Toric Varieties of Schröder Type

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Abstract. A dissection of a polygon is obtained by drawing diagonals such that no two diagonals intersect in their interiors. In this paper, we define a toric variety of Schröder type as a smooth toric variety associated with a polygon dissection. Toric varieties of Schröder type are Fano generalized Bott manifolds, and they are isomorphic if and only if the associated Schröder trees are the same as unordered rooted trees. We describe the cohomology ring of a toric variety of Schröder type using the associated Schröder tree and discuss the cohomological rigidity problem.

1. Introduction

Let $P_{n+2}$ be a regular polygon with $n+2$ vertices. A dissection $D$ of $P_{n+2}$ is obtained by drawing diagonals such that no two diagonals intersect in their interiors. A dissection $D$ of $P_{n+2}$ is called a $k$-dissection if it divides $P_{n+2}$ into $k$ polygons of smaller size. Note that $k$ varies from 1 to $n$, and an $n$-dissection of $P_{n+2}$ is called a triangulation.

Let $f(n,k)$ be the number of $k$-dissections of $P_{n+2}$ for $1 \leq k \leq n$. Cayley [3] proved that $f(n,k) = \frac{1}{2} \binom{n-1}{k-1} \binom{n+k}{k}$, and the number $f(n,k)$ is called the Kirkman-Cayley number. There are a number of equivalent ways to represent polygon dissections: as binary bracketings, Lukasiewicz words, standard Young tableaux of shape $(k,k,1^{n-k})$, and rooted plane trees whose non-leaf vertices have at least two children. (See [12, 2, 17, 20, 22].) The last one is called a Schröder tree because the total number of these trees with $n+1$ leaves, $\sum_{1 \leq k \leq n} f(n,k)$, is the $(n+1)$st small Schröder number $s_{n+1}$. Note that $f(n,n)$ is just the Catalan number $C_n$, and the Schröder trees corresponding to triangulations of a polygon are full binary rooted trees.

A toric variety of (complex) dimension $n$ is a normal complex algebraic variety containing a torus $T = (\mathbb{C}^*)^n$ as an open dense subset, such that the action of $T$ on itself extends to the whole variety. A typical example of a smooth compact toric variety is the projective space $\mathbb{C}P^n$ of complex dimension $n$ with the standard action of $T$.

We construct a smooth toric variety $X_D$ associated with a dissection $D$ of the polygon $P_{n+2}$ inductively. We first associate the projective space $\mathbb{C}P^n$ with the polygon $P_{n+2}$. If $\tilde{D}$ is a dissection of $P_{n+2}$ obtained from $D$ by drawing a new diagonal, then the toric variety $X_{\tilde{D}}$ associated with $\tilde{D}$ is the blow up of $X_D$ along

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a certain $T$-invariant submanifold, see Section 3 for the precise definition. We say that $X_D$ is of Schröder type.

Recall that a toric variety of Catalan type introduced in [15] is the toric variety associated with a triangulation of a polygon. When $D$ is a triangulation of a polygon, the fan $\Sigma_D$ in our construction is the same as that in [15], so our toric varieties of Schröder type contain the toric varieties of Catalan type.

Every toric variety of Schröder type is a generalized Bott manifold, the total space of an iterated complex projective space bundle over a point. If $D$ is a $k$-dissection of $\mathbb{P}_{n+2}$, then $X_D$ is a $k$-stage generalized Bott manifold of dimension $n$. Furthermore, each toric variety of Schröder type is Fano.

Let $\tau(D)$ be the Schröder tree associated with a polygon dissection $D$. Using the Fano condition of $X_D$, we can recover the Schröder tree $\tau(D)$ and prove the following.

**Theorem 1.1** (Theorem 3.4 and Corollary 3.7). For two dissections $D$ and $\tilde{D}$, the toric varieties $X_D$ and $X_{\tilde{D}}$ of Schröder type are isomorphic as varieties if and only if the associated Schröder trees $\tau(D)$ and $\tau(\tilde{D})$ are isomorphic as unordered rooted trees, and the number of isomorphism classes of $n$-dimensional toric varieties of Schröder type is equal to the number of series-reduced rooted trees with $n+1$ leaves.

Cohomological rigidity problem for toric varieties asks whether two smooth compact toric varieties are diffeomorphic when their cohomology rings are isomorphic as graded rings, see [18]. This problem is still open even for generalized Bott manifolds and many results have been produced in support of the affirmative answer to the problem, see [5, 6, 7, 13].

It is proved in [16] that two toric varieties of Catalan type are isomorphic as varieties if and only if their integral cohomology rings are isomorphic as graded rings. Hence we may ask the following.

**Problem 1.2.** When two toric varieties $X_D$ and $X_{\tilde{D}}$ of Schröder type have the same cohomology ring, are they isomorphic?

We explicitly describe the integral cohomology ring of $X_D$ from the associated Schröder tree $\tau(D)$, see Theorem 4.1 for more details. We give a partial affirmative answer to the above problem when we restrict the number of partitions for polygon dissections.

**Theorem 1.3.** Let $D$ and $\tilde{D}$ be $k$-dissections of the polygon $\mathbb{P}_{n+2}$. When $k \leq 3$ or $k = n$, toric varieties $X_D$ and $X_{\tilde{D}}$ are isomorphic as varieties if and only if $H^*(X_D)$ and $H^*(X_{\tilde{D}})$ are isomorphic as graded rings.

This paper is organized as follows. In Section 2, we introduce the Etherington’s bijection between polygon dissections and Schröder trees and review some basic definitions and properties of toric varieties. In Section 3, we construct a toric variety of Schröder type and show that toric varieties of Schröder type are Fano generalized Bott manifolds. In Section 4, we give an explicit description of the cohomology ring of a toric variety of Schröder type in terms of the associated Schröder tree. Section 5 is devoted to prove Theorem 1.3. In Section 6, we give another evidence for the affirmative answer to Problem 1.2.
2. Preliminaries

In this section, we introduce the Etherington’s bijection between the set of polygon dissections and that of Schröder trees in \cite{12} and review some basic facts on toric varieties.

2.1. Polygon dissections and Schröder trees. We give the labels 0, 1, \ldots, n+1 to the vertices of a regular polygon $P_{n+2}$ with $n + 2$ vertices counterclockwise. We call the edge $\{0, n+1\}$ the distinguished edge of $P_{n+2}$. For simplicity, we also denote by $D$ the set of all polygons in the dissection $D$ of $P_{n+2}$. For each polygon $P(i)$ in $D$, we call the edge connecting the minimal and the maximal vertices the distinguished edge of $P(i)$ in $D$. Let $E(D)$ denote the set of all the edges in $D$ except the distinguished edge of $P_{n+2}$. Then $E(D)$ is decomposed into the edge sets $E(P(i))$, $i = 1, \ldots, k$.

Example 2.1. Let $D$ be the 4-dissection of $P_{10}$ in Figure 1. Then $\{0,9\}$ is the distinguished edge of $P_{10}$ and it is also the distinguished edge of $P(1)$ in $D$. The edges $\{0, 7\}, \{0, 3\}$, and $\{3, 7\}$ are the distinguished edges of $P(2)$, $P(3)$, and $P(4)$, respectively. Note that in Figure 1, the distinguished edges are represented by dashed lines.

![Figure 1](image-url)  

**Figure 1.** An example of a 4-dissection of $P_{10}$
In order to consider the inverse from Schröder trees to polygon dissections, we define recursively a labeling of Schröder trees. Given a Schröder tree $T$ with $n + 1$ leaves, let
\[
\phi : V(T) \to \{(i, j) \mid 0 \leq i < j \leq n + 1\},
\]
where
\[
\phi(v) = \begin{cases} 
0, n + 1 & \text{if } v \text{ is the root,} \\
(i - 1, i) & \text{if } v \text{ is the } i\text{th leaf in the preorder listing of } T, \\
(i, j) & \text{if } v \text{ is an internal vertex whose left-most and right-most children are labeled by } \{i, \bullet\} \text{ and } \{\bullet, j\}, \text{ respectively.}
\end{cases}
\]

Note that for a dissection $D$, there is a polygon $P(i)$ in $D$ with vertices labeled by $v_0 < v_1 < \cdots < v_\ell$ counterclockwise if and only if there is an internal vertex $v \in V(\tau(D))$, which is labeled by $\{v_0, v_\ell\}$ and has $\ell$ children labeled by $\{v_{j-1}, v_j\}$ for $j = 1, 2, \ldots, \ell$ from left to right. Hence, given a Schröder tree $T$, we get the corresponding dissection $\tau^{-1}(T)$. Sometimes, we use the simple notation $\phi(v) = ij$ instead of $\phi(v) = \{i, j\}$. Figure 2 shows a simple way of finding the correspondence between $D$ and $\tau(D)$. Note that in Figure 2, the root is represented by $\odot$.

**Figure 2.** A dissection of $P_{10}$ and its corresponding Schröder tree

2.2. Toric varieties. A toric variety of complex dimension $n$ is a normal complex algebraic variety with an algebraic action of $(\mathbb{C}^*)^n$ having an open dense orbit. A fundamental result of toric geometry is that there is a bijective correspondence between toric varieties of complex dimension $n$ and rational fans of real dimension $n$. We can determine geometric or topological properties of toric varieties combinatorially.

Let $\Sigma$ be a fan whose ray generators are $v_1, \ldots, v_m$ in $\mathbb{R}^n$. The fan $\Sigma$ is complete if $\bigcup_{\sigma \in \Sigma} \sigma = \mathbb{R}^n$; $\Sigma$ is nonsingular if the $v_i$’s for the generators of $\sigma$ form a part of a basis of $\mathbb{Z}^n$ for every $\sigma \in \Sigma$; and $\Sigma$ is polytopal if it is the normal fan of a lattice polytope. Then a toric variety $X$ is compact (respectively, smooth and projective) if and only if the associated fan $\Sigma_X$ is complete (respectively, nonsingular and polytopal).
We can also describe the cohomology ring of a smooth compact toric variety explicitly using the associated fan.

**Theorem 2.2 ([11, 14]).** Let \( \Sigma \) be a complete nonsingular fan whose ray generators are \( v_1, \ldots, v_m \). Let \( X_\Sigma \) be the toric variety associated with the fan \( \Sigma \). Then the (integral) cohomology ring \( H^*(X_\Sigma) \) is isomorphic to \( \mathbb{Z}[x_1, \ldots, x_m]/I \) as a graded ring, where \( I \) is the ideal generated by the following two types of elements:

1. \( \prod_{i \in I} x_i \), where \( \{ v_i \mid i \in I \} \) does not form a cone in \( \Sigma \), and
2. \( \sum_{i=1}^{m} \langle u, v_i \rangle x_i \) for any \( u \in \mathbb{Z}^n \),

where \( \langle , \rangle \) denotes the standard scalar product on \( \mathbb{Z}^n \).

A projective smooth variety \( X \) is Fano if the anticanonical divisor \( -K_X \) is ample. There is a combinatorial way to determine whether a smooth projective toric variety is Fano. For a projective fan \( \Sigma \), a subset \( R \) of the primitive ray vectors is called a **primitive collection** of \( \Sigma \) if

\[
\text{Cone}(R) \notin \Sigma \quad \text{but} \quad \text{Cone}(R \setminus \{ u \}) \in \Sigma \quad \text{for every} \quad u \in R.
\]

Note that if \( \Sigma_p \) is the normal fan of a lattice polytope \( P \), then primitive collections of \( \Sigma_p \) correspond to the minimal non-faces of \( P \). For a primitive collection \( R = \{ u'_1, \ldots, u'_r \} \), we get \( u'_1 + \cdots + u'_r = 0 \) or there exists a unique cone \( \sigma \) such that \( u'_1 + \cdots + u'_r \) is in the interior of \( \sigma \). That is,

\[
(2.1) \quad u'_1 + \cdots + u'_r = \begin{cases} 0, & \text{or} \\ a_1 u_1 + \cdots + a_r u_r, & \end{cases}
\]

where \( u_1, \ldots, u_r \) are the primitive generators of \( \sigma \) and \( a_1, \ldots, a_r \) are positive integers. We call \( (2.1) \) a primitive relation, and the **degree** \( \text{deg}(R) \) of a primitive collection \( R \) is defined to be \( \ell - (a_1 + \cdots + a_r) \). Batyrev [1] gave a criterion for a smooth projective toric variety to be Fano.

**Proposition 2.3 ([1, Proposition 2.3.6]).** A projective toric variety \( X_\Sigma \) is Fano when \( \text{deg}(R) > 0 \) for every primitive collection \( R \) of \( \Sigma \).

We can also classify smooth Fano toric varieties up to isomorphism using the primitive relations.

**Proposition 2.4 ([1, Proposition 2.1.8 and Theorem 2.2.4]).** Two smooth Fano toric varieties \( X_\Sigma \) and \( X_{\Sigma'} \) are isomorphic as toric varieties if and only if there is a bijection between the sets of rays of \( \Sigma \) and \( \Sigma' \) inducing a bijection between maximal cones and preserving the primitive relations.

### 3. Smooth toric varieties associated with polygon dissections

In this section, we construct a toric variety of Schröder type and show that they are Fano generalized Bott manifolds. We first construct a nonsingular polytopal fan \( \Sigma_D \) associated with a dissection \( D \) and then show that the toric variety associated with \( \Sigma_D \) is a Fano generalized Bott manifold.

We first recall the notion of generalized Bott manifolds.

**Definition 3.1 ([7]).** A generalized Bott tower \( B_\bullet \) is an iterated \( \mathbb{C} \mathbb{P}^n \)-bundle:

\[
(3.1) \quad B_m \xrightarrow{\pi_m} B_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0,
\]

\[
P(\mathbb{C} \oplus \mathbb{C}^{m-1} \oplus \cdots \oplus \mathbb{C}^{n_{m-1}}) \quad \mathbb{C} \mathbb{P}^{n_1} \quad \{ \text{a point} \}
\]
where each $B_i$ is the complex projectivization of the Whitney sum of holomorphic line bundles $\xi_k^{k-1}$ $(1 \leq k \leq n_i)$ and the trivial line bundle $\mathbb{C}$ over $B_{-1}$. We call $B_m$ an $m$-stage generalized Bott manifold.

When $n_i = 1$ for all $i = 1, \ldots, m$, we omit ‘generalized.’ That is, the tower in (3.1) is called a Bott tower and the total space $B_m$ is called a Bott manifold.

Note that a generalized Bott manifold in (3.1) is a projective smooth toric variety, and the associated lattice polytope is combinatorially equivalent to $\prod_{i=1}^{m} \Delta^{n_i}$. Conversely, if a smooth lattice polytope $P$ is combinatorially equivalent to a product of simplices, then the toric variety associated with the polytope $P$ is a generalized Bott manifold. See [7, 8] for more details.

Recall that we give the labels $0, 1, \ldots, n+1$ to the vertices of $P_{n+2}$ counterclockwise. Let us construct the fan $\Sigma_0$ as follows.

1. To each edge $\{i, j\}$ ($i < j$) in $E(P_{n+2})$, we associate the ray generated by the vector $e_j - e_i$, where $e_0 = e_{n+1} = \mathbf{0}$ and $e_1, \ldots, e_n$ are the standard basis vectors of $\mathbb{R}^n$.

2. Every proper subset of $E(P_{n+2})$ constructs the cone generated by the corresponding rays.

Note that the empty set corresponds to the zero-dimensional cone and the toric variety associated with the fan $\Sigma_0$ is the complex projective space $\mathbb{C}P^n$.

Now we associate a dissection of $P_{n+2}$ to an iterated blowing up of the fan $\Sigma_0$. We give an order on the edges in the interior of $D$ according to the preorder listing of the associated Schröder tree $\tau(D)$. Suppose that the edges in the interior of a $k$-dissection $D$ of $P_{n+2}$ are ordered as follows:

\[
\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_{k-1}, j_{k-1}\}
\]

according to the preorder listing of $\tau(D)$. For simplicity, we assume that $i_\ell < j_\ell$ for all $\ell = 1, \ldots, k-1$. Then we define the fan $\Sigma_D$ associated with $D$ as follows. We first subdivide the cone of $\Sigma_0$ corresponding to the set $\{\{i_1, i_1 + 1\}, \ldots, \{j_1 - 1, j_1\}\}$ by adding the ray generated by $e_{j_1} - e_{i_1}$. Let $\Sigma_1$ be the resulting fan. Now we subdivide the cone of $\Sigma_1$ corresponding to the set generated by $\{\{i_2, i_2 + 1\}, \ldots, \{j_2 - 1, j_2\}\}$ by adding the ray generated by $e_{j_2} - e_{i_2}$. Continuing this process until the last edge $\{i_{k-1}, j_{k-1}\}$, we get a fan $\Sigma_D$. Note that the vector $e_{j_\ell} - e_{i_\ell}$ is the sum of the primitive ray generators corresponding to $\{i_\ell, i_\ell + 1\}, \ldots, \{j_\ell - 1, j_\ell\}$, and $\Sigma_D$ is also smooth.

**Definition 3.2.** For a dissection $D$, we say that the toric variety $X_D$ associated with the fan $\Sigma_D$ is of Schröder type.

Note that $\Sigma_D$ is also polytopal since $\Sigma_0$ is polytopal. Hence there is a lattice polytope $P_D$ whose normal fan is $\Sigma_D$. We can get the polytope $P_D$ by truncating faces of the simplex $\Delta^n$. We denote the facets of $\Delta^n$ by $F_{i,i+1}$ for $i = 0, 1, \ldots, n$. Then the polytope $P_D$ is obtained from $\Delta^n$ by truncating the face $F_{i_\ell, i_{\ell+1}} \cap \cdots \cap F_{j_{\ell-1}, j_\ell}$ repeatedly, where $\ell$ starts from 1 and ends at $k - 1$. We denote by $F_{i_\ell,k}$ the new facet obtained from the truncation of the face $F_{i_\ell, i_{\ell+1}} \cap \cdots \cap F_{j_{\ell-1}, j_\ell}$. See Figure 3.

**Lemma 3.3.** For a dissection $D$ of $P_{n+2}$, the polytope $P_D$ is combinatorially equivalent to a product of simplices.
Proof. Let $D$ be a $k$-dissection of $P_{n+2}$ and $E(D) = \cup_{i=1}^{k} E(P^{(i)})$. If $k = 1$, then every proper subset of $E(P_{n+2})$ corresponds to a face of $\Delta^n$. When $k = 2$, assume

\[
E(P^{(1)}) = \{0, 1\}, \ldots, \{i_1 - 1, i_1\}, \{i_1, j_1\}, \{j_1, j_1 + 1\}, \ldots, \{n, n + 1\},
\]

\[
E(P^{(2)}) = \{i_1, i_1 + 1\}, \ldots, \{j_1 - 1, j_1\}.
\]

Then every proper subset of $E(D)$ forms a cone in $\Sigma_D$ unless it is $E(P^{(1)})$ or $E(P^{(2)})$. Hence every proper subset of $E(D)$ corresponds to a face of $P_D$ except $E(P^{(1)})$ and $E(P^{(2)})$. Thus $P_D = \Delta^{[E(P^{(1)})]-1} \times \Delta^{[E(P^{(2)})]-1}$. In general, for arbitrary $k$, a subset $S$ of $E(D)$ forms a cone in $\Sigma_D$ if and only if $S$ does not contain $E(P^{(i)})$ for any $i \in [k]$. Thus the polytope $P_D$ is combinatorially equivalent to the product of simplices $\prod_{i=1}^{k} \Delta^{[E(P^{(i)})]-1}$.

Since every smooth projective toric variety over a product of simplices is a generalized Bott manifold ([8, Theorem 6.4]), every toric variety of Schröder type is a generalized Bott manifold. More precisely, for a $k$-dissection of the polygon $P_{n+2}$, the toric variety $X_D$ is a $k$-stage generalized Bott manifold of dimension $n$. If $k = n$, we get an $n$-stage Bott manifold.

Note that the primitive integral generator of the ray corresponding to $\{i, j\}$, $i < j$, is $e_j - e_i$, and we denote it by $u_{ij}$.

**Theorem 3.34.** Every toric variety of Schröder type is a Fano generalized Bott manifold.

Proof. Let $D$ be a $k$-dissection of $P_{n+2}$ and $E(D) = \cup_{i=1}^{k} E(P^{(i)})$. Lemma 3.3 together with [8, Theorem 6.4] implies that $X_D$ is a $k$-stage generalized Bott manifold. Hence it is enough to show that $X_D$ is Fano.

From the proof of Lemma 3.3, the primitive collections of $\Sigma_D$ correspond to the edge sets $E(P^{(i)})$. Let $E(P^{(i)}) = \{i_1, i_2\}, \ldots, \{i_{\ell - 1}, i_{\ell}\}$, where $i_1 < i_2 < \cdots < i_{\ell}$. Then the associated primitive relation is

\[
u_{i_1 i_2} + \cdots + \nu_{i_{\ell - 1} i_{\ell}} = (e_{i_2} - e_{i_1}) + \cdots + (e_{i_{\ell}} - e_{i_{\ell - 1}}) = e_{i_{\ell}} - e_{i_1}.
\]
Note that \( \{i_1, i_\ell\} \) is the distinguished edge of \( P^{(i)} \) in \( D \). Hence the associated primitive relation is

\[
(3.2) \quad u_{i_1i_2} + \cdots + u_{i_{\ell-1}i_\ell} = \begin{cases} 
0 & \text{if } i_1 = 0 \text{ and } i_\ell = n + 1, \\
u_{i_{i_1i_\ell}} & \text{otherwise.}
\end{cases}
\]

Note that \( \{i_1, i_\ell\} \) is an edge in \( E(P^{(j)}) \) for some \( j \neq i \). Hence every primitive collection of \( \Sigma_D \) has a positive degree. By Proposition 2.3, we conclude that the toric variety \( X_D \) is a Fano generalized Bott manifold.

It should be noted that the primitive relations in (3.2) define the Schröder tree \( \tau(D) \) as follows:

1. the vertices \( v \) such that \( \phi(v) \in \{i_1i_2, \ldots, i_{\ell-1}i_\ell\} \) have the same parent, and
2. if \( u_{i_1i_2} + \cdots + u_{i_{\ell-1}i_\ell} = 0 \), then the parent is the root of \( \tau(D) \); otherwise, the parent is the vertex \( v_{i_1i_\ell} \).

By applying Proposition 2.4 to the toric varieties \( X_D \), we get the following.

**Theorem 3.5.** The toric varieties \( X_D \) and \( X_D^\sim \) of Schröder type are isomorphic as varieties if and only if \( \tau(D) \) and \( \tau(D) \) are isomorphic as unordered rooted trees.

Let

\[
\mathcal{SR}(n, k) = \{ \text{Schröder trees with } k \text{ internal vertices and } n \text{ leaves} \} / \sim,
\]

\[
\mathcal{SR}_n = \bigcup_{k=0}^{n-1} \mathcal{SR}(n, k)
\]

where two Schröder trees are equivalent if they are the same as unordered rooted trees. Theorem 3.5 says that \( |\mathcal{SR}_{n+1}| \) is the number of isomorphism classes of \( n \)-dimensional toric varieties of Schröder type.

Riordan studied on the numbers \( |\mathcal{SR}(n, k)| \) and \( |\mathcal{SR}_n| \), and he used the term series-reduced in [19]. That is, each element of \( \mathcal{SR}_n \) is called a series-reduced rooted tree. Let

\[
s_1(y) := 1 \quad \text{and} \quad s_n(y) := \sum_{k \geq 1} s(n, k) y^k \quad (n \geq 2),
\]

where \( s(n, k) \) denotes the size of \( \mathcal{SR}(n, k) \). A recurrence relation for \( s_n(y) \) is given as follows.

**Proposition 3.6 ([19, Equation (11)])**. For \( n \geq 2 \),

\[
n(1 + y)s_n(y) = y s_n(y) - s_{n-1}(y) + (1 + y) \sum_{i=1}^{n-1} s_i(y) s_{n-i}(y),
\]

where

\[
s_1(y) := 1 \quad \text{and} \quad s_n(y) := \sum_d d s_d(y^{n/d})
\]

with the sum over all divisors of \( n \).

The numbers \( s(n, k) \) and \( s_n(1) \) for \( n \leq 10 \) are given in Table 1. Clearly, \( s(n, 1) = 1 \) and \( s(n, 2) = n - 2 \), and it is known that \( s(n, 3) = \frac{(n-3)(n-2)}{2} \). However, for general \( k \), the explicit formula for \( s(n, k) \) is unknown. See sequences A106179 and A000699 in OEIS [21] for more numbers \( s(n, k) \) and \( s_n(1) \), respectively. Note that \( s_n(1) \) is the total number of series-reduced rooted trees with \( n \) leaves.
Corollary 3.7. The number of isomorphism classes of \(n\)-dimensional toric varieties of Schröder type is equal to \(s_{n+1}(1)\), the number of series-reduced rooted trees with \(n+1\) leaves.

Example 3.8. Figure 4 shows the equivalent classes of Schröder trees with four leaves and the corresponding dissections of \(P_5\). This agrees that \(s(4,1) = 1\), \(s(4,2) = 2\), and \(s(4,3) = 2\) so that \(s_4(y) = y + 2y^2 + 2y^3\) as in Table 1. By using this and Theorem 4.1, we will see that the cohomology ring of \(X_D\) for a dissection \(D\) of \(P_5\) can be written as one of the following forms:

\[
\begin{align*}
\mathbb{Z}[x_1]/(x_1^4), & \quad \mathbb{Z}[x_1, x_2]/(x_1^2, x_2(x_1 + x_2)^2), & \quad \mathbb{Z}[x_1, x_2]/(x_1^3, x_2(x_1 + x_2)), \\
\mathbb{Z}[x_1, x_2, x_3]/(x_1^2, x_2(x_1 + x_2), x_3(x_1 + x_2 + x_3)), & \quad \mathbb{Z}[x_1, x_2, x_3]/(x_1^3, x_2^2, x_3(-x_1 + x_2 + x_3)).
\end{align*}
\]

Remark 3.9. If a dissection \(D\) is a triangulation, then \(X_D\) becomes a toric variety of Catalan type in \([15]\) and the associated Schröder tree \(\tau(D)\) is a full binary tree. The number of isomorphism classes of toric varieties of Catalan type is \(s(n+1, n)\), which is known as the Wedderburn-Etherington number \(b_{n+1}\).

Note that not every Fano generalized Bott manifold is associated with a polygon dissection. For instance, the total space of the projective bundle \(P(\mathbb{C}^2 \oplus \mathbb{C}\gamma)\) over \(\mathbb{C}P^3\) is a Fano generalized Bott manifold, but there is no polygon dissection associated with it, where \(\gamma\) is the tautological line bundle over \(\mathbb{C}P^3\).

4. Cohomology ring of the toric variety from a polygon dissection

In this section, we describe the cohomology ring of \(X_D\) using the Schröder tree \(\tau(D)\).

Let \(v\) be a vertex of \(\tau(D)\) with \(\phi(v) = \{a, b\}\). Then we denote by \(S(v)\) the set of descendants of \(v\) whose label is \(\{\bullet, b\}\). For instance, if \(v\) is the vertex labeled by \(\{0, 7\}\) in Figure 2, then \(S(v) = \{u \mid \phi(u) = \{3, 7\}\} \cup \{6, 7\}\}. Note that \(S(v)\) is empty for a leaf \(v\) of \(\tau(D)\).

Theorem 4.1. Given a \(k\)-dissection \(D\) of a polygon, consider the corresponding Schröder tree \(\tau(D)\). For \(1 \leq i \leq k\), let \(v_i\) be the \(i\)th internal vertex in the preorder

| \(k\backslash n\) | 1| 2| 3| 4| 5| 6| 7| 8| 9| 10 | \ldots |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1| 1| 1| 1| 1| 1| 1| 1| 1| 1  | \ldots |
| 2 | 1| 2| 3| 4| 5| 6| 7| 8| 9| 10  | \ldots |
| 3 | 2| 5| 10| 16| 24| 33| 44 | 57| 71| 115 | \ldots |
| 4 | 3| 12| 29| 57| 99| 157|  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |
| 5 | 6| 28| 84| 192| 382|  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |
| 6 | 11| 66| 231|  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |
| 7 | 23| 157| 634|  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |
| 8 | 46| 373|  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |
| 9 |  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |
| 10 |  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |  \ldots |

\(s_{n}(1)\)

Table 1. The numbers \(s(n, k)\) and \(s_{n}(1)\) for \(1 \leq n \leq 10\)
listing of $\tau(D)$. For each $i$, suppose that $v_i$ has $\ell_i$ children $w_{i1}, w_{i2}, \ldots, w_{i\ell_i}$ from left to right, and $\phi(w_{i\ell_i}) = \{a_i, b_i\}$ with $a_i < b_i$. Then the cohomology ring of $X_D$ can be written as

$$H^*(X_D) = \mathbb{Z}[x_{a_1 b_1}, x_{a_2 b_2}, \ldots, x_{a_k b_k}] / \langle p_1, \ldots, p_k \rangle,$$

where

$$p_i := x_{a_i b_i} \prod_{j=1}^{\ell_i-1} \left( - \sum_{u \in S(w_{ij})} x_{\phi(u)} + \sum_{u \in S(v_i)} x_{\phi(u)} \right).$$

Proof. Let $D$ be a $k$-dissection of $P_{n+2}$ and $E(D) = \bigcup_{i=1}^k E(P^{(i)})$. Let $E(P^{(i)}) = \{i_1 i_2, i_2 i_3, \ldots, i_{\ell-1} i_\ell\}$, where $i_1 < i_2 < \cdots < i_\ell$. Then there are $k$ generators of first type of $I$ in Theorem 2.2:

$$x_{i_1 i_2} \cdots x_{i_{\ell-1} i_\ell} x_{i_1 i_\ell},$$

corresponding to $E(P^{(i)})$ for $i = 1, \ldots, k$. For each $i \in [n]$, we have the linear relation $q_i = 0$, where

$$q_i := \sum_{(j, i) \in E(P_{n+2})} x_{ji} - \sum_{(i, j) \in E(P_{n+2})} x_{ij},$$

which are generators of the second type of $I$ in Theorem 2.2. For simplicity, for $e \in E(P_{n+2})$, let $v_e$ be the vertex in $\tau(D)$ satisfying $\phi(v_e) = e$ and let $r_e$ be the parent of $v_e$. We need to show that for $e \in E(P_{n+2})$, if $v_e$ is not the right-most child of $r_e$, then

$$x_e = - \sum_{u \in S(v_e)} x_{\phi(u)} + \sum_{u \in S(r_e)} x_{\phi(u)},$$

and $\phi(u) \in \{a_1 b_1, a_2 b_2, \ldots, a_k b_k\}$ for all $u \in S(v_e) \cup S(r_e)$. Suppose that $\phi(v_e) = bc$ and $\phi(r_e) = ad$. By the construction of labeling, it follows that $a \leq b < c \leq d$. In
Theorem 4.1. The cohomology ring of the toric variety $X_D$ is

$$H^*(X_D) = \mathbb{Z}[x_{23}, x_{37}, x_{67}, x_{89}] / \langle x_{23}, x_{37}(x_{23} + x_{37} + x_{67}), x_{67}, x_{89}(x_{37} - x_{67} + x_{89}) \rangle.$$ 

The following is a direct consequence of Theorem 4.1.

Corollary 4.3. Let $D$ be a dissection of a polygon and let $\tau(D)$ be its associated Schröder tree. For a vertex $w$ with $\phi(w) = \{a_i, b_i\}$ in Theorem 4.1, if the vertex $w$ is a leaf of $\tau(D)$ and $v$ is the parent of $w$, then $x_{a_i b_i} = 0$ in $H^*(X_D)$, where $t_i$ is the number of children of $v$.

Note that the combinatorial type of the associated lattice polytope with a generalized Bott manifold is determined by the cohomology ring. That is, for two generalized Bott manifolds $B$ and $B'$, if the lattice polytope associated with $B$ is combinatorially equivalent to $\prod_{i=1}^{m} \Delta^{n_i}$ and $H^*(B) \cong H^*(B')$, then the lattice polytope associated with $B'$ is also combinatorially equivalent to $\prod_{i=1}^{m} \Delta^{n_i}$. See [10, Theorem 5.3] for more details. Therefore, for two dissections $D$ and $\tilde{D}$, if $H^*(X_D) \cong H^*(X_{\tilde{D}})$, then $P_D$ and $P_{\tilde{D}}$ are combinatorially equivalent, so we get the following.

Example 4.2. Let $D$ be the 4-dissection of $P_{10}$ in Figure 1. Then $\tau(D)$ with the label $\phi(v)$ for $v \in V(\tau(D))$ is given in Figure 2. Let us compute the cohomology ring $H^*(X_D)$ using Theorem 4.1. Then

- $x_{01} = x_{12} = x_{23},$
- $x_{34} = x_{45} = x_{56} = x_{67},$
- $x_{03} = -x_{23} + x_{37} + x_{67},$
- $x_{07} = -x_{37} - x_{67} + x_{89},$
- $x_{78} = x_{89}.$

Hence we get

$$x_{01} x_{12} x_{23} = x_{23}^3,$$
$$x_{34} x_{45} x_{56} x_{67} = x_{67}^4,$$
$$x_{03} x_{37} = x_{37} (x_{23} + x_{37} + x_{67}),$$
$$x_{07} x_{78} x_{89} = x_{89}^2 (x_{37} - x_{67} + x_{89}).$$

Therefore, the cohomology ring of $X_D$ is

$$H^*(X_D) = \mathbb{Z}[x_{23}, x_{37}, x_{67}, x_{89}] / \langle x_{23}, x_{37}(x_{23} + x_{37} + x_{67}), x_{67}, x_{89}(x_{37} - x_{67} + x_{89}) \rangle.$$

The following is a direct consequence of Theorem 4.1.
Corollary 4.4. Let $D$ and $\tilde{D}$ be $k$-dissection and $\tilde{k}$-dissection, respectively, if $H^*(X_D) \cong H^*(X_{\tilde{D}})$, then $k = \tilde{k}$ and the sets $\{|E(P^{(i)})| \mid i = 1, \ldots, k\}$ and $\{|E(\tilde{P}^{(i)})| \mid i = 1, \ldots, \tilde{k}\}$ are the same as multi-sets.

5. Proof of Theorem 1.3

In this section, we prove Theorem 1.3. When $k = n$, it is proved in [15]. If $k \leq 2$, then it is clear from Theorem 3.4 and Corollary 4.4. We prove when $k = 3$ by dividing three cases. We first divide the three-dissections into two types and show that a cohomology ring determines the type of a three-dissection. After that, we show that for each type, toric varieties associated with three-dissections are classified by cohomology rings up to variety isomorphism.

Let $D$ be a three-dissection of a polygon and let $m_1, m_2,$ and $m_3$ be the out-degrees of the internal vertices of the Schröder tree $\tau(D)$. It suffices to consider the two types of Schröder trees with three internal vertices as in Figure 5. Note that $m_i \geq 2$ for all $i = 1, 2, 3$ since $\tau(D)$ is a Schröder tree. By abuse of notation, we use the same letter $D$ for the Schröder tree $\tau(D)$ and we say that $D$ is of the first (respectively, second) type if its corresponding Schröder path is of the first (respectively, second) type.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{Two types of Schröder trees $D^I$ and $D^H$.}
\end{figure}

For a three dissection of each type, we let
\begin{align*}
I_{D^I} &= \langle x_1^{m_1}, x_2^{m_2}, x_3(-x_1 + x_2 + x_3)(x_2 + x_3)^{m_3-2} \rangle, \\
I_{D^H} &= \langle x_1^{m_1}, x_2(x_1 + x_2)^{m_2-1}, x_3(x_1 + x_2 + x_3)^{m_3-1} \rangle.
\end{align*}

Then from Theorem 4.1, we have
\begin{align*}
H^*(X_{D^I}) &= \mathbb{Z}[x_1, x_2, x_3]/I_{D^I}, \quad \text{and} \quad H^*(X_{D^H}) = \mathbb{Z}[x_1, x_2, x_3]/I_{D^H}.
\end{align*}

Let $D$ and $\tilde{D}$ be three-dissections of a polygon. For $i = 1, 2, 3$, let $m_i$ (respectively, $\tilde{m}_i$) denote the out-degrees of the internal vertices in $D$ (respectively, $\tilde{D}$). For $m = \min\{m_1, m_2, m_3\}$, let $n(D)$ denote the number of independent linear terms $\alpha$ in $H^*(X_D)$ such that $\alpha^n = 0$. Note that if $H^*(X_D) \cong H^*(X_{\tilde{D}})$, then $n(D) = n(\tilde{D})$, and the multi-sets $\{m_1, m_2, m_3\}$ and $\{\tilde{m}_1, \tilde{m}_2, \tilde{m}_3\}$ are the same by Corollary 4.4.

We first prove that if $H^*(X_D) \cong H^*(X_{\tilde{D}})$, then $D$ and $\tilde{D}$ are of the same type. To prove this, we prepare three lemmas about the cohomology of the toric varieties associated with the second type three-dissections.
Lemma 5.1. Let $D = D^H(m_1, m_2, m_3)$ and let $m := \min\{m_2, m_3\}$. If $m < m_1$, then there is no nonzero element in $H^2(X_D)$ whose $m$th power vanishes in $H^*(X_D)$.

Proof. Suppose that there exists a nonzero element $px_1 + qx_2 +(rx_3$ in $H^2(X_D)$ such that $(px_1 + qx_2 + rx_3)^m = 0$ in $H^*(X_D)$. Since $m < m_1$, we have

$$ (px_1 + qx_2 + rx_3)^m = \alpha x_2(x_1 + x_2)^{m_2 - 1} + \beta x_3(x_1 + x_2 + x_3)^{m_3 - 1} $$

as polynomials. In any case of $m = m_2$ and $m = m_3$, comparing the coefficients of $x_i^m$ in both sides of \((5.1)\), we get $p = 0$. Since $m_2, m_3 \geq 2$ it follows that $\alpha = \beta = 0$, which is a contradiction to $px_1 + qx_2 + rx_3 \neq 0$.

Lemma 5.2. Let $D = D^H(m_1, m_2, m_3)$. If $m_1 = 2$, then $\{(m_2 - 1)x_1 + m_2 x_2\}^{m_2} = 0$ in $H^*(X_D)$.

Proof. Since the coefficient of $x_1 x_2^{m_2 - 1}$ in $\{(m_2 - 1)x_1 + m_2 x_2\}^{m_2}$ is

$$ m_2 (m_2 - 1) m_2^{m_2 - 1} = m_2^{m_2} (m_2 - 1), $$

there exists a polynomial $\alpha$ such that

$$ \{(m_2 - 1)x_1 + m_2 x_2\}^{m_2} = \alpha x_2^2 + m_2^{m_2} x_2 (x_1 + x_2)^{m_2 - 1}. $$

This proves the lemma.

Lemma 5.3. Let $D = D^H(m_1, m_2, m_3)$. If $m_1 > 2$, then there is no element $px_1 + qx_2 + rx_3 \in H^2(X_D)$ with $(q, r) \neq (0, 0)$ such that $(px_1 + qx_2 + rx_3)^{m_2}$ or $(px_1 + qx_2 + rx_3)^{m_3}$ vanishes in $H^*(X_D)$.

Proof. By Lemma 5.1, it is enough to check when $m_1 \leq m_2, m_3$. Suppose that

$$ (px_1 + qx_2 + rx_3)^{m_2} = 0 \text{ in } H^*(X_D), \text{ where } (q, r) \neq (0, 0). $$

Then we have

$$ (px_1 + qx_2 + rx_3)^{m_2} = \alpha x_1^{m_1} + q^{m_2} x_2 (x_1 + x_2)^{m_2 - 1} + \gamma x_3 (x_1 + x_2 + x_3)^{m_2 - 2} $$

as polynomials. Here, we may assume that $q \neq 0$. Indeed, if $q = 0$, then $\gamma = 0$, so $r = 0$. This is a contradiction to $(q, r) \neq (0, 0)$. Since $m_1 > 2$, by comparing the coefficients of $x_1^{m_2 - 1} x_2$ and $x_1^{m_2 - 2} x_2^2$ in the above, respectively, we get

$$ m_2 p q^{m_2 - 1} = q^{m_2} (m_2 - 1), $$

$$ \left( \begin{array}{c} m_2 \\ 2 \end{array} \right) p^2 q^{m_2 - 2} = q^{m_2} \left( \frac{m_2 - 1}{2} \right). $$

Hence we have

$$ m_2 p = q (m_2 - 1) \quad \text{and} \quad m_2 p^2 = q^2 (m_2 - 2). $$

Thus $(m_2 - 1) p = q (m_2 - 2)$. There is no integer $m_2$ satisfying both $m_2 p = q (m_2 - 1)$ and $(m_2 - 1) p = q (m_2 - 2)$. Hence $(px_1 + qx_2 + rx_3)^{m_2} \neq 0$ in $H^*(X_D)$. From a similar computation, one can show that $(px_1 + qx_2 + rx_3)^{m_3} \neq 0$ if $(p, q) \neq (0, 0)$.

Now, we are ready to prove that the cohomology ring $H^*(X_D)$ determines the type of $D$.

Proposition 5.4. Let $D = D^I(m_1, m_2, m_3)$ and $\tilde{D} = D^H(m_1, m_2, \tilde{m}_3)$. Then $H^*(X_D)$ and $H^*(X_{\tilde{D}})$ cannot be isomorphic as graded rings.
Proposition 5.6. Let \( \phi(x_1), \phi(x_2), \phi(x_3) \) be \( (\phi) \) and \( \alpha \). It follows from Lemma 5.3 that \( \tilde{m}_1 = 1 \). We may further assume that \( m_1 = 2 \). Then \( m_2 = \tilde{m}_2 \) and the isomorphism \( \phi \) should send the set \( \{ \pm x_1, \pm x_2 \} \) to \( \{ \pm y_1, \pm (m_2 - 1)y_1 + m_2 y_2 \} \). Then \( |\det G| = |y_{13}|m_2 \). Since \( m_2 \geq 2 \), we get \( |\det G| \neq 1 \), which is a contradiction.

Let us consider the three-dissections of the first type. We prepare one lemma about the cohomology of the toric varieties of the first type dissections.

Lemma 5.5. Let \( D = D^I(m_1, m_2, m_3) \). There is no element \( px_1 + qx_2 + rx_3 \in H^2(X_D) \) with \( r \neq 0 \) such that \( (px_1 + qx_2 + rx_3)^m = 0 \) in \( H^*(X_D) \).

Proof. Suppose that \( (px_1 + qx_2 + rx_3)^m \neq 0 \) in \( H^*(X_D) \) and \( r \neq 0 \). Then \( (px_1 + qx_2 + rx_3)^m = \alpha x_1^m + \beta x_2^m + \gamma x_3^m(-x_1 + x_2 + x_3) \) as polynomials, where \( \alpha \) and \( \beta \) are homogeneous polynomials of degree \( m_3 - 1 \) and \( m_3 - 2 \), respectively. Note that if \( m_3 - m_1 \) and \( m_3 - m_2 \) are negative, then \( \alpha \) and \( \beta \) are zero, respectively. Since we assume \( r \neq 0 \), neither \( p \) nor \( q \) is zero. However, this is impossible. If \( m_1 > 2 \), then we get \( p = 0 \) by comparing the coefficients of \( x_1 x_3^{m_1 - 1} \) and \( x_3^{m_3 - 2} \) in (5.2). If \( m_2 > 2 \), then we get \( q = 0 \) by comparing the coefficients of \( x_3 x_3^{m_2 - 1} \) and \( x_3^{m_3 - 2} \) in (5.2). If \( m_1 = m_2 = 2 \), then by comparing the coefficients of \( x_1 x_3^{m_1 - 1} \), \( x_2 x_3^{m_2 - 1} \), \( x_1 x_3^{m_3 - 2} \), \( x_2 x_3^{m_3 - 2} \), \( m_3 p = -r \), and \( m_3 q = r(m_3 - 1) \), and \( m_3(m_3 - 1)pq = -r^2(m_3 - 2) \). After doing some calculation, we get \( q = 0 \).

Proposition 5.6. Let \( D = D^I(m_1, m_2, m_3) \) and \( \tilde{D} = D^J(m_1, m_2, \tilde{m}_3) \). If \( H^*(X_D) \cong H^*(X_{\tilde{D}}) \), then \( D = \tilde{D} \).

Proof. We may assume that \( m_1 \leq m_2 \) and \( \tilde{m}_1 \leq \tilde{m}_2 \). From Theorem 4.1, we have
\[
H^*(X_D) = \mathbb{Z}[x_1, x_2, x_3]/\langle x_1^{m_1}, x_2^{m_2}, x_3(-x_1 + x_2 + x_3)(x_2 + x_3)^{m_3 - 2} \rangle,
\]
\[
H^*(X_{\tilde{D}}) = \mathbb{Z}[y_1, y_2, y_3]/\langle y_1^{\tilde{m}_1}, y_2^{\tilde{m}_2}, y_3(-y_1 + y_2 + y_3)(y_2 + y_3)^{\tilde{m}_3 - 2} \rangle.
\]

By assuming \( (m_1, m_2, m_3) \neq (\tilde{m}_1, \tilde{m}_2, \tilde{m}_3) \), we show that \( H^*(X_D) \ncong H^*(X_{\tilde{D}}) \).

We consider two cases according to the set of out-degrees.

Case 1. \( \{ m_1, m_2, m_3 \} = \{ m_1, \tilde{m}_2, \tilde{m}_3 \} = \{ a, b \} \) with \( a \neq b \)

We may assume that \( m_3 = b \) and \( \tilde{m}_3 = a \). Then \( n(\tilde{D}) = 1 \) while \( n(D) = 2 \) if \( a < b \), and \( n(D) = 0 \) otherwise. It follows that \( H^*(X_D) \ncong H^*(X_{\tilde{D}}) \).

Case 2. \( \{ m_1, m_2, m_3 \} = \{ \tilde{m}_1, \tilde{m}_2, \tilde{m}_3 \} = \{ a, b, c \} \) with \( a < b < c \)

In this case, \( (m_1, m_2) \) is \( (a, b) \), \( (a, c) \), or \( (b, c) \). Hence \( m_1 = \tilde{m}_1 \) if and only if \( n(D) = n(\tilde{D}) \). Thus it suffices to check when \( m_3 = b \) and \( \tilde{m}_3 = c \). Then \( H^*(X_{\tilde{D}}) \) has a linear term \( \alpha \) such that \( \alpha^a = 0 \), but \( H^*(X_D) \) has no such term by Lemma 5.5. It follows that \( H^*(X_D) \ncong H^*(X_{\tilde{D}}) \).

In any case, we conclude that \( H^*(X_D) \ncong H^*(X_{\tilde{D}}) \) as desired.

Finally, we consider the three-dissections of the second type.

Proposition 5.7. Let \( D = D^H(m_1, m_2, m_3) \) and \( \tilde{D} = D^H(\tilde{m}_1, \tilde{m}_2, \tilde{m}_3) \). If \( H^*(X_D) \cong H^*(X_{\tilde{D}}) \), then \( D = \tilde{D} \).
Proof. If $H^*(X_D) \cong H^*(X_D)$, there exists a graded isomorphism $\varphi$ from $\mathbb{Z}[x_1, x_2, x_3]$ to $\mathbb{Z}[y_1, y_2, y_3]$ such that $I_D$ sends to $I_D$. Then there is a matrix $G = [g_{i,j}] \in \text{GL}_3(\mathbb{Z})$ satisfying that $\det G = 1$ and $[\varphi(x_1), \varphi(x_2), \varphi(x_3)]^t = G[y_1, y_2, y_3]^t$. In the following, we show that the assumptions $(m_1, m_2, m_3) \neq (\tilde{m}_1, \tilde{m}_2, \tilde{m}_3)$ and $H^*(X_D) \cong H^*(X_D)$ lead to a contradiction to the existence of a matrix $G$.

Now we consider two cases according to the set of out-degrees.

Case 1. $\{m_1, m_2, m_3\} = \{\tilde{m}_1, \tilde{m}_2, \tilde{m}_3\} = \{a, a, b\}$ with $a \neq b$

In order to have that $n(D) = n(D)$, we set $m_1 = \tilde{m}_1$. We may assume that $(m_1, m_2, m_3) = (a, b, a)$ and $(\tilde{m}_1, \tilde{m}_2, \tilde{m}_3) = (a, a, b)$. Then we have

$$I_D = \langle x_1^a, x_2(x_1 + x_2)^{b-1}, x_3(x_1 + x_2 + x_3)^{a-1} \rangle,$$

$$I_D = \langle y_1^a, y_2(y_1 + y_2)^{a-1}, y_3(y_1 + y_2 + y_3)^{b-1} \rangle.$$

We divide two sub-cases according to the size of $a$ and $b$.

Case 1-1. ($a < b$)

Since $a < b$, $\varphi(x_1^a) = \pm y_1^a$. It follows that

$$g_{12} = g_{13} = 0, \quad g_{11} = 1, \quad g_{22}g_{33} - g_{23}g_{32} = 1.$$

Since $\varphi(x_3(x_1 + x_2 + x_3)^{a-1})$ and $\varphi(x_2(x_1 + x_2)^{b-1})$ belong to $I_D$, it follows from (5.3) that

$$(5.5) \quad (g_{31}y_1 + g_{32}y_2 + g_{33}y_3)(\langle g_{11} + g_{21} + g_{31} \rangle y_1 + \langle g_{22} + g_{32} \rangle y_2 + \langle g_{23} + g_{33} \rangle y_3)^{a-1}$$

$$= g_{31}(g_{11} + g_{21} + g_{31})^{a-1}y_1^a + g_{32}(g_{22} + g_{32})^{a-1}y_2(y_1 + y_2)^{a-1}$$

and

$$(5.6) \quad (g_{21}y_1 + g_{22}y_2 + g_{23}y_3)(\langle g_{11} + g_{21} \rangle y_1 + g_{22}y_2 + g_{23}y_3)^{b-1}$$

$$= \alpha y_1^a + \beta y_2(y_1 + y_2)^{a-1} + g_{23}y_3(y_1 + y_2 + y_3)^{b-1}$$

as polynomials, where $\alpha$ and $\beta$ are homogeneous polynomials of degree $b - a$. Comparing the coefficients of $y_3^2$ in (5.5), we have $g_{33}(g_{23} + g_{33}) = 0$. From this fact together with (5.4), it follows that

$$|g_{23}| = 1.$$

Comparing the coefficients of $y_2y_3^{b-1}$ in both sides of (5.6), we have $bg_{22} = (b-1)g_{23}$. By (5.7), we get $|g_{22}| = (b - 1)/b$, which contradicts to $g_{22} \in \mathbb{Z}$. Therefore, we conclude that there is no such $G$.

Case 1-2. ($b < a$)

Since $\varphi(x_1^a)$ belongs to $I_D$, we have

$$(5.8) \quad (g_{11}y_1 + g_{12}y_2 + g_{13}y_3)^a = g_{11}^ay_1^a + g_{12}y_2^a(y_1 + y_2)^{a-1} + \gamma_3y_3(y_1 + y_2 + y_3)^{b-1},$$

where $\gamma$ is a homogeneous polynomial of degree $a - b$. Comparing the coefficients of $y_1y_2^{a-1}$ and $y_1^{a-1}y_2$ in both sides of (5.8), we obtain

$$a(g_{11}g_{12}^a - (a - 1)g_{11}^a), \quad a(g_{11}^{-1}g_{12} = g_{12}^a).$$

From (5.9), we get $g_{12} = 0$ or $ag_{11} = (a - 1)g_{12}$. However, if $g_{12} \neq 0$, then $g_{11} = g_{12}$ yields that $a = 2$ by (5.10), which contradicts to $a > 2$. Thus, we have
$g_{12} = 0$. Applying this to (5.8), we get $\gamma = 0$ since there is no term containing $y_2$ on the left-hand side of (5.8). Thus $g_{13} = 0$. Then $g_{11} \neq 0$ since $\det G \neq 0$. Since $g_{12} = g_{13} = 0$ and $\phi(x_2(x_1 + x_2)^b) \in I_D$, we have

$$
(g_{21} y_1 + g_{22} y_2 + g_{23} y_3) \cdot ((g_{11} + g_{21}) y_1 + g_{22} y_2 + g_{23} y_3)^{b-1} = g_{23} y_3(y_1 + y_2 + y_3)^{b-1}.
$$

(5.11)

Comparing the coefficients of $y_2^b$ in both sides of (5.11), we get $g_{22} = 0$. Then there is no term containing $y_2$ on the left side of (5.11), so $g_{23} = 0$. This is a contradiction to $\det G \neq 0$.

**Case 2.** $\{(m_1, m_2, m_3) = \{m_1, m_2, m_3\} = \{a, b, c\}$ with $a < b < c$

We consider three sub-cases.

**Case 2-1.** $(m_1, m_2, m_3) = (a, b, c)$ and $(\bar{m}_1, \bar{m}_2, \bar{m}_3) = (a, c, b)$

In this case we have

$$H^*(X_D) = \mathbb{Z}[x_1, x_2, x_3]/\langle x_1^b, x_2(x_1 + x_2)^b - 1, x_3(x_1 + x_2 + x_3)^{c-1} \rangle,$$

$$H^*(X_{\bar{D}}) = \mathbb{Z}[y_1, y_2, y_3]/\langle y_1^b, y_2 (y_1 + y_2)^{c-1}, y_3 (y_1 + y_2 + y_3)^{b-1} \rangle.$$  

Since $a < b < c$, we have $\phi(x_2^b) = \pm \gamma y_1^b$, so $|g_{11}| = 1$ and $g_{12} = g_{13} = 0$. Since $\phi(x_2(x_1 + x_2)^b) \in I_D$, we have

$$
(g_{21} y_1 + g_{22} y_2 + g_{23} y_3) \cdot ((g_{11} + g_{21}) y_1 + g_{22} y_2 + g_{23} y_3)^{b-1} = \alpha y_1^b + g_{23} y_3(y_1 + y_2 + y_3)^{b-1},
$$

where $\alpha$ is a homogeneous polynomial of degree $b - a$. Comparing the coefficients of $y_2^b$, we have $g_{22} = 0$. However, this yields that $g_{23} = 0$ so that $\det G = 0$, which is a contraction. There is no such $G$.

**Case 2-2.** $(m_1, m_2, m_3) = (b, a, c)$ and $(\bar{m}_1, \bar{m}_2, \bar{m}_3) = (b, c, a)$

In this case, we have

$$H^*(X_D) = \mathbb{Z}[x_1, x_2, x_3]/\langle x_1^b, x_2(x_1 + x_2)^b - 1, x_3(x_1 + x_2 + x_3)^{c-1} \rangle,$$

$$H^*(X_{\bar{D}}) = \mathbb{Z}[y_1, y_2, y_3]/\langle y_1^b, y_2 (y_1 + y_2)^{c-1}, y_3 (y_1 + y_2 + y_3)^{a-1} \rangle.$$  

Since $\phi(x_2^b) \in I_D$, we have

$$
(g_{11} y_1 + g_{12} y_2 + g_{13} y_3)^b = g_{11} y_1^b + \gamma y_3(y_1 + y_2 + y_3)^{a-1},
$$

where $\gamma$ is a homogeneous polynomial of degree $b - a$. Comparing the coefficients of $y_2^b$ in both sides of (5.12), we get $g_{12} = 0$. Then there is no term having $y_2$ on the left side of (5.12), so $\gamma$ should be zero. Thus $|g_{11}| = 1$ and $g_{13} = 0$. Since $\phi(x_2(x_1 + x_2)^a) \in I_D$, we have

$$
(g_{21} y_1 + g_{22} y_2 + g_{23} y_3) \cdot ((g_{11} + g_{21}) y_1 + g_{22} y_2 + g_{23} y_3)^{a-1} = g_{23} y_3(y_1 + y_2 + y_3)^{a-1}.
$$

(5.13)

Comparing the coefficients of $y_2^a$ in both sides of (5.13), we have $g_{22} = 0$. Then there is no term having $y_2$ on the left-hand side of (5.13), so $g_{23} = 0$. This is a contradiction to $\det G \neq 0$. There is no such $G$.

**Case 2-3.** $(m_1, m_2, m_3) = (c, a, b)$ and $(\bar{m}_1, \bar{m}_2, \bar{m}_3) = (c, b, a)$

In this case we have

$$H^*(X_D) = \mathbb{Z}[x_1, x_2, x_3]/\langle x_1^c, x_2(x_1 + x_2)^a - 1, x_3(x_1 + x_2 + x_3)^{b-1} \rangle,$$

$$H^*(X_{\bar{D}}) = \mathbb{Z}[y_1, y_2, y_3]/\langle y_1^c, y_2 (y_1 + y_2)^{b-1}, y_3 (y_1 + y_2 + y_3)^{a-1} \rangle.$$
Since \( \varphi(x_2(x_1 + x_2)^{a-1}) \) belongs to \( \mathcal{I}_D \), we have
\[
(5.14) \quad (g_1 y_1 + g_2 y_2 + g_3 y_3) \{(g_1 + g_2)y_1 + (g_1 + g_2)y_2 + (g_1 + g_2)y_3\}^{a-1} = g_3 g_2 (g_1 + g_2)^{a-1} y_3 (y_1 + y_2 + y_3)^{a-1}.
\]

Here, \( g_3 (g_1 + g_2) \neq 0 \). If not, it contradicts to \( \det G \neq 0 \).

Comparing the coefficients of \( y_i^2 \) and \( y_j^2 \) in both sides of (5.14), we have
\[
(5.15) \quad g_21 (g_11 + g_21) = 0 \quad \text{and} \quad g_22 (g_12 + g_22) = 0.
\]

There are four possibilities.

i) If \( g_21 = 0 \) and \( g_22 \neq 0 \), then (5.14) becomes
\[
(g_2 y_2 + g_3 y_3) \{(g_1 + g_2)y_1 + (g_1 + g_2)y_2 + (g_1 + g_2)y_3\}^{a-1} = g_3 (g_1 + g_2)^{a-1} y_3 (y_1 + y_2 + y_3)^{a-1}.
\]
Comparing the coefficients of \( y_1^{a-1} y_2 \) in both sides of the above equation, we have \( g_11 = 0 \). Then there is no term having \( y_1 \) on the left-hand side, which is a contradiction.

ii) If \( g_21 \neq 0 \) and \( g_22 = 0 \), then (5.14) becomes
\[
(g_2 y_2 + g_3 y_3) \{(g_1 + g_2)y_1 + (g_1 + g_2)y_2 + (g_1 + g_2)y_3\}^{a-1} = g_3 (g_1 + g_2)^{a-1} y_3 (y_1 + y_2 + y_3)^{a-1}.
\]
Comparing the coefficients of \( y_1^{a-1} y_2 \) in both sides of the above equation, we have \( g_12 = 0 \). Then there is no term having \( y_2 \) on the left-hand side, which is a contradiction.

iii) If \( g_21 = g_22 = 0 \), then (5.14) becomes
\[
y_3 \{(g_11 y_1 + g_12 y_2 + (g_1 + g_2 + g_3)y_3)\}^{a-1} = (g_1 + g_2 + g_3) y_3 (y_1 + y_2 + y_3)^{a-1},
\]
so we have
\[
g_11 = g_12 = g_13 + g_23.
\]

From the fact that \( \varphi(x_3(x_1 + x_2 + x_3)^{b-1}) \) belongs to \( \mathcal{I}_D \), we have
\[
(5.16) \quad (g_3 y_1 + g_3 y_2 + g_3 y_3) \{(g_1 + g_3)y_1 + (g_1 + g_3)y_2 + (g_1 + g_3 + g_3)y_3\}^{b-1} = g_3 g_3 (g_1 + g_3)^{b-1} y_2 (y_1 + y_2 + y_3)^{b-1} + \alpha g_3 (y_1 + y_2 + y_3)^{a-1},
\]
where \( \alpha \) is a homogeneous polynomial of degree \( b - a \). Plugging \( y_1 = 1 \), \( y_2 = -1 \), and \( y_3 = 0 \) into (5.16), we get \( g_31 = g_32 \). This contradicts to \( \det G \neq 0 \).

iv) If \( g_21 \neq 0 \) and \( g_22 \neq 0 \), then by (5.15), we get
\[
g_11 + g_21 = 0 \quad \text{and} \quad g_22 + g_22 = 0.
\]

Hence (5.14) becomes
\[
(g_2 y_1 + g_2 y_2 + g_3 y_3) y_3^{a-1} = g_3 g_2 (g_1 + g_2)^{a-1} y_1 (y_1 + y_2 + y_3)^{a-1}.
\]

On the left-hand side of the above equation, the degree of \( y_1 \) or \( y_2 \) in each term is at most one, so \( a = 2 \), and \( g_21 = g_22 = g_23 \). Since \( g_11 + g_21 = 0 \) and \( g_12 + g_22 = 0 \), we get
\[
g_11 = g_12 = -g_21 = -g_22 = -g_23 \neq 0.
\]

From the fact that \( \varphi(x_3(x_1 + x_2 + x_3)^{b-1}) \) belongs to \( \mathcal{I}_D \), we have
\[
(5.17) \quad (g_3 y_1 + g_3 y_2 + g_3 y_3) \{(g_1 y_1 + g_3 y_2 + (g_1 + g_3 + g_3)) y_3\}^{b-1} = g_3 g_3 y_3 (y_1 + y_2)^{b-1} + \alpha y_3 (y_1 + y_2 + y_3)^{a-1},
\]
where $\alpha$ is a homogeneous polynomial of degree $b-a$. Plugging $y_1 = 1$, $y_2 = -1$, and $y_3 = 0$ into (5.17), we get $g_{31} = g_{32}$, which is a contradiction to $\det G \neq 0$.

Hence, in any case, there is no isomorphism between $H^*(X_D)$ and $H^*(X_{\bar{D}})$. \qed

Combining Propositions 5.4, 5.6, and 5.7, we prove Theorem 1.3.

6. FURTHER DISCUSSION

In this section, we discuss Problem 1.2 and introduce a problem about a relation between toric varieties of Schröder type and flag varieties.

Let $D$ be a $k$-dissection of $\mathbb{P}_{n+2}$. Using Corollary 4.3, we define a set $L(D)$ by the set of leaves $w$ in $\tau(D)$ such that $\phi(w) = \{a_i, b_i\}$ in Theorem 4.1. Then for every $w \in L(D)$, if $v$ is the parent of $w$, then $x^\ell_i_{a_i, b_i} = 0$ in $H^*(X_D)$, where $\ell_i$ is the number of children of $v$.

The following proposition gives another evidence for the affirmative answer to Problem 1.2.

**Proposition 6.1.** Let $\tau(D)$ and $\tau(\bar{D})$ be the Schröder trees with at least four internal vertices such that all the internal vertices have the same number of children. If $|L(D)| \neq |L(\bar{D})|$, then $H^*(X_D) \neq H^*(X_{\bar{D}})$.

**Proof.** Assume that each vertex of $\tau(D)$ and $\tau(\bar{D})$ has $\ell$ children. If $\ell = 2$, then $X_D$ and $X_{\bar{D}}$ are toric varieties of Catalan type. It is shown in [16] that two toric varieties of Catalan type are isomorphic if and only if their integral cohomology rings are isomorphic as graded rings. Thus it is enough to prove the case when $\ell > 2$.

Suppose that $|L(D)| < |L(\bar{D})|$ and $H^*(X_D) \cong H^*(X_{\bar{D}})$. Then there exists a non-monomial element $\alpha = \sum_{i=1}^{k} c_i x_{a_i, b_i}$ in $H^2(X_D)$ such that $\alpha^\ell = 0$ in $H^*(X_D)$. Let $v_{i_1}, v_{i_2}, v_{i_3}$ be three adjacent internal vertices of $\tau(D)$. Consider the Schröder subtree $T$ of $\tau(D)$ by taking $v_{i_1}, v_{i_2}, v_{i_3}$ and their children. Then $T$ defines a toric variety $X_T$ and $H^*(X_T)$ is obtained from $H^*(X_{\bar{D}})$ by substituting $x_{a_i, b_i} = 0$ whenever the parent of the vertex labeled by $\{a_i, b_i\}$ is not in $\{v_{i_1}, v_{i_2}, v_{i_3}\}$. Since the restriction $\alpha_T$ of $\alpha$ to $H^*(X_T)$ also satisfies that $\alpha^\ell = 0$, Lemmas 5.3 and 5.5 imply that at most one of $c_{i_1}, c_{i_2}$, and $c_{i_3}$ is nonzero. Hence if $c_{i_1}$ and $c_{i_2}$ are nonzero, then the distance between $v_{i_1}$ and $v_{i_2}$ is greater than two. Let $\mathcal{H}$ be the subring of $H^*(X_D)$ obtained by substituting $x_{a_j, b_j} = 0$ for $j \neq i, i'$. Then $(c_i x_{a_i, b_i} + c_{i'} x_{a_{i'}, b_{i'}})^\ell = 0$ in $\mathcal{H}$, where $\mathcal{H}$ is

$$Z[x_{a_i, b_i}, x_{a_{i'}, b_{i'}}]/\langle x^\ell_{a_i, b_i}, x_{a_i, b_i} (\pm x_{a_i, b_i} + x_{a_{i'}, b_{i'}})^{\ell-1} \rangle$$

or

$$Z[x_{a_i, b_i}, x_{a_{i'}, b_{i'}}]/\langle x^\ell_{a_i, b_i}, x^\ell_{a_{i'}, b_{i'}} \rangle.$$

Since $\ell > 2$, in any case, there is no linear element $c_i x_{a_i, b_i} + c_{i'} x_{a_{i'}, b_{i'}}$ such that $c_i c_{i'} \neq 0$ and $(c_i x_{a_i, b_i} + c_{i'} x_{a_{i'}, b_{i'}})^\ell = 0$ in $\mathcal{H}$. Therefore, there is no non-monomial element $\alpha \in H^2(X_D)$ such that $\alpha^\ell = 0$ in $H^*(X_D)$. This proves the proposition. \qed

We close this section by introducing a problem about a relation between toric varieties of Schröder type and flag varieties. It is shown in [15] that every toric variety of Catalan type appears as a certain Richardson variety in the full flag variety. Since the family of toric varieties of Schröder type includes the toric varieties of Catalan type, one can ask naturally whether toric varieties of Schröder type can be realized as subvarieties of flag variety like Richardson varieties.
Problem 6.2. For each dissection $D$ of a polygon, does the toric variety $X_D$ appear in a (partial) flag variety as a Richardson variety?

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