Characterization of Objects based on the Polarization Tensor: Nature versus Artificial Intelligence

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Abstract. Describing the perturbation in electric or electromagnetic fields due to conductivity contrast could be essential to improve many industrial applications. The applications include electrical imaging such as electrical impedance tomography, electrical resistivity tomography and metal detectors. In this case, understanding the perturbation helps, for examples, to improve reconstruction of images for medical purposes or reduce the possibility of detecting nonthreat objects during security screening with metal detectors. One way to describe the perturbation in electric or electromagnetic fields due to the presence of a conducting object in the region of the field is to use the terminology called as polarization tensor, where, polarization tensor can then be used to describe and characterize the presented object. Mathematically, polarization tensor can be defined in terms of boundary value problems of a PDE or also as integral equations in an asymptotic series. In this paper, the applications of polarization tensor are highlighted specifically to characterize object. The examples included are in the natural electric fish and also in an artificial intelligence. It is proposed to relate all studies in the future to improve the related applications using polarization tensor.

1. The Generalized Polarization Tensor

Polarization Tensor (PT) actually originates from classical studies for example by [1] during their investigation on solid and fluid mechanics before being adapted in other areas such as in scattering problems (see [2]). In the 21st century, the applications of PT are more specified in science and engineering. It is used by Milton [3] in the study of material science, particularly, theoretical composites. Besides, Ammari and Kang [4] have generalized the study by [1] and used PT to improve electrical imaging for medical and industrial purposes. They [5-7] also conducted mathematical studies and simulation on cloaking device based on that PT. The same PT is also used by Ahmad Khairuddin and Lionheart [8] to investigate characterization of objects by weakly electric fish. On the other hand, a slightly different PT is studied by Ledger and Lionheart [9] to improve metal detection, specifically, for landmine clearance.

In this paper, our focus is on the PT that can describe the perturbation in electric field when a conducting object is presented in the region of the field. Here, the conductivity of the region itself is different than the conductivity of the object. Now, the mathematical formulation of the Generalized Polarization Tensor (GPT) is firstly presented as follows (see [4]).
Consider a Lipschitz bounded domain $B$ in $\mathbb{R}^3$ such that the origin $O$ is in $B$. Let $k$ be the conductivity of $B$, where $0<k \neq 1<+\infty$ and the conductivity of $\mathbb{R}^3 - B$ is assumed equal to 1. Suppose that $H$ is a harmonic function in $\mathbb{R}^3$ and $u$ is the solution to the following problem

\[
\begin{align*}
\Delta u &= 0 \text{ in } \mathbb{R}^3, \\
u \cdot (1 + (k - 1)\chi(B)\nabla u) &= 0 \text{ in } \mathbb{R}^3, \\
u \cdot H + \frac{1}{k - 1} \nabla u &= O(\lvert x \rvert^{-2}) \text{ as } \lvert x \rvert \to +\infty,
\end{align*}
\]

(1)

where, $\chi$ denotes the characteristic function of $B$. Equation (1) has been used to mathematically model many industrial applications such as electrical resistivity tomography (ERT) of geoscience, electrical impedance tomography (EIT) for medical imaging or also metal detection. The PT is then defined by [4] in the following asymptotic expansion.

\[
(u - H)(x) = \sum_{i,j \geq 1} \frac{(-1)^{i+j}}{i!j!} \Gamma(x) M_{ij}(k, B) \delta^{i+j} H(0) \text{ as } \lvert x \rvert \to +\infty
\]

(2)

for $i, j$ multi indices, $\Gamma$ is the fundamental solution of the Laplacian and the dominant term, $M_{ij}(k, B)$ is the GPT. In physics, The GPT could be similar to the dipole in electromagnetic applications, where, both of them show the electric or electromagnetic distribution of the object. Equation (2) suggests that the GPT can be determined if $u$ and $H$ are known, regardless of $B$.

On the other hand, Ammari and Kang [4] also extends the definition of GPT in (2) through an integral equation over the boundary of $B$ by

\[
M_{ij}(k, B) = \int_{\partial B} y^i \phi_j(y) d\sigma(y)
\]

(3)

where, $\phi_j(y)$ is given by

\[
\phi_j(y) = \left( \lambda I - K_{ij}^* \right)^{-1} \left( \nu \cdot \nabla x^j \right)(y)
\]

(4)

for $x, y \in \partial B$ with identity $I$ and $\nu_x$ is the outer unit normal vector to the boundary $\partial B$ at $x$. Here, $\lambda$ is defined by $\lambda = (k + 1)/2(k - 1)$ and $K_{ij}^*$ is a singular integral operator defined with Cauchy principal value $P.V.$ by

\[
K_{ij}^*(\nu)(x) = \frac{1}{4\pi} \text{p.v.} \int_{\partial B} \frac{x - y \cdot \nu_x}{\lvert x - y \rvert} \phi_j(y) d\sigma(y)
\]

(5)

Therefore, if $B$ is known such that its geometry and its conductivity $k$ are given, the GPT due to the presence of $B$ can be directly determined without solving (2), as given by (3), (4) and (5). Thus, the notation $M_{ij}(k, B)$ is used and the GPT associated with $B$ at conductivity $k$ is commonly referred to as the GPT for $B$.

Alternatively, the GPT can also be defined from a transmission problem of a PDE with the following set of equations (see [4])

\[
\begin{align*}
\Delta \psi_i &= 0, \quad x \in B \cup (\mathbb{R}^3 \setminus \overline{B}), \\
\psi_i \bigg|_{\partial B} &= 0, \quad x \in \partial B, \\
\frac{\partial \psi_i}{\partial \nu} \bigg|_{\partial B} &= -k \frac{\partial \psi_i}{\partial \nu} \bigg|_{\partial B} = \nu \cdot \nabla \psi_i, \quad x \in \partial B, \\
\psi_i(x) &\to 0 \text{ as } \lvert x \rvert \to +\infty \text{ if } d = 3,
\end{align*}
\]

(6)

where, $\psi_i$ satisfies $\psi_i(x) - \Gamma(x) \int_{\partial B} \nu \cdot \nabla y^i d\sigma(y) = O(\lvert x \rvert^{-2}) \text{ as } \lvert x \rvert \to +\infty$. By finding the unique solution $\psi_i$ to the problem (6), Ammari and Kang [4] proved that

\[
\left( \lambda I - K_{ij}^* \right) \left( \frac{\partial \psi_i}{\partial \nu} \right)(x) = \frac{1}{k - 1} \left( -\frac{1}{2} I + K_{ij}^* \right) (\nu \cdot \nabla \psi_i)(x)
\]

(7)
for \( x \in \partial B \). The GPT can then be alternatively determined according to [4] using

\[
M_{\phi}(k,B) = (k-1) \int_{\partial B} x' \frac{\partial x'}{\partial \nu} d\sigma(x) + (k-1)^2 \int_{\partial B} x' \frac{\partial \psi_{\phi}}{\partial V} (x)d\sigma(x).
\]

(8)

In some applications of PT, describing and characterizing objects based on their PT offers lower computational cost than finding the whole image of the objects. Although PT can be implemented in the algorithm for reconstructing image of any objects during electrical imaging (see [4]), most of the time, the process requires high performance computation. As can be seen in (2), the GPT is build up by several coefficients depending on \( i \) and \( j \), but, our interest at the moment is only the GPT for the case \(| i | = j | = 1\) called as the first order GPT.

2. The First Order Generalized Polarization Tensor

The first order GPT (or simply the first order PT) is the simplest form of the GPT and can be obtained by using (2), (3) and (4) when \(| i | = j | = 1\). Considering the combination of all related indices \( i \) and \( j \), the first order PT for \( B \) at conductivity \( k \), denoted by \( M(k,B) \), can be written as \( 3 \times 3 \) matrix and is given by

\[
M(k,B) = [M_{(1,0,0)(1,0,0)} M_{(1,0,0)(1,0,1)} M_{(1,0,0)(0,0,1)} M_{(1,0,1)(1,0,0)} M_{(1,0,1)(1,0,1)} M_{(1,0,1)(0,0,1)} M_{(0,0,1)(1,0,0)} M_{(0,0,1)(1,0,1)} M_{(0,0,1)(0,0,1)}] \]

where, every element of (9) can be evaluated using (3), (4) and (5). Like all other GPTs, the first order PT does not depend on the position of \( B \). A method to compute the first order PT (9) based on (3), (4) and (5) can be found for examples in [10-11]. In both [10-11], the object \( B \) were approximated by a mesh consisting of triangular elements before numerical integrations were performed to compute the corresponding first order PT. Specifically, in [10], each triangle was represented as a linear element. On the other hand, quadratic element was used to represent each triangle in [11].

Moreover, the first order PT is also symmetry [4]. It also transforms as \( B \) transforms so, the first order PT also depends on size and orientation of \( B \). If \( B' \) is the rotation of \( B \) then \( M(k,B') = RM(k,B)R' \), where, \( R \) is the appropriate rotation matrix. Besides, the first order PT is a positive definite matrix if \( k > 1 \) whereas it is a negative definite matrix when \( 0 < k < 1 \). In addition, an explicit formula of the first order PT for ellipsoid is also given in Ammari and Kang (2007) by

\[
M(k,B) = (k-1)|B| \begin{bmatrix}
1/(1-P+kP) & 0 & 0 \\
0 & 1/(1-Q+kQ) & 0 \\
0 & 0 & 1/(1-R+kR)
\end{bmatrix}
\]

(10)

where, \(|B|\) is the volume of the ellipsoid while \( P, Q \) and \( R \) are constants defined by

\[
P = \frac{bc}{a^2} \int_{t^2}^{\infty} \frac{1}{t^2 - 1 + (b/a)^2} \sqrt{t^2 - 1 + (c/a)^2} dt,
\]

(11)

\[
Q = \frac{bc}{a^2} \int_{(b/a)^2}^{\infty} \frac{1}{(t^2 - 1 + (b/a)^2)^{\frac{3}{2}}} \sqrt{t^2 - 1 + (c/a)^2} dt,
\]

(12)

\[
R = \frac{bc}{a^2} \int_{(b/a)^2}^{\infty} \frac{1}{(t^2 - 1 + (b/a)^2)^{\frac{3}{2}}} \left( t^2 - 1 + (c/a)^2 \right)^{\frac{3}{2}} dt.
\]

(13)

The previous information are very useful to validate the first order PT obtained based on (3), (4) and (5), when \( B \) and its conductivity are given. In other applications, the PT could be determined by field measurements during experiments in the lab or field work. In this case, the PT produced is also a \( 3 \times 3 \) matrix, but \( B \) and its conductivity might be known or unknown. If the PT obtained is either a
positive or a negative definite matrix, the PT is also equivalent to the first order PT for an ellipsoid and hence, an ellipsoid that has the same first order PT with \( B \) can be determined (see [12]). That ellipsoid is called as the best fitting ellipsoid to the given first order PT. Moreover, the first order PT of an ellipsoid has a close relationship with the classical terminology depolarization factors (see details in [13] and [14]). By investigating further properties of depolarization factors for ellipsoid [15], a few results on characterizing an ellipsoid based on a given first order PT have been proposed.

Recently, in order to describe an unknown object based on a given first order PT, it is firstly assumed that the given first order PT is also the first order PT of an ellipsoid. After that, the ellipsoid is characterized based on that first order PT. Here, the ellipsoid could have the same size and conductivity with the unknown object originally represented by the given first order PT. It is found in [16] that if the given first order PT is a positive definite matrix then the conductivity of the ellipsoid must be greater than 1 whereas if the given first order PT is a negative definite matrix then the conductivity must be between 0 and 1. It is also proven in [16] that the semi principal axes of the ellipsoid could be characterized according to the elements of the given first order PT.

Particularly, if the given first order PT is either a positive or a negative definite matrix with only one or two eigenvalues, additional information about the best fitting ellipsoid to this first order PT can be obtained. These include the volume, the depolarization factors and also all semi axes of the ellipsoid. As presented in [17], given the first order PT with one or two eigenvalues, the volume and the depolarization factors of the best fitting ellipsoid to the given first order PT can be analytically determined. The values for the volume and the depolarization factors obtained will then be used to compute the values for all semi axes. Unlike the method proposed in [12], it is now only an option to determine the semi axes of the ellipsoid. For instance, the volume obtained can already be used to estimate the size of the ellipsoid. So, the semi axes can then be determined for any other reasons than describing the size of the ellipsoid.

3. The First Order PT in Electric Fish

A weakly electric fish has an organ to discharge electric. It uses sensing cells on the surface of its body to measure the difference between transmitting and receiving electrical waves when performing electrosensing in order to move or identify objects in the river [18]. While performing electrosensing and moving in the water before approaching or avoiding objects, its single electric source seems to act in a similar way with switching driven electrodes in an EIT system [19]. This suggests to adapt the study on the EIT into the study of electrosensing fish. Due to the complexity of the process, it assumed that the fish with a small brain do not perform reconstruction of images in the real time. They might not clearly see the object until at a certain distance especially in a deep water with dark surrounding. In this situation, one possible mechanism for the fish to identify an object during electrosensing is by recognizing the first order PT for the object.

Following [8], let the electrical conductivity in the region exterior to a weakly electric fish be \( \sigma \) and suppose that there is an isolated object \( B \) which is assumed to be a Lipschitz bounded domain in \( \mathbb{R}^3 \) at some distance from the fish. Consider the domain \( \Omega = \mathbb{R}^3 - F \) where \( F \) is the fish and for any point \( x \in \mathbb{R}^3 \),

\[
\sigma(x) = \begin{cases} 
1, & x \in \Omega - B \\
\sigma, & x \in B
\end{cases}
\]

(14)

where \( \sigma \) constant. If \( u \) is the voltage in the region \( \Omega \) then the perturbation in the voltage due to the small object \( B \) can be approximated by an asymptotic expansion where, the dominant term of the expansion is determined by the GPT. Assuming that a harmonic function \( H \) is the voltage in \( \Omega \) without the object \( B \) then from [4],

\[
(u - H)(x) = -\nabla \Gamma(x) \cdot M(k, B) \nabla H(O) + O(1/|x|^2)
\]

(15)

where the origin \( O \in B \), \( \Gamma(x) = -(4\pi |x|)^{-3} \) and \( M(k, B) \) is the first order PT for \( B \). The derivation is the same as (2) (see [4]) except that only the first order PT is considered.
In order to examine the role of the first order PT in electrosensing by the fish, a few experimental studies conducted by von der Emde and Fetz [20] are reinvestigated by a few previous researches. In these experiments [20], a few fish were trained to distinguish two different objects, where, each fish was given a reward if it chose the correct object and punished when they chose the wrong object. They were trained for three consecutive days until they would only choose the object that they were asked to choose at 75% success rate. In [8], the first order PT for objects with specified dimension used during the training in [20] were determined. After that, the PT obtained in the matrix from were analyzed and related to the time taken by the fish to complete the training. It is found that in [8] fish take longer time to accomplish the training when they have been asked to choose two objects where, the first order PT for both objects are almost the same. Several future experiments have also been suggested in [8] to further test the ability of electric fish in describing object based on the first order PT.

A similar study which relates the first order PT to a few more results in [20] can be found in [21]. Some comparisons and similarities on object characterization based on the PT in electrosensing fish with metal detection are also given in [22] and [23]. In addition, more advanced studies about GPT and electrical imaging in electrosensing fish have also been conducted for examples by [24] and [25].

4. The First Order PT in an Artificial Intelligence

Based on the ability of electric fish in distinguishing objects using electrosensing, Lanneau et al [26] have developed an underwater sensor to further investigate how electrical sense can be further used to describe an object. In this study, the underwater sensor was arranged at a distance from an object inside a tank. After wave was transmitted by the sensor, electrical data collected by the electrical sense of the sensor was analyzed by a specified algorithm to locate the object independently of the shape of the object. In addition, the electrical response from the object was assumed to be initially described by the first order polarization tensor for an ellipsoid. The ellipsoid was then determined to estimate the actual size and the actual orientation of the object.

The study initiated by [26] was then extended by Bazeille et al [27] where, the underwater sensor was upgraded to an underwater robot equipped with artificial electrical sense. In their experiment [27], the robot was not static like [26] and was set to move underwater along a trajectory and at some distance from an object. Similar with [26], the electrical response of the object as the robot moved was initially described by the first order PT for an ellipsoid. As the robot moved along the trajectory, electrical data was collected and this data was used in their proposed algorithm with optimization technique to simultaneously detect the correct location and determine the shape of the object. Thus, unlike [26], the location and shape of an object were estimated in one algorithm. This was done to mimic electrosensing fish when detecting object. Their promising findings showed 18% of error in shape of the object, 25° of error in orientation of the object and 1 cm of error within a range of 1 to 11 cm for the distance between the object and the robot.

An improved model for [27] with heuristic approach was developed later on by [28]. This time, the error, on average, for location and size were below 1 cm and 15% respectively. Moreover, the experiments by [28] have also been conducted in two different types of water (fresh and salt) which will influence the electrical sense and response in the study. It was shown in their numerical simulation that the results obtained in the fresh water was less good than the salt water but the results in the fresh water were still consistent with other previously conducted studies. On the other hand, the promising results obtained when salt water is used might contribute in the future study on electrical sense not only in a river but also in a sea water.

5. Summary and Recommendation

In this article, the GPT are reviewed together with some of its properties in characterizing a presented object which causes perturbation in electrical field. After that, the paper also discusses the applications of the first order GPT for describing object in electrosensing fish and also an artificial intelligence based on the electric fish. In electrosensing fish, the first order PT was computed and related to the experimental studies conducted to the fish to justify the role of the PT to the fish in choosing object.
On the other hand, the electrical response to the artificial electrical sense for the underwater robot was firstly described by the first order polarization tensor for an ellipsoid, before the ellipsoid was determined to describe an object at some distance with the robot. As both studies have the same origin, it is our hope that the both studies can be further linked to improve characterization of object based on electrical sense using PT, which could be very useful to solve real world problems in the future.

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