Non-Embedding Theorems of Nilpotent Lie Groups and Sub-Riemannian Manifolds

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Abstract

In this paper, we prove that there do not exist quasi-isometric embeddings of connected non-abelian nilpotent Lie groups equipped with left invariant Riemannian metrics into a metric measure space satisfying the curvature-dimension condition $RCD(0,N)$ with $N \in \mathbb{R}$ and $N > 1$. In fact, we can prove that a sub-Riemannian manifold whose generic degree of nonholonomy is not smaller than 2 cannot be bi-Lipschitzly embedded in any Banach space with the Radon-Nikodým property. We also get that every regular sub-Riemannian manifold does not satisfy the curvature-dimension condition $CD(K,N)$, where $K,N \in \mathbb{R}$ and $N > 1$. Along the way to the proofs, we show that the minimal weak upper gradient (Def. 6.5) and the horizontal gradient (Def. 2.17) coincide on the Carnot-Carathéodory spaces which may have independent interests (Thm. 6.15).

Keywords: Nilpotent Lie Group; sub-Riemannian manifold; Curvature-dimension condition; Bi-Lipschitz embedding.

1 Introduction

In [22], Pauls proved that there do not exist quasi-isometric embeddings of any connected, simply connected nonabelian nilpotent Lie groups equipped with left invariant Riemannian metric into either a $CAT_0$ metric space or an Alexandrov metric space with nonnegative curvature bounded. Inspired by Pauls' work, we address ourselves the following problem whether such connected nilpotent Lie groups equipped with left invariant Riemannian metrics can be quasi-isometrically embedded into metric measure spaces. A metric measure space is a triple $(M,d,m)$, where $(M,d)$ is a complete and separable metric space and $m$ is a locally finite (i.e., $m(B_r(x)) < \infty$ for all $x \in M$ and for all sufficiently small $r > 0$) nonnegative complete Borel measure on $M$ equipped with its Borel $\sigma$-algebra. To avoid pathologies, we exclude the case $m(M) = 0$. Let $(G,g)$ be a connected nilpotent Lie group with a left invariant Riemannian metric and $d$ be the induced distance function on $G$. If $(X,d_X)$ is a complete metric space, then $f : G \to X$ is called an $(L,C)$-quasi-isometric

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embedding if, for all $x, y \in G$,
\[
\frac{1}{L}d(x, y) - C \leq d_X(f(x), f(y)) \leq Ld(x, y) + C.
\]

We get the following theorem

**Theorem 1.1.** There does not exist a quasi-isometric embedding of any nonabelian nilpotent Lie group equipped with a left invariant Riemannian metric into any metric measure space satisfying $RCD(0, N)$, with $K, N \in \mathbb{R}$ and $N > 1$, where $(M, d)$ is a length space and $\text{supp}(m) = M$.

It is very challenging to define metric spaces with a lower bound on the curvature. Alexandrov [1] introduced the notion of lower sectional curvature bound for metric spaces in terms of comparison properties for geodesic triangles. For Ricci curvature bound, an amazing theory has recently been developed independently by Sturm [25] and Lott and Villani [16] in terms of curvature-dimension condition $CD(K, N)$, where $K$ is the lower bound of Ricci curvature and $N$ is the upper bound of Hausdorff dimension. Sub-Riemannian manifolds with their Carnot-Carathéodory metrics and Riemannian volumes are natural metric measure spaces. Through the method from metric geometry, we can prove the following result

**Theorem 1.2.** Let $(M, \Delta, g)$ be a regular sub-Riemannian manifold. Then $M$ with the Carnot-Carathéodory metric $d$ induced by $g$ and a Riemannian volume, as a metric measure space does not satisfy any $CD(K, N)$, where $K, N \in \mathbb{R}$ and $N > 1$.

Our main observation to prove the theorem is that in this case, the curvature-dimension condition $CD(K, N)$ implies $RCD(K, N)$ (Def. 6.8). In fact, we prove that for any sub-Riemannian manifold with Carnot-Carathéodory metric and fixed Riemannian volume, the horizontal gradient and the minimal weak gradient coincide. More precisely, we prove the following

**Theorem 1.3.** Let $(M, \Delta, g)$ be a regular sub-Riemannian manifold with measure $m$ induced by Riemannian volume and Carnot-Carathéodory distance $d$. Then the two spaces $W^{1,2}(M, d, m)$ and $W^{1,2}_H(M, m)$ (Def. 2.17) coincide. So $(M, d, m)$ is an infinitesimally Hilbert space (Def. 6.7).

This is our Theorem 6.15 which together with the following Theorem 1.4 forms the key ingredient to the proof of Theorem 1.2.

Cheeger and Kleiner [9] proved that the Heisenberg group with the Carnot-Carathéodory metric does not admit a bi-Lipschitz embedding into any Banach space satisfying the Radon-Nikodym property (see Def. 2.18). By refining their methods and ideas, we prove that the same result holds in the case of Carnot groups. Then, applying a blow up argument, we have the following theorem

**Theorem 1.4.** Any sub-Riemannian manifold whose generic degree of non-holonomy is not smaller than 2 can not be bi-Lipschitzly embedded into a Banach space with the Radon-Nikodym property.

The following corollary is a byproduct of the Theorem 1.4. Its proof will be given in [4].

**Corollary 1.5.** Any complete regular sub-Riemannian manifold can not be bi-Lipschitzly embedded into a metric measure space $(M, d, m)$ satisfying $RCD(K, N)$, with $K, N \in \mathbb{R}$ and $N > 1$, where $(M, d)$ is a length space and $\text{supp}(m) = M$. 
By direct calculations, Juillet [14] showed that no curvature-dimension bound $CD(K, N)$ holds in any Heisenberg group $\mathbb{H}_n$ with its Carnot-Carathéodory distance and Lebesgue measure. The Heisenberg group with Carnot-Carathéodory metric is a basic example of sub-Riemannian manifold (see §2). Our proof of Theorem 1.2 is based on a blow-up argument and by contradiction: if a sub-Riemannian manifold satisfies $CD(K, N)$ (see Def. 5.1), then it would be a $RCD(K, N)$ space (see Def. 6.5) and thus by structure theory of $RCD(K, N)$ spaces the tangent cones would be Euclidean. On the other hand the tangent cones to sub-Riemannian manifolds are Carnot groups, and the contradiction is due to our Theorem 1.4. The Gromov-Hausdorff convergence theory for $L$-biLipschitz maps between locally compact length spaces plays an essential role in this argument. Concerning Theorem 1.4, for the sub-Riemannian manifold whose generic degree of non-holonomy is 1, it may have a bi-Lipschitz embedding into some Banach space with the Radon-Nikodym property. In fact, in a preprint [24], Seo proved that the Grushin plane can be bi-Lipschitzly embedded in some Euclidean space.

The paper is organized as follows. In §2 we review the nilpotent Lie groups and sub-Riemannian geometry. We prove Theorem 1.4 in §3 for the special case that the sub-Riemannian manifold is a Carnot group (Theorem 3.6) to which the general case will be reduced to in §4 by Gromov-Hausdorff convergence theory for $L$-biLipschitz maps between locally compact length spaces. The section §5 is a review of some basic properties of curvature-dimension condition for later use. Finally, in §6 we recall the definitions of Cheeger energy and Sobolev spaces on metric measure spaces and then prove Theorem 1.3, Theorem 1.2 and Theorem 1.1.

All the main results of the paper were announced in [13]. In fact, Theorem 1.2 and Theorem 1.4 were derived and firstly announced at a seminar run by Professor Fuquan Fang in 2014. While we submitted [13] to the arXiv, we also noticed that Ambrosio and Stefani claimed a similar result (Proposition 3.6 in [5]) as our Theorem 1.2 using different method of proof.

2 Nilpotent Lie groups and Sub-Riemannian manifolds

In this section, we recall some basic definitions of nilpotent Lie groups and sub-Riemannian geometry. For a detailed discussions of these topics, please refer to [2].

Definition 2.1 (Nilpotent Lie algebra). Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie algebra over $\mathbb{R}$. We define the lower central sequence $C^k(\mathfrak{g})$ as follows: for any natural number $k$,

$$C^0(\mathfrak{g}) := \mathfrak{g},$$
$$C^{k+1}(\mathfrak{g}) := [\mathfrak{g}, C^k(\mathfrak{g})].$$

(1)

We say $\mathfrak{g}$ is a nilpotent Lie algebra, if $C^k(\mathfrak{g}) = 0$ for some $k$.

Nilpotent Lie groups are defined via their Lie algebras.

Definition 2.2 (Nilpotent Lie group). Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$. Then $G$ is called nilpotent if its Lie algebra $\mathfrak{g}$ is nilpotent.

Carnot groups are a special class of nilpotent Lie groups which play a central role in this paper.

Definition 2.3 (Carnot group). A Carnot group is a simply connected Lie group $G$ with a distinguished vector space $V_1$ such that the Lie algebra $\mathfrak{g}$ of the group has the direct sum decomposition
\[ g = \sum_{k=1}^{m} V_k, \quad V_{k+1} = [V_1, V_k]. \]  

(2)

The number \( m \) is called the step of the group.

For more motivations and backgrounds on the Carnot groups, please refer to Pansu \[20, 21\] and Mitchell \([18]\).

**Remark 2.4.** In the following, all Carnot groups that we will use are **nonabelian**.

**Example 2.5** (Heisenberg group). The Heisenberg group \( \mathbb{H} \) is the group of \( 3 \times 3 \) upper triangular matrices of the form

\[
\begin{pmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix}
\]

under the operation of matrix multiplication. The Lie algebra \( \mathfrak{h} \) of \( \mathbb{H} \) is generated by

\[
X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

with the Lie bracket the usual commutator of matrices.

The Heisenberg group is a Carnot group. In fact, the generators of the Lie algebra satisfy the unique nontrivial relation \([X, Y] = Z\). Thus the Lie algebra \( \mathfrak{h} \) has a direct sum decomposition

\[ \mathfrak{h} = V_1 \oplus V_2, \text{where } V_1 = \text{span}\{X, Y\}, \text{ } V_2 = \text{span}\{Z\}. \]

**Definition 2.6.** A sub-Riemannian manifold is a triple \((M, \Delta, g)\) such that \( M \) is a smooth manifold, \( \Delta \) is a bracket-generating distribution and \( g \) is an inner product defined on \( \Delta \). The metric \( g \) is also called a sub-Riemannian metric of manifold \( M \).

Recall that a distribution \( \Delta \) is said to be a bracket-generating distribution if for any point \( p \in M, \exists \ k = k(p) \in \mathbb{N} \) such that

\[
\begin{align*}
\Delta^1 &= \Delta, \\
\Delta^2 &= \Delta^1 + [\Delta, \Delta], \\
\Delta^3 &= \Delta^2 + [\Delta, \Delta^2], \\
\Delta^4 &= \Delta^3 + [\Delta, \Delta^3], \\
& \quad \vdots \\
\Delta^k &= \Delta^{k-1} + [\Delta, \Delta^{k-1}], \\
\Delta^k_p &= T_pM.
\end{align*}
\]

**Remark 2.7.** The smallest integer \( k = k(p) \) such that \( \Delta^k_p = T_pM \) is called the degree of non-holonomy at \( p \).

**Definition 2.8.** On a sub-Riemannian manifold \((M, \Delta, g)\), we say that a point \( p \in M \) is a regular point if \( \dim \Delta^i \) remains constant in some neighborhood of \( p \) for \( 0 \leq i \leq k(p) \). Otherwise we call \( p \) a singular point.
Remark 2.9. The regular point is defined by open condition. So the set of regular points is open and dense in $M$. For more details, the reader is referred to [6] (the arguments in the last paragraph of p.31). If all points are regular, we call the sub-Riemannian manifold is regular.

Here are some well known examples of sub-Riemannian manifolds.

**Example 2.10.** By Example 2.5, the Heisenberg group $\mathbb{H}$ with the distribution $\Delta$ generated by $X$ and $Y$, and Euclidean metric given by $\langle X, Y \rangle = 0, \langle X, X \rangle = 1, \langle Y, Y \rangle = 1$ is a sub-Riemannian manifold.

**Example 2.11 (Hopf fiberation).** The special unitary group $SU(2)$ is the Lie group consisting of $2 \times 2$ unitary matrices of determinant $1$. Its Lie algebra $su(2)$ consists of skew Hermitian matrices of trace zero. The distribution $\Delta$ generated by left invariant vector fields of the following two elements in $su(2)$:

$$v_1 = \begin{bmatrix} 0 & 1/2 \\ -1/2 & 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 & i/2 \\ i/2 & 0 \end{bmatrix},$$

is a bracket-generating distribution. The standard sub-Riemannian metric $g_0$ on $\Delta$ is given by $\langle v_i, v_j \rangle = \delta_{ij}$, where $i, j = 1, 2$. We denote the sub-Riemannian manifold by $(SU(2), \Delta, g_0)$. The one parameter subgroup generated by the left invariant vector field $v_3 = \begin{bmatrix} i/2 & 0 \\ 0 & -i/2 \end{bmatrix}$ gives a (right) $S^1$-action on $SU(2)$. The quotient of $SU(2)$ by the $S^1$-action is the standard 2-sphere $S^2$, which is the Hopf fiberation.

**Example 2.12 (Special linear group).** The special linear group $SL(2, \mathbb{R})$ consists of $2 \times 2$ matrices of determinants $1$. The distribution $\Delta$ is defined by

$$X = \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}.$$ 

And the standard sub-Riemannian metric on $\Delta$ is given by $\langle X, Y \rangle = 0, \langle X, X \rangle = 1, \langle Y, Y \rangle = 1$.

**Example 2.13 (The Grushin plane).** The Grushin plane is the plane $\mathbb{R}^2$ with the horizontal distribution spanned by two vector fields

$$X_1 = \frac{\partial}{\partial x} \quad \text{and} \quad X_2 = x \frac{\partial}{\partial y}.$$ 

The vector fields $X_1$ and $X_2$ form an orthonormal basis for the tangent space at each point in $\mathbb{R}^2 \setminus \{(x, y) : x = 0\}$.

**Definition 2.14.** An absolutely continuous curve $\gamma : [a, b] \to M$ on $(M, \Delta, g)$ is called horizontal if it is almost everywhere tangent to the distribution $\Delta$.

Be aware that here the notion of absolute continuity is the usual one on differential manifolds. Later in §6 [7], we will define absolutely continuous curves in metric measure space.

Using the sub-Riemannian metric, one can introduce the notion of length for horizontal curves.
Definition 2.15. Let $\gamma : [a, b] \to M$ be a horizontal curve in $(M, \Delta, g)$. The length $L(\gamma)$ of $\gamma$ is defined by
$$
\int_a^b \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))}dt.
$$
The sub-Riemannian or Carnot-Carathéodory distance $d(x, y)$ between two points $x$ and $y$ on $(M, \Delta, g)$ is defined to be $d(x, y) = \inf \{L(\gamma)\}$, where the infimum is taken over all horizontal curves $\gamma$ connecting $x$ and $y$.

The notion of distance is well-defined due to the following Theorem 2.16.

Theorem 2.16. (Chow-Rashevskii, see also [2]) Suppose that $M$ is a connected smooth manifold and $\Delta$ is a bracket-generating distribution on $M$. Then

1. for any two points on $M$, there is a piecewise smooth horizontal curve connecting them;

2. the Carnot-Carathéodory distance $d$ is finite and continuous on $M$;

3. the Carnot-Carathéodory distance $d$ induces the topology of the manifold.

Definition 2.17 (Horizontal gradient). Let $\{X_1, \cdots, X_m\}$ be an orthogonal frame of $\Delta$ in a sub-Riemannian manifold $(M, \Delta, g)$. For a Lipschitz function $f$ with respect to the sub-Riemannian distance, its horizontal gradient is defined to be
$$
\nabla_H f = X_1(f) \cdot X_1 + \cdots + X_m(f) \cdot X_m.
$$

The corresponding horizontal Sobolev space $W^{1,2}_H(M)$ with norm
$$
\|f\|_H = \left( \int_M f^2 + \langle \nabla_H f, \nabla_H f \rangle dV \right)^{\frac{1}{2}}
$$
is a Hilbert space, where $dV$ is a fixed Riemannian volume.

Definition 2.18 (Radon-Nikodym property). A Banach space $V$ is said to have Radon-Nikodym property, if every Lipschitz function $f : \mathbb{R}^k \to V$ is differentiable almost everywhere.

According to an early result of Gelfand, separable dual spaces and reflexive spaces have the Radon-Nikodym property.

Finally, we recall the definition of Baire set and Baire category theorem, which will be used in §3.

Definition 2.19 (First category set and second category set). A set $A$ in a topological space $S$ is called nowhere dense if for every nonempty open set $U \subset S$, there is a nonempty open set $V \subset U$ with $A \cap V = \emptyset$. A union of countably many nowhere dense sets is called a set of first category. Sets which are not of first category are said to be of second category.

Definition 2.20. A subset of any topological space is said to have the property of Baire if it can be represented in the form $G \Delta A = (G - A) \cup (A - G)$, where $G$ is an open set and $A$ is a first category set.

Definition 2.21. A map $f$ between two topological spaces is said to have the property of Baire if $f^{-1}(V)$ has the property of Baire for every open set $V$.

Theorem 2.22 (Baire category theorem). Let $(S, d)$ be any complete metric space. Suppose $N$ is a first category set of $S$, then $S \setminus N$ is dense in $S$.

For more details about Baire category theory, one can see [19].
3 Embeddings of the Carnot groups

In this section, all the Carnot groups are assumed to be nonabelian as said in Remark 2.4. Our main goal is to prove Theorem 3.6 which is one of the main ingredients of the proofs of our main results.

**Theorem 3.1.** Let $g$ be a Borel map from a complete metric space $(M_1, d_1)$ to a separable metric space $(M_2, d_2)$. Then there is a first category set $F \subseteq M_1$ such that $g$ is continuous on $M_1 \setminus F$ (dense in $M_1$).

**Proof.** Take a countable topological basis $\{U_1, U_2, \cdots, U_n, \cdots\}$ of $M_2$. Because $g$ is a Borel map, it has property of Baire (see Def. 2.21). Then

$$g^{-1}(U_i) = G_i \triangle F_i = (G_i \bigcup F_i) \setminus (G_i \bigcap F_i),$$

where $G_i$ are open sets of $M_1$ and $F_i$ are first category sets of $M_1$. Let

$$F = \bigcup_{i=1}^{\infty} F_i,$$

then $F$ is a first category set of $M_1$. Set

$$g_1 = g \mid_{M_1 \setminus F},$$

then $g_1$ is continuous on $M_1 \setminus F$. In fact,

$$g_1^{-1}(U_i) = g^{-1}(U_i) \cap (M_1 \setminus F) = (G_i \triangle F_i) \cap (M_1 \setminus F) = G_i \cap (M_1 \setminus F)$$

i.e., $g_1^{-1}(U_i)$ are open sets of $M_1 \setminus F$. By Baire category Theorem 2.22, $M_1 \setminus F$ is dense in $M_1$. \qed

Let $G$ be a Carnot group with Lie algebra

$$\mathfrak{g} = \sum_{k=1}^{s} V_k, \quad V_{k+1} = [V_1, V_k]$$

(4)

where the dimension of $V_k = m_k - m_{k-1}$, $2 \leq k \leq s$ and the dimension of $V_1 = m_1$.

We can construct a basis $\{X_1, X_2 \cdots X_n\}$ for $\mathfrak{g}$ with respect to the above decomposition, where $n = \dim G$. Firstly, fix a basis $\{X_1, \ldots, X_{m_1}\}$ of $V_1$, then consider all brackets $[X_i, X_j]$ for $i, j = 1, \ldots, m_1$. Since $[V_1, V_1] = V_2$, we can find among such brackets a basis for $V_2$. Pick such a basis and denote by $\{X_{m_1+1}, \ldots, X_{m_2}\}$. Similarly, extract a basis $\{X_{m_2+1}, \ldots, X_{m_3}\}$ for $V_{m_3}$ from the set of $[X_i, X_j]$, for $i = 1, \ldots, m_1$ and $j = m_1 + 1, \ldots, m_2$. And so on. In such a way, we have constructed a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ such that

1. For any $1 \leq j \leq s$, $\{X_{m_{j-1}+1}, \ldots, X_{m_j}\}$ is a basis of $V_j$.
2. For any $i = m_1 + 1, \ldots, n$, there are integers $d_i, l_i, k_i$, such that $X_i \in V_{d_i}$, $X_l \in V_1$, $X_k \in V_{d_i-1}$, and $X_i = [X_l, X_k]$. 7
We can now define a sub-Riemannian structure on $G$ with respect to the Lie algebra decomposition given above. Let $\Delta$ be the distribution on $G$ which is generated by the left invariant vector fields, still denoted by $X_1, \ldots, X_{m_1} \in V_1$, where $m_1 = \dim V_1$. We define a left invariant sub-Riemannian metric on $M$ by

$$\langle X_i, X_j \rangle = \delta_{i,j}, \quad 0 \leq i, j \leq m_1.$$ 

Then $(G, \Delta, \langle \cdot, \cdot \rangle)$ is a sub-Riemannian manifold. We denote the induced Carnot-Carathéodory distance function on $G$ by $d$.

Using the basis $\{X_1, X_2 \cdots X_n\}$, we can construct a privileged coordinate system at $x \in G$ as follows: we define a map

$$\Psi: \mathbb{R}^n \to G$$

by

$$(s_1, s_2, \cdots, s_n) \mapsto \exp(s_nX_n) \cdot \exp(s_{n-1}X_{n-1}) \cdots \exp(s_1X_1) \cdot x.$$ 

Obviously, $(s_1, s_2, \cdots, s_n)$ gives a local coordinate system at $x$.

**Theorem 3.2.** Let $(s_1, s_2, \cdots, s_n)$ be a privileged coordinate system at $x$ in a Carnot group $G$, constructed as above. There exist constants $c$ and $e$ such that if $d(x, q) < e$, then

$$\frac{1}{c} \|s\| \leq d(x, q) \leq c \|s\|,$$

where $s = (s_1, s_2, \cdots, s_n)$ is the coordinate of $q$, and

$$\|s\| = |s_1|^{\frac{1}{w_1}} + |s_2|^{\frac{1}{w_2}} + \cdots + |s_n|^{\frac{1}{w_n}}$$

with $w_i = k$ if $m_{k-1} < i \leq m_k$ and $m_0 = 0$.

The proof and the literature about the theorem can be found in [2].

With these preparations, we now consider the Lipschitz maps between a Carnot group and a Banach space $(V, \|\cdot\|)$ with Radon-Nikodym property. If there is a Lipschitz map

$$f: G \to V,$$

i.e.,

$$d(f(x_1), f(x_2)) \leq Ld(x_1, x_2), \quad \forall x_1, x_2 \in G,$$

then we have the following:

**Lemma 3.3.** The directional derivatives $X_i(f), 1 \leq i \leq m_1$ of the Lipschitz map $f: G \to V$ exist almost everywhere, and they are Borel maps from $G$ to $V$.

**Proof.** By definition, $X_i(f), 1 \leq i \leq m_1$ exists almost everywhere.

Let $E_i = \{x \in G \mid X_i(f) \text{ does not exist at } x\}, 1 \leq i \leq m_1$. Then

$$E_i = \bigcup_{k=1}^{\infty} \bigcap_{m=1}^{\infty} E_{k,m}, \quad 1 \leq i \leq m_1,$$

where

$$E_{k,m} = \bigcup_{0\leq|t_1|,|t_2|<\frac{1}{k}} \left\{ x \in G \mid \frac{f(\exp(t_1X_i) \cdot x) - f(x)}{t_1} - \frac{f(\exp(t_2X_i) \cdot x) - f(x)}{t_2} \right\} > \frac{1}{k}. \right\}$$
It is easy to see that $E_{k,m}$ are open sets, so $E_i, 1 \leq i \leq m_1$ is Borel set. By the definition of Radon-Nikodym property and Fubini theorem, we have $\mu(E_i) = 0$ for the Lebesgue measure $\mu$ on $G$.

Take a sequence \{t_n\} satisfying $t_n > 0$ and $\lim_{n \to \infty} t_n = 0$. Let

$$F_n(x) = \frac{f(\exp(t_nX_i) \cdot x) - f(x)}{t_n}.$$  \hspace{1cm} (6)

For any $n$, $F_n$ are continuous functions on $G$, so they are Borel maps. Because $F_n(x) \to X_i(f)(x)$ as $n \to \infty$, for $x \in G \setminus E_1$, $X_i(f)$ are Borel maps from $G$ to $V$ for $1 \leq i \leq m_1$.

**Lemma 3.4.** There is a first category set $F$ such that $X_i(f)$ is continuous on the dense subset $G \setminus F$ of $G$ for $1 \leq i \leq m_1$.

**Proof.** Because $(G,d)$ is a complete separable metric space, the Lipschitz map $f$ takes values in a closed separable subspace $V^1$ of $V$. Hence, $X_i(f) (1 \leq i \leq m_1)$ takes values in $V^1$ as well. Then the conclusion is deduced from Theorem 3.1.

The proof of the next lemma is inspired by the method used by Cheeger and Kleiner in their proof of Theorem 6.1 in [9].

**Lemma 3.5.** If $x \in G \setminus F$, then for any $0 < \varepsilon < 1$, we have

$$\|f(x') - f(x)\| \leq h\varepsilon \cdot d(x', x),$$

where $c$ and $h$ are constants and $x' = \exp(t^2[X_i, X_j]) \cdot x$, $[X_i, X_j] = X_k$ for some $m_1 < k \leq m_2$ and given $i, j \in [1, m_1]$.

**Proof.** For any $1 \leq i \leq m_1$, $X_i(f)|_{G \setminus F}$ are continuous and $G \setminus F$ is dense in $G$ (Theorem 3.1). So there exists $\delta > 0$ such that if $x_1 \in G \setminus F$ and $d(x, x_1) < \delta$, then we have

$$|X_i(f)(x) - X_i(f)(x_1)| < \varepsilon, \ 1 \leq i \leq m_1.$$  \hspace{1cm}

Take a neighborhood $B_r(x) \subset G$ and $r < \delta$, then $(G \setminus F) \cap B_r(x)$ is dense in $B_r(x)$.

Using $\eta_1, \eta_2, \eta_3, \gamma_4$ to denote the four vector fields: $X_i, X_j, -X_i, -X_j$, where $1 \leq i, j \leq m_1$. Then we have the following equality:

$$\exp(t^2[X_i, X_j])(x) = \exp(-tX_i) \circ \exp(-tX_j) \circ \exp(tX_i) \circ \exp(tX_j)(x)$$

$$= \exp(t\eta_4) \circ \exp(t\eta_3) \circ \exp(t\eta_2) \circ \exp(t\eta_1)(x).$$

Let $\gamma_i(t) = \exp(t\eta_i) \circ \exp(\eta_{i-1}) \circ \cdots \circ \exp(\eta_1)(x)$, then $\gamma_i$ is an integral curve of $\eta_i, 1 \leq i \leq 4$.

By the denseness of $G \setminus F$, when $t < 1$, there exists an integral curve $\tau_i$ of $\eta_i$ starting from $\tau_i(0) \in (G \setminus F) \cap B_r(x)$ such that

$$d(\tau_i(s), \gamma_i(s)) < \varepsilon t, \ s \in [0,t].$$

We claim that

$$\frac{1}{t} \int_0^t \|\eta_i f(\tau_i(s)) - \eta_i f(x)\| ds < 2\varepsilon.$$
In fact, by the Lebesgue’s lemma,
\[
\lim_{t \to 0} \frac{1}{t} \int_0^t \| f(\tau_i(s)) - f(\tau_i(0)) \| ds = 0.
\]
So when \( t \) is sufficiently small, we have
\[
\frac{1}{t} \int_0^t \| f(\tau_i(s)) - f(x) \| ds \leq \frac{1}{t} \int_0^t \| f(\tau_i(s)) - f(\tau_i(0)) \| ds + \| f(\tau_i(0)) - f(x) \| ds
\]
\[
= \frac{1}{t} \int_0^t \| f(\tau_i(s)) - f(\tau_i(0)) \| ds + \| f(\tau_i(0)) - f(x) \| ds
\]
\[
< 2\varepsilon.
\]
Therefore
\[
\| f(x') - f(x) \| = \| f \circ \gamma_4(t) - f \circ \gamma_1(0) \|
\]
\[
\leq \| f \circ \tau_4(t) - f \circ \tau_1(0) \| + 2L\varepsilon t
\]
\[
\leq \sum_{i=1}^{4} \| f \circ \tau_i(t) - f \circ \tau_i(0) \| + 8L\varepsilon t
\]
\[
= \sum_{i=1}^{4} \int_0^t \eta_i(t) \| ds \| + 8L\varepsilon t
\]
\[
\leq \sum_{i=1}^{4} \int_0^t \eta_i(f)(\tau_i(s)) - \eta_i(f)(x) \| ds \| + \sum_{i=1}^{4} \| \eta_i(f)(x) \| t \| + 8L\varepsilon t
\]
\[
\leq \sum_{i=1}^{4} \| \eta_i(f)(x) \| t \| + 8L\varepsilon t + 8\varepsilon
\]
\[
= 8tL\varepsilon + 8\varepsilon = h\varepsilon,
\]
where \( h = 8L + 8 \).

For sufficiently small \( t \), we have
\[
\frac{t}{c} \leq d(x', x) \leq ct,
\]
where \( x' = \exp(t^2[X_i, X_j]) \cdot x \) and the constant \( c \) depends only on \( x \). This is a direct consequence of Theorem 3.2 bearing in mind the condition that \( [X_i, X_j] = X_k \) for some \( k \in (m_1, m_2] \).

By the above results, we get
\[
\| f(x') - f(x) \| \leq h\varepsilon
\]
\[
\leq h\varepsilon \cdot d(x', x).
\]

\[\Box\]

**Theorem 3.6.** Let \((G, d)\) be a Carnot group with a Carnot-Carathéodory metric \( d \). Then it does not admit a bi-Lipschitz embedding into any Banach space with the Radon-Nikodym property.

**Proof.** It follows from the Lemmata 3.3, 3.4 and 3.5. \[\Box\]
4 Gromov-Hausdorff Convergence

The goal of this section is to complete the proof of Theorem 1.4. We first prove the following
Theorem 4.1 which is a refined version of a similar result which we learned from Rong [23]. By
direct application of Theorem 4.1 to the case that \((X, p)\) and \((Y, q)\) are metric tangent cones, we
get corollary 4.2 which is important in the proof of Theorem 1.4.

Theorem 4.1 (Convergence of \(L\)-biLipschitz maps). Suppose \(\lim_{i \to \infty} (X_i, p_i) = (X, p)\) and \(\lim_{i \to \infty} (Y_i, q_i) = (Y, q)\) are two sequences of locally compact length spaces which converge in the Gromov-Hausdorff sense to two locally compact length spaces respectively. If \(\{f_i : (X_i, p_i) \to (Y_i, q_i)\}_{i=1}^{\infty}\) is a sequence of \(L\)-biLipschitz maps and \(f_i(p_i) = q_i\) for all \(i\) with \(L\) a fixed constant. Then there is a subsequence of \(f_i\) converging to an \(L\)-biLipschitz map \(f : (X, p) \to (Y, q)\) and \(f(p) = q\) in the sense that if \(\lim_{i \to \infty} x_i = x\), then \(\lim_{i \to \infty} f_i(x_i) = f(x)\).

Proof. Firstly, we prove the theorem in the case that \(X\) and \(Y\) are compact metric spaces. In this
case, we can omit the base points. Let \(A = \{a_1, a_2, a_3, a_4, \ldots\}\) denote a countable dense subset of \(X\). We first define an \(L\)-biLipschitz map from \(A\) to \(Y\):

\[
 f : A \to Y.
\]

This extends uniquely to a map from \(X\) to \(Y\). For \(a_1\), let \(x_i \in X_i\) such that \(\lim_{i \to \infty} x_i = a_1\), in \(\coprod_{i=1}^{\infty} X_i \coprod X\). By the equicontinuity, there exist a subsequence \(\{X_{i_1}\}\) of \(\{X_i\}\) and \(x_{i_1} \in X_{i_1}\) such that \(\lim_{i_1 \to \infty} x_{i_1} = a_1\) and \(\lim_{i_1 \to \infty} f_{i_1}(x_{i_1}) = b_1\). We define \(f(a_1) = b_1\). Repeating this process, and by the standard diagonal argument, we can find a subsequence \(\{X_{i_j}\}\) of \(\{X_i\}\) such that \(\lim_{j \to \infty} X_{i_j} = X\) and if \(\lim_{l \to \infty} x_{k,l} = a_k\), then \(\lim_{l \to \infty} f_l(x_{k,l}) = b_k\), and we define \(f(a_k) = b_k\), where \(k \in \mathbb{N}\). Obviously, the map \(f : A \to Y\) is \(L\)-biLipschitz and can be extended uniquely to an \(L\)-biLipschitz map \(f : X \to Y\).

We now consider the case when either \(X\) or \(Y\) is not compact. By the definition of Gromov-
Hausdorff convergence of metric spaces, for any \(r > 0\), we can assume that the closed balls
\(\lim_{i \to \infty} B_r(p_i) = B_r(p)\) and \(\lim_{i \to \infty} B_r(q_i) = B_r(q)\). Applying the above result, we conclude that there is an \(L\)-biLipschitz map \(f_r : B_r(p) \to B_r(q)\) such that \(f_r(p) = q\). Note that \(f_r\) depends only on \(A_r = B_r(p) \cap A\). Let \(X = \bigcup_{s=1}^{\infty} B_s(p)\). Clearly, for \(s, t \in \mathbb{N}\), \(s < t\), \(B_s(p) \subseteq B_t(p)\) and \(f_s = f_t|B_s(p)\).

We get the desired \(L\)-biLipschitz map \(f : (X, p) \to (Y, q)\).

Corollary 4.2. Suppose \(f : (X, d_X) \to (Y, d_Y)\) is an \(L\)-biLipschitz map between two metric spaces
\((X, d_X)\) and \((Y, d_Y)\). If \((T_{x}X, d_1)\) is a metric tangent cone of \((X, d_X)\) at \(x\) and \((T_{f(x)}Y, d_2)\) is the unique metric tangent cone of \((Y, d_Y)\) at \(f(x)\), then there exists an \(L\)-biLipschitz map:

\[
 D_xf : (T_{x}X, d_1) \to (T_{f(x)}Y, d_2).
\]

Theorem 4.3. (Mitchell, [12]; Bellalche, [6]) For a sub-Riemannian manifold \((M, \Delta, g)\), the
tangent cone of \((M, d)\) at every regular point is \((G, d^c)\) where \(G\) is a Carnot Lie group with left invariant Carnot-Carathéodory metric \(d^c\).

Remark 4.4. If the generic nonholonomy degree of \((M, \Delta, g)\) is not smaller than 2, then the set of
the regular points of \(M\) is open and dense in \(M\) as pointed out by Bellaihe ([6], the last paragraph
of p.31). By the Theorem 1.3, the tangent cone \((G, d^c)\) of \((M, d)\) at a regular point is a nonabelian
Carnot Lie group with left invariant Carnot-Carathéodory metric \(d^c\).
Proof of Theorem 1.4. Let $(M, d)$ be a sub-Riemannian manifold with generic nonholonomy degree $\geq 2$ and sub-Riemannian distance $d$. Suppose $(V, \| \cdot \|)$ is a Banach space satisfying the Radon-Nikodym property with the distance function $d^*$ induced by the norm $\| \cdot \|$.

We proceed by contradiction, suppose there is a bi-Lipschitz map

$$f : (M, d) \to (V, d^*)$$

By Remark 2.9, the set of the regular points of $M$ with the degree of non-holonomy satisfying $\geq 2$ is dense in $M$. So we can get a bi-Lipschitz map $D_x f$ from the tangent cone $(T_x M, d_1)$ at a regular point $x$ of $M$ to the tangent cone $(T_{f(x)} V, d_2)$ of $V$, i.e.,

$$D_x f : (T_x M, d_1) \to (T_{f(x)} V, d_2)$$

Because $(V, d^*)$ is a Banach space, the tangent cone of $V$ at $f(x)$ is

$$(T_{f(x)} V, d_2) = (V, d^*)$$

By Remark 4.4, $(T_x M, d_1)$ is a non-commutative Carnot group with left invariant Carnot-Carathéodory metric $d_1$. However, it is impossible by Theorem 3.6. Therefore, every sub-Riemannian manifold whose generic degree of non-holonomy is not smaller than 2 can not be bi-Lipschitzly embedded in any Banach space with Radon-Nikodym property.

Proof of Corollary 1.5. We still go on by contradiction. If there is a biLipschitz embedding of a complete sub-Riemannian manifold into a metric measure space $(M, d, m)$ satisfying $RCD(K, N)$ (Def. 6.8), where $(M, d)$ is a length space and $\text{supp}(m) = M$. As in the above argument, through the blowup analysis (Theorem 4.1), we get a biLipschitz embedding from a Carnot group with Carnot-Carathéodory metric to a Euclidean space (Mondino-Naber, [17], see also Theorem 6.9). However, it is impossible by Theorem 1.4.

Remark 4.5. In the above proof of Corollary 1.5, the condition that the metric measure space $(M, d, m)$ satisfies $\text{supp}(m) = M$, is essential. Because, by the Corollary 2.4 in [25](II), under the condition, the $CD(K, N)$ space $(M, d, m)$ is locally compact.

5 Curvature-dimension condition

In this section, we recall the definition of curvature-dimension condition and some of its basic properties.

Given a metric measure space $(M, d, m)$ and a number $N \in \mathbb{R}$ with $N \geq 1$, we define the Renyi entropy functional $S_N(\cdot | m) : \mathcal{P}_2(M, d) \to \mathbb{R}$ with respect to $m$ by

$$S_N(\nu | m) = - \int_M \rho^{\frac{1}{N}} d\nu$$

where $\mathcal{P}_2(M, d)$ denotes the $L_2$-Wasserstein space of probability measures on $M$ and $\rho$ is the density of the absolutely continuous part $\nu^c$ in the Lebesgue decomposition $\nu = \nu^c + \nu^s = gm + \nu^s$ of $\nu \in \mathcal{P}_2(M, d)$.  

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Definition 5.1. Given two numbers \( K, N \in \mathbb{R} \) with \( N \geq 1 \), we say that a metric measure space \((M, d, m)\) satisfies the curvature-dimension condition \(\text{CD}(K, N)\) if for each pair \( \nu_0, \nu_1 \in \mathcal{P}(M, d) \), there is an optimal coupling \( q \) of \( \nu_0 \) and \( \nu_1 \), and a geodesic \( \Gamma : [0, 1] \to \mathcal{P}_2(M, d)\) connecting \( \nu_0 \) and \( \nu_1 \), with

\[
\mathcal{S}_{N'}(\Gamma(t)|m) \leq -\int_{M\times M} \left[ t^{(1-t)}_{K,N'}(d(x_0,x_1)) \frac{\partial}{\partial t} q_{t}^{\nabla^*}(x_0) + t^{(1-t)}_{K,N'}(d(x_0,x_1)) \frac{\partial}{\partial t} q_{t}^{\nabla^*}(x_1)\right] dq(x_0,x_1)
\]

for all \( t \in [0, 1] \) and all \( N' \geq N \), where

\[
\tau^{(1-t)}_{K,N'}(\theta) = \begin{cases} 
\infty & \text{if } K\theta \geq (N-1)\pi^2 \\
\frac{1}{N} \frac{\sin(t\theta \sqrt{K(N-1)})}{\sin(\theta \sqrt{K(N-1)})} \left(1 - \frac{1}{N} \right) & \text{if } 0 < K\theta < (N-1)\pi^2 \\
t & \text{if } K\theta^2 = 0 \text{ or if } K\theta^2 < 0, N = 1 \\
\frac{1}{N} \frac{\sinh(t\theta \sqrt{-K(N-1)})}{\sinh(\theta \sqrt{-K(N-1)})} \left(1 - \frac{1}{N} \right) & \text{if } K\theta^2 < 0, N > 1.
\end{cases}
\]

Remark 5.2. A coupling \( q \) of two probability measures \( \mu \) and \( \nu \) on a metric space \( M \) is a probability measure on the product space \( M \times M \) whose marginals are the given measures \( \mu \) and \( \nu \). For the condition of an coupling being optimal, please refer to \cite{25} and \cite{16}.

We list some properties of curvature-dimension condition to be used later in this paper, and refer to \cite{24} for more complete discussions.

Lemma 5.3. Let \((M, d, m)\) be a metric measure space which satisfies the \(\text{CD}(K, N)\) condition for some pair of real numbers \( K \) and \( N \). Then the following properties hold.

1. Each metric measure space \((M', d', m')\) which is isomorphic to \((M, d, m)\) satisfies the \(\text{CD}(K, N)\) condition.

2. For each \( \alpha, \beta > 0 \), the metric measure space \((M, \alpha d, \beta m)\) satisfies the \(\text{CD}(\alpha^{-2} K, N)\) condition.

3. For each convex subset \( M' \) of \( M \), the metric measure space \((M', d, m)\) satisfies the same \(\text{CD}(K, N)\) condition.

6 Cheeger energy and Sobolev space on metric measure spaces

In this section, we will give the proof of Theorem 1.1, Theorem 1.2 and Theorem 6.15. We firstly recall some basics about the differential structures of metric measure spaces, which will be used in this section.

In order to define the Sobolev space \(W^{1,2}(M, d, m)\) on metric measure space, we need to introduce the weak gradients. As far as we know, there are four notions of weak gradients. However, under our assumptions, they are all equivalent (\cite{4}, Theorem 7.4 and Further comments and extensions). In the following, we only need two of them, namely the Cheeger’s gradient \(|\nabla f|_{C,2}\) and 2-weak upper gradient \(|Df|_{w,2}\).

Firstly, recall the notion of absolute continuity of curves on the metric measure spaces which plays an important role in the definition of weak gradients.
A curve \( \gamma : [0, 1] \to M \) is said to be absolutely continuous if

\[
d(\gamma(t), \gamma(s)) \leq \int_s^t g(r)dr \quad \forall s, t \in [0, 1], s \leq t
\]

for some \( g \in L^1(0, 1) \). If \( \gamma \) is absolutely continuous, the metric derivative \( |\dot{\gamma}| : [0, 1] \to [0, \infty] \) is defined by

\[
|\dot{\gamma}| := \lim_{h \to 0} \frac{d(\gamma(t + h), \gamma(t))}{h}.
\]

One can show that the limit exists for almost every \( t \), and \( |\dot{\gamma}| \in L^1(0, 1) \), and it is the minimal \( L^1 \)-function (up to Lebesgue negligible sets) for which the bound (7) holds.

**Remark 6.1.** For the case of regular sub-Riemannian manifold with Carnot-Carathéodory distance, the absolutely continuous curves \( \gamma \) are almost differentiable and Horizontal. The metric derivatives \( |\dot{\gamma}| \) of \( \gamma \) at the differentiable points are the usual norm of the tangent vectors of the curves ([15], Corollary 2.8.6, Property 3.2.2 and Theorem 3.2.10).

Following [1], we give the following definitions.

**Definition 6.2.** (Upper gradient) Suppose \( f \) is a Borel function on \( X \), we say that a Borel function \( h \) is an upper gradient of \( f \), if the inequality

\[
|f(0) - f(1)| \leq \int \gamma h(\gamma) |\dot{\gamma}|ds
\]

holds for all absolutely continuous curves \( \gamma : [0, 1] \to X \).

The next notion is due to Cheeger, which is defined via upper gradient by a relaxation procedure.

**Definition 6.3.** (2-relaxed upper gradient) We say that \( h \in L^2(M, m) \) is a 2-relaxed upper gradient of \( f \in L^2(M, m) \) if there exist \( \tilde{h} \in L^2(M, m) \), functions \( f_n \in L^2(M, m) \) and upper gradient \( h_n \) of \( f_n \) such that:

1. \( f_n \to f \) in \( L^2(M, m) \) and \( h_n \) weakly converge to \( \tilde{h} \) in \( L^2(M, m) \);
2. \( \tilde{h} \leq h \).

We say that \( h \) is a minimal 2-relaxed upper gradient of \( f \) if its \( L^2(M, m) \) norm is minimal among 2-relaxed upper gradients. We denote by \( |\nabla f|_{C,2} \) the minimal 2-relaxed upper gradient.

We will denote by \( AC^2([0, 1], M) \) the class of absolutely continuous curves with metric derivative in \( L^2(0, 1) \).

In order to introduce the concept of weak upper gradient, some special probability measure is defined on the space of continuous paths of \( M \).

**Definition 6.4.** Let \( (M, d, m) \) be a metric measure space and \( \pi \) be a probability of \( C([0, 1], M) \). We say that \( \pi \) has bounded compression provided there exists \( N > 0 \) such that

\[
(e_t)_2 \pi \leq N \cdot m, \quad \forall t \in [0, 1],
\]

where \( e_t \) is the evaluation function from \( C([0, 1], M) \) to \( M \), at \( t \).

We say that \( \pi \) is a test plan if it has bounded compression and is concentrated on \( AC^2([0, 1], M) \) such that

\[
\int \int_0^1 |\dot{\gamma}|^2 dt d\pi(\gamma) < \infty.
\]

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Definition 6.5. Let $f : M \rightarrow \mathbb{R}$ be any Borel function. A function $G \in L^2(M, m)$ is called a weak upper gradient of $f$ if
\[
|f(\gamma(0)) - f(\gamma(1))| \leq \int_0^1 G(\gamma(t))|\dot{\gamma}(t)|dt < \infty
\]
for almost every curve $\gamma : [0, 1] \rightarrow M$ in $AC^2([0, 1]; M)$, under every test plan $\pi$.

It turns out ([3], P.321) that there is a weak upper gradient $|D(f)|_w : M \rightarrow [0, \infty]$ having the property that
\[
|D(f)|_w(x) \leq G(x)
\]
for $m$-a.e $x \in M$, where $G$ is any weak upper gradient of $f$. The function $|D(f)|_w$ will be called the minimal weak upper gradient of $f$.

In terms of minimal weak upper gradient, one defines the Cheeger energy of $f : M \rightarrow \mathbb{R}$ by
\[
\text{Ch}(f) := \frac{1}{2} \int_M |Df|^2_w dm
\]

The Sobolev space $W^{1,2}(M, d, m)$ is by definition the space of $L^2(M, m)$- functions having finite Cheeger energy, and it is endowed with the natural norm $\|f\|_{W^{1,2}}^2 := \|f\|_{L^2}^2 + 2\text{Ch}(f)$ which makes it into a Banach space.

Remark 6.6. In general, $W^{1,2}(M, d, m)$ is not a Hilbert space. For instance, on a smooth Finsler manifold the space $W^{1,2}$ is Hilbert if and only if it is actually Riemannian manifold.

Definition 6.7 (Infinitesimally Hilbertian space). A metric measure space $(M, d, m)$ is said to be an infinitesimally Hilbertian space if $(M, d)$ is a complete separable metric space, $m$ is a locally finite nonnegative measure on $M$, and the Sobolev space $W^{1,2}(M, d, m)$ is a Hilbert space.

Definition 6.8. If an infinitesimally Hilbertian space $(M, d, m)$ satisfies curvature-dimension condition $CD(K,N)$, we say $(M,d,m)$ is a $RCD(K,N)$ space.

In [17], Mondino and Naber proved that

Theorem 6.9. (Mondino-Naber, [17]) Suppose $(X,d,m)$ is a complete and separable metric space with $m$ a locally finite nonnegative complete Borel measure satisfying $RCD(K,N)$ with $N > 1$, then for $m$-a.e. point the tangent cone is an Euclidean space of dimension at most $N$.

Remark 6.10. In fact, reduced curvature-dimension condition $CD(K,N)^*$ is used in [17], which is slightly weaker than the curvature-dimension condition $CD(K,N)$.

We will prove that the minimal weak upper gradient and the horizontal gradient (Def. 2.17) coincide on the Carnot-Caratheodory spaces. The following lemma tells us that if the function is continuous, then the two concepts are the same up to a measure zero set. However it is still not enough, and we need a stronger result to prove the coincidence.

Lemma 6.11. Let $(M,\Delta,g)$ be a regular sub-Riemannian manifold with a Riemannian volume $m$. Suppose $f \in L^1_{loc}(M,m)$ is a nonnegative function. If $f$ is an upper gradient of a function $u$ which is continuous. Then the distributional derivatives $\nabla_H(u)$ is locally integrable and $|\nabla_H(u)| \leq f$ $m$-a.e in $M$. 

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Proof. Obviously, we only need to check the conclusion locally. Without loss of generality, we may assume that $M$ is an open domain of Euclidean space, and the horizontal distribution $\Delta$ is generated by global orthogonal vector fields $\{X_1, \ldots, X_r\}$, where $r$ is the rank of $\Delta$. The measure $m$ can be written in the form $\rho dV$, where $\rho$ is a smooth function and $dV$ is the Lebesgue measure. By the definition of distributional derivatives, we have:

$$\int_M X_i(u) \cdot h \cdot \rho dV = \int_M u \cdot X_i^*(h) \cdot \rho dV,$$

where $h$ is any smooth function with compact support on $M$, and $X_i^*$ is the adjoint operator of $X_i$ with respect to $\rho dV$. This formula can be rewritten as:

$$\int_M (\rho \cdot X_i)(u) \cdot h \cdot dV = \int_M u \cdot (\rho \cdot X_i^*)(h) \cdot dV$$

So, $\rho \cdot X_i^*$ is the adjoint operator of $\rho \cdot X_i$ with respect to $dV$.

Let $Y_i = \rho \cdot X_i$, then we get a new basis of the horizontal distribution $\Delta$. Define a new sub-Riemannian metric $g'$ on $\Delta$, such that $\{Y_1, \ldots, Y_r\}$ is orthogonal. On this new sub-Riemannian manifold $(M, \Delta, g')$, $\rho \cdot f$ is an upper gradient of $u$, and the distributional horizontal gradient is

$$\nabla^H_H(u) = \rho \cdot \nabla_H(u).$$

Hence, we reduce the problem to the case: $M$ is a domain of Euclidean space with Carnot-Carathéodory metric and the Lebesgue measure. This is exactly the Theorem 11.7 in [12].

In [4], Ambrosio, Gigli and Savare proved the following density in energy of Lipschitz functions.

**Proposition 6.12.** (Ambrosio-Gigli-Savare, [4]) If $f \in W^{1,2}(M, d, m)$, then there exist Lipschitz functions $f_n$ satisfying

$$\lim_{n \to \infty} \left[ \int_M |f_n - f|^2 dm + \int_M (|\nabla f_n| - |Df|_w)^2 dm \right] = 0,$$

where $|\nabla f_n|$ is defined by

$$|\nabla f_n|(x) = \lim_{y \to x} \frac{|f(x) - f(y)|}{d(x, y)}.$$

**Remark 6.13.** In general, the Lipschitz functions are not dense in $W^{1,2}(M, d, m)$ w.r.t $W^{1,2}$ norm, as pointed out by Gigli ([10]). So we cannot apply Lemma 6.11 directly to prove Theorem 6.15.

A further lemma known to the experts is needed, and we give a proof here for the completeness.

**Lemma 6.14.** Let $(M, g)$ be a Riemannian manifold with Riemannian measure $m$, which is induced from the Riemannian volume. Suppose $f$ is an $m$-measurable function, then there exists a Borel function $\tilde{f}$ such that

$$f \leq \tilde{f} \quad \text{and} \quad f = \tilde{f} \quad m \text{-a.e.}.$$

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Proof. Because $f$ is measurable, for any $r \in \mathbb{Q}$, the set
\[ E_r = \{ x \in M : f(x) \geq r \} \] (16)
is measurable. By the regularity of the measure $m$, we have a Borel set $A_r \supseteq E_r$ such that $m(A_r \setminus E_r) = 0$. We define
\[ \tilde{f}(x) := \sup \{ r \in \mathbb{R} : x \in \bigcap_{q \in \mathbb{Q}, q < r} A_q \}. \] (17)
By the definition of $E_r$, we have that
\[ f(x) = \sup \{ r \in \mathbb{R} : x \in \bigcap_{q \in \mathbb{Q}, q < r} E_q \}. \] (18)
Since $E_q \subset A_q$, for all $q \in \mathbb{Q}$, we see that $f(x) \leq \tilde{f}(x)$. If there is an $x \in M$, such that $f(x) < \tilde{f}(x)$, then there is an $e \in \mathbb{R}$ satisfying $f(x) < e < \tilde{f}(x)$. Then we have
\[ x \in \bigcap_{q \in \mathbb{Q}, q < r} A_q \subset \bigcup_{q \in \mathbb{Q}, q < e} (A_q \setminus E_q). \] (19)
Since $m(A_q \setminus E_q) = 0$ for all $q \in \mathbb{Q}$, $f(x) = \tilde{f}(x)$ $m$-a.e.

By the definition of $\tilde{f}(x)$, if $x \in \bigcap_{q \in \mathbb{Q}, q < r} A_q$, then $\tilde{f}(x) \geq r$. On the other hand, if $\tilde{f}(x) \geq r$ then
\[ \bigcap_{q \in \mathbb{Q}, q < r} A_q = \tilde{f}^{-1}([r, +\infty]) \] (20)
i.e., $\tilde{f}$ is a Borel function. □

Theorem 6.15. Let $(M, \Delta, g)$ be a regular sub-Riemannian manifold with measure $m$ induced by Riemannian volume and Carnot-Carathéodory distance $d$. Then the two spaces $W^{1,2}(M, d, m)$ and $W^{1,2}_H(M, m)$ (Def. 2.17) coincide. So $(M, d, m)$ is an infinitesimally Hilbert space.

Proof. Suppose $f \in W^{1,2}(M, d, m)$, then there is a sequence of Lipschitz functions $\{f_n\}$ converging to $f$ in $L^2(M, m)$ by Proposition 6.12. Because $f_n$ is Lipschitz function for any $n$, $\nabla_H f$ exists and $|\nabla f_n|$ is an upper gradient of $f_n$. By Lemma 6.11,
\[ |\nabla_H f_n| \leq |\nabla f_n|. \] (21)
Hence,
\[ \int_M |\nabla_H f_n|^2 dm \leq \int_M |\nabla f_n|^2 dm. \] (22)
And also,
\[ \lim_{n \to \infty} \int_M |\nabla f_n| - |Df|_w^2 dm = 0, \]
\[ \int_M |Df|_w^2 dm < \infty, \] (23)
then
\[ \int_M |\nabla_H f_n|^2 dm < \infty. \] (24)
So \( \{\nabla_H f_n\} \) is a bounded set in the space \( L^2(M, \Delta) \) of square integrable sections of \( \Delta \). On the other hand, \( L^2(M, \Delta) \) is a Hilbert space, so \( L^2(M, \Delta) \) is reflexive. Hence, there is a subsequence of \( \{\nabla_H f_n\} \) converging to a horizontal section \( \alpha \), and \( \nabla_H f = \alpha \). We can assume that

\[
\lim_{n \to \infty} \nabla_H f_n = \alpha, \quad (25)
\]

\[
|\nabla_H f_n| \to |\alpha| \quad \text{in} \quad L^2(M, m). \quad (26)
\]

By Lemma 6.14 for each \( n \), we can find a Borel function \( \tilde{f}_n \) such that

\[
|\nabla_H f_n| \leq \tilde{f}_n \quad \text{and} \quad |\nabla_H f_n| = \tilde{f}_n \quad m-a.e.. \quad (27)
\]

For any \( n \), the \( \tilde{f}_n \) is an upper gradient of \( f_n \). By Remark 6.1 any absolutely continuous path \( \gamma : [0,1] \to M \) is almost differentiable and horizontal. Then,

\[
|f(\gamma(0)) - f(\gamma(1))| \leq \int_0^1 |\nabla_H f_n(\gamma(s))| \cdot |\gamma(s)| ds \leq \int_0^1 |\tilde{f}_n(\gamma(s))| \cdot |\gamma(s)| ds. \quad (28)
\]

Therefore, we get a sequence of Lipschitz functions \( \{f_n\} \) and a sequence of Borel functions \( \{\tilde{f}_n\} \), satisfying

1. \( f_n \to f \) in \( L^2(M, m) \) and \( \tilde{f}_n \to |\alpha| \) in \( L^2(M, m) \);
2. \( \tilde{f}_n \) is a upper gradient of \( f_n \), for each \( n \).

By Definition 6.3 we have \( \nabla_H f = |\alpha| \) is a 2-relax upper gradient of \( f \). And also, the \( |\nabla f|_{C,2} \) is minimal. So,

\[
\int_M |\nabla f|^2_{C,2} dm \leq \int_M |\nabla_H f|^2 dm. \quad (29)
\]

By the formula (5.22) and (5.23),

\[
\int_M |\nabla_H f|^2 dm \leq \int_M |Df|^2_{w,m} dm. \quad (30)
\]

By \( \mathbb{H} \) (Theorem 7.4 and the section "Further comments and extensions"), we get \( |\nabla f|_{C,2} = |Df|_w \) \( m-a.e. \) in \( M \). So

\[
\int_M |\nabla f|^2_{C,2} dm = \int_M |\nabla_H f|^2 dm. \quad (31)
\]

And also, the minimal relaxed upper gradient is unique up to a set of measure zero (\( \mathbb{S} \), Theorem 2.10), then

\[
|\nabla_H f| = |Df|_w \quad m-a.e.. \quad (32)
\]

Therefore, we get an embedding of Banach space \( W^{1,2}(M, d, m) \) into Banach space \( W_H^{1,2}(M, m) \), which preserves the norms. But the space \( C_0^\infty(M) \) of smooth functions with compact support is
dense in $W^{1,2}_H(M, m)$ and $W^{1,2}(M, d, m)$ contains $C_0^\infty(M)$. So $W^{1,2}(M, d, m) = W^{1,2}_H(M, m)$. On the other hand, the Sobolev space $W^{1,2}_H(M, m)$ with norm
\[ \|f\|_H = \left( \int_M (f^2 + \langle \nabla_H f, \nabla_H f \rangle) dm \right)^{\frac{1}{2}} \] (33)
is a Hilbert space. Therefore, $(M, d, m)$ is an infinitesimally Hilbertian space. □

**Proof of Theorem 1.2.** Suppose $(M, d, m)$ satisfies curvature-dimension condition $CD(K, N)$, where $K, N \in \mathbb{R}$ and $N > 1$. By Theorem 6.15, the metric measure space $(M, d, m)$ is a $RCD(K, N)$ space. It follows from Theorem 6.9 that for $m$-a.e. point the tangent cone is a Euclidean space. By Theorem 4.3, we get a bi-Lipschitz map between Carnot group and Euclidean space. This contradicts to Theorem 1.4. □

**Proof of Theorem 1.1.** Suppose a complete separable metric measure space $(M, d, m)$ satisfies $RCD(0, N)$ for $N > 1$. Fixed a point $x \in M$, consider the family of pointed metric measure spaces:
\[ \{(M, \lambda \cdot d, m, x) : \lambda \in \mathbb{R}_{>0}\} \] (34)

Then, by Lemma 5.3, the above family of pointed metric measure spaces also satisfies $RCD(0, N)$. And also, by the Theorem 2.3 and Corollary 2.4 in (II), the family is uniformly doubling. So, by Lemma 3.32 in [11], there is a sequence $\lambda_n$ of positive real numbers and $\lim n \to \infty \lambda_n = 0$ such that the $(CM, d_0, m) := \lim n \to \infty (M, \lambda_n \cdot d, m, x)$ exists. Then, by the stability of lower Ricci curvature bounds under pointed measured Gromov-Hausdorff convergence [11], $(CM, d_0, m)$ satisfies $RCD(0, N)$.

$(N, g)$ is a non-abelian nilpotent Lie group with a left invariant Riemannian metric. Pansu (21) proved that the asymptotic cone of $(N, g)$ is unique and isometric to Carnot group with a left invariant Carnot-Carathéodory metric. By Corollary 1.5, we complete the proof. □

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