BALL PACKINGS IN HYPERBOLIC SPACE

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Abstract. In hyperbolic space density cannot be defined by a limit as we define it in Euclidean space. We describe the local density bounds for sphere packings and we discuss the different attempts to define optimal arrangements in hyperbolic space.

It is natural to extend the study of packing and covering problems to the hyperbolic plane, as well as hyperbolic spaces of higher dimension. However, we encounter here the problem of defining a “reasonable” notion of global density. Since in hyperbolic space the volume and surface area of a ball of radius $r$ are of the same order of magnitude, density cannot be defined by a limit as we define it in Euclidean space. Böröczky [1974] exposed that the problem is much deeper than one would expect. Namely, he constructed an arrangement of congruent circles along with two different decompositions $Z_1$ and $Z_2$ of the hyperbolic plane into congruent cells, each containing one circle, with the property that the circles’ density in each cell of $Z_i$ presents the same value $d_i$ ($i = 1, 2$), and yet $d_1 ≠ d_2$.

Two ways were considered to overcome the aforementioned difficulties with defining global density in hyperbolic geometry. One way was to consider local density relative to Dirichlet cells; the other was to come up with suitable notions alternate to arrangements of optimal density.

1. The simplex bound

Recall the simplex bound of Böröczky [1978]: For any packing of balls of radius $r$ in $n$-dimensional hyperbolic space the density of a ball in its Dirichlet cell cannot exceed the density $d_n(r)$ of $n + 1$ mutually touching balls of radius $r$ with respect to the simplex spanned by the centers of the balls. Böröczky stated this for balls of finite radius, however the proof can be extended to packings of horoballs (balls of infinite radius). As in the case of balls of finite radius, the bound refers to the density of the horoballs in their Dirichlet cell in this case as well. It should be mentioned that the volume of the horoballs, as well as their Dirichlet cells, is infinite. However, as the hyperbolic metric on a horosphere is Euclidean, the density of a horoball in its Dirichlet cell can be defined by a limit of the density in growing sectors of the horoball.

The definition of the simplex bound also needs further explanation in the case of horoball packings. Let $S$ be a totally asymptotic regular simplex in $n$-dimensional hyperbolic space. Take $n + 1$ mutually tangent horoballs centered at the vertices of
S and consider their density relative to S. In contrast to balls of finite radius, the condition that the horoballs are mutually tangent does not determine the configuration. We get the simplex bound \( d_n(\infty) \) occurring in Böröczky’s theorem if the symmetry group of the arrangement of the horoballs coincides with that of S.

Szirmai [2012, 2013] investigated the density of mutually tangent horoballs centered at the vertices of a totally asymptotic regular simplex S in the case when the symmetry group of the arrangement of the horoballs does not coincide with that of S. It turned out that for \( n \geq 3 \) there are arrangement of horoballs that produce local density in S higher than \( d_n(\infty) \). Of course, these configurations are only locally optimal and cannot be extended to the entire hyperbolic space. But their existence shows that there might also be arrangements of mutually tangent balls of finite radius \( r \) centered at the vertices of a simplex with density in the simplex higher than \( d_n(r) \).

We recall that for a packing of circles of radius \( r \) in a plane of constant curvature the density of each circle in its Dirichlet cell, as well the density of the circles in each Delone cell, is at most \( d_2(r) \). It appears that in higher-dimensional hyperbolic space \( d_n(r) \) is an upper bound only for the local density in the Dirichlet cells and not for the density in the Delone simplices, although no explicit example is known. Also, it is an interesting open question whether, for some \( n \geq 3 \), in \( n \)-dimensional Euclidean or spherical space the density of a packing of balls of radius \( r \) in the Delone simplices can exceed \( d_n(r) \).

In 3-dimensional hyperbolic space there is a remarkable tiling \( \{6,3,3\} \) whose cells are degenerate Euclidean polyhedra \( \{6,3\} \), circumscribed about horoballs. The horoballs inscribed in the cells of this tiling form a packing while the horoballs circumscribed about the cells form a covering. Coxeter [1954] calculated the density of the horoballs in the cells and found that it is

\[
d_3(\infty) = \left( 1 + \frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} - \ldots \right)^{-1} \approx 0.853
\]

in the case of packing, and

\[
D_3(\infty) = \left( 1 - \frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{8^2} + \ldots \right)^{-1} \approx 1.280
\]

in the case of covering (see also Zeitler [1965] for the case of covering). Of course, these densities are the same as the density of the respective horoballs in the asymptotic tetrahedral cells of the dual tiling \( \{3,3,6\} \), that is, the bound proved by Böröczky for packings and the still conjectured corresponding tetrahedral density bound for coverings. Also, we have \( \lim_{r \to \infty} d_3(r) = d_3(\infty) \) and \( \lim_{r \to \infty} D_3(r) = D_3(\infty) \).

Florian (see Böröczky and Florian [1964]) showed that in hyperbolic space the tetrahedral density bound \( d_3 = d_3(r) \) is a strictly increasing function of \( r \). Therefore, for an arbitrary packing of congruent balls of finite or infinite radius in hyperbolic space, the density of each ball in its Dirichlet cell is at most \( d_3(\infty) \). We can therefore say that in 3-dimensional hyperbolic space the packing of the balls inscribed in the cells of the tiling \( \{6,3,3\} \) is the densest among all packings with congruent balls. It is conjectured that \( d_n(r) \) is a strictly increasing function of \( r \) for all \( n \). Marshall [1999] gave a partial verification of this conjecture by proving that it holds for sufficiently large values of \( n \). Kellerhals [1995] proved that \( d_n(r) > d_{n+1}(r) \) for all \( r > 0 \).
Szirmai [2005, 2007, 2018] and Kozma and Szirmai [2012, 2015, 2020] determined the minimum density of certain classes of periodic packings of horoballs in low dimensions.

2. Hyperspheres

Besides spheres of finite radius and horospheres, there is a third type of “sphere” in hyperbolic space, namely hyperspheres. In n-dimensional hyperbolic space a hypersphere is the set of points of the space that are at the same distance r from an (n−1)-dimensional hyperplane. A hypersphere consists of two disjoint surfaces that bound a connected component of the space called the hyperball of radius r. The study of packing and covering by hyperballs was initiated by Vermes. In 1979 he investigated packings of congruent hypercircles of radius r, and gave an upper bound for the local density of the hypercircles in the cells of the dual subdivision of the plane into Dirichlet cells. His bound is sharp for all values of r, increases monotonously, and its limit at infinity is \( \frac{3}{\pi} \), the density of the densest packing of horocircles. Unaware of Vermes’s result, Marshall and Martin [2003] discovered the same bound.

Przeworski [2013] proved an upper bound for the density of packings of congruent hyperballs. His bound is an analogue of the simplex bound for ball packings. In the proof he uses Delone cells for the base-hyperplanes of the hyperballs. These cells, which he introduced in his paper [2012], are truncated ultraideal simplices. The truncation faces are the base-hyperplanes of the hyperballs, and they are perpendicular to all non-truncation faces that intersect them. The maximum density of the hyperballs in the truncated simplices occurs for a regular simplex. Hence he gets the following: In \( H^n \), let \( d_n(r) \) denote the density of \( n + 1 \) pairwise touching hyperballs in the truncated simplex bounded by the base hyperplanes of the hyperballs and the hyperplanes orthogonal to the base planes though the touching points of the hyperballs. Then the density of any packing of hyperballs of radius r within every Delone cell, as well as in every Dirichlet cell, is at most \( d_n(r) \). The same result was proved by Miyamoto [1994] in another context. For 3-dimensional space an alternative decomposition into truncated tetrahedra was suggested by Szirmai [2019].

Vermes [1981] described for every \( r > 0 \) a class of regular packings of hypercircles of radius r, calculated the density of each of them and determined the minimum density for all values of r. It turned out that the minimum density \( \Theta(r) \) is an increasing function of r with limit value \( \sqrt{12}/\pi \) as \( r \to \infty \). In 1988 he considered general coverings of hypercircles of radius r subject to the condition that there exists a number \( 0 < \rho < r \) such that the hypercircles of radius \( \rho \) around the same base lines form a packing. This condition guarantees that no point of the plane is covered by infinitely many members of the covering. For the density of coverings of the plane by hypercircles of radius r satisfying the above property, Vermes [1988] gave a lower bound that is sharp for the thinnest regular coverings. The infimum of the density of coverings by hyperspheres is \( \sqrt{12}/\pi \), attained by the thinnest covering of horospheres.

A hyperball is the parallel body of an \( (n-1) \)-dimensional hyperplane. It is natural to consider packings of parallel bodies of lower dimensional planes as well. Marshall and Martin [2005] and Przeworski [2004, 2006a] proved upper bounds for
the density of packings of tubes, that is of parallel bodies of lines in 3-dimensional hyperbolic space.

3. Solid arrangements

As one of the possible substitutes for the notions of densest packing and thinnest covering, L. Fejes Tóth [1968] defined solidity of packings and coverings. We say that a circle packing (covering) is solid if no finite subset of the set of circles can be rearranged so that the resulting packing (covering) is not congruent to the original one. Roughly speaking, if we remove any finite number of arbitrarily chosen circles from a solid circle packing (covering), and we wish to place them again so that we still obtain a packing (covering), then they have to be returned to their original places.

It follows from a theorem of Imre [1964] that the circles inscribed in the faces of the tiling \{k, 3\} (\(k = 2, 3, \ldots\)) form a solid packing, and the circles circumscribed about the faces of the tiling \{k, 3\} (\(k = 3, 4, \ldots\)) form a solid covering. Moreover, L. Fejes Tóth [1968] conjectured that the circles inscribed in, and the circles circumscribed about the faces of every three-valent Archimedean tiling form a solid packing and a solid covering, respectively. On the other hand, he conjectured that the face-incircles and the face-circumcircles of a more than trihedral uniform tiling never form a solid arrangement. In many cases this has been confirmed, see L. Fejes Tóth [1968], Heppes [1992], Heppes and Kertész [1997], Florian [1999, 2000, 2001a, 2001b, 2007], Florian and Heppes [2000] and G. Fejes Tóth [1974]. In particular, both conjectures were confirmed for the face-incircles of all spherical and Euclidean uniform tilings. As a particularly appealing example we mention the “football”, i.e., the tiling of the truncated icosahedron \{5, 6, 6\}, whose face-incircles form a solid packing. In some cases, e.g., for the incircles of the tilings \{8, 8, 4\} and \{4, 4, n\} for \(n \geq 6\), in addition to the solidity of the packing it is shown that the arrangement has greatest possible density among all packings with any collection of circles of the given radii (see Florian [2001a], Florian and Heppes [1999], Heppes [2000, 2003b] and Heppes and Kertész [1997]).

We emphasize the following special case of the above conjecture: Augmenting the incircles of a regular (spherical, Euclidean or hyperbolic) tiling by circles inscribed in the holes results in a solid circle packing. This indeed occurs in the case of the Euclidean tilings \{6, 3\} and \{3, 6\}, since the inscribed circles in \{6, 3\} form a solid packing by themselves, and augmenting the incircles of \{3, 6\} results in the incircles of \{6, 3\}. For the case of the third regular Euclidean tiling, namely \{4, 4\}, the solidity of the corresponding tiling was proved by Heppes [1992] using the idea of weighted density.

The solidity of a packing \(P\) follows if we can show that in every packing of circles with the given radii the local density in the Delone triangles cannot exceed the density of the circles of \(P\) in the Delone triangles. For the face incircles of the tiling \{4, 4, 8\} this is not true. Heppes observed that the solidity of \(P\) follows if some weights can be assigned to the circles so that the above statement holds for the weighted density. Using appropriate weights he could prove, besides the incircles of the tiling \{4, 4, 8\}, the solidity of several other packings. The work of Heppes inspired further research. Weighted density of packings has also been studied for its own sake (see Hárs [1990, 1992]).
A notion weaker than solidity was introduced and investigated by A. Bezdek, K. Bezdek and Connelly [1995, 1998]. A packing or a covering is uniformly stable if there is an $\varepsilon > 0$ such that no finite subset of the arrangement can be rearranged so that each member is moved by a distance less than $\varepsilon$ and the rearranged members, together with the rest, form a packing or covering, respectively, different from the original arrangement. Using techniques from rigidity theory they proved uniform stability of certain packings.

A solid circle packing is strongly solid if it remains solid even after any one of the circles is removed. L. Fejes Tóth conjectured that the incircles of the faces of the tilings $\{4, 3\}$ and $\{5, 3\}$ are strongly solid. A. Bezdek [1979] confirmed the conjecture for all $p \geq 8$. The two remaining cases, $p = 6$ (in the Euclidean plane) and $p = 7$ (in the hyperbolic plane) remain unsolved. A partial result supporting the case $p = 6$ of the conjecture is due to Barágy and Dolbilin [1988]. They proved that the packing obtained by removing one circle from the densest lattice packing of unit circles is uniformly stable with $\varepsilon = 1/40$. Another result supporting the conjecture was given by Heppes [1994]. He proved that the hexagonal tiling $\{6, 3\}$ is strongly translationally solid. This means that in the packing of hexagons arising by omitting one tile from the hexagonal tiling, every rearrangement of a finite number of hexagons by translations results in a packing congruent to the original one.

L. Fejes Tóth [1980] strengthened Bezdek’s result in the following sense. For any packing of congruent circles, a packing obtained from it by removing a finite number $k$ of circles is called a $k$-truncation of the packing. We say that a circle packing is solid of order $k$ if every $k$-truncation has the property that when finitely many additional circles are removed from it and then placed back in any place where there is room for them, the resulting packing is again some $k$-truncation of the original packing. The grade of saturation of a packing of congruent circles is the maximum number $g$ such that every circle congruent to, but not identical with, a circle of the packing intersects at least $g$ circles of the packing. L. Fejes Tóth [1980] proved that for $p \geq 8$, the order of solidity of the face-incircles of $\{p, 3\}$ equals the grade of saturation minus 2. In particular, it follows that the order of solidity of these packings becomes arbitrarily large as $p \to \infty$.

4. Completely saturated packings and completely reduced coverings

G. Fejes Tóth, G. Kuperberg and W. Kuperberg [1998] introduced the notion of a $k$-saturated packing ($k = 1, 2, \ldots$) as a packing with congruent copies of a set $K$, such that deleting $k - 1$ members of the packing never creates a void large enough to pack in $k$ copies of $K$. Similarly, a covering with congruent copies of $K$ is $k$-reduced if deleting $k$ members of the covering always creates a void too large to be covered by $k - 1$ copies of $K$. A packing that is $k$-saturated for every $k$ is completely saturated. Completely reduced coverings are defined similarly. Obviously, in Euclidean spaces a completely saturated packing with congruent copies of a convex body $K$ must have density $\delta(K)$; similarly, a completely reduced covering with congruent copies of $K$ must be of density $\vartheta(K)$. This suggested that the notions of completely saturated packings and completely reduced coverings could serve in hyperbolic spaces as a substitute for packings and coverings of maximum
and minimum density, respectively. The question of existence of completely saturated packings and completely reduced coverings for every convex body turned out to be non-trivial by any means, and it was answered affirmatively for Euclidean spaces in G. Fejes Tóth, G. Kuperberg and W. Kuperberg [1998] and for hyperbolic spaces by Bowen [2003a]. The notion of completely reduced coverings found applications in approximation theory, see Hinrichs and Richter [2004].

5. A probabilistic approach to optimal arrangements and their density

Bowen and Radin [2003, 2004] proposed a probabilistic approach to define optimal arrangements and their density, especially useful in hyperbolic geometry. Their main idea is that with a properly defined probability measure on the set of all packings with copies of a body $K$, the density of a specific packing is the same as the probability that its randomly chosen congruent copy contains the origin. Below we sketch some details.

Let $S$ denote an $n$-dimensional space of constant curvature, namely the Euclidean $n$-space, the $n$-sphere, or the $n$-dimensional hyperbolic space. Instead of studying individual packings in $S$, Bowen and Radin consider the space $\Sigma_K$ consisting of all packings of $S$ by congruent copies of a body $K$. A suitable topology derived from the Hausdorff metric on $\Sigma_K$ is introduced which makes $\Sigma_K$ compact and makes the natural action of the group $\mathcal{G}$ of rigid motions of $S$ on $\Sigma_K$ continuous. We consider Borel probability measures on $\Sigma_K$ invariant under $\mathcal{G}$. For such an invariant measure $\mu$ the density of $\mu$, $d(\mu)$, is defined as $d(\mu) = \mu(A)$, where $A$ is the set of packings $P \in \Sigma_K$ for which the origin of $S$ is contained in some member of $P$. It follows easily from the invariance of $\mu$ that this definition is independent from the choice of the origin.

A measure $\mu$ is ergodic if it cannot be expressed as the positive linear combination of two other invariant measures. The relationship between density of measures and density of packings is established by the following theorem.

Suppose that $\mu$ is an ergodic invariant Borel probability measure on $\Sigma_K$. If a packing $P$ is chosen $\mu$-randomly, then with probability 1, for every $p \in S$,

$$d(\mu) = \lim_{\lambda \to \infty} \frac{1}{V(B_\lambda(p))} \sum_{P \in \mathcal{P}} V(P \cap (B_\lambda(p))).$$

The packing density $\delta(K)$ of $K$ can then be defined as the supremum of $d(\mu)$ for all ergodic invariant measures on $\Sigma_K$. A packing $\mathcal{P} \in \Sigma_K$ is optimally dense if the closure of its orbit under $\mathcal{G}$ is the support of an ergodic invariant measure whose density reaches this supremum.

It is shown in Bowen and Radin [2003, 2004] that, for every $K$, an ergodic invariant measure $\mu$ with $d(\mu) = \delta(K)$ exists whose support contains a set of full $\mu$-measure of optimally dense packings. The existence of completely saturated packings with copies of $K$ was proved in the same way as for optimally dense packings. In fact, with probability 1, a $\mu$-randomly chosen optimally dense packing is completely saturated.

Bowen and Radin [2003, 2004] proved several statements justifying that they proposed a workable notion of optimal density and optimally dense packings. In particular, in the Euclidean case, that is $S = \mathbb{E}^n$, the Bowen-Radin notion of $\delta(K)$ coincides with the corresponding traditional notion of the packing density of $K$. 
The probabilistic approach of Bowen-Radin can be naturally applied to coverings, or more generally, to locally finite arrangements of congruent copies of $K$. The definition of density in such setting, however, requires a modification: the “measure” $\mu(A)$ should be replaced by another quantity that takes into account the multiplicity with which portions of the areas of the union of $A$ are covered. More precisely, $\mu(A)$ should be replaced with a combination of weighted measures, assigning the weight of $w$ to the regions in $S$ consisting of points covered exactly $w$ times by the given arrangement of copies of $K$.

The advantage of this probabilistic approach is that it focuses on periodic packings and neglects pathological packings such as Böröczky’s example. Another important feature of this approach is that we get bounds for the packing density defined in this framework by considering the local density in the Dirichlet or the Delone partitions.

Concerning packings of balls in hyperbolic $n$-space, it was shown by Bowen and Radin [2003] that there are only countably many radii admitting an optimally dense periodic packing of balls. Thus, for most radii $r$, no periodic packing is densest. For the hyperbolic plane, Bowen [2003b] proved that the packing density of circles of radius $r$ is a continuous function of $r$, and it is the supremum of densities of periodic packings.

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