Localization of elastic waves in heterogeneous media with off-diagonal disorder and long-range correlations

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Using the Martin-Siggia-Rose method, we study propagation of acoustic waves in strongly heterogeneous media which are characterized by a broad distribution of the elastic constants. Gaussian-white distributed elastic constants, as well as those with long-range correlations with non-decaying power-law correlation functions, are considered. The study is motivated in part by a recent discovery that the elastic moduli of rock at large length scales may be characterized by long-range power-law correlation functions. Depending on the disorder, the renormalization group (RG) flows exhibit a transition to localized regime in any dimension. We have numerically checked the RG results using the transfer-matrix method and direct numerical simulations for one- and two-dimensional systems, respectively.

Understanding how waves propagate in heterogeneous media is fundamental to such important problems as earthquakes, underground nuclear explosions, the morphology of oil and gas reservoirs, oceanography, and medical and materials sciences [1]. For example, seismic wave propagation and reflection are used to not only estimate the hydrocarbon content of a potential oil or gas field, but also to image structures located over a wide area, ranging from the Earth’s near surface to the deeper crust and upper mantle. The same essential concepts and techniques are used in such diverse fields as materials science and medicine.

In condensed matter physics, a related problem, namely, the nature of electronic states in disordered materials, has been studied for several decades and shown to depend strongly on the spatial dimensionality d of the materials [2]. It was rigorously shown that, for one-dimensional (1D) systems, even infinitesimally small disorder is sufficient for localizing the wave function, irrespective of the energy [3], and that the envelop of the wave function \( \psi(r) \) decays exponentially at large distances r from the domain’s center, \( \psi(r) \sim \exp(-r/\xi) \), with \( \xi \) being the localization length. The most important results for \( d > 1 \) follow from the scaling theory of localization [4] which predicts that, for \( d \leq 2 \), all electronic states are localized for any degree of disorder, while a transition to extended states - the metal-insulator transition - occurs for \( d > 2 \) if disorder is sufficiently strong. The transition between the two states is characterized by divergence of the localization length, \( \xi \propto |W - W_c|^{-\nu} \), where \( W_c \) is the critical value of the disorder intensity. Wegner [5] derived a field-theoretic formulation for the localization problem which, together with the scaling theory [4], predict a lower critical dimension, \( d_c = 2 \), for the localization problem. These predictions have been confirmed by numerical simulations [6].

Wave characteristics of electrons suggest that the localization phenomenon may occur in other wave propagation processes. For example, consider propagation of seismic waves in heterogeneous rock. In this case, the interference of the waves that have undergone multiple scattering, caused by the heterogeneities of the medium, may cause their localization. Unlike electrons, however, the classical waves do not interact with one another and, therefore, propagation of such waves in heterogeneous media (such as porous rock) provides [7] an ideal model for studying the classical Anderson localization [8-12] in strongly disordered media. This is the focus of this Letter. We study localization of acoustic waves in strongly heterogeneous media, and formulate a field-theoretic method to investigate the problem in the media that are characterized by a broad distribution of the elastic constants. Localization of acoustic waves was previously studied by several groups [13], although not in the strongly disordered media that we consider in this Letter. The system that we study is the continuum limit of an acoustic system with off-diagonal disorder. Our approach is based on the method first introduced by Martin, Siggia and Rose [14] for analyzing dynamical critical phenomena. We calculate the one-loop beta functions [8,14] for both spatially delta-correlated and power-law correlated disorder in the elastic constants, and show that in any
case there is a disorder-induced transition from delocalized to localized states for any $d$. In addition to being interesting on its own, our study is motivated in part by the recent discovery [15] that the distribution of the elastic moduli of heterogeneous rock contains long-range correlations characterized by a *nondecaying* power-law correlation function. Baluni and Willemen [16] studied propagation of acoustic waves in a 1D layered system, which can be thought of as a simple model of rock (although their goal was not to study acoustic wave propagation in rock), and showed that the waves are localized. However, they did not consider higher-dimensional systems, nor did they study the type of disorder that we consider in the present Letter. The possibility of wave localization in disordered media with long-range, nondecaying correlations has important practical implications which will we discuss briefly. However, our results are completely general and apply to any material in which the local elastic constants are distributed broadly. We confirm the analytical predictions for 1D and 2D systems using numerical simulations.

Wave propagation in a medium with a distribution of elastic constants is described by the following equation (for simplicity we consider the scalar wave equation):

$$\frac{\partial^2 \psi(x,t)}{\partial t^2} - \nabla \cdot [\lambda(x) \nabla \psi(x,t)] = 0 ,$$  

(1)

where $\psi(x,t)$ is the wave amplitude, and $\lambda(x) = C(x)/m$ is the ratio of the elastic stiffness $C(x)$ and the mean density $m$ of the medium. We then write $\lambda$ as,

$$\lambda(x) = \lambda_0 + \eta(x) ,$$  

(2)

where $\lambda_0 = \langle \lambda(x) \rangle$. In this Letter we assume $\eta(x)$ to be a Gaussian random process with zero mean and the covariance,

$$\langle \eta(x) \eta(x') \rangle = 2K(|x-x'|) = 2D_0 \delta^d(x-x')$$  

$$+ 2D_\rho |x-x'|^{2\rho-d} .$$  

(3)

in which $D_0$ and $D_\rho$ represent the strength of the disorder due to the delta-correlated and power-law correlated parts of the disorder. Previously, Souillard and co-workers [17] studied wave propagation in disordered fractal media, which is characterized by a *decaying* power-law correlation function. Their study is not, however, directly related to our work. Consider a wave component $\psi(x,\omega)$ with angular frequency $\omega$, which is obtained by taking the temporal Fourier transform of Eq. (1) which yields the following equation for propagation of a wave component in a disordered medium,

$$\nabla^2 \psi(x,\omega) + \frac{\omega^2}{\lambda_0} \psi(x,\omega) + \nabla \cdot \left[ \frac{\eta(x)}{\lambda_0} \nabla \psi(x,\omega) \right] = 0 .$$  

(4)

Since $\eta(x)$ is a Gaussian variable, we obtain a Martin-Siggia-Rose effective action $S_c$ for the probability density functional of the wave function $\psi(x,\omega)$, given by

$$S_c(\psi_I, \psi_R, \psi, \chi, \chi^*) =$$

$$\int dx dx' [ i \psi_I(x') (\nabla^2 + \frac{\omega^2}{\lambda_0}) \psi_I(x)$$

$$+ i \psi_R(x') (\nabla^2 + \frac{\omega^2}{\lambda_0}) \psi_R(x) ,$$

$$+ \chi^*(x') (\nabla^2 + \frac{\omega^2}{\lambda_0}) \chi(x) \delta(x-x')$$

$$+ (i \nabla \tilde{\psi}_I \nabla \psi_I + i \nabla \tilde{\psi}_R \nabla \psi_R + \nabla \chi \nabla \chi) \frac{K(x-x')}{\lambda_0^2}$$

$$\times \left[ i \nabla \tilde{\psi}_I \nabla \psi_I + i \nabla \tilde{\psi}_R \nabla \psi_R + \nabla \chi \nabla \chi \right] .$$  

(5)

Here, $\tilde{\psi}_I(x)$, $\tilde{\psi}_R(x)$, $\chi$ and $\chi^*$ are the auxiliary and Grassmanian fields of the field-theoretic formulation, respectively. Two coupling constants, $g_0 = D_0/\lambda_0^2$ and, $g_\rho = D_\rho/\lambda_0^2$, appear in $S_c$. Thus, we carry out a renormalization group (RG) analysis in the critical limit, $\omega^2/\lambda_0 \rightarrow 0$, to derive, to one-loop order, the beta functions [7,12] that govern the two couplings under the RG transformation. The results are given by,

$$\beta(g_0) = \frac{\partial g_0}{\partial \ln l} = -d g_0 + 8g_0^2 + 10g_0^2 + 20g_0 g_\rho ,$$  

(6)

$$\beta(g_\rho) = \frac{\partial g_\rho}{\partial \ln l} = (2\rho - d) g_\rho + 12g_0 g_\rho + 16g_\rho ,$$  

(7)

where $l > 1$ is the re-scaling parameter, and $\tilde{g}_0$ and $\tilde{g}_\rho$ are given by,

$$\tilde{g}_0 = k_d \left[ \frac{d + 5}{2(d+2)} \right] g_0 ,$$  

(8)

$$\tilde{g}_\rho = k_d \left[ \frac{d + 5}{2(d+2)} \right] g_\rho ,$$  

(9)

with $k_d = S_d/(2\pi^d)$, and $S_d$ being the surface area of the $d$-dimensional unit sphere. Examining the RG flows, Eqs. (6) and (7), reveals that, depending on $\rho$, there are two distinct regimes:

(i) For $0 < \rho < d/2$ there are three fixed points: The trivial Gaussian fixed point ($g_0^* = g_\rho^* = 0$) which is stable, and two non-trivial fixed points and eigenvalues. One is, $\{ g_0^* = d/8, g_\rho^* = 0 \}$, while the other set is given by,

$$g_0^* = - \frac{4}{41} \left[ \frac{d + 5}{16} (2\rho - d) \right]$$

$$- \frac{4}{41} \sqrt{\left[ \frac{d + 5}{16} (2\rho - d) \right]^2 + \frac{205}{256} (2\rho - d)^2} .$$
which is stable in one eigendirection but unstable in the other eigendirection.

The corresponding RG flow diagram is shown in Figure 1. Therefore, for $0 < \rho < d/2$ the one-loop RG calculation indicates that the system with uncorrelated disorder is unstable against long-range correlated disorder toward a new fixed point in the coupling constants space, for which there is a phase transition from delocalized to localized states with increasing the disorder intensity.

(ii) For $\rho > d/2$ there are two fixed points: the Gaussian fixed point which is stable on the $g_0$ axis but unstable on the $g_\rho$ axis, and the non-trivial fixed point, $\{g_0^* = d/8, g_\rho^* = 0\}$, which is unstable in all directions. The RG flow diagram for this case is shown in Figure 2. The implication is that, while the power-low correlated disorder is relevant, no new fixed point exists to one-loop order and, therefore, the long-wavelength behavior of the system is determined by the long-range component of the disorder. This means that for $\rho > d/2$ the waves are localized for any $d$. In addition, in both cases (i) and (ii) the system undergoes a disorder-induced transition when only the uncorrelated disorder is present. Let us mention that the above results are general so long as $D_\rho > 0$ (which is the only physically acceptable limit). For $D_\rho < 0$ the above phase space is valid for $\rho > \frac{1}{2}(d + 1)$.

To test these predictions, we have carried out numerical simulations of the problem in both 1D and 2D. Consider first the 1D disordered systems. In this case, the waves are localized when the wave functions are of the form, $\psi(x) = f(x) \exp(-|x - x_0|/\xi)$, where $f(x)$ is a stochastic function which depends on the particular realization of the disordered chain, and $\xi$ is the localization length. Experimentally, the simplest 1D model that exhibits wave localization is [18] a 15 m long steel wire with a 0.178 mm diameter, suspended vertically. The tension in the wire is maintained with a weight attached at its
lower end. The function $\psi(x,t)$ consists of transverse waves in the wire with an electromechanical actuator at one end of the wire.

It was shown [18] that, even for very small deviations (less than 1%) from periodicity, the diagonal disorder (e.g., variations in the resonance frequencies of the oscillators) produces localization (which is in agreement with Furstenberg’s theorem [19]), while variations (up to 13%) in the sizes of the masses (off-diagonal disorder) result in localization lengths that are much larger than the size of the system. This is in agreement with our theoretical prediction.

To reproduce this result numerically and to calculate the localization length $\xi$, we used the transfer-matrix (TM) method [12]. Discretizing Eq. (1) and writing down the result for site $n$ of a linear chain yields,

$$(\omega + \lambda_n)\psi_n + \lambda_{n+1}\psi_{n+2} - (\lambda_{n+1} - \lambda_n)\psi_{n+1} = 0,$$  \hspace{1cm} (11)

which can be rewritten in the recursive form

$$M_n \begin{pmatrix} \psi_{n+2} \\ \psi_{n+1} \end{pmatrix} = \begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix}$$  \hspace{1cm} (12)

with

$$M_n = \begin{pmatrix} -\omega^2 - \lambda_n + \lambda_{n-2} & \lambda_{n-2} \\ \lambda_n & 1 \end{pmatrix}.$$  \hspace{1cm} (13)

The localization length $\xi(\omega)$ is then defined by, $\xi(\omega)^{-1} = \lim_{N \to \infty} N^{-1}|\psi_N/\psi_0|$, where $N$ is the chain’s length. For every realization of the disorder we computed $\psi_N$ and, hence, $\xi(\omega)$. We chose, $\psi_0 = \psi_1 = 1/\sqrt{2}$, and averaged $\xi$ over a large ensemble of realizations for a fixed system size $N$ and frequency $\omega$. We then repeated this procedure for several values of $N$ and $\omega$. The extended states correspond to having, $\lim_{N \to \infty} \xi/N = \text{constant} > 1$. As $\xi$ is also a function of $\omega$, we chose, $\omega = 2\pi \sqrt{\lambda_0}/N$, which is the smallest mode of the system. Our RG analysis indicates that this mode most likely passes through the 1D disordered chain.

The TM computations indicate that the coupling constant $g(N)$ follows a finite-size scaling, $g(N) = g_0 + 1/N$, where $g_0$ is the coupling constant in the thermodynamic ($N \to \infty$) limit; we find that, $g_0 \approx 0.117$. In addition, when $\xi \to N$, one can write, $N \propto (g - g_0)^{-\nu}$ and obtain an estimate of the localization exponent $\nu$. Our analytical results indicated [20] that in $d$ dimensions, $\nu = 1/d$, which agrees with the TM calculations that yield $\nu = 1$. Figure 3 shows the results for the localization length $\xi$ as a function of $\omega$. These results confirm the RG predictions for disordered linear chains.

To further check the RG results, we also solved Eq. (1) in 2D using the finite-difference method with second-order discretization for both the space and time variables. Such approximations are acceptable as we work in the low frequencies or long wavelengths. For short wavelengths we should use higher-order discretizations for the spatial variables [20]. A $L_x \times L_y$ grid was used with $L_x = 8000$ and $L_y = 400$. The parameter $\lambda(x)$, representing the local, gridblock-scale elastic constant, was distributed with a power-law correlation function, of the type considered above, with its spatial distribution generated using the midpoint displacement method [21]. We also simulated the case in which the local values of $\lambda(x)$ were uncorrelated and uniformly distributed, with the same variance as that of the power-law case. By inserting a wave source on a line on one side of the grid, we simulated numerically 2D wave propagation through the system. Periodic boundary conditions were imposed in the lateral direction, which did not distort the nature of the wave propagation, as we used large system sizes. The decay in the amplitude of the wave is caused by scattering from heterogeneities of system generated by the distribution of the local elastic constants. The accuracy of the solution was checked by considering the stability criterion and the wavelength of the source [22], and using higher-order finite-difference discretizations. To compute the amplitude decay in the medium, we collected the numerical results at 80 receivers (grid points), distributed evenly throughout the grid, along the direction of wave propagation. The results were averaged over 32 realizations of the system.

Figure 4 presents the decay in the wave amplitude through the uniformly random medium, and that of a medium with a nondecaying power-law correlation function for the local elastic constants $\lambda(x)$, with $\rho = 1.3$ and 1.8. The wave amplitudes for the correlated cases decline much faster than those in the uniformly random medium. In particular, for $\rho = 1.3$, which corresponds to negative correlations (that is, a large local elastic modulus is likely to be neighbor to a small one, and vice versa), the amplitude decreases rather sharply. These results confirm the RG predictions for 2D systems.

In summary, we show that, depending on the nature of disorder, acoustic waves in strongly disordered media can be localized or delocalized in any dimensions. In particular, they can be extended in disordered 1D systems if the correlation function for the distribution of the local elastic constants is of nondecaying power-law type, and that the waves are localized in any dimension if the exponent $\rho$ of the power-law correlation function is larger than $d/2$. These results, which contradict the generally-accepted view that off-diagonal disorder has a much weaker effect on localization than the diagonal disorder, have important practical implications. For example, in order for seismic records to contain meaningful information on the geology and content of a natural porous formation of linear size $L$, the localization length $\xi$ must be larger $L$. Otherwise, propagation and scattering of such waves can provide information on the formation only up to length scale $\xi$; one cannot obtain meaningful information at larger length scales [23]. The localization
length $\xi$ is, clearly, a function of the dimensionality of the system, the exponent $\rho$ and amplitude $D_\rho$, and other relevant physical parameters of the system. Its determination remains a major task [20].

We thank John Cardy for useful comments. The work of SMVA was supported by the NIOC.

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