is symmetry of Eq.(1) if and only if \([3]\)
\[
\partial G/\partial t = -\{F, G\}. \tag{3}
\]
In many examples Eq.(1) is quasilinear, so let
\[
n_0 = \begin{cases} 
\max(1-j, 0), & \text{if } \partial F/\partial u_{n-i} = \phi_i(x, t), \quad i = 0, \ldots, j, \\
2 & \text{otherwise}. 
\end{cases} \tag{4}
\]
It is known \([3]\) that for any \(G \in S\), ord \(G = k \geq n_0\), we have
\[
\partial G/\partial u_k = c_k(t)\Phi^{k/n}, \tag{5}
\]
where \(c_k(t)\) is a function of \(t\) and \(\Phi = \partial F/\partial u_n\).

It is also well known \([3]\) that for any sufficiently smooth functions \(P, Q\) of \(x, t, u, u_1, u_2, \ldots\) the relation \(R = \{P, Q\}\) implies
\[
R_* = \nabla_P(Q_*) - \nabla_Q(P_*) + [Q_*, P_*], \tag{6}
\]
where \(\nabla_P(Q_*) = \sum_{i,j=0}^{\infty} D^j(P) \frac{\partial^2 Q}{\partial u_j \partial u_i} D^i\) and likewise for \(\nabla_Q(P_*)\); \([, \cdot, \cdot]\) stands for the usual commutator of linear differential operators.
In particular, Eq. (3) yields
\[ \partial G^* / \partial t \equiv (\partial G / \partial t)^* = \nabla_G F^* - \nabla_F (G^*) + [F^*, G^*]. \] (7)

Equating the coefficients at \( D_s, s = 0, 1, 2, \ldots \) on right and left hand sides of Eq. (7), we obtain
\[ \frac{\partial^2 G}{\partial u \partial t} = \sum_{m=0}^{n} D^m(G) \frac{\partial^2 F}{\partial u_n \partial u_l} - \sum_{r=0}^{k} D^r(F) \frac{\partial^2 G}{\partial u_r \partial u_l} \]
\[ + \sum_{j=\max(0, l+1-n)}^{i=\max(l+1-j, 0)} \sum_{i}^{m} \left[ C^{i+j-l} \frac{\partial F}{\partial u_i} D^{i+j-l} \left( \frac{\partial G}{\partial u_j} \right) \right], \quad l = 0, \ldots, n + k - 1, \] (8)
where \( C^p = \frac{q!}{p!(q-p)!} \) and we assume that \( 1/p! = 0 \) for negative integer \( p \).

Let us also remind some facts concerning the formal series in powers of \( D \) (see e.g. [2], [4] for more information), i.e. the expressions of the form
\[ H = \sum_{j=-\infty}^{m} h_j(x, t, u, u_1, \ldots) D^j. \] (9)
The greatest integer \( m \) such that \( h_m \neq 0 \) is called the degree of formal series \( H \) and is denoted by \( \deg H. \) For any formal series \( H \) of degree \( m \) there exists unique (up to the multiplication by \( m \)-th root of unity) formal series \( H^{1/m} \) of degree 1 such that \((H^{1/m})^m = H.\) Now we can define the fractional powers of \( H \) as \( H^{l/m} = (H^{1/m})^l \) for all integer \( l.\) The key result here is that
\[ [H^{p/m}, H^{q/m}] = 0 \] (10)
for all integer \( p \) and \( q.\)

The formal symmetry of Eq. (4) of rank \( l \) is the formal series \( R, \) satisfying the relation
\[ \deg(\partial R / \partial t + \nabla_F (R) - [F^*, R]) \leq \deg F^* + \deg R - l. \] (11)
The commutator of two formal symmetries of ranks \( l \) and \( m \) obviously is again a formal symmetry of rank not lower than \( \min(l, m) \), and thus the set \( FS_r \) of all formal symmetries of given equation (6) of rank not lower than \( r \) is a Lie algebra. Like for the case of symmetries, we shall denote by \( FS_r^{(k)} \) the set of formal symmetries of degree not higher than \( k \) and of rank not lower than \( r.\) Note that if \( G \) is symmetry of order \( k, \) then by virtue of Eq. (6) \( G^* \) is the formal symmetry of degree \( k \) and rank \( k + n - \deg \nabla_G (F^*).\)
3 Explicit form of symmetries and formal symmetries

In this section we shall consider a symmetry $G$ of order $k \geq n_0$. The successive solving of Eq. (8) for $l = k + n - 1, \ldots, n_0 + n - 1$ yields Eq. (5) and

$$\frac{\partial G}{\partial u_i} = c_i(t)\Phi^{i/n} + \sum_{p=i+1}^{k} \sum_{r=0}^{\frac{k-i}{n-1}} \chi_{i,p,r}(x,t,u,u_1,\ldots,u_k)\frac{\partial^r c_p}{\partial t^r},$$

where $i = n_0, \ldots, k - 1$ and $c_i(t)$ are some functions of $t$. In particular, it may be easily shown that for $k > n + n_0 - 2$

$$\chi_{k-n+1,k,1} = (1/n)\Phi^{(k-n+1)/n}D^{-1}(\Phi^{-1/n}).$$

Using Flach's theorem [8] and Eqs. (8), (10), (13), we have obtained

\textbf{Theorem 1} For any symmetry $G$ of Eq. (4) of order $k > n + n_0 - 2$

$$G_* = N + \sum_{j=\max(k-n+1,n_0)}^{k-1} d_j(t)F_*^{j/n} + c_k(t)F_*^{k/n}$$

$$+ \frac{k}{n} c_k(t)D^{-1}(\Phi^{-1/n})F_*^{(k-n+1)/n}$$

$$+(1/n)c_k(t)D^{-1}(\Phi^{-1/n})F_*^{(k-n+1)/n},$$

where $d_i(t)$ are some functions of $t$ (in fact they are linear combinations of $c_{\max(k-n+1,n_0)}(t), \ldots, c_k(t)$) and $N$ is some formal series, deg $N < \max(k-n+1,n_0)$. Likewise, for $n_0 \leq k \leq n + n_0 - 2$ Eq. (14) remains true, if two last terms on its right hand side are rejected.

Dot here and below stands for the partial derivative with respect to $t$.

The analysis of Eq. (11), similar to the above analysis of Eq. (7), yields

\textbf{Theorem 2} For any formal symmetry $R$ of Eq. (4) of degree $k$ and of rank $r > n$

$$R = \tilde{R} + \sum_{j=k-n+1}^{k} d_j(t)F_*^{j/n} + \frac{k}{n} d_k(t)D^{-1}(\Phi^{-1/n})F_*^{(k-n+1)/n}$$

$$+(1/n)d_k(t)D^{-1}(\Phi^{-1/n})F_*^{(k-n+1)/n},$$

where $\tilde{R}$ is some formal series, deg $\tilde{R} < k - n + 1$.

4 Structure of algebras of symmetries and formal symmetries

Let us consider the Lie bracket $R = \{P, Q\}$ of two symmetries $P$ and $Q$. Obviously, ord $R = \deg R_* \equiv r$, and we can find from Eq. (10) $\partial R/\partial u_r$, which equals to the sum of coefficients at $D^r$ on the right hand side of Eq. (7).

The substitution of representations (14) for $P_*$ and $Q_*$ into Eq. (3) yields after some computations the following results:
Theorem 3 Let $P, Q \in S/S^{(n+n_0-2)}$, ord $P = p$, ord $Q = q$. By virtue of Eq. (3) \( \partial P/\partial u_p = c_p(t)\Phi^{p/n} \), \( \partial Q/\partial u_q = d_q(t)\Phi^{q/n} \).

Then ord \{P, Q\} \leq p + q - n and

\[
\{P, Q\} = \frac{1}{n} \Phi^{\frac{p+q}{n}} u_{p+q-n} \left( q\dot{c}_p(t)d_q(t) - pc_p(t)\dot{d}_q(t) \right) + \tilde{R},
\]

(16)

where ord $\tilde{R} < p + q - n$.

Corollary 1 For all integer $p \geq n+n_0 - 1$ the spaces $S^{(p)}$ are invariant under the adjoint action of $S^{(n)}$, i.e. the Lie bracket of any symmetry from $S^{(p)}$ with any symmetry from $S^{(n)}$ again belongs to $S^{(p)}$.

Theorem 4 For all $p = 0, \ldots, n$ $S^{(p)}$ are Lie subalgebras in $S$.

Note that for $p = 0, 1$ the result of Theorem 4 is well known, while for $p = 2, \ldots, n$ it is essentially new.

Theorem 3 shows that Lie algebra $S$ has Virasoro type structure (if we forget about low order symmetries and consider just the leading terms of symmetries). In particular, one may easily establish the existence of Virasoro (or hereditary) algebra $\mathfrak{h}$ of time-independent symmetries and mastersymmetries for many integrable equations $\mathfrak{h}$, using Eq. (10). More generally, Theorem 3 is very useful in the proof of existence of infinite number of symmetries for given evolution equation, starting from few initially found ones and analyzing their commutators, as described in [10]. Note that for particular cases of KdV and Burgers equations it was proved in [10].

Likewise, for formal symmetries we have proved the following results:

Theorem 5 Let $P, Q$ be formal symmetries of Eq. (3), deg $P = p$, deg $Q = q$, and the ranks of $P$ and $Q$ are greater than $n$. By virtue of Theorem 2

\[ P = c_p(t)\Phi^{p/n} + \tilde{P} \quad \text{and} \quad Q = d_q(t)\Phi^{q/n} + \tilde{Q}, \]

then deg $[P, Q] \leq p + q - n$ and

\[
[P, Q] = -\frac{1}{n} \Phi^{\frac{p+q}{n}} \left( q\dot{c}_p(t)d_q(t) - pc_p(t)\dot{d}_q(t) \right) + \tilde{R},
\]

(17)

where deg $\tilde{R} < p + q - n$.

Corollary 2 For all integer $p$ the spaces $FS_r^{(p)}$ are invariant under the adjoint action of $FS_r^{(n)}$, provided $r > n$.

Corollary 3 For all integer $p \leq n$, $r > n$ $FS_r^{(p)}$ are Lie subalgebras in $FS_r$. 

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5 Some possible generalizations

Our results may be easily generalized to the case of systems of evolution equations of the form (1), when \(u, u_1, u_2, \ldots, F, G, P, Q\) become \(m\)-component vectors, while \(c_p(t), d_q(t)\) become \(m \times m\) matrices, which should commute with the \(m \times m\) matrix \(\Phi = \partial F/\partial u_n\), etc. (see [3] for more information). Namely, the second part of Theorem 1, concerning the symmetries of order \(k, n_0 \leq k \leq n + n_0 - 2\), Theorem 3, Theorem 4 for \(n_0 \leq 1\), Corollary 1, Corollary 3 for \(p < n\) hold true, provided all the eigenvalues of \(\Phi\) are distinct. The part of Theorem 1, concerning the symmetries of order \(k > n + n_0 - 2\), Theorems 2, 4, 5, Corollaries 2 and 3 hold true if, in addition to the above, \(\det \Phi \neq 0\). For the case, when \(\det \Phi = 0\), we have proved instead of the first part of Theorem 1 the following result:

**Theorem 6** If \(\det \Phi = 0\) and all the eigenvalues of \(\Phi\) are distinct, for any symmetry \(G\) of system (1) of order \(k > n + n_0 - 2\)

\[
G = N + \sum_{j=k-n+2}^{k} d_j(t) F_j^{1/n},
\]

where \(d_j(t)\) are some matrices, commuting with \(\Phi\), and \(N\) is some formal series (with matrix coefficients), \(\deg N < k - n + 2\).

In its turn, Theorem 2 is replaced by the following

**Theorem 7** If \(\det \Phi = 0\) and all the eigenvalues of \(\Phi\) are distinct, for any formal symmetry \(R\) of system (1) of degree \(k \geq 0\) and of rank \(r > n\)

\[
R = \tilde{R} + \sum_{j=\max(k-n+2,0)}^{k} d_j(t) F_j^{1/n},
\]

where \(d_j(t)\) are some matrices, commuting with \(\Phi\), and \(\tilde{R}\) is some formal series (with matrix coefficients), \(\deg \tilde{R} < \max(k - n + 2, 0)\).

As a final remark, let us note that Theorems 1 - 7 and Corollaries 1 - 3 may be also extended (under some extra conditions) to the symmetries and formal symmetries, involving nonlocal variables. We shall discuss this in more detail in separate paper.

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References

[1] V.V. Sokolov, *Russian Math. Surveys* 43, no.5, 165 (1988).
[2] A V Mikhailov, A B Shabat and V V Sokolov in *What is integrability?*, ed. V E Zakharov (Springer, N.Y., 1991).

[3] P Olver, *Applications of Lie Groups to Differential Equations* (Springer, N.Y., 1986).

[4] A.V. Mikhailov, R.I. Yamilov, *J. Phys. A* 31, 6707 (1998).

[5] B. Fuchssteiner, *Progr. Theor. Phys.* 70, 1508 (1983).

[6] B. Fuchssteiner, *J. Math. Phys.* 34, 5140 (1993).

[7] W.X. Ma, P.K. Bullough, P.J.Caudrey and W.I. Fushchych, *J. Phys. A* 30, 5141 (1997).

[8] B. Flach, *Lett. Math. Phys.* 17, 321 (1989).

[9] Magadeev B A 1994 *St. Petersburg Math. J.* 5, no.2, 345 (.)

[10] A M Vinogradov, I S Krasil’shchik, V V Lychagin, *Introduction to Geometry of Nonlinear Differential Equations* (Nauka, Moscow, 1986).