RADICAL-INJECTIVITY IN THE CATEGORY S-ACT

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Abstract. Various generalizations of the concept of injectivity, in particular injectivity with respect to a specific class of morphisms, have been intensively studied throughout the years in different categories. One of the important kinds of injectivity studied in the category \textbf{R-Mod} of \(R\)-modules is \(\tau\)-injectivity, for a torsion theory \(\tau\), or in the other words \(r\)-injectivity, where \(r\) is the induced idempotent radical by \(\tau\).

In this paper, we introduce the notion of \(r\)-injectivity, for a Hoehnke radical \(r\) in the category \textbf{S-Act} of \(S\)-acts and we study the main properties of this kind of injectivity. Indeed, we show that this kind of injectivity is well behavior and also we present a Bear Theorem for \(r\)-injective \(S\)-acts. We then consider \(r\)-injectivity for a Kurosh-Amitsur radical \(r\) and we give stronger results in this case. Finally we present conditions under which \(r\)-injective \(S\)-acts are exactly injective ones and we give a characterization for injective \(S\)-acts.

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1. Introduction and Preliminaries

Injectivity and its various generalizations, important and interesting for their own and also tightly related to certain concepts such as purity and etc, have been intensively studied throughout the years in different categories [2, 3, 8, 9, 14, 15, 17]. One of the important kinds of injectivity for module theorists is \(\tau\)-injectivity, for a torsion theory \(\tau\), or in the other words \(r\)-injectivity in which \(r\) is the induced idempotent radical by \(\tau\), [4, 6, 7, 13, 16].

In this paper first, with every Hoehnke radical \(r\) we associate a closure operator \(c^r\) and consider the class of all \(c^r\)-dense monomorphisms so-called \(r\)-monomorphisms. We then, in Section 3, study the properties of the class of \(r\)-monomorphisms. In Sections 4 and 5 we consider the injective \(S\)-acts relative to \(r\)-monomorphisms, \(r\)-injective \(S\)-acts, and we study the main properties of this kind of injectivity and we establish the well behavior theorems for \(r\)-injectivity. We then give Bear-Skornjakov criterion for \(r\)-injective \(S\)-acts and weakly injective \(S\)-acts in Section 6. Then, in section 7, we investigate \(r\)-injectivity when \(r\) is a Kurosh-Amitsur radical and we get stronger results in this case. Finally, the relationship between \(r\)-injectivity and usual injectivity is analyzed. Indeed, we present conditions under which \(r\)-injective \(S\)-acts are exactly injective ones and we give a characterization for the usual injective \(S\)-acts.

Now Let us recall some necessary notions. An \(S\)-act over a monoid \(S\) is a set \(A\) together with an action \((s,a) \mapsto as\), for \(a \in A, s \in S\), subject to the rules

\[ t(sa) = (ts)a \quad \text{and} \quad 1a = a, \]

where 1 is the identity element of the monoid \(S\), for all \(a \in A\) and \(s, t \in S\). A \textit{homomorphism of S-acts} is a map \(f : A \rightarrow B\) subject to
$f(sa) = sf(a)$, for all $a \in A$ and $s \in S$. We will work in the category of all $S$-acts and homomorphisms between them. An $S$-act $A$ is said to be trivial, if $|A| \leq 1$.

An equivalence relation $\rho$ on an $S$-act $A$ is called a congruence on $A$, if $apa'$ implies $(sa)\rho(sa')$, for all $s \in S$. We denote the set of all congruences on $A$ by $\Con(A)$ which forms a bounded lattice in which the diagonal relation $\Delta_A = \{(a, a) \mid a \in A\}$ is the smallest element and the total relation $\nabla_A = \{(a, b) \mid a, b \in A\}$ is the greatest one. Any congruence $\chi \in \Con(A)$ determines a partition of $A$ into $\chi$-classes and a system $\Sigma_\chi$ of those $\chi$-classes each of which is a non-trivial subact of $A$. Of course, $\Sigma_\chi$ may be empty. Throughout this paper we use the general notion of Rees congruence, as well as $[\chi \Sigma]$ instead of the usual Rees congruence defined in [12], meaning that a congruence $\rho$ is a Rees congruence if the $\rho$-cosets either are subacts or consist of one element. So every system $\Sigma$ of disjoint non-trivial subacts of an $S$-act $A$ determines a Rees congruence $\rho_\Sigma$ given by

$$(a, b) \in \rho_\Sigma \iff \begin{cases} a, b \in B & \text{for some } B \in \Sigma \\ a = b & \text{otherwise.} \end{cases}$$

We call $\rho_\Sigma$ to be the generated Rees congruence by $\Sigma$ on $A$ and $A/\rho_\Sigma$ a Rees factor of $A$ over $\rho_\Sigma$. Also we use the notion $\rho_B$ instead of $\rho_\Sigma$ when $\Sigma$ is the singleton set $\{B\}$ and denote the Rees factor of over $\rho_B$ by $A/B$ instead of $A/\rho_\Sigma$.

A congruence $\chi_B$ of a subact $B$ of an $S$-act $A$ may extend to a congruence of the $S$-act $A$. There is always the smallest extension $\chi_A$ given by

$$(a, b) \in \chi_A \iff \begin{cases} (a, b) \in \chi_B \\ a = b & \text{otherwise.} \end{cases}$$

Therefore we may consider each congruence $\chi_B \in \Con(B)$ as a congruence of $\Con(A)$ by identifying $\chi_B$ and $\chi_A$. In particular, $\nabla_B$ can be considered as the generated Rees congruence by $B$, $\rho_B \in \Con(A)$.

Now we give some different types of radicals in $S$-Act which is usually considered.

- An assignment $r : A \rightarrow r(A)$ assigning each $S$-act $A$ to a congruence $r(A) \in \Con(A)$ is called a Hoehnke radical or simply a radical whenever,
  
  (i) every homomorphism $f : A \rightarrow B$ induces the a homomorphism $r(f) : r(A) \rightarrow r(B)$; meaning that $(f(a), f(a')) \in r(B)$ if $(a, a') \in r(A)$, for every homomorphism $f : A \rightarrow B$.
  
  (ii) $r(A/r(A)) = \Delta_A/r(A)$.

- A radical $r$ is said to be hereditary, if $r(B) = r(A) \wedge \Delta_B$, for all $B \leq A$ and all $S$-acts $A$.

- A radical $r$ of $S$-acts is called a Kurosh-Amitsur radical, if
  
  (i) $r(A)$ is a Rees congruence, for all $S$-acts $A$,
  
  (ii) for every $B \in \Sigma_{r(A)}$, $r(B) = \nabla_B$.

With every radical $r$ one can associate two classes of $S$-acts, namely radical class (or torsion class) $\Re_r = \{A \mid r(A) = \nabla_A\}$ and semisimple class (or torsion-free class) $\Se_r = \{A \mid r(A) = \Delta_A\}$. We call the members of $\Re_r$ to be the radical $S$-acts and the members of $\Se_r$ to be the semisimple $S$-acts. It is worth noting that $\Se_r$ is closed under taking subacts, product, isomorphic copies and contains all trivial $S$-acts. Also every subclass $S$ of $S$-acts which is closed under taking subacts, product,
isomorphic copies and contains all trivial $S$-acts, determines a radical $r_S$ defined by $r_S(A) = \wedge(\chi \in \text{Con}(A) \mid A/\chi \in S)$. Moreover, $S = S_r$ if and only if $r = r_S$, see [20].

We recall, from [20, 11], that a subclass $S$ of $S$-acts is a semisimple class of a radical $r$ if and only if

1. $S$ contains all trivial $S$-acts,
2. $S$ is closed under isomorphic copies,
3. $S$ is closed under taking subacts,
4. $S$ is closed under products,
5. $S$ is closed under congruence extensions. That is, $A/\chi \in S$ and $\Sigma \chi \subseteq S$ imply $A \in S$, for every $A \in S$-Act and every congruence $\chi$ on $A$.

Also a subclass $R$ of $S$-acts is a radical class of a radical $r$ if and only if

1. $R$ contains all trivial $S$-acts,
2. $R$ is homomorphically closed,
3. $R$ has the inductive property; that is $\bigcup_{i \in I} A_i \in R$, for every ascending chain $\{A_i\}_{i \in I} \subseteq R$,
4. $R$ is closed under Rees extensions. That is $A/\rho \in R$ and $\Sigma \rho \subseteq R$ imply $A \in R$, for every $A \in S$-Act and every congruence $\rho$ on $A$.

It is worth noting the following remark concerning $\Sigma r(A)$, for every $S$-act $A$, where $r$ is a radical.

Remark 1.1. (i) For every subact $B$ of $A$ with $B \in R_r$, there exists $X \in \Sigma r(A)$ such that $B \subseteq X$.

(ii) Each $r(A)$-class $X$ containing a subact $B$ of $A$ is itself a subact of $A$, and so $X \in \Sigma r(A)$.

Now we recall the following lemma from [10] which is used in the sequel.

Lemma 1.2. Let $r$ be a radical and $\chi \subseteq r(A)$ be a congruence on an $S$-act $A$. Then $r(A/\chi) = r(A)/\chi$.

In particular for a Kurosh-Amitsur radical $r$ and a set $\Sigma$ of disjoint subacts of an $S$-act $A$ with $\Sigma \leq \Sigma r(A)$, we have $r(A/\rho \Sigma) = r(A)/\rho \Sigma$.

Also we recall, given a subclass of monomorphisms $M$, an $M$-morphism $m$ is called to be $M$-essential if for every homomorphism $f : B \to C$, $fm \in M$ implies $f \in M$. In this paper we use the terminology of B. Banaschewski [2, 8, 9] and we say that injectivity relative to a class $M$ is well behaviour in the category $S$-Act if the following propositions are stablished.

Proposition 1.3 (First well behaviour Theorem [2]). The following conditions are equivalent, for an $S$-act $A$:

(i) $A$ is $M$-injective.

(ii) $A$ is an $M$-absolute retract.

(iii) $A$ has no proper $M$-essential extension.

Proposition 1.4 (Second well behaviour Theorem [2]). Every $S$-act $A$ has an $M$-injective hull.

Proposition 1.5 (Third well behaviour Theorem [2]). The following conditions are equivalent, for an $M$-morphism $m : A \to B$:

(i) $B$ is an $M$-injective hull of $A$.

(ii) $B$ is a maximal $M$-essential extension of $A$.

(iii) $B$ is a minimal $M$-injective extension of $A$. 

A family \( C = (C_A)_{A \in S-\text{Act}} \), with \( C_A : \text{Sub}(A) \to \text{Sub}(A) \), assigning every subact \( B \leq A \) to a subact \( C_A(B) \) (or simply \( C(A) \) when no confusion arises) is called a closure operator on \( S-\text{Act} \) if it satisfies the following properties:

1. (Extension) \( B \leq C(B) \),
2. (Monotonicity) \( B_1 \leq B_2 \leq A \) implies \( C(B_1) \leq C(B_2) \),
3. (Continuity) \( f(C_A(B)) \leq C_C(f(B)) \), for all homomorphisms \( f : A \to C \).

A closure \( C \) is called weakly hereditary if \( C_A(B) = C_{C_A(B)}(B) \), for every subact \( B \) of every \( S \)-act \( A \). A closure operator \( C \) is called hereditary if it holds \( C_B(D) = C_A(D) \cap B \), for every subact \( D \leq B \leq A \).

The readers may consult [1, 5, 12] for the general facts about category theory and universal algebra used in this paper. Here we also follow the notations and terminologies used there.

2. The induced closure operator from a radical

Usually radicals are a rich supply for the closure operators, See [18]. Hence we introduce a closure operator \( c^r \), associated with a radical \( r \) and we describe the interrelationship of these two notions.

**Definition 2.1.** For a given radical \( r \) of \( S-\text{Act} \), we define a closure operator \( c^r \) in the category of \( S-\text{Act} \) by \( c^r_A(B) = \pi^{-1}(\lceil B \rceil_{r/A(B)}) \) in which \( \pi : A \to A/B \) is the canonical epimorphism.

A subact \( B \) of \( S \)-act \( A \) is said to be \( r \)-closed if \( c^r(B) = B \) and it is said to be \( r \)-dense if \( c^r(B) = A \). A monomorphism \( m : B \to A \) is said \( r \)-monomorphism if \( m(B) \) is \( r \)-dense in \( A \).

One can easily check that \( c^r \) is an idempotent closure operator, that is \( c^r_A(c^r_A(B)) = c^r_A(B) \), for every \( S \)-act \( A \) and every subact \( B \) of \( A \). So the class of \( r \)-closed subacts of an \( S \)-act \( A \) is of the form \( \{c^r_A(B) \mid B \leq A \} \).

It is worth noting that the class of \( r \)-monomorphisms is closed under composition. Also we have the following Lemma.

**Lemma 2.2.** Let \( B \) be an \( r \)-dense subact of an \( S \)-act \( A \) and \( \chi \) be a congruence on \( B \). Then \( B/\chi \) is \( r \)-dense in \( A/(\chi \lor \Delta_A) \).

**Proof.** The result can easily follow from the following equations.

\[
A/(\chi \lor \Delta_A)/B/\chi = A/B \quad \& \quad r(A/B) = \nabla_{A/B}
\]

**Proposition 2.3.** Let \( r \) be a radical and \( B \) be an \( r \)-closed subact of an \( S \)-act \( A \). Then, for every \( X_A \in \Sigma_{r(A)} \) and \( X_B \in \Sigma_{r(B)} \), we have \( X_A \cap X_B = \emptyset \) or \( X_B \leq X_A \leq B \).

**Proof.** To prove, we assume \( X_A \cap X_B \neq \emptyset \) and we show \( X_B \leq X_A \leq B \). To do so, we consider the canonical epimorphism \( \pi : A \to A/B \) and we have \( \pi(r(A)) \subseteq r(A/B) \). Hence there exists \( Y_{A/B} \in \Sigma_{r(A/B)} \) such that \( \pi(X_A) \subseteq Y_{A/B} \), since \( X_A \in \Sigma_{r(A)} \). But since \( X_A \cap X_B \leq X_A \cap B \) is non-empty, the homomorphic image of \( B \) under \( \pi : A \to A/B \) is a zero element of \( Y_{A/B} \). So \( Y_{A/B} \leq [B]_{r(A/B)} \). Therefore \( X_A \subseteq c^r_A(B) = B \).
Now by considering the canonical epimorphism $\pi : B \to B/X_A$, we have $\pi(r(B)) \subseteq r(B/X_A)$. Hence there exists $Y_{B/X_A} \in \Sigma_{r(B/X_A)}$ with $\pi(X_B), \pi(X_A) \subseteq Y_{B/X_A}$ since $X_B \in \Sigma_{r(B)}$ and $X_A \cap X_B \neq \emptyset$. Also $[X_A]_{r(B/X_A)}$ is singleton since, by Lemma 1.2, $r(B/X_A) \leq r(A/X_A) \setminus \sigma_{B/X_A} = r(A)/X_A$. Therefore $X_B \leq \pi^{-1}(Y_{B/X_A}) = X_A$. \hfill \Box

One is tempted to assume that the radical class of a radical $r$ is closed under coproduct. But this is not true in general, see example 3.1 from [11], we recall some equivalent conditions with closedness of $\mathbb{R}_r$ under coproducts, for a Kurosh-Amitsur radical $r$, from [11], and then using the mentioned closure operator we give another characterization for the closedness of $\mathbb{R}_r$ under coproduct in Theorem 2.6.

**Theorem 2.4.** Given a Kurosh-Amitsur radical $r$, the following statements are equivalent

1. The class $\mathbb{R}_r$ is closed under coproduct.
2. For every $A \in S\text{-}\text{Act}$, $|\Sigma_{r(A)}| \leq 1$ and $\mathbb{R}_r$ contains a non-trivial $S$-act.
3. For the trivial $S$-act $\Theta$, $\Theta \sqcup \Theta \in \mathbb{R}_r$.
4. For $B \leq A$, $\pi^{-1}([B]_{r(A/B)})$ contains all zeros of $A$ where $\pi : A \to A/B$ is the canonical epimorphism.

As a corollary of the above theorem, we have:

**Corollary 2.5.** Given a Kurosh-Amitsur radical $r$, the class $\mathbb{R}_r$ is closed under coproducts if and only if every $S$-act $A$ has a subact such as $A_r \in \mathbb{R}_r$ such that $A_r$ contains all zeros of $A$ and $r(A) = \rho_{A_r}$.

**Theorem 2.6.** Let $r$ be a Kurosh-Amitsur radical. Then $\mathbb{R}_r$ is closed under coproducts if and only if $r(A/B) = \Delta_{A/B}$, for every $S$-act $A$ and every $r$-closed subact $B$ of $A$.

**Proof.** ($\Rightarrow$) Let $B$ be an $r$-closed subact of an $S$-act $A$. Then, by Corollary 2.5, $A/B$ has a subact such as $X_{A/B} \in \mathbb{R}_r$ such that $X_{A/B}$ contains all the zeros of $A/B$ and $r(A/B) = \rho_{A_{A/B}}$. But since the image of $B$ under the canonical homomorphism $\pi : A \to A/B$ is a zero element of $A/B$, $[B]_{r(A/B)} = X_{A/B}$. Also since $B$ is an $r$-closed subact of $A$, we have $B = \pi^{-1}([B]_{r(A/B)}) = \pi^{-1}(X_{A/B})$. Therefore $X_{A/B} = B/B$ and hence $r(A/B) = \Delta_{A/B}$.

($\Leftarrow$) For the converse we show the second assertion of Theorem 2.4. Let $A$ be a non semisimple $S$-act and $B \subseteq \Sigma_{r(A)}$. Then, by Lemma 1.2, $r(A/B) = r(A)/B$ and hence $B = \pi^{-1}([B]_{r(A/B)})$, where $\pi : A \to A/B$ is the canonical homomorphism. That is, $B$ is an $r$-closed subact of $A$. Now $r(A/B) = \Delta_{A/B}$ follows from the hypothesis. This means that $r(A) = \rho_B$. So, for every $S$-act $A$, $|\Sigma_{r(A)}| \leq 1$. \hfill \Box

**Proposition 2.7.** Let $r$ be a radical whose semisimple class is closed under coproducts and $B$ be a proper $r$-dense subact of an $S$-act $A \subseteq S_r$. Then there exists $x \in A \setminus B$ and $s \in S$ such that $sx \in B$.

**Proof.** To prove, we suppose $sx \notin B$, for every $s \in S$ and $x \in A \setminus B$, and we get a contradiction. Indeed, if $sx \notin B$, for every $s$ and $x$, then $A \setminus B$ is a subact of $A$ and so, $A/B \cong (A \setminus B)/\Theta$, where $\Theta$ is a singleton trivial $S$-act. But since $S_r$ is closed under taking subacts and $A \subseteq S_r$, $A \setminus B \subseteq S_r$. So $A/B = (A \setminus B)/\Theta \subseteq S_r$ follows from the closedness of $S_r$ under coproduct. Therefore $A/B \subseteq S_r \cap \mathbb{R}_r$ and hence $A/B$ is a trivial $S$-act. So $A = B$ and this contradicts the hypothesis. \hfill \Box
For some especial kind of radical more relations between $r$ and $c^r$ will display. In the following, we give some of them.

**Definition 2.8.** A radical $r$ is called
1. **pre-hereditary** if for every $S$-act $A$ and $Y \leq X \in \Sigma_{r(A)}$, $Y \in R_r$.
2. **weakly-hereditary** if, for every $S$-act $A$ with a zero element $\theta$ and $X \in \Sigma_{r(A)}$ with $\theta \in X$, $X \in R_r$.
3. **zero-hereditary** if, for every $S$-act $A$ with a zero element $\theta$ and $Y \leq X \in \Sigma_{r(A)}$ with $\theta \in Y$, $Y \in R_r$.
4. **pre-Kurosh** if, for every $S$-act $A$ and $X \in \Sigma_{r(A)}$, $X \in R_r$.

Figure 1 present the relation between the different kinds of radicals.

**Theorem 2.9.** Given a radical $r$, $c^r$ is weakly hereditary if and only if, $r$ is weakly hereditary.

*Proof.* $(\Rightarrow)$ Let $A$ be an $S$-act with a zero element $\theta$ and $X$ be an $r$-class of $A$ with $\theta \in X$. Then one can easily see that $X = c^r_A(\{\theta\})$. Now weakly heredity of $r$ implies $X = c^r_A(\{\theta\}) = c^r_A(\{\theta\})(\{\theta\}) = c^r_X(\{\theta\})$. Hence $X = \pi^{-1}(\theta)_r(X/\{\theta\})$. That is $r(X/\{\theta\}) = \nabla_{X/\{\theta\}}$. Therefore $X \cong X/\{\theta\} \in R_r$. 

**Figure 1.**
Let \( r \) be a weakly hereditary radical. Then since \( c_A^r(B)/B \in \Sigma_{r(A/B)} \) and the image of \( B \) under canonical epimorphism is a zero element of \( c_A^r(B)/B \), for every subact \( B \) of an \( S \)-act \( A \). So, we have \( c_A^r(B)/B \in R_r \). That is \( r(c_A^r(B)/B) = \nabla c_A^r(B)/B \). Therefore \( c_A^r(B) = \pi^{-1}[B] r(c_A^r(B)/B) = c_A^r(B) \) and we are done. \( \square \)

**Proposition 2.10.** Let \( r \) be a pre-Kurosh radical and \( B \) be an \( r \)-closed subact of an \( S \)-act \( A \). Then \( \Sigma_{r(B)} \subseteq \Sigma_{r(A)} \).

**Proof.** We know that \( r(B) \leq r(A) \land \nabla B \), for every radical \( r \) and a subact \( B \) of an \( S \)-act \( A \). So, for every \( X_B \in \Sigma_{r(B)} \), there exists an \( r(A) \)-class \( X_A \) such that \( X_B \subseteq X_A \). Since \( X_B \) is a subact of \( A \), \( X_A \) is a subact of \( A \). So \( X_A \in \Sigma_{r(A)} \) and, By Proposition 2.3, we have \( X_B \leq X_A \leq B \). Now since \( r(X_A) \leq r(B) \land \nabla X_A \), there exists an \( r(B) \)-class \( C \in \Sigma_B \) such that \( X_A \subseteq C \). Therefore \( X_B = C = X_A \) since \( \Sigma_{r(B)} \) is a set of disjoint subacts of \( B \). \( \square \)

Now since for every Kurosh-Amitsur radical \( r \) we have \( r(A) = \rho_{\Sigma_{r(A)}} \), by the above proposition, one can easily see that every Kurosh-Amitsur radical is hereditary for \( r \)-closed subacts. See the following corollary.

**Corollary 2.11.** Let \( r \) be a Kurosh-Amitsur radical and \( B \) be an \( r \)-closed subact of an \( S \)-act \( A \). Then \( r(B) = r(A) \land \nabla B \).

We denote the injective hull of an \( S \)-act \( A \) by \( E(A) \) and give the following lemma.

**Lemma 2.12.** Let \( r \) be a radical and \( \chi \) be a congruence on an \( S \)-act \( B \). Then \( \rho_B \leq \pi^{-1}(r(E(B)/\chi)) \) in which \( \pi_{E(A)} : E(B) \rightarrow E(B)/\chi \) is the canonical epimorphism if and only if there exists an extension \( A \) of \( B \) with \( \rho_B \leq \pi^{-1}(r(A/\chi)) \), where \( \pi_A : A \rightarrow A/\chi \) is the canonical epimorphism.

**Proof.** \( (\Rightarrow) \) It is enough to take \( A = E(B) \).

\( (\Leftarrow) \) Let \( B \leq A \) and \( \rho_B \leq \pi_A^{-1}(r(A/\chi)) \). Then there exists a homomorphism \( f : A \rightarrow E(B) \) which commutes the following diagram.

\[
\begin{array}{ccc}
B & \xrightarrow{\iota_B} & A \\
\downarrow \iota_E(A) & & \downarrow f \\
E(B) & & \\
\end{array}
\]

Now we consider the homomorphism \( \overline{f} : A/\chi \rightarrow E(B)/\chi \), mapping each \( [a]_\chi \) to \( [f(a)]_\chi \). Then we have \( \rho_B/\chi = \overline{f}(\rho_B/\chi) \subseteq \overline{f}(r(A/\chi)) \subseteq r(\overline{f}(A/\chi)) \leq r(E(B)/\chi) \) and so, \( \rho_B \leq \pi_{E(B)}^{-1}(\rho_B/\chi) \leq \pi^{-1}(r(E(B)/\chi)) \). \( \square \)

**Theorem 2.13.** Given a subact \( C \) of an \( S \)-act \( B \), \( B \leq c_{E(B)}^r(C) \) if and only if there exists an extension \( A \) of \( B \) with \( B \leq c_A^r(C) \).

**Proof.** \( (\Rightarrow) \) It is clear.

\( (\Leftarrow) \) Let \( B \leq c_A^r(C) \). Then \( \rho_B \leq \pi_A^{-1}(r(A/C)) \), since \( c_A^r(C) = \pi_A^{-1}([C]_{r(A/C)}) \), for the canonical epimorphism \( \pi_A : A \rightarrow A/C \). Hence Lemma 2.12 implies \( \rho_B \leq \pi_{E(B)}^{-1}(r(E(B)/C)) \) where \( \pi_{E(B)} : E(B) \rightarrow E(B)/C \) is the canonical epimorphism. But \( C \leq B \). So, \( \pi_{E(B)}(B) \leq [C]_{r(A/C)} \), and hence \( B \leq \pi_{E(B)}^{-1}([C]_{r(A/C)}) = c_{E(B)}^r(C) \). \( \square \)
Proposition 2.14. For a radical \( r \) of \( S\text{-Act} \), the following conditions are equivalent.

(a) The radical \( r \) is zero-hereditary.

(b) A subact \( C \) of an \( S\text{-act} \) \( B \) is \( r \)-dense in \( B \) if and only if there exists an extension \( A \) of \( B \) with \( B \leq c_A(C) \).

Proof. (a) \( \Rightarrow \) (b) To prove necessity it is enough to consider \( B = A \).

To prove sufficiency first we note that \( B \leq c_{E(B)}(C) \), by Theorem 2.13. So, \( \pi_{E(B)}(B) = B/C \leq [C]_{r[E(B)/C]} \) in which \( \pi_{E(B)} \) is canonical homomorphism from \( E(B) \) to \( E(B)/C \). Also, \( [C]_{r[E(B)/C]} \in \Sigma_{r[E(B)/C]} \) and the homomorphic image of \( C \) under \( \pi_{E(B)} \) is a zero element of \( B/C \). Thus, by the hypothesis, we have \( r(B/C) = \nabla_{B/C} \). This means that \( C \) is \( r \)-dense in \( B \).

(b) \( \Rightarrow \) (a) Suppose \( A \in S\text{-Act} \), \( B \in \Sigma_{r(A)} \) and \( C \leq B \) with \( \theta \in C \). Then \( \rho_C \leq r(A/\theta) \). and hence \( \{ \theta \} \) is \( r \)-dense in \( C \), by Condition (b). Therefore \( r(C) = \nabla_C \). □

In the following we give a definition of intersection large subacts in a more general meaning than it is in [19].

Definition 2.15. A non-trivial subact \( B \) of \( A \) is called intersection large (\( \cap \)-large) in \( A \) if \( |B \cap X| \geq 2 \), for all non-trivial subact \( X \) of \( A \).

Lemma 2.16. Let \( B \) be a large subact of an \( S\text{-act} \) \( A \). Then \( B \) is \( \cap \)-large in \( A \).

Theorem 2.17. Let \( r \) be a pre-hereditary radical, \( A \in S_r \) and \( B \) be an \( r \)-dense subact of \( A \). Then \( B \) is \( \cap \)-large in \( A \).

Proof. Let \( X \) be a non-trivial subact of \( A \). We have to show that \( |B \cap X| \geq 2 \). But since \( X/(B \cap X) \leq A/B \in \mathbb{R}_r \), \( X/(B \cap X) \in \mathbb{R}_r \) follows from being pre-hereditary of \( r \). Therefore \( X/(B \cap X) \neq X \) because otherwise \( X \in \mathbb{R}_r \cap S_r \) which implies that \( X \) is trivial \( S\text{-act} \) which is a contradiction. So \( \rho_{B \cap X} \neq \Delta_X \) which means \( |B \cap X| \geq 2 \) and we are done. □

Proposition 2.18. Let \( r \) be a zero-hereditary radical whose semisimple class is closed under coproducts and \( B \) be an \( r \)-dense subact of a semisimple \( S\text{-act} \) \( A \). Then \( B \) is \( \cap \)-large in \( A \).

Proof. To prove, we show that, for every \( x \in A \setminus B \), \( |B \cap Sx| \geq 2 \). To do so, we suppose there exists \( x \in A \setminus B \) such that \( |B \cap Sx| \leq 2 \) and we get a contradiction. So let \( |B \cap Sx| \leq 2 \), then two possible cases may occur;

(i) \( |B \cap Sx| = 0 \) and (ii) \( |B \cap Sx| = 1 \). In both cases \( (Sx \cup B)/B \in S_r \), because, in case (i), \( (Sx \cup B)/B \cong Sx \sqcup \Theta \) in which \( \Theta \) is a singleton trivial \( S\text{-act} \). Hence \( (Sx \cup B)/B \in S_r \), follows from the closedness of \( S_r \) under coproducts and this fact that \( Sx, \Theta \in S_r \). Also, in case (ii), we have \( (Sx \cup B)/B \cong Sx \in S_r \). Also since \( B \leq Sx \cup B \leq A \) and \( B \) is \( r \)-dense in \( A \). Proposition 2.14 implies that \( (Sx \cup B)/B \in \mathbb{R}_r \). So \( (Sx \cup B)/B \in S_r \cap \mathbb{R}_r \) which means \( (Sx \cup B)/B \) is a trivial \( S\text{-act} \) and so \( (Sx \cap B)/B = B \). Therefore \( Sx \leq B \) which contradicts \( x \in A \setminus B \). □

3. Banaschewski’s condition on \( r \)-monomorphisms

Because of the crucial role of Banaschewski’s condition in the study of the well-behaviour of injectivity, we dedicate this short section to verify this condition concerning \( r \)-monomorphisms. To do so, we use the notion of essential congruence as
introduced in [20]. this notion is tightly related to the notion of essential monomor-
phisms which is important to study injective hull, see for example [8, 9]. Now let us give the definition of essential congruence in S-Act.

**Definition 3.1.** A congruence $\chi$ on an $S$-act $A$ is said to be essential if $\chi \land \theta \neq \Delta_A$, for every congruence $\theta \neq \Delta_A$ on $A$.

**Definition 3.2.** A family $\{A_i\}_{i \in I}$ of subacts of an $S$-act $A$ is called collectively large in $A$ if any homomorphism $g : A \to C$ whose restriction to $A_i$'s is a monomorphism, is itself a monomorphism.

In the following we give the relation between two former defined notion.

**Theorem 3.3.** A family $\Sigma = \{A_i\}_{i \in I}$ of disjoint subacts of an $S$-act $A$ is collectively large in $A$ if and only if the Rees congruence $\rho_\Sigma$ is an essential congruence on $A$.

**Proof.** ($\Rightarrow$) Suppose that $\Sigma = \{A_i\}_{i \in I}$ is collectively large in $A$ and $\rho_\Sigma \land \chi = \Delta_A$, for some $\chi \in \text{Con}(A)$. Then $\pi_{|A_i} : A_i \to A/\chi$ is a monomorphism, for every $i \in I$ where $\pi : A \to A/\chi$ is the canonical epimorphism. Hence $\pi : A \to A/\chi$ is a monomorphism and so $\chi = \Delta_A$.

($\Leftarrow$) Let $\Sigma = \{A_i\}_{i \in I}$ be a family of disjoint subacts of $S$-act $A$ such that $\rho_\Sigma$ is an essential congruence on $A$ and also let $g : A \to C$ be a homomorphism such that $g|_{A_i}$ is a monomorphism, for every $i \in I$. Then $\ker(g) \land \rho_\Sigma = \Delta_A$. So $\ker(g) = \Delta_A$ follows from essentiality of $\rho_\Sigma$. That is $\{A_i\}_{i \in I}$ is collectively large in $A$.  

**Corollary 3.4.** A homomorphism $f : A \to B$ is an essential monomorphism if and only if the generated Rees congruence by $f(A)$, that is $\rho_f(A)$, is essential on $B$.

**Theorem 3.5.** Let $A$ be an $S$-act, $\chi \in \text{Con}(A)$ and $\kappa$ be a maximal congruence with $\chi \land \kappa = \Delta_A$. Then, $(\kappa \lor \chi)/\kappa$ is an essential congruence on $A/\kappa$.

**Proof.** First we claim that $\tau \land \chi \neq \Delta_A$, for every congruence $\tau/\kappa \in \text{Con}(A/\kappa)$. This follows from the maximality of $\kappa$ with respect to $\chi \land \kappa = \Delta_A$, and the fact that every congruence on $A/\kappa$ is in the form of $\tau/\kappa$ in which $\tau \in \text{Con}(A)$ contains $\kappa$.

Therefore, for every $\tau/\kappa \in \text{Con}(A/\kappa)$, there exist $x \neq y$ in $A$ such that $(x, y) \in \chi \land \tau$. But since $(x, y) \in \chi$ and $\chi \land \kappa = \Delta_A$, we have $(x, y) \notin \kappa$. So $[x]_\kappa \neq [y]_\kappa$ and $([x]_\kappa, [y]_\kappa) \in \tau/\kappa \land (\chi \lor \kappa)/\kappa$. Hence we have $(\chi \lor \kappa)/\kappa \land \tau/\kappa \neq \Delta_{A/\kappa}$, for every $\tau/\kappa \neq \Delta_{A/\kappa}$ in $\text{Con}(A/\kappa)$. This means that $(\chi \lor \kappa)/\kappa$ is an essential congruence on $A/\kappa$.

**Lemma 3.6.** Let $A$ be an $S$-act, $\chi \in \text{Con}(A)$ and $\kappa$ be a maximal congruence with the property $\chi \land \kappa = \Delta_A$. Then $\pi|_B : B \to B/\kappa|_B$, mapping each $b \in B$ to $[b]_\kappa$, is an isomorphism, for every $B \in \Sigma_\chi$.

**Proof.** To prove it is enough to show that the map $\pi|_B$ is injective. Indeed, if $b \neq b'$ in $B$ then $(b, b') \in \chi$ and hence $(b, b') \notin \kappa$ follows from $\chi \land \kappa = \Delta_A$. That is $[b]_\kappa \neq [b']_\kappa$, for every $b \neq b'$ in $B$.

**Lemma 3.7.** Let $A$ be an $S$-act, $\rho_B$ be the Rees congruence on the subact $B$ of $A$, and $\kappa_B$ be a maximal congruence with the property $\rho_B \land \kappa_B = \Delta_A$. Then $\pi(\rho_B) = (\rho_B \lor \kappa_B)/\kappa_B$, in which $\pi : A \to A/\kappa_B$ is the canonical homomorphism.
Proof. To prove we show that \( \pi(\rho_B) = \{(a)_{\kappa_B}, (b)_{\kappa_B}\} | (a, b) \in \rho_B \} \) is a congruence on \( A/\kappa_B \), for every subact \( B \) of \( A \), then Correspondence Theorem gives the result. The reflexive and symmetric properties of \( \pi(\rho_B) \) easily follows from being epimorphism of \( \pi \) and the reflexive and symmetric properties of \( \rho_B \). To check the transitive property suppose that \( (a)_{\kappa_B}, (b)_{\kappa_B}, (c)_{\kappa_B}, (d)_{\kappa_B} \in \pi(\rho_B) \) and \( (b)_{\kappa_B} = (c)_{\kappa_B} \). Then we have the following possible cases:

1. \( c = d \), then \( (a)_{\kappa_B}, (d)_{\kappa_B} = (a)_{\kappa_B}, (b)_{\kappa_B} \) \( \in \pi(\rho_B) \);
2. \( a = b \), then \( (a)_{\kappa_B}, (d)_{\kappa_B} = (b)_{\kappa_B}, (b)_{\kappa_B} = (a)_{\kappa_B}, (b)_{\kappa_B} \) \( \in \pi(\rho_B) \);
3. \( a \neq b \) and \( c \neq d \), then \( a, b, c, d \in B \), since \( (a, b), (c, d) \in \rho_B \). So \( (a)_{\kappa_B}, (d)_{\kappa_B} \in \pi(\rho_B) \).

The compatibility of \( \pi(\rho_B) \) with the action is obvious. Hence \( \pi(\rho_B) \) is a congruence on \( A/\kappa \) and we are done. \( \square \)

Now we give the Banaschewski’s condition for \( r \)-monomorphisms, but first let us note the following definition.

**Definition 3.8.** A subact \( B \) of an \( S \)-act \( A \) is called to be \( r \)-large if \( B \) is both large and \( r \)-dense in \( A \). Then we call \( A \) to be \( r \)-essential extension of \( B \).

Also, an \( r \)-monomorphism \( \iota : B \to A \) is called \( r \)-essential monomorphism if \( \iota(B) \) in \( A \) is \( r \)-large.

**Theorem 3.9** (Banaschewski’s \( r \)-condition). Given an \( r \)-monomorphism \( f : B \to A \), there exists a homomorphism \( g : A \to X \) such that \( g \circ f : B \to X \) is an \( r \)-essential monomorphism.

Proof. To prove, it is enough to show that there exists a congruence \( \kappa \) on the \( S \)-act \( A \) such that \( \pi(f(B)) \) is \( r \)-large in \( A/\kappa \), for the canonical homomorphism \( \pi : A \to A/\kappa \). But, from Lemma 3.1 of [20], we know that there exists a maximal congruence \( \kappa \) on \( A \) with respect to \( \rho(f(B)) \cap \kappa = \Delta_A \). So \( \pi(f(B)) \) is large in \( A/\kappa \), by Theorem 3.5 and Lemma 3.7. Also \( \pi(f(B)) \) is an \( r \)-dense subact of \( A/\kappa \), by Lemma 2.2. Therefore \( \pi(f(B)) \) is \( r \)-large in \( A/\kappa \). \( \square \)

### 4. \textit{r}-injective \( S \)-acts

In this section we discuss the notion of \( r \)-injectivity in \textbf{S-Act}, where \( r \) is a radical, and give some properties concerning \( r \)-injective \( S \)-acts to identify this kind of injectivity. Let us begin with the following definition.

**Definition 4.1.** Let \( r \) be a radical. An \( S \)-act \( Q \) is called \( r \)-injective if, for every \( r \)-monomorphism \( \iota : A \to B \), every homomorphism \( f : A \to Q \) can be extended to the homomorphism \( \overline{f} : B \to Q \) thorough \( i : A \to B \), that is \( f = \overline{f}i \). Moreover \( Q \) is called orthogonal \( r \)-injective if \( \overline{f} \) is unique.

**Theorem 4.2.** Let \( r \) be a radical. Then a subact \( F \) of an \( r \)-injective \( S \)-act \( E \) is \( r \)-injective if \( E/F \in \mathbb{S}_r \).

Proof. Suppose \( E/F \in \mathbb{S}_r \) and consider the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{m} & B \\
\downarrow{f} & & \downarrow{\overline{f}} \\
F & \xrightarrow{c} & E
\end{array}
\]
in which \( m \) is an \( r \)-monomorphism. Then there exists a homomorphism \( \overline{f} : B \to E \) which commutes the above diagram. Now consider the homomorphism \( f' : B/A \to E/F \) which maps each \([b]_A \in B/A\) to \([\overline{f}(b)]_F\). Since \( B/A \in \mathcal{R}_r \) and \( E/F \in \mathcal{S}_r \), and also \( \mathcal{R}_r \) is closed under homomorphic image, \( f' \) is a zero homomorphism. This implies that \( \overline{f}(B) \subseteq F \). That is, \( \overline{f} : B \to F \) is a homomorphism with \( \overline{f} \circ m = f \), and we are done. \( \square \)

To give a characterization of \( r \)-injective \( S \)-acts, first we give the following lemma.

**Lemma 4.3.** Given a radical \( r \), the class \( \mathbb{L}_r = \{ A/B \mid B \text{ is } r \text{-dens in } A \} \) is the radical class of a Kurosh-Amitsur radical.

**Proof.** To prove, we use Lemma 2.4 of [20] and we show that \( \mathbb{L}_r \) is closed under homomorphic image and Rees extension, and has inductive property.

The closedness of \( \mathbb{L}_r \) under homomorphic image: since each \( X \in \mathbb{L}_r \) is a member of \( \mathcal{R}_r \) and has a zero element, every homomorphic image of \( X \) such as \( Y \) belongs to \( \mathcal{R}_r \) and has an element such as \( \theta_0 \). So, \( \{ \theta_0 \} \) is \( r \)-dense in \( Y \). Therefore \( Y/\{ \theta_0 \} \cong Y \) is in \( \mathbb{L}_r \) and this means that \( \mathbb{L}_r \) is closed under homomorphic image.

The closedness of \( \mathbb{L}_r \) under Rees extension: let \( A \) be an \( S \)-act and \( \rho \) be a Rees congruence on \( A \) such that \( \Sigma_\rho \subseteq \mathbb{L}_r \) and \( A/\rho \in \mathbb{L}_r \). Then \( A \) has a zero element such as \( \theta_0 \), since every \( B \in \Sigma_\rho \) has a zero element. Also \( A \) belongs to \( \mathcal{R}_r \), since \( \Sigma_\rho \subseteq \mathbb{L}_r \subseteq \mathcal{R}_r \) and \( A/\rho \in \mathbb{L}_r \subseteq \mathcal{R}_r \). Therefore \( A/\{ \theta_0 \} \cong A \) is in \( \mathbb{L}_r \) and this means that \( \mathbb{L}_r \) is closed under Rees extension.

Inductive property: let \( \{ A_i \}_{i \in I} \) be an ascending chain in \( \mathbb{L}_r \). Then \( \bigcup_{i \in I} A_i \) has a zero element such as \( \theta_0 \) and belongs to \( \mathcal{R}_r \). Hence \( \{ \theta_0 \} \) is \( r \)-dense in \( \bigcup_{i \in I} A_i \). Therefore \( \bigcup_{i \in I} A_i / \{ \theta_0 \} \cong \bigcup_{i \in I} A_i \) is in \( \mathbb{L}_r \) and this means that \( \mathbb{L}_r \) has inductive property. \( \square \)

**Theorem 4.4.** Given a radical \( r \), the class of \( r \)-injective \( S \)-acts is exactly the class of \( t_{\mathbb{L}_r} \)-injective \( S \)-acts, where \( t_{\mathbb{L}_r} \) is the induced Kurosh-Amitsur radical by \( \mathbb{L}_r \).

**Proof.** One can easily see that a subact \( B \) of an \( S \)-act \( A \) is \( r \)-dense if and only if \( A/B \in \mathcal{R}_r \). So \( \mathbb{L}_r \subseteq \mathcal{R}_r \) and this implies that every \( t_{\mathbb{L}_r} \)-dense subact of \( A \) is \( r \)-dense. Hence every \( r \)-injective \( S \)-act is \( t_{\mathbb{L}_r} \)-injective. Conversely let \( I \) be a \( t_{\mathbb{L}_r} \)-injective \( S \)-act. Then since, for every \( r \)-monomorphism \( m : A \to B, B/m(A) \) belongs to \( \mathbb{L}_r \), every homomorphism \( f : A \to I \) can be extended to \( \overline{f} : B \to I \). Therefore \( I \) is \( r \)-injective. \( \square \)

**Theorem 4.5.** Given a radical \( r \), every orthogonal \( r \)-injective \( S \)-act belongs to \( \mathcal{S}_{t_{\mathbb{L}_r}} \), where \( t_{\mathbb{L}_r} \) is the Kurosh-Amitsur radical given by \( \mathbb{L}_r \).

**Proof.** Let \( I \) be an orthogonal \( r \)-injective \( S \)-act and \( I \notin \mathcal{S}_{t_{\mathbb{L}_r}} \). Then there exists a non-trivial homomorphism \( f \) from an \( S \)-act \( A \in \mathbb{L}_r \) to \( I \). But since each \( A \in \mathbb{L}_r \) has a zero element such as \( \theta_A \), \( I \) has a zero element \( \theta_I \), and hence the zero homomorphism \( 0_{\theta_I} : \{ \theta_A \} \to I \), which maps \( \theta_A \) to \( \theta_I \), has at least two extension \( f \) and the zero homomorphism \( 0_{\theta_I}(\theta_A) = \theta_I \). This contradict orthogonally of \( I \). \( \square \)

We end this section by expressing an interesting property of \( r \)-closed subacts of an \( r \)-injective \( S \)-act wherewith we shall give a characterization of \( r \)-injective \( S \)-acts in Section 7.
**Theorem 4.6.** Let $I$ be an $r$-injective $S$-act and $A$ be an $r$-dense subact of an $S$-act $B$. Then the image of $B$ under every extension $\overline{f} : B \to I$ of a homomorphism $f : A \to I$ is a subact of $c_r^{\overline{f}}(f(A))$.

**Proof.** Let $A$ be $r$-dense subact of an $S$-act $B$ and $f : A \to I$ be a homomorphism. Then there exists $\overline{f} : B \to I$ which commutes the following rectangle.

But we have by the third property of a closure operator $\overline{f}(c_r^B(A)) \leq c_r^{\overline{f}}(\overline{f}(A))$. Now since $c_r^B(A) = B$ and $\overline{f}(A) = f(A)$, so $\overline{f}(B) \leq c_r^{\overline{f}}(f(A))$ and we are done. $\square$

One can easily get the following corollary from the above theorem.

**Corollary 4.7.** (i) Let $A$ be an $S$-act and $I$ be an $r$-injective extension of $A$. Then $c_r^I(A)$ is $r$-injective.

(ii) Let $E(A)$ be an injective hull of an $S$-act $A$. Then $c_r^{E(A)}(A)$ is $r$-injective.

(iii) An $S$-act $A$ is $r$-injective if and only if every $r$-closed subact of $A$ is $r$-injective.

5. **The well-Behaviour of $r$-injectivity**

Different sets of conditions are sufficient, although not always necessary, for the well-Behaviour of injectivity. The crucial conditions to verify whether injectivity is well-Behavior are so-called Banaschewski’s condition, which is given in the previous section, $r$-transferability condition, and Direct limit condition, see 5.4. In this section to verify the well-behaviour of $r$-injectivity, for a given radical $r$, we first check these conditions.

**Lemma 5.1** ($r$-transferability condition). The category $S$-Act satisfies the $r$-transferability property. That is, every diagram

with the $r$-monomorphism $m$ can be completed to a commutative square as follows in which $u$ is an $r$-monomorphism.

**Proof.** Consider $D = (B \setminus m(A)) \cup C$ together with the action

$$s.x = \begin{cases} s'x & x \in C \\ s \ast x & x \in B \setminus m(A) \text{ and } s \ast x \in B \setminus m(A) \\ f(m^{-1}(s \ast x)) & x \in B \setminus m(A) \text{ and } s \ast x \in m(A) \end{cases}$$
in which $\ast$ is the action of $B$ and $\ast'$ is the action of $C$. Clearly,
\[ v : B \rightarrow D \]
\[ b \mapsto v(b) = \begin{cases} 
  b & b \in B \setminus A \\
  f(b) & b \in A
\end{cases} \]
and the inclusions map $u : C \rightarrow D$ makes the following diagram commutative.

![Diagram](image)

Also, since $D/u(C) \cong B/m(A)$ and $m(A)$ is $r$-dense in $B$, $u(C)$ is $r$-dense in $D$. \qed

We recall that a directed family of $S$-acts is a family $(A_i)_{i \in I}$ of $S$-acts indexed by an up-directed set $(I, \leq)$ endowed by a family $(f_{ij} : A_i \rightarrow A_j)_{i \leq j \in I}$ of monomorphisms such that given $i \leq j \leq k \in I$ we have $f_{jk} \circ f_{ij} = f_{ik}$, also $f_{ii} = id_{A_i}$, for every $i \in I$. Note that the direct limit of a directed family $((A_i)_{i \in I}, (f_{ij})_{i \leq j \in I})$ in $S$-Act is given as $\lim(A_i)_{i \in I} = \bigsqcup_{i \in I} A_i / \chi$, where the congruence $\chi$ is given by $a_i \chi a_j$, if and only if there exists $k \geq i, j$ such that $u_k f_{ik}(a_i) = u_k f_{jk}(a_j)$ in which each $u_i : A_i \rightarrow \bigsqcup_{i \in I} A_i$ is an injection map of the coproduct.

To establish the direct limit condition for $r$-injectivity, or for short $r$-direct limit, we need the following lemma.

**Lemma 5.2.** Let $\mathbb{R}$ be a subclass of $S$-Act which is closed under homomorphic image and Rees congruence extension. Then $\mathbb{R}$ is a radical class of a radical if and only if $\lim(A_i)_{i \in I} \in \mathbb{R}$, for every directed family $((A_i)_{i \in I}, (f_{ij})_{i \leq j \in I})$ of $\mathbb{R}$.

**Proof.** ($\Rightarrow$) Let $r$ be a radical and $((A_i)_{i \in I}, (f_{ij})_{i \leq j \in I})$ be a directed family in $\mathbb{R}_r$. Then since $\mathbb{R}_r$ is closed under homomorphic image, $\pi \circ u_i(A_i)$ is a radical subact of $\lim(A_i)_{i \in I}$ for the epimorphism $\pi : \bigsqcup_{i \in I} A_i \rightarrow \lim(A_i)_{i \in I}$ and every $i \in I$. So, by Remark 1.1, there exists $X_i \in \Sigma_r(\lim(A_i)_{i \in I})$ such that $\pi \circ u_i(A_i) \leq X_i$, for every $i \in I$.

Now we show that for a fixed $j_0 \in I$, $X_i = X_{j_0}$, for every $i \in I$. Because, for every $i \in I$, there exist $k \in I$ with $f_{ik}(A_i), f_{j_0}(A_{j_0}) \leq A_k$. So we have $\pi \circ u_i(A_i), \pi \circ u_{j_0}(A_{j_0}), \pi \circ u_k(A_k) \leq X_k$. Therefore $X_i = X_{j_0} = X_k$ follows from this fact that $\Sigma_r(\lim(A_i)_{i \in I})$ consist of some disjoint subacts of $\lim(A_i)_{i \in I}$ and hence $\lim(A_i)_{i \in I} = \bigcup_{i \in I} \pi \circ u_i(A_i) = X_{j_0} \in \mathbb{R}_r$.

($\Leftarrow$) Conversely, let $R$ be a subclass of $S$-acts which is closed under homomorphic image and Rees congruence extension. Then since every chain in $\mathbb{R}$ is a directed family, $\mathbb{R}$ has the inductive property. So, by Theorem 2.4 of [20], $\mathbb{R}$ is a radical class of a radical. \qed

**Definition 5.3.** A directed family $D = ((A_i)_{i \in I}, (f_{ij})_{i \leq j \in I})$ of $S$-acts is called $r$-directed if each $f_{ij}$ is an $r$-monomorphism, for every $i \leq j \in I$.

**Theorem 5.4** ($r$-direct limit condition). Let $I$ be an up-directed set with the first element 0 and $((A_i)_{i \in I}, (f_{ij})_{i \leq j \in I})$ be an $r$-directed family of $S$-acts indexed by $I$. Then $\pi \circ u_i$ is an $r$-monomorphism, where $u_i : A_i \rightarrow \bigsqcup_{i \in I} A_i$ is the injection map, for every $i \in I$, and $\pi : \bigsqcup_{i \in I} A_i \rightarrow \lim(A_i)_{i \in I}$ is the canonical epimorphism.
Proof. Given an $r$-directed family $\mathcal{D} = ((A_i)_{i \in I}, (f_{ij})_{i \leq j \in I})$ of $S$-acts, we get the $r$-directed family $((A_i/f_{0i}(A_0))_{i \in I}, (f_{ij})_{i \leq j \in I})$ in $\mathbb{R}_r$. So, Lemma 5.2 implies that $\varprojlim (A_i/f_{0i}(A_0))_{i \in I} \in \mathbb{R}_r$.

But since

$$\varprojlim (A_i/f_{0i}(A_0))_{i \in I} \cong \bigcup_{i \in I} \pi' \circ u_i(A_i/f_{0i}(A_0))$$

$$\cong \left( \bigcup_{i \in I} \pi \circ u_i(A_i) \right)/\left( \pi \circ u_0(A_0) \right)$$

where $\pi : \prod_{i \in I} A_i \to \varprojlim (A_i)_{i \in I}$ and $\pi' : \prod_{i \in I} (A_i/f_{0i}) \to \varprojlim (A_i/f_{0i}(A_0))_{i \in I}$ are the canonical epimorphisms, $(\bigcup_{i \in I} \pi \circ u_i(A_i))/(\pi \circ u_0(A_0)) \in \mathbb{R}_r$ follows from the closedness of $\mathbb{R}_r$ under homomorphic image, for every $j \in I$. That is $\pi \circ u_j(A_j)$ is $r$-dense in $\varprojlim (A_i)_{i \in I}$, for every $j \in I$. □

Remark 5.5. Now, as it is mentioned in [2], in the present of conditions $B_1$-$B_6$, which are stated as follows, we have the well-Behaviour of $r$-injectivity.

$B_1$ - The class of $r$-monomorphisms is composition closed. Because $c'$ is an idempotent closure operator, see Section 2.4 of [18].

$B_2$ - The class of $r$-monomorphisms is trivially isomorphism closed and left regular; that is, for $f \in M$ with $fg = f$ we have $g$ is an isomorphism.

$B_3$ - Banaschewski’s $r$-condition, see Theorem 3.9.

$B_4$ - $S$-Act satisfies $r$-transferability conditions, see Lemma 5.1.

$B_5$ - $S$-Act has $r$-direct limit of well ordered direct systems, See Theorem 5.4.

$B_6$ - $S$-Act is $r^*$-cowell powered; that is for every $S$-act $A$, the class

$$\{ m : A \to B \mid B \in S$-Act, $m$ is an $r$-essential monomorphism. $$

up to isomorphism, is a set. It is trivial.

6. BEAR CRITERION FOR $r$-INJECTIVITY

An important point of study in injectivity is to investigate where there is any relation between the desired injectivity and injectivity with respect to another subclass of monomorphisms, the result of which may be called the Bear type criterion. In this section we give the counterpart of Bear-Skornjakov criterion for $r$-injectivity. We also give another criterion to characterize the weakly injective $S$-acts. We also give a Bear criterion for injective $S$-act in corollary 7.8.

Theorem 6.1. Let $r$ be a radical whose radical class $\mathbb{R}_r$ is closed under coproduct. Then

(i) every $r$-injective $S$-act contains a zero.

(ii) The product $\prod_{i \in I} Q_i$ is $r$-injective if and only if $Q_i$ is an $r$-injective $S$-act, for all $i \in I$.

Proof. One can easily prove the part (ii). To prove part (i), first we note that $A$ is $r$-dense in $A \sqcup \Theta$. Now the result is immediately follows from the following completed commutative diagram, by $g : A \sqcup \Theta \to A$.

$$\begin{array}{ccc}
A & \xrightarrow{\subseteq} & A \sqcup \Theta \\
\downarrow{id_A} & & \downarrow{g} \\
A & & A
\end{array}$$
Theorem 6.2 (Bear-Skornjakov). Let $r$ be a zero-hereditary radical of $S$-Act and $Q$ be an $S$-act with a zero element $\theta$. Then $Q$ is $r$-injective if and only if each homomorphism $f : A_0 \rightarrow Q$ in which $A_0$ is an $r$-dense subact of a cyclic $S$-act $A$ can be extended to $A$.

Proof. To prove it is enough to show that injectivity with respect to $r$-dense subacts of cyclic $S$-acts implies $r$-injectivity, to do so, we follow the standard prove of Skornjakov. So assume $Q$ is an $S$-act with a zero which satisfies the hypothesis and consider the following diagram

\[
\begin{array}{ccc}
B & \rightarrow & A \\
\downarrow f & & \downarrow \\
Q & & \end{array}
\]

in which $B$ is $r$-dense in $A$. Then we take the poset

\[
T = \{ h : C \rightarrow Q \mid B \leq C \leq A, \text{ and } h|_B = f \}
\]

together the partial order

\[
h_1 \leq h_2 \Leftrightarrow \text{Dom}(h_1) \leq \text{Dom}(h_2) \text{ and } h_2|_{\text{Dom}(h_1)} = h_1
\]

But $\text{Dom}(h)$ is $r$-dense subact of $A$, for every $h \in T$, because $A/\text{Dom}(h)$ is homomorphic image of $A/A_0$ and $R_r$ homomorphically closed. Also one can easily see that every ascending chain \{ $h_i : C_i \rightarrow Q$ \}_{i \in I} of $(T, \leq)$ has the upper bond $h : A_1 \rightarrow Q$, by the Zorn’s lemma. Now we show that $A = A_1$. To do so, suppose on the contrary that $A_1 \not\leq A$. Then there exists $a \in A \setminus A_1$ for which we define $D_a = A_1 \cap S_a$. If $D = \emptyset$, then

\[
\overline{f} : A \rightarrow Q \\
a \rightarrow \begin{cases} 
  h(a) & a \in A_1 \\
  \theta & a \in A \setminus A_1
\end{cases}
\]

is an extension of $f$ which commutes the diagram (*) and we get the result.

If $D \neq \emptyset$ then, $D$ is an $r$-dense subact of $S_a$. Because kernel of the homomorphism $k : Sa \rightarrow A/A_1$ defined by $k(sa) = sa/A_1$ is $\rho_D$. So Homomorphism Theorem for $S$-acts implies that $Sa/D$ is isomorphic to a subact $H$ of $A/A_1$. Now since $r$ is a zero hereditary radical and $H$ is a subact with a zero element of the radical $S$-act $A/A_1$, we have $r(Sa/D) \cong r(H) = \nabla_H \cong \nabla_{Sa/D}$.

Therefore there exists an extension $\overline{g} : Sa \rightarrow Q$ of the homomorphism $g : D \rightarrow Q$ defined by $g(sa) = h(sa)$, for every $sa \in D$. Thus this means that

\[
\overline{g} : A_1 \cup Sa \rightarrow Q \\
x \rightarrow \begin{cases} 
  h(x) & x \in A_1 \\
  s\overline{g}(a) & x = sa \in Sa
\end{cases}
\]

is an extension of $h$ and it contradicts the maximality of $h$. So $A_1 = A$ and we are done. \qed
Corollary 6.3. Let \( r \) be a zero-hereditary radical of \( S\text{-}\text{Act} \) and \( E \) be an \( S\text{-}\text{act} \) with a zero element \( \theta \). Then \( E \) is \( r \)-injective if and only if it is injective with respect to the \( r \)-large monomorphisms into cyclic \( S\text{-}\text{acts} \).

Proof. One way is clear. To prove converse, using Theorem 6.2, we show that every \( S\text{-}\text{act} \) with a zero which satisfies the hypothesis is an injective \( S\text{-}\text{act} \) with respect to \( r \)-monomorphisms into the cyclic \( S\text{-}\text{acts} \). To do so, consider the following diagram

\[
\begin{array}{ccc}
B & \xrightarrow{m} & C \\
\downarrow f & & \downarrow f \\
E & & \end{array}
\]

in which \( m \) is an \( r \)-monomorphism and \( C \) is a cyclic \( S\text{-}\text{act} \). Then, by Theorem 3.9, \( m : B \rightarrow C \) can be extend to an \( r \)-large monomorphism \( g \circ m : B \rightarrow C \rightarrow A \). Now existence of a homomorphism \( f : A \rightarrow E \) with \( f \circ m = f \) follows from hypothesis. Hence we get \( f | C : C \rightarrow E \) which completes the designed diagram. \( \square \)

Theorem 6.4. Given a hereditary radical \( r \), a semisimple \( S\text{-}\text{act} \) \( I \) is weakly injective if and only if it is injective relative to all inclusions into \( S/r( S) \).

Proof. (\( \Rightarrow \)) For an arbitrary weakly injective \( S\text{-}\text{act} \) \( I \), consider \( f' \) to be a homomorphism from a subact \( K/r( K) \) of \( S/r( S) \) to \( I \). Then, by the hypothesis, there exists an extension homomorphism \( f : S \rightarrow I \) for \( f' \circ \pi_K = f \) which commutes the left triangle of the following diagram,

\[
\begin{array}{ccc}
K & \xrightarrow{\pi_K} & S \\
\downarrow f' & & \downarrow f \\
K/r( K) & \xrightarrow{\pi_S} & S/r( S) \\
\end{array}
\]

in which \( \pi_S \) and \( \pi_K \) are the canonical epimorphisms. But the property of the radical, implies \( f( r( S)) \leq r( I) = \Delta_I \) and hence \( r( S) \leq \ker( f) \). So, Homomorphism Theorem for \( S\text{-}\text{acts} \), implies that there exists a homomorphism \( f' \) from \( S/r( S) \) to \( I \) which completes the above diagram and we are done.

(\( \Leftarrow \)) Let \( I \in S_r \) be injective relative to all inclusions into \( S/r( S) \), and \( f \) be a homomorphism from a left ideal \( K \) to \( I \). Then by \( f( r( K)) \subseteq r( I) = \Delta_I \) we have \( r( K) \leq \ker( f) \). Hence Homomorphism Theorem for \( S\text{-}\text{acts} \) implies the existence of a homomorphism \( f' \) from \( K/r( K) \) to \( I \) such that \( f = f' \pi_K \) where \( \pi_K : K \rightarrow K/r( K) \) is the canonical epimorphism. Now, by the hypothesis, there exists a homomorphism \( f' \) from \( S/r( S) \) to \( I \) which commutes the bottom triangle of the following diagram,

\[
\begin{array}{ccc}
K & \xrightarrow{\pi_K} & S \\
\downarrow f' & & \downarrow f' \\
K/r( K) & \xrightarrow{\pi_S} & S/r( S) \\
\end{array}
\]
where $\pi_S$ and $\pi_K$ are the canonical epimorphisms. Now $\mathcal{J} = \hat{f} \circ \pi_S$ is an extension of $f$ and commutes the desired diagram, meaning that $I$ is weakly injective. \hfill \Box

7. $r$-injectivity for a Kurosh-Amitsur radical

In this section we discuss $r$-injectivity when $r$ is a Kurosh-Amitsur radical rather than a radical to improve the results hereof. We then construct an Kurosh-Amitsur radical $r_G$ whose associated $r_G$-injective $S$-acts are exactly injective $S$-act. Throughout this section, we assume that $E(A)$ and $E_r(A)$ are respectively, the usual injective hull and $r$-injective hull of the $S$-act $A$.

**Proposition 7.1.** Let $r$ be a Kurosh-Amitsur radical. Then $E_r(A) = c_{E_r(A)}^r(A)$, for every $A \in S$-Act.

**Proof.** Given an $S$-act $A$, then considering $A$ as a subact of $E(A)$, three possible cases may occur:

Case (i): $A$ is $r$-dense in $E(A)$, that is $E(A) = c_{E(A)}^r(A)$. Then $E(A)$ is a maximal $r$-essential extension of $A$. Therefore $E(A) = c_{E(A)}^r(A)$ is $r$-injective hull of $A$.

Case (ii): $A$ is $r$-closed in $E(A)$, that is $A = c_{E_r(A)}^r(A)$. Then $A$ is not $r$-dense in any extension of itself, by lemma 2.13, and hence $E_r(A) = A = c_{E_r(A)}^r(A)$.

Case (iii): $A < c_{E_r(A)}^r(A) < E(A)$. Then, since $c_{E_r(A)}^r$ is weakly hereditary, by Lemma 2.9, $A$ is $r$-dense in $c_{E_r(A)}^r(A)$. Also $c_{E_r(A)}^r(A)$ is an $r$-essential extension of $A$ since $A \leq c_{E_r(A)}^r(A) \leq E(A)$. So, to prove, it is enough we verify the maximality of $c_{E_r(A)}^r(A)$ among all $r$-essential extensions of $A$. To do so, let $B$ be an $r$-essential extension of $A$ with $c_{E_r(A)}^r(A) \leq B$. Then $B/A \in \mathbb{R}$, and $c_{E_r(A)}^r(A)/A \leq B/A \leq E(A)/A$. Thus $\nabla_{B/A} = r(B/A) \leq r(E(A)/A)$. Hence, by Remark 1.1, $C/A \in \Sigma_{E_r(A)/A}$ exists such that $c_{E_r(A)}^r(A)/A = B/A \leq C/A$. But since the subacts in $\Sigma_{E_r(A)/A}$ are disjoint and $c_{E_r(A)}^r(A)/A \in \Sigma_{E_r(A)/A}$ ($c_{E_r(A)}^r(A)/A = [A]_{r(E_r(A)/A)}$), we have $c_{E_r(A)}^r(A)/A = B/A = C/A$. Thus $c_{E_r(A)}^r(A) = B$. That is $c_{E_r(A)}^r(A)$ is the maximal $r$-essential extension of $A$. \hfill \Box

We use the above proposition to give a characterization of the $r$-injective $S$-acts, see the following corollary.

**Corollary 7.2.** Given a Kurosh-Amitsur radical $r$, an $S$-act $A$ is $r$-injective if and only if $A$ is an $r$-closed subact of $E(A)$.

**Proof.** ($\Rightarrow$) If $A$ be an $r$-injective $S$-act, then $E_r(A) = A$. But since $E_r(A) = c_{E(A)}^r(A)$, by Proposition 7.1, $A = c_{E(A)}^r(A)$. That is, $A$ is $r$-closed in $E(A)$.

($\Leftarrow$) Immediately follows from Lemma 4.6. \hfill \Box

In the following we give a characterization of the hereditary Kurosh-Amitsur radicals by injective hull and $r$-injective hull. But first we recall the lemma below from [11] which is used in the sequel.

**Lemma 7.3.** A pair $(\mathbb{R}, S)$ of subclasses of $S$-acts is the radical class and the semisimple class of a Kurosh-Amitsur radical $r$ if and only if

1. $\mathbb{R} \cap S$ consists of trivial $S$-acts,
2. $\mathbb{R}$ is homomorphically closed,
3. $S$ is closed under taking subacts,
(4) every $S$-act $A$ has an $\mathbb{R}$-system such as $\Sigma$ whose Rees factor, $A/\rho_r$, belong to $\mathcal{S}$. 

**Theorem 7.4.** For a Kurosh-Amitsur radical $r$ of $S$-Act, the following conditions are equivalent.

1. The radical $r$ is hereditary.
2. Given an $S$-act $B$, the homomorphic image $B$ under a homomorphism $f$ is a radical $S$-act if and only if there exists an extension $A$ of $B$ such that $\nabla_B \subseteq \pi^{-1}(r(A/\ker(f) \vee \Delta_A))$ where $\pi : A \to A/(\ker(f) \vee \Delta_A)$ is the canonical epimorphism.
3. The radical class $\mathbb{R}_r$ is closed under taking subacts.
4. The semisimple class $\mathcal{S}_r$ is closed under $r$-injective hulls.
5. The semisimple class $\mathcal{S}_r$ is closed under essential Rees extensions.

**Proof.** (1) $\Rightarrow$ (2) Necessity: Follows from Homomorphism Theorem, for $S$-acts, when we take $B = A$.

Sufficiency: From Lemma 2.12, we know that the hypothesis implies $\nabla_B \subseteq \pi^{-1}(r(E(B)/(\ker(f) \vee \Delta_E(B))))$, where $E(B)$ is the injective hull of $B$ and $\pi : E(B) \to E(B)/(\ker(f) \vee \Delta_E(B))$ is the canonical epimorphism. So, we have $\nabla_B/\ker(f) \subseteq r(E(B)/(\ker(f) \vee \Delta_E(B)))$. Hence

$$r(f(B)) \cong \frac{B}{\ker(f)} = \frac{E(B)}{\ker(f) \vee \Delta_E(B)} \Rightarrow \nabla_B \subseteq \pi^{-1}(r(A/\Delta_A)),$$

since $r$ is hereditary.

(2) $\Rightarrow$ (3) Suppose $A \in \mathbb{R}_r$ and $i : B \to A$ is the inclusion map. Then $\nabla_B = \nabla_{i(B)} \subseteq \nabla_A = r(A) = r(A/\Delta_A)$. Hence hypothesis implies that $B \in \mathbb{R}_r$.

(3) $\Rightarrow$ (4) To prove, we show that the $r$-injective hull $E_r(A)$ of each semisimple $S$-act $A$ is a semisimple $S$-act. Indeed, The largeness of $A$ in $E_r(A)$ implies that $A \cap X \neq \emptyset$, for every non-trivial subact $X \in \Sigma_r(E_r(A))$. But we know that $A \in \mathcal{S}_r$, $X \in \mathbb{R}_r$ and both $\mathcal{S}_r$ and $\mathbb{R}_r$ are closed under taking subacts. So, we have $A \cap X \in \mathbb{R}_r \cap \mathcal{S}_r$. Hence $A \cap X$ is a trivial $S$-act since $\mathbb{R}_r \cap \mathcal{S}_r$ consists of the trivial $S$-acts. Thus $X$ is a trivial $S$-act which means $\Sigma_{r(E_r(A))} = \emptyset$. Therefore $E_r(A) \in \mathcal{S}_r$ since $r$ is a Kurosh-Amitsur radical.

(4) $\Rightarrow$ (5) To prove, we show that the injective hull $E(A)$ of each semisimple $S$-act $A$ is a semisimple $S$-act. Indeed, The largeness of $E_r(A)$ in $E(A)$ implies that $E_r(A) \cap \pi^{-1}(X) \neq \emptyset$, for every non-trivial subact $X \in \Sigma_r(E(A)/E_r(A))$ and the canonical epimorphism $\pi : E(A) \to E(A)/E_r(A)$. Thus $X = [E_r(A)]_{r(E(A)/E_r(A))}$, for every non-trivial subact $X \in \Sigma_r(E(A)/E_r(A))$, since $X$ and $[E_r(A)]_{r(E(A)/E_r(A))}$ are $r(E(A)/E_r(A))$-classes. But $[E_r(A)]_{r(E(A)/E_r(A))}$ is singleton since $E_r(A)$ is $r$-closed in $E(A)$. So, $X$ is a trivial $S$-act. Thus $\Sigma_r(E(A)/E_r(A))$ is empty, and hence $E(A)/E_r(A)$ belongs to $\mathcal{S}_r$ since $r$ is a Kurosh-Amitsur radical. Also $E_r(A) \in \mathcal{S}_r$, by hypothesis. Therefore $E(A) \in \mathcal{S}_r$ follows from the closedness of $\mathcal{S}_r$ under Rees congruence extension. This means that $\mathcal{S}_r$ is closed under injective hulls.

(5) $\Rightarrow$ (6) Suppose $\rho$ is an essential Rees congruence on an $S$-act $A$ with $\Sigma_r \in \mathcal{S}_r$. We Show that $A \in \mathcal{S}_r$. To do so, we contrary assume on the $A \notin \mathcal{S}_r$. Then $r(A) \neq \Delta_A$ and $\rho \cap r(A) \neq \Delta_A$ follows from essentiality of $\rho$. Thus there exists a non-trivial subact $B \leq C \in \Sigma_r(A)$ such that $\rho_B \leq \rho \cap r(A)$ since $r(A)$ and $\rho$ are
Rees congruence. The closedness of \( \mathcal{S}_r \) under taking subacts implies that \( B \in \mathcal{S}_r \), and we have \( E(B) \in \mathcal{S}_r \), by hypothesis. Now consider the following commutative diagram.

\[
\begin{array}{ccc}
B & \xrightarrow{c} & C \\
\subseteq & \downarrow f & \subseteq \\
E(B) & & \\
\end{array}
\]

We note that \( f(C) \in \mathcal{S}_r \cap \mathbb{R}_r \), since \( C \in \mathbb{R}_r \) and \( E(B) \in \mathcal{S}_r \). Thus, by Lemma 7.3, \( f(C) \) is a trivial \( S \)-act. The commutativity of the above diagram implies that \( B \) is trivial and this is a contradiction. Therefore \( \mathcal{S}_r \) is closed under essential Rees extension.

(6) \( \Rightarrow \) (1) Proposition 3.3 of [20] implies that \( \mathbb{R}_r \) is closed under taking subacts and this implies (1) by Proposition 4.1 of [20].

**Theorem 7.5.** Given a Kurosh-Amitsur radical \( r \), an \( r \)-injective \( S \)-act \( A \) is a semisimple \( S \)-act if and only if \( E(A) \) is a semisimple \( S \)-act.

**Proof.** \(( \Rightarrow \) \) Let \( A \) be a semisimple \( r \)-injective \( S \)-act. Then \( A \) is \( r \)-closed in \( E(A) \), by Corollary 7.2. Also \( \Delta_A = r(A) \subseteq r(E(A)) \), by Theorem 2.11. Hence \( \rho_{A \cap B} \leq r(E(A)) \), for all non-trivial subact \( B \in \Sigma_{r(E(A))} \). Thus \( |A \cap B| \leq 2 \). But, for all non-trivial subact \( B \in \Sigma_{r(E(A))} \), we have \( |A \cap B| \geq 2 \) since \( A \) is large in \( E(A) \). So every \( B \in \Sigma_{r(E(A))} \) is a trivial \( S \)-act. Hence \( \Sigma_{r(E(A))} = \emptyset \). Therefore \( r(E(A)) = \Delta_{E(A)} \) since \( r \) is a Kurosh-Amitsur radical. That is \( E(A) \in \mathcal{S}_r \).

\(( \Leftarrow \) \) It follows from the closedness of \( \mathcal{S}_r \) under taking subact. \( \square \)

**Theorem 7.6.** Let \( r \) be a Kurosh-Amitsur radical. Then

1. the radical class \( \mathbb{R}_r \) is closed under \( r \)-injective hulls.
2. Every \( B \in \Sigma_{r(A)} \) is \( r \)-injective if \( A \) is an \( r \)-injective \( S \)-act.

**Proof.**

1. By Proposition 7.1, we have \( E_r(A) = c^r_{E(A)}(A) \), for \( A \in \textbf{S-Act} \). So, to prove, it is enough to show that \( c^r_{E(A)}(A) \in \mathbb{R}_r \), for every \( A \in \mathbb{R}_r \). But, since \( c^r \) is weakly hereditary, see Lemma 2.9, we have \( c^r_{E(A)}(A) = c^r_{E(A)}(A) = \pi^{-1}([A]_{r(c^r_{E(A)}(A))}) \), for the canonical epimorphism \( \pi : c^r_{E(A)}(A) \to c^r_{E(A)}(A)/A \).

Hence \( c^r_{E(A)}(A)/A = [A]_{r(c^r_{E(A)}(A))} \). Also, since \( \pi(A) \) is a zero element of \( c^r_{E(A)}(A)/A \), \( [A]_{r(c^r_{E(A)}(A))} \subseteq \Sigma_r(c^r_{E(A)}(A)) \). Therefore \( r(c^r_{E(A)}(A)) = \nabla c^r_{E(A)}(A)/A \). That is \( c^r_{E(A)}(A) \in \mathbb{R}_r \).

2. Let \( A \) be an \( r \)-injective \( S \)-act and \( B \in \Sigma_{r(A)} \). Then we show that \( E_r(B) = B \).

Indeed, \( E_r(B) \in \mathbb{R}_r \), since \( B \in \Sigma_{r(A)} \subseteq \mathbb{R}_r \). And \( \mathbb{R}_r \) is closed under taking \( r \)-injective hull, by the former part. So \( E_r(B) \leq A \) since \( A \) is an \( r \)-injective \( S \)-act, containing \( B \). Thus there exists \( C \in \Sigma_{r(A)} \) such that \( E_r(B) \leq C \). Since \( E_r(B) = r(E_r(B)) \), \( r(A) = r(E_r(B)) \), and \( r \) is a Kurosh-Amitsur radical. Now \( B = E_r(B) = C \) follows from this fact that the subacts in \( \Sigma_{r(A)} \) are disjoint and \( B \leq E_r(A) \leq C \).

\( \square \)

In the sequel we are going to define a Kurosh-Amitsur radical \( r_G \) such that the injective \( S \)-acts, with respect to \( r_G \)-monomorphisms are exactly the injective \( S \)-acts.
For every $S$-act $A$ and a zero element $\theta$ of $A$, we define

$$X_\theta := \bigcup \{ C_\theta \mid C_\theta \text{ is a cyclic subact of } A \text{ such that } \forall c \in C_\theta \exists s \in S, sc = \theta \}$$

and $Z_A = \{ \theta \mid \theta \text{ is a zero element of } A \}$. We claim that the following assignment is a Kurosh-Amitsur radical.

$$r_G : A \mapsto r_G(A) = \begin{cases} \bigvee_{\theta \in Z_A} \rho_{X_\theta} Z(A) \neq \emptyset \\ \Delta_A \text{ otherwise} \end{cases}$$

Indeed, for

$$S_{rg} = \{ A \mid \text{every non-trivial subact of } A \text{ has a cyclic subact without zero} \}$$

and

$$\mathbb{R}_{rg} = \{ A \mid A \text{ has a zero element } \theta_A \text{ such that } \forall a \in A \exists s \in S, sa = \theta \},$$

we have

1. $\mathbb{R}_{rg} \cap S_{rg}$ consists of trivial $S$-acts,
2. $\mathbb{R}_{rg}$ is homomorphically closed,
3. $S_{rg}$ is closed under taking subacts,
4. every $S$-act $A$ has $\mathbb{R}_{rg}$-system $\Sigma = \{ X_\theta \}_{\theta \in Z_A}$ whose Rees factor, $A/\rho_\kappa$, belong to $S_{rg}$.

Therefore $r_G$ is a Kurosh-Amitsur radical, by Lemma 7.3.

It worth noting that since the radical class $\mathbb{R}_{rg}$ is closed under taking subacts, $r_G$ is hereditary, by Theorem 7.4.

**Theorem 7.7.** The $r_G$-injective $S$-acts are exactly the injective $S$-acts.

**Proof.** Let $I$ be an $r_G$-injective $S$-act. Then, using Skornjakov criterion, we show that $I$ is injective with respect to the cyclic subacts. So consider the following diagram in which $B$ is a cyclic subact of $A$.

$$\begin{array}{c}
B \\
\downarrow f \\
I
\end{array} \quad A$$

Then there exists a maximal congruence $\kappa$ with $\kappa \land \rho_B = \Delta$. By Theorem 3.5 and Lemma 3.7, the image of $B$ under canonical epimorphism $\pi : A \to A/\kappa$ is large in $A/\kappa$. Hence, for every $[a]_\kappa \in A/\kappa$, there is $s \in S$ such that $[sa]_\kappa \in A/\kappa$, by Lemma 2.16. Thus from the definition of $\mathbb{R}_{rg}$ we have $A/\kappa/\pi(B) \in \mathbb{R}_{rg}$. Now since, by Lemma 3.6, $B$ is isomorph with $\pi(B)$, there exists an extension $\hat{f} : A/\kappa \to I$ of $f$. Therefore the map $\bar{f} : A \to I$ with $\bar{f}(\bar{a}) = \hat{f}([a]_\kappa)$ is an extension of $f$. So $I$ is injective. \qed

With Corollary 6.3 and Theorem 7.7 in mind we give an stronger version of Bear-Skornjakov criterion for injectivy, see the following corollary.

**Corollary 7.8.** An $S$-act $I$ is injective if and only if $I$ has a zero element and each homomorphism $f : B \to I$ in which $B$ is a large subact of a cyclic $S$-act $A$ can be extended to $A$. 
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