Theory of melting of vortex lattice in high $T_c$ superconductors

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Abstract

Theory of melting of the vortex lattice in type II superconductors in the framework of Ginzburg - Landau approach is presented. The melting line location is determined and magnetization and specific heat jumps along it are calculated. The magnetization of liquid is larger than that of solid by 1.8% irrespective of the melting temperature, while the specific heat jump is about 6% and decreases slowly with temperature. The magnetization curves agrees with experimental results on $YBCO$ and Monte Carlo simulations.

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I. INTRODUCTION AND THE MAIN IDEA

Abrikosov vortices created by magnetic field in type II superconductors strongly interact with each other creating highly correlated configurations like the vortex lattice. In high $T_c$ cuprates thermal fluctuations at relatively large temperatures are strong enough to melt the lattice. Several recent remarkable experiments discovered that the vortex lattice melting in high $T_c$ superconductors is first order with magnetization jumps [1] and spikes in specific heat [2]. Magnetization and entropy jumps were measured using great variety of techniques: local Hall probes [1], SQUID [3,4], torque magnetometry [5,6] and integrating the specific heat spike [2]. It was found that in addition to the spike there is also a jump in specific heat which was measured as well [2,7]. These precise measurements pose a question of accurate quantitative theoretical description of thermal fluctuations in vortex matter.

The Ginzburg - Landau (GL) approach is appropriate to describe thermal fluctuations near $T_c$. The GL model is however highly nontrivial even within the lowest Landau level (LLL) approximation valid at high fields. In this simplified model the only parameter is the dimensionless scaled temperature $a_T \sim (T - T_{mf}(H))/(TH)^{2/3}$ (defined more precisely in eq.(12a) below). Over last twenty years great variety of theoretical methods were applied to study this model. Brezin, Nelson and Thiaville [8] applied the renormalization group (RG) on the one loop level. No fixed points of the (functional) RG equations were found and they concluded therefore that the transition from liquid to the solid is first order [9]. This approach however does not provide a quantitative theory of melting.

Two perturbative approaches were developed and greatly improved recently to describe the solid phase and the liquid phase respectively. The perturbative approach on the liquid side was pioneered by Thouless and Ruggeri [10]. They developed a perturbative expansion around a homogeneous (liquid) state in which all the "bubble" diagrams are resummed. Unfortunately they found that the series are asymptotic and although first few terms provide accurate results at very high temperatures, the series become inapplicable for $a_T$ less than $-2$ which is quite far above the melting line (believed to be located around $a_T \sim -10$). We recently obtained the optimized gaussian series [11] which are convergent rather than asymptotic with radius of convergence of $a_T = -5$ ($a_T = -5$ is still above the melting point).

On the solid side, Eilenberger and Maki and Takayama [12] calculated the fluctuations spectrum around Abrikosov’s mean field solution. They noticed that the vortex lattice phonon modes are softer than that of the acoustic phonons in atomic crystals and this leads to infrared divergences in certain quantities. This was initially interpreted as destruction of the vortex solid by thermal fluctuations and the perturbation theory was abandoned. However the divergence looks suspiciously similar to "spurious" IR divergences in the critical phenomena theory and recently it was shown that all these IR divergences cancel in physical quantities [13]. The series therefore are reliable, and were extended to two loops, so that the LLL GL theory on the solid side is now precise enough even at melting point. However on the liquid side one needs a theory in the region $-10 < a_T < -5$. Moreover this theory should be very precise since free energies of solid and liquid differ only by few percents near melting. This requires a better qualitative understanding of the metastable phases of the theory. It is clear that the overheated solid becomes unstable at some finite temperature. It not clear however whether overcooled liquid becomes unstable at some
finite temperature (like water) or exists all the way down to \( T = 0 \) as a metastable state. The gaussian (Hartree - Fock) variational calculation, although perhaps of a limited precision, is usually a very good guide as far as qualitative features of the phase diagram are concerned. Such a calculation in the liquid was performed long ago [10], while more complicated one sampling also inhomogeneous states (vortex lattice) was obtained recently [11]. The gaussian results are as follows. The free energy of the solid state is lower than that of the liquid for temperatures lower than melting temperature. The solid state is therefore the stable one below it, becomes metastable at somewhat higher temperatures and is destabilized at \( a_T = -5 \). The liquid state becomes metastable below the melting temperature, but unlike the solid, does not loose metastability all the way down to \( T = 0 \).

The excitation energy of the overcooled liquid approaches zero as a power \( \varepsilon \sim 1/a_T^2 \). Meantime in different area of physics similar qualitative results were obtained. It was shown by variety of analytical and numerical methods that liquid (gas) phase of the classical one component Coulomb plasma exists as a metastable state down to \( T = 0 \) with energy gradually approaching that of the Madelung solid and excitation energy diminishing [14]. It seems plausible to speculate that the same would happen in any system of particles interacting via long range repulsive forces. In fact the vortices in the London approximation become a sort of repelling particles with the force even more long range than Coulombic. This was an additional strong motivation to consider the above scenario in vortex matter. Below we provide both theoretical and phenomenological evidence that the above scenario is the correct one.

Assuming absence of singularities on the liquid branch allows to develop an essentially precise theory of the LLL GL model in vortex liquid (even including overcooled liquid) using the Borel - Padé (BP) [15] method at any temperature. First we clarify several issues which prevented the use and acceptance of the BP method in the past and then combine it with the recently developed LLL theory of solids to calculate the melting line and magnetization and specific heat jumps. Very early Ruggeri and Thouless [10] tried to use BP to calculate the specific heat without much success because their series were too short [16]. Attempts to use BP for the calculation of melting also ran into problems. Hikami, Fujita and Larkin [17] tried to find the melting point by comparing the BP energy with the one loop solid energy and obtained \( a_T = -7 \). However their one loop solid energy was incorrect and in any case it was not precise enough.

The LLL GL model was also studied numerically in both 3D [18] and 2D [19] and by a variety of nonperturbative analytical methods. Among them the density functional [20], 1/N [21], dislocation theory of melting [22] and others [23].

As we show in this paper, the BP energy combined with the correct two loop solid energy computed recently gives scaled melting temperature \( a_T^m = -9.5 \) and in addition predicts other characteristics of the model. The melting line location is determined and magnetization and specific heat jumps are calculated. The magnetization of liquid is larger than that of solid by 1.8% irrespective of the melting temperature, while the specific heat jump is about 6% and decreases slowly with temperature.

In addition to theory of melting we calculated magnetization and specific heat curves. The magnetization curves agree quite well with Monte Carlo simulations of the LLL GL [18], and almost perfectly for specific heat in 2D by Kato and Nagaosa in ref. [19]. However to describe experimental results like \( YBCO \) at lower fields and lower temperature, higher
Landau levels (HLL) corrections are required. Experimentally it was claimed that one can establish the LLL scaling for fields above $3T_c$ \cite{24}. A glance at the data however shows that above $T_c$ the scaling for magnetization curves is generally very bad: the magnetization due to the LLL contribution is much larger that the experimental one above $T_c$. Therefore we calculated the leading correction by the HLLs and then compare with experiments.

The paper is organized as follows. The models are defined in section II and the melting theory in the LLL is described in section III. In section IV, the HLL correction is discussed and the magnetization curves are compared with experiments. We conclude in section V.

II. THE MODEL AND BASIC ASSUMPTIONS

A. The model

Thermal fluctuations of 3D materials with not very strong asymmetry along the $z$ axis are effectively described by the following Ginzburg-Landau free energy:

$$F = \int d^3x \left[ \frac{\hbar^2}{2m_{ab}} |D\psi|^2 + \frac{\hbar^2}{2m_c} |\partial_z \psi|^2 - a(T)|\psi|^2 + \frac{b'}{2} |\psi|^4 + \frac{(B - H)^2}{8\pi} \right]$$

(1)

where $A = (By, 0)$ describes magnetic field (considered constant and nonfluctuating, see below) in Landau gauge and covariant derivative is defined by $D \equiv \nabla - i\frac{2\pi}{\Phi_0} A, \Phi_0 \equiv \frac{hc}{e^*}(e^* = 2e)$. Statistical physics is described by the statistical sum:

$$Z = \int D\psi D\bar{\psi} \exp \left(-\frac{F}{T}\right)$$

(2)

Our aim is to quantitatively describe the effects of thermal fluctuations of high $T_c$ cuprates of the YBCO type on the few percent precision level (such a precision is required for the theory of melting since energies of liquid and solid near melting differ only by a few percent).

B. Assumptions

The use of the above GL energy makes several physical assumptions. They are listed below.

(1) Continuum model

We use anisotropic GL model despite the layered structure of the high $T_c$ cuprates for which the Lawrence - Donihah model is more appropriate model. Effects of layered structure are dominant in BSCCO or Tl compounds ($\gamma \equiv \sqrt{m_c/m_{ab}} > 1000$) and noticeable for cuprates with anisotropy of order $\gamma = 50$ like LaBaCuO or Hg1223. The requirement, that the GL can be effectively used, therefore limits us to optimally doped YBCO$_{7-\delta}$ (or slightly overdoped or underdoped) for which the anisotropy parameter is not very large $\gamma = 4 - 8$), DyBCO or possibly Hg1221 which has a slightly larger anisotropy.

(2) Range of validity of using the mesoscopic (GL) approach

The GL approach generally is an effective mesoscopic approach applicable when we can neglect higher order terms generated when one "integrates out" microscopic degrees of
freedom typically but not always near second order phase transitions. The leading higher dimensional terms we neglect (as ”irrelevant”) are $|\psi|^6$ and higher (four) derivative terms. After that the model practically becomes rotationally symmetric in the $ab$ plane. We can rescale it to $m_a \simeq m_b = m_{ab}$. For several physical questions this assumption is not valid because irrelevant terms neglected might become ”dangerous”. For example the question of the structural phase transition into the square lattice is clearly of this type \[25\]. It is known that even assuming $m_a/m_b = 1$ in low temperature vortex lattices in $YBCO$ rotational symmetry is broken down to the fourfold symmetry by the four derivative terms. However there is no significant correction to the magnetization from the those higher dimension terms.

(3) Expansion of parameters around $T_c$
Generally parameters of the GL model of eq.(1) are complicated functions of temperature which is determined by the details of the microscopic theory. We expand the coefficient $a(T)$ near $T_c$:

$$a(T) = T_c[\alpha(1-t) - \alpha'(1-t)^2 + ...], \quad (3)$$

where $t \equiv T/T_c$. The second and higher terms in the expansion are omitted and therefore when temperature deviates significantly from $T_c$ one cannot expect the model to have a good precision.

(4) Constant nonfluctuating magnetic field
For strongly type II superconductors like the high $T_c$ cuprates not very far from $H_{c2}(T)$ (this easily covers the range of interest in this paper, for the detailed discussion of the range of applicability beyond it see ref. \[26\]) magnetic field is homogeneous to a high degree due to superposition from many vortices. Inhomogeneity is of order $1/\kappa^2 \sim 10^{-3}$. Since the main subject of this study is the thermal fluctuations effects of the order parameter field, one might ask whether thermal fluctuations of the electromagnetic field should be also taken into account. Halperin, Lubensky and Ma considered this question long ago \[27\]. The conclusion was that they are completely negligible for very large $\kappa$. Upon discovery of the high $T_c$ cuprates, the issue was reconsidered \[28\] and the same result was obtained to a very high precision. Therefore here magnetic field is treated both as constant and nonfluctuating $(B = H)$ and the last term in eq.(1) can be omitted (to precision of order $1/\kappa^2$). However when we calculate the magnetization, $M = (B - H)/4\pi$ which is of order $1/\kappa^2$, high order correction must be considered.

(5) Disorder.
Point - like disorder is always present in $YBCO$. For example when the melting line becomes the second order transition and magnetization becomes irreversible the systems show clear disorder effects. However in some samples the disorder effects are minor. In the maximally oxidized sample \[2\] the second order transition is not seen even at highest available fields $(30T)$. In the optimally doped sample the same is true up to $12T$ \[24\]. The situation is believed to be different in other materials like $DyBa_2Cu_3O_7$ \[7,30\] or twinned $YBCO$. We will address the disorder problem in our future publication.
C. Landau level modes in the quasimomentum basis

Assuming that all the requirements are met we now divide the fluctuations into the LLL and HLL modes. Throughout most of the paper will use the coherence length \( \xi = \sqrt{\hbar^2 / (2m_{ab}\alpha T_c)} \) as a unit of length, \( T_c \) as unit of temperature, \( T = tT_c \), and \( \frac{dH_c(T)}{dT} = \frac{2\alpha}{2\pi^2} \) as a unit of magnetic field, \( B = bH_c \). After rescaling eq.(1) by \( x \rightarrow \xi x, y \rightarrow \xi y, z \rightarrow \xi z \), one obtains:

\[
f = \frac{F}{T} = \frac{1}{\omega} \int d^3x \left[ \frac{1}{2} |D\psi|^2 + \frac{1}{2} |\partial_z \psi|^2 - \left( \frac{a_h}{2} + \frac{b}{2} \right) |\psi|^2 + \frac{1}{2} |\psi|^4 + \frac{\kappa^2 (b - h)^2}{4} \right],
\]

where

\[
\omega = \sqrt{2Gi\pi^2 t}, \quad a_h = \frac{1 - t - b}{2}
\]

The Ginzburg number is given by

\[
Gi \equiv \frac{1}{2} \left( \frac{32\pi e^2 \kappa^2 T_c \gamma}{c^2 h^2} \right)^2
\]

It is convenient to expand the order parameter field in a complete basis of noninteracting theory: Landau levels. In the hexagonal lattice phase the most convenient basis is the quasimomentum basis:

\[
\psi(x) = \frac{1}{\sqrt{2}} \int_k e^{-ik \cdot x} \sum_{n=0}^{\infty} \frac{\varphi_k^n(x)}{\sqrt{2\pi}} \psi^n(k).
\]

Here \( \varphi_k^n(x) \) the \( n^{th} \) Landau level with quasi-momentum \( k \):

\[
\varphi_k^n = \sqrt{\frac{2\pi}{\sqrt{\pi} 2^n n! a}} \sum_{l=-\infty}^{\infty} H_n \left( y \sqrt{b} + \frac{k_x \sqrt{b} - 2\pi l}{a} \right) \times \exp \left\{ i \left[ \frac{\pi l (l - 1)}{2} + \frac{2\pi (\sqrt{b} x - k_x \sqrt{b} - 2\pi l)}{a} l - x k_x \right] - \frac{1}{2} \left( y \sqrt{b} + \frac{k_x \sqrt{b} - 2\pi l}{a} \right)^2 \right\}.
\]

Even in the liquid state which is more symmetric than the hexagonal lattice we find it convenient to use this basis.

III. MELTING LINE, MAGNETIZATION AND SPECIFIC HEAT IN LLL APPROXIMATION

The LLL contributes significantly the thermal fluctuation which in particular leads to the melting (for example, ref. (3), and Pierson and Valls in ref. (24) et al). In this section, we will essentially solve the LLL GL model.
A. The LLL scaling

If the magnetic field is quite high, we can keep only the $n = 0$ LLL modes (in eq.(4), we use the LLL condition $|D\psi|^2 = b|\psi|^2$):

$$f = \frac{F}{T} = \frac{1}{\omega} \int d^3x \left[ \frac{1}{2} |\partial_z \psi|^2 - a_h |\psi|^2 + \frac{1}{2} |\psi|^4 + \frac{\kappa^2 (b - h^2)}{4} \right].$$  \hfill (9)

Within the LLL approximation the problem simplifies (no gradient term in directions perpendicular to the field) and possesses an LLL scaling \[31]. After additional rescaling $x \rightarrow x/\sqrt{b}, y \rightarrow y/\sqrt{b}, z \rightarrow z \left( \frac{b\omega}{4\pi \sqrt{2}} \right)^{-1/3}, |\psi|^2 \rightarrow \left( \frac{b\omega}{4\pi \sqrt{2}} \right)^{2/3} |\psi|^2$, the dimensionless free energy becomes:

$$f = \frac{1}{4\pi \sqrt{2}} \int d^3x \left[ \frac{1}{2} |\partial_z \psi|^2 + a_T |\psi|^2 + \frac{1}{2} |\psi|^4 + \left( \frac{b\omega}{4\pi \sqrt{2}} \right)^{-4/3} \kappa^2 \frac{b - h^2}{4} \right]$$ \hfill (10)

and it can be approximated to order \( \frac{1}{\kappa^2} \)

$$\frac{1}{4\pi \sqrt{2}} \int d^3x \left[ \frac{1}{2} |\partial_z \psi|^2 + a_T |\psi|^2 + \frac{1}{2} |\psi|^4 \right]$$ \hfill (11)

where the scaled temperature

$$a_T = - \left( \frac{b\omega}{4\pi \sqrt{2}} \right)^{-2/3} a_h. \hfill (12a)$$

Free energy density in the newly scaled model is:

$$f_{eff} = -\frac{4\pi \sqrt{2}}{\sqrt{V}} \ln \int D\psi D\bar{\psi} \exp (-f)$$ \hfill (13)

is a function of $a_T$ only to order \( \frac{1}{\kappa^2} \).

B. Free energy of liquid and solid

Now we specify the solution of the LLL GL model. The liquid LLL (scaled) free energy is written as

$$f_{liq} = 4\varepsilon^{1/2} [1 + g(x)]. \hfill (14)$$

The function $g$ can be expanded as

$$g(x) = \sum c_n x^n, \hfill (15)$$

where the high temperature small parameter $x = \frac{1}{2} \varepsilon^{-3/2}$ is defined as a solution of the Gaussian gap equation.
for the excitation energy $\varepsilon$. The coefficients $c_n$ can be found in [17]. We will denote by $g_k(x)$ the $[k, k - 1]$ BP transform [15] of $g(x)$ (other BP approximants clearly violate the correct low temperature asymptotics). The BP transform is defined as

$$
\int_0^{\infty} g_k'(xt) \exp(-t) dt
$$

where $g_k$ is the $[k, k-1]$ Padé transform of $\sum_{n=1}^{2k-1} c_n x^n$.

For $k = 4$ and $k = 5$, the liquid energy converges to required precision (0.1%). In what follows we will use $g_5$.

The solid energy to two loops is [13,32]:

$$
f_{sol} = -\frac{a_T^2}{2\beta_A} + 2.848 |a_T|^{1/2} + \frac{2.4}{a_T},
$$

(17)

where $\beta_A = 1.1596$. On Fig.1 we plot the energies of solid and liquid. They are very close (see the difference on inset). The liquid energy completely agrees with the optimized gaussian expansion results [11] till its radius of convergence at $a_T = -5$. The energy of the overcooled liquid at low temperatures approaches that of the vortex solid. We obtained similar results in 2D and found that they also agree with Monte Carlo simulations [13] and optimized gaussian expansion [11] (see a brief discussion in the next section).

C. Melting line. Comparison with experiments, Monte Carlo simulations and Lindemann criterion

Comparing solid and liquid energy (inset in Fig.1), we find that $a_m T = -9.5$. This is in accord with experimental results. As an example on Fig.2 we present fitting of the melting line of fully oxidized $YBa_2Cu_3O_7$ [6] which gives $T_c = 88.2, H_{c2} = 175.9, Gi = 7.0 \times 10^{-5}$. Melting lines of optimally doped untwinned $YBa_2Cu_3O_7$ and $DyBa_2Cu_3O_7$ [7] are also fitted extremely well. The results of fitting are given in Table 1.

| material          | $T_c$  | $H_{c2}$ | Gi     | $\kappa$ | $\gamma$ | reference |
|-------------------|--------|----------|--------|----------|----------|-----------|
| $YBCO_{7-\delta}$| 93.07  | 167.53   | 1.910^{-4} | 48.5     | 7.76     | [2]       |
| $YBCO_7$          | 88.16  | 175.9    | 7.010^{-5} | 50       | 4        | [8]       |
| $DyBCO_{6.7}$     | 90.14  | 163      | 3.210^{-3} | 33.77    | 5.3      | [7]       |

The available 3D Monte Carlo simulations [18] unfortunately are not precise enough to provide an accurate melting point since the LLL scaling is violated and one gets values of $a_m^T = -14.5, -13.2, -10.9$ at magnetic fields 1, 2, 5$T$ respectively. We found also that the theoretical magnetization calculated by using parameters given by ref. [18] is in a very good agreement with the Monte Carlo simulation result of ref. [18]. However the determination of melting temperature needs higher precision, and the sample size ($\sim$100 vortices) used in ref. [18] may be not large enough to give an accurate determination of the melting temperature (due to boundary effects, LLL scaling will be violated too). The situation in 2D is better since the sample size is much larger. We performed similar calculation for the 2D LLL GL liquid free energy, combined it with the earlier solid energy calculation [13,32].
and find that the melting point \( a_T^m = -13.2 \). It is in good agreement with MC simulations \[19\].

Phenomenologically melting line can be located using Lindemann criterion or its more refined version using Debye - Waller factor. The more refined definition is required since vortices are not pointlike. It was found numerically for Yukawa gas \[33\] that the Debye - Waller factor \( e^{-2W} \) (ratio of the structure function at the second Bragg peak at melting to its value at \( T = 0 \)) is about 60\%. Using methods of \[34\], one obtains for the 3D LLL GL model

\[ e^{-2W} = 0.59. \] (19)

D. Magnetization jump at melting

The scaled magnetization is defined by:

\[ m(a_T) = -\frac{d}{da_T} f_{eff}(a_T). \] (20)

At the melting point \( a_T^m = -9.5 \) the magnetization jump \( \Delta M \) divided by the magnetization at the melting on the solid side is

\[ \frac{\Delta M}{M_s} = \frac{\Delta m}{m_s} = .018 \] (21)

This is compared on the inset of Fig.2 with experimental results of fully oxidized \( YBa_2Cu_3O_7 \) \[3\] (rhombs) and optimally doped untwinned \( YBa_2Cu_3O_{7-\delta} \) \[4\] (stars). The agreement is quite good.

E. Specific heat jump at melting

In addition to the delta function like spike at melting for specific heat experiments shows also a specific heat jump. The theory allows to quantitatively estimate it.

The specific heat of the vortex lattice is \( C = -T \frac{\partial^2}{\partial T^2} F_{eff}. \) The scaled specific heat is defined as

\[ c = \frac{C}{C_{mf}} = -\frac{16\beta_A}{9t^2} \left( \frac{b\omega}{4\pi \sqrt{2}} \right)^{4/3} f(a_T) \]
\[ + \frac{4\beta_A}{3t^2} (b - 1 - t) \left( \frac{b\omega}{4\pi \sqrt{2}} \right)^{2/3} f'(a_T) \]
\[ - \frac{\beta_A}{9t^2} (2 - 2b + t)^2 f''(a_T) \] (22)

where \( C_{mf} \) is the mean field specific heat of solid, \( \frac{H^2_T}{4\pi \kappa T^2}. \)
Thus the specific heat jump is:

\[ \Delta c = 0.0075 \left( \frac{2 - 2b + t}{t} \right)^2 - 0.20Gt^{1/3} (b - 1 - t) \left( \frac{b}{t^2} \right)^{2/3} \]  

(23)

Using the parameters obtained fitting the melting line, we compare eq.(23) with the experimental result of ref. [2] in the inset of Fig.2.

IV. HIGHER LANDAU LEVELS CONTRIBUTIONS

A. Where is the LLL approximation really valid?

Contributions of HLL are important phenomenologically in two regions of the phase diagram. The first is at temperatures above the mean field \( T_c \) inside the liquid phase. The second is far below the melting point deep inside the solid phase.

Naively when ”distance from the mean field transition line” is smaller than the ”inter Landau level gap”, \( 1 - t - b < 2b \) one expects that higher Landau harmonics can be neglected. More careful examination shows that a weaker condition \( 1 - t - b < 12b \) should be used for a validity test of the LLL approximation [26] to calculate the mean field contributions in vortex solid. Additional factor 6 comes from the hexagonal symmetry of the lattice since contributions of higher Landau levels (HLL), first to fifth HLL do not appear in perturbative calculation. How what about contributions beyond mean field? The question has been studied by Lawrie [26] in the framework of the Hartree - Fock (gaussian) approximation in the vortex liquid. The result was that the region of validity is limited. In this section we will incorporate the leading HLL correction using gaussian approximation and then compare the theoretical results with experimental magnetization curves.

B. Gaussian Approximation in the liquid phase

The free energy density is:

\[ F_{\text{eff}} = -\frac{\omega H_0^2}{2\pi \kappa^2 V} \ln \int D\psi \bar{D}\psi \exp (-f) \]  

(24)

where \( f, V \) is the scaled full GL model eq.(4) and the scaled volume. In gaussian approximation, \( f \) is divided into an optimized quadratic part \( K \), and a ”small” part \( v \). Then \( K \) is chosen in such a way that the gaussian energy is minimal. The gaussian energy is a rigorous lower bound on energy. Due to translational symmetry of the vortex liquid an arbitrary symmetric quadratic part \( K \) has only one variational parameter \( \varepsilon \),

\[ K = \frac{1}{\omega} \left( \frac{1}{2} (|D\psi|^2 - b|\psi|^2) + \frac{1}{2} |\partial_z \psi|^2 + \varepsilon |\psi|^2 \right) \]  

(25)

The small perturbation is therefore:
\[ v = \frac{1}{\omega} \left[ (-a_h - \varepsilon) |\psi|^2 + \frac{1}{2} |\psi|^4 \right]. \] (26)

The gaussian energy consists of two parts. The first is the Trlog term:
\[ -\frac{\omega H^2}{2\pi\kappa^2} \log \left[ \int D\psi \exp(-K) \right] = \frac{\omega H^2}{2\pi\kappa^2} \frac{1}{\sqrt{2\pi}} b \sum_{n=0}^{\infty} \sqrt{nb + \varepsilon}, \] (27)
The second is proportional to the expectation value of \( v \) in a solvable model defined by \( K \)
\[ \frac{\omega H^2}{2\pi\kappa^2} \langle v \rangle = \frac{\omega H^2}{2\pi\kappa^2} \left[ (-a_h - \varepsilon) \frac{b}{2\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{nb + \varepsilon}} + \omega \left( \frac{b}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{nb + \varepsilon}} \right)^2 \right]. \] (28)
Both are divergent in the ultraviolet. Introducing a UV momentum cutoff which effectively limits the number of Landau levels to \( N_f = \frac{A}{b} - 1 \), the Trlog term diverges as:
\[ \frac{1}{\sqrt{2\pi}} b \sum_{n=0}^{\infty} \frac{1}{\sqrt{nb + \varepsilon}} = \frac{1}{\sqrt{2\pi}} \Lambda^{1/2} + \frac{\partial}{\partial \varepsilon} f_{\text{Tr-log}}(\varepsilon, b) \] (29)
with the last term finite. The ”bubble” integral diverges logarithmically:
\[ \frac{b}{2\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{nb + \varepsilon}} = \frac{1}{\sqrt{2\pi}} \Lambda^{1/2} + \frac{\partial}{\partial \varepsilon} f_{\text{Tr-log}}(\varepsilon, b) \] (30)
Substituting eq. (29) into the gaussian energy one obtains: (in units of \( \frac{\omega H^2}{2\pi\kappa^2} \))
\[ \frac{1}{\sqrt{2\pi}} \frac{2}{3} \Lambda^{3/2} + \omega \left( \frac{1}{\sqrt{2\pi}} \Lambda^{1/2} \right)^2 + \left( -a_h - \frac{b}{2} \right) \frac{1}{\sqrt{2\pi}} \Lambda^{1/2} - a_h \partial_{\varepsilon} f_{\text{Tr-log}}(\varepsilon, b) + 2\omega \frac{1}{\sqrt{2\pi}} \Lambda^{1/2} \partial_{\varepsilon} f_{\text{Tr-log}}(\varepsilon, b) \] (31)
\[ -\varepsilon \partial_{\varepsilon} f_{\text{Tr-log}}(\varepsilon, b) + \omega (\partial_{\varepsilon} f_{\text{Tr-log}}(\varepsilon, b))^2 + f_{\text{Tr-log}}(\varepsilon, b). \]
\( T_c \) will be renormalized. We define \( a_h = a^r_h + 2\omega \frac{1}{\sqrt{2\pi}} \Lambda^{1/2} \), the above energy becomes (in units of \( \frac{\omega H^2}{2\pi\kappa^2} \)):
\[ \frac{1}{\sqrt{2\pi}} \frac{2}{3} \Lambda^{3/2} - \omega \left( \frac{1}{\sqrt{2\pi}} \Lambda^{1/2} \right)^2 + \left( -a^r_h - \frac{b}{2} \right) \frac{1}{\sqrt{2\pi}} \Lambda^{1/2} - a^r_h \partial_{\varepsilon} f_{\text{Tr-log}}(\varepsilon, b) \] (32)
\[ -\varepsilon \partial_{\varepsilon} f_{\text{Tr-log}}(\varepsilon, b) + \omega (\partial_{\varepsilon} f_{\text{Tr-log}}(\varepsilon, b))^2 + f_{\text{Tr-log}}(\varepsilon, b). \]
Thus the temperature and vacuum energy will be renormalized. The first three terms are divergent and linear in temperature, they will not contribute to any physical quantities, like magnetization and specific heat. Minimizing the energy eq. (32), we get the gap equation:
\[ \varepsilon = -a^r_h - 2\omega \partial_{\varepsilon} f_{\text{Tr-log}}(\varepsilon, b) \] (33)
Superscript ”r” will be dropped later on. Function \( f_{\text{Tr-log}}(\varepsilon, b) \) can be written
\[ f_{\text{Tr-log}}(\varepsilon, b) = \frac{1}{\sqrt{2\pi}} b^{3/2} g \left( \frac{\varepsilon}{b} \right) \] (34)
where

$$g(x) = \sum_{n=0}^{\infty} \left[ \sqrt{n+x} - \frac{2}{3} (x+n+\frac{1}{2})^\frac{3}{2} + \frac{2}{3} (x+n-\frac{1}{2})^\frac{3}{2} \right] - \frac{2}{3} (x-\frac{1}{2})^\frac{3}{2} $$

(35)

For the LLL model in the gaussian approximation, $g(x) = \sqrt{x}$. For Prange limit $Gi \rightarrow 0$, the free energy will be $\frac{\omega H^2_c}{2\pi \kappa^2} \frac{1}{\sqrt{2\pi}} b^{3/2} g \left( -\frac{a_h}{b} \right)$.

C. Magnetization

When $\kappa$ is quite large and magnetization can be approximated by

$$M = -\frac{\partial}{\partial B} F_{\text{eff}}(B).$$

(36)

The HLL correction will be calculated as follows. We numerically solve the gap equation (33), and $F_{\text{eff}}(B)$ can be obtained. Then eq.(33) is used to calculate the magnetization of the full GL model in gaussian approximation. The HLL correction is thus the magnetization of the full GL model in gaussian approximation minus the magnetization of the LLL contribution in gaussian approximation. We compare the experiments using following approximation. The LLL contribution (for example, see ref. (11)) is

$$M_{\text{LLL}} = \frac{H_{c2}}{4\pi \kappa^2} \frac{a_h}{a_T} m(a_T),$$

(37)

where $m(a_T)$ is defined in eq.(20) is taken exactly (calculated by the BP method to highest order), while the corrections due to HLL are taken in gaussian approximation. The comparison of the theoretical predictions with the experiments for fully oxidized $YBa_2Cu_3O_7$ [3], is shown on Fig. 3. We use the experimental asymmetry value $\gamma = 4$ and values of $T_c$, $H_{c2}$ and $Gi$ from the fitting of the melting curve (see Fig.2 and Table 1). The agreement is fair at high magnetic fields, while at low magnetic fields is not good. We comment that the theory of the full GL model (higher Landau levels included) beyond Gaussian approximation is required at low magnetic fields.

V. SUMMARY

The problem of calculating the fluctuations effects in the framework of the Ginzburg - Landau approach is discussed and essentially solved in the LLL limit. The melting line location is determined and magnetization and specific heat jumps along it are calculated. The magnetization of liquid is larger than that of solid by 1.8% irrespective of the melting temperature, while the specific heat jump is about 6% and decreases slowly with temperature. The LLL results for melting line, magnetization jump, specific heat jump and the magnetization curves are in good agreement with experiments and Monte Carlo simulations. The leading corrections due to higher Landau levels have been also calculated and lead to agreement with experimental results in the whole region in which the GL approach is applicable.
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\[ \int dz \int dk \left[ \left( -\frac{1}{2} \frac{\partial^2}{\partial z^2} + \varepsilon \right) \psi_k^*(z) \psi_k(z) + c \left[ \psi_k^*(z) \psi_k(z) \right]^2 \right]. \]
Different momenta in directions xy are decoupled, thus the fixed point is not related to second order melting. This also explains their critical exponents.
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**Figure captions**

**Fig. 1**
Free energy of solid (line) and liquid (dashed line) as function of the scaled temperature. The solid line ends at a point (dot) indicating the loss of metastability. Inset shows a tiny difference between liquid and solid near the melting point at $a_T^m = -9.5$.

**Fig. 2**
Comparison of the experimental melting line for fully oxidized $YBa_2Cu_3O_7$ [6] with our fitting. Inset on the right shows the relative universal magnetization jump of 1.8% (line) and experimental results for fully oxidized $YBa_2Cu_3O_7$ (rhombs) and optimally doped untwinned $YBa_2Cu_3O_{7-\delta}$ (stars). Inset on the left shows the relative nonuniversal specific heat jump (line) and experimental results for optimally doped untwinned $YBa_2Cu_3O_{7-\delta}$ [2].

**Fig. 3**
Comparison of the theoretical magnetization curves (lines) of fully oxidized $YBa_2Cu_3O_7$ utilizing parameters obtained by fitting the melting line on Fig. 2 with torque magnetometry experimental results [6] (dots). Arrows indicate melting points while at low magnetic field the experimental data start from the point in which the magnetization is reversible indicating low disorder.